Discrete Structures (Monsoon 2021)

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Countable Set

Countable or Denumerable Set

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Definition

A set is said to be *countable*, if it is finite or denumerable.

- If a set A be cardinally equivalent to the set of natural numbers $N = \{1, 2, 3, \ldots\}$, A is called a denumerable (or enumerable) set.
- In other words, there exists a mapping $f : N \rightarrow A$, which is one-one and onto, that is, there exists an one-one correspondence between N and A.
- Thus, A is coutable if there exists a bijection $f : N \rightarrow A$.

Countable or Denumerable Set

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Theorem

The set of all integers is countable.

Proof.

Let Z be the set of all intergers, that is,
$Z = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ and N be the set of natural numbers.
Claim: There exists one-one correspondence between N and Z.
Construct the mapping $f : N \to Z$ that is defined by

$$f(n) = \begin{cases} \frac{n}{2}, \text{ if } n \in N \text{ is even} \\ -\frac{n-1}{2}, \text{ if } n \in N \text{ is odd} \end{cases}$$

□

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- To prove that f is one-one, let $n_1, n_2 \in N$ such that $f(n_1) = f(n_2)$.
  Case 1. n is even.
  Then, $f(n_1) = f(n_2)$
  $\Rightarrow \frac{n_1}{2} = \frac{n_2}{2}$
  $\Rightarrow n_1 = n_2$.
  Case 2. n is odd.
  Then, $f(n_1) = f(n_2)$
  $\Rightarrow -\frac{n_1-1}{2} = -\frac{n_2-1}{2}$
  $\Rightarrow 1 - n_1 = 1 - n_2$
  $\Rightarrow n_1 = n_2$

In both the cases, $f(n_1) = f(n_2) \Rightarrow n_1 = n_2$, for all $n_1, n_2 \in N$.
Hence, f is one-one.

Countable or Denumerable Set

- To prove that f is onto, let $m \in Z$.
  Then, there exists $n \in N$ such that $m = f(n) = \frac{n}{2}$ when n is even and $m = f(n) = -\frac{n-1}{2}$ when n is odd.
  Case 1. n is even.
  $n = 2m \in N$, where m is a positive integer, that is, $m > 0$
  Case 2. n is odd.
  $1 - n = 2m$ or $n = 1 - 2m \in N$, where m is a non-negative integer, that is, $m \leq 0$

Combing both the cases, f is onto.
Since f is both one-one and onto, f is bijective. As a result, Z is countable.

Countable or Denumerable Set

Theorem

The union of a countable collection of countable sets is again countable.

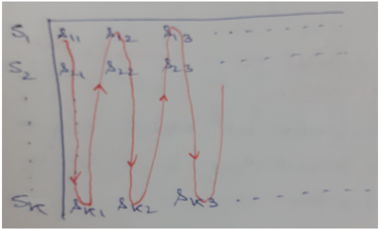
Proof. Consider the following two cases:

- **Case 1.** Let $S_1, S_2, \ldots, S_k$ be a finite collection of countable sets, which are finite sets. Then, by the principle of inclusion-exclusion of sets,

$$|S_1 \cup S_2 \cup \cdots \cup S_k| \leq |S_1| + |S_2| + \cdots + |S_k|$$

  Thus, the union of S_1 $S_2, \ldots, S_k$ is *countably finite*.

Countable or Denumerable Set



- **Case 2.** Let $S_1, S_2, \ldots, S_k$ be a finite collection of countable sets, where each S_i is an infinite set.
  * Let $S_i = \{s_{i1}, s_{i2}, s_{i3}, \ldots\}$. Then, construct the table as shown in figure.
  * For union of $S_1, S_2, \ldots, S_k$, we traverse the elements of $S_1, S_2, \ldots, S_k$ in column-wise which are finite. To check if an element is repeated or not, we can go back to check all the previous elements traverse.
  * Again, this is done by checking a finite number of elements. Thus, the union is countably infinite.

Countable or Denumerable Set

Theorem

Every subset of a countable set is either finite or countable.

Theorem

Every infinite set has a countable subset.

Proof. Let A be an infinite set. Then, A is non-empty, so choose $a_1 \in A$.
Let $A_1 = A - \{a_1\}$. Since A is infinite, therefore $A_1 \neq \emptyset$.
Choose $a_2 \in A_1$ and let $A_2 = A - \{a_1, a_2\}$. Again, $A_2 \neq \emptyset$.
Choose $a_3 \in A_2$ and let $A_3 = A - \{a_1, a_2, a_3\}$.
Continuing in this way, we obtain a countably subset $\{a_1, a_2, a_3, \ldots\}$ of A.

Countable or Denumerable Set

Problem. The set of real numbers between 0 and 1 is not a countable set. Hence, the set of real numbers, R, is not also a countable set.

Proof [Proof by Contradiction].

Part 1: If the real numbers between 0 and 1 were countable, they could be written as a succession:

$$x_1, x_2, x_3, x_4, \cdots, x_n, \cdots \tag{1}$$

Let us express each x_n as a decimal. If we agree not to use recurring 9's, this can be done in only one way. Let us agree to this and write the decimals as:

$$\begin{aligned} x_1 &= 0.a_1 a_2 a_3 a_4 \cdots \\ x_2 &= 0.b_1 b_2 b_3 b_4 \cdots \\ x_3 &= 0.c_1 c_2 c_3 c_4 \cdots \\ \cdots & \quad \cdots \end{aligned}$$

where $a_i, b_i, c_i \in \{0, 1, 2, 3, \cdots, 9\}$

Countable or Denumerable Set

Now, let us take the diagonal $a_1 b_2 c_3 \cdots$ and form a decimal:

$$0.\alpha\beta\gamma\cdots$$

by defining

$$\alpha = \begin{cases} 1, \text{ if } a_1 \neq 1 \\ 2, \text{ if } a_1 = 1 \end{cases}$$

$$\beta = \begin{cases} 1, \text{ if } b_2 \neq 1 \\ 2, \text{ if } b_2 = 1 \end{cases}$$

$$\gamma = \begin{cases} 1, \text{ if } c_3 \neq 1 \\ 2, \text{ if } c_3 = 1 \end{cases}$$

and so on.

Countable or Denumerable Set

Then, $y = 0.\alpha\beta\gamma\cdots$ (which contains no $\dot{9}$) denotes a real number between 0 and 1, and so must itself appear somewhere in the succession in (1), if this succession is to contain all the real numbers between 0 and 1.
But,

- $y \neq x_1$ as it differs from x_1 in the first place after the decimal point;
- $y \neq x_2$ as it differs from x_2 in the second place after the decimal point;
- $y \neq x_3$ as it differs from x_3 in the third place after the decimal point; and so on.

Hence, $y \notin$ the succession (1) at all, but y is a real number between 0 and 1.
Thus, no succession $x_1, x_2, x_3, \cdots$ can include all the real numbers between 0 and 1, i.e., the set of real numbers between 0 and 1 is not countable, that is, $(0, 1)$ is uncountable.

Countable or Denumerable Set

Part 2:

Let us consider any open interval (a, b), with $b > a$. Then, we can write $w = \frac{x-a}{b-a}$, $a < x < b$.
Thus, $0 < w < 1$. Since $(0, 1)$ is uncountable, (a, b) is not also countable.
Now, $R =$ the set of real numbers can be expressed as

$$R = \bigcup_i (a_i, b_i)$$

where $(a_i, b_i) \subset R$.
Since (a_i, b_i) is not countable, R is not also countable.