Discrete Structures (MA5.101)

Quiz - 3 (Monsoon 2021)

International Institute of Information Technology, Hyderabad

Time: 60 Minutes Total Marks: 30

Instructions: This is online examination.

Write at the top of your answer book the following:

Discrete Structures (MA5.101)

Quiz - 3 (Monsoon 2021)

Date: 7-Feb-2022

Name:

Roll Number:

Submit your scanned hand-written answer script in the moodle with the file name: RollNo_Quiz3_SecNo_7Feb2022.pdf

1. Answer the following questions:

   (a) Let $\alpha = \frac{1+\sqrt{5}}{2}$ and F_n the n^{th} Fibonacci's number. Then $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} =$ __________.
   Ans: α.

   (b) Let S be an infinite set and $x \notin S$. Then, the connection between the cardinalities of S and $S \cup \{x\}$ is __________.
   Ans: S and $S \cup \{x\}$ have the same cardinality.

   (c) If a is a discrete numeric function, then $S^{-1}(\nabla a) =$ __________.
   Ans: $\triangle a$.

   (d) The permutation

   $$p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$$

   is __________ (even/odd).
   Ans: odd

   (e) The generating function of the recurrence relation $a_k - 7a_{k-1} + 10a_{k-2} = 0$ is __________.
   Ans: $A(z) = \frac{(1-7z)a_0 + a_1 z}{1-7z+10z^2}$

$[5 \times 1 = 5]$

2. (a) **(Pigeonhole Principle)** Given a set of $(n+1)$ positive integers, none of which exceeds $2n$, show that at least one number of the set must divide another member of the set.

   [Hint: Any positive integer p can be expressed uniquely as $p = 2^k.m$ where $k \geq 0$ and m is an odd positive integer]

3. Given a set of $(n+1)$ positive integers, none of whi exceeds $2n$, show that at least one member of the set must divide another member of the set.

Solution:- we know that any positive integer can be expressed uniquely as

$p = 2^{k} \cdot m$, where $k \geq 0$ and m is an odd positive integer.

Let us consider the function

$f : P \longrightarrow O$

$: p \longmapsto m$, where P is the set of positive integers and O is the set of odd integers.

If we consider the domain set of f consisting of $(n+1)$ elements, then the range of f $= \{1, 3, 5, \dots, (2n-1)\}$ which consists only n elements.

Let $\left.\begin{array}{l} n_1 = 2^{k_1} \cdot m \\ n_2 = 2^{k_2} \cdot m \end{array}\right\} k_1 > k_2$

But, according to the problem, $|P| = n+1$, $|O| = n$, $\left\lceil \frac{|P|}{|O|} \right\rceil =$

