MA 6.101
Probability and Statistics

Tejas Bodas

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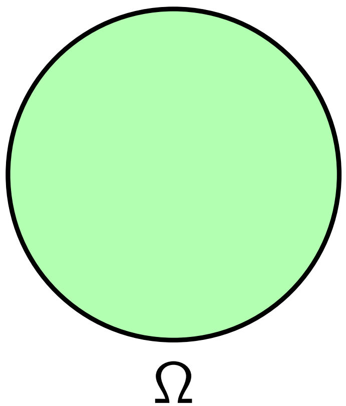
Random variables ($\Omega' = \mathbb{R}$)



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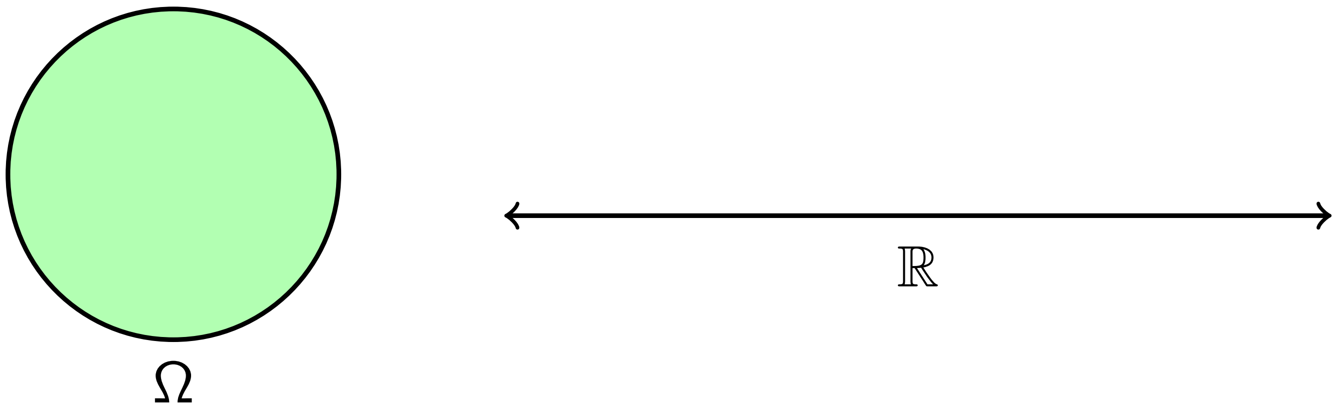
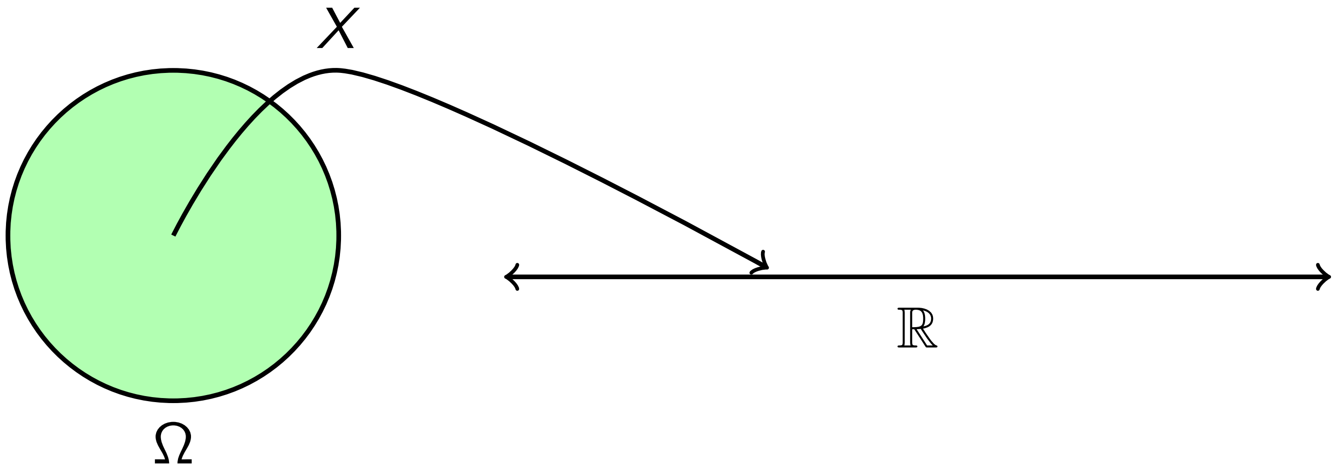
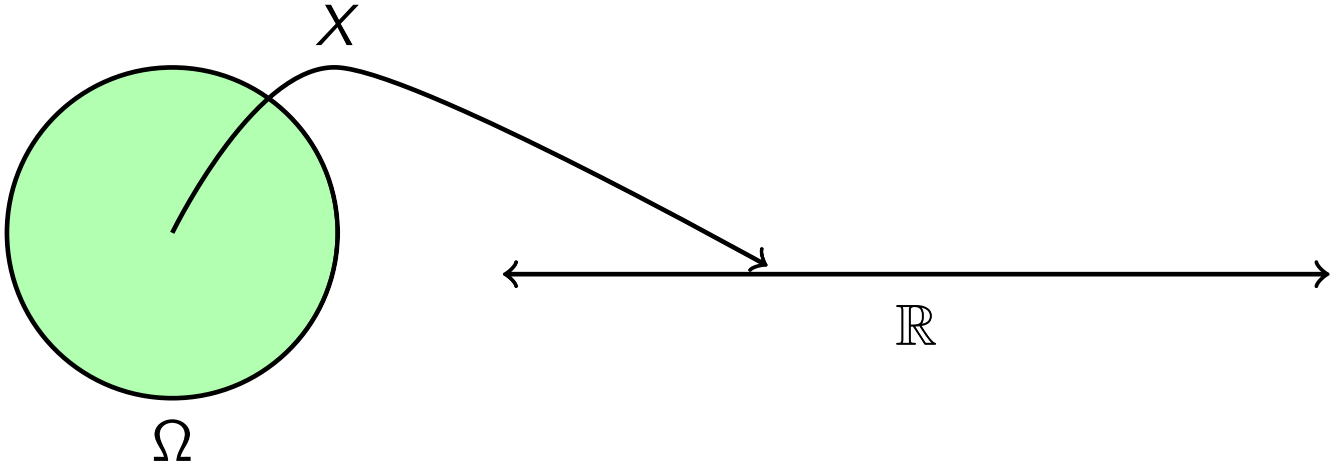


