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The method of characteristics:

$$a u_x + b u_y = c u + d \quad \text{——— (1)}$$

a, b, c, d are functions of $(x,y) \in \Omega$

Initial curve : $\Gamma_0 : \{(x,y) : x = x_0(s),\ y = y_0(s),\ 0 \le s \le 1\}$.

$(x_0(s), y_0(s))$ given $\subset \Omega$

Given $u_0(s)$ the value of the solⁿ u on the initial curve.

i.e. $u(s) = u(x_0(s), y_0(s))$ ——— (2).

Note that $\Gamma_0 \subset \Omega$.

Problem: To find continuously differentiable fxⁿ $u(x,y)$ defined in some nbhd of Γ_0 satisfying eqn (1) in Ω_0 and satisfying eqⁿ (2).

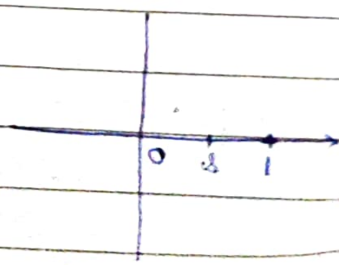
Theorem: Let a, b, c, d be continuously differentiable on Ω. Let $x_0(s), y_0(s), u_0(s)$ $0 \le s \le 1$ be continuously differentiable in $[0,1]$. Suppose further that the following transversality condition is satisfied $\forall\ s \in [0,1]$,

$$b(x_0(s), y_0(s)) \frac{dx_0}{ds} - a(x_0(s), y_0(s)) \frac{dy_0}{ds} \neq 0$$

Then $\exists$ a nbhd Ω_0 of Γ_0 and a unique solution $u(x,y)$ continuously differentiable in Ω_0 and satisfying (1) and (2).

Proof: (Sketch)

$$(3)\begin{cases} \dfrac{dx}{dt} = a(x,y), & x_0 = x_0(s) \\[2ex] \dfrac{dy}{dt} = b(x,y), & y_0 = y_0(s) \end{cases}$$



Continuously differentiable → locally Lipschitz.

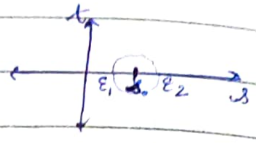
Choose a neighbourhood $D(x_0, y_0)$ st a, b, c, d are Lipschitz in $D(x_0, y_0)$.

Hence, the system (3) has a unique solution for every $(x,y) \in D(x_0, y_0)$.

If $(x_0, y_0) = (x_0(s), y_0(s))$ then denote the solution of (3) by $x(s,t)$, $y(s,t)$ where $x(s,0) = x_0(s)$

$$y(s,0) = y_0(s)$$

where ~~(s,t)~~ $(s,t) \in (\varepsilon_1, \varepsilon_2) \times (-\varepsilon_0, \varepsilon_0)$



<u>Fact:</u> Solutions of (3) as a function of (x_0, y_0) are continuously differentiable.

This implies $x(s,t)$ which is a composition of the solution of (3) with $x_0(s)$, $y_0(s)$ is continuously differentiable in (s,t).

The Jacobian of $(s,t) \longrightarrow (x(s,t), y(s,t))$

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \frac{\partial(x(s,t))}{\partial s} & \frac{\partial(x(s,t))}{\partial t} \\ \frac{\partial(y(s,t))}{\partial s} & \frac{\partial(y(s,t))}{\partial t} \end{vmatrix}$$

$$= \frac{\partial x(s,t)}{\partial s} \frac{\partial y(s,t)}{\partial t} - \frac{\partial(x(s,t))}{\partial t} \frac{\partial(y(s,t))}{\partial s}$$

~~At t = 0~~

By transversality,

$$\left.\frac{\partial(x,y)}{\partial(s,t)}\right|_{t=0} = b(x_0(s), y_0(s)) \frac{dx_0}{ds} - a(x_0(s), y_0(s)) \frac{dy_0}{ds} \neq 0$$

By choosing a sufficiently small neighbourhood and $D(s,0) \subset (\varepsilon_1 \times \varepsilon_2) \times (-\varepsilon_0, \varepsilon_0)$.

$$\left.\frac{\partial(x,y)}{\partial(s,t)}\right|_{t=0} \neq 0 \quad \forall (s,t) \in D(s,0)$$

