Lecture 2 Echelon

- Row-reduced ~~Echleon~~ echelon matrices

Def^n: An $m \times n$ matrix R is called a row-reduced ~~echleon~~ echelon matrix if

(a) R is row-reduced;

(b) every row of R which has all its entries 0 occurs below every row which has a non-zero entry;

(c) if rows $1, \ldots, r$ are the non-zero rows of R, and if the leading non-zero entry of row i occurs in column k_i, $i = 1, \ldots, r$, then $k_1 < k_2 < \ldots < k_r$.

An $m \times n$ row-reduced echleon matrix R can be also described as follows.

Either every entry in R is 0, or $\exists$ a $r \in \mathbb{Z}^+$, $1 \leq r \leq m$, and r positive integers $k_1, \ldots, k_r$ with $1 \leq k_i \leq n$ &

(a) $R_{ij} = 0$ for $i > r$, and $R_{ij} = 0$ if $j < k_i$.

(b) $R_{ik_j} = \delta_{ij}$, $1 \leq i \leq r$, $1 \leq j \leq r$.

(c) $k_1 < \ldots < k_r$.

Examples: $\mathbb{1}_{n \times n}$, $O_{n \times n}$, $\begin{bmatrix} 0 & 1 & -3 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

<u>Thm</u>: Every $m \times n$ matrix A is row-equivalent to a row-reduced echleon matrix.

Notice the significance of row-reduced matrices in solving homogenous linear eq^{n}s $RX = 0$. Now we discuss ~~the~~ system $RX = 0$, when R is row-reduced echleon ~~form~~ matrix.

Thm: To each elementary row operation e there corresponds an elementary row operation as e, such that $e_1(e(A)) = e(e_1(A)) = A$ for each A. I.e., the inverse operation (function) of an elementary row operation exists and is an elementary row operation of the same type.

Proof: ———

Def^{n}: If A & B are $m \times n$ matrices over the field F, we say that B is row-equivalent to A if B can be obtained from A by a finite sequence of elementary row operations.

Remark: Row-equivalence is an equivalence relation.

A binary relation $\sim$ on a set X is said to be an equivalence relation iff it is reflexive, symmetric and transitive. I.e., $\forall a, b, c \in X$:

1. $a \sim a$ (reflexivity)
2. $a \sim b$ iff $b \sim a$ (symmetry)
3. If $a \sim b$ and $b \sim c$ then $a \sim c$ (transitivity)

Equivalence class of a under $\sim$, denoted $[a]$ is defined as $[a] = \{x \in X : x \sim a\}$.

Thm: If A & B are row-equivalent $m \times n$ matrices, the homogenous systems of linear equations $AX=0$ & $BX=0$ have exactly the same solutions.

Proof: $A = A_0 \to A_1 \to A_2 \to \dots \to A_k = B$. $\quad \xrightarrow{e} \quad \xleftarrow[e]{}$

Example: Let F be the field of rational numbers, and

$$A = \begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{bmatrix}.$$ We perform elementary row operations.

$$\begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{bmatrix} \xrightarrow{(1)-2\times(2)} \begin{bmatrix} 0 & -9 & 3 & 4 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{bmatrix} \xrightarrow{(3)-2\times(2)} \begin{bmatrix} 0 & -9 & 3 & 4 \\ 1 & 4 & 0 & -1 \\ 0 & -2 & -1 & +7 \end{bmatrix}$$

$$\xrightarrow{(3)/-2} \begin{bmatrix} 0 & -9 & 3 & 4 \\ 1 & 4 & 0 & -1 \\ 0 & 1 & 1/2 & -7/2 \end{bmatrix} \xrightarrow{(2)-4\times(3)} \begin{bmatrix} 0 & -9 & 3 & 4 \\ 1 & 0 & -2 & 13 \\ 0 & 1 & 1/2 & -7/2 \end{bmatrix} \xrightarrow{9\times(3)+(1)} \begin{bmatrix} 0 & 0 & 15/2 & -55/2 \\ 1 & 0 & -2 & 13 \\ 0 & 1 & 1/2 & -7/2 \end{bmatrix}$$

