Lecture 2.

We now consider various techniques for solving the first order equation

$$\frac{dy}{dx} = f(x,y).$$

This is of the form $F(x, y, y') = 0$ where $F(x, y_0, y_1) \equiv y_1 - f(x, y_0)$, $(x, y_0, y_1) \in [a,b] \times E \subset [a,b] \times \mathbb{R}^2$.

We will consider the case when $f(x,y)$

$$= \frac{M(x,y)}{N(x,y)}$$

Case (i) Suppose $M(x,y) \equiv M(x)$ and $N(x,y) \equiv N(y)$. We assume that $N(y) \neq 0$.

We then have the equation

$$N(y)\frac{dy}{dx} = M(x).$$

Hence

$$\int_{x_0}^{x} N(y(r))\, y'(r)\, dr = \int_{x_0}^{x} M(r)\,dr + c.$$

Making a change of variable we can rewrite the above as

$$\int_{y(x_0)}^{y(x)} N(t)\, dt = \int_{x_0}^{x} M(r)\, dr + c$$

Introducing the function (2).

$$\bar{N}(y) = \int_{y_0}^{y} N(t)\,dt, \quad \bar{M}(x) = \int_{x_0}^{x} M(t)\,dt$$

with $y_0 = y(x_0)$. Thus the solution $y(x)$ when it exists will satisfy

$$\bar{N}(y(x)) = \bar{M}(x) + c$$

When $\bar{N}(\cdot)$ is an invertible function then

$$y(x) = \bar{N}^{-1}(\bar{M}(x) + c)$$

One can also use the implicit function theorem to show the existence of $y(\cdot)$.

Remark. The above method is often summarised by saying that we can 'separate variables' and 'integrate' to obtain the solution $y(\cdot)$ by solving the equation

$$\int N(y)\,dy = \int M(x)\,dx + c.$$

Case 2. Suppose that $f = \frac{M}{N}$ is homogeneous of degree zero. i.e. for $t > 0$, $f(tx, ty) = f(x, y)$.

Then $f(x, y) = f(1, y/x)$ (3)

$= f(1, z)$ where $y = zx$, $x \neq 0$.

Hence $z + x\frac{dz}{dx} = \frac{dy}{dx} = f(1, z)$.

or $\frac{dz}{dx} = \frac{f(1,z) - z}{x}$

We can use Case (1) to solve the above equation.

Example $f(x, y) = \frac{x + y}{x - y} = \frac{1+z}{1-z}$

where $z = y/x$. Our ODE becomes

$$\frac{dz}{dx} = \frac{1+z^2}{1-z} \cdot \frac{1}{x}.$$

Seperating variables and integrating and substituting $y = zx$ we get

$$\tan^{-1}\left(\frac{y}{x}\right) = \log\sqrt{x^2+y^2} + c.$$

which defines the solution y implicitly as a function of x.

Case 3. Suppose that $\frac{dy}{dx} = f(x,y) = \frac{M}{N}$

$M(x,y) = \frac{\partial g}{\partial x}$ and $N(x,y) = \frac{\partial g}{\partial y}$

for some function $g(x,y)$. Our ODE becomes

$$\frac{\partial g}{\partial x}(x,y) - \frac{\partial g}{\partial y}(x,y)\frac{dy}{dx} = 0.$$

If we define $h(x) := g(x, -y(x))$ then our ODE reduces to $\frac{dh}{dx} = 0$ or $h(x) = g(x, -y(x)) = c$. Thus the solutions of $\frac{dy}{dx} = \frac{\partial g/\partial x}{\partial g/\partial y}$ are defined implicitly by the family of curves $g(x,y) = c$ (see lecture 1, example 4 with $F(x,y,y_0)$ replaced by $F(x,y_0,y_1)$!)

Remark. Note that the conditions $\frac{\partial M}{\partial}$ $\frac{\partial g}{\partial x} = M$, $\frac{\partial g}{\partial y} = N$ can be restated as $(M,N) = \nabla g$ i.e. the vector field (M,N) is given by a potential. Note that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ is a necessary condition for the existence of a potential g.

On an open convex set this condition is also sufficient.

Linear Equations

$$\frac{dy}{dx} = p(x)y + q(x)$$

Here $F(x, y_0, y_1) := y_1 - p(x)y_0 - q(x)$.
so that $F(x, y, y') = 0$. It is easily verified that the solution of this equation is given by

$$y(x) = e^{\int_{x_0}^{x} p(t)\,dt} \int_{x_0}^{x} e^{-\int_{x_0}^{r} p(t)dt} q(r)\,dr + c.$$

Reduction of order.

(1) Suppose we have a 2^{nd} order ODE of the form $F(x, y', y'') = 0$.

example: $xy'' - y' = 3x^2$

Take $p = p(x) \equiv y'(x)$. Then $p'(x) = y''(x)$.
Then we have $F(x, p, p') = 0$, whose solution is obtained as a function of x

viz $p(x)$. Then $y(x) = \int_{x_0}^{x} p(t)\,dt + c$ will give a solution of $F(x, y', y'') = 0$.

(2). Suppose the second order equation

$$F(y, y', y'') = 0$$

does not depend on x. We wish to determine y' as a function of y i.e. $y' = p(y)$, so that $y'(x) = p(y(x))$.

Note that $y''(x) = \frac{dp}{dy}(y(x)) \frac{dy}{dx}$

$$= \frac{dp}{dy}(y(x))\, p(y(x)).$$

Then our 2^{nd} order equation reduces to

$$F\left(y, p, p\frac{dp}{dy}\right) = 0.$$

i.e. $F_0(y, p, p') = 0$.

which maybe solved to obtain p as a function of y.

Example $y'' + k^2 y = 0$ reduces to

$p\frac{dp}{dy} + k^2 y = 0 \Rightarrow p^2(y) + k^2(y^2 - y_0^2) = 0$ on integrating from y_0 to y and taking $p(y_0) = 0$.