$\Rightarrow$ By the inverse function theorem, we have the inverse map ~~f(x,y)~~ $s(x,y)$, $t(x,y)$; $(x,y) \in D(x,y)$

Recall: $\frac{d\bar{u}(s,t)}{dt} = c(x(t), y(t))\, \bar{u}(t) + d(x(t), y(t))$

$$\bar{u}(s,0) = u_0(s)$$

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Define $u(x,y) := \bar{u}(s(x,y), t(x,y))$

Note that inverse function theorem, $s(x,y)$ and $t(x,y)$ are continuously differentiable and by hypothesis $\bar{u}(\cdot\,\cdot)$ is cont differentiable $\Rightarrow u(x,y)$ is cont. differentiable.

Note that $(x,y) = (x(s,t), y(s,t))$

$$a(x,y)\, u_x(x,y) + b(x,y)\, u_y(x,y)$$
$$= a(x(t), y(t))\, u_x(x(t), y(t)) + b(x(t), y(t))\, u_y(x(t), y(t))$$
$$= \cancel{\frac{d\bar{u}}{dt}(s,t)} = \frac{d}{dt}(u(x(t), y(t))) = \frac{d}{dt}(\bar{u}(t))$$
$$= c(x(t), y(t))\, \bar{u}(t) + d(x(t), y(t))$$

Note that from the definition, $u(x(t), y(t)) = \bar{u}(s,t)$.
Since, $(x,y) = (x(t), y(t))$, we have (1).
Note that $u(x_0(s), y_0(s)) = u(x(s,0), y(s,0)) = \bar{u}(s,0) = u_0(s)$.

Uniqueness: Suppose $V(x,y)$ is another solution in $D(x_0, y_0)$ satisfying $V(x_0(s), y_0(s)) = u_0(s)$ $\quad (x_0(s), y_0(s)) \in D(x_0, y_0)$

Define $\bar{V}(t) := V(x(t), y(t))$
$$\bar{V}(0) = u_0(s)$$
Then since V satisfies (1),
$$\frac{d\bar{V}(t)}{dt} = c(x(t), y(t))\,\bar{V}(t) + d(x(t), y(t))$$

By uniqueness of solutions of ODE, $\bar{V}(t) = \bar{u}(t)$
$(s,t) \in D(s,0)$

Suppose $(x,y) \in D(x_0, y_0)$ ~~$\forall (x,y)$~~
$(x,y) = (x(t), y(t))$ $\quad \therefore V(x,y) = V(x(t), y(t)) = \bar{V}(t) = \bar{u}(t)$
$$= u(x(t), y(t))$$
$$= u(x,y)$$

Extension of U to $\Omega_0 \supset \Gamma_0$:
for each $s \in [0,1]$, get a nbhd $D(s,0)$ as above and a solⁿ

$u_s(x,y)$ in $D(x_0(s), y_0(s))$

Then $T_s \subset \bigcup_s D(x_0(s), y_0(s))$

By compactness, $T_0 \subset \bigcup_{i=1}^{n} D(x_0(s_i), y_0(s_i)) =: \Omega_0$

~~Let $U_{s_1}, \ldots, U$~~

Let $U^1, \ldots, U^n$ be the corresponding solutions.

By uniqueness, if $D(x_0(s_i), y_0(s_i)) \cap D(x_0(s_j), y_0(s_j)) \neq \phi$
then $U^i = U^j$ on $D(x_0(s_i), y_0(s_i)) \cap D(x_0(s_j), y_0(s_j))$.

Define $U(x,y)$ on $\Omega_0 = \bigcup_{i=1}^{n} D(x_0(s_i), y_0(s_i))$

as $u(x,y) = u^i(x,y)$ if $(x,y) = D(x_0(s_i), y_0(s_i))$

Then U is the unique solⁿ on Ω_0.

~~Flows~~

<u>Remark</u>: Note that for each $(x,y) \in \Omega_0$, there is exactly one characteristic $(x(t), y(t))$ s.t. $(x,y) = (x(t), y(t))$.
This is because of the 'homeomorphism' property of the solutions as a function of the initial point s_0
and hence $x(s_1, t) = x(s_2, t)$, $y(s_1, t) = y(s_2, t)$, $s_1 \neq s_2$
not possible.
[as one-one]

<u>Quasi-linear 1st order PDE.</u>

$$a(x,y,u)\, u_x + b(x,y,u)\, u_y = c(x,y,u)$$

The main difference with the linear case is that the equation for the characteristic curves include x, y, u.

