Combinatorics

1 Dec 21

Lecture 16

$(X, \mathcal{F})$; $\mathcal{F}$ a family of subsets of X.

satisfying
1. closed under intersections
2. no ∞ chains
3. ϕ, X, $\{x\}$ $\forall x$ are in $\mathcal{F}$.
4. $\forall F \in \mathcal{F}$ the flats that cover F partition $X \setminus F$.

Lattice :- POSET where every finite set has a meet & a join.

Geometric lattices - lattices without ∞ chains, that are atomic & semi-modular.

Thm: Geometric lattice gives a comb. geometry & vice versa.

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rk of a flat = cardinality of maximal ind. set.

rank $AG_n = n+1$ (not n).

Semi-modular law :- $rk\,E + rk\,F \geq rk(E \cap F) + rk(E \vee F)$

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Theorem :- $\boxed{\text{Assume that every flat of rank } i \text{ has } k_i \text{ points in a geometry } (X, \mathcal{F}),\ \forall\, 0 \leq i \leq rk(X).}$

Then ① the total no. of rank r flats in $(X, \mathcal{F})$ is

$$\prod_{i=0}^{r-1} \frac{(v-k_i)}{(k_r-k_i)}$$

② Further, the set of points together with all rank r flats is a 2-design.

Proof :- ① If $r=1$, then the only rk 1 flat is a singleton set $\{x\}$. ($\because$ if it contains two pts $x \neq y$ then $\{x, y\}$ is independent!)

$\Rightarrow$ # rk 1 flats equals $v = \prod_{i=0}^{0} \frac{v-k_i}{k_1-k_i} = \frac{v-0}{1-0}$.

$\therefore$ the counting is true for $rk = 1$.

Induction Hypothesis :- For any geometry (with fixed flat size) the no. of rk s flats are

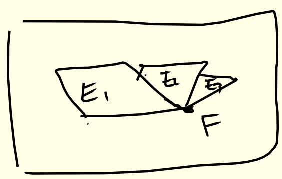
$$\prod_{i=0}^{s-1}\frac{(v-k_i)}{(k_s-k_i)} \quad \text{where } |X|=v.$$

$\forall s\le r$. (say).

Now look at $S=\{(F,E)\mid F$ is a rk r flat, E - rk $r+1$ & $F\subset E\}$

① Fixing F first, $\#F=\displaystyle\prod_{i=0}^{r-1}\frac{(v-k_i)}{(k_r-k_i)}$.

Also for a fixed F (rk r flat), the no. of rk $r+1$ flats that contain F are $\dfrac{(v-k_r)}{(k_{r+1}-k_r)}$



X

($\because$ E - rank $r+1$ $\Rightarrow$ $E\supset F$.
$\Rightarrow$ all such E's partition $X\setminus F$.
$|X\setminus F| = v-k_r$ & $|E\setminus F| = k_{r+1}-k_r$.

$$\Rightarrow |S| = \left(\prod_{i=0}^{r-1}\frac{v-k_i}{k_r-k_i}\right)\cdot\left(\frac{v-k_r}{k_{r+1}-k_r}\right). \quad \text{——— } (*)$$

② Fixing E first, $|S| = (\#\,(r+1)\text{-flats})\cdot(\#\, r\text{-flats inside a fixed } (r+1)\text{-flat})$

Since $|E|=k_{r+1}$ $\forall$ rk $r+1$ flats E, looking at the induced geometry on E & applying ind. hypothesis on $(E, \mathcal{B}\cap E)$ we get that $v=k_{r+1}$ & $\#\, r$-flats $=\displaystyle\prod_{i=0}^{r-1}\frac{(k_{r+1}-k_i)}{(k_r-k_i)}$.

$$\Rightarrow |S| = (\#(r+1)\text{-flats})\left(\prod_{i=0}^{r-1}\frac{k_{r+1}-k_i}{k_r-k_i}\right) \quad \text{——— } (**)$$

$$\Rightarrow \#\,(r+1)\text{-flats} = \prod_{i=0}^{r-1}\frac{v-k_i}{\cancel{k_r-k_i}}\cdot\frac{v-k_r}{(k_{r+1}-k_r)}\cdot\prod_{i=0}^{r-1}\frac{\cancel{(k_r-k_i)}}{(k_{r+1}-k_i)}$$

$$= \prod_{i=0}^{r}\frac{v-k_i}{k_{r+1}-k_i}.$$

$\therefore$ The formula is true for rk $r+1$ flats too.
This proves the first part.

② Consider $\mathcal{B}_r$ = set of r-flats in $(X, \mathcal{F}s)$ as above

look at the incidence system $(X, \mathcal{B}_r)$.

TPT it forms a 2-design.

By assumption block size is constant.

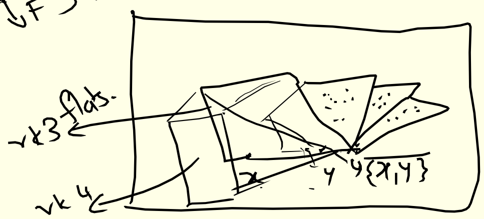
∴ only need to prove that any 2-set is in a fixed no. of blocks.

