

Convex Optimization II

Lecture 11: Generalized Network Utility Maximization

Hamed Shah-Mansouri

Department of Electrical Engineering
Sharif University of Technology

1400-2

REFERENCES

- [1] M. Chiang, S. H. Low, A. R. Calderbank, and J. C. Doyle, “Layering as optimization decomposition: A mathematical theory of network architectures,” *Proceedings of the IEEE*, vol. 95, no. 1, pp. 255-312, January 2007.
- [2] M. Chiang, “Balancing transport and physical layers in wireless multihop networks: Jointly optimal congestion control and power control,” *IEEE Journal of Selected Areas in Communications*, vol. 23, no. 1, pp. 104-116, January 2005.

Acknowledgment

Thanks to Prof. Vincent Wong for the slides.

MOTIVATIONS

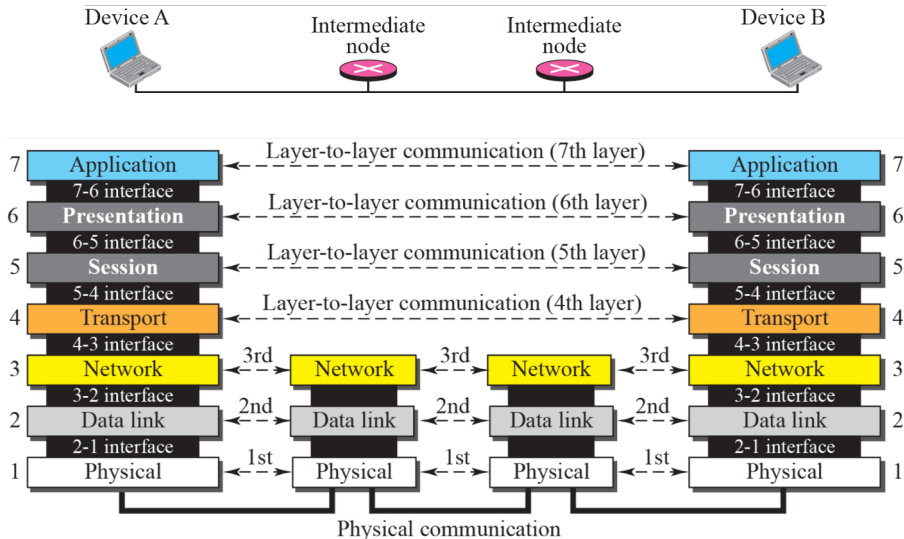
- Recall the network utility maximization problem for congestion control

$$\begin{aligned} & \underset{\mathbf{x}}{\text{maximize}} && \sum_{s \in \mathcal{S}} U_s(x_s) \\ & \text{subject to} && R\mathbf{x} \preceq \mathbf{c} \end{aligned} \tag{1}$$

where R is the **fixed** routing matrix, \mathbf{c} is the **fixed** link capacity vector, and \mathbf{x} is the **variable** vector of data rates.

- Problem (1) formulates the protocol stack only at the transport layer.
- Question:** Is it possible to extend the basic network utility maximization problem to design a layered architecture?

NETWORK OSI MODEL



[1] BA. Forouzan, *TCP/IP Protocol Suite*. McGraw-Hill Inc., 4th edition, Jun, 2002.

LAYERING AS OPTIMIZATION DECOMPOSITION [1]

- Network architecture determines functionality allocation: *Who does what, and how to connect them*

Systematic Approach

- **Network:** Generalized network utility maximization
- **Layering architecture:** Vertical decomposition
- **Layer:** Decomposed subproblems
- **Interfaces:** Functions of primal or dual variables
- **Distributed computation** and **control** of a **functionality module** over geographically disparate network elements: **Horizontal decomposition**

