# Lecture 5: Exterior products

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⊛ Let $V$ be a $k$-v.s. A multilinear map $\varphi : \underbrace{V \times \cdots \times V}_{a \text{ copies}} \longrightarrow U$ (a v.s.) is called alternating or anti symmetric or skew-symm if $\varphi(v_1, \ldots, v_a) = \mathrm{sgn}(\sigma)\, \varphi(v_{\sigma 1}, \ldots, v_{\sigma a})$ $\forall\, \sigma \in S_a$.

HW: The subspaces $H = \langle v_1 \otimes v_2 \otimes \cdots \otimes v_a \mid v_i \in V \ \&\ v_i = v_j \text{ for some } i \neq j \rangle$

& $H' = \langle v_1 \otimes v_2 \otimes \cdots \otimes v_a - \mathrm{sgn}(\sigma)\, v_{\sigma 1} \otimes v_{\sigma 2} \otimes \cdots v_{\sigma a} \mid v_i \in V \ \&\ \sigma \in S_a \rangle$

of $T^a V$ are same.

Def$^n$: The $a^{th}$ exterior power of $V$ $\Lambda^a V$ or $\mathrm{Ext}^a V$ is defined to be $T^a V / H = T^a V / H'$

The map $\varphi : V \times \cdots \times V \longrightarrow \Lambda^a V$

$(v_1, \ldots, v_a) \longmapsto v_1 \wedge v_2 \wedge \cdots \wedge v_a := \overline{v_1 \otimes \cdots \otimes v_a}$

is an alternating multilinear map.

Prop: Let $\theta : V \times \cdots \times V \longrightarrow U$ be a alternating multilinear map. Then $\exists!\ \hat{\theta} : \Lambda^a V \longrightarrow U$ s.t.

$$\begin{tikzcd} \Lambda^a V \arrow[r, "\hat{\theta}"] & U \\ V \times \cdots \times V \arrow[u, "\varphi"] \arrow[ur, "\theta"'] & \end{tikzcd}$$

$\hat{\theta} \circ \varphi = \theta$

Pf same as Sym.

$\theta$ induces a $k$-linear unique $\theta' : T^a V \to U$ s.t.

$\theta'(v_1 \otimes \cdots \otimes v_a) \longmapsto \theta(v_1, \ldots, v_a) \quad \forall v_i \in V$

Since $\theta$ is alternating

$H' \subseteq \ker(\theta')$. Hence by 1st isom thm

$$\left.\begin{array}{l} \exists!\ \tilde{\theta} : T^a V / H' \longrightarrow U \quad \text{s.t.} \\ \overline{v_1 \otimes \cdots \otimes v_a} \longmapsto \theta'(v_1 \otimes \cdots \otimes v_a) \\ \text{Hence } \hat{\theta} : \Lambda^a V \longrightarrow U \\ \text{s.t. } \hat{\theta}(v_1 \wedge \cdots \wedge v_a) = \theta(v_1, \ldots, v_a) \\ \Rightarrow \tilde{\theta} \circ \phi(v_1, \ldots, v_a) = \theta(v_1, \ldots, v_a) \end{array}\right\} \forall v_i \in V$$

$\blacksquare$

$\circledast$ Note $\Lambda^a V = 0$ for $a > \dim V$. Let $\{e_1, \ldots, e_n\}$ be a basis of $V$ then $e_{i_1} \wedge \cdots \wedge e_{i_a} = 0$ ($\because 1 \le i_1, \ldots, i_a \le n$ and $a > n$ hence at least two are same)

But $\{e_{i_1} \wedge \cdots \wedge e_{i_a} \mid 1 \le i_1, \ldots, i_a \le n\}$ gen $\Lambda^a V$.

$n = \dim V \quad \Lambda^n V = \langle e_1 \wedge e_2 \wedge \cdots \wedge e_n \rangle$ is one dim'l.

$\circledast$ $\Lambda^a V \times \Lambda^b V \longrightarrow \Lambda^{a+b} V$ is well-defined bilinear map.

$\alpha, \beta \longmapsto \alpha \wedge \beta$

$\Lambda^{\cdot} V = \bigoplus_{a=0}^{\dim V} \Lambda^a V$ is an algebra with product defined by the above. This called the exterior algebra.

