SURFACES - II

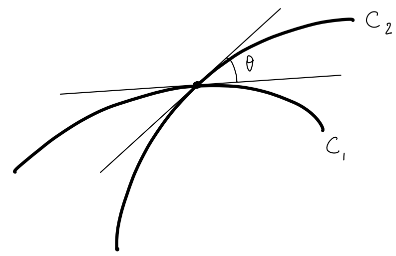
MEASUREMENTS

On a curve, the only thing we measure is length along it. On a surface S, there are more things to measure such as lengths of curves on S, angle between 2 curves on S and area of suitable regions in S. While the first two are easier to define, computing area can give rise to some subtleties.

Let $S \subseteq \mathbb{R}^3$ be a smooth surface.

A regular curve $\gamma(t)$ in S is, by definition, a regular curve $\gamma(t): I \longrightarrow \mathbb{R}^3$ whose trace lies in S. Its length is calculated as before: $\int_a^b \|\dot{\gamma}(t)\| \, dt$ = length of $\gamma([a,b])$ (or of $\gamma((a,b))$). It equals the supremum of the lengths of all the piece-wise linear approximations of $\gamma(t)$ over $[a,b]$.

Let $p \in S \subseteq \mathbb{R}^3$ and suppose there are two regular curves in S through p. We define the <u>angle between them at p</u> to be the angle between the corresponding tangent lines at p. Thus if v, w are nonzero tangent vectors on the corresponding curves, then the angle between the curves is $\theta = \cos^{-1}\left(\frac{v \cdot w}{\|v\| \|w\|}\right)$.

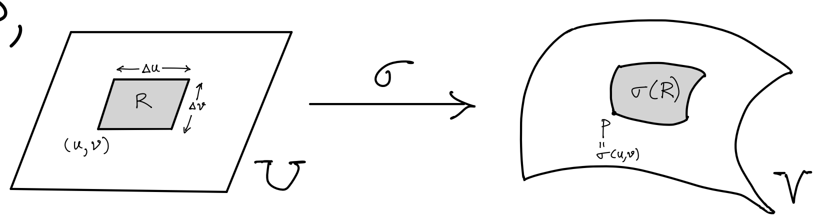


Defining area is a little trickier. One problem is that of determining which regions are suitable for computing area. Even in $\mathbb{R}^2$, closed or open connected sets are complicated in general. For instance, for a bounded open connected set $U \subseteq \mathbb{R}^2$ it may be tempting to set the area as $\iint_U 1 \, dx\,dy$, but, in general, U may have a complicated boundary and in that case the integral itself may be tricky to define. (The integral in the Riemannian sense need not exist.)

Even when we stick to 'reasonable' surfaces a second problem is that defining the area of a surface S as a limit of areas of piece-wise polygonal (planar) approximations of S may not give the expected answer. You may look up the 'Schwarz

Lantern' where even for a finite cylinder, the area of the polygonal approximations may approach infinity!

One remedy to the above situation is to first project the polygonal approximations to a nearby tangent plane and then calculate the area. Let us work this out for a parametrisation, say $\sigma(u,v) : U \to V \subseteq S$, with a rectangle $R \subseteq U$

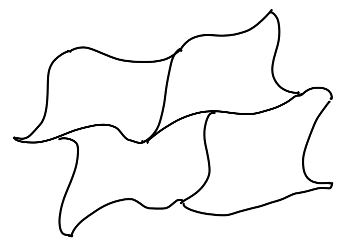


as shown. We see that the area of the corresponding region $\sigma(R)$ is approximately that of the parallelogram (in T_pS) of sides $\sigma_u \Delta u$ and $\sigma_v \Delta v$ approximating $\sigma(R)$, namely

$$\|\sigma_u \Delta u \times \sigma_v \Delta v\| \;=\; \|\sigma_u \times \sigma_v\| \,\Delta u \Delta v.$$

For our purpose, it will be enough to calculate area of 'regions' in S defined as follows. A <u>basic region</u> in S is a compact set $\overline{V'}$ where V' is an open subset of a patch $V \subseteq S$ containing $\overline{V'}$ such that its boundary is a simple closed curve in S which is regular at all but finitely many points on it (e.g. $\sigma(R)$ above).

A <u>region in S</u> is a compact subset which is a finite union of basic regions whose interiors are mutually disjoint.



For a basic region $\overline{V'} \subseteq S$, we calculate its area by using the formula given above: Let $V' = \sigma(U')$, so that $\overline{V'} = \sigma(\overline{U'})$. Then we set $\boxed{\text{area}\left(\overline{V'}\right) := \iint_{\overline{U'}} \|\sigma_u \times \sigma_v\| \, du\, dv.}$ As long as we work with single patches containing $\overline{V'}$, we can see that this expression for the area is independent of the choice of parametrisation (using the change of variables formula from calculus).

For an arbitrary region in S, we define its area to be the sum of the areas of finitely many basic regions in it. It is more technical to prove that this expression for the area is independent of how we decompose the region into basic ones.