Tutorial Exercise 3

MA2.101: Linear Algebra (Spring 2022)

April 11, 2022

Problem 1

Let V be the set of all real numbers x such that $x > 0$. Define an operation of "addition" by $x \oplus y = xy$ for all $x, y \in V$. Define an operation of "scalar multiplication" by $\alpha \otimes x = x^\alpha$ for all $\alpha \in \mathbb{R}$ and $x \in V$.

Prove that under the operations $\{\oplus, \otimes\}$, the set V is a vector space. ++++

Problem 2

$$\mathbf{F} = \begin{bmatrix} 0 & -\iota \\ \iota & 0 \end{bmatrix}$$

Bonus:

$$\mathbf{F} = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(N-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \omega^{3(N-1)} & \cdots & \omega^{(N-1)(N-1)} \end{bmatrix}$$

Given $\mathbf{F}$, where $\omega = e^{-2\pi i/N}$ is a primitive N^{th} root of unity. Now, please solve the following.

- Find the inverse of $\mathbf{F}$.
- Multiply $\mathbf{F}$ and $\mathbf{F}^\dagger$ to prove that $\mathbf{F}\mathbf{F}^\dagger = \mathbb{I}$.

Problem 3

Which of the following pairs of sets V and F form a valid vector space? Prove all your claims. (Addition and multiplication operations are defined as usual arithmetic on real numbers. Note that you have to check if V is a vector space over F.)

1. $V = \mathbb{Q}$ and $F = \mathbb{R}$
2. $V = \mathbb{R}$ and $F = \mathbb{Q}$
3. $V = \mathbb{R}$ and $F = \mathbb{C}$

Problem 4

Prove that a field F is a vector space over itself. Also show that the direct sums of a field F will form a vector space V over F. *Do you see a pattern here relating to the previous question?*