Discrete Structures (Monsoon 2021)

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Topic: **Relations**

RELATIONS

Definition

- A relation between two sets A and B is a subset of the cartesian product $A \times B$ and is defined by R (or ρ or r).
  $R \subseteq A \times B$.
- We write ${}_xR_y$ or ${}_x\rho_y$ if and only if (iff) $(x, y) \in R$ (or ρ).
- We also write ${}_x(\sim R)_y$ when x is NOT related to y in R.
- **Empty Relation:** A relation R on a set A is called Empty if the set A is empty set.
- **Full Relation:** A binary relation R on the sets A and B is called full if $A \times B = R$.

RELATIONS

Examples

- **Example.** Consider the relation $R = \{(x, y) \in I \times I : x > y\}$, where I is the set of all integers.
  Clearly, $R \subseteq I \times I$ and R is a relation in I.
  We write ${}_7R_5$ as $(7, 5) \in I \times I$ and $7 > 5$.
- **Example.** Consider the relation $R = \{(x, y) \in N \times N : x = 3y\}$, where N is the set of natural numbers.
  Clearly, $R \subseteq N \times N$ and R is a relation on the set N.
  We write ${}_{15}R_5$, ${}_{18}R_6$, and ${}_{27}R_9$.

RELATIONS

Inverse Relation

- If R be the relation from A to B, then the inverse relation of R is the relation from B to A and is denoted and defined by $R^{-1} = \{(y, x) : y \in B, x \in A, (x, y) \in R\}$.
$\Longrightarrow (x, y) \in R \Leftrightarrow (y, x) \in R^{-1}$
- **Example.** If $A = \{1, 2\}$, $B = \{2, 3\}$ and R be the relation from A to B, $R = \{(1, 2), (2, 3)\}$, then $R^{-1} = \{(2, 1), (3, 2)\}$.

Theorem

If R be a relation from A to B, then the domain of R is the range of R^{-1} and the range of R is the domain of R^{-1}.

Theorem

If R be a relation from A to B, then $(R^{-1})^{-1} = R$.

RELATIONS

Reflexive relation

- Let A be a set and R the relation defined in it (i.e., $R \subseteq A \times A$). R is said to be *reflexive*, if $(a, a) \in R, \forall a \in A$
  $\implies {}_aR_a$ holds for every $a \in A$.
- **Example.** Consider the relation $R = \{(a, a), (a, c), (b, b), (c, c), (d, d)\}$ in the set $A = \{a, b, c, d)\}$. Then R is reflexive, since $(x, x) \in R, \forall x \in A$, that is, ${}_xR_x$ holds for every $x \in A$.
- **Example.** Consider the relation $S = \{(a, a), (a, c), (b, c), (b, d), (c, d)\}$ in the set $A = \{a, b, c, d)\}$. Verfify whether S is reflexive.

RELATIONS

Symmetric relation

- Let A be a set and R the relation defined in it (i.e., $R \subseteq A \times A$). R is said to be *symmetric*, if $(a, b) \in R \Rightarrow (b, a) \in R, \forall a, b \in A$
  In other words, ${}_aR_b \Rightarrow {}_bR_a$ for every $a, b \in A$.
- **Example.** Let N be the set of natural numbers and R the relation defined in it such that ${}_xR_y$ if x is a divisor of y (that is, $x|y$), $x, y \in N$.
  Then R is NOT symmetric, since ${}_xR_y \not\Rightarrow {}_yR_x, \forall x, y \in N$.
  For example, ${}_3R_9 \not\Rightarrow_9 R_3$.
- **Example.** Consider the relation S in the set of natural numbers N as $R = \{(x, y) \in N \times N : x + y = 5\}$. Verfify whether S is symmetric.

RELATIONS

Theorem

For a symmetric relation R, $R^{-1} = R$.

Proof.

Required to prove (RTP) (i) $R \subseteq R^{-1}$, and (ii) $R^{-1} \subseteq R$.
(i) Let $(x, y) \in R$.
Then $(x, y) \in R \Rightarrow (y, x) \in R$, since R is symmetric
$\Rightarrow (x, y) \in R^{-1}$, by definition of R^{-1}
Thus, $R \subseteq R^{-1}$.
(ii) Let $(x, y) \in R^{-1}$.
Then $(y, x) \in (R^{-1})^{-1} = R$, by definition of R^{-1}
$\Rightarrow (x, y) \in R$, since R is symmetric
Thus, $R^{-1} \subseteq R$.

$\square$