Assignment 1

MA2.101: Linear Algebra (Spring 2022)

June 8, 2022

Problem 1

Find the orthogonal basis for $\mathbb{R}^3$ that contains the vector

1. $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ 2. $v = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$

Part 1

given $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and given $\beta = \{v, v_1, v_2\}$ form a orthogonal basis for $\mathbb{R}^3$. As the basis β is orthogonal we have to find v_1, v_2 such that $\langle v, v_1 \rangle = 0, \langle v, v_2 \rangle = 0, \langle v_1, v_2 \rangle = 0$.

lets try to find v_1 first. As v, v_1 are orthogonal, $v^T v_1 = 0$.

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

one possible solution for v_1 such that it is orthogonal to v is $v_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

now using the vectors v, v_1 we can find the vector $v2$ which is orthogonal to both $v, v1$. So, $v^T v_2 = 0, v_1^T v_1 = 0$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\begin{bmatrix} -1 & -1 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

by solving the above two equations, we find the value of $v_2 = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}$

As the vectors $v, v1, v2$ are linearly independent, they form a valid basis for $\mathbb{R}^3$.

So, the set of vectors $\left\{ \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \right\}$ form a valid basis

Part 2

given $v = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$ and given $\beta = \{v, v_1, v_2\}$ form a orthogonal basis for $\mathbb{R}^3$. As the basis β is orthogonal we have to find v_1, v_2 such that $\langle v, v_1 \rangle = 0, \langle v, v_2 \rangle = 0, \langle v_1, v_2 \rangle = 0$.

lets try to find v_1 first. As v, v_1 are orthogonal, $v^T v_1 = 0$.

$$\begin{bmatrix} 3 & 1 & 5 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

one possible solution for v_1 such that it is orthogonal to v is $v_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

now using the vectors v, v_1 we can find the vector $v2$ which is orthogonal to both $v, v1$. So, $v^T v_2 = 0, v_1^T v_1 = 0$

$$\begin{bmatrix} 3 & 1 & 5 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

by solving the above two equations, we find the value of $v_2 = \begin{bmatrix} -11 \\ 8 \\ 5 \end{bmatrix}$

As the vectors $v, v1, v2$ are linearly independent, they form a valid basis for $\mathbb{R}^3$.

So, the set of vectors $\left\{ \begin{pmatrix} 3 \\ 1 \\ -11 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix}, \begin{pmatrix} 5 \\ -1 \\ 5 \end{pmatrix} \right\}$ form a valid basis

Problem 2

Find the orthogonal of $\mathbb{R}^4$ that contains the vector

$$v = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}$$

Let $w = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix}$ be orthogonal to $v \Rightarrow m_1 + 2m_2 - m_3 = 0$

$\therefore w = \begin{pmatrix} m_1 \\ m-2 \\ m_1 + 2m_2 \\ m_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} m_1 + \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix} m_2 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} m_4$

Now, consider the set:

$B = \{b_1, b_2, b_3, b_4\} = \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$. For $B = \{b_1, b_2, b_3, b_4\}$ to be linearly independent,

$$\begin{aligned} c_1b_1 + c_2b_2 + c_3b_3 + c_4b_4 &= 0 \\ \Rightarrow c_1 + c_2 &= 0 \\ \Rightarrow 2c_1 + c_3 &= 0 \\ \Rightarrow -c_1 + c_2 + 2c_3 &= 0 \\ \Rightarrow \boxed{c_4 = 0} \end{aligned}$$

Using $c_2 = -c_1$ and $c_3 = -2c_1$ we get:

$$\boxed{c_1 = c_2 = c_3 = 0}$$

$\therefore$ The vectors b_1, b_2, b_3, b_4 are linearly independent. B forms a basis of $\mathbb{R}^4$

