Convex Optimization II

Lecture 4: Convex Sets

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MOTIVATION

- The watershed between tractable and intractable problems is not linearity, but convexity.
- Only the very basic concepts and results in convex sets are covered without proofs.
- This lecture and the next two lectures on convex functions and problems are primarily mathematical, but a wide range of applications will soon follow.

References

- All materials and figures in this lecture are from [1].

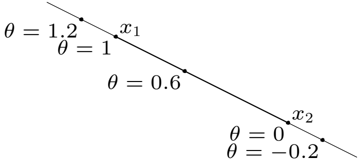
[1] S. Boyd and L. Vandenberghe, *Convex Optimization*, first edition, Cambridge University Press, 2004.

- Thanks to Prof. Vincent Wong and Prof. Stephen Boyd for all the slides used in this lecture.

Line and Affine Set

line through x_1, x_2: all points

$$x = \theta x_1 + (1-\theta)x_2 \qquad (\theta \in \mathbf{R})$$



affine set: contains the line through any two distinct points in the set

example: solution set of linear equations $\{x \mid Ax = b\}$

(conversely, every affine set can be expressed as solution set of system of linear equations)

Line Segment and Convex Set

line segment between x_1 and x_2: all points

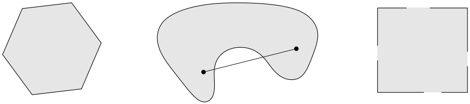
$$x = \theta x_1 + (1-\theta) x_2$$

with $0 \le \theta \le 1$

convex set: contains line segment between any two points in the set

$$x_1, x_2 \in C, \quad 0 \le \theta \le 1 \quad \Longrightarrow \quad \theta x_1 + (1-\theta) x_2 \in C$$

examples (one convex, two nonconvex sets)



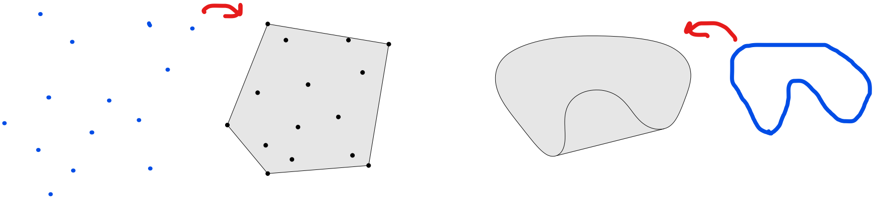
Convex Combination and Convex Hull

convex combination of x_1,. . . , x_k: any point x of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$$

with $\theta_1 + \cdots + \theta_k = 1$, $\theta_i \geq 0$

convex hull $\mathbf{conv}\, S$: set of all convex combinations of points in S



Cone and Convex Cone

- A set C is called a **cone**, if for every $x \in C$ and $\theta > 0$, we have $\theta x \in C$.

- A set C is a **convex cone** if it is convex and a cone, which means that for any $x_1, x_2 \in C$ and $\theta_1, \theta_2 \geq 0$, we have

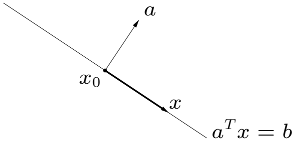
$$\theta_1 x_1 + \theta_2 x_2 \in C$$

- A point of the form $\theta_1 x_1 + \cdots + \theta_k x_k$ with $\theta_1, \ldots, \theta_k \geq 0$ is called a conic combination of $x_1, \ldots, x_k$.

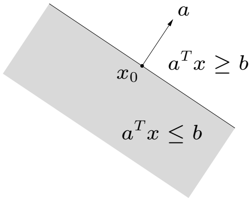
- A set C is a convex cone if and only if it contains all conic combinations of its elements.

Hyperplane and Halfspace

hyperplane: set of the form $\{x \mid a^T x = b\}$ $(a \neq 0)$



halfspace: set of the form $\{x \mid a^T x \leq b\}$ $(a \neq 0)$



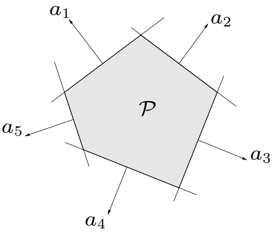
- a is the normal vector
- hyperplanes are affine and convex; halfspaces are convex

Polyhedra

solution set of finitely many linear inequalities and equalities

$$Ax \preceq b, \qquad Cx = d$$

$(A \in \mathbf{R}^{m\times n}$, $C \in \mathbf{R}^{p\times n}$, $\preceq$ is componentwise inequality)



polyhedron is intersection of finite number of halfspaces and hyperplanes

Euclidean Ball

- A Euclidean ball with center $x_c \in \mathbf{R}^n$ and radius $r > 0$

$$
\begin{aligned}
B(x_c, r) &= \{x \mid ||x - x_c||_2 \le r\} \\
&= \{x \mid (x - x_c)^T (x - x_c) \le r^2\} \\
&= \{x_c + ru \mid ||u||_2 \le 1\}.
\end{aligned}
$$

- $B(x_c, r)$ consists of all points within a distance r of the center x_c.

- A Euclidean ball is a convex set.

