Lecture 1: Multilinear algebra, tensor products

19 September 2021
13:58

Syllabus: Introduction to multilinear algebra: Review of linear algebra, multilinear forms, tensor products, wedge product, Grassmann ring, symmetric product. Representation of finite groups: Complete reducibility, Schurs' lemma, characters, projection formulae. Induced representation, Frobenius reciprocity. Representations of permutation groups.

References:
(1) J.P. Serre: Linear representations of finite groups
(2) W. Fulton and J. Harris: Representation Theory, Part I

From <https://www.isibang.ac.in/~manish/teaching/index.html>

Homework + quizes : 60% Final Exam : 40%

Homework on Moodle and website. Grader : Satyendra

Let k be a field (usually $\mathbb{C}$). Let V, W be k-vector spaces.
A map $\varphi: V \longrightarrow W$ is called linear if $\varphi(av_1 + v_2) = a\varphi(v_1) + \varphi(v_2)$ $\forall a \in k$ & $v_1, v_2 \in V$.

Similarly if $V_1, V_2, \ldots, V_n$ are vector space, a map
$\varphi: V_1 \times V_2 \longrightarrow W$ is bilinear if $\varphi(v_1, \cdot)$ and $\varphi(\cdot, v_2)$ are linear $\forall$ $v_1 \in V_1$ & $v_2 \in V_2$.

$\varphi: V_1 \times V_2 \times \cdots \times V_n \longrightarrow W$ is multilinear if it is linear in each variable.

Lecture 9: Tensor product of modules

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23:03

Recall R a comm ring with unity & M an R-module then we constructed $S^{-1}R$-mod $S^{-1}M$ and $\varphi : M \longrightarrow S^{-1}M$, $m \longmapsto \frac{m}{1}$ is an R-module homo s.t.

N a $S^{-1}R$-mod & $M \xrightarrow{\alpha} N$ an R-mod homo then $\exists !\ \tilde{\alpha} : S^{-1}M \to N$ s.t. $\tilde{\alpha} \circ \varphi = \alpha$.

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[dr, "\varphi"'] & N \\ & S^{-1}M \arrow[u, "\tilde{\alpha}"'] \end{tikzcd}$$

One can use universal property to define localization as well.

For defining tensor product we use this strategy.

<u>Def^n</u> Let M & N be R-modules. An R-module T together with an R-bilinear map $\varphi : M \times N \longrightarrow T$

(i.e. $\varphi(m, n_1 + r n_2) = \varphi(m, n_1) + r\varphi(m, n_2)$ $\forall\, m \in M$ $\forall\, n_1, n_2 \in N$ & $\forall\, r \in R$

$|||^{ly}$ $\varphi(m_1 + r m_2, n) = \varphi(m_1, n) + r\varphi(m_2, n)$ $\forall\, m_1, m_2 \in M$, $\forall\, n \in N$ & $\forall\, r \in R$)

is said to be a <u>tensor product of M & N over R</u> if given any R-bilin map $\psi : M \times N \longrightarrow A$ where A is an R-mod there exists a unique R-mod homo $\theta : T \longrightarrow A$ s.t. $\theta \circ \varphi = \psi$.

$$\begin{tikzcd} M \times N \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & T \arrow[d, "\exists ! \theta"] \\ & A \end{tikzcd}$$

T

<u>Prop:</u> T exist and is unique upto unique isomorphism and it is denoted by $M \otimes_R N$.

Pf: Uniqueness:

Let $\varphi' : M \times N \longrightarrow T'$ be another tensor product of M & N. Then want to show that $\exists!$ isom $T \xrightarrow{\alpha} T'$ s.t.

$$\alpha \circ \varphi = \varphi' .$$

By Universal property of T

$$\begin{tikzcd}
M \times N \arrow[r, "\varphi"] \arrow[dr, "\varphi'"] \arrow[ddr, "\varphi"'] & T \arrow[d, dashed, "\exists! \alpha \; R\text{-lin}"] \arrow[dd, dashed, bend left=60, "id"] \\
& T' \arrow[d, dashed, "\exists! \alpha'"] \\
& T
\end{tikzcd}$$

s.t. $\alpha \circ \varphi = \varphi'$ ...(i)

$\alpha' \circ \varphi' = \varphi$ ...(ii)

$$(\alpha' \circ \alpha) \circ \varphi \overset{(i)}{=} \alpha' \circ \varphi' \overset{(ii)}{=} \varphi$$

$$id_T \circ \varphi = \varphi$$

By uniqueness $\alpha' \circ \alpha = id_T$

Similarl $\alpha \circ \alpha' = id_{T'}$

Hence α is an isom.

