For $b = \nabla a$, we have:

$$b_r = \begin{cases} a_0, & r = 0 \\ a_r - a_{r-1}, & r \geq 1. \end{cases}$$

Then, $B(z) =$ G.f. of n.f. $b$

$$= \sum_{r=0}^{\infty} b_r z^r$$

$$= b_0 + b_1 z + b_2 z^2 + \cdots + b_r z^r + \cdots$$

$$= a_0 + (a_1 - a_0) z + (a_2 - a_1) z^2 + \cdots + (a_r - a_{r-1}) z^r + \cdots$$

$$= [a_0 + a_1 z + a_2 z^2 + \cdots + a_r z^r + \cdots] - z[a_0 + a_1 z + a_2 z^2 + \cdots + a_{r-1} z^{r-1} + a_r z^r + \cdots]$$

$$= A(z) - z \cdot A(z)$$

$$\therefore \boxed{B(z) = (1 - z) A(z).}$$

If $c = a * b$, then $C(z) = A(z) \cdot B(z)$

$$a = (a_0, a_1, a_2, \ldots, a_r, \ldots)$$

$$b = (b_0, b_1, b_2, \ldots, b_r, \ldots)$$

$$c = (c_0, c_1, c_2, \ldots, c_r, \ldots)$$

$$\text{RHS} = A(z) \cdot B(z)$$

$$= [a_0 + a_1 z + a_2 z^2 + \cdots + a_r z^r + \cdots] \times [b_0 + b_1 z + b_2 z^2 + \cdots + b_r z^r + \cdots]$$

$$= (a_0 b_0) z^0 + (a_0 b_1 + a_1 b_0) z^1 + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + \cdots + (a_0 b_r + a_1 b_{r-1} + \cdots + a_r b_0) z^r + \cdots$$

$$= c_0 + c_1 z + c_2 z^2 + \cdots + c_r z^r + \cdots$$

where $c_r = a_0 b_r + a_1 b_{r-1} + \cdots + a_r b_0$

$$= \sum_{i=0}^{r} a_i b_{r-i}$$

$$= C(z) = \text{LHS}.$$

$$* \quad C(z) = \frac{\alpha}{1-2z} + \frac{\beta}{1-3z} = \frac{1+0.z}{(1-3z)(1-2z)}$$

$$= \frac{\alpha(1-3z) + \beta(1-2z)}{(1-2z)(1-3z)}$$

$$= \frac{(\alpha+\beta) + (-3\alpha - 2\beta)z}{(1-2z)(1-3z)}$$

$\therefore \quad \alpha + \beta = 1 \quad \cdots (1) \Rightarrow \beta = 1 - \alpha$

$-3\alpha - 2\beta = 0 \quad \cdots (2) \Rightarrow 2\beta = -3\alpha$

$\Rightarrow 2(1-\alpha) = -3\alpha$

$\Rightarrow \alpha = -2, \quad \underline{\underline{\beta = 3}}.$

$$\therefore \quad C(z) = \sum_{r=0}^{\infty} c_r z^r$$

$$= -\frac{2}{1-2z} + \frac{3}{1-3z}$$

$$= -2 \sum_{r=0}^{\infty} 2^r z^r + 3 \cdot \sum_{r=0}^{\infty} 3^r z^r$$

$$= \sum_{r=0}^{\infty} \underbrace{\left[ 3^{r+1} - 2^{r+1} \right]}_{c_r} \cdot z^r$$

$$\Rightarrow \boxed{c_r = 3^{r+1} - 2^{r+1}, \quad r \geq 0}.$$

• First determine the G.f. of the numeric function $a = (0^2, 1^2, 2^2, 3^2, \dots, r^2, \dots)$.

