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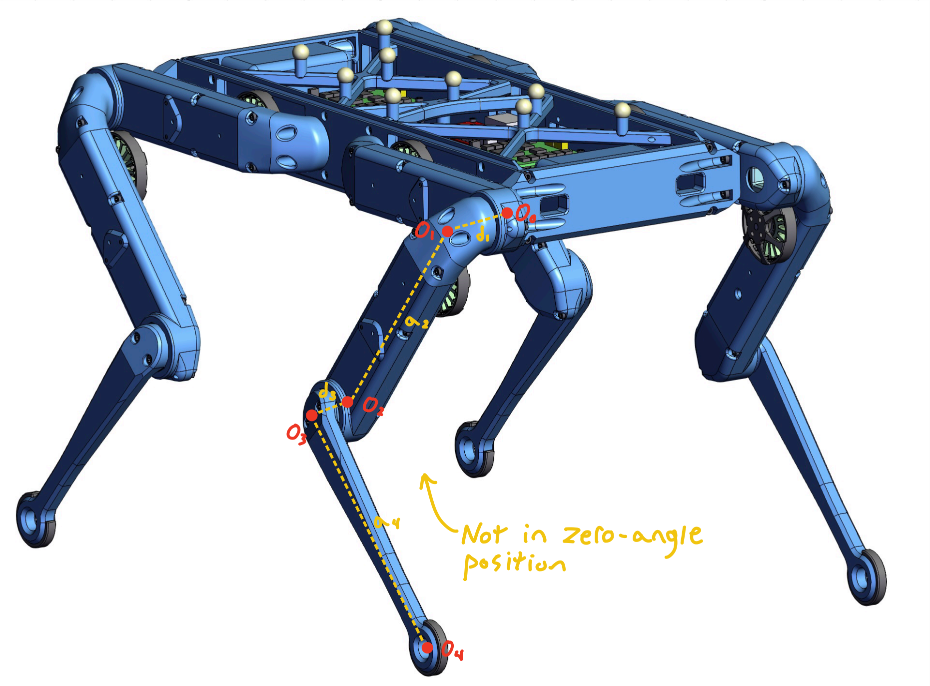
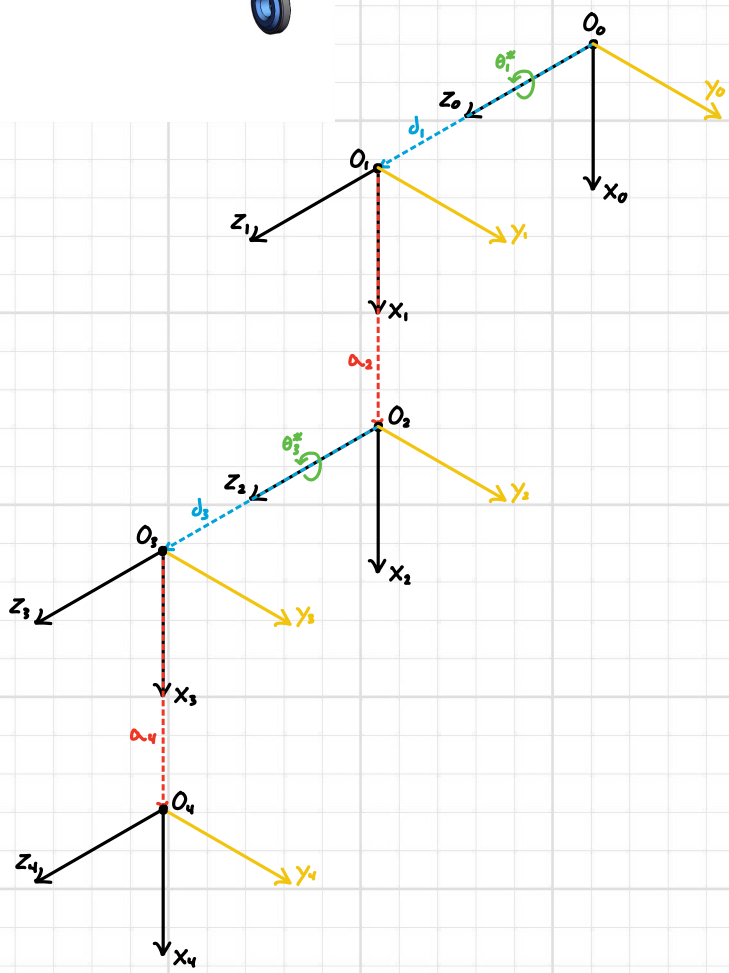
Diagram of leg in zero-angle position using Denavit-Hartenberg (DH) representation.



