Calculus, Algebra, and Analysis for JMC

Marie Amellie, Frank Berkshire

January 15, 2020

Contents

1 Group theory **2**
1.1 Binary operations and groups 2
1.2 Consequences of the axioms of group 6

2 Applied Mathematical Methods **7**
2.1 Differential Equations 7
2.1.1 Definitions and examples 7
2.1.2 First Order Differential Equations 10
2.1.3 'Special' Second Order Differential Equations 14

3 Linear Algebra **15**
3.1 System of linear equations 19

4 Analysis **20**

Chapter 1

Group theory

Study of the simplest algebraic structure on a set.

1.1 Binary operations and groups

Definition 1. *Set* is a collection of distinct elements. Let G be a set. ***Binary operation on G*** is a function

$$*: G \times G \to G(\text{Closure is included})$$

Example 2.

- $(\mathbb{N}, +), (\mathbb{Z}, +), (\mathbb{R}, \cdot)$
- $(\mathbb{N}, -)$ not a binary op. Not closed.
- $g, h \in G, g * h = h$
- Find a certain $c \in G$, define $g * h = c \; \forall g, h \in G$

Example 3. Cayley table: Draw a table of all the possible binary operations on a set. How many possible binary operations on a finite set with n elements? In general, there are ∞-many biniary operatiions. In this case, there are n^{n^2} possible binary operations. *In general,* $g_i * g_j \neq g_j * g_i$ (Not commutative!)

Definition 4. A binary operation $*$ on a set G is called associative if

$$(g * h) * k = g * (h * k) \; \forall g, h, k \in G$$

Example 5.

- $+$ on $\mathbb{N}, \mathbb{Z}, \mathbb{R}$? Yes
- $-$ on $\mathbb{R}$? No
- $g * h = g^h$ on $\mathbb{N}$? No

Definition 6. A binary operation is called commutative if

$$\forall g, h \in G, g * h = h * g$$

Example 7.

- $+, \cdot$ on $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$
- matrix multiplication ($AB \neq BA$ in general for A, B in $M(\mathbb{R}^n)$)
- let $g, h \in \mathbb{R}$, $g * h = 1 + g \cdot h$: commutative but *not associative*!

Definition 8. Let $(G, *)$ be a set. An element e is called *left identity* (respectively *right identity*) if:

$$e * g = g (\text{resp. } g * e = g) \; \forall \; g \in G$$

Caution: There might be *many* left/right identities or none.

Example 9.

1. let $(G, *)$ be a set with $g * h := g$. Find the left/right identities.

   ∞-many (or equal to the number of elements) right identities since h satisfies definition $\forall h$. No left identities: wanted $e * g = g = e$ by definitioin of $*$ (*unless only one element*).

2. $(G, *)$, $g * h = 1 + gh$. Ex: No right/left identities.

Idea: We want a good unique identity.

Theorem 10. let $(G, *)$ be set, such that $*$ has both a left identity e_1 *and* a right identity e_2, then

$$e_1 = e_2 =: e \quad \text{and} \quad e \text{ is unique.}$$

Proof.

- $e_1 = e_2$

$$\Rightarrow \left\{ \begin{array}{l} e_1 * g = g \Rightarrow e_1 * e_2 = e_2 \\ g * e_2 = g \Rightarrow e_1 * e_2 = e_1 \end{array} \right\} \forall\, g \in G \Rightarrow \; e_1 = e_2$$

- Unicity: Assume there exists another identity e'.

$$\Rightarrow e' * g = g * e' = g$$

$$e' * g = e' * e = e$$

$$g * e' = e * e' = e'$$

Therefore

$$e = e'$$

$\square$

As soon as you get one left and one right identity, you have a unique identity e.

Definition 11. let $(G, *)$ be a set. Let $g \in G$. An element $h \in G$ is called left (resp. right) inverse if

$$h * g = e \text{ (resp. } g * h = e)$$

Caution: Again inverses might not exist, there might be many, or *not* the same on both sides.

Example 12.

(1) $(\mathbb{N}, \cdot)$ 1 has an inverse, otherwise *no* inverse.

(2) Find a binary operation on a set of 4 elements with left/right inverses not the same but identity e.

