DIFFERENTIAL TOPOLOGY - LECTURE 3

1. Introduction

Having defined the notion of a manifold and smooth maps between manifolds, we shall now define the notion of the derivative of a smooth map between manifolds. Towards this, given a k-manifold X, we first associate to each $x \in X$ a (real) vector space $T_x(X)$ called the tangent space to X at x. Given a smooth map $f : X \longrightarrow Y$ between manifolds the derivative of f at $x \in X$ will then be defined to be a certain linear map

$$df_x : T_x(X) \longrightarrow T_{f(x)}(Y).$$

This map we shall see has the usual properties that one would expect.

2. Tangent spaces and derivatives

Recall that a subset $X \subseteq \mathbb{R}^N$ is a k-manifold if each $x \in X$ is contained in an open set V which is diffeomorphic to an open subset $U \subseteq \mathbb{R}^k$. If X is a k-manifold and $x \in V \subseteq X$ is open and

$$\varphi : U \underset{\text{open}}{\subseteq} \mathbb{R}^k \longrightarrow U$$

a diffeomorphism, then φ is called a parametrization about x. Recall that it is possible to choose the above parametrization so that $0 \in U$ and that $\varphi(0) = x$.

We shall now discuss the definition of the derivative of a smooth function between manifolds. Towards defining this we shall first define the notion of *tangent spaces.*

Start with a k-manifold $X \subseteq \mathbb{R}^N$. Let

$$\varphi : U \subseteq \mathbb{R}^k \longrightarrow X$$

be a local parametrization about x such that $0 \in U$ and $\varphi(0) = x$. We think of φ as a (smooth) map

$$\varphi : U \subseteq \mathbb{R}^k \longrightarrow \mathbb{R}^N.$$

Hence $d\varphi_0$, the derivative of φ at 0, is then a linear map

$$d\varphi_0 : \mathbb{R}^k \longrightarrow \mathbb{R}^N.$$

Then $\mathrm{im}(d\varphi_0)$, the image of $d\varphi_0$, is a subspace of $\mathbb{R}^N$. This subspace is called the *tangent space* to X at x.

Definition 2.1. If $X \subseteq \mathbb{R}^N$ is a k-manifold and $x \in X$, then the tangent space $T_x(X)$ to X at x is defined to be the vector space $\mathrm{im}(d\varphi_0) \subseteq \mathbb{R}^N$ where $\varphi : U \subseteq \mathbb{R}^k \longrightarrow X$ is a local parametrization about x with $\varphi(0) = x$. The vectors $v \in \mathbb{R}^N$ such that $v \in T_x(X)$ are called *tangent vectors* to X at x. Notice that, by definition, $T_x(X)$ is a subspace of $\mathbb{R}^N$.

A priori, the definition depends upon the choice of the local parametrization about $x \in X$. We shall see that this is not the case. Indeed, suppose that

$$\varphi : U \underset{\text{open}}{\subseteq} \mathbb{R}^k \longrightarrow X$$

$$\psi : V \underset{\text{open}}{\subseteq} \mathbb{R}^k \longrightarrow X$$

are two local parametrizations about $x \in X$ with $\varphi(0) = x = \psi(0)$. We may arrange (by shrinking the open sets) so that $\varphi(U) = \psi(V)$. Then the composition

$$h = \psi^{-1} \circ \varphi : U \longrightarrow V$$

is a diffeomorphism. Thus,

$$\varphi = \psi \circ h$$

is now a composition of smooth maps smooth maps defined on open subsets of the euclidean space. By the chain rule we have that

$$d\varphi_0 = d\psi_0 \circ dh_0.$$

This essentially shows that $\operatorname{im}(d\varphi_0) = \operatorname{im}(d\psi_0)$ (since dh_0 is an isomorphism). Hence the tangent space $T_x(X)$ is well defined.

Here is an example.

