Analysis IV – Compact metric space

Let (X, d) be a metric space.
For any $r > 0$, $x \in X$
$B_r(x) = \{ y \in X \mid d(x,y) < r \}$

Open sets : $U \subseteq X$ is called open if for each $x \in U$ $\exists r > 0$ such that
$$B_r(x) \subseteq U.$$

Closed sets : $F \subseteq X$ is called closed if F^c is open.

Eg $B_r(x)$ are open, $\{x\}$ are closed.

A subset $F \subseteq X$ is called compact if every open cover of F has a finite subcover.

F is compact

every sequence in F has a convergent subsequence in F (limit is also in F)

Any collection $\{F_\alpha\}$ closed sets with fip (i.e., $\bigcap_{i=1}^{n} F_{\alpha_i} \neq \emptyset$) has nonempty intersection
$$\bigcap F_\alpha \neq \emptyset$$

Any compact set is closed

X - compact, $F \overset{cl}{\subseteq} X \Rightarrow F$ is compact

$X = \mathbb{R}^d$, $F \subseteq X$ is compact $\Leftrightarrow F$ is closed and bounded.

Need not be true in general.

A metric space (X, d) is called ~~separable~~ separable if X contains a countable dense set.

Recall $A \subseteq X$ is called dense if $\bar{A} = X$.

$X = \mathbb{R}$, $A = \mathbb{Q}$, $\bar{\mathbb{Q}} = \mathbb{R}$

<u>Proposition</u>: Any compact metric space is separable.

<u>Proof</u>: Let X be a compact metric space.

Let $n > 0$. Then $\{B_{\frac{1}{n}}(x)\}_{x \in X}$ ~~its~~ is an open cover X.

X is compact $\Rightarrow \exists\, x_{n,1}, x_{n,2}, \dots, x_{n,k_n}$ in X such that

$$X \subseteq \bigcup_{i=1}^{n} B_{\frac{1}{n}}(x_{n,k_i}).$$

$F_n = \{x_{n,1}, x_{n,2}, \ldots, x_{n,k_n}\}$

$F = \bigcup F_n$ is a countable set

Let U open set in X.

Then $U \subseteq X$ $\exists\, y \in U$ and $r > 0$

such that $B_r(y) \subseteq U$

Choose $n \geq 1$ such that $\frac{1}{n} < r$

$$y \in U \subseteq X = \bigcup_{x \in F_n} B_{\frac{1}{n}}(x)$$

$\exists\, x \in F_n$ such that $y \in B_{\frac{1}{n}}(x)$

$$d(x, y) < 1/n < r$$

$$\Rightarrow x \in B_r(y) \subseteq U$$

$$x \in F \cap U$$

Thus, every open sets meets F.

$$\therefore \overline{F} = X$$

X is separable.

Eg $X = [0,1]$, $X = \{0, 1, \frac{1}{2}, \ldots, 1/n, \ldots\}$

$$[0,1] \times [0,1] \times \cdots \times [0,1]$$

A metric space X is called complete if every Cauchy sequence in X converges.

Eg $\mathbb{R}^d$, $[a,b]$ are complete metric spaces.

$\mathbb{Q}$ is not complete.

Ex F of a complete metric space is complete if and only if F is closed.

Proposition: Every compact metric space is complete.

Proof: Let (x_n) be a Cauchy sequence in (a compact metric space) X.

(x_n) has a subsequence (x_{k_n}) such that $x_{k_n} \longrightarrow x \in X$.

for $\varepsilon > 0$ $\exists N$ such that

$$d(x_{k_n}, x) < \varepsilon/2 \quad \forall n \geq N$$

and $d(x_n, x_m) < \varepsilon/2 \quad \forall n, m \geq N$

For $n \geq N$, $k_n \geq N$

$$d(x_n, x) \leq d(x_n, x_{k_n}) + d(x_{k_n}, x)$$

$$\leq \varepsilon/2 + \varepsilon/2 = \varepsilon$$

$\Rightarrow x_n \longrightarrow x$.

Function spaces

eg $X = [a,b]$ or $[0,1]$

Let X be a compact metric space.

Let $C(X) = \{ f : X \to \mathbb{C}$ is continuous$\}$.

$C_{\mathbb{R}}(X) = \{ f : X \to \mathbb{R}$ is continuous$\}$.

$C_{\mathbb{R}}(X) \subseteq C(X)$.

Let $f \in C_{\mathbb{R}}(X)$.

$\exists\, a \in X$ such that $f(a) \geq f(x)\ \forall x \in X$

$\sup_{x \in X} f(x)$ is attained.

If $f \in C(X)$, then $|f| \in C_{\mathbb{R}}(X)$

$|f|(x) = |f(x)|$
$\forall x \in X$.

$\exists\, a \in X$, $|f(a)| \geq |f(x)|\ \forall x \in X$

$\sup |f|$ is attained

Convergence in $C(X)$.

$(f_n) \subseteq C(X)$.

Point-wise : $f_n(x)$ converging

Disadvantage : limit need not be continuous

Uniformly : f_n converging uniformly

Advantage : limit is continuous

For a general metric space (not necessarily compact), uniform convergence is required only on compact sets, not on the whole space.

In our study X is compact.

<u>Metric on $C(X)$</u>:

Let $f, g \in C(X)$.

$$d_\infty(f,g) = \sup_{x \in X} |f(x) - g(x)|$$

d_∞ is a metric on $C(X)$. [Ex]

$f_n \to f$ uniformly $\iff d_\infty(f_n, f) \to 0$

d_∞ — uniform metric, sup metric, L_∞ metric

$(C(X), d)$ is compact?

$f_n : X \to \mathbb{C}$ $\quad f_n(x) = n \quad \forall x \in X$

(f_n) has no subsequence that converges.

$C(X)$ is not compact.

$\mathbb{R}^d = \{(x_1, x_2, \dots, x_d)\}$, $E \subseteq \mathbb{R}^d$ is compact $\iff$ E is closed & bdd

$C(X)$ is a complete <u>separable</u> metric space.

$E \subseteq C(X)$ is compact, what is E?