$$\xrightarrow{(1)\times 2/15} \begin{bmatrix} 0 & 0 & 1 & -11/3 \\ 1 & 0 & -2 & 13 \\ 0 & 1 & 1/2 & -7/2 \end{bmatrix} \xrightarrow{2\times(1)+(2)} \begin{bmatrix} 0 & 0 & 1 & -11/3 \\ 1 & 0 & 0 & 17/3 \\ 0 & 1 & 1/2 & -7/2 \end{bmatrix} \xrightarrow{-1/2\times(1)+(3)} \begin{bmatrix} 0 & 0 & 1 & -11/3 \\ 1 & 0 & 0 & 17/3 \\ 0 & 1 & 0 & -5/3 \end{bmatrix}$$

Row equivalence of A w/ the final matrix above tells us that the two systems are equivalent, i.e., have the same solutions.

$$\begin{aligned} 2x_1 - x_2 + 3x_3 + 2x_4 &= 0 \\ x_1 + 4x_2 \qquad\quad - x_4 &= 0 \\ 2x_1 + 6x_2 - x_3 + 5x_4 &= 0 \end{aligned} \qquad \Bigg| \qquad \begin{aligned} x_3 - \tfrac{11}{3}x_4 &= 0 \\ x_1 \qquad + \tfrac{17}{3}x_4 &= 0 \\ x_2 \quad - \tfrac{5}{3}x_4 &= 0. \end{aligned}$$

Def^n: An $m\times n$ matrix R is called <u>row-reduced</u> if

(a) the 1st non-zero entry in each non-zero row of R is equal to 1;

(b) each column of R which contains the leading non-zero entry of some row has all its other entries 0.

<u>Thm</u>: Every $m\times n$ matrix over the field is row-equivalent to a row-reduced matrix.

Proof: ____________ (as an exercise).

Let $1, \dots, r$ be the non-zero rows of R, & suppose that the leading non-zero entry of row i occurs in column k_i. The system $RX=0$ then consists of r non-trivial eq^{n}s. Also the unknown x_{k_i} will occur (with non-zero coeff.) only in the i^{th} eq^n. Let $u_1, \dots, u_{n-r}$ be the $(n-r)$ unknowns which are different from $x_{k_1}, \dots, x_{k_r}$, then the r- non-trivial eq^{n}s in $RX=0$ are of the form:

$$
\left.
\begin{aligned}
x_{k_1} + \sum_{j=1}^{n-r} C_{1j} u_j &= 0 \\
\vdots \qquad & \quad \vdots \\
x_{k_r} + \sum_{j=1}^{n-r} C_{rj} u_j &= 0
\end{aligned}
\right] \quad (1.3)
$$

We assign any possible values to $u_1, \dots, u_r$ and compute $x_{k_1}, \dots, x_{k_r}$ from (1.3).

Example: Consider
$$
\begin{bmatrix} 0 & 1 & -3 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

$r=2$, $k_1=2$, $k_2=4$ and 2 non-trivial eq^{n}s are

$$x_2 - 3x_3 + \tfrac{1}{2}x_5 = 0 \quad \text{or} \quad x_2 = 3x_2 - \tfrac{1}{2}x_5$$
$$x_4 + 2x_5 = 0 \quad \text{or} \quad x_4 = -2x_5$$

Assign any values to x_1, x_3, x_5: $x_1 = a$, $x_3 = b$, $x_5 = c$, then sol^{n}s are $(a, 3b - \tfrac{1}{2}c, b, -2c, c)$ $X=0$ is trivial sol^n

Remark: If the no. r of non-zero rows in R is less than n $(R_{m\times n} X_{n\times 1} = O_{m\times 1})$, then $RX=0$ has a non trivial sol^n (i.e., not all x_i's will be 0

Thm: If A is an $m\times n$ matrix and $m<n$, then the homogenous systems of linear eq^{n}s $AX=0$ has a non-trivial sol^n.

