DIFFERENTIAL TOPOLOGY - LECTURE 4 - I

1. Introduction

Given a function defined on an open set $U \subseteq \mathbb{R}^N$, whether or not the function is differentiable at a point $x \in U$ depends only on the nature of the function in a neighborhood of the point x. In this sense differentiability is local in nature. This is, therefore, also true about functions defined between manifolds. Since manifolds are locally like euclidean spaces one would expect that any local phenomena exhibited by functions in euclidean space should also have an analogue in the manifold world. Indeed we have already seen two examples of this, namely, composition of two smooth functions is smooth and that the chain rule holds for composition of smooth maps between manifolds.

Another result that is local in nature and we are familiar with, at least in the euclidean setup, is the Inverse function theorem. It is quite believable and infact not difficult to prove that the Inverse function theorem should be true for maps between manifolds. We shall quickly prove this below.

Recall that the Inverse function theorem gives a local topological/differential conclusion about the function based on certain algebraic assumption on the derivative of the function. This is a recurrent theme in differential topology: the algebriac properties of the (higher) derivatives of a function implies a lot of the local behaviour of the function.

Below we shall discuss what local conclusions about the function can be drawn when different restrictions are imposed on the derivative. This will lead us to two important results namely the local immersion theorem and the local submersion theorem. We shall discuss these theorems and their consequences in this and the next set of notes.

2. Inverse function theorem and the Local immersion theorem

Let us begin by trying to understand the statement of the Inverse function theorem for maps between manifolds. We first remind ourselves of certain definitions.

Recall that a map $f : X \longrightarrow Y$ between manifolds is a local diffeomorphism at $x \in X$ if f maps a neighborhood[1] of x diffeomorphically onto a neghborhood of $f(x)$. The map f itself is a local diffeomorphism if it is a local diffeomorphism at each $x \in X$.

Suppose that $f : X \longrightarrow Y$ is a local diffeomorphism at x. Then there exist neighborhoods U of x and V of $f(x)$ such that $f : U \longrightarrow V$ is a diffeomorphism. This implies that the derivative

$$df_x : T_x(X) \longrightarrow T_{f(x)}(Y)$$

is an isomorphism. The Inverse function theorem is the statement that the converse is also true.

Theorem 2.1. (Inverse function theorem) Suppose that $f : X \longrightarrow Y$ is such that

$$df_x : T_x(X) \longrightarrow T_{f(x)}(Y)$$

is an isomorphism for some $x \in X$. Then f is a local diffeomorphism at x.

[1] Neighborhoods are always open.

Proof. We shall use the Inverse function theorem in euclidean spaces to prove this. Fix parametrizations

$$\varphi : U \longrightarrow X; \quad \psi : V \longrightarrow Y$$

about x and $f(x) = y$ respectively with $\varphi(0) = x$ and $\psi(0) = y$. As before, we set $g = \psi^{-1} \circ f \circ \varphi$ to get commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \uparrow{\scriptstyle\varphi} & & \uparrow{\scriptstyle\psi} \\ U & \xrightarrow[g]{} & V \end{array} \qquad \qquad \begin{array}{ccc} T_x(X) & \xrightarrow{df_x} & T_{f(x)}Y \\ \uparrow{\scriptstyle d\varphi_0} & & \uparrow{\scriptstyle d\psi_0} \\ \mathbb{R}^k & \xrightarrow[dg_0]{} & \mathbb{R}^k \end{array}$$

Since $d\varphi_0, df_x$ and $d\psi_0$ are all isomorphisms we conclude that dg_0 is also an isomorphism. Since g is a map between open subsets of euclidean spaces we conclude, by the Inverse function theorem, that g is a local diffeomorphism at 0. Thus there exist open sets $0 \in U_1 \subseteq U$ and $0 \in V_1 \subseteq V$ such that

$$g : U_1 \longrightarrow V_1$$

is a diffeomorphism. Keeping in mind that φ and ψ are themselves diffeomorphisms, the commutativity of the first diagram now implies that

$$f : \varphi(U_1) \longrightarrow \psi(V_1)$$

is now a diffeomorphism. This shows that f is a local diffeomorphism at x. This completes the proof. $\square$

