Fundamentals of Operational Research

0 Introduction to the Course

School: Mathematics
Session: 2025/26, Semester 1
Course:
MATH10065 Fundamentals of Operational Research (FuOR)
Lecturers:

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Exams: Exams after Semester 1 (December).

Workshops: There will be workshopss in addition to the lectures in weeks 2, 4, 6, 8 and 10 (on Tuesdays).

Assignments: There will be 4 assignments during the course. Each will be handed out at the start of weeks 2, 4, 6 and 8 and has to be submitted by Monday, 2pm of the following week (3, 5, 7, 9). The marked assignment will be returned a week after that. Each assignment will contribute 5% to the total course mark.

Piazza forum. Online discussion forum. Students can post questions either under their name or anonymously. Piazza encourages answers by students. However teaching staff will monitor, comment and answer questions that have been left unanswered.

Books

1. "Operations Research", Frederick S. Hillier and Gerald J. Lieberman, 10th Edition, McGraw Hill (7th edition available online).
   General textbook on Operations Research with chapters on Dynamic Programming, Integer Programming and Game Theory (only introductory for this)

2. "Operations Research", Wayne L. Winston, 4th edition, Duxbury (available online).

Another general textbook, again with chapters on Dynamic Programming, Integer Programming and an introduction to Game Theory.

3. "Model Building in Mathematical Programming", H.P. Williams 5th edition, Wiley (available online).
   Textbook on formulation mathematical programming (optimization) problems. Covers some items of the Integer Programming part (eg. modelling logical constraints) that are not covered elsewhere. Does not cover solution algorithms.

4. "Fun and Games", Ken Binmore, D.C. Heath and Co (a newer and re-organised version of the book called "Playing for Real" is available online).

5. "A Primer in Game Theory", Robert Gibbons, Harvester Wheatsheaf
   While the above Operations Research books each have a chapter on Game Theory, these books will give much more detail. They will also cover much more ground than will be covered by the lecture.

The course does not assume previous knowledge about optimization, but occasionally a linear programming formulation of a problem will be written down (but not solved). If you have not met linear programming before (and are confused by this) you can read more about it in references 1 or 2. Note that the course does **not** require any knowledge about solution methods for linear programming (such as the simplex method).

0.1 Course Structure

There are three sections to these courses:

1. Game Theory

2. Integer Optimization

3. Dynamic Optimization

These notes cover the same material as the lecture but they will sometimes use different examples. This is by design, so that you see the same method applied to different example. Examples often come in groups with the same number but with the additional letter **N**, **L** or **S**. **N** examples are in the notes only and will not be done in lectures, **L** examples will be done in lectures, and **S** examples will not be done in lectures and are for you to try yourself.

1 Game Theory

1.1 Introduction

Game Theory is a mathematical framework to study situations of conflict and/or cooperation between rational decision makers. Players are typically people or companies. It can also be applied for example to model evolution in which case players can be nature or genes. "Rational" means that players are expected to try to make best decisions in order to maximize their individual payoff (this may be profit, utility, pervasion of the gene pool, etc).

More formally a *game* is played between n players, each of which has a portfolio of possible behaviours (called *strategies*) $\mathcal{S}_i$. All player simultaneously and individually each choose a strategy $p_i \in \mathcal{S}_i$. After that each player receives a payoff $\pi_i = \pi_i(p_1, ..., p_n)$ which depends on the strategy choice of all players. It is usually assumed that each player knows the complete setup of the game (i.e all strategy sets and each players payoff function). The goal is for each player to maximize their payoff.

An example of a simple (2-player) game is the *Prisoners Dilemma*, with the payoffs given by the following pay-off matrix

	C	D
C	(-1, -1)	(-5, 0)
D	(0, -5)	(-3, -3)

The two players taking part in this game are referred to as the *row player* (R) and the *column* player (C). Each row of the matrix corresponds to a strategy for the row player, each column is a strategy for the column player. Both players have two possible strategies: C (cooperate) and D (defect). The entries in the matrix give the payoff pair (π_R, π_C) for the row and column player respectively. If, for example, R plays C and C plays D, then R receives a payoff of -5 and C receives a payoff of 0.

