

# Condensed Notes for Maths40010

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# Contents

<b>1</b>	<b>Numbers</b>	<b>2</b>
1.1	Countability . . . . .	2
1.2	The Completeness Axiom . . . . .	2
1.3	Dedekind cuts . . . . .	3
1.4	triangle inequalities . . . . .	4
<b>2</b>	<b>Sequences</b>	<b>5</b>
2.1	convergence of sequences . . . . .	5
2.2	Cauchy Sequences . . . . .	6
2.3	Subsequences . . . . .	6
2.4	Series . . . . .	7
2.5	Convergence of Series . . . . .	7
2.6	Absolute convergence . . . . .	8
2.7	Tests for convergence . . . . .	8
2.8	Rearrangement of series . . . . .	9
2.9	Power Series . . . . .	9
2.9.1	Products of Series . . . . .	10
2.10	Exponential Power Series . . . . .	10
<b>3</b>	<b>Continuity</b>	<b>11</b>
3.1	Limits . . . . .	11
3.2	Continuity . . . . .	11

# Chapter 1

## Numbers

### 1.1 Countability

**Definition 1.** A set  $S$  is *countable* iff  $\exists$  bijection  $f : \mathbb{N} \rightarrow S$ .

**Theorem 2.** Suppose  $S \subset \mathbb{N}$  is infinite. Then  $S$  is *countable*.

**Theorem 3.**  $\mathbb{Z}$  is countable.

**Theorem 4.**  $\mathbb{Q}$  is countable.

**Theorem 5.**  $\mathbb{R}$  is uncountable.

### 1.2 The Completeness Axiom

**Definition 6.**  $\emptyset \neq S \subset \mathbb{R}$  is *bounded above* if

$$\exists M \in \mathbb{R} \text{ s.t. } \forall x \in S, x \leq M$$

Such an  $M$  is called an *upper bound* for  $S$ . In addition, we say  $x \in \mathbb{R}$  is a *least upper bound* for  $S$  or **supremum** of  $S$  iff

- $x$  is an upper bound for  $S$  (i.e.  $x \geq s \forall s \in S$ )
- $x \leq y \forall$  upper bounds  $y$  of  $S$  (i.e.  $y \geq s \forall s \in S \Rightarrow y \geq x$ )

**Theorem 7.**  $\emptyset \neq S \subset \mathbb{R}$  is *bounded below* if

$$\exists M \in \mathbb{R} \text{ s.t. } \forall x \in S, x \geq M$$

Such an  $M$  is called an *lower bound* for  $S$ . In addition, we say  $x \in \mathbb{R}$  is a *greatest lower bound* for  $S$  or **infimum** of  $S$  iff

- $x$  is a lower bound for  $S$  (i.e.  $x \leq s \forall s \in S$ )
- $x \geq y \forall$  lower bounds  $y$  of  $S$  (i.e.  $y \leq s \forall s \in S \Rightarrow y \leq x$ )

**Theorem 8.** Suppose  $S \subseteq \mathbb{R}$  is nonempty, bounded above, then  $\exists \sup S \in \mathbb{R}$

### 1.3 Dedekind cuts

**Definition 9.** We say a nonempty subset  $s \subset \mathbb{Q}$  is a *Dedekind cut* if it satisfy

- (i)  $\forall s \in S, [t < s \Rightarrow t \in S]$ , i.e.  $S$  is a semi-infinite interval to the left.
- (ii)  $S$  has an upper bound but no maximum

**Definition 10.** New Definition of  $\mathbb{R}$ :

$$\mathbb{R} := \{\text{Dedekind cuts } S \subset \mathbb{Q}\}$$

## 1.4 triangle inequalities

**Theorem 11.**  $\forall a, b \in \mathbb{R}$ , we have

$$|a + b| \leq |a| + |b|$$

$$|a + b| \geq \left| |a| - |b| \right|$$

$$|a| \leq |b| + |a - b|$$

$$|a| \geq |b| - |a - b|$$

$$|a - b| \leq |a - c| + |b - c|$$

# Chapter 2

## Sequences

**Definition 12.** A *sequence* is a function  $a : \mathbb{N} \rightarrow \mathbb{R}$

### 2.1 convergence of sequences

**Definition 13.** We say that  $a_n \rightarrow a$  as  $n \rightarrow \infty$  iff

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } |a_n - a| < \varepsilon \forall n > N$$

**Definition 14.** We say that  $a_n$  *converges* iff  $\exists a \in \mathbb{R}$  s.t.  $a_n \rightarrow a$ , i.e.  $a_n$  converges iff

$$\exists a \in \mathbb{R} \text{ s.t. } \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N, |a_n - a| < \varepsilon$$

