Physics 4

Lecture 20-21

Recall Lorentz transformations of force.

Force measured in S, F and in S' F'. S' is moving w.r.t S with uniform vel v along $x-x'$ axis. The particle is moving along its worldline with instantaneous speed u.

We had seen that F could be written as.

$$F = \begin{pmatrix} \frac{\gamma_u}{c} \frac{dE}{dt} \\ \gamma_u \vec{f} \end{pmatrix} = \begin{pmatrix} \frac{\gamma_u}{c} \vec{f} \cdot \vec{u} \\ \gamma_u \vec{f} \end{pmatrix}.$$

L.T matrix

$$L = \begin{pmatrix} \gamma_v & -\beta_v \gamma_v & 0 & 0 \\ -\beta_v \gamma_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \; ; \; F' = \begin{pmatrix} \gamma_{u'} \dfrac{\vec{f}' \cdot \vec{u}'}{c} \\ \gamma_{u'} \vec{f}' \end{pmatrix}$$

$$\boxed{F'^{\mu} = L^{\mu}{}_{\nu} F^{\nu}}$$

$$\begin{pmatrix} \frac{\gamma_{u'}}{c} \vec{f}' \cdot \vec{u}' \\ \gamma_{u'} f_x' \\ \gamma_{u'} f_y' \\ \gamma_{u'} f_z' \end{pmatrix} = \begin{pmatrix} \gamma_v & -\beta_v \gamma_v & 0 & 0 \\ -\beta_v \gamma_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_u \frac{\vec{f} \cdot \vec{u}}{c} \\ \gamma_u f_x \\ \gamma_u f_y \\ \cancel{f} \gamma_u f_z \end{pmatrix}$$

$$\gamma_{u'} f'_x = -\beta_v \gamma_v \gamma_u \left(\frac{\vec{f} \cdot \vec{u}}{c} \right) + \gamma_v \gamma_u f_x \quad \text{— (1)}$$

$$\gamma_{u'} f'_y = \gamma_u f_y . \quad \text{— (2)}$$

$$\gamma_u f_z' = \gamma_u f_z \quad \text{— (2)}$$

you can prove

$$\boxed{\frac{\gamma_u \gamma_v}{\gamma_{u'}} = \frac{1}{1 - \frac{u_x v}{c^2}}}$$

→ identity

$$\gamma_u \gamma_v = \frac{\gamma_{u'}}{\left(1 + u'_x v/c^2\right)}.$$

→ L.H.S $u \to u'$ using vel. addition.

Resnick 2. prob. 37.

→ using this + (1), (2), (3)

$$f_x' = \frac{f_x - \frac{v}{c^2}(\vec{u} \cdot \vec{f})}{1 - \frac{u_x v}{c^2}}$$

$$f_y' = \frac{f_y}{\gamma_v\left(1 - u_x v/c^2\right)}$$

$$f_z' = \frac{f_z}{\gamma_v\left(1 - u_x v/c^2\right)}$$

$$f_x = \frac{f_x' + \frac{v}{c^2}(\vec{u}' \cdot \vec{f}')}{1 + u'_x v/c^2}$$

$$f_y = \frac{f_y'}{\gamma_v\left(1 + u'_x \frac{v}{c^2}\right)}$$

$$f_z = \frac{f_z'}{\gamma_v\left(1 + u'_x v/c^2\right)}$$

In the instantaneous rest frame $u' = 0$.

$$\boxed{\begin{aligned} f_x &= f_x' \\ f_y &= \frac{f_y'}{\gamma} \\ f_z &= \frac{f_z'}{\gamma} \end{aligned}}$$

$$\boxed{\vec{f} = \vec{f}'_{\parallel} + \frac{\vec{f}'_{\perp}}{\gamma}}$$

Electromagnetism

- Apparent paradox.

Charge q_s in motion in S. q_s: source charge.

$\downarrow$ set up magnetic field $\vec{B}$.

