Lecture 12: Orthogonality of characters

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Recall: Given a repr $\rho : G \to GL(V)$, the character of V

$$\chi_V : G \to k \text{ the base field, we assume } k = \mathbb{C}.$$
$$g \mapsto tr(\rho(g))$$

- $\chi(1) = \dim V$; $\chi(g^{-1}) = \overline{\chi(g)}$; χ is a class function $\chi(g^{-1}hg) = \chi(h)$
  $\chi_{V\oplus W} = \chi_V + \chi_W$; $\chi_{V\otimes W} = \chi_V \cdot \chi_W$, $\chi_{V^*} = \overline{\chi_V}$; $\chi_{Hom(V,W)} = \overline{\chi_V} \cdot \chi_W$

- $\chi_{k[G]}(1) = |G|$, $\chi_{k[G]}(g) = 0 \quad \forall g \neq 1$.

<u>Defn</u>: Let V_1 & V_2 be two repr of a group G. Let χ_1, χ_2 be corresponding characters.

$$(\chi_1 | \chi_2) = \frac{1}{|G|} \sum_{g\in G} \chi_1(g) \overline{\chi_2(g)} \in \mathbb{C}$$

Note that $(\, | \,)$ is linear in first variable & conjugate linear in 2^{nd} var. $= \langle \chi_1, \chi_2 \rangle$

- Note $(\chi_1 | \chi_2) = \frac{1}{|G|} \sum_{g\in G} \chi_1(g) \chi_2(g^{-1})$. Hence it is bilinear on set of characters of G.

- Note that $\chi_{Hom(V_2, V_1)} = \chi_1 \overline{\chi_2}$.

⊛ Also if V is a repr of G then the map

$\varphi = \frac{1}{|G|}\sum_{g\in G} g : V \longrightarrow V$ is a G-equivar map.

Claim: φ is a projection map onto $V^G = \{v\in V \mid gv = v \ \forall g\in G\}$

Pf: $v\in V^G \Rightarrow \varphi(v) = v$ & $\mathrm{Im}(\varphi) = \{w = \frac{1}{|G|}\sum_{g\in G} gv \mid v\in V\}$

$g'w = \frac{1}{|G|}\sum_{g\in G} g'gv = w \quad \forall g'\in G$

$\Rightarrow \mathrm{Im}(\varphi) \subseteq V^G$ & $\varphi_{V^G} = id$

$\Rightarrow \mathrm{Im}(\varphi) = V^G$.

$$\mathrm{Tr}(\varphi) = \frac{1}{|G|}\sum_{g\in G}\chi(g)$$

$V = V^G \oplus W$ where $W = \ker(\varphi)$ is some subrepr of V.

$$\chi_V = \chi_{V^G} + \chi_W$$

$\chi_{V^G}(g) = \dim(V^G) \quad \forall g\in G$.

$\mathrm{Tr}(\varphi) = \dim(V^G)$ ($\because \varphi$ is projection on V^G.

$$\frac{1}{|G|}\sum_{g\in G}\chi(g) = \dim(V^G)$$

Prop: If V_1 & V_2 are irred then

$$(\chi_1 \mid \chi_2) = \begin{cases} 1 & \text{if } V_1 \cong V_2 \\ 0 & \text{o.w.} \end{cases}$$

Pf: $\mathrm{Hom}(V_1, V_2)$ is also a G-repr. $\mathrm{Hom}(V_1,V_2)^G$ is the set G-equiv lin maps from V_1 to V_2.

$(g\varphi)(gv) = g\cdot\varphi(v)$

$$\begin{array}{ccc} V_1 & \xrightarrow{\varphi} & V_2 \\ g\downarrow & & \downarrow g \\ V_1 & \xrightarrow{g\varphi} & V_2 \end{array}$$

So $g\varphi = \varphi \ \forall g\in G$ iff

$\varphi(gv) = g\cdot\varphi(v) \quad \forall g\in G$

$\Leftrightarrow \varphi$ is G-equiv.

$\iff \varphi$ is G-equiv.

By Schurs' lemma if $V_1 \cong V_2$ then $\mathrm{Hom}(V_2, V_1)^G$ is one dim'l.

