Exercises

1. A set Ω is *pathwise connected* if any two points in Ω can be joined by a piecewise smooth curve entirely contained in Ω. We show that an open set Ω is pathwise connected if and only if Ω is connected.

   (a) Suppose first that Ω is open and pathwise connected, and that it can be written as a disjoint union of non-empty open sets $\Omega = \Omega_1 \cup \Omega_2$. Let $w_1 \in \Omega_1$ and $w_2 \in \Omega_2$ and let $z : [0,1] \to \Omega$ be a curve with $z(0) = w_1$ and $z(1) = w_2$. Let

   $$t^* = \sup_{0 \leq t \leq 1} \{t : z(s) \in \Omega_1 \quad \text{for all } 0 \leq s \leq t\}.$$

   Arrive at a contradiction by considering the point $z(t^*)$.

   (b) Conversely, suppose that Ω is open and connected. Fix a point $w \in \Omega$ and let $\Omega_1 \subset \Omega$ denote the set of all points that can be joined to w by a curve contained in Ω. Also, let $\Omega_2 \subset \Omega$ denote the set of all points that cannot be joined to w by a curve in Ω. Show that both Ω_1 and Ω_2 are open, disjoint and their union is Ω. Conclude that $\Omega = \Omega_1$.

2. Let Ω be an open set in $\mathbb{C}$ and $z \in \Omega$. The connected component C_z of z is the set of all points in w that can be reached from z by a curve entirely contained in Ω.

   (a) Show that C_z is open and connected, and any two connected components are either disjoint or coincide.

   (b) Show that Ω can have only countably many distinct connected components.

   (c) Prove that if Ω is the complement of a compact set, then Ω has only one unbounded component.

3. For a function $f : \mathbb{C} \to \mathbb{C}$ and a curve γ in the complex plane define the integral with respect to $\bar{z}$ as $\int_\gamma f d\bar{z} = \overline{\int_\gamma \bar{f} dz}$. From this the line integral with respect to x and y can be defined as

   $$\int_\gamma f dx = \frac{1}{2}\left(\int_\gamma f dz + \int_\gamma f d\bar{z}\right)$$
   $$\int_\gamma f dy = \frac{1}{2i}\left(\int_\gamma f dz - \int_\gamma f d\bar{z}\right)$$

   Check that for $f = u + iv$

   $$\int_\gamma f dz = \int_\gamma (u dx - v dy) + i \int_\gamma (u dy + v dx). \tag{0.1}$$

   If we instead start by defininng for any $p, q : \mathbb{R}^2 \to \mathbb{R}$ and $\gamma : [a,b] \to \mathbb{R}^2$ the line integral $\int_\gamma p dx + q dy$ by

   $$\int_\gamma p dx + q dy := \int_a^b p\left(x(t), y(t)\right) \cdot x'(t) dt + q\left(x(t), y(t)\right) \cdot y'(t) dt$$

   then show that the right hand side of (**??**) gives $\int_\gamma f dz$.

The integral *with respect to the arc length* is

$$\int_\gamma f ds = \int_\gamma f|dz| := \int_\gamma f(z(t))|z'(t)|dt$$

With $f \equiv 1$ one gets the arc length. In this case $\int_{-\gamma} f|dz| = \int_\gamma f|dz|$ and

$$\left|\int_\gamma f dz\right| \leq \int_\gamma |f| \cdot |dz|.$$

Show the following **Theorem**: If p and q are (possibly complex valued) continuous functions in a region Ω, then for any curve γ in Ω the line integral $\int_\gamma pdx + qdy$ depends only on the endpoints of γ if and only if there exists a function $U(x, y)$ in Ω with the partial derivatives $\partial U/\partial x = p$, $\partial U/\partial y = q$.
Hint: For the only if part fix a point (x_0, y_0) and let $U(x', y') = \int_\gamma pdx + qdy$ for any(?) curve γ which starts at (x_0, y_0) and ends at (x', y').
Thus $\int_\gamma f(z)dz = \int_\gamma f(z)dx + i\int_\gamma f(z)dy$ is dependent only on the endpoints for any γ if there is a function F on Ω such that

$$\frac{\partial F(z)}{\partial x} = f(z), \quad \frac{\partial F(z)}{\partial y} = if(z).$$

