Physics 4

Lecture 18-19

Relativistic Dynamics

$$u^\mu = \frac{dx^\mu}{d\tau} = (c\gamma, \vec{v}\gamma) .$$

$$\Big\} \quad \frac{dt}{d\tau} = \gamma .$$

$$a^\mu = \frac{du^\mu}{d\tau} = \frac{d}{d\tau}(c\gamma, \vec{v}\gamma) .$$

$$= \gamma\left(c\frac{d\gamma}{dt}, \gamma\frac{d\vec{v}}{dt} + \vec{v}\frac{d\gamma}{dt}\right) .$$

$$\frac{d\gamma}{dt} = \frac{d}{dt}\frac{1}{\sqrt{1-v^2/c^2}} = \gamma^3 \frac{v\dot{v}}{c^2} .$$

$$a^\mu = \left(\gamma^4 \frac{v\dot{v}}{c}, \gamma^4 \frac{v\dot{v}}{c^2}\vec{v} + \gamma^2\vec{a}\right) .$$

Generalization of Newton's Law

$$F^\mu = \frac{dP^\mu}{d\tau} = m\frac{du^\mu}{d\tau}$$

where we assume mass does not vary.

Recall $P^\mu = mu^\mu$

$$= (mu^0, m\gamma\vec{v}).$$

$$= (m\gamma c, m\gamma\vec{v}).$$

$$= \left(\frac{E}{c}, m\gamma\vec{v}\right)$$

$$= \left(\frac{E}{c}, \vec{p}\right).$$

$$\boxed{\begin{aligned} F &= \gamma\left(\frac{1}{c}\frac{dE}{dt}, \frac{d\vec{p}}{dt}\right) \\ &= \gamma\left(\frac{1}{c}\frac{dE}{dt}, \vec{f}\right) \end{aligned}}$$

$$F^{\mu} = \frac{dp^{\mu}}{d\tau} = m\frac{du^{\mu}}{d\tau} = ma^{\mu}.$$

$$F = m\left(\gamma^4 \frac{v\dot{v}}{c^2}, \gamma^4 \frac{v\dot{v}}{c^2}\vec{v} + \gamma^2\vec{a}\right).$$

In the special case of $\vec{v}$ pointing in the x-direction

$$\vec{v} = (v_x, 0, 0), \quad v = v_x, \quad \dot{v} = \dot{v}_x = a_x.$$

$$F = m\left(\gamma^4 \frac{v_x a_x}{c^2}, \gamma^4 a_x, \gamma^2 a_y, \gamma^2 a_z\right).$$

must reduce to 0, when $a^{\mu} = 0$.

must reduce to $\vec{F} = m\vec{a}$ for $v/c \ll 1$

$F^\mu = m a^\mu \longrightarrow$ 4 equations.

but all are not independent.

$u^\mu u_\mu = c^2$.

$\hookrightarrow a^\mu u_\mu = 0$.

$\hookrightarrow \boxed{F^\mu u_\mu = 0}$. $\rightarrow$ constraint.

3 independent equations

$$F = \gamma\left(\frac{1}{c}\frac{dE}{dt}, \vec{f}\right).$$

We can write the zeroth component in a slightly different form.

we saw

$$F^\mu P_\mu = 0.$$

$$F^0 P_0 - \vec{F}\cdot\vec{P} = 0. \qquad [\ \vec{P} = \gamma m \vec{v}\ ].$$

$$\frac{\gamma}{c}\frac{dE}{dt}\cdot\gamma m c - \gamma^2 m \vec{f}\cdot\vec{v} = 0$$

$$\frac{dE}{dt} = \vec{f}\cdot\vec{v}.$$

$$\boxed{F = \left(\frac{\gamma}{c}\vec{f}\cdot\vec{v}, \gamma\vec{f}\right)}$$

Transformations for momentum & force

case where S' moves along x-axis with const vel.

