Theorem: $C(X)$ is a complete metric space.

Proof: Let (f_n) be a Cauchy sequence in $C(X)$.

For $\varepsilon > 0$ $\exists N$ such that $\quad f_n(x) \to f(x)$

$$|f_n(x) - f_m(x)| < \varepsilon \quad \forall n, m \geq N \quad \forall x \in X.$$

By Cauchy condition in uniform convergence (f_n) converges uniformly to a $\mathbb{C}$-valued function f on X.

f_n is continuous, convergence is uniform $\Rightarrow f$ is continuous on X.

$$d_\infty(f_n, f) \longrightarrow 0, \quad f \in C(X)$$

$\Rightarrow C(X)$ is complete.

Stone-Weierstrass theorem: $f \in C([a,b])$ and $\varepsilon > 0$ Then $P \in \mathbb{C}[x]$ such that

$$|P(x) - f(x)| < \varepsilon \quad \forall x \in [a,b].$$

$$\overline{\mathbb{C}[X]} = C([a,b]).$$

Let $\mathcal{A}$ be a collection of complex-valued functions on X.

We say that $\mathcal{A}$ is an algebra if $f+g \in \mathcal{A}$ $\forall f, g \in \mathcal{A}$

$\alpha f \in \mathcal{A}$ $\forall \alpha \in \mathbb{C}$

$fg \in \mathcal{A}$

Suppose $\mathcal{B}$ is a collection real-valued functions on X. We say that $\mathcal{B}$ is an algebra if for $f, g \in \mathcal{B}$, $t \in \mathbb{R}$

$f+g \in \mathcal{B}$, $fg \in \mathcal{B}$, $tf \in \mathcal{B}$.

<u>Proposition</u> If $\mathcal{A}$ is an algebra in $C(X)$, then $\overline{\mathcal{A}}$ is also an algebra.

<u>Proof</u>: Let $f, g \in \overline{\mathcal{A}} \subseteq C(X)$

$\Rightarrow (f_n), (g_n) \in \mathcal{A}$ such that

$f_n \longrightarrow f$ and $g_n \longrightarrow g$

$d_\infty(f_n + g_n, f+g)$

$$= \sup_{x \in X} \left| f_n(x) + g_n(x) - f(x) - g(x) \right|$$

$$\leq \sup_{x \in X} \left[|f_n(x) - f(x)| + |g_n(x) - g(x)| \right]$$

$$\leq d_\infty(f_n, f) + d_\infty(g_n, g) \longrightarrow 0$$

$$\Rightarrow f_n + g_n \longrightarrow f + g$$

$\mathcal{A}$ is an algebra $\quad f_n + g_n \in \mathcal{A}$

$$\Rightarrow f + g \in \overline{\mathcal{A}}$$

$$f_n g_n \longrightarrow f g$$

$$|f_n(x) g_n(x) - f(x) g(x)|$$

$$= | f_n(x) g_n(x) - f_n(x) g(x) + f_n(x) g(x) - f(x) g(x) |$$

$$\leq |f_n(x)| \, |g_n(x) - g(x)| + |g(x)| \, |f_n(x) - f(x)|$$

$$\leq M_1 \, d_\infty(g_n, g) + M_2 \, d(f_n, f) \longrightarrow 0$$

$f_n \to f \quad \exists M_1 > 0$ such that $|f_n(x)| \leq M_1 \; \forall x \in X$
$\forall n$
(Ex)

$$M_2 = \sup_x |g(x)|$$

$f_n g_n \longrightarrow f g \;;\; \mathcal{A}$ is an algebra
$f_n g_n \in \mathcal{A} \Rightarrow fg \in \overline{\mathcal{A}}$

$\alpha f \in \bar{\mathcal{A}}$ if $\alpha \in \mathbb{C}$, $f \in \bar{\mathcal{A}}$.

Hence $\bar{\mathcal{A}}$ is also an algebra.

