DIFFERENTIAL TOPOLOGY - LECTURE 11

1. Introduction

In our definition, a manifold is a subset of some euclidean space $\mathbb{R}^N$. However, the N that we have may not be the optimal one, in the sense that there could exist $N' < N$ such that $\mathbb{R}^{N'}$ contains a diffeomorphic copy of the manifold. We have, for example, that $S^1 \times S^1$ is naturally a subset of $\mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$. But we also know that $S^1 \times S^1$ embeds in $\mathbb{R}^3$ as the set of points at a distance b from the circle of radius a, $0 < b < a$. On the other hand we know that a compact k-manifold cannot be embedded in $\mathbb{R}^k$

The question that naturally arises is : Given k, is there an integer M such that every k-dimensional manifold embeds in $\mathbb{R}^M$. The answer is provided by the Whitney embedding theorem. It states that every k-manifold can be embedded in $\mathbb{R}^{2k}$. We shall prove the weak version : Every k-manifold can be embedded in $\mathbb{R}^{2k+1}$. This is again a beautiful application of Sard's theorem.

Let us remind ourselves that $X, Y, Z \ldots$ always denote manifolds and all maps are smooth.

2. Whitney embedding theorem

A fundamental construction in differential topology is that of the tangent bundle[1] of a manifold. This is formal way of keeping track of tangent spaces at different points of the manifold.

Here is the formal definition. Let $X \subseteq \mathbb{R}^N$ be a k-manifold. We look at

$$T(X) = \{(p, v) \, : \, p \in X, v \in T_x(X)\} \subseteq X \times \mathbb{R}^N$$

with the subspace topology. $T(X)$ is called the *tangent bundle* of X. Thus, $T(X)$ is the disjoint union of tangent space at point of X and consists of pairs (p, v) where $p \in X$ and the second coordinate v is a tangent vector at p.

It is convenient to think of the tangent bundle in the following informal way. The tangent bundle $T(X)$ is the disjoint union

$$T(X) = \cup_{p \in X} T_p(X)$$

of tangent spaces and every element $v \in T(X)$ comes with an additional data, namely, the point p at which v is a tangent vector.

Observe that we have two canonical maps. The map

$$\pi : T(X) \longrightarrow X$$

defined by

$$\pi(p, v) = p$$

is called the *projection* of the tangent bundle. The map $s : X \longrightarrow T(X)$ defined by

$$s(x) = (x, 0)$$

[1]The tangent bundle along with a host of of other bundles that can be associated to a manifod form a set of powerful tools for understanding the geometry and topology of manifolds.

is called the 0-section. Notice that $\pi \circ s$ is the identity map of X. Also notice that both π and s are smooth and hence s is a diffeomorphism onto its image. Thus $T(X)$ contains a diffeomorphic copy of X as the (image of the) 0-section. It is again straightforward to verify that $s \circ \pi$ is homotopic to the identity map of $T(X)$ (see the exercises). Thus, π is a homotopy equivalence and $T(X)$ and X have the same homotopy type.

Before looking at some examples, we collect some more facts about the tangent bundle. Let $f : X \longrightarrow Y$ be a smooth map between manifolds. This induces a map

$$df : T(X) \longrightarrow T(Y)$$

from the tangent bundle of X to that of Y. The map df is defined in the obvious manner by

$$df(p, v) = (f(p), df_p(v)).$$

The map df is called the *total derivative* of f. If we think of the tangent bundle as the union of tangent spaces, then df is the map whose restriction to $T_p(X)$ is just df_p, the derivative of f at p. It is an exercise to show that the total derivative df is smooth. Further, if $X \xrightarrow{f} Y \xrightarrow{g} Z$ are smooth maps between manifolds, then $d(g \circ f) = dg \circ df$. Using this chain rule it immediately follows that if f is a diffeomorphism, then df is also a diffeomorphism.

Here is an example.

Example 2.1. Let us try to understand the tangent bundle of $\mathbb{R}^n$. By defintion,

$$T(\mathbb{R}^n) = \{(p, v) \, : \, p \in \mathbb{R}^n, v \in T_p(\mathbb{R}^n) = \mathbb{R}^n\}.$$

Thus $T(\mathbb{R}^n) = \mathbb{R}^n \times \mathbb{R}^n$. Similarly, if $U \subseteq \mathbb{R}^n$ is open, then $T(U) = U \times \mathbb{R}^n$.

