Mathematical Logic

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Proposition 12.0.1 (Tarski-Vaught Test):

Suppose $\mathcal{M}$ is a substructure of $\mathcal{N}$. Then $\mathcal{M}$ is an elementary substructure if and only if for any formula $\varphi(v, \vec{w})$ and $\vec{a} \in \mathcal{M}^n$, if there is a $b \in \mathcal{N}$ such that $\mathcal{N} \vDash \varphi(b, \vec{a})$, then there is a $c \in \mathcal{M}$ such that $\mathcal{N} \vDash \varphi(c, \vec{a})$.

Proof:

If $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$, then since

$$\mathcal{N} \vDash \exists x(\varphi(x, \vec{a}))$$

we have, by definition,

$$\mathcal{M} \vDash \exists x(\varphi(x, \vec{a}))$$

as required.

To show the converse, we must show that for every $\vec{a} \in \mathcal{M}^n$,

$$\mathcal{M} \vDash \varphi(\vec{a}) \iff \mathcal{N} \vDash \varphi(\vec{a})$$

we will prove this by formula induction. If φ is quantifier-free, then this is due to $\mathcal{M}$ being a substructure of $\mathcal{N}$. The induction step for boolean combinations is trivial. Now suppose

$$\mathcal{M} \vDash \exists x \varphi(x, \vec{a})$$

then there is a $b \in \mathcal{M}$ such that $\mathcal{M} \vDash \varphi(b, \vec{a})$ and since $b \in \mathcal{N}$, and so inductively $\mathcal{N} \vDash \varphi(b, \vec{a})$, which means that $\mathcal{N} \vDash \exists x \varphi(x, \vec{a})$, as required. And if

$$\mathcal{N} \vDash \exists x \varphi(x, \vec{a})$$

then there exists a $c \in \mathcal{N}$ such that $\mathcal{N} \vDash \varphi(c, \vec{a})$ which by our assumption means that $\mathcal{N} \vDash \varphi(b, \vec{a})$ for $b \in \mathcal{M}$ and thus $\mathcal{M} \vDash \varphi(b, \vec{a})$ so $\mathcal{M} \vDash \exists x \varphi(x, \vec{a})$ as required. ■

Definition 12.0.2:

An $\mathcal{L}$-theory T has **built-on Skolem functions** if for every $\mathcal{L}$-formula $\varphi(v, w_1, \dots, w_n)$ there is a function symbol f such that

$$T \vdash \forall \vec{w} \big((\exists v \varphi(v, \vec{w})) \to \varphi(f(\vec{w}), \vec{w}) \big)$$

Or in other words, if $\varphi(\cdot, \vec{w})$ can be witnessed, there is a function symbol f so that it can be witnessed by $f(\vec{w})$.

Lemma 12.0.3:

Let T be an $\mathcal{L}$-theory. Then there exists a signature $\mathcal{L} \subseteq \mathcal{L}^*$ and an $\mathcal{L}^*$-theory $T \subseteq T^*$ such that T^* has built-in Skolem functions. Furthermore, if $\mathcal{M} \vDash T$ then we can extend $\mathcal{M}$ to an $\mathcal{L}^*$-model $\mathcal{M}^*$ such that $\mathcal{M}^* \vDash T^*$. Even further, $\mathcal{L}^*$ can be chosen such that

$$|\mathcal{L}*| = |\mathcal{L}| + \aleph_0$$

Proof:

Let us construct an ascending sequence of languages $\{\mathcal{L}_i\}_{i=0}^{\infty}$, and an ascending sequence of theories $\{T_i\}_{i=0}^{\infty}$ where T_i is an $\mathcal{L}_i$-theory.

