Lecture 23: Regular and rational functions

24 March 2021
14:30

Definition: Let X be a affine variety over an alg closed field k.

The elements of the coordinate ring are called regular functions on X.

Another way: regular function on X is a function from X to k given by polynomials
$\overset{\shortparallel}{\mathbb{A}^1}$

i.e. reg functions on X is same as morphisms of affine varieties from X to $\mathbb{A}^1_k$

$\Updownarrow$

k-alg homo $k[x] \longrightarrow \mathcal{O}(X)$ (poly ring)

$\Updownarrow$

elements of $\mathcal{O}(X)$

⊛ $\mathcal{O}(X)$ is also called ring of regular functions on X.

⊛ More generally, let $f_1, \dots, f_m$ be regular functions on X. Then they define a morphism

$$P \longmapsto (f_1(P), \dots, f_m(P))$$

$$\underset{\sim}{f} : X \longrightarrow \mathbb{A}^m_k = k^m$$

Because $f^{\#} : k[y_1, \dots, y_m] \longrightarrow \mathcal{O}(X)$ defines a

$$y_i \longmapsto f_i \qquad 1 \le i \le m$$

k-alg homo.

Let $P \in X$ i.e. m_P is a max ideal of $\mathcal{O}(X)$

Claim: $f^{\#^{-1}}(m_P) = (y_1 - f_1(P), \dots, y_m - f_m(P))$

$$f^{\#} : k[y_1, \dots, y_m] \longrightarrow \overset{k[x_1,\dots,x_n]/\mathcal{Q}}{\mathcal{O}(X)} \longrightarrow \mathcal{O}(X)/m_P = k$$

$$y_1 \longmapsto f_1 \longmapsto f_1 \pmod{m_P} = f_1(P)$$

$\Rightarrow y_1 - f_1(P) \longmapsto 0$

$\Rightarrow y_1 - f_1(P) \in f^{\#^{-1}}(m_P)$, $|||^{ly}$ $i \ge 2$

Hence claim.

Def^n: A rational function on an affine variety X is an element of the field of fractions of $\mathcal{O}(X)=k[X]$. This fraction field is also called the function field of X and is denoted by $k(X)$.

Eg: $X=\mathbb{A}^2$, $\mathcal{O}(X)=k[x_1,x_2]$, $k(X)=k(x_1,x_2)$

$f=\frac{x_1}{x_2}\in k(X)$. Note f is not a function on X

$f: \mathbb{A}^2\setminus\{x_2=0\}\longrightarrow k$ is a function.

(*) So a rational function on X is a regular function on a nonempty open affine variety of X.

Note $\mathbb{A}^2\setminus\{x_2=0\}$ is an affine variety with coordinate ring

$$\frac{k[x_1,x_2,x_3]}{(x_2x_3-1)}=k[x_1,x_2,\tfrac{1}{x_2}]$$

More generally, a rational map from an affine variety X to an affine variety Y is a morphism from a nonempty affine open subset of X to Y and it is denoted by $f: X \dashrightarrow Y$.

(*) $f_1,\dots,f_m$ be rat'l function on an affine variety X then they define a rational map from X to $\mathbb{A}^m$.

(*) Every nonempty open subset of an affine variety is dense.

X is irred. Then $U\subseteq \underset{\text{is closed}}{\overline{U}} \subsetneq X$ then

& $X\setminus U$ is closed diff from X & $X=X\setminus U\cup\overline{U}$ contradicting X is irred.

⊛ If we think of $X = \operatorname{mspec}(R)$ for some k-algebra R. Then for $f \in R$, the function defined by f from X to k is given by

$$\begin{aligned} f : X &\longrightarrow k \\ m &\longmapsto f \pmod{m} \end{aligned} \qquad \text{i.e. } \begin{aligned} R &\longrightarrow R/m \\ f &\longmapsto f \pmod{m} \end{aligned}$$

Def^n: Let f be a rat'l function on a variety X. f is said to be regular at a point $P \in X$ if $\exists\, g, h \in \mathcal{O}(X)$ s.t. $f = \frac{g}{h}$ and $h(P) \neq 0$.

Domain of $f := \{ P \in X \mid f \text{ is regular at } P \}$

Prop: Let X be an affine variety and $f \in k(X)$.

1) Domain of f is an open dense subset of X.

2) $\operatorname{domain}(f) = X \iff f \in \mathcal{O}(X)$.

3) $\operatorname{domain}(f) \supseteq X_h := \{P \in X \mid h(P) \neq 0\}$ for $h \in \mathcal{O}(X)$

iff $f \in \mathcal{O}(X)[h^{-1}] \subseteq k(X)$

Pf: $f \in k(X)$, let $I = (k[X] : f) := \{g \in k[X] \mid gf \in k[X]\}$

I is a nonzero ideal of $\mathcal{O}(X) (= k[X])$. if $f = \frac{f_1}{f_2}$ where $f_1, f_2 \in k[X]$, $f_2 \neq 0$ then $f_2 \in I$

Claim: $P \in V(I)$ iff f is not regular at P.

($\Rightarrow$:) Suppose $f = \frac{g}{h}$ with $g, h \in k[X]$ & $h(P) \neq 0$. But $h \in I$ ($\because hf \in k[X]$)

$\overset{(\because P \in V(I))}{\Longrightarrow}$ $h(P) = 0$. So f is not regular at P.

Conversely, if f is not regular at P and $h \in I$

then $\underset{g}{\underbrace{hf}} \in k[X]$. Hence $h(P) = 0$ ($\because f = \frac{g}{h}$ & f is not reg at P)

$\Rightarrow$ $P \in V(I)$.

Claim $\Rightarrow$ 1 since $I \neq 0 \Rightarrow V(I) \subsetneq X$.

② Domain$(f) = X$, let $I = (k[X] : f)$ then

$V(I) = \phi \xRightarrow{HN} I = k[X]$

$X = V(P) \quad P \subseteq k[x_1, \dots, x_n]$, $k = \bar{k}$

$k[X] = k[x_1, \dots, x_n]/P \leftarrow$

$\underset{\text{ideal}}{I \subseteq k[X]} \longleftrightarrow \tilde{I} \subset k[x_1, \dots, x_n]$ containing P

$$\begin{array}{ccc} V(I) & = & V(\tilde{I}) \\ \cap & & \\ X & & \\ \cap & & \\ \mathbb{A}^n & & \end{array}$$

if $V(\tilde{I}) = \phi \Longrightarrow \tilde{I} = k[x_1, \dots, x_n]$
$\Downarrow$
$I = k[X]$

③ Exc