Lecture 5

We consider the differential equation

$$\frac{d^2x}{dt^2} + p\frac{dx}{dt} + qx = R(t).$$

Example (the simple harmonic oscillator)

We first consider the case $p = 0$ and $R(t) \equiv 0$. This relates to the motion of a particle which at position $x = x(t)$ at time t experiences a restoring force $F(x) = -qx$ directed towards the origin. Then Newton's 2nd law 'F = ma', becomes

$$m\frac{d^2x}{dt^2}(t) = F(x(t)) = -q\,x(t).$$

The solution is

$$x(t) = c_1 \sin\sqrt{\frac{q}{m}}\,t + c_2 \cos\sqrt{\frac{q}{m}}\,t$$

Under the 'initial conditions' $x(0) = x_0 > 0$, and $v(0) = \frac{dx}{dt}(0) = 0$ at time $t = 0$.

This implies $C_1 = 0$ and $C_2 = x_0$.

Hence $x(t) = x_0 \cos \sqrt{\frac{q}{m}}\, t$.

Here $x_0 > 0$ is the 'amplitude' of the oscillation. The period T for one complete oscillation is $T = 2\pi \sqrt{\frac{m}{q}}$ and the frequency $f = \frac{1}{2\pi}\sqrt{\frac{q}{m}}$ where

$f\,T = 1$.

<u>Example</u>. (damped oscillations). In addition to the 'restoring force' which is $-qx$ there can also be a force due to viscosity or friction. This force is proportional to the velocity and hence equal to $-p\,v(t) = -p \frac{dx}{dt}(t)$.

Thus Newton's 2nd law becomes

$$m \frac{d^2x}{dt^2}(t) = F(x(t)) = -q\,x(t) - p \frac{dx}{dt}(t)$$

or

$$\frac{d^2x}{dt^2} + \frac{p}{m}\frac{dx}{dt} + \frac{q}{m}x = 0.$$

The auxillary equation is

$$\cancel{m^2 + \frac{p}{m}} \quad a^2 + \frac{p}{m} a + \frac{q}{m} = 0$$

The roots a_1, a_2 are given

$$a_1, a_2 = -\, p/2m \pm \frac{\sqrt{\left(\frac{p}{m}\right)^2 - 4\frac{q}{m}}}{2}$$

Here a_1 and a_2 are negative numbers. With the initial conditions $x(0) = x_0$ and $v(0) = \frac{dx}{dt}(0) = 0$ we get

$$x(t) = \frac{x_0}{a_1 - a_2}\left(a_1 e^{a_2 t} - a_2 e^{a_1 t}\right).$$

When the roots a_1, a_2 are equal i.e. when $\left(\frac{p}{m}\right)^2 = 4\frac{q}{m}$, then the solution is given as

$$x(t) = x_0 e^{-at}(1 + at)$$

where $a = -\frac{p}{m}$ (see L3, p.7).

Note that in both the above cases, $x(t) \to 0$ as $t \to \infty$ i.e. the motion is overdamped. Note that $x(\infty) \equiv 0$ is the 'equilibrium' position.

When $\left(\frac{p}{m}\right)^2 < \frac{4q}{m}$, the $a_1 = -\frac{p}{2m} + i\frac{1}{2}\sqrt{\frac{4q}{m} - \left(\frac{p}{m}\right)^2}$ and $a_2 = \frac{-p}{2m} - i\frac{1}{2}\sqrt{\frac{4q}{m} - \left(\frac{p}{m}\right)^2}$.