Thus it turns out that that whenever $X = \mathbb{R}^n$ or U, where U is open in $\mathbb{R}^n$, then $T(X) = X \times \mathbb{R}^n$ is a product.

Here is some idea about the topology of the tangent bundle. Suppose that $X \subseteq \mathbb{R}^N$ is a k-manifold. Let $U \subseteq X$ be an open set. Then U is also a k-manifold. As

$$T(U) = \{(p, v) \, : \, p \in U, v \in T_p(U) = T_p(X)\}$$

we see that if $(p, v) \in T(U)$, then $(p, v) \in T(X)$. Hence,

$$T(U) \subseteq T(X).$$

We claim that $T(U)$ is an open subset of $T(X)$. Remeber that

$$T(X) \subseteq X \times \mathbb{R}^N \subseteq \mathbb{R}^N \times \mathbb{R}^N$$

has the subspace topology. The claim will follow if we can check that

$$T(U) = T(X) \cap (U \times \mathbb{R}^N).$$

Clearly, $T(U) \subseteq T(X) \cap (U \times \mathbb{R}^N)$. On the other hand suppose that

$$(p, v) \in T(X) \cap (U \times \mathbb{R}^N).$$

Then clearly $p \in U$ and $v \in T_p(X) = T_p(U)$. This forces $(p, v) \in T(U)$. Thus $T(U)$ is indeed open in $T(X)$. Thus for every open set U in X, the tangent bundle $T(U)$ of U sits as an open subset of $T(X)$. Now recall that total derivative of a diffeomorphism is again a diffeomorphism between the tangent bundles. Thus if X is a k-manifold and $\varphi : V \longrightarrow U \subseteq X$ is a local parametrization, then

$$d\varphi : T(V) = V \times \mathbb{R}^k \longrightarrow T(U)$$

is a diffeomorphism. This shows that the tangent bundle of a k-manifold X is a manifold. We record this below.

Proposition 2.2. Let X be a k-manifold. Then the tangent bundle $T(X)$ is a manifold of dimension $2k$. $\square$

Remark 2.3. It is very important note the local nature of the tangent bundle that is evident in the discussion above. Let X be a k-manifold. The above discussion tells us that if U is an open subset of the manifold X, then $T(U)$ is an open subset of the tangent bundle $T(X)$. Next, the open set $T(U)$ is diffeomorphic to the product $U \times \mathbb{R}^k$ if U is small enough. Thus the tangent bundle is locally a product. It may not be a product globally. That is, $T(X)$ may not be diffeomorphic to the product $X \times \mathbb{R}^k$.

Here is an example.

Example 2.4. Let us try to understand the tangent bundle of $S^1 \subseteq \mathbb{R}^2$. We claim that $T(S^1)$ is diffeomorphic to the product $S^1 \times \mathbb{R}$. To see this consider the map $f : T(S^1) \longrightarrow S^1 \times \mathbb{R}$ by

$$f(p, v) = (p, p_2 v_1 - p_1 v_2).$$

Here $p \in S^1$ is the point $p = (p_1, p_2)$. The map f is smooth and actually a diffeomorphism for the map $g : S^1 \times \mathbb{R} \longrightarrow T(S^1)$ defined by

$$g(p, t) = (p, tp_2, -tp_1)$$

is the inverse of f.

Thus the tangent bundle of S^1 is also a product. It is well known, and we shall see a proof if time permits, that tangent bundle of S^2 is not a product[2].

The following theorem will imply the (weak form of) Whitney embedding theorem.

Theorem 2.5. Every k-manifold X admits a one-one immersion into $\mathbb{R}^{2k+1}$. Thus if X is a compact k-manifold, then X embeds in $\mathbb{R}^{2k+1}$.

Proof. The proof is a beautiful application of Sard's theorem. We assume that there is a one-one immersion $f : X \longrightarrow \mathbb{R}^N$ with $N > 2k+1$. Using this we construct a one-one immersion of X into $\mathbb{R}^{N-1}$ cutting down the dimension (into which X immerses) by one. If $N - 1 > 2k + 1$, then following the same procedure will show that the dimesnion again can be cut down by one. This will complete the proof.

