Calculus, Algebra, and Analysis for JMC

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Contents

Chapter 1

Group theory

Study of the simplest algebraic structure on a set.

1.1 Basic Definitions and Examples

1.1.1 Binary operations and groups

> **Definition 1.** *Set* is a collection of distinct elements. Let G be a set.
> ***Binary operation on G*** is a function
> $$* : G \times G \to G \text{(Closure is included)}$$

Example 2.

- $(\mathbb{N}, +), (\mathbb{Z}, +), (\mathbb{R}, \cdot)$

- $(\mathbb{N}, -)$ not a binary op. Not closed.

- $g, h \in G, g * h = h$

- Find a certain $c \in G$, define $g * h = c \ \forall g, h \in G$

Example 3. Cayley table: Draw a table of all the possible binary operations on a set. How many possible binary operations on a finite set with n elements? In general, there are ∞-many biniary operatiions. In this case, there are n^{n^2} possible binary operations. *In general, $g_i * g_j \neq g_j * g_i$* (Not commutative!)

Definition 4. A binary operation $*$ on a set G is called associative if
$$(g * h) * k = g * (h * k) \ \forall g, h, k \in G$$

Example 5.

- $+$ on $\mathbb{N}, \mathbb{Z}, \mathbb{R}$? Yes

- $-$ on $\mathbb{R}$? No

- $g * h = g^h$ on $\mathbb{N}$? No

Definition 6. A binary operation is called commutative if
$$\forall g, h \in G, g * h = h * g$$

Example 7.

- $+, \cdot$ on $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$

- matrix multiplication ($AB \neq BA$ in general for A, B in $M(\mathbb{R}^n)$)

- let $g, h \in \mathbb{R}$, $g * h = 1 + g \cdot h$: commutative but *not associative*!

Definition 8. Let $(G, *)$ be a set. An element e is called *left identity* (respectively *right identity*) if:
$$e * g = g(\text{resp. } g * e = g) \ \forall \ g \in G$$

<u>Caution</u>: There might be *many* left/right identities or none.

Example 9.

1. let $(G, *)$ be a set with $g * h := g$. Find the left/right identities.

   ∞-many (or equal to the number of elements) right identities since h satisfies definition $\forall h$. No left identities: wanted $e * g = g = e$ by definitioin of $*$ (*unless only one element*).

2. $(G, *)$, $g * h = 1 + gh$. Ex: No right/left identities.

Idea: We want a good unique identity.

Theorem 10. let $(G, *)$ be set, such that $*$ has both a left identity e_1 *and* a right identity e_2, then

$$e_1 = e_2 =: e \quad \text{and} \quad e \text{ is unique.}$$

Proof.

- $e_1 = e_2$

$$\Rightarrow \left\{ \begin{array}{l} e_1 * g = g \Rightarrow e_1 * e_2 = e_2 \\ g * e_2 = g \Rightarrow e_1 * e_2 = e_1 \end{array} \right\} \forall g \in G \Rightarrow e_1 = e_2$$

- Unicity: Assume there exists another identity e'.

$$\Rightarrow e' * g = g * e' = g$$

$$e' * g = e' * e = e$$

$$g * e' = e * e' = e'$$

Therefore

$$e = e'$$

$\square$

As soon as you get one left and one right identity, you have a unique identity e.

Definition 11. let $(G, *)$ be a set. Let $g \in G$. An element $h \in G$ is called left (resp. right) inverse if

$$h * g = e \text{ (resp. } g * h = e)$$

Caution: Again inverses might not exist, there might be many, or *not* the same on both sides.

Example 12.

(1) $(\mathbb{N}, \cdot)$ 1 has an inverse, otherwise *no* inverse.

(2) Find a binary operation on a set of 4 elements with left/right inverses not the same but identity e.

Theorem 13. Let $(G, *)$ be a set with associative binary operation and identity e. Then if h_1 is left inverse, and h_2 is right inverse, then

$$h_1 = h_2 = g^{-1} \quad \text{and } \textit{it is unique}$$

Proof.

- $h_1 = h_2$

  $h_1 * g = e, g * h_2 = e$. Therefore

  $$h_2 = e * h_2 = (h_1 * g) * h_2 = h_1 * (g * h_2) = e = h_1$$

- unicity: Assume $\exists g'^{-1}$ another inverse.

  $$g'^{-1} = e * g'^{-1} = (g^{-1} * g) * g'^{-1} = g^{-1} * (g * g'^{-1}) = g^{-1} * e = g^{-1}$$

  $\square$

(Group) Definition 14. A set $(G, *)$ with binary operatioin $*$ is called a *group* if:

(1) $*$ is associative

(2) $\exists e \in G$ an identity $\forall g \in G$

(3) All elements $g \in G$ have an inverse g^{-1}

<u>Attention</u>: The identity and inverses are *unique* by our previous results.

