Lecture 23: Mackey's criterion

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12:45

(*) Let G be group, H & K be subgrps of G. Let W be an H-mod.

What is $\mathrm{Res}_K^G(\mathrm{Ind}_H^G W)$?

Let $V = \mathrm{Ind}_H^G W$ then

$$V = \bigoplus_{x \in G/H} xW$$

$\{KsH \mid s \in G\}$ be the collection of double cosets.

$H_s = sHs^{-1} \cap K$ for $s \in G$.

$$\rho^s : H_s \to GL(W) \quad \rho^s(x) = \rho(s^{-1}xs)$$

$$x \cdot w := \underbrace{s^{-1}xs}_{\in H} \cdot w \quad \text{for } x \in H_s$$

Let W_s denote the underlying v.s for ρ^s

<u>Prop</u> $$\mathrm{Res}_K^G \mathrm{Ind}_H^G W \cong \bigoplus_{\substack{\text{sum over double cosets} \\ K\backslash G/H}} \mathrm{Ind}_{H_s}^K W_s \cong \bigoplus_{s \in S} \mathrm{Ind}_{H_s}^K W_s$$

where S is a collection of representatives of the double coset $\{KsH \mid s \in G\}$
$= K\backslash G/H$

<u>Pf:</u> Let $V = \mathrm{Ind}_H^G W$

$$V = \bigoplus_{x \in G/H} xW$$

For $s \in S$, let

$$V(s) = \bigoplus_{\substack{x \in G/H \\ x \subseteq KsH}} xW \subseteq V$$

Note if $y \in K$

$$y \cdot V(s) = \bigoplus_{\substack{x \in G/H \\ x \subseteq KsH}} yxW = \bigoplus_{\substack{yx \in G/H \\ yx \subseteq KsH}} yxW = V(s)$$

So $V(s)$ is K-stable subreps of $\mathrm{Res}_K V$

Also $s' \in S$ & $s' \neq s$ then

$V(s') \cap V(s) = 0$. So

$$\mathrm{Res}^G_K V = \bigoplus_{s \in S} V(s) \text{ and this is a } K\text{-reps.}$$

So enough to show $V(s) \cong \mathrm{Ind}^K_{H_s} W_s$

$$V(s) = \bigoplus_{\substack{x \in G/H \\ x \subseteq KsH}} xW$$

For $g \in H_s = sHs^{-1} \cap K$ $\quad g(sW) = sW \quad \left(\because s^{-1}gs \in H \text{ \& } W \text{ is an } H\text{-mod} \right)$

Hence sW is an H_s-mod. Moreover

$$V(s) = \bigoplus_{\substack{x \in G/H \\ x \subseteq KsH}} x(sW) = \bigoplus_{x \in K/H_s} x(sW) \quad \left(\because g \in K \text{ s.t. } g(sW) = sW \Rightarrow s^{-1}gs \in H \Rightarrow g \in H_s \right)$$

$$\Rightarrow V(s) = \mathrm{Ind}^K_{H_s}(sW)$$

$H_s \neq xH_s$ then $xsW \cap sW = (0)$

Finally $sW \cong W_s$ as H_s-reps. In fact $W_s \xrightarrow{\varphi} sW \subseteq V$ is the isom

$w \longmapsto s.w = \mathrm{Ind}^G_H \rho(s)(w)$

For $g \in H_s$ $\quad \varphi(\rho^s(g)(w)) = \varphi(\rho(s^{-1}gs)(w)) = \mathrm{Ind}\rho(gs)w = \mathrm{Ind}\rho(g)\,\varphi(w)$

$= g \cdot \varphi(w)$

WTS $\varphi(\underset{\rho^s \text{ in } W_s}{g \cdot w}) = \underset{\text{in } sW}{g \cdot \varphi(w)}$

$\blacksquare$

Apply this to $K = H$. Then for $s \in G$.

$H_s = sHs^{-1} \cap H$ & Prop becomes $\rho : H \to GL(W)$

$$\boxed{\mathrm{Res}^G_H \mathrm{Ind}^G_H W \cong \bigoplus_{s \in S} \mathrm{Ind}^H_{H_s} W_s}$$

where S is the collection of representative of double cosets $H \backslash G / H = \{ HsH \mid s \in G \}$

& W_s is the H_s-mod via $g \in H_s$

$$\rho^s(g)(w) = \rho(s^{-1} g s)(w) \quad \text{for } w \in W_s = W$$

Note $\rho|_{H_s}$ is $\mathrm{Res}^H_{H_s} W$ is also an H_s-repr.

