Lecture 27: Projective Hilbert Nullstellensatz

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Defⁿ: Let $f \in k[X_0, \dots, X_n]$ be a polynomial. It is said to be homogeneous if every monomial of f has the same degree. e.g. $f(X_0, \dots, X_n) = X_0^2 + X_0X_1 + X_n^2$

Lemma: $f \in k[X_0, \dots, X_n]$ is homogen $\Rightarrow$ $f(\lambda a_0, \dots, \lambda a_n) = \lambda^d f(a_0, \dots, a_n)$ $\forall \lambda \in k$ & $(a_0, \dots, a_n) \in k^{n+1}$ where $d = \deg(f)$.

converse holds if k is infinite.

Defⁿ: An ideal I in $k[X_0, \dots, X_n]$ is said to be homogen if I is generated by homogeneous polynomials.

Lemma: $I \subseteq k[X_0, \dots, X_n]$ is homogen iff $\forall f \in I$ $f = f_0 + f_1 + \dots + f_d$ with f_i homogen of deg i then $f_i \in I$ $\forall 0 \le i \le d$.

Defⁿ: A subset $X \subseteq \mathbb{P}^n$ is said to be an algebraic set if there exist a coll of homogen polys S s.t.

$$X = \{P \in \mathbb{P}^n \mid f(P) = 0 \ \forall f \in S\} = Z(S) = Z_{\mathbb{P}^n}(S)$$

If I is homogen ideal of $k[X_0, \dots, X_n]$ then

$$Z(I) = \{P \in \mathbb{P}^n \mid f(P) = 0 \ \forall \text{ homogen } f \text{ in } I\}.$$

$$Z(I + J) = Z(I) \cap Z(J) \qquad I, J \text{ homogen}$$

$$Z(I \cap J) = Z(I) \cup Z(J) \qquad "$$

Defⁿ: $X \subseteq \mathbb{P}^n$

$$\mathcal{I}(X) = \langle\{f \in k[X_0, \dots, X_n] \mid f \text{ homogen } \& \ f(P) = 0 \ \forall P \in X\}\rangle.$$

$\mathcal{I}(X)$ is by defⁿ homogen ideal.

Ⓡ Zariski top on X (an alg subset of $\mathbb{P}^n$): closed sets are algebraic subsets of X. $Z(\mathcal{I}(X)) = \overline{X}$

(*) Note that the only maximal ideal which is homogen in $k[X_0, \dots, X_n]$ is $(X_0, \dots, X_n)$. Also

irrelevant maximal ideal

$Z_{\mathbb{A}^{n+1}}((X_0, \dots, X_n)) = (0, \dots, 0)$. Hence

$Z_{\mathbb{P}^n}((X_0, \dots, X_n)) = \phi$.

Also note that for a point $P = ([a_0, \dots, a_n])$ in $\mathbb{P}^n$ the corresponding homogen ideal is $(a_0X_1 - a_1X_0, \dots, a_0X_n - a_nX_0)$

$a_iX_j - a_jX_i$

$CP \subseteq \mathbb{A}^{n+1}$

$\{(\lambda a_0, \dots, \lambda a_n) \mid \lambda \in k\}$

$I_{\mathbb{A}^{n+1}}(CP) = (a_0X_1 - a_1X_0, \dots, a_0X_n - a_nX_0)$

$\supseteq$ ✓

$$k[X_0, \dots, X_n] \big/ (a_0X_1 - a_1X_0, \dots, a_0X_n - a_nX_0) \cong k[X_0]$$

(if $a_0 \neq 0$)

Hence $I(CP) = (a_0X_1 - a_1X_0, \dots, a_0X_n - a_nX_0)$ ← prime ideal

($\because Z_{\mathbb{A}^{n+1}}(a_0X_1 - a_1X_0, \dots, a_0X_n - a_nX_0) = CP$)

Note that $I(P)$ is maximal among homogen ideals whose radical is properly contained in the irrelevant maximal ideal $(X_0, \dots, X_n)$.

Def: Such an ideal of $k[X_0, \dots, X_n]$ is called homogen maximal ideal. i.e. maximal elements of $\{$homogen ideals J of $k[X_0, \dots, X_n]$ s.t. $\sqrt{J} \subsetneq (X_0, \dots, X_n)\}$

Hilbert Null

Thm $I \subseteq k[x_0, \dots, x_n]$ homogen ideal.

(1) $\mathcal{I}(Z_{\mathbb{P}^n}(I)) = \sqrt{I}$ if $\sqrt{I} \subsetneq (x_0, \dots, x_n)$

(2) $Z_{\mathbb{P}^n}(I) = \phi$ iff $\sqrt{I} = (x_0, \dots, x_n)$ or $1 \in I$.

Lemma: I homogen $\Rightarrow \sqrt{I}$ is homogen.