Example 15.

- $(\mathbb{Z}, +), (\mathbb{Z}_n, +)$ (will see this later) are groups.

- $(\mathbb{N}, +)$ not a group $\Rightarrow$ no inverses.

- $(\mathbb{C}, \cdot)$ not a group (0 has no multiplicative inverse), but $(\mathbb{C}^*, \cdot)$ is. ($\mathbb{C}^* = \mathbb{C} \backslash \{0\}$)

- $(G = \{e\}, *)$ with $e * e = e$ is a group called the *trivial group*.

- Empty set $\varnothing$ is not a group (No identity element.)

Definition 16. Let G be a group. It is called <u>finite</u> if it has finitely many elements.

    <u>Notation</u>: $|G| = n$ (number of elements)

    We say that G has **order** n. If $|G| = \infty$, the G is called an infinite group.

Example 17.

- the trivial group is finite, $|G| = 1$

- let $G = \{1, -1, i, -i\} \subset \mathbb{C}$, with $* = \cdot$. Is it a group? Yes. Check associativity, identity, and inverses.

(Abelian Group) Definition 18. A group is called *Abelian* if $*$ is commutative.

Example 19.

- previous example, trvial group, $(\mathbb{Z}, +), (\mathbb{C}^*, \cdot)$

- let $GL(\mathbb{R}^n)$ be the set of all invertible $n \times n$ matrices, $* =$ matrix multiplication. It is associative: $(AB)C = A(BC)$; It has identity: I_n. It has inverses: yes since we asked for it. So this is a group of matrices. But this is not Abelian since $AB \neq BA$.

- let G be the set of *invertible* functions with $* = \circ$, the composition of functions. Identity is $F(x) = x$; they are associative, invertible, but *not Abelian*.

1.1.2   Consequences of the axioms of group

Theorem 20. Let $(G, *)$ be a group, $g, h \in G$. Then

$$(g * h)^{-1} = h^{-1} * g^{-1}$$

Proof. To show: $(g * h) * (h^{-1} * g^{-1}) = e$.

    Using assocativity, we have

$$g * (h * h^{-1}) * g^{-1} = g * g^{-1} = e$$

$\square$

Definition 21. Let $n \in \mathbb{Z}$, let $(G, *)$ be a group and let $g \in G$. Then we definie g^n as follows:

$$g^n = \begin{cases} g * g * \cdots * g & n > 0 \\ g^{-1} * g^{-1} * \cdots * g^{-1} & n < 0 \\ e & n = 0 \end{cases}$$

where in the first case there are n copies of g in the product and ni the second there are $-n$ copies of g^{-1}, so that $g^n = (g^{-1})^{-n}$.

Theorem 22. Let $n, m \in \mathbb{Z}$ and let $G, *$ be a group. Then

1. $g^n * g^m = g^{n+m}$

2. $(g^n)^m = g^{nm}$

Proof. Exercise! (Hint: Induction.) □

1.1.3 Modular Artihmetic and the group $\mathbb{Z}_n$

Definition 23. let $n > 0$, $n \in \mathbb{Z}$ fixed, $a, b \in \mathbb{Z}$. a and b are called **congruent modulo** n if $n | a - b$.

Definition 24. $\forall a, b, c \in \mathbb{Z}$, $n > 0$ fixed in $\mathbb{Z}$:

(1) $a \equiv a \mod n$ (reflexivity)

(2) If $a \equiv b \mod n \iff b \equiv a \mod n$ (symmetry)

(3) if $a \equiv b \mod n$ and $b \equiv c \mod n \implies a \equiv c \mod n$ (transitivity)

Definition 25. Given a set S and an equivalence relation $\sim$ on S, the **equivalence class** of an element a in S is the set $\{x \in S \mid x \sim a\}$.

