Discrete Structures (Monsoon 2021)

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The Language of Logic

Propositions

- A declarative sentence that is either true or false, but not both, is a proposition (or a statement), which we will denote by the lowercase letter p, q, r, s, or t.
- The variable p, q, r, s, or t are boolean variable (or logic variable).

Example 1: $3 + 5 = 8$.
Example 2: Socrates was a Greek philosopher.
Example 3: If $1 = 2$, then roses are red.
Example *: Is $x + 5 = 8$ proposition? (NO, because x can take any value)

Conjunction

The conjunction of two arbitrary propositions p and q, denoted by $p \wedge q$, is the proposition p *and* q. It is formed by combining the propositions using the word *and*, called a connective.

Example 1: Consider the statements
p: Socrates was a Greek philosopher
and
q: Euclid was a Chinese musician
Their conjunction is given by
$p \wedge q$: Socrates was a Greek philosopher *and* Euclid was a Chinese musician.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Table: Truth table for $p \wedge q$

Disjunctions

A second way of combining two propositions p and q is by using the connective *or*. The resulting proposition p *or* q is the disjunction of p and q and is denoted by $p \vee q$.

Example 1: Consider the statements

p: Harry likes pepperoni pizza for lunch

and

q: Harry likes mushroom pizza for lunch

Their conjunction is given by

$p \vee q$: Harry likes pepperoni pizza for lunch *or* Harry likes mushroom pizza for lunch.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Table: Truth table for $p \vee q$

Negation

- The negation of a proposition p is *It is not the case that p*, denoted by $\neg p$.
- Read $\neg p$ as the *negation* of p or simply *not* p.

Example 1: Consider the statement
p: Apollo is a Hindu god.
The negation of p is
$\neg p$: Apollo is not Hindu god.

p	$\neg p$
T	F
F	T

Table: Truth table for $\neg p$

Implication

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Two propositions p and q can be combined to form statements of the form: *If p, then q*.
Such a statement is an implication, denoted by $p \to q$. Since it involves a condition, it is also called a conditional statement. The component p is the hypothesis (or premise) of the implication and q the conclusion.

Example : Let
p: $\triangle$ ABC is equilateral and q: $\triangle$ ABC is isosceles.
Then
$p \to q$: If $\triangle$ ABC is equilateral, then it is isosceles.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Table: Truth table for $p \to q$

Converse, Inverse, and Contrapositive

The converse of the implication $p \rightarrow q$ is $q \rightarrow p$. The inverse of $p \rightarrow q$ is $\neg p \rightarrow \neg q$ (negate the premise and the conclusion).
The contrapositive of $p \rightarrow q$ is $\neg q \rightarrow \neg p$ (negate the premise and the conclusion, and then switch them).

Example : Let
$p \rightarrow q$: If $\triangle$ ABC is equilateral, then it is isosceles. Then

- *Conserve* ($q \rightarrow p$): If $\triangle$ ABC is isosceles, then it is equilateral.
- *Inverse* ($\neg p \rightarrow \neg q$): If $\triangle$ ABC is not equilateral, then it is not isosceles.
- *Contrapositive* ($\neg q \rightarrow \neg p$): If $\triangle$ ABC is not isosceles, then it is not equilateral.

Continued...

Example: Construct a truth table for $(p \to q) \wedge (q \to p)$

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Table: Truth table for $(p \to q) \wedge (q \to p)$

Example: Construct the truth table of the compound proposition $(p \vee \neg q) \to (p \wedge q)$

Biconditional Statement

Two propositions p and q can be combined using the connective *if and only if*. The resulting proposition, *p if and only if q*, is the conjunction of two implications: (1) p only if q, and (2) p if q, that is, $p \to q$ and $q \to p$. Accordingly, it is called biconditional statement, symbolized by $p \leftrightarrow q$

Example: Let
p: $\triangle$ ABC is equilateral and q: $\triangle$ ABC is equiangular.
Then the biconditional statement is given by
$p \leftrightarrow q$: $\triangle$ ABC is equilateral iff (if and only if) it is equiangular.
Note: The statement $p \leftrightarrow q$ is true if both p and q have the same truth value.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Table: Truth table for $p \leftrightarrow q$

Tautology

A tautology is a compound statement which always results are true in Truth value.

Example: Construct a truth table for $(p \to q) \leftrightarrow (\neg p \vee q)$.

p	q	$p \to q$	$\neg p$	$(\neg p \vee q)$	$(p \to q) \leftrightarrow (\neg p \vee q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Table: Truth table for $(p \to q) \leftrightarrow (\neg p \vee q)$

Logical Equivalences

Compound propositions that have the same truth values in all possible cases are called logically equivalent.

The compound propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

Example: Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\vee q$	$(\neg p \wedge \neg q)$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Table: Truth table for $\neg(p \vee q)$ and $\neg p \wedge \neg q$

Home Works

- **Example 1**: Show that $p \to q$ and $\neg p \vee q$ are logically equivalent.
- **Example 2**: Show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent. This is the *distributive law* of disjunction over conjunction.
- **Example 3**: Show that $\neg(p \to q)$ and $p \wedge \neg q$ are logically equivalent.
- **Example 4**: Show that $p \to q \equiv \neg q \to \neg p$; that is, an implication is logically equivalent to its contrapositive.

Some important equivalences

Let p, q, and r be any three propositions, and T denotes the compound proposition that is always true and F denotes the compound proposition that is always false.

Equivalence	Name
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws