Lecture 22: Going down theorem, regular functions

23 March 2021
17:18

Thm: (Going down thm) Let $A \subseteq B$ be an integral ext of integral domains. Assume A is normal. Let $P_0 \subseteq P_1$ be prime ideals in A and Q_1 be a prime ideal of B s.t. $Q_1 \cap A = P_1$. Then there exist a prime ideal Q_0 of B s.t. $Q_0 \cap A = P_0$ & $Q_0 \subseteq Q_1$.

Lemma Let $A \subseteq B$ int extⁿ & P a prime ideal of A. Let $x \in PB$ then $x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0$ for some $n \geq 1$ & $a_0, \dots, a_{n-1} \in P$.

Pf: $x = \sum_{i=1}^{m} p_i b_i \quad p_i \in P$ & $b_i \in B$. Replacing B by the subring $A[b_1, \dots, b_m]$ may assume B is a finite A-module.

say $B = Ay_1 + \dots + Ay_n \quad y_i \in B$.

Now $xy_i = \sum_{i=1}^{n} r_{ij} y_i \quad 1 \leq i \leq n \quad r_{ij} \in P \quad \forall ij \in \{1, \dots, n\}$ $\left(\because xy_i \in PB \Rightarrow xy_i = \sum \alpha_k b_k \quad \alpha_k \in P, \ b_k \in B, \ b_k = \sum r'_{kj} y_j \right)$

$(xI - ((r_{ij})))\underset{\sim}{y} = \underset{\sim}{0} \qquad \underset{\sim}{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$\cdot \operatorname{Adj}(xI - ((r_{ij}))) \rightarrow$

$\Rightarrow \det(xI - ((r_{ij}))) = 0$

$\Rightarrow x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0$ where a_i involve r_{ij} hence are in P. ∎

Prop: Let A be normal domain & x be integral over A then the min poly of x over $\operatorname{frac}(A)$ has coeff in A.

Pf: $m_x(z)$ is the min poly of x in $K[z]$ where $K = \operatorname{frac}(A)$

& $f(z) \in A[z]$ is monic s.t. $f(x) = 0$. Then $m_x(z) | f(z) \Rightarrow$ all conjugates of x are int over $A \Rightarrow$ coeff of m_x are integral over A.

$\Rightarrow$ coeff m_x are in A, since A is normal. ∎

Pf of going down thm

Let $S_1 = A \setminus P_0$ & $S_2 = B \setminus Q_1$

Then $S := S_1 S_2 := \{ab \mid a \in S_1, b \in S_2\}$ is a mult subset.

Want : $P_0 S^{-1} B$ to be proper ideal of $S^{-1}B$. Then choose $\widetilde{Q}_0$ to be a maximal ideal of $S^{-1}B$ containing $P_0 S^{-1} B$ and let $Q_0 = \widetilde{Q}_0 \cap B$. Finally check $Q_0 \cap A = P_0$.

Claim: $P_0 B \cap S = \phi$

Pf: Suppose not and let $a \in S_1$ & $b \in S_2$ be s.t. $ab \in P_0 B$.

Now, $(ab)^n + r_{n-1}(ab)^{n-1} + \dots + r_0 = 0$ for some $r_i \in P_0$

(by Lemma)

Let $f(x) = x^n + r_{n-1}x^{n-1} + \dots + r_0$ & $g(x)$ be the minimal poly of ab over $K = \mathrm{frac}(A)$ then $g(x) \in A[x]$ and $f(x) = g(x) h(x)$ for some $h(x) \in A[x]$

(by Prop.)

In fact coeff of $g(x)$ & $h(x)$ are in P_0

because $g(x) = x^m + c_{m-1}x^{m-1} + \dots + c_0$ &

$h(x) = x^k + d_{k-1}x^{k-1} + \dots + d_0$ then

$$f(x) \pmod{P_0} = g(x)h(x) \pmod{P_0}$$

$$\Rightarrow \quad x^n \equiv g(x)h(x) \pmod{P_0}$$

$$\Rightarrow \quad g(x) \equiv x^m \pmod{P_0} \text{ \& } h(x) \equiv x^k \pmod{P_0}$$

$$\Rightarrow \quad c_i \text{ \& } d_j \in P_0 \quad \forall \ i, j.$$

$$g(x)h(x) \pmod{P_0} = x^n$$

i_0 be smallest s.t. $c_{i_0} \notin P_0$

& j_0 " " s.t. $d_{j_0} \notin P_0$

$$g(x) \pmod{P_0} \equiv x^m + \bar{c}_{m-1}x^{m-1} + \dots + \underset{\neq 0}{\bar{c}_{i_0}} x^{i_0}$$

$$h(x) \pmod{P_0} \equiv x^{\cancel{k}} + \bar{d}_{k-1}x^{k-1} + \dots + \underset{\neq 0}{\bar{d}_{j_0}} x^{k_0}$$

$$g(x)h(x) \pmod{P_0} \equiv x^n + \dots + \underset{\neq 0}{\bar{c}_{i_0}\bar{d}_j} x^{i_0+k_0}$$

