Exercises

1. A set Ω is *pathwise connected* if any two points in Ω can be joined by a piecewise smooth curve entirely contained in Ω. We show that an open set Ω is pathwise connected if and only if Ω is connected.

   (a) Suppose first that Ω is open and pathwise connected, and that it can be written as a disjoint union of non-empty open sets $\Omega = \Omega_1 \cup \Omega_2$. Let $w_1 \in \Omega_1$ and $w_2 \in \Omega_2$ and let $z : [0,1] \to \Omega$ be a curve with $z(0) = w_1$ and $z(1) = w_2$. Let

   $$t^* = \sup_{0 \le t \le 1} \{t : z(s) \in \Omega_1 \quad \text{for all } 0 \le s \le t\}.$$

   Arrive at a contradiction by considering the point $z(t^*)$.

   (b) Conversely, suppose that Ω is open and connected. Fix a point $w \in \Omega$ and let $\Omega_1 \subset \Omega$ denote the set of all points that can be joined to w by a curve contained in Ω. Also, let $\Omega_2 \subset \Omega$ denote the set of all points that cannot be joined to w by a curve in Ω. Show that both Ω_1 and Ω_2 are open, disjoint and their union is Ω. Conclude that $\Omega = \Omega_1$.

2. Let Ω be an open set in $\mathbb{C}$ and $z \in \Omega$. The connected component C_z of z is the set of all points in w that can be reached from z by a curve entirely contained in Ω.

   (a) Show that C_z is open and connected, and any two connected components are either disjoint or coincide.

   (b) Show that Ω can have only countably many distinct connected components.

   (c) Prove that if Ω is the complement of a compact set, then Ω has only one unbounded component.

3. For a function $f : \mathbb{C} \to \mathbb{C}$ and a curve γ in the complex plane define the integral with respect to $\bar{z}$ as $\int_\gamma f d\bar{z} = \overline{\int_\gamma \bar{f} dz}$. From this the line integral with respect to x and y can be defined as

   $$\int_\gamma f dx = \frac{1}{2}\left(\int_\gamma f dz + \int_\gamma f d\bar{z}\right)$$
   $$\int_\gamma f dy = \frac{1}{2i}\left(\int_\gamma f dz - \int_\gamma f d\bar{z}\right)$$

   Check that for $f = u + iv$

   $$\int_\gamma f dz = \int_\gamma (u dx - v dy) + i \int_\gamma (u dy + v dx). \tag{0.1}$$

   If we instead start by defininng for any $p, q : \mathbb{R}^2 \to \mathbb{R}$ and $\gamma : [a,b] \to \mathbb{R}^2$ the line integral $\int_\gamma p dx + q dy$ by

   $$\int_\gamma p dx + q dy := \int_a^b p\left(x(t), y(t)\right) \cdot x'(t) dt + q\left(x(t), y(t)\right) \cdot y'(t) dt$$

   then show that the right hand side of (0.1) gives $\int_\gamma f dz$.

The integral *with respect to the arc length* is

$$\int_\gamma f ds = \int_\gamma f|dz| := \int_\gamma f(z(t))|z'(t)|dt$$

With $f \equiv 1$ one gets the arc length. In this case $\int_{-\gamma} f|dz| = \int_\gamma f|dz|$ and

$$\left|\int_\gamma f dz\right| \leq \int_\gamma |f| \cdot |dz|.$$

Show the following **Theorem**: If p and q are (possibly complex valued) continuous functions in a region Ω, then for any curve γ in Ω the line integral $\int_\gamma pdx + qdy$ depends only on the endpoints of γ if and only if there exists a function $U(x, y)$ in Ω with the partial derivatives $\partial U/\partial x = p$, $\partial U/\partial y = q$.
Hint: For the only if part fix a point (x_0, y_0) and let $U(x', y') = \int_\gamma pdx + qdy$ for any(?) curve γ which starts at (x_0, y_0) and ends at (x', y').

Thus $\int_\gamma f(z)dz = \int_\gamma f(z)dx + i\int_\gamma f(z)dy$ is dependent only on the endpoints for any γ if there is a function F on Ω such that

$$\frac{\partial F(z)}{\partial x} = f(z), \quad \frac{\partial F(z)}{\partial y} = if(z).$$

Conclude then that $\int_\gamma f dz$ with f continuous, depends only on the endpoints of γ if and only if f is the derivative of a holomorphic function in Ω. (note that we proved only one direction in class)

4. These calculations provide some insight into Cauchy's theorem

   (a) Evaluate $\int_\gamma z^n dz$ for all integers n. Here γ is any circle centered at the origin with positive orientation. What if γ is a circle not containing the origin?