The general solution is

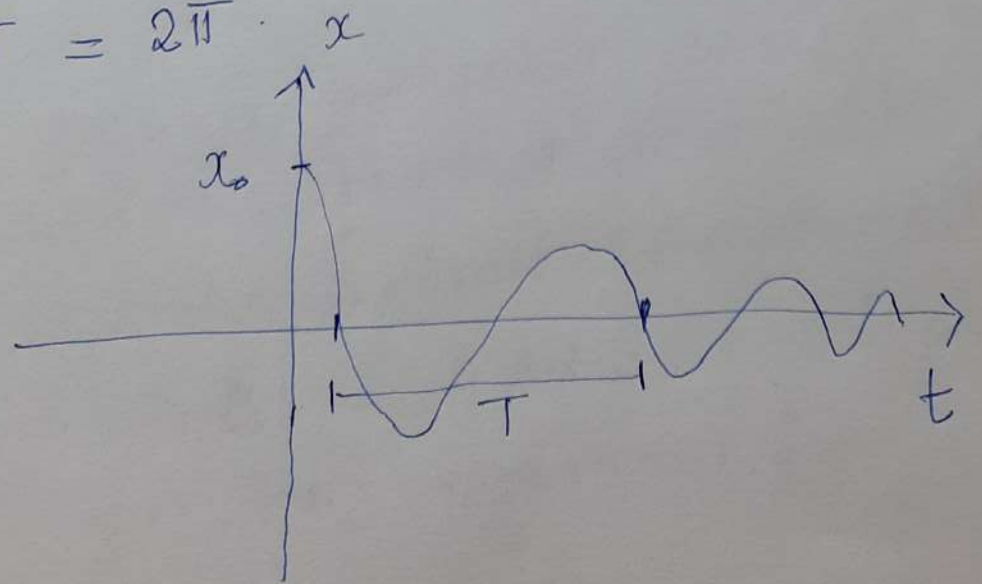
$$x(t) = e^{-\left(\frac{p}{2m}\right)t}\left(c_1 \cos\alpha t + c_2 \sin\alpha t\right)$$

where $\alpha := \frac{1}{2}\sqrt{\frac{4q}{m} - \left(\frac{p}{m}\right)^2}$ (see L 3, p.7).

With the same initial conditions as above we get

$$x(t) = \frac{x_0}{\alpha} e^{-\frac{p}{2m}t}\left(\alpha \cos\alpha t + \frac{p}{2m}\sin\alpha t\right).$$

The function $f(t) \equiv \alpha\cos\alpha t + \frac{p}{2m}\sin\alpha t$ is periodic with period T given by $\alpha T = 2\pi$.



Hence $T = \frac{2\pi}{\alpha} = \frac{2\pi}{\left(\frac{q}{m} - \left(\frac{p}{2m}\right)^2\right)^{1/2}}$.

and the frequency $f = \frac{1}{T}$.

<u>Example</u>. (forced oscillations).

In addition to the above forces (i.e. linear + damping) there can also be an external force acting on the system so that Newton's 2^{nd} law becomes $m\frac{d^2x}{dt^2} = -qx - p\frac{dx}{dt} + f(t)$

where $f(t)$ is the external force acting at time t. If we take $f(t) = F_0 \cos\omega t$ then we have the equation

$$m\frac{d^2x}{dt^2} + p\frac{dx}{dt} + qx = F_0 \cos\omega t.$$

To get a particular solution of this equation we proceed as in L4, p.4.
i.e. we take a trial solution of the form

$$x(t) = A\sin\omega t + B\cos\omega t$$

and get two equations for A and B:

$$\omega \frac{p}{m} A + \left(\frac{k}{m} - \omega^2\right) B = F_0/m$$

and $$\left(\frac{k}{m} - \omega^2\right) A - \omega \frac{p}{m} B = 0$$

where $k \equiv q$. Solving we get

$$A = \frac{\omega (p/m)(F_0/m)}{\left(\frac{k}{m} - \omega^2\right)^2 + \omega^2 \left(\frac{p}{m}\right)^2} \; ; \; B = \frac{\left(\frac{k}{m} - \omega^2\right)(F_0/m)}{\left(\frac{k}{m} - \omega^2\right)^2 + \omega^2 \left(\frac{p}{m}\right)^2}$$

Thus our particular solution is

$$x_p(t) = \frac{F_0}{(k - \omega^2 m)^2 + \omega^2 p^2} \left[\omega p \sin \omega t + (k - \omega^2 m) \cos \omega t\right]$$

