Discrete Structures (Monsoon 2021)

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Pigeonhole Principle

Pigeonhole Principle

Theorem

If n pigeons are assigned to m pigeonholes, and $m < n$, then at least one pigeonhole contains two or more pigeons.

Proof.

Suppose that each pigeonhole contains at most one pigeon. Then, at most m pigeons have been assigned.
But, since $m < n$, not all pigeons have been assigned pigeonholes.
This is a contradiction. Hence, at least one pigeonhole contains two or more pigeons. $\square$

Pigeonhole Principle

- Mathematically, we can express the Pigeonhole Principle as follows:
  There exists a function $f : D \to C$, D is the set of pigeons (domain set) and C is the set of pigeonholes (co-domain set) such that $|D| > |C|$ and $\exists d_1, d_2 \in D$ such that $f(d_1) = f(d_2)$, where $d_1 \neq d_2$.
- In general, if $\left\lceil \frac{|D|}{|C|} \right\rceil = k$, a positive integer, then $\exists d_1, d_2, \ldots, d_k \in D$ such that $f(d_1) = f(d_2) = \cdots = f(d_k)$.

Pigeonhole Principle

Problem: Given a sequence of (n^2+1) distinct integers. Then to prove that:
there is either an increasing sub-sequence of length $(n+1)$, or a decreasing sub-sequence of length $(n+1)$.

Solution: Let $S = \{45, 25, 39, 16, 11, 7, 120, 63, 94, 56\}$ be a sequence of distinct integers. Then, $\{45, 63, 94\}$ is an increasing sub-sequence, whereas $\{25, 16, 11, 7\}$ is a decreasing sub-sequence.
Let $a_1, a_2, \cdots, a_{n^2+1}$ be the distinct integers. Consider the ordered pairs (x_k, y_k) for a_k where
$x_k =$ maximum length of an increasing sequence starting from a_k,
$y_k =$ maximum length of a decreasing sequence starting from a_k.
Assume that there be NO sub-sequence of length $(n+1)$ neither increasing nor decreasing.
Therefore, the values of x_k and y_k lie between 1 and n, that is, $1 \leq x_k \leq n$ and $1 \leq y_k \leq n$, for $k = 1, 2, \ldots, n^2+1$.

Pigeonhole Principle

Then, there are $n \times n = n^2$ possible distinct ordered pairs. But, the total distinct integers are $n^2 + 1$. By the pigeonhole principle, the ordered pairs must be same.
Let these ordered pairs be $\langle x_i, y_i \rangle$ and $\langle x_j, y_j \rangle$.
Without any loss of generality, let $i < j$.
Since $\langle x_i, y_i \rangle = \langle x_j, y_j \rangle$, we have two cases:

- If $a_i < a_j$, then $x_i > x_j$.
- If $a_i > a_j$, then $y_i > y_j$. This is impossible

Hence, there is either an increasing sub-sequence of length $(n + 1)$ or a decreasing sub-sequence of length $(n + 1)$.

The Generalized Pigeonhole Principle

Theorem

If m pigeons are assigned to n pigeonholes, there must be a pigeonhole containing at least $\left\lfloor \frac{m-1}{n} \right\rfloor + 1$ *pigeons.*

Proof.

(Proof by Contradiction)
Suppose no pigeonhole contains more than $\left\lfloor \frac{m-1}{n} \right\rfloor$ pigeons. Then, maximum number of pigeons

$$= n * \left\lfloor \frac{m-1}{n} \right\rfloor \leq n * \frac{m-1}{n} = m - 1$$

This contradicts our assumption that there are m pigeons. Thus, one pigeonhole must contain at least $\left\lfloor \frac{m-1}{n} \right\rfloor + 1$ pigeons. $\square$

The Generalized Pigeonhole Principle

Problem: If we select any group of 1000 students on Campus, show that at least three of them must have the same birthday.

Solution: The maximum number of days in a year is 366 (including the leap year, 29 days in February).
Think of students as pigeons and days of the year as pigeonholes.
Then, by the *Generalized Pigeonhole Principle*, the minimum number of students having the same birthday is $\lfloor \frac{1000-1}{366} \rfloor + 1 = 2 + 1 = 3$, where $m = 1000$ and $n = 366$.

The Generalized Pigeonhole Principle

Problem: Ten people came forward to volunteer for a three person committee. Every possible committee of three that can be formed from these ten names is written on a slip of paper, one slip for each possible committee and the slips are put in 10 hats. Show that at least one hat contains 12 or more slips of paper.

Solution: A committee of three (3) people can be chosen from 10 names in ${}^{10}C_3 = \frac{10!}{3!7!} = 120$ ways.
Thus, there are 120 slips (pigeons) in which these committees are written.
The slips are put in 10 hats (pigeonholes).
So, by the *Generalized (Extended) Pigeonhole Principle*, one hat must contain at least $\lfloor \frac{120-1}{10} \rfloor + 1 = 11 + 1 = 12$ or more slips of paper.