Calculus, Algebra, and Analysis for JMC

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Contents

Chapter 1

Group theory

Study of the simplest algebraic structure on a set.

1.1 Basic Definitions and Examples

1.1.1 Binary operations and groups

Definition 1. *Set* is a collection of distinct elements. Let G be a set. ***Binary operation on G*** is a function

$$* : G \times G \to G (\text{Closure is included})$$

Example 2.

- $(\mathbb{N}, +), (\mathbb{Z}, +), (\mathbb{R}, \cdot)$
- $(\mathbb{N}, -)$ not a binary op. Not closed.
- $g, h \in G, g * h = h$
- Find a certain $c \in G$, define $g * h = c \; \forall g, h \in G$

Example 3. Cayley table: Draw a table of all the possible binary operations on a set. How many possible binary operations on a finite set with n elements? In general, there are ∞-many biniary operatiions. In this case, there are n^{n^2} possible binary operations. *In general,* $g_i * g_j \neq g_j * g_i$ (Not commutative!)

Definition 4. A binary operation $*$ on a set G is called associative if

$$(g * h) * k = g * (h * k)\ \forall g, h, k \in G$$

Example 5.

- $+$ on $\mathbb{N}, \mathbb{Z}, \mathbb{R}$? Yes
- $-$ on $\mathbb{R}$? No
- $g * h = g^h$ on $\mathbb{N}$? No

Definition 6. A binary operation is called commutative if

$$\forall g, h \in G, g * h = h * g$$

Example 7.

- $+, \cdot$ on $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$
- matrix multiplication ($AB \neq BA$ in general for A, B in $M(\mathbb{R}^n)$)
- let $g, h \in \mathbb{R}$, $g * h = 1 + g \cdot h$: commutative but *not associative*!

Definition 8. Let $(G, *)$ be a set. An element e is called *left identity* (respectively *right identity*) if:

$$e * g = g(\text{resp. } g * e = g)\ \forall\ g \in G$$

<u>Caution</u>: There might be *many* left/right identities or none.

Example 9.

1. let $(G, *)$ be a set with $g * h := g$. Find the left/right identities.

   ∞-many (or equal to the number of elements) right identities since h satisfies definition $\forall h$. No left identities: wanted $e * g = g = e$ by definitioin of $*$ (*unless only one element*).

2. $(G, *)$, $g * h = 1 + gh$. Ex: No right/left identities.

Idea: We want a good unique identity.

Theorem 10. let $(G,*)$ be set, such that $*$ has both a left identity e_1 *and* a right identity e_2, then

$$e_1 = e_2 =: e \quad \text{and} \quad e \text{ is unique.}$$

Proof.

- $e_1 = e_2$

$$\Rightarrow \left\{ \begin{array}{l} e_1 * g = g \Rightarrow e_1 * e_2 = e_2 \\ g * e_2 = g \Rightarrow e_1 * e_2 = e_1 \end{array} \right\} \forall\, g \in G \Rightarrow\ e_1 = e_2$$

- Unicity: Assume there exists another identity e'.

$$\Rightarrow e' * g = g * e' = g$$

$$e' * g = e' * e = e$$

$$g * e' = e * e' = e'$$

Therefore

$$e = e'$$

$\square$

As soon as you get one left and one right identity, you have a unique identity e.

Definition 11. let $(G,*)$ be a set. Let $g \in G$. An element $h \in G$ is called left (resp. right) inverse if

$$h * g = e \text{ (resp. } g * h = e)$$

<u>Caution</u>: Again inverses might not exist, there might be many, or *not* the same on both sides.

Example 12.

(1) $(\mathbb{N}, \cdot)$ 1 has an inverse, otherwise *no* inverse.

(2) Find a binary operation on a set of 4 elements with left/right inverses not the same but identity e.

Theorem 13. Let $(G, *)$ be a set with associative binary operation and identity e. Then if h_1 is left inverse, and h_2 is right inverse, then

$$h_1 = h_2 = g^{-1} \text{ and } \textit{it is unique}$$

Proof.

- $h_1 = h_2$

  $h_1 * g = e, g * h_2 = e$. Therefore

  $$h_2 = e * h_2 = (h_1 * g) * h_2 = h_1 * (g * h_2) = e = h_1$$

- unicity: Assume $\exists g'^{-1}$ another inverse.

  $$g'^{-1} = e * g'^{-1} = (g^{-1} * g) * g'^{-1} = g^{-1} * (g * g'^{-1}) = g^{-1} * e = g^{-1}$$

□

(Group) Definition 14. A set $(G, *)$ with binary operatioin $*$ is called a *group* if:

(1) $*$ is associative

(2) $\exists e \in G$ an identity $\forall g \in G$

(3) All elements $g \in G$ have an inverse g^{-1}

Attention: The identity and inverses are *unique* by our previous results.

Example 15.

- $(\mathbb{Z}, +), (\mathbb{Z}_n, +)$ (will see this later) are groups.
- $(\mathbb{N}, +)$ not a group $\Rightarrow$ no inverses.
- $(\mathbb{C}, \cdot)$ not a group (0 has no multiplicative inverse), but $(\mathbb{C}^*, \cdot)$ is. ($\mathbb{C}^* = \mathbb{C}\backslash\{0\}$)
- $(G = \{e\}, *)$ with $e * e = e$ is a group called the *trivial group*.
- Empty set $\varnothing$ is not a group (No identity element.)

Definition 16. Let G be a group. It is called finite if it has finitely many elements.

Notation: $|G| = n$ (number of elements)

We say that G has ***order*** n. If $|G| = \infty$, the G is called an infinite group.

Example 17.

- the trivial group is finite, $|G| = 1$
- let $G = \{1, -1, i, -i\} \subset \mathbb{C}$, with $* = \cdot$. Is it a group? Yes. Check associativity, identity, and inverses.

(Abelian Group) Definition 18. A group is called *Abelian* if $*$ is commutative.

Example 19.

- previous example, trvial group, $(\mathbb{Z}, +), (\mathbb{C}^*, \cdot)$
- let $GL(\mathbb{R}^n)$ be the set of all invertible $n \times n$ matrices, $*$ = matrix multiplication. It is associative: $(AB)C = A(BC)$; It has identity: I_n. It has inverses: yes since we asked for it. So this is a group of matrices. But this is not Abelian since $AB \neq BA$.
- let G be the set of *invertible* functions with $* = \circ$, the composition of functions. Identity is $F(x) = x$; they are associative, invertible, but *not Abelian*.

1.1.2 Consequences of the axioms of group

Theorem 20. Let $(G, *)$ be a group, $g, h \in G$. Then

$$(g * h)^{-1} = h^{-1} * g^{-1}$$

Proof. To show: $(g * h) * (h^{-1} * g^{-1}) = e$.

