Lecture 20: More examples

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<u>Representations of S_4</u>

$|S_4| = 24 \quad S_4 = \{ \underbrace{\{e\}, \{(1\,2), \dots\}}_{}$, 6 transpositions, 8 3-cycles, 6 4-cycles, $\{(1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\} \}$

$$H = \{\{e\}, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\} \trianglelefteq S_4$$

$$S_4/H \cong S_3$$

$$S_4 = S_3 \ltimes H$$

$S_3 \hookrightarrow S_4$

$S_3 = \{\sigma \in S_4 \mid \sigma(4) = 4\}$

5 conjugacy classes in S_4, so there are 5 irred repr.

$\theta : S_4 \longrightarrow S_3 \quad \ker(\theta) = H$. Since θ is surj

Every irrd. repr of S_3 is an irr rep of S_4.

$\psi_1 : S_3 \longrightarrow \mathbb{C}^*$ trivial

$\psi_2 : S_3 \longrightarrow \mathbb{C}^*$ is sgn

$\psi_3 : S_3 \longrightarrow GL_2$ is the 2-dim irred repr.

$\chi_{\psi_1}(\sigma) = 1 \quad \forall \sigma \in S_3$

$\chi_{\psi_2}(\sigma) = \mathrm{sgn}(\sigma)$

S_3

	$\{e\}$	transp	3-cycles
ψ_1	1	1	1
ψ_2	1	-1	1
ψ_3	2	0	-1

Let $\varphi_i = \psi_i \circ \theta$ are 3 irred rep of S_4.

character table of S_4

S_4	e	trans	3 cycles	4 cycles	prod of trans
φ_1	1	1	1	1	1
φ_2	1	-1	1	-1	1
φ_3	2	0	-1	0	2
φ_4	3	1	0	-1	-1
φ_5	3	-1	0	1	-1

$$\chi_{\varphi_i}(\sigma) = tr(\varphi_i(\sigma)) = tr(\psi_i(\theta(\sigma))) = \chi_{\psi_i}(\theta(\sigma))$$

$(1234)H = (\ ,\ )H$ transposition

$$\boxed{\begin{array}{c}(1234)(12)(34)\\ \shortparallel \\ (13)(2)(4)\end{array}}$$

$(1234)H = (13)H$

S_4 acts on $\mathbb{C}^4 = \{(z_1, z_2, z_3, z_4) \mid z_i \in \mathbb{C}\}$

$(z_1, \dots, z_4) \mapsto (z_{\sigma 1}, \dots, z_{\sigma 4})$

$\mathbb{C}^4 = (\text{trivial reps}) \oplus V$

std reps.

V is irred reps of S_4.

$$\left.\begin{array}{l} n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2 = 24 \\ \ 1 \quad\ \ 1 \quad\ \ 4 \quad\ \ 9 \end{array}\right\} \Rightarrow n_5 = 3$$

$\chi_{\mathbb{C}^4}(e) = 4$, $\chi_{\mathbb{C}^4}(\text{trans}) = 2$ $\chi_{\mathbb{C}^4}(3\text{ cycles}) = 1$, $\chi_{\mathbb{C}^4}(4\text{ cycles}) = 0$

$\chi_{\mathbb{C}^4}(\text{prod disj trans}) = 0$

Since V is irred as A_4-reps it is irred as S_4-reps.

$$\varphi_4 : S_4 \longrightarrow GL(V)$$

$$V = \{(z_1, z_2, z_3, z_4) \mid z_1 + z_2 + z_3 + z_4 = 0\}$$

$\varphi_5 = \varphi_4 \otimes \varphi_2$ is a 3-dim'l reps.

These all the irred reps of S_4

(*) Let G be group. $H \leq G$ & $K \leq H$.

Let W be a K-repr.

$$\operatorname{Ind}_H^G\left(\operatorname{Ind}_K^H W\right) \cong \operatorname{Ind}_K^G W$$

Let $h_1, \dots, h_r \in H$ be s.t.
$h_1K, \dots, h_rK$ are the distinct left cosets of K in H.
$r = [H:K]$

Let $g_1, \dots, g_s \in G$ be s.t.
$g_1H, \dots, g_sH$ are the distinct left cosets of H in G. $s = [G:H]$

The left cosets of K in G are

$$\left.\begin{matrix} g_1h_1K, & g_1h_2K, & \dots, & g_1h_rK \\ g_2h_1K, & g_2h_2K, & \dots, & g_2h_rK \\ \vdots & & & \\ g_sh_1K, & \dots & , & g_sh_rK \end{matrix}\right\} \quad [G:K] = rs$$

If $V = \operatorname{Ind}_K^H W$ & $U = \operatorname{Ind}_K^G W$.

$$V = \bigoplus_{i=1}^{r} h_iW \qquad U = \bigoplus g_ih_jW$$

$$\operatorname{Ind}_H^G V = U \qquad \text{by taking } g_1 = e$$