Assignment 2 Solutions

Problem 1

Let R be a row-reduced echelon matrix which is row equivalent to A. Then the systems $AX = 0$ and $RX = 0$ have the same solutions from the following Lemma.

Lemma: If A and B are row-equivalent $m \times n$ matrices, the homogeneous systems of linear equations $AX = 0$ and $BX = 0$ have exactly the same solutions.

Now, if r is the number of non-zero rows in R, then certainly $r \leq m$, and since $m < n$, we have $r < n$. Thus, $AX = 0$ has a non-trivial solution.

Problem 2

We have to prove the `if` and the `only if` parts seprarately.

- Proof of if A is row equivalent to I_n then $AX = 0$ has only the trivial solution:

  If A is row equivalent to I_n then, A is invertible which implies that A only has the trivial solution.

$$AX = 0 \implies A^{-1}AX = 0 \implies IX = 0 \implies X = 0$$

- Proof of if $AX = 0$ has only the trivial solution, then A is row equivalent to I_n:

  Let $R_{n \times n}$ be a row-reduced echelon matrix which is row equivalent to A. Then $AX = 0 \iff RX = 0 \implies RX = 0$ has only the trivial solution.

  Thus, $r \geq n$ where r is the number of non-zero rows in R. But, given there are only n rows in the matrix, we have $r = n$.

  Thus, $R = I_n$ and A is row equivalent to I_n.

Problem 3

Let $\mathbf{A} = [a]_{mn}, \mathbf{B} = [b]_{np}, \mathbf{C} = [c]_{pq}$ be matrices.

From inspection of the subscripts, we can see that both $(\mathbf{AB})\mathbf{C}$ and $\mathbf{A}(\mathbf{BC})$ are defined such that $\mathbf{A}$ has n columns and $\mathbf{B}$ has n rows, while $\mathbf{B}$ has p columns and $\mathbf{C}$ has p rows.

Consider $(\mathbf{AB})\mathbf{C}$. Let $\mathbf{R} = [r]_{mp} = \mathbf{AB}, \ \ \mathbf{S} = [s]_{mq} = \mathbf{A}(\mathbf{BC})$.

Then we have the following:

$$
\begin{aligned}
s_{ij} &= \sum_{k=1}^{p} r_{ik} \circ c_{kj} \\
r_{ik} &= \sum_{l=1}^{n} a_{il} \circ b_{lk} \\
\rightsquigarrow \quad s_{ij} &= \sum_{k=1}^{p} \left(\sum_{l=1}^{n} a_{il} \circ b_{lk} \right) \circ c_{kj} \\
&= \sum_{k=1}^{p} \sum_{l=1}^{n} (a_{il} \circ b_{lk}) \circ c_{kj}
\end{aligned}
$$

Now consider $\mathbf{A}(\mathbf{BC})$. Let $\mathbf{R} = [r]_{nq} = \mathbf{BC}, \ \ \mathbf{S} = [s]_{mq} = \mathbf{A}(\mathbf{BC})$.

Then we have the following:

$$
\begin{aligned}
s_{ij} &= \sum_{l=1}^{n} a_{il} \circ r_{lj} \\
r_{lj} &= \sum_{k=1}^{p} b_{lk} \circ c_{kj} \\
\rightsquigarrow \quad s_{ij} &= \sum_{l=1}^{n} a_{il} \circ \left(\sum_{k=1}^{p} b_{lk} \circ c_{kj} \right) \\
&= \sum_{l=1}^{n} \sum_{k=1}^{p} a_{il} \circ (b_{lk} \circ c_{kj})
\end{aligned}
$$

Using associativity of product we finally have the following:

$$
s_{ij} = \sum_{k=1}^{p} \sum_{l=1}^{n} (a_{il} \circ b_{lk}) \circ c_{kj} = \sum_{l=1}^{n} \sum_{k=1}^{p} a_{il} \circ (b_{lk} \circ c_{kj}) = s'_{ij}
$$

Thus, it is concluded that $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.

Problem 4

The three types of elementary row operations should be taken up separately for the proof.

- Operation I

$$E_{ik} = \begin{cases} \delta_{ik}, \ i \neq r \\ \delta_{rk} + c\delta_{sk}, \ i = r \end{cases}$$

$$(EA)_{ij} = \sum_{k=1}^{m} E_{ik} A_{kj} = \begin{cases} A_{ij}, \ i \neq r \\ A_{rj} + cA_{sj}, \ i = r \end{cases} = e(A)_{ij}$$

- Operation II

$$E_{ik} = \begin{cases} \delta_{ik}, \ i \neq r \\ c\delta_{rk}, \ i = r \end{cases}$$

$$(EA)_{ij} = \sum_{k=1}^{m} E_{ik} A_{kj} = \begin{cases} A_{ij}, \ i \neq r \\ cA_{rj}, \ i = r \end{cases} = e(A)_{ij}$$

- Operation III

$$E_{ik} = \begin{cases} \delta_{ik}, \ i \neq r, \ i \neq s \\ \delta_{rk}, \ i = s \\ \delta_{sk}, \ i = r \end{cases}$$

$$(EA)_{ij} = \sum_{k=1}^{m} E_{ik} A_{kj} = \begin{cases} A_{ij}, \ i \neq r, \ i \neq s \\ A_{rj}, \ i = s \\ A_{sj}, \ i = r \end{cases} = e(A)_{ij}$$

Thus, $e(A) = EA$.