Using assocativity, we have

$$g * (h * h^{-1}) * g^{-1} = g * g^{-1} = e$$

$\square$

Definition 21. Let $n \in \mathbb{Z}$, let $(G, *)$ be a group and let $g \in G$. Then we definie g^n as follows:

$$g^n = \begin{cases} g * g * \cdots * g & n > 0 \\ g^{-1} * g^{-1} * \cdots * g^{-1} & n < 0 \\ e & n = 0 \end{cases}$$

where in the first case there are n copies of g in the product and ni the second there are $-n$ copies of g^{-1}, so that $g^n = (g^{-1})^{-n}$.

Theorem 22. Let $n, m \in \mathbb{Z}$ and let $G, *$ be a group. Then

1. $g^n * g^m = g^{n+m}$
2. $(g^n)^m = g^{nm}$

Proof. Exercise! (Hint: Induction.) □

1.1.3 Modular Artihmetic and the group $\mathbb{Z}_n$

Definition 23. let $n > 0$, $n \in \mathbb{Z}$ fixed, $a, b \in \mathbb{Z}$. a and b are called ***congruent modulo*** n if $n | a - b$.

Definition 24. $\forall a, b, c \in \mathbb{Z}$, $n > 0$ fixed in $\mathbb{Z}$:

(1) $a \equiv a \mod n$ (reflexivity)

(2) If $a \equiv b \mod n \iff b \equiv a \mod n$ (symmetry)

(3) if $a \equiv b \mod n$ and $b \equiv c \mod n \implies a \equiv c \mod n$ (transitivity)

Definition 25. Given a set S and an equivalence relation $\sim$ on S, the ***equivalence class*** of an element a in S is the set $\{x \in S \,|\, x \sim a\}$.

Definition 26. Define the equivalence class of $a \in \mathbb{Z}$ in the relation of congruence modulo n as:

$$[a]_n := \{b \in \mathbb{Z} \mid b \equiv a \mod n\}$$

Definition 27. Define equivalence classes $\mathbb{Z}_n$ as

$$\mathbb{Z}_n := \{[0]_n, [1]_n, \dots [n-1]_n\}$$

with 2 binary operations on $\mathbb{Z}_n$:

$+$: $\mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}_n$, $([a]_n, [b]_n) \mapsto [a+b]_n$

$\cdot$: $\mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}_n$, $([a]_n, [b]_n) \mapsto [ab]_n$

As we can see from the following lemma, the two operations are well-defined.

Lemma 28. Let $a, a', b, b' \in \mathbb{Z}$ s.t. $[a]_n = [a']_n, [b]_n = [b']_n$. Then $[a+b]_n = [a'+b']_n$, $[a \cdot b]_n = [a' \cdot b']_n$.

Proof. Exercise! □

Theorem 29. $(\mathbb{Z}_n, +)$ is an Abelian group.

Proof.

(1) Associativity:

$$\begin{aligned}([a]_n + [b]_n) + [c]_n &= [a+b]_n + [c]_n \\ &= [a+b+c]_n \\ &= [a]_n + [b+c]_n \\ &= [a]_n + ([b]_n + [c]_n)\end{aligned}$$

(2) Commutativity:

$$\begin{aligned}[a]_n + [b]_n &= [a+b]_n \\ &= [b+a]_n \\ &= [b]_n + [a]_n\end{aligned}$$

(3) Identity element: $[0]_n$

(4) Inverse: Any element $[a]_n$ has an inverse $[-a]_n$.

$\square$

Example 30. $(\mathbb{Z}_n, \cdot)$ is an Abelian group?
Similary to above for associative, commutative, and identity.
Inverses:

Draw Caley table for $(\mathbb{Z}_3, \cdot)$. We realize that $[0]_3$ has no inverses. But $(\mathbb{Z}_3 \backslash \{[0]_3\}, \cdot)$ is.

Similarly, for $(\mathbb{Z}_4, \cdot)$, it does not have inverses for all classes.

Caution: In general $(\mathbb{Z}_n, \cdot)$ is *not* a group. The idea then is to make it a group by removing non-invertible elements.

Lemma 31. The element $[a]_n \in \mathbb{Z}_n$ has an inverse $\iff$ $(a, n) = 1$.

Proof. $(a, n) = 1 \iff \exists b, c \in \mathbb{Z}$, s.t $ab + cn = 1 \iff cn = 1 - ab \iff \exists [b]_n$ s.t. $[a]_n [b]_n = [1]_n$. $\square$

Definition 32. $\mathbb{Z}_n^* := \{[a]_n \in \mathbb{Z}_n \mid \exists b \in \mathbb{Z} \text{ s.t. } [a]_n [b]_n = [1]_n\}$.

Theorem 33. $(\mathbb{Z}_n^*, \cdot)$ is an Abelian group.

Proof. To Show: if $[a]_n, [b]_n \in (\mathbb{Z}_n^*, \cdot) \Rightarrow [a]_n \cdot [b]_n \in (Z_n^*, \cdot)$.
$\Rightarrow$ $(a, n) = (b, n) = 1 \Rightarrow$ $(ab, n) = 1 \Rightarrow$ $[ab]_n$ has inverse $[a]_n [b]_n$.

Alternatively: if g, h have inverse, $h^{-1} g^{-1}$ is inverse of gh. $\square$

Definition 34. Let G be a group, $g \in G$. The ***order*** of g is the smallest positive integer $n > 0$ such that $g^n = e$.

Notation: ord $g = n$. If $n = \infty$, then g is called of infinite order.

Example 35. $G = (\mathbb{C}^*, \cdot)$, ord $(-1) = 2$, ord $i = 4$, ord $2 = \infty$

Lemma 36. Let G be a finite group. Then every element $g \in G$ has finite orders.

Proof. Assume $g \in G$ has infinite orders. Write the list: $g^0, g^1, g^2, \ldots$

Since $|G| = n < \infty$, there are two elements g^k, g^l s.t. $g^k = g^l$, $k > l$. $\iff \quad g^k g^{-l} = e \iff g^{k-l} = e$.

But then ord $g \leq k - l < \infty$. $\square$

Chapter 2

Applied Mathematical Methods

2.1 Differential Equations

2.1.1 Definitions and examples

Definition 37. An ***ordinary differential equation*** (ODE) for $y(x)$ is an equation involving <u>derivatives</u> of y.

$$f(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}, \ldots, \frac{\mathrm{d}^ny}{\mathrm{d}x^n}) = 0 \tag{2.1}$$

$$\frac{\mathrm{d}^ny}{\mathrm{d}x^n} = F(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \ldots, \frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}})$$

and we seek a solution (or solutions) for $y(x)$ satisfying the equations. (If there are more independent variables then we have a partial differential equation (PDE).)

Definition 38.

Order is the order of the highest derivative present.

Degree is the power of the highest derivative when fractional powers have been removed.

