MA 6.101
Probability and Statistics

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Conditioning with random variables

Today's class

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- Conditioning X on an event $A \in \mathcal{F}$.

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- What is $E[X], E[Y]$ and $E[XY]$?

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- Given/If $\bar{\omega} \in [0, 0.5]$ what is the probability that $\bar{\omega} \in [0, 0.25]$?
- The conditional probability of event B given event A is defined as $\mathbb{P}(B/A) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$ when $\mathbb{P}(A) > 0$.

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- How do we know that it is consistent, i.e., $\sum_x p_{X|A}(x) = 1$?

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- From theorem of total probability, this implies that $\{X = x\} \cap A$ are disjoint sets for all x.
- $\sum_x p_{X|A}(x) = \frac{\mathbb{P}(\bigcup_x \{\{X=x\}\cap A)\}}{\mathbb{P}(A)} = \frac{\mathbb{P}(A)}{\mathbb{P}(A)} = 1.$ $\square$

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- Given that event A has happened, what is the average value of the dice, i.e., $E[X|A]$?

$$E[X/A] = \sum_x x p_{X|A}(x).$$

Using LOTUS,

$$E[g(X)/A] = \sum_x g(x) p_{X|A}(x).$$

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$$E[X] = \sum_{i=1}^{n} \mathbb{P}(A_i) E[X|A_i]$$

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- Otherwise (when $x \in A$,), we have $p_{X|A}(x) = \frac{p_X(x)}{\mathbb{P}(X \in A)}$.
- Running example: Suppose we are given $X \in A$ where $A = \{2, 3\}$. What is $p_{X|A}(x)$?

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- What is $p_{N|A}(k)$?
- For $k > n$, $p_{N|A}(k) = \frac{P\{(N>n) \cap N=k\}}{P(N>n)} = (1-p)^{k-1-n}p$. For $k \leq n$, we have $p_{N|A}(k) = 0$.

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- If N denotes number of tosses till you first get a head, and having already tossed more than n times, the probability of having to toss more than $n + m$ is same as starting the experiment (forgetting that you have already tossed more than n times) fresh and having to toss more than m times.

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- How much you have tossed till now has no bearing on how much you will be required to toss.

$$\boxed{P(N > n + m | N > n) = P(N > m) \text{ (Memoryless property).}}$$

HW: Find $E[N|A]$ where event $A = \{N > n\}$ and $n > 0$.

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- Law of iterated expectation $E[X|Y]$

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- Conditional expectation $E[X|Y = y]$.
- Law of iterated expectation $E[X|Y]$
- Bayes rule revisited

Today's class

- Conditioning X on an event $A \in \mathcal{F}$.
- Conditional Expectation $E[X|A]$.
- Conditioning X with disjoint partitions $\{A_i\}$ of Ω.
- Conditioning X on an event $\{X \in A\} \in \mathcal{F}'$
- Conditioning X on another random variable Y.
- Conditional expectation $E[X|Y = y]$.
- Law of iterated expectation $E[X|Y]$
- Bayes rule revisited
- Sums of random variables.

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- This is essentially same as $P(A \cap B) = P(A|B)P(B)$
- Is $p_{X|Y}(x|y)$ consistent?
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- Notice similarity to the law of total probabiltiy. $P(A) = \sum_i P(A|B_i)P(B_i)$.

Conditional expectation $E[X|Y = y]$

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Proof:

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How about all this for continuous X & Y?

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- $E[X|Y]$ is a function of Y, say $g(Y)$.
- When Y takes the value y,(this happens with probability $p_Y(y)$)$E[X|Y]$ takea the value $E[X|Y=y]$.
- What is the expectation of $E[X|Y]$?

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- $E[g(Y)] = \sum_y g(y)p_Y(y) = \sum_y E[X|Y = y]p_Y(y)$.
- This implies $E[g(Y)] = E[E[X|Y]] = E[X]$. This is the law of iterated expectation.

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$$E[E[X|Y]] = E[X]$$

Conditional expectation $E[X|Y]$ – Example

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- Consider $Y = \begin{cases} \lambda_1 \text{ with prob } p \\ \lambda_2 \text{ with prob } 1-p \end{cases}$.

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- Consider $Y = \begin{cases} \lambda_1 & \text{with prob } p \\ \lambda_2 & \text{with prob } 1-p \end{cases}$.
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- We have $X \sim Exp(\lambda_1)$ with probability p when $Y = \lambda_1$.

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- Consider $Y = \begin{cases} \lambda_1 \text{ with prob } p \\ \lambda_2 \text{ with prob } 1-p \end{cases}$.
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- $E[X] = E[E[X|Y]] \sum_y E[X|Y = y] p_Y(y)$
- We have $X \sim Exp(\lambda_1)$ with probability p when $Y = \lambda_1$.
- Similarly $X \sim Exp(\lambda_2)$ with probability $1-p$ when $Y = \lambda_2$.

Conditional expectation $E[X|Y]$ – Example

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For continuous random variables X and Y

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HW: What if X and Y are not independent?

MGF of Sums of independent random variable

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