Lecture 9: Tensor product of modules

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Recall R a comm ring with unity & M an R-module then we constructed $S^{-1}R$-mod $S^{-1}M$ and $\varphi : M \longrightarrow S^{-1}M$, $m \longmapsto \frac{m}{1}$ is an R-module homo s.t.

N a $S^{-1}R$-mod & $M \xrightarrow{\alpha} N$ an R-mod homo then $\exists!\ \tilde{\alpha} : S^{-1}M \to N$ s.t. $\tilde{\alpha} \circ \varphi = \alpha$.

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[dr, "\varphi"'] & N \arrow[d, phantom, "\circlearrowleft"] \\ & S^{-1}M \arrow[u, "\tilde{\alpha}"'] \end{tikzcd}$$

One can use universal property to define localization as well.

For defining tensor product we use this strategy.

<u>Def^n</u> Let M & N be R-modules. An R-module T together with an R-bilinear map $\varphi : M \times N \longrightarrow T$

(i.e. $\varphi(m, n_1 + r n_2) = \varphi(m, n_1) + r\varphi(m, n_2)$ $\forall\, m \in M$ $\forall\, n_1, n_2 \in N$ & $\forall\, r \in R$

$|||^{ly}$ $\varphi(m_1 + r m_2, n) = \varphi(m_1, n) + r\varphi(m_2, n)$ $\forall\, m_1, m_2 \in M$, $\forall\, n \in N$ & $\forall\, r \in R$)

is said to be a <u>tensor product of M & N over R</u> if given any R-bilin map $\psi : M \times N \longrightarrow A$ where A is an R-mod there exists a unique R-mod homo $\theta : T \longrightarrow A$ s.t. $\theta \circ \varphi = \psi$.

$$\begin{tikzcd} M\times N \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & T \arrow[d, "\exists!\theta"] \arrow[dl, phantom, "\circlearrowleft"] \\ & A \end{tikzcd}$$

T

<u>Prop:</u> T exist and is unique upto unique isomorphism and it is denoted by $M \otimes_R N$.

Pf: Uniqueness:

Let $\varphi' : M \times N \longrightarrow T'$ be another tensor product of M & N. Then want to show that $\exists!$ isom $T \xrightarrow{\alpha} T'$ s.t.

$$\alpha \circ \varphi = \varphi'.$$

By Universal property of T

$$\begin{tikzcd}
M \times N \arrow[r, "\varphi"] \arrow[dr, "\varphi'" description] \arrow[ddr, "\varphi"'] & T \arrow[d, dashed, "\exists! \alpha \ R\text{-lin}"] \arrow[dd, dashed, bend left=60, "id"] \\
& T' \arrow[d, dashed, "\exists! \alpha'"] \\
& T
\end{tikzcd}$$

s.t. $\alpha \circ \varphi = \varphi'$ ..(i)

$$\alpha' \circ \varphi' = \varphi \quad ..(ii)$$

$$(\alpha' \circ \alpha) \circ \varphi \overset{(i)}{=} \alpha' \circ \varphi' \overset{(ii)}{=} \varphi$$

$$id_T \circ \varphi = \varphi$$

By uniqueness $\alpha' \circ \alpha = id_T$

Similarl $\alpha \circ \alpha' = id_{T'}$

Hence α is an isom.

Existence: Let

$F_{M\times N}$ be the free R-module over $M\times N$.

i.e. $F = F_{M\times N} = \bigoplus_{(m,n)\in M\times N} R(m,n)$

Let $i : M\times N \longrightarrow F$
$(m,n) \longmapsto 1(m,n)$

$q : F \longrightarrow F \Big/ \Big\langle \big((m, n_1 + rn_2) - (m,n_1) - r(m,n_2)\big),\ (m_1 + rm_2, n) - (m_1, n) - r(m_2, n) \quad \forall\, m_1, m_2, m \in M,\ \forall\, r\in R \ \&\ n, n_1, n_2 \in N \Big\rangle$

$\left(\overset{\shortparallel}{T} \text{ say}\right)$ $\quad (\overset{\shortparallel}{K})$

$\varphi = q\circ i : M\times N \longrightarrow T$.

WTS φ is bilinear & it has the universal property.

$K = \ker(q) \subseteq F$

$$\varphi(m, n_1 + r n_2) = q \circ i(m, n_1 + r n_2) = q(m, n_1 + r n_2)$$

$$= \overline{(m, n_1 + r n_2)}$$

$$= \overline{(m, n_1)} + r\overline{(m, n_2)}$$

$$\left(\because \ \overline{(m, n_1 + r n_2) - (m, n_1) - r(m, n_2)} = 0 \text{ in } T \right)$$

$$= \varphi(m, n_1) + r\varphi(m, n_2)$$

Ill'y lin in 1st var holds.

Let $\psi : M \times N \longrightarrow A$ be R-bilin map.

Want $\theta : T \longrightarrow A$ R-linear s.t.

$\theta \circ \varphi = \psi$.

Let $\tilde{\theta} : F \longrightarrow A$ in R-lin

$$(m,n) \longmapsto \Psi(m,n)$$

$$\sum_{\substack{(m,n)\in M\times N \\ \text{finite}}} r_{(m,n)}(m,n) \longmapsto \sum_{\substack{(m,n)\in M\times N \\ \text{finite}}} r_{(m,n)}\Psi(m,n)$$

Note that Ψ is bilin. and hence

$$\tilde{\theta}\big((m, n_1 + r n_2) - (m, n_1) - r(m, n_2)\big)$$
$$= \Psi(m, n_1 + r n_2) - \Psi(m, n_1) - r\Psi(m, n_2)$$
$$= 0 \quad (\Psi \text{ is R-bilin})$$

$\overset{1^{st}\text{ isom}}{\Longrightarrow} K \subseteq \ker(\tilde{\theta})$

So $\exists\, \theta : T \longrightarrow A$ s.t.

$$\theta \circ q = \tilde{\theta}$$

$$\begin{tikzcd} F \arrow[d, "q"'] \arrow[dr, "\tilde{\theta}"] & \\ T \arrow[r, "\exists! \theta"'] & A \end{tikzcd}$$

$$\theta \circ \varphi = \theta \circ q \circ i$$
$$= \tilde{\theta} \circ i \; : \; M\times N \xrightarrow{i} F \xrightarrow{\tilde{\theta}} A$$
$$(m,n) \longmapsto \Psi(m,n)$$
$$= \Psi$$

$\boxtimes$