$$p^{0'} = \gamma(p^0 - \beta p^1)$$

$$p^{1'} = \gamma(p^1 - \beta p^0).$$

$$p^{2'} = p^2$$

$$p^{3'} = p^3.$$

$$\frac{E'}{c} = \gamma\left(\frac{E}{c} - \beta p_x\right).$$

$$p_x' = \gamma\left(p_x - \beta \frac{E}{c}\right)$$

Analyzing relativistic collision and decays

strategy is the same as analyzing non-relativistic collisions, i.e conserving energy & momentum except that now we use relativistic definitions of these quantities and work with 4-vectors.

$$P_{before} = P_{after}.$$

We will also make heavy use of the norm invariance

$$\boxed{P_\mu P^\mu = m^2c^2} \qquad P_0 P^0 - \vec{p}^{\,2} = m^2c^2$$ (calculate in rest frame).

$$P = (\gamma mc, \gamma m v_x, \gamma m v_y, \gamma m v_z)$$

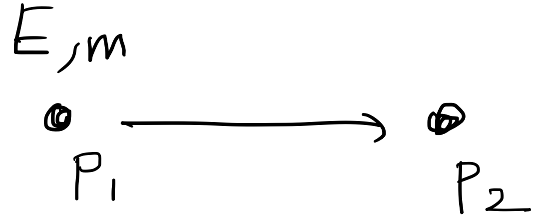
Ex 1 Relativistic Billiards.

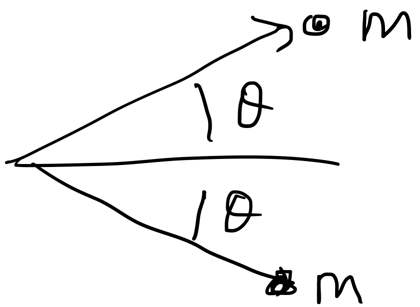
Before

After

Particle with mass m, energy E approaches identical particle at rest, after an elastic collision both scatter at angle θ relative to incident direction.

find θ in terms of E, m, Look at extreme relativistic and non rel. limit.





First step: write down before/after 4 momenta.

Before

$$P_1 \equiv \left(\frac{E}{c}, p, 0, 0\right), \quad P_2 \equiv (mc, 0, 0, 0).$$

where $E^2 = p^2c^2 + m^2c^4$

After

$$P_1' = \left(\frac{E'}{c}, p'\cos\theta, p'\sin\theta, 0\right)$$

$$P_2' = \left(\frac{E'}{c}, p'\cos\theta, -p'\sin\theta, 0\right)$$

Conservation of energy.

$$\frac{2E'}{c} = \frac{E+mc^2}{c}.$$

$$\boxed{E' = \frac{E+mc^2}{2}}$$

Conservation of momentum.

$$2p'\cos\theta = p.$$

$$\boxed{p'\cos\theta = \frac{p}{2}}$$

$$P_\mu P^\mu = m^2c^2.$$

$$P_0 P^0 - \vec{P}^2$$

$$\frac{E^2}{c^2} - \vec{p}^{\,2} = m^2c^2$$

$$\boxed{E^2 = \vec{p}^{\,2}c^2 + m^2c^4}$$

$$P'_{1,2} = \left(\frac{E + mc^2}{2c} , \frac{p}{2} , \pm \frac{p}{2} \tan\theta , 0 \right) \text{ --- } (*)$$

$$P'^{\mu}_{1,2} \, P'_{1,2\mu} = m^2 c^2 \text{ --- } (*)(*) \quad \longrightarrow \quad P^0 P_0 + P^1 P_1 + P_2 P^2 + P_3 P^3$$

$$= p_0^2 - p_1^2 - p_2^2 - p_3^2 .$$

Using $(*)$ in $(**)$

$$m^2 c^2 = \left(\frac{E + mc^2}{2c} \right)^2 - \left(\frac{p}{2} \right)^2 \left(1 + \tan^2\theta \right)$$

