Group

Problem: For each g in a group $[G, .]$, the set $N_g = \{h | h.g.h^{-1} = g\}$ is called the *normalizer* of g. Show that N_g is a subgroup of G for every g.

Subgroup

Theorem

The left (right) cosets of a group G relative to a subgroup H form a partition of G. Moreover, all of the left or right cosets of G relative to H have equal number of elements.

Subgroup

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Definition (Left coset relation)

Let G be a group with subgroup H. The **left coset relation** on G with respect to H is the relation R with the property that ${}_{g_1}R_{g_2}$ iff $g_1^{-1} \cdot g_2 \in H, \forall g_1, g_2 \in G$.

Definition (Right coset relation)

Let G be a group with subgroup H. The **right coset relation** on G with respect to H is the relation R with the property that ${}_{g_1}R_{g_2}$ iff $g_1 \cdot g_2^{-1} \in H, \forall g_1, g_2 \in G$.

Subgroup

Theorem

The left (right) coset relation is an equivalence relation on a group G, and the equivalence classes are the left (right) cosets of G with respect to a subgroup H of G.

Normal Subgroup

Definition (Normal Subgroup)

A subgroup H of a group G is said to be a **normal subgroup** if the left coset partition induced by H is identical to the right coset partition induced by H.
Equivalently, H is normal if

$$g \cdot H = H \cdot g, \forall g \in G.$$

Theorem

*A subgroup H of a group G is **normal** if and only if*

$$g^{-1} \cdot H \cdot g \subseteq H, \forall g \in G.$$

*In other words, a subgroup H of a group G is **normal** if and only if*

$$g^{-1} \cdot h \cdot g \in H, \forall g \in G \text{ and } h \in H.$$

Quotient group

Theorem

If H is a normal subgroup of a group $\langle G, \cdot \rangle$, then the quotient structure $\langle G/H, \circ \rangle$ is a group, where $\circ$ is the composition of cosets defined by

$$[g] \circ [h] = [g \cdot h]$$

where $[g]$ denotes a left (right) coset of G relative to H and it is defined by $[g] = g \cdot H, \forall g \in G$, with respect to the left coset operation.

The group $\langle G/H, \circ \rangle$ is called the "quotient group" or "factor group" of G relative to the normal subgroup H.