Combinatorics

8 DEC 21

Lecture 18

Pasch Axiom

Modular geometries iff $rk\,E + rk\,F = rk(E \vee F) + rk(E \wedge F)$ $\quad E \cap F$

$\forall\, E, F \in \mathcal{F}s.$

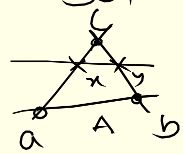
(*) — Two rk 2 flats in a rk 3 flats must intersect in a modular geometry.

(P) :- A lines that meets two sides of a triangle (in distinct pts) must meet the third line.

$(X, \mathcal{L})$ - linear space - satisfies (P) or not.

Thm: A finite linear space $(X, \mathcal{L})$ satisfies (P) iff it consists of points & lines of some modular geometry.
rk 1 flats, rk 2 flats

pf :- • First part that pts & lines of a modular geom. satisfies (P) is due to (*)



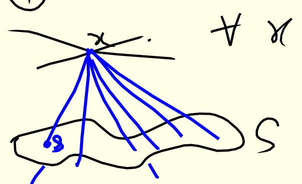
$\Rightarrow A \cap \overline{\{x,y\}} \neq \phi$ as both are in $\overline{\{a,b,c\}}$.

• Define $\mathcal{F}s$ on the set of pts of $(X, \mathcal{L})$ (which is X !)
by $\mathcal{F}s = \{ S \mid S \supset L$ whenever $|S \cap L| \geq 2 \}$.

1. No infinite chains ; $|X| < \infty$ ✓
2. $\phi, \{x\}; x \in X$, X satisfy that this condn trivially. ✓
3. Closed under intersection ✓
4. flats that cover a given flat S partition $X \setminus S$??

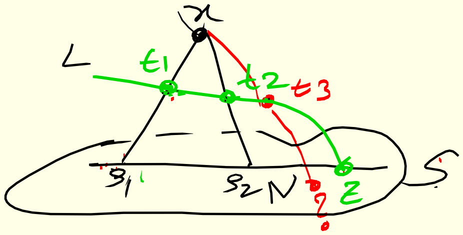
Pf of (4) :- We give a description of flats that cover S. (cone over S)

$\forall\, x \notin S$ let $S_x = \bigcup_{\substack{L \ni x \\ L \cap S \neq \phi}} L \;.=\; \bigcup_{s \in S} L_s$ $\quad$ L_s is the unique line joining x & s.



tpt. (a) S_x is a flat & (b) if $S_x \supseteq U \supseteq S$ with $U \in \mathcal{F}_S$ then $U = S$ or $U = S_x$.

Last time we proved that S_x is a flat.



WLOG assume $t_1 \notin S$.

look at triangle z, t, s_1.

Line joining x & t_3 intersects L_{s,t_1} in x. intersects $L_{t,z}$ in t_3

$\because$ x satisfies (P), L_{x,t_3} must intersect $L_{s,z} \subseteq S$

$\Rightarrow L_{x,t_3} \cap S \neq \phi$. $\Rightarrow L_{x,t_3} \subseteq S_x$.

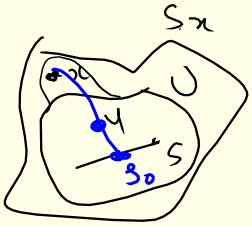
$\Rightarrow t_3 \in S_x$.

$\Rightarrow \underset{L_{t_1,t_2}}{\overset{\shortparallel}{L}} \subseteq S_x$.

To prove (b) Assume that $U \in \mathcal{F}_S$ & $S \subsetneqq U \subseteq S_x$ for some $x \notin S$.

take any pt. $u \in U \setminus S$ then $U \subseteq S_x \Rightarrow$ line joining x & u must intersect S say in s_0.

$\Rightarrow L_{u,s_0}$ line joining u & s_0 (which equals L_{x,s_0})



intersects U in at least u & s_0

$\Rightarrow L_{u,s_0} \subseteq U$. $\Rightarrow x \in U$.

$\Rightarrow$ any line $L_{x,s} \subseteq U$ $\forall s \in S$. $\Rightarrow S_x \subseteq U$.

$\Rightarrow S_x = U$.

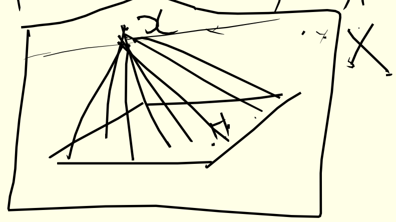
Flats that cover S partition $X \setminus S$.

$\forall x \notin S$ there is a unique flat (namely S_x) that covers S & contains x. ($\because S_x = S_y$ $\forall x, y \in S_x \setminus S$).

$\Rightarrow$ if $w \in S_x \cap S_y \Rightarrow S_w = S_x$, $S_w = S_y$

Is this geometry modular?

