Lecture 29: Rational functions on projective varieties

08 April 2021
09:56

<u>Def^n</u>: A <u>projective variety</u> is an irred alg subset of $\mathbb{P}^n$ for some n.

Ⓐ Let $X \subseteq \mathbb{P}^n$ be a proj variety. Want to define functions on X. Let $F \in k[X_0, \dots, X_n]$ homogen

then
$$\Longrightarrow \begin{cases} F(a_0, \dots, a_n) = F(\lambda a_0, \dots, \lambda a_n) & (\because [a_0, \dots, a_n] = [\lambda a_0, \dots, \lambda a_n]) \\ \qquad = \lambda^d F(a_0, \dots, a_n) & \forall \lambda \in k^\times \text{ where } d = \deg F) \\ F \text{ is homogen of deg } 0. \end{cases}$$

i.e. F is const.

<u>Def^n</u>: A rat'l function on $\mathbb{P}^n$ is $F/G \in k(X_0, \dots, X_n)$ where $F, G \in k[X_0, \dots, X_n]$ are homogen of the same degree. Note that $\frac{F}{G}([a_0, \dots, a_n]) = \frac{F(a_0, \dots, a_n)}{G(a_0, \dots, a_n)}$ is well-defined on $\underset{\text{open}}{\mathbb{P}^n \setminus \{G = 0\}}$.

Let $X \subseteq \mathbb{P}^n$ be a variety. Then a <u>rat'l func</u> on X is a function defined on a nonempty open subset U of X given by

$\frac{F}{G} : X \dashrightarrow k$ where F, G are in $k[X_0, \dots, X_n]$ homogen of same degree. & $G \notin I_{\mathbb{P}^n}(X)$

here $U = X \setminus \{G = 0\}$

Example: Let $y^2 = x(x-1)(x-2)$ be affine equation i.e.

Let $C = Z(y^2 - x(x-1)(x-2)) \subseteq \mathbb{A}^2_{x,y} \subseteq \mathbb{P}^2_{X_0,X_1,X_2}$

$X_2 \neq 0$

$$\frac{X_0}{X_2} = x, \quad \frac{X_1}{X_2} = y$$

$$f(X_0,X_1,X_2) = X_2^3\left(\frac{X_1^2}{X_2^2} - \frac{X_0}{X_2}\left(\frac{X_0}{X_2}-1\right)\left(\frac{X_0}{X_2}-2\right)\right)$$

$$= X_1^2 X_2 - X_0(X_0 - X_2)(X_0 - 2X_2)$$

$\overline{C} = Z_{\mathbb{P}^2}(f) \subseteq \mathbb{P}^2$ and $C = U_2 \cap \overline{C}$ where $U_2 = \{X_2 \neq 0\} \subseteq \mathbb{P}^2$

$[a_0, a_1, a_2] \in \overline{C}$ & $a_2 \neq 0$ then $[a_0,a_1,a_2] = \left[\frac{a_0}{a_2}, \frac{a_1}{a_2}, 1\right]$

then $f\left(\frac{a_0}{a_2}, \frac{a_1}{a_2}, 1\right) = 0 \Rightarrow \left(\frac{a_0}{a_2}, \frac{a_1}{a_2}\right) \in C$.

if $a_2 = 0 \Rightarrow [a_0,a_1,a_2] \notin U_2$ $\quad f(a_0,a_1,0) = 0$

i.e. $a_0(a_0)(a_0) = 0$

$a_0^3 = 0 \Rightarrow a_0 = 0$

$\Rightarrow a_1 = 1$ & $\{[0,1,0]\} = \overline{C} \setminus C$

$\varphi_1 : \mathbb{A}^1 \subseteq \mathbb{P}^1$

$x \longmapsto [x, 1]$

$[1,0] \in \mathbb{P}^1$ is called the point at ∞.

Def^n Let $X \subseteq \mathbb{P}^n$ be a projective variety and $\mathcal{I}(X)$ be ideal of definition of X then the ring $k[X_0, \dots, X_n] / \mathcal{I}(X)$ is called the homogeneous coordinate ring of X. Note that $\mathcal{I}(X)$ is a prime ideal and hence the homogen coord ring is an integral domain.

(*) Let $f : X \dashrightarrow k$ be a rat'l function on a projective variety. If $X \subseteq \mathbb{P}^n$ then $\exists\ F, G \in k[X_0, \dots, X_n]$ homogen with $\deg F = \deg G$ and $G \notin \mathcal{I}(X)$ s.t. $\forall\ P = [a_0, \dots, a_n] \in X$, $G(a_0, \dots, a_n) \neq 0$

$$f(P) = \frac{F(a_0, \dots, a_n)}{G(a_0, \dots, a_n)}.$$

Also if $F \equiv F' \mod \mathcal{I}(X)$,
$G \equiv G' \mod \mathcal{I}(X)$
$\deg F' = \deg G'$ then

$$f(P) = \frac{F'(a_0, \dots, a_n)}{G'(a_0, \dots, a_n)}$$

Moreover f is said to be regular at P if we can find F, G as above with $G(P) \neq 0$.

