Complex Functions

Lecture 3, Wednesday March 29, 2023
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3.1 The Cauchy-Riemann Equations

Recall that if $f = u + iv$ is differentiable at z then the partial derivatives of u and v exist at z and satisfy the Cauchy-Riemann equations:

$$\begin{matrix} u_x(z) = v_y(z) \\ u_y(z) = -v_x(z) \end{matrix} \iff f_y(z) = if_x(z)$$

But this is not a sufficient condition, for example:

Example 3.1.1:

Take the function

$$f(z) = f(x,y) = \begin{cases} 0 & z = 0 \\ \frac{xy(x+iy)}{x^2+y^2} & z \neq 0 \end{cases}$$

this satisfies the Cauchy-Riemann equations at $z = 0$ since:

$$f_x(0) = \lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x} = \lim_{x \to 0} \frac{0-0}{x} = 0$$

And similarly $f_y(0) = 0$, so the Cauchy-Riemann equations hold. But for f to be differentiable at 0 it must satisfy:

$$f(x) = f(0) + \nabla f(0) \cdot x + \varepsilon(x) \iff f(x) = \varepsilon(x)$$

So f must satisfy $\frac{f(x)}{\sqrt{x^2+y^2}}$ converges to 0 as (x,y) does. If we take the path $x = y$ then this means

$$\frac{xy(x+iy)}{(x^2+y^2)^{1.5}} = \frac{x^3(1+i)}{2^{1.5}x^3} = \frac{1+i}{2^{1.5}}$$

must converge to 0, which it doesn't.

But what this does tell us is that if f is differentiable at z then its Jacobian (as a function $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$) is of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ by the Cauchy-Riemann inequalities, which is a representation of a complex number. So if a function is complex differentiable it should act like complex multiplication. This makes sense since by the chain rule, $(f \circ g)'(z) = f'(g(z))g'(z)$, but by calculus we know that it should also be $Df(g(z)) \cdot Dg(z)$ where Df is the differential of f. So Df should act like f', ie. its Jacobian should represent a complex number.
Recall that in order for $f(x,y)$ to be differentiable at (x_0, y_0) it must satisfy

$$f(x_0 + r, y_0 + s) - f(x_0, y_0) = Ar + Bs + \alpha(r,s)r + \beta(r,s)s$$

where $\alpha(r,s), \beta(r,s) \xrightarrow[(r,s)\to 0]{} 0$ (we sometimes write $\varepsilon(r,s) = \alpha(r,s)r + \beta(r,s)s$ and require $\frac{\varepsilon(r,s)}{\sqrt{r^2+s^2}} \xrightarrow[(r,s)\to 0]{} 0$.

Theorem 3.1.2:

$f = u + iv$ is differentiable at $z_0 \in \mathbb{C}$ if and only if u and v are differentiable at z_0 and u and v satisfy the Cauchy-Riemann equations.

Proof:

We showed that if f is differentiable at z_0 then u and v have partial derivatives and satisfy the Cauchy-Riemann equations. So all we need to show is that u and v are differentiable. Since f is (complex) differentiable, we have that

$$\frac{f(z_0+h)-f(z_0)}{h} = f'(z_0) + \alpha(h) + i\beta(h)$$

Since the right hand side approaches $f'(z_0)$ (so subtracting $f'(z_0)$ from both sides gives a function $\alpha+i\beta$ which satisfies that α and β converge to 0 as h does). And so

$$f(z_0+h)-f(z_0) = f'(z_0)h + \alpha(h)h + i\beta(h)h$$

And so if we have $z_0 = x_0 + iy_0$ and $h = h_1 + ih_2$ and $f'(z_0) = A + iB$ then:

$$\big(u(x_0+h_1, y_0+h_2) - u(x_0,y_0)\big) + i\big(v(x_0+h_1,y_0+h_2) - v(x_0,y_0)\big) = \big(A + iB + \alpha(h_1,h_2) + i\beta(h_1,h_2)\big)(h_1+ih_2)$$

And so we have that

$$\Delta u = Ah_1 - Bh_2 + \alpha(h_1,h_2)h_1 - \beta(h_1,h_2)h_2$$

which means u is differentiable at (x_0,y_0) and similar for v.
To show the converse, suppose u and v are differentiable and satisfy the Cauchy-Riemann equations. Notice that by differentiability:

$$\begin{aligned}\Delta u &= u_x(x_0,y_0)h_1 + u_y(x_0,y_0)h_2 + \alpha_1(h_1,h_2)h_1 + \beta_1(h_1,h_2)h_2 \\ \Delta v &= v_x(x_0,y_0)h_1 + v_y(x_0,y_0)h_2 + \alpha_2(h_1,h_2)h_1 + \beta_2(h_1,h_2)h_2\end{aligned}$$

Where α_i and β_i approach 0 as their input does. Now note

$$\frac{f(z_0+h)-f(z_0)}{h} = \frac{\Delta u + i\Delta v}{h_1+ih_2} = \frac{u_xh_1 + u_yh_2 + i(v_xh_1+v_yh_2)}{h_1+ih_2} + \frac{\alpha_1h_1+\beta_1h_2+i(\alpha_2h_1+\beta_2h_2)}{h_1+ih_2}$$