Writing $\sin\phi = \dfrac{\omega p}{\left((k-\omega^2 m)^2 + \omega^2 p^2\right)^{1/2}}$; $\cos\phi = \dfrac{(k - \omega^2 m)}{\left((k-\omega^2 m) + \omega^2 p^2\right)^{1/2}}$

we get

$$x_p(t) = \frac{F_0}{\left((k-\omega^2 m)^2 + \omega^2 p^2\right)^{1/2}} \cos(\omega t - \phi)$$

When $\left(\frac{p}{m}\right)^2 < 4\frac{q}{m}$ (underdamped solutions)

The general solution of our equation is given by the formula (see on p.5 L4, p.2) $x(t, c_1, c_2) = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

where $x_1(t)$ and $x_2(t)$ are linearly independent solutions of the homogeneous equation. Thus the general solution in the case of underdamped, forced oscillations is given by (see p.4)

$$x(t) = e^{-\left(\frac{p}{2m}\right)t}\left(c_1 \cos\alpha t + c_2 \sin\alpha t\right) + x_p(t).$$

Note that as $t \to \infty$ $x(t) \to x_p(t)$.

The period $T = \frac{2\pi}{\omega}$ and frequency $f = \frac{\omega}{2\pi}$ for the forced vibrations and the amplitude is given by

$$\frac{F_0}{\left((q-\omega^2 m)^2 + \omega^2 p^2\right)^{1/2}}$$

when p is small and ω is close to $\sqrt{\frac{q}{m}}$ then the amplitude of the vibrations is large. Note that these conditions imply that the impressed frequency $f = \frac{\omega}{2\pi}$ is close to the frequency $\frac{1}{2\pi}\sqrt{\frac{q}{m} - \frac{p^2}{4m}}$ of the under-damped system. This phenomenon is called resonance.

<u>Remark</u>. Consider the differential equation satisfied by the charge $Q(t)$ in a circuit containing a resistor, an inductor and a capacitor viz.

$$L \frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E_0 \cos\omega t$$

This is similar to the mechanical oscillators that we considered in the previous example viz.

$$m\frac{d^2x}{dt^2} + p\frac{dx}{dt} + qx = F_0 \cos\omega t$$

<u>Example</u>. (Planetary motion) We consider the motion of a planet of mass m moving in a planar orbit around a star (like the sun) under the force of gravity (or more generally a central force) according to Newton's second law $\vec{F} = m\vec{a}$ where $\vec{F}$ and $\vec{a}$ are vectors, $\vec{a} = \frac{d\vec{v}}{dt}$ is the acceleration.

Since the motion is planar we take the position vector $\vec{r}$ of the planet to be $\vec{r} = r\,\vec{u_r}$ where $\vec{u_r} = \vec{i}\cos\theta + \vec{j}\sin\theta$. Let $\vec{u_\theta} = -\vec{i}\sin\theta + \vec{j}\cos\theta$

Here r is the distance from the star (the origin) to the planet and θ is the angle made between $\vec{r}$ and $\vec{i}$ (x-axis).

So (r, θ) are the polar coordinates of the planet. Then

$$\vec{V} = \frac{d\vec{r}}{dt} = \vec{u_r}\frac{dr}{dt} + r\frac{d\vec{u_r}}{dt}$$

$$= \vec{u_r}\frac{dr}{dt} + r\vec{u_\theta}\frac{d\theta}{dt}.$$

Hence

$$\vec{a} = \frac{d\vec{v}}{dt} = \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\vec{u_\theta}$$

$$+ \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\vec{u_r}.$$

Writing $\vec{F} = F_\theta\vec{u_\theta} + F_r\vec{u_r}$ and using $\vec{F} = m\vec{a}$ we get

$$m\left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right) = F_\theta \quad \text{and} \quad m\left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right) = F_r$$

$\vec{F}$ is called a central force when $\bar{F}_\theta = 0$. In that case we get

$$r \frac{d^2\theta}{dt^2} + 2 \frac{dr}{dt} \frac{d\theta}{dt} = 0.$$

$$\Rightarrow \frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right) = 0 \quad \text{or} \quad r^2 \frac{d\theta}{dt} = h$$

where h is a constant. Let $A(t)$ denote the area swept out by the radius vector $\vec{r}(\cdot)$ from time 0 to t.