Remark 2.2. Observe that since $f : \varphi(U_1) \longrightarrow \psi(V_1)$ is a diffeomorphism, we have that df_p is an isomorphism for all $p \in \varphi(U_1)$. This shows that if for a smooh map $f : X \longrightarrow Y$, df_x is an isomorphism for some $x \in X$, then the derivative continues to be an isomorphism in a neighborhood of x. It is important to note that the Inverse function theorem is a local theorem in that the function may be a local diffeomorphism at every point but fail to be a global diffeomorphism.

Here is an example that illustrates one of the points made in the above remark.

Example 2.3. Consider the map $p : \mathbb{R} \longrightarrow \mathbb{S}^1$ defined by $p(t) = (\cos t, \sin t)$. Then p is smooth and a local diffeomorphism but fails to be one-one. Let us try to prove that the derivative

$$dp_a : T_a(\mathbb{R}) = \mathbb{R} \longrightarrow T_{p(a)}(\mathbb{S}^1)$$

is an isomorphism for every $a \in \mathbb{R}$. To prove this let us look at the commutative diagrams

$$\begin{tikzcd} \mathbb{R} \arrow[d, "p"'] \arrow[dr, "f"] & \\ \mathbb{S}^1 \arrow[r, "i"'] & \mathbb{R}^2 \end{tikzcd} \qquad \qquad \begin{tikzcd} T_a(\mathbb{R}) \arrow[d, "dp_a"'] \arrow[dr, "df_a"] & \\ T_{p(a)}(\mathbb{S}^1) \arrow[r, "di_{p(a)}"'] & T_{i(a)}(\mathbb{R}^2) \end{tikzcd}$$

Here $i : \mathbb{S}^1 \longrightarrow \mathbb{R}^2$ is the inclusion map of the submanifold $\mathbb{S}^1$ of $\mathbb{R}^2$ and $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ is the map $f(t) = (\cos t, \sin t)$. By (Exercise 2.9, Lecture 3), the derivatve di_a is injective. Observe that

$$df_a = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$$

Thus df_a is also injective. This implies the composition $di_{p(a)} \circ dp_a$ is also injective. This forces dp_a to be injective. Since $T_a(\mathbb{R})$ and $T_{p(a)}(\mathbb{S}^1)$ are both 1-dimensional, dp_a must be an isomorphism.

By the Inverse function theorem p is a local diffeomorphism at a. Since a is arbitrary, p is a local diffeomorphism.

Given a smooth map $f : X \longrightarrow Y$ between manifolds there are other algebraic restrictions that one can impose on the derivative

$$df_x : T_x(X) \longrightarrow T_{f(x)}(Y)$$

of f at x. This leads us to the following definitions.

Definition 2.4. Let $f : X \longrightarrow Y$ be a smooth map between manifolds. We say that f is an *immersion* (respectively a *submersion*) at x if the derivative

$$df_x : T_x(X) \longrightarrow T_{f(x)}(Y)$$

at x is injective (respectively surjective). The map f itself is called an immersion (respectively a submersion) if it is an immersion (respectively a submersion) at each $x \in X$.

Before looking at examples some remarks are in order. Observe that if $f : X \longrightarrow Y$ is an immersion, then $\dim(X) \leq \dim(Y)$. If the dimensions of X and Y are equal, then a map $f : X \longrightarrow Y$ is an immersion if and only if it is a submersion if and only if it is a local diffeomorphism.

Here are some examples.

Example 2.5. The maps p and f in Example 2.3 above are immersions. p is in addition a submersion too.

Example 2.6. Let $k \leq N$. The inclusion map $i : \mathbb{R}^k \longrightarrow \mathbb{R}^N$ defined by

$$i(x_1, \dots, x_k) = (x_1, \dots x_k, 0, \dots, 0)$$

is an immersion. The map i is called the *canonical immersion* of $\mathbb{R}^k$ into $\mathbb{R}^N$.

