Mathematical Logic

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3.1 Soundness and Completeness

We prove some corollaries of the deduction theorem:

Corollary 3.1.1:

For any well-formed formulas φ and ψ of $\mathcal{L}$:

(1) $\varphi \to \psi, \varphi \to \mu \vdash \varphi \to \mu$

(2) $\varphi \to (\psi \to \mu), \psi \vdash \varphi \to \mu$

Proof:

(1) By the deduction theorem, this is equivalent to proving

$$\varphi \to \psi, \varphi \to \mu, \varphi \vdash \mu$$

Since we have φ and $\varphi \to \psi$, by modus ponens we have ψ and since $\psi \to \mu$, we have μ as required.

(2) Again, this is equivalent to proving

$$\varphi \to (\psi \to \mu), \psi, \varphi \vdash \mu$$

And by modus ponens we have $\psi \to \mu$ and again by modus ponens we have μ as required. ■

Lemma 3.1.2:

For any well-formed formulas φ and ψ of $\mathcal{L}$, the following are theorems:

(1) $\neg\neg\varphi \to \varphi$

(2) $\varphi \to \neg\neg\varphi$

(3) $\neg\varphi \to (\varphi \to \psi)$

(4) $(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$

(5) $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$

(6) $\varphi \to \big(\neg\psi \to \neg(\varphi \to \psi)\big)$

(7) $(\varphi \to \psi) \to \big((\neg\varphi \to \psi) \to \psi\big)$

Proof:

(1) $\neg\neg\varphi \to \varphi$:

(i) $(\neg\varphi \to \neg\neg\varphi) \to \big((\neg\varphi \to \neg\varphi) \to \varphi$; Axiom 3 (**A3**) for φ and $\neg\varphi$.

(ii) $\neg\varphi \to \neg\varphi$; Lemma 2.2.3.

(iii) $(\neg\varphi \to \neg\neg\varphi) \to \varphi$; (i), (ii), Corollary 3.1.1 (2).

(iv) $\neg\neg\varphi \to (\neg\varphi \to \neg\neg\varphi)$; **A1** for $\neg\varphi$ and $\neg\neg\varphi$.

(v) $\neg\neg\varphi \to \varphi$; (iii), (iv), Corollary 3.1.1 (1).

(2) $\varphi \to \neg\neg\varphi$

(i) $(\neg\neg\neg\varphi \to \neg\varphi) \to \big((\neg\neg\neg\varphi \to \varphi) \to \neg\neg\varphi\big)$; **A3** for φ and $\neg\neg\varphi$.

(ii) $\neg\neg\neg\varphi \to \neg\varphi$; Part (1).

(iii) $(\neg\neg\neg\varphi \to \varphi) \to \neg\neg\varphi$; (i), (ii), modus ponens (*MP*).

(iv) $\varphi \to (\neg\neg\neg\varphi \to \varphi)$; **A3** for φ and $\neg\neg\varphi$.

(v) $\varphi \to \neg\neg\varphi$; (iii), (iv), Corollary 3.1.1 (1).

(3) $\neg\varphi \to (\varphi \to \psi)$

(i) $\neg\varphi$; hypothesis (meaning we are showing $\neg\varphi \vdash \varphi \to \psi$).

(ii) φ; hypothesis.

(iii) $\varphi \to (\neg\psi \to \varphi)$; **A1**.

(iv) $\neg\varphi \to (\neg\psi \to \neg\varphi)$; **A1**.

(v) $\neg\varphi \to \psi$; (ii), (iii), *MP*.

(vi) $\neg\psi \to \neg\varphi$; (i), (iv), *MP*.

(vii) $(\neg\psi \to \neg\varphi) \to \big((\neg\psi \to \varphi) \to \psi\big)$; **A3**.

(viii) $(\neg\psi \to \varphi) \to \psi$; (vi), (vii), *MP*.

(ix) ψ; (v), (viii), *MP*.

(x) $\varphi, \neg\varphi \vdash \psi$; (i)–(ix).

(xi) $\neg\varphi \vdash \varphi \to \psi$; (x), deduction theorem.

(xii) $\neg\varphi \to (\varphi \to \psi)$; (xi), dedcution theorem.

(4) $(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$

(i) $\neg\psi \to \neg\varphi$; hypothesis.

(ii) $(\neg\psi \to \neg\varphi) \to \big((\neg\psi \to \varphi) \to \psi\big)$; **A3**.

(iii) $\varphi \to (\neg\psi \to \varphi)$; **A1**.

(iv) $(\neg\psi \to \varphi) \to \psi$; (i), (ii), *MP*.

(v) $\varphi \to \psi$; (iii), (iv), Corollary 3.1.1 (1).

(vi) $\neg\psi \to \neg\varphi \vdash \varphi \to \psi$; (i)–(v).

(vii) $(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$; (vi), deduction theorem.

