Digital Image Processing

Two Dimensional Signals Processing

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Two Dimensional Systems:

› General Definition:

$$f(x,y) \longrightarrow \boxed{\begin{array}{c}\text{System}\\ \text{H}\end{array}} \longrightarrow g(x,y)$$

$$g(x,y) = \mathcal{H}\{f(x,y)\}$$

System Properties:

- Linearity:

$$\mathcal{H}\{af_1(x,y) + bf_2(x,y)\} = a\mathcal{H}\{f_1(x,y)\} + b\mathcal{H}\{f_2(x,y)\}$$

- Spatial Invariant:

$$\mathcal{H}\{f(x - x_0, y - y_0)\} = g(x - x_0, y - y_0)$$

- Causality: We do not care about it!
- Stability: Same as before.

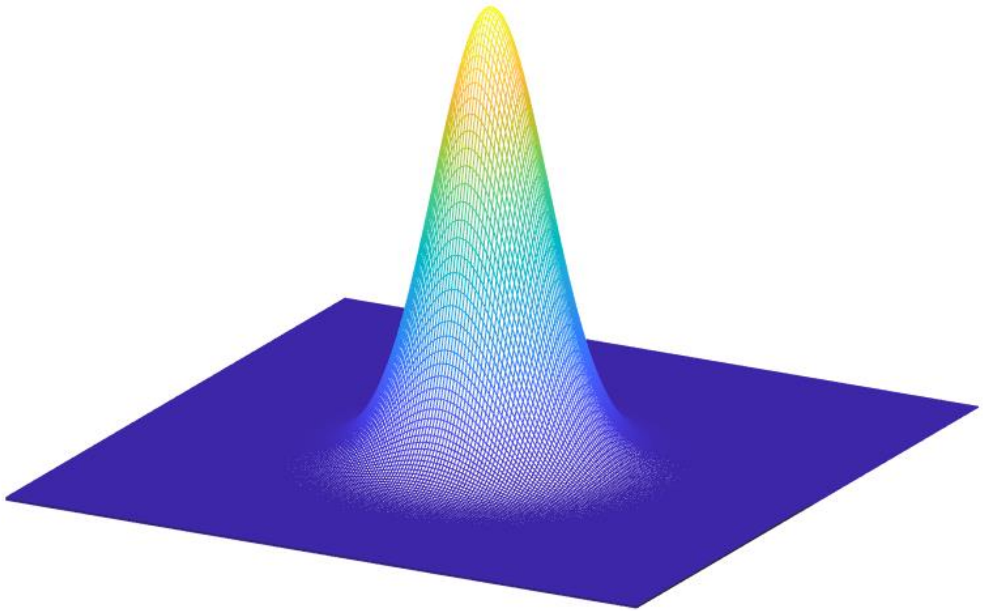
Unit Impulse (pinhole)

› Mathematical Definition:

$$\delta(x,y) = \begin{cases} 0, & (x,y) \neq (0,0) \\ \infty, & (x,y) = (0,0) \end{cases}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x,y) dx dy = 1$$

› Approximation:



Point Spread Function (PSF)

- Definition:

$$H(x, y; x_0, y_0) = \mathcal{H}\{\delta(x - x_0, y - y_0)\}$$

- Linear Shift Invariant (LSI):

$$H(x, y; x_0, y_0) = \mathcal{H}\{\delta(x - x_0, y - y_0)\} = H(x - x_0, y - y_0)$$

$$H(x, y) = \mathcal{H}\{\delta(x, y)\}$$

Convolution and Correlation

- Discrete Convolution:

$$(f \bigstar h)(x,y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n) h(x-m, y-n)$$

- Discrete Correlation:

$$(f \star h)(x,y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f\ (m,n) h(x+m, y+n)$$

Discrete Fourier Transform (DFT):

› Forward Transform

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux/M + vy/N)}$$

› Inverse Transform:

$$f(x,y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) e^{j2\pi(ux/M + vy/N)}$$

DFT (Definitions)

› Useful definitions:

Name	Expression(s)
1) Discrete Fourier transform (DFT) of $f(x,y)$	$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$
2) Inverse discrete Fourier transform (IDFT) of $F(u,v)$	$f(x,y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$
3) Spectrum	$\lvert F(u,v)\rvert = \left[R^2(u,v) + I^2(u,v)\right]^{1/2} \quad R = \text{Real}(F);\ I = \text{Imag}(F)$
4) Phase angle	$\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$
5) Polar representation	$F(u,v) = \lvert F(u,v)\rvert e^{j\phi(u,v)}$
6) Power spectrum	$P(u,v) = \lvert F(u,v)\rvert^2$
7) Average value	$\bar{f} = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) = \frac{1}{MN}F(0,0)$
8) Periodicity (k_1 and k_2 are integers)	$F(u,v) = F(u+k_1M,v) = F(u,v+k_2N) = F(u+k_1,v+k_2N)$ $f(x,y) = f(x+k_1M,y) = f(x,y+k_2N) = f(x+k_1M,y+k_2N)$

