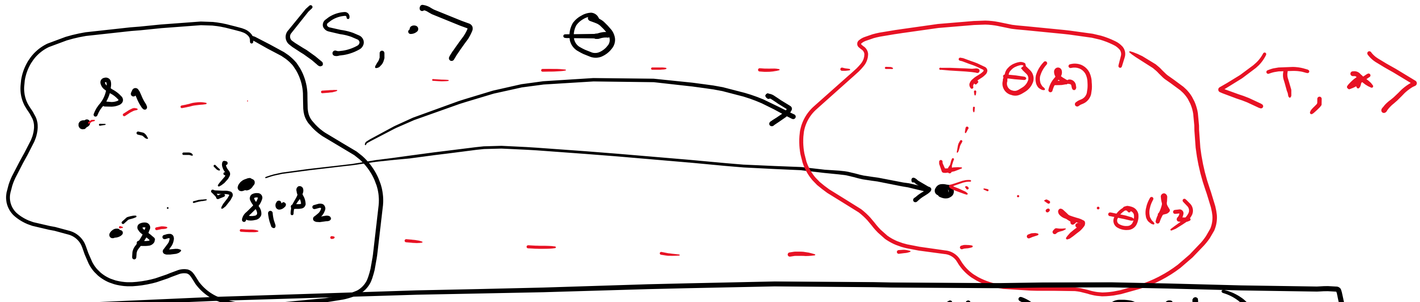
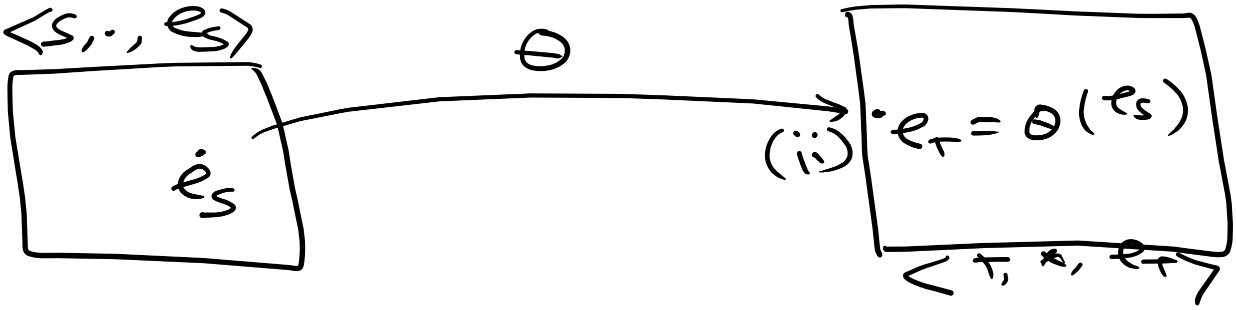


$$\boxed{\forall s_1, s_2 \in S, \quad \Theta(s_1 \cdot s_2) = \Theta(s_1) * \Theta(s_2)}$$

(i)



<u>Theorem</u>: $\mu : [G, \cdot] \to [G', *]$ is a morphism of groups $[G, \cdot]$ and $[G', *]$ iff $\mu(g \cdot g') = \mu(g) * \mu(g'), \ \forall g, g' \in G$.

<u>Proof.</u>

$(\Rightarrow)$: Given $\mu : [G, \cdot] \to [G', *]$ is a homomorphism.

Therefore, (i) $\mu(g \cdot g') = \mu(g) * \mu(g')$ $\forall g, g' \in G$

(ii) $\mu(1_G) = 1_{G'}$, where, 1_G and $1_{G'}$ are the identities of G & G', respectively

(iii) $\boxed{\mu(g)}^{-1} = \mu(g^{-1}), \ \forall g \in G$.

So, $\mu(g \cdot g') = \mu(g) * \mu(g'), \ \forall g, g' \in G$ holds.

$(\Leftarrow)$: Given $\mu : [G, \cdot] \to [G', *]$ is a function of two groups $[G, \cdot]$ and $[G', *]$.

Given $\mu(g \cdot g') = \mu(g) * \mu(g'), \ \forall g, g' \in G$.

RTP: μ is morphism

i.e, RTP: (i) $\mu(g \cdot g') = \mu(g) * \mu(g')$, $\forall g, g' \in G$ holds from given condition.

