Def^n. Let S be a set of vectors in a vector space V. The subspace spanned by S is defined to be the intersection W of all subspaces of V which contain S. When S is a finite set of vectors, $S = \{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n\}$, we shall simply call W the subspace spanned by the vectors $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n$.

<u>Thm</u>. The subspace spanned by a non-empty subset S of a vector space V is the set of all linear combinations of vectors in S.

Proof: Let W be the subspace spanned by S. Then each linear combination

$$\bar{\alpha} = x_1\bar{\alpha}_1 + x_2\bar{\alpha}_2 + \ldots + x_m\bar{\alpha}_m$$

of vectors $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_m$ in S is clearly in W. Thus W contains the set L of all linear combinations of vectors in S. The set L,

If $\bar{\alpha}, \bar{\beta}$ belong to L then $\bar{\alpha}$ is a linear combination,

$$\bar{\alpha} = x_1 \bar{\alpha}_1 + x_2 \bar{\alpha}_2 + \dots + x_m \bar{\alpha}_m$$

of $\bar{\alpha}_i \in S$, and $\bar{\beta}$ is a linear combination,

$$\bar{\beta} = y_1 \bar{\beta}_1 + y_2 \bar{\beta}_2 + \dots + y_n \bar{\beta}_n$$

of $\bar{\beta}_j \in S$. For each scalar c,

$$c\bar{\alpha} + \bar{\beta} = \sum_{i=1}^{m} (c x_i) \bar{\alpha}_i + \sum_{j=1}^{n} y_j \bar{\beta}_j .$$

Hence, $c\bar{\alpha} + \bar{\beta} \in L$. Thus L is a subspace of V.

Now we showed that L is a subspace of V which contains S, and also that any subspace which contains S contains L. It follows that L is the intersection of all subspaces containing S, i.e., that L is the subspace spanned by the set S.

Def^n: If $S_1, S_2, \ldots, S_k$ are subsets of a vector space V, the set of all sums

$$\alpha_1 + \alpha_2 + \ldots + \alpha_k$$

of vectors $\alpha_i \in S_i$ is called the sum of the subsets $S_1, S_2, \ldots, S_k$ and is denoted by $S_1 + S_2 + \ldots + S_k$

or by $$\sum_{i=1}^{k} S_i .$$

If $W_1, W_2, \ldots, W_k$ are subspaces of V, then the sum

$$W = W_1 + W_2 + \ldots + W_k$$

is also a subspace of V.

W is the subspace spanned by the union of $W_1, W_2, \ldots, W_k$.

Example: Let F be subfield of the field $\mathbb{C}$. Suppose,

$$\bar{\alpha}_1 = (1,2,0,3,0)$$
$$\bar{\alpha}_2 = (0,0,1,4,0)$$
$$\bar{\alpha}_3 = (0,0,0,0,1).$$

We know that $\bar{\alpha}$ is in the subspace W of F^5 spanned by $\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3$ iff $\exists$ $c_1, c_2, c_3 \in F$ s.t.

$$\bar{\alpha} = c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + c_3\bar{\alpha}_3.$$

$\Rightarrow$ W contains of all $\bar{\alpha}$'s of form:

$$\bar{\alpha} = (c_1, 2c_1, c_2, 3c_1 + 4c_2, c_3),$$

$\forall$ $c_1, c_2, c_3 \in F$.

I.e., $\bar{\alpha} = (x_1, x_2, x_3, x_4, x_5)$ where

$$x_2 = 2x_1, \quad x_3 = 3x_1 + 4x_3 \quad \Big\} \text{ can't be arbitrary.}$$

Bases & Dimension (Vector spaces)

Let V be a vector space over F. A subset S of V is said to be linearly dependent if $\exists$ distinct vectors $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n \in S$ & scalars $c_1, c_2, \ldots, c_n \in F$, not all of which are 0, such that

$$c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + \ldots + c_n\bar{\alpha}_n = 0.$$

A set which is not linearly dependent is called linearly independent. If the set S contains only finitely many vectors $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n$, we ~~sometimes~~ say that $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n$ are dependent (or independent) instead of saying S is dependent (or independent).

- Any set which contains a linearly dependent set is linearly dependent.
- Any subset of a linearly independent set is linearly independent.

- A set S of vectors is linearly independent if and only if each finite subset of S is linearly independent, i.e., iff for any distinct vectors $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n$ of S, $c_1\bar{\alpha}_1 + \ldots + c_n\bar{\alpha}_n = 0$ implies each $c_i = 0$.

