پردازش سیگنال‌های حیاتی
مبحث هشتم- فیلتر کالمن

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مبحث هشتم– فیلتر کالمن

- مقدمه و فرضیات
- دستگاه معادلات حالت/مشاهدات
- روش اول برای بدست آوردن فیلتر کالمن
- روش دوم برای بدست آورد فیلتر کالمن
- تفسیر معادلات فیلتر کالمن
- مثال‌ها
- فیلتر کالمن تعمیم یافته برای مدل غیرخطی

Introduction

- Kalman Filter

  - Recursive solution of an IIR causal Wiener filtering problem

  - Signals to be estimated are considered as the states of a linear discrete time dynamical system

$$\underline{X}_k = \underline{X}[k] = \begin{pmatrix} X_1[k] \\ X_2[k] \\ \vdots \\ X_N[k] \end{pmatrix} \qquad \underline{X}_k = F_k \underline{X}_{k-1} + \underline{W}_k$$

  - Observations are linear combinations of states

$$\underline{Z}_k = \underline{Z}[k] = \begin{pmatrix} Z_1[k] \\ Z_2[k] \\ \vdots \\ Z_M[k] \end{pmatrix} \qquad \underline{Z}_k = H_k \underline{X}_k + \underline{V}_k$$

  - Linearity
  - Real random process

Introduction

- General formulation

  - Process equation (Linear state space equation) $\underline{X}_k = F_k \underline{X}_{k-1} + \underline{W}_k$

    - Sate vector (state) $\underline{X}_k \quad N \times 1$
    - Transition matrix $F_k \quad N \times N$
    - Process noise (modeling noise): White in time, Gaussian

$$\underline{W}_k \quad N \times 1, \quad E\{\underline{W}_k\} = \underline{0}, \quad E\{\underline{W}_k \underline{W}_l^T\} = Q_k \delta[k-l]$$

  - Measurement equation (Observation equation) $\underline{Z}_k = H_k \underline{X}_k + \underline{V}_k$

    - Measurement vector (observation) $\underline{Z}_k \quad M \times 1$
    - Observation matrix $H_k \quad M \times N$
    - Observation noise: White in time, Gaussian, uncorrelated with process noise

$$\underline{V}_k \quad M \times 1, \quad E\{\underline{V}_k\} = \underline{0}, \quad E\{\underline{V}_k \underline{V}_l^T\} = R_k \delta[k-l], \quad E\{\underline{W}_k \underline{V}_l^T\} = 0$$

  - Initial condition of state vector: Gaussian and uncorrelated with process noise and observation noise

$$\underline{X}_0 \quad N \times 1, \quad \underline{m}_0 = E\{\underline{X}_0\} = \underline{0}, \quad P_0 = E\{(\underline{X}_0 - \underline{m}_0)(\underline{X}_0 - \underline{m}_0)^T\}, \quad E\{(\underline{X}_0 - \underline{m}_0)\underline{V}_K^T\} = 0, \quad E\{(\underline{X}_0 - \underline{m}_0)\underline{W}_K^T\} = 0$$

Introduction

- Important remark: at each instant k, state vector and observation vector are zero mean Gaussian

  - Linear combination of independent/mutual Gaussian RVs

$$\underline{X}_k = F_k \underline{X}_{k-1} + \underline{W}_k$$
$$\underline{X}_1 = F_1 \underline{X}_0 + \underline{W}_1$$
$$\underline{X}_2 = F_2 \underline{X}_1 + \underline{W}_2$$
$$\underline{X}_3 = F_3 \underline{X}_2 + \underline{W}_3$$
$$\underline{X}_4 = F_4 \underline{X}_3 + \underline{W}_4$$

$$\underline{Z}_k = H_k \underline{X}_k + \underline{V}_k$$
$$\underline{Z}_0 = H_0 \underline{X}_0 + \underline{V}_0$$
$$\underline{Z}_1 = H_1 \underline{X}_1 + \underline{V}_1$$
$$\underline{Z}_2 = H_2 \underline{X}_2 + \underline{V}_2$$
$$\underline{Z}_3 = H_3 \underline{X}_3 + \underline{V}_3$$
$$\underline{Z}_4 = H_4 \underline{X}_4 + \underline{V}_4$$

Introduction

- Goal

  - Linear MMSE estimate of state vector $\underline{X}_i$ using all the past observations $\underline{Z}_0, \underline{Z}_1, \underline{Z}_2, \cdots, \underline{Z}_k \quad \Rightarrow \quad \underline{Z}_{0:k}$

    - Prediction $\quad \hat{\underline{X}}_i \quad i > k$
    - Filtering $\quad \hat{\underline{X}}_i \quad i = k$
    - Smoothing (non causal filtering) $\quad \hat{\underline{X}}_i \quad i < k$

  - Under Gaussian assumption for zero mean Random Variables:

    - Linear MMSE estimate is equivalent to optimal estimate (conditional mean)

$$\hat{\underline{X}}_i = E\{\underline{X}_i | \underline{Z}_{0:k}\}$$

- Recursive filtering

  - Relation between $\hat{\underline{X}}_k = E\{\underline{X}_k | \underline{Z}_{0:k}\}, \quad \underline{e}_k = \underline{X}_k - \hat{\underline{X}}_k, \quad P_k = E\{(\underline{X}_k - \hat{\underline{X}}_k)(\underline{X}_k - \hat{\underline{X}}_k)^T | \underline{Z}_{0:k}\}$

    and $\hat{\underline{X}}_{k-1} = E\{\underline{X}_{k-1} | \underline{Z}_{0:k-1}\}, \quad \underline{e}_{k-1} = \underline{X}_{k-1} - \hat{\underline{X}}_{k-1}, \quad P_{k-1} = E\{(\underline{X}_{k-1} - \hat{\underline{X}}_{k-1})(\underline{X}_{k-1} - \hat{\underline{X}}_{k-1})^T | \underline{Z}_{0:k-1}\}$

- Method

  - Prediction (a priori estimate) $\quad \hat{\underline{X}}_k^- = E\{\underline{X}_k | \underline{Z}_{0:k-1}\}, \quad \underline{e}_k^- = \underline{X}_k - \hat{\underline{X}}_k^-, \quad P_k^- = E\{(\underline{X}_k - \hat{\underline{X}}_k^-)(\underline{X}_k - \hat{\underline{X}}_k^-)^T | \underline{Z}_{0:k-1}\}$
  - Update using the last observation $\underline{Z}_k$ or the innovation signal $\quad \underline{I}_k = \underline{Z}_k - \hat{\underline{Z}}_k^- = \underline{Z}_k - E\{\underline{Z}_k | \underline{Z}_{0:k-1}\}$
  - Two interpretation of $P_k = E\{(\underline{X}_k - \hat{\underline{X}}_k)(\underline{X}_k - \hat{\underline{X}}_k)^T | \underline{Z}_{0:k}\} = E\{(\underline{X}_k - E\{\underline{X}_k | \underline{Z}_{0:k}\})(\underline{X}_k - E\{\underline{X}_k | \underline{Z}_{0:k}\})^T | \underline{Z}_{0:k}\}$

    - Covariance matrix of error / Covariance matrix of conditional pdf

Kalman Filter: First Approach

- Step One: Prediction

$$\underline{\hat{X}}_k^- = E\left\{\underline{X}_k \middle| \underline{Z}_{0:k-1}\right\} = E\left\{(F_k \underline{X}_{k-1} + \underline{W}_k) \middle| \underline{Z}_{0:k-1}\right\} = F_k E\left\{\underline{X}_{k-1} \middle| \underline{Z}_{0:k-1}\right\} + E\left\{\underline{W}_k \middle| \underline{Z}_{0:k-1}\right\} = F_k \underline{\hat{X}}_{k-1} + \underline{0}$$

$$P_k^- = E\left\{(\underline{X}_k - \underline{\hat{X}}_k^-)(\underline{X}_k - \underline{\hat{X}}_k^-)^T\right\} = E\left\{(F_k \underline{X}_{k-1} + \underline{W}_k - F_k \underline{\hat{X}}_{k-1})(F_k \underline{X}_{k-1} + \underline{W}_k - F_k \underline{\hat{X}}_{k-1})^T\right\} = F_k P_{k-1} F_k^T + \underline{0} + \underline{0} + Q_k$$

- Step Two: Update with last observation

$$\underline{\hat{X}}_k = E\left\{\underline{X}_k \middle| \underline{Z}_{0:k}\right\} = E\left\{\underline{X}_k \middle| \underline{Z}_{0:k-1}, \underline{Z}_k\right\}$$

$$\underline{I}_k = \underline{Z}_k - \underline{\hat{Z}}_k^- = \underline{Z}_k - E\left\{\underline{Z}_k \middle| \underline{Z}_{0:k-1}\right\} = \underline{Z}_k - E\left\{(H_k \underline{X}_k + \underline{V}_k) \middle| \underline{Z}_{0:k-1}\right\} = \underline{Z}_k - H_k \underline{\hat{X}}_k^- + \underline{0} = H_k (\underline{X}_k - \underline{\hat{X}}_k^-) + \underline{V}_k$$

$$\left(\underline{I}_k = \underline{Z}_k - \underline{\hat{Z}}_k^-\right) \perp \underline{Z}_{0:k-1} \quad \Rightarrow \quad \underline{\hat{X}}_k = E\left\{\underline{X}_k \middle| \underline{Z}_{0:k}\right\} = E\left\{\underline{X}_k \middle| \underline{Z}_{0:k-1}, \underline{Z}_k\right\} = E\left\{\underline{X}_k \middle| \underline{Z}_{0:k-1}\right\} + E\left\{\underline{X}_k \middle| \underline{I}_k\right\} = \underline{\hat{X}}_k^- + G_k \underline{I}_k = \underline{\hat{X}}_k^- + G_k \left(\underline{Z}_k - H_k \underline{\hat{X}}_k^-\right)$$

$$E\left\{\underline{I}_k\right\} = \underline{0}$$

$$\text{Kalman Gain}: G_k = R_{X_k I_k} R_{I_k}^{-1}$$

$$\underline{\hat{X}} = A\underline{Z} \Rightarrow A = R_{xz} R_z^{-1}$$

Kalman Filter: First Approach

- Step Two: Update with last observation

$$\underline{\hat{X}} = A\underline{Z} \Rightarrow A = R_{xz}R_z^{-1}$$

$$\text{Kalman Gain}: G_k = R_{X_k I_k} R_{I_k}^{-1}$$

$$R_{X_k I_k} = E\left\{\underline{X}_k \underline{I}_k^T\right\} = E\left\{(\underline{X}_k - \underline{\hat{X}}_k^- + \underline{\hat{X}}_k^-)(H_k(\underline{X}_k - \underline{\hat{X}}_k^-) + \underline{V}_k)^T\right\}$$

$$= E\left\{(\underline{X}_k - \underline{\hat{X}}_k^-)(\underline{X}_k - \underline{\hat{X}}_k^-)^T\right\}H_k^T + E\left\{(\underline{X}_k - \underline{\hat{X}}_k^-)\underline{V}_k^T\right\} + E\left\{\underline{\hat{X}}_k^-(\underline{X}_k - \underline{\hat{X}}_k^-)^T\right\}H_k + E\left\{\underline{\hat{X}}_k^-\underline{V}_k^T\right\} = P_k^- H_k^T + \underline{0} + \underline{0} + \underline{0}$$

$$R_{I_k} = E\left\{\left(H_k(\underline{X}_k - \underline{\hat{X}}_k^-) + \underline{V}_k\right)\left(H_k(\underline{X}_k - \underline{\hat{X}}_k^-) + \underline{V}_k\right)^T\right\} = H_k P_k^- H_k^T + \underline{0} + \underline{0} + R_k$$

$$\Rightarrow G_k = R_{X_k I_k} R_{I_k}^{-1} = P_k^- H_k^T (H_k P_k^- H_k^T + R_k)^{-1}$$

$$P_k = E\left\{(\underline{X}_k - \underline{\hat{X}}_k)(\underline{X}_k - \underline{\hat{X}}_k)^T \,|\underline{Z}_{0:k}\right\} = E\left\{(\underline{X}_k - \underline{\hat{X}}_k^- - G_k \underline{I}_k)(\underline{X}_k - \underline{\hat{X}}_k^- - G_k \underline{I}_k)^T \,|\underline{Z}_{0:k}\right\}$$

$$= E\left\{(\underline{X}_k - \underline{\hat{X}}_k^-)(\underline{X}_k - \underline{\hat{X}}_k^-)^T \,|\underline{Z}_{0:k}\right\} - E\left\{\underline{X}_k \underline{I}_k^T \,|\underline{Z}_{0:k}\right\}G_k^T + E\left\{\underline{\hat{X}}_k^- \underline{I}_k^T \,|\underline{Z}_{0:k}\right\}G_k^T - G_k E\left\{\underline{I}_k \underline{X}_k^T \,|\underline{Z}_{0:k}\right\} + G_k E\left\{\underline{I}_k \underline{\hat{X}}_k^{-T} \,|\underline{Z}_{0:k}\right\} + G_k E\left\{\underline{I}_k \underline{I}_k^T \,|\underline{Z}_{0:k}\right\}G_k^T$$

