DIFFERENTIAL TOPOLOGY - LECTURE 6

1. Introduction

In the previous set of notes we discussed the preimage theorem. In this set of notes we shall discuss some examples. Recall that the preimage theorem states that if $y \in Y$ is a regular value of a smooth map $f : X \longrightarrow Y$ between manifolds of dimensions k and ℓ respectively, then the preimage $Z = f^{-1}(y)$ is submanifold of X of codimension equal to the codimension of y in Y which is ℓ. Thus the dimension of Z equals $k - \ell$. Moreover we can also say something about the tangent space to Z at $z \in Z$. By (Proposition 3.4, Lecture 5), we know that

$$T_z(Z) = \text{kernel}(df_z : T_z(X) \longrightarrow T_{f(z)}(Y)).$$

We shall discuss several examples.

2. Examples

Using the preimage theorem to prove a certain subset of the euclidean space is a manifold bypasses the often tricky task of constructing explicit parametrizations. To begin with we look at the simplest examples. First the sphere.

Example 2.1. The unit sphere $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ is the level set $f^{-1}(1)$ of the smooth function $f : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ defined by

$$f(x_1, \ldots, x_{n+1}) = x_1^2 + x_2^2 + \cdots + x_{n+1}^2.$$

The derivative at $x \in \mathbb{R}^{n+1}$ is given by the linear transformation (a $(1 \times (n+1))$ matrix)

$$df_x = (2x_1, 2x_2, \cdots, 2x_{n+1}) : T_x(\mathbb{R}^{n+1}) = \mathbb{R}^{n+1} \longrightarrow T_{f(x)}(\mathbb{R}) = \mathbb{R}$$

which is not zero unless $x = 0$. Thus, in particular, f is a submersion at each $x \in f^{-1}(1) = \mathbb{S}^n$. Hence by the preimage theorem $\mathbb{S}^n$ is a manifold of codimension 1 in $\mathbb{R}^{n+1}$. Thus it has dimension n. What about the tangent space at $x \in \mathbb{S}^n$? We know that, for $x \in \mathbb{S}^n$

$$T_x(\mathbb{S}^n) = \text{kernel}(df_x)$$

where

$$df_x : T_x(\mathbb{R}^{n+1}) = \mathbb{R}^{n+1} \longrightarrow T_1(\mathbb{R}) = \mathbb{R}.$$

is as above. Therefore $T_x(\mathbb{S}^n)$ consists of all those vectors $(a_1, \ldots, a_{n+1}) \in \mathbb{R}^{n+1}$ such that

$$df_x(a) = (2x_1, \ldots, 2x_{n+1}) \begin{pmatrix} a_1 \\ \vdots \\ a_{n+1} \end{pmatrix} = 2x_1a_1 + \cdots + 2x_{n+1}a_{n+1} = 0.$$

Thus the tangent space $T_x(\mathbb{S}^n)$ consists of all those vectors $a \in \mathbb{R}^{n+1}$ whose inner product with x is zero. Hence the tangent space $T_x(\mathbb{S}^n)$ equals the orthogonal complement of x in $\mathbb{R}^{n+1}$.

Example 2.2. Consider the function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ given by $f(x, y, z) = x^2 + y^2$. Then $Z = f^{-1}(1)$ is the (infinite) cylinder. It is clear that 1 is a regular value of f. Hence Z is a codimension 1 submanifold of $\mathbb{R}^3$. Given $p = (a, b, c) \in Z$, we know that the tangent space $T_p(Z)$ equals the kernel of

$$df_p : T_p(\mathbb{R}^3) = \mathbb{R}^3 \longrightarrow \mathbb{R}.$$

The derivative df_p is given by the (1×3)-matrix

$$df_p = (2a, 2b, 0)\,.$$

Hence

$$T_p(Z) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,:\, 2ax + 2by = 0 \right\}.$$

Example 2.3. Consider the function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ defined by $f(x, y, z) = x^2 + y^2 - z^2$. Every $r \neq 0$ is a regular value of f. Hence, fixing $r \neq 0$, we have that the hyperboloid $Z = f^{-1}(r)$ is a codimension 1 submanifold of $\mathbb{R}^3$. It is clear that if $p = (a, b, c) \in Z$, then

$$T_p(Z) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,:\, 2ax + 2by - 2cz = 0 \right\}.$$

If $r < 0$, then $f^{-1}(r)$ is not connected.

