MA 6.101
Probability and Statistics

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Conditioning with random variables

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- What is $E[X], E[Y]$ and $E[XY]$?

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- The conditional probability of event B given event A is defined as $\mathbb{P}(B/A) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$ when $\mathbb{P}(A) > 0$.

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- How do we know that it is consistent, i.e., $\sum_x p_{X|A}(x) = 1$?

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- $\sum_x p_{X|A}(x) = \frac{\mathbb{P}(\bigcup_x \{\{X=x\}\cap A)\}}{\mathbb{P}(A)} = \frac{\mathbb{P}(A)}{\mathbb{P}(A)} = 1.$ □

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- Given that event A has happened, what is the average value of the dice, i.e., $E[X|A]$?

$$E[X/A] = \textstyle\sum_x x p_{X|A}(x).$$

Using LOTUS,

$$E[g(X)/A] = \textstyle\sum_x g(x) p_{X|A}(x).$$

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$$E[X] = \sum_{i=1}^{n} \mathbb{P}(A_i) E[X|A_i]$$

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- Running example: Suppose we are given $X \in A$ where $A = \{2, 3\}$. What is $p_{X|A}(x)$?

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- What is $p_{N|A}(k)$?
- For $k > n$, $p_{N|A}(k) = \frac{P\{(N>n)\cap N=k\}}{P(N>n)} = (1-p)^{k-1-n}p$. For $k \leq n$, we have $p_{N|A}(k) = 0$.

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- If N denotes number of tosses till you first get a head, and having already tossed more than n times, the probability of having to toss more than $n + m$ is same as starting the experiment (forgetting that you have already tossed more than n times) fresh and having to toss more than m times.

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- How much you have tossed till now has no bearing on how much you will be required to toss.

$$P(N > n + m | N > n) = P(N > m) \text{ (Memoryless property)}.$$

HW: Find $E[N|A]$ where event $A = \{N > n\}$ and $n > 0$.