So $\dim \mathrm{Hom}(V_2, V_1)^G = 1 = \frac{1}{|G|} \sum_{g \in G} \chi_{\mathrm{Hom}(V_2,V_1)}(g)$

$= \frac{1}{|G|} \sum \chi_1(g) \bar{\chi}_2(g)$

$= (\chi_1 | \chi_2)$

Finally if $V_1 \ncong V_2$ then $\mathrm{Hom}(V_1, V_2)^G = 0$ by schur's lemma

Hence $(\chi_1 | \chi_2) = \dim \mathrm{Hom}(V_2, V_1)^G = 0$. $\boxed{A}$

<u>Cor</u>: Let V_1 & V_2 be irred G-repr & $h : V_1 \longrightarrow V_2$ be any linear map. Let

$$h_0 = \frac{1}{|G|} \sum_{g \in G} \rho_{V_2}^{-1}(g) h \rho_{V_1}(g) : V_1 \longrightarrow V_2 \quad \text{- - - - -} \quad (*)$$

Then 1) If V_1 & V_2 are not isom then $h_0 = 0$

2) If $V_1 = V_2$ & $\rho_{V_1} = \rho_{V_2}$ then $h_0 = \frac{1}{n} \mathrm{Tr}(h)$ where $n = \dim V$

<u>Matrix interpretation</u>

Let $\rho_{V_1}(g)$ be given by a matrix $A(g) = ((a_{ij}(g)))$ for $g \in G$ & $\rho_{V_2}(g)$ be given by $B(g) = ((b_{ij}(g)))$. Let matrix of h be $C = ((c_{ij}))$ & of h_0 be $C_0 = ((c_{ij}^0))$

Then $$C_0 = \frac{1}{|G|} \sum_{g \in G} B(g^{-1}) C A(g)$$

1) V_1 & V_2 are not isom then $C_0 = 0$ i.e.

$$\sum_{g\in G} B(g^{-1}) C A(g) = 0$$

$$\sum_{g\in G} \sum_{j,k} b_{ij}(g^{-1}) c_{jk} a_{kl}(g) = 0 \quad \forall\ i,l$$

Since h is arbitrary C is arbitrary i.e. the above holds for all c_{jk}

Hence $\frac{1}{|G|} \sum_{g\in G} b_{ij}(g^{-1}) a_{kl}(g) = 0 \quad \forall\ i,j,k,l.$

$$\overset{\shortparallel}{\langle a_{kl}, b_{ij}\rangle} = 0$$

2) If $V_1 = V_2$ & $B = A$ then $C_0 = \frac{1}{n} \mathrm{Tr}(C) I$ $\qquad \delta_{jk} = \begin{cases} 0 & j\neq k \\ 1 & j=k \end{cases}$

$$\frac{1}{|G|} \sum_{g\in G} \sum_{j,k} a_{ij}(g^{-1}) c_{jk} a_{kl}(g) = \left(\frac{1}{n} \sum_{j,k} c_{j,k}\, \delta_{jk}\right) \delta_{il}$$

$$\Rightarrow \quad \frac{1}{|G|} \sum_{g\in G} a_{ij}(g^{-1}) a_{kl}(g) = \frac{1}{n} \delta_{jk} \delta_{il} = \begin{cases} 1/n & j=k \ \& \ i=l \\ 0 & \text{o.w.} \end{cases}$$

$$\overset{\shortparallel}{\langle a_{kl}, a_{ij}\rangle}$$

Now if V is an irred repr with char χ then

$$(\chi|\chi) = \frac{1}{|G|} \sum_{g\in G} \chi(g) \chi(g^{-1}) = \frac{1}{|G|} \sum_{g\in G} \sum_{i,j} a_{ii}(g) a_{jj}(g^{-1})$$

$$= \sum_{i,j} \langle a_{ii}, a_{jj}\rangle$$

$$= \sum_{i=j} \frac{1}{n} = 1$$

& $V_1 \neq V_2$ irred reps then

$$(\chi_1|\chi_2) = \frac{1}{|G|} \sum_{g\in G} \chi_1(g) \chi_2(g^{-1}) = \frac{1}{|G|} \sum_{g\in G} \sum_{i,j} a_{ii}(g) b_{jj}(g^{-1})$$

$$= \sum_{i,j} \langle a_{ii}, b_{jj}\rangle$$

$$= 0$$