Lecture 21: Going up theorem and applications

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<u>Lemma</u>: $A \subseteq B$ be integral ext^n.
Let $Q \subseteq B$ be a prime ideal of B & $P = Q \cap A$. Then Q is a maximal ideal of B iff P is a maximal ideal of A.

<u>Going up theorem</u>: Let $A \subseteq B$ be rings with B integral over A. Let

$$P_1 \subseteq P_2 \subseteq \dots P_m \subseteq P_{m+1} \subseteq \dots \subseteq P_n$$

be a chain of prime ideals in A & $Q_1 \subseteq Q_2 \subseteq \dots \subseteq Q_m$ be a chain of prime ideals in B s.t. $Q_i \cap A = P_i$ $1 \leq i \leq m$. Then $\exists$ $Q_m \subseteq Q_{m+1} \subseteq \dots \subseteq Q_n$ prime ideals of B s.t. $Q_i \cap A = P_i$ $\forall 1 \leq i \leq n$.

<u>Prop</u>: (Lying over thm): Let $A \subseteq B$ be an integral ext^n of rings. Let $P \subseteq A$ be a prime ideal then $\exists$ a prime ideal Q of B s.t. $Q \cap A = P$

Pf: Let $S = A \setminus P$ then S is a mult set.
Since B/A is integral ext^n, $S^{-1}B / S^{-1}A$ is an integral ext^n. Let $\tilde{Q}$ be a maximal ideal of $S^{-1}B$. (Note $S^{-1}A = A_P$)

Now, $\tilde{Q} \cap A_P = PA_P$. (by lemma)

$B \longrightarrow S^{-1}B$

Let $Q = \tilde{Q} \cap B$.

Claim: $Q \cap A = P$

Pf: $Q \cap A = \tilde{Q} \cap B \cap A \supseteq P$

($\because PS^{-1}B \subseteq \tilde{Q}$)

Let $x \in A \setminus P$ then $x \in S \Rightarrow$

$\frac{x}{1} \notin \tilde{Q}$ ($\because \tilde{Q}$ is a proper ideal)

$\Rightarrow x \notin Q \cap A$

Hence $Q \cap A \subseteq P$.

$\blacksquare$

Pf of Going up theorem

Let $\overline{B} = B/Q_m$

$$A \xhookrightarrow{i} B \xrightarrow{q} \overline{B}$$

$\ker(q \circ i) = Q_m \cap A = P_m$

Hence $\overline{A} := A/P_m \subseteq \overline{B}$

Moreover $\overline{B}$ is integral over $\overline{A}$. Since $\overline{b} \in \overline{B}$ then its lift b satisfy $b^k + a_{k-1} b^{k-1} + \dots + a_0 = 0$ for some $a_i \in A$. Hence

$$\overline{b}^k + \overline{a}_{k-1} \overline{b}^{k-1} + \dots + \overline{a}_0 = 0 \text{ is int over } \overline{A}.$$

where $\overline{x} = q(x)$ for $x \in B$.

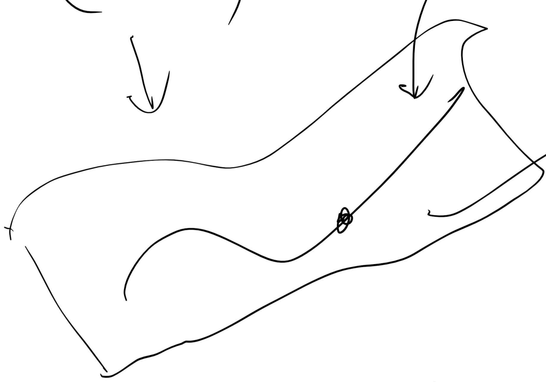
Hence by lying over $\exists\, \overline{Q}_{m+1} \subseteq \overline{B}$ prime ideal of $\overline{B}$ s.t. $\overline{Q}_{m+1} \cap A = \overline{P}_{m+1}$. Let $Q_{m+1} = q^{-1}(\overline{Q}_{m+1})$ then Q_{m+1} is a prime ideal of B and

$$Q_{m+1} \cap A = q^{-1}(\overline{Q}_{m+1}) \cap A = q|_A^{-1}(\overline{Q}_{m+1} \cap \overline{A}) = q|_A^{-1}(\overline{P}_{m+1}) = P_{m+1}$$

