In effect, the Corollary says that $f_1, \dots, f_r$ is a partial sequence of coordinate functions at p iff their linear approximations at p are homogeneous and linearly independent.

Example: Verify that the function $f(x,y,z) = yz - \cos x + 1$ is a partial sequence of coordinate functions at $p = (0,0,1)$ and extend it to a full sequence using some of the standard coordinate functions at p, namely, $x, y, z-1$.

Solution: We have $f(p) = 0$ and $Df = [\sin x \;\; z \;\; y]$, so $Df(p) = [0 \;\; 1 \;\; 0]$.

As $Df(p) \neq 0$ (i.e., has rank 1) f forms a partial sequence. To extend it to a full sequence we use $f, x, z-1$ because for $F = (f, x, z-1)$, we have $DF(p) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ which is invertible.

In the above example, if we 'translate' p to the origin by using $\tilde{z} := z-1$, then $f = y(\tilde{z}+1) - \cos x + 1 = y + \underbrace{y\tilde{z} + 1 - \cos x}_{\text{quadratic}}$, so that its linear part is just y. Clearly $y, x, \tilde{z}$ form a full system of coordinate functions and so do $f, x, \tilde{z}$.

We can use the above ideas and concepts to discuss the implicit function theorem, mostly in $\mathbb{R}^2$ and $\mathbb{R}^3$. Let us first look at the linear picture which is quite neat and clean.

(i) Consider a line in $\mathbb{R}^2$, e.g., the x-axis. It is parametrised by $t \mapsto (t,0)$, while as a level curve it is given by $y=0$.

(ii) Consider a line in $\mathbb{R}^3$, e.g., the z-axis. It is parametrised by $t \mapsto (0,0,t)$, while as a level curve it is given by $x=0=y$.

(iii) Consider a plane in $\mathbb{R}^3$, e.g., the $(x-y)$-plane. It is parametrised by $(s,t) \mapsto (s,t,0)$ and it is the level surface $z=0$.

The implicit function theorem for a general curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ or for a general surface in $\mathbb{R}^3$ works by locally reducing to the linear situations (i), (ii), (iii) above via a suitable change of coordinates. In particular, a curve in $\mathbb{R}^2$ or a surface in $\mathbb{R}^3$ is locally given by one equation in each case while a curve in $\mathbb{R}^3$ is locally given by 2 independent equations.

Let $S \subseteq \mathbb{R}^m$ be a subset. A function $f : S \to \mathbb{R}^n$ is said to be <u>smooth</u> if for every $x \in S$, $\exists$ an open neighbourhood U of x in $\mathbb{R}^m$ and a smooth function $F : U \to \mathbb{R}^n$ such that $F|_{U \cap S} = f|_{U \cap S}$. In other words, around any $x \in S$, f admits an extension to a smooth function around x in $\mathbb{R}^m$. The local extension F and the open set U are not uniquely determined by f. (For example, let $S = \{p\}$ where $p \in \mathbb{R}^n$.)

Let $S \subseteq \mathbb{R}^m$, $T \subseteq \mathbb{R}^n$ be subsets. A function $f : S \to T$ is said to be a <u>diffeomorphism</u> if f is bijective and both f, f^{-1} are smooth.

<u>Examples:</u>

(i) Any line in $\mathbb{R}^m$ is diffeomorphic to any line in $\mathbb{R}^n$ (Exercise).

(ii) The map $t \mapsto (t, t^2)$ from $\mathbb{R}$ to the parabola $y = x^2$ is a diffeomorphism with the inverse being given by $(x, y) \mapsto x$.

(iii) There is no diffeomorphism from $\mathbb{R}$ to the singular curve C given by $y^2 = x^3$. Indeed, if $\mathbb{R} \underset{g}{\overset{f}{\rightleftarrows}} C$ are smooth inverse bijections with $f(0) = (0,0)$, then $\underset{\text{(Exercise 6, page 19)}}{D_0(f) = \vec{0}}$, so that $\underset{D_{\vec{0}}(g) \cdot D_0(f)}{\overset{}{D_0(gf)}} = 0$, a contradiction.

Implicit function theorem for curves in $\mathbb{R}^n$ (Part I)

Let $p \in \mathbb{R}^n$ and let $f_1, \dots, f_{n-1}$ be a partial system of coordinate functions at p (i.e., $f_i(p) = 0$ $\forall i$ and $Df_i(p)$ are linearly independent). Let C denote the locus $f_1 = f_2 = \cdots = f_{n-1} = 0$. Then there is an open neighbourhood U of p in $\mathbb{R}^n$ such $U \cap C$ is diffeomorphic to an open interval in $\mathbb{R}$. In particular, $U \cap C$ admits a regular parametrisation.

Proof: Let f_n be a smooth function to $\mathbb{R}$ around p so that $f_1, \dots, f_n$ form a full sequence of coordinate functions at p, i.e., for $F = (f_1, \dots, f_n)$, $D_p(F)$ is invertible and $f_n(p) = 0$. Then $F(p) = \vec{0}$ and there exists a diffeomorphism $F : U \to V$ for suitable open neighbourhood U, V of $p, \vec{0}$ respectively. If $\tilde{x}_1, \dots, \tilde{x}_n$ are standard coordinate functions on $V \subseteq \mathbb{R}^n$, then it follows from the definition of F that $F(U \cap C)$ is the locus $\tilde{x}_1 = \cdots = \tilde{x}_{n-1} = 0$ in V, i.e., $F(U \cap C) = V \cap$ the $\tilde{x}_n$-axis, the latter being an open subset of the $\tilde{x}_n$-axis $(= \mathbb{R})$. Its connected component through $\vec{0}$ is an open interval and hence by shrinking U, V if necessary, we may arrange that $U \cap C$ is diffeomorphic (via F) to an open interval around $\vec{0}$ in the $\tilde{x}_n$-axis.

We may parametrise the open interval in the $\tilde{x}_n$-axis by $t \overset{\alpha}{\longmapsto} (0, 0, \dots, 0, t)$. Composing $\alpha(t)$ with F^{-1} gives a smooth parametrisation $\gamma(t) = F^{-1} \circ \alpha(t)$ of $U \cap C$ with $\gamma(0) = p$. Since $F \circ \gamma = \alpha$ we see by Chain Rule that $\gamma'(0) \neq \vec{0}$. Q.E.D.

Since F^{-1} in the above proof may be hard to calculate, hence the parametrisation γ may be difficult to write down in terms of the initially given functions $f_1, \dots, f_{n-1}$.

However the tangent vector at p can be calculated since

$$\gamma'(0) = D_{\vec{0}}(F^{-1}) \circ D_0(\alpha) = D_p(F)^{-1} \cdot e_n \quad \left(\text{where } e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}\right).$$

To calculate $D_p(F)$ we still need to find f_n. Here we may use the fact that f_n can be chosen from one among the standard coordinate functions $\{x_1 - c_1, \dots, x_n - c_n\}$, $c_i = x_i(p)$. (Choose the one that makes $D_p(F)$ invertible).