Tutorial 5

Generating Functions

Resources

1. Generating Functionology
2. Discrete Mathematics and Its Applications

Basic Questions

1. Find the number of solutions of

$$e_1 + e_2 + e_3 = 17$$

   where $2 \leq e_1 \leq 5$, $3 \leq e_2 \leq 6$ and $4 \leq e_3 \leq 7$.
2. Find the number of ways to select r objects of n different kinds if we must select at least one kind object of each kind.
3. Solve $a_k = 4a_{k-1} - 4a_{k-2} + k^2$, with $a_0 = 2$ and $a_1 = 5$.

Answers

1. $G(x) = (x^2 + \ldots + x^5)(x^3 + \ldots + x^6)(x^4 + \ldots + x^7)$
2. $G(x) = (x + x^2 + \ldots)^n = \dfrac{x^n}{(1-x)^n}$
3. $a_k = k^2 + 8k + (6k - 18)2^k$

Solution 3

$$\begin{aligned}
\text{Let } G(x) &= \sum_{i=0}^{\infty} a_i x^i, \\
xG(x) &= \sum_{i=0}^{\infty} a_i x^{i+1} = \sum_{i=1}^{\infty} a_{i-1} x^i, \\
x^2 G(x) &= \sum_{i=0}^{\infty} a_k x^{i+2} = \sum_{i=2}^{\infty} a_{k-2} x^i. \\
\implies & G(x) - 4xG(x) + 4x^2 G(x) \\
&= a_0 + a_1 x + -4a_0 x + \sum_{k=2}^{\infty} k^2.\, x^k \\
&= 2 - 4x + \frac{2}{(1-x)^3} - \frac{3}{(1-x)^2} + \frac{1}{(1-x)}
\end{aligned}$$

Solution 3

$$\begin{aligned}
\implies & G(x).\,(1 - 4x + 4x^2) \\
&= 2 - 4x + \frac{2}{(1-x)^3} - \frac{3}{(1-x)^2} + \frac{1}{(1-x)}
\end{aligned}$$

Solve the resulting partial fractions and seperate them to get

$$\begin{aligned}
G(x) = \frac{13}{1-x} + \frac{5}{(1-x)^2} + \frac{2}{(1-x)^3} \\
-\frac{24}{1-2x} + \frac{6}{(1-2x)^2}
\end{aligned}$$

Catalan Numbers

Find a recurrence relation for C_n, the number of ways to parenthesize the product of $n+1$ numbers, $x_0, \ldots, x_n$ to specify the order of multiplication.
Example: $x_0.\,(x_1.\,x_2)$, $(x_0.\,x_1).\,x_2$

- **Solution**: $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$, with the initial conditions $C_0 = 1$ and $C_1 = 1$.
- **Generating Function**:

$$f(x) = \frac{1 - \sqrt{1-4x}}{2x}$$

Analogy b/w generating function and actual structure

The generating function is

$$C(x) = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \cdots,$$

where C_n, the nth Catalan number, is the number of ways of bracketing products together, as mentioned above.

Thus

$$\begin{aligned} C(x) &= x + x^2 + 2x^3 + 5x^4 + \cdots \\ &= x + x^2 + [x^3 + x^3] + [x^4 + x^4 + x^4 + x^4 + x^4] + \cdots \\ &= x + (x.x) + ((x.x)x) + (x(x.x)) \\ &\quad + (((x.x)x)x) + ((x(x.x))x) + ((x.x)(x.x)) + (x((x.x)x)) + (x(x(x.x))) + \cdots \end{aligned}$$

where each term coresponds to multiplying x by itself using all the possible bracketings. Observe that, apart from the first term, if we strip off the outer pair of brackets, every term is naturally the product of two smaller terms in the series for $C(x)$:

$$\begin{aligned} C(x) &= x + x.x + (x.x)x + x(x.x) \\ &\quad + ((x.x)x)x + (x(x.x))x + (x.x)(x.x) + x((x.x)x) + x(x(x.x)) + \cdots \end{aligned}$$

and a term like $((x.x)x)x$ is the product of the two terms $((x.x)x)$ and x. In fact, it is easy to see that, apart from the first term, the other terms are exactly $C(x)^2$ (all terms apart from the first are a product of two smaller terms, and any product of two smaller terms will arise as a way of multiplying x by itself a suitable number of times, so will appear in $C(x)$). Thus

$$C(x) = x + C(x)^2,$$

Group Theory

> **Resources**
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> [Contemporary Abstract Algebra - Gallian]