Schwarz's inequality:

$f, g \in \mathcal{R}[a,b]$ Then

$$\left|\int f\bar{g}\right|^2 \le \int |f|^2 \int |g|^2$$

Proof:

$$|f(x)g(y) - g(x)f(y)|^2$$

$$= |f(x)|^2 |g(y)|^2 - g(x)f(y)\overline{f(x)}\,\overline{g(y)}$$

$$- \overline{g(x)}\,\overline{f(y)} f(x) g(y)$$

$$+ |g(x)|^2 |f(y)|^2$$

$$0 \le \int |f(x)|^2 dx \int |g(y)|^2 - \int g(x)\overline{f(x)} \int \overline{g(y)} f(y)$$

$$- \int \overline{g(x)} f(x) \int g(y)\overline{f(y)}$$

$$+ \int |g(x)|^2 \int |f(y)|^2$$

$$= 2\int |f(x)|^2 \int |g(x)|^2 - 2\left|\int f(x)\overline{g(x)}\right|^2$$

$$\Rightarrow \left|\int f\bar{g}\right|^2 \le \int |f|^2 \int |g|^2 .$$

Notation: $\|f\|_2 = \left\{\int_a^b |f|^2\right\}^{1/2}$

— L^2 — norm

1) $\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$

2) $\|f-g\|_2 \leq \|f-h\|_2 + \|h-g\|_2$

Proposition: $f \in Q[-\pi, \pi]$ and $\varepsilon > 0$

Then $g \in C[-\pi, \pi]$ such that

$$\|f-g\|_2 < \varepsilon.$$

Further if $f(-\pi) = f(\pi)$, then

g may chosen so that

$$g(-\pi) = g(\pi).$$

Parseval's Theorem: Let f, g be in $R[-\pi, \pi]$ and 2π-periodic.

$f \sim \sum c_n e^{inx}$, $g \sim \sum \gamma_n e^{inx}$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} f(x)\,dx, \quad \gamma_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} g(x)\,dx$$

Then

1) $$\lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} |S_n(f) - f(x)|^2\,dx = 0$$

2) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} f\bar{g} = \sum_{n=-\infty}^{\infty} c_n \bar{\gamma}_n$$

3) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

Recall $$S_n(f)(x) = \sum_{k=-n}^{n} c_k e^{ikx}.$$

$$\|f\|_2^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx$$

<u>Proof:</u> Let $\varepsilon > 0$. Then

$h \in C([-\pi, \pi])$, $h(-\pi) = h(\pi)$

such that $\|f - h\|_2 < \frac{\varepsilon}{3}$

$\exists$ trig poly P of degree N_0

such that

$$\sup_{x \in [-\pi, \pi]} |h(x) - P(x)| < \frac{\varepsilon}{3}$$

$$\Rightarrow \|h - P\|_2 < \frac{\varepsilon}{3}.$$

$$P = \sum_{n=-N_0}^{N_0} d_n e^{inx}$$

$$\|h - S_N(h)\|_2 \le \|h - P\|_2$$

$\forall N \ge N_0$

$$\|S_N(f) - S_N(h)\|_2 \le \|S_N(f-h)\|_2 \quad \boxed{\int |S_N|^2 \le \int |f|^2}$$

$$\le \|f - h\|_2 \le \varepsilon/3$$

$$\|f - S_N(f)\|_2 \le \|f - h\|_2 + \|h - S_N(h)\|_2 + \|S_N(h) - S_N(f)\|_2$$

$< \varepsilon$ $\forall N \ge N_0$

$$\frac{1}{2\pi}\int S_N(f)\,\bar{g} = \sum_{k=-N}^{N} c_k \frac{1}{2\pi}\int e^{ikx}\,\bar{g}(x)\,dx$$

$$= \sum_{k=-N}^{N} c_k \overline{\gamma_k}$$

$$\left|\int f\bar{g} - \int S_N(f)\,\bar{g}\right|^2$$

$$\leq \int |f - S_N(f)|^2 \int |g|^2$$

$$\longrightarrow 0 \quad \text{as } N \longrightarrow \infty$$

$$\sum_{k=-n}^{n} c_k \overline{\gamma_k} \longrightarrow \frac{1}{2\pi}\int f\bar{g}$$

Take $g = f$, $\displaystyle\frac{1}{2\pi}\int |f|^2 = \sum_{k=-\infty}^{\infty} |c_k|^2$.

