Dominant maps, closed subvarieties, projective spaces

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⊛ X an affine variety & $f \in \mathcal{O}(X)$, $f \neq 0$. Then $Z(f)$ is a closed subset of X. Let $U_f = X \setminus Z(f)$. These open sets are called basic open subsets of X. $\{U_f \mid f \in k[X]\}$ is a basis for the Zariski top on X.

Prop: U_f is an affine variety with coordinate ring $\mathcal{O}_X(U_f) \cong \mathcal{O}_X(X)[\frac{1}{f}] = k[X][\frac{1}{f}]$.

⊛ Let X be an affine variety and $U \subseteq X$ be a nonempty affine open subset s.t. $\mathcal{O}_X(U)$ is the coordinate ring of U then U is called an open affine subvariety of X.

⊛ Let X be an affine variety and $P \in X$ be a point. Let $f \in k(X)$ be regular at P then $\exists\, U \subseteq X$ affine open containing P s.t. $f \in \mathcal{O}_X(U)$.

Hence $\mathcal{O}_{X,P} = \bigcup_{P \in U \subseteq X} \mathcal{O}_X(U) := \varinjlim_{P \in U \subseteq X} \mathcal{O}_X(U)$

affine open   affine open

$f = \frac{a}{b}$, $a, b \in k[X]$ s.t. $b(P) \neq 0$. Then $U = X \setminus Z(b) = U_b$

⊛ Dominant rat'l maps: A rational map $f: X \dashrightarrow Y$ is said to be dominant if the image (f) is dense in Y. Eg: 1) $U \overset{i}{\hookrightarrow} X$ U is open nonempty

2) $\mathbb{A}^2 \longrightarrow \mathbb{A}^1$, $(x,y) \longmapsto x$

3) $X \dashrightarrow U$

⊛ Composition of rat'l maps. Let $f: X \dashrightarrow Y$ & $g: Y \dashrightarrow Z$ be rational maps. If f is dominant then $g \circ f$ as a rational map from X to Y make sense.

$$X \overset{f}{\dashrightarrow} Y \overset{g}{\dashrightarrow} Z$$

U open

$$U \xrightarrow{f} Y$$

V open

$$V \xrightarrow{g} Z$$

$\text{im}(f)$ is dense in $Y \Rightarrow \text{im}(f) \cap V$ is nonempty ($\because V$ is dense open in Y)

$U_0 = f^{-1}(V)$ open & nonempty and hence dense in U.

U_0 is open nonempty subset of X.

$g \circ f : U_0 \longrightarrow Z$ is a morphism. Hence

$g \circ f$ is rat'l map from X to Y.

So can find a basic open U_1 nonempty subset of U_0 s.t. $g \circ f : U_1 \longrightarrow Z$.

Eg: $\mathbb{A}^1 \xrightarrow{f} \mathbb{A}^2 \overset{i}{\dashrightarrow} \mathbb{A}^2 \setminus \{y = 0\}$

$x \longmapsto (x, 0)$

$i \circ f$ does not make sense.

Prop: Let $f : X \dashrightarrow Y$ be a rat'l map. f is dominant iff the induced map $f^{\#} : k[Y] \longrightarrow k(X)$ is injective.

Note: $\exists\, U \subseteq X$ affine open nonempty s.t. $f : U \longrightarrow Y$ is a morphism. Hence $f^{\#} : k[Y] \longrightarrow k[U] \subseteq k(X)$ is a k-alg homo

Note $k[X] \subseteq k[U] \subseteq k(X)$. Hence function field of U & X are same.

Pf of prop: $(\Rightarrow)$: Let $I = \ker(f^{\#})$. If $I \neq 0$ then $Z(I) \subsetneq Y$ is a proper closed subset.

Let $x \in \text{Domain}(f)$ & $g \in I$ then $g(f(x)) = f^{\#}(g)(x)$
$= 0$

Hence $\overset{\in Y}{f(x)} \in Z(I)$. Hence $\text{image}(f) \subseteq Z(I)$

contradicting $\text{image}(f)$ is dense.

$(\Leftarrow)$: Assume $f^{\#} : k[Y] \longrightarrow k(X)$ is injective

Let $J = \mathcal{I}_Y(\text{Image}(f)) = \{ g \in k[Y] \mid g(f(x)) = 0 \ \forall x \in \text{dom}(f) \}$

$= \{ g \in k[Y] \mid f^{\#}(g)(x) = 0 \ \forall x \in \text{dom}(f) \}$

$= \{ g \in k[Y] \mid f^{\#}(g) = 0 \}$ ($\because$ $\text{dom}(f)$ is an open dense subset of X)

$= (0)$ ($\because f^{\#}$ is inj)

$\Rightarrow \quad V(J) = V(\mathcal{I}(\text{Image}(f))) = Y$

i.e. $\overline{\text{Image}(f)} = Y$.

