Assignment 5

Physics III: Electricity and Magnetism
B. Math. Year 3,
September - December 2021.

Due on: December 27^{th}, 2021.

Please give arguments where necessary. If it is unclear from your answer why a particular step is being taken, full credit will not be awarded. Please feel free to discuss amongst yourselves; however, copying the assignment solutions from someone else is strictly prohibited and both persons involved will be penalized. Each one of you must submit your own answers. Total: 55 points.

1. A current I starts at $z = -\infty$ and travels along $\hat{e}_z$ until it hits a sphere (centered at the origin) of radius R. The current then spreads out uniformly and flows up from the south pole to the north pole along lines of longitude. Then it recombines at the north pole and continues along $\hat{e}_z$ to $z = +\infty$.
   (a) Find the surface current density of the current distribution. [4]
   (b) Find the magnetic field everywhere. [8]
   (c) Show that the magnetic field satisfies the boundary matching conditions everywhere on the spherical surface [3]

2. Consider a infinite cylindrical wire with a uniform cross-section of radius R. Along the axial direction a current flows with the following radial dependence:
$$\vec{J}(\vec{r}) = \begin{cases} J_0 \frac{\rho}{R} \hat{e}_z; & \rho \leq R \\ 0; & \text{otherwise} \end{cases}$$
   Find the magnetic vector potential and the magnetic field everywhere. Make sure you handle singularities and boundary conditions effectively for both quantities. [8+7]

3. A uniformly charged solid sphere with total charge Q rotates with a uniform angular velocity ω.
   (a) Find the current density inside the sphere. [4]
   (b) In full analogy with the electrostatic case, the magnetic dipole moment of a steady current density $\vec{j}(\vec{r}) \in \mathbb{V}$ is given by
$$\vec{m} = \frac{1}{2} \int_{\mathbb{V}} dV \vec{r} \times \vec{j}(\vec{r})$$

Find the magnetic dipole moment of the rotating sphere. [5]

(c) In the Coulomb gauge, carry out the multipole expansion for $\vec{A}(\vec{r})$ for $r \gg r'$ ($\vec{r}' \in \mathbb{V}'$, where $\mathbb{V}'$ is the volume where the current density has support) in terms of the current density. Find the magnetic dipole contribution to the vector potential for a localized (i.e., $\mathbb{V}'$ is finite) current distribution with dipole moment $\vec{m}$. Explain why or why not is the choice of the Coulomb gauge necessary. [6 + 2]

(d) For the rotating sphere in this problem, find the vector potential and the magnetic field for $r \gg R$, up to and including the magnetic dipole contribution. [4+4]