# Problem Set 3

MA2.101: Linear Algebra (Spring 2022)

April 19, 2022

Due date: April 18, 2022

**General Instructions:** All symbols have the usual meanings (example: $F$ is an arbitrary field, $\mathbb{R}$ is the set of reals, $\mathbb{N}$ the set of natural numbers, and so on.) Remember to prove all your intermediate claims, starting from basic definitions and theorems used in class to show whatever is being asked. You may use any other non-trivial theorems not used in class, as long as they are well known and a part of basic linear algebra texts. It is always best to try to prove everything from definitions. Arguments should be mathematically well formed and concise.

---

1. On $\mathbb{R}^n$, define two properties: $\bar{\alpha} \oplus \bar{\beta} = \bar{\alpha} - \bar{\beta}$ and $c\bar{\alpha} = -c\bar{\alpha}$,
   Which of the axioms for the vector space are satisfied by $(\mathbb{R}^n, \oplus, \cdot)$?

   **Answer:**

   1. $\oplus$ is not commutative since (0, . . . , 0) $\oplus$ (1, . . . , 1) = ($-$1, . . . , $-$1) while (1, . . . , 1) $\oplus$ (0, . . . , 0) = (1, . . . , 1). And (1, . . . , 1) $\neq$ ($-$1, . . . , $-$1).
   2. $\oplus$ is not associative since ((1, . . . , 1) $\oplus$ (1, . . . , 1)) $\oplus$ (2, . . . , 2) = (0, . . . , 0) $\oplus$ (2, . . . , 2) = ($-$2, . . . ,$-$2) while (1, . . . , 1) $\oplus$ ((1, . . . , 1) $\oplus$ (2, . . . , 2)) = (1, . . . , 1) $\oplus$ ($-$1, . . . ,$-$1) = (2, . . . , 2).
   3. There does exist a right additive identity, i.e. a vector 0 that satisfies $\alpha$ + 0 = $\alpha$ for all $\alpha$.There does not exist a left additive identity.
   4. There do exist right additive inverses. For the vector $\alpha$ = (x1, . . . , xn) clearly only $\alpha$ itself satisfies $\alpha \oplus \alpha$ = (0, . . . , 0).
   5. The element 1 does not satisfy 1$\cdot\alpha = \alpha$ for any non-zero $\alpha$ since otherwise we would have 1$\cdot$(x1, . . . , xn) = ($-x1$, . . . , $-xn$) = (x1, . . . , xn) only if xi = 0 for all i.
   6. The property (c1c2) $\cdot\ \alpha$ = c1 $\cdot$ (c2 $\cdot\ \alpha$) does not hold since (c1c2)$\alpha$ = ($-c$1c2)$\alpha$ while c1(c2$\alpha$) = c1($-c2\alpha$) = ($-c1(-c2\alpha)$) = +c1c2$\alpha$. Since c1c2 , $-c$1c2 for all c1, c2 they are not always equal.
   7. It does hold that c $\cdot$ ($\alpha \oplus\beta$) = c $\cdot\ \alpha \oplus$ c $\cdot\ \beta$. Firstly, c $\cdot$ ($\alpha \oplus\beta$) = c $\cdot$ ($\alpha - \beta$) = $-$c($\alpha - \beta$) = $-c\alpha$ + c$\beta$. And secondly c $\cdot\ \alpha \oplus c\beta$ = ($-c\alpha$) $\oplus$(-c$\beta$)) = $-c\alpha$ $-$ ($-c\beta$) = $-c\alpha$ + c$\beta$. Thus they are equal.
   8. It does not hold that (c1 + c2) $\cdot\ \alpha$ = (c1 $\cdot\ \alpha$) $\oplus$($c2 \cdot \alpha$). Firstly, (c1 + c2) $\cdot\ \alpha$ = $-$(c1 + c2)$\alpha$ = $-$c1$\alpha$ $-$ c2$\alpha$. Secondly, c1 $\cdot\ \alpha \oplus$ c2 $\cdot\ \alpha$ = ($-$c1 $\cdot\ \alpha$) $\oplus$(-$c2 \cdot \alpha$) = $-$c1$\alpha$ + c2$\alpha$. Since $-$c1$\alpha$ $-$ c2$\alpha$ $\neq$-c1$\alpha$ + c2$\alpha$ for all c1, c2 they are not equal.