Conclude then that $\int_\gamma f dz$ with f continuous, depends only on the endpoints of γ if and only if f is the derivative of a holomorphic function in Ω. (note that we proved only one direction in class)

4. These calculations provide some insight into Cauchy's theorem

   (a) Evaluate $\int_\gamma z^n dz$ for all integers n. Here γ is any circle centered at the origin with positive orientation. What if γ is a circle not containing the origin?

   (b) show that if $|a| < r < |b|$ then

   $$\int_\gamma \frac{dz}{(z-a)(z-b)} = \frac{2\pi i}{a-b},$$

   where γ denotes the circle centered at the origin, of radius r, with the positive orientation.

5. Suppose that f is holomorphic in an open set Ω. Prove that in any one of the following cases:

   (a) $\mathrm{Re}(f)$ is constant;

   (b) $\mathrm{Im}(f)$ is constant;

   (c) $|f|$ is constant;

   one can conclude that f is constant.

6. Suppose f is continuous in a region Ω. Prove that any two primitives of f (if they exist) differ by a constant.

7. [HW 1, due 5 Oct] Consider a holomorphic function f on a region Ω. Let C be a circle inside Ω whose interior is also contained in Ω. Here is another way to show that $\int_C f(z)dz = 0$.

(a) Consider any regular polygon P_n of n sides inscribed inside the circle. Argue that $\int_{P_n} f(z)dz = 0$.

(b) Show that $\lim_{n\to\infty} \int_{P_n} f(z)dz = \int_C f(z)dz$.

8. The next few exercises show how complex integration can help us compute complicated real integrals.

   (a) [HW 2, due Oct 11] Prove

$$\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{\sqrt{2\pi}}{4}.$$

   The integral $\int_0^\infty$ is interpreted as $\lim_{R\to\infty} \int_0^R$.
   HINT: Integrate e^{-z^2} from 0 to R, then along the circular arc from R to $Re^{i\pi/4}$ and then along the straight line from $Re^{i\pi/4}$ to 0.

   (b) Show $\int_0^\infty \frac{\sin x}{x} = \frac{\pi}{2}$.
   HINT: The integral equals $\frac{1}{2i}\int_{-\infty}^\infty \frac{e^{ix}-1}{x}dx$. Use the indented semicircle.

   (c) [HW 2, due Oct 11] Evaluate the integrals

$$\int_0^\infty e^{-ax}\cos(bx)dx \quad \text{and} \quad \int_0^\infty e^{-ax}\sin(bx)dx, \quad a > 0$$

   by integrating e^{-Ax}, $A = \sqrt{a^2+b^2}$, over an appropriate sector with angle ω, with $\cos\omega = a/A$.

   (d) Prove that for all $\xi \in \mathbb{C}$ we have $e^{-\pi\xi^2} = \int_{-\infty}^\infty e^{-\pi x^2} e^{2\pi i x\xi}$.

9. Suppose f is continuously complex differentiable on Ω, and $T \subset \Omega$ is a triangle whose interior is also contained in Ω. Apply Green's theorem to show that $\int_T f(z)dz = 0$. this provides a proof of Goursat's theorem under the additional assumption that f' is continuous.

10. Show that every non-constant polynomial $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$ with complex coefficients has a root in $\mathbb{C}$. From this conclude that $P(z)$ has n roots $w_1, w_2, \cdots, w_n$ and $P(z) = a_n(z-w_1)(z-w_2)\cdots(z-w_n)$.

    HINT: Suppose not. Then note that $P(z)^{-1}$ is entire.

11. HW 3 (Due Monday 25 October) Let f be a holomorphic function on the disc D_{R_0} centered at the origin and of radius R_0.