Pf: Let $f \in \sqrt{I}$ & $f = f_{i_1} + f_{i_2} + \dots + f_{i_d}$

then for some m $\quad i_1 < i_2 < \dots < i_d$

$$f^m \in I \Rightarrow f_{i_1}^m + \text{higher degree terms} \in I$$

$$\Rightarrow f_{i_1}^m \in I \quad (\because I \text{ is homogen})$$

$$\Rightarrow f_{i_1} \in \sqrt{I}$$

$$\Rightarrow f_{i_2} + \dots + f_{i_d} \in \sqrt{I}$$

continue in the same way to get $f_{i_j} \in \sqrt{I} \ \forall j$.

Hence $\sqrt{I}$ is homogen.

Pf of thm:

$$\sqrt{I} = \mathcal{I}_{\mathbb{A}^{n+1}}(Z_{\mathbb{A}^{n+1}}(I)) = \mathcal{I}_{\mathbb{P}^n}(Z_{\mathbb{P}^n}(I))$$

HN

$\supseteq$: f homogen $f(P)=0 \ \forall P \in Z_{\mathbb{P}^n}(I)$

$\Rightarrow f(\underset{\sim}{a})=0 \ \forall \ \underset{\sim}{a} \in Z_{\mathbb{A}^{n+1}}(I)$

$\Rightarrow f \in \mathcal{I}_{\mathbb{A}^{n+1}}(Z_{\mathbb{A}^{n+1}}(I))$

$\subseteq$: f homogen & $f \in \sqrt{I} \Rightarrow$

$f^m \in I$ for some m

$\Rightarrow f^m(P)=0 \ \forall \ P \in Z_{\mathbb{P}^n}(I)$

$\Rightarrow f(P)=0 \ \forall P \in Z_{\mathbb{P}^n}(I)$

$\Rightarrow f \in \mathcal{I}_{\mathbb{P}^n}(Z_{\mathbb{P}^n}(I))$

But $\sqrt{I}$ is homogen. Hence

$$\sqrt{I} \subseteq \mathcal{I}_{\mathbb{P}^n}(Z_{\mathbb{P}^n}(I))$$

(2) Exc

Remark: $X \subseteq \mathbb{P}^n$ alg subset $\quad CX \subseteq \mathbb{A}^{n+1}$ alg subset

$$\mathcal{I}_{\mathbb{A}^{n+1}}(CX) = \mathcal{I}_{\mathbb{P}^n}(X)$$

$Y \subseteq \mathbb{A}^{n+1}$ is a cone then

$$Y = Z\Big(\bigcap_{P \in Y \setminus \{\underset{\sim}{0}\}/\sim} \mathcal{I}_{\mathbb{P}^n}(P)\Big) = Z\Big(\bigcap_{\substack{CP \\ P\in Y}} \mathcal{I}_{\mathbb{A}^{n+1}}(CP)\Big)$$

$\mathcal{I}(Y)$ is homogen ideal.

Hilbert Null: Let k be an alg closed field; $m \subseteq k[x_0,..,x_n]$ a homogen ideal.

m is homogen maximal iff $\exists P \in \mathbb{P}^n$ s.t. $m = \mathcal{I}_{\mathbb{P}^n}(P)$.

Pf: ($\Leftarrow$): $P = [a_0, a_1, .., a_n]$ then $\mathcal{I}(P) = (a_0 X_1 - a_1 X_0, .., a_0 X_n - a_n X_0)$

$\mathcal{I}(P)$ is a prime ideal $k[x_0,..,X_n]$

$Z_{\mathbb{A}^{n+1}}(\mathcal{I}(P))$ = line joining $(a_0,..,a_n)$ & $\underset{\sim}{0}$ ($= l$ say)

Let $J \supsetneq \mathcal{I}(P)$ & J homogen

$Z_{\mathbb{A}^{n+1}}(J) \subsetneq l$ & $Z_{\mathbb{A}^{n+1}}(J)$ is a cone.

$$Z_{\mathbb{A}^{n+1}}(J) = \{\underset{\sim}{0}\}$$

$$\Rightarrow \sqrt{J} = \mathcal{I}(Z_{\mathbb{A}^{n+1}}(J)) = (X_0,..,X_n).$$

Hence $\mathcal{I}(P)$ homogen maximal.

($\Rightarrow$): Let m be homogen maximal in $k[x_0,..,X_n]$.

$m \subset (X_0,..,X_n)$ & $\sqrt{m} \subsetneq (X_0,..,X_n)$

By HN $Z_{\mathbb{A}^{n+1}}(m) \supsetneq \{\underset{\sim}{0}\}$ ("=" $\overset{HN}{\Rightarrow} \sqrt{m} = (X_0,..,X_n)$)

Let $\underset{\sim}{a} \in Z_{\mathbb{A}^{n+1}}(m)$ $\underset{\sim}{a} \neq \underset{\sim}{0}$. Then

$[a_0,..,a_n] \in Z_{\mathbb{P}^n}(m) \Rightarrow \mathcal{I}_{\mathbb{P}^n}([a_0,..,a_n]) \supseteq m$

$$\Rightarrow m = \mathcal{I}_{\mathbb{P}^n}([a_0,..,a_n])$$

$\therefore$ m is homo maximal.