Inverse Function theorem :

Let E be an open set in $\mathbb{R}^n$ and $f : E \longrightarrow \mathbb{R}^n$ be a C^1-map.
If $f'(x)$ is invertible at $x_0 \in E$, then there exists an open set $U \subseteq E$ such that

1) $x_0 \in U$, $f(U) = V$, say, is open

2) $f|_U$ is $1-1$

3) g is the inverse of f on V,

then g is a C^1-map.

Proof : Let $A = f'(x_0)$.

Let λ be such that $\lambda \|A^{-1}\| = \frac{1}{2}$.

$\exists$ open ~~set~~ U containing x_0 such that $\|f'(y) - A\| < \lambda$ $\forall y \in U$.

For $y \in \mathbb{R}^n$, $\varphi_y : E \longrightarrow \mathbb{R}^n$ by

$$\varphi_y(x) = x + A^{-1}(y - f(x)) \quad \forall x \in E$$

$$\varphi_y'(x) = I + A^{-1}(-f'(x)) = A^{-1}(A - f'(x))$$

$$\|\varphi_y'(x)\| \le \|A^{-1}\| \, \|A - f'(x)\| < \frac{1}{2}$$

$$\|\varphi_y(x_1) - \varphi_y(x_2)\| \le \frac{1}{2}\|x_1 - x_2\|$$

$$\forall x_1, x_2 \in U$$

Let $V = f(U)$, $y_0 = f(x_0)$.

Let $y_1 \in V$. Then $f(x_1) = y_1$ for some $x_1 \in U$

$\exists\, r > 0$ such that $\overline{B_r(x_1)} \subseteq U$

<u>Claim</u> $B_{\lambda r}(y_1) \subseteq V$

Let y be such that: $\|y - y_1\| < \lambda r$

$\|\varphi_y(x_1) - x_1\| \leq \|A^{-1}(y - f(x_1))\|$
$\leq \|A^{-1}\|\,\|y - f(x_1)\| < \|A^{-1}\|\lambda r$
$< \frac{r}{2}$

$x \in \overline{B_r(x_1)}$

$\|\varphi_y(x) - x_1\| \leq \frac{1}{2}\|x - x_1\| + \frac{r}{2} \leq r$

$\varphi_y(x) \in \overline{B_r(x_1)}$

$\varphi_y : \overline{B_r(x_1)} \longrightarrow \overline{B_r(x_1)}$ is a contraction

By CMP, there is a unique point
$x \in \overline{B_r(x_1)}$ such that $\varphi_y(x) = x$

$\Rightarrow y = f(x) \in f(U) = V$

$\therefore B_{\lambda r}(y_1) \subseteq V$

$\Rightarrow V$ is open.

$f : U \to V$ is onto and one-one (Ex)

f is invertible, say $g = f^{-1}$.

Let $y \in V$, $y + k \in V$. Then $x \in U$, $x+h \in U$ such that $f(x) = y$ and $f(x+h) = y+k$.

$$\varphi_y(x+h) - \varphi_y(x) = h + A^{-1}(f(x) - f(x+h))$$
$$= h - A^{-1}(k)$$

$$\|h - A^{-1}(k)\| \leq \frac{1}{2}\|h\|$$

$$\Rightarrow \quad -\frac{1}{2}\|h\| \leq \|A^{-1}(k)\| - \|h\|$$

$$\Rightarrow \quad \|h\| \leq 2\|A^{-1}(k)\| \leq \frac{\|k\|}{\lambda}$$

Let $y \in U$.