A few remarks can be made regarding this game

- It is not sensible for either player to play C: no matter what the opponent plays, each player will receive less pay-off for playing C than playing D. The strategy C is **strictly dominated** by strategy D for each player.

- If both players play D, neither will regret their choice: They can each argue: "*now that I know my opponent was going to play D, my choice of D was the best that I could have done*". A strategy pair with this property is called a **Nash Equilibrium** (NE).

- If both players had played (C, C) they would each have received a higher payoff (-1), then they receive for the NE choice (-3). Indeed (C, C) has the highest combined payoff (-2). It is the **pareto optimum**. The NE actually gives the worst possible combined payoff.

1.1.1 Definitions

A n**-person game** is a game between n players. In one play of the game each player receives a payoff which depends on the actions chosen by the other players. (In this course we will concentrate on 2-person games.)

A **zero sum game** is a game in which *in every play of the game* the sum of the payoffs to the players is zero.

Clearly in a 2-person zero-sum game there is no advantage in co-operation, because what one player wins the other loses. Such games are called **strictly competitive**. However co-operation between groups of players is possible in n-person zero sum games, where $n > 2$.

A constant sum game is a game in which *in every play of the game* the sum of the payoffs to the players is the same amount, K say.

A constant sum game may be converted to an equivalent zero sum game by subtracting K from one of the player's payoff each time he plays the game. That player is then paid K each time he plays the game.

A 2-person game in which the "row" player has m actions and the column player has n actions is called a $m \times n$ **matrix game**. In a zero sum game each entry in the payoff matrix is a single number giving the row player's payoff, *i.e. gain*, which in this case is also what the column player *loses*. In a non-zero sum game each entry in the payoff matrix is a pair of numbers, the first of which is what the row player *gains* and the second of which is what the column player *gains*. The nonzero-sum game notation can also be used for zero-sum games.

Zero-Sum Game

	C_1	C_2
R_1	0	-2
R_2	-3	5

$\equiv$

	C_1	C_2
R_1	(0,0)	(-2,2)
R_2	(-3,3)	(5,-5)

Nonzero-sum Game

	C_1	C_2
R_1	(0,1)	(-2,10)
R_2	(-3,-4)	(5,6)

Constant-Sum Game (K=4) Zero-sum game Extra reward

$$\begin{array}{c|cc} & C_1 & C_2 \\ \hline R_1 & (4,0) & (2,2) \\ R_2 & (1,3) & (9,-5) \end{array} \quad \equiv \quad \begin{array}{c|cc} & C_1 & C_2 \\ \hline R_1 & (0,0) & (-2,2) \\ R_2 & (-3,3) & (5,-5) \end{array} \quad + \quad (4,0)$$

Note that it is often useful to calculate the maximum guaranteed expected payoff, corresponded to a 2-person zero sum game between the row player and his "opponent", even when the "opponent" is not necessarily motivated to minimize the row player's gain. It covers the worst which could happen to the row player in an unknown environment. (This is the case in the farmer versus weather example in the lecture.)

If a player has m possible actions he can choose from in each play of the game, and if in repeated plays of the game he chooses action i a proportion p_i of the time, we define his **strategy** as the vector $\underline{p} = (p_1, p_2, ..., p_m)$, where $\sum_{i=1}^{m} p_i = 1$ and $p_i \geq 0, 1 \leq i \leq m$. If some $p_i = 1$ (and so all the rest are 0), the strategy is called a **pure strategy**. Otherwise there is at least 2 $p_i > 0$ and the strategy is called a **mixed strategy**. Pure strategies will sometimes be denoted by the number of the action which is always chosen, *e.g.* (0,0,1,0) could be denoted by 3.