$$= P_k^- - R_{X_k I_k} G_k^T + 0 - G_k R_{X_k I_k}^T + 0 + G_k R_{I_k} G_k^T = P_k^- - 2G_k R_{X_k I_k}^T + G_k R_{I_k}\left(R_{X_k I_k} R_{I_k}^{-1}\right)^T = P_k^- - 2G_k R_{X_k I_k}^T + G_k R_{I_k}\left(R_{I_k}^{-1}\right)^T R_{X_k I_k}^T = P_k^- - G_k R_{X_k I_k}^T = P_k^- - G_k H_k P_k^-$$

$$\Rightarrow P_k = P_k^- - G_k H_k P_k^- = P_k^- - P_k^- H_k^T G_k^T$$

$$E\left\{(\underline{X}_k - \underline{\hat{X}}_k^-)(\underline{X}_k - \underline{\hat{X}}_k^-)^T \,|\underline{Z}_{0:k}\right\} = E\left\{(\underline{X}_k - \underline{\hat{X}}_k^-)(\underline{X}_k - \underline{\hat{X}}_k^-)^T \,|\underline{Z}_{0:k-1}, \underline{Z}_k\right\} = E\left\{(\underline{X}_k - \underline{\hat{X}}_k^-)(\underline{X}_k - \underline{\hat{X}}_k^-)^T \,|\underline{Z}_{0:k-1}\right\} = P_k^-$$

$$\left.\begin{cases} R_{X_k I_k} G_k^T = R_{X_k I_k}\left(R_{X_k I_k} R_{I_k}^{-1}\right)^T = R_{X_k I_k}\left(\left(R_{I_k}^{-1}\right)^T R_{X_k I_k}^T\right) = R_{X_k I_k} R_{I_k}^{-1} R_{X_k I_k}^T \\ G_k R_{X_k I_k}^T = R_{X_k I_k} R_{I_k}^{-1} R_{X_k I_k}^T \end{cases}\right\} \Rightarrow R_{X_k I_k} G_k^T + G_k R_{X_k I_k}^T = 2G_k R_{X_k I_k}^T$$

$$\left(R_{I_k}^{-1}\right)^T = R_{I_k}^{-1}$$

$$\left(P_k^-\right)^T = P_k^-$$

Kalman Filter: Second Approach

- Alternative method

$$\hat{\underline{X}}_k = E\left\{\underline{X}_k \middle| \underline{Z}_{0:k}\right\} = E\left\{\underline{X}_k \middle| \underline{Z}_{0:k-1}, \underline{Z}_k\right\} = A_k \hat{\underline{X}}_k^- + G_k \underline{Z}_k$$

$$(\underline{X}_k - \hat{\underline{X}}_k) \perp \underline{Z}_i \quad i = 0,1,\cdots,k-1 \quad \Rightarrow \quad E\left\{(\underline{X}_k - \hat{\underline{X}}_k)\underline{Z}_i^T\right\} = 0 \quad i = 0,1,\cdots,k-1$$

$$\Rightarrow \quad E\left\{\left(\underline{X}_k - A_k \hat{\underline{X}}_k^- - G_k \underline{Z}_k\right)\underline{Z}_i^T\right\} = E\left\{\left(\underline{X}_k - A_k \underline{X}_k + A_k \underline{X}_k - A_k \hat{\underline{X}}_k^- - G_k (H_k \underline{X}_k + \underline{V}_k)\right)\underline{Z}_i^T\right\} = 0$$

$$\Rightarrow \quad (I - A_k - G_k H_k) E\left\{\underline{X}_k \underline{Z}_i^T\right\} + A_k E\left\{(\underline{X}_k - \hat{\underline{X}}_k^-)\underline{Z}_i^T\right\} - G_k E\left\{\underline{V}_k \underline{Z}_i^T\right\} = 0 \quad i = 0,1,\cdots,k-1$$

$$\Rightarrow \quad (I - A_k - G_k H_k) E\left\{\underline{X}_k \underline{Z}_i^T\right\} + 0 + 0 = 0 \quad i = 0,1,\cdots,k-1$$

$$\Rightarrow \quad (I - A_k - G_k H_k) = 0 \quad \Rightarrow \quad A_k = I - G_k H_k \quad \Rightarrow \quad \hat{\underline{X}}_k = \hat{\underline{X}}_k^- + G_k (\underline{Z}_k - H_k \hat{\underline{X}}_k^-)$$

$$\left.\begin{array}{l}
(\underline{X}_k - \hat{\underline{X}}_k) \perp \underline{Z}_k \quad \Rightarrow \quad E\left\{(\underline{X}_k - \hat{\underline{X}}_k)\underline{Z}_k^T\right\} = 0 \quad \Rightarrow \quad E\left\{(\underline{X}_k - \hat{\underline{X}}_k)(H_K \underline{X}_k + \underline{V}_k)^T\right\} = 0 \\
\hat{\underline{Z}}_k^- = E\left\{(H_K \underline{X}_k + \underline{V}_k) \middle| \underline{Z}_{0:k-1}\right\} = H_K \hat{\underline{X}}_k^- \\
(\underline{X}_k - \hat{\underline{X}}_k) \perp \underline{Z}_{0:k-1} \quad \Rightarrow \quad (\underline{X}_k - \hat{\underline{X}}_k) \perp \hat{\underline{Z}}_k^- \quad \Rightarrow \quad E\left\{(\underline{X}_k - \hat{\underline{X}}_k)\hat{\underline{Z}}_k^{-T}\right\} = 0 \quad \Rightarrow \quad E\left\{(\underline{X}_k - \hat{\underline{X}}_k)(H_K \hat{\underline{X}}_k^-)^T\right\} = 0
\end{array}\right\} \Rightarrow$$

$$\Rightarrow \quad E\left\{(\underline{X}_k - \hat{\underline{X}}_k)\left(H_k (\underline{X}_k - \hat{\underline{X}}_k^-) + \underline{V}_k\right)^T\right\} = 0 \quad \Rightarrow \quad E\left\{\left(\underline{X}_k - \hat{\underline{X}}_k^- + \hat{\underline{X}}_k^- - \hat{\underline{X}}_k\right)\left(H_k (\underline{X}_k - \hat{\underline{X}}_k^-) + \underline{V}_k\right)^T\right\} = 0$$

$$\Rightarrow \quad E\left\{\left(\underline{X}_k - \hat{\underline{X}}_k^- - G_k (\underline{Z}_k - H_k \hat{\underline{X}}_k^-)\right)\left(H_k (\underline{X}_k - \hat{\underline{X}}_k^-) + \underline{V}_k\right)^T\right\} = 0$$

$$\Rightarrow \quad E\left\{\left(\underline{X}_k - \hat{\underline{X}}_k^- - G_k H_k (\underline{X}_k - \hat{\underline{X}}_k^-) + G_k \underline{V}_k\right)\left(H_k (\underline{X}_k - \hat{\underline{X}}_k^-) + \underline{V}_k\right)^T\right\} = 0$$

$$\Rightarrow \quad E\left\{\left((I - G_k H_k)(\underline{X}_k - \hat{\underline{X}}_k^-) + G_k \underline{V}_k\right)^T \left(H_k (\underline{X}_k - \hat{\underline{X}}_k^-) + \underline{V}_k\right)^T\right\} = 0$$

$$\Rightarrow \quad (I - G_k H_k) P_k^- H_k^T + 0 + 0 + G_k R_k = 0 \quad \Rightarrow \quad G_k = P_k^- H_k^T (H_k P_k^- H_k^T + R_k)^{-1}$$

Kalman Filter: Summary

- Algorithm
  - Process equation and observation equation $\begin{cases} \underline{X}_k = F_k \underline{X}_{k-1} + \underline{W}_k \\ \underline{Z}_k = H_k \underline{X}_k + \underline{V}_k \end{cases}$
  - Recursive algorithme $\hat{\underline{X}}_{k-1} \Rightarrow \hat{\underline{X}}_k^- \Rightarrow \hat{\underline{X}}_k$
    - Prediction $\begin{cases} \hat{\underline{X}}_k^- = F_k \hat{\underline{X}}_{k-1} \\ P_k^- = F_k P_{k-1} F_k^T + Q_k \end{cases}$
    - Update $\begin{cases} \hat{\underline{X}}_k = \hat{\underline{X}}_k^- + G_k(\underline{Z}_k - H_k \hat{\underline{X}}_k^-) \\ G_k = P_k^- H_k^T (H_k P_k^- H_k^T + R_k)^{-1} \\ P_k = P_k^- - G_k H_k P_k^- \end{cases}$
    - Initial condition $\hat{\underline{X}}_0 = E\{\underline{X}_0\}$ $\quad \hat{P}_0 = E\{(\underline{X}_0 - \hat{\underline{X}}_0)(\underline{X}_0 - \hat{\underline{X}}_0)^T\}$
- Interpretation of Innovation/Gain
  - Model
  - Observation

$$\hat{\underline{X}}_k = \hat{\underline{X}}_k^- + G_k(\underline{Z}_k - H_k \hat{\underline{X}}_k^-) = F_k \hat{\underline{X}}_{k-1} + \left(P_k^- H_k^T (H_k P_k^- H_k^T + R_k)^{-1}\right) I_k \cong F_k \hat{\underline{X}}_{k-1} + \left(P_k H_k R_k^{-1}\right) I_k$$

  - Particular case1: scaler case $\hat{X}_k = \hat{X}_k^- + G_k(Z_k - H_k \hat{X}_k^-) = F_k \hat{X}_{k-1} + \dfrac{P_k^- H_k}{H_k P_k^- H_k + R_k} I_k \cong F_k \hat{X}_{k-1} + \dfrac{P_k H_k}{R_k} I_k \cong F_k \hat{X}_{k-1} + \dfrac{\sigma_w^2}{\sigma_v^2} H_k I_k$
  - Particular case2

$$\begin{cases} R_k = \sigma_v^2 I \\ P_k = \sigma_e^2 I \cong \sigma_w^2 I \end{cases} \Rightarrow \hat{\underline{X}}_k \cong F_k \hat{\underline{X}}_{k-1} + \frac{\sigma_w^2}{\sigma_v^2} H_k \underline{I}_k$$

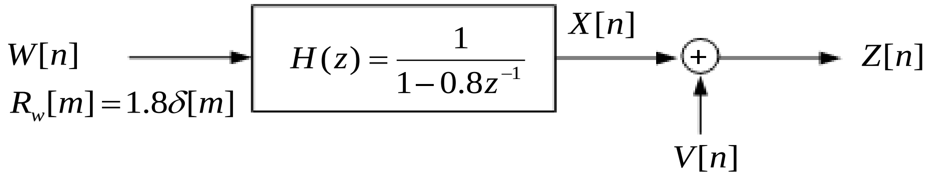
Example

- Exemple 1
  - White additive noise
  - Establishment of state space equation

$$Z[n] = X[n] + V[n]$$

$$R_x[m] = 5(0.8)^{|m|}, \quad R_v[m] = 5\delta[m], \quad R_{xv}[m] = 0$$

$$R_x[m] = 5(0.8)^{|m|} \quad \Rightarrow \quad AR(1) \quad \Rightarrow \quad X[n] - 0.8X[n-1] = W[n]$$



$$X[n] - 0.8X[n-1] = W[n] \quad \Rightarrow \quad X_k = 0.8X_{k-1} + W_k$$

$$Z[n] = X[n] + V[n] \quad \Rightarrow \quad Z_k = X_k + V_k$$

$$F_k = 0.8 \quad Q_k = \sigma_w^2 = 1.8$$

$$H_k = 1 \quad R_k = \sigma_v^2 = 5$$

- Prediction

$$\begin{cases} \hat{X}_k^- = 0.8\hat{X}_{k-1} \\ P_k^- = 0.64P_{k-1} + 1.8 \end{cases}$$

- Update

$$\begin{cases} \hat{X}_k = \hat{X}_k^- + G_k(Z_k - \hat{X}_k^-) \\ G_k = \dfrac{P_k^-}{P_k^- + 5} \\ P_k = P_k^- - G_K P_k^- \end{cases}$$

- Initial condition

$$\hat{X}_0 = E\{X_0\} = 0$$

$$\hat{P}_0 = E\{X_0^2\} = \sigma_x^2 = R_x[0] = 5$$

- Steady state

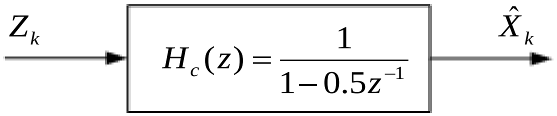
$$P_k = 0.64P_{k-1} + 1.8 - \frac{(0.64P_{k-1} + 1.8)^2}{(0.64P_{k-1} + 1.8) + 5}$$

$$k \to +\infty \quad \Rightarrow \quad P_k = P_{k-1} = P_\infty$$

$$\Rightarrow \quad P_\infty = 0.64P_\infty + 1.8 - \frac{(0.64P_\infty + 1.8)^2}{(0.64P_\infty + 1.8) + 5} \quad \Rightarrow \quad P_\infty = 1.875$$

$$\Rightarrow \quad G_\infty = 0.375 \quad \Rightarrow \quad \hat{X}_k = 0.8\hat{X}_{k-1} + 0.375(Z_k - 0.8\hat{X}_{k-1})$$

$$\Rightarrow \quad \hat{X}_k = 0.5\hat{X}_{k-1} + 0.375Z_k \quad \Rightarrow \quad \hat{X}_k - 0.5\hat{X}_{k-1} = 0.375Z_k$$

