Lecture 9: Dual representation and the character of a representation

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(*) Let V be G-reps.

$V^* = \mathrm{Hom}(V, k)$

Let $\varphi \in V^*$, What is $g\cdot\varphi$?

$$V^* \times V \longrightarrow k$$
$$(\varphi, v) \longmapsto \varphi(v)$$

We want $(g\cdot\varphi, g\cdot v) = (\varphi, v)$

i.e. $g\cdot\varphi(g\cdot v) = \varphi(v)$

$$\Downarrow$$

$$(g\varphi)(v) = \varphi(g^{-1}v)$$

$$(hg\varphi)(v) = \varphi((hg)^{-1}\cdot v) = \varphi(g^{-1}h^{-1}\cdot v)$$
$$= (g\varphi)(h^{-1}v)$$
$$= (h\cdot(g\varphi))(v)$$

i.e. $(hg)\cdot\varphi = h\cdot(g\cdot\varphi)$. Hence it is an action

$\rho_{V^*}(g) \in GL(V^*)$

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "g"'] \arrow[dr, phantom, "\#"] & k \arrow[d] \\ V \arrow[r, "g\varphi"] & k \end{tikzcd}$$

$v \in V$

$g\varphi(g\cdot v) = \varphi(v)$

More generally, if V, W are two G-reps

$\mathrm{Hom}(V, W)$

$(g\cdot\varphi)(g\cdot v) = g\cdot(\varphi(v))$

$(g\cdot\varphi)(v) = g\cdot(\varphi(g^{-1}v))$

$\rho_{\mathrm{Hom}}(g)(\varphi)(v) = \rho_W(g)(\varphi(\rho_V(g^{-1})(v)))$

Let $\{e_1, \dots, e_n\}$ be a basis of V & $\{e_1^*, \dots, e_n^*\}$ the dual basis of V^*

i.e. $e_i^*(e_j) = \delta_{ij} \quad \forall i, j$

If $\rho_V(g) = A$ w.r.t $\{e_1, \dots, e_n\}$

Let $\varphi \in V^*$ be $\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \underset{\sim}{u}$ w.r.t. $\{e_1^*, \dots, e_n^*\}$ i.e. $\varphi = \sum_{j=1}^{n} u_j e_j^*$, $u_i \in k$

$$(g\varphi)(e_i) = \varphi(g^{-1}e_i) = \varphi(A^{-1}e_i)$$
$$= \sum_{j=1}^{n} u_j e_j^*(A^{-1}e_i) = \sum_{j=1}^{n} u_j e_j^*\left(\sum_{k=1}^{n} A^{-1}_{ki} e_k\right)$$
$$= \sum_j u_j (A^{-1})_{ji}$$

if $g\varphi = \sum_{j=1}^{n} v_j e_j^* \Rightarrow g\varphi(e_i) = v_i$

$\sum_{j=1}^{n} u_j (A^{-1})_{ji} = v_i$, $1 \le i \le n$ $\Rightarrow (A^{-1})^T \cdot \underset{\sim}{u} = \underset{\sim}{v}$

$$\boxed{\text{Hence } \rho_{V^*}(g) \text{ w.r.t. } \{e_1^*, \dots, e_n^*\} \text{ is } \left(\rho_V(g)^{-1}\right)^T.}$$

(*) HW a) $\mathrm{Hom}_k(V, W) \cong V^* \otimes_k W$

b) If V & W are G-rep then show that the two rep are isomorphic.

Def^n: Let $\rho : G \longrightarrow GL(V_\rho)$ be a G-repr. The <u>character</u> of ρ, χ_ρ is function from $G \to \mathbb{C}$ given by $\chi_\rho(g) = Tr(\rho(g)) \quad \forall g \in G$.

(*) Note that if the repr is one dimensional the it is its character. So they are determined by the character.

Prop: Let χ be a character of a G-repr V. Then

(i) $\chi(1) = \chi(e_G) = \dim V$

(ii) $\chi(g^{-1}) = \overline{\chi(g)}$

(iii) $\chi(g^{-1}hg) = \chi(h)$ i.e. χ is class function.

Pf: (i) $\chi(1) = Tr(\rho_V(1)) = Tr(id_V) = \dim(V) = n$

(ii) $\chi(g^{-1}) = Tr(\rho(g^{-1}))$

Let $\{\lambda_1, \dots, \lambda_n\}$ be eigen values of $\rho(g)$

Then $\chi(g^{-1}) = \lambda_1^{-1} + \dots + \lambda_n^{-1} = \sum_{i=1}^{n} \overline{\lambda_i} = \overline{\chi(g)}$

(iii) WTS $Tr(\rho(g^{-1}hg)) = Tr(\rho(h))$

$$\begin{aligned} Tr(\rho(g^{-1}hg)) &= Tr(\rho(g^{-1})\rho(h)\rho(g)) \\ &= Tr(\rho(h)\rho(g)\rho(g^{-1})) = Tr(\rho(h)) \end{aligned}$$

($\because Tr(AB) = T(BA)$)

⊛ V & W are G-reps. Then

a) $\chi_{V \oplus W} = \chi_V + \chi_W$

b) $\chi_{V \otimes W} = \chi_V \cdot \chi_W$

Pf: a) $\mathrm{Tr}\left(\rho_{V\oplus W}(g)\right) = \mathrm{Tr}\left(\rho_V(g)\right) + \mathrm{Tr}\left(\rho_W(g)\right)$

The matrix of $\rho_{V\oplus W}(g) = \begin{bmatrix} \rho_V(g) & \\ & \rho_W(g) \end{bmatrix}$

$\forall g \in G$

b) $\mathrm{Tr}\left(\rho_{V\otimes W}(g)\right) = \mathrm{Tr}\left(\rho_V(g) \otimes \rho_W(g)\right)$

$= \mathrm{Tr}\left(\rho_V(g)\right) \cdot \mathrm{Tr}\left(\rho_W(g)\right)$

$\blacksquare$