Lecture 26: Projective space, homogeneous ideals

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Projective space $\mathbb{P}^n$

On $k^{n+1} \setminus \{\underline{0}\}$ define an equivalence relation.

$$(a_0, \dots, a_n) \sim (\lambda a_0, \dots, \lambda a_n) \quad \forall \lambda \in k^\times \ (\lambda \neq 0)$$

$$\mathbb{P}^n = k^{n+1} \setminus \{\underline{0}\} / \sim \qquad \text{i.e. lines in } \mathbb{A}^{n+1} \text{ passing through } \underline{0}.$$

Projective n-space.

$$\varphi_i : \mathbb{A}^n \longrightarrow \mathbb{P}^n \qquad 0 \leq i \leq n$$
$$(b_1, \dots, b_n) \longmapsto [b_1, \dots, b_i, 1, b_{i+1}, \dots, b_n]$$

$$\mathbb{P}^n \setminus \varphi_0(\mathbb{A}^n) = \{[0, a_1, \dots, a_n] \mid (a_1, \dots, a_n) \neq \underline{0}\} \hookrightarrow \mathbb{P}^{n-1}$$

So $\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1}$; $\mathbb{P}^n = \varphi_0(\mathbb{A}^n) \cup \varphi_1(\mathbb{A}^n) \cup \quad \cup \varphi_n(\mathbb{A}^n)$

$\mathbb{P}^0 = \{pt\}$, $\mathbb{A}^0 = \{pt\}$.

Topology on $\mathbb{P}^n$

$U \subseteq \mathbb{P}^n$ is open iff $\varphi_i^{-1}(U) \cap \mathbb{A}^n$ is open in $\mathbb{A}^n$ $\forall\ 0 \leq i \leq n$.

$X \subseteq \mathbb{P}^n$ is closed iff $\varphi_i^{-1}(X) \cap \mathbb{A}^n$ is closed in $\mathbb{A}^n$ $\forall$ " .

So by def^n φ_i's are cont. maps.

Alternatively one uses graded rings, homogen. ideals, etc.

Def^n: Let $f \in k[X_0, \dots, X_n]$ be a polynomial. It is said to be homogeneous if every monomial of f has the same degree. e.g. $f(X_0, \dots, X_n) = X_0^2 + X_0 X_1 + X_n^2$

Lemma: $f \in k[X_0, \dots, X_n]$ is homogen $\Rightarrow$ $f(\lambda a_0, \dots, \lambda a_n) = \lambda^d f(a_0, \dots, a_n)$ $\forall \lambda \in k$ & $(a_0, \dots, a_n) \in k^{n+1}$ where $d = \deg(f)$.

converse holds if k is infinite.

Pf: $f(X_0, \dots, X_n) = \sum_{\underset{\text{finite}}{\underline{i}}} a_{\underline{i}} X_0^{i_0} \cdots X_n^{i_n}$. $\underline{i} = (i_0, \dots, i_n)$ f is homogen $\Rightarrow$ $a_{\underline{i}} \neq 0$ only if $i_0 + \dots + i_n = d$ where $\underline{i} = (i_0, \dots, i_n)$

$$f(\lambda a_0, \dots, \lambda a_n) = \sum_{\underline{i}} a_{\underline{i}} (\lambda a_0)^{i_0} \cdots (\lambda a_n)^{i_n} = \sum_{\underline{i}} \lambda^{i_0 + \dots + i_n} a_{\underline{i}} a_0^{i_0} \cdots a_n^{i_n}$$
$$= \lambda^d \sum_{\underline{i}} a_{\underline{i}} a_0^{i_0} \cdots a_n^{i_n}$$
$$= \lambda^d f(a_0, \dots, a_n).$$

If k is infinite & $f = f_d + f_{d-1} + \dots + f_0$ where f_i is homogen of deg i

then $f(\lambda a_0, \dots, \lambda a_n) = \lambda^d f_d(a_0, \dots, a_n) + \lambda^{d-1} f_{d-1}(a_0, \dots, a_n) + \dots + \lambda f_1(a_0, \dots, a_n) + f_0$

$$f(\lambda a_0, \dots, \lambda a_n) - \lambda^d f(a_0, \dots, a_n) = \lambda^{d-1} f_{d-1}(a_0, \dots, a_n)(\lambda - 1) + \lambda^{d-2} f_{d-2}(a_0, \dots, a_n)(\lambda^2 - 1) + \dots + f_0(\lambda^d - 1)$$

if $|k| = \infty$ can find $a_0, \dots, a_n$ and λ s.t. this is nonzero

$\square$

Cor: f homogen & k infinite field then $f(a_0, \dots, a_n) = 0 \iff f(\lambda a_0, \dots, \lambda a_n) = 0$ $\forall \lambda \in k$. where $(a_0, \dots, a_n) \neq \underline{0}$

Def^n: An ideal I in $k[x_0, \dots, x_n]$ is said to be homogen if I is generated by homogeneous polynomials.