Definition 26. Define the equivalence class of $a \in \mathbb{Z}$ in the relation of congruence modulo n as:

$$[a]_n := \{b \in \mathbb{Z} \mid b \equiv a \mod n\}$$

Definition 27. Define equivalence classes $\mathbb{Z}_n$ as

$$\mathbb{Z}_n := \{[0]_n, [1]_n, \ldots [n-1]_n\}$$

with 2 <u>binary operations</u> on $\mathbb{Z}_n$:

$$+: \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}_n, ([a]_n, [b]_n) \mapsto [a+b]_n$$

$$\cdot: \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}_n, ([a]_n, [b]_n) \mapsto [ab]_n$$

As we can see from the following lemma, the two operations are well-defined.

Lemma 28. Let $a, a', b, b' \in \mathbb{Z}$ s.t. $[a]_n = [a']_n, [b]_n = [b']_n$. Then $[a+b]_n = [a'+b']_n, [a \cdot b]_n = [a' \cdot b']_n$.

Proof. Exercise! □

Theorem 29. $(\mathbb{Z}_n, +)$ is an Abelian group.

Proof.

(1) Associativity:

$$
\begin{aligned}
([a]_n + [b]_n) + [c]_n &= [a+b]_n + [c]_n \\
&= [a+b+c]_n \\
&= [a]_n + [b+c]_n \\
&= [a]_n + ([b]_n + [c]_n)
\end{aligned}
$$

(2) Commutativity:

$$
\begin{aligned}
[a]_n + [b]_n &= [a+b]_n \\
&= [b+a]_n \\
&= [b]_n + [a]_n
\end{aligned}
$$

(3) Identity element: $[0]_n$

(4) Inverse: Any element $[a]_n$ has an inverse $[-a]_n$.

$\square$

Chapter 2

Applied Mathematical Methods

2.1 Differential Equations

2.1.1 Definitions and examples

Definition 30. An ***ordinary differential equation*** (ODE) for $y(x)$ is an equation involving <u>derivatives</u> of y.

$$f(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}, \ldots, \frac{\mathrm{d}^n y}{\mathrm{d}x^n}) = 0 \qquad (2.1)$$

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = F(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \ldots, \frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}})$$

and we seek a solution (or solutions) for $y(x)$ satisfying the equations. (If there are more independent variables then we have a partial differential equation (PDE).)

Definition 31.
    Order is the order of the highest derivative present.
    Degree is the power of the highest derivative when fractional powers have been removed.
    Linear *differential equation* is a differential equation that is defined by a *linear polynomial* in the unknown function and its derivative in each term of equation(2.1).

Example 32.

(a) <u>Particle moving along a line</u> with a given force → $x(t)$ position as function of time t.

$$\frac{d^2x}{dt^2} = f\left(t, x, \frac{dx}{dt}\right)$$

e.g.

$$\frac{d^2x}{dt^2} = -\omega^2 x - 2k\frac{dx}{dt}$$

The first term is regarding the restoring force, while the second term is regarding the damping/friction. The function is of order 2, degree 1, and linear.

(b) <u>Radius of curvature</u> of a curve

It can be shown that

$$R(x, y) = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

The function is of order 2 and degree 2.

(c) <u>Simple growth and decay</u>

$$\frac{dQ}{dt} = kQ$$

The function is of order 1, degree 1, and linear. e.g.

(1) $k > 0$. Q as the quantity of money, and $k = (1 + \frac{r}{100})$, and r being the rate of interest.

(2) $k < 0$. Q as the amount of radioactive material, and k as the decay rate.

Hence, obviously $Q(t) = Q_0 e^{kt}$ where $Q_0 = Q(0)$ at $t = 0$.

(d) <u>Population dynamics</u>

$P(t)$ as population over time and $F(t)$ as food over time, with

$$\frac{dP}{dt} = aP(a > 0) \tag{2.2}$$

$$\frac{\mathrm{d}F}{\mathrm{d}t} = c \, (c > 0)$$

These two equations form a linear system, with both being of order 1, degree 1.

So $P(t) = P_0 e^{at}, F(t) = ct + F_0$. Misery! Population outgrows food supply.

Pierre Verhulst (1845) replaced a in equation(2.2) with $(a - bP)$ so that growth decreases as P increases:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = aP - bP^2 \tag{2.3}$$

This is in fact a *logistic ODE*, with order 1, degree 1, and nonlinear.

Note: Equation(2.3) is *separable*. Alternatively we can note that equation(2.3) is an example of a *Bernoulli differential equation*

$$\frac{\mathrm{d}y}{\mathrm{d}x} + F(x)y = H(x)y^n \tag{2.4}$$

with $n \neq 0, 1$ Substitution on $z(x) = (y(x))^{1-n} \Rightarrow$ a *linear* equation for $z(x) \rightarrow$ solution. (See below)

(e) Predator-Prey System

$x(t)$ as prey and $y(t)$ as predators, we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax - bxy, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -cy + \hat{d}xy \tag{2.5}$$

Note: Equation(2.5) is *separable* when written in principle

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \Rightarrow y(x) \Rightarrow x(t), y(t)$$

This is of order 1, degree 1, and a nonlinear system.

