Combinatorics

13 oct 21

Lecture 7

Steiner triple systems.

$\lambda = 1$, $k = 3$ & $t = 2$. v is the only unknown parameter. formulae of $b_i \Rightarrow v \equiv 1$ or $3 \pmod 6$.

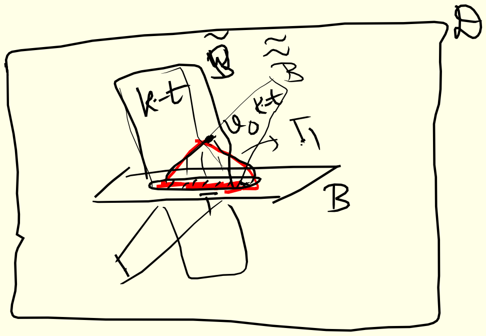
For each such v, $\exists$ Steiner triple system.

Remark:- For $v = 25$; $\exists$ at least 1639 2992 9318400 non-iso. steiner triple systems.

The following theorem shows that in general the nec. conditions (b_i, b_i^j are integers $\forall\ 0 \le i, j \le t$) are NOT sufficient.

Theorem (Jacques Tits, 1964) In any non-trivial Steiner system, we must have $v \geq (t+1)(k-t+1)$

Proof :- (Geometrical proof) Let $\mathcal{D} = S(t, k, v)$ be a Steiner design. Fix a block B in $\mathcal{D}$. Let T be a subset of B with $|T| = t$. Choose a point $v_0 \notin B$. Let $T_1 = T \cup \{v_0\}$.



Note that no block B_1 contains T_1. ($\because B_1 \supset T_1 \Rightarrow B_1 \supset T \Rightarrow B_1 = B$ $\Rightarrow v_0 \in B$ contradiction)

$\exists\ t+1$ t-subsets of T_1 because $|T_1| = t+1$.

& each one of them is contained in a unique block. clearly any two t-subsets of a $t+1$-set must contain $t-1$ elements common.

$\Rightarrow$ Each of these $t+1$ blocks have $t-1$ elts in common.
This is the max. intersection two distinct blocks can have since $\lambda = 1$.

$\therefore$ These $t+1$ blocks can not have any intersection outside T_1!

$\Rightarrow$ number of elements of $\mathcal{D} \geq \underbrace{(t+1)(k-t)}_{\text{elts in blocks containing a } t\text{-subset of } T_1 \text{ that are outside } T_1} + \underbrace{(t+1)}_{\text{elts of } T_1}$

$$\Rightarrow \boxed{v \geq (t+1)(k-t+1)}$$

QED.

<u>Example</u> :- $S(\underset{t}{10}, \underset{k}{16}, \underset{v}{72})$ does not exist.

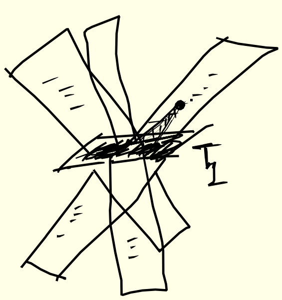
since $\underset{v}{72} \not\geq \underset{(t+1)(k-t+1)}{11 \cdot 7 = 77}$

<u>Please</u> check that all b_i's are integers for $0 \leq i \leq 10$.

$\therefore$ b_i's are integers is not a suff. cond^n for Steiner design to exist.

———— x ———— x ———— x ————

<u>Q</u>. What happens when $v = (t+1)(k-t+1)$.



T_1 is cleverly chosen.

? Not clear (as of now) ?

———— x ———— x ———— x ————

$\boxed{\text{Next Friday, Wed, Friday — No class}}$ We will meet after two weeks on 27/10/21.

———— x ———— x ———— x ————

$\longrightarrow$ Defined designs, elementary computations examples & insufficiency of the computations. $\longleftarrow$

Introduce linear algebra.