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viz we have $\frac{dx}{dt} = a(x, y, \bar{u})$

$\frac{dy}{dt} = b(x, y, \bar{u})$

$\frac{d\bar{u}}{dt} \cancel{\frac{dt}{dt}} = c(x, y, \bar{u})$

$\left.\begin{aligned} x(0) &= x_0(s) \\ y(0) &= y_0(s) \end{aligned}\right\} \Gamma_0$

$\bar{u}(0) = u_0(s)$

Burger's Equation : $(t, x)\ t > 0\quad x \in \mathbb{R}$

inviscid ~~Implicity~~: $u_t + uu_x = 0$

viscosity : $u_t + uu_x = \nu u_{xx}$.

Navier Stokes $u_t + uu_x = \nu u_{xx} + \partial p$

$u_t + uu_x = 0$

$b(x, y, u) = 1 \qquad a(x, y, u) = u \qquad c(x, y, u) = 0$

$\frac{dx}{dt} = u(t) \qquad x(0) = s$

$y(0) = 0$

$\frac{dy}{dt} = 1 = \frac{dt}{dt} \qquad u(0) = u_0(s)$

$\frac{du}{dt} = 0$

~~$\Gamma_0 = \{(x, t) : t = 0\}$~~

$\Gamma_0 = \{(s, 0) : s \in \mathbb{R}\}$.

Given $u_0(s)$, $s \in \mathbb{R}$

$\frac{dx}{dt} = u(x(t), t) \qquad y = t$

$\frac{du}{dt} = 0 \qquad u(x(t), t) = u(x_0(0), 0) = u_0(s)$

$\Rightarrow x(s, t) = u_0(s)\, t + s$.

Given (x, t), we have $x = u_0(s)\, t + s = x(t)$

$u(x, t) = u_0(s)$

Obtain $s = s(x,t)$

$$u(x,t) = u_0(s(x,t))$$

$$F(x,t,s) = x - u_0(s)t - s$$

$$F_s(x,t,s) = -u_0'(s)t - 1 \neq 0$$

- Wave Eq^n
- Heat Eq^n
- Laplace Eq^n.

11-04-22

Remarks on system of ODE's:

$y' = Ay$ $\quad y' = \begin{pmatrix} y_1' \\ \vdots \\ y_n' \end{pmatrix}$ $\quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$y' : [a,b] \to \mathbb{R}$ $\qquad$ A is $n \times n$ matrix

example: $y'' + P(x)y' + Q(x)y = 0$

$z = y'$ $\qquad z' = -P(x)z - Q(x)y$

$$A \overset{?}{=} \begin{pmatrix} 1 & 0 \\ -P(x) & -Q(x) \end{pmatrix}$$

$y' = Ay$ $\qquad y(x_0) = y_0$

$$\Rightarrow y(x) = \begin{pmatrix} y_1(x) \\ \vdots \\ y_n(x) \end{pmatrix} = e^A (y_0)$$

where $e^A := \sum_{n=0}^{\infty} \frac{A^n}{n!} = \lim_{n\to\infty} \sum_{n=0}^{m} \frac{A^n}{n!}$

$\|A\| = \max_{ij} |(a_{ij})|$

$\Rightarrow \left\| \sum_{k+1}^{m} \frac{A^m}{m!} \right\| \xrightarrow{k,m \to 0} 0$

Wave equation:

Initial position. $y_0 = f(x)$

$y(x,t)$ = position of string at time t.

