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Discrete Structures (Monsoon 2021)

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Group Theory

Group

Definition

Let $(S, \circ)$ be a structure. An element $x \in S$ is said to be an *idempotent* if $x \circ x = x$.

Theorem

A finite monoid $(M, \circ, e)$ is a group if and only if the identity element $e \in M$ is its only idempotent.

Proof.

$(\Rightarrow)$: Given M is a finite monoid and it is a group.
R.T.P. If $x \circ x = x$, then $x = e$ is the identity in M, for $x \in M$.
Since M is a group, so x^{-1} exists for each $x \in M$.
Now, $x \circ x = x$. Then, $x^{-1} \circ (x \circ x) = x^{-1} \circ x$
$\Rightarrow (x^{-1} \circ x) \circ x = x^{-1} \circ x$
$\Rightarrow e \circ x = e$, since $x^{-1} \circ x = x \circ x^{-1} = e$, the identity in M
$\Rightarrow x = e$. $\square$

Subgroup

Definition

A subgroup of a group G is a subset of the elements of the set G that forms a group under the composition of the group G.

Theorem

Let H be a subgroup of a group G. Then, the identity of H is the same as the identity of G.

Theorem

Let H be a subset of a group G. Then, H forms a subgroup of the group G if and only if $(h_1 . h_2^{-1}) \in H$, for every $h_1, h_2 \in H$.

Subgroup

Theorem

Let $H \subseteq \langle G, \cdot \rangle$ be a finite subset of a group G which is closed under the binary composition '$\cdot$'. Then, H is a subgroup of G.

Proof. Given $H \subseteq \langle G, \cdot \rangle$ is a finite subset of a group G, and $\forall h_1, h_2 \in H, (h_1 \cdot h_2) \in H$.
RTP: H is a subgroup of G, that is,

$$\forall h_1, h_2 \in H, (h_1 \cdot h_2^{-1}) \in H.$$

In other words, it is sufficient to prove that

$$\forall h_2 \in H, h_2^{-1} \in H.$$

Let $h \in H$. Then start generating its positive powers. We have: $h^1, h^2, h^3, \cdots, h^{m+n} = h^m$, for some $n > 0$ as H is a finite subset.

Subgroup

Now,

$$\begin{aligned} h^{m+n} &= h^m \\ \Rightarrow h^m \cdot h^n &= h^m \\ \Rightarrow h^n &= e, \text{identity element in } G \\ \Rightarrow h^{n-1} \cdot h = h \cdot h^{n-1} &= e, \text{ for } n-1 \geq 0. \end{aligned}$$

Note that $h^0 = e$ is the identity in H, since $h^0 \cdot h = h \cdot h^0 = h$. Hence, h^{n-1} is the left as well as right inverse of $h \in H$. Thus, $h^{-1} = h^{n-1}$.
Since $\forall h \in H, h^{-1} \in H$, take $h_2 = h$.
Therefore, $\forall h_1, h_2 \in H, (h_1 \cdot h_2^{-1}) \in H$, since H is closed under $\cdot$. As a result, H is a subgroup of G.

Subgroup

Problem:

- Prove that the intersection of two subgroups of a group G is also a subgroup.
- Discover whether the following statement is true or false:
  "The union of two subgroups of a group is also a subgroup."

Subgroup

Problem:

Prove that a group $\langle G, \cdot \rangle$ is abelian, if and only if $(a \cdot b)^{-1} = a^{-1} \cdot b^{-1}$, for all $a, b \in G$.

Cosets

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Definition (Left Coset)

Let H be a subgroup of a group $\langle G, \cdot \rangle$. The left cosets of G relative of H are defined by

$$g \cdot H = \{g \cdot h | h \in H\}, \forall g \in G.$$

If $\cdot = +$, then

$$g \cdot H = g + H = \{g + h | h \in H\}.$$

Definition (Right Coset)

Let H be a subgroup of a group $\langle G, \cdot \rangle$. The right cosets of G relative of H are defined by

$$H \cdot g = \{h \cdot g | h \in H\}, \forall g \in G.$$

Cosets

Example

Let $\underline{3} = \{1, 2, 3\}$ be a finite set. Considering all $3! = 6$ permutations on $\underline{3}$, define a set $S_3 = \{e, (1\,2), (1\,3), (2\,3), (1\,2\,3), (1\,3\,2)\}$. Then, S_3 forms a group under permutation composition (multiplication). Also, S_3 is called a symmetric group of degree 3. Find the left and right cosets of S_3 relative to a subgroup $H = \{e, (1\,2)\} \subseteq S_3$, where e is the identity permutation defined on $\underline{3}$.

Group

Problem: If H be a subgroup of a group $\langle G, \circ \rangle$ and $h \in H$, then $h \circ H = H \circ h = H$.

Group

Problem: For each g in a group $[G, .]$, the set $N_g = \{h | h.g.h^{-1} = g\}$ is called the *normalizer* of g. Show that N_g is a subgroup of G for every g.

Subgroup

Theorem

The left (right) cosets of a group G relative to a subgroup H form a partition of G. Moreover, all of the left or right cosets of G relative to H have equal number of elements.

