Lecture 20: Noether normalization and Going up theorem

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Noether Normalization : Let k be a field, R be a f.g. k-algebra then R contains a subring $S = k[y_1, \dots, y_d]$ isomorphic to the poly ring in r-variables and R is integral over S.

Pf:

Let $R = k[x_1, \dots, x_n]$ for some $x_1, \dots, x_n \in R$.

Proof by ind on n. If $n = 1$ then take $S = k$ if x_1 is alg over k else $S = R = k[x_1]$. Now for general n, if $x_1, \dots, x_n$ are alg indep over k then again $S = R = k[x_1, \dots, x_n]$ works. So may assume $x_1, \dots, x_n$ are alg dependent.

Then $\exists$ $f(X_1, \dots, X_n) \overset{\neq 0}{\in} k[X_1, \dots, X_n]$ s.t.

$$f(x_1, \dots, x_n) = 0 \quad , \quad f(X_1, \dots, X_n) = \sum a_{\underset{\sim}{i}} X^{\underset{\sim}{i}} \qquad \underset{\sim}{i} = (i_1, \dots, i_n) \in \mathbb{N}^n, \; a_{\underset{\sim}{i}} \in k$$

Let $z_i = x_n^{r^i} - x_i \in R \qquad 1 \leq i \leq n-1$

$$0 = f(x_1, \dots, x_n) = f(x_n^{r} - z_1, x_n^{r^2} - z_2, \dots, x_n^{r^{n-1}} - z_{n-1}, x_n)$$

$$= \sum_{\text{finite}} a_{\underset{\sim}{i}} (x_n^{r} - z_1)^{i_1} (x_n^{r^2} - z_2)^{i_2} \cdots (x_n^{r^{n-1}} - z_{n-1})^{i_{n-1}} x_n^{i_n}$$

if $r > \max\{i_1, \dots, i_n\}$ then $\downarrow$

is almost monic poly eq in x_n with coeff in $k[z_1, \dots, z_n]$

if we choose $r > \max\{i_1, \dots, i_n \mid a_{\underset{\sim}{i}} \neq 0 \text{ in } f(X_1, \dots, X_n)\}$

then x_n is integral over $z_1, \dots, z_{n-1}$

$\Rightarrow$ x_i $1 \le i \le n-1$ are also integral over $z_1, \dots, z_{n-1}$

$x_i = x_n^{r^i} - z_i$

Hence R is integral over $R' = k[z_1, \dots, z_{n-1}]$.

Now use induction to get $S \subseteq R'$ with $S = k[y_1, \dots, y_d]$ with $y_1, \dots, y_d$ alg indep over k, and R' integral over S. Hence R is integral over S. $\blacksquare$

$$\varphi : A = k[y_1, \dots, y_d] \overset{\text{integral}}{\subseteq} R = \frac{k[X_1, \dots, X_n]}{I}$$

$$f : \operatorname{mspec}(R) \longrightarrow \mathbb{A}^d \quad \text{finite morphism}$$

then f is finite to 1 map.

by HN m a maximal ideal of R then $\varphi^{-1}(m)$ is a maximal ideal of A. $f(m) = \varphi^{-1}(m)$

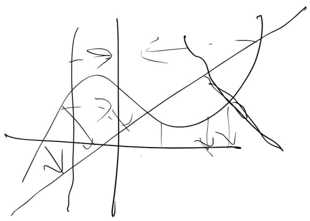
Def^n: A ring R is said to be of <u>dim d</u> if $\exists$ a chain of prime ideals $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_d$ in R and any other chain of prime ideals in R is of length $\leq d$.

krull-dimension.

Ex: 1) $$\frac{k[x,y,z]}{(xz^2 - x^3 y)} \supseteq k[u, v]$$

$$x(z^2 - x^2 y) \qquad \frac{k[x,y]}{(xy)} \cong \frac{k[u,v]}{(u^2 - v^2)} \supseteq k[u]$$

$u + v = x$ &
$u - v = y$



(2) $k(x)$ is a k-alg
$\underset{k[f]}{\overset{\cup\!\mid}{}}$ not integral $\forall f \in k(x)$

Lemma: $A \subseteq B$ be integral ext^n.
Let $Q \subseteq B$ be a prime ideal of B & $P = Q \cap A$. Then Q is a maximal ideal of B iff P is a maximal ideal of A.

Pf: $A/P \subseteq B/Q$ $\left(\because Q \cap A = \ker(q \circ i) \quad A \xhookrightarrow{i} B \xrightarrow{q} B/Q \right)$

& B/Q is int over A/P

So if A/P is a field then B/Q is a field.

Conversely if B/Q is a field & $\overset{\neq 0}{x} \in A/P$ then $\frac{1}{x} \in B/Q \implies \frac{1}{x}$ is int over A/P. Hence

$$\left(\frac{1}{x}\right)^n + a_{n-1}\left(\frac{1}{x}\right)^{n-1} + \cdots + a_0 = 0 \quad \text{for some } a_i \in A/P$$

$$\overset{\cdot x^{n-1}}{\implies} \frac{1}{x} + a_{n-1} + a_{n-2}x + \cdots + a_0 x^{n-1} = 0 \quad \text{in } B/Q$$

$\implies \frac{1}{x} \in A/P$. Hence A/P is a field. ∎

Going up theorem: Let $A \subseteq B$ be rings with B integral over A. Let

$P_1 \subseteq P_2 \subseteq \dots P_m \subseteq P_{m+1} \subseteq \dots \subseteq P_n$ be a chain of prime ideals in A & $Q_1 \subseteq Q_2 \subseteq \dots \subseteq Q_m$ be a chain of prime ideals in B s.t. $Q_i \cap A = P_i$ $1 \le i \le m$. Then $\exists$ $Q_{m+1}, \dots, Q_n$ prime ideals of B s.t. $Q_i \cap A = P_i$ $\forall 1 \le i \le n$.