

Exercises

1. A set Ω is *pathwise connected* if any two points in Ω can be joined by a piecewise smooth curve entirely contained in Ω . We show that an open set Ω is pathwise connected if and only if Ω is connected.

- (a) Suppose first that Ω is open and pathwise connected, and that it can be written as a disjoint union of non-empty open sets $\Omega = \Omega_1 \cup \Omega_2$. Let $w_1 \in \Omega_1$ and $w_2 \in \Omega_2$ and let $z : [0, 1] \rightarrow \Omega$ be a curve with $z(0) = w_1$ and $z(1) = w_2$. Let

$$t^* = \sup_{0 \leq t \leq 1} \{t : z(s) \in \Omega_1 \text{ for all } 0 \leq s \leq t\}.$$

Arrive at a contradiction by considering the point $z(t^*)$.

- (b) Conversely, suppose that Ω is open and connected. Fix a point $w \in \Omega$ and let $\Omega_1 \subset \Omega$ denote the set of all points that can be joined to w by a curve contained in Ω . Also, let $\Omega_2 \subset \Omega$ denote the set of all points that cannot be joined to w by a curve in Ω . Show that both Ω_1 and Ω_2 are open, disjoint and their union is Ω . Conclude that $\Omega = \Omega_1$.
2. Let Ω be an open set in \mathbb{C} and $z \in \Omega$. The connected component C_z of z is the set of all points in Ω that can be reached from z by a curve entirely contained in Ω .
 - (a) Show that C_z is open and connected, and any two connected components are either disjoint or coincide.
 - (b) Show that Ω can have only countably many distinct connected components.
 - (c) Prove that if Ω is the complement of a compact set, then Ω has only one unbounded component.
3. For a function $f : \mathbb{C} \rightarrow \mathbb{C}$ and a curve γ in the complex plane define the integral with respect to \bar{z} as $\int_{\gamma} f d\bar{z} = \overline{\int_{\gamma} f dz}$. From this the line integral with respect to x and y can be defined as

$$\begin{aligned} \int_{\gamma} f dx &= \frac{1}{2} \left(\int_{\gamma} f dz + \int_{\gamma} f d\bar{z} \right) \\ \int_{\gamma} f dy &= \frac{1}{2i} \left(\int_{\gamma} f dz - \int_{\gamma} f d\bar{z} \right) \end{aligned}$$

Check that for $f = u + iv$

$$\int_{\gamma} f dz = \int_{\gamma} (u dx - v dy) + i \int_{\gamma} (u dy + v dx). \quad (0.1)$$

If we instead start by defining for any $p, q : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $\gamma : [a, b] \rightarrow \mathbb{R}^2$ the line integral $\int_{\gamma} p dx + q dy$ by

$$\int_{\gamma} p dx + q dy := \int_a^b p(x(t), y(t)) \cdot x'(t) dt + q(x(t), y(t)) \cdot y'(t) dt$$

then show that the right hand side of (0.1) gives $\int_{\gamma} f dz$.

The integral *with respect to the arc length* is

$$\int_{\gamma} f ds = \int_{\gamma} f |dz| := \int_{\gamma} f(z(t)) |z'(t)| dt$$

With $f \equiv 1$ one gets the arc length. In this case $\int_{-\gamma} f |dz| = \int_{\gamma} f |dz|$ and

$$\left| \int_{\gamma} f dz \right| \leq \int_{\gamma} |f| \cdot |dz|.$$

Show the following **Theorem**: If p and q are (possibly complex valued) continuous functions in a region Ω , then for any curve γ in Ω the line integral $\int_{\gamma} p dx + q dy$ depends only on the endpoints of γ if and only if there exists a function $U(x, y)$ in Ω with the partial derivatives $\partial U / \partial x = p$, $\partial U / \partial y = q$.

Hint: For the only if part fix a point (x_0, y_0) and let $U(x', y') = \int_{\gamma} p dx + q dy$ for any(?) curve γ which starts at (x_0, y_0) and ends at (x', y') .

Thus $\int_{\gamma} f(z) dz = \int_{\gamma} f(z) dx + i \int_{\gamma} f(z) dy$ is dependent only on the endpoints for any γ if there is a function F on Ω such that

$$\frac{\partial F(z)}{\partial x} = f(z), \quad \frac{\partial F(z)}{\partial y} = if(z).$$

Conclude then that $\int_{\gamma} f dz$ with f continuous, depends only on the endpoints of γ if and only if f is the derivative of a holomorphic function in Ω . (note that we proved only one direction in class)

4. These calculations provide some insight into Cauchy's theorem

- (a) Evaluate $\int_{\gamma} z^n dz$ for all integers n . Here γ is any circle centered at the origin with positive orientation. What if γ is a circle not containing the origin?
- (b) show that if $|a| < r < |b|$ then

$$\int_{\gamma} \frac{dz}{(z-a)(z-b)} = \frac{2\pi i}{a-b},$$

where γ denotes the circle centered at the origin, of radius r , with the positive orientation.