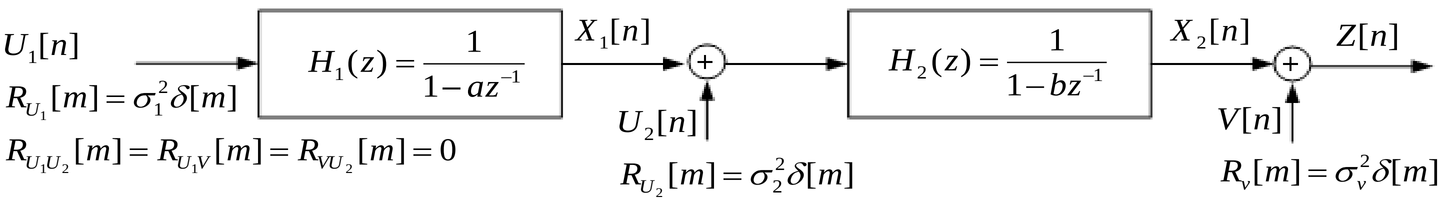


$$h_c[n] = 0.375(0.5)^n u[n]$$

Causal IIR Wiener Filter

Example

- Exemple 2



- Establishment
of state space equation

$$X_1[n]=aX_1[n-1]+U_1[n]$$

$$X_2[n]=bX_2[n-1]+(X_1[n]+U_2[n])=aX_1[n-1]+bX_2[n-1]+U_1[n]+U_2[n]$$

$$\begin{cases}\underline{X}_k=\begin{pmatrix}a & 0\\ a & b\end{pmatrix}\underline{X}_{k-1}+\begin{pmatrix}U_1[k]\\ U_1[k]+U_2[k]\end{pmatrix}\\ Z_k=\begin{pmatrix}0 & 1\end{pmatrix}\underline{X}_k+V_k\end{cases}$$

$$F_k=\begin{pmatrix}a & 0\\ a & b\end{pmatrix}\qquad Q_k=\begin{pmatrix}\sigma_1^2 & \sigma_1^2\\ \sigma_1^2 & \sigma_1^2+\sigma_2^2\end{pmatrix}\qquad H_k=\begin{pmatrix}0 & 1\end{pmatrix}\qquad R_k=\sigma_v^2$$

- Prediction

$$\begin{cases}\underline{\hat{X}}_k^-=F_k\underline{\hat{X}}_{k-1}\\ P_k^-=F_kP_{k-1}F_k^T+Q_k\end{cases}$$

- Update

$$\begin{cases}\underline{\hat{X}}_k=\underline{\hat{X}}_k^-+G_k(\underline{Z}_k-H_k\underline{\hat{X}}_k^-)\\ G_k=P_k^-H_k^T(H_kP_k^-H_k^T+R_k)^{-1}\\ P_k=P_k^--G_kH_kP_k^-\end{cases}$$

- Initial condition

$$\underline{\hat{X}}_0=E\{\underline{X}_0\}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\hat{P}_0=E\{\underline{X}_0\underline{X}_0^T\}=\begin{pmatrix}\frac{\sigma_1^2}{1-a^2} & \frac{\sigma_1^2}{(1-a^2)(1-ab)}\\ \frac{\sigma_1^2}{(1-a^2)(1-ab)} & \frac{\sigma_1^2+(1-a^2)\sigma_2^2}{(1-a^2)(1-b^2)}\end{pmatrix}$$

Example

- Exemple 3
  - 50 Hz notch filter by Kalman filter

$$Z[n] = X[n] + V[n]$$
$$X[n] = A\cos(\omega_0 n + \phi)$$

$$\cos\left(n\alpha + \phi\right) + \cos\left((n-2)\alpha + \phi\right) = 2\cos\left((n-1)\alpha + \phi\right).\cos\left(\alpha\right)$$

$$\Rightarrow \quad X[n] = 2\cos\left(\omega_0\right) X[n-1] - X[n-2]$$

$$\Rightarrow \quad \underline{X}_n = \begin{pmatrix} X_1[n] \\ X_2[n] \end{pmatrix} = \begin{pmatrix} X[n] \\ X[n-1] \end{pmatrix} = \begin{pmatrix} 2\cos\left(\omega_0\right) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} X[n-1] \\ X[n-2] \end{pmatrix}$$

$$= \begin{pmatrix} 2\cos\left(\omega_0\right) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} X_1[n-1] \\ X_2[n-1] \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2\cos\left(\omega_0\right) & -1 \\ 1 & 0 \end{pmatrix} \underline{X}_{n-1} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$Z_n = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} X[n] \\ X[n-1] \end{pmatrix} + V_n = \begin{pmatrix} 1 & 0 \end{pmatrix} \underline{X}_n + V_n$$

Extended Kalman Filter (EKF)

- General formulation

  - Nonlinear process equation and observation equation

$$\begin{cases} \underline{X}_k = f_k\left(\underline{X}_{k-1}, \underline{W}_k, k\right) \\ \underline{Z}_k = h_k\left(\underline{X}_k, \underline{V}_k, k\right) \end{cases}$$

  - Assumptions:

    - Process noise: White in time, Gaussian

    - Observation noise: White in time, Gaussian, uncorrelated with process noise

    - Initial condition of state vector: Gaussian and uncorrelated with process noise and observation noise

- Important remark

    - State vector and Observations vector are not Gaussian

    - Linear MMSE estimate is not optimal estimate (conditional mean)

    - Approximation of optimal estimation

Extended Kalman Filter (EKF)

- Method

  - Linearization of state space model and observation model at each time instant around the most recent state estimate

$$\begin{cases} \underline{X}_k = f_k(\underline{X}_{k-1}, \underline{W}_k, k) \cong f_k\left(\hat{\underline{X}}_{k-1}, \overline{\underline{W}}_k, k\right) + A_k(\underline{X}_{k-1} - \hat{\underline{X}}_{k-1}) + B_k(\underline{W}_k - \overline{\underline{W}}_k) \\ \underline{Z}_k = h_k(\underline{X}_k, \underline{V}_k, k) \cong h_k\left(\hat{\underline{X}}_k^-, \overline{\underline{V}}_k, k\right) + C_k(\underline{X}_k - \hat{\underline{X}}_k^-) + D_k(\underline{V}_k - \overline{\underline{V}}_k) \end{cases}$$

$$A_k = \left.\frac{\partial f_k}{\partial \underline{X}_{k-1}}\right|_{\hat{\underline{X}}_{k-1}, \overline{\underline{W}}_k} = F_k \qquad B_k = \left.\frac{\partial f_k}{\partial \underline{W}_k}\right|_{\hat{\underline{X}}_{k-1}, \overline{\underline{W}}_k}$$

$$C_k = \left.\frac{\partial h_k}{\partial \underline{X}_k}\right|_{\hat{\underline{X}}_k^-, \overline{\underline{V}}_k} = H_k \qquad D_k = \left.\frac{\partial h_k}{\partial \underline{V}_k}\right|_{\hat{\underline{X}}_k^-, \overline{\underline{V}}_k}$$

  - Recursive algorithme $\quad \hat{\underline{X}}_{k-1} \Rightarrow \hat{\underline{X}}_k^- \Rightarrow \hat{\underline{X}}_k$

    - Prediction

$$\begin{cases} \hat{\underline{X}}_k^- = f_k\left(\hat{\underline{X}}_{k-1}, \overline{\underline{W}}_k, k\right) \\ P_k^- = A_k P_{k-1} A_k^T + B_k Q_k B_k^T \end{cases}$$

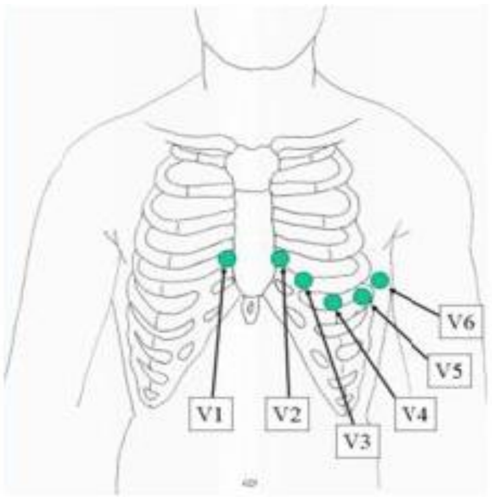
    - Update

$$\begin{cases} \hat{\underline{X}}_k = \hat{\underline{X}}_k^- + G_k\left(\underline{Z}_k - h_k(\hat{\underline{X}}_k^-, \overline{\underline{V}}_k, k)\right) \\ G_k = P_k^- C_k^T (C_k P_k^- C_k^T + D_k R_k D_k^T)^{-1} \\ P_k = P_k^- - G_k C_k P_k^- \end{cases}$$

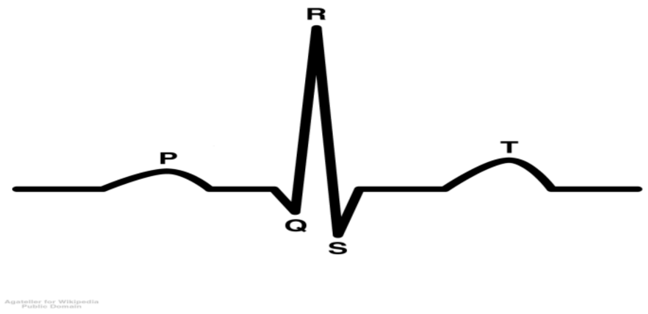
    - Initial condition $\quad \hat{\underline{X}}_0 = E\{\underline{X}_0\} \qquad \hat{P}_0 = E\left\{(\underline{X}_0 - \hat{\underline{X}}_0)(\underline{X}_0 - \hat{\underline{X}}_0)^T\right\}$

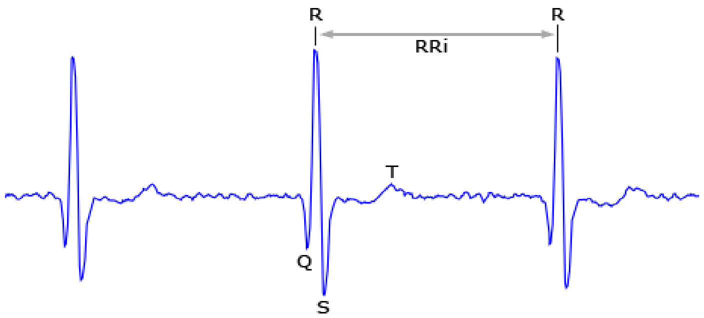
Biomedical Application: Model based ECG Signal Processing

- Electrocardiogram (ECG)
  - Recording the electrical activity of heart (Precordial leads)



  - Characteristic waveforms PQRST

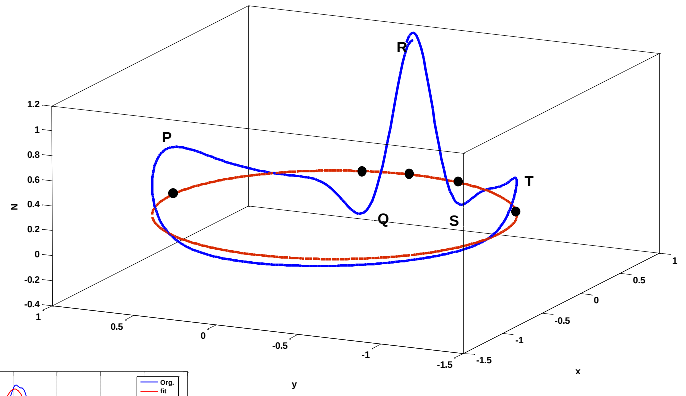


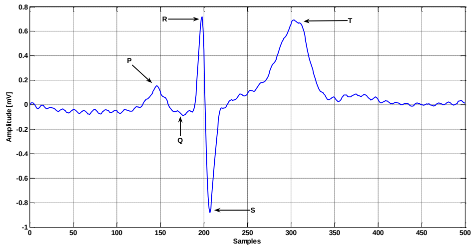


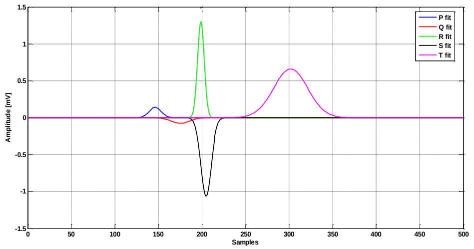
The Whole ECG - A Really Basic ECG Primer, Online available: http://www.anaesthetist.com/icu/organs/heart/ecg/Findex.htm

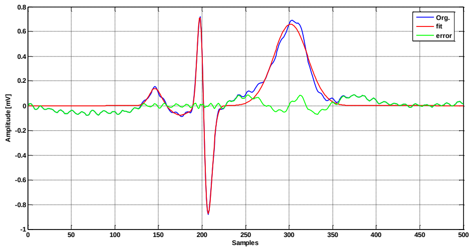
McSharry's Model

- McSharry's synthetic ECG model (2003)
  - Basic Idea: Representing Each beat by the sum of 5 Gaussain Functions
  - Cardiac phase signal: $\theta(t) \in (-\pi, \pi)$
  - The R-peak is considered at $\theta(t) = 0$
  - Pulse Timing of Caracteristique Waves:
    - The position of Gaussain Functions
  - Morphology of Caracteristique Waves :
    - Amplitude of Gaussain Functions
    - Standard Deviation of Gaussain Functions
  - Pseudo-peridicity: Rotating around a unit circle





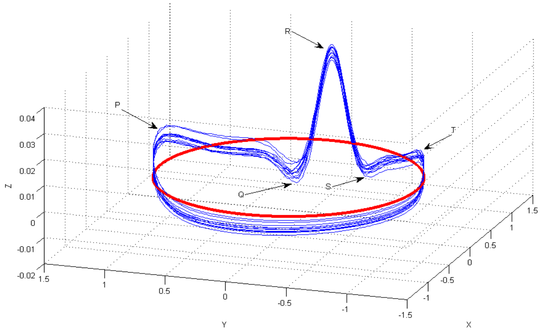




P. E. McSharry, G. D. Clifford, L. Tarassenko, and L. A. Smith, "A Dynamic Model for Generating Synthetic Electrocardiogram Signals," IEEE Trans. Biomed. Eng., vol. 50, pp. 289–294, March 2003.