Random variables ($\Omega' = \mathbb{R}$)



Random variables ($\Omega' = \mathbb{R}$)



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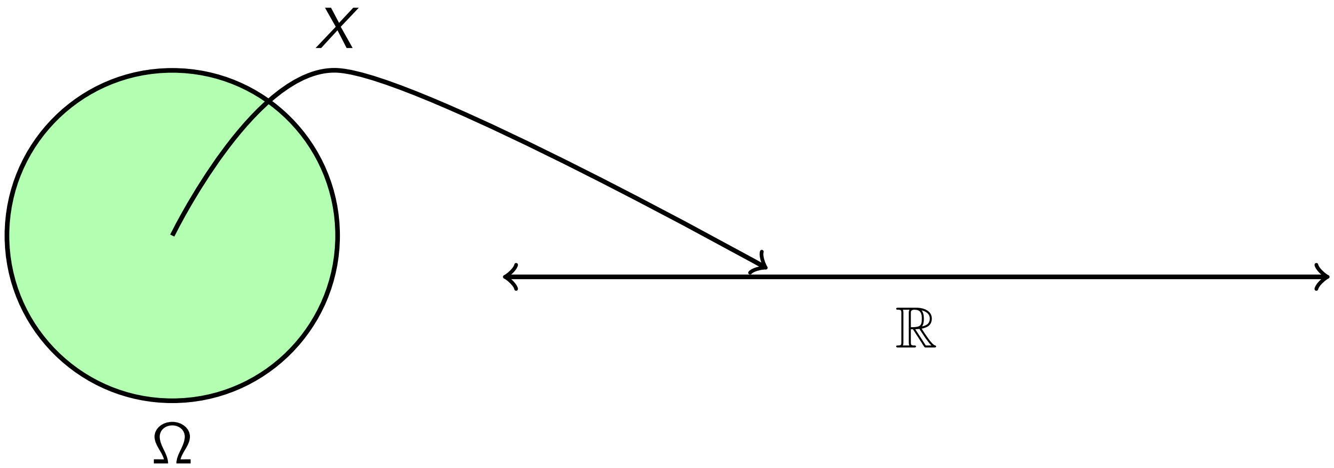
Random variables ($\Omega' = \mathbb{R}$)