2. Let $\mathbf{V}$ be the set of pairs $(x, y)$ of real numbers and let $\mathbf{F}$ be the field of real numbers.

$$\begin{aligned} (x, y) + (x_1, y_1) &= (x + x_1, 0) \\ c(x, y) &= (cx, 0) \end{aligned}$$

Is $\mathbf{V}$ a vector space?

**Answer:** This is not a vector space because there would have to be an additive identity element (a, b) which has the property that (a, b) + (x, y) = (x, y) for all (x, y)$\in V$. But this is impossible, because (a, b) + (0, 1) = (a, 0) $\neq (0,1)$ no matter what (a, b) is. Thus V does not satisfy the third requirement of having an additive identity element.

3. Let $\mathbf{V}$ be the set of all complex valued functions $f$ on real line such that $\forall t \in \mathbb{R}$, $f(-t) = f((t))^* = f^*(t)$ Note: *f*(t) denotes the complex conjugation of f(t). For given problem, f(-t) is equal to the complex conjugation of f(t)*
Show that $\mathbf{V}$ with operations $(f+g)(t) = f(t) + g(t)$ and $(cf)(t) = cf(t)$ is a vector space over the field $\mathbb{R}$.
Give an example of a function $f^n$ in $\mathbf{V}$ which is not real valued.

**Answer:** Note: We will be using the Notation $A^* = \bar{A}$ to denote complex conjugation for easy readability.
We will use the basic fact that $\overline{a+b} = \bar{a} + \bar{b}$ and $\overline{ab} = \bar{a} \cdot \bar{b}$. Before we show $V$ satisfies the eight properties we must first show vector addition and scalar multiplication as defined are actually well-defined in the sense that they are indeed operations on $V$ . In other words if $f$ and $g$ are two functions in $V$ then we must show that $f+g$ is in $V$. In other words if $f(-t) = \overline{f(t)}$ and $g(-t) = \overline{g(t)}$ then we must show that $(f+g)(-t) = \overline{(f+g)(t)}$. This is true because

$$(f+g)(-t) = f(-t) + g(-t) = \overline{f(t)} + \overline{g(t)} = \overline{(f(t) + g(t)} = \overline{(f+g)(t)}$$

Similary, if $c \in \mathbb{R}, (cf)(-t) = cf(-t) = c\overline{f(t)} = \overline{cf(t)}$ since $\bar{c} = c$ when $c \in \mathbb{R}$. Thus the operations are well defined. We now show the eight properties hold:

1. Addition on functions in $V$ is defined by adding in $\mathbb{C}$ to the values of the functions in $\mathbb{C}$. Thus since $\mathbb{C}$ is commutative, addition in $V$ inherits this commutativity.
2. Similar to 1 , since $\mathbb{C}$ is associative, addition in $V$ inherits this associativity.
3. The zero function $g(t) = 0$ is in $V$ since $-0 = \bar{0}$. And $g$ satisfies $f + g = f$ for all $f \in V$. Thus $V$ has a right additive identity.
4. Let $g$ be the function $g(t) = -f(t)$. Then

$$g(-t) = -f(-t) = -\overline{f(t)} = \overline{-f(t)} = \overline{g(t)}$$

Thus $g \in V$. And
$$(f+g)(t) = f(t) + g(t) = f(t) - f(t) = 0.$$
Thus $g$ is a right additive inverse for $f$.
5. Clearly $1 \cdot f = f$ since 1 is the multiplicative identity in $\mathbb{R}$.
6. As before, associativity in $\mathbb{C}$ implies $(c_1 c_2) f = c_1 (c_2 f)$.
7. Similarly, the distributive property in $\mathbb{C}$ implies $c(f+g) = cf + cg$.