DFT Pairs

› Useful Pairs:

Name	DFT Pairs
1) Symmetry properties	See Table 4.1
2) Linearity	$af_1(x,y) + bf_2(x,y) \Leftrightarrow aF_1(u,v) + bF_2(u,v)$
3) Translation (general)	$f(x,y)e^{j2\pi(u_0x/M + v_0y/N)} \Leftrightarrow F(u-u_0, v-v_0)$ $f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(ux_0/M + vy_0/N)}$
4) Translation to center of the frequency rectangle, $(M/2, N/2)$	$f(x,y)(-1)^{x+y} \Leftrightarrow F(u-M/2, v-N/2)$ $f(x-M/2, y-N/2) \Leftrightarrow F(u,v)(-1)^{u+v}$
5) Rotation	$f(r, \theta+\theta_0) \Leftrightarrow F(\omega, \varphi+\theta_0)$ $r = \sqrt{x^2+y^2}$ $\theta = \tan^{-1}(y/x)$ $\omega = \sqrt{u^2+v^2}$ $\varphi = \tan^{-1}(v/u)$
6) Convolution theorem†	$f \bigstar h)(x,y) \Leftrightarrow (F \bullet H)(u,v)$ $(f \bullet h)(x,y) \Leftrightarrow (1/MN)[(F \bigstar H)(u,v)]$
7) Correlation theorem†	$(f \star h)(x,y) \Leftrightarrow (F^* \bullet H)(u,v)$ $(f^* \bullet h)(x,y) \Leftrightarrow (1/MN)[(F \star H)(u,v)]$

DFT Pairs

› Useful Pairs:

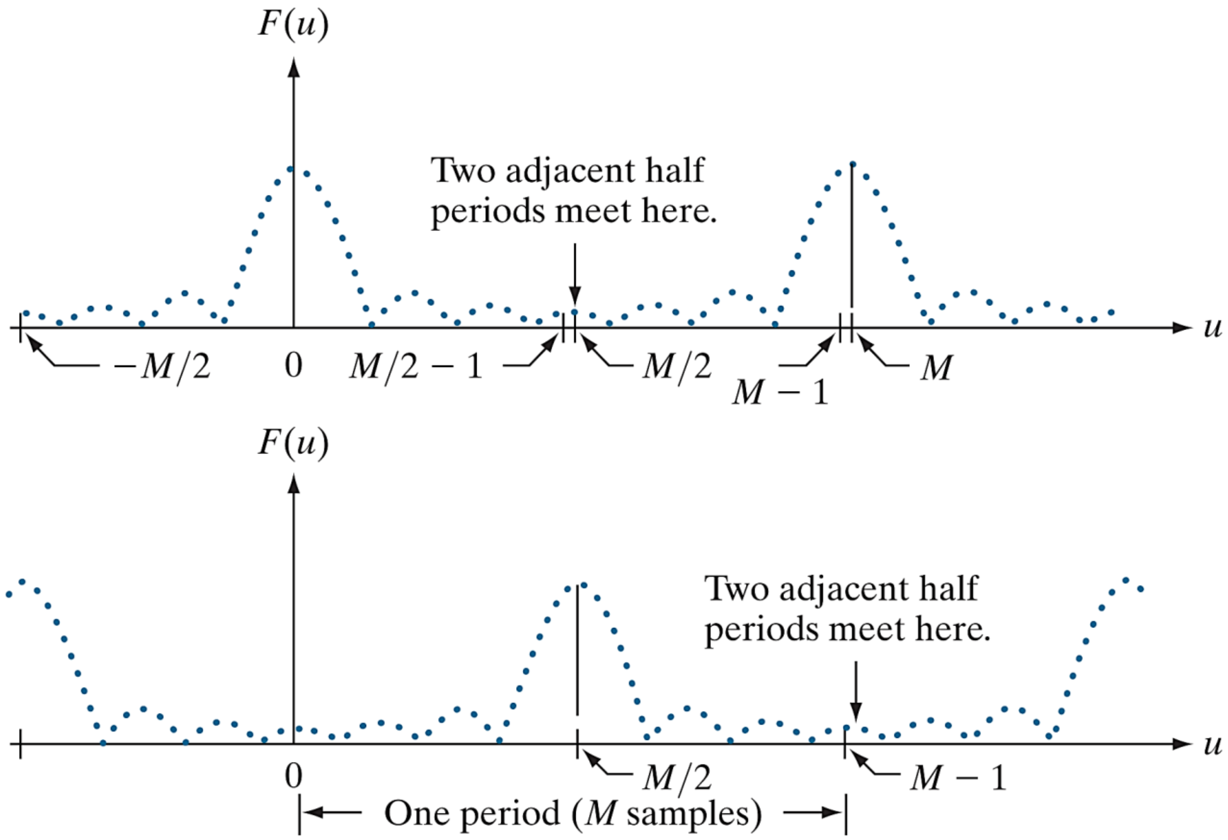
8)	Discrete unit impulse	$\delta(x,y) \Leftrightarrow 1$ $1 \Leftrightarrow MN\delta(u,v)$
9)	Rectangle	$\mathrm{rec}[a,b] \Leftrightarrow ab\dfrac{\sin(\pi ua)}{(\pi ua)}\dfrac{\sin(\pi vb)}{(\pi vb)}e^{-j\pi(ua+vb)}$
10)	Sine	$\sin(2\pi u_0 x/M + 2\pi v_0 y/N) \Leftrightarrow \dfrac{jMN}{2}\left[\delta(u+u_0, v+v_0) - \delta(u-u_0, v-v_0)\right]$
11)	Cosine	$\cos(2\pi u_0 x/M + 2\pi v_0 y/N) \Leftrightarrow \dfrac{1}{2}\left[\delta(u+u_0, v+v_0) + \delta(u-u_0, v-v_0)\right]$

