Topology

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3.1 Homeomorphisms and closed sets

Definition 3.1.1:

Let X be a topological space, then $\mathcal{F} \subseteq X$ is called **closed** if its complement is open.

It is immediate from the definition of open sets that

(1) $\varnothing$ and X are closed.

(2) If $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ is an arbitrary set of closed sets then $\bigcap_{\lambda\in\Lambda} \mathcal{F}_\lambda$ is also closed.

(3) If $\{\mathcal{F}_k\}_{k=1}^n$ is a finite set of closed sets, then $\bigcup_{k=1}^n \mathcal{F}_k$ is also closed.

Proposition 3.1.2:

A function between topological spaces $f\colon X \longrightarrow Y$ is continuous if for every $\mathcal{F} \subseteq Y$ closed, $f^{-1}(\mathcal{F}) \subseteq X$ is also closed.

This is trivial: if f is continuous, let $F \subseteq Y$ be closed then F^c is open and so $f^{-1}(\mathcal{F}^c) = f^{-1}(\mathcal{F})^c \subseteq X$ is open and so $f^{-1}(\mathcal{F})$ is closed. And for the converse, let $\mathcal{U} \subseteq Y$ be open then $\mathcal{U}^c$ is closed so $f^{-1}(\mathcal{U}^c) = f^{-1}(\mathcal{U})^c$ is closed and therefore $f^{-1}(\mathcal{U})$ is open.

Definition 3.1.3:

Let X and Y be topological spaces, a mapping $f\colon X \longrightarrow Y$ is an **open mapping** if for every $\mathcal{U} \subseteq X$ open, $f(\mathcal{U})$ is open. And f is a **closed mapping** if for every $\mathcal{F} \subseteq X$ closed, $f(\mathcal{F})$ is closed.

Proposition 3.1.4:

Let X be a topological space, and $A \subseteq X$ then $\mathcal{F} \subseteq A$ is closed relative to A (closed in τ_A) if and only if there is a $Q \subseteq X$ closed such that $\mathcal{F} = Q \cap A$.

This is again trivial: if $\mathcal{F}$ is closed relative to A then $A \setminus \mathcal{F}$ is open in τ_A so $A \setminus \mathcal{F} = A \cap \mathcal{U}$ for some $\mathcal{U} \subseteq X$ open. Then $\mathcal{F} = A \cap \mathcal{U}^c$ so $Q = \mathcal{U}^c$ satisfies this. And if $\mathcal{F} = Q \cap A$ for Q closed in X then $A \setminus \mathcal{F} = A \setminus Q = A \cap Q^c$ and Q^c is open so $A \setminus \mathcal{F}$ is open relative to A so $\mathcal{F}$ is closed relative to A.

Example 3.1.5:

If X is a toplogical space and $A \subseteq X$, then τ_A is not necessarily a subset of τ_X. That is, there can be a set open relative to A which is not open relative to X.
For example, take $X = \mathbb{R}^2$ and $A = (0,1) \times \{0\}$. Then A is open relative to itself but not relative to X (in general if $A \subseteq X$ is not open then obviously $\tau_A \not\subseteq \tau_X$ since $A \in \tau_A$ is not in τ_X).
Similarly we can gave $S \subseteq A$ which is closed in A but not in X. For example take a non-closed set $A \subseteq X$ then A is closed in A but not in X.

So we see that if $A \notin \tau_X$, then $\tau_A \not\subseteq \tau_X$. But what if A is open in X?

Proposition 3.1.6:

If A is open in X then and $\mathcal{U}$ is open in A then $\mathcal{U}$ is open in X. That is, if $A \in \tau_X$ then $\tau_A \subseteq \tau_X$.
Similarly, if A is closed in X and $\mathcal{F}$ is closed in A then $\mathcal{F}$ is closed in X.

This is yet again trivial: if $\mathcal{U} = A \cap \mathcal{U}'$ is open in τ_A, since the finite intersection of open sets is open, $A \cap \mathcal{U}'$ is open in X, ie. $A \cap \mathcal{U}' \in \tau_X$. And if A is closed then if $\mathcal{F} = A \cap \mathcal{F}'$ is closed in A, then $\mathcal{F}$ is closed in X as the intersection of closed sets.
This condition is actually equivalent, since as we said earlier, if A is not open in X then $\tau_A \not\subseteq \tau_X$. And for closed sets if A is not closed, then A is closed in A but not in X. And this is an equality if and only if $A = X$ of course, since otherwise $X \notin \tau_A$.

Theorem 3.1.7:

If X and Y are toplogical spaces and $f\colon X \longrightarrow Y$ is a mapping then the following are equivalent:

(1) f is bijective, continuous, and f^{-1} is continuous.

(2) f is bijective, continuous, and an open mapping.

