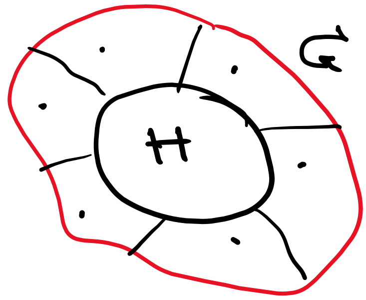
PART-1: Let $\{g\cdot H, \forall g \in G\}$ be the left cosets of G relative to a subgroup H.



R.T.P: $\{g\cdot H, \forall g \in G\}$ forms a partition of G

ie, RTP: (i) $\bigcup_{g \in G} g\cdot H = G$, and

(ii) Distinct left cosets are disjoint

ie, $(g_1\cdot H \cap g_2\cdot H \neq \phi)$

$\Rightarrow g_1\cdot H = g_2\cdot H \quad , \forall g_1, g_2 \in G.$

(i) Since H is a subgroup of G, so the identity e in G is also the identity of H.

$\therefore e \in H$

Now, let $g \in G$. Then, $g = g\cdot e \in g\cdot H$

$\therefore \forall g \in G, \; g \in g\cdot H$

Hence, $\boxed{\bigcup_{g\in G} g\cdot H = G}$

(ii) If two left cosets $g_1\cdot H$ and $g_2\cdot H$ are NOT disjoint, there is some g such that g is in both $g_1\cdot H$ and $g_2\cdot H$

$\therefore \exists \, h_1, h_2 \in H$, s.t.

$$g = g_1\cdot h_1 = g_2\cdot h_2$$

Since H is a subgroup of G, so

$h\cdot H = H = H\cdot h \quad , \forall h \in H$

Now, $g_1\cdot H = g_1\cdot(h_1\cdot H)$

$= (g_1\cdot h_1)\cdot H$

$= (g_2\cdot h_2)\cdot H$

$$= g_2 \cdot (h_2 \cdot H)$$
$$= g_2 \cdot H$$

$\therefore$ The left cosets $g_1 \cdot H$ and $g_2 \cdot H$ must be identical.

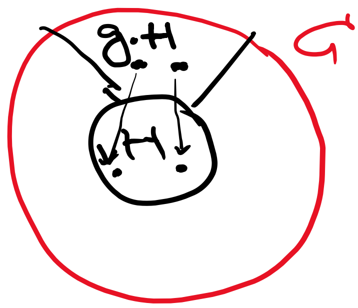
PART 2:

R.T.P. $g \cdot H$ and H are equinumerous

i.e, $|g \cdot H| = |H|$

Consider $g \cdot h_1 = g \cdot h_2$, $h_1, h_2 \in H$

Since G is a group, $g^{-1} \in G$ exists.



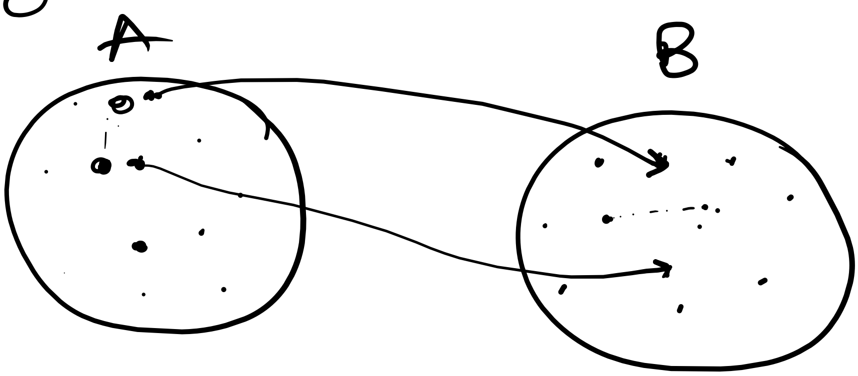
$\therefore$ $g^{-1} \cdot (g \cdot h_1) = g^{-1} \cdot (g \cdot h_2)$

$\Rightarrow (g^{-1} \cdot g) \cdot h_1 = (g^{-1} \cdot g) \cdot h_2$

$\Rightarrow e \cdot h_1 = e \cdot h_2$

$\Rightarrow h_1 = h_2$

$\therefore$ $|g \cdot H| = |H|$.



$|A| = |B|$

Proof:

PART-1. Let R be a left coset relation on G w.r.to H, the subgroup of G.

Then, $g_1 R g_2$ iff $g_1^{-1} \cdot g_2 \in H$, $\forall g_1, g_2 \in G$.

R.T.P.: R is an equivalence relation

ie, (i) R is reflexive

Let $g_2 = g_1$. Then, $g_1^{-1} \cdot g_2 = g_1^{-1} \cdot g_1 = e \in H$,

where e is the identity in G

$\Rightarrow g_1 R g_1$ holds, $\forall g_1 \in G$.