Linear *differential equation* is a differential equation that is defined by a *linear polynomial* in the unknown function and its derivative in each term of equation(2.1).

Example 39.

(a) Particle moving along a line with a given force $\to x(t)$ position as function of time t.

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = f\left(t, x, \frac{\mathrm{d}x}{\mathrm{d}t}\right)$$

e.g.

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\omega^2 x - 2k\frac{\mathrm{d}x}{\mathrm{d}t}$$

The first term is regarding the restoring force, while the second term is regarding the damping/friction. The function is of order 2, degree 1, and linear.

(b) Radius of curvature of a curve

It can be shown that

$$R(x,y) = \frac{\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{\frac{3}{2}}}{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}}$$

The function is of order 2 and degree 2.

(c) Simple growth and decay

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = kQ$$

The function is of order 1, degree 1, and linear. e.g.

(1) $k > 0$. Q as the quantity of money, and $k = (1 + \frac{r}{100})$, and r being the rate of interest.

(2) $k < 0$. Q as the amount of radioactive material, and k as the decay rate.

Hence, obviously $Q(t) = Q_0 e^{kt}$ where $Q_0 = Q(0)$ at $t = 0$.

(d) Population dynamics

$P(t)$ as population over time and $F(t)$ as food over time, with

$$\frac{\mathrm{d}P}{\mathrm{d}t} = aP(a > 0) \tag{2.2}$$

$$\frac{\mathrm{d}F}{\mathrm{d}t} = c(c > 0)$$

These two equations form a linear system, with both being of order 1, degree 1.

So $P(t) = P_0 e^{at}, F(t) = ct + F_0$. Misery! Population outgrows food supply.

Pierre Verhulst (1845) replaced a in equation(2.2) with $(a - bP)$ so that growth decreases as P increases:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = aP - bP^2 \tag{2.3}$$

This is in fact a *logistic ODE*, with order 1, degree 1, and nonlinear.

Note: Equation(2.3) is *separable*. Alternatively we can note that equation(2.3) is an example of a *Bernoulli differential equation*

$$\frac{\mathrm{d}y}{\mathrm{d}x} + F(x)y = H(x)y^n \tag{2.4}$$

with $n \neq 0, 1$ Substitution on $z(x) = (y(x))^{1-n} \Rightarrow$ a *linear* equation for $z(x) \to$ solution. (See below)

(e) Predator-Prey System

$x(t)$ as prey and $y(t)$ as predators, we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax - bxy, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -cy + dxy \tag{2.5}$$

Note: Equation(2.5) is *separable* when written in principle

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \Rightarrow y(x) \Rightarrow x(t), y(t)$$

This is of order 1, degree 1, and a nonlinear system.

(f) Combat Model System

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -ay, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -bx \tag{2.6}$$

This is of order 1, degree 1, and linear system.

Note: Again equation(2.6) is *separable* when written as $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{bx}{ay} \Rightarrow y(x) \Rightarrow x(t), y(t)$

In general the solution of a differential equation of order n contains a number n of *arbitrary constants*. This general solution can be specialised to a particular solution by assigninig definite values to these constants.

Example 40.

(a) Family or parabolae $y = Cx^2$ as constant C takes different values.

On a particular curve of the family $\frac{\mathrm{d}y}{\mathrm{d}x} = 2Cx$. By substitutiion, eliminate $C \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2y}{x}$. This is a geometrical statement about slopes.

Note: 1st order differential equation $\leftrightarrow$ 1 arbitrary constant in general solution.

(b)

$$\left.\begin{aligned} x &= A\sin\omega t + B\cos\omega t \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= A\omega\cos\omega t - B\omega\sin\omega t \\ \frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= -A\omega^2\sin\omega t - B\omega^2\cos\omega t \end{aligned}\right\} \Rightarrow \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega^2 x = 0$$

Note: 2nd order differential equation $\leftrightarrow$ 2 arbitrary constants in general solution.

Of course it's the reverse of this process we normally want to perform in order to get the general solution. We then often need a particular solution — which satisfieis certain other conditions — *boundary* or *initial condition.* These allow us to find the arbitrary constants in the solutions.

2.1.2 First Order Differential Equations

Properties and approaches

There are essentially 4 types we can solve *analytically*:

- ***separable***
- ***homogeneous***
- ***linear***
- ***exact*** (in Chapter "Partial Differentiation and Multivariable Calculus" later)

Let's look at them one by one:

(a) **Separable**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = G(x) \cdot H(y)$$

Solve by rearrangement and integration

$$\int^y \frac{\mathrm{d}y}{H(y)} = \int^x G(x)\mathrm{d}x$$

E.g.

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= xy^2e^{-x} \\ \int \frac{1}{y^2}\mathrm{d}y &= \int xe^{-x}\mathrm{d}x \\ -\frac{1}{y} &= -xe^{-x} - e^{-x} + C \end{aligned}$$

Or singular solution $y = 0$.

If we want the particular solution which passes through $x = 1, y = 1$, then of course we need

$$C = -1 + 2e^{-1} \quad \text{and} \quad \frac{1}{y} = (x+1)e^{-x} + 1 - 2e^{-1}$$

(b) **Homogeneous**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right)$$

Substitution $\frac{y}{x} = u(x)$, i.e. a new dependent variable,

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= u + x\frac{\mathrm{d}u}{\mathrm{d}x}(= f(u)) \quad \textit{\textbf{(Remember!)}} \\ f(u) - u &= \frac{x\mathrm{d}u}{\mathrm{d}x} \\ \int \frac{\mathrm{d}u}{f(u) - u} &= \int \frac{\mathrm{d}x}{x} \\ &\vdots \end{aligned}$$

E.g.

(i)

$$
\begin{aligned}
x^2\frac{\mathrm{d}y}{\mathrm{d}x} + xy - y^2 = 0& \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \left(\frac{y}{x}\right)^2 - \frac{y}{x} \\
\frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{u^2-2u}{x} \\
&\vdots
\end{aligned}
$$

(ii)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y-3}{x-y+1}$$

This does not look homogeneous as it stands, but can be made so by substituting $x = 1+X, y = 2+Y$, and the expression becomes

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{X+Y}{X-Y} = \frac{1+\left(\frac{Y}{X}\right)}{1-\left(\frac{Y}{X}\right)}$$

Then let $\frac{Y}{X} = u(X)$,

$$\Rightarrow \int \left(\frac{1-u}{1+u^2}\right)\mathrm{d}u = \int \frac{\mathrm{d}X}{X}$$

Eventually, the equation becomes

$$\tan^{-1}\frac{Y}{X} - \frac{1}{2}\ln\left(1+\frac{Y^2}{X^2}\right) = \ln X + C$$

$$\tan^{-1}\left(\frac{y-2}{x-1}\right) - \frac{1}{2}\ln\left[(x-1)^2+(y-2)^2\right] = C$$

Note: If we have e.g. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y-3}{2(x+y)-7}$, then substitute $v(x) = x+y$ will work!

