Tutorial Exercise 6 solutions

MA2.101: Linear Algebra (Spring 2022)

May 15, 2022

1 Matrix Representation of Linear Transformation

Problem 1

Let $T : R^3 \to P_2$ be a linear transformation, where P_2 is the vector space of polynomials in x with real coefficients having degree at most 2, given by

$$T(\begin{bmatrix} a \\ b \\ c \end{bmatrix}) = (a-b)x^2 + cx + (a+b+c)$$

Let $\tau = (\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix})$ and $\Omega = (x+1, x^2-x, x^2+x-1)$ be the respective bases. Find $[T]_\tau^\Omega$.

Answer: Please find the solution for this in this *link*.

Problem 2

Let $T : R^2 \to R^2$ be a linear transformation. Let $\tau = (\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix})$ and $\Omega = (\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix})$ be ordered basis for R^2. Suppose $T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 2 \\ 1 \end{bmatrix})$ and $T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 1 \\ -1 \end{bmatrix})$. Find $[T]_\tau^\Omega$.

Answer: Please find the solution for this in this *link*.

2 Isomorphism

Problem 3

let V be he set of complex numbers and let F be the field of real numbers. With the usual operators V is a vector space over F. Describe explicitly an Isomorphism of this space onto R^2.

Answer: The natural isomorphism from V to R^2 is given a+bi $\to$ (a,b). Since i acts like a placeholder for addition in C, (a+bi) + (c+di) = (a+c) +(b+d)i $\to$ (a+c,b+d) = (a,b) + (c,d). And c(a+bi) = c(a,b).Thus this is a linear transformation.Hence this is an isomorphism of this space onto R^2.

Problem 4

Let V be the set of complex numbers regarded as a vector space over the field of real numbers. We define a function T from V into the space of 2 x 2 real matrices as follows, If $z = x + yi$ with x and y real numbers then,

$$T(z) = \begin{bmatrix} x+7y & 5y \\ -10y & x-7y \end{bmatrix}$$

1. Verify that T is a one-one linear transformation of V into the space of 2 x 2 matrices.
2. Verify that $T(z_1 z_2) = T(z_1)T(z_2)$.

Answer:

1. The four coordinates of T(z) are written as linear combinations of the coordinates of z(as a vector space over R).Thus T is clearly a linear transformation. Let z=x+yi and w=a+bi,in order to prove that T is one-one,we have to prove that if T(z)=T(w) then z=w. Assume T(z)=T(w). Considering the top right entry of the matrix we see that 5y= 5b which implies b=y. It now follows from the top left entry of the matrix that x=a. Thus T(z)= T(w) $\Longrightarrow$ z=w, Thus T is one-to-one.

2. Let z_1=x+yi and z_2=a+bi. Then
   T($z_1 z_2$) = T((ax-by)+i(ay+bx)) = $\begin{bmatrix} (ax-by)+7(ay+bx) & 5(ay+bx) \\ -10(ay+bx) & (ax-by)-7(ay+bx) \end{bmatrix}$.

   On the other hand,T(z_1)T(z_2) $= \begin{bmatrix} x+7y & 5y \\ -10y & x-7y \end{bmatrix} \begin{bmatrix} a+7b & 5b \\ -10b & a-7b \end{bmatrix}$

   $= \begin{bmatrix} (ax-by)+7(ay+bx) & 5(ay+bx) \\ -10(ay+bx) & (ax-by)-7(ay+bx) \end{bmatrix}$.Thus T($z_1 z_2$)=T($z_1$)T($z_2$)