

Lemma: If p is prime and $p|ab$, for $a, b \in \mathbb{Z}$, then either $p|a$ or $p|b$.

Proof. Case 1. Let $p \nmid a$.
RTP: $p|b$.

Since $p \nmid a$, $\gcd(p, a) = 1$.

Lemma: If $g = \gcd(a, b)$, then
 $g = a \cdot x + b \cdot y$ for some integers x and y .

$$\left. \begin{array}{l} a = 2 \\ b = 7 \end{array} \right\} \quad \begin{array}{l} 7 = 3 \times 2 + \textcircled{1} \rightarrow \textcircled{1} \\ 2 = 2 \times 1 + \boxed{0} \rightarrow \textcircled{2} \end{array}$$

$$\begin{aligned} \therefore 1 = \gcd(2, 7) &= 7 - 3 \times 2 \\ &= (-3) \times 2 + 1 \times 7 \\ &= a \cdot x + b \cdot y, \quad x = -3, y = 1 \end{aligned}$$

Since $1 = \gcd(p, a)$, $p \cdot x + a \cdot y = 1 \dots \textcircled{3}$
for some integers x and y .

Given: $p|ab \Rightarrow ab = p \cdot z$ for some integer z .
 $\dots \textcircled{4}$

$$\begin{aligned} \text{Now, } b &= b \cdot 1 = b(p \cdot x + a \cdot y) \\ &= (bp)x + (ab)y \\ &= (bp)x + pz \cdot y \\ &= (bx + yz) \cdot p \\ &= l \cdot p, \quad l = bx + yz \in \mathbb{Z}. \end{aligned}$$

$$\Rightarrow \boxed{p|b.}$$

Case 2. Let $p \nmid b$.
RTP: $p|a$.

Similar to Case 1.

$$\begin{array}{l} 5 \mid \overset{a}{3}(\overset{b}{2b+3a}) \\ 5 \leftarrow \text{prime} \\ p|ab \\ p|a \text{ or } p|b \\ 5 \nmid 3 \\ 5 \mid (2b+3a) \\ \Rightarrow b \text{ is divisible by } 5 \end{array}$$