Def^n: Let V be a vector space. A basis for V is a linearly independent set of vectors in V which spans the space V. The space V is finite-dimensional if it has a finite basis.

Example: F be a field. $S \subseteq F^n$, S consists vectors $\bar{\epsilon}_1, \bar{\epsilon}_2, \ldots, \bar{\epsilon}_n$ defined by

$$\bar{\epsilon}_1 = (1, 0, 0, \ldots, 0)$$
$$\bar{\epsilon}_2 = (0, 1, 0, \ldots, 0)$$
$$\vdots \qquad \vdots$$
$$\bar{\epsilon}_n = (0, 0, 0, \ldots, 1)$$

Let $x_1, x_2, \ldots, x_n$ be scalars in F & put $\bar{\alpha} = x_1\bar{\epsilon}_1 + x_2\bar{\epsilon}_2 + \ldots + x_n\bar{\epsilon}_n$. Then

This shows that $\bar{\epsilon}_1, \ldots, \bar{\epsilon}_n$ span F^n.

$\because$ $\alpha = 0$ iff $x_1 = x_2 = \ldots = x_n = 0$, $\bar{\epsilon}_1, \bar{\epsilon}_2, \ldots, \bar{\epsilon}_n$ are linearly independent.

The set $S = \{\bar{\epsilon}_1, \ldots, \bar{\epsilon}_n\}$ is accordingly a basis for F^n.

$\downarrow$

the standard basis of F^n.

<u>Theorem</u>: Let V be a vector space which is spanned by a finite set of vectors $\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m$. Then any ~~set~~ independent set of vectors in V is finite and contains no more than m elements.

Proof: To prove the theorem it suffices to show that every subset S of V which contains more than m vectors is linearly dependent. S be such a set. In S there are distinct vectors $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n$ where $n > m$. Since $\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m$ span V, there exist scalars $A_{ij} \in F$ s.t.

$$\alpha_i = \sum^{m} A_{ij} \beta_j .$$

For any n scalars $x_1, x_2, \ldots, x_n$ we have

$$x_1\alpha_1 + \cdots + x_n\alpha_n = \sum_{j=1}^{n} x_j \alpha_j$$

$$= \sum_{j=1}^{n} x_j \sum_{i=1}^{m} A_{ij}\beta_i$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{m} (A_{ij}x_j)\beta_j$$

$$= \sum_{i=1}^{m}\left(\sum_{j=1}^{n} A_{ij}x_j\right)\beta_i .$$

Since $n > m$, $AX = 0$ will have non-trivial sol^n's as well, i.e., not all x_i's are zero for

$$\sum_{j=1}^{n} A_{ij}x_j = 0 \quad , \; 1 \le i \le m .$$

Hence $x_1\bar{\alpha}_1 + x_2\bar{\alpha}_2 + \cdots + x_n\bar{\alpha}_n = 0$. This shows that S is a linearly dependent set.

Corollary: If V is a finite-dimensional vector space, then any two bases of V have the same (finite) no. of elements.

Corollary: Let V be a finite-dimensional vector space and let $n = \dim V$. Then

(a) any subset of V which contains more than n vectors is linearly dependent;

(b) no subset of V which contains less than n vectors can span V.

Lemma: Let S be a linearly independent subset of a vector space V. Suppose $\bar{\beta}$ is a vector in V which is not in the subspace spanned by S. Then the set obtained by adjoining $\bar{\beta}$ to S is linearly independent.

<u>Thm</u>. If W is a subspace of a finite-dimensional vector space V, every linearly independent subset of W is a finite and is part of a (finite) basis for W.

Corollary: If W is a proper subspace of a finite-dimensional vector space V, then W is finite-dimensional ~~vector space V,~~ and $\dim W < \dim V$.

Corollary: In a finite-dimensional vector space V every non-empty linearly independent set of vectors is a part of a basis.

Corollary: Let A be an $n \times n$ matrix over a field F, and suppose the row vectors of A form a linearly set of vectors in F^n. Then A is invertible.

<u>Thm</u>. If W_1 & W_2 are finite-dimensional subspaces of a vector space V, then

Coordinates

Coordinates will be defined relative to sequences of vectors rather than the set of vectors.

Def^{n} If V is a finite-dimensional vector space, an ordered basis for V is a finite sequence of vectors which is linearly independent and spans V.