We define $\mathcal{L}_0 = \mathcal{L}$, and recursively

$$\mathcal{L}_{i+1} = \mathcal{L}_i \cup \{ f_\varphi \mid \varphi(v, w_1, \dots, w_n) \text{ is an } \mathcal{L}_i\text{-formula for } n = 1, 2, \dots \}$$

where f_φ is a function symbol. Then for an $\mathcal{L}_i$-formula $\varphi(v, \vec{w})$, we define

$$\Phi_\varphi = \forall \vec{w}\big((\exists v \varphi(v, \vec{w})) \to \varphi(f_\varphi(\vec{w}), \vec{w})\big)$$

Then we define

$$T_{i+1} = T_i \cup \{\Phi_\varphi \mid \varphi \text{ is an } \mathcal{L}_i\text{-formula}\}$$

Now we claim that if $\mathcal{M} \vDash T_i$, it can be extended to an $\mathcal{L}_{i+1}$-model of T_{i+1}. Let $c \in \mathcal{M}$, then if $\varphi(v, w_1, \dots, w_n)$ is an $\mathcal{L}_i$-formula, we define a function $g\colon \mathcal{M}^n \longrightarrow \mathcal{M}$ such that for every $\vec{a} \in \mathcal{M}^n$ if

$$X_{\vec{a}} = \{b \in \mathcal{M} \mid \mathcal{M} \vDash \varphi(b, \vec{a})\}$$

is non-empty ($\varphi(\cdot, \vec{a})$ has a witness), then let $g(\vec{a}) \in X_{\vec{a}}$. Otherwise $g(\vec{a}) = c$. Such a function is guaranteed by the axiom of choice.

Thus if $\mathcal{M} \vDash \exists v \varphi(v, \vec{a})$ then $X_{\vec{a}}$ is non-empty and so $\mathcal{M} \vDash \varphi(g(\vec{a}), \vec{a})$. So we interpret f_φ as g. And thus $\mathcal{M} \vDash \Phi_\varphi$.

Let us define

$$\mathcal{L}^* = \bigcup_{i=0}^\infty \mathcal{L}_i, \qquad T^* = \bigcup_{i=0}^\infty T_i$$

And $\mathcal{M}^*$ is the extension of $\mathcal{M}$ we have defined above. And if we have $\Phi \in T^*$, either $\Phi \in T$ in which case $\mathcal{M}^* \vDash \Phi$ as it extends $\mathcal{M}$, and otherwise it is equal to Φ_φ for some $\mathcal{L}^*$-formula φ, which must be an $\mathcal{L}_i$-formula for some i, and we showed that $\mathcal{M} \vDash \Phi_\varphi$. Thus $\mathcal{M}^* \vDash T^*$.

Then if $\varphi(v, \vec{w})$ is an $\mathcal{L}^*$-formula, it is a $\mathcal{L}_i$-formula for some i and so $\Phi_\varphi \in T_{i+1} \subseteq T^*$, and this states exactly the property of φ having a built-in Skolem function. (Note that φ may be Φ_ψ for some other $\mathcal{L}^*$-formula ψ).

Furthermore, note that since we've added function symbols to $\mathcal{L}_{i+1}$ for every formula of $\mathcal{L}_i$, we have that $|\mathcal{L}_{i+1}| = |\mathcal{L}_i| + \aleph_0$ (if $\mathcal{L}_i$ is uncountable then the number of functions added is $|\mathcal{L}_i| = |\mathcal{L}_i| + \aleph_0$, so this still holds). And so every $\mathcal{L}_i$ has the same cardinality for each $i > 0$, which is $|\mathcal{L}| + \aleph_0$. Thus their union, as a countable union, also has this cardinality. ■

Definition 12.0.4:

The T^* defined in the proof above is called the **skolemization** of T.

Theorem 12.0.5 (Downward Lowenheim-Skolem Theorem):

Suppose $\mathcal{M}$ is an $\mathcal{L}$-structure and $X \subseteq \mathcal{M}$. Then there exists an elementary substructure $\mathcal{N}$ of $\mathcal{M}$ such that $X \subseteq \mathcal{N}$ and $|\mathcal{N}| \leq |X| + |\mathcal{L}| + \aleph_0$.

Proof:

By the above lemma, we can assume $\mathrm{Th}(\mathcal{M})$ has built-in Skolem functions. Let $X_0 = X$, then we recursively define

$$X_{i+1} = X_i \cup \left\{f^{\mathcal{M}}(\vec{a}) \;\middle|\; f \text{ is an } n\text{-ary function symbol and } \vec{a} \in X_i^n \text{ for } n = 0, 1, 2, \dots\right\}$$

Then let

$$\mathcal{N} = \bigcup_{i=0}^\infty X_i$$

Notice that $|X_{i+1}| \leq |X_i| + |\mathcal{L}| \cdot \varkappa_{X_i}$ where

$$\varkappa_X = \left|\bigcup_{n\in\mathbb{N}} X^n\right|$$

We can split this into cases, but it is not hard to show that we get $|\mathcal{N}| \leq |X| + |\mathcal{L}| + \aleph_0$.

