Quadratic Forms.

Recall that, $Y'AY$ is called a quadratic form of Y when Y is a random vector.

Result. If $X \sim N_p(\mu, \Sigma)$, $\Sigma > 0$, then $(X-\mu)'\Sigma^{-1}(X-\mu) \sim \chi^2_p$.

Proof. $Z = \Sigma^{-1/2}(X-\mu) \sim N_p(0, I_p)$. i.e., $Z_1, Z_2, \ldots, Z_p$ are i.i.d. $N(0,1)$. Therefore $Z'Z = \sum_{i=1}^p Z_i^2 \sim \chi^2_p$. Note that $(X-\mu)'\Sigma^{-1}(X-\mu) = Z'Z$.

Result. If $X_1, X_2, \ldots, X_n$ is a random sample from $N(\mu, \sigma^2)$, then $\bar{X}$ and $S^2 = \sum_{i=1}^n (X_i - \bar{X})^2$ are independent, $\bar{X} \sim N(\mu, \sigma^2/n)$ and $S^2/\sigma^2 \sim \chi^2_{n-1}$.

Proof. First note that $X = (X_1, X_2, \ldots, X_n)' \sim N_n(\mu\mathbf{1}, \sigma^2 I_n)$. Now consider an orthogonal matrix $A_{n\times n} = ((a_{ij}))$ with the first row being $\mathbf{a}_1' = (\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}) = \frac{1}{\sqrt{n}}\mathbf{1}'$. (Simply consider a basis for $\mathcal{R}^n$ with $\mathbf{a}_1$ as the first vector, orthogonalize the rest.) Now let $Y = AX$. i.e., $Y_i = a_i'X$, $i = 1, 2, \ldots, n$. Since $X \sim N_n(\mu\mathbf{1}, \sigma^2 I_n)$, we have that $Y \sim N_n(\mu A\mathbf{1}, \sigma^2 AA') = N_n(\mu A\mathbf{1}, \sigma^2 I_n)$. Therfeore, Y_i are independent normal with variance σ^2. Further, $E(Y_i) = E(a_i'X) = \mu a_i'\mathbf{1}$. Thus, $E(Y_1) = \mu a_1'\mathbf{1} = \mu\frac{1}{\sqrt{n}}\mathbf{1}'\mathbf{1} = \sqrt{n}\mu$. For $i > 1$, $E(Y_i) = \mu a_i'\mathbf{1} = \mu\sqrt{n}a_i'a_1 = 0$. i.e., $Y_2, \ldots, Y_n$ are i.i.d $N(0, \sigma^2)$. Therefore, $\sum_{i=2}^n Y_i^2 \sim \chi^2_{n-1}$. Further, $Y_1 = a_1'X = \frac{1}{\sqrt{n}}\sum_{i=1}^n X_i = \sqrt{n}\bar{X} \sim N(\sqrt{n}\mu, \sigma^2)$ and is independent of $(Y_2, \ldots, Y_n)$. Also, $S^2 = \sum_{i=1}^n (X_i - \bar{X})^2 = \sum_{i=1}^n X_i^2 - n\bar{X}^2 = X'X - Y_1^2 = Y'Y - Y_1^2 = \sum_{i=2}^n Y_i^2 \sim \chi^2_{n-1}$ which is independent of Y_1, and therefore of $\bar{X}$.

If $X \sim N_p(0, I)$, then $X'X = \sum_{i=1}^p X_i^2 \sim \chi^2_p$. i.e., $X'IX \sim \chi^2_p$. Also, note $X'(\frac{1}{\sqrt{p}}\mathbf{1}\frac{1}{\sqrt{p}}\mathbf{1}')X = p\bar{X}^2 \sim \chi^2_1$ and $X'(I - \frac{1}{p}\mathbf{1}\mathbf{1}')X \sim \chi^2_{p-1}$.

