Let $K = \{ z \in \mathbb{C} \,/\, |z| = 1 \}$

Let $\mathcal{A}$ = the algebra generated by i

where i is defined by $i(z) = z$

OR

$= \left\{ z \longmapsto \sum_{k=0}^{n} c_k z^k \,/\, \begin{array}{l} c_k \in \mathbb{C} \\ n \in \mathbb{N} \cup \{0\} \end{array} \right\}$

$\mathcal{A}$ is an algebra that separates points of K and nowhere vanishes.

$\mathcal{A}$ is not dense in $C(K)$.

$f(z) = \bar{z} \quad \forall z \in K$

$f \notin \overline{\mathcal{A}}$

If $P_n \in \mathcal{A}$ such that $P_n \longrightarrow f$

$|P_n(z) - \bar{z}| < \frac{1}{2} \quad \forall z \in K$ for all large n

$|z P_n(z) - 1| < \frac{1}{2} \quad \forall z \in K$

$\int_0^{2\pi} e^{i\theta} P_n(e^{i\theta})\, d\theta = 0$ because for $k \geq 1$

$$\int_0^{2\pi} e^{ik\theta}\, d\theta = \left. \frac{e^{ik\theta}}{ik} \right]_0^{2\pi} = 0$$

$$\left| \int_0^{2\pi} [z P_n - 1]\, d\theta \right| = 2\pi$$

$$\leq \frac{1}{2} \times 2\pi = \pi \quad \Rightarrow \Leftarrow$$

$\therefore \mathcal{A}$ is not dense in $C(K)$

Let A be an algebra in $C(X)$.
We say that A is self-adjoint if for any $f \in A$, $\bar{f} \in A$
where $\bar{f}(x) = \overline{f(x)}$ for all $x \in X$.

Cor: If A is an algebra in $C(X)$ that separates points of X and nowhere vanishes, then $\bar{A} = C(X)$ provided A is self-adjoint.

Proof:
Let $f \in C(X)$. Then $f_r, f_i \in C_{\mathbb{R}}(X)$
such that $f = f_r + i f_i$.

$$f_r = \frac{f + \bar{f}}{2}, \quad f_i = \frac{f - \bar{f}}{2}.$$

A is a self-adjoint algebra
for any $f \in A$, $f_r, f_i \in A$
and $f = f_r + i f_i$.
Let $A_{\mathbb{R}} = A \cap C_{\mathbb{R}}(X)$.

Verify $\mathcal{A}_{\mathbb{R}}$ is an algebra in $C_{\mathbb{R}}(X)$.

Let $x \neq y$ in X. Then $f \in \mathcal{A}$ such that $f(x) \neq f(y)$

$f_r(x) + i f_s(x) \neq f_r(y) + i f_s(y)$

$\Rightarrow f_r(x) \neq f_r(y)$ or $f_s(x) \neq f_s(y)$.

$f \in \mathcal{A}$, $f_r, f_s \in \mathcal{A}_{\mathbb{R}}$

$\Longrightarrow \mathcal{A}_{\mathbb{R}}$ separates points of X.

Let $x \in X$. Then $f \in \mathcal{A}$

$f(x) \neq 0$

$\Rightarrow f_r(x) \neq 0$ or $f_s(x) \neq 0$

$\mathcal{A}_{\mathbb{R}}$ nowhere vanishes.

$\overline{\mathcal{A}_{\mathbb{R}}} = C_{\mathbb{R}}(X)$.

$f \in C(X)$ Then f_n, g_n in $\mathcal{A}_{\mathbb{R}}$ such that $f_n \to f_r$ $g_n \to f_s$.

$f_n + i g_n \in \mathcal{A}$

$\downarrow$

$f_r + i f_s = f$

$\therefore \overline{\mathcal{A}} = C(X)$.

Theorem $C(X)$ is separable

Proof: X is separable

$\Rightarrow \{x_n\} \subseteq X$, $\overline{\{x_n\}} = X$.

Let $f_n(x) = d(x_n, x)$ for all $x \in X$.

Then $f_n \in C(X)$.

$$\mathcal{A} = \left\{ a + \sum_{k=1}^{m} a_{n_1, n_2, \dots, n_k} f_{n_1} f_{n_2} \cdots f_{n_k} \;\middle|\; \begin{array}{l} a \text{ and } a_{n_1,n_2,\dots,n_k} \\ \text{are in } \mathbb{C} \end{array} \right\}$$

$\mathcal{A}$ is a self-adjoint algebra in $C(X)$.

