SET THEORY

Symmetric Difference

Let A and B be two sets.

- The symmetric difference of A and B is denoted and defined by

$$\begin{aligned} A \bigtriangleup B &= (A - B) \cup (B - A) \\ &= \{x | [x \in A \text{ and } x \notin B] \text{ or } [x \in B \text{ and } x \notin A]\}. \end{aligned}$$

- Example: If $A = \{1, 2, 4, 7, 9\}$ and $B = \{2, 3, 7, 8, 9\}$, then $A - B = \{1, 4\}$, $B - A = \{3, 8\}$.
  Thus, $A \bigtriangleup B = \{1, 4\} \cup \{3, 8\} = \{1, 3, 4, 8\}$.
- It can be easily verified that
  (i) $A \bigtriangleup \emptyset = A$,
  (ii) $A \bigtriangleup A = \emptyset$,
  (iii) $A \bigtriangleup B = \emptyset \Rightarrow A = B$.

SET THEORY

Cartesian product of sets

- The Cartesian product of two sets A and B is denoted and defined by

$$A \times B = \{(a, b) | a \in A \, \text{and} \, b \in B\}.$$

More generally, the Cartesian product of n sets A_1, A_2, ..., A_n is

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i, 1 \leq i \leq n\}.$$

- Example: If $A = \{a, b, c\}$ and $B = \{m, n\}$, then $A \times B = \{(a, m), (a, n), (b, m), (b, n), (c, m), (c, n)\}$.
- It can be easily verified that if $|A| = m$ and $|B| = n$, then $|A \times B| = mn$.
- In general, $A \times B \neq B \times A$.

SET THEORY

The Inclusion-Exclusion Principle

- Let $A_1, A_2, \ldots, A_n$ be n finite sets. Then

$$
\begin{aligned}
|\cup_{i=1}^{n} A_i| &= \sum_{i=1}^{n} |A_i| - \sum_{i,j=1; i\neq j}^{n} |A_i \cap A_j| \\
&+ \sum_{i,j,k=1; i\neq j\neq k}^{n} |A_i \cap A_j \cap A_k| - \ldots \\
&+ (-1)^{n+1} |\cap_{i=1}^{n} A_i|
\end{aligned}
$$

- Special cases
  - When $n = 2$, $|A \cup B| = |A| + |B| - |A \cap B|$
  - When $n = 3$,
    $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

SET THEORY

Problem: Prove that $(A - B)$, $(B - A)$ and $A \cap B$ are disjoint, where A and B are two sets.

Two sets X and Y are disjoint, if $X \cap Y = \emptyset$.
Now,

$$
\begin{aligned}
(A - B) \cap (A \cap B) &= (A \cap B') \cap (A \cap B), \text{by De Morgan's laws} \\
&= (A \cap B') \cap (B \cap A), \text{by Commutative laws} \\
&= A \cap (B' \cap B) \cap A, \text{by Associative laws} \\
&= A \cap (\emptyset \cap A) \\
&= A \cap \emptyset \\
&= \emptyset
\end{aligned}
$$

Similarly, it can be shown that
$(B - A) \cap (A \cap B) = \emptyset$
$(A - B) \cap (B - A) = \emptyset$

SET THEORY

Problem

- The number of elements in a finite set S is denoted by $|S|$.
  (a) Starting from the fact that $|A \cup B| = |A| + |B|$ when A and B are two disjoint sets, show that in general, $|A \cup B| = |A| + |B| - |A \cap B|$.
  (b) For any three sets A, B, and C, show that
  $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$.

SET THEORY

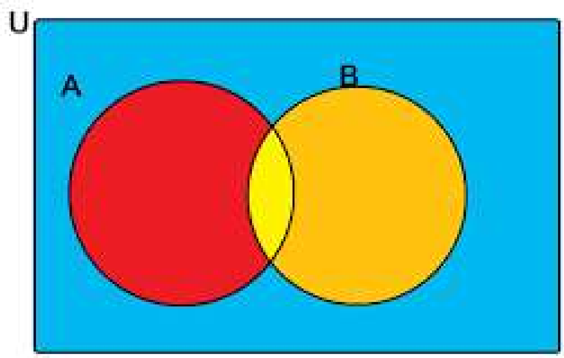

Figure: a) $A - B = A \cap B'$, b) $A \cap B$, c) $B - A = B \cap A'$

- Note that $A \cap B'$, $A \cap B$ and $A' \cap B$ are pairwise disjoint, and we have:

$$\begin{aligned} A &= (A \cap B') \cup (A \cap B) \\ |A| &= |A \cap B'| + |A \cap B| \end{aligned} \tag{5}$$

as $A \cap B'$ and $A \cap B$ are disjoint.

