Combinatorics

10 DEC 21

Lecture 19 — Desargues Theorem

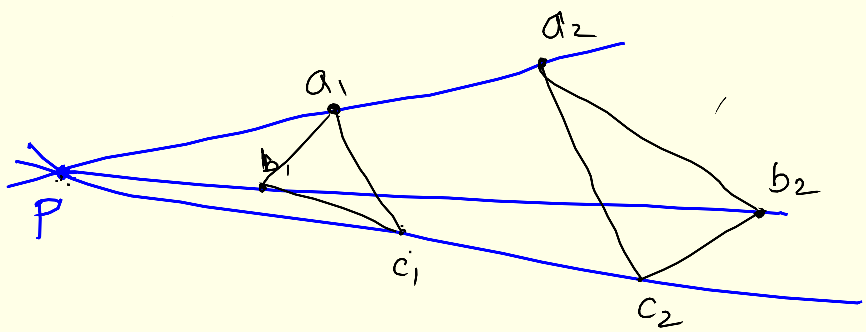
Thm: Subgeometry induced on a flat of a proj. geometry is also projective.

Remark: Any line in a projective geometry must contain at least three points! (otherwise it will be a union of two 1-flats)

Desargues theorem (1648) (Girard Desargues French math. 1593-1662)

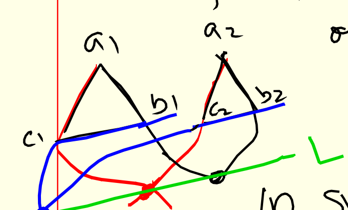
About perspectivity in Geometry.

Def^n: ① Two triangles $\{a_1, b_1, c_1\}$ & $\{a_2, b_2, c_2\}$ are said to be "perspective from a point P" if the lines $\{a_1, a_2\}$, $\{b_1, b_2\}$ & $\{c_1, c_2\}$ all pass thru' P (in a geometry $(X, \mathcal{F})$)



P is called the point of perspectivity.

② $\{a_1, b_1, c_1\}$ & $\{a_2, b_2, c_2\}$ are said to be "perspective from a line" if $\exists$ a line L containing the three points of intersections $\overline{\{a_1, b_1\}} \cap \overline{\{a_2, b_2\}}$; $\overline{\{a_1, c_1\}} \cap \overline{\{a_2, c_2\}}$ & $\overline{\{b_1, c_1\}} \cap \overline{\{b_2, c_2\}}$.

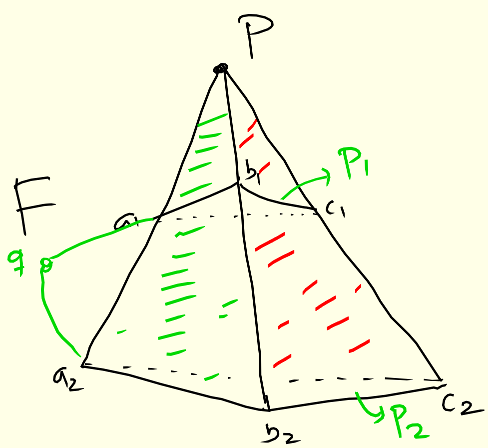


In such a case L is called "the line or the axis of perspectivity"

Theorem In a projective geometry of $\underline{\text{rank} \geq 4}$, if two triangles are perspective from a point, then are perspective from a line.

pf. Assume that P is the point of perspectivity for triangles T_1 & T_2.

$\boxed{\text{case 1}}$ The planes $P_1 = \overline{\{a_1, b_1, c_1\}}$; $P_2 = \overline{\{a_2, b_2, c_2\}}$ are distinct.



Let $F = \overline{\{P, a_1, b_1, c_1\}}$, then both P_1 & P_2 are contained in F.

F has rk 4 & $(X, \mathcal{F})$ is modular.

$$\underbrace{rk P_1 + rk P_2}_{= 6} = \underbrace{rk(P_1 \vee P_2)}_{= 4 = rk F} + rk(P_1 \cap P_2)$$

$$6 = 4 + x$$

$\Rightarrow$ $P_1 \cap P_2$ is a line say L.

Claim L is the axis of perspectivity for T_1 & T_2.

$\overline{\{a_1, b_1\}}$ & $\overline{\{a_2, b_2\}}$ are both contained in the plane $\{P, a_1, b_1\}$ & hence, due to modularity of X, must intersect say at a point q_1. Clearly $q_1 \in P_1 \cap P_2$