McSharry's Model

- McSharry's synthetic ECG model (2003)
  - A 3D state space model
  - A unit radius limit cycle, with Gaussian push up and down

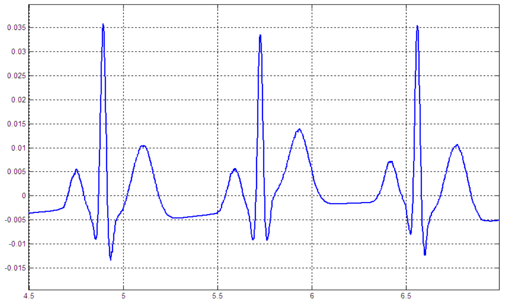


$$\dot{x} = \alpha x - \omega y$$
$$\dot{y} = \alpha y + \omega x$$
$$\dot{z} = -\sum_{i\in\{P,Q,R,S,T\}} a_i \Delta\theta_i \exp\left(-\frac{\Delta\theta_i^2}{2b_i^2}\right) - (z - z_0)$$

$$\xrightarrow[(\alpha_i = \frac{a_i b_i^2}{\omega})]{\int} \quad z = \sum_i \alpha_i \exp(-\Delta\theta_i^2 / 2b_i^2)$$

$$\alpha = 1 - \sqrt{x^2 + y^2}$$
$$\Delta\theta_i = (\theta - \theta_i) \bmod 2\pi$$
$$-\pi \le \theta = atan2(y, x) \le \pi$$

$$1+15 \quad parametrs$$
$$\omega = 2\pi \times Heart\ reate$$
$$= 2\pi / (RR\ interval)$$
$$\theta_P, \quad \theta_Q, \quad \theta_R, \quad \theta_S, \quad \theta_T$$
$$a_P, \quad a_Q, \quad a_R, \quad a_S, \quad a_T$$
$$b_P, \quad b_Q, \quad b_R, \quad b_S, \quad b_T$$



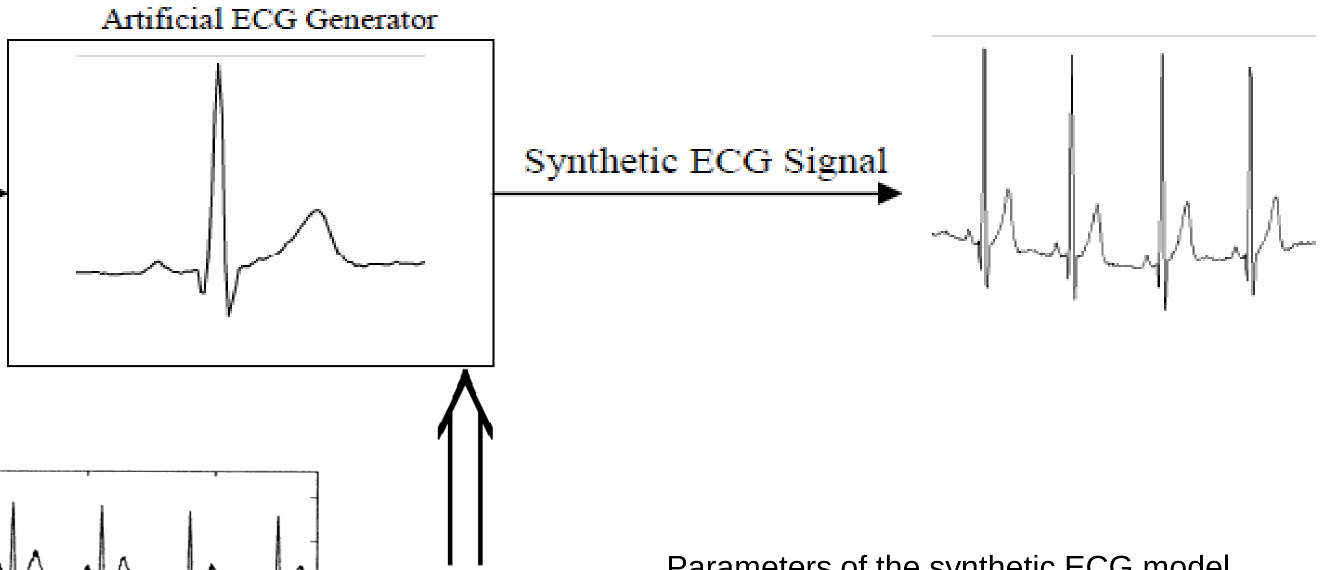
P. E. McSharry, G. D. Clifford, L. Tarassenko, and L. A. Smith, "A Dynamic Model for Generating Synthetic Electrocardiogram Signals," IEEE Trans. Biomed. Eng., vol. 50, pp. 289–294, March 2003.

McSharry's Model

- Synthetic ECG vs Real ECG
  - RR interval generator: $T(t)$
  - An ECG beat generator

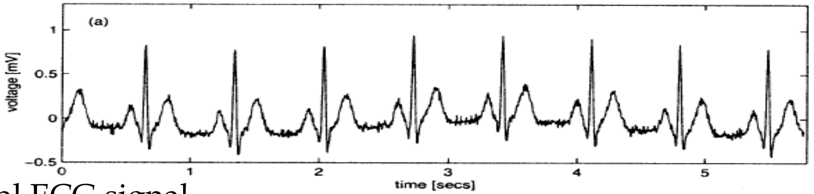
Pulse Generator

$$\omega(t) = \frac{2\pi}{T(t)}$$

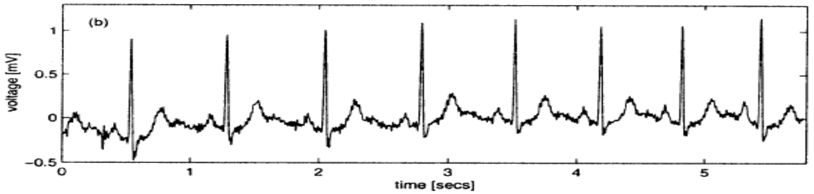


  - A typical example

(a) Synthetic ECG with additive noise



(b) Real ECG signal



Parameters of the synthetic ECG model

Index (i)	P	Q	R	S	T
Time (Sec.)	-0.2	-0.05	0	0.05	0.3
θ_i (rads.)	-π/3	-π/12	0	π/12	π/2
a_i	1.2	-5.0	30.0	-7.5	0.75
b_i	0.25	0.1	0.1	0.1	0.4

P. E. McSharry, G. D. Clifford, L. Tarassenko, and L. A. Smith, "A Dynamic Model for Generating Synthetic Electrocardiogram Signals," IEEE Trans. Biomed. Eng., vol. 50, pp. 289–294, March 2003.

Biomedical Application: Model based ECG Signal Processing

- Model Based ECG Signal Processing

  - Gari CLIFFORD approach

    - Representation of each pattern by a Gaussian with 3 parameters

$$z(t) = \sum_i \alpha_i \exp(-\Delta\theta_i^2 / 2b_i^2)$$

    - Fitting the parameters by minimizing the squared error for each beat (lsqnonlin.m)

$$\min_{\alpha_i, b_i, \theta_i} \int_t \{\|ECG(t) - z(t)\|_2^2\} dt = \min_{\alpha_i, b_i, \theta_i} \int_t \left(\left\| ECG(t) - \sum_i \alpha_i \exp(-\Delta\theta_i^2 / 2b_i^2) \right\|_2^2 \right) dt$$

    - Denoising and Compression

  - New approach

    - McShary's Dynamical Model is cosidered as state space equations
    - ECG is a state variable
    - Extended Kalman Filter can be used

G. D. Clifford, et al.,"Model-based filtering, compression and classification of the ECG," in Proc. BEM & NFSI, 2005, pp. 1-4.

Kalman Based Framework for ECG Signal Processing

- From Mcsharry model to a Bayesian estimation procedure
  - McShary's Dynamical Model as state space equations
  - Necessity of Observations for a Bayesian Estimation Procedure
  - ECG is a state variable => Recording ECG can be considered as Observation
  - A Bayesian Framework using Extended Kalman Filter

- Modification of McShary Model and EKF
  - Polar coordinates

$$\begin{cases} r' = r(1-r) \\ \theta' = \omega \\ z' = -\sum_{i\in\{P,Q,R,S,T\}} a_i \Delta\theta_i \exp\left(-\frac{\Delta\theta_i^2}{2b_i^2}\right) - (z - z_0) \end{cases} \Rightarrow \begin{cases} \theta' = \omega \\ z' = -\sum_{i\in\{P,Q,R,S,T\}} a_i \Delta\theta_i \exp\left(-\frac{\Delta\theta_i^2}{2b_i^2}\right) - (z - z_0) \end{cases}$$

  - Discrete form

$$\begin{cases} \theta_{k+1} = (\theta_k + \omega\delta) mod(2\pi) \\ z_{k+1} = -\sum_{i\in\{P,Q,R,S,T\}} \delta \frac{\alpha_i \omega}{b_i^2} \Delta\theta_i \exp(-\frac{\Delta\theta_i^2}{2b_i^2}) + z_k + \eta \end{cases}$$

Pseudo-Periodicity

Morphology

First Framework: EKF2

- Process equation

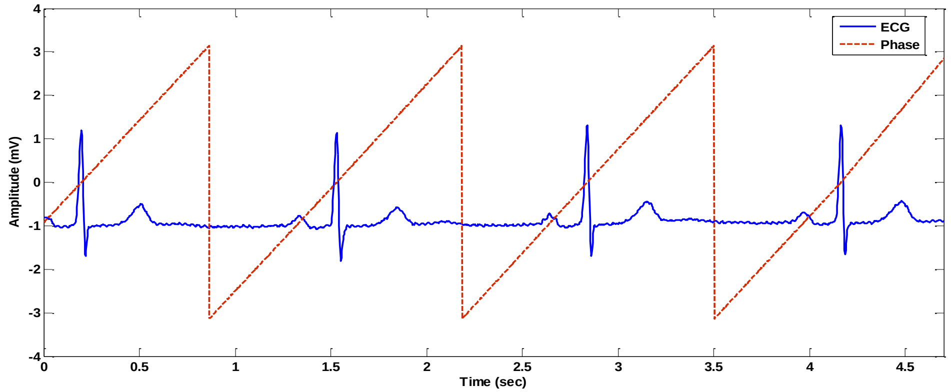
$$\begin{cases} \theta_{k+1} = (\theta_k + \omega\delta) mod(2\pi) = f_1(\theta_k, \omega, k) \\ z_{k+1} = -\sum_{i\in\{P,Q,R,S,T\}} \delta \frac{\alpha_i \omega}{b_i^2} \Delta\theta_i \exp(-\frac{\Delta\theta_i^2}{2b_i^2}) + z_k + \eta = f_2(\theta_k, z_k, \omega, \alpha_i, b_i, \theta_i, \eta, k) \end{cases} \Rightarrow \underline{X}_{k+1} = f_k(\theta_k, z_k, \omega, \alpha_i, b_i, \theta_i, \eta, k) = f_k(\underline{X}_k, \underline{W}_k, k)$$

$$\underline{X}_k = [\theta_k \quad z_k]^T$$

$$\underline{W}_k = [\alpha_P, \ldots, \alpha_T, b_P, \ldots, b_T, \theta_P, \ldots, \theta_T, \omega, \eta]^T$$

