Lecture 18: Primary decomposition of ideals

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Definition: Let **R** be a ring. A proper ideal **Q** is called primary if for $x, y \in \boldsymbol{R}, xy \in \boldsymbol{Q}$ implies $x \in \boldsymbol{Q}$ or $y^n \in \boldsymbol{Q}$, for some n.

Not: Q is P-primary if $P = \sqrt{Q}$

Lemma: I a primary R-ideal then $\sqrt{I}$ is a prime ideal.

Lemma: Q an R-ideal s.t. $\sqrt{Q}$ is a maximal ideal of R then Q is primary ideal.

Def^n: Let I be an R-ideal. If there exists P_i-primary ideals Q_i $1 \le i \le n$ s.t. $I = \bigcap_{i=1}^{n} Q_i$ then we say that $I = \bigcap_{i=1}^{n} Q_i$ is a primary decomposition of I. Note $P_i = \sqrt{Q_i}$ are prime ideals.

A primary decomposition $I = \bigcap_{i=1}^{n} Q_i$ is called minimal if $P_i \neq P_j$ for $i \neq j$ & $Q_i \not\supseteq \bigcap_{\substack{j=1 \\ j \neq i}}^{n} Q_j$.

Lemma: Let Q_1, Q_2 be P-primary ideals of a ring R for a prime ideal P then $Q_1 \cap Q_2$ is also P-primary. In particular, if I has a primary decomposition then it has a minimal primary decomp.

Pf: Let $Q = Q_1 \cap Q_2$ and $xy \in Q$ if $x \in Q$ done

otherwise $x \notin Q_1$ or $x \notin Q_2$ (say $x \notin Q_1$)

$\Rightarrow y^{n_1} \in Q_1$ for some n_1 ($\because xy \in Q_1$ & Q_1 is primary)

$\Rightarrow y \in P = \sqrt{Q_1} = \sqrt{Q_2}$

$\Rightarrow y^{n_2} \in Q_2$ for some n_2 ($\because y \in \sqrt{Q_2}$)

$\Rightarrow y^{\max(n_1, n_2)} \in Q$.

Hence Q is primary. Moreover $\sqrt{Q} = \sqrt{Q_1 \cap Q_2} \overset{\subseteq \checkmark}{\underset{\supseteq}{=}} \sqrt{Q_1} \cap \sqrt{Q_2} = P$

($z \in \sqrt{Q_1} \cap \sqrt{Q_2} \Rightarrow z^n \in Q_1 \cap Q_2 = Q$ for some n)

Hence Q is P-primary.

If $I = \bigcap_{i=1}^{n} Q_i$ is a primary decomp then by what we have proved we may assume $P_i \neq P_j$ for $i \neq j$. Now throw away Q_i's s.t. $Q_i \supseteq \bigcap_{\substack{j=1 \\ j \neq i}}^{n} Q_j$ one by one to get minimal primary decomp.

$\blacksquare$

<u>Theorem</u>: Every proper ideal of a noetherian ring has a primary decomposition (& hence a minimal primary decomp)

<u>Def^n</u>: A proper ideal I of a ring R is called <u>irred</u> if $I = I_1 \cap I_2 \Rightarrow I = I_1$ or $I = I_2$.
I_1, I_2 proper ideals

<u>Prop 1</u>: Every proper ideal in a noeth ring is an intersection of irred. ideals

<u>Prop 2</u> Every irred. ideal in a noeth ring is a primary ideal.

So clearly Prop 1 & Prop 2 $\Rightarrow$ Theorem

<u>Pf of Prop 1</u>: Suppose not. Let R be such a noeth &

$S = \{ I \subsetneq R \text{ ideal} \mid I$ is not an inter of finitely many irred ideals$\}$. Then $S \neq \phi$ & hence has a maximal element I. $\because I \in S$

Then I is reducible $\Rightarrow I = I_1 \cap I_2$ for some proper ideals I_1 & I_2 with $I \subsetneq I_1$ & $I \subsetneq I_2$.

The maximality of $I \Rightarrow I_1, I_2 \notin S \Rightarrow I_1$ & I_2 are inter of finitely irred ideals $\Rightarrow I$ is the inter of irred ideals. ∎

($I_1 = \bigcap_{i=1}^{n_1} Q_i$, $I_2 = \bigcap_{i=1}^{n_2} Q_i'$ then $I = Q_1 \cap Q_2 \cap \cdots \cap Q_{n_1} \cap Q_1' \cap Q_2' \cap \cdots \cap Q_{n_2}'$

Proof of Prop 2 : Let R be a noetherian ring and $I \subseteq R$ be an irred ideal.
Then in R/I the zero ideal is irred and to show I is primary ideal of R it is enough to show every zero divisor in R/I is nilpotent.

