Combinatorics

Lecture 10

Bruck-Ryser-Chowla theorem

Thm: Assume $\exists$ a symmetric $2-(v,k,\lambda)$ design. Then,

→ ① If v is even, then $k-\lambda$ is a perfect square

② If v is odd, then the equation
$$z^2 = (k-\lambda)x^2 + (-1)^{\left(\frac{v-1}{2}\right)} \cdot \lambda y^2$$
has a non-zero integral solution.

Application :- A proj. plane of order n is a $2-(n^2+n+1, n+1, 1)$ design.

① Proj. plane of order 6 does not exist.

If it does, then $v = 43$, $k=7$. $43 \equiv 3 \pmod 4$

$\therefore$ $z^2 = 6x^2 - y^2$ must have non-zero integral sol^n.

If non-zero sol^n exists, then remove the common factor to assume that $(x,y,z)=1$.

If z is even, the power of 2 on LHS is even but $\Downarrow$ y even. But the power of 2 on the left is 1.

$\Rightarrow$ z is odd. $\Rightarrow$ y is odd. $\therefore$ $z^2 \,\&\, y^2 \equiv 1 \pmod 8$.

1, 3, 5, 7   2, 4, 6, 8

$6x^2 \pmod 8$ if x is even $x^2 \equiv 0$ or $4 \bmod 8$

x is odd $x^2 \equiv 1 \bmod 8$.

$6x^2$ will be $6, 0, 0 \bmod 8$.

$\Rightarrow$ $\nexists$ proj. plane of order 6.

② If $\exists$ a proj. plane of order $n \equiv 3$ or $2 \pmod 4$ then n is a sum of two squares.

pf. Fact :- $n = p_1^{e_1} \cdots p_r^{e_r}$ square free part of $n = \prod p_{i_\ell}^{e_{i_\ell}}$ where $e_{i_\ell} \not\equiv 0 \pmod 2$

n is a sum of 2 squares iff the square free part of n has no prime that is $\equiv 3 \pmod 4$

BRC thm: $\Rightarrow$ $z^2 = nx^2 - y^2$ has soln $n \equiv 1 \bmod 4$

nx^2 is a sum of two squares.

In nx^2 the square free part comes from n.

$\therefore$ square free part of n has no prime that is $\equiv 3 \pmod 4$ $\Rightarrow$ n is a sum of two squares.

ex. 15 is not a sum of 2 squares.

1, 4, 9.

—— x —— x —— x ——

Recall

$\boxed{\begin{array}{l}\text{Design theory — parameters, cond}^{\text{ns}}\text{ on parameters,} \\ \text{counting techniques.} \\ b \geq v \quad \text{Fisher's inequality.} \\ \text{Symm. design - BRC thm.}\end{array}}$

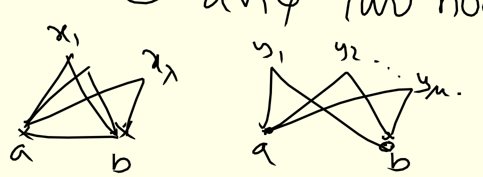
Chapter 21 Partial Geometries

Defn $\longrightarrow$ $Srg(v, k, \lambda, \mu)$.

k → deg. of each vertex

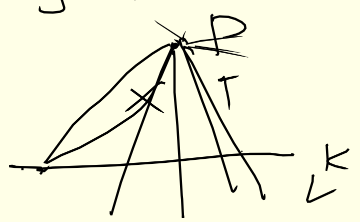
v → no. of vertices

① any two adj. vertices have λ common neighbours
② any two non-adj. vertices have μ common neighbours



Def^n :- A partial geometry $pg(k, R, T)$ is an incidence structure of points & lines s.t.

1. every line has k points & every point lies on R lines
2. Any 2 points belong to at most 1 line
3. If L is a line & $p \notin L$ then $\exists$ T lines containing p that intersect L.



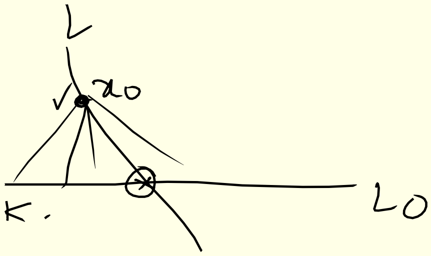
(Note $T \leq k$)
$T \leq R$

Problem 21 G Find the number of pts & lines in $pg(k, R, T)$.

(Remark: this explains why the no. of points is not one of the parameters!)