If f is an n-ary function symbol of $\mathcal{L}$ and $\vec{a} \in \mathcal{N}^n$, then there exists some i such that $\vec{a} \in X_i^n$, and so $f^{\mathcal{M}}(\vec{a}) \in X_{i+1} \subseteq \mathcal{N}$. Thus $f^{\mathcal{M}}$ can be restricted on $\mathcal{N}^n$, ie. $\mathcal{N}$ is a substructure of $\mathcal{M}$.

If $\varphi(v, \vec{w})$ is an $\mathcal{L}$-structure and $\mathcal{M} \vDash \varphi(b, \vec{a})$ then since $\mathcal{M}$ has built-in skolem functions, there exists some function f such that $\mathcal{M} \vDash \varphi(f(\vec{a}), \vec{a})$. But since $f^{\mathcal{M}}(\vec{a}) \in \mathcal{N}$, thus by **Tarski-Vaught Test**, $\mathcal{N}$ is an elementary substructure of $\mathcal{M}$. ■

12.1 Ehrenfeucht-Fraïssé Games

Definition 12.1.1:

If $\mathcal{M}$ and $\mathcal{N}$ are two $\mathcal{L}$-interpretations and $A \subseteq \mathcal{M}$ and $B \subseteq \mathcal{N}$, then a function $f: A \longrightarrow B$ is a **partial embedding** if we can extend f such that $f(c^{\mathcal{M}}) = c^{\mathcal{N}}$ for every constant symbol $c \in \mathcal{L}$ and f is a bijection which preserves relations and functions of $\mathcal{L}$.

If $\mathcal{M}$ and $\mathcal{N}$ are two $\mathcal{L}$-interpretations, we define the two-player game $G_\omega(\mathcal{M}, \mathcal{N})$ like so: during each round, player I either plays an element $m_i \in \mathcal{M}$ or an element $n_i \in \mathcal{N}$. If player I plays an element from $\mathcal{M}$, then player II must play an element $n_i \in \mathcal{N}$, and if player I played ane element $m_i \in \mathcal{M}$. Player II wins if the function $f(m_i) = n_i$ is a partial embedding, and player I wins otherwise.

For simplicity, we assume that $\mathcal{M}$ and $\mathcal{N}$ are disjoint.

We define a *strategy* for player II to be a function τ which maps finite sequences of elements in $\mathcal{M} \cup \mathcal{N}$ to an element of $\mathcal{M} \cup \mathcal{N}$. And player II *uses* the strategy if for each round if player I plays c_i, then player II plays $\tau(c_1, \ldots, c_n)$ (meaning in the first round player II plays $\tau(c_1)$, then $\tau(c_1, c_2)$, and so on).

A strategy τ is called a **winning strategy** if for any sequence $c_1, c_2, \ldots$ of moves by player I, if player II uses the strategy τ, player II will win. Strategies and winning strategies for player I are defined similarly.

Example 12.1.2:

If $\mathcal{M}, \mathcal{N} \vDash DLO$, then player II has a winning strategy. Suppose that for the nth round, a partial embedding $g: A \longrightarrow B$ has been created. If player I plays $a \in \mathcal{M}$ then player II plays $b \in \mathcal{N}$ such that the cut b induces in B (the partition of B into elements greater and less than b) is the image of the cut a induces in A. And if player I plays $b \in \mathcal{N}$ then player II plays $a \in \mathcal{M}$ such that the cut a induces in A is the preimage of the cut b induces in B. Such elements exist precisely because $\mathcal{M}$ and $\mathcal{N}$ are dense linear orders without endpoints.

This is a winning strategy since at every step, g is a partial embedding (as it preserves order between A and B, and order is the only symbol in $\mathcal{L}$).

Proposition 12.1.3:

If $\mathcal{M}$ and $\mathcal{N}$ are countable then player II has a winning if and only if $\mathcal{M}$ and $\mathcal{N}$ are isomorphic.

Proof:

If the interpretations are isomorphic, then player II can simply play according to the isomorphism.

