سری چهارم 97/10/16 امیررضا حاتمی پور

11

C: convex subset of $\mathbb{R}^n$

$f=(f_1,\dots,f_m)$ $\quad f_i: C\to\mathbb{R}$, $i=1,\dots,m$ are convex functions

$g:\mathbb{R}^m\to\mathbb{R}$ $\quad$ convex and monotonically nondecreasing

$u_1\le u_2 \rightsquigarrow g(u_1)\le g(u_2)$

$h(x)=g(f(x))$ is convex ?

$h(x)=g(f_1(x),\dots,f_m(x))$

$h'(x)=f'(x)\,\underbrace{\nabla g(f(x))}_{\geq 0}$ → چون در هر فرسیشن الزاما باید تولید g که پس ≥ 0

$$h''(x)=f'(x)^T\underbrace{\nabla^2 g(f(x))}_{\substack{\geq 0\\ g\text{ convex}}}f'(x)+\underbrace{\nabla g(f(x))^T}_{\substack{\geq 0\\ \text{non decreasing}}}\underbrace{f''(x)}_{\substack{\geq 0\\ f_i\text{ are convex}}}$$

لذا

$h''(x)\geq 0$

پس طبق second order : h هم convex میباشد

$m=1$, g is monotonically increasing, f is strictly convex

$m=1 \rightsquigarrow h(x)=g(f(x))$

$\rightsquigarrow h'(x)=f'(x)\,g'(f(x))$

$$\rightsquigarrow h''(x)=\underbrace{f''(x)}_{\substack{>0\\ \text{strictly convex}}}\underbrace{g'(f(x))}_{\substack{>0\\ \text{increasing}}}+\underbrace{(f'(x))^2}_{\substack{\geq 0\\ \text{nonnegative}}}\underbrace{g''(f(x))}_{\substack{\geq 0\\ \text{convex}}}$$

لذا $h''(x) = >0 + \geq 0$

$\Rightarrow h''(x)>0$

پس h هم strictly convex میباشد.

$f(x): \mathbb{R}^n \to \mathbb{R}$ convex

$h(x) = x^T w - h_0 \quad , h_0 = \sup_x x^T w - f(x) \quad , h$ affine

for any $x \in \mathbb{R}^n$: $\quad f(x) \geq h(x)$

$$h_0 = x^T w - h_0$$

$$= x^T w - (\sup_x x^T w - f(x))$$

$$= x^T w - \sup_x x^T w - \underbrace{\sup_x - f(x)}$$

$$\stackrel{\mathrm{I}}{=} \underbrace{x^T w - \sup_x x^T w}_{\leq 0} + \inf_x f(x) \qquad \mathrm{I}\ f(x)$$

$$\Rightarrow \begin{cases} \sup_x x^T w \geq x^T w \Rightarrow \sup_x x^T w - x^T w \geq 0 \quad \mathrm{I} \\ \inf_x f(x) \leq f(x) \end{cases}$$

$$\Rightarrow \quad \inf_x \leq f(x) \xrightarrow[\text{عدد منفی جمع میکنیم}]{\text{طرف اول را با I}} x^T w - \sup_x x^T w + \inf_x f(x) \leq f(x)$$

$$\Rightarrow h(x) \leq f(x)$$

$$f(x) = \frac{\|Ax-b\|_2^2}{1-x^Tx} \quad \text{on } \{x \mid \|x\|_2 < 1\}$$

3

$Ax - b \rightarrow$ affine function $\rightarrow$ convex

$\|Ax-b\|_2^2 \rightarrow$ Norm $\longrightarrow$ convex

$f(x) = \dfrac{\|Ax-b\|_2^2}{1-x^Tx}$ $\rightarrow$ linear-fractional $\rightarrow$ quasilinear

