Differential and Analytic Geometry

Assignment 4
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Exercise 4.1:

A torus is obtained by rotating the curve

$$(x-a)^2 + z^2 = b^2$$

about the y axis. Suppose that $b < a$, then

(1) Find a parameterization for the torus.

(2) Find the first fundamental form of the torus.

(3) Find the Christoffel symbols of the torus.

(4) Find the geodesic equation of the torus.

(1) We can parameterize the curve by

$$\gamma(\theta) = (b\cos\theta + a, 0, b\sin\theta)$$

Thus rotating it about the y axis gives the parameterization

$$\sigma(\theta, \varphi) = (b\cos\theta\cos\varphi + a\cos\varphi, b\cos\theta\sin\varphi, b\sin\theta)$$

(2) Find the partial derivatives of σ:

$$\begin{aligned} \sigma_1 &= (-b\sin\theta\cos\varphi, -b\sin\theta\sin\varphi, b\cos\theta) \\ \sigma_2 &= (-(b\cos\theta + a)\sin\varphi, (b\cos\theta + a)\cos\varphi, 0) \end{aligned}$$

Thus we get, by $g_{ij} = \langle\sigma_i, \sigma_j\rangle$,

$$g = \begin{pmatrix} b^2 & 0 \\ 0 & (b\cos\theta + a)^2 \end{pmatrix}$$

(In general for rotations of the curve $\gamma(\theta) = (r(\theta), 0, z(\theta))$, a simple computation yields $g_{11} = r'(\theta)^2 + z'(\theta)^2$, $g_{12} = 0$, $g_{22} = r(\theta)^2$).

(3) Recall from lecture

$$\Gamma^k_{ij} = \frac{1}{2} g^{km}(g_{mi,j} + g_{jm,i} - g_{ij,m})$$

For a diagonal metric (first fundamental form), g^{-1} is also diagonal so we can focus only coordinates which are equal (ie. $g_{ij} = g^{ij} = 0$ for $i \neq j$). Thus we have

$$\Gamma^k_{ij} = \frac{1}{2} g^{kk}(g_{ki,j} + g_{jk,i} - g_{ij,k})$$

So if $i = j = k$ then we have

$$\Gamma^k_{kk} = \frac{1}{2} g^{kk}(g_{kk,k} + g_{kk,k} - g_{kk,k}) = \frac{1}{2} g^{kk} g_{kk,k}$$

And if $i = j \neq k$ then $g_{ki} = g_{jk} = 0$ and so

$$\Gamma^k_{ij} = -\frac{1}{2} g^{kk} g_{ij,k}$$

Finally if only one of i and j are equal to k, we can assume $i = k$ and $i \neq j$ (since $\Gamma^k_{ij} = \Gamma^k_{ji}$), so

$$\Gamma^k_{ij} = \frac{1}{2} g^{kk} g_{kk,j}$$

Let us summarize this, using the notation $\neg k = 3-k$ (ie. swapping one and two),

$$\Gamma^k_{ij} = \begin{cases} \frac{1}{2}g^{kk}g_{kk,k} & \\ -\frac{1}{2}g^{kk}g_{\neg k\neg k,k} & i=j=\neg k \\ \frac{1}{2}g^{kk}g_{kk,\neg k} & \text{only one between } i \text{ and } j \text{ is equal to } k \end{cases}$$

Now, since our first fundamental form is diagonal, we can utilize this.

$$\begin{aligned} \Gamma^1_{11} =& \frac{1}{2}g^{11}g_{11,1} = 0 \\ \Gamma^1_{12} = \Gamma^1_{21} =& \frac{1}{2}g^{11}g_{11,2} = 0 \\ \Gamma^1_{22} =& -\frac{1}{2}g^{11}g_{22,1} = \frac{\sin\theta\cdot(b\cos\theta+a)}{b} \\ \Gamma^2_{11} =& -\frac{1}{2}g^{22}g_{11,2} = 0 \\ \Gamma^2_{12} = \Gamma^2_{21} =& \frac{1}{2}g^{22}g_{22,1} = -\frac{b\sin\theta}{b\cos\theta+a} \\ \Gamma^2_{22} =& \frac{1}{2}g^{22}g_{22,2} = 0 \end{aligned}$$

(4) The geodesic equations of a surface are, for $i=1,2$

$$\ddot{\beta}^i + \dot{\beta}^k\dot{\beta}^j\Gamma^i_{kj} = 0$$

So for $i=1$, we can focus on only $k=j=2$,

$$\ddot{\beta}^1 + \left(\dot{\beta}^2\right)^2 \cdot \frac{\sin\theta\cdot(b\cos\theta+a)}{b}$$

And for $i=2$, we can focus on only when one of k and j are equal to 2, then we get

$$\ddot{\beta}^2 - 2\dot{\beta}^1\dot{\beta}^2 \cdot \frac{b\sin\theta}{b\cos\theta+a}$$

Exercise 4.2:

Suppose we have the function $f(x,y) = \frac{9}{x}$, and the metric of a surface is given by

$$g_{ij} = f(x,y)^2\delta^i_j$$

find the Christoffel symbols of σ_1.

