Linear Algebra 2022

Assignment 1 Solutions

1) Why study linear algebra? Mention your motivations or potential applications, etc

   Linear algebra is vital in multiple areas of science in general. Because linear equations are so easy to solve, practically every area of modern science contains models where equations are approximated by linear equations (using Taylor expansion arguments) and solving for the system helps the theory develop. Some applications of Linear Algebra are :

   - It is useful in machine learning.
   - It is helpful in optimization techniques.
   - It is used for flow in a network of pipes.
   - It us used in computer vision.
   - The equations are used in an LCR circuit.

   And any other application that has some linear algebra component.

2) Given: $x, y \in \mathbb{Q}$

$K \subseteq C$ such that $K = \{x + y\sqrt{2}\}$

We need to prove that K is a sub-field of $(C, +, \cdot)$. This can be done by proving the following

Let $a, b \in C$

Let $a = x + y\sqrt{2}$; $b = p + q\sqrt{2}$

(i) Closure:

We need to show that $a+b \in S$

$a+b = (x + y\sqrt{2}) + (p + q\sqrt{2}) = (x+p) + (y+q)\sqrt{2}$ (1)

Clearly, $(x+p)$ & $(y+q)$ both belong to Q.

$\therefore a+b \in K$

$\therefore$ Closure is satisfied.

(ii) Commutativity

Let $a, b \in K$

We need to show that $a+b = b+a$

$b+a = (p+q\sqrt{2}) + (x+y\sqrt{2}) = (p+x) + (q+y)\sqrt{2}$

$= (x+p) + (y+q)\sqrt{2} = a+b$ (Using (1))

$\therefore$ Commutativity is also satisfied.

(iii) If $a \in K$, $-a \in K$ also.

$-a = -(x+y\sqrt{2}) = -x - y\sqrt{2}$

$\therefore -x, y \in \mathbb{Q} \quad \therefore -a \in C$

$\therefore$ This property is also satisfied.

Note here that $a + (-a) = 0 + 0\sqrt{2}$ (which is the zero element).

Proof for zero element:

$$a + 0_c = x + y\sqrt{2} + 0 + 0\sqrt{2} = x + y\sqrt{2} = a$$

$\therefore 0_c$ exists.

$\therefore -a$ is the additive inverse.

(iv) If $x = y = 0 \Rightarrow a = 0 \in K$ } Both the identity & the inverse

If $x = 1$ & $y = 0 \Rightarrow a = 1 \in K$ } elements $\in K$

(v) Let $m = x^2 - 2y^2$

Let $b = \frac{x}{m} + \left(\frac{-y}{m}\right)\sqrt{2} \in C$

$$\text{Now, } ab = \frac{1}{m}(x + y\sqrt{2})(x - \sqrt{2}y)$$

$$= \frac{1}{m}(x^2 - 2y^2)$$

$$= 1 \qquad \therefore b = a^{-1} \text{ and } \therefore \text{ the inverse exists.}$$

$\therefore K$ is a subfield of $(C, +, \cdot)$ (SHP)

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3) Let K be a subfield of C. K has a zero element (property of a sub-field) which is $1_K \in K$. Now, let $n \in K$. Now, $n \cdot 1_K = 0_K$. This means that $n \cdot 1_K \in C$ $\Rightarrow n = 0$. $\therefore S \equiv N$ are all distinct elements of K. $\because K$ is a field the set of all negative integers, call it P, are also distinct elements of K. $\because 0_K$ is already in K. $\therefore$ All the elements in $\mathbb{Z}$ are in K.

Now, let $m \in \mathbb{Z} - \{0\}$. A multiplicative inverse of n exists $\forall n \in \mathbb{Z} - \{0\}$

$\therefore 1/n \in \mathbb{Q} \Rightarrow \frac{m}{n} \in K$ & $\in \mathbb{Q}$ as well

$\therefore$ Any subfield of complex numbers contains every rational number.

(SHP)

4) Let ω be the inverse of an elementary row operation f.

Let K be a non-zero scalar & a row of any matrix be R.

(a) If f is the operation $R \to cR$ $\Rightarrow \omega$ is the operation $R \to \frac{R}{c}$

(b) If f is the operation $R' \to R' + cR''$ $\Rightarrow \omega$ is the operation $R' \to R' - cR''$

(c) If f is the operation that exchanges R and R'
$\Rightarrow$ ω is the operation that exchanges R and R'

There are no other cases and any other case is a linear combination of these cases and it can be seen that $\omega(f(A)) = f(\omega(A))$ and ω, f are of the same type.

$\therefore$ The inverse function of an elementary row operation exists and is of the same type

(SHP).