Lecture 17: Induced representation contd.

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Def^n: Let $\rho : G \to GL(V)$ be a repr, let $H \leq G$ and let $W \subseteq V$ be a H-subrepr of V, i.e. $\rho(h)(w) \in W$ $\forall$ $w \in W$ & $h \in H$.
Let $\theta : H \to GL(W)$ denote this repr. i.e. $\theta(h) = \rho(h)|_W$ for $h \in H$. For $g \in G$, let $W_g := \rho(g)(W)$. Note that if g, g' are in the same left coset then $W_g = W_{g'}$. For $\bar{g} \in G/H$ ← collection of left cosets define $W_{\bar{g}} := W_g$ where $gH = \bar{g}$.
We say that ρ or V is induced from θ or W if $V = \bigoplus_{\bar{g} \in G/H} W_{\bar{g}}$.

$g = g'h$ for some $h \in H$
$\Rightarrow \rho(g) = \rho(g')\rho(h)$
$\Rightarrow \rho(g)(W) = \rho(g')(W)$
$(\because \rho(h)(W) = W)$

Notation: $V = \mathrm{Ind}_H^G W$

- $W_{\bar{g}} \cap \bigoplus_{\substack{\bar{g}' \in G/H \\ \bar{g}' \neq \bar{g}}} W_{\bar{g}'} = \{0\}$ $\forall \bar{g} \in G/H$
- $\langle W_{\bar{g}} \rangle = V$.

$\iff V = \bigoplus_{\bar{g} \in G/H} W_{\bar{g}}$

Note $\dim(V) = [G:H] \dim(W)$

- For W, W' H-repr
- $\mathrm{Ind}_H^G W \oplus \mathrm{Ind}_H^G W' = \mathrm{Ind}_H^G (W \oplus W')$ - - - 1
- $W' \subseteq W$ subrepr $\mathrm{Ind}_H^G W' \subseteq \mathrm{Ind}_H^G W$ as subrepr. - - - 2
- $(\mathrm{Ind}_H^G W) \otimes V' = \mathrm{Ind}_H^G (\underbrace{W \otimes V'}_{H\text{-repr}})$ here V' is a G-repr.
- $\mathrm{Ind}_H^G k[H] = k[G]$.

Proof of Existence of induced repr.

Let $\theta : H \to GL(W)$ be a repr.
Want $\rho : G \to GL(V)$ s.t. W is a H-stable subspace with repr θ of V & ρ is induced from θ.
Using 1 it is enough to show the above for W irreducible.
So $W \subseteq k[H]$ as subrepr. & $k[H]$ as induced repr $k[G]$. Hence by 2, W has a induced G-repr (which is infact a subrepr of $k[G]$.) ∎

Prop (Universal property) Let G be a group & $H \leq G$. Let $\rho : G \to GL(V)$ be a repr induced from $\theta : H \to GL(W)$. Let $\rho' : G \to GL(V')$ be another repr. Let $f : W \to V'$ be an H-equivariant map. Then f extends uniquely to a G-equivariant map $F : V \to V'$.

Pf: For $h \in H$ & $w \in W$,

$$f(\theta(h)(w)) = \rho'(h)(f(w))$$

i.e. $f \circ \theta(h) = \rho'(h) \circ f$

$$V = \bigoplus_{\bar{g} \in G/H} W_{\bar{g}} = \bigoplus_{\bar{g} \in G/H} \rho(g)(W) \quad \text{where } g \in \bar{g}$$

Since V is a direct sum enough to define F on $W_{\bar{g}}$.

For $w \in W_{\bar{g}}$ define

$$w = \rho(g)\rho(g^{-1})w \quad \text{for } g \in \bar{g}$$

$$F(w) := \rho'(g) f(\rho(g^{-1})w) \quad (\because \rho(g^{-1})(w) \in W \;\because w \in \rho(g)(W))$$

For $h \in H$, note that

$$\rho'(gh) f(\rho((gh)^{-1})(w))$$
$$= \rho'(g)\rho'(h) f(\rho(h^{-1})\rho(g^{-1})(w))$$
$$= \rho'(g)\cancel{\rho'(h)\rho'(h^{-1})} f(\rho(g^{-1})(w))$$
$$= F(w).$$

This defines F on V.

