Lecture 19: Examples

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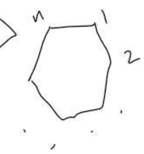
$$D_n = \langle r, s \mid r^n = 1,\ s^2 = 1,\ rs = sr^{-1} \rangle$$ 

$$= \{ r^i, sr^i \mid 1 \leq i \leq n \} \leq S_n$$

1-dim'l repr $\leftrightarrow$ group homo $D_n \longrightarrow \mathbb{C}^*$

$\varphi : D_n \longrightarrow \mathbb{C}^*$ grp homo $\Rightarrow \varphi(r^2) = 1$

$\& \ \varphi(s^2) = 1$

n even

$\varphi(r) = 1$ or -1 & $\varphi(s) = 1$ or -1

So there 4 1-dim repr of D_n

$\psi_0(r) = 1$
$\psi_0(s) = 1$
$\psi_1(r) = -1$
$\psi_1(s) = 1$
$\psi_2(r) = 1$
$\psi_2(s) = -1$
$\psi_3(r) = \psi_3(s) =$

n odd then

$\varphi(r) = 1$ & $\varphi(s) = 1$ or -1

$\Rightarrow$ 2 1-dim'l repr of D_n.

ψ_0 & ψ_2

2-dim'l irr repr of D_n

Even case: $4 + a4 = 2n$ where $a = \#$ 2-dim'l irr repr

$$1 + a = \frac{n}{2}$$

$$a = \frac{n}{2} - 1$$

Odd case: $2 + 4a = 2n$

$$\Rightarrow a = \frac{n-1}{2}$$

2-dim'l repr $\leftrightarrow \varphi : D_n \longrightarrow GL_2(\mathbb{C})$

2-dim'l repr of D_n
$$\begin{cases} r \longmapsto \text{rotation by } k\left(\frac{2\pi i}{n}\right) & k \in \mathbb{Z} \\ s \longmapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{cases}$$

$$\begin{bmatrix} (e^{2\pi i/n})^k & 0 \\ 0 & (e^{-2\pi i/n})^k \end{bmatrix}$$

$0 \le k \le n$

To verify $\varphi_k : r \mapsto R_k = \begin{bmatrix} e^{2\pi i k/n} & 0 \\ 0 & e^{-2\pi i k/n} \end{bmatrix}$ extends to a group homo $D_n \longrightarrow GL_2(\mathbb{C})$

$s \mapsto \underset{S}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$

<u>Note</u>: $R_k^n = I$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^2 = 1$, so need to verify $R_k S = S R_k^{-1}$ $\quad LHS = \begin{bmatrix} 0 & e^{2\pi i k/n} \\ e^{-2\pi i k/n} & 0 \end{bmatrix} = RHS$

$\varphi_k(sr^j) = SR_k^j$ is a group homo.

So φ_k is a 2-dim'l repr of D_n.

$$\chi_{\varphi_k}(r^j) = e^{2\pi i k j/n} + e^{-2\pi i k j/n} = 2\cos\left(\frac{2\pi k j}{n}\right) = 2\cos\left(\frac{2\pi(n-k)j}{n}\right)$$

$$\chi_{\varphi_k}(sr^j) = 0$$

$$\chi_{\varphi_{n-k}} = \chi_{\varphi_k} \Rightarrow \varphi_k = \varphi_{n-k}$$

$\varphi_k \quad 0 \le k \le n/2$ are distinct 2-dim'l reps.

φ_0 & $\varphi_{n/2}$ are reducible $\quad \chi_{\varphi_{n/2}}(r^j) = 2\cos(\pi j) = 2(-1)^j = \psi_1(r^j) + \psi_3(r^j)$

$\chi_{\varphi_{n/2}} = \psi_1 + \psi_3 = \chi_{\psi_1} + \chi_{\psi_3}$

III'ly $\chi_{\varphi_0} = \chi_{\psi_0} + \chi_{\psi_2}$

$\varphi_k \quad 1 \le k < n/2$ are irred (No line is invariant under R_k & S)

Hence we get $\frac{n}{2} - 1$ irred 2-dim'l reps of D_n.

