Lecture 2: Review of rings, algebraic sets

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Recall: $a \in R$ is a <u>zero divisor</u> if $ab = 0$ for some $b \in R$, $b \neq 0$
$a \in R$ is <u>nilpotent</u> if $a^n = 0$ for some $n \geq 1$.

Def^n: Let R be a nonzero comm ring with unity. It is said to be **reduced** if it does not contain nonzero nilpotents.

Def^n: A ring R is said to be an **integral domain** if it is a nonzero comm ring with unity and it does not contain any nonzero zero divisors. i.e. every nonzero element of R is a nonzerodivisor.

<u>Jacobson radical</u>: Let R be a nonzero comm ring with identity. The Jackson radical of R is defined to be intersection of all maximal ideals of R.

$$J(R) := \bigcap_{\substack{m \text{ maximal} \\ \text{ideal of } R}} m$$

<u>Nil radical</u>: $nil(R) := \{x \in R \mid x^n = 0\}$
= set of nilpotents of R

* $\mathrm{nil}(R) = \bigcap_{\substack{P \subset R \\ \text{prime ideal}}} P$

* $\mathrm{Jac}(R) = \{ x \in R \mid 1 + xy \text{ is a unit } \forall y \in R \}$

$\boxed{\begin{array}{l} \text{Eg: 1) } \mathbb{Z},\ \mathbb{Z}[t] \quad \mathrm{Jac}(R) = \mathrm{nil}(R) \\ \text{2) } k[t], \quad S = \mathbb{Z} \setminus (p) \\ \qquad\qquad\quad S^{-1}\mathbb{Z} = \mathbb{Z}_{(p)} \end{array}}$

* $\mathrm{nil}(R) \subseteq \mathrm{Jac}(R)$

* $\mathrm{rad}(I)$ for an R-ideal I is $\mathrm{rad}(I) = \sqrt{I} = \{ f \in R \mid f^n \in I \text{ for } n \geq 1 \}$

$$= \bigcap_{\substack{P \subseteq R \text{ prime ideal} \\ \& \ I \subseteq P}} P$$

$\varphi : R \to R/I$

$\sqrt{I} = \varphi^{-1}(\mathrm{nil}(R/I))$

* $I \subseteq \sqrt{I}$; $I \subseteq J \Rightarrow \sqrt{I} \subseteq \sqrt{J}$

  & $\sqrt{I} = \sqrt{I^n}$ for any $n \geq 1$.

* I is called a radical ideal if $I = \sqrt{I}$.

* I is a radical ideal iff R/I is reduced

$\varphi : R \to R/I$

Pf: $\sqrt{I} = I \iff \varphi^{-1}(\mathrm{nil}(R/I)) = I$

$\iff \mathrm{nil}(R/I) = 0$

$(\because \varphi^{-1}(0) = I)$

Affine space

Def^n: Let k be an alg closed field, like $\mathbb{C}$, $\overline{\mathbb{Q}}$, $\overline{\mathbb{F}_p}$.
The set n-tuples of k, k^n, is called the affine n-space over k and it is denoted by $\mathbb{A}^n_k$ or simply $\mathbb{A}^n$.

Def^n: A subset X of $\mathbb{A}^n_k$ is called an affine algebraic set if X is the zero set of a collection S of polynomials in $k[x_1, \dots, x_n]$.

$$X = Z(S).$$

Prop: $Z(S) = Z(\langle S \rangle)$

Pf:

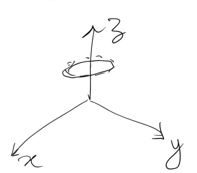
$$\underset{\in \mathbb{A}^n_k}{p} \in Z(S) \iff f(p) = 0 \quad \forall f \in S$$

$$\iff g(p) = 0 \quad \forall g = \sum_{i=1}^{m} a_i f_i \quad \& \ f_i \in S,\ a_i \in k[x_1, \dots, x_n]$$

$$\iff g(p) = 0 \quad \forall g \in \langle S \rangle.$$

Eg: $\mathbb{A}^3_{x,y,z}$ $\quad S = \{x^2 + y^2 - 1,\ z - 2\}$

$$Z(S) = \{\underset{\in \mathbb{R}^3}{(a,b,c)} \mid a^2 + b^2 = 1 \ \&\ c = 2\}$$



⊛1 $I \subseteq J$ ideals in $k[x_1,\dots,x_n]$ then $Z(I) \supseteq Z(J)$

Def^n X be an (alg) subset of $\mathbb{A}^n_k$ then define

$I(X) = \{ f \in k[x_1,\dots,x_n] : f(a_1,\dots,a_n) = 0 \ \forall (a_1,\dots,a_n) \in X \}$.

Then $I(X)$ is an ideal in $k[x_1,\dots,x_n]$.

This is called the defining ideal of X.

⊛2 $X \subseteq Y$ affine alg sets in $\mathbb{A}^n_k$ then $I(X) \supseteq I(Y)$

⊛3 $Z(I) = Z(\sqrt{I})$

Prop: $I(Z(J)) = \sqrt{J}$ for any ideal $J \subseteq k[x_1,\dots,x_n]$.

Pf: "$\supseteq$": Let $f \in \sqrt{J} \Rightarrow f^m \in J$ for some $m \geq 1$

$\Rightarrow f^m(a_1,\dots,a_n) = 0 \ \forall (a_1,\dots,a_n) \in Z(J)$

$\Rightarrow f(a_1,\dots,a_n) = 0 \ \forall (a_1,\dots,a_n) \in Z(J)$

$\Rightarrow f \in I(Z(J))$

"$\subseteq$" Hilbert Nullstellensatz.

④ $Z(I+J) = Z(I \cup J) = Z(I) \cap Z(J)$

⑤ $Z(I \cap J) = Z(I) \cup Z(J) = Z(IJ)$

Pf: ⑤ $I \cap J \subseteq I$

$\Rightarrow Z(I) \subseteq Z(I \cap J)$
$\quad Z(J) \subseteq$

$\Rightarrow Z(I) \cup Z(J) \subseteq Z(I \cap J)$ ⊛

$(a_1, \dots, a_n) \in Z(I \cap J)$

$\Rightarrow f(a_1, \dots, a_n) = 0 \quad \forall f \in I \cap J$

$\Rightarrow f(a_1, \dots, a_n) = 0 \quad \forall f \in IJ$

$(\because IJ \subseteq I \cap J)$

$f = gh \quad g \in I, h \in J$

$\Rightarrow g(a_1, \dots, a_n) = 0$ or $h(a_1, \dots, a_n) = 0 \quad \forall g \in I$
or $\forall h \in J$

For $g \in I \quad g(a_1, \dots, a_n) \neq 0$
then $\forall h \in J \quad h(a_1, \dots, a_n) = 0$

$\Rightarrow (a_1, \dots, a_n) \in Z(I)$ or $(a_1, \dots, a_n) \in Z(J)$

$\Rightarrow (a_1, \dots, a_n) \in Z(I) \cup Z(J)$

$(a_1, \dots, a_n) \in Z(IJ) \Rightarrow (a_1, \dots, a_n) \in Z(I) \cup Z(J)$

$Z(I \cap J) \subseteq Z(IJ) \subseteq Z(I) \cup Z(J) \overset{⊛}{\subseteq} Z(I \cap J)$

$\blacksquare$