Lecture 2: Tensor, symmetric and alternating powers

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(*) V & W be k-vector spaces then $V \otimes W$ is also a k-vector space.

There is a k-bilinear map $\varphi : V \times W \longrightarrow V \otimes W$.

$$(v, w) \longmapsto v \otimes w$$

Elements of $V \otimes W$ are of the form $\sum_{\text{finite}} a_\alpha v_\alpha \otimes w_\alpha$ $\quad v_\alpha \in V$ & $w_\alpha \in W$. $a_\alpha \in k$.

In fact if $\{v_1, \ldots, v_n\}$ & $\{w_1, \ldots, w_m\}$ are bases of V & W resp. then $\left\{ v_i \otimes w_j \;\middle|\; \begin{matrix} 1 \le i \le n \\ 1 \le j \le m \end{matrix} \right\}$ is a basis of $V \otimes W$.

(*) $T^2 V := V \otimes V \longleftarrow$ is $(\dim V)^2$

$T^3 V := V \otimes (V \otimes V) \cong (V \otimes V) \otimes V \cong V \otimes V \otimes V$ is a vector space with basis $\left\{ e_i \otimes e_j \otimes e_k \;\middle|\; \begin{matrix} 1 \le i \le n \\ 1 \le j \le n \\ 1 \le k \le n \end{matrix} \right\}$ if $\{e_1, \ldots, e_n\}$ is a basis of V.

There is a k-multilinear map

$$V \times V \times V \xrightarrow{\varphi_3} T^3 V$$
$$(v_1, v_2, v_3) \longmapsto v_1 \otimes v_2 \otimes v_3$$

$$T^n V := \begin{cases} k & \text{if } n = 0 \\ V & \text{if } n = 1 \end{cases} \Big\} \text{ by convention}$$
$$\underbrace{V \otimes \cdots \otimes V}_{n \text{ times}} \text{ for } n \ge 2$$

$\varphi_n : V \times V \times \cdots \times V \longrightarrow T^n V$ k-multilinear map

Note there is a natural map k-bilinear

$$T^n V \times T^m V \longrightarrow T^{(n+m)} V$$
$$(\alpha, \beta) \longmapsto \alpha \otimes \beta$$

Let V be r-dim vs. & $\{e_1, \ldots, e_r\}$ is a basis.

$$\alpha = \sum_{\substack{\underline{i} = (i_1, \ldots, i_n) \\ 1 \le i_j \le r,\ 1 \le j \le n}} a_{\underline{i}}\, e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}$$

$$\beta = \sum_{\substack{\underline{l} = (l_1, \ldots, l_m) \\ 1 \le l_j \le r,\ 1 \le j \le m}} b_{\underline{l}}\, e_{l_1} \otimes \cdots \otimes e_{l_m}$$

Let $T^{\cdot} V := \bigoplus_{n \ge 0} T^n V = k \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots$ then $T^{\cdot} V$ is a k-algebra with the multiplication defined by the above map.

$\alpha \in T^{\cdot} V$ then $\alpha = \alpha_0 + \alpha_1 + \alpha_2 + \cdots$ where $\alpha_i \in T^i V$

Let $\beta \in T^{\cdot} V$ then $\beta = \sum \beta_i$ $\quad \beta_i \in T^i V$ uniquely.

In $T^{\cdot} V$, $\alpha \cdot \beta := \alpha_0 \beta_0 + \underbrace{(\alpha_0 \beta_1 + \alpha_1 \beta_0)}_{\in\, T^1 V} + \underbrace{(\alpha_0 \beta_2 + \alpha_1 \otimes \beta_1 + \alpha_2 \beta_0)}_{\in\, T^2 V} + \cdots$

$$= \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} (\alpha_j \otimes \beta_{i-j}) \right) \in T^{\cdot} V.$$

<u>Prop:</u> Let V be a k-vs. Let A be a k-alg & $\varphi : V \to A$ be a k-li- map then $\exists !$ k-alg homo $\tilde{\varphi} : T^{\cdot} V \longrightarrow A$ s.t. $\tilde{\varphi}\big|_{T^1 V} : V \longrightarrow A$ is same as φ.

Pf: Want to define

$$\tilde{\varphi} : T^{\cdot}V \longrightarrow A$$

Let $s : k \longrightarrow A$, $1 \longmapsto 1_A$ be the map which makes A a k-algebra.

$\varphi_0 : T^0 V = k \longrightarrow A$ to be s.

$\varphi_1 : T^1 V \longrightarrow A$ is φ.

$\varphi_2 : T^2 V \longrightarrow A$, $T^2V = V \otimes V$

$$\psi : V \times V \longrightarrow A$$
$$(v_1, v_2) \longmapsto \varphi(v_1)\varphi(v_2)$$

So it induces $\tilde{\psi} : V \otimes V \to A$ which sends $\underline{v_1 \otimes v_2 \longmapsto \varphi(v_1)\varphi(v_2)}$

So $\varphi_2 = \tilde{\psi}$

So if $\alpha_2 \in V \otimes V$ then $\alpha_2 = \sum_{\text{finite}} v_i \otimes v_i'$

$$\varphi_2(\alpha_2) = \sum \varphi(v_i)\varphi(v_i')$$

$\varphi_n : T^n V \longrightarrow A$. Let $\{e_1, \dots, e_r\}$ be a basis of V.

then $\varphi_n(e_{i_1} \otimes \cdots \otimes e_{i_n}) = \varphi(e_{i_1}) \cdot \varphi(e_{i_2}) \cdots \varphi(e_{i_n})$

φ_n are k-lin maps from $T^n V$ to A.

$$\tilde{\varphi} : T^{\cdot} V \longrightarrow A$$

$$\tilde{\varphi}(\alpha) := \varphi_0(\alpha_0) + \varphi_1(\alpha_1) + \cdots$$

where $\alpha \in T^{\cdot} V$, $\alpha = \alpha_0 + \alpha_1 + \cdots$, $\alpha_i \in T^i V$.

Note that φ_i are k-lin, hence $\tilde{\varphi}$ behaves well with addition.

Let $\alpha, \beta \in T^{\circ} V$ then

$$\tilde{\varphi}(\alpha \cdot \beta) = \tilde{\varphi}\left(\sum_{i=0}^{\infty} \sum_{j=0}^{i} \alpha_j \otimes \beta_{i-j}\right)$$

$$= \sum_{i=0}^{\infty} \varphi_i \left(\sum_{j=0}^{i} (\alpha_j \otimes \beta_{i-j}) \right)$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{i} \varphi_i (\alpha_j \otimes \beta_{i-j})$$

check

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{i} \varphi_j(\alpha_j)\, \varphi_{i-j}(\beta_{i-j})$$

$$= \left(\varphi_0(\alpha_0) + \varphi_1(\alpha_1) + \cdots\right)\left(\varphi_0(\beta_0) + \varphi_1(\beta_1) + \cdots\right)$$

$$= \tilde{\varphi}(\alpha)\, \tilde{\varphi}(\beta)$$

So $\tilde{\varphi}$ is a ring homo.

Check Uniqueness of $\tilde{\varphi}$

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