Implicit function Theorem

Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$. Then
$(x, y) \in \mathbb{R}^{n+m}$.

If $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_m)$

then $(x, y) = (x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m)$

Let $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$.

We define $A_x \in L(\mathbb{R}^n, \mathbb{R}^n)$, $A_y \in L(\mathbb{R}^m, \mathbb{R}^n)$

$$A_x(h) = A(h, 0)$$

$$A_y(k) = A(0, k)$$

A_x and A_y are linear

$$\begin{aligned} A(h, k) &= A[(h, 0) + (0, k)] \\ &= A(h, 0) + A(0, k) \\ &= A_x(h) + A_y(k). \end{aligned}$$

$h \in \mathbb{R}^n$
$k \in \mathbb{R}^m$

Theorem: If $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$ and A_x is invertible, then for each $k \in \mathbb{R}^m$ $\exists\, h \in \mathbb{R}^n$ such that $A(h, k) = 0$.
Moreover $h = -A_x^{-1} A_y(k)$.

Proof: $h = -A_x^{-1} A_y(k)$

$$A(h,k) = A_x(h) + A_y(k)$$
$$= -A_y(k) + A_y(k) = 0$$

Implicit function Theorem:

Let E be an open set in $\mathbb{R}^{n+m}$ and $f : E \to \mathbb{R}^n$ be a C'-map. Assume $f(a,b) = 0$ for some $(a,b) \in E$

Let $A = f'(a,b)$. Assume A_x is invertible.

Then there exists open sets $U \subseteq \mathbb{R}^{n+m}$ and $W \subseteq \mathbb{R}^m$ such that for each $y \in W$ there exists a unique $x \in \mathbb{R}^n$ such that $(x,y) \in U$ and $f(x,y) = 0$.

Suppose $g(y)$ denotes the unique $x \in \mathbb{R}^n$ such that $(x,y) \in U$ and $f(x,y) = 0$.

Then $g \in C'(W)$ and

$$g'(y) = A_x^{-1} A_y.$$

$\frac{\partial f_i}{\partial x_j}$ is conts $\Longleftrightarrow f$ is C'

Proof: Let $F : E \to \mathbb{R}^{n+m}$ such that

$$F(x,y) = (f(x,y), y)$$

$f \in C'$. Let $f_1, f_2, \ldots, f_n$ be the co-ordinate functions of f.

Then $\frac{\partial f_i}{\partial x_j}$, $\frac{\partial f_i}{\partial y_k}$ $(1 \le i, j \le n,\ 1 \le k \le m)$

are continuous

Let $p_j : \mathbb{R}^m \to \mathbb{R}$ be the projections onto the jth co-ordinate.

$$\frac{\partial p_j}{\partial y_k} = \delta_{jk}$$

F has co-ordinate functions

$f_1, f_2, \ldots, f_n, p_1, p_2, \ldots, p_m$.

Then $F \in C'$ on E.

$F'(a,b) = (f'(a,b), I_m)$

$= (A, I_m)$

Suppose $F'(a,b)(h,k) = 0$

$A(h,k) = 0$ and $I_m(k) = 0$

$\Downarrow$

$A(h,0) = 0$ $\qquad k = 0$

$A_x(h) = 0 \implies h = 0$

$\therefore \dot{F}'(a,b)$ is 1-1

$\Rightarrow F'(a,b)$ is invertible.

By Inverse function theorem there exists a open set $W \subseteq E$ such that $(a,b) \in W$, $F(W)$ is open, $F|_W$ is bijective.

$F(W) = \{(f(x,y), y) \,/\, (x,y) \in W\}$

$(0,b) \in F(W)$

Let $U = \{ y \in \mathbb{R}^m \,|\, (0,y) \in F(W)\}$

$\Rightarrow U$ is open set containing b.

For $y \in U \Rightarrow (0,y) \in F(W)$

$\Rightarrow \exists\, x$ such that

$(x,y) \in W$, $f(x,y) = 0$

Suppose $x' \in \mathbb{R}^n$ is such that $(x',y) \in W$ and $f(x',y) = 0$

$F(x',y) = F(x,y)$

Since $F|_W$ is bijective, $(x',y) = (x,y)$

$\Rightarrow x' = x$

Thus for each $y \in U$ there exists a unique $x \in \mathbb{R}^n$ such that $(x,y) \in W$ and $f(x,y) = 0$.

Let $g(y)$ denotes the unique x.

Then $f(g(y), y) = 0$

Let $G = (F|_W)^{-1}$.

$$y \mapsto (0, y) \xrightarrow{G} (g(y), y) \xrightarrow{P_1} g(y)$$

$G(0, y) = (g(y), y)$

g is a C^1 - map

Let $\Phi(y) = (g(y), y) \in \mathbb{R}^{n+m}$.

Then $\Phi \in C^1$, $\Phi'(y) = (g'(y), I_m)$

$$f(\Phi(y)) = 0$$

$$f'(\Phi(y))\, \Phi'(y) = 0$$

for $y = b$, $\Phi(y) = (a, b)$

$$A\, \Phi'(b) = 0$$

$$\left.\begin{aligned} &A\, \Phi'(b)\,(k) = 0 \\ &A\,(g'(b)(k), k) = 0 \\ &A_x\, g'(b)\,(k) + A_y(k) = 0 \end{aligned}\right\} \; \forall\, k \in \mathbb{R}^m$$

$$A_x\, g'(b) = -A_y$$

$$g'(b) = -A_x^{-1} A_y.$$