Problem: The fibonacci's sequence is defined by

$$F_0 = 0$$
$$f_1 = 1$$
$$F_r = F_{r-1} + F_{r-2}, \quad r \geq 2.$$

Using the generating function, find F_r.

$$\sum_{r=2}^{\infty} F_r z^r = \sum_{r=2}^{\infty} F_{r-1} z^r + \sum_{r=2}^{\infty} F_{r-2} z^r$$

$$\Rightarrow \left(\sum_{r=0}^{\infty} F_r z^r - F_0 - f_1 z \right) = z \cdot \sum_{r=2}^{\infty} F_{r-1} z^{r-1} + z^2 \cdot \sum_{r=2}^{\infty} F_{r-2} z^{r-2}$$

$$\Rightarrow F(z) - F_0 - f_1 z = z \left[\sum_{r=1}^{\infty} F_{r-1} z^{r-1} - F_0 \right] + z^2 \cdot F(z)$$

$$\Rightarrow F(z) - 0 - 1.z = z\left[F(z) - 0\right] + z^2 . f(z)$$

$$\Rightarrow (1 - z - z^2) . f(z) = z$$

$$\Rightarrow F(z) = \text{G.f. of } F = (F_0, F_1, F_2, \cdots, F_r, \cdots)$$

$$= \frac{z}{1-z-z^2}$$

$$= \frac{z}{(1-\alpha z)(1-\beta z)}, \text{ say} \qquad z = 0 + 1.z$$

$$= \frac{A}{1-\alpha z} + \frac{B}{1-\beta z}.$$

$$= \frac{(A+B) + (-\beta A - \alpha B) \cdot z}{(1-\alpha z)(1-\beta z)}$$

$$\therefore \left. \begin{array}{l} A + B = 0 \\ -\beta A - \alpha B = 1 \end{array} \right\}$$

Solving:

$$\boxed{\begin{array}{l} A = -\dfrac{1}{\beta - \alpha} \\ B = \dfrac{1}{\beta - \alpha} \end{array}}$$

We have:

$$(1-\alpha z)(1-\beta z) = 1-z-z^2$$

$$\Rightarrow 1-(\alpha+\beta).z + \alpha\beta.z^2 = 1-z-z^2$$

Then, $\left.\begin{array}{l}\alpha+\beta = 1\\ \alpha\beta = -1\end{array}\right\}$ $\alpha-\beta = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$

$$= \sqrt{1+4} = \sqrt{5}$$

$$\left.\begin{array}{l}\alpha+\beta = 1\\ \alpha-\beta = \sqrt{5}\end{array}\right\} \Rightarrow \boxed{\begin{array}{l}\alpha = \dfrac{1+\sqrt{5}}{2}\\ \beta = \dfrac{1-\sqrt{5}}{2}\end{array}}$$

Now, $F(z) = \sum_{r=0}^{\infty} F_r . z^r$

$$= \frac{A}{1-\alpha z} + \frac{B}{1-\beta z}$$

$$= A(1-\alpha z)^{-1} + B(1-\beta z)^{-1}$$

$$= A\sum_{r=0}^{\infty} \alpha^r z^r + B\sum_{r=0}^{\infty} \beta^r z^r$$

$$= \sum_{r=0}^{\infty} (A\alpha^r + B\beta^r) z^r$$

$\Rightarrow F_r$ = coefficient of z^r

$$= A\alpha^r + B\beta^r$$

$$\boxed{\begin{array}{l}A = -\dfrac{1}{\beta-\alpha} = +\dfrac{1}{\sqrt{5}}\\ B = -\dfrac{1}{\sqrt{5}}\end{array}}$$

$$= +\frac{1}{\sqrt{5}}.\left(\frac{1+\sqrt{5}}{2}\right)^r - \frac{1}{\sqrt{5}}.\left(\frac{1-\sqrt{5}}{2}\right)^r$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^r - \left(\frac{1-\sqrt{5}}{2}\right)^r\right], \quad r \geq 0.$$

$\frac{1+\sqrt{5}}{2}$ is known as the "Golden Ratio"!

Problem: Let a, b and c be the numeric functions such that $c = a * b$. Given that

$$a_r = \begin{cases} 1, & r = 0 \\ 2, & r = 1 \\ 0, & r \geq 2 \end{cases} \qquad c_r = \begin{cases} 1, & r = 0 \\ 0, & r \geq 1 \end{cases}$$

determine b.

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We have $c = a * b$

$$\therefore \quad C(z) = A(z) \cdot B(z)$$

$$\Rightarrow \quad B(z) = \frac{C(z)}{A(z)} \quad \text{- - - -} \quad (1)$$

$$\left.\begin{aligned} a &= (a_0, a_1, a_2, \ldots) \\ b &= (b_0, b_1, b_2, \ldots) \\ c &= (c_0, c_1, c_2, \ldots, c_r, \ldots) \end{aligned}\right\}$$

Now, $A(z) = \sum_{r=0}^{\infty} a_r z^r$ = G.F. of a

$$= a_0 + a_1 z + a_2 z^2 + \cdots\cdots$$

$$= 1 + 2z \quad \text{- - - -} \quad (2)$$

$C(z)$ = G.f. of c

$$= \sum_{r=0}^{\infty} c_r z^r$$

$$= c_0 + c_1 z + c_2 z^2 + \cdots$$

$$= 1 \quad \text{- - -} \quad (3)$$

Eqs. (1), (2) and (3) give:

$$B(z) = \frac{C(z)}{A(z)}$$

$$\Rightarrow \sum_{r=0}^{\infty} b_r z^r = \frac{1}{1+2z}$$

$$= (1+2z)^{-1}$$

$$= 1 + \sum_{r=1}^{\infty} (-1)^r \cdot \frac{1.2.3\ldots r}{r!} (2z)^r$$

$$= 1 + \sum_{r=1}^{\infty} (-1)^r \cdot 2^r \cdot z^r$$

$\therefore \quad b_r = (-1)^r \cdot 2^r, \quad r \geq 0$

i.e, $$\boxed{b_r = \begin{cases} 1, & r = 0 \\ 2^r, & r \text{ is even} \\ -2^r, & r \text{ is odd} \end{cases}}$$