So let us start with a one-one immersion $f : X \longrightarrow \mathbb{R}^N$ with $N > 2k + 1$. Consider the map

$$h : X \times X \times \mathbb{R} \longrightarrow \mathbb{R}^N$$

defined by

$$h(x, y, t) = t(f(x) - f(y)).$$

Next, consider the map $g : T(X) \longrightarrow \mathbb{R}^N$ defined by

$$g(x, v) = df_x(v).$$

Observe that $\dim(X \times X \times \mathbb{R}) = 2k+1 < N$ and $\dim(T(X)) = 2k < N$. By Sard's theorem, almost every $a \in \mathbb{R}^N$ is a regular value of both h and g. In particular there exists $a \in \mathbb{R}^N$, $a \neq 0$, that is not in the image of both h and g. We now write $\mathbb{R}^N$ as a direct sum

$$\mathbb{R}^N =< a > \oplus a^{\perp}$$

and let $p : \mathbb{R}^N \longrightarrow a^{\perp}$ denote the projection.

[2]It is a deep theorem that $T(S^n)$ is diffeomorphic to $S^n \times \mathbb{R}^n$ if and only if $n = 0, 1, 3, 7$. This is due to several mathematicians the final step being taken by J Frank Adams in his celebrated work "Vector fields on spheres".

We first claim that the map $p \circ f : X \longrightarrow a^{\perp}$ is one-one. For if we assume that

$$p \circ f(x) = p \circ f(y),$$

$x \neq y$, then as

$$p(f(x) - f(y)) = 0$$

the vector $f(x) - f(y)$ has no component in $a^{\perp}$. Hence we must have

$$f(x) - f(y) = ta$$

for some $t \neq 0$. If $x \neq y$, then t must be non zero. But this now means that $h(x, y, 1/t) = a$ contradicting the choice of a. Thus $p \circ f$ is one-one.

Next we claim that $p \circ f$ is an immersion. Suppose that

$$d(p \circ f)_x(v) = 0$$

for some $x \in X$ and $v \in T_x(X)$, $v \neq 0$. Then

$$dp_{f(x)}(df_x(v)) = p(df_x(v) = 0.$$

This means that $df_x(v)$ has no component in $a^{\perp}$ and therefor as before we must have that

$$df_x(v) = ta$$

for some $t \neq 0$ (f is an immersion). But then,

$$g(x, 1/t) = a$$

contradicting the choice of a. Thus $p \circ f : X \longrightarrow a^{\perp} = \mathbb{R}^{N-1}$ is both one-one and an immersion. If $N - 1 > 2k + 1$, the precedure will allow us to construct an immersion $X \longrightarrow \mathbb{R}^{N-2}$. We can finally get an immersion of X into $\mathbb{R}^{2k+1}$. This completes the proof. □

This theorem shows that if X is compact, then X can be embedded in $\mathbb{R}^{2k+1}$. To complete the proof of the Whitney embedding theorem when the manifold is not compact one additional condition that we need to ensure is that of properness. We should therefore have a method of modifying a one-one immersion to a one-one immersion that is proper. The technical tool required to achieve this is the existence of smooth partitions of unity on a manifold. We have already seen the proof of the continuous version and the smooth version is not too different. We therefore just state it without proof.

Theorem 2.6. (Partitions of unity) Let $A \subseteq \mathbb{R}^N$ be an arbitrary subset and $\{U_\alpha\}$ an open cover of X. There exists a sequence $\{\theta_i\}$ of smooth functions on A such that

(1) $0 \leq \theta_i(x) \leq 1$ for all $x \in X$ and for all i.
(2) Each $x \in X$ has a neighbourhood on which all but finitely many θ_i are identically zero.
(3) Each θ_i is identically zero except possibly on a closed set contained inside some U_α.
(4) $\sum_i \theta_i(x) =$ for each $x \in X$.

Note that U_α is open in A. The sequence of functions $\{\theta_i\}$ are called a partition of unity subordinate to the cover $\{U_\alpha\}$. An immediate corollary is the existence of a proper map on any manifold.

Corollary 2.7. On any manifold X there exists a proper function $\rho : X \longrightarrow \mathbb{R}$.

