Lecture 15: Representation of subgroups and product of groups.

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(*) Let V be a G-repr. Then

$V = r_1 W_1 + r_2 W_2 + \dots + r_m W_m$ where W_i's are irred repr & $r_i \geq 0$.

$V = V_1 \oplus V_2 \oplus \dots \oplus V_m$ where the subspace $V_i \underset{\text{as repr.}}{\cong} W_i^{r_i}$ is the image of the projection map $p_i = \frac{n_i}{|G|} \sum_{g \in G} \overline{\chi_i(g)} \rho_V(g)$ where $n_i = \dim(W_i)$ & $\chi_i = \chi_{W_i}$

3) $S_3 = \langle (1,2), (1\ 2\ 3) \rangle = \langle \tau, \sigma \rangle$

τ: transposition

conjugacy classes $\{e\}, \{\text{trans}\}, \{3\text{-cycles}\}$

$\chi_{\text{trivial}} = \chi_0(g) = 1 \quad \forall g \in S_3$

$1 + a^2 + b^2 = 6 \implies a = 1, b = 2$

$\chi_{sgn} = \chi_1(g) = sgn(g)$ is a character

Irr $\chi_2(e) = 2$, $\chi_2(\text{transp}) = 0$

$\chi_2(3\text{-cycles}) = -1$

$\{e, (1 2), (2 3), (3 1), (1 2 3), (3 2 1)\}$

$S_3 \curvearrowright \mathbb{C}^3 \quad \sigma(z_1, z_2, z_3) = (z_{\sigma 1}, z_{\sigma 2}, z_{\sigma 3})$

$$p_0 = \frac{1}{6} \sum_{g \in S_3} \rho(g)$$

$$= \frac{1}{6}\left[I + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \dots \right]$$

$$= \frac{1}{6}\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$\operatorname{Im}(p_0) = \Delta$

$$p_1 = \frac{1}{6}\left(I - \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right)$$

$$= \frac{1}{6}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$p_2 = \frac{2}{6}\left[2I - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right]$$

$$= \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \quad \operatorname{Im}(p_2) = \{(z_1, z_2, z_3) \mid z_1 + z_2 + z_3 = 0\}$$

$\operatorname{Im}(p_2)$ is a repr with character χ_2.

<u>Cor:</u> G is abelian iff every irred repr of G is one dimensional.

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Def^n: Let G be a group & $H \leq G$. Let V be a G-repr. then V is also an H-repr via restriction. $\rho : G \to GL(V)$
then $\rho_{|H}$ is the H-repr.

(*) If V is irred G-repr & $H \leq G$ then is V an irred H-repr? $\{e\} \leq G$ then V is direct sum of one dim'l repr of $\{e\}$.

(*) Let G be a grp & $H \leq G$ with H abelian. Let V be an irred G-repr then $\dim(V) \leq [G:H]$.

Pf: V be an irred G-repr. Since H is abelian
$V = W_1 \oplus W_2 \oplus \cdots \oplus W_m$ where W_i's are irred. H-modules are 1-dim'l

$g_1, g_2 \in G$. $\rho_V(g_1)(W_1) = g_1 W_1 \subseteq V$

$g_2 W_1 \subseteq V$. Note if $g_2 = g_1 h$ for some $h \in H$
then $g_2 W_1 = (g_1 h) \cdot W_1 = g_1 W$

So $g_1 W_1 = g_2 W_1$ if g_1, g_2 are in same left coset.

$\# \{ g W_1 \mid g \in G \} \leq [G:H]$

The subspace $\langle \{ g W_1 \mid g \in G \} \rangle$ is G-stable & hence a G-subrep of V

Since V is irr. This subspace is V.

$\Longrightarrow \dim(V) \leq [G:H]$ ($\because$ gW_1 are 1-dim'l)

$\boxed{q}$

(*) G-repr V & $H \leq G$, then V is H-repr.