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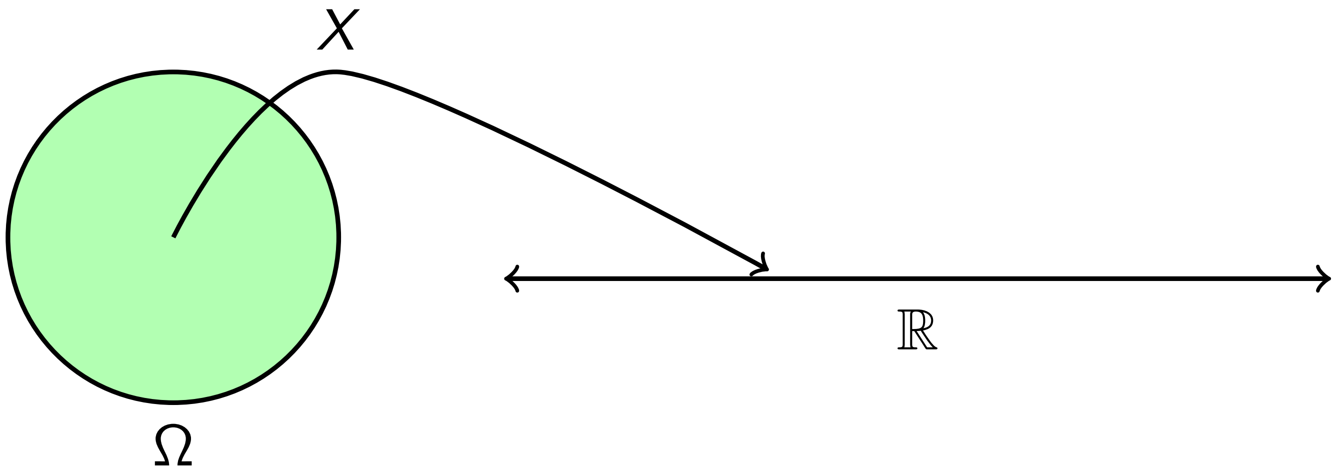

- $\Omega \xrightarrow{X} \mathbb{R}, \quad \mathcal{F} \xrightarrow{X} \mathcal{B}(\mathbb{R}), \;$ and $\; P(.) \xrightarrow{X} P_X(.)$

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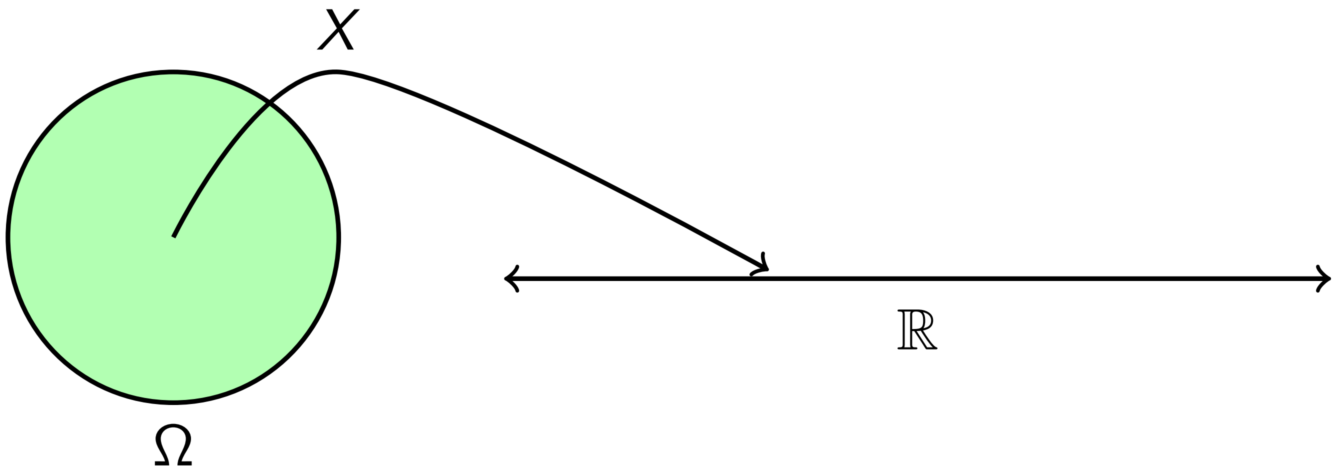

