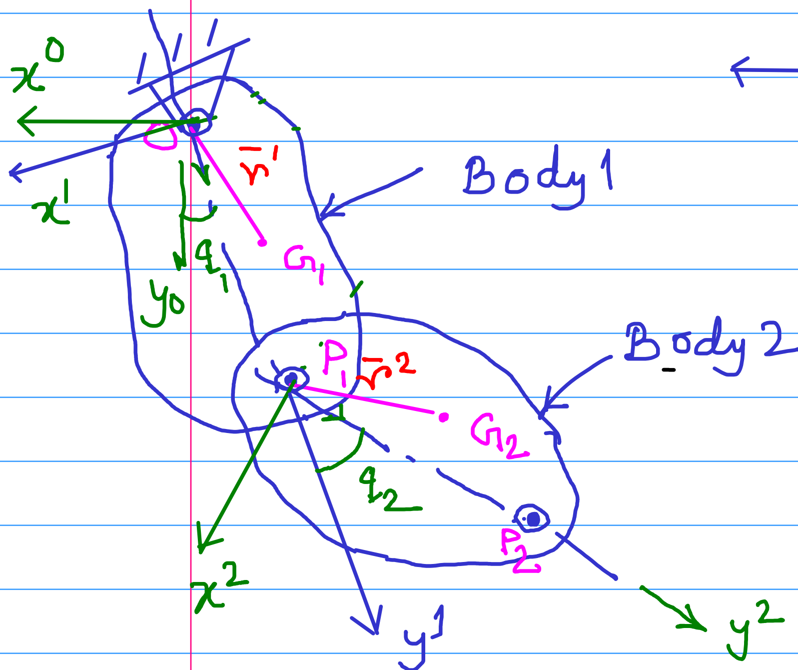
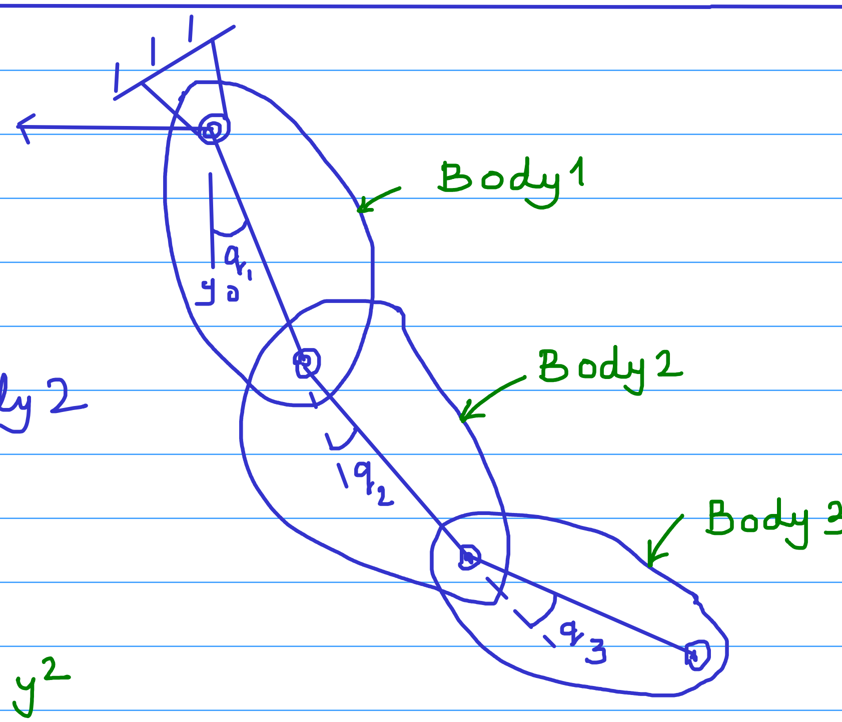
MBD

Plane motion of pin connected rigid bodies





Generate $[M]\{\ddot{q}\} = \{f\}$

Inputs

$r^1_{x^1}$ $r^1_{y^1}$ $r^2_{x^2}$ $r^2_{y^2}$

$\overline{OP_1} = \bar{p}^1$, $\overline{P_1P_2} = \bar{p}^2$

$p^1_{x^1}$ $p^1_{y^1}$

at cm of bodies $\left\{ F^1_{x^0}, F^1_{y^0}, F^2_{x^0}, F^2_{y^0} \right.$

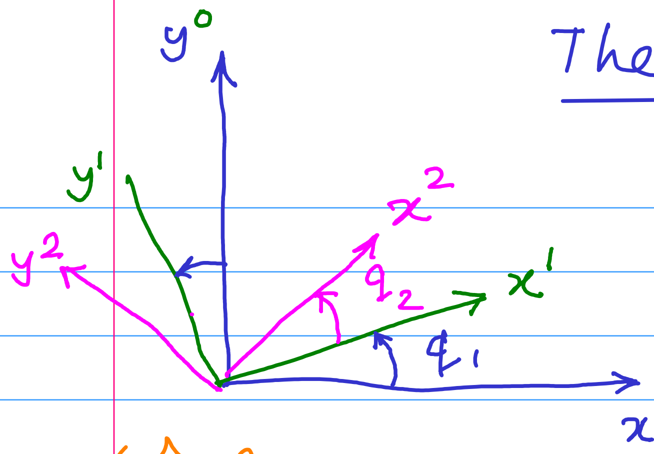
M_1, M_2

m_1, m_2, I_1, I_2

Eqn generated with $\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$, $\begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}$ & $\begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix}$

