Let X be a metric space and $f : X \to X$ be a function. We say that f is a contraction if there exists a constant $0 < c < 1$ such that
$d(f(x), f(y)) \leq cd(x, y)$ for all $x, y \in X$.
Iterations serve an approximation. For instance, Taylor's series provides an approximation. $e^x = 1 + \frac{x}{1!} + \ldots$. In other words, $a_n = \sum_{k=0}^{n} \frac{x^k}{k!} \to e^x$.
A fixed point of $f : X \to X$ is a point $x \in X$ such that $f(x) = x$.
Example:

- Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{x}{2}$ is a contraction. 0 is the only fixed point.
- Let $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (\frac{x}{3} + y, \frac{y}{2})$. Is this a contraction? Work with a suitable metric. $(0, 0)$ is the only fixed point.
- Let $f : C[0, 1] \to C[0, 1]$ given by $f(u)(x) = \frac{1}{2} \int_0^x u(t)\, dt$. f is a contraction and 0 is the only fixed point.

Contraction Mapping Principle:

Let X be a complete metric space and $f : X \to X$ be a contraction. Then, f has a unique fixed point x_0 such that $f^n(x) \to x_0$.

Proof. Let x_1, x_2 be fixed points. $d(x_1, x_2) = d(f(x_1), f(x_2)) \leqslant cd(x_1, x_2)$ for some $0 < c < 1$. This implies that $d(x_1, x_2) = 0$
Let $x \in X$. Consider an "orbit" - $(f^n(x))_{n \geq 1}$. Let $x_n = f^n(x)$, $n \geqslant 1$. For $m > n$, $d(x_m, x_n) = d(f^m(x), f^n(x)) \leqslant cd(f^{m-1}(x), f^{n-1}(x)) \leqslant \ldots \leqslant c^{n-1}d(f^{m-n-1}(x), x)$. Therefore, $d(x_m, x_n) \leqslant c^{n-1}d(f^{m-n-1}(x), x)$. Let $n > 1$. $d(x_1, x_n) \leqslant d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{n-1}, x_n) \leqslant (1 + c + c^2 + \ldots + c^{n-1})d(x^2, x_1) \leqslant \frac{d(x_2,x_1)}{1-c}$. Similarly, for $m > n$, $d(x_m, x_n) \leqslant \sum_{n-1}^{m-2} d(x_k, x_{k-1}) \leqslant (\sum_{n-1}^{m-2} c^k)d(x_2, x_1) \leqslant \frac{c^{n-1}}{1-c} d(x_1, x_2) \overset{n\to\infty}{\longrightarrow} 0$. Therefore, (x_n) is Cauchy. Since X is complete, $x_n \to x_0$ (let). That is, $\lim_{n\to\infty} f^n(x) = x_0$. $f(x_n) = x_{n+1} \to x_0$. If f is a contraction, then f is continuous. Therefore, $f(x_0) = x_0$. Since the fixed point is unique, $f^n(y) \to x_0$ for any $y \in X$. □

Without the completeness assumption, contraction mapping principle need not hold.

Differentiation of several variables

Let U be an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^m$. We say that f is differentiable at a point $x \in U$ if there is a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ such that $\lim_{h\to 0} \frac{f(x+h)-f(x)-T(h)}{\|h\|} = 0$. In that case, T is denoted by $f'(x)$. If f is differentiable at all points $x \in U$, we say that f is differentiable on U.
$L(\mathbb{R}^n, \mathbb{R}^m) := \{\alpha : \mathbb{R}^n \to \mathbb{R}^m \text{ is a linear map}\} \cong \mathbb{R}^{nm}$. Let $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, $\|A\| = \text{norm of } A := \sup_{\|x\|\leqslant 1} \|Ax\| = \sup_{\|x\|=1} \|Ax\| < \infty$.

