Lecture 18: Character of induced representation

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14:04

Let G be a group & $H \leq G$. Let V be a G-repr & $W \subseteq V$ a H-stable subspace. We say $V = \mathrm{Ind}_H^G W$ if

$$V = \bigoplus_{i=1}^{r} g_i W$$ where $g_1, \dots, g_r \in G$ are representatives of the left cosets of H in G/H. Note $r = [G:H]$

Note: $G \xrightarrow{\rho} GL(V)$ & $H \xrightarrow{\theta} GL(W)$

For $g \in G$

$$\chi_V(g) = tr(\rho(g)) = \sum_{i=1, j_i=i}^{r} tr(\rho(g)|_{g_i W})$$

$$= \sum_{\substack{i=1 \\ \& j_i = i}}^{r} tr(\rho(g_i h_i g_i^{-1})|_{g_i W})$$

$$= \sum_{1 \leq i \leq r \ \& \ g_i^{-1} g g_i \in H} tr(\theta(h_i))$$

Note if $j_i = i$
$\Rightarrow g = g_i h_i g_i^{-1}$

For $1 \leq i \leq r$

$g \cdot g_i = g_{j_i} h_i$ for some $1 \leq j_i \leq r$ & $h_i \in H$

Let $\{w_1, \dots, w_m\}$ be a basis of W then $\{g_i w_1, \dots, g_i w_m\}$ is a basis of $g_i W$.

$\rho(g) : \bigoplus_{i=1}^{r} g_i W \circlearrowleft$ Consider basis $\mathcal{B} = \{g_1 w_1, g_1 w_2, \dots, g_1 w_m, \dots, g_r w_1, \dots, g_r w_m\}$

($\because$ $\rho(g_i)^{-1} \circ \rho(g_i h_i g_i^{-1}) \circ \rho(g_i) : W \to W$
$= \rho(h_i)|_W = \theta(h_i)$)

$\Rightarrow tr(g_i h_i g_i^{-1})|_W = tr(\theta(h_i))$

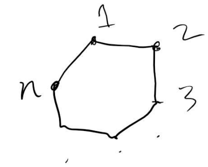
($\because h_i = g_i^{-1} g g_i$)

$$\boxed{\chi_V(g) = \sum_{1 \leq i \leq r,\ g_i^{-1} g g_i \in H} \chi_W(g_i^{-1} g g_i)}$$

$$\boxed{\chi_V(g) = \frac{1}{|H|} \sum_{y \in G,\ y^{-1} g y \in H} \chi_W(y^{-1} g y)}$$

($\because$ $g_i^{-1} g g_i \in H$ then for $y = g_i h$ $\forall h \in H$ $y^{-1} g y \in H$)

Representations of D_n, dihedral group of order $2n$.



$$D_n = \langle r, s \mid r^n = 1,\ s^2 = 1,\ rs = sr^{-1} \rangle$$
$$= \{ r^j, sr^j \mid 1 \le j \le n \} \le S_n$$

<u>n even</u>

1-dim'l repr $\leftrightarrow$ group homo $D_n \longrightarrow \mathbb{C}^*$

$\varphi : D_n \longrightarrow \mathbb{C}^*$ grp homo $\Rightarrow \varphi(r^2) = 1$

$$\left\{ \begin{array}{l} \varphi(rs) = \varphi(sr^{-1}) \\ \quad \shortparallel \\ \varphi(r)\cancel{\varphi(s)} = \varphi(r^{-1})\cancel{\varphi(s)} \\ \varphi(r^2) = 1 \end{array} \right.$$

$\&\ \varphi(s^2) = 1$

$\varphi(r) = 1$ or -1 $\&$ $\varphi(s) = 1$ or -1

φ	r	s
trivial	1	1
	1	-1
	-1	1
	-1	-1

So there 4 1-dim repr of D_n

n odd then

$\varphi(r) = 1$ & $\varphi(s) = 1$ or -1

$\Rightarrow$ 2 1-dim'l reps of D_n.

Even case

2-dim'l reps of D_n $\begin{cases} r \longmapsto \text{rotation by } k\left(\frac{2\pi i}{n}\right) & k \in \mathbb{Z} \\ s \longmapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{cases}$

$\begin{bmatrix} (e^{2\pi i/n})^k & 0 \\ 0 & (-e^{2\pi i/n})^k \end{bmatrix}$

$1 \le k \le n-1$

$4 + a4 = 2n$

$1 + a = \frac{n}{2}$

$a = \frac{n}{2} - 1$ 2-dim'l irr reps of D_n