Lecture 19: Uniqueness of primary decomposition

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Def^n: Let I be an R-ideal. If there exists P_i-primary ideals Q_i, $1 \le i \le n$ s.t. $I = \bigcap_{i=1}^{n} Q_i$ then we say that $I = \bigcap_{i=1}^{n} Q_i$ is a primary decomposition of I. Note $P_i = \sqrt{Q_i}$ are prime ideals.

A primary decomposition $I = \bigcap_{i=1}^{n} Q_i$ is called minimal if $P_i \neq P_j$ for $i \neq j$ & $Q_i \not\supseteq \bigcap_{j=1, j\neq i}^{n} Q_j$.

Theorem: Every proper ideal of a noetherian ring has a primary decomposition (& hence a minimal primary decomp)

Def^n: Minimal primes Let I be a proper ideal of a ring R. A prime ideal P of R is called a minimal prime of I if $I \subseteq P$ and $I \subseteq P_1 \subseteq P$ with P_1 prime implies $P_1 = P$.

Prop: Let R be a ring, I an R-ideal & $I = \bigcap_{i=1}^{n} Q_i$ a minimal primary decomposition of I then every minimal prime of I belongs to $\underbrace{\{\sqrt{Q_1}, \dots, \sqrt{Q_n}\}}_{= \mathrm{Assoc}(I)}$ ← is called the associated prime ideals of I.

Thm: Let $I = \bigcap_{i=1}^{n} Q_i$ be a minimal primary decomp of I then the set $S = \{P_i := \sqrt{Q_i} \mid i = 1, \dots, n\}$ is independent of the choice of the primary decomposition. Moreover if P_i is minimal prime of I then Q_i is also determined by I.

Def^n: Let R be a ring, I an ideal & $x \in R$ then $(I:x) := \{r \in R \mid rx \in I\}$

Note: $(I:x) = q^{-1}(\mathrm{ann}(q(x)))$ where $q: R \to R/I$ is the quot map and $(I:x) = R$ iff $x \in I$.

Lemma: Let Q be a P-primary R-ideal.

1. $(Q:x) = R$ iff $x \in Q$
2. $(Q:x)$ is P-primary if $x \notin Q$
3. $(Q:x) = Q$ if $x \notin P$

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So $S = \{ \sqrt{(I:x)} \mid x \in R \ \& \ \sqrt{(I:x)} \text{ prime ideal of } R \}$ depends only on I.

Finally let $I = \bigcap_{i=1}^{n} Q_i = \bigcap_{i=1}^{n} Q_i'$ be two minimal primary decomposition of I with Q_i & Q_i' P_i-primary ideals.

If P_1 is a minimal prime of I then $Q_1 = Q_1'$.

<u>Claim</u>: $P_1 \not\supseteq \bigcap_{i=2}^{n} Q_i$ (and similarly $P_1 \not\supseteq \bigcap_{i=2}^{n} Q_i'$)

Suppose $P_1 \supseteq \bigcap_{i=2}^{n} Q_i \Rightarrow P_1 \supseteq \bigcap_{i=2}^{n} \sqrt{Q_i} \Rightarrow P_1 \supseteq \bigcap_{i=2}^{n} P_i$

$\Rightarrow P_1 \supseteq P_i$ for some $2 \le i \le n$ contradicting P_1 is a minimal prime of I. (as $P_i \supseteq I \ \forall i$)

Let $x \in \bigcap_{i=2}^{n} Q_i \setminus P_1$ & $x' \in \bigcap_{i=2}^{n} Q_i' \setminus P_1$

then $xx' \in \bigcap_{i=2}^{n} Q_i \cap Q_i'$ & $xx' \notin P_1$. ($\because P_1$ is prime)

$$(I : xx') = \bigcap_{i=1}^{n} (Q_i : xx') = \bigcap_{i=1}^{n} (Q_i' : xx') \quad \text{(by Lemma 1)}$$

$$(Q_1 : xx') = (Q_1' : xx')$$

$$Q_1 = Q_1' \quad \text{(by Lemma 3)}$$

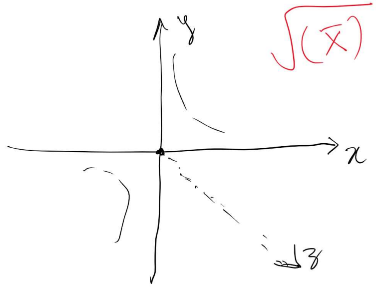
$\blacksquare$

(*) Let P be a prime ideal in R. Then P^n need not be primary ideal.
Though true in $\mathbb{Z}$, $k[X,Y]$

Ex: $R = \dfrac{k[X,Y,Z]}{(Z^2 - XY)}$ $\quad P = (\bar{X}, \bar{Z})$ $\quad$ then $P^2 = (\bar{X}^2, \bar{X}\bar{Z}, \bar{Z}^2, \bar{X}\bar{Y})$

$\bar{X}\bar{Y} \in P^2$, $\bar{X} \notin P^2$ & $\bar{Y}^n \notin P^2$ $\forall n$. Hence P^2 is not primary. In fact $P^2 = (\bar{X}) \cap (\bar{X}^2, \bar{Y}, \bar{Z}^2, \bar{X}\bar{Z})$

$\sqrt{(\bar{X})} = P$ $\quad$ $(\bar{X})$: P-primary $\quad$ $(\bar{X}^2, \bar{Y}, \bar{Z}^2, \bar{X}\bar{Z})$: $(\bar{X}, \bar{Y}, \bar{Z})$-primary



$a, \dots, f, g \in R$

$\subseteq$ ✓

$\supseteq$: $\alpha = a\bar{X}^2 + b\bar{Y} + c\bar{Z}^2 + d\bar{X}\bar{Z}$

$\alpha \in (\bar{X}) \Rightarrow b\bar{Y} \in (\bar{X})$

$\Rightarrow \tilde{b}Y - gX = h(Z^2 - XY)$ in $k[X,Y,Z]$

$\Rightarrow \tilde{b} \in (X, Z) \Rightarrow b \in (\bar{X})$

$\Rightarrow \alpha \in P^2$

$Z^2 - X^2 - Y^2$

$(\bar{X}) = P^2 R_P \cap R$

Prop: Let R be a ring P a prime ideal. $P^n R_P \cap R$ is a P-primary ideal.

Pf: Note $\varphi : R \to R_P$ is the loc map. $P^n R_P \cap R := \varphi^{-1}(P^n R_P)$

$P^n R_P$ is PR_P-primary ($\because$ PR_P is a max ideal)

$xy \in P^n R_P \cap R \Rightarrow x \in P^n R_P$ or $y^m \in PR_P$ for some m

$\Rightarrow x \in P^n R_P \cap R$ or $y^m \in PR_P \cap R$.

Hence $P^n R_P \cap R$ is primary. check that $\sqrt{P^n R_P \cap R} = P$

(*) $P^{(n)} := P^n R_P \cap R$ are called symbolic powers of P.

The P-primary component of P^n is $P^{(n)}$.