Existence: Let

$F_{M\times N}$ be the free R-module over $M\times N$.

i.e. $F = F_{M\times N} = \bigoplus_{(m,n)\in M\times N} R(m,n)$

Let $i : M\times N \longrightarrow F$
$(m,n) \longmapsto 1(m,n)$

$$q : F \longrightarrow F \Big/ \Big\langle \big((m, n_1 + r n_2) - (m,n_1) - r(m,n_2)\big),\ (m_1 + r m_2, n) - (m_1, n) - r(m_2,n) \ \ \forall m_1, m_2, m \in M,\ \forall r \in R \ \&\ n, n_1, n_2 \in N \Big\rangle$$

$\Big(\overset{\shortparallel}{T}$ say$\Big)$ $\Big(\overset{\shortparallel}{K}\Big)$

$\varphi = q \circ i : M\times N \longrightarrow T$.

WTS φ is bilinear & it has the universal property.

$K = \ker(q) \subseteq F$

$$\varphi(m, n_1 + r n_2) = q \circ i(m, n_1 + r n_2) = q(m, n_1 + r n_2)$$

$$= \overline{(m, n_1 + r n_2)}$$

$$= \overline{(m, n_1)} + r\overline{(m, n_2)}$$

$$\left(\because \overline{(m, n_1 + r n_2) - (m, n_1) - r(m, n_2)} = 0 \text{ in } T \right)$$

$$= \varphi(m, n_1) + r\varphi(m, n_2)$$

Ill'y lin in 1st var holds.

Let $\psi : M \times N \longrightarrow A$ be R-bilin map.

Want $\theta : T \longrightarrow A$ R-line s.t.

$\theta \circ \varphi = \psi$.

Let $\tilde{\Theta} : F \longrightarrow A$ in R-lin

$$(m,n) \longmapsto \Psi(m,n)$$

$$\sum_{\substack{(m,n)\in M\times N \\ \text{finite}}} r_{(m,n)}(m,n) \longmapsto \sum_{\substack{(m,n)\in M\times N \\ \text{finite}}} r_{(m,n)}\Psi(m,n)$$

Note that Ψ is bilin. and hence

$$\tilde{\Theta}\big((m, n_1+rn_2) - (m, n_1) - r(m,n_2)\big)$$
$$= \Psi(m, n_1+rn_2) - \Psi(m,n_1) - r\Psi(m,n_2)$$
$$= 0 \quad (\Psi \text{ is R-bilin})$$

$\overset{1^{st}\text{ isom}}{\Longrightarrow}$ $K \subseteq \ker(\tilde{\Theta})$

So $\exists\, \Theta : T \longrightarrow A$ s.t.

$$\Theta \circ q = \tilde{\Theta}$$

$$\begin{tikzcd} F \arrow[d, "q"'] \arrow[dr, "\tilde{\Theta}"] & \\ T \arrow[r, "\exists!\Theta"'] & A \end{tikzcd}$$

$$\Theta\circ\varphi = \Theta\circ q\circ i$$
$$= \tilde{\Theta}\circ i \quad : M\times N \xrightarrow{i} F \xrightarrow{\tilde{\Theta}} A$$
$$(m,n) \longmapsto \Psi(m,n)$$
$$= \Psi$$

$\boxtimes$

Check θ is unique.

$$\begin{tikzcd} & & M\times N \arrow[d, "\varphi"] \arrow[dr, "\psi"] & \\ M\times N \arrow[r, "i"] & F \arrow[r, "q"] & T \arrow[r, "\theta"] \arrow[r, bend right, "\theta'"'] & A \end{tikzcd}$$

Let $\overline{(m,n)} \in T$

$$\theta\left(\overline{(m,n)}\right) = \theta \circ q \circ i(m,n)$$
$$= \theta \circ \varphi$$
$$= \psi(m,n)$$

$$\theta'\left(\overline{(m,n)}\right) = \theta' \circ q \circ i\,(m,n)$$
$$= \theta' \circ \varphi(m,n)$$
$$= \psi(m,n)$$

$$\Rightarrow \theta = \theta' \quad \left(\because \left\{ \overline{(m,n)} \ \ \forall m \in M \ \& \ n \in N \right\} \text{ gen } T \right)$$

Examples: ① $R = \mathbb{Z}$, $M = \mathbb{Z}$, $N = \mathbb{Z}$

$\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} = ?$

$\cong \mathbb{Z}$

$$\boxed{\begin{aligned} \varphi : \mathbb{Z} \times \mathbb{Z} &\longrightarrow \mathbb{Z} \\ (a,b) &\longmapsto ab \end{aligned}}$$

$\psi : \mathbb{Z} \times \mathbb{Z} \longrightarrow M$ is a bil map & M a $\mathbb{Z}$-mod

$\theta : \mathbb{Z} \longrightarrow M$

$1 \longmapsto \psi(1,1)$

$a \longmapsto \psi(a,1)$ $\quad \mathbb{Z}$-lin

$\theta \circ \varphi(a,b) = \theta(ab) = \psi(ab,1) = b\psi(a,1) = \psi(a,b)$ $\quad \forall a,b \in \mathbb{Z}$

$\Rightarrow \theta \circ \varphi = \psi$. Check uniqueness of θ

Hence $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z}$.

