Contraction Mapping Principle:

Let X be a metric space and $f: X \longrightarrow X$ be a function.

We say that f is a contraction if there exists a constant $c < 1$ such that $d(f(x), f(y)) \leq c\, d(x,y)$ $\forall x, y \in X$

Iterations serves an approximation

Taylors series provides an approximation

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$a_n = \sum_{k=0}^{n} \frac{x^k}{k!} \longrightarrow e^x$$

A fixed point of $f: X \longrightarrow X$ is a point $x \in X$ such that $f(x) = x$.

Eg 1) $f: \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = \frac{x}{2}$ is a contraction. fixed point 0

2) $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$f(x,y) = \left(\frac{x}{3} + y, \frac{y}{2}\right)$ for a choice of suitable equivalent metric

Is f contraction? yes

'0' is the only fixed point

3) $f : C[0,1] \longrightarrow C[0,1]$ given by

$$f(u)(x) = \frac{1}{2} \int_0^x u(t)\, dt$$

f is a contraction

'0' is the only fixed point.

Contraction Mapping principle:

Let X be a complete metric space and $f : X \longrightarrow X$ be a contraction. Then f has a unique fixed point x_0 such that $f^n(x) \longrightarrow x_0$.

Proof: Let x_1, x_2 be fixed points.

$d(f(x_1), f(x_2)) \leq c\, d(x_1, x_2)$ $\quad (c < 1)$

$d(x_1, x_2) \leq c\, d(x_1, x_2)$

$\Rightarrow d(x_1, x_2) = 0 \Rightarrow x_1 = x_2$

Let $x \in X$. $(f^n(x))_{n \geq 1}$ — orbit

Let $x_n = f^n(x)$, $n \geq 1$. For $m > n$

$d(x_m, x_n) = d(f^m(x), f^n(x))$
$\leq c\, d(f^{m-1}(x), f^{n-1}(x))$

$$\leq c^2 d(f^{m-2}(x), f^{n-2}(x))$$

$$\leq c^{n-1} d(f^{m-n-1}(x), x)$$

$\therefore$ $d(x_m, x_n) \leq c^{n-1} d(f^{m-n-1}(x), x)$

$n > 1$

$$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \dots + d(x_{n-1}, x_n)$$

$$\leq [1 + c + c^2 + \dots + c^{n-1}] \, d(x_2, x_1)$$

$$\leq \frac{d(x_2, x_1)}{1-c}$$

for $m > n$

$$d(x_m, x_n) \leq \sum_{n-1}^{m-2} d(x_k, x_{k-1})$$

$$\leq \left(\sum_{n-1} c^k \right) d(x_2, x_1)$$

$$\leq \frac{c^{n-1}}{1-c} d(x_1, x_2)$$

$$\longrightarrow 0 \text{ as } n \to \infty$$

$\therefore$ (x_n) is Cauchy

Since X is complete, $x_n \longrightarrow x_0$, say.

i.e., $\lim_{n \to \infty} f^n(x) = x_0$

$f(x_n) = x_{n+1} \longrightarrow x_0$.

f is contraction $\Longrightarrow$ f is continuous

$\Longrightarrow f(x_n) \longrightarrow f(x_0)$

$\Longrightarrow x_{n+1} \longrightarrow f(x_0)$

$\therefore f(x_0) = x_0.$

Since the fixed point is unique,

$f^n(y) \longrightarrow x_0$ ~~for~~ for any $y \in X$.

———— x ————

Without completeness the CMP need not be true.

———— x ————

<u>Differentiation of several variables</u>:

Let U be an open subset of $\mathbb{R}^n$ and $f : U \longrightarrow \mathbb{R}^m$.

We say that f is differentiable at $x \in U$ if there is a linear map $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - T(h)}{\|h\|} = 0.$$

In that case, T is denoted by $f'(x)$.

If f is diff at all points $x \in U$, then we say that f is diff on U.