(f) Combat Model System

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -ay, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -bx \tag{2.6}$$

This is of order 1, degree 1, and linear system.

Note: Again equation(2.6) is *separable* when written as $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{bx}{ay} \Rightarrow$ $y(x) \Rightarrow x(t), y(t)$

In general the solution of a differential equation of order n contains a number n of *arbitrary constants*. This general solution can be specialised to a particular solution by assigninig definite values to these constants.

Example 33.

(a) Family or parabolae $y = Cx^2$ as constant C takes different values.

On a particular curve of the family $\frac{dy}{dx} = 2Cx$. By substitutiion, eliminate $C \Rightarrow \frac{dy}{dx} = \frac{2y}{x}$. This is a geometrical statement about slopes.

<u>Note</u>: 1st order differential equation $\leftrightarrow$ 1 arbitrary constant in general solution.

(b)
$$\left. \begin{aligned} x &= A \sin \omega t + B \cos \omega t \\ \frac{dx}{dt} &= A\omega \cos \omega t - B\omega \sin \omega t \\ \frac{d^2 x}{dt^2} &= -A\omega^2 \sin \omega t - B\omega^2 \cos \omega t \end{aligned} \right\} \Rightarrow \frac{d^2 x}{dt^2} + \omega^2 x = 0$$

<u>Note</u>: 2nd order differential equation $\leftrightarrow$ 2 arbitrary constants in general solution.

Of course it's the reverse of this process we normally want to perform in order to get the general solution. We then often need a particular solution — which satisfieis certain other conditions — *boundary* or *initial condition*. These allow us to find the arbitrary constants in the solutions.

2.1.2 First Order Differential Equations

Properties and approaches

There are essentially 4 types we can solve *analytically*:

- ***separable***

- ***homogeneous***

- ***linear***

- ***exact*** (in Chapter "Partial Differentiation and Multivariable Calculus" later)

Let's look at them one by one:

(a) **Separable**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = G(x) \cdot H(y)$$

Solve by rearrangement and integration

$$\int^y \frac{\mathrm{d}y}{H(y)} = \int^x G(x)\mathrm{d}x$$

E.g.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy^2 e^{-x}$$

$$\int \frac{1}{y^2}\mathrm{d}y = \int xe^{-x}\mathrm{d}x$$

$$-\frac{1}{y} = -xe^{-x} - e^{-x} + C$$

Or singular solution $y = 0$.

If we want the particular solution which passes through $x = 1, y = 1$, then of course we need

$$C = -1 + 2e^{-1} \quad \text{and} \quad \frac{1}{y} = (x+1)e^{-x} + 1 - 2e^{-1}$$

(b) **Homogeneous**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right)$$

Substitution $\frac{y}{x} = u(x)$, i.e. a new dependent variable,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u + x\frac{\mathrm{d}u}{\mathrm{d}x}(= f(u)) \quad \textbf{\textit{(Remember!)}}$$

$$f(u) - u = \frac{x\mathrm{d}u}{\mathrm{d}x}$$

$$\int \frac{\mathrm{d}u}{f(u) - u} = \int \frac{\mathrm{d}x}{x}$$

$$\vdots$$

E.g.

(i)

$$x^2 \frac{\mathrm{d}y}{\mathrm{d}x} + xy - y^2 = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{y}{x}\right)^2 - \frac{y}{x}$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{u^2 - 2u}{x}$$

$$\vdots$$

(ii)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y-3}{x-y+1}$$

This does not look homogeneous as it stands, but can be made so by substituting $x = 1 + X, y = 2 + Y$, and the expression becomes

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{X+Y}{X-Y} = \frac{1 + \left(\frac{Y}{X}\right)}{1 - \left(\frac{Y}{X}\right)}$$

Then let $\frac{Y}{X} = u(X)$,

$$\Rightarrow \int \left(\frac{1-u}{1+u^2}\right) \mathrm{d}u = \int \frac{\mathrm{d}X}{X}$$

Eventually, the equation becomes

$$\tan^{-1} \frac{Y}{X} - \frac{1}{2} \ln\left(1 + \frac{Y^2}{X^2}\right) = \ln X + C$$

$$\tan^{-1}\left(\frac{y-2}{x-1}\right) - \frac{1}{2} \ln\left[(x-1)^2 + (y-2)^2\right] = C$$

<u>Note</u>: If we have e.g. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y-3}{2(x+y)-7}$, then substitute $v(x) = x + y$ will work!

