Heat Equation

$$\frac{\partial w}{\partial t} = a^2 \frac{\partial^2 w}{\partial x^2} \qquad 0 < x < \pi \qquad t > 0$$

$w(0, x) = g(x)$ initial temperature distribution between $0 < x < \pi$

Boundary condition $\quad w(t, 0) = w(t, \pi) = 0$

Method of separation of variables:

Assume a solution ~~$y(t,x)$~~ $= w(t, x) = u(x)\, v(t)$

$$\frac{\partial w}{\partial t} = u(x)\, v'(t)$$

$$\frac{\partial^2 w}{\partial x^2} \cancel{= \partial v(t)\, u'(x)}$$

$$\frac{\partial w}{\partial x} = v(t)\, u'(x)$$

$$\frac{\partial^2 w}{\partial x^2} = v(t)\, u''(x)$$

$$\frac{u''(x)}{u(x)} = \frac{v'(t)}{a^2 v(t)} = -\lambda^2$$

Note that $\dfrac{v'(t)}{a^2 v(t)} = -\lambda^2 \Rightarrow v(t) = c e^{-\lambda^2 a^2 t}$

$u''(x) + \lambda^2 u(x) = 0$

$\Rightarrow u(x) = A \sin \lambda x + B \cos \lambda x$

$w(t, 0) = 0 \quad \Rightarrow u(0) = 0 \quad \Rightarrow B = 0$

$w(t, x) = u(x)\, v(t) = A \sin \lambda x \,(c e^{-\lambda^2 a^2 t})$

$\cancel{=} A$

$w(t, \pi) = 0 \Rightarrow \lambda = n \in \mathbb{Z}$

$w(t, x) = A e^{-n^2 a^2 t} \sin nx.$

By linearity,

$w(t, x) = \sum_{n} A_n e^{-n^2 a^2 t} \sin nx$ is a sol^n of the heat equation

initial condition. $w(0, x) = \sum_{n=0} A_n \sin nx = g(x)$

$$A_n = \frac{2}{\pi} \int_0^{\pi} g(x) \sin nx \, dx$$

Ex: solve: $\partial_t w = a^2 \partial_x^2 w \qquad 0 < x < L, \ t > 0$

$w(0, x) = g(x)$

$w(t, 0) = w(t, L) = 0$

Ex: $\partial_t w = a^2 \partial_x^2 w$ $\quad 0 < x < L, \ t > 0$

$w(0, x) = g(x)$

$w(t, 0) = u_1, \quad w(t, L) = u_2$

Heat Eqⁿ on infinite rod:

$$\partial_t w(t,x) = a^2 \partial_x^2 w(t,x) \qquad 0 < x < \infty, \ t > 0$$

$$w(0,x) = g(x)$$

$$w(t,x) = g * p_{a^2 t}(x) = \int g(y)\, p_{a^2 t}(x-y)\, dy$$

$$p_{a^2 t}(x) = \frac{1}{\sqrt{2\pi a^2 t}}\, e^{-\frac{|x|^2}{2a^2 t}}$$

Separation of variable $w(t,x) = X(x)\,T(t)$

$$\frac{X''(x)}{X(x)} = \frac{1}{a^2}\frac{T'(t)}{T(t)} = -\lambda^2 \qquad \cancel{T(t) = ce^{-\lambda^2 a^2 t}}$$

$$\frac{T'(t)}{T(t)} = -\lambda^2 a^2 \qquad T(t) = c e^{-\lambda^2 a^2 t}$$

$$\frac{X''(x)}{X(x)} = -\lambda^2 \qquad X(x) = A \sin \lambda x + B \cos \lambda x$$

$$w(t,x) = e^{-\lambda^2 a^2 t}(A_\lambda \sin \lambda x + B_\lambda \cos \lambda x)$$

By linearity,
any finite linear combination is also a solution.

$$\Rightarrow w(t,x) = \int_0^\infty e^{-\lambda^2 a^2 t}(A_\lambda \sin \lambda x + B_\lambda \cos \lambda x)\, d\lambda$$

Assumption: $g(x) = \dfrac{1}{\pi}\displaystyle\int_0^\infty \int_{-\infty}^\infty g(y) \cos \lambda (x-y)\, dy\, d\lambda$

$$\cos \lambda (x-y) = \cos \lambda x \cos \lambda y + \sin \lambda x \sin \lambda y$$

$$RHS = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty g(y) \cos \lambda x \cos \lambda y\, dy + g(y) \sin \lambda x \sin \lambda y\, dy\, d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \left(\cos \lambda x \int_{-\infty}^\infty g(y) \cos \lambda y\, dy + \sin \lambda x \int_{-\infty}^\infty g(y) \sin \lambda y\, dy\right) \quad \text{— } (*)$$

Initial condⁿ: $\Rightarrow g(x) = w(0,x) = \displaystyle\int_0^\infty A_\lambda \sin \lambda x + B_\lambda \cos \lambda x\, d\lambda$

Comparing with (*), $A_\lambda = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^\infty g(y) \sin \lambda y\, dy \qquad B_\lambda = \dfrac{1}{\pi}\int_{-\infty}^\infty g(y) \cos \lambda y\, dy$

③

$$\Rightarrow w(t,x) = \frac{1}{\pi}\int_0^\infty e^{-\lambda^2 a^2 t}\left(\left(\int_{-\infty}^{\infty} g(y)\sin\lambda y\,dy\right)\sin\lambda x + \left(\int_{-\infty}^{\infty} g(y)\cos\lambda y\,dy\right)\cos\lambda x\right)d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty e^{-\lambda^2 a^2 t}\int_{-\infty}^{\infty} g(y)\,\cancel{\lambda(y-x)}\cos\lambda(y-x)\,dy\,d\lambda$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} g(y)\int_0^\infty e^{-\lambda^2 a^2 t}\cos\lambda(y-x)\,d\lambda\,dy$$

$$\Rightarrow \int_0^\infty e^{-\lambda^2 a^2 t}\cos\lambda(y-x)\,d\lambda\,\cancel{dy} = \frac{c}{\sqrt{t}}\,e^{-\frac{(x-y)^2}{4a^2 t}}$$