Here g is once again a diagonal matrix, so we can utilize the formulas we computed in the previous question.

$$\begin{aligned} \Gamma^1_{11} =& \frac{1}{2}g^{11}g_{11,1} = \frac{f_x}{f} \\ \Gamma^1_{12} = \Gamma^1_{21} =& \frac{1}{2}g^{11}g_{11,2} = \frac{f_y}{f} \\ \Gamma^1_{22} =& -\frac{1}{2}g^{11}g_{22,1} = -\frac{f_x}{f} \end{aligned}$$

Now since $f = \frac{9}{x}$ we have

$$\begin{aligned} \Gamma^1_{11} &= -\frac{1}{x} \\ \Gamma^1_{12} = \Gamma^1_{21} &= 0 \\ \Gamma^1_{22} &= \frac{1}{x} \end{aligned}$$

Exercise 4.3:

Using the coordinates (r, φ), suppose we have a surface whose metric is

$$g = \begin{pmatrix} r^2 & 0 \\ 0 & r^4 \end{pmatrix}$$

Prove that every geodesic which isn't constant in φ (ie. the geodesic, as a curve on the surface, is of the form $\sigma \circ \beta$. So we require β^2 is not constant) is isolated from the origin. This means that there is a neighborhood of the origin which is disjoint from the curve.

Let us first find the geodesic equations of this surface. To do so we must find the Christoffel symbols, and since the metric is diagonal, we can use the above formulas.

$$\begin{aligned} \Gamma^1_{11} &= \frac{1}{2} g^{11} g_{11,1} = r^{-1} \\ \Gamma^1_{12} = \Gamma^1_{21} &= \frac{1}{2} g^{11} g_{11,2} = 0 \\ \Gamma^1_{22} &= -\frac{1}{2} g^{11} g_{22,1} = -2r \\ \Gamma^2_{11} &= -\frac{1}{2} g^{22} g_{11,2} = 0 \\ \Gamma^2_{12} = \Gamma^2_{21} &= \frac{1}{2} g^{22} g_{22,1} = 2r^{-1} \\ \Gamma^2_{22} &= \frac{1}{2} g^{22} g_{22,2} = 0 \end{aligned}$$

The geodesic equations are, for $i = 1, 2$

$$\ddot{\beta}^i + \dot{\beta}^j \dot{\beta}^k \Gamma^i_{jk} = 0$$

So if $\beta(t) = (r(t), \varphi(t))$, we have

$$\begin{aligned} \ddot{r} + \dot{r}^2 r^{-1} - 2\dot{\varphi}^2 r &= 0 \\ \ddot{\varphi} + 4\dot{r}\dot{\varphi} r^{-1} &= 0 \end{aligned}$$

So we have that $2\dot{\varphi}^2 = \ddot{r} r^{-1} + \dot{r}^2 r^{-2}$ and differentiating this gives

$$4\dot{\varphi}\ddot{\varphi} = \dddot{r} r^{-1} - \ddot{r}\dot{r} r^{-2} + 2\ddot{r}\dot{r} r^{-2} - 2\dot{r}^3 r^{-3} = \dddot{r} r^{-1} + \ddot{r}\dot{r} r^{-2} - 2\dot{r}^3 r^{-3}$$

And multiplying the second equation by $4\dot{\varphi}$ gives

$$4\dot{\varphi}\ddot{\varphi} + 16\dot{r}\dot{\varphi}^2 r^{-1} = \dddot{r} r^{-1} + \ddot{r}\dot{r} r^{-2} - 2\dot{r}^3 r^{-3} + 8\ddot{r}\dot{r} r^{-2} + 8\dot{r}^3 r^{-3} = \dddot{r} r^{-1} + 9\ddot{r}\dot{r} r^{-2} + 6\dot{r}^3 r^{-3} = 0$$

So we have that

$$\dddot{r} r^2 + 9\ddot{r}\dot{r} r + 6\dot{r}^3 = 0$$

Suppose now that β is not isolated from the origin, then there exists a sequence $t_1 < t_2 < \dots$ such that $r(t_n), \varphi(t_n) \to 0$. But notice then that since r is positive, and so $r(t_n)$ converges to a minimum/lower bound, we have that $\dot{r} \leq 0$ on this sequence. And since this is a minimum, we have $\ddot{r} \geq 0$ And so we have that

$$9\ddot{r}\dot{r}r, 6\dot{r}^3 \leq 0$$

which means that by the ODE, $\dddot{r} \geq 0$.

