Lecture 11: More on tensor products

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<u>Prop:</u> R a ring, A, B, C R-modules. Then following holds.

(1) $R \otimes_R A \cong A$ $\quad r \otimes a \longmapsto ra \quad \forall r \in R \;\&\; a \in A$

(2) $A \otimes_R B \cong B \otimes_R A$ $\quad a \otimes b \longmapsto b \otimes a \quad \forall a \in A \;\&\; b \in B$

HW (3) $(A \otimes_R B) \otimes_R C \cong A \otimes_R (B \otimes_R C)$ $\quad (a \otimes b) \otimes c \longmapsto a \otimes (b \otimes c)$

Exc (4) $(A \oplus B) \otimes_R C \cong (A \otimes_R C) \oplus (B \oplus_R C)$ $\quad (a, b) \otimes c \longmapsto (a \otimes c, b \otimes c)$

(5) $S \subseteq R$ multiplicative subset then
$S^{-1}A \cong S^{-1}R \otimes_R A$ $\quad \frac{ra}{s} \longleftarrow \frac{r}{s} \otimes a$

HW (6) $I \subseteq R$ ideal then $R/I \otimes_R M \cong M/IM$

Examples:

1) $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$

2) $\mathbb{Z}/15\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/12\mathbb{Z} \overset{(6)}{\cong} \mathbb{Z}/12\mathbb{Z} \Big/ 15(\mathbb{Z}/12\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$

$\mathbb{Z}/n \otimes_{\mathbb{Z}} \mathbb{Z}/m \cong \mathbb{Z}/(n,m)$

(3) $R^n \otimes_R M = M^n$

④ $\mathbb{Q} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}/n\mathbb{Z} = 0$

$a \otimes \bar{k} = n\frac{a}{n} \otimes \bar{k} = \frac{a}{n} \otimes n\bar{k} = \frac{a}{n} \otimes 0 =$

$\mathbb{Q}/n\mathbb{Q} = 0$

⑤ $k[x] \underset{k}{\otimes} k[y] \cong k[x,y]$

$$\varphi : k[x] \times k[y] \longrightarrow k[x,y]$$
$$(f, g) \longmapsto fg$$

So φ is clearly k-bilinear.

~~So~~ we get a k-lin map

$$\theta : k[x] \underset{k}{\otimes} k[y] \longrightarrow k[x,y]$$
$$f(x) \otimes g(y) \longmapsto f(x)g(y)$$

$$\theta' : k[x,y] \longrightarrow k[x] \underset{k}{\otimes} k[y]$$
$$x \longmapsto x \otimes 1$$
$$y \longmapsto 1 \otimes y$$
$$\sum a_{ij} x^i y^j \longmapsto \sum a_{ij} x^i \otimes y^j$$

Check θ' is k-linear

$$\theta \circ \theta'\left(\sum a_{ij} x^i y^j\right) = \theta\left(\sum a_{ij}\, x^i \otimes y^j\right)$$
$$= \sum a_{ij}\, \theta(x^i \otimes y^j)$$
$$= \sum a_{ij}\, x^i y^j$$

$$\theta' \circ \theta\left(x^i \otimes y^j\right) = \theta'\left(x^i y^j\right)$$
$$= x^i \otimes y^j$$

Hence $\theta' \circ \theta$ is id on

$\{x^i \otimes y^j \mid j, i \geq 0\}$ But this gen

$k[x] \otimes_k k[y]$.

$\circledast$ $\varphi : A \to B$ be a ring homo then B is an A-module. In fact B is called A-algebra

(via $a \cdot b = \varphi(a) b$ for $a \in A$ & $b \in B$.)

