DIFFERENTIAL TOPOLOGY - LECTURE 7

1. Introduction

The preimage theorem gives us a sufficient condition under which a level set $f^{-1}(y)$ of a smooth map $f : X \longrightarrow Y$, $y \in Y$ is a manifold. Thinking of the point y as a 0-dimensional manifold we now wish to understand, given a submanifold Z of Y, when is the inverse image $f^{-1}(Z)$ a submanifold of X. This leads to a fundamental notion in differential topology, namely, the notion of *transversality.*

The concept of transversality was developed by René Thom in 1954 in his PhD thesis[1] and contains the celebrated theorem which is now called the Thom transversality theorem.

The notion of transversality captures in a precise way the nature in which two manifolds intersect in space. The power of transversality lies in the fact that it is a "stable property"[2]. We shall make precise definitions soon, but here is a simple description of what is meant by transversality and its stabilty.

Consider the submanifolds

$$X = \text{graph}(f(t) = t^2)$$

$$X' = \text{graph}(g(t) = t^2 - 1)$$

$$Y = \{(t, 0) \, : \, t \in \mathbb{R}\}$$

of $\mathbb{R}^2$.

The manifolds X and Y intersect at the origin. The tangent spaces at the point of intersection do not add up to the tangent space of the ambient space, that is,

$$T_{(0,0)}(X) + T_{(0,0)}(Y) \neq T_{(0,0)}(\mathbb{R}^2) = \mathbb{R}^2.$$

One says that the intersection (of X and Y) is not transversal (or X and Y are not transversal). Notice that if we slightly perturb X or Y, then the two manifolds don't intersect at all. So this type of (tangential, or non transversal) intersection is not "stable". The two manifolds can be pulled apart (into a non intersecting situation) by a slight deformation of X or Y.

On the other hand the manifolds X' and Y intersect at two points $x = (-1, 0), (1, 0)$. At both these points the tangent spaces add up tp the tangent space of the ambient space, that is,

$$T_x(X') + T_x(Y) = T_x(\mathbb{R}^2) = \mathbb{R}^2.$$

This condition is summed up by saying that the intersection (of X' and Y) is transversal. Now notice that even if we slightly perturb X' or Y, the two manifolds continue to intersect and intersect transversally. We cannot pull them apart by slightly deforming either X' or Y. Thus in this sense transverse intersection is stable. Drawing pictures will help in visualizing the above situation.

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[1] Quelques propriétés globales des variétés différentiables, Comment. Math. Helv. 28, (1954). 17-86. English translation is available on the web.

[2] Actually, a stronger statement is true : transversality is *generic.* This means that even if transversality does not hold, it can be made to hold by a small deformation.

It is important to note that transverse intersection is as much a property of the intersecting manifolds as it is of the ambient space in which the intersection is occuring. For example, if we think of the manifolds X' and Y as submanifolds of $\mathbb{R}^3$, then for dimensional reasons the tangent spaces at the points of intersection cannot add up to the tangent space of the ambient space. So when considered as submanifolds of $\mathbb{R}^3$ the manifolds X' and Y cannot intersect transversally. Indeed, by a slight perturbation (in the z direction) the two manifolds can be pulled apart so as to not intersect at all.

Transversality captures and makes precise these geometric observations and more. We shall discuss this in the present set of notes.

Remember that $X, Y, Z, X', \ldots$ always denote manifolds and that all maps/functions that we consider are always smooth.

2. Transversality

We begin with the definition of when a map $f : X \longrightarrow Y$ is transversal to a submanifold Z of Y.

Definition 2.1. Let $f : X \longrightarrow Y$ be a smooth map between manifolds and Z a submanifold of Y. We say that f is *transversal* to the submanifold Z if the equality

$$\mathrm{im}(df_x) + T_{f(x)}(Z) = T_{f(x)}(Y) \tag{2.1.1}$$

holds for all $x \in f^{-1}(Z)$.

We use the notation $f \pitchfork Z$ to mean that f is transversal to Z. Thus for $f \pitchfork Z$ to hold we must have

$$\dim(X) + \dim(Z) \geq \dim(Y).$$

The sum of the vector spaces in the definition need not be a direct sum. Observe that if $Z = \{y\}$ is a point, then $f \pitchfork Z$ if and only if y is a regular value of f. Thus transversality generalizes the notion of a regular value.

Here are some examples.

