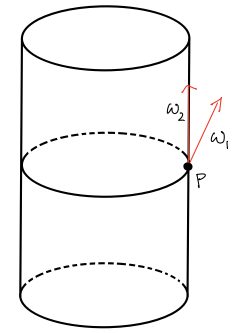
tangent plane $\parallel$

Then $\mathbb{N}_\sigma(p) = (1,0,0)$ corresponding to the y-z plane.



Let $\omega_1 = (0,1,0)$. Then $\left(\nabla_{\omega_1} \mathbb{N}_\sigma\right)(p) = \frac{\partial}{\partial \theta} \mathbb{N}_\sigma(p) = (-\sin\theta, \cos\theta, 0)_p$

$= (0,1,0) = \omega_1$. Let $\omega_2 = (0,0,1)$. Then $\left(\nabla_{\omega_2} \mathbb{N}_\sigma\right)(p)$

$= \frac{\partial}{\partial z} \mathbb{N}_\sigma(p) = (0,0,0)$. Thus $D_p\mathbb{N}_\sigma(\omega_1) = \omega_1$, $D_p \mathbb{N}_\sigma(\omega_2) = 0$.

The negative of $D_p\mathbb{N}$, i.e., $-D_p\mathbb{N} : T_pS \longrightarrow T_pS$ is called the <u>Weingarten map</u> or the <u>shape operator</u> of S at p. We also denote this map by $\mathcal{S}h_p$ or just $\mathcal{S}h$.

To summarize, for any $\omega \in T_pS$, we have

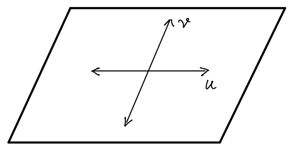
$\mathcal{S}h_p(\omega) = -D_p\mathbb{N}(\omega) = -\nabla_\omega \mathbb{N}$ (directional derivative of $\mathbb{N}$ along ω)

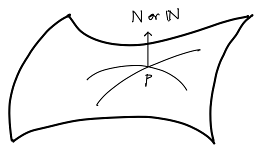
$=$ $-$ derivative of $\mathbb{N}$ along any curve through p with velocity ω.

We should think of $\mathcal{S}h_p$ as capturing the shape of S near p.

To compute using $\mathcal{S}h_p$ and further describe its properties we must express $\mathcal{S}h_p$ in terms of a suitable basis of T_pS. If $\sigma(u,v) : U \longrightarrow V \subseteq S$ is a surface patch, then there is a natural choice available: σ_u, σ_v, i.e., we need to express $\mathcal{S}h_p(\sigma_u)$, $\mathcal{S}h_p(\sigma_v)$ as linear combinations of σ_u, σ_v respectively.

It turns out it is more convenient and useful to first compute the dot products of $Sh_p(\sigma_u)$, $Sh_p(\sigma_v)$ with σ_u, σ_v. Let us choose $\mathbb{N}$ to be $\mathbb{N}_\sigma$, the normal induced by σ, so that $N_\sigma (= \mathbb{N}_\sigma \circ \sigma) = \dfrac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$. We use N for N_σ. We have the following relations:





$$N \cdot \sigma_u = 0 = N \cdot \sigma_v, \quad Sh(\sigma_u) = -\nabla_{\sigma_u} \mathbb{N} = -N_u, \quad Sh(\sigma_v) = -N_v.$$

Using this we obtain these calculations:

(i) $Sh(\sigma_u) \cdot \sigma_u = \sigma_{uu} \cdot N$ because $(N \cdot \sigma_u)_u = 0$ implies that

$$\underbrace{N_u \cdot \sigma_u}_{-Sh(\sigma_u)\cdot\sigma_u} + N \cdot \sigma_{uu} = 0.$$

(ii) $Sh(\sigma_v) \cdot \sigma_v = \sigma_{vv} \cdot N$ (similar to (i), using $(N \cdot \sigma_v)_v = 0$).

(iii) $Sh(\sigma_u) \cdot \sigma_v = \sigma_{vu} \cdot N$ because $(N \cdot \sigma_v)_u = 0$ implies that

$$N_u \cdot \sigma_v + N \cdot \sigma_{vu} = 0.$$

(iv) $Sh(\sigma_v) \cdot \sigma_u = \sigma_{uv} \cdot N$ (use $(N \cdot \sigma_u)_v = 0$).