Prop: (HW) The set of rat'l functions on a projective variety $X \subseteq \mathbb{P}^n$ is a field denoted $k(X)$. In fact,

$$k(X) = \left\{ \frac{f}{g} \;\middle|\; \begin{array}{l} F, G \in k[X_0, \dots, X_n] \text{ homogen of the same degree,} \\ G \notin \mathcal{I}(X) \; ; \quad f = F \pmod{\mathcal{I}(X)}\ \& \\ \qquad\qquad\qquad\quad g = G \pmod{\mathcal{I}(X)} \end{array} \right\}$$

$$\subseteq \operatorname{frac}\left(k[X_0, \dots, X_n] / \mathcal{I}(X) \right)$$

Pf: Let φ_1 & φ_2 be two rat'l functions on X. Then $\exists$ $F_1, G_1, F_2, G_2 \in k[X_0, \dots, X_n]$ s.t. $\varphi_i(P) = \frac{F_i(P)}{G_i(P)}$ $\forall P$ in a nonempty open subset U_i of X.

homogen with $\deg F_i = \deg G_i$

if $\frac{\overline{F_1}}{\overline{G_1}} = \frac{\overline{F_2}}{\overline{G_2}}$ then $\varphi_1(P) = \frac{F_1(P)}{G_1(P)} = \frac{F_2(P)}{G_2(P)} = \varphi_2(P)$ $\forall P$ in an open subset of X

$\Downarrow$

$G_2F_1 - F_2G_1 \in I(X) \implies G_2(P)F_1(P) - F_2(P)G_1(P) = 0 \ \forall P \in X$

Hence $\varphi_1 = \varphi_2$. Conversely, let $\frac{\overline{F_1}}{\overline{G_1}}, \frac{\overline{F_2}}{\overline{G_2}} \in k(X)$ s.t. $\frac{\overline{F_1}(P)}{\overline{G_1}(P)} = \frac{\overline{F_2}(P)}{\overline{G_2}(P)}$ $\forall P$ in an nonempty open subset of X

$\implies (\overline{G_2}\,\overline{F_1} - \overline{F_2}\,\overline{G_1})(P) = 0$ $\forall P$ in a nonempty open subset of X

Hence $\overline{G_2}\,\overline{F_1} - \overline{F_2}\,\overline{G_1} = 0$ in the homogen coord ring of X. ($\because$ nonempty open subset of X is dense in X as X is irred.)

$\implies \frac{\overline{F_1}}{\overline{G_1}} = \frac{\overline{F_2}}{\overline{G_2}}$ in $k(X)$.

Hence elements of $k(X)$ are in bijection with rat'l functions on X.

Check that $k(X)$ is a subfield of $\mathrm{frac}\left(\frac{k[X_0, \dots, X_n]}{I(X)}\right)$!

HW Remark: Let $U_i \subseteq \mathbb{P}^n$ be given by $\{X_i \neq 0\}$. If $U_i \cap X$ is nonempty then $k(X) = k(U_i \cap X)$, i.e. $k(X)$ is the fraction field of the coordinate ring of $U_i \cap X$.

Prop: Let $P \in X \subseteq \mathbb{P}^n$ be a point of a proj var X. Then $\mathcal{O}_{X,P} = \{f \in k(X) \mid f \text{ is regular at } P\}$ is a local ring with maximal ideal $m = \{f \in \mathcal{O}_{X,P} \mid f(P) = 0\}$.

Pf: $f, g \in \mathcal{O}_{X,P}$ then $f+g$ & $f \cdot g$ are $\mathcal{O}_{X,P}$ ($\because$ $f = \frac{\overline{F_1}}{\overline{F_2}}$; $F_1, F_2 \in k[X_0, \dots, X_n]$ of same degree

$g = \frac{\overline{G_1}}{\overline{G_2}}$ $G_1, G_2 \in k[X_0, \dots, X_n]$ of same degree

then $f+g = \frac{\overline{G_2}\,\overline{F_1} + \overline{G_1}\,\overline{F_2}}{\overline{F_2 G_2}}$ Note $\deg(G_2F_1 + G_1F_2) = \deg(F_2G_2)$

Also $f \cdot g$ is a rat'l function)

m is an ideal. Also if $f \notin m$

then $f = \frac{\overline{F}}{\overline{G}}$ with $F(P) \neq 0$, $G(P) \neq 0$ $F, G \in k[X_0, \dots, X_n]$ homogen of same degree

then $\frac{\overline{G}}{\overline{F}} \in \mathcal{O}_{X,P}$

& $\frac{\overline{G}}{\overline{F}} \cdot \frac{\overline{F}}{\overline{G}} = 1$. $\boxtimes$

(#) Let f be a rat'l func on $X \subseteq \mathbb{P}^n$ a proj var s.t. f is regular at all points of X.
Then f is const.

First assume $X = \mathbb{P}^n$

then $f = \frac{F}{G}$ where F, G are homo poly in $k[x_0, \dots, x_n]$ of same degree.

$(F, G) = 1$

If G is not const

then let $P \in Z(G)$. Then f is not regular at P. Suppose

$f = \frac{F'}{G'}$ s.t. $G'(P) \neq 0$, F', G' homogen of same degree

$G'F = F'G$

$\Rightarrow F \mid F'G$

$\Rightarrow F \mid F' \quad (\because (F, G) = 1)$

$\Rightarrow G' = \frac{F'}{F} G$

$\Rightarrow G \mid G'$ contradicting $G(P) = 0$.

Complete the proof ~~for~~ for proj variety.