INTERVAL OF DEFINITION You cannot think *solution* of an ordinary differential equation without simultaneously thinking *interval*. The interval I in Definition 1.1.2 is variously called the **interval of definition,** the **interval of existence,** the **interval of validity,** or the **domain of the solution** and can be an open interval (a, b), a closed interval $[a, b]$, an infinite interval (a, ∞), and so on.

EXAMPLE 5 Verification of a Solution

Verify that the indicated function is a solution of the given differential equation on the interval $(-\infty, \infty)$.

(a) $\dfrac{dy}{dx} = xy^{1/2}; \quad y = \frac{1}{16}x^4$ **(b)** $y'' - 2y' + y = 0; \quad y = xe^x$

SOLUTION One way of verifying that the given function is a solution is to see, after substituting, whether each side of the equation is the same for every x in the interval.

(a) From

$$\textit{left-hand side}: \quad \frac{dy}{dx} = \frac{1}{16}(4 \cdot x^3) = \frac{1}{4}x^3,$$

$$\textit{right-hand side}: \quad xy^{1/2} = x \cdot \left(\frac{1}{16}x^4\right)^{1/2} = x \cdot \left(\frac{1}{4}x^2\right) = \frac{1}{4}x^3,$$

we see that each side of the equation is the same for every real number x. Note that $y^{1/2} = \frac{1}{4}x^2$ is, by definition, the nonnegative square root of $\frac{1}{16}x^4$.

(b) From the derivatives $y' = xe^x + e^x$ and $y'' = xe^x + 2e^x$ we have, for every real number x,

$$\textit{left-hand side}: \quad y'' - 2y' + y = (xe^x + 2e^x) - 2(xe^x + e^x) + xe^x = 0,$$

$$\textit{right-hand side}: \quad 0.$$

Note, too, that each differential equation in Example 5 possesses the constant solution $y = 0$, $-\infty < x < \infty$. A solution of a differential equation that is identically zero on an interval I is said to be a **trivial solution.**

Initial-Value Problems:

INTRODUCTION We are often interested in problems in which we seek a solution $y(x)$ of a differential equation so that $y(x)$ also satisfies certain prescribed side conditions, that is, conditions that are imposed on the unknown function $y(x)$ and its derivatives at a number x_0. On some interval I containing x_0 the problem of solving an nth-order differential equation subject to n side conditions specified at x_0:

$$\begin{aligned} &\textit{Solve:} && \frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}) \\ &\textit{Subject to:} && y(x_0) = y_0,\ y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}, \end{aligned} \qquad (1)$$

where $y_0, y_1, \ldots, y_{n-1}$ are arbitrary constants, is called an ***n*th-order initial-value problem (IVP).** The values of $y(x)$ and its first $n-1$ derivatives at x_0, $y(x_0) = y_0$, $y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$ are called **initial conditions (IC).**

Solving an nth-order initial-value problem such as (1) frequently entails first finding an n-parameter family of solutions of the differential equation and then using the initial conditions at x_0 to determine the n constants in this family. The resulting particular solution is defined on some interval I containing the number x_0.

GEOMETRIC INTERPRETATION The cases $n = 1$ and $n = 2$ in (1),

$$\begin{aligned} &\textit{Solve:} && \frac{dy}{dx} = f(x, y) \\ &\textit{Subject to:} && y(x_0) = y_0 \end{aligned} \qquad (2)$$

and

$$\begin{aligned} &\textit{Solve:} && \frac{d^2 y}{dx^2} = f(x, y, y') \\ &\textit{Subject to:} && y(x_0) = y_0,\ y'(x_0) = y_1 \end{aligned} \qquad (3)$$

are examples of **first-** and **second-order** initial-value problems, respectively. These two problems are easy to interpret in geometric terms. For (2) we are seeking a solution $y(x)$ of the differential equation $y' = f(x, y)$ on an interval I containing x_0 so that its graph passes through the specified point (x_0, y_0). A solution curve is shown in blue in Figure 1.2.1. For (3) we want to find a solution $y(x)$ of the differential equation $y'' = f(x, y, y')$ on an interval I containing x_0 so that its graph not only passes through (x_0, y_0) but the slope of the curve at this point is the number y_1. A solution curve is shown in blue in Figure 1.2.2. The words *initial conditions* derive from physical systems where the independent variable is time t and where $y(t_0) = y_0$

