

# EE 25088 Convex Optimization II

## Problem Set III

**Due Date: Khordad 10, 1401**

- 1) Derive the Lagrange dual of the optimization problem

$$\begin{aligned} & \underset{x \in \mathbf{R}^n}{\text{minimize}} && \sum_{i=1}^n \phi(x_i) \\ & \text{subject to} && A\mathbf{x} = \mathbf{b}, \end{aligned}$$

where  $\phi(x) = \frac{|x|}{c-|x|}$  and  $\text{dom}\phi = (-c, c)$ , while  $c$  is a positive parameter.

- 2) *Step sizes that guarantee moving closer to the optimal set.* Consider the subgradient method iteration  $x_{k+1} = x_k - \alpha g$ , where  $g \in \partial f(x)$ . Show that if  $\alpha < 2(f(x) - f^*)/\|g\|_2^2$ , we have

$$\|x_{k+1} - x^*\|_2 < \|x_k - x^*\|_2, \quad (2)$$

for any optimal point  $x^*$ .

- 3) *Numerical perturbation analysis.* Consider the following problem

$$\begin{aligned} & \text{minimize} && x_1^2 + 2x_2^2 - x_1x_2 - x_1 \\ & \text{subject to} && x_1 + 2x_2 \leq u_1 \\ & && x_1 - 4x_2 \leq u_2 \\ & && 5x_1 + 76x_2 \leq 1 \end{aligned}$$

with variables  $x_1, x_2$ , and parameters  $u_1, u_2$ .

- a) Solve this problem, for parameter values  $u_1 = -2, u_2 = -3$ , to find optimal primal variable values  $x_1^*$  and  $x_2^*$ , and optimal dual variable values  $\lambda_1^*, \lambda_2^*$ , and  $\lambda_3^*$ . Let  $p^*$  denote the optimal objective value. Verify that the KKT conditions hold for the optimal primal and dual variables you found. (You need to find the answers via CVX and also verify your results using KKT conditions).

*Hint:* To specify the quadratic-form functions, you can use `quad_form()`

$\delta_1$	$\delta_2$	$p_{\text{pred}}^*$	$p_{\text{exact}}^*$
0	0		
0	-0.1		
0	0.1		
-0.1	0		
-0.1	-0.1		
-0.1	0.1		
0.1	0		
0.1	-0.1		
0.1	0.1		

Table I: Sample Table for problem 3

- ✓ b) We will now solve some perturbed versions of this problem, with

$$u_1 = -2 + \delta_1, u_2 = -3 + \delta_2$$

where  $\delta_1$  and  $\delta_2$  each take values from  $\{-0.1, 0, 0.1\}$ . (There are a total of nine such combinations, including the original problem with  $\delta_1 = \delta_2 = 0$ .) For each combination of  $\delta_1$  and  $\delta_2$ , make a prediction  $p_{\text{pred}}^*$  of the optimal value of the perturbed  $QP$ , and compare it to  $p^*$  exact, the exact optimal value of the perturbed problem. Put your results in the two righthand columns in a table with the form shown below. Check that the inequality  $p_{\text{pred}}^* \leq p_{\text{exact}}^*$  holds. Express your insights about this problem.

- 4) *When strong duality fails!* Consider the optimization problem

$$\begin{aligned} &\text{minimize} && e^{-x} \\ &\text{subject to} && x^2/y \leq 0 \end{aligned}$$

with variables  $x$  and  $y$ , and domain  $\mathcal{D} = \{(x, y) | y > 0\}$ .

- ✓ a) Verify that this is a convex optimization problem. Then find the optimal value and verify your answers using CVX.
- ✓ b) Derive the Lagrange dual problem, and find the optimal solution  $\lambda^*$ . Also find the optimal value  $d^*$  of the dual problem. What is the optimal duality gap? Verify your answers using CVX.
- ✓ c) Does Slater's condition hold for this problem?

