Lecture 24: More on rational maps; open and closed subvarieties.

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Def^n: A rational function on an affine variety X is an element of the field of fractions of $\mathcal{O}(X) = k[X]$. This fraction field is also called the function field of X and is denoted by $k(X)$.

More generally, a rational map from an affine variety X to an affine variety Y is a morphism from a nonempty affine open subset of X to Y and it is denoted by $f: X \dashrightarrow Y$.

(*) $f_1, \dots, f_m$ be rat'l function on an affine variety X then they define a rational map from X to $\mathbb{A}^m$.

(*) Every nonempty open subset of an affine variety is dense.

Def^n: Let f be a rat'l function on a variety X. f is said to be regular at a point $P \in X$ if $\exists\, g, h \in \mathcal{O}(X)$ s.t. $f = \frac{g}{h}$ and $h(P) \neq 0$.

Domain of $f := \{ P \in X \mid f \text{ is regular at } P \}$

Prop: Let X be an affine variety and $f \in k(X)$.

1) Domain of f is an open dense subset of X.
2) $\text{domain}(f) = X \iff f \in \mathcal{O}(X)$.
3) $\text{domain}(f) \supseteq X_h := \{ P \in X \mid h(P) \neq 0 \}$ for $h \in \mathcal{O}(X)$ iff $f \in \mathcal{O}(X)[h^{-1}]$.

$f : X \dashrightarrow Y \subseteq \mathbb{A}^n$ rational map then $\exists f_1, \ldots, f_n \in k(X)$ s.t.
$f(P) = (f_1(P), \ldots, f_n(P))$ for P in an open dense subset of X.
f is said to be regular at P if f_i is regular at P $\forall 1 \le i \le n$.

(*) $\varphi : X \to Y$ morphism of affine varieties. Let $\varphi^{\#} : k[Y] \to k[X]$ be the k-alg homo which induce φ. Let g be regular fn on Y then $\varphi^{\#}(g)(P) = g(\varphi(P))$ for $P \in X$.

Pf: g regular fn on Y, $y \in Y$ point then m_y be the maximal ideal of $k[Y]$ associated to the point y. Then $g(y)$ is the image of g under the map

$$k[Y] \longrightarrow k[Y]/m_y = k$$
$$g \longmapsto g(y) = g \pmod{m_y}$$

$\varphi^{\#}(g)$ is regular on X & $P \in X$ a point corresponding to maximal ideal m_P of $k[X]$. Then $\varphi^{\#}(g)(P) = \varphi^{\#}(g) \pmod{m_P}$

$$k[Y] \xrightarrow{\varphi^{\#}} k[X] \xrightarrow{q_P} k[X]/m_P = k$$

So $\varphi^{\#}(g)(P) = q_P \circ \varphi^{\#}(g)$ $\longleftarrow$

max ideal $k[Y]$ $\to$ $m_{\varphi(P)} = \varphi^{\#^{-1}}(m_P) = \ker q_P \circ \varphi^{\#}$

$$q_{\varphi(P)} : \begin{tikzcd} k[Y] \arrow[rr] \arrow[dr, "\varphi^{\#}"'] & & k[Y]/m_{\varphi(P)} \\ & k[X] \arrow[ur, "q_P"'] & \end{tikzcd}$$

Hence $g(\varphi(P)) = q_{\varphi(P)}(g) = q_P \circ \varphi^{\#}(g)$
$= \varphi^{\#}(g)(P)$

(*) Let X be a variety & $P \in X$ a point.
$\mathcal{O}_{X,P} := \{f \in k(X) \mid f \text{ regular at } P\}$ is a local ring with maximal ideal $\{f \in k(X) \mid f \text{ reg at } P \text{ \& } f(P)=0\} = m_P\mathcal{O}_{X,P}$

Pf: f, g are in $\mathcal{O}_{X,P}$. $f = \frac{a}{b}$, $a, b \in k[X]$ with $b(P) \neq 0$ & $g = \frac{a'}{b'}$, $a', b' \in k[X]$ & $b'(P) \neq 0$.

$$f+g = \frac{b'a + a'b}{bb'} \in k(X) \text{ \& } bb'(P) = b(P)b'(P) \neq 0$$

So $f+g \in \mathcal{O}_{X,P}$ & $fg \in \mathcal{O}_{X,P}$

Note $f, g \in m_P\mathcal{O}_{X,P}$ then $(f+g)(P) = f(P) + g(P) = 0$

$\Rightarrow f+g \in m_P\mathcal{O}_{X,P}$

& $r \in \mathcal{O}_{X,P}$ & $f \in m_P\mathcal{O}_{X,P}$ then $rf(P) = 0$

So $m_P\mathcal{O}_{X,P}$ is a proper ideal of $\mathcal{O}_{X,P}$.

Let $f \in \mathcal{O}_{X,P} \setminus m_P\mathcal{O}_{X,P}$.

$f = \frac{a}{b}$ where $a, b \in k[X]$, $b(P) \neq 0$ and $a(P) \neq 0$.

