Combinatorics 3 DEC 21

Lecture 17

Greene's Theorem

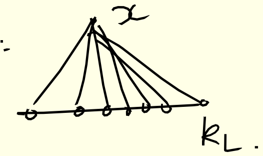
If $(X, \mathcal{F}s)$ is a finite combinatorial geometry ~~of~~ rank n then the no. of $(n-1)$-flats are at least as many as the no. of rk 1 flats. (# hyperplanes $\geq$ # pts)

(Recall: Erdös-DeBruijn thm, Fisher's inequality)

Exact copy of Erdös-DeBruijn theorem's proof.

Pf. Exact copy of Erdös-DeBruijn theorem's proof.

There:

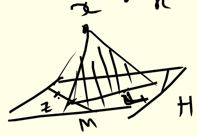


$r_x \geq k_L$. $\forall x \notin L$.

Here: $r_x \geq k_H$ $\forall x \notin H$

r_x = # hyperplanes cont. x.

k_H = # pts in H.



$\boxed{\text{Assume } b \leq v}$ where b = # hyperplanes & v = # pts of X.

then we have: $1 = \sum\sum_{S=\{(x,H) \mid x \notin H\}} \frac{1}{v(b-r_x)} \geq \sum\sum_{S} \frac{1}{b(v-k_H)} = 1$

$\Rightarrow$ equality holds & we get $\boxed{v = b \ \& \ r_x = k_H}$

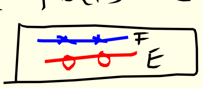
QED

E, F two flats then $rk(E) + rk(F) \geqq rk(E \vee F) + rk(E \cap F)$ in general.

Def^n:- A geometry is called modular iff equality holds in the semi-modular law for any two flats E & F.

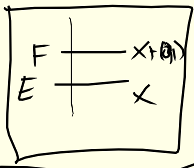
ie $\forall E, F \in \mathcal{F}s$; $rk(E) + rk(F) = rk(E \vee F) + rk(E \cap F)$.

Remark :- If $(X, \mathcal{F}s)$ is modular of rk. 3 then any two rk 2 flats (lines) must meet.



$X \Rightarrow$ ($\{o, o, *, *\}$ is ind. in X)

$\Rightarrow rk\,X = rk(E\vee F)$ & $\underbrace{rk\,E + rk\,F}_{4} > \underbrace{3}_{rk(E\vee F)} + 0 = rk(E\cap F)$ contradiction.

ex. 

$A^2(\mathbb{F})$ is not modular.

? ② If $X = E\cup F$ (proper union) & X - modular, then $E\cap F = \phi$.

(Is this true?)

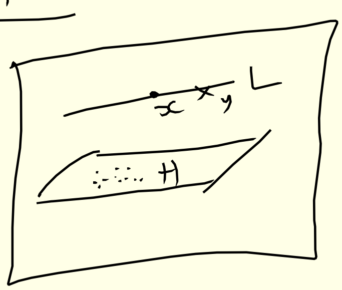
Def^n :- A projective geometry is a modular geometry whose point set cannot be expressed as a union of two proper flats. (ie $X = E\cup F$; $E, F \in \mathcal{F}_s$ $\Rightarrow E = X$ or $F = X$.)

Def^n :- Geometry is said to be connected if it is not a union of two proper flats.

Theorem (Veblen? 1907) :- Any projective geometry of rank $n \geq 4$ is isomorphic to $PG_n(\mathbb{F})$ for some division ring $\mathbb{F}$.

Lemma :- A geometry is modular iff any line and any hyperplane meet non-trivially.

pf. :- ① If $\exists\, L$ - line; H - hyperplane such that $L\cap H = \phi$, then



$$\underbrace{rk\,L + rk\,H}_{(2)+(n-1)} > \underbrace{rk(L\vee H)}_{n} + \underbrace{\cancel{rk}(L\cap H)}_{\phi} \xrightarrow{} 0$$

$\Rightarrow$ If X is modular $L\cap H \neq \phi$ $\forall$ line L & hyp. H.

(2) Assume that $(X, \mathcal{F}_1)$ is a geo. of rk. n s.t. any rk 2 flat L & any rk$(n-1)$-flat H meet.

TPT X is modular.

Induction on rank of X.

