Medical Image Analysis and Processing

Medical Image Segmentation
Deformable Model - Geometric

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Distance/online Course: Session 23

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- Smooth Curve Theory
- Curve Evolution Theory
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- Geodesic Active Contour

Geometric Deformable Model

- Preliminary#2 Smooth Curve Theory
- Consider a family of smoothed curve,

$$C(\boldsymbol{x}, t): [x(p,t), y(p,t)]: \mathbb{R} \times [0, t_{max}] \longmapsto \mathbb{R}^2$$

- Where t is family (evolving) index and p parametrized the curve.
- Example:

$$x(p,t) = tcos(2\pi p)$$
$$y(p,t) = tsin(2\pi p)$$

- For $0 \leq p \leq 1$ is a family of circles with increasing radius

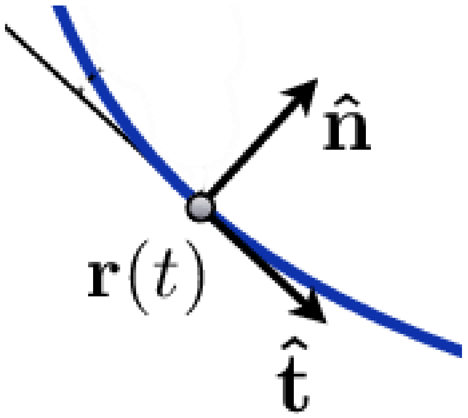
Geometric Deformable Model

- Tangent vector:

$$\vec{T} = \frac{\partial C(p,t)}{\partial p} = C'(p,t) = [x'(p,t), y'(p,t)]$$

- Unit tangent vector is

$$\vec{\mathcal{T}} = \frac{\vec{T}}{|\vec{T}|} = \frac{C'(p,t)}{|C'(p,t)|}$$



Geometric Deformable Model

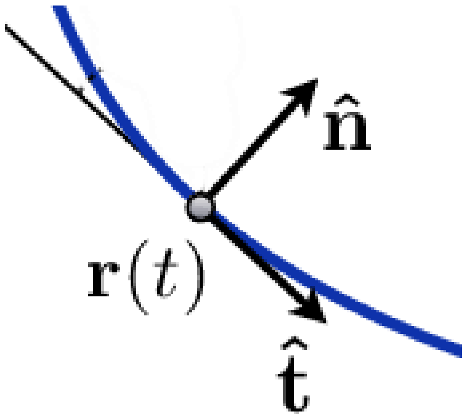
› Normal vector:

$$\vec{N} = [-y'(p,t), x'(p,t)]$$

$$\vec{N} = \frac{\partial \hat{T}(p,t)}{\partial p}$$

› Unit normal vector is:

$$\vec{\mathcal{N}} = \frac{\vec{N}}{|\vec{N}|} = \frac{\frac{\partial \hat{T}(p,t)}{\partial p}}{\left|\frac{\partial \hat{T}(p,t)}{\partial p}\right|}$$



Geometric Deformable Model

› Arc length and normalization :

$$s(p) = \int_0^p \|C'(r,t)\|\,dr \Rightarrow \frac{\partial s}{\partial p} = \|C'(p,t)\|$$

› It tangent vector magnitude, thus

$$\frac{\partial C(p,t)}{\partial s} = \frac{\partial C(p,t)}{\partial p}\frac{\partial p}{\partial s} = \frac{1}{\|C'(p,t)\|}\frac{\partial C(p,t)}{\partial p} = \frac{\vec{T}}{|\vec{T}|} = \vec{\mathcal{T}}$$

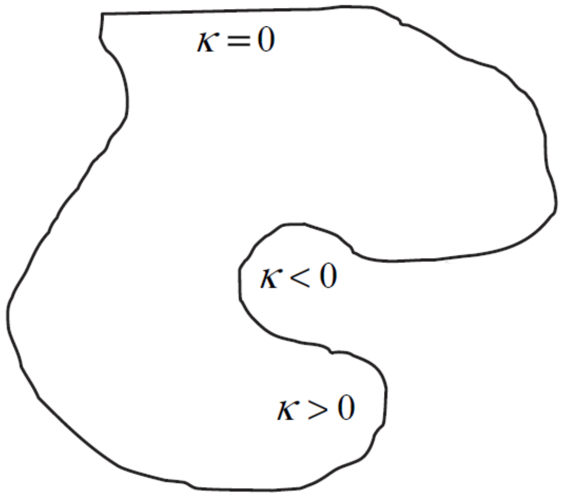
$$\frac{\partial C(p,t)}{\partial s} = \vec{\mathcal{T}}$$

