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- $\mathbb{P}$ is a set-function.
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- Formal definition of probability measure with its axioms.

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- Set theory 101
- $\mathbb{P}$ is a set-function.
- All sets in $\mathcal{P}(\Omega)$ need not be measurable.
- We restrict domain of $\mathbb{P}$ to sigma-algebra of measurable sets.
- $(\Omega, \mathcal{F}, \mathbb{P})$ is known as probability space.
- Formal definition of probability measure with its axioms.
- We looked at $\mathcal{B}([0,1])$ and $\mathcal{B}(\mathbb{R})$.

# This class ...

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- Continuity of set-function $\mathbb{P}$.

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- Continuity of set-function $\mathbb{P}$.
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- Independence and Correlation between sets.
- Conditional independence.

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- When do you say a function is continuous ?

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- How do you define continuity of a function $f(x)$ at $c$ ?
- When do you say a function is continuous ?
- $(\epsilon, \delta)$-definition of limits and continuity?

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- Recall that $\mathbb{P}$ is a set-function.

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- Recall that $\mathbb{P}$ is a set-function. Is it continuous?
- We will see the proof shortly.

# Sequence of sets

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- Given $(\Omega, \mathcal{F})$, If $A_1 \subset A_2 \ldots$ is an increasing sequence of events defined on $\mathcal{F}$ and $\cup_{n=1}^{\infty} A_n = A \in \mathcal{F}$, then we say thats the sequence of sets $A_n$ are increasing to $A$ ($A_n \uparrow A$).

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- Alternative notation: For an increasing sequence of sets $A_n$ we often write $\lim\limits_{n\to\infty} A_n$ for $\cup_{n=1}^{\infty} A_n$ and for a decreasing sequence of sets $A_n$ that $\lim\limits_{n\to\infty} A_n = \cap_{n=1}^{\infty} A_n$.

# Continuity of set-function $\mathbb{P}$

## Lemma

*For sequence of events of the type $A_n \uparrow A$ or $A_n \downarrow A$, we have $\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(A)$.*

Proof

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- $\cup_{n=1}^{\infty} A_n = \cup_{n=1}^{\infty} F_n$.
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- But $\mathbb{P}(\cup_{n=1}^{\infty} F_n) = \lim_{n\to\infty} \sum_{i=1}^{n} P(F_i)$

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Equivalently if $An \to \emptyset$, then $\mathbb{P}(A_n) \to 0$.

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> HW: Prove the chain rule for conditional probability given by
>
> $P(A_1 \cap A_2 \ldots A_n) = P(A_1)P(A_2/A_1)P(A_3/A_1A_2)\ldots P(A_n/A_{n-1}\ldots A_1)$.

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- $P(B/A) = \frac{|A \cap B|}{|A|}$.

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- This formula is useful when $P(A)$ is not given or is difficult to find but $P(B)$ or $P(A/B)$ is readily available.

Let $B_1, B_2, \ldots B_n$ be the partition of the sample space $\Omega$. Then for any event $A$ we have

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A/B_i)P(B_i).$$

# Example

1. If an item is defective, a robot can spot it with 98% accuracy.
2. If an item is not defective, a robot will declare it so with 99% accuracy.
3. A total of 0.1% items are defective.
4. If the robot says that the item you drew at random is defective, what is the probability that the robot is correct?

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