✓ d) \*(Bonus) What is the optimal value  $p^*(u)$  of the perturbed problem

$$\begin{aligned} &\text{minimize} && e^{-x} \\ &\text{subject to} && x^2/y \leq u \end{aligned}$$

as a function of  $u$ ? Verify that the following inequality does not hold.

$$p^*(u) \geq p^*(0) - \lambda^* u.$$

5) *A distributed approach to the bi-commodity network flow problem!* Consider a network, i.e., a directed graph, with  $n$  arcs and  $p$  nodes, described by the incidence matrix  $A \in \mathbf{R}^{p \times n}$ , where

$$A_{ij} = \begin{cases} 1, & \text{if arc } j \text{ enters node } i \\ -1 & \text{if arc } j \text{ leaves node } i \\ 0 & \text{otherwise} \end{cases}$$

Two commodities flow in the network. Commodity 1 has source vector  $s \in \mathbf{R}^p$ , and commodity 2 has source vector  $t \in \mathbf{R}^p$ , which satisfy  $\mathbf{1}^T s = \mathbf{1}^T t = 0$ . The flow of commodity 1 on arc  $i$  is denoted by  $x_i$ , and the flow of commodity 2 on arc  $i$  is denoted by  $y_i$ . Each of the flows must satisfy flow conservation, which can be expressed as  $Ax + s = 0$  (for commodity 1), and  $Ay + t = 0$  (for commodity 2).

Arc  $i$  has associated flow cost  $\phi_i(x_i, y_i)$ , where  $\phi_i : \mathbf{R}^2 \rightarrow \mathbf{R}$  is convex. We can impose constraints such as nonnegativity of the flows by restricting the domain of  $\phi_i$  to  $\mathbf{R}_+^2$ . One natural form for  $\phi_i$  is a function of only the total traffic on the arc, i.e.,  $\phi_i(x_i, y_i) = f_i(x_i + y_i)$ , where  $f_i : \mathbf{R} \rightarrow \mathbf{R}$  is convex. In this form, however,  $\phi$  is not strictly convex, which will complicate things. Therefore, to avoid this complications, we assume that  $\phi_i$  is strictly convex.

The problem of choosing the minimum cost flows that satisfy the flow conservation can be expressed as

$$\begin{aligned} &\text{minimize} && \sum_{i=1}^n \phi_i(x_i, y_i) \\ &\text{subject to} && Ax + s = 0, \quad Ay + t = 0 \end{aligned}$$

with variables  $x, y \in \mathbf{R}^n$ . This is the *bi-commodity network flow problem*.

- a) Propose a distributed solution to the bi-commodity flow problem using dual decomposition.
- b) Use your algorithm to solve the particular problem instance with

$$\phi_i(x_i, y_i) = (x_i + y_i)^2 + \epsilon(x_i^2 + y_i^2), \quad \text{dom } \phi_i = \mathbf{R}_+^2,$$

with  $\epsilon = 0.1$ . The other data for this problem can be found in `bicommodity_data.m`. To check that your method works, compute the optimal value  $p^*$ , using CVX. For the subgradient updates use a constant step-size of 0.1. Run the algorithm for 200 iterations and plot the dual lower bound versus iteration. With a logarithmic vertical axis, plot the norms of the residuals for each of the two flow conservation equations, i.e.,  $\|Ax + s\|_2$  and  $\|Ay + t\|_2$ , respectively, versus iteration number, on the same plot.

*Hint:* A function `[x, y] = quad2_min(eps, alpha, beta)` is posted, which analytically computes

$$(x^*, y^*) = \underset{x \geq 0, y \geq 0}{\operatorname{argmin}} ((x + y)^2 + \epsilon(x^2 + y^2) + \alpha x + \beta y).$$

You might find this function useful.

- 6) Problem 9.10 of the text book. <sup>1</sup>
- 7) (Bonus points\*) Problem 9.30 (a) of the text book.

<sup>1</sup>Boyd, Stephen, Stephen P. Boyd, and Lieven Vandenberghe. Convex optimization. Cambridge university press, 2004.