Assumptions:
- The coordinate system at O_0 is aligned with the global coordinate system.
- Straight, fully extended leg is zero-angle position.
- The coordinate system for each origin is orientated in the same direction.



DH Parameters

i	a_i	d_i	α_i	θ_i
1	0	d_1	0	*
2	a_2	0	0	0
3	0	d_3	0	*
4	a_4	0	0	0

All rotary joints

Find the position equation of the leg's endpoint relative to the global coordinate system.

$${}^0d_{0n} = \sum_{i=1}^{n} {}^0d_{i-1,i} = \sum_{i=1}^{n} \left(d_i\,{}^0z_{i-1} + a_i\,{}^0x_i\right)$$

$${}^0d_{04} = {}^0d_{01} + {}^0d_{12} + {}^0d_{23} + {}^0d_{34} = d_1\,{}^0z_0 + a_2\,{}^0x_2 + d_3\,{}^0z_2 + a_4\,{}^0x_4$$

$$\boxed{{}^0d_{04} = d_1\,{}^0z_0 + a_2\,{}^0x_2 + d_3\,{}^0z_2 + a_4\,{}^0x_4}$$

Find the velocity equation of the leg's endpoint relative to the global coordinate system.

$${}^0\dot{d}_{0n} = \sum_{i=1}^{n} {}^0\dot{d}_{i-1,i} = \sum_{i=1}^{n} \begin{cases} {}^0\omega_{0i} \times {}^0d_{i-1,i} & ;\ \text{For rotary joint } i \\ \dot{d}_i\,{}^0z_{i-1} + {}^0\omega_{0i} \times {}^0d_{i-1,i} & ;\ \text{For prismic joint } i \end{cases}$$

$${}^0\dot{d}_{04} = {}^0\dot{d}_{01} + {}^0\dot{d}_{12} + {}^0\dot{d}_{23} + {}^0\dot{d}_{34} = {}^0\omega_{01} \times {}^0d_{01} + {}^0\omega_{02} \times {}^0d_{12} + {}^0\omega_{03} \times {}^0d_{23} + {}^0\omega_{04} \times {}^0d_{34}$$

$${}^0\dot{d}_{04} = {}^0\omega_{01} \times d_1\,{}^0z_0 + {}^0\omega_{02} \times a_2\,{}^0x_2 + {}^0\omega_{03} \times d_3\,{}^0z_2 + {}^0\omega_{04} \times a_4\,{}^0x_4$$

Find the acceleration equation of the leg's endpoint relative to the global coordinate system.

$${}^0\ddot{d}_{0n} = \sum_{i=1}^{n} {}^0\ddot{d}_{i-1,i} = \sum_{i=1}^{n} \begin{cases} {}^0\dot{\omega}_{0i} \times {}^0d_{i-1,i} + {}^0\omega_{0i} \times \left({}^0\omega_{0i} \times {}^0d_{i-1,i}\right) & ;\ \text{For rotary joint } i \\ \ddot{d}_i\,{}^0z_{i-1} + 2\dot{d}_i\,{}^0\omega_{0,i-1} \times {}^0z_{i-1} & ;\ \text{For prismic joint } i \end{cases}$$

$${}^0\ddot{d}_{04} = {}^0\ddot{d}_{01} + {}^0\ddot{d}_{12} + {}^0\ddot{d}_{23} + {}^0\ddot{d}_{34}$$