8. Similarly, the distributive property in $\mathbb{C}$ implies $(c_1 + c_2)\,f = c_1 f + c_2 f$. An example of a function in $V$ which is not real valued is $f(x) = ix$. Since $f(1) = i, f$ is not realvalued. And $f(-x) = -ix = \overline{ix}$ since $x \in \mathbb{R}$, so $f \in V$.

4. Prove the given theorems:

1. A non empty subset $\mathbf{W}$ of vector space $\mathbf{V}$ is a subspace of $\mathbf{V}$ if and only if, for each pair of vectors $\bar{\alpha}, \beta \in \mathbf{W}$ and each scalar $c \in \mathbf{F}$, the vector $c\bar{\alpha} + \beta \in \mathbf{W}$

   **Answer:** Suppose that W is a non-empty subset of V such that c$\alpha$+$\beta$ belongs to W for all vectors $\alpha$,$\beta$ in W and all scalars c in F. Since W is non-empty, there is a vector p in W, and hence (- 1)p + p = 0 is in W. Then if ($\alpha$ is any vector in W and c any scalar, the vector c$\alpha$ = c$\alpha$ + 0 is in W.In particular, (-1)$\alpha = -\alpha$ is in W.Finally,if $\alpha$ and $\beta$ are in W, then $\alpha$ + $\beta$ =1$\alpha$+$\beta$ is in W. Thus W is a subspace of V. Conversely, if W is a subspace of V, $\alpha$ and $\beta$ are in W, and c is a scalar,certainly c$\alpha$+$\beta$ is in W.

2. Let $\mathbf{V}$ be a vector space over the field $\mathbf{F}$. The intersection of any collection of subspaces of the $\mathbf{V}$ is a subspace of $\mathbf{V}$.

   **Answer:** Proof. To prove that the intersection $U \cap V$ is a subspace of $\mathbb{R}^n$, we check the following subspace criteria:

   (a) The zero vector $\mathbf{0}$ of $\mathbb{R}^n$ is in $U \cap V$.
   (b) For all $\mathbf{x}, \mathbf{y} \in U \cap V$, the sum $\mathbf{x} + \mathbf{y} \in U \cap V$.
   (c) For all $\mathbf{x} \in U \cap V$ and $r \in \mathbb{R}$, we have $r\mathbf{x} \in U \cap V$.

   - As $U$ and $V$ are subspaces of $\mathbb{R}^n$, the zero vector $\mathbf{0}$ is in both $U$ and $V$. Hence the zero vector $\mathbf{0} \in \mathbb{R}^n$ lies in the intersection $U \cap V$. So condition 1 is met.

   - Suppose that $\mathbf{x}, \mathbf{y} \in U \cap V$. This implies that $\mathbf{x}$ is a vector in $U$ as well as a vector in $V$. Similarly, $\mathbf{y}$ is a vector in $U$ as well as a vector in $V$. Since $U$ is a subspace and $\mathbf{x}$ and $\mathbf{y}$ are both vectors in $U$, their sum $\mathbf{x} + \mathbf{y}$ is in $U$. Similarly, since $V$ is a subspace and $\mathbf{x}$ and $\mathbf{y}$ are both vectors in $V$, their sum $\mathbf{x} + \mathbf{y} \in V$. Therefore the sum $\mathbf{x} + \mathbf{y}$ is a vector in both $U$ and $V$. Hence $\mathbf{x} + \mathbf{y} \in U \cap V$. Thus condition 2 is met.

   - To verify condition 3 , let $\mathbf{x} \in U \cap V$ and $r \in \mathbb{R}$. As $\mathbf{x} \in U \cap V$, the vector $\mathbf{x}$ lies in both $U$ and $V$. Since both $U$ and $V$ are subspaces, the scalar multiplication is closed in $U$ and $V$, respectively. Thus $r\mathbf{x} \in U$ and $r\mathbf{x} \in V$. It follows that $r\mathbf{x} \in U \cap V$. This proves condition 3 , and hence the intersection $U \cap V$ is a subspace of $\mathbb{R}^n$.