Lecture 31: Morphism of projective varieties.

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13:00

Def^n A map $\varphi: X \to Y$ between varieties is said to be a morphism if $\exists$ an affine open cover $\{U_i\}$ of X and affine open cover $\{V_i\}$ of Y s.t. $\varphi(U_i) \subseteq V_i$ and $\varphi_{|U_i} : U_i \to V_i$ is a morphism of affine varieties. $\Leftrightarrow \left\{ \begin{array}{l} \forall P \in X \ \exists \text{ open affine subsets } U \subseteq X \ \& \ V \subseteq Y \\ \text{s.t. } P \in U, \varphi(P) \in V \ \& \ \varphi_{|U} : U \to V \text{ is a morphism of affine var.} \end{array} \right\}$

Ex: $\mathbb{P}^1$ the homogen coordinate ring $k[X_0, X_1]$.

(a) $\mathbb{P}^1 \to \mathbb{P}^2$, $[a_0, a_1] \mapsto [a_0, a_1, 0]$

(b) $\varphi: \mathbb{P}^1 \to \mathbb{P}^2$, $[a_0, a_1] \mapsto [a_0^2, a_1^2, a_0 a_1]$

$\text{Image}(\varphi) = Y = Z(X_0^2 + X_1^2 - X_2^2)$

homogen coord ring of Y is $k[X_0, X_1, X_2]/(X_0^2+X_1^2-X_2^2)$

Prop: Let $X \subseteq \mathbb{P}^n$ & $Y \subseteq \mathbb{P}^m$ be proj varieties. Let $\dfrac{k[X_0, \dots, X_n]}{I}$ & $\dfrac{k[Y_0, \dots, Y_m]}{J}$ be the homogen coord ring of X and Y resp. Let $\varphi: \dfrac{k[Y_0, \dots, Y_m]}{J} \to \dfrac{k[X_0, \dots, X_n]}{I}$ be a "graded" ring homo (i.e. $\exists d > 0$ s.t. $\varphi(\overline{Y_i})$ is homogen of deg d $\forall 0 \le i \le m$) (convention: 0 poly is homogen of every degree) s.t. $Z(\varphi(\overline{Y_0}), \dots, \varphi(\overline{Y_m}), I) = \phi$. Then φ defines a morphism of proj varieties $\hat{\varphi}: X \to Y$

$$[a_0, \dots, a_n] \mapsto [\varphi(Y_0)(a_0, \dots, a_n), \dots, \varphi(Y_m)(a_0, \dots, a_n)]$$

Pf: Note that $\tilde{\varphi}$ is well-defined since for $\lambda \in k^\times$,

$$\varphi(\bar{Y}_i)(\lambda a_0, \dots, \lambda a_n) = \lambda^d \varphi(\bar{Y}_i)(a_0, \dots, a_n) \quad \forall\, 1 \le i \le m$$

Hence $[\varphi(\bar{Y}_0)(\lambda \underline{a}), \dots, \varphi(Y_m)(\lambda \underline{a})] = [\varphi(\bar{Y}_0)(\underline{a}), \dots, \varphi(\bar{Y}_m)(\underline{a})]$

Let $V_i = Y \cap \{Y_i \neq 0\}$ is an open affine subset of Y. $0 \le i \le m$

$$\tilde{\varphi}^{-1}(V_i) = \{ [a_0, \dots, a_n] \in X \mid \varphi(\bar{Y}_i)(a_0, \dots, a_n) \neq 0 \} = \bigcup_{j=0}^{n} \underbrace{\left\{ [a_0, \dots, a_n] \in X \,\middle|\, \begin{array}{l} \varphi(\bar{Y}_i)(a_0, \dots, a_n) \neq 0 \\ \& \ a_j \neq 0 \end{array} \right\}}_{= U_{ij}}$$

$$U_{ij} = \overbrace{(X \cap U_j)}^{\text{affine open in } X} \setminus Z\left(\varphi(\bar{Y}_i)\left(\frac{X_0}{X_j}, \dots, \frac{X_{j-1}}{X_j}, 1, \frac{X_{j+1}}{X_j}, \dots, \frac{X_n}{X_j}\right)\right)$$

where $\underbrace{U_j = \{X_j \neq 0\} \subset \mathbb{P}^n}$

Note that $\varphi(\bar{Y}_i)(x_0, x_1, \dots, x_{j-1}, 1, x_{j+1}, \dots, x_n) \in k[U_j] = k[x_0, \dots, \hat{x}_j, \dots, x_n]$, $x_i = \frac{X_i}{X_j}$

$U_{ij} \cong$ $\underbrace{k[x_0, \dots, \hat{x}_j, \dots, x_n, \tfrac{1}{\varphi(\bar{Y}_i)(x_0, \dots, x_n)}]}_{I^a}$ is again affine.