$\boxtimes$

(*) Let X be an affine variety. Let $Y \subseteq X \subseteq \mathbb{A}^n$ be an irred closed subset, then $\mathcal{I}_X(Y) \subseteq k[X]$ is prime ideal. Then Y is said to be a closed subvariety of X with coord ring $k[X]/\mathcal{I}(Y)$. Conversely any prime ideal P of $k[X]$ defines a closed subvar of X.

$$\mathcal{I}_{\mathbb{A}^n}(X) \subseteq \mathcal{I}_{\mathbb{A}^n}(Y)$$

$$k[X] = k[x_1, \dots, x_n] / \mathcal{I}_{\mathbb{A}^n}(X)$$

$$k[Y] = \frac{k[x_1, \dots, x_n]}{\mathcal{I}_{\mathbb{A}^n}(Y)}$$

$$\mathcal{I}_X(Y) = \overline{\mathcal{I}_{\mathbb{A}^n}(Y)} \text{ in } k[X]$$

0) points of var $\longleftrightarrow$ max ideal of coord ring

Remark: 1) morp of var $\longleftrightarrow$ k-alg homo.

1) closed subvar $\longleftrightarrow$ taking quot of coord ring

2) open subvar $\longleftrightarrow$ localization of coord ring

3) Product of var $\longleftrightarrow$ $k[X] \otimes_k k[Y]$ tensor products

(*) $f: X \to Y$ of affine var s.t. $f^{\#}: k[Y] \to k[X]$ k alg homo

if $k[X]$ is finite $k[Y]$-alg. then $f^{-1}(y)$ is finite $\forall y \in Y$.

Projective space $\mathbb{P}^n$

On $k^{n+1} \setminus \{\underset{\sim}{0}\}$ define an equivalence relation.

$$(a_0, \dots, a_n) \sim (\lambda a_0, \dots, \lambda a_n) \quad \forall \ \underset{\neq}{\overset{0}{\lambda}} \in k^*$$

$$\mathbb{P}^n = k^{n+1} \setminus \{\underset{\sim}{0}\} \big/ \sim$$ i.e. lines in $\mathbb{A}^{n+1}$ passing through $\underset{\sim}{0}$.

Projective n-space.

There are maps

$$\varphi_0 : \mathbb{A}^n \longrightarrow \mathbb{P}^n$$
$$(b_1, \dots, b_n) \longmapsto [(1, b_1, \dots, b_n)]$$

$$(1, b_1, \dots, b_n) \sim (1, b_1', \dots, b_n') \Rightarrow b_i = b_i'$$

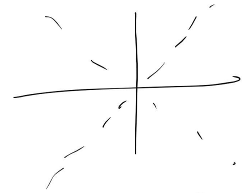
$$\varphi_i : \mathbb{A}^n \longrightarrow \mathbb{P}^n \qquad 0 \leq i \leq n$$
$$(b_1, \dots, b_n) \longmapsto [b_1, \dots, b_i, 1, b_{i+1}, \dots, b_n]$$

$$\mathbb{P}^n \setminus \varphi_0(\mathbb{A}^n) = \{ [0, a_1, \dots, a_n] \mid (a_1, \dots, a_n) \neq \underset{\sim}{0} \} \leftrightsquigarrow \mathbb{P}^{n-1}$$

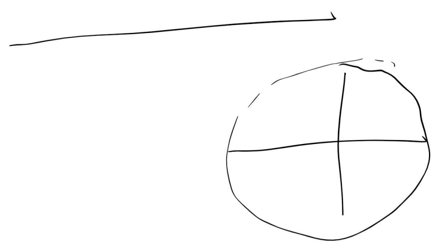
So $\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1}$; $\mathbb{P}^n = \varphi_0(\mathbb{A}^n) \cup \varphi_1(\mathbb{A}^n) \cup \quad \cup \varphi_n(\mathbb{A}^n)$

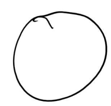
$\mathbb{A}^0 = pt$, $\mathbb{P}^0 = pt = \mathbb{A}^0$

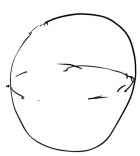
∞ $\qquad$ $\mathbb{P}^1$



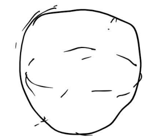
$\mathbb{P}^1 = \mathbb{A}^1 \cup \mathbb{P}^0$



$\mathbb{RP}^1 =$ 

$\mathbb{P}^1_{\mathbb{C}} =$ 

$\mathbb{RP}^2$



$$\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{P}^1$$
$$= \mathbb{A}^2 \cup \mathbb{A}^1 \cup \{pt\}$$