- Observation equations

  - Noisy ECG Recording $s_k = z_k + noise$
  - Coarse ECG phase calculated using R peaks (linear approximation between $-\pi$ et π



$$\varphi_k = \theta_k + noise$$

  - Observation vector

$$\begin{pmatrix} \varphi_k \\ s_k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \theta_k \\ z_k \end{pmatrix} + \begin{pmatrix} v_{1k} \\ v_{2k} \end{pmatrix}$$

First Framework: EKF2

- Extended Kalman Filter
  - Nonlinear process equation and observation equation

$$\begin{cases} \underline{X}_k = f_k(\underline{X}_{k-1}, \underline{W}_k, k) \\ \underline{Z}_k = h_k(\underline{X}_k, \underline{V}_k, k) \end{cases}$$

- Method
  - Linearization of state space model and observation model at each time instant around the most recent state estimate

$$\begin{cases} \underline{X}_k = f_k(\underline{X}_{k-1}, \underline{W}_k, k) \cong f_k(\hat{\underline{X}}_{k-1}, \overline{\underline{W}}_k, k) + A_k(\underline{X}_{k-1} - \hat{\underline{X}}_{k-1}) + B_k(\underline{W}_k - \overline{\underline{W}}_k) \\ \underline{Z}_k = h_k(\underline{X}_k, \underline{V}_k, k) \cong h_k(\hat{\underline{X}}_k^-, \overline{\underline{V}}_k, k) + C_k(\underline{X}_k - \hat{\underline{X}}_k^-) + D_k(\underline{V}_k - \overline{\underline{V}}_k) \end{cases}$$

$$A_k = \left.\frac{\partial f_k}{\partial \underline{X}_{k-1}}\right|_{\hat{\underline{X}}_{k-1}, \overline{\underline{W}}_k} = F_k \qquad B_k = \left.\frac{\partial f_k}{\partial \underline{W}_k}\right|_{\hat{\underline{X}}_{k-1}, \overline{\underline{W}}_k}$$

$$C_k = \left.\frac{\partial h_k}{\partial \underline{X}_k}\right|_{\hat{\underline{X}}_k^-, \overline{\underline{V}}_k} = H_k \qquad D_k = \left.\frac{\partial h_k}{\partial \underline{V}_k}\right|_{\hat{\underline{X}}_k^-, \overline{\underline{V}}_k}$$

$$\frac{\partial F1}{\partial z} = 0 \qquad \frac{\partial F1}{\partial \theta} = \frac{\partial F2}{\partial z} = 1$$

$$\frac{\partial F2}{\partial \theta} = -\sum_{i \in \{P,Q,R,S,T\}} \delta \frac{\alpha_i \omega}{b_i^2}[1 - \frac{\Delta\theta_i^2}{b_i^2}]exp(-\frac{\Delta\theta_i^2}{2b_i^2})$$

  - Recursive algorithme $\hat{\underline{X}}_{k-1} \Rightarrow \hat{\underline{X}}_k^- \Rightarrow \hat{\underline{X}}_k$

$$\frac{\partial F1}{\partial \omega} = \delta \qquad \frac{\partial F2}{\partial \eta} = 1 \qquad i \in \{P, Q, R, S, T\}$$

$$\frac{\partial F1}{\partial \alpha_i} = \frac{\partial F1}{\partial b_i} = \frac{\partial F1}{\partial \theta_i} = \frac{\partial F1}{\partial \eta} = 0$$

$$\frac{\partial F2}{\partial \alpha_i} = -\delta \frac{\omega \Delta\theta_i}{b_i^2} exp(-\frac{\Delta\theta_i^2}{2b_i^2})$$

$$\frac{\partial F2}{\partial b_i} = 2\delta \frac{\alpha_i \omega \Delta\theta_i}{b_i^3}[1 - \frac{\Delta\theta_i^2}{2b_i^2}]exp(-\frac{\Delta\theta_i^2}{2b_i^2})$$

$$\frac{\partial F2}{\partial \theta_i} = \delta \frac{\alpha_i \omega}{b_i^2}[1 - \frac{\Delta\theta_i^2}{b_i^2}]exp(-\frac{\Delta\theta_i^2}{2b_i^2})$$

$$\frac{\partial F2}{\partial \omega} = -\sum_i \delta \frac{\alpha_i \Delta\theta_i}{b_i^2} exp(-\frac{\Delta\theta_i^2}{2b_i^2})$$

    - Prediction

$$\begin{cases} \hat{\underline{X}}_k^- = f_k(\hat{\underline{X}}_{k-1}, \overline{\underline{W}}_k, k) \\ P_k^- = A_k P_{k-1} A_k^T + B_k Q_k B_k^T \end{cases}$$

    - Update

$$\begin{cases} \hat{\underline{X}}_k = \hat{\underline{X}}_k^- + G_k\left(\underline{Z}_k - h_k(\hat{\underline{X}}_k^-, \overline{\underline{V}}_k, k)\right) \\ G_k = P_k^- C_k^T (C_k P_k^- C_k^T + D_k R_k D_k^T)^{-1} \\ P_k = P_k^- - G_k C_k P_k^- \end{cases}$$

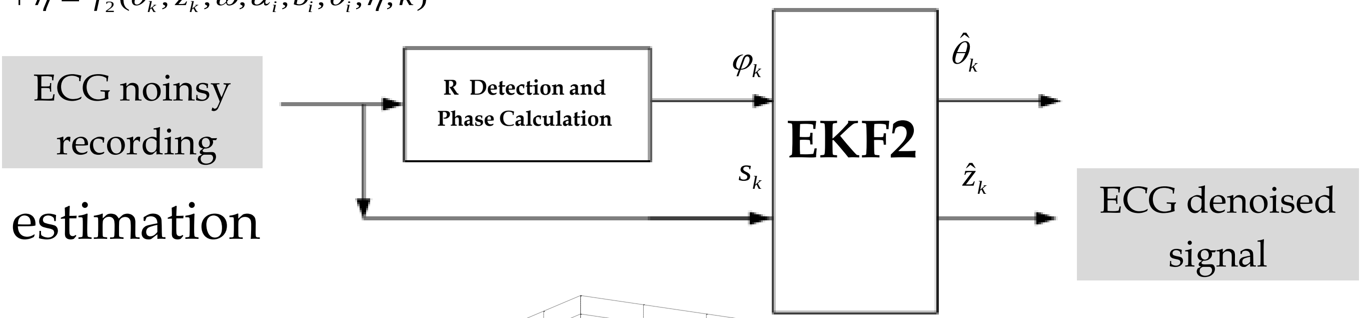
    - Initial condition $\hat{\underline{X}}_0 = E\{\underline{X}_0\}$ $\hat{P}_0 = E\{(\underline{X}_0 - \hat{\underline{X}}_0)(\underline{X}_0 - \hat{\underline{X}}_0)^T\}$

First Framework: EKF2

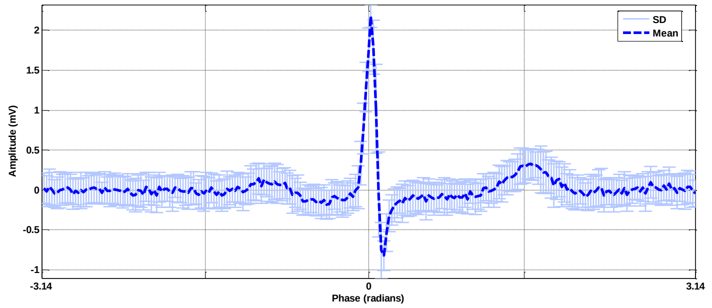
- Application 1: Single Channel ECG Denoising
  - EKF2 inputs: Noisy ECG and Approximate Phase
  - EKF2 outputs: Estimate of true ECG (Denoised ECG) and true Phase
  - Function of KF: using a priori information from ECG dynamics and noisy observations

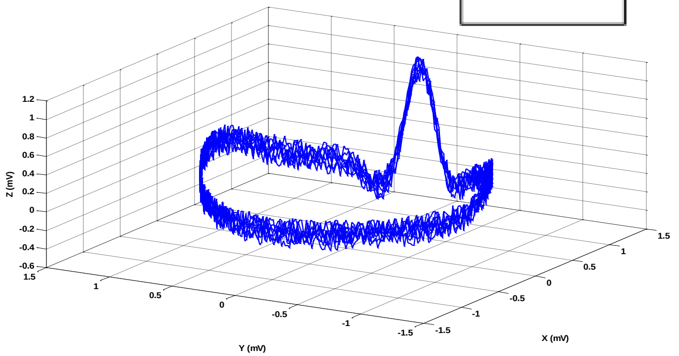
$$\begin{cases} \theta_{k+1} = (\theta_k + \omega\delta) mod(2\pi) = f_1(\theta_k, \omega, k) \\ z_{k+1} = -\sum_{i\in\{P,Q,R,S,T\}} \delta \frac{\alpha_i \omega}{b_i^2} \Delta\theta_i \exp(-\frac{\Delta\theta_i^2}{2b_i^2}) + z_k + \eta = f_2(\theta_k, z_k, \omega, \alpha_i, b_i, \theta_i, \eta, k) \end{cases}$$

$$\begin{cases} \varphi_k = \theta_k + v_{1k} \\ s_k = z_k + v_{2k} \end{cases}$$



- Kalman Filter parameters estimation
  - 30 beats of noisy ECG

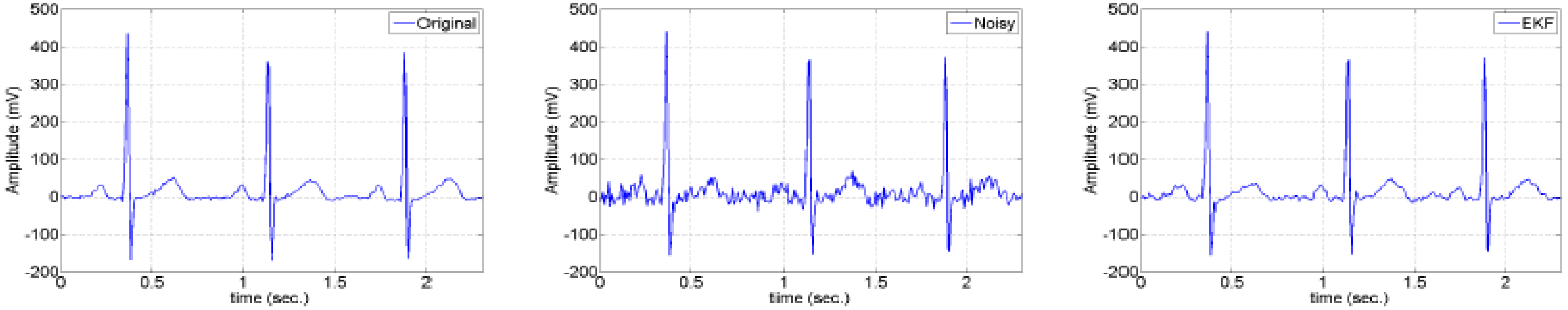




R. Sameni, M.B Shamsollahi, C. Jutten, and G.D Clifford, "A Nonlinear Bayesian Filtering Framework for ECG Denoising", IEEE Transactions on Biomedical Engineering, Vol.54, No. 12, Dec. 2007.

First Framework: EKF2

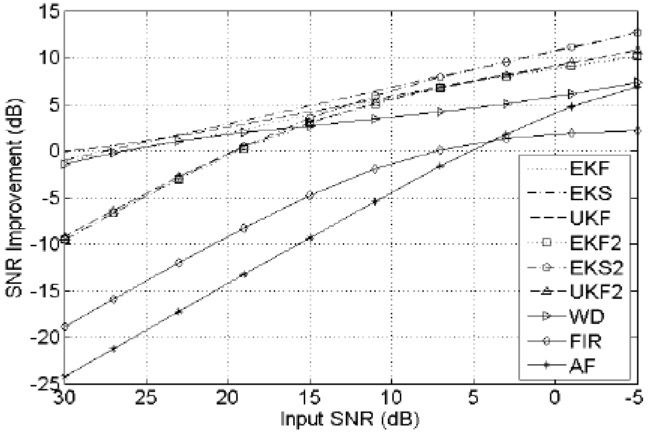
- Performance evaluation: Typical filtering results for an input signal of 6dB



(a) Original (b) Noisy (c) EKF

- The mean of the SNR improvements versus different input SNRs

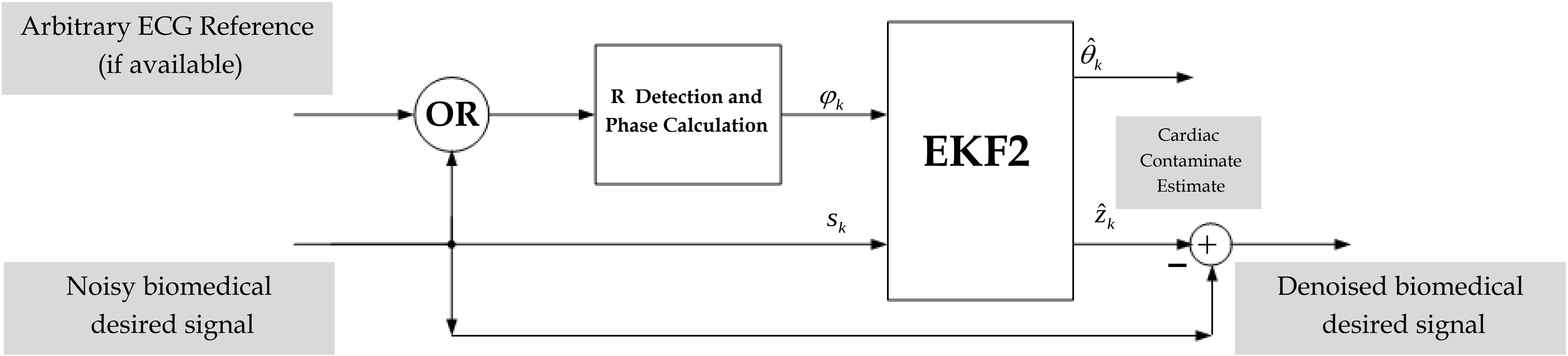
$$imp[dB] = SNR_{output} - SNR_{input} = 10\log\left(\frac{\sum_i |x_n(i)-x(i)|^2}{\sum_i |x_d(i)-x(i)|^2}\right)$$



R. Sameni, M.B Shamsollahi, C. Jutten, and G.D Clifford, "A Nonlinear Bayesian Filtering Framework for ECG Denoising", IEEE Transactions on Biomedical Engineering, Vol.54, No. 12, Dec. 2007.

