Lecture 28: Functions on projective spaces

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<u>Recall</u>: $\mathbb{P}^n := \mathbb{A}^{n+1} \setminus \{\underset{\sim}{0}\}/\sim$ $\quad \underset{\sim}{a} \sim \lambda \underset{\sim}{a} \ \forall \lambda \in k\setminus\{0\}$ & $\underset{\sim}{a} \in \mathbb{A}^{n+1}\setminus\{\underset{\sim}{0}\}$

Also $\quad \varphi_i : \mathbb{A}^n \longrightarrow \mathbb{P}^n \qquad 0 \le i \le n$

$\qquad (b_1, \dots, b_n) \longmapsto [(b_1, \dots, b_{i-1}, 1, b_{i}, \dots, b_n)]$

$U_i := \varphi_i(\mathbb{A}^n)$ is an open subset of $\mathbb{P}^n$

& $\bigcup_{i=0}^{n} U_i = \mathbb{P}^n \qquad [a_0, \dots, a_n] \in \mathbb{P}^n$ and $a_0 \neq 0$ then $[a_0, \dots, a_n] = [1, \frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}] \in \varphi_0(\mathbb{A}^n) = U_0$

(*) $X \subseteq \mathbb{P}^n$ is closed (in Zariski top) iff $X \cap U_i$ is closed in U_i (in Zariski top) (i.e. $\varphi_i^{-1}(X)$ is closed in $\mathbb{A}^n$)

i.e. the Zariski top on $\mathbb{P}^n$ is same as the top induced by the cover $\{U_i\}_{i=0}^{n}$.

($\Longrightarrow$) is an immediate consequence of the following:

<u>Prop</u>: $X \subseteq \mathbb{P}^n$ an alg subset & $I = \mathcal{I}(X) \subseteq k[x_0, \dots, x_n]$ be the homogen ideal defining X. Let $I^a = \{ f(x_0, x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n) \mid f \in I \} \subseteq k[x_0, \dots, \hat{x}_i, \dots, x_n] = k[x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n]$.

Then I^a is an ideal & $V(I^a) = X \cap U_i \ (= \varphi_i^{-1}(X))$

<u>Pf</u>: I^a is an ideal ✓ $\quad f, g \in I^a$ then $\exists\, F, G \in I$ s.t. $f = F(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n)$ & $g = G(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n)$ then $f + g = (F + G)(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots)$

Let $(a_0, \dots, \hat{a}_i, \dots, a_n) \in V(I^a) \iff f(a_0, a_1, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_n) = 0 \ \forall f \in I$

$\iff [a_0, a_1, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_n] \in X \cap U_i$

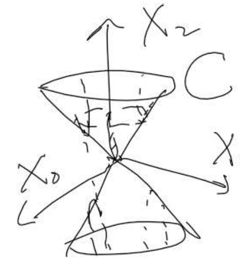
$\boxtimes$

(*) Let $X \subseteq \mathbb{P}^n$ be an alg subset. Since Zariski top on X is the subspace top induced from $\mathbb{P}^n$, $X \cap U_i$ is open in X. Moreover $\{X \cap U_i\}_{i=0}^n$ is an open affine cover of X. (Note U_i is Zariski open in $\mathbb{P}^n$ $\because U_i = \{X_i = 0\}^c$)

(*) $X = V(f_1, \dots, f_m)$ $f_1, \dots, f_m$ homogen poly in $k[x_0, \dots, x_n]$. Then
$X \cap U_i = V(f_1^a, \dots, f_n^a)$.

<u>Example</u>: $f = X_0^2 + X_1^2 - X_2^2 \subseteq k[X_0, \dots, X_n]$

$C = V(f) = Z(f)$ in $\mathbb{A}^3$ is a cone



$V_{\mathbb{P}^2}(f) \subseteq \mathbb{P}^2$

$C \setminus \{x_0 \neq 0\} = U_0 \cap C = V_{\mathbb{A}^2}(f(1, x_1, x_2))$

$f(1, x_1, x_2) = 1 + x_1^2 - x_2^2 = 0$

$x_2^2 - x_1^2 = 1$

$C \setminus \{x_1 \neq 0\} = U_1 \cap C = V_{\mathbb{A}^2}(x_0^2 + 1 - x_1^2)$

$C \setminus \{x_2 \neq 0\} = U_2 \cap C = V_{\mathbb{A}^2}(x_0^2 + x_1^2 - 1)$

<u>Prop</u> Let $X \subseteq \mathbb{A}^n$ be an algebraic subset defined by an ideal $I \subseteq k[x_1, \dots, x_n]$. $\underset{U_0}{\mathbb{A}^n} \subseteq \mathbb{P}^n$

