Lecture 17: Normal domains, Primary ideals

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Def: A map $f : X \to Y$ between affine varieties over k is said to be a morphism of affine varieties if there exist a k-alg homo $f^{\#} : \mathcal{O}_Y \to \mathcal{O}_X$ s.t. $f(m) = f^{\#^{-1}}(m)$ where m is a maximal ideal of $\mathcal{O}_X$. Note $X = \text{mspec}(\mathcal{O}_X) = \{ m \mid m \text{ maximal ideals of } \mathcal{O}_X \}$.

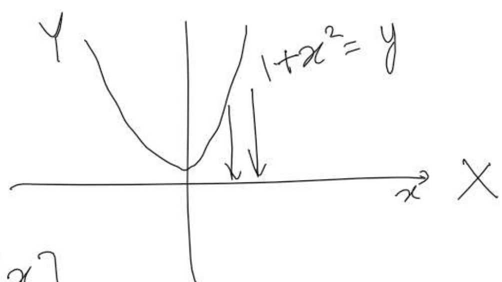
By Prop $f^{\#^{-1}}(m)$ is a max ideal of $\mathcal{O}_Y$ hence $f(m) \in Y$.

Prop: $f : X \to Y$ is a morphism of affine varieties over alg closed field k iff it is defined by polynomials. i.e. $\mathbb{A}^n \supseteq X \xrightarrow{f} Y \subseteq \mathbb{A}^m$

Let $Y = Z(I)$ where $I \subseteq k[y_1, \dots, y_m]$ & $X = Z(J)$, $J \subseteq k[x_1, \dots, x_n]$ then $\exists$ polynomials $F_1, \dots, F_m \in k[x_1, \dots, x_n]$

s.t. $f(a_1, \dots, a_n) = (F_1(a_1, \dots, a_n), \dots, F_m(a_1, \dots, a_n)) \quad \forall (a_1, \dots, a_n) \in X$

Ex:



$k = \mathbb{C}$

$\mathcal{O}_X = k[x]$

$\mathcal{O}_Y = \dfrac{k[x,y]}{(x^2 - y + 1)}$

$Y = \left\{ (a,b) \;\middle|\; \begin{array}{l} b = 1 + a^2 \\ a, b \in \mathbb{C} \end{array} \right\}$

$\mathbb{A}^2_{x,y} \supseteq Y$

$f : Y \longrightarrow X \subseteq \mathbb{A}^1_x$

$(a,b) \longmapsto a$

$f(x,y) = x$

$\mathcal{O}_X \longrightarrow \mathcal{O}_Y$

$k[x] \longrightarrow \dfrac{k[x,y]}{(x^2 - y + 1)}$

$x \longmapsto \bar{x}$ here $F(x,y) = x$

Definition: An integral domain **A** is called normal or integrally closed if the integral closure of **A** in its field of fraction **K** is **A** itself.

Example: Fields, integers ($\mathbb{Z}$), Dedekind domain. $\mathbb{Q}(\sqrt[3]{2})$. Let $R = \overline{\mathbb{Z}}^{\mathbb{Q}(\sqrt[3]{2})}$

Theorem: **A** UFD is a normal domain.

Pf: Let $K = \mathrm{frac}(A)$. Let $\frac{a}{b} \in K \setminus A$, $a, b \in A$ & $(a,b) = 1$ — since A is UFD make sense.

Suppose $\frac{a}{b}$ is int over A: $\left(\frac{a}{b}\right)^n + c_{n-1}\left(\frac{a}{b}\right)^{n-1} + \dots + c_0 = 0$ for some $c_i \in A$, $1 \le i \le n-1$

$\overset{\cdot b^n}{\Longrightarrow}$ $a^n + c_{n-1}a^{n-1}b + \dots + c_1 a b^{n-1} + c_0 b^n = 0$ in A.

Suppose $p \mid b$ for some prime $p \in A$.

Then $p \mid a^n \Rightarrow p \mid a$ contradicting $(a,b) = 1$.

(*) A normal domain, S mult subset $\Rightarrow S^{-1}A$ is normal.

(*) If A_p is normal $\forall$ prime $P \in \mathrm{spec}(A)$ then A is normal.