    (a) Prove that whenever $0 < R < R_0$ and $|z| < R$, then

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\phi}) \cdot Re\left(\frac{Re^{i\phi}+z}{Re^{i\phi}-z}\right) d\phi.$$

    HINT: Note that if $w = R^2/\bar{z}$, then the integral of $f(\zeta)/(\zeta - w)$ around the circle of radius R centered at the origin is 0. Use this, together with the Cauchy integral formula.

    (b) Show that

$$Re\left(\frac{Re^{i\gamma}+r}{Re^{i\gamma}-r}\right) = \frac{R^2-r^2}{R^2-2Rr\cos\gamma+r^2}.$$

12. HW 3 (Due Monday 25 October) Say that a twice continuosly differentiable real-valued function is harmonic if $\Delta u(x,y) = 0$ where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

   (a) If f is holomorphic in an open set Ω, then show that the real and imaginary parts of f are harmonic.

   (b) Let u be a real-valued function defined on the unit disc $\mathbb{D}$. Suppose that u is twice continuously differentiable and harmonic.

      i. Prove that there exists a holomorphic function f on $\mathbb{D}$ such that $Re(f) = u$. Also show that the imaginary part of f is uniquely defined up to an additive (real) constant.
      HINT: If there is such an f then $f'(z) = 2\partial u/\partial z$. Therefore, let $g(z) = 2\partial u/\partial z$ and prove that g is holomorphic. Why can one find F with $F' = g$? Prove that $Re(f)$ differs from u by a real constant.

      ii. Deduce from this result, and the above exercise, the Poisson integral representation formula from the Cauchy integral formula: If u is harmonic in $\mathbb{D}$ and continuous on its closure, then if $z = re^{i\theta}$ one has

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - \phi)u(\phi)d\phi$$

      where $P_r(\gamma)$ is the Poisson kernel for the unit disc given by

$$P_r(\gamma) = \frac{1 - r^2}{1 - 2r\cos\gamma + r^2}.$$

13. Suppose f is an analytic function defined everywhere in $\mathbb{C}$ and such that for each $z_0 \in \mathbb{C}$ at least one coefficient in the expansion

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

   is equal to 0. Prove that f is a polynomial.
   HINT: Use the fact that $c_n n! = f^{(n)}(z_0)$ and use a countability argument.

14. Let Ω be a bounded open subset of $\mathbb{C}$, and $\phi : \Omega \to \Omega$ a holomorphic function. Prove that if there exists a point $z_0 \in \Omega$ such that

$$\phi(z_0) = z_0 \quad \text{and} \quad \phi'(z_0) = 1$$

   then ϕ is linear.
   HINT: Why can one assume that $z_0 = 0$? Write $\phi(z) = z + a_n z^n + O(z^{n+1})$ near 0, and prove that if $\phi_k = \phi \circ \cdots \circ \phi$ (k times) then $\phi_k(z) = z + k a_n z^n + O(z^{n+1})$. Apply the Cauchy inequailties and let $k \to \infty$ to conclude the proof.

15. [HW 4, Due Wednesday 3 November] This exercise shows that one cannot always extend a holomorphic function from a smaller set to a larger one (see the seciton on Schwarz reflection principle in Stein-Shakarchi for some cases in which one can extend). The following definition is needed. Let f be a function defined in the unit disc $\mathbb{D}$, with boundary circle C. A point w on C is said to be *regular* for f if there is an open neighbourhood U of w and an analytic function g on U, so that $f = g$ on $\mathbb{D} \cap U$. A function f defined on $\mathbb{D}$ cannot be continued analytically past the unit circle if no point of C is regular for f.

(a) Let

$$f(z) = \sum_{n=0}^{\infty} z^{2^n} \quad \text{for} \quad |z| < 1.$$

Notice that the radius of convergence of the above series is 1. Show that f cannot be continued analytically past the unit disc.
HINT: Suppose $\theta = 2\pi p/2^k$, whre p and k are positive integers. Let $z = re^{i\theta}$; then $|f(re^{i\theta})| \to \infty$ as $r \to 1$.