SET THEORY

Problem: $|A \cup B| = |A| + |B| - |A \cap B|$ (Continued...)

- Similarly,

$$\begin{aligned} B &= (A \cap B) \cup (A' \cap B) \\ |B| &= |A \cap B| + |A' \cap B| \end{aligned} \tag{6}$$

as $A \cap B$ and $A' \cap B$ are disjoint.

$$\begin{aligned} A \cup B &= (A \cap B') \cup (A \cap B) \cup (A' \cap B) \\ |A \cup B| &= |A \cap B'| + |A \cap B| + |A' \cap B| \end{aligned} \tag{7}$$

as $A \cap B'$, $A \cap B$ and $A' \cap B$ are disjoint.

Eqs. (5), (6) and (7) give:

$$\begin{aligned} |A \cup B| &= |A \cap B'| + |A \cap B| + |A' \cap B| \\ &= (|A| - |A \cap B|) + |A \cap B| + (|B| - |A \cap B|) \\ &= |A| + |B| - |A \cap B| \end{aligned}$$

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Problem:
$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$.

- We use the generalization of Part (a).
  Take $X = A$ and $Y = B \cup C$.
  Then,

$$\begin{aligned} |X \cup Y| &= |X| + |Y| - |X \cap Y| \\ |A \cup B \cup C| &= |A| + |B \cup C| - |A \cap (B \cup C)| \\ &= |A| + |B| + |C| - |B \cap C| - |A \cap (B \cup C)| \end{aligned} \quad (9)$$

$$\begin{aligned} |A \cap (B \cup C)| &= |(A \cap B) \cup (A \cap C)|, \text{using distributive law} \\ &= |A \cap B| + |A \cap C| - |(A \cap B) \cap (A \cap C)| \\ &= |A \cap B| + |A \cap C| - |(A \cap B \cap C)| \end{aligned} \quad (10)$$

- Eqs. (9) and (10) give the following result:

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |B \cap C| - (|A \cap B| + |A \cap C| - |(A \cap B \cap C)|) \\ &= |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|. \end{aligned}$$

SET THEORY

Problem [The Inclusion-Exclusion Principle]: Find the number of positive integers ≤ 2076 and divisible by neither 4 nor 5.

Let $A = \{x \in N | x \leq 2076 \text{ and divisible by } 4\}$, and
$B = \{x \in N | x \leq 2076 \text{ and divisible by } 5\}$.
By the Inclusion-Exclusion Principle, we have,

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= \lfloor \frac{2076}{4} \rfloor + \lfloor \frac{2076}{5} \rfloor - \lfloor \frac{2076}{4 \times 5} \rfloor \\ &= 519 + 415 - 103 \\ &= 831. \end{aligned}$$

Thus, among the first 2076 positive numbers, there are $2076 - 831$ $= 1245$ integers NOT divisible by neither 4 nor 5.

SET THEORY: An Application

INTERNATIONAL INSTITUTE OF INFORMATION TECHNOLOGY HYDERABAD

A Number-Theoretic Function

- An integer $p(> 1)$ is called a prime number or simply a prime, if its only positive divisors are 1 and itself. In other words, p does not have any non-trivial divisor d such that $1 < d < p$.
- Let x be a positive real number. Then $\pi(x)$ denotes the number of primes $\leq x$.
- **Prime Number Theorem:** $\pi(x) \rightarrow \frac{x}{ln(x)}$ as $x \rightarrow \infty$
- **Theorem:** Let $p_1, p_2, \ldots, p_t$ be the primes $\leq \sqrt{n}$. Then $\pi(n) = n - 1 + \pi(\sqrt{n}) - \sum_i \lfloor \frac{n}{p_i} \rfloor + \sum_{i<j} \lfloor \frac{n}{p_i p_j} \rfloor - \sum_{i<j<k} \lfloor \frac{n}{p_i p_j p_k} \rfloor + \ldots + (-1)^t \lfloor \frac{n}{p_1 p_2 \ldots p_t} \rfloor$

Problem: Find the number of primes ≤ 100.