Theorem 13. Let $(G, *)$ be a set with associative binary operation and identity e. Then if h_1 is left inverse, and h_2 is right inverse, then

$$h_1 = h_2 = g^{-1} \text{ and \textit{it is unique}}$$

Proof.

- $h_1 = h_2$

  $h_1 * g = e, g * h_2 = e$. Therefore

  $$h_2 = e * h_2 = (h_1 * g) * h_2 = h_1 * (g * h_2) = e = h_1$$

- unicity: Assume $\exists g'^{-1}$ another inverse.

  $$g'^{-1} = e * g'^{-1} = (g^{-1} * g) * g'^{-1} = g^{-1} * (g * g'^{-1}) = g^{-1} * e = g^{-1}$$

□

(Group) Definition 14. A set $(G, *)$ wth binary operatioin $*$ is called a *group* if:

(1) $*$ is associative

(2) $\exists e \in G$ an identity $\forall g \in G$

(3) All elements $g \in G$ have an inverse g^{-1}

Attention: The identity and inverses are *unique* by our previous results.

Example 15.

- $(\mathbb{Z}, +), (\mathbb{Z}_n, +)$ are groups.
- $(\mathbb{N}, +)$ not a group $\Rightarrow$ no inverses.
- $(\mathbb{C}, \cdot)$ not a group (0 has no multiplicative inverse), but $(\mathbb{C}^*, \cdot)$ is. $(\mathbb{C}^* = \mathbb{C}\backslash\{0\})$
- $(G = \{e\}, *)$ with $e * e = e$ is a group called the *trivial group*.
- Empty set $\varnothing$ is not a group (No identity element.)

Definition 16. Let G be a group. It is called finite if it has finitely many elements.

Notation: $|G| = n$ (number of elements)

If $|G| = \infty$, the G is called an infinite group.

Example 17.

- the trivial group is finite, $|G| = 1$
- let $G = \{1, -1, i, -i\} \subset \mathbb{C}$, with $* = \cdot$. Is it a group? Yes. Check associativity, identity, and inverses.

(Abelian Group) Definition 18. A group is called *Abelian* if $*$ is commutative.

Example 19.

- previous example, trvial group, $(\mathbb{Z}, +), (\mathbb{C}^*, \cdot)$
- let $GL(\mathbb{R}^n)$ be the set of all invertible $n \times n$ matrices, $*$ = matrix multiplication. It is associative: $(AB)C = A(BC)$; It has identity: I_n. It has inverses: yes since we asked for it. So this is a group of matrices. But this is not Abelian since $AB \neq BA$.
- let G be the set of *invertible* functions with $* = \circ$, the composition of functions. Identity is $F(x) = x$; they are associative, invertible, but *not Abelian.*

1.2 Consequences of the axioms of group

Chapter 2

Applied Mathematical Methods

2.1 Differential Equations

2.1.1 Definitions and examples

Definition 20. An *ordinary differential equation* (ODE) for $y(x)$ is an equation involving <u>derivatives</u> of y.

$$f(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}, \dots, \frac{\mathrm{d}^ny}{\mathrm{d}x^n}) = 0 \tag{2.1}$$

$$\frac{\mathrm{d}^ny}{\mathrm{d}x^n} = F(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \dots, \frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}})$$

and we seek a solution (or solutions) for $y(x)$ satisfying the equations. (If there are more independent variables then we have a partial differential equation (PDE).)

Definition 21.

Order is the order of the highest derivative present.

Degree is the power of the highest derivative when fractional powers have been removed.

Linear *differential equation* is a differential equation that is defined by a *linear polynomial* in the unknown function and its derivative in each term of equation(2.1).

Example 22.

(a) Particle moving along a line with a given force $\rightarrow x(t)$ position as function of time t.

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = f\left(t, x, \frac{\mathrm{d}x}{\mathrm{d}t}\right)$$

e.g.

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\omega^2 x - 2k\frac{\mathrm{d}x}{\mathrm{d}t}$$

The first term is regarding the restoring force, while the second term is regarding the damping/friction. The function is of order 2, degree 1, and linear.

(b) Radius of curvature of a curve

It can be shown that

$$R(x, y) = \frac{\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{\frac{3}{2}}}{\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}}$$

The function is of order 2 and degree 2.

