(CS1.406) Advanced Algorithms (Spring 2023) 02.03.2023

Mid-Semester Exam

Alloted time: 90 minutes [34 marks]

Instructions:

- There are five questions with sub-parts in some.
- Discussions amongst the students are not allowed. Any dishonesty shall be penalized heavily.
- Be clear in your arguments. Vague arguments shall not be given full credit.

Question 1 [1+2+3 marks]

Suppose we flip a fair coin n times to obtain n random bits. Consider all $m = \binom{n}{2}$ pairs of these bits in some order. Let Y_i be the exclusive-OR of the ith pair of bits. Let $Y = \sum_{i=1}^{m} Y_i$.

1. Compute $\mathbb{P}[Y_i = 1]$.
2. Are Y_i's mutually independent? Show that Y_i's satisfy the property $\mathbb{E}\left[Y_i \cdot Y_j\right] = \mathbb{E}[Y_i] \cdot \mathbb{E}\left[Y_j\right]$.
3. Prove a concentration bound on $\mathbb{P}[|Y - \mathbb{E}[Y]| \geqslant n]$.

Question 2 [6 marks]

Let $a_1, a_2, \ldots, a_n$ be a list of n distinct numbers. We say that a_i and a_j are inverted if $i < j$ but $a_i > a_j$. The bubblesort algorithm swaps pairwise adjacent inverted numbers in a list until there are no more inversions, so the list is in sorted order. Suppose that the input to Bubblesort is a random permutation, equally likely to be any of the $n!$ permutations of n distinct numbers. Determine the expected number of inversions that need to be corrected by Bubblesort.

Question 3 [6 marks]

Let $X_1, \ldots, X_n$ be independent random variables that take $\{-1, 1\}$ values such that the following holds. For all $i \in \{1, \ldots, n\}$, $\mathbb{P}[X_i = 1] = \mathbb{P}[X_i = -1] = \frac{1}{2}$. Let $X = \sum_{i=1}^{n} X_i$. Then for any $a > 0$, show that $\mathbb{P}[X \geqslant a] \leqslant e^{-\frac{a^2}{2}}$.

Question 4 [2 + 6 marks]

One could consider the following approach for estimating the value of the constant π. Let (X, Y) be a point chosen uniformly at random in a 2×2 square centered at origin. That is, X and Y are chosen independently from a uniform distribution on $[-1, 1]$ (continuous space). A circle of radius 1 centered at $(0, 0)$ lies inside the square, and has area π. Define the random variable Z dependent on X and Y is defined as follows.

$$Z = \begin{cases} 1 & \text{if } \sqrt{X^2 + Y^2} \leqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

1. What is the expected value of Z.

2. Let this experiment be run m times by sampling X and Y independently among the runs. Let Z_i be the value of Z in the ith run, and $W = \sum_{i=1}^{m} Z_i$. Using this set-up and the information provided, estimate the value of π as closely[1] as possible.

Question 5 [8 marks]

Two rooted trees T_1 and T_2 are said to be isomorphic if there exists a one-to-one onto mapping f from the vertices of T_1 to the vertices of T_2 satisfying the following condition: for each vertex v in T_1 with children $v_1, \ldots, v_k$, the vertex $f(v)$ has exactly the children $f(v_1), \ldots, f(v_k)$. Further, no ordering is assumed on the children of any of the internal nodes. Device an efficient randomized algorithm for testing the isomorphism of the rooted trees.

Hints

- **Hints for question 3:** You may need the information that $e^t = \sum_{i \geqslant 0} \frac{t^i}{i!}$ and $(2i)! \geqslant (2^i) \cdot (i!)$.

- **Hints for question 5:** Associate a polynomial P_V with each vertex v in a tree T. The polynomials are defined recursively, the base case being the leaf verices all have $P = X_0$. An internal vertex v of height h with the children $v_1, \ldots, v_k$ has it polynomial defined to be $(x_h - P_{v_1})(x_h - P_{v_2}) \ldots (x_h - P_{v_k})$. Note that there is exactly one indeterminate for each level in the tree.

[1] closeness in terms of a parameter ε and m