Lecture 8.

The inequality used in the proof of existence and uniqueness of first order ODEs gives rise to the following definition.

<u>Definition</u>. Let $\Omega \subseteq \mathbb{R}^n$ and let $f : \Omega \to \mathbb{R}$. We say that f is Lipschitz continuous on Ω iff $\exists\ K = K(\Omega)$ such that

$$|f(x) - f(y)| \leq K\,|x - y|$$

for all $x, y \in \Omega$. Say that f is locally Lipschitz on Ω iff $\forall\ x \in \Omega$ $\exists$ a neighbourhood N_x and constant K_x such that $\forall\ y_1, y_2 \in N_x \cap \Omega$

$$|f(y_1) - f(y_2)| \leq K_x\,|y_1 - y_2|.$$

<u>Example</u>. $f(x) = x^2$. This function is locally Lipschitz but not Lipschitz on $\mathbb{R}$. $f(x) - f(y) = (x+y)(x-y)$

<u>Example</u>. Let $f : \Omega \to \mathbb{R}$ where $\Omega \subset \mathbb{R}^n$ is open. If f is continuosly differentiable

on Ω then f is locally Lipschitz on Ω. This follows from the mean value theorem

$$f(x) - f(x_0) = \nabla f(\bar{x}) \cdot (x - x_0)$$

Hence $|f(x) - f(x_0)| \leq K_{x_0} |x - x_0|$ for every $x \in N_{x_0}$ and $K_{x_0} = \sup_{x \in N_{x_0}} |\nabla f(x)|$.

Example. Suppose $f : \Omega \to \mathbb{R}$ has bounded derivatives $\frac{\partial f}{\partial x_i}(x)$ on the open set $\Omega \subset \mathbb{R}^n$, $i = 1, \cdots n$. Then f is (globally) Lipschitz on Ω.

Remark. The function $f(x,y)$ in the Theorem (L7, p.1) is Lipschitz continuous on the rectangle R (see inequality (8) on p.4, L7).

Remark. Note that a Lipschitz continuous function on $\mathbb{R}$ has linear growth:

$$|f(x)| \leq c(1 + |x|)$$

$\forall x \in \mathbb{R}$ and some constant c. (use $f(x) = (f(x) - f(0)) + f(0)$)

<u>Example</u>. Consider the first order, linear, non homogeneous equation

$$\frac{dy}{dx} + P(x)y = Q(x).$$

Here $f(x,y) = -P(x)y + Q(x)$. This satisfies a (global) Lipschitz condition on $R = [a,b] \times \mathbb{R}$:

$$|f(x,y_1) - f(x,y_2)| = |P(x)(y_1 - y_2)|$$
$$\leq K|y_1 - y_2| \quad \text{—— (1)}$$

where $K = \sup_{x\in[a,b]} |P(x)|$.

<u>Theorem</u>. Suppose $f(x,y)$ is Lipschitz continuous on $[a,b]\times\mathbb{R}$. Then the initial value problem

$$y' = f(x,y), \quad y(x_0) = y_0$$

has a unique solution for every $(x_0,y_0) \in R := [a,b]\times\mathbb{R}$.

<u>proof</u>: We consider the iteration scheme

as in L 7. viz

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\,dt$$

for $x \in [a,b]$

Let $M = |y_0| + \sup_{x \in [a,b]} |y_1(x)|$.

Then $|y_1(x) - y_0(x)| \leq M \quad \forall x \in [a,b]$.

Let $x_0 \leq x \leq b$. Then

$$|y_2(x) - y_1(x)| = \left| \int_{x_0}^{x} [f(t, y_1(t)) - f(t, y_0(t))]\,dt \right|$$

$$\leq \int_{x_0}^{x} |f(t, y_1(t)) - f(t, y_0(t))|\,dt$$

$$\leq K \int_{x_0}^{x} |y_1(t) - y_0(t)|\,dt$$

$$\leq KM\,|x - x_0|. \left(= KM(x - x_0)\right)$$

Similarly,

$$|y_3(x) - y_2(x)| \leq \int_{x_0}^{x} |f(t, y_2(t)) - f(t, y_1(t))|\,dt$$

$$\leq K \int_{x_0}^{x} |y_2(t) - y_1(t)|\,dt$$

$$\leq K^2 M \int_{x_0}^{x} (t - x_0)\,dt$$

$$= K^2 M \frac{(x-x_0)^2}{2}$$

Hence if $|y_n(x) - y_{n-1}(x)| \leq K^{n-1} M \frac{(x-x_0)^{n-1}}{(n-1)!}$

for $x_0 \leq x \leq b$ then

$$|y_{n+1}(x) - y_n(x)| \leq K \int_{x_0}^{x} |y_n(t) - y_{n-1}(t)| \, dt$$

$$\leq K^n M \int_{x_0}^{x} \frac{(t-x_0)^{n-1}}{(n-1)!} \, dt$$

$$= K^n M \frac{(x-x_0)^n}{n!}$$

If $a \leq x \leq x_0$ then

$$|y_n(x) - y_{n-1}(x)| = \left| \int_{x}^{x_0} \left(f(t, y_{n-1}(t)) - f(t, y_{n-2}(t)) \right) dt \right|$$

$$\leq K \int_{x}^{x_0} |y_{n-1}(t) - y_{n-2}(t)| \, dt$$

If $|y_{n-1}(x) - y_{n-2}(x)| \leq K^{n-2} M \frac{(x_0-x)^{n-2}}{(n-2)!}$

then

$$|y_n(x) - y_{n-1}(x)| \leq K^{n-1} M \int_{x}^{x_0} \frac{(x_0-t)^{n-2}}{(n-2)!} \, dt$$

$$= K^{n-1} M \frac{(x_0-x)^{n-1}}{(n-1)!}.$$

The above holds for $a \leq x \leq x_0$

Combining the 2 cases $a \leq x \leq x_0$ and $x_0 \leq x \leq b$ we get

$$|y_n(x) - y_{n-1}(x)| \leq K^{n-1} M \frac{|x-x_0|^{n-1}}{(n-1)!}$$

for every $x \in [a,b]$. Hence the series

$$\sum_{n=1}^{\infty} |y_n(x) - y_{n-1}(x)| \leq \sum_{n=1}^{\infty} M \frac{(K(b-a))^{n-1}}{(n-1)!}$$

$$\leq Me^{K(b-a)}.$$

By the Weierstrass theorem

$$y_n(x) = y_0 + \sum_{K=1}^{n} y_K(x) - y_{K-1}(x)$$

converges uniformly on $[a,b]$ to a continuous function $y(x)$ on $[a,b]$.

As in L7 uniform convergence allows us to conclude that for every $x \in [a,b]$

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt.$$