1.1.2 Nash Equlibria & Reaction Sets

In a 2-person game, a pair of actions, (r, c), (*i.e.* a pair of pure strategies) is a **pure strategy Nash Equilibrium** if there is no better action than r (*i.e.* no better row than r) for the row player when the column player plays action c (*i.e.* chooses column c), and also there is no better action than c for the column player when the row player plays action r. (It is therefore not in either player's interest to move away unilaterally from a Nash Equilibrium). A pure strategy Nash Equilibrium is also called a **Saddle Point**.

Reaction Sets and Curves (2 person case). These 'curves' show the best response for one player to each possible strategy for the other player.

$\mathcal{R}_R(\underline{q})$ = set of strategies which are the best response for player R to C's strategy $\underline{q}$

$\mathcal{R}_C(\underline{p})$ = set of strategies which are the best response for player C to R's strategy $\underline{p}$

A pair of strategies $(\underline{p}^*, \underline{q}^*)$ for the row and column player is a **Nash equilibrium** $\iff$ $\underline{p}^* \in \mathcal{R}_R(\underline{q}^*)$ and $\underline{q}^* \in \mathcal{R}_C(\underline{p}^*)$, i.e. each is an optimal response to the other.

1.1.3 Eliminating Dominated Strategies

In a 2-person zero-sum game, where the row player's expected payoff is $\pi(\underline{p}, \underline{q})$, the row player's strategy $\underline{p}$ is (strongly) **dominated** if there exists another strategy for the row player $\hat{\underline{p}}$ such that

$$\pi(\underline{p}, \underline{q}) < \pi(\hat{\underline{p}}, \underline{q}) \text{ for all strategies } \underline{q} \text{ of the column player,}$$

and the column player's strategy $\underline{q}$ is (strongly) dominated if there exists another strategy for the column player $\hat{\underline{q}}$ such that

$$\pi(\underline{p}, \underline{q}) < \pi(\underline{p}, \hat{\underline{q}}) \text{ for all strategies } \underline{p} \text{ of the row player.}$$

In both cases we speak of **weakly dominated** strategies if the above inequalities hold with '$\leq$' instead of '$<$'.

It is easy to find pure strategies which are dominated by other pure strategies, and this can be used to eliminate actions from a game, thus simplifying it. If the original game has Nash equilibria, (some of) these will remain in the reduced game. When eliminating *weakly dominated* strategies it may be the case, however, that a NE is eliminated (as long as they are others left in the game):

- eliminating strongly dominated strategies will not loose *any* Nash equilibria, that is all NE of the original game are still present in the reduced game.
- eliminating weakly dominated strategies *may* loose Nash equilibria. More precisely any NE of the reduced game is also a NE of the original game, but the reduced game may not have any NE left(!).

1.2 Zero-Sum Games

In Zero-Sum games one persons profit is the other persons loss. We will use the one-matrix notation. Here the row player has to maximize pay-off whereas the column player minimizes.

1.2.1 Farmer's "Game" against nature

Ex 1L. Part (i) (Primal)
A farmer can plant any proportion p and $1-p$ of crops C_1 and C_2.
The the probability that the weather is dry is q and the probability its wet is $1-q$.
The farmer's profit per hectare is given in the following table

			Weather	
			q	$1-q$
			Dry	Wet
Farmer	p	C_1	5	2
	$1-p$	C_2	1	8

Note that by modelling this as a (zero-sum) game, we are assuming that the weather is sentinent and malicious: it will always do whatever is worst for the farmer.

We could ask what should the farmer plant assuming that the weather turns out to be whatever would be worst for the farmer. In other words: What is the maximum profit the farmer can guarantee whatever the weather is (and what should the farmer play in order to achieve this):

Notation: Let

$$\pi^{\downarrow C}(p) = \min_{0\le q\le 1} \pi(p,q), \qquad \pi^{\uparrow R\downarrow C} = \max_{0\le p\le 1} \pi^{\downarrow C}(p)$$

$\pi^{\downarrow C}(p)$ is the worst payoff the farmer could get if he *played* p (i.e. planted a proportion of p of C_1). $\pi^{\uparrow R\downarrow C}$ is the best the farmer could do, if he knew the weather would be the worst possible. Due to the way it is defined this value is also known as the *maximin*.

