Life (in hrs) data for the battery design experiment:

material	temperature (°F)					
type	15		70		125	
1	130	155	34	40	20	70
	74	180	80	75	82	58
2	150	188	126	122	25	70
	159	126	106	115	58	45
3	138	110	174	120	96	104
	168	160	150	139	82	60

Let y_{ijk} be the observed response when factor A is at the ith level ($i = 1, 2, \ldots, I$) and factor B is at the jth level ($j = 1, 2, \ldots, J$) for the kth replicate ($k = 1, 2, \ldots, K$).

$$y_{ijk} = \mu_{ij} + \epsilon_{ijk}, i = 1, 2, \ldots, I; j = 1, 2, \ldots, J; k = 1, 2, \ldots, K.$$

Now, $\hat{\mu}_{ij} = \bar{y}_{ij.}$ under no constraints, and hence
RSS $= \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} (y_{ijk} - \bar{y}_{ij.})^2$ has $IJ(K-1)$ d.f.
Reparametrization: $\mu_{ij} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij}$, where
$\sum_{i=1}^{I} \alpha_i = 0$, $\sum_{j=1}^{J} \beta_j = 0$, $\sum_{i=1}^{I} (\alpha\beta)_{ij} = 0$ for all j and $\sum_{j=1}^{J} (\alpha\beta)_{ij} = 0$ for all i are the identifiability conditions.

To investigate the existence of interaction, we should test,
$H_{AB} : (\alpha\beta)_{ij} = 0 (i = 1, 2, \ldots, I; j = 1, 2, \ldots, J)$ as the restricted model without interaction. Estimation of $(\alpha\beta)_{ij}$ can also be considered. Now, consider the main effects of factors A and B.
To test for lack of difference in levels of factor A, use, $H_A : \alpha_i = 0$ for all i.
To test for lack of difference in levels of factor B, use, $H_B : \beta_j = 0$ for all j. If $H_{AB} : (\alpha\beta)_{ij} = 0$ has been rejected, there is evidence for significant interaction, so main effects cannot be non-existent.

To find estimates, confidence intervals and to conduct tests, we proceed as follows. Since
$\mu_{ij} = \bar{\mu}_{..} + (\bar{\mu}_{i.} - \bar{\mu}_{..}) + (\bar{\mu}_{.j} - \bar{\mu}_{..}) + (\mu_{ij} - \bar{\mu}_{i.} - \bar{\mu}_{.j} + \bar{\mu}_{..}) = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij}$,
we use a similar representation for ϵ_{ijk}:

$$\epsilon_{ijk} = \bar{\epsilon}_{...} + (\bar{\epsilon}_{i..} - \bar{\epsilon}_{...}) + (\bar{\epsilon}_{.j.} - \bar{\epsilon}_{...}) + (\bar{\epsilon}_{ij.} - \bar{\epsilon}_{i..} - \bar{\epsilon}_{.j.} + \bar{\epsilon}_{...}) + (\epsilon_{ijk} - \bar{\epsilon}_{ij.}).$$

Therefore, as in one-way classification,

$$\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\epsilon_{ijk}^2 = IJK\bar{\epsilon}_{...}^2 + JK\sum_{i=1}^{I}(\bar{\epsilon}_{i.} - \bar{\epsilon}_{..})^2 + IK\sum_{j=1}^{J}(\bar{\epsilon}_{.j} - \bar{\epsilon}_{..})^2$$
$$+K\sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{\epsilon}_{ij.} - \bar{\epsilon}_{i..} - \bar{\epsilon}_{.j.} + \bar{\epsilon}_{...})^2 + \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(\epsilon_{ijk} - \bar{\epsilon}_{ij.})^2,$$

since cross products vanish. Noting that $\epsilon_{ijk} = y_{ijk} - \mu - \alpha_i - \beta_j - (\alpha\beta)_{ij}$, with $\sum_{i=1}^{I}\alpha_i = 0$, $\sum_{j=1}^{J}\beta_j = 0$, $\sum_{i=1}^{I}(\alpha\beta)_{ij} = 0$ for all j and $\sum_{j=1}^{J}(\alpha\beta)_{ij} = 0$ for all i, we get $\bar{\epsilon}_{...} = \bar{y}_{...} - \mu$, $\bar{\epsilon}_{i..} = \bar{y}_{i..} - \mu - \alpha_i$, $\bar{\epsilon}_{.j.} = \bar{y}_{.j.} - \mu - \beta_j$, $\bar{\epsilon}_{ij.} = \bar{y}_{ij.} - \mu - \alpha_i - \beta_j - (\alpha\beta)_{ij}$. Hence,

$$\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(y_{ijk} - \mu - \alpha_i - \beta_j - (\alpha\beta)_{ij})^2$$
$$= IJK(\bar{y}_{...} - \mu)^2 + JK\sum_{i=1}^{I}(\bar{y}_{i..} - \bar{y}_{...} - \alpha_i)^2 + IK\sum_{j=1}^{J}(\bar{y}_{.j.} - \bar{y}_{...} - \beta_j)^2$$
$$+K\sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...} - (\alpha\beta)_{ij})^2 + \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(y_{ijk} - \bar{y}_{ij.})^2.$$

Subject to the identifiability conditions, we obtain the least squares estimates:
$\hat{\mu} = \bar{y}_{...}$, $\hat{\alpha}_i = \bar{y}_{i..} - \bar{y}_{...}$, $\hat{\beta}_j = \bar{y}_{.j.} - \bar{y}_{...}$ and $(\hat{\alpha\beta})_{ij} = \bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}$. Therefore, RSS $= \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(y_{ijk} - \bar{y}_{ij.})^2$, as seen earlier.