$\Rightarrow f(x) \longrightarrow$ convex ✓

$\lambda_1, \ldots, \lambda_n > 0 \quad , \quad f: \mathbb{R}^n \to \mathbb{R}$

$$f(x) = \prod_{i=1}^{n} (1-e^{-x_i})^{\lambda_i} \qquad \text{concave?}$$

$* \quad x \in \mathbb{R}^n_{++} \rightsquigarrow x_i \in \mathbb{R}_{++} \quad 0 < e^{-x_i} < 1 \rightsquigarrow 0 < 1-e^{-x_i} < 1$

$$\rightsquigarrow 0 \leq (1-e^{-x_i})^{\lambda_i} < 1 \Rightarrow 0 < f(x) < 1$$

$**$ فرض: $\sum_{i=1}^{n} \lambda_i e^{-x_i} \leq 1 \rightsquigarrow \sum_{i=1}^{n} (\lambda_i e^{-x_i})^2 \leq \sum_{i=1}^{n} \lambda_i e^{-x_i}$

$$f(x) = \prod_{i=1}^{n} (1-e^{-x_i})^{\lambda_i}$$

$$\rightsquigarrow \nabla f = \left[\lambda_1 e^{-x_1}(1-e^{-x_1})^{\lambda_1 - 1} \cdots (1-e^{-x_n})^{\lambda_n}, \ldots, \lambda_n e^{-x_n}(1-e^{-x_1})^{\lambda_1} \cdots (1-e^{-x_n})^{\lambda_n - 1}\right]$$

$$= \prod_{i=1}^{n} (1-e^{-x_i})^{\lambda_i} \left[\frac{\lambda_1 e^{-x_1}}{1-e^{-x_1}}, \ldots, \frac{\lambda_n e^{-x_n}}{1-e^{-x_n}}\right]$$

$$= \underbrace{f(x)}_{a} \underbrace{\left[\frac{\lambda_1 e^{-x_1}}{1-e^{-x_1}}, \ldots, \frac{\lambda_n e^{-x_n}}{1-e^{-x_n}}\right]}_{b}$$

$$\rightsquigarrow \nabla^2 f = \underbrace{f(x)}_{a} \operatorname{diag} \underbrace{\left[\frac{-\lambda_1 e^{-x_1}(1-e^{-x_1}) - \lambda_1 e^{-x_1}(+e^{-x_1})}{(1-e^{-x_1})^2}, \ldots, \frac{\sim}{(1-e^{-x_n})^2}\right]}_{b'}$$

$$+ \underbrace{f(x) \operatorname{diag}\left[\frac{\lambda_1 e^{-x_1}}{1-e^{-x_1}}, \ldots, \frac{\lambda_n e^{-x_n}}{1-e^{-x_n}}\right]}_{a'} \times \operatorname{diag} \underbrace{\left[\frac{\lambda_1 e^{-x_1}}{(1-e^{-x_1})}, \ldots, \frac{\lambda_n e^{-x_n}}{(1-e^{-x_n})}\right]}_{b}$$

$$f(x) \circ \operatorname{diag}\left[\frac{-\lambda_1 e^{-x_1} + (\lambda_1 e^{-x_1})^2}{\underbrace{(1-e^{-x_1})^2}_{>0}}, \ldots, \frac{-\lambda_n e^{-x_n} + (\lambda_n e^{-x_n})^2}{\underbrace{(1-e^{-x_n})^2}_{>0}}\right]$$

$$\Rightarrow v^T \nabla^2 f \, v = \underbrace{d_{(x_i)}}_{* > 0} \left(\sum_{i=1}^{n} \underbrace{\left(-\lambda_i e^{-x_i} + (\lambda_i e^{-x_i})^2\right)}_{** \leq 0} \underbrace{v_i^2}_{\geq 0} \right)$$

$$\leq 0 \rightsquigarrow \text{concave}$$

* از مخرج بردارها $(1-e^{-x_i})^2$ صرف نظر کردیم - چون همه‌ی اعداد مثبت و بین صفر و یک بودن.