First Framework: EKF2

- Application 2: Filtering of Cardiac Contaminants from biomedical recordings
  - **Recorded signal** = desired biomedical signal + ECG + other components
  - Step 1: EKF2 Based ECG denoising
  - Step 2: Estimation of Biomedical desired signal

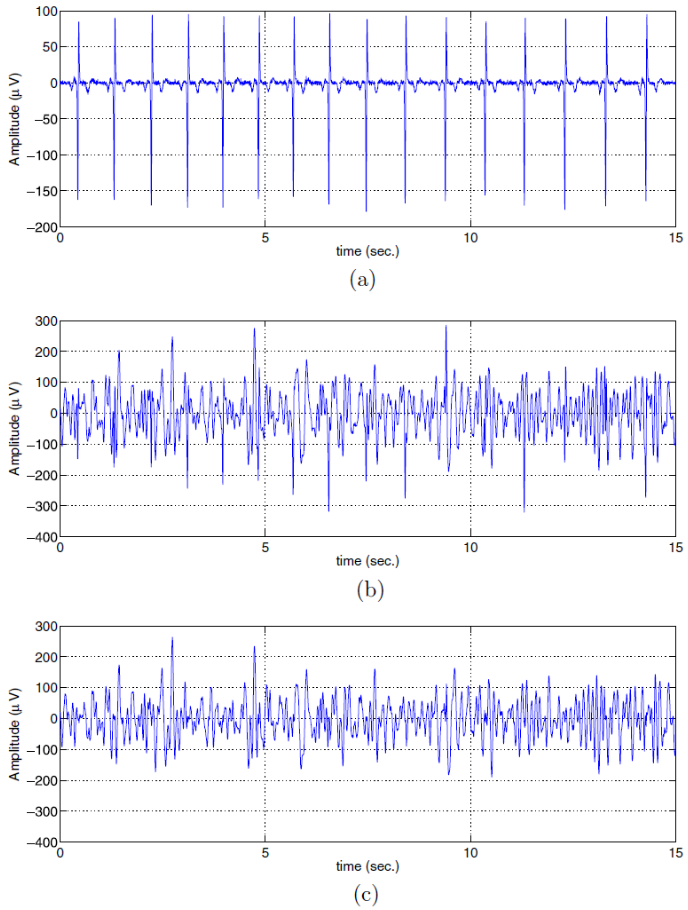


R. Sameni, M.B Shamsollahi, C. Jutten, "Model-based Bayesian filtering of cardiac contaminants from biomedical recordings", Physiological Measurement, Vol. 29, pp. 595-613, 2008.

First Framework: EKF2

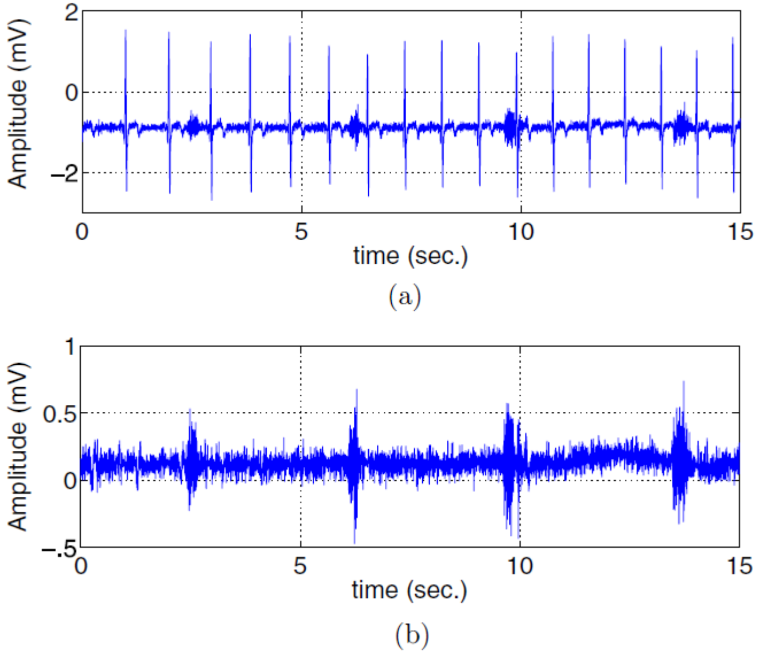
- Example: EEG Denoinsing:

(a) Reference ECG channel, (b) noisy EEG channel ,
(c) EEG channel after ECG removal



- Example: EMG Denoising:

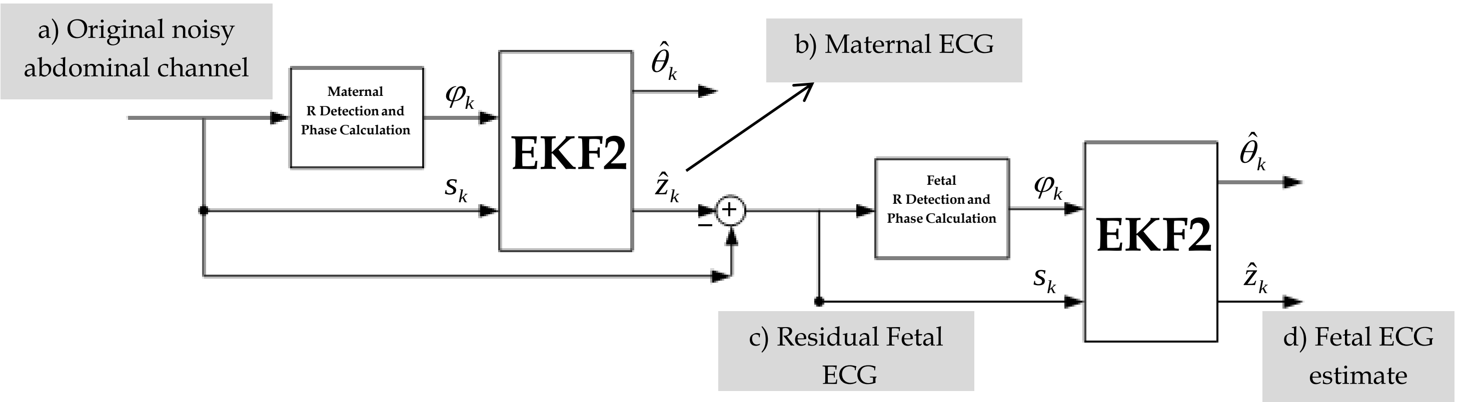
(a) Original noisy EMG channel
(b) Residual EMG bursts

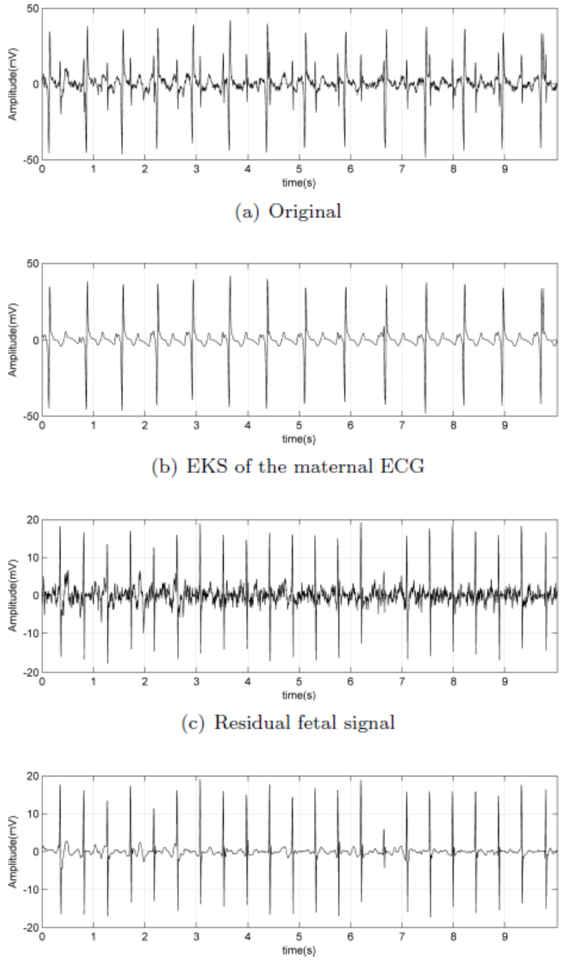


R. Sameni, M.B Shamsollahi, C. Jutten, "Model-based Bayesian filtering of cardiac contaminants from biomedical recordings", Physiological Measurement, Vol. 29, pp. 595-613, 2008.

First Framework: EKF2

- Application 3: Single Channel fetal ECG Extraction
  - First step: mECG extraction and mECG elimination
  - Second step: fECG extraction



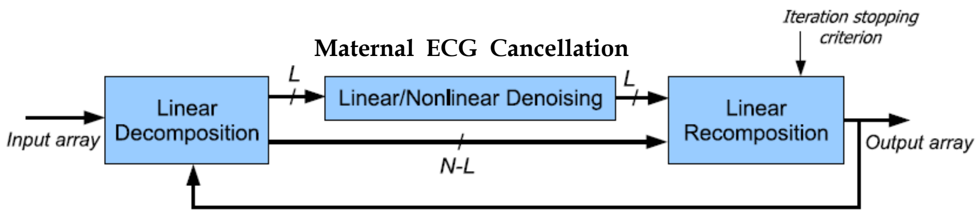


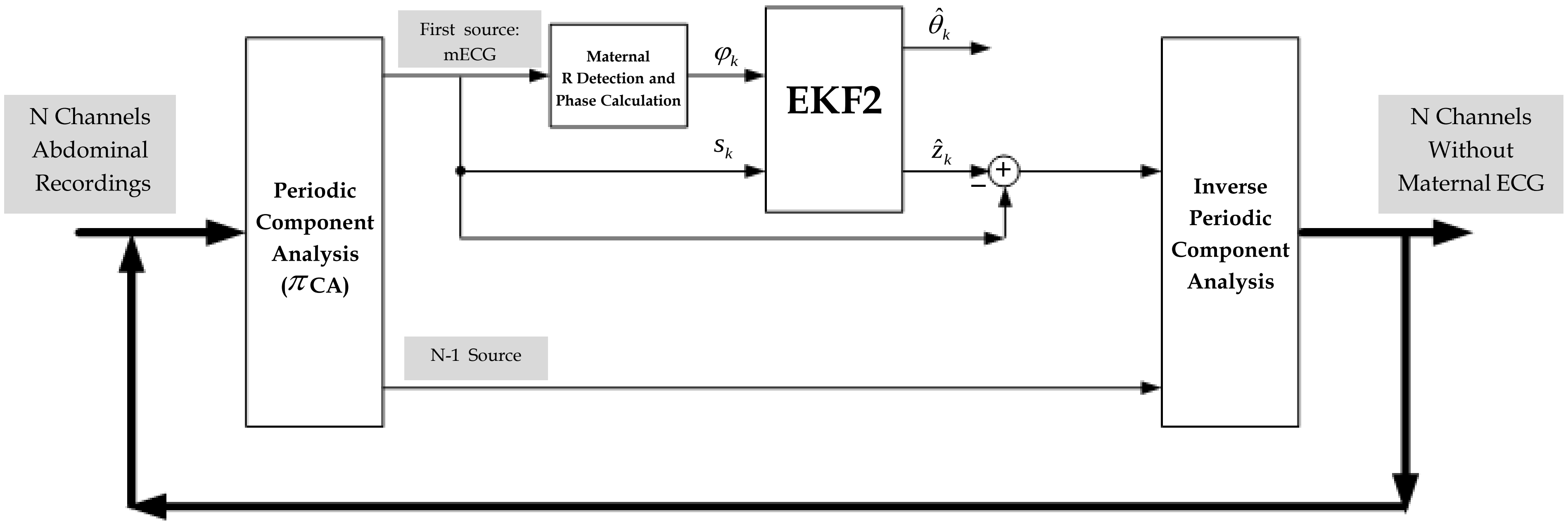
(a) Original
(b) EKS of the maternal ECG
(c) Residual fetal signal
(d) Fetal signal after post-processing

R. Sameni, M.B Shamsollahi, C. Jutten, "Model-based Bayesian filtering of cardiac contaminants from biomedical recordings", Physiological Measurement, Vol. 29, pp. 595-613, 2008.

