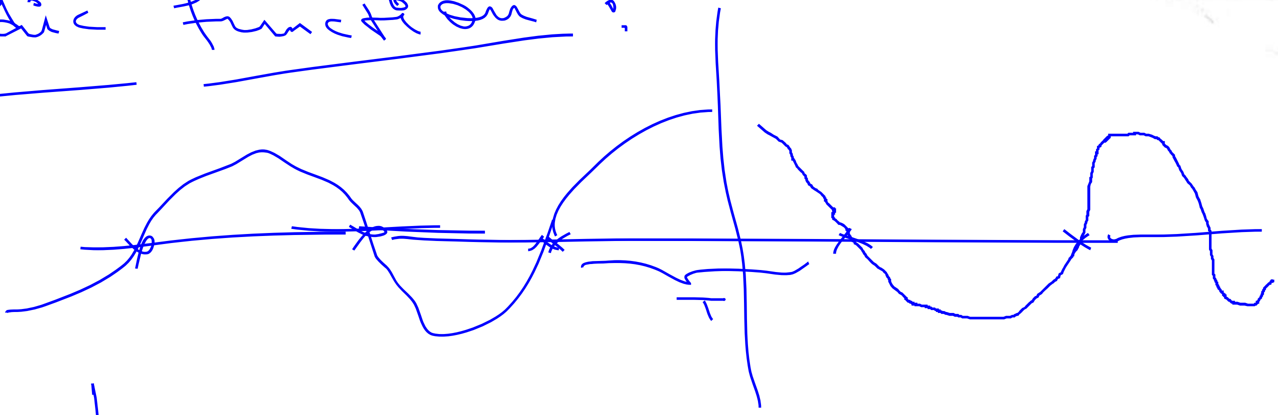
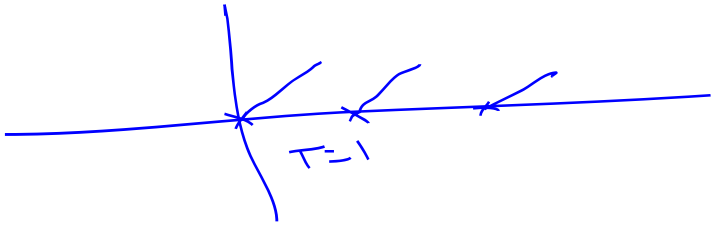
Periodic function:





A real-valued (or complex-valued) function f on $\mathbb{R}$ is called periodic if there exists $T > 0$ such that $f(x+T) = f(x)$ $\forall x \in \mathbb{R}$. In such case, T is called the period of f.

Let $P(f) = \{ t \in \mathbb{R} \,/\, f(x+t) = f(x) \text{ for all } x \in \mathbb{R} \}$

Ex $P(f)$ is a additive group.

Ex If f is continuous on $\mathbb{R}$, then $P(f)$ is a closed set and $P(f) = \mathbb{Z}T$ or f is constant.
$\{nT \mid n \in \mathbb{Z}\}$
for some $T > 0$.

If f is a continuous periodic function, then f has a maximum and a minimum

Eg (1) $\sin x$, $\cos x$ are periodic functions of period 2π

(2) $e^{ix} = \cos x + i \sin x$ is a periodic function of period 2π.

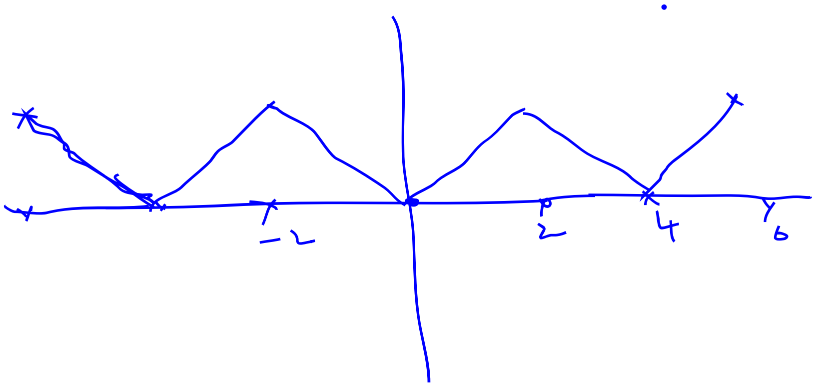
Suppose f is a periodic function and $L>0$ is called the least period of f if. L is a period of f. and no $0<T<L$ is a period of f.

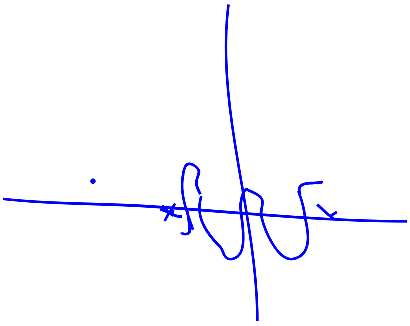
Ex If f is a non-constant continuous periodic function, then f has the least periodic.

Eg $f(x) = |x|$ if $|x| \leq 2$

f on $\mathbb{R}$ is defined by

$f(x+4) = f(x) = f(x-4)$





Let $P_L(\mathbb{R})$ denote the set of all periodic functions of period L.

