Def^n: If V is a vector space over the field F, a linear operator on V is a linear tranformation from V to V.

$L(V,V)$ has a 'multiplication' defined on it by composition. Suppose $T: V \to V$ and $U: V \to V$ for $T, U \in L(V,V)$ are distinct, UT & TU are well-defined. However, in general $UT \neq TU$. $\underbrace{T \circ T \circ \dots \circ T}_{n \text{ times}} = T^n$.

For $T \neq 0$, we define $T^0 = \mathbb{1}$.

Lemma: Let V be a vector space over the field F; let $U, T_1, T_2 \in L(V,V)$; let $c \in F$.

(a) $\mathbb{1}U = U\mathbb{1} = U$;

(b) $U(T_1 + T_2) = UT_1 + UT_2$;

$(T_1 + T_2)U = T_1U + T_2U$;

(c) $c(UT_1) = (cU)T_1 = U(cT_1)$.

Proof: (a) Obvious.

(b) $[U(T_1+T_2)](\bar{\alpha}) = U[(T_1+T_2)(\bar{\alpha})]$

$$= U(T_1\bar{\alpha} + T_2\bar{\alpha})$$

$$= U(T_1\bar{\alpha}) + U(T_2\bar{\alpha})$$

$$= (UT_1)(\bar{\alpha}) + (UT_2)(\bar{\alpha})$$

so that $U(T_1+T_2) = UT_1 + UT_2$.

$$[(T_1+T_2)U](\bar{\alpha}) = (T_1+T_2)(U\bar{\alpha})$$

$$= T_1(U\bar{\alpha}) + T_2(U\bar{\alpha})$$

$$= (T_1U)(\bar{\alpha}) + (T_2U)(\bar{\alpha})$$

so that $(T_1+T_2)U = T_1U + T_2U$.

Note that the proofs of these two distributive laws do not use the fact that T_1 & T_2 are linear, and the proof of the 2nd one does not use the fact that U is linear either.

(c) exercise.

For which linear operators $T: V \to V$ does there exist a linear operator T^{-1} s.t. $TT^{-1} = T^{-1}T = \mathbb{1}$?

f^n $T: V \to W$ is called invertible if $\exists$ a f^n $U: W \to V$ s.t. UT is the identity f^n on V & TU is the identity f^n on W. If T is invertible, the f^n U is unique & is denoted by T^{-1}.

Furthermore, T is invertible iff

1. T is 1:1, i.e., $T\bar{\alpha} = T\bar{\beta} \Rightarrow \bar{\alpha} = \bar{\beta}$.
2. T is onto, i.e., range$(T) = W$.

<u>Thm.</u> Let V & W be vector spaces over the field F. Let $T: V \to W$ be a linear trans. If T is invertible, then the inverse f^n T^{-1} is a linear transformation from W to V, i.e., $T^{-1}: W \to V$.

Proof: When T is one-one and onto, there is uniquely determined

inverse f^n T^{-1} which maps W onto V s.t. $T^{-1}T$ is the identity f^n on V, and TT^{-1} is the identity f^n on W. We now prove here that if T is a linear f^n that is invertible, the the inverse T^{-1} is also linear.

Let $\bar{\beta}_1, \bar{\beta}_2 \in W$, $c \in F$. We need to show, $T^{-1}(c\bar{\beta}_1 + \bar{\beta}_2) = cT^{-1}\bar{\beta}_1 + T^{-1}\bar{\beta}_2$

Let $\bar{\alpha}_i = T^{-1}\bar{\beta}_i$ $\forall i \in \{1,2\}$, i.e., $\bar{\alpha}_i \in V$ is the unique vector s.t. $T\bar{\alpha}_i = \bar{\beta}_i$.

$\because$ T is linear,

$$T(c\bar{\alpha}_1 + \bar{\alpha}_2) = cT\bar{\alpha}_1 + T\bar{\alpha}_2$$
$$= c\bar{\beta}_1 + \bar{\beta}_2.$$

$\therefore$ $c\bar{\alpha}_1 + \bar{\alpha}_2 \in V$ is the unique vector which is sent by T into $c\bar{\beta}_1 + \bar{\beta}_2$, & so

$$T^{-1}(c\bar{\beta}_1 + \bar{\beta}_2) = c\bar{\alpha}_1 + \bar{\alpha}_2$$
$$= cT^{-1}\bar{\beta}_1 + T^{-1}\bar{\beta}_2$$

& T is linear.

For invertible linear transformations:

$T: V \to W$, $U: W \to Z$,

we have

invertible linear transformation

$UT: V \to Z$ and

$(UT)^{-1} = T^{-1}U^{-1}$. Verification of $(UT)^{-1} = T^{-1}U^{-1}$ requires that $T^{-1}U^{-1}$ is both a left and a right inverse of UT.

If T is linear, then $T(\bar{\alpha} - \bar{\beta}) = T\bar{\alpha} - T\bar{\beta}$. Hence, $T\bar{\alpha} = T\bar{\beta}$ iff $T(\bar{\alpha} - \bar{\beta}) = 0$.

$\to$ verifies that T is 1:1.

A linear transformation T is non-singular if $T\bar{\gamma} = 0$ implies $\bar{\gamma} = 0$, i.e., if the null space of T is $\{0\}$.

T is 1:1 iff T is non-singular.

<u>Thm</u>: Linear tran. $T: V \to W$. T is non-singular iff T carries each linearly independent subset of V onto a linearly indep. subset of W.

A linear transformation may be non-singular w/o being onto and maybe onto w/o being non-singular.

<u>Thm</u>. Let V & W be finite-dim. vector spaces over the field F s.t. $\dim V = \dim W$. If $T: V \to W$ is a linear transf., the following are equiv.

(i) T is invertible.

(ii) T is non-singular.

(iii) T is onto, i.e., $\text{range}(T) = W$.

(iv) If $\{\bar{\alpha}_1, \ldots, \bar{\alpha}_n\}$ is a basis for V, then $\{T\bar{\alpha}_1, \ldots, T\bar{\alpha}_n\}$ is a basis for W.

(v) There is some basis $\{\bar{\alpha}_1, \ldots, \bar{\alpha}_n\}$ for V s.t. $\{T\bar{\alpha}_1, \ldots, T\bar{\alpha}_n\}$ is a basis for W.

The set of invertible linear operators on a space V, w/ the operations of composition, provides an example of a 'group'.

Def^n: A group consists of

1. A set G;
2. A rule (or operation) which associates w/ each pair of elements $x, y \in G$ an element $xy \in G$ in such a way that

   (a) $x(yz) = (xy)z \ \forall x, y, z \in G$ (associativity);

   (b) $\exists$ an element $e \in G$ s.t. $ex = xe = x \ \forall x \in G$;

   (c) to each $x \in G$ there corresponds an element $x^{-1} \in G$ s.t. $xx^{-1} = x^{-1}x = e$.

Isomorphism

V & W be vector spaces over the field F, any <u>one-one</u> linear transformation $T: V \to W$ is called an <u>isomorphic</u> of V <u>onto</u> W. If $\exists$ an <u>isomorphism</u> of V <u>onto</u> W, we say V is isomorphic to W.

Isomorphism is an equivalence rel^{n} on the class of vector spaces.

<u>Thm</u>. Every n-dim. vector space over the field F is isomorphic to the space F^n.

Proof: V be an n-dim. space over the field F. $\mathcal{B} = \{\bar{\alpha}_1, \ldots, \bar{\alpha}_n\}$ be an ordered basis for V. Define a f^{n} $T: V \longrightarrow F^n$ as follows: If $\bar{\alpha} \in V$, let $T\bar{\alpha} = (x_1, \ldots, x_n)$, coordinates of $\bar{\alpha}$ relative to $\mathcal{B}$, i.e.,

$$\bar{\alpha} = x_1\bar{\alpha}_1 + \ldots + x_n\bar{\alpha}_n .$$

<u>T is</u> linear, one-one, and maps V onto F^n.

<u>Isomorphism</u> is dimension preserving.

Representation of Transformations by Matrices.

V be an n-dimensional vector space over the field F and let W be an m-dimensional vector space over F. Let $\mathcal{B} = \{\bar{\alpha}_1, \ldots, \bar{\alpha}_n\}$ be an ordered basis for V and $\mathcal{B}' = \{\bar{\beta}_1, \ldots, \bar{\beta}_m\}$ an ordered basis for W.

If T is any linear transformation, $T: V \to W$, then T is determined by its action on $\bar{\alpha}_j$'s. Each of the n vectors $T\bar{\alpha}_j$ is uniquely expressible as a linear combination

$$T\bar{\alpha}_j = \sum_{i=1}^{n} A_{ij} \bar{\beta}_i \qquad (1)$$

of $\bar{\beta}_i$, scalars $A_{1j}, \ldots, A_{mj}$ being the coordinates of $T\bar{\alpha}_j$ in the ordered basis $\mathcal{B}'$. A/cly, the transformation T is determined by mn scalars A_{ij} via eq^n (1).

The $m \times n$ matrix A defined by $A(i,j) = A_{ij}$ is called the matrix of T relative to the pair of ordered basis $\mathcal{B}$ and $\mathcal{B}'$.