If $rk(X) = 3$ then lines are hyperplanes

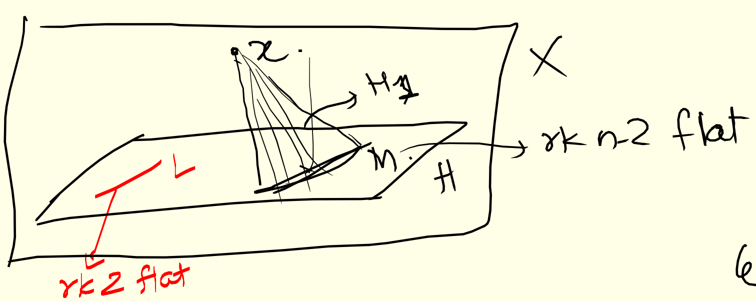
ϕ, $\{x\}$, Lines. are the only flats & case by case analysis shows that $\forall E, F \in \mathcal{F}_1$ equality holds in the semi-modular law.

Assume that the lemma is true for all $(Y, \mathcal{F}_1')$ with $rk Y < rk X$.

Let $(X, \mathcal{F}_1)$ be st. $L \cap H \neq \phi$ $\forall$ lines L, hyperplanes H.

Claim :- look at the induced geometry on H for any hyperplane H. It is modular.

Enough to check $L \cap M \neq \phi$ $\forall$ hyperplanes in H.
L line in H



If not, $\exists$ L, M s.t. $L \cap M = \phi$ & $rk L = 2$, $rk M = n-2$.

Let $x \in X \setminus H$.

$H_1 = M \vee \{x\}$. Then H_1 is hyperplane
the "unique flat that covers M & contains x.

then $H_1 \cap H = M \Rightarrow H_1 \cap L = \phi$. contradiction!

$\Rightarrow M \cap L \neq \phi$. $\Rightarrow H$ is modular (by ind. hypothesis)

Claim :- X is modular.

let $E, F \in \mathcal{F}_1$. If $E \vee F \neq X$. Then $E \vee F \subset H$ for some hyperplane H.

(extend base of $E \vee F$ to that of X &

remove one of the added points!)

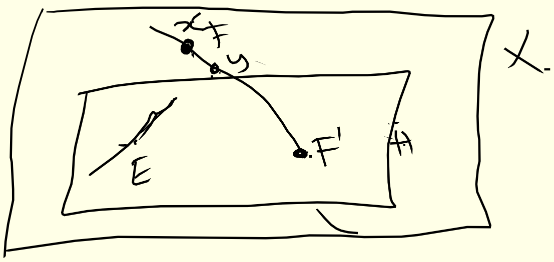
$\therefore$ E, F are flats in H which is modular!

$\therefore$ $rk\,E + rk\,F = rk(E \vee F) + rk(E \cap F)$.

$\therefore$ Assume that $E \vee F = X$. ; If E or $F = X$ then modular law holds!

$\therefore$ assume E, F are proper.

Let H be a hyperplane in X s.t. $E \subset H$. & $F' = F \cap H$.



Claim $rk\,F' = rk\,F - 1$.

if not then $H \vee \{x\} = X$.

$\Rightarrow$ $F' \vee \{x\} = F$.

If not $\exists\, y \notin F' \vee \{x\}$.

$\Rightarrow$ $\overline{\{x, y\}} \cap H = \phi$

because $\overline{x, y} \subset F$.

& $\overline{\{x, y\}} \subset H \vee \{x\}$.

$\Rightarrow$ $y \in \overline{H \vee \{x\}}$

If $rk\,F' \leq rk\,F - 2$ then let $\{x, y\} \subseteq F \setminus F'$ be independent. Then $\overline{\{x, y\}} \subset F$. & $\overline{x, y} \cap H$ must contain z.

(since $L \cap H \neq \phi$ $\forall$ lines in X)

$\Rightarrow z \in F'$.

$\overline{\{x, z\}} = \overline{\{x, y\}}$. as rk is well defined.

$\Rightarrow$ $y \in \overline{F' \cup x}$ $\Rightarrow$ y is not independent ~~from~~ from $B \vee \{x\}$ where B is basis of F'.

Let B be a basis of F' & extend it to a basis of F.

let (x, y) be two elts outside the basis of F' in this base.

$\Rightarrow$ $rk\,F' = rk\,F - 1$.