The following Fourier transform pairs are derivable only for continuous variables, denoted as before by t and z for spatial variables and by μ and ν for frequency variables. These results can be used for DFT work by sampling the continuous forms.

12)	Differentiation (the expressions on the right assume that $f(\pm\infty, \pm\infty) = 0$.	$\left(\dfrac{\partial}{\partial t}\right)^m\left(\dfrac{\partial}{\partial z}\right)^n f(t,z) \Leftrightarrow (j2\pi\mu)^m (j2\pi\nu)^n F(\mu,\nu)$ $\dfrac{\partial^m f(t,z)}{\partial t^m} \Leftrightarrow (j2\pi\mu)^m F(\mu,\nu);\ \dfrac{\partial^n f(t,z)}{\partial z^m} \Leftrightarrow (j2\pi\nu)^n F(\mu,\nu)$
13)	Gaussian	$A2\pi\sigma^2 e^{-2\pi^2\sigma^2(t^2+z^2)} \Leftrightarrow Ae^{-(\mu^2+\nu^2)/2\sigma^2}$ (A is a constant)

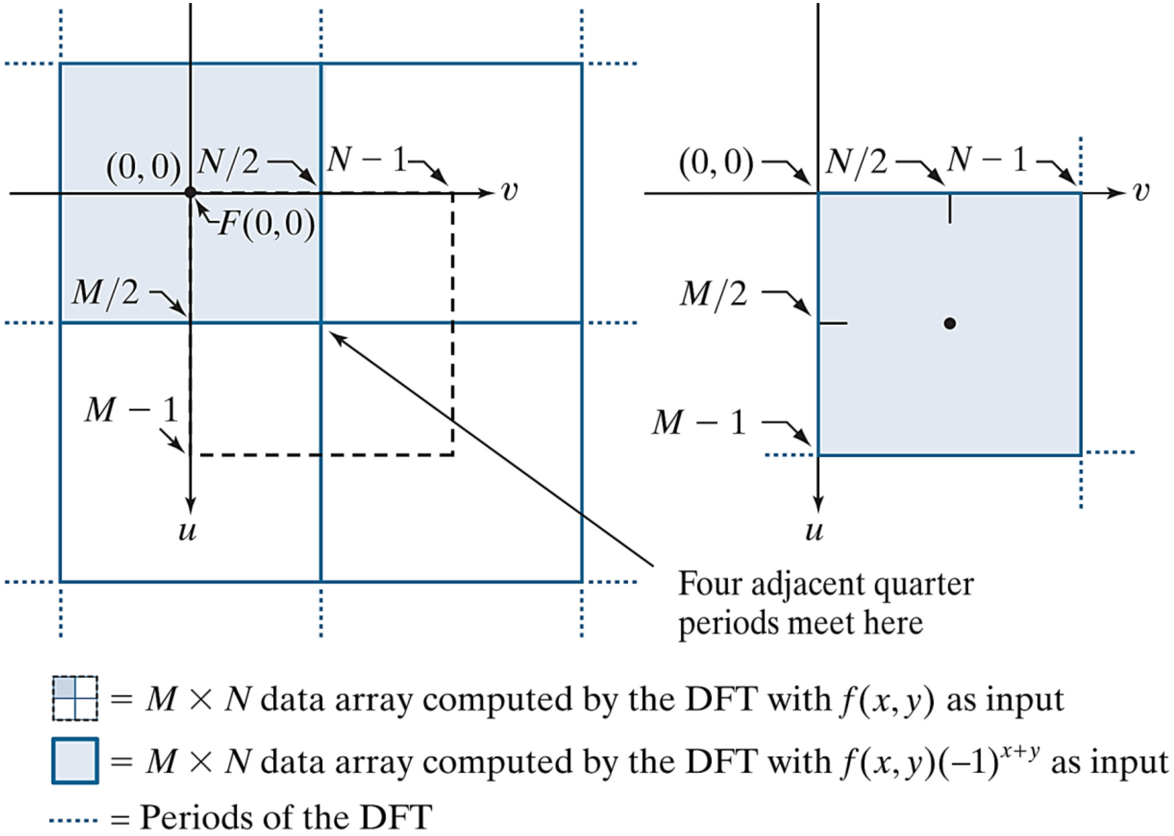
DFT Centering

› From DSP:



DFT Centering

› For DIP (*fftshift*):



DFT Centering

- Example:

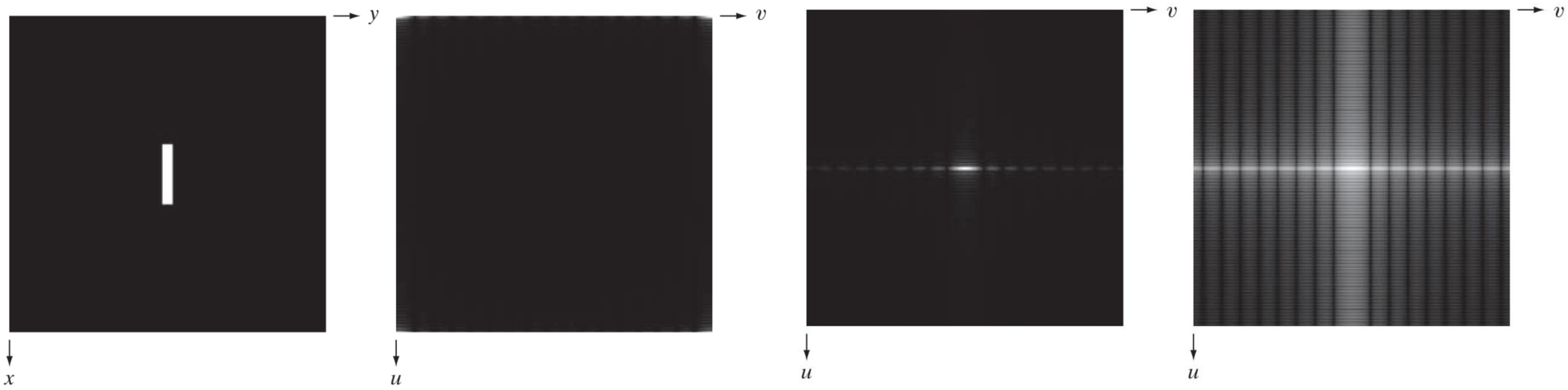
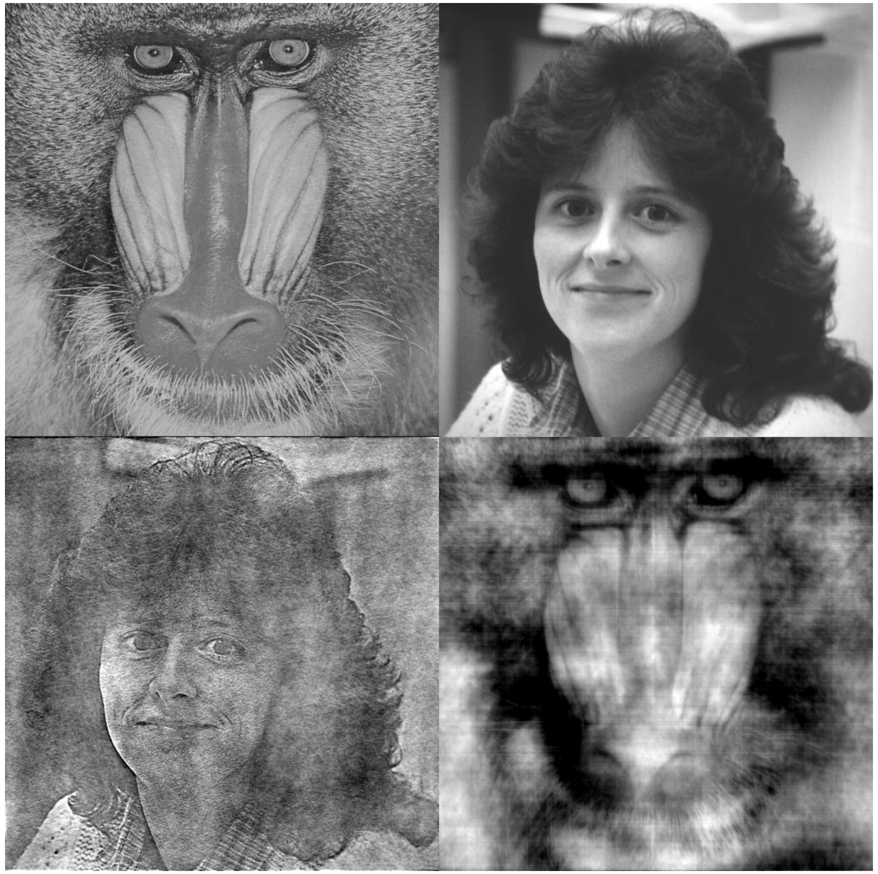


