Analysis 4 - Lecture 3

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Example: Let $K = \{z \in \mathbb{C} | |z| = 1\}$. Let $\mathcal{A}$ be the algebra generated by $\iota : z \mapsto z$ and the constant functions. That is,
$\mathcal{A} = \{\phi : z \mapsto \sum_{k=0}^{n} c_k z^k | c_k \in \mathbb{C}, n \in \mathbb{N} \cup \{0\}\}$. $\mathcal{A}$ is an algebra that separates points of K and nowhere vanishes. $\mathcal{A}$ is not dense in $C(K)$. Let $f \in C(K)$, $f : z \mapsto \overline{z}$. We claim that $f \notin \overline{\mathcal{A}}$. If $P_n \in \mathcal{A}$ such that $P_n \to f$. Then, for all large n and for all $z \in K$, $|P_n(z) - \overline{z}| = |zP_n(z) - 1| < \frac{1}{2}$. Then,
$\int_0^{2\pi} e^{i\theta} P_n(e^{i\theta}) d\theta = 0$. $|\int_0^{2\pi} (zP_n - 1) d\theta| = 2\pi \leq \frac{1}{2}.2\pi = \pi$. This is a contradiction. Therefore, $\mathcal{A}$ is not dense in $C(K)$.
Let $\mathcal{A}$ be an algebra in $C(X)$. We say that $\mathcal{A}$ is self adjoint if for any $f \in \mathcal{A}$, $\overline{f} \in \mathcal{A}$ where $\overline{f}(x) = \overline{f(x)}$ for all $x \in X$.
Proposition: If $\mathcal{A}$ is an algebra in $C(X)$ that separates points of X and nowhere vanishes, then $\overline{\mathcal{A}} = C(X)$ provided $\mathcal{A}$ is self adjoint.

Proof. Let $f \in C(X)$. Then, there are two functions $f_r, f_s \in C_{\mathbb{R}}(X)$ such that $f = f_r + if_s$. We have $f_r = \frac{f+\overline{f}}{2}$ and $f_s = \frac{f-\overline{f}}{2}$. Since $\mathcal{A}$ is a self adjoint algebra, for any $f \in \mathcal{A}$, $f_r, f_s \in \mathcal{A}$ and $f = f_r + if_s$. Let $\mathcal{A}_{\mathbb{R}} = \mathcal{A} \cap C_{\mathbb{R}}(X)$. Verify that $\mathcal{A}_{\mathbb{R}}$ is an algebra in $C_{\mathbb{R}}(X)$. Let $x, y \in X$ be distinct. Then, there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$. Either $f_r(x) \neq f_r(y)$ or $f_s(x) \neq f_s(y)$. This implies that $\mathcal{A}_{\mathbb{R}}$ separates points of X. Let $x \in X$. Then, there exists $f \in \mathcal{A}$ such that $f(x) \neq 0$. This implies that $f_r(x) \neq 0$ or $f_s(x) \neq 0$. So, $\mathcal{A}_{\mathbb{R}}$ nowhere vanishes. So, $\overline{\mathcal{A}_{\mathbb{R}}} = C_{\mathbb{R}}(X)$. Given $f \in C(X)$ there exist f_n, g_n in $\mathcal{A}_{\mathbb{R}}$ such that $f_n \to f_r$ and $g_n \to f_s$. And, $f_n + ig_n$ converges to f. So, $\overline{\mathcal{A}} = C(X)$ □

Theorem: $C(X)$ is separable.

Proof. X is separable. This implies that there is a subset $\{x_n\} \subseteq X$ such that $\overline{\{x_n\}} = X$. Let $f_n(x) = d(x_n, x)$ for all $x \in X$. Then, $f_n \in C(X)$. Let $\mathcal{A} = \{a + \sum_{k=1}^{m} a_{n_1,\ldots,n_k} f_{n_1} \ldots f_{n_k} | a, a_{n_1}, \ldots, a_{n_k} \in \mathbb{C}\}$. $\mathcal{A}$ is a self-adjoint algebra in $C(X)$. Let $x, y \in X$ be distinct and let $d(x, y) = \delta > 0$. There exists x_n such that $d(x, x_n) < \frac{\delta}{2}$. $f_n(x) < \frac{\delta}{2}$. $\delta = d(x, y) \leq d(x, x_n) + d(x_n, y)$. So, $\delta < \frac{\delta}{2} + f_n(y)$. So, $f_n(x) < \frac{\delta}{2} < f_n(y)$. $\mathcal{A}$ separates points of X. It is easy to see $\mathcal{A}$ nowhere vanishes. $\overline{\mathcal{A}} = C(X)$ Let
$E = \{a + \sum a_{n_1,n_2,\ldots,n_k} f_{n_1} \ldots f_{n_k} | a, a_{n_1}, \ldots, a_{n_k} \in \mathbb{Q} + i\mathbb{Q}\}$. Then, E is a dense set in $\mathcal{A}$ is countable and dense in $\mathcal{A}$ and hence also in $C(X)$. □

$E \subseteq \mathbb{R}^n$ or $\mathbb{C}^n$ is compact if and only if E is closed and bounded.

Let E be a collection of functions in X. We say that E is pointwise bounded if to each $x \in X$, there exists a constant $M_x > 0$ such that $|f(x)| \leqslant M_x$ for all $f \in E$. We say that E is equicontinuous if to each $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $d(x, y) < \delta$ for all $f \in E$.
Exercise: If $E \subseteq C(X)$ is such that $\overline{E}$ is compact, then E is pointwise bounded and E is equicontinuous.

Theorem:(Arzèla-Arcoli) Let $E \subseteq C(X)$. Suppose that E is pointwise bounded and equicontinuous. Then, $\overline{E}$ is compact.

Proof. X has a countable dense subset D. Let (f_n) be a sequence in E. Let $E \subseteq C(X)$ be pointwise bounded and quicontinuous. There exists a subsequence (f_{k_n}) of (f_n) such that $(f_{k_n}(x))$ converges for all $x \in D$. Given $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{3}$ for all $f \in E$. $\{B(x_n, \delta) | x_n \in D\}$ is an open cover for X. Since X is compact, $X = \bigcup_{i=1}^{m} B(x_i, \delta)$ for $x_1, x_2, \ldots, x_m \in D$. We can find N such that $|f_{k_n}(x_i) - f_{k_m}(x_i)| < \frac{\varepsilon}{3}$ for $n, m \geq N$ and all i. Let $x \in X$ Then, there exists x_i such that $d(x, x_i) < \delta$. This implies that $|f_{k_n}(x) - f_{k_m}(x)| \leqslant$ $|f_{k_n}(x) - f_{k_n}(x_i)| + |f_{k_n}(x_i) - f_{k_m}(x_i)| + |f_{k_m}(x_i) - f_{k_m}(x)| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$. That is, $\exists N$ such that $|f_{k_n}(x) - f_{k_m}(x)| < \varepsilon$ for all $n, m \geq N$ and for all x. So, (f_{k_n}) is Cauchy in $C(X)$. Therefore, f_{k_n} converges in $C(X)$. Thus, for any sequence in $E \subseteq C(X)$ that is pointwise bounded and equicontinuous, has a convergent subsequence. □

Exercise: $\overline{E}$ is compact.