<u>Step - function</u>:

Let $[a, b]$ and

$a = x_0 < x_1 < x_2 < \cdots < x_n = b$

be a partition.

Let φ be a function on $[a, b]$.
Then φ is called a step-function
if φ is constant on $[x_{k-1}, x_k)$.

$$\varphi = \sum a_k \chi_{[x_{k-1}, x_k)}$$

$$\int_a^b \varphi = \sum a_k \Delta_k \qquad \Delta_k = x_k - x_{k-1}.$$

<u>Proposition</u>: If $f \in R[a, b]$ and $\varepsilon > 0$, then there exists a step-function φ on $[a, b]$ such that

$$\int |f - \varphi| < \varepsilon.$$

<u>Proof</u>: there exists a partition $a = x_0 < x_1 < \dots < x_n = b$ such that

$$\left| \sum_{k=1}^{n} m_k \Delta_k - \int f(t)\,dt \right| < \varepsilon$$

$$m_k = \inf_{[x_{k-1}, x_k)} f$$

$\varphi(x) = m_k$ on $[x_{k-1}, x_k)$

φ is a step function

$\int |f - \varphi|$

$$= \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} [f(t) - m_k] \, dt$$

$$= \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(t) - \sum_{k=1}^{n} m_k \Delta_k$$

$$= \int_a^b f - \sum_{k=1}^{n} m_k \Delta_k < \varepsilon.$$

Riemann-Lebesgue Lemma :

Theorem : Let $f \in R[a,b]$. Then for each real β, we have

$$\lim_{\alpha \to \infty} \int_a^b f(t) \sin(\alpha t + \beta)\, dt = 0.$$

Proof : $f = \varphi$ — a step function

$$\int_x^y \sin(\alpha t+\beta)\, dt = \frac{-\cos(\alpha y+\beta)+\cos(\alpha x+\beta)}{\alpha}$$

$$\longrightarrow 0 \quad \text{as } \alpha \to \infty$$

$$\int_a^b \varphi(t) \sin(\alpha t+\beta)\, dt \longrightarrow 0 \quad \text{as } \alpha \to \infty$$

Let $\varepsilon > 0$. Then $\exists$ step function φ such that

$$\int_a^b |f - \varphi| < \frac{\varepsilon}{2(b-a)}$$

$$\left|\int_a^b f(t)\sin(\alpha t + \beta)\,dt\right|$$

$$\leq \int_a^b |f(t) - \varphi(t)|\,|\sin(\alpha t+\beta)|\,dt + \left|\int_a^b \varphi(t)\sin(\alpha t + \beta)\,dt\right|$$

$$< \frac{\varepsilon \cancel{(b-a)}}{2\cancel{(b-a)}} + \frac{\varepsilon}{2} \quad \text{for all large } \alpha$$

$$< \varepsilon \quad \text{for all large } \alpha.$$

$$\therefore \int_a^b f(t)\sin(\alpha t+\beta)\,dt \to 0 \quad \text{as } \alpha \to \infty.$$

Remark : $\beta = 0$

$$\lim_{\alpha \to \infty} \int_{-\pi}^{\pi} f(t) \sin \alpha t \, dt$$

$$= 0$$

$$= \lim_{\alpha \to \infty} \int_{-\pi}^{\pi} f(t) \cos \alpha t \, dt$$

$\beta = \frac{\pi}{2}$