Exercise 4.4:

Find the Gaussian curvature of the rotation of $t \mapsto (\cosh(t), 0, t)$ at every point using

(1) Theorema Egregium

(2) The Laplace-Beltrami operator

(1) We know that the metric of this surface is

$$g = \begin{pmatrix} r'(t)^2 + z'(t)^2 & 0 \\ 0 & r(t)^2 \end{pmatrix} = \begin{pmatrix} \sinh(t)^2 + 1 & 0 \\ 0 & \cosh(t)^2 \end{pmatrix} = \begin{pmatrix} \cosh(t)^2 & 0 \\ 0 & \cosh(t)^2 \end{pmatrix}$$

Theorema Egregium gives us

$$K = g^{11}\Big(\Gamma^2_{11,2} - \Gamma^2_{12,1} + \Gamma^2_{11}\Gamma^2_{22} + \Gamma^1_{11}\Gamma^2_{12} - \Gamma^1_{12}\Gamma^2_{11} - (\Gamma^2_{12})^2\Big)$$

Stunning. So now lets compute the Christoffel symbols.

$$\begin{aligned} \Gamma^1_{11} =& \ \frac{1}{2}g^{11}g_{11,1} = \tanh(t) \\ \Gamma^1_{12} = \Gamma^1_{21} =& \ \frac{1}{2}g^{11}g_{11,2} = 0 \\ \Gamma^1_{22} =& -\frac{1}{2}g^{11}g_{22,1} = -\tanh(t) \\ \Gamma^2_{11} =& -\frac{1}{2}g^{22}g_{11,2} = 0 \\ \Gamma^2_{12} = \Gamma^2_{21} =& \ \frac{1}{2}g^{22}g_{22,1} = \tanh(t) \\ \Gamma^2_{22} =& \ \frac{1}{2}g^{22}g_{22,2} = 0 \end{aligned}$$

So we get

$$K = \cosh(t)^{-2}\Big(-\cosh(t)^{-2} + \tanh(t)^2 - \tanh(t)^2\Big) = -\frac{1}{\cosh(t)^4}$$

as required.

(2) Since $g = \cosh(t)\delta^i_j$, we have that, by computing the Laplace-Beltrami operator (which we will do in question six):

$$K = \frac{f_t^2 - f \cdot f_{tt}}{f^4} = \frac{\sinh(t)^2 - \cosh(t)^2}{\cosh(t)^4} = -\frac{1}{\cosh(t)^4}$$

Exercise 4.5:

Represent the following in terms $\Gamma^k_{ij}, S^i_j, b_{ij}, g_{ij}$:

(1) $\langle x_{ij}, x_\ell \rangle g^{\ell i}$

(2) $\langle x_{ij\ell}, n \rangle$

(3) $\langle x_{ij}, n_\ell \rangle \delta^j_m$

(4) $g_{pq}\delta^q_s g^{st}\delta^p_t$

(5) $\langle x_{ij}, n_k \rangle \delta^k_m g^{m\ell}$

(6) $\langle n_i, x_j \rangle g^{i\ell}$

(7) $\langle n_i, n_j \rangle$

(8) $\langle n, n_{ab} \rangle \delta^a_c$

(9) $|x_{ij}|^2$

(10) $\langle x_{ij}, x_k\rangle \delta^k_m g^{m\ell}$

(1) Recall that the definition of the Christoffel symbols (and b_{ij}) are

$$x_{ij} = \Gamma^k_{ij} x_k + b_{ij} N$$

so

$$\langle x_{ij}, x_\ell\rangle = \Gamma^k_{ij}\langle x_k, x_\ell\rangle = \Gamma^k_{ij} g_{k\ell}$$

Thus

$$\langle x_{ij}, x_\ell\rangle g^{\ell i} = \Gamma^k_{ij} g_{k\ell} g^{\ell i} = \Gamma^k_{ij}\delta^i_k = \Gamma^i_{ij}$$

(2) Since $x_{ij} = \Gamma^k_{ij} x_k + b_{ij} n$, we have

$$x_{ij\ell} = \Gamma^k_{ij,\ell} x_k + \Gamma^k_{ij} x_{k\ell} + b_{ij,\ell} n + b_{ij} n_\ell$$

Since n is a unit vector, its derivative is orthogonal to it (and so are x_k), so

$$\langle x_{ij\ell}, n\rangle = \Gamma^k_{ij}\langle x_{k\ell}, n\rangle + b_{ij,\ell} = \Gamma^k_{ij} b_{k\ell} + b_{ij,\ell}$$