(c) Simple growth and decay

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = kQ$$

The function is of order 1, degree 1, and linear. e.g.

(1) $k > 0$. Q as the quantity of money, and $k = (1 + \frac{r}{100})$, and r being the rate of interest.

(2) $k < 0$. Q as the amount of radioactive material, and k as the decay rate.

Hence, obviously $Q(t) = Q_0 e^{kt}$ where $Q_0 = Q(0)$ at $t = 0$.

(d) Population dynamics

$P(t)$ as population over time and $F(t)$ as food over time, with

$$\frac{\mathrm{d}P}{\mathrm{d}t} = aP(a > 0) \tag{2.2}$$

$$\frac{\mathrm{d}F}{\mathrm{d}t} = c(c > 0)$$

These two equations form a linear system, with both being of order 1, degree 1.

So $P(t) = P_0 e^{at}, F(t) = ct + F_0$. Misery! Population outgrows food supply.

Pierre Verhulst (1845) replaced a in equation(2.2) with $(a - bP)$ so that growth decreases as P increases:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = aP - bP^2 \tag{2.3}$$

This is in fact a *logistic ODE*, with order 1, degree 1, and nonlinear.

Note: Equation(2.3) is *separable*. Alternatively we can note that equation(2.3) is an example of a *Bernoulli differential equation*

$$\frac{\mathrm{d}y}{\mathrm{d}x} + F(x)y = H(x)y^n \tag{2.4}$$

with $n \neq 0, 1$ Substitution on $z(x) = (y(x))^{1-n} \Rightarrow$ a *linear* equation for $z(x) \rightarrow$ solution. (See below)

(e) Predator-Prey System

$x(t)$ as prey and $y(t)$ as predators, we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax - bxy, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -cy + \hat{d}xy \tag{2.5}$$

Note: Equation(2.5) is *separable* when written in principle

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \Rightarrow y(x) \Rightarrow x(t), y(t)$$

This is of order 1, degree 1, and a nonlinear system.

(f) Combat Model System

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -ay, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -bx \tag{2.6}$$

This is of order 1, degree 1, and linear system.

Note: Again equation(2.6) is *separable* when written as $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{bx}{ay} \Rightarrow y(x) \Rightarrow x(t), y(t)$

In general the solution of a differential equation of order n contains a number n of *arbitrary constants*. This general solution can be specialised to a particular solution by assigninig definite values to these constants.

Example 23.

(a) Family or parabolae $y = Cx^2$ as constant C takes different values.

On a particular curve of the family $\frac{\mathrm{d}y}{\mathrm{d}x} = 2Cx$. By substitutiion, eliminate $C \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2y}{x}$. This is a geometrical statement about slopes.

Note: 1st order differential equation $\leftrightarrow$ 1 arbitrary constant in general solution.

(b)

$$\left.\begin{aligned} x &= A\sin\omega t + B\cos\omega t \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= A\omega\cos\omega t - B\omega\sin\omega t \\ \frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= -A\omega^2\sin\omega t - B\omega^2\cos\omega t \end{aligned}\right\} \Rightarrow \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega^2 x = 0$$

Note: 2nd order differential equation $\leftrightarrow$ 2 arbitrary constants in general solution.

Of course it's the reverse of this process we normally want to perform in order to get the general solution. We then often need a particular solution — which satisfieis certain other conditions — *boundary* or *initial condition.* These allow us to find the arbitrary constants in the solutions.

2.1.2 First Order Differential Equations

Properties and approaches

There are essentially 4 types we can solve *analytically*:

- ***separable***
- ***homogeneous***
- ***linear***
- ***exact*** (in Chapter "Partial Differentiation and Multivariable Calculus" later)

Let's look at them one by one:

(a) **Separable**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = G(x) \cdot H(y)$$

Solve by rearrangement and integration

$$\int^y \frac{\mathrm{d}y}{H(y)} = \int^x G(x)\mathrm{d}x$$

E.g.

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= xy^2e^{-x} \\ \int \frac{1}{y^2}\mathrm{d}y &= \int xe^{-x}\mathrm{d}x \\ -\frac{1}{y} &= -xe^{-x} - e^{-x} + C \end{aligned}$$

Or singular solution $y = 0$.