Subgroup

Definition (Left coset relation)

Let G be a group with subgroup H. The **left coset relation** on G with respect to H is the relation R with the property that ${}_{g_1}R_{g_2}$ iff $g_1^{-1} \cdot g_2 \in H, \forall g_1, g_2 \in G$.

Definition (Right coset relation)

Let G be a group with subgroup H. The **right coset relation** on G with respect to H is the relation R with the property that ${}_{g_1}R_{g_2}$ iff $g_1 \cdot g_2^{-1} \in H, \forall g_1, g_2 \in G$.

Subgroup

Theorem

The left (right) coset relation is an equivalence relation on a group G, and the equivalence classes are the left (right) cosets of G with respect to a subgroup H of G.

Normal Subgroup

Definition (Normal Subgroup)

A subgroup H of a group G is said to be a **normal subgroup** if the left coset partition induced by H is identical to the right coset partition induced by H.
Equivalently, H is normal if

$$g \cdot H = H \cdot g, \forall g \in G.$$

Theorem

*A subgroup H of a group G is **normal** if and only if*

$$g^{-1} \cdot H \cdot g \subseteq H, \forall g \in G.$$

*In other words, a subgroup H of a group G is **normal** if and only if*

$$g^{-1} \cdot h \cdot g \in H, \forall g \in G \, and \, h \in H.$$

Quotient group

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Theorem

If H is a normal subgroup of a group $\langle G, \cdot \rangle$, then the quotient structure $\langle G/H, \circ \rangle$ is a group, where $\circ$ is the composition of cosets defined by

$$[g] \circ [h] = [g \cdot h]$$

where $[g]$ denotes a left (right) coset of G relative to H and it is defined by $[g] = g \cdot H, \forall g \in G$, with respect to the left coset operation.

The group $\langle G/H, \circ \rangle$ is called the “quotient group” or “factor group” of G relative to the normal subgroup H.

Homomorphism

Definition (Homomorphism of semigroups)

Let $[S, \cdot]$ and $[T, *]$ be two semigroups. A mapping (function) $\theta : [S, \cdot] \rightarrow [T, *]$ is called a morphism (or homomorphism) of two semigroups $[S, \cdot]$ and $[T, *]$, if $\forall s_1, s_2 \in S$, $\theta(s_1 \cdot s_2) = \theta(s_1) * \theta(s_2)$.

Definition (Homomorphism of monoids)

Let $[S, \cdot, e_S]$ and $[T, *, e_T]$ be two monoids. A mapping (function) $\theta : [S, \cdot, e_S] \rightarrow [T, *, e_T]$ is called a morphism (or homomorphism), if the following conditions are met:

- (i) $\forall s_1, s_2 \in S$, $\theta(s_1 \cdot s_2) = \theta(s_1) * \theta(s_2)$.
- (ii) $\theta(e_S) = e_T$, where e_S and e_T denote the identity elements in the monoids $[S, \cdot, e_S]$ and $[T, *, e_T]$, respectively.

Homomorphism

Definition (Homomorphism of groups)

Let $[G, \cdot]$ and $[G', *]$ be two groups. A mapping (function) $\mu : [G, \cdot] \to [G', *]$ is called a morphism (or homomorphism), if the following conditions are met:

- (i) $\forall g, g' \in G$, $\mu(g \cdot g') = \mu(g) * \mu(g')$.
- (ii) $\mu(e_G) = e_{G'}$, where e_G and $e_{G'}$ denote the identity elements in the groups $[G, \cdot]$ and $[G', *]$, respectively.
- (iii) $[\mu(g)]^{-1} = \mu(g^{-1})$, $\forall g \in G$.

Homomorphism

Definition

Let g be a homomorphism from a structure $[X, \cdot]$ to another structure $[Y, *]$.

- If $g : X \to Y$ is onto (surjective), then g is called an **epimorphism**.
- If $g : X \to Y$ is one-one (injective), then g is called an **monomorphism**.
- If $g : X \to Y$ is one-one (injective) and onto (surjective) (that is, g is bijective), then g is called an **isomorphism**.
- If $g : X \to Y$ is called an **automorphism**, if $X = Y$ and g is a bijection.

Homomorphism

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Theorem

Let $[G, \cdot]$ and $[G', *]$ be two groups. A mapping (function) $\mu : [G, \cdot] \rightarrow [G', *]$ is called a morphism (or homomorphism) of the groups $[G, \cdot]$ and $[G', *]$ if and only if

$$\mu(g \cdot g') = \mu(g) * \mu(g'), \forall g, g' \in G.$$

Homomorphism

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Example

Let G be the group of non-zero real numbers under the multiplication operation. Determine whether the following functions are morphisms or not:

- (i) $\phi : G \to G$, where $\phi(x) = x^2$, for all $x \in G$.
- (ii) $\psi : G \to G$, where $\psi(x) = 2^x$, for all $x \in G$.

