Discrete Structures (Monsoon 2021)

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Discrete Numeric Functions and Generating Functions

Numeric Function

Definition

A numeric function a is written as $a_0, a_1, a_2, \cdots, a_r, \cdots$ to denote the values of the function at $0, 1, 2, \cdots, r, \cdots$

Example: $a_r = 7r^3 + 1, r \geq 0$.
Then, $a = (1, 8, 57, 190, 449, 876, 1513, 2402, 3585, 5104, 7001, \cdots)$

Numeric Function

Right Shift

- Let $a = (a_0, a_1, a_2, \cdots, a_r, \cdots)$ be a numeric function and i be a positive integer.
- $S^i.a$ denotes a numeric function such that its value at r is 0 for $r = 0, 1, 2, \cdots, i-1$; and is a_{r-i} for $r \geq i$.
- If $b = S^i.a$, then

$$b_r = \begin{cases} 0, & 0 \leq r \leq i-1 \\ a_{r-i}, & r \geq i \end{cases}$$

- $S =$ shift; $S^i \leftarrow$ right shift

Numeric Function

Left Shift

- Let $a = (a_0, a_1, a_2, \cdots, a_r, \cdots)$ be a numeric function and i be a positive integer.
- $S^{-i}.a$ denotes a numeric function such that its value at r is a_{r+i} for $r \geq 0$.
- If $c = S^{-i}.a$, then

$$c_r = a_{r+i}, r \geq 0.$$

Numeric Function

Forward Difference

- Let $a = (a_0, a_1, a_2, \cdots, a_r, \cdots)$ be a numeric function.
- The forward difference of a is defined as $\triangle a$.
- If $b = \triangle a$, then

$$b_r = a_{r+1} - a_r, r \geq 0.$$

Thus, we have:

$$\begin{aligned} b_0 &= a_1 - a_0 \\ b_1 &= a_2 - a_1 \\ b_2 &= a_3 - a_2 \\ \vdots & \quad \vdots \end{aligned}$$

Numeric Function

Backward Difference

- Let $a = (a_0, a_1, a_2, \cdots, a_r, \cdots)$ be a numeric function.
- The backward difference of a is defined as ∇a.
- If $c = \nabla a$, then

$$c_r = \begin{cases} a_0, & r = 0 \\ a_r - a_{r-1}, & r \geq 1 \end{cases}$$

Thus, we have:

$$\begin{aligned} c_0 &= a_0 \\ c_1 &= a_1 - a_0 \\ c_2 &= a_2 - a_1 \\ \vdots & \quad \vdots \end{aligned}$$

Numeric Function

Problem: Let a be a numeric function such that

$$a_r = \begin{cases} 2, & 0 \leq r \leq 3 \\ 2^{-r} + 5, & r \geq 4 \end{cases}$$

(a) Determine $S^2 a$ and $S^{-2} a$.
(b) Determine $\triangle a$ and ∇a.

Numeric Function

Convolution

Definition

Let a and b be two numeric functions. The *convolution* of a and b, defined by $a * b$, is a numeric function c such that $c = a * b$, where

$$\begin{aligned} c_r &= a_0 b_r + a_1 b_{r-1} + \cdots + a_{r-1} b_1 + a_r b_0 \\ &= \sum_{i=0}^{r} a_i b_{r-i}. \end{aligned}$$

Numeric Function

Problem: Consider the problem of determining c_r, the number of sequences of length r that are made up of the letters $\{x, y, z, \alpha, \beta\}$, with the first portion of each sequence made up of English letters and the second portion made up of Greek letters.

