Assignment 1 , Differential Equations 2022, B.Math 3rd Year

Maximum Marks -100

Submission Date-25th April, 2022

1. (a) (5 points) Find the exact solution of the initial value problem :

$$y' = y^2, \quad y(0) = 1$$

Starting with $y_0(x) = 1$, apply Picard's method to calculate $y_n(x)$ and show that it converges to the exact solution as $n \to \infty$.

(b) Let (x_0, y_0) be an arbitrary point in the plane and considerthe initial value problem

$$y' = y^2, \quad y(x_0) = y_0$$

i. (5 points) Show that it has an unique local solution $y = y(x)$ on the interval $|x - x_0| \leq h$.

ii. (5 points) Show that, by considering the solutions through $(0, 0)$ and $(0, 1)$, it may not have a global solution.

(c) (5 points) Show that $f(x, y) = y^{\frac{1}{2}}$

i. Doesn't satisfy a Lipschitz condition on the rectangle $|x| \leq 1$ and $0 \leq y \leq 1$;

ii. Does satisfy a Lipschitz condition on the rectangle $|x| \leq 1$ and $c \leq y \leq d$ where $0 < c < d$.

(d) (5 points) Show that $f(x, y) = |x|^2 y$ satisfies a Lipschitz condition on the rectangle $|x| \leq 1$ and $|y| \leq 1$ but that $\frac{\partial f}{\partial y}$ fails to exist at many points of this rectangle.

2. Consider the Hermite's equation :

$$y'' - 2xy' + 2py = 0$$

where p is a constant.

(a) (10 points) Show that it's general solution is $y(x) = c_1 y_1(x) + c_2 y_2(x)$, where

$$y_1(x) = 1 - \frac{2p}{2!}x^2 + \frac{2^2 p(p-2)}{4!}x^4 - \frac{2^3 p(p-2)(p-4)}{6!}x^6 + \ldots,$$

and

$$y_2(x) = x - \frac{2(p-1)}{3!}x^3 + \frac{2^2(p-1)(p-3)}{5!}x^5 - \frac{2^3(p-1)(p-3)(p-5)}{7!}x^7 + \dots$$

Also show that both the series converges for all x.

(b) (4 points) Show that, when p is a nonnegative integer then one of the two series terminates and becomes a polynomial.

(c) (8 points) Let $y_p(x)$ be the polynomial solution when p is a nonnegative integer. Show that

$$H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2, \quad H_3(x) = 8x^3 - 12x$$

satisfy

$$H_1(x) = \alpha_1 y_{p_1}(x), \quad H_2(x) = \alpha_2 y_{p_2}(x), \quad H_3(x) = \alpha_3 y_{p_3}(x),$$

for some constants α_i and nonnegative integers p_i, for $i = 1, 2, 3$.

3. Consider the following Chebyshev's equation :

$$(1 - x^2)y'' - xy' + p^2 y = 0$$

where p is a constant.

(a) (4 points) Find two linearly independent solutions, for $|x| < 1$.

(b) (4 points) Show that if $p = n \in \mathbb{Z}_+$, then there is a polynomial solution of degree n.

4. (8 points) The differential equation :

$$x^2 y'' + (3x - 1)y' + y = 0 \tag{1}$$

has an irregular singular point at $x = 0$. By putting

$$y = x^m(a_0 + a_1 x + a_2 x^2 + \dots)$$

into (1) show that $m = 0$ and the corresponding Frobenius series solution is the power series

$$y = \sum_{n=0}^{\infty} n! x^n$$

which converges only at $x = 0$.

5. Verify by examining the series expansions of the functions on l.h.s.

(a) (5 points)

$$(1 + x)^p = F(-p, b, b, -x)$$

(b) (5 points)

$$\sin^{-1}(x) = xF\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, x^2\right)$$

6. Validate the following statements without attempting to justify the limit processes involved

   (a) (5 points)

   $$e^x = \lim_{b\to\infty} F\left(a, b, a, \frac{x}{b}\right)$$

   (b) (5 points)

   $$\sin x = x\left[\lim_{a\to\infty} F\left(a, a, \frac{3}{2}, \frac{-x^2}{4a^2}\right)\right]$$

7. (10 points) Consider the following Chebyshev equation

   $$(1-x^2)y'' - xy' + p^2 y = 0 \tag{2}$$

   where p is a nonnegative constant. Transform it into a hypergeometric equation by replacing x by $t = \frac{1}{2}(1-x)$, and show that its general solution near $x = 1$ is

   $$y = c_1 F\left(p, -p, \frac{1}{2}, \frac{1-x}{2}\right) + c_2\left(\frac{1-x}{2}\right)^{1/2} F\left(p + \frac{1}{2}, -p + \frac{1}{2}, \frac{3}{2}, \frac{1-x}{2}\right)$$

8. (a) (5 points) Show that

   $$F'(a, b, c, x) = \frac{ab}{c} F(a+1, b+1, c+1, x) \tag{3}$$

   (b) (5 points) Applying formula (3) , show that the only solutions of Chebyshev's equation (2), whose derivatives are bounded near $x = 1$ are

   $$y = c_1 F\left(p, -p, \frac{1}{2}, \frac{1-x}{2}\right)$$

   Conclude that the only polynomial solutions of Chebyshev's equation are constant multiples of $F\left(n, -n, \frac{1}{2}, \frac{1-x}{2}\right)$, where n is a non-negative integer.