If we want the particular solution which passes through $x = 1, y = 1$, then of course we need

$$C = -1 + 2e^{-1} \quad \text{and} \quad \frac{1}{y} = (x+1)e^{-x} + 1 - 2e^{-1}$$

(b) **Homogeneous**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right)$$

Substitution $\frac{y}{x} = u(x)$, i.e. a new dependent variable,

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= u + x\frac{\mathrm{d}u}{\mathrm{d}x}(= f(u)) \quad \textbf{\textit{(Remember!)}} \\ f(u) - u &= \frac{x\mathrm{d}u}{\mathrm{d}x} \\ \int \frac{\mathrm{d}u}{f(u) - u} &= \int \frac{\mathrm{d}x}{x} \\ &\vdots \end{aligned}$$

E.g.

(i)

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} + xy - y^2 = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{y}{x}\right)^2 - \frac{y}{x}$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{u^2 - 2u}{x}$$

$$\vdots$$

(ii)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x + y - 3}{x - y + 1}$$

This does not look homogeneous as it stands, but can be made so by substituting $x = 1 + X, y = 2 + Y$, and the expression becomes

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{X + Y}{X - Y} = \frac{1 + \left(\frac{Y}{X}\right)}{1 - \left(\frac{Y}{X}\right)}$$

Then let $\frac{Y}{X} = u(X)$,

$$\Rightarrow \int \left(\frac{1 - u}{1 + u^2}\right) \mathrm{d}u = \int \frac{\mathrm{d}X}{X}$$

Eventually, the equation becomes

$$\tan^{-1}\frac{Y}{X} - \frac{1}{2}\ln\left(1 + \frac{Y^2}{X^2}\right) = \ln X + C$$

$$\tan^{-1}\left(\frac{y - 2}{x - 1}\right) - \frac{1}{2}\ln\left[(x - 1)^2 + (y - 2)^2\right] = C$$

Note: If we have e.g. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y-3}{2(x+y)-7}$, then substitute $v(x) = x + y$ will work!

(c) **Linear**

$$\frac{\mathrm{d}y}{\mathrm{d}x} + F(x)y = G(x)$$

1st power only for y and $\frac{\mathrm{d}y}{\mathrm{d}x}$. We apply an *integrating factor* $R(x)$:

$$R(x) = \exp\left[\int^x F(x)\mathrm{d}x\right]$$

This allows us to form the expression

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[y\exp\left(\int^x F(x)\mathrm{d}x\right)\right] = G(x)\exp\left(\int^x F(x)\mathrm{d}x\right)$$

and then integrate...

E.g.

$$(x+2)\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = (x+2)^6$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{4}{x+2} = (x+2)^5$$

$$\Rightarrow F(x) = -\frac{4}{x+2}, G(x) = (x+2)^5$$

Therefore,

$$R(x) = \exp\left[-\int^x \left(\frac{4}{x+2}\right)\mathrm{d}x\right] = \cdots = K(x+2)^{-4}$$

Subsequently, take $K = 1$ W.L.O.G.:

$$(x+2)^{-4}\frac{\mathrm{d}y}{\mathrm{d}x} - 4(x+2)^{-5}y = \frac{\mathrm{d}}{\mathrm{d}x}\left[y(x+2)^{-4}\right] = x+2$$

As such,

$$y(x+2)^{-4} = \frac{1}{2}x^2 + 2x + C \quad \text{(Put } C \text{ at the right time!)}$$

$$y(x) = \left(\frac{1}{2}x^{2+2x+C}\right)(x+2)^4$$

(So e.g. $y(0) = 8 \Rightarrow C = \frac{1}{2}$)

Novelties!

(i) Bernoulli equation (See Equation(2.4))
A nonlinear equation rendered linear by a substitution $u = y^{1-n}$...

(ii) E.g.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x+e^y}$$

It is <u>nonlinear</u> for $y(x)$ but <u>linear</u> for $x(y)$:

$$\frac{\mathrm{d}x}{\mathrm{d}y} - x = e^y \Rightarrow \ldots$$

2.1.3 'Special' Second Order Differential Equations

Definition 24. General Explicit form is

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = F\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

(a) $y, \frac{\mathrm{d}y}{\mathrm{d}x}$ **missing**, i.e.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = f(x)$$

Just integrate twice!