(c) **Linear**

$$\frac{\mathrm{d}y}{\mathrm{d}x} + F(x)y = G(x)$$

1st power only for y and $\frac{\mathrm{d}y}{\mathrm{d}x}$. We apply an *integrating factor* $R(x)$:

$$R(x) = \exp\left[\int^x F(x)\mathrm{d}x\right]$$

This allows us to form the expression

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[y\exp\left(\int^x F(x)\mathrm{d}x\right)\right] = G(x)\exp\left(\int^x F(x)\mathrm{d}x\right)$$

and then integrate...

E.g.

$$(x+2)\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = (x+2)^6$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{4}{x+2} = (x+2)^5$$

$$\Rightarrow F(x) = -\frac{4}{x+2}, G(x) = (x+2)^5$$

Therefore,

$$R(x) = \exp\left[-\int^x\left(\frac{4}{x+2}\right)\mathrm{d}x\right] = \cdots = K(x+2)^{-4}$$

Subsequently, take $K = 1$ W.L.O.G.:

$$(x+2)^{-4}\frac{\mathrm{d}y}{\mathrm{d}x} - 4(x+2)^{-5}y = \frac{\mathrm{d}}{\mathrm{d}x}\left[y(x+2)^{-4}\right] = x+2$$

As such,

$$y(x+2)^{-4} = \frac{1}{2}x^2 + 2x + C \quad \text{(Put } C \text{ at the right time!)}$$

$$y(x) = \left(\frac{1}{2}x^{2+2x+C}\right)(x+2)^4$$

(So e.g. $y(0) = 8 \Rightarrow C = \frac{1}{2}$)

Novelties!

(i) Bernoulli equation (See Equation(2.4))
A nonlinear equation rendered linear by a substitution $u = y^{1-n}$...

(ii) E.g.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x+e^y}$$

It is <u>nonlinear</u> for $y(x)$ but <u>linear</u> for $x(y)$:

$$\frac{\mathrm{d}x}{\mathrm{d}y} - x = e^y \Rightarrow \ldots$$

2.1.3 'Special' Second Order Differential Equations

Definition 41. General Explicit form is

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = F\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

(a) $y, \frac{\mathrm{d}y}{\mathrm{d}x}$ **missing**, i.e.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = f(x)$$

Just integrate twice!

(b) $x, \frac{\mathrm{d}y}{\mathrm{d}x}$ **missing**, i.e.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = f(y)$$

Warning: Do not write $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{1}{\frac{\mathrm{d}^2x}{\mathrm{d}y^2}}$. However, it may be true, but for what class of functions $y(x)$?

Let $\frac{\mathrm{d}y}{\mathrm{d}x} = p$,

$$\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}p}{\mathrm{d}x} = \frac{\mathrm{d}p}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = p\frac{\mathrm{d}p}{\mathrm{d}y} = \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{1}{2}p^2\right)$$

This substitution is effective because it eliminates x, so that the equation becomes separable for p and y.

Then we can integrate $\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{1}{2}p^2\right) = f(y)$ w.r.t. y to get $p(y)$. Then using the definition of p,

$$x = \int \frac{\mathrm{d}y}{p(y)}$$

The same is obtained by multiplying the original equation by $\frac{\mathrm{d}y}{\mathrm{d}x}$ and recognizing $\frac{\mathrm{d}y}{\mathrm{d}x} \cdot \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]$

Example:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\omega^2 y$$

with ω being a real constant. (It is a simple harmonic motion.)

$$\Rightarrow \frac{1}{2}p^2 = -\frac{1}{2}\omega^2 y^2 + C$$

Let $C = \frac{1}{2}\omega^2\overline{A}^2$. We therefore get

$$\frac{1}{p} = \frac{\mathrm{d}x}{\mathrm{d}y} = \pm\frac{1}{\omega(\overline{A}^2 - y^2)^{\frac{1}{2}}}$$

$$\Rightarrow \omega x + \overline{B} = \pm\sin^{-1}\frac{y}{\overline{A}}$$
$$y = \overline{A}\sin(\omega x + \overline{B}) \text{ W.L.O.G}$$
$$= A\sin\omega x + B\cos\omega x$$

(c) y **missing**, i.e.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = f\left(x, \frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

We put $\frac{\mathrm{d}y}{\mathrm{d}x} = p$, so

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}p}{\mathrm{d}x} = f(x, p)$$

i.e. First order $p(x)$. This substitution is effective because it eliminates y, so that the equation becomes separable for p and x.

Solve for $p(x)$ then integrate $\Rightarrow y(x)$.

Example: Radius of curvature

$$\frac{\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{\frac{3}{2}}}{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}} = a \quad (a \text{ is an arbitrary constant})$$

$$\Rightarrow \frac{\mathrm{d}p}{\mathrm{d}x} = \frac{1}{a}(1+p^2)^{\frac{3}{2}}$$

$$\Rightarrow \frac{x}{a} + C = \int\frac{\mathrm{d}p}{(1+p^2)^{\frac{3}{2}}} \quad \text{i.e.} \quad \frac{x}{a} - \frac{A}{a} = \frac{p}{(1+p^2)^{\frac{1}{2}}}$$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = p = \pm\frac{x-A}{[a^2 - (x-A)^2]^{\frac{1}{2}}}$$

$$\Rightarrow y = B \mp [a^2 - (x-A)^2]^{\frac{1}{2}} \quad \text{i.e.} \quad (x-A)^2 + (y-B)^2 = a^2$$

So they are all circles of radius a!

(d) x **missing**, i.e.

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = f\left(y, \frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

Yet again, let $\frac{\mathrm{d}y}{\mathrm{d}x} = p$, so

$$p\frac{\mathrm{d}p}{\mathrm{d}y} = f(y, p)$$

i.e. First order $p(y)$. So we solve for $p(y)$, then find $x = \int \frac{\mathrm{d}y}{p(y)}$.

Example:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\omega^2 y \mp 2k\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$$

SHM with resistance proportional to $(\text{speed})^2$.
Hint: Solving this equation is the perfect application for solving Bernoulli Equation!

(e) **Linear Equations**, i.e. $y, \frac{\mathrm{d}y}{\mathrm{d}x}$ only occur to 1st power, if at all. So no products of y and $\frac{\mathrm{d}y}{\mathrm{d}x}$. The following section is dedicated to explaining the approach to solve linear differential equations.

