Lecture 10: Schur's lemma

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<u>Def^n</u>: Let $\rho : G \longrightarrow GL(V)$ be a G-repr. The <u>character</u> of ρ, χ_ρ is function from $G \to \mathbb{C}$ given by $\chi_\rho(g) = Tr(\rho(g)) \quad \forall\, g \in G$.

<u>Prop</u>: Let χ be a character of a G-repr V. Then

(i) $\chi(1) = \chi(e_G) = \dim V$

(ii) $\chi(g^{-1}) = \overline{\chi(g)}$

(iii) $\chi(g^{-1}hg) = \chi(h)$ i.e. χ is class function.

(*) V & W are G-reps. Then

a) $\chi_{V \oplus W} = \chi_V + \chi_W$

b) $\chi_{V \otimes W} = \chi_V \cdot \chi_W$

c) $\chi_{V^*} = \overline{\chi_V}$ $\left[\because \rho_{V^*}(g) = (\rho_V(g))^{-1\,T}\right]$ matrix reps.

$$\left[\begin{aligned} \chi_{V^*}(g) &= Tr(\rho_{V^*}(g)) = Tr(\rho_V(g)^{-1}) \\ &= \sum \lambda_i^{-1} \quad \text{(where } \lambda_i \text{ are eigenvalues of } \rho_V(g)) \\ &= \sum \overline{\lambda_i} \quad (\because |\lambda_i| = 1) \\ &= \overline{\chi_V(g)} \end{aligned}\right]$$

(d) $\chi_{\text{Hom}(V,W)} = \chi_{V^*\otimes W}$ $\left(\because \text{Hom}(V,W) \underset{\text{as repr.}}{\cong} V^*\otimes W\right)$

$= \overline{\chi_V} \cdot \chi_W$

(e) $\chi_{T^n V} = \chi_V^n$

Example: Let G be a finite group then $V = k[G]$ is a regular repr.

$\chi_V(1) = |G|$

$g \in G$ s.t. $g \neq 1$. then $\chi_V(g) =$

G acts on G via left multiplication

$G \longrightarrow \text{Sym}(G)$

$g \longmapsto \sigma_g : G \longrightarrow G$

$h \longmapsto gh$

$\sigma_g(h) = h \Rightarrow gh = h \Rightarrow g = e$

Hence $g \neq e$ then $\sigma_g(h) \neq h \ \forall h \in G$.

w.r.t the basis $\{1g \mid g \in G\}$ of $V = k[G]$

$\rho_V(g)$ is a permutation matrix whose diagonal entries are 0. Hence $\text{Tr}(\rho_V(g)) = 0$.

Hence $\chi_V(g) = 0 \quad \forall g \neq 1$. $\boxtimes$

(*) More generally, let X be a G-set &

$\rho : G \to V_X$ be the permutation repr.

Then $\chi_{V_X}(g) = \#$ fixed points of σ_g where $\sigma_g : X \to X$, $x \mapsto g\cdot x$

(*) $\text{Sym}^2 V \overset{i}{\hookrightarrow} T^2 V$. Let $V = \{e_1, \dots, e_n\}$

$e_i \cdot e_j \longmapsto e_i \otimes e_j + e_j \otimes e_i$

$$\begin{aligned} \mathrm{Ext}^2 V &\xhookrightarrow{i_2} T^2 V \\ e_i \wedge e_j &\longmapsto e_i \otimes e_j - e_j \otimes e_i \end{aligned}$$

Note $\{e_i \cdot e_j \mid i \leq j\}$ is a basis of $\mathrm{Sym}^2 V$

$\{e_i \wedge e_j \mid i < j\}$ is a basis of $\mathrm{Ext}^2 V$

Moreover $\mathrm{Sym}^2 V \oplus \mathrm{Ext}^2 V \xrightarrow[\sim]{i_1 \oplus i_2} T^2 V$ which is G-equivar (HW)

$\Rightarrow \{e_i \otimes e_j + e_j \otimes e_i \mid i \leq j\} \cup \{e_i \otimes e_j - e_j \otimes e_i \mid i < j\}$

is a basis of $T^2 V$.

$$g \cdot (e_i \otimes e_j + e_j \otimes e_i) = g \cdot e_i \otimes g e_j + g e_j \otimes g e_i$$

Now assume $\{e_1, \ldots, e_n\}$ are eigen vectors of $\rho(g)$ with eigen values $\lambda_1, \ldots, \lambda_n$

$$= \lambda_i \lambda_j (e_i \otimes e_j + e_j \otimes e_i)$$

$$\chi_{\mathrm{Sym}^2 V}(g) = \sum_{i \leq j} \lambda_i \lambda_j = \left(\sum \lambda_i\right)^2 - \sum_{j > i} \lambda_i \lambda_j$$

$$= \chi_V(g)^2 - \sum_{j > i} \lambda_i \lambda_j$$

$$g \cdot (e_i \otimes e_j - e_j \otimes e_i) = g e_i \otimes g e_j - g e_j \otimes g e_i$$

$$= \lambda_i \lambda_j (e_i \otimes e_j - e_j \otimes e_i)$$

$$\Rightarrow \chi_{\mathrm{Ext}^2 V}(g) = \sum_{j < i} \lambda_i \lambda_j$$

$$= \frac{\left(\sum \lambda_i\right)^2 - \sum \lambda_i^2}{2}$$

$$= \left[(\chi_V(g))^2 - \chi_V(g^2)\right] / 2$$

Note $\chi_{T^2 V} = \chi_{\mathrm{Sym}^2 V} + \chi_{\mathrm{Ext}^2 V}$

Schur's lemma: Let V & W be two irred G-representa

Let $f: V \to W$ be a G-equivariant map.

Then $f = 0$ or f is an isom. Moreover

if $W = V$ then f is a scalar multiple of identity

Pf: Let $V_0 = \ker(f)$. Then V_0 is

a subrep of V ($\because$ $v \in V_0$ & $g \in G$

$f(g \cdot v) = g \cdot f(v)$ (as f is G-equiv

$= 0$

$\Rightarrow g \cdot v \in V_0$)

But V is irred so $V_0 = 0 \neq V$ or $V_0 = V$

$\Downarrow$ $\Downarrow$

$f = 0$

If $V_0 = 0$ then f is injective.

$\mathrm{Im}(f) = f(V)$ is a subrep of W, since f is G-equi

Hence by irred of W $\mathrm{Im}(f) = W$.

$\Rightarrow f$ is an isom.

Now assume $V = W$.

$f : V \longrightarrow V$ is an endomorphism

Let λ be an eigen value of f.

$f - \lambda \mathrm{Id} : V \longrightarrow V$ is G-equivar.

and not an isom. Hence by prev part

$f - \lambda \mathrm{Id} = 0 \implies f = \lambda \mathrm{Id}$.

$\blacksquare$