If the weather was dry ($q = 1$) the farmers payoff would be

$$\pi(p,1) = 5p + 1(1-p) = 1 + 4p$$

If the weather was wet ($q = 0$) the farmers payoff would be

$$\pi(p,0) = 2p + 8(1-p) = 8 - 6p$$

If the weather was dry a proportion of $\frac{2}{3}$ of the time the payoff would be

$$\pi(p,q) = \tfrac{2}{3}\pi(p,1) + \tfrac{1}{3}\pi(p,0) = \tfrac{10}{3} + \tfrac{2}{3}p$$

In general

$$\pi(p,q) = \begin{pmatrix} p \\ 1-p \end{pmatrix}^\top \begin{pmatrix} 5 & 2 \\ 1 & 8 \end{pmatrix} \begin{pmatrix} q \\ 1-q \end{pmatrix} = 10pq - 6p - 7q + 8$$

$\pi^{\uparrow C}(q)$ is the best payoff the farmer could get if he knew that the weather was going to play q (i.e. be dry a proportion of q of the time)

Whatever the farmer plays the worst response of the weather is always either completely dry or completely wet. We have therefore

$$\pi^{\downarrow C}(p) = \min\{1 + 4p, 8 - 6p\}$$

The max of which is taken at $p = \frac{7}{10}$ yielding $\pi^{\uparrow R \downarrow C} = 3.8$

Ex 1 L Part (ii) (Dual)

Now assume the farmer's goal is to maximise the expected profit for a given weather state, *i.e.* for a known value of q find the value of p to maximize $\pi(p, q)$. Analogous to above this value is denoted by $\pi^{\uparrow R}(q) = \max_{0 \le p \le 1} \pi(p, 1)$.

Again we can argue that for every given weather state the farmers best response is either to plant only C_1 or C_2. We have

$$\pi(1, q) = 2 + 3q$$
$$\pi(0, q) = 8 - 7q$$

and hence

$$\pi^{\uparrow R}(q) = \max\{2 + 3q, 8 - 7q\}$$

Again we could ask the question "what weather condition is worst for the farmer", *i.e.* what value of q minimizes $\pi^{\uparrow R}(q)$. This minimum is denoted by $\pi^{\downarrow C \uparrow R}$ and is called the *minimax*).
$\pi^{\uparrow R}(q)$ is minimized for $q^* = \frac{6}{10}$ leading to $\pi^{\downarrow C \uparrow R} = \pi^{\uparrow R}(\frac{6}{10}) = 3.8$

We have seen that for this example *minimax* = *maximin*. This is a general result for zero-sum games.

1.2.2 LP formulations of MAXIMIN and MINIMAX

Assume two players, R and C, repeatedly play a $m \times n$ zero sum game with payoff matrix A. (So if R chooses R_i and C chooses C_j then R wins a_{ij} and C loses this same amount.) Assume R chooses R_i with probability p_i and C chooses C_j with probability q_j and the choices are independent of one another. Also assume that each player's goal is to maximize the worst expected payoff that could occur.

R's Problem

$$\begin{aligned} \mathcal{P}: \qquad \pi^{\uparrow R \downarrow C} \quad &= \quad \max_{p_i, 1 \le i \le m} \min_j \sum_{i=1}^m a_{ij} p_i \\ &\quad \text{Subject to} \sum_{i=1}^m p_i = 1, \\ &\quad p_i \ge 0, \quad 1 \le i \le m. \end{aligned}$$