Note: $\bigcup_{i=0}^{m} \bigcup_{j=0}^{n} U_{ij} = \bigcup_{i=0}^{m} \tilde{\varphi}^{-1}(V_i) = X$

$\tilde{\varphi}|_{U_{ij}} : U_{ij} \longrightarrow V_i$ is defined by regular functions

$$\left[\frac{a_0}{a_j}, \dots, \frac{a_j}{a_j}, \dots, \frac{a_n}{a_j}\right] \longmapsto \left[\frac{\varphi(Y_0)[\frac{a_0}{a_j}, \dots, 1, \dots, \frac{a_n}{a_j}]}{\varphi(Y_i)[\quad " \quad]}, \dots, \underset{i\text{th spot}}{1}, \dots, \frac{\varphi(Y_n)[\cdots]}{\varphi(Y_i)[\cdots]} \right]$$

$$[a_0, \dots, a_n] \longmapsto [\varphi(Y_0)[a_0, \dots, a_n], \dots, \varphi(Y_n)(a_0, \dots, a_n)]$$

$\varphi(Y_0)(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n)$ is a reg fun on U_{ij}

$\varphi(\bar{Y}_i)[a_0, \dots, a_n] \neq 0$ on U_{ij}

the rat'l maps $\dfrac{\varphi(\bar{Y}_0)}{\varphi(\bar{Y}_i)}$ are regular on U_{ij}.

Hence $\tilde{\varphi}|_{U_{ij}}$ is a morphism of affine varieties.

<u>Example:</u> $Y = Z(X_0^2 + X_1^2 - X_2^2) \subseteq \mathbb{P}^2$

$$\varphi : Y \longrightarrow \mathbb{P}^1$$
$$[a_0, a_1, a_2] \longmapsto [a_0, a_1]$$

$[0,0,1]$

$a_0 = 0 = a_1$

$a_2 = 1$

φ is a morphism. Is it an isomorphism?

$$\varphi^{-1}([1,0]) = \{ [1,0,a_2] \mid 1 - a_2^2 = 0 \}$$
$$= \{ [1,0,1], [1,0,-1] \}$$

$Y_0 = X_0 \qquad Y_1 = X_1 - X_2$ & $Y_2 = X_1 + X_2$

$\mathbb{P}^2 \longrightarrow \mathbb{P}^2$

then $Y = Z(Y_0^2 + Y_1 Y_2)$ in new homogen coord Y_0, Y_1, Y_2

Define $\varphi : Y \longrightarrow \mathbb{P}^1$

$[a_0, a_1, a_2] \longmapsto [a_0, a_1]$ if $a_1 \neq 0$

$[0,0,1] \longmapsto [1,0]$

$\varphi : Y \longrightarrow \mathbb{P}^1$

$$\varphi([a_0, a_1, a_2]) = \begin{cases} [a_0, a_1] & \text{if } a_1 \neq 0 \\ [a_2, a_0] & \text{if } a_2 \neq 0 \end{cases}$$

Note: $a_0^2 + a_1 a_2 = 0$ if $[a_0, a_1, a_2] \in Y$. If $a_1 \neq 0$ & $a_2 \neq 0$ then $a_0 \neq 0$

So $[a_0, a_1] = [a_0^2, a_0 a_1] = [a_1 a_2, a_0 a_1]$

$= [-a_2, a_0]$

Hence φ is well-defined.

$$Y = \underbrace{Y \cap \{Y_1 \neq 0\}}_{= U_1} \cup \underbrace{Y \cap \{Y_2 \neq 0\}}_{= U_2}$$

Let Z_0, Z_1 be homogen coord on $\mathbb{P}^1$

then $\mathbb{P}^1 = \underbrace{V_0}_{= \{Z_0 \neq 0\}} \cup \underbrace{V_1}_{= \{Z_1 \neq 0\}}$

$\varphi(U_1) \subseteq V_1$ and $\varphi|_{U_1} : U_1 \longrightarrow V_1$

$$\left[\frac{a_0}{a_1}, 1, \frac{a_2}{a_1}\right] \mapsto \left[\frac{a_0}{a_1}, 1\right]$$

is a morphism of affine var.

$|||^{ly}$ $\varphi(U_2) \subseteq V_0$ and $\varphi|_{U_2} : U_2 \longrightarrow V_0$

$$\underbrace{\left[\frac{a_0}{a_2}, \frac{a_1}{a_2}, 1\right]}_{= [a_0, a_1, a_2]} \longmapsto \left[-1, \frac{a_0}{a_2}\right] = [-a_2, a_0]$$

is also a morphism of affine varieties

Note that $\varphi|_{U_1}$ & $\varphi|_{U_2}$ are isom.

$\left(\because \frac{a_2}{a_1} = -\left(\frac{a_0}{a_1}\right)^2\right)$ $\left(\because \frac{a_1}{a_2} = -\left(\frac{a_0}{a_2}\right)^2\right)$