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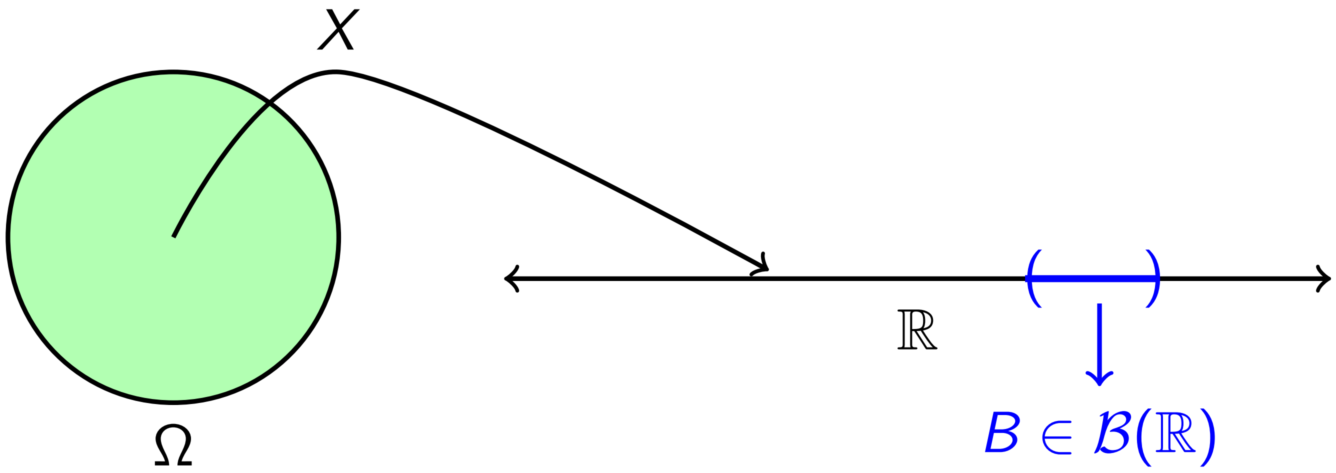

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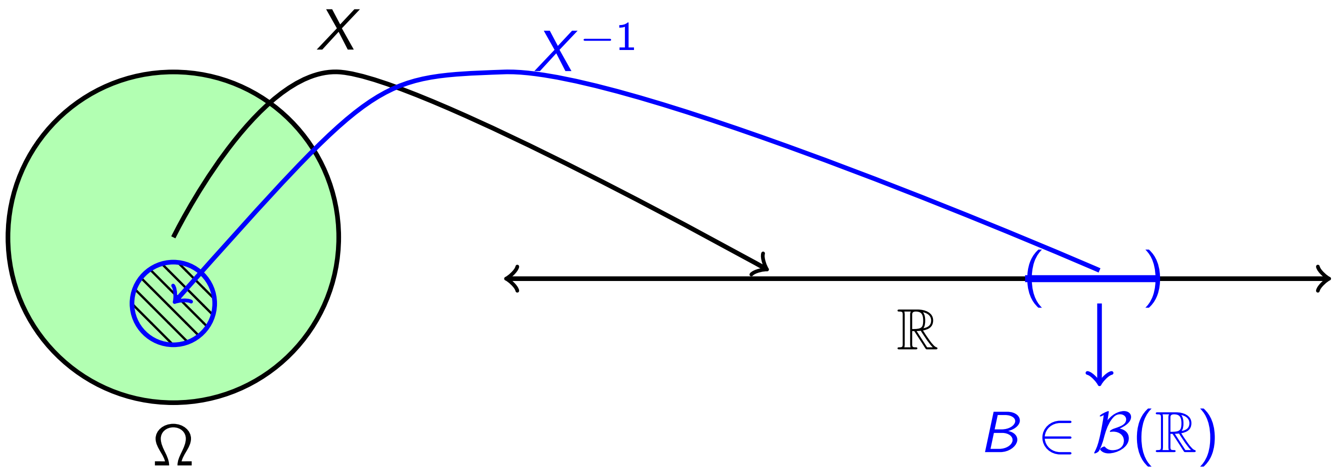