Proof. Convince youselves that there exists an open cover $\{U_\alpha\}$ of A such that the closure of each U_α is compact. Fix a partition of unity $\{\theta_n\}$ subordinate to one such open cover. The function $\rho : X \longrightarrow \mathbb{R}$ defined by

$$\rho = \sum_n n\theta_n$$

is smooth and has the required properties. To see this first observe that the function $n\theta_n \geq 0$ on X. Now suppose that $\rho(x) \leq j$. We claim that for some i, $1 \leq i \leq j$, we must have that $\theta_i(x) \neq 0$. If not, there exists finitely many indices $i_1, \cdots, i_r > j$, $i_1 < \cdots < i_r$, such that $\theta_t(x) = 0$ for $t \neq i_1, i_2, \ldots, i_r$ so that

$$\theta_{i_1}(x) + \cdots + \theta_{i_r}(x) = 1.$$

Now

$$j \geq \rho(x) = i_1\theta_{i_1}(x) + \cdots + i_r\theta_{i_r}(x) \geq i_1(\theta_{i_1}(x) + \cdots + \theta_{i_r}(x)) = i_1$$

which is a contradiction. Thus $\rho(x) \leq j$ implies that one of $\theta_1, \ldots, \theta_j$ is non-zero at x. Consequently the inclusion

$$\rho^{-1}([-j, j]) \subseteq \cup_{i=1}^{j}\{x : \theta_i(x) \neq 0\}.$$

holds. The closure of the right hand side is compact for each set on the right hand side is contained in some U_α which has compact closure. Thus each inverse image $\rho^{-1}[-j, j]$ is compact. Hence the ρ inverse image of every compact set is compact. □

Theorem 2.8. (Whitney embedding theorem) Every k-manifold X embeds in $\mathbb{R}^{2k+1}$.

Proof. We have already seen that every compact k-manifold embeds in $\mathbb{R}^{2k+1}$. Given a k-manifold X we first fix a one-one immersion $f : X \longrightarrow \mathbb{R}^{2k+1}$ and a proper function $\rho : X \longrightarrow \mathbb{R}$. Composing with a suitable diffeomorphism, if necessary, we may assume that f maps into $\mathrm{Int}(\mathbb{D}^n)$ so that

$$f : X \longrightarrow \mathbb{R}^{2k+1}$$

is a one-one immersion with image contained in the open unit ball. Clearly the map $F : X \longrightarrow \mathbb{R}^{2k+2}$ defined by

$$F(x) = (f(x), \rho(x))$$

is also a one-one immersion. Since $2k+2 > 2k+1$, we may use the methods in the proof of Theorem 2.6 to cut down the dimension by one by composing F with a suitable projection p to obtain

$$p \circ F : X \longrightarrow a^{\perp}$$

which is again a one-one immersion and $a \in \mathbb{R}^{2k+2}$ is a suitable unit vector. Recall from Theorem 2.6, that $p \circ F$ is a one-one immersion for almost all choices of of $a \in \mathbb{R}^{2k+1}$. We choose $a \in S^{2k+1}$ so that a does not equal either the "north pole" or the "south pole". Since $p \circ F$ is an one-one immersion, the proof will be complete if we can show that $p \circ F$ is proper. Towards this it is enough to check that the inverse image of any closed ball in $a^{\perp}$ (under $p \circ F$) is compact. To prove this we claim that given $c > 0$, there exists $d > 0$ such that

$$\{x \in X : ||p \circ F(x)|| \leq c\} \subseteq \{x \in X : ||\rho(x)|| \leq d\}.$$

Since the set on the right hand side is compact (ρ is proper), the set on the left hand side is compact thereby completing the proof. If the claim is not true we can find a sequence x_i of points in X such that $||p \circ F(x)|| \leq c$ but $||\rho(x_i)|| \to \infty$. Now note that, by the very definition of p, we must have

$$w_i = \frac{1}{\rho(x_i)}(F(x_i) - p \circ F(x_i))$$

is a multiple of a (since $F(x_i) - p \circ F(x_i)$ is a multiple of a). Now we observe several things. First as $i \to \infty$ we have that

$$\frac{F(x_i)}{\rho(x_i)} = \left(\frac{f(x_i)}{\rho(x_i)}, 1\right) \to (0, \ldots 0, 1)$$

as $||f(x_i)|| < 1$. Next, as $i \to \infty$ we have that

$$\frac{p \circ F(x_i)}{\rho(x_i)} \to 0$$

as the numerator has bounded norm. The previous two observations imply that

$$w_i \to (0, \dots, 0, 1)$$

as $i \to \infty$. Since each w_i is a multiple of a, so is the limit. So a must either be the north pole or the south pole. This contradiction completes the proof. $\square$

The strong Whitney embedding theorem states that every k-manifold embeds in $\mathbb{R}^{2k}$. We end this section with brief discussion of vector fields.