We know H is modular. $\therefore$ $rk\,E + rk\,F' = rk(E \vee F') + rk(E \cap F')$

$$rk\,E + rk\,F - \cancel{1} = rk(E \vee F) \cancel{-1} + rk(E \cap F)$$

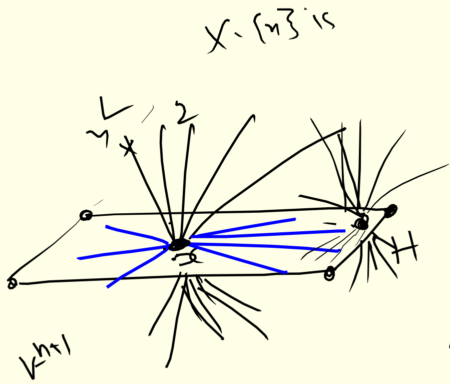
$\Rightarrow$ modular law holds for E & F in X.

Q. L a line, H hyperplane $|L \cap H| = ?$

if $L \subset H$ then $L \cap H = L$.

if $L \not\subset H$ & if $|L \cap H| \geq 2$

then $\{x, y\} \subseteq L \cap H$



— x — x — x —

In affine geometry all cosets of subspaces are flats.

* Proj. geometry pts are lines in vector space & lines are planes. hyperplane in $PG_n(\mathbb{F})$ are all lines in n-diml subspace of $\mathbb{F}^{n+1}$.

— x — x — x — x —

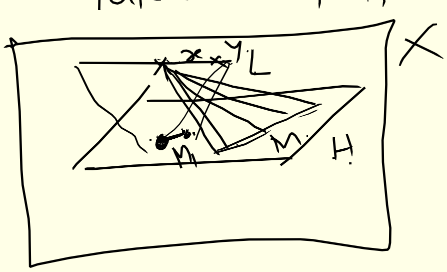
Remark :- (Relationship between this lemma & Greene's theorem)

$\gamma_x \geq k_H \quad \forall x \notin H$ $\quad \gamma_x = \#$ hyperplanes thru' x

$k_H = \#$ pts in H. H hyperplane.

Assume equality holds in Greene's theorem $\forall x \notin H$. then X is modular. $\gamma_x = k_H$ then we need to prove that $L \cap H \neq \phi$ $\forall$ line L & hyperplane H.

take $L \not\subset H$. assume $L \cap H = \phi$.



take $x \in L$.

then every hyperplane must be obtained as $M \vee \{x\}$ for some M a hyperplane in H.

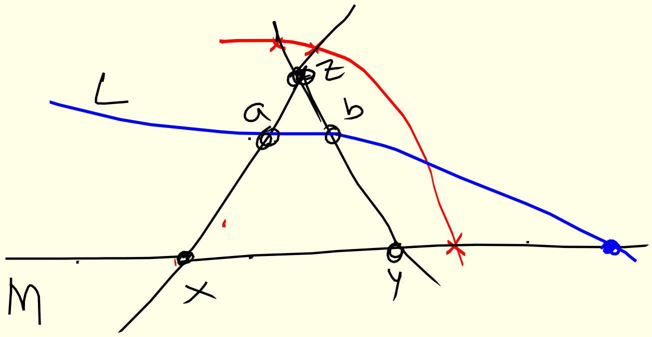
But if $L \cap H = \phi$ then given any rk $n-3$ flat M_1 in H, we $L \vee M_1$ has rk $n-1$.

$\Rightarrow \gamma_x > k_H$. contradiction.

$\Rightarrow L \cap H \neq \phi \Rightarrow X$ is modular.

Remark :- If $(X, \mathcal{F})$ is modular, then the linear space formed by pts & rk 2 flats (lines) will have following property :

(P) : "A line L that meets two sides a triangle (in distinct points) also meets the third line"



pf of (P) . $\overline{\{x,y,z\}} \supset \{a,b\}$.

$\Rightarrow \overline{\{x,y,z\}} \supset \underbrace{\overline{\{a,b\}}}_{=L}$

$\Rightarrow$ L & M are contained in a rk.3 flat $\overline{\{x,y,z\}}$.

$\therefore$ Modular law $\Rightarrow L \cap M \neq \phi$.

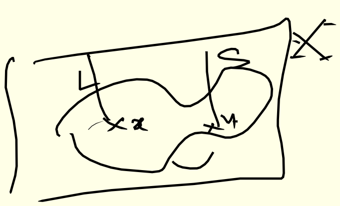
(P) is called the

Pasch Axiom

Theorem :- A finite linear space $(X, \mathcal{L})$ for which Pasch Axiom hold consists of points & lines of ~~some~~ modular geometry.

Pf :- We need to get entire $\mathcal{F}$, family of flats from just rank 2 flats

(*) $S \subset X$ is a flat iff for any line $L \in \mathcal{L}$ with $|L \cap S| \geq 2$ we must have $L \subseteq S$.