(3) Here, this is equal to $\langle x_{ij}, n_\ell\rangle$. Now, we have that

$$b_{ij,\ell} = \langle x_{ij,\ell}, n\rangle + \langle x_{ij}, n_\ell\rangle$$

And so we have that from the previous question,

$$\langle x_{ij}, n_\ell\rangle = b_{ij,\ell} - \langle x_{ij,\ell}, n\rangle = -\Gamma^k_{ij} b_{k\ell}$$

(4) Simplifying gives

$$g_{pq}\delta^q_s g^{st}\delta^p_t = g_{ps} g^{sp} = \delta^p_p = 3$$

(5) We showed already that $\langle x_{ij}, n_k\rangle = -\Gamma^k_{ij} b_{k\ell}$ so this is equal to

$$-\Gamma^k_{ij} b_{k\ell} g^{k\ell}$$

Recall that $S = g^{-1}B$ and so

$$= -\Gamma^k_{ij} S^k_k$$

(6) We know that $\langle n, x_j\rangle = 0$ and so $\langle n_i, x_j\rangle + \langle n, x_{ij}\rangle = 0$ meaning $\langle n_i, x_j\rangle = -b_{ij}$. So this is equal to

$$-b_{ij} g^{i\ell} = -g^{\ell i} b_{ij} = -S^\ell_j$$

(g and therefore g^{-1} is symmetric and so $g^{i\ell} = g^{i\ell}$).

(7) Recall that $S(x_i) = -n_i$ and so the ith column of S represents $-n_i$ with respect to the basis x_1, x_2 meaning

$$-n_i = S^j_i x_j$$

Thus

$$\langle n_i, n_j\rangle = \langle -n_i, -n_j\rangle = \left\langle S^k_i x_k, S^\ell_j x_\ell\right\rangle = S^k_i S^\ell_j g_{k\ell}$$

(To satisy Einstein notation perhaps I should've swapped the placement of the indexes on S, which is symmetric.) Now, we know that $S^\ell_j = g^{\ell t} b_{tj}$ and so this is equal to

$$g_{k\ell} g^{\ell t} b_{tj} S^k_i = \delta^k_t b_{tj} S^k_i = b_{kj} S^k_i$$

(8) This is equal to $\langle n, n_{ab}\rangle$. We know that $\langle n, n_a\rangle = 0$ as n is a unit vector and thus orthogonal to its derivative. Thus $\langle n_b, n_a\rangle + \langle n, n_{ab}\rangle = 0$ and so

$$\langle n, n_{ab}\rangle = -\langle n_a, n_b\rangle = -b_{kb}S_a^k$$

(9) Since $x_{ij} = \Gamma_{ij}^k x_k + b_{ij}n$ we get

$$|x_{ij}|^2 = \langle x_{ij}, x_{ij}\rangle = \left\langle \Gamma_{ij}^k x_k + b_{ij}n, \Gamma_{ij}^\ell x_\ell + b_{ij}n\right\rangle = \Gamma_{ij}^k\Gamma_{ij}^\ell g_{k\ell} + b_{ij}^2$$

Since n is orthogonal to x_k and is a unit vector.

(10) This is equal to

$$\langle x_{ij}, x_k\rangle g^{k\ell}$$

Now, again we have that $x_{ij} = \Gamma_{ij}^t x_t + b_{ij}n$ and so this is equal to

$$\left\langle \Gamma_{ij}^t x_t + b_{ij}n, x_k\right\rangle g^{k\ell} = \Gamma_{ij}^t g_{tk} g^{k\ell} = \Gamma_{ij}^t \delta_\ell^t = \Gamma_{ij}^\ell$$

Exercise 4.6:

Suppose we have a surface whose metric is given by

$$g_{ij} = \lambda(x,y)\delta_j^i = f(x,y)^2\delta_j^i$$

Show that the Gaussian curvature of the surface is given by

$$K = -\frac{1}{2}\Delta_{\mathrm{LB}}(\log\lambda) = -\Delta_{\mathrm{LB}}(\log f)$$

Where Δ_{LB} is the Laplace-Beltrami operator

$$\Delta_{\mathrm{LB}}(h) = \frac{1}{\lambda}\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right)$$

Firstly, it is obvious that Δ_{LB} is linear and so

$$-\frac{1}{2}\Delta_{\mathrm{LB}}(\log\lambda) = -\frac{1}{2}\Delta_{\mathrm{LB}}(2\log f) = -\Delta_{\mathrm{LB}}(\log f)$$

So we need only to show that this is equal to the Gaussian curvature. We know that

$$-\Delta_{\mathrm{LB}}(\log f) = -\frac{1}{f^2}\left(\frac{\partial^2}{\partial x^2}\log(f) + \frac{\partial^2}{\partial y^2}\log(f)\right) = -\frac{1}{f^2}\left(\frac{\partial}{\partial x}\frac{f_x}{f} + \frac{\partial}{\partial y}\frac{f_y}{f}\right) = -\frac{f_{xx}f + f_{yy}f - f_x^2 - f_y^2}{f^4}$$