GENERALIZED NETWORK UTILITY MAXIMIZATION

- Generalized NUM (GNUM) can be formulated as [1]:

$$\begin{aligned} & \underset{\mathbf{x}, \mathbf{w}, R}{\text{maximize}} && \sum_{s \in \mathcal{S}} U_s(x_s) \\ & \text{subject to} && R\mathbf{x} \leq \mathbf{c}(\mathbf{w}), \\ & && R \in \mathcal{R}, \quad \mathbf{w} \in \mathcal{W} \end{aligned}$$

where

- \mathbf{x} is vector of data rates (**variable**)
- \mathbf{w} is vector of network (**lower layer**) resources (**variable**)
- R is routing matrix (**variable**)
- \mathcal{W} is feasible set for network resources
- \mathcal{R} is feasible set for routing matrices

GENERALIZED NETWORK UTILITY MAXIMIZATION

- For the choices of routing matrix, we distinguish two different routing models:
 - ▶ Single-path routing
 - ▶ Multipath routing
- **Question:** What are the differences between the two cases above in terms of the corresponding feasible sets?
- We will see a lot about the routing problem in the next lecture.

CROSS-LAYER DESIGN

From cross-layer design point of view, x , w , and R are usually associated to different network layers.

- Data rate x :
 - ▶ Transport Layer
- Routing matrix R :
 - ▶ Network Layer
- Network resources w :
 - ▶ Data Link or MAC Layer
 - ▶ Physical Layer
- **Question:** Does GNUM provide a framework for cross-layer design?

CASE STUDY: JOINT CONGESTION CONTROL & POWER CONTROL [2]

- Assume that \mathbf{w} denotes the vector of **transmission power** in a wireless network.
- In particular, we replace \mathbf{w} by vector $\mathbf{p} = (p_l, \forall l \in \mathcal{L})$, where p_l denotes the transmission power of link l .
- Assume that routing paths are fixed. That is, the matrix R is given.
- We would like to solve the following GNUM [2]:

$$\begin{aligned} & \underset{\mathbf{x} \succeq 0, \mathbf{p} \succeq 0}{\text{maximize}} && \sum_{s \in \mathcal{S}} U_s(x_s) \\ & \text{subject to} && \sum_{s : l \in \mathcal{L}_s} x_s \leq c_l(\mathbf{p}), \quad \forall l \in \mathcal{L}. \end{aligned} \tag{2}$$

- **Question:** What is set \mathcal{W} (or \mathcal{P}) in problem (2)?

CAPACITY MODEL

- For each link $l \in \mathcal{L}$, the link capacity can be obtained as

$$\begin{aligned}c_l(\mathbf{p}) &= B \log(1 + K \text{ SINR}_l(\mathbf{p})) \\ &\approx B \log(K \text{ SINR}_l(\mathbf{p}))\end{aligned}$$

where

SINR_l : signal to interference plus noise ratio for link l

K : a constant, depending on the modulation and required bit-error rate

B : channel bandwidth, assumed to be one unit (i.e., $B = 1$)

- Here, we assume that $K \text{ SINR}_l$ is much larger than one for all links.

SINR MODEL

- For each link $l \in \mathcal{L}$, we have

$$\text{SINR}_l(\mathbf{p}) = \frac{p_l G_{ll}}{\sum_{k \in \mathcal{L}, k \neq l} p_k G_{lk} + n_l}$$

where

p_l : transmission power of link l

G_{lk} : path loss from the **transmitter** of link k to the **receiver** of link l

n_l : noise power at the receiver of link l

PROBLEM FORMULATION

- Joint congestion control and power control problem can be formulated as

$$\begin{aligned} & \underset{\mathbf{x} \succeq 0, \mathbf{p} \succeq 0}{\text{maximize}} && \sum_{s \in \mathcal{S}} U_s(x_s) \\ & \text{subject to} && \sum_{s : l \in \mathcal{L}_s} x_s \leq \log \left(\frac{p_l K G_{ll}}{\sum_{k \in \mathcal{L}, k \neq l} p_k G_{lk} + n_l} \right), \quad \forall l \in \mathcal{L}. \end{aligned}$$

- Question:** Is this problem convex?
- Question:** Is this problem separable?

IS THE PROBLEM CONVEX?

