## Compatible Relation

### Definition (Compatibility Relation)

Let $R$ be a relation in a non-empty set $A$ (i.e., $R \subseteq A \times A$). Then, $R$ is said to be a *compatibility relation* if it is both reflexive and symmetric.

- **Problem:** Let $A$ be a set of people, and $R$ a binary relation on $A$ such that $(a, b) \in R$ if $a$ is a friend of $b$. Verify whether $R$ is a compatibility relation.
  - ***Solution:*** (i) $R$ is reflexive, since $a$ is always a friend of $a \in A$ (i.e., himself/herself), that is, ${}_aR_a$ holds, $\forall a \in A$.
    (ii) $R$ is symmetric, since, if $a$ is a friend of $b$, then obviously $b$ is also a friend of $a$, that is,
    if ${}_aR_b$ holds, then ${}_bR_a$ also holds, $\forall a, b \in A$.
    Hence, $R$ is a compatibility relation.

## Compatible Relation (Continued...)

- **Important Observations**
  - All equivalence relations are compatibility relations.
  - Let $R$ and $S$ be two compatibility relations on a set $A$. Then $R \cap S$ is a compatibility relation, but $R \cup S$ may or may not be a compatibility relation (True/False).

## Closure of Relations

### Definition (Reflexive Closure)

A relation $R'$ is the reflexive closure of a relation $R$ if and only if
(a) $R'$ is reflexive,
(b) $R \subseteq R'$,
(c) For any relation $R''$, if $R \subseteq R''$ and $R''$ is reflexive, then $R' \subseteq R''$,
i.e., $R'$ is the smallest relation that satisfies the conditions (a) and (b).

The reflexive closure of a relation $R$ is denoted by $r(R)$.

**Problem (Closure of Relations):** Given the relation $R = \{(a,b),(b,a),(b,b),(c,b)\}$ on the set $A = \{a,b,c\}$. Compute the reflexive closure $r(R)$ of $R$.

- It is clear that $R$ is not reflexive, since $(a,a) \notin R$ and $(c,c) \notin R$.
- Consider a relation $R'$ which contains $R$ as well as the tuples $(a,a)$ and $(c,c)$, that is,

$$\begin{aligned} R' &= R \cup \{(a,a),(c,c)\} \\ &= \{(a,a),(a,b),(b,a),(b,b),(c,b),(c,c)\} \end{aligned}$$

  Then, clearly $R'$ is reflexive and $R \subseteq R'$.
- Furthermore, any other relation, say $R''$, containing $R$ must also contain $(a,a)$ and $(c,c)$; otherwise it will not be reflexive. So, $R' \subseteq R''$. As $R'$ contains $R$, and $R'$ is reflexive, and is contained in every reflexive relation that contains $R$, so $R'$ is the smallest relation satisfies conditions (a) and (b). Hence, $r(R) = R'$.

## Closure of Relations (Continued...)

### Definition (Symmetric Closure)

A relation $R'$ is the symmetric closure of a relation $R$ if and only if
(a) $R'$ is symmetric,
(b) $R \subseteq R'$,
(c) For any relation $R''$, if $R \subseteq R''$ and $R''$ is symmetric, then $R' \subseteq R''$,
i.e., $R'$ is the smallest relation that satisfies the conditions (a) and (b).

The symmetric closure of a relation $R$ is denoted by $s(R)$.

**Problem (Closure of Relations): Given the relation $R = \{(a, a), (a, b), (c, c), (b, c), (b, a), (a, c)\}$ on the set $A = \{a, b, c\}$. Compute the symmetric closure $s(R)$ of $R$.**

- It is clear that $R$ is not symmetric.
- To be symmetric, $R$ needs the pairs $(c, b)$ and $(c, a)$. Consider a relation $R'$ which contains $R$ as well as the tuples $(c, b)$ and $(c, a)$, that is,

$$\begin{aligned} R' &= R \cup \{(c, b), (c, a)\} \\ &= \{(a, a), (a, b), (c, c), (b, c), (b, a), (a, c), (c, b), (c, a)\} \end{aligned}$$

  Then, clearly $R'$ is symmetric and $R \subseteq R'$.
- Furthermore, any other relation, say $R''$, containing $R$ must also contain $(c, b)$ and $(c, a)$; otherwise it will not be symmetric. So, $R' \subseteq R''$. So, $R'$ is the smallest relation satisfies conditions (a) and (b). Hence, $s(R) = R'$.

## Closure of Relations

### Definition (Transitive Closure)

A relation $R'$ is the transitive closure of a relation $R$ if and only if
(a) $R'$ is transitive,
(b) $R \subseteq R'$,
(c) For any relation $R''$, if $R \subseteq R''$ and $R''$ is transitive, then $R' \subseteq R''$,
i.e., $R'$ is the smallest relation that satisfies the conditions (a) and (b).

The transitive closure of a relation $R$ is denoted by $t(R)$ or $R^t$.

Problem (Closure of Relations): Let $R$ be the less than $(<)$ relation on the set $Z$ of integers. Compute the transitive closure $t(R)$ of $R$.

- The transitive closure of the less than $(<)$ relation on $Z$ is the less than $(<)$ relation itself.

### How to find Transitive Closure of a given Relation $R$ ?

- We need to add the minimum number of tuples to $R$ giving us $R^t$ such that if $(a, b) \in R^t$ and $(b, c) \in R^t$, then $(a, c) \in R^t$.
- Thus, $R^t = R \cup \{(a, b) \in R^t \wedge (b, c) \in R^t \Rightarrow (a, c) \in R^t\}$.

Problem (Closure of Relations): Let $A = \{1, 2, 3\}$ and $R = \{(1, 1), (1, 2), (1, 3), (2, 3), (3, 1)\}$ be a relation on $A$. Compute the transitive closure $R^t$ of $R$.

## Solution

- Clearly, $R$ is not transitive. For example, $(2,3) \in R \wedge (3,1) \in R \nRightarrow (2,1) \in R$.
- Add the following minimum number of tuples in $R$ to construct $R'$ such that $R \subseteq R'$ and $R'$ is transitive:

$$\begin{aligned}
(2,3) \in R \wedge (3,1) \in R &\Rightarrow (2,1) \in R^t \\
(3,1) \in R \wedge (1,2) \in R &\Rightarrow (3,2) \in R^t \\
(3,1) \in R \wedge (1,3) \in R &\Rightarrow (3,3) \in R^t \\
(2,1) \in R^t \wedge (1,2) \in R &\Rightarrow (2,2) \in R^t
\end{aligned}$$

- Thus, $R^t = t(R) = R' = R \cup \{(2,1),(2,2),(3,2),(3,3)\}$.