Hom-tensor duality

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(*) Let $f : A \to B$ be R-lin map of R-mod & M be an R-mod then

$\exists$ $f \otimes M : A \otimes M \longrightarrow B \otimes M$ which is R-lin satisfying
$a \otimes m \longmapsto f(a) \otimes m$

if $g : B \to C$ is R-lin then

$(g \otimes M) \circ (f \otimes M) = g \circ f \otimes M$.

Moreover if $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is an exact seq of R-mod

then $A \otimes M \xrightarrow{\tilde{f}} B \otimes M \xrightarrow{\tilde{g}} C \otimes M \to 0$ is exact

Caution! Even if f is injective $f \otimes M$ need not be injective.

Pf: (contd.) We saw $\tilde{g}$ is surjective & $\mathrm{Im}(\tilde{f}) \subseteq \ker(\tilde{g})$.

Finally to see $\mathrm{Im}(\tilde{f}) = \ker(\tilde{g})$, it is enough to

show that the map $B \otimes M / \mathrm{Im}(\tilde{f}) \xrightarrow{\bar{g}} C \otimes M$ (induced by 1st isom thm)

$\bar{g}(\bar{x}) = \tilde{g}(x)$ where $x \in B \otimes M$

is an isomorphism.

For this we will define an R-lin map $\theta : C \otimes M \longrightarrow B \otimes M / \mathrm{Im}(\tilde{f})$

$$\varphi : C \times M \longrightarrow B \otimes M / \mathrm{Im}(\tilde{f})$$
$$(x, m) \longmapsto \underbrace{\tilde{x} \otimes m + \mathrm{Im}(\tilde{f})}_{= \overline{\tilde{x} \otimes m}} \quad \text{where } \tilde{x} \in g^{-1}(x)$$

$g : B \twoheadrightarrow C$

Claim φ is well-defined: $\tilde{x}, \tilde{x}' \in g^{-1}(x)$ then $\tilde{x} \otimes m - \tilde{x}' \otimes m = (\tilde{x} - \tilde{x}') \otimes m$

($\because$ $A \to B \to C$ exact)

But $\tilde{x} - \tilde{x}' \in \ker(g) = \mathrm{Im}(f)$. Hence $\exists\, a \in A$ s.t. $f(a) = \tilde{x} - \tilde{x}'$
$\Rightarrow (\tilde{x} - \tilde{x}') \otimes m = \tilde{f}(a \otimes m)$

Hence φ is well-defined. Moreover, for $r \in R$, $x, x' \in C$ and $m \in M$

$$\varphi(x + r x', m) = (\tilde{x} + r \tilde{x}') \otimes m \quad (\because\ g(\tilde{x} + r\tilde{x}') = x + r x' \text{ if } g(\tilde{x}) = x \ \&\ g(\tilde{x}') = x')$$
$$= \tilde{x} \otimes m + r\, \tilde{x}' \otimes m = \varphi(x, m) + r\, \varphi(x', m)$$

Also φ is R-lin in 2nd variable. Hence $\exists$ R-lin map

$$\theta : C \otimes M \longrightarrow B \otimes M / \mathrm{Im}(\tilde{f}) \quad \text{s.t.}$$
$$c \otimes m \longmapsto b \otimes m + \mathrm{Im}(\tilde{f}) \quad \text{where } b \in g^{-1}(c).$$

$$\theta \circ \bar{g}\,(b \otimes m + \mathrm{Im}(\tilde{f})) = \theta(\tilde{g}(b \otimes m)) = \theta(g(b) \otimes m) = b \otimes m + \mathrm{Im}(\tilde{f}).$$

& $\bar{g} \circ \theta\,(c \otimes m) = \bar{g}\,(b \otimes m + \mathrm{Im}(\tilde{f}))$ for some $b \in g^{-1}(c)$.
$= g(b) \otimes m = c \otimes m$.

Hence $\theta \circ \bar{g}$ & $\bar{g} \circ \theta$ are identity (as they are id on a gen set). ☒

Example: $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$ of $\mathbb{Z}$-modules

$\otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ $\quad 0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}/2\mathbb{Z} \to 0$

not injective!

$\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$

$a \otimes \bar{b} \longmapsto 2a \otimes \bar{b} = 2(a\otimes \bar{b}) = a \otimes 2\bar{b} = 0$

Def^n: An R-module M is said to be flat if for every ses of R-modules

$$0 \to A \to B \to C \to 0$$

the induced seq

$$0 \to A \otimes M \to B \otimes M \to C \otimes M \to 0$$

is exact.

1) Projective R-modules are flat.

2) S a mult subset of R then $S^{-1}R$ is a flat R-module.

Thm: Hom-tensor duality: Let M, N, K be R-modules then

$$\mathrm{Hom}_R(M, \mathrm{Hom}_R(N,K)) \cong \mathrm{Hom}_R(M \otimes_R N, K)$$

as R-modules

Pf: $\mathrm{Hom}_R(M \otimes_R N, K) \xleftrightarrow{1-1} \{ R\text{-bilin maps } M \times N \longrightarrow K \}$

$(?) \updownarrow 1-1$

$\mathrm{Hom}_R(M, \mathrm{Hom}_R(N, K))$

$\theta \in$

$\varphi_\theta(m, n) = \theta(m)(n)$ $\forall\, m \in M \,\&\, n \in N$

φ_θ is R-bilin

Conversely,

$\varphi : M \times N \longrightarrow K$ be R-bilin.

