Medical Image Analysis and Processing

Medical Image Segmentation
Deformable Model - Parametric

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Distance/online Course: Session 22

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Contents

Parametric Deformable Model - External Force

- Interactive Force (user):
- Allow an operator to interact with the deformable model as it is deforming
- For example, the user may want to *pull* the model toward significant *image features*, or would like to constrain the model so that it must pass through a set of landmark points identified by an expert.
- Two types:
  - Spring force
  - Volcano force

Parametric Deformable Model - External Force

› Interactive Force (user) – Spring Force:

$$F_s(X) = \omega_s(p - X)$$

X: points on model (selected by finding the closest point on the model to p using a heuristic search around a local neighborhood of p)

p: user specified point(s)

Parametric Deformable Model - External Force

› Interactive Force (user) – Volcano Force:

› Push the model away from a local region around a “volcano” point p.

$$F_v(X) = \begin{cases} \omega_s \dfrac{X - p}{|X - p|^3}, & X \in \mathcal{N}(p) \\ 0, & X \notin \mathcal{N}(p) \end{cases}$$

$$F_v(X) = \begin{cases} \omega_s exp\left(-\dfrac{|X - p|^2}{\sigma_v^2}\right)\dfrac{X - p}{|X - p|}, & X \in \mathcal{N}(p) \\ 0, & X \notin \mathcal{N}(p) \end{cases}$$

Parametric Deformable Model – Numerical Implementation

› Recall that:

$$\lambda \frac{\partial X}{\partial t} = \frac{\partial}{\partial s}\left(\alpha \frac{\partial X}{\partial s}\right) - \frac{\partial^2}{\partial s^2}\left(\beta \frac{\partial^2 X}{\partial s^2}\right) + F_{ext}(X)$$

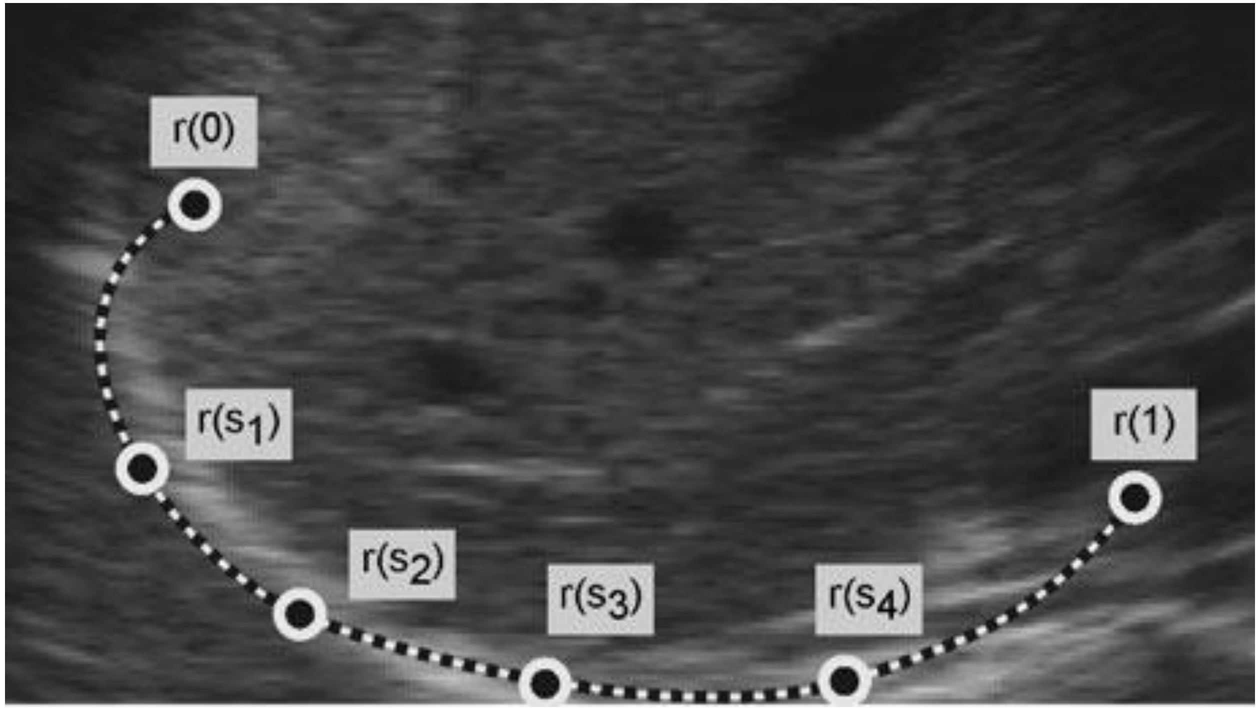
› Discretize contour (model) as:

$$\boldsymbol{X}_i^n = (X_i^n, Y_i^n) = \big(X(ih, n\Delta T), Y(ih, n\Delta T)\big)$$