Def^n :- Let $\mathcal{D} = S_\lambda(t,k,v)$ be a design. The incidence matrix N have v rows & b columns (with rows indexed by elements & columns by blocks)

$x_1, x_2, \ldots, x_v$ are elements

& $B_1, B_2, \ldots, B_b$ are blocks

$$(x_i, B_j)^{th} \text{ entry} = \begin{cases} 1 & \text{if } x_i \in B_j \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{array}{c|c} & \begin{matrix} B_1 & B_2 & \cdots & B_b \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_v \end{matrix} & \begin{bmatrix} 0 & 1 & \cdots & 1 \\ & & & \\ & & & \\ & & & \end{bmatrix}_{v \times b} \end{array}$$

Row sum of $N = b_1 = r$

Column sum of $N = k = $ ~~#~~ elts of blocks

Q. $NN^T = ?$ This is square matrix of size $v \times v$. $\therefore$ Rows & columns are indexed by elements.

$$(x_i, x_j)^{th} \text{ entry} = \begin{cases} b_2 & i \neq j \\ b_1 & i = j \end{cases}$$

$N^T =$ transpose of T.

If $\mathcal{D}$ is a 2-design then

$$NN^T = \begin{cases} \lambda & i \neq j \\ r & i = j \end{cases} = (r - \lambda) I + \lambda J$$

where $J =$ all 1 matrix (of correct size!)

Theorem (Fisher's Inequality) For a $S_\lambda(2,k,v)$, with b blocks & $v > k$, we have $\underline{b \geq v}$

Proof :- For any 2-design, we must have

$$\lambda(v-1) = r(k-1)$$

{ Fix $x \in P$, $T \ni x$ $|T|=2$. $\{(T,B) \mid T \subset B\}$.

Since $v > k$, we must have $\boxed{r > \lambda}$.

look at the incidence matrix of the give design, say N.

$NN^T = (r-\lambda)I + \lambda J$.

Claim :- 0 is not an eigen value of NN^T.

1. Every vector is an eigen vector with eigen value $r-\lambda$ of $(r-\lambda)I$.

2. $J = \begin{bmatrix} 1 & \cdots & 1 \\ 1 & \cdots & 1 \\ 1 & \cdots & 1 \end{bmatrix}$ $\Rightarrow$ rk $J = 1$

$\Rightarrow J$ has $v-1$ eigen vectors with eigen value 0

& 1 eigen vector namely $j = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

with eigen value v.

$\Rightarrow NN^T$ has $v-1$ eigen vectors with eigen value $r-\lambda$ & j is an eigen vector with eigen value $\underline{\lambda v + (r-\lambda)}$.

$\Rightarrow \boxed{\det(NN^T) = (r-\lambda)^{v-1} \cdot ((r-\lambda)+\lambda v)} > 0$.

$\therefore$ rk $NN^T = v$. $\Rightarrow$

$$V_1 \xrightarrow{N} V_2 \xrightarrow{N^T} V_3$$

Image has dim v

$\Rightarrow \operatorname{Im} N$ has $\geq v$ dim.

& N is $v \times b$ mx. $\Rightarrow$ rk $N = v$.

$\Rightarrow$ col^n rank $N = v \leq$ # col^{n}s of N

$\therefore \boxed{v \leq b}$ QED !

Cor. If $S_\lambda(t,k,v)$ design has $b = v$ & if v is even, then $k-\lambda$ must be a perfect square

proof :- $\det NN^T = (\det N)^2 = (r-\lambda)^{v-1} \cdot \underline{((r-\lambda)+\lambda v)}$

But $bk = v r$ in a 2-design $\{(x, B) \mid x \in B\}$.

$\Rightarrow k = r$

$\Rightarrow (\det N)^2 = r^2 \cdot (r - \lambda)^{v-1}$

$$\left\{ \begin{aligned} (r-\lambda) + \lambda v &= r + \lambda(v-1) \\ &= r + r(k-1) \\ &= r \cdot k = r^2. \end{aligned} \right\}$$

But v is even $\Rightarrow v-1$ is odd.

$\Rightarrow$ $r - \lambda$ is a perfect square

$\Rightarrow$ $\underline{k - \lambda}$ is a perfect square.

$\therefore$ For a symmetric $(b = v)$ 2-design to exist we must have $k - \lambda$ a perfect square!

QED.