SURFACES - III

SHAPES AND CURVATURE

Our discussion on measurements in the previous chapter concerned the internal geometry of surfaces. It ignored the shape of these surfaces in $\mathbb{R}^n$. For instance the flat plane and the curved cylinder (cut open along a slit) are isometric to each other, yet they look different because of their shapes. We now discuss quantities that describe the shape of surfaces.

For a curve in $\mathbb{R}^n$, we determined its rate of bending in $\mathbb{R}^n$ by computing the derivative of unit tangent vectors, i.e., the rate of change of the tangent lines. Also, for a curve in $\mathbb{R}^2$, the tangent spaces have codimension 1 in $\mathbb{R}^2$ and so the amount of of bending is also captured by differentiating the unit normals.

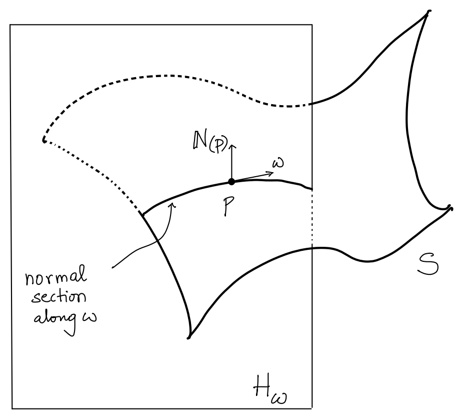
For a surface S we have the following approaches at our disposal and we use them both:

(A) At any $p \in S$, we look at various curves in S passing through p and compute their curvature.

(B) Compute the rate of change of tangent planes of S. For a surface in $\mathbb{R}^3$, the tangent spaces have codimension 1, so we may differentiate the unit normals.

We restrict to surfaces in $\mathbb{R}^3$.

Let $S \subseteq \mathbb{R}^3$ be a surface and let $p \in S$. <u>Fix</u> a unit normal $\mathbb{N}(p)$ at p. For any <u>unit</u> vector $w \in T_pS$, the two dimensional subspace of $\mathbb{R}^3$ spanned by $\mathbb{N}(p)$ and w, when translated to p, forms a plane through p, say H_w.



By the implicit function theorem, $H_w \cap S$ is a regular curve near p: [If S is locally given by $f=0$ near p, then D_pf is parallel to $\mathbb{N}(p)$, while if H_w is given by a linear equation $g=0$, then D_pg is orthogonal to $\mathbb{N}(p)$; Thus f, g are a partial sequence of coordinate functions at p.]

We call the set $S \cap H_\omega$ the <u>normal section of S at p along ω</u>. It remains unchanged upon replacing $N(p)$ by $-N(p)$ or ω by $-\omega$. In what follows we only deal with an open subset around p of the normal section where it is regular and we continue to call it the normal section.

For $p \in S$ and $\omega \in T_pS$ as above, the normal section $H_\omega \cap S$ is a plane curve where we may identify H_ω with $\mathbb{R}^2$ via the ordered orthonormal basis $\omega, N(p)$. As the normal section is regular, it admits a unique unit-speed parametrisation $\gamma(t)$ such that $\gamma(0) = p$, $\dot{\gamma}(0) = \omega$. We set $\kappa_n(\omega)$ to be its (normal) signed curvature and we call it the <u>normal curvature of S at p along ω</u>. Changing $N(p)$ by $-N(p)$ changes the sign of $\kappa_n(\omega)$, but $\kappa_n(-\omega) = \kappa_n(\omega)$.

<u>Example:</u> Let S be the sphere $X^2+Y^2+Z^2=R^2$. Let $p=(0,0,R)$. Pick $N(p)=(0,0,1)$. Let $\omega = (\cos\varphi, \sin\varphi, 0) \in T_pS$. The plane H_ω is given by $\sin\varphi \cdot X - \cos\varphi \cdot Y = 0$. Set $X = W\cos\varphi$, $Y = W\sin\varphi$, so that

$W = \cos\varphi \cdot X + \sin\varphi \cdot Y$. We use W, Z as coordinate functions on H_ω and the equation of $S \cap H_\omega$ is $\underbrace{W^2}_{X^2+Y^2} + Z^2 = R^2$.

Note that $p = (0,0,R)$ has coordinates $(0,R)$ in H_ω. We parametrise the circle as $\theta \mapsto (R\sin\frac{\theta}{R}, R\cos\frac{\theta}{R})$ in W-Z coordinates, so that its velocity at the point $(0,R)$ is $(1,0)$ in W-Z coordinates or $\omega = (\cos\varphi, \sin\varphi, 0)$ in X-Y-Z coordinates. The signed curvature is $-1/R$.