Now introduce a new free variable V (*i.e.* V is not constrained to be positive) and make it less than or equal to each of the sums inside the above minimization. If V is maximized and this is the only occurrence of it in the problem, it will take a value when maximized equal to the min of all the sums. This is true for any fixed value of the p_i variables. Hence $\mathcal{P}$ can be rewritten as

$$\begin{aligned}
\mathcal{P}: \qquad \pi^{\uparrow R \downarrow C} = \quad & \max_{V, p_i, 1 \le i \le m} \quad V \\
& \text{Subject to} \qquad \sum_{i=1}^{m} p_i \quad = 1, \\
& \qquad\qquad V - \sum_{i=1}^{m} a_{ij} p_i \le 0, \quad 1 \le j \le n, \\
& \qquad\qquad V \text{ free}, \quad p_i \ge 0, \quad 1 \le i \le m.
\end{aligned}$$

C's Problem

$$\begin{aligned}
\mathcal{D}: \qquad \pi^{\downarrow C \uparrow R} \quad = \quad & \min_{q_j, 1 \le i \le n} \max_{i} \sum_{j=1}^{n} a_{ij} q_j \\
& \text{Subject to} \sum_{j=1}^{n} q_j = 1, \\
& q_j \ge 0, \quad 1 \le j \le n.
\end{aligned}$$

Now introduce a new free variable v and make it greater than or equal to each of the sums inside the above maximization. If v is minimized and this is the only occurrence of it in the problem, it will take a value when minimized equal to the max of all the sums. This is true for any fixed value of the q_j variables. Hence $\mathcal{D}$ can be rewritten as

$$\begin{aligned}
\mathcal{D}: \qquad \pi^{\downarrow C \uparrow R} = \quad & \min_{v, q_j, 1 \le j \le n} \quad v \\
& \text{Subject to} \qquad \sum_{j=1}^{n} q_j \quad = 1, \\
& \qquad\qquad v - \sum_{j=1}^{n} a_{ij} q_j \ge 0, \quad 1 \le i \le m \\
& \qquad\qquad v \text{ free}, \quad q_j \ge 0, \quad 1 \le j \le n
\end{aligned}$$

The second formulations of $\mathcal{P}$ and $\mathcal{D}$ are a dual pair of linear programming problems. (Note that the free variables, V and v, correspond to equality constraints in the other problem, the non-negative p_i variable in the max problem correspond to the ith $\geq$ constraint in the min problem, and the non-negative q_j variable in the min problem correspond to the jth $\leq$ constraint in the max problem.)

Theorem: MAXIMIN = MINIMAX
For a zero sum 2 person game $\pi^{\uparrow R \downarrow C} = \pi^{\downarrow C \uparrow R}$, *i.e.* MAXIMIN = MINIMAX
Proof
It remains to be shown that $\mathcal{P}$ and $\mathcal{D}$ are indeed dual to each other. The Theorem the follows from the strong duality theorem for linear programming (which is not covered in the course). $\square$

Theorem: (Nash Theorem for zero sum games)
Let (p^*, q^*) be the pair of optimal strategies that achieve $\pi^{\uparrow R \downarrow C} = \pi^{\downarrow C \uparrow R}$,i.e p^* maximizes $\pi^{\downarrow C}(p)$ and q^* minimizes $\pi^{\uparrow R}(q)$. Then (p^*, q^*) is a Nash Equilibrium.

In particular every (2-person) zero-sum game has a NE.
Proof
For any strategy p of the row player let $q(p)$ be (one of) the optimal responses of the column player. Similarly for any strategy q of the column player let $p(q)$ be (one of) the optimal responses of the row player. Now

$$\pi^{\uparrow R \downarrow C} = \pi(p^*, q(p^*)) \leq \pi(p^*, q^*)$$

$q(p^*)$ is by definition the best response of the column player to p^*. q^* *could be* a worse response and the column player is seeking to minimize

$$\pi(.,.)$$

. Also analogous

$$\pi^{\downarrow C \uparrow R} = \pi(p(q^*), q^*) \geq \pi(p^*, q^*)$$

Since $\pi^{\downarrow C \uparrow R} = \pi^{\uparrow R \downarrow C}$ we have equality everywhere, that is q^* is also one of the best responses to p^* (and vice versa). This means that (p^*, q^*) is a NE as claimed $\square$