$$= {}^0\dot{\omega}_{01} \times {}^0d_{01} + {}^0\omega_{01} \times \left({}^0\omega_{01} \times {}^0d_{01}\right) + {}^0\dot{\omega}_{02} \times {}^0d_{12} + {}^0\omega_{02} \times \left({}^0\omega_{02} \times {}^0d_{12}\right) + {}^0\dot{\omega}_{03} \times {}^0d_{23} + {}^0\omega_{03} \times \left({}^0\omega_{03} \times {}^0d_{23}\right) + {}^0\dot{\omega}_{04} \times {}^0d_{34} + {}^0\omega_{04} \times \left({}^0\omega_{04} \times {}^0d_{34}\right)$$

$${}^0\ddot{d}_{04} = {}^0\dot{\omega}_{01} \times d_1\,{}^0z_0 + {}^0\omega_{01} \times \left({}^0\omega_{01} \times d_1\,{}^0z_0\right) + {}^0\dot{\omega}_{02} \times a_2\,{}^0x_2 + {}^0\omega_{02} \times \left({}^0\omega_{02} \times a_2\,{}^0x_2\right) + {}^0\dot{\omega}_{03} \times d_3\,{}^0z_2 + {}^0\omega_{03} \times \left({}^0\omega_{03} \times d_3\,{}^0z_2\right) \ldots$$

$$+ {}^0\dot{\omega}_{04} \times a_4\,{}^0x_4 + {}^0\omega_{04} \times \left({}^0\omega_{04} \times a_4\,{}^0x_4\right)$$

Find the total angular velocities of each link. These are the sum of the angular velocity contributions due to all joints from 1 to i.

$${}^0\omega_{0n} = \sum_{i=1}^{n} {}^0\omega_{i-1,i} = \sum_{i=1}^{n} \begin{cases} \dot{\theta}_i\,{}^0z_{i-1} & ;\ \text{For rotary joint } i \\ 0 & ;\ \text{For prismic joint } i \end{cases}$$

$${}^0\omega_{01} = \dot{\theta}_1\,{}^0z_0$$

$${}^0\omega_{02} = {}^0\omega_{01} + {}^0\omega_{12} = \dot{\theta}_1\,{}^0z_0 + \cancelto{0}{\dot{\theta}_2\,{}^0z_1} = \dot{\theta}_1\,{}^0z_0$$

$${}^0\omega_{03} = {}^0\omega_{01} + {}^0\omega_{12} + {}^0\omega_{23} = \dot{\theta}_1\,{}^0z_0 + \cancelto{0}{\dot{\theta}_2\,{}^0z_1} + \dot{\theta}_3\,{}^0z_2 = \dot{\theta}_1\,{}^0z_0 + \dot{\theta}_3\,{}^0z_2$$

$${}^0\omega_{04} = {}^0\omega_{01} + {}^0\omega_{12} + {}^0\omega_{23} + {}^0\omega_{34} = \dot{\theta}_1\,{}^0z_0 + \cancelto{0}{\dot{\theta}_2\,{}^0z_1} + \dot{\theta}_3\,{}^0z_2 + \cancelto{0}{\dot{\theta}_4\,{}^0z_3} = \dot{\theta}_1\,{}^0z_0 + \dot{\theta}_3\,{}^0z_2$$

$${}^0\omega_{01} = {}^0\omega_{02}$$

$${}^0\omega_{03} = {}^0\omega_{04}$$

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Find the total angular accelerations of each link. These are the sum of the angular acceleration contributions due to all joints from 1 to i.

$$ {}^0\dot{\omega}_{0n} = \sum_{i=1}^{n} {}^0\dot{\omega}_{i-1,i} = \sum_{i=1}^{n} \begin{cases} \ddot{\theta}_i {}^0Z_{i-1} + \dot{\theta}_i {}^0\omega_{0,i-1} \times {}^0Z_{i-1} & ;\ \text{For rotary joint } i \\ 0 & ;\ \text{For prismic joint } i \end{cases} $$

$$ {}^0\dot{\omega}_{01} = \ddot{\theta}_1 {}^0Z_0 + 0 = \ddot{\theta}_1 {}^0Z_0 $$

