Lecture 20

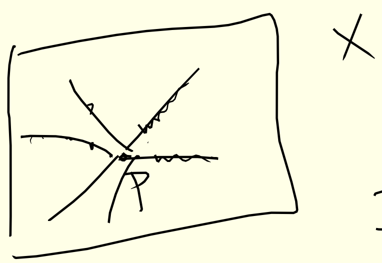
Witt Designs

This is going to be the last lecture of this course. Today we will briefly introduce Witt designs & describe their automorphism groups, called the Mathieu groups.

Theorem :- The projective plane $PG_2(4)$ over the field of four elements can be "extended" three times.

Def^n :- Witt design is an extension of $\mathbb{P}_2(4)$.

Recall: • Given a t-(v,k,λ)-design $\mathcal{D}$ its derived design $\mathcal{D}_P$ at a point P is the design whose pts are $Y = X \setminus P$ & blocks are all blocks containing $P \cap Y$. (without P) It becomes a $(t-1, v-1, k-1, \lambda)$ design.



This always exist.

It may happen that $D_P \not\cong D_Q$ for distinct points P & Q.

Def^n :- Let $\mathcal{D}$ be a design. A design $(X, \mathcal{B})$ is called an extension of $\mathcal{D}$ if for all points P of X, the derived design of $(X,\mathcal{B})$ at P is isomorphic $\mathcal{D}$.

Proposition :- If $(X, \mathcal{B})$ is an extension of a t-(v,k,λ) design with b blocks, then $|\mathcal{B}| = \frac{b\,(v+1)}{(k+1)}$.

Proof: We know that block size of $(X, \mathcal{B})$ is $k+1$, the no. of pts are $v+1$, if $r =$ no. of blocks containing a given pt, then

we have $|\mathcal{B}| \cdot (k+1) = (v+1) \cdot r$. $\Leftarrow S = \{(x, B) \mid x \in B \in \mathcal{B}\}$

$$\Rightarrow |\mathcal{B}| = \frac{b(v+1)}{k+1} \quad \text{as } r = b$$

Corollory: If a projective plane of order q has an extension, then $q = 2, 4$ or 10.

Pf. (Recall: a projective plane of order n is an $2\text{-}(n^2+n+1, n+1, 1)$ symmetric 2-design.

pts of $\mathbb{P}_2(n) = \mathbb{F}_n^3 / \mathbb{F}_n^*$ & pts are indexed by 2-diml subspaces of $\mathbb{F}_n^3$ if $\exists$ a field of order n.)

Assume $(X, \mathcal{B})$ is an extn of $\mathbb{P}_2(q)$.

then, $v+1 = q^2+q+2$, $k+1 = q+2$, $b = q^2+q+1$

$$\Rightarrow |\mathcal{B}| = \frac{(q^2+q+2)(q^2+q+1)}{q+2}$$

$$\Rightarrow (q+2) \mid \left((q+2-2)^2 + (q+2)\right)\left((q+2-2)^2 + (q+2) - 1\right).$$

$$\Rightarrow (q+2) \mid 4 \cdot 3 = 12. \Rightarrow q = 2, 4 \text{ or } \cancel{10}$$

QED.

Remark :- (using computers) Lam & others proved that a proj. plane of order 10 can not exists.

$\nexists$ $2\text{-}(111, 11, 1)$ design. $\Rightarrow N(10) < 9$.

$N(10)$ = max. no. of MOLS of order 10

Thm: there exists unique $2\text{-}(21, 5, 1)$-design.

$21 = 4^2+4+1$, $5 = 4+1$

$\therefore$ any 3-$(22,6,1)$-design will be an extension of $\mathbb{P}_2(4)$. denoted by W_{22}.

$\|^{ly}$ any two 3-$(22,6,1)$ designs are iso. to each other.

$\therefore$ any 4-$(23,7,1)$ design will be an ext^n of W_{22} denoted by W_{23}.

Also W_{23} is unique (any two 4-$(23,7,1)$ designs are iso. to each other)

$\therefore$ if $\exists$ a 5-$(24,8,1)$ design, it will be an ext^n of W_{23}.

It is denoted by W_{24}.

Group of auto. of W_{22} is denoted by M_{22} (Mathieu groups)

— " — W_{23} — " — M_{23}

— " — W_{24} — " — M_{24}.

Thm: ① $|M_{24}| = 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 16 \cdot 3$ & acts 5-transitively on W_{24}.

ie given x_1, x_2, x_3, x_4, x_5 & y_1, y_2, y_3, y_4, y_5 $\exists$ an auto. σ s.t. $\sigma(x_i) = y_i$.

② $M_{23} = \mathrm{Aut}(W_{23})$ & $M_{22} = \mathrm{Aut}(W_{22})$ are obtained as stabilizers of pts of W_{24}.

$\Rightarrow |M_{23}| = 23 \cdot 22 \cdot 21 \cdot 20 \cdot 16 \cdot 3$

& $|M_{22}| = 22 \cdot 21 \cdot 20 \cdot 16 \cdot 3$.

($|G| = |O| \, |\mathrm{Stab}_x|$, if G acts transitively)

Recall: $G \times X \to X$ is an action

ie $e.x = x \ \forall x$

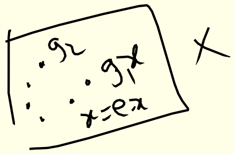
& $(g_1 g_2) \cdot x = g_1 \cdot (g_2 . x) \quad \forall g_i \in G, x \in X$.

$\Updownarrow$

An action is a gp. hom. $\varphi : G \to \mathrm{Perm}(X)$.

$O_x = \{ g.x \mid g \in G \}$ orbit of x.

$|G| = |O_x| \cdot |Stab_x(G)|$; $Stab_x(G) = \{ g \in G \mid g.x = x \}$.



Ref :- The Mathieu groups & Designs
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End of the course!

All the best ☺