<u>Claim:</u> F is G-equivariant

$$V = W_{\bar{g}_1} \oplus \cdots \oplus W_{\bar{g}_r}$$

For $g \in G$ & $w \in W_{\bar{g}_i} = \rho(g_i)(W)$

$$F(\rho(g)(w)) = \rho'(gg_i) f(\rho(g_i^{-1}g^{-1})\rho(g)(w))$$

$$(\because \rho(g)(w) \in W_{\overline{gg_i}})$$

$$= \rho'(g)\rho'(g_i) f(\rho(g_i^{-1})(w))$$

$$= \rho'(g) F(w)$$

Since F is linear, the identity holds $\forall v \in V$.

$$v \in V \qquad v = w_1 + \cdots + w_r \qquad w_i \in W_{\bar{g}_i}$$

$$\rho(g)(v) = \rho(g)w_1 + \cdots + \rho(g)(w_r)$$

$$
\begin{aligned}
F(\rho(g)(v)) &= \sum_{i=1}^{r} F(\rho(g) w_i) \\
&= \sum_{i=1}^{r} \rho'(g)(F(w_i)) \\
&= \rho'(g)\left(F\left(\sum_{i=1}^{r} w_i\right)\right) \\
&= \rho'(g)(F(v))
\end{aligned}
$$

Note that for $w \in W = W_{\bar{e}}$

$$
\begin{aligned}
F(w) &= \rho'(e) f(\rho(e)^{-1} w) \\
&= f(w) \quad ,
\end{aligned}
$$

So F extends f.

Uniqueness is also trivial.

⊛ Proof of uniqueness of induced reps.

Let $\theta : H \to GL(W)$ be a H-rep &

V & V' are G-rep induced from W.

Since V is induced rep of W

& $i_1 : W \hookrightarrow V'$ which is H-equivariant

($\because$ W is a H-stable subspace of V' as well)

So i_1 extends to a unique G-equiv map

$\varphi_1 : V \to V'$ by the prev prop.

Similarly $\varphi_2 : V' \to V$ is the G-equivariant extension of

$i_2 : W \hookrightarrow V$

$\varphi_1 \circ \varphi_2 : V' \to V'$ is G-equivariant

& $\varphi_1 \circ \varphi_2 |_W = id$ & $Id_{V'}|_W = id$

Uniqueness of the extension $\implies \varphi_1 \circ \varphi_2 = Id_{V'}$

Direct construction of $\mathrm{Ind}_H^G W$. Let $\theta : H \to GL(W)$ be a repr.

Let $g_1, \dots, g_r$ be representatives in G of all the left cosets of H in G/H and let $g_1 = e$.

Let $W^{(i)}$ be a copy of W for $i = 1, \dots, r$ & $V := W^{(1)} \oplus \dots \oplus W^{(r)}$ as vector space. Define the G-action as follow: for $g \in G$ & $w \in W^{(i)}$

Note $g \cdot g_i = g_j h$ for some $1 \le j \le r$ & $h \in H$

$$g \cdot w = \theta(h)(w) \in W^{(j)}$$

$v \in V$ then

$$v = (w_1, w_2, \dots, w_r) \qquad w_i \in W^{(i)}$$

$$g \cdot g_i = g_{j_i} h_i \quad \text{for } 1 \le j_i \le r \;\&\; h_i \in H$$

$$g \cdot v = (\; , \; , h_1 \cdot w_1 , \; \ldots)$$

$\uparrow$ j^{th} component

One has to check that

$$g' \cdot (g \cdot w) = (g'g) \cdot w \quad \text{(Exc.)}$$

Note $W^{(1)} \subseteq V$ is H-stable

$w \in W^{(1)}$ & $h \in H$ then

$h \cdot e = e \cdot h$ &

$h \cdot w = \theta(h)(w) \in W^{(1)}$ &

Claim $g \cdot W^{(1)} = W^{(j)}$ where $g \in \bar{g}_j$

$g \cdot e = g_j h$ for some $h \in H$

& hence for $w \in W^{(1)}$

$$g \cdot w = \theta(h)(w) \in W^{(j)}$$

Since $V = \bigoplus_{j=1}^{r} W^{(j)}$, it is induced from θ.

$\blacksquare$