Exercise 15: Let notation and assumptions be as in the implicit function theorem above.

(i) Prove that $T_p(C) :=$ the subspace of $\mathbb{R}^n$ generated by $\gamma'(0)$ is given by the vanishing of $n-1$ homogeneous linear polynomials, namely

$$T_p(C) = \{\vec{x} = (x_1, \dots, x_n) \mid D_p(f_1)\cdot\vec{x} = 0,\ D_p(f_2)\cdot\vec{x} = 0, \dots, D_p(f_{n-1})\cdot\vec{x} = 0\}.$$

(*Hint*: Either use $D_p(F)\cdot\gamma'(0) = e_n$ or differentiate $f_i \circ \gamma = 0$).

(ii) Deduce that the tangent line L to C at p is given by the vanishing of $n-1$ equations

$$L = \{\vec{x} = (x_1, \dots, x_n) \mid D_p(f_1)(\vec{x} - p) = 0, \ \dots, \ D_p(f_{n-1})(\vec{x} - p) = 0\}.$$

Thus, the exercise above says that for a point $p = (a, b)$ on a plane curve C given by $f(x, y) = 0$, the tangent line L at p is given by the equation $f_x(p)\cdot(x - a) + f_y(p)(y - b) = 0$ while the tangent space T_pC is generated by the vector $(f_y(p), -f_x(p))$.

It is worth noting that for the curve C in the implicit function theorem, the parametrisation map $\gamma(t)$ is a diffeomorphism (and hence a homeomorphism) as $\gamma = F^{-1} \circ \alpha$.

Examples:

(i) Let $f_\lambda(x,y) = x^2+y^2-\lambda$ where λ is a constant. Then $Df = (2x, 2y)$. Let C_λ denote the locus $\{f_\lambda = 0\}$. If $\lambda<0$, then $C_\lambda = \phi$. If $\lambda=0$, then $C_\lambda = \{(0,0)\}$ clearly does not have any open subset homeomorphic to an open interval and indeed the implicit function theorem does not apply as $D_{(0,0)}(f_\lambda) = (0,0)$.

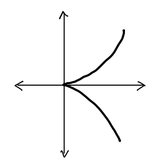
For $\lambda>0$, C_λ is non-empty and $D_p f_\lambda \neq (0,0)$ for any $p \in C_\lambda$.

Let $\lambda=1$. For $p=(1,0)$, $D_p(f_\lambda) = (2,0)$, so that f_λ, y form a full system of coordinates at p. The map $F = (\underset{\tilde{x}}{\underset{\shortparallel}{f_\lambda}}, \underset{\tilde{y}}{\underset{\shortparallel}{y}})$ gives a diffeomorphism near p sending C_λ near p to the locus $\tilde{x}=0$. As we may use $\tilde{y}$ itself to parametrise the $\tilde{y}$-axis, we see that we may use y to parametrise C_λ near $p=(1,0)$. For $p=(0,1)$, $D_p(f_\lambda) = (0,2)$ and we use f_λ, x (or $\underset{\tilde{x}}{\underset{\shortparallel}{x}}, \underset{\tilde{y}}{\underset{\shortparallel}{f_\lambda}}$) as a full system of coordinates. In this case we may use x to parametrise C_λ near $p=(0,1)$.

(ii) Let $f = (x^2+y^2-1)^k$ for some integer $k>1$.

Then $Df = k(x^2+y^2-1)^{k-1}(2x, 2y)$. Thus, on the locus C given by $\{f=0\}$ we see that $Df = 0$ everywhere. In particular, f cannot be part of a full sequence of coordinate functions at any point of C.

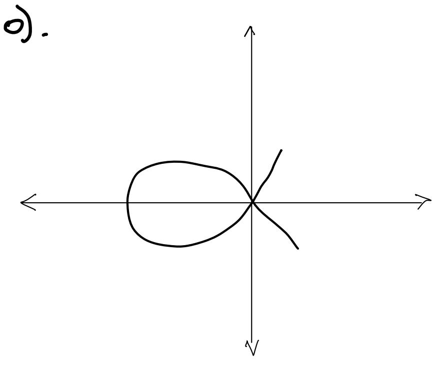
(iii) Let $f = y^2 - x^3$ and let C be the curve $\{f=0\}$. Since $Df = [-3x^2, 2y]$, we see that $D_p(f) \neq \vec{0}$ unless $p = (0,0)$. Away from $(0,0)$, the implicit function theorem applies and we may use x or y for parametrising C away from $(0,0)$. In this case the parametrisation of C given by $t \overset{\gamma}{\mapsto} (t^2, t^3)$ is a homeomorphism (the inverse being given by $(x,y) \mapsto y^{1/3}$) but C is not diffeomorphic to $\mathbb{R}$ near $(0,0)$ (see page 89).