Ex If $\mathcal{A}$ is an algebra of $\mathbb{R}$-valued functions in $C_{\mathbb{R}}[X]$, then $\bar{\mathcal{A}} \subset C_{\mathbb{R}}(X)$ is an algebra.

Let $\mathcal{A}$ be a collection of functions on X (real or complex-valued).

We say that $\mathcal{A}$ separates points of X if for any $x, y \in X$, $x \neq y$, $\exists f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

We say that $\mathcal{A}$ is said to vanish nowhere if for any $x \in X$, $\exists f \in \mathcal{A}$ such that $f(x) \neq 0$.

Eg: 1) $\mathcal{A}$ = polynomial : $\mathcal{A}$ separates points and nowhere vanishes.

$$\left.\begin{array}{l} P(x) \neq P(y) \\ x+b \neq y+b \end{array}\right\} \text{ find } P? \quad P(x) = x.$$

2) A = all even polynomials

Can A separate points of $X = [-1,1]$?

$P(-1) = P(1) \quad \forall P \in A$

$\Rightarrow A$ does not separate points of $[-1,1]$

Proposition: Assume that $|X| \geq 2$. Let A be an algebra of functions on X. Then the following are :

(1) A separates points of X and vanishes nowhere.

(2) for any $x, y \in X$, $x \neq y$ and constants c_1, c_2 $\exists g \in A$ such that $g(x) = c_1$ and $g(y) = c_2$.

Proof: Assume (2) :

Let $x, y \in X$, $x \neq y$.

by (2) $\exists g \in A$ such that $g(x) = 0 \neq 1 = g(y)$

A separates points of X.

Let $x \in X$. By (2) $\exists g \in A$ such that

$g(x) = \frac{1}{2} \neq 0$

$\therefore A$ nowhere vanishes.

Assume (1):

Let $x, y \in X$, $x \neq y$.

$\exists\, g, h, k \in \mathcal{A}$ such that

$g(x) \neq g(y)$, $h(x) \neq 0$ and $k(y) \neq 0$

Let $u = gk - g(x)k$ and

$v = gh - g(y)h$.

$u, v \in \mathcal{A}$

$u(x) = g(x)k(x) - g(x)k(x) = 0$

$v(y) = 0$

$u(y) = g(y)k(y) - g(x)k(y) = k(y)(g(y) - g(x)) \neq 0$

$v(x) = h(x)(g(x) - g(y)) \neq 0$

Let $f = c_1 \dfrac{v}{v(x)} + c_2 \dfrac{u}{u(y)} \in \mathcal{A}$

$f(x) = c_1$ and $f(y) = c_2$.

Theorem: Let $\mathcal{A}$ be an algebra of real-valued continuous functions on a compact metric space X. Suppose $\mathcal{A}$ separates points of X and $\mathcal{A}$ nowhere vanishes. Then $\overline{\mathcal{A}} = C_{\mathbb{R}}(X)$.

Proof: Let $\mathcal{B} = \overline{\mathcal{A}}$.
Then $\mathcal{B}$ is also an algebra.

claim (1) $f \in \mathcal{B} \Rightarrow |f| \in \mathcal{B}$.

Let $f \in \mathcal{B}$. Let $a = \sup |f|$

Let $\varepsilon > 0$. Stone-Weierstrass Theorem gives $c_1, c_2, \ldots, c_n \in \mathbb{R}$ such that

$$\left| \sum_{i=1}^{n} c_i y^i - |y| \right| < \varepsilon \quad \forall y \in [-a, a]$$

Let $g = \sum_{k=1}^{n} c_k f^k \in \mathcal{B}$, take $y = |f(x)|$.

for $x \in X$, $|f(x)| \leq a$,

$$|g(x) - |f(x)|| < \varepsilon \quad \forall x \in X$$

$d_\infty(|f|, g) < \varepsilon$. for $g \in \mathcal{B}$

$|f| \in \overline{\mathcal{B}} = \mathcal{B}$.

