DIFFERENTIAL TOPOLOGY - LECTURE 5

1. Introduction

In this set of notes we shall see two new ways that manifolds arise[1]. Here is a brief discussion of the the two methods that we shall see.

Recall that in the previous notes (Lecture 4 - I) we have seen that if $f : X \longrightarrow Y$ is an immersion then $f(X)$ need not be a manifold. Even if f is injective. This raises the question about what additional conditions are needed to ensure that image of an immersion is a manifold. Let us recall what could go wrong.

Suppose $f : X \longrightarrow Y$ is an immersion. Then given $x \in X$, we know that there exists a (parametrizable) neighbourhood U of x restricted to which f is just the canonical immersion. Since the canonical immersion maps open sets diffeomorphically onto its image, we have that f maps U diffeomorphically onto $f(U)$. Now what could happen is that $f(U)$ may not be open in $f(X)$. If all such $f(U)$ were open, then as U is parametrizable $f(U)$ would be too and then, by definition, $f(X)$ would be a manifold. So we need to look for a condition (on f) that would ensure that $f(U)$ is open. The condition turns out to be topological and leads us to the notion of proper maps.

The next situation is the following. Given a smooth map $f : X \longrightarrow Y$ between manifolds and $p \in Y$ we wish to understand under what conditions is the level set $f^{-1}(p)$ a submanifold of X. This is answered by the preimage theorem. This is a powerful tool to construct examples of manifolds. We shall use this to show that many familiar spaces are manifolds.

Conventions. Recall that $X, Y, Z, \ldots$ will always denote manifolds and all maps/functions are always smooth.

2. Embeddings

We shall discuss when the image of an immersion is a manifold. We first make the following definition.

Definition 2.1. A smooth map $f : X \longrightarrow Y$ between manifolds is said to be *proper*[2] if the inverse image of every compact set is compact.

The definition of course makes sense for maps between topological spaces. Clearly, if X is compact and Y is Hausdorff, then every map $f : X \longrightarrow Y$ is proper. It is easy to see that every polynomial with real coefficients thought of as a function $\mathbb{R} \longrightarrow \mathbb{R}$ is proper. On the other hand polynomials in more than one variables need not be proper.

Definition 2.2. A smooth map $f : X \longrightarrow Y$ between manifolds is called an *embedding* if f is one-one, immersion and proper.

[1]We already know of two : Open subsets of manifolds and products of manfolds.

[2]The notion/theory of proper maps is extremely important. Proper maps make their appearance in many situations. One encounters proper maps in the theory of compactifications. A very important invariant that makes its appearance in topology/geomtery/ group theory is that of ends (of a space) and the right maps to look at are proper maps. Proper maps are also the right maps to look at when dealing with (co)homology with compact supports.

The result that we are interested in is the following.

Theorem 2.3. Suppose $f : X \longrightarrow Y$ is an embedding. Then $f(X)$ is a submanifold of Y.

Proof. Since f is an embedding we have that f is injective, proper and an immersion. As we noted above it suffices to show that if U is open in X, then $f(U)$ is open in $f(X)$. Here $f(X)$ has the subspace topology of Y. We assume that for some open set $U \subseteq X$, $f(U)$ is not open in $f(X)$ and derive a contradiction as follows.

Since $f(U)$ is not open we have that $(f(X)-f(U))$ is not closed and therefore must fail to contain one of its limit points which is forced to now lie in $f(U)$. Hence there exists a sequence $y_i \in (f(X)-f(U))$ that converges to a point $y \in f(U)$. Since f is proper and the set $\{y_i, y\}$ compact, we have that $f^{-1}\{y_i, y\}$ is compact. Since f is injective there exist unique $x_i, x \in X$ such that

$$f(x_i) = y_i, \quad f(x) = y.$$

Since $y \in f(U)$, and f is injective it follows that $x \in U$. Without loss of generality we may assume (using compactness) that $x_i \to x_0$, $x_0 \in X$. But as $f(x_i) = y_i$ converges to $f(x_0)$ as well as $y = f(x)$ we conclude that $f(x_0) = f(x)$. As f is injective, we have that $x = x_0$. Hence $x_i \to x$. As $x \in U$ and U is open almost every $x_i \in U$ and therefore almost every $f(x_i) = y_i \in f(U)$. This contradiction completes the proof that $f(U)$ must be open in $f(X)$ and hence $f(X)$ is a submanifold of Y. □

We have actually proved something stronger. We have actually showed that f maps X diffeomorphically onto $f(X)$. Notice that the fact that f is one-one has been crucially used to show that $f(U)$ is open in $f(X)$.

