Lecture 7: Irreducible representation

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Def^n: Let G be a finite group & V be a finite dim'l vector space over k. Then a representation of G is a group homo. $\rho : G \longrightarrow GL(V)$. In this case V is said to be a representation of G.

Equivalently, it is an action of a group G on a v.s. V s.t. $g\cdot(v_1 + a v_2) = g\cdot v_1 + a\, g\cdot v_2$ $\forall g \in G$, $v_1, v_2 \in V$ & $a \in k$.

⊛ Giving a one dim'l representation of G is equivalent to giving a grp homo $\rho : G \longrightarrow k^*$. Here take $V = k$ & $g\cdot a = \rho(g)a$.

⊛ Let $\rho : G \longrightarrow GL(V)$ & $\rho' : G \longrightarrow GL(V')$ be two reps of G. A linear map $\tau : V \longrightarrow V'$ is said to be G-equivariant or a homo of representations if $\tau \circ \rho(g) = \rho'(g) \circ \tau$ $\forall g \in G$. $\longrightarrow$ $\boxed{\rho(g) = \tau^{-1}\rho'(g)\circ\tau \ \forall g \in G.}$

Equivalently, $\tau(g\cdot v) = g\cdot\tau(v)$ $\forall g \in G$ & $v \in V$. If τ is an isom then ρ & ρ' are said to isomorphic or similar.

Def^n: Let V be a rep of a group G. A subspace W of V is called a subrepresentation of V if $\forall g \in G$ $\rho(g)(w) \in W$ $\forall w \in W$. i.e. W is stable under G-action. ⊛

$$\rho : G \longrightarrow GL(V)$$

⊛ $\Rightarrow$ $\rho_W : G \longrightarrow GL(W)$ where $\rho_W(g) = \rho(g)|_W$

(*) $i : W \hookrightarrow V$ the inclusion map is G-equivariant. ($\because$ $i(g\cdot w) = g\cdot i(w)$ as i inclusion)

(*) $V = k[G]$, the regular representation of G.

$W = \langle \sum_{g\in G} g \rangle$ is a subrepresentation.

$W' = k$, $g.a = a$ $\forall a \in k$ & $g \in G$.

$$\tau : W' \longrightarrow V = k[G] \quad \text{is a } G\text{-equi map.}$$
$$a \longmapsto a\left(\sum_{g\in G} g\right)$$

$G = \mathbb{Z}/n\mathbb{Z}$ cyclic group.

$\rho : G \longrightarrow \mathbb{C}^*$ is a 1-dim'l non trivial representation.

$\bar{1} \longmapsto e^{2\pi i/n}$

$$v = \sum_{l=0}^{n-1} e^{2\pi i l/n} \cdot \bar{l} \in \mathbb{C}[\mathbb{Z}/n\mathbb{Z}]$$

$\oplus$ in $\mathbb{Z}/n\mathbb{Z}$

$$\bar{m} \cdot v = \sum_{l=0}^{n-1} e^{2\pi i l/n} \cdot (\bar{m} \oplus \bar{l}) = \sum_{l=0}^{n-1} e^{2\pi i (l+m)/n} \cdot e^{\frac{-2\pi i m}{n}} (\bar{m}\oplus\bar{l}) = e^{\frac{-2\pi i m}{n}} v$$

Hence $W = \langle v \rangle$ is a subrepresentation of $\mathbb{C}[\mathbb{Z}/n\mathbb{Z}]$.

$$\tau : \mathbb{C} \longrightarrow \mathbb{C}[\mathbb{Z}/n\mathbb{Z}] \quad \text{is } \mathbb{Z}/n\mathbb{Z}\text{-equivariant.}$$
$$1 \longmapsto v$$

$$\mathbb{C}[\mathbb{Z}/n\mathbb{Z}] = a_0\bar{0} + a_1\bar{1} + \dots + a_{n-1}\overline{n-1} \qquad (\mathbb{Z}/n\mathbb{Z}, \oplus)$$

v.s

$$\bar{m} \cdot (a_0\bar{0} + \dots + a_{n-1}\bar{n}) = a_0(\bar{m}\oplus\bar{0}) + a_1(\bar{m}\oplus\bar{1}) + \dots + a_{n-1}(\bar{m}\oplus\overline{n-1})$$

<u>Def</u>

Def^n: Let V be a representation of a group G. It is said to be <u>irreducible</u> if it does not have any nontrivial subrepresentation.

<u>Example</u>: Every one dimensional representation is irreducible.

Thm: Let $\rho : G \to GL(V)$ be a representation & $W \subseteq V$ be a subrepresentation of G.
Then $\exists\, W_0 \subseteq V$ a subrepresentation of G s.t. $V = W \oplus W_0$. i.e. W_0 is a complement of W in V & W_0 is stable under G-action.

Pf: Let $p : V \to W \hookrightarrow V$ be a projection map to W. i.e. $p(x) = x$ $\forall\, x \in W$ & p is linear.

Let $W' = \ker(p)$ then $W' \oplus W = V$.

Let $p_0 = \frac{1}{|G|} \sum_{g \in G} \rho(g)\, p\, \rho(g^{-1})$ ($\frac{1}{|G|}$ make sense as $\mathrm{char}(k) = 0$ ← base field)

Note that p_0 is a projection as for $x \in V$, $p \circ \rho(g^{-1})(x) \in W$ & W is a subrep $\Rightarrow$ $\rho(g) \circ p \circ \rho(g^{-1})(x) \in W$

Hence $p_0(x) \in W$ $\forall\, x \in V$.

Moreover for $x \in W$ ($\because W$ is a subrep)

$$\rho(g)\, p\, \rho(g^{-1})(x) = \rho(g)\rho(g^{-1})(x)$$
$$= x$$

$\Rightarrow \quad p_0(x) = x.$

Hence p_0 is a projection. Let $W_0 = \ker(p_0)$

Claim: W_0 is a subrepresentation.

Note $\rho(g_0^{-1}) p_0 \rho(g_0) = p_0$ by def^n of p_0 $\forall g_0 \in G$

$$\rho(g_0^{-1}) \frac{1}{|G|} \sum_{g \in G} \rho(g) p \rho(g^{-1}) \rho(g_0) = \frac{1}{|G|} \sum_{g \in G} \rho(g_0^{-1} g) p \rho(g^{-1} g_0)$$
$$= p_0$$

Hence $\rho(g) p_0 = p_0 \rho(g)$

$\Rightarrow$ $x \in W_0$ then

$p_0 \circ \rho(g) x = \rho(g) \circ p_0(x) = 0$

$\Rightarrow \rho(g)(x) \in W_0 = \ker(p_0) \quad \forall g \in G$

$\blacksquare$