Now suppose player II has a winning strategy. Let us enumerate $\mathcal{M}$ and $\mathcal{N}$ as $\{m_i\}_{i=0}^{\infty}$ and $\{n_i\}_{i=0}^{\infty}$ respectively. Then suppose player I plays $m_0, n_0, m_1, n_1, \ldots$ (if player II plays, say, n_0 after player I plays m_0 then player I skips it so the strategy is well-defined). Then the resulting partial embedding's domain must be $\mathcal{M}$ and its codomain must be $\mathcal{N}$, and so the partial embedding is a function $\mathcal{M} \to \mathcal{N}$. Since it is a bijection preserving functions, constants, and relations, it is an isomorphism as required. ∎

Now let $\mathcal{L}$ be a signature with no function symbols, and let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-interpretations. For $n \in \mathbb{N}$ we define the game $G_n(\mathcal{M}, \mathcal{N})$ as a game with n rounds with the same rules for each round as before, and the condition for winning is the same as well. (Meaning all we've changed is that there is a finite number of rounds.) The game $G_n(\mathcal{M}, \mathcal{N})$ is called an *Ehrenfeucht-Fraïsse Game*.

Lemma 12.1.4:

One of the players has a winning strategy in $G_n(\mathcal{M}, \mathcal{N})$.

Proof:

Suppose player II does not have a winning strategy, then this means that there exists a move player I can make on the first round for which player II cannot force a win. So player I starts with this move, and player II responds, and the game continues where for each round player I makes a move where player II cannot force a win. For the last round, there still must be a move player I can make for which player II has no winning move and so player I plays this and player II does not win and therefore player I does. ∎

Let us define some things we should've before:

Definition 12.1.5:

If $\mathcal{L}$ is a signature then let us define $\mathcal{L}^n$ to be the set of all $\mathcal{L}$-formulas φ such that $\text{free}(\varphi) \subseteq \{x_1, \dots, x_n\}$.

For example, $\mathcal{L}^0$ is the set of all closed formulas/sentences.

Definition 12.1.6:

Let us define the **quantifier-depth** of an $\mathcal{L}$-formula φ. This is done recursively on the construction of φ.

(1) $\text{depth}(\varphi) = 0$ for prime/atomic formulas φ (formulas consisting of only a relational symbol on terms).

(2) $\text{depth}(\neg\varphi) = \text{depth}(\varphi)$

(3) $\text{depth}(\varphi \wedge \psi) = \max\{\text{depth}(\varphi), \text{depth}(\psi)\}$

(4) $\text{depth}(\forall x\varphi) = \text{depth}(\varphi) + 1$

Notice that $\text{depth}(\varphi * \psi) = \max\{\text{depth}(\varphi), \text{depth}(\psi)\}$ for all logical operations $*$ by (2) and (3). And $\text{depth}(\exists x\varphi) = \text{depth}(\varphi) + 1$. Finally notice that $\text{depth}(\varphi) = 0$ if and only if φ is quantifier-free.

Definition 12.1.7:

If $\mathcal{M}$ and $\mathcal{N}$ are two $\mathcal{L}$-interpretations, then we say $\mathcal{M} \equiv_n \mathcal{N}$ if for every $\mathcal{L}$-sentence φ where $\text{depth}(\varphi) \leq n$ we have $\mathcal{M} \vDash \varphi$ if and only if $\mathcal{N} \vDash \varphi$.

Our goal is to show that player II has a winning strategy in $G_n(\mathcal{M}, \mathcal{N})$ if and only if $\mathcal{M} \equiv_n \mathcal{N}$.

Lemma 12.1.8:

For each n and m, there is a finite list of formulas $\varphi_1, \dots, \varphi_k \in \mathcal{L}^m$ whose depth is at most n, such that for every $\varphi \in \mathcal{L}^m$ whose depth is at most n is equivalent to some φ_i.

In other words there are only finitely many unique formulas of depth $\leq n$.