General case — Linear Equations

The general form is, for order n,

$$\begin{aligned} \mathscr{L}y = a_0(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_1(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + a_2(x)\frac{\mathrm{d}^{n-2} y}{\mathrm{d}x^{n-2}} + \cdots \\ + a_{n-1}(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_n(x)y = f(x) \end{aligned} \tag{2.7}$$

where $a_0, a_1, \ldots, a_n$ and $f(x)$ are known functions of x only.
$\mathscr{L}$ is a **linear operator**, operating on $y(x)$:

$$\mathscr{L} \equiv \left[a_0\frac{\mathrm{d}^n}{\mathrm{d}x^n} + a_1\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} + \cdots + a_n\right]$$

The equation(2.7) is called ***homogeneous*** iff $f(x) = 0$ and ***inhomogeneous*** iff $f(x) \neq 0$.
The homogeneous equation $\mathscr{L}y = 0$ has n independent solutions $y_1(x), y_2(x)$,

$\ldots, y_n(x)$ apart from *trivial* $y(x) = 0$. That is to say that $\mathscr{L} y_i(x) = 0$ for $i = 1, 2, \ldots, n$. (***Independence*** is an algebraic property. . .) Because of the linearity of $y_i(x)$ we find that the most general solution of the homogeneous equation $\mathscr{L} y = 0$ is given by

$$y(x) = A_1 y_1(x) + A_2 y_2(x) + \cdots + A_n y_n(x) \tag{2.8}$$

with $A_1, A_2, \ldots, A_n$ being arbitrary constants. This is because

$$\mathscr{L} y = \mathscr{L} \left(\sum_{i=1}^{n} A_i y_i(x) \right) = \sum_{i=1}^{n} A_i (\mathscr{L} y_i(x)) = 0$$

Of course equation(2.8) contains n arbitrary constants in accord with the order n of the differential equation.

For the inhomogeneous equation ($\mathscr{L} y = f(x)$(2.7)), the expression(2.8) is called the ***complementary functions*** (CF) of equation(2.7). Any solution of the inhomogeneous equation(2.7), say $Y(x)$, is called a ***particular integral*** (PI) of equation(2.7). The most general solution of equation(2.7) is thus

$$y(x) = (\text{CF}) + (\text{PI})$$

This contains n arbitrary constants as required/expected!

The constants can be specified in practice to produce a particular solution which satisfies (n) initial/boundary conditions.
Note

(a) For any two solutions $Y_1(x), Y_2(x)$ of equation(2.7), their difference satisfies

$$\mathscr{L}(Y_1 - Y_2) = \mathscr{L} Y_1 - \mathscr{L} Y_2 = f(x) - f(x) = 0$$

(b) Generally, finding $y_1(x), y_2(x), \ldots, y_n(x)$ functions might be very tough — our differential equation has generally variable coefficients after all! So we look at the most common case we need to study — constant coefficients! W.L.O.G.:

$$a_0(x) = 1, a_1(x) = a_1, a_2(x) = a_2, \ldots, a_n(x) = a_n$$

Linear Equations — Second Order, Constant Coefficients

Consider

$$\mathscr{L}y = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}y}{\mathrm{d}x} + a_2 y = f(x) \tag{2.9}$$

Alternatively, in terms of notation,

$$\mathscr{L}y = y'' + a_1 y' + a_2 y = f(x)$$

Overall flow of solving the equation is to firstly find CF then PI,

$$\Rightarrow y(x) = \mathrm{CF} + \mathrm{PI}$$

Finding the CF We need to solve

$$\mathscr{L}y = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}y}{\mathrm{d}x} + a_2 y = 0 \tag{2.10}$$

Try a solution of the form $y = e^{\lambda x}$ where λ is a constant — which we need to find! (It works by demonstration.) Evidently,

$$(\lambda^2 + a_1\lambda + a_2)e^{\lambda x} = 0$$

The exponential cannot help — for any λ let alone for all x. So

$$\lambda^2 + a_1\lambda + a_2 = 0 \tag{2.11}$$

as the auxiliary equations. In general, there are two distinct roots λ_1, λ_2 of this quadratic, so that $e^{\lambda_1 x}, e^{\lambda_2 x}$ are solutions of equation(2.10), i.e.

$$\mathscr{L}\left(e^{\lambda_1 x}\right) = 0 = \mathscr{L}\left(e^{\lambda_2 x}\right)$$

Because of the linearity property of $\mathscr{L}$ we have

$$y_{\mathrm{CF}} = A_1 e^{\lambda_1} + A_2 e^{\lambda_2 x}$$

where A_1, A_2 are two arbitrary constants and $\mathscr{L}y_{\mathrm{CF}} = 0$ as required.

If the roots of (2.11) are equal, i.e. $\lambda_1 = \lambda_2 = \lambda$, then certainly $A_1 e^{\lambda x}$ is a solution of (2.10) with *one* arbitrary constant — we need *another*! A second linearly independent solution is given by $A_2 x e^{\lambda x}$, so that we have

$$y_{\mathrm{CF}} = A_1 e^{\lambda x} + A_2 x e^{\lambda x}$$

We can see this easily: (2.11) must take the form $(\lambda + \frac{a_1}{2})^2 = 0$ since $a_2 = \frac{a_1^2}{4}$ and $\lambda = -\frac{a_1}{2}$ (repeated root). Then substituting $xe^{\lambda x}$ into (2.10) we have

$$\mathscr{L}\left(xe^{\lambda x}\right) = (2\lambda + a_1)e^{\lambda x} + (\lambda^2 + a_1\lambda + a_2)xe^{\lambda x} = 0$$

as required. Here, n in $\mathscr{L}$ is 2.

Example 42.

1.
$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 0$$
$\Rightarrow \lambda^2 + 5\lambda + 6 = 0,\ \lambda = -3, -2.$ So
$$y(x) = A_1 e^{-3x} + A_2 e^{-2x}$$

2.
$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 0$$
$\Rightarrow \lambda^2 + 4\lambda + 4 = 0,\ \lambda = -2, -2.$ So
$$y(x) = A_1 e^{-2x} + A_2 x e^{-2x}$$

What about *complex roots* of (2.11)? (assuming $a_1, a_2 \in \mathbb{R}$) We know that the roots are complex conjugates, i.e. $\lambda_{1,2} = \alpha \pm i\beta, \alpha, \beta \in \mathbb{R}$. Now, formally our solution is, as above,

$$y = A_1 e^{(\alpha + i\beta)x} + A_2 e^{(\alpha - i\beta)x}$$

Since $\beta \neq 0$ here since the roots cannot be equal! so we can rewrite in alternative forms:

$$y = e^{\alpha x}\left[A_1 e^{i\beta x} + A_2 e^{-i\beta x}\right] = e^{\alpha x}\left[C_1 \cos\beta x + C_2 \sin\beta x\right]$$

where A_1, A_2 or C_1, C_2 can be taken as our arbitrary constants. (Naturally, $C_1 = A_1 + A_2, C_2 = (A_1 - A_2)i$ by De Moivre.)

Example 43.

