DIFFERENTIAL TOPOLOGY - LECTURE 1

1. Introduction

The subject of differential topology, as the name suggests, is the study of spaces using the methods of calculus. This entails understanding which properties of spaces are invariant under a suitable notion of equivalence (diffeomorphism). Often, a tremendous amount of algebraic topology is involved in understanding (and defining) the invariants and finding solutions to problems in differential topology.

A class of objects that yield to methods of calculus are the so-called (smooth) manifolds. Examples of such objects first arose as solutions to equations, spaces with group actions and their quotients, etc. Although a formal definition of a topological space was only given around 1906 (Frechet) and 1914 (Hausdorff), mathematicians worked with an intuitive understanding of what a smooth manifold meant. The now accepted definition of a smooth manifold took some time to take shape. The reader interested in this history can have a look at the following wonderful references :

(1) http://www.quantum-gravitation.de/media/3a2a81c0493b7f72ffff8061ffffff0.pdf
(2) The history of algebraic and differential topology, by J Dieudonne.
(3) Algebraic Topology, by S Lefschetz. Here, 9 different definitions of a manifold appear.
(4) Foundations of Differential geometry, by Veblen and Whitehead. This more or less gave what is now the standard definition of a smooth manifold.

The first two references[1] above takes the reader through an interesting journey of the initial development of the subject of differential topology.

Our plan is to study some basic properties of manifolds inside the euclidean space. There will be some exercises at the end of each note. It is important to work these out or at the very least be familiar with the statements. These will be used in the sequel, often implicitly. The topics and material are mostly borrowed from the wonderful book *Differential Topology* by V Guillemin and A Pollack which we shall closely follow.

2. Review of Calculus of several variables

As we shall be dealing with several variable calculus throughout the course, we shall begin by recalling the basic definitions from several variable calculus and fixing notations. We shall state some basic theorems (without proofs).

As usual $\mathbb{R}^n$ will denote the euclidean n-space and the functions

$$x_i : \mathbb{R}^n \longrightarrow \mathbb{R}$$

$1 \leq i \leq n$, defined by

$$x_i(a_1, \ldots, a_n) = a_i$$

will be called the i-th coordinate function on $\mathbb{R}^n$. $\mathbb{S}^n$ is the unit sphere of length 1 vectors in $\mathbb{R}^{n+1}$ and $\mathbb{D}^n$ (sometimes $\mathbb{B}^n$) the unit ball of vectors of length at most 1 in $\mathbb{R}^n$.

[1] The first reference above appears in the book "History of Topology" edited by I. M. James.

Suppose $f : U \underset{\text{open}}{\subseteq} \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a function and $x \in U$. Given a vector $v \in \mathbb{R}^n$ the change in the function f along the line segment from x to $x+v$ may be measured by the limit

$$\lim_{h\to 0} \frac{f(x+hv)-f(x)}{h}.$$

Observe that for small h, $x + hv \in U$. If the above limit exists it is denoted by

$$f'(x;v) = \lim_{h\to 0} \frac{f(x+hv)-f(x)}{h}$$

and is called the *directional derivative* of f at x in the direction v. Note that $f'(x;v) \in \mathbb{R}^m$ (if it exists). The directional derivative can therefore be thought of as a (partial) function

$$f'(x;-) : \mathbb{R}^n \longrightarrow \mathbb{R}^m. \tag{2.0.1}$$

The above function is defined on a (non-empty) subset of $\mathbb{R}^n$ and may not be globally defined. Note that the zero vector 0 is always in the domain of definition of $f'(x;\,-\,)$.

A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ may be written in coordinates as $f = (f_1, \ldots, f_m)$ where $f_i = x_i \circ f$. We may now write

$$\lim_{h\to 0} \frac{f(x+hv)-f(x)}{h} = \lim_{h\to 0} \left(\frac{f_1(x+hv)-f_1(x)}{h}, \ldots, \frac{f_m(x+hv)-f_m(x)}{h} \right)$$

Thus the directional derivative $f'(x;v)$ exists if and only if each $f_i'(x;v)$ $(i = 1, \ldots, m)$ exists and in this case we have

$$f'(x;v) = (f_1'(x;v), \ldots, f_m'(x;v)). \tag{2.0.2}$$

If $m = 1$, then the directional derivative $f'(x;e_i)$ is traditionally denoted[2] by either of

$$\left.\frac{\partial f}{\partial x_i}\right|_x; \qquad \frac{\partial f}{\partial x_i}(x),$$

and is called the *partial derivative* of f with respect to the coordinate function x_i. Here e_i is the standard i-th basis vector of $\mathbb{R}^n$.

As we noted above, the directional derivative function is usually only defined on a subset of $\mathbb{R}^n$. If the function f is differentiable, then the directional derivative is globally defined.

We now recall the definition of a differentiable function. As before, let $f : U \underset{\text{open}}{\subseteq} \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function and $x \in U$. We say that f is differentiable at x if there exists a $\mathbb{R}$-linear transformation

$$T_x : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

such that

$$f(x+v) - f(x) - T_x(v) = ||v|| E_x(v) \tag{2.0.3}$$

where E_x is a function defined in a neighborhood of 0 and $E_x(v) \to 0$ as $v \to 0$. Thus we have

$$\lim_{v\to 0} \frac{f(x+v)-f(x)-T_x(v)}{||v||} = 0.$$

Equation (2.0.3) is called a first order Taylor's formula for f. The linear transformation T_x is called the (in anticipation of its uniqueness) derivative of f at x and henceforth will be denoted by df_x.

Now assume that f is differentiable at x and let $u \in \mathbb{R}^n$. Then setting $v = hu$ we have

$$f(x+hu) - f(x) = hT_x(u) + |h| ||v|| E_x(hu).$$

[2]The notation ∂ for the partial derivative goes back to Marquis de Condorcet, about 1770.

Hence, dividing by h and letting $h \to 0$ we have

$$T_x(u) = f'(x; u).$$

Thus, if f is differentiable at x, then directional derivative $f'(x; u)$ exists for all $u \in \mathbb{R}^n$ and

$$f'(x; u) = T_x(u).$$

This in particular proves that the derivative $T_x = df_x$ is unique. It is now easy to describe the derivative df_x which is a linear map

$$df_x : \mathbb{R}^n \longrightarrow \mathbb{R}^m.$$

The matrix of df_x can be computed by understanding the action of df_x on the standard basis vectors $e_j \in \mathbb{R}^n$. We note that, by equation (2.0.2), we have

$$\begin{aligned} df_x(e_j) = f'(x; e_j) &= \left(\tfrac{\partial f_1(x)}{\partial x_j}, \ldots, \tfrac{\partial f_m(x)}{\partial x_j}\right) \\ &= \textstyle\sum_1^m \tfrac{\partial f_i(x)}{\partial x_j} e_i \end{aligned}$$

Hence the matrix of df_x in the standard bases is

$$Jf(x) = \left(\frac{\partial f_i(x)}{\partial x_j}\right).$$

and is called the *Jacobian matrix* of f at x.

The *chain rule* for the derivative of the composition of two differentiable maps $f : U \underset{\text{open}}{\subseteq} \mathbb{R}^n \longrightarrow V \underset{\text{open}}{\subseteq} \mathbb{R}^m$ and $g : V \underset{\text{open}}{\subseteq} \mathbb{R}^m \longrightarrow \mathbb{R}^\ell$ states that

$$d(g \circ f)_x = dg_{f(x)} \circ df_x$$

and in terms of the Jacobian matrices we have

$$J(g \circ f)(x) = Jg(f(x))Jf(x)$$

where the product on the right is a matrix product. It is therefore implicit (in the statement of the chain rule) that the composition of two differentiable functions is differentiable.

A function $f : U \underset{\text{open}}{\subseteq} \mathbb{R}^n \longrightarrow \mathbb{R}^1$ is said to be *smooth* (or sometimes C^∞) if the iterated partial derivatives

$$\frac{\partial^k f(x)}{\partial x_{i_1} \cdots \partial x_{i_k}}$$

exist and are continuous for all $k = 1, 2, 3, \ldots$. A function $f : U \underset{\text{open}}{\subseteq} \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be smooth if each component function f_i of f is smooth. Throughout these notes all functions will be assumed to be smooth.

Remark 2.1. Let $f : U \underset{\text{open}}{\subseteq} \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a a function. If f is continuous we say that f is a C^0 function on U. If f is differentiable, then df_x exists for each $x \in U$ and

$$df_x : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

is a linear map. This gives rise to a function

$$Df : U \longrightarrow \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m).$$

defined by $x \mapsto df_x$. The continuity of Df is equivalent to the continuity of each partial derivative

$$\frac{\partial f_i}{\partial x_j}$$

$1 \le i \le m, 1 \le j \le n$. The function f is said to be C^1 if Df exists and is continuous. If the function Df is differentiable we may then talk of $D(Df) = D^2 f$ which is now a map

$$D(Df) : U \longrightarrow \mathrm{Hom}(\mathbb{R}^n, \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)).$$

A moments thought tells us that the coordinate functions of $D^2 f$ are the mixed (second order) partial derivatives

$$\frac{\partial^2 f_i}{\partial x_j^2}.$$

Thus $D^2 f$ is continuous if and only if the second order partial derivatives of f are continuous. We say f is of class C^2 on U if $D^2 f$ exists and is continuous. Iterating this process, we say that f is of class C^k on U if $D^k f$ exists and is continuous. Finally, the function f is of class C^∞ (or smooth) on U if it is of class C^k for each $k = 1, 2, \ldots$. This is easily seen to be equivalent to the definition of smoothness given above.

A smooth function $f : U \underset{\text{open}}{\subseteq} \mathbb{R}^n \longrightarrow V \underset{\text{open}}{\subseteq} \mathbb{R}^n$ is said to be a *diffeomorphism* if f is $1-1$, onto and f^{-1} is smooth. In this case we say that U and V are *diffeomorphic*. Observe that, by the chain rule, df_x is a linear isomorphism for each $x \in U$.

A fundamental result which will be used often is the Inverse function theorem. We state it without proof.

Theorem 2.2. (Inverse function theorem) Let $f : U \underset{\text{open}}{\subseteq} \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a smooth function. Assume that df_x is invertible as a linear transformation (equivalently, the Jacobian matrix $J(f(x)$ is invertible), then there exist open sets $U_1 \subset U$ and $V_1 \subset f(U)$ with $x \in U_1$ such that

$$f/U_1 : U_1 \longrightarrow V_1$$

is a diffeomorphism (onto V_1). $\square$

The importance of this theorem cannot be overstated. This theorem transforms alegebraic information of the derivative (invertibility of df_x) into a topological/analytical conclusion, namely, that f is $1-1$ in a neighborhood of x and the smoothness of the inverse (restricted to that neighborhood). We shall have occasion to understand what conclusions can be drawn from other (algebraic) restriction(s) on the derivative.

At this point we recall the definition of a local diffeomorphism. Given a smooth function $f : U \underset{\text{open}}{\subseteq} \mathbb{R}^n \longrightarrow \mathbb{R}^n$ we say that f is a *local diffeomorphism* at x if f maps a neighborhood of x diffeomorphically onto a neighborhood of $f(x)$. The map f itself will be called a local diffeomorphism if it is a local diffeomorphism at each $x \in U$. With this notion, the Inverse function theorem simply states that if df_x is invertible, then f is a local diffeomorphism at x. Observe that every diffeomorphism is a local diffeomorphism.

One now extends the definition of smoothness of maps to maps that are defined on subsets[3] that are not necessarily open. Given $X \subseteq \mathbb{R}^N$ and a function

$$f : X \longrightarrow \mathbb{R}^m$$

we say that f is smooth if for every $x \in X$ there exists an open set $U \subseteq \mathbb{R}^N$, $x \in U$, and a smooth function

$$F : U \longrightarrow \mathbb{R}^m$$

[3]When one talks of smooth maps between subsets of euclidean spaces it is implicitly assumed that the subsets have the subspace topology.

with $F/(U \cap X) = f$. That is, f must locally be the restriction of a smooth function defined on an open set in $\mathbb{R}^N$. As an example, the function

$$f : S^1 \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$$

given by $f(x, y) = x$ is smooth. It is the restriction of the projection which is smooth.
A function

$$f : X \subseteq \mathbb{R}^N \longrightarrow Y \subseteq \mathbb{R}^M$$

is said to be a diffeomorphism if f is bijective, smooth with f^{-1} also smooth. The subsets X and Y are then said to be *diffeomorphic*. The notion of a local diffeomorphism between subsets of euclidean spaces may be similarly defined.

Example 2.3. Suppose that we have composable smooth functions

$$X \subseteq \mathbb{R}^N \xrightarrow{f} Y \subseteq \mathbb{R}^M \xrightarrow{g} Z \subseteq \mathbb{R}^L$$

defined on subsets of euclidean spaces. Then the composition $g \circ f$ is also smooth. To see this, given $x \in X$ find open sets U and V about x and $f(x)$ respectively and smooth functions

$$F : U \longrightarrow \mathbb{R}^M; \quad G : V \longrightarrow \mathbb{R}^L$$

that extend f and g respectively. We may assume that $F(U) \subseteq V$. Then clearly

$$(G \circ F)/(U \cap X) = g \circ f$$

and hence $g \circ f$ is smooth.

Thus the composition of two smooth functions is smooth. Here is another example.

Example 2.4. Let $f : X \subseteq \mathbb{R}^N \longrightarrow \mathbb{R}^M$ be a smooth function. The set

$$Z = \text{graph}(f) = \{(x, f(x)) \ : \ x \in X\} \subseteq \mathbb{R}^N \times \mathbb{R}^M$$

is called the graph of f. We claim that the function

$$F : X \longrightarrow Z$$

defined by $F(x) = (x, f(x))$ is a diffeomorphism. It is clear that F is bijective. We shall check that both F and F^{-1} are smooth. The smoothness of F can be checked as follows. We first observe that the diagonal map

$$\triangle : X \longrightarrow X \times X; \quad \triangle(x) = (x, x)$$

is smooth. This is because it is the restriction of the diagonal map of the euclidean space which we know is smooth. Next suppose we have smooth maps $h_i : X_i \longrightarrow Y_i$, $i = 1, 2$ between subsets of euclidean spaces. We claim that the map

$$h_1 \times h_2 : X_1 \times X_2 \longrightarrow Y_1 \times Y_2$$

defined by

$$h_1 \times h_2(x, y) = (h_1(x), h_2(y))$$

is smooth. This is because the map $h_1 \times h_2$ is locally the restriction of product of two smooth functions defined on open subsets of the euclidean space (use the definition of smoothness of h_1 and h_2). Finally, F is the composition

$$F = (id \times f) \circ \triangle : X \longrightarrow X \times X \longrightarrow Z$$

of two smooth functions and hence is smooth. The inverse function F^{-1} is evidently just the projection to the first factor and hence is smooth. Thus F is indeed a diffeomorphism.

Conventions. Throughout our course of discussions a function/map will always mean a smooth function/map. There should be no confusion about this.

Here are some exercises.

Exercise 2.5. Picturise the set

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 \,:\, \Sigma_i x_i^2 = 1, \Sigma x_i = 0\}.$$

Exercise 2.6. Let $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be the function

$$g(x) = ||x|| = \sqrt{x_1^2 + \cdots + x_n^2}.$$

Discuss the differentiability of the functions $g(x)$ and $h(x) = ||x||^2$. Compute dg_p and dh_p at points p where they are defined.

Exercise 2.7. Consider the function

$$f(x, y) = \begin{cases} 0 & y \le 0 \text{ or } y \ge x^2 \\ \left[\frac{y}{x^2}\left(1 - \frac{y}{x^2}\right)\right]^2 & 0 < y < x^2 \end{cases}$$

Show that f is continuous at all points other than $(0, 0)$. Show that $f'((0, 0); v)$ exists for all $v \in \mathbb{R}^2$.

Exercise 2.8. Show that the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \le 0 \end{cases}$$

is smooth.

Exercise 2.9. Construct a diffeomorphism $f : \mathrm{Int}(\mathbb{D}^n) \longrightarrow \mathbb{R}^n$. This raises several questions. The most general question would be : what are the open subsets of $\mathbb{R}^n$ that are diffeomorphic to $\mathbb{R}^n$. One could also ask if two open subsets of $\mathbb{R}^n$ are homeomorphic, then are they are diffeomorphic? The answers to these questions involves understanding some beautiful and deep mathematics. One can show that certain well behaved open subsets of $\mathbb{R}^n$ are indeed diffeomorphic to $\mathbb{R}^n$. For example, it is a theorem that every open star shaped subset of $\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$. In particular, every convex open subset of $\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$.

Exercise 2.10. Let $\triangle^n$ and A_n be the sets

$$\triangle^n = \{(t_1, \ldots, t_{n+1}) \,:\, \Sigma t_i = 1, t_i \ge 0\}$$
$$A_n = \{(x_1, \ldots, x_n) \,:\, 0 \le x_1 \le x_2 \le \cdots \le x_n \le 1\}.$$

Construct a diffeomorphism $f : A_n \longrightarrow \triangle^n$. It might be helpful to draw the spaces for $n = 1, 2$. $\triangle^n$ is called the standard n-simplex.

Exercise 2.11. Let $f : U \longrightarrow V$ be a diffeomorphism where U is open in $\mathbb{R}^n$ and V open in $\mathbb{R}^m$. Show that $n = m$. Give an example to show that the conclusion is not true if the "open" condition is dropped.

Exercise 2.12. If $k \le \ell$, we can consider $\mathbb{R}^k$ to be the subset $\{(a_1, \ldots, a_k, 0, \ldots, 0)\}$ in $\mathbb{R}^\ell$. Show that smooth functions on $\mathbb{R}^k$ considered as a subset of $\mathbb{R}^\ell$ are the same as usual.

Exercise 2.13. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a smooth map and

$$U = \{p \in \mathbb{R}^n \,:\, df_p \text{ is invertible}\}.$$

Show that U is open.

Exercise 2.14. Suppose that $Z \subseteq X \subseteq \mathbb{R}^N$. Show that the restriction to Z of any smooth function on X is a smooth function on Z.

Exercise 2.15. Show that if $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is a local diffeomorphism, then $m = n$. Thus $\mathbb{R}^m$ and $\mathbb{R}^n$ are diffeomorphic if and only if $m = n$. This statement with diffeomorphism replaced by homeomorphism is harder to prove.

Exercise 2.16. Define the notion of a local diffeomorphism between two subsets of euclidean spaces. Show that a local diffeomorphism is an open map.

Exercise 2.17. Show that a smooth map $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a local diffeomorphism if and only if it is a diffeomorphism onto its image. Is this true for a smooth map $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$?

Exercise 2.18. Let

$$U = \mathbb{S}^n - (0, \ldots, 0, 1) \quad V = \mathbb{S}^n - (0, \ldots, 0, -1)$$

and

$$\varphi(x_1, , \ldots, x_n, x_{n+1}) = \frac{(x_1, \ldots, x_n)}{1 - x_{n+1}}; \quad \psi(x_1, , \ldots, x_n, x_{n+1}) = \frac{(x_1, \ldots, x_n)}{1 + x_{n+1}}$$

Check that φ and ψ are diffeomorphisms onto $\mathbb{R}^n$. They are called the *stereographic projections.*

Exercise 2.19. Let X denote the subspace

$$X = \{[0,1] \times \{y\} \cup \{x\} \times [0,1] \, : \, x, y \in \{0,1\}\}$$

of $\mathbb{R}^2$. Show that no neighborhood of $(0,0) \in X$ is diffeomorphic to an open set in $\mathbb{R}$.

Exercise 2.20. Let $U \subseteq \mathbb{R}^n$ be an open set. Fix a point $x \in U$ and a vector $v \in \mathbb{R}^n$. We can now construct a function

$$v(x; -) : \text{smooth in a neighborhood of } x \longrightarrow \mathbb{R} \tag{2.20.1}$$

by setting

$$v(x; f) = f'(x; v)$$

where f is a smooth (real valued) function defined in a neighborhood of x. The domain of $v(x; -)$ consists of the set of smooth functions that are defined in some (varying) neighborhood of x. This gives another way of looking at the directional derivative function in (2.0.1) where the function f was fixed and v was varying. Thus the vectors $v \in \mathbb{R}^n$ "operate" on smooth functions defined in a neighborhood of a point to produce real numbers. Check the following.

(1) If f, g are two smooth functions that are defined in some (possibly different) neighborhood of x and such that they agree in a (smaller) neighborhood of x, then

$$v(x; f) = v(x; g)$$

for all $v \in \mathbb{R}^n$.

(2) If $v, u \in \mathbb{R}^n$, then

$$(v + u)(x; f) = v(x; f) + u(x; f), \quad (av)(x; f) = a \cdot v(x; f)$$
$$v(x; f + ag) = v(x; f) + a \cdot v(x; g)$$
$$v(x; f \cdot g) = f(x) \cdot v(x; g) + g(x) \cdot v(x; f)$$

for all $a \in \mathbb{R}$ and all smooth functions f, g defined in a neighborhood of x. The domains of the various functions involved are suitably chosen. The last property above is called the *derivation* property of $v(x; -)$.

