Infinitesimal Calculus 3

Lecture 21, Wednsday January 11, 2023
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Theorem 21.1:

Our goal is to find a crtitical point for $f(x,y,z) \in C^1$ with constraint $h(x,y,z) = 0$. Then there is a $\lambda \in \mathbb{R}$ such that $\nabla f = \lambda \nabla g$ at the critical point.

Proof:

By the implicit function theorem there is a function $\varphi(x,y) \in C^1$ such that $z = \varphi(x,y)$ and the constraint is $h(x,y,\varphi(x,y)) = 0$ in a neighborhood. So our function becomes $f(x,y,\varphi(x,y))$. And so by the chain rule at the crtitical point

$$\begin{cases} f_x + f_z \cdot \dfrac{\partial \varphi}{\partial x} = 0 \\ f_y + f_z \cdot \dfrac{\partial \varphi}{\partial y} = 0 \end{cases}$$

So

$$f_x = -f_z \frac{\partial \varphi}{\partial x}$$
$$f_y = -f_z \frac{\partial \varphi}{\partial y}$$

since $h(x,y,\varphi(x,y)) = 0$ in a neighborhood, it is constant and therefore we must have

$$h_x = -h_z \frac{\partial \varphi}{\partial x}$$
$$h_y = -h_z \frac{\partial \varphi}{\partial y}$$

And so we have that

$$\frac{\partial \varphi}{\partial x} = -\frac{h_x}{h_z} \qquad \frac{\partial \varphi}{\partial y} = -\frac{h_y}{h_z}$$

And so

$$f_x = f_z \cdot \frac{h_x}{h_z} = h_x \cdot \frac{f_z}{h_z}$$
$$f_y = f_z \cdot \frac{h_y}{h_z} = h_y \cdot \frac{f_z}{h_z}$$

So if we define $\lambda = \frac{f_z}{h_z}$ at the critical point then we have that

$$f_x = \lambda h_x$$
$$f_y = \lambda h_y$$
$$f_z = \lambda h_z$$

at the critical point, so $\nabla f = \lambda \nabla h$.

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Lemma 21.2:

Suppose a curve $\lambda \in C^1$ is parameterized by $x(t) = (x_1(t), \dots, x_n(t))$, and its tangent is $(x_1'(t), \dots, x_n'(t))$ (λ is the image of the parameterization).

(1) Suppose $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ is defined and in C^1 in some neighborhood of λ. If the restriction of f onto λ has a crtitical point at $x(t_0)$, then $x'(t_0) \perp \nabla f(x(t_0))$.

(2) If $k \in C^1$ is constant in λ then $\nabla \perp x'(t)$ in λ.

Proof:

(1) Let $g(t) = f(x(t))$, so g has a critical point at t_0, so $\frac{d}{dt} g(t_0) = 0$ so by the chain rule

$$J_f(x(t_0)) J_x(t_0) = \nabla f(x(t_0)) \cdot x'(t_0) = 0$$

as required.

(2) This is true since every point in λ is a crtitical point of k's.

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Theorem 21.3:

If $f, h_1, h_2 \colon \mathbb{R}^3 \longrightarrow \mathbb{R}$ are functions in C^1, if f has a crtitical point under the constraints $h_1, h_2 = 0$, if $\nabla h_1\big|_P$ and $\nabla h_1\big|_P$ are linearly independent then $\nabla f\big|_P = \lambda_1 \nabla h_1 + \lambda_2 \nabla h_2$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$.

Proof:

Let $\lambda = \left\{ x \in \mathbb{R}^3 \;\middle|\; h_1(x) = h_2(x) = 0 \right\}$, then this is the contour of $x = (h_1, h_2)$, we can