# Class Exercise

MA2.101: Linear Algebra (Spring 2022)

June 1, 2022

## 1 Problem Set 1

Determine which sets of vectors are orthogonal.

### 1.1

$$\begin{bmatrix}-3\\1\\2\end{bmatrix}, \begin{bmatrix}2\\4\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\2\end{bmatrix}$$

### 1.2

$$\begin{bmatrix}4\\2\\-2\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}-2\\5\\1\end{bmatrix}$$

### 1.3

$$\begin{bmatrix}3\\1\\-1\end{bmatrix}, \begin{bmatrix}-1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\-2\\4\end{bmatrix}$$

### 1.4

$$\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\1\\-2\end{bmatrix}, \begin{bmatrix}-2\\6\\1\end{bmatrix}$$

### 1.5

$$\begin{bmatrix}2\\3\\-1\\4\end{bmatrix}, \begin{bmatrix}-2\\1\\-1\\0\end{bmatrix}, \begin{bmatrix}-4\\-6\\2\\7\end{bmatrix}$$

### 1.6

$$\begin{bmatrix}1\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}-1\\1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\1\\1\end{bmatrix},$$

## 2 Problem Set 2

Show that the given vectors form an orthogonal basis for $\mathbb{R}^2$ or $\mathbb{R}^3$. Then express $\mathbf{w}$ as a linear combination of these basis vectors. Give the coordinate vector $[\mathbf{w}]_{\mathcal{B}}$ of $\mathbf{w}$ with respect to the basis $\mathcal{B} = \{\mathbf{v}, \mathbf{v}_2\}$ of $\mathbb{R}^2$ or $\mathcal{B} = \{\mathbf{v}, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathbb{R}^3$.

### 2.1

$$\mathbf{v}_1 = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}; \quad \mathbf{w} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

### 2.2

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 6 \end{bmatrix}; \quad \mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

### 2.3

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}; \quad \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

### 2.4

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}; \quad \mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

## 3 Problem Set 3

Determine whether the given orthogonal set of vectors is orthonormal. If it is not, normalize the vectors to form an orthonormal set.

### 3.1

$$\begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}, \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$$

### 3.2

$$\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

### 3.3

$$\begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -\frac{5}{2} \end{bmatrix}$$

### 3.4

$$\begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/6 \\ 1/6 \\ -1/6 \end{bmatrix}$$

### 3.5

$$\begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 0 \\ \sqrt{6}/3 \\ 1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} \sqrt{3}/2 \\ -\sqrt{3}/6 \\ \sqrt{3}/6 \\ -\sqrt{3}/6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

## 4 Problem Set 4

The given vectors form a basis for $\mathbb{R}^2$ or $\mathbb{R}^3$. Apply the Gram-Schmidt Process to obtain an orthogonal basis. Then normalize this basis to obtain an orthonormal basis.

### 4.1

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

### 4.2

$$\mathbf{x}_1 = \begin{bmatrix} 3 \\ -3 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

### 4.3

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$

### 4.4

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

## 5 Problem Set 5

The given vectors form a basis for a subspace $W$ of $\mathbb{R}^3$ or $\mathbb{R}^4$. Apply the Gram-Schmidt Process to obtain an orthogonal basis for $W$.

### 5.1

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$$

### 5.2

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 4 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

## 6 Problem Set 6

Find the orthogonal decomposition of $\mathbf{v}$ with respect to the subspace $W$.

### 6.1

$\mathbf{v} = \begin{bmatrix} 4 \\ -4 \\ 3 \end{bmatrix}$, $W$ as in **5.1**

### 6.2

$\mathbf{v} = \begin{bmatrix} 1 \\ 4 \\ 0 \\ 2 \end{bmatrix}$, $W$ as in **5.2**

## 7 Problem Set 7

Let $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$. Compute **(a)** $\langle \mathbf{u}, \mathbf{v} \rangle$ **(b)** $||\mathbf{u}||$ **(c)** $d(\mathbf{u}, \mathbf{v})$
**(d)** Find a non-zero vector orthogonal to $\mathbf{u}$.

### 7.1

$\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 3u_2v_2$ for $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.

### 7.2

$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T A \mathbf{v}$ for $A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$.

## 8 Problem Set 8

Let $p(x) = x^2 - 3x + 2$ and $q(x) = -3x^2 + 1$. Compute **(a)** $\langle p(x), q(x) \rangle$ **(b)** $||p(x)||$ **(c)** $d(p(x), q(x))$
**(d)** Find a non-zero vector orthogonal to $p(x)$.

### 8.1

For the vector space $\mathcal{P}_2$ and vectors (polynomials) $p(x) = a_0 + a_1x + a_2x^2$ and $q(x) = b_0 + b_1x + b_2x^2$, define $\langle p(x), q(x) \rangle = a_0b_0 + a_1b_1 + a_2b_2$.

### 8.2

For the vector space $\mathcal{P}_2[0,1]$ and vectors $p(x), q(x)$, define $\langle p(x), q(x)\rangle = \displaystyle\int_0^1 p(x)q(x)dx$.

## 9 Problem Set 9

Determine which of the four inner product axioms do not hold. Give a specific example in each case.

### 9.1

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ in $\mathbb{R}^2$. Define $\langle \mathbf{u}, \mathbf{v}\rangle = u_1 v_1$.

### 9.2

In $\mathbb{M}_{2\times 2}$, define $\langle A, B\rangle = \det(AB)$.