APPENDIX A: POLAR COORDINATES

Theorem: Let $\gamma(t) = (\gamma_1(t), \gamma_2(t)) : \mathbb{R} \to \mathbb{R}^2$ be a (periodic) simple closed curve. Then there exist smooth functions $\theta(t)$ and $r(t)$ on $\mathbb{R}$ such that

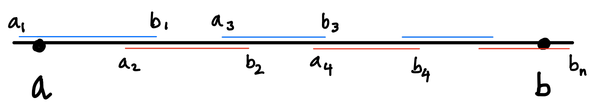
$$\gamma_1(t) = r(t)\cos(\theta(t)), \qquad \gamma_2(t) = r(t)\sin(\theta(t)).$$

Sketch Proof: Let C be the trace of the curve and let $I = [a,b]$ be an interval consisting of one period for γ, i.e., $\gamma|_{[a,b)}$ is one-one with $\gamma([a,b)) = C$. Set $c := b-a$.

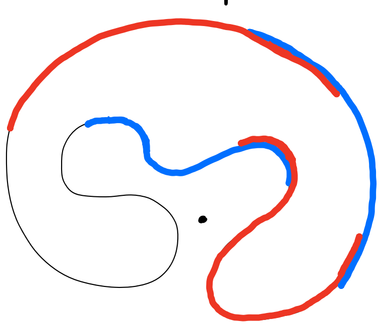
(a) We first construct $\theta(t)$.

For simplicity, first assume that $(0,0) \notin C$.

(i) Find intervals $[a_i, b_i]$ as shown:



(i.e., find $a_i, b_i \in \mathbb{R}$ such that $a_1 < a < a_2 < a_3 \cdots$, $b_1 < b_2 < \cdots b_{n-1} < b < b_n$ and $a_{i+1} < b_i < a_{i+2} < b_{i+1}$) such that for any i, with $J_i = [a_i, b_i]$, either $\gamma_1(t) \neq 0 \ \forall\, t \in J_i$ or $\gamma_2(t) \neq 0 \ \forall\, t \in J_i$.



(We may additionally arrange that $b_{n-1} < a_1 + c$ and $b_n < a_2 + c$.)

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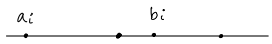
(ii) For any i, if for all $t \in J_i = [a_i, b_i]$ we have $\gamma_1(t) \neq 0$, then we set $\theta_i(t) := \tan^{-1}\left(\dfrac{\gamma_2(t)}{\gamma_1(t)}\right) \quad \forall\, t \in J_i$

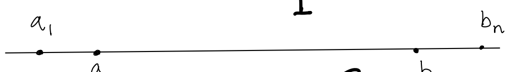
else we set $\theta_i(t) := \cot^{-1}\left(\dfrac{\gamma_1(t)}{\gamma_2(t)}\right) \quad \forall\, t \in J_i$.

(iii) On any overlap $[a_{i+1}, b_i] = J_i \cap J_{i+1}$ we must



have $\theta_{i+1}(t) - \theta_i(t)$ is a constant integer multiple of π. Inductively starting from $i=2$ onwards, add a suitable multiple of π to each θ_i, so that the modified functions $\tilde{\theta}_i(t)$ have the property that $\tilde{\theta}_{i+1} = \tilde{\theta}_i$ on any overlap $J_i \cap J_{i+1}$. (Here $\tilde{\theta}_1 := \theta$.)

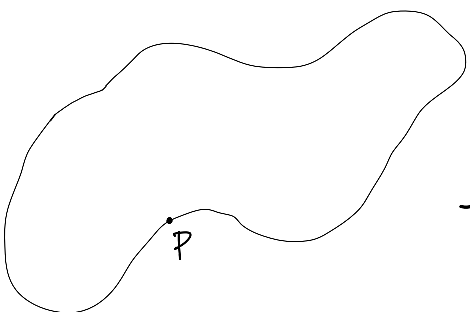
(iv) For any $t \in I = [a,b]$, set $\theta_I(t) := \tilde{\theta}_i(t)$ keeping in mind that t can belong to at most 2 intervals J_i and by construction, the values of $\tilde{\theta}_i$ agree on overlaps. Clearly $\theta_I(t)$ is a smooth function and extends to one over $I' = [a_1, b_n]$.