› where i is contour point index, n is time index, h the step size in space, and ΔT the step size in time

Parametric Deformable Model – Numerical Implementation

› Contour discretization:



Parametric Deformable Model – Numerical Implementation

- Rewrite the following:

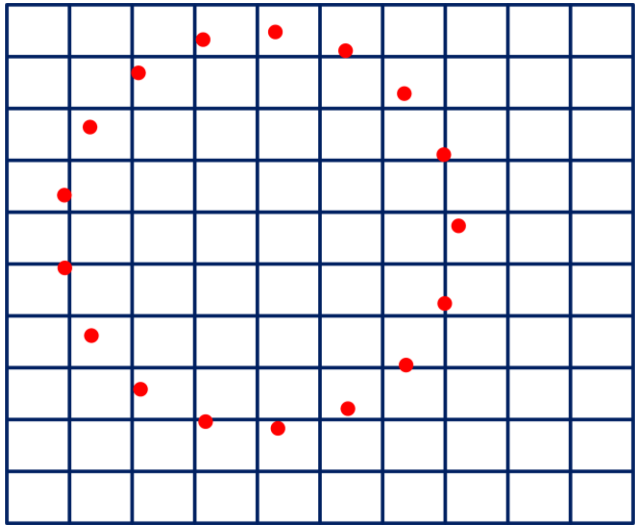
$$\lambda \frac{\partial X}{\partial t} = \frac{\partial}{\partial s}\left(\alpha \frac{\partial X}{\partial s}\right) - \frac{\partial^2}{\partial s^2}\left(\beta \frac{\partial^2 X}{\partial s^2}\right) + F_{ext}(X)$$

- As (semi-implicit):

$$\lambda \frac{\boldsymbol{X}_i^n - \boldsymbol{X}_i^{n-1}}{\Delta T} = \frac{1}{h^2}\left[\alpha_{i+1}\left(\boldsymbol{X}_{i+1}^n - \boldsymbol{X}_i^n\right) - \alpha_i\left(\boldsymbol{X}_i^n - \boldsymbol{X}_{i-1}^n\right)\right]$$
$$- \frac{1}{h^4}\left[\beta_{i-1}\left(\boldsymbol{X}_{i-2}^n - \boldsymbol{2X}_{i-1}^n + \boldsymbol{X}_i^n\right) - 2\beta_i\left(\boldsymbol{X}_{i-1}^n - \boldsymbol{2X}_i^n + \boldsymbol{X}_{i+1}^n\right)\right.$$
$$\left. + \beta_{i+1}\left(\boldsymbol{X}_i^n - \boldsymbol{2X}_{i+1}^n + \boldsymbol{X}_{i+2}^n\right)\right] + F_{ext}\left(\boldsymbol{X}_i^{n-1}\right)$$

Parametric Deformable Model – Numerical Implementation

- In general, the external force $F_{ext}(X)$ is defined on an image grid.
- The value of $F_{ext}(X)$ at any location $\boldsymbol{X}_i^n$ can be obtained through a bilinear interpolation of the external force values at the grid points near $\boldsymbol{X}_i^n$.



Parametric Deformable Model – Numerical Implementation

› This equation

$$\lambda \frac{\boldsymbol{X}_i^n - \boldsymbol{X}_i^{n-1}}{\Delta T} = \frac{1}{h^2}\left[\alpha_{i+1}\left(\boldsymbol{X}_{i+1}^n - \boldsymbol{X}_i^n\right) - \alpha_i\left(\boldsymbol{X}_i^n - \boldsymbol{X}_{i-1}^n\right)\right]$$
$$- \frac{1}{h^4}\left[\beta_{i-1}\left(\boldsymbol{X}_{i-2}^n - \boldsymbol{2X}_{i-1}^n + \boldsymbol{X}_i^n\right) - 2\beta_i\left(\boldsymbol{X}_{i-1}^n - \boldsymbol{2X}_i^n + \boldsymbol{X}_{i+1}^n\right)\right.$$
$$\left. + \beta_{i+1}\left(\boldsymbol{X}_i^n - \boldsymbol{2X}_{i+1}^n + \boldsymbol{X}_{i+2}^n\right)\right] + F_{ext}\left(\boldsymbol{X}_i^{n-1}\right)$$

› May be rewrite as:

$$\frac{\boldsymbol{X}^n - \boldsymbol{X}^{n-1}}{\tau} = \boldsymbol{A}\boldsymbol{X}^n + \boldsymbol{F}_{\boldsymbol{ext}}\left(\boldsymbol{X}^{n-1}\right)$$

› Where $\{\boldsymbol{X}^n, \boldsymbol{X}^{n-1}, \boldsymbol{F}_{\boldsymbol{ext}}(\boldsymbol{X}^{n-1})\} \in \mathbb{R}^{m\times 2}$, $\boldsymbol{A} \in \mathbb{R}^{m\times m}$, with m being the number of sample points.

