Lecture 11.

Power Series Solutions of Differential Equations.

The solutions of differential equations are in general not given by the elementary 'transendental' functions like $\sin x$, $\cos x$ or e^x. We will now develop techniques to solve ODE's by means of power series and in turn leading to higher transendental functions (or special functions).

Consider first a simple example.

Example.

$$y' = y \quad \text{—} \quad (1)$$

We know that $y = e^x$ solves this equation. We also know that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

where the series converges for all x.

Consider now a power series of the form

(2) $\quad y(x) = a_0 + a_1 x + \dots + a_n x^n + \dots$

and assume that it converges for $|x| < R$, $R > 0$. Suppose that $y(x)$ satisfies $y'(x) = y(x)$. We know that a power series is diffentiable and the derivative is given by term wise differentiation of the series:

(3) $\quad y'(x) = a_1 + 2a_2 x + \dots + n a_n x^{n-1} + \dots$

The equation (1) viz $y' = y$ now leads us to the equality of the RHSs in eqns. (2) and (3): Thus

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

We can rewrite this as

$$\sum_{n=0}^{\infty} (a_n - (n+1) a_{n+1}) x^n = 0$$

If a power series vanishes for all

x in an interval, then the coefficients must be zero. Since the above power series is zero in $|x| < R$ we should have $a_n - (n+1)\, a_{n+1} = 0 \quad \forall\, n \geq 0$.

Thus $a_{n+1} = \dfrac{a_n}{n+1} = \dfrac{a_{n-1}}{n\,(n+1)}$

$$\cdots = \frac{a_0}{1 \cdot 2 \cdots (n+1)} = \frac{a_0}{(n+1)!}.$$

Thus $y(x) = \displaystyle\sum_{n=0}^{\infty} a_0 \frac{x^n}{n!}$

$$= a_0\, e^{x}$$

where a_0 is any real number, to be determined by an initial condition.

<u>Example.</u> Consider the equation

$$(1+x)\,y' = p\,y, \quad y(0) = 1 \quad \text{--- (4)}$$

where p is any real number. We proceed as in the previous example

(4)

by assuming a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad , \quad |x| < R$$

We now try to find a recurrence relation for the a_n's.

$$y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$\Rightarrow \quad x y'(x) = \sum_{n=0}^{\infty} n a_n x^n$$

$$\Rightarrow \quad (1+x) y'(x) = \sum_{n=0}^{\infty} (n a_n + (n+1) a_{n+1}) x^n$$

The equation $(1+x) y' = py$ now implies

$$\sum_{n=0}^{\infty} (n a_n + (n+1) a_{n+1} - p a_n) x^n = 0$$

holds identically for $|x| < R$. This gives $n a_n + (n+1) a_{n+1} - p a_n = 0 \ \forall n$.

Hence we get the recurrence relation

$$a_{n+1} = \frac{(p-n) a_n}{n+1}$$

$$= \frac{(p-n)(p-n+1)\, a_{n-1}}{n(n+1)}$$

$$\cdots\cdots$$

$$= \frac{a_0\, p(p-1)\cdots(p-n)}{(n+1)!}$$

Thus $$y(x) = a_0 \sum_{n=1}^{\infty} \frac{p(p-1)\cdots(p-n+1)}{n!} x^n + a_0$$

for $|x| < R$. From the initial condition $y(0) = 1$ we get $a_0 = 1$. By the ratio test the series

$$y(x) = 1 + \sum_{n=1}^{\infty} \frac{p(p-1)\cdots(p-n+1)}{n!} x^n$$

converges for $|x| < 1$ and diverges for $|x| > 1$. We note that if $p = m \geq 1$ an integer then

$$y(x) = 1 + \sum_{n=1}^{m} \frac{m(m-1)\cdots(m-n+1)}{n!} x^n$$

$$= (1+x)^m$$

This leads us to define $(1+x)^p := 1 + \sum_{n=1}^{\infty} \frac{p(p-1)\cdots(p-n+1)}{n!} x^n$.