Modern Analysis

Homework 2

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2.1 Exercise

Let $X = S^1 \times I$, show that $A = S^1 \times \left\{\frac{1}{2}\right\}$ is a deformation retract.

Define $r\colon X \longrightarrow A$ by $r(x,t) = (x, 1/2)$. This is continuous (the composition of $(x,t) \mapsto x$ and $x \mapsto (x, 1/2)$) and $r(x, 1/2) = (x, 1/2)$ for all $(x, 1/2) \in A$ so it is a retraction. Now we must show that $\iota \circ r \overset{A}{\sim} \mathrm{id}_X$, so we need to define a homotopy $H\colon X \times I \longrightarrow X$ such that $H((x,t), 0) = r(x,t) = (x, 1/2)$, $H((x,t), 1) = (x,t)$ and $H((x, 1/2), s) = (x, 1/2)$. So let us define $H((x,t), s) := \left(x, \frac{1}{2} - s\left(\frac{1}{2} - t\right)\right)$. This is continuous and indeed we have

$$H((x,t), 0) = \left(x, \frac{1}{2}\right), \qquad H((x,t), 1) = (x,t), \qquad H\left(\left(x, \frac{1}{2}\right), s\right) = \left(x, \frac{1}{2}\right)$$

meaning r is a deformation retract, as required.

2.2 Exercise

Let X be a contractible space and Y an arbitrary topological space.

(1) show that all morphisms $Y \longrightarrow X$ are null-homotopic, and all are homotopic to one another.

(2) show that all morphisms $X \longrightarrow Y$ are null-homotopic, and if Y is path connected they are all homotopic to one another.

(1) Since X is contractible, the identity map is homotopic to some constant map, suppose $\mathrm{id}_X \sim p$. Now let $f\colon Y \longrightarrow X$, we claim that $f \sim p$ (here $y \mapsto p$ is a map $Y \longrightarrow X$, above it is $X \longrightarrow X$). Suppose $H\colon X \times I \longrightarrow X$ is a homotopy from id_X to p, so $H(x, 0) = x$ and $H(x, 1) = p$. Then define $K\colon Y \times I \longrightarrow X$ by $K(y,t) = H(f(y), t)$ which is continuous as the composition of continuous functions. And

$$K(y, 0) = H(f(y), 0) = f(y), \qquad K(y, 1) = H(f(y), 1) = p$$

so K is a homotopy from f to p, meaning all maps $Y \longrightarrow X$ are homotopic to the constant map $y \mapsto p$, meaning they are all homotopic to one another and null-homotopic.

(2) Let $f\colon X \longrightarrow Y$ and H be a homotopy from id_X to p as above, then define $K\colon X \times I \longrightarrow Y$ by $K = f \circ H$ which is continuous and

$$K(x, 0) = f \circ H(x, 0) = f(x), \qquad K(x, 1) = f \circ H(x, 1) = f(p)$$

So f is homotopic to the constant map $x \mapsto f(p)$, and is therefore null-homotopic.

If Y is path connected, then every two constant maps $x \mapsto p$ and $x \mapsto q$ (which are morphisms $X \longrightarrow Y$) are homotopic. This is since if γ is a path from p to q, define $H(x,t) = \gamma(t)$. So $H(x, 0) = \gamma(0) = p$ and $H(x, 1) = \gamma(1) = q$, so $p \sim q$ as required. And since all maps $X \longrightarrow Y$ are homotopic to a constant map, which are all homotopic, all maps $X \longrightarrow Y$ are homotopic.

2.3 Exercise

(1) Show that a contractible space is path connected.

(2) X is called **simply connected** if it is path connected and every morphism $S^1 \longrightarrow X$ is null-homotopic. Show that a contractible space is simply connected.

