Question. Given X, how to find P such that $\mathcal{M}_C(X) = \mathcal{M}_C(P)$?

Result. If $\Omega = \mathcal{M}_C(X)$, then $P_\Omega = X(X'X)^- X'$, where $X'X)^-$ is any generalized inverse of $X'X$.

Definition. If $B_{m\times n}$ is any matrix, *a* generalized inverse of B is any $n \times m$ matrix B^- satisfying $BB^-B = B$.

Existence. From singular value decomposition of B, there exist orthogonal matrices $P_{m\times m}$ and $Q_{n\times n}$ such that

$$P_{m\times m} B_{m\times n} Q_{n\times n} = \Delta_{m\times n} = \begin{pmatrix} D_{r\times r} & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{pmatrix},$$

where $r = \text{Rank}(B)$. Define $\Delta^-_{n\times m} = \begin{bmatrix} D_r^{-1} & 0 \\ 0 & 0 \end{bmatrix}$ and let $B^- = Q\Delta^- P$. First,

$$\Delta\Delta^-\Delta = \begin{pmatrix} D_r & 0 \\ 0 & 0 \end{pmatrix} \begin{bmatrix} D_r^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} D_r & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} D_r & 0 \\ 0 & 0 \end{pmatrix} = \Delta.$$

Further, $B = P'\Delta Q'$, so that

$$BB^-B = P'\Delta Q' Q\Delta^- P P'\Delta Q' = P'\Delta\Delta^-\Delta Q' = P'\Delta Q' = B.$$

Proof of Result. Let $B = X'X$. Find B^- such that $BB^-B = B$. For any $Y \in \mathcal{R}^n$, let $c = X'Y$, and let $\tilde\beta$ be any solution of $X'X\beta = X'Y$, or that of $B\beta = c$. Then

$$B(B^-c) = BB^-B\tilde\beta = B\tilde\beta = c,$$

so that $\hat\beta = B^-c$ is a particular solution of $B\beta = c$. Let $\hat\theta = X\hat\beta = XB^-c$. Then, $Y = \hat\theta + (Y - \hat\theta)$, where

$$\hat\theta'(Y - \hat\theta) = \hat\beta' X'(Y - X\hat\beta) = \hat\beta'(X'Y - X'X\hat\beta) = 0.$$

Therefore we have an orthogonal decomposition of Y such that $\hat\theta \in \mathcal{M}_C(X)$ and $(Y - \hat\theta) \perp \mathcal{M}_C(X)$. Now note that $\hat\theta = X\hat\beta = X(X'X)^- X'Y$. i.e., for Y, its projection onto $\mathcal{M}_C(X)$ is given by $X(X'X)^- X'Y$. Therefore, $P_\Omega = X(X'X)^- X'$ since P_Ω is unique.

Techniques for finding B^- are needed: if $B = X'X$, then $P_\Omega = X(X'X)^- X'$; if we want to solve $X'X\beta = X'Y$, or $B\beta = c$, then $\hat\beta = B^-c$.

For $B_{p\times m}$ with rank $r<p$ and $B=\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$, where B_{11} (which is $r\times r$ of rank r) is nonsingular, if we take $B^- = \begin{pmatrix} B_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix}$, then note that

$$\begin{aligned} BB^-B &= \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}\begin{pmatrix} B_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \\ &= \begin{pmatrix} I_r & 0 \\ B_{21}B_{11}^{-1} & 0 \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{21}B_{11}^{-1}B_{12} \end{pmatrix}. \end{aligned}$$

Now note that $(B_{21}|B_{22})$ is a linear function of $(B_{11}|B_{12})$, or $(B_{21}|B_{22}) = K(B_{11}|B_{12}) = (KB_{11}|KB_{12})$ for some matrix K. Therefore, $KB_{11}=B_{21}$, or $K=B_{21}B_{11}^{-1}$, so $B_{22}=KB_{12}=B_{21}B_{11}^{-1}B_{12}$.

Example. Let $B=\begin{pmatrix} 1 & 2 & 5 & 2 \\ 3 & 7 & 12 & 4 \\ 0 & 1 & -3 & -2 \end{pmatrix}$. Then rank of B is 2 since 2nd row - 3× 1st row = 3rd row. Partition B as: $B=\left(\begin{array}{cc|cc} 1 & 2 & 5 & 2 \\ 3 & 7 & 12 & 4 \\ \hline 0 & 1 & -3 & -2 \end{array}\right) = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$. Take

$$B^- = \begin{pmatrix} B_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 7 & -2 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Example. Consider the model:
$y_1 = \beta_1 + \beta_2 + \epsilon_1$
$y_2 = \beta_1 + \beta_2 + \epsilon_2$
$y_3 = \beta_1 + \beta_2 + \epsilon_3$

This is equivalent to

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix}.$$

X has rank 1; $X'X = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} = \left(\begin{array}{c|c} 3 & 3 \\ \hline 3 & 3 \end{array}\right)$, so choose $(X'X)^- = \begin{pmatrix} 1/3 & 0 \\ 0 & 0 \end{pmatrix}$.

Then check that $(X'X)(X'X)^{-}(X'X) = X'X$. We have then

$$\begin{aligned}
X\hat{\beta} &= \hat{\theta} = X(X'X)^{-}X'Y \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1/3 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \\
&= \begin{pmatrix} 1/3 & 0 \\ 1/3 & 0 \\ 1/3 & 0 \end{pmatrix} \begin{pmatrix} y_1 + y_2 + y_3 \\ y_1 + y_2 + y_3 \end{pmatrix} = \begin{pmatrix} (y_1 + y_2 + y_3)/3 \\ \bar{y} \\ \bar{y} \end{pmatrix} \\
&= \begin{pmatrix} \widehat{\beta_1 + \beta_2} \\ \widehat{\beta_1 + \beta_2} \\ \widehat{\beta_1 + \beta_2} \end{pmatrix}.
\end{aligned}$$

Only $\beta_1 + \beta_2$ can be estimated? Note $\beta_1 + \beta_2 = (1 \;\; 1) \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ and $\mathcal{M}_C \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \mathcal{M}_C(X')$. More on this later.