Example 2.2. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$ and $Z = \{(x, 0) : t \in \mathbb{R}\}$. Let $f, g : \mathbb{R} \longrightarrow Y$ be the maps defined by $f(x) = (x, x^2)$ and $g(x) = (x, x^2 - 1)$. Let us check that $g \pitchfork Z$. Let $x \in g^{-1}(Z)$. Then $x = \pm 1$. Let $x = 1$. Then,

$$dg_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Hence,

$$\mathrm{im}(dg_1) = \mathrm{span}\{(1, 2)\}.$$

It is clear that

$$T_{(1,0)}(Z) = \mathrm{span}\{(1, 0)\}.$$

Thus with $x = 1$, the equation (2.1.1) holds. Similarly, equation (2.1.1) holds with $x = -1$. Hence $g \pitchfork Z$. It is an exercise to check that f is not transversal to Z.

Example 2.3. Consider the map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$f(x, y) = e^x(\cos y, \sin y).$$

Let $Z = S^1$. Then one checks easily that $f \pitchfork Z$.

Example 2.4. Let X, Z be submanifolds of Y. Let $i : X \hookrightarrow Z$ be the inclusion map. Then $i \pitchfork Z$ if and only if the equality

$$T_x(X) + T_x(Z) = T_x(Y)$$

holds for all $x \in i^{-1}(Z) = X \cap Z$.

Some remarks are in order.

Remark 2.5. Observe that if $f : X \longrightarrow Y$ is a submersion, then f is transversal to every submanifold Z of Y because in this case $\operatorname{im}(df_x) = T_{f(x)}(Y)$ for every $x \in X$ and so equation (2.1.1) always holds. We already have an idea of what stability means. In the examples above we have noted that the manifolds X and Y do not intersect transversally. If we slightly pull the manifold X in the positive (or negative) y direction the two manifolds now intersect transversally. This is what is meant by saying transversality is generic : If the intersection is not transversal, then by a slight deformation the intersection can be made transversal (see Footnote 2 on page 1). The motivation for the definition of transversality will become clear while proving the next theorem.

The following theorem generalizes the preimage theorem[3].

Theorem 2.6. Suppose $f : X \longrightarrow Y$ is a smooth map between manifolds. Let Z be a submanifold of Y. If f is transversal to Z, then $f^{-1}(Z)$ is a submanifold of X of codimension equal to the codimension of Z in Y. □

Before proving the theorem we remind ourselves of some facts that we have discussed earlier. More precisely, we recall here the content and conclusion of (Proposition 3.7, Lecture 5). Suppose that

$$g_1, \ldots, g_\ell : X \longrightarrow \mathbb{R}$$

are smooth functions defined on a manifold X (or on an open subset U of X). Let

$$Z = \{x \in X \,:\, g_i(x) = 0, i = 1, \ldots, \ell\}$$

be the set of common zeros of the functions g_is. We then have a map

$$g = (g_1, \ldots, g_\ell)) : X \longrightarrow \mathbb{R}^\ell$$

which exhibits Z as the level set

$$Z = g^{-1}(0).$$

By the preimage theorem, Z is a submanifold of codimension equal to ℓ (the number of functions g_i) if 0 is a regular value of g. By (Lemma 3.6, Lecture 5), 0 is a regular value of g if and only if the functions

$$d(g_1)_x, \ldots, d(g_\ell)_x \in T_x(X)^* \tag{2.6.1}$$

are linearly independent for each $x \in Z$. Recall that if equation (2.4.1) holds for all $x \in Z$ then we say that the functions g_i are independent and that Z is cut out by independent functions.

Finally we recall (Proposition 3.10, Lecture 5) which says that every submanifold Z of a manifold Y is locally cut out by independent functions. What this means is the following. Given a submanifold Z of Y of codimension ℓ and $y \in Z$, there is an open set U, $y \in U$, U open in Y and independent functions

$$g_1, \ldots, g_\ell : U \longrightarrow \mathbb{R}$$

such that $U \cap Z$ is the set of common zeros of the functions g_is.

We shall use these facts to prove the above theorem.