<u>Thm</u> (Mackey's criterion): Let $\rho : H \to GL(W)$ repr.

$V = \mathrm{Ind}^G_H W$ is irred iff

① W is irred &

② ρ^s & $\mathrm{Res}^H_{H_s} \rho$ have no common nonzero subreps for $s \in G \setminus H$.

<u>Pf</u>: $$\langle V, V \rangle_G = \langle V, \mathrm{Ind}^G_H W \rangle_G$$

$$\overset{\text{Frobenius Reciprocity}}{=} \langle \mathrm{Res}^G_H \mathrm{Ind}^G_H W, W \rangle_H$$

$$= \sum_{s \in S} \langle \mathrm{Ind}_{H_s}^{H} W_s, W \rangle_H$$

$$= \sum_{s \in S} \langle \rho^s, \mathrm{Res}_{H_s}^{H} \rho \rangle_{H_s}$$

If $s = e$ then $H_s = H$ & $\rho^s = \rho$

$$H_e = eHe^{-1} \cap H = H$$

$$= \langle \rho, \rho \rangle_H + \sum_{\substack{s \in S \\ s \notin G \setminus H}} \langle \rho^s, \mathrm{Res}_{H_s}^{H} \rho \rangle$$

$$\langle \rho, \rho \rangle_H \geq 1$$

$\langle V, V \rangle_G = 1$ iff $\langle \rho, \rho \rangle_H = 1$ &

$\langle \rho^s, \mathrm{Res}_{H_s}^{H} \rho \rangle = 0$

$\forall\ s \in S$ & $s \notin G \setminus H$

$\Rightarrow$ V is irr iff W is irr & 2 of the thm holds.

$\blacksquare$

Cor: If $H \trianglelefteq G$ then $\mathrm{Ind}_H^G W$ is irr iff W is irred. & the every conjugate repr of H is not isom to ρ.

Conjugate repr is ρ^s for $s \notin H$.

Pf: $H_s = H$ if H is normal & $\mathrm{Res}_s \rho = \rho$ $\boxtimes$

Representation of Symm group S_n

(*) Irred repr are in bijection with conjugacy classes in S_n.

$()()\dots()$ n 1-cycles

$(\ )()\dots()$ 1 2-cycle & $n-2$ 1-cycles

Conjugacy classes are in bijection with partitions of n.

i.e. $\lambda_1, \dots, \lambda_k$ s.t. $\lambda_i \geq 1$ &

$\sum \lambda_i = n$ $\quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$

λ_1 boxes

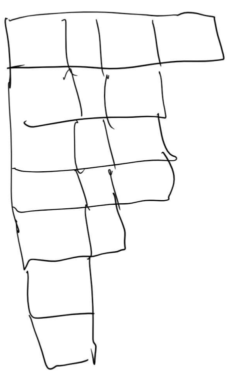
1	2	3	4	5	6	7
8	9	10	11	12		
13	14	15	16			
17						

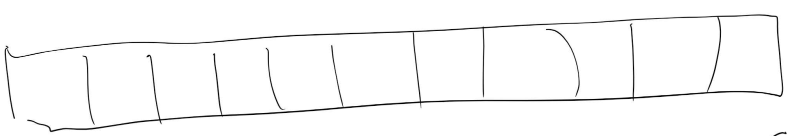
young tableaux

(1 2 3 4 5 6 7)(8 9 10 11 12)(13 14 15 16)(17)

$n = 17$.

conjugate tableaux



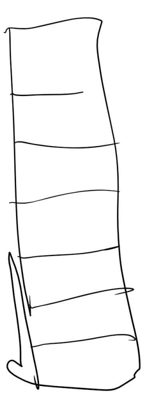


trivial

$\lambda \longleftrightarrow c_\lambda \in \mathbb{C}S_n$

$\cup\!|$

$\mathbb{C}S_n c_\lambda$



sgn repr.