Now $(ab)^m + c_{m-1}(ab)^{m-1} + \dots + c_0 = 0 \leftarrow$

$$\Rightarrow \quad b^m + \frac{c_{m-1}}{a} b^{m-1} + \dots + \frac{c_1 b}{a^{m-1}} + \frac{c_0}{a^m} = 0$$

is in P_0

is the min poly of b over K

$$\big(\because [K(b):K] = [K(ab):K] \big)$$

By Prop.

$$\Rightarrow \quad s_{m-1} = \frac{c_{m-1}}{a}, \dots, s_0 = \frac{c_0}{a^m} \in A$$

$$\Rightarrow \quad \underbrace{a s_{m-1}}_{= c_{m-1}}, \dots, \underbrace{a^m s_0}_{= c_0} \in P_0$$

($\because P_0$ is prime)

But $a \in A \setminus P_0$, hence $s_0, \dots, s_{m-1} \in P_0$

$\Rightarrow \quad b^m \in P_0 B \subseteq Q_1$ contradicting

$b \notin Q_1$ ..

Let $\widetilde{Q}_0$ be the maximal ideal of $S^{-1}B$ containing $P_0S^{-1}B$.

Let $Q_0 = \widetilde{Q}_0 \cap B$. Then $P_0 \subseteq Q_0 \cap A$.

Let $x \in Q_0 \cap A$ if $x \notin P_0$ then x is a unit in $S^{-1}B$ & $x \in Q_0 \subseteq \widetilde{Q}_0$ contradicting $\widetilde{Q}_0$ is a proper ideal.

Hence $Q_0 \cap A = P_0$.

<u>Cor</u>: $A \subseteq B$ int ext^n, A normal domain, B domain. Let $P \in \mathrm{spec}(A)$ & $Q \in \mathrm{spec}(B)$. Then $Q \cap A = P$ iff Q is a minimal prime of PB.

Pf: ($\Rightarrow$): $Q \cap A = P$, suppose Q_1 is a prime ideal of B s.t.

$PB \subseteq Q_1 \subseteq Q$

$\Rightarrow P \subseteq Q_1 \cap A \subseteq Q \cap A = P$

$\Rightarrow Q_1 = Q$ ($\because A \subseteq B$ is int ext^n)

($\Leftarrow$): Q is a minimal prime of PB

$Q \cap A \supseteq P$ if $Q \cap A \supsetneq P$

then by going down $\exists Q_0 \subsetneq Q$ prime ideal of B s.t. $Q_0 \cap A = P$.

This contradicts Q is minimal prime of PB. $\blacksquare$

<u>Def</u>^{n}: Let X be a affine variety over a field k. Let $\mathcal{O}(X)$ or $\mathcal{O}_X$ or $k[X]$ be its coordinate ring.

$$k[X] = k[x_1, \dots, x_n]/P \quad \text{where } Z(P) = X$$

P a prime ideal of $k[x_1, \dots, x_n]$

Let $f \in \mathcal{O}(X)$ then f defines a function from X to k.

Let $\underline{a} = (a_1, \dots, a_n) \in X$ & $\tilde{f} \in k[x_1, \dots, x_n]$ $\tilde{f} \pmod P = f$

$f(\underline{a}) := \tilde{f}(\underline{a}) \in k$ $\qquad f : X \longrightarrow k$

Note it is well-defined function $\tilde{g} \pmod P = f$ then

$$\tilde{f} - \tilde{g} \in P \Rightarrow (\tilde{f} - \tilde{g})(\underline{a}) = 0 \Rightarrow \tilde{g}(a) = \tilde{f}(a).$$

These elements of the coordinate ring are called regular functions on X.

Another way: regular function on X is a function from X to $k = \mathbb{A}^1$ given by polynomials

i.e. reg functions on X is same as morphisms of varieties from X to $\mathbb{A}^1_k$

$$\Updownarrow$$

k-alg homo $k[x] \rightarrow \mathcal{O}(X)$

$$\Updownarrow$$

elements of $\mathcal{O}(X)$