Proof of Theorem 2.6. Let $x \in f^{-1}(Z)$ and $y = f(x) \in Z$. Since submanifolds are locally cut out by independent functions, we can find an open set U, $y \in U$, U open in Y and independent functions

$$g_1, \ldots, g_\ell : U \longrightarrow \mathbb{R}$$

[3] The preimage theorem is due to Lev Pontryagin and was proved in the 1930s. It took almost 20 years more for the notion of transversality to be developed. Theorem 2.6 appears in Thom's thesis.

such that $U \cap Z$ is the set of common zeros of the functions g_is. This means that if we set

$$g = (g_1, \ldots, g_\ell) : U \longrightarrow \mathbb{R}^\ell,$$

then 0 is a regular value of g and $g^{-1}(0) = U \cap Z$. In particular, g is a submersion at y. We assume without loss of generality that g is a submersion on the whole of U. Set $V = f^{-1}(U)$. Then V is open in X and $W = V \cap f^{-1}(Z)$ is open in $f^{-1}(Z)$. We now look at the composition

$$g \circ f : V \longrightarrow \mathbb{R}^\ell.$$

Observe that as

$$\begin{aligned}(g \circ f)^{-1}(0) &= f^{-1}g^{-1}(0) \\ &= f^{-1}(U \cap Z) \\ &= f^{-1}(U) \cap f^{-1}(Z) \\ &= V \cap f^{-1}(Z) = W\end{aligned}$$

we have that W is a manifold of codimension ℓ if 0 is a regular value of $g \circ f$. We shall check that for all $p \in W$, the derivative

$$d(g \circ f)_p = dg_{f(p)} \circ df_p : T_p(X) \longrightarrow \mathbb{R}^\ell$$

is surjective. This will prove that W is a codimension ℓ submanifold of X. Observe that as g is a submersion on U,

$$dg_{f(p)} : T_{f(p)}(Y) \longrightarrow \mathbb{R}^\ell$$

is onto with

$$\text{kernel}(dg_{f(p)}) = T_{f(p)}(Z). \tag{2.6.2}$$

Now transversality tells us that

$$\text{im}(df_p) + T_{f(p)}(Z) = T_{f(p)}(Y).$$

Hence

$$dg_{f(p)}\left(\text{im}(df_p) + T_{f(p)}(Z)\right) = dg_{f(p)}(T_{f(p)}(Y)) = \mathbb{R}^\ell.$$

Since (2.7.2) holds, we have

$$dg_{f(p)} \circ df_p = d(g \circ f)_p$$

is onto. This shows that W is a codimension ℓ submanifold of X. Since W is open in $f^{-1}(Z)$ and $f^{-1}(Z)$ can be covered by such open sets, we conclude that $f^{-1}(Z)$ is a codimension ℓ submanifold of X. This completes the proof. $\square$

Observe that the transversality condition (2.1.1) is precisely the condition required to show that 0 is a regular value of $g \circ f$ above.

Here is a definition of when two submanifolds intersect transversally.

Definition 2.7. Suppose X, Z are two submanifolds of Y. We say that X and Y are transversal (or intersect transversally) if $i \pitchfork Z$, where $i : X \hookrightarrow Y$ is the inclusion map. In this case we write $X \pitchfork Z$ to denote that X and Z are transversal.

For example it is easy to see that the two coordinate axes in $\mathbb{R}^2$ intersect transversally.

So now suppose that X, Z are submanifolds of Y and let $i : X \hookrightarrow Y$ denote the inclusion map. As we had observed in Example 2.4, $i \pitchfork Z$ if and only if

$$\text{im}(di_x) + T_x(Z) = T_x(X) + T_x(Z) = T_x(Y)$$

for all $x \in i^{-1}(Z) = X \cap Z$. In particular, for X and Z to intersect transversally we must have

$$\dim(X) + \dim(Z) \geq \dim(Y).$$

Assume that X, Z, Y are as above and that $X \pitchfork Z$, that is, $i \pitchfork Z$ where $i : X \hookrightarrow Y$ is the inclusion map. By Theorem 2.6

$$i^{-1}(Z) = X \cap Z$$

is a submanifold of X. Further, the codimension of $(X \cap Z)$ in X equals codimension of Z in Y. Hence the codimension of $(X \cap Z)$ in Y is the sum of the codimension of $(X \cap Z)$ in X and the codimension of X in Y. We have thus proved the following.

Proposition 2.8. Suppose X, Z are submanifolds of Y. If $X \pitchfork Z$, then $X \cap Z$ is a submanifold of Y and

$$\operatorname{codim}(X \cap Z) = \operatorname{codim}(X) + \operatorname{codim}(Z).$$

All the codimensions in the above equality are with respect to Y. □

A very interesting situation occurs under following conditions. Assume that X, Z are compact submanifolds of Y. Suppose that X, Z have complementary dimensions, that is,

$$\dim(X) + \dim(Z) = \dim(Y).$$

Assume now that X and Z intersect transversally. Then $X \cap Z$ is a (compact) submanifold of Y. It follows from our observations above that

$$\dim(X \cap Z) = 0$$

so that $X \cap Z$ must consist of just a finite set of points.