Also γ induces a diffeomorphism $\mathbb{R} \setminus \{0\} \to C \setminus \{(0,0)\}$ (inverse: $(x,y) \mapsto \frac{y}{x}$).

(iv) Let $f = y^2 - x^2(x+1)$. Then $Df = (-2x - 3x^2, 2y)$ and hence the only point p of $f=0$ where $D_p(f) = 0$ is $p = (0,0)$.

Near p, the curve $f=0$ is not even homeomorphic to an open interval in $\mathbb{R}$.



Implicit function theorem for curves in $\mathbb{R}^n$ (part II):

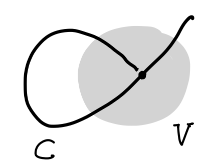
Let $\gamma = (\gamma_1, \dots, \gamma_n) : I \to \mathbb{R}^n$ be a curve with trace C and let $p = \gamma(a)$ $(a \in I)$ be a regular point on C. Then there exists an open neighbourhood $I' \subseteq I$ of a and a partial sequence of coordinate functions $f_1, \dots, f_{n-1}$ at p such that $C' := \gamma(I') = \bigcap_{i=1}^{n-1} \{f_i = 0\} \cap V$ where V is an open neighbourhood of p in $\mathbb{R}^n$ over which f_i are defined and smooth.

Proof: Since $\gamma(t)$ is regular at $t = a$, there is an i for which $\gamma_i'(a) \neq 0$. By the inverse function theorem (in one variable), γ_i induces a diffeomorphism $I' \to J$ where I', J are open intervals around a, $\gamma_i(a)$ respectively. Let $u(\tilde{t})$ be the inverse map $J \to I'$. Using the substitution $t = u(\tilde{t})$ we reparametrise C near p by $\tilde{\gamma}(\tilde{t}) = (\underbrace{\gamma_1 \circ u(\tilde{t})}_{= \tilde{\gamma}_1(\tilde{t})}, \dots, \underbrace{\gamma_n \circ u(\tilde{t})}_{= \tilde{\gamma}_n(\tilde{t})})$.

Clearly $\tilde{\gamma}_i(\tilde{t}) = \tilde{t}$. Setting $f_j(x_1, \dots, x_n) := x_j - \tilde{\gamma}_j(x_i)$, we see that on $V = \mathbb{R} \times \cdots \times \underset{(i^{\text{th}} \text{ spot})}{J} \times \cdots \times \mathbb{R}$, the locus $\bigcap_{j=1}^{n} \{f_j = 0\} \cap V$ is $\gamma(I')$. (Clearly f_j are defined and smooth over V). For $j \neq i$, since Df_j has only 2 nonzero components, namely, a 1 in the j-th coordinate and something in the i-th coordinate, we see that for $j \neq i$ the $D_p(f_j)$'s form a linearly independent set. As $f_i = 0$, we drop it and re-number the indices. Q.E.D.

In particular, the theorem says that if $\gamma(t) = (\underset{x(t)}{\underset{\shortparallel}{\gamma_1(t)}}, \underset{y(t)}{\underset{\shortparallel}{\gamma_2(t)}})$ is a plane curve regular at $t=a$, then near $t=a$, we may write (on the curve) y as a function of x (if $\gamma_1'(a) \neq 0$) or x as a function of y (if $\gamma_2'(a) \neq 0$). Therefore, in each of these cases we may think of a piece of the curve near $\gamma(a)$ as being the graph of a function in x or y. (So Exercise 8 describes a fairly general situation).

While it may look like the part II version of the theorem is a converse to part I, there is a subtle difference. In part II, $C' = \left(\bigcap_j \{f_j = 0\} \cap V\right)$ may not be an open subset of C and in particular, $C \cap V$ may have more points



than C' and the parametrising map γ may not be a homeomorphism.