(b) Fix $0 < \alpha < \infty$. show that the analytic function f defined by

$$f(z) = \sum_{n=0}^{\infty} 2^{-n\alpha} z^{2^n} \quad \text{for} \quad |z| < 1$$

extends continuously ot the unit circle, but cannot be analytically continued past the unit circle.

16. [HW 4, Due Wednesday 3 November] Prove the converse to Runge's theorem: if K is a compact set whose compleemnt is not connected, then there exists a function f holomorphic in a neighborhood of K which cannot be appromixated uniformly by polynomials on K.
HINT: Pick a point z_0 in a bounded component of K^c, and let $f(z) = 1/(z - z_0)$. If f can be approximated uniformly by polynomials on K, show that there exists a polynomial p such that $|(z - z_0)p(z) - 1| < 1$. Use the maximum modulus principle (see below) to show that this inequality continues to hold for all z in the component of K^c that contains z_0. *The maximum modulus principle (which we learn later) states that if h is a non-constant holomorphic function in a region Ω, then $|h|$ cannot attain a maximum in Ω.*

17. Using Euler's formula

$$\sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i}$$

show that the complex zeroes of $\sin \pi z$ are exactly at the integers, and that they are each of order 1.
Calculate the residue of $\frac{1}{\sin \pi z}$ at $z = n \in \mathbf{Z}$.

18. [HW 5, Due Monday 15 November] Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^4}.$$

19. [HW 5, Due Monday 15 November] Show that

$$\int_{-\infty}^{\infty} \frac{\cos x \, dx}{x^2 + a^2} = \frac{\pi e^{-a}}{a}, \qquad \text{for all } a > 0.$$

20. [HW 5, Due Monday 15 November] Prove that

$$\int_0^{2\pi} \frac{d\theta}{(a + \cos\theta)^2} = \frac{2\pi a}{(a^2 - 1)^{3/2}}, \qquad \text{whenever } a > 1.$$

21. (HW 6, Due Monday 22 November) Morera's theorem states that if f is continuous in $\mathbb{C}$, and $\int_T f(z)dz = 0$ for all triangles T, then f is holomorphic in $\mathbb{C}$. We may ask if the conclusion still holds if we replace triangles by other sets.

(a) Suppose that f is continuous on $\mathbb{C}$, and

$$\int_C f(z)dz = 0 \tag{0.2}$$

for every circle C. Prove that f is holomorphic.

(b) More generally, let Γ be any toy contour, and $\mathcal{F}$ the collection of all translates and dilates of Γ. Show that if f is continuous on $\mathbb{C}$, and

$$\int_\gamma f(z)dz = 0 \qquad \text{for all } \gamma \in \mathcal{F}$$

then f is holomorphic. In particular, Morera's theorem holds under the weaker assumption that $\int_T f(z)dz = 0$ for all equilateral triangles.

[HINT (for part (a)): As a first step, assume that f is twice real differentiable, and write $f(z) = f(z_0) + a(z - z_0) + b(\overline{z - z_0}) + O(|z - z_0|^2)$ for z near z_0. Integrate this expression over small circles around z_0 to conclude $\partial f/\partial \bar{z} = b = 0$ at z_0. Alternatively, suppose only that f is differentiable and apply Green's theorem to conclude that the real and imaginary parts of f satisfy the Cauchy-Riemann equations.
In general, let $\phi(w) = \phi(x, y)$ (when $w = x + iy$) denote a smooth function with $0 \leq \phi(w) \leq 1$, and $\int_{\mathbb{R}^2} \phi(w)dV(w) = 1$, wehre $dV(w) = dxdy$, and $\int$ denotes the usual integral of a function of two variables in $\mathbb{R}^2$. For each $\epsilon > 0$, let $\phi_\epsilon(z) = \epsilon^{-2}\phi(\epsilon^{-1}z)$, as well as

$$f_\epsilon(z) = \int_{\mathbb{R}^2} f(z - w)\phi_\epsilon(w)dV(w),$$

where the integral denotes the usual integral of functions of two variables, with $dV(w)$ the area element of $\mathbb{R}^2$. Then f_ϵ is smooth, satisfies condition (**??**), and $f_\epsilon \to f$ uniformly on any compact subset of $\mathbb{C}$.]

