$$H = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}_{3\times 7}$$

$$x = \langle \underset{x_1}{1}\ \underset{x_2}{1}\ \underset{x_3}{0}\ \underset{x_4}{1} \rangle$$

Why its null space $N(H)$ is a single error correcting code.

$N(H)$ is a single error correcting code if H generates a code of minimum weight at least 3.

H has minimum weight at least 3, since it has the following properties:

(i) No column is all 0's;

(ii) No two columns are identical; and

(iii) $\exists$ three columns, say

$$c_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } c_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

such that

$$c_1 \oplus c_2 \oplus c_3 = \begin{pmatrix} 1 \oplus 0 + 1 \\ 0 \oplus 1 \oplus 1 \\ 0 \oplus 0 \oplus 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0$$

$\therefore$ $N(H)$ is a single-error correcting code.

$$H \sim \begin{array}{c} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \\ \begin{matrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 \end{matrix} \end{array}$$

$\sim$ $c_1 \leftrightarrow c_5$, $c_2 \leftrightarrow c_6$, $c_4 \leftrightarrow c_7$

$$\left(\begin{array}{cccc:ccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$\underbrace{}_{P_{3 \times 4}}$ $\underbrace{}_{I_3}$

✓ $r \times (n-r)$ $\underbrace{}_{I_3\ (r \times r)}$