(A) Let $\phi: A \to B$ be a ring homo & M be an A-mod. Then $B \otimes_A M$ is a B-module via

$$B \times B \otimes_A M \xrightarrow{s} B \otimes_A M$$

where s is s.t. $s(b, b' \otimes m) = bb' \otimes m$

- $s(1, \alpha) = \alpha \quad \forall\, \alpha \in B \otimes_A M$
- $(b+b') \cdot \alpha = b \cdot \alpha + b' \cdot \alpha$
- $b \cdot (\alpha + \alpha') = b \cdot \alpha + b\alpha'$
- $b \cdot (b' \cdot \alpha) = bb' \cdot \alpha$

$\left.\right\} \forall\, b, b' \in B$ & $\alpha, \alpha' \in B \otimes_A M$

For each $b \in B$ $\quad \psi_b : B \times M \longrightarrow B \otimes_A M$
$$(b', m) \longmapsto bb' \otimes m$$

ψ_b is bilin so $\theta_b : B \otimes M \to B \otimes M$
$$b' \otimes m \mapsto bb' \otimes m$$

Set $s(b, \alpha) := \theta_b(\alpha)$.

Check s satisfy module properties.

Recall $k[x] = \mathcal{O}(\mathbb{A}^1_k)$

$k[x,y] = \mathcal{O}(\mathbb{A}^2_k)$

$\overset{\otimes}{\Longrightarrow} \mathcal{O}(\mathbb{A}^1_k \times \mathbb{A}^1_k) = \mathcal{O}(\mathbb{A}^1_k) \otimes_k \mathcal{O}(\mathbb{A}^1_k)$

$\otimes$ More generally, X & Y are affine varieties then $\mathcal{O}(X \times Y) = \mathcal{O}(X) \otimes \mathcal{O}(Y)$

$\otimes$ Follows from $k[x_1, \dots, x_n] \otimes_k k[y_1, \dots, y_m]$

$\overset{\text{rings}}{\cong} k[x_1, \dots, x_n, y_1, \dots, y_m]$

$\otimes$M
H/$\otimes$m

HW: A & B are k-alg. k a ring.

Then $A \otimes_k B$ is also a k-alg.

(*) Let $f : A \to B$ be R-lin map of R-mod & M be an R-mod then

$\exists$ $f \otimes M : A \otimes M \longrightarrow B \otimes M$ which is R-lin satisfying

if $g : B \to C$ is R-lin then

$$(g \otimes M) \circ (f \otimes M) = g \circ f \otimes M .$$

Moreover if $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is an exact seq of R-mod

then $A \otimes M \xrightarrow{\tilde{f}} B \otimes M \xrightarrow{\tilde{g}} C \otimes M \longrightarrow 0$ is exact

Cation! Even if f is injective $f \otimes M$ need not be injective.

Pf: $\varphi : A \times M \longrightarrow B \otimes M$
$(a, m) \longmapsto f(a) \otimes m$

Check that φ is R-bilin.

Hence φ induces

$\tilde{f} : A \otimes M \longrightarrow B \otimes M$ which in R-li
$a \otimes m \longmapsto f(a) \otimes m$

$g : B \to C$ then $\tilde{g} : B \otimes M \to C \otimes M$
$b \otimes m \mapsto g(b) \otimes m$

$\widetilde{g \circ f} : A \otimes M \longrightarrow C \otimes M$
$a \otimes m \longmapsto g \circ f(a) \otimes m$

$\tilde{g} \circ \tilde{f}(a \otimes m) = \tilde{g}(f(a) \otimes m) = g \circ f(a) \otimes m$

So $\tilde{g} \circ \tilde{f} = \widetilde{g \circ f}$ ($\because$ they agree on a gen set)

To show $\tilde{g}$ is surj enough to show
$c \otimes m \in \mathrm{Im}(\tilde{g})$ $\forall c \in C$ & $m \in M$.
$\exists b \in B$ s.t. $g(b) = c$ ($\because$ g is surj)

So $\tilde{g}(b \otimes m) = c \otimes m$.

$\widetilde{g \circ f} = \tilde{g} \circ \tilde{f}$ & $g \circ f = 0$ & $\tilde{0} = 0$

$\Rightarrow \mathrm{Im}(\tilde{f}) \subseteq \ker(\tilde{g})$ ($\because \tilde{g} \circ \tilde{f} = 0$)