Lecture 33: Proof of Bezout's theorem

20 April 2021
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<u>Thm</u> (Bezout's thm): Let C_1 & C_2 be two distinct irred curves in $\mathbb{P}^2_k$ where k is alg closed of deg m & n resp.
i.e. $C_1 = Z(f_1)$ & $C_2 = Z(f_2)$ where f_1 & f_2 are irred. homo poly in three var of deg m & n resp. Then
$1 \le |C_1 \cap C_2| \le mn$. In fact $|C_1 \cap C_2| = mn$ "if each point is counted with right multiplicity".

<u>Lemma</u>: Let k be an algebraically closed and $f, g \in k[X, Y, Z]$ be homogen poly of deg m & $n \geq 1$. After a nonsing linear change of variable X, Y, Z, we may assume f & g are monic in Z.

Example: Suppose C_1 is a line i.e. $f = aX + bY + cZ$ for some a, b, c. May assume $c = -1$, i.e. $C_1 \equiv \{Z = aX + bY\}$

$g(X, Y, Z)$ is homogen of deg n.

$1 \le |C_1 \cap C_2| \le n$. So if $[a_0, a_1, a_2] \in C_1 \cap C_2$ then

(a_0, a_1) should satisfy $g(X, Y, aX + bY) = 0$ & $a_2 = a a_0 + b a_1$.

$\overset{\shortparallel}{h}(X, Y)$ is a homogen poly in X, Y of deg n. over the field k.

$h(X, Y)$ is product of linear homogen factors as k is alg closed

$$\overset{\shortparallel}{X^n h\left(1, \frac{Y}{X}\right)} = X^n \left(\frac{Y}{X} - \alpha_1\right) \cdots \left(\frac{Y}{X} - \alpha_d\right) = X^{n-d} (Y - \alpha_1 X) \cdots (Y - \alpha_d X)$$

$X = 0$ if $d < n \Longleftrightarrow [0, 1, b] \leftarrow$ with multiplicity $n-d$

$X = 1, Y = \alpha_i \Longleftrightarrow [1, \alpha_i, a + b\alpha_i] \quad 1 \le i \le d$

Lemma: Let R be a UFD and f, g be nonconstant polynomials in $R[Z]$. Then f, g have a common nontrivial factor iff $\operatorname{Res}(f,g) = 0$ in R.

Here if $f = a_0 Z^m + a_1 Z^{m-1} + \dots + a_{m-1} Z + a_m$ for $a_i \in R$

& $g = b_0 Z^n + b_1 Z^{n-1} + \dots + b_{n-1} Z + b_n$ $b_j \in R$

then

$$\operatorname{Res}(f,g) = \det \underbrace{\begin{pmatrix} a_0 & \cdots & a_m & & 0 \\ 0 & a_0 & \cdots & a_m & \\ 0 & & \ddots & & \\ & & a_0 & \cdots & a_m \\ b_0 & \cdots & b_n & & 0 \\ & b_0 & \cdots & b_n & \\ & & \ddots & & \\ 0 & & b_0 & \cdots & b_n \end{pmatrix}}_{M =} \begin{matrix} \left.\right\} n \text{ rows} \\ \\ \left.\right\} m \text{ rows} \end{matrix}$$

is $(m+n) \times (m+n)$ matrix

Pf: Let $f = uh$ & $g = vh$ where $u, v, h \in R[Z]$ with degree h at least 1.

$\Rightarrow$ $vf - ug = 0$

Conversely if $\exists\, u, v \in R[Z]$ nonzero with $\deg(u) < \deg f$ & $\deg v < \deg h$

& $vf - ug = 0$ then $vf = ug$

Since $\deg(u) < \deg f$, $f \nmid u$ hence some irred factor of f divides g. (Note $R[Z]$ is a UFD)

