Discrete Structures

Tutorial 4

Question 1:

Show that if A is a set, then there does not exist an onto function f from A to P(A), the power set of A. Conclude that $|A| < P(A)$. This result is known as Cantor's theorem.

Solution:

Proof

Consider the set $B = \{x \in A \mid x \notin f(x)\}$. Suppose to the contrary that f is surjective. Then there exists $\xi \in A$ such that $f(\xi) = B$. But by construction, $\xi \in B \iff \xi \notin f(\xi) = B$. This is a contradiction. Thus, f cannot be surjective. On the other hand, $g : A \to \mathcal{P}(A)$ defined by $x \mapsto \{x\}$ is an injective map. Consequently, we must have $\mathbf{card}(A) < \mathbf{card}(\mathcal{P}(A))$. Q.E.D.

Question 2:

Show that there is a one-to-one correspondence from the power set of positive integers, $P(Z_+)$, to the set of real numbers, R.

Solution:

Step 1- (should have been bijection between $\mathcal{P}(\mathbb{Z}^+)$ to $[0,1]$)

A subset of the set of positive integers can be thought of as an infinite bit string with the i^{th} bit 1 if i belongs to the subset and 0 otherwise. With any such bit string we can associate$r \in (0,1)$ by attaching a decimal point at the beginning of the string and viewing it as the binary expansion of r. Conversely, given a $r \in (0,1)$ we write the binary expansion of r and make it an infinite bit string by removing the decimal point and appending an infinite sequence of 0's. This infinite bit sring corresponds to a set of positive integers. This gives a bijection between the set $(0,1)$ and $P(Z^+)$. In question

Step 2 - (Prove bijection between $(0,1)$ and $[0,1]$ by Schroder-Bernstein Theorem)

The identity mapping ($h(x) = x$) is an injective mapping from (0,1) to [0,1]. The mapping $h(x) = (2x+1)/4$ is a bijection from [0,1] to [1/4, 3/4] and hence an injection from [0,1] to (0,1).

Step 3 - (Again use Schroder-Bernstein Theorem to prove bijection between $(0,1)$ to $\mathbb{R}$)

Consider the mapping defined as follows:

$$h(x) = \begin{cases} \frac{1}{4x} + \frac{3}{4}, & x > 1 \\ \frac{x}{4} + \frac{1}{2}, & -1 \le x \le 1 \\ \frac{1}{4x} + \frac{1}{4}, & x < -1 \end{cases}$$

h is an injective mapping from $\Re$to (0,1). The identity mapping is an injective mapping from (0,1) to $\Re$.

Question 3:

a) **E8.** *In the sequence* $1, 1, 2, 3, 5, 8, 3, 1, 4, \ldots$ *each term starting with the third is the sum of the two preceding terms. But addition is done mod 10. Prove that the sequence is purely periodic. What is the maximum possible length of the period?*

b) **E9.** *Consider the Fibonacci sequence defined by*

$$a_1 = a_2 = 1, \quad a_{n+1} = a_{n-1} + a_n, \; n > 1.$$

Prove that, for any n*, there is a Fibonacci number ending with* n *zeros.*

Solutions:

a)

Solution. Any two consecutive terms of the sequence determine all succeeding terms and all preceding terms. Thus the sequence will become periodic if any pair (a, b) of successive terms repeats, and the first repeating pair will be $(1, 1)$.

Consider 101 successive terms $1, 1, 2, 3, 5, 8, 3, \ldots$. They form 100 pairs $(1, 1)$, $(1, 2)$, $(2, 3)$, $\ldots$. Since the pair $(0, 0)$ cannot occur, there are only 99 possible distinct pairs. Thus two pairs will repeat, and the period of the sequence is at most 99.

b)

Solution. A term a_p ends in n zeros if it is divisible by 10^n, or, if $a_p \equiv 0 \bmod 10^n$. Thus we consider the Fibonacci sequence modulo 10^n, and we prove that the term 0 will occur in the sequence. Take $(10^{2n} + 1)$ terms of the sequence $a_1, a_2, \ldots \bmod 10^n$. They form 10^{2n} pairs $(a_1, a_2), (a_2, a_3), \ldots$, but the pair $(0, 0)$ cannot occur. Thus there are only $(10^{2n} - 1)$ possible pairs. Hence one pair will repeat. So the period length is at most $(10^{2n} - 1)$. As in **E8**, the first pair to repeat is $(1, 1)$.