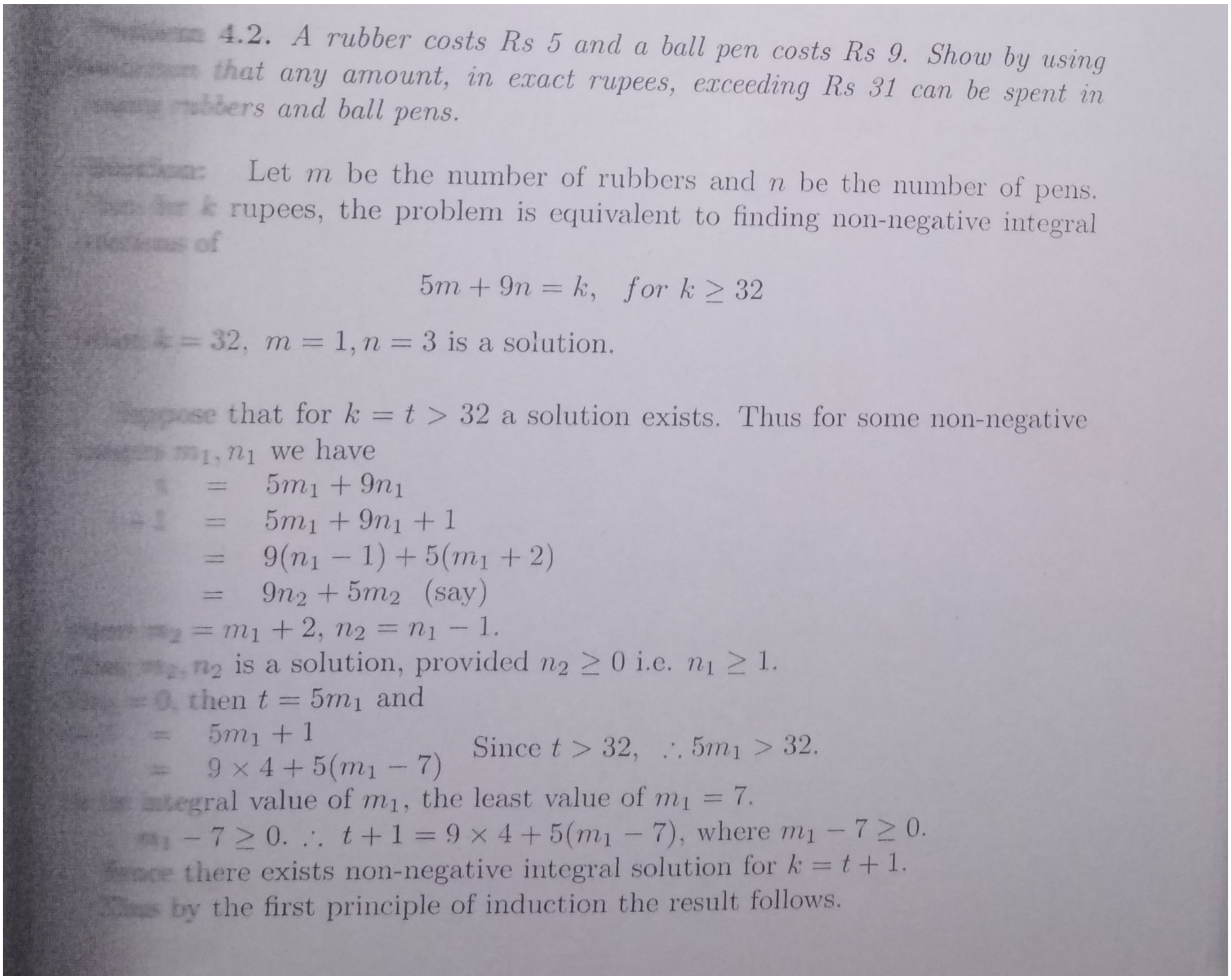
$\therefore$, by pigeonhole principle, there exists two n_1, n_2 s.t. $f(n_1) = f(n_2)$. $\therefore n_2 \mid n_1$ since $\frac{n_1}{n_2} = 2^{k_1 - k_2}$. If $k_2 > k_1$

ie, n_1 & n_2 have the same m.

tha.. n_2 k_2-k_1

Figure 1: Answer of the question 2-a

(b) **(Mathematical Induction)** A rubber costs Rs. 5 and a ball pen costs Rs. 9. Show by using the principle of mathematical induction that any amount, in exact rupees, exceeding Rs. 31 can be spent in buying rubbers and ball pens.



4.2. *A rubber costs Rs 5 and a ball pen costs Rs 9. Show by using that any amount, in exact rupees, exceeding Rs 31 can be spent in rubbers and ball pens.*

Let m be the number of rubbers and n be the number of pens. rupees, the problem is equivalent to finding non-negative integral of

$$5m + 9n = k, \quad for\ k \geq 32$$

$= 32$, $m = 1, n = 3$ is a solution.

that for $k = t > 32$ a solution exists. Thus for some non-negative n_1 we have

$$\begin{aligned} &= 5m_1 + 9n_1 \\ &= 5m_1 + 9n_1 + 1 \\ &= 9(n_1 - 1) + 5(m_1 + 2) \\ &= 9n_2 + 5m_2 \quad \text{(say)} \end{aligned}$$

$= m_1 + 2,\ n_2 = n_1 - 1.$

n_2 is a solution, provided $n_2 \geq 0$ i.e. $n_1 \geq 1$.

then $t = 5m_1$ and

$$\begin{aligned} &= 5m_1 + 1 \\ &= 9 \times 4 + 5(m_1 - 7) \end{aligned} \qquad \text{Since } t > 32, \ \therefore 5m_1 > 32.$$

tegral value of m_1, the least value of $m_1 = 7$.