Initial condition

$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$ & $\begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}$ at $t=0$



The shifter matrix

Inertial

$$\begin{Bmatrix} \hat{i}^1 \\ \hat{j}^1 \end{Bmatrix} = [S_{10}] \begin{Bmatrix} \hat{i}^0 \\ \hat{j}^0 \end{Bmatrix}$$

$$\begin{Bmatrix} \hat{i}^1 \\ \hat{j}^1 \end{Bmatrix} = \underbrace{\begin{bmatrix} cq_1 & sq_1 \\ -sq_1 & cq_1 \end{bmatrix}}_{[S_{10}]} \begin{Bmatrix} \hat{i}^0 \\ \hat{j}^0 \end{Bmatrix}$$

$$\begin{Bmatrix} \hat{i}^2 \\ \hat{j}^2 \end{Bmatrix} = [S_{21}] \begin{Bmatrix} \hat{i}^1 \\ \hat{j}^1 \end{Bmatrix}$$

$$= [S_{21}][S_{10}] \begin{Bmatrix} \hat{i}^0 \\ \hat{j}^0 \end{Bmatrix}$$

$$= [S_{20}] \begin{Bmatrix} \hat{i}^0 \\ \hat{j}^0 \end{Bmatrix}$$

The angular velocity matrix

$$\begin{Bmatrix} \hat{i}^1 \\ \hat{j}^1 \end{Bmatrix} = \begin{bmatrix} cq_1 & sq_1 \\ -sq_1 & cq_1 \end{bmatrix} \begin{Bmatrix} \hat{i}^0 \\ \hat{j}^0 \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{\hat{i}}^1 \\ \dot{\hat{j}}^1 \end{Bmatrix} = \dot{q}_1 \begin{bmatrix} -sq_1 & cq_1 \\ -cq_1 & -sq_1 \end{bmatrix} \begin{Bmatrix} \hat{i}^0 \\ \hat{j}^0 \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{\hat{i}}^1 \\ \dot{\hat{j}}^1 \end{Bmatrix} = \underbrace{\dot{q}_1 \begin{bmatrix} -sq_1 & cq_1 \\ -cq_1 & -sq_1 \end{bmatrix}}_{[\dot{S}_{10}]} \underbrace{\begin{bmatrix} cq_1 & -sq_1 \\ sq_1 & cq_1 \end{bmatrix}}_{[S_{10}]^T} \begin{Bmatrix} \hat{i}^1 \\ \hat{j}^1 \end{Bmatrix}$$

$$= \dot{q}_1 \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix} \begin{Bmatrix} \hat{i}^1 \\ \hat{j}^1 \end{Bmatrix}$$

$$= \underbrace{\begin{bmatrix} 0 & \omega_1 \\ -\omega_1 & 0 \end{bmatrix}}_{[\omega^1]} \begin{Bmatrix} \hat{i}^1 \\ \hat{j}^1 \end{Bmatrix} \qquad \omega_1 = \dot{q}_1$$

$$[\omega^1] = [\overset{\circ}{S}_{10}][S_{10}]^T$$

Similarly it can be shown that

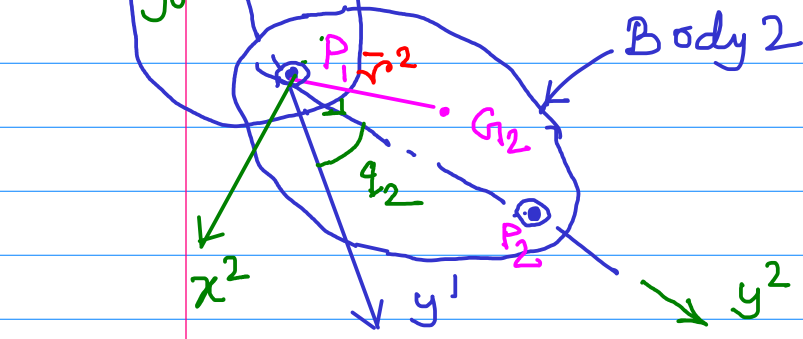
$$\begin{Bmatrix} \overset{\circ}{\hat{i}}{}^2 \\ \overset{\circ}{\hat{j}}{}^2 \end{Bmatrix} = [\overset{\circ}{S}_{20}][S_{20}]^T \begin{Bmatrix} \hat{i}^2 \\ \hat{j}^2 \end{Bmatrix}$$

where, $[S_{20}] = \begin{bmatrix} cq_2 & sq_2 \\ -sq_2 & cq_2 \end{bmatrix} \begin{bmatrix} cq_1 & sq_1 \\ -sq_1 & cq_1 \end{bmatrix}$

$$= \begin{bmatrix} c(q_1+q_2) & s(q_1+q_2) \\ -s(q_1+q_2) & c(q_1+q_2) \end{bmatrix}$$

$$[\omega^2] = [\overset{\circ}{S}_{20}][S_{20}]^T = \begin{bmatrix} 0 & \omega_2 \\ -\omega_2 & 0 \end{bmatrix}$$