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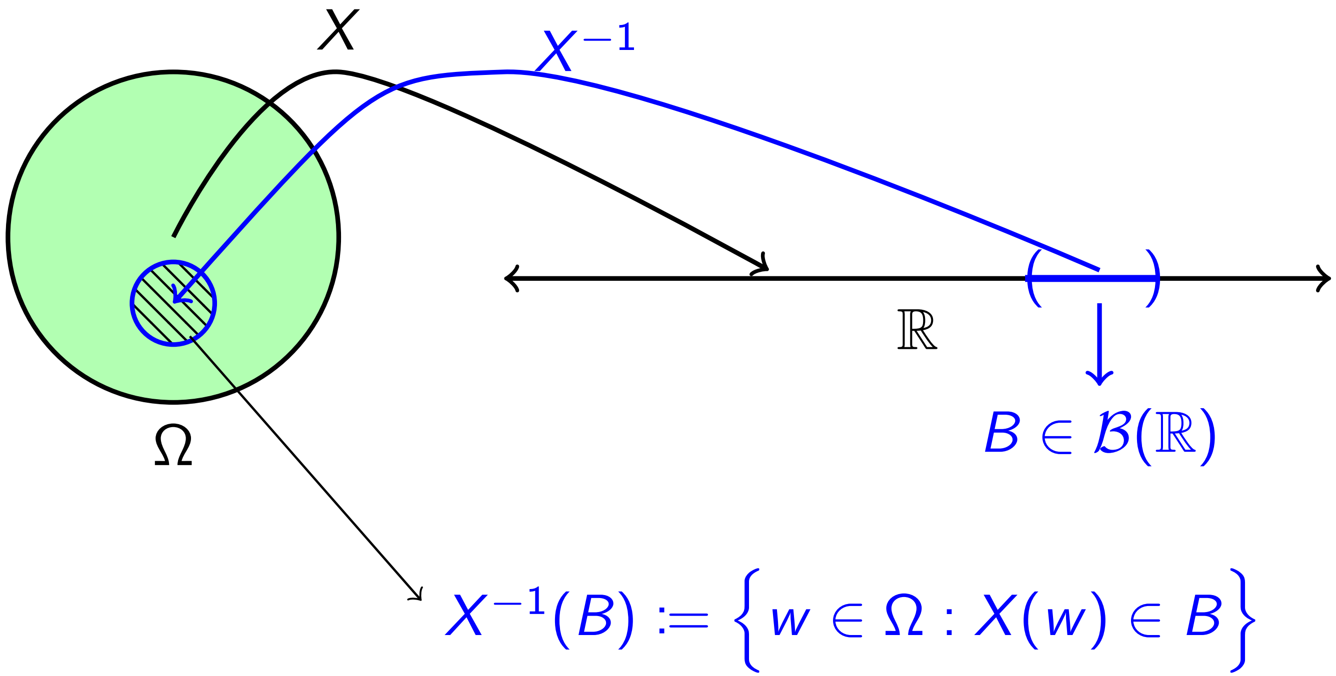

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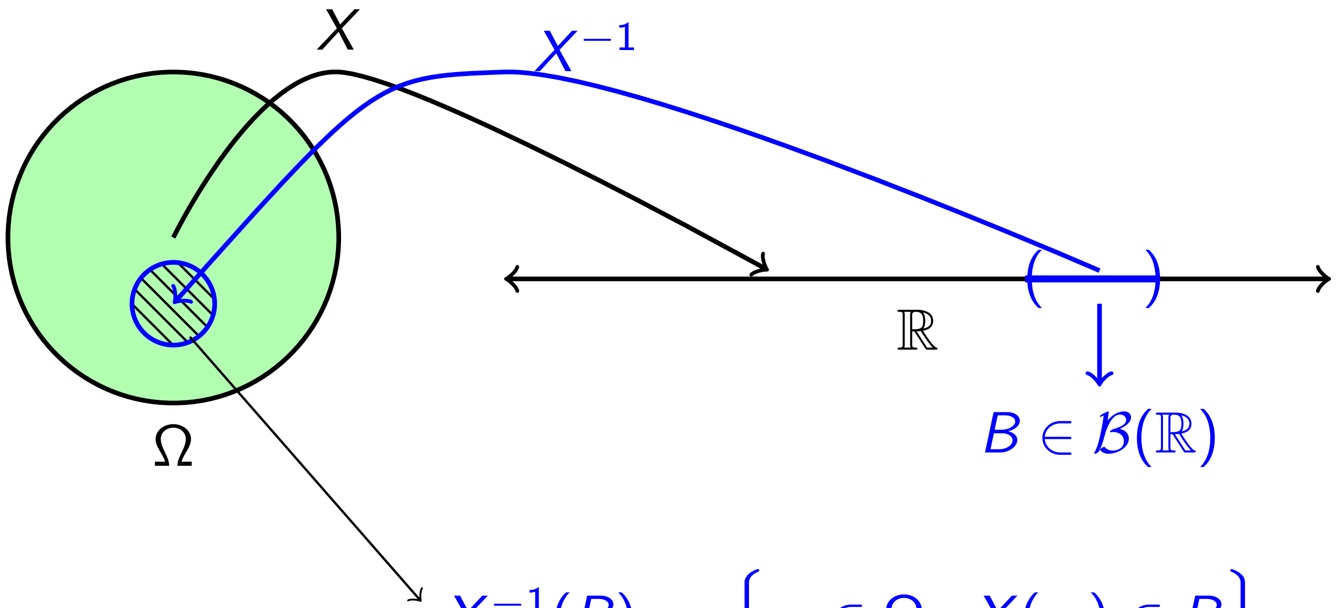