② M an R-mod $\quad M \otimes_R 0 = 0$

Prop: R a ring, A, B, C R-modules. Then following holds.

① $R \otimes_R A \cong A$ $\quad r\otimes a \longmapsto ra \quad \forall r\in R \,\&\, a\in A$

② $A \otimes_R B \cong B \otimes_R A$ $\quad a\otimes b \longmapsto b\otimes a \quad \forall a\in A \,\&\, b\in B$

HW ③ $(A \otimes_R B) \otimes_R C \cong A \otimes_R (B \otimes_R C)$ $\quad (a\otimes b)\otimes c \mapsto a\otimes(b\otimes c)$

Exc ④ $(A\oplus B)\otimes_R C \cong (A\otimes_R C) \oplus (B \otimes_R C)$ $\quad (a,b)\otimes c \longmapsto (a\otimes c, b\otimes c)$

⑤ $S\subseteq R$ multiplicative subset then
$S^{-1}A \cong S^{-1}R \otimes_R A$ $\quad \frac{ra}{s} \longleftarrow \frac{r}{s}\otimes a$

HW ⑥ $I \subseteq R$ ideal then $R/I \otimes_R M \cong M/IM$
$(r+I)\otimes m \longmapsto rm + IM$

Pf: ② $A \otimes_R B \cong B \otimes_R A$

$\varphi_1 : A\times B \to A\otimes B$, $\quad (a,b) \longmapsto a\otimes b$

$\varphi_2 : B\times A \to B\otimes A$, $\quad (b,a) \longmapsto b\otimes a$

$$\psi : A\times B \longrightarrow B\otimes_R A$$
$$(a,b) \longmapsto \underset{\varphi_2(b,a)}{b\otimes a}$$

It is trivial to check that ψ is bilinear.
(as φ_2 is bilinear)

Hence by defn of tensor product $\exists$,

$\theta : A\otimes_R B \longrightarrow B\otimes_R A$ $\quad R$-linear

s.t. $\theta\circ\varphi_1(a,b) = \psi(a,b)$

$\theta(a\otimes b) = b\otimes a$

|||ly $\theta' : B\otimes_R A \longrightarrow A\otimes_R B$ $\quad R$-linear

$b\otimes a \longmapsto a\otimes b$

Note $\left.\begin{array}{l}\theta\circ\theta'(b\otimes a) = b\otimes a \\ \& \ \theta'\circ\theta(a\otimes b) = a\otimes b\end{array}\right\} \forall\, b\in B \,\&\, a\in A$

But $\{a\otimes b \mid a\in A, b\in B\}$ generate $A\otimes B$
& $\{b\otimes a \mid a\in A, b\in B\}$ " $B\otimes A$

Hence $\theta\circ\theta' = id_{B\otimes A}$ & $\theta'\circ\theta = id_{A\otimes B}$

Let V & W be f.d k-vs then

$V \otimes W$ is also a f.d vector space.

V has a basis $\{v_1, \dots, v_n\}$ & W has a basis $\{w_1, \dots, w_m\}$ then

$\circledast$ $V \otimes W$ has a basis $\{ v_i \otimes w_j \mid 1 \le i \le n \ \& \ 1 \le j \le m \} = \mathcal{B}$.

$\varphi : V \times W \longrightarrow \underline{V \otimes W}$ is the bilinear

<u>Notation:</u> $\varphi(v, w) = v \otimes w$

A general element of $V \otimes W$ looks like $\sum_{\text{finite}} r_\alpha (v_\alpha \otimes w_\alpha)$ where $r_\alpha \in k$, $v_\alpha \in V$ & $w_\alpha \in W$.

Note that $\mathcal{B}$ generates $V \otimes W$

<u>Properties</u>: (i) $v \otimes (w_1 + a w_2) = v \otimes w_1 + a(v \otimes w_2)$ $\forall v \in V$, $w_1, w_2 \in W$ & $a \in k$.

(ii) $v \otimes 0_W = 0_V \otimes w = 0_{V \otimes W}$

(iii) $a(v \otimes w) = av \otimes w = v \otimes aw$ $\forall v \in V$ & $w \in W$ & $a \in k$

<u>HW</u>: Show $\mathcal{B}$ is lin independent.

Since V is of dim n, $V \cong k^n$ & $|||^{ly}$ $W \cong k^m$

$$V \otimes W \cong k^n \otimes W \cong (k \oplus k^{n-1}) \otimes W \cong (k \otimes W) \oplus (k^{n-1} \otimes W)$$

$$\cong W \oplus W^{n-1} \cong W^n \quad \text{induction on } n$$

$k^n \otimes W \cong W^n$

$\Longrightarrow \dim_k(V \otimes W) = mn$.