Result. Suppose $Y \sim N_p(0, I_p)$ and let $Y'Y = Y'AY + Y'BY$. If $Y'AY \sim \chi^2_r$, then $Y'BY \sim \chi^2_{p-r}$ independent of $Y'AY$.

Proof. Note that $Y'Y \sim \chi^2_p$. Since $Y'AY \sim \chi^2_r$, A is symmetric idempotent of rank r. Therefore, $B = I - A$ is symmetric and $B^2 = (I - A)^2 = I - 2A + A^2 = I - A = B$, so that B is idempotent also. Further, Rank(B) = trace(B) = trace($I - A$) = $p - r$. Therefore, $Y'BY \sim \chi^2_{p-r}$. Independence is shown later.

Result. Let $Y \sim N_p(0, I_p)$ and let $Q_1 = Y'P_1Y$, $Q_2 = Y'P_2Y$, $Q_1 \sim \chi^2_r$, and $Q_2 \sim \chi^2_s$. Then Q_1 and Q_2 are independent iff $P_1P_2 = 0$.

Corollary. In the result before the above one, $A(I - A) = 0$, so $Y'AY$ and $Y'(I - A)Y$ are independent.

Proof. P_1 and P_2 are symmetric idempotent. If $P_1P_2 = 0$ then $Cov(P_1Y, P_2Y) = 0$ so that $Q_1 = (P_1Y)'(P_1Y) = Y'P_1^2Y = Y'P_1Y$ is independent of $Q_2 = (P_2Y)'(P_2Y) = Y'P_2Y$. Conversely, if Q_1 and Q_2 are independent χ^2_r and χ^2_s, then $Q_1 + Q_2 \sim \chi^2_{r+s}$. Since $Q_1 + Q_2 = Y'(P_1 + P_2)Y$, $P_1 + P_2$ is symmetric idempotent. Hence, $P_1 + P_2 = (P_1 + P_2)^2 = P_1^2 + P_2^2 + P_1P_2 + P_2P_1$, implying $P_1P_2 + P_2P_1 = 0$. Multiplying by P_1 on the left, we get, $P_1^2P_2 + P_1P_2P_1 = P_1P_2 + P_1P_2P_1 = 0\,(*)$. Similarly, multiplying by P_1 on the right yields, $P_1P_2P_1 + P_2P_1 = 0$. Subtracting, we get, $P_1P_2 - P_2P_1 = 0$. Combining this with $(*)$ above, we get $P_1P_2 = 0$.

Result. Let $Q_1 = Y'P_1Y$, $Q_2 = Y'P_2Y$, $Y \sim N_p(0, I_p)$. If $Q_1 \sim \chi^2_r$, $Q_2 \sim \chi^2_s$ and $Q_1 - Q_2 \geq 0$, then $Q_1 - Q_2$ and Q_2 are independent, $r \geq s$ and $Q_1 - Q_2 \sim \chi^2_{r-s}$.

Proof. $P_1^2 = P_1$ and $P_2^2 = P_2$ are symmetric idempotent. $Q_1 - Q_2 \geq 0$ means that $Y'(P_1 - P_2)Y \geq 0$, hence $P_1 - P_2$ is p.s.d. Therefore, from Lemma shown below, $P_1 - P_2$ is a projection matrix and also $P_1P_2 = P_2P_1 = P_2$. Thus $(P_1 - P_2)P_2 = 0$. Also, Rank($P_1 - P_2$) = tr($P_1 - P_2$) = tr(P_1) - tr(P_2) = Rank(P_1) - Rank(P_2) = $r - s$. Hence, $Q_1 - Q_2 = Y'(P_1 - P_2)Y \sim \chi^2_{r-s}$, and is independent of $Q_2 = Y'P_2Y \sim \chi^2_s$.

Lemma. If P_1 and P_2 are projection matrices such that $P_1 - P_2$ is p.s.d., then (a) $P_1P_2 = P_2P_1 = P_2$ and (b) $P_1 - P_2$ is also a projection matrix.

Proof. (a) If $P_1x = 0$, then $0 \leq x'(P_1 - P_2)x = -x'P_2x \leq 0$, implying $0 = x'P_2x = x'P_2^2x = (P_2x)'P_2x$, so $P_2x = 0$. Therefore, for any y, $P_2(I - P_1)y = 0$ since $P_1(I - P_1)y = 0$ (Take $x = (I - P_1)y$.) Thus, for any y, $P_2P_1y = P_2y$ or $P_2P_1 = P_2$, and so $P_2 = P_2' = (P_2P_1)' = P_1P_2$.
(b) $(P_1 - P_2)^2 = P_1^2 + P_2^2 - P_1P_2 - P_2P_1 = P_1 + P_2 - P_2 - P_2 = P_1 - P_2$.

Result. Any orthogonal projection matrix (i.e., symmetric idempotent) is p.s.d.

Proof. If P is a projection matrix, $x'Px = x'P^2x = (Px)'Px \geq 0$.

