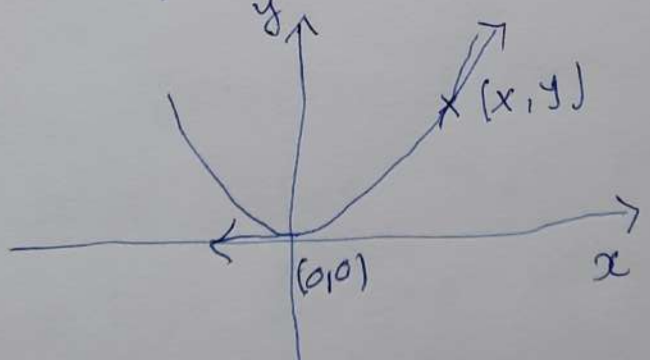
Lecture 3.

<u>Example</u> (reduction of order). The
equation satisfied by the coordinates
of a chain suspended between two
points, hanging under its own weight
in the x-y plane can be shown to
be
$$y'' = a\sqrt{1+(y')^2}$$
where a is a constant and the density
of the chain is assumed to be a con-
stant. Take $y' = p$; $y'' = p'$. The
equation becomes
$$p' = a\sqrt{1+p^2}$$



Note that when $x = 0$, $p = \frac{dy}{dx} = 0$.
Integrating the equation for p

we get $\log(p + \sqrt{1+p^2}) = ax$.

$$\Longrightarrow \quad p(x) = \frac{e^{ax} - e^{-ax}}{2}$$

Hence $y(x) = \int_0^x p(t)\,dt = \frac{e^{ax} - e^{-ax}}{2a}$.

i.e. $y(x) = \frac{1}{a} \cosh ax$.

This curve is called the Catenary.

<u>Remark</u>. When the arc of a curve lying above the x-axis is rotated about the x-axis then the resulting surface of revolution has the smallest area when the curve is a Catenary.

<u>Example</u>. (Electrical circuits). Kirchoff's law (which states that the sum of the electromotive forces around a closed circuit is zero) can be written as an ODE:

$$L \frac{dI}{dt} + RI + \frac{Q}{C} = E$$

where L is the inductance, R the resistance, Q the charge, C the capacitance, E the EMF and $I \equiv I(t)$ the current at time t. When there is no capacitor, $Q = 0$ and we have the first order (linear) ODE

$$L \frac{dI}{dt} + RI = E_0$$

where E_0 is the emf at $t = 0$. If I_0 is the initial condition at $t=0$, we can separate variables and integrate to obtain

$$I(t) = \frac{E_0}{R} + \left(I_0 - \frac{E_0}{R}\right) e^{-\frac{Rt}{L}}.$$

This solution has two parts. The steady state part is E_0/R. The transient part is $\left(I_0 - \frac{E_0}{R}\right) e^{-\frac{Rt}{L}}$, which goes to zero as $t \to \infty$. Note that, as $t \to \infty$, $E_0 \simeq RI$, which is Ohm's law

Second Order (linear) Equations.

We now consider second order equations whose determining funtion F has the form

$$F(x, y_0, y_1, y_2) = y_2 + P(x)y_1 + Q(x)y_0 - R(x)$$

where $P(x), R(x)$ and $Q(x)$ are given functions on $[a, b]$. Accordingly the ODE corresponding to F ie

$$F(x, y, y', y'') = 0$$

takes the form

$$(1) \text{ — } \quad y'' + P(x)y' + Q(x)y = R(x).$$

The corresponding homogeneous equation is

$$y'' + P(x)y' + Q(x)y = 0 \text{ — } (2)$$

Definition. We say that two functions f and g (both continuous) on $[a, b]$ are linearly dependent iff $\exists$ c such that $f = cg$ on $[a, b]$ (c a constant).

If no such constant exists then we say that f and g are linearly independent.

Remark. Note that if $f \equiv 0$ on $[a,b]$, then f and g are linearly dependent for any g.

Propn. Let y_1 be a solution of (2) with $y_1(x) \neq 0$ $\forall x \in [a,b]$. Let

$$V(x) := \int_{x_0}^{x} \frac{1}{y_1^2(r)} e^{-\int_{x_0}^{r} P(t)\,dt} \, dr$$

Then $y_2 = V y_1$ is also a solution of (2) which is linearly idependent of y_1.

proof. That y_2 is a solution is verified by differentiation. That y_2 and y_1 are linearly independent is proved by contradiction i.e. assume $\exists$ c $\ni$ $y_2 = c y_1$ etc.

The Homogeneous Equation with Constant Coefficients.

$$y'' + py' + qy = 0$$

Here $R(x) \equiv 0$ and $P(x) = p$, $Q(x) = q$, where p and q are constants.

We consider a candidate solution of the form $y = e^{mx}$. On substitution into the equation we get

$$(m^2 + pm + q)e^{mx} = 0.$$

This leads to $m^2 + pm + q = 0$

which gives two roots m_1, m_2

where $m_1, m_2 = \dfrac{-p \pm \sqrt{p^2 - 4q}}{2}$.

Case 1 $p^2 - 4q > 0$. We get two real and distinct roots and the corresponding solutions $e^{m_1 x}$ and $e^{m_2 x}$. It is easy to see that these are linearly independent.