(3) f is bijective, continuous, and a closed mapping.

(4) f is bijective, and for every $S \subseteq X$, S is open in X if and only if $f(S)$ is open in Y.

(5) f is bijective, and for every $S \subseteq Y$, S is open in Y if and only if $f^{-1}(S)$ is open in X.

Proof:

We show the first implication: suppose $\mathcal{U} \subseteq X$ is open then $f(\mathcal{U}) = \left(f^{-1}\right)^{-1}(\mathcal{U})$ and since $\mathcal{U}$ is open and f^{-1} is continuous, $f(\mathcal{U})$ is open. Now for the second, suppose f is an open mapping, and let $\mathcal{F} \subseteq X$ is closed then $f(\mathcal{F}) = f(\mathcal{F}^{cc}) = f(\mathcal{F}^c)^c$ is closed since $f(\mathcal{F}^c)$ is open (the last equivalence is due to the bijectivity of f). This also shows that the third implies the second. To show the third implication, we can assume f is an open mapping then this is a direct consequence of the continuity and open-mapping nature of f. The fourth property obviously implies the fifth. To show the fifth implies the first, notice that this obviously implies continuity of f and f^{-1}.

■

Definition 3.1.8:

A mapping which satisfies any of the above equivalent conditions is called a **homeomorphism**. If there exists a homeomorphism between two toplogical spaces X and Y then X and Y are considered **homeomorphic**, which is denoted $X \cong Y$.

It is obvious that if f is a homeomorphism then so is f^{-1}. And the composition of homeomorphisms is also a homeomorphism. Thus the homeomorphic relation is symmetric and transitive, it is also obviously reflexive since it is trivial to see that id_X is a homeomorphism on X. So the homeomorphic relation is essentially an equivalence relation (barring the issue of constructing a set of all the toplogical spaces).

Example 3.1.9:

We can show that $[a, b] \cong [c, d]$ by defining $f(x) = \alpha x + \beta$ where $f(a) = c$ and $f(b) = d$ (this has a solution), and it can be shown that this is continuous and so is its inverse. We use the same function to show $(a, b) \cong (c, d)$. Similarly $(-\infty, a) \cong (-\infty, b) \cong (b, \infty)$, the (correct) bounds can also be closed.
And we can use tan as a homeomorphism between $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\mathbb{R}$ and thus $\mathbb{R}$ is homeomorphic with every open interval. We can also restrict $\tan^{-1}$ from $[0, \infty)$ to $\left[0, \frac{\pi}{2}\right)$ to show that $[a, \infty) \cong [a, b)$. And we can take $-x$ to show $(0, 1] \cong [-1, 0)$ so all half-open intervals are homeomorphic, independent of which side is open.
So we can classify the intervals in homeomorphism classes: closed intervals, open intervals (including open intervals including infinite bounds), and half open intervals (including infinite bound intervals with closed finite bounds like $[a, \infty)$).
And we know this is the strictest possible classification, as injective continuous functions between intervals must be monotonic. So (a, b) cannot be homeomorphic to $(a, b]$ as nothing can map to b, and neither can be homeomorphic to $[a, b]$ since nothing can map to a.

Definition 3.1.10:

Let X be a toplogical space, and $A \subseteq X$ then the **closure** of A is defined as

$$\overline{A} = \bigcap_{\substack{\mathcal{F} \text{ is closed} \\ A \subseteq \mathcal{F}}} \mathcal{F}$$

And the **interior** of A is defined as

$$\mathring{A} = \bigcup_{\substack{\mathcal{U} \text{ is open} \\ \mathcal{U} \subseteq A}} \mathcal{U}$$

The closure of a set has the following properties:

(1) $\overline{A}$ is closed

(2) $A \subseteq \overline{A}$

(3) For any set $\mathcal{F}$ which satisfies the above two properties, $\overline{A} \subseteq \mathcal{F}$ (as $\mathcal{F}$ is in the intersection then). In other words, the closure of a set is the minimal closed set which contains the set (the closure is minimal).

(4) If two sets B and C satisfy the above three properties, then $B = C = \overline{A}$ as $B \subseteq C$ and $C \subseteq B$ by the third property (and also for $\overline{A}$) (the closure is unique).

The interior of a set has the following properties:

(1) $\mathring{A}$ is open

(2) $\mathring{A} \subseteq A$

(3) For any set $\mathcal{U}$ which satisfies the above two properties, $\mathcal{U} \subseteq \mathring{A}$ (the interior is maximal).

(4) If two sets B and C satisfy the above three properties, then $B = C = \mathring{A}$ (the interior is unique).