If the sequence $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n$ is an ordered basis for V, then the set $\{\bar{\alpha}_1, \ldots, \bar{\alpha}_n\}$ is a basis for V. The ordered basis is the set, together w/ the specified ordering. W/ slight abuse of notation,

$\mathcal{B} = \{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n\}$ is an ordered basis for V.

Suppose V is a finite-dim. vector space over the field F &

$\mathcal{B} = \{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n\}$ is an ordered basis for

$(x_1, x_2, \ldots, x_n)$ of scalars s.t.

$$\bar{\alpha} = \sum_{i=1}^{n} x_i \bar{\alpha}_i .$$

[If we also had $\bar{\alpha} = \sum_{i=1}^{n} z_i \bar{\alpha}_i$ ~~and~~ along w/ $\bar{\alpha} = \sum_{i=1}^{n} x_i \bar{\alpha}_i$ for ~~n-tuplt~~ distinct n-tupples $(z_1, \ldots, z_n)$ and $(x_1, \ldots, x_n)$ then

$$\bar{\alpha} - \bar{\alpha} = \sum_{i=1}^{n} (x_i - z_i) \bar{\alpha}_i = 0 .$$ But $\mathcal{B}$ is a basis. So, $x_i = z_i \ \forall i \in \{1, \ldots, n\}$.]

x_i is called i^{th} coordinate of $\bar{\alpha}$ relative to the ordered basis $\mathcal{B} = \{\bar{\alpha}_1, \ldots, \bar{\alpha}_n\}$.

~~If $\beta =$~~

Each ordered basis for V determines a one-one correspondence

$\bar{\alpha} \longrightarrow (x_1, \ldots, x_n)$ Let the set of all vectors in V & the set of all n-tupples in F^n.

coordinate $\leftarrow [\alpha]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ \vdots \\ \vdots \end{bmatrix}$

Suppose V is n-dimensional & that $\mathcal{B} = \{\bar{\alpha}_1, \ldots, \bar{\alpha}_n\}$ and $\mathcal{B}' = \{\bar{\alpha}_1', \ldots, \bar{\alpha}_n'\}$ are two ordered bases for V. There are unique scalars P_{ij} s.t.

$$\bar{\alpha}_j' = \sum_{i=1}^{n} P_{ij} \bar{\alpha}_i \quad \text{for } 1 \le j \le n.$$

If $x_1', \ldots, x_n'$ are coordinates of $\bar{\alpha}$ in $\mathcal{B}'$, then

$$\bar{\alpha} = \sum_{j=1}^{n} x_j' \bar{\alpha}_j' = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} P_{ij} x_j' \right) \bar{\alpha}_i .$$

For coordinates $x_1, \ldots, x_n$ of $\bar{\alpha}$ in $\mathcal{B}$,

$$x_i = \sum_{j=1}^{n} P_{ij} x_j' \quad , 1 \le j \le n.$$

$$\left.\begin{aligned} [\alpha]_{\mathcal{B}} &= P\,[\alpha]_{\mathcal{B}'} \\ [\alpha]_{\mathcal{B}} &= P^{-1}[\alpha]_{\mathcal{B}} \end{aligned}\right\} \begin{array}{c} [\alpha]_{\mathcal{B}'} = 0 \\ \text{iff} \\ [\alpha]_{\mathcal{B}} = 0. \end{array}$$

<u>Thm</u>. Let V be an n-dimensional vector space over the field F, & let $\mathcal{B}$ & $\mathcal{B}'$ be two ordered bases of V. Then $\exists$ a unique, necessarily invertible, $n \times n$ matrix P w/

(i) $[\alpha]_{\mathcal{B}} = P[\alpha]_{\mathcal{B}'}$

(ii) $[\alpha]_{\mathcal{B}'} = P^{-1}[\alpha]_{\mathcal{B}}$

$\forall\, \bar{\alpha} \in V$. The columns of P are given by

$$P_j = [\alpha_j]_{\mathcal{B}} \quad , \; j = 1, \ldots, n.$$

<u>Thm</u>. Suppose P is an $n \times n$ invertible matrix over F. Let V be an n-dimensional vector space over F, and let $\mathcal{B}$ be an ordered basis of V. Then $\exists$ a unique ordered basis $\mathcal{B}'$ of V s.t.

(i) $[\alpha]_{\mathcal{B}} = P[\alpha]_{\mathcal{B}'}$,

(ii) $[\alpha]_{\mathcal{B}'} = P^{-1}[\alpha]_{\mathcal{B}}$,

for every $\bar{\alpha} \in V$.