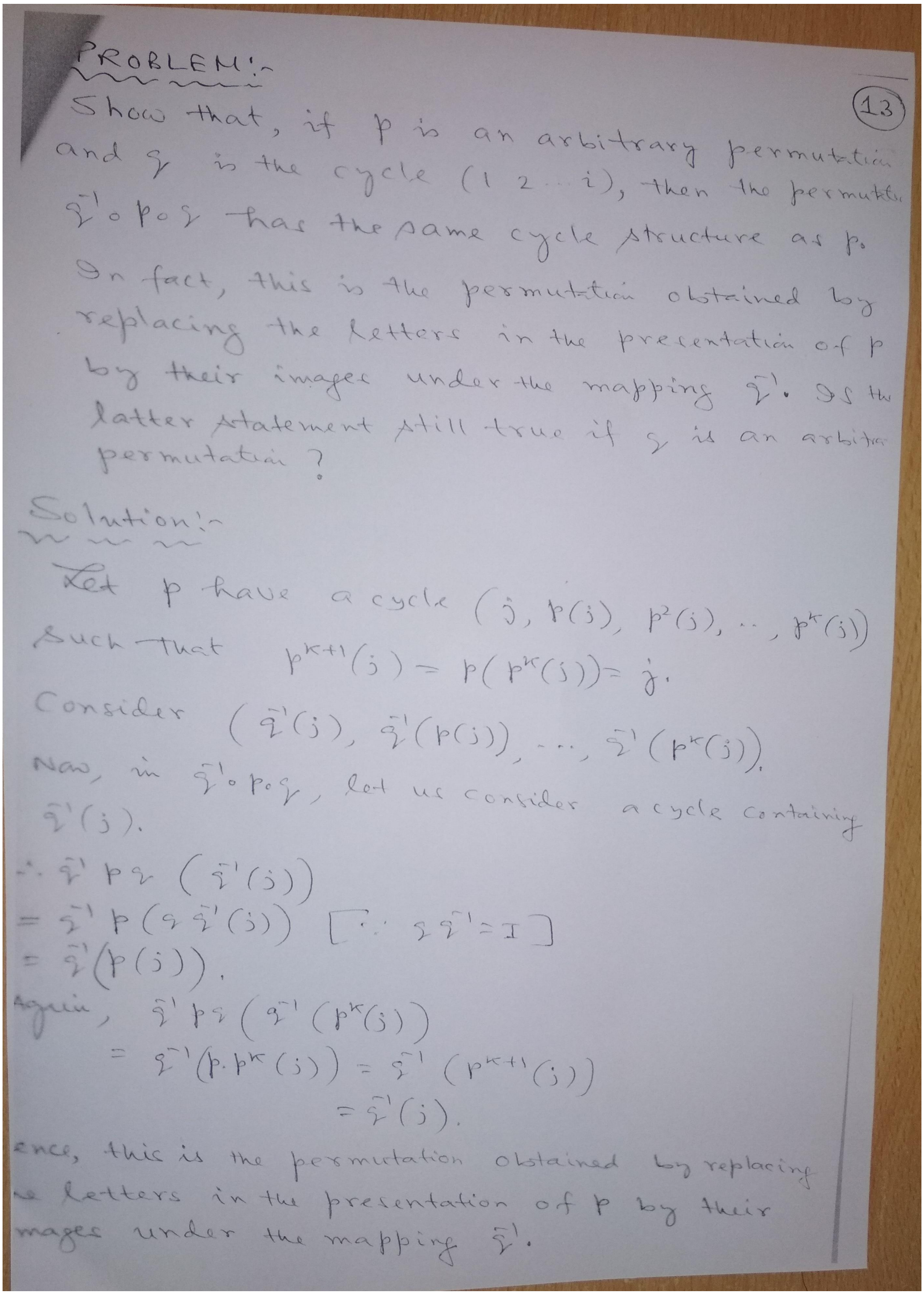
$- 7 \geq 0$. $\therefore$ $t + 1 = 9 \times 4 + 5(m_1 - 7)$, where $m_1 - 7 \geq 0$.

there exists non-negative integral solution for $k = t + 1$.

by the first principle of induction the result follows.

Figure 2: Answer of the question 2-b

(c) **(Permutations)** Show that, if p is an arbitrary permutation and q is the cycle $(1\ 2 \cdots \mathrm{i})$, then the permutation $q^{-1} \circ p \circ q$ has the same cycle structure as p.



PROBLEM:-

Show that, if p is an arbitrary permutation and q is the cycle $(1\ 2 \cdots i)$, then the permutation $q^{-1} \circ p \circ q$ has the same cycle structure as p. In fact, this is the permutation obtained by replacing the letters in the presentation of p by their images under the mapping q^{-1}. Is the latter statement still true if q is an arbitrary permutation?