   (b) show that if $|a| < r < |b|$ then

   $$\int_\gamma \frac{dz}{(z-a)(z-b)} = \frac{2\pi i}{a-b},$$

   where γ denotes the circle centered at the origin, of radius r, with the positive orientation.

5. Suppose that f is holomorphic in an open set Ω. Prove that in any one of the following cases:

   (a) $\mathrm{Re}(f)$ is constant;

   (b) $\mathrm{Im}(f)$ is constant;

   (c) $|f|$ is constant;

   one can conclude that f is constant.

6. Suppose f is continuous in a region Ω. Prove that any two primitives of f (if they exist) differ by a constant.

7. [HW 1, due 5 Oct] Consider a holomorphic function f on a region Ω. Let C be a circle inside Ω whose interior is also contained in Ω. Here is another way to show that $\int_C f(z)dz = 0$.

(a) Consider any regular polygon P_n of n sides inscribed inside the circle. Argue that $\int_{P_n} f(z)dz = 0$.

(b) Show that $\lim_{n\to\infty} \int_{P_n} f(z)dz = \int_C f(z)dz$.

8. The next few exercises show how complex integration can help us compute complicated real integrals.

   (a) [HW 2, due Oct 11] Prove

$$\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{\sqrt{2\pi}}{4}.$$

   The integral $\int_0^\infty$ is interpreted as $\lim_{R\to\infty} \int_0^R$.
   HINT: Integrate e^{-z^2} from 0 to R, then along the circular arc from R to $Re^{i\pi/4}$ and then along the straight line from $Re^{i\pi/4}$ to 0.

   (b) Show $\int_0^\infty \frac{\sin x}{x} = \frac{\pi}{2}$.
   HINT: The integral equals $\frac{1}{2i}\int_{-\infty}^\infty \frac{e^{ix}-1}{x}dx$. Use the indented semicircle.

   (c) [HW 2, due Oct 11] Evaluate the integrals

$$\int_0^\infty e^{-ax}\cos(bx)dx \quad \text{and} \quad \int_0^\infty e^{-ax}\sin(bx)dx, \quad a > 0$$

   by integrating e^{-Ax}, $A = \sqrt{a^2+b^2}$, over an appropriate sector with angle ω, with $\cos\omega = a/A$.

   (d) Prove that for all $\xi \in \mathbb{C}$ we have $e^{-\pi\xi^2} = \int_{-\infty}^\infty e^{-\pi x^2} e^{2\pi i x\xi}$.

9. Suppose f is continuously complex differentiable on Ω, and $T \subset \Omega$ is a triangle whose interior is also contained in Ω. Apply Green's theorem to show that $\int_T f(z)dz = 0$. this provides a proof of Goursat's theorem under the additional assumption that f' is continuous.

10. Show that every non-constant polynomial $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$ with complex coefficients has a root in $\mathbb{C}$. From this conclude that $P(z)$ has n roots $w_1, w_2, \cdots, w_n$ and $P(z) = a_n(z-w_1)(z-w_2)\cdots(z-w_n)$.

    HINT: Suppose not. Then note that $P(z)^{-1}$ is entire.

11. HW 3 (Due Monday 25 October) Let f be a holomorphic function on the disc D_{R_0} centered at the origin and of radius R_0.

    (a) Prove that whenever $0 < R < R_0$ and $|z| < R$, then

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\phi}) \cdot Re\left(\frac{Re^{i\phi}+z}{Re^{i\phi}-z}\right) d\phi.$$

    HINT: Note that if $w = R^2/\bar{z}$, then the integral of $f(\zeta)/(\zeta - w)$ around the circle of radius R centered at the origin is 0. Use this, together with the Cauchy integral formula.

    (b) Show that

$$Re\left(\frac{Re^{i\gamma}+r}{Re^{i\gamma}-r}\right) = \frac{R^2-r^2}{R^2-2Rr\cos\gamma+r^2}.$$

12. HW 3 (Due Monday 25 October) Say that a twice continuosly differentiable real-valued function is harmonic if $\Delta u(x,y) = 0$ where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

    (a) If f is holomorphic in an open set Ω, then show that the real and imaginary parts of f are harmonic.

