Theorem (The isoperimetric inequality)

Let γ be a simple closed curve in $\mathbb{R}^2$ of length L and let A be the area of its interior. Then $A \leqslant \frac{L^2}{4\pi}$ with equality iff γ is a circle.

Proof: Let $\gamma = (\gamma_1, \gamma_2)$. As in ① from page 56, we use x, y interchangeably with γ_1, γ_2. Recall from ① that

$$A = \frac{1}{2} \int_{C = \mathrm{trace}(\gamma)} (xy' - yx')\,dt. \quad \text{-----------} \quad (i)$$

Without loss of generality we may assume that $\gamma(t)$ is periodic and of constant speed such that the period is π. This means that the speed is $\frac{L}{\pi}$, i.e., $\|\gamma'\| = \frac{L}{\pi}$. Also, by translating the curve if necessary, we may assume that the origin $\vec{0}$ is in C and that $\gamma(0) = \vec{0} = \gamma(\pi)$. Now,

$$L^2/\pi = \int_0^\pi \|\gamma'(t)\|^2\,dt = \int_0^\pi (x'^2 + y'^2)\,dt. \quad \text{-----------} \quad (ii)$$

By (i) and (ii), it is enough to prove that

$$\int_0^\pi \left[\frac{1}{4}(x'^2 + y'^2) - \frac{1}{2}(xy' - x'y)\right] dt \geqslant 0,$$

i.e., $\displaystyle\int_0^\pi [x'^2 + y'^2 - 2(xy' - x'y)]\,dt \geqslant 0.$ ----------- (iii)

Now we switch to polar coordinates, i.e., we put $x = r\cos\theta$ and $y = r\sin\theta$. Therefore, with $(\cdot)' = \frac{d}{dt}(\cdot)$,

$$x' = r'\cos\theta - r\sin\theta\cdot\theta' \quad \text{and} \quad y' = r'\sin\theta + r\cos\theta\cdot\theta'.$$

$$\therefore\ x'^2 + y'^2 = r'^2 + r^2\theta'^2; \quad xy' - x'y = r^2\theta'.$$

Now (iii) reduces to

$$\int_0^{\pi}[r'^2 + r^2\theta'^2 - 2r^2\theta']\,dt \geq 0, \quad \text{i.e.,}$$

$$\underbrace{\int_0^{\pi}(r'^2 - r^2)\,dt}_{(\geq 0 \text{ by Wirtinger})} + \underbrace{\int_0^{\pi} r^2(\theta'-1)^2\,dt}_{(\text{obviously } \geq 0)} \geq 0.$$

Finally, equality holds $\Longleftrightarrow$ both the integrals are 0, i.e.,

$r(t) = A\sin t$ for some A and $\theta(t) = t + \text{constant } (=\theta_0)$.

This implies that $r = A\sin(\theta - \theta_0)$. If $\theta_0 = 0$, i.e., $r = A\sin\theta$, then

$$x = (A\sin\theta)\cos\theta = \frac{A}{2}\sin 2\theta, \quad y = (A\sin\theta)\sin\theta = \frac{A}{2}(1-\cos 2\theta),$$

i.e., $x^2 + (y - A/2)^2 = (A/2)^2$ which is a circle of diameter $|\frac{A}{2}|$ and centre $(0, A/2)$. $\therefore$ $r = A\sin(\theta - \theta_0)$ is a circle of diameter $|\frac{A}{2}|$ rotated by θ_0.

Centre $= \left(-\frac{A\sin\theta_0}{2}, \frac{A\cos\theta_0}{2}\right)$.