Derivative of ${}^0Z_0$ is zero since it's fixed in place

$$ {}^0\dot{\omega}_{02} = {}^0\dot{\omega}_{01} + {}^0\dot{\omega}_{12} = \ddot{\theta}_1 {}^0Z_0 + \cancelto{0}{\ddot{\theta}_2 {}^0Z_1} + \cancelto{0}{\dot{\theta}_2 {}^0\omega_{01} \times {}^0Z_1} = \ddot{\theta}_1 {}^0Z_0 $$

$$ {}^0\dot{\omega}_{03} = {}^0\dot{\omega}_{01} + {}^0\dot{\omega}_{12} + {}^0\dot{\omega}_{23} = \ddot{\theta}_1 {}^0Z_0 + \cancelto{0}{\ddot{\theta}_2 {}^0Z_1} + \cancelto{0}{\dot{\theta}_2 {}^0\omega_{01} \times {}^0Z_1} + \ddot{\theta}_3 {}^0Z_2 + \dot{\theta}_3 {}^0\omega_{02} \times {}^0Z_2 = \ddot{\theta}_1 {}^0Z_0 + \ddot{\theta}_3 {}^0Z_2 + \dot{\theta}_3 {}^0\omega_{02} \times {}^0Z_2 $$

$$ {}^0\dot{\omega}_{04} = {}^0\dot{\omega}_{01} + {}^0\dot{\omega}_{12} + {}^0\dot{\omega}_{23} + {}^0\dot{\omega}_{34} = \ddot{\theta}_1 {}^0Z_0 + \cancelto{0}{\ddot{\theta}_2 {}^0Z_1} + \cancelto{0}{\dot{\theta}_2 {}^0\omega_{01} \times {}^0Z_1} + \ddot{\theta}_3 {}^0Z_2 + \dot{\theta}_3 {}^0\omega_{02} \times {}^0Z_2 + \cancelto{0}{\ddot{\theta}_4 {}^0Z_3} + \cancelto{0}{\dot{\theta}_4 {}^0\omega_{03} \times {}^0Z_3} = \ddot{\theta}_1 {}^0Z_0 + \ddot{\theta}_3 {}^0Z_2 + \dot{\theta}_3 {}^0\omega_{02} \times {}^0Z_2 $$

$$ {}^0\dot{\omega}_{01} = {}^0\dot{\omega}_{02} $$
$$ {}^0\dot{\omega}_{03} = {}^0\dot{\omega}_{04} $$

Total Angular Velocities	Total Angular Accelerations
${}^0\omega_{01} = \dot{\theta}_1 {}^0Z_0$	${}^0\dot{\omega}_{01} = \ddot{\theta}_1 {}^0Z_0$
${}^0\omega_{02} = \dot{\theta}_1 {}^0Z_0$	${}^0\dot{\omega}_{02} = \ddot{\theta}_1 {}^0Z_0$
${}^0\omega_{03} = \dot{\theta}_1 {}^0Z_0 + \dot{\theta}_3 {}^0Z_2$	${}^0\dot{\omega}_{03} = \ddot{\theta}_1 {}^0Z_0 + \ddot{\theta}_3 {}^0Z_2 + \dot{\theta}_3 {}^0\omega_{02} \times {}^0Z_2$
${}^0\omega_{04} = \dot{\theta}_1 {}^0Z_0 + \dot{\theta}_3 {}^0Z_2$	${}^0\dot{\omega}_{04} = \ddot{\theta}_1 {}^0Z_0 + \ddot{\theta}_3 {}^0Z_2 + \dot{\theta}_3 {}^0\omega_{02} \times {}^0Z_2$

Endpoint Velocity Equation

$$ {}^0\dot{d}_{04} = \dot{\theta}_1 {}^0Z_0 \times d_1 {}^0Z_0 + \dot{\theta}_1 {}^0Z_0 \times a_2 {}^0X_2 + (\dot{\theta}_1 {}^0Z_0 + \dot{\theta}_3 {}^0Z_2) \times d_3 {}^0Z_2 + (\dot{\theta}_1 {}^0Z_0 + \dot{\theta}_3 {}^0Z_2) \times a_4 {}^0X_4 $$