Let $I^h = \left(\left\{ X_0^d f\left(\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}\right) \,\middle|\, f \in I \ \&\ d = \deg f \right\} \right)$

$\subseteq k[X_0, \dots, X_n]$.

Then I^h is a homogen ideal. $V(I^h) \subseteq \mathbb{P}^n$

and $V(I^h) \cap U_0 = X$. $V(I^h) = \overline{X}$ in $\mathbb{P}^n$

<u>Pf</u>: $[1, \frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}]$

Let $[a_0, \dots, a_n] \in V(I^h) \cap U_0 \Rightarrow a_0 \neq 0$ & $F(a_0, \dots, a_n) = 0$ $\forall$ homogen F in I^h

Let $d = \deg F$

$F(X_0, \dots, X_n) = \sum_{\text{finite}} g_i X_0^{d_i} f_i\left(\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}\right)$ $d_i = \deg f_i$, $f_i \in I$ & g_i homogen of deg $d - d_i$

Let $f(x_1, \dots, x_n) := F(1, x_1, \dots, x_n) \in k[x_1, \dots, x_n]$

then $\boxed{X_0^d f\left(\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}\right) = F(X_0, \dots, X_n)}$

$f(x_1, \dots, x_n) = \sum g_i(1, x_1, \dots, x_n) f_i(x_1, \dots, x_n) \in I$

$F(a_0, \dots, a_n) = 0 \Rightarrow a_0^d f\left(\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}\right) = 0$ $\forall f \in I$ where $d = \deg(f)$
$\forall F$ homogen in I^h

$\underset{a_0 \neq 0}{\Longrightarrow} f\left(\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}\right) = 0$ $\forall f \in I$

$\Longleftrightarrow \left(\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}\right) \in V(I) = X$

$V(I^h) \cap U_0 \subseteq X$

$\blacksquare$

(*) Alg subsets of $\mathbb{P}^n$ $\xrightarrow{I}$ radical homogen ideals in $k[X_0, \dots, X_n]$

$\xleftarrow{V(-)}$

irred alg subsets of $\mathbb{P}^n$ $\longleftrightarrow$ homogen prime ideals in $k[X_0, \dots, X_n]$. (HW)

Def^n: A projective variety is an irred alg subset of $\mathbb{P}^n$ for some n.

(*) Let $X \subseteq \mathbb{P}^n$ be a proj variety. Want to define functions on X. Let $F \in k[X_0, \dots, X_n]$ homogen

then $\left\{ \begin{array}{l} F(a_0, \dots, a_n) = F(\lambda a_0, \dots, \lambda a_n) \quad (\because [a_0, \dots, a_n] = [\lambda a_0, \dots, \lambda a_n]) \\ \quad = \lambda^d F(a_0, \dots, a_n) \quad \forall \lambda \in k^{\times} \text{ where } d = \deg F) \\ \Rightarrow F \text{ is homogen of deg } 0. \end{array} \right.$

i.e. F is const.

Def^n: A rat'l function on $\mathbb{P}^n$ is $F/G \in k(X_0, \dots, X_n)$ where $F, G \in k[X_0, \dots, X_n]$ are homogen of the same degree. Note that $\frac{F}{G}([a_0, \dots, a_n]) = \frac{F(a_0, \dots, a_n)}{G(a_0, \dots, a_n)}$ is well-defined on $\mathbb{P}^n \setminus \{G=0\}$ open.

Let $X \subseteq \mathbb{P}^n$ be an alg subset. Then a rat'l func on X is a function defined on an open subset U of X given by

$\frac{F}{G} : X \dashrightarrow k$ where F, G are in $k[X_0, \dots, X_n]$ homogen of same degree. & $G \notin I_{\mathbb{P}^n}(X)$

here $U = X \setminus \{G = 0\}$