So we need to show

$$K = \frac{f_x^2 + f_y^2 - f(f_{xx} + f_{yy})}{f^4}$$

Computing the Christoffel symbols:

$$\begin{aligned}
\Gamma^1_{11} =& \ \frac{1}{2}g^{11}g_{11,1} = \frac{f_x}{f} \\
\Gamma^1_{12} = \Gamma^1_{21} =& \ \frac{1}{2}g^{11}g_{11,2} = \frac{f_y}{f} \\
\Gamma^1_{22} =& -\frac{1}{2}g^{11}g_{22,1} = -\frac{f_x}{f} \\
\Gamma^2_{11} =& -\frac{1}{2}g^{22}g_{11,2} = -\frac{f_y}{f} \\
\Gamma^2_{12} = \Gamma^2_{21} =& \ \frac{1}{2}g^{22}g_{11,2} = \frac{f_x}{f} \\
\Gamma^2_{22} =& \ \frac{1}{2}g^{22}g_{22,2} = \frac{f_y}{f}
\end{aligned}$$

Now recall that the Gaussin curvature is given by

$$K = g^{11}\Big(\Gamma^2_{11,2} - \Gamma^2_{12,1} + \Gamma^2_{11}\Gamma^2_{22} + \Gamma^1_{11}\Gamma^2_{12} - \Gamma^1_{12}\Gamma^2_{11} - (\Gamma^2_{12})^2\Big)$$

Computing the derivatives of the necessary Christoffel symbols:

$$\begin{aligned}
\Gamma^2_{11,2} &= -\frac{f_{yy}f - f_y^2}{f^2} \\
\Gamma^1_{12,1} &= \frac{f_{xx}f - f_x^2}{f^2}
\end{aligned}$$

And so plugging this in gives

$$K = \frac{1}{f^2}\left(\frac{f_x^2 + f_y^2 - f(f_{xx} + f_{yy})}{f^2} - \frac{f_y^2}{f^2} + \frac{f_x^2}{f^2} + \frac{f_y^2}{f^2} - \frac{f_x^2}{f^2}\right) = \frac{f_x^2 + f_y^2 - f(f_{xx} + f_{yy})}{f^4}$$

as required.

Exercise 4.7:

Given the metric

$$g_{ij} = \frac{1}{y}\delta^i_j$$

for $y > 0$, find the Gaussian curvature.

We know, by above,

$$K = -\frac{1}{2}\Delta_{\mathrm{LB}}(-\log y) = \frac{1}{2}y\left(\frac{\partial^2}{\partial x^2}\log(y) + \frac{\partial^2}{\partial y^2}\log(y)\right) = -\frac{1}{2y}$$

Exercise 4.8:

Given the metric

$$g = \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix}$$

for $y > 0$, find the Gaussian curvature.

This is in polar coordinates, so we can apply the simplified formula from Theorem Egregium:

$$K = \frac{1}{G}\Big(\Gamma^1_{22,1} - \Gamma^2_{12}\Gamma^1_{22}\Big)$$

Here we have

$$\Gamma^1_{22} = -\frac{1}{2}g^{11}g_{22,1} = 0, \quad \Gamma^2_{12} = \frac{1}{2}g^{22}g_{22,1} = 0$$

and so $K = 0$.

Exercise 4.9:

Find the normal and geodesic curvatures of the curve

$$\beta(t) = (\cos t, \sin t, 1)$$

on the parabaloid

$$z = x^2 + y^2$$

(1) By taking the inner product of β'' with $\beta' \times n$ and using the Pythagorean theorem.

(2) Using the second fundamental form, $\kappa_n = \mathrm{II}(\beta', \beta')$ and the Pythagorean theorem.

(1) Notice that

$$\beta'(t) = (-\sin t, \cos t, 0), \quad \beta''(t) = (-\cos t, -\sin t, 0)$$

And so $\|\beta'\| = 1$, so β is a natural parameterization. Thus recall that

$$\kappa_g = |\langle \beta'', \beta' \times n\rangle|$$

So we must find n. We parameterize the surface by

$$\sigma(u, v) = (u, v, u^2 + v^2)$$

Thus it has partial derivatives

$$\sigma_1 = (1, 0, 2u), \quad \sigma_2 = (0, 1, 2v)$$

And so we have

$$\sigma_1 \times \sigma_2 = (-2u, -2v, 1) \implies n = \frac{1}{\sqrt{4u^2 + 4v^2 + 1}}(-2u, -2v, 1)$$