Definition 2.9. A *vector field* on a manifold $X \subseteq \mathbb{R}^N$ is a smooth map $\mathbf{v} : X \longrightarrow \mathbb{R}^N$ such that $\mathbf{v}(x) \in T_x(X)$ for all $x \in X$. A point $x \in X$ such that $\mathbf{v}(x) = 0$ is called a *zero* of the vector field $\mathbf{v}$.

A *cross-section* (or simply a section) of the tangent bundle of X is a smooth map $s : X \longrightarrow T(X)$ with $p \circ s = id$ map of X. We already know of the 0-section of a manifold. Thus every section s of the tangent bundle is of the form $s(p) = (p, s'(p))$ where $s' : X \longrightarrow \mathbb{R}^N$ is a smooth map. It is therefore immediate that a vector field on X determines a section of the tangent bundle and every section of the tangent bundle determines a vector field on the manifold. Because of this, a section of the tangent bundle is also called a vector field. We could have considered a section defined on an open subset U of X in which case s is called a vector field over U.

Here are some exercises.

Exercise 2.10. Show that on any manifold X

(1) given $p \in X$, there exists a non-constant smooth function $f : X \longrightarrow \mathbb{R}$ such that $f(p) = 0$,
(2) given $p \in X$, there exists a neighborhood U of p and a smooth function $f : U \longrightarrow \mathbb{R}$ that f has no critical points[3].

Exercise 2.11. Let $f : X \longrightarrow Y$ be a smooth map. Show the totao derivative map $df : T(X) \longrightarrow T(Y)$ is smooth.

Exercise 2.12. Let g be a everywhere positive function on X. Show that the map $T(X) \longrightarrow T(X)$ defined by $(p, v) \mapsto (p, g(p)v)$ is smooth.

Exercise 2.13. Show that the projection $\pi : T(X) \longrightarrow X$ of the tangent bundle is a submersion. Hence p has no critical points. Also note that $T(X)$ is always non-compact. Finally show that if s is the zero section, then $s \circ \pi$ is homotopic to the identity map of $T(X)$. Thus X and $T(X)$ have the same homotopy type.

Exercise 2.14. Show that $T(X \times Y) = T(X) \times T(Y)$.

Exercise 2.15. Let X be a manifold and $x \in X$. Show that there exists a vector field s (think of the vector field s as a section of the tangent bundle) such that $s(x) \neq 0$.

Exercise 2.16. Show that if k is odd, then there exists a vector field on S^k with no zeros.

Exercise 2.17. Show that if S^k admits a nowhere zero vector field, then its antipodal map is homotopic to the identity.

Exercise 2.18. Let X be a k-manifold. Let $S(X)$ be the set of points $(x, v) \in T(X)$ with $||v|| = 1$. Prove that $S(X)$ is a submanifold of $T(X)$. $S(X)$ is called the sphere bundle of X. Observe that the we can restrict the projection π of the tangent bundle to $S(X)$ to get a map $\pi' : S(X) \longrightarrow X$. Now observe that for each $x \in X$ the fiber $\pi'^{-1}(x) = S^{k-1}$ is a sphere. This is the reason $S(X)$ is called the sphere bundle of X.

[3]We know that on any compact manifold every real valued smooth function has a critical point. It is a theorem of Morris Hirsch that if X is a non-compact manifold then there always exists a smooth function $f : X \longrightarrow \mathbb{R}$ having no critical points. In other words the neighborhood U can be taken to be the whole of X if X is non-compact.

Exercise 2.19. (Whitney immersion theorem) Prove that every k-manifold X immerses in $\mathbb{R}^{2k}$. (Go through the proof of Theorem 2.5 carefully).

Exercise 2.20. Let X be a compact k-manifold. Show that there exists a a map $f : X \longrightarrow \mathbb{R}^{2k-1}$ that is an immersion except at finitely many points of X. (See the hint in G and P).