⊛ Let $V, W$ be two v.s then

$$\Lambda^a(V\oplus W) \cong \bigoplus_{i=0}^{a} \Lambda^i V \otimes \Lambda^{a-i} W$$

Pf:

$$\Lambda^i V \times \Lambda^{a-i} W \xrightarrow{\ \Psi_i\ } \Lambda^a(V\oplus W)$$
$$(\alpha, \beta) \longmapsto \alpha\wedge\beta$$

This is well defined bilinear map

For $\alpha\in\Lambda^i V$, $\theta_\alpha : \underbrace{W\times\dots\times W}_{a-i \text{ times}} \longrightarrow \Lambda^a(V\oplus W)$

$$\theta_\alpha(w_1,\dots,w_{a-i}) = \alpha\wedge w_1\wedge\dots\wedge w_{a-i}$$

$\theta_\alpha$ is multilinear & alternating map & hence induces a unique map $\tilde\theta_\alpha : \Lambda^{a-i} W \longrightarrow \Lambda^a(V\oplus W)$ which is $k$-linear

$\Psi_i(\alpha,\beta) = \tilde\theta_\alpha(\beta)$. Then $\Psi_i$ is linear in 1st var (i.e. $\alpha$)

$$\tilde\theta_{\alpha+\alpha'}(\beta) = (\alpha+\alpha')\wedge\beta = \alpha\wedge\beta+\alpha'\wedge\beta = \tilde\theta_\alpha(\beta)+\tilde\theta_{\alpha'}(\beta)$$

Hence $\Psi_i$ induces unique $k$-linear $\theta_i : \Lambda^i V\otimes\Lambda^{a-i} W \longrightarrow \Lambda^a(V\oplus W)$
$$\alpha\otimes\beta \longmapsto \alpha\wedge\beta.$$

Set
$$\bigoplus_{i=0}^{a}\Lambda^i V\otimes\Lambda^{a-i} W \xrightarrow{\ \theta\ } \Lambda^a(V\oplus W)$$
$$\theta = \bigoplus_{i=0}^{a}\theta_i \quad \text{is } k\text{-linear.}$$

$$\underbrace{V\oplus W\times V\oplus W\times\dots\times V\oplus W}_{a \text{ copies}} \xrightarrow{\ \Psi\ } \bigoplus_{i=0}^{a}\Lambda^i V\otimes\Lambda^{a-i} W$$

$$((v_1,w_1),(v_2,w_2),\dots,(v_a,w_a)) \longmapsto w_1\wedge\dots\wedge w_a + \begin{cases} v_1\wedge w_2\wedge\dots\wedge w_a \\ + w_1\wedge v_2\wedge\dots\wedge w_a \\ + \vdots \\ w_1\wedge\dots\wedge w_{a-1}\wedge v_a \end{cases}$$

$$+ v_1\otimes w_2\wedge\dots\wedge w_a$$
$$- v_2\otimes w_1\wedge\dots\wedge w_a$$
$$+ v_3\otimes w_1\wedge\dots\wedge w_a$$
$$\vdots$$
$$+ v_1\wedge v_2\otimes w_3\wedge\dots\wedge w_a - \dots + \dots$$
$$\vdots$$

$$+ v_1\wedge v_2\otimes w_3\wedge w_4\dots\wedge w_a$$
$$- v_1\wedge v_3\otimes w_2\wedge w_4\wedge\dots\wedge w_a$$
$$+ \mathrm{sgn}(\sigma)\, v_{j_1}\wedge v_{j_2}\otimes w_{j_3}\wedge\dots\wedge w_{j_a}$$
where $\sigma = \begin{matrix} 1 & \dots & a \\ j_1 & \dots & j_a \end{matrix}$
$j_1 < j_2$
$j_3 < j_4 < \dots < j_a$

$$+ \dots$$
$$+ \sum_{\substack{1\le j_1<\dots<j_i\le n \\ 1\le j_{i+1}<\dots<j_a\le n \\ \sigma(l)=j_l}} \mathrm{sgn}(\sigma)\, v_{j_1}\wedge\dots\wedge v_{j_i}\otimes w_{j_{i+1}}\wedge w_{j_{i+2}}\wedge\dots\wedge w_{j_a}$$

check $\Psi$ is alternating & multilin map

it induces a map
$$\tilde\Psi : \Lambda^a(V\oplus W) \longrightarrow \bigoplus_{i=0}^{a}\Lambda^i V\otimes\Lambda^{a-i} W$$

One checks $\theta$ & $\tilde\Psi$ are inverses to each other.