where $\omega_2 = \overset{\circ}{q}_1 + \overset{\circ}{q}_2$



$$\overline{OG_1} = \bar{r}^1 = r^1_{x^1}\hat{i}^1 + r^1_{y^1}\hat{j}^1$$

$$\overline{P_1G_2} = \bar{r}^2 = r^2_{x^2}\hat{i}^2 + r^2_{y^2}\hat{j}^2$$

$$\overline{OP_1} = \bar{P}^1 = P^1_{x^1}\hat{i}^1 + P^1_{y^1}\hat{j}^1$$

$$\overline{P_1P_2} = \bar{P}^2 = P^2_{x^2}\hat{i}^2 + P^2_{y^2}\hat{j}^2$$

Velocity of centre of mass

For body 1

$$\bar{v}^1 = \omega_1 \hat{k}^1 \times \left(r^1_{x^1} \hat{i}^1 + r^1_{y^1} \hat{j}^1 \right)$$

$$= \omega_1 \left(- r^1_{y^1} \hat{i}^1 + r^1_{x^1} \hat{j}^1 \right)$$

$$\lfloor v^1_{x^0} \quad v^1_{y^0} \rfloor \underline{\begin{Bmatrix} \hat{i}_0 \\ \hat{j}_0 \end{Bmatrix}} = \omega_1 \lfloor - r^1_{y^1} \quad r^1_{x^1} \rfloor \begin{Bmatrix} \hat{i}^1 \\ \hat{j}^1 \end{Bmatrix}$$

$$= \omega_1 \lfloor - r^1_{y^1} \quad r^1_{x^1} \rfloor [S_{10}] \underline{\begin{Bmatrix} \hat{i}^0 \\ \hat{j}^0 \end{Bmatrix}}$$

$$\lfloor v^1_{x^0} \quad v^1_{y^0} \rfloor = \omega_1 \lfloor - r^1_{y^1} \quad r^1_{x^1} \rfloor [S_{10}]$$

For body 2

$$\bar{v}^2 = \omega_1 \hat{k}^1 \times \left(P^1_{x^1} \hat{i}^1 + P^1_{y^1} \hat{j}^1 \right)$$

$$+ \omega_2 \hat{k}^2 \times \left(r^2_{x^2} \hat{i}^2 + r^2_{y^2} \hat{j}^2 \right)$$

$$\lfloor v^2_{x^0} \quad v^2_{y^0} \rfloor \begin{Bmatrix} \hat{i}^0 \\ \hat{j}^0 \end{Bmatrix} = \omega_1 \lfloor - P^1_{y^1} \quad P^1_{x^1} \rfloor [S_{10}] \begin{Bmatrix} \hat{i}^0 \\ \hat{j}^0 \end{Bmatrix}$$

$$+ \omega_2 \lfloor - r^2_{y^2} \quad r^2_{x^2} \rfloor [S_{20}] \begin{Bmatrix} \hat{i}_0 \\ \hat{j}^0 \end{Bmatrix}$$

$$\lfloor v^2_{x0} \quad v^2_{y0} \rfloor = \omega_1 \lfloor -p^1_{y1} \quad p^1_{x1} \rfloor [S_{10}]$$

$$+ \; \omega_2 \lfloor -r^2_{y2} \quad r^2_{x2} \rfloor [S_{20}]$$

$$\lfloor v^1_{x0} \quad v^1_{y0} \quad v^2_{x0} \quad v^2_{y0} \rfloor$$

$$= \underbrace{\lfloor \omega_1 \quad \omega_2 \rfloor}_{1\times 2} \underbrace{\left[\begin{array}{c|c} \overbrace{\underbrace{\lfloor -r^1_{y1} \quad r^1_{x1} \rfloor}_{1\times 2} \underbrace{[S_{10}]}_{2\times 2}}^{1\times 2} & \overbrace{\lfloor -p^1_{y1} \quad p^1_{x1} \rfloor [S_{10}]}^{1\times 2} \\ \hline \lfloor 0 \rfloor & \lfloor -r^2_{y2} \quad r^2_{x2} \rfloor [S_{20}] \end{array}\right]}_{2\times 4}$$

$$= \lfloor \omega_1 \quad \omega_2 \rfloor \left[\begin{array}{c|c} [S_{r^1}][S_{10}] & [S_{p^1}][S_{10}] \\ \hline 0 & [S_{r^2}][S_{20}] \end{array}\right]$$

$$\lfloor \omega_1 \quad \omega_2 \rfloor = \lfloor \dot{q}_1 \quad \dot{q}_1 + \dot{q}_2 \rfloor$$

$$= \lfloor \dot{q}_1 \quad \dot{q}_2 \rfloor \left[\begin{array}{c|c} 1 & 1 \\ \hline 0 & 1 \end{array}\right]$$

$$= \lfloor \dot{q}_1 \quad \dot{q}_2 \rfloor [T_{\omega q}]$$

$$\lfloor v^1_{x0} \quad v^1_{y0} \quad v^2_{x0} \quad v^2_{y0} \rfloor$$

$$= \lfloor \dot{q}_1 \quad \dot{q}_2 \rfloor [T_{\omega q}] \left[\begin{array}{c|c} [S^1_r][S_{10}] & [S^1_p][S_{10}] \\ \hline 0 & [S^2_r][S_{20}] \end{array}\right]$$

$$\lfloor v^1_{x0} \quad v^1_{y0} \quad v^2_{x0} \quad v^2_{y0} \rfloor$$

$$= \lfloor \dot{q}_1 \quad \dot{q}_2 \rfloor \underbrace{\underset{2\times 2}{[T_{\omega q}]} \left[\begin{array}{c|c} [S^1_r][S_{10}] & [S^1_p][S_{10}] \\ \hline 0 & [S^2_r][S_{20}] \end{array}\right]}_{2\times 4}$$