Let f be differential on U and $f'(x) \in L(\mathbb{R}^n, \mathbb{R}^m)$. Let $x \mapsto f'(x)$. $L(\mathbb{R}^n, \mathbb{R}^m)$ is a metric space with metric $d(A,B) = \|A - B\|$. We say that $f \in C^1(U)$ if f is differentiable on U and $x \mapsto f'(x)$ is continuous.

Proposition 1 Let U be a convex open set in $\mathbb{R}^n$ and $f : U \to \mathbb{R}^m$ be a differentiable function such that $\|f'(x)\| \leqslant M$ for all $x \in U$. Then, $\|f(x) - f(y)\| \leqslant M\|x - y\|$ for all $x, y \in U$. (U is convex if $\lambda x + (1-\lambda)y \in U$ for all $\lambda \in [0,1]$ and $x, y \in U$)

Proposition 2 Let U be an open set in $\mathbb{R}^n$ and V be an open set in $\mathbb{R}^m$. Let $f : U \to \mathbb{R}^m$ is such that $f(U) \subseteq V$ and $g : V \to \mathbb{R}^k$. If f is differentiable at x and g is differentiable at $f(x)$, then $g \circ f$ is differentiable at x and $(g \circ f)'(x) = g'(f(x))f'(x)$.

Example: If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, then $f \in C^1$. $f' = f$.

Inverse Function Theorem:

Let E be an open set in $\mathbb{R}^n$ and $f : E \to \mathbb{R}^n$ be C^1. If $f'(x)$ is invertible at some point $x \in E$, then there exists an open set $U \subseteq E$ such that

- $x \in U$
- V:=$f(U)$ is open in $\mathbb{R}^n$
- $f|_U$ is injective
- If g is the inverse of f defined on V, then g is C^1.

Proof: Deferred to the next lecture.

Lemma: Let $\Omega = \{A \in L(\mathbb{R}^n, \mathbb{R}^n) | A \text{ is ivertible}\}$.

1. If $A \in \Omega$, $B \in L(\mathbb{R}^n, \mathbb{R}^n)$ such that $\|A - B\| \|A^{-1}\|\| < 1$, then $B \in \Omega$.
2. Ω is open.
3. $A \mapsto A^{-1}$ is continuous on Ω

Proof. $A \in \Omega$. $B \in L(\mathbb{R}^n, \mathbb{R}^n)$. Let $\alpha = \frac{1}{\|A^{-1}\|}$. Let $\beta = \|A - B\|$. We assume that $\beta < \alpha$. $\|x\| \leqslant \|A^{-1}\|\|Ax\| \leqslant \frac{1}{\alpha}\|(A-B)(x)\| + \frac{1}{\alpha}\|B(x)\| \leqslant \frac{\beta}{\alpha}\|x\| + \frac{1}{\alpha}\|Bx\|$. Hence, we have $\|x\|(\alpha - \beta) \leqslant \frac{1}{\alpha}\|Bx\|$. $Bx \neq 0$ if $x \neq 0$. Thus, B is injective. This implies that B is invertible. Statement *(2)* is left as an exercise. For $A \neq B$, $\|A^{-1}\|^{-1} = \alpha, \beta = \|A - B\|$. $(\alpha - \beta)\|B^{-1}y\| \leqslant \|y\|$. So, for any B such that $\beta < \alpha$ $\|B^{-1}\| \leqslant \frac{1}{\alpha - \beta}$.
$\|A^{-1} - B^{-1}\| \leqslant \|A^{-1}\|\|A - B\|\|B^{-1}\| \leqslant \frac{1}{\alpha(\alpha-\beta)}\|A - B\| = \frac{\beta}{\alpha(\alpha-\beta)}$. So, as $B \longrightarrow A$, $\beta \longrightarrow 0$. This implies that $\|A^{-1} - B^{-1}\| \longrightarrow 0$. Therefore, $A \mapsto A^{-1}$ on Ω is continuous. $\square$