Lecture 6

We want to solve the ordinary differential equation

$$(0) \quad \text{———} \quad \frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0.$$

Here $x_0 \in [a,b]$ and $y : [a,b] \to \mathbb{R}$ so that $y'(x) = f(x, y(x))$. We consider the following equation:

$$(1) \quad \text{———} \quad y(x) = y_0(\cancel{x_0}) + \int_{x_0}^{x} f(t, y(t))\, dt.$$

The above equation is an <u>integral equation</u>. Suppose that the given function $f(\cdot,\cdot)$ is defined on a rectangle $[a,b] \times [c,d] \equiv R$ with $(x_0, y_0) \in R$. We assume that f is continuous on R. Hence if $y : [a,b] \to [c,d]$ is any continuous function, $t \to f(t, y(t))$ is a a continuous function on $[a, b]$ and hence the integral

$$\int_{x_0}^{x} f(t, y(t))\, dt$$

is well defined as a Rieman integral and moreover is a continuous function of the upper limit i.e. x. Hence the RHS of (1) is a continuous function of $x \in [a, b]$. We now describe the 'method of successive approximation'. This involves finding successive approximations to the solution $y(x)$ by <u>iteration</u>. Thus we define inductively

$$(2) \quad y_n(x) := y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt$$

with

$$y_1(t) := y_0 + \int_{x_0}^{x} f(t, y_0)\, dt$$

The idea is to show that under suitable assumptions on f the limit

$y(x) = \lim_{n \to \infty} y_n(x)$ exists and solves equation (1). Note that any continuous function $y(x)$ that solves equation (1) is automatically differentiable, by the fundamental theorem of calculus and satisfies equation (0). Conversely any solution of (0) is also a solution of (1):

$$y(x) - y(x_0) = \int_{x_0}^{x} \frac{dy}{dt}(t)\,dt = \int_{x_0}^{x} f(t, y(t))\,dt.$$

Thus (0) and (1) are equivalent. Thus it is sufficient to solve (1). Notice that the iteration scheme (2) defines a sequence of functions $\{y_n(\cdot)\}_{n \geq 1}$ given $x_0 \in [a,b]$ and the function $f(\cdot,\cdot)$. Moreover each $y_n(\cdot)$ is a continuously differentiable function on $[a,b]$. We now consider a couple of examples of this iteration scheme

Example 1. We consider the (simplest !) first order ODE $y' = ay$, with $a \in \mathbb{R}$ and $y(0) = 1$. Here $f(x,y) = ay$. Hence by our iteration scheme (2)

$$y_1(x) = 1 + \int_0^x f(x, y_0)\, dt$$

$$= 1 + \int_0^x a y_0 \, dt$$

$$= 1 + ax$$

$$y_2(x) = 1 + \int_0^x f(t, y_1(t))\, dt$$

$$= 1 + \int_0^x a\, y_1(t)\, dt$$

$$= 1 + \int_0^x a(1 + at)\, dt$$

$$= 1 + ax + \frac{a^2 t^2}{2}$$

It is easy to verify that if

$$y_{n-1}(x) = 1 + ax + \cdots + \frac{a^{n-1} x^{n-1}}{(n-1)!}$$

then $\quad y_n(x) = 1 + ax + \cdots + \frac{(ax)^n}{n!}$.

Thus $\quad y_n(x) \longrightarrow e^{ax}$ which we know is the solution of eqn. (0).

<u>Remark</u>. $\quad y(x) = e^{ax}$ is the <u>unique</u> solution of (0) with $y_0 = 1$, $x_0 = 0$.

If there are two solutions $y_1(x)$ and $y_2(x)$ then let $\bar{y}(x) := y_1(x) - y_2(x)$.

Then $\bar{y}$ solves $\bar{y}' = a\bar{y}$ and $\bar{y}(0) = 0$.

Thus $\bar{y}$ satisfies

$$\bar{y}(x) = \int_0^x a\,\bar{y}(t)\,dt .$$

Then by <u>iteration</u> it is easy to see that

$$|\bar{y}(x)| \leq \int_0^x |a|\,|\bar{y}(s)|\,ds$$

$$\leq C.\ \frac{|a|^n |x|^n}{n!}$$

for every $n \geq 1$ for a suitable constant C. This implies $\bar{y}(x) = 0$ or $y_1(x) = y_2(x)$.