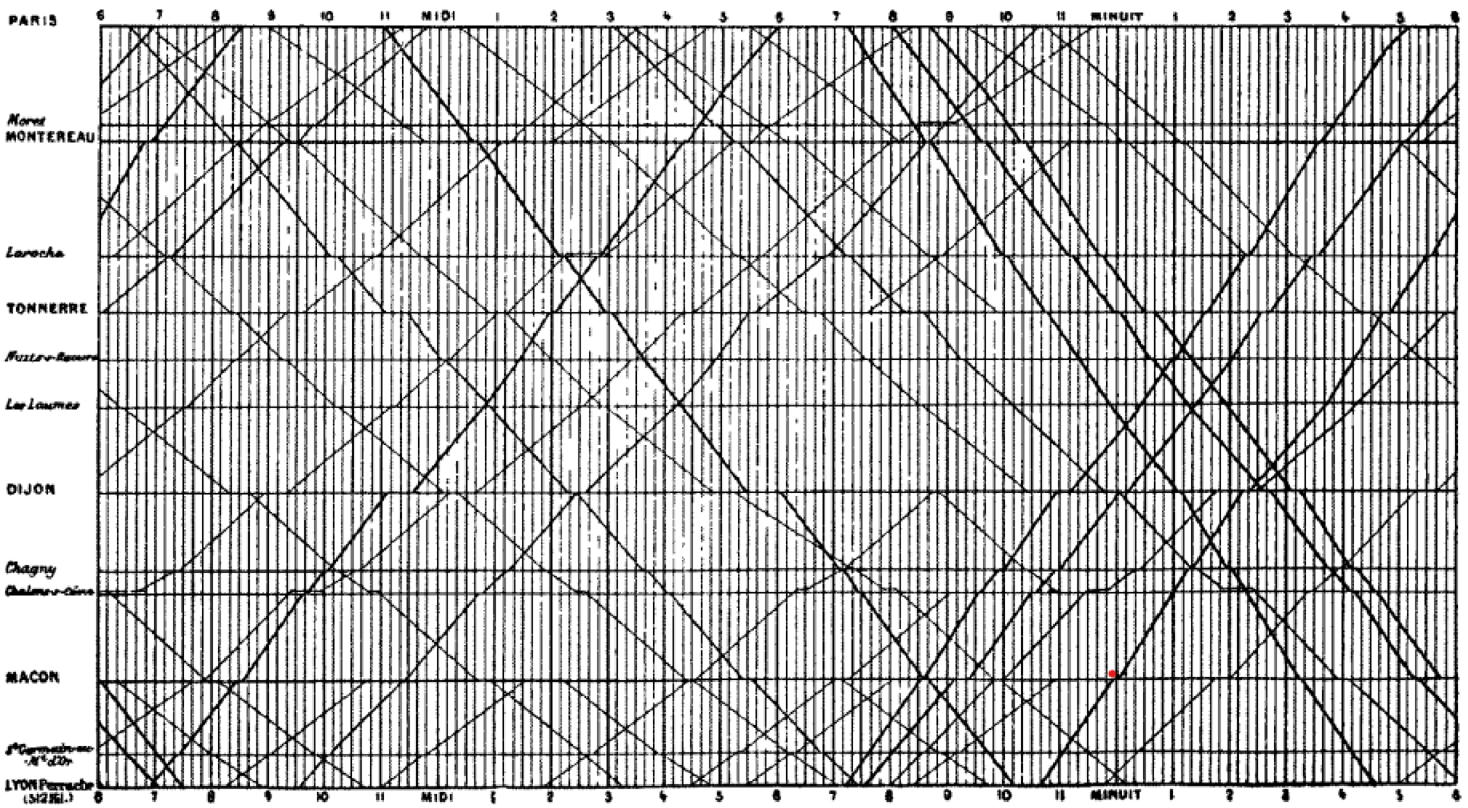
Physics 4

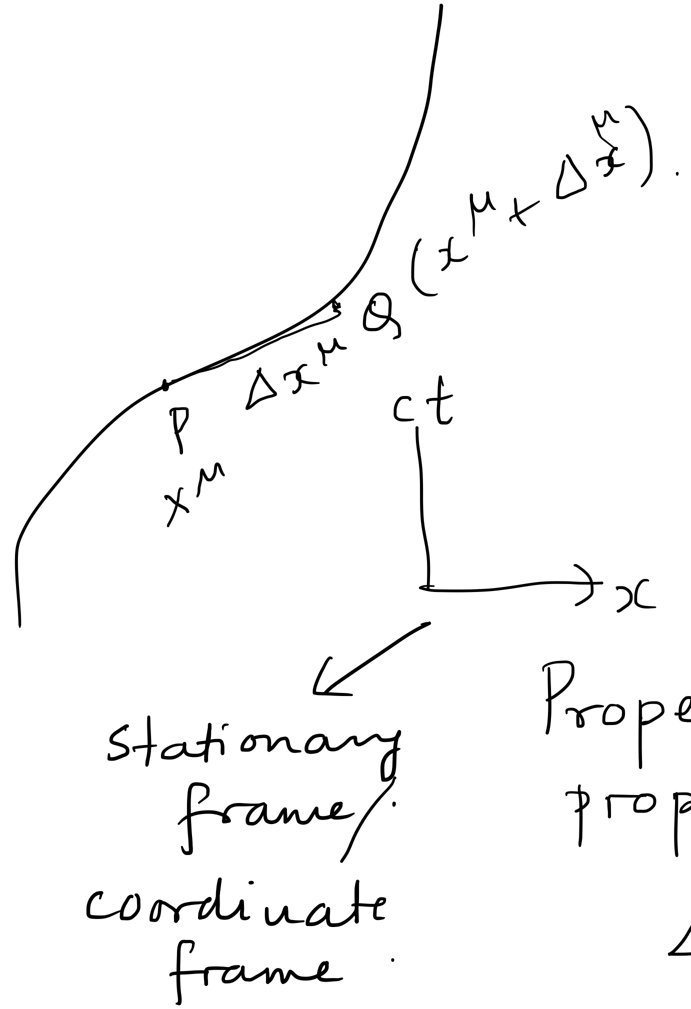
Lecture 16-17

BOX 4.2 Railway Trains in Spacetime



Spacetime diagrams were in use before the advent of special relativity, as this timetable for the railway trains on the Paris–Lyon line reproduced from Marey (1885) shows. Unfortunately the designer of the timetable did not anticipate the convention of relativity and plotted time horizontally. The world lines of the stations (at rest) are horizontal lines. The slanting lines are trains of various speeds moving in between stations and halting at them. Faster trains have steeper slopes, but the time axis is measured in hours, so the 45° lines are not at the speed of light. Rotate the diagram by 90° to view it with the conventions of special relativity.

Proper velocity and momentum



world line of particle is timelike.
Consider world line in spacetime with coordinates x^μ

P, Q on world line close together

can define a small displacement four vector Δx^μ

Proper length of Δx^μ, is in fact the proper time. $\Delta \tau$

$$\Delta \tau = \sqrt{\Delta x_\mu \Delta x^\mu}$$

invariant.

Since $\Delta\tau$ is invariant, the set of qties

$$\frac{\Delta x^\mu}{\Delta\tau}$$ is a four vector.

define <u>proper velocity</u> $u^\mu = \lim_{\Delta\tau \to 0} \frac{\Delta x^\mu}{\Delta\tau}$.

$$u^\mu = \frac{dx^\mu}{d\tau}.$$

We can calculate the proper velocity in terms of the Newtonian velocity.

$$\frac{d\vec{x}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = (v_x, v_y, v_z).$$

for some path $x^\mu \to (ct, x, y, z)$

$$x^\mu = (ct, x, y, z)$$

$$u^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}\,\frac{dt}{d\tau} \rightarrow \frac{dt}{d\tau}\left(c, v_x, v_y, v_z\right).$$

