DIFFERENTIAL TOPOLOGY - LECTURE 9

1. Introduction

We begin by recalling certain definitions and theorems that we have discussed earlier. Recall that for a function $f : X \longrightarrow Y$ between manifolds a point $y \in Y$ is called a regular value (of f) if f is a submersion at every $x \in f^{-1}(y)$. Thus if $y \notin f(X)$, then y is always a regular value. A point $y \in Y$ that is not a regular value is called a critical value. Thus the only way a point $y \in Y$ can be a critical value is that there exists (at least one) $x \in f^{-1}(y)$ such that

$$df_x : T_x(X) \longrightarrow T_y(Y)$$

is not surjective.

For a map $f : X \longrightarrow Y$ between manifolds a point $x \in X$ is called a regular point (of f) if f is a submersion at x. If f is not a submersion at x, that is,

$$df_x : T_x(X) \longrightarrow T_{f(x)}(Y)$$

is not onto, then x is called a critical point of f. Let C_f denote the set of critical points of f. Then the set of critical values of f equals $f(C_f)$, the image of the set of set of critical points.

It is of interest to understand, given a function $f : X \longrightarrow Y$, the nature of the set of critical points, the set of regular values (and therefore the set of critical values). In particular one would like to know (under what condtions) are these sets nonempty and how large are these sets if nonempty? Why should one be interested in knowing whether the above sets are nonempty, large? One reason is the preimage theorem which states that the inverse image of a regular value is a submanifold. Thus knowing whether a smooth function has a regular value is important.

Let us remind ourselves of certain possibilities. Recall that we had shown that if X is a compact manifold, then any smooth function $f : X \longrightarrow \mathbb{R}$ must have a critical point.

For dimensional reasons, if $\dim(Y) > \dim(X)$ and $f : X \longrightarrow Y$ is smooth, then $C_f = X$. The image $f(X) = f(C_f)$ consists entirely of critical values. So in this case regular values are those that are not in $f(X)$. The question that one can ask here is : how "large" can the set $f(C_f)$ of critical values be? Largeness, for example, could be the question: can $f(C_f)$ contain an open set?

On the other hand it is easy to find smooth maps that have no critical points. The simplest being a projection of the euclidean space. Given manifolds X, Y, the projection $X \times Y \longrightarrow X$ is a submersion and thus has no critical points. The map $p : \mathbb{R}^2 \longrightarrow S^1 \times S^1$ defined by

$$p(x, y) = (\cos x, \sin x, \cos y, \sin y)$$

is a local diffeomorphism and hence has no critical points. Finally there also exists maps $f : X \longrightarrow Y$ between compact manifolds (of different dimensions) having no critical points[1]. In the examples considered in the present paragraph, the set of critical values is as small as it can be. It is empty.

[1] Examples of such maps are provided by a class of maps called fibre bundles which are always submersions. For example there is a fibre bundle $S^3 \longrightarrow S^2$, called the Hopf map. The projection $X \times Y \longrightarrow Y$, the tangent and normal bundles (which we shall soon encounter) of a manifold are examples of fibre bundles.

The answer to the question of how large can the set of critical values be is provided by Sard's theorem. We shall state this without proof and concentrate on some of its easy applications here. Sard's theorem has far reaching generalizations and is at the heart of transversality, Morse theory and intersection theory. We shall soon use this to understand some interesting facts about Morse functions.

2. Sard's theorem

In this section we shall state Sards theorem. We shall not prove this. We begin by discussing some technical definitions that all of us are familiar with.

Definition 2.1. A product of open intervals of the form

$$S = \Pi_{i=1}^{n}(a_i, b_i) \subseteq \mathbb{R}^n$$

is called a *rectangular solid.*

The volume of the rectangular solid S is then defined to be

$$\mathrm{vol}(S) = \Pi_i(b_i - a_i).$$

Definition 2.2. A subset $A \subseteq \mathbb{R}^n$ is said to have *measure zero* in $\mathbb{R}^n$ if given $\varepsilon > 0$, there exists a countable collection $\{S_i\}$ of rectangular solids such that

$$\Sigma_i \mathrm{vol}(S_i) < \varepsilon$$

and $A \subseteq \cup_i S_i$.

It is therefore clear that if A has measure zero and $B \subseteq A$, then B also has measure zero. Further if S is a rectangular solid, then S is not a set of measure zero (see Exercise 2.10). In particular, an open subset of $\mathbb{R}^n$ is not of measure zero.

One can extend this notion to manifolds via parametrizations.

