1) $f(x) = \begin{cases} 1 & \text{if } |x| < \delta < \pi \\ 0 & \delta \leq |x| \leq \pi \end{cases}$

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\delta}^{\delta} e^{-inx}\, dx$$

$$= \frac{-(in)^{-1}}{2\pi}\left[e^{-inx}\right]_{-\delta}^{\delta}$$

$$= \frac{-1}{2\pi i n}\left[e^{-in\delta} - e^{in\delta}\right]$$

$$= \frac{-1}{2\pi i n}\; -2i \sin n\delta$$

$$= \frac{\sin(n\delta)}{n\pi}, \quad n \neq 0$$

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{\delta}{\pi}$$

$$C_n = \frac{\sin n\delta}{n\pi}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\, dx = \frac{\delta}{\pi}$$

$$\sum |c_n|^2 = \frac{1}{\pi^2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\sin^2 n\delta}{n^2} + \left(\frac{\delta}{\pi}\right)^2$$

$$= \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin^2 n\delta}{n^2} + \frac{\delta^2}{\pi^2}$$

By Parseval's identity

$$\frac{1}{2\pi}\int |f(x)|^2 \, dx = \sum |c_n|^2$$

$$\frac{\delta}{\pi} = \frac{2}{\pi^2} \sum_{1}^{\infty} \frac{\sin^2 n\delta}{n^2} + \frac{\delta^2}{\pi^2}$$

$$\sum_{n=1}^{\infty} \frac{\sin^2 n\delta}{n^2} = \frac{\pi}{2}\left(-\frac{\delta^2}{\pi} + \delta\right)$$

$$= \frac{\delta\pi}{2}\left(-\frac{\delta}{\pi} + 1\right)$$

$$= \frac{\delta}{2}(\pi - \delta)$$

$$\sum c_n = \sum_{n\neq 0} \frac{\sin(n\delta)}{n\pi} + \frac{\delta}{\pi}$$

$$= \frac{\delta}{\pi} + \frac{1}{\pi} \sum_{n\neq 0} \frac{\sin(n\delta)}{n}$$

$$= \frac{\delta}{\pi} + \frac{1}{\pi}\left(\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} + \sum_{n=1}^{\infty} \frac{\sin(-n\delta)}{-n}\right)$$

$$= \frac{\delta}{\pi} + \frac{2}{\pi} \sum_{1}^{\infty} \frac{\sin(n\delta)}{n}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{\delta}{\pi}$$

$f(x) = 1 \qquad |x| < \delta$

$\Rightarrow f$ is Lipschitz

i.e., $|f(0) - f(x)| = 0 \qquad |x| < \delta$.

$$\Rightarrow S_N(f)(0) \longrightarrow f(0)$$

$$\sum_{n=-N}^{N} c_n e^{in0} = \sum_{n=-N}^{N} c_n \longrightarrow f(0) = 1$$

$$\sum_{n=-\infty}^{\infty} c_n = 1$$

$$\frac{\delta}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\delta)}{n} = 1$$

$$\sum_{n=1}^{\infty}\frac{\sin(n\delta)}{n} = \left(1 - \frac{\delta}{\pi}\right)\frac{\pi}{2}$$

$$= \frac{\pi - \delta}{2}$$

Frame.

$$\boxed{\sum_{n=1}^{\infty}\frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2}}$$

Ex Prove $\sum_{1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Theorem: Let $f \in \mathcal{R}[-\pi, \pi]$ be 2π-periodic and $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$.

Let $s_n(x) = \sum_{k=-n}^{n} c_k e^{ikx}$ and

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_{n-1}}{n}.$$

Then $$\sigma_n(x) = \frac{1}{2n\pi} \int_{-\pi}^{\pi} \frac{f(x+t) + f(x-t)}{2} \frac{\sin^2 \frac{1}{2}nt}{\sin^2 \frac{1}{2}t} dt$$

Proof: Let $$D_n(x) = \sum_{k=-n}^{n} e^{ikx}$$

$$= \frac{\sin (n+\frac{1}{2})x}{\sin \frac{x}{2}}$$

$$= D_n(-x)$$

$$s_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_n(x-t) dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) D_n(t) dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t+x) D_n(t) dt$$

$$S_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(x+t)+f(x-t)}{2} D_n(t)\,dt$$

$$\sigma_n(x) = \frac{S_0(x) + S_1(x) + \cdots + S_{n-1}(x)}{n}$$

$$= \frac{1}{n} \int_{-\pi}^{\pi} \frac{f(x+t)+f(x-t)}{2} \sum_{k=0}^{n-1} D_k(t)\,dt$$

$$\sum_{k=0}^{n-1} D_k(t) = \sum_{k=0}^{n-1} \frac{\sin(2k+1)t/2}{\sin\frac{t}{2}}$$

$$= \frac{\sin^2 \frac{nt}{2}}{\sin^2 \frac{t}{2}} \cdot \left(\underline{Ex}\right)$$

$$\therefore \quad \sigma_n(x) = \frac{1}{2n\pi} \int_{-\pi}^{\pi} \frac{f(x+t)+f(x-t)}{2} \frac{\sin^2\frac{nt}{2}}{\sin^2\frac{t}{2}}\,dt$$