$$u_\alpha(x) = \frac{x^\alpha - 1}{\alpha} \qquad 0 < \alpha \le 1$$

$u_0 = \text{?}$

$u_0(x) = \cancel{\text{lim}} \log x$

a) $u_0(x) = \lim_{\alpha \to 0} u_\alpha(x)$ for $x > 0$

$$u_0(x) = \lim_{\alpha \to 0} u_\alpha(x) = \lim_{\alpha \to 0} \frac{x^\alpha - 1}{\alpha} = \frac{0}{0}$$

$$\lim_{\alpha \to 0} \frac{x^\alpha - 1}{\alpha} = \lim_{\alpha \to 0} \frac{d(x^\alpha - 1)/d\alpha}{d(\alpha)/d\alpha} = \lim_{\alpha \to 0} \frac{\log x \times x^\alpha}{1}$$

$= \log x$ ✓

b) u_α concave

monotone increasing

$u_\alpha(1) = 0$

$0 < \alpha \le 1 \rightsquigarrow -1 \le \alpha - 1 \le 0$

$$u_\alpha(x) = \frac{x^\alpha - 1}{\alpha}$$

$$u'_\alpha(x) = \frac{\alpha x^{\alpha - 1}}{\alpha} = x^{\alpha - 1}$$

$$u''_\alpha(x) = \underbrace{(\alpha - 1)}_{* \le 0} \underbrace{x^{\alpha - 2}}_{> 0} \rightsquigarrow \text{concave} ✓$$

*

$$u'_\alpha(x) = x^{\alpha - 1} \xrightarrow[x > 0]{0 < \alpha \le 1} > 0 \rightsquigarrow \text{increasing} ✓$$

*

$$u_\alpha(1) = \frac{1^\alpha - 1}{\alpha} = \frac{1 - 1}{\alpha} = 0 \; ✓$$

a) $f(x) = -\log(-\log(\sum_{i=1}^{m} e^{a_i^T x + b_i}))$, $f = \{x \mid \sum_{i=1}^{m} e^{a_i^T x + b_i} < 1\}$

$\underline{x} \in \mathbb{R}^n$, $\log(\sum_{i=1}^{n} e^{y_i})$ is convex

$\log(\sum_{j=1}^{n} e^{x_j}) \longrightarrow$ convex ✓ $\xrightarrow{\text{affine}} \sum_{j=1}^{n} e^{a_j x_j + b_j} \rightarrow$ convex

$= e^{a^T \underline{x} + b} \xrightarrow{\text{composition}} \sum_{i=1}^{m} e^{a_i^T x + b_i} \rightarrow$ convex

$-\log(x)$ is convex

$\longrightarrow -\log(\sum_{i=1}^{m} e^{a_i^T x + b_i}) \rightarrow$ convex

$-\log(x)$ is convex

$\longrightarrow -\log(-\log(\sum_{i=1}^{m} e^{a_i^T x + b_i})) \rightarrow$ convex ✓

b) $f(x, u, v) = -\sqrt{uv - x^T x}$ $f = \{(x, u, v) \mid uv > x^T x, u, v > 0\}$

$= -\sqrt{u(v - x^T x / u)}$

$= -\sqrt{g(u, v, x)} = -\sqrt{g_1(u, v, x) g_2(u, v)}$

$\frac{x^T x}{u}$ is convex $(x, u), u > 0$ $\rightarrow -\sqrt{x_1 x_2}$ is convex on R^2_{++}

$-\frac{x^T x}{u} \rightarrow$ convex $\longrightarrow -\frac{x^T x}{u} \rightarrow$ concave $\rightarrow v - \frac{x^T x}{u} \rightarrow$ concave

$-u \rightarrow$ concave

$- h(x_1, x_2) = -\sqrt{x_1 x_2} \rightarrow$ convex

$\Rightarrow f(x, u, v) = h(g(u, v, x)) \longrightarrow$ convex

h: convex & decreasing

g: concave

Scanned by CamScanner

c) $f(x, u, v) = -\log(uv - x^T x)$

$$= -\log\left(u\left(v - \frac{x^T x}{u}\right)\right)$$

$$= \underbrace{-\log u}_{\text{convex}} + \log\Big(v - \underbrace{\frac{x^T x}{u}}_{\text{concave}}\Big)$$

concave $\rightarrow$ from previous part

$f(x) = h(g(x))$

$$\Rightarrow \begin{cases} h(x) = -\log x \rightarrow \text{convex, decreasing} \\ g(x) = v - \frac{x^T x}{u} \rightarrow \text{concave} \end{cases} \Big\} \; f \text{ is convex}$$

$$\Rightarrow \; = \underbrace{-\log u}_{\text{convex}} \underbrace{- \log\left(v - \frac{x^T x}{u}\right)}_{\text{convex}}$$