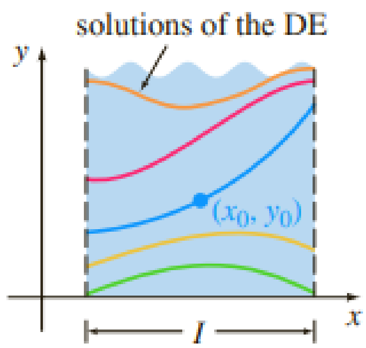

FIGURE 1.2.1 Solution curve of first-order IVP

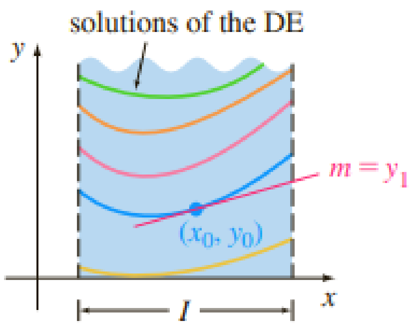

FIGURE 1.2.2 Solution curve of second-order IVP

and $y'(t_0) = y_1$ represent the position and velocity, respectively, of an object at some beginning, or initial, time t_0.

EXAMPLE 1 Two First-Order IVPs

(a) In Problem 45 in Exercises 1.1 you were asked to deduce that $y = ce^x$ is a one-parameter family of solutions of the simple first-order equation $y' = y$. All the solutions in this family are defined on the interval $(-\infty, \infty)$. If we impose an initial condition, say, $y(0) = 3$, then substituting $x = 0$, $y = 3$ in the family determines the constant $3 = ce^0 = c$. Thus $y = 3e^x$ is a solution of the IVP

$$y' = y, \quad y(0) = 3.$$

(b) Now if we demand that a solution curve pass through the point $(1, -2)$ rather than $(0, 3)$, then $y(1) = -2$ will yield $-2 = ce$ or $c = -2e^{-1}$. In this case $y = -2e^{x-1}$ is a solution of the IVP

$$y' = y, \quad y(1) = -2.$$

The two solution curves are shown in dark blue and dark red in Figure 1.2.3. ■

EXAMPLE 3 Second-Order IVP

In Example 9 of Section 1.1 we saw that $x = c_1 \cos 4t + c_2 \sin 4t$ is a two-parameter family of solutions of $x'' + 16x = 0$. Find a solution of the initial-value problem

$$x'' + 16x = 0, \quad x\left(\frac{\pi}{2}\right) = -2, \quad x'\left(\frac{\pi}{2}\right) = 1. \tag{4}$$

SOLUTION We first apply $x(\pi/2) = -2$ to the given family of solutions: $c_1 \cos 2\pi + c_2 \sin 2\pi = -2$. Since $\cos 2\pi = 1$ and $\sin 2\pi = 0$, we find that $c_1 = -2$. We next apply $x'(\pi/2) = 1$ to the one-parameter family $x(t) = -2 \cos 4t + c_2 \sin 4t$. Differentiating and then setting $t = \pi/2$ and $x' = 1$ gives $8 \sin 2\pi + 4c_2 \cos 2\pi = 1$, from which we see that $c_2 = \frac{1}{4}$. Hence $x = -2 \cos 4t + \frac{1}{4} \sin 4t$ is a solution of (4). ■