Parametric Deformable Model – Numerical Implementation

- Can be solved iteratively

$$\boldsymbol{X}^n = (\boldsymbol{I} - \tau\boldsymbol{A})^{-\mathbf{1}}[\boldsymbol{X}^{n-1} + \tau\boldsymbol{F}_{\boldsymbol{ext}}(\boldsymbol{X}^{n-1})]$$

- Another version (explicate):

$$\frac{\boldsymbol{X}^n - \boldsymbol{X}^{n-1}}{\tau} = \boldsymbol{A}\boldsymbol{X}^{n-1} + \boldsymbol{F}_{\boldsymbol{ext}}(\boldsymbol{X}^{n-1})$$

- or:

$$\boldsymbol{X}^n = (\boldsymbol{I} + \tau\boldsymbol{A})\boldsymbol{X}^{n-1} + \tau\boldsymbol{F}_{\boldsymbol{ext}}(\boldsymbol{X}^{n-1})$$

Parametric Deformable Model – Numerical Implementation

- Variational Calculus Free Approach:
- The snake evolution equations can also be derived completely avoiding the route of variational calculus if we make the energy functional *discrete* even before minimizing it:

$$\mathcal{E}(X) = \frac{1}{2}\int_0^1 \left\{ \alpha(s)\left|\frac{\partial X(s)}{\partial s}\right|^2 + \beta(s)\left|\frac{\partial^2 X(s)}{\partial s^2}\right|^2 + E_{external}\big(X(s)\big) \right\} ds$$

Parametric Deformable Model – Numerical Implementation

› Variational Calculus Free Approach:

$$E(\mathbf{X}) = \frac{1}{2}\sum_{i=1}^{N} \alpha\{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2\} +$$

$$\sum_{i=1}^{N} \beta\{(x_{i+1} - 2x_i + x_{i-1})^2 + (y_{i+1} - 2y_i + y_{i-1})^2\} + \sum_{i=1}^{N} f(x_i, y_i)$$

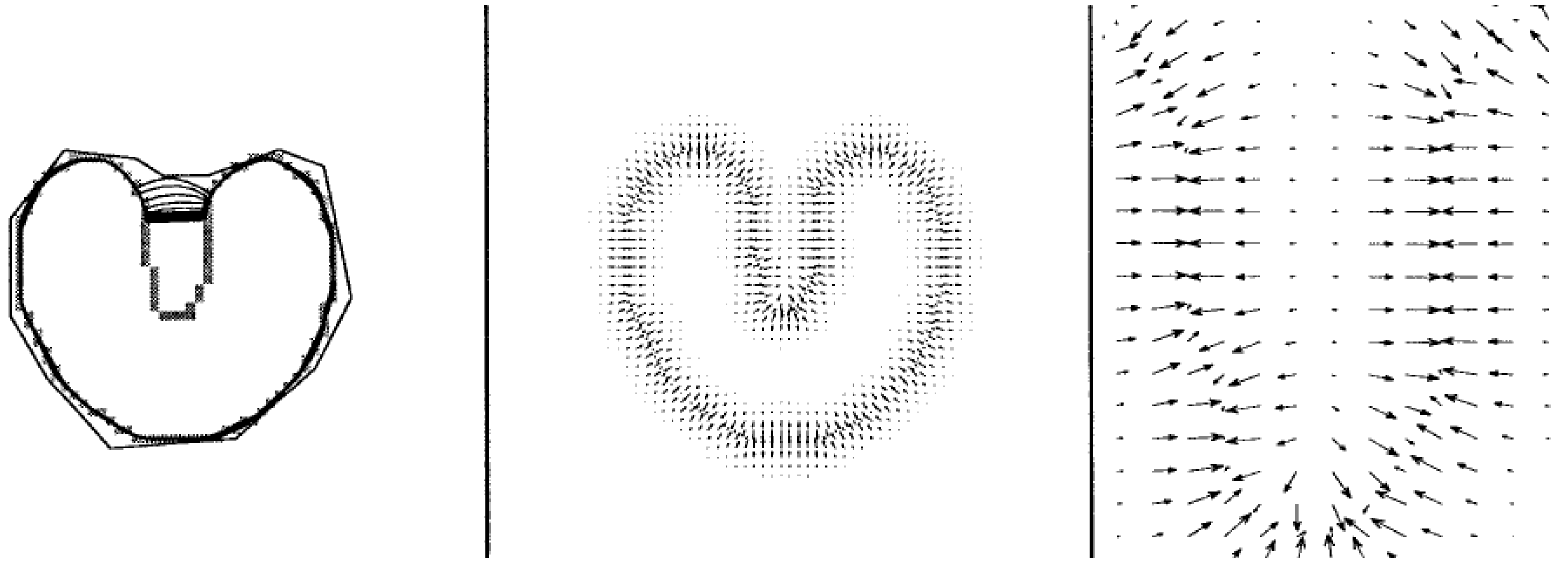
$$\frac{\partial E}{\partial x_i} = -\alpha(x_{i+1} + x_{i-1} - 2x_i) + \beta(x_{i+2} - 4x_{i+1} + 6x_i - 4x_{i-1} + x_{i-2}) + f_x(x_i, y_i)$$

$$\frac{\partial E}{\partial y_i} = -\alpha(y_{i+1} + y_{i-1} - 2y_i) + \beta(y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}) + f_y(x_i, y_i)$$