$\lambda = 1$

Let $x \neq y$ be two points. $\exists!$ rk 2 flat $\overline{\{x,y\}}$ containing it.

Then $\exists\, \lambda = \dfrac{v-k_2}{k_3-k_2}$ no. of rk 3 flats containing $\{x,y\}$.

(∵ any 3-flat containing $\{x,y\}$ must contain $\overline{\{x,y\}}$ a 2-flat)



∴ (X, rk3 flats) is a 2-design.

Each of these rk 3 flats are contained in $\dfrac{v-k_3}{k_4-k_3}$ rk-4 flats.

⇒ fixed no. of # rk 4 flats that contain $\{x,y\}$.

||ly for rk 5, I get ~~some~~ fixed number.

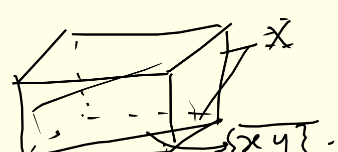
∴ For any $r \geq 2$, $(X, \mathcal{B}_r)$ is a 2-design.

QED.

Remark. $x \neq y \in X$, let $F_0 = \overline{\{x,y\}}$.

Let E be a 4-flat that contain $\overline{\{x,y\}}$

then # 3-flats that are contained in E & containing $\overline{\{x,y\}}$ are $\dfrac{k_4-k_2}{(k_3-k_2)}$. $k_4 = v$.



This is a fixed no. (not depending on $\{x,y\}$)

$$S = \{ (L,K) \mid L \subset K . \ \begin{matrix} L - 3\text{ flat containing } \{x,y\} \\ K - 4\text{-flat} \end{matrix} \}.$$

fixing L first.

$$|S| = \left(\frac{v-k_2}{k_3-k_2}\right) \cdot \frac{v-k_3}{(k_4-k_3)}.$$

Fixing K-first $=$ # 4-flats containing $\{x,y\}$ $\cdot \dfrac{k_4-k_2}{(k_3-k_2)}$.

$$\Rightarrow \quad \# \text{ 4-flats containing } x,y = \frac{v-k_2}{\cancel{(k_3-k_2)}} \cdot \frac{v-k_3}{(k_4-k_3)} \cdot \frac{\cancel{k_3-k_2}}{k_4-k_2}$$

$$= \frac{(v-k_2)\cdot(v-k_3)}{(k_4-k_2)(k_4-k_3)}.$$

Exercise:

Ily

$$\boxed{\# \text{ 5-flats containing } \{x,y\} = \frac{(v-k_2)(v-k_3)(v-k_4)}{(k_5-k_2)(k_5-k_3)(k_5-k_4)}}$$

& so on.

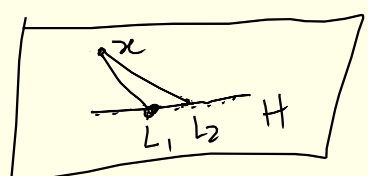
Theorem (Greene 1970) The no. of hyperplanes in ($rk(X)-1$)-flat in X) a finite combi. geometry is at least the no. of pts.

Pf :- Induction on rk of X.

- if $rk\, X = 1$ then $X = \{x\}$. hyperplane $= \phi$.
  $\therefore$ thm. is trivially satisfield.
- Assume by way of ind. hyp. that the thm. is true $\forall$ $rk < n$.

Now, let X be a rk n geometry.

let H be a hyper plane in X, $x \in X \setminus H$



X.

Claim :- no. of hyperplanes that contain x (say γ_x) is atleast $|H|$.

pf. by ind. hyp. $\exists$ at least $|H|$ hyperplanes in H.

$\forall$ $L \subset H$ a hyperplane,

$rk\, L = rk\, H - 1 = n-2$. $\therefore$ $L \vee \{x\}$ has rk $n-1$.

Also $L_1 \neq L_2$ hyperplanes in $H \Rightarrow L_1 \vee \{x\} \neq L_2 \vee \{x\}$.

$\therefore$ Because if $L_1 \vee \{x\} = L_2 \vee \{x\} \supset L_2$

Look at any $y \in L_2 \setminus L_1$ then

$y \in L_1 \vee \{x\}$

but $y \notin L_1 \Rightarrow x \in L_1 \vee \{y\}$ exchange lemma.

$\Rightarrow x \in H$ contradiction

QED of claim.

$$\boxed{\begin{array}{l} x \in \overline{A} \text{ but } x \in \overline{A \vee \{y\}} \\ \Rightarrow y \in \overline{A \cup \{x\}} \end{array}}$$

let $k_H = |H|$.

then we have proved $\boxed{r_x \geq k_H}$. $b =$ # hyperplanes. $\boxed{\text{Assume } b \leq v.}$

$\boxed{\underline{\text{Exercise}}}$:- Prove that
$$1 = \sum_{x \in X} \sum_{\substack{x \ni H \\ H \text{ hyperplane}}} \frac{1}{v(b - r_x)} \geq \sum_{\substack{H \in \mathcal{H} \\ \text{set of hyperplanes}}} \sum_{x \notin H} \frac{1}{b(v - k_H)}$$

$b =$ # hyperplanes

$\sum_{H \in \mathcal{H}} \sum_{x \notin H} \frac{1}{b(v-k_H)} = 1$

$\therefore$ equality holds & we get the result.