Then $\frac{b}{a} \in k(X)$ & $\frac{b}{a}$ is regular at P.
$b/a \in \mathcal{O}_{X,P}$ & $\frac{b}{a} \cdot f = 1$. Hence f is a unit. Hence $\mathcal{O}_{X,P}$ is a local.

Def^n: Let X be a variety $U \subseteq X$ a nonempty open subset $\mathcal{O}_X(U)$, the set of regular functions on U, is $\{f \in k(X) \mid f \text{ is regular at } P \ \forall P \in U\}$.

Ex: 1) $X = \mathbb{A}^1$, $U = \mathbb{A}^1 \setminus \{0, 1\}$, k, $\underset{\mathcal{O}(X)}{\overset{k[x]}{\shortparallel}}$

$$\mathcal{O}_X(U) = \left\{ f \in k(x) \;\middle|\; f = \frac{a}{b},\ a, b \in k[x],\ b(\alpha) \neq 0 \ \forall \alpha \in k \setminus \{0,1\} \right\}$$

$$= k\left[x, \frac{1}{x}, \frac{1}{x-1}\right] = k\left[x, \frac{1}{x(x-1)}\right]$$

$$= S^{-1} k[x]$$

$S = \{1, x, x-1, \text{their products}\}$

(*) X an affine variety & $f \in \mathcal{O}(X)$, $f \neq 0$. Then $Z(f)$ is a closed subset of X. Let $U_f = X \setminus Z(f)$. These open sets are called basic open subsets of X.

Ex: $X = \mathbb{A}^2$, $U = \mathbb{A}^2 \setminus \{(0,0)\}$

But U is not affine variety $\mathcal{O}_X(U) = \left\{ \frac{f}{g} \in k(X) \,\middle|\, \begin{array}{l} f, g \in k[x,y] \\ g(a,b) \neq 0 \\ \forall (a,b) \neq (0,0) \end{array} \right\}$

$\mathcal{O}_X(X) = k[x,y]$

$g \in$ g is not const.

then there exist infinitely many point in $\mathbb{A}^2$ s.t. g vanishes at them. Hence g is a nonzero const. $\mathcal{O}_X(U) = k[x,y]$.

(*) X an affine variety & $f \in \mathcal{O}(X)$, $f \neq 0$. Then $Z(f)$ is a closed subset of X. Let $U_f = X \setminus Z(f)$. These open sets are called basic open subsets of X.

Prop: U_f is an affine variety with coordinate ring $\mathcal{O}_X(U_f) \cong \mathcal{O}_X(X)[\frac{1}{f}] = k[X][\frac{1}{f}]$.

Pf: $X \subseteq \mathbb{A}^n$, $k[X] = \dfrac{k[x_1,\dots,x_n]}{I}$; $X = Z(I)$; $U_f = X \setminus Z(f)$

Want: $U_f \subseteq \mathbb{A}^{n+1}$, $J \subseteq k[x_1,\dots,x_n,x_{n+1}]$; $x_{n+1} = \frac{1}{f}$
$J = (I, \tilde{f}x_{n+1} - 1)$ where $\tilde{f} \in k[x_1,\dots,x_n]$ is s.t. $\tilde{f} \pmod I = f$

Claim: $U_f \approx Z(J) = Y$ homeomorphism.

$(x_1,\dots,x_{n+1}) \in Z(J) \implies (x_1,\dots,x_n) \in Z(I)$ & $x_{n+1}\tilde{f}(x_1,\dots,x_n) = 1$

$\implies (x_1,\dots,x_n) \in X \setminus Z(f) = U_f$

So the map $\underset{(x_1,\dots,x_{n+1}) \longmapsto (x_1,\dots,x_n)}{Y = Z(J) \longrightarrow U_f}$ is well-defined.

the map $(x_1,\dots,x_n) \longmapsto (x_1,\dots,x_n, \frac{1}{f(x_1,\dots,x_n)})$ from U_f to $Z(J) = Y$ is the inverse map. Both are cont. and hence a homeo.

Let $Y = V(J)$ then $k[Y] = \dfrac{k[x_1,\dots,x_{n+1}]}{J} \cong k[X][\frac{1}{f}]$ (Use 1st isom thm)
$x_i \longmapsto x_i$, $1 \le i \le n$
$x_{n+1} \longmapsto \frac{1}{f}$

$\varphi^{\#}: k[X] \hookrightarrow k[Y]$ is a k-alg homo.

which defines $\varphi: Y \longrightarrow X$, $(a_1,\dots,a_{n+1}) \longmapsto (a_1,\dots,a_n)$. Infact image φ is U_f.

$\varphi^{\#}$ is inclusion map.

$g \in k[X][\frac{1}{f}]$ then g is regular on U_f.

Conversely let $g \in k(X)$ be regular on U_f.

$k(Y) = k(X)$

$\Rightarrow g$ is a rat'l on Y. Let $P \in Y$, $P = (a_1, \dots, a_{n+1})$

Then $\phi(P) = (a_1, \dots, a_n) \in U_f$ and g is regular on U_f.

Hence g is regular on whole of Y. Hence

$g \in k[Y] = k[X][\frac{1}{f}]$. (by Prop)