(5) $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$

(i) $\varphi \to \psi$; hypothesis.

(ii) $\neg\neg\varphi \to \varphi$; part (1).

(iii) $\neg\neg\varphi \to \psi$; (i), (ii), Corollary 3.1.1 (1).

(iv) $\psi \to \neg\neg\psi$; part (4).

(v) $\neg\neg\varphi \to \neg\neg\psi$; (iii), (iv), Corollary 3.1.1 (1).

(vi) $(\neg\neg\varphi \to \neg\neg\psi) \to (\neg\psi \to \neg\varphi)$; part (4).

(vii) $\neg\psi \to \neg\varphi$; (v), (vi), *MP*.

(viii) $\varphi \to \psi \vdash \neg\psi \to \neg\varphi$; (i)–(vii).

(ix) $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$; (viii), deduction theorem.

(6) $\varphi \to \big(\neg\psi \to \neg(\varphi \to \psi)\big)$

(i) $\varphi \to \psi, \varphi \vdash \psi$; this is clear by *MP*.

(ii) $\varphi \to \big((\varphi \to \psi) \to \psi\big)$; (i), deduction theorem (twice).

(iii) $\big((\varphi \to \psi) \to \psi\big) \to \big(\neg\psi \to \neg(\varphi \to \psi)\big)$; (ii), part (5).

(iv) $\varphi \to \big(\neg\psi \to \neg(\varphi \to \psi)\big)$; (ii), (iii), Corollary 3.1.1 (1).

(7) $(\varphi \to \psi) \to \big((\neg\varphi \to \psi) \to \psi\big)$

(i) $\varphi \to \psi$; hypothesis.

(ii) $\neg\varphi \to \psi$; hypothesis.

(iii) $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$; part (5).

(iv) $\neg\psi \to \neg\varphi$; (i), (iii), *MP*.

(v) $(\neg\varphi \to \psi) \to (\neg\psi \to \neg\neg\varphi)$; part (5).

(vi) $\neg\psi \to \neg\neg\varphi$; (ii), (v), *MP*.

(vii) $(\neg\psi \to \neg\neg\varphi) \to \big((\neg\psi \to \neg\varphi) \to \psi\big)$; **A3**.

(viii) $(\neg\psi \to \neg\varphi) \to \psi$; (vi), (vii), *MP*.

(ix) ψ; (iv), (viii), *MP*.

(x) $\varphi \to \psi, \neg\varphi \to \psi \vdash \psi$; (i)–(ix).

(xi) $(\varphi \to \psi) \to \big((\neg\varphi \to \psi) \to \psi\big)$; (x), deduction theorem (twice). ■

Definition 3.1.3:

We define the following connectives as shorthands:

(1) $(\varphi \wedge \psi)$ for $\neg(\varphi \to \neg\psi)$.

(2) $(\varphi \vee \psi)$ for $(\neg\varphi) \to \psi$.

(3) $(\varphi \leftrightarrow \psi)$ for $(\varphi \to \psi) \wedge (\psi \to \varphi)$, meaning $\neg\big((\varphi \to \psi) \to \neg(\psi \to \varphi)\big)$.

Exercise 3.1.4:

Show the following:

(1) $\varphi \to (\varphi \vee \psi)$

(2) $\varphi \to (\psi \vee \varphi)$

(3) $(\varphi \vee \psi) \to (\psi \vee \varphi)$

(4) $(\varphi \wedge \psi) \to \varphi$

(5) $(\varphi \wedge \psi) \to \psi$

(6) $(\varphi \to \mu) \to \Big((\psi \to \mu) \to (\varphi \vee \psi) \to \mu\Big)$

(7) $\big((\varphi \to \psi) \to \varphi\big) \to \varphi$

(8) $\varphi \to \big(\psi \to (\varphi \wedge \psi)\big)$

Our goal now is to show that a well-formed formula of $\mathcal{L}$ is a theorem if and only if it is a tautology in the sense of statement forms. The first part of this is simple.

Theorem 3.1.5 (Soundness Theorem):

Every theorem of $\mathcal{L}$ is a tautology.

Proof:

It can be shown with relative ease that every axiom of $\mathcal{L}$ is a tautology. Given some theorem φ, we must have a proof of length n, which we induct on. For $n = 1$, φ is an axiom. Otherwise, either φ is an we must have that $\psi \to \varphi$ and ψ are well-formed formulas in the proof. Thus they can both be proven in fewer than n steps and by our inductive hypothesis are thus tautologies. Therefore since ψ is always true and $\psi \to \varphi$ is always true, we can infer that φ is always true (a tautology). This last step takes place entirely in the world of statement forms/boolean functions. ■

What this means is that propositional calculus is *sound*: everything that can be proven is true. We know continue with the other half.