(something like $S \subset V$ is a subspace iff given any $x, y \in S$ $\langle x, y \rangle \subset S$. $\alpha x + \beta y \in S$)

Claim $(X, \mathcal{F})$ where $\mathcal{F} = \{ S \mid S \text{ satisfies } * \}$

is a modular geometry.

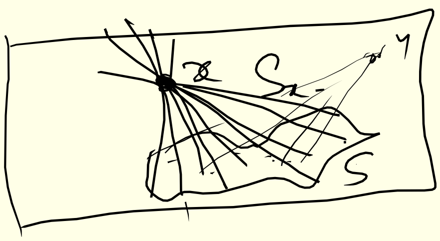
(1) Clearly $\mathcal{F}_S$ is closed under intersection. S_1, S_2 satisfy * then so does $S_1 \cap S_2$ (obvious)

(2) No infinite chain because X is finite.

(3) $\phi, \{x\}, X$ trivially satisfy (*)

(4) Given $S \in \mathcal{F}_S$ we need to show that $X \setminus S$ is partitioned by flats T that "cover" S.

let $x \notin S$ in X.

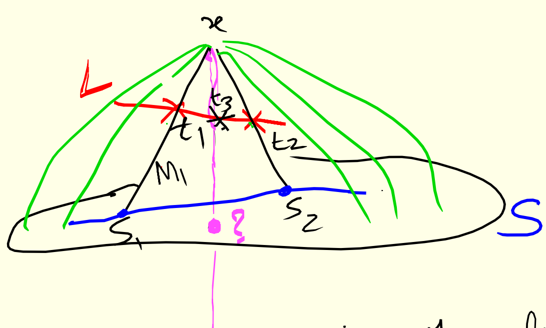


S_x = union of all lines thru' x that intersects S.

"cone over S with x as a vertex"

<u>Claim</u> S_x is a flat, ~~S_x~~ cover S & $S_x \cap S_y \in S$ if $x \notin S, y \notin S$.

(1) S_x is a flat? Let L be a line s.t. $|L \cap S_x| \geq 2$.



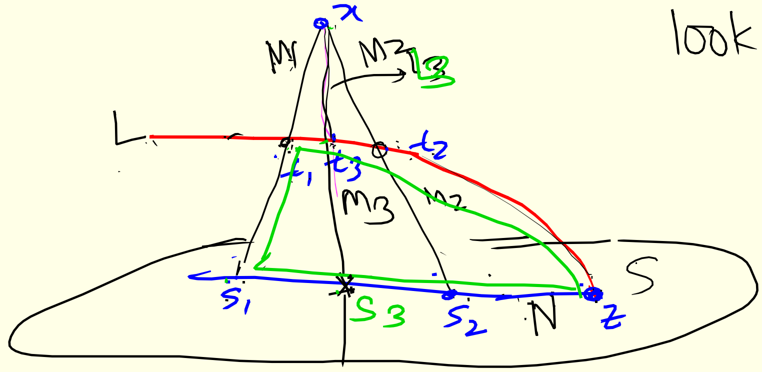
- if those two points lie in S then $L \subseteq S \subseteq S_x$.
- If two pts lie on the same line thru' x then that line is the line joining those two points & hence is in S_x.

* $L \cap S_x$ lie on two diff. lines thru' x & $t_1 \notin S$.

if $L = \{t_1, t_2\}$ then $L \subseteq S_x$!

$\therefore$ assume that $\exists$ $t_3 \in L \setminus \{t_1, t_2\}$.

If the line L_3 joining x & t_3 intersects S then $L_3 \subset S_x \Rightarrow t_3 \in S_3 \Rightarrow L \subseteq S_x$ & we are done.



look at the $\triangle$ formed by s_1, s_2, x

First L meets two sides of $\triangle$ formed by (x, s_1, s_2)

$\Rightarrow L \cap N \neq \phi$ by Pasch's axiom.

Let $z = L \cap N$.

Now look at the $\triangle$ formed by s_1, t_1 & z

the pt $x \in \overline{\{s_1 t_1\}}$, $t_3 \in \overline{\{t_1 t_2\}} = \overline{\{t_1, z\}}$

$\Rightarrow$ line joining x & t_3 must meet the line N joining s_1, s_2. but $N \subseteq S$.

$\Rightarrow$ line joining x & t_3 intersects S.

$\Rightarrow t_3 \in S_x$. $\Rightarrow L \subseteq S_x$.

$\Rightarrow S_x$ is a flat!

Now