DIFFERENTIAL TOPOLOGY - LECTURE 2

1. Introduction

Here we discuss the definition of a manifold and look at some examples. Manifolds have been around for some time now, although the right definition took some time to arrive at. Manifolds can be (and are) studied basically in two ways, one in the abstract setting and in the second case they are studied as certain subsets of some $\mathbb{R}^N$. Both the methods are essentially the same.

The book G and P adopts the second viewpoint and studies manifolds as subsets of $\mathbb{R}^N$. Manifolds are subspaces of $\mathbb{R}^N$ that locally are diffeomorphic to open subsets of (a fixed) $\mathbb{R}^k$. Because of this the methods of calculus (more generally most local phenomena) in the euclidean space can be transferred to the world of manifolds.

2. Definition of a manifold

Recall that for $X \subseteq \mathbb{R}^N$ a function $f : X \longrightarrow \mathbb{R}^m$ is said to be smooth if for each $x \in X$ there is a neighborhood $x \in U \subseteq \mathbb{R}^N$ and a smooth function $F : U \longrightarrow \mathbb{R}^m$ such that

$$F/(U \cap X) = f.$$

Definition 2.1. A subset $X \subseteq \mathbb{R}^N$ is said to be a k-dimensional manifold (or simply a k-manifold) if each $x \in X$ has a neighborhood V (V open in X) and a diffeomorphism

$$\varphi : U \subseteq \mathbb{R}^k \longrightarrow V$$

where U is open in $\mathbb{R}^k$. φ is called a *parametrization* of the open set V. We often (forget to mention V and) say that $\varphi : U \longrightarrow X$ is a *local parametrization about* x.

The inverse diffeomorphism

$$\varphi^{-1} : V \longrightarrow U \subseteq \mathbb{R}^k$$

is called a *coordinate system* (or a *chart*) on U. The reason being that if we write φ^{-1} in coordinates, that is,

$$\varphi^{-1} = (x_1, \ldots, x_k)$$

then φ provides a way of associating coordinates to points in V. For example, we may think of a point $p \in V$ as having coordinates $(x_1(p), \ldots, x_k(p))$[1]. The real valued functions x_i are called the *coordinate functions* on V.

If X is a k-manifold, then k is the dimension of X and we write $\dim(X) = k$. One can make wise choices of the parametrizing open set U. For example, we may choose U to be either the open unit ball centered at the origin or the whole of $\mathbb{R}^k$ and ensure that $\varphi(0) = x$.

The parametrizations allow us to transfer local data back and forth between the manifold and the euclidean space. Note that a subspace $X \subseteq \mathbb{R}^N$ is a k-manifold if and only if there is a covering of X by open sets each of which is diffeomorphic to an open subset of $\mathbb{R}^k$.

Here are some examples.

[1] This is what we usually do in the euclidean spaces where there are global coordinates.

Example 2.2. $\mathbb{R}^k$ is a k-manifold. Every open subset U of a k-manifold X is a k-manifold. The checking of these facts is left as an exercise.

Example 2.3. The product $X \times Y$ of two manifolds is again a manifold. To see this, suppose that $X \subseteq \mathbb{R}^N$ is a k-manifold and $Y \subseteq \mathbb{R}^M$ is a ℓ-manifold. Fix parametrizations

$$\varphi : U \subseteq \mathbb{R}^k \longrightarrow V \subseteq X$$
$$\psi : U' \subseteq \mathbb{R}^\ell \longrightarrow V' \subseteq Y$$

about $p \in X$ and $q \in Y$ respectively. Here U, U', V, V' are open in the respective ambient spaces. We claim that the map

$$\varphi \times \psi : U \times U' \longrightarrow V \times V'$$

defined by

$$(\varphi \times \psi)(x, y) = (\varphi(x), \psi(y))$$

is a parametrization about $(p, q) \in X \times Y$ showing that $X \times Y$ is a $(k + \ell)$-manifold. By (Example 2.2, Lecture 1) we know that $\varphi \times \psi$ is smooth. We only need to check that the inverse of $\varphi \times \psi$ is smooth. We fix open sets $W \subseteq \mathbb{R}^N$, $W' \subseteq \mathbb{R}^M$ about p, q respectively and smooth functions

$$F : W \longrightarrow \mathbb{R}^k; \quad G : W' \longrightarrow \mathbb{R}^\ell$$

such that

$$F/(W \cap V) = \varphi^{-1}; \quad G/(W' \cap V') = \psi^{-1}.$$

We can do this as both φ^{-1} and ψ^{-1} are smooth. It is now clear that the smooth function

$$F \times G : W \times W' \longrightarrow \mathbb{R}^k \times \mathbb{R}^\ell$$

restricts to $\varphi^{-1} \times \psi^{-1}$ on $(W \times W') \cap (V \times V')$ completing the proof that $\varphi^{-1} \times \psi^{-1}$ is smooth.

Example 2.4. The n-sphere

$$\mathbb{S}^n = \{(x_1, \ldots, x_{n+1}) : x_1^2 + \cdots + x_{n+1}^2 = 1\} \subseteq \mathbb{R}^{n+1}$$

is a n-manifold. Let us try to work through the details of this example. Define subsets U_i^+ and U_i^-, $i = 1, 2, \ldots, n+1$ of $\mathbb{S}^n$ by setting

$$U_i^+ = \{(x_1, \ldots, x_{n+1}) \in \mathbb{S}^n \ : \ x_i > 0\}$$
$$U_i^- = \{(x_1, \ldots, x_{n+1}) \in \mathbb{S}^n \ : \ x_i < 0\}.$$

Each of the above set is open (why?) in $\mathbb{S}^n$ and they cover $\mathbb{S}^n$. For each $i = 1, 2, \ldots n+1$, consider the maps

$$\varphi_i^+ : U_i^+ \longrightarrow V$$

where $V = \mathrm{Int}(\mathbb{D}^n)$ is the open unit ball in $\mathbb{R}^n$ defined by.

$$\varphi_i^+(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_{i-1}, \hat{x_i}, \ldots, x_{n+1}).$$

In other words, φ_i^+ drops the i-th coordinate. This map is clearly smooth, as it is a projection and is bijective. We claim that this is a chart on $\mathbb{S}^n$. We need to check that ψ_i^+, the inverse of φ_i^+ is also smooth. Observe that

$$\psi_i^+ : \mathrm{V} \longrightarrow U_i^+$$

is given by

$$\psi_i^+(y_1, \ldots, y_n) = (y_1, \ldots, y_{i-1}, \sqrt{1 - (y_1^2 + \cdots + y_n^2)}, y_i, \ldots, y_n)$$

which is clearly smooth. Thus φ_i^+ is a chart. The maps

$$\varphi_i^- : U_i^- \longrightarrow V$$

which drop the i-th coordinate are also charts for their inverse ψ_i^- is the smooth map

$$\psi_i^-(y_1, \ldots, y_n) = (y_1, \ldots, y_{i-1}, -\sqrt{1-(y_1^2+\cdots+y_n^2)}, y_i, \ldots, y_n).$$

This shows that $\mathbb{S}^n$ is a n-manifold.

Example 2.5. Let V be a k-dimesnsional vector subspace of $\mathbb{R}^N$. Then V is a k-manifold. Here is one way to see this. Fix a basis $v_1, \ldots, v_k$ of V. Consider the linear transformation

$$\varphi : \mathbb{R}^k \longrightarrow \mathbb{R}^N$$

defined by $\varphi(e_i) = v_i$. Since φ is linear, it is smooth. Also, by construction, φ maps $\mathbb{R}^k$ injectively onto its image which is V. We claim that φ is a diffeomorphism onto its image. We need to show that φ^{-1} is smooth. To see this we extend the basis of V to a basis

$$v_1, \ldots, v_k, v_{k+1}, \ldots, v_N$$

of $\mathbb{R}^N$. The linear map $T : \mathbb{R}^N \longrightarrow \mathbb{R}^k$ defined by

$$T(v_i) = \begin{cases} e_i & i \leq k \\ 0 & i > k \end{cases}$$

clearly restricts to φ^{-1} on V showing that φ^{-1} is smooth. Thus φ is a global parametrization for V.

Example 2.6. The set $M_n(\mathbb{R})$ of $(n \times n)$ matrices with real entries can be identified with $\mathbb{R}^{n^2}$ by writing the rows of the matrix one after another to get a point of $\mathbb{R}^{n^2}$. The determinant function

$$\det : M_n(\mathbb{R}) = \mathbb{R}^{n^2} \longrightarrow \mathbb{R}$$

being a polynomial in the coordinates of $\mathbb{R}^{n^2}$ is a smooth function. Then $GL_n(\mathbb{R})$ (which is the set of invertible matrices in $M_n(\mathbb{R})$) is open in $M_n(\mathbb{R})$ being the inverse image $\det^{-1}(\mathbb{R} - 0)$. Hence $GL_n(\mathbb{R}) \subseteq \mathbb{R}^{n^2}$ is a n^2-manifold.

Example 2.7. Let $X \subseteq \mathbb{R}^N$ be a manifold and $f : X \longrightarrow \mathbb{R}$ a smooth function. Then, Z the graph of f

$$Z = \mathrm{graph}(f) = \{(x, f(x)) : x \in X\} \subseteq \mathbb{R}^N \times \mathbb{R}$$

is a manifold of dimension equal to $\dim(X)$. This is because the map

$$F : X \longrightarrow \mathrm{graph}(f)$$

defined by $F(x) = (x, f(x))$ is a diffeomorphism as we have already seen (Example 2.3, Lecture 1). Thus graph(f), being diffeomorphic to the manifold X, is itself a manifold. .

Example 2.8. (Level set of smooth funtions) Let $f : U \longrightarrow \mathbb{R}$ be a smooth function defined on an open set in $\mathbb{R}^{n+1}$. Given $c \in \mathbb{R}$, the subset

$$f^{-1}(c) = \{x \in U \;:\; f(x) = c\}$$

is called the level set of f of height c. If $\nabla f(x) \neq 0$ for all $x \in f^{-1}(c)$, then the level set $f^{-1}(c)$ is a n-manifold. We shall see a proof of this soon.

At this point we know two ways of constructing new manifolds : by taking open subsets of manifolds and taking products of manifolds. We shall see several more ways of constructing manifolds once we develop further techniques. Using these methods, we shall show that many familiar matrix groups inside $GL_n(\mathbb{R})$, for example, are manifolds.

Here is an example of a subset of $\mathbb{R}^2$ that is not a manifold.

Example 2.9. (Compare with Exercise 2.19, Lecture 1) Let X be the subspace of $\mathbb{R}^2$ defined by

$$X = \{[0,1] \times \{y\} \cup \{x\} \times [0,1] \, : \, x, y \in \{0,1\}\}.$$

Thus X is the boundary of the unit square. We claim that no neighborhood of $(0,0)$ in X is diffeomorphic to an open subset of $\mathbb{R}$. We assume that there exists a diffeomorphism of a neighborhood U of $(0,0)$, say,

$$\psi : U \longrightarrow (-1,1)$$

with $\psi((0,0)) = 0$ and derive a contradiction. Let $\varphi : (-1,1) \longrightarrow U$ be the inverse of ψ. In coordinates, let

$$\varphi(t) = (\varphi_1(t), \varphi_2(t)).$$

Then as $\psi \circ \varphi = \mathrm{id}$ we have by the chain rule that

$$\left(\frac{\partial\psi}{\partial x}(0,0), \frac{\partial\psi}{\partial y}(0,0)\right)\left(\begin{array}{c} \varphi_1'(0) \\ \varphi_2'(0) \end{array}\right) = \mathrm{id}.$$

We assume without loss of generality that $\varphi_1'(0) \neq 0$. Since $\varphi_1(0) = 0$ it follows that the image of φ_1 contains an interval $(-\varepsilon, \varepsilon)$ (why?). Thus the image of $\varphi = (\varphi_1, \varphi_2)$ contains points with negative first coordinate. This is a contradiction.

We end this section by defining the notion of a submanifold. If X, Y are manifolds in $\mathbb{R}^N$ and $X \subseteq Y$, then we say that X is a *submanifold* of Y.

Here are some problems. The letters $X, Y, Z, \ldots$ will always denote a manifold.

Exercise 2.10. If X is a k-manifold show that every x has a neighborhood diffeomorphic to whole of $\mathbb{R}^k$.

Exercise 2.11. Prove the claims made in Example 2.2.

Exercise 2.12. Show that the projection $X \times Y \longrightarrow X$ is smooth.

Exercise 2.13. Explicity exhibit enough parametrizations to cover $S^1 \times S^1 \subseteq \mathbb{R}^4$.

Exercise 2.14. Find a subset of $\mathbb{R}^2$ that is diffeomorphic to $S^1 \times \mathbb{R}$. Use this to show that $\mathbb{R}^3$ must contain a subset that is diffeomorphic to $S^1 \times S^1$. Generalize. An interesting point to note here is that since $S^1 \subseteq \mathbb{R}^2$, the product $S^1 \times S^1$ is naturally a subset of $\mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$. The exercise shows that there is a copy of $S^1 \times S^1$ in $\mathbb{R}^3$, one dimension lower. Finding the smallest N such that $\mathbb{R}^N$ contains a copy of a given manifold is an interesting and important question in differential topology.

Exercise 2.15. The torus is the set of points in $\mathbb{R}^3$ at a distance b from the circle of radius a in the xy-plane, $0 < b < a$. Prove that these tori are all diffeomorphic to $S^1 \times S^1$.

Exercise 2.16. Prove that the hyperboloid in $\mathbb{R}^3$ defined by $x^2 + y^2 - z^2 = a$ is a manifold if $a > 0$. Why doesn't $x^2 + y^2 - z^2 = 0$ define a manifold? Note that paraboloids are graphs of quadratic functions, for example, $z = x^2 + y^2$.

Exercise 2.17. Fill in the details in Example 2.9.

DIFFERENTIAL TOPOLOGY - LECTURE 3

1. Introduction

Having defined the notion of a manifold and smooth maps between manifolds, we shall now define the notion of the derivative of a smooth map between manifolds. Towards this, given a k-manifold X, we first associate to each $x \in X$ a (real) vector space $T_x(X)$ called the tangent space to X at x. Given a smooth map $f : X \longrightarrow Y$ between manifolds the derivative of f at $x \in X$ will then be defined to be a certain linear map

$$df_x : T_x(X) \longrightarrow T_{f(x)}(Y).$$

This map we shall see has the usual properties that one would expect.

2. Tangent spaces and derivatives

Recall that a subset $X \subseteq \mathbb{R}^N$ is a k-manifold if each $x \in X$ is contained in an open set V which is diffeomorphic to an open subset $U \subseteq \mathbb{R}^k$. If X is a k-manifold and $x \in V \subseteq X$ is open and

$$\varphi : U \underset{\text{open}}{\subseteq} \mathbb{R}^k \longrightarrow U$$

a diffeomorphism, then φ is called a parametrization about x. Recall that it is possible to choose the above parametrization so that $0 \in U$ and that $\varphi(0) = x$.

We shall now discuss the definition of the derivative of a smooth function between manifolds. Towards defining this we shall first define the notion of *tangent spaces*.

Start with a k-manifold $X \subseteq \mathbb{R}^N$. Let

$$\varphi : U \subseteq \mathbb{R}^k \longrightarrow X$$

be a local parametrization about x such that $0 \in U$ and $\varphi(0) = x$. We think of φ as a (smooth) map

$$\varphi : U \subseteq \mathbb{R}^k \longrightarrow \mathbb{R}^N.$$

Hence $d\varphi_0$, the derivative of φ at 0, is then a linear map

$$d\varphi_0 : \mathbb{R}^k \longrightarrow \mathbb{R}^N.$$

Then $\mathrm{im}(d\varphi_0)$, the image of $d\varphi_0$, is a subspace of $\mathbb{R}^N$. This subspace is called the *tangent space* to X at x.

Definition 2.1. If $X \subseteq \mathbb{R}^N$ is a k-manifold and $x \in X$, then the tangent space $T_x(X)$ to X at x is defined to be the vector space $\mathrm{im}(d\varphi_0) \subseteq \mathbb{R}^N$ where $\varphi : U \subseteq \mathbb{R}^k \longrightarrow X$ is a local parametrization about x with $\varphi(0) = x$. The vectors $v \in \mathbb{R}^N$ such that $v \in T_x(X)$ are called *tangent vectors* to X at x. Notice that, by definition, $T_x(X)$ is a subspace of $\mathbb{R}^N$.

A priori, the definition depends upon the choice of the local parametrization about $x \in X$. We shall see that this is not the case. Indeed, suppose that

$$\varphi : U \underset{\text{open}}{\subseteq} \mathbb{R}^k \longrightarrow X$$

$$\psi : V \underset{\text{open}}{\subseteq} \mathbb{R}^k \longrightarrow X$$

are two local parametrizations about $x \in X$ with $\varphi(0) = x = \psi(0)$. We may arrange (by shrinking the open sets) so that $\varphi(U) = \psi(V)$. Then the composition

$$h = \psi^{-1} \circ \varphi : U \longrightarrow V$$

is a diffeomorphism. Thus,

$$\varphi = \psi \circ h$$

is now a composition of smooth maps smooth maps defined on open subsets of the euclidean space. By the chain rule we have that

$$d\varphi_0 = d\psi_0 \circ dh_0.$$

This essentially shows that $\mathrm{im}(d\varphi_0) = \mathrm{im}(d\psi_0)$ (since dh_0 is an isomorphism). Hence the tangent space $T_x(X)$ is well defined.

Here is an example.

Example 2.2. Consider the n-manifold $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$. The tangent space $T_x(\mathbb{S}^n)$ at $x = (1, 0, \ldots, 0) \in \mathbb{S}^n$ can be described as follows. By definition we need to look at a parametrization

$$\varphi : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{S}^n$$

with $0 \in U$ and $\varphi(0) = x$ about x and then the tangent space $T_x(\mathbb{S}^n)$ is

$$T_x(\mathbb{S}^n) = \mathrm{im}(d\varphi_0).$$

We know that (see Example 2.4, Lecture 2)

$$\varphi : \mathrm{Int}(\mathbb{D}^n) \longrightarrow \mathbb{S}^n$$

defined by

$$\varphi(x_1, \ldots, x_n) = \left(\sqrt{1 - (x_1^2 + x_2^2 + \cdots + x_n^2)}, x_1, \ldots, x_n\right)$$

is a parametrization about $x \in \mathbb{S}^n$ with $\varphi(0) = x$. The Jacobian matrix $d\varphi_0$ can be checked to be the $(n+1) \times n$ matrix

$$J\varphi(0) = d\varphi_0 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ I_n \end{pmatrix}$$

Thus the first row is zero. It is then clear that the tangent space $\mathrm{im}(d\varphi_0) = T_x(\mathbb{S}^n)$ equals the subspace of those vectors $v \in \mathbb{R}^{n+1}$ whose first coordinate is 0. It is important to note that any vector $v \in \mathbb{R}^{n+1}$ with first coordinate 0 is actually a tangent vector to $\mathbb{S}^n$ at x.

Describing the tangent space from first principles, as in the above example, often turns out to be messy. In general writing down parametrizations for a manifold is difficult There are ways to determine the tangent space without using explicit parametrizations. Exercise 2.15 below describes one such way which is very useful in concrete situations. Later, we shall see yet another way of determining the tangent space to a manifold.

The question that we now wish to discuss is : for a k-manifold X, what is the dimension of the tangent space $T_x(X)$ at $x \in X$?

Lemma 2.3. If $X \subseteq \mathbb{R}^N$ is a k-manifold, then $\dim(T_x(X)) = k$ for all $x \in X$.

Proof. Fix a local parametrization

$$\varphi : U \subseteq \mathbb{R}^k \longrightarrow X$$

about $x \in X$. Since φ is a diffeomorphism onto its image, we have that φ^{-1} is smooth. By definition, this means that there exists an open set $W \subseteq \mathbb{R}^N$, $x \in W$ and a smooth map $F : W \longrightarrow \mathbb{R}^k$ with

$$F/(W \cap V) = \varphi^{-1}.$$

Assuming $\varphi(U) \subseteq W$ we have $F \circ \varphi = \mathrm{id}$. Invoking the chain rule we have that

$$dF_x \circ d\varphi_0 = \mathrm{id}.$$

This shows that $d\varphi_0$ is injective. This completes the proof. □

Remark 2.4. The proof of the above lemma shows that if X is a k-manifold and $\varphi : U \subseteq \mathbb{R}^k \longrightarrow X$ is a local parametrizaion about x with $\varphi(0) = x$, then the linear map

$$d\varphi_0 : \mathbb{R}^k \longrightarrow T_x(X) \subseteq \mathbb{R}^N$$

is a linear isomorphism (onto $T_x(X)$). This observation will now help us in defining the derivative of a smooth function between two manifolds. The method of the proof of the above lemma appears in many situation as we shall see. Finally note that if $X \subseteq \mathbb{R}^N$ is a k-manifold, then $k \leq N$.