Consider a test charge q_t moving through $\vec{B}$ with vel $\vec{u}$.

$$\vec{F_m} = q_t(\vec{u} \times \vec{B}) \quad \text{in the S frame.}$$

S' frame moving either with vel. of charge q_s or with vel $\vec{u}$

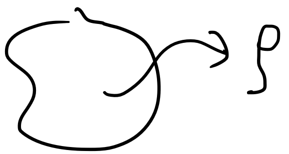
In first case $\vec{B} = 0$, $\vec{F}_m' = 0$

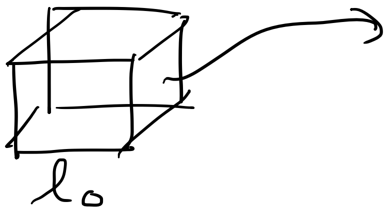
second case also $\vec{F}_m' = 0$.

$\boxed{\text{Are S, S' equivalent as inertial frames?}}$

- Electric & mag fields by themselves do not have relativistically invariant meaning

- A current carrying wire viewed from two inertial frames.





cube contains N electrons

charge Ne.

$$\text{charge density} = \rho_0 = \frac{Ne}{l_0^3}.$$

charges are at rest in frame S', no current

$j_0 = 0$.

View vol. element from S which moves with vel $\vec{u}$ along one edge of cube.

length of that side in S $= \dfrac{l_0}{\gamma} = l_0\sqrt{1-u^2/c^2}$.

# of electrons and charge don't change

vol. of cube $= l_0^3\sqrt{1-u^2/c^2}$.

Observer in S observes a charge density.

$$\rho = \frac{Ne}{l_0^3\sqrt{1-u^2/c^2}} = \gamma\rho_0 .$$

charges move with vel u in S, so the measured

Current density $= \rho u$

$= \rho_0 u \gamma$.

$\vec{j} = (j_x, j_y, j_z)$, ρ .

$u^\mu = (\gamma c, \gamma \vec{u})$.

combine into 4 vector .

$j = (\rho_0 \gamma c , \rho_0 \gamma \vec{u})$ . .

$$\boxed{j^\mu = \rho_0 u^\mu .}$$

$$\boxed{j^\mu j_\mu = \rho_0 u^\mu u_\mu = \rho_0 c^2}$$

no charge density in one frame

$j^0 = 0$; $\vec{j} \neq 0$. , charge density + current density in another frame

Transformations of $\vec{E}$ and $\vec{B}$

Lorentz force : $\vec{F} = q(\vec{E} + \vec{u} \times \vec{B})$.

Let us consider particle instantaneously at rest in S',

$$F_x = F_x' \quad \text{— (1)}$$

$$F_y = F_y' \sqrt{1 - v^2/c^2} = \frac{F_y'}{\gamma} \quad \text{— (2)}$$

$$F_z = \frac{F_z'}{\gamma} \quad \text{— (3)}$$

In S', electric & mag fields $\vec{E}'$, $\vec{B}'$

$$\vec{F}' = q\vec{E}' \quad \text{no magnetic field} \quad \text{— (4)}$$

In S

$$\vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right) \quad \text{— (5)}$$

Let us take v along common $x-x'$ axis

Using (4) & (5) in (1).

$$E'_x = E_x \quad \text{—— (6)}$$

in (2) and (3).

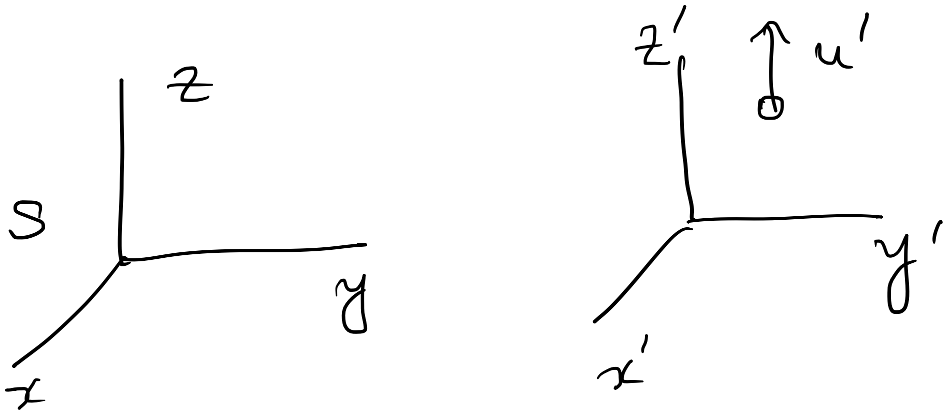
$$\frac{qE'_y}{\gamma} = q\left(E_y - (\vec{v}\times\vec{B})_y\right).$$

$$\boxed{\begin{aligned} E_y' &= \gamma\left(E_y - v B_z\right) \\ E_z' &= \gamma\left(E_z + v B_y\right) \\ E_x' &= E_x \end{aligned}}$$

can break into components $\parallel$ or $\perp$ to $\vec{v}$

$$\boxed{\begin{aligned} E'_{\parallel} &= E_{\parallel} \\ E'_{\perp} &= \gamma\left(E_{\perp} + \left(\vec{v} \times \vec{B}\right)_{\perp}\right) \end{aligned}}$$

