Lecture 22: Towards Mackey's criterion

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Thm: (Frobenius reciprocity): Let $H \leq G$.

$$\langle \varphi, \mathrm{res}\,\psi \rangle_H = \langle \mathrm{Ind}\,\varphi, \psi \rangle_G$$

where φ is a class function on H & ψ is a class function on G.

The proof reduces to the following adjointness statement using linearity & orthogonality of charac.

(*) Let $\rho' : G \to GL(V')$ be another repr. Let $f : W \to V'$ be an H-equivariant map. Then f extends uniquely to a G-equivariant map $F : V \to V'$. Here $V = \mathrm{Ind}\,W$

$$\mathrm{Hom}^H(W, \mathrm{Res}_H V') =: \boxed{\mathrm{Hom}^H(W, V') \cong \mathrm{Hom}^G(V, V')}$$

(*) $$\left(\mathrm{Ind}_H^G \varphi\right) \cdot \psi = \mathrm{Ind}_H^G\left(\varphi \cdot \mathrm{Res}_H^G \psi\right)$$

(follows for characters from $(\mathrm{Ind}_H^G W) \otimes V' \cong \mathrm{Ind}_H^G (W \otimes \mathrm{Res}_H^G V')$

Use linearity for arbitrary class function)

Cor: Let $H \leq G$, W be an irred. H-repr & E be an irred G repr. Then the number of copies of E in $\mathrm{Ind}_H^G W$ is same as the number of " of W in $\mathrm{Res}_H^G E$ (i.e. E as an H-repr).

Pf: $$\#\text{ copies of } E \text{ in } \mathrm{Ind}_H^G W = (\chi_E \mid \chi_{\mathrm{Ind}_H^G W})$$

($\because E$ is irred)

$$= \langle \chi_E, \mathrm{Ind}_H^G \chi_W \rangle_G$$

$$\overset{F.R.}{=} \langle \mathrm{Res}_H^G \chi_E, \chi_W \rangle_H$$

$$= (\chi_W \mid \chi_{\mathrm{Res}_H^G E})$$

$$= \#\text{ copies of } W \text{ in } \mathrm{Res}_H^G E$$

since W is irred.

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(*) Let G be group, H & K be subgrps of G. Let W be an H-mod.

What is $\mathrm{Res}^G_K(\mathrm{Ind}^G_H W)$?

Let $V = \mathrm{Ind}^G_H W$ then

$$V = \bigoplus_{x \in G/H} xW$$

$\{KsH \mid s \in G\}$ be the collection of double cosets.

For $s \in G$, $KsH = \{xsy \mid x \in K \ \& \ y \in H\}$

$KsH \cap Ks'H = \phi$ or $KsH = Ks'H$ for $s, s' \in G$.

If $s' \in KsH \Rightarrow s' = x_0 s y_0$ for $x_0 \in K$ & $y_0 \in H$

$\Rightarrow Ks'H \subseteq KsH$

$$xs'y = \underbrace{x x_0}_{\in K} s \underbrace{y_0 y}_{\in H} \quad \text{for } x \in K \ \& \ y \in H$$

Hence $Ks'H = KsH$

$$KsH = \bigcup_{x \in K} xsH$$

(Note $xsH = x'sH$ if $(x's)^{-1}xs \in H$ i.e. $s^{-1}x'^{-1}xs \in H$, $x'^{-1}x \in sHs^{-1}$), $x, x' \in K$

$H_s = sHs^{-1} \cap K$ for $s \in G$.

$\rho^s : H_s \to GL(W)$ $\quad \rho^s(x) = \rho(sxs^{-1})$

$x \cdot w := \underbrace{sxs^{-1}}_{\in H} \cdot w$ for $x \in H_s$

Let W_s denote the underlying v.s for ρ^s

Prop $\quad \mathrm{Res}^G_K \mathrm{Ind}^G_H W \cong \bigoplus \mathrm{Ind}^K_{H_s} W_s \cong \bigoplus_{s \in S} \mathrm{Ind}^K_{H_s} W_s$

sum over double cosets. $K \backslash G / H$

where S is a collection of representatives of the double coset $\{KsH \mid s \in G\}$

$= K \backslash G / H$

Pf: Let $V = \mathrm{Ind}^G_H W$

$$V = \bigoplus_{x \in G/H} xW$$

For $s \in S$, let

$$V(s) = \bigoplus_{\substack{x \in G/H \\ x \subseteq KsH}} xW \subseteq V$$

Note if $y \in K$

$$y \cdot V(s) = \bigoplus_{\substack{x \in G/H \\ x \subseteq KsH}} yxW = \bigoplus_{\substack{yx \in G/H \\ yx \subseteq KsH}} yxW = V(s)$$

So $V(s)$ is K-stable subrep of $\mathrm{Res}_K V$

Also $s' \in S$ & $s' \neq s$ then

$V(s') \cap V(s) = 0$. So

$\mathrm{Res}^G_K V = \bigoplus_{s \in S} V(s)$ and this is a K-reps.

So enough to show $V(s) \cong \mathrm{Ind}^K_{H_s} W_s$