Solution: Let $a_r =$ the number of sequences of length r that are made up from English letters $\{x, y, z\}$;
$b_r =$ the number of sequences of length r that are made up from Greek letters $\{\alpha, \beta\}$.
Then, we have,

$$\begin{aligned} a_r &= 3^r, r \geq 0 \\ b_r &= 2^r, r \geq 0 \end{aligned}$$

Then, for $c = a * b$, we have,

$$c_r = \sum_{i=0}^{r} a_i b_{r-i} = \sum_{i=0}^{r} 3^i 2^{r-i}, r \geq 0.$$

Generating Function

Tests for Convergence

Whether an infinite series is convergent or not, the following tests are available (see `http://home.iitk.ac.in/~psraj/mth101/lecture_notes/Lecture11-13.pdf`):

- Comparison Test
- Cauchy Test
- Ratio Test
- Root Test
- Leibniz Test

Generating Function

Definition

For a numeric function $a = (a_0, a_1, a_2, \cdots, a_r, \cdots)$, define an infinite series

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_r z^r + \cdots$$

which is called generating function (G.F.) of the numeric function a and denoted by

$$A(z) = \sum_{r=0}^{\infty} a_r z^r$$

provided that the series $A(z)$ is convergent, where z is a variable.

Generating Function

Properties

Let a, b, c be the numeric functions.

- If $a_r = z^r$, $r \geq 0$, then $A(z) = \frac{1}{1-z^2}$.
- If $b = \alpha a$, where α is a constant, then $B(z) = \alpha A(z)$.
- If $c = a + b$, then $C(z) = A(z) + B(z)$.
- If a is a numeric function and $A(z)$ is its generating function and $b_r = \alpha^r a_r$ for a numeric function b and α is a constant, then $B(z) = A(\alpha z)$.
- If $b = S^i.a$, then $B(z) = z^i.A(z)$
- If $c = S^{-i}.a$, then

$$C(z) = z^{-i}[A(z) - a_0 - a_1 z - a_2 z^2 - \cdots - a_{i-1} z^{i-1}]$$

Generating Function

Properties

Let a, b, c be the numeric functions.

- If $b = \triangle a$, then

$$B(z) = \frac{1}{z}\left[A(z) - a_0\right] - A(z)$$

- If $c = \nabla a$, then

$$C(z) = (1 - z)A(z)$$

- If $c = a * b$, that is, c is the convolution of a and b, then

$$C(z) = A(z).B(z)$$

Generating Function

Problem: Consider the problem of determining c_r, the number of sequences of length r that are made up of the letters $\{x, y, z, \alpha, \beta\}$, with the first portion of each sequence made up of English letters and the second portion made up of Greek letters.

Solution: Let $a_r =$ the number of sequences of length r that are made up from English letters $\{x, y, z\}$;
$b_r =$ the number of sequences of length r that are made up from Greek letters $\{\alpha, \beta\}$.
Then, we have,

$$\begin{aligned} a_r &= 3^r, r \geq 0 \\ b_r &= 2^r, r \geq 0 \end{aligned}$$

Then, for $c = a * b$, we have,

$$c_r = \sum_{i=0}^{r} a_i b_{r-i} = \sum_{i=0}^{r} 3^i 2^{r-i}, r \geq 0.$$

Generating Function

Now,

$$C(z) = A(z).B(z), \tag{1}$$

where

$$A(z) = \sum_{r=0}^{\infty} a_r z^r = \sum_{r=0}^{\infty} 3^r z^r = \sum_{r=0}^{\infty} (3z)^r = \frac{1}{1-3z} \tag{2}$$

$$B(z) = \sum_{r=0}^{\infty} b_r z^r = \sum_{r=0}^{\infty} 2^r z^r = \sum_{r=0}^{\infty} (2z)^r = \frac{1}{1-2z} \tag{3}$$

Hence,

$$\begin{aligned} C(z) &= A(z).B(z) \\ &= \frac{1}{1-3z}.\frac{1}{1-2z} = \frac{\alpha}{1-2z} + \frac{\beta}{1-3z}, \textit{ say} \end{aligned}$$

Generating Function

Solving, we have, $\alpha = -2$ and $\beta = 3$. Thus,

$$\begin{aligned} C(z) &= \sum_{r=0}^{\infty} c_r z^r \\ &= -\frac{2}{1-2z} + \frac{3}{1-3z} \\ &= \sum_{r=0}^{\infty} [3.3^r - 2.2^r] z^r \\ &= \sum_{r=0}^{\infty} [3^{r+1} - 2^{r+1}] z^r \end{aligned}$$

As a result, we obtain:

$$c_r = 3^{r+1} - 2^{r+1}, \; r \geq 0$$

Generating Function

Problem: Evaluate the sum

$$1^2 + 2^2 + 3^3 + \cdots + n^2$$

using the generating function.