We now turn to the definition of the derivative of a smooth map between manifolds. Suppose $X \subseteq \mathbb{R}^N$ is a k-manifold and $Y \subseteq \mathbb{R}^M$ is an ℓ-manifold. Let $f : X \longrightarrow Y$ be a smooth function. Let $x \in X$. The derivative df_x of f at x will be defined to be a linear map

$$df_x : T_x(X) \longrightarrow T_{f(x)}(Y)$$

from the tangent space to X at x to the tangent space of Y at $f(x)$. This is done as follows.

Fix parametrizations

$$\varphi : U \subseteq \mathbb{R}^k \longrightarrow X$$
$$\psi : V \subseteq \mathbb{R}^\ell \longrightarrow Y$$

about x and $f(x)$ respectively. We assume that $0 \in U$ and $0 \in V$ with $\varphi(0) = x$ and $\psi(0) = f(x)$. We now let[1]

$$g = \psi^{-1} \circ f \circ \varphi : U \longrightarrow V.$$

Observe that $g(0) = 0$ and that g is smooth and we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\varphi"] \arrow[r, "g=\psi^{-1}\circ f\circ\varphi"'] & V \arrow[u, "\psi"'] \end{tikzcd}$$

Taking derivatives of the two vertical maps and the lower horizontal map we get a diagram

$$\begin{tikzcd} T_x(X) \arrow[r, dashed, "df_x"] & T_{f(x)}Y \\ \mathbb{R}^k \arrow[u, "d\varphi_0"] \arrow[r, "dg_0"'] & \mathbb{R}^\ell \arrow[u, "d\psi_0"'] \end{tikzcd}$$

[1]Such compositions make sense on suitably chosen open sets and we shall not mention this explicitly.

where the the top horizontal dotted arrow is the map that we need to define. The observations that we made above will now help us in defining df_x. First observe that all the maps in the solid arrows are linear maps of vector spaces. By the Remark above we know that the two vertical maps are linear isomorphisms. We now define the derivative df_x of f at x to be the composition

$$df_x = d\psi_0 \circ dg_0 \circ d\varphi_0^{-1}.$$

Clearly

$$df_x : T_x(X) \longrightarrow T_{f(x)}(Y)$$

is a linear map. With this definition, the second diagram also commutes.

Having defined the derivative, what now remains is to check that the definition of df_x does not depend upon the local parametrizations chosen. We shall check this in some detail. The proof uses the method of Lemma 3.2 above. Similar checking in the future will be left as an exercise. Fix two parametrizations

$$\varphi : U \longrightarrow X; \quad \varphi' : U' \longrightarrow X$$

about x with $\varphi(0) = x = \varphi'(0)$. Also fix two parametrizations

$$\psi : V \longrightarrow Y; \quad \psi : V' \longrightarrow Y$$

with $\psi(0) = f(x) = \psi'(0)$. Set, as before,

$$g = \psi^{-1} \circ f \circ \varphi$$

and

$$g' = \psi'^{-1} \circ f \circ \varphi'.$$

As usual we arrange so that $\varphi(U) = \varphi'(U')$ and $\psi(V) = \psi(V')$. The two setups give rise to two competing definitions of the derivative. To resolve this we must show

$$d\psi_0 \circ dg_0 \circ d\varphi_0^{-1} = d\psi_0' \circ dg_0' \circ d{\varphi'}_0^{-1}. \tag{2.4.1}$$

To prove this we first note that there is a diagram

$$\begin{tikzcd}
U \arrow[r, "g=\psi^{-1}\circ f\circ\varphi"] \arrow[d, "\varphi"] & V \arrow[d, "\psi"] \\
X \arrow[r, "f"] & Y \\
U' \arrow[u, "\varphi'"] \arrow[r, "g'=\psi'^{-1}\circ f\circ\varphi'"'] & V' \arrow[u, "\psi'"]
\end{tikzcd}$$

in which all the three squares commute. In particular

$$g \circ \varphi^{-1} \circ \varphi' = \psi^{-1} \circ \psi' \circ g'.$$

We now replace the maps φ^{-1} and ψ^{-1} by maps F and G respectively that are smooth extensions of φ^{-1} and ψ^{-1} (see the proof of Lemma 2.3, and the remark following the lemma) to write

$$g \circ F \circ \varphi' = G \circ \psi' \circ g'.$$

Since all the maps are smooth maps on open subsets of the euclidean space, we may now apply the chain rule to conclude that

$$dg_0 \circ dF_x \circ d\varphi_0' = dG_{f(x)} \circ d\psi_0' \circ dg_0'. \tag{2.4.2}$$

Just to remind ourselves, the above linear maps fit into the sequences

$$\mathbb{R}^k \xrightarrow{d\varphi_0'} \mathbb{R}^N \xrightarrow{dF_x} \mathbb{R}^k \xrightarrow{dg_0} \mathbb{R}^\ell$$

and

$$\mathbb{R}^k \xrightarrow{dg_0'} \mathbb{R}^\ell \xrightarrow{d\psi_0'} \mathbb{R}^M \xrightarrow{dG_{f(x)}} \mathbb{R}^\ell$$

The well definedness of the tangent space tells us that

$$\mathrm{im}(d\varphi_0') = T_x(X) = \mathrm{im}(d\varphi_0).$$

Now as

$$F \circ \varphi = \mathrm{id}$$

it follows that

$$dF_x/T_x(X) = d\varphi_0^{-1}.$$

Similarly,

$$dG_{f(x)}/T_{f(x)}(Y) = d\psi_0^{-1}.$$

The equation (2.4.2) may now be written as

$$dg_0 \circ d\varphi_0^{-1} \circ d\varphi_0' = d\psi_0^{-1} \circ d\psi_0' \circ dg_0'.$$

Rewriting this immediately implies that equality in equation (2.4.1) holds. This shows that the derivative is well defined.

We end this discussion by showing that the chain rule holds.

Proposition 2.5. (Chain Rule) The chain rule holds.

Proof. Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

be smooth maps between manifolds. Then the chain rule states that for $x \in X$ we have that

$$d(g \circ f)_x = dg_{f(x)} \circ df_x.$$

The proof is an exercise in understanding and unravelling the definition of the derivative. The proof is left as an exercise. $\square$

Remark 2.6. Let $f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^M$ be a smooth map defined on an open set U. Then as f is defined on an open set we have the usual derivative of f. On the other hand we may think of U as a manifold and then there is the "manifold derivative" of f. Both the derivatives coincide. Observe that we can choose the parametrizations φ, ψ to be translations. Thus the function $g = \psi^{-1} \circ f \circ \varphi$ is essentially f upto a translation. The derivatives of φ and ψ are the identity linear transformations. Hence the usual derivative equals the manifold derivative.

Here are some problems.

Exercise 2.7. Let $f : S^1 \longrightarrow \mathbb{R}$ be the map $f(x,y) = x$. Use the definition to find $df_{(x,y)}$ when $(x,y) = (1,0), (0,1)$.

Exercise 2.8. Consider the map $f : S^1 \longrightarrow S^1$ defined by $f(z) = z^2$. Use the definition to find df_i where $i = (0,1)$.

Exercise 2.9. Let $Z \subseteq X \subseteq \mathbb{R}^N$ be a submanifold of X and let $i : Z \hookrightarrow X$ denote the inclucion map. Show that i is smooth and the derivative $di_x : T_x(Z) \longrightarrow T_x(X)$, $x \in Z$, equals the inclusion map of tangent spaces. Note that both $T_x(Z)$ and $T_x(X)$ are subspaces of $\mathbb{R}^N$. This exercise shows that $T_x(Z)$ is a subspace of $T_x(X)$.

Exercise 2.10. We know that if U is an open subset of a k-manifold X, then U is a manifold of the same dimension as X. Show that for all $x \in U$, we have $T_x(U) = T_x(X)$.

Exercise 2.11. Let V be a vector subspace of $\mathbb{R}^N$. Then show that $T_x(V) = V$ for all $x \in V$.

Exercise 2.12. The tangent space to S^1 at (a, b) is a one-dimensional subspace of $\mathbb{R}^2$. Explicitly describe the subspace in terms of a and b. Similarly, exhibit a basis of $T_p(\mathbb{S}^2)$ at an arbitrary point $p = (a, b, c)$.

Exercise 2.13. Prove the following for manifolds X, X', Y, Y'.

(1) Show that $T_{(x,y)}(X \times Y) = T_x(X) \times T_y(Y)$.
(2) If $f : X \times Y \longrightarrow X$ is the projection to the first factor, then
$$df_{(x,y)} : T_x(X) \times T_y(Y) \longrightarrow T_x(X)$$
is also projection to the first factor.
(3) Fix $y \in Y$ and let $f : X \longrightarrow X \times Y$ be defined by $f(x) = (x, y)$. Show that $df_x(v) = (v, 0)$.
(4) Let $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$ be two maps. Show that
$$d(f \times g)_{(x,y)} = df_x \times dg_y.$$

Exercise 2.14. Let $f : X \longrightarrow X \times X$ be the map $f(x) = (x, x)$. Show that $df_x(v) = (v, v)$.

Exercise 2.15. Given a map $f : X \longrightarrow Y$ define $F : X \longrightarrow X \times Y$ by $F(x) = (x, f(x))$. Show that $dF_x(v) = (v, df_x(v)$. Further show that the tangent space to the graph of f at $(x, f(x))$ equals the graph of $df_x : T_x(X) \longrightarrow T_{f(x)}(Y)$.

Exercise 2.16. (Equivalent definition of tangent space). A curve in a manifold X is a smooth map
$$\sigma : I \longrightarrow X$$
where I is an interval. The derivative at $t_0 \in I$ is then a linear map
$$d\sigma_{t_0} : T_{t_0}(I) = \mathbb{R} \longrightarrow T_{\sigma(t_0)}(X).$$
The vector
$$d\sigma_{t_0}(1) \in T_{\sigma(t_0)}(X)$$
is called the *velocity vector* of σ at time t_0. We also make use of the notation s
$$\sigma'(t_0);\ \frac{d\sigma}{dt}(t_0);\ \left.\frac{d\sigma}{dt}\right|_{t_0}$$
to denote the velocity vector. Show that every tangent vector at $x \in X$ is the velocity vector of some curve <u>in X</u> and conversely. This definition of the tangent space is very useful in concrete situations. For example many exercises above follow easily from this definition of the tangent space. However it is instructive to solve the preceding exercises without using this alternate definition of the tangent space.

Exercise 2.17. Let $p \in \mathbb{S}^n$. Use Exercise 2.16 to describe $T_p(\mathbb{S}^n)$.

Exercise 2.18. Complete the proof of Proposition 2.5.

Exercise 2.19. Let Z be a submanifold of X, $f : X \longrightarrow Y$ smooth and $g = f/Z$. For $z \in Z$ show that $dg_z = df_z/T_z(Z)$.

3. Solutions to selected problems

Here are solutions to selected problems. Often there is more than one way to solve a problem.

(1) (2.7) Let $\varphi : (-1,1) \longrightarrow S^1$ be the map
$$\varphi(t) = (\sqrt{1-t^2}, t).$$
Then φ is a parametrization about $(1,0)$ and $\varphi(0) = (1,0)$. Let $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ be the parametrization about $f(1,0) = 1$ given by $\psi(t) = t+1$. Then $\psi(0) = f(1,0) = 1$. Let $g = \psi^{-1} \circ f \circ \varphi$. So that $g : (-1,1) \longrightarrow \mathbb{R}$ is given by
$$g(t) = \sqrt{1-t^2} - 1.$$
By definition,
$$df_{(1,0)} = d\psi_0 \circ dg_0 \circ d\varphi_0^{-1}.$$
It is clear that $dg_0 = 0$. Hence
$$df_{(1,0)} : T_{(1,0)}(S^1) \longrightarrow T_1(\mathbb{R}) = \mathbb{R}$$
is the zero linear map. Another way to see this is to consider the maps
$$S^1 \hookrightarrow \mathbb{R}^2 \xrightarrow{\pi_1} \mathbb{R}.$$
The composition is f. Now apply chain rule.

(2) (2.8) The map f in cartesian coordinates is
$$f(x,y) = (x^2 - y^2, 2xy).$$
Now we may use parametrizations as in the previous problem.

(3) (2.9) Let Z be a submanifold of $X \subseteq \mathbb{R}^N$ and let $i : Z \longrightarrow X$ be the inclusion map. This is clearly smooth. Given $x \in Z$ fix parametrizations
$$\varphi : U \longrightarrow Z; \quad \psi : V \longrightarrow X$$
about $x \in Z$ and $x \in X$ with $\varphi(0) = x = \psi(0)$. . First observe that
$$g = \psi^{-1} \circ i \circ \varphi = \psi^{-1} \circ \varphi.$$
By definition,
$$di_x = d\psi_0 \circ dg_0 \circ d\varphi_0^{-1} = d\psi_0 \circ (d\psi_0^{-1} \circ d\varphi_0) \circ d\varphi_0^{-1}$$
and the result follows.

(4) (2.10) The previous exercise already says that $T_x(U) \subseteq T_x(X)$. Since both have the same dimension equality follows.

(5) (2.11) This follows for there exists a global parametrization of V by a linear map. The image of the linear map is V.

(6) (2.12) Let $(a,b) \in S^1$ with $b > 0$. Fix $\varepsilon > 0$ such that $(a-\varepsilon, a+\varepsilon) \subseteq (-1,1)$. Consider the parametrization $\varphi : (-\varepsilon, \varepsilon) \longrightarrow S^1$ about (a,b) defined by
$$\varphi(t) = (a+t, \sqrt{1-(a+t)^2})$$
so that $\varphi(0) = (a,b)$. We think of φ as a map $\varphi : (\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^2$ and then, by definition,
$$T_{(a,b)}(S^1) = \operatorname{im}(d\varphi_0).$$
Clearly,
$$d\varphi_0 = \left(1, \frac{-a}{\sqrt{1-a^2}}\right)^t$$

and hence

$$d\varphi_0(v) = \left(v, \frac{-av}{\sqrt{1-a^2}}\right)^t$$

where $v \in \mathbb{R}$. For dimension reasons,

$$T_{(a,b)}(S^1) = \operatorname{span}(d\varphi_0(1)) = \operatorname{span}\left(1, \frac{-a}{\sqrt{1-a^2}}\right)^t.$$

Thus,

$$T_{(ab)}(S^1) = (a,b)^{\perp}.$$

The other cases can be dealt with similarly. This, somewhat long, calculation shows that a hands on computation with parametrizations to compute the tangent space can be messy.

(7) (2.13) We will discuss the proof of (1). The other proofs are similar. First note that the dimensions of the vector spaces involved are equal. Fix parametrizations

$$\varphi : \mathbb{R}^k \longrightarrow X; \quad \psi : \mathbb{R}^\ell \longrightarrow Y$$

about x and y such that $\varphi(0) = x, \psi(0) = y$. By definition

$$\operatorname{im}(d(\varphi \times \psi)_{(0,0)}) = T_{(x,y)}(X \times Y).$$

By definition

$$\begin{aligned} d(\varphi \times \psi)_{(0,0)}(v,w) &= \lim_{h\to 0} \tfrac{(\varphi\times\psi)(hv,hw)-(\varphi\times\psi)(0,0)}{h} \\ &= \lim_{h\to 0} \tfrac{(\varphi(hv),\psi(hw))-(\varphi(0),\psi(0))}{h} \\ &= \lim_{h\to 0} \tfrac{(\varphi(hv)-\varphi(0),\psi(hw)-\psi(0))}{h} \\ &= (d\varphi_0(v), d\psi_0(w)) \end{aligned}$$

This shows $T_{(x,y)}(X \times Y) \subseteq T_x(X) \times T_y(Y)$ and hence equality follows.

(8) (2.17) Let $p \in S^n$ and let $\sigma : (-\varepsilon, \varepsilon) \longrightarrow S^n$ be a curve with $\sigma(0) = p$. Let $f : \mathbb{S}^n \longrightarrow \mathbb{R}$ be the function $f(x) = ||x||^2$. Then $f \circ \sigma = 1$. Thus

$$0 = d(f \circ \sigma)_0(1) = df_{\sigma(0)}(d\sigma_0(1)) = \nabla f(\sigma(0)) \cdot \sigma'(0) = 2p \cdot \sigma'(0).$$

Thus every tangent vector at p is orthogonal to p. Thus

$$T_p(\mathbb{S}^n) \subseteq p^{\perp}$$

and since the dimensions are equal we have that the above inclusion is an equality.

(9) (2.18) Use Chain rule and Exercise 2.9.

DIFFERENTIAL TOPOLOGY - LECTURE 4 - I

1. Introduction

Given a function defined on an open set $U \subseteq \mathbb{R}^N$, whether or not the function is differentiable at a point $x \in U$ depends only on the nature of the function in a neighborhood of the point x. In this sense differentiability is local in nature. This is, therefore, also true about functions defined between manifolds. Since manifolds are locally like euclidean spaces one would expect that any local phenomena exhibited by functions in euclidean space should also have an analogue in the manifold world. Indeed we have already seen two examples of this, namely, composition of two smooth functions is smooth and that the chain rule holds for composition of smooth maps between manifolds.

Another result that is local in nature and we are familiar with, at least in the euclidean setup, is the Inverse function theorem. It is quite believable and infact not difficult to prove that the Inverse function theorem should be true for maps between manifolds. We shall quickly prove this below.

Recall that the Inverse function theorem gives a local topological/differential conclusion about the function based on certain algebraic assumption on the derivative of the function. This is a recurrent theme in differential topology: the algebriac properties of the (higher) derivatives of a function implies a lot of the local behaviour of the function.

Below we shall discuss what local conclusions about the function can be drawn when different restrictions are imposed on the derivative. This will lead us to two important results namely the local immersion theorem and the local submersion theorem. We shall discuss these theorems and their consequences in this and the next set of notes.

2. Inverse function theorem and the Local immersion theorem

Let us begin by trying to understand the statement of the Inverse function theorem for maps between manifolds. We first remind ourselves of certain definitions.

Recall that a map $f : X \longrightarrow Y$ between manifolds is a local diffeomorphism at $x \in X$ if f maps a neighborhood[1] of x diffeomorphically onto a neghborhood of $f(x)$. The map f itself is a local diffeomorphism if it is a local diffeomorphism at each $x \in X$.

Suppose that $f : X \longrightarrow Y$ is a local diffeomorphism at x. Then there exist neighborhoods U of x and V of $f(x)$ such that $f : U \longrightarrow V$ is a diffeomorphism. This implies that the derivative

$$df_x : T_x(X) \longrightarrow T_{f(x)}(Y)$$

is an isomorphism. The Inverse function theorem is the statement that the converse is also true.

Theorem 2.1. (Inverse function theorem) Suppose that $f : X \longrightarrow Y$ is such that

$$df_x : T_x(X) \longrightarrow T_{f(x)}(Y)$$

is an isomorphism for some $x \in X$. Then f is a local diffeomorphism at x.

[1] Neighborhoods are always open.

Proof. We shall use the Inverse function theorem in euclidean spaces to prove this. Fix parametrizations

$$\varphi : U \longrightarrow X; \quad \psi : V \longrightarrow Y$$

about x and $f(x) = y$ respectively with $\varphi(0) = x$ and $\psi(0) = y$. As before, we set $g = \psi^{-1} \circ f \circ \varphi$ to get commutative diagrams

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\varphi"] \arrow[r, "g"'] & V \arrow[u, "\psi"'] \end{tikzcd} \qquad \begin{tikzcd} T_x(X) \arrow[r, "df_x"] & T_{f(x)}Y \\ \mathbb{R}^k \arrow[u, "d\varphi_0"] \arrow[r, "dg_0"'] & \mathbb{R}^k \arrow[u, "d\psi_0"'] \end{tikzcd}$$

Since $d\varphi_0, df_x$ and $d\psi_0$ are all isomorphisms we conclude that dg_0 is also an isomorphism. Since g is a map between open subsets of euclidean spaces we conclude, by the Inverse function theorem, that g is a local diffeomorphism at 0. Thus there exist open sets $0 \in U_1 \subseteq U$ and $0 \in V_1 \subseteq V$ such that

$$g : U_1 \longrightarrow V_1$$

is a diffeomorphism. Keeping in mind that φ and ψ are themselves diffeomorphisms, the commutativity of the first diagram now implies that

$$f : \varphi(U_1) \longrightarrow \psi(V_1)$$

is now a diffeomorphism. This shows that f is a local diffeomorphism at x. This completes the proof. $\square$

Remark 2.2. Observe that since $f : \varphi(U_1) \longrightarrow \psi(V_1)$ is a diffeomorphism, we have that df_p is an isomorphism for all $p \in \varphi(U_1)$. This shows that if for a smooh map $f : X \longrightarrow Y$, df_x is an isomorphism for some $x \in X$, then the derivative continues to be an isomorphism in a neighborhood of x. It is important to note that the Inverse function theorem is a local theorem in that the function may be a local diffeomorphism at every point but fail to be a global diffeomorphism.

Here is an example that illustrates one of the points made in the above remark.