Result. Let C be a symmetric p.s.d. matrix. If $X \sim N_p(0, I_p)$, then AX and $X'CX$ are independent iff $AC = 0$.

Proof. (i) If part: Since C is symmetric p.s.d., $C = TT'$. If $AC = 0$, then $ATT' = 0$, so $ATT'A' = (AT)(AT)' = 0$ and hence $AT = 0$. Thus AX and $T'X$ are independent, so AX and $(T'X)(T'X)' = X'CX$ are independent.

(ii) Only if: If AX and $X'CX$ are independent, then $X'A'AX$ and $X'CX$ are independent. But the mgf of $X'BX$ for any B is $E(\exp(tX'BX)) = |I - 2tB|^{-1/2}$ for an interval of values of t. Therefore, the joint mgf of $X'CX$ and $X'A'AX$ is $|I - 2(t_1C + t_2A'A)|^{-1/2}$, but because of independence this is given to be equal to

$$|I - 2t_1C|^{-1/2}|I - 2t_2A'A|^{-1/2} = |I - 2t_1C - 2t_2A'A + 4t_1t_2CA'A|^{-1/2}.$$

Show that, for this to hold on an open set, we must have $CA'A = 0$, implying $CA'AC' = 0$, and thus $AC' = 0$. But $C' = C$.

Lemma. If $X \sim N_p(\mu, \Sigma)$, then $Cov(AX, X'CX) = 2A\Sigma C\mu$.

Proof. Note that $(X-\mu)'C(X-\mu) = X'CX + \mu'C\mu - 2X'C\mu = X'CX - 2((X-\mu)'C\mu - \mu'C\mu$ and $E(X'CX) = tr(C\Sigma) + \mu'C\mu$. Therefore $X'CX - E(X'CX) = X'CX - \mu'C\mu - tr(C\Sigma) = (X-\mu)'C(X-\mu) + 2(X-\mu)'C\mu - tr(C\Sigma)$. Hence,

$$\begin{aligned}
&Cov(AX, X'CX) \\
&\quad = E\left[(AX - A\mu)(X'CX - E(X'CX))\right] \\
&\quad = AE\left\{(X-\mu)\left[(X-\mu)'C(X-\mu) + 2(X-\mu)'C\mu - tr(C\Sigma)\right]\right\} \\
&\quad = 2AE\left\{(X-\mu)(X-\mu)'C\mu\right\} - tr(C\Sigma)AE(X-\mu) \\
&\quad\quad +AE\left\{(X-\mu)(X-\mu)'C(X-\mu)\right\} \\
&\quad = 2A\Sigma C\mu,
\end{aligned}$$

since $E(X-\mu) = 0$ and $E\left\{(X-\mu)(X-\mu)'C(X-\mu)\right\} =$
$E\left\{(X-\mu)\left[\sum_i \sum_j C_{ij}(X_i-\mu_i)(X_j-\mu_j)\right]\right\} = 0$. To prove this last equality, it is enough to show that $E\left\{(X_l-\mu_l)(X_i-\mu_i)(X_j-\mu_j)\right\} = 0$ for all i, j, l. For this note:
(i) if $i = j = l$, $E(X_i - \mu_i)^3 = 0$.
(ii) if $i = j \neq l$, $E\left\{(X_i-\mu_i)^2(X_l-\mu_l)\right\} = 0$ since $X_l - \mu_l = \frac{\sigma_{il}}{\sigma_{ll}}(X_i - \mu_i) + \epsilon$,

where $\epsilon \sim N(0,.)$ is independent of X_i, so this case reduces to (i).
(iii) if i, j and l are all different, the case reduces to (i) and (ii). Alternatively, consider $Y = (Y_1, Y_2, Y_3)' \sim N_3(0, \Sigma)$. Then $Y = \Sigma^{1/2}(Z_1, Z_2, Z_3)$, where Z_i are i.i.d. $N(0,1)$. Then to show that $E(Y_1Y_2Y_3) = 0$, simply note that $Y_1Y_2Y_3$ is a linear combination of Z_i^3, $Z_i^2Z_j$ and $Z_1Z_2Z_3$, all of which have expectation 0.

Loynes' Lemma. If B is symmetric idempotent, Q is symmetric p.s.d. and $I - B - Q$ is p.s.d., then $BQ = QB = 0$.

Proof. Let x be any vector and $y = Bx$. Then $y'By = y'B^2x = y'Bx = y'y$, so $y'(I-B-Q)y = -y'Qy \leq 0$. But $I-B-Q$ is p.s.d., so $y'(I-B-Q)y \geq 0$, implying $-y'Qy \geq 0$. Since Q is also p.s.d., we must have $y'Qy = 0$. (Note, y is not arbitrary, but Bx for some x.) In addition, since Q is symmetric p.s.d., $Q = L'L$ for some L, and hence $y'Qy = y'L'Ly = 0$, implying $Ly = 0$. Thus $L'Ly = Qy = QBx = 0$ for all x. Therefore, $QB = 0$ and hence $(QB)' = B'Q' = BQ = 0$.