Proof:

Let us prove this for $n = 0$, ie. quantifier-free formulas. Since $\mathcal{L}$ contains no function symbols, the only terms in $\mathcal{L}$ using variables $x_1, \dots, x_n$ are these variables as well as the constant symbols in $\mathcal{L}$. Since $\mathcal{L}$ is finite, this means there are only finitely many terms in $\mathcal{L}$ using only the variables $x_1, \dots, x_n$. And so there are only finitely many atomic $\mathcal{L}^n$-formulas, which we enumerate as $\sigma_1, \dots, \sigma_t$. Let φ be a quantifier-free formula using only the variables $x_1, \dots, x_n$ then it has a disjunctive normal form, ie. there exists a set $S \subseteq \mathcal{P}(1, \dots, t)$ such that

$$\vdash \varphi \leftrightarrow \bigvee_{x \in S} \left(\bigwedge_{i \in X} \sigma_i \wedge \bigwedge_{i \notin X} \neg\sigma_i \right)$$

This gives 2^{2^t} formulas where every quantifier-free formula using only these variables is equivalent to one of them, as required.

Since formulas of depth $n+1$ are equivalent to boolean combinations of formulas of the form $\forall x\varphi$, where $\text{depth}(\varphi) \leq n$, and quantifier-free formulas, the rest of the proof follows by induction. ■

Lemma 12.1.9:

Let $\mathcal{L}$ be a finite signature without function symbols and $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-interpretations. Then player II has a winning strategy in $G_n(\mathcal{M}, \mathcal{N})$ if and only if $\mathcal{M} \equiv_n \mathcal{N}$.

Proof:

We prove this by induction on n. Suppose $\mathcal{M} \equiv_n \mathcal{N}$, and suppose player I plays $a \in \mathcal{M}$ on the first round. Then we claim there exists a $b \in \mathcal{N}$ where

$$\mathcal{M} \vDash \varphi(a) \iff \mathcal{N} \vDash \varphi(b)$$

for every formula where $\text{depth}(\varphi) < n$.

Let $\varphi_1(x), \ldots, \varphi_m(x)$ be the formulas of depth $< n$, and let

$$X = \{i \leq n \mid \mathcal{M} \vDash \varphi_i(a)\}$$

and let

$$\Phi(x) = \bigwedge_{i \in X} \varphi_i(x) \wedge \bigwedge_{i \notin X} \neg\varphi_i(x)$$

then $\operatorname{depth}(\exists x\Phi) \leq n$ and $\mathcal{M} \vDash \Phi(a)$ and so $\mathcal{M} \vDash \exists x\Phi$, and since $\mathcal{M} \equiv_n \mathcal{N}$ this means that $\mathcal{N} \vDash \exists x\Phi$ and so there exists a $b \in \mathcal{N}$ where $\mathcal{N} \vDash \Phi(b)$. Notice that if $\mathcal{M} \vDash \varphi_i(a)$ then $i \in X$ and so $\varphi_i(b)$ is part of the conjunction of $\Phi(b)$ and so $\mathcal{N} \vDash \varphi_i(b)$. And if $\mathcal{M} \vDash \neg\varphi_i(a)$, then $i \notin X$ and so $\neg\varphi_i(b)$ is part of the conjunction of $\Phi(b)$ so $\mathcal{N} \vDash \neg\varphi_i(b)$. So $\mathcal{M} \vDash \varphi_i(a)$ if and only if $\mathcal{N} \vDash \varphi_i(b)$ for every i. Since φ is equivalent to some φ_i, $\mathcal{M} \vDash \varphi(a)$ if and only if $\mathcal{N} \vDash \varphi(b)$, as required.

So have player II play b in the first round.

If $n = 1$, then $a \mapsto b$ is a partial embedding (since it preserves all $\mathcal{L}^1$-formulas), and so the game has finished and player II has won. Otherwise suppose $n > 1$, then let $\mathcal{L}^* = \mathcal{L} \cup \{c\}$ where c is a new constant symbol, and extend $\mathcal{M}$ and $\mathcal{N}$ as $\mathcal{L}^*$-interpretations where they interpret c as a and b respectively. Let these extensions be $\mathcal{M}^*$ and $\mathcal{N}^*$. Since

$$\mathcal{M} \vDash \varphi(a) \iff \mathcal{N} \vDash \varphi(b)$$

for $\operatorname{depth}(\varphi) < n$, $\mathcal{M}^* \equiv_{n-1} \mathcal{N}^*$. So by our inductive assumption, player II has a winning strategy in $G_{n-1}(\mathcal{M}^*, \mathcal{N}^*)$.