$\cos \frac{2n\pi}{L}x$, $\sin \frac{2n\pi}{L}x \in P_L(\mathbb{R})$

$$a_0 + \sum_{k=1}^{n} a_k \sin \frac{2k\pi}{L}x + b_k \cos \frac{2k\pi}{L}x$$

$\in P_L(\mathbb{R})$.

Main Theorem:

Prove that for well-behaved functions,

$$f(x) = a_0 + \sum_{m=1}^{\infty} \left(a_m \cos \frac{2m\pi}{L}x + b_m \sin \frac{2m\pi}{L}x\right]$$

The infinite sum is called Fourier series or Fourier expansion.

If f is a piecewise C^1-function in $P_L(\mathbb{R})$. Then there are constants

a_0, a_m, and b_m such that at each point of continuity of f f has Fourier expansion.

if f has discontinuity at $y \in \mathbb{R}$, then series converges to $\frac{1}{2}[f(y_-)+f(y_+)]$

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

$$a_m = \frac{2}{L}\int_0^L f(x) \cos \frac{2m\pi}{L} x \,dx$$

$$b_m = \frac{2}{L}\int_0^L f(x) \sin \frac{2m\pi}{L} x \,dx .$$

$$\int_0^L \cos\left(\frac{2m\pi}{L}\right)x \cos \frac{2n\pi}{L} x \,dx = \begin{cases} 0 & \text{if } n \neq m \\ L/2 & \text{if } n = m \neq 0 \\ L & \text{if } n = 0 = m \end{cases}$$

$$\int_0^L \cos \frac{2m\pi}{L} x \sin \frac{2n\pi}{L} x = 0 \quad \forall\, n, m$$

$$\int_0^L \sin \frac{2m\pi}{L} x \sin \frac{2m\pi}{L} x \,dx = \begin{cases} 0 & \text{if } n \neq m \text{ or } m = n = 0 \\ \frac{L}{2} & \text{if } n = m \neq 0. \end{cases}$$

Let A be a collection of complex-valued functions on $[a,b]$.

We say that A consists of orthogonal functions or A is an orthogonal system if

$$\int_a^b \varphi(x)\overline{\psi(x)}\,dx = 0 \quad \text{if } \varphi \neq \psi.$$

In addition, if $\int_a^b |\varphi(x)|^2 = 1$

We say that A is a orthonormal system.

Eg 1) $\left\{ \frac{e^{inx}}{\sqrt{2\pi}} \mid n \in \mathbb{Z} \right\}$ on $[-\pi, \pi]$.

(2) $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin nx}{\sqrt{\pi}} \right\}$ is also a orthonormal system.

Let $\{\varphi_n\}$ be an orthonormal system.

Then $c_n = \int_a^b f(x)\overline{\varphi_n(x)}\,dx$.

nth Fourier coefficient of f.

In this case, we write $f \sim \sum c_n\varphi_n$

$\sum c_n\varphi_n$ is called the Fourier series of f w.r.t the orthonormal system

Theorem: Let $\{\varphi_n\}$ be an orthonormal system in $[a,b]$. Let $f \in R[a,b]$.

Assume $f \sim \sum c_n \varphi_n$. Then

Let $s_n = \sum_{k=1}^{n} c_k \varphi_k$.

$$\int_a^b |f - s_n|^2 dx \leq \int_a^b |f - t_n|^2 dx$$

for any $t_n = \sum_{k=1}^{n} \gamma_k \varphi_k$, γ_k are scalars.

Further the equality occurs if $\gamma_k = c_k$.

Proof: $\int f \bar{t}_n = \sum c_m \bar{\gamma}_m$

$\int \bar{f} t_n = \sum \bar{c}_m \gamma_m$

$\int |t_n|^2 = \sum |\gamma_m|^2$, $\sum |c_m|^2 = \int |s_n|^2$.

$$\int |f - t_n|^2 = \int |f|^2 - \int f \bar{t}_n - \int \bar{f} t_n + \int |t_n|^2$$

$$= \int |f|^2 - \sum c_m \bar{\gamma}_m - \sum \bar{c}_m \gamma_m + \sum \gamma_m \bar{\gamma}_m$$

$$= \int |f|^2 + \sum |c_m - \gamma_m|^2 - \sum |c_m|^2.$$

$$\int |f - s_n|^2 = \int |f|^2 - \sum |c_m|^2$$

$$\int |f - t_n|^2 = \int |f - s_n|^2 + \sum |c_m - \gamma_m|^2$$

$$\geq \int |f - s_n|^2$$

and the equality occurs iff $\sum |c_m - \gamma_m|^2 = 0$ ie, $c_m = \gamma_m$.

Corollary $\sum_{n=1}^{\infty} |c_n|^2 \leq \int |f|^2$

and $c_n \longrightarrow 0$ as $n \to \infty$.

Proof: (Ex) $\int |s_n|^2 \leq \int |f|^2 \quad \forall n$

$$\sum_{m=1}^{n} |c_m|^2 \leq \int |f|^2 \quad \forall n$$

$$\sum_{m=1}^{\infty} |c_m|^2 \leq \int |f|^2$$

$c_m \longrightarrow 0$ as $n \to \infty$.
