Lecture 30: Morphisms of varieties

08 April 2021
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Prop: The set of rat'l functions on a projective variety $X \subseteq \mathbb{P}$ is a field denoted $k(X)$. In fact,

$$k(X) = \left\{ \frac{f}{g} \;\middle|\; \begin{array}{l} F, G \in k[x_0, \dots, x_n] \text{ homogen of the same degree,} \\ G \notin \mathcal{I}(X) \,; \quad f = F \pmod{\mathcal{I}(X)} \;\&\; g = G \pmod{\mathcal{I}(X)} \end{array} \right\}$$

$$\subseteq \operatorname{frac}\left(k[x_0, \dots, x_n] / \mathcal{I}(X) \right)$$

Prop: Let $P \in X \subseteq \mathbb{P}^n$ be a point of a proj var X. Then $\mathcal{O}_{X,P} = \{ f \in k(X) \mid f \text{ is regular at } P \}$ is a local ring with maximal ideal $m = \{ f \in \mathcal{O}_{X,P} \mid f(P) = 0 \}$.

Let f be a rat'l func on $\mathbb{P}^n$ s.t. f is regular at all points. Then f is const.

Example: 1) $X = \mathbb{P}^n$ then homoge coord ring $k[x_0, \dots, x_n]$ then $\frac{X_0}{X_1}$ is a rational function, it is defined on $U_1 = \{x_1 \neq 0\}$, $U_1 \longrightarrow k$, $[a_0, a_1, \dots, a_n] \mapsto \frac{a_0}{a_1}$. But $X_0 \in k(X_0, \dots, X_n)$ is not a rational function on $\mathbb{P}^n$.

HW $X \subseteq \mathbb{P}^n$ a proj var & $U_0 \subseteq \mathbb{P}^n$ given by $X_0 \neq 0$ then $k(U_0 \cap X) \cong k(X)$ if $U_0 \cap X \neq \phi$.

<u>Ex 2</u>: $X \subseteq \mathbb{P}^2$

$X = Z(X_0^2 X_1 + X_1^2 X_2 + X_0 X_2^2)$

homogen coordinate ring of X is

$$\frac{k[X_0, X_1, X_2]}{(X_0^2 X_1 + X_1^2 X_2 + X_0 X_2^2)} = k \oplus (kX_0 + kX_1 + kX_2) \oplus \{\text{deg 2 terms}\} \oplus \frac{\text{deg 3 terms}}{k(X_0^2X_1 + X_1^2X_2 + X_0X_1X_2)} \oplus \cdots$$

<u>Note</u>: $I \subseteq k[X_0,\dots,X_n]$ homogen ideal then $I = I_0 \oplus I_1 \oplus \cdots$, $k[X_0,\dots,X_n] = k \oplus R_1 \oplus R_2 \oplus \cdots$

$\frac{k[X_0,\dots,X_n]}{I} = k/I_0 \oplus R_1/I_1 \oplus \cdots \oplus R_d/I_d \oplus \cdots$

$$k(X) = \left\{ \frac{f}{g} \;\middle|\; f, g \in \frac{k[X_0, X_1, X_2]}{(X_0^2X_1 + X_1^2X_2 + X_0X_2^2)} \text{ homo.}, \; \deg f = \deg g, \; g \neq 0 \right\}$$ is the field of ratl functions

Let $x_1 = \frac{X_1}{X_0}$ & $x_2 = \frac{X_2}{X_0}$

$f/g \in k(X)$ & $\deg f = d$ then $\frac{f}{g} = \frac{f}{\bar{X}_0^d} \cdot \frac{\bar{X}_0^d}{g}$

$$= f(1, \bar{x}_1, \bar{x}_2) / g(1, \bar{x}_1, \bar{x}_2)$$

where $f(1, \bar{x}_1, \bar{x}_2)$ & $g(1, \bar{x}_1, \bar{x}_2)$

$$\in \frac{k\left[\frac{X_1}{X_0}, \frac{X_2}{X_0}\right]}{\left(\frac{X_1}{X_0} + \left(\frac{X_1}{X_0}\right)^2 \frac{X_2}{X_0} + \frac{X_2^2}{X_0^2}\right)}$$

