DIFFERENTIAL TOPOLOGY - LECTURE 2

1. Introduction

Here we discuss the definition of a manifold and look at some examples. Manifolds have been around for some time now, although the right definition took some time to arrive at. Manifolds can be (and are) studied basically in two ways, one in the abstract setting and in the second case they are studied as certain subsets of some $\mathbb{R}^N$. Both the methods are essentially the same.

The book G and P adopts the second viewpoint and studies manifolds as subsets of $\mathbb{R}^N$. Manifolds are subspaces of $\mathbb{R}^N$ that locally are diffeomorphic to open subsets of (a fixed) $\mathbb{R}^k$. Because of this the methods of calculus (more generally most local phenomena) in the euclidean space can be transferred to the world of manifolds.

2. Definition of a manifold

Recall that for $X \subseteq \mathbb{R}^N$ a function $f : X \longrightarrow \mathbb{R}^m$ is said to be smooth if for each $x \in X$ there is a neighborhood $x \in U \subseteq \mathbb{R}^N$ and a smooth function $F : U \longrightarrow \mathbb{R}^m$ such that

$$F/(U \cap X) = f.$$

Definition 2.1. A subset $X \subseteq \mathbb{R}^N$ is said to be a k-dimensional manifold (or simply a k-manifold) if each $x \in X$ has a neighborhood V (V open in X) and a diffeomorphism

$$\varphi : U \subseteq \mathbb{R}^k \longrightarrow V$$

where U is open in $\mathbb{R}^k$. φ is called a *parametrization* of the open set V. We often (forget to mention V and) say that $\varphi : U \longrightarrow X$ is a *local parametrization about* x.

The inverse diffeomorphism

$$\varphi^{-1} : V \longrightarrow U \subseteq \mathbb{R}^k$$

is called a *coordinate system* (or a *chart*) on U. The reason being that if we write φ^{-1} in coordinates, that is,

$$\varphi^{-1} = (x_1, \dots, x_k)$$

then φ provides a way of associating coordinates to points in V. For example, we may think of a point $p \in V$ as having coordinates $(x_1(p), \dots, x_k(p))$[1]. The real valued functions x_i are called the *coordinate functions* on V.

If X is a k-manifold, then k is the dimension of X and we write $\dim(X) = k$. One can make wise choices of the parametrizing open set U. For example, we may choose U to be either the open unit ball centered at the origin or the whole of $\mathbb{R}^k$ and ensure that $\varphi(0) = x$.

The parametrizations allow us to transfer local data back and forth between the manifold and the euclidean space. Note that a subspace $X \subseteq \mathbb{R}^N$ is a k-manifold if and only if there is a covering of X by open sets each of which is diffeomorphic to an open subset of $\mathbb{R}^k$.

Here are some examples.

[1] This is what we usually do in the euclidean spaces where there are global coordinates.

Example 2.2. $\mathbb{R}^k$ is a k-manifold. Every open subset U of a k-manifold X is a k-manifold. The checking of these facts is left as an exercise.

Example 2.3. The product $X \times Y$ of two manifolds is again a manifold. To see this, suppose that $X \subseteq \mathbb{R}^N$ is a k-manifold and $Y \subseteq \mathbb{R}^M$ is a ℓ-manifold. Fix parametrizations

$$\varphi : U \subseteq \mathbb{R}^k \longrightarrow V \subseteq X$$
$$\psi : U' \subseteq \mathbb{R}^\ell \longrightarrow V' \subseteq Y$$

about $p \in X$ and $q \in Y$ respectively. Here U, U', V, V' are open in the respective ambient spaces. We claim that the map

$$\varphi \times \psi : U \times U' \longrightarrow V \times V'$$

defined by

$$(\varphi \times \psi)(x, y) = (\varphi(x), \psi(y))$$

is a parametrization about $(p, q) \in X \times Y$ showing that $X \times Y$ is a $(k + \ell)$-manifold. By (Example 2.2, Lecture 1) we know that $\varphi \times \psi$ is smooth. We only need to check that the inverse of $\varphi \times \psi$ is smooth. We fix open sets $W \subseteq \mathbb{R}^N$, $W' \subseteq \mathbb{R}^M$ about p, q respectively and smooth functions

$$F : W \longrightarrow \mathbb{R}^k; \quad G : W' \longrightarrow \mathbb{R}^\ell$$

such that

$$F/(W \cap V) = \varphi^{-1}; \quad G/(W' \cap V') = \psi^{-1}.$$

We can do this as both φ^{-1} and ψ^{-1} are smooth. It is now clear that the smooth function

$$F \times G : W \times W' \longrightarrow \mathbb{R}^k \times \mathbb{R}^\ell$$

restricts to $\varphi^{-1} \times \psi^{-1}$ on $(W \times W') \cap (V \times V')$ completing the proof that $\varphi^{-1} \times \psi^{-1}$ is smooth.