Example 2.3. Consider the map $p : \mathbb{R} \longrightarrow \mathbb{S}^1$ defined by $p(t) = (\cos t, \sin t)$. Then p is smooth and a local diffeomorphism but fails to be one-one. Let us try to prove that the derivative

$$dp_a : T_a(\mathbb{R}) = \mathbb{R} \longrightarrow T_{p(a)}(\mathbb{S}^1)$$

is an isomorphism for every $a \in \mathbb{R}$. To prove this let us look at the commutative diagrams

$$\begin{tikzcd} \mathbb{R} \arrow[d, "p"'] \arrow[dr, "f"] & \\ \mathbb{S}^1 \arrow[r, "i"'] & \mathbb{R}^2 \end{tikzcd} \qquad \begin{tikzcd} T_a(\mathbb{R}) \arrow[d, "dp_a"'] \arrow[dr, "df_a"] & \\ T_{p(a)}(\mathbb{S}^1) \arrow[r, "di_{p(a)}"'] & T_{i(a)}(\mathbb{R}^2) \end{tikzcd}$$

Here $i : \mathbb{S}^1 \longrightarrow \mathbb{R}^2$ is the inclusion map of the submanifold $\mathbb{S}^1$ of $\mathbb{R}^2$ and $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ is the map $f(t) = (\cos t, \sin t)$. By (Exercise 2.9, Lecture 3), the derivatve di_a is injective. Observe that

$$df_a = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$$

Thus df_a is also injective. This implies the composition $di_{p(a)} \circ dp_a$ is also injective. This forces dp_a to be injective. Since $T_a(\mathbb{R})$ and $T_{p(a)}(\mathbb{S}^1)$ are both 1-dimensional, dp_a must be an isomorphism.

By the Inverse function theorem p is a local diffeomorphism at a. Since a is arbitrary, p is a local diffeomorphism.

Given a smooth map $f : X \longrightarrow Y$ between manifolds there are other algebraic restrictions that one can impose on the derivative

$$df_x : T_x(X) \longrightarrow T_{f(x)}(Y)$$

of f at x. This leads us to the following definitions.

Definition 2.4. Let $f : X \longrightarrow Y$ be a smooth map between manifolds. We say that f is an *immersion* (respectively a *submersion*) at x if the derivative

$$df_x : T_x(X) \longrightarrow T_{f(x)}(Y)$$

at x is injective (respectively surjective). The map f itself is called an immersion (respectively a submersion) if it is an immersion (respectively a submersion) at each $x \in X$.

Before looking at examples some remarks are in order. Observe that if $f : X \longrightarrow Y$ is an immersion, then $\dim(X) \leq \dim(Y)$. If the dimensions of X and Y are equal, then a map $f : X \longrightarrow Y$ is an immersion if and only if it is a submersion if and only if it is a local diffeomorphism.

Here are some examples.

Example 2.5. The maps p and f in Example 2.3 above are immersions. p is in addition a submersion too.

Example 2.6. Let $k \leq N$. The inclusion map $i : \mathbb{R}^k \longrightarrow \mathbb{R}^N$ defined by

$$i(x_1, \ldots, x_k) = (x_1, \ldots x_k, 0, \ldots, 0)$$

is an immersion. The map i is called the *canonical immersion* of $\mathbb{R}^k$ into $\mathbb{R}^N$.

Example 2.7. Let $k \leq N$. The projection map $j : \mathbb{R}^N \longrightarrow \mathbb{R}^k$ defined by

$$j(x_1, \ldots, x_N) = (x_1, \ldots, x_k)$$

is a submersion. The map j is called the *canonical submersion* of $\mathbb{R}^N$ onto $\mathbb{R}^k$.

The Inverse function theorem describes the behavior of a function in a neighborhood of point at which the derivative is invertible. We now wish to understand what can one say about the local behaviour of a funtion in a neighborhood of a point at which the function is an immersion. The behavior of a function in a neighborhood of a point at whiich the function is an immersion is described by the local immersion theorem.

Theorem 2.8. (Local Immersion theorem) Let X, Y be manifolds of dimension k, ℓ respectively. Suppose that $f : X \longrightarrow Y$ is an immersion at $x \in X$. Then there exist parametrizations

$$\varphi : U \longrightarrow X; \quad \psi : V \longrightarrow Y$$

about x and $f(x) = y$ respectively such that

$$\psi^{-1} \circ f \circ \varphi = i.$$

Here i is the canonical immersion of $\mathbb{R}^k$ into $\mathbb{R}^\ell$.

Proof. A composition of the form $\psi^{-1} \circ f \circ \varphi$ as in the theorem is called a *local representation* of the function f about $x \in X$. Thus the local immersion theorem basically says that if f is an immersion at x, then some local representation of f is the canonical immersion.

The proof is in some ways an exercise in linear algebra. We start with arbitrary local parametrizations

$$\varphi : U \longrightarrow X; \quad \psi : V \longrightarrow Y$$

about x and $y = f(x)$ with $\varphi(0) = x$ and $\psi(0) = y$ to get a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\varphi"] \arrow[r, "g"'] & V \arrow[u, "\psi"'] \end{tikzcd}$$

Remember that $g = \psi^{-1} \circ f \circ \varphi$. If we modify φ and ψ, the function g changes accordingly. The whole point is that we can do this, that is modify φ and ψ, in such a way so that g gets modified to the canonical immersion. We proceed as follows.

First observe that the composition

$$dg_0 = d\psi_0^{-1} \circ df_x \circ d\varphi_0$$

is injective since f is an immersion at x. Thus the matrix of

$$d\psi_0^{-1} \circ df_x \circ d\varphi_0 : \mathbb{R}^k \longrightarrow T_x(X) \longrightarrow T_y(Y) \longrightarrow \mathbb{R}^\ell$$

has k independent rows. We may now post compose with a change of basis isomorphism $T : \mathbb{R}^\ell \longrightarrow \mathbb{R}^\ell$ (a composition of elementary row operations) to get a composition

$$T \circ d\psi_0^{-1} \circ df_x \circ d\varphi_0 : \mathbb{R}^k \longrightarrow T_x(X) \longrightarrow T_y(Y) \longrightarrow \mathbb{R}^\ell \longrightarrow \mathbb{R}^\ell$$

where now the matrix of the composition has the first k rows independent. Observe that T is a diffeomorphism. Setting $V' = T(V)$ we get a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\varphi"] \arrow[r, "g'"'] & V' \arrow[u, "\psi' = \psi \circ T^{-1}"'] \end{tikzcd}$$

Observe that ψ' is still a parametrization and $g' = \psi'^{-1} \circ f \circ \varphi$. What we have now achieved (after modifying the parametrization ψ to ψ') is that the matrix of

$$dg'_0 = d\psi_0'^{-1} \circ df_x \circ d\varphi_0 = T \circ d\psi_0^{-1} \circ df_x \circ d\varphi_0$$

has first k rows independent and equals (say)

$$\begin{pmatrix} A \\ B \end{pmatrix}$$

where A is a $(k \times k)$ (invertible) matrix and B is a $((\ell - k) \times k)$ matrix. We now post compose

$$T \circ d\psi_0^{-1} \circ df_x \circ d\varphi_0 : \mathbb{R}^k \longrightarrow T_x(X) \longrightarrow T_y(Y) \longrightarrow \mathbb{R}^\ell \longrightarrow \mathbb{R}^\ell$$

with the linear isomorphism

$$S = \begin{pmatrix} A^{-1} & 0 \\ -BA^{-1} & I_{\ell-k} \end{pmatrix} : \mathbb{R}^\ell \longrightarrow \mathbb{R}^\ell.$$

The composition

$$S \circ T \circ d\psi_0^{-1} \circ df_x \circ d\varphi_0 : \mathbb{R}^k \longrightarrow T_x(X) \longrightarrow T_y(Y) \longrightarrow \mathbb{R}^\ell \longrightarrow \mathbb{R}^\ell \longrightarrow \mathbb{R}^\ell$$

now has a matrix of the form

$$\begin{pmatrix} I_k \\ 0 \end{pmatrix}.$$

If we now set

$$V'' = S(V')$$

we get a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\varphi"] \arrow[r, "g''"'] & V'' \arrow[u, "\psi''=\psi\circ T^{-1}\circ S^{-1}"'] \end{tikzcd}$$

where g'' is as usual. Observe that ψ'' is a parametrization. It is also clear that the matrix of

$$dg_0'' = d\psi_0''^{-1} \circ df_x \circ d\varphi_0 = S \circ T \circ d\psi_0^{-1} \circ df_x \circ d\varphi_0$$

has the form

$$\begin{pmatrix} I_k \\ 0 \end{pmatrix}.$$

The upshot of the above discussion is that we might as well have assumed (and we do so now) that the matrix of dg_0 has the form

$$\begin{pmatrix} I_k \\ 0 \end{pmatrix}.$$

Now consider the map $G : U \times \mathbb{R}^{\ell-k} \longrightarrow \mathbb{R}^\ell$ defined by

$$G(x, z) = g(x) + (0, z).$$

Since

$$dG_0 = I_\ell$$

we have, by the Inverse function theorem, that G maps a neighborhood W of $0 \in \mathbb{R}^\ell$ diffeomorphically onto a neighborhood W' of $0 \in \mathbb{R}^\ell$. Now observe that

$$g = G \circ i$$

where

$$i : U \subseteq \mathbb{R}^k \longrightarrow \mathbb{R}^\ell$$

is the canonical immersion. Thus,

$$\psi^{-1} \circ f \circ \varphi = G \circ i.$$

Using the fact that G is invertible, after shrinking the open set U and V if required wehave

$$(\psi \circ G)^{-1} \circ f \circ \varphi = i$$

So finally we modify ψ to the parametrization $(\psi\circ G)$ to ensure that g equals the canonical immersion. This completes the proof. □

Remark 2.9. In the Local immersion theorem let the coordinate system φ^{-1} be given by

$$\varphi^{-1} = (x_1, \ldots, x_k).$$

Thus a point $p \in V$ maybe thought of as the point $(x_1(p), \ldots, x_k(p))$ The function f, in a neighbourhood of x now looks like

$$f(x_1, \ldots x_k) = (x_1, \ldots x_k, 0, \ldots 0).$$

The local immersion theorem can therefore be also stated as: Suppose that $f : X \longrightarrow Y$ is an immersion at x. Then there exist local coordinates around x and $f(x)$ such that

$$f(x_1, \ldots x_k) = (x_1, \ldots x_k, 0, \ldots 0).$$

Although the proof might seem long, the major part of the proof involved convincing ourselves that even though we start with arbitrary parametrizations φ and ψ we can assume without loss of generality that the matrix of of dg_0 has a required suitable form. Having achieved this, the remaining of the proof is a somewhat routine application of the Inverse function theorem. Note that we never modified the parametrization φ.

Here is an interesting consequence of the Local immersion theorem.

Corollary 2.10. Suppose $f : X \longrightarrow Y$ is an immersion at x. Then f is an immersion in a neighborhood of x.

Proof. By the Local immersion theorem some local representation $\psi^{-1} \circ f \circ \varphi$ equals the canonical immersion i. Thus the equality

$$f = \psi \circ i \circ \varphi^{-1}$$

is valid in a neighborhood of x. Taking derivatives (at proper points) we have that

$$df = d\psi \circ i \circ d\varphi^{-1}$$

is valid in a neighborhood of x. Since all the maps on the right hand side are injective, so is df. This completes the proof. □

A subtle point is discussed in the following remark.

Remark 2.11. Suppose that $f : X \longrightarrow Y$ is an immersion at $x \in X$. Then there is a local representation $\psi^{-1} \circ f \circ \varphi$ that equals the canonical immserion i. Hence the equality

$$\psi^{-1} \circ f \circ \varphi = i$$

is valid in a neighborhood of 0 so that we have

$$f = \psi \circ i \circ \varphi^{-1}$$

in a neghborhood of x. Now observe that the canonical immersion maps open sets diffeomorphically onto its image. Since the parametrizations φ and ψ are already diffeomorphisms, we conclude that f maps a small enough neighborhood U of x diffeomorphically onto its image $f(U)$. This means every point in $f(X)$ is contained in a *subset* that is diffeomorphic to an open set in the euclidean space. However, this image $f(U)$ may not be open in $f(X)$. What we are pointing out is that the image of an immersion may not be a manifold.

Here are two examples that show that the image of a $1-1$ immersion need not be a manifold.

Example 2.12. Consider the map $p : \mathbb{R} \longrightarrow \mathbb{S}^1$ defined by

$$p(t) = (\cos 2\pi t, \sin 2\pi t).$$

We know that (see Example 2.3) p is a local diffeomorphism. The map

$$G : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{S}^1 \times \mathbb{S}^1$$

defined by $G(t, s) = (p(t), p(s))$ is therefore a local diffeomorphism (see exercises Lecture 4 - II). Now let L be a line through the origin in $\mathbb{R}^2$ having irrational slope. Then L is a manifold. Since G is a local diffeomorphism

$$G/L : L \longrightarrow \mathbb{S}^1 \times \mathbb{S}^1$$

is an immersion (see exercises Lecture 4 - II). Let $(t, at), (t', at') \in L$ where a is the slope of L. Assume that $(G/L)(t, at) = (G/L)(t', at')$. So that

$$(\cos 2\pi t, \sin 2\pi t, \cos 2\pi at, \sin 2\pi at)) = (\cos 2\pi t', \sin 2\pi t', \cos 2\pi at', \sin 2\pi at').$$

This implies that $(t - t') \in \mathbb{Z}$ and $a(t - t') \in \mathbb{Z}$. This forces $t = t'$ as a is irrational. Thus G/L is one-one. It is an exercise to show that the image of L under G is not locally connected. Hence the image is not a manifold. This example shows that the image of a one-one immersion need not be a manifold.

Example 2.13. Consider the map $\beta : (-\pi, \pi) \longrightarrow \mathbb{R}^2$ defined by

$$\beta(t) = (\sin 2t, \sin t).$$

Clearly β is $1 - 1$ and since the derivative

$$d\beta_t = (2\cos 2t, \cos t)^t$$

is injective, we conclude that f is an immersion. The image of f is compact and is the figure of eight[2] and so cannot be a manifold. The justifications are left as an exercise.

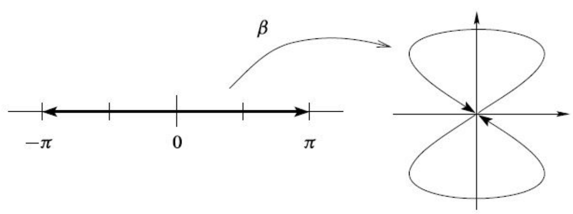

FIGURE 1. Lemniscate of Bernoulli.

Exercises for this set of notes will appear in the next set of of notes after we state the Local submersion theorem.

[2]Image source : Wikipedia.

DIFFERENTIAL TOPOLOGY - LECTURE 4 - II

1. Introduction

In the previous set of notes we proved the local immersion theorem. The local immersion theorem states that if $f : X \longrightarrow Y$ is an immersion at $x \in X$, then there are parametrizations $\varphi : U \longrightarrow X$ about x and $\psi : V \longrightarrow Y$ about $f(x)$ so that

$$\psi^{-1} \circ f \circ \varphi = i$$

where i is the canonical immersion. We remind ourselves that the composition on the left is called a local representation of the function f.

We also noted one consequence of the local immersion theorem, namely, if $f : X \longrightarrow Y$ is an immersion at x, then f is an immersion in a neighborhood of x. We shall see several more applications of the local immersion theorem.

2. Local submersion theorem

The local immersion theorem gives us information about function in a neighborhood of a point at which it is an immersion. Having understood this, we turn our attention to understanding the behaviour of a function in a neighborhood of a point at which it is a submersion. This is described by the theorem below. The proof is similar to the proof of the local immersion theorem and we omit the proof.

Theorem 2.1. (Local submersion theorem) Let X, Y be manifolds and suppose that $f : X \longrightarrow Y$ is a submersion at $x \in X$. Then there exists a local representation of f that equals the canonical projection. In other words, there exist local parametrizations about x and $f(x)$ such that $f(x_1, \dots, x_k) = (x_1, \dots x_\ell)$, the projection to the first ℓ coordinates. Here $\dim(X) = k$ and $\dim(Y) = \ell$. $\square$

The proof is similar the proof of the Local immersion theorem. We start with arbitrary parametrizations

$$\varphi : U \longrightarrow X; \quad \psi : V \longrightarrow Y$$

about x and $f(x)$ and this time keep modifying just φ so that finally the local representation $\psi^{-1} \circ f \circ \varphi$ equals the canonical projection. That ψ does not need to be modified and ψ can be chosen arbitrarily can be helpful in some situations. For example, if $f : X \longrightarrow \mathbb{R}$ is a submersion at x, then as a parametrization about $f(x)$ we have the liberty to choose $\psi = id : \mathbb{R} \longrightarrow \mathbb{R}$.

We now record some important consequences of the Local submersion theorem. The Local submersion theorem says that if $f : X \longrightarrow Y$ is a submersion at $x \in X$, then there are parametrizations

$$\varphi : \longrightarrow X; \quad \psi : V \longrightarrow Y$$

about x and $f(x)$ respectively such that

$$\psi^{-1} \circ f \circ \varphi = j$$

where j is the canonical submersion. Thus the expression

$$f = \psi \circ j \circ \varphi^{-1} \tag{2.1.1}$$

holds in a neighborhood of x. Taking derivatives we have that the equality

$$df = d\psi \circ dj \circ d\psi^{-1}$$

holds at all points in a neighborhood of x. Since every map on the right hand side is surjective we see that f is a submersion in a neighborhood of x. We have thus proved the following.

Corollary 2.2. Suppose that $f : X \longrightarrow Y$ is a submersion at $x \in X$. Then f is a submersion in a neighborhood of x. □

In Equation 2.12.1, all the maps on the right hand side are open maps. Thus we have the following important observation about submersions.

Corollary 2.3. Suppose that $f : X \longrightarrow Y$ is a submersion. Then f is an open map. □

Here is another important consequence.

Corollary 2.4. Let X be compact and $f : X \longrightarrow \mathbb{R}$ a function. Then $df_p = 0$ for some $p \in X$.

Proof. Since X is compact, there exists $p \in X$ such that $f(x) \leq f(p)$ for all $x \in X$. We claim $df_p = 0$. Assume, if possible, that $df_p \neq 0$. This implies that f is a submersion at p. By the local submersion theorem, there is a local representation $\psi^{-1} \circ f \circ \varphi$ so that

$$\psi^{-1} \circ f \circ \varphi = j$$

the canonical submersion which is now the projection to the first factor. Since we may assume $\psi = \text{id}$ (see the paragraph after the statement of the local submersion theorem) we have that

$$f \circ \varphi = j$$

the canonical immersion. Since f has a maximum at p, the composition $f \circ \varphi$ has a maximum at 0 (since $\varphi(0) = p$). But $f \circ \varphi$ is the projection to the first factor defined on an open set and as such cannot have a maximum. This contradiction forces $df_p = 0$. □

Thus a non-constant smooth function $f : X \longrightarrow \mathbb{R}$ on a compact manifold X has at least two points at which the derivative vanishes.

Suppose that X is a compact manifold and Y a non-compact connected manifold. We claim that there is no submersion $f : X \longrightarrow Y$. For if such a submersion exists, then $f(X)$ would be both open and closed in Y. Since Y is connected, this would force $f(X) = Y$ and consequently Y would be compact, a contradiction. Thus we have the following.

Corollary 2.5. If X is a compact manifold, then X cannot be submersed in $\mathbb{R}^N$. □

On the other hand it is a theorem (and we shall prove a version of this later) that every manifold can be immersed in some $\mathbb{R}^N$. It is of great interest (and an active area of research) to know, given a manifold X, what is the smallest n for which there exists an immersion (not necessarily one-one)

$$f : X \longrightarrow \mathbb{R}^n.$$

A tremendous amount of literature[1] exists about individual spaces and the smallest dimension in which they immerse.

[1] See https://www.lehigh.edu/ dmd1/imms.html for details of immersion and embeddings of projective spaces.

A very intersting question about immersions is the following. Given a positive integer n, is there a N isuch that every n-manifold immerses in $\mathbb{R}^N$. A complete answer is known. Given n, let $\alpha(n)$ denote the number of 1's appearing in the binary expansion of n. The Immersion conjecture (which is now a theorem) states that every n-manifold immerses in $\mathbb{R}^{2n-\alpha(n)}$. This was proved by Ralph Cohen[2] in 1985. On the other hand it is a theorem of William Massey[3] that for every n there exists a n-manifold X that does not immerse in $\mathbb{R}^{2n-\alpha(n)-1}$.

Here are some problems. Most are from G and P.

Exercise 2.6. Let Z be an ℓ-dimensional submanifold of X and let $x \in X$. Show that there exists a local coordinate system $\{x_1, \dots, x_k\}$ defined in a neighborhood of $x \in X$ such that $Z \cap U$ is defined by the equations $x_{\ell+1} = 0, \dots, x_k = 0$.

