INDIAN STATISTICAL INSTITUTE, BANGALORE CENTRE
B.MATH - Third Year, 2021-22
Statistics - III, Semesteral Examination, January 5, 2022
Marks are shown in square brackets. Total Marks: 50
Time: $2\frac{1}{2}$ Hours; submission must be complete by 1 pm
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You may freely consult the lecture notes, but no other books or resources may be consulted. You may use any of the results stated and discussed in the lecture notes, by stating them explicitly. Results from the assignments may not be used without establishing them. Calculators may be used.

1. Let $Z_i, 1 \leq i \leq n$, $n \geq 5$, be independent $N(0, \sigma^2)$ random variables. Define $Y_1 = Z_1$, $Y_2 = Y_1 + Z_2$, $Y_3 = Z_3$, $Y_4 = Y_3 + Z_4$, and $Y_j = Y_{j-1} + Z_j$ for $5 \leq j \leq n$. Let $\mathbf{Y} = (Y_1, Y_2, Y_3, Y_4, \ldots, Y_n)'$.
(a) Find the probability distribution of $\mathbf{Y}$.
(b) Find the partial correlation coefficients $\rho_{12.3}$ and $\rho_{12.34}$ (between elements of $\mathbf{Y}$).
(c) Find the multiple correlation coefficient between Y_1 and (Y_2, Y_4). [2+4+3]

2. Consider the following model:
$y_1 = \alpha + \gamma + \epsilon_1$
$y_2 = \alpha + \delta + \epsilon_2$
$y_3 = \delta - \gamma + \epsilon_3$
$y_4 = \alpha - \gamma + 2\delta + \epsilon_4$
where α, γ and δ are unknown constants, and ϵ_i are uncorrelated random variables having mean 0 and variance σ^2.
(a) Is $\alpha - \gamma$ estimable? Justify. If it is estimable, find its BLUE.
(b) Is $\gamma - \delta$ estimable? Justify. If it is estimable, find its BLUE. [5+5]

3. Consider the model:

$$y_{ij} = \mu + \alpha_i + \epsilon_{ij},$$

$1 \leq i \leq 4$, $j = 1, 2$, where ϵ_{ij} are i.i.d. $N(0, \sigma^2)$ and $\sum_{i=1}^{4} \alpha_i = 0$.
(a) Show that $\alpha_k - \alpha_l$, $1 \leq k < l \leq 4$ are estimable.
(b) Find best linear unbiased estimators of the above mentioned linear contrasts.
(c) Find simultaneous 95% confidence intervals for $\alpha_1 - \alpha_2$ and $\alpha_2 - \alpha_3$. [2+3+4]

4. Consider the model:

$$y_j = \mathbf{x}_j'\beta + \epsilon_j,\ E(\epsilon_j) = 0,\ Var(\epsilon_j) = j\sigma^2,\ j = 1, 2, \cdots, n; \quad \mathbf{x}_j,\ \beta \in R^p.$$

(a) Find a solution $\hat{\beta}$ for β by solving

$$\min_{\beta \in \mathbb{R}^p} \sum_{j=1}^{p} \frac{1}{j} \left(y_j - \mathbf{x}_j'\beta\right)^2.$$

(b) What is the condition on $\mathbf{a} \in R^p$ which makes the linear parametric function $\mathbf{a}'\beta$ estimable under this model?
(c) What is the BLUE of $\mathbf{a}'\beta$ if it is estimable under this model? [4+2+4]

5. Given below are two linear models under consideration:

$$\textbf{Model I:} \qquad y_i = \beta x_i + \epsilon_i, \quad 1 \leq i \leq n,$$

where ϵ_i are uncorrelated errors with mean 0 and common variance σ^2. Additionally, x_i's are not all equal to each other.

$$\textbf{Model II:} \qquad y_i = \alpha + \beta x_i + \epsilon_i, \quad 1 \leq i \leq n,$$

with the same assumptions on ϵ_i and x_i as given above.

Assume that **Model I** is the correct model (from which y_is arise). However, suppose one computes the least squares estimate of β using the incorrect model (i.e., **Model II**).
(a) Compute the mean and variance of this estimate of β under the correct model.
(b) Compare results in (a) above with those of the *best linear unbiased* estimate of β under the correct model. [6+6]