Let $x \in R/I$ be a zero divisor.
$\Rightarrow \exists\, y \in R/I$, $y \neq 0$ s.t. $xy = 0$ in R/I.

$$\underbrace{\{r \in R/I \mid rx = 0\}}_{} = (0:x)$$
$$\operatorname{ann}(x) \subseteq \operatorname{ann}(x^2) \subseteq \dots \quad \text{is an inc chain of ideals}$$
so $\exists\, n$ s.t. $\operatorname{ann}(x^n) = \operatorname{ann}(x^{n+1})$

Claim: $(y) \cap (x^n) = (0) \Rightarrow x^n = 0$ since (0) is irred in R/I

Let $a \in (y) \cap (x^n)$

$a = ry$ for some $r \in R/I$ &
$= r'x^n$ for some $r' \in R/I$

$ax = 0$ (as $xy = 0$)
$\Rightarrow r'x^{n+1} = 0 \Rightarrow r' \in \operatorname{ann}(x^{n+1})$
$\Rightarrow r'x^n = 0$ ($\because \operatorname{ann}(x^n) = \operatorname{ann}(x^{n+1})$)
$\Rightarrow a = 0$

$\blacksquare$

Example: 1) $R = k[x,y]$, $I = (xy(y^2-2x+3))$

$$I \subseteq (x),\ (y)\ \&\ (y^2-2x+3)$$

$$\sqrt{I} \subseteq (x) \cap (y) \cap (y^2-2x+3)$$

Question: Is $I \overset{\subseteq \checkmark}{\underset{\supseteq}{=}} (x) \cap (y)(y^2-2x+3)$?

Let $f \in RHS$

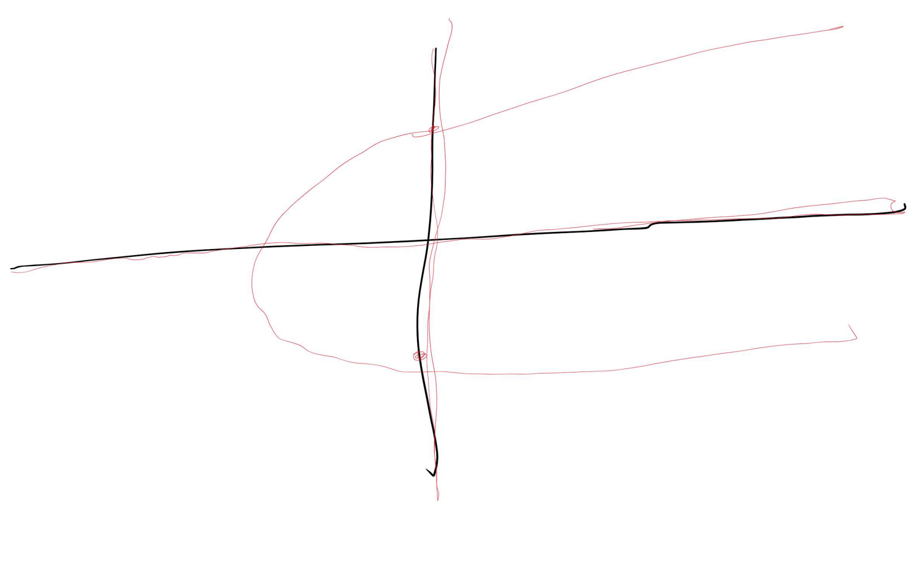
$\Rightarrow x \mid f\ \&\ y \mid f\ \&\ y^2-2x+3 \mid f$

($\because k[x,y]$ is a UFD)

$\Rightarrow xy(y^2-2x+3) \mid f$

$$J = (x^2) \cap (y^3) \cap (y^2-2x+3) \neq I$$

$Z(I)$ in $\mathbb{A}^2$



Def^n: Minimal primes Let I be a proper ideal of a ring R. A prime ideal P of R is called a minimal prime of I if $I \subseteq P$ and $I \subseteq P_1 \subseteq P$ with P_1 prime implies $P_1 = P$.

Prop: Let R be a ring, I an R-ideal & $I = \bigcap_{i=1}^{n} Q_i$ a minimal primary decomposition of I then every minimal prime of I belongs to $\underbrace{\{\sqrt{Q_1}, \dots, \sqrt{Q_n}\}}$ ← is called the associated prime ideals of I.

Pf: Let $P_i = \sqrt{Q_i}$ $1 \le i \le n$. Let P be a minimal prime of I then $I = \bigcap_{i=1}^{n} Q_i \subseteq P$

$$\Rightarrow \sqrt{I} = \bigcap_{i=1}^{n} \sqrt{Q_i} \subseteq P$$

$$\Rightarrow \bigcap_{i=1}^{n} P_i \subseteq P$$

$$\Rightarrow P_i \subseteq P \text{ for some } i$$ (if not $\exists x_i \in P_i \setminus P$ $\forall 1 \le i \le n$ $\Rightarrow x_1 \cdots x_n \in \bigcap_{i=1}^{n} P_i \subseteq P$. But $x_i \notin P$ $\forall 1 \le i \le n$ contradicts P is prime ideal)

$$\Rightarrow P = P_i \text{ for some } i$$ ($\because P$ is a minimal prime of I.)

Example: $R = k[X,Y]$, $I = (X^2, XY) \subsetneq (X)$ $\quad I \neq (X^2)$

$XY \in I$

(X) or (Y)

$I = (X) \cap \underbrace{(Y, X^2)}$

$\subseteq$ ✓

$\supseteq$: $f \in$ RHS

$X \mid f$ & $f = aX^2 + bY$ for $a, b \in k[X,Y]$

$\Rightarrow X \mid b \Rightarrow f = aX^2 + b'XY \in I$.

Hence this is a minimal primary decomp.

Assoc primes of $I = \{(X), (X,Y)\}$

↑ Embedded prime

(Y^n, XY, X^2)