Discrete Structures (Monsoon 2021)

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Topic: **Relations**

RELATIONS

Definition

- A relation between two sets A and B is a subset of the cartesian product $A \times B$ and is defined by R (or ρ or r).
  $R \subseteq A \times B$.
- We write ${}_xR_y$ or ${}_x\rho_y$ if and only if (iff) $(x, y) \in R$ (or ρ).
- We also write ${}_x(\sim R)_y$ when x is NOT related to y in R.
- **Empty Relation:** A relation R on a set A is called Empty if the set A is empty set.
- **Full Relation:** A binary relation R on the sets A and B is called full if $A \times B = R$.

RELATIONS

Examples

- **Example.** Consider the relation $R = \{(x, y) \in I \times I : x > y\}$, where I is the set of all integers.
  Clearly, $R \subseteq I \times I$ and R is a relation in I.
  We write ${}_7R_5$ as $(7, 5) \in I \times I$ and $7 > 5$.
- **Example.** Consider the relation $R = \{(x, y) \in N \times N : x = 3y\}$, where N is the set of natural numbers.
  Clearly, $R \subseteq N \times N$ and R is a relation on the set N.
  We write ${}_{15}R_5$, ${}_{18}R_6$, and ${}_{27}R_9$.

RELATIONS

Inverse Relation

- If R be the relation from A to B, then the inverse relation of R is the relation from B to A and is denoted and defined by
  $R^{-1} = \{(y, x) : y \in B, x \in A, (x, y) \in R\}$.
  $\implies (x, y) \in R \Leftrightarrow (y, x) \in R^{-1}$
- **Example.** If $A = \{1, 2\}$, $B = \{2, 3\}$ and R be the relation from A to B, $R = \{(1, 2), (2, 3)\}$, then $R^{-1} = \{(2, 1), (3, 2)\}$.

Theorem

If R be a relation from A to B, then the domain of R is the range of R^{-1} and the range of R is the domain of R^{-1}.

Theorem

If R be a relation from A to B, then $(R^{-1})^{-1} = R$.

RELATIONS

Reflexive relation

- Let A be a set and R the relation defined in it (i.e., $R \subseteq A \times A$). R is said to be *reflexive*, if $(a, a) \in R$, $\forall a \in A$
  $\implies {}_aR_a$ holds for every $a \in A$.
- **Example.** Consider the relation $R = \{(a, a), (a, c), (b, b), (c, c), (d, d)\}$ in the set $A = \{a, b, c, d\}$. Then R is reflexive, since $(x, x) \in R$, $\forall x \in A$, that is, ${}_xR_x$ holds for every $x \in A$.
- **Example.** Consider the relation $S = \{(a, a), (a, c), (b, c), (b, d), (c, d)\}$ in the set $A = \{a, b, c, d\}$. Verfify whether S is reflexive.

RELATIONS

Symmetric relation

- Let A be a set and R the relation defined in it (i.e., $R \subseteq A \times A$). R is said to be *symmetric*, if $(a, b) \in R \Rightarrow (b, a) \in R, \forall a, b \in A$
  In other words, ${}_aR_b \Rightarrow {}_bR_a$ for every $a, b \in A$.
- **Example.** Let N be the set of natural numbers and R the relation defined in it such that ${}_xR_y$ if x is a divisor of y (that is, $x|y$), $x, y \in N$.
  Then R is NOT symmetric, since ${}_xR_y \not\Rightarrow {}_yR_x, \forall x, y \in N$.
  For example, ${}_3R_9 \not\Rightarrow {}_9R_3$.
- **Example.** Consider the relation S in the set of natural numbers N as $R = \{(x, y) \in N \times N : x + y = 5\}$. Verfify whether S is symmetric.

RELATIONS

Theorem

For a symmetric relation R, $R^{-1} = R$.

Proof.

Required to prove (RTP) (i) $R \subseteq R^{-1}$, and (ii) $R^{-1} \subseteq R$.
(i) Let $(x, y) \in R$.
Then $(x, y) \in R \Rightarrow (y, x) \in R$, since R is symmetric
$\Rightarrow (x, y) \in R^{-1}$, by definition of R^{-1}
Thus, $R \subseteq R^{-1}$.
(ii) Let $(x, y) \in R^{-1}$.
Then $(y, x) \in (R^{-1})^{-1} = R$, by definition of R^{-1}
$\Rightarrow (x, y) \in R$, since R is symmetric
Thus, $R^{-1} \subseteq R$.

$\square$

RELATIONS

Anti-symmetric relation

- Let A be a set and R the relation defined in it (i.e., $R \subseteq A \times A$). R is said to be *anti-symmetric*, if ${}_aR_b$ and ${}_bR_a \Rightarrow a = b$, for every $a, b \in A$.
- **Example.** Let A be the set of real numbers and R the relation defined in it such that ${}_xR_y$ if $x \leq y$, that is,
  $R = \{(x, y) \in A \times A : x \leq y\}$.
  Then R is anti-symmetric, since
  ${}_xR_y$ and ${}_yR_x$
  $\Rightarrow x \leq y$ and $y \leq x$
  $\Rightarrow x = y$.

RELATIONS

Transitive relation

- Let A be a set and R the relation defined in it (i.e., $R \subseteq A \times A$). R is said to be *transitive*, if ${}_aR_b$ and ${}_bR_c \Rightarrow {}_aR_c, \forall a, b, c \in A$.
- **Example.** Let N be the set of natural numbers and R the relation defined in it such that ${}_xR_y$ if $x < y$, that is,
  $R = \{(x, y) \in N \times N : x < y\}$.
  Then R is transitive, since
  ${}_xR_y$ and ${}_yR_z$
  $\Rightarrow x < y$ and $y < z$
  $\Rightarrow x < z$
  $\Rightarrow {}_xR_z$.