Example 2.2. Consider the n-manifold $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$. The tangent space $T_x(\mathbb{S}^n)$ at $x = (1, 0, \ldots, 0) \in \mathbb{S}^n$ can be described as follows. By definition we need to look at a parametrization

$$\varphi : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{S}^n$$

with $0 \in U$ and $\varphi(0) = x$ about x and then the tangent space $T_x(\mathbb{S}^n)$ is

$$T_x(\mathbb{S}^n) = \operatorname{im}(d\varphi_0).$$

We know that (see Example 2.4, Lecture 2)

$$\varphi : \operatorname{Int}(\mathbb{D}^n) \longrightarrow \mathbb{S}^n$$

defined by

$$\varphi(x_1, \ldots, x_n) = \left(\sqrt{1 - (x_1^2 + x_2^2 + \cdots + x_n^2)}, x_1, \ldots, x_n\right)$$

is a parametrization about $x \in \mathbb{S}^n$ with $\varphi(0) = x$. The Jacobian matrix $d\varphi_0$ can be checked to be the $(n+1) \times n$ matrix

$$J\varphi(0) = d\varphi_0 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ I_n \end{pmatrix}$$

Thus the first row is zero. It is then clear that the tangent space $\operatorname{im}(d\varphi_0) = T_x(\mathbb{S}^n)$ equals the subspace of those vectors $v \in \mathbb{R}^{n+1}$ whose first coordinate is 0. It is important to note that any vector $v \in \mathbb{R}^{n+1}$ with first coordinate 0 is actually a tangent vector to $\mathbb{S}^n$ at x.

Describing the tangent space from first principles, as in the above example, often turns out to be messy. In general writing down parametrizations for a manifold is difficult There are ways to determine the tangent space without using explicit parametrizations. Exercise 2.15 below describes one such way which is very useful in concrete situations. Later, we shall see yet another way of determining the tangent space to a manifold.

The question that we now wish to discuss is : for a k-manifold X, what is the dimension of the tangent space $T_x(X)$ at $x \in X$?

Lemma 2.3. If $X \subseteq \mathbb{R}^N$ is a k-manifold, then $\dim(T_x(X)) = k$ for all $x \in X$.

Proof. Fix a local parametrization
$$\varphi : U \subseteq \mathbb{R}^k \longrightarrow X$$
about $x \in X$. Since φ is a diffeomorphism onto its image, we have that φ^{-1} is smooth. By definition, this means that there exists an open set $W \subseteq \mathbb{R}^N$, $x \in W$ and a smooth map $F : W \longrightarrow \mathbb{R}^k$ with
$$F/(W \cap V) = \varphi^{-1}.$$
Assuming $\varphi(U) \subseteq W$ we have $F \circ \varphi = \text{id}$. Invoking the chain rule we have that
$$dF_x \circ d\varphi_0 = \text{id}.$$
This shows that $d\varphi_0$ is injective. This completes the proof. □

Remark 2.4. The proof of the above lemma shows that if X is a k-manifold and $\varphi : U \subseteq \mathbb{R}^k \longrightarrow X$ is a local parametrizaion about x with $\varphi(0) = x$, then the linear map
$$d\varphi_0 : \mathbb{R}^k \longrightarrow T_x(X) \subseteq \mathbb{R}^N$$
is a linear isomorphism (onto $T_x(X)$). This observation will now help us in defining the derivative of a smooth function between two manifolds. The method of the proof of the above lemma appears in many situation as we shall see. Finally note that if $X \subseteq \mathbb{R}^N$ is a k-manifold, then $k \leq N$.

We now turn to the definition of the derivative of a smooth map between manifolds. Suppose $X \subseteq \mathbb{R}^N$ is a k-manifold and $Y \subseteq \mathbb{R}^M$ is an ℓ-manifold. Let $f : X \longrightarrow Y$ be a smooth function. Let $x \in X$. The derivative df_x of f at x will be defined to be a linear map
$$df_x : T_x(X) \longrightarrow T_{f(x)}(Y)$$
from the tangent space to X at x to the tangent space of Y at $f(x)$. This is done as follows.

