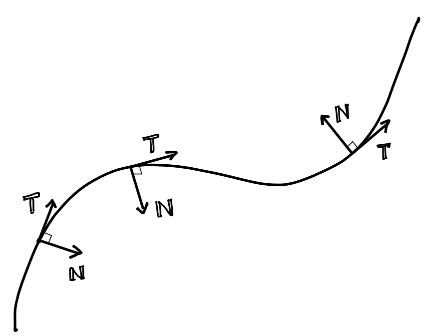
Let $\gamma(s)$ be a unit-speed curve in $\mathbb{R}^n$. For any $a \in I$, we call $\underline{\mathbb{T}(a) := \dot{\gamma}(a)}$ as the <u>unit tangent vector</u> at $s=a$. Clearly $\mathbb{T}(s)$ is a smooth no-where zero function in s. We call $\underline{\kappa(a) := \|\dot{\mathbb{T}}(a)\| = \|\ddot{\gamma}(a)\|}$ the <u>curvature</u> of γ at $s=a$. By continuity, the set $\{x \in I \mid \kappa(x) > 0\}$ is open in I and over this set, κ is a smooth function in s. Suppose $\kappa(a) \neq 0$. We then call $\underline{\mathbb{N}(a) := \frac{1}{\kappa(a)} \dot{\mathbb{T}}(a)}$, the <u>principal normal vector</u> at $s=a$. By the above lemma, $\mathbb{T}(a) \cdot \mathbb{N}(a) = 0$.

In particular, if $n=2$, then wherever $\kappa(a) \neq 0$ (so that $\mathbb{N}(a)$ is defined), $\{\mathbb{T}(a), \mathbb{N}(a)\}$ form an orthonormal basis of $\mathbb{R}^2$.



[Note that it is intuitively more meaningful to draw $\mathbb{T}, \mathbb{N}$ at the corresponding point than at, say, the origin.]

<u>Examples:</u>

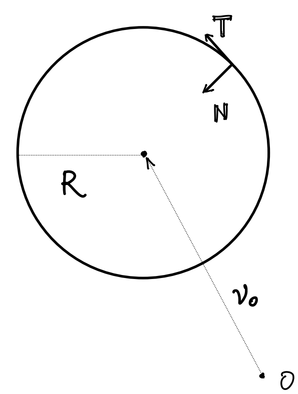
(i) Let $\gamma(s)$ be a unit-speed line in $\mathbb{R}^n$, say $\gamma(s) = v + su$ where $u, v \in \mathbb{R}^n$ and $\|u\| = 1$. Then $\mathbb{T}(s) = \dot{\gamma}(s) = u$, a constant vector. Thus $\dot{\mathbb{T}}(s) = 0$, i.e. $\kappa(s) \equiv 0$.

Thus a line has curvature 0 everywhere and the principal normal vector on it is undefined.

(ii) Conversely, suppose $\gamma(s)$ is a unit-speed curve with $\kappa(s) \equiv 0$. Then $\dot{\mathbb{T}}(s) \equiv 0$, hence $\mathbb{T}(s)$ is constant. Since $\dot{\gamma}(s)$ is a constant, $\gamma(s)$ is a linear function with unit speed (by hypothesis), i.e., $\gamma(s)$ is a line travelled at unit speed.

(iii) Let $\gamma(s)$ be a unit-speed circle in $\mathbb{R}^2$ of radius $R>0$ with motion anti-clockwise, say $\gamma(s) = \gamma_0 + \left(R\cos(\frac{s}{R}),\ R\sin(\frac{s}{R})\right)$, $\gamma_0 \in \mathbb{R}^2$.

Then $\dot{\gamma}(s) = \left(-\sin(\frac{s}{R}), \cos(\frac{s}{R})\right) = \mathbb{T}(s)$



$\therefore\ \dot{\mathbb{T}}(s) = \left(\frac{-1}{R}\cos(\frac{s}{R}), \frac{-1}{R}\sin(\frac{s}{R})\right)$.

$\boxed{\text{Thus, } \kappa(s) = \frac{1}{R}.}$ Also,

$$\mathbb{N}(s) = \frac{1}{\kappa(s)}\dot{\mathbb{T}}(s) = \left(-\cos(\tfrac{s}{R}), -\sin(\tfrac{s}{R})\right),$$

i.e., $\mathbb{N}$ is a radially inward vector.

In the above example, as $R \to \infty$, the curvature $\kappa \equiv \frac{1}{R}$ approaches 0. Thus, locally a circle of large radius resembles a line :

(iv) Let $\gamma(s)$ be a helix in $\mathbb{R}^3$ of unit speed, say

$\gamma(s) = \gamma_0 + \left(a\cos\left(\frac{s}{c}\right),\ a\sin\left(\frac{s}{c}\right),\ \frac{bs}{c}\right)$ where $\gamma_0 \in \mathbb{R}^3$, $a, b \in \mathbb{R}$, $c = \sqrt{a^2+b^2}$.