$\underline{\underline{\text{Cor}}}$ $$\frac{1}{2n\pi} \int_{-\pi}^{\pi} \frac{\sin^2\frac{nt}{2}}{\sin^2\frac{t}{2}} = 1.$$

$\underline{\text{Proof:}}$ $f(x) = 1 \quad \forall x.$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}\,dx = \begin{cases} 0 & \text{if } n \neq 0 \\ 1 & \text{if } n = 0 \end{cases}$$

$$S_n(x) = \sum_{k=-n}^{n} c_k e^{ikx} = 1$$

$$\sigma_n(x) = 1$$

By the previous Theorem

$$1 = \frac{1}{2n\pi} \int_{-\pi}^{\pi} \frac{\sin^2 nt/2}{\sin^2 \frac{t}{2}}\, dt.$$

<u>Fejer's Theorem</u>:

Let $f \in R[-\pi, \pi]$ be 2π-periodic.

For $x \in [-\pi, \pi]$, if

$$S(x) = \lim_{t \to 0} \frac{f(x+t) + f(x-t)}{2}$$ exists, then

$$\sigma_n(x) \longrightarrow S(x).$$

Moreover, if f is continuous in $[-\pi, \pi]$, then $\sigma_n(x) \to f(x)$ uniformly on $[-\pi, \pi]$.

<u>Proof:</u> Let $g_x(t) = \frac{f(x+t) + f(x-t)}{2} - s(x)$, $\forall t \in [-\pi, \pi]$

$$\lim_{t \to 0} g_x(t) = 0$$

for $\varepsilon > 0$ $\exists \delta > 0$ such that

$$|g_x(t)| < \frac{\varepsilon}{2} \quad \forall |t| < \delta.$$

$$\left| \int_{-\delta}^{\delta} \left[\frac{f(x+t) + f(x-t)}{2} - s(x) \right] \frac{\sin^2 \frac{nt}{2}}{\sin^2 \frac{t}{2}} dt \right|$$

$$\leq \int_{-\delta}^{\delta} |g_x(t)| \left| \frac{\sin^2 \frac{nt}{2}}{\sin^2 \frac{t}{2}} \right| dt$$

$$< \frac{\varepsilon}{2} \int_{-\delta}^{\delta} \frac{\sin^2 \frac{nt}{2}}{\sin^2 \frac{t}{2}} dt \leq \varepsilon \, 2n\pi$$

$$\left| \int_{\delta}^{\pi} g_x(t) \frac{\sin^2 \frac{nt}{2}}{\sin^2 \frac{t}{2}} \right| dt$$

$$\leq \frac{1}{\sin^2 \frac{\delta}{2}} \int_{\delta}^{\pi} |g_x(t)| \, dt$$

Let $M_x = \int_{-\pi}^{\pi} |g(t)|\, dt$

$$g(-t) = \frac{f(x-t) + f(x+t)}{2} - \cancel{g}\,(x)$$

$$= g(t)$$

$$\left|\frac{1}{2n\pi} \int_{-\pi}^{\pi} g(t) \frac{\sin^2 \frac{nt}{2}}{\sin^2 \frac{t}{2}}\, dt\right| = \left|\frac{1}{n\pi} \int_0^{\pi} g(t) \frac{\sin^2 \frac{nt}{2}}{\sin^2 \frac{t}{2}}\right|.$$

$$\leq \frac{1}{n\pi}\left|\int_0^{\delta} g_x(t) \frac{\sin^2 \frac{nt}{2}}{\sin^2 \frac{t}{2}}\right| + \frac{1}{n\pi}\left|\int_{\delta}^{\pi} g_x(t) \frac{\sin^2 \frac{nt}{2}}{\sin^2 \frac{t}{2}}\right|$$

$$< \frac{\varepsilon}{2} + \frac{1}{n\pi} \frac{M_x}{\sin^2 \frac{\delta}{2}}$$

For n large enough $\frac{1}{n\pi} \frac{M_x}{\sin^2 \frac{\delta}{2}} < \frac{\varepsilon}{2}$

$$\Rightarrow \sigma_n(x) \longrightarrow s(x).$$

If f is continuous on $[-\pi, \pi]$

then $s(x) = f(x)$.

f is uniformly continuous

$\therefore$

$\therefore$ δ may be chosen indepen-dent of x, M_x also may be chosen independent of x, hence the convergence is uniform.