(**1**) First we claim that X is contractible if and only if $\mathrm{id}_X \sim p$ for all $p \in X$. The right-to-left direction is trivial. Now suppose $\mathrm{id}_X \sim p$ and $q \in X$, then we claim $p \sim q$ and so $\mathrm{id}_X \sim q$ as required. Let $H\colon X \times I \longrightarrow X$ be a homotopy from id_X to p, then define $K(x,t) = H(q,t)$ which is continuous. Then $K(x,0) = H(q,0) = q$ and $K(x,1) = H(q,1) = p$, so K is a homotopy from p to q as required.

Now let $p, q \in X$, then there exists a homotopy $H\colon X \times I \longrightarrow X$ between them. Define $\gamma(t) = H(x_0, t)$ for any $x_0 \in X$. This is continuous and $\gamma(0) = H(x_0, 0) = p$ and $\gamma(1) = H(x_0, 1) = q$ so p and q are path connected.

(**2**) By 2(1), every morphism $Y \longrightarrow X$ is null-homotopic, and so in particular every morphism $S^1 \longrightarrow X$.

2.4 Exercise

Show that a retract of a contractible space is contractible.

Let X be contractible, $r\colon X \longrightarrow A$ a retraction. Take $a \in A$, then by the previous question $\mathrm{id}_X \sim a$, so let $H\colon X \times I \longrightarrow X$ be a homotopy from id_X to a. Define $K\colon A \times I \longrightarrow A$ by $K = r \circ H$, then

$$K(x,0) = r \circ H(x,0) = r(x) = x \text{ (since } x \in A \text{ so } r(x) = x\text{)}, \qquad K(x,1) = r \circ H(x,1) = r(a) = a$$

and so $\mathrm{id}_A \sim a$, meaning A is contractible as well.

2.5 Exercise

Show that a space is contractible if and only if it is homotopically equivalent to a space with a single point.

Suppose X is contractible, and let $a \in X$, we claim $X \simeq \{a\}$. Since X is contractible, by above we have $\mathrm{id}_X \sim a$ and so there exists a homotopy $H\colon X \times I \longrightarrow X$ from id_X to a. Define $f\colon X \longrightarrow \{a\}$ and $g\colon \{a\} \longrightarrow X$ by $f(x) = a$ and $g(a) = a$. Then $f \circ g = \mathrm{id}_{\{a\}}$ and $g \circ f = a \sim \mathrm{id}_X$, so f is a homotopic equivalence between X and $\{a\}$ as required.

Now suppose $X \simeq \{a\}$, so there exists homotopic equivalences $f\colon X \longrightarrow \{a\}$ and $g\colon \{a\} \longrightarrow X$. Necessarily $f(x) = a$, and so $g \circ f = a$. But since we are given that these are homotopic equivalences, $g \circ f \sim \mathrm{id}_X$, so $\mathrm{id}_X \sim a$, meaning X is contractible.

2.6 Exercise

Let $f, g\colon (X,a) \longrightarrow (Y,b)$ two morphisms homotopic relative to $\{a\}$. Show that $f_* = g_*$.

Recall that $f_*([\gamma]) = [f \circ \gamma]$, so we must show that $f \circ \gamma \overset{\partial I}{\sim} g \circ \gamma$ for every $\gamma \in \Gamma_{aa}$. Since $f \overset{\{a\}}{\sim} g$, there exists a homotopy relative to $\{a\}$: $H(x,0) = f(x)$, $H(x,1) = g(x)$, and $H(a,t) = f(a) = g(a)$. So define $K\colon I \times I \longrightarrow Y$ by $K(t,s) = H(\gamma(t), s)$. Then

$$K(t,0) = H(\gamma(t),0) = f \circ \gamma(t), \qquad K(t,1) = H(\gamma(t),1) = g \circ \gamma(t),$$
$$K(0,s) = H(\gamma(0),s) = H(a,s) = f(a) = f \circ \gamma(0), \qquad K(1,s) = H(a,s) = f \circ \gamma(1)$$

So K is a homotopy relative to $\partial I = \{0,1\}$ between $f \circ \gamma$ and $g \circ \gamma$ as required.