Now using Gram Schmidt Process on B, we get orthogonal basis $\{v_1, v_2, v_3, v_4\}$ such that

$$v_1 = b_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}$$

$$v_2 = b_2 - \left(\frac{\langle v_1, b_2\rangle}{\langle v_1, v_1\rangle}\right) v_1$$

$$= \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - 0v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$v_3 = b_3 - \left(\frac{\langle v_1, b_3\rangle}{\langle v_1, v_1\rangle}\right) v_1 - \left(\frac{\langle v_2, b_3\rangle}{\langle v_2, v_2\rangle}\right) v_2$$

$$= \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix} - 0v_1 - 1v_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$v_4 = b_4 - \left(\frac{\langle v_1, b_4\rangle}{\langle v_1, v_1\rangle}\right) v_1 - \left(\frac{\langle v_2, b_4\rangle}{\langle v_2, v_2\rangle}\right) v_2 - \left(\frac{\langle v_3, b_4\rangle}{\langle v_3, v_3\rangle}\right) v_3$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} - 0v_1 - 0v_2 - 0v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Hence, the orthogonal basis set containing v is: $\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

Problem 3

Apply Gram Schmidt process to construct an orthonormal basis for the subspace W = span(x_1, x_2, x_3) where $x_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$, $x_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $x_3 = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 2 \end{pmatrix}$ **.**

Here as no inner product is specified we consider the dot product as the inner product.
From Gram schmidt process we know that to get orthogonal basis $\alpha_1, \alpha_2, \alpha_3$ we have

$\alpha_1 = x_1$, $\alpha_2 = x_2 - \frac{(x_2|\alpha_1)}{||\alpha_1||^2}\alpha_1$, $\alpha_3 = x_3 - \frac{(x_3|\alpha_1)}{||\alpha_1||^2}\alpha_1 - \frac{(x_3|\alpha_2)}{||\alpha_2||^2}\alpha_2$

$$\implies \alpha_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$(x_2|\alpha_1) = (2)(1) + (1)(-1) + (0)(-1) + (1)(1) = 2 - 1 + 0 + 1 = 2$
$||\alpha_1||^2 = (\alpha_1|\alpha_1) = (1)^2 + (-1)^2 + (-1)^2 + (1)^2 = 4$
$||\alpha_1|| = 2$

$$\implies \alpha_2 = x_2 - \tfrac{2}{4}\alpha_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0.5 \\ -0.5 \\ -0.5 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 1.5 \\ 1.5 \\ 0.5 \\ 0.5 \end{pmatrix}$$

$(x_3|\alpha_1) = (2)(1) + (2)(-1) + (1)(-1) + (2)(1) = 2 - 2 - 1 + 2 = 1$
$||\alpha_1||^2 = (\alpha_1|\alpha_1) = 1^2 + (-1)^2 + (-1)^2 + (1^2) = 4$
$(x_3|\alpha_2) = (2)(1.5) + (2)(1.5) + (1)(0.5) + (2)(0.5) = 3 + 3 + 0.5 + 1 = 7.5$
$||\alpha_2||^2 = (\alpha_2|\alpha_2) = (1.5)^2 + (1.5)^2 + (0.5)^2 + (0.5)^2 = 2.25 + 2.25 + 0.25 + 0.25 = 5$
$||\alpha_2|| = \sqrt{5}$

$$\implies \alpha_3 = x_3 - \tfrac{1}{4}\alpha_1 - \tfrac{7.5}{5}\alpha_2 = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0.25 \\ -0.25 \\ -0.25 \\ 0.25 \end{pmatrix} - \begin{pmatrix} 2.25 \\ 2.25 \\ 0.75 \\ 0.75 \end{pmatrix} = \begin{pmatrix} -0.5 \\ 0 \\ 0.5 \\ 1 \end{pmatrix}$$

$||\alpha_3||^2 = (\alpha_3|\alpha_3) = (-0.5)^2 + (0)^2 + (0.5)^2 + (1)^2 = 0.25 + 0 + 0.25 + 1 = 1.5$
$||\alpha_3|| = \sqrt{1.5} = \sqrt{\frac{3}{2}}$
Now to get the orthonormal basis we divide each α_i with the respective $||\alpha_i||$ where i= 1,2,3.