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2k\frac{\mathrm{d}x}{\mathrm{d}t} + \omega^2 x = 0$$

which is the equation for damped harmonic oscillator ($k > 0$).

$$\lambda^2 + 2k\lambda + \omega^2 = 0, \quad \lambda_{1,2} = -k \pm \sqrt{k^2 - \omega^2}$$

and

$$x(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

in general. This can be broken down into different cases.

(1) $k = 0$, i.e. *No Damping.*

$$x = A_1 e^{i\omega t} + A_2 e^{-i\omega t} = C_1 \cos \omega t + C_2 \sin \omega t$$

(2) $k^2 < \omega^2$, i.e. *Light Damping.*

$$x = A_1 e^{-kt+i\omega t} + A_2 e^{-kt-i\omega t} = (C_1 \cos \omega t + C_2 \sin \omega t)e^{-kt}$$

with $\omega = (\omega^2 + k^2)^{\frac{1}{2}}$.

(3) $k^2 > \omega^2$, i.e. *Heavy Damping.*

$$x = A_1 e^{-|\lambda_1|t + A_2 e^{-|\lambda_2|t}}$$

since λ_1, λ_2 are each neagative real.

(4) $k^2 = \omega^2$, i.e. *Critical Damping.*

$$\lambda_1 = \lambda_2 = -k \Rightarrow x = (A_1 + A_2 t)e^{-kt}$$

Note: $x(t)$ behaviours for various cases!

Finding a PI Now we have the CF we need any particular solution of (2.9), in order to complete the job of finding the general solution. The PI is *not* unique! Our guide is the form of the function $f(x)$ on RHS.

(a) *polynomial in x*
Try a polynomial for the PI and choose the coefficients to fit! Example:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = x$$

Try PI $= ax^2 + bx + c$, where we need to find a, b, c. This method is often known as the method of undetermined coefficients.
We now determine them! (SIAS — Suck It And See)

$$2a - 3(2ax + b) + 2(ax^2 + bx + c) = x$$

By comparing the coefficients, we can obtain

$$a = 0, b = \frac{1}{2}, c = \frac{3}{4} \Rightarrow y_{\mathrm{PI}} = \frac{1}{2}x + \frac{3}{4}$$

Since $y_{\mathrm{CF}} = A_1e^x + A_2e^{2x}$ for this equation, then the general solution can be written as

$$y(x) = A_1e^x + A_2e^{2x} + \frac{1}{2}x + \frac{3}{4}$$

Note: Our inclusion of ax^2 term in our trial PI has been self-correcting since it emerged that $a = 0$. This is always so; the method gives what is needed!

(b) *multiple of e^{bx}*
The obvious choice for the PI is Ae^{bx}, since the linear operator $\mathscr{L}$ generates only terms of this type — choose A to fit! But there are two cases to consider:

(i) e^{bx} *not* in y_{CF}, i.e. $\mathscr{L}(e^{bx}) \neq 0$
Example:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 7e^{8x}$$

with

$$y_{\mathrm{CF}} = A_1e^{-3x} + A_2e^{-2x}$$

Try $y_{\mathrm{PI}} = Ae^{8x}$, then

$$Ae^{8x}[64 + 40 + 6] = 7e^{8x} \Rightarrow A = \frac{7}{110}$$

and general solution is

$$y(x) = y_{\mathrm{CF}} + \frac{7}{110}e^{8x}$$

(ii) e^{bx} is *contained* in y_{CF}, i.e. $\mathscr{L}e^{bx} = 0$

Our trial solution in (i) now does not work! We might hope (anticipate) that xe^{bx} might be involved, and just try it... (SIAS)
A more 'automatic' approach is to take the Ae^{bx} from the CF (where A was constant) and try a PI of the form $A(x)e^{bx}$ — called ***variation of parameters***. We expect that $A(x)$ will be a polynomial in x!
Example:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3x + 2y = e^{-x}$$

with

$$y_{\mathrm{CF}} = A_1e^{-x} + A_2e^{-2x}$$

Try $y_{PI} = A(x)e^{-x}$.

$$\Rightarrow (A'' - 2A' + A)e^{-x} + 3(A' - A)e^{-x} + 2Ae^{-x} = e^{-x}$$

By comparing the coefficients, we get

$$A'' + A' = 1$$

Afterwards, integrate with respect to x once and we get

$$A' + A = x + \overline{C_1}$$

Solving this first-order linear equation, and we get

$$A = x + C_1 + C_2e^{-x}$$

$$\Rightarrow y_{\mathrm{PI}} = A(x)e^{-x} = xe^{-x} + C_1e^{-x} + C_2e^{-2x}$$

Take PI $= xe^{-x}$ (W.L.O.G), we can obtain

$$y(x) = A_1e^{-x} + A_2e^{-2x} + xe^{-x}$$

Of course if the auxiliary equation has equal roots then y_{CF} has xe^{bx} too! However the variabtion of parameters still works — or alternatively (a trial polynomial)(e^{bx}).

Example:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = e^{-2x}$$

with

$$y_{\mathrm{CF}} = A_1e^{-2x} + A_2xe^{-2x}$$

We can then set PI as

$$y_{\mathrm{PI}} = A(x)e^{-2x} \Rightarrow \ldots A'' = 1 \Rightarrow A = \frac{x^2}{2} + [\overline{A_1} + \overline{A_2}x]$$

$$\Rightarrow y(x) = A_1e^{-2x} + A_2xe^{-2x} + \frac{x^2}{2}e^{-2x}$$

(c) e^{bx} is *polynomial* in x

Try PI $= C(x)e^{bx}$ where $C(x)$ is a polynomial with coefficients to be found — as in (a), (b) above.

(d) sines, cosines, sinh, cosh

We *either* just recognize the pattern and put e.g. $A\cos() + B\sin()$ *or* $A\cosh() + B\sinh()$, etc.

OR

Make use of exponentials — maybe complex ones using $e^{ix} = \cos x + i\sin x$, etc.

Example:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = e^x\cos x$$

with

$$y_{\mathrm{CF}} = A_1e^{-x} + A_2e^{-2x}.$$

There is no obvious trouble with this CF...

(1) Try $y_{\mathrm{PI}} = Be^x\cos x + Ce^x\sin x$ because $\mathscr{L}(y_{\mathrm{PI}})$ produces terms of a similar type. Substitute in and equate coefficents of $e^x\cos x, e^x\sin x$ on the two sides $\Rightarrow B = \frac{1}{10}, C = \frac{1}{10}$.

