Let f be a 2π-periodic function on $\mathbb{R}$ and $f \in R[-\pi, \pi]$.

$\left\{\frac{1}{\sqrt{2\pi}} e^{inx}\right\}_{n \in \mathbb{Z}}$ is an orthonormal system on $[-\pi, \pi]$.

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\bar{g}$$

$$\langle f, g\rangle = 0, \quad f \neq g$$
$$= 1$$

Let $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$.

Let $S_N(f; x) = S_N = \sum_{n=-N}^{N} c_n e^{inx}$.

Let $D_N(x) = \sum_{n=-N}^{N} e^{inx}$.

<u>Lemma</u>: $D_N(x) = \dfrac{\sin\left(N+\frac{1}{2}\right)x}{\sin\frac{x}{2}}$.

<u>Proof</u>:

$$\frac{(e^{ix}-1)\, D_N(x)}{e^{i\frac{x}{2}}} = \left(\sum_{-N}^{N} e^{inx}\right)\left(\frac{e^{ix}-1}{e^{i\frac{x}{2}}}\right)$$

$$= \frac{1}{e^{ix/2}} \sum_{n=-N}^{N}\left[e^{i(n+1)x} - e^{inx}\right]$$

$$= \frac{e^{i(N+1)x} - e^{-iNx}}{e^{i\frac{x}{2}}}.$$

$$= e^{i(N+\frac{1}{2})x} - e^{-i(N+\frac{1}{2})x}$$

$$D_N(x) \; 2i \sin\frac{x}{2} = 2i \sin(N+\tfrac{1}{2}x)$$

$$\Rightarrow D_N(x) = \frac{\sin(N+\frac{1}{2})x}{\sin\frac{1}{2}x}$$

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Proposition:

$$S_N(f;x) = \overbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, D_N(x-t)\, dt}^{f * D_N(x)}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\, D_N(t)\, dt$$

and $$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x) = 1 .$$

Proof:

$$S_N = \sum_{n=-N}^{N} c_n e^{inx}$$

$$= \frac{1}{2\pi} \sum_{-N}^{N} \int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt\; e^{inx} .$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, D_N(x-t)\, dt$$

$$= f * D_N(x)$$

$s = x - t$
$ds = -dt$

$$= \frac{1}{2\pi} \int_{x-\pi}^{x+\pi} f(x-s)\, D_N(s)\, ds$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-s)\, D_N(s)\, ds \qquad (D_N(-x) = D_N(x))$$

$$= D_N * f(x)$$

$\times$

<u>Theorem</u> For $x \in [-\pi, \pi]$ $\exists\, \delta > 0$ and $M > 0$ such that

$$|f(x+t) - f(x)| \leq M|t| \quad \text{for all } t \in (-\delta, \delta).$$

Then $S_N(f; x) \longrightarrow f(x)$.

<u>Proof:</u> Let
$$g(t) = \begin{cases} \dfrac{f(x-t) - f(x)}{\sin\left(\frac{t}{2}\right)}, & 0 < |t| \leq \pi \\ 0, & t = 0 \end{cases}$$

$S_N(f; x) - f(x)$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} [f(x-t) - f(x)]\, D_N(t)\, dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \sin\left(N + \tfrac{1}{2}\right)t\, dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[g(t) \cos\frac{t}{2} \right] \sin Nt + \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[g(t) \sin\frac{t}{2} \right] \cos Nt$$

<u>Ex</u> $g(t)\cos\frac{t}{2}$; $g(t)\sin\frac{t}{2}$ are bounded.

$\longrightarrow 0$ as $N \to \infty$ (since $c_n \to 0$ as $n \to \infty$)

$\therefore S_N(f; x) \longrightarrow f(x)$.

<u>Corollary</u>: If $f = 0$ on an open interval J, then $\lim_{n\to\infty} S_N = 0$ $\forall x \in J$.

<u>Ex</u>

<u>Remark</u>: Two fourier series may have same behaviour in some interval but have different behaviour in some other interval.

Lemma: Let f be a continuous 2π-periodic function. Then there exists $F \in C(S^1)$ $\left(S^1 = \{z \in \mathbb{C} \mid |z|=1\} \cong \mathbb{R}/\mathbb{Z}\right)$ such that $f(x) = F(e^{ix})$.

Proof: $x \mapsto e^{ix}$ is bijective map from $[-\pi, \pi)$ onto S^1.

$z_n \in S^1$
$\parallel$
e^{ix_n}

Define $F(e^{ix}) = f(x)$.

$e^{ix_n} \longrightarrow e^{ix}$, $x_n, x \in [-\pi, \pi)$

(x_n) is a bounded sequence.

Case (i) $x \neq -\pi$

If (y_n) is a convergent subsequence and $a = \lim y_n$.

$e^{ix} \leftarrow e^{iy_n} \longrightarrow e^{ia}$.

$$a - x = 2n\pi, \quad n \in \mathbb{Z}$$

$|a| \leq \pi$ $\quad |x| < \pi$

$$-2\pi = -\pi - \pi < a - x < \pi + \pi = 2\pi$$

$$a - x = 0$$

$$a = x \quad \therefore\ y_n \to x$$

$\therefore\ x_n \longrightarrow x$

$\therefore\ F(e^{ix_n}) = f(x_n) \longrightarrow f(x) = F(e^{ix}).$

<u>Case (ii)</u> $x = -\pi$

Let (y_n) be a convergent subsequence of x_n and $\lim y_n = a$.

$$a - (-\pi) \in 2\cancel{\pi}\,\pi$$

$$a + \pi \in 2\mathbb{Z}\pi$$

$$a = \pm\pi$$

$f(y_n) \longrightarrow \begin{cases} f(\pi) \\ f(-\pi) \end{cases}$ But $f(\pi) = f(-\pi)$
$= f(x)$

$\Rightarrow\ f(x_n) \longrightarrow f(x)$

i.e., $F(e^{ix_n}) \longrightarrow F(e^{ix})$.

Thus, F is continuous on S^1.

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