Exercise 2.7. Prove that a local diffeomorphism f is actually a diffeomorphism onto an open subset provided that f is one-one.

Exercise 2.8. Show that there does not exist an immersion $f : \mathbb{S}^n \longrightarrow \mathbb{R}^n$.

Exercise 2.9. Suppose f, g are immersions, then show that so is $f \times g$. Show that composition of two immersions is an immersion. Finally show that the restriction of an immersion to a submanifold is an immersion.

Exercise 2.10. Let $x_1, \dots, x_N$ be the standard coordinate function on $\mathbb{R}^N$ and let X be a k-dimensional submanifold of $\mathbb{R}^N$. Show that the every point $x \in X$ has a neighborhood (in X) on which the restriction of some k coordinate functions

$$x_{i_1}, \dots, x_{i_k}$$

form a coordinate system.

Exercise 2.11. In continuation of the previous exercise, assume that $x_1, \dots, x_k$ form a local coordinate system in a neighborhood V of $x \in X$. Prove that there are smooth functions

$$g_{k+1}, \dots, g_N$$

on an open set U in $\mathbb{R}^k$ such that V may be be taken to be the set

$$\{(a_1, \dots, a_k, g_{k+1}(a), \dots, g_N(a_k)) \,:\, a = (a_1, \dots, a_k) \in U\}.$$

Thus if we define $g : U \longrightarrow \mathbb{R}^{N-k}$ by $(g = (g_{k+1}, \dots, g_N)$, then V is the graph of g. Thus every manifold is locally expressble as the graph of a function.

Exercise 2.12. (Generalization of IFT) Let $f : X \longrightarrow Y$ be a smooth map that is one-one on a compact submanifold Z of X. Assume that for each $x \in Z$,

$$df_x : T_x(X) \longrightarrow T_{f(x)}(Y)$$

is an isomorphism. Prove that f maps Z diffeomorphically onto $f(Z)$. Moreover show that f maps an open set containing Z diffeomorphically onto an open set containing $f(Z)$. Observe that this reduces to the Inverse function theorem when Z is a point.

Exercise 2.13. At what points is the map $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by $f(x, y, z) = (x^2 - y^2, y^2 - z^2)$ a submersion?

Exercise 2.14. Show that if $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a submersion, then f cannot be one-one.

[2]The immersion conjecture for differentiable manifolds, *Annals of Maths.*, 122 (2) 237-328.

[3]On the Stiefel-Whitney classes of a manifold, *American J. Math.*, 82 (1) 92-102.

Exercise 2.15. Construct a smooth surjective map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that is an immersion but not a diffeomorphism.

Exercise 2.16. Let $n > 1$. Suppose that X is a compact connected n-manifold. Let $f : X \longrightarrow \mathbb{S}^n$ be an immersion. Show that f is a diffeomorphism.

Exercise 2.17. Construct an immersion $f : \mathbb{R} \longrightarrow \mathbb{S}^2$.

Exercise 2.18. Does there exist an immersion $S^1 \times S^1 \longrightarrow S^2$? Does there exist an immersion $S^1 \times S^1 \longrightarrow S^3$?

Exercise 2.19. Complete the proof of the final claim in Example 2.12.

Exercise 2.20. For the function $p : \mathbb{R} \longrightarrow \mathbb{R}^2$, $p(t) = (\cos t, \sin t)$, find a local representation about 0 that equals the canonical immersion.

Exercise 2.21. For the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$f(x, y) = \sqrt{x^2 + 2xy + 2y^2}$$

find a local representation about $(1, 0)$ so that the local representation equals the function $(u, v) \mapsto u^2 + v^2$.

Exercise 2.22. Suppose that $g : \mathbb{R} \longrightarrow \mathbb{R}$ is smooth. Determine if the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $f(x, y) = g(x^2 + y^2)$ is a submersion.

DIFFERENTIAL TOPOLOGY - LECTURE 5

1. Introduction

In this set of notes we shall see two new ways that manifolds arise[1]. Here is a brief discussion of the the two methods that we shall see.

Recall that in the previous notes (Lecture 4 - I) we have seen that if $f : X \longrightarrow Y$ is an immersion then $f(X)$ need not be a manifold. Even if f is injective. This raises the question about what additional conditions are needed to ensure that image of an immersion is a manifold. Let us recall what could go wrong.

Suppose $f : X \longrightarrow Y$ is an immersion. Then given $x \in X$, we know that there exists a (parametrizable) neighbourhood U of x restricted to which f is just the canonical immersion. Since the canonical immersion maps open sets diffeomorphically onto its image, we have that f maps U diffeomorphically onto $f(U)$. Now what could happen is that $f(U)$ may not be open in $f(X)$. If all such $f(U)$ were open, then as U is parametrizable $f(U)$ would be too and then, by definition, $f(X)$ would be a manifold. So we need to look for a condition (on f) that would ensure that $f(U)$ is open. The condition turns out to be topological and leads us to the notion of proper maps.

The next situation is the following. Given a smooth map $f : X \longrightarrow Y$ between manifolds and $p \in Y$ we wish to understand under what conditions is the level set $f^{-1}(p)$ a submanifold of X. This is answered by the preimage theorem. This is a powerful tool to construct examples of manifolds. We shall use this to show that many familiar spaces are manifolds.

Conventions. Recall that $X, Y, Z, \ldots$ will always denote manifolds and all maps/functions are always smooth.

2. Embeddings

We shall discuss when the image of an immersion is a manifold. We first make the following definition.

Definition 2.1. A smooth map $f : X \longrightarrow Y$ between manifolds is said to be *proper*[2] if the inverse image of every compact set is compact.

The definition of course makes sense for maps between topological spaces. Clearly, if X is compact and Y is Hausdorff, then every map $f : X \longrightarrow Y$ is proper. It is easy to see that every polynomial with real coefficients thought of as a function $\mathbb{R} \longrightarrow \mathbb{R}$ is proper. On the other hand polynomials in more than one variables need not be proper.

Definition 2.2. A smooth map $f : X \longrightarrow Y$ between manifolds is called an *embedding* if f is one-one, immersion and proper.

[1] We already know of two : Open subsets of manifolds and products of manfolds.

[2] The notion/theory of proper maps is extremely important. Proper maps make their appearance in many situations. One encounters proper maps in the theory of compactifications. A very important invariant that makes its appearance in topology/geomtery/ group theory is that of ends (of a space) and the right maps to look at are proper maps. Proper maps are also the right maps to look at when dealing with (co)homology with compact supports.

The result that we are interested in is the following.

Theorem 2.3. Suppose $f : X \longrightarrow Y$ is an embedding. Then $f(X)$ is a submanifold of Y.

Proof. Since f is an embedding we have that f is injective, proper and an immersion. As we noted above it suffices to show that if U is open in X, then $f(U)$ is open in $f(X)$. Here $f(X)$ has the subspace topology of Y. We assume that for some open set $U \subseteq X$, $f(U)$ is not open in $f(X)$ and derive a contradiction as follows.

Since $f(U)$ is not open we have that $(f(X)-f(U))$ is not closed and therefore must fail to contain one of its limit points which is forced to now lie in $f(U)$. Hence there exists a sequence $y_i \in (f(X)-f(U))$ that converges to a point $y \in f(U)$. Since f is proper and the set $\{y_i, y\}$ compact, we have that $f^{-1}\{y_i, y\}$ is compact. Since f is injective there exist unique $x_i, x \in X$ such that

$$f(x_i) = y_i, \quad f(x) = y.$$

Since $y \in f(U)$, and f is injective it follows that $x \in U$. Without loss of generality we may assume (using compactness) that $x_i \to x_0$, $x_0 \in X$. But as $f(x_i) = y_i$ converges to $f(x_0)$ as well as $y = f(x)$ we conclude that $f(x_0) = f(x)$. As f is injective, we have that $x = x_0$. Hence $x_i \to x$. As $x \in U$ and U is open almost every $x_i \in U$ and therefore almost every $f(x_i) = y_i \in f(U)$. This contradiction completes the proof that $f(U)$ must be open in $f(X)$ and hence $f(X)$ is a submanifold of Y. □

We have actually proved something stronger. We have actually showed that f maps X diffeomorphically onto $f(X)$. Notice that the fact that f is one-one has been crucially used to show that $f(U)$ is open in $f(X)$.

It would be interesting to find examples to show that the theorem is false if any one of the conditions of an embedding is dropped. Notice that none of the maps in (Examples 2.11 and 2.12, Lecture 4) are proper.

Finally, the converse of the above theorem is clearly true and is relegated to the exercises.

3. The preimage theorem

We shall now discuss the preimage theorem which states conditions under which a level set of a smooth function is a manifold. We shall state some definitions before proving the theorem.

Definition 3.1. Let $f : X \longrightarrow Y$ be a smooth function between manifolds and $y \in Y$. We say that y is a *regular value* of f if f is a submersion at each $x \in f^{-1}(y)$.

Points $y \in Y$ that are not regular values are said to be *critical values*. Thus the only way a point $y \in Y$ can be a critical value is that there exists $x \in f^{-1}(y)$ such that f is not a submersion at x. Observe that a point $y \notin f(X)$ automatically becomes a regular value.

Definition 3.2. Let $f : X \longrightarrow Y$ be a smooth map between manifolds. A point $x \in X$ is called a *critical point* of f if the derivative

$$df_x : T_x(X) \longrightarrow T_{f(x)}(Y)$$

is *not surjective*.

In other words, x is a critical point of f if f is not a submersion at x. We have already seen that a real valued function on a compact manifold must have a critical point. Points that are not critical points are called *regular points*. Thus regular points are precisely those points at which f is a submersion.

We are now in a position to state and prove the pre-image theorem. The proof is another application of the Local submersion theorem.

Theorem 3.3. (preimage theorem) Suppose that $y \in Y$ is a regular value of a smooth function $f : X \longrightarrow Y$. Then $Z = f^{-1}(y)$ is a submanifold of X and $\dim(Z) = \dim(X) - \dim(Y)$.

Proof. Suppose that $\dim(X) = k$ and $\dim(Y) = \ell$. Then $k \geq \ell$. Let $Z = f^{-1}(y)$ be non-empty and let $x \in Z$. We shall exhibit a neighborhood of x in Z that is diffeomorphic to an open set in the euclidean space $\mathbb{R}^{k-\ell}$. This will complete the proof.

Now as f is a submersion at x, by the local submersion theorem, there exists parametrizations

$$\varphi : U \longrightarrow X; \quad \psi : V \longrightarrow Y$$

with $\varphi(0) = x, \psi(0) = y$ such that the local representation $\psi \circ f \circ \varphi^{-1}$ is

$$\psi^{-1} \circ f \circ \varphi = j$$

the canonical immersion j. Suppose that $\varphi(U) = U'$, then the coordinate system φ^{-1} is defined on U'. We claim that the restriction $\varphi^{-1}/(U' \cap Z)$ is a coordinate system on the open set $U' \cap Z$ about $x \in Z$. This will complete the proof. Observe that $U' \cap Z$ is an open set in Z. Let us write

$$\varphi^{-1} = (x_1, \ldots, x_k).$$

We know that $\varphi^{-1}/(U' \cap Z)$ is bijective onto its image and is smooth (being the restriction of a smooth function). So we only need to check that the image of $\varphi^{-1}/(U \cap Z)$ is open in $\mathbb{R}^{k-\ell}$ and that the inverse of $\varphi^{-1}/(U \cap Z)$ is smooth.

Suppose that $p \in U' \cap Z$. Then as $f(p) = y$ we have

$$\psi \circ j \circ \varphi^{-1}(p) = y$$

But as $\psi(0) = y$ and ψ is bijective, we must have

$$j \circ \varphi^{-1}(p) = (0, \ldots, 0).$$

Since j is the canonical submersion of $\mathbb{R}^k$ into $\mathbb{R}^\ell$ it follows that the coordinate functions

$$x_1, \ldots, x_\ell$$

are identically zero on $U' \cap Z$. Thus $\varphi^{-1}/(U' \cap Z)$ maps $(U' \cap Z)$ bijectively onto $U \cap \mathbb{R}^{k-\ell}$ which is open in $\mathbb{R}^{k-\ell}$. Here we are identifying $\mathbb{R}^{k-\ell}$ as the subspace of $\mathbb{R}^k$ with the first ℓ coordinates zero. Clearly, the inverse of $\varphi^{-1}/(U' \cap Z)$ equals $\varphi/(U \cap \mathbb{R}^{k-\ell})$ which is smooth, being the restriction of a smooth function. This completes the proof. □

Proposition 3.4. Let $f : X \longrightarrow Y$ be a smooth map between manifolds and let $y \in Y$ be a regular value of f. Let $Z = f^{-1}(y)$ and $z \in Z$. Then the tangent space $T_z(Z)$ equals the kernel of the derivative $df_z : T_z(X) \longrightarrow T_y(Y)$.

Proof. By the preimage theorem, we know that Z is a manifold of dimension equal to $(k - \ell)$ where k, ℓ are respectively the dimensions of X and Y. Consider the commutative diagrams

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[dr, "f'"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd} \qquad \begin{tikzcd} T_z(Z) \arrow[r, "di_z"] \arrow[dr, "df'_z"'] & T_z(X) \arrow[d, "df_z"] \\ & T_y(Y) \end{tikzcd}$$

where $f/Z = f'$. Since f' is constant, we have that $df'_z = 0$. We know that di_z is the inclusion map, thus $T_z(Z)$ is contained in the kernel of df_z. But since the kernel of df_z and $T_z(Z)$ have the same dimension, the proposition follows. $\square$

The preimage theorem gives us a sufficient condition under which a level set of a smooth function is a manifold. Without much extra effort we can extend the methods to understand when a set of common zeros of several functions defines a submanifold.

More precisely, suppose we have a collection smooth functions

$$g_1, \ldots, g_\ell : X \longrightarrow \mathbb{R}$$

defined on a manifold X. We would like to know under what conditions is the set

$$Z = \{x \in X \; : \; g_i(x) = 0, i = 1, 2, \ldots, \ell\}$$

of common zeros of the functions g_i a submanifold of X. We quickly realise that this is not very different from the situation in the preimage theorem once we have the function

$$g = (g_1, g_2, \ldots, g_\ell) : X \longrightarrow \mathbb{R}^\ell$$

in front of us. We immediately note that

$$Z = g^{-1}(0)$$

and therefore Z will be a submanifold if 0 is a regular value of g! This condition, that is whether 0 is a regular value of g or not, can be checked in terms of the functions g_i as follows.

Suppose $x \in Z$. since the functions g_i are smooth we have the derivatives

$$d(g_i)_x : T_x(X) \longrightarrow \mathbb{R}.$$

Hence the linear functionals $d(g_i)_x \in T_x(X)^*$ the linear dual of $T_x(X)$. We prove the easy fact.

Lemma 3.5. With the above notations 0 is a regular value of g, that is, $dg_x : T_x(X) \longrightarrow \mathbb{R}^\ell$ is onto if and only if the linear functionals

$$d(g_1)_x, \ldots, d(g_\ell)_x \in T_x(X)^*$$

are linearly independent in $T_x(X)^*$.

Proof. Suppose that 0 is a regular value of g. Assume that $\Sigma_i a_i d(g_i)_x = 0$ with $a_i \in \mathbb{R}$. Now as $g_i = x_i \circ g$ we have $d(g_i)_x = d(x_i)_0 \circ dg_x$. So that

$$\Sigma_i a_i d(g_i)_x = (\Sigma_i a_i d(x_i)_0) \circ dg_x.$$

Since dg_x is assumed to be onto, we fix $v_j \in T_x(X)$ such that $dg_x(v_j) = e_j$. Then

$$0 = (\Sigma a_i d(g_i)_0)(v_j) = (\Sigma_i a_i d(x_i)_0) \circ dg_x(v_j) = (\Sigma_i a_i d(x_i)_0)(e_j) = a_j.$$

Thus each $a_j = 0$. Conversely let us assume that the linear functionals are linearly independent. Let $v_1, \ldots, v_\ell \in T_x(X)$ be the basis dual to $d(g_1)_x, \ldots, d(g_\ell)_x$. We claim that the vectors $dg_x(v_1), \ldots, dg_x(v_\ell) \in \mathbb{R}^\ell$ are linearly independent. This will prove that dg_x is onto. Assume that

$$\Sigma_i a_i dg_x(v_i) = 0$$

with $a_i \in \mathbb{R}$. Then

$$0 = dx_j (\Sigma_i a_i dg_x(v_i)) = \Sigma_i dx_j dg_x(v_i) = \Sigma_i a_i d(g_j)_x(v_i) = a_j.$$

Thus all $a_j = 0$. This completes the proof. $\square$

Here is a definition.

Definition 3.6. Suppose $g_1, \ldots, g_\ell : X \longrightarrow \mathbb{R}$ are smooth functions on the manifold X. We say that these functions are independent at $x \in X$ if the linear functionals $d(g_1)x, \ldots, d(g_\ell)_x$ are linearly independent.

We have therefore proved the following proposition.

Proposition 3.7. Suppose that the functions $g_1, \ldots, g_\ell : X \longrightarrow \mathbb{R}$ are independent at each point where they all vanish, then Z the set of common zeros is a submanifold of X of dimension equal to $\dim(X) - \ell$. $\square$

We then say that Z is *cut out* by independent functions. Various questions immediately crop up. The obvious one being : Given a submanifold Z of X, is Z cut out by independent functions? This means do there exist smooth functions

$$g_1, \ldots, g_\ell : X \longrightarrow \mathbb{R}$$

with Z being the set of common zeros and the functions independent on Z? In general the answer is no. We shall see examples later if time permits.

A convenient way to keep track of the dimensions is to introduce the *codimension* of a submanifold.

Definition 3.8. Let Z be a submanifold of X. Then $\operatorname{codim}(Z) = \dim(X) - \dim(Z)$.

If we now see the statement of the pre-image theorem the claim about the dimension of Z can be replaced by $\operatorname{codim}(Z) = \ell = \operatorname{codim}\{y\}$. The codimension on the left is that of Z in X and on the right is the codimension of the 0-dimensional submanifold $\{y\}$ in Y. Similarly, we observe that in Proposition 3.7, the ℓ-many independentent functions cut out a submanifold Z of codimension ℓ.

As was pointed out above, in general the converse of Proposition 3.7 is not true. However in specific situations the converse holds.

Proposition 3.9. Suppose that $f : X \longrightarrow Y$ is a smooth map and $y \in Y$ a regular value of f. Then $Z = f^{-1}(y)$ can be cut out by independent functions.

Proof. The proposition says that if a submanifold Z of X is already the inverse image of a regular value of some function, then it can be cut out by independent functions.

First observe that Z being the inverse image of a regular value is a submanifold of codimension $\ell = \dim(Y)$. Let $\psi = (x_1, \ldots, x_\ell)$ be a coordinate system defined on an open set Uabout y with $\psi(y) = 0$. Then the functions $g_i = x_i \circ f$ are defined on the open set $V = f^{-1}(U)$ containing Z. Z is precisely the set of common zeros of $g_i's$. By Lemma 3.5 the functions g_i are independent on Z (since $\psi \circ f$ is a submersion on Z). This completes the proof. $\square$

Proposition 3.10. Every submanifold of X is locally cut out by independent functions.

Proof. Exercise. The statement of the proposition means the following. Let Z be a submanifold of X of codimension ℓ. Given $z \in Z$, there exists an open set $U \subseteq X$ with $z \in U$ and smooth functions

$$g_1, \ldots, g_\ell : U \longrightarrow \mathbb{R}$$

with $U \cap Z$ as the set of common zeros of the g_i's. It might be of help to look at the Exercise 2.6 of Lecture 4 - II. $\square$

Here are some exercises.

Exercise 3.11. Show that $f : \mathbb{R} \longrightarrow \mathbb{R}^3$ defined by $f(t) = (t, t^2, t^3)$ is an embedding. Find two independent functions that globally define the image. Are your functions independent on the whole of $\mathbb{R}^3$ or just on a neighborhood of the image?

Exercise 3.12. Prove the following extension of Proposition 3.10. Let $Z \subseteq X \subseteq Y$ be submanifolds and $z \in Z$. Show that there exists a neighborhood U of z in Y and independent functions $g_1, \ldots, g_\ell$ on U such that

$$Z \cap U = \{y \in U \,:\, g_1(y) = 0, \ldots, g_\ell(y) = 0\}$$

and

$$X \cap U = \{y \in U \,:\, g_1(y) = 0, \ldots, g_m(y) = 0\}$$

where $\ell - m$ is the codimension of Z in X.

Exercise 3.13. Show that 0 is the only critical value of the map $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ given by

$$f(x, y, z) = x^2 + y^2 - z^2.$$

Prove that if s and b are either both positive or both negative, then $f^{-1}(a)$ and $f^{-1}(b)$ are both diffeomorphic.

