Discrete Structures Tutorial-1 (Extra Practice Problems solutions)

(SECTION-A)

Question-6

Determine whether these statements are true or false.

a) $\varnothing \in \{\varnothing\}$

b) $\varnothing \in \{\varnothing, \{\varnothing\}\}$

c) $\{\varnothing\} \in \{\varnothing\}$

d) $\{\varnothing\} \in \{\{\varnothing\}\}$

e) $\{\varnothing\} \subset \{\varnothing, \{\varnothing\}\}$

f) $\{\{\varnothing\}\} \subset \{\varnothing, \{\varnothing\}\}$

g) $\{\{\varnothing\}\} \subset \{\{\varnothing\}, \{\varnothing\}\}$

Solution-6

a) True

b) True

c) False

d) True

e) True

f) True

g) True

K.H. Rosen 2.1 Exercise Q10

Question-7

If A, B, C, and D are sets, does it follow that $(A \oplus B) \oplus (C \oplus D) = (A \oplus D) \oplus (B \oplus C)$?

Solution-7

	$(A \oplus B) \oplus (C \oplus D)$	$(A \oplus D) \oplus (B \oplus C)$
$A \cap B \cap C \cap D$	Excluded	Excluded
$A \cap B \cap C \cap D^C$	Included	Included
$A \cap B \cap C^C \cap D$	Included	Included
$A \cap B \cap C^C \cap D^C$	Excluded	Excluded
$A \cap B^C \cap C \cap D$	Included	Included
$A \cap B^C \cap C \cap D^C$	Excluded	Excluded
$A \cap B^C \cap C^C \cap D$	Excluded	Excluded
$A \cap B^C \cap C^C \cap D^C$	Included	Included
$A^C \cap B \cap C \cap D$	Included	Included
$A^C \cap B \cap C \cap D^C$	Excluded	Excluded
$A^C \cap B \cap C^C \cap D$	Excluded	Excluded
$A^C \cap B \cap C^C \cap D^C$	Included	Included
$A^C \cap B^C \cap C \cap D$	Excluded	Excluded
$A^C \cap B^C \cap C \cap D^C$	Included	Included
$A^C \cap B^C \cap C^C \cap D$	Included	Included
$A^C \cap B^C \cap C^C \cap D^C$	Excluded	Excluded

With this table of Inlcusion/Exclusion of partitions, we can conclusively prove that
$$(A \oplus B) \oplus (C \oplus D) = (A \oplus D) \oplus (B \oplus C)$$

Question-8

Suppose that the relation R is irreflexive. Is R^2 necessarily irreflexive? Give a reason for your answer.

Solution-8

It is not necessarily true that R^2 is irreflexive when R is. We might have pairs (a, b) and (b, a) both in R, with $a \neq b$; then it would follow that $(a, a) \in R^2$, preventing R^2 from being irreflexive. As the simplest example, let $A = \{1, 2\}$ and let $R = \{(1, 2), (2, 1)\}$. Then R is clearly irreflexive. In this case $R^2 = \{(1, 1), (2, 2)\}$, which is not irreflexive.

K.H. Rosen 9.1 Exercise Q9

Question-9

Find the error in the "proof" of the following "theorem."

"*Theorem*": Let R be a relation on a set A that is symmetric and transitive. Then R is reflexive.

"*Proof*": Let $a \in A$. Take an element $b \in A$ such that $(a, b) \in R$. Because R is symmetric, we also have $(b, a) \in R$. Now using the transitive property, we can conclude that $(a, a) \in R$ because $(a, b) \in R$ and $(b, a) \in R$.

Solution-9

The second sentence of the proof asks us to "take an element $b \in A$ such that $(a, b) \in R$." There is no guarantee that such an element exists for the taking. This is the only mistake in the proof. If one could be guaranteed that each element in A is related to at least one element, then symmetry and transitivity would indeed imply reflexivity. Without this assumption, however, the proof and the proposition are wrong. As a simple example, take the relation $\emptyset$ on any nonempty set. This relation is vacuously symmetric and transitive, but not reflexive. Here is another counterexample: the relation $\{(1, 1), (1, 2), (2, 1), (2, 2)\}$ on the set $\{1, 2, 3\}$.

K.H. Rosen 9.1 Exercise Q49

Question-10

Show that the relation R on a set A is antisymmetric if and only if $R \cap R^{-1}$ is a subset of the diagonal relation $\Delta = \{(a, a) \mid a \in A\}$.

Solution-10

Let R be a relation on set A, then R is anti-symmetric if and only if for $a, b \in A$, $_a R_b,_b R_a \Rightarrow a = b$
and $R^{-1} = \{(b, a) | a, b \in A, _a R_b\}$
Also note that $R, R^{-1} \subseteq A \times A$
Now if possible, let $\exists x, y \in A$ such that $x \neq y, _x R_y, _y R_x$
Then, $_x R_y \Rightarrow_y R_x^{-1}$ and $_y R_x \Rightarrow_x R_y^{-1}$
$\therefore (x, y), (y, x) \in R \cap R^{-1}$
But it is given that $R \cap R^{-1} \subseteq \Delta$ where $\Delta = \{(a, a) | a \in A\}$
$\therefore (x, y), (y, x) \notin \Delta$ and thus, $(x, y), (y, x) \notin R \cap R^{-1}$
Hence, if $_x R_y, _y R_x$ then $x = y$. And R is anti-symmetric

K.H. Rosen 9.1 Exercise Q52