› Now we can apply gradient descent for contour location update

Parametric Deformable Model – Advanced version - GVF

- Gradient Vector Flow (GVF):
- Problem with traditional snake:



Parametric Deformable Model – Advanced version - GVF

- Gradient Vector Flow (GVF):

$$\lambda \frac{\partial X}{\partial t} = \frac{\partial}{\partial s}\left(\alpha \frac{\partial X}{\partial s}\right) - \frac{\partial^2}{\partial s^2}\left(\beta \frac{\partial^2 X}{\partial s^2}\right) + \mathbf{v}(X)$$

- Using definition of edge map, $f(x, y)$, any energy, $-E_{ext}(X)$ with following properties:
  - The gradient of an edge map, ∇f, has vectors pointing toward the edges, which are normal to the edges at the edges.
  - These vectors generally have large magnitudes only in the immediate vicinity of the edges (undesirable for ∪ shape contour)
  - Third, in homogeneous regions, where $I(x, y)$ is nearly constant, $f(x, y)$ is nearly zero (undesirable for ∪ shape contour)

Parametric Deformable Model – Advanced version - GVF

- In GVF the gradient map *extended* farther away from the edges and into *homogeneous* regions using a computational *diffusion* process.
- A gradient vector flow field $\mathbf{v}(x,y) = [u(x,y), v(x,y)]$ minimize the following energy:

$$\iint \mu\left(u_x^2 + u_y^2 + v_x^2 + v_y^2\right) + |\nabla f|^2 |\mathbf{v} - \nabla f|^2 \, dxdy$$

- Using calculus of variation:

$$\begin{cases} \mu\nabla^2 u - (u - f_x)\left(f_x^2 + f_y^2\right) = 0 \\ \mu\nabla^2 v - \left(v - f_y\right)\left(f_x^2 + f_y^2\right) = 0 \end{cases}$$

- In homogeneous region, the second term is zero and therefore: $\nabla^2 u = \nabla^2 v = 0$, which is Laplace equation (solution is interpolation of boundary condition)

Parametric Deformable Model – Advanced version - GVF

- How to solve:

$$\begin{cases} \dfrac{\partial u(x,y,t)}{\partial t} = \mu\nabla^2 u(x,y,t) - \left(u(x,y,t) - f_x\right)\left(f_x^2 + f_y^2\right) \\ \dfrac{\partial v(x,y,t)}{\partial t} = \mu\nabla^2 v(x,y,t) - \left(v(x,y,t) - f_x\right)\left(f_x^2 + f_y^2\right) \end{cases}$$

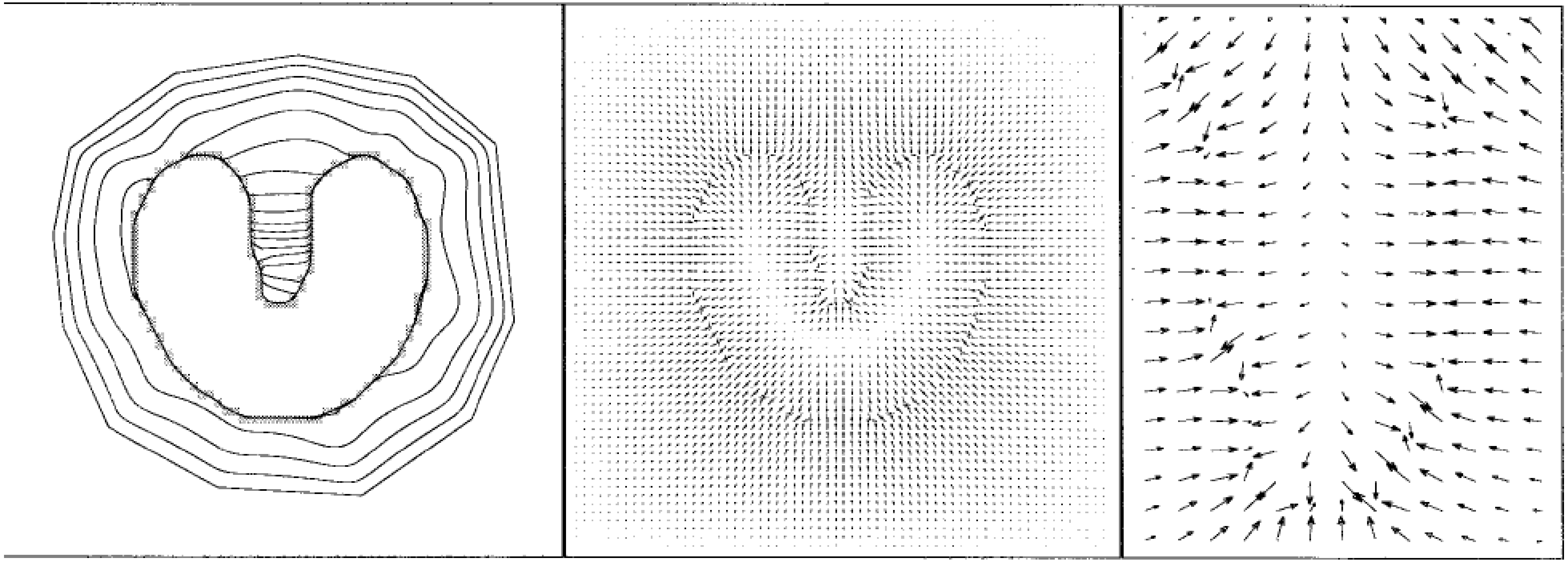
- Generalized GVF (GGVF):

$$\begin{cases} \dfrac{\partial u(x,y,t)}{\partial t} = g(|\nabla f|)\nabla^2 u(x,y,t) - h(|\nabla f|)\left(u(x,y,t) - f_x\right) \\ \dfrac{\partial v(x,y,t)}{\partial t} = g(|\nabla f|)\nabla^2 v(x,y,t) - h(|\nabla f|)\left(v(x,y,t) - f_x\right) \end{cases}$$

- A good choice is: $g(|\nabla f|) = e^{-\alpha|\nabla f|}$ and $h(|\nabla f|) = 1 - g(|\nabla f|)$

Parametric Deformable Model – Advanced version - GVF

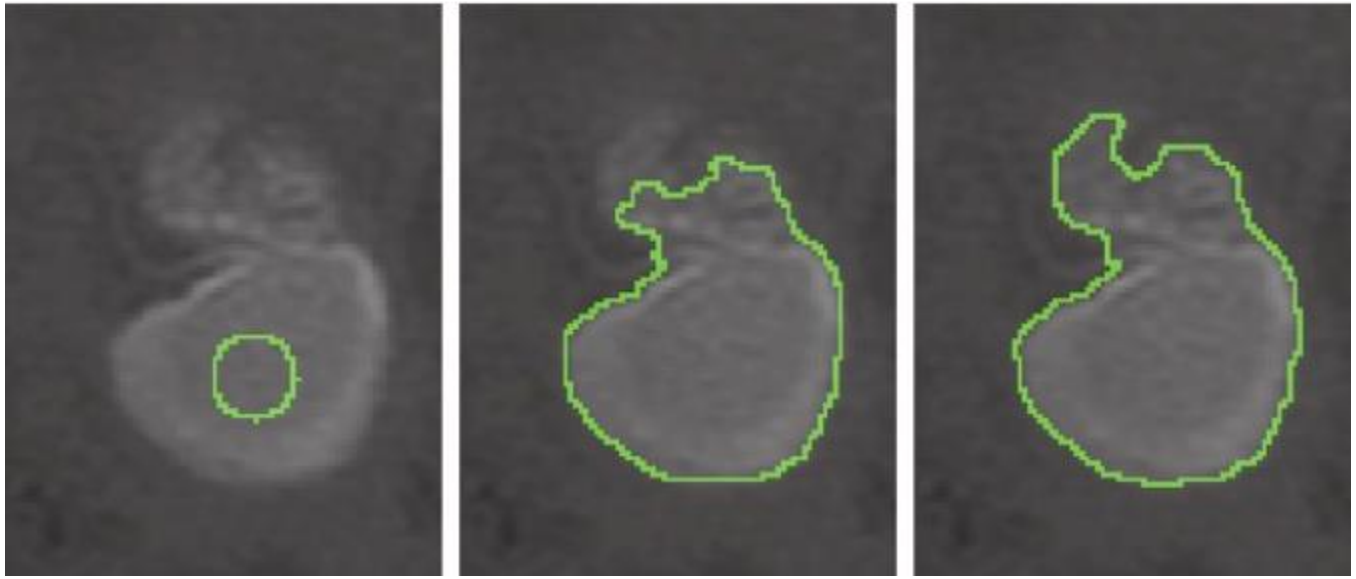
› GVF for U shape contour:



