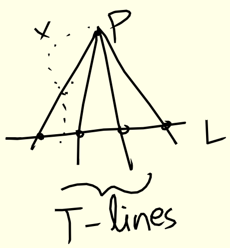
Combinatorics

17 Nov 21.

Lecture 13 — MOLS & partial geometries.

Recall: ① A partial geometry $pg(k,R,T)$ is an incidence system with block size k, every elt in R blocks, any 2 points in ≤ 1 blocks & $\forall\, p \notin$ block L $\exists$ T blocks containing p that intersect L.



② When $T = R-1$, $pg(k,R,R-1)$ is called a <u>net</u>.

In a net we have $\forall\, p \notin L$, $\exists!$ line L_p thru' p that does not intersect L.

Theorem (22.2) A set of k MOLS of order n exists iff a $(n, k+2, k+1)$ - net exists.

Pf:- (I) Suppose $L_i : R \times C \to S$ $\quad 1 \le i \le k$ are MOLS.

(Recall: $v = \dfrac{k(R-1)(k-1)}{T} + k$

if $T = R-1$, $v = k^2$)

$\Rightarrow$

$\Rightarrow$ we need a set n^2 points.

Let $\mathcal{P} = R \times C$. ($\because$ points are of the type $(x,y) \mid x \in R, y \in C$).

Construct blocks as follows: (block size has to be n)

$|A_1| = n. \Leftarrow \quad A_1 = \{ (x, b) \mid b \in C \}$

$L_1 \to \{(1,1)\ (1,2)\ (1,3), \cdots (1,n)\}$

$L_2 \to \{(2,1), (2,2) \cdots \cdots (2,n)\}$

$|A_2| = n \Leftarrow \quad A_2 = \{ (a, y) \mid a \in R \}$

$|A_{i+2}| = n \Leftarrow A_{i+2} = \{ \{(x,y) \mid L_i(x,y) = s\} \mid s \in S \}$. $1 \le i \le k$.

$(1,10)$

$$L_4 = \begin{bmatrix} a_{11}^{(4)} & \cdots & a_{1n}^{(4)} \\ \vdots & & \vdots \\ a_{n1}^{(4)} & \cdots & a_{nn}^{(4)} \end{bmatrix} \Rightarrow A_6.$$

($b = \#$ blocks $= n + n + kn$
$= n(k+2)$.

k R T
$n, k+2, k+1$

$$b = \frac{R\left((R-1)(n-1)+T\right)}{T}$$
$= n(k+2)$ for $D(n, k+2, k+1)$.)

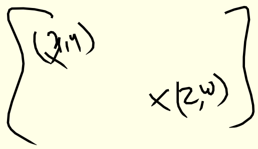
① Clearly block size is n $\forall$ lines in A_1 & A_2.
& also for A_j $j \ge 3$ because every L_{j-2} is a Latin square!

② Fix $(x,y) \in P$. TPT $\exists$ $k+2$ lines thru' (x,y)
clearly x^{th} row in A_1 & y^{th} col^n in A_2 contain (x,y).
Further for all $1 \le i \le k$ let $L_i(x,y) = s_i$.
then the line corr. to s_i in A_{i+2} contains (x,y) $\Rightarrow \exists$ precisely one line in each of the A_i's $1 \le i \le k+2$ containing (x,y).
$\Rightarrow R = k+2$.

③ if (x,y) & (z,w) are two distinct point
& if ③a $x = z$ OR $y = w$ then they belong to same row or column. Hence $L_i(x,y) \ne L_i(z,w)$
for any $1 \le i \le k$.
$\Rightarrow$ no line from A_{i+2} can contain both of them. $1 \le i \le k$.

Further $x = z \Rightarrow x^{th}$ row
& $y = w \Rightarrow y^{th}$ col^n contain both of them.
$\Rightarrow \exists!$ line containing both of them.

③b Assume $x \ne z$ & $y \ne w$.



$\Rightarrow$ no line from $\mathcal{A}_1$ & $\mathcal{A}_2$ can contain both.

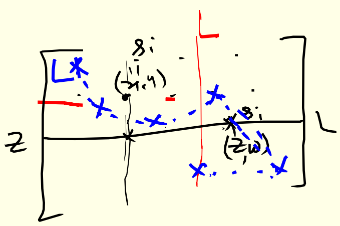
Further if $L_i(x,y) = L_i(z,w)$ for some i, then
$= s$ the line corr. to s in $\mathcal{A}_{i+2}$ contains both of them.

Also since L_i, L_j are orthogonal $\forall j \neq i$
we can not have $L_j(x,y) = L_j(z,w)$ for any $j \neq i$
(otherwise (s,s) occurs twice in $L_i \times L_j$!)

$\Rightarrow$ at most one line contain both of them.

(4) If $(x,y) \notin L$ then to prove that $\exists$ precisely $k+1$ lines thru' (x,y) that intersects with L. Let L be z^{th} row

(4a) If $L \in \mathcal{A}_1$. then $L \neq x^{th}$ row.



(o) Clearly line corr. to y^{th} col^n in $\mathcal{A}_2$ contains (x,y) & intersects L.