Transformation of the magnetic field



$$\vec{u}' = u'_y \hat{y}$$

$$\vec{F}' = q(\vec{E}' + \vec{u}' \times \vec{B}') \quad \text{— (7)}$$

$$F_x' = q(E_x' + u_y' B_z') \quad \text{— (8)}$$

$$F_y' = qE_y' \quad \text{— (9)}$$

$$F_z' = q(E_z' - u'_y B_x') \quad \text{— (10)}$$

To get the force in S, we must know what the particles velocity is in the S frame.

vel. addition/transfn eqns.

$$u_x = v.$$
$$u_y = \frac{u'_y}{\gamma}$$
$$u_z = 0.$$

$$\left\{ u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}} \right\}$$

$$\vec{F} = q(\vec{E} + \vec{u} \times \vec{B}).$$

$$F_x = q(E_x + u_y B_z) \quad \text{--- (11)}$$
$$F_y = q(E_y - v B_z) \quad \text{--- (12)}$$
$$F_z = q(E_z + v B_y - u_y B_x) \quad \text{--- (13)}.$$

Force transformation eqn.s

$$F_x' = \frac{F_x - v/c^2 (\vec{u} \cdot \vec{F})}{(1 - u_x v/c^2)} \qquad F_y' = \frac{F_y}{\gamma\left(1 - u_x \frac{v}{c^2}\right)}$$

$$F_z' = \frac{F_z}{\gamma\left(1 - u_x \frac{v}{c^2}\right)} .$$

+ E field transformation.

we get

$$\boxed{\begin{aligned} B_x' &= B_x \\ B_y' &= \gamma\left(B_y + \frac{v}{c^2} E_x\right) \\ B_z' &= \gamma\left(B_z - v/c^2 \, E_y\right) \end{aligned}}$$

$$B_{\parallel}' = B_{\parallel}$$

$$B_{\perp}' = \gamma\left(B_{\perp} - \frac{1}{c^2}(\vec{v} \times \vec{E})_{\perp}\right)$$

What remains invariant ?

You can check that the following quantity is invariant

$$\boxed{E^2 - c^2B^2 = E'^2 - c^2B'^2}$$

- a pure electric field cannot be transformed into a pure mag field through L.T.
- $E > cB$ in one frame $E' > cB'$ in another.

A second invariant

$$\boxed{\vec{E}\cdot\vec{B} = \vec{E}'\cdot\vec{B}'}$$

- If E & B are orthogonal in one frame.
  $\longrightarrow$ orthogonal in all frame.

- For a given EM field we can find an inertial frame where $\vec{E} = 0$ (if $E < cB$) or $\vec{B} = 0$ if ($E > cB$) at a given pt iff

$$\vec{E}\cdot\vec{B} = 0.$$

Ex. Suppose an EM field is purely electric in S.
i.e. $\vec{E} \neq 0$, $\vec{B} = 0$.
what does it look like in S'

$$E'_{\parallel} = E_{\parallel} \quad , \quad B'_{\parallel} = 0$$

$$E'_{\perp} = \gamma E_{\perp} . \qquad B'_{\perp} = -\frac{\gamma}{c^2}(\vec{v} \times \vec{E})_{\perp} .$$

$$\vec{v} \times \vec{E} = \vec{v} \times \vec{E}_{\perp} = \frac{\vec{v} \times \vec{E}_{\perp}'}{\gamma} = \frac{\vec{v} \times \vec{E}'}{\gamma} .$$

$$\vec{B}' = B'_{\perp} = -\frac{\gamma}{c^2}(\vec{v} \times \vec{E}) = -\frac{\vec{v} \times \vec{E}}{c^2} .$$