Let the rightmost fraction be $\gamma(h_1,h_2)$. The left fraction is equal, by the Cauchy-Riemann equations, to:

$$\frac{u_xh_1 - v_xh_2 + i(v_xh_1+u_xh_2)}{h_1+ih_2} = \frac{u_x(h_1+ih_2) + v_x(ih_1-h_2)}{h_1+ih_2} = u_x + iv_x$$

So we get that

$$\frac{f(z_0+h)-f(z_0)}{h} = u_x + iv_x + \gamma(h_1,h_2)$$

So all that is left to show is that $\gamma(h_1,h_2)$ approaches 0 as h does. By the triangle inequality, for every $(h_1,h_2) \neq 0$, $|h_1| \leq |h_1+ih_2|$ by the triangle inequality and so $\left|\frac{h_1}{h_1+ih_2}\right| \leq 1$. So

$$|\gamma| \leq |\alpha_1| + |\beta_1| + |\alpha_2| + |\beta_2|$$

which converges to 0 as h does, and that means so does γ as required.

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Example 3.1.3:

Take $f(z) = |z|^2$ and so $f(x,y) = x^2+y^2$ so $v=0$ and $u=0$. Notice that $f_x = 2x$ and $f_y = 2y$ which are differentiable over all $\mathbb{R}^2$. So f is differentiable at $z = x+iy$ if and only if $f_y = if_x$, which is if and only if $2y = 2ix$ which means that $x=y=0$ (since x and y are real).

This should make sense since as we said, if a function is differentiable, its Jacobian should act like a complex number which it doesn't.

Definition 3.1.4:

A complex function f is **analytic** at $z \in \mathbb{C}$ if it is (complex) differential in a neighborhood of z. And f is analytic over a set $S \subseteq \mathbb{C}$ if it is analytic at every point in S. If f is analytic over all of $\mathbb{C}$, then f is an **entire function**.

z^n is entire, and therefore so is every complex polynomial.
And the division of two analytic functions $\frac{p}{q}$ is analytic in $\{z \in \mathbb{C} \mid q(z) \neq 0\}$.

Proposition 3.1.5:

If $f = u + iv$ is analytic over the domain D and u is constant, then f is constant over D.

This is true by Cauchy-Riemann since we get that $v_x = v_y = 0$, so v is constant over D and therefore so is f.

Proposition 3.1.6:

If f is analytic over D and $|f|$ is constant, then so is f.

Definition 3.1.7:

If $z \in \mathbb{C}$ where $z = x + iy$ then we define $e^z = e^x e^{iy} = e^x\big(\cos(y) + i\sin(y)\big)$.

Notice then that

(1) $e^{z_1+z_2} = e^{x_1+x_2+i(y_1+y_2)} = e^{x_1}e^{x_2}\big(\cos(y_1+y_2) + i\sin(y_1+y_2)\big) = e^{x_1}e^{x_2}\operatorname{cis}(y_1)\cdot\operatorname{cis}(y_2) = e^{z_1}e^{z_2}$.

(2) $|e^z| = e^x = e^{\operatorname{Re}(z)} \leq e^{|z|}$

(3) $e^z = \alpha = re^{i\theta}$ has infinite solutions for $0 \neq \alpha \in \mathbb{C}$ since $e^x = |\alpha|$ (which defines x) and then $e^{iy} = e^{i\theta}$ so $y = \theta + 2\pi k$, which gives a countably infinite number of solutions.

(4) By our proof above $e^z = e^{z+2\pi k}$ for every $k \in \mathbb{Z}$, so exp (which is another notation for $e^{\cdot}$) is periodic with period $2\pi i$.

(5) e^z is entire.

(6) $f(z) = e^z$ is the only function satisfying $f'(z) = f(z)$ and $f(0) = 1$.

(7) It is also the only analytic function satisfying $f(z_1 + z_2) = f(z_1)f(z_2)$ and $f(x) = e^x$ for $x \in \mathbb{R}$.

We define the complex trigonometric functions:

$$\sin(z) = \frac{1}{2i}\big(e^{zi} - e^{-iz}\big), \qquad \cos(z) = \frac{1}{2}\big(e^{iz} + e^{-iz}\big)$$

These are analytic since the linear combinations of analytic functions are analytic. Furthermore

$$\sin^2 z + \cos^2 z = 1$$

But notice that these functions aren't bounded on $\mathbb{C}$, for example

$$\cos(ir) = \frac{1}{2}\big(e^{-r} + e^r\big) \xrightarrow[r\to\infty]{} \infty$$

3.2 Power Series

Definition 3.2.1:

A **complex power series** is a function of the form

$$f(z) = \sum_{k=0}^{\infty} c_k z^k$$

where $c_k \in \mathbb{C}$. For a power series about $z_0 \in \mathbb{C}$ this is of the form

$$f(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k$$

which is just a shift of a power series about 0.

Definition 3.2.2:

The **domain of convergence** of a power series $\sum c_k z^k$ is the set $\left\{w \in \mathbb{C} \;\middle|\; \sum_{k=0}^{\infty} c_k w^k \in \mathbb{C}\right\}$. And the **radius of convergence** is $R = \sup\left\{|w| \;\middle|\; \sum_{k=0}^{\infty} c_k w^k \in \mathbb{C}\right\}$.

It can be shown that

$$R = \frac{1}{\limsup\limits_{k\to\infty} |c_k|^{\frac{1}{k}}}$$

and if the limit exists:

$$R = \lim_{k\to\infty} \left|\frac{c_k}{c_{k+1}}\right|$$