5. Suppose that f is holomorphic in an open set Ω . Prove that in any one of the following cases:

- (a) $\operatorname{Re}(f)$ is constant;
- (b) $\operatorname{Im}(f)$ is constant;
- (c) $|f|$ is constant;

one can conclude that f is constant.

6. Suppose f is continuous in a region Ω . Prove that any two primitives of f (if they exist) differ by a constant.

7. **[HW 1, due 5 Oct]** Consider a holomorphic function f on a region Ω . Let C be a circle inside Ω whose interior is also contained in Ω . Here is another way to show that $\int_C f(z) dz = 0$.

- (a) Consider any regular polygon P_n of n sides inscribed inside the circle. Argue that $\int_{P_n} f(z)dz = 0$.
- (b) Show that $\lim_{n \rightarrow \infty} \int_{P_n} f(z)dz = \int_C f(z)dz$.
8. The next few exercises show how complex integration can help us compute complicated real integrals.

- (a) [HW 2, due Oct 11] Prove

$$\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{\sqrt{2\pi}}{4}.$$

The integral \int_0^∞ is interpreted as $\lim_{R \rightarrow \infty} \int_0^R$.

HINT: Integrate e^{-z^2} from 0 to R , then along the circular arc from R to $Re^{i\pi/4}$ and then along the straight line from $Re^{i\pi/4}$ to 0.

- (b) Show $\int_0^\infty \frac{\sin x}{x} = \frac{\pi}{2}$.

HINT: The integral equals $\frac{1}{2i} \int_{-\infty}^\infty \frac{e^{ix}-1}{x} dx$. Use the indented semicircle.

- (c) [HW 2, due Oct 11] Evaluate the integrals

$$\int_0^\infty e^{-ax} \cos(bx)dx \quad \text{and} \quad \int_0^\infty e^{-ax} \sin(bx)dx, \quad a > 0$$

by integrating e^{-Ax} , $A = \sqrt{a^2 + b^2}$, over an appropriate sector with angle ω , with $\cos \omega = a/A$.

- (d) Prove that for all $\xi \in \mathbb{C}$ we have $e^{-\pi\xi^2} = \int_{-\infty}^\infty e^{-\pi x^2} e^{2\pi i x \xi} dx$.

9. Suppose f is continuously complex differentiable on Ω , and $T \subset \Omega$ is a triangle whose interior is also contained in Ω . Apply Green's theorem to show that $\int_T f(z)dz = 0$. this provides a proof of Goursat's theorem under the additional assumption that f' is continuous.
10. Show that every non-constant polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$ with complex coefficients has a root in \mathbb{C} . From this conclude that $P(z)$ has n roots w_1, w_2, \dots, w_n and $P(z) = a_n(z - w_1)(z - w_2) \dots (z - w_n)$.
- HINT: Suppose not. Then note that $P(z)^{-1}$ is entire.
11. HW 3 (Due Monday 25 October) Let f be a holomorphic function on the disc D_{R_0} centered at the origin and of radius R_0 .

- (a) Prove that whenever $0 < R < R_0$ and $|z| < R$, then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\phi}) \cdot Re \left(\frac{Re^{i\phi} + z}{Re^{i\phi} - z} \right) d\phi.$$

HINT: Note that if $w = R^2/\bar{z}$, then the integral of $f(\zeta)/(\zeta - w)$ around the circle of radius R centered at the origin is 0. Use this, together with the Cauchy integral formula.

- (b) Show that

$$Re \left(\frac{Re^{i\gamma} + r}{Re^{i\gamma} - r} \right) = \frac{R^2 - r^2}{R^2 - 2Rr \cos \gamma + r^2}.$$

12. **HW 3 (Due Monday 25 October)** Say that a twice continuously differentiable real-valued function is harmonic if $\Delta u(x, y) = 0$ where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

- (a) If f is holomorphic in an open set Ω , then show that the real and imaginary parts of f are harmonic.
- (b) Let u be a real-valued function defined on the unit disc \mathbb{D} . Suppose that u is twice continuously differentiable and harmonic.
- Prove that there exists a holomorphic function f on \mathbb{D} such that $\operatorname{Re}(f) = u$. Also show that the imaginary part of f is uniquely defined up to an additive (real) constant.
HINT: If there is such an f then $f'(z) = 2\partial u/\partial z$. Therefore, let $g(z) = 2\partial u/\partial z$ and prove that g is holomorphic. Why can one find F with $F' = g$? Prove that $\operatorname{Re}(f)$ differs from u by a real constant.
 - Deduce from this result, and the above exercise, the Poisson integral representation formula from the Cauchy integral formula: If u is harmonic in \mathbb{D} and continuous on its closure, then if $z = re^{i\theta}$ one has

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - \phi) u(\phi) d\phi$$

where $P_r(\gamma)$ is the Poisson kernel for the unit disc given by

$$P_r(\gamma) = \frac{1 - r^2}{1 - 2r \cos \gamma + r^2}.$$

13. Suppose f is an analytic function defined everywhere in \mathbb{C} and such that for each $z_0 \in \mathbb{C}$ at least one coefficient in the expansion

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

is equal to 0. Prove that f is a polynomial.