So f, g have a common factor iff $\exists\, u, v \in R[Z]$ nonzero

$u = c_0 Z^{m-1} + c_1 Z^{m-2} + \dots + c_{m-1}$ $c_i, d_i \in R$

$v = d_0 Z^{n-1} + d_1 Z^{n-2} + \dots + d_{n-1}$

s.t. $vf - ug = 0$

$$vf - ug = a_0 d_0 Z^{m+n-1} + (a_0 d_1 + a_1 d_0) Z^{m+n-2} + \dots + a_m d_{n-1}$$
$$- b_0 c_0 Z^{m+n-1} - (b_0 c_1 + b_1 c_0) Z^{m+n-2} - \quad - b_n c_{m-1} = 0$$

$$vf - ug = 0 \iff \underbrace{\left\{\begin{array}{llll}
a_0 d_0 & - b_0 c_0 & = 0 & \leftarrow \text{coeff of } Z^{m+n-1} \text{ is zero} \\
a_1 d_0 + a_0 d_1 & - b_1 c_0 - b_0 c_1 & = 0 & \text{'' } Z^{m+n-2} \text{ is zero} \\
a_2 d_0 + a_1 d_1 + a_0 d_2 & - b_2 c_0 - b_1 c_1 - b_0 c_2 & = 0 & \\
\vdots & & & \\
a_m d_0 + a_{m-1} d_1 + \dots + a_0 d_m & - b_m c_0 - b_{m-1} c_1 \dots - b_1 c_{m-1} & = 0 & Z^m \\
a_m d_1 + \dots + a_1 d_m + a_0 d_{m+1} & - b_{m+1} c_0 - b_m c_1 \quad - b_2 c_{m-1} & = 0 & \\
\vdots & & & \\
a_m d_{n-1} & - b_n c_{m-1} & = 0 &
\end{array}\right.}_{m+n \text{ equations}}$$

$$M^T = \begin{bmatrix}
a_0 & & & b_0 & & \\
a_1 & a_0 & & b_1 & b_0 & \\
a_2 & a_1 & \ddots & \vdots & \vdots & \ddots \\
\vdots & & a_0 & b_n & \vdots & b_0 \\
a_m & & \vdots & & b_n & \vdots \\
 & a_m & \vdots & & & \vdots \\
 & & a_m & & & b_n
\end{bmatrix} \quad \underbrace{M^T \begin{bmatrix} d_0 \\ \vdots \\ d_{n-1} \\ -c_0 \\ \vdots \\ -c_{m-1} \end{bmatrix} = 0}_{}$$

This has a nontrivial sol^n iff $\det(M) = 0$

$\Updownarrow$

$\exists\, u, v$ nonzero in $R[Z]$ with $\deg u < m$ & $\deg v < n$

s.t. $vf - ug = 0$.

$\Leftrightarrow$

f, g have a common nontrivial factor.

Thm: (Weak Bezout's thm) $f, g \in k[X,Y,Z]$ homogen of deg m & n resp. Suppose f, g have no common irred factor. Let $C = Z(f)$ & $D = Z(g)$ then $1 \le |C \cap D| \le mn$.

Pf: $f = Z^m + a_1(X,Y) Z^{m-1} + a_2(X,Y) Z^{m-2} + \dots + a_m(X,Y)$

$g = Z^n + b_1(X,Y) Z^{n-1} + \dots + b_n(X,Y)$

where a_i & b_i are homogen of degree i

Claim: $\mathrm{Res}(f,g)$ is a nonzero homogen poly of deg mn.

$$M = \begin{pmatrix} 1 & a_1 & a_2 & \cdots & a_m & & & 0 \\ & 1 & a_1 & a_2 & \cdots & a_m & & \\ 0 & & & 1 & a_1 & \cdots & & a_m \\ 1 & b_1 & \cdots & b_n & & & 0 & \\ 0 & 1 & \cdots & & b_n & & & \\ 0 & & & & 1 & \cdots & & b_n \end{pmatrix} \in M_{(m+n)\times(m+n)}(k[X,Y])$$

