Combinatorics

Lecture 5

Conditions on parameters

Grading will be 60% - home assignments
40% - final exam.
NO Mid-term. (like last sem)

→ Incidence Structure. $(\mathcal{P}, \mathcal{B}, I)$.

$\mathcal{P}$ - set. elts are called points
$\mathcal{B}$ - set elts are called blocks
$I \subset \mathcal{P} \times \mathcal{B}$ elts are called flats. ←

Linear space. $\mathcal{B}$ is a collection of subsets of $\mathcal{P}$ s.t. any two points are in a unique block
& $|B| \geq 2 \ \forall B \in \mathcal{B}$.

(Erdös + De Bruijn) Either $|\mathcal{B}| = 1$
or $|\mathcal{B}| \geq |\mathcal{P}|$.

Conway's proof.

t - design.

Defn :- Let v, k, t, λ be integers s.t. $v \geq k \geq t > 0$

A t-design is an incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$
s.t. ① $|\mathcal{P}| = v$,
② $|B| = k \ \forall B \in \mathcal{B}$
③ Any subset of t points is contained in exactly λ blocks.

$\underline{Ex}$:- $\mathcal{P} = \mathbb{P}^n(GF(q)) = \{$ set of all 1-dim'l subspaces of $GF(q)^{n+1}\}$

$\mathcal{B} = \{ L_W \mid W$ - 2 dim'l subspace of $GF(q)^{n+1}\}$

& $L_W = \{ v \in \mathcal{P} \mid v \subseteq W \}$.

$|L_W| = \frac{(q^2-1)}{q-1} = q+1 = k$. , $v = \frac{q^{n+1}-1}{q-1} = 1+q+\cdots+q^n$.

$t=2, \lambda=1$. • Any two distinct one-dimensional subspaces of a v. space span a unique 2-dim'l subspace.

<u>Aim of today</u> :-

To give conditions that v, k, t, λ must satisfy if a $S_\lambda(t,k,v)$-design exists.

<u>Theorem 1</u>. The number of blocks (i.e. $|\mathcal{B}|$) in an $S_\lambda(t,k,v)$ equals $\frac{\lambda\binom{v}{t}}{\binom{k}{t}} =: b$

(we denote this number by b).

($b = |\mathcal{B}|$, B - a block.
$\mathcal{B}$ - set of all blocks)

<u>proof</u> :- • If one knows how to double count, then one has a very good understanding of combinatorics

① Typically one construct a set of tuples $S = \{(a,b) \mid a \in \cdots, b \in \cdots\}$ — cleverness

② Count $|S|$ by fixing 1^{st} coordinate
③ count $|S|$ by fixing 2^{nd} coordinate } technical.

② = ③ gives the required result.

① $S = \{(T, B) \mid B \in \mathcal{B},\ T \subset \mathcal{P} \text{ s.t. } |T| = t\}$

② Fixing T first we get $\binom{v}{t}$ - T's.
& for each such T, $\exists\ \lambda$ - B's

$\Rightarrow |S| = \lambda \cdot \binom{v}{t}$.

③ Fixing B first we see that $\exists\ |\mathcal{B}|$ such B's. & each one contains exactly $\binom{k}{t}$ subsets of size t.

$\therefore |S| = |\mathcal{B}| \cdot \binom{k}{t}$.

$\Rightarrow \lambda \binom{v}{t} = |\mathcal{B}| \binom{k}{t} \Rightarrow |\mathcal{B}| = \dfrac{\lambda \binom{v}{t}}{\binom{k}{t}}$.

$\boxed{\text{QED}}$

→ If one knows the result, it is easy to find a proof ←

Theorem 2. (Every t-design is also an i-design for every $0 \leq i \leq t$.)