Ex: 1) $R = \mathbb{Z}[\sqrt{-3}]$ is not normal

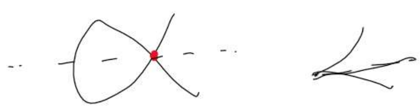
$\overline{R}^{\mathbb{Q}(\sqrt{-3})} = \mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$

$\alpha = \frac{1+\sqrt{-3}}{2}$

$(2\alpha - 1)^2 = -3$

$4\alpha^2 - 4\alpha + 1 = -3$

$\alpha^2 - \alpha + 1 = 0$

2) 

$y^2 = x^2 + x^3$ $\qquad$ $y^2 = x^3$

$R = \frac{k[x,y]}{(y^2 - x^2 - x^3)}$ & $\frac{k[x,y]}{(y^2 - x^3)}$ are not normal

$\alpha = \frac{\bar{y}}{\bar{x}} \in \mathrm{frac}(R) \setminus R$

$\left(\frac{\bar{y}}{\bar{x}}\right)^2 = \frac{\bar{y}^2}{\bar{x}^2} = \frac{\bar{x}^2 + \bar{x}^3}{\bar{x}^2} = 1 + \bar{x}$

$\alpha^2 - \bar{x} - 1$

So R is not normal

Definition: Let **R** be a ring. A proper ideal **Q** is called primary if for $x, y \in \boldsymbol{R}, xy \in \boldsymbol{Q}$ implies $x \in \boldsymbol{Q}$ or $y^n \in \boldsymbol{Q}$, for some n.

Lemma: I a primary R-ideal then $\sqrt{I}$ is a prime ideal.

Pf: $xy \in \sqrt{I} \Rightarrow x^m y^m \in I \Rightarrow x^m \in I$ or $(y^m)^n \in I$

for some m   for some n

$\Rightarrow x \in \sqrt{I}$ or $y \in \sqrt{I}$.

(*) I is primary R-ideal & $P = \sqrt{I}$ then I is called P-primary.

Prop: An ideal **Q** of **R** is primary iff every zero divisor in **R/Q** is nilpotent.

Pf: $P = \sqrt{Q}$. Let $\bar{x} \in R/Q$ be a zero divisor. Then $\bar{x}\bar{y} = 0$ in R/Q for some $\bar{y} \neq 0$

$\Rightarrow xy \in Q$ & $y \notin Q$ $x, y \in R$ & $x + Q = \bar{x}$, $y + Q = \bar{y}$

$\Rightarrow x^n \in Q$ for some n

$\Rightarrow \bar{x}^n = 0$ in R/Q

Let $x, y \in R$ s.t. $xy \in Q$

if $x \notin Q \Rightarrow \bar{x}\bar{y} = 0$ in R/Q

but $\bar{x} \neq 0$ in R/Q

$\Rightarrow \bar{y}$ is a zero divisor in R/Q

$\Rightarrow \bar{y}^n = 0$ in R/Q

$\Rightarrow y^n \in Q$

$\Rightarrow Q$ is primary ideal. ∎

(*) Examples are prime ideals.

In $\mathbb{Z}$, (p^n), p prime $n \geq 1$.

Lemma: Q an R-ideal s.t. $\sqrt{Q}$ is a maximal ideal of R then Q is primary ideal.

Pf: Note that $\sqrt{Q} = \bigcap_{\substack{P \text{ prime} \\ \text{ideal containing} \\ Q}} P = \underbrace{m}_{\text{a maximal ideal of } R}$.

$\Rightarrow$ There is only one maximal ideal of R containing Q, i.e. R/Q is a local ring with max ideal $\bar{m} = m/Q$.

Let $\bar{x} \in R/Q$ then if $\bar{x} \notin \bar{m} \Rightarrow \bar{x}$ is a unit. If $\bar{x} \in \bar{m} \in \bar{x} \in \sqrt{Q}/Q$

$\Rightarrow \bar{x}^n = 0$ in R/Q for some n.

Hence every zero divisor is nilpotent.

$\Rightarrow Q$ is primary.

<u>Example</u> : 1) $R = k[X, Y, Z, W]$, k is a field

$I = (X^n)$ is (X)-primary

$fg \in I$
$\Rightarrow X^n \mid fg$
$f \notin I$
$X^n \nmid f$
$\Rightarrow X \mid g$
$\Rightarrow g^n \in I$

2) $I = (X, Y, Z, W)^n$ is primary ideal.

$\sqrt{I} = (X, Y, Z, W)$

3) $R = k[X, Y]$, $I = (X^2, XY)$

I is not primary ideal though

$\sqrt{I} = (X)$.

$(X) \subseteq \sqrt{I} \subseteq (X)$ ($\because$ (X) is a prime ideal containing I)

$XY \in I$ but $X \notin I$ &

$Y^n \notin I \quad \forall n \geq 1$

Hence I is not primary.

$I = (X^2, Y) \cap (X)$