The remaining examples will concern matrix groups. Let $M_{m\times n}(\mathbb{R})$ denote the real vector space of $(m \times n)$-matrices with real entries. $M_{n\times n}(\mathbb{R})$ will be written as $M_n(\mathbb{R})$. We shall identify $M_{m\times n}(\mathbb{R})$ with the euclidean space $\mathbb{R}^{mn}$ by writing the rows of a matrix A one after another to get a point in $\mathbb{R}^{mn}$. Thus $M_{m\times n}(\mathbb{R}) = \mathbb{R}^{mn}$ is a manifold and clearly, for any $A \in M_{m\times n}(\mathbb{R})$, we have

$$T_A(M_{m\times n}(\mathbb{R})) = M_{m\times n}(\mathbb{R}).$$

The notation $GL_n(\mathbb{R})$ will stand for the group of invertible matrices in $M_n(\mathbb{R})$. If $\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ denotes the determinant function, then as

$$GL_n(\mathbb{R}) = \det{}^{-1}(\mathbb{R} - 0)$$

we have that $GL_n(\mathbb{R})$ is open in $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$ and hence is a manifold of dimension n^2. Notice that the function det is smooth.

Here are some familiar subgroups of $GL_n(\mathbb{R})$. The special linear group $SL_n(\mathbb{R})$ is defined to be

$$SL_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) \,:\, \det(A) = 1\}.$$

Thus $SL_n(\mathbb{R}) = \det^{-1}(1)$.

For a matrix A, let A^t denote its transpose. The orthogonal group $O(n)$ is defined to be

$$O(n) = \{A \in GL_n(\mathbb{R}) \,:\, AA^t = A^tA = I_n\}$$

where I_n denotes the identity $(n\times n)$-matrix. A matrix A is said to be *symmetric* if $A = A^t$. Clearly, for any square matrix A, we have that AA^t is symmetric. Let $\mathrm{Sym}(n)$ denote the vector space of symmetric $(n \times n)$-matrices. The map

$$f : M_n(\mathbb{R}) \longrightarrow \mathrm{Sym}(n) = \mathbb{R}^{\frac{n(n+1)}{2}} \tag{2.3.1}$$

defined by $f(A) = AA^t$ is clearly smooth and

$$O(n) = f^{-1}(I). \tag{2.3.2}$$

Since for any $A \in O(n)$, $\det(A) = \pm 1$, the space $O(n)$ has at least two components. The special orthogonal group $SO(n)$ is defined to be

$$SO(n) = \{A \in O(n) \,:\, \det(A) = 1\}.$$

$SO(n)$ is an index two subgroup of $O(n)$. It can be shown that $SO(n)$ is connected[1] (infact path connected). It then follows that $O(n)$ has exactly two components and $SO(n)$ is the component containing I.

Lemma 2.4. The orthogonal group $O(n)$ is compact.

Proof. Let $A \in O(n)$ be an orthogonal matrix. If $v_1, \ldots, v_n$ denote the row vectors of A, then as $AA^t = I$ we have that the inner product

$$v_i \cdot v_i = 1.$$

Thus, as a subset of $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$, the orthogonal group $O(n)$ is bounded. The Equation (2.3.2) shows that $O(n)$ is also closed. $\square$

Thus the row vectors (also the column vectors) of an orthogonal matrix are unit vectors in $\mathbb{R}^n$ Thus $O(n)$ is actually a subset of

$$\mathbb{S}^{n-1} \times \cdots \times \mathbb{S}^{n-1}$$

a n-fold product.

We now show that the matrix groups defined above and some that we shall define below are manifolds.

