RELATIONS

Problem: Z be the set of all integers. Define a relation R on the set $Z \times Z$ by ${}_{(a,b)}R_{(c,d)}$ if and only if $ad = bc$, $\forall a, b, c, d \in Z$. Prove or disprove: R is a partial-order relation.

- Claim 1: Verify whether R is ***reflexive***. (Yes/No)
- Claim 2: Verify whether R is ***anti-symmetric***. (Yes/No)
- Claim 3: Verify whether R is ***transitive***. (Yes/No)

Partial-Order Set (POSET)

- A non-empty set in which the partial-order relation is defined, is called the partial-order set (poset/POSET).
- Example: In the above example, the set N is POSET under which partial-order relation R is defined.

Equivalence classes

- Let A be a non-empty set and R be an equivalence relation defined in A.
- Let $a \in A$ be an arbitrary element. Then the elements $x \in A$ which satisfy ${}_xR_a$ form a subset of A which is called the *equivalence class* of a in A with respect to (w.r.to) R.
- Thus, A_a or $[a]$ or $cl(a)$ or $\bar{a}$
  $= \{x | {}_xR_a, x \in A\}$
  is called the equivalence class of a in A w.r.to R.

RELATIONS

Important properties of equivalence classes

- Let A be a non-empty set and R be an equivalence relation defined in A.
- Let $a \in A$ and $b \in A$ be two arbitrary elements. Then,
  1. $a \in [a]$;
  2. $b \in [a] \Rightarrow [b] = [a]$;
  3. $[a] = [b] \Leftrightarrow (a, b) \in R$;
  4. either $[a] = [b]$ or $[a] \cap [b] = \emptyset$, that is, either two equivalence classes are identical or disjoint.

RELATIONS

Problem(Equivalence classes): Let A be the set of triangles in a plane. Let R be a relation in A defined by "x is similar to y", where $x, y \in A$. Verify whether R is an equivalence relation. If so, find the equivalence classes.

- **Part 1.** *Claim:* R is an equivalence relation.
- **Part 2.** Here $R = \{(x, y) | x, y \in A, x \text{ is similar to } y\}$.
  Let $a \in A$ be an arbitrary triangle in the plane.
  Then,

$$\begin{aligned} [a] &= \{x | x \in A \text{ and } {}_xR_a\} \\ &= \{x | x \in A, x \text{ is similar to } a\} \end{aligned}$$

  is an equivalence class of $a \in A$.

Partitions

- Let S be a non-empty set. Then a *partition* of S is a collection of non-empty disjoint sub-sets of S whose union is S.
- In other words, if $A_1, A_2, \ldots, A_n$ be the non-empty sub-sets of S, then the set $\mathcal{P} = \{A_1, A_2, \ldots, A_n\}$ is said to be a partition of S, if
  1. $A_1 \cup A_2 \cup \ldots \cup A_n = S$,
  2. either $A_i = A_j$ or $A_i \cap A_j = \emptyset$, for all $i, j = 1, 2, \ldots, n$.

Example (Partitions)

- Consider a set $S = \{1, 2, 3, \ldots, 22\}$. Now consider three subsets A, B and C of S as follows:
  $A = \{1, 4, 7, \ldots, 22\}$,
  $B = \{2, 5, 8, \ldots, 20\}$,
  $C = \{3, 6, 9, \ldots, 21\}$.
  See that
  1. $A \cup B \cup C = S$, and
  2. $A \cap B = B \cap C = C \cap A = \emptyset$.

  Hence, the set $(P) = \{A, B, C\}$ forms a partition of S.

Relationship between Partitions and Equivalence relations

Theorem (Fundamental Theorem on Equivalence Relations)

An equivalence relation R in a non-empty set A partitions A and conversely, a partition of A defines an equivalence relation.

RELATIONS

Problem(Equivalence classes): Let Z be the set of integers. Let R be a relation in Z defined by the open sentence "$(x - y)$ is divisible by m", where $x, y \in Z$. Verify whether R is an equivalence relation. If so, find the equivalence classes.

- **Part 1.** *Claim:* R is an equivalence relation.
- **Part 2.** Equivalence classes.