# Numeric Function

## Convolution

### Definition

Let $a$ and $b$ be two numeric functions. The *convolution* of $a$ and $b$, defined by $a * b$, is a numeric function $c$ such that $c = a * b$, where

$$\begin{aligned} c_r &= a_0 b_r + a_1 b_{r-1} + \cdots + a_{r-1} b_1 + a_r b_0 \\ &= \sum_{i=0}^{r} a_i b_{r-i}. \end{aligned}$$

# Numeric Function

**Problem:** Consider the problem of determining $c_r$, the number of sequences of length $r$ that are made up of the letters $\{x, y, z, \alpha, \beta\}$, with the first portion of each sequence made up of English letters and the second portion made up of Greek letters.

**Solution:** Let $a_r =$ the number of sequences of length $r$ that are made up from English letters $\{x, y, z\}$;
$b_r =$ the number of sequences of length $r$ that are made up from Greek letters $\{\alpha, \beta\}$.
Then, we have,

$$\begin{aligned} a_r &= 3^r, r \geq 0 \\ b_r &= 2^r, r \geq 0 \end{aligned}$$

Then, for $c = a * b$, we have,

$$c_r = \sum_{i=0}^{r} a_i b_{r-i} = \sum_{i=0}^{r} 3^i 2^{r-i}, r \geq 0.$$

# Generating Function

## Tests for Convergence

Whether an infinite series is convergent or not, the following tests are available (see `http://home.iitk.ac.in/~psraj/mth101/lecture_notes/Lecture11-13.pdf`):

- Comparison Test
- Cauchy Test
- Ratio Test
- Root Test
- Leibniz Test

# Generating Function

INTERNATIONAL INSTITUTE OF INFORMATION TECHNOLOGY HYDERABAD

### Definition

For a numeric function $a = (a_0, a_1, a_2, \cdots, a_r, \cdots)$, define an infinite series

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_r z^r + \cdots$$

which is called generating function (G.F.) of the numeric function $a$ and denoted by

$$A(z) = \sum_{r=0}^{\infty} a_r z^r$$

provided that the series $A(z)$ is convergent, where $z$ is a variable.

# Generating Function

## Properties

Let $a, b, c$ be the numeric functions.

- If $a_r = z^r$, $r \geq 0$, then $A(z) = \frac{1}{1-z^2}$.
- If $b = \alpha a$, where $\alpha$ is a constant, then $B(z) = \alpha A(z)$.
- If $c = a + b$, then $C(z) = A(z) + B(z)$.
- If $a$ is a numeric function and $A(z)$ is its generating function and $b_r = \alpha^r a_r$ for a numeric function $b$ and $\alpha$ is a constant, then $B(z) = A(\alpha z)$.
- If $b = S^i.a$, then $B(z) = z^i.A(z)$
- If $c = S^{-i}.a$, then

$$C(z) = z^{-i}[A(z) - a_0 - a_1 z - a_2 z^2 - \cdots - a_{i-1} z^{i-1}]$$

# Generating Function

## Properties

Let $a, b, c$ be the numeric functions.

- If $b = \triangle a$, then

$$B(z) = \frac{1}{z}\Big[A(z) - a_0\Big] - A(z)$$

- If $c = \nabla a$, then

$$C(z) = (1 - z)A(z)$$

- If $c = a * b$, that is, $c$ is the convolution of $a$ and $b$, then

$$C(z) = A(z).B(z)$$

# Generating Function

**Problem:** Consider the problem of determining $c_r$, the number of sequences of length $r$ that are made up of the letters $\{x, y, z, \alpha, \beta\}$, with the first portion of each sequence made up of English letters and the second portion made up of Greek letters.

**Solution:** Let $a_r =$ the number of sequences of length $r$ that are made up from English letters $\{x, y, z\}$;
$b_r =$ the number of sequences of length $r$ that are made up from Greek letters $\{\alpha, \beta\}$.
Then, we have,

$$\begin{aligned} a_r &= 3^r, r \geq 0 \\ b_r &= 2^r, r \geq 0 \end{aligned}$$

Then, for $c = a * b$, we have,

$$c_r = \sum_{i=0}^{r} a_i b_{r-i} = \sum_{i=0}^{r} 3^i 2^{r-i}, r \geq 0.$$

## Generating Function

Now,

$$C(z) = A(z).B(z), \tag{1}$$

where

$$A(z) = \sum_{r=0}^{\infty} a_r z^r = \sum_{r=0}^{\infty} 3^r z^r = \sum_{r=0}^{\infty} (3z)^r = \frac{1}{1-3z} \tag{2}$$

$$B(z) = \sum_{r=0}^{\infty} b_r z^r = \sum_{r=0}^{\infty} 2^r z^r = \sum_{r=0}^{\infty} (2z)^r = \frac{1}{1-2z} \tag{3}$$

Hence,

$$\begin{aligned} C(z) &= A(z).B(z) \\ &= \frac{1}{1-3z}.\frac{1}{1-2z} = \frac{\alpha}{1-2z} + \frac{\beta}{1-3z}, \textit{ say} \end{aligned}$$

# Generating Function

Solving, we have, $\alpha = -2$ and $\beta = 3$. Thus,

$$\begin{aligned} C(z) &= \sum_{r=0}^{\infty} c_r z^r \\ &= -\frac{2}{1-2z} + \frac{3}{1-3z} \\ &= \sum_{r=0}^{\infty} [3.3^r - 2.2^r] z^r \\ &= \sum_{r=0}^{\infty} [3^{r+1} - 2^{r+1}] z^r \end{aligned}$$

As a result, we obtain:

$$c_r = 3^{r+1} - 2^{r+1},\ r \geq 0$$

# Generating Function

**Problem:** Evaluate the sum

$$1^2 + 2^2 + 3^3 + \cdots + n^2$$

using the generating function.