First Framework: EKF2

- Application 4: A Deflation based Multichannel Fetal ECG Extraction
  - Improvement of BSS based approaches
  - π CA: A new linear decomposition method
  - EKF based denoising for mECG elimination





R. Sameni, C. Jutten, M.B Shamsollahi, "A Deflation Procedure for Subspace Decomposition", IEEE Transactions on Signal Processing, Vol. 58, No. 4, pp 2363-2374, 2010.

Second Framework: EKF17

- Process equations with 17 states

$$\begin{cases}\theta[k+1]=\theta[k]+\omega.\delta \\ z[k+1]=-\sum_{i\in\{P,Q,R,S,T\}}\delta.\frac{\alpha_i[k]\omega}{b_i[k]^2}.\Delta\theta_i[k]\exp(-\frac{\Delta\theta_i[k]^2}{2b_i[k]^2})+z[k]+\eta \\ \alpha_P[k+1]=\alpha_P[k]+u_1[k] \\ \vdots \\ b_P[k+1]=b_P[k]+u_6[k] \\ \vdots \\ \theta_P[k+1]=\theta_P[k]+u_{11}[k] \\ \vdots \\ \theta_T[k+1]=\theta_T[k]+u_{15}[k]\end{cases} \Rightarrow \begin{cases}\theta_{k+1} = f_1(\theta_k,\omega,k) \\ z_{k+1} = f_2(\theta_k,z_k,\alpha_{ik},b_{ik},\theta_{ik},\eta,k) \\ \alpha_{P_{k+1}} = f_3(\alpha_{P_k},u_1,k) \\ \vdots \\ b_{P_{k+1}} = f_8(b_{P_k},u_6,k) \\ \vdots \\ \theta_{P_{k+1}} = f_{13}(\theta_{P_k},u_{11},k) \\ \vdots \\ \theta_{T_{k+1}} = f_{17}(\theta_{T_k},u_{15},k)\end{cases}$$

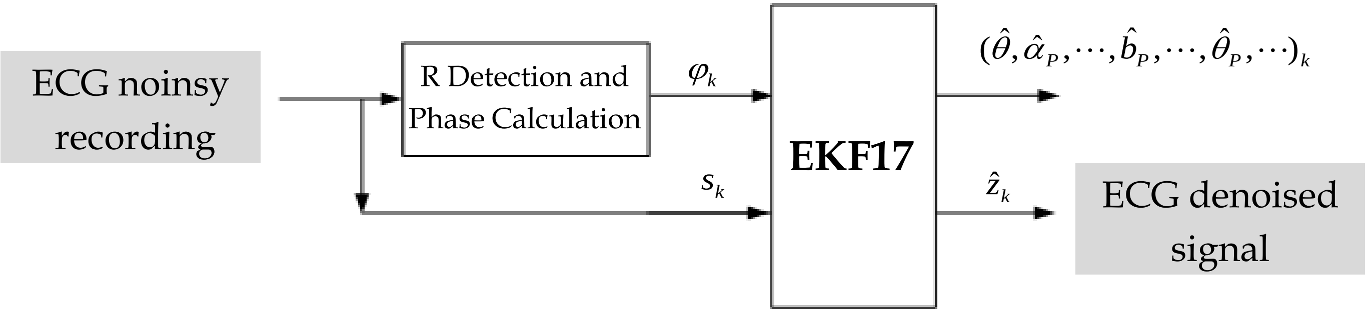
$$\begin{cases}\underline{X}_k = [\theta \quad z \quad \alpha_P \ldots b_P \ldots \theta_P \ldots \theta_T]_k^T \\ \underline{W}_k = [\omega,\eta,u_1,\ldots,u_{15}]^T\end{cases}$$

- Observation equations $\begin{bmatrix}\varphi_k \\ s_k\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0\end{bmatrix}\cdot\underline{X}_k+\begin{bmatrix}v_{1k} \\ v_{2k}\end{bmatrix}$

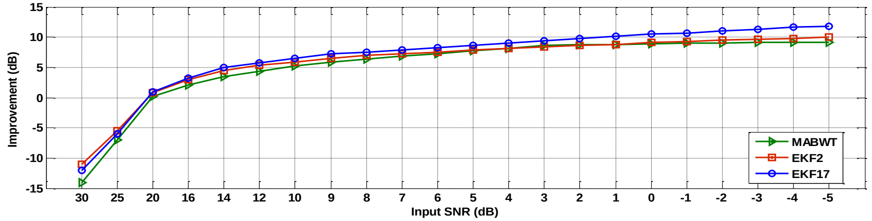
O. Sayadi and M.B. Shamsollahi, "ECG Denoising and Compression Using a Modified Extended Kalman Filter Structure", IEEE Transactions on Biomedical Engineering, Vol. 55, No. 9, pp. 2240-2248, 2008.

Second Framework: EKF17

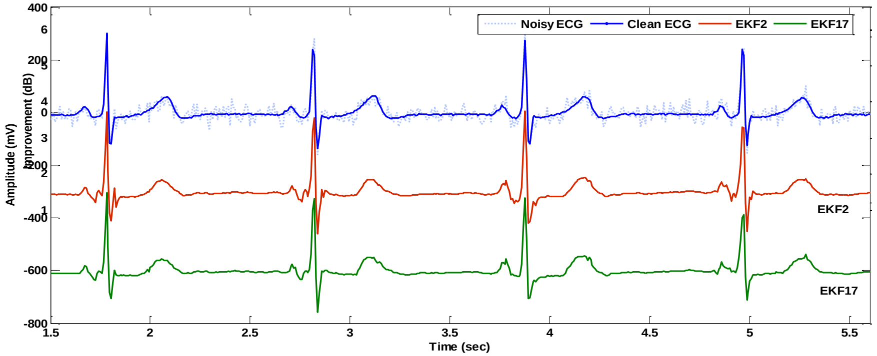
- Application 1: EKF17 based Denoising



- The mean of the SNR improvements versus different input SNRs



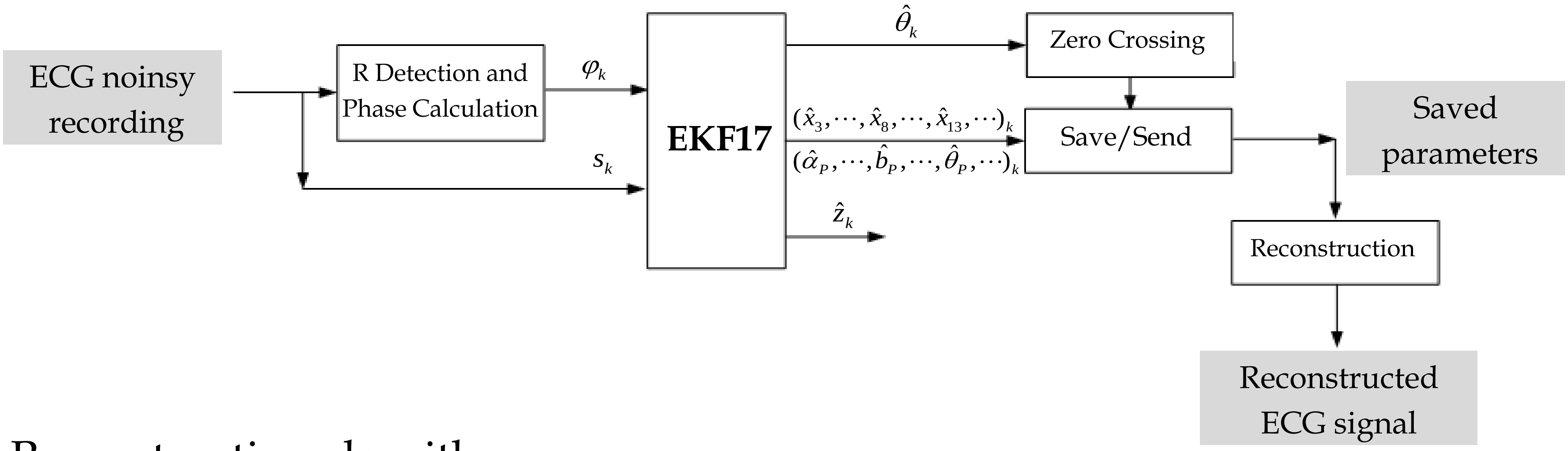
- Typical filtering results for an input signal of 5dB



O. Sayadi and M.B. Shamsollahi, "ECG Denoising and Compression Using a Modified Extended Kalman Filter Structure", IEEE Transactions on Biomedical Engineering, Vol. 55, No. 9, pp. 2240-2248, 2008.

Second Framework: EKF17

- Application 2: EKF17 based Compression



- Reconstruction algorithm

$$z_{rec} = \sum_{i=3:7} \hat{x}_i \exp(-\Delta\hat{\theta} / 2\hat{x}_{i+5}^2) \qquad , \quad \Delta\hat{\theta} = (\hat{x}_1 - \hat{x}_{i+10}) \bmod(2\pi)$$

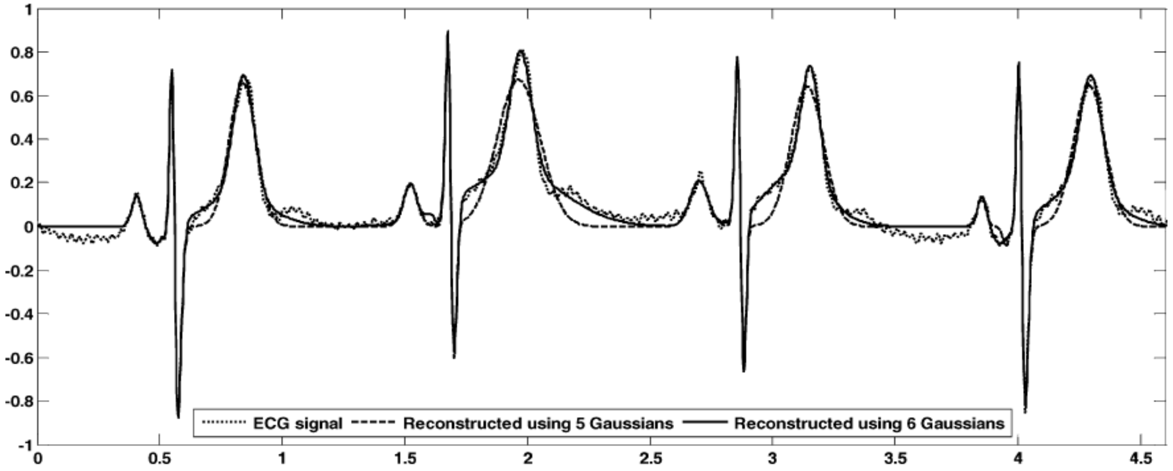
O. Sayadi and M.B. Shamsollahi, "ECG Denoising and Compression Using a Modified Extended Kalman Filter Structure", IEEE Transactions on Biomedical Engineering, Vol. 55, No. 9, pp. 2240-2248, 2008.

Second Framework: EKF17

- Compression performance

$$PAD\,(\%) = \frac{\left| \int_{t_i}^{t_f} x(t)dt - \int_{t_i}^{t_f} \hat{x}(t)dt \right|}{(t_f - t_i).(x_{\max} - x_{\min})} \times 100$$

Percentage Area Difference



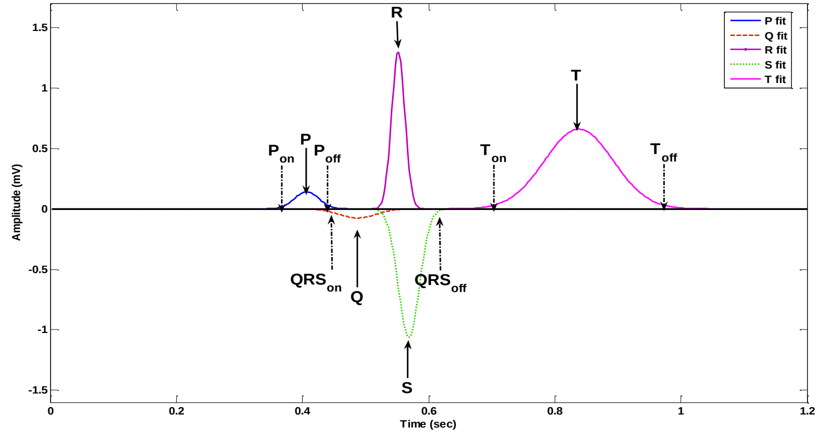
$$CR = \frac{n_{bc}}{n_{ac}} = \frac{L}{n_{param}} = \frac{L}{(n_{beat} \times (3n_{Gauss})) + n_R}$$

Algorithm		Compression Performance Measures		
		CR:1	PAD (%)	WDD (%)
EZW		6.85±1.06	4.86±2.18	5.12±3.09
SPIHT		9.24±0.75	1.42±0.85	1.35±1.00
EKF	5 Gaussians (EKF17)	13.65±2.92	3.90±1.59	4.53±1.06
	6 Gaussians (EKF20)	11.37±2.48	1.07±0.74	1.73±0.71
	7 Gaussians (EKF23)	9.75±2.13	0.65±0.43	1.54±0.55
	8 Gaussians (EKF26)	8.53±1.87	0.42±0.24	0.96±0.33

O. Sayadi and M.B. Shamsollahi, "ECG Denoising and Compression Using a Modified Extended Kalman Filter Structure", IEEE Transactions on Biomedical Engineering, Vol. 55, No. 9, pp. 2240-2248, 2008.

