Tutorial 5 solutions

Problem 1

First, we need to prove that the vectors in $\tilde{\mathbf{v}}$ are linearly independent.

Suppose that $\exists\ c_1, \ldots, c_n$ such that

$$c_1(v_1) + c_2(v_2 - v_1) + \ldots + c_n(v_n - v_{n-1}) = 0$$
$$\implies (c_1 - c_2)v_1 + \ldots + (c_{n-1} - c_n)v_{n-1} + (c_n)v_n = 0$$

Now, since $v_1, \ldots v_n$ are linearly independent (given) hence we will have $c_1 - c_2 = c_2 - c_3 = \ldots c_{n-1} - c_n = c_n = 0$.

This reduces to $c_1 = c_2 = \ldots = c_n = 0$. Thus, the vectors in $\tilde{\mathbf{v}}$ are linearly independent.

Now, we know that the maximally linearly independent set of vectors in a vector space form the basis of the space. Now, the size of the maximally linearly independent set of vectors for $V = \text{size}(\mathbf{v}) = n$.

Now, the size of $\tilde{\mathbf{v}}$ is n as well. Thus, $\tilde{\mathbf{v}}$ is also a set of maximally linearly independent set of vectors of V.

Thus, $\tilde{\mathbf{v}}$ is a basis for V.

Problem 2

If P_4 is the vector space of all polynomials of degree atmost 4 then, the this space can be spanned by the vectors $\{x^0, x^1, x^2, x^3\}$ such that the polynomials are parametrized by the variable x.

Any $p(x) \in P_4$ can be represented as $p(x) = ax^0 + bx^1 + cx^2 + dx^3$ where $a, b, c, d \in F$.

Now, each of the "basic functions" $\{x^0, x^1, x^2, x^3\}$ are linearly independent (you can check this from your knowledge of linear independence) and hence forms the basis of P_4.

- Thus, dimension of P_4 is $|\{x^0, x^1, x^2, x^3\}| = 4$.
- Not possible. This is because, if none of the polynomials have degree equal to 3, there no possible linear combinations of them will yield x^3 which belongs to P_4. Hence, such a set of vectors won't span P_4.

Problem 3

To be updated.

Problem 4

3. $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

$$e = \left\{ \overset{e_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \overset{e_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}, \overset{e_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \right\}$$

standard basis

Find $P_{e \to B}$ & then find $P_{e \to B} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$[\alpha]_B = P_{e \to B} [\alpha]_e$$

$$P_j = [e_j]_B$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{P_1} \quad \text{--- (1)}$$

$$\Rightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{P_2} \quad \text{--- (2)}$$

$$\Rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{P_3} \quad \text{--- (3)}$$

You can solve the above 3 individually or else you can solve them simultaneously in the following way.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$\downarrow$ ~~$R_1 \to R_1 - R_2$~~ $R_3 \to R_3 + R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 & 1 \end{array}\right]$$

$\downarrow R_1 \to R_1 - R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 & 1 \end{array}\right]$$

$\downarrow R_3 \to R_3 - 2R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

this is P matrix as the first three columns is an identity matrix

$$P = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

$$[\alpha]_B = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b-c \\ b \\ a-2b+c \end{bmatrix}$$

You can solve the above 3 individually or else you can solve them simultaneously in the following way.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$\downarrow$ ~~$R_1 \to R_1 - R_2$~~ $R_3 \to R_3 + R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 & 1 \end{array}\right]$$

$\downarrow$ $R_1 \to R_1 - R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 & 1 \end{array}\right]$$

$\downarrow$ $R_3 \to R_3 - 2R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

this is P matrix as the first three columns is an identity matrix

$$P = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

$$[\alpha]_B = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b - c \\ b \\ a - 2b + c \end{bmatrix}$$

4. $\cancel{P_{B\to A}}$

1. $$\begin{bmatrix} 1 & 1 & 2 & 3 & 0 & -1 \\ 0 & 1 & 2 & 1 & 0 & 2 \\ 1 & 0 & -3 & 1 & 1 & 0 \end{bmatrix}.$$

convert it to RREF

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 0 & -3 \\ 0 & 1 & 0 & \frac{1}{3} & \frac{2}{3} & 4 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & -1 \end{bmatrix}$$

$P_{B\to A}$.

$$P_{B\to A} = \begin{bmatrix} 2 & 0 & -3 \\ \frac{1}{3} & \frac{2}{3} & 4 \\ \frac{1}{3} & -\frac{1}{3} & -1 \end{bmatrix}$$

2. $$\begin{bmatrix} 3 & 0 & -1 & 1 & 1 & 2 \\ 1 & 0 & 2 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 & 0 & -3 \end{bmatrix}$$

convert it to RREF

$$\begin{bmatrix} 1 & 0 & 0 & \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ 0 & 1 & 0 & \frac{5}{7} & -\frac{3}{7} & -\frac{27}{7} \\ 0 & 0 & 1 & -\frac{1}{7} & \frac{2}{7} & \frac{4}{7} \end{bmatrix}$$

$P_{A\to B}$

3. $X = 2a_1 - a_2 + 3a_3$.

$$[X]_A = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$[X]_B = \frac{1}{7}\begin{bmatrix} 2 & 3 & 6 \\ 5 & -3 & -27 \\ -1 & 2 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 19 \\ -68 \\ 8 \end{bmatrix}.$$