Lecture 11: Applications of Schur's lemma

13 October 2021
10:02

<u>Schur's lemma</u>: Let V & W be two irred G-representa

Let $f: V \longrightarrow W$ be a G-equivariant map.

Then $f = 0$ or f is an isom. Moreover if $W = V$ then f is a scalar multiple of identity

<u>Cor</u>: Let V be G-repr then $V \cong V_1^{r_1} \oplus \dots \oplus V_m^{r_m}$ where V_i's are irred reprs of G and $r_i > 0$. This unique in the sense that if $V \cong \bigoplus_{i=1}^{m'} W_i^{r_i'}$ then $m' = m$ & after a permutation of $\{1, \dots, m\}$ $V_i \cong W_i$ as G-repr & $r_i = r_i'$.

<u>Cor</u>: Let V_1 & V_2 be irred G-repr & $h: V_1 \longrightarrow V_2$ be any linear map. Let

$$h_0 = \frac{1}{|G|} \sum_{g \in G} \rho_{V_2}(g)^{-1} h \rho_{V_1}(g) : V_1 \longrightarrow V_2 \quad \text{- - - - - } (*)$$

Then 1) If V_1 & V_2 are not isom then $h_0 = 0$

2) If $V_1 = V_2$ & $\rho_{V_1} = \rho_{V_2}$ then $h_0 = \frac{1}{n} \mathrm{Tr}(h)$ where $n = \dim V$

<u>Pf</u>: <u>Claim</u>: h_0 is G-equivar. $\left(h_0(g \cdot_{V_1} v) = g \cdot_{V_2} h_0(v) \right)$

$$h_0 \circ \rho_{V_1}(g') = \rho_{V_2}(g') \circ h_0 \quad \forall g' \in G \quad (**)$$

This follows by plugging in the formula for h_0.

By Schur's lemma 1 is immediate.

And 2 we know that $h_0 = \lambda \mathrm{Id}_{V_1}$ where $\lambda \in \mathbb{C}$.

$\Rightarrow \mathrm{Tr}(h_0) = \lambda n$. By $(*)$,

$$\mathrm{Tr}(h_0) = \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}\left(\rho_{V_1}(g)^{-1} h \rho_{V_1}(g) \right)$$

$$= \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}(h)$$

$$= \mathrm{Tr}(h)$$

$\lambda n = \mathrm{Tr}(h) \Longrightarrow \lambda = \frac{\mathrm{Tr}(h)}{n}$ ∎

Def^n: Let V_1 & V_2 be two repr of a group G. Let χ_1, χ_2 be corresponding characters.

$$(\chi_1 | \chi_2) = \frac{1}{|G|} \sum_{g \in G} \chi_1(g) \overline{\chi_2(g)} \in \mathbb{C}$$

Note that $(\, | \,)$ is linear in first variable & conjugate linear in 2^{nd} var.

Prop: If V_1 & V_2 are irred then

$$(\chi_1 | \chi_2) = \begin{cases} 1 & \text{if } V_1 \cong V_2 \\ 0 & \text{o.w.} \end{cases}$$

Pf: next class

Pf of Cor 1:

Existence ✓

Let V_1 be an irred repr. of G.

$(\chi_{V_1}, \chi_V) = r_1$ $\left(\because \chi_V = r_1\chi_{V_1} + r_2\chi_{V_2} + \dots + r_m\chi_{V_m} \right)$

$\parallel$

$\left(\chi_{V_1}, \chi_{W_1^{r_1'} \oplus \dots \oplus W_{m'}^{r_{m'}'}}\right)$

V_1 is isom to one of $W_1, \dots, W_{m'}$, say W_1 and then $r_1' = r_1$.

Now use induction on dim repr. to conclude that $m' = m$ & $r_i' = r_i$ after reordering. $\boxed{}$