Convex Optimization II

Lecture 12: Routing Optimization

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Motivations

We would like to answer the following questions:

- How can we include routing in the generalized NUM?
- What are the choices for the objective functions in optimization-based routing?

References

[1] J. He, M. Bresler, M. Chiang, and J. Rexford, “Towards robust multi-layer traffic engineering: Optimization of congestion control and routing,” *IEEE Journal of Selected Areas in Communications*, vol. 25, no. 5, pp. 868-880, June 2007.

Acknowledgment

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Generalized Network Utility Maximization

- Recall from our previous lecture that the generalized NUM can be formulated as

$$\begin{array}{ll} \underset{\boldsymbol{x},\boldsymbol{w},R}{\text{maximize}} & \sum_{s\in\mathcal{S}} U_s(x_s) \\ \text{subject to} & R\boldsymbol{x} \preceq \boldsymbol{c}(\boldsymbol{w}), \\ & R \in \mathcal{R}, \quad \boldsymbol{w} \in \mathcal{W}. \end{array}$$

- Let us consider the case where $\boldsymbol{w}$ is fixed, but $\boldsymbol{x}$ and R are variables.

Single-Path vs. Multipath Routing

- In general, we may consider two types of routing models:
  - Single-path routing:

$$R_{ls} = \begin{cases} 1, & \text{if flow } s \text{ is routed through link } l, \\ 0, & \text{otherwise.} \end{cases}$$

  - Multipath routing:

    R_{ls} = portion of the traffic load of flow s which is routed through link l.

$$0 \leq R_{ls} \leq 1$$

- Optimization-based single-path routing $\Rightarrow$ Discrete optimization
- Optimization-based multipath routing $\Rightarrow$ Continuous optimization
- In this lecture, we will only consider multipath routing.

Flow Conservation Constraint

- The routing matrix R (either single-path or multipath) should satisfy the flow conservation constraint.
- For each node n, let $\mathcal{L}_{in}^{n}$ and $\mathcal{L}_{out}^{n}$ denote the set of incoming and outgoing links.
- For each flow $s \in \mathcal{S}$ and each node $n \in \mathcal{N}$, we have

$$\sum_{l \in \mathcal{L}_{in}^{n}} R_{ls} - \sum_{l \in \mathcal{L}_{out}^{n}} R_{ls} = \begin{cases} 1, & \text{if } n \text{ is the destination of flow } s, \\ -1, & \text{if } n \text{ is the source of flow } s, \\ 0, & \text{otherwise.} \end{cases}$$

Generalized Network Utility Maximization

- GNUM for joint congestion control and multipath routing can be formulated as

$$
\begin{aligned}
\underset{\boldsymbol{x}, R}{\text{maximize}} \quad & \sum_{s \in \mathcal{S}} U_s(x_s) \\
\text{s. t.} \quad & \sum_{s \in \mathcal{S}} R_{ls} x_s \le c_l, \quad \forall\, l \in \mathcal{L}, \\
& x_s \ge 0, \quad \forall\, s \in \mathcal{S}, \\
& R_{ls} \ge 0, \quad \forall\, s \in \mathcal{S},\ l \in \mathcal{L}, \\
& \sum_{l \in \mathcal{L}_{in}^n} R_{ls} - \sum_{l \in \mathcal{L}_{out}^n} R_{ls} = \begin{cases} 1, & \text{if } n \text{ is dest of } s, \\ -1, & \text{if } n \text{ is src of } s, \\ 0, & \text{otherwise,} \end{cases} \quad \forall n \in \mathcal{N},\ s \in \mathcal{S}.
\end{aligned} \tag{1}
$$

- Question: Is this problem a convex optimization problem?

- Question: Reformulate the problem for single-path routing.

Generalized Network Utility Maximization

- Problem (1) is not a convex optimization problem.
- Question: Do you have any suggestion to transform this problem?
- Next, we study two other alternatives:
  - Reformulating GNUM to end up with a convex optimization problem.
  - Formulating some problems (other than GNUM) which are tractable and work for a similar model as in problem (1) –> Network cost minimization (NCM).

