DIFFERENTIAL TOPOLOGY - LECTURE 8

1. Introduction

In the last set of notes we tried to understand the basics of the notion of transversality. We also tried to understand, albeit in a non precise way, the notion of stability. We shall now make precise the notion of stability of certain properties that a function may have. A property of a map is said to be stable if the function continues to have the property when deformed by a small amount. The notion of deformation of a function is captured by homotopy. We shall recall these definitions and try to understand which properties of a function are preseved under a slight deformation.

We remind ourselves that $X, Y, Z, X' \ldots$ shall always denote manifolds and all maps/functions will be smooth.

2. Stability

We begin with notion of homotopy and then define stability.

Definition 2.1. Let $f, g : X \longrightarrow Y$ be smooth maps between manifolds. We say f is *homotopic* to g and write $f \sim g$ if there exists a smooth map $F : X \times [0,1] \longrightarrow Y$ with $F(x,0) = f(x)$ and $F(x,1) = g(x)$.

The map F is called a homotopy from f to g. The homtopy F gives rise to a parametrized family

$$F_t : X \longrightarrow X$$

of smooth maps defined by $F_t(x) = F(x,t)$. Observe that $F_0 = f$ and $F_1 = g$. At each time $t \in [0,1]$, the map $F_t : X \longrightarrow X$ is smooth. One interprets the homotopy F as a way of deforming f over time to g.

Let P be a property that a map can possess. For example P could be any one of the following properties : immersion, submersion, local diffeomorphism, embedding, being transverse to a given manifold etc. We shall define when such a property is stable.

Definition 2.2. A property P of a function is said to be a stable property if whenever a map $f : X \longrightarrow Y$ has the property P and F_t is a homotopy of f (that is, $F_t : X \longrightarrow X$ is such that $F_0 = f$) then there exists $\varepsilon > 0$ such that F_t has the property P for all $t < \varepsilon$.

In other words, a property of a map is stable if when we start deforming the map by a homotopy, then then for some length of time the function continues to have that property. It is important to note that stability means that the condition in the definition should hold for *every* homotopy of the map f.

To begin with let us try to see some properties that are not stable. Let Z be the x-axis in $\mathbb{R}^2$ and let P be the property that a submanifold X of $\mathbb{R}^2$ intersects Z. We wish to know whether this property of a submanifold intersecting Z is stable? This can be translated in terms of maps as follows. Let X be a submanifold of $\mathbb{R}^2$ and $i : X \hookrightarrow \mathbb{R}^2$ be the inclusion such that

$$i(X) \cap Z = X \cap Z \neq \emptyset.$$

We wish to know if this is stable? That is we wish to know if we take any homotopy F_t of i with $F_0 = i$, then does there exist $\varepsilon > 0$ such that

$$F_t(X) \cap Z \neq \emptyset$$

for all $t < \varepsilon$. Here is an example to show that this is not true.

Exercise 2.3. Let X be the submanifold defined by

$$X = \{(x, x^2) : x \in \mathbb{R}\}$$

and let $i : X \longrightarrow \mathbb{R}^2$ be the inclusion map. Then $i(X) \cap Z \neq \emptyset$. Let F_t be the homotopy of i given by

$$F_t(x, x^2) = (x, x^2 + t).$$

Then $F_0 = i$ and for every $t > 0$

$$F_t(X) \cap Z = \emptyset.$$

Thus the property of two manifolds intersecting is not a stable property.

The above example also shows (by altering the homotopy) that non transversal intersection is not a stable property. This is relegated to the exercises.

The following terminology is often used : If a property P is stable, then the class of maps having the property P is said to form a *stable class*. Our aim in this set of notes is to discuss the following beautiful theorem.

Theorem 2.4. (Stability theorem) The following classes of maps $f : X \longrightarrow Y$ from a compact manifold X to a manifold Y form a stable class :

(1) local diffeomorphisms,
(2) immersions,
(3) submersions,
(4) maps transversal to a given closed submanifold Z of Y,
(5) embeddings,
(6) diffeomorphisms.

Proof. Before discussing the proof let us be sure that we understand the statement of the theorem. For example, the statement that local diffeomorphisms form a stable class when considering maps defined on compact domains means that whenever X is a compact manifold and $f : X \longrightarrow Y$ a local diffeomorphism and F_t is a homotopy of f, then there exists $\varepsilon > 0$ such that $F_t : X \longrightarrow Y$ is a local diffeomorphism for all $t < \varepsilon$.