Proof: Assume R be a row-reduced echleon matrix which is row-equivalent to A. $AX=0$ and $RX=0$ will have exactly same sol^{n}s. If r is no. of non-zero rows in R, then $r\leq m$, $\because m<n$; we have $r<n$. $\Rightarrow AX=0$ has a non-trivial sol^n.

Thm: If A is an $n\times n$ (square) matrix, then A is row-equivalent to $\mathbb{1}_{n\times n}$ if and only if the system $AX=0$ has only the trivial sol^n.

Proof: If A is row-equivalent to $\mathbb{1}_{n\times n}$, then $AX=0$ and $\mathbb{1}X=0$ have the same sol^n.
Conversely, suppose $AX=0$ has only the trivial sol^n $X=0$. $R_{n\times n}$ be row-reduced echleon matrix that is row-equiv. to A, & let r be no. of non-zero rows of R. Given that $RX=0$ has no non-trivial sol^n, $r\geq n$. That implies, $r=n$ since $R_{n\times n}$. Row-reduced echleon matrix $R_{n\times n}$ with n non-zero rows is $\mathbb{1}_{n\times n}$.

– $AX=0$ <u>always</u> has a trivial sol^n

What about systems $AX = Y$? non-homogenous systems.

→ While $AX = 0$ always has a trivial solⁿ, systems $AX = Y$ for $Y \neq 0$ need not have a solⁿ.

How to find solutions for $AX = Y$, $Y \neq 0$?

→ Form the augmented matrix A' of the system $AX = Y$. A' is the $m \times (n+1)$ matrix whose 1st n columns are the columns ~~are~~ of A and whose last column is Y.

$$A'_{ij} = A_{ij} \quad \forall\, j \leq n$$

$$A'_{i(n+1)} = y_i \,.$$

$$A \xrightarrow{} R \qquad A' \xrightarrow{} R' \qquad Y \xrightarrow{} Z$$

sequence of elementary row operations

(same for both A & A')

Then, $R' = \left[\begin{array}{c|c} R & Z_{m\times 1} \end{array}\right]$, $Z = \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix}$

$AX = Y$ and $RX = Z$ are equivalent and hence have same solutions.

Whether $RX = Z$ has any solutions? To determine all the sol^{n}s if any exist.

If R has r non-zero rows, with leading non-zero entry of row i occuring in column k_i, $i = 1, \ldots, r$, then the first r eq^{n}s of $RX = Z$ effectively express $x_{k_1}, \ldots, x_{k_r}$ in the terms of the $(n-r)$ remaining x_j and the scalars $z_1, \ldots, z_r$. The last $(m-r)$ eq^{n}s are:

$$\begin{aligned} 0 &= z_{r+1} \\ &\vdots \\ 0 &= z_m \end{aligned}$$

and accordingly the cond^n for the system to have a sol^n is $z_i = 0$ for $i > r$. If this cond^n is satisfied, all sol^{n}s to the system are found as in the homogenous case, by assigning arbitrary values to $(n-r)$ of the x_j and then computing x_{k_i} from the i^{th} eq^n.

<u>Example</u>: F be a field of $\mathbb{Q}$ and

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 1 \\ 0 & 5 & -1 \end{bmatrix}.$$ Solve for $AX = Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$.

We perform a sequence of row operations on the augmented matrix A' which row-reduces A:

$$\begin{bmatrix} 1 & -2 & 1 & y_1 \\ 2 & 1 & 1 & y_2 \\ 0 & 5 & -1 & y_3 \end{bmatrix} \xrightarrow{2 - 2 \times 1} \begin{bmatrix} 1 & -2 & 1 & y_1 \\ 0 & 5 & -1 & (y_2 - 2y_1) \\ 0 & 5 & -1 & y_3 \end{bmatrix}$$