EXAMPLE 4 An IVP Can Have Several Solutions

Each of the functions $y = 0$ and $y = \frac{1}{16}x^4$ satisfies the differential equation $dy/dx = xy^{1/2}$ and the initial condition $y(0) = 0$, so the initial-value problem

$$\frac{dy}{dx} = xy^{1/2}, \quad y(0) = 0$$

has at least two solutions. As illustrated in Figure 1.2.5, the graphs of both functions, shown in red and blue pass through the same point (0, 0). ■

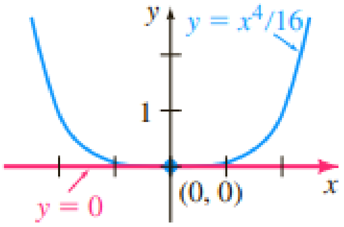

FIGURE 1.2.5 Two solution curves of the same IVP in Example 4

Question:

In Problems 21–24 verify that the indicated family of functions is a solution of the given differential equation. Assume an appropriate interval *I* of definition for each solution.

23. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0; \quad y = c_1e^{2x} + c_2xe^{2x}$

Solution:

From $y = c_1e^{2x} + c_2xe^{2x}$ we obtain $\frac{dy}{dx} = (2c_1 + c_2)e^{2x} + 2c_2xe^{2x}$ and $\frac{d^2y}{dx^2} = (4c_1 + 4c_2)e^{2x} + 4c_2xe^{2x}$, so that

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = (4c_1 + 4c_2 - 8c_1 - 4c_2 + 4c_1)e^{2x} + (4c_2 - 8c_2 + 4c_2)xe^{2x} = 0.$$

Question:

In Problems 35 and 36 find values of m so that the function $y = x^m$ is a solution of the given differential equation.

35. $xy'' + 2y' = 0$

Solution:

Substitute the function $y = x^m$ into the equation $xy'' + 2y' = 0$ to get

$$\begin{aligned} x \cdot (x^m)'' + 2(x^m)' &= 0 \\ x \cdot m(m-1)x^{m-2} + 2mx^{m-1} &= 0 \\ (m^2 - m)x^{m-1} + 2mx^{m-1} &= 0 \\ x^{m-1}[m^2 + m] &= 0 \\ x^{m-1}[m(m+1)] &= 0 \end{aligned}$$

The last line implies that $m = 0$ or $m = -1$ therefore $y = x^0 = 1$ and $y = x^{-1}$ are solutions.

Question:

In Problems 7 10, $x - c_1 \cos t + c_2 \sin t$ is a two-parameter family of solutions of the second-order DE $x'' + x = 0$. Find a solution of the second-order IVP consisting of this differential equation and the given initial conditions.

9. $x(\pi/6) = \frac{1}{2}, \quad x'(\pi/6) = 0$

Solution:

From the initial conditions we obtain

$$\frac{\sqrt{3}}{2}c_1 + \frac{1}{2}c_2 = \frac{1}{2} \quad -\frac{1}{2}c_2 + \frac{\sqrt{3}}{2} = 0$$

Solving, we find $c_1 = \sqrt{3}/4$ and $c_2 = 1/4$. The solution of the initial-value problem is

$$x = (\sqrt{3}/4)\cos t + (1/4)\sin t.$$

Question:

In Problems 39–44, $y = c_1 \cos 2x + c_2 \sin 2x$ is a two-parameter family of solutions of the second-order DE $y'' + 4y = 0$. If possible, find a solution of the differential equation that satisfies the given side conditions. The conditions specified at two different points are called boundary conditions.

42. $y(0) = 1, y'(\pi) = 5$

Solution:

From the boundary conditions $y(0) = 1$ and $y'(\pi) = 5$ we obtain

$$y(0) = c_1 = 1$$

$$y'(\pi) = 2c_2 = 5.$$

Thus, $c_1 = 1$, $c_2 = \frac{5}{2}$, and the solution of the boundary-value problem is $y = \cos 2x + \frac{5}{2} \sin 2x$.