$$\implies \frac{\alpha_1}{|\alpha_1||} = \begin{pmatrix} 0.5 \\ -0.5 \\ -0.5 \\ 0.5 \end{pmatrix}, \frac{\alpha_2}{|\alpha_2||} = \begin{pmatrix} 1.5/\sqrt{5} \\ 1.5/\sqrt{5} \\ 0.5/\sqrt{5} \\ 0.5/\sqrt{5} \end{pmatrix}, \frac{\alpha_3}{|\alpha_3||} = \begin{pmatrix} -1/\sqrt{6} \\ 0 \\ 1/\sqrt{6} \\ \sqrt{2}/\sqrt{3} \end{pmatrix}$$

$\therefore$ The orthonormal basis for the subspace W is $\{ \begin{pmatrix} 0.5 \\ -0.5 \\ -0.5 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 1.5/\sqrt{5} \\ 1.5/\sqrt{5} \\ 0.5/\sqrt{5} \\ 0.5/\sqrt{5} \end{pmatrix}, \begin{pmatrix} -1/\sqrt{6} \\ 0 \\ 1/\sqrt{6} \\ \sqrt{2}/\sqrt{3} \end{pmatrix} \}$

Problem 4

In the following given are the vectors from R^2 and R^3. Apply Gram Schmidt process to obtain the orthogonal basis. Then normalize the basis to obtain orthonormal basis.

1. $x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $x_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\begin{aligned} v_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ v_2 &= x_2 - proj_{v_1} x_2 \\ &= \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix} \end{aligned}$$

The orthonormal basis obtained after converting v_1 and v_2 to orthonormal vectors are

$$\begin{aligned} e_1 &= \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \\ e_2 &= \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \end{aligned}$$

2. $x_1 = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$, $x_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$
Applying the procedure similar to the above, we obtain the following orthonormal basis vectors:

$$\begin{aligned} e_1 &= \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \\ e_2 &= \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \end{aligned}$$

3. $x_1=\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$, $x_2=\begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}$, $x_2=\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$

$$\begin{aligned} v_1 &= \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \\ v_2 &= x_2 - proj_{v_1} x_2 \\ &= \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \\ v_3 &= x_3 - proj_{v_1} x_3 - proj_{v_2} x_3 \\ &= \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \end{aligned}$$

The orthonormal basis obtained after converting v_1 and v_2 and v_3 to orthonormal vectors are

$$\begin{aligned} e_1 &= \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ -1/\sqrt{3} \end{pmatrix} \\ e_2 &= \begin{pmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix} \\ e_3 &= \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \end{aligned}$$

4. $x_1=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $x_2=\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $x_2=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Applying the procedure similar to the above, we obtain the following orthonormal basis vectors:

$$\begin{aligned} e_1 &= \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \\ e_2 &= \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{pmatrix} \\ e_3 &= \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} \end{aligned}$$

Problem 5

Show that in an inner product space there can not be unit vectors u and v with $\langle u, v\rangle < -1$

Since u and v are unit vectors,

$$||u|| = 1 = ||v||$$

Using Cauchy - Schwartz Inequality,

$$\begin{aligned} |\langle u, v\rangle| &\leq ||u||||v|| \\ &\leq 1 \\ \Rightarrow -1 \leq &\langle u, v\rangle \leq 1 \end{aligned}$$

Hence it is impossible to have $\langle u, v\rangle < -1$ for any unit vectors u and v. Q.E.D.

Problem 6

Let u and v are two vectors in the inner product space V. Then show that $||u+v|| \leq ||u||+||v||$.

$$\begin{aligned} ||u+v||^2 &= \langle u+v, u+v\rangle \\ &= \langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle \\ &= ||u||^2 + 2\text{Re}(\langle u, v\rangle) + ||v||^2 \\ &\leq ||u||^2 + 2\langle u, v\rangle + ||v||^2 \\ \text{(Using Cauchy - Schwartz inequality on } \langle u, v\rangle) & \\ &\leq ||u||^2 + 2||u||||v|| + ||v||^2 \\ &\leq (||u|| + ||v||)^2 \end{aligned}$$

Since $||u||, ||v||$ and $||u+v||$ are all non-negative,

$$\begin{aligned} ||u+v||^2 &\leq (||u|| + ||v||)^2 \\ \Rightarrow ||u+v|| &\leq ||u|| + ||v|| \end{aligned}$$

Q.E.D.