$$\xrightarrow{3 - 1} \begin{bmatrix} 1 & -2 & 1 & y_1 \\ 0 & 5 & -1 & (y_2 - 2y_1) \\ 0 & 0 & 0 & (y_3 - y_2 + 2y_1) \end{bmatrix} \xrightarrow{2 \cdot \frac{1}{5}} \begin{bmatrix} 1 & -2 & 1 & y_1 \\ 0 & 1 & -\frac{1}{5} & \frac{1}{5}(y_2 - y_1) \\ 0 & 0 & 0 & (y_3 - y_2 + 2y_1) \end{bmatrix}$$

$$\xrightarrow{1 + 2 \times 2} \begin{bmatrix} 1 & 0 & \frac{3}{5} & \frac{1}{5}(y_1 + 2y_2) \\ 0 & 1 & -\frac{1}{5} & \frac{1}{5}(y_2 - 2y_1) \\ 0 & 0 & 0 & (y_3 - y_2 + 2y_1) \end{bmatrix}$$

Cond^n that $AX = Y$ has a sol^n is

$$2y_1 - y_2 + y_3 = 0$$

and if scalars y_i satisfy this cond^n, all sol^{n}s are obtained by assigning a value c to x_3 & then computing

$$x_1 = -\frac{3}{5}c + \frac{1}{5}(y_1 + 2y_2)$$

$$x_2 = \frac{1}{5}c + \frac{1}{5}(y_2 - 2y_1)$$

Lecture 3:
Matrix multiplication

Suppose B is an $n\times p$ matrix over a field F with rows $\beta_1, \dots, \beta_n$ and that from B we construct a matrix C with rows $\gamma_1, \dots, \gamma_m$ by forming certain linear combinations:

$$\gamma_i = A_{i1}\beta_1 + A_{i2}\beta_2 + \dots + A_{in}\beta_n \quad (1.4)$$

The rows of C are determined by the ~~$m\times n$~~ mn scalars A_{ij} which are themselves the entries of an $m\times n$ matrix A.

From (1.4),

$$(C_{i1} \; \dots \; C_{ip}) = \sum_{r=1}^{n} (A_{ir}B_{r1} \dots A_{ir}B_{rp}),$$

entries of C: $\quad C_{ij} = \sum_{r=1}^{n} A_{ir}B_{rj}$.

Def^n: Let A be an $m\times n$ matrix over the field F and let B be an $n\times p$ matrix over F. The product AB is the $m\times p$ matrix C whose i,j entry is

$$C_{ij} = \sum_{r=1}^{n} A_{ir}B_{rj}.$$

Example: (a) Consider $\begin{bmatrix} 5 & -1 & 2 \\ 0 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 5 & -1 & 2 \\ 15 & 4 & 8 \end{bmatrix}$

Here, $\gamma_1 = (5 \;\; -1 \;\; 2) = 1\cdot(5 \;\; -1 \;\; 2) + 0\cdot(15 \;\; 4 \;\; 8)$

$\gamma_2 = (0 \;\; 7 \;\; 2) = -3\cdot(5 \;\; -1 \;\; 2) + 1\cdot(15 \;\; 4 \;\; 8)$.

(b) $\begin{bmatrix} 0 & 6 & 1 \\ 9 & 12 & -8 \\ 12 & 62 & -3 \\ 3 & 8 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 3 \\ 5 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 6 & 1 \\ 3 & 8 & -2 \end{bmatrix}$

$\gamma_3 = 5\cdot(0 \;\; 6 \;\; 1) + 4\cdot(3 \;\; 8 \;\; -2)$

B is an $n \times p$ matrix, $B = [B_1, \ldots, B_p]$,

$$B_j = \begin{bmatrix} B_{1j} \\ \vdots \\ B_{nj} \end{bmatrix}, \quad 1 \leq j \leq p.$$

B_j is $1 \times n$ matrix. column matrix.

Check that $AB = [AB_1, \ldots, AB_p]$.

