Combinatorics

26 NOV 21

Lecture 15

Combinatorial Geometry

$(X, \mathcal{F}s)$. X - set of points.

$\mathcal{F}s$ a family of subsets of X.

① $\not\exists$ infinite chain in $\mathcal{F}s$ (w.r.t. inclusion)

② closed under intersections

③ ϕ, X & all singletons are in $\mathcal{F}s$

④ Given $E \in \mathcal{F}s$ (a flat) $X \setminus E$ is a disjoint union of elts $F \setminus E$ for $F \in \mathcal{F}s$ that "cover" E.

cover of E is a flat s.t. $F \supset E$ & $F \supseteq G \supseteq E$ $\Updownarrow$ $G = F$ or $G = E$. $G \in \mathcal{F}s$.

Typical example • $\mathbb{F}$ - field, V f.d. v. space over $\mathbb{F}$. $V = X$.

$$\mathcal{F}s = \{ v + W \mid v \in V,\ W \text{ a subspace of } V \}.$$

—x—x—x—

Lattices & Geometry

A lattice L is a partially ordered set s.t. given any finite subset $S \subseteq L$ has g.l.b. & l.u.b.

ie Given $S \subseteq L$, $|S| < \infty$ $\exists$ a & b s.t.

$\underset{\text{g.l.b.}}{\underline{a}} \leq s \ \forall s \in S$ & if $c \leq s \ \forall s \in S$ then $c \leq a$.

& $\underset{\text{l.u.b.}}{\underline{b}} \geq s \ \forall s \in S$ s.t. if $d \geq s \ \forall s \in S$ then $d \geq b$.

ex: ① $\mathbb{F}$ - field, V - v. space. L consists of all subspaces of V. (explain!)

Def^n :- • greatest lower bound is called the "meet" of S

• lowest upper bound is called the "join" of S.

meet of $\{a,b\}$ is denoted by $a \wedge b$.

join of $\{a,b\}$ —"— $a \vee b$

& a & $b \in L$ a lattice we say a covers b if $a > b$ & $a > c > b \Rightarrow c = b$ or $c = a$.

& denote it by $a \gtrdot b$

Exercise : ① check that $\wedge$ & $\vee$ are commutative & associative binary operations.

② If a lattice L has no infinite chain then there exists an elt in L that is minimum, it is denoted by 0_L & an elt that is maximum, it is denoted by 1_L

Def^n :- A point in L with minimum elt 0_L is an elt of L that covers 0_L.

(all 1-diml subspaces will be points of lattice of subspaces of V)

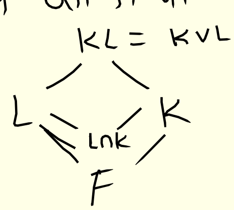
(ie $0_L \lessdot x$)

Def^n :- A geometric lattice is a lattice L having no infinite chains and such that

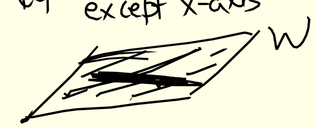
① L is atomic ie each elt $l \in L$ is the join of points that are in l.

② L is semi-modular ie $a \neq b$ & $a \gtrdot c$, $b \gtrdot c$ then $a \vee b \gtrdot a$ & $a \vee b \gtrdot b$

F - field. L consists of all finite extensions of F. $F = 0_L$. ext^n of deg p are pts.



$\mathbb{R}^3$, All subspaces of $\mathbb{R}^3$ except those in X-Y plane except X-axis



Theorem :- The set of flats of a geometry ordered by inclusion is a geometric lattice. Conversely, given a geometric lattice L with set of pts X then $(X, \{F_y \mid y \in L\})$ is a comb. geometry where $F_y = \{x \in X = \text{pts of } L \mid x \le y\}$.

$(X, \mathcal{F})$
$\parallel$
L

Proof. $\Rightarrow$ ① there is no infinite chain in $(X, \mathcal{F})$ & hence in associated L. Note: $\phi = 0_L$ & $\{x\}$ are points of L.

& since $\mathcal{F}$ is closed under intersections, the join of pts in F (for some $F \in \mathcal{F}$) is F.

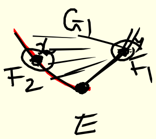
$\Rightarrow$ associated lattice is atomic.

Also: join of $S := \bigcap_{\substack{F \in \mathcal{F} \\ F \supset S}} F \in \mathcal{F}$.

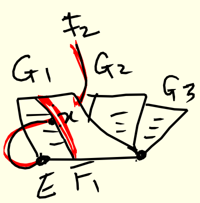
To prove semi-modularity

let $E \in \mathcal{F}$ & $F_1, F_2 \in \mathcal{F}$, $F_1 \ne F_2$ s.t. $F_1 \gtrdot E$, $F_2 \gtrdot E$.

to prove that $F_1 \vee F_2 \gtrdot F_1$ & $F_1 \vee F_2 \gtrdot F_2$



Choose $x \in F_2 \setminus F_1$.