(c) **<u>Linear</u>**

$$\frac{\mathrm{d}y}{\mathrm{d}x} + F(x)y = G(x)$$

1st power only for y and $\frac{\mathrm{d}y}{\mathrm{d}x}$. We apply an *integrating factor* $R(x)$:

$$R(x) = \exp\left[\int^x F(x)\mathrm{d}x\right]$$

This allows us to form the expression

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[y\exp\left(\int^x F(x)\mathrm{d}x\right)\right] = G(x)\exp\left(\int^x F(x)\mathrm{d}x\right)$$

and then integrate...

E.g.

$$(x+2)\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = (x+2)^6$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{4}{x+2} = (x+2)^5$$

$$\Rightarrow F(x) = -\frac{4}{x+2}, G(x) = (x+2)^5$$

Therefore,

$$R(x) = \exp\left[-\int^x \left(\frac{4}{x+2}\right)\mathrm{d}x\right] = \cdots = K(x+2)^{-4}$$

Subsequently, take $K = 1$ W.L.O.G.:

$$(x+2)^{-4}\frac{\mathrm{d}y}{\mathrm{d}x} - 4(x+2)^{-5}y = \frac{\mathrm{d}}{\mathrm{d}x}\left[y(x+2)^{-4}\right] = x+2$$

As such,

$$y(x+2)^{-4} = \frac{1}{2}x^2 + 2x + C \quad \text{(Put C at the right time!)}$$

$$y(x) = \left(\frac{1}{2}x^{2+2x+C}\right)(x+2)^4$$

(So e.g. $y(0) = 8 \Rightarrow C = \frac{1}{2}$)

Novelties!

(i) Bernoulli equation (See Equation(2.4))
A nonlinear equation rendered linear by a substitution $u = y^{1-n}$...

(ii) E.g.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x+e^y}$$

It is <u>nonlinear</u> for $y(x)$ but <u>linear</u> for $x(y)$:

$$\frac{\mathrm{d}x}{\mathrm{d}y} - x = e^y \Rightarrow \ldots$$

2.1.3 'Special' Second Order Differential Equations

Definition 34. General Explicit form is

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = F\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

(a) $y, \frac{\mathrm{d}y}{\mathrm{d}x}$ **missing**, i.e.

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = f(x)$$

Just integrate twice!

(b) $x, \frac{\mathrm{d}y}{\mathrm{d}x}$ **missing**, i.e.

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = f(y)$$

Warning: Do not write $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{1}{\frac{\mathrm{d}^2 x}{\mathrm{d}y^2}}$. However, it may be true, but for what class of functions $y(x)$?

Let $\frac{\mathrm{d}y}{\mathrm{d}x} = p$,

$$\Rightarrow \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}p}{\mathrm{d}x} = \frac{\mathrm{d}p}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = p\frac{\mathrm{d}p}{\mathrm{d}y} = \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{1}{2}p^2\right)$$

This substitution is effective because it eliminates x, so that the equation becomes separable for p and y.

Then we can integrate $\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{1}{2}p^2\right) = f(y)$ w.r.t. y to get $p(y)$. Then using the definition of p,

$$x = \int \frac{\mathrm{d}y}{p(y)}$$

The same is obtained by multiplying the original equation by $\frac{\mathrm{d}y}{\mathrm{d}x}$ and recognizing $\frac{\mathrm{d}y}{\mathrm{d}x} \cdot \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]$

Example:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\omega^2 y$$

with ω being a real constant. (It is a simple harmonic motion.)

$$\Rightarrow \frac{1}{2}p^2 = -\frac{1}{2}\omega^2 y^2 + C$$

Let $C = \frac{1}{2}\omega^2 \overline{A}^2$. We therefore get

$$\frac{1}{p} = \frac{\mathrm{d}x}{\mathrm{d}y} = \pm \frac{1}{\omega(\overline{A}^2 - y^2)^{\frac{1}{2}}}$$

$$\Rightarrow \omega x + \overline{B} = \pm \sin^{-1} \frac{y}{\overline{A}}$$

$$y = \overline{A} \sin(\omega x + \overline{B}) \quad \text{W.L.O.G}$$

$$= A \sin \omega x + B \cos \omega x$$

(c) y **missing**, i.e.