Fix parametrizations
$$\varphi : U \subseteq \mathbb{R}^k \longrightarrow X$$
$$\psi : V \subseteq \mathbb{R}^\ell \longrightarrow Y$$
about x and $f(x)$ respectively. We assume that $0 \in U$ and $0 \in V$ with $\varphi(0) = x$ and $\psi(0) = f(x)$. We now let[1]
$$g = \psi^{-1} \circ f \circ \varphi : U \longrightarrow V.$$
Observe that $g(0) = 0$ and that g is smooth and we have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \\
U \arrow[u, "\varphi"] \arrow[r, "g=\psi^{-1}\circ f\circ\varphi"'] & V \arrow[u, "\psi"']
\end{tikzcd}$$

Taking derivatives of the two vertical maps and the lower horizontal map we get a diagram

$$\begin{tikzcd}
T_x(X) \arrow[r, dashed, "df_x"] & T_{f(x)}Y \\
\mathbb{R}^k \arrow[u, "d\varphi_0"] \arrow[r, "dg_0"'] & \mathbb{R}^\ell \arrow[u, "d\psi_0"']
\end{tikzcd}$$

[1] Such compositions make sense on suitably chosen open sets and we shall not mention this explicitly.

where the the top horizontal dotted arrow is the map that we need to define. The observations that we made above will now help us in defining df_x. First observe that all the maps in the solid arrows are linear maps of vector spaces. By the Remark above we know that the two vertical maps are linear isomorphisms. We now define the derivative df_x of f at x to be the composition

$$df_x = d\psi_0 \circ dg_0 \circ d\varphi_0^{-1}.$$

Clearly

$$df_x : T_x(X) \longrightarrow T_{f(x)}(Y)$$

is a linear map. With this definition, the second diagram also commutes.

Having defined the derivative, what now remains is to check that the definition of df_x does not depend upon the local parametrizations chosen. We shall check this in some detail. The proof uses the method of Lemma 3.2 above. Similar checking in the future will be left as an exercise. Fix two parametrizations

$$\varphi : U \longrightarrow X; \quad \varphi' : U' \longrightarrow X$$

about x with $\varphi(0) = x = \varphi'(0)$. Also fix two parametrizations

$$\psi : V \longrightarrow Y; \quad \psi : V' \longrightarrow Y$$

with $\psi(0) = f(x) = \psi'(0)$. Set, as before,

$$g = \psi^{-1} \circ f \circ \varphi$$

and

$$g' = \psi'^{-1} \circ f \circ \varphi'.$$

As usual we arrange so that $\varphi(U) = \varphi'(U')$ and $\psi(V) = \psi(V')$. The two setups give rise to two competing definitions of the derivative. To resolve this we must show

$$d\psi_0 \circ dg_0 \circ d\varphi_0^{-1} = d\psi'_0 \circ dg'_0 \circ d{\varphi'}_0^{-1}. \qquad (2.4.1)$$

To prove this we first note that there is a diagram

$$\begin{tikzcd}
U \arrow[r, "g=\psi^{-1}\circ f\circ\varphi"] \arrow[d, "\varphi"] & V \arrow[d, "\psi"] \\
X \arrow[r, "f"] & Y \\
U' \arrow[u, "\varphi'"] \arrow[r, "g'=\psi'^{-1}\circ f\circ\varphi'"'] & V' \arrow[u, "\psi'"]
\end{tikzcd}$$

in which all the three squares commute. In particular

$$g \circ \varphi^{-1} \circ \varphi' = \psi^{-1} \circ \psi' \circ g'.$$

We now replace the maps φ^{-1} and ψ^{-1} by maps F and G respectively that are smooth extensions of φ^{-1} and ψ^{-1} (see the proof of Lemma 2.3, and the remark following the lemma) to write

$$g \circ F \circ \varphi' = G \circ \psi' \circ g'.$$

Since all the maps are smooth maps on open subsets of the euclidean space, we may now apply the chain rule to conclude that

$$dg_0 \circ dF_x \circ d\varphi'_0 = dG_{f(x)} \circ d\psi'_0 \circ dg'_0. \qquad (2.4.2)$$

