

## Exercises

1. A set  $\Omega$  is *pathwise connected* if any two points in  $\Omega$  can be joined by a piecewise smooth curve entirely contained in  $\Omega$ . We show that an open set  $\Omega$  is pathwise connected if and only if  $\Omega$  is connected.

- (a) Suppose first that  $\Omega$  is open and pathwise connected, and that it can be written as a disjoint union of non-empty open sets  $\Omega = \Omega_1 \cup \Omega_2$ . Let  $w_1 \in \Omega_1$  and  $w_2 \in \Omega_2$  and let  $z : [0, 1] \rightarrow \Omega$  be a curve with  $z(0) = w_1$  and  $z(1) = w_2$ . Let

$$t^* = \sup_{0 \leq t \leq 1} \{t : z(s) \in \Omega_1 \text{ for all } 0 \leq s \leq t\}.$$

Arrive at a contradiction by considering the point  $z(t^*)$ .

- (b) Conversely, suppose that  $\Omega$  is open and connected. Fix a point  $w \in \Omega$  and let  $\Omega_1 \subset \Omega$  denote the set of all points that can be joined to  $w$  by a curve contained in  $\Omega$ . Also, let  $\Omega_2 \subset \Omega$  denote the set of all points that cannot be joined to  $w$  by a curve in  $\Omega$ . Show that both  $\Omega_1$  and  $\Omega_2$  are open, disjoint and their union is  $\Omega$ . Conclude that  $\Omega = \Omega_1$ .
2. Let  $\Omega$  be an open set in  $\mathbb{C}$  and  $z \in \Omega$ . The connected component  $C_z$  of  $z$  is the set of all points in  $\Omega$  that can be reached from  $z$  by a curve entirely contained in  $\Omega$ .
  - (a) Show that  $C_z$  is open and connected, and any two connected components are either disjoint or coincide.
  - (b) Show that  $\Omega$  can have only countably many distinct connected components.
  - (c) Prove that if  $\Omega$  is the complement of a compact set, then  $\Omega$  has only one unbounded component.
3. For a function  $f : \mathbb{C} \rightarrow \mathbb{C}$  and a curve  $\gamma$  in the complex plane define the integral with respect to  $\bar{z}$  as  $\int_{\gamma} f d\bar{z} = \overline{\int_{\gamma} f dz}$ . From this the line integral with respect to  $x$  and  $y$  can be defined as

$$\begin{aligned} \int_{\gamma} f dx &= \frac{1}{2} \left( \int_{\gamma} f dz + \int_{\gamma} f d\bar{z} \right) \\ \int_{\gamma} f dy &= \frac{1}{2i} \left( \int_{\gamma} f dz - \int_{\gamma} f d\bar{z} \right) \end{aligned}$$

Check that for  $f = u + iv$

$$\int_{\gamma} f dz = \int_{\gamma} (u dx - v dy) + i \int_{\gamma} (u dy + v dx). \quad (0.1)$$

If we instead start by defining for any  $p, q : \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $\gamma : [a, b] \rightarrow \mathbb{R}^2$  the line integral  $\int_{\gamma} p dx + q dy$  by

$$\int_{\gamma} p dx + q dy := \int_a^b p(x(t), y(t)) \cdot x'(t) dt + q(x(t), y(t)) \cdot y'(t) dt$$

then show that the right hand side of (0.1) gives  $\int_{\gamma} f dz$ .

The integral *with respect to the arc length* is

$$\int_{\gamma} f ds = \int_{\gamma} f |dz| := \int_{\gamma} f(z(t)) |z'(t)| dt$$

With  $f \equiv 1$  one gets the arc length. In this case  $\int_{-\gamma} f |dz| = \int_{\gamma} f |dz|$  and

$$\left| \int_{\gamma} f dz \right| \leq \int_{\gamma} |f| \cdot |dz|.$$

Show the following **Theorem**: If  $p$  and  $q$  are (possibly complex valued) continuous functions in a region  $\Omega$ , then for any curve  $\gamma$  in  $\Omega$  the line integral  $\int_{\gamma} p dx + q dy$  depends only on the endpoints of  $\gamma$  if and only if there exists a function  $U(x, y)$  in  $\Omega$  with the partial derivatives  $\partial U / \partial x = p$ ,  $\partial U / \partial y = q$ .

**Hint:** For the only if part fix a point  $(x_0, y_0)$  and let  $U(x', y') = \int_{\gamma} p dx + q dy$  for any(?) curve  $\gamma$  which starts at  $(x_0, y_0)$  and ends at  $(x', y')$ .