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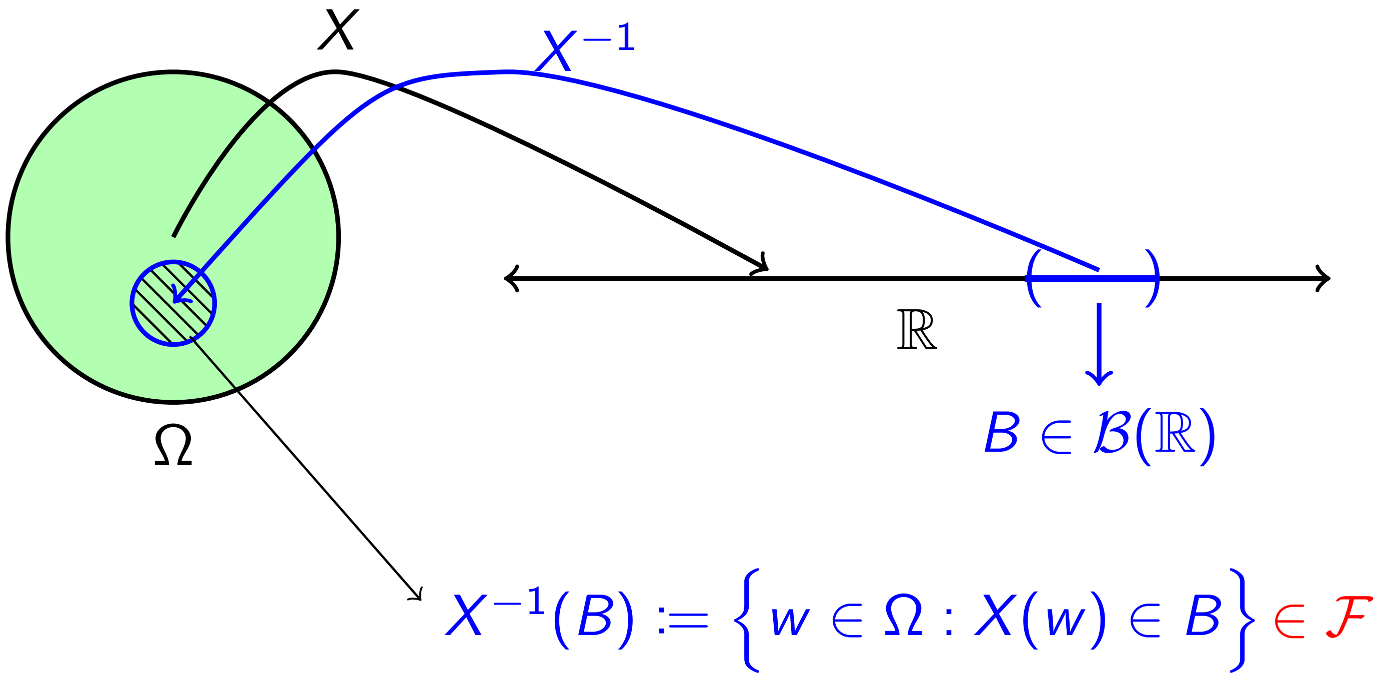

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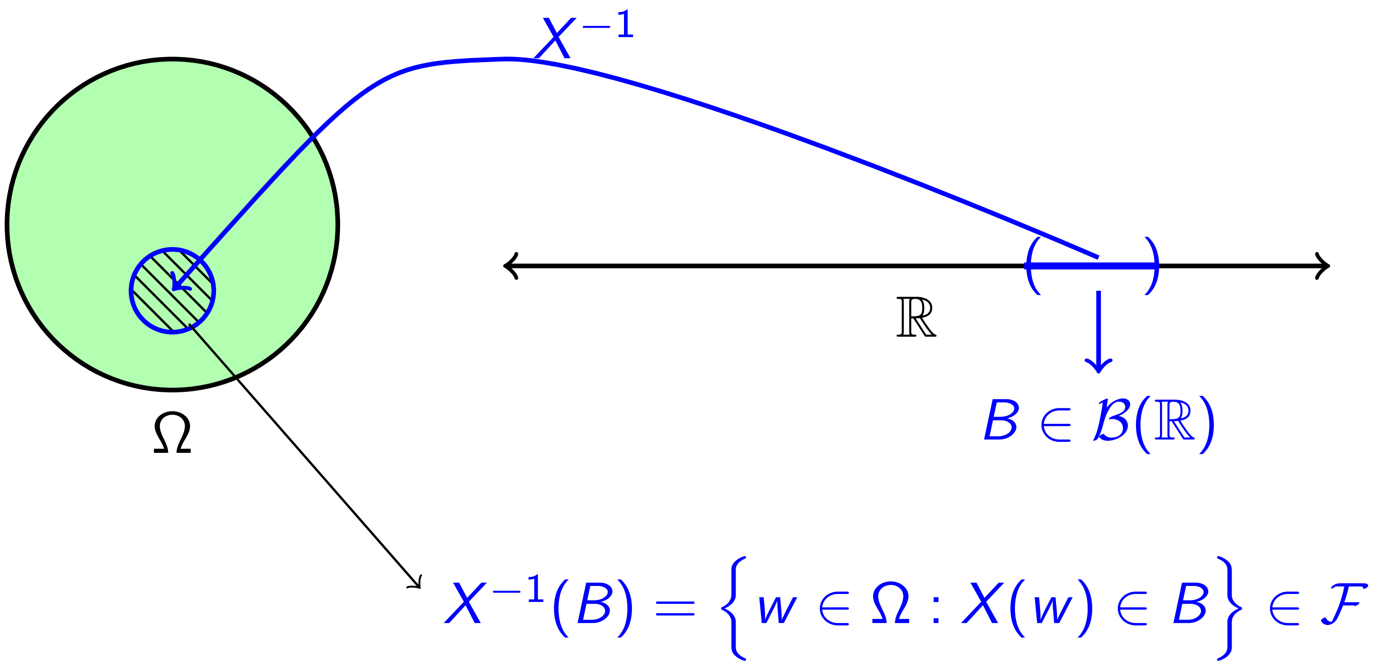
Definition of a random variables