HINT: Use the fact that $c_n n! = f^{(n)}(z_0)$ and use a countability argument.

14. Let Ω be a bounded open subset of \mathbb{C} , and $\phi : \Omega \rightarrow \Omega$ a holomorphic function. Prove that if there exists a point $z_0 \in \Omega$ such that

$$\phi(z_0) = z_0 \quad \text{and} \quad \phi'(z_0) = 1$$

then ϕ is linear.

HINT: Why can one assume that $z_0 = 0$? Write $\phi(z) = z + a_n z^n + O(z^{n+1})$ near 0, and prove that if $\phi_k = \phi \circ \cdots \circ \phi$ (k times) then $\phi_k(z) = z + k a_n z^n + O(z^{n+1})$. Apply the Cauchy inequalities and let $k \rightarrow \infty$ to conclude the proof.

15. **[HW 4, Due Wednesday 3 November]** This exercise shows that one cannot always extend a holomorphic function from a smaller set to a larger one (see the section on Schwarz reflection principle in Stein-Shakarchi for some cases in which one can extend). The following definition is needed. Let f be a function defined in the unit disc \mathbb{D} , with boundary circle C . A point w on C is said to be *regular* for f if there is an open neighbourhood U of w and an analytic function g on U , so that $f = g$ on $\mathbb{D} \cap U$. A function f defined on \mathbb{D} cannot be continued analytically past the unit circle if no point of C is regular for f .

(a) Let

$$f(z) = \sum_{n=0}^{\infty} z^{2^n} \quad \text{for } |z| < 1.$$

Notice that the radius of convergence of the above series is 1. Show that f cannot be continued analytically past the unit disc.

HINT: Suppose $\theta = 2\pi p/2^k$, where p and k are positive integers. Let $z = re^{i\theta}$; then $|f(re^{i\theta})| \rightarrow \infty$ as $r \rightarrow 1$.

(b) Fix $0 < \alpha < \infty$. show that the analytic function f defined by

$$f(z) = \sum_{n=0}^{\infty} 2^{-n\alpha} z^{2^n} \quad \text{for } |z| < 1$$

extends continuously to the unit circle, but cannot be analytically continued past the unit circle.

16. [HW 4, Due Wednesday 3 November] Prove the converse to Runge's theorem: if K is a compact set whose complement is not connected, then there exists a function f holomorphic in a neighborhood of K which cannot be approximated uniformly by polynomials on K .

HINT: Pick a point z_0 in a bounded component of K^c , and let $f(z) = 1/(z - z_0)$. If f can be approximated uniformly by polynomials on K , show that there exists a polynomial p such that $|(z - z_0)p(z) - 1| < 1$. Use the maximum modulus principle (see below) to show that this inequality continues to hold for all z in the component of K^c that contains z_0 . *The maximum modulus principle (which we learn later) states that if h is a non-constant holomorphic function in a region Ω , then $|h|$ cannot attain a maximum in Ω .*

17. Using Euler's formula

$$\sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i}$$

show that the complex zeroes of $\sin \pi z$ are exactly at the integers, and that they are each of order 1.

Calculate the residue of $\frac{1}{\sin \pi z}$ at $z = n \in \mathbf{Z}$.

18. [HW 5, Due Monday 15 November] Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^4}.$$

19. [HW 5, Due Monday 15 November] Show that

$$\int_{-\infty}^{\infty} \frac{\cos x \, dx}{x^2 + a^2} = \frac{\pi e^{-a}}{a}, \quad \text{for all } a > 0.$$

20. [HW 5, Due Monday 15 November] Prove that

$$\int_0^{2\pi} \frac{d\theta}{(a + \cos \theta)^2} = \frac{2\pi a}{(a^2 - 1)^{3/2}}, \quad \text{whenever } a > 1.$$

21. (HW 6, Due Monday 22 November) Morera's theorem states that if f is continuous in \mathbb{C} , and $\int_T f(z) dz = 0$ for all triangles T , then f is holomorphic in \mathbb{C} . We may ask if the conclusion still holds if we replace triangles by other sets.

- (a) Suppose that f is continuous on \mathbb{C} , and

$$\int_C f(z) dz = 0 \quad (0.2)$$

for every circle C . Prove that f is holomorphic.

