Theorem. Suppose X_i are $n \times n$ symmetric matrices with rank k_i, $i = 1, 2, \ldots, p$. Let $X = \sum_{i=1}^p X_i$ have rank k. (It is symmetric.) Then, of the conditions
(a) X_i idempotent for all i
(b) $X_i X_j = 0$, $i \neq j$
(c) X idempotent
(d) $\sum_{i=1}^p k_i = k$,
it is true that
I. any two of (a), (b), and (c) imply all of (a), (b), (c) and (d)
II. (c) and (d) imply (a) and (b)
III. (c) and $\{X_1, \ldots, X_{p-1}$ idempotent, X_p p.s.d.$\}$ imply that X_p idempotent and hence (a), and therefore (b) and (d).

Proof. I (i): Show (a) and (c) imply (b) and (d). For this, note, given (c), $I - X$ is idempotent and hence p.s.d. Now, given (a), $X - X_i - X_j = \sum_{r \neq i,j} X_r$ is p.s.d, being the sum of p.s.d matrices. Therefore, $(I - X) + (X - X_i - X_j) = I - X_i - X_j$ is p.s.d., hence $X_i X_j = 0$ from Loynes' Lemma. i.e., (b). Also, given (c), $\text{Rank}(X) = \text{tr}(X) = \text{tr}(\sum X_i) = \sum \text{tr}(X_i) = \sum k_i$, if (a) is also given. i.e., (d).

(ii): Show (b) and (c) imply (a) and (d). Let λ be an eigen value of X_1 and u be the corresponding eigen vector. Then $X_1 u = \lambda u$. Either $\lambda = 0$, or, if $\lambda \neq 0$, $u = X_1 \frac{1}{\lambda} u$. Therefore, for $i \neq 1$, $X_i u = X_i X_1 \frac{1}{\lambda} u = 0$ given (b). Therefore, given (b), $Xu = X_1 u = \lambda u$, and so λ is an eigen value of X. But given (c), X is idempotent, and hence $\lambda = 0$ or 1. Therefore eigen values of X_1 are 0 or 1, or X_1 is idempotent. Similarly for the other X_i's. i.e., (a).

(iii): (a) and (b) together imply (c). (Note that then they imply (d) also, since (a) and (c) give (d).) Given (b) and (a), $X^2 = (\sum X_i)^2 = \sum X_i^2 = \sum X_i = X$, which is (c).

II. Show (c) and (d) imply (a) and (b). Given (c), $I - X$ is idempotent and hence has rank $n - k$. Therefore rank of $X - I$ is also $n - k$. i.e., $X - I$ has $n - k$ linearly independent rows. i.e.,

$(X - I)x = 0$ has $n - k$ linearly independent equations. Further,
$X_2 x = 0$ has k_2 linearly independent equations,
$\vdots$
$X_p x = 0$ has k_p linearly independent equations.

Therefore the maximum number of linearly independent equations in

$$\begin{pmatrix} X - I \\ X_2 \\ \vdots \\ X_p \end{pmatrix} x = 0 \qquad \text{is } n - k + k_2 + \ldots + k_p = n - k_1.$$

i.e., the dimension of the solution space is at least $n-(n-k_1) = k_1$. However, this space is exactly $X_1 x = x$ because the above equations reduce to that. Thus $X_1 x = 1x$ has at least k_1 linearly independent solutions, or 1 is an eigen value of X_1 with multiplicity at least k_1. But since the rank of X_1 is k_1, multiplicity must be exactly k_1. Also, the other eigen values must be 0. Therefore X_1 is idempotent. Similar argument for the other X_i's. So, (a). Now combine it with (c) to get (b).

III. Given (c), X is idempotent, so p.s.d. Therefore, $I - X$ is idempotent and hence p.s.d. If $X_1, \ldots, X_{p-1}$ are idempotent, hence p.s.d., and X_p is also p.s.d., then $\sum_{r \neq i,j} X_r = X - X_i - X_j$ is p.s.d., so $(I - X) + (X - X_i - X_j) = I - X_i - X_j$ is p.s.d. Then $X_i X_j = 0$ from Loynes', giving (b). Now (b) and (c) give (a) and (d).

The above theorem in linear algebra translates into a powerful result called Fisher-Cochran theorem on the question of: when are quadratic forms independent χ^2?

Theorem. Suppose $Y \sim N_n(0, I_n)$, A_i, $i = 1, \ldots, p$ are symmetric $n \times n$ matrices of rank k_i, and $A = \sum_{i=1}^p A_i$ is symmteric with rank k. Then
(i) $Y' A_i Y \sim \chi^2_{k_i}$, (ii) $Y' A_i Y$ are pairwise independent, and (iii) $Y' A Y \sim \chi^2_k$ iff
I. any two of (a) A_i are idempotent for all i, (b) $A_i A_j = 0$, $i \neq j$, (c) A is idempotent, are true, or
II. (c) is true and (d) $k = \sum_i k_i$, or
III. (c) is true and
(e) $A_1, \ldots, A_{p-1}$ are idempotent and A_p is p.s.d. is true.

Proof. Follows from the previous theorem.