A random variable X is a map $X : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}), P_X)$ such that for each $B \in \mathcal{B}(\mathbb{R})$, the inverse image $X^{-1}(B) := \{w \in \Omega : X(w) \in B\}$ satisfies

$$X^{-1}(B) \in \mathcal{F} \text{ and}$$
$$P_X(B) = \Pr(w \in \Omega : X(w) \in B)$$

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- This is a general definition of CDF (applicable for both continuous or discrete).
- If $F_X(\cdot)$ is continuous (resp. piecewise continuous), then X is continuous (resp. discrete) random variable.

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- This proves the non-decreasing part.

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- Is $A_n \uparrow A$ or $A_n \downarrow A$? Clearly, $A_n \downarrow A$.
- From continuity of probability, $\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(A)$.
- This implies $\lim_{x_n \downarrow x} F_X(x_n) = F_X(x)$. $\square$
- You cannot prove the other way by considering $x_n \uparrow x$ because $\cup_n(-\infty, x_n] = (-\infty, x)$ and $P_X(-\infty, x) \neq F_X(x)$.

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- In this case, $\mathcal{F}' = \mathcal{B}(\mathbb{R})$.
- Intuitively, in a continuous random variable, the unit probability measure is spread continuously (like spreading a fluid) over the range of the random variable.

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- Travel time from office to home.

Examples of Continuous random variables

- Pick a number uniformly from $[a, b]$.
- Time interval between successive customers entering DMart.
- Travel time from office to home.
- Level of water in a dam or pending workload on a server.

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- For $Y = aX + b$ and $a < 0$, $F_Y(y) = 1 - F_X(\frac{y-b}{a})$.

Standard Examples

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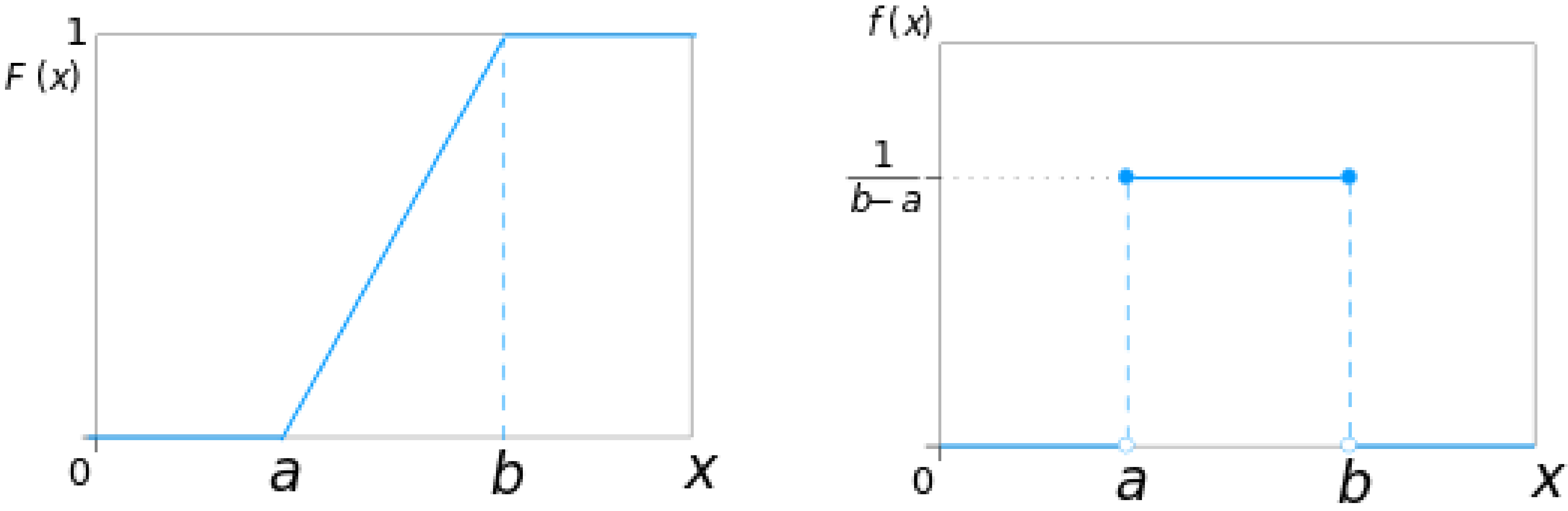
- HW: Verify $E[X] = \frac{a+b}{2}$ and $Var(X) = \frac{(b-a)^2}{12}$