جمع دوتا convex ، convex ن

$\Longrightarrow$ f - convex ✓

Scanned by CamScanner

17

minimize $f(x)$
$x \in \mathbb{R}^n$
s.t $h(x) = 0$

a) $f(x) = \|x\|^2$, $h(x) = \sum_{i=1}^{n} x_i - 1$

$\Rightarrow$ minimize $\|x\|^2$
s.t $\sum_{i=1}^{n} x_i - 1 = 0$

- $f_i(x^*) \leq 0 \rightarrow \times$
- $h_i(x^*) = 0 \Rightarrow \sum_{i=1}^{n} x_i - 1 = 0$
- $\lambda_i^* \geq 0$
- $\lambda_i^* f_i(x^*) = 0 \Rightarrow \times$
- $\nabla L = 0 \rightsquigarrow L = \|x\|^2 + \nu \left(\sum_{i=1}^{n} x_i - 1 \right)$

$\Rightarrow \nabla L = \nabla(x^T x) + \nabla\left(\nu\left(\sum_{i=1}^{n} x_i - 1\right)\right)$

$= 2x + \nu[1, \ldots, 1]$

$= [2x_1 + \nu, \ldots, 2x_n + \nu]$

$$\Rightarrow \begin{cases} \sum_{i=1}^{n} x_i - 1 = 0 \\ \nabla L = 0 \Rightarrow x_i = -\frac{\nu}{2} \end{cases} \Rightarrow n \times \left(-\frac{\nu}{2}\right) - 1 = 0 \Rightarrow \boxed{\nu = -\frac{2}{n}}$$

So, $x_i = \frac{1}{n}$

$\Rightarrow x = \begin{bmatrix} \frac{1}{n} \\ \vdots \\ \frac{1}{n} \end{bmatrix}$

b) $f(x) = \sum_{i=1}^{n} x_i - 1$, $h(x) = \|x\|^2 - 1$

$L = \sum_{i=1}^{n} x_i - 1 + v(\|x\|^2 - 1) \Rightarrow \nabla L = [1, \ldots, 1] + 2vx$

$\rightarrow \begin{cases} \|x\|^2 - 1 = 0 \\ \forall i: 2vx_i + 1 = 0 \Rightarrow x_i = -\frac{1}{2v} \end{cases}$

$\Rightarrow n \times (-\frac{1}{2v})^2 - 1 = 0$

$\Rightarrow \frac{n}{4v^2} = 1$

$\Rightarrow v = \pm\sqrt{\frac{n}{4}}$

$v \rightarrow v = \sqrt{\frac{n}{4}} \rightarrow x = -\frac{1}{\sqrt{n}} \rightarrow ✓$

$v \rightarrow v = -\sqrt{\frac{n}{4}} \rightarrow x = \frac{1}{\sqrt{n}} \rightarrow \times$

minimize $\|z-x\|^2$ $\quad A \in \mathbb{R}^{m\times n}$ 18

s.t $\quad Ax=0$ $\quad z \in \mathbb{R}^n$

$\rightarrow$ Lagrangian:

$$L(x,\lambda,v) = \|z-x\|^2 + v^T A x$$

$\rightarrow$ dual func

$$g(\lambda,v) = \inf_{x \in D} L(x,\lambda,v)$$

$\rightarrow$ dual problem

maximize $g(\lambda,v)$

s.t $\quad \lambda \geq 0$

dual func

$$\Rightarrow g(v) = \inf L(x,v)$$

$$\Rightarrow \frac{\partial L}{\partial x_i} = 0 \Rightarrow -2(z-x) + A^T v = 0$$

