Lecture 13: Integral extensions

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(A) X, Y irred affine alg sets $\Rightarrow X \times Y$ irred.

$X \subseteq \mathbb{A}^n \qquad Y \subseteq \mathbb{A}^m$

(B) $\mathcal{O}_X = \dfrac{k[x_1, \dots, x_n]}{P}$, $\mathcal{O}_Y = \dfrac{k[y_1, \dots, y_m]}{Q}$, $X \times Y \subseteq \mathbb{A}^{n+m}$

Let $I = (P \cup Q) \subseteq k[x_1, \dots, x_n, y_1, \dots, y_m]$ then $Z(I) = X \times Y$

So $\mathcal{O}_{X \times Y} = \dfrac{k[x_1, \dots, x_n, y_1, \dots, y_m]}{\mathcal{I}(X \times Y)} = \dfrac{k[\underset{\sim}{x}, \underset{\sim}{y}]}{\mathcal{I}(Z(I))} \overset{HN}{=} \dfrac{k[\underset{\sim}{x}, \underset{\sim}{y}]}{\sqrt{I}}$ $\quad \mathcal{I}(Z(I)) \overset{?}{=} \sqrt{I}$

$$\mathcal{O}_X \otimes_k \mathcal{O}_Y = \left(\frac{k[x_1, \dots, x_n]}{P} \otimes_{k[x_1, \dots, x_n]} k[x_1, \dots, x_n]\right) \otimes_k \left(k[y_1, \dots, y_m] \otimes_{k[y_1, \dots, y_m]} k[\underset{\sim}{y}]/Q\right)$$

$$= \frac{k[\underset{\sim}{x}]}{P} \otimes_{k[\underset{\sim}{x}]} \left(k[\underset{\sim}{x}, \underset{\sim}{y}] \otimes_{k[\underset{\sim}{y}]} k[\underset{\sim}{y}]/Q\right)$$

$$= \frac{k[\underset{\sim}{x}]}{P} \otimes_{k[\underset{\sim}{x}]} \left(\frac{k[\underset{\sim}{x}, \underset{\sim}{y}]}{Q\,k[\underset{\sim}{x}, \underset{\sim}{y}]}\right)$$

$$= \frac{k[\underset{\sim}{x}, \underset{\sim}{y}]}{Q\,k[\underset{\sim}{x}, \underset{\sim}{y}]} \Big/ P\left(k[\underset{\sim}{x}, \underset{\sim}{y}]/Q\,k[\underset{\sim}{x}, \underset{\sim}{y}]\right)$$

$$\cong \frac{k[\underset{\sim}{x}, \underset{\sim}{y}]}{(P, Q)} = \frac{k[\underset{\sim}{x}, \underset{\sim}{y}]}{I}$$

> Let R, R' be rings & B be a (R, R')-bimod. i.e. $r(b r') = (rb) r'$ $\forall r \in R, r' \in R'$ & $b \in B$
> Let A be an R-mod & C be an R'-mod then $(A \otimes_R B) \otimes_{R'} C \cong A \otimes_R (B \otimes_{R'} C)$ as R, R'-bimod

Thm: $\mathcal{O}_X \otimes_k \mathcal{O}_Y$ is an integral domain if k is alg closed field.
(A consequence of Hilbert-Nullstellensatz)

Cor: I is a prime ideal of $k[\underset{\sim}{x}, \underset{\sim}{y}]$ & hence $\mathcal{I}(Z(I)) = I$
Hence $\mathcal{O}_{X \times Y} = \mathcal{O}_X \otimes_k \mathcal{O}_Y$.

Example: $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ $\qquad \mathbb{C} = \dfrac{\mathbb{R}[x]}{(x^2+1)}$

$\mathbb{C} \otimes_{\mathbb{R}} \dfrac{\mathbb{R}[x]}{(x^2+1)} \cong \dfrac{\mathbb{C}[x]}{(x^2+1)}$ is not an int domain.

$\cong$

$\left(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}[x]\right) \otimes_{\mathbb{R}[x]} \mathbb{R}[x]/(x^2+1)$

$\cong$

$\mathbb{C}[x] \otimes_{\mathbb{R}[x]} \dfrac{\mathbb{R}[x]}{(x^2+1)}$

⊛ Recall for a ring A an A-algebra is a ring homo $\varphi: A \to B$; B is called an A-alg.

B is said to be <u>finitely generated</u> A-algebra if $\exists\, b_1, \dots, b_n \in B$ s.t. $\forall\, b \in B$ $\exists$ a poly $f(x_1, \dots, x_n) \in A[x_1, \dots, x_n]$ s.t. $b = f(b_1, \dots, b_n)$. i.e. the map

$$A[x_1, \dots, x_n] \longrightarrow B \quad \text{sending}$$

$$\sum_{\text{finite}} a_{\underset{\sim}{i}}\, \underset{\sim}{x}^{\underset{\sim}{i}} \longmapsto \varphi(a_{\underset{\sim}{i}})\, \underset{\sim}{b}^{\underset{\sim}{i}} \quad \text{is surj ring homo.}$$

where $a_{\underset{\sim}{i}} \in A$ where $\underset{\sim}{x} = (x_1, \dots, x_n)$, $\underset{\sim}{i} = (i_1, \dots, i_n) \in \mathbb{N}^n$

$\underset{\sim}{b} = (b_1, \dots, b_n)$ & $\underset{\sim}{x}^{\underset{\sim}{i}} = x_1^{i_1} \cdots x_n^{i_n}$.

