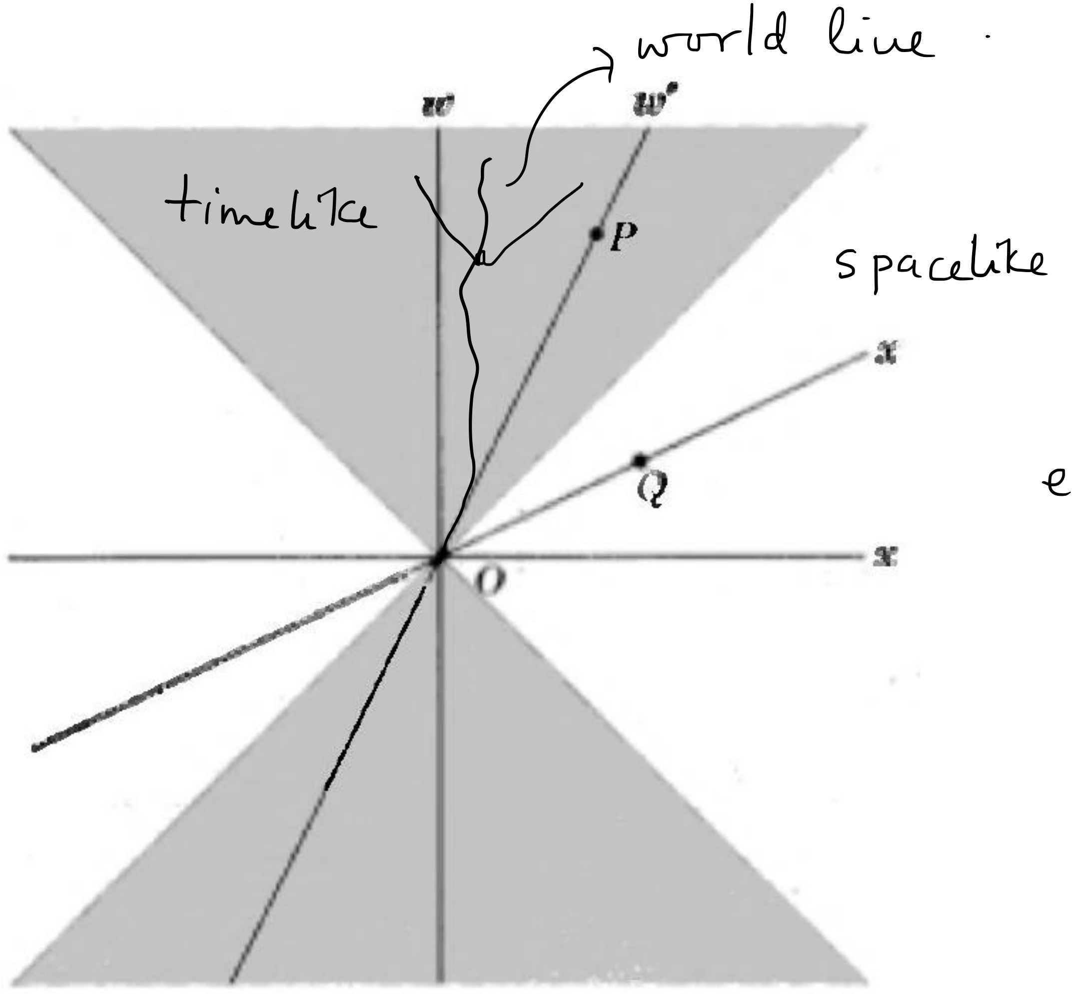
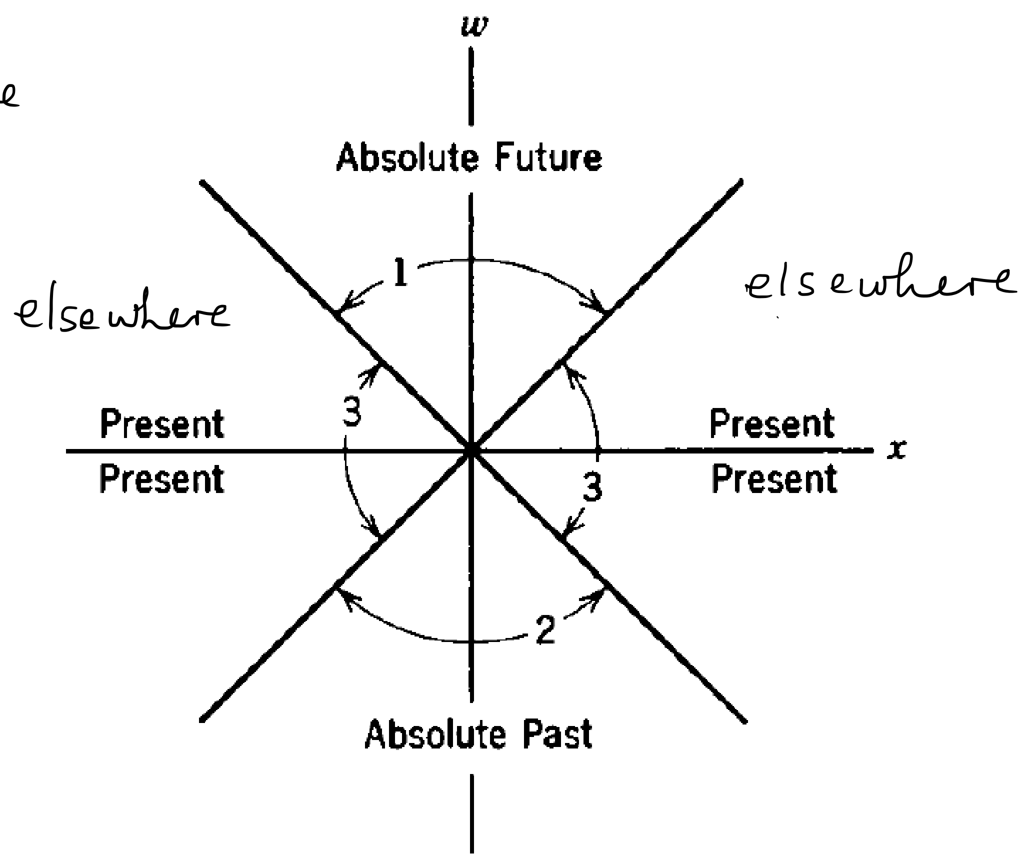
Physics 4

