Lecture 16: Induced representation

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Def^n: Let $\rho : G \to GL(V)$ be a repr, let $H \le G$ and let $W \subseteq V$ be a H-subrepr of V, i.e. $\rho(h)(w) \in W \ \forall\, w \in W \,\&\, h \in H$.

Let $\theta : H \to GL(W)$ denote this repr. i.e. $\theta(h) = \rho(h)|_W$ for $h \in H$. For $g \in G$, let $W_g := \rho(g)(W)$. Note that if g, g' are in the same left coset then $W_g = W_{g'}$. For $\bar{g} \in G/H$ ← collection of left cosets define $W_{\bar{g}} := W_g$ where $gH = \bar{g}$.

We say that ρ or V is induced from θ or W if $V = \bigoplus_{\bar{g} \in G/H} W_{\bar{g}}$.

$g = g'h$ for some $h \in H$
$\Rightarrow \rho(g) = \rho(g')\rho(h)$
$\Rightarrow \rho(g)(W) = \rho(g')(W)$
$(\because \rho(h)(W) = W)$

Notation: $V = \mathrm{Ind}_H^G W$

- $W_{\bar{g}} \cap \bigoplus_{\substack{\bar{g}' \in G/H \\ \bar{g}' \neq \bar{g}}} W_{\bar{g}'} = \{0\} \quad \forall\, \bar{g} \in G/H$
- $\langle W_{\bar{g}} \rangle = V$.

$\iff V = \bigoplus_{\bar{g} \in G/H} W_{\bar{g}}$

Note $\dim(V) = [G:H] \dim(W)$

Example: 0) $k[G]$ be the regular repr. $H = \{e\}$ & $W = ke \subseteq k[G]$

$W_g = kg$ & $k[G] = \bigoplus_{g \in G} W_g$. Hence $k[G]$ is induced from H-repr $\theta : \{e\} \to GL(ke) = G_1$

1) $H \le G$ then $W = \bigoplus_{h \in H} kh \subseteq k[G]$. Note W is H-stable

in fact $W = k[H]$. $W_{\bar{g}} = \rho(g)(W) = \bigoplus_{h \in H} kgh$ where $g \in G$ is s.t. $\bar{g} = gH$.

$= \bigoplus_{g' \in \bar{g}} kg'$

Hence $k[G] = \bigoplus_{\bar{g} \in G/H} W_{\bar{g}}$. So $k[G]$ is induced repr of $W = k[H]$.

2) $H \le G$ & let W be the trivial H-repr i.e. $\{ W = k \ \& \ h \cdot w = w \ \forall\, w \in W.$

Note G acts on G/H via left multiplication.

Let V be the permutation repr., i.e. $V = \bigoplus_{\bar{g} \in G/H} k\bar{g}$

$$g \cdot \left(\sum a_i \bar{g}_i\right) = \sum a_i \overline{gg_i}$$

Let $\bar{e}$ denote the coset H. Then $W = k\bar{e}$ is H-stable subspace of V. Moreover $W_g = k\,\overline{ge} = k\bar{g} = W_{\bar{g}}$ &

$V = \bigoplus_{\bar{g} \in G/H} W_{\bar{g}} = \bigoplus_{\bar{g} \in G/H} k\bar{g}$. Hence V is induced repr from W.

(*) If ρ_1 is induced by θ_1 & ρ_2 is induced θ_2 then $\rho_1 \oplus \rho_2$ is induced by $\theta_1 \oplus \theta_2$.

$\rho_i : G \to GL(V_i)$

$W_i \subseteq V_i$ are H-stable

$\theta_i : H \to GL(W_i)$

Note $\bigoplus_{\bar{g} \in G/H} W_{i\bar{g}} = V_i$ (ρ_i is induced from θ_i)

$$\Rightarrow V_1 \oplus V_2 = \bigoplus_{\bar{g} \in G/H} W_{1\bar{g}} \oplus \bigoplus_{\bar{g} \in G/H} W_{2\bar{g}} = \bigoplus_{\bar{g} \in G/H} (W_1 \oplus W_2)_{\bar{g}}$$

$$\Rightarrow V_1 \oplus V_2 = \mathrm{Ind}_H^G (W_1 \oplus W_2)$$

$\left(\because (W_1 \oplus W_2)_{\bar{g}} = W_{1\bar{g}} \oplus W_{2\bar{g}} \right)$

(*) Let V be induced by W & W_1 be a H-subrepr of W. Let $V_1 = \bigoplus_{\bar{g} \in G/H} \rho(g)(W_1) = \bigoplus_{\bar{g} \in G/H} W_{1\bar{g}}$ where $g \in G$ is any representative of $\bar{g}$ in G/H.

Then V_1 is a G-subrepr of V & V_1 is induced by W_1. (Since $W_{1\bar{g}} \subseteq W_{\bar{g}}$ &

$$W_{\bar{g}} \cap \bigoplus_{\substack{\bar{g}' \in G/H \\ \bar{g}' \neq \bar{g}}} W_{\bar{g}'} = 0 \Rightarrow W_{1\bar{g}} \cap \bigoplus_{\substack{\bar{g}' \in G/H \\ \bar{g}' \neq \bar{g}}} W_{1\bar{g}'} = 0)$$

$V = \mathrm{Ind}_H^G W$ & W_1 is subrepr then $\mathrm{Ind}_H^G W_1$ is subrepr of V.

(*) Let ρ be induced by θ & $\rho' : G \to GL(V')$ be another rep. then $\rho \otimes \rho'$ is induced by $\theta \otimes \rho'|_H$.

$$V = \mathrm{Ind}_H^G W \qquad \theta : H \to GL(W)$$

$$V \otimes V' = \mathrm{Ind}_H^G (W \otimes V')$$ as H-repn

$$\boxed{(A \oplus B) \otimes C = A \otimes C \oplus B \otimes C}$$

$$V = \bigoplus_{\bar{g} \in G/H} W_{\bar{g}}$$

$$V \otimes V' = \left(\bigoplus_{\bar{g} \in G/H} W_{\bar{g}} \right) \otimes V'$$

$$= \bigoplus_{\bar{g} \in G/H} (W_{\bar{g}} \otimes V')$$

Note
$$W_{\bar{g}} \otimes V' = \rho(g)(W) \otimes V'$$
$$= \rho(g)(W \otimes V') = g \cdot W \otimes \overset{V'}{\overset{\parallel}{g \cdot V'}}$$
$$= (W \otimes V')_g$$

Hence $V \otimes V' = \mathrm{Ind}_H^G (W \otimes V')$