Example 2.7. Let $k \leq N$. The projection map $j : \mathbb{R}^N \longrightarrow \mathbb{R}^k$ defined by

$$j(x_1, \dots, x_N) = (x_1, \dots, x_k)$$

is a submersion. The map j is called the *canonical submersion* of $\mathbb{R}^N$ onto $\mathbb{R}^k$.

The Inverse function theorem describes the behavior of a function in a neighborhood of point at which the derivative is invertible. We now wish to understand what can one say about the local behaviour of a funtion in a neighborhood of a point at which the function is an immersion. The behavior of a function in a neighborhood of a point at whiich the function is an immersion is described by the local immersion theorem.

Theorem 2.8. (Local Immersion theorem) Let X, Y be manifolds of dimension k, ℓ respectively. Suppose that $f : X \longrightarrow Y$ is an immersion at $x \in X$. Then there exist parametrizations

$$\varphi : U \longrightarrow X; \quad \psi : V \longrightarrow Y$$

about x and $f(x) = y$ respectively such that

$$\psi^{-1} \circ f \circ \varphi = i.$$

Here i is the canonical immersion of $\mathbb{R}^k$ into $\mathbb{R}^\ell$.

Proof. A composition of the form $\psi^{-1} \circ f \circ \varphi$ as in the theorem is called a *local representation* of the function f about $x \in X$. Thus the local immersion theorem basically says that if f is an immersion at x, then some local representation of f is the canonical immersion.

The proof is in some ways an exercise in linear algebra. We start with arbitrary local parametrizations

$$\varphi : U \longrightarrow X; \quad \psi : V \longrightarrow Y$$

about x and $y = f(x)$ with $\varphi(0) = x$ and $\psi(0) = y$ to get a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\varphi"] \arrow[r, "g"'] & V \arrow[u, "\psi"'] \end{tikzcd}$$

Remember that $g = \psi^{-1} \circ f \circ \varphi$. If we modify φ and ψ, the function g changes accordingly. The whole point is that we can do this, that is modify φ and ψ, in such a way so that g gets modified to the canonical immersion. We proceed as follows.

First observe that the composition

$$dg_0 = d\psi_0^{-1} \circ df_x \circ d\varphi_0$$

is injective since f is an immersion at x. Thus the matrix of

$$d\psi_0^{-1} \circ df_x \circ d\varphi_0 : \mathbb{R}^k \longrightarrow T_x(X) \longrightarrow T_y(Y) \longrightarrow \mathbb{R}^\ell$$

has k independent rows. We may now post compose with a change of basis isomorphism $T : \mathbb{R}^\ell \longrightarrow \mathbb{R}^\ell$ (a composition of elementary row operations) to get a composition

$$T \circ d\psi_0^{-1} \circ df_x \circ d\varphi_0 : \mathbb{R}^k \longrightarrow T_x(X) \longrightarrow T_y(Y) \longrightarrow \mathbb{R}^\ell \longrightarrow \mathbb{R}^\ell$$

where now the matrix of the composition has the first k rows independent. Observe that T is a diffeomorphism. Setting $V' = T(V)$ we get a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\varphi"] \arrow[r, "g'"'] & V' \arrow[u, "\psi' = \psi \circ T^{-1}"'] \end{tikzcd}$$

Observe that ψ' is still a parametrization and $g' = \psi'^{-1} \circ f \circ \varphi$. What we have now achieved (after modifying the parametrization ψ to ψ') is that the matrix of

$$dg'_0 = d\psi_0'^{-1} \circ df_x \circ d\varphi_0 = T \circ d\psi_0^{-1} \circ df_x \circ d\varphi_0$$

has first k rows independent and equals (say)

$$\begin{pmatrix} A \\ B \end{pmatrix}$$

where A is a $(k \times k)$ (invertible) matrix and B is a $((\ell - k) \times k)$ matrix. We now post compose

$$T \circ d\psi_0^{-1} \circ df_x \circ d\varphi_0 : \mathbb{R}^k \longrightarrow T_x(X) \longrightarrow T_y(Y) \longrightarrow \mathbb{R}^\ell \longrightarrow \mathbb{R}^\ell$$

with the linear isomorphism

$$S = \begin{pmatrix} A^{-1} & 0 \\ -BA^{-1} & I_{\ell - k} \end{pmatrix} : \mathbb{R}^\ell \longrightarrow \mathbb{R}^\ell.$$

