Lecture 14: More on characters and decomposition of representation.

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$$\boxed{k[G] = n_1 V_1 + \dots + n_m V_m}$$

where $n_i = \dim(V_i)$.
& V_i's are irred.

Cor: 1) $\chi_{k[G]} = n_1 \chi_{V_1} + \dots + n_m \chi_{V_m}$

where V_i's are all the irr repr of G of degree n_i.

2) $|G| = n_1^2 + \dots + n_m^2$

3) $n_1 \chi_{V_1}(g) + \dots + n_m \chi_{V_m}(g) = 0$
$\forall\, g \neq e$ in G.

Thm: Let H be the space of class functions on a group G. Then the characters of irred repr of G form an orthonormal basis of H w.r.t $\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \varphi(g) \psi(g^{-1})$.

Cor: Let G be finite group. The number of irred G-repr is same as the number of conjugacy classes in G.

Pf: Note that H, the space of class function of G is a vector space of dimension same as number of conjugacy classes. By previous thm the set of irr characters is a basis of H. Hence the result. ($\because$ G-repr are in bijection of characters)

Cor: G is abelian iff every irred repr of G is one dimensional.

Pf: G abelian $\iff$ # conj classes in G = $|G|$
$\iff$ There are $|G|$ irred G-repr.
$\iff$ All irred repr are 1-dim'l
$\left(\because \sum_{i=1}^{|G|} n_i^2 = |G|\right)$
$\Rightarrow n_i = 1$

Prop: For $g \in G$ let $c(g) = \# \{ h^{-1} g h \mid h \in G \}$. Let $\chi_1, \dots, \chi_m$ be all the irred characters of G. Then

(1) $$\sum_{i=1}^{m} |\chi_i(g)|^2 = \frac{|G|}{c(g)}$$

(2) For g' not a conjugate of g
$$\sum_{i=1}^{m} \overline{\chi_i(g)}\, \chi_i(g') = 0$$

Pf: Let f be ~~the~~ the class function which is 1 on $\{ h^{-1} g h \mid h \in G \}$ & 0 otherwise.

By thm
$$f = \lambda_1 \chi_1 + \dots + \lambda_m \chi_m \quad \text{for some } \lambda_i \in \mathbb{C}$$

$$\overline{\lambda_j} = (\chi_j \| f) = \frac{1}{|G|} \sum_{g' \in G} \overline{\chi_j(g')} f(g')$$
$$= \frac{c(g)}{|G|} \chi_j(g)$$

$$\Rightarrow f = \frac{c(g)}{|G|} \sum_{j=1}^{m} \overline{\chi_j(g)}\, \chi_j \quad \dots \text{(*)}$$

$$1 = f(g) = \frac{c(g)}{|G|} \sum_{j=1}^{m} |\chi_j(g)|^2$$

$$\Rightarrow \sum_{j=1}^{m} |\chi_j(g)|^2 = \frac{|G|}{c(g)}$$

g' not a conj of g: plug in (*)

$$0 = f(g') = \sum_{j=1}^{m} \overline{\chi_j(g)}\, \chi_j(g')$$

Examples: 1) $\mu_n = \{\sigma \mid \sigma^n = 1\} \cong \mathbb{Z}/n\mathbb{Z}$

Irred. Characters are same as μ_n-repr.

$$\left.\begin{aligned} \chi_\ell(\sigma) &= e^{2\pi i \ell/n} \\ \chi_\ell(\sigma^j) &= e^{2\pi i \ell j/n} \end{aligned}\right\} \qquad 0 \le \ell \le n-1$$

2) G abelian group. The set of irred. characters is $\mathrm{Hom}(G, \mathbb{C}^*) = \mathrm{Hom}(G, \mathbb{C}^{|z|=1})$

3) $S_3 = \langle (1,2), (1\ 2\ 3) \rangle = \langle \underset{\text{transposition}}{\tau}, \overset{\text{3 cycle}}{\sigma} \rangle$

conjugacy classes $\{e\}, \{\text{trans}\}, \{\text{3-cycles}\}$

$\chi_{\text{trivial}} = \chi_0(g) = 1 \quad \forall g \in S_3$

$1 + a^2 + b^2 = 6 \Rightarrow a = 1, b = 2$

$\chi_{\text{sgn}} = \chi_1(g) = \text{sgn}(g)$ is a character

Irr $\chi_2(e) = 2$, $\chi_2(\text{transp}) = 0$, $\chi_2(\text{3-cycles}) = -1$ $\quad \Big(\chi_0(\tau) + \chi_1(\tau) + 2\chi_2(\tau) = 0 \Rightarrow \chi_2(\tau) = 0$

$$k[S_3] = ke + k\tau + k\sigma + k\sigma^2 + k\sigma\tau + k\sigma^2\tau$$

S_3 also acts on $\mathbb{C}^3$

via $\sigma \in S_3 \quad \sigma(z_1, z_2, z_3) = (z_{\sigma 1}, z_{\sigma 2}, z_{\sigma 3})$

$\triangle \subseteq \mathbb{C}^3$

$\|$

$\ell = \{(z, z, z) \mid z \in \mathbb{C}\}$ is fixed by S_3.

$$\mathbb{C}^3 = \ell \oplus \underbrace{\{(z_1, z_2, z_3) \mid z_1 + z_2 + z_3 = 0\}}_{\text{is irred.}}$$

(*) Let V be a G-repr. Then

$V = r_1 W_1 + r_2 W_2 + \dots + r_m W_m$ where W_i's are irred reps & $r_i \geq 0$.

$V = V_1 \oplus V_2 \oplus \dots \oplus V_m$ where the subspace $V_i \cong W_i^{r_i}$ as reps. is

the image of the projection map $p_i = \frac{n_i}{|G|} \sum_{g \in G} \overline{\chi_i(g)} \rho_V(g)$ where

$n_i = \dim(W_i)$ & $\chi_i = \chi_{W_i}$

Pf: Let $q_{ij} = p_i|_{W_j}$ where $W_j \subseteq V$ isom to the irred rep W_j. Note that p_i is G-equivar. (as $\overline{\chi_i}$ is a class function.)

$w \in W_j$

$$q_{ij}(w) = p_i(w) = \boxed{\frac{n_i}{|G|} \sum_{g \in G} \overline{\chi_i(g)}\, g} \cdot w \in W_j$$

$q_{ij} : W_j \to W_j$ which is G-equiv

Hence by Schur's lemma q_{ij} is a homothety with scalar λ_{ij} s.t.

$$n_j \lambda_{ij} = \mathrm{Tr}(q_{ij}) = \frac{n_i}{|G|} \sum_{g \in G} \overline{\chi_i(g)}\, \mathrm{Tr}(g|_{W_j}) = n_i (\chi_i | \chi_j)$$

$$\Rightarrow \lambda_{ij} = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{o.w.} \end{cases}$$

$$p_i|_{V_i} = id \quad \& \quad p_i|_{V_j} = 0 \quad \text{for } j \neq i$$

Hence p_i is a projection map. $\blacksquare$