$[V_{pd}] \rightarrow$ Partial velocity matrix

$$= \lfloor \dot{q}_1 \quad \dot{q}_2 \rfloor [V_{pd}]$$

$$\left\lfloor \frac{\partial v^1_{x0}}{\partial \dot{q}_1} \quad \frac{\partial v^1_{y0}}{\partial \dot{q}_1} \quad \frac{\partial v^2_{x0}}{\partial \dot{q}_1} \quad \frac{\partial v^2_{y0}}{\partial \dot{q}_1} \right\rfloor$$

$$= \underset{1\times 2}{\lfloor 1 \quad 0 \rfloor} \underset{2\times 4}{[V_{pd}]}$$

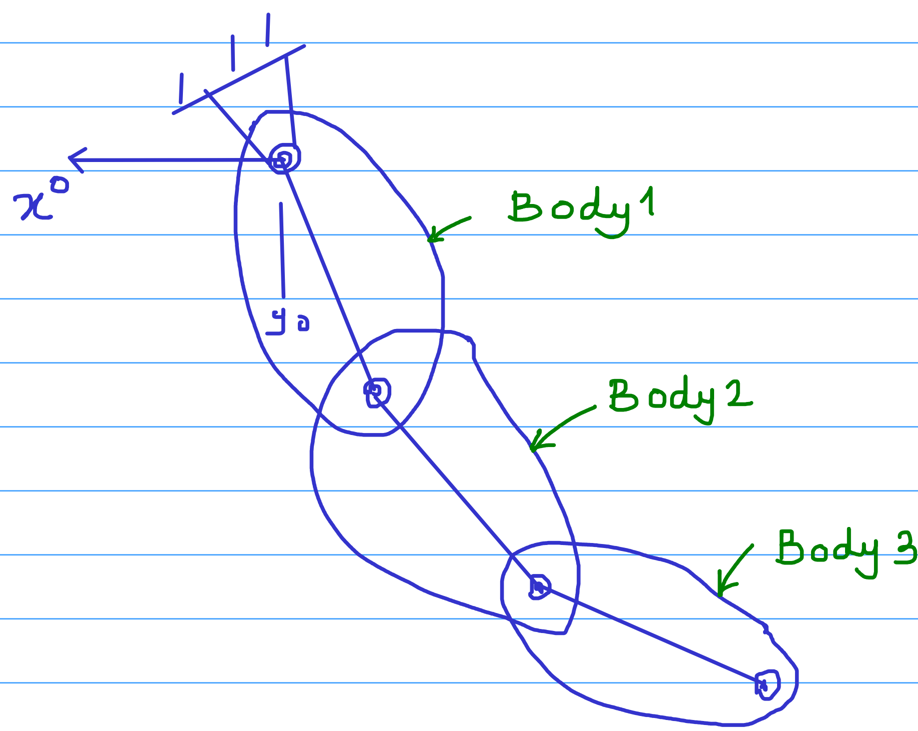
$= \lfloor$ First row of $V_{pd} \rfloor$

$$\begin{bmatrix} \dfrac{\partial v^1_{x^0}}{\partial \dot{q}_2} & \dfrac{\partial v^1_{y^0}}{\partial \dot{q}_2} & \dfrac{\partial v^2_{x^0}}{\partial \dot{q}_2} & \dfrac{\partial v^2_{y^0}}{\partial \dot{q}_2} \end{bmatrix}$$

$$= \left[\text{Second row of } V_{pd}\right]$$

$$\begin{array}{cc}
 & \begin{array}{cc} \overbrace{\qquad\qquad\qquad\qquad}^{\text{Body 1}} & \overbrace{\qquad\qquad\qquad\qquad}^{\text{Body 2}} \end{array} \\
\begin{array}{c} \dfrac{\partial}{\partial \dot{q}_1} \\[3ex] \dfrac{\partial}{\partial \dot{q}_2} \end{array} &
\begin{bmatrix}
\dfrac{\partial v^1_{x^0}}{\partial \dot{q}_1} & \dfrac{\partial v^1_{y^0}}{\partial \dot{q}_1} & \dfrac{\partial v^2_{x^0}}{\partial \dot{q}_1} & \dfrac{\partial v^2_{y^0}}{\partial \dot{q}_1} \\[2ex]
\dfrac{\partial v^1_{x^0}}{\partial \dot{q}_2} & \dfrac{\partial v^1_{y^0}}{\partial \dot{q}_2} & \dfrac{\partial v^2_{x^0}}{\partial \dot{q}_2} & \dfrac{\partial v^2_{y^0}}{\partial \dot{q}_2}
\end{bmatrix}
\end{array} = \left[V_{pd}\right]$$

3 pin jointed bodies



$$\lfloor v^1_{x^0} \quad v^1_{y^0} \quad v^2_{x^0} \quad v^2_{y^0} \quad v^3_{x^0} \quad v^3_{y^0} \rfloor$$

$$= \lfloor \omega_1 \quad \omega_2 \quad \omega_3 \rfloor \begin{bmatrix} [S_r^1][S_{10}] & [S_p^1][S_{10}] & [S_p^1][S_{10}] \\ [0] & [S_r^2][S_{20}] & [S_p^2][S_{20}] \\ [0] & [0] & [S_p^3][S_{30}] \end{bmatrix}$$