RELATIONS

Equivalence relation

- Let A be a set and R the relation defined in it (i.e., $R \subseteq A \times A$). R is said to be an *equivalence* relation, if and only if
  1. R is reflexive, that is, ${}_aR_a$ holds, for every $a \in A$.
  2. R is symmetric, that is, ${}_aR_b \Rightarrow {}_bR_a, \forall a, b \in A$.
  3. R is transitive, that is, ${}_aR_b$ and ${}_bR_c \Rightarrow {}_aR_c, \forall a, b, c \in A$.

RELATIONS

Number of relations

- The total number of relations on a set A containing n elements is $2^{n.n} = 2^{n^2}$ since a binary relation on A is precisely a subset of $A \times A$ and $|\mathcal{P}(A \times A)| = 2^{n^2}$.
- The total number of reflexive relations defined in A is $2^{n(n-1)}$.
- The total number of symmetric relations defined in A is $2^{\frac{n(n+1)}{2}}$.
- The total number of anti-symmetric relations defined in A is $2^n.3^{\frac{n(n-1)}{2}}$.
- The total number of both reflexive and symmetric relations defined in A is 2^p, where $p = {}^nC_2$.
- The total number of equivalence relations defined in A is same as counting the total number of partitions of a set A of sie n, which is given by $\sum_{r=1}^{n} S(n,r)$, where

$$S(n,1) = 1 = S(n,n),$$

$$S(n,r) = S(n-1,r-1) + r.S(n-1,r), 1 < r < n$$

Problem: A relation ρ is defined on the set Z (set of all integers) by ${}_a\rho_b$ if and only if $(2a + 3b)$ is divisible by 5. Prove or disprove: ρ is an equivalence relation.

- Claim 1: Let $a \in Z$. Then, $2a + 3a = 5a$ is divisible by 5. Hence, ${}_a\rho_a$ holds, $\forall a \in Z$.
  $\Rightarrow \rho$ is ***reflexive***.
- Claim 2: **Lemma:** If $a(\neq 0)$ divides b (i.e., $a|b$), $a, b \in Z$ being integers, then $\exists\ x \in Z$ such that $b = ax$.
  Lemma: If p be prime and a, b are integers such that $p|ab$, then either $p|a$ or $p|b$.

Problem (Continued...)

- Let $a, b \in Z$. Assume that ${}_a\rho_b$ holds. Then, $(2a+3b)$ is divisible by 5. By the Euclid's division algorithm, we have,
  $2a+3b = 5k_1$, for some integer $k_1 \in Z$.
  $\Rightarrow 2(2a+3b) = 10k_1$
  $\Rightarrow 4a+6b = 10k_1$
  $\Rightarrow 3(2b+3a) - 5a = 10k_1$
  $\Rightarrow 3(2b+3a) = 5(a+2k_1) = 5k_2$, say, where $k_2 = (a+2k_1)$ is an integer
  If p is prime and $p|ab$, then either $p|a$ or $p|b$. Thus, $5|(2b+3a) \Rightarrow$ ${}_b\rho_a$ holds. Hence, ρ is ***symmetric***.

Problem (Continued...)

- Claim 3: Let ${}_a\rho_b$ and ${}_b\rho_c$ hold, for every $a, b, c \in Z$. Then
  $(2a + 3b)$ is divisible by 5
  $\Rightarrow 2a + 3b = 5l_1$, for some $l_1 \in Z$, and
  $(2b + 3c)$ is divisible by 5
  $\Rightarrow 2b + 3c = 5l_2$, for some $l_2 \in Z$.
  Now $2(2a + 3b) - 3(2b + 3c) = 10l_1 - 15l_2$
  $\Rightarrow 4a - 9c = 10l_1 - 15l_2$
  $\Rightarrow 2(2a + 3c) = 10l_1 - 15l_2 + 15c = 5(2l_1 - 3l_2 + 3c) = 5l_3$, say,
  where $l_3 = 2l_1 - 3l_2 + 3c \in Z$
  $\Rightarrow 5|(2a + 3c)$
  $\Rightarrow_a \rho_c$ holds and ρ is also ***transitive***.
  Since ρ is reflexive, symmetric and transitive, so ρ is an equivalence relation.

RELATIONS

Partial-order relation

- Let S be a non-empty set and R the relation defined in it (i.e., $R \subseteq S \times S$). R is said to be an *partial-order* relation, if and only if it satisfies the following three conditions:
  1. R is reflexive, that is, ${}_aR_a$ holds, for every $a \in S$.
  2. R is anti-symmetric, that is, ${}_aR_b$ and ${}_bR_a \Rightarrow a = b$, $\forall a, b \in S$.
  3. R is transitive, that is, ${}_aR_b$ and ${}_bR_c \Rightarrow {}_aR_c$, $\forall a, b, c \in S$.

RELATIONS

Problem: A relation R is defined on the set N (set of natural numbers) by ${}_aR_b$ if and only if a divides b, that is, $R = \{(a, b) \in N \times N : a|b\}$. Prove or disprove: R is a partial-order relation.

- Claim 1: Verify whether R is ***reflexive***. (Yes/No)
- Claim 2: Verify whether R is ***anti-symmetric***. (Yes/No)
- Claim 3: Verify whether R is ***transitive***. (Yes/No)