# Homomorphism

## Definition (Homomorphism of semigroups)

Let $[S, \cdot]$ and $[T, *]$ be two semigroups. A mapping (function) $\theta : [S, \cdot] \rightarrow [T, *]$ is called a morphism (or homomorphism) of two semigroups $[S, \cdot]$ and $[T, *]$, if $\forall s_1, s_2 \in S$, $\theta(s_1 \cdot s_2) = \theta(s_1) * \theta(s_2)$.

## Definition (Homomorphism of monoids)

Let $[S, \cdot, e_S]$ and $[T, *, e_T]$ be two monoids. A mapping (function) $\theta : [S, \cdot, e_S] \rightarrow [T, *, e_T]$ is called a morphism (or homomorphism), if the following conditions are met:

- (i) $\forall s_1, s_2 \in S$, $\theta(s_1 \cdot s_2) = \theta(s_1) * \theta(s_2)$.
- (ii) $\theta(e_S) = e_T$, where $e_S$ and $e_T$ denote the identity elements in the monoids $[S, \cdot, e_S]$ and $[T, *, e_T]$, respectively.

### Definition (Homomorphism of groups)

Let $[G, \cdot]$ and $[G', *]$ be two groups. A mapping (function) $\mu : [G, \cdot] \rightarrow [G', *]$ is called a morphism (or homomorphism), if the following conditions are met:

- (i) $\forall g, g' \in G$, $\mu(g \cdot g') = \mu(g) * \mu(g')$.
- (ii) $\mu(e_G) = e_{G'}$, where $e_G$ and $e_{G'}$ denote the identity elements in the groups $[G, \cdot]$ and $[G', *]$, respectively.
- (iii) $[\mu(g)]^{-1} = \mu(g^{-1})$, $\forall g \in G$.

# Homomorphism

## Definition

Let $g$ be a homomorphism from a structure $[X, \cdot]$ to another structure $[Y, *]$.

- If $g : X \to Y$ is onto (surjective), then $g$ is called an **epimorphism**.
- If $g : X \to Y$ is one-one (injective), then $g$ is called an **monomorphism**.
- If $g : X \to Y$ is one-one (injective) and onto (surjective) (that is, $g$ is bijective), then $g$ is called an **isomorphism**.
- If $g : X \to Y$ is called an **automorphism**, if $X = Y$ and $g$ is a bijection.

# Homomorphism

**Theorem**

*Let $[G, \cdot]$ and $[G', *]$ be two groups. A mapping (function) $\mu : [G, \cdot] \rightarrow [G', *]$ is called a morphism (or homomorphism) of the groups $[G, \cdot]$ and $[G', *]$ if and only if*

$$\mu(g \cdot g') = \mu(g) * \mu(g'), \forall g, g' \in G.$$

# Homomorphism

### Example

Let $G$ be the group of non-zero real numbers under the multiplication operation. Determine whether the following functions are morphisms or not:

- (i) $\phi : G \to G$, where $\phi(x) = x^2$, for all $x \in G$.
- (ii) $\psi : G \to G$, where $\psi(x) = 2^x$, for all $x \in G$.

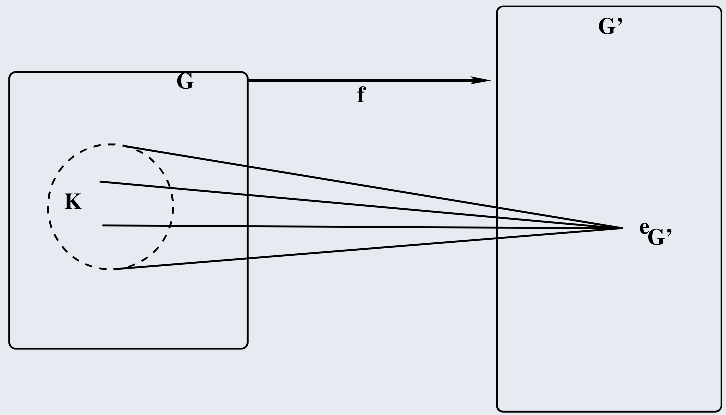
#### Theorem

*Let $H$ be a normal subgroup of $G$. Then, the mapping $f : G \to G/H$, $f(g) = [g]$, is a group epimorphism. Here, $[g]$ denotes a left (right) coset of $G$ relative to $H$ and it is defined by $[g] = g \cdot H, \forall g \in G$, with respect to the left coset operation.*

# Kernal of group homomorphism

## Definition

The **kernal** of a group homomorphism is the set of domain elements that is mapped onto the identity element in the range.




If $f : G \to G'$ be a group homomorphism and $K \subseteq G$ is the kernal of $f$, then $f(K) = \{e'_G\}$, where $G$ and $G'$ are groups and $e_{G'}$ is the identity in $G'$. In other words, $f(x) = e_{G'}, \forall x \in K$.

# Kernal of group homomorphism

**Theorem (Fundamental theorem of group homomorphism)**

*Let $f : G \to G'$ be any group homomorphism, where $G$ and $G'$ be two groups. Then, the kernal of the homomorphism $f$ is a* ***normal subgroup*** *of $G$.*