Geometric Deformable Model

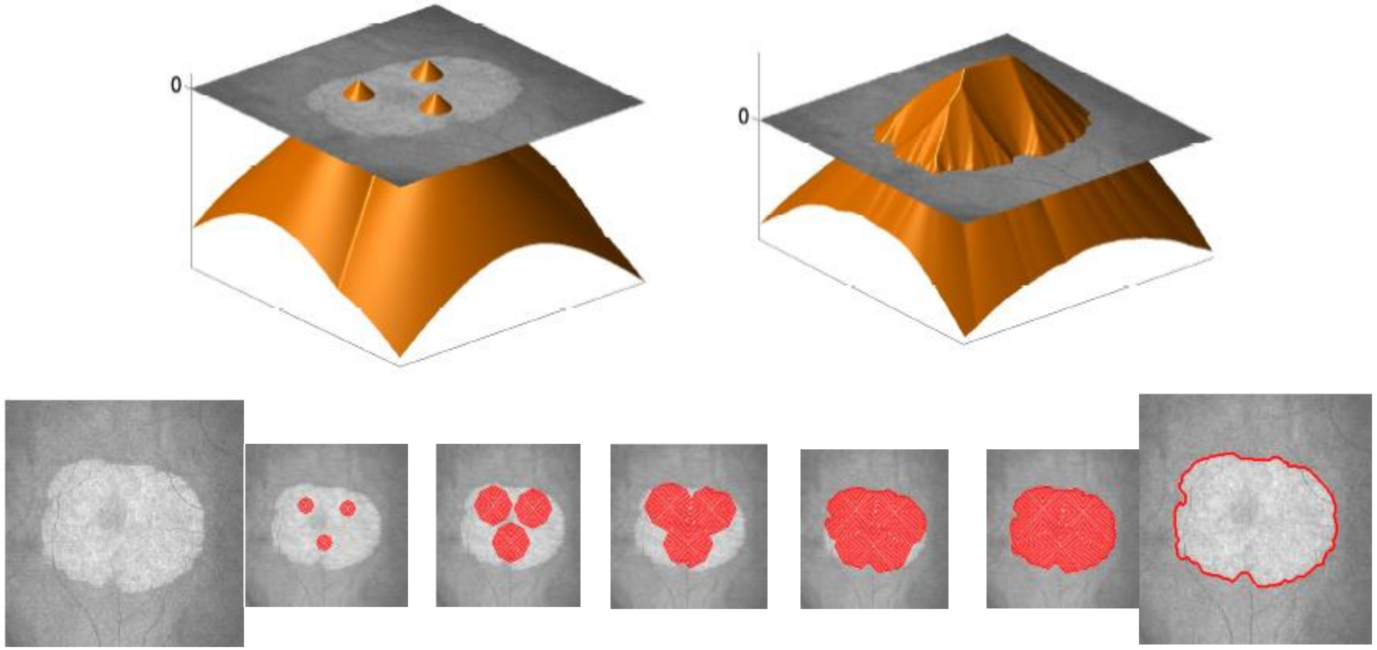
› Geometric Deformable Model (Level-Set):

› User view:



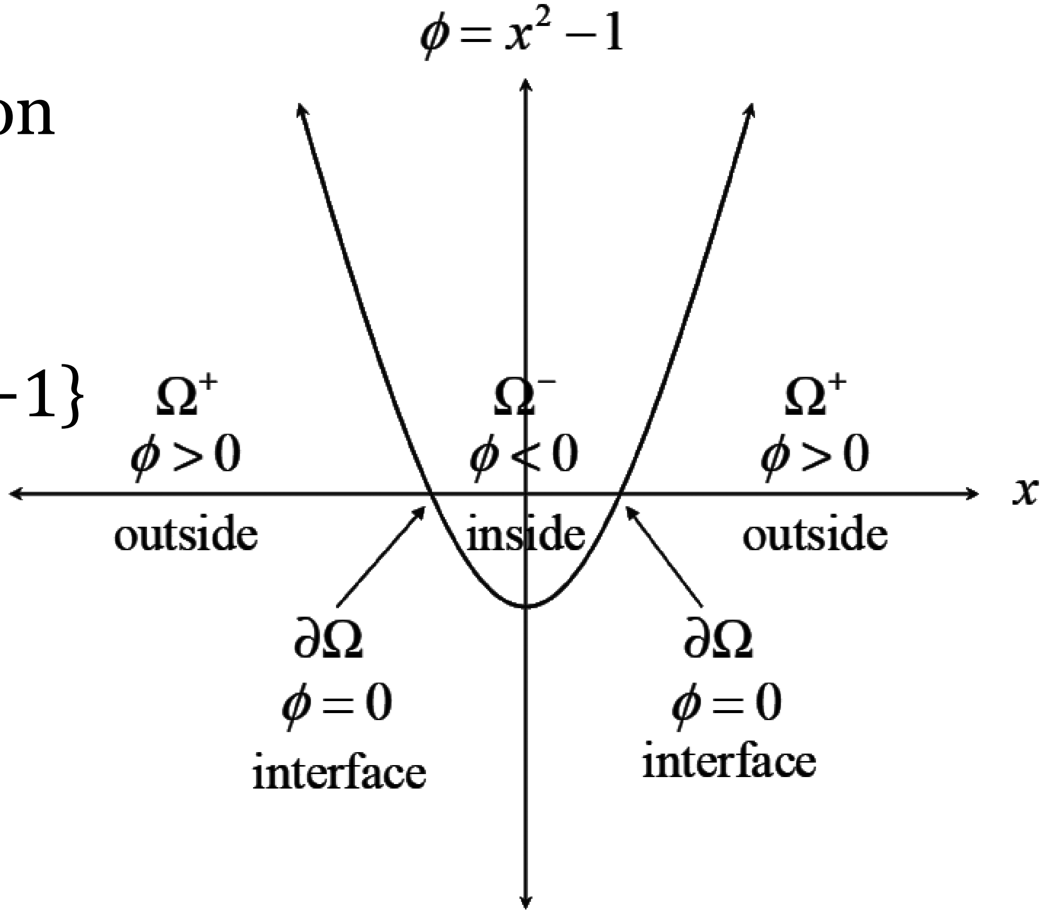
Geometric Deformable Model

- Geometric Deformable Model (Level-Set):
- Algorithmic view:



Geometric Deformable Model

- Preliminary#1 Implicit functions
- 1- Points Implicit Representation
- $\phi(x) = x^2 - 1$
- Its zero isocontour: $\partial\Omega = \{-1, +1\}$
- $\Omega^+: (-\infty, -1) \cup (+1, +\infty)$
- $\Omega^-: (-1, +1)$
- $\partial\Omega: \{-1, +1\}$



Geometric Deformable Model

› Preliminary#1 Implicit functions

› 2- Curve Implicit Representation

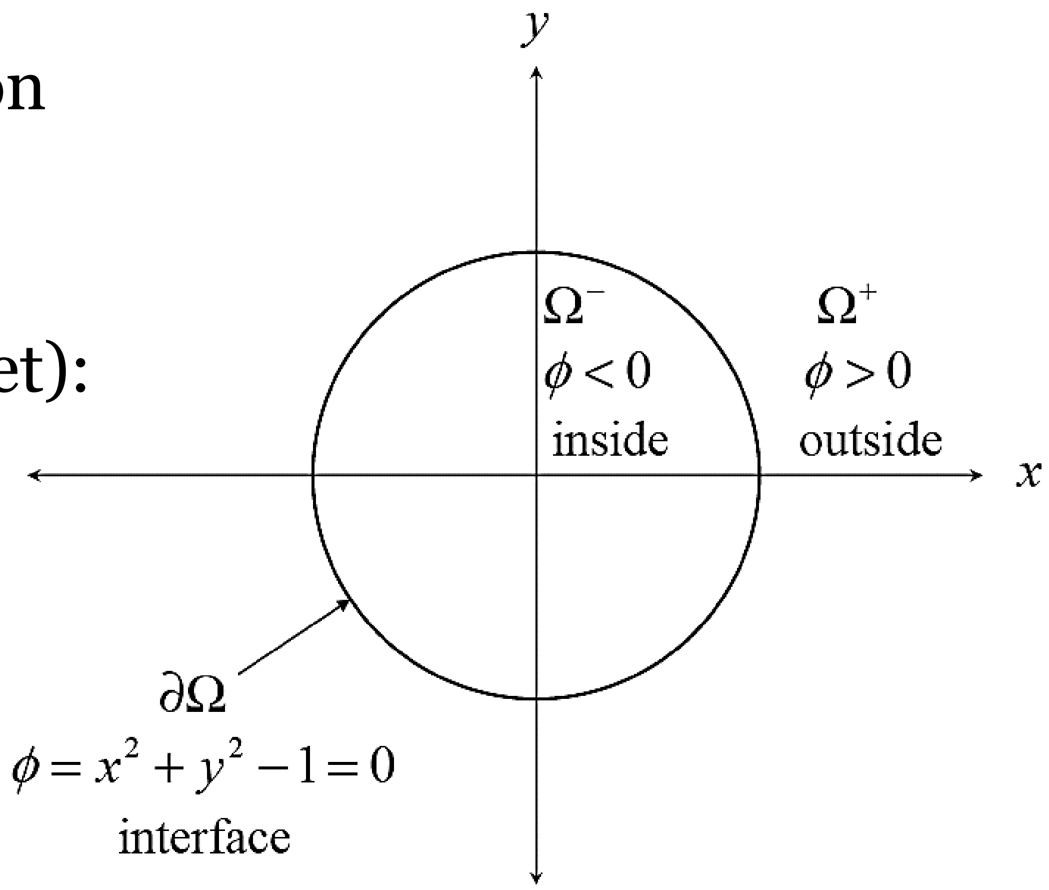
› $\phi(x, y) = x^2 + y^2 - 1$

› Its zero isocontour (zero level set):