Geometric Deformable Model

› Curvature:

$$\kappa = \nabla . \vec{\mathcal{N}} = \mathrm{div}(\vec{\mathcal{N}}) = \frac{\partial n_1}{\partial x} + \frac{\partial n_2}{\partial y}$$

› Convex region: $\kappa > 0$, concave region: $\kappa < 0$, plane: $\kappa = 0$



Geometric Deformable Model

- Curvature:
- For any arbitrary parameterization:

$$\kappa(p) = \frac{x'(p)y''(p) - x''(p)y'(p)}{(x'(p)^2 + y'(p)^2)^{3/2}}$$

- Convex region: $\kappa > 0$, concave region: $\kappa < 0$, plane: $\kappa = 0$
- It can be shown:

$$\frac{1}{|C'(p,t)|}\frac{\partial}{\partial p}\left(\frac{C'(p,t)}{|C'(p,t)|}\right) = \kappa(p)\frac{\vec{N}}{|\vec{N}|} \Rightarrow \frac{1}{|C'(p,t)|}\frac{\partial}{\partial p}(\vec{\mathcal{T}}) = \kappa(p)\vec{\mathcal{N}}$$

Geometric Deformable Model

- Curvature:
- For arc length parametrization:

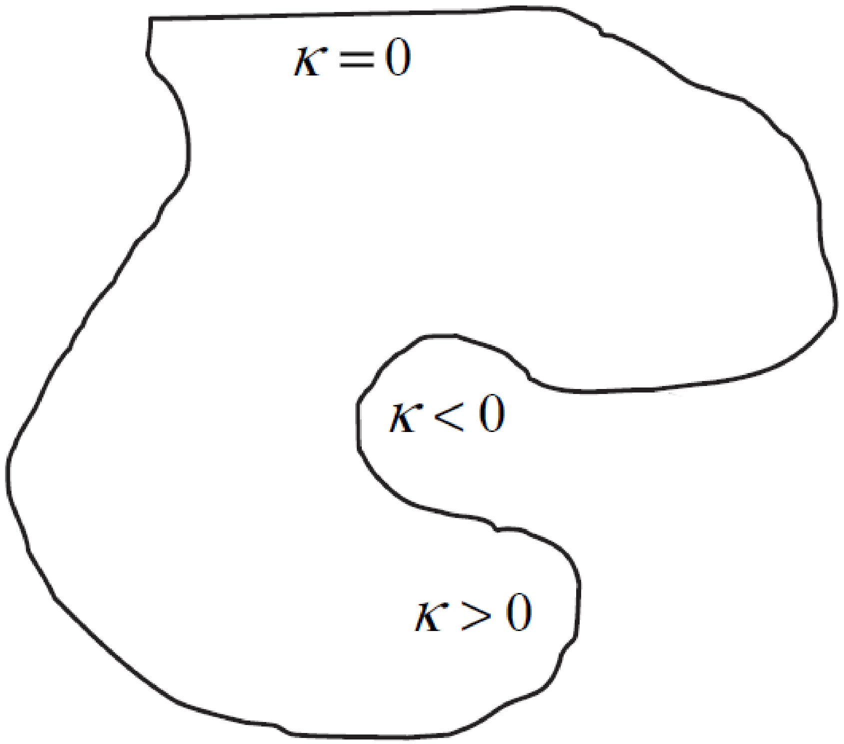
$$\frac{\partial C}{\partial s} = \vec{\mathcal{T}}$$

