Combinatorics

27 oct 21

Lecture 8 — Symmetric Designs

Fisher's inequality. For 2-(v, k, λ) design with $v > k$, we must have $b \geq v$

(b = no. of blocks in a design).

Def^n A 2-design with $b = v$ is called a symmetric design.

If N is the incidence matrix of a symmetric design, (up to an order of pts & blocks) then N is a $v \times v$ square mx.

symmetric design does not mean that N is symmetric !!

Examples (1) $\mathbb{P}^2(GF(q))$ — pts = 1 dim^l subspaces of $GF(q)^3 = V$

blocks = 2-dim^l subsp. of V.

$$v = \frac{q^3 - 1}{q - 1} = q^2 + q + 1.$$

(Since $2 = 3 - 1$, given any 1-dim^l subspace $\rightsquigarrow$ orth. compl^t. is a 2-dim^l subsp.)

$b = q^2 + q + 1$; k = size of the block $= \frac{q^2 - 1}{q - 1} = \underline{q + 1}$.

In this example $\lambda = 1$.

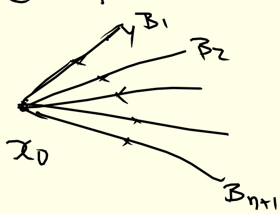
Def^n:- Any symmetric design with $\lambda = 1$ is called a projective plane. If the block size is $n + 1$ we say that the given proj. plane has order n.

$\boxed{\text{Open}}$:- Does there exist a proj. plane of order $\neq$ prime power.

There does not exist any proj. plane of order $\underline{10}$.

Any proj. plane of order n must contain n^2+n+1 pts.

block size $= n+1$.



$\frac{(n+1)\cdot n+1}{\text{pts} \neq x_0}$ $\ast x_0$

$r = k$ if $b = v$

$bk = vr$

$\rightarrow$ $\nexists$ a collection of 111 subsets of size 11 of an 111 size set such that any 2-subset occurs exactly once in the chosen collection $\leftarrow$

—×—×—×—

Example 2 :- Latin Squares

Let $S = \{1, 2, \ldots, n\}$

Any $n \times n$ array with entries from S such that every row & column contains all n numbers is a Latin square.

$$\rightarrow \begin{bmatrix} 1 & 2 & \cdots & n \\ 2 & 3 & \cdots n & 1 \\ & & & \\ n & 1 & 2 \cdots & n-1 \end{bmatrix} \text{ - example.}$$

multiplicative table of a gp of order n $\rightarrow$

$$\begin{bmatrix} e & g_1 & \cdots & g_{n-1} \\ g_1 & g_1^2 & \cdots & g_1 g_{n-1} \\ & & & \\ & & & \end{bmatrix}$$

$g_1 \cdot g_j \neq g_1 \cdot g_k$ if $j \neq k$

Problem 19G (van Lint-Wilson)

Consider a Latin square of order 6 say L.

Let $\mathcal{P} = \mathcal{R} \times \mathcal{C}$ where $\mathcal{R}$ = set of rows $= \{1, \ldots, 6\}$

$\mathcal{C}$ = set of columns $= \{1, \ldots, 6\}$

$= \{(i,j) \mid 1 \leq i,j \leq 6\}$ $\quad |\mathcal{P}| = 36.$

Given a pair (i,j) construct a block as follows:

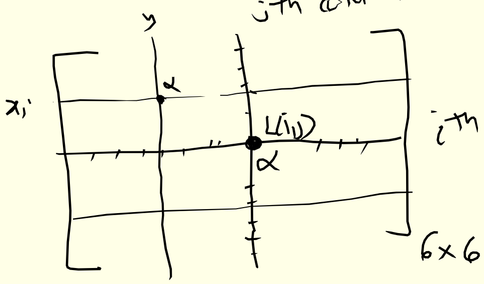
$$B_{i,j} = \{(x,y) \mid x = i \text{ or } y = j \text{ or } L(x,y) = L(i,j)\} \setminus \{(i,j)\}$$

$\underline{b = 36.}$ 1. Is block size constant? If yes then what is it?