$$ {}^0Z_0 = {}^0Z_1 = {}^0Z_2 = {}^0Z_3 = {}^0Z_4 $$

$$ = \cancelto{0}{\dot{\theta}_1 {}^0Z_0 \times d_1 {}^0Z_0} + \dot{\theta}_1 {}^0Z_2 \times a_2 {}^0X_2 + \cancelto{0}{(\dot{\theta}_1 {}^0Z_2 + \dot{\theta}_3 {}^0Z_2) \times d_3 {}^0Z_2} + (\dot{\theta}_1 {}^0Z_4 + \dot{\theta}_3 {}^0Z_4) \times a_4 {}^0X_4 $$

$$ {}^0\dot{d}_{04} = a_2 {}^0Y_2 \dot{\theta}_1 + a_4 {}^0Y_4 \dot{\theta}_1 + a_4 {}^0Y_4 \dot{\theta}_3 $$

$$ \boxed{{}^0\dot{d}_{04} = (a_2 {}^0Y_2 + a_4 {}^0Y_4)\dot{\theta}_1 + a_4 {}^0Y_4 \dot{\theta}_3} $$

Endpoint Acceleration Equation

$$ {}^0\ddot{d}_{04} = \cancelto{0}{(\ddot{\theta}_1 {}^0Z_0) \times d_1 {}^0Z_0} + \cancelto{0}{(\dot{\theta}_1 {}^0Z_0) \times (\dot{\theta}_1 {}^0Z_0 \times d_1 {}^0Z_0)} + (\ddot{\theta}_1 {}^0Z_0) \times a_2 {}^0X_2 + (\dot{\theta}_1 {}^0Z_0) \times (\dot{\theta}_1 {}^0Z_0 \times a_2 {}^0X_2) + (\cancelto{0}{\ddot{\theta}_1 {}^0Z_0} + \cancelto{0}{\ddot{\theta}_3 {}^0Z_2} + \dot{\theta}_3 \cancelto{0}{(\dot{\theta}_1 {}^0Z_0) \times {}^0Z_2}) \times \cancelto{0}{d_3 {}^0Z_2} + \dots $$

$$ \cancelto{0}{(\dot{\theta}_1 {}^0Z_0 + \dot{\theta}_3 {}^0Z_2)} \times ((\cancelto{0}{\dot{\theta}_1 {}^0Z_0} + \cancelto{0}{\dot{\theta}_3 {}^0Z_2}) \times \cancelto{0}{d_3 {}^0Z_2}) + (\ddot{\theta}_1 {}^0Z_0 + \ddot{\theta}_3 {}^0Z_2 + \dot{\theta}_3 \cancelto{0}{(\dot{\theta}_1 {}^0Z_0) \times {}^0Z_2}) \times a_4 {}^0X_4 + (\dot{\theta}_1 {}^0Z_0 + \dot{\theta}_3 {}^0Z_2) \times ((\dot{\theta}_1 {}^0Z_0 + \dot{\theta}_3 {}^0Z_2) \times a_4 {}^0X_4) $$

$$ {}^0Z_0 = {}^0Z_1 = {}^0Z_2 = {}^0Z_3 = {}^0Z_4 $$

$$ = (\ddot{\theta}_1 {}^0Z_2) \times a_2 {}^0X_2 + (\dot{\theta}_1 {}^0Z_2) \times (\dot{\theta}_1 {}^0Z_2 \times a_2 {}^0X_2) + (\ddot{\theta}_1 {}^0Z_4 + \ddot{\theta}_3 {}^0Z_4) \times a_4 {}^0X_4 + (\dot{\theta}_1 {}^0Z_4 + \dot{\theta}_3 {}^0Z_4) \times ((\dot{\theta}_1 {}^0Z_4 + \dot{\theta}_3 {}^0Z_4) \times a_4 {}^0X_4) $$

$$ = \ddot{\theta}_1 a_2 {}^0Y_2 - \dot{\theta}_1^2 a_2 {}^0X_2 + a_4 {}^0Y_4 (\ddot{\theta}_1 + \ddot{\theta}_3) - a_4 \dot{\theta}_1 {}^0X_4 (\dot{\theta}_1 + \dot{\theta}_3) - a_4 \dot{\theta}_3 {}^0X_4 (\dot{\theta}_1 + \dot{\theta}_3) $$