<u>Thm</u>: If A, B, C are matrices over the field F such that the products BC and $A(BC)$ are defined, then so are the products AB, $(AB)C$ and

$$(AB)C = A(BC).$$

Proof: ——

Remark: For a square matrix A, A^n is well-defined.

$$A^p A^q A^r = A^s A^t A^u \quad \text{for all } p+q+r = s+t+u = n.$$

$A(BC) = (AB)C \longrightarrow$ linear combinations of linear combinations of the rows of C are again linear combinations of the rows of C.

If $B \xrightarrow[\text{row operations}]{\text{elementary}} C$, then each row of C is a linear combination of the rows of B, and so $\exists$ a matrix A s.t. $AB = C$.

(There can be many such A's in general.)

Def$^{\underline{n}}$: An $m\times m$ matrix is said to be an elementary matrix if it can be obtained from the $m\times m$ identity matrix $\mathbb{1}_{m\times n}$ by means of a single elementary row operation.

Example: 2×2 elementary matrices:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix},$$

$$\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix} \text{ for } c\neq 0, \quad \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} \text{ for } c\neq 0.$$

Thm: Let e be an elementary row operation and ~~let~~ $\mathcal{E}$ be the $m\times n$ elementary matrix $\mathcal{E} = e(\mathbb{1})$. Then, for every $m\times n$ matrix A,

$$e(A) = \mathcal{E}A.$$

Proof: Type (ii)

$$\mathcal{E}_{ik} = \begin{cases} \delta_{ik}, & i\neq r \\ \delta_{rk} + c\,\delta_{sk}, & i = r \end{cases}$$

(To replace row r with row $r + c\cdot$ row s)

$$(\mathcal{E}A)_{ij} = \sum_{k=1}^{m} \mathcal{E}_{ik} A_{kj} = \begin{cases} A_{ik}, & i\neq r \\ A_{rj} + cA_{sj}, & i = r. \end{cases}$$

Check for other types.

$$\begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & c & & \\ & & & \ddots & \\ 0 & & & & 1 \end{bmatrix} \mathcal{E}_{rr}, \quad \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & \cdots & c \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \mathcal{E}_{rs}, \quad \begin{bmatrix} 1 & & & & \\ & \ddots & & 1 & \\ & & & & \\ & 1 & & \ddots & \\ & & & & 1 \end{bmatrix}$$

$r \to r + c\cdot s$

Corollary: Let A and B be $m \times n$ matrices over the field F. Then B is row-equivalent to A if and only if $B = PA$, where P is a product of $m \times n$ elementary operation.

1.5 Invertible matrices.

Def^n: Let A be an $n \times n$ matrix over the field F. An $n \times n$ matrix B such that $BA = I$ is called a left inverse of A; an $n \times n$ matrix B such that $AB = I$ is called a right inverse of A. If $AB = BA = I$ then B is called a two-sided inverse of A and A is said to be invertible.

Lemma: If A has a left inverse B and a right inverse C, then $B = C$.

Proof: $B = BI = BAC = IC = C$.

Thm: Let A and B be $n \times n$ matrices over the field F.

(i) If A is invertible, so is A^{-1} and $(A^{-1})^{-1} = A$.

(ii) If both A and B are invertible, so is AB and $(AB)^{-1} = B^{-1}A^{-1}$.

Corollary: A product of invertible matrix is invertible.

Theorem: An elementary matrix is invertible.

Proof: ______

Thm. If A is an $n \times n$ matrix, the following are equivalent.

(I) A is invertible.

(II) A is row-equivalent to $\mathbb{1}_{n \times n}$.

(III) A is product of elementary operations.

Proof: ______

Thm: For an $n \times n$ matrix A, the following are equivalent.

(I) A is invertible.

(II) The homogenous system $AX=0$ has only the trivial solⁿ.

(III) The system of eqⁿs $AX=Y$ has a solⁿ X for each $n \times 1$ matrix Y.

Proof: ______

Column-equivalent

Column-reduced echelon matrix

~~Column~~ Elementary column operations: $A\underline{E}$

$$AX = Y$$
$$X^T A^T = Y^T$$