Tutorial 7 solutions

▼ Question 1

$\{1, 2, 3\}$ is not a group under $\times_4$ because it is not closed. e.g. $2 \times_4 2 = 0$

$\{1, 2, 3, 4\}$ is closed under $\times_5$ as it is closed, identity element (1) and inverse of all elements ($1^{-1} = 1, 2^{-1} = 3, 3^{-1} = 2, 4^{-1} = 4$) exist in the set, and multiplication is known to be associative.

▼ Question 2

$\mathbb{Z}_9^* = \langle\{1, 2, 4, 5, 7, 8\}, \times_9\rangle$ and the identity element is 1

$\therefore 2^{-1} = 5, 7^{-1} = 4, 8^{-1} = 8$

▼ Question 3

Under $\times_{91}$ $22^{-1} = 29$

Hence, 29 was left out of the list.

▼ Question 4

For some $a, b \in G$,

$$\begin{aligned} aba = aba &\Rightarrow a(ba) = (ab)a \\ &\Rightarrow ab = ba \qquad\qquad \text{by taking } x = a, y = ba, z = ab \end{aligned}$$

Hence, G is commutative on its operation and thus it is Abelian.

▼ Question 5

The group $G = \langle\mathbb{Z}_n, +_n\rangle$ is cyclic, and hence, any subgroup H of G is also cyclic.

Thus, $H = \langle h\rangle$

If h is even, all elements of H are even (as n is even)

If h is odd, let $2k = \frac{n}{h}$. Then,

$$
\begin{aligned}
H &= \{0, h, 2h, ..., (2k-1)h\} \\
&= \{0, 2h, ..., (2k-2)h\} \cup \{h, ..., (2k-1)h\} \\
&= H_1 \cup H_2
\end{aligned}
$$

and $|H_1| = |H_2| = k$

Hence, either every member of H is even or exactly half of the members of H are even

▼ Question 6

Associativity: Since we assume H, K are subgroups of G, then $H \cap K$ inherits associativity from G

Closure: $\forall x, y \in H \cap K; x, y \in H$ and $x, y \in K$. And since H and K are subgroups they are closed. $\therefore x, y \in H \Rightarrow xy \in H$ and $x, y \in K \Rightarrow xy \in K$.
Hence, $\forall x, y \in H \cap K; xy \in H \cap K$, and $H \cap K$ is closed

Identity: Since H and K are subgroups, $e \in H$ and $e \in K$. Thus, $e \in H \cap K$

Inverse: $\forall x \in H \cap K; x \in H$ and $x \in K$. And since H and K are subgroups they contain the inverses of their elements. $\therefore x \in H \Rightarrow x^{-1} \in H$ and $x \in K \Rightarrow x^{-1} \in K$.
Hence, $\forall x \in H \cap K; x^{-1} \in H \cap K$, and $H \cap K$ contains the inverses of their elements.

Hence, $H \cap K$ is a subgroup.

▼ Question 7

Associativity: Since H is a subset of G, then H inherits associativity from G

Closure: $\forall x, y \in H$ let $|x| = 2k+1, |y| = 2n+1$.
$\therefore x^{2k+1} = e$ and $\left(x^{2k+1}\right)^{2n} = x^{4kn+2n} = e$. Similarly, $y^{4kn+2k} = e$

$$
\begin{aligned}
x^{4kn+2n}y^{4kn+2k} = e &\Rightarrow (xy)^{4kn+2k+2n+1}x^{-2k-1}y^{-2n-1} = e \\
&\Rightarrow (xy)^{(2k+1)(2n+1)} = x^{2k+1}y^{2n+1} \\
&\Rightarrow (xy)^{(2k+1)(2n+1)} = e
\end{aligned}
$$

Thus, $(|xy|) \mid (|x|)\,(|y|)$. And since $|x|$ and $|y|$ are odd, all of its divisors are odd.
Hence, $|xy|$ is also odd and $\forall x, y \in H; xy \in H$ and H is closed.

Identity: $|e| = 1 \Rightarrow e \in H$

Inverse: $\forall x \in G, |x^{-1}| = |x|$. And $x \in H \Rightarrow |x|$ is odd. $\therefore |x^{-1}|$ is also odd and $x^{-1} \in H$, and H contains the inverses of their elements.

Hence, H is a subgroup.

▼ Question 10

$\mathbb{Z}_8^* = \langle\{1,3,5,7\}, \times_8\rangle$ and $\mathbb{Z}_{12}^* = \langle\{1,5,7,11\}, \times_{12}\rangle$

Consider a morphism $F : \mathbb{Z}_8^* \to \mathbb{Z}_{12}^*$ such that $F(1) = 1, F(3) = 11, F(5) = 5. F(7) = 7$

1 is the identity element of $\mathbb{Z}_8^*$, 1 is the identity element of $\mathbb{Z}_{12}^*$ and $F(1) = 1$. Hence, identity mapping satisfied.

$\times_8$ and $\times_{12}$ are commutative operators, and $F(1 \times_8 1) = F(1) \times_{12} F(1) = 1$,
$F(1 \times_8 3) = F(1) \times_{12} F(3) = 11$, $F(1 \times_8 5) = F(1) \times_{12} F(5) = 5$,
$F(1 \times_8 7) = F(1) \times_{12} F(7) = 7$, $F(3 \times_8 3) = F(3) \times_{12} F(3) = 1$,
$F(3 \times_8 5) = F(3) \times_{12} F(5) = 7$, $F(3 \times_8 7) = F(3) \times_{12} F(7) = 5$,
$F(5 \times_8 5) = F(5) \times_{12} F(5) = 1$, $F(5 \times_8 7) = F(5) \times_{12} F(7) = 11$ and
$F(7 \times_8 7) = F(7) \times_{12} F(7) = 1$. Hence, mapping of operation on any two elements satisfied.

$F(1^{-1}(\text{w.r.t. } \times_8)) = F(1) = 1 = 1^{-1}(\text{w.r.t. } \times_{12})$,
$F(3^{-1}(\text{w.r.t. } \times_8)) = F(3) = 11 = 11^{-1}(\text{w.r.t. } \times_{12})$,
$F(5^{-1}(\text{w.r.t. } \times_8)) = F(5) = 5 = 5^{-1}(\text{w.r.t. } \times_{12})$ and
$F(7^{-1}(\text{w.r.t. } \times_8)) = F(7) = 7 = 7^{-1}(\text{w.r.t. } \times_{12})$. Hence, mapping of inverses satisfied.

Hence, $\mathbb{Z}_8^*$ is isomorphic to $\mathbb{Z}_{12}^*$