We have:

$$\frac{1}{1-z} = 1 + z + z^2 + \dots + z^r + \dots \quad \text{[converges if } |z| < 1 \text{ i.e., } -1 < z < 1]$$

$$\frac{d}{dz}\left(\frac{1}{1-z}\right) = 0 + 1 + 2z + 3z^2 + \dots + rz^{r-1} + \dots$$

$$\Rightarrow \frac{z}{(1-z)^2} = 0 \cdot z^0 + 1 \cdot z + 2z^2 + 3z^3 + \dots + rz^r + \dots$$

$$\frac{d}{dz}\left[\frac{z}{(1-z)^2}\right] = 1 + 2^2 z + 3^2 z^2 + \dots + r^2 z^{r-1} + \dots$$

$$\Rightarrow \frac{1}{(1-z)^2} + z \cdot (-2)(1-z)^{-3} \cdot (-1)$$

$$= \frac{1+z}{(1-z)^3} = 1 + 2^2 z + 3^2 z^2 + \dots + r^2 z^{r-1} + \dots$$

$$\Rightarrow \frac{z(1+z)}{(1-z)^3} = 1 \cdot z + 2^2 \cdot z^2 + 3^2 \cdot z^3 + \dots + r^2 \cdot z^r + \dots$$

$$= 0^2 \cdot z^0 + 1^2 \cdot z^1 + 2^2 \cdot z^2 + \dots + r^2 \cdot z^r + \dots$$

$$\Rightarrow \text{G.f. of } (0^2, 1^2, 2^2, \dots, r^2, \dots) = \frac{z(1+z)}{(1-z)^3}.$$

Now, $$\frac{z(1+z)}{(1-z)^4} = \frac{z(1+z)}{(1-z)^3 \cdot (1-z)}$$

$$= \frac{z(1+z)}{(1-z)^3} \cdot (1-z)^{-1}$$

$$= (0^2 \cdot z^0 + 1^2 \cdot z^1 + 2^2 \cdot z^2 + \dots + r^2 z^r + \dots) \times (1 + z + z^2 + \dots + z^r + \dots)$$

$$= 0^2 \cdot z^0 + (0^2 + 1^2) z^1 + (0^2 + 1^2 + 2^2) \cdot z^2 + \dots + (\underline{0^2 + 1^2 + 2^2 + \dots + r^2}) z^r + \dots$$

Then, $$\frac{z(1+z)}{(1-z)^4} = \sum_{r=0}^{\infty} a_r z^r,$$ then $a_r = 0^2 + 1^2 + 2^2 + \dots + r^2 = 1^2 + 2^2 + \dots + r^2.$

Again, $\frac{z(1+z)}{(1-z)^4} = z(1+z)(1-z)^{-4}$

$$= (z+z^2)\cdot\left[1+(-4)\cdot z+\frac{(-4)(-4-1)}{2!}z^2+\cdots + (-1)^r\cdot\frac{(-4)(-4-1)\cdots(-4-\overline{r-1})}{r!}z^r+\cdots\right]$$

Then,

$$(-1)^r\cdot\frac{(-4)(-4-1)\cdots(-4-\overline{r-1})}{r!}$$

$$= (-1)^r\cdot(-1)^r\cdot\frac{4\cdot5\cdot6\cdots(r+3)}{r!}$$

$$= (-1)^{2r}\cdot\frac{(1.2.3.\cancel{4.5.6\cdots..r}).(r+1)(r+2)(r+3)}{1.2.3.\ \cancel{r!}}$$

$$= \frac{1}{6}(r+1)(r+2)(r+3)$$

$$A(z) = \sum_{r=0}^{\infty} a_r z^r$$

$$= (z+z^2)\left[1-4z+\frac{4.5}{2!}z^2-\cdots + \frac{1}{6}.(r+1)(r+2)(r+3)z^r+\cdots\right]$$

$\therefore$ $a_r$ = coefficient of $z^r$

$$= \frac{1}{6}(r-1+1)(r-1+2)(r-1+3) + \frac{1}{6}(r-2+1)(r-2+2)(r-2+3)$$

$$= \frac{1}{6}(r)(r+1)(r+2)+\frac{1}{6}(r-1)(r)(r+1)$$

$$= \frac{1}{6}r(r+1)[r+2+r-1]$$

$$= \frac{1}{6}r(r+1)(2r+1)$$

Hence,

$$\boxed{\begin{aligned} a_n &= 0^2+1^2+2^2+\cdots+n^2 \\ &= 1^2+2^2+\cdots+n^2 \\ &= \frac{1}{6}\,n\,(n+1)(2n+1) \end{aligned}}$$