Exercise 3.14. (Stack of Records theorem) Suppose that y is a regular value of $f : X \longrightarrow Y$ with X compact and $\dim(X) = \dim(Y)$. Show that

$$f^{-1}(y) = \{x_1, \ldots, x_N\}$$

is a finite set. Prove that there exists a neighborhood U 0f y such that $f^{-1}(U)$ is a disjoint union

$$f^{-1}(U) = V_1 \cup \cdots \cup V_N$$

where V_i is a neighborhood of x_i and f maps each V_i diffeomorphically onto U. In particular the Exercise shows that if $f : X \longrightarrow Y$ is an immersion between manifolds of the same dimension with X compact and Y connected, then f is a covering map.

DIFFERENTIAL TOPOLOGY - LECTURE 6

1. Introduction

In the previous set of notes we discussed the preimage theorem. In this set of notes we shall discuss some examples. Recall that the preimage theorem states that if $y \in Y$ is a regular value of a smooth map $f : X \longrightarrow Y$ between manifolds of dimensions k and ℓ respectively, then the preimage $Z = f^{-1}(y)$ is submanifold of X of codimension equal to the codimension of y in Y which is ℓ. Thus the dimension of Z equals $k - \ell$. Moreover we can also say something about the tangent space to Z at $z \in Z$. By (Proposition 3.4, Lecture 5), we know that

$$T_z(Z) = \text{kernel}(df_z : T_z(X) \longrightarrow T_{f(z)}(Y)).$$

We shall discuss several examples.

2. Examples

Using the preimage theorem to prove a certain subset of the euclidean space is a manifold bypasses the often tricky task of constructing explicit parametrizations. To begin with we look at the simplest examples. First the sphere.

Example 2.1. The unit sphere $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ is the level set $f^{-1}(1)$ of the smooth function $f : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ defined by

$$f(x_1, \ldots, x_{n+1}) = x_1^2 + x_2^2 + \cdots + x_{n+1}^2.$$

The derivative at $x \in \mathbb{R}^{n+1}$ is given by the linear transformation (a $(1 \times (n+1))$ matrix)

$$df_x = (2x_1, 2x_2, \cdots, 2x_{n+1}) : T_x(\mathbb{R}^{n+1}) = \mathbb{R}^{n+1} \longrightarrow T_{f(x)}(\mathbb{R}) = \mathbb{R}$$

which is not zero unless $x = 0$. Thus, in particular, f is a submersion at each $x \in f^{-1}(1) = \mathbb{S}^n$. Hence by the preimage theorem $\mathbb{S}^n$ is a manifold of codimension 1 in $\mathbb{R}^{n+1}$. Thus it has dimension n. What about the tangent space at $x \in \mathbb{S}^n$? We know that, for $x \in \mathbb{S}^n$

$$T_x(\mathbb{S}^n) = \text{kernel}(df_x)$$

where

$$df_x : T_x(\mathbb{R}^{n+1}) = \mathbb{R}^{n+1} \longrightarrow T_1(\mathbb{R}) = \mathbb{R}.$$

is as above. Therefore $T_x(\mathbb{S}^n)$ consists of all those vectors $(a_1, \ldots, a_{n+1}) \in \mathbb{R}^{n+1}$ such that

$$df_x(a) = (2x_1, \ldots, 2x_{n+1}) \begin{pmatrix} a_1 \\ \vdots \\ a_{n+1} \end{pmatrix} = 2x_1a_1 + \cdots + 2x_{n+1}a_{n+1} = 0.$$

Thus the tangent space $T_x(\mathbb{S}^n)$ consists of all those vectors $a \in \mathbb{R}^{n+1}$ whose inner product with x is zero. Hence the tangent space $T_x(\mathbb{S}^n)$ equals the orthogonal complement of x in $\mathbb{R}^{n+1}$.

Example 2.2. Consider the function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ given by $f(x, y, z) = x^2 + y^2$. Then $Z = f^{-1}(1)$ is the (infinite) cylinder. It is clear that 1 is a regular value of f. Hence Z is a codimension 1 submanifold of $\mathbb{R}^3$. Given $p = (a, b, c) \in Z$, we know that the tangent space $T_p(Z)$ equals the kernel of

$$df_p : T_p(\mathbb{R}^3) = \mathbb{R}^3 \longrightarrow \mathbb{R}.$$

The derivative df_p is given by the (1×3)-matrix

$$df_p = (2a, 2b, 0)\,.$$

Hence

$$T_p(Z) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;:\; 2ax + 2by = 0 \right\}.$$

Example 2.3. Consider the function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ defined by $f(x, y, z) = x^2 + y^2 - z^2$. Every $r \neq 0$ is a regular value of f. Hence, fixing $r \neq 0$, we have that the hyperboloid $Z = f^{-1}(r)$ is a codimension 1 submanifold of $\mathbb{R}^3$. It is clear that if $p = (a, b, c) \in Z$, then

$$T_p(Z) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;:\; 2ax + 2by - 2cz = 0 \right\}.$$

If $r < 0$, then $f^{-1}(r)$ is not connected.

The remaining examples will concern matrix groups. Let $M_{m\times n}(\mathbb{R})$ denote the real vector space of $(m \times n)$-matrices with real entries. $M_{n\times n}(\mathbb{R})$ will be written as $M_n(\mathbb{R})$. We shall identify $M_{m\times n}(\mathbb{R})$ with the euclidean space $\mathbb{R}^{mn}$ by writing the rows of a matrix A one after another to get a point in $\mathbb{R}^{mn}$. Thus $M_{m\times n}(\mathbb{R}) = \mathbb{R}^{mn}$ is a manifold and clearly, for any $A \in M_{m\times n}(\mathbb{R})$, we have

$$T_A(M_{m\times n}(\mathbb{R})) = M_{m\times n}(\mathbb{R}).$$

The notation $GL_n(\mathbb{R})$ will stand for the group of invertible matrices in $M_n(\mathbb{R})$. If $\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ denotes the determinant function, then as

$$GL_n(\mathbb{R}) = \det{}^{-1}(\mathbb{R} - 0)$$

we have that $GL_n(\mathbb{R})$ is open in $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$ and hence is a manifold of dimension n^2. Notice that the function det is smooth.

Here are some familiar subgroups of $GL_n(\mathbb{R})$. The special linear group $SL_n(\mathbb{R})$ is defined to be

$$SL_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) \;:\; \det(A) = 1\}.$$

Thus $SL_n(\mathbb{R}) = \det^{-1}(1)$.

For a matrix A, let A^t denote its transpose. The orthogonal group $O(n)$ is defined to be

$$O(n) = \{A \in GL_n(\mathbb{R}) \;:\; AA^t = A^tA = I_n\}$$

where I_n denotes the identity $(n\times n)$-matrix. A matrix A is said to be *symmetric* if $A = A^t$. Clearly, for any square matrix A, we have that AA^t is symmetric. Let $\mathrm{Sym}(n)$ denote the vector space of symmetric $(n \times n)$-matrices. The map

$$f : M_n(\mathbb{R}) \longrightarrow \mathrm{Sym}(n) = \mathbb{R}^{\frac{n(n+1)}{2}} \tag{2.3.1}$$

defined by $f(A) = AA^t$ is clearly smooth and

$$O(n) = f^{-1}(I). \tag{2.3.2}$$

Since for any $A \in O(n)$, $\det(A) = \pm 1$, the space $O(n)$ has at least two components. The special orthogonal group $SO(n)$ is defined to be

$$SO(n) = \{A \in O(n) \,:\, \det(A) = 1\}.$$

$SO(n)$ is an index two subgroup of $O(n)$. It can be shown that $SO(n)$ is connected[1] (infact path connected). It then follows that $O(n)$ has exactly two components and $SO(n)$ is the component containing I.

Lemma 2.4. The orthogonal group $O(n)$ is compact.

Proof. Let $A \in O(n)$ be an orthogonal matrix. If $v_1, \ldots, v_n$ denote the row vectors of A, then as $AA^t = I$ we have that the inner product

$$v_i \cdot v_i = 1.$$

Thus, as a subset of $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$, the orthogonal group $O(n)$ is bounded. The Equation (2.3.2) shows that $O(n)$ is also closed. $\square$

Thus the row vectors (also the column vectors) of an orthogonal matrix are unit vectors in $\mathbb{R}^n$ Thus $O(n)$ is actually a subset of

$$\mathbb{S}^{n-1} \times \cdots \times \mathbb{S}^{n-1}$$

a n-fold product.

We now show that the matrix groups defined above and some that we shall define below are manifolds.

Example 2.5. We shall show that the space $O(n)$ of orthogonal matrices is a manifold. We have seen above that $O(n) = f^{-1}(I)$ where

$$f : M_n(\mathbb{R}) \longrightarrow \mathrm{Sym}(n)$$

is defined by $f(A) = AA^t$. We wish to use the preimage theorem to prove that $O(n)$ is manifold. Towards this we shall check that $I \in \mathrm{Sym}(n)$ is a regular value of f. Since f is clearly smooth, given $A \in M_n(\mathbb{R})$ the derivative

$$df_A : T_A(M_n(\mathbb{R})) = M_n(\mathbb{R}) \longrightarrow T_{f(a)}\mathrm{Sym}(n) = \mathrm{Sym}(n)$$

is given by

$$\begin{aligned} df_A(B) &= \lim_{h\to 0} \tfrac{f(A+hB)-f(A)}{h} \\ &= \lim_{h\to 0} \tfrac{(A+hB)(A+hB)^t - AA^t}{h} \\ &= BA^t + AB^t. \end{aligned}$$

This gives a complete description of the derivative. Now let $A \in f^{-1}(I) = O(n)$. Given $C \in \mathrm{Sym}(n)$ it is easy to see that

$$df_A\left(\frac{CA}{2}\right) = C.$$

Thus df_A is surjective and hence I is a regular value of f. Hence by the preimage theorem, $O(n)$ is a submanifold of $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$ of codimension equal to $n(n+1)/2$. Thus

$$\dim(O(n)) = n^2 - (n(n+1))/2 = n(n-1)/2.$$

The tangent space to $O(n)$ at A is the kernel of the map

$$df_A : T_A(M_n(\mathbb{R})) = M_n(\mathbb{R}) \longrightarrow T_{f(a)}\mathrm{Sym}(n) = \mathrm{Sym}(n).$$

---

[1] There are several ways to prove this. One way is to use group actions. It turns out that the homogeneous space $SO(n)/SO(n-1)$ is the sphere $\mathbb{S}^{n-1}$ and then use induction. Another way is to look at the CW decomposition of $SO(n)$. It is a fact that $SO(n)$ has a CW structure with one 0-cell and hence is path connected.

Thus

$$T_A(O(n)) = \{B \in M_n(\mathbb{R}) \,:\, AB^t + BA^t = 0\}.$$

In particular, if $A = I_n = I$, then

$$T_I(O(n)) = \{B \in M_n(\mathbb{R}) \,:\, B^t + B = 0\}$$

the vector space of skew-symmetric matrices in $M_n(\mathbb{R})$.

Thus $O(n)$ is a manifold and hence its components are (also) open[2]. Since $SO(n)$ is a connected component of $O(n)$, it is open in $O(n)$ and hence is a manifold of the same dimension as $O(n)$ and $T_A(SO(n)) = T_A(O(n))$ for all $A \in SO(n)$.

Example 2.6. We now turn our attention to the special linear group $SL_n(\mathbb{R})$. Recall that

$$SL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \,:\, \det(A) = 1\}.$$

It is also the level set $\det^{-1}(1)$ for the determinant defined on $M_n(\mathbb{R})$. We compute the value of the derivative

$$d\det_A : T_A(M_n(\mathbb{R})) \longrightarrow \mathbb{R}$$

at $A \in M_n(\mathbb{R})$ on the tangent vector $A \in T_A(M_n(\mathbb{R})) = M_n(\mathbb{R})$. We have

$$\begin{aligned} d\det_A(A) &= \lim_{h\to 0} \tfrac{f(A+hA)-f(A)}{h} \\ &= \lim_{h\to 0} \tfrac{\det(A+hA)-\det(A)}{h} \\ &= \lim_{h\to 0} \tfrac{(1+h)^n\det(A)-\det(A)}{h} \\ &= n\cdot\det(A). \end{aligned}$$

which is nonzero if $A \in \det^{-1}(1) = SL_n(\mathbb{R})$. So det is a submersion at each $A \in SL_n(\mathbb{R})$. By the preimage theorem, $SL_n(\mathbb{R})$ is a submanifold of $M_n(\mathbb{R})$ of codimension 1. Hence the dimension of $SL_n(\mathbb{R})$ is $n^2 - 1$.

To understand the tangent space at $A \in SL_n(\mathbb{R})$ we need to understand the derivative of the determinant completely. Let $f : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ denote the determinant function. Then for $B \in T_A(M_n(\mathbb{R}))$ and $A \in SL_n(\mathbb{R})$ we have

$$\begin{aligned} df_A(B) &= \lim_{h\to 0} \tfrac{f(A+hB)-f(A)}{h} \\ &= \lim_{h\to 0} \tfrac{\det(A+hB)-\det(A)}{h} \\ &= \lim_{h\to 0} \tfrac{\det(A(I+hA^{-1}B)-\det(A)}{h} \\ &= \lim_{h\to 0} \tfrac{\det(A)\det(I+hA^{-1}B)-\det(A)}{h} \\ &= \det(A)\lim_{h\to 0} \tfrac{\det(I+hA^{-1}B)-1}{h} \\ &= \operatorname{tr}(A^{-1}B). \end{aligned}$$

The justification for the last step above is the following. The expansion of $\det(I + hA^{-1}B)$ is a polynomial in h. It is not difficult to see that the constant term is 1 and the coefficient of h is $\operatorname{tr}(A^{-1}B)$. Once we have this complete description of the derivative we can immediately compute the tangent spaces. Observe that for $A \in SL_n(\mathbb{R})$,

$$T_A(SL_n(\mathbb{R})) = \{B \in M_n(\mathbb{R})\,;\, \operatorname{tr}(A^{-1}B) = 0\}.$$

In particular the tangent space at $A = I$ equals

$$T_I(SL_n(\mathbb{R})) = \{B \in M_n(\mathbb{R})\,;\, \operatorname{tr}(B) = 0\}$$

the vector space of trace zero matrices. Notice that we did not need the complete description of the derivative (of determinant) to show that $SL_n(\mathbb{R})$ is a manifold.

[2] Components are always closed. They are also open if the space is locally connected.

Let I_n denote the $(n \times n)$ identity matrix and let J denote the $(2n \times 2n)$ matrix

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

then $\det(J) = 1$ and $J^{-1} = J^t = -J$. The *symplectic group* $Sp(2n)$ is defined to be

$$Sp(2n) = \{A \in M_{2n}(\mathbb{R}) \,:\, A^t J A = J\}.$$

It is well known[3] that if $A \in Sp(2n)$, then $\det(A) = 1$. Thus $Sp(2n)$ is a subgroup of the special linear group $SL_{2n}(\mathbb{R})$.

Recall that a matrix A is skew-symmetric if $A^t = -A$.

Example 2.7. The symplectic group $Sp(2n)$ is a manifold. To see this let $\mathrm{SkSym}(n)$ denote the vector space of skew-symmetric $(n \times n)$-matrices. Then

$$\mathrm{SkSym}(n) = \mathbb{R}^{\frac{n(n-1)}{2}}.$$

Observe that J is a skew-symmetric matrix. Now consider the smooth function

$$f : M_{2n}(\mathbb{R}) \longrightarrow \mathrm{SkSym}(2n)$$

given by

$$f(A) = A^t J A.$$

Then $Sp(2n) = f^{-1}(J)$. We claim that J is a regular value of f. Given $A \in Sp(2n)$ we compute the derivative

$$df_A : T_A(M_{2n}(\mathbb{R})) = M_{2n}(\mathbb{R}) \longrightarrow T_J(\mathrm{SkSym}(2n)) = \mathrm{SkSym}(2n).$$

as follows. For $B \in T_A(M_{2n}(\mathbb{R}))$ we have

$$\begin{aligned} df_A(B) &= \lim_{h \to 0} \tfrac{f(A+hB) - f(A)}{h} \\ &= \lim_{h \to 0} \tfrac{(A+hB)^t J (A+hB) - (A+hB)}{h} \\ &= A^t J B + B^t J A. \end{aligned}$$

So now given $C \in \mathrm{SkSym}(2n)$ we can quickly check using the facts mentioned above that if we let

$$B = (1/2)(AJ^{-1}C)$$

then

$$df_A(B) = C.$$

This shows that df_A is surjective and hence J is a regular value of f. Thus $Sp(2n)$ is a submanifold of $M_{2n}(\mathbb{R})$ of codimension equal to $n(n-1)/2$.

Here are some exercises. Remember that $X, Y, Z, \ldots$ will always denote manifolds and all maps/functions are smooth.

Exercise 2.8. Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ be the map defined by

$$f(x_1, x_2, x_3, x_4) = (x_1^2 + x_2^2 - x_3^2 - x_4^2, x_1^2 + x_2^2 + x_3^2 + x_4^2 - 4).$$

Show that $Z = f^{-1}(0)$ is a submanifold of $\mathbb{R}^4$. Find its dimension. Find a basis of the tangent space to Z at $p = (1, 1, -1, -1)$.

[3] See, for example, https://arxiv.org/pdf/1505.04240.pdf.

Exercise 2.9. Consider the function $f : \mathbb{R}^3 - \{z\text{-axis}\} \longrightarrow \mathbb{R}$ defined by

$$f(x, y, z) = (2 - \sqrt{x^2 + y^2})^2 + z^2.$$

Show that 1 is a regular value of f. Identify the manifold $Z = f^{-1}(1)$.

Exercise 2.10. Prove that the set of real (2×2)-matrices of rank 1 is a 3-dimensional submanifold of $M_2(\mathbb{R})$.

Exercise 2.11. Prove that the set of $(m \times n)$ matrices of rank r is a submanifold of $\mathbb{R}^{mn}$ of codimension equal to $(m - r)(n - r)$.

Exercise 2.12. Let Ω denote the $((n + 1) \times (n + 1))$ matrix

$$\Omega = \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix}$$

Let

$$X = \{A \in M_{n+1}(\mathbb{R}) \, : \, A^t \Omega A = \Omega\}.$$

Show that X is a manifold. Find its dimension.

Exercise 2.13. The product $S^2 \times S^2$ is a manifold that is a subset of $\mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6$. Prove that there is a submanifold $X \subseteq \mathbb{R}^5$ that is diffeomorphic to $S^2 \times S^2$. Now generalize. Show that there is no subset of $\mathbb{R}^4$ that is diffeomorphic to $S^2 \times S^2$. You could try out similar questions with other manifolds familiar to you.

Exercise 2.14. Let $X \subseteq \mathbb{R}^3$ be a compact 2-manifold. Prove that there exist at least two (distinct) points $x, y \in X$ such that both the tangent spaces $T_x(X)$ and $T_y(X)$ are spanned by the vectors $(1, 0, 0)$ and $(0, 1, 0)$. Is this true if the compactness assumption is dropped?

Exercise 2.15. Let $F : M_n(\mathbb{R}) \longrightarrow \mathbb{R}^n$ be the map that sends a matrix to the first column vector. This restricts to a smooth map

$$f = F/O(n) : O(n) \longrightarrow \mathbb{S}^{n-1}.$$

Show that f is a submersion. Is $f/SO(n)$ a submersion?

Exercise 2.16. Use (Exercise 2.10, Lecture 4 - II) to show that the boundary of the unit square is not a submanifold of $\mathbb{R}^2$.

Exercise 2.17. Use (Exercise 2.10, Lecture 4 - II) to show that the cone $x^2 + y^2 - z^2 = 0$, $z \geq 0$, is not a submanifold of $\mathbb{R}^3$.

Exercise 2.18. Convince yourself by an example that the inverse image of a critical value can be a manifold.

Exercise 2.19. Let A be a symmetric real $(n \times n)$-matrix and $c \in \mathbb{R}$. Set

$$X = \{x \in \mathbb{R}^n \, : \, x^t A x = c\}.$$

Is X a manifold?

Exercise 2.20. Convince yourself that there is an immersion of $(S^1 \times S^1)$ minus a point into $\mathbb{R}^2$. Can any such immersion be one-one? Note that there does not exist an immersion of $S^1 \times S^1$ into $\mathbb{R}^2$.

Exercise 2.21. Suppose $f : S^1 \longrightarrow \mathbb{R}$ is smooth and $y \in \mathbb{R}$ is a regular value. Show that $f^{-1}(y)$ has even number of elements.

Exercise 2.22. We identify S^1 with the "equator" in S^2, that is, with the the set of points $(x, y, z) \in S^2$ with $z = 0$. Is there a smooth function $f : S^2 \longrightarrow \mathbb{R}$ with $f^{-1}(y) = S^1$ where y is a regular value of f? What is the answer if $\mathbb{R}$ is replaced by S^1?

Exercise 2.23. Let X be the subset of $\mathrm{Sym}(2) = \mathbb{R}^3$ defined by

$$X = \left\{ A = \begin{pmatrix} x & y \\ y & z \end{pmatrix} \ : \ \det(A) = -1, \ \mathrm{tr}(A) = 0 \right\} \subseteq \mathrm{Sym}(2).$$

Show that X is a submanifold of $\mathrm{Sym}(2)$. Is X a familiar manifold?