Example 2.5. We shall show that the space $O(n)$ of orthogonal matrices is a manifold. We have seen above that $O(n) = f^{-1}(I)$ where

$$f : M_n(\mathbb{R}) \longrightarrow \mathrm{Sym}(n)$$

is defined by $f(A) = AA^t$. We wish to use the preimage theorem to prove that $O(n)$ is manifold. Towards this we shall check that $I \in \mathrm{Sym}(n)$ is a regular value of f. Since f is clearly smooth, given $A \in M_n(\mathbb{R})$ the derivative

$$df_A : T_A(M_n(\mathbb{R})) = M_n(\mathbb{R}) \longrightarrow T_{f(a)}\mathrm{Sym}(n) = \mathrm{Sym}(n)$$

is given by

$$\begin{aligned} df_A(B) &= \lim_{h\to 0} \tfrac{f(A+hB)-f(A)}{h} \\ &= \lim_{h\to 0} \tfrac{(A+hB)(A+hB)^t - AA^t}{h} \\ &= BA^t + AB^t. \end{aligned}$$

This gives a complete description of the derivative. Now let $A \in f^{-1}(I) = O(n)$. Given $C \in \mathrm{Sym}(n)$ it is easy to see that

$$df_A\left(\frac{CA}{2}\right) = C.$$

Thus df_A is surjective and hence I is a regular value of f. Hence by the preimage theorem, $O(n)$ is a submanifold of $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$ of codimension equal to $n(n+1)/2$. Thus

$$\dim(O(n)) = n^2 - (n(n+1))/2 = n(n-1)/2.$$

The tangent space to $O(n)$ at A is the kernel of the map

$$df_A : T_A(M_n(\mathbb{R})) = M_n(\mathbb{R}) \longrightarrow T_{f(a)}\mathrm{Sym}(n) = \mathrm{Sym}(n).$$

---

[1] There are several ways to prove this. One way is to use group actions. It turns out that the homogeneous space $SO(n)/SO(n-1)$ is the sphere $\mathbb{S}^{n-1}$ and then use induction. Another way is to look at the CW decomposition of $SO(n)$. It is a fact that $SO(n)$ has a CW structure with one 0-cell and hence is path connected.

Thus
$$T_A(O(n)) = \{B \in M_n(\mathbb{R}) \,:\, AB^t + BA^t = 0\}.$$
In particular, if $A = I_n = I$, then
$$T_I(O(n)) = \{B \in M_n(\mathbb{R}) \,:\, B^t + B = 0\}$$
the vector space of skew-symmetric matrices in $M_n(\mathbb{R})$.

Thus $O(n)$ is a manifold and hence its components are (also) open[2]. Since $SO(n)$ is a connected component of $O(n)$, it is open in $O(n)$ and hence is a manifold of the same dimension as $O(n)$ and $T_A(SO(n)) = T_A(O(n))$ for all $A \in SO(n)$.

Example 2.6. We now turn our attention to the special linear group $SL_n(\mathbb{R})$. Recall that
$$SL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \,:\, \det(A) = 1\}.$$
It is also the level set $\det^{-1}(1)$ for the determinant defined on $M_n(\mathbb{R})$. We compute the value of the derivative
$$d\det_A : T_A(M_n(\mathbb{R})) \longrightarrow \mathbb{R}$$
at $A \in M_n(\mathbb{R})$ on the tangent vector $A \in T_A(M_n(\mathbb{R})) = M_n(\mathbb{R})$. We have
$$\begin{aligned} d\det_A(A) &= \lim_{h\to 0} \tfrac{f(A+hA)-f(A)}{h} \\ &= \lim_{h\to 0} \tfrac{\det(A+hA)-\det(A)}{h} \\ &= \lim_{h\to 0} \tfrac{(1+h)^n\det(A)-\det(A)}{h} \\ &= n\cdot\det(A). \end{aligned}$$
which is nonzero if $A \in \det^{-1}(1) = SL_n(\mathbb{R})$. So det is a submersion at each $A \in SL_n(\mathbb{R})$. By the preimage theorem, $SL_n(\mathbb{R})$ is a submanifold of $M_n(\mathbb{R})$ of codimension 1. Hence the dimension of $SL_n(\mathbb{R})$ is $n^2 - 1$.