Example 2.4. The n-sphere

$$\mathbb{S}^n = \{(x_1, \ldots, x_{n+1}) : x_1^2 + \cdots + x_{n+1}^2 = 1\} \subseteq \mathbb{R}^{n+1}$$

is a n-manifold. Let us try to work through the details of this example. Define subsets U_i^+ and U_i^-, $i = 1, 2, \ldots, n+1$ of $\mathbb{S}^n$ by setting

$$U_i^+ = \{(x_1, \ldots, x_{n+1}) \in \mathbb{S}^n \ : \ x_i > 0\}$$
$$U_i^- = \{(x_1, \ldots, x_{n+1}) \in \mathbb{S}^n \ : \ x_i < 0\}.$$

Each of the above set is open (why?) in $\mathbb{S}^n$ and they cover $\mathbb{S}^n$. For each $i = 1, 2, \ldots n+1$, consider the maps

$$\varphi_i^+ : U_i^+ \longrightarrow V$$

where $V = \mathrm{Int}(\mathbb{D}^n)$ is the open unit ball in $\mathbb{R}^n$ defined by.

$$\varphi_i^+(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_{i-1}, \hat{x_i}, \ldots, x_{n+1}).$$

In other words, φ_i^+ drops the i-th coordinate. This map is clearly smooth, as it is a projection and is bijective. We claim that this is a chart on $\mathbb{S}^n$. We need to check that ψ_i^+, the inverse of φ_i^+ is also smooth. Observe that

$$\psi_i^+ : \mathrm{V} \longrightarrow U_i^+$$

is given by

$$\psi_i^+(y_1, \ldots, y_n) = (y_1, \ldots, y_{i-1}, \sqrt{1 - (y_1^2 + \cdots + y_n^2)}, y_i, \ldots, y_n)$$

which is clearly smooth. Thus φ_i^+ is a chart. The maps

$$\varphi_i^- : U_i^- \longrightarrow V$$

which drop the i-th coordinate are also charts for their inverse ψ_i^- is the smooth map

$$\psi_i^-(y_1,\dots,y_n) = (y_1,\dots,y_{i-1}, -\sqrt{1-(y_1^2+\cdots+y_n^2)}, y_i,\dots,y_n).$$

This shows that $\mathbb{S}^n$ is a n-manifold.

Example 2.5. Let V be a k-dimesnsional vector subspace of $\mathbb{R}^N$. Then V is a k-manifold. Here is one way to see this. Fix a basis $v_1,\dots,v_k$ of V. Consider the linear transformation

$$\varphi : \mathbb{R}^k \longrightarrow \mathbb{R}^N$$

defined by $\varphi(e_i) = v_i$. Since φ is linear, it is smooth. Also, by construction, φ maps $\mathbb{R}^k$ injectively onto its image which is V. We claim that φ is a diffeomorphism onto its image. We need to show that φ^{-1} is smooth. To see this we extend the basis of V to a basis

$$v_1,\dots,v_k,v_{k+1},\dots,v_N$$

of $\mathbb{R}^N$. The linear map $T : \mathbb{R}^N \longrightarrow \mathbb{R}^k$ defined by

$$T(v_i) = \begin{cases} e_i & i \le k \\ 0 & i > k \end{cases}$$

clearly restricts to φ^{-1} on V showing that φ^{-1} is smooth. Thus φ is a global parametrization for V.

Example 2.6. The set $M_n(\mathbb{R})$ of $(n \times n)$ matrices with real entries can be identified with $\mathbb{R}^{n^2}$ by writing the rows of the matrix one after another to get a point of $\mathbb{R}^{n^2}$. The determinant function

$$\det : M_n(\mathbb{R}) = \mathbb{R}^{n^2} \longrightarrow \mathbb{R}$$

being a polynomial in the coordinates of $\mathbb{R}^{n^2}$ is a smooth function. Then $GL_n(\mathbb{R})$ (which is the set of invertible matrices in $M_n(\mathbb{R})$) is open in $M_n(\mathbb{R})$ being the inverse image $\det^{-1}(\mathbb{R}-0)$. Hence $GL_n(\mathbb{R}) \subseteq \mathbb{R}^{n^2}$ is a n^2-manifold.

Example 2.7. Let $X \subseteq \mathbb{R}^N$ be a manifold and $f : X \longrightarrow \mathbb{R}$ a smooth function. Then, Z the graph of f

$$Z = \mathrm{graph}(f) = \{(x, f(x)) : x \in X\} \subseteq \mathbb{R}^N \times \mathbb{R}$$

is a manifold of dimension equal to $\dim(X)$. This is because the map

$$F : X \longrightarrow \mathrm{graph}(f)$$

defined by $F(x) = (x, f(x))$ is a diffeomorphism as we have already seen (Example 2.3, Lecture 1). Thus $\mathrm{graph}(f)$, being diffeomorphic to the manifold X, is itself a manifold. .