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2.$$

$$\left(\frac{d\tau}{dt}\right)^2 = 1 - \frac{v^2}{c^2}.$$

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1 - v^2/c^2}} = \gamma.$$

$$\boxed{u^\mu = (\gamma c, \gamma \vec{v})} = (\gamma c, \gamma v_x, \gamma v_y, \gamma v_z).$$

$u^\mu \Rightarrow$ tangent to the world line

$$u^\mu u_\mu = \frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau} = \frac{dx^\mu dx_\mu}{d\tau^2} = \frac{c^2 d\tau^2}{d\tau^2} = c^2$$

$$\boxed{u^\mu u_\mu = c^2}$$

In units where $c = 1$

$u^\mu u_\mu = 1 \longrightarrow$ unit tangent to the world line.

This is analogous to the unit spatial tangent

$$\vec{u} = \frac{d\vec{x}}{ds} = \frac{d\vec{x}}{dt}\frac{dt}{ds} = \frac{1}{v}\frac{d\vec{x}}{dt}.$$

to the curve $\vec{x} = \vec{x}(s)$

Analogous to the three momentum $\vec{p} = m\vec{v}$

the proper 4 momentum

$$p^{\mu} = m u^{\mu} .$$

m → mass of the particle
scalar invariant.

components of 4 mom in stationary frame.

$$p^{\mu} = (m\gamma c, m\gamma v_x, m\gamma v_y, m\gamma v_z).$$

the time component becomes.

$$p^0 = \frac{mc}{\sqrt{1 - v^2/c^2}} .$$

We notice that if we multiply both sides by c, we get a qty with dimensions of energy.

$$[P^0 c] = [mc^2] = \text{mass} \times \text{speed}^2 .$$

$$\frac{1}{\sqrt{1-v^2/c^2}} \simeq 1 + \frac{1}{2} v^2/c^2 + O\left(\frac{v^4}{c^4}\right) .$$

$$P^0 c = \gamma m c^2 \simeq mc^2\left(1 + \frac{1}{2} v^2/c^2\right) .$$

$$= \underbrace{mc^2} + \frac{1}{2} m v^2 .$$

which apart from a const. is the K.E

rest energy.

Distinguish two frames.

particle frame, we really mean the 'momentarily co-moving reference frame'.