- Let us rewrite the constraint as

$$\sum_{s: l \in \mathcal{L}_s} x_s \leq \log(p_l K G_{ll}) - \log\left(\sum_{k \in \mathcal{L}, k \neq l} p_k G_{lk} + n_l\right), \quad \forall l \in \mathcal{L}.$$

- After reordering the terms, we have

$$\sum_{s: l \in \mathcal{L}_s} x_s - \log(p_l K G_{ll}) + \log\left(\sum_{k \in \mathcal{L}, k \neq l} p_k G_{lk} + n_l\right) \leq 0, \quad \forall l \in \mathcal{L}.$$

Handwritten notes in blue: "خطی" (linear) under x_s , "عذب" (convex) under $-\log(p_l K G_{ll})$, and "مغفر" (concave) under $\log(\dots)$.

- Question:** Is this set of constraints convex over x and p ?

CHANGE OF VARIABLES

- **Question:** How to solve the joint congestion control and power control problem?
In particular, how to solve it in a distributed manner?

- Let us define some new variables

$$\tilde{p}_l = \log p_l, \quad \forall l \in \mathcal{L}.$$

- In that case, we have

$$p_l = \exp \tilde{p}_l = e^{\tilde{p}_l}, \quad \forall l \in \mathcal{L}.$$

- **Question:** In case we can solve the problem with respect to $\tilde{\mathbf{p}}$, then would it be possible to **recover** optimal \mathbf{p} from optimal $\tilde{\mathbf{p}}$? Why?

CHANGE OF VARIABLES

- Let us rewrite the constraint for the new choice of variables

$$\sum_{s: l \in \mathcal{L}_s} x_s - \log(e^{\tilde{p}_l} K G_{ll}) + \log\left(\sum_{k \in \mathcal{L}, k \neq l} e^{\tilde{p}_k} G_{lk} + n_l\right) \leq 0, \quad \forall l \in \mathcal{L}.$$

- After reordering the terms, the constraints become

ثابت متغیر متغیر

$$\sum_{s: l \in \mathcal{L}_s} x_s - \tilde{p}_l - \log(K G_{ll}) + \log\left(\sum_{k \in \mathcal{L}, k \neq l} e^{\tilde{p}_k} G_{lk} + n_l\right) \leq 0, \quad \forall l \in \mathcal{L}.$$

- Question:** Are these new constraints convex in \mathbf{x} and $\tilde{\mathbf{p}}$?

MODIFIED PROBLEM FORMULATION

- Joint congestion control and power control problem can be reformulated as

$$\begin{aligned} & \underset{x \succeq 0, \tilde{\mathbf{p}}}{\text{maximize}} && \sum_{s \in \mathcal{S}} U_s(x_s) \\ & \text{subject to} && \sum_{s : l \in \mathcal{L}_s} x_s - \tilde{p}_l - \log(KG_{ll}) + \log \left(\sum_{k \in \mathcal{L}, k \neq l} e^{\tilde{p}_k} G_{lk} + n_l \right) \leq 0, \forall l \in \mathcal{L}. \end{aligned}$$

- Question:** Is this problem convex?
- Question:** Is this problem separable?

DUAL FUNCTION

- We can write the Lagrangian as x_s

$$L(\mathbf{x}, \tilde{\mathbf{p}}, \boldsymbol{\lambda}) = \sum_{s \in \mathcal{S}} U_s(x_s) - \sum_{l \in \mathcal{L}} \lambda_l \sum_{s : l \in \mathcal{L}_s} x_s + \sum_{l \in \mathcal{L}} \lambda_l \left(\tilde{p}_l + \log(K G_{ll}) - \log \left(\sum_{k \in \mathcal{L}, k \neq l} e^{\tilde{p}_k} G_{lk} + n_l \right) \right)$$

- The dual function is obtained as p_l

$$D(\boldsymbol{\lambda}) = \sup_{\mathbf{x} \succeq 0, \tilde{\mathbf{p}}} L(\mathbf{x}, \tilde{\mathbf{p}}, \boldsymbol{\lambda})$$

- **Question:** Is the above maximization separable?