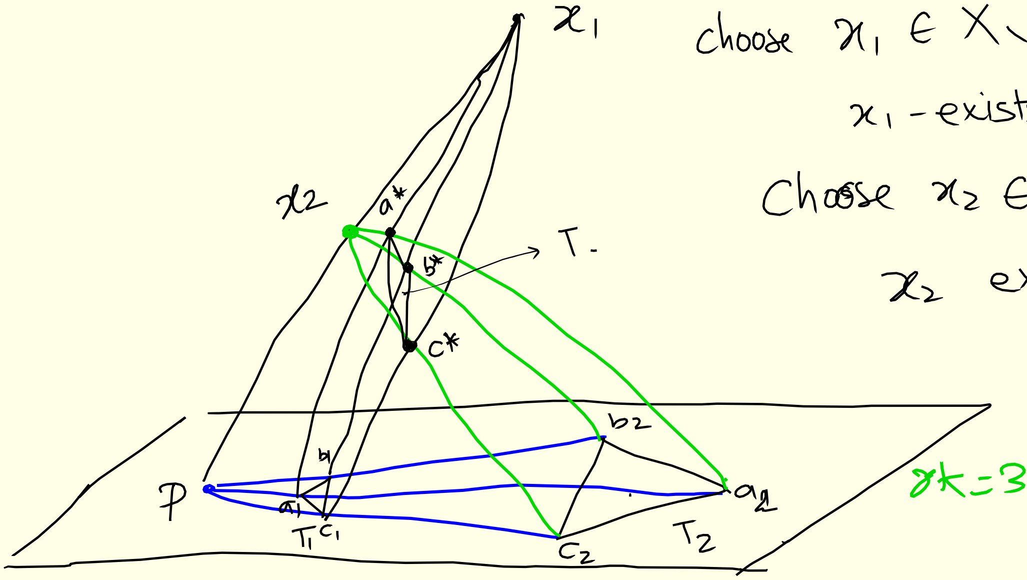
$\Rightarrow q_1 \in L$.

||ly $\overline{\{b_1, c_1\}} \cap \overline{\{b_2, c_2\}} \neq \phi$ & belongs to L

& $\overline{\{a_1, c_1\}} \cap \overline{\{a_2, c_2\}} \neq \phi$ & belongs to L.

$\therefore$ L is the axis of perspectivity.

Case 2 $\exists$ a plane containing both triangles



choose $x_1 \in X \setminus \overline{\{a_1,b_1,c_1\}}$

x_1 - exists as $rk\, X \geq 4$.

Choose $x_2 \in \overline{\{x_1,P\}} - \{x_1,P\}$.

x_2 exists as any line in a projective geom. contains at least three points.

$x_2 \notin \overline{\{a_1,b_1,c_1\}}$

because in that case, $\overline{\{x_2,P\}}$ will lie in $\overline{\{a_1,b_1,c_1\}} \Rightarrow$ $x_1 \in \overline{\{a_1,b_1,c_1\}}$ contra.

The lines $\overline{\{x_1,a_1\}}$ & $\overline{\{x_2,a_2\}}$ are contained in the plane $\{x_1,a_1,P\}$

let $a^* = \overline{\{x_1,a_1\}} \cap \overline{\{x_2,a_2\}}$. in $\overline{\{x_1,P,a_1\}}$

lly let $b^* = \overline{\{x_1,b_1\}} \cap \overline{\{x_2,b_2\}}$ in $\overline{\{x_1,P,b_1\}}$

$c^* = \overline{\{x_1,c_1\}} \cap \overline{\{x_2,c_2\}}$ in $\overline{\{x_1,P,c_1\}}$

Clearly the triangles $\{a^*,b^*,c^*\}$ & $\{a_1,b_1,c_1\}$ have x_1 as the point of persp.

& $\{a^*,b^*,c^*\}$ & $\{a_2,b_2,c_2\}$ has x_2 as pt. of persp.

Further the plane formed by $\{a^*,b^*,c^*\}$ is diff. from $\overline{\{a_1,b_1,c_1\}} \underset{\{a_2,b_2,c_2\}}{\shortparallel}$.

$\therefore$ by $\boxed{\text{Case 1}}$ $\exists$ an axis of perspectivity $L = \{a^*,b^*,c^*\} \cap \overline{\{a_1,b_1,c_1\}}$

for both T^* & T_1 & T^* & T_2.

(Exercise) $\therefore$ it is the axis ~~of a~~ perspectivity for T_1 & T_2.

QED.

End of part 1

Def^n:- ($\exists$ rank 3 projective geometries where the statement of the above thm is not true.)

$\hookrightarrow$ Any proj. geometry for which the statement of the above thm. is true is called a "desarguesian geometry".

Thm: (Desargues ~1648)

Let $\mathbb{F}$ be a field. Then two triangles that are perspective from a point in $\mathbb{P}^n(\mathbb{F})$ has axis of perspectivity.

Exercise - $\mathbb{P}^n(\mathbb{F})$ is a projective geometry $\forall n$.

pf. If $n \geq 3$ then previous theorem (along with exercise) proves the theorem. $\therefore$ $n=2$ is the case to be proved.

pts are 1-diml subspace of $\mathbb{F}^3$

& lines are - i.e. all 1-diml subspaces in a fixed 2-diml subspace

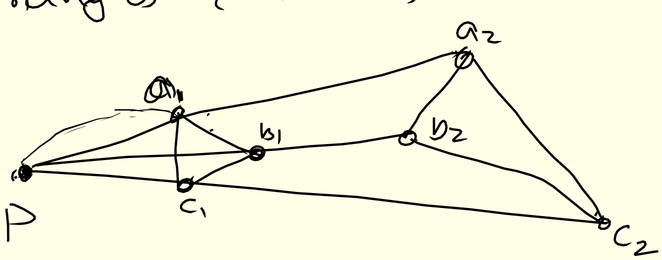
indexed 2diml subspaces

"coordinate system" on $\mathbb{P}^2$ let $x=(x_1, x_2, x_3) \in \mathbb{F}^3 \setminus \{0\}$. then the line $\lambda \cdot x \mid \lambda \in \mathbb{F}$ is denoted by $[x_1 : x_2 : x_3]$. "homogeneous coordinates"

($x \sim y$ iff $\exists \lambda \in \mathbb{F}^*$ s.t. $x = \lambda y$.)

proof Let P be the point of perspectivity of two triangles $\{a_1, b_1, c_1\}$ & $\{a_2, b_2, c_2\}$.