Recall: $dA = \frac{1}{2} r^2 d\theta$. Integrating between t_1 and t_2 we get

$$A(t_2) - A(t_1) = \frac{h}{2}(t_2 - t_1).$$

This is Kepler's 2^{nd} law: The radius vector from the sun to the planet sweeps out equal areas in equal intervals of time.

When $\vec{F}$ is a gravitational force then

$$\bar{F}_r = -G \frac{mM}{r^2} = -\frac{km}{r^2} \quad \text{where } k := GM.$$

Hence

$$\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{k}{r^2}.$$

when $\theta = 0$. These imply $A = 0$ and $B > 0$. Thus we get

$$\frac{1}{r} = z = B\cos\theta + \frac{k}{h^2}$$

or
$$r = \frac{h^2/k}{1 + \left(B\frac{h^2}{k}\right)\cos\theta} = \frac{pe}{1 + e\cos\theta}$$

where $e = \frac{Bh^2}{k}$ and $p = \frac{1}{B}$. This is the equation for a conic section with eccentricity e. To determine e we use the law of conservation of energy: The total energy $E =$ K.E + P.E is a constant. Hence

$$\frac{1}{2}m\left[r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dr}{dt}\right)\right]^2 - \frac{km}{r} = E$$

when $\theta = 0$, $\frac{dr}{dt}(t_0) = 0$. We get

$$\frac{1}{2}mr^2\frac{h^2}{r^4} - \frac{km}{r} = E$$

at $t = t_0$ and $r(t_0) = \frac{h^2/k}{1+e}$.

$$\Rightarrow \quad e = \sqrt{1 + E\left(\frac{2h^2}{mk^2}\right)}.$$

For a closed orbit $E < 0$ and hence $e < 1$. This gives Kepler's first law: The orbit is an ellipse with the sun as the focus.

If a is the length of the major axis then $2a = r_{min} + r_{max}$. Hence

$$2a = \frac{h^2/k}{1+e} + \frac{h^2/k}{1-e} = \frac{h^2}{k(1-e^2)}$$

$$= \frac{2h^2}{k}\frac{a^2}{b^2}$$

where b is the ~~major ax~~ length of the minor axis. $\Rightarrow b^2 = \frac{h^2 a}{k}$.

Let T = period of the planet. Then from Kepler's second law we have $A(T) = \frac{hT}{2} = \pi a b$. Hence

$$T^2 = \frac{4}{h^2}\pi^2 a^2 b^2 = \frac{4\pi^2}{k} a^3$$

This gives us Kepler's third law: the squares of the periods of a planet is proportional to the cubes of their mean distances from the sun.

Let $r := \frac{1}{z}$. Then

$$\frac{dr}{dt} = -\frac{1}{z^2}\frac{dz}{dt} = -\frac{1}{z^2}\frac{dz}{d\theta}\frac{d\theta}{dt} = -h\frac{dz}{d\theta}.$$

$$\frac{d^2r}{dt^2} = -h\frac{d}{dt}\frac{dz}{d\theta} = -h\frac{d^2z}{d\theta^2}\frac{d\theta}{dt}$$

$$= -h^2z^2\frac{d^2z}{d\theta^2}.$$

Then the radial component of our differential equation becomes

$$\frac{d^2z}{d\theta^2} + z = \frac{k}{h^2}.$$

The general solution of this can be written using the formula $z = z_g + z_p$ where z_p is a particular solution of the non-homogeneous equation. Then

$$z(\theta) = A\sin\theta + B\cos\theta + \frac{k}{h^2}$$

We choose the axis $\theta = 0$ when $r(t_0)$ is closest to the star i.e. when $r(t)$ is a minimum or $z(\theta)$ is a maximum at $\theta = 0$. Thus $\frac{dz}{d\theta} = 0$ and $\frac{d^2z}{d\theta^2} < 0$.