(ii) $\mu(1_G) = 1_{G'}$

(iii) $[\mu(g)]^{-1} = \mu(g^{-1}), \forall g \in G$.

(ii) Idempotent element:

An element $x \in G'$ is idempotent if $x * x = x$.

$$\mu(1_G) = \mu(1_G \cdot 1_G)$$
$$= \mu(1_G) * \mu(1_G)$$

$\Rightarrow$ $\mu(1_G)$ is the idempotent in G'.
Since all the idempotents in G' is $1_{G'}$ itself, so $\boxed{\mu(1_G) = 1_{G'}}$.

(iii) Since $g \in G$, $g^{-1} \in G$

and $1_G = g \cdot g^{-1} = g^{-1} \cdot g$

Now, $1_G = g \cdot g^{-1}$

$\Rightarrow \mu(1_G) = \mu(g \cdot g^{-1}) = \mu(g) * \mu(g^{-1})$

$\Rightarrow \mu(g) * \mu(g^{-1}) = 1_{G'}$ ... (1)

Again, $1_G = g^{-1} \cdot g$

$\Rightarrow \mu(g^{-1} \cdot g) = \mu(1_G)$

$\Rightarrow \mu(\delta^{-1}) * \mu(\delta) = 1_{G'}$ ... (2)

(1) & (2): $\mu(g^{-1})$ is the inverse of $\mu(\delta)$

$\Rightarrow [\mu(\delta)]^{-1} = \mu(\delta^{-1})$.

(i) $\phi : G \rightarrow G$

where $\phi(x) = x^2$, for all $x \in G$,

G is the group of non-zero reals under multiplication.

Let $x_1, x_2 \in G$

Then, $\phi(x_1 \cdot x_2) = (x_1 \cdot x_2)^2$

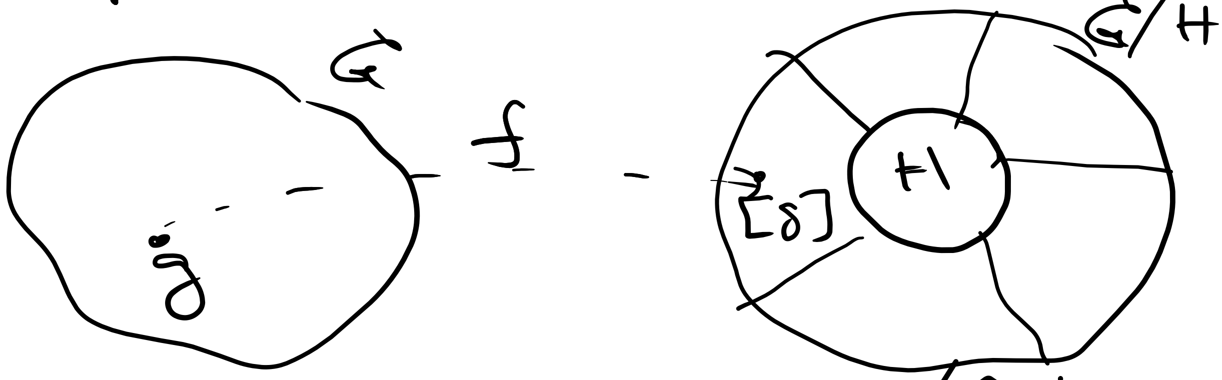
$= x_1^2 \cdot x_2^2$

$= \phi(x_1) \cdot \phi(x_2)$

$\therefore \phi : G \rightarrow G$ is homomorphism.

Theorem: If H is a normal subgroup of a group G, then the mapping $f: G \rightarrow G/H$, $f(g) = [g]$, $\forall g \in G$, where $[g] = g \cdot H$, is a group epimorphism.

Proof.



$\langle G/H, \circ \rangle$ is a quotient/factor group.

$\therefore [g] \circ [k] = [g \cdot k]$, $\forall g, k \in G$

where '$\cdot$' and '$\circ$' are the operations defined in G and G/H, respectively;

$$[g] = g \cdot H$$

RTP: (i) f is homomorphism

(ii) f is onto.

(i) RTP: $f(g_1 \cdot g_2) = f(g_1) \circ f(g_2)$, $\forall g_1, g_2 \in G$.

Now, $f(g_1) \circ f(g_2)$

$= [g_1] \circ [g_2]$, by def^n.

