Proof.

$(\Rightarrow)$: Given that $H \subseteq [G, \cdot]$ is a subgroup of the group $[G, \cdot]$.

R.T.P: $\forall h_1, h_2 \in H$, $(h_1 \cdot h_2^{-1}) \in H$.

H forms a group under the composition '$\cdot$' of the group G.

Let $h_1, h_2 \in H$.

Since $h_2 \in H$, so $h_2^{-1} \in H$.

$\therefore$ by the closure property, $(h_1 \cdot h_2^{-1}) \in H$.

$(\Leftarrow)$: Given $\forall h_1, h_2 \in H$, $h_1 \cdot h_2^{-1} \in H$.

R.T.P. H is a subgroup.

ie, RTP: (i) The identity $1_G \in H$,
(ii) $\forall h \in H$, $h^{-1} \in H$, and
(iii) $\forall h_1, h_2 \in H$, $h_1 \cdot h_2 \in H$.

(i) Since $1_G \in G$ is the identity element, $1_G \in H$ because

$h_1 \cdot h_1^{-1} \in H \Rightarrow 1_G \in H$, by choosing $h_2 = h_1$.

(ii) RTP: $\forall h_2 \in H$, $h_2^{-1} \in H$.

Let $h_1 = 1_G \in H$.

Then, $(h_1 \cdot h_2^{-1}) \in H$

$\Rightarrow 1_G \cdot h_2^{-1} \in H$

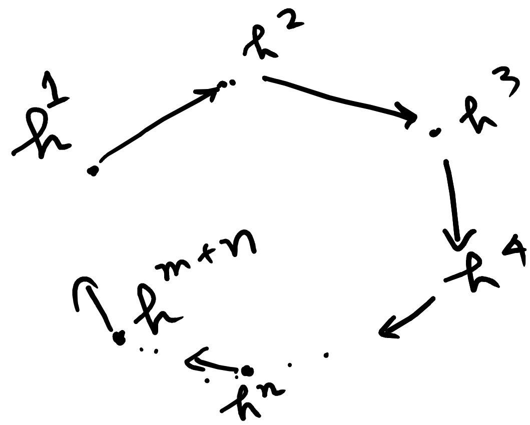
$\Rightarrow h_2^{-1} \in H$.

(iii) Given $\forall h_1, h_2 \in H$, $h_1 \cdot h_2^{-1} \in H$

$\Rightarrow h_2^{-1} \in H$ $\quad$ [$\forall x \in G$, $(x^{-1})^{-1} = x$]

$\therefore h_1 \cdot (h_2^{-1})^{-1} = h_1 \cdot h_2 \in H$

[closure property]



a) Let S_1 and S_2 be two subgroups of a group $\langle G, \cdot \rangle$.

R.T.P. $S_1 \cap S_2$ is a subgroup

ie, R.T.P.: $\forall a, b \in S_1 \cap S_2$, $a \cdot b^{-1} \in S_1 \cap S_2$.

Let $a \in S_1 \cap S_2 \Rightarrow a \in S_1$ and $a \in S_2$

Again, $b \in S_1 \cap S_2 \Rightarrow b \in S_1$ and $b \in S_2$

Now, $a \in S_1$ and $b \in S_1 \Rightarrow a \cdot b^{-1} \in S_1$ since S_1 is a subgroup

$a \in S_2$ and $b \in S_2 \Rightarrow a \cdot b^{-1} \in S_2$ since S_2 is also a subgroup.

$\therefore$ $a \cdot b^{-1} \in S_1$ and $a \cdot b^{-1} \in S_2 \Rightarrow \boxed{a \cdot b^{-1} \in S_1 \cap S_2.}$

(b) The union of two subgroups of a group $\langle G, \cdot \rangle$ is a subgroup if and only if one is contained in the other.

<u>Proof.</u> Let S_1 and S_2 be two subgroups of $\langle G, \cdot \rangle$.

