$R = \{ (a,b) \in \mathbb{Z}\times\mathbb{Z}, (c,d) \in \mathbb{Z}\times\mathbb{Z} : ad = bc \}.$

- **Claim 1.** Let $(c,d) = (a,b)$.

  Then, $a \cdot b = b \cdot a$, $\forall a, b \in \mathbb{Z}$

  $\Rightarrow (a,b) R (a,b)$ holds.

  $\therefore$ R is reflexive.

- **Claim 2.** Let $(a,b) R (c,d)$, $\forall a,b,c,d \in \mathbb{Z}$

  such that $ad = bc$

  $\Rightarrow bc = ad$

  $\Rightarrow c \cdot b = d \cdot a$

  $\Rightarrow (c,d) R (a,b)$, $\forall a,b,c,d \in \mathbb{Z}$

  $\therefore$ R is symmetric.

- **Claim 3.** Let $(a,b) R (c,d)$ & $(c,d) R (e,f)$,

  $\forall a,b,c,d,e,f \in \mathbb{Z}$.

Then, $ad = bc \quad \cdots (1)$

and $cf = de \quad \cdots (2)$

from (1): $adf = bcf$

$\Rightarrow adf = b \cdot de$, using (2)

$\Rightarrow a \cdot f = b \cdot e \Rightarrow (a,b) R (e,f)$ holds.

$\therefore$ R is transitive.

$\boxed{\text{R is equivalence relation.}}$

Theorem: Let R be an equivalence relation on a set A. ~~Let~~ $a \in A$. Then for any $b \in A$, bR_a if and only if $[b] = [a]$.

Proof: $[a] = \{ x \in A : xR_a \text{ or } aR_x \}$

~~Let~~ $b \in A$ such that bRa.

we show that $[a] = [b]$ ie, (i) $[a] \subseteq [b]$
(ii) $[b] \subseteq [a]$.

Part 1.

(i) Let $x \in [a]$. Then, aR_x holds.

Now, bR_a and $aR_x \Rightarrow bR_x$, since R is transitive.

$$\Rightarrow x \in [b]$$

$$\therefore [a] \subseteq [b] \quad \text{---- (1)}$$

(ii) Let $x \in [b]$. Then, bR_x holds.

Now, bR_a and bR_x

$\Rightarrow aR_b$ and bR_x, since R is symmetric

$\Rightarrow aR_x$, since R is transitive

$\Rightarrow x \in [a]$.

$$\therefore [b] \subseteq [a] \quad \text{--- (2)}$$

From (1) & (2): $[a] = [b]$.

Part 2. Let $[b] = [a]$

Then, $b \in [b]$, since bR_b holds due to R is reflexive

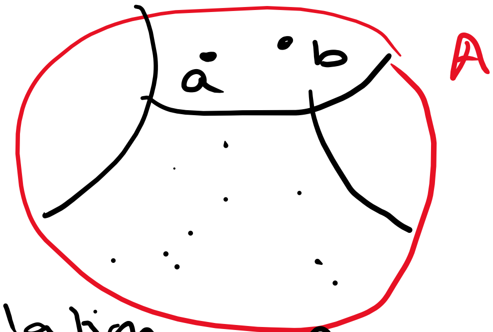
$\Rightarrow b \in [b] = [a]$

$\Rightarrow b \in [a]$

$\Rightarrow aR_b$ holds $\Rightarrow bR_a$ also holds, since R is symmetric.

Theorem: Every partition of a set induces an equivalence relation on it.

Proof. Let $P = \{A_1, A_2, \ldots\}$ be a partition on a set A.
Define a relation R on A as:
" $a R_b$ if a belongs to the same block as b".



(Required to Prove)
RTP: R is an equivalence relation on A.

* Since every element in A belongs to the block block as itself, R is reflexive.

* Let $a R_b$ hold, $\forall a, b \in A$.
Then, a belongs to the same block as b
$\Rightarrow$ b also ,, ,, ,, ,, a
$\Rightarrow$ $b R_a$ holds.
$\therefore$ R is symmetric.

* Let $a R_b$ and $b R_c$ hold, $a, b, c \in A$.
Then, a belongs to the same block as b
and b ,, ,, ,, ,, c
$\Rightarrow$ a also ,, ,, ,, ,, c
$\Rightarrow$ $a R_c$ holds.
$\therefore$ R is transitive.

Problem: Let R be a relation defined in the set Z of all integers s.t. xRy if $(x-y)$ is divisible by 6 ($m = 6$).

PART-1

Claim 1. R is an equivalence relation.

- R is reflexive, since $a-a=0$ is divisible by $6 \Rightarrow aRa$ holds, $\forall a \in Z$.
- R is symmetric

$$
\begin{aligned}
aRb &\Rightarrow (a-b) \text{ is divisible by } 6 \\
&\Rightarrow a-b = 6k, \text{ for some } k \in Z \\
&\Rightarrow -(b-a) = 6k \\
&\Rightarrow b-a = -6k = 6k', \; k' = -k \in Z \\
&\Rightarrow bRa \text{ holds}; \; \forall a, b \in Z.
\end{aligned}
$$

- R is transitive

Let aRb and bRc hold, $\forall a, b, c \in Z$

Then, $a-b = 6k_1$ and $b-c = 6k_2$, for some $k_1, k_2 \in Z$

$$
\begin{aligned}
&\Rightarrow (a-b)+(b-c) = 6k_1 + 6k_2 \\
&\Rightarrow a-c = 6(k_1+k_2) = 6k_3, \; k_3 = k_1+k_2 \in Z \\
&\Rightarrow aRc \text{ holds.}
\end{aligned}
$$

PART-2

For this relation R, let the equivalence classes be

$S_p = \{6k+p : p = 0,1,2,3,4,5$ and k is any integer, ie, $k = 0, \pm 1, \pm 2, \pm 3, \dots\}$

$[\, xRy \Rightarrow x-y = 6k$, for some $k \in Z$
$\Rightarrow x = 6k+y \,]$