2. Is it a 2-design? If yes, then what is λ?

Q. : Show that $(\mathcal{P}, \mathcal{B})$ defines a $2-(36, 15, 6)$-design.
$v \quad k \quad \lambda$

Sol :- Fix (i,j) & look at B_{ij}.



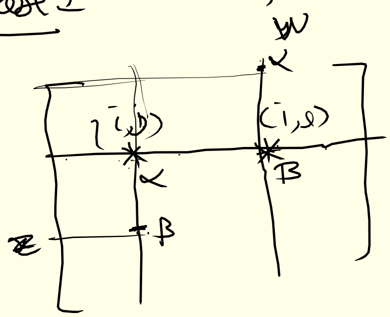
$$B_{i,j} \supset \{(i,k) \mid k \neq j\}^{5}$$
$$\cup$$
$$\{(l,j) \mid l \neq i\}^{5}$$
$$+ \{(x,y) \mid L(x,y) = L(i,j)\}_{5}$$

$$\Rightarrow \boxed{|B_{i,j}| = 15}$$

$$\Rightarrow k = 15.$$

Let $(i,j), (k,l)$ be two distinct elts of $\mathcal{P}$.

Case 1 $i = k$, case 2 : $j = l$ case 3 $i \neq k$ & $j \neq l$.



$B_{ix} \supset \{(i,j), (i,l)\} \ \forall x \neq j, l$

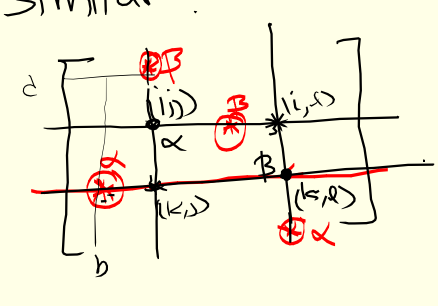
also $B_{x,j} \supset (i,j)$ if $L(x,j) = \beta$.
$\supset (i,l)$

||ly if $L(y,l) = \alpha$ then
$B_{y,l} \supset (i,j)$
$\supset (i,l)$

$\Rightarrow$ this pair occurs in at least 6 blocks.

Case 2 - Similar!

Case 3.



case 3(a) $\alpha = \beta$
$\boxed{3(b) \quad \alpha \neq \beta}$

$B_{i,l} \supset \{(i,j), (k,l)\} \subset B_{k,j}$

In the ith row β must

occur say at $(i,a)^{th}$ place then $B_{i,a} \supseteq \{(i,j),(k,l)\}$

Note that $a \neq l$ since β already occured at $(k,l)^{th}$ place in the l^{th} column. $\Rightarrow (i,a) \neq (i,l)$

||ly one can find b, c & d such that $B_{k,b}$; $B_{c,l}$ & $B_{d,j}$ contains both elts. $\Rightarrow$ this pair is in at least 6 blocks.

$\Rightarrow$ Every pair of elts occur is ≥ 6 blocks.

However $\exists \binom{36}{2}$ pairs. & $\binom{15}{2} \cdot 36$ pairs that occur in blocks

Since $\binom{36}{2} \cdot 6 = \binom{15}{2} \cdot 36$

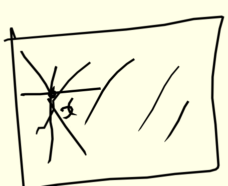
we see that every pair can occur in ≤ 6 blocks.

$\Rightarrow$ it is a $2-(36,15,6)$ design.

Theorem :- If $\mathcal{D}$ is a symmetric $2-(v,k,\lambda)$ design, then $\bar{\mathcal{D}}$ is also a symmetric $2-(v,k,\lambda)$ design

where pts of $\bar{\mathcal{D}}$ = blocks of $\mathcal{D}$

blocks of $\bar{\mathcal{D}}$ = pts of $\mathcal{D}$ = $\left\{ B_x \;\middle|\; \begin{array}{l} x \in \mathcal{P} \\ B_x = \{B \ni x\} \end{array} \right\}$.



$\mathcal{P}$

OR If N is the incidence matrix of a symmetric $2-(v,k,\lambda)$ design, $\mathcal{D}$, then N^T is also the incidence matrix of a symmetric $2-(v,k,\lambda)$ design.

Def^n:- The design associated with N^T is called the dual design of $\mathcal{D}$.