Just to remind ourselves, the above linear maps fit into the sequences

$$\mathbb{R}^k \xrightarrow{d\varphi_0'} \mathbb{R}^N \xrightarrow{dF_x} \mathbb{R}^k \xrightarrow{dg_0} \mathbb{R}^\ell$$

and

$$\mathbb{R}^k \xrightarrow{dg_0'} \mathbb{R}^\ell \xrightarrow{d\psi_0'} \mathbb{R}^M \xrightarrow{dG_{f(x)}} \mathbb{R}^\ell$$

The well definedness of the tangent space tells us that

$$\mathrm{im}(d\varphi_0') = T_x(X) = \mathrm{im}(d\varphi_0).$$

Now as

$$F \circ \varphi = \mathrm{id}$$

it follows that

$$dF_x/T_x(X) = d\varphi_0^{-1}.$$

Similarly,

$$dG_{f(x)}/T_{f(x)}(Y) = d\psi_0^{-1}.$$

The equation (2.4.2) may now be written as

$$dg_0 \circ d\varphi_0^{-1} \circ d\varphi_0' = d\psi_0^{-1} \circ d\psi_0' \circ dg_0'.$$

Rewriting this immediately implies that equality in equation (2.4.1) holds. This shows that the derivative is well defined.

We end this discussion by showing that the chain rule holds.

Proposition 2.5. (Chain Rule) The chain rule holds.

Proof. Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

be smooth maps between manifolds. Then the chain rule states that for $x \in X$ we have that

$$d(g \circ f)_x = dg_{f(x)} \circ df_x.$$

The proof is an exercise in understanding and unravelling the definition of the derivative. The proof is left as an exercise. □

Remark 2.6. Let $f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^M$ be a smooth map defined on an open set U. Then as f is defined on an open set we have the usual derivative of f. On the other hand we may think of U as a manifold and then there is the "manifold derivative" of f. Both the derivatives coincide. Observe that we can choose the parametrizations φ, ψ to be translations. Thus the function $g = \psi^{-1} \circ f \circ \varphi$ is essentially f upto a translation. The derivatives of φ and ψ are the identity linear transformations. Hence the usual derivative equals the manifold derivative.

Here are some problems.

Exercise 2.7. Let $f : S^1 \longrightarrow \mathbb{R}$ be the map $f(x, y) = x$. Use the definition to find $df_{(x,y)}$ when $(x, y) = (1, 0), (0, 1)$.

Exercise 2.8. Consider the map $f : S^1 \longrightarrow S^1$ defined by $f(z) = z^2$. Use the definition to find df_i where $i = (0, 1)$.

Exercise 2.9. Let $Z \subseteq X \subseteq \mathbb{R}^N$ be a submanifold of X and let $i : Z \hookrightarrow X$ denote the inclucion map. Show that i is smooth and the derivative $di_x : T_x(Z) \longrightarrow T_x(X)$, $x \in Z$, equals the inclusion map of tangent spaces. Note that both $T_x(Z)$ and $T_x(X)$ are subspaces of $\mathbb{R}^N$. This exercise shows that $T_x(Z)$ is a subspace of $T_x(X)$.

Exercise 2.10. We know that if U is an open subset of a k-manifold X, then U is a manifold of the same dimension as X. Show that for all $x \in U$, we have $T_x(U) = T_x(X)$.

Exercise 2.11. Let V be a vector subspace of $\mathbb{R}^N$. Then show that $T_x(V) = V$ for all $x \in V$.

Exercise 2.12. The tangent space to S^1 at (a, b) is a one-dimensional subspace of $\mathbb{R}^2$. Explicitly describe the subspace in terms of a and b. Similarly, exhibit a basis of $T_p(\mathbb{S}^2)$ at an arbitrary point $p = (a, b, c)$.

Exercise 2.13. Prove the following for manifolds X, X', Y, Y'.