For $x \neq y$ in X, $d(x,y) = \delta > 0$

$\exists\, x_n$ such that $d(x, x_n) < \frac{\delta}{2}$

$f_n(x) < \frac{\delta}{2}$

$\delta = d(x,y) \le d(x, x_n) + d(x_n, y)$

$\delta < f_n(x) + f_n(y) < \delta/2 + f_n(y)$

$f_n(x) < \frac{\delta}{2} < f_n(y)$

$\mathcal{A}$ separates points of X.

Easy to see $\mathcal{A}$ nowhere vanishes.

$\overline{\mathcal{A}} = C(X)$.

Let $E = \left\{ a + \sum a_{n_1, n_2, \ldots, n_k} f_{n_1} f_{n_2} f_{n_k} \;\middle|\; a, a_{n_1, n_2, \ldots, n_k} \in \mathbb{Q} + i\mathbb{Q} \right\}$

E is a dense in $\mathcal{A}$ and countable

$\Rightarrow$ E is a countable dense set in $C(X)$

as $\overline{\mathcal{A}} = C(X)$.

$\overset{\cdot}{\overline{\quad\quad\quad}}$ $\times$ $\overline{|F_n| = n}$

$E \subseteq \mathbb{R}^n$ or $\mathbb{C}^n$ $C(F_n)$ is compact $\Leftrightarrow$ E is closed and bounded

—— $\times$ ——

Let E be a collection of functions on X.

We say that E is pointwise bounded if to each $x \in X$ $\exists M_x > 0$ such that

$$|f(x)| \leq M_x \quad \forall f \in E.$$

We say E is equicontinuous if to each $\varepsilon > 0$ $\exists \delta > 0$ such that

$|f(x) - f(y)| < \varepsilon$ whenever $d(x,y) < \delta$ $\forall f \in E$.

<u>Ex</u> $E \subseteq C(X)$ is such that $\overline{E}$ is compact then E is pointwise bounded and E is equicontinuous.

<u>Theorem</u> (Arzela–Ascoli Theorem)

Let $E \subseteq C(X)$. Suppose E is pointwise bounded and equicontinuous. Then E is compact.

<u>Proof:</u> X has a countable dense set, D. Let $E \subseteq C(X)$ be pointwise bounded and equicontinuous. Let (f_n) be a sequence in E.

$\exists (f_{k_n})$ of (f_n) such that $(f_{k_n}(x))$ converges for all $x \in D$

Let $\varepsilon > 0$. $\exists \delta > 0$ such that

$d(x, y) < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{3} \quad \forall f \in E$

$\{B(x_n, \delta) \mid x_n \in D\}$ is an open cover for X $[x \in X \quad \exists x_n \in D, \ d(x, x_n) < \delta$
$x \in B(x_n, \delta)]$

$$X = \bigcup_{i=1}^{m} B(x_i, \delta) \quad \text{for } x_1, x_2, \ldots, x_m \in D$$

$\exists N$ such that $|f_{k_n}(x_i) - f_{k_m}(x_i)| < \frac{\varepsilon}{3}$ for $n, m \geq N$ and all i.

$x \in X \quad \exists x_i \in \{x_1, \ldots, x_m\}$
$d(x, x_i) < \delta$

$\implies |f_{k_n}(x) - f_{k_n}(x_i)| < \varepsilon/3 \quad \forall n$

$$|f_{k_n}(x) - f_{k_m}(x)| \leq |f_{k_n}(x) - f_{k_n}(x_i)| + |f_{k_n}(x_i) - f_{k_m}(x_i)| + |f_{k_m}(x_i) - f_{k_m}(x)|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \quad \forall n, m \geq N$$

i.e., $\exists N$ such that

$$|f_{k_n}(x) - f_{k_m}(x)| < \varepsilon \quad \forall n, m \geq N \quad \forall x$$

(f_{k_n}) is Cauchy in $C(X)$

$\therefore$ f_{k_n} converges in $C(X)$

Thus, any sequence in $E \subseteq C(X)$ that is pointwise bounded and equicontinuous has convergent subsequence.

$\therefore$ $\overline{E}$ is compact.

—— x ——