We begin by proving (2). We claim that for each $x \in X$, there exists a neighborhood U_x of $(x, 0)$ in $X \times I$ such that $d(f_t)_y$ is injective whenever $(y, t) \in U_x$. We may now cover $X \times \{0\}$ by finitely many open sets

$$X \times \{0\} \subseteq \cup_{i=1}^{s} U_{x_i} = U$$

as X is compact. By construction, $d(f_t)_y$ is injective whenever $(y, t) \in U$. Now (by tube Lemma), we can find $\varepsilon > 0$ such that

$$X \times [0, \varepsilon) \subseteq U.$$

Thus $f_t : X \longrightarrow Y$ is an immersion for $t < \varepsilon$. The proof of the claim is seen as follows. We may locally work in the euclidean space. Since $d(f_0)_x$ is injective and $(x, t) \mapsto d(f_t)_x$ is continuous, there exists a neghborhood $U_x = V_x \times [0, \varepsilon_x)$ of (x, o) such that whenever $(y, t) \in U_x$, $d(f_t)_y$ is injective.

The proof of (3) is similar to (2). The proof of (1) is a consequence of (2) (or (3)).

The condition (4) is proved as follows. We keep the proof of the transversality theorem in mind. We start with a smooth map $f : X \longrightarrow Y$ with $f \pitchfork Z$. We then know that $f^{-1}(Z)$ is a submanifold of X. Let f_t be a homotopy of f. Now given $x \in f^{-1}(Z)$ we can find a neighborhood U of $f(x)$ in Y and a submersion

$$g : U \longrightarrow \mathbb{R}^\ell$$

such that $g^{-1}(0) = U \cap Z$. If $V = f^{-1}(U)$ and $W = V \cap f^{-1}(Z)$, then we know that 0 is a regular value of

$$g \circ f : W \longrightarrow \mathbb{R}^\ell.$$

In other words $g \circ f$ is a submersion on W. Since $g \circ f_t$ is a homotopy of $g \circ f$, there exists $\varepsilon_x > 0$ such that $g \circ f_t$ is a submersion on W for all $t < \varepsilon_x$. We may now cover the compact set $f^{-1}(Z)$ by finitely many such open sets and get a suitable ε.

We now turn to the proof of (5). We assume that $f : X \longrightarrow Y$ is an embedding and f_t a homotopy of f. Then $f_0 = f$ is one-one, an immersion and proper. We already know that f stays an immersion for a small interval of time. We only neeed to check that f stays one-one for a small interval of time. The proof is by contradition. We assume there exists a sequence $t_n \to 0$ and distinct points $x_n, y_n \in X$ such that

$$f_{t_n}(x_n) = f_{t_n}(y_n).$$

We assume that $x_n \to x$ and $y_n \to y$. Then

$$f(x) = f_0(x) = \lim_{n\to\infty} f_{t_n}(x_n) = \lim_{n\to\infty} f_{t_n}(y_n) = f_0(y) = f(y).$$

Then as f is injective, we have $x = y$.

Now consider the function $G : X \times I \longrightarrow Y \times I$ defined by

$$G(x,t) = (f_t(x), t).$$

We then have that

$$G(x_n, t_n) = G(y_n, t_n). \tag{2.4.1}$$

We may work locally in the euclidean space. The matix of $dG_{(x,0)}$ has the (block) form

$$\begin{pmatrix} d(f_0)_x & * \\ 0 & 1 \end{pmatrix}$$

The Jacobian matrix $d(f_0)_x$ has rank k (the dimension of X) and hence the above matrix has rank equal to $k+1$. The map G therefore is injective in a neighborhood of $(x, 0)$. But this is a contradiction to (2.4.1).