(oo) If $L_i(x,y) = s_i$ then $\exists!$ elt. in z^{th} row say (z,w) s.t. $L_i(z,w) = s$.
$\Rightarrow$ the line corr. to symbol s in $\mathcal{A}_{i+2}$ contains (x,y) & (z,w) ($\therefore$ intersects with L)
$\Rightarrow$ precise 1 line from each $\mathcal{A}_{i+2}$ $1 \le i \le k$ contain (x,y) & intersects L. QED.

(ooo) Similar arguments if $L \in \mathcal{A}_2$.

(oooo) $L \in \mathcal{A}_{i+2}$ for some $1 \le i \le k$.
If $L_i(x,y) = s$ then L can not correspond to s but for some $t \neq s$.

x^{th} row & y^{th} col^n of L_i must have symbol t occuring in them. $\Rightarrow$ x^{th} row $\in \mathcal{A}_1$ contains (x,y) & intersects L.
& y^{th} col^n $\in \mathcal{A}_2$ —— " —— .

No two lines in $\mathcal{A}_j$ intersect with each other $\forall 1 \le j \le k+2$
$\Rightarrow$ $\nexists$ any line in $\mathcal{A}_i$ (containing the given L) contains (x,y) & intersects L.

L_i

$L_1, \cdots, L_k$

$\downarrow$

L_j

But $\forall j \neq i$ let $L_j(x,y) = a$ $\quad a \in S'$

Consider the line corresponding to a in $\mathcal{A}_{j+2}$.

Since L_i & L_j are orthogonal to each other, the pair (t, a) must occur in $L_i \times L_j$

$\Rightarrow \exists$ z^{th} row & w^{th} col^n s.t.

$L_i(z,w) = t \Rightarrow (z,w) \in L.$

& $L_j(z,w) = a. \Rightarrow (z,w) \in$ unique line in $\mathcal{A}_j$, that contains (x,y)

$\Rightarrow$ each $\mathcal{A}_{j+2}$ $j \neq i$

contains unique line that passes thru' (x,y) & intersects with $L \in \mathcal{A}_{i+2}$, $L \not\ni (x,y)$.

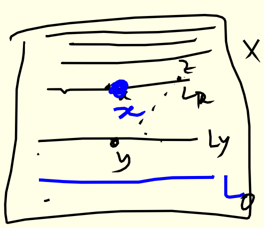
$$\Rightarrow \left(\mathcal{P}, \mathcal{B} = \bigsqcup_{i=1}^{k+2} \mathcal{A}_i\right) \text{ is a } pg(n, k+2, k+1).$$

— x —

II

Assume that $(\mathcal{P}, \mathcal{B})$ is an incidence system that is a $pg(n, k+2, k+1)$ (or a $(n, k+2)$-net)

Given $L \in \mathcal{B}$. & $x \notin L$, $\exists!$ line thru' x not intersecting with L.



$\underline{|X| = n^2}$

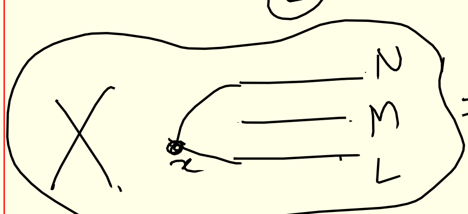
$\Rightarrow$ Given any L_0 $\exists$ $n-1$ mutually parall lines $L_1, \cdots L_{n-1}$ that are $\parallel$ to L_0.

Define $\sim$ on $\mathcal{B}$ by.

$L \sim M$ iff $L \cap M = \emptyset$. or $L = M$.

(1) $\sim$ is reflexive (2) $\sim$ is symmetric

(3) $\sim$ is transitive. $L \sim M$ & $M \sim N$ then $L \sim N$.

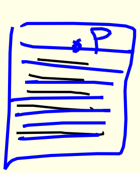


$\Longrightarrow$ Note that L & N are both $\parallel$ to M.

$\therefore$ they can't intersect !!!

$\Rightarrow L \parallel N$.

$\Rightarrow$ $\sim$ is an equivalence relation



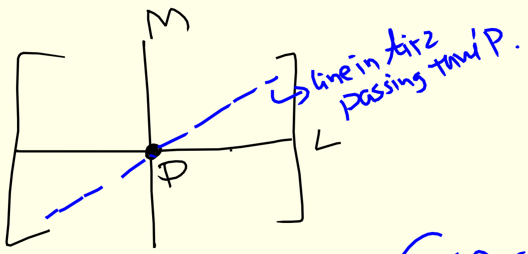
$\Rightarrow$ $\mathcal{B}$ gets partitioned in eq. classes of mutually parallel lines. Each eq. class containing n such lines.

But $|\mathcal{B}| = n(k+2)$. $\Rightarrow \exists$ $k+2$ such classes!

say $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{k+2}$.

Define $L_1, \ldots, L_k$ as follows:

$$L_i : \underset{\substack{\downarrow \\ \text{rows}}}{\mathcal{A}_1} \times \underset{\substack{\downarrow \\ \text{columns}}}{\mathcal{A}_2} \longrightarrow \underset{\substack{\downarrow \\ \text{symbols}}}{\mathcal{A}_{i+2}}.$$



$(L, M) \longmapsto$ the line passing thru $L \cap M$ in the $\mathcal{A}_{i+2}$.

Exercise :- ① Check that each L_i is a Latin square $1 \le i \le k$

② (L_i, L_j) are ortho. $\forall\, i \neq j$.

$\boxed{\text{QED}}$

No lecture on 19^{th} Nov 21.