**Probability and Statistics**
**Tutorial 3**

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# Continuous Random Variables and MGF

**Question 1.**
Let X be a continuous random variable with PDF given by

$$f_X(x) = \frac{1}{2}e^{-|x|}, \quad \text{for all } x \in \mathbb{R}$$

If $Y = X^2$, find the CDF of $Y$.

**Question 2.**
Let X be a random variable with PDF given by

$$f_X(x) = \begin{cases} cx^2 & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

a. Find the constant $c$.
b. Find $E[X]$ and $Var(X)$.
c. Find $P(X \geq \frac{1}{2})$.

**Question 3.**
Let X be a random variable with PDF given by

$$f_X(x) = \begin{cases} cxe^{-x} & \text{if } x \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

a. Find the constant $c$.
b. Find $E[X]$.

**Question 4.** (Exponential as the limit of Geometric)
Let $Y \sim \mathit{Geometric}(p)$, where $p = \lambda\Delta$. Define $X = Y\Delta$, where $\lambda, \Delta > 0$. Prove that for any $x \in (0, \infty)$, we have

$$\lim_{\Delta \to \infty} F_X(x) = 1 - e^{-\lambda x}$$

**Question 5.**
Prove that exponential random variable is memoryless. If $X$ is exponential with parameter $\lambda > 0$, then $X$ is a memoryless random variable, that is

$$P(X > x + a | X > a) = P(X > x), \quad \text{for } a, x \geq 0.$$

**Question 6.**
The Moment Generating Function (MGF) for a random variable $X$, $mgf_X(t)$ is given.

$$mgf_x(t) = \frac{1}{10}e^{-20t} + \frac{1}{5}e^{-3t} + \frac{3}{10}e^{4t} + \frac{2}{5}e^{5t}$$

Find the probability:

$$P(|X| \leq 2)$$

Hint: use the uniqueness theorem for MGFs. If any two MGFs are same, then their distributions are also exactly same. [Wikipedia]

**Question 7.**
Given a random variable X with Moment Generating function M(t), compute the Moment generating functions for the following random variables (in terms of M). In each subpart, k is a scalar.
a. kX
b. X + k
c. $\Sigma_{i=0}^{N} X_i$ where all $X_i$ are sampled independently from X
d. A random variable Y with PDF(Y) = PDF (X + k), both PDF defined over $\mathbb{R}$.
e. A random variable Y with PDF(Y) = PDF (2X).

**Question 8.**
A city's temperature is modelled as a normal random variable with mean and standard deviation both equal to 10 degrees Celsius. What is the probability that the temperature at a randomly chosen time will be less than or equal to 59 degrees Fahrenheit?

**Question 9.**
If $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$ are independent, then prove that $X + Y \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$.

**Question 10.**
You are going from College (Point A) to some distant eastern part of Hyderabad(Point B) which is 25 kms away. While your friend starts from (Point B) towards (Point A) with the goal of meeting you. Both of you travel at 50 km/h towards each other. Both your starting time is truly random and uniformly distributed from 1 pm to 2 pm and both your starting time is independent of each other. Let the random variable X denote the distance between college and the point where both Question of you meet. Find $F_X = P(X \leq x)$.

# Questions on Discrete Random Variables

## Question 11:

Let $X$ be a discrete random variable with the following PMF:

$$P_X(k) = \begin{cases} 1/4 & \text{for } k = -2 \\ 1/8 & \text{for } k = -1 \\ 1/8 & \text{for } k = 0 \\ 1/4 & \text{for } k = 1 \\ 1/4 & \text{for } k = 2 \\ 0 & otherwise \end{cases}$$

We define a new random variable $Y$ as $Y = (X+1)^2$. Answer the following:

1. Find $R_Y$ range of $Y$.
2. Find the PMF of $Y$.

## Question 12:

A human rolls two dice and observes two numbers $X$ and $Y$. Find:

1. $R_X$, $R_Y$ and PMFs of $X$ and $Y$.
2. $P(X = 2, Y = 6)$
3. $P(X > 3 | Y = 2)$
4. Let $Z = X + Y$. Find the range and PMF of $Z$.
5. $P(X = 4 | Z = 8)$

## Question 13:

Let $X$ be a discrete random variable witht he following PMF

$$P_X(x) = \begin{cases} 0.3 & \text{for } k = 3 \\ 0.2 & \text{for } k = 5 \\ 0.3 & \text{for } k = 8 \\ 0.2 & \text{for } k = 10 \\ 0 & otherwise \end{cases}$$

Find and plot the CDF of $X$.

## Question 14:

Let $X$ and $Y$ be two independent random variables. Suppose that we know $Var(2X - Y) = 6$ and $Var(X + 2Y) = 9$. Find $Var(X)$ and $Var(Y)$.

## Question 15:

Let $X$ be a discrete random variable with

$$R_X \subset \{0, 1, 2, ...\}$$

Prove that

$$EX = \sum_{k=0}^{\infty} P(X > k)$$

## Question 16:

Let $X$ have range [0,3] and density $f_x(x) = kx^2$. Let $Y = X^3$.

1. Find $k$ and the cumulative distribution function of $X$.
2. Compute $E[Y]$.
3. Compute $Var[Y]$.
4. Find the probability density function $f_y(y)$ for $Y$.

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