$$\underbrace{1, 1, 2, 3, \ldots, a_p}_{\text{period}}, 1, 1.$$

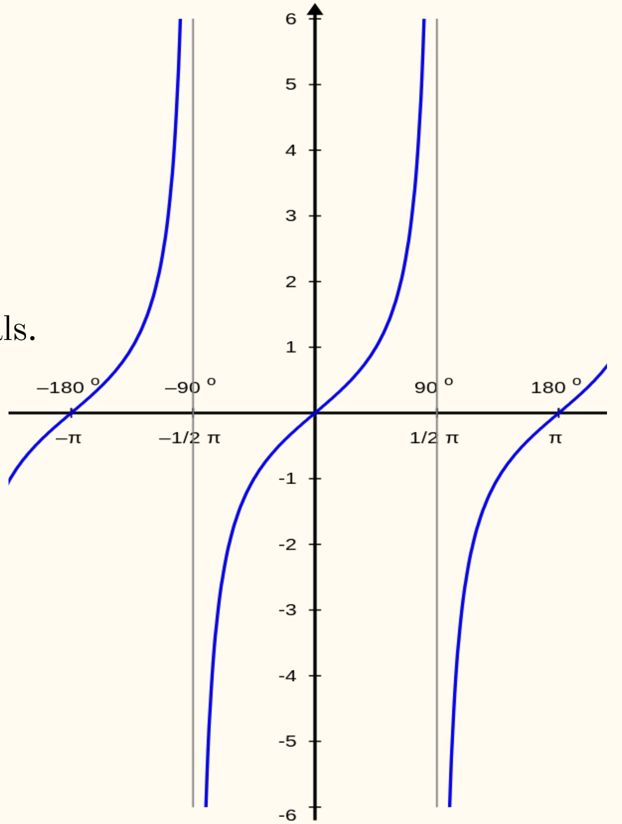
Then $a_p = 1 - 1 = 0$. Thus, the term 0, will occur in the sequence. In fact, it is the last term of the period.

Question 4: Given any five distinct real numbers, prove that there are two of them, say x and y, such that $0 < \frac{(x-y)}{(1+xy)} \leq 1$

Solution: Given a real number r, we can always find a unique real number θ in $(-\pi/2, \pi/2)$ such that $\tan\theta = r$ because tan is onto over R in this interval.

We take 5 real numbers r_1, r_2, r_3, r_4, r_5 in the interval $(-\pi/2, \pi/2)$ and

by previous argument, we have $\tan\theta_i = r_i$ for i in $\{1,2,3,4,5\}$.



Divide the open interval $(-\pi/2, \pi/2)$ into four equal intervals, each of length $\pi/4$.

By Pigeonhole principle at least two of the θ_i's must lie in one of the four intervals.

Let θ_s and θ_t lie in the same interval, then

$0 < \theta_s - \theta_t \leq \pi/4 \Rightarrow \tan 0 < \tan(\theta_s - \theta_t) < \tan \pi/4$

[$\tan\theta$ is strictly increasing over the interval $(-\pi/2, \pi/2)$]

$\Rightarrow 0 < (\tan(\theta_s) - \tan(\theta_t))/(1-\tan(\theta_s)\tan(\theta_t)) < 1$

$\Rightarrow 0 < (r_s - r_t)/(1-r_s * r_t) < 1$

Hence, there exist two real numbers $x = r_s$, $y = r_t$ such that $0 < \frac{(x-y)}{(1+xy)} \leq 1$

Question 5:

Problem 46*: Prove that among any 2^{n+1} natural numbers there are 2^n numbers whose sum is divisible by 2^n

Solution:

46. The inductive step can be proved as follows: split the given 2^{n+1} numbers into two halves each containing 2^n numbers. In each of these halves we can find 2^{n-1} numbers with the sum divisible by 2^{n-1} Then, out of the remaining 2^n numbers, we can choose the third set of 2^{n-1} numbers whose sum is divisible by 2^{n-1} Let the sums of the numbers in the three chosen sets be $2^{n-1}a$, $2^{n-1}b$, and $2^{n-1}c$. Among the numbers a, b, and c we can find two numbers of the same parity. The union of corresponding sets is a set of 2^n numbers whose sum is divisible by 2^n

Question 6:

Prove **Zeckendorf's theorem**: *Any positive integer N can be expressed uniquely as a sum of distinct Fibonacci numbers containing no neighbors:*

$$N = \sum_{j=1}^{m} F_{i_j+1}, \quad |i_j - i_{j-1}| \geq 2.$$

Here $F_1 = 1,\ F_2 = 2,\ F_{n+2} = F_{n+1} + F_n,\ n \geq 1.$

Solution:

If N is a Fibonacci number, the theorem is trivial. For small N, we check it by inspection. Assume it to be true for all integers up to and including F_n, and let $F_{n+1} \geq N > F_n$. Now, $N = F_n + (N - F_n)$, and $N \leq F_{n+1} < 2F_n$, i.e., $N - F_n < F_n$. Thus $N - F_n$ can be written in the form

$$N - F_n = F_{t_1} + \cdots + F_{t_r}, \quad t_{i+1} \leq t_i - 2,\ t_r \geq 2,$$

and $N = F_n + F_{t_1} + F_{t_2} + \cdots F_{t_r}$. We can be certain that $n \geq t_1 + 2$, because, if we had $n = t_1 + 1$, then $F_n + F_{t_1+1} = 2F_n$. But this is larger than N. In fact, F_n must appear in the representation of N because no sum of smaller Fibonacci numbers, obeying $k_{i+1} \leq k_i - 2\,(i = 1, 2, \ldots r - 1)$ and $k_r \geq 2$, could add up to N. This follows, if n is even, say $2k$, from

$$F_{2k-1} + F_{2k-3} + \cdots + F_3 = (F_{2k} - F_{2k-2}) + (F_{2k-2} - F_{2k-4}) + \cdots + (F_4 - F_2),$$

which is $F_{2k} - 1$, and if n is odd, say $2k - 1$, it follows from

$$F_{2k} + F_{2k-2} + \cdots + F_2 = (F_{2k+1} - F_{2k-1}) + \cdots + (F_3 - F_1) = F_{2k-1} - 1.$$

Again, the largest F_i not exceeding $N - F_n$ must appear in the representation of $N - F_n$, and it cannot be F_{n-1}. This proves uniqueness by induction.