Lecture 32: Bezout's theorem

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<u>Example</u>: Let $f_1, \dots, f_m \in k[X_0, \dots, X_n]$ be homogen poly. Let $Y_0, \dots, Y_n$ be lin change of variable

i.e. $\begin{pmatrix} Y_0 \\ \vdots \\ Y_n \end{pmatrix} = A \begin{pmatrix} X_0 \\ \vdots \\ X_n \end{pmatrix}$ $A \in GL_{n+1}(k)$.

Let $g_i(Y_0, \dots, Y_n)$ be s.t. $g_i\left(A\begin{pmatrix} X_0 \\ \vdots \\ X_n \end{pmatrix}\right) = f_i(X_0, \dots, X_n)$

i.e. $g_i(Y_0, \dots, Y_n) = f_i(B_{0,*}\underline{Y}, B_{1,*}\underline{Y}, \dots, B_{n,*}\underline{Y})$

where $B = A^{-1}$ & $B = \begin{bmatrix} B_{0,*} \\ B_{1,*} \\ \vdots \\ B_{n,*} \end{bmatrix}$

Then $V_{\mathbb{P}^n}(f_1, \dots, f_m) \cong V_{\mathbb{P}^n}(g_1, \dots, g_m)$ $A : k[X_0, \dots, X_n] \xrightarrow{\sim} k[Y_0, \dots, Y_n]$

if $(f_1, \dots, f_m)$ is a prime ideal.

<u>Lemma</u>: Let k be an algebraically closed and $f, g \in k[X, Y, Z]$ be homogen poly of deg m & $n \geq 1$. After a nonsing linear change of variable X, Y, Z, we may assume f & g are monic in Z.

<u>Pf</u>: Let $f(X,Y,Z) = a_0(X,Y)Z^m + a_1(X,Y)Z^{m-1} + \dots + a_m(X,Y)$

where $a_i(X,Y)$ is homogen of deg i.

$(\Rightarrow)$ ✓

$a_0(X,Y) \equiv 0 \iff [0,0,1] \in Z(f)$

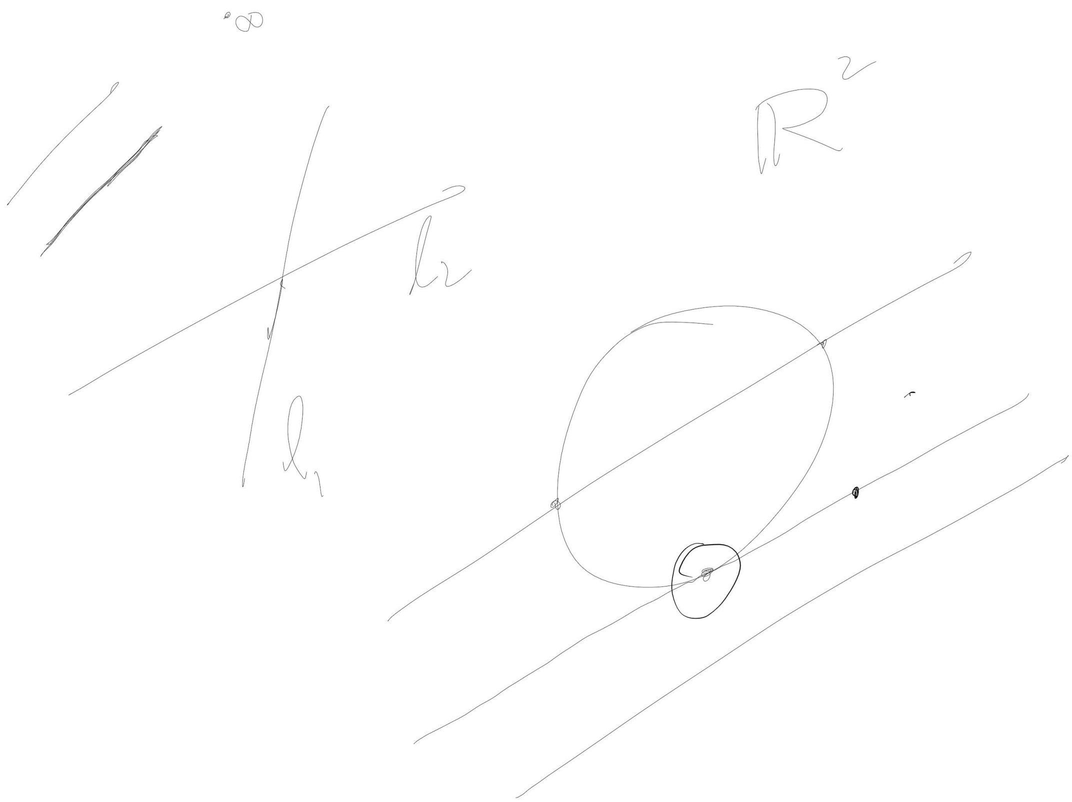
$\Leftarrow$ $a_0 1 = 0$ $\Leftarrow$

$\|$ly $g(X,Y,Z) = b_0(X,Y)Z^n + \dots + b_n(X,Y)$

$b_i(X,Y)$ homogen of deg i.

& $b_0(X,Y) \equiv 0 \iff [0,0,1] \in Z(g)$

So make a linear change of var so that $[0,0,1] \notin Z(f) \cup Z(g)$.



<u>Thm</u> (Bezout's thm): Let C_1 & C_2 be two distinct irred curves in $\mathbb{P}^2_k$ where k is alg closed of deg m & n resp. i.e. $C_1 = Z(f_1)$ & $C_2 = Z(f_2)$ where f_1 & f_2 are irred. homo poly in three var of deg m & n resp. Then $1 \leq |C_1 \cap C_2| \leq mn$. In fact $|C_1 \cap C_2| = mn$ "if each point is counted with right multiplicity".