Since we are on the curve β, we have $u = \cos(t)$ and $v = \sin(t)$, thus

$$n = \frac{1}{\sqrt{5}}(-2\cos t, -2\sin t, 1)$$

And finally

$$\beta' \times n = \frac{1}{\sqrt{5}}(\cos t, \sin t, 2)$$

Thus we have

$$\kappa_g = \left|\frac{1}{\sqrt{5}}(-cos^2 t - sin^2 t)\right| = \frac{1}{\sqrt{5}}$$

And since $\kappa = |\beta''| = 1$ and $\kappa_n^2 + \kappa_g^2 = \kappa^2$ we have

$$\kappa_n = \frac{2}{\sqrt{5}}$$

(2) Here we need to represent β' with respect to the basis σ_1 and σ_2 and compute the second fundamental form (b_{ij}). Firstly, we have

$$\sigma_1 = (1, 0, 2\cos t), \quad \sigma_2 = (0, 1, 2\sin t)$$

Thus

$$\beta' = (-\sin t, \cos t, 0) = -\sin(t)\sigma_1 + \cos(t)\sigma_2$$

So β''s representation is $(-\sin(t), \cos(t))$. Now, $b_{ij} = \langle \sigma_{ij}, n \rangle$ and so

$$\sigma_{11} = (0,0,2), \quad \sigma_{12} = (0,0,0), \quad \sigma_{22} = (0,0,2)$$

And thus

$$b_{11} = b_{22} = \frac{2}{\sqrt{5}}, \qquad b_{12} = b_{21} = 0$$

So we have

$$\kappa_n = \frac{2}{\sqrt{5}}(-\sin(t), \cos(t)) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} = \frac{2}{\sqrt{5}}$$

And since $\kappa = 1$ we get by the Pythagorean theorem, $\kappa_g = \frac{1}{\sqrt{5}}$.

Exercise 4.10:

Suppose two surfaces have the same second-order Taylor series at a point. Show then that they have the same Gaussian curvature.

Suppose our two surfaces are σ and τ, and our point is $p = \sigma(a,b) = \tau(c,d)$. Then the Taylor series of σ at p is

$$\sigma(a+h, b+k) = p + (h\sigma_1 + k\sigma_2) + \frac{1}{2}\left(h^2\sigma_{11} + 2hk\sigma_{12} + k^2\sigma_{22}\right)$$

and τ's is

$$\tau(c+h, d+k) = p + (h\tau_1 + k\tau_2) + \frac{1}{2}\left(h^2\tau_{11} + 2hk\tau_{12} + k^2\tau_{22}\right)$$

Since these are equal, this means we can equate the factors and so we have that

$$\sigma_i = \tau_i, \qquad \sigma_{ij} = \tau_{ij}$$

for all i, j. Since the first fundamental form is determined by the values of σ_i (it is $g_{ij} = \langle \sigma_i, \sigma_j \rangle$), this means that at p, both f and g have the same first fundamental form. This in and of itself is not enough to say that they have the same Gaussian curvature, as while the curvature is determined by the first fundamental form and its derivatives, we only have equality at a point, not a neighborhood. So we cannot take derivatives. But, the unit normal n (which is the normalized $\sigma_1 \times \sigma_2$) are also the same for σ and τ (as in we can choose an orientation where they are the same, but since Gaussian curvature is intrinsic, this choice doesn't affect the result). And since the second fundamental form is dependent only on σ_{ij} and n (it is $b_{ij} = \langle \sigma_{ij}, n \rangle$), we have that the second fundamental form of both surfaces at p are equal (as in we can choose an orientation for them to be equal).
Since the Gaussian curvature is given by the ratio between the determinants of the second and first fundamental forms, which are the same at p for both σ and τ, the Gaussian curvature of σ and τ at p are equal.

Exercise 4.11:

(1) Suppose β is the natural parameterization of a curve on the surface M. Suppose $\beta(0) = p$ and $\beta'(0) = v$. Prove that at p, $\kappa_n = \mathrm{II}(v, v) = \langle Sv, v \rangle$.

(2) Suppose v is a unit vector which is an eigenvector of the shape operator with an eigenvalue of κ in the tangent space T_pM. Suppose $\beta(s)$ is a geodesic on M which satisfies $\beta(0) = p$ and $\beta'(0) = v$. Prove that $\kappa_\beta(0) = |\kappa|$, that the curvature of β at 0 is equal to $|\kappa|$.

(3) Suppose $\kappa_1 \leq \kappa_2$ are the eigenvalues of the shape operator. Let $\beta(s)$ be a natural parameterization geodesic, where $\beta(0) = p$. Prove that $\kappa_1 \leq \kappa_n \leq \kappa_2$.