If player I's second play is d, then player II responds as if d was player I's first play in $G_{n-1}(\mathcal{M}^*, \mathcal{N}^*)$ and continues using this winning strategy. The resulting function $f^* \colon X \longrightarrow \mathcal{N}$ is a partial $\mathcal{L}^*$-embedding, and so it preserves constants and relations of $\mathcal{L}$. Let us extend f^* to $f \colon X \cup \{a\} \longrightarrow \mathcal{N}$ by $f(a) = b$, this is the function created in the game $G_n(\mathcal{M}, \mathcal{N})$. Since f^* is a partial $\mathcal{L}^*$-embedding, it can be extended to a bijection preserving the relations and constants of $\mathcal{L}^*$, and in particular since $c^{\mathcal{M}^*} = a$ and $c^{\mathcal{N}^*} = b$, its extension must map a to b. And thus this extension is also an extension of f, and so f can be extended to a $\mathcal{L}$-preserving bijection as required.

Now suppose $\mathcal{M} \not\equiv_n \mathcal{N}$. Since formulas of depth $\leq n$ are boolean combinations of formulas of the form $\exists x\varphi(x)$ where $\operatorname{depth}(\varphi) < n$, $\mathcal{M}$ and $\mathcal{N}$ must disagree on a formula of this type. So we can assume $\mathcal{M} \vDash \exists x\varphi(x)$ and $\mathcal{N} \vDash \neg\exists x\varphi(x) = \forall x\neg\varphi(x)$ where $\operatorname{depth}(\varphi) < n$. We claim that player I has a winning strategy.

For the first round, player I plays $a \in \mathcal{M}$ such that $\mathcal{M} \vDash \varphi(a)$. If player II responds with $b \in \mathcal{N}$, let us again extend $\mathcal{M}$ and $\mathcal{N}$ to $\mathcal{M}^*$ and $\mathcal{N}^*$. Then $\mathcal{M}^* \not\equiv_{n-1} \mathcal{N}^*$ since $\mathcal{M}^* \vDash \varphi(c)$ and $\mathcal{N}^* \vDash \neg\varphi(c)$. So inductively, player I has a winning strategy in $G_{n-1}(\mathcal{M}^*, \mathcal{N}^*)$, and the resulting function f^* is not a partial embedding, and so no extension of it is not a partial embedding, as required. ■

Since $\mathcal{M} \equiv \mathcal{N}$ if and only if $\mathcal{M} \equiv_n \mathcal{N}$ for all n, the following theorem follows trivially from this lemma.

Theorem 12.1.10:

If $\mathcal{M}$ adn $\mathcal{N}$ are two $\mathcal{L}$-interpretations, then $\mathcal{M} \equiv \mathcal{N}$ if and only if player II has a winning strategy in $G_n(\mathcal{M}, \mathcal{N})$ for every n.

Let us look at an example usage of Ehrenfeucht-Fraïsé Games. Let $\mathcal{L} = \{<\}$ and T be the theory of discrete linear orders without endpoints, ie. the set of axioms for linear orders as well as the axiom

$$\forall x \exists y \exists z \big(y < x < z \wedge \forall w(w \leq y \vee w = x \vee w \geq z)\big)$$

z is the *successor* of x in this theory. Suppose $\mathcal{N} \vDash T$ then we define the equivalence relation E on $\mathcal{N}$ where aEb if and only if b the nth successor or predecessor of a for some n. Then each equivalence class of $\mathcal{N}$ is itself a model of T and is isomorphic to $(\mathbb{Z}, <)$ (an isomorphism can be constructed by choosing some a in the equivalence class and mapping it to 0, and mapping its nth successor to n and nth predecessor to $-n$).

Notice that if aEb and $\neg(aEc)$, and if $a < c$ then $b < c$ (since if $c < b$ it would have to be a successor of a as well), and so the equivalence classes are linearly ordered as well. Thus every model of T is isomorphic to $(L \times \mathbb{Z}, <)$ where L is some linearly ordered set and $<$ is the lexicographic ordering on $L \times \mathbb{Z}$. And every structure of this form is a model of T.

Proposition 12.1.11:

The theory of discrete linear orders without endpoints is complete.

