Lecture 4

Consider $y'' + P(x)y' + Q(x)y = R(x)$ (1)

where $x \in [a,b]$ and P, Q and R are continuous functions on $[a,b]$.

<u>Definition</u>. The general solution of the above equation is a function $y = y(x, c_1, c_2)$ $x \in [a,b]$, $(c_1, c_2) \in E \subset \mathbb{R}^2$ such that for each $(c_1, c_2) \in E$, $y(x, c_1, c_2)$ solves equation (1) on $[a,b]$. By a particular solution of (1) we mean the unique solution of (1) given by the Theorem on p.12, L3 for some y_0 and y_0'.

<u>Remark</u>. By Theorem p.12, L3, given y_0 and y_0' and the general solution, we can determine a particular solution by solving the equations

$$y_0 = y(x_0, c_1, c_2)$$

$$y_0' = y'(x_0, c_1, c_2)$$

for c_1 and c_2 for some $x_0 \in [a,b]$.

Consider the homogeneous equation (2).

$$y'' + P(x)y' + Q(x)y = 0 \quad \text{---} (2)$$

<u>Proposition</u>. Let y_1 and y_2 be two linearly independent solutions of (2) and y_p be a particular solution of (1). The $y(x, c_1, c_2) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$ is the general solution of (1).

<u>Proof</u>. Let $y(x) \equiv y(x, c_1, c_2)$ and $y_g(x) = c_1 y_1(x) + c_2 y_2(x)$, c_1 and c_2 fixed. Then

$$y = y_g + y_p$$

and

$$y'' + P(x)y' + Q(x)y = y_g'' + P(x)y_g' + Q(x)y_g + y_p'' + P(x)y_p' + Q(x)y_p = R(x). \qquad \square$$

<u>Remark</u>. Given y_0, y_0' and $x_0 \in [a,b]$ we determine c_1 and c_2 by solving

$$c_1 y_1(x_0) + c_2 y_2(x_0) = y_0 - y_p(x_0)$$

$$c_1 y_1'(x_0) + c_2 y_2'(x_0) = y_0' - y_p'(x_0).$$

Finding a Particular Solution. (3).

We first consider some special cases with $P(x) \equiv p$ and $Q(x) \equiv q$. So we are looking at

$$y'' + py + q = R(x) \quad \text{--- (3)}$$

Case (1). Suppose $R(x) = e^{ax}$. Then we look for a solution of (3) of the form $y(x) = Ae^{ax}$. We can determ-ine the constant A by substituting in (3):

$$A(a^2 + pa + q)e^{ax} = e^{ax}$$

Thus if $a^2 + pa + q \neq 0$ we get

$$A = \frac{1}{a^2 + pa + q}$$

Hence for $R(x) = e^{ax}$, we get the particular solution $y(x) = \dfrac{e^{ax}}{a^2+pa+q}$.

Exercise. Verify that there exists A such that $y(x) = Axe^{ax}$ is a is a particular solution of (3)

when $a^2 + pa + q = 0$, $a \neq -\frac{p}{2}$.

In the latter case show that there exists A such that $y(x) = Ax^2e^{ax}$ is a particular solution.

<u>Case 2.</u> when $R(x) = \sin bx$, $b \neq 0$

Then we can take $y_p(x) = A\sin bx + B\cos bx$. By equating $y_p'' + py_p' + qy_p$ to $\sin bx$ we can determine A and B by equating coefficients of $\sin bx$ and $\cos bx$ on either side, provided $y_p'' + py_p' + qy_p \neq 0$. then when $y_p'' + py_p' + qy_p = 0$ the method breaks down and we have to consider other possible solutions like $y_p = x(A\sin bx + B\cos bx)$.

<u>Case 3.</u> $R(x) = a_0 + a_1x + \cdots + a_nx^n$.

Consider $y_p = A_0 + A_1x + \cdots + A_nx^n$, for $q \neq 0$. Then if $y_p'' + py_p' + qy_p \neq 0$

Then we can equate coefficients (5)
of x^k on either side of the equation
$y_p'' + p y_p' + q y_p = a_0 + a_1 x + \cdots + a_n x^n$
and get $n+1$ equations for the $n+1$
unknowns $A_0, \ldots, A_{n+1}$.

Method of Variation of Parameters

The general solution of the homogeneous equation (2) is $y = c_1 y_1 + c_2 y_2$
where y_1 and y_2 are linearly independent solutions. To get particular solns.
we have to specify $y(x_0)$ and $y'(x_0)$
for some $x_0 \in [a,b]$. We get c_1 and c_2
depending on $x_0 \in [a,b]$ ie. $c_i = c_i(x_0)$
$i = 1, 2$. We now determine a particular solution of (1) by taking

$$y(x) = v_1(x)\, y_1(x) + v_2(x)\, y_2(x)$$

The idea is to get 2 equations involving the derivatives $v_1'(x)$ and $v_2'(x)$ and
Solve them so that y solves equation (1).

Integrating the resulting expressions (6).
for v_1' and v_2' we can get v_1 and v_2.

We have $\quad y' = v_1' y_1 + v_2' y_2 + v_1 y_1' + v_2 y_2'$

We set $\quad v_1' y_1 + v_2' y_2 = 0.$

Hence $\quad y'' = v_1 y_1'' + v_1' y_1' + v_2' y_2' + v_2 y_2''$

Hence if $\quad y'' + P(x) y' + Q(x) y = R(x)$

we should have $\quad v_1' y_1' + v_2' y_2' = R(x).$

Thus we get the pair of equations

$$v_1' y_1 + v_2' y_2 = 0$$
$$v_1' y_1' + v_2' y_2' = R(x)$$

Solving these we get

$$v_1' = \frac{-y_2 R(x)}{W(y_1, y_2)} \; ; \quad v_2' = \frac{-y_1 R(x)}{W(y_1, y_2)}$$

Hence
$$v_1(x) = v_1(x_{01}) + \int_{x_{01}}^{x} \frac{(-y_2(t) R(t))}{W(y_1, y_2)(t)} \, dt$$

$$v_2(x) = v_2(x_{02}) + \int_{x_{02}}^{x} \frac{(-y_1(t) R(t))}{W(y_1, y_2)(t)} \, dt$$

where $v_1(x_{01})$ and $v_2(x_{02})$ are arbitrarily specified for $x_{0i} \in [a, b]$ $i = 1, 2$.

Example. $y'' + y = \operatorname{cosec} x$.

So $R(x) = \operatorname{Cosec} x$, $P(x) \equiv 0$, $Q(x) \equiv 1$.

Two linearly independent solutions are given by $\sin x$ and $\cos x$. With $y_1 = \sin x$ and $y_2 = \cos x$ we get

Then

$$W(y_1, y_2)(x) = -1.$$

$$V_1(x) = \int_{\pi/2}^{x} \frac{-\cos t \operatorname{Cosec} t}{-1}\, dt$$

$$= \cancel{\log \cos x} = \log \sin x$$

where $x \in [a,b]$ and $\frac{\pi}{2} \in [a,b]$.

Similarly

$$V_2(x) = -\left(x - \frac{\pi}{2}\right).$$

Thus a particular solution of $y'' + y = \operatorname{Cosec} x$ is given by

$$y(x) = \sin x \log \cos x - x \cos x.$$