$= [g_1 \cdot g_2]$

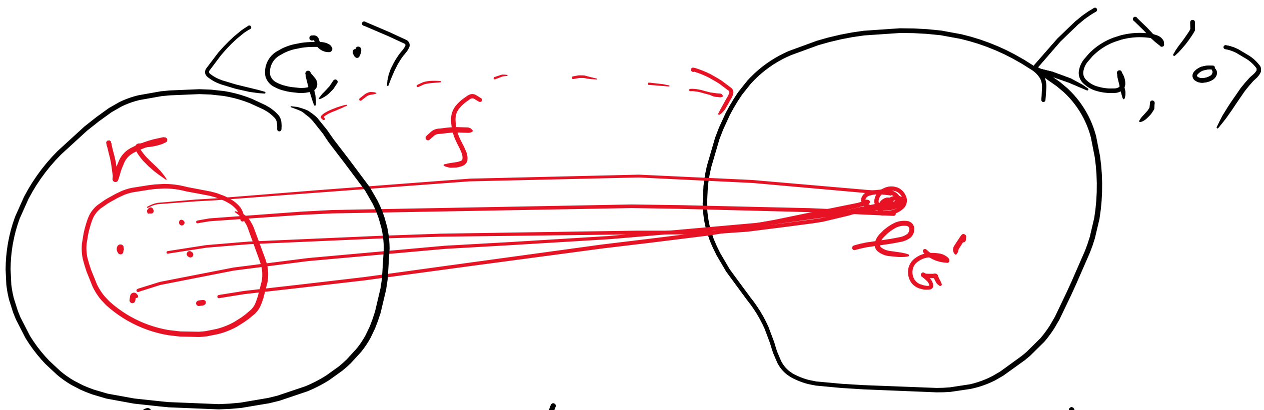
$= f(g_1 \cdot g_2)$, $\forall g_1, g_2 \in G$

$\therefore f$ is homomorphism.

(ii) f is onto, because $\forall [g] \in G/H$, $\exists g \in G$ such that
$f(g) = [g]$, by defⁿ.

Theorem: Let $f : G \to G'$ be any group homomorphism. Then the kernal of the homomorphism, f is a normal subgroup of G.

Proof.



Given: $f : G \to G'$ be a group homomorphism.

$$\therefore \quad f(g_1 \cdot g_2) = f(g_1) \circ f(g_2) \quad \cdots (1)$$
$$\forall g_1, g_2 \in G$$

$f(e_G) = e_{G'}$, where e_G and $e_{G'}$ $\cdots (2)$ are the identities in G and G', respectively.

$$[f(g)]^{-1} = f(g^{-1}), \ \forall g \in G \quad \cdots (3)$$

Let K be a kernal of f.

$\therefore f(K) = \{e_{G'}\}$,

i.e, $f(x) = e_{G'}$, $\forall x \in K$. ---(4)

R.T.P. K is a normal subgroup of G

i.e, (i) K is a subgroup of G

(ii) K is a normal subgroup of G.

(i) RTP: $\forall k_1, k_2 \in K$, $\boxed{k_1 \cdot k_2^{-1} \in K}$

i.e, **RTP**: $f(k_1 \cdot k_2^{-1}) = e_{G'}$

Now, $f(k_1 \cdot k_2^{-1})$

$= f(k_1) \circ f(k_2^{-1})$ by (1)

$= f(k_1) \circ [f(k_2)]^{-1}$, by (3)

$= e_{G'} \circ [e_{G'}]^{-1}$

$= e_{G'} \circ e_{G'} = e_{G'}$

$\therefore$ K is a subgroup of G.

(ii) **RTP**: K is a normal subgroup of G

i.e, $\forall g \in G$, $g^{-1} \cdot K \cdot g \subseteq K$

i.e, $g^{-1} \cdot k \cdot g \in K, \ \forall k \in K$.

i.e, $f(g^{-1} \cdot k \cdot g) = e_{G'}$

Now, $f(g^{-1} \cdot k \cdot g)$

$= f(g^{-1}) \circ f(k \cdot g)$ by (1)

$= f(g^{-1}) \circ \underline{f(k)} \circ f(g)$, by (1)

$= f(g^{-1}) \circ e_{G'} \circ f(g)$

$= f(g^{-1}) \circ f(g)$

$= f(\underline{g^{-1} \cdot g})$, by (1)

$= f(e_G)$

$= e_{G'}$

$\therefore$ K is a normal subgroup of G.

———— x ———— x ————