Thus  $\int_{\gamma} f(z) dz = \int_{\gamma} f(z) dx + i \int_{\gamma} f(z) dy$  is dependent only on the endpoints for any  $\gamma$  if there is a function  $F$  on  $\Omega$  such that

$$\frac{\partial F(z)}{\partial x} = f(z), \quad \frac{\partial F(z)}{\partial y} = if(z).$$

Conclude then that  $\int_{\gamma} f dz$  with  $f$  continuous, depends only on the endpoints of  $\gamma$  if and only if  $f$  is the derivative of a holomorphic function in  $\Omega$ . (note that we proved only one direction in class)

4. These calculations provide some insight into Cauchy's theorem

- (a) Evaluate  $\int_{\gamma} z^n dz$  for all integers  $n$ . Here  $\gamma$  is any circle centered at the origin with positive orientation. What if  $\gamma$  is a circle not containing the origin?
- (b) show that if  $|a| < r < |b|$  then

$$\int_{\gamma} \frac{dz}{(z-a)(z-b)} = \frac{2\pi i}{a-b},$$

where  $\gamma$  denotes the circle centered at the origin, of radius  $r$ , with the positive orientation.

5. Suppose that  $f$  is holomorphic in an open set  $\Omega$ . Prove that in any one of the following cases:

- (a)  $\operatorname{Re}(f)$  is constant;
- (b)  $\operatorname{Im}(f)$  is constant;
- (c)  $|f|$  is constant;

one can conclude that  $f$  is constant.

6. Suppose  $f$  is continuous in a region  $\Omega$ . Prove that any two primitives of  $f$  (if they exist) differ by a constant.

7. [HW 1, due 5 Oct] Consider a holomorphic function  $f$  on a region  $\Omega$ . Let  $C$  be a circle inside  $\Omega$  whose interior is also contained in  $\Omega$ . Here is another way to show that  $\int_C f(z) dz = 0$ .

- (a) Consider any regular polygon  $P_n$  of  $n$  sides inscribed inside the circle. Argue that  $\int_{P_n} f(z)dz = 0$ .
- (b) Show that  $\lim_{n \rightarrow \infty} \int_{P_n} f(z)dz = \int_C f(z)dz$ .
8. The next few exercises show how complex integration can help us compute complicated real integrals.
- (a) [HW 2, due Oct 11] Prove

$$\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{\sqrt{2\pi}}{4}.$$

The integral  $\int_0^\infty$  is interpreted as  $\lim_{R \rightarrow \infty} \int_0^R$ .

HINT: Integrate  $e^{-z^2}$  from 0 to  $R$ , then along the circular arc from  $R$  to  $Re^{i\pi/4}$  and then along the straight line from  $Re^{i\pi/4}$  to 0.

- (b) Show  $\int_0^\infty \frac{\sin x}{x} = \frac{\pi}{2}$ .

HINT: The integral equals  $\frac{1}{2i} \int_{-\infty}^\infty \frac{e^{ix}-1}{x} dx$ . Use the indented semicircle.

- (c) [HW 2, due Oct 11] Evaluate the integrals

$$\int_0^\infty e^{-ax} \cos(bx)dx \quad \text{and} \quad \int_0^\infty e^{-ax} \sin(bx)dx, \quad a > 0$$

by integrating  $e^{-Ax}$ ,  $A = \sqrt{a^2 + b^2}$ , over an appropriate sector with angle  $\omega$ , with  $\cos \omega = a/A$ .

- (d) Prove that for all  $\xi \in \mathbb{C}$  we have  $e^{-\pi\xi^2} = \int_{-\infty}^\infty e^{-\pi x^2} e^{2\pi i x \xi} dx$ .

9. Suppose  $f$  is continuously complex differentiable on  $\Omega$ , and  $T \subset \Omega$  is a triangle whose interior is also contained in  $\Omega$ . Apply Green's theorem to show that  $\int_T f(z)dz = 0$ . this provides a proof of Goursat's theorem under the additional assumption that  $f'$  is continuous.
10. Show that every non-constant polynomial  $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$  with complex coefficients has a root in  $\mathbb{C}$ . From this conclude that  $P(z)$  has  $n$  roots  $w_1, w_2, \dots, w_n$  and  $P(z) = a_n(z - w_1)(z - w_2) \dots (z - w_n)$ .
- HINT: Suppose not. Then note that  $P(z)^{-1}$  is entire.
11. (Due Monday 25 October) Let  $f$  be a holomorphic function on the disc  $D_{R_0}$  centered at the origin and of radius  $R_0$ .

- (a) Prove that whenever  $0 < R < R_0$  and  $|z| < R$ , then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\phi}) \cdot Re \left( \frac{Re^{i\phi} + z}{Re^{i\phi} - z} \right) d\phi.$$

HINT: Note that if  $w = R^2/\bar{z}$ , then the integral of  $f(\zeta)/(\zeta - w)$  around the circle of radius  $R$  centered at the origin is 0. Use this, together with the Cauchy integral formula.

- (b) Show that

$$Re \left( \frac{Re^{i\gamma} + r}{Re^{i\gamma} - r} \right) = \frac{R^2 - r^2}{R^2 - 2Rr \cos \gamma + r^2}.$$

12. (Due Monday 25 October) Say that a twice continuously differentiable real-valued function is harmonic if  $\Delta u(x, y) = 0$  where  $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ .

- (a) If  $f$  is holomorphic in an open set  $\Omega$ , then show that the real and imaginary parts of  $f$  are harmonic.
- (b) Let  $u$  be a real-valued function defined on the unit disc  $\mathbb{D}$ . Suppose that  $u$  is twice continuously differentiable and harmonic.
- Prove that there exists a holomorphic function  $f$  on  $\mathbb{D}$  such that  $\operatorname{Re}(f) = u$ . Also show that the imaginary part of  $f$  is uniquely defined up to an additive (real) constant.  
HINT: If there is such an  $f$  then  $f'(z) = 2\partial u/\partial z$ . Therefore, let  $g(z) = 2\partial u/\partial z$  and prove that  $g$  is holomorphic. Why can one find  $F$  with  $F' = g$ ? Prove that  $\operatorname{Re}(f)$  differs from  $u$  by a real constant.
  - Deduce from this result, and the above exercise, the Poisson integral representation formula from the Cauchy integral formula: If  $u$  is harmonic in  $\mathbb{D}$  and continuous on its closure, then if  $z = re^{i\theta}$  one has

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - \phi) u(\phi) d\phi$$

where  $P_r(\gamma)$  is the Poisson kernel for the unit disc given by

$$P_r(\gamma) = \frac{1 - r^2}{1 - 2r \cos \gamma + r^2}.$$

13. Suppose  $f$  is an analytic function defined everywhere in  $\mathbb{C}$  and such that for each  $z_0 \in \mathbb{C}$  at least one coefficient in the expansion

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

is equal to 0. Prove that  $f$  is a polynomial.

HINT: Use the fact that  $c_n n! = f^{(n)}(z_0)$  and use a countability argument.

14. Let  $\Omega$  be a bounded open subset of  $\mathbb{C}$ , and  $\phi : \Omega \rightarrow \Omega$  a holomorphic function. Prove that if there exists a point  $z_0 \in \Omega$  such that

$$\phi(z_0) = z_0 \quad \text{and} \quad \phi'(z_0) = 1$$

then  $\phi$  is linear.

HINT: Why can one assume that  $z_0 = 0$ ? Write  $\phi(z) = z + a_n z^n + O(z^{n+1})$  near 0, and prove that if  $\phi_k = \phi \circ \cdots \circ \phi$  ( $k$  times) then  $\phi_k(z) = z + k a_n z^n + O(z^{n+1})$ . Apply the Cauchy inequalities and let  $k \rightarrow \infty$  to conclude the proof.

15. [Due Wednesday 3 November] This exercise shows that one cannot always extend a holomorphic function from a smaller set to a larger one (see the section on Schwarz reflection principle in Stein-Shakarchi for some cases in which one can extend). The following definition is needed. Let  $f$  be a function defined in the unit disc  $\mathbb{D}$ , with boundary circle  $C$ . A point  $w$  on  $C$  is said to be *regular* for  $f$  if there is an open neighbourhood  $U$  of  $w$  and an analytic function  $g$  on  $U$ , so that  $f = g$  on  $\mathbb{D} \cap U$ . A function  $f$  defined on  $\mathbb{D}$  cannot be continued analytically past the unit circle if no point of  $C$  is regular for  $f$ .

(a) Let

$$f(z) = \sum_{n=0}^{\infty} z^{2^n} \quad \text{for } |z| < 1.$$

Notice that the radius of convergence of the above series is 1. Show that  $f$  cannot be continued analytically past the unit disc.

HINT: Suppose  $\theta = 2\pi p/2^k$ , where  $p$  and  $k$  are positive integers. Let  $z = re^{i\theta}$ ; then  $|f(re^{i\theta})| \rightarrow \infty$  as  $r \rightarrow 1$ .

(b) Fix  $0 < \alpha < \infty$ . show that the analytic function  $f$  defined by

$$f(z) = \sum_{n=0}^{\infty} 2^{-n\alpha} z^{2^n} \quad \text{for } |z| < 1$$

extends continuously to the unit circle, but cannot be analytically continued past the unit circle.

16. **[Due Wednesday 3 November]** Prove the converse to Runge's theorem: if  $K$  is a compact set whose complement is not connected, then there exists a function  $f$  holomorphic in a neighborhood of  $K$  which cannot be approximated uniformly by polynomials on  $K$ .

HINT: Pick a point  $z_0$  in a bounded component of  $K^c$ , and let  $f(z) = 1/(z - z_0)$ . If  $f$  can be approximated uniformly by polynomials on  $K$ , show that there exists a polynomial  $p$  such that  $|(z - z_0)p(z) - 1| < 1$ . Use the maximum modulus principle (see below) to show that this inequality continues to hold for all  $z$  in the component of  $K^c$  that contains  $z_0$ . *The maximum modulus principle (which we learn later) states that if  $h$  is a non-constant holomorphic function in a region  $\Omega$ , then  $|h|$  cannot attain a maximum in  $\Omega$ .*