Lecture 3: Zariski topology, Affine variety

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Recall: ⊛ k alg closed then affine n-space over k is k^n, denoted by $\mathbb{A}^n_k$.

⊛ $S \subseteq k[x_1, \dots, x_n]$, $Z(S)$ simultaneous zeros of elements of S. $Z(S) \subseteq \mathbb{A}^n_k$ are affine alg subsets of $\mathbb{A}^n_k$.

⊛ $Z(S) = Z(\underbrace{\langle S \rangle}_{\text{ideal gen by } S})$

⊛ $X \subseteq \mathbb{A}^n_k$, $I(X) = \{ f \mid f(\underset{\sim}{a}) = 0 \ \forall \underset{\sim}{a} \in X \} \subseteq k[x_1, \dots, x_n]$ is an ideal.

⊛ $I \subseteq J$ ideals of $k[x_1, \dots, x_n]$ $Z(I) \supseteq Z(J)$
$X \subseteq Y \subseteq \mathbb{A}^n_k \Rightarrow I(X) \supseteq I(Y)$

I, J ideals in $k[x_1, \dots, x_n]$

⊛ $Z(I+J) = Z(I) \cap Z(J) = Z(I \cup J)$

⊛ $Z(IJ) = Z(I \cap J) = Z(I) \cup Z(J)$

⊛ $Z(I) = Z(\sqrt{I})$

⊛ $I(Z(J)) \supseteq \sqrt{J}$; $J \subseteq k[x_1, \dots, x_n]$ ideal
($=$ holds by HN)

⊛ $X \subseteq \mathbb{A}^n_k$ subset $Z(I(X)) = X$? yes if X is algebraic

Let $\underset{\sim}{a} \in X$, $f \in I(X) \Rightarrow f(\underset{\sim}{a}) = 0 \Rightarrow \underset{\sim}{a} \in Z(I(X))$

So $X \subseteq Z(I(X))$

If $\exists J$ s.t. $X = Z(J)$ i.e. X is affine alg then

$$I(X) = I(Z(J)) \supseteq \sqrt{J}$$
$$\Rightarrow Z(I(X)) \subseteq Z(\sqrt{J}) = Z(J) = X$$

Zariski topology on affine n-space.

Defⁿ: All alg affine subsets of $\mathbb{A}^n_k$ are closed.

Note: $\phi = Z(1)$

& $\mathbb{A}^n_k = Z(0)$

$Z(IJ) = Z(I) \cup Z(J)$

implies finite union of alg sets are alg.

Let $X_\alpha = Z(I_\alpha)$ $\alpha \in \Omega \leftarrow$ indexing set

$I_\alpha \subseteq k[x_1, \dots, x_n]$ ideals

then $\bigcap_{\alpha \in \Omega} X_\alpha = Z\left(\bigcup_{\alpha \in \Omega} I_\alpha\right)$ easy.

Hence this is topology on $\mathbb{A}^n_k$

And it is called the Zariski topology on $\mathbb{A}^n_k$.

$\underset{\sim}{a} \in \bigcap_{\alpha \in \Omega} X_\alpha$

$\Rightarrow f(\underset{\sim}{a}) = 0$

$\forall f \in I_\alpha$

$\forall \alpha \in \Omega$

$\Rightarrow \underset{\sim}{a} \in Z\left(\bigcup_{\alpha \in \Omega} I_\alpha\right)$

$Z\left(\bigcup_{\alpha \in \Omega} I_\alpha\right) \subseteq Z(I_\alpha)$

$\forall \alpha \in \Omega$

$\Rightarrow Z\left(\bigcup_{\alpha \in \Omega} I_\alpha\right) \subseteq \bigcap_{\alpha \in \Omega} X_\alpha$

Ex: 1) $\mathbb{A}^1_{\mathbb{C}} = \mathbb{C}$. What is the Zariski top on $\mathbb{A}^1_{\mathbb{C}}$? ϕ, $\mathbb{C}$ are closed

$S = \{f(x)\} \subseteq \mathbb{C}[x]$ then $X = Z(\langle S \rangle)$
$= Z((f(x)))$

So X is set of zeros of $f(x)$ which is finite. So top on $\mathbb{A}^1_{\mathbb{C}}$ is co-finite topology. Not Hausdorff

2) $\mathbb{A}^2_{\mathbb{C}} = \mathbb{C}^2$. What are closed sets in Zariski topology?