Problem 7

In $\mathcal{P}_2$, let $p(x) = a_0+a_1x+a_2x^2$ and $q(x) = b_0+b_1x+b_2x^2$. Show that $< p(x), q(x) >= a_0b_0+a_1b_1+a_2b_2$ defines an inner product on P_2

Given: $p(x) = a_0 + a_1x + a_2x^2$, $q(x) = b_0 + b_1x + b_2x^2$ and $< p(x), q(x) >= a_0b_0 + a_1b_1 + a_2b_2$

Assumption: Let $p(x) = a_0 + a_1x + a_2x^2 =$ u and $q(x) = b_0 + b_1x + b_2x^2 =$ v.

To prove:
(i) $\langle u, v\rangle = \langle v, u\rangle$
(ii) $\langle av + bv, w\rangle = a\langle v, w\rangle + b\langle v, w\rangle$.
(iii) $\langle u, u\rangle \geqslant 0 \quad | \langle u, u\rangle = 0.$ iff $u = 0$

Condition 1:

$$\langle p(x), q(x)\rangle = a_0b_0 + a_1b_1 + a_2b_2 - \ (1.)$$
$$\langle q(x), p(x)\rangle = b_0a_0 + b_1a_1 + b_2a_2 - \ (2.)$$

as multiplication is commutative, (1) = (2)
$\therefore$ **Condition 1 is satisfied.**

Condition 2:

$$\langle au + bv, w\rangle = a\langle u, \omega\rangle + b\langle v, w\rangle - \ (1.)$$
$$\langle au + bv, w\rangle = (a, u_0 + bv_0)\, w_0 + (au_1 + bv_1)\, w_1 + (au_2 + bv_2)\, w_2 - \ (2.)$$
$$\langle au + bv, w\rangle = a\,(u_0w_0 + u_1w_1 + u_2w_2) + b\,(v_0w_0 + v_1w_1 + v_2w_2)$$
$$\langle au + bv, w\rangle = a\langle u, \omega\rangle + b\,\langle v, \omega\rangle$$

as (1) = (2)
Hence condition 2 is satisfied.

Condition 3:

$\langle u, u\rangle \geq 0$ and $\langle u, u\rangle = 0$ iff $u = 0$
Let $u > a_0 + a_1x^2 + a_2x^2$

$$\langle u, u\rangle = a_0 \cdot a_0 + a_1 \cdot a_1 + a_2 \cdot a_2$$
$$= a_0^2 + a_1^2 + a_2^2 \geq 0$$

for $\langle u, u\rangle = 0$ iff

$$a_0 = 0, a_1 = 0, a_2 = 0$$
$$u = 0 \cdot x + 0 + 0 \cdot x^2 = 0$$

Similarly,
for $\langle v, v\rangle = 0$ iff

$$b_0 = 0, b_1 = 0, b_2 = 0$$

Axiom 1 holds,
Hence condition 3 is satisfied.

$$\therefore P_2 isainnerproduct$$

Problem 8

Prove that $d(u, v) = \sqrt{|u||^2 + ||v||^2}$ iff u and v are orthogonal.
Assumption: The field of scalars is $\mathbb{R}$.

$$d(u, v) = ||u - v||$$

Part i. ($\Longrightarrow$)
Given: u and v are orthogonal
To prove: $d(u, v) = \sqrt{|u||^2 + ||v||^2}$

Since u and v are orthogonal,

$$\langle u, v\rangle = 0 = \langle v, u\rangle$$

$$\begin{aligned} d(u, v) &= ||u - v|| \\ &= \sqrt{\langle u, u\rangle + \langle v, u\rangle + \langle u, v\rangle + \langle v, v\rangle} \\ &= \sqrt{||u||^2 + 0 + 0 + ||v||^2} \\ \Rightarrow d(u, v) &= \sqrt{||u||^2 + ||v||^2} \end{aligned}$$

Q.E.D.

