Notes on Quantum Mechanics

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Incomplete, cryptic, typo-ridden notes and (elementary) questions on quantum information and quantum computation. I'm writing these *for me* as I'm reading Nielsen and Chuang. Why did I put it on GitHub then? I don't have an answer, sorry. The pdf is meant to be printed on A3 paper in landscape mode.

1 Linear algebra

1. **Inner product**

$$(|v\rangle, \lambda|w\rangle) = \lambda(|v\rangle, |w\rangle) \tag{1}$$
$$(\lambda|v\rangle, |w\rangle) = \lambda^*(|v\rangle, |w\rangle) \tag{2}$$
$$(|v\rangle, |w\rangle) = (|w\rangle, |v\rangle)^* \tag{3}$$

2. A **dual vector** is a linear operator from $V \to \mathbb{C}$ since $(\langle v|)\,|w\rangle = \langle v|w\rangle$. So, it's a row vector. *Check*: So, row vector can act as an operator? I think 3blue1brown mentioned something on this.

3. **Orthogonal vectors**: $(|v\rangle, |w\rangle) = 0$

4. **Orthonormal basis set** is complete $\sum_i |i\rangle\langle i| = I$

5. **Gram–Schmidt method** makes a basis set orthonormal. *Check*: right? what else it does?

6. **Outer-product representation** of operators

$$A = I_W A I_V = \sum_{ij} |w_j\rangle\langle w_j|\, A\, |v_i\rangle\langle v_i| = \sum_{ij} \langle w_j|A|v_i\rangle\, |w_j\rangle\, \langle v_i| \tag{4}$$

7. Outer-product representation reduces to **diagonal representation** (eigendecomposition) when $|v_i\rangle, |w_j\rangle$ are orthonormal eigenvalues of A (do they need to form a basis?)

   (a) $A = \sum_i \lambda_i |i\rangle\langle i|$

   (b) Example: Hamiltonian operator! $H = \sum_E E\,|E\rangle\langle E|$

8. **Adjoint vector** $|v\rangle^\dagger \equiv \langle v|$

   (a) Properties

$$A^\dagger = (A^*)^{\mathrm{T}} \tag{5}$$
$$(AB)^\dagger = B^\dagger A^\dagger \tag{6}$$
$$(A^\dagger)^\dagger = A \tag{7}$$
$$\left(\sum_i a_i A_i\right)^\dagger = \left(\sum_i a_i^* A_i^\dagger\right) \tag{8}$$

9. **Self-adjoint/Hermitian operator**: $A^\dagger = A$

10. **Simultaneous diagonalization theorem**: $[A, B, =]0$ iff there exists an orthonormal basis in which A and B are both diagonal. Important for uncertainty relation. (Check: is this the complete set of commutating operators Cohen Tannoudji keeps mentioning?)

11. Projector operator: Hermitian operator that projects to a subspace

$$P = \sum_{i=1}^{k} |i\rangle\langle i|\,,\ P^\dagger = P,\ P^2 = P \tag{9}$$

   (a) Special case: Measurement operator $M = \sum_i \lambda_i P_i$

12. **Normal operator**: A is normal if

$$A^\dagger A = AA^\dagger \tag{10}$$

13. **Spectral decomposition** A is normal if and only if it's diagonalizable.

14. A **normal matrix is Hermitian if and only if it has real eigenvalues**.

15. **Unitary operator**: normal with $A^\dagger A = AA^\dagger = I$. Since U is normal, it's diagonalizable, too.

   (a) U preserves inner product $(U|v\rangle, U|w\rangle) = \langle v|w\rangle$.

   (b) $U = \sum_i |w_i\rangle\,\langle v_i|$ *Check*: how

16. **Positive operator**: $(|v\rangle, A|v\rangle) \geq 0$. If it is > 0, then A is called positive definite.

17. **Tensor products**: If V and W are m and n dimensional vector spaces, then $V \otimes W$ (read as 'V tensor W') is an mn dimensional vector space.

   (a) If $|i\rangle$ and $|j\rangle e$ are orthonormal bases in V and W then $|i\rangle \otimes |j\rangle e$ is a basis for $V \otimes W$.

   (b) $z(|v\rangle \otimes |w\rangle) = (z\,|v\rangle) \otimes |w\rangle = |v\rangle \otimes (z\,|w\rangle)$

   (c) $(|v_1\rangle + |v_2\rangle) \otimes |w\rangle = |v_1\rangle \otimes |w\rangle + |v_2\rangle \otimes |w\rangle$

   (d) $|v\rangle \otimes (|w_1\rangle + |w_2\rangle) = |v\rangle \otimes |w_1\rangle + |v\rangle \otimes |w_2\rangle$

   (e) Linear operators on $V \otimes W$: If $A : V \to V'$ and $B : W \to W'$ are two linear operators, then $A \otimes B : V \otimes W \to V' \otimes W'$ is defined as

$$(A \otimes B)(|v\rangle \otimes |w\rangle) \equiv A\,|v\rangle \otimes B\,|w\rangle\,. \tag{11}$$

   It obeys all properties of linear operators.

   (f) Inner product in tensor-product space:

$$\sum_i a_i |v_i\rangle \otimes |w_i\rangle\,, \sum_j b_j |v_j'\rangle \otimes |w_j'\rangle = \sum_{ij} a_i^* b_j\, \langle v_i|\, v_j'\rangle\langle w_i|w_j'\rangle \tag{12}$$

   (g) **Kronecker product** as matrix representation for $A \otimes B$

$$A \otimes B \equiv \begin{bmatrix} A_{11}B & A_{12}B \\ A_{21}B & A_{22}B \end{bmatrix} \tag{13}$$

18. **Pauli matrices**

$$\sigma_0 \equiv I \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{14}$$
$$\sigma_1 \equiv \sigma_x \equiv X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{15}$$
$$\sigma_2 \equiv \sigma_y \equiv Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{16}$$
$$\sigma_3 \equiv \sigma_z \equiv Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{17}$$

19. **Functions of operators** Let $A = \sum_a a|a\rangle\langle a|$ be a spectral decomposition of the *normal operator* A, then $f(A) = \sum_a f(a)|a\rangle\langle a|$. Example:

$$\exp(\theta Z) = \begin{bmatrix} \exp(\theta) & 0 \\ 0 & \exp(-\theta) \end{bmatrix} \tag{18}$$

Check: does it only work for normal operators? need to think

20. **Trace**: $\mathrm{tr}(A) = \sum_i A_{ii}$, $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$, $\mathrm{tr}(zA) = z\,\mathrm{tr}(A)$

21. **Similarity transformation** $A \to UAU^\dagger$. *Preserves trace.*