What is the distribution of $X'AX$ for any arbitrary A which is p.s.d.? Without loss of generality we can assume that A is symmetric since

$$X'AX = X'(\frac{1}{2}(A+A'))X = X'BX, \text{ where } B = \frac{1}{2}(A+A') \text{ is always symmetric.}$$

Since A is symmetric p.s.d., $A = \Gamma D_\lambda \Gamma'$, so $X'AX = X'\Gamma D_\lambda \Gamma' X = Y'D_\lambda Y$, where $Y = \Gamma'X \sim N_p(0, \Gamma'\Gamma = I)$. Therefore $X'AX = \sum_{i=1}^p d_i Y_i^2$, where d_i are eigen values of A and Y_i are i.i.d $N(0,1)$. Therefore $X'AX$ has the χ^2 distribution if $d_i = 1$ or 0. Equivalently, $X'AX \sim \chi^2$ if $A^2 = A$ or A is symmetric idempotent or A is an orthogonal projection matrix. The equivalence may be seen as follows. If $d_1 \geq d_2 \geq \ldots \geq d_p \geq 0$ are such that

$d_1 = d_2 = \ldots = d_r = 1$ and $d_{r+1} = \ldots = d_p = 0$, then

$$\begin{aligned} A &= \Gamma \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \Gamma', \\ A^2 &= \Gamma \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \Gamma'\Gamma \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \Gamma' = A. \end{aligned}$$

If $A^2 = A$ then $\Gamma D_\lambda \Gamma' \Gamma D_\lambda \Gamma' = \Gamma D_\lambda^2 \Gamma' = \Gamma D_\lambda \Gamma'$ implies that $D_\lambda^2 = D_\lambda$, or that $d_i^2 = d_i$, or that $d_i = 0$ or 1.

We will show the converse now. Suppose $X'AX \sim \chi_r^2$ and A is symmetric p.s.d. Then the mgf of $X'AX$ is:

$$\begin{aligned} M_{X'AX}(t) &= \int_0^\infty \exp(tu) \frac{\exp(-u/2) u^{r/2-1}}{2^{r/2}\Gamma(r/2)}\, du \\ &= \int_0^\infty \frac{\exp(-\frac{u}{2}(1-2t)) u^{r/2-1}}{2^{r/2}\Gamma(r/2)}\, du \\ &= (1-2t)^{-r/2}, \text{ for } 1-2t>0. \end{aligned}$$

But in distribution, $X'AX = \sum_{i=1}^p d_i Y_i^2$, Y_i i.i.d. $N(0,1)$, so

$$\begin{aligned} M_{X'AX}(t) &= E\left[\exp(t\sum_{i=1}^p d_i Y_i^2)\right] = E\left[\prod_{i=1}^p \exp(t d_i Y_i^2)\right] \\ &= \prod_{i=1}^p E\left[\exp(t d_i Y_i^2)\right] = \prod_{i=1}^p (1-2td_i)^{-1/2}, \text{ for } 1-2td_i>0. \end{aligned}$$

Now note that $X'AX \sim \chi_r^2$ implies $X'AX > 0$ wp 1. i.e., $\sum_{i=1}^p d_i Y_i^2 > 0$ wp 1, which in turn imples that $d_i \geq 0$ for all i. (This is because, if $d_l < 0$, since $Y_l^2 \sim \chi_1^2$ independently of Y_i, $i \neq l$, we would have $\sum_{i=1}^p d_i Y_i^2 < 0$ with positive probability.) Therefore, for $t < \min_i \frac{1}{2d_i}$, equating the two mgf's, we have $(1-2t)^{-r/2} = \prod_{i=1}^p (1-2td_i)^{-1/2}$, or $(1-2t)^{r/2} = \prod_{i=1}^p (1-2td_i)^{1/2}$, or $(1-2t)^r = \prod_{i=1}^p (1-2td_i)$. Equality of two polynomials mean that their roots must be the same. Check that r of the d_i's must be 1 and rest 0. Thus the following result follows.

Result. $X'AX \sim \chi_r^2$ iff A is a symmetric idempotent matrix or an orthogonal projection matrix of rank r.