Lecture 21: Frobenius reciprocity

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Let G be a group & $H \leq G$. Let V be a G-repr & $W \subseteq V$ an H-stable subspace. We say $V = \mathrm{Ind}_H^G W$ if

$V = \bigoplus_{i=1}^{r} g_i W$ where $g_1, \dots, g_r \in G$ are representatives of the left cosets of H in G/H. Note $r = [G:H]$

(*) Let $\rho' : G \to GL(V')$ be another repr. Let $f : W \to V'$ be an H-equivariant map. Then f extends uniquely to a G-equivariant map $F : V \to V'$.

$$\mathrm{Hom}^H(W, V') \cong \mathrm{Hom}^G(V, V')$$
$$\left(:= \mathrm{Hom}(W, V')^H\right)$$

(*) For W, W' H-repr

(*) $\mathrm{Ind}_H^G W \oplus \mathrm{Ind}_H^G W' = \mathrm{Ind}_H^G (W \oplus W')$ - - - ①

(*) $W' \subseteq W$ subrepr $\quad \mathrm{Ind}_H^G W' \subseteq \mathrm{Ind}_H^G W$ as subrepr. - - ②

(*) $(\mathrm{Ind}_H^G W) \otimes V' = \mathrm{Ind}_H^G (\underbrace{W \otimes V'}_{H\text{-repr}})$ here V' is a G-repr.

(*) $\mathrm{Ind}_H^G k[H] = k[G]$.

$$\boxed{\mathrm{Ind}_H^G W \cong k[G] \otimes_{k[H]} W}$$

(*) Let G be group. $H \leq G$ & $K \leq H$.

Let W be a K-repr.

$$\mathrm{Ind}_H^G(\mathrm{Ind}_K^H W) \cong \mathrm{Ind}_K^G W$$

$$\boxed{\chi_V(g) = \sum_{1\le i\le r,\ g_i^{-1}gg_i\in H} \chi_W(g_i^{-1}gg_i)}$$

$$\boxed{\chi_V(g) = \frac{1}{|H|}\sum_{y\in G,\ y^{-1}gy\in H} \chi_W(y^{-1}gy)}$$

<u>Defⁿ</u>: Let f be a class function on H. Define

$$f' = \mathrm{Ind}_H^G f : G \longrightarrow \mathbb{C}$$

$$f'(g) = \frac{1}{|H|}\sum_{\substack{y\in G\\ y^{-1}gy\in H}} f(y^{-1}gy) \;\textcircled{=}\; |Z_G(g)|\frac{1}{|H|}\sum_{a\in C(g)\cap H} f(a)$$

⊛ ① $\mathrm{Ind}_H^G \chi_W = \chi_{\mathrm{Ind}_H^G W}$

② $f' = \mathrm{Ind}_H^G f$ is a class function G.

Note that class functions on H (& G) are $\mathbb{C}$-linear combinations of characters of H (& G respectively)

{ & f_1 & f_2 are class functions on H & $a_1, a_2 \in \mathbb{C}$
then $\mathrm{Ind}(a_1 f_1 + a_2 f_2) = a_1 \mathrm{Ind} f_1 + a_2 \mathrm{Ind} f_2$

⊛ Let f' be a class function on G then

$f'|_H = \mathrm{Res}\, f'$ is a class function G.

Recall

$$\langle \varphi_1, \varphi_2 \rangle_G = \frac{1}{|G|} \sum_{g \in G} \varphi_1(g)\varphi_2(g^{-1}) \left(= (\varphi_1 | \varphi_2) \text{ if } \varphi_2 \text{ is a character} \right)$$

φ_i class function on G.

Let V, V' be G-repr.

$$\langle V, V' \rangle_G := \dim \mathrm{Hom}^G(V, V') = \dim \mathrm{Hom}(V, V')$$

<u>Prop</u>: $\langle V, V' \rangle_G = \langle \chi_V, \chi_{V'} \rangle_G$

<u>Proof</u>: $V = n_1 W_1 + \dots + n_r W_r$ where W_i are irr G-reps

$V' = n_1' W_1 + \dots + n_r' W_r$

$$\dim \mathrm{Hom}^G\left(\bigoplus_{i=1}^{r} W_i^{n_i}, \bigoplus_{j=1}^{r} W_j^{n_j'} \right) = \sum_{i=1}^{r} n_i \dim \mathrm{Hom}\left(W_i, \bigoplus_{j=1}^{r} W_j^{n_j'} \right)$$

$$= \sum_{i=1}^{r} n_i n_i' = \langle \chi_V, \chi_{V'} \rangle$$

orthogonality of chars

<u>Thm</u>: (Frobenius <u>reciprocity</u>):

$$\langle \varphi, \mathrm{res}\, \psi \rangle_H = \langle \mathrm{Ind}\, \varphi, \psi \rangle_G$$

where φ is a class function on H & ψ is a class function on G.

Pf: Since $\langle \, , \rangle$ is bilinear & the characters generate the space of class function & as a vector space. We may assume $\varphi = \chi_W$ & $\psi = \chi_{V'}$ where W is an H-repn & V' is a G-repn.

Then $\langle \varphi, \operatorname{Res} \psi \rangle_H = \langle W, V' \rangle_H$

& $\langle \operatorname{Ind} \varphi, \psi \rangle = \langle \operatorname{Ind}_H^G W, V' \rangle_G$

But $\operatorname{Hom}^H(W, V') = \operatorname{Hom}^G(\operatorname{Ind}_H^G W, V')$

$$\Rightarrow \dim \operatorname{Hom}^H(W, V') = \dim \operatorname{Hom}^G(\operatorname{Ind} W, V')$$

$$\begin{array}{ccc} \| & & \| \\ \langle W, V' \rangle_H & & \langle \operatorname{Ind} W, V' \rangle_G \end{array}$$

$\boxtimes$