Combinatorics

24 NOV 21

Lecture 14 — Combinatorial Geometries

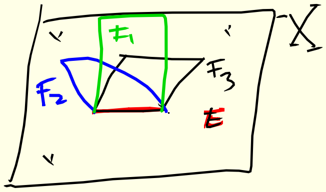
Thm :- $N(n) \geq k$ iff $\exists$ an $(n, k+2)$-net.

Remark :- Easy exercise to check that $(n, n+1)$-net is an affine plane of order n. It exists iff a projective plane of order n exists.
$\therefore N(n) = n-1$. iff $\exists$ a proj. plane of order n

Ref: Chapter 23 (van Lint & Wilson)

Def^n :- A combinatorial geometry is a pair $(X, \mathcal{F})$ where X is a set of "points" & $\mathcal{F}$ is a family of subsets of X whose elements are called 'flats' such that

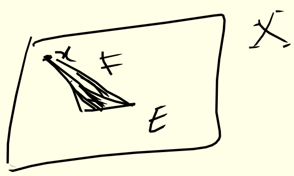
1. $\mathcal{F}$ is closed under (pairwise) intersection.
2. $(\mathcal{F}, \text{inclusion})$ is POSET (partially ordered set) without infinite chains.
   (chain is a totally ordered subset of a poset.)
   ex. $\mathcal{P}(\mathbb{N}) = \{$
   $\{1\} \subseteq \{1,2\} \subseteq \{1,2,3\} \subseteq \{1,2,3,307\}, \subseteq$
3. $\mathcal{F}$ contains empty set ϕ, whole set X, & $\{x\}$ $\forall x \in X$.
4. For every flat $E \in \mathcal{F}$; $E \neq X$, that flats that "cover" E partition $X \setminus E$.



we say F "covers" E if $F \supsetneq E$ & there is no flat between F & E.
i.e. if $\exists\, G \in \mathfrak{F}$ s.t. $F \supseteq G \supseteq E$ then $G = E$ or $G = F$.

Note that if the flats that cover E contain every elt of $X \setminus E$
then they must partition because
$$F_1 \supseteq F_1 \cap F_2 \supseteq E \Rightarrow F_1 \cap F_2 = F_1 \text{ or } E.$$
for all F_1, F_2 covering E

Another way of saying (4) is $\forall\, x \notin E$ & $E \in \mathfrak{F}$, $\exists!$
flat $F \in \mathfrak{F}$ s.t. F covers E & $F \ni x$.



Remark: Note that no infinite chains
$\Rightarrow \mathfrak{F}$ is closed under arbitrary intersection.

Examples

(1) Linear space $(X, \mathcal{L})$, any two points are in a unique line & $|L| \geq 2$ $\forall\, L \in \mathcal{L}$.

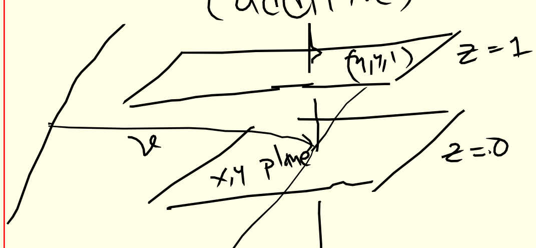
ϕ, X, singletons & L s.t. $L \in \mathcal{L}$.

(2) $S(t, k, v)$ (i.e. $\lambda = 1$) is a comb. geometry $t \geq 2$. $X = \mathcal{P}$ set of pts.

$\mathfrak{F} = \{\phi, \mathcal{P}, \text{all subsets of size} \leq t-1, \{B ; B \in \mathcal{B}\}\}$.

(3) $V = \mathbb{F}^n$ for a field $\mathbb{F}$.

An affine subspace of V is either ϕ or a (additive) coset of a subspace of V. ($x + W$, $x \in V$, W subspace)



(V, all affine subspaces) is a combinatorial geometry.

Denoted by $AG_n(\mathbb{F})$; $\mathbb{F} = GF(q)$ we write $AG_n(q)$.

(4) Projective Geometry $\mathbb{P}G_n(\mathbb{F})$.

$V = \mathbb{F}^{n+1}$; X = all 1-dimensional subspaces of V.

$= V \setminus \{0\} / \mathbb{F}^*$.

$\mathbb{F}^*$ acts on $V \setminus \{0\}$; $(\alpha, v) \mapsto \alpha . v$.

$\forall$ linear subspace W associate a subset of X denoted by F_W defined as $F_W := \{x \in X \mid \text{the 1-diml subspace corr. to } x \text{ is contained in } W\}$

Ex. Check that (3) & (4) are combi. geometries.

Def^n:- $Y \subseteq X$, $(X, \mathcal{F})$ a geometry then $(Y, \mathcal{F}_Y)$ is also a geometry where $\mathcal{F}_Y = \{F \cap Y \mid F \in \mathcal{F}\}$.

called the subgeometry on Y.

(Recall: $v \neq \sum \alpha_i w_i$ for any $\alpha_i \in \mathbb{F}$ then v is independent of w_i.)

Def^n:- Let $S \subset X$, $(X, \mathcal{F})$ - geometry then $\overline{S}$, the closure of S, is the smallest flat containing S. ie $\overline{S} = \bigcap_{\substack{F \in \mathcal{F} \\ F \supseteq S}} F$.

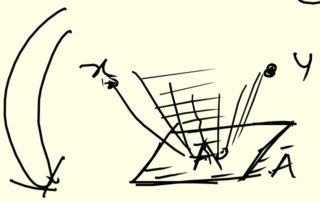
Every flat is closed!

Def^n:- A subset $S \subseteq X$ is said to be 'independent' iff $\forall x \in S$; $x \notin \overline{S \setminus \{x\}}$.

Example 3 non-collinear points are 'independent' in AG_n

Lemma let x, y be points of a geometry

1. If $x \notin \bar{A}$ but $x \in \overline{A \cup \{y\}}$ then $y \in \overline{A \cup \{x\}}$



$\overline{A \cup \{y\}} = \overline{A \cup \{x\}}$ (exchange lemma)

$\overline{A \cup \{y\}} = \overline{A \cup \{x\}}$

2. S is independent, $x \notin \bar{S}$ $\Rightarrow$ $x \cup S$ is independent.

3. if $F = \bar{S}$ then $F = \bar{A}$ for any maximal independent subset of S.

proof $\mapsto$ Exercise

— x — x — x —

Next: Lattices in set theory & "Certain" lattices are combinatorial geometries.