Linear Transformations

Defⁿ: Let V & W be vector spaces over the field F. A linear transformation from V into W is a fⁿ T from V into W, i.e., $T: V \longrightarrow W$, s.t.

$$T(c\bar{\alpha}+\bar{\beta}) = c(T\bar{\alpha}) + T\bar{\beta}$$

$\forall\ \bar{\alpha}, \bar{\beta} \in V$ and $\forall\ c \in F$.

Example: If V is any vector space, the identity transformation I, defined by $I\bar{\alpha} = \bar{\alpha}$, is a linear transformation from V into V. The zero transformation 0, defined by $0\bar{\alpha} = 0$, is a linear transformation from V into V.

Remark: If T is a linear transformation $T: V \rightarrow W$, then $T(0) = 0$.

$$T(0) = T(0+0) = T(0) + T(0) \Rightarrow T(0) = 0.$$

linear transformation preserves linear combinations; i.e., $T: V \longrightarrow W$

$\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_n \in V, \quad c_1, c_2, \dots, c_n \in F$

$$T(c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + \dots + c_n\bar{\alpha}_n) = c_1T(\bar{\alpha}_1) + c_2T\bar{\alpha}_2 + \dots + c_nT\bar{\alpha}_n$$

<u>Thm</u>. Let V be a finite-dim. vector space over the field F. Let $\{\bar{\alpha}_1, \dots, \bar{\alpha}_n\}$ be an ordered basis for V. Let W be a vector space over the same field F. Let $\{\bar{\beta}_1, \dots, \beta_n\}$ be any vectors in W. Then there is precisely one linear transformation $T: V \longrightarrow W$ s.t. $T\bar{\alpha}_j = \bar{\beta}_j \quad j = 1, \dots, n$.

Proof: First we prove $\exists\ T: V \to W$ w/ $T\bar{\alpha}_j = \bar{\beta}_j$.

Given $\bar{\alpha} \in V$, $\exists$ unique n-tupple $(x_1, \dots, x_n)$ s.t. $\bar{\alpha} = x_1\bar{\alpha}_1 + \dots + x_n\alpha_n$.

For $\bar{\alpha}$, we define

$$T\bar{\alpha} = x_1 \bar{\beta}_1 + \dots + x_n \bar{\beta}_n.$$

Then T is a well-defined rule for associating w/ each vector $\bar{\alpha} \in V$ a vector $T\bar{\alpha} \in W$.

From def^n: $T\bar{\alpha}_j = \bar{\beta}_j$ for each j.

To check if T is linear, let

$$\bar{\beta} = y_1 \bar{\alpha}_1 + \dots + y_n \bar{\alpha}_n \in V, \quad c \in F.$$

Now

$$c\bar{\alpha} + \bar{\beta} = (c x_1 + y_1)\bar{\alpha}_1 + \dots + (c x_n + y_n)\bar{\alpha}_n$$

& so by def^n:

$$T(c\bar{\alpha} + \bar{\beta}) = (c x_1 + y_1)\beta_1 + \dots + (c x_n + y_n)\bar{\beta}_n.$$

On the other hand,

$$c(T\bar{\alpha}) + T\bar{\beta} = c\sum_{i=1}^{n} x_i \bar{\beta}_i + \sum_{i=1}^{n} y_i \beta_i$$

$$= \sum_{i=1}^{n} (c x_i + y_i)\bar{\beta}_i$$

Thus, $T(c\bar{\alpha} + \bar{\beta}) = cT\bar{\alpha} + T\bar{\beta}$.

If U is a linear transformation $T: V \to W$ w/ $U\bar{\alpha}_i = \bar{\beta}_j$, $j = 1, \dots, n$, then for $\bar{\alpha} = \sum_{i=1}^{n} x_i \bar{\alpha}_i$

~~then for~~ we have,

$$U\bar{x} = U\left(\sum_{i=1}^{n} x_i \alpha_i\right)$$

$$= \sum_{i=1}^{n} x_i (U\bar{\alpha}_i)$$

$$= \sum_{i=1}^{n} x_i \bar{\beta}_i ,$$

so that U is exactly the rule T which we defined above. This shows that the linear transformation T w/ $T\bar{\alpha}_j = \bar{\beta}_j$ is unique.

$T: V \to W$ Image of T is a subspace of W

Def^n: Let V & W be vector spaces over the field F & let T be a linear trans. from V into W. The null space of T is the set of all vectors $\bar{\alpha} \in V$ s.t. $T\bar{\alpha} = 0$.

If V is fin.-dimensional, the rank of T is the dimension of the range of T & the nullity of T is the dim. of the null space of T.

<u>Thm.</u> Let V & W be vector spaces over the field F & $T: V \to W$ be a linear transformation. Suppose V is fin. dim. Then,

$$\text{rank}(T) + \text{nullity}(T) = \dim V.$$

<u>Thm</u>. If A is an $m \times n$ matrix w/ entries in the field F, then

$$\text{row rank}(A) = \text{column rank}(A).$$

<u>~~Thm~~</u>

The Algebra of linear transformations

<u>Thm</u>. Let V & W be vector spaces over the field F. Let $T: V \to W$, $U: V \to W$ be linear transformations. The f^n $(T+U)$ defined by

$$(T+U)(\bar{\alpha}) = T(\bar{\alpha}) + U(\bar{\alpha})$$

is a linear transformation $(T+U): V \to W$.

If $c \in F$, the f^n (cT) defined by

$$(cT)(\bar{\alpha}) = c(T\bar{\alpha})$$

is a linear transformation $(cT): V \to W$.

The set of all linear transformations from V into W, together w/ the add^n & scalar multiplication defined above, is a vector space over the field F.

- The space of linear transformations $T: V \to W$ to be denoted as $L(V, W)$.

<u>Thm</u>. Let V be an n-dim. vector space over the field F, and let W be an m-dim vector space over F. Then the space $L(V,W)$ is finite dim. & has dim. mn ($=\dim V \times \dim W$).

Proof. Let $\mathcal{B} = \{\bar{\alpha}_1, \dots, \bar{\alpha}_n\}$ and $\mathcal{B}' = \{\bar{\beta}_1, \dots, \bar{\beta}_m\}$ be ordered bases for V & W, resp. For each pair of integers (p,q) with $1 \le p \le m$ & $1 \le q \le n$, we define a linear transformation $E^{p,q}$ from V into W by

$$E^{p,q}(\bar{\alpha}_i) = \begin{cases} 0, & \text{if } i \neq q \\ \bar{\beta}_p, & \text{if } i = q \end{cases}$$

$$= \delta_{iq} \bar{\beta}_p.$$

A/c to a theorem earlier, $\exists$ a unique linear transformation from $V \to W$ satisfying these cond$\underline{n}$s. The claim is ~~that~~ the mn transformations $E^{p,q}$ form a basis for $L(V,W)$.

Thm. Let V, W, & Z be vector spaces over the field F. Let $T: V \to W$ & $U: W \to Z$ be linear transformations. Then the composed f^n UT defined by $(UT)(\bar{\alpha}) = U(T(\bar{\alpha}))$ is a linear trans.

$$UT : V \longrightarrow Z.$$

Proof:
$$\begin{aligned}(UT)(c\bar{\alpha}+\bar{\beta}) &= U(T(c\bar{\alpha}+\bar{\beta})) \\ &= U(cT\bar{\alpha}+T\bar{\beta}) \\ &= c\,UT\bar{\alpha} + UT\bar{\beta} \\ &= c(UT)(\bar{\alpha}) + (UT)(\bar{\beta}).\end{aligned}$$