Intuitively speaking, 'bending' preserves all measurements.

4) For the unit sphere S with its polar, or more precisely, latitude-longitude parametrisation (where $U = (-\pi/2, \pi/2) \times (0, 2\pi)$)

$$\sigma(\underset{\text{latitude}}{\theta}, \underset{\text{longitude}}{\varphi}) = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta), \quad (\theta, \varphi) \in U,$$

we saw that the FFF is $\begin{bmatrix} 1 & 0 \\ 0 & \cos^2\theta \end{bmatrix}$, i.e., $(d\theta)^2 + \cos^2\theta (d\varphi)^2$.

This means that the diffeomorphism $\sigma : U \to \sigma(U) \subseteq S$ is not an isometry over any open subset of U. Is there some other diffeomorphism from an open subset of the flat plane $\mathbb{R}^2$ to an open subset of the sphere which is also an isometry?

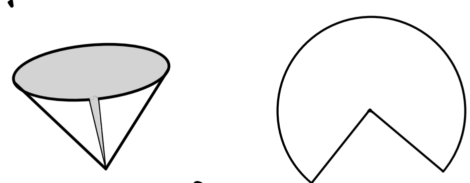
Such a question is important for our mathematical understanding of map-making wherein we want to faithfully reproduce the surface geography of our 'spherical' earth onto a flat sheet of paper. Intuitively, we can sense that mere bending of paper will never yield any part of the sphere; some shrinking / stretching must be made everywhere. Thus, we expect that no open subset of the sphere can be isometric to an open subset of $\mathbb{R}^2$

with its default Euclidean metric. We shall discuss a general method to prove such statements for arbitrary surfaces later.

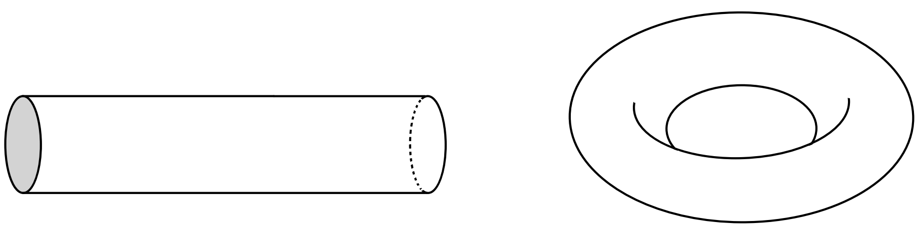
Exercise 28: Prove that the isoperimetric inequality can be violated over any open subset of the unit sphere in $\mathbb{R}^3$. Thus no open subset of the sphere is isometric to an open subset of $\mathbb{R}^2$.

5) We saw earlier that the FFF of the generalised cone $\sigma(t,\lambda) = \lambda(\gamma_1, \gamma_2, 1)$, $\lambda > 0$, $\|\dot{\gamma}(t)\| = 1$, is $\begin{bmatrix} \lambda^2 & \lambda \gamma\cdot\dot{\gamma} \\ \lambda\gamma\cdot\dot{\gamma} & \|\gamma\|^2+1 \end{bmatrix}$.

Thus σ is not an isometry. But it turns out that the cone is locally isometric to an open subset of the plane. (See Exercises 5.5, 5.7 of the text-book). For instance the circular cone, $(\|\gamma(t)\| \equiv 1)$ can be flattened by unbending.



Question: Can you bend the cylinder and make the open ends meet to get a torus in $\mathbb{R}^3$ without shrinking or stretching?



Isometries preserve all the internal measurements we have looked at. Are there diffeomorphisms that preserve only some of the measurements? (To what extent can we salvage map-making?)

Proposition: Let $f : S_1 \to S_2$ be a diffeomorphism of surfaces that preserves lengths, i.e., for any segment of a curve $C \subseteq S_1$, its length equals the length of its image under f. Then f is an isometry.

Proof: Let
$$\begin{tikzcd} S_1 \arrow[rr, "f"] & & S_2 \\ & U \arrow[ul, "\sigma_1"] \arrow[ur, "\sigma_2"'] & \end{tikzcd}$$
be a common parametrising domain with maps $\sigma_i : U \to V_i \subseteq S_i$ such that $f \circ \sigma_1 = \sigma_2$. Let $B_i = \begin{bmatrix} E_i & F_i \\ F_i & G_i \end{bmatrix}$ be the corresponding FFF's. It suffices to show that $B_1 = B_2$.

Let $\|-\|_i$ be the associated norms (over the tangent spaces in U), i.e., at any $p \in U$, and for any $w \in T_pU$, we have $\|w\|_i := w^t B_i w$. For each i, $\|\cdot\|_i$, when viewed as a function $\underbrace{U \times \mathbb{R}^2}_{TU} \longrightarrow \mathbb{R}$ is smooth.

Since norms determine inner-products (polarisation identity), it suffices to prove that $\|-\|_1 = \|-\|_2$ at every $p \in U$. Suppose $\|-\|_1 \neq \|-\|_2$ at some $p \in U$, i.e., say, for some $v \in T_pU$, $\|v\|_1 < \|v\|_2$. Then, by continuity, we see that along the line L through p parallel to v, we have $\|v\|_1 < \|v\|_2$ at every point q of L in a neighbourhood of p. Hence, if $\alpha(t)$ denotes a linear parametrisation of L with $\dot{\alpha} \equiv v$, then $\int_L \|\dot{\alpha}\|_1 \, dt < \int_L \|\dot{\alpha}\|_2 \, dt$. The corresponding curves $\sigma_i(\alpha(t))$ in S_i exhibit the same inequality for lengths, a contradiction.

Q.E.D.

Let $(V_i, \langle -,- \rangle_i)$, for $i = 1, 2$, be two inner-product spaces. A linear isomorphism $\varphi : V_1 \xrightarrow{\sim} V_2$ is said to be <u>conformal</u> if it preserves angles, i.e., $\forall\, v, w \in V_1$, $\dfrac{\langle v, w \rangle_1}{\|v\|_1 \|w\|_1} = \dfrac{\langle \varphi(v), \varphi(w) \rangle_2}{\|\varphi(v)\|_2 \|\varphi(w)\|_2}$

Clearly an isometry is conformal. A composition of conformal maps is conformal. An example of a conformal map that is not an isometry is the scaling map $\begin{array}{c} V \longrightarrow V \\ v \longmapsto \lambda v \end{array}$ for some fixed $\lambda \in \mathbb{R} \setminus \{0\}$.