Example 2.8. (Level set of smooth funtions) Let $f : U \longrightarrow \mathbb{R}$ be a smooth function defined on an open set in $\mathbb{R}^{n+1}$. Given $c \in \mathbb{R}$, the subset

$$f^{-1}(c) = \{x \in U \;:\; f(x) = c\}$$

is called the level set of f of height c. If $\nabla f(x) \neq 0$ for all $x \in f^{-1}(c)$, then the level set $f^{-1}(c)$ is a n-manifold. We shall see a proof of this soon.

At this point we know two ways of constructing new manifolds : by taking open subsets of manifolds and taking products of manifolds. We shall see several more ways of constructing manifolds once we develop further techniques. Using these methods, we shall show that many familiar matrix groups inside $GL_n(\mathbb{R})$, for example, are manifolds.

Here is an example of a subset of $\mathbb{R}^2$ that is not a manifold.

Example 2.9. (Compare with Exercise 2.19, Lecture 1) Let X be the subspace of $\mathbb{R}^2$ defined by

$$X = \{[0,1] \times \{y\} \cup \{x\} \times [0,1] \, : \, x, y \in \{0,1\}\}.$$

Thus X is the boundary of the unit square. We claim that no neighborhood of $(0,0)$ in X is diffeomorphic to an open subset of $\mathbb{R}$. We assume that there exists a diffeomorphism of a neighborhood U of $(0,0)$, say,

$$\psi : U \longrightarrow (-1,1)$$

with $\psi((0,0)) = 0$ and derive a contradiction. Let $\varphi : (-1,1) \longrightarrow U$ be the inverse of ψ. In coordinates, let

$$\varphi(t) = (\varphi_1(t), \varphi_2(t)).$$

Then as $\psi \circ \varphi = \mathrm{id}$ we have by the chain rule that

$$\left(\frac{\partial \psi}{\partial x}(0,0), \frac{\partial \psi}{\partial y}(0,0) \right) \begin{pmatrix} \varphi_1'(0) \\ \varphi_2'(0) \end{pmatrix} = \mathrm{id}.$$

We assume without loss of generality that $\varphi_1'(0) \neq 0$. Since $\varphi_1(0) = 0$ it follows that the image of φ_1 contains an interval $(-\varepsilon, \varepsilon)$ (why?). Thus the image of $\varphi = (\varphi_1, \varphi_2)$ contains points with negative first coordinate. This is a contradiction.

We end this section by defining the notion of a submanifold. If X, Y are manifolds in $\mathbb{R}^N$ and $X \subseteq Y$, then we say that X is a *submanifold* of Y.

Here are some problems. The letters $X, Y, Z, \ldots$ will always denote a manifold.

Exercise 2.10. If X is a k-manifold show that every x has a neighborhood diffeomorphic to whole of $\mathbb{R}^k$.

Exercise 2.11. Prove the claims made in Example 2.2.

Exercise 2.12. Show that the projection $X \times Y \longrightarrow X$ is smooth.

Exercise 2.13. Explicity exhibit enough parametrizations to cover $S^1 \times S^1 \subsetneq \mathbb{R}^4$.

Exercise 2.14. Find a subset of $\mathbb{R}^2$ that is diffeomorphic to $S^1 \times \mathbb{R}$. Use this to show that $\mathbb{R}^3$ must contain a subset that is diffeomorphic to $S^1 \times S^1$. Generalize. An interesting point to note here is that since $S^1 \subsetneq \mathbb{R}^2$, the product $S^1 \times S^1$ is naturally a subset of $\mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$. The exercise shows that there is a copy of $S^1 \times S^1$ in $\mathbb{R}^3$, one dimension lower. Finding the smallest N such that $\mathbb{R}^N$ contains a copy of a given manifold is an interesting and important question in differential topology.

Exercise 2.15. The torus is the set of points in $\mathbb{R}^3$ at a distance b from the circle of radius a in the xy-plane, $0 < b < a$. Prove that these tori are all diffeomorphic to $S^1 \times S^1$.

Exercise 2.16. Prove that the hyperboloid in $\mathbb{R}^3$ defined by $x^2 + y^2 - z^2 = a$ is a manifold if $a > 0$. Why doesn't $x^2 + y^2 - z^2 = 0$ define a manifold? Note that paraboloids are graphs of quadratic functions, for example, $z = x^2 + y^2$.

Exercise 2.17. Fill in the details in Example 2.9.