To understand the tangent space at $A \in SL_n(\mathbb{R})$ we need to understand the derivative of the determinant completely. Let $f : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ denote the determinant function. Then for $B \in T_A(M_n(\mathbb{R}))$ and $A \in SL_n(\mathbb{R})$ we have
$$\begin{aligned} df_A(B) &= \lim_{h\to 0} \tfrac{f(A+hB)-f(A)}{h} \\ &= \lim_{h\to 0} \tfrac{\det(A+hB)-\det(A)}{h} \\ &= \lim_{h\to 0} \tfrac{\det(A(I+hA^{-1}B)-\det(A)}{h} \\ &= \lim_{h\to 0} \tfrac{\det(A)\det(I+hA^{-1}B)-\det(A)}{h} \\ &= \det(A)\lim_{h\to 0} \tfrac{\det(I+hA^{-1}B)-1}{h} \\ &= \mathrm{tr}(A^{-1}B). \end{aligned}$$
The justification for the last step above is the following. The expansion of $\det(I + hA^{-1}B)$ is a polynomial in h. It is not difficult to see that the constant term is 1 and the coefficient of h is $\mathrm{tr}(A^{-1}B)$. Once we have this complete description of the derivative we can immediately compute the tangent spaces. Observe that for $A \in SL_n(\mathbb{R})$,
$$T_A(SL_n(\mathbb{R})) = \{B \in M_n(\mathbb{R})\,;\, \mathrm{tr}(A^{-1}B) = 0\}.$$
In particular the tangent space at $A = I$ equals
$$T_I(SL_n(\mathbb{R})) = \{B \in M_n(\mathbb{R})\,;\, \mathrm{tr}(B) = 0\}$$
the vector space of trace zero matrices. Notice that we did not need the complete description of the derivative (of determinant) to show that $SL_n(\mathbb{R})$ is a manifold.

[2]Components are always closed. They are also open if the space is locally connected.

Let I_n denote the $(n \times n)$ identity matrix and let J denote the $(2n \times 2n)$ matrix

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

then $\det(J) = 1$ and $J^{-1} = J^t = -J$. The *symplectic group* $Sp(2n)$ is defined to be

$$Sp(2n) = \{A \in M_{2n}(\mathbb{R}) \,:\, A^t J A = J\}.$$

It is well known[3] that if $A \in Sp(2n)$, then $\det(A) = 1$. Thus $Sp(2n)$ is a subgroup of the special linear group $SL_{2n}(\mathbb{R})$.

Recall that a matrix A is skew-symmetric if $A^t = -A$.

Example 2.7. The symplectic group $Sp(2n)$ is a manifold. To see this let $\mathrm{SkSym}(n)$ denote the vector space of skew-symmetric $(n \times n)$-matrices. Then

$$\mathrm{SkSym}(n) = \mathbb{R}^{\frac{n(n-1)}{2}}.$$

Observe that J is a skew-symmetric matrix. Now consider the smooth function

$$f : M_{2n}(\mathbb{R}) \longrightarrow \mathrm{SkSym}(2n)$$

given by

$$f(A) = A^t J A.$$

Then $Sp(2n) = f^{-1}(J)$. We claim that J is a regular value of f. Given $A \in Sp(2n)$ we compute the derivative

$$df_A : T_A(M_{2n}(\mathbb{R})) = M_{2n}(\mathbb{R}) \longrightarrow T_J(\mathrm{SkSym}(2n)) = \mathrm{SkSym}(2n).$$

as follows. For $B \in T_A(M_{2n}(\mathbb{R}))$ we have

$$\begin{aligned} df_A(B) &= \lim_{h \to 0} \tfrac{f(A+hB) - f(A)}{h} \\ &= \lim_{h \to 0} \tfrac{(A+hB)^t J (A+hB) - (A+hB)}{h} \\ &= A^t J B + B^t J A. \end{aligned}$$

So now given $C \in \mathrm{SkSym}(2n)$ we can quickly check using the facts mentioned above that if we let

$$B = (1/2)(AJ^{-1}C)$$

then

$$df_A(B) = C.$$

This shows that df_A is surjective and hence J is a regular value of f. Thus $Sp(2n)$ is a submanifold of $M_{2n}(\mathbb{R})$ of codimension equal to $n(n-1)/2$.

Here are some exercises. Remember that $X, Y, Z, \ldots$ will always denote manifolds and all maps/functions are smooth.

Exercise 2.8. Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ be the map defined by

$$f(x_1, x_2, x_3, x_4) = (x_1^2 + x_2^2 - x_3^2 - x_4^2, x_1^2 + x_2^2 + x_3^2 + x_4^2 - 4).$$

Show that $Z = f^{-1}(0)$ is a submanifold of $\mathbb{R}^4$. Find its dimension. Find a basis of the tangent space to Z at $p = (1, 1, -1, -1)$.

[3] See, for example, https://arxiv.org/pdf/1505.04240.pdf.