Wave eq^n: $a^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{dt^2}$ $\qquad \forall t > 0.$

Boundary conditions: $y(0,t) = y(\pi,t) = 0$

Initial condition: $y_0 = f(x) = y(x,0)$

$\cancel{\frac{\partial y(x,t)}{\partial t}}$ $\frac{\partial y(x,t)}{\partial t}\Big|_{t=0} = 0$

Method of Separation of variables:

We assume that $y(x,t) = u(x)v(t)$

$a^2 (u''(x)v(t)) = (u(x)v''(t))$

$\frac{u''(x)}{u(x)} - \left(\frac{1}{a^2}\right)\frac{v''(t)}{v(t)} = -\lambda$ $\qquad \lambda > 0$

$\Rightarrow \frac{u''(x)}{u(x)} = -\lambda$ $\qquad \left(\frac{1}{a^2}\right)\frac{v''(t)}{v(t)} = -\lambda$

~~$\frac{u''(x)}{u(x)} = -\lambda a$~~

$\Rightarrow u''(x) + \lambda u(x) = 0$ —— (1)

~~$v''(t) + \lambda v(t) = 0$~~

$v''(t) + \lambda a^2 v(t) = 0$ —— (2)

$u(x) = \sin(rx)$
$u'(x) = r\cos(rx)$
$u''(x) = -r^2 \sin(rx)$
$u''(x) = -r^2 u(x)$

Refer Lecture-10
$\lambda > 0$ ~~say~~ $y(0,t) = y(\pi, t)$
for $\lambda < 0$. (1) has atmost one solution.

General solⁿ of (1): $u(x) = c_1 \sin(\sqrt{\lambda}\, x) + c_2 \cos(\sqrt{\lambda}\, x)$

$u(0) = 0 = c_2 = 0$

~~$0 = u(\pi) = c_2 \cos(\sqrt{\lambda}\pi) = 0$~~ ~~∴ Boundary con~~.

~~$u(\pi) = 0 \Rightarrow$~~

∴ ~~$u(x) = c_1 \sin(\sqrt{\lambda}\, x)$~~.

$u(\pi) = 0 \Rightarrow c_1 \sin(\sqrt{\lambda}\, x) = 0 \Rightarrow \sqrt{\lambda} \in \mathbb{Z}$

$\lambda = n^2 \quad n \in \mathbb{Z}$

General solⁿ of (2) = $v(t) = c_1' \sin(\sqrt{\lambda a^2}\, t) + c_2' \cos(\sqrt{\lambda a^2}\, t)$

$= c_1' \sin(\sqrt{a^2} nt) + c_2' \cos(\sqrt{a^2} nt)$

$v'(t) = c_1' \cos(\sqrt{a^2} nt)(\sqrt{a^2} n) + c_2'(-\sin(\sqrt{a^2} nt))(\sqrt{a^2} n)$

$v'(0) = c_1' \sqrt{a^2}\, n \cos(\sqrt{a^2}\, n) = 0$

$\Rightarrow c_1' = 0$

$y(x,t)$ ~~$y(t)$~~ $= c_1 \sin(\sqrt{\lambda}\, x)\, c_2' \cos(\sqrt{a^2} nt)$

$= c_1 \sin(nx)\, c_2' \cos(\sqrt{a^2} nt)$

$= C \sin(nx) \cos(\sqrt{a^2} nt) = C \sin(nx) \cos(ant)$

Linearity, (i.e if y_1 and y_2 are sol's $\Rightarrow y_1 + y_2$ is also a solⁿ.) of wave eqⁿ

$\Rightarrow y(x,t) = \sum_{k=1}^{m} b_k \sin(kx) \cos(\sqrt{a^2} kt)$

$y(x,0) = f(x) \Rightarrow f(x) = \sum_{k=1}^{m} b_k \sin kx$ ~~$\cos(k\sqrt{a^2}t)$~~

Since f is arbitrary, m cannot be finite

$$f(x) = \sum_{k=1}^{\infty} b_k \sin kx.$$

$$y(x,t) = \sum_{k=1}^{\infty} b_k \sin kx \cos(\sqrt{a^2} kt)$$

From Fourier Analysis $\quad b_k = \frac{2}{\pi} \int_0^{\pi} f(x) \sin kx \, dx$

since $\{\sin kx, k \geq 1\}$ form an orthogonal family in $[0, \pi]$.

$$\int_0^{\pi} \sin kx \sin lx \, dx = \delta_{kl} c_k.$$

Ex: Find c_k.

Remark: If f ~~is~~ has finite no. of discontinuities, i.e. $f(x_+) \neq f(x_-)$ and a finite no. of maxima and minima, f bounded, $f: [-\pi, \pi] \to \mathbb{R}$ then $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx =$ ~~$f(x)$~~ $\frac{f(x+) + f(x-)}{2}$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$ $\qquad$ $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$

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