› $\Omega^+ : x^2 + y^2 > 1$

› $\Omega^- : x^2 + y^2 < 1$

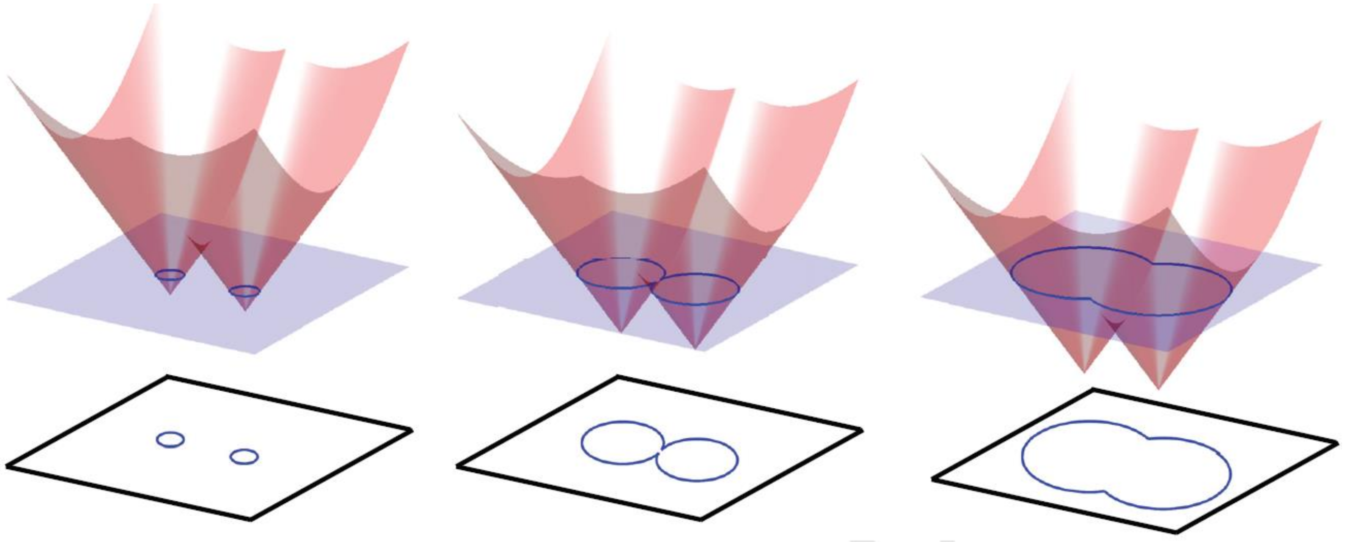
› $\partial\Omega : x^2 + y^2 = 1$



Geometric Deformable Model

› Preliminary#1 Implicit functions

› 2- Curve Implicit Representation



Geometric Deformable Model

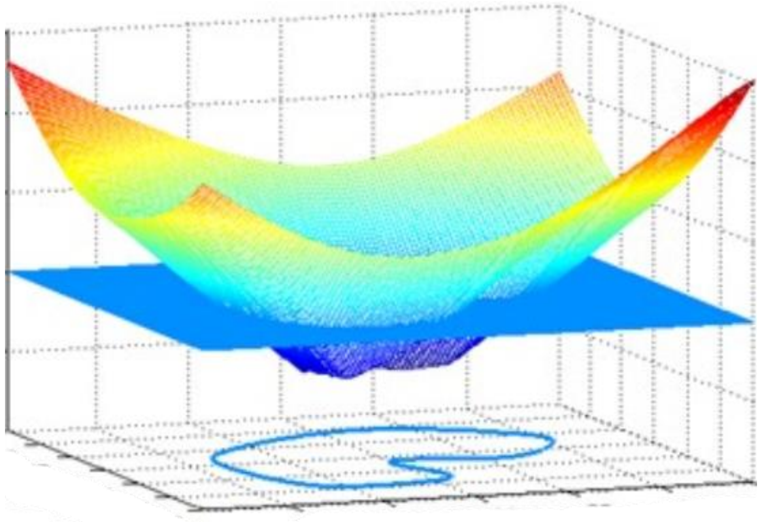
› Preliminary#1 Zero Level-Set (Level-Set)

› Consider a deformable high dimension function,$\phi(\boldsymbol{x}, t) \in \mathbb{R}^n$

› $\partial\Omega$: $\phi(\boldsymbol{x}, t) = 0, \partial\Omega \in \mathbb{R}^{n-1}$

› Level-Set function of a contour $\mathcal{C}(\boldsymbol{x}, t)$:

› $\phi(\boldsymbol{x}, t) = \begin{cases} d(\boldsymbol{x}, \mathcal{C}(\boldsymbol{x}, t)) & x \textit{ is outside the } \mathcal{C} \\ -d(\boldsymbol{x}, \mathcal{C}(\boldsymbol{x}, t)) & x \textit{ is inside the } \mathcal{C} \\ 0 & o.w. \end{cases}$



› $d(\boldsymbol{x}, \mathcal{C}(\boldsymbol{x}, t))$ is the shortest distance of $\boldsymbol{x}$ to this curve.

The End

- AnY QuEsTiOn?