$U[a, b]$



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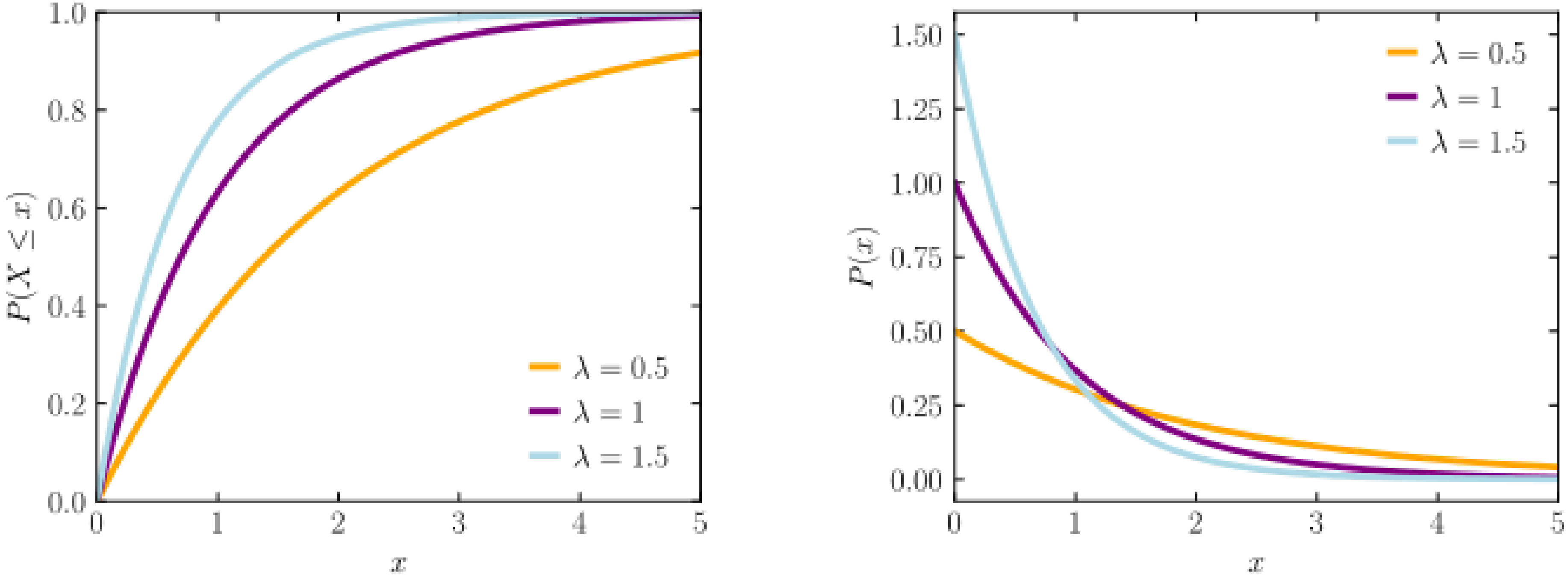
- $E[X^n] = \frac{n!}{\lambda^n}$

$Exp(\lambda)$



$P(X \leq x)$
1.0
0.8
0.6
0.4
0.2
0.0
0
1
2
3
4
5
x
$\lambda = 0.5$
$\lambda = 1$
$\lambda = 1.5$
$P(x)$
1.50
1.25
1.00
0.75
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- Used extensively in Queueing theory to model inter-arrival time and service time of jobs.