Lecture 10: Tensor products

16 February 2021
16:45

<u>Def^n</u> Let M & N be R-modules. An R-module T together with an R-bilinear map $\varphi: M \times N \longrightarrow T$

(i.e. $\varphi(m, n_1 + r n_2) = \varphi(m, n_1) + r\varphi(m, n_2)$ $\forall m \in M$ $\forall n_1, n_2 \in N$ & $\forall r \in R$

$|||^{ly}$ $\varphi(m_1 + r m_2, n) = \varphi(m_1, n) + r\varphi(m_2, n)$ $\forall m_1, m_2 \in M$, $\forall n \in N$ & $\forall r \in R$)

is said to be a <u>tensor product of M & N over R</u> if given any R-bilin map $\psi: M \times N \longrightarrow A$ where A is an R-mod there exists a unique R-mod homo $\theta: T \longrightarrow A$ s.t. $\theta \circ \varphi = \psi$.

$$\begin{tikzcd} M \times N \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & T \arrow[d, "\exists!\theta"] \\ & A \end{tikzcd}$$

<u>Prop:</u> T exist and is unique upto unique isomorphism. $\boxed{T}$ and it is denoted by $M \underset{R}{\otimes} N$.

$$M \times N \xrightarrow{i} F_{M \times N} \xrightarrow{q} M \underset{R}{\otimes} N$$

$$M \underset{R}{\otimes} N \ni \alpha = \sum_{i \text{ finite}} r_i \, m_i \otimes n_i$$

<u>Basic notation & properties</u>

$\varphi: M \times N \longrightarrow M \underset{R}{\otimes} N$ be tensor product of M & N over R.

Then $\varphi(m, n)$ is denoted by <u>$m \otimes n$</u> for $m \in M$ & $n \in N$.

1) $m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$; $(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$ $\forall m_1, m_2, m \in M$, $n_1, n_2, n \in N$

2) $r(m \otimes n) = (rm \otimes n) = m \otimes rn$ $\forall m \in M$, $n \in N$ & $r \in R$.

3) $0_M \otimes n = m \otimes 0_N = 0_{M \underset{R}{\otimes} N}$ $\forall m \in M$ & $n \in N$.

<u>Examples:</u> ① $R = \mathbb{Z}$, $M = \mathbb{Z}$, $N = \mathbb{Z}$

$\mathbb{Z} \underset{\mathbb{Z}}{\otimes} \mathbb{Z} = ?$

$\cong \mathbb{Z}$

$$\varphi: \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$$
$$(a, b) \longmapsto ab$$

$\psi: \mathbb{Z} \times \mathbb{Z} \longrightarrow M$ is a bil map & M a $\mathbb{Z}$-mod

$$\theta: \mathbb{Z} \longrightarrow M$$
$$1 \longmapsto \psi(1,1)$$
$$a \longmapsto \psi(a,1) \quad \mathbb{Z}\text{-lin}$$

$\theta \circ \varphi(a, b) = \theta(ab) = \psi(ab, 1) = b\psi(a, 1) = \psi(a, b)$ $\forall a, b \in \mathbb{Z}$

$\Longrightarrow \theta \circ \varphi = \psi$. Check uniqueness of θ

Hence $\mathbb{Z} \underset{\mathbb{Z}}{\otimes} \mathbb{Z} \cong \mathbb{Z}$.

② M an R-mod $M \underset{R}{\otimes} 0 = 0$

Prop: R a ring, A, B, C R-modules. Then following holds.

① $R \otimes_R A \cong A$ $\quad r\otimes a \longmapsto ra \quad \forall r\in R \,\&\, a\in A$

② $A \otimes_R B \cong B \otimes_R A$ $\quad a\otimes b \longmapsto b\otimes a \quad \forall a\in A \,\&\, b\in B$

HW ③ $(A \otimes_R B) \otimes_R C \cong A \otimes_R (B \otimes_R C)$ $\quad (a\otimes b)\otimes c \longmapsto a\otimes(b\otimes c)$

Ex ④ $(A\oplus B) \otimes_R C \cong (A \otimes_R C) \oplus (B \otimes_R C)$ $\quad (a,b)\otimes c \longmapsto (a\otimes c, b\otimes c)$

⑤ $S\subseteq R$ multiplicative subset then

$S^{-1}A \cong S^{-1}R \otimes_R A$ $\quad \frac{ra}{s} \longleftarrow \frac{r}{s}\otimes a$

