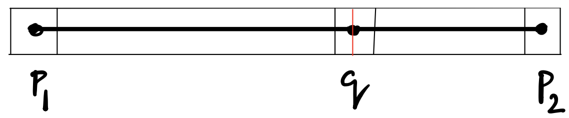
APPENDIX B : NOTES ON CONVEX GEOMETRY

1. Let $X \subseteq \mathbb{R}^2$ be a convex subset. Then $\overline{X}$ is also convex. (Hint: use ε-boxes as shown where $p_i \in \overline{X}$.)



2. If $C \subseteq \mathbb{R}^2$ is a convex simple closed curve, then $D := C \cup \operatorname{Int}(C)$ is convex and compact. (Use $\overline{\operatorname{Int}(C)} = D$).
Jordan curve theorem

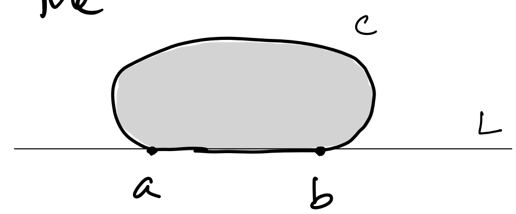
3. If $X \subseteq \mathbb{R}^2$ is a convex subset and $L \subseteq \mathbb{R}^2$ a line, then $X \cap L$ is an interval in L.

4. For D as in 2, and $L \subseteq \mathbb{R}^2$ a line, $D \cap L$ is either empty or a bounded closed interval, say $[a,b]$, (after identifying $L = \mathbb{R}$) while $\operatorname{Int}(C) \cap L$ is either empty or an open subinterval of (a,b). From now on, we only consider the case $D \cap L \neq \emptyset$, i.e., $D \cap L = [a,b]$.

B2

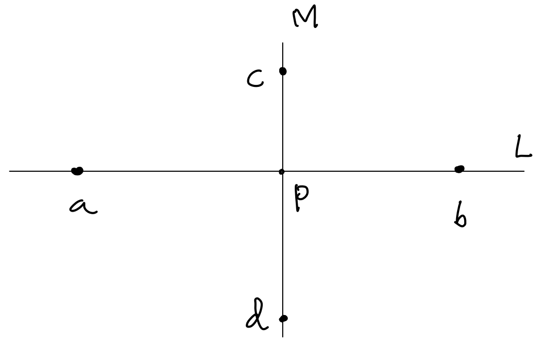
5. <u>Key claim</u>: With notation as in 4, either $\mathrm{Int}(C) \cap L = \emptyset$ or $\mathrm{Int}(C) \cap L = (a,b)$. Let's assume this for now.

6. If $\mathrm{Int}(C) \cap L = \emptyset$, then $C \cap L = D \cap L \ (= [a,b])$ and in this situation we have convexity condition (A) of page 67 (new number). Moreover, since the rest of the curve $C \setminus (C \cap L)$ is connected (image of an interval) it lies entirely in one of the two connected components of $\mathbb{R}^2 \setminus L$ (half planes). Thus C lies on one side of L.



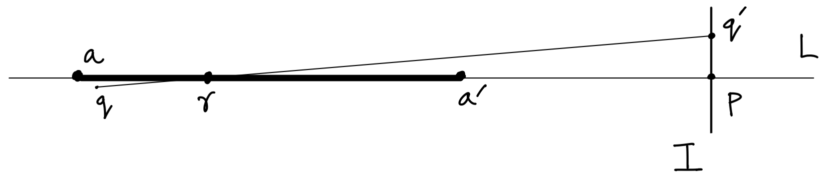
7. Suppose $\mathrm{Int}(C) \cap L = (a,b)$. Then $C \cap L = \{a,b\}$ and we have convexity condition (B) of page 68 (new number). Now $C \setminus \{a,b\}$ has at most 2 connected components and each component lies entirely in one of the two

open half-planes of $\mathbb{R}^2 \setminus L$. To see that there are 2 components lying on the opposite sides of $\mathbb{R}^2 \setminus L$, pick $p \in [a, b]$ and consider a line M through p different from L. Then, as above, $C \cap M$ also consists of 2 points, say $\{c, d\}$ with $p \in \mathrm{Int}(C)$ in between them. Thus, the curve lies on either side of L.



It now remains to prove the key claim in 5. Suppose $\mathrm{Int}(C) \cap L = (a', b') \subsetneq (a, b)$, say $a < a'$. We shall achieve a contradiction.

Pick $p \in (a', b')$, M a line through p perpendicular to L and $I \subseteq M \cap \mathrm{Int}(C)$ an open interval around p in M. Then one can choose a point $q \in \mathrm{Int}(C)$ "sufficiently close" to a and $q' \in I$ on the opposite side of L such that $\overline{qq'}$ intersects $\overline{aa'}$ in a point r. By convexity, $r \in \mathrm{Int}(C)$, a contradiction as $r \in \overline{aa'} \subseteq C$.



Postscript:

In 6 above, the assertion that $C \setminus C \cap L$ is connected is justified by claiming that it is the trace of an (open) interval. This is easily proven if we show that $C \cap L = [a,b]$ is the trace of a closed interval. This, in turn, is an easy consequence of Exercise 13 below.

Exercise 12: Let $\gamma : I \to \mathbb{R}^n$ be one-one and regular.

(i) Prove that if I is a closed interval, say $I = [c,d]$, then the continuous bijection $I \longrightarrow \gamma(I)$ is a homeomorphism.

(ii) Show that the conclusion in (i) is false, if I is assumed to be an open or semi-open interval. (You may use any exercise or example from the notes.)

Exercise 13:

Let $\gamma : [c,d] \to \mathbb{R}^n$ be a (regular) simple closed curve. Prove that γ induces a homeomorphism from a circle (obtained by identifying the endpoints of $[c,d]$) to the trace $\gamma([c,d])$.