$L(\mathbb{R}^n, \mathbb{R}^m) = \{ \alpha : \mathbb{R}^n \to \mathbb{R}^m \text{ is a linear map} \}$

$\simeq \mathbb{R}^{nm}$

$A \in L(\mathbb{R}^n, \mathbb{R}^m)$, $\|A\|$ = norm of A

$= \sup_{\|x\| \leq 1} \|Ax\| < \infty$

$\|A(x)\| \leq \|A\| \, \|x\|$

$= \sup_{\|x\| = 1} \|Ax\|$

f is diff on U, $f'(x) \in L(\mathbb{R}^n, \mathbb{R}^m)$ $\forall x \in U$

$x \mapsto f'(x) \in L(\mathbb{R}^n, \mathbb{R}^m)$

$L(\mathbb{R}^n, \mathbb{R}^m)$ is a metric space with metric $d(A, B) = \|A - B\|$ $\forall A, B \in L(\mathbb{R}^n, \mathbb{R}^m)$.

We say that $f \in C^1(U)$ if f is diff on U and $x \mapsto f'(x)$ is continuous.

<u>Proposition(1)</u> Let U be a convex open set in $\mathbb{R}^n$ and $f : U \to \mathbb{R}^m$ be a differentiable function such that $\|f'(x)\| \leq M$ $\forall x \in U$. Then $\|f(x) - f(y)\| \leq M \|x - y\|$ for all $x, y \in U$.

U is convex, if $\lambda x + (1-\lambda) y \in U$, $x, y \in U$, $\lambda \in [0,1]$

Proposition (2) Let U be an open set in $\mathbb{R}^n$, Let V be a open set in $\mathbb{R}^m$. Let $f : U \longrightarrow \mathbb{R}^m$ be such that $f(U) \subseteq V$ and $g : V \longrightarrow \mathbb{R}^k$. If f is diff at x and g is diff at $f(x)$, then $g \circ f$ is diff at x and

$$(g \circ f)'(x) = g'(f(x))\, f'(x).$$

Eg: If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear map then $f \in C^1$, $f' = f$.

Inverse function Theorem:

Let E be an open set in $\mathbb{R}^n$ and $f : E \longrightarrow \mathbb{R}^n$ be C^1. If $f'(x)$ is invertible at $x \in E$, then there exists an open set $U \subseteq E$ such that

(i) $x \in U$; (ii) $V = f(U)$ is open in $\mathbb{R}^n$

(iii) $f\big|_U$ is one-one

(iv) g is the inverse of f defined on V, then g is C^1.

Lemma : Let $\Omega = \{ A \in L(\mathbb{R}^n, \mathbb{R}^n) \mid A$ is invertible$\}$

(1) If $A \in \Omega$, $B \in L(\mathbb{R}^n, \mathbb{R}^n)$ such that $\|A - B\| \, \|A^{-1}\| < 1$, then $B \in \Omega$

(2) Ω is open

(3) $A \longmapsto A^{-1}$ is continuous on Ω.

Proof: $A \in \Omega$, $B \in L(\mathbb{R}^n, \mathbb{R}^n)$

$\|Ax\| \leq \|A\| \, \|x\| \quad \forall x \in \mathbb{R}^n$

Let $\alpha = \dfrac{1}{\|A^{-1}\|}$.

Let $\beta = \|A - B\|$

We assume that $\beta < \alpha$.

$$\|x\| \leq \|A^{-1}\| \, \|Ax\| = \frac{1}{\alpha} \|(A-B)(x)\| + \frac{1}{\alpha} \|B(x)\|$$

$$\leq \frac{\beta}{\alpha} \|x\| + \frac{1}{\alpha} \|Bx\|$$

$$\|x\| (\alpha - \beta) \leq \frac{1}{\alpha} \|Bx\|$$

$Bx \neq 0$ if $x \neq 0$

B is one-one.

$\Rightarrow B$ is invertible

(2) is an exercise

for $A \neq B$, $\|A^{-1}\|^{-1} = \alpha$, $\beta = \|A - B\|$

$(\alpha - \beta) \|B^{-1} y\| \leq \|y\|$

$\|B^{-1}\| \leq \dfrac{1}{(\alpha - \beta)}$ for any B with $\beta < \alpha$

$$\|\bar{A} - \bar{B}\| \leq \|\bar{A}\| \, \|A - B\| \, \|\bar{B}\|$$

$$\leq \frac{1}{\alpha(\alpha-\beta)} \|A - B\| = \frac{\beta}{\alpha(\alpha-\beta)}$$

So as $B \longrightarrow A$

$\beta \longrightarrow 0 \implies \|\bar{A} - \bar{B}\| \longrightarrow 0$

$\therefore A \longmapsto \bar{A}$ on Ω is continuous.