$$\|f'(y) - A\| < \lambda$$

$$\Rightarrow \quad f'(y) \text{ is invertible}$$

Let $T = f'(x_0)^{-1}$

$$\frac{1}{\|k\|}\|g(y+k) - g(y) - T(k)\|$$

$$= \frac{1}{\|k\|}\|x+h - x - f'(x)^{-1}(k)\|$$

$$= \frac{1}{\|k\|}\|h - T(f(x+h) - f(x))\|$$

$$\leq \frac{1}{\|k\|}\|T\| \, \|T^{-1}(h) - f(x+h) + f(x)\|$$

$$\leq \frac{1}{\lambda} \|T\| \frac{\| f(x+h) - f(x) - f'(x)(h) \|}{\|h\|}$$

$\longrightarrow 0$ as $k \to 0$ ($\Rightarrow h \to 0$)

Thus, g is differentiable and

$$g'(f(x)) = f'(x)^{-1}$$

$\Rightarrow g$ is a C^1-map.

<u>Cor</u>: $f : U \to \mathbb{R}^n$ be a C^1-map

If $f'(x)$ is invertible for all $x \in U$, then f is open map, that is, f takes open sets to open sets.

<u>Eg</u> <u>Algorithm for $\sqrt{a}$ $(a>0)$</u>

Let $a > 0$

$$x^2 - a = 0$$

$$x = \frac{1}{2}\left(x + \frac{a}{x}\right)$$

$f : (0, \infty) \longrightarrow (0, \infty)$

$$f(t) = \frac{1}{2}\left(t + \frac{a}{t}\right)$$

$$|f(x) - f(y)| = \frac{1}{2}\left|x - y + \frac{a}{x} - \frac{a}{y}\right|$$

$$= \frac{1}{2}\left|1 - \frac{a}{xy}\right| |x-y|$$

$$\frac{1}{2}\left|1-\frac{a}{xy}\right| < 1$$

$$|xy - a| < 2xy$$

$$-xy < a < 3xy$$

If $x, y \geq \sqrt{a}$;

then $a < 3xy$

$$f(x) = \frac{1}{2}\left(x + \frac{a}{x}\right) \geq \sqrt{2}\,(\sqrt{a} + \sqrt{a}) \geq \sqrt{a}$$

$f : [\sqrt{a}, \infty) \longrightarrow [\sqrt{a}, \infty)$

By CMP $\quad f^n(x) \longrightarrow \sqrt{a} \quad (x \geq \sqrt{a})$

<u>Ex</u> $\quad f^n(x) \longrightarrow \sqrt{a}$ for $x > 0$

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2) <u>Newton method</u> :

Root of differentiable function.

Let f be a diff function $f' \neq 0$

$$g(x) = x - \frac{f(x)}{f'(x)}$$

If $g^n(x) \longrightarrow x_0$

$f(x) = 0$

If g is a contraction, then f has a root, which is that unique fixed point of g.

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There are also other methods such as Runge-Kutta methods etc.,

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Inverse function theorem gives a solution to the equation

$$y = f(x)$$

in a nbhd of the point $f(x_0)$ provided $f'(x_0)$ is invertible.

<u>Eg</u>: (local inverse but no global inverse)

Consider $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} x^2 & x \geq 0 \\ x^2 - 1 & x < 0 \end{cases}$$

for $x > 0$, $g(y) = \sqrt{y}$ is a inverse of f.

for $x < 0$, $y \mapsto \sqrt{y+1}$ is inverse if $|y| < 1$

$$g(y) = \begin{cases} -\sqrt{y+1}, & y \in (-1, 0) \\ \sqrt{y} & y \in (0, 1) \end{cases}$$

But f has no global inverse.

<u>Remark</u>: Under some additional conditions global invertibility is true.

$f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1, $f'(x)$ is invertible for all $x \in \mathbb{R}^n$. Then f has a inverse provided f is proper. (Hadamard – Caccioppoli Theorem)

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In case the equation $y = f(x)$ is not given explicitly i.e.,

$\boxed{0 = 3 + xy^2 - yx^3 + x}$

$\boxed{x^2 + y^2 - xy = 0}$

and

if we are given $f(x, y) = 0$ required to solve y in terms of x. This is called implicit equation.

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