$$\Rightarrow \boxed{x = z - \frac{1}{2}A^T v}$$

$$g(v) = \|z - z + \frac{1}{2}A^T v\|^2 + v^T A(z - \frac{1}{2}A^T v)$$

$$= \frac{1}{4}\|A^T v\|^2 + v^T A z - \frac{1}{2}\|A^T v\|^2$$

dual prob

$$= v^T A z - \frac{1}{4}\|A^T v\|^2$$

$\Rightarrow$ maximize $v^T A z - \frac{1}{4}\|A^T v\|^2 = v^T A z - \frac{1}{4}\|A^T v\|^2 + z^T z - z^T z$

$$= \underbrace{z^T z}_{\text{ثابت}} - \|\frac{1}{2}A^T v - z\|_2^2$$

دوگان $\Rightarrow \boxed{\text{minimize } \|\frac{1}{2}A^T v - z\|_2^2}$

12

Relative entropy: $x, y \in \mathbb{R}_{++}$, $A \in \mathbb{R}^{m\times n}_{++}$, $b \in \mathbb{R}^{m}_{++}$

$$\sum_{k=1}^{n} x_k \log(x_k / y_k)$$

minimize $\sum_{k=1}^{n} x_k \log(x_k / y_k)$

s.t. $Ax = b$

$1^T x = 1$

a) $v_1 \in \mathbb{R}^m, v_2 \in \mathbb{R}$

$$L(x, v_1, v_2) = \sum_{k=1}^{n} x_k \log(x_k / y_k) + v_1^T (Ax - b) + v_2 (1^T x - 1)$$

$\Rightarrow$ dual func:

$$g(v_1, v_2) = \inf_{x \in D} L(x, v_1, v_2)$$

$$= \inf_{x \in D} \sum_{k=1}^{n} x_k \log(x_k / y_k) + v_1^T (Ax - b) + v_2 (1^T x - 1)$$

$$\Rightarrow \frac{\partial L}{\partial x_k} = 0 \Rightarrow \log(x_k / y_k) + x_k \times \frac{1/y_k}{x_k / y_k} + v_1^T a_k + v_2 = 0$$

$$\Rightarrow x_k = y_k \exp(-1 - v_1^T a_k - v_2)$$

$$g(v_1, v_2) = \sum_{k=1}^{n} \Big[\underbrace{y_k \exp(-1 - v_1^T a_k - v_2)}_{x_k} (-1 - v_1^T a_k - v_2) \Big]$$

$$+ v_1^T A x - v_1^T b + v_2 1^T x - v_2$$

$$= \sum_{k=1}^{n} - x_k - \underbrace{\sum_{k=1}^{n} \cancel{x_k v_1^T a_k}}_{= v_1^T A x} - \underbrace{\sum_{k=1}^{n} x_k v_2}_{= v_2 1^T x}$$

$$+ \cancel{v_1^T A x} - v_1^T b + \cancel{v_2 1^T x} - v_2$$

$$= -v_1^T b - v_2 + \sum_{k=1}^{n} y_k \exp(-1 - v_1^T a_k - v_2)$$

$\Rightarrow$ dual prob: $\underset{\nu_1, \nu_2}{\text{maximize}} \quad g(\nu_1, \nu_2)$

$$= \underset{\nu_1, \nu_2}{\text{maximize}} \quad -\nu_1^T b - \nu_2 - \sum_{k=1}^{n} y_k \exp(-1 - \nu_1^T a_k - \nu_2)$$

$\boxed{b}$ برای سادگی، نسبت به ν_2 مسئله را حل می‌کنیم.

$$\frac{\partial g}{\partial \nu_2} = 0 \Rightarrow \quad -1 + \sum_{k=1}^{n} y_k \exp(-1 - \nu_1^T a_k - \nu_2) = 0$$

$$\Rightarrow \quad \nu_2 = 1 - \log \sum_{k=1}^{n} y_k \exp(a_k^T \nu_1$$

$$\Rightarrow \text{maximize} \quad -\nu_1^T b - \log \sum_{k=1}^{n} y_k \exp(a_k^T \nu_1)$$