OR

(2) Put RHS $= \frac{1}{2}e^{(1+i)x} + \frac{1}{2}e^{(1-i)x} (= \Re\left(e^{(1+i)x}\right))$. Then try

$$y_{\text{PI}} = C_1 e^{(1+i)x} \Rightarrow [(1+i)^2 + 3(1+i) + 2]C_1 = 1$$

and $C_1 = \frac{1}{5(1+i)} = \frac{1}{10}(1-i)$, and

$$y_{\text{PI}} = \Re\left[\frac{1}{10}(1-i)e^{(1+i)x}\right] = \frac{1}{10}e^x \cos x + \frac{1}{10}e^x \sin x$$

Naturally, we might need to be adaptable if we find polynomials on RHS in $f(x)$ as well, or the 'equal roots' case... However something to beware:

<u>Example</u>:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + e\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \cosh 2x$$

with

$$y_{\text{CF}} = A_1 e^{-x} + A_2 e^{-2x}$$

If we try $y_{\text{PI}} = C_1 \cosh 2x + C_2 \sinh 2x$, we would find C_1, C_2 not defined...

$$\begin{cases} 6C_1 + 6C_2 = 1 \\ 6C_1 + 6C_2 = 0. \end{cases}$$

Why?! Well $\cosh 2x = \frac{1}{2}(e^{2x} + e^{-2x})$ and one of these exponentials *is* in y_{CF}. The better one is

$$y_{\text{PI}} = \frac{1}{24}e^{2x} - \frac{1}{2}xe^{-2x}$$

using earlier results.

Chapter 3

Linear Algebra

3.1 Introduction to Matrices and Vectors

3.1.1 Column vectors

Definition 44. A *column vector* (n-column vector) $\boldsymbol{v}_n$ is a tuple of n real numbers written as a single columnn, with $a_1, a_2, a_3, \ldots, a_n \in \mathbb{R}$:

$$\boldsymbol{v}_n := \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{pmatrix}$$

Definition 45. $\mathbb{R}^n$ is the set of all column vectors of height n whose entries are real numbers. In symbols:

$$\mathbb{R}^n = \{ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} : a_1, a_2, \ldots, a_n \in \mathbb{R}\}$$

Example 46. $\mathbb{R}^2$ can be seen as Euclidean plane. $\mathbb{R}^3$ can be seen as Euclidean space.
Caution: Our vectors always “start” at the origin.

Definition 47. The ***zero vector*** $\mathbf{0}_n$ is the height n-column vector all of whose entries are 0.

Definition 48. The ***standard basis vectors*** in $\mathbb{R}^n$ are the vectors

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots, \quad \boldsymbol{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

i.e. $\boldsymbol{e}_k$ is the vector with kth entry equal to 1 and all other entries equal to 0.

Operations on column vectors

$$\boldsymbol{v} := \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}, \quad \boldsymbol{u} := \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

be column vectors $\mathbb{R}^n$, and let λ be a (real or complex) number.

(1) Addition on vectors in $\mathbb{R}^n$ is given by:

$$\begin{pmatrix} v_1 + u_1 \\ v_2 + u_2 \\ \vdots \\ v_n + u_n \end{pmatrix}$$

$+ : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ (binary operatioin). $(\mathbb{R}^n, +)$ is a group.

(2) ***Scalar multiplication*** $\lambda \boldsymbol{v}$ on $\mathbb{R}^n$:

$$\begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \vdots \\ \lambda v_n \end{pmatrix}$$

$s : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, so not binary operation.

(3) ***Dot product*** $v \cdot u$ is defined to be the number $v_1u_1+v_2u_2+\cdots+v_nu_n$. $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, so not binary.

Example 49. Show that $(\mathbb{R}^n, +)$ is an Abelian group.

- Identity: $\mathbf{0}_n$ $(v + \mathbf{0}_n = \boldsymbol{v})$
- $-\boldsymbol{v}$ are inverses, where
$$-\boldsymbol{v} := \begin{pmatrix} -v_1 \\ -v_2 \\ \vdots \\ -v_n \end{pmatrix}$$
- associativity: $(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w} = \boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})$.
- commutative: $\boldsymbol{u}+\boldsymbol{v} = \boldsymbol{v}+\boldsymbol{u}$

Caution: + only makes sense for vectors of the *same size.* e.g. $\boldsymbol{v} \cdot \mathbf{0}_n = 0 \in \mathbb{R}$.

Definition 50. let $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \dots, \boldsymbol{v}_n \in \mathbb{R}^n, \lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n \in \mathbb{R}$, then
$$\lambda_1\boldsymbol{v}_1 + \lambda_2\boldsymbol{v}_2 + \cdots + \lambda_n\boldsymbol{v}_n$$
is called a ***linear combination*** of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \dots, \boldsymbol{v}_n$.

Definition 51. The set of all linear combinatioins of a collection of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$ is called the ***span*** of the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$.
Notation:
$\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\} := \{\lambda_1\boldsymbol{v}_1 + \lambda_2\boldsymbol{v}_2 + \cdots + \lambda_n\boldsymbol{v}_n | \lambda_1, \dots, \lambda_n \in \mathbb{R}\}$

Example 52. compute the span of

- $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$, $\boldsymbol{e}_1, \boldsymbol{e}_2 \in \mathbb{R}^2$.
$$\operatorname{span}\{\boldsymbol{e}_1, \boldsymbol{e}_2\} = \{\lambda_1\boldsymbol{e}_1 + \lambda_2\boldsymbol{e}_2 | \lambda_1, \lambda_2 \in \mathbb{R}\} = \{\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} | \lambda_1, \lambda_2 \in \mathbb{R}\}$$
- $\operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \right\} = \{ \begin{pmatrix} \lambda_1 \\ 2\lambda_2 \\ 0 \end{pmatrix} | \lambda_1, \lambda_2 \in \mathbb{R}\}$

Definition 53. let $\boldsymbol{v} \in \mathbb{R}^n$. The ***length*** of $\boldsymbol{v}$, a.k.a. the ***norm*** of $\boldsymbol{v}$, is the non-negative real number $\|\boldsymbol{v}\|$ defined by

$$\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}$$

Note: $\|\mathbf{0}\| = 0$, and conversely if $\boldsymbol{v} \neq 0$ then $\|\boldsymbol{v}\| > 0$. This definition agrees with out usual ideas about the length of a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$, which follows from Pythagoras' theorem.

Definition 54. A vector $\boldsymbol{v} \in \mathbb{R}^n$ is called a ***unit vector*** if $\|\boldsymbol{v}\| = 1$.

Example 55.

(1) Any non-zero vector $\boldsymbol{v}$ can be made into a unit vector $u := \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}$. This process is called ***normalizing***.

(2) The standard basis vectors are unit vectors.

3.1.2 Basic Matrix Operations

Definition 56. An $n \times m$-matrix is a rectangular grid of numbers called the *entries* of the matrix with n rows and m columns. A real matrix is onne whose entries are real numbers, and a complex matrix is one whose entries are complex numbers.

Notations: $M_{n\times m}(\mathbb{R}), M_{n,m}(\mathbb{R}), \mathrm{Mat}_{n\times m}(\mathbb{R}), \mathbb{R}^{n\times m}$.