(1) The normal curvature of β is defined to be the coefficient of n in β's representation in the orthonormal basis $\beta', n, \beta' \times n$. So

$$\beta'' = c\beta' + \kappa_n n + \kappa_g \beta' \times n$$

We know that $\langle \beta', n\rangle = 0$ and so $\kappa_n = \langle \beta'', n\rangle = -\langle \beta', n'\rangle$. So at p, we have $\kappa_n = -\langle v, n'(0)\rangle$ which is equal to $\mathrm{II}(v,v)$.

(2) Since β is a geodesic, $\|\beta'\|$ is constant. Since $\|\beta'(0)\| = \|v\| = 1$, β is a natural parameterization. Therefore $\kappa_\beta(0)^2 = \kappa_g^2 + \kappa_n^2$. Since β is a geodesic, its geodesic curvature is zero, and thus we have $\kappa_\beta(0) = |\kappa_n|$. Now, by the previous question

$$\kappa_n = \mathrm{II}(v,v) = \langle S(v), v\rangle = \langle \kappa v, v\rangle = \kappa\langle v, v\rangle = \kappa$$

which means that

$$\kappa_\beta(0) = |\kappa|$$

as required.

(3) We showed in lecture that

$$\kappa_1 = \min_{\|v\|=1} \langle Sv, v\rangle, \qquad \kappa_2 = \max_{\|v\|=1} \langle Sv, v\rangle$$

Since we know that

$$\kappa_n = \langle Sv, v\rangle$$

we have that $\kappa_1 \leq \kappa_n \leq \kappa_2$ as required.

Exercise 4.12:

Let us define the surface M to be the graph of the function

$$f(x,y) = 3x^2 + 8xy - 3y^2$$

Let e_1, e_2, e_3 be the standard basis for $\mathbb{R}^3$.

(1) Find the Hessian of f at the origin.

(2) Let λ_1, λ_2 be H_f's eigenvalues and v_1, v_2 corresponding eigenvectors. We define $E_i \subseteq \mathbb{R}^3$ as the hyperplane spanned by v_i and e_3. We define $\gamma_i = M \cap E_i$.
Find the curvature of the planar curves γ_i at the origin.

(3) Find the shape operator of M at the origin and find the Gaussian curvature of M at the origin.

(4) Find the mean curvature of M at the origin.

(1) Computing the second derivatives of f yields

$$H_f = \begin{pmatrix} 6 & 8 \\ 8 & -6 \end{pmatrix}$$

this is true for every point, particularly at the origin.

(2) The characteristic polynomial of H_f is

$$(x-6)(x+6) - 64 = x^2 - 100$$

and so $\lambda_i = \pm 10$. The eigenvectors are

$$v_1 = (1, -2), \qquad v_2 = (1, 2)$$

So we have that

$$E_1 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} u \\ -2u \\ v \end{pmatrix} \right\}$$

The intersection of E_1 with M requires

$$v = 3u^2 - 16u^2 - 12u^2 = -25u^2$$

So

$$\gamma_1 = \begin{pmatrix} t \\ -2t \\ -25t^2 \end{pmatrix}$$

Since the curvature of a curve is given by

$$\kappa = \frac{\|\gamma' \times \gamma''\|}{\|\gamma'\|^3}$$

we compute γ_1's derivatives:

$$\gamma_1' = (1, -2, -50t), \qquad \gamma_2' = (0, 0, -50)$$

The cross product of these gives $(100, 50, 0) = 50(2, 1, 0)$ and so

$$\kappa = \frac{50\sqrt{5}}{(5 + 2500t^2)^{3/2}} = \frac{10}{(1 + 500t)^{3/2}}$$

So when $t = 0$, $\gamma_1(0) = 0$ and thus the curvature is equal to 10.
Now for E_2:

$$E_2 = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} u \\ 2u \\ v \end{pmatrix} \right\}$$

And so the intersection requires

$$v = 3u^2 + 16u^2 - 12u^2 = 7u^2$$

So

$$\gamma_2(t) = \begin{pmatrix} t \\ 2t \\ 7t^2 \end{pmatrix}$$

And

$$\gamma_2'(t) = (1, 2, 14t), \qquad \gamma_2''(t) = (0, 0, 14)$$

And so the cross product yields $(28, -14, 0) = 14(2, -1, 0)$ and we get

$$\kappa = \frac{14\sqrt{5}}{(5 + 196t^2)^{3/2}}$$

So at the origin,

$$\kappa = \frac{14\sqrt{5}}{5\sqrt{5}} = \frac{14}{\sqrt{5}}$$

(3) We showed that for the graph of a function,

$$g = \begin{pmatrix} 1 + f_x^2 & f_x f_y \\ f_x f_y & 1 + f_y^2 \end{pmatrix}, \qquad B = \frac{1}{\sqrt{1 + f_x^2 + f_y^2}} \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}$$