- (b) More generally, let Γ be any toy contour, and \mathcal{F} the collection of all translates and dilates of Γ . Show that if f is continuous on \mathbb{C} , and

$$\int_{\gamma} f(z) dz = 0 \quad \text{for all } \gamma \in \mathcal{F}$$

then f is holomorphic. In particular, Morera's theorem holds under the weaker assumption that $\int_T f(z) dz = 0$ for all equilateral triangles.

[HINT (for part (a)): As a first step, assume that f is twice real differentiable, and write $f(z) = f(z_0) + a(z - z_0) + b(\bar{z} - \bar{z}_0) + O(|z - z_0|^2)$ for z near z_0 . Integrate this expression over small circles around z_0 to conclude $\partial f / \partial \bar{z} = b = 0$ at z_0 . Alternatively, suppose only that f is differentiable and apply Green's theorem to conclude that the real and imaginary parts of f satisfy the Cauchy-Riemann equations.

In general, let $\phi(w) = \phi(x, y)$ (when $w = x + iy$) denote a smooth function with $0 \leq \phi(w) \leq 1$, and $\int_{\mathbb{R}^2} \phi(w) dV(w) = 1$, where $dV(w) = dx dy$, and \int denotes the usual integral of a function of two variables in \mathbb{R}^2 . For each $\epsilon > 0$, let $\phi_{\epsilon}(z) = \epsilon^{-2} \phi(\epsilon^{-1} z)$, as well as

$$f_{\epsilon}(z) = \int_{\mathbb{R}^2} f(z - w) \phi_{\epsilon}(w) dV(w),$$

where the integral denotes the usual integral of functions of two variables, with $dV(w)$ the area element of \mathbb{R}^2 . Then f_{ϵ} is smooth, satisfies condition (0.2), and $f_{\epsilon} \rightarrow f$ uniformly on any compact subset of \mathbb{C} .]

22. [HW 7, due Saturday Dec 4] Suppose $f(z)$ is holomorphic in a punctured disc $D_r(z_0) - \{z_0\}$. Suppose also that

$$|f(z)| \leq A|z - z_0|^{-1+\epsilon}$$

for some $\epsilon > 0$, and all z near z_0 . Show that the singularity of f at z_0 is removable.

23. Use the Cauchy inequalities or the maximum modulus principle to solve the following problems.

- (a) Prove that if f is an entire function that satisfies

$$\sup_{|z|=R} |f(z)| \leq AR^k + B$$

for all $R > 0$, and for some integer $k \geq 0$ and some constants $A, B > 0$, then f is a polynomial of degree $\leq k$.

- (b) Show that if f is holomorphic in the unit disc, is bounded, and converges uniformly to zero in the sector $\theta < \arg z < \phi$ as $|z| \rightarrow 1$, then $f = 0$.
- (c) Let w_1, w_2, \dots, w_n be points on the unit circle in the complex plane. Prove that there exists a point z on the unit circle such that the product of the distances from z to the points $w_j, 1 \leq j \leq n$, is at least 1. Conclude that there exists a point w on the unit circle such that the product of the distances from w to the points $w_j, 1 \leq j \leq n$, is exactly equal to 1.

(d) Show that if the real part of an entire function f is bounded, then f is constant.

24. [HW 7, due Saturday Dec 4] Suppose f and g are holomorphic in a region containing the disc $\{|z| \leq 1\}$. Suppose that f has a simple zero at $z = 0$ and vanishes nowhere else in $\{|z| \leq 1\}$. Let

$$f_\epsilon(z) = f(z) + \epsilon g(z).$$

Show that if ϵ is sufficiently small, then

- (a) $f_\epsilon(z)$ has a unique zero in $\{|z| \leq 1\}$, and
- (b) If z_ϵ is this zero, the mapping $\epsilon \mapsto z_\epsilon$ is continuous.

25. Give another proof of the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta$$

using homotopy of curves.

HINT: Deform the circle C to a small circle centered at z , and note that the quotient $(f(\zeta) - f(z))/(\zeta - z)$ is bounded.

26. Prove the maximum principle for harmonic functions, that is:

- (a) If u is a non-constant real-valued harmonic function in a region Ω , then u cannot attain a maximum (or minimum) in Ω .
- (b) Suppose that Ω is a region with compact closure $\bar{\Omega}$. If u is harmonic in Ω and continuous in $\bar{\Omega}$, then

$$\sup_{z \in \Omega} |u(z)| \leq \sup_{z \in \bar{\Omega} - \Omega} |u(z)|.$$

HINT: To prove the first part, assume that u attains a local maximum at z_0 . Let f be holomorphic near z_0 with $u = \operatorname{Re}(f)$, and show that f is not open.