Lecture 4: Symmetric and Exterior products

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Let V be a vector space of dim n over a field k of char 0.

$$\mathrm{Sym}^a V := T^a V \big/ \langle v_1 \otimes \cdots \otimes v_a - v_{\sigma 1} \otimes \cdots \otimes v_{\sigma a} \mid v_i \in V \ \&\ \sigma \in \underset{\text{symm group}}{S_a} \rangle$$

$$\boxed{\begin{aligned} \mathrm{Sym}^0(V) &= k \\ \mathrm{Sym}^1(V) &= V \end{aligned}}$$

There is a symmetric multilinear map

$$\varphi : \underbrace{V \times V \times \cdots \times V}_{a \text{ copies}} \longrightarrow \mathrm{Sym}^a V$$

$$(v_1, \ldots, v_a) \longmapsto v_1 \cdot \ldots \cdot v_a := \overline{v_1 \otimes \cdots \otimes v_a}$$

Note $\varphi(v_1, \ldots, v_a) = \varphi(v_{\sigma 1}, \ldots, v_{\sigma a})$ $\forall \sigma \in S_a$. Hence φ is symmetric multilinear map.

(*) $\mathrm{Sym}^a V$ is the universal object for symmetric multi linear maps from $\underbrace{V \times \cdots \times V}_{a \text{ times}}$ to a v.s. i.e.

$\theta : \underbrace{V \times V \times \cdots \times V}_{a \text{ times}} \longrightarrow \underset{\text{v.s.}}{U}$ be a sym multilinear map then θ uniquely factors through φ i.e.

$$\begin{tikzcd} V \times \cdots \times V \arrow[r, "\theta"] \arrow[d, "\varphi"'] \arrow[r, phantom, shift right=3, "\#"] & U \\ \mathrm{Sym}^a V \arrow[ur, dashed, "\exists! \tilde{\theta}"'] & \end{tikzcd} \qquad \theta = \tilde{\theta} \circ \varphi$$

$\rightsquigarrow$ $\tilde{\theta}_1 : T^a V \longrightarrow U$

$v_1 \otimes \cdots \otimes v_a \longmapsto \theta(v_1, \ldots, v_a)$

Now use 1st isom.

$$\tilde{\theta}(v_1 \cdot \ldots \cdot v_a) = \theta(v_1, \ldots, v_a)$$

(*) Let $\{e_1, \ldots, e_n\}$ be a basis of V then for $\mathrm{Sym}^a V$

$\{(e_{i_1} \cdot e_{i_2} \cdot \ldots \cdot e_{i_a}) \mid 1 \le i_1 \le i_2 \le \cdots \le i_a \le n\}$ is a basis of $\mathrm{Sym}^a V$ } HW

$$\dim(\mathrm{Sym}^a(V)) = \binom{n+a-1}{a} = \binom{n+a-1}{n-1}$$

⊛ Let $V \& W$ be v.s. then

$$\mathrm{Sym}^a(V \oplus W) \cong \bigoplus_{i=0}^{a} \mathrm{Sym}^i(V) \otimes \mathrm{Sym}^{a-i}(W) = (\mathrm{Sym}^a W) \oplus (V \otimes \mathrm{Sym}^{a-1} W) \oplus (\mathrm{Sym}^2 V \otimes \mathrm{Sym}^{a-2} W) \cdots$$

$$\mathrm{Sym}^i(V) \otimes \mathrm{Sym}^{a-i}(W) \xrightarrow{\theta_i} \mathrm{Sym}^a(V \oplus W)$$

θ_i corresponds to the bilin map $\mathrm{Sym}^i V \times \mathrm{Sym}^{a-i}(W) \to \mathrm{Sym}^a(V \oplus W)$

$$v_1 \cdots v_i \otimes w_1 \cdots w_{a-i} \longmapsto v_1 \cdots v_i \cdot w_1 \cdots w_{a-i}$$

$$\theta := \bigoplus_{i=0}^{a} \theta_i : \bigoplus_{i=0}^{a} \mathrm{Sym}^i(V) \otimes \mathrm{Sym}^{a-i}(W) \longrightarrow \mathrm{Sym}^a(V \oplus W)$$

Let $\{v_1, \dots, v_n\}$ be a basis of V & $\{w_1, \dots, w_m\}$ a basis of W then

$\alpha \in \mathrm{Sym}^a(V \oplus W)$

$$\alpha = \sum_{\text{finite}} a_{\underline{j}} \overline{v_{j_1} \otimes \cdots \otimes v_{j_i} \otimes w_{j_{i+1}} \otimes \cdots \otimes w_{j_a}} \qquad \begin{matrix} 1 \le j_1 \le j_2 \le \cdots \le j_i \le n \\ 1 \le j_{i+1} \le \cdots \le j_a \le m \end{matrix}$$

$a_{\underline{j}} \in k$

$$\Psi(\alpha) = \sum a_{\underline{j}} \big(\overline{v_{j_1} \otimes \cdots \otimes v_{j_i}} \otimes \overline{w_{j_{i+1}} \otimes \cdots \otimes w_{j_a}} \big)$$

check Ψ is k-linear

$$\mathrm{Sym}^i(V) \otimes \mathrm{Sym}^{a-i}(W)$$

check $\theta \& \Psi$ are inverses.

$$\begin{tikzcd} \times_{i=1}^{a} (V \oplus W) \arrow[r] \arrow[dr] & \mathrm{Sym}^a(V \oplus W) \\ & \bigoplus_{i=0}^{a} \mathrm{Sym}^i(V) \otimes \mathrm{Sym}^{a-i}(W) \end{tikzcd}$$

$$\mathrm{Sym}^a V \times \mathrm{Sym}^b V \longrightarrow \mathrm{Sym}^{a+b}(V)$$

$$\big((v_1 \cdots v_a), (v_1' \cdots v_b') \big) \longmapsto v_1 \cdots v_a \cdot v_1' \cdots v_b'$$

$\mathrm{Sym}^{\bullet} V$ is the symmetric algebra with product defined above.

Let V be as above.

A multilinear map

$$\phi : \underbrace{V \times \cdots \times V}_{a \text{ copies}} \longrightarrow U$$

is called <u>antisymmetric</u> if

$$\phi(v_1, \dots, v_a) = \mathrm{sgn}(\sigma)\phi(v_{\sigma 1}, \dots, v_{\sigma a}) \quad \forall \sigma \in \underset{\text{sym group}}{S_a}$$

The subsp gen by

$$H = \left\langle v_1 \otimes \cdots \otimes v_a - \mathrm{sgn}(\sigma)\, v_{\sigma 1} \otimes \cdots \otimes v_{\sigma a} \;\middle|\; \begin{matrix} \sigma \in S_a \\ v_i \in V \end{matrix} \right\rangle$$

$$H' = \left\langle v_1 \otimes \cdots \otimes v_a \;\middle|\; v_i = v_j \text{ for some } 1 \le i \ne j \le a \right\rangle$$

$$T^a V / H = T^a V / H'$$

Check:

$$H' / H \cap H' = 0 \quad \& \quad H / H \cap H' = 0$$

$$\Lambda^a V = T^a V / H' = T^a V / H$$

$$\boxed{\begin{aligned} &(v+w) \otimes (v+w) = v \otimes w + w \otimes v + v \otimes v + w \otimes w \\ &v \otimes v + w \otimes w - (v+w) \otimes (v+w) = v \otimes w + w \otimes v \end{aligned}}$$