Lecture 12-13


world line
w
w'
timelike
P
spacelike
x
Q
x
O


w
Absolute Future
1
elsewhere
elsewhere
3
Present
Present
3
Present
Present
x
2
Absolute Past

General Lorentz transformations

S, S', S' moves with respect to S with vel $\vec{v} = (v_x, v_y, v_z)$

We know LT when frames are moving in x-direction

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Strategy

1. Rotate S so that x aligns with the direction of moving frame.
2. Now apply the standard LT.
3. Then rotate back to original orientation of frames

$$\mathcal{L} = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & R^T & \\ 0 & & & \end{array}\right) \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & R & \\ 0 & & & \end{array}\right)$$

R is the 3×3 rotation matrix.

$$R \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix}$$; To determine R and $\mathcal{L}$.

One way to determine it is to do 2 successive rotation, one setting z-component to zero, and then the y-comp. to zero.

$$R = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

$$\tan\theta = \frac{v_z}{v_y}.$$

$$= \begin{pmatrix} \cos\phi & \sin\theta\cos\theta & \sin\phi\sin\theta \\ -\sin\phi & \cos\phi\cos\theta & \cos\phi\sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}.$$

$$\tan\phi = \frac{\sqrt{v_y^2 + v_z^2}}{v_x}.$$

Applying this to

$$\mathcal{L} = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & R^T & \\ 0 & & & \end{array}\right) L \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & R & \\ 0 & & & \end{array}\right).$$

symmetric

$$\mathcal{L} = \begin{bmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ \cdots & 1+(\gamma-1)\frac{\beta_x^2}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_y}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_z}{\beta^2} \\ \cdots & \cdots & 1+(\gamma-1)\frac{\beta_y^2}{\beta^2} & (\gamma-1)\frac{\beta_y\beta_z}{\beta^2} \\ \cdots & \cdots & \cdots & 1+(\gamma-1)\frac{\beta_z^2}{\beta^2} \end{bmatrix}$$

Approach 2

Let us decompose the position vector $\vec{r}$ along components along the direction $\parallel$ to velocity and $\perp$ to it

$$\vec{r} = \vec{r}_{\parallel} + \vec{r}_{\perp} \quad \text{--- (1)}$$

$$\left.\begin{aligned} \vec{r}_{\perp}' &= \vec{r}_{\perp} && \text{--- (2)} \\ \vec{r}_{\parallel}' &= \gamma(\vec{r}_{\parallel} - \vec{v}t) && \text{--- (3)} \end{aligned}\right\} \text{L.T.}$$

$$ct' = \gamma\left[ct - \frac{(\vec{v}\cdot\vec{r})}{c}\right] \quad \text{--- (4)}$$

$$\therefore \quad \vec{r}' = \vec{r}_{\perp}' + \vec{r}_{\parallel}' = \vec{r}_{\perp} + \gamma(\vec{r}_{\parallel} - \vec{v}t)$$

$$= -\gamma\vec{v}t + \vec{r} + (\gamma - 1)\vec{r}_{\parallel} \quad \text{--- (5)}$$

But $\vec{V}$ and $\vec{r}_{\parallel}$ are parallel.

$$\vec{r}_{\parallel} = \frac{(\vec{r}\cdot\vec{V})}{v^2}\vec{V} \quad \text{--- (6)}$$

Combining these,

$$\left.\begin{aligned} ct' &= \gamma\left[ct - \frac{(\vec{v}\cdot\vec{r})}{c}\right] \\ \vec{r}\,' &= -\gamma\vec{v}t + \vec{r} + (\gamma-1)\frac{\vec{r}\cdot\vec{v}}{v^2}\vec{v} \end{aligned}\right\} \quad (7)$$

Previously we had explicitly found.