New Constants and Variables

- For each flow $s \in \mathcal{S}$, let $\mathcal{J}_s$ denote the set of available paths for flow s.
- For each flow $s \in \mathcal{S}$ and path $j \in \mathcal{J}_s$, let $\mathcal{L}_s^j$ denote the set of links along path j.

  The multipath routing matrix is defined such that

  $$R_{j,s} = \text{portion of the traffic load of flow } s \text{ which is routed via path } j.$$

- The capacity constraints become

$$\sum_{s \in \mathcal{S}} \sum_{j \,:\, l \in \mathcal{L}_s^j} R_{j,s} x_s \leq c_l, \quad \forall l \in \mathcal{L}.$$

- It is also required that (Question: Why?)

$$\sum_{j \in \mathcal{J}_s} R_{j,s} = 1, \quad \forall s \in \mathcal{S}.$$

Generalized Network Utility Maximization

- GNUM for joint congestion control and multipath routing can be formulated as

$$
\begin{array}{lll}
\underset{\boldsymbol{x},\, R}{\text{maximize}} & \displaystyle\sum_{s \in \mathcal{S}} U_s(x_s) & \\
\text{subject to} & \displaystyle\sum_{s \in \mathcal{S}} \sum_{j\,:\, l \in \mathcal{L}_s^j} R_{j,s} x_s \leq c_l, & \forall\, l \in \mathcal{L}, \\
& \displaystyle\sum_{j \in \mathcal{J}_s} R_{j,s} = 1, & \forall\, s \in \mathcal{S}, \\
& x_s \geq 0, & \forall\, s \in \mathcal{S}, \\
& R_{j,s} \geq 0, & \forall\, s \in \mathcal{S},\ j \in \mathcal{J}_s.
\end{array}
\tag{2}
$$

- Question: What is the key difference between problem (2) and problem (1)? What do the rows and columns represent in R?

- Question: Is problem (2) a convex optimization problem?

Generalized Network Utility Maximization

- Let us introduce some new variables as

$$y_{j,s} = R_{j,s} x_s,$$

where $y_{j,s}$ denotes the data rate of source s over path j.

- Problem (2) can be reformulated as

$$\begin{aligned} \underset{\boldsymbol{y}}{\text{maximize}} \quad & \sum_{s \in \mathcal{S}} U_s \left(\sum_{j \in \mathcal{J}_s} y_{j,s} \right) \\ \text{subject to} \quad & \sum_{s \in \mathcal{S}} \sum_{j : l \in \mathcal{L}_s^j} y_{j,s} \leq c_l, \quad \forall\, l \in \mathcal{L}, \\ & y_{j,s} \geq 0, \qquad\qquad\qquad \forall\, j \in \mathcal{J}_s,\ s \in \mathcal{S}. \end{aligned} \tag{3}$$

- Question: What is the difference between problem (3) and problem (2)?

- Question: Is problem (3) a convex optimization problem?

Other Objectives

- Problems (2) and (3) are more tractable compared to problem (1). However, they require us to first find all possible paths before formulating the problem.

- Finding all possible paths can be difficult in practice, particularly in large-scale networks.

- Recall that the key difficulty in problem (1) is having product forms of R and x variables.

- Question: What if we only focus on routing, rather than joint routing and congestion control? What would be a good design objective in that case? Of course, we cannot think of NUM or GNUM anymore!

Other Objectives

- Assume that end-to-end traffic demand is known from each source to each destination. That is, assume that the traffic matrix X is given:

$$X_{s,d} = \text{traffic demand (data rate) from source node } s \text{ to destination node } d.$$

- Routing variables are defined as

$$R_l^{s,d} = \text{portion of traffic load of source } s \text{ to destination } d \text{ over link } l.$$

- In this case, the traffic load on each link $l \in \mathcal{L}$ becomes

$$\sum_{s,d \in \mathcal{N}} R_l^{s,d} X_{s,d}$$

- Question: Given the above, what would be a good objective function?