$$ = a_2 {}^0Y_2 \ddot{\theta}_1 - a_2 {}^0X_2 \dot{\theta}_1^2 + a_4 {}^0Y_4 \ddot{\theta}_1 + a_4 {}^0Y_4 \ddot{\theta}_3 - a_4 {}^0X_4 \dot{\theta}_1^2 - a_4 {}^0X_4 \dot{\theta}_1 \dot{\theta}_3 - a_4 {}^0X_4 \dot{\theta}_1 \dot{\theta}_3 - a_4 {}^0X_4 \dot{\theta}_3^2 $$

$$ \boxed{{}^0\ddot{d}_{04} = (a_2 {}^0Y_2 + a_4 {}^0Y_4)\ddot{\theta}_1 + a_4 {}^0Y_4 \ddot{\theta}_3 - (a_2 {}^0X_2 + a_4 {}^0X_4)\dot{\theta}_1^2 - a_4 {}^0X_4 \dot{\theta}_3^2 - 2a_4 {}^0X_4 \dot{\theta}_1 \dot{\theta}_3} $$

Endpoint Position Equation

$$\boxed{{}^0d_{04} = d_1\,{}^0z_0 + a_2\,{}^0x_2 + d_3\,{}^0z_2 + a_4\,{}^0x_4}$$

Endpoint Velocity Equation

$$\boxed{{}^0\dot{d}_{04} = \left(a_2\,{}^0y_2 + a_4\,{}^0y_4\right)\dot{\theta}_1 + a_4\,{}^0y_4\,\dot{\theta}_3}$$

Endpoint Acceleration Equation

$$\boxed{{}^0\ddot{d}_{04} = \left(a_2\,{}^0y_2 + a_4\,{}^0y_4\right)\ddot{\theta}_1 + a_4\,{}^0y_4\,\ddot{\theta}_3 - \left(a_2\,{}^0x_2 + a_4\,{}^0x_4\right)\dot{\theta}_1^2 - a_4\,{}^0x_4\,\dot{\theta}_3^2 - 2a_4\,{}^0x_4\,\dot{\theta}_1\dot{\theta}_3}$$

Solve for angles, angular velocity, and angular acceleration for each actuator using inverse kinematics.

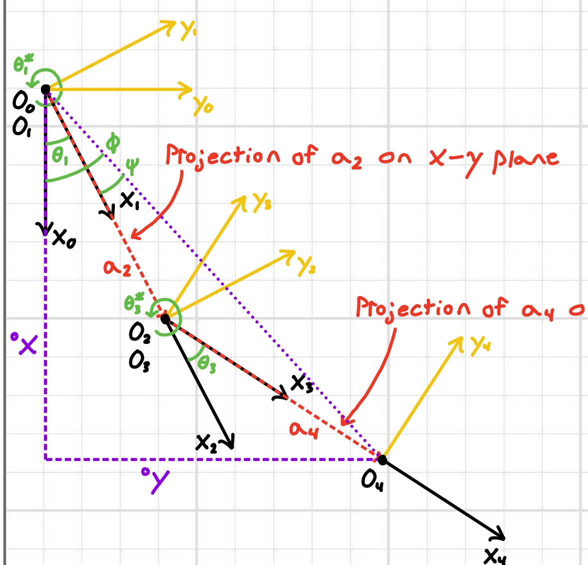
$${}^0d_{04} = \begin{bmatrix} X & Y & Z \end{bmatrix}^T$$

Projection Approach

Variable	X	Y	Z
θ_1	*	*	
θ_3	*	*	

There's no obvious direction to continue in using the projection approach.

Non-zero angle front view:



Because there are no actuators that can change the z-coordinate (rotate the leg into and out of the page), the lengths of a_2 and a_4 don't change in the x-y plane.

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Assuming the leg can be treated as a two-link planar manipulator like the one shown in Figure 5.1 below.