(b) $x, \frac{\mathrm{d}y}{\mathrm{d}x}$ **missing**, i.e.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = f(y)$$

Warning: Do not write $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{1}{\frac{\mathrm{d}^2x}{\mathrm{d}y^2}}$

Chapter 3

Linear Algebra

Definition 25. A *column vector* (n-column vector) $\boldsymbol{v}_n$ is a tuple of n real numbers written as a single columnn, with $a_1, a_2, a_3, \ldots, a_n \in \mathbb{R}$:

$$\boldsymbol{v}_n := \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{pmatrix}$$

Definition 26. $\mathbb{R}^n$ is the set of all column vectors of height n whose entries are real numbers. In symbols:

$$\mathbb{R}^n = \{ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} : a_1, a_2, \ldots, a_n \in \mathbb{R} \}$$

Example 27. $\mathbb{R}^2$ can be seen as Euclidean plane. $\mathbb{R}^3$ can be seen as Euclidean space.
<u>Caution</u>: Our vectors always "start" at the origin.

Definition 28. The ***zero vector*** $\mathbf{0}_n$ is the height n-column vector all of whose entries are 0.

Definition 29. The ***standard basis vectors*** in $\mathbb{R}^n$ are the vectors

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad \boldsymbol{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

i.e. $\boldsymbol{e}_k$ is the vector with kth entry equal to 1 and all other entries equal to 0.

(Operations on column vectors) Definition 30. Let

$$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}, \quad \boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

be column vectors $\mathbb{R}^n$, and let λ be a (real or complex) number.

(1) Addition on vectors in $\mathbb{R}^n$ is given by:

$$\begin{pmatrix} v_1 + u_1 \\ v_2 + u_2 \\ \vdots \\ v_n + u_n \end{pmatrix}$$

$+ : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ (binary operatioin). $(\mathbb{R}^n, +)$ is a group.

(2) ***Scalar multiplication*** $\lambda \boldsymbol{v}$ on $\mathbb{R}^n$:

$$\begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \vdots \\ \lambda v_n \end{pmatrix}$$

$s : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, so not binary operation.

(3) ***Dot product*** $v \cdot u$ is defined to be the number $v_1u_1 + v_2u_2 + \cdots + v_nu_n$. $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, so not binary.

Example 31. Show that $(\mathbb{R}^n, +)$ is an Abelian group.

- Identity: $\mathbf{0}_n$ $(v + \mathbf{0}_n = \boldsymbol{v})$
- $-\boldsymbol{v}$ are inverses, where
$$-\boldsymbol{v} := \begin{pmatrix} -v_1 \\ -v_2 \\ \vdots \\ -v_n \end{pmatrix}$$
- associativity: $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$.
- commutative: $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$

Caution: + only makes sense for vectors of the *same size.* e.g. $\boldsymbol{v} \cdot \mathbf{0}_n = 0 \in \mathbb{R}$.

Definition 32. let $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \ldots, \boldsymbol{v}_n \in \mathbb{R}^n, \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n \in \mathbb{R}$, then

$$\lambda_1\boldsymbol{v}_1 + \lambda_2\boldsymbol{v}_2 + \cdots + \lambda_n\boldsymbol{v}_n$$

is called a ***linear combination*** of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \ldots, \boldsymbol{v}_n$.

Definition 33. The set of all linear combinatioins of a collection of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ is called the ***span*** of the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$.
Notation:
$$\mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\} := \{\lambda_1\boldsymbol{v}_1 + \lambda_2\boldsymbol{v}_2 + \cdots + \lambda_n\boldsymbol{v}_n | \lambda_1, \ldots, \lambda_n \in \mathbb{R}\}$$

Example 34. compute the span of

- $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$, $\boldsymbol{e}_1, \boldsymbol{e}_2 \in \mathbb{R}^2$.
$$\mathrm{span}\{\boldsymbol{e}_1, \boldsymbol{e}_2\} = \{\lambda_1\boldsymbol{e}_1 + \lambda_2\boldsymbol{e}_2 | \lambda_1, \lambda_2 \in \mathbb{R}\} = \{\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} | \lambda_1, \lambda_2 \in \mathbb{R}\}$$
- $\mathrm{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}\right\} = \{\begin{pmatrix} \lambda_1 \\ 2\lambda_2 \\ 0 \end{pmatrix} | \lambda_1, \lambda_2 \in \mathbb{R}\}$