Part ii. ($\Longleftarrow$)
Given: $d(u, v) = \sqrt{|u||^2 + ||v||^2}$
To prove: u and v are orthogonal

$$||u - v||^2 = \langle u, u\rangle + \langle v, u\rangle + \langle u, v\rangle + \langle v, v\rangle$$

But $||u - v|| = d(u, v)$, and it is given that $d(u, v) = \sqrt{|u||^2 + ||v||^2}$, hence

$$\begin{aligned} ||u - v||^2 &= (d(u, v))^2 \\ \langle u, u\rangle + \langle v, u\rangle + \langle u, v\rangle + \langle v, v\rangle &= |u||^2 + ||v||^2 \\ \Rightarrow |u||^2 + \langle v, u\rangle + \langle u, v\rangle + ||v||^2 &= |u||^2 + ||v||^2 \\ \Rightarrow \langle v, u\rangle + \langle u, v\rangle &= 0 \end{aligned}$$

Since the field of scalars is taken to be $\mathbb{R}$,

$$\Rightarrow \langle u, v\rangle = \langle v, u\rangle = 0$$
$$\Rightarrow u \text{ and } v \text{ are orthogonal}$$

Q.E.D.

Problem 9

Prove that $||u+v|| = ||u-v||$ **iff** u **and** v **are orthogonal.**
Assumption: The field of scalars is $\mathbb{R}$.

Part i. ($\Longrightarrow$)
Given: u and v are orthogonal
To prove: $||u+v|| = ||u-v||$

Since u and v are orthogonal,

$$\langle u, v\rangle = 0 = \langle v, u\rangle$$

$$\begin{aligned} ||u+v||^2 &= ||u||^2 + ||v||^2 \\ ||u-v||^2 = ||u||^2 + ||-v||^2 &= ||u||^2 + ||v||^2 \end{aligned}$$

Therefore, $||u-v||^2 = ||u+v||^2$ and since the norm is always positive, $||u+v|| = ||u-v||$
Q.E.D.

Part ii. ($\Longleftarrow$)
Given: $||u+v|| = ||u-v||$
To prove: u and v are orthogonal

$$||u+v|| = ||u-v||$$

Squaring both sides,

$$\begin{aligned} ||u+v||^2 &= ||u-v||^2 \\ ||u||^2 + 2u\cdot v + ||v||^2 &= ||u||^2 - 2u\cdot v + ||v||^2 \\ 2u\cdot v &= -2u\cdot v \\ &\Longleftrightarrow 4u\cdot v = 0 \\ &\Longleftrightarrow u\cdot v = 0 \end{aligned}$$

This means that u and v are orthogonal.

Q.E.D.

Problem 10

Verify that if W is a subspace of an inner product space V and v $\in$ V , then $perp_w(v)$ is orthogonal to all w in W .

We will use the following two properties in our proof.
Property-1 : $\langle u, \lambda v\rangle = \overline{\lambda}\langle u, v\rangle$
Property-2 : $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$
From the above two properties we can say that $\langle u, v + \lambda w\rangle = \langle u, v\rangle + \overline{\lambda}\langle u, w\rangle$

Proof:
We have to prove that $\langle w, perp_w(v)\rangle = 0$.

$$
\begin{aligned}
\langle w, perp_w(v)\rangle &= \langle w, v - proj_w(v)\rangle \\
&= \langle w, v - \frac{\langle v, w\rangle w}{\langle w, w\rangle}\rangle \\
&= \langle w, v\rangle - \langle w, \frac{\langle v, w\rangle w}{\langle w, w\rangle}\rangle && \text{(obtained using property-2)} \\
&= \langle w, v\rangle - \overline{\left(\frac{\langle v, w\rangle}{\langle w, w\rangle}\right)}\langle w, w\rangle && \text{(obtained using property-1)} \\
&= 0 && \text{(properties of conjugate symmetry of inner product)}
\end{aligned}
$$

Q.E.D.