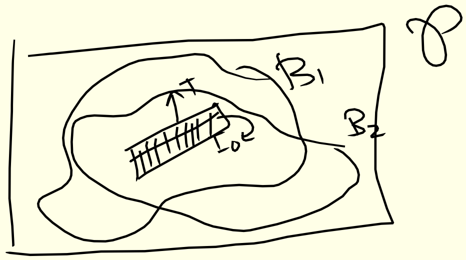
Given $0 \leq i \leq t$, the number of blocks that contain a subset of size i is :

$$b_i := \frac{\lambda \binom{v-i}{t-i}}{\binom{k-i}{t-i}} \qquad 0 \leq i \leq t$$

$\left(\Rightarrow S_\lambda(t, k, v) = S_{\frac{\lambda \binom{v-i}{t-i}}{\binom{k-i}{t-i}}}(i, k, v) \text{ design.} \right)$

proof Fix $I_0 \subseteq \mathcal{P}$ s.t. $|I_0| = i$

① $S_1 = \{(T,B) \mid B \in \mathcal{B},\ T \subseteq \mathcal{P};\ |T| = t \ \&\ T \supseteq I_0\}$



② Fixing T first:

$\exists \binom{v-i}{t-i}$ such T's & for each such $\exists\ \lambda$ B's.

$$\therefore |S_1| = \lambda \binom{v-i}{t-i}.$$

③ On the other hand, Fixing B first we get # blocks that contain I_0 say b_{I_0} & for each such block, $\exists \binom{k-i}{t-i}$ T's of size t inside it that contain I_0.

$$\therefore |S_1| = b_{I_0} \cdot \binom{k-i}{t-i}$$

② = ③ $\Rightarrow$ $b_{I_0} = \dfrac{\lambda \binom{v-i}{t-i}}{\binom{k-i}{t-i}}$. $\therefore b_{I_0} = b_{I_1}$ whenever $|I_0| = |I_1| = i$, $i \le t$

$\boxed{\text{QED}}$

$\boxed{\text{Corollary}}$ (Prob. 19B in van Lint + Wilson) If $S_1(3,6,v)$ exists, then $v \equiv 2$ or $6 \pmod{20}$.

Nomenclature: $S_\lambda(t,k,v)$ will be denoted as $S(t,k,v)$ if $\lambda = 1$.

S → Steiner.

$S(3,6,v)$ = collection of 6-subsets of a set of size v such that every 3-subset occurs in exactly one of these chosen 6-subsets.

$\Rightarrow$ simplest non-trivial design must have atleast 22 elts!

pf: $b_2 \Rightarrow v \equiv 2 \pmod 4$; $\underbrace{b_1 \Rightarrow}_{4s}$ $20 \mid (v-1)(v-2)$.

$$5 \mid (v-1) \cdot \frac{(v-2)}{4}$$

if $5 \mid v-1 \Rightarrow 5 \mid v-6. \Rightarrow 5 \,\&\, 4$ divides $v-6$.

$\Rightarrow$ 20 divides $v-6$

$\Rightarrow v \equiv 6 \pmod{20}$ — (*)

If $5 \mid \left(\frac{v-2}{4}\right) \Rightarrow v \equiv 2 \pmod{20}$. — (**)

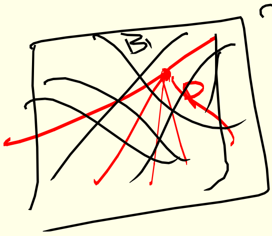
——— x ——— x ——— x ———

Remark: Since any t-design (for $t \geq 2$) is also a 2-design, theorems regarding 2-designs are applicable for all designs too. For 2-design.

$$b_1 = \frac{\lambda \binom{v-1}{1}_{2-1}}{\binom{k-1}{1}} = b_1 = \frac{\lambda(v-1)}{(k-1)}$$

$\|$

no. of blocks containg a subset of size 1. i.e # blocks containing an element.



$\mathcal{P}$

It is generally accepted to write $b_1 = r$ (the no. of times an elt. repeats in $\mathcal{B}$)

$\therefore \boxed{r(k-1) = \lambda(v-1)} \longrightarrow$ true in a 2-design.

Count in two ways the set $S_x := \{(x, B) \mid x \in \mathcal{P},\ B \in \mathcal{B},\ x \in B\}$ to get $\boxed{v \cdot r = b \cdot k}$.