$$m^2 c^2 = \frac{(E + mc^2)^2}{4c^2} - \frac{p^2}{4\cos^2\theta} .$$

$$\boxed{\cos^2\theta = \frac{E^2 - m^2 c^4}{E^2 + 2mEc^2 - 3m^2 c^4} = \frac{E + mc^2}{E + 3mc^2}}$$

$$\cos^2\theta = \frac{E + mc^2}{E + 3mc^2} .$$

Rel. limit $E \gg mc^2$

$\cos^2\theta \simeq 1$, both particles are scattered almost directly forward.

Non rel. limit

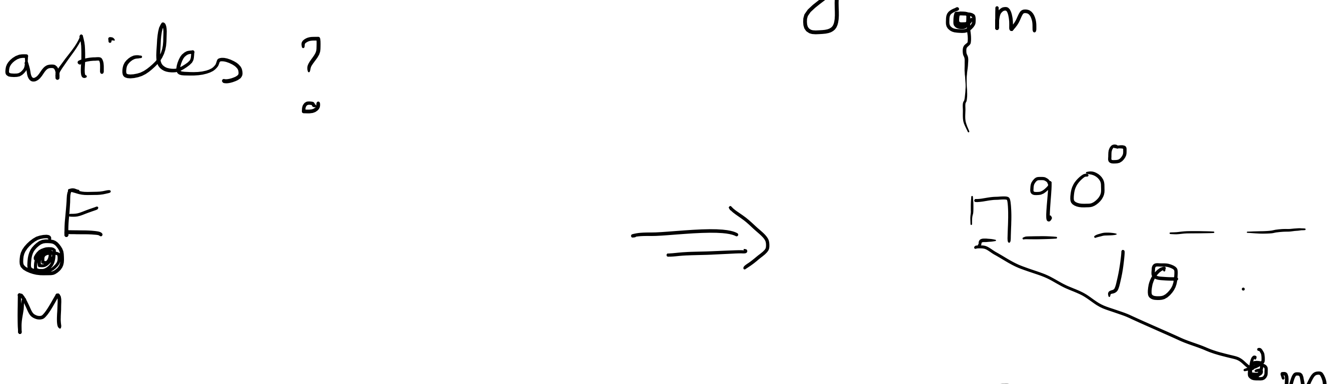
$E \approx mc^2$.

$\cos^2\theta = \frac{1}{2}$, $\cos\theta = \frac{1}{\sqrt{2}}$, $\theta = 45^\circ$.

particles scatter with a 90° angle between them

Decay at an angle

A particle with mass M and energy E decays into two identical particles. In the lab frame, one of them is emitted at a $90°$ angle. What are the energies of the particles?



Before $P = \left(\frac{E}{c}, p, 0, 0\right)$.

After

$$P_1 = \left(\frac{E_1}{c}, 0, p_1, 0\right), \quad P_2 = \left(\frac{E_2}{c}, p_2\cos\theta, -p_2\sin\theta, 0\right)$$

$$\boxed{P = P_1 + P_2} \longrightarrow \text{Cons. of 4-momentum.}$$

$$P = P_1 + P_2 .$$

$$\left[\begin{array}{l} P_1 = \left(\frac{E_1}{c}, 0, p, 0\right) \\ P_2 = \left(\frac{E_2}{c}, p_1\cos\theta, -p_1\sin\theta, 0\right) \end{array}\right.$$

$$(P - P_1) = P_2 .$$

$$(P-P_1)^\mu (P-P_1)_\mu = P_2^\mu P_{2\mu} . \qquad P = \left(\frac{E}{c}, p, 0, 0\right).$$

$$P^\mu P_\mu - 2 P_1^\mu P_\mu + P_1^\mu P_{1\mu} = P_2^\mu P_{2\mu} .$$

$$M^2c^2 - 2\left(\frac{E_1 E}{c^2} - 0\right) + \cancel{m^2c^2} = \cancel{m^2c^2} .$$

$$M^2c^2 - \frac{2E_1E}{c^2} = 0$$

$$\boxed{E_1 = \frac{M^2c^4}{2E}}$$

$$\boxed{E_2 = E - E_1}$$

Compton Scattering.

