Calculus, Algebra, and Analysis for JMC

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Contents

Chapter 1

Group theory

Study of the simplest algebraic structure on a set.

1.1 Binary operations and groups

Definition 1. *Set* is a collection of distinct elements. Let G be a set. ***Binary operation on G*** is a function

$$* : G \times G \to G \text{(Closure is included)}$$

Example 2.

- $(\mathbb{N}, +), (\mathbb{Z}, +), (\mathbb{R}, \cdot)$
- $(\mathbb{N}, -)$ not a binary op. Not closed.
- $g, h \in G, g * h = h$
- Find a certain $c \in G$, define $g * h = c \; \forall g, h \in G$

Example 3. Cayley table: Draw a table of all the possible binary operations on a set. How many possible binary operations on a finite set with n elements? In general, there are ∞-many biniary operatiions. In this case, there are n^{n^2} possible binary operations. *In general,* $g_i * g_j \neq g_j * g_i$ (Not commutative!)

Definition 4. A binary operation $*$ on a set G is called associative if

$$(g * h) * k = g * (h * k) \ \forall g, h, k \in G$$

Example 5.

- $+$ on $\mathbb{N}, \mathbb{Z}, \mathbb{R}$? Yes
- $-$ on $\mathbb{R}$? No
- $g * h = g^h$ on $\mathbb{N}$? No

Definition 6. A binary operation is called commutative if

$$\forall g, h \in G, g * h = h * g$$

Example 7.

- $+, \cdot$ on $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$
- matrix multiplication ($AB \neq BA$ in general for A, B in $M(\mathbb{R}^n)$)
- let $g, h \in \mathbb{R}$, $g * h = 1 + g \cdot h$: commutative but *not associative*!

Definition 8. Let $(G, *)$ be a set. An element e is called *left identity* (respectively *right identity*) if:

$$e * g = g (\text{resp. } g * e = g) \ \forall \ g \in G$$

<u>Caution</u>: There might be *many* left/right identities or none.

Example 9.

1. let $(G, *)$ be a set with $g * h := g$. Find the left/right identities.

   ∞-many (or equal to the number of elements) right identities since h satisfies definition $\forall h$. No left identities: wanted $e * g = g = e$ by definitioin of $*$ (*unless only one element*).

2. $(G, *)$, $g * h = 1 + gh$. Ex: No right/left identities.

Idea: We want a good unique identity.

Theorem 10. let $(G, *)$ be set, such that $*$ has both a left identity e_1 *and* a right identity e_2, then

$$e_1 = e_2 =: e \quad \text{and} \quad e \text{ is unique.}$$

Proof.

- $e_1 = e_2$

$$\Rightarrow \left\{ \begin{array}{l} e_1 * g = g \Rightarrow e_1 * e_2 = e_2 \\ g * e_2 = g \Rightarrow e_1 * e_2 = e_1 \end{array} \right\} \forall\, g \in G \Rightarrow\ e_1 = e_2$$

- Unicity: Assume there exists another identity e'.

$$\Rightarrow e' * g = g * e' = g$$

$$e' * g = e' * e = e$$

$$g * e' = e * e' = e'$$

Therefore

$$e = e'$$

∎

As soon as you get one left and one right identity, you have a unique identity e.

Definition 11. let $(G, *)$ be a set. Let $g \in G$. An element $h \in G$ is called left (resp. right) inverse if

$$h * g = e \text{ (resp. } g * h = e)$$

Caution: Again inverses might not exist, there might be many, or *not* the same on both sides.

Example 12.

(1) $(\mathbb{N}, \cdot)$ 1 has an inverse, otherwise *no* inverse.

(2) Find a binary operation on a set of 4 elements with left/right inverses not the same but identity e.

Theorem 13. Let $(G, *)$ be a set with associative binary operation and identity e. Then if h_1 is left inverse, and h_2 is right inverse, then

$$h_1 = h_2 = g^{-1} \text{ and } \textit{it is unique}$$

Proof.

- $h_1 = h_2$

  $h_1 * g = e, g * h_2 = e$. Therefore

  $$h_2 = e * h_2 = (h_1 * g) * h_2 = h_1 * (g * h_2) = e = h_1$$

- unicity: Assume $\exists g'^{-1}$ another inverse.

  $$g'^{-1} = e * g'^{-1} = (g^{-1} * g) * g'^{-1} = g^{-1} * (g * g'^{-1}) = g^{-1} * e = g^{-1}$$

$\square$

(Group) Definition 14. A set $(G, *)$ wth binary operatioin $*$ is called a *group* if:

(1) $*$ is associative

(2) $\exists e \in G$ an identity $\forall g \in G$

(3) All elements $g \in G$ have an inverse g^{-1}

Attention: The identity and inverses are *unique* by our previous results.

Example 15.

- $(\mathbb{Z}, +), (\mathbb{Z}_n, +)$ are groups.
- $(\mathbb{N}, +)$ not a group $\Rightarrow$ no inverses.
- $(\mathbb{C}, \cdot)$ not a group (0 has no multiplicative inverse), but $(\mathbb{C}^*, \cdot)$ is. $(\mathbb{C}^* = \mathbb{C}\backslash\{0\})$
- $(G = \{e\}, *)$ with $e * e = e$ is a group called the *trivial group*.
- Empty set $\varnothing$ is not a group (No identity element.)