Partial velocity

$$\lfloor \omega_1 \quad \omega_2 \quad \omega_3 \rfloor = \lfloor \dot{q}_1 \quad \dot{q}_1 + \dot{q}_2 \quad \dot{q}_1 + \dot{q}_2 + \dot{q}_3 \rfloor$$

$$= \lfloor \dot{q}_1 \quad \dot{q}_2 \quad \dot{q}_3 \rfloor \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{[T_{\omega q}]}$$

$$[V_{pd}] = \underset{3\times3}{[T_{\omega q}]} \underset{3\times 6}{\begin{bmatrix} [S_r^1][S_{10}] & [S_p^1][S_{10}] & [S_p^1][S_{10}] \\ 0 & [S_r^2][S_{20}] & [S_p^2][S_{20}] \\ 0 & & [S_r^3][S_{30}] \end{bmatrix}}$$

$$
\begin{array}{cl}
 & \quad \overbrace{\hphantom{\frac{\partial v^1_{x0}}{\partial \dot q_1} \quad \frac{\partial v^1_{y0}}{\partial \dot q_1}}}^{\text{Body 1}} \quad \overbrace{\hphantom{\frac{\partial v^2_{x0}}{\partial \dot q_1} \quad \frac{\partial v^2_{y0}}{\partial \dot q_1}}}^{\text{Body 2}} \\
\begin{array}{c}
\frac{\partial}{\partial \dot q_1} \\[2.5em]
\frac{\partial}{\partial \dot q_2} \\[2.5em]
\frac{\partial}{\partial \dot q_3}
\end{array}
&
\begin{bmatrix}
\frac{\partial v^1_{x0}}{\partial \dot q_1} & \frac{\partial v^1_{y0}}{\partial \dot q_1} & \frac{\partial v^2_{x0}}{\partial \dot q_1} & \frac{\partial v^2_{y0}}{\partial \dot q_1} & \frac{\partial v^3_{x0}}{\partial \dot q_1} & \frac{\partial v^3_{y0}}{\partial \dot q_1} \\[1em]
\frac{\partial v^1_{x0}}{\partial \dot q_2} & \frac{\partial v^1_{y0}}{\partial \dot q_2} & \frac{\partial v^2_{x0}}{\partial \dot q_2} & \frac{\partial v^2_{y0}}{\partial \dot q_2} & \frac{\partial v^3_{x0}}{\partial \dot q_2} & \frac{\partial v^3_{y0}}{\partial \dot q_2} \\[1em]
\frac{\partial v^1_{x0}}{\partial \dot q_3} & \frac{\partial v^1_{y0}}{\partial \dot q_3} & \frac{\partial v^2_{x0}}{\partial \dot q_3} & \frac{\partial v^2_{y0}}{\partial \dot q_3} & \frac{\partial v^3_{x0}}{\partial \dot q_3} & \frac{\partial v^3_{y0}}{\partial \dot q_3}
\end{bmatrix}
= \begin{bmatrix} V_{pd} \end{bmatrix}
\end{array}
$$

Partial angular velocity

For 2 bodies

$$\begin{bmatrix} \omega_1 & \omega_2 \end{bmatrix} = \begin{bmatrix} \dot{q}_1 & \dot{q}_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$[W_{pd}] = [T_{wq}]$

$$= \begin{bmatrix} \dot{q}_1 & \dot{q}_2 \end{bmatrix} [W_{pd}]$$

$$\begin{bmatrix} \dfrac{\partial \omega_1}{\partial q_1} & \dfrac{\partial \omega_2}{\partial q_1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} [W_{pd}]$$

$=$ First row of $[W_{pd}]$

$$\begin{bmatrix} \dfrac{\partial \omega_1}{\partial q_2} & \dfrac{\partial \omega_2}{\partial q_2} \end{bmatrix} = \text{Second row of } [W_{pd}]$$

$$\text{Partial ang vel} = [W_{pd}] = \begin{array}{c c} & \begin{array}{cc} \text{Body 1} & \text{Body 2} \end{array} \\ \begin{array}{c} \dfrac{\partial}{\partial \dot{q}_1} \\ \dfrac{\partial}{\partial \dot{q}_2} \end{array} & \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \end{array}$$

For 3 bodies connected as shown

$$[W_{pd}] = \begin{array}{c c} & \begin{array}{ccc} \text{Body 1} & \text{Body 2} & \text{Body 3} \end{array} \\ \begin{array}{c} \partial/\partial \dot{q}_1 \\ \partial/\partial \dot{q}_2 \\ \partial/\partial \dot{q}_3 \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Acceleration of centre of mass