Flats that cover F_1 partition $X \setminus F_1$.

$\therefore \exists!\, G_1 \in \mathcal{F}$ s.t. $G_1 \ni x$.

Claim: $F_2 \subset G_1$ if not look at

$$F_2 \supseteq \underbrace{F_2 \cap G_1}_{\in \mathcal{F}} \underset{x}{\supsetneq} E. \quad \text{qed.}$$

G_1 contains both F_1 & F_2.

$G_1 \gtrdot F_1$. by defn.

$\|$ly $\exists!\, G_2 \gtrdot F_2$ s.t. G_2 contains $y \in F_1 \setminus F_2$. This G_2 must contain F_1.

$$G_1 \supseteq G_1 \cap G_2 \supsetneq F_1 \qquad \Rightarrow G_1 = G_2.$$
$$G_2 \supseteq \{x, y\} \supsetneq F_2$$

$\Rightarrow G_1$ covers F_1 & F_2 both.

$\therefore G_1$ is the join.

$\Rightarrow$ join of F_1 & F_2 covers both F_1 & F_2 !!

($\Leftarrow$) (2) Conversely let L be a geometric lattice with point set X.

$\forall y \in L$, let $F_y = \{x \in X \mid x \leq y\}$.

(a) then no point is $\leq 0_L \Rightarrow \phi \in \mathcal{F}_S$. ✓

if $y = x \in X \Rightarrow \{x\} \in \mathcal{F}_S \ \forall x \in X$. ✓

Also, L contains max. elt also (∵ it has no infinite chains) say 1_L

if $y = 1_L$ then $F_y = X$. ✓

(b) Check that $F_y \cap F_z = F_{y \wedge z} \Rightarrow$ closed under intersection.

(c) Check that $\forall F_y \in \mathcal{F}_S$ $X \setminus F_y$ is partitioned by flats that cover F_y.

(Hint: $\forall y \in L$ construct a chain

$$0_L \lessdot y_1 \lessdot y_2 \lessdot \cdots y_k = y$$

such that y_{i+1} is join of y_i & a pt in X

Now $\forall x \in X \setminus F_y$ $x \neq y_i$'s

$F_{y \vee x}$ will cover y & contain x.)

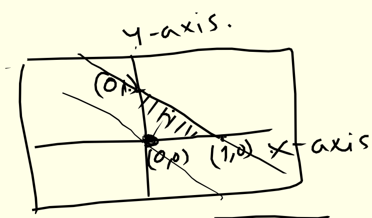
$\Rightarrow (X, \{F_y \mid y \in L\})$ is a geometry.

QED.

Def^n:- By (a) (or the?) rank of a combinatorial geometry on X we mean the size of a maximal independent subset of X.

Since there does not exist infinite chain in $\mathcal{F}_S$, we see that any geometry has finite rank.

example AG_2



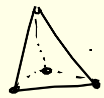
S is ind. $\Rightarrow x \notin \overline{S \setminus \{x\}} \quad \forall x \in S$.

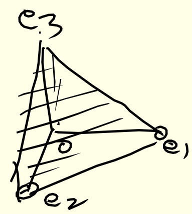
$$\overline{S \setminus \{x\}} = \bigcap_{\substack{F \supset S \setminus \{x\} \\ F \in \mathcal{F}_S}} F$$

$\Rightarrow$ $rk\ A^2 = 3$

in general rank of A^n is $n+1$.

namely take the vertices of a n-simplex !!





convex polytope formed by $\underset{\to}{0}$ & $\{e_i \mid 1 \le i \le n\}$.

lly the size of maximal ind-set in a flat $F \in \mathcal{F}_S$ is called a rank of F.

A basis of a flat F is an ind. subset $B \subseteq F$ s.t. $\overline{B} = F$.

A spanning subset C of F is any set C s.t. $\overline{C} = F$.

basis = minimal spanning subset
= maximal independent subset.

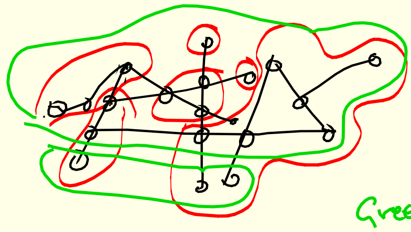
Example (using lattices to construct geometries).

G is any simple connected graph with n vertices

Associate a geometric lattice $L(G)$ to G as follows:

The elements of L are all partitions of $V(G)$ such that the induced subgraph on each part is connected.

G.



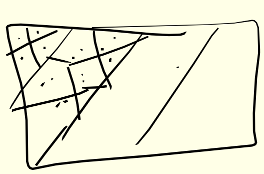
Green is not allowed.

Red is allowed

S a set. Π_S = all partitions of S

$\le$ on partitions is as follows:

A partition $\Pi_1 \le \Pi_2$ if each block of Π_2 is a union of blocks of Π_1.



lowest partition is $\{\{x\} \mid x \in S\} = 0_{\Pi_n}$.