Thus if a circle and a sphere intersect transversally in $\mathbb{R}^3$ then they must intersect only at a finite set of points. If two 2-spheres intersect tranversally in $\mathbb{R}^4$, then they must intersect only at a finite set of points. Two spheres intersecting transversally in $\mathbb{R}^3$ can have 1-dimensional intersection as you can easily verify.

Here are some exercises. It is important to solve the exercises to make our understanding firm. Most of the exercises are from G and P.

Exercise 2.9. Suppose that $A : \mathbb{R}^k \longrightarrow \mathbb{R}^m$ is a linear map. Let U, V be subspaces of $\mathbb{R}^m$. Show that $A \pitchfork V$ means $A(\mathbb{R}^k) + V = \mathbb{R}^m$. Further show that $U \pitchfork V$ means $U + V = \mathbb{R}^m$.

Exercise 2.10. Let V_1, V_2, V_3 be linear subspaces of $\mathbb{R}^m$. We say that they have normal intersection if

$$V_i \pitchfork (V_j \cap V_k)$$

for $i \neq j$ and $j \neq k$. Prove that this holds if and only if

$$\operatorname{codim}(V_1 + V_2 + V_3) = \operatorname{codim}(V_1) + \operatorname{codim}(V_2) + \operatorname{codim}(V_3).$$

Exercise 2.11. Prove the claim made in Example 2.3.

Exercise 2.12. Suppose that X, Z are transversal submanifolds of Y. Show that for $x \in X \cap Z$ we have

$$T_x(X \cap Z) = T_x(X) \cap T_x(Z).$$

Note that all the tangent spaces above are subspaces of $T_x(Y)$.

Exercise 2.13. Let $f : \mathbb{R}^2 - 0 \longrightarrow \mathbb{R}^2$ be the map defined by

$$f(x, y) = \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{x}{\sqrt{x^2 + y^2}} \right).$$

Show that f is not transverse to the submanifold S^1 of $\mathbb{R}^2$. Construct a map $g : \mathbb{R}^2 - 0 \longrightarrow \mathbb{R}^2$. such that g is homotopic to f and $f \pitchfork S^1$. This exercise shows the possibility of modifying a non transverse function to a function which is transverse within the homotopy class.

Exercise 2.14. Let Z be a submanifold of Y and let $f : X \longrightarrow Y$ be such that $f \pitchfork Z$. Show that if $x \in f^{-1}(Z)$, then

$$T_x(f^{-1}(Z)) = df_x^{-1}(T_{f(x)}(Z)).$$

Here $df_x : T_x(X) \longrightarrow T_{f(x)}(Y)$.

Exercise 2.15. Suppose that $X \xrightarrow{f} Y \xrightarrow{g} Z$ are maps between manifolds. Let W be a submanifold of Z. Let $g \pitchfork W$. Show that $f \pitchfork (g^{-1}(W)$ if and only if $(g \circ f) \pitchfork W$.

Exercise 2.16. Let $A : V \longrightarrow V$ be a linear map where V is a vector space. Let $\triangle \subseteq V \times V$ be the diagonal. Show that

$$(\mathrm{graph}(A)) \pitchfork \triangle$$

if and only if $+1$ is not an eigenvalue of A.

Exercise 2.17. Let $f : X \longrightarrow X$. A fixed point x of f is called a Lefschetz fixed point if $+1$ is not an eigenvalue of df_x. f itself is called a Lefschetz map if all its fixed points are Lefschetz. Prove that if X is compact and f Lefschetz, then f has only finitely many fixed points.

Exercise 2.18. Identify $\mathbb{C}^2$ with $\mathbb{R}^4$ by identifying $(z_1, z_2) \in \mathbb{C}^2$ with $(x, y, u, v) \in \mathbb{R}^4$ where $z_1 = x + iy$ and $z_2 = u + iv$. Let

$$X = \{(z_1, z_2) \,:\, z_1^3 + z_2^2 = 0\}.$$

Show that $X - (0, 0)$ is a manifold. What is its dimension?

Exercise 2.19. Is it true that every smooth map $f : S^2 \longrightarrow S^1$ has a critical point?

Exercise 2.20. Let $X \subseteq \mathbb{R}^6$ be the subset defined by the equations

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 = 1$$
$$x_4^2 - x_5^2 - x_6^2 = -1.$$

Show that X is a manifold. Find its dimension. Let X_1 denote the manifold defined by the first equation and X_2 denote the manifold defined by the second. Do X_1 and X_2 intersect transversally?

Exercise 2.21. Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be tha map $f(x, y, z) = (xy, yz)$. Is f transverse to S^1?