$\Theta_\varphi(m) = \varphi(m, -) : N \longrightarrow K$ is R-lin
$\in \mathrm{Hom}_R(N, K)$

For $m_1, m_2 \in M \,\&\, r \in R$

$$\Theta_\varphi(m_1 + r m_2) = \varphi(m_1 + r m_2, -) = \varphi(m_1, -) + r\varphi(m_2, -)$$
$$= \Theta_\varphi(m_1) + r\,\Theta_\varphi(m_2)$$

Hence $\Theta_\varphi : M \longrightarrow \mathrm{Hom}_R(N, K)$ is R-linear

$\Rightarrow \Theta_\varphi \in \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, K))$

Claim: These are inverses & R-lin.

For $\theta_1, \theta_2 \in \mathrm{Hom}_R(M, \mathrm{Hom}_R(N,K))$ & $r \in R$

For $m \in M$ & $n \in N$.

$$\varphi_{\theta_1 + r\theta_2}(m,n) = (\theta_1 + r\theta_2)(m)(n)$$

(∵ $\mathrm{Hom}_R(M, \mathrm{Hom}(N,K))$ is R-Mod)

$$= \theta_1(m)(n) + r\theta_2(m)(n)$$

$$= \varphi_{\theta_1}(m,n) + r\varphi_{\theta_2}(m,n)$$

$$\Rightarrow \varphi_{\theta_1 + r\theta_2} = \varphi_{\theta_1} + r\varphi_{\theta_2}$$

So the map $\mathrm{Hom}_R(M, \mathrm{Hom}_R(N,K)) \to \mathrm{Hom}_R(M \otimes_R N, K)$

is R linear.

For $\theta \in \mathrm{Hom}_R(M, \mathrm{Hom}_R(N,K))$, let

$\varphi = \varphi_\theta$ and $\theta' = \theta_\varphi$. WTS $\theta' = \theta$

For $m \in M$

$$\theta'(m) = \theta_\varphi(m) = \varphi(m,-) = \varphi_\theta(m,-) = \theta(m)(-)$$

$$\Rightarrow \theta' = \theta$$

$|||^{ly}$ φ be R-bilin $M \times N \longrightarrow K$, let $\theta = \theta_\varphi$

& $\varphi' = \varphi_\theta$. WTS $\varphi = \varphi'$

For $m \in M$ & $n \in N$,

$$\varphi'(m,n) = \varphi_\theta(m,n) = \theta(m)(n) = \theta_\varphi(m)(n) = \varphi(m,n)$$

$\Rightarrow \varphi = \varphi'$ & Hence the claim ∎

Prop: Let X & Y be affine varieties over an alg closed field k. i.e. X and Y are irr alg sets. Then $X \times_k Y$ is also an irred alg set i.e. an affine variety.

Pf: $X \subseteq \mathbb{A}^n$ & $Y \subseteq \mathbb{A}^m$, then $X \times Y \subseteq \mathbb{A}^{n+m}$

$X = V(I)$ $\quad Y = V(J)$ $\quad$ then $X \times Y = V((I, J))$

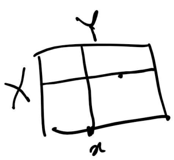
$I \subseteq k[x_1, \dots, x_n]$ $\quad J \subseteq k[y_1, \dots, y_m]$ $\quad (I, J) \subseteq k[x_1, \dots, x_n, y_1, \dots, y_m]$

Let $f \in I$ $\quad V(I) = X \times \mathbb{A}^m$

$f(\underline{a}, \underline{b}) = 0$ if $\underline{a} \in X$ $\quad V(J) = \mathbb{A}^n \times Y$

$$V(I \cup J) = (X \times \mathbb{A}^m) \cap (\mathbb{A}^n \times Y)$$
$$= X \times Y$$

Hence $X \times Y$ is alg set.



Suppose $X \times Y = F_1 \cup F_2$ $\quad F_1$ & F_2 proper closed sets of $X \times Y$.

For $x \in X$

$$Y \cong x \times Y = (x \times Y \cap F_1) \cup (x \times Y \cap F_2)$$

$$\Rightarrow x \times Y \subseteq F_1 \text{ or } x \times Y \subseteq F_2$$

$$X_1 = \{x \in X \mid x \times Y \subseteq F_1\} \text{ and } X_2 = \{x \in X \mid x \times Y \subseteq F_2\}$$

Hence $X = X_1 \cup X_2$

So if we show X_1 & X_2 are closed then $X_i = X$ for some i. Hence $F_i = X \times Y$.

For $y \in Y$ $\quad X \xrightarrow{i_y} X \times Y$ is continuous $\left(\because i_y : X \xrightarrow{\cong} X \times y \underset{\text{closed}}{\subseteq} X \times Y\right)$

$x \mapsto (x, y)$

$$\bigcap_{y \in Y} i_y^{-1}(X \times y \cap F_1) = X_1 \quad \text{\& } |||^{ly} \ X_2 \text{ is closed.}$$

$$x \Leftrightarrow (x, y) \in F_1 \ \forall y \in Y \Leftrightarrow x \times Y \subseteq F_1 \Leftrightarrow x \in X_1$$

∎