Definition 35. let $\boldsymbol{v} \in \mathbb{R}^n$. The ***length*** of $\boldsymbol{v}$, a.k.a. the ***norm*** of $\boldsymbol{v}$, is the non-negative real number $||\boldsymbol{v}||$ defined by

$$||\boldsymbol{v}|| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}$$

Note: $||\mathbf{0}|| = 0$, and conversely if $\boldsymbol{v} \neq 0$ then $||\boldsymbol{v}|| > 0$. This definition agrees with out usual ideas about the length of a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$, which follows from Pythagoras' theorem.

Definition 36. A vector $\boldsymbol{v} \in \mathbb{R}^n$ is called a ***unit vector*** if $||\boldsymbol{v}|| = 1$.

Example 37.

(1) Any non-zero vector $\boldsymbol{v}$ can be made into a unit vector $u := \frac{\boldsymbol{v}}{||\boldsymbol{v}||}$. This process is called ***normalizing***.

(2) The standard basis vectors are unit vectors.

3.0.1 Basic Matrix Operations

Definition 38. An $n \times m$-matrix is a rectangular grid of numbers called the *entries* of the matrix with n rows and m columns. A real matrix is onne whose entries are real numbers, and a complex matrix is one whose entries are complex numbers.

Notations: $M_{n\times m}(\mathbb{R}), M_{n,m}(\mathbb{R}), \mathrm{Mat}_{n\times m}(\mathbb{R}), \mathbb{R}^{n\times m}$.

Operations on matrices:

Definition 39. let $A = (a_{ij})$ and $B = (b_{ij})$ are $n \times m$-matrix, $\lambda \in \mathbb{R}$. Then:

(1) $A + B = n \times m$-matrix $(a_{ij} + b_{ij})$. $+ : M_{n\times m}(\mathbb{R}) \times M_{n\times m}(\mathbb{R}) \to M_{n\times m}(\mathbb{R})$

(2) $\lambda A = n \times m$-matrix (λa_{ij})

Theorem 40. $(M_{n\times m}(\mathbb{R}), +)$ is an Abelian grouop.

Definition 41. The *transpose* A^T of an $n \times m$-matrix (a_{ij}) is the $m \times n$-matrix (a_{ij}).

(Multiplying matrices with vectors) Definition 42. Let $A = (a_{ij})$ be an $n \times m$-matrix, $v \in \mathbb{R}^m$. Then Av is the vector in $\mathbb{R}^n$ with i-th row entry $\sum_{j=1}^{m} a_{ij} v_j$

Example 43.

- Prove that for $A \in M_{n\times m}(\mathbb{R}), e_k \in \mathbb{R}^m$, $Ae_k = k$-th column of A.

  Proof: let $A = (a_{ij})$. By definition the i-th entry of Ae_k is

  $$\sum_{j=1}^{m} a_{ij} (e_k)_j = a_{ik}$$

  since $(e_k)_j = 0$ whenever $j \neq k$, 1 for $j = k$

- $\nu \cdot v = \nu^T v$

- Let I_n be the identity matrix. Show formally that $I_n \nu = \nu$, $\forall \nu \in \mathbb{R}^n$.

- let $\nu_1, \nu_2, \nu_3 \in \mathbb{R}^3$. Write the linear combination $3\nu_1 - 5\nu_2 + 7\nu_3$ as a multiplication of matrix $A \in M_{3\times 3}(\mathbb{R})$ with a vector $x \in \mathbb{R}^3$

3.1 System of linear equations

Definition 44. A *linear equation* in the variables $x_1, x_2, \ldots, x_n \in \mathbb{R}$ is an equation of the form:

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = c, \text{ with } \lambda_1, \ldots, \lambda_n \subset \textit{ Fixed} \text{ real numbers}$$

Caution: In particular, no powers/multiplications/function of one or more variables.

Definition 45. A system of linnear equations is a simultaneous list of linear equations

Chapter 4

Analysis