Lemma 3.1.6:

Let φ be a well-formed formula and $B_1, \ldots, B_k$ the statement letters which occur in φ. For some assignment of truth values to these statement letters, define B_j' to be B_j if it is true, and $\neg B_j$ if it is false. Then let φ' be φ if it is true under this assignment, and $\neg\varphi$ if φ is false. Then

$$B_1', \ldots, B_k' \vdash \varphi'$$

Proof:

We induct on n, the number of occurrences of $\neg$ or $\to$ in φ (we assume that φ is written without shorthands). If $n = 0$, then φ is just a single statement letter $\varphi = B_1$ then this reduces to $B_1 \vdash B_1$ and $\neg B_1 \vdash \neg B_1$ which are both trivial.

Otherwise, we split into two cases:

For the first case, $\varphi = \neg\psi$. Then ψ has fewer than n occurrences of $\neg$ and $\to$. Under the given truth value assignments, we again have two possibilities: if ψ is true then φ is false. Thus ψ' is ψ and $\varphi' = \neg\varphi = \neg\neg\psi$. By our inductive hypothesis

$$B_1', \ldots, B_n' \vdash \psi' = \psi$$

By **lemma 3.1.2** (2), $\psi \to \neg\neg\psi$ so

$$B_1', \ldots, B_n' \vdash \neg\neg\psi = \varphi'$$

as required. And if ψ is true, then φ is true and ψ' is $\neg\psi$ and φ' is φ, so $\psi' = \varphi$. And by our inductive hypothesis:

$$B_1', \ldots, B_n' \vdash \psi' = \varphi$$

as required.

For the second case, $\varphi = \psi \to \mu$. We have three possibilities here: if ψ is false then φ takes on the value true. So $\varphi' = \neg\varphi$ and $\varphi' = \varphi$, so by our inductive hypothesis:

$$B_1', \ldots, B_n' \vdash \neg\psi$$

and by **lemma 3.1.2** (3), $\neg\psi \to (\psi \to \mu)$, so

$$B_1', \ldots, B_n' \vdash \psi \to \mu = \varphi$$

as required. And if μ is true then again φ takes the value true. So $\mu' = \mu$ and $\varphi' = \varphi$ and so by our inductive hypothesis

$$B_1', \ldots, B_n' \vdash \mu$$

and by **A1** $\mu \to (\psi \to \mu)$, so $\mu \to \varphi$ so by modus ponens

$$B_1', \ldots, B_n' \vdash \varphi$$

as required. The final possibility is that ψ is true and μ is false, then φ is false. So $\psi' = \psi$ and $\mu' = \neg\mu$ and $\varphi' = \neg\varphi = \neg(\psi \to \mu)$, thus by our inductive hypothesis

$$B_1', \ldots, B_n' \vdash \psi, \neg\mu$$

By **lemma 3.1.2** (6) $\psi \to \big(\neg\mu \to \neg(\psi \to \mu)\big)$, thus by applying modus ponens twice we have $\neg(\psi \to \mu) = \neg\varphi = \psi'$ as required. ■

Theorem 3.1.7 (Completeness Theorem):

If a well-formed formula φ of $\mathcal{L}$ is a tautology, then it is a theorem of $\mathcal{L}$.

Proof:

Let $B_1, \ldots, B_n$ be the statement letters in φ. For any truth value assignment to these letters, we have $B_1', \ldots, B_n' \vdash \varphi$ since $\varphi' = \varphi$ as φ is always true. So when $B_n = \mathsf{true}$ we have $B_1', \ldots, B_n, B_n \vdash \varphi$ and when $B_n = \mathsf{false}$ we have $B_1', \ldots, B_{n-1}', \neg B_n \vdash \varphi$, so by the deduction theorem

$$B_1', \ldots, B_{n-1}' \vdash (B_n \to \varphi),\ (\neg B_n \to \varphi)$$

for any truth value assignment to $B_1, \ldots, B_{n-1}$. Thus by **lemma 3.1.2** (7) we have $(B_n \to \varphi) \to \big((\neg B_n \to \varphi) \to \varphi\big)$ and so applying modus ponens twice gives

$$B_1', \ldots, B_n' \vdash \varphi$$

We can continue inductively and after n steps we have

$$\vdash \varphi$$

as required. ■

Corollary 3.1.8:

If ψ is an expression involving the signs $\neg$, $\to$, $\vee$, $\wedge$, and $\leftrightarrow$ which is a shorthand for a well-formed formula φ of $\mathcal{L}$, then ψ is a tautology if and only if φ is a theorem of $\mathcal{L}$.

Proof:

Since ψ is a tautology if and only if φ is (it remains an exercise to show that the definitions of the shorthands are logically equivalent to the connectives) by the **Soundness Theorem**, if φ is a theorem then it is a tautology and therefore so is ψ. And if ψ is a tautology then so is φ and therefore φ is a theorem by the **Completeness Theorem**.

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