$$= \frac{k[x_1, x_2]}{(x_1 + x_1^2 x_2 + x_2^2)}$$

coordinate ring of $U_0 \cap X$

$U_0 = \{X_0 \neq 0\}$ in $\mathbb{P}^2$

Def^n A map $\varphi : X \to Y$ between varieties is said to be a morphism if $\exists$ an affine open cover $\{U_i\}$ of X and affine open cover $\{V_i\}$ of Y s.t. $\varphi(U_i) \subseteq V_i$ and $\varphi_{|U_i} : U_i \longrightarrow V_i$ is a morphism of affine varieties.

$\left\{ \begin{array}{l} \forall\ P \in X\ \exists \text{ open affine subsets } U \subseteq X \ \&\ V \subseteq Y \\ \text{s.t. } P \in U,\ \varphi(P) \in V \ \&\ \varphi_{|U} : U \longrightarrow V \text{ is a morphism of affine var.} \end{array} \right\}$

$\varphi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$

(i) $[a_0, a_1] \longmapsto [a_1, a_0]$

(ii) $[a_0, a_1] \longmapsto [a_0 + a_1, a_1]$

(iii) $\cancel{[a_0, a_1] \longmapsto [a_0 a_1, a_1]}$ Not a morphism

$[a_0, a_1] \mapsto [a_0 a_1, a_1]$
$\quad \shortparallel \qquad\qquad \not=$
$[\lambda a_0, \lambda a_1] \mapsto [\lambda^2 a_0 a_1, \lambda a_1]$

(i) $U_0 \cup U_1 = \mathbb{P}^1 \qquad V_0 \cup V_1 = \mathbb{P}^1$

$\varphi(U_0) = V_1$

$\varphi(U_1) = V_0$

$\varphi_{|U_0} : U_0 \longrightarrow V_1$

$[a_0, a_1] \longmapsto [a_1, a_0]$

$\shortparallel$

$\left[1, \frac{a_1}{a_0}\right] \longmapsto \left[\frac{a_1}{a_0}, 1\right]$

So $\varphi_{|U_0}$ is indeed a morphism of var.

(ii) $\varphi([a_0, a_1]) \longrightarrow [a_0+a_1, a_1]$

$\varphi(U_1) \subseteq V_1$

$\varphi(\tilde{U}_0) \subseteq V_0 \qquad \tilde{U}_0 = \{X_0+X_1 \neq 0\}$

if $a_1 \neq 0$

$$[a_0, a_1] = \left[\frac{a_0}{a_1}, 1\right] \mapsto \left[\frac{a_0+a_1}{a_1}, 1\right] = \left[\frac{a_0}{a_1}+1, 1\right]$$

$$\varphi|_{U_0} : \mathbb{A}^1 \longrightarrow \mathbb{A}^1$$
$$\alpha \longmapsto \alpha+1$$

$$\varphi|_{\tilde{U}_0} : [a_0, a_1] \longmapsto [a_0+a_1, a_1]$$

$$[a_0, a_1] = \left[\frac{a_0}{a_0+a_1}, \frac{a_1}{a_0+a_1}\right] \qquad [a_0+a_1, a_1] = \left[1, \frac{a_1}{a_0+a_1}\right]$$

if $a_0 \neq 0$

$$\left[1, \frac{a_1}{a_0}\right] \longmapsto \left[1, \frac{a_1/a_0}{1+a_1/a_0}\right]$$

$$\alpha \longmapsto \frac{\alpha}{1+\alpha}$$

$$\frac{a_1}{a_0} = -1 \implies a_0+a_1 = 0 \cap \tilde{U}_0 = \phi$$

So $\tilde{\varphi}(\tilde{U}_0) \subseteq V_0$.