Question 1. What effects do material type and temperature have on the life of the battery?
Question 2. Is there a choice of material that would give uniformly long life regardless of temperature? (Robust product design?)

Life (in hrs) data for the battery design experiment:

material type	temperature (°F) 15		70		125	
1	130	155	34	40	20	70
	74	180	80	75	82	58
2	150	188	126	122	25	70
	159	126	106	115	58	45
3	138	110	174	120	96	104
	168	160	150	139	82	60

Both factors, material type and temperature are important and there may be interaction between the two also. Let us denote the row factor as factor A and column factor as factor B (in general). Then the model for the data may be developed as follows.

Let y_{ijk} be the observed response when factor A is at the ith level ($i = 1, 2, \ldots, I$) and factor B is at the jth level ($j = 1, 2, \ldots, J$) for the kth replicate ($k = 1, 2, \ldots, K$). In the example, $I = 3, J = 3, K = 4$. This design is like having IJ different cells each of which has K observations, and one wants to see if the IJ cell means are different or not (in various ways).

$$y_{ijk} = \mu_{ij} + \epsilon_{ijk}, i = 1, 2, \ldots, I; j = 1, 2, \ldots, J; k = 1, 2, \ldots, K.$$

Therefore it is also called a completely randomized 2-factor design. We assume, ϵ_{ijk} are i.i.d. $N(0, \sigma^2)$. As before, this is a linear model, and hence various linear hypotheses can be tested. Let
$\bar{y}_{ij.} = \frac{1}{K}\sum_{k=1}^{K} y_{ijk}, i = 1, 2, \ldots, I; j = 1, 2, \ldots, J$
$\bar{y}_{i..} = \frac{1}{JK}\sum_{j=1}^{J}\sum_{k=1}^{K} y_{ijk} = \frac{1}{J}\sum_{j=1}^{J}\bar{y}_{ij.}, i = 1, 2, \ldots, I$
$\bar{y}_{.j.} = \frac{1}{IK}\sum_{i=1}^{I}\sum_{k=1}^{K} y_{ijk} = \frac{1}{I}\sum_{i=1}^{I}\bar{y}_{ij.}, j = 1, 2, \ldots, J$
$\bar{y}_{...} = \frac{1}{IJK}\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} y_{ijk} = \frac{1}{IJ}\sum_{i=1}^{I}\sum_{j=1}^{J}\bar{y}_{ij.} = \frac{1}{I}\sum_{i=1}^{I}\bar{y}_{i..} = \frac{1}{J}\sum_{j=1}^{J}\bar{y}_{.j.}$

Now, $\hat{\mu}_{ij} = \bar{y}_{ij.}$ under no constraints, and hence
RSS $= \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(y_{ijk} - \bar{y}_{ij.})^2$ has $IJ(K-1)$ d.f. To consider interesting questions, it is best to adopt the reparametrization,
$\mu_{ij} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij}$, where
$\mu = \bar{\mu}_{..} = \frac{1}{IJ}\sum_{i=1}^{I}\sum_{j=1}^{J}\mu_{ij}$, $\alpha_i = \bar{\mu}_{i.} - \bar{\mu}_{..}$, $\beta_j = \bar{\mu}_{.j} - \bar{\mu}_{..}$ and
$(\alpha\beta)_{ij} = \mu_{ij} - \bar{\mu}_{i.} - \bar{\mu}_{.j} + \bar{\mu}_{..}$.

Then note that $\sum_{i=1}^{I}\alpha_i = 0$, $\sum_{j=1}^{J}\beta_j = 0$, $\sum_{i=1}^{I}(\alpha\beta)_{ij} = 0$ for all j and $\sum_{j=1}^{J}(\alpha\beta)_{ij} = 0$ for all i.
(Note, $\sum_{i=1}^{I}(\alpha\beta)_{ij} = \sum_{i=1}^{I}\mu_{ij} - \sum_{i=1}^{I}\bar{\mu}_{i.} - I\bar{\mu}_{.j} + I\bar{\mu}_{..} = \sum_{i=1}^{I}(\mu_{ij} - \bar{\mu}_{.j}) = 0$.)
These are the conditions required for identifiability of the parameters under reparametrization.

Now consider the interpretation of these parameters. $\mu = \bar{\mu}_{..}$ is the overall effect. $\alpha_i = \bar{\mu}_{i.} - \bar{\mu}_{..}$ = main effect of factor A at level i since eliminating the effect of level j by averaging over it leaves the departure of effect i (of factor A) from overall, and similarly, $\beta_j = \bar{\mu}_{.j} - \bar{\mu}_{..}$ = main effect of factor B at level j. What does $(\alpha\beta)_{ij} = \mu_{ij} - \bar{\mu}_{i.} - \bar{\mu}_{.j} + \bar{\mu}_{..}$ measure?

Suppose we want to see if the effect of factor A at level i depends on the level of factor B. If there were no such interaction, we would expect the difference in means $\mu_{i_1 j} - \mu_{i_2 j}$ depend on i_1 and i_2 and not on j. i.e.,

$$\begin{aligned}
\mu_{i_1 j} - \mu_{i_2 j} &= \phi(i_1, i_2) = \frac{1}{J}\sum_{j=1}^{J}\phi(i_1, i_2) \\
&= \frac{1}{J}\sum_{j'=1}^{J}(\mu_{i_1 j'} - \mu_{i_2 j'}) = \bar{\mu}_{i_1 .} - \bar{\mu}_{i_2 .},
\end{aligned}$$

for all i_1, i_2. Or, equivalently, $\mu_{i_1 j} - \bar{\mu}_{i_1 .} = \mu_{i_2 j} - \bar{\mu}_{i_2 .}$ for all i_1, i_2. i.e.,

$$\begin{aligned}
\mu_{ij} - \bar{\mu}_{i.} &= \Phi(j) \text{ (independent of } i) \\
&= \frac{1}{I}\sum_{i'=1}^{I}\Phi(j) = \frac{1}{I}\sum_{i'=1}^{I}(\mu_{i'j} - \bar{\mu}_{i'.}) \\
&= \bar{\mu}_{.j} - \bar{\mu}_{..}, \text{ for all } i, j.
\end{aligned}$$

i.e., $\mu_{ij} - \bar{\mu}_{i.} - \bar{\mu}_{.j} + \bar{\mu}_{..} = 0$ for all i, j. Because of symmetry, we could have begun with $\mu_{ij_1} - \mu_{ij_2}$ depending on j_1, j_2, but not on i. Thus, we see that $(\alpha\beta)_{ij} = \mu_{ij} - \bar{\mu}_{i.} - \bar{\mu}_{.j} + \bar{\mu}_{..}$ measures the interaction of i and j. Therefore, to investigate the existence of interaction, we should test,
$H_{AB} : (\alpha\beta)_{ij} = 0$ $(i = 1, 2, \ldots, I; j = 1, 2, \ldots, J)$ as the restricted model without interaction. Estimation of $(\alpha\beta)_{ij}$ can also be considered. Now, consider the main effects of factors A and B.
To test for lack of difference in levels of factor A, use, $H_A : \alpha_i = 0$ for all i.
To test for lack of difference in levels of factor B, use, $H_B : \beta_j = 0$ for all j. If $H_{AB} : (\alpha\beta)_{ij} = 0$ has been rejected, there is evidence for significant interaction, so main effects cannot be non-existent.