To prove that any hyperplane & a line intersect (criterion for modularity)



for any hyperplane H & $x \notin H$ "the cone"

$H_x = X \Rightarrow$ any line thru x must meet H.

$\Rightarrow$ any line meets H.

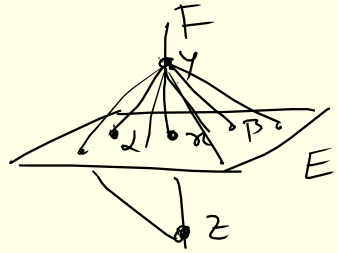
$\Rightarrow$ X is modular.

QED.

Remark :- If a modular geometry is a union of two flats then it is also a disjoint union of two flats.

$E \neq X, F \neq X$

Pf. (sketch)



let $X = E \cup F$ X - modular

let $x = E \cap F$.

let $y \in F \setminus E$.

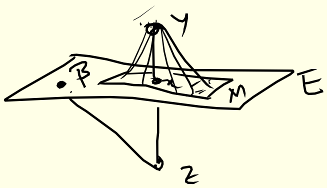
If $F = \{x, y\}$ then $X = E \cup \{y\}$ is a disjoint union.

$\Rightarrow \exists\, z \in E \setminus \{x, y\}$.

The two flat $\overline{\{\alpha, y\}} = \{\alpha, y\}$. ($\because |\{\alpha, y\} \cap E| \leq 1$ as $y \notin E$)

Let M be a hyperplane in E $M \ni x$. look at $\overline{\{M, y\}}$

$= M \cup \{y\} =: H$ hyperplane in X.



But then the line $\{z, \beta\}$ for $\beta \in E \setminus H$ does not intersect H. $\therefore X$ is not modular. contradiction.

Exercise :- Complete the proof for $\mathrm{rk}\, F > 1$.

Def^n :- Projective geometry is a modular geometry that is not a union of two flats.

iff it is not a union of two disjoint flats.

iff it is a connected modular geometry.

Thm: Let $(X, \mathcal{F}s)$ be projective geometry. The subgeometry induced on any of $F \in \mathcal{F}s$ is also projective.

pf. (1) Modularity holds because flat of a flat is flat!!

ie let $F \in \mathcal{F}s$ & E_1, E_2 be two flats in F.

$rk_F E_1 = rk_X E_1$ & so on!! $E_1 \vee_F E_2 = E_1 \vee_X E_2$

$\therefore$ modular law holds!

(2) Only thing to prove is that F is connected; knowing that X is connected.

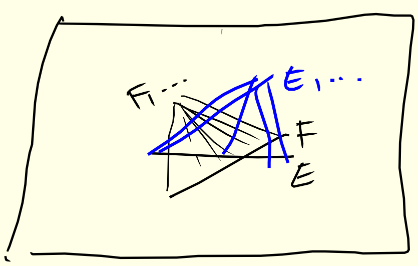
$\to$ Downward ~~induction on rk.~~

(**) — ie first we prove that any hyperplane is connected.

then extending the basis of F & removing one elt we see that $F \subset H$ for some hyperplane & keep applying (**) to H, to a hyperplane in H that contains F & so on till F is a hyperplane in some proj. geometry.

then (**) gives the result.

Let $H \subset X$ be a hyperplane & assume $H = E \cup F$



X

Let $E_1, \cdots, E_s$ be flats that partition $X \setminus F$

& $F_1, \cdots, F_r$ be flats that partition $X \setminus H$.

note that $E_i \cap H = E$

& $F_j \cap H = F$

(since H is a flat itself.)

further if $s = 1 \Rightarrow X = E_1 \cup F. \Rightarrow X$ is not projective!

$\therefore s \geq 2$. & $r \geq 2$.

look at $\boxed{\begin{array}{l} x_1 \in E_1 \cap F_1 \\ x_2 \in E_2 \cap F_2 \end{array}}$. $\quad x_1 \notin H$, $x_2 \notin H$.

Q. why such x_i exist? (if not then $rk(E_1) + rk(F_1) = rk(E_1 \vee F_1) \leq rk X$.

$rk E + 1 + rk F + 1$

but $rk E + rk F = rk X - 1$
since $E \cup F = H$ is a hyperplane
which is a contradiction.)

Now, look at $\overline{\{x_1, x_2\}}$. $\overline{\{x_1, x_2\}} \cap H \neq \phi$ (since X is modular)

$\therefore$ either $\overline{\{x_1, x_2\}} \cap E \neq \phi$ or $\overline{\{x_1, x_2\}} \cap F \neq \phi$.

$\Downarrow$ $\cdot e$

$x_1 \& x_2$ belong to same flat that cover E contradiction

$\Downarrow$

$x_1, x_2 \in$ same flat that cover F.

$\Rightarrow$ $H \neq E \cup F$ for any two flats E, F in H.

$\therefore$ H is projective. QED.