$$\frac{\partial \vec{\mathcal{T}}}{\partial s} = \kappa \vec{\mathcal{N}}$$

$$\frac{\partial \vec{\mathcal{N}}}{\partial s} = -\kappa \vec{\mathcal{T}}$$

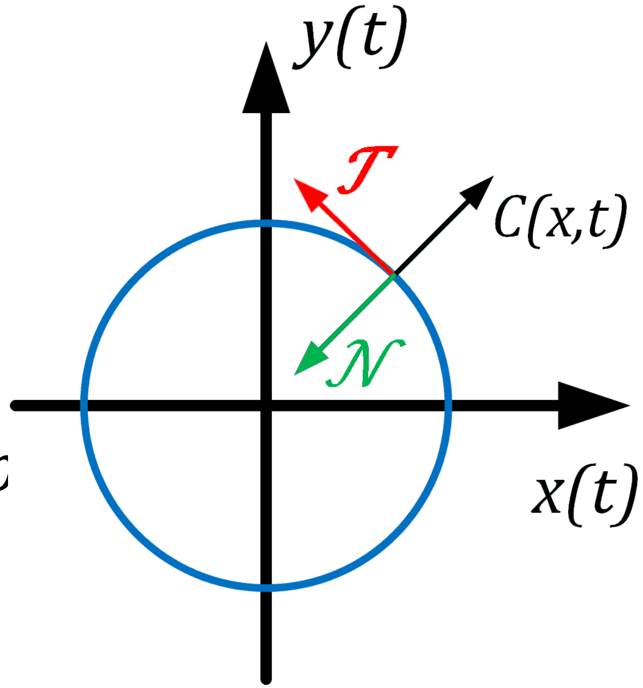
Geometric Deformable Model

› Curvature Illustration:



Geometric Deformable Model

- Curvature Illustration:
- $x(p,t) = cos(2\pi p)$
- $y(p,t) = sin(2\pi p)$
- $\vec{\mathcal{T}} = [-sin(2\pi p), cos(2\pi p)]$
- $\vec{\mathcal{N}} = [-cos(2\pi p), -sin(2\pi p$



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Geometric Deformable Model

› Curve Evolution Theory:

› Main formulation:

$$\frac{\partial C(p,t)}{\partial t} = V(\kappa)\vec{\mathcal{N}} + U(\kappa)\vec{\mathcal{T}}$$

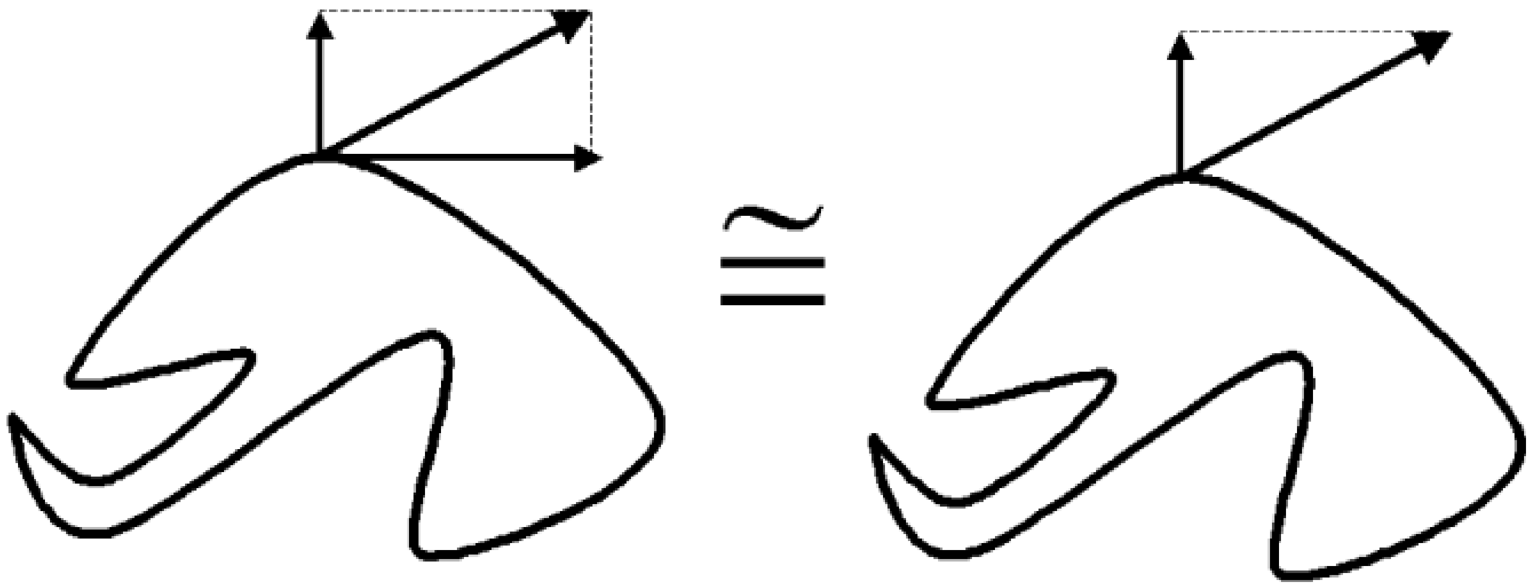
› which is equivalent to:

$$\frac{\partial C(p,t)}{\partial t} = V(\kappa)\vec{\mathcal{N}}$$

› where $V(\kappa)$ is speed function.

Geometric Deformable Model

› Curve Evolution Theory:



Geometric Deformable Model

- Curve Evolution Theory:
- Curvature deformation

$$\frac{\partial C(p,t)}{\partial t} = \alpha\kappa\vec{\mathcal{N}}$$

- where $\alpha > 0$, and $V(\kappa)$ is speed function.
- This equation will smooth a curve, eventually shrinking it to a circular point. (line as internal force)

Geometric Deformable Model

- Curve Evolution Theory:
- Constant deformation

$$\frac{\partial C(p,t)}{\partial t} = V_0 \vec{\mathcal{N}}$$

- where V_0 is a coefficient determining the speed and direction of deformation (like as pressure force)

Geometric Deformable Model

› Curve Evolution Theory:

› Hybrid deformation

$$\frac{\partial C(p,t)}{\partial t} = c(\kappa + V_0)\vec{\mathcal{N}}$$

› Image segmentation:

$$c = \frac{1}{1 + |\nabla(G_\sigma * I)|}$$

Geometric Deformable Model

› Level-Set approach for curve evolution:

› Suppose the evolving curve. $C(p,t)$, has level-set function:

$$\phi(\boldsymbol{x},t) \Leftrightarrow C(p,t)$$

› which means:

$$\phi(C(p,t),t) = \phi(x(p,t),y(p,t),t) = 0$$

Geometric Deformable Model

› Mathematics of level-set:

$$\phi(C(p,t),t) = \phi(x(p,t),y(p,t),t) = 0$$

1) Derivative with respect to t:

› $\frac{d\phi}{dt} = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial\phi}{\partial t} = 0$

› $\frac{d\phi}{dt} = \begin{bmatrix} \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \end{bmatrix} + \frac{\partial\phi}{\partial t} = 0$

$$\nabla\phi \cdot \frac{\partial C(p,t)}{\partial t} + \frac{\partial\phi}{\partial t} = 0$$

Geometric Deformable Model

› Mathematics of level-set:

$$\phi(C(p,t),t) = \phi(x(p,t),y(p,t),t) = 0$$

2) Derivative with respect to p:

› $\frac{d\phi}{dp} = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial p} = 0 \Rightarrow \frac{d\phi}{dt} = \begin{bmatrix}\frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y}\end{bmatrix}\begin{bmatrix}\frac{\partial x}{\partial p} \\ \frac{\partial y}{\partial p}\end{bmatrix} = 0 \Rightarrow \nabla\phi \cdot \vec{T} = 0 \Rightarrow$

$$\vec{\mathcal{N}} = -\frac{\nabla\phi}{|\nabla\phi|}$$

› $\frac{\nabla\phi}{|\nabla\phi|}$ is outward unit normal vector, we need inward for compatibility with curve evolution theory

Geometric Deformable Model

- Mathematics of level-set:

$$\phi(C(p,t),t) = \phi(x(p,t),y(p,t),t) = 0$$

3) curvature:

$$\kappa = \nabla.\left(\frac{\nabla\phi}{|\nabla\phi|}\right) = \frac{\phi_x^2\phi_{yy} - 2\phi_x\phi_y\phi_{xy} + \phi_y^2\phi_{xx}}{\left(\phi_x^2 + \phi_y^2\right)^{3/2}}$$