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \underset{\sim}{\mathcal{L}} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \equiv (7)$$

was explicitly determined

Four vector formalism

Recall, Newton's Laws $\vec{F} = m\vec{a}$, Physical Laws written in terms of vectors - 3 vectors

But 3 vectors are not invariant under L.T, so we have to use different objects to write our physical Laws.

event $= (ct, x, y, z)$. $\longrightarrow (x^0, x^1, x^2, x^3)$.

change notation to

$$ct \to x^0$$

$$x \to x^1.$$

$$y \to x^2$$

$$z \to x^3.$$

x^μ $\quad \mu = 0, 1, 2, 3$.

$\hookrightarrow$ four vector, in particular, a contravariant vector.

Define a second set

x_μ $\quad \mu = 0, 1, 2, 3 \longrightarrow$ covariant vector.

related to x^μ in the following way.

$$x^\mu \Rightarrow (ct, x, y, z) : (x^0, x^1, x^2, x^3).$$

$$x_\mu \Rightarrow (ct, -x, -y, -z)$$

$$x_0 = x^0$$
$$x_1 = -x^1$$
$$x_2 = -x^2.$$
$$x_3 = -x^3.$$

Now consider

$$\sum_{\mu=0}^{3} x_\mu x^\mu = x_0 x^0 + x_1 x^1 + x_2 x^2 + x_3 x^3 .$$

$$= x^0 x^0 - x^1 x^1 - x^2 x^2 - x^3 x^3$$

$$= c^2 t^2 - x^2 - y^2 - z^2 .$$

$\rightsquigarrow$ invariant interval.

Einstein summation convention

$\Rightarrow$ repeated indices $\to$ summation

$$x_\mu x^\mu \equiv \sum_{\mu=0}^{3} x_\mu x^\mu$$

<u>Conventions</u>

- Greek indices are used for 4-vectors $\mu, \nu \ldots \in \{0,1,2,3\}$
- Latin " " " for 3-vectors $m, n \ldots \in \{1,2,3\}$

Recall that the full set of symmetric transformations include boosts, rotations and any compositions.

will write these as transfn matrices.

$$L = \left[L^{\mu}{}_{\nu} \right]_{\mu,\nu = 0,..3}.$$

e.g

$$L^{\mu}{}_{\nu} \longrightarrow \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

L. Transfn from unprimed frame to primed frame

$$\boxed{x'^{\mu} = L^{\mu}{}_{\nu}\, x^{\nu}}$$

Consider position 4-vectors x^μ as the archetypal contravariant 4-vector, since we know how they behave under L.T.

Generalize

A^μ are the components of a contravariant 4-vector if they transform from one reference frame to another in the same way as the components of a position vector.

$$\boxed{A'^\mu = L^\mu{}_\nu A^\nu}$$

$$\boxed{\begin{aligned} A^{0'} &= \gamma(A^0 - \beta A^1) \\ A^{1'} &= \gamma(-\beta A^0 + A^1) \\ A^{2'} &= A^2 \\ A^{3'} &= A^3 \end{aligned}}$$

The 4-vectors do indeed form a vector space.

If A^μ, B^μ are 4 vectors.

then $$C^\mu = \alpha A^\mu + \beta B^\mu.$$

is also a four vector for any $\alpha, \beta \in \mathbb{R}$.

$$\begin{aligned}
C^{\mu'} &= (\alpha A^\mu + \beta B^\mu)' \\
&= \alpha A^{\mu'} + \beta B^{\mu'}. \\
&= \alpha (L^\mu_\nu A^\nu) + \beta (L_\nu{}^\mu B^\nu). \\
&= L^\mu_\nu (\alpha A^\nu + \beta B^\nu). \\
&= L^\mu_\nu C^\nu.
\end{aligned}$$

Proper Length (square) of A^μ.

$$A_\mu A^\mu = A_0 A^0 + A_1 A^1 + A_2 A^2 + A_3 A^3.$$

$$\equiv (A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2$$

$$= A'^0 A'_0 + A'^1 A'_1 + A'^2 A'_2 + A'^3 A'_3.$$

$$= A'_\mu A'^\mu \longrightarrow \text{invariant}.$$

Inner Product/Dot product between A^μ, B^μ.

$$A_\mu B^\mu \equiv A^\mu B_\mu = A^0 B^0 - A^1 B^1 - A^2 B^2 - A^3 B^3.$$

$\longrightarrow$ can check that gives a Lorentz scalar. invariant.