HW ⑥ $I\subseteq R$ ideal then $R/I \otimes_R M \cong M/IM$

$(r+I)\otimes m \longmapsto rm + IM$

Pf: ② $A \otimes_R B \cong B \otimes_R A$

$(a,b) \longmapsto a\otimes b$

$\varphi_1 : A\times B \to A\otimes B$

$\varphi_2 : B\times A \to B\otimes A$

$(b,a) \longmapsto b\otimes a$

$\psi : A\times B \longrightarrow B \otimes_R A$

$(a,b) \longmapsto b\otimes a$

$\varphi_2(b,a)$

It is trivial to check that ψ is bilinear.
(as φ_2 is bilinear)

Hence by defn of tensor product $\exists$,

$$\theta : A \underset{R}{\otimes} B \longrightarrow B \underset{R}{\otimes} A \quad R\text{-linear}$$

s.t. $\theta \circ \varphi_1(a,b) = \psi(a,b)$

$$\theta(a \otimes b) = b \otimes a$$

$|||^{ly}$ $\theta' : B \underset{R}{\otimes} A \longrightarrow A \underset{R}{\otimes} B$ R-linear

$$b \otimes a \longmapsto a \otimes b$$

Note $\left.\begin{aligned} \theta \circ \theta'(b \otimes a) &= b \otimes a \\ \& \ \theta' \circ \theta(a \otimes b) &= a \otimes b \end{aligned}\right\} \forall\ b \in B \ \& \ a \in A$

But $\{a \otimes b \mid a \in A, b \in B\}$ generate $A \otimes B$
& $\{b \otimes a \mid a \in A, b \in B$ " $B \otimes A$

Hence $\theta \circ \theta' = id_{B \otimes A}$ & $\theta' \circ \theta = id_{A \otimes B}$

(15) $\varphi : S^{-1}R \times A \longrightarrow S^{-1}A$

$$\left(\frac{r}{s}, a\right) \longmapsto \frac{r}{s}\cdot\frac{a}{1} = \frac{ra}{s}$$

φ is well-defined and bilinear

$$\varphi\left(\frac{r_1}{s_1} + \frac{r_2}{s_2}, a\right) = \varphi\left(\frac{s_2 r_1 + s_1 r_2}{s_1 s_2}, a\right)$$

$$= \frac{(s_2 r_1 + s_1 r_2)\cdot a}{s_1 s_2}$$

$$= \frac{s_2 r_1 a}{s_1 s_2} + \frac{s_1 r_2 a}{s_1 s_2}$$

$$= \frac{r_1 a}{s_1} + \frac{r_2 a}{s_2}$$

$$= \varphi\left(\frac{r_1}{s_1}, a\right) + \varphi\left(\frac{r_2}{s_2}, a\right)$$

check the rest.

Hence $\exists$ R-lin map

$$\theta : S^{-1}R \otimes A \longrightarrow S^{-1}A$$

$$\left(\frac{r}{s}, a\right) \longmapsto \frac{ra}{s}$$

Want: $\theta' : S^{-1}A \longrightarrow S^{-1}R \otimes_R A$ $\quad \theta'\left(\frac{a}{s}\right) = \frac{1}{s} \otimes a$

check θ' is well-defined.

$\frac{a'}{s'} = \frac{a}{s} \Rightarrow \exists u \in S$ s.t. $u(sa' - s'a) = 0$ ......

$$\frac{1}{s} \otimes a = \frac{us'}{us's} \otimes a$$
$$= \frac{1}{us's} \otimes us'a$$
$$= \frac{1}{us's} \otimes usa'$$
$$= \frac{us}{us's} \otimes a'$$
$$= \frac{1}{s'} \otimes a'$$

& θ' is R-linear.

$$\theta \circ \theta'\left(\frac{a}{s}\right) = \theta\left(\frac{1}{s} \otimes a\right)$$
$$= \frac{a}{s}$$

$\theta \circ \theta' = id_{S^{-1}A}$

$$\theta' \circ \theta\left(\frac{r}{s} \otimes a\right) = \theta'\left(\frac{ra}{s}\right)$$
$$= \frac{1}{s} \otimes ra$$
$$= \frac{r}{s} \otimes a$$

So $\theta' \circ \theta = id_{S^{-1}R \otimes A}$

$\left(\because \left\{ \frac{r}{s} \otimes a \mid r \in R, s \in R, a \in A \right\} \right.$ gen $S^{-1}R \otimes_R A$. $\square$