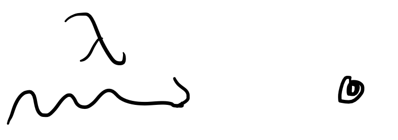
photon collides with a bound (stationary) electron.

the photon scatters at angle θ.

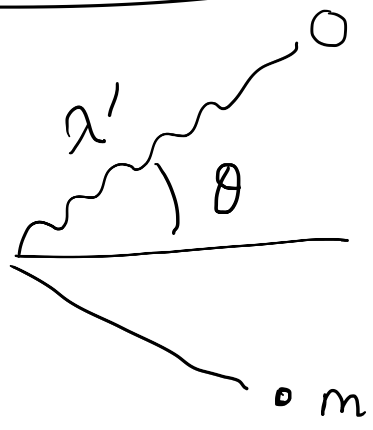
Show that the resulting wavelength λ' is given in terms of the original wavelength λ

$$\boxed{\lambda' = \lambda + \frac{h}{mc}(1-\cos\theta)}$$

$$\boxed{\begin{aligned} E &= h\nu = \frac{hc}{\lambda}. \\ p &= \frac{E}{c} \end{aligned}}$$



before



$$P_\gamma = \left(\frac{E}{c}, p, 0, 0\right). \qquad p = \frac{E}{c}.$$

$$= \left(\frac{h}{\lambda}, \frac{h}{\lambda}, 0, 0\right).$$

$$P_m = (mc, 0, 0, 0).$$

$$\left.\begin{array}{l} P_\gamma \\ P_m \end{array}\right\} \text{before}.$$

After.

$$P'_\gamma = \left(\frac{h}{\lambda'}, \frac{h}{\lambda'}\cos\theta, \frac{h}{\lambda'}\sin\theta, 0\right), \quad P'_m = (\cdots\cdots).$$

Conservation eqn.

$$P'_m + P'_\gamma = P_\gamma + P_m.$$

$$P'^2_m = p'^\mu_m P_{m\mu}.$$

$$P'^2_m = (P_\gamma + P_m - P'_\gamma)^2.$$

$$P_m'^2 = (P_r + P_m - P_r')^2 .$$

$$P_r^2 = P_r'^2 = 0$$

$$m^2c^2 = P_r^2 + P_m^2 + P_r'^2 + 2P_r \cdot P_m$$

$$- 2P_r P_r' \cancel{+} - 2P_m P_r' .$$

$$P_r = \left(\frac{h}{\lambda}, \frac{h}{\lambda}, 0, 0\right)$$

$$P_m = (mc, 0, 0, 0)$$

$$P_r' = \left(\frac{h}{\lambda'}, \frac{h}{\lambda'}\cos\theta, \frac{h}{\lambda'}\sin\theta, 0\right)$$

$$= 0 + m^2c^2 + 0 + 2P_m(P_r - P_r')$$

$$- 2P_r P_r' .$$

$$\cancel{m^2c^2} = \cancel{m^2c^2} + 2mc\left(\frac{h}{\lambda} - \frac{h}{\lambda'}\right)$$

$$- 2\frac{h}{\lambda}\frac{h}{\lambda'}(1 - \cos\theta) .$$

$$2mc\left(\frac{h}{\lambda}-\frac{h}{\lambda'}\right) = 2\frac{h}{\lambda}\frac{h}{\lambda'}(1-\cos\theta)$$

$$\boxed{\lambda' = \lambda + \frac{h}{mc}(1-\cos\theta)}$$

In the limit

$h \longrightarrow 0$

$\lambda = \lambda'$