Let β_0 be a solution of $A\beta = c$. Then $Y - X\beta_0 = X(\beta - \beta_0) + \epsilon$ or $\tilde{Y} = X\gamma + \epsilon$ with $A\gamma = A(\beta - \beta_0) = 0$. i.e.,

$$\tilde{Y} = \theta + \epsilon, \quad \theta \in \mathcal{M}_C(X) = \Omega, \quad \text{and}$$

$$A(X'X)^{-1}X'\theta = A(X'X)^{-1}X'X(\beta - \beta_0) = A(\beta - \beta_0) = A\gamma = 0.$$

Set $A_1 = A(X'X)^{-1}X'$ and $\omega = \mathcal{N}(A_1) \cap \Omega$. Then $A_1\theta = A\gamma = 0$ and we want the projection of $\tilde{Y}$ onto ω since we want:

$$\min_{\theta \in \mathcal{M}_C(X)} ||\tilde{Y} - \theta||^2 \text{ subject to } A_1\theta = 0.$$

We need the following series of results to solve this.

Result A. If $\mathcal{N}(C)$ is the null space of C, then $\mathcal{N}(C) = \mathcal{M}^{\perp}(C')$.

Proof. If $x \in \mathcal{N}(C)$, then $Cx = 0$ so that x is orthogonal to each row of C. i.e., $x \perp \mathcal{M}(C')$. Conversely, if $x \perp \mathcal{M}(C')$, then $x'C' = (Cx)' = 0$, or $Cx = 0$, hence $x \in \mathcal{N}(C)$.

Result B. $(\Omega_1 \cap \Omega_2)^{\perp} = \Omega_1^{\perp} + \Omega_2^{\perp}$.

Proof. Let $\Omega_i = \mathcal{N}(C_i), i = 1, 2$. Then,

$$(\Omega_1 \cap \Omega_2)^{\perp} = \left(\mathcal{N}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}\right)^{\perp} = \mathcal{M}\left(C_1'|C_2'\right) = \mathcal{M}(C_1') + \mathcal{M}(C_2') = \Omega_1^{\perp} + \Omega_2^{\perp}.$$

Result C. If $\omega \subset \Omega$, then $P_\Omega P_\omega = P_\omega P_\Omega = P_\omega$.

Proof. Show that $P_\Omega P_\omega$ and $P_\omega P_\Omega$ both satisfy the defining properties of P_ω: If $x \in \omega \subset \Omega$, then $P_\Omega P_\omega x = P_\Omega x = x$; if $\xi \in \omega^{\perp}$, $P_\Omega P_\omega \xi = P_\Omega 0 = 0$. Similar is the other case.

Result D. If $\omega \subset \Omega$, then $P_\Omega - P_\omega = P_{\omega^{\perp} \cap \Omega}$.

Proof. $\Omega = \mathcal{M}_C(P_\Omega)$, so each $x \in \Omega$ can be written $x = P_\Omega y$. Consider the decomposition, $P_\Omega y = P_\omega y + (P_\Omega - P_\Omega)y$. Now $P_\omega y \in \omega \subset \Omega$, and already $P_\Omega y \in \Omega$, so $(P_\Omega - P_\Omega)y = P_\Omega y - P_\omega y \in \Omega$. Further, $P_\omega(P_\Omega - P_\omega) = P_\omega P_\Omega - P_\omega = P_\omega - P_\omega = 0$, so that $(P_\omega y)'(P_\Omega - P_\omega)y = y'P_\omega(P_\Omega - P_\omega)y = 0$. Therefore, $P_\Omega y = P_\omega y \oplus (P_\Omega - P_\omega)y$ is the orthogonal decomposition of Ω into $\omega \oplus (\omega^{\perp} \cap \Omega)$.

Result E. If A_1 is any matrix such that $\omega = \mathcal{N}(A_1) \cap \Omega$, then $\omega^{\perp} \cap \Omega = \mathcal{M}_C(P_\Omega A_1')$.

Proof. Note that

$$\omega^{\perp} \cap \Omega = (\Omega \cap \mathcal{N}(A_1))^{\perp} \cap \Omega = \left(\Omega^{\perp} \oplus \mathcal{N}^{\perp}(A_1)\right) \cap \Omega = \left(\Omega^{\perp} \oplus \mathcal{M}_C(A_1')\right) \cap \Omega.$$

Now, let $x \in \omega^\perp \cap \Omega \left(= \left(\Omega^\perp \oplus \mathcal{M}_C(A_1')\right) \cap \Omega.\right)$. Then $x \in \Omega$, so $x = P_\Omega x$. Also, $x \in \Omega^\perp \oplus \mathcal{M}_C(A_1')$, so $x = (I - P_\Omega)\alpha + A_1'\beta$. Therefore,

$$x = P_\Omega x = P_\Omega \{(I - P_\Omega)\alpha + A_1'\beta\} = P_\Omega A_1'\beta \in \mathcal{M}_C(A_1').$$

Conversely, if $x \in \mathcal{M}_C(P_\Omega A_1')$, then $x = P_\Omega A_1'\beta = P_\Omega(A_1'\beta) \in \mathcal{M}_C(P_\Omega) = \Omega$. For any $\xi \in \omega(\subset \Omega)$, we have $x'\xi = \beta' A_1 P_\Omega \xi = \beta' A_1 \xi = 0$ since $\omega = \mathcal{N}(A_1) \cap \Omega$. Therefore, $x \in \omega^\perp$.

