Lecture 7.

We prove a basic existence and uniqueness theorem for the ODE $y' = f(x,y)$, $y(x_0) = y_0$ — (1)

Theorem (Picard). Let $R = [a,b] \times [c,d]$ be a closed rectangle in $\mathbb{R}^2$. Let $f : R \longrightarrow \mathbb{R}$ be a continuous function such that $\partial_y f(x,y)$ exists at each $(x,y) \in R^\circ$ = interior of R and $\partial_y f(x,y)$ extends as a continuous function to the whole of R. If $(x_0, y_0) \in R^\circ$ then there exists $h > 0$ with the property that the initial value problem (1) has a unique solution $y = y(x)$ on the interval $[x_0 - h, x_0 + h] \subset [a,b]$.

Remark. By a solution $y = y(x)$ of (1) we mean a continuous function

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$y : [x_0 - h, x_0 + h] \longrightarrow \mathbb{R}$ that satisfies the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt \quad \text{— (2)}$$

for $x \in [x_0 - h, x_0 + h]$.

Proof of Theorem. Let $y_0(x) = y_0$ $\forall\, x \in [a, b]$. Then for $n \geq 1$ we define iteratively

$$y_n(x) := y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt \quad \text{— (3)}$$

The sequence $\{y_n\}$ is a sequence of continuous functions on $[a, b]$. We want to show the existence of a limit $y(x) = \lim_{n \to \infty} y_n(x)$ in some interval around x_0. To do this we write

$$\text{(4) —} \quad y_n(x) = y_0(x) + \sum_{k=1}^{n} \left(y_k(x) - y_{k-1}(x) \right)$$

It follows that the convergence of

of the sequence $\{y_n(x)\}$ is equivalent to the convergence of the series

$$\sum_{k=1}^{\infty} \left(y_k(x) - y_{k-1}(x) \right) \quad \text{---} \ (5)$$

It then follows that $y(x)$ is given by

$$y(x) = y_0 + \sum_{k=1}^{\infty} \left(y_k(x) - y_{k-1}(x) \right) \quad \text{---} \ (6).$$

To show convergence of the series (5), it suffices to show, by the Weierstrass M-test that $\exists\ h > 0$ and a sequence $\{M_k\}$ with $\sum_{k=1}^{\infty} M_k < \infty$ $(M_k > 0\ \forall\ k)$ and such that

$$|y_k(x) - y_{k-1}(x)| \leq M_k \quad \text{---} \ (7).$$

for $x \in [x_0 - h, x_0 + h]$. The convergence in (5) will in fact be uniform. Since $y_k(x)$ are continuous functions, and since the uniform limit of continu-

-ous functions is continuous, the limit $y(x)$ will be a continuous function.

We first determine the number $h > 0$ as follows: Since $f(x,y)$ and $\partial_y f(x,y)$ are continuous on the closed rectangle R, $\exists$ constants M and K (both positive) such that

(7a) — $|f(x,y)| \leq M$ and $|\partial_y f(x,y)| \leq K$ for every $(x,y) \in R$. Next, note that if (x,y_1) and (x,y_2) belong to R, then by the mean value theorem,

$$|f(x,y_1) - f(x,y_2)| = |\partial_y f(x,\bar{y})|\,|y_2 - y_1|$$

for some $\bar{y}$ in the interval between y_1 and y_2. In particular it follows that $|f(x,y_1) - f(x,y_2)| \leq K\,|y_2 - y_1|$ — (8) for all y_1, y_2 lying on a vertical line in R.

We now choose $h > 0$ such that $kh < 1$ and such that the rectangle $R' = [x_0 - h, x_0 + h] \times [y_0 - Mh, y_0 + Mh]$ $\subset R$. This is possible because $(x_0, y_0) \in R^\circ$. To prove the estimate in (7) we first observe that

$$|y_k(x) - y_0| = \left| \int_{x_0}^{x} f(t, y_{k-1}(t))\, dt \right|$$

$$\leq \int_{x_0}^{x} |f(t, y_{k-1}(t))|\, dt$$

$$\leq M(x - x_0) \leq Mh$$

for $x \in [x_0 - h, x_0 + h]$. It follows that $y_k(x)$ has a graph lying in R' ie. $(x, y_k(x)) \in R'$ for $x \in [x_0 - h, x_0 + h]$.

Suppose $\sup\limits_{x \in [x_0 - h, x_0 + h]} |y_1(x) - y_0| = a$.

Then $|f(t, y_1(t)) - f(t, y_0(t))| \leq k |y_1(t) - y_0|$

$\leq ka$.

Hence,

$$|y_2(x) - y_1(x)| \leqslant \int_{x_0}^{x} |f(t, y_1(t)) - f(t, y_0(t))| \, dt$$

$$\leqslant \quad Kah.$$

[Note : we are using the fact that if $0 \leqslant f(t) \leq g(t)$, then

$$\int_a^b f(t)\,dt \leq \int_a^b g(t)\,dt \quad]$$

More generally, if

$$|y_k(x) - y_{k-1}(x)| \leqslant a(Kh)^{k-1}$$

for $x \in [x_0 - h, x_0 + h]$, then

$$|f(t, y_k(t)) - f(t, y_{k-1}(t))| \leqslant K|y_k(t) - y_{k-1}(t)|$$

$$\leqslant aK(Kh)^{k-1}$$

and hence

$$|y_{k+1}(x) - y_k(x)| \leqslant \int_{x_0}^{x} |f(t, y_k(t)) - f(t, y_{k-1}(t))| \, dt$$

$$\leqslant \quad a(Kh)^k.$$

Hence taking $M_k = (Kh)^k$ and

noting that by our choice of h, $Kh < 1$ and hence $\sum_K M_K = \sum_K (Kh)^K < \infty$. Hence

$$y_n(x) = y_0(x) + \sum_{K=1}^{n} \left(y_K(x) - y_{K-1}(x)\right)$$

converges uniformly to some continuous function $y(x)$ on the interval $[x_0 - h, x_0 + h]$.