Exercise 2.24. Show that $SL_2(\mathbb{R})$ is diffeomorphic to $S^1 \times \mathbb{R}^2$.

DIFFERENTIAL TOPOLOGY - LECTURE 7

1. Introduction

The preimage theorem gives us a sufficient condition under which a level set $f^{-1}(y)$ of a smooth map $f : X \longrightarrow Y$, $y \in Y$ is a manifold. Thinking of the point y as a 0-dimensional manifold we now wish to understand, given a submanifold Z of Y, when is the inverse image $f^{-1}(Z)$ a submanifold of X. This leads to a fundamental notion in differential topology, namely, the notion of *transversality.*

The concept of transversality was developed by René Thom in 1954 in his PhD thesis[1] and contains the celebrated theorem which is now called the Thom transversality theorem.

The notion of transversality captures in a precise way the nature in which two manifolds intersect in space. The power of transversality lies in the fact that it is a "stable property"[2]. We shall make precise definitions soon, but here is a simple description of what is meant by transversality and its stabilty.

Consider the submanifolds

$$X = \mathrm{graph}(f(t) = t^2)$$

$$X' = \mathrm{graph}(g(t) = t^2 - 1)$$

$$Y = \{(t, 0)\ :\ t \in \mathbb{R}\}$$

of $\mathbb{R}^2$.

The manifolds X and Y intersect at the origin. The tangent spaces at the point of intersection do not add up to the tangent space of the ambient space, that is,

$$T_{(0,0)}(X) + T_{(0,0)}(Y) \neq T_{(0,0)}(\mathbb{R}^2) = \mathbb{R}^2.$$

One says that the intersection (of X and Y) is not transversal (or X and Y are not transversal). Notice that if we slightly perturb X or Y, then the two manifolds don't intersect at all. So this type of (tangential, or non transversal) intersection is not "stable". The two manifolds can be pulled apart (into a non intersecting situation) by a slight deformation of X or Y.

On the other hand the manifolds X' and Y intersect at two points $x = (-1, 0), (1, 0)$. At both these points the tangent spaces add up tp the tangent space of the ambient space, that is,

$$T_x(X') + T_x(Y) = T_x(\mathbb{R}^2) = \mathbb{R}^2.$$

This condition is summed up by saying that the intersection (of X' and Y) is transversal. Now notice that even if we slightly perturb X' or Y, the two manifolds continue to intersect and intersect transversally. We cannot pull them apart by slightly deforming either X' or Y. Thus in this sense transverse intersection is stable. Drawing pictures will help in visualizing the above situation.

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[1] Quelques propriétés globales des variétés différentiables, Comment. Math. Helv. 28, (1954). 17-86. English translation is available on the web.

[2] Actually, a stronger statement is true : transversality is *generic*. This means that even if transversality does not hold, it can be made to hold by a small deformation.

It is important to note that transverse intersection is as much a property of the intersecting manifolds as it is of the ambient space in which the intersection is occuring. For example, if we think of the manifolds X' and Y as submanifolds of $\mathbb{R}^3$, then for dimensional reasons the tangent spaces at the points of intersection cannot add up to the tangent space of the ambient space. So when considered as submanifolds of $\mathbb{R}^3$ the manifolds X' and Y cannot intersect transversally. Indeed, by a slight perturbation (in the z direction) the two manifolds can be pulled apart so as to not intersect at all.

Transversality captures and makes precise these geometric observations and more. We shall discuss this in the present set of notes.

Remember that $X, Y, Z, X', \ldots$ always denote manifolds and that all maps/functions that we consider are always smooth.

2. Transversality

We begin with the definition of when a map $f : X \longrightarrow Y$ is transversal to a submanifold Z of Y.

Definition 2.1. Let $f : X \longrightarrow Y$ be a smooth map between manifolds and Z a submanifold of Y. We say that f is *transversal* to the submanifold Z if the equality

$$\operatorname{im}(df_x) + T_{f(x)}(Z) = T_{f(x)}(Y) \tag{2.1.1}$$

holds for all $x \in f^{-1}(Z)$.

We use the notation $f \pitchfork Z$ to mean that f is transversal to Z. Thus for $f \pitchfork Z$ to hold we must have

$$\dim(X) + \dim(Z) \geq \dim(Y).$$

The sum of the vector spaces in the definition need not be a direct sum. Observe that if $Z = \{y\}$ is a point, then $f \pitchfork Z$ if and only if y is a regular value of f. Thus transversality generalizes the notion of a regular value.

Here are some examples.

Example 2.2. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$ and $Z = \{(x, 0) : t \in \mathbb{R}\}$. Let $f, g : \mathbb{R} \longrightarrow Y$ be the maps defined by $f(x) = (x, x^2)$ and $g(x) = (x, x^2 - 1)$. Let us check that $g \pitchfork Z$. Let $x \in g^{-1}(Z)$. Then $x = \pm 1$. Let $x = 1$. Then,

$$dg_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Hence,

$$\operatorname{im}(dg_1) = \operatorname{span}\{(1, 2)\}.$$

It is clear that

$$T_{(1,0)}(Z) = \operatorname{span}\{(1, 0)\}.$$

Thus with $x = 1$, the equation (2.1.1) holds. Similarly, equation (2.1.1) holds with $x = -1$. Hence $g \pitchfork Z$. It is an exercise to check that f is not transversal to Z.

Example 2.3. Consider the map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$f(x, y) = e^x(\cos y, \sin y).$$

Let $Z = S^1$. Then one checks easily that $f \pitchfork Z$.

Example 2.4. Let X, Z be submanifolds of Y. Let $i : X \hookrightarrow Z$ be the inclusion map. Then $i \pitchfork Z$ if and only if the equality

$$T_x(X) + T_x(Z) = T_x(Y)$$

holds for all $x \in i^{-1}(Z) = X \cap Z$.

Some remarks are in order.

Remark 2.5. Observe that if $f : X \longrightarrow Y$ is a submersion, then f is transversal to every submanifold Z of Y because in this case $\mathrm{im}(df_x) = T_{f(x)}(Y)$ for every $x \in X$ and so equation (2.1.1) always holds. We already have an idea of what stability means. In the examples above we have noted that the manifolds X and Y do not intersect transversally. If we slightly pull the manifold X in the positive (or negative) y direction the two manifolds now intersect transversally. This is what is meant by saying transversality is generic : If the intersection is not transversal, then by a slight deformation the intersection can be made transversal (see Footnote 2 on page 1). The motivation for the definition of transversality will become clear while proving the next theorem.

The following theorem generalizes the preimage theorem[3].

Theorem 2.6. Suppose $f : X \longrightarrow Y$ is a smooth map between manifolds. Let Z be a submanifold of Y. If f is transversal to Z, then $f^{-1}(Z)$ is a submanifold of X of codimension equal to the codimension of Z in Y. □

Before proving the theorem we remind ourselves of some facts that we have discussed earlier. More precisely, we recall here the content and conclusion of (Proposition 3.7, Lecture 5). Suppose that

$$g_1, \dots, g_\ell : X \longrightarrow \mathbb{R}$$

are smooth functions defined on a manifold X (or on an open subset U of X). Let

$$Z = \{x \in X \, : \, g_i(x) = 0, i = 1, \dots, \ell\}$$

be the set of common zeros of the functions g_is. We then have a map

$$g = (g_1, \dots, g_\ell)) : X \longrightarrow \mathbb{R}^\ell$$

which exhibits Z as the level set

$$Z = g^{-1}(0).$$

By the preimage theorem, Z is a submanifold of codimension equal to ℓ (the number of functions g_i) if 0 is a regular value of g. By (Lemma 3.6, Lecture 5), 0 is a regular value of g if and only if the functions

$$d(g_1)_x, \dots, d(g_\ell)_x \in T_x(X)^* \tag{2.6.1}$$

are linearly independent for each $x \in Z$. Recall that if equation (2.4.1) holds for all $x \in Z$ then we say that the functions g_i are independent and that Z is cut out by independent functions.

Finally we recall (Proposition 3.10, Lecture 5) which says that every submanifold Z of a manifold Y is locally cut out by independent functions. What this means is the following. Given a submanifold Z of Y of codimension ℓ and $y \in Z$, there is an open set U, $y \in U$, U open in Y and independent functions

$$g_1, \dots, g_\ell : U \longrightarrow \mathbb{R}$$

such that $U \cap Z$ is the set of common zeros of the functions g_is.

We shall use these facts to prove the above theorem.

Proof of Theorem 2.6. Let $x \in f^{-1}(Z)$ and $y = f(x) \in Z$. Since submanifolds are locally cut out by independent functions, we can find an open set U, $y \in U$, U open in Y and independent functions

$$g_1, \dots, g_\ell : U \longrightarrow \mathbb{R}$$

[3]The preimage theorem is due to Lev Pontryagin and was proved in the 1930s. It took almost 20 years more for the notion of transversality to be developed. Theorem 2.6 appears in Thom's thesis.

such that $U \cap Z$ is the set of common zeros of the functions g_is. This means that if we set

$$g = (g_1, \ldots, g_\ell) : U \longrightarrow \mathbb{R}^\ell,$$

then 0 is a regular value of g and $g^{-1}(0) = U \cap Z$. In particular, g is a submersion at y. We assume without loss of generality that g is a submersion on the whole of U. Set $V = f^{-1}(U)$. Then V is open in X and $W = V \cap f^{-1}(Z)$ is open in $f^{-1}(Z)$. We now look at the composition

$$g \circ f : V \longrightarrow \mathbb{R}^\ell.$$

Observe that as

$$\begin{aligned} (g \circ f)^{-1}(0) &= f^{-1}g^{-1}(0) \\ &= f^{-1}(U \cap Z) \\ &= f^{-1}(U) \cap f^{-1}(Z) \\ &= V \cap f^{-1}(Z) = W \end{aligned}$$

we have that W is a manifold of codimension ℓ if 0 is a regular value of $g \circ f$. We shall check that for all $p \in W$, the derivative

$$d(g \circ f)_p = dg_{f(p)} \circ df_p : T_p(X) \longrightarrow \mathbb{R}^\ell$$

is surjective. This will prove that W is a codimension ℓ submanifold of X. Observe that as g is a submersion on U,

$$dg_{f(p)} : T_{f(p)}(Y) \longrightarrow \mathbb{R}^\ell$$

is onto with

$$\text{kernel}(dg_{f(p)}) = T_{f(p)}(Z). \tag{2.6.2}$$

Now transversality tells us that

$$\text{im}(df_p) + T_{f(p)}(Z) = T_{f(p)}(Y).$$

Hence

$$dg_{f(p)}\left(\text{im}(df_p) + T_{f(p)}(Z)\right) = dg_{f(p)}(T_{f(p)}(Y)) = \mathbb{R}^\ell.$$

Since (2.7.2) holds, we have

$$dg_{f(p)} \circ df_p = d(g \circ f)_p$$

is onto. This shows that W is a codimension ℓ submanifold of X. Since W is open in $f^{-1}(Z)$ and $f^{-1}(Z)$ can be covered by such open sets, we conclude that $f^{-1}(Z)$ is a codimension ℓ submanifold of X. This completes the proof. □

Observe that the transversality condition (2.1.1) is precisely the condition required to show that 0 is a regular value of $g \circ f$ above.

Here is a definition of when two submanifolds intersect transversally.

Definition 2.7. Suppose X, Z are two submanifolds of Y. We say that X and Y are transversal (or intersect transversally) if $i \pitchfork Z$, where $i : X \hookrightarrow Y$ is the inclusion map. In this case we write $X \pitchfork Z$ to denote that X and Z are transversal.

For example it is easy to see that the two coordinate axes in $\mathbb{R}^2$ intersect transversally.

So now suppose that X, Z are submanifolds of Y and let $i : X \hookrightarrow Y$ denote the inclusion map. As we had observed in Example 2.4, $i \pitchfork Z$ if and only if

$$\text{im}(di_x) + T_x(Z) = T_x(X) + T_x(Z) = T_x(Y)$$

for all $x \in i^{-1}(Z) = X \cap Z$. In particular, for X and Z to intersect transversally we must have

$$\dim(X) + \dim(Z) \geq \dim(Y).$$

Assume that X, Z, Y are as above and that $X \pitchfork Z$, that is, $i \pitchfork Z$ where $i : X \hookrightarrow Y$ is the inclusion map. By Theorem 2.6

$$i^{-1}(Z) = X \cap Z$$

is a submanifold of X. Further, the codimension of $(X \cap Z)$ in X equals codimension of Z in Y. Hence the codimension of $(X \cap Z)$ in Y is the sum of the codimension of $(X \cap Z)$ in X and the codimension of X in Y. We have thus proved the following.

Proposition 2.8. Suppose X, Z are submanifolds of Y. If $X \pitchfork Z$, then $X \cap Z$ is a submanifold of Y and

$$\mathrm{codim}(X \cap Z) = \mathrm{codim}(X) + \mathrm{codim}(Z).$$

All the codimensions in the above equality are with respect to Y. □

A very interesting situation occurs under following conditions. Assume that X, Z are compact submanifolds of Y. Suppose that X, Z have complementary dimensions, that is,

$$\dim(X) + \dim(Z) = \dim(Y).$$

Assume now that X and Z intersect transversally. Then $X \cap Z$ is a (compact) submanifold of Y. It follows from our observations above that

$$\dim(X \cap Z) = 0$$

so that $X \cap Z$ must consist of just a finite set of points.

Thus if a circle and a sphere intersect transversally in $\mathbb{R}^3$ then they must intersect only at a finite set of points. If two 2-spheres intersect tranversally in $\mathbb{R}^4$, then they must intersect only at a finite set of points. Two spheres intersecting transversally in $\mathbb{R}^3$ can have 1-dimensional intersection as you can easily verify.

Here are some exercises. It is important to solve the exercises to make our understanding firm. Most of the exercises are from G and P.

Exercise 2.9. Suppose that $A : \mathbb{R}^k \longrightarrow \mathbb{R}^m$ is a linear map. Let U, V be subspaces of $\mathbb{R}^m$. Show that $A \pitchfork V$ means $A(\mathbb{R}^k) + V = \mathbb{R}^m$. Further show that $U \pitchfork V$ means $U + V = \mathbb{R}^m$.

Exercise 2.10. Let V_1, V_2, V_3 be linear subspaces of $\mathbb{R}^m$. We say that they have normal intersection if

$$V_i \pitchfork (V_j \cap V_k)$$

for $i \neq j$ and $j \neq k$. Prove that this holds if and only if

$$\mathrm{codim}(V_1 + V_2 + V_3) = \mathrm{codim}(V_1) + \mathrm{codim}(V_2) + \mathrm{codim}(V_3).$$

Exercise 2.11. Prove the claim made in Example 2.3.

Exercise 2.12. Suppose that X, Z are transversal submanifolds of Y. Show that for $x \in X \cap Z$ we have

$$T_x(X \cap Z) = T_x(X) \cap T_x(Z).$$

Note that all the tangent spaces above are subspaces of $T_x(Y)$.

Exercise 2.13. Let $f : \mathbb{R}^2 - 0 \longrightarrow \mathbb{R}^2$ be the map defined by

$$f(x, y) = \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{x}{\sqrt{x^2 + y^2}} \right).$$

Show that f is not transverse to the submanifold S^1 of $\mathbb{R}^2$. Construct a map $g : \mathbb{R}^2 - 0 \longrightarrow \mathbb{R}^2$. such that g is homotopic to f and $f \pitchfork S^1$. This exercise shows the possibility of modifying a non transverse function to a function which is transverse within the homotopy class.

Exercise 2.14. Let Z be a submanifold of Y and let $f : X \longrightarrow Y$ be such that $f \pitchfork Z$. Show that if $x \in f^{-1}(Z)$, then

$$T_x(f^{-1}(Z)) = df_x^{-1}(T_{f(x)}(Z)).$$

Here $df_x : T_x(X) \longrightarrow T_{f(x)}(Y)$.

Exercise 2.15. Suppose that $X \xrightarrow{f} Y \xrightarrow{g} Z$ are maps between manifolds. Let W be a submanifold of Z. Let $g \pitchfork W$. Show that $f \pitchfork (g^{-1}(W)$ if and only if $(g \circ f) \pitchfork W$.

Exercise 2.16. Let $A : V \longrightarrow V$ be a linear map where V is a vector space. Let $\triangle \subseteq V \times V$ be the diagonal. Show that

$$(\text{graph}(A)) \pitchfork \triangle$$

if and only if $+1$ is not an eigenvalue of A.

Exercise 2.17. Let $f : X \longrightarrow X$. A fixed point x of f is called a Lefschetz fixed point if $+1$ is not an eigenvalue of df_x. f itself is called a Lefschetz map if all its fixed points are Lefschetz. Prove that if X is compact and f Lefschetz, then f has only finitely many fixed points.

Exercise 2.18. Identify $\mathbb{C}^2$ with $\mathbb{R}^4$ by identifying $(z_1, z_2) \in \mathbb{C}^2$ with $(x, y, u, v) \in \mathbb{R}^4$ where $z_1 = x + iy$ and $z_2 = u + iv$. Let

$$X = \{(z_1, z_2) \, : \, z_1^3 + z_2^2 = 0\}.$$

Show that $X - (0, 0)$ is a manifold. What is its dimension?

Exercise 2.19. Is it true that every smooth map $f : S^2 \longrightarrow S^1$ has a critical point?

Exercise 2.20. Let $X \subseteq \mathbb{R}^6$ be the subset defined by the equations

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 = 1$$
$$x_4^2 - x_5^2 - x_6^2 = -1.$$

Show that X is a manifold. Find its dimension. Let X_1 denote the manifold defined by the first equation and X_2 denote the manifold defined by the second. Do X_1 and X_2 intersect transversally?

Exercise 2.21. Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be tha map $f(x, y, z) = (xy, yz)$. Is f transverse to S^1?

DIFFERENTIAL TOPOLOGY - LECTURE 8

1. Introduction

In the last set of notes we tried to understand the basics of the notion of transversality. We also tried to understand, albeit in a non precise way, the notion of stability. We shall now make precise the notion of stability of certain properties that a function may have. A property of a map is said to be stable if the function continues to have the property when deformed by a small amount. The notion of deformation of a function is captured by homotopy. We shall recall these definitions and try to understand which properties of a function are preseved under a slight deformation.

We remind ourselves that $X, Y, Z, X' \dots$ shall always denote manifolds and all maps/functions will be smooth.

2. Stability

We begin with notion of homotopy and then define stability.

Definition 2.1. Let $f, g : X \longrightarrow Y$ be smooth maps between manifolds. We say f is *homotopic* to g and write $f \sim g$ if there exists a smooth map $F : X \times [0,1] \longrightarrow Y$ with $F(x,0) = f(x)$ and $F(x,1) = g(x)$.

The map F is called a homotopy from f to g. The homtopy F gives rise to a parametrized family

$$F_t : X \longrightarrow X$$

of smooth maps defined by $F_t(x) = F(x,t)$. Observe that $F_0 = f$ and $F_1 = g$. At each time $t \in [0,1]$, the map $F_t : X \longrightarrow X$ is smooth. One interprets the homotopy F as a way of deforming f over time to g.

Let P be a property that a map can possess. For example P could be any one of the following properties : immersion, submersion, local diffeomorphism, embedding, being transverse to a given manifold etc. We shall define when such a property is stable.

Definition 2.2. A property P of a function is said to be a stable property if whenever a map $f : X \longrightarrow Y$ has the property P and F_t is a homotopy of f (that is, $F_t : X \longrightarrow X$ is such that $F_0 = f$) then there exists $\varepsilon > 0$ such that F_t has the property P for all $t < \varepsilon$.

In other words, a property of a map is stable if when we start deforming the map by a homotopy, then then for some length of time the function continues to have that property. It is important to note that stability means that the condition in the definition should hold for *every* homotopy of the map f.

To begin with let us try to see some properties that are not stable. Let Z be the x-axis in $\mathbb{R}^2$ and let P be the property that a submanifold X of $\mathbb{R}^2$ intersects Z. We wish to know whether this property of a submanifold intersecting Z is stable? This can be translated in terms of maps as follows. Let X be a submanifold of $\mathbb{R}^2$ and $i : X \hookrightarrow \mathbb{R}^2$ be the inclusion such that

$$i(X) \cap Z = X \cap Z \neq \emptyset.$$

We wish to know if this is stable? That is we wish to know if we take any homotopy F_t of i with $F_0 = i$, then does there exist $\varepsilon > 0$ such that

$$F_t(X) \cap Z \neq \emptyset$$

for all $t < \varepsilon$. Here is an example to show that this is not true.

Exercise 2.3. Let X be the submanifold defined by

$$X = \{(x, x^2) : x \in \mathbb{R}\}$$

and let $i : X \longrightarrow \mathbb{R}^2$ be the inclusion map. Then $i(X) \cap Z \neq \emptyset$. Let F_t be the homotopy of i given by

$$F_t(x, x^2) = (x, x^2 + t).$$

Then $F_0 = i$ and for every $t > 0$

$$F_t(X) \cap Z = \emptyset.$$

Thus the property of two manifolds intersecting is not a stable property.