It would be interesting to find examples to show that the theorem is false if any one of the conditions of an embedding is dropped. Notice that none of the maps in (Examples 2.11 and 2.12, Lecture 4) are proper.

Finally, the converse of the above theorem is clearly true and is relegated to the exercises.

3. The preimage theorem

We shall now discuss the preimage theorem which states conditions under which a level set of a smooth function is a manifold. We shall state some definitions before proving the theorem.

Definition 3.1. Let $f : X \longrightarrow Y$ be a smooth function between manifolds and $y \in Y$. We say that y is a *regular value* of f if f is a submersion at each $x \in f^{-1}(y)$.

Points $y \in Y$ that are not regular values are said to be *critical values*. Thus the only way a point $y \in Y$ can be a critical value is that there exists $x \in f^{-1}(y)$ such that f is not a submersion at x. Observe that a point $y \notin f(X)$ automatically becomes a regular value.

Definition 3.2. Let $f : X \longrightarrow Y$ be a smooth map between manifolds. A point $x \in X$ is called a *critical point* of f if the derivative

$$df_x : T_x(X) \longrightarrow T_{f(x)}(Y)$$

is *not surjective*.

In other words, x is a critical point of f if f is not a submersion at x. We have already seen that a real valued function on a compact manifold must have a critical point. Points that are not critical points are called *regular points*. Thus regular points are precisely those points at which f is a submersion.

We are now in a position to state and prove the pre-image theorem. The proof is another application of the Local submersion theorem.

Theorem 3.3. (preimage theorem) Suppose that $y \in Y$ is a regular value of a smooth function $f : X \longrightarrow Y$. Then $Z = f^{-1}(y)$ is a submanifold of X and $\dim(Z) = \dim(X) - \dim(Y)$.

Proof. Suppose that $\dim(X) = k$ and $\dim(Y) = \ell$. Then $k \geq \ell$. Let $Z = f^{-1}(y)$ be non-empty and let $x \in Z$. We shall exhibit a neighborhood of x in Z that is diffeomorphic to an open set in the euclidean space $\mathbb{R}^{k-\ell}$. This will complete the proof.

Now as f is a submersion at x, by the local submersion theorem, there exists parametrizations

$$\varphi : U \longrightarrow X; \quad \psi : V \longrightarrow Y$$

with $\varphi(0) = x, \psi(0) = y$ such that the local representation $\psi \circ f \circ \varphi^{-1}$ is

$$\psi^{-1} \circ f \circ \varphi = j$$

the canonical immersion j. Suppose that $\varphi(U) = U'$, then the coordinate system φ^{-1} is defined on U'. We claim that the restriction $\varphi^{-1}/(U' \cap Z)$ is a coordinate system on the open set $U' \cap Z$ about $x \in Z$. This will complete the proof. Observe that $U' \cap Z$ is an open set in Z. Let us write

$$\varphi^{-1} = (x_1, \ldots, x_k).$$

We know that $\varphi^{-1}/(U' \cap Z)$ is bijective onto its image and is smooth (being the restriction of a smooth function). So we only need to check that the image of $\varphi^{-1}/(U \cap Z)$ is open in $\mathbb{R}^{k-\ell}$ and that the inverse of $\varphi^{-1}/(U \cap Z)$ is smooth.

Suppose that $p \in U' \cap Z$. Then as $f(p) = y$ we have

$$\psi \circ j \circ \varphi^{-1}(p) = y$$

But as $\psi(0) = y$ and ψ is bijective, we must have

$$j \circ \varphi^{-1}(p) = (0, \ldots, 0).$$

Since j is the canonical submersion of $\mathbb{R}^k$ into $\mathbb{R}^\ell$ it follows that the coordinate functions

$$x_1, \ldots, x_\ell$$

are identically zero on $U' \cap Z$. Thus $\varphi^{-1}/(U' \cap Z)$ maps $(U' \cap Z)$ bijectively onto $U \cap \mathbb{R}^{k-\ell}$ which is open in $\mathbb{R}^{k-\ell}$. Here we are identifying $\mathbb{R}^{k-\ell}$ as the subspace of $\mathbb{R}^k$ with the first ℓ coordinates zero. Clearly, the inverse of $\varphi^{-1}/(U' \cap Z)$ equals $\varphi/(U \cap \mathbb{R}^{k-\ell})$ which is smooth, being the restriction of a smooth function. This completes the proof. □

Proposition 3.4. Let $f : X \longrightarrow Y$ be a smooth map between manifolds and let $y \in Y$ be a regular value of f. Let $Z = f^{-1}(y)$ and $z \in Z$. Then the tangent space $T_z(Z)$ equals the kernel of the derivative $df_z : T_z(X) \longrightarrow T_y(Y)$.

