Lecture 4:

02 February 2021
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Def^{n}: An nonempty algebraic set X in $\mathbb{A}^n$ is said to be irreducible if it not reducible and X is reducible if $X = X_1 \cup X_2$ where X_1 & X_2 are algebraic subsets of $\mathbb{A}^n$ properly contained in X.

Prop: Let k be alg closed field. $X \subseteq \mathbb{A}^n_k$ alg subset is irred iff $I(X) \subseteq k[x_1, .., x_n]$ is a prime ideal.

Pf: $(\Rightarrow)$: Let $fg \in I(X) \Rightarrow Z(fg) \supseteq Z(I(X)) \underset{\substack{\uparrow \\ \text{Last time}}}{=} X$

$$\Rightarrow Z(f) \cup Z(g) \supseteq X$$

$$\Rightarrow (Z(f) \cap X) \cup (Z(g) \cap X) = X$$

$$\xRightarrow{X \text{ irred}} X \subseteq Z(f) \text{ or } X \subseteq Z(g)$$

$$\Rightarrow f \in I(X) \text{ or } g \in I(X)$$

$X \neq \phi$
$\Rightarrow I(X) \neq k[x_1, .., x_n]$

$(\Leftarrow)$: Let $X = X_1 \cup X_2$ X_1, X_2 alg subset

Let $f \in I(X_1)$ & $g \in I(X_2)$ then $fg \in I(X_1 \cup X_2) = I(X)$

$$\xRightarrow{I(X) \text{ is prime}} f \in I(X) \text{ or } g \in I(X)$$

So if $I(X_1) \nsubseteq I(X) \Rightarrow I(X_2) \subseteq I(X)$

$$\Rightarrow X_2 = Z(I(X_2)) \supseteq Z(I(X)) = X$$

o.w. $X_1 \supseteq X$. Hence X is irred.

(*) X an alg set in $\mathbb{A}^n$ then $I(X) \subseteq k[x_1, \dots, x_n]$ is a radical ideal.

$$f^n \in I(X) \quad \text{for some } f \in k[x_1, \dots, x_n]$$

$$\Rightarrow f^n(a_1, \dots, a_n) = 0 \quad \forall\ (a_1, \dots, a_n) \in X$$

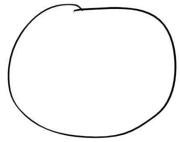
$$\Rightarrow f(a_1, \dots, a_n) = 0 \qquad ''$$

$$\Rightarrow f \in I(X)$$

<u>Def^n</u>: Let X be an affine algebraic set. Then <u>the coordinate ring</u> of X is defined to be $k[x_1, \dots, x_n]/I(X)$ if X is an alg subset of $\mathbb{A}^n_k$. It is denoted by $\mathcal{O}(X)$ or $k[X]$ or $\mathcal{O}_X$.

(*) Note that the coordinate ring of an affine algebraic set is a reduced ring and the coordinate ring of an affine variety is an integral domain.

<u>Example</u> 1)

$X_1 =$  $x^2+y^2=1$ in $\mathbb{A}^2_{\mathbb{C}}$

$$k[X_1] = \mathcal{O}(X_1) = \frac{\mathbb{C}[x,y]}{(x^2+y^2-1)}$$

2) $X_2 =$ 

$X_2 = \mathbb{A}^1_k$ or $y=0$ in $\mathbb{A}^2_k$

$I(X_2) = (0) \subseteq k[x]$ $\qquad$ $I(X_2) \subseteq k[x,y]$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\| $

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (y)

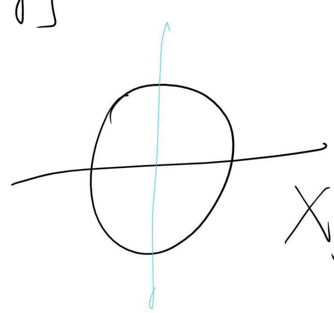
$$\mathcal{O}(X_2) = \frac{k[\cancel{x}]}{(0)} = k[x] \qquad \mathcal{O}(X_2) = \frac{k[x,y]}{(y)} \cong k(x)$$

③ $X_3 = X_1 \cup X_2$; $X_4 = X_1 \cap X_2$

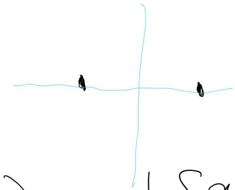
$$\mathcal{O}(X_3) = \frac{k[x,y]}{I(X_3)}$$

$$\begin{aligned} I(X_3) &= I(X_1) \cap I(X_2) \\ &= (x^2+y^2-1) \cap (y) \subseteq k[x,y] \\ &= (y(x^2+y^2-1)) \subseteq k[x,y] \end{aligned}$$

$$\mathcal{O}(X_3) = \frac{k[x,y]}{(y(x^2+y^2-1))}$$



X_3

$X_4 =$ 

$$\mathcal{O}(X_4) = \frac{k[x,y]}{I(X_4)}$$

$$I(X_4) = (x-1, y) \cap (x+1, y)$$

$$\mathcal{O}(X_4) = \frac{k[x,y]}{(x-1, y) \cap (x+1, y)}$$

$$\overset{CRT}{\cong} \frac{k[x,y]}{(x-1, y)} \times \frac{k[x,y]}{(x+1, y)}$$

$$\cong k \times k$$

$$I(X_4) = \sqrt{I(X_1) + I(X_2)}$$

$$= \sqrt{(x^2+y^2-1, y)}$$

$$= \sqrt{(x^2-1, y)} = (x^2-1, y)$$

$$\mathcal{O}(X_4) = \frac{k[x,y]}{(x^2-1, y)}$$

$$= \frac{k[x]}{(x^2-1)}$$

$$= k \times k$$

$\circledast$ X_1, X_2 alg subsets of $\mathbb{A}^n_k$ then

1) $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$

2) $I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$

Pf: 1) $f \in I(X_1 \cup X_2) \Longleftrightarrow f(a) = 0 \;\; \forall a \in X_1 \cup X_2$

$\Longleftrightarrow f \in I(X_1) \cap I(X_2)$

$X_1 \cap X_2 = Z(I(X_1 \cap X_2))$

$Z(\underbrace{I(X_1) + I(X_2)}_{J}) = Z(I(X_1)) \cap Z(I(X_2))$

$= X_1 \cap X_2$

$I(X_1 \cap X_2) = I(Z(J)) \overset{\uparrow}{=} \sqrt{J}$

HN

Ex $X_1 = Z(x^2+y^2-1)$; $X_2 = Z(y-1)$

$x^2 \in (x^2+y^2-1, y-1) = (x^2, y-1)$

But $x \notin (x^2+y^2-1, y-1)$

HN: If k is alg closed field. Then any maximal ideal of $k[x_1, \dots, x_n]$ is of the form $(x_1 - a_1, \dots, x_n - a_n)$ for some $a_1, \dots, a_n \in k$.