And finally it is easy to see

$$\big(\mathring{A}\big)^c = \overline{A^c}, \qquad \big(\overline{A}\big)^c = \big(\mathring{A^c}\big)$$

Proposition 3.1.11:

Let X be a toplogical space and $A \subseteq X$ then $p \in X$ is in $\overline{A}$ if and only if for every neighborhood $\mathcal{U}$ of p, $A \cap \mathcal{U} \neq \varnothing$.

Proof:

If $p \in \overline{A}$ then let $p \in \mathcal{U}$ be a neighborhood of p, then $p \in \mathcal{U} \cap \overline{A}$ so the intersection is non-empty. If $p \in \overline{A}$ then suppose there is a neighborhood $\mathcal{U}$ of p which is disjoint from A. Thus $A \subseteq \mathcal{U}^c$ and $\mathcal{U}^c$ is closed so $\overline{A} \subseteq \mathcal{U}^c$, and since $p \in \overline{A}$ then $p \in \mathcal{U}^c$ so $p \notin \mathcal{U}$ in contradiction. ■

Definition 3.1.12:

If X is a toplogical space and $A \subseteq X$. We say that A is **dense** in X if $\overline{A} = X$.

Thus A is dense if and only if for every $x \in X$ and every neighborhood $\mathcal{U}$ of x, $\mathcal{U} \cap A \neq \varnothing$. Thus A is dense if and only if it intersects with every non-empty open set (as every non-empty open set is a neighborhood of its elements).

Proposition 3.1.13:

Suppose $A \subseteq B \subseteq X$ where X is a topological space, then $\overline{A}^B = \overline{A}^X \cap B$ where $\overline{A}^X$ is the closure of A with respect to X.

Proof:

We know that

$$\overline{A}^X \cap B = \bigcap_{A \subseteq \mathcal{F} \text{ closed}} \mathcal{F} \cap B = \bigcap_{A \subseteq \mathcal{F} \text{ closed in } B} \mathcal{F} = \overline{A}^B$$

since the sets which are closed relative to B are exactly $\mathcal{F} \cap B$ where $\mathcal{F}$ is closed in X. ■

Thus A is dense in B if and only if $\overline{A}^X \supseteq B$.

Theorem 3.1.14:

Let X be a toplogical space and $\{\mathcal{U}_\lambda\}_{\lambda \in \Lambda}$ be an open cover of X. Let Y be another toplogical space and $f\colon X \longrightarrow Y$. Then if for every $\lambda \in \Lambda$, $f\big|_{\mathcal{U}_\lambda} \colon \mathcal{U}_\lambda \longrightarrow Y$ is continuous, then f is continuous.

Proof:

Let $f_\lambda = f\big|_{\mathcal{U}_\lambda}$. Notice that for any $S \subseteq Y$:

$$x \in f^{-1}(S) \iff x \in \bigcup_{\lambda \in \Lambda} f_\lambda^{-1}(S)$$

since if $x \in f^{-1}(S)$ then $x \in \mathcal{U}_\lambda$ for some $\mathcal{U}_\lambda$ and since $f(x) \in S$, $f_\lambda(x) \in S$ so $x \in f_\lambda^{-1}(S)$. And if x is in the union then $x \in f_\lambda^{-1}(S)$ so $f_\lambda(x) \in S$ and so $f(x) \in S$. Thus for any $S \subseteq Y$:

$$f^{-1}(S) = \bigcup_{\lambda \in \Lambda} f_\lambda^{-1}(S)$$

And so if $S = \mathcal{U}$ is open, then $f_\lambda^{-1}(\mathcal{U})$ is open in $\mathcal{U}_\lambda$ and since $\mathcal{U}_\lambda$ is open, $f_\lambda^{-1}(\mathcal{U})$ is open in X. So $f^{-1}(\mathcal{U})$ is the union of open sets $(f_\lambda^{-1}(\mathcal{U}))$ and is therefore itself open. Thus f is continuous. ■

Theorem 3.1.15:

Let X be a toplogical space and $\{\mathcal{F}_k\}_{k=1}^n$ be a finite closed cover of X. Let Y be another topological space and $f\colon X \longrightarrow Y$. Then if for every $1 \leq k \leq n$ $f\big|_{\mathcal{F}_k} = f_k\colon \mathcal{F}_k \longrightarrow Y$ is continuous, then f is continuous.

Proof:

By above we have

$$f^{-1}(\mathcal{F}) = \bigcup_{k=1}^{n} f_k^{-1}(\mathcal{F})$$

for every closed set $\mathcal{F} \subseteq Y$. Since $f_k^{-1}(\mathcal{F})$ is closed in $\mathcal{F}_k$ since f_k is continuous, and since $\mathcal{F}_k$ is closed in X, $f_k^{-1}(\mathcal{F})$ is closed in X for every k. Thus $f^{-1}(\mathcal{F})$ is a finite union of closed sets and is therefore also closed. So f is continuous. ■