Eg: 1) Every f.g. ring is a f.g. $\mathbb{Z}$-algebra

(2) Poly ring over a ring k is a f.g. k-alg

(3) If B is f.g. A-alg & $I \subseteq B$ ideal then B/I is a f.g. A-alg.

(4) $A \subseteq B$ as subring then B is gen by $A, x_1, \dots, x_n$ as a ring $\iff$ B is f.g. A-alg.

Defn: Let B & C be A-algebras. A ring homo $f: B \to C$ is said to be A-alg homo if $f \circ \varphi_1 = \varphi_2$

$$\begin{tikzcd} B \arrow[rr, "f"] & & C \\ & A \arrow[ul, "\varphi_1"] \arrow[ur, "\varphi_2"'] \arrow[u, phantom, "\#"] & \end{tikzcd}$$

i.e. f is also an A-mod homo.

$a \in A$ & $b \in B$

$$f(a \cdot b) = f(\varphi_1(a) b) = f(\varphi_1(a)) f(b) = \varphi_2(a) f(b) = a \cdot f(b)$$

Defn: B is said to be a finite A-alg if B is a finitely generated A-module.

Examples

1) $k[x_1, \dots, x_n]$, the poly ring, is a f.g. k-alg but not a finite k-alg.

⊛ A finite k-alg is a f.g. k-alg

⊛ B is a finite A-alg $\Rightarrow \exists\, b_1, \dots, b_n \in B$ s.t. $\forall\, b \in B$, $\exists\, a_1, \dots, a_n \in A$ s.t.

$b = a_1 b_1 + \dots + a_n b_n$, i.e.

$$B = Ab_1 + \dots + Ab_n$$

⊛ $\mathbb{Z}$, finite $\mathbb{Z}$-alg; $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{Z} \times \dots \times \mathbb{Z}$ are finite $\mathbb{Z}$-alg.

$\mathbb{Z}[i] \subseteq \mathbb{C}$

$\{a + bi \mid a, b \in \mathbb{Z}\}$

* $\mathbb{Z}[\frac{1}{2}]$ is not a finite $\mathbb{Z}$-alg
But $\mathbb{Z}[\frac{1}{2}]$ is f.g. $\mathbb{Z}$-alg.

* $\mathbb{Q}$ as $\mathbb{Z}$-alg. is not finite $\mathbb{Z}$-alg

* $\mathbb{Q}$ is not even f.g. $\mathbb{Z}$-alg.

Let $\frac{a_1}{b_1}, \dots, \frac{a_n}{b_n}$ be elements of $\mathbb{Q}$.
$(b_i, a_i) = 1$, $\exists\, p \in \mathbb{Z}$ prime s.t.
$(p, b_1 \cdots b_n) = 1$.
$\frac{1}{p}$ is not in $\mathbb{Z}[\frac{a_1}{b_1}, \dots, \frac{a_n}{b_n}]$

* k is field then a finite k-alg.
l/k finite field ext^n then l is finite k-alg
$l_1 \times l_2 \times \dots \times l_n$ where l_i/k finite field ext^n then this finite k-alg.

$\frac{k[x]}{x^2}$ is finite k-alg.

Integral extension

Def^n: Let B be a ring & A be a subring of B then B is called an extension of A. An element $b \in B$ is said to be integral over A if $\exists$ a monic poly $f(x) \in A[x]$ s.t. $f(b)=0$. B is said to be integral over A if $\forall x \in B$, x is integral over A.

(*) B is finite A-alg $\Rightarrow$ B is integral over A.
Converse holds if B is a f.g. A-alg.

Thm: Let $A \subseteq B$ be rings & $x \in B$. TFAE

1. x is integral over A.
2. $A[x]$ is a finite A-module
3. There exist $C \subseteq B$ subring s.t $A[x] \subseteq C$ & C is a finite A-module.

Pf: 1 $\Rightarrow$ 2 like field

2 $\Rightarrow$ 3 trivial $C = A[x]$

3 $\Rightarrow$ 1 $C = Ac_1 + \dots + Ac_n$ for some $c_1, \dots, c_n \in C$

$$xc_1 = a_{11}c_1 + \dots + a_{1n}c_n$$
$$xc_2 = a_{21}c_1 + \dots + a_{2n}c_n$$
$$\vdots$$
$$xc_n = a_{n1}c_1 + \dots + a_{nn}c_n$$

$\Rightarrow$ $(xI - M)\underset{\sim}{c} = 0$ where $M = ((a_{ij}))$

$\cdot\, \mathrm{Adj}(xI-A)$ $\Rightarrow$ $\det(xI-M)\underset{\sim}{c} = 0$

$\Rightarrow$ $\det(xI-M) = 0$ i.e. $x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0$ for som $a_{n-1}, \dots, a_0 \in A$.

i.e. x is int over A. ∎