Proof. By the preimage theorem, we know that Z is a manifold of dimension equal to $(k - \ell)$ where k, ℓ are respectively the dimensions of X and Y. Consider the commutative diagrams

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[dr, "f'"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd} \qquad \begin{tikzcd} T_z(Z) \arrow[r, "di_z"] \arrow[dr, "df'_z"'] & T_z(X) \arrow[d, "df_z"] \\ & T_y(Y) \end{tikzcd}$$

where $f/Z = f'$. Since f' is constant, we have that $df'_z = 0$. We know that di_z is the inclusion map, thus $T_z(Z)$ is contained in the kernel of df_z. But since the kernel of df_z and $T_z(Z)$ have the same dimension, the proposition follows. □

The preimage theorem gives us a sufficient condition under which a level set of a smooth function is a manifold. Without much extra effort we can extend the methods to understand when a set of common zeros of several functions defines a submanifold.

More precisely, suppose we have a collection smooth functions

$$g_1, \ldots, g_\ell : X \longrightarrow \mathbb{R}$$

defined on a manifold X. We would like to know under what conditions is the set

$$Z = \{x \in X \; : \; g_i(x) = 0, i = 1, 2, \ldots, \ell\}$$

of common zeros of the functions g_i a submanifold of X. We quickly realise that this is not very different from the situation in the preimage theorem once we have the function

$$g = (g_1, g_2, \ldots, g_\ell) : X \longrightarrow \mathbb{R}^\ell$$

in front of us. We immediately note that

$$Z = g^{-1}(0)$$

and therefore Z will be a submanifold if 0 is a regular value of g! This condition, that is whether 0 is a regular value of g or not, can be checked in terms of the functions g_i as follows.

Suppose $x \in Z$. since the functions g_i are smooth we have the derivatives

$$d(g_i)_x : T_x(X) \longrightarrow \mathbb{R}.$$

Hence the linear functionals $d(g_i)_x \in T_x(X)^*$ the linear dual of $T_x(X)$. We prove the easy fact.

Lemma 3.5. With the above notations 0 is a regular value of g, that is, $dg_x : T_x(X) \longrightarrow \mathbb{R}^\ell$ is onto if and only if the linear functionals

$$d(g_1)_x, \ldots, d(g_\ell)_x \in T_x(X)^*$$

are linearly independent in $T_x(X)^*$.

Proof. Suppose that 0 is a regular value of g. Assume that $\Sigma_i a_i d(g_i)_x = 0$ with $a_i \in \mathbb{R}$. Now as $g_i = x_i \circ g$ we have $d(g_i)_x = d(x_i)_0 \circ dg_x$. So that

$$\Sigma_i a_i d(g_i)_x = (\Sigma_i a_i d(x_i)_0) \circ dg_x.$$

Since dg_x is assumed to be onto, we fix $v_j \in T_x(X)$ such that $dg_x(v_j) = e_j$. Then

$$0 = (\Sigma a_i d(g_i)_0)(v_j) = (\Sigma_i a_i d(x_i)_0) \circ dg_x(v_j) = (\Sigma_i a_i d(x_i)_0)(e_j) = a_j.$$

Thus each $a_j = 0$. Conversely let us assume that the linear functionals are linearly independent. Let $v_1, \ldots, v_\ell \in T_x(X)$ be the basis dual to $d(g_1)_x, \ldots, d(g_\ell)_x$. We claim that the vectors $dg_x(v_1), \ldots, dg_x(v_\ell) \in \mathbb{R}^\ell$ are linearly independent. This will prove that dg_x is onto. Assume that

$$\Sigma_i a_i dg_x(v_i) = 0$$

with $a_i \in \mathbb{R}$. Then

$$0 = dx_j (\Sigma_i a_i dg_x(v_i)) = \Sigma_i dx_j dg_x(v_i) = \Sigma_i a_i d(g_j)_x(v_i) = a_j.$$

Thus all $a_j = 0$. This completes the proof. □

Here is a definition.

Definition 3.6. Suppose $g_1, \ldots, g_\ell : X \longrightarrow \mathbb{R}$ are smooth functions on the manifold X. We say that these functions are independent at $x \in X$ if the linear functionals $d(g_1)x, \ldots, d(g_\ell)_x$ are linearly independent.

We have therefore proved the following proposition.