Thus, the dot-products are the entries of the second fundamental form $\begin{bmatrix} L & M \\ M & N \end{bmatrix}$. We deduce some important properties from this.

Proposition: Let notation and assumptions be as above.

(i) The bilinear form $(\omega_1, \omega_2) := Sh_p(\omega_1) \cdot \omega_2$ on T_pS is symmetric, i.e., $(\omega_1, \omega_2) = (\omega_2, \omega_1)$ $\forall\, \omega_i \in T_pS$.

T_pS
$\cong$

(ii) The corresponding induced symmetric bilinear form on T_qU (where $q \in U$ is such that $\sigma(q) = p$) is $(z_1, z_2) = z_1^t \begin{bmatrix} L & M \\ M & N \end{bmatrix} z_2$.

Proof: (i) Writing $\omega_i = a_i \sigma_u + b_i \sigma_v$ we see that

$$(\omega_1, \omega_2) = a_1 a_2 (\sigma_u, \sigma_u) + a_1 b_2 (\sigma_u, \sigma_v) + b_1 a_2 (\sigma_v, \sigma_u) + b_1 b_2 (\sigma_v, \sigma_v).$$

Since $(\sigma_u, \sigma_v) = (\sigma_v, \sigma_u)$ $(= M)$, the result follows.

(ii) It suffices to calculate the induced symmetric bilinear form on the standard basis e_1, e_2 of T_qU. Now $(e_1, e_1) := (D_q\sigma(e_1), D_q\sigma(e_1))$ $= (\sigma_u, \sigma_u) = L$. The other calculations are similar. Q.E.D.

Viewing T_pS and T_qU as inner-product spaces (via the standard dot-product and the FFF respectively) we deduce that Sh induces a self-adjoint operator on these spaces:

$(\omega_1, \omega_2) := Sh(\omega_1) \cdot \omega_2$, $\quad (z_1, z_2) = Sh(D_q\sigma\, z_1) \cdot D_q\sigma\, z_2$. Just like the FFF is the form on T_qU induced by the dot-product on T_pS, the SFF is the form on T_qU induced by the shape.

Examples: (i) For a plane in $\mathbb{R}^3$, $Sh \equiv 0$ everywhere and hence at every point the corresponding symmetric bilinear form is also zero. The image of the Gauss map in S^2 is a single point corresponding to the (unique) normal of the plane.

(ii) For the unit sphere in $\mathbb{R}^3$, Sh is minus of the identity map at each tangent space (if we use the latitude-longitude parametrisation) and the corresponding symmetric bilinear form is negative of the usual inner-product. The image of the Gauss map is the whole sphere.

(iii) For the right circular cylinder, Sh induces $\pm$ projection to the tangent vector along the circular section. The image of the Gauss map is the equator of the sphere.

Proposition: With notation as before, for any unit vector $w \in T_pS$, we have $Sh(w) \cdot w = \kappa_n(w)$.

Proof: Pick a unit-speed normal section $\gamma(s)$ on S such that

$r(0) = p$, $\dot{r}(0) = \omega$. Differentiate the relation $\overset{\text{tangent}}{\dot{r}} \cdot \overset{\text{normal}}{\mathbb{N}} = 0$ with respect to s (along the curve $r(s)$) to get $\ddot{r} \cdot \mathbb{N} + \dot{r} \cdot \dot{\mathbb{N}} = 0$. Hence,

$$\kappa_n(\omega) = \text{signed curvature of the normal section } r(s)$$

$$= \ddot{r}(0) \cdot (\text{Normal}_{(\text{signed})} \text{ to } r(s) \text{ at } p = r(0))$$

$\left(\ddot{r}(s) = \kappa_{\text{signed}}(s)\, \text{Normal}_{\text{signed}}(s)\right)$

Both have angle $\pi/2$ anticlockwise from $\omega = \dot{r}(0)$ in the oriented basis $\omega, \mathbb{N}$ of the normal section, hence are equal.

$$= \ddot{r}(0) \cdot \mathbb{N}(s=0)$$

$$= -\dot{r}(0) \cdot \dot{\mathbb{N}}(0) = -\omega \cdot \nabla_\omega \mathbb{N} = Sh_p\omega \cdot \omega. \qquad \text{Q.E.D.}$$