(Note : A sequence of functions $f_n(x)$ $x \in [a,b]$ converges to $f(x)$ uniformly iff $\forall\ \varepsilon > 0$, $\exists\ n_0 = n_0(\varepsilon)$ such that

$$|f_n(x) - f(x)| < \varepsilon$$

for all $x \in [a,b]$ and $\forall\ n \geq n_0$)

We need to show that the limit function $y(x)$ satisfies our integral equation. To see this, note that we have

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt .$$

For $x \in [x_0 - h, x_0 + h]$, $y_n(x) \to y(x)$.

Since $f : R \to \mathbb{R}$ is continuous, we have $f(t, y_{n-1}(t)) \to f(t, y(t))$ for every $t \in [x_0 - h, x_0 + h]$. Note that $\{\text{graph of } y_{n-1}(t)\} \in R' \subset R$. In fact $f(t, y_{n-1}(t)) \to f(t, y(t))$ uniformly for $t \in [x_0 - h, x_0 + h]$. This can be seen from the estimate

$$|f(t, y(t)) - f(t, y_{n-1}(t))| \le K\, |y(t) - y_{n-1}(t)|$$

for every $t \in [x_0 - h, x_0 + h]$.

Exersise. Let $R \subset \mathbb{R}^2$ be a rectangle viz $R = [a,b] \times [c,d]$ and $y_n : [a,b] \to [c,d]$. Let $f : R \to \mathbb{R}$ be a continuous function. If $y_n(x) \to y(x)$ uniformly on $[a,b]$ show that $f(t, y_n(t))$ converges to $f(t, y(t))$ uniformly on $[a,b]$.

Back to the proof : Since $f(t, y_{n-1}(t))$

Converges to $f(t, y(t))$ <u>uniformly</u> we have $\int_{x_0}^{x} f(t, y_{n-1}(t))\,dt \to \int_{x_0}^{x} f(t, y(t))\,dt$

Hence taking limits in the equation

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\,dt$$

we get that $y(t)$ satisfies our integral equation. This proves existence of a solution to our ODE (1).

We now show uniqueness of solutions of equation (1). ie we need to show that if $\bar{y}(x)$ is any other solution of (1) on $[x_0 - h, x_0 + h]$, then $y(x) = \bar{y}(x)$ $\forall\, x \in [x_0 - h, x_0 + h]$. To see this we first claim that

$\{(x, \bar{y}(x)) : x_0 - h \le x \le x_0 + h\} \subset R'$

Recall that $R' := [x_0 - h, x_0 + h] \times [y_0 - Mh, y_0 + Mh]$

$|f(x, y)| \le M$, $Kh < 1$

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Once the claim is proved, we can complete the proof of uniqueness as follows:

$$y(x) - \bar{y}(x) = \int_{x_0}^{x} (f(t, y(t)) - f(t, \bar{y}(t)))\, dt$$

$$\Rightarrow |y(x) - \bar{y}(x)| \leq \int_{x_0}^{x} |f(t, y(t)) - f(t, \bar{y}(t))|\, dt$$

$$\leq \int_{x_0}^{x} K\, |y(t) - \bar{y}(t)|\, dt$$

$$\leq K h \sup_{t \in [x_0 - h, x_0 + h]} |y(t) - \bar{y}(t)|$$

Since $x \in [x_0 - h, x_0 + h]$. Hence we get

$$\sup_{x \in [x_0 - h, x_0 + h]} |y(x) - \bar{y}(x)| \leq K h \sup_{t \in [x_0 - h, x_0 + h]} |y(t) - \bar{y}(t)|$$

But $K h < 1 \Rightarrow y(x) = \bar{y}(x)$ for all $x \in [x_0 - h, x_0 + h]$.

proof of claim : The claim is proved by showing that there does not exist any point $x_1 \in (x_0 - h, x_0 + h)$ such that $\bar{y}(x_1) = y_0 \pm Mh$. For, if such a

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a point exists, [Then for a point x such that $|x - x_0| \leq |x_1 - x_0|$] we have
$$\frac{|\bar{y}(x_1) - y_0|}{|x_1 - x_0|} = \frac{Mh}{|x_1 - x_0|} > M.$$

On the other hand for a point x such that $|x - x_0| \leq |x_1 - x_0|$,
$$\frac{|\bar{y}(x) - y_0|}{|x - x_0|} = \frac{|\bar{y}(x) - \bar{y}(x_0)|}{|x - x_0|}$$
$$= |\bar{y}'(\bar{x})| = |f(\bar{x}, \bar{y}(\bar{x}))|$$
$$\leq M$$

where $\bar{x}$ is in the interval between x and x_0. This contradiction shows that $x_1 \notin (x_0 - h, x_1 + h)$. This proves our claim. $\square$