Other Objectives

- One design objective is to balance traffic in the network

$$
\begin{aligned}
\underset{R}{\text{minimize}} \quad & \max_{l \in \mathcal{L}} \left(\sum_{s,d \in \mathcal{N}} R_l^{s,d} X_{s,d} \right) / c_l \\
\text{s.t.} \quad & \sum_{s,d \in \mathcal{N}} R_l^{s,d} X_{s,d} \le c_l, \quad \forall l \in \mathcal{L}, \\
& \sum_{l \in \mathcal{L}_{in}^n} R_l^{s,d} - \sum_{l \in \mathcal{L}_{out}^n} R_l^{s,d} = \begin{cases} 1, & \text{if } n = s, \\ -1, & \text{if } n = d, \\ 0, & \text{otherwise}, \end{cases} \qquad \forall n, s, d \in \mathcal{N}, \\
& R_l^{s,d} \ge 0, \quad \forall n, s, d \in \mathcal{N}.
\end{aligned} \tag{4}
$$

- Question: Is problem (4) a convex optimization problem? Is it a linear program?

Other Objectives

- We can replace the objective and introduce a new class of routing problems

$$
\begin{aligned}
\underset{R}{\text{minimize}} \quad & \sum_{l\in\mathcal{L}} f_l\left(\sum_{s,d\in\mathcal{N}} R_l^{s,d} X_{s,d}/c_l\right) \\
\text{s.t.} \quad & \sum_{s,d\in\mathcal{N}} R_l^{s,d} X_{s,d} \le c_l, \quad \forall l \in \mathcal{L}, \\
& \sum_{l\in\mathcal{L}_{in}^n} R_l^{s,d} - \sum_{l\in\mathcal{L}_{out}^n} R_l^{s,d} = \begin{cases} 1, & \text{if } n = s, \\ -1, & \text{if } n = d, \\ 0, & \text{otherwise}, \end{cases} \qquad \forall n, s, d \in \mathcal{N}, \\
& R_l^{s,d} \ge 0, \quad \forall n, s, d \in \mathcal{N},
\end{aligned} \tag{5}
$$

where $f_l(\cdot)$ is a convex cost function.

- Problem (5) can be interpreted as network cost minimization (NCM) problem.
- Question: Is problem (5) a convex optimization problem?

NUM vs. NCM

- NUM (and GNUM) are usually seen as a design from network users' viewpoint.
  - Maximizing user utilities is what the users want.

- NCM is usually seen as a design from network managers' viewpoint.
  - Minimizing the cost of the network is what the network managers want.

- Question: How can we combine NCM and NUM to have a trade-off between network users and managers interest?

NUM vs. NCM

- For example, we may formulate the following problem:

$$
\begin{aligned}
\underset{\boldsymbol{y}}{\text{maximize}} \quad & \sum_{s\in\mathcal{S}} U_s\left(\sum_{j\in\mathcal{J}_s} y_{j,s}\right) - \omega \sum_{l\in\mathcal{L}} f_l\left(\sum_{s\in\mathcal{S}} \sum_{j:l\in\mathcal{L}_s^j} y_{j,s}/c_l\right) \\
\text{subject to} \quad & \sum_{s\in\mathcal{S}} \sum_{j:l\in\mathcal{L}_s^j} y_{j,s} \le c_l, \quad \forall l \in \mathcal{L}, \\
& y_{j,s} \ge 0, \quad \forall s \in \mathcal{S},\ j \in \mathcal{J}_s
\end{aligned} \tag{6}
$$

where ω is a tuning parameter.

- Question: Is problem (6) convex?

- Question: Why optimal solution of problem (6) can be a good design?

SUMMARY

- Routing problem can be considered as
  - Single-path routing
  - Multipath routing
- Flow conservation
- Generalized Network Utility Maximization (GNUM)
- Network Cost Minimization (NCM)
- Joint NUM and NCM