Geometric Deformable Model

- Level-Set equation:
- $\frac{\partial C(p,t)}{\partial t} = V(\kappa)\vec{\mathcal{N}}$
- $\vec{\mathcal{N}} = -\frac{\nabla\phi}{|\nabla\phi|}$
- $\nabla\phi \cdot \frac{\partial C(p,t)}{\partial t} + \frac{\partial\phi}{\partial t} = 0 \Rightarrow \nabla\phi \cdot \left(V(\kappa)\vec{\mathcal{N}}\right) + \frac{\partial\phi}{\partial t} = 0$
- $\frac{\partial\phi}{\partial t} = -\nabla\phi \cdot \left(V(\kappa)\left(-\frac{\nabla\phi}{|\nabla\phi|}\right)\right) = V(\kappa)\nabla\phi \cdot \left(\frac{\nabla\phi}{|\nabla\phi|}\right) \Rightarrow \frac{\partial\phi}{\partial t} = V(\kappa)|\nabla\phi|$

Geometric Deformable Model

- Level-Set for image segmentation:
- $\frac{\partial\phi}{\partial t} = c(\kappa + V_0)|\nabla\phi|$
- $c = \frac{1}{1+|\nabla(G_\sigma * I)|}$
- $\phi(C(p,0),0) = \phi_0(x,y)$
- where $\phi_0(x,y)$ is initial *signed distance function*

Geometric Deformable Model

› Level-Set equation:

› Note all previous equation are based on level-set equation:

$$\phi(C(p,t),t) = \phi(x(p,t),y(p,t),t) = 0$$

› Thus:

$$\frac{\partial \phi}{\partial t} = \delta(\phi)V(\kappa)|\nabla\phi|$$

› where $\delta(\phi)$ is delta Dirac function

Geometric Deformable Model

- How deal with $\delta(\phi)$:
- Let $H_\varepsilon(r)$ be a smooth approximation of Heaviside (unit step):
  - $H_\varepsilon(r) = \frac{1}{1+e^{-r/\varepsilon}}$
  - $H_\varepsilon(r) = 0.5\left(1 + \frac{2}{\pi}\tan^{-1}\frac{r}{\varepsilon}\right)$
  - and …
- then we may approximate $\delta(r)$ as $H'_\varepsilon(r)$

Geometric Deformable Model

- Since in general the level-set function does not retain its signed-distance-function property as it evolves in time through equation:

$$\frac{\partial \phi}{\partial t} = \delta(\phi)V(\kappa)|\nabla\phi|$$

- the following reinitialization equation has been introduced:

$$\frac{\partial \phi}{\partial \tau} = sign(\phi)(1 - |\nabla\phi|)$$

- Here, $sign$ is a smoothed-sign function and τ represents a fictitious time that controls the width of the band around the zero-level set where φ will be sign-distanced.

Geometric Deformable Model

- Variation #1: Geodesic Active Contour
- Recall curve length:

$$L = \int_0^1 \|C'(p,t)\| \, dp$$

- Let's define a new image-weighted Length:

$$L_w = \int_0^1 g\left(\left|\nabla I\left(C(p,t)\right)\right|\right)\|C'(p,t)\| \, dp$$

- Where $g(\cdot)$ is strictly decreasing function, and our aim is to minimize new curve length.

Geometric Deformable Model

- Variation #1: Geodesic Active Contour
- Calculus of variation gives:

$$\frac{\partial C(p,t)}{\partial t} = g(I)\kappa\vec{\mathcal{N}} - \left(\nabla g(I)\cdot\vec{\mathcal{N}}\right)\vec{\mathcal{N}}$$

- level-set equation become:

$$\frac{\partial \phi}{\partial t} = g(I)\kappa|\nabla\phi| + \nabla g(I)\cdot\nabla\phi$$

- variation for image segmentation:

$$\frac{\partial \phi}{\partial t} = g(I)(\kappa + V_0)|\nabla\phi| + \nabla g(I)\cdot\nabla\phi$$

The End

- AnY QuEsTiOn?