III'ly in the odd:

$\varphi_0 = \varphi_n$ are reducible

$\varphi_0 = \psi_0 + \psi_2$

So there are $\underset{\varphi_{n-1}}{\varphi_1}, \underset{\varphi_{n-2}}{\varphi_2}, \dots, \underset{\varphi_{n+1/2}}{\varphi_{\frac{n-1}{2}}}$ are distinct, irred 2-dim'l rep of D_n

Irreducible representations of A_4.

$$A_4 = \Big\{ \overbrace{e}^{C_0},\ (1\,2)(3\,4),\ (2\,3)(1\,4),\ (1\,3)(2\,4), \Big\}$$

3-cycles ← 8 of them

$C_1 = \{(1\ 2\ 3), \cdots \}$

$C_2 = \{(3\,2\,1), \cdots \}$

So A_4 has 4 irred repr

$H = \{e, (1\,2)(3\,4), (2\,3)(1\,4), (1\,3)(2\,4)\}$ is a normal subgroup of A_4.

& $A_4/H \cong \mathbb{Z}/3\mathbb{Z}$

Any repr of $\mathbb{Z}/3\mathbb{Z}$ is also a repr of A_4 ($\because$ $A_4 \xrightarrow{\theta} \mathbb{Z}/3\mathbb{Z}$ is homo.)

$(1\,2\,3) \longmapsto \bar{1}$

⊛ Also distinct irred reps of $\mathbb{Z}/3\mathbb{Z}$ are distinct irred repr of A_4. (HW) ($\because$ $A_4 \to \mathbb{Z}/3\mathbb{Z}$ is surjective)

Since $\mathbb{Z}/3\mathbb{Z}$ has 3 1-dim'l irred repr.

characters of $\mathbb{Z}/3\mathbb{Z}$ $\begin{cases} \chi_1(\bar{1}) = 1 \\ \chi_2(\bar{1}) = \omega \\ \chi_3(\bar{1}) = \omega^2 \end{cases}$ where $\omega = e^{2\pi i/3}$

Let $\theta_i = \chi_i \circ \Theta$ are repre of A_4

then θ_i are 1-dim'l repr of A_4

χ_{θ_1} is the trivial.

$\chi_{\theta_2}(e) = 1 \quad \chi_{\theta_2}(C_0) = 1$

	e	C_0	C_1	C_2
χ_{θ_1}	1	1	1	1
χ_{θ_2}	1	1	ω	ω^2
χ_{θ_3}	1	1	ω^2	ω
χ_4	3	-1	0	0

$\sum n_i \chi_i(g) = 0$

So the 3rd irred repr is 3-dim'l & its character is χ_4.

$A_4 \subseteq S_4$ acts on $\mathbb{C}^4$ via

$$\sigma \cdot (z_1, z_2, z_3, z_4) = (z_{\sigma 1}, z_{\sigma 2}, z_{\sigma 3}, z_{\sigma 4})$$

it has 1-dim'l trivial repr.

$$V = \{(z, z, z, z) \mid z \in \mathbb{C}\} \subseteq \mathbb{C}^4$$

$$\chi_{\mathbb{C}^4}(e) = 4 \quad \Big| \quad \chi_{\mathbb{C}^4}(C_1) = 1$$

$$\chi_{\mathbb{C}^4}(C_0) = 0 \quad \Big| \quad \chi_{\mathbb{C}^4}(C_2) = 1$$

$$\mathbb{C}^4 = V \oplus W$$

$\leftarrow$ std repr

$\nwarrow$ some repr of A_4

$$\chi_W = \chi_{\mathbb{C}^4} - \chi_V$$

$$\chi_W(e) = 3, \ \chi_W(C_0) = -1$$

$$\chi_W(C_1) = 0 = \chi_W(C_2) = 0$$

So $\chi_W = \chi_4$