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = f\left(x, \frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

We put $\frac{\mathrm{d}y}{\mathrm{d}x} = p$, so

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}p}{\mathrm{d}x} = f(x, p)$$

i.e. First order $p(x)$. This substitution is effective because it eliminates y, so that the equation becomes separable for p and x.

Solve for $p(x)$ then integrate $\Rightarrow y(x)$.

Example: Radius of curvature

$$\frac{\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{\frac{3}{2}}}{\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}} = a \quad (a \text{ is an arbitrary constant})$$

$$\Rightarrow \frac{\mathrm{d}p}{\mathrm{d}x} = \frac{1}{a}(1 + p^2)^{\frac{3}{2}}$$

$$\Rightarrow \frac{x}{a} + C = \int \frac{\mathrm{d}p}{(1 + p^2)^{\frac{3}{2}}} \quad \text{i.e.} \quad \frac{x}{a} - \frac{A}{a} = \frac{p}{(1 + p^2)^{\frac{1}{2}}}$$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = p = \pm \frac{x - A}{[a^2 - (x - A)^2]^{\frac{1}{2}}}$$

$$\Rightarrow y = B \mp [a^2 - (x - A)^2]^{\frac{1}{2}} \quad \text{i.e.} \quad (x - A)^2 + (y - B)^2 = a^2$$

So they are all circles of radius a!

(d) x **missing**, i.e.

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = f\left(y, \frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

Yet again, let $\frac{\mathrm{d}y}{\mathrm{d}x} = p$, so

$$p\frac{\mathrm{d}p}{\mathrm{d}y} = f(y, p)$$

i.e. First order $p(y)$. So we solve for $p(y)$, then find $x = \int \frac{\mathrm{d}y}{p(y)}$.

Example:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\omega^2 y \mp 2k\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$$

SHM with resistance proportional to $(\text{speed})^2$.

(e) **Linear Equations**, i.e. $y, \frac{\mathrm{d}y}{\mathrm{d}x}$ only occur to 1st power, if at all. So no products of y and $\frac{\mathrm{d}y}{\mathrm{d}x}$. The following section is dedicated to explaining the approach to solve linear differential equations.

General case — Linear Equations

The general form is, for order n,

$$\begin{aligned}
\mathscr{L}y = a_0(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_1(x)\frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}} + a_2(x)\frac{\mathrm{d}^{n-2}y}{\mathrm{d}x^{n-2}} + \cdots \\
+ a_{n-1}(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_n(x)y = f(x)
\end{aligned} \tag{2.7}$$

where $a_0, a_1, \ldots, a_n$ and $f(x)$ are known functions of x only.

$\mathscr{L}$ is a **linear operator**, operating on $y(x)$:

$$\mathscr{L} \equiv \left[a_0\frac{\mathrm{d}^n}{\mathrm{d}x^n} + a_1\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} + \cdots + a_n\right]$$

The equation(2.7) is called **homogeneous** iff $f(x) = 0$ and **inhomogeneous** iff $f(x) \neq 0$.

The homogeneous equation $\mathscr{L}y = 0$ has n independent solutions $y_1(x), y_2(x),$ $\ldots, y_n(x)$ apart from *trivial* $y(x) = 0$. That is to say that $\mathscr{L}y_i(x) = 0$ for $i = 1, 2, \ldots, n$. (**Independence** is an algebraic property...) Because of the

linearity of $y_i(x)$ we find that the most general solution of the homogeneous equation $\mathscr{L}y = 0$ is given by

$$y(x) = A_1 y_1(x) + A_2 y_2(x) + \cdots + A_n y_n(x) \tag{2.8}$$

with $A_1, A_2, \ldots, A_n$ being arbitrary constants. This is because

$$\mathscr{L}y = \mathscr{L}\left(\sum_{i=1}^{n} A_i y_i(x)\right) = \sum_{i=1}^{n} A_i(\mathscr{L}y_i(x)) = 0$$

Of course equation(2.8) contains n arbitrary constants in accord with the order n of the differential equation.

For the inhomogeneous equation ($\mathscr{L}y = f(x)$(2.7)), the expression(2.8) is called the **complementary functions** (CF) of equation(2.7). Any solution of the inhomogeneous equation(2.7), say $Y(x)$, is called a **particular integral** (PI) of equation(2.7). The most general solution of equation(2.7) is thus

$$y(x) = (\text{CF}) + (\text{PI})$$

This contains n arbitrary constants as required/expected!