**Definition 15.** We say  $a_n$  *diverges* iff it does not converge (to any  $a \in \mathbb{R}$ ), i.e.

$$\forall a \in \mathbb{R}, \exists \varepsilon > 0 \text{ s.t. } \forall N \in \mathbb{N}, \exists n \geq N \text{ s.t. } |a_n - a| \geq \varepsilon$$

**Definition 16.** We say  $a_n \rightarrow +\infty$  iff

$$\forall R > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N, a_n > R$$

**Definition 17.**  $a_n \in \mathbb{C}, \forall \geq 1$ . We say  $a_n \rightarrow a \in \mathbb{C}$  iff

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } n \geq N \Rightarrow |a_n - a| < \varepsilon$$

**Theorem 18.** Limits are unique. If  $a_n \rightarrow a$  and  $a_n \rightarrow b$ , then  $a = b$ .

**Theorem 19.** if  $a_n \rightarrow a$  and  $b_n \rightarrow b$  then:

1.  $a_n + b_n \rightarrow a + b$
2.  $a_n b_n \rightarrow ab$
3.  $\frac{a_n}{b_n} \rightarrow \frac{a}{b}$  if  $b \neq 0$ .

**Theorem 20.** If  $(a_n)$  is *bounded above* and *monotonically increasing* then  $a_n$  converges to  $a := \sup \{a_i : i \in \mathbb{N}\}$ . We write  $a_n \uparrow a$ .

## 2.2 Cauchy Sequences

**Definition 21.**  $(a_n)_{n \geq 1}$  is called a *Cauchy* sequence iff

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n, m \geq N, |a_n - a_m| < \varepsilon$$

**Theorem 22.**  $(a_n)$  is Cauchy  $\Rightarrow (a_n)$  is bounded.

**Theorem 23.**  $(a_n)$  is Cauchy  $\iff (a_n)$  is convergent.

## 2.3 Subsequences

**Definition 24.** A *subsequence* of  $a_n$  is a new sequence  $b_i = a_{n(i)} \forall i \in \mathbb{N}$ , where  $n(1) < n(2) < \dots < n(i) < \dots \forall i$ .

**(Bolzano-Weiestrass) Theorem 25.** If  $(a_n)$  is a *bounded* sequence of real numbers, then it has a convergent subsequence.

**Theorem 26.** If  $a_n \rightarrow a$  then any subsequence  $a_{n(i)} \rightarrow a$  as  $i \rightarrow \infty$ .

## 2.4 Series

**Definition 27.** An (infinite) series is an expression

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad a_1 + a_2 + a_3 + \dots,$$

where  $(a_i)_{i \geq 1}$  is a sequence.

**Definition 28.**  $n^{\text{th}}$  partial sum is

$$S_n := \sum_{i=1}^n a_i \in \mathbb{R}$$

## 2.5 Convergence of Series

**Definition 29.** We say that the series  $\sum a_n$  “converges to  $A \in \mathbb{R}$ ” iff the sequence  $(S_n)$  of partial sums converges to  $A$ :

$$\sum_{n=1}^{\infty} a_n = A \in \mathbb{R} \iff S_n \rightarrow A \text{ as } n \rightarrow \infty$$



**Theorem 30.**  $\sum_{n=1}^{\infty} a_n$  is convergent  $\Rightarrow a_n \rightarrow 0$ . In other words,  $a_n \not\rightarrow 0 \Rightarrow \sum_{n=1}^{\infty} a_n$  is divergent.

**Theorem 31.** Suppose  $a_n \geq 0 \forall n$  ( $\iff S_n = \sum_{i=1}^n a_i$  monotonically increasing), then  $S_{\infty} = \sum_{n=1}^{\infty} a_n$  convergent  $\iff (S_n)$  bounded above. Similarly,  $\sum_{n=1}^{\infty} a_n \rightarrow +\infty \iff (S_n)$  is unbounded.

**Theorem 32.** if  $\sum a_n, \sum b_n$  are convergent then so is  $\sum(\lambda a_n + \mu b_n)$ , to

$$\sum_{n=1}^{\infty} (\lambda a_n + \mu b_n) = \lambda \sum_{n=1}^{\infty} a_n + \mu \sum_{n=1}^{\infty} b_n$$

## 2.6 Absolute convergence

**Definition 33.** For  $a_n \in \mathbb{R}$  or  $\mathbb{C}$ , we say the series  $\sum_{n=1}^{\infty} a_n$  is *absolutely convergent* iff the series  $\sum_{n=1}^{\infty} |a_n|$  is convergent.

**Theorem 34.** If  $\sum a_n$  is absolutely convergent, then it is convergent.

## 2.7 Tests for convergence

**Theorem 35.** if  $0 \leq a_n \leq b_n$  and  $\sum_{n=1}^{\infty} b_n$  is convergent, then  $\sum a_n$  convergent and  $0 \leq \sum a_n \leq \sum b_n$ .