Proof:

Let $\mathcal{M}$ be the model $(\mathbb{Z}, <)$ and let $\mathbb{N} = (L \times \mathbb{Z}, <)$ where L is a linear order and $<$ is the lexicographic ordering on $L \times \mathbb{Z}$. We claim that $\mathcal{M} \equiv \mathbb{N}$, and we will prove this by developing a winning strategy for player II in $G_n(\mathcal{M}, \mathcal{N})$.

For $a, b \in \mathbb{Z}$ let us define

$$\operatorname{dist}(a, b) = |a - b|$$

And for (i, a), $(j, b) \in L \times \mathbb{Z}$, let us define

$$\operatorname{dist}((i,a),\,(j,b)) = \begin{cases} |a-b| & i = j \\ \infty & i \neq j \end{cases}$$

Our goal is to try and ensure that after m rounds of the game, if $a_1 < a_2 < \cdots < a_m$ are the elements of $\mathcal{N}$ which have been played, and $b_1 < b_2 < \cdots < b_m$ are the elements in $\mathbb{Z}$ which have been played, then the mapping $a_i \mapsto b_i$ is the partial embedding corresponding to the play of the game. Furthermore, if $\operatorname{dist}(a_i, a_{i+1}) > 3^{n-m}$ then $\operatorname{dist}(b_i, b_{i+1}) > 3^{n-m}$, and if $\operatorname{dist}(a_i, a_{i+1}) \leq 3^{n-m}$ then $\operatorname{dist}(b_i, b_{i+1}) = \operatorname{dist}(a_i, a_{i+1})$, for $i = 1, \ldots, m-1$.

Obviously since $a_i < a_{i+1}$ and $b_i < b_{i+1}$, the function will preserve the relations of the theory and thus be a partial embedding.

We claim that player II can always make a move to preserve this condition. In round 1, player II can choose any arbitrary element and the condition will hold. Now suppose we have played m rounds and $a_1 < \cdots < a_m$ and $b_1 < \cdots < b_m$ be defined as above. Now suppose player I plays $b \in L \times \mathbb{Z}$, then there are several cases

(1) If $b < b_1$ then if $\operatorname{dist}(b, b_1) = k < \infty$ then player II plays $a_1 - k$. If $\operatorname{dist}(b, b_1) = \infty$ then player II plays $a_1 - 3^n$, but in any case the condition holds.

(2) If $b_i < b < b_{i+1}$ and $\operatorname{dist}(b_i, b_{i+1}) \leq 3^{n-m}$ then $\operatorname{dist}(a_i, a_{i+1}) = \operatorname{dist}(b_i, b_{i+1})$. Play $a = a_i + \operatorname{dist}(b, b_i)$, then $\operatorname{dist}(a, a_{i+1}) = \operatorname{dist}(b, b_{i+1})$ as required.

(3) If $b_i < b < b_{i+1}$ and $\operatorname{dist}(b_i, b_{i+1}) > 3^{n-m}$ and $\operatorname{dist}(b, b_i) < 3^{n-m-1}$ then $\operatorname{dist}(a_i, a_{i+1}) > 3^{n-m}$. Play $a = a_i + \operatorname{dist}(b, b_i)$, then $\operatorname{dist}(a, a_{i+1})$ and $\operatorname{dist}(a_i, a)$ are greater than 3^{n-m-1} as required.

(4) If $b_i < b < b_{i+1}$ and $\operatorname{dist}(b_i, b_{i+1}) > 3^{n-m}$ and $\operatorname{dist}(b, b_{i+1}) < 3^{n-m-1}$, play $a = a_{i+1} - \operatorname{dist}(b, b_{i+1})$.

(5) If $b_i < b < b_{i+1}$ and $\operatorname{dist}(b_i, b_{i+1}) > 3^{n-m}$, $\operatorname{dist}(b, b_i) > 3^{n-m-1}$, and $\operatorname{dist}(b, b_{i+1}) < 3^{n-m-1}$. Then $\operatorname{dist}(a_i, a_{i+1}) > 3^{n-m}$ and so choose an a such that $a_i < a < a_{i+1}$ and the distance of a between them both is greater than 3^{n-m-1}. Playing a satisfies the condition.

(6) If $b > b_m$, this is similar to the first condition.

Thus player II has a winning strategy, and so $\mathcal{M} \equiv \mathcal{N}$, meaning all models of T are elementarily equivalent so T is complete. ■