Vector cross product in matrix notation

$$\omega \hat{k} \times (r_x \hat{i} + r_y \hat{j})$$

$$= \omega \begin{bmatrix} -r_y & r_x \end{bmatrix} \begin{Bmatrix} \hat{i} \\ \hat{j} \end{Bmatrix}$$

$$= \begin{bmatrix} c_x & c_y \end{bmatrix} \begin{Bmatrix} \hat{i} \\ \hat{j} \end{Bmatrix}$$

$$\omega \hat{k} \times \{ \omega \hat{k} \times (r_x \hat{i} + r_y \hat{j}) \}$$

$$= \omega \hat{k} \times (c_x \hat{i} + c_y \hat{j})$$

$$= \begin{bmatrix} c_x & c_y \end{bmatrix} \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \begin{Bmatrix} \hat{i} \\ \hat{j} \end{Bmatrix}$$

$$= \omega \begin{bmatrix} -r_y & r_x \end{bmatrix} \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \begin{Bmatrix} \hat{i} \\ \hat{j} \end{Bmatrix}$$

$$= \omega \begin{bmatrix} S r \end{bmatrix} \begin{bmatrix} \omega \end{bmatrix} \begin{Bmatrix} \hat{i} \\ \hat{j} \end{Bmatrix}$$

$\swarrow$ accl vector of cm of body 1

$$\bar{a}^1 = a^1_{x^0}\,\underline{\hat{i}_0} + a^1_{y^0}\,\underline{\hat{j}_0}$$

$$= \lfloor a^1_{x^0} \;\; a^1_{y^0} \rfloor \begin{Bmatrix} \hat{i}^0 \\ \hat{j}^0 \end{Bmatrix}$$

$$\alpha_1 = \mathring{\omega}_1$$

$$= \omega_1 \hat{k}^1 \times (\omega_1 \hat{k}^1 \times \bar{r}^1) + \alpha_1 \hat{k}^1 \times \bar{r}^1$$

$$= \omega_1 \lfloor -r^1_{y^1} \;\; r^1_{x^1} \rfloor \left[\begin{array}{c|c} 0 & \omega_1 \\ \hline -\omega_1 & 0 \end{array}\right] \begin{Bmatrix} \hat{i}^1 \\ \hat{j}^1 \end{Bmatrix}$$

$$+ \alpha_1 \lfloor -r^1_{y^1} \;\; r^1_{x^1} \rfloor \begin{Bmatrix} \hat{i}^1 \\ \hat{j}^1 \end{Bmatrix}$$

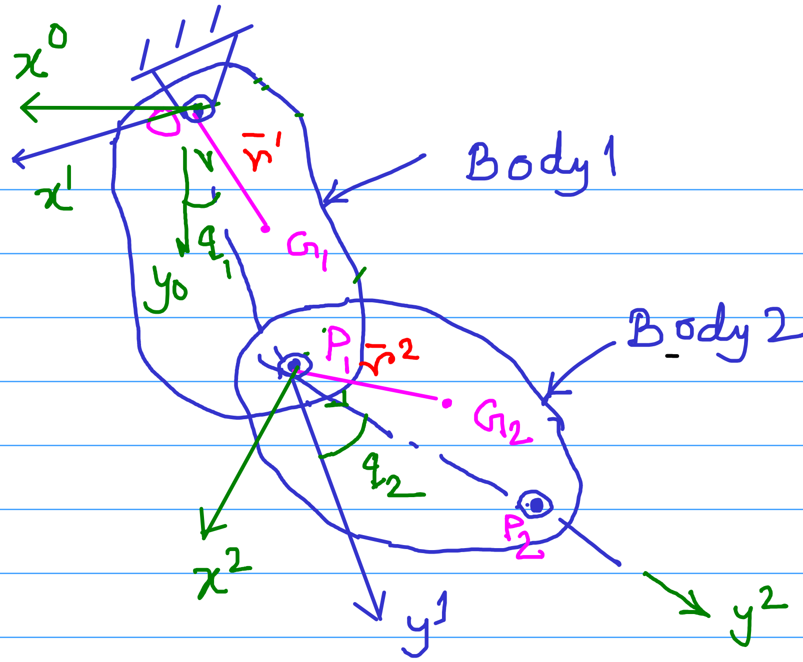
$$= \underbrace{\omega_1 [S_r^1] \; [\omega^1] [S_{10}] \begin{Bmatrix} \hat{i}^0 \\ \hat{j}^0 \end{Bmatrix}}_{n} + \underbrace{\alpha_1 [S_r^1] [S_{10}] \begin{Bmatrix} \hat{i}^0 \\ \hat{j}^0 \end{Bmatrix}}_{t}$$

$$\underline{\lfloor a^1_{x^0} \;\; a^1_{y^0} \rfloor = \omega_1 [S_r^1][\omega^1][S_{10}] + \alpha_1 [S_r^1][S_{10}]}$$

$$\alpha_2 = \mathring{\omega}_2$$

$$\bar{a}^2 = \bar{a}^1_p + \omega_2 \hat{k}^2 \times (\omega_2 \hat{k}^2 \times \bar{r}^2) + \alpha_2 \hat{k}^2 \times \bar{r}^2$$

$$\lfloor a^1_{p x^0} \;\; a^1_{p y^0} \rfloor = \omega_1 [S_p^1][\omega^1][S_{10}] + \alpha_1 [S_p^1][S_{10}]$$



$$\lfloor a^2_{x^0} \; a^2_{y^0} \rfloor = \lfloor a^1_{p x^0} \; a^1_{p y^0} \rfloor + \omega_2 [S_r^2][\omega^2][S_{20}] + \alpha_2 [S_r^2][S_{20}]$$