$$Z(f)$$

$f \in \mathbb{C}[x, y]$

$\shortparallel$

$x^2 + y^2 - 1$

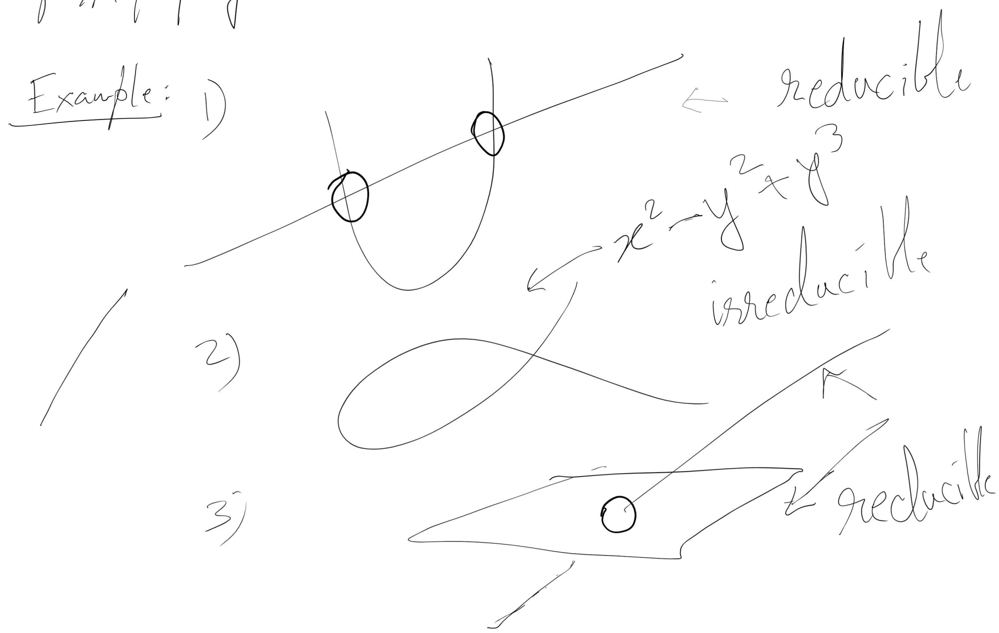
Every finite set is closed.

$(a, b) \in \mathbb{C}^2$

$$Z((X - a, Y - b)) = \{(a, b)\}$$

Def^n: An algebraic set X in $\mathbb{A}^n$ is said to be irreducible if it not reducible and X is reducible if $X = X_1 \cup X_2$ where X_1 & X_2 are algebraic subsets of $\mathbb{A}^n$ properly contained in X.

Example:

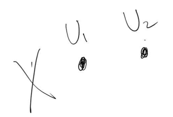


(*) If $X \subseteq \mathbb{A}^n_k$ is irred alg set then X is connected (w.r.t subsp top)

Pf: Suppose $X = U_1 \cup U_2$ $\quad U_1, U_2$ disjoint proper open subset of X

$U_i = V_i \cap X \quad V_i$ open in $\mathbb{A}^n_k$

$Z_i = \mathbb{A}^n \setminus V_i$ is closed in $\mathbb{A}^n_k$



Claim: $U_1 \subseteq Z_2$, $U_2 \subseteq Z_1$

$$\begin{aligned} U_1 = X \setminus U_2 &= X \setminus (V_2 \cap X) \\ &= X \cap (V_2 \cap X)^c \\ &= X \cap (Z_2 \cup X^c) \\ &= X \cap Z_2 \subseteq Z_2 \end{aligned}$$

$\|^{ly}$ & $U_2 \subseteq Z_1$

$$X = \underbrace{(X \cap Z_1)}_{\text{closed}} \cup \underbrace{(X \cap Z_2)}_{\text{closed}}$$

Also $\quad Z_i \cap U_i = \phi$

$\Rightarrow X \cap Z_i \subsetneq X$

Contradicting X is irreducible.

$\underline{\text{Def}^n}$: An affine variety over a field k is an irreducible alg. subset of $\mathbb{A}^n_k$ for some n together with subspace topology coming from Zariski topology on $\mathbb{A}^n_k$.

$\underline{\text{Ex}}$: 1) $Z(x^2-y^2+y^3)$

2) $Z(x^2+y^2-1)$

$\underline{\text{Prop}}$: Let k be alg closed field. $X \subseteq \mathbb{A}^n_k$ alg subset is irred iff $I(X) \subseteq k[x_1,\dots,x_n]$ is a prime ideal.

In $\underline{\text{Ex 1}}$: $(x^2-y^2+y^3) \subseteq k[x,y]$

(x^2+y^2-1) are prime ideals

$x^2-y^2(y-1)$