The constants can be specified in practice to produce a particular solution which satisfies (n) initial/boundary conditions.
<u>Note</u>

(a) For any two solutions $Y_1(x), Y_2(x)$ of equation(2.7), their difference satisfies

$$\mathscr{L}(Y_1 - Y_2) = \mathscr{L}Y_1 - \mathscr{L}Y_2 = f(x) - f(x) = 0$$

(b) Generally, finding $y_1(x), y_2(x), \ldots, y_n(x)$ functions might be very tough — our differential equation has generally variable coefficients after all! So we look at the most common case we need to study — constant coefficients! W.L.O.G.:

$$a_0(x) = 1, a_1(x) = a_1, a_2(x) = a_2, \ldots, a_n(x) = a_n$$

Linear Equations — Second Order, Constant Coefficients

Consider

$$\mathscr{L}y = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}y}{\mathrm{d}x} + a_2 y = f(x)$$

Alternatively, in terms of notation,

$$\mathscr{L}y = y'' + a_1 y' + a_2 y = f(x)$$

Overall flow of solving the equation is to firstly find CF then PI,

$$\Rightarrow y(x) = \mathrm{CF} + \mathrm{PI}$$

Finding the CF We need to solve

$$\mathscr{L}y = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}y}{\mathrm{d}x} + a_2 y = 0 \tag{2.9}$$

Try a solution of the form $y = e^{\lambda x}$ where λ is a constant — which we need to find!

Chapter 3

Linear Algebra

Definition 35. A *column vector* (*n*-column vector) $\boldsymbol{v}_n$ is a tuple of n real numbers written as a single columnn, with $a_1, a_2, a_3, \ldots, a_n \in \mathbb{R}$:

$$\boldsymbol{v}_n := \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{pmatrix}$$

Definition 36. $\mathbb{R}^n$ is the set of all column vectors of height n whose entries are real numbers. In symbols:

$$\mathbb{R}^n = \{ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} : a_1, a_2, \ldots, a_n \in \mathbb{R} \}$$

Example 37. $\mathbb{R}^2$ can be seen as Euclidean plane. $\mathbb{R}^3$ can be seen as Euclidean space.
<u>Caution</u>: Our vectors always "start" at the origin.

Definition 38. The ***zero vector*** $\boldsymbol{0}_n$ is the height n-column vector all of whose entries are 0.

Definition 39. The ***standard basis vectors*** in $\mathbb{R}^n$ are the vectors

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad \boldsymbol{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

i.e. $\boldsymbol{e}_k$ is the vector with kth entry equal to 1 and all other entries equal to 0.

(Operations on column vectors) Definition 40. Let

$$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}, \quad \boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

be column vectors $\mathbb{R}^n$, and let λ be a (real or complex) number.

(1) Addition on vectors in $\mathbb{R}^n$ is given by:

$$\begin{pmatrix} v_1 + u_1 \\ v_2 + u_2 \\ \vdots \\ v_n + u_n \end{pmatrix}$$

$+ : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ (binary operatioin). $(\mathbb{R}^n, +)$ is a group.

(2) ***Scalar multiplication*** $\lambda \boldsymbol{v}$ on $\mathbb{R}^n$:

$$\begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \vdots \\ \lambda v_n \end{pmatrix}$$

$s : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, so not binary operation.

(3) **Dot product** $v \cdot u$ is defined to be the number $v_1 u_1 + v_2 u_2 + \cdots + v_n u_n$. $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, so not binary.

Example 41. Show that $(\mathbb{R}^n, +)$ is an Abelian group.

- Identity: $\mathbf{0}_n$ $(v + \mathbf{0}_n = \mathbf{v})$

- $-\mathbf{v}$ are inverses, where

$$-\mathbf{v} := \begin{pmatrix} -v_1 \\ -v_2 \\ \vdots \\ -v_n \end{pmatrix}$$

- associativity: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

- commutative: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

<u>Caution</u>: $+$ only makes sense for vectors of the *same size.* e.g. $\mathbf{v} \cdot \mathbf{0}_n = 0 \in \mathbb{R}$.

Definition 42. let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n \in \mathbb{R}^n$, $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n \in \mathbb{R}$, then

$$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_n \mathbf{v}_n$$

is called a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n$.