$\blacksquare$

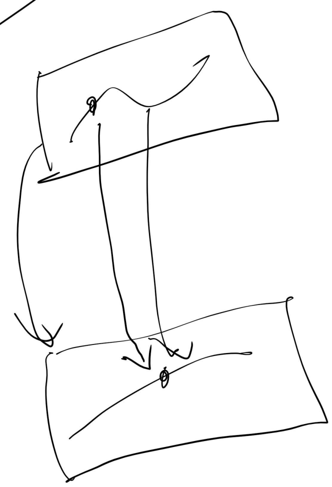
$$P_0 \subsetneq P_1 \subsetneq P_2$$

$$Z(P_0) \supseteq Z(P_1) \supseteq Z(P_2)$$



$$A \subseteq B$$

$$\begin{array}{c} X_B \\ \downarrow \\ X_A \end{array}$$



$Z(IB) = \phi \Rightarrow Z(I) = \phi$

Cor: $A \subseteq B$ be integral ext^n & I be A-ideal. If $IB = B$ then $I = A$.

Pf: Suppose $I \subsetneq A$ then $\exists\, m \supseteq I$ a maximal ideal of A.
By lying over $\exists\, Q \subseteq B$ prime ideal s.t. $Q \cap A = m$.
Hence $I \subseteq Q \Rightarrow IB \subseteq Q$ contradicting $IB = B$.

Prop: $A \subseteq B$ be integral ext^n $Q_0 \subsetneq Q_1 \subseteq B$ be prime ideals s.t.
$Q_0 \cap A \subsetneq Q_1 \cap A$.

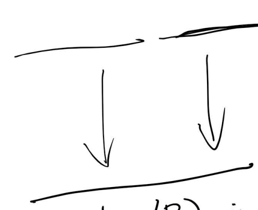
Pf: $A \subseteq B$ integral $\Rightarrow$ $A/_{Q_0 \cap A} \subseteq B/_{Q_0}$ is integral ext^n of integral domains. $\overline{Q}_1$ is a nonzero prime ideal of B/Q_0.
Hence $\exists\, \underset{\neq 0}{\bar{x}} \in \overline{Q}_1$ which is integral over $A/_{Q_0 \cap A}$ $(= \overline{A})$ i.e.
$\underset{1 \le i \le n-1}{\exists\, a_i \in \overline{A}}$ s.t. $\bar{x}^n + a_{n-1}\bar{x}^{n-1} + \dots + a_0 = 0$ with $a_0 \neq 0$. ($\because \overline{A}$ is int domain)
So $\underset{\neq 0}{a_0} \in \overline{Q}_1 \cap \overline{A} \Rightarrow Q_1 \cap A \supsetneq Q_0 \cap A$. ∎

Cor: $\dim(A) = \dim(B)$ if $B \supseteq A$ is integral ext^n.

Pf:
$P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_m$ in A chain of prime ideals
$Q_0 \subsetneq Q_1 \subsetneq \dots \subsetneq Q_m$ in B " " ".
$\Rightarrow \dim(B) \geqslant \dim(A)$ (Going up thm)

$Q_0 \subsetneq \dots \subsetneq Q_m$ in B
(Prev prob) $Q_0 \cap A \subsetneq \dots \subsetneq Q_m \cap A$ is a chain of prime ideals in A.
$\Rightarrow \dim(A) \geqslant \dim(B)$ ∎

Ex: $\underset{A}{k[x]} \subseteq \underset{B}{k[x, \tfrac{1}{x}]} = \dfrac{k[x,y]}{(xy-1)}$



mspec(B) is an open set of mspec(A)

$(0) \subseteq (x)$

$(0) \subseteq B$ is prime ideal
& Q is any prime ideal of B then
$Q \cap A \neq (x)$ ($\because x$ is a unit in B)