Lecture 1.

Ordinary differential Equations are equations of the form

$$F(x, y, y', \ldots, y^{(n)}) = 0.$$

$y = y(x)$, $x \in [a, b]$. $y'(x) = \frac{dy}{dx}(x)$

$\ldots\ y^{(n)}(x) = \frac{d^n y}{dx^n}(x)$.

This is an ODE of order $n \geq 1$.

$F : [a,b] \times E \to \mathbb{R}$, $E = E_0 \times \cdots \times E_n$ $\subseteq \mathbb{R}^{n+1}$ is a given map. And $y^{(i)} \in E_i$

Example 1. $y'' - 5y' + 6y = 0$.

Suppose we want to solve this on an interval $[a, b]$. ie we want to find $y = y(x)$, $x \in [a,b]$ such that

$$y''(x) - 5y'(x) + 6y(x) = 0.$$

for $x \in [a,b]$. Here $F : [a,b] \times E \to \mathbb{R}$ and is given by : $E = \mathbb{R}^3$

$$F(x, y_0, y_1, y_2) := y_2 - 5y_1 + 6y_0.$$

Example 2. $\dfrac{dy}{dx} = f(x)$

Here $f(x)$, $x \in [a, b]$ is a given (continuous) function. This is a first order equation i.e. $n = 1$ and $F : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ is $F(x, y_0, y_1^*) := y_1 - f(x)$.

Here the equation $F(x, y, y') = 0$ can be solved by integration as

$$y(x) = y(x_0) + \int_{x_0}^{x} f(t)\, dt$$

where $x_0 \in [a,b]$ and provided $y(x_0)$ is given. If $y(x_0)$ is not given then the solution is determined upto a constant as

$$y = c + \int f(t)\, dt \quad \text{or}$$

$$y_c(x) = c + \int_{x_0}^{x} f(t)\, dt$$

Example 3. $\dfrac{dy}{dx} = f(x, y)$.

Here $f(\cdot,\cdot)$ is a given and $F : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ is $F(x, y_0, y_1) = y_1 - f(x, y_0)$.

The solution $y = y(x)$ $x \in [a,b]$, if it exists satisfies

$$y(x) = y(x_0) + \int_{x_0}^{x} f(t, y(t))\, dt$$

for any $x_0 \in [a,b]$. Note that when a continuous solution $y(t)$, $t \in [a,b]$ exists and $(t, y_0) \to f(t, y_0) : [a,b] \times E_0 \to \mathbb{R}$ is continuous, then $t \to f(t, y(t))$ is continuous and the integral

$$\int_{x_0}^{x} f(t, y(t))\, dt$$

is well defined as a Rieman integral.

Geometrically what this means is that we are trying to find a curve $y(t)$ such that for each $(s, y_0) \in [a,b] \times E_0$ the curve $y(t)$ passes through the point (s, y_0) i.e. $y(s) = y_0$ and has a slope $y'(s) = f(s, y_0)$.

Remark. We use the notation $y(x; x_0, c) \equiv y_c(x; x_0)$ to represent the solution in Example 3 which satisfies $y(x_0) = c$. Note that Thus $y(x_0; x_0, c) = c$.

The solution may not exist for arbitrary values of c. When such solutions exist for $c \in E_0$ (say) then the solutions $y(x; x_0, c)$ represent a parametrised family of curves $x \longrightarrow y(x; x_0, c)$.

Example 4. Starting with a family of curves parametrised by $c \in E_0$ viz. $f(x, y, c) = 0$ we can work backwards to arrive at the differential equation satisfied by these curves viz. $F(x, y, y') = 0$ by differentiation :

$$F(x, y, y_0) = \frac{\partial f}{\partial x}(x, y, c) + \frac{\partial f}{\partial y}(x, y, c)\, y_0$$
$$= g_1(x, y) + g_2(x, y)\, y_0$$

where in the 2nd equality we have eliminated c using the given equation. For example when $f(x, y, c) = x^2 + y^2 - 2cx$ so that $f(x, y, c) = 0$ represent circles tangent to the the y-axis at the origin and centres at $(c, 0)$. Then, using the above method, the differential equation for this family of curves is given by

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}.$$

Prove this.

Exercise : Prove this.

Remark. (Geometric interpretation contd.) Given a one parameter family of curves $y(x; x_0, c)$ we can obtain a family of curves 'orthogonal' to the given family as follows: Suppose $y(x; x_0, c)$ satisfies

$$\frac{dy}{dx} = f(x, y).$$

and suppose that $f(x, y) \neq 0 \ \forall (x, y) \in [a, b] \times E_0$. Then the orthogonal family of curves is given by the solutions $z(x; x_0, c)$ of the equation

$$\frac{dz}{dx} = -\frac{1}{f(x,z)} \qquad (6)$$

This is because the product of the slopes $\frac{dy}{dx}\frac{dz}{dx} = -1$. Hence at a point $x \in [a,b]$ where the curves meet i.e. $y(x; x_0, c) = z(x; x_0, c)$, the tangent to the curves at x are orthogonal.

<u>Exersise</u>: Determine the family of curves orthogonal to the family of curves $x^2 + y^2 = c^2$.

<u>Remark</u>. Another important generalisation of Example 3 is as follows:

We are given a vector field $f: \mathbb{R}^n \to \mathbb{R}^n$, $f(z) = (f_1(z), \ldots, f_n(z))$ where $f_i : \mathbb{R}^n \to \mathbb{R}$. This gives rise to a <u>System of ODEs</u> viz

$$\frac{dy}{dx} = f(y), \quad y(x) = (y^1(x), \ldots, y^n(x))$$

or equivalently, $\frac{dy^i}{dx} = f_i(y) \quad i = 1, \ldots n.$

Here $x \in [a,b]$ and $y = y(x)$ represents a curve in $\mathbb{R}^n$ with $\frac{dy}{dx}(x) = (y^{1'}(x), \ldots y^{n'}(x))$ the tangent vector at x, specified by the vector field f at $y(x)$ i.e. $f(y(x)) = (f_1(y(x)), \ldots f_n(y(x)))$.

<u>Example 5.</u> $\frac{dx}{dt} = -kx \quad t \in [a,b]$.

This is a simple but important equation and is an example of a <u>dynamical system</u>. Here t represents time $x = x(t)$ represents the state of the system at time t. The RHS is given by the vector field $f(x) = -kx$ $(n=1)$ $f: \mathbb{R} \to \mathbb{R}$. Note that $F(t, x, x') = x'(t) - k\,x(t)$. The solution of the above eqn. is

$$x(t) = x_0 e^{-kt}$$

which represents a 1-parameter family of curves where the parameter 'c' is the initial value x_0 at $t = 0$. When

$k > 0$ (resp. $k < 0$) the system represents the decay (resp. growth) of an initial amount x_0 of some substance.

Example 6. We now consider examples of equations of order 2 i.e. $F(x, y, y', y'') = 0$. Typically they arise as time evolution of a system in some 'force field' that accelerates or retards the system as in a 'gravitational field'. The equation of motion of the system can be written as

$$\frac{d^2y}{dt^2} = g - c\frac{dy}{dt}.$$

If $c = 0$ then the system represents the height of a falling body from a fixed point, under the influence of gravity alone and the solution is

$$y(t) = \frac{1}{2}gt^2 + c_1 t + c_2 .$$