Sol^n :- Fix a line L_0. Look at

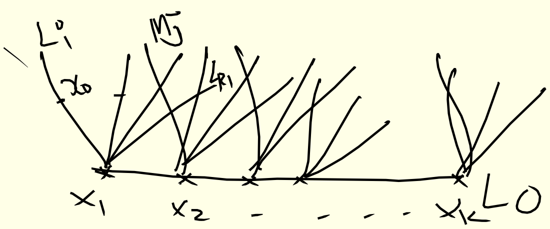
$$S = \{(x_0, L) \mid L \cap L_0 \neq \phi \ \& \ x_0 \in L \setminus L_0\}$$



Fix x_0 first $v = \#$ pts.

we get $|S| = (v-k) \cdot T$

Fix L first. $\exists$ $k(R-1)$ lines.



$\therefore |S| = k(R-1)(k-1)$.

$$(v-k)T = k(R-1)(k-1)$$

$$v = \frac{k(R-1)(k-1) + kT}{T}$$

$$\boxed{v = \frac{k\left((R-1)(k-1)+T\right)}{T}}$$

Let $b = \#$ lines. $S_1 = \{(x, L) \mid x \in L\}$

$$bk = vR.$$

$$\Rightarrow b = \frac{vR}{k}$$

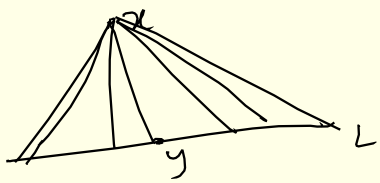
$$\Rightarrow \boxed{b = \frac{(R-1)(k-1)+T}{T} \cdot R}$$

Partial geometries are divided into four classes:

$pg(k, R, T)$

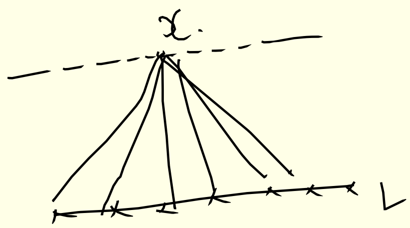
① $T = k$. $pg(k, R, k)$

2-(v, k, λ) design.



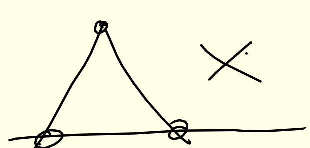
② $T = R - 1$. This called a <u>net</u>

given $x \notin L$, $\exists !$ line L_x containing x & $L \cap L_x = \emptyset$.



dually $T = k - 1$, called a <u>transversal design</u>

③ $T = 1$. <u>generalised quadrangle</u>



no triangles.

usually, $pg(k, R, 1)$ is denoted as $GQ(k-1, R-1)$.

(4) If $1 < T < \min\{R-1, R-1\}$ then such a partial geometry is called <u>proper</u>.

—— x —— x —— x ——

22. Mutually orthogonal Latin squares.

<u>Def^n</u> A latin square is a function $L : R \times C \to S$

(real $n \times n$ matrix $M : \{1 \cdots n\} \times \{1, \cdots n\} \to \mathbb{R}$
$(i,j) \to M_{ij}$)

where $|R| = |C| = |S| = n$ (for some $n > 0$)

such that $L(i, -) : C \to S$ is onto.
$c \to L(i,c)$

& $L(-, j) : R \to S$ is on-to.

order of L $:= |R|$.

(i.e every symbol occurs in each row & col^n).

(2) L, M two latin squares order n. They are called orthogonal.
$L : R_1 \times C_1 \to S_1$,
$M : R_2 \times C_2 \to S_2$.

if $\{(L(r_i, c_j), M(\tilde{r}_k, \tilde{c}_\ell))\}$
$\subseteq$
$S_1 \times S_2$.

i.e. $\forall$ elts $(s_1, s_2) \in S_1 \times S_2$
$\exists!$ entry (i,j) s.t. $(L_{ij}, M_{ij}) = (s_1, s_2)$

$$\begin{bmatrix} L_{11} & \cdots & L_{1n} \\ L_{n1} & & L_{nn} \end{bmatrix}, \begin{bmatrix} M_{11} & \cdots & M_{1n} \\ M_{n1} & & M_{nn} \end{bmatrix} \longrightarrow \begin{bmatrix} (L_{11}, M_{11}) & (L_{12}, M_{12}) & \cdots \\ & & (L_{nn}, M_{nn}) \end{bmatrix}$$

i.e. Front cover of the book shows a pair of 4×4 m.o.l.s.

♠, ♡, ♣, ◇ $= S_1$

A, K, Q, J $= S_2$.