HN (Alg version) k a field

$k[x_1, \dots, x_n] = k(x_1, \dots, x_n) \Rightarrow x_1, \dots, x_n$ is alg over k.

k-alg

poly ring

Cor: Let k be an alg closed field. Then $m \subseteq k[z_1, \dots, z_n]$ is a maximal ideal iff $\exists a_1, \dots, a_n \in k$ s.t. $m = (z_1 - a_1, \dots, z_n - a_n)$

Thm: k alg closed field. Let $I \subseteq k[x_1, \dots, x_n]$ be an ideal then $I(Z(I)) = \sqrt{I}$.

Prop: Let k be a field and $f : A \to B$ be a k-alg homo between f.g. k-algebras. Then $f^{-1}(m)$ is a maximal ideal of A for m a maximal ideal of B.

Pf:

$$A \xrightarrow{f} B \xrightarrow{q} B/m = l$$ note that $k \subseteq l$

l is a field, $q \circ f(k) = k$

$\exists x_1, \dots, x_n \in A$ s.t. $A = k[x_1, \dots, x_n]$ as k-alg.

Let $n = \ker(q \circ f)$
$= f^{-1}(m)$

By 1st isom thm $q \circ f$ induces an inj ring homo

$B = k[y_1, \dots, y_m]$ for some $y_1, \dots, y_m \in B$

$B/m = l$ is a field & let $\bar{y}_i = q(y_i)$

$\Rightarrow k[\bar{y}_1, \dots, \bar{y}_m]$ is a field

$\xRightarrow{\text{HN alg ver}}$ $\bar{y}_1, \dots, \bar{y}_m$ are alg over k. $\Rightarrow l/k$ is a finite field ext^n

But $k \subseteq A/n \subseteq B/m = l$

$\overset{l/k \text{ is finite field ext}^n}{\Longrightarrow}$ A/n is a field.

$\Longrightarrow$ $n = f^{-1}(m)$ is a maximal ideal of A. ∎

Example: ① $\mathbb{Z} \overset{i}{\hookrightarrow} \mathbb{Q}$

$i^{-1}(0) = (0)$ is not a maximal ideal of $\mathbb{Z}$

② $\underset{A}{k[x,y]} \overset{i}{\hookrightarrow} \underset{B}{k[x,y]_{(x)}}$

$i^{-1}(xB) = xB \cap A = xA$

which is not a maximal ideal.

k alg field	k arb.	Arbitrary Ring R
Affine space $\mathbb{A}^n_k = k^n$	$\mathrm{mspec}(k[x_1,\dots,x_n])$	$\mathrm{Spec}(R)$ is set prime ideals of
Affine var $Z(P)$ P prime ideal of $k[x_1,\dots,x_n]$	$\mathrm{mspec}(\mathcal{O}_X) \leftrightarrow Z(P)$ where $\mathcal{O}_X = k[x_1,\dots,x_n]/P$	$I \subseteq R$ ideal $Z(I) = \{P \in \mathrm{Spec}(R) \mid P \supseteq I\}$
		$X \subseteq \mathrm{Spec}(R)$ $\mathcal{I}(X) = \bigcap_{P \in X} P$
$\sqrt{I} = \mathcal{I}(Z(I))$	$\sqrt{I} = \mathcal{I}(Z(I))$	$\mathcal{I}(Z(I)) = \sqrt{I}$

Def^n: A map $f : X \to Y$ between affine varieties over k is said to be a morphism of affine varieties if there exist a k-alg homo $f^{\#}: \mathcal{O}_Y \to \mathcal{O}_X$ s.t. $f(m) = f^{\#^{-1}}(m)$ where m is a maximal ideal of $\mathcal{O}_X$. Note $X = \text{mspec}(\mathcal{O}_X) = \{m \mid m \text{ maximal ideals of } \mathcal{O}_X\}$.

By Prop $f^{\#^{-1}}(m)$ is a max ideal of $\mathcal{O}_Y$ hence $f(m) \in Y$.

Prop: $f : X \to Y$ is a morphism of affine varieties over alg closed field k iff it is defined by polynomials. i.e. $\mathbb{A}^n \supseteq X \xrightarrow{f} Y \subseteq \mathbb{A}^m$

Let $Y = Z(I)$ where $I \subseteq k[y_1, \dots, y_m]$ & $X = Z(J)$

$J \subseteq k[x_1, \dots, x_n]$ then $\exists$ polynomials $F_1, \dots, F_m \in k[x_1, \dots, x_n]$

s.t. $f(a_1, \dots, a_n) = (F_1(a_1, \dots, a_n), \dots, F_m(a_1, \dots, a_n)) \quad \forall (a_1, \dots, a_n) \in X$

Pf: ($\Rightarrow$:) Let $f^\#: \mathcal{O}_Y \longrightarrow \mathcal{O}_X$ be the ring homo. Note $\mathcal{O}_Y = \dfrac{k[y_1,\dots,y_m]}{I}$ &