The above example also shows (by altering the homotopy) that non transversal intersection is not a stable property. This is relegated to the exercises.

The following terminology is often used : If a property P is stable, then the class of maps having the property P is said to form a *stable class.* Our aim in this set of notes is to discuss the following beautiful theorem.

Theorem 2.4. (Stability theorem) The following classes of maps $f : X \longrightarrow Y$ from a compact manifold X to a manifold Y form a stable class :

(1) local diffeomorphisms,
(2) immersions,
(3) submersions,
(4) maps transversal to a given closed submanifold Z of Y,
(5) embeddings,
(6) diffeomorphisms.

Proof. Before discussing the proof let us be sure that we understand the statement of the theorem. For example, the statement that local diffeomorphisms form a stable class when considering maps defined on compact domains means that whenever X is a compact manifold and $f : X \longrightarrow Y$ a local diffeomorphism and F_t is a homotopy of f, then there exists $\varepsilon > 0$ such that $F_t : X \longrightarrow Y$ is a local diffeomorphism for all $t < \varepsilon$.

We begin by proving (2). We claim that for each $x \in X$, there exists a neighborhood U_x of $(x, 0)$ in $X \times I$ such that $d(f_t)_y$ is injective whenever $(y, t) \in U_x$. We may now cover $X \times \{0\}$ by finitely many open sets

$$X \times \{0\} \subseteq \cup_{i=1}^{s} U_{x_i} = U$$

as X is compact. By construction, $d(f_t)_y$ is injective whenever $(y, t) \in U$. Now (by tube Lemma), we can find $\varepsilon > 0$ such that

$$X \times [0, \varepsilon) \subseteq U.$$

Thus $f_t : X \longrightarrow Y$ is an immersion for $t < \varepsilon$. The proof of the claim is seen as follows. We may locally work in the euclidean space. Since $d(f_0)_x$ is injective and $(x, t) \mapsto d(f_t)_x$ is continuous, there exists a neghborhood $U_x = V_x \times [0, \varepsilon_x)$ of (x, o) such that whenever $(y, t) \in U_x$, $d(f_t)_y$ is injective.

The proof of (3) is similar to (2). The proof of (1) is a consequence of (2) (or (3)).

The condition (4) is proved as follows. We keep the proof of the transversality theorem in mind. We start with a smooth map $f : X \longrightarrow Y$ with $f \pitchfork Z$. We then know that $f^{-1}(Z)$ is a submanifold of X. Let f_t be a homotopy of f. Now given $x \in f^{-1}(Z)$ we can find a neighborhood U of $f(x)$ in Y and a submersion

$$g : U \longrightarrow \mathbb{R}^\ell$$

such that $g^{-1}(0) = U \cap Z$. If $V = f^{-1}(U)$ and $W = V \cap f^{-1}(Z)$, then we know that 0 is a regular value of

$$g \circ f : W \longrightarrow \mathbb{R}^\ell.$$

In other words $g \circ f$ is a submersion on W. Since $g \circ f_t$ is a homotopy of $g \circ f$, there exists $\varepsilon_x > 0$ such that $g \circ f_t$ is a submersion on W for all $t < \varepsilon_x$. We may now cover the compact set $f^{-1}(Z)$ by finitely many such open sets and get a suitable ε.

We now turn to the proof of (5). We assume that $f : X \longrightarrow Y$ is an embedding and f_t a homotopy of f. Then $f_0 = f$ is one-one, an immersion and proper. We already know that f stays an immersion for a small interval of time. We only neeed to check that f stays one-one for a small interval of time. The proof is by contradition. We assume there exists a sequence $t_n \to 0$ and distinct points $x_n, y_n \in X$ such that

$$f_{t_n}(x_n) = f_{t_n}(y_n).$$

We assume that $x_n \to x$ and $y_n \to y$. Then

$$f(x) = f_0(x) = \lim_{n\to\infty} f_{t_n}(x_n) = \lim_{n\to\infty} f_{t_n}(y_n) = f_0(y) = f(y).$$

Then as f is injective, we have $x = y$.

Now consider the function $G : X \times I \longrightarrow Y \times I$ defined by

$$G(x,t) = (f_t(x), t).$$

We then have that

$$G(x_n, t_n) = G(y_n, t_n). \tag{2.4.1}$$

We may work locally in the euclidean space. The matix of $dG_{(x,0)}$ has the (block) form

$$\begin{pmatrix} d(f_0)_x & * \\ 0 & 1 \end{pmatrix}$$

The Jacobian matrix $d(f_0)_x$ has rank k (the dimension of X) and hence the above matrix has rank equal to $k+1$. The map G therefore is injective in a neighborhood of $(x,0)$. But this is a contradiction to (2.4.1).

Finally we prove (6). We suppose that $f : X \longrightarrow Y$ is a diffeomorphism with X compact and that we are a given a homotopy $F_t : X \longrightarrow Y$ of f so that $F_0 = f$. We shall show that there exists $\varepsilon > 0$ such that F_t is a diffeomorphism for all $t < \varepsilon$. To begin with since X is compact, X has only finitely many components. We may therefore assume that X is connected (for we can find ε for each component and then take the smallest). Thus Y is also connected and both have the same dimension. Now, as f is also an embedding, we can (by (5)) find an $\varepsilon > 0$ such that F_t is an embedding for all $t < \varepsilon$. The proof will be complete if we can show that F_t is onto for all $t < \varepsilon$. This is clearly so (see the paragraph before Corollary 2.5, Lecture 4 - II). $\square$

This is a fundamental theorem. Such stability theorems concerning stability of various differential properties are of importance. Of particular interest is the part (4) of the above theorem. Here is an example to show that the compactness assumption in the above theorem is crucial.

Example 2.5. Let $\rho : \mathbb{R} \longrightarrow \mathbb{R}$ be a smooth function with the property

$$\rho(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 2 \end{cases}$$

Define $F_t : \mathbb{R} \longrightarrow \mathbb{R}$ by $F_t(x) = x\rho(tx)$. Observe that

$$F_0(x) = x\rho(0) = x \cdot 1 = x$$

is the identity map which is a diffeomorphism (also an embedding, immersion, submersion, local diffeomorphism). Notice that $f = F_0$ is transversal to every submanifold of $\mathbb{R}$. In particular, $f \pitchfork \{0\}$. Now let $t > 0$ and $|x| > 2/t$ so that $F_t(x) = 0$. Then as $dF_t(x) = 0$ we see that (by definition) F_t cannot be transversal to the submanifold $\{0\}$. This shows that for functions defined non compact domains the condition (4) of the Stability theorem can fail.

Since for $t > 0$ and $|x|, |y| > (2/t)$ we have $F_t(x) = F_t(y) = 0$, the functions F_t cannot be an embedding (since they are not one-one) for $t > 0$. This shows that for functions defined on non compact domains condition (5) of the Stability theorem can fail.

It is also clear from the above observations that for $t > 0$, F_t is not an immersion (therefore not a submersion, local diffeomorphism, diffeomorphism). Thus for functions defined on non compact domains all of (1)-(6) in the Stability theorem can fail.

Here are some problems.

Exercise 2.6. Show by an example that non transversal intersection is not stable.

Exercise 2.7. Show by an example that being one-one, onto is not a stable property.

Exercise 2.8. Suppose that $f_0, f_1 : X \longrightarrow Y$ are homotopic. show that the exists a homotopy

$$F : X \times [0, 1] \longrightarrow Y$$

such that $F(x, t) = f_0(x)$ for all $t \in [0, 1/4]$ and $F(x, t) = f_1(x)$ for all $t \in [3/4, 1]$.

Exercise 2.9. Show that homotopy is an equivalence relation.

Exercise 2.10. Show that every connected manifold is path connected. Further show that if $x, y \in X$ (X is connected), then there is a diffeomorphism $f : X \longrightarrow Y$ such that $f(x) = y$[1].

Exercise 2.11. A manifold X is contractible if its identity map is homotopic to some constant map. Show that $\mathbb{R}^k$ is contractible.

Exercise 2.12. A connected manifold X is simply connected if every map $S^1 \longrightarrow X$ is homotopic to a constant. Show that contractible spaces are simply connected.

Exercise 2.13. Show that the map $a : S^k \longrightarrow S^k$, $a(x) = -x$ is homotopic to the identity map if k is odd.

Exercise 2.14. A *deformation* of a submanifold Z of Y is a smooth homotopy $i_t : Z \longrightarrow Y$ where i_0 is the inclusion map of $Z \hookrightarrow Y$ and each i_t is an embedding. Thus $Z_t = i_t(Z)$ is a smoothly varying submanifold of Y with $Z_0 = Z$. Show that if Z is compact, then any homotopy i_t of its inclusion map is a deformation for small t. Give a counterexample in the noncompact case.

Exercise 2.15. Let

$$X = \{(A, v) \in M_{3\times 2}(\mathbb{R}) \times (\mathbb{R}^2 - 0) \; : \; Av = o\}.$$

Show that X is a manifold. Also exhibit X as the set of common zeros of independent functions.

[1]This exercise shows that the diffeomorphism group $\mathrm{Diff}(X)$ of a connected manifold X acts transitively on X. This has generalizations.

Exercise 2.16. Construct a smooth function $\rho : \mathbb{R} \longrightarrow \mathbb{R}$ having the prescribed properties as in Example 2.5.

DIFFERENTIAL TOPOLOGY - LECTURE 9

1. Introduction

We begin by recalling certain definitions and theorems that we have discussed earlier. Recall that for a function $f : X \longrightarrow Y$ between manifolds a point $y \in Y$ is called a regular value (of f) if f is a submersion at every $x \in f^{-1}(y)$. Thus if $y \notin f(X)$, then y is always a regular value. A point $y \in Y$ that is not a regular value is called a critical value. Thus the only way a point $y \in Y$ can be a critical value is that there exists (at least one) $x \in f^{-1}(y)$ such that

$$df_x : T_x(X) \longrightarrow T_y(Y)$$

is not surjective.

For a map $f : X \longrightarrow Y$ between manifolds a point $x \in X$ is called a regular point (of f) if f is a submersion at x. If f is not a submersion at x, that is,

$$df_x : T_x(X) \longrightarrow T_{f(x)}(Y)$$

is not onto, then x is called a critical point of f. Let C_f denote the set of critical points of f. Then the set of critical values of f equals $f(C_f)$, the image of the set of set of critical points.

It is of interest to understand, given a function $f : X \longrightarrow Y$, the nature of the set of critical points, the set of regular values (and therefore the set of critical values). In particular one would like to know (under what condtions) are these sets nonempty and how large are these sets if nonempty? Why should one be interested in knowing whether the above sets are nonempty, large? One reason is the preimage theorem which states that the inverse image of a regular value is a submanifold. Thus knowing whether a smooth function has a regular value is important.

Let us remind ourselves of certain possibilities. Recall that we had shown that if X is a compact manifold, then any smooth function $f : X \longrightarrow \mathbb{R}$ must have a critical point.

For dimensional reasons, if $\dim(Y) > \dim(X)$ and $f : X \longrightarrow Y$ is smooth, then $C_f = X$. The image $f(X) = f(C_f)$ consists entirely of critical values. So in this case regular values are those that are not in $f(X)$. The question that one can ask here is : how "large" can the set $f(C_f)$ of critical values be? Largeness, for example, could be the question: can $f(C_f)$ contain an open set?

On the other hand it is easy to find smooth maps that have no critical points. The simplest being a projection of the euclidean space. Given manifolds X, Y, the projection $X \times Y \longrightarrow X$ is a submersion and thus has no critical points. The map $p : \mathbb{R}^2 \longrightarrow S^1 \times S^1$ defined by

$$p(x, y) = (\cos x, \sin x, \cos y, \sin y)$$

is a local diffeomorphism and hence has no critical points. Finally there also exists maps $f : X \longrightarrow Y$ between compact manifolds (of different dimensions) having no critical points[1]. In the examples considered in the present paragraph, the set of critical values is as small as it can be. It is empty.

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[1] Examples of such maps are provided by a class of maps called fibre bundles which are always submersions. For example there is a fibre bundle $S^3 \longrightarrow S^2$, called the Hopf map. The projection $X \times Y \longrightarrow Y$, the tangent and normal bundles (which we shall soon encounter) of a manifold are examples of fibre bundles.

The answer to the question of how large can the set of critical values be is provided by Sard's theorem. We shall state this without proof and concentrate on some of its easy applications here. Sard's theorem has far reaching generalizations and is at the heart of transversality, Morse theory and intersection theory. We shall soon use this to understand some interesting facts about Morse functions.

2. Sard's theorem

In this section we shall state Sards theorem. We shall not prove this. We begin by discussing some technical definitions that all of us are familiar with.

Definition 2.1. A product of open intervals of the form

$$S = \Pi_{i=1}^{n}(a_i, b_i) \subseteq \mathbb{R}^n$$

is called a *rectangular solid.*

The volume of the rectangular solid S is then defined to be

$$\text{vol}(S) = \Pi_i(b_i - a_i).$$

Definition 2.2. A subset $A \subseteq \mathbb{R}^n$ is said to have *measure zero* in $\mathbb{R}^n$ if given $\varepsilon > 0$, there exists a countable collection $\{S_i\}$ of rectangular solids such that

$$\Sigma_i \text{vol}(S_i) < \varepsilon$$

and $A \subseteq \cup_i S_i$.

It is therefore clear that if A has measure zero and $B \subseteq A$, then B also has measure zero. Further if S is a rectangular solid, then S is not a set of measure zero (see Exercise 2.10). In particular, an open subset of $\mathbb{R}^n$ is not of measure zero.

One can extend this notion to manifolds via parametrizations.

Definition 2.3. A subset $A \subseteq X$ is said to have measure zero in the manifold X if for every parametrization

$$\varphi : U \longrightarrow X$$

the set $\varphi^{-1}(A)$ has measure zero in $\mathbb{R}^n$.

It is clear that subsets of sets of measure zero in a manifold have measure zero and that an open subset of a manifold is not a set of measure zero. In particular a set of measure zero cannot contain an open set (see Exercise 2.10).

Here is one more familiar terminology before we state the Sard's theorem. We say that almost every point in a manifold has a property P if the set of points that do not have the property P has measure zero. We are now in a position to state Sard's theorem.

Theorem 2.4. (Sard's theorem) Let $f : X \longrightarrow Y$ be a smooth map between manifolds. Then the set of critical values is a set of measure zero. Equivalently, if $f : X \longrightarrow Y$ is a smooth map between manifolds, then almost every point of Y is a regular value of f. □

The set of critical values of a smooth map f is the set $f(C_f)$ where C_f is the set of critical points of f.

Thus Sard's theorem tell us that any smooth map $f : X \longrightarrow Y$ has an abundance of regular values. Sometimes (for example, for the constant function) the regular values may lie outside the image of the function. An interesting situation is when the function is onto : then the theorem guarantees that there are points in the image that are regular values. This is a very powerful fact, as we shall see. Here are some consequences of the Sard's theorem.

Corollary 2.5. Let $f : X \longrightarrow Y$ be a smooth map between manifolds. Then the set of regular values of f is dense in Y.

Proof. Let C_f denote the set of critical points of f. Then as $f(C_f)$ has measure zero in Y, it cannot contain an open set. Thus every open set in Y must intersect the complement of $f(C_f)$. Hence the set of regular values (which is the complement of $f(C_f)$) is dense in Y. □

A much stronger statement is true.

Corollary 2.6. Let $f_i : X_i \longrightarrow Y$ be a countable family of smooth maps. Then the set

$$R = \{y \in Y \;:\; y \text{ is a regular value of each } f_i\}$$

is dense in Y.

Proof. Let C_i denote the set of critical points of f_i. Then by Sard's theorem $f(C_i)$ has measure zero in Y. Note that R is the complement of $\cup_i f(C_i)$. The corollary now follows as $\cup_i f(C_i)$ has measure zero in Y (see Exercise 2.9 below). □

Thus the corollary says that given a family f_i as above, the set of points $y \in Y$ that are simultaneously regular values of each f_i is dense in Y.

Corollary 2.7. Suppose that $f : X \longrightarrow Y$ is a smooth map between manifolds. If $\dim(X) < \dim(Y)$, then f is not surjective.

Proof. Let C_f denote the set of critical points of f. Clearly, by dimension asuumptions, $C_f = X$. Then as $f(C_f) = f(X)$ has measure zero in Y, we have that $f(X) \neq Y$. □

In particular, there cannot exist a smooth surjective map $f : \mathbb{R} \longrightarrow \mathbb{R}^2$. Observe that continuous surjections $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ do exist. Here is another corollary.

Corollary 2.8. If $k < n$, then every smooth map $f : S^k \longrightarrow S^n$ is null homotopic[2].

Proof. Recall that null homotopic means homotopic to constant. By Sard's theorem f cannot be onto. Thus f factors as a composition

$$S^k \longrightarrow \mathbb{R}^n \hookrightarrow S^n.$$

Since $\mathbb{R}^n$ is contractible, f must be null homotopic. □

Hence the sphere S^n is simply connected if $n > 1$ Here are some exercises.

Exercise 2.9. Show that a countable union of sets of measure zero is again a set of measure zero.

Exercise 2.10. Show that a rectangular solid is not a set of measure zero (see Appendix A in G and P).

Exercise 2.11. Let X be a compact manifold. Is it true that every smooth function $f : X \longrightarrow \mathbb{R}^n$ has a critical point if $n < \dim(X)$.

Exercise 2.12. Show that $\mathbb{R}^k$ is of measure zero in $\mathbb{R}^n$, $k < n$.

Exercise 2.13. Show that any submanifold Z of lower dimension in X is of measure zero.

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[2]It is a fact that given a continuous map $f : X \longrightarrow Y$ between manifolds, there exists a smooth map $g : X \longrightarrow Y$ such that f and g are continuously homotopic. In other words, every continuous map between manifolds is homotopic to a smooth map. This shows that every continuous map $f : S^k \longrightarrow S^n$, $k < n$, is also null homotopic.

th Let Ω be an open set of $\mathbb{R}^n$, starshaped (at o) then Ω is C^∞-diffeomorph to $\mathbb{R}^n$.

Proof Let $F = \mathbb{R}^n \setminus \Omega$ and $\varphi : \mathbb{R}^n \to \mathbb{R}^+$ be a C^∞ function such that $F = \varphi^{-1}(\{0\})$.

We set $f : \Omega \to \mathbb{R}^n$

$$x \mapsto \overbrace{\left[1 + \left(\int_0^1 \frac{dv}{\varphi(vx)}\right)^2 \|x\|_2^2\right]}^{\lambda(x)} . x = \left[1 + \left(\int_0^{\|x\|_2} \frac{dt}{\varphi(t\frac{x}{\|x\|_2})}\right)^2\right] . \|x\|_2 \times \frac{x}{\|x\|_2}$$

(where $\|x\|_2 = (\sum_i x_i^2)^{1/2}$)

f is smooth on $\mathbb{R}^n$. We set $A(x) = \sup\{t > 0,\ \frac{tx}{\|x\|_2} \in \Omega\}$. f send injectively $[0, A(x)[. \frac{x}{\|x\|_2}$ into $\mathbb{R}_+ . \frac{x}{\|x\|_2}$ moreover, if we set $u = \frac{x}{\|x\|_2}$, then

$\|f(0.u)\|_2 = 0$ and $\lim_{t \to A(x)} \|f(t.u)\| = \left[1 + \left(\int_0^{A(x)} \frac{dt}{\varphi(tu)}\right)^2\right] \times A(x) = +\infty$

indeed, if $A(x) = +\infty$ it is obvious

~~if $A(x) < \infty$ then~~ $\begin{cases} \varphi(A(x)\,x) = 0 \\ \varphi \text{ smooth} \end{cases} \Rightarrow \varphi(tu) \underset{t \to A(x)}{=} O(t - A(x))$

and so $\int_0^{A(x)} \frac{ds}{\varphi(su)}$ diverges.

~~We infer that $\varphi([0, A(x)[. \frac{x}{\|x\|_2}) = \mathbb{R}_+ \frac{x}{\|x\|_2}$ and so $\varphi(\Omega) = \mathbb{R}^n$.~~

To conclude, we have $d_x f(h) = \lambda(x) h + d_x \lambda(h) x$

so if $h \in \operatorname{Ker} d_x f \setminus \{0\}$ then there exists $\mu \in \mathbb{R}^*$ such that $h = \mu x$

and we get $[\lambda(x) + d_x \lambda(x)] = 0$ (note that $\lambda(0) = 1$ so $x \neq 0$).

but we have $\lambda(x) \geq 1$ and $g(t) = \lambda(tx)$ increasing so $g'(1) = d_x \lambda(x) x$

wich gives a contradiction.

Nota bene - The withney Theorem is a classical result.

- in the case $n = 2$ the Riemann theorem implies that Ω is holomorphically diffeomorph to $\mathbb{R}^2 \simeq \mathbb{C}$.