(Though we assume $a \neq 0 \neq b$, we see that as $a \to 0$ we obtain a line (the z-axis) while as $b \to 0$ we obtain a circle of radius $|a|$ in the x-y plane.) Then

$$\mathbb{T}(s) = \dot{\gamma}(s) = \left(-\frac{a}{c}\sin\left(\frac{s}{c}\right),\ \frac{a}{c}\cos\left(\frac{s}{c}\right),\ \frac{b}{c}\right)$$

$$\therefore\ \dot{\mathbb{T}}(s) = \left(\frac{-a}{c^2}\cos\left(\frac{s}{c}\right),\ \frac{-a}{c^2}\sin\left(\frac{s}{c}\right),\ 0\right).$$

Hence $\kappa(s) = \|\dot{\mathbb{T}}(s)\| = \frac{|a|}{c^2}$. Assume that $a > 0$, which means that we travel anti-clockwise as s increases. Then

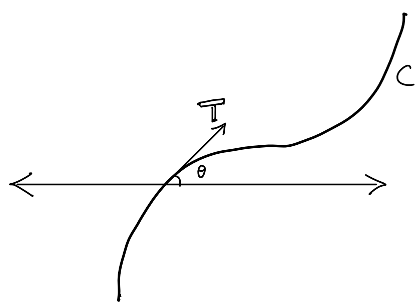
$$\mathbb{N}(s) = \frac{1}{\kappa(s)}\dot{\mathbb{T}}(s) = \left(-\cos\left(\frac{s}{c}\right),\ -\sin\left(\frac{s}{c}\right),\ 0\right). \text{ (compare with circle)}$$

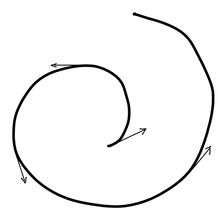
Exercise 7:

Let $\gamma(s)$ be a unit-speed curve in $\mathbb{R}^2$ such that <u>$\kappa(s)$ is a positive constant. Prove that $\gamma(s)$ is a part of a circle.</u>

Thus, any curve in $\mathbb{R}^2$ other than a line or a circle, has non-constant curvature. We shall look for formulas which do not rely on unit-speed parametrisations later.

For a curve $\gamma(s)$ in $\mathbb{R}^2$, we give a geometric description of $\kappa(s)$. Recall that $\mathbb{T}(s) = \dot{\gamma}(s)$ is the unit tangent vector. Let $\dot{\gamma}(s) = (\dot{\gamma}_1(s), \dot{\gamma}_2(s))$. Then we may write $\dot{\gamma}_1(s) = \cos(\theta(s))$, $\dot{\gamma}_2(s) = \sin(\theta(s))$ for some $\theta(s)$ uniquely determined upto translation by 2π. (Indeed $\theta(s)$ is just the angle formed by $\dot{\gamma}(s)$ with respect to the positive X-axis)





Note that we may not be able to define $\theta(s)$ as a continuous function for all s (going around the unit circle increases θ by 2π) but we can <u>locally</u> define it as a continuous function, which is moreover smooth, by using $\theta(s) = \cos^{-1}(\dot{\gamma}_1(s))$ or $\theta(s) = \sin^{-1}(\dot{\gamma}_2(s))$ with range in a suitable interval of length 2π. Finally note that for all these definitions of $\theta(s)$, the function $\frac{d\theta}{ds}$ is globally continuous (since the different definitions of $\theta(s)$ only differ by a multiple of 2π) and hence globally smooth.

So let $\dot{\gamma}(s) = \big(\cos(\theta(s)), \sin(\theta(s))\big)$. Upon differentiating we get $\ddot{\gamma}(s) = \dot{\theta}(s)\big(-\sin(\theta(s)), \cos(\theta(s))\big)$. Thus $\kappa(s) = \|\ddot{\gamma}(s)\| = \left|\frac{d\theta}{ds}\right|$, i.e., $\kappa(s)$ is the absolute value of the rate at which the unit tangent vector is turning (or how the curve is turning).

For the plane curve $\gamma(s)$ as above, we call $\frac{d\theta}{ds}$ as the <u>signed curvature</u> $\kappa_s(s)$ of $\gamma(s)$ and we call the unit vector $\big(-\sin(\theta(s)), \cos(\theta(s))\big)$ as the <u>signed unit normal</u> $\mathbb{N}_s(s)$ of $\gamma(s)$. The following facts are readily seen:

(i) $\kappa(s) = |\kappa_s(s)|$.

(ii) $\mathbb{N}(s) = \pm\,\mathbb{N}_s(s)$.

(iii) $\ddot{\gamma} = \kappa\,\mathbb{N} = \kappa_s\,\mathbb{N}_s$.

(iv) $\mathbb{N}_s$ is obtained by rotating $\mathbb{T} = \dot{\gamma}$ anticlockwise by $\pi/2$.

<u>Example / Exercise</u>:

For a unit-speed circle of radius R, one easily checks that $\kappa = \frac{1}{R}$ and $\mathbb{N}$ points inward. For anti-clockwise motion, we have $\kappa_s = \kappa$, $\mathbb{N}_s = \mathbb{N}$, while for clockwise we have $\kappa_s = -\kappa$, $\mathbb{N}_s = -\mathbb{N}$.