Operations on matrices:

Definition 57. let $A = (a_{ij})$ and $B = (b_{ij})$ are $n \times m$-matrix, $\lambda \in \mathbb{R}$. Then:

(1) $A + B = n \times m$-matrix $(a_{ij} + b_{ij})$. $+ : M_{n\times m}(\mathbb{R}) \times M_{n\times m}(\mathbb{R}) \to M_{n\times m}(\mathbb{R})$

(2) $\lambda A = n \times m$-matrix (λa_{ij})

Theorem 58. $(M_{n\times m}(\mathbb{R}), +)$ is an Abelian grouop.

Definition 59. The ***transpose*** A^T of an $n\times m$-matrix (a_{ij}) is the $m\times n$-matrix (a_{ij}). The ***leading diagonal*** of a matrix is the $(1,1),(2,2),\dots$ entries. So the transpose is obtained by doing a reflection in the leading diagonal.

(Multiplying matrices with vectors) Definition 60. Let $A = (a_{ij})$ be an $n\times m$-matrix, $\boldsymbol{v} \in \mathbb{R}^m$. Then $A\boldsymbol{v}$ is the vector in $\mathbb{R}^n$ with i-th row entry $\sum_{j=1}^m a_{ij}\boldsymbol{v}_j$

Example 61.

- Prove that for $A \in M_{n\times m}(\mathbb{R}), \boldsymbol{e}_k \in \mathbb{R}^m$, $A\boldsymbol{e}_k = k$-th column of A.

  Proof: let $A = (a_{ij})$. By definition the i-th entry of $A\boldsymbol{e}_k$ is

  $$\sum_{j=1}^m a_{ij}(\boldsymbol{e}_k)_j = a_{ik}$$

  since $(\boldsymbol{e}_k)_j = 0$ whenever $j \neq k$, 1 for $j = k$

- Let I_n be the identity matrix. Show formally that $I_n\boldsymbol{\nu} = \boldsymbol{\nu}$, $\forall\boldsymbol{\nu} \in \mathbb{R}^n$.

- $\boldsymbol{\nu}\cdot\boldsymbol{v} = \boldsymbol{\nu}^T\boldsymbol{v}$

- let $\boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \boldsymbol{\nu}_3 \in \mathbb{R}^3$. Write the linear combination $3\boldsymbol{\nu}_1 - 5\boldsymbol{\nu}_2 + 7\boldsymbol{\nu}_3$ as a multiplication of matrix $A \in M_{3\times 3}(\mathbb{R})$ with a vector $\boldsymbol{x} \in \mathbb{R}^3$. Then

  $$A\boldsymbol{x} = \begin{pmatrix}\boldsymbol{\nu}_1 & \boldsymbol{\nu}_2 & \boldsymbol{\nu}_3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = x_1\boldsymbol{\nu}_1 + x_2\boldsymbol{\nu}_2 + x_3\boldsymbol{\nu}_3$$

  with $\boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \boldsymbol{\nu}_3$ written as a column vector to form a matrix in the above expression, thus using matrix multiplication to express linear combination of vectors.

3.2 Systems of linear equations

Definition 62. A ***linear equation*** in the variables $x_1, x_2, \ldots, x_n \in \mathbb{R}$ is an equation of the form:

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = c, \text{ with } \lambda_1, \ldots, \lambda_n \subset \textit{ Fixed} \text{ real numbers}$$

Caution: In particular, no powers/multiplications/function of one or more variables.

Definition 63. A system of n linear equations is a list of simultaneous linear equations. It can be converted to $A\boldsymbol{x} = \boldsymbol{b} \in \mathbb{R}^m$,with

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \in \mathbb{R}^{m \times n}$$

Caution: Thee $m \times n$-matrix A is called coefficient matrix. The matrix $(A|\boldsymbol{b})$ where the vector $\boldsymbol{b}$ is added as a column on the right is called ***augmented matrix***.

Definition 64. A system is called ***consistent*** (resp. inconsistent) if it has a solution $(s_1, s_2, \ldots, s_m)$ (resp. no solution).

Example 65.

$$\begin{cases} x_1 + x_3 - x_4 = 1 \\ x_2 - x_4 = 6 \\ x_1 + x_2 + 6x_3 - 3x_4 = 0 \end{cases}$$

Augmented matrix form:

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 6 \\ 1 & 1 & 6 & -3 & 0 \end{array}\right)$$

Definition 66. A ***row operation*** is one of the following procedures on a $n \times m$-matrix (a_{ij}):

(1) $r_i(\lambda)$: multiply row i by a scalar $\lambda \in \mathbb{R}, \lambda \neq 0$.

(2) r_{ij}: swap row i with row j.

(3) $r_{ij}(\lambda)$: multiply row i by $\lambda \neq 0$, $\lambda \in \mathbb{R}$ and add it to row j.

Example 67. let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, so

$$r_{12} \Rightarrow \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$$

$$r_2(2) \Rightarrow \begin{pmatrix} 1 & 2 \\ 6 & 8 \end{pmatrix}$$

$$r_{12}(2) \Rightarrow \begin{pmatrix} 1 & 2 \\ 5 & 8 \end{pmatrix}$$

Proposition 68. Let $A\boldsymbol{x} = \boldsymbol{b}$ be a system of linear equations in matrix form, $(A|\boldsymbol{b})$ the augmented matrix, $(A'|\boldsymbol{b}')$ the augmented matrix of the system after row operation. Show that x is solution of $A\boldsymbol{x} = \boldsymbol{b} \iff x$ is solution of $A'\boldsymbol{x} = \boldsymbol{b}'$.

Proof. row operations of type (1) and (2) $\Rightarrow$ trivial.

(3) Take equation i, multiply it by λ, add it to equation j. $\Rightarrow (a_{j1} + \lambda a_{i1})x_1 + \cdots + (a_{jm} + \lambda a_{im})x_m = b_j + \lambda b_i$. ∎

Caution: Every row operation is invertible:

$$[r_i(\lambda)]^{-1} = r_i(\frac{1}{\lambda}),\ [r_{ij}]^{-1} = r_{ij},\ [r_{ij}(\lambda)]^{-1} = r_{ij}(-\lambda)$$

3.2.1 Gauss algorithm

Definition 69. The left most non-zero entry in a non-zero row is called ***leading entry***. A matrix is called in ***echelon form*** if:

(1) The leading entry in each non-zero row is 1.

(2) The leading 1 of each row is to *the right* of the leading 1 in the row above.

(3) The zero-rows are *below* all other rows.

Example 70.

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 3 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Only the last one is in echelon form.

Definition 71. A matrix is ***row reduced echelon form*** if:

(1) It is in echelon form.

(2) The leading 1 in each row is the *only* non-zero entry in its column.

Example 72.

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \alpha & \beta & 2 \\ 0 & 0 & 1 & -2 \end{pmatrix}.$$

The second one is not.

Chapter 4

Analysis