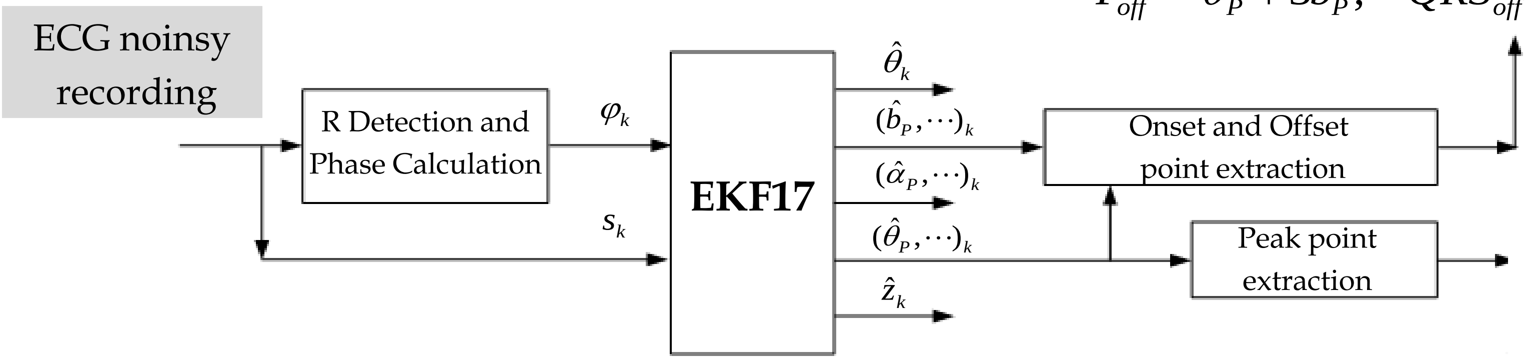
Second Framework: EKF17

- Application 3: EKF17 based Fiducial point extraction
  - Morphological features



  - Idea: using the Gaussian kernel parameters

$$P_{on} = \theta_P - 3b_P,\quad QRS_{on} = \theta_Q - 3b_Q,\quad T_{on} = \theta_T - 3b_T$$
$$P_{off} = \theta_P + 3b_P,\quad QRS_{off} = \theta_S + 3b_s,\quad T_{off} = \theta_T + 3b_T$$



$$P_{peak} = \theta_P$$
$$Q_{peak} = \theta_Q, R_{peak} = \theta_R, S_{peak} = \theta_S$$
$$T_{peak} = \theta_T$$

O. Sayadi and M.B. Shamsollahi, "A model-based Bayesian framework for ECG beat segmentation", Physiological Measurement, Vol. 30, pp. 335-352, 2009.

Second Framework: EKF17

- Application 3: EKF17 based Fiducial point extraction
  - Evaluation performance

		Label assigned by algorithm	
		Fiducial	Non Fiducial
Original label	Fiducial	TP	FN
	Non Fiducial	FP	TN

$$Sn = \frac{TP}{TP + FN}$$

$$Sp = \frac{TN}{TN + FP}$$

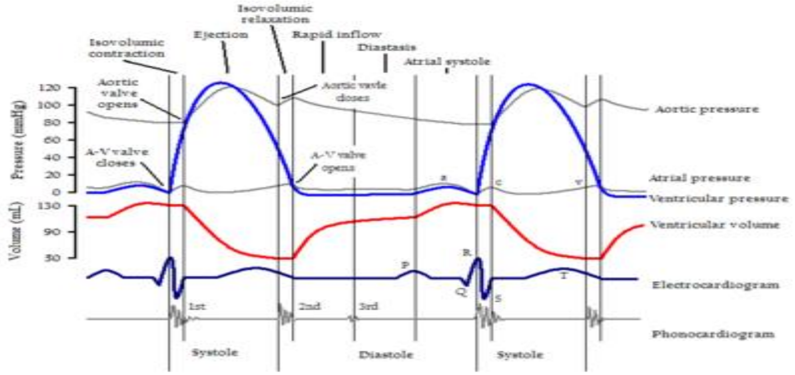
$$+P = \frac{TP}{TP + FP}$$

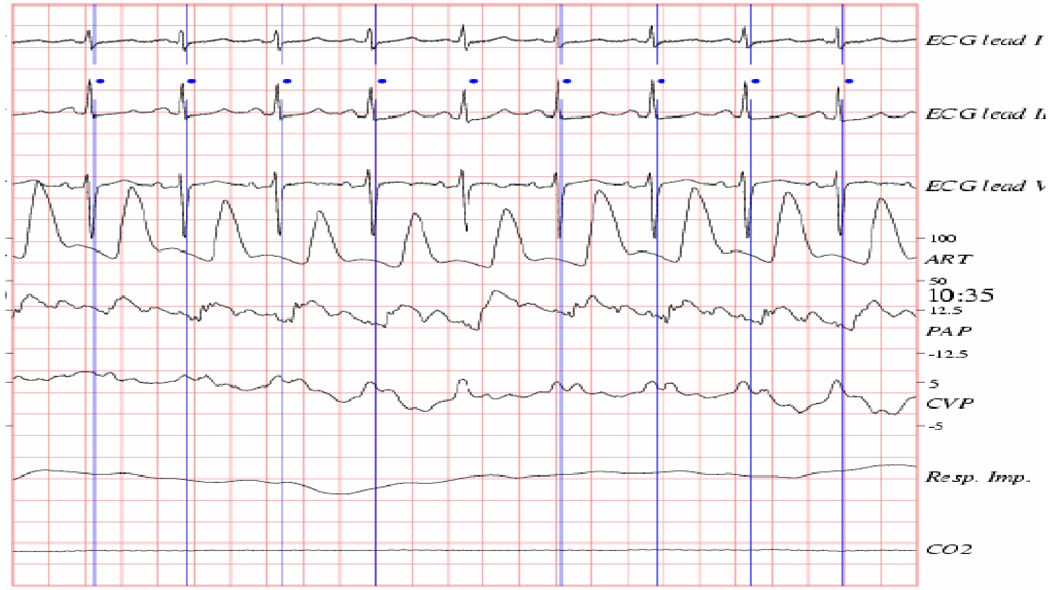
	Benchmark methods[a]			Proposed method		
Components	Sn (%)	Sp (%)	+P (%)	Sn (%)	Sp (%)	+P (%)
P wave	90.24	91.08	84.18	100	98.76	99.11
QRS complex	98.79	99.91	99.90	100	99.93	99.96
T wave	95.32	98.80	99.25	100	99.06	99.39

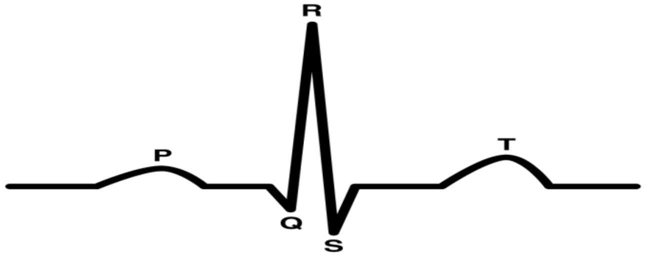
O. Sayadi and M.B. Shamsollahi, "A model-based Bayesian framework for ECG beat segmentation", Physiological Measurement, Vol. 30, pp. 335-352, 2009.

Cardiovascular Signals

- Cardiac Electro-Mechanical Activity
  - Electrocardiogram (ECG)
  - Phonocardiogram (PCG)
  - Arterial Blood Pressure (ABP)
  - Central Venous Pressure (CVP)
  - Pulmonary Artery Pressure (PAP)
  - Photoplethysmogram (PPG)
  - Pulse oximetry (POX)
- Electrocardiogram (ECG)
  - Recording the electrical activity of heart
  - Characteristic waveforms PQRST







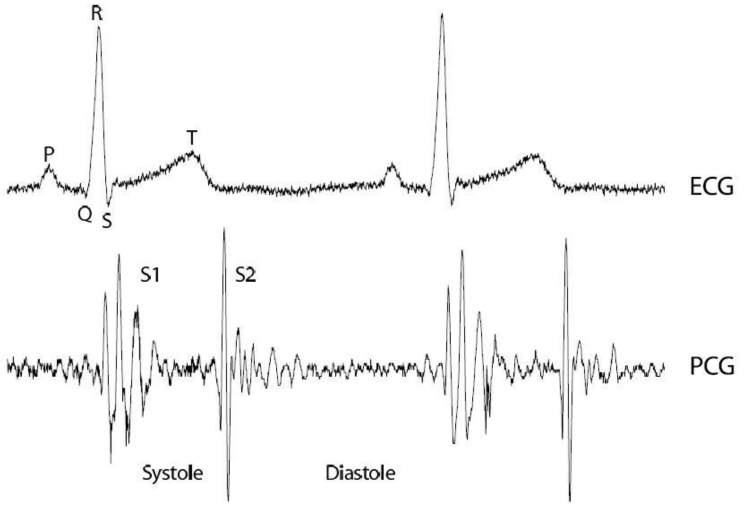
M. D. Cheitlin, M. Sokolow, and M. B. Mcllroy, Clinical Cardiology, Appleton & Lange, 1993.

Extension of McSharry Model to other CV signals

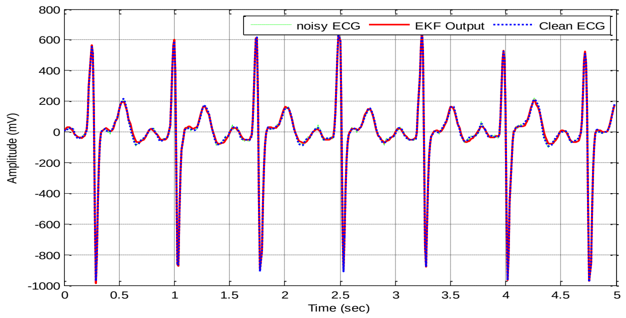
- Again McSharry's Model

$$\dot{x} = \alpha x - \omega y$$
$$\dot{y} = \alpha y + \omega x$$
$$\dot{z} = -\sum_{i \in \{P,Q,R,S,T\}} a_i \Delta\theta_i \exp\left(-\frac{\Delta\theta_i^2}{2b_i^2}\right) - (z - z_0)$$

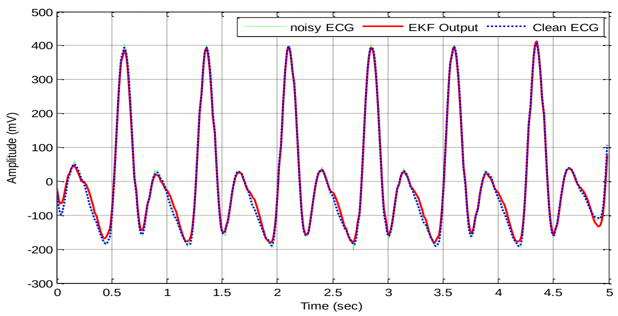
Applicable to some other CV signals, separately but not to PCG



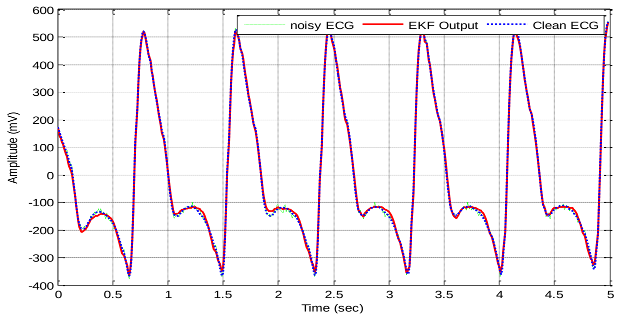
ECG



ABP



PPG



G. D. Clifford and P. E. McSharry, "Generating 24-Hour ECG, BP and Respiratory Signals with Realistic Linear and Nonlinear Clinical Characteristics Using a Nonlinear Model", Computers in Cardiology, vol. 31, pp. 709-712, 2004.