Finally we prove (6). We suppose that $f : X \longrightarrow Y$ is a diffeomorphism with X compact and that we are a given a homotopy $F_t : X \longrightarrow Y$ of f so that $F_0 = f$. We shall show that there exists $\varepsilon > 0$ such that F_t is a diffeomorphism for all $t < \varepsilon$. To begin with since X is compact, X has only finitely many components. We may therefore assume that X is connected (for we can find ε for each component and then take the smallest). Thus Y is also connected and both have the same dimension. Now, as f is also an embedding, we can (by (5)) find an $\varepsilon > 0$ such that F_t is an embedding for all $t < \varepsilon$. The proof will be complete if we can show that F_t is onto for all $t < \varepsilon$. This is clearly so (see the paragraph before Corollary 2.5, Lecture 4 - II). $\square$

This is a fundamental theorem. Such stability theorems concerning stability of various differential properties are of importance. Of particular interest is the part (4) of the above theorem. Here is an example to show that the compactness assumption in the above theorem is crucial.

Example 2.5. Let $\rho : \mathbb{R} \longrightarrow \mathbb{R}$ be a smooth function with the property

$$\rho(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 2 \end{cases}$$

Define $F_t : \mathbb{R} \longrightarrow \mathbb{R}$ by $F_t(x) = x\rho(tx)$. Observe that

$$F_0(x) = x\rho(0) = x \cdot 1 = x$$

is the identity map which is a diffeomorphism (also an embedding, immersion, submersion, local diffeomorphism). Notice that $f = F_0$ is transversal to every submanifold of $\mathbb{R}$. In particular, $f \pitchfork \{0\}$. Now let $t > 0$ and $|x| > 2/t$ so that $F_t(x) = 0$. Then as $dF_t(x) = 0$ we see that (by definition) F_t cannot be transversal to the submanifold $\{0\}$. This shows that for functions defined non compact domains the condition (4) of the Stability theorem can fail.

Since for $t > 0$ and $|x|, |y| > (2/t)$ we have $F_t(x) = F_t(y) = 0$, the functions F_t cannot be an embedding (since they are not one-one) for $t > 0$. This shows that for functions defined on non compact domains condition (5) of the Stability theorem can fail.

It is also clear from the above observations that for $t > 0$, F_t is not an immersion (therefore not a submersion, local diffeomorphism, diffeomorphism). Thus for functions defined on non compact domains all of (1)-(6) in the Stability theorem can fail.

Here are some problems.

Exercise 2.6. Show by an example that non transversal intersection is not stable.

Exercise 2.7. Show by an example that being one-one, onto is not a stable property.

Exercise 2.8. Suppose that $f_0, f_1 : X \longrightarrow Y$ are homotopic. show that the exists a homotopy

$$F : X \times [0,1] \longrightarrow Y$$

such that $F(x,t) = f_0(x)$ for all $t \in [0, 1/4]$ and $F(x,t) = f_1(x)$ for all $t \in [3/4, 1]$.

Exercise 2.9. Show that homotopy is an equivalence relation.

Exercise 2.10. Show that every connected manifold is path connected. Further show that if $x, y \in X$ (X is connected), then there is a diffeomorphism $f : X \longrightarrow Y$ such that $f(x) = y$[1].

Exercise 2.11. A manifold X is contractible if its identity map is homotopic to some constant map. Show that $\mathbb{R}^k$ is contractible.

Exercise 2.12. A connected manifold X is simply connected if every map $S^1 \longrightarrow X$ is homotopic to a constant. Show that contractible spaces are simply connected.

Exercise 2.13. Show that the map $a : S^k \longrightarrow S^k$, $a(x) = -x$ is homotopic to the identity map if k is odd.

Exercise 2.14. A *deformation* of a submanifold Z of Y is a smooth homotopy $i_t : Z \longrightarrow Y$ where i_0 is the inclusion map of $Z \hookrightarrow Y$ and each i_t is an embedding. Thus $Z_t = i_t(Z)$ is a smoothly varying submanifold of Y with $Z_0 = Z$. Show that if Z is compact, then any homotopy i_t of its inclusion map is a deformation for small t. Give a counterexample in the noncompact case.

Exercise 2.15. Let

$$X = \{(A, v) \in M_{3\times 2}(\mathbb{R}) \times (\mathbb{R}^2 - 0) \, : \, Av = o\}.$$

Show that X is a manifold. Also exhibit X as the set of common zeros of independent functions.

[1]This exercise shows that the diffeomorphism group $\mathrm{Diff}(X)$ of a connected manifold X acts transitively on X. This has generalizations.

Exercise 2.16. Construct a smooth function $\rho : \mathbb{R} \longrightarrow \mathbb{R}$ having the prescribed properties as in Example 2.5.