22. [HW 7, due Saturday Dec 4] Suppose $f(z)$ is holomorphic in a punctured disc $D_r(z_0) - \{z_0\}$. Suppose also that

$$|f(z)| \leq A|z - z_0|^{-1+\epsilon}$$

for some $\epsilon > 0$, and all z near z_0. Show that the singularity of f at z_0 is removable.

23. Use the Cauchy inequalities or the maximum modulus principle to solve the following problems.

(a) Prove that if f is an entire function that satisfies

$$\sup_{|z|=R} |f(z)| \leq AR^k + B$$

for all $R > 0$, and for some integer $k \geq 0$ and some constants $A, B > 0$, then f is a polynomial of degree $\leq k$.

(b) Show that if f is holomorphic in the unit disc, is bounded, and converges uniformly to zero in the sector $\theta < argz < \phi$ as $|z| \to 1$, then $f = 0$.

(c) Let $w_1, w_2, \cdots w_n$ be points on the unit circle in the complex plane. Prove that there exists a point z on the unit circle such that the product of the distances from z to the points $w_j, 1 \leq j \leq n$, is at least 1. Conclude that there exists a point w on the unit circle such that the product of the distances from w to the points $w_j, 1 \leq j \leq n$, is exactly equal to 1.

(d) Show that if the real part of an entire function f is bounded, then f is constant.

24. [HW 7, due Saturday Dec 4] Suppose f and g are holomorphic in a region containing the disc $\{|z| \leq 1\}$. Suppose that f has a simple zero at $z = 0$ and vanishes nowhere else in $\{|z| \leq 1\}$. Let
$$f_\epsilon(z) = f(z) + \epsilon g(z).$$
Show that if ϵ is sufficiently small, then

   (a) $f_\epsilon(z)$ has a uniques zero in $\{|z| \leq 1\}$, and

   (b) If z_ϵ is this zero, the mapping $\epsilon \mapsto z_\epsilon$ is continuous.

25. Give another proof of the Cauchy integral formula
$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z} d\zeta$$
using homotopy of curves.
HINT: Deform the circel C to a small circle centered at z, and note that the quotient $(f(\zeta) - f(z))/(\zeta - z)$ is bounded.

26. Prove the maximum principle for harmonic functions, that is:

   (a) If u is a non-constant real-valued harmonic function in a region Ω, then u cannot attain a maximum (or minimum) in Ω.

   (b) Suppose that Ω is a region with compact closure $\bar{\Omega}$. If u is harmonic in Ω and continuous in $\bar{\Omega}$, then
$$\sup_{z\in\Omega} |u(z)| \leq \sup_{z\in\bar{\Omega}-\Omega} |u(z)|.$$
   HINT: To prove the first part, assume that u attains a local maximum at z_0. Let f be holomorphic near z_0 with $u = \mathrm{Re}(f)$, and show that f is not open.

27. Prove that all entire functions that are alos injetive take the form $f(z) = az + b$ with $a, b \in \mathbb{C}$, and $a \neq 0$.
HINT: Apply the Casorati-Weierstrass theorem to $f(1/z)$.

28. Let f be non-constant and holomorphic in an open set containing the closed unit disc. Show that if $|f(z)| = 1$ whenever $|z| = 1$, then the image of f contains the unit disc.
HINT: One must show that $f(z) = w_0$ has a root for every $w_0 \in \mathbb{D}$. To do this, it suffices to show that $f(z) = 0$ has a root (why?). Use the maximum modulus principle to conclude.

29. [HW 8, due Dec 13] This exercise is borrowed from Chapter 4, Section 2 of Ahlfors. You can consult that section. There is no need to prove parts (a) and (d) below since the proofs are already there in the book.

    For a curve γ define
$$n(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - a}$$
to be the *index of the point a with respect to the curve γ*. It is also called the *winding number* of γ with respect to a.