In this case, at the origin $f_x = 0$, $f_y = 0$, $f_{xx} = 6$, $f_{xy} = 8$, and $f_{yy} = -6$. So we get

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 6 & 8 \\ 8 & -6 \end{pmatrix}$$

So we get that

$$S = g^{-1}B = \begin{pmatrix} 6 & 8 \\ 8 & -6 \end{pmatrix}$$

The Gaussian curvature is the determinant of this (the product of the eigenvalues)

$$K = -36 - 64 = -100$$

(4) The mean curvature is half the trace of the shape operator, so

$$H = 0$$

Exercise 4.13:

Suppose $C > 0$ is a positive constant, and $f(x, y)$ is a function satisfying

$$f(x, y) \geq C(x^2 + y^2), \quad f(0,0) = 0$$

(1) Find a lower bound on the eigenvalues of $H_f(0,0)$.

(2) Find a lower bound on the Gaussian curvature of the graph of f at the origin.

(1) Notice that for every $(x, y) \neq (0,0)$,

$$f(x, y) \geq C(x^2 + y^2) > 0 = f(0,0)$$

So $(0,0)$ is a minimum of f, and in particular it is a critical point. We know that at critical points, the second fundamental form and the shape operator of the graph of f is equal to the Hessian of f. So the eigenvalues of the Hessian at the origin are the principal directions of the graph at the origin.
Recall that the principal directions are the minimum and maximum curvatures of the intersection of the graph with a normal plane. Meaning for every unit vector $w \in T_pM$ (here $p = (0, 0, f(0,0)) = (0, 0, 0)$), we define γ_w to be the natural parameterization of the curve obtained by intersecting M with the plane $p + \text{span}\{w, n\}$. Suppose that $\gamma_w(0) = p$, then the principal directions are

$$\kappa_1 = \min_{\substack{w \in T_pM \\ \|w\|=1}} \kappa_{\gamma_w}(0), \qquad \kappa_2 = \max_{\substack{w \in T_pM \\ \|w\|=1}} \kappa_{\gamma_w}(0)$$

We showed this in Euler's theorem. So we must find a lower bound of $\kappa_{\gamma_w}(0)$. By definition,

$$\kappa_{\gamma_w}(0) = \|\gamma_w''(0)\|$$

Now, suppose we define M_1 to be the graph of f, and M_2 to be the graph of $C(x^2 + y^2)$. Both f and $C(x^2 + y^2)$ have minimums at $(0,0)$ and so the tangent plane at $(0,0,0)$ for both of them is $T_0M_1 = T_0M_2 = \text{span}\{(1,0,0), (0,1,0)\}$. So for every unit vector w in this plane we have the curve γ_w^1 on M_1 and γ_w^2 on M_2. Since $f \geq C(x^2 + y^2)$ we have (since γ_w^i are planar curves, we can view them as real functions)

$$\gamma_w^1(t) \geq \gamma_w^2(t), \qquad \gamma_w^1(0) = \gamma_w^2(0) = (0,0,0)$$

And so we get that, using this equality and inequality (if $f \geq g$ and $f(0) = g(0)$ then $f'(t) \geq g'(t)$ in a neighbrohood of 0)

$$\gamma_w^1{}'(t) \geq \gamma_w^2{}'(t), \qquad \gamma_w^1{}'(0) = \gamma_w^2{}'(0)$$

The derivatives at 0 are equal since $(\gamma_w^i)'(0) = w$. So again, we get that

$$\gamma_w^1{}''(t) \geq \gamma_w^2{}''(t)$$

And thus the curvature of γ_w^1 is greater than that of γ_w^2. And so we have

$$\kappa_1^1 = \min_{\|w\|=1} \kappa_{\gamma_w^1} \geq \min_{\|w\|=1} \kappa_{\gamma_w^2} = \kappa_1^2$$

$$\kappa_2^1 = \max_{\|w\|=1} \kappa_{\gamma_w^1} \geq \max_{\|w\|=1} \kappa_{\gamma_w^2} = \kappa_2^2$$

where κ_j^i is the jth principal curvature of the surface M_i.
So if we can find the principal curvatures of the graph of $C(x^2 + y^2)$, then we can find lower bounds for the principal curvatures of the graph of f. Since $(0,0,0)$ is a critical point of $C(x^2 + y^2)$, its shape operator at the origin is its Hessian:

$$S = H_f = C\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

So the principal curvatures of $C(x^2 + y^2)$ are the eigenvalues of S, which are both $2C$. So we have that the lower bounds for both κ_1 and κ_2 (the principal curvatures of the graph of f) are $2C$.

(2) The Gaussian curvature of a manifold is equal to the product of its principal curvatures. Thus we have

$$K = \kappa_1 \kappa_2 \geq 2C \cdot 2C = 4C^2$$