Lecture 13: Consequences of Schur's lemma and character theory

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φ, ψ class functions on G.

$$(\chi_1 | \chi_2) = \frac{1}{|G|}\sum_{g\in G} \chi_1(g)\overline{\chi_2(g)} \in \mathbb{R} \quad ; \quad \langle \varphi, \psi \rangle = \frac{1}{|G|}\sum_{g\in G} \varphi(g)\psi(g^{-1})$$

<u>Prop</u>: If V_1 & V_2 are irred then
$$(\chi_1 | \chi_2) = \begin{cases} 1 & \text{if } V_1 \cong V_2 \\ 0 & \text{o.w.} \end{cases}$$
} Orthogonality of irred characters

<u>Cor</u>: Let V be G-repr then $V \cong V_1^{r_1} \oplus \dots \oplus V_m^{r_m}$ where V_i's are irred reprs of G and $r_i > 0$. This unique in the sense that if $V \cong \bigoplus_{i=1}^{m'} W_i^{r_i'}$ then $m' = m$ & after a permutation of $\{1, \dots, m\}$ $V_i \cong W_i$ as G-repr & $r_i = r_i'$.

$\circledast$ In fact, let $V = V_1 \oplus V_2 \oplus \dots \oplus V_N$ as direct sum of irred repr of G & W_j is an irred repr of G. Then

$$\begin{aligned}
(\chi_{W_j} | \chi_V) &= (\chi_{W_j} | \chi_{V_1} + \dots + \chi_{V_N}) \\
&= \sum_{i=1}^{N} (\chi_{W_j} | \chi_{V_i}) \\
&= r_j = \#\{ V_i \mid V_i \cong W_j, 1 \le i \le N \}
\end{aligned}$$

We also write
$$\boxed{V = r_1 W_1 + \dots + r_m W_m}$$
instead $V = W_1^{r_1} \oplus \dots \oplus W_m^{r_m}$

<u>Cor</u> Let V & V' be two G-repr, if $\chi_V = \chi_{V'}$ then $V \cong V'$.

<u>Pf</u>: Let $W_1, \dots, W_m$ be all the irred subreps of $V \oplus V'$ & $r_i = (\chi_{W_i}, \chi_V) = (\chi_{W_i}, \chi_{V'})$. Then by the above argument $V = r_1 W_1 + \dots + r_m W_m = V'$

<u>Cor</u>: Let V be a G-repr then $(\chi_V | \chi_V) = \sum_{i=1}^{m} r_i^2$ where r_i's are the multiplicities of irr repr in V. In part $(\chi_V | \chi_V)$ is a positive integer & V is irr iff $(\chi_V | \chi_V) = 1$.

Decomposition of regular representation

Let $V = k[G]$ be the regular repr of G.

$$\chi_V(g) = \begin{cases} |G| & g = e \\ 0 & g \neq e \end{cases}$$

Let V_i be an irred reps of G then

$$(\chi_{V_i} \mid \chi_V) = \frac{1}{\cancel{|G|}} \chi_{V_i}(e) \cancel{|G|}$$

$$= \dim(V_i) = \underset{\substack{\uparrow \\ \text{degree of } V_i}}{n_i} \geq 1$$

$$\boxed{V = n_1 V_1 + \cdots + n_m V_m}$$

where $V_1, \ldots, V_m$ are all the irred reprs of G.

$$\boxed{k[G] = n_1 V_1 + \cdots + n_m V_m}$$

where $n_i = \dim(V_i)$ & V_i's are irred.

<u>Cor</u>: 1) $\chi_{k[G]} = n_1 \chi_{V_1} + \cdots + n_m \chi_{V_m}$

where V_i's are all the irr repr of G of degree n_i.

2) $|G| = n_1^2 + \cdots + n_m^2$

3) $n_1 \chi_{V_1}(g) + \cdots + n_m \chi_{V_m}(g) = 0$

$\forall g \neq e$ in G.

Thm: Let H be the space of class functions on a group G. Then the characters of irred repr of G form an orthonormal basis of H w.r.t $\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \varphi(g) \psi(g^{-1})$.

(*) We already know that $\{\chi_1, \dots, \chi_r\}$ the set of irreducible characters of G is an orthogonal set. So need to show the generate H. Equivalently let $f \in H$ s.t. $\langle f, \overline{\chi_i} \rangle = 0 \quad \forall 1 \le i \le r$ then $f = 0$.

Lemma: Given a repr $\rho: G \to GL(V)$ & a class function f on G. Define the endomor.

$$\rho_f : V \longrightarrow V, \quad \rho_f = \sum_{g \in G} f(g) \rho(g).$$

If V is irr then ρ_f is a scalar multiple by the scalar $\lambda = \frac{1}{n} \sum_{g \in G} f(g) \chi_V(g) = \frac{|G|}{n} \langle f, \overline{\chi_V} \rangle$

Here $n = \dim V$.

Pf of lemma: Claim: ρ_f is G-equivariant map.

i.e. $\rho_f(g \cdot v) = g \cdot \rho_f(v) \quad \forall v \in V$

i.e. $\rho_f \circ \rho(g) = \rho(g)\rho_f$

i.e. $\rho(g^{-1})\rho_f \circ \rho(g) = \rho_f$

LHS: $= \sum_{g' \in G} f(g')\, \rho(g^{-1}) \circ \rho(g') \circ \rho(g)$

$= \sum_{g' \in G} f(g')\, \rho(g^{-1} g' g)$

$= \sum_{h \in G} f(h) \rho(h)$ where $h = g^{-1} g' g$

$= \rho_f$

By Schur's lemma ρ_f is a scalar multiple by (say) λ.

$Tr(\rho_f) = n\lambda$

Also $T(\rho_f) = Tr\left(\sum_{g \in G} f(g)\rho(g)\right)$

$= \sum_{g \in G} f(g)\chi(g)$

$= |G| \langle f, \bar{\chi} \rangle$

$\Longrightarrow \lambda = \frac{|G|}{n} \langle f, \bar{\chi} \rangle$

$\boxtimes$

Now back to the theorem,

Since $\langle f, \bar{\chi}_i \rangle = 0 \quad \forall$ irr characters χ_i.

$\Rightarrow \langle f, \bar{\chi}_V \rangle = 0 \quad \forall$ repr V of G.

$\Rightarrow \rho_f = \sum_{g \in G} f(g) \rho(g) = 0$ for any repr V of G.

Take $V = k[G]$ then

$$\rho_f = \sum_{g \in G} f(g) \rho(g)$$

$$0 = \rho_f(e) = \sum_{g \in G} f(g)\, g \in V$$

$\Rightarrow f(g) = 0 \quad \forall g \in G$.

$\Rightarrow f = 0$

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