Case 2. $p^2 - 4q < 0$. In this case (7)

m_1 and m_2 are distinct complex roots $m_1, m_2 = a \pm ib$.

$$e^{m_1 x} = e^{ax}(\cos bx + i \sin bx)$$

$$e^{m_2 x} = e^{ax}(\cos bx - i \sin bx).$$

$$\Longrightarrow \quad e^{ax}\cos bx = \frac{e^{m_1 x} + e^{m_2 x}}{2}$$

$$e^{ax}\sin bx = \frac{e^{m_1 x} - e^{m_2 x}}{2i}$$

Using the linearity of the equation it follows that $e^{ax}\cos bx$ and $e^{ax}\sin bx$ are also solutions of our equation. Again it is easy to see that these are linearly independent.

Case 3. $p^2 - 4q = 0$. In this case we get only one solution e^{mx}, $m = \frac{-p}{2}$. Using the Propn. on p.5 (with $x_0 = 0$) we get xe^{mx} is another lin. indep. soln.

Theorem. Let y_1 and y_2 be two linearly independent solutions of eqn. (2) (p.4). on the interval $[a,b]$. Then any solution y of eqn. (2) may be written as $y = c_1 y_1 + c_2 y_2$ for some constants c_1 and c_2.

Any solution of (2), say $y(x)$, is uniquely determined by two numbers viz. $y(x_0)$ and $y'(x_0)$ where $x_0 \in [a,b]$ is some fixed value. Thus to determine c_1 and c_2 in the theorem, we are lead to solving the system of eqns.

$$y(x_0) = c_1 y_1(x_0) + c_2 y_2(x_0)$$

$$y'(x_0) = c_1 y_1'(x_0) + c_2 y_2'(x_0).$$

This in turn leads us to the determinant

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = y_1(x_0) y_2'(x_0) - y_2(x_0) y_1'(x_0).$$

Given two differentiable functions (9)
y_1 and y_2 on $[a,b]$, their Wronskian
$W(y_1, y_2)$ is the function on $[a,b]$
defined by

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2$$

Lemma. Two solutions y_1 and y_2 of eqn. (2) are linearly independent iff $W(y_1, y_2)(x_0) \neq 0$ for some $x_0 \in [a,b]$.

Lemma. If y_1 and y_2 are two solutions of eqn. (2) on $[a,b]$, then $W(y_1, y_2)(x) = 0 \;\forall\, x \in [a,b]$
or $W(y_1, y_2)(x) \neq 0 \;\forall\, x \in [a,b]$.

Proof. Since y_1 and y_2 solve (2),

$$\frac{dW}{dx} = y_1 y_2'' - y_2 y_1'' = -P(y_1 y_2' - y_1' y_2)$$
$$= -PW.$$

The W satisfies the first order (10).
(linear) ODE $\dfrac{dW}{dx} + PW = 0$.
The general solution of this eqn
is $W(x) = c\, e^{-\int_{x_0}^{x} P(t)\,dt}$, $x \in [a,b]$. $\square$

<u>Exercise</u>. Give a rigorous proof of the last statement.

<u>Proof of the first Lemma</u> (P.9)

If y_1 and y_2 are linearly dependent then $\exists$ c such that $y_1 = c y_2$.
Hence $y_1' = c y_2'$. $\Rightarrow$ If $c \neq 0$ then
if $y_2(x) \neq 0$ $\forall x \in [a,b]$, then eliminating c, we get $W(y_1, y_2)(x) = 0$
$\forall x \in [a,b]$. If $y_2(x) = 0$ for some $x \in [a,b]$, then $y_1(x) = c y_2(x) = 0$
and hence $W(y_1, y_2)(x) = 0$. Hence
by the second lemma on p.9, $W(y_1, y_2) \equiv 0$ on $[a,b]$.

Conversley suppose $W(y_1, y_2) = 0$ (11).
on $[a, b]$. If $y_1(x) = 0 \ \forall x \in [a, b]$,
then $y_1 = c y_2$, where $c = 0$.
Suppose $y_1(x) \neq 0 \ \forall x \in [c, d] \subset [a, b]$.
[We are assuming y_1 and y_2 are differentiable on $[a, b]$.] Then

$$\frac{W(y_1, y_2)}{y_1^2} = \frac{y_1 y_2' - y_1' y_2}{y_1^2} = 0$$

on $[c, d]$. Hence $\frac{d}{dx}\left(\frac{y_2}{y_1}\right) = 0$
on $[c, d]$ or $y_2 = k y_1$ on $[c, d]$
for some constant k. Hence
$y_2' = k y_1'$ on $[c, d]$. However, as
follows from the next theorem, any
two solutions of eqn. (2) on $[a, b]$
(Here y_2 and $k y_1$) for which their
values and the values of their derivatives
agree at some $x_0 \in [a, b]$ must be

identically equal on $[a,b]$. This completes the proof of the first lemma. $\square$

<u>Proof of Theorem on p. 8:</u>

We can solve the two equations on p. 8, because the determinant of the coefficients, given by the Wronskian is non zero since y_1 and y_2 are linearly independent. For c_1 and c_2 thus obtained, the two solutions y and $c_1 y_1 + c_2 y_2$ and their derivatives y' and $c_1 y_1' + c_2 y_2'$ have the same value at $x_0 \in [a,b]$. Hence by the following theorem, $y = c_1 y_1 + c_2 y_2$. $\square$

<u>Theorem</u>. Let P, Q and R be continuous on $[a,b]$. Let $x_0 \in [a,b]$ and $y_0, y_0' \in \mathbb{R}$. Then $\exists$ one and only one solution y of eqn. (1) on $[a,b]$ satisfying $y(x_0) = y_0$ and $y'(x_0) = y_0'$.