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10 min

Partial Differential Equations:

example: Transport equation:

$$\partial_t U + c\partial_x U = 0$$

$u = u(t,x) \quad x \in \mathbb{R} \; t > 0.$

$u(0,x) = u_0 x.$

$\partial_t = \dfrac{\partial}{\partial t} \qquad \partial_x = \dfrac{\partial}{\partial x}.$

Generalisation: $\partial_t u + c \cdot \nabla U = 0$

where $c = (c_1, \ldots, c_d)$

$\nabla = (\partial_{x_1}, \ldots, \partial_{x_d})$

$u(x,t) := u_0(x - ct).$

example: $\partial_t u + c(x) \cdot \nabla u = 0$

$u(0,x) = u_0(x)$

Example: Burger's equation: $u_t + u u_x = 0$

$u(0,x) = u_0(x)$

$$\partial_t u(t,x) + u(t,x)\,\partial_x u(t,x) = 0$$

$u(0,x) = u_0(x) \quad 0 < t < T_0$

$x \in E$

$$\partial_t u + c(u)\,\partial_x u = 0$$

$c(z) = z$

$$\partial_t u(t,x) + c(u(t,x))\,\partial_x(u(t,x)) = 0$$

Example: Hamiltonian Equation:

$$\partial_t U + (\partial_x U)^2 = 0$$

$u(0,x) = u_0(x)$

$$\Leftrightarrow \partial_t U(t,x) + (\partial_x U(t,x))(\partial_x U(t,x)) = 0$$

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~~Example~~ Remark: ODE $\leftrightarrow F(x,y,y') = 0$

$$\frac{dy}{dx} = f(x,y)$$

$$F(x,y,y') = y' - f(x,y)$$
$$F(x,y_0,y_1) = y_1 - f(x,y_0)$$

$$F((t,x), u, \partial_t u, \partial_x u) = \partial_t u + c\,\partial_x u$$
$$F((t,x), u_0, u_1, u_2) = u_1 + c\,u_2$$

2nd order equations. $\qquad \Delta = \sum_{i=1}^{d} \partial^2_{x_i}$

Example: (Laplace equation)

$$\Delta u(x) = 0 \qquad x \in B \subset \mathbb{R}^d$$

B is an open set.

Dirichlet's problem: Given $g : \partial B \longrightarrow \mathbb{R}$
Find $u(x)$ twice continuously differentiable in B satisfying

Boundary Value Problem.
$$\begin{cases} \Delta u(x) = 0 & x \in B \\ u(x) = g(x) & x \in \partial B \end{cases}$$

Example: (The Heat Equation) $\quad \partial_t U = \frac{1}{2}\Delta U \qquad (t,x) \in (0,\infty)\times\mathbb{R}^d$ ~~$(t,x) \in (0,\infty)\times\mathbb{R}^d$~~

$$u(0,x) = u_0(x)$$

$$\Longrightarrow u(t,x) := \frac{1}{(2\pi t)^{d/2}} \int u_0(y)\, e^{-\frac{|x-y|^2}{2t}}\, dy$$

$\longleftrightarrow \quad u(t,x) = u_0 * p_t(x)$ where

$$p_t(x) = \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x|^2}{2t}}$$

$$= E\,u_0(X_t)$$
$$= \int u_0(y)\, P_{X_t}(dy) \qquad X_t \sim N(x, t I_d)$$
$$= \int u_0(y)\, p_t(x-y)\, dy$$

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Example: Wave equation

$$\partial_t^2 U = c \Delta U = c \sum_{i=1}^{d} \partial_i^2 u$$

The method of characteristics