Lecture 8: Tensor products of representations.

05 October 2021
09:02

Def^n: Let V be a representation of a group G. It is said to be irreducible if it doesnot have any nontrivial subrepresentation.

Example: Every one dimensional representation is irreducible.

Thm: Let $\rho: G \longrightarrow GL(V)$ be a representation & $W \subseteq V$ be a subrepresentation of G. Then $\exists\, W_0 \subseteq V$ a subrepresentation of G s.t. $V = W \oplus W_0$ i.e. W_0 is a complement of W in V & W_0 is stable under G-action.

(*) Let V & W be G-representations (also called modules). Then $V \oplus W$ is also a G-representation via the natural action.

$$g\cdot(v,w) = (g\cdot v, g\cdot w) \quad \forall\, v \in V,\ w \in W\ \&\ g \in G.$$

$$\rho_{V\oplus W}(g) = \rho_V(g) \oplus \rho_W(g) : V\oplus W \longrightarrow V \oplus W \quad \in GL(V\oplus W)$$

$\rho_V : G \longrightarrow GL(V)$ & $\rho_W : G \longrightarrow GL(W)$.

$\mathcal{B} = \{v_1, \ldots, v_n\}$ a basis of V & $\mathcal{B}' = \{w_1, \ldots, w_m\}$ a basis of W.

So $\rho_V(g)$ w.r.t $\mathcal{B}$ is A_g & $\rho_W(g) = B_g$ w.r.t. $\mathcal{B}'$.

$\rho_{V\oplus W}(g)$ w.r.t $\mathcal{B} \cup \mathcal{B}'$ is the matrix $\begin{bmatrix} A_g & 0 \\ 0 & B_g \end{bmatrix}$

(*) Let V be a G-repr & W be a subrepr of V. Then V/W is also a G-repr.

$$\rho_{V/W}(g)(\bar{v}) := \overline{\rho_V(g)(v)}$$ is well-defined.

$\bar{v} = \bar{v'} \Rightarrow v - v' \in W$

$\rho_V(g)(v - v') \in W \quad \forall g \in G$

$\Rightarrow \rho_V(g)(v) - \rho_V(g)(v') \in W$

check $\rho_{V/W} : G \to GL(V/W)$ is homo.

In fact, V is a G-repr then V is $k[G]$-module via $k[G] \times V \longrightarrow V$

$$\Big(\sum_{g_i \in G} a_i g_i\Big)\cdot v = \sum a_i\, \rho_V(g_i)(v) \quad \text{for } a_i \in k,\ g_i \in G\ \&\ v \in V.$$

& conversely if a v.s. V is a $k[G]$-module then the restriction is a G-representation.

$$\rho_V(g) = \{ v \longmapsto 1g\cdot v \} \in GL(V)$$

& note that :- $k[G]$

$1g^{-1}\cdot 1g = 1e_G$ which is the multiplicative identity of $k[G]$

V is irred. repr means V is simple $k[G]$-module.

Cor Every group representation is a direct sum irred representation.

Pf: V a repr. If V is irred stop. Otherwise let W be a proper nonzero subrep. Then by the thm $V = W \oplus W'$. Since $\dim(W) < \dim(V)$ & $\dim(W') < \dim(V)$. By induction on dim W & W' are direct sum irred. reps & hence V is " " " ". ∎

⊛ Let V & W be G-representation. Then $V \otimes W$ also has a natural G-repr given by $g.(v \otimes w) = g.v \otimes g.w$ for $v \in V$ & $w \in W$

Want a map $G \times V \otimes W \xrightarrow{\theta} V \otimes W$ s.t. $\theta(g, -) \in GL(V \otimes W)$

For $g \in G$
$$\psi_g : V \times W \longrightarrow V \otimes W$$
$$(v, w) \longmapsto \rho_V(g)(v) \otimes \rho_W(g)(w)$$

Since ψ_g is bilinear, we get a map

$$\tilde{\Psi}_g : V \otimes W \longrightarrow V \otimes W$$
$$v \otimes w \longmapsto \rho_V(g)(v) \otimes \rho_W(g)(w)$$

Define $\Theta : G \times V \otimes W \longrightarrow V \otimes W$

as $\Theta(g, \alpha) = \tilde{\Psi}_g(\alpha)$ for $\alpha \in V \otimes W$ & $g \in G$

Want to verify Θ defines a reps on $V \otimes W$

Note that $\Theta(g, v \otimes w) = \tilde{\Psi}_g(v \otimes w)$
$$= g \cdot v \otimes g \cdot w$$

$$\Theta(gg', v \otimes w) = gg' \cdot v \otimes gg' \cdot w$$
$$= g \cdot (g' \cdot v) \otimes g \cdot (g' \cdot w)$$
$$= \Theta(g, g' \cdot v \otimes g' \cdot w)$$
$$= \Theta(g, \Theta(g', v \otimes w))$$

Since $\{v \otimes w \mid v \in V \,\&\, w \in W\}$ gen $V \otimes W$

$\Theta(gg', \alpha) = \Theta(g, \Theta(g', \alpha)) \quad \forall \alpha \in V \otimes W$

Hence Θ is an action. (Note $\Theta(e, \cdot)$ is identity)

This also implies that $\Theta(g, -) = \tilde{\Psi}_g$ is in $GL(V \otimes W)$ with $\tilde{\Psi}_{g^{-1}}$ as the inverse.

$\boxtimes$

Tensor product of matrices.

A a $n\times n$ matrix & B a $m\times m$ matrix

then $A\otimes B := \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & & & \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{bmatrix}$

is of order $mn\times mn$

HW

$\rho_{V\otimes W}(g) = \rho_V(g)\otimes\rho_W(g)$ in terms of matrices.

Let $\{v_1,\dots,v_n\}$ be a basis of V & $\{w_1,\dots,w_m\}$ a basis of W. Then $\{v_1\otimes w_1, v_1\otimes w_2,\dots, v_1\otimes w_m, v_2\otimes w_1,\dots, v_2\otimes w_m,\dots, v_n\otimes w_1,\dots, v_n\otimes w_m\}$ is a basis $V\otimes W$.

(*) Let V be a repr of a group G. Then

$T^n V$ is a G-repr $\forall n$.

$$T^nV = \underbrace{V\otimes\cdots\otimes V}_{n \text{ times}}$$

where

$$g\cdot(v_1\otimes\cdots\otimes v_n) = g\cdot v_1\otimes\cdots\otimes g\cdot v_n.$$

Similarly $\mathrm{Sym}^n V$ is also a G-rep & $\Lambda^n V$ is a G-repr. via

$$\rho_{\mathrm{Sym}^n(V)}(g)(v_1\cdots v_n) = \rho_V(g)(v_1)\cdot\cdots\cdot\rho_V(g)(v_n)$$

and $$\rho_{\Lambda^n V}(g)(v_1\wedge\cdots\wedge v_n) = \rho_V(g)(v_1)\wedge\cdots\wedge\rho_V(g)(v_n)$$

respectively.

(*) $$T^2V = V\otimes V \underset{\substack{\uparrow \\ G\text{-equivariant}}}{\cong} \mathrm{Sym}^2 V\oplus\Lambda^2 V$$ (HW)