**Theorem 36.** If  $c_n \leq a_n \leq b_n \forall n$  and  $\sum c_n, \sum b_n$  both convergent, then  $\sum a_n$  convergent and  $\sum c_n \leq \sum a_n \leq \sum b_n$ .

**Theorem 37.** If  $\frac{a_n}{b_n} \rightarrow L \in \mathbb{R} (b_n \neq 0 \forall n)$ , then if  $\sum b_n$  is absolutely convergent, then  $\sum a_n$  is absolutely convergent.

**Theorem 38.** If  $(a_n)$  is alternating and  $|a_n| \downarrow 0$ , then  $\sum a_n$  is convergent.

**Theorem 39.** If  $\left| \frac{a_{n+1}}{a_n} \right| \rightarrow r < 1$ , then  $\sum a_n$  is absolutely convergent.

**Theorem 40.** If  $|a_n|^{\frac{1}{n}} \rightarrow r < 1$ , then  $\sum a_n$  is absolutely convergent.

## 2.8 Rearrangement of series

**Definition 41.** Given a bijection  $n : \mathbb{N} \rightarrow \mathbb{N}$ , define  $b_i := a_{n(i)}$ . Then  $(b_i)_{i \geq 1}$  is a *rearrangement* or *reordering* of  $(a_n)_{n \geq 1}$ .

**Theorem 42.**  $\sum a_n$  is absolutely convergent  $\iff (1) + (2) \Rightarrow (3) + (4)$ , where

1.  $\sum_{a_n \geq 0} a_n$  is convergent (to  $A$  say),
2.  $\sum_{a_n < 0} a_n$  is convergent (to  $B$  say),
3.  $\sum a_n = A + B$ ,
4.  $\sum b_m = A + B$  where  $(b_m)$  is any rearrangement of  $(a_n)$ .

## 2.9 Power Series

**Theorem 43.** Fix a real complex series  $(a_n)$  and consider the series  $\sum a_n z^n$  for  $z \in \mathbb{C}$ . Then  $\exists R \in [0, \infty]$  s.t.

- $|z| < R \Rightarrow \sum a_n z^n$  is absolutely convergent, and
- $|z| > R \Rightarrow \sum a_n z^n$  is divergent.

### 2.9.1 Products of Series

**Definition 44.** Given series  $\sum a_n$ ,  $\sum b_n$ , their *Cauchy Product* is the series  $\sum c_n$  where  $c_n := \sum_{i=0}^n a_i b_{n-i}$ .

**Theorem 45.** If  $\sum a_n$ ,  $\sum b_n$  are absolutely convergent, then their Cauchy Product  $\sum c_n$  is absolutely convergent to  $(\sum a_n) \cdot (\sum b_n)$ .

## 2.10 Exponential Power Series

**Definition 46.** For any  $z \in \mathbb{C}$  set

$$E(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

**Theorem 47.**  $E(x)$  has the following properties for  $x \in \mathbb{R}$ .

1.  $E(x) > 0 \forall x \in \mathbb{R}$
2.  $x \geq 0 \Rightarrow E(x) \geq 1$  and  $x > 0 \Rightarrow E(x) > 1$
3.  $E(x)$  is strictly increasing for  $x \in \mathbb{R}$
4.  $|E(x) - 1| < \frac{|x|}{1-|x|} \forall |x| < 1$
5.  $x \mapsto E(x)$  is a continuous bijection  $\mathbb{R} \rightarrow (0, \infty)$

# Chapter 3

## Continuity

### 3.1 Limits

**Definition 48.** Fix a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  and points  $a, b \in \mathbb{R}$ . We say that  $f(x) \rightarrow b$  as  $x \rightarrow a$  (or “ $\lim_{x \rightarrow a} f(x) = b$ ”) iff

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \Rightarrow |f(x) - b| < \varepsilon$$

### 3.2 Continuity

**Definition 49.** Given a function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , we say that  $f$  is *continuous* at  $a \in \mathbb{R}$  iff

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \text{ that are } |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$$

We say that  $f$  is continuous on  $\mathbb{R}$  (or just “continuous”) if it is continuous at all  $a \in \mathbb{R}$ .

**Definition 50.** Given a function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , we say that  $f$  is *discontinuous* at  $a \in \mathbb{R}$  iff

$$\exists \varepsilon > 0 \text{ s.t. } \forall \delta > 0, \exists x \text{ with } |x - a| < \delta \Rightarrow |f(x) - f(a)| \geq \varepsilon$$

**Theorem 51.**  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous at  $a \in \mathbb{R} \iff f(x_n) \rightarrow f(a) \quad \forall$  sequences  $(x_n)$  which tends to  $a$ .

In other words,  $f : \mathbb{R} \rightarrow \mathbb{R}$  is *not* continuous at  $a \in \mathbb{R} \iff f(x_n) \nrightarrow f(a) \quad \forall$  sequences  $(x_n)$  which tends to  $a$ .