The composition

$$S \circ T \circ d\psi_0^{-1} \circ df_x \circ d\varphi_0 : \mathbb{R}^k \longrightarrow T_x(X) \longrightarrow T_y(Y) \longrightarrow \mathbb{R}^\ell \longrightarrow \mathbb{R}^\ell \longrightarrow \mathbb{R}^\ell$$

now has a matrix of the form

$$\begin{pmatrix} I_k \\ 0 \end{pmatrix}.$$

If we now set

$$V'' = S(V')$$

we get a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\varphi"] \arrow[r, "g''"'] & V'' \arrow[u, "\psi''=\psi\circ T^{-1}\circ S^{-1}"'] \end{tikzcd}$$

where g'' is as usual. Observe that ψ'' is a parametrization. It is also clear that the matrix of

$$dg''_0 = d\psi''^{-1}_0 \circ df_x \circ d\varphi_0 = S \circ T \circ d\psi_0^{-1} \circ df_x \circ d\varphi_0$$

has the form

$$\begin{pmatrix} I_k \\ 0 \end{pmatrix}.$$

The upshot of the above discussion is that we might as well have assumed (and we do so now) that the matrix of dg_0 has the form

$$\begin{pmatrix} I_k \\ 0 \end{pmatrix}.$$

Now consider the map $G : U \times \mathbb{R}^{\ell-k} \longrightarrow \mathbb{R}^\ell$ defined by

$$G(x, z) = g(x) + (0, z).$$

Since

$$dG_0 = I_\ell$$

we have, by the Inverse function theorem, that G maps a neighborhood W of $0 \in \mathbb{R}^\ell$ diffeomorphically onto a neighborhood W' of $0 \in \mathbb{R}^\ell$. Now observe that

$$g = G \circ i$$

where

$$i : U \subseteq \mathbb{R}^k \longrightarrow \mathbb{R}^\ell$$

is the canonical immersion. Thus,

$$\psi^{-1} \circ f \circ \varphi = G \circ i.$$

Using the fact that G is invertible, after shrinking the open set U and V if required wehave

$$(\psi \circ G)^{-1} \circ f \circ \varphi = i$$

So finally we modify ψ to the parametrization $(\psi\circ G)$ to ensure that g equals the canonical immersion. This completes the proof. □

Remark 2.9. In the Local immersion theorem let the coordinate system φ^{-1} be given by

$$\varphi^{-1} = (x_1, \ldots, x_k).$$

Thus a point $p \in V$ maybe thought of as the point $(x_1(p), \ldots, x_k(p))$ The function f, in a neighbourhood of x now looks like

$$f(x_1, \ldots x_k) = (x_1, \ldots x_k, 0, \ldots 0).$$

The local immersion theorem can therefore be also stated as: Suppose that $f : X \longrightarrow Y$ is an immersion at x. Then there exist local coordinates around x and $f(x)$ such that

$$f(x_1, \dots x_k) = (x_1, \dots x_k, 0, \dots 0).$$

Although the proof might seem long, the major part of the proof involved convincing ourselves that even though we start with arbitrary parametrizations φ and ψ we can assume without loss of generality that the matrix of of dg_0 has a required suitable form. Having achieved this, the remaining of the proof is a somewhat routine application of the Inverse function theorem. Note that we never modified the parametrization φ.

Here is an interesting consequence of the Local immersion theorem.

Corollary 2.10. Suppose $f : X \longrightarrow Y$ is an immersion at x. Then f is an immersion in a neighborhood of x.