DUAL DECOMPOSITION

- We can write

$$\begin{aligned} D(\boldsymbol{\lambda}) = & \underset{\mathbf{x} \succeq 0}{\text{maximize}} \sum_{s \in \mathcal{S}} \left(U_s(x_s) - \sum_{l \in \mathcal{L}_s} \lambda_l x_s \right) \\ & + \underset{\tilde{\mathbf{p}}}{\text{maximize}} \sum_{l \in \mathcal{L}} \lambda_l \left(\tilde{p}_l + \log(K G_{ll}) - \log \left(\sum_{k \in \mathcal{L}, k \neq l} e^{\tilde{p}_k} G_{lk} + n_l \right) \right) \end{aligned}$$

- In fact, the maximization problem is **decomposed** into two subproblems.

CONGESTION CONTROL SUBPROBLEM

- The congestion control **subproblem** becomes

$$\underset{x \succeq 0}{\text{maximize}} \sum_{s \in \mathcal{S}} \left(U_s(x_s) - \sum_{l \in \mathcal{L}_s} \lambda_l x_s \right)$$

- **Question:** Is this problem different from the problem we saw in TCP forward engineering? How can we solve this problem?
- Also, notice that

$$\underset{x \succeq 0}{\text{maximize}} \sum_{s \in \mathcal{S}} \left(U_s(x_s) - \sum_{l \in \mathcal{L}_s} \lambda_l x_s \right) = \sum_{s \in \mathcal{S}} \underset{x_s \geq 0}{\text{maximize}} \left(U_s(x_s) - \sum_{l \in \mathcal{L}_s} \lambda_l x_s \right)$$

POWER CONTROL SUBPROBLEM

- The power control **subproblem** becomes

$$\underset{\tilde{\mathbf{p}}}{\text{maximize}} \sum_{l \in \mathcal{L}} \lambda_l \left(\tilde{p}_l + \log(KG_{ll}) - \log \left(\sum_{k \in \mathcal{L}, k \neq l} e^{\tilde{p}_k} G_{lk} + n_l \right) \right)$$

- **Question:** How do we solve the above problem?
- To answer this question, let us replace $\tilde{p}_l = \log p_l$. The power control subproblem becomes

$$\underset{\mathbf{p}}{\text{maximize}} \sum_{l \in \mathcal{L}} \lambda_l c_l(\mathbf{p}).$$

- **Question:** Which links do you think should assign a higher capacity?

POWER CONTROL SUBPROBLEM

- **Question:** How can we solve the power control subproblem?
- We can use the **gradient projection method**.
- In particular, we can update the transmission power of link $l \in \mathcal{L}$ as

$$p_l(t+1) = \left[p_l(t) + \kappa \left(\frac{\lambda_l}{p_l(t)} - \sum_{j \in \mathcal{L}, j \neq l} \frac{\lambda_j G_{jl}}{\sum_{k \in \mathcal{L}, k \neq j} G_{jk} p_k + n_j} \right) \right]^+$$

where $\kappa > 0$ is a fixed stepsize. The above is equivalent to

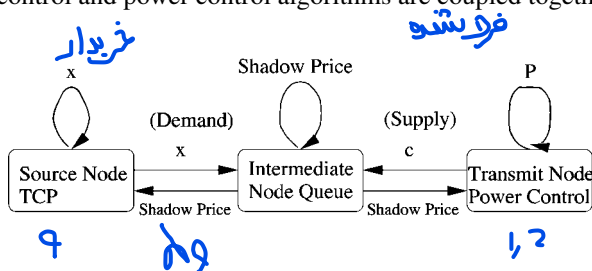
$$p_l(t+1) = \left[p_l(t) + \frac{\kappa \lambda_l}{p_l(t)} - \kappa \sum_{j \in \mathcal{L} \setminus \{l\}} G_{jl} m_j(t) \right]^+$$

where $m_j(t)$ are **messages passed** from link j :

$$m_j(t) = \frac{\lambda_j \text{SINR}_j(t)}{p_j(t) G_{jj}}.$$

COUPLING BETWEEN CONGESTION AND POWER CONTROL

- Congestion control and power control algorithms are coupled together as [2]:



SUMMARY

- Network utility maximization problem can be extended to include various other network design problems in various other layers of the network protocol stack.
- Joint congestion control and power control problem
- Congestion control subproblem
- Power control subproblem
- Layering as optimization decomposition