$$\lfloor a^1_{x^0} \; a^1_{y^0} \; a^2_{x^0} \; a^2_{y^0} \rfloor =$$

$$\lfloor \omega_1 \;\; \omega_2 \rfloor \left[\begin{array}{c|c} [S_r^1][\omega^1][S_{10}] & [S_p^1][\omega^1][S_{10}] \\ \hline [0] & [S_r^2][\omega^2][S_{20}] \end{array}\right] +$$

$$\lfloor \alpha_1 \;\; \alpha_2 \rfloor \left[\begin{array}{c|c} [S_r^1][S_{10}] & [S_p^1][S_{10}] \\ \hline [0] & [S_r^2][S_{20}] \end{array}\right]$$

$$= \lfloor \dot{q}_1 \;\; \dot{q}_2 \rfloor [\underline{T_{wq}}] \left[\begin{array}{c|c} [S_r^1][\omega^1][S_{10}] & [S_p^1][\omega^1][S_{10}] \\ \hline [0] & [S_r^2][\omega^2][S_{20}] \end{array}\right]$$

$$+ \lfloor \ddot{q}_1 \;\; \ddot{q}_2 \rfloor [T_{wq}] \left[\begin{array}{c|c} [S_r^1][S_{10}] & [S_p^1][S_{10}] \\ \hline [0] & [S_r^2][S_{20}] \end{array}\right]$$

If there are 3 connected bodies

$$\lfloor \underline{a^1_{x0} \; a^1_{y0}} \;\; \underline{a^2_{x0} \; a^2_{y0}} \;\; \underline{a^3_{x0} \; a^3_{y0}} \rfloor =$$

$$\underset{1\times 3}{\lfloor \omega_1 \;\; \omega_2 \;\; \omega_3 \rfloor} \underset{3\times 6}{\left[\begin{array}{c|c|c} [S_r^1][\omega^1][S_{10}] & [S_p^1][\omega^1][S_{10}] & [S_p^1][\omega^1][S_{10}] \\ \hline [0] & [S_r^2][\omega^2][S_{20}] & [S_p^2][\omega^2][S_{20}] \\ \hline [0] & [0] & [S_r^3][\omega^3][S_{30}] \end{array}\right]}$$

$$+ \underset{1\times 3}{\lfloor \alpha_1 \;\; \alpha_2 \;\; \alpha_3 \rfloor} \underset{3\times 6}{\left[\begin{array}{c|c|c} [S_r^1][S_{10}] & [S_p^1][S_{10}] & [S_p^1][S_{10}] \\ \hline [0] & [S_r^2][S_{20}] & [S_p^2][S_{20}] \\ \hline [0] & [0] & [S_r^3][S_{30}] \end{array}\right]}$$

$$\left[-m_1 a^1_{x^\circ} \quad -m_1 a^1_{y^\circ} \quad -m_2 a^2_{x^\circ} \quad -m_2 a^2_{y^\circ}\right]$$

$$= \left[a^1_{x^\circ} \quad a^1_{y^\circ} \quad a^2_{x^\circ} \quad a^2_{y^\circ}\right] \begin{bmatrix} -m_1 & 0 & 0 & 0 \\ 0 & -m_1 & 0 & 0 \\ 0 & 0 & -m_2 & \\ 0 & 0 & 0 & -m_2 \end{bmatrix}$$

$$= -\underset{1\times 2}{\left[\dot{q}_1 \quad \dot{q}_2\right]} \overbrace{\underset{2\times 2}{\left[T_{wq}\right]} \underset{2\times 4}{\begin{bmatrix} S_r^1 \omega^1 S_{10} & S_p^1 \omega^1 S_{10} \\ 0 & S_r^2 \omega^2 S_{20} \end{bmatrix}} \underset{4\times 4}{\begin{bmatrix} +m_1 & 0 & 0 & 0 \\ 0 & +m_1 & 0 & 0 \\ 0 & 0 & +m_2 & 0 \\ 0 & 0 & 0 & +m_2 \end{bmatrix}}}^{[A_1]}$$

$$-\underset{1\times 2}{\left[\ddot{q}_1 \quad \ddot{q}_2\right]} \overbrace{\underset{2\times 2}{\left[T_{wq}\right]} \underset{2\times 4}{\begin{bmatrix} S_r^1 S_{10} & S_p^1 S_{10} \\ 0 & S_r^2 S_{20} \end{bmatrix}} \underset{4\times 4}{\begin{bmatrix} +m_1 & 0 & 0 & 0 \\ 0 & +m_1 & 0 & 0 \\ 0 & 0 & +m_2 & 0 \\ 0 & 0 & 0 & +m_2 \end{bmatrix}}}^{[A_2]}$$

$$\left[F^1_{in\,x^\circ} \quad F^1_{in\,y^\circ} \quad F^2_{in\,x^\circ} \quad F^2_{in\,y^\circ}\right]$$

$$= -\left[\dot{q}_1 \quad \dot{q}_2\right]\left[A_1\right] - \left[\ddot{q}_1 \quad \ddot{q}_2\right]\left[A_2\right]$$

$$\left[M^1_{in} \quad M^2_{in}\right] = \left[\alpha_1 \quad \alpha_2\right] \begin{bmatrix} -I_1 & \\ & -I_2 \end{bmatrix}$$

$$= -\left[\ddot{q}_1 \quad \ddot{q}_2\right] \underbrace{\left[T_{wq}\right] \begin{bmatrix} +I_1 & 0 \\ 0 & +I_2 \end{bmatrix}}_{[A_3]}$$

