Case of X having less than full column rank

Rank$(X_{n\times p}) = r < p$. Since only estimable linear functions $a'\beta$ can be estimated, assume $a_i'\beta$, $i = 1, 2, \ldots, q$ are estimable and $A_{q\times p} = \begin{pmatrix} a_1' \\ \vdots \\ a_q' \end{pmatrix}$.
However, since $a_i' = m_i'X$ for some m_i', we have $A = M_{q\times n}X_{n\times p}$. Since A has rank q, M also has rank q ($\leq r$). Proceeding as before, let β_0 be any solution of $A\beta = c$. Then consider: $\tilde{Y} = Y - X\beta_0 = X(\beta - \beta_0) + \epsilon$ or $\tilde{Y} = X\gamma + \epsilon$ or

$$\tilde{Y} = \theta + \epsilon, \theta \in \mathcal{M}_C(X) = \Omega, \text{ and}$$

$M\theta = MX\gamma = A\gamma = 0$. We want to find $\hat{\beta}_H$, the least squares solution subject to $H : A\beta = c$. If $\omega = \Omega \cap \mathcal{N}(M)$, then $\omega^\perp \cap \Omega = \mathcal{M}_C(P_\Omega M')$, and $P_\Omega M' = X(X'X)^- X'M' = X(X'X)^- A'$. Further,
$MP_\Omega M' = MX(X'X)^- X'M' = A(X'X)^- A'$ is nonsingular. This is because, (since $X'P_\Omega = X'$)

$$\begin{aligned} q &= \text{Rank}(M') \geq \text{Rank}(P_\Omega M') \geq \text{Rank}(X'P_\Omega M') \\ &= \text{Rank}(X'M') = \text{Rank}(A') = q. \end{aligned}$$

Therefore

$$\begin{aligned} P_\Omega - P_\omega &= P_{\omega^\perp\cap\Omega} = P_{\mathcal{M}_C(P_\Omega M')} \\ &= P_\Omega M'(MP_\Omega M')^{-1} MP_\Omega \\ &= X(X'X)^- A' \left(A(X'X)^- A'\right)^{-1} A(X'X)^- X'. \end{aligned}$$

Hence,

$$\begin{aligned} &X\hat{\beta}_H - X\beta_0 = X\hat{\gamma}_H = P_\omega\tilde{Y} = P_\Omega\tilde{Y} - P_{\omega^\perp\cap\Omega}\tilde{Y} \\ &\quad = P_\Omega Y - X\beta_0 - P_\Omega M'(MP_\Omega M')^{-1} MP_\Omega(Y - X\beta_0), \text{ so that} \end{aligned}$$

$$X'X\hat{\beta}_H - X'X\beta_0 = X'P_\Omega Y - X'X\beta_0 - X'P_\Omega M'(MP_\Omega M')^{-1} MP_\Omega(Y - X\beta_0).$$

Thus,

$$\begin{aligned} X'X\hat{\beta}_H &= X'Y - X'M'(MP_\Omega M')^{-1} \{MP_\Omega Y - MP_\Omega X\beta_0\} \\ &= X'Y - X'M'(MP_\Omega M')^{-1} \left\{MX(X'X)^- X'Y - MX\beta_0\right\} \\ &= X'Y - X'M'(MP_\Omega M')^{-1} \left\{A(X'X)^- X'Y - A\beta_0\right\} \\ &= X'Y - A' \left(A(X'X)^- A'\right)^{-1} \left\{A\hat{\beta} - c\right\} \\ &= X'X\hat{\beta} - A' \left(A(X'X)^- A'\right)^{-1} \left\{A\hat{\beta} - c\right\}. \end{aligned}$$

Now recall, a solution of $Bu = d$ is $\hat{u} = B^{-}d$. Therefore, from above, since

$$X'X(\hat{\beta}_H - \hat{\beta}) = -A'\left(A(X'X)^{-}A'\right)^{-1}\left\{A\hat{\beta} - c\right\},$$

we have that

$$\hat{\beta}_H = \hat{\beta} - (X'X)^{-}A'\left(A(X'X)^{-}A'\right)^{-1}\left\{A\hat{\beta} - c\right\}.$$

Also, these two together yield,

$$\begin{aligned}
&(\hat{\beta}_H - \hat{\beta})'X'X(\hat{\beta}_H - \hat{\beta}) \\
&= (A\hat{\beta} - c)'\left(A(X'X)^{-}A'\right)^{-1}A(X'X)^{-}A'\left(A(X'X)^{-}A'\right)^{-1}(A\hat{\beta} - c) \\
&= (A\hat{\beta} - c)'\left(A(X'X)^{-}A'\right)^{-1}(A\hat{\beta} - c).
\end{aligned}$$