(ii) R is symmetric

$g_1 R g_2 \Rightarrow g_1^{-1} \cdot g_2 \in H$

$\Rightarrow (g_1^{-1} \cdot g_2)^{-1} \in H$, since H is a subgroup of G

$\Rightarrow g_2^{-1} \cdot (g_1^{-1})^{-1} \in H$

$\Rightarrow g_2^{-1} \cdot g_1 \in H$

$\Rightarrow g_2 R g_1$, $\forall g_1, g_2 \in G$

(iii) R is transitive

$g_1 R g_2$ & $g_2 R g_3$

$\Rightarrow g_1^{-1} \cdot g_2 \in H$ and $g_2^{-1} \cdot g_3 \in H$

$\Rightarrow (g_1^{-1} \cdot g_2) \cdot (g_2^{-1} \cdot g_3) \in H$, since H is closed under '$\cdot$'

$\Rightarrow g_1^{-1} \cdot (g_2 \cdot g_2^{-1}) \cdot g_3 \in H$

$\Rightarrow g_1^{-1} \cdot e \cdot g_3 \in H$

$\Rightarrow g_1^{-1} \cdot g_3 \in H$

$\Rightarrow g_1 R g_3$, $\forall g_1, g_2, g_3 \in G$.

PART - 2.

To prove the equivalence classes are the left cosets, $g \cdot H$ of G w.r.to H

The equivalence classes are the left cosets, because if $g_1 \cdot H = g_2 \cdot H$, then

[$h \cdot H = H = H \cdot h$, $\forall h \in H$]

$$H = e \cdot H$$
$$= (g_1^{-1} \cdot g_1) \cdot H$$
$$= g_1^{-1} \cdot (g_1 \cdot H)$$
$$= g_1^{-1} \cdot (g_2 \cdot H)$$
$$= (g_1^{-1} \cdot g_2) \cdot H$$

$\Rightarrow \quad g_1^{-1} \cdot g_2 \in H$

$\Rightarrow \quad g_1 R g_2.$

↳ Conversely, if $g_1^{-1} \cdot g_2 \in H$, then

$$g_1 \cdot (g_1^{-1} \cdot g_2) \in g_1 \cdot H$$
$$\Rightarrow (g_1 \cdot g_1^{-1}) \cdot g_2 \in g_1 \cdot H$$
$$\Rightarrow g_2 \in g_1 \cdot H$$

$$g_2 = g_2 \cdot e \in g_2 \cdot H$$

$\therefore \; g_1 \cdot H = g_2 \cdot H$

(1) R is left coset relation on G w.r.to H if

$$\boxed{g_1 R g_2 \text{ iff } g_1 \cdot \underbrace{H = g_2 \cdot H}_{g_1^{-1} \cdot g_2 \in H},} \quad \forall g_1, g_2 \in G.$$

(2) R is right coset relation on G w.r.to. H if $g_1 R g_2$ iff $H \cdot g_1 = H \cdot g_2$, $\forall g_1, g_2 \in G$

Let H be a normal subgroup of a group G.

Then, $g \cdot H = H \cdot g, \ \forall g \in G$

$\Rightarrow H \cdot g = g \cdot H$

$\Rightarrow g^{-1} \cdot H \cdot g = (g^{-1} \cdot g) \cdot H = e \cdot H = H$

$\Rightarrow \boxed{g^{-1} \cdot H \cdot g \subseteq H}$

R.T.P. $\langle G/H, \circ \rangle$ is group.

R.T.P.

(i) Closure: It is ensured by the relation $[g] \circ [h] = [g \cdot h], \forall g, h \in G$, that defines '$\circ$' of cosets.

The coset $[g]$ is defined by w.r.to. left coset as $[g] = g \cdot H, \ \forall g \in G$.

(ii) Associativity:

Let $[g], [h], [k] \in G/H$.

R.T.P. $([g] \circ [h]) \circ [k] = [g] \circ ([h] \circ [k])$

$$LHS = [g \cdot h] \circ [k]$$

$$= [(g \cdot h) \cdot k]$$

$$= [g \cdot (h \cdot k)]$$

$$= [g] \circ [h \cdot k]$$

$$= [g] \circ ([h] \circ [k]) = RHS$$

(iii) Existence of Identity

Let $e \in G$ be the identity.

Then, $[e] \circ [g] = [e \cdot g] = [g]$

Again, $[g] \circ [e] = [g \cdot e] = [g]$

$\therefore$ $[e]$ is the identity in $\langle G/H, \circ \rangle$

$$\boxed{[e] = e \cdot H = H}$$

(iv) **Existence of Inverse:**

$[e] = [g \cdot g^{-1}] = [g] \circ [g^{-1}]$

$[e] = [g^{-1} \cdot g] = [g^{-1}] \circ [g]$

$\therefore$ $[g^{-1}]$ is the inverse of $[g] \in G/H$.

Hence, $\langle G/H, \circ \rangle$ is a group.