Physics 4

Lecture 14-15

Quick recap of 4-vectors

Position 4 vector $x^\mu = (ct, x, y, z) \equiv (x^0, x^1, x^2, x^3)$

contravariant $\mu = 0, 1, 2, 3$.

$x_\mu = (ct, -x, -y, -z) \equiv (x_0, x_1, x_2, x_3)$

Covariant $= (x^0, -x^1, -x^2, -x^3)$.

$x_\mu x^\mu = c^2t^2 - x^2 - y^2 - z^2 = x'^\mu x'_\mu$ → invariant

L.T. $x'^\mu = L^\mu{}_\nu x^\nu$

Generalize to 4 vectors.

- A^μ, A_μ that transform like x^μ, x_μ under L.T.
- form a vector space.
- Proper length (square) $A^\mu A_\mu$.
- Scalar product $A^\mu B_\mu$.
- Sometimes useful to define the matrix

$$\eta_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

→ transforms a contravariant vector → covariant

$$A_\mu = \eta_{\mu\nu} A^\nu$$

→ metric tensor.

Classification of 4 vectors

Timelike : $A^{\mu} A_{\mu} > 0$

Spacelike : $A^{\mu} A_{\mu} < 0$

Null : $A^{\mu} A_{\mu} = 0$

In addition two 4vectors are said to be orthogonal

$$A^{\mu} B_{\mu} = 0 .$$

examples :

timelike : $(1, 0, 0, 0)$

spacelike : $(0, 1, 0, 0)$

null : $(1, 1, 0, 0)$

$\longrightarrow$ need not be the zero vector

A 4 vector whose spatial/temporal part vanishes in some inertial frame must be timelike/spacelike

Converse given by following proposition

<u>Proposition 1</u> : If A is timelike, then $\exists$ an inertial frame in which $A^1 = A^2 = A^3 = 0$. If A is space like $\exists$ an inertial frame in which $A^0 = 0$.

<u>Proof</u> : Consider components A^0, A^1, A^2, A^3 in an inertial coordinate system (t, x, y, z).

- By rotating the spatial coordinates axes make the x axis parallel to $\vec{A}$: $A : (\overset{\circ}{A}, \vec{A})$ ensures $A^2 = A^3 = 0$

We can make A^1 vanish by making a standard L.T. chosen such that

$$\begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} A^0 \\ A^1 \end{pmatrix} = \begin{pmatrix} A^{0\prime} \\ 0 \end{pmatrix}$$

i,e we must choose v such that

$$-\beta A^0 + A^1 = 0 .$$

$$\frac{v}{c} = \left| \frac{A^1}{A^0} \right| \longrightarrow$$ always possible, because A timelike $\left| \frac{A^1}{A^0} \right| < 1$.

Similarly in the case of A spacelike, then $\left| \frac{A^1}{A^0} \right| > 1$ and one can find v with $|v| < c$ to make $A^0 = 0$.

In the case of timelike and null vectors (but not spacelike vectors) the sign of the time component A^0 is invariant.

<u>Proposition 2</u>

Suppose $A \neq 0$ is timelike or null, If $A^0 > 0$ in some inertial coordinate system, then $A^0 > 0$ in <u>every</u> inertial coordinate system.

Proof :

Since rotations do not alter A^0, it is sufficient to consider what happens under Lorentz boosts.

$$\begin{pmatrix} A^{0'} \\ A^{1'} \\ A^{2'} \\ A^{3'} \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix}$$

$$A'^0 = \gamma A^0 \left(1 - \frac{v}{c}\frac{A^1}{A^0}\right)$$

expression in brackets is +ve.

since $\left|\frac{v}{c}\right| < 1$ and $\left|\frac{A^1}{A^0}\right| \leq 1$.

$$\boxed{A'^0 \text{ and } A^0 \text{ must have same sign}}$$

future pointing $A^0 > 0$.

past pointing $A^0 < 0$.

Claim: If one of the components of a four vector is zero in every frame, it must be a zero vector.

Proof: $A^1 = 0$ in all frames, then A^2, A^3 also must be zero otherwise a rotation could make $A^1 \neq 0$.

A^0 must be zero, otherwise a boost could make $A^1 \neq 0$.

Your can argue similarly for $A^0 = 0$.