Let $\{v_1,\dots,v_n\}$ be a basis of $V$ & $\{w_1,\dots,w_m\}$ a basis of $W$ then

$$\tilde\Psi(v_{j_1}\wedge\dots\wedge v_{j_i}\wedge w_{j_{i+1}}\wedge\dots\wedge w_{j_a}) = v_{j_1}\wedge v_{j_2}\wedge\dots\wedge v_{j}\otimes w_{j_{i+1}}\wedge\dots\wedge w_{j_a}$$

$$= \Psi((v_{j_1},0),(v_{j_2},0),\dots,(v_{j_i},0),(0,w_{j_{i+1}}),\dots,(0,w_{j_a})) \quad \Big|\quad \tilde\Psi\circ\theta_i(\ \downarrow\ ) =$$

$$\Rightarrow \theta\circ\tilde\Psi = \mathrm{id}$$

(*) Let $\{e_1, \dots, e_n\}$ be a basis of $V$ & $a \leq n$ then

$\{ e_{i_1} \wedge \cdots \wedge e_{i_a} \mid 1 \leq i_1 < i_2 < \cdots < i_a \leq n \}$ is a basis of

$\Lambda^a V$.

$V_2 = \{e_2, \dots, e_n\}$, $V_1 = \{e_1\}$

$$\Lambda^a V = \Lambda^a(V_1 \oplus V_2) \cong k \otimes \Lambda^a V_2 \oplus V_1 \otimes \Lambda^{a-1} V_2 \oplus \overbrace{\underbrace{\Lambda^2 V_1}_{=0} \otimes \Lambda^{a-2} V_2 \oplus \cdots}^{0}$$

$$\cong \Lambda^a V_2 \oplus V_1 \otimes \Lambda^{a-1} V_2$$

By ind on dim $V$ $\{ e_{i_1} \wedge \cdots \wedge e_{i_a} \mid 2 \leq i_1 < i_2 < \cdots < i_a \leq n \}$ is a basis of $\Lambda^a V_2$

& $\{ e_1 \otimes e_{i_1} \wedge \cdots \wedge e_{i_{a-1}} \mid 2 \leq i_1 < i_2 < \cdots < i_{a-1} \leq n \}$ is

a basis of $V_1 \otimes \Lambda^{a-1} V_2$.

Hence their images in $\Lambda^a(V_1 \oplus V_2)$ is a basis.

This set is $\{ e_{i_1} \wedge \cdots \wedge e_{i_a} \mid 1 \leq i_1 < i_2 < \cdots < i_a \leq n \}$ ∎

HW $\mathrm{Sym}^{\cdot} V \cong T^{\cdot} V / ( v_1 \otimes v_2 - v_2 \otimes v_1 \mid v_1, v_2 \in V )$

$\mathrm{Ext}^{\cdot} V = \Lambda^{\cdot} V \cong T^{\cdot} V / ( v \otimes v \mid v \in V )$

(b) $i : \mathrm{Sym}^a V \longrightarrow T^a V$

$$\begin{tikzcd} v_1 \cdots v_a \arrow[r, mapsto] & \sum_{\sigma \in S_a} v_{\sigma 1} \otimes \cdots \otimes v_{\sigma a} \\ V \times \cdots \times V \arrow[u, "\varphi"] \arrow[ur, bend right, "i'"] & \\ (v_1, \ldots, v_a) \arrow[uur, mapsto] & \end{tikzcd}$$

$i'$ is multilinear sym map

and hence induces $i$.

$T^a V \xrightarrow{\pi} \mathrm{Sym}^a V$

$\frac{1}{a!} \pi \circ i = id_{\mathrm{Sym}^a V}$

& $\frac{1}{a!} i \circ \pi$ is the projection on image$(i)$.

image$(i)$ is invariants of $S_a$ action on $T^a V$

via for $\alpha \in T^a V$, $\sigma \in S_a$

$\alpha = \sum_{\text{finite}} v_1 \otimes \cdots \otimes v_a$

$\sigma . \alpha = \sum v_{\sigma 1} \otimes \cdots \otimes v_{\sigma a}$

$\tau . (\sigma . \alpha) = \sum v_{\tau \sigma 1} \otimes \cdots \otimes v_{\tau \sigma a} = (\tau \sigma) . \alpha$

$|||^{ly}$ $i : \Lambda^a V \longrightarrow T^a V$

$v_1 \wedge \cdots \wedge v_a \longmapsto \sum_{\sigma \in S_a} \mathrm{sgn}(\sigma)\, v_{\sigma 1} \otimes \cdots \otimes v_{\sigma a}$

$i$ is $k$-linear

$\frac{1}{a!} \pi \circ i = id_{\Lambda^a V}$

& $\frac{1}{a!} i \circ \pi$ is the proj on image$(i)$

image$(i) = \{ z \in T^a V \mid \sigma . z = \mathrm{sgn}(\sigma) z \}$

HW: $T^2 V = \mathrm{Sym}^2 V \oplus \mathrm{Ext}^2 V$ if $V$ finite dim.