(v) Set $c := b-a$. For any $k \in \mathbb{Z}$, set $I_k := [a+kc, b+kc]$. Define $\theta_{I_k}(t)$ as in (iv) with θ_{I_k} also defined over $I_k' := [a_1+kc, b_n+kc]$. On any overlap $I_k' \cap I_{k+1}'$ these functions differ by an integer multiple of π. By proceeding inductively on $|k|$ we modify θ_{I_k}'s (by adding a suitable multiple of π) to obtain a smooth function $\theta(t)$ on $\mathbb{R}$.

Now suppose $p=(0,0)\in C$. We may assume that a, b are chosen such that $\gamma(a)=p=\gamma(b)$. Now we modify the steps (i) — (v) above to construct $\theta(t)$ as follows:

(i)$'$ Find sequentially overlapping intervals $[a_i, b_i]$ for $1\le i\le n$ as before such that if $J=[a_i, b_i]$ for $1<i<n$ or if $J=[a_1, a)$ or $(a, b_1]$ or $[a_n, b]$ or $(b, b_n]$, then either $\gamma_1\neq 0$ everywhere on J or $\gamma_2\neq 0$ everywhere on J.



(ii)$'$ For any $t\in J$, we set $\theta_J(t):=\tan^{-1}\left(\dfrac{\gamma_2(t)}{\gamma_1(t)}\right)$ or $\theta_J(t):=\cot^{-1}\left(\dfrac{\gamma_1(t)}{\gamma_2(t)}\right)$ accordingly. Set $\theta_1(a):=\tan^{-1}\left(\dfrac{\gamma_2'(a)}{\gamma_1'(a)}\right)$ if $\gamma_1'(a)\neq 0$, else set $\theta_1(a):=\pi/2$. Note that if $\gamma_1'(a)\neq 0$, then

$$\frac{\gamma_2'(a)}{\gamma_1'(a)}=\frac{\lim_{t\to a}\gamma_2(t)/t-a}{\lim_{t\to a}\gamma_1(t)/t-a}=\lim_{t\to a}\frac{\gamma_2(t)}{\gamma_1(t)}$$ and that

$\dfrac{\gamma_i(t)}{t-a}$ are smooth by ③ on page 57. Define $\theta_n(b)$ likewise.
$(i=1,2)$

(iii)$'$ The values $\lim_{t\to a^-} \theta_{[a_1,a)}(t)$, $\theta_1(a)$ and $\lim_{t\to a^+} \theta_{(a,b_1]}(t)$ all differ from each other by an integer multiple of π. Hence by adding a suitable multiple of π to $\theta_{[a_1,a)}(t)$ and $\theta_1(a)$, we obtain a continuous function $\tilde{\theta}_1(t)$ on $[a_1, b_1]$ which is moreover smooth at a as $\frac{r_j(t)}{t-a}$ is smooth for $j=1,2$. Now define $\tilde{\theta}_i(t)$ for $1<i<n$ as before as in (iii). Finally, define $\tilde{\theta}_n(t)$ similar to the way we have defined $\tilde{\theta}_1(t)$.

(iv)$'$ and (v)$'$: same as (iv) and (v) above.

Thus we have constructed a smooth function $\theta(t)$ on $\mathbb{R}$.

(b) Now $r(t)$ is easy to define.

By construction of $\theta(t)$, for any $t\in\mathbb{R}$, $r_1(t)\sin\theta(t) = r_2(t)\cos\theta(t)$.

We define $r(t) := \frac{r_1(t)}{\cos\theta(t)}$ or $\frac{r_2(t)}{\sin\theta(t)}$, whichever is defined. Since $\theta(t)$, $r_1(t)$ and $r_2(t)$ are smooth, so is $r(t)$.

Q.E.D.