Image	DFT (abs)	Centered DFT	Centered DFT Log

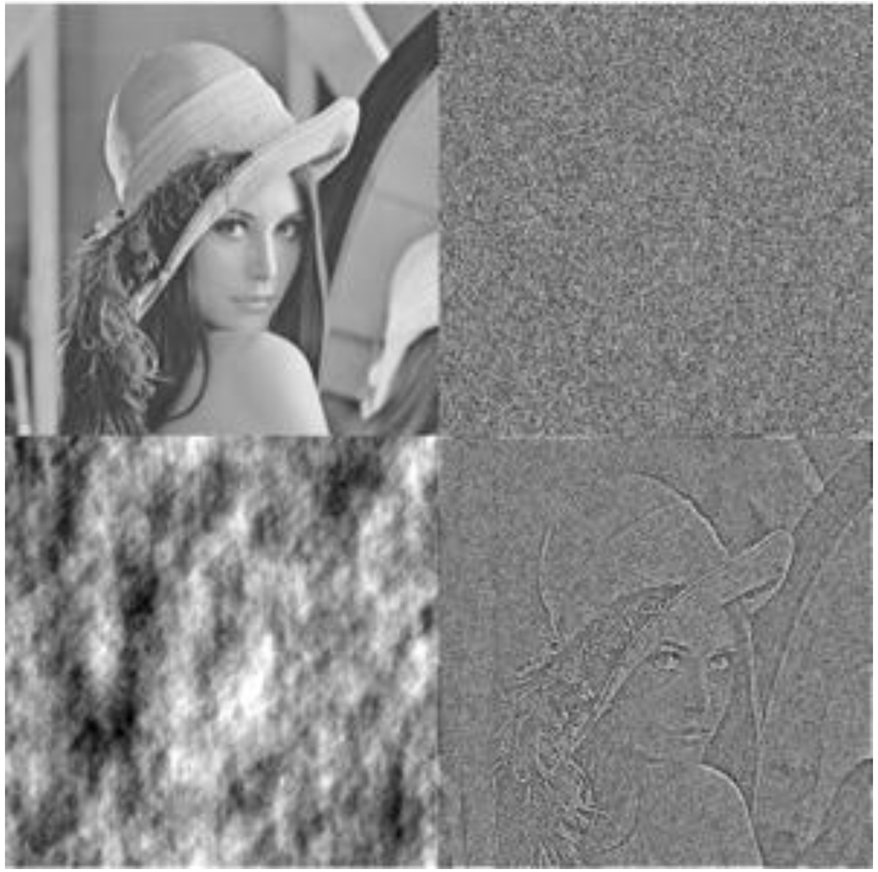
The Importance of Phase

› Let swap phase and magnitude of DFT of two images:



The Importance of Phase

› Let swap phase and magnitude of DFT of two images:



The End

- AnY QuEsTiOn?