Π_n := set of all partitions of $\{1, 2, \dots, n\}$.

$0_{\Pi_n} \lessdot \Pi_1$ iff Π_1 has one part of size 2 & all other parts of size 1

$\therefore$ edges of G are "points" of $L(G)$.

the partiton $V(G) = V(G)$ is the largest elt. $1_{L(G)}$

<u>check</u> :- This is geometric lattice!

<u>Remark</u> :- $L(G)$ corresponding K_n is just the lattice Π_n of partitions of $\{1, 2, \cdots, n\}$

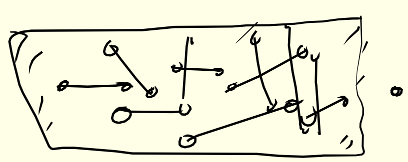
any partition of $\{1, \cdots n\}$ is allowed.

—x—x—x—

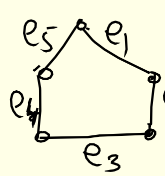
<u>Problem 23 B.</u> Let G be a simple connected graph. Show that any basis of the comb. geom. associated to $L(G)$ consists of <u>edges of a spanning tree</u>

<u>Sol^n</u> :- pts of $L(G)$ are edges (i.e. a partition that consists of a single 2-set & all other 1-sets with induced graph on 2-set connected)

$\therefore$ bases consists of edges. If the set of edges in a basis does not <u>have a vertex as one of its end points</u> then its closure will miss that vertex & hence can't be whole of G



$\Rightarrow$ end pts of edges in a basis $= V(G)$.



$\Rightarrow$ $\{e_1, e_2, e_3, e_4, e_5\}$ not independent! (check this).

$\Rightarrow$ these edges must form a spanning tree! QED.

$\boxed{\text{Theorem}}$:- $(X, \mathcal{F})$ be a comb. geometry

① All bases of a flat F in a comb. geom. have same (finite) cardinality (the common size is called "the" rank of F)

② $E, F \in \mathcal{F} \Rightarrow$

$\circledast$ —— $rk(E) + rk(F) \geq rk(E \cap F) + rk(E \cup F)$

Note: ($*$) is called semi-modular law.

In case equality holds for all $E, F \in \mathcal{F}_S$ then $(X, \mathcal{F}_S)$ is called a <u>modular</u> geometry

Pf (1) If B_1, B_2 are two bases of $F \in \mathcal{F}_S$.

$\to$ Assume $|B_1|$ is max. amongst all bases of F.

This is how it is proved in lin. algebra.

$\{y_1, \ldots, y_n\}$ $\{z_1, \ldots, z_k\}$ are bases of subspace.

$z_1 = \sum a_i y_i$. WLOG assume $a_1 \neq 0$.

$\Rightarrow \quad y_1 = a_1^{-1} z_1 - \sum_{i=2} a_i y_i$

$\Rightarrow \langle y_1, \ldots, y_n \rangle = \langle z_1, y_2, \ldots y_n \rangle = V$

$z_2 = \sum a_i y_i + b_1 z_1$

$x \in B_1 \setminus B_2$ Then $B_1 \setminus \{x\}$ is independent & its closure can not contain whole of B_2.

$$\overline{B_1 \setminus \{x\}} \supset B_2 \Rightarrow \overline{B_1 \setminus \{x\}} \supset \overline{B_2} = F \ni x$$

let $y \in B_2$ s.t. $y \notin \overline{B_1 \setminus \{x\}}$

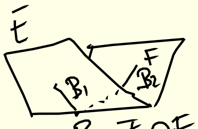
$B_1 \setminus \{x\} \cup \{y\}$ is independent. claim $\overline{B_1 \setminus \{x\} \cup \{y\}} = F$.

$\longrightarrow$ this can be guaranteed because basis B_1 is maximum size of all bases of F. $\longleftarrow$

Keep replacing elements of B_1 by that of B_2.

if $|B_2| < |B_1|$ then we get contradiction.

(2) Let B be a basis of $E \cap F$. Extend B by adding one elt at a time to bases of B_1 of E & B_2 of F.



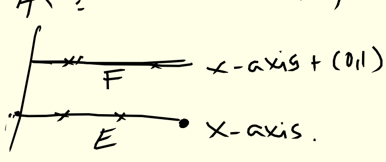
Then any flat containing $B_1 \cup B_2$ contains both E & F. $\Rightarrow \overline{B_1 \cup B_2} = E \vee F$.

$$\therefore \ rk(E \vee F) \leq |B_1 \cup B_2| = rk(E) + rk(F) - rk(E \cap F)$$

$$\Rightarrow rk(E) + rk(F) \geq rk(E \vee F) + rk(E \cap F).$$

QED.

example A^2:



$\underbrace{4}_{rkE + rkF} > \underbrace{3}_{rk A^2} + 0 \to rk(\phi)$