(1) Show that $T_{(x,y)}(X \times Y) = T_x(X) \times T_y(Y)$.
(2) If $f : X \times Y \longrightarrow X$ is the projection to the first factor, then
$$df_{(x,y)} : T_x(X) \times T_y(Y) \longrightarrow T_x(X)$$
is also projection to the first factor.
(3) Fix $y \in Y$ and let $f : X \longrightarrow X \times Y$ be defined by $f(x) = (x, y)$. Show that $df_x(v) = (v, 0)$.
(4) Let $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$ be two maps. Show that
$$d(f \times g)_{(x,y)} = df_x \times dg_y.$$

Exercise 2.14. Let $f : X \longrightarrow X \times X$ be the map $f(x) = (x, x)$. Show that $df_x(v) = (v, v)$.

Exercise 2.15. Given a map $f : X \longrightarrow Y$ define $F : X \longrightarrow X \times Y$ by $F(x) = (x, f(x))$. Show that $dF_x(v) = (v, df_x(v)$. Further show that the tangent space to the graph of f at $(x, f(x))$ equals the graph of $df_x : T_x(X) \longrightarrow T_{f(x)}(Y)$.

Exercise 2.16. (Equivalent definition of tangent space). A curve in a manifold X is a smooth map
$$\sigma : I \longrightarrow X$$
where I is an interval. The derivative at $t_0 \in I$ is then a linear map
$$d\sigma_{t_0} : T_{t_0}(I) = \mathbb{R} \longrightarrow T_{\sigma(t_0)}(X).$$
The vector
$$d\sigma_{t_0}(1) \in T_{\sigma(t_0)}(X)$$
is called the *velocity vector* of σ at time t_0. We also make use of the notation s
$$\sigma'(t_0);\ \frac{d\sigma}{dt}(t_0);\ \left.\frac{d\sigma}{dt}\right|_{t_0}$$
to denote the velocity vector. Show that every tangent vector at $x \in X$ is the velocity vector of some curve <u>in X</u> and conversely. This definition of the tangent space is very useful in concrete situations. For example many exercises above follow easily from this definition of the tangent space. However it is instructive to solve the preceding exercises without using this alternate definition of the tangent space.

Exercise 2.17. Let $p \in \mathbb{S}^n$. Use Exercise 2.16 to describe $T_p(\mathbb{S}^n)$.

Exercise 2.18. Complete the proof of Proposition 2.5.

Exercise 2.19. Let Z be a submanifold of X, $f : X \longrightarrow Y$ smooth and $g = f/Z$. For $z \in Z$ show that $dg_z = df_z/T_z(Z)$.

3. Solutions to selected problems

Here are solutions to selected problems. Often there is more than one way to solve a problem.

(1) (2.7) Let $\varphi : (-1,1) \longrightarrow S^1$ be the map
$$\varphi(t) = (\sqrt{1-t^2}, t).$$
Then φ is a parametrization about $(1,0)$ and $\varphi(0) = (1,0)$. Let $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ be the parametrization about $f(1,0) = 1$ given by $\psi(t) = t+1$. Then $\psi(0) = f(1,0) = 1$. Let $g = \psi^{-1} \circ f \circ \varphi$. So that $g : (-1,1) \longrightarrow \mathbb{R}$ is given by
$$g(t) = \sqrt{1-t^2} - 1.$$
By definition,
$$df_{(1,0)} = d\psi_0 \circ dg_0 \circ d\varphi_0^{-1}.$$
It is clear that $dg_0 = 0$. Hence
$$df_{(1,0)} : T_{(1,0)}(S^1) \longrightarrow T_1(\mathbb{R}) = \mathbb{R}$$
is the zero linear map. Another way to see this is to consider the maps
$$S^1 \hookrightarrow \mathbb{R}^2 \xrightarrow{\pi_1} \mathbb{R}.$$
The composition is f. Now apply chain rule.

(2) (2.8) The map f in cartesian coordinates is
$$f(x,y) = (x^2 - y^2, 2xy).$$
Now we may use parametrizations as in the previous problem.

(3) (2.9) Let Z be a submanifold of $X \subseteq \mathbb{R}^N$ and let $i : Z \longrightarrow X$ be the inclusion map. This is clearly smooth. Given $x \in Z$ fix parametrizations
$$\varphi : U \longrightarrow Z; \quad \psi : V \longrightarrow X$$
about $x \in Z$ and $x \in X$ with $\varphi(0) = x = \psi(0)$. . First observe that
$$g = \psi^{-1} \circ i \circ \varphi = \psi^{-1} \circ \varphi.$$
By definition,
$$di_x = d\psi_0 \circ dg_0 \circ d\varphi_0^{-1} = d\psi_0 \circ (d\psi_0^{-1} \circ d\varphi_0) \circ d\varphi_0^{-1}$$
and the result follows.