    (b) Let u be a real-valued function defined on the unit disc $\mathbb{D}$. Suppose that u is twice continuously differentiable and harmonic.

        i. Prove that there exists a holomorphic function f on $\mathbb{D}$ such that $Re(f) = u$. Also show that the imaginary part of f is uniquely defined up to an additive (real) constant.
        HINT: If there is such an f then $f'(z) = 2\partial u/\partial z$. Therefore, let $g(z) = 2\partial u/\partial z$ and prove that g is holomorphic. Why can one find F with $F' = g$? Prove that $Re(f)$ differs from u by a real constant.

        ii. Deduce from this result, and the above exercise, the Poisson integral representation formula from the Cauchy integral formula: If u is harmonic in $\mathbb{D}$ and continuous on its closure, then if $z = re^{i\theta}$ one has

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - \phi)u(\phi)d\phi$$

        where $P_r(\gamma)$ is the Poisson kernel for the unit disc given by

$$P_r(\gamma) = \frac{1 - r^2}{1 - 2r\cos\gamma + r^2}.$$

13. Suppose f is an analytic function defined everywhere in $\mathbb{C}$ and such that for each $z_0 \in \mathbb{C}$ at least one coefficient in the expansion

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

    is equal to 0. Prove that f is a polynomial.
    HINT: Use the fact that $c_n n! = f^{(n)}(z_0)$ and use a countability argument.

14. Let Ω be a bounded open subset of $\mathbb{C}$, and $\phi : \Omega \to \Omega$ a holomorphic function. Prove that if there exists a point $z_0 \in \Omega$ such that

$$\phi(z_0) = z_0 \quad \text{and} \quad \phi'(z_0) = 1$$

    then ϕ is linear.
    HINT: Why can one assume that $z_0 = 0$? Write $\phi(z) = z + a_n z^n + O(z^{n+1})$ near 0, and prove that if $\phi_k = \phi \circ \cdots \circ \phi$ (k times) then $\phi_k(z) = z + k a_n z^n + O(z^{n+1})$. Apply the Cauchy inequailties and let $k \to \infty$ to conclude the proof.

15. [HW 4, Due Wednesday 3 November] This exercise shows that one cannot always extend a holomorphic function from a smaller set to a larger one (see the seciton on Schwarz reflection principle in Stein-Shakarchi for some cases in which one can extend). The following definition is needed. Let f be a function defined in the unit disc $\mathbb{D}$, with boundary circle C. A point w on C is said to be *regular* for f if there is an open neighbourhood U of w and an analytic function g on U, so that $f = g$ on $\mathbb{D} \cap U$. A function f defined on $\mathbb{D}$ cannot be continued analytically past the unit circle if no point of C is regular for f.

(a) Let

$$f(z) = \sum_{n=0}^{\infty} z^{2^n} \quad \text{for} \quad |z| < 1.$$

Notice that the radius of convergence of the above series is 1. Show that f cannot be continued analytically past the unit disc.
HINT: Suppose $\theta = 2\pi p/2^k$, whre p and k are positive integers. Let $z = re^{i\theta}$; then $|f(re^{i\theta})| \to \infty$ as $r \to 1$.

(b) Fix $0 < \alpha < \infty$. show that the analytic function f defined by

$$f(z) = \sum_{n=0}^{\infty} 2^{-n\alpha} z^{2^n} \quad \text{for} \quad |z| < 1$$

extends continuously ot the unit circle, but cannot be analytically continued past the unit circle.

16. [HW 4, Due Wednesday 3 November] Prove the converse to Runge's theorem: if K is a compact set whose compleemnt is not connected, then there exists a function f holomorphic in a neighborhood of K which cannot be appromixated uniformly by polynomials on K.
HINT: Pick a point z_0 in a bounded component of K^c, and let $f(z) = 1/(z - z_0)$. If f can be approximated uniformly by polynomials on K, show that there exists a polynomial p such that $|(z - z_0)p(z) - 1| < 1$. Use the maximum modulus principle (see below) to show that this inequality continues to hold for all z in the component of K^c that contains z_0. *The maximum modulus principle (which we learn later) states that if h is a non-constant holomorphic function in a region Ω, then $|h|$ cannot attain a maximum in Ω.*

17. Using Euler's formula

$$\sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i}$$

show that the complex zeroes of $\sin \pi z$ are exactly at the integers, and that they are each of order 1.

Calculate the residue of $\frac{1}{\sin \pi z}$ at $z = n \in \mathbf{Z}$.

18. [HW 5, Due Monday 15 November] Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^4}.$$

19. [HW 5, Due Monday 15 November] Show that

$$\int_{-\infty}^{\infty} \frac{\cos x \, dx}{x^2 + a^2} = \frac{\pi e^{-a}}{a}, \qquad \text{for all } a > 0.$$

20. [HW 5, Due Monday 15 November] Prove that

$$\int_0^{2\pi} \frac{d\theta}{(a + \cos\theta)^2} = \frac{2\pi a}{(a^2 - 1)^{3/2}}, \qquad \text{whenever } a > 1.$$