Lecture 12.

Power Series Solutions (Contd.)

Consider the linear second order equation $y'' + P(x)y' + Q(x)y = 0$ — (1)

A point $x_0 \in \mathbb{R}$ is said to be an <u>ordinary</u> point of (1) iff

$$\left.\begin{aligned} P(x) &= \sum_{n=0}^{\infty} p_n x^n \quad \text{and} \\ Q(x) &= \sum_{n=0}^{\infty} q_n x^n \end{aligned}\right\} \quad (2)$$

for $x \in N(x_0)$, some open neighbourhood of x_0.

A point $x_0 \in \mathbb{R}$ is said to be a singular point of (1) if either $P(x)$ or $Q(x)$ fails to be analytic at x_0.

<u>Remark</u>. For $f : \mathbb{R} \to \mathbb{R}$ is said to be analytic at x_0 iff for $x \in N(x_0)$,

$$f(x) = \sum_{n=0}^{\infty} f_n x^n$$

in some neighbourhood $N(x_0)$ of x_0.

A singular point x_0 of eqn. (1) is said to be a regular (singular) point iff $(x-x_0)P(x)$ and $(x-x_0)^2 Q(x)$ are analytic at x_0 i.e.

$$(x-x_0)P(x) = \sum_0^\infty p_n' x^n$$

and $$(x-x_0)^2 Q(x) = \sum_0^\infty q_n' x^n.$$

Recall that if $f(x)$ is analytic at x_0 then we can write for $x \in N(x_0)$

$$f(x) = \sum_{n=0}^\infty f_n x^n$$
$$= \sum_{n=0}^\infty f_n' (x-x_0)^n$$

Hence we can write

$$(x-x_0)P(x) = \sum_0^\infty p_n'' (x-x_0)^n$$

$$\Rightarrow \quad P(x) = \frac{p_0''}{(x-x_0)} + p_1'' + p_2''(x-x_0) + \dots$$

Similarly $$Q(x) = \frac{q_0''}{(x-x_0)^2} + \frac{q_1''}{(x-x_0)} + q_2''$$
$$+ q_3''(x-x_0) + \dots$$

Example 1. $y'' + y = 0.$

Here $P(x) = 0$ and $Q(x) = 1$. There are no singular points for this equation.

Exercise In the above example let $y = \sum_{n=0}^{\infty} a_n x^n$. Show that the general solution for that equation

$$y(x) = a_0 \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \right) + a_1 \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \right)$$

$$= a_0 \cos x + a_1 \sin x.$$

Example 2. (Legendre's equation).

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0.$$

Here $P(x) = \dfrac{-2x}{(1-x^2)}$ and $Q(x) = \dfrac{p(p+1)}{(1-x^2)}$.

The origin is an ordinary point. The point $x = 1$ is a singular point.

Because $(x-1)P(x) = \dfrac{2x}{x+1}$ and

$$Q(x)(x-1)^2 = -\frac{(x-1)p(p+1)}{x+1} \qquad (4)$$

Note that $\frac{1}{x+1} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}(x-1)^n$

for $|x-1| < 2$.

<u>Example 3</u>. Consider the Bessel's equation of order p, $p \geq 0$

$$x^2y'' + xy' + (x^2-p^2)y = 0.$$

Here $P(x) = \frac{1}{x}$ and $Q(x) = \frac{x^2-p^2}{x^2}$.

Hence the origin is a singular point.

Since $xP(x) = 1$ and $x^2Q(x) = x^2-p^2$ are both analytic at $x=0$, the origin is a regular singular point.

[Now consider the case when x_0 is an ordinary point of eqn. (1) ie. $P(x)$ and $Q(x)$ are analytic at x_0. We then have the following existence and uniqueness theorem for eqn. (1).]

Example 4. (Gauss's Hypergeometric equation)

$$x(1-x)y'' + (c-(a+b+1)x)y' - aby = 0$$

where a, b and c are constants. We have

$$P(x) = \frac{c-(a+b+1)x}{x(1-x)} \; ; \; Q(x) = \frac{-ab}{x(1-x)}.$$

Hence $x = 0$ and $x = 1$ are singular points of this equation.

$$xP(x) = \frac{c-(a+b+1)x}{(1-x)}$$

$$= (c-(a+b+1)x)\left(\sum_{n=0}^{\infty} x^n\right)$$

$$= \sum_{n=0}^{\infty} \alpha_n x^n$$

Similarly $$x^2 Q(x) = \frac{-abx}{(1-x)}$$

$$= (-abx)\sum_{n=0}^{\infty} x^n$$

$$= \sum_{n=1}^{\infty} (-ab)x^n$$

Hence $x = 0$ is a regular singular point.

Similarly $x=1$ is also a regular (b). singular point :

$$(x-1)P(x) = \frac{(a+b+1)x-c}{x} = \frac{(a+b+1)x-c}{1+(x-1)}$$

$$= ((a+b+1)x-c)\sum_{n=0}^{\infty}(-1)^n(x-1)^n$$

$$= \sum_{n=0}^{\infty}\alpha_n(x-1)^n$$

is valid in $|x-1|<1$. Similarly

$$(x-1)^2Q(x) = \frac{ab(x-1)}{x} = \frac{ab(x-1)}{1+(x-1)}$$

$$= \sum_{n=0}^{\infty}\beta_n(x-1)^n$$

is valid in $|x-1|<1$. Note that α_n is determined in terms of a, b and c and β_n is determined in terms of a and b. The solutions of the hypergeometric differential equation is given by the hypergeometric series :

<u>If</u> c is <u>not</u> zero or a negative intege

$$y(x) \equiv F(a, b, c, x)$$
$$:= 1 + \sum_{n=1}^{\infty} \frac{a(a+1)\cdots(a+n-1)\, b(b+1)\cdots(b+n-1)}{n!\, c(c+1)\cdots(c+n-1)} x^n$$
$$= \sum_{n=0}^{\infty} a_n x^n .$$

Then
$$\frac{|a_{n+1}|\, |x|^{n+1}}{|a_n|\, |x|^n} = \frac{|a+n|\, |b+n|}{(n+1)\, |c+n|} |x|$$
$$\to |x| \text{ as } n \to \infty .$$

Hence by the ratio test the series $F(a, b, c, x)$ converges for $|x| < 1$.
Note that when $a = 1$ and $b = c$
$$F(1, b, b, x) = 1 + \sum_{n=1}^{\infty} x^n = \frac{1}{1-x} .$$

Thus, it generalises the geometric series. Note also that when either a or b is zero or a negative integer then the series terminates after a finite stage and hence $F(a, b, c, x)$ is a polynomial.