

Incomplete, cryptic, typo-ridden notes and (elementary) questions on quantum information and quantum computation. I'm writing these *for me* as I'm reading Nielsen and Chuang. Why did I put it on GitHub then? I don't have an answer, sorry. The pdf is meant to be printed on A3 paper in landscape mode. Last compiled on August 17, 2023.—Arghya

## 1 Linear algebra

- **Inner product**

$$(|v\rangle, \lambda |w\rangle) = \lambda (|v\rangle, |w\rangle) \quad (1)$$

$$(\lambda |v\rangle, |w\rangle) = \lambda^* (|v\rangle, |w\rangle) \quad (2)$$

$$(|v\rangle, |w\rangle) = (|w\rangle, |v\rangle)^* \quad (3)$$

- A **dual vector** is a linear operator from  $V \rightarrow \mathbb{C}$  since  $(\langle v|) |w\rangle = \langle v|w\rangle$ . So, it's a row vector. *Check: So, row vector can act as an operator? I think 3blue1brown mentioned something on this.*

- **Orthogonal vectors:**  $(|v\rangle, |w\rangle) = 0$

- **Orthonormal basis set** is complete  $\sum_i |i\rangle\langle i| = I$

- **Gram–Schmidt method** makes a basis set orthonormal. *Check: right? what else it does?*

- **Outer-product representation** of operators

$$A = I_W A I_V = \sum_{ij} |w_j\rangle\langle w_j| A |v_i\rangle\langle v_i| = \sum_{ij} \langle w_j| A |v_i\rangle |w_j\rangle\langle v_i| \quad (4)$$

- Outer-product representation reduces to **diagonal representation** (eigendecomposition) when  $|v_i\rangle, |w_j\rangle$  are orthonormal eigenvalues of A (do they need to form a basis?)

$$A = \sum_i \lambda_i |i\rangle\langle i|$$

$$\text{Example: Hamiltonian operator! } H = \sum_E E |E\rangle\langle E|$$

- **Adjoint vector**  $|v\rangle^\dagger \equiv \langle v|$

Properties

$$A^\dagger = (A^*)^T \quad (5)$$

$$(AB)^\dagger = B^\dagger A^\dagger \quad (6)$$

$$(A^\dagger)^\dagger = A \quad (7)$$

$$\left( \sum_i a_i A_i \right)^\dagger = \left( \sum_i a_i^* A_i^\dagger \right) \quad (8)$$

- **Self-adjoint/Hermitian operator:**  $A^\dagger = A$

- **Simultaneous diagonalization theorem:**  $[A, B, =]0$  iff there exists an orthonormal basis in which  $A$  and  $B$  are both diagonal. Important for uncertainty relation. (Check: is this the complete set of commuting operators Cohen Tannoudji keeps mentioning?)

- **Projector operator:** Hermitian operator that projects to a subspace

$$P = \sum_{i=1}^k |i\rangle\langle i|, \quad P^\dagger = P, \quad P^2 = P \quad (9)$$

Special case: Measurement operator  $M = \sum_i \lambda_i P_i$

- **Normal operator:**  $A$  is normal if

$$A^\dagger A = A A^\dagger \quad (10)$$

- **Spectral decomposition**  $A$  is normal if and only if it's diagonalizable.

- A **normal matrix is Hermitian if and only if it has real eigenvalues.**

- **Unitary operator:** normal with  $A^\dagger A = A A^\dagger = I$ . Since  $U$  is normal, it's diagonalizable, too.

$$U \text{ preserves inner product } (U|v\rangle, U|w\rangle) = \langle v|w\rangle.$$

$$U = \sum_i |w_i\rangle\langle v_i| \quad \text{Check: how}$$

- **Positive operator:**  $(|v\rangle, A|v\rangle) \geq 0$ . If it is  $> 0$ , then  $A$  is called positive definite.

- **Tensor products:** If  $V$  and  $W$  are  $m$  and  $n$  dimensional vector spaces, then  $V \otimes W$  (read as 'V tensor W') is an  $mn$  dimensional vector space.

If  $|i\rangle$  and  $|j\rangle$  are orthonormal bases in  $V$  and  $W$  then  $|i\rangle \otimes |j\rangle$  is a basis for  $V \otimes W$ .

$$z(|v\rangle \otimes |w\rangle) = (z|v\rangle) \otimes |w\rangle = |v\rangle \otimes (z|w\rangle)$$

$$(|v_1\rangle + |v_2\rangle) \otimes |w\rangle = |v_1\rangle \otimes |w\rangle + |v_2\rangle \otimes |w\rangle$$

$$|v\rangle \otimes (|w_1\rangle + |w_2\rangle) = |v\rangle \otimes |w_1\rangle + |v\rangle \otimes |w_2\rangle$$

Linear operators on  $V \otimes W$ : If  $A : V \rightarrow V'$  and  $B : W \rightarrow W'$  are two linear operators, then  $A \otimes B : V \otimes W \rightarrow V' \otimes W'$  is defined as

$$(A \otimes B)(|v\rangle \otimes |w\rangle) \equiv A|v\rangle \otimes B|w\rangle. \quad (11)$$

It obeys all properties of linear operators.

Inner product in tensor-product space:

$$\sum_i a_i |v_i\rangle \otimes |w_i\rangle, \sum_j b_j |v'_j\rangle \otimes |w'_j\rangle = \sum_{ij} a_i^* b_j \langle v_i | v'_j \rangle \langle w_i | w'_j \rangle \quad (12)$$

**Kronecker product** as matrix representation for  $A \otimes B$

$$A \otimes B \equiv \begin{bmatrix} A_{11}B & A_{12}B \\ A_{21}B & A_{22}B \end{bmatrix} \quad (13)$$

- **Pauli matrices**

$$\sigma_0 \equiv I \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (14)$$

$$\sigma_1 \equiv \sigma_x \equiv X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (15)$$

$$\sigma_2 \equiv \sigma_y \equiv Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad (16)$$

$$\sigma_3 \equiv \sigma_z \equiv Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (17)$$

- **Functions of operators** Let  $A = \sum_a a|a\rangle\langle a|$  be a spectral decomposition of the *normal operator*  $A$ , then  $f(A) = \sum_a f(a)|a\rangle\langle a|$ . Example:

$$\exp(\theta Z) = \begin{bmatrix} \exp(\theta) & 0 \\ 0 & \exp(-\theta) \end{bmatrix} \quad (18)$$

*Check: does it only work for normal operators? need to think*

- **Trace:**  $\text{tr}(A) = \sum_i A_{ii}$ ,  $\text{tr}(AB) = \text{tr}(BA)$ ,  $\text{tr}(A+B) = \text{tr}(A) + \text{tr}(B)$ ,  $\text{tr}(zA) = z \text{tr}(A)$

- **Similarity transformation**  $A \rightarrow UAU^\dagger$ . Preserves trace.