(a) **(Lemma 1:)** If the piecewise differentiable closed (not necessarily simple) curve γ does not pass through the point a, then the value of the integral

$$\int_\gamma \frac{dz}{z-a}$$

is a multiple of $2\pi i$. In particular $n(\gamma, a)$ is an integer. For example if γ goes around the circle $C_r(0)$ twice then $n(\gamma, 0) = 2$ (the number of times γ *winds* around 0 is 2.)
Look at the proof in Ahlfors, no need to reproduce it here.

(b) Show that if γ lies inside of a circle then $n(\gamma, a) = 0$ for all points a outside of the same circle.

(c) Show that as a function of a the index $n(\gamma, a)$ is constant in each of the regions determined by γ, and zero in the unbounded region.

(d) **(Lemma 2:)** Let z_1, z_2 be two points on a closed curve γ which does not pass through the origin. Denote the subarc from z_1 to z_2 in the direction of the curve by γ_1, and the subarc from z_2 to z_1 by γ_2. Suppose that z_1 lies in the lower half plane and z_2 in the upper half plane. If γ_1 does not meet the negative real axis and γ_2 does not meet the positive real axis, then $n(\gamma, 0) = 1$.
Look at the proof in Ahlfors, no need to reproduce it here.

(e) Give an alternate proof of Lemma 1 by dividing γ into a finite number of subarcs such that there exists a single-valued branch of $\arg(z-a)$ on each subarc. Pay particular attention to the compactness argument that is needed to prove the existence of such a subdivision.

(f) It is possible to define $n(\gamma, a)$ for any continuous closed curve γ that does not pass through a, whether piecewise differentiable or not. For this purpose γ is divided into subarcs $\gamma_1, \gamma_2, \cdots, \gamma_n$, each contained in a circular disc that does not include a. Let σ_k be the directed line segment from the initial to the terminal point of γ_k, and set $\sigma = \sigma_1 + \cdots \sigma_n$. We define $n(\gamma, a)$ to be the value $n(\sigma, a)$. To justify the definition, prove the following.

   i. The result is independent of the subdivisions.
   ii. If γ is piecewise differentiable the new definition is equivalent to the old.
   iii. The properties (b) and (c) continue to hold.

(g) The Jordan curve theorem asserts that every Jordan curve in the plane determines exactly two regions. The notion of winding number leads to a quick proof of one part of the theorem, namely that the complement of a Jordan curve γ has at least two components. This will be so if there exists a point a with $n(\gamma, a) \neq 0$.

We may assume that $Re(z) > 0$ on γ, and that there are points $z_1, z_2 \in \gamma$ with $Im z_1 < 0$, $Im z_2 > 0$. These points may be chosen so that there are no other points of γ on the line segments from 0 to z_1 and from 0 to z_2. Let γ_1 and γ_2 be the arcs of γ from z_1 to z_2 (excluding the end points).

Let σ_1 be the closed curve that consists of the line segment from 0 to z_1 followed by γ_1 and the segment from z_2 to 0, and let σ_2 be constructed in the same way with γ_2 in the place of γ_1. Then $\sigma_1 - \sigma_2 = \pm\gamma$.

The positive real axis intersects both γ_1 and γ_2 (why?). Choose the notation so that the intersection x_2 farthest to the right is with γ_2. Prove the following

i. $n(\sigma_1, x_2) = 0$, hence $n(\sigma_1, z) = 0$ for $z \in \gamma_2$.

ii. $n(\sigma_1, x) = n(\sigma_2, x) = 1$ for small $x > 0$ (see Lemma 2).

iii. the first intersection x_1 of the positive real axis with γ lies on $\gamma_!$.

iv. $n(\sigma_2, x_1) = 1$, hence $n(\sigma_2, z) = 1$ for $z \in \gamma_1$.

v. there exists a segment of the positive real axis with one end point on γ_1, the other on γ_2, and no other points on γ. The points x between the end points satisfy $n(\gamma, x) = \pm 1$.