Proposition 3.7. Suppose that the functions $g_1, \ldots, g_\ell : X \longrightarrow \mathbb{R}$ are independent at each point where they all vanish, then Z the set of common zeros is a submanifold of X of dimension equal to $\dim(X) - \ell$. □

We then say that Z is *cut out* by independent functions. Various questions immediately crop up. The obvious one being : Given a submanifold Z of X, is Z cut out by independent functions? This means do there exist smooth functions

$$g_1, \ldots, g_\ell : X \longrightarrow \mathbb{R}$$

with Z being the set of common zeros and the functions independent on Z? In general the answer is no. We shall see examples later if time permits.

A convenient way to keep track of the dimensions is to introduce the *codimension* of a submanifold.

Definition 3.8. Let Z be a submanifold of X. Then $\mathrm{codim}(Z) = \dim(X) - \dim(Z)$.

If we now see the statement of the pre-image theorem the claim about the dimension of Z can be replaced by $\mathrm{codim}(Z) = \ell = \mathrm{codim}\{y\}$. The codimension on the left is that of Z in X and on the right is the codimension of the 0-dimensional submanifold $\{y\}$ in Y. Similarly, we observe that in Proposition 3.7, the ℓ-many independentent functions cut out a submanifold Z of codimension ℓ.

As was pointed out above, in general the converse of Proposition 3.7 is not true. However in specific situations the converse holds.

Proposition 3.9. Suppose that $f : X \longrightarrow Y$ is a smooth map and $y \in Y$ a regular value of f. Then $Z = f^{-1}(y)$ can be cut out by independent functions.

Proof. The proposition says that if a submanifold Z of X is already the inverse image of a regular value of some function, then it can be cut out by independent functions.

First observe that Z being the inverse image of a regular value is a submanifold of codimension $\ell = \dim(Y)$. Let $\psi = (x_1, \ldots, x_\ell)$ be a coordinate system defined on an open set Uabout y with $\psi(y) = 0$. Then the functions $g_i = x_i \circ f$ are defined on the open set $V = f^{-1}(U)$ containing Z. Z is precisely the set of common zeros of $g_i' s$. By Lemma 3.5 the functions g_i are independent on Z (since $\psi \circ f$ is a submersion on Z). This completes the proof. □

Proposition 3.10. Every submanifold of X is locally cut out by independent functions.

Proof. Exercise. The statement of the proposition means the following. Let Z be a submanifold of X of codimension ℓ. Given $z \in Z$, there exists an open set $U \subseteq X$ with $z \in U$ and smooth functions

$$g_1, \ldots, g_\ell : U \longrightarrow \mathbb{R}$$

with $U \cap Z$ as the set of common zeros of the g_i's. It might be of help to look at the Exercise 2.6 of Lecture 4 - II. □

Here are some exercises.

Exercise 3.11. Show that $f : \mathbb{R} \longrightarrow \mathbb{R}^3$ defined by $f(t) = (t, t^2, t^3)$ is an embedding. Find two independent functions that globally define the image. Are your functions independent on the whole of $\mathbb{R}^3$ or just on a neighborhood of the image?

Exercise 3.12. Prove the following extension of Proposition 3.10. Let $Z \subseteq X \subseteq Y$ be submanifolds and $z \in Z$. Show that there exists a neighborhood U of z in Y and independent functions $g_1, \dots, g_\ell$ on U such that

$$Z \cap U = \{y \in U \, : \, g_1(y) = 0, \dots, g_\ell(y) = 0\}$$

and

$$X \cap U = \{y \in U \, : \, g_1(y) = 0, \dots, g_m(y) = 0\}$$

where $\ell - m$ is the codimension of Z in X.

Exercise 3.13. Show that 0 is the only critical value of the map $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ given by

$$f(x, y, z) = x^2 + y^2 - z^2.$$

Prove that if s and b are either both positive or both negative, then $f^{-1}(a)$ and $f^{-1}(b)$ are both diffeomorphic.

Exercise 3.14. (Stack of Records theorem) Suppose that y is a regular value of $f : X \longrightarrow Y$ with X compact and $\dim(X) = \dim(Y)$. Show that

$$f^{-1}(y) = \{x_1, \dots, x_N\}$$

is a finite set. Prove that there exists a neighborhood U 0f y such that $f^{-1}(U)$ is a disjoint union

$$f^{-1}(U) = V_1 \cup \cdots \cup V_N$$

where V_i is a neighborhood of x_i and f maps each V_i diffeomorphically onto U. In particular the Exercise shows that if $f : X \longrightarrow Y$ is an immersion between manifolds of the same dimension with X compact and Y connected, then f is a covering map.