Paired differences - example of a block design

Sometimes independent samples, such as the ones in a completely randomized design, from two (or $k > 2$) populations is not an efficient way for comparisons. Consider the following example.

Example. It is of interest to compare an enriched formula with a standard formula for baby food. Weights of infants vary significantly and this influences weight gain more than the difference in food quality. Therefore, independent samples (with infants having very different weights) for the two formulas will not be very efficient in detecting the difference. Instead, pair babies of similar weight and feed one of them the standard formula, and the other the enriched formula. Then observe the gain in weight:

pair	1	2	3	...	n
enriched	e_1	e_2	e_3	...	e_n
standard	s_1	s_2	s_3	...	s_n

However, the samples may not be treated as independent but correlated. The n pairs of observations, $(e_1, s_1), \ldots, (e_n, s_n)$ may still be treated to be uncorrelated (or even independent). These n pairs are like n independent blocks, inside each of which we can compare enriched with standard. This is the idea of blocking and block designs. Blocks are supposed to be homogeneous inside, so comparison of treatments within blocks becomes efficient. We assume that
$\begin{pmatrix} e_i \\ s_i \end{pmatrix} \sim N_2\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}\right)$. In the above example, we want to test $H_0 : \mu_D \equiv \mu_1 - \mu_2 = 0$, so consider $y_i = e_i - s_i$. Then,
$y_i = \mu_D + \epsilon_i$, $E(\epsilon_i) = 0$, $Var(\epsilon_i) = \sigma_D^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2 = Var(y_i)$. If normality is assumed, then we have, $y_1, \ldots, y_n$ i.i.d. $N(\mu_D, \sigma_D^2)$ and we want to test $H_0 : \mu_D = 0$. Consider the test statistic,

$$\frac{\sqrt{n}\bar{y}}{\sqrt{\sum_{i=1}^n (y_i - \bar{y})^2/(n-1)}} \sim t_{n-1},$$

if H_0 is true, or equivalently,

$$\frac{n\bar{y}^2}{\sum_{i=1}^n (y_i - \bar{y})^2/(n-1)} \sim F_{1,n-1}.$$

Note that,

$$
\begin{aligned}
& \frac{n\bar{y}^2}{\sum_{i=1}^{n}(y_i-\bar{y})^2/(n-1)} \\
= & \quad \frac{n(\bar{e}-\bar{s})^2}{\frac{1}{n-1}\sum_{i=1}^{n}\left[(e_i-\bar{e})-(s_i-\bar{s})\right]^2} \\
= & \quad \frac{n(\bar{e}-\bar{s})^2}{\frac{1}{n-1}\left[\sum_{i=1}^{n}(e_i-\bar{e})^2+\sum_{i=1}^{n}(s_i-\bar{s})^2-2\sum_{i=1}^{n}(e_i-\bar{e})(s_i-\bar{s})\right]} \\
= & \quad \frac{(\bar{e}-\bar{s})^2/(\frac{1}{n}+\frac{1}{n})}{\frac{1}{2(n-1)}\left[\sum_{i=1}^{n}(e_i-\bar{e})^2+\sum_{i=1}^{n}(s_i-\bar{s})^2-2\sum_{i=1}^{n}(e_i-\bar{e})(s_i-\bar{s})\right]}.
\end{aligned}
$$

Compare this test statistic with the one used for independent samples. $Cov(e,s)$ is expected to be positive (due to blocking), so the variance in the denominator above is typically less than $\frac{1}{2(n-1)}\left[\sum_{i=1}^{n}(e_i-\bar{e})^2+\sum_{i=1}^{n}(s_i-\bar{s})^2\right]$, which appears there. This is the positive effect due to blocking.

Confounding of effects.

Example. Consider two groups of similar students and two teachers. It is of interest to compare two different training methods. Consider the design where teacher A teaches one group using method I, whereas teacher B teaches the other group using method II. Later the results are analyzed. The problem with this design is that, if one group performs better it may be due to teacher effect or due to method effect, but it is not possible to separate the effects. We say then that the two effects are confounded. Sometimes we may not be interested in certain effects, in which case we may actually look for designs that will confound their effects. This will reduce the number of parameters to be estimated.

Experiments with a single factor – One-way ANOVA

We want to compare $k > 2$ treatments. Treatment i produces a population of y values with mean μ_i, $i = 1, 2, \ldots, k$. Or, if treatment i is applied, then the response $Y \sim N(\mu_i, \sigma^2)$, $i = 1, 2, \ldots, k$. Are these k populations different?

Design. n_i observations are made independently from population i, so the k samples are independent. Equivalently, we may look at this experiment as a design where N subjects are available to study the k treatments. n_1 of these are randomly selected and assigned to a group which will get treatment 1, n_2 of the remaining for treatment 2, and so on. Such a design is called a *completely randomized design* (as mentioned previously). Model for such a

design is as follows.
Let y_{ij} = response of the jth individual in the ith group (ith treatment), $j = 1, 2, \ldots, n_i$; $i = 1, 2, \ldots, k$. Then,
$y_{ij} = \mu_i + \epsilon_{ij},\ j = 1, 2, \ldots, n_i;\ i = 1, 2, \ldots, k$. $E(\epsilon_{ij}) = 0$, $Var(\epsilon_{ij}) = \sigma^2$, uncorrelated errors; $\epsilon_{ij} \sim N(0, \sigma^2)$ i.i.d. for testing and confidence statements.
In the usual linear model formulation:

$$\begin{pmatrix} y_{11} \\ \vdots \\ y_{1n_1} \\ y_{21} \\ \vdots \\ y_{2n_2} \\ \vdots \\ y_{k1} \\ \vdots \\ y_{kn_k} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & 1 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_k \end{pmatrix} + \epsilon.$$

Since $(X'X)^{-1} = \begin{pmatrix} \frac{1}{n_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{n_2} & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & \frac{1}{n_k} \end{pmatrix}$ and $X'Y = \begin{pmatrix} \sum_{j=1}^{n_1} y_{1j} \\ \vdots \\ \sum_{j=1}^{n_1} y_{kj} \end{pmatrix}$, we get

$\begin{pmatrix} \hat{\mu}_1 \\ \vdots \\ \hat{\mu}_k \end{pmatrix} = \begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_k \end{pmatrix}$ and

RSS $= \sum_{i=1}^{k} \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_i)^2 = \sum \sum \hat{\epsilon}_{ij}^2 = \sum \sum (y_{ij} - \hat{\mu}_i)^2$.

Questions.

(i) Are the group means μ_i equal? i.e., test $H_0 : \mu_1 = \mu_2 = \cdots = \mu_k$.
(ii) If not, how are they different?