(A)



Three of $\{P, a_1, b_1, c_1\}$ are collinear. say $\{P, a_1, b_1\}$ are collinear.

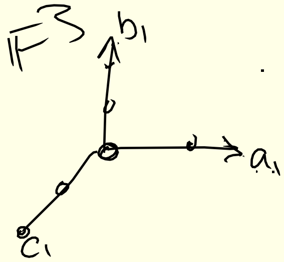
then the points a_2, b_2 belong to $\overline{\{a_1,b_1\}}$.

$$\Rightarrow \overline{\{a_1,b_1\}} \cap \overline{\{a_2,b_2\}} = \overline{\{a_1,b_1\}} = \overline{\{a_2,b_2\}}$$

Let $q_1 = \overline{\{a_1,c_1\}} \cap \overline{\{a_2,c_2\}}$ & $q_2 = \overline{\{b_1,c_1\}} \cap \overline{\{b_2,c_2\}}$ will define a line $\overline{\{q_1,q_2\}}$ which intersects with $\overline{\{a_1,b_1\}} = \overline{\{a_2,b_2\}}$ (because of modularity) say in q_3. $\therefore \overline{\{q_1,q_2\}}$ is the axis of perspectivity.

(B) Therefore we assume that no three of $\{P, a_1, b_1, c_1\}$ & no three of $\{P, a_2, b_2, c_2\}$ are collinear.

Choose a basis of $\mathbb{F}^3$ so that
$$a_1 = [1:0:0]$$
$$b_1 = [0:1:0]$$
$$c_1 = [0:0:1]$$



Since the point of perspectivity does not lie in span of $\overline{\{a_1,b_1\}}$, $\overline{\{a_1,c_1\}}$, $\overline{\{b_1,c_1\}}$
$\Rightarrow$ all three coord of P are nonzero.
Let $P = [\alpha, \beta, \gamma)$. Change the basis of of $\mathbb{F}^3$ so that we have
$$a_1 = [1:0:0]$$
$$b_1 = [0:1:0]$$
$$c_1 = [0:0:1]$$
$$\& \ P = [1:1:1]$$

Note that $\alpha . a_1 + \beta P = [\alpha+\beta : \beta : \beta] \simeq [\underset{\frac{\gamma+\beta}{\beta}}{\gamma} : 1 : 1]$

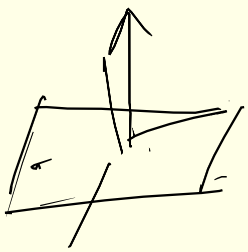
$$\therefore \left.\begin{aligned} a_2 &= [\alpha:1:1] \text{ for some } \alpha \\ b_2 &= [1:\beta:1] \text{ for some } \beta \\ c_2 &= [1:1:\gamma] \text{ for some } \gamma \end{aligned}\right\} \in \mathbb{F}$$

The line joining (a_1, b_1) is of the type

$$\alpha(1,0,0) + \beta(0,1,0) = \{[x:y:0] \mid x, y \in \mathbb{F} \text{ not both zero}\}$$

Line joining a_2, b_2 is of the type

$$\{[x:y:z] \mid \underline{(1-\beta)x + (1-\alpha)y + (\alpha\beta-1)z} = 0\}$$



Denote this line by $\langle 1-\beta, 1-\alpha, \alpha\beta-1\rangle$.

(i.e. the orth. comp of $(1-\beta, 1-\alpha, \alpha\beta-1)$ is the line)

lly the line given by b_2, c_2 is $\langle 1-\beta\gamma, \gamma-1, \beta-1\rangle$.

& line given by a_2, c_2 is $\langle 1-\gamma, \alpha\gamma-1, 1-\alpha\rangle$.

Lines joining (a_1, b_1), (a_1, c_1) & (b_1, c_1) are far easier to determine!!

$= \langle c_1\rangle$, $\langle b_1\rangle$, $\langle a_1\rangle$

The intersection of $(a_1,b_1), (a_2,b_2)$; $(a_1,c_1), (a_2,c_2)$; & $(b_1,c_1), (b_2,c_2)$ are respectively, $[1-\alpha : \beta-1 : 0]$; $[\alpha-1 : 0 : 1-\gamma]$; $[0 : 1-\beta : \gamma-1]$.

$= P_1$, P_2, P_3

Clearly $\underline{\underline{P_1 + P_2 + P_3 = 0}}$. $\therefore$ they are collinear!!

QED !!