Solution:-

Let p have a cycle $(j, p(j), p^2(j), \dots, p^k(j))$ such that $p^{k+1}(j) = p(p^k(j)) = j$.

Consider $(q^{-1}(j), q^{-1}(p(j)), \dots, q^{-1}(p^k(j)))$.

Now, in $q^{-1} \circ p \circ q$, let us consider a cycle containing $q^{-1}(j)$.

$\therefore q^{-1} p q (q^{-1}(j))$
$= q^{-1} p (q q^{-1}(j))$ $[\because q q^{-1} = I]$
$= q^{-1}(p(j))$.

Again, $q^{-1} p q (q^{-1}(p^k(j))$
$= q^{-1}(p \cdot p^k(j)) = q^{-1}(p^{k+1}(j))$
$= q^{-1}(j)$.

Hence, this is the permutation obtained by replacing the letters in the presentation of p by their images under the mapping q^{-1}.

Figure 3: Answer of the question 2-c

[5 + 5 + 5 = 15]

3. **(Recurrence Relations and Generating Functions)**

(a) Let a, b be the numeric functions such that $b = \triangle a$. Derive the generating function of b.

Ans:

$$B(z) = \frac{1}{z}\Big[A(z) - a_0\Big] - A(z)$$

(b) Every particle inside a nuclear reactor splits into two particles in each second. Suppose one particle is injected into the reactor every second beginning at time $t = 0$.
(i) Express the number of particles a_n in the reactor at the n^{th} second as a discrete numeric function.
(ii) Derive a_n using the generating function.

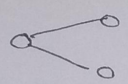
[3 + (2 + 5) = 10]

Every particle inside a nuclear reactor splits into two particles in each second. Suppose one particle is injected into the reactor every second begining at $t=0$. Express the number of particle in the reactor at the n^{th} second as a discrete numeric function and give a closed form for it.

Solution:- Let a_n be the number of particles in the reactor at the n^{th} second.

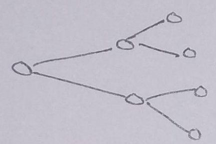
Then, $a_0 = 1$.

for $t=1$,



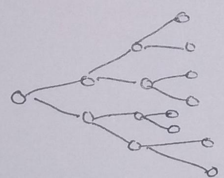
$\therefore a_1 = 3$

For $t=2$,



$a_2 = 7$

For $t=3$,



$a_3 = 7+8 = 15$

For $t=4$, $a_4 = 15+16 = 31, \ldots$

Thus, we have the discrete numeric function a as

$$a = (a_0, a_1, a_2, a_3, a_4, \ldots)$$
$$= (1, 3, 7, 15, 31, \ldots).$$

we have seen that

$$a_1 - a_0 = 3-1 = 2 = 2^1$$
$$a_2 - a_1 = 7-3 = 4 = 2^2$$
$$a_3 - a_2 = 15-7 = 8 = 2^3 \ldots$$

$$\therefore a_r = a_{r-1} + 2^r, \quad r \geq 1 \quad \text{----} (1)$$

Figure 4: Answer of the question 3-b-(i)

Part-2:- we have to solve the recurrence relation

$$a_r = a_{r-1} + 2^r, \quad r \geq 1$$ with the initial condition $a_0 = 1$.

$$\therefore \sum_{r=1}^{\infty} a_r z^r = \sum_{r=1}^{\infty} a_{r-1} z^r + \sum_{r=1}^{\infty} 2^r z^r$$

$$\text{or, } \left(\sum_{r=0}^{\infty} a_r z^r - a_0 \right) = z \sum_{r=1}^{\infty} a_{r-1} z^{r-1} + \left[\sum_{r=0}^{\infty} (2z)^r - 1 \right]$$

$$\text{or, } A(z) - 1 = z A(z) + \frac{1}{1-2z} - 1$$

$$\text{or, } A(z)(1-z) = \frac{1}{1-2z}$$

$$\text{or, } A(z) = \frac{1}{(1-2z)(1-z)}$$

$$= \frac{2}{1-2z} - \frac{1}{1-z}$$

$$\therefore a_r = 2 \cdot 2^r - 1$$

$$\text{or, } a_r = 2^{r+1} - 1, \quad r \geq 0.$$

Figure 5: Answer of the question 3-b-(ii)