Definition 2.3. A subset $A \subseteq X$ is said to have measure zero in the manifold X if for every parametrization

$$\varphi : U \longrightarrow X$$

the set $\varphi^{-1}(A)$ has measure zero in $\mathbb{R}^n$.

It is clear that subsets of sets of measure zero in a manifold have measure zero and that an open subset of a manifold is not a set of measure zero. In particular a set of measure zero cannot contain an open set (see Exercise 2.10).

Here is one more familiar terminology before we state the Sard's theorem. We say that almost every point in a manifold has a property P if the set of points that do not have the property P has measure zero. We are now in a position to state Sard's theorem.

Theorem 2.4. (Sard's theorem) Let $f : X \longrightarrow Y$ be a smooth map between manifolds. Then the set of critical values is a set of measure zero. Equivalently, if $f : X \longrightarrow Y$ is a smooth map between manifolds, then almost every point of Y is a regular value of f. $\square$

The set of critical values of a smooth map f is the set $f(C_f)$ where C_f is the set of critical points of f.

Thus Sard's theorem tell us that any smooth map $f : X \longrightarrow Y$ has an abundance of regular values. Sometimes (for example, for the constant function) the regular values may lie outside the image of the function. An interesting situation is when the function is onto : then the theorem guarantees that there are points in the image that are regular values. This is a very powerful fact, as we shall see. Here are some consequences of the Sard's theorem.

Corollary 2.5. Let $f : X \longrightarrow Y$ be a smooth map between manifolds. Then the set of regular values of f is dense in Y.

Proof. Let C_f denote the set of critical points of f. Then as $f(C_f)$ has measure zero in Y, it cannot contain an open set. Thus every open set in Y must intersect the complement of $f(C_f)$. Hence the set of regular values (which is the complement of $f(C_f)$) is dense in Y. □

A much stronger statement is true.

Corollary 2.6. Let $f_i : X_i \longrightarrow Y$ be a countable family of smooth maps. Then the set

$$R = \{y \in Y \,:\, y \text{ is a regular value of each } f_i\}$$

is dense in Y.

Proof. Let C_i denote the set of critical points of f_i. Then by Sard's theorem $f(C_i)$ has measure zero in Y. Note that R is the complement of $\cup_i f(C_i)$. The corollary now follows as $\cup_i f(C_i)$ has measure zero in Y (see Exercise 2.9 below). □

Thus the corollary says that given a family f_i as above, the set of points $y \in Y$ that are simultaneously regular values of each f_i is dense in Y.

Corollary 2.7. Suppose that $f : X \longrightarrow Y$ is a smooth map between manifolds. If $\dim(X) < \dim(Y)$, then f is not surjective.

Proof. Let C_f denote the set of critical points of f. Clearly, by dimension asuumptions, $C_f = X$. Then as $f(C_f) = f(X)$ has measure zero in Y, we have that $f(X) \neq Y$. □

In particular, there cannot exist a smooth surjective map $f : \mathbb{R} \longrightarrow \mathbb{R}^2$. Observe that continuous surjections $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ do exist. Here is another corollary.

Corollary 2.8. If $k < n$, then every smooth map $f : S^k \longrightarrow S^n$ is null homotopic[2].

Proof. Recall that null homotopic means homotopic to constant. By Sard's theorem f cannot be onto. Thus f factors as a composition

$$S^k \longrightarrow \mathbb{R}^n \hookrightarrow S^n.$$

Since $\mathbb{R}^n$ is contractible, f must be null homotopic. □

Hence the sphere S^n is simply connected if $n > 1$ Here are some exercises.

Exercise 2.9. Show that a countable union of sets of measure zero is again a set of measure zero.

Exercise 2.10. Show that a rectangular solid is not a set of measure zero (see Appendix A in G and P).

Exercise 2.11. Let X be a compact manifold. Is it true that every smooth function $f : X \longrightarrow \mathbb{R}^n$ has a critical point if $n < \dim(X)$.

Exercise 2.12. Show that $\mathbb{R}^k$ is of measure zero in $\mathbb{R}^n$, $k < n$.

Exercise 2.13. Show that any submanifold Z of lower dimension in X is of measure zero.

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[2]It is a fact that given a continuous map $f : X \longrightarrow Y$ between manifolds, there exists a smooth map $g : X \longrightarrow Y$ such that f and g are continuously homotopic. In other words, every continuous map between manifolds is homotopic to a smooth map. This shows that every continuous map $f : S^k \longrightarrow S^n$, $k < n$, is also null homotopic.