$$M(\lambda X, \lambda Y) = \begin{array}{c} \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \\ \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{m-1} \end{array} \begin{bmatrix} 1 & \lambda a_1 & \lambda^2 a_2 & \cdots & \lambda^m a_m & & & \\ & 1 & \lambda a_1 & \cdots & & \lambda^m a_m & & \\ & & \ddots & & & & & \\ & & & 1 & \lambda a_1 & \cdots & & \lambda^m a_m \\ 1 & \lambda b_1 & \cdots & \lambda^n b_n & & & & \\ & 1 & \cdots & & \lambda^n b_n & & & \\ & & & & & \ddots & & \\ & & & & 1 & \cdots & & \lambda^n b_n \end{bmatrix}$$

$$\det(M(\lambda X, \lambda Y)) \frac{\lambda^{1+2+\dots+n-1+1+2+\dots+m-1}}{\lambda^{1+2+\dots+m+n-1}} = \det M(X,Y)$$

$$\mathrm{Res}(f,g)(\lambda X, \lambda Y) \cdot \overbrace{\lambda^{\frac{n(n-1)}{2} + \frac{m(m-1)}{2} - \frac{(m+n)(m+n-1)}{2}}}^{-mn} = \mathrm{Res}(f,g)(X,Y)$$

$$\mathrm{Res}(f,g)(\lambda X, \lambda Y) = \lambda^{mn} \mathrm{Res}(f,g)(X,Y)$$

Hence claim

Now if $aX - bY$ is a factor of $\mathrm{Res}(f,g)$
$(a,b) \neq (0,0)$

then $f(b,a,Z)$ & $g(b,a,Z)$ has a common factor say $Z-c$ then

$[b,a,c] \in C \cap D$.

i.e. $|C \cap D| \geq 1$.

Suppose there are at least $mn+1$ points in $C \cap D$. We can ensure that $[0,0,1]$ is not on any line joining any two of these $mn+1$ points by changing coordinates linearly.

Let $[a,b,c]$ be a point in $C \cap D$ then $bX - aY$ is a factor of $\mathrm{Res}(f,g)$

If $[a,b,c]$ & $[a,b,c'] \in C \cap D$

$c \neq c'$ then line $bX - aY = 0$ in $\mathbb{P}^2$ passes through $[0,0,1]$ contradicting the hypo.

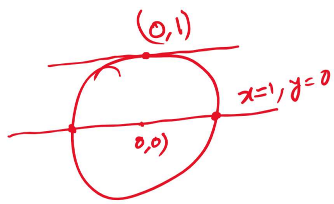
Hence $\mathrm{Res}(f,g)$ has $mn+1$ distinct linear factor contradicting $\mathrm{Res}(f,g)$ is homogen of degree mn.

Intersection multiplicity

$k[x,y]$ $\quad x^2+y^2=1$

$y=0$

$y=1$



$$\dim_k\left(\frac{k[x,y]_{(x-1,y)}}{(x^2+y^2-1,\, y)}\right) \cong \frac{k[x]_{(x-1)}}{((x-1)(x+1))} \cong \frac{k[x]_{(x-1)}}{(x-1)} \cong k$$

$$\dim_k\left(\frac{k[x,y]_{(x-1,y)}}{(x^2+y^2-1,\, y)}\right) = \dim_k k = 1$$

$$\dim_k\left(\left(\frac{k[x,y]}{(x^2+y^2-1,\, y)}\right)_{(\bar{x}-1,\, \bar{y})}\right) = I_{(1,0)}(C,D)$$

where $C = x^2+y^2-1$
$D = y$

$$\left(\frac{k[x,y]}{(x^2+y^2-1,\, y-1)}\right)_{(x,\, y-1)} \cong \frac{k[x]_{(x)}}{(x^2)} \cong \frac{k[x]}{(x^2)} \leftarrow \text{dim } 2 \text{ over } k.$$

Let $P \in C \cap D$ where C & D have no common component.

$C \leftrightarrow Z(f)$, $\quad D = Z(g)$

then $I_p(C \cap D) = \dim_k \dfrac{\mathcal{O}_{\mathbb{A}^2,p}}{(f,g)}$ is intersection multiplicity of P in $C \cap D$.