Lecture 13.

Legendre Polynomials (Contd.)

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$t = \frac{1}{2}(1-x)$

$$t(1-t)y'' + (1-2t)y' + n(n+1)y = 0$$

$$a = -n, \quad b = n+1, \quad c = 1.$$

$$P_n(x) := F\left(-n, n+1, 1 \; \tfrac{1}{2}(1-x)\right)$$

$$F(a,b,c,x) = 1 + \sum_{n=1}^{\infty} \frac{a(a+1)\cdots(a+n-1)\, b(b+1)\cdots(b+n-1)}{n!\, c(c+1)\cdots(c+n-1)} x^n.$$

$$P_n(x) = F\left(-n, n+1, 1, \tfrac{1}{2}(1-x)\right)$$

$$= 1 + \frac{(-n)(n+1)}{1!\,1!} \frac{1}{2}(1-x) + \frac{(-n)(-n+1)(n+1)(n+2)}{2!\;2!}\left(\frac{1-x}{2}\right)^2$$

$$+ \cdots \frac{(-n)(-n+1)\cdots(-n+n-1)(n+1)(n+2)\cdots(n+n-1)}{n!\;n!}\left(\frac{1-x}{2}\right)^n$$

$$= 1 + \frac{n(n+1)}{2}(x-1) + \frac{n(n-1)(n+1)(n+2)}{(2!)^2\, 2^2}(x-1)^2$$

$$+ \cdots + \frac{(2n)!}{(n!)^2\, 2^n}(x-1)^n$$

We know that (2).

$$P_n(x) = a_n x^n + a_{n-2} x^{n-2} + \cdots + a_{n-2k} x^{n-2k} + \cdots + a_0 .$$

From above we know that

$$a_n = \frac{(2n)!}{(n!)^2 \, 2^n}$$

We also know from p.10 that

$$a_{k+2} = -\frac{(p-k)(p+k+1)}{(k+1)(k+2)} a_k$$

Or with $p = n$,

$$a_k = -\frac{(n-k+2)(n+k-1)}{(k-1)k} a_{k-2}$$

<u>Proposition</u> (Rodrigue's formula)

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

for $n \geq 0$ with $P_0(x) \equiv 1$.

<u>Corollary</u>. $\displaystyle\int_{-1}^{1} P_m(x) P_n(x)\, dx = \delta_{mn} \cdot \frac{2}{2n+1}$

Remark Let $f, g : [-1,1] \to \mathbb{R}$

Introduce the inner product

$$\langle f, g\rangle := \int_{-1}^{1} f(x)\, g(x)\, dx .$$

Then the above corollary says $P_m(x)$ and $P_n(x)$, $m \neq n$ are "orthogonal vectors" ie. $\langle P_m, P_n\rangle = 0$ with respect to the inner product $\langle \cdot, \cdot \rangle$ defined above.

Proof of Corollary. Suppose $m < n$.

Then from Rodrigue's formula

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = \frac{1}{2^n n!} \int_{-1}^{1} P_m(x) \frac{d^n}{dx^n} (x^2-1)^n\, dx$$

$$= \frac{(-1)^n}{2^n n!} \int_{-1}^{1} \frac{d^n}{dx^n} P_m(x)\, (x^2-1)^n\, dx$$

$$= 0$$

where we have integrated by parts to get the second equality and that $\frac{d^k}{dx^k}(x^2-1)^n$ vanishes at $x = \pm 1$ for $k < n$.

On the other hand when $m=n$

$$\int_{-1}^{1} (P_n(x))^2 dx = \frac{1}{2^n n!}\int_{-1}^{1} P_n(x) \frac{d^n}{dx^n}(x^2-1)^n dx$$

$$= \frac{(-1)^n}{2^n n!}\int_{-1}^{1} \frac{d^n}{dx^n} P_n(x)\,(x^2-1)^n dx$$

$$= \frac{(2n)!}{(2^n n!)^2}\int_{-1}^{1} (1-x^2)^n . dx$$

$$= \frac{2\,(2n)!}{2^{2n}(n!)^2}\int_{0}^{1} (1-x^2)^n dx$$

$$= \frac{2\,(2n)!}{2^{2n}(n!)^2}\int_{0}^{\pi/2} \cos^{2n+1}\theta \, d\theta$$

$$= \frac{2}{2n+1} . \qquad \square$$

Proof of Rodrigue's formula :

$$P_n(x) = a_n x^n + a_{n-2} x^{n-2} + \dots + a_0$$

where $a_n = \frac{(2n)!}{(n!)^2 2^n}$ and

$$a_{k-2} = -\frac{k(k-1)}{(n-k+2)(n+k-1)} a_k$$

Hence $P_n(x) = \frac{(2n)!}{(n!)^2 2^n}\left[x^n + \dots + \frac{a_{n-2k}}{a_n} x^{n-2k} + \dots + a_0\right]$

By iterating the relationship

$a_{k-2} = d_{n,k}\, a_k$ with $k = n, n-2 \cdots$

we get an expression for $\frac{a_{n-2k}}{a_n}$.

$$\frac{a_{n-2k}}{a_n} = (-1)^k \frac{n(n-1)\cdots(n-2k+1)}{2^k\, k!\,(2n-1)(2n-3)\cdots(2n-2k+1)}$$

$$= (-1)^k \frac{(n!)^2\,(2n-2k)!}{k!\,(2n)!\,(n-k)!\,(n-2k)!}$$

Hence $P_n(x) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{(2n-2k)!}{2^n k!\,(n-k)!\,(n-2k)!} x^{n-2k}$

where $[m]$ = Greatest integer $\leq m$.

We note that

$$\frac{(2n-2k)!}{(n-2k)!} x^{n-2k} = \frac{d^n}{dx^n} x^{2n-2k}$$

Hence $\sum_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{(2n-2k)!}{2^n k!\,(n-k)!\,(n-2k)!} x^{n-2k}$

$$= \frac{1}{2^n n!} \frac{d^n}{dx^n} \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{k} (x^2)^{n-k} (-1)^k$$

$$= \frac{1}{2^n n!} \frac{d^n}{dx^n} \sum_{k=0}^{n} \binom{n}{k} (x^2)^{n-k} (-1)^k$$

$$= \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n \qquad \square$$