Lecture 15: Hilbert-Nullstellensatz and its applications

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HN (Alg version) k a field

$k[x_1,\dots,x_n] = k(x_1,\dots,x_n) \Rightarrow x_1,\dots,x_n$ is alg over k.

k-alg

poly ring

Cor: Let k be an alg closed field. Then $m \subseteq k[z_1,\dots,z_n]$ is a maximal ideal iff $\exists\, a_1,\dots,a_n \in k$ s.t. $m = (z_1-a_1,\dots,z_n-a_n)$

Pf: $(\Leftarrow)$ ✓

$(\Rightarrow)$ $k[z_1,\dots,z_n]/m$ is field. Let $a_i = \bar{z}_i$

then $k[a_1,\dots,a_n]$ is a field hence by HN $a_1,\dots,a_n$ are alg over k. But k is alg closed $\Rightarrow a_i \in k$.

$\Rightarrow \bar{z}_i - a_i = 0$ in $k[z_1,\dots,z_n]/m$ $\quad 1 \le i \le n$

$\Rightarrow z_i - a_i \in m \quad 1 \le i \le n$

$\Rightarrow (z_1-a_1,\dots,z_n-a_n) \subseteq m$

But LHS is maximal $\Rightarrow m = (z_1-a_1,\dots,z_n-a_n)$. ∎

Thm: k alg closed field. Let $I \subseteq k[x_1,\dots,x_n]$ (poly ring) be an ideal then

$$\mathcal{I}(Z(I)) = \sqrt{I}.$$

Pf: $\sqrt{I} \subseteq \mathcal{I}(Z(I))$ ✓

Let $f \in \mathcal{I}(Z(I))$, suppose $f \notin \sqrt{I}$ then

$\underbrace{\{1, f, f^2, \dots\}}_{S} \cap I = \phi$.

$\Rightarrow I S^{-1}k[x_1,\dots,x_n]$ is a proper ideal of $S^{-1}k[x_1,\dots,x_n] = k[x_1,\dots,x_n,\frac{1}{f}] \underset{\varphi}{\cong} k[x_1,\dots,x_n,y]/(fy-1)$

Let m be maximal ideal of $k[x_1,\dots,x_n,\frac{1}{f}]$ containing $IS^{-1}k[x_1,\dots,x_n]$ then $\varphi(m)$ is a maximal ideal of $k[x_1,\dots,x_n,y]/(fy-1)$

$q: k[x_1,\dots,x_n,y] \to k[x_1,\dots,x_n,y]/(fy-1)$

By H.N, $q^{-1}(\varphi(m)) = (x_1-a_1,\dots,x_n-a_n, y-a_{n+1})$ for some $a_1,\dots,a_{n+1} \in k$.

$\dfrac{k[x,y]}{(fy-1)} \Big/ \varphi(m)$

since $I \subseteq q^{-1}\varphi(m)$, $(a_1,\dots,a_n,a_{n+1}) \in Z(I) \subseteq \mathbb{A}^{n+1}$

$\Rightarrow (a_1,\dots,a_n) \in Z(I) \subseteq \mathbb{A}^n$

$Z(I) \supseteq Z(q^{-1}\varphi(m)) = (a_1,\dots,a_n,a_{n+1})$

$\Downarrow f \in \mathcal{I}(Z(I))$

$f(a_1,\dots,a_n) = 0$. Also $fy - 1 \in q^{-1}(\varphi(m))$

$\Rightarrow f(a_1,\dots,a_n)a_m - 1 = 0$

$\Rightarrow f(a_1,\dots,a_n) \neq 0$

contradiction! ∎

Prop: Let k be a field and $f: A \to B$ be a k-alg homo between f.g. k-algebras. Then $f^{-1}(m)$ is a maximal ideal of A for m a maximal ideal of B.

$\circledast$ $f: A \to B$ be a ring homo & P a prime ideal of B then $f^{-1}(P)$ is a prime ideal of A.

$\circledast$ Let $\underset{\text{ideal}}{I} \subseteq k[x_1, \dots, x_n]$ poly ring.

then $Z(I) = \{(a_1, \dots, a_n) \in \mathbb{A}^n_k \mid f(a_1, \dots, a_n) = 0 \ \forall f \in I\}$

= set of maximal ideals of $k[x_1, \dots, x_n]$ containing I

= set of max ideals of $\underbrace{k[x_1, \dots, x_n]/I}_{\text{coord ring of } Z(I)}$

(*) If k is not alg closed then $\mathbb{A}^n_k$ is the set of maximal ideals of $k[x_1, \dots, x_n]$ and $Z(I)$ for $I \subseteq k[x_1, \dots, x_n]$ is the set max ideals cont. I.

An affine var is an irred alg set with subsp top.

Equivalently $\downarrow$ is $\operatorname{mspec}(R)$ where R is a f.g. k-alg which is an integral domain.

$$\mathcal{J}_m(X) = \bigcap_{m \in X} m \quad X \subseteq \mathbb{A}^n_k$$

$$\boxed{\mathcal{J}_m(Z(I)) = \sqrt{I}}$$

$$q : k[x_1, \dots, x_n] \to k[x_1, \dots, x_n]/I$$

$$Z(I) = q^{-1}\left(\operatorname{mspec}\left(k[x_1, \dots, x_n]/I\right)\right)$$

$$\mathcal{J}_m(Z(I)) = q^{-1}\left(\operatorname{Jac}\left(k[x_1, \dots, x_n]/I\right)\right)$$

$$q^{-1}\left(\operatorname{Jac}\left(k[x_1, \dots, x_n]/I\right)\right) = \sqrt{I}$$

$$\Rightarrow \operatorname{nil}\left(k[x_1, \dots, x_n]/I\right) = \operatorname{Jac}\left(k[x_1, \dots, x_n]/I\right)$$

Cor Let A be a f.g. k-alg for a field k. then $\mathrm{Jac}(A) = \mathrm{nil}(A)$.

Pf: $\sqrt{0} \subseteq \mathrm{Jac}(A)$ ✓

$\bar{f} \in \mathrm{Jac}(A)$ if $\bar{f} \notin \sqrt{0} \Rightarrow \{1, \bar{f}, \bar{f}^2, \dots\} \cap \{0\} = \phi$

poly ring

$A = k[x_1, \dots, x_n]/I$, $\exists f \in k[x_1, \dots, x_n]$ s.t. $f + I = \bar{f}$ in A.

$S = \{1, f, f^2, \dots\} \cap I = \phi$

$\Rightarrow (I) \subseteq k[x_1, \dots, x_n, \frac{1}{f}]$ is a proper ideal.

Let m be max ideal of $k[x_1, \dots, x_n, \frac{1}{f}]$ containing I

Prop $\Rightarrow m' = m \cap k[x_1, \dots, x_n]$ is a maximal ideal of $k[x_1, \dots, x_n]$ containing I, but not f $\Rightarrow m' + I$ is a max ideal of A.

$f \notin m' \Rightarrow \bar{f} \notin m' + I$. But $\bar{f} \in m' + I$ since $\bar{f} \in \mathrm{Jac}(A)$ & $m' + I$ is a ~~max~~ ideal of A. contra.

applied to $k[x_1, \dots, x_n] \xhookrightarrow{i} k[x_1, \dots, x_n, \frac{1}{f}]$