Result F. If A_1 is a $q \times n$ matrix of rank q, then $\text{Rank}(P_\Omega A_1') = q$ iff $\mathcal{M}_C(A_1') \cap \Omega^\perp = \{0\}$.

Proof. $\text{Rank}(P_\Omega A_1') \le \text{Rank}(A_1') = \text{Rank}(A_1) = q$. Suppose $\text{Rank}(P_\Omega A_1') < q$. Let the rows of A_1 (i.e., columns of A_1') be $a_1', \ldots, a_q'$. Columns of $P_\Omega A_1'$ are linearly dependent, so $\sum_{i=1}^q c_i P_\Omega a_i = P_\Omega(\sum_{i=1}^q c_i a_i) = 0$ for some $\mathbf{c} \neq \mathbf{0}$. Then there exists a vector $\sum_{i=1}^q c_i a_i \in \mathcal{M}_C(A_1')$ ($\neq 0$ since rank of A_1 is q) such that $\sum_{i=1}^q c_i a_i \perp \Omega$. i.e., $\mathcal{M}_C(A_1') \cap \Omega^\perp \neq \{0\}$. If $\text{Rank}(P_\Omega A_1') = q = \text{Rank}(A_1')$ then $\mathcal{M}_C(A_1') = \mathcal{M}_C(P_\Omega A_1') = \omega^\perp \cap \Omega \subset \Omega$.

Now let us return to the problem of finding the projection of $\tilde{Y}$ onto $\omega = \mathcal{N}(A_1) \cap \Omega$ which achieves:

$$\min_{\theta \in \mathcal{M}_C(X)} ||\tilde{Y} - \theta||^2 \text{ subject to } A_1\theta = 0.$$

From Results A and B, $\omega^\perp \cap \Omega = (\mathcal{N}(A_1) \cap \Omega)^\perp \cap \Omega = (\mathcal{M}_C(A_1') + \Omega^\perp) \cap \Omega$ and from Result E, $\omega^\perp \cap \Omega = \mathcal{M}_C(P_\Omega A_1')$. Now note that

$$P_\Omega A_1' = (X(X'X)^{-1}X')X(X'X)^{-1}A' = X(X'X)^{-1}A' = A_1'.$$

Therefore, $\text{Rank}(P_\Omega A_1') = \text{Rank}(A_1') \le q$. However, since $\text{Rank}(P_\Omega A_1') = \text{Rank}(X(X'X)^{-1}A') \ge \text{Rank}(X'X(X'X)^{-1}A') = \text{Rank}(A') = q$, we must have $\text{Rank}(P_\Omega A_1') = q$. Therefore, from Result D,

$$\begin{aligned}
P_\Omega - P_\omega &= P_{\omega^\perp \cap \Omega} = P_{\mathcal{M}_C(P_\Omega A_1')} \\
&= P_\Omega A_1'(A_1 P_\Omega^2 A_1')^{-1}(P_\Omega A_1')' \\
&= X(X'X)^{-1}A'\left[A(X'X)^{-1}X'X(X'X)^{-1}A'\right]^{-1} A(X'X)^{-1}X' \\
&= X(X'X)^{-1}A'\left(A(X'X)^{-1}A'\right)^{-1} A(X'X)^{-1}X'.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&X\hat{\beta}_H - X\beta_0 = X\hat{\gamma}_H = P_\omega \tilde{Y} = P_\Omega \tilde{Y} - P_{\omega^\perp \cap \Omega}\tilde{Y} \\
&= P_\Omega Y - X\beta_0 - X(X'X)^{-1}A'\left(A(X'X)^{-1}A'\right)^{-1} A(X'X)^{-1}X'(Y - X\beta_0) \\
&= P_\Omega Y - X\beta_0 - X(X'X)^{-1}A'\left(A(X'X)^{-1}A'\right)^{-1} A\left((X'X)^{-1}X'Y - \beta_0\right) \\
&= P_\Omega Y - X\beta_0 - X(X'X)^{-1}A'\left(A(X'X)^{-1}A'\right)^{-1}\left(A\hat{\beta} - c\right).
\end{aligned}$$

Therefore,

$$X\hat{\beta}_H = X\hat{\beta} - X(X'X)^{-1}A'\left(A(X'X)^{-1}A'\right)^{-1}\left(A\hat{\beta} - c\right).$$

Multiplying by $(X'X)^{-1}X'$ on the left, we get,

$$\hat{\beta}_H = \hat{\beta} - (X'X)^{-1}A'\left(A(X'X)^{-1}A'\right)^{-1}\left(A\hat{\beta} - c\right).$$

This yields the minimum since $||Y - X\hat{\beta}_H||^2 = ||\tilde{Y} - X\hat{\gamma}_H||^2$.