claim (2) $f, g \in \mathcal{B} \Rightarrow \max\{f, g\}, \min\{f, g\} \in \mathcal{B}$

$$\max\{f(x), g(x)\} = \frac{f(x)+g(x)}{2} + \frac{|f(x) - g(x)|}{2} \quad \forall x \in X$$

$$\min\{f(x), g(x)\} = \frac{f(x)+g(x)}{2} - \frac{|f(x) - g(x)|}{2}$$

By claim 1, $\max\{f,g\}$, $\min\{f,g\} \in \mathcal{B}$.

<u>Claim (3)</u> $f \in C_{\mathbb{R}}(X)$ and $\varepsilon > 0$, $x \in X$ such that there exists $g_x \in \mathcal{B}$ $g_x(x) = f(x)$, $g_x(y) > f(y) - \varepsilon$ $\forall y \in X$.

Let $x \in X$, $\varepsilon > 0$ and $f \in C_{\mathbb{R}}(X)$.
for any $y \in X$ $\exists f_y \in \mathcal{B}$ such that

$f_y(x) = f(x)$, $f_y(y) = f(y)$.

$f_y(y) > f(y) - \varepsilon$ $\exists$ open set U_y in X

such that $y \in U_y$, $f_y(t) > f(t) - \varepsilon$

$\{U_y\}_{y \in X}$ is an open cover.

$\exists\, y_1, y_2, \dots, y_n$ such that $X = \bigcup_{k=1}^{n} U_{y_k}$.

Let $g_x = \max\{f_{y_1}, f_{y_2}, \dots, f_{y_n}\} \in \mathcal{B}$.

$g_x(x) = \max\limits_{1 \le k \le 1}\{f_{y_k}(x)\} = \max\{f(x)\} = f(x)$.

$y \in X \implies y \in U_{y_k}$ for some k

$g_x(y) \geq f_{y_k}(y) > f(y) - \varepsilon$

thus, $g_x(x) = f(x)$, $g_x(y) > f(y) - \varepsilon \;\; \forall y \in X$.

$$g_x(x) < f(x) + \varepsilon$$

$(g_x - f)(x) < \varepsilon$ & $g_x - f$ is continuous

$\therefore \exists W_x$ — open in X such that

$x \in W_x$ and $(g_x - f)(t) < \varepsilon \;\; \forall t \in W_x$

$$g_x(t) < f(t) + \varepsilon \;\; \forall t \in W_x$$

$\exists x_1, \dots, x_m$ such that $X = \bigcup_{k=1}^{m} W_{x_k}$

$$g = \min_{1 \le k \le m} \{g_{x_k}\} \in \mathcal{B} = \overline{\mathcal{A}}$$

Let $x \in X$.

$x \in W_{x_k}$ for some k

$$g(x) \le g_{x_k}(x) < f(x) + \varepsilon$$

$$g(x) < f(x) + \varepsilon \;\; \forall x \in X$$

$$g_{x_k}(x) > f(x) - \varepsilon \;\; \forall k = 1, 2, \dots, n \;\; \forall x$$

$$g(x) = \min_{1 \le k \le n} \{g_{x_k}(x)\} > f(x) - \varepsilon \;\; \forall x$$

$$\Rightarrow \; f(x) - \varepsilon < g(x) < f(x) + \varepsilon \;\; \forall x$$

$$\Rightarrow \quad |g(x) - f(x)| < \varepsilon \quad \forall x$$

$$d_\infty(f, g) < \varepsilon \ , \ g \in \mathcal{B}$$

$$B_\varepsilon(f) \cap \mathcal{B} \neq \emptyset \qquad \forall \varepsilon > 0$$

$$f \in \overline{\mathcal{B}} = \mathcal{B}$$

$$C_{\mathbb{R}}(X) = \mathcal{B} = \overline{\mathcal{A}} .$$