$$= -\left[\ddot{q}_1 \quad \ddot{q}_2\right]\left[A_3\right]$$

$$\begin{array}{c} \\ \frac{\partial}{\partial \dot{q}_1} \rightarrow \\ \frac{\partial}{\partial \dot{q}_2} \rightarrow \end{array} \overbrace{\underbrace{\begin{array}{cc} x_0 \text{ comp} & y_0 \text{ comp} \end{array}}_{}}^{\text{Body 1}} \overbrace{\begin{array}{cc} x_0 \text{ comp} & y_0 \text{ comp} \end{array}}^{\text{Body 2}}$$

$$\left[\begin{array}{c|c|c|c} V_{pd}(1,1) & V_{pd}(1,2) & V_{pd}(1,3) & V_{pd}(1,4) \\ \hline V_{pd}(2,1) & V_{pd}(2,2) & V_{pd}(2,3) & V_{pd}(2,4) \end{array}\right]$$

Partial (linear) velocity

$$\left[\begin{array}{c|c|c|c} F^1_{x^0} & F^1_{y^0} & F^2_{x^0} & F^2_{y^0} \\ = F(1) & = F(2) & = F(3) & = F(4) \end{array}\right]$$

Active force at CM

$$\left[\begin{array}{c|c|c|c} F^1_{in\,x^0} & F^1_{in\,y^0} & F^2_{in\,x^0} & F^2_{in\,y^0} \\ = F_{in}(1) & = F_{in}(2) & = F_{in}(3) & = F_{in}(4) \\ = -m_1 a^1_{x^0} & = -m_1 a^1_{y^0} & = -m_2 a^2_{x^0} & = -m_2 a^2_{y^0} \end{array}\right]$$

Inertia force at CM

$$\left[\begin{array}{c|c} W_{pd}(1,1) & W_{pd}(1,2) \\ \hline W_{pd}(2,1) & W_{pd}(2,2) \end{array}\right]$$

Partial angular velocity

$$\left[\begin{array}{c|c} M^1 = M(1) & M^2 = M(2) \end{array}\right]$$

Active moment

$$\left[\begin{array}{c|c} M^1_{in} = M_{in}(1) & M^2_{in} = M_{in}(2) \\ = -I_1 \alpha_1 & = -I_2 \alpha_2 \end{array}\right]$$

Inertia moment about CM

$$V_{pd}(1,1)\{F(1)+F_{in}(1)\}+V_{pd}(1,2)\{F(2)+F_{in}(2)\}$$
$$+V_{pd}(1,3)\{F(3)+F_{in}(3)\}+V_{pd}(1,4)\{F(4)+F_{in}(4)\}$$
$$+W_{pd}(1,1)\{M(1)+M_{in}(1)\}$$
$$+W_{pd}(1,2)\{M(2)+M_{in}(2)\}=0 \longrightarrow \text{Eqn }(1)$$

$$V_{pd}(2,1)\{F(1)+F_{in}(1)\}+V_{pd}(2,2)\{F(2)+F_{in}(2)\}$$
$$+V_{pd}(2,3)\{F(3)+F_{in}(3)\}+V_{pd}(2,4)\{F(4)+F_{in}(4)\}$$
$$+W_{pd}(2,1)\{M(1)+M_{in}(1)\}$$
$$+W_{pd}(2,2)\{M(2)+M_{in}(2)\}=0 \text{ —— Eqn }(2)$$

Inputs

r^1_{x1} r^1_{y1} r^2_{x2} r^2_{y2}

P^1_{x1} P^1_{y1}

F^1_{x0}, F^1_{y0}, F^2_{x0}, F^2_{y0}

M_1, M_2

m_1, m_2, I_1, I_2

Eqn generated with $\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$, $\begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}$ & $\begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix}$

Initial condition

$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$ & $\begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}$ at $t=0$

$$[V_{pd}]\begin{Bmatrix} F^1_{x0} \\ F^1_{y0} \\ F^2_{x0} \\ F^2_{y0} \end{Bmatrix} + [W_{pd}]\begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix}$$

$$+ [V_{pd}]\begin{Bmatrix} F^1_{inx0} \\ F^1_{iny0} \\ F^2_{inx0} \\ F^2_{iny0} \end{Bmatrix} + [W_{pd}]\begin{Bmatrix} M^1_{in} \\ M^2_{in} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\underbrace{\{f_{ext}\} - [V_{pd}][A_1]^T\begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}}_{\{f\}} -$$

$$\underbrace{\left([V_{pd}][A_2]^T + [W_{pd}][A_3]^T\right)}_{[M]}\begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$[M]\begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} = \left\{ f(q_1, q_2, \dot{q}_1, \dot{q}_2, t) \right\}$$

$$\{f_{ext}\} = [V_{pd}] \begin{Bmatrix} F^1_{x0} \\ F^1_{y0} \\ F^2_{x0} \\ F^2_{y0} \end{Bmatrix} + [W_{pd}] \begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix}$$

$$\{f\} = \{f_{ext}\} - [V_{pd}][A_1]^T \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}$$

$$[M] = \left([V_{pd}][A_2]^T + [W_{pd}][A_3]^T \right)$$

Ref :- Farid Amirouche

Fundamentals of multibody dynamics

– theory & applications