Lecture 6: Definition of group representation

30 September 2021
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(*) $f: V \to W$ a lin map of v.s. Then f induces maps

$$\left.\begin{array}{l} \text{(i) } T^n V \xrightarrow{T^n f} T^n W \\ \text{(ii) } Ext^n V \xrightarrow{\Lambda^n f} Ext^n W \\ \text{(iii) } Sym^n V \xrightarrow{Sym^n f} Sym^n W \end{array}\right\}$$

$$\begin{array}{l} V \times \cdots \times V \xrightarrow{f} W \times \cdots \times W \xrightarrow{\varphi_W} T^n W \\ (a_1, \cdots, a_n) \longmapsto (f(a_1), \cdots, f(a_n)) \xrightarrow{\varphi^e_W} Ext^n W \\ \qquad\qquad\qquad\qquad \xrightarrow{\varphi^s_W} Sym^n W \end{array}$$

So T^n, Ext^n, Sym^n are functors. One has to check than $T^n(f \circ g) = T^n f \circ T^n g$.

$A \xrightarrow{f} B$ · $Hom_R(M, -)$ & $Hom_R(-, N)$, localization are functors
$S \subseteq R$ $\quad M \xrightarrow{f} N$, $\; S^{-1}M \to S^{-1}N$

$$Hom_R(M, A) \longrightarrow Hom_R(M, B)$$
$$\varphi \longmapsto f \circ \varphi$$

<u>Defn</u>: Let G be a finite group & V be a finite dim'l vector space over k. Then a <u>representation of G</u> is a group homo. $\rho: G \to GL(V)$.
In this case V is said to be a representation of G.

(*) If V is a representation of G, Then for $g \in G$ & $v \in V$

$$g \cdot v = \rho(g)(v). \quad \text{This is group action}$$

because
$$e_G \cdot v = \rho(e_G)(v) = v \quad (\because \rho \text{ is a grp homo})$$
$$\begin{aligned} g \cdot (h \cdot v) &= \rho(g)(\rho(h)(v)) \\ &= \rho(g) \circ \rho(h)(v) \\ &= \rho(gh)(v) \\ &= (gh) \cdot v \end{aligned}$$

Moreover $g \cdot (v_1 + a v_2) = g \cdot v_1 + a g \cdot v_2$ for $\forall v_1, v_2 \in V$ & $a \in k$. $\forall g \in G$.

Conversely, Given an action of a group G on a v.s. V s.t. $g\cdot(v_1 + a v_2) = g\cdot v_1 + a\, g\cdot v_2 \quad \forall g \in G, v_1, v_2 \in V$ & $a \in k$. --- ⊛

One obtains a map
$$\rho : G \longrightarrow GL(V)$$
$$g \longmapsto \rho(g) : v \longrightarrow g\cdot v$$

Since G acts on V $\rho(g) \in S(V) \leftarrow$ set of bijections of V.

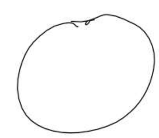
The condition ⊛ ensures that $\rho(g)$ is linear.

And hence $\rho(g) \in GL(V)$

Examples 1) Let $V = \mathbb{C}$ & G a finite group.

$\rho : G \rightarrow GL(V) = \mathbb{C}^*$ is group homo.

$$\rho(g)^{|G|} = \rho(g^{|G|}) = \rho(e) = 1$$



$\Rightarrow \rho(g)$ is a root of unity.

$$|\rho(g)| = 1$$

a) ρ is the trivial homo. Then G acts trivially on V. This is called the trivial representation.

(2) G acts on a finite set X. Let V_X be a $\mathbb{C}$-vs with a basis consisting of elements of X.

$$V_X = \mathbb{C}x_1 \oplus \mathbb{C}x_2 \oplus \cdots \oplus \mathbb{C}x_n \quad \text{where } X = \{x_1, \ldots, x_n\}$$

$$g \cdot 1x_i = \underset{\substack{\nwarrow \\ G\text{-action on } X}}{g \cdot x_i} = x_j \text{ for some } j$$

$$g \cdot \left(\sum_{i=1}^{n} a_i x_i\right) = \sum a_i \, g \cdot x_i$$

So V_X is a repr. of G. It is called a permutation repr.

$$\rho : G \longrightarrow GL(V_X) \overset{\text{is fixed w.r.t basis } \{x_1,\ldots,x_n\}}{\cong} GL_n(\mathbb{C})$$

$\rho(g)$ is a permutation matrix.

(3) Let G be a finite group $k[G]$ the group ring

$$k[G] = \left\{ \sum_{g \in G} a_g g \mid a_g \in k \right\}$$

$$G \times k[G] \longrightarrow k[G]$$

$$\left(h, \sum_{g \in G} a_g g\right) \longmapsto \underbrace{\sum_{g \in G} a_g \, hg}_{\text{coeff of } g \text{ is } a_{h^{-1}g}}$$

Note that action is linear.

So $k[G]$ is a representation of G. This is called the regular representation of G.

<u>Def^n</u>: Let V be a repr of a group G. A subspace W of V is called a subrepresentation of V if

(*) $\begin{cases} \forall g \in G \quad \rho(g)(w) \in W \quad \forall w \in W. \quad \text{i.e. } W \text{ is stable} \\ \text{under } G\text{-action.} \end{cases}$

$$\rho : G \longrightarrow GL(V)$$

$$(*) \Rightarrow \quad \rho_W : G \longrightarrow GL(W)$$

<u>Remark</u>: $$W = \left\{ a\left(\sum_{g \in G} g\right) \,\middle|\, a \in k \right\}$$

$$h \cdot w = w \quad \forall \, w \in W.$$

So W is a trivial rep of G & it is a subrep of $k[G]$.