Definition 16. Let G be a group. It is called finite if it has finitely many elements.

Notation: $|G| = n$ (number of elements)

If $|G| = \infty$, the G is called an infinite group.

Example 17.

- the trivial group is finite, $|G| = 1$
- let $G = \{1, -1, i, -i\} \subset \mathbb{C}$, with $* = \cdot$. Is it a group? Yes. Check associativity, identity, and inverses.

(Abelian Group) Definition 18. A group is called *Abelian* if $*$ is commutative.

Example 19.

- previous example, trvial group, $(\mathbb{Z}, +), (\mathbb{C}^*, \cdot)$
- let $GL(\mathbb{R}^n)$ be the set of all invertible $n \times n$ matrices, $*$ = matrix multiplication. It is associative: $(AB)C = A(BC)$; It has identity: I_n. It has inverses: yes since we asked for it. So this is a group of matrices. But this is not Abelian since $AB \neq BA$.
- let G be the set of *invertible* functions with $* = \circ$, the composition of functions. Identity is $F(x) = x$; they are associative, invertible, but *not Abelian.*

1.2 Consequences of the axioms of group

Chapter 2

Applied Mathematical Methods

2.1 Differential Equations

2.1.1 Definitions and examples

Definition 20. An *ordinary differential equation* (ODE) for $y(x)$ is an equation involving derivatives of y.

$$f(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}, \ldots, \frac{\mathrm{d}^ny}{\mathrm{d}x^n}) = 0 \tag{2.1}$$

$$\frac{\mathrm{d}^ny}{\mathrm{d}x^n} = F(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \ldots, \frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}})$$

and we seek a solution (or solutions) for $y(x)$ satisfying the equations. (If there are more independent variables then we have a partial differential equation (PDE).)

Definition 21.

Order is the order of the highest derivative present.

Degree is the power of the highest derivative when fractional powers have been removed.

Linear *differential equation* is a differential equation that is defined by a *linear polynomial* in the unknown function and its derivative in each term of equation(2.1).

Example 22.

(a) Particle moving along a line with a given force $\rightarrow x(t)$ position as function of time t.

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = f\left(t, x, \frac{\mathrm{d}x}{\mathrm{d}t}\right)$$

e.g.

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\omega^2 x - 2k\frac{\mathrm{d}x}{\mathrm{d}t}$$

The first term is regarding the restoring force, while the second term is regarding the damping/friction. The function is of order 2, degree 1, and linear.

(b) Radius of curvature of a curve

It can be shown that

$$R(x, y) = \frac{\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{\frac{3}{2}}}{\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}}$$

The function is of order 2 and degree 2.

(c) Simple growth and decay

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = kQ$$

The function is of order 1, degree 1, and linear. e.g.

(1) $k > 0$. Q as the quantity of money, and $k = (1 + \frac{r}{100})$, and r being the rate of interest.

(2) $k < 0$. Q as the amount of radioactive material, and k as the decay rate.

Hence, obviously $Q(t) = Q_0 e^{kt}$ where $Q_0 = Q(0)$ at $t = 0$.

(d) Population dynamics

$P(t)$ as population over time and $F(t)$ as food over time, with

$$\frac{\mathrm{d}P}{\mathrm{d}t} = aP(a > 0) \tag{2.2}$$

$$\frac{\mathrm{d}F}{\mathrm{d}t} = c(c > 0)$$

These two equations form a linear system, with both being of order 1, degree 1.

So $P(t) = P_0 e^{at}, F(t) = ct + F_0$. Misery! Population outgrows food supply.

Pierre Verhulst (1845) replaced a in equation(2.2) with $(a-bP)$ so that growth decreases as P increases:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = aP - bP^2 \tag{2.3}$$

This is in fact a *logistic ODE*, with order 1, degree 1, and nonlinear.

Note: Equation(2.3) is *separable*. Alternatively we can note that equation(2.3) is an example of a *Bernoulli differential equation*

$$\frac{\mathrm{d}y}{\mathrm{d}x} + F(x)y = H(x)y^n$$

with $n \neq 0, 1$ Substitution on $z(x) = (y(x))^{1-n} \Rightarrow$ a *linear* equation for $z(x) \rightarrow$ solution. (See below)

(e) Predator-Prey System

$x(t)$ as prey and $y(t)$ as predators, we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax - bxy, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -cy + \hat{d}xy \tag{2.4}$$

Note: Equation(2.4) is *separable* when written in principle

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \Rightarrow y(x) \Rightarrow x(t), y(t)$$

This is of order 1, degree 1, and a nonlinear system.

(f) Combat Model System

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -ay, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -bx \tag{2.5}$$

This is of order 1, degree 1, and linear system.

Note: Again equation(2.5) is *separable* when written as $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{bx}{ay} \Rightarrow y(x) \Rightarrow x(t), y(t)$

2.1.2 General Statement

Chapter 3

linear algebra

Chapter 4

Analysis