$\mathcal{O}_X = \dfrac{k[x_1,\dots,x_n]}{J}$

$$\begin{tikzcd}
 & & k[x_1,\dots,x_n] \ni F_i \arrow[d, "q_X"] \arrow[dd, bend left] \\
k[y_1,\dots,y_m] \ni y_i \arrow[urr, dashed, red] \arrow[r, "q_Y"] & \dfrac{k[y_1,\dots,y_m]}{I} \arrow[r, "f^\#"] & \dfrac{k[x_1,\dots,x_n]}{J} = \mathcal{O}_X \arrow[d] \\
 & & \dfrac{k[x_1,\dots,x_n]}{(x_1-a_1,\dots,x_n-a_n)} \cong k
\end{tikzcd}$$

Let $F_i \in q_X^{-1}\left(f^\# \circ q_Y(y_i)\right) \quad 1 \le i \le m$.

then for $(a_1,\dots,a_n) \in X$

$f(a_1,\dots,a_n) = (b_1,\dots,b_m) \in Y \Rightarrow$

$f^{\#-1}\left((x_1-a_1,\dots,x_n-a_n)\right) = (y_1-b_1,\dots,y_m-b_m)$

$f(a_1,\dots,a_n) = (F_1(a_1,\dots,a_n),\dots,F_m(a_1,\dots,a_n))$

WTS: $F_i(a_1,\dots,a_n) = b_i \quad 1 \le i \le m$

$$\begin{aligned}
F_i(a_1,\dots,a_n) &= F_i + (x_1-a_1,\dots,x_n-a_n) \\
&= f^\# \circ q_Y(y_i) + (x_1-a_1,\dots,x_n-a_n) \\
&= y_i + q_Y^{-1} f^{\#-1}(x_1-a_1,\dots,x_n-a_n) \\
&= y_i + (y_1-b_1,\dots,y_m-b_m) \\
&= b_i
\end{aligned}$$

Conversely

Given $F_1,\dots,F_m \in k[x_1,\dots,x_n]$ s.t.

$$f(a_1,\dots,a_n) = (F_1(a_1,\dots,a_n),\dots,F_n(a_1,\dots,a_n)) \in Y$$

$$\forall (a_1,\dots,a_n)\in X$$

Define

$$k[y_1,\dots,y_m] \xrightarrow{\tilde{f}} k[x_1,\dots,x_n] \xrightarrow{q} \mathcal{O}_X = \frac{k[x_1,\dots,x_n]}{J}$$

$$\tilde{f}(y_i) = F_i \qquad 1\le i\le m$$

$\tilde{f}$ is a ring homo.

For $s\in I \subseteq k[y_1,\dots,y_m]$, $s(b_1,\dots,b_m)=0 \ \forall (b_1,\dots,b_m)\in Y$

$$\implies s(F_1(a_1,\dots,a_n), F_2(a_1,\dots,a_n),\dots,F_m(a_1,\dots,a_n))=0$$

$\tilde{f}(s) = s(F_1,\dots,F_m)\in k[x_1,\dots,x_n]$ $\quad \forall (a_1,\dots,a_n)\in X$

$f(a_1,\dots,a_n) = (F_1(\underset{\sim}{a}),\dots,F_m(\underset{\sim}{a}))\in Y$

$$\implies \tilde{f}(s)\in \mathcal{J}(X) = J$$

$$\implies q\circ\tilde{f}(s)=0 \implies I\subseteq \ker(q\circ\tilde{f})$$

$$\overset{1^{st}\text{ isom.}}{\implies} \exists\, f^{\#}: \mathcal{O}_Y \to \mathcal{O}_X$$

$$\bar{y}_i \longmapsto F_i$$

s.t.

```tikzcd
k[\underset{\sim}{y}] \arrow[r, "\tilde{f}"] \arrow[d, "q_y"'] & k[\underset{\sim}{x}] \arrow[d, "q"] \arrow[dd, bend left, "\tilde{e}_a"] \\
\frac{k[\underset{\sim}{y}]}{I} \arrow[r, dashed, "f^{\#}"] & k[\underset{\sim}{x}]/J \arrow[d, dotted, "e_a"'] \\
 & k[\underset{\sim}{x}]/\mathfrak{m}
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Let $\mathfrak{m} \subseteq \mathcal{O}_X$ be a max ideal then

$\mathfrak{m} = (x_1-a_1,\dots,x_n-a_n)$ for some $(a_1,\dots,a_n)\in X$

Let $f(\underset{\sim}{a}) = (F_1(\underset{\sim}{a}),\dots,F_m(\underset{\sim}{a})) = (b_1,\dots,b_m)\in Y$

WTS: $f^{\#-1}(\mathfrak{m}) = (y_1-b_1,\dots,y_m-b_m) \subseteq \mathcal{O}_Y$

$$q_y^{-1}\circ f^{\#-1}(\mathfrak{m}) = \tilde{f}^{-1}\circ q^{-1}(\mathfrak{m})$$

So enough to show $y_i-b_i \in \ker(e_a\circ q\circ\tilde{f})$

$$e_a\circ q\circ\tilde{f}(y_i-b_i) = e_a\circ q(F_i-b_i) = \tilde{e}_a(F_i-b_i)$$

$$= F_i(\underset{\sim}{a}) - b_i = 0$$

$$\forall\, 1\le i\le m$$

∎