Homomorphism

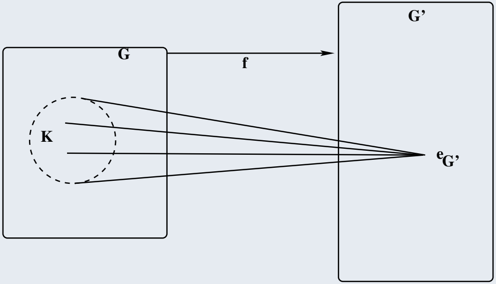
Theorem

Let H be a normal subgroup of G. Then, the mapping $f : G \to G/H$, $f(g) = [g]$, is a group epimorphism. Here, $[g]$ denotes a left (right) coset of G relative to H and it is defined by $[g] = g \cdot H, \forall g \in G$, with respect to the left coset operation.

Kernal of group homomorphism

Definition

The **kernal** of a group homomorphism is the set of domain elements that is mapped onto the identity element in the range.



If $f : G \to G'$ be a group homomorphism and $K \subseteq G$ is the kernal of f, then $f(K) = \{e'_G\}$, where G and G' are groups and $e_{G'}$ is the identity in G'. In other words, $f(x) = e_{G'}, \forall x \in K$.

Kernal of group homomorphism

Theorem (Fundamental theorem of group homomorphism)

*Let $f : G \to G'$ be any group homomorphism, where G and G' be two groups. Then, the kernal of the homomorphism f is a **normal subgroup** of G.*

Order of a group

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Theorem (Lagrange's theorem)

The order of a finite group G is divided by the order of its subgroup H.

Proof. Let G be a finite group of order n and $H \subseteq G$ be its subgroup of order m.
Then, $|G| = n$ and $|H| = m$.
RTP: $m|n$, that is, $n = mk$ for some positive integer k.
Let $H = \{h_1, h_2, \ldots, h_m\} \subseteq G$ be a subgroup of G. Then,

$$a \cdot H = \{a \cdot h_1, a \cdot h_2, \ldots, a \cdot h_m\}, a \in G$$

contains m elements and these elements are distinct, since

$$a \cdot h_i = a \cdot h_j \Rightarrow h_i = h_j,$$

by the left cancellation law in G.

Order of a group

$$a \cdot h_i = a \cdot h_j \Rightarrow (a^{-1} \cdot a) \cdot h_i = (a^{-1} \cdot a) \cdot h_j \Rightarrow e \cdot h_i = e \cdot h_j \Rightarrow h_i = h_j,$$

where $e \in G$ as well as $e \in H$ is the identity.
Now, G is a finite group. Therefore, the number of distinct left (right) cosets is also finite. Let the number of distinct left cosets be k, that is, $a_1 \cdot H, a_2 \cdot H, \cdots a_k \cdot H$ so that the number of elements of the k cosets is km, and this is the total number of elements of G. Since the disjoint left (right) cosets of G form a partition of G, so

$$G = (a_1 \cdot H) \cup (a_2 \cdot H) \cup \cdots \cup (a_k \cdot H).$$

Therefore,

$$|G| = |a_1 \cdot H| + |a_2 \cdot H| + \cdots + |a_k \cdot H|$$

and $n = km$. This proves that the order of H, i.e., m, is a divisor of n, which is the order of G.

Order of a group

Example

Let $G = S_3$ be a symmetric group of order 3 on the set $\underline{3} = \{1, 2, 3\}$, which contains $3! = 6$ permutations, and $H = \{e, (1\,2)\} \subseteq S_3$ is subgroup order 2.
Thus, $|G| = 6$ and $|H| = 2$. Hence, $2|6$.

Order of a group

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Corollary

The index k of a subgroup H of a finite group G is a divisor of the order of G.

Proof. Since $n = mk$, where $|G| = n$ and $|H| = m$, so $k|n$.
Note: The index of H under G, $[G : H] = k$ is the number of distinct left (right) cosets of G relative to H.

Order of a group

Corollary

The order of every element of a finite group G is a divisor of the order of the group G.

Proof. Let $a \in G$ and order of a in G is $Ord_G(a) = m$.
Then, m is the least positive integer such that $a^m = e$, the identity in G.
Therefore,

$$a^1, a^2, a^3, \cdots, a^{m-1}, a^m = e$$

are all distinct elements in G.
Now, construct a subset $H = \{a^1, a^2, a^3, \cdots, a^{m-1}, a^m = e\}$.
We see that $|H| = m$ and it is a subgroup of G. Since the order of H divides the order of G, so $n = mk$, for some positive integer k, $|G| = n$.
Thus, the order of $a \in G$ divides the order of the group G.

Order of a group

Corollary

If G be a finite group of order n and $a \in G$, then $a^n = e$, where $e \in G$ is the identity element in G.

Proof. Given $|G| = n$.
If the order of an element a in G is $Ord_G(a) = m$, then $m|n$, that is, $n = mk$ for some positive integer k.
Since $Ord_G(a) = m$, so $a^m = e$.
Now,

$$\begin{aligned} a^n &= a^{mk} \\ &= (a^m)^k \\ &= e^k \\ &= e. \end{aligned}$$