$\chi_{(\rho_H : H \to GL(V))} = \chi_V|_H$

$\underline{\text{Def}^n}$ Let G_1, G_2 be groups & V_i be a G_i-repr.
Then $\underline{V_1 \otimes V_2 \text{ is } G_1 \times G_2 \text{ repr}}$ via

$$\rho(g_1, g_2) = \rho_1(g_1) \otimes \rho_2(g_2) \quad \text{where } \rho_i : G_i \to GL(V_i),\ i = 1, 2 \text{ are repr.}$$

$$(g_1, g_2) \cdot (v_1 \otimes v_2) = g_1 v_1 \otimes g_2 v_2$$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} & & b_{1m} \\ b_{m1} & \cdots & b_{mm} \end{bmatrix}$$

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ a_{21}B & \cdots & a_{2n}B \\ \vdots & & \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix}$$

$\underline{\text{Prop}}$: If V_1 & V_2 are irr G_1 & G_2 repr resp.
Then $V_1 \otimes V_2$ is irr $G_1 \times G_2$ repr. Conversely every irred. $G_1 \times G_2$ repr is isom to $V_1 \otimes V_2$ for some irr G_i-repr V_i.

(*)
$$\chi_{V_1 \otimes V_2}(g_1, g_2) = \operatorname{tr}(\rho_1(g_1) \otimes \rho_2(g_2)) = \operatorname{tr}(\rho_1(g_1)) \operatorname{tr}(\rho_2(g_2)) = \chi_{V_1}(g_1) \chi_{V_2}(g_2)$$

$\underline{\text{Pf}}$:
$$(\chi_{V_1 \otimes V_2} | \chi_{V_1 \otimes V_2})_{G_1 \times G_2} = \frac{1}{|G_1 \times G_2|} \sum_{(g_1, g_2) \in G_1 \times G_2} |\chi_{V_1 \otimes V_2}(g_1, g_2)|^2$$

$$= \frac{1}{|G_1||G_2|} \sum_{\substack{g_1 \in G_1 \\ g_2 \in G_2}} |\chi_{V_1}(g_1)|^2 |\chi_{V_2}(g_2)|^2 = \left(\frac{1}{|G_1|} \sum_{g_1 \in G_1} |\chi_{V_1}(g_1)|^2 \right) \left(\frac{1}{|G_2|} \sum_{g_2 \in G_2} |\chi_{V_2}(g_2)|^2 \right)$$

$$= (\chi_{V_1} | \chi_{V_1})_{G_1} (\chi_{V_2} | \chi_{V_2})_{G_2}$$

Now V_1 & V_2 are irr the $(\chi_{V_1} | \chi_{V_1})$ & $(\chi_{V_2} | \chi_{V_2})$ are 1 $\Rightarrow (\chi_{V \otimes V} | \chi_{V_1 \otimes V_2}) = 1 \Leftrightarrow V_1 \otimes V_2$ is irred.

For converse, let $V_{11}, \ldots, V_{1m_1}$ be all irr reps of G_1

& " $V_{21}, \ldots, V_{2m_2}$ " " " & " G_2

Let $n_{ij} = \dim V_{ij}$ $i = 1$ or 2

Then $|G_i| = \sum_{j=1}^{m_i} n_{ij}^2$ (proved in class earlier using regular reps)

$$|G_1 \times G_2| = |G_1||G_2| = \left(\sum_{j_1=1}^{m_1} n_{1j_1}^2\right)\left(\sum_{j_2=1}^{m_2} n_{2j_2}^2\right)$$

$$= \sum_{j_1=1}^{m_1}\sum_{j_2=1}^{m_2} \left(n_{1j_1} n_{2j_2}\right)^2$$

$$= \sum_{j_1, j_2} \dim\left(V_{1j_1} \otimes V_{2j_2}\right)^2$$ & $V_{1j} \otimes V_{2j}$ & irr $G_1 \times G_2$-reps

Hence these are all the irred reps of $G_1 \times G_2$. $\blacksquare$