Lemma: $I \subseteq k[x_0, \dots, x_n]$ is homogen iff $\forall f \in I$ $f = f_0 + f_1 + \dots + f_d$ with f_i homogen of deg i then $f_i \in I$ $\forall 0 \le i \le d$. $\leftarrow$ homogen parts

Pf: $(\Leftarrow)$ ✓ $I = (g_1, \dots, g_m)$ then $g_i = g_{i,0} + \dots + g_{i,d_i}$ $\Rightarrow g_{i,j} \in I$ & $I = (g_{ij})$

$(\Rightarrow)$: Let $f_1, \dots, f_m$ be homogen in I with f_i of deg d_i s.t. $I = (f_1, \dots, f_m)$.

Let $h \in I$, then $h = \sum_{i=1}^m g_i f_i$ for some $g_1, \dots, g_m \in k[x_0, \dots, x_m]$

$$h_0 = \sum_{i=1}^m g_{i,0} f_{i,0} = \sum_{\substack{1 \le i \le m \\ i:\, d_i = 0}} g_{i,0} f_i \in I$$

$$h_l = \left(\sum_{i=1}^m g_i f_i\right)_l = \sum_{\substack{1 \le i \le m \\ d_i \le l}} g_{i,l-d_i} f_i \in I$$

where $g_{i,l-d_i}$ is homogen deg $l - d_i$ part of g_i

Examples 1) $k[x_0, \dots, x_n]$, (0)

$I = (x_0^2 x_1 + x_2^3, \; x_2^2 - x_1^2)$

$I = (x_0, \dots, x_n)$

(*) $I \subseteq k[x_0, \dots, x_n]$ homogen ideal then

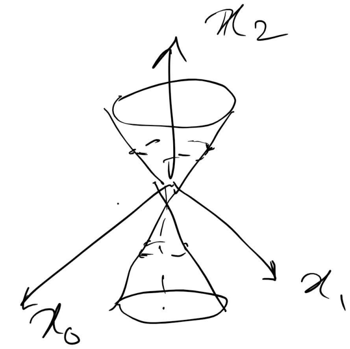
$(a_0, \dots, a_n) \in Z_{\mathbb{A}^{n+1}}(I) \subseteq \mathbb{A}^{n+1}$ iff $(\lambda a_0, \dots, \lambda a_n) \in Z(I)$ $\forall \lambda \in k$.

So $\underset{\underset{Z_{\mathbb{P}^n}(I)}{\shortparallel}}{X} = Z(I) \setminus \{\underline{0}\} / \sim \subseteq \mathbb{P}^n$.

$C(X) = Z_{\mathbb{A}^{n+1}}(I)$ is called the <u>cone over</u> X.

<u>Ex</u> $I = (x_0^2 + x_1^2 - x_2^2) \subseteq k[X_0, X_1, X_2]$

$X = Z(I) \setminus \{0\} / \sim$ is circle (in one chart)



(*) Let $f \in k[x_0, \dots, x_n]$ be homogen &

$P = [a_0, \dots, a_n] \in \mathbb{P}^n$ then $f(P) = 0$ or nonzero

does not depend on the choice of repr $(a_0, \dots, a_n)$

of P.

"$f(P) = 0$" $:=$ "$f(a_0, \dots, a_n) = 0$"

Def^n: A subset $X \subseteq \mathbb{P}^n$ is said to be an algebraic set if there exist a coll of homogen polys S s.t.

$$X = \{ P \in \mathbb{P}^n \mid f(P) = 0 \ \forall f \in S \} = Z(S)$$

If I is homogen ideal of $k[x_0, \dots, x_n]$ then

$$Z(I) = \{ P \in \mathbb{P}^n \mid f(P) = 0 \ \forall \text{ homogen } f \text{ in } I \}.$$

$$Z(I+J) = Z(I) \cap Z(J) \qquad I, J \text{ homogen}$$

$$Z(I \cap J) = Z(I) \cup Z(J) \qquad \text{"}$$

Def^n: $X \subseteq \mathbb{P}^n$

$$I(X) = (\{ f \in k[x_0, \dots, x_n] \mid f \text{ homogen } \& \ f(P) = 0 \ \forall P \in X \}).$$

$I(X)$ is by def^n homogen ideal.

Zariski top on X (an alg subset of $\mathbb{P}^n$): closed sets are algebraic subsets of X. $Z(I(X)) = \overline{X}$

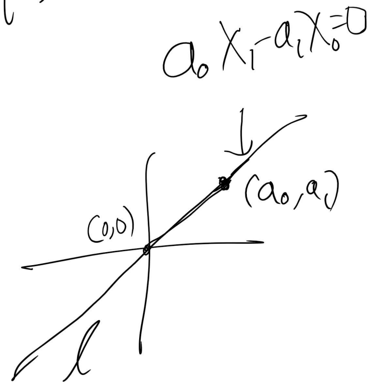
Example: $\mathbb{P}^1$, alg subsets: ϕ, $\mathbb{P}^1$

$[a_0, a_1]$ where $(a_0, a_1) \neq (0,0) \in \mathbb{A}^2$

$\in \mathbb{P}^1$

Find $I \subseteq k[X_0, X_1]$, I homogen s.t.

$Z(I) = \{[a_0, a_1]\}$



$(X_0 - a_0, X_1 - a_1)$

$f = a_0 X_1 - a_1 X_0 \in$

$Z_{\mathbb{A}^2}(f) = \ell$ & hence $Z_{\mathbb{P}^1}(f) = [a_0, a_1]$

$I = (f)$

So the top on $\mathbb{P}^1$ is cofinite topology.