Here $n = 100$. Then $\pi(\sqrt{n}) = \pi(\sqrt{100}) = \pi(10) = 4$. The four primes $\leq \sqrt{n} = 10$ are $2, 3, 5$ and 7. Let $p_1 = 2, p_2 = 3, p_3 = 5$ and $p_4 = 7$, $t = 4$. From the previous theorem, we have,

$$
\begin{aligned}
\pi(100) &= 100 - 1 + 4 - (\lfloor \frac{100}{2} \rfloor + \lfloor \frac{100}{3} \rfloor + \lfloor \frac{100}{5} \rfloor + \lfloor \frac{100}{7} \rfloor) \\
&\quad + (\lfloor \frac{100}{2.3} \rfloor + \lfloor \frac{100}{2.5} \rfloor + \lfloor \frac{100}{2.7} \rfloor + \lfloor \frac{100}{3.5} \rfloor + \lfloor \frac{100}{3.7} \rfloor + \lfloor \frac{100}{5.7} \rfloor) \\
&\quad - (\lfloor \frac{100}{2.3.5} \rfloor + \lfloor \frac{100}{2.3.7} \rfloor + \lfloor \frac{100}{2.5.7} \rfloor + \lfloor \frac{100}{3.5.7} \rfloor) + \lfloor \frac{100}{2.3.5.7} \rfloor \\
&= 103 - (50 + 33 + 20 + 14) + (16 + 10 + 7 + 6 + 4 + 2) \\
&\quad - (3 + 2 + 1 + 0) + 0 = 25.
\end{aligned}
$$

Note: Using the sieve of Eratosthenes, the primes ≤ 100 are:
$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47,$
$53, 59, 61, 67, 71, 73, 79, 83, 89, 97.$

Problem: Find the number of primes in between 50 and 100.

Solution

- Step 1. Find the number of primes ≤ 50. We have $\pi(50) = 15$.
- Step 2. Find the number of primes ≤ 100. We have $\pi(100) = 25$.
- Step 3. Finally, calculate the number of primes ≥ 50 and ≤ 100, which is $\pi(100) - \pi(50) = 25 - 15 = 10$.

This is consistent with the sieve of Eratosthenes.
Note: Using the sieve of Eratosthenes, the primes ≥ 50 and ≤ 100 are:
$53, 59, 61, 67, 71, 73, 79, 83, 89, 97$.

Problem: If $A = \{1, 2\}$, $B = \{2, 3\}$ and $C = \{3, 4\}$, then find $A \times (B \cup C)$. Further verifies whether $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Part 1: We have, $B \cup C = \{2, 3, 4\}$.
Now,

$$\begin{aligned} A \times (B \cup C) &= \{1, 2\} \times \{2, 3, 4\} \\ &= \{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4)\} \end{aligned}$$

Part 2: We also have,

$$\begin{aligned} (A \times B) \cup (A \times C) &= \{(1, 2), (1, 3), (2, 2), (2, 3)\} \\ &\quad \cup \{(1, 3), (1, 4), (2, 3), (2, 4)\} \\ &= \{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4)\} \\ &= A \times (B \cup C) \end{aligned}$$

SET THEORY

Problem: Let X, A and B be three sets such that $X \cap A = X \cap B$ and $X \cup A = X \cup B$. Prove that $A = B$.

We have to prove $A \subseteq B$ and $B \subseteq A$.
Let $x \in A$.
We have then two cases:

- Case 1: Let $x \in X$.
  Then $x \in A \cap X = X \cap B$.
  Thus, $x \in B$.
- Case 2: Let $x \notin X$.
  Then $x \in A$
  $\Rightarrow x \in A \cup X = X \cup B$.
  Thus, $x \in B$, since $x \notin X$.
  Hence, for each case, we have $x \in A$
  $\Rightarrow x \in B$.
  As a result, $A \subseteq B$.
  Similarly, one can prove that $B \subseteq A$.