In this frame, we have

Position $x'^{\mu} \longrightarrow (c\tau, 0, 0, 0)$.

velocity $v'^{\mu} \longrightarrow (c, 0, 0, 0)$.

Momentum $p'^{\mu} \longrightarrow (mc, 0, 0, 0)$.

We shall refer to the coordinate (unprimed) frame relative to which the particle moves at a Newtonian velocity $\vec{v} = (v_x, v_y, v_z)$ as the stationary frame

Positon $x^\mu \rightarrow (ct, x, y, z)$

Velocity $u^\mu = \frac{dx^\mu}{d\tau} \rightarrow (\gamma c, \gamma v_x, \gamma v_y, \gamma v_z)$.

Momentum $P^\mu = m u^\mu = (m\gamma c, m\gamma v_x, m\gamma v_y, m\gamma v_z)$

where $\gamma = \frac{dt}{d\tau} = \frac{1}{\sqrt{1 - v^2/c^2}}$ is related to

the "instantous speed" $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$

of the particle as measured in the stationary.

In either frame,

$$u^\mu u_\mu = u'^\mu u'_\mu = c^2 .$$

Proper Acceleration

$$a^\mu = \frac{du^\mu}{d\tau} .$$

We can show that the proper velocity is orthogonal to the proper acceleration.

$$u^\mu u_\mu = c^2 .$$

$$\frac{d}{d\tau}\left(u^\mu \cdot u_\mu\right) = 0$$

$$\frac{du^\mu}{d\tau} \cdot u_\mu = 0 \implies \boxed{a^\mu u_\mu = 0}$$

In the particle frame,

$$u'^{\mu} = (c, 0, 0, 0).$$

The only way that acceleration can be orthogonal to velocity in this frame is that

$$a'^{\mu} \longrightarrow (0, a'^1, a'^2, a'^3)$$

In the stationary frame, time comp of a^{μ}.

$$a^0 = c\frac{d\gamma}{d\tau} = c\frac{d\gamma}{dv}\frac{dv}{d\tau} = c\gamma^3\frac{v}{c^2}\cdot\frac{dv}{dt}\cdot\frac{dt}{d\tau}.$$

$$\frac{d\gamma}{dv} =$$

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$

$$= \frac{v}{c}\gamma^3\gamma\frac{dv}{dt} = \gamma^4\frac{v}{c}\cdot\frac{dv}{dt}.$$

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

$$a = \sqrt{\left(\frac{dv_x}{dt}\right)^2 + \left(\frac{dv_y}{dt}\right)^2 + \left(\frac{dv_z}{dt}\right)^2}$$

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$!$

x-component

$$a^1 = \frac{du^1}{d\tau}.$$

$$= \frac{d(\gamma v_x)}{d\tau}.$$

$$= \frac{d\gamma}{d\tau} v_x + \gamma \frac{dv_x}{d\tau}.$$

$$= \frac{\gamma^4 v}{c^2} v_x \frac{dv}{dt} + \gamma^2 \frac{dv_x}{dt}.$$

$$= \gamma^2 \left(\gamma^2 \frac{v v_x}{c^2} \frac{dv}{dt} + a_x \right).$$

$$a^\mu = \gamma^2 \left(\gamma^2 \frac{v}{c} \frac{dv}{dt}, \; \gamma^2 \frac{v \vec{V}}{c^2} \frac{dv}{dt} + \vec{a} \right)$$

Newtonian limit

$\frac{v}{c} \to 0$.

$a^0 = 0$.

$a^i = \vec{a}$

$\vec{a} \to$ Newtonian 3 accln.

Consider a simple case, where motion is restricted to x-axis

$$v_x = v, \quad a_x = a, \quad v_y = v_z = a_y = a_z = 0.$$

$$a^x = \gamma^2 \left(\gamma^2 \frac{v v_x}{c^2} \frac{dv}{dt} + a_x \right).$$

$$= \gamma^2 \underbrace{\left(\gamma^2 \frac{v^2}{c^2} + 1 \right)} a.$$

$$= \gamma^2 \left(\frac{1}{(1 - v^2/c^2)} \times v^2/c^2 + 1 \right) a.$$

$$= \gamma^2 \frac{1}{(1 - v^2/c^2)} \cdot a = \gamma^4 a.$$

p^μ : momentum 4-vector.

↳ often referred to as energy-mom 4 vector.

$$p^\mu = \left(\frac{E}{c}, \vec{p}\right) = (m\gamma c, m\gamma \vec{v})$$

particle frame $p^\mu = (mc, 0, 0, 0)$.

Force 4-vector

$$F^\mu = \frac{dp^\mu}{d\tau} = \gamma\left(\frac{1}{c}\frac{dE}{dt}, \frac{d\vec{p}}{dt}\right)$$

$$= \gamma\left(\frac{1}{c}\frac{dE}{dt}, \vec{f}\right)$$

→ Newtonian 3force

$$\vec{f} = \frac{d}{dt}(\gamma m \vec{v}).$$

$$\boxed{F^\mu = \frac{d}{d\tau}(m u^\mu) = m a^\mu}$$