Assignment 3

Physics III: Electricity and Magnetism
B. Math. Year 3,
September - December 2021.

Due on: December 02^{nd}, 2021.

Please give arguments where necessary. If it is unclear from your answer why a particular step is being taken, full credit will not be awarded. Please feel free to discuss amongst yourselves; however, copying the assignment solutions from someone else is strictly prohibited and both persons involved will be penalized. Each one of you must submit your own answers. Total: 45 points.

1. In lectures, we derived that the effective charge density that can be used to mimic the electrostatic behaviour of a point quadrupole located at $\vec{r} = \vec{r}_0$ with Cartesian components Q_{ij} is given by

$$\rho_Q^{\text{eff}}(\vec{r}) = \epsilon_0 Q_{ij} \nabla_i \nabla_j \delta(\vec{r} - \vec{r}_0)$$

   The point quadrupole is now placed in an arbitrary external field $\vec{E}(\vec{r})$. Find the force, the net torque and the potential energy of the point quadrupole due the external electric field. [3+4+3].

2. Imagine a short straight charged wire of finite length placed along the z-axis (take the origin to be the midpoint of the wire) which has a linear charge density $\lambda(z)$ with the following properties:

   - $\lambda(z) = \lambda(-z)$.
   - Carries a total charge Q.
   - $Ql_0^2 = \int_{-\infty}^{\infty} z^2 \lambda(z)\, dz$.

   Please answer the following questions:

   (a) Evaluate the dipole moment of the charge distribution w.r.t the origin. Without explicit calculation, argue why or why not the dipole moment will be the same when calculated w.r.t the positive z end of the wire. [3]

   (b) The quadrupole moment of any charge distribution is a tensor having nine components $Q_{ij},\ i, j = \{x, y, z\}$. A component is defined as, given a charge distribution $\rho(\vec{r'})$.

$$Q_{ij} = \int_{\mathbb{V}'} d^3r' \rho(\vec{r'}) \left(3r'_i r'_j - \delta_{ij} r'^2\right)$$

Of course, for a surface or a line charge distribution it should be suitably modified. Evaluate the quadrupole moment tensor components for the given charge distribution. [5]

(c) Evaluate the electrostatic potential $\phi(r,\theta)$ far away from this charge distribution, keeping terms up to and including the quadrupole contribution. In terms of the azimuthally symmetric Legendre polynomials that was set as part of the reading homework in preparation for Laplace and Poisson's equations, interpret the various terms that appear in the expansion in spherical co-ordinates. [4+3]

3. (a) Consider a cube of side a filled with a simple dielectric material and having uniform polarization $\vec{P}$ inside, with the direction of polarization parallel to one of its face. Find the electric field at the center of the cube. *Hint:* You may find the following formula useful. Suppose $d\vec{S}(\vec{r})$ is an infinitesimal area vector centered around $\vec{r}$ ($\vec{r} \neq 0$). Then the solid angle subtended by the infinitesimal area at the origin is given by $d\Omega(\vec{r}) = \frac{d\vec{S}(\vec{r})\cdot\vec{r}}{r^3}$. [5]

   (b) Evaluate and compare this field with the case of uniformly polarized sphere with the same dielectric material and same magnitude of uniform polarization inside. [5]

4. Consider a positive point charge q lying at the origin surrounded by a sphere of radius R. On the spherical surface there is a uniform (in magnitude) dipole layer of magnitude τ and a uniform surface charge density σ. The potential vanishes everywhere for $r > R$. For $r < R$, the potential is identical to the potential of the point charge sitting at the origin without even the spherical surface. Find σ and τ. [5+5]