Proof. By the Local immersion theorem some local representation $\psi^{-1} \circ f \circ \varphi$ equals the canonical immersion i. Thus the equality

$$f = \psi \circ i \circ \varphi^{-1}$$

is valid in a neighborhood of x. Taking derivatives (at proper points) we have that

$$df = d\psi \circ i \circ d\varphi^{-1}$$

is valid in a neighborhood of x. Since all the maps on the right hand side are injective, so is df. This completes the proof. $\square$

A subtle point is discussed in the following remark.

Remark 2.11. Suppose that $f : X \longrightarrow Y$ is an immersion at $x \in X$. Then there is a local representation $\psi^{-1} \circ f \circ \varphi$ that equals the canonical immserion i. Hence the equality

$$\psi^{-1} \circ f \circ \varphi = i$$

is valid in a neighborhood of 0 so that we have

$$f = \psi \circ i \circ \varphi^{-1}$$

in a neghborhood of x. Now observe that the canonical immersion maps open sets diffeomorphically onto its image. Since the parametrizations φ and ψ are already diffeomorphisms, we conclude that f maps a small enough neighborhood U of x diffeomorphically onto its image $f(U)$. This means every point in $f(X)$ is contained in a *subset* that is diffeomorphic to an open set in the euclidean space. However, this image $f(U)$ may not be open in $f(X)$. What we are pointing out is that the image of an immersion may not be a manifold.

Here are two examples that show that the image of a $1-1$ immersion need not be a manifold.

Example 2.12. Consider the map $p : \mathbb{R} \longrightarrow \mathbb{S}^1$ defined by

$$p(t) = (\cos 2\pi t, \sin 2\pi t).$$

We know that (see Example 2.3) p is a local diffeomorphism. The map

$$G : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{S}^1 \times \mathbb{S}^1$$

defined by $G(t, s) = (p(t), p(s))$ is therefore a local diffeomorphism (see exercises Lecture 4 - II). Now let L be a line through the origin in $\mathbb{R}^2$ having irrational slope. Then L is a manifold. Since G is a local diffeomorphism

$$G/L : L \longrightarrow \mathbb{S}^1 \times \mathbb{S}^1$$

is an immersion (see exercises Lecture 4 - II). Let $(t, at), (t', at') \in L$ where a is the slope of L. Assume that $(G/L)(t, at) = (G/L)(t', at')$. So that

$$(\cos 2\pi t, \sin 2\pi t, \cos 2\pi at, \sin 2\pi at)) = (\cos 2\pi t', \sin 2\pi t', \cos 2\pi at', \sin 2\pi at').$$

This implies that $(t - t') \in \mathbb{Z}$ and $a(t - t') \in \mathbb{Z}$. This forces $t = t'$ as a is irrational. Thus G/L is one-one. It is an exercise to show that the image of L under G is not locally connected. Hence the image is not a manifold. This example shows that the image of a one-one immersion need not be a manifold.

Example 2.13. Consider the map $\beta : (-\pi, \pi) \longrightarrow \mathbb{R}^2$ defined by

$$\beta(t) = (\sin 2t, \sin t).$$

Clearly β is $1 - 1$ and since the derivative

$$d\beta_t = (2\cos 2t, \cos t)^t$$

is injective, we conclude that f is an immersion. The image of f is compact and is the figure of eight[2] and so cannot be a manifold. The justifications are left as an exercise.

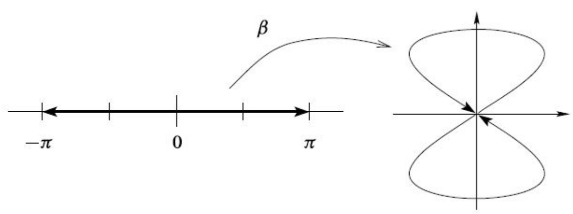

FIGURE 1. Lemniscate of Bernoulli.

Exercises for this set of notes will appear in the next set of of notes after we state the Local submersion theorem.

[2]Image source : Wikipedia.