Exercise 2.9. Consider the function $f : \mathbb{R}^3 - \{z\text{-axis}\} \longrightarrow \mathbb{R}$ defined by

$$f(x, y, z) = (2 - \sqrt{x^2 + y^2})^2 + z^2.$$

Show that 1 is a regular value of f. Identify the manifold $Z = f^{-1}(1)$.

Exercise 2.10. Prove that the set of real (2×2)-matrices of rank 1 is a 3-dimensional submanifold of $M_2(\mathbb{R})$.

Exercise 2.11. Prove that the set of $(m \times n)$ matrices of rank r is a submanifold of $\mathbb{R}^{mn}$ of codimension equal to $(m - r)(n - r)$.

Exercise 2.12. Let Ω denote the $((n + 1) \times (n + 1))$ matrix

$$\Omega = \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix}$$

Let

$$X = \{A \in M_{n+1}(\mathbb{R}) \,:\, A^t \Omega A = \Omega\}.$$

Show that X is a manifold. Find its dimension.

Exercise 2.13. The product $S^2 \times S^2$ is a manifold that is a subset of $\mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6$. Prove that there is a submanifold $X \subseteq \mathbb{R}^5$ that is diffeomorphic to $S^2 \times S^2$. Now generalize. Show that there is no subset of $\mathbb{R}^4$ that is diffeomorphic to $S^2 \times S^2$. You could try out similar questions with other manifolds familiar to you.

Exercise 2.14. Let $X \subseteq \mathbb{R}^3$ be a compact 2-manifold. Prove that there exist at least two (distinct) points $x, y \in X$ such that both the tangent spaces $T_x(X)$ and $T_y(X)$ are spanned by the vectors $(1, 0, 0)$ and $(0, 1, 0)$. Is this true if the compactness assumption is dropped?

Exercise 2.15. Let $F : M_n(\mathbb{R}) \longrightarrow \mathbb{R}^n$ be the map that sends a matrix to the first column vector. This restricts to a smooth map

$$f = F/O(n) : O(n) \longrightarrow \mathbb{S}^{n-1}.$$

Show that f is a submersion. Is $f/SO(n)$ a submersion?

Exercise 2.16. Use (Exercise 2.10, Lecture 4 - II) to show that the boundary of the unit square is not a submanifold of $\mathbb{R}^2$.

Exercise 2.17. Use (Exercise 2.10, Lecture 4 - II) to show that the cone $x^2 + y^2 - z^2 = 0$, $z \geq 0$, is not a submanifold of $\mathbb{R}^3$.

Exercise 2.18. Convince yourself by an example that the inverse image of a critical value can be a manifold.

Exercise 2.19. Let A be a symmetric real $(n \times n)$-matrix and $c \in \mathbb{R}$. Set

$$X = \{x \in \mathbb{R}^n \,:\, x^t A x = c\}.$$

Is X a manifold?

Exercise 2.20. Convince yourself that there is an immersion of $(S^1 \times S^1)$ minus a point into $\mathbb{R}^2$. Can any such immersion be one-one? Note that there does not exist an immersion of $S^1 \times S^1$ into $\mathbb{R}^2$.

Exercise 2.21. Suppose $f : S^1 \longrightarrow \mathbb{R}$ is smooth and $y \in \mathbb{R}$ is a regular value. Show that $f^{-1}(y)$ has even number of elements.

Exercise 2.22. We identify S^1 with the "equator" in S^2, that is, with the the set of points $(x, y, z) \in S^2$ with $z = 0$. Is there a smooth function $f : S^2 \longrightarrow \mathbb{R}$ with $f^{-1}(y) = S^1$ where y is a regular value of f? What is the answer if $\mathbb{R}$ is replaced by S^1?

Exercise 2.23. Let X be the subset of $\mathrm{Sym}(2) = \mathbb{R}^3$ defined by

$$X = \left\{ A = \begin{pmatrix} x & y \\ y & z \end{pmatrix} : \det(A) = -1,\ \mathrm{tr}(A) = 0 \right\} \subseteq \mathrm{Sym}(2).$$

Show that X is a submanifold of $\mathrm{Sym}(2)$. Is X a familiar manifold?

Exercise 2.24. Show that $SL_2(\mathbb{R})$ is diffeomorphic to $S^1 \times \mathbb{R}^2$.