# Probability & Statistics Tutorial I

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August 5, 2022

### Problem 1

Consider the sample space  $\Omega = \{a, b, c, d, e\}$ . Given that  $\{a, b, e\}$  and  $\{b, c\}$  are both events (elements of the  $\sigma$ -algebra), what other events are implied by taking union, intersection and complement?

#### Problem 2

Let  $\{A_i : i \in I\}$  be a collection of sets. Prove **De Morgan's Laws**:

$$\left(\bigcap_{i} A_{i}\right)^{c} = \bigcup_{i} A_{i}$$
and
$$\left(\bigcup_{i} A_{i}\right)^{c} = \bigcap_{i} A_{i}$$

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### Problem 3

Prove Boole's inequality

$$P\left(\bigcup_{i=1}^{\infty} B_i\right) \le \sum_{i=1}^{\infty} P(B_i)$$

#### Problem 4

State and prove the Inclusion-Exclusion principle for  $P(\bigcup_{i=1}^{n} A_i)$ .

### Problem 5

Let  $\mathcal{F}$  be a a  $\sigma$ -algebra of subsets of  $\Omega$ , and suppose  $P: \mathcal{F} \to [0,1]$  satisfies: (i)  $P(\Omega) = 1$ , and (ii) P is additive, in that  $P(A \cup B) = P(A) + P(B)$  whenever  $A \cap B = \phi$ . Show that  $P(\phi) = 0$ .

### Problem 6

If  $P(A) = \frac{3}{4}$  and  $P(B) = \frac{1}{3}$ . Then show that

$$\frac{1}{12} \le P(A \cap B) \le \frac{1}{3}$$

#### Problem 7

Prove that

$$P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(E^cFG) - P(EF^cG) - P(EFG^c) - 2P(EFG)$$

#### Problem 8

Express [a, b] as a countable union or intersection over open intervals.

#### Problem 9

Let A, B and C be three events in the sample space S. Suppose we know

- $A \cup B \cup C = S$
- P(A) = 1/2
- P(B) = 2/3
- $P(A \cup B) = 5/6$

Answer the following questions:

- a) Find  $P(A \cap B)$
- b) Do A, B, C form a partition of S?
- c) Find  $P(C (A \cup B))$
- d) If  $P(C \cap (A \cup B)) = 5/12$ , find P(C)

#### Problem 10

Show that if  $P(A_i) = 1$  for all  $i \geq 1$ , then

$$P\left(\bigcap_{i=1}^{\infty} A_i\right) = 1$$

### Problem 11

Let  $\mathcal{F}$  be a  $\sigma$ -algebra on  $\Omega$  and suppose that  $B \in \mathcal{F}$ .

Show that  $\mathcal{G} = \{A \cap B : A \in \mathcal{F}\}$  is a  $\sigma$ -algebra of subsets of B.

#### Problem 12

There are n urns of which the  $r^{th}$  contains r-1 red balls and n-r magenta balls. You pick an urn at random and remove two balls at random without replacement. Find the probability that:

- (a) the second ball is magenta
- (b) the second ball is magenta, given that the first is magenta.

#### Problem 13

A box contains three coins: two regular coins and one fake two-headed coin (P(H) = 1),

- (i) You pick a coin at random and toss it. What is the probability that it lands heads up?
- (ii) You pick a coin at random and toss it, and get heads. What is the probability that it is the two-headed coin?

### Problem 14

Given the following:

- 60% of all the emails is spam
- 20% of spam has the word 'DEA'
- 1% of non-spam has the word 'DEA'

You get an email with the word 'DEA'. Find the probability of it being spam.

## Problem 15

I roll a fair die. Let A be the event that the outcome is an odd number, i.e.  $A = \{1, 3, 5\}$ . Also, let B be the event that the outcome is less than or equal to 3, i.e.  $B = \{1, 2, 3\}$ . What is P(A)? What is P(A|B)?

### Problem 16

Show that for any events E and F,

$$P(E|E \cup F) \ge P(E|F)$$

### Problem 17

Three coins are tossed at the same time. We say A as the event of receiving at least 2 heads. Likewise, B denotes the event of getting no heads and C is the event of getting heads on the second coin. Which of these is mutually exclusive?