Definition 43. The set of all linear combinatioins of a collection of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is called the **span** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.
Notation:
$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} := \{\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_n \mathbf{v}_n | \lambda_1, \ldots, \lambda_n \in \mathbb{R}\}$$

Example 44. compute the span of

- $\{\mathbf{e}_1, \mathbf{e}_2\}$, $\mathbf{e}_1, \mathbf{e}_2 \in \mathbb{R}^2$.

$$\text{span}\{\mathbf{e}_1, \mathbf{e}_2\} = \{\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 | \lambda_1, \lambda_2 \in \mathbb{R}\} = \{\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} | \lambda_1, \lambda_2 \in \mathbb{R}\}$$

- $\text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \right\} = \{\begin{pmatrix} \lambda_1 \\ 2\lambda_2 \\ 0 \end{pmatrix} | \lambda_1, \lambda_2 \in \mathbb{R}\}$

Definition 45. let $v \in \mathbb{R}^n$. The **length** of v, a.k.a. the **norm** of v, is the non-negative real number $\|v\|$ defined by

$$\|v\| = \sqrt{v \cdot v}$$

Note: $\|\mathbf{0}\| = 0$, and conversely if $v \neq 0$ then $\|v\| > 0$. This definition agrees with out usual ideas about the length of a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$, which follows from Pythagoras' theorem.

Definition 46. A vector $v \in \mathbb{R}^n$ is called a **unit vector** if $\|v\| = 1$.

Example 47.

(1) Any non-zero vector v can be made into a unit vector $u := \frac{v}{\|v\|}$. This process is called **normalizing**.

(2) The standard basis vectors are unit vectors.

3.0.1 Basic Matrix Operations

Definition 48. An $n \times m$-matrix is a rectangular grid of numbers called the *entries* of the matrix with n rows and m columns. A real matrix is onne whose entries are real numbers, and a complex matrix is one whose entries are complex numbers.

    Notations: $M_{n \times m}(\mathbb{R}), M_{n,m}(\mathbb{R}), \mathrm{Mat}_{n \times m}(\mathbb{R}), \mathbb{R}^{n \times m}$.

  Operations on matrices:

Definition 49. let $A = (a_{ij})$ and $B = (b_{ij})$ are $n \times m$-matrix, $\lambda \in \mathbb{R}$. Then:

(1) $A + B = n \times m$-matrix $(a_{ij} + b_{ij})$. $+ : M_{n \times m}(\mathbb{R}) \times M_{n \times m}(\mathbb{R}) \rightarrow M_{n \times m}(\mathbb{R})$

(2) $\lambda A = n \times m$-matrix (λa_{ij})

Theorem 50. $(M_{n \times m}(\mathbb{R}), +)$ is an Abelian grouop.

Definition 51. The *transpose* A^T of an $n \times m$-matrix (a_{ij}) is the $m \times n$-matrix (a_{ij}).

(Multiplying matrices with vectors) Definition 52. Let $A = (a_{ij})$ be an $n \times m$-matrix, $v \in \mathbb{R}^m$. Then Av is the vector in $\mathbb{R}^n$ with i-th row entry $\sum_{j=1}^m a_{ij} v_j$

Example 53.
- Prove that for $A \in M_{n \times m}(\mathbb{R}), e_k \in \mathbb{R}^m$, $Ae_k = k$-th column of A.

  <u>Proof</u>: let $A = (a_{ij})$. By definition the i-th entry of Ae_k is

  $$\sum_{j=1}^m a_{ij}(e_k)_j = a_{ik}$$

  since $(e_k)_j = 0$ whenever $j \neq k$, 1 for $j = k$

- $\nu \cdot v = \nu^T v$

- Let I_n be the identity matrix. Show formally that $I_n \nu = \nu$, $\forall \nu \in \mathbb{R}^n$.

- let $\nu_1, \nu_2, \nu_3 \in \mathbb{R}^3$. Write the linear combination $3\nu_1 - 5\nu_2 + 7\nu_3$ as a multiplication of matrix $A \in M_{3 \times 3}(\mathbb{R})$ with a vector $x \in \mathbb{R}^3$

3.1 System of linear equations

Definition 54. A *linear equation* in the variables $x_1, x_2, \ldots, x_n \in \mathbb{R}$ is an equation of the form:

$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = c$, with $\lambda_1, \ldots, \lambda_n \subset$ *Fixed* real numbers

<u>Caution</u>: In particular, no powers/multiplications/function of one or more variables.

Definition 55. A system of linnear equations is a simultaneous list of linear equations

Chapter 4

Analysis