Generating Function

Problem (Simultaneous Recurrence): There are two kinds of particles inside a nuclear reactor. In every second, an α particle will split into three β particles, and a β particle will split into an α particle and two β particles. Assume that there is a single α particle in the reactor at time $t = 0$. Let a_r and b_r denote the number of α particles and β particles at the r-th second in the reactor, respectively.
(i) Construct the simultaneous recurrence relations for a_r and b_r.
(ii) Show that

$$\begin{aligned} a_r &= \frac{3}{4}(3^{r-1} + (-1)^r), r \geq 0, \\ b_r &= \frac{3}{4}(3^r - (-1)^r), r \geq 0. \end{aligned}$$

Solution:

(i) Let a_r and b_r denote the number of α particles and β particles at the r-th second in the reactor, respectively. According to the initial condition, $a_0 = 1$ and $b_0 = 0$.

α particle $\rightarrow$ 0(α particle)&3(β particles)

β particle $\rightarrow$ 1(α particle)&2(β particles)

We have the following simultaneous recurrence relations:

$$\begin{aligned} a_r &= 0.a_{r-1} + 1.b_{r-1} \\ &= b_{r-1} \qquad (4) \\ b_r &= 3.a_{r-1} + 2.b_{r-1}, r \geq 1, \qquad (5) \end{aligned}$$

with the initial condition $a_0 = 1$ and $b_0 = 0$.

Solution:

(ii) From Equation (4), using the generating function both sides, we have,

$\sum_{r=1}^{\infty} a_r z^r = \sum_{r=1}^{\infty} b_{r-1} z^r$

or, $(\sum_{r=0}^{\infty} a_r z^r - a_0) = z.\sum_{r=1}^{\infty} b_{r-1} z^{r-1}$

or, $A(z) - 1 = zB(z)$

$$A(z) = zB(z) + 1. \tag{6}$$

Again, from Equation (5), using the generating function both sides, we have, $\sum_{r=1}^{\infty} b_r z^r = 3\sum_{r=1}^{\infty} a_{r-1} z^r + 2\sum_{r=1}^{\infty} b_{r-1} z^r$

or, $(\sum_{r=0}^{\infty} b_r z^r - b_0) = 3z.\sum_{r=1}^{\infty} a_{r-1} z^{r-1} + 2z\sum_{r=1}^{\infty} b_{r-1} z^{r-1}$

or, $B(z) - 0 = 3zA(z) + 2zB(z)$

$$A(z) = \frac{1-2z}{3z} B(z). \tag{7}$$

Simultaneous Recurrence

Solving Equations (6) and (7), we obtain,

$$B(z) = \frac{3z}{1-2z-3z^2}. \tag{8}$$

$$A(z) = \frac{1-2z}{3z} \times \frac{3z}{(1-3z)(1+z)}. \tag{9}$$

Now, from Equation (8),
$B(z) = \frac{3z}{(1-3z)(1+z)}$
$= \frac{3}{4}\frac{1}{1-3z} - \frac{3}{4}\frac{1}{1+z}$
$= \frac{3}{4}\sum_{r=0}^{\infty} 3^r z^r - \frac{3}{4}\sum_{r=0}^{\infty}(-1)^r z^r.$
Hence, we have, $b_r = \frac{3}{4}3^r - \frac{3}{4}(-1)^r$, that is,

$$b_r = \frac{3}{4}(3^r - (-1)^r), r \geq 0.$$

Simultaneous Recurrence

Similarly, we have,
$A(z) = \frac{1-2z}{3z} \times \frac{3z}{(1-3z)(1+z)}$
$= \frac{1}{4}\frac{1}{1-3z} + \frac{3}{4}\frac{1}{1+z}$
$= \frac{1}{4}\sum_{r=0}^{\infty} 3^r z^r + \frac{3}{4}\sum_{r=0}^{\infty}(-1)^r z^r.$
Thus,
$a_r = \frac{1}{4}3^r + \frac{3}{4}(-1)^r$, that is,

$$a_r = \frac{3}{4}(3^{r-1} + (-1)^r), r \geq 0.$$

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