$1^2 + 2^2 + 3^2 + \cdots + r^2$ using G.F.

~~Let~~ us consider a numeric function

$a = (0^2, 1^2, 2^2, 3^2, \ldots, r^2, (r+1)^2, (r+2)^2, \ldots)$

and determine the generating function (G.F.) of a.

We know,

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots + z^r + \cdots \quad [\text{Converges if } |z|<1]$$

$$\frac{d}{dz}\left(\frac{1}{1-z}\right) = 0 + 1 + 2z + 3z^2 + \cdots + rz^{r-1} + \cdots$$

$$\frac{1}{(1-z)^2} = 0 + 1 + 2z + 3z^2 + \cdots + rz^{r-1} + \cdots$$

$$\boxed{\frac{z}{(1-z)^2} = 0 \cdot z^0 + 1 \cdot z^1 + 2 \cdot z^2 + 3z^3 + \cdots + rz^r + \cdots}$$

$$\boxed{\frac{d}{dz}\left(\frac{1}{z}\right) = -\frac{1}{z^2}}$$

$$\frac{d}{dz}\left(\frac{z}{(1-z)^2}\right) = 1 + 2^2 z + 3^2 z^2 + \cdots + r^2 z^{r-1} + \cdots$$

$$\frac{1+z}{(1-z)^3} = 1 + 2^2 z + 3^2 z^2 + \cdots + r^2 z^{r-1} + \cdots$$

$$\frac{z(1+z)}{(1-z)^3} = 0^2 \cdot z^0 + 1^2 \cdot z^1 + 2^2 \cdot z^2 + 3^2 \cdot z^3 + \cdots + \cdots + r^2 \cdot z^r + \cdots$$

$\therefore$ G.F. of $a = (0^2, 1^2, 2^2, \ldots, r^2, \ldots)$

$$= \frac{z(1+z)}{(1-z)^3}.$$

$$\left[A(z) = \sum_{r=0}^{\infty} a_r z^r \right]$$

Now, $\frac{z(1+z)}{(1-z)^3}\cdot\frac{1}{(1-z)}$

$$= \left[0^2\cdot z^0 + 1^2\cdot z^1 + 2^2\cdot z^2 + \dots + r^2 z^r + \dots\right] \times \left[1 + z + z^2 + \dots + z^r + \dots\right]$$

$$= 0^2\cdot z^0 + (0^2+1^2)z^1 + (0^2+1^2+2^2)z^2 + \dots + (0^2+1^2+2^2+\dots+r^2)z^r + \dots$$

$\therefore\ 0^2+1^2+2^2+\dots+r^2 =$ coefficient of z^r in $\frac{z(1+z)}{(1-z)^4}$.

Again, $\frac{z(1+z)}{(1-z)^4} = z(1+z)(1-z)^{-4}$

$$= (z+z^2)\cdot\left[1 + (-4)z + \frac{(-4)(-4-1)}{2!}z^2 + \dots + (-1)^r\cdot\frac{(-4)(-4-1)\dots(-4-r-1)}{r!}z^r + \dots\right]$$

$b_r z^r$

$= + \dots \left(z\cdot b_{r-1} z^{r-1} + z^2\cdot b_{r-2} z^{r-2}\right) + \dots$

Now,

$$b_r = (-1)^r\,\frac{(-1)^r\cdot 4\cdot 5\cdot 6\cdots\cdots(r+3)}{r!}$$

$$= (-1)^{2r}\cdot\frac{(1\cdot2\cdot3\cdot\cancel{4\cdot5\cdot6\cdots r})\cdot(r+1)(r+2)(r+3)}{\cancel{r!}\cdot 1\cdot2\cdot3}$$

$$= \frac{1}{6}(r+1)(r+2)(r+3)$$

$\therefore\ a_r = 0^2+1^2+2^2+\dots+r^2$

$=$ coefficient of z^r

$=$ coefficient of z^{r-1} + coefficient of z^{r-2}

$$= \frac{1}{6}(r-1+1)(r-1+2)(r-1+3)\,(b_{r-1}) + \frac{1}{6}(r-2+1)(r-2+r)(r-2+3)\,(b_{r-2})$$

$$= \frac{1}{6}r(r+1)(r+2) + \frac{1}{6}(r-1)(r)(r+1)$$

$$= \frac{1}{6}r(r+1)\left[r+2+r-1\right]$$

$$= \frac{1}{6} r(r+1)(2r+1)$$

Hence, $\boxed{1^2 + 2^2 + \cdots + r^2 = \frac{1}{6} r(r+1)(2r+1)}$.