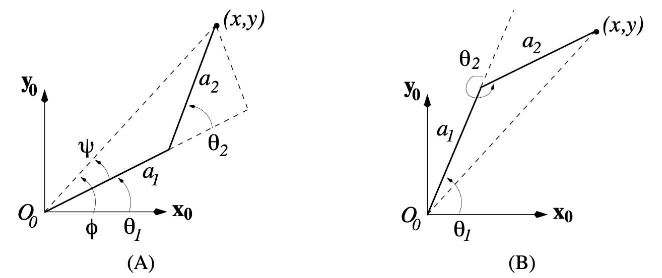

Figure 5.1: Inverse kinematics for two-link planar manipulator. Elbow down (A) and elbow up (B) solutions.

Solving for θ_3 using the cosine rule:

$$x^2+y^2=a_2^2+a_4^2-2a_2a_4\cos(\pi-\theta_3)$$

$$\cos(\theta_3)=\frac{x^2+y^2-a_2^2-a_4^2}{2a_2a_4}$$

The function $\cos^{-1}()$ is inaccurate, so instead I'm using the half-angle tangent formula.

$$\tan^2\left(\frac{\theta_3}{2}\right)=\frac{1-\cos(\theta_3)}{1+\cos(\theta_3)}=\frac{2a_2a_4-x^2-y^2+a_2^2+a_4^2}{2a_2a_4+x^2+y^2+a_2^2+a_4^2}=\frac{(a_2+a_4)^2-(x^2+y^2)}{(x^2+y^2)-(a_2-a_4)^2}$$

$$\boxed{\theta_3=\pm 2\tan^{-1}\sqrt{\frac{(a_2+a_4)^2-(x^2+y^2)}{(x^2+y^2)-(a_2-a_4)^2}}}$$

Solving for θ_1 using ϕ, ψ, and the atan2() function:

$$\phi=\operatorname{atan2}(y,x)$$

$$\psi=\operatorname{atan2}(a_4\sin(\theta_3),\ a_2+a_4\cos(\theta_3))$$

$$\theta_1=\phi-\psi$$

$$\boxed{\theta_1=\operatorname{atan2}(y,x)-\operatorname{atan2}(a_4\sin(\theta_3),\ a_2+a_4\cos(\theta_3))}$$

Alternate solution based on Gaussian elimination that's more efficient on a computer

$$x=a_2\cos(\theta_1)+a_4\cos(\theta_1+\theta_3)$$

$$y=a_2\sin(\theta_1)+a_4\sin(\theta_1+\theta_3)$$

$$x\cos(\theta_1)+y\sin(\theta_1)=a_2+a_4\cos(\theta_3)$$

$$x\cos(\theta_1)-y\sin(\theta_1)=-a_4\sin(\theta_3)$$

$$\boxed{\cos(\theta_1)=\frac{x(a_2+a_4\cos(\theta_3))+y(a_4\sin(\theta_3))}{x^2+y^2}}$$

$$\boxed{\sin(\theta_1)=\frac{y(a_2+a_4\cos(\theta_3))-x(a_4\sin(\theta_3))}{x^2+y^2}}$$

Apply the 4-quadrant arctangent to find θ_1

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Workspace of the two-link planar manipulator shown in Figure 5.11

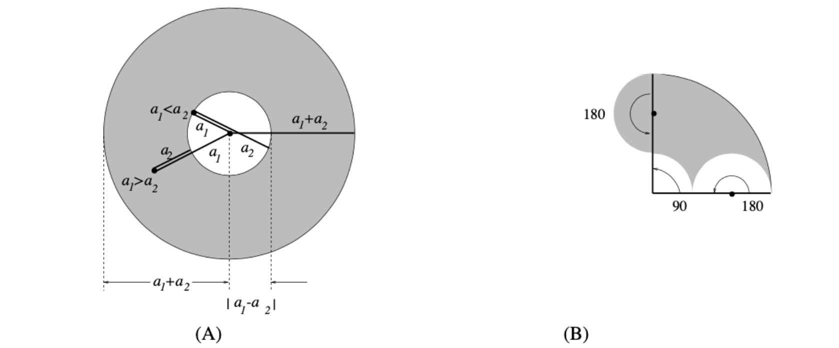

Figure 5.11: Workspace of two-link planar manipulator: (A) no joint limits. (B) Joint limits of $0 \leq \theta_1 \leq \pi/2$ and $0 \leq \theta_2 \leq \pi$.