Consider $H_{AB} : (\alpha\beta)_{ij} = 0$ for all i, j. Due to the identiability constraints on these parameters, namely, $0 = \sum_{i=1}^{I}(\alpha\beta)_{ij} = \sum_{j=1}^{J}(\alpha\beta)_{ij} = \sum_{i=1}^{I}\sum_{j=1}^{J}(\alpha\beta)_{ij}$, there are $IJ - I - J + 1 = (I-1)(J-1)$ linearly independent equations, so the A matrix used to express this as a linear hypothesis has rank $IJ - I - J + 1 = (I-1)(J-1)$. Further, by inspection,

$$\text{RSS}_{H_{AB}} = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(y_{ijk} - \bar{y}_{ij.})^2 + K\sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...})^2,$$

since $\hat{\mu}, \hat{\alpha}_i$ and $\hat{\beta}_j$ remain as before. Hence

$$\text{RSS}_{H_{AB}} - \text{RSS} = K\sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...})^2 = K\sum_{i=1}^{I}\sum_{j=1}^{J}(\hat{\alpha\beta})_{ij}^2,$$

which has d.f. $(I-1)(J-1)$. To test H_{AB}, use

$$F_{AB} = \frac{(\mathrm{RSS}_{H_{AB}} - \mathrm{RSS})/\{(I-1)(J-1)\}}{\mathrm{RSS}/\{IJ(K-1)\}} \sim F_{(I-1)(J-1),IJ(K-1)}$$

under H_{AB}. Now consider $H_A : \alpha_i = 0$ for all i. There are $I-1$ linearly independent equations here, so the rank of A matrix is $I-1$. Again, by inspection, note that estimates of the remaining parameters, μ, β_j and $(\alpha\beta)_{ij}$ remain unchanged, so

$$\mathrm{RSS}_{H_A} = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(y_{ijk} - \bar{y}_{ij.})^2 + JK\sum_{i=1}^{I}(\bar{y}_{i..} - \bar{y}_{...})^2, \text{ so}$$

$$\mathrm{RSS}_{H_A} - \mathrm{RSS} = JK\sum_{i=1}^{I}(\bar{y}_{i..} - \bar{y}_{...})^2 = JK\sum_{i=1}^{I}\hat{\alpha}_i^2$$

with d.f. $I-1$. Similarly,

$$\mathrm{RSS}_{H_B} = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(y_{ijk} - \bar{y}_{ij.})^2 + IK\sum_{j=1}^{J}(\bar{y}_{.j.} - \bar{y}_{...})^2, \text{ so}$$

$$\mathrm{RSS}_{H_B} - \mathrm{RSS} = IK\sum_{j=1}^{J}(\bar{y}_{.j.} - \bar{y}_{...})^2 = IK\sum_{j=1}^{J}\hat{\beta}_j^2$$

with d.f. $J-1$. Therefore, for the respective tests use,

$$F_A = \frac{(\mathrm{RSS}_{H_A} - \mathrm{RSS})/(I-1)}{\mathrm{RSS}/\{IJ(K-1)\}} \sim F_{I-1,IJ(K-1)}$$

under H_A and

$$F_B = \frac{(\mathrm{RSS}_{H_B} - \mathrm{RSS})/(J-1)}{\mathrm{RSS}/\{IJ(K-1)\}} \sim F_{J-1,IJ(K-1)}$$

under H_B. The decomposition of the total sum of squares along with its d.f. is as follows.

$$\begin{aligned}\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}y_{ijk}^2 &= IJK\bar{y}_{...}^2 + JK\sum_{i=1}^{I}(\bar{y}_{i..} - \bar{y}_{...})^2 + IK\sum_{j=1}^{J}(\bar{y}_{.j.} - \bar{y}_{...})^2 \\ &\quad + K\sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...})^2 + \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(y_{ijk} - \bar{y}_{ij.})^2. \\ IJK &= 1 + (I-1) + (J-1) + (IJ - I - J + 1) + (IJK - IJ).\end{aligned}$$

ANOVA table for 2-factor analysis:

source	d.f	SS	MS	F
A main effects	$I-1$	$\mathrm{SS}_A = JK\sum_{i=1}^{I}\hat{\alpha}_i^2$	$\mathrm{MS}_A =$	$F_A = \mathrm{MS}_A/\mathrm{MSE}$
B main effects	$J-1$	$\mathrm{SS}_B = IK\sum_{j=1}^{J}\hat{\beta}_j^2$	$\mathrm{MS}_B =$	$F_B = \mathrm{MS}_B/\mathrm{MSE}$
AB interactions	$(I-1)(J-1)$	$\mathrm{SS}_{AB} = K\sum\sum\hat{(\alpha\beta)}_{ij}^2$	$\mathrm{MS}_{AB} =$	$F_{AB} = \mathrm{MS}_{AB}/\mathrm{MSE}$
Error	$IJ(K-1)$	$\mathrm{RSS} = \sum\sum\sum(y_{ijk}-\bar{y}_{ij.})^2$	$\mathrm{MSE} = \frac{\mathrm{RSS}}{IJ(K-1)}$	
Total (c)	$IJK-1$	$\sum\sum\sum(y_{ijk}-\bar{y}_{...})^2$		
Mean	1	$IJK\bar{y}_{...}^2$		
Total	IJK	$\sum\sum\sum y_{ijk}^2$		

ANOVA for the battery example:

source	d.f	SS	MS	F
plate	2	10684	5342	7.91 (2, 27)
temperature	2	39119	19559	28.97 (2, 27)
interactions	4	9614	2413	3.56 (4, 27)
error	27	18231	675	
total (c)	35	77647		