DIFFERENTIAL TOPOLOGY - LECTURE 4 - II

1. Introduction

In the previous set of notes we proved the local immersion theorem. The local immersion theorem states that if $f : X \longrightarrow Y$ is an immersion at $x \in X$, then there are parametrizations $\varphi : U \longrightarrow X$ about x and $\psi : V \longrightarrow Y$ about $f(x)$ so that

$$\psi^{-1} \circ f \circ \varphi = i$$

where i is the canonical immersion. We remind ourselves that the composition on the left is called a local representation of the function f.

We also noted one consequence of the local immersion theorem, namely, if $f : X \longrightarrow Y$ is an immersion at x, then f is an immersion in a neighborhood of x. We shall see several more applications of the local immersion theorem.

2. Local submersion theorem

The local immersion theorem gives us information about function in a neighborhood of a point at which it is an immersion. Having understood this, we turn our attention to understanding the behaviour of a function in a neighborhood of a point at which it is a submersion. This is described by the theorem below. The proof is similar to the proof of the local immersion theorem and we omit the proof.

Theorem 2.1. (Local submersion theorem) Let X, Y be manifolds and suppose that $f : X \longrightarrow Y$ is a submersion at $x \in X$. Then there exists a local representation of f that equals the canonical projection. In other words, there exist local parametrizations about x and $f(x)$ such that $f(x_1, \dots, x_k) = (x_1, \dots x_\ell)$, the projection to the first ℓ coordinates. Here $\dim(X) = k$ and $\dim(Y) = \ell$. □

The proof is similar the proof of the Local immersion theorem. We start with arbitrary parametrizations

$$\varphi : U \longrightarrow X; \quad \psi : V \longrightarrow Y$$

about x and $f(x)$ and this time keep modifying just φ so that finally the local representation $\psi^{-1} \circ f \circ \varphi$ equals the canonical projection. That ψ does not need to be modified and ψ can be chosen arbitrarily can be helpful in some situations. For example, if $f : X \longrightarrow \mathbb{R}$ is a submersion at x, then as a parametrization about $f(x)$ we have the liberty to choose $\psi = id : \mathbb{R} \longrightarrow \mathbb{R}$.

We now record some important consequences of the Local submersion theorem. The Local submersion theorem says that if $f : X \longrightarrow Y$ is a submersion at $x \in X$, then there are parametrizations

$$\varphi : \longrightarrow X; \quad \psi : V \longrightarrow Y$$

about x and $f(x)$ respectively such that

$$\psi^{-1} \circ f \circ \varphi = j$$

where j is the canonical submersion. Thus the expression

$$f = \psi \circ j \circ \varphi^{-1} \tag{2.1.1}$$

holds in a neighborhood of x. Taking derivatives we have that the equality

$$df = d\psi \circ dj \circ d\psi^{-1}$$

holds at all points in a neighborhood of x. Since every map on the right hand side is surjective we see that f is a submersion in a neighborhood of x. We have thus proved the following.

Corollary 2.2. Suppose that $f : X \longrightarrow Y$ is a submersion at $x \in X$. Then f is a submersion in a neighborhood of x. □

In Equation 2.12.1, all the maps on the right hand side are open maps. Thus we have the following important observation about submersions.

Corollary 2.3. Suppose that $f : X \longrightarrow Y$ is a submersion. Then f is an open map. □

Here is another important consequence.

Corollary 2.4. Let X be compact and $f : X \longrightarrow \mathbb{R}$ a function. Then $df_p = 0$ for some $p \in X$.

Proof. Since X is compact, there exists $p \in X$ such that $f(x) \leq f(p)$ for all $x \in X$. We claim $df_p = 0$. Assume, if possible, that $df_p \neq 0$. This implies that f is a submersion at p. By the local submersion theorem, there is a local representation $\psi^{-1} \circ f \circ \varphi$ so that

$$\psi^{-1} \circ f \circ \varphi = j$$

the canonical submersion which is now the projection to the first factor. Since we may assume $\psi = \mathrm{id}$ (see the paragraph after the statement of the local submersion theorem) we have that

$$f \circ \varphi = j$$

the canonical immersion. Since f has a maximum at p, the composition $f \circ \varphi$ has a maximum at 0 (since $\varphi(0) = p$). But $f \circ \varphi$ is the projection to the first factor defined on an open set and as such cannot have a maximum. This contradiction forces $df_p = 0$. □

Thus a non-constant smooth function $f : X \longrightarrow \mathbb{R}$ on a compact manifold X has at least two points at which the derivative vanishes.

Suppose that X is a compact manifold and Y a non-compact connected manifold. We claim that there is no submersion $f : X \longrightarrow Y$. For if such a submersion exists, then $f(X)$ would be both open and closed in Y. Since Y is connected, this would force $f(X) = Y$ and consequently Y would be compact, a contradiction. Thus we have the following.

Corollary 2.5. If X is a compact manifold, then X cannot be submersed in $\mathbb{R}^N$. □

On the other hand it is a theorem (and we shall prove a version of this later) that every manifold can be immersed in some $\mathbb{R}^N$. It is of great interest (and an active area of research) to know, given a manifold X, what is the smallest n for which there exists an immersion (not necessarily one-one)

$$f : X \longrightarrow \mathbb{R}^n.$$

A tremendous amount of literature[1] exists about individual spaces and the smallest dimension in which they immerse.