If $\bar{\alpha} = x_1\bar{\alpha}_1 + \dots + x_n\bar{\alpha}_n \in V$,

$$T\bar{\alpha} = T\left(\sum_{j=1}^{n} x_j\bar{\alpha}_j\right) = \sum_{j=1}^{n} x_j\left(T\bar{\alpha}_j\right)$$

$$= \sum_{j=1}^{n} x_j \sum_{i=1}^{m} A_{ij}\bar{\beta}_i = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} A_{ij}x_j\right)\bar{\beta}_i.$$

If X is the coordinate matrix of $\bar{\alpha}$ in the ordered basis $\mathcal{B}$, then the computation above shows AX is the coordinate matrix of the vector $T\bar{\alpha}$ in ordered basis $\mathcal{B}'$, as scalar $\sum_{j=1}^{n} A_{ij}x_j$ is the entry in the ith row of the column matrix AX. Let us also observe that if A is any $m \times n$ matrix over F, then

$$T\left(\sum_{j=1}^{n} x_j \bar{\alpha}_j\right) = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} A_{ij} x_j\right)\bar{\beta}_i$$

defines a linear transformation $T: V \to W$, the matrix of which is A, relative to $\mathcal{B}, \mathcal{B}'$.

<u>Thm.</u> Let V be an n-dim. vector space over the field F & W an m-dim. vector space over F. $\mathcal{B}$ be an ordered basis for V and $\mathcal{B}'$ an ordered basis for W. For each linear transformation $T: V \to W$, $\exists$ an $m \times n$ matrix A w/ entries in F s.t.

$$[T\bar{\alpha}]_{\mathcal{B}'} = A[\bar{\alpha}]_{\mathcal{B}}$$

$\forall\ \bar{\alpha} \in V$. Furthermore, $T \to A$ is a one-one correspondence betn the set of all linear transf. $L(V, W)$ and the set of all $m \times n$ matrices over F.

The matrix A associated w/ T is called the matrix of T relative to the ordered bases $\mathcal{B}, \mathcal{B}'$.

From eq^n ①, the matrix A has colum A_j given by

$$A_j = [T\bar{\alpha}_j]_{\mathcal{B}'} \qquad j \in \{1, \dots, n\}.$$

If U is another linear trans. from V to W and $B = [B_1, \dots, B_n]$ is the matrix of U relative to ordered bases $\mathcal{B}$ and $\mathcal{B}'$ then $cA + B$ is the matrix of $cT + U$ relative to $\mathcal{B}$ and $\mathcal{B}'$.

$$\begin{aligned} cA_j + B_j &= c[T\bar{\alpha}_j]_{\mathcal{B}'} + [U\bar{\alpha}_j]_{\mathcal{B}'} \\ &= [cT\bar{\alpha}_j + U\bar{\alpha}_j]_{\mathcal{B}'} \\ &= [(cT + U)(\bar{\alpha}_j)]_{\mathcal{B}'}. \end{aligned}$$

<u>Thm</u> Let V be an n-dim. vector space over the field F and let W be an m-dim. vector space over F. For each pair of ordered bases $\mathcal{B}, \mathcal{B}'$ for V, W resp., the f^n which assigns to a linear transformation T its matrix relative to $\mathcal{B}, \mathcal{B}'$ is an isomorphism bet^n the space $L(V,W)$ and the space of all $m \times n$ matrices over the field F.

For $T \in L(V,V)$, matrix of T relative to the ordered basis $\mathcal{B}$.

If $T \in L(V,V)$ for n-dim vector space V w/ an ordered basis $\mathcal{B} = \{\bar{\alpha}_1, \dots, \bar{\alpha}_n\}$, then associated matrix A is $n \times n$ with

$$T\bar{\alpha}_j = \sum_{i=1}^{n} A_{ij} \bar{\alpha}_i \quad , \quad j \in \{1, \dots, n\}.$$

$$A = [T]_{\mathcal{B}}.$$

$$[T\bar{\alpha}]_{\mathcal{B}} = [T]_{\mathcal{B}} [\bar{\alpha}]_{\mathcal{B}}.$$

Example: Let F be a field and let T be the operator on F^2 defined by

$$T(x_1, x_2) = 0.$$

T is a linear operator on F^2. Let $\mathcal{B} = \{\epsilon_1, \epsilon_2\}$ be std. basis for F^2.

Now $T\epsilon_1 = T(1,0) = (1,0) = 1\epsilon_1 + 0\epsilon_2$

$T\epsilon_2 = T(0,1) = (0,0) = 0\epsilon_1 + 0\epsilon_2$

so, $[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$

<u>Thm.</u> V, W, Z be fin. dim. vector spaces over F. $T \in L(V,W)$, $U \in L(W,Z)$. $\mathcal{B}, \mathcal{B}', \mathcal{B}''$ be ordered bases for V, W, Z resp. If $A = [T]_{\mathcal{B} \to \mathcal{B}'}$ and $U = [U]_{\mathcal{B}' \to \mathcal{B}''}$, then

$$[UT]_{\mathcal{B} \to \mathcal{B}''} = BA.$$

$UT = TU = \mathbb{1} \iff [U]_{\mathcal{B}} [T]_{\mathcal{B}} = [T]_{\mathcal{B}} [U]_{\mathcal{B}} = \mathbb{1}.$

$\Rightarrow \quad [T^{-1}]_{\mathcal{B}} = [T]_{\mathcal{B}}^{-1}.$