(4) (2.10) The previous exercise already says that $T_x(U) \subseteq T_x(X)$. Since both have the same dimension equality follows.

(5) (2.11) This follows for there exists a global parametrization of V by a linear map. The image of the linear map is V.

(6) (2.12) Let $(a,b) \in S^1$ with $b > 0$. Fix $\varepsilon > 0$ such that $(a-\varepsilon, a+\varepsilon) \subseteq (-1,1)$. Consider the parametrization $\varphi : (-\varepsilon, \varepsilon) \longrightarrow S^1$ about (a,b) defined by
$$\varphi(t) = (a+t, \sqrt{1-(a+t)^2})$$
so that $\varphi(0) = (a,b)$. We think of φ as a map $\varphi : (\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^2$ and then, by definition,
$$T_{(a,b)}(S^1) = \operatorname{im}(d\varphi_0).$$
Clearly,
$$d\varphi_0 = \left(1, \frac{-a}{\sqrt{1-a^2}}\right)^t$$

and hence

$$d\varphi_0(v) = \left(v, \frac{-av}{\sqrt{1-a^2}}\right)^t$$

where $v \in \mathbb{R}$. For dimension reasons,

$$T_{(a,b)}(S^1) = \operatorname{span}(d\varphi_0(1)) = \operatorname{span}\left(1, \frac{-a}{\sqrt{1-a^2}}\right)^t.$$

Thus,

$$T_{(ab)}(S^1) = (a,b)^\perp.$$

The other cases can be dealt with similarly. This, somewhat long, calculation shows that a hands on computation with parametrizations to compute the tangent space can be messy.

(7) (2.13) We will discuss the proof of (1). The other proofs are similar. First note that the dimensions of the vector spaces involved are equal. Fix parametrizations

$$\varphi : \mathbb{R}^k \longrightarrow X; \quad \psi : \mathbb{R}^\ell \longrightarrow Y$$

about x and y such that $\varphi(0) = x, \psi(0) = y$. By definition

$$\operatorname{im}(d(\varphi \times \psi)_{(0,0)}) = T_{(x,y)}(X \times Y).$$

By definition

$$\begin{aligned} d(\varphi \times \psi)_{(0,0)}(v,w) &= \lim_{h\to 0} \tfrac{(\varphi\times\psi)(hv,hw)-(\varphi\times\psi)(0,0)}{h} \\ &= \lim_{h\to 0} \tfrac{(\varphi(hv),\psi(hw))-(\varphi(0),\psi(0))}{h} \\ &= \lim_{h\to 0} \tfrac{(\varphi(hv)-\varphi(0),\psi(hw)-\psi(0))}{h} \\ &= (d\varphi_0(v), d\psi_0(w)) \end{aligned}$$

This shows $T_{(x,y)}(X \times Y) \subseteq T_x(X) \times T_y(Y)$ and hence equality follows.

(8) (2.17) Let $p \in S^n$ and let $\sigma : (-\varepsilon, \varepsilon) \longrightarrow S^n$ be a curve with $\sigma(0) = p$. Let $f : \mathbb{S}^n \longrightarrow \mathbb{R}$ be the function $f(x) = ||x||^2$. Then $f \circ \sigma = 1$. Thus

$$0 = d(f \circ \sigma)_0(1) = df_{\sigma(0)}(d\sigma_0(1)) = \nabla f(\sigma(0)) \cdot \sigma'(0) = 2p \cdot \sigma'(0).$$

Thus every tangent vector at p is orthogonal to p. Thus

$$T_p(\mathbb{S}^n) \subseteq p^\perp$$

and since the dimensions are equal we have that the above inclusion is an equality.

(9) (2.18) Use Chain rule and Exercise 2.9.