($\Rightarrow$): Given S_1 and S_2 are subgroups and $S_1 \cap S_2$ is also a subgroup.

R.T.P. $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$.

If possible, let $S_1 \nsubseteq S_2$ and $S_2 \nsubseteq S_1$.

Now, $S_1 \nsubseteq S_2 \Rightarrow \exists\, a \in S_1$ and $a \notin S_2$

$S_2 \nsubseteq S_1 \Rightarrow \exists\, b \in S_2$ and $b \notin S_1$.

$\therefore$ $a \in S_1 \cup S_2$ and $b \in S_1 \cup S_2$.

Since $S_1 \cup S_2$ is a subgroup of $\langle G, \cdot \rangle$,

$a \cdot b \in S_1 \cup S_2$ (by closure property)

Let $c = a \cdot b \in S_1 \cup S_2$

$\Rightarrow c = a \cdot b \in S_1$

or

$c = a \cdot b \in S_2$

Let $c = a \cdot b \in S_1$

Then, $a^{-1} \cdot c = (\underline{\underline{a^{-1} \cdot a}}) \cdot b \in S_1$

$\Rightarrow b = a^{-1} \cdot c \in S_1$.

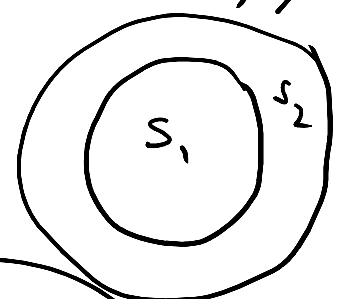
This is a contradiction.

Hence, either $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$.

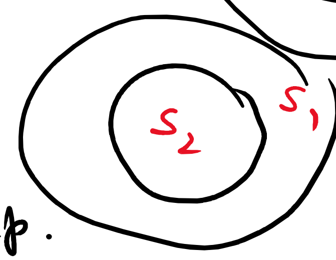
$(\Leftarrow)$: Given $\underline{S_1 \subseteq S_2}$ or $S_2 \subseteq S_1$

R.T.P. $S_1 \cup S_2$ is a subgroup of $\langle G, \cdot \rangle$

Since $S_1 \subseteq S_2$, so $S_2 = S_1 \cup S_2$ is also a subgroup, because S_2 is a subgroup.



Again, if $S_2 \subseteq S_1$, then $S_1 = S_1 \cup S_2$ is a subgroup of $\langle G, \cdot \rangle$ as S_1 is also a subgroup.



$S_1 = S_1 \cup S_2$

[General statement]

The union of two subgroups of a group is NOT a subgroup.

[Counter-example]

1) Prove that $\langle Z, + \rangle$, $\langle 2Z, + \rangle$, $\langle 3Z, + \rangle$ are groups under '$+$', where

$Z = \{ \ldots -3, -2, -1, 0, 1, 2, 3 \ldots \}$ is the set of all integers.

Let $G = \langle \mathbb{Z}, + \rangle$

$$S_1 = 2\mathbb{Z} = \{0, \pm 2, \pm 4, \pm 6, \cdots\}$$

$$S_2 = 3\mathbb{Z} = \{0, \pm 3, \pm 6, \pm 9, \cdots\}$$

$\langle 2\mathbb{Z}, + \rangle$ and $\langle 3\mathbb{Z}, + \rangle$ are subgroups of the group $\langle \mathbb{Z}, + \rangle$.

$$S_1 \cup S_2 = 2\mathbb{Z} \cup 3\mathbb{Z}$$
$$= \{0, \pm 2, \pm 3, \pm 4, \pm 6, \cdots\}$$

But, $S_1 \cup S_2$ does NOT form a subgroup of $\langle \mathbb{Z}, + \rangle$, since

$2, 3 \in S_1 \cup S_2$, then

$2 + 3 = 5 \notin S_1 \cup S_2$.