Norm Balls and Norm Cones

norm: a function $\|\cdot\|$ that satisfies

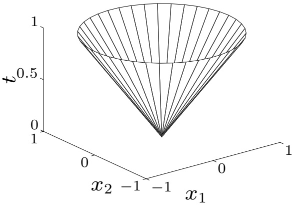
- $\|x\| \geq 0$; $\|x\| = 0$ if and only if $x = 0$
- $\|tx\| = |t|\,\|x\|$ for $t \in \mathbf{R}$
- $\|x + y\| \leq \|x\| + \|y\|$

notation: $\|\cdot\|$ is general (unspecified) norm; $\|\cdot\|_{\text{symb}}$ is particular norm

norm ball with center x_c and radius r: $\{x \mid \|x - x_c\| \leq r\}$

norm cone: $\{(x,t) \mid \|x\| \leq t\}$

Euclidean norm cone is called second-order cone



norm balls and cones are convex

How to Investigate Convexity of a Set

practical methods for establishing convexity of a set C

1. apply definition

$$x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \Longrightarrow \quad \theta x_1 + (1-\theta) x_2 \in C$$

2. show that C is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, . . .) by operations that preserve convexity
   - intersection
   - affine functions
   - perspective function
   - linear-fractional functions

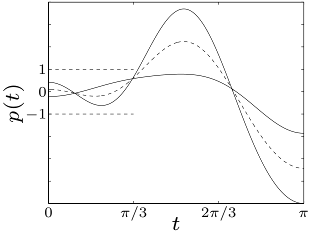
the intersection of (any number of) convex sets is convex

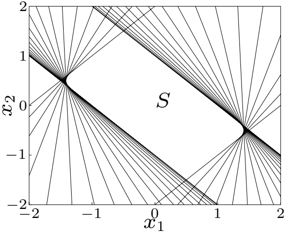
example:

$$S = \{x \in \mathbf{R}^m \mid |p(t)| \leq 1 \text{ for } |t| \leq \pi/3\}$$

where $p(t) = x_1 \cos t + x_2 \cos 2t + \cdots + x_m \cos mt$

for $m = 2$:





Operations that Preserve Convexity
Affine Function

suppose $f : \mathbf{R}^n \to \mathbf{R}^m$ is affine ($f(x) = Ax + b$ with $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$)

- the image of a convex set under f is convex

$$S \subseteq \mathbf{R}^n \text{ convex} \quad \Longrightarrow \quad f(S) = \{f(x) \mid x \in S\} \text{ convex}$$

- the inverse image $f^{-1}(C)$ of a convex set under f is convex

$$C \subseteq \mathbf{R}^m \text{ convex} \quad \Longrightarrow \quad f^{-1}(C) = \{x \in \mathbf{R}^n \mid f(x) \in C\} \text{ convex}$$

examples

- scaling, translation, projection
- solution set of linear matrix inequality $\{x \mid x_1 A_1 + \cdots + x_m A_m \preceq B\}$ (with $A_i, B \in \mathbf{S}^p$)
- hyperbolic cone $\{x \mid x^T P x \le (c^T x)^2,\ c^T x \ge 0\}$ (with $P \in \mathbf{S}^n_+$)

perspective function $P : \mathbf{R}^{n+1} \to \mathbf{R}^n$:

$$P(x,t) = x/t, \qquad \mathbf{dom}\, P = \{(x,t) \mid t > 0\}$$

images and inverse images of convex sets under perspective are convex

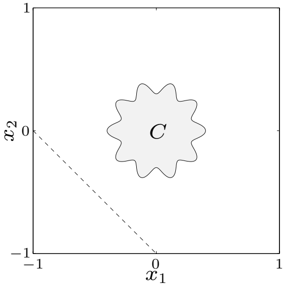
linear-fractional function $f : \mathbf{R}^n \to \mathbf{R}^m$:

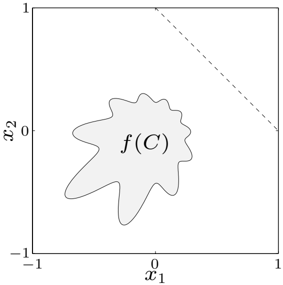
$$f(x) = \frac{Ax+b}{c^T x + d}, \qquad \mathbf{dom}\, f = \{x \mid c^T x + d > 0\}$$

images and inverse images of convex sets under linear-fractional functions are convex

Operations that Preserve Convexity
Linear-Fractional Function

$$f(x) = \frac{1}{x_1 + x_2 + 1} x$$

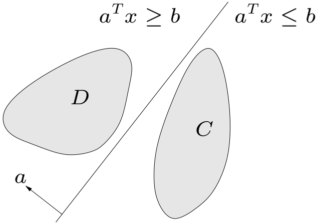




Separating Hyperplane Theorem

if C and D are disjoint convex sets, then there exists $a \neq 0$, b such that

$$a^T x \leq b \text{ for } x \in C, \qquad a^T x \geq b \text{ for } x \in D$$



the hyperplane $\{x \mid a^T x = b\}$ separates C and D

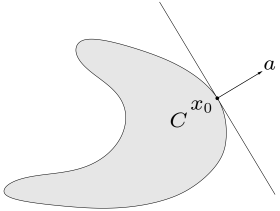
strict separation requires additional assumptions ($e.g.$, C is closed, D is a singleton)

Supporting Hyperplane Theorem

supporting hyperplane to set C at boundary point x_0:

$$\{x \mid a^T x = a^T x_0\}$$

where $a \neq 0$ and $a^T x \leq a^T x_0$ for all $x \in C$



supporting hyperplane theorem: if C is convex, then there exists a supporting hyperplane at every boundary point of C

SUMMARY

- Definition of line, line segment, affine set, convex set, convex combination, convex hull, and convex cone.
- Definition of hyperplane, halfspace, and polyhedron.
- Operations that preserve convexity.
- Separating hyperplane theorem and supporting hyperplane theorem.
- Convexity is the watershed between easy and hard optimization problems. Recognize convexity.
- Reading: Sections 2.1 - 2.3, and 2.5 in [1] by Boyd and Vandenberghe.