[1] See https://www.lehigh.edu/ dmd1/imms.html for details of immersion and embeddings of projective spaces.

A very intersting question about immersions is the following. Given a positive integer n, is there a N isuch that every n-manifold immerses in $\mathbb{R}^N$. A complete answer is known. Given n, let $\alpha(n)$ denote the number of 1's appearing in the binary expansion of n. The Immersion conjecture (which is now a theorem) states that every n-manifold immerses in $\mathbb{R}^{2n-\alpha(n)}$. This was proved by Ralph Cohen[2] in 1985. On the other hand it is a theorem of William Massey[3] that for every n there exists a n-manifold X that does not immerse in $\mathbb{R}^{2n-\alpha(n)-1}$.

Here are some problems. Most are from G and P.

Exercise 2.6. Let Z be an ℓ-dimensional submanifold of X and let $x \in X$. Show that there exists a local coordinate system $\{x_1, \ldots, x_k\}$ defined in a neighborhood of $x \in X$ such that $Z \cap U$ is defined by the equations $x_{\ell+1} = 0, \ldots, x_k = 0$.

Exercise 2.7. Prove that a local diffeomorphism f is actually a diffeomorphism onto an open subset provided that f is one-one.

Exercise 2.8. Show that there does not exist an immersion $f : \mathbb{S}^n \longrightarrow \mathbb{R}^n$.

Exercise 2.9. Suppose f, g are immersions, then show that so is $f \times g$. Show that composition of two immersions is an immersion. Finally show that the restriction of an immersion to a submanifold is an immersion.

Exercise 2.10. Let $x_1, \ldots, x_N$ be the standard coordinate function on $\mathbb{R}^N$ and let X be a k-dimensional submanifold of $\mathbb{R}^N$. Show that the every point $x \in X$ has a neighborhood (in X) on which the restriction of some k coordinate functions

$$x_{i_1}, \ldots, x_{i_k}$$

form a coordinate system.

Exercise 2.11. In continuation of the previous exercise, assume that $x_1, \ldots, x_k$ form a local coordinate system in a neighborhood V of $x \in X$. Prove that there are smooth functions

$$g_{k+1}, \ldots, g_N$$

on an open set U in $\mathbb{R}^k$ such that V may be be taken to be the set

$$\{(a_1, \ldots, a_k, g_{k+1}(a), \ldots, g_N(a_k)) \,:\, a = (a_1, \ldots, a_k) \in U\}.$$

Thus if we define $g : U \longrightarrow \mathbb{R}^{N-k}$ by $(g = (g_{k+1}, \ldots, g_N)$, then V is the graph of g. Thus every manifold is locally expressble as the graph of a function.

Exercise 2.12. (Generalization of IFT) Let $f : X \longrightarrow Y$ be a smooth map that is one-one on a compact submanifold Z of X. Assume that for each $x \in Z$,

$$df_x : T_x(X) \longrightarrow T_{f(x)}(Y)$$

is an isomorphism. Prove that f maps Z diffeomorphically onto $f(Z)$. Moreover show that f maps an open set containing Z diffeomorphically onto an open set containing $f(Z)$. Observe that this reduces to the Inverse function theorem when Z is a point.

Exercise 2.13. At what points is the map $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by $f(x, y, z) = (x^2 - y^2, y^2 - z^2)$ a submersion?

Exercise 2.14. Show that if $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a submersion, then f cannot be one-one.

[2]The immersion conjecture for differentiable manifolds, *Annals of Maths.*, 122 (2) 237-328.

[3]On the Stiefel-Whitney classes of a manifold, *American J. Math.*, 82 (1) 92-102.

Exercise 2.15. Construct a smooth surjective map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that is an immersion but not a diffeomorphism.

Exercise 2.16. Let $n > 1$. Suppose that X is a compact connected n-manifold. Let $f : X \longrightarrow \mathbb{S}^n$ be an immersion. Show that f is a diffeomorphism.

Exercise 2.17. Construct an immersion $f : \mathbb{R} \longrightarrow \mathbb{S}^2$.

Exercise 2.18. Does there exist an immersion $S^1 \times S^1 \longrightarrow S^2$? Does there exist an immersion $S^1 \times S^1 \longrightarrow S^3$?

Exercise 2.19. Complete the proof of the final claim in Example 2.12.

Exercise 2.20. For the function $p : \mathbb{R} \longrightarrow \mathbb{R}^2$, $p(t) = (\cos t, \sin t)$, find a local representation about 0 that equals the canonical immersion.

Exercise 2.21. For the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$f(x, y) = \sqrt{x^2 + 2xy + 2y^2}$$

find a local representation about $(1, 0)$ so that the local representation equals the function $(u, v) \mapsto u^2 + v^2$.

Exercise 2.22. Suppose that $g : \mathbb{R} \longrightarrow \mathbb{R}$ is smooth. Determine if the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $f(x, y) = g(x^2 + y^2)$ is a submersion.