Theorem: Let f be a continuous 2π-periodic function. Then for $\varepsilon > 0$ there exists a periodic polynomial P

$P = \sum_{i=-N}^{N} \alpha_n e^{inx}$ such that

$$|P(x) - f(x)| < \varepsilon \quad \forall x \in \mathbb{R}.$$

Proof: By Lemma 2π-periodic, continuous function f can be identified with a function in $C(S^1)$.

$$A = \left\{ \sum_{n=-N}^{N} a_n z^n \in C(S^1) \Big/ a_n \in \mathbb{C} \right\}.$$

A is an algebra

Since $\bar{z} = z^{-1}$, $z \in S^1$, we have A is a self-adjoint algebra

A nowhere vanishes and A separates points of S^1.

$$\overline{A} = C(S^1).$$

So for $\varepsilon > 0$ $\exists$ $P \in \mathcal{A}$ such that

$$|f(x) - P(e^{ix})| < \varepsilon \quad \forall x \in \mathbb{R}.$$

<u>Schwarz's inequality</u>:

Let f, g ~~be~~ functions on $[a,b]$ such that $\int_a^b |f|^2$, $\int_a^b |g|^2 < \infty$, then

$f\bar{g} \in \mathcal{R}([a,b])$ and

$$\left|\alpha \int f\bar{g}\right|^2 \leq \alpha\int |f|^2 \; \alpha\int |g|^2 . \qquad \alpha > 0$$

<u>Proof</u>:

<u>claim</u> $u, v \geq 0 \Rightarrow uv \leq \frac{u^2}{2} + \frac{v^2}{2}$

$$(u-v)^2 = u^2 + v^2 - 2uv \geq 0$$

$$\Rightarrow uv \leq \frac{u^2}{2} + \frac{v^2}{2}.$$

If $\int |f|^2 = 1$ and $\int |g|^2 = 1$,

$$\int |f||g| \leq \int \frac{|f|^2}{2} + \frac{|g|^2}{2} = \frac{1}{2} + \frac{1}{2} = 1$$

Consider any f, g $\int |f|^2, \int |g|^2 \neq 0$

$$F = \frac{f}{\sqrt{\int |f|^2}}, \quad G = \frac{g}{\sqrt{\int |g|^2}}$$

$$\int |F|^2 = 1 \qquad \int |G|^2 = 1$$

$$\int |F G| \leq 1$$

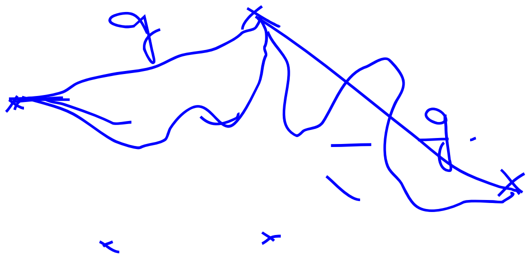
$$\left[\int |f|\,|g|\right]^2 \leq \int |f|^2 \int |g|^2 .$$

<u>Proposition</u>: Let $f \in \mathcal{R}[-\pi, \pi]$ and $\varepsilon > 0$.
Then there exists a $g \in C[-\pi, \pi]$
such that $\int |f - g|^2 < \varepsilon$.
Moreover if $f(-\pi) = f(\pi)$, then g may be chosen
so that $g(-\pi) = g(\pi)$.

<u>Proof</u>: $a = \pi$, $b = -\pi$. $M > 0$ be such that $|f(x)| \leq M$ $\forall x$.

Let P be a partion $\{a = x_0 < x_1 < \ldots < x_n = b\}$
of $[a, b]$ such that

$$U(P, f) - L(P, f) < \frac{\varepsilon}{2M}$$



$\alpha \geq 0 \qquad \alpha + \beta = 1 \qquad \beta \geq 0$

Let $g(t) = \dfrac{x_i - t}{x_i - x_{i-1}} f(x_{i-1}) + \dfrac{t - x_{i-1}}{x_i - x_{i-1}} f(x_i)$

$\forall t \in [x_{i-1}, x_i]$

$g \in C[a,b]$.

$$\int_{x_{i-1}}^{x_i} |f - g| \leq \int_{x_{i-1}}^{x_i} \left| \frac{x_i - t}{x_i - x_{i-1}} (f(t) - g(x_{i-1})) \right|$$

$$+ \int_{x_{i-1}}^{x_i} \frac{t - x_{i-1}}{x_i - x_{i-1}} |f(t) - g(x_i)|$$

$$\leq \frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} (x_i - t) |f(t) - g(x_{i-1})| + (t - x_{i-1}) |f(t) - g(x_i)|$$

$$\leq \frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} [(x_i - t) + (t - x_{i-1})] [M_i - m_i]$$

where $M_i = \sup_{[x_{i-1}, x_i]} f$

$m_i = \inf_{[x_{i-1}, x_i]} f$

$$\leq (x_i - x_{i-1})(M_i - m_i)$$

$$\int_a^b |f-g| < \frac{\varepsilon}{2M}$$

$$\int_a^b |f-g|^2 \leq 2M \int_a^b |f-g| < \varepsilon.$$

Proposition: For f, g on $[a,b]$ with $|f|^2, |g|^2 \in \mathcal{R}[a,b]$. Then

$$\left(\int |f+g|^2\right)^{1/2} \leq \sqrt{\int_a^b |f|^2} + \sqrt{\int_a^b |g|^2}$$

Proof: Let $\|f\|_2 = \sqrt{\int |f|^2}$.

$$\|f+g\|_2^2 = \int [f(x)+g(x)][\bar{f}(x) + \bar{g}(x)]$$

$$\leq \int [|f|^2 + |g|^2 + f\bar{g} + \bar{g}f]$$

$$= \int [|f|^2 + |g|^2 + 2\,\mathrm{Re}(f\bar{g})]$$

$$\leq \int |f|^2 + |g|^2 + 2|f||g|$$

$$\leq \int |f|^2 + \int |g|^2 + 2\sqrt{\int |f|^2}\sqrt{\int |g|^2}$$

$$= \left[\|f\|_2 + \|g\|_2 \right]^2$$

$$\|f+g\|_2 \leq \|f\|_2 + \|g\|_2 .$$

__Cor__ f, g, h be such functions

Then $\|f-g\|_2 \leq \|f-h\|_2 + \|h-g\|_2$

__Cauchy-Schwarz :__ $|f|^2 \in R[a,b]$ f is ... integrable over $[a,b]$

$$0 \leq \int\int \left[|f(x)\, g(y) - g(x)\, f(y)| \right]^2 dx\, dy$$

$$\leq \int_a^b \int_a^b \left[|f(x)|^2 |g(y)|^2 + |g(x)|^2 |f(y)|^2 \right] dx\, dy$$

$$- \int_a^b \int_a^b f(x)\, g(y)\, \overline{g(x)}\, \overline{f(y)}\, dx\, dy$$

$$- \int_a^b \int_a^b f(y)\, g(x)\, \overline{f(x)}\, \overline{g(y)}$$

$$= 2 \int |f|^2 \int |g|^2 - 2 \left| \int_a^b f(x)\, \overline{g(x)} \right|^2$$

$$\left| \int f \bar{g} \right|^2 \leq \int |f|^2 \int |g|^2$$