Infinitesimal Calculus 3

Lecture 5, Sunday November 6, 2022
Ari Feiglin

4.1 Connected Spaces

Definition 4.1.1:

If X is a metric space, $E \subseteq X$ is **disconnected** if there exists two disjoint open sets $\mathcal{O}_1$ and $\mathcal{O}_2$ such that $E \subseteq \mathcal{O}_1 \cup \mathcal{O}_2$ and $E \cap \mathcal{O}_1, E \cap \mathcal{O}_2 \neq \varnothing$. A **connected** space is a space which is not disconnected.

This is equivalent to saying that if $E \subseteq \mathcal{O}_1 \uplus \mathcal{O}_2$ then $S \subseteq \mathcal{O}_1$ or $S \subseteq \mathcal{O}_2$.

Proposition 4.1.2:

A metric space is connected if and only if the only clopen sets is the entire space and the empty set.

Proof:

Suppose X is disconnected, then there exists open sets $\mathcal{O}_1$ and $\mathcal{O}_2$ such that $X = \mathcal{O}_1 \uplus \mathcal{O}_2$. Therefore the complement of $\mathcal{O}_1$ is $\mathcal{O}_2$ and so it is also closed. And since $\mathcal{O}_1$ and $\mathcal{O}_2$ are disjoint and have non-empty intersection with X, they are neither X nor the empty set. Therefore if X is disconnected there exists clopen sets other than X and $\varnothing$.
Suppose $\mathcal{O}_1$ is a clopen set in X not equal to X or $\varnothing$, then $\mathcal{O}_2 = \mathcal{O}_1^c$ is open and disjoint from $\mathcal{O}_1$. Since $\mathcal{O}_1 \neq X, \varnothing$, $\mathcal{O}_2 \neq X, \varnothing$ since $X = \mathcal{O}_1 \uplus \mathcal{O}_2$, X is disconnected.

■

So $\mathbb{R}$ for example is connected. And $\mathbb{R} \setminus \{0\}$ is disconnected since $(-\infty, 0)$ and $(0, \infty)$ are open, disjoint, and cover $\mathbb{R} \setminus \{0\}$.

Definition 4.1.3:

A **line segment** between two vectors P and Q in a linear space X is

$$\overleftrightarrow{PQ} = \{P + t(Q - P) \mid t \in [0, 1]\}$$

Notice that $\overleftrightarrow{PQ} = \overleftrightarrow{QP}$, and both P and Q are in this segment. If the line segment in focus is understood, x_t is understood to be the point $P + t(Q - P)$.

Proposition 4.1.4:

A line segment in a normed linear space X is connected.

Proof:

Suppose $P, Q \in X$, suppose for the sake of a contradiction that A and B are open such that $\overleftrightarrow{PQ} \subseteq A \uplus B$ as well as having non-empty intersections with the segment. We know $x_1 = Q$ and suppose $x_1 \in B$, we define:

$$K = \{t \in [0, 1] \mid x_t \in A\} \qquad u = \sup K$$

K is non-empty since A has non-empty intersection with $\overleftrightarrow{PQ}$ and K is bound by 1, so it has a supremum. We will show that $x_u \notin A$. If $u = 1$ then $x_u = Q \in B$ which is disjoint from A, so $x_u \notin A$. If $u < 1$ and $x_u \in A$, since A is open there is an $\varepsilon > 0$ such that $B_\varepsilon(x_u) \subseteq A$. Notice then that $x_{t'} = x_{t + \frac{\varepsilon}{2\|Q-P\|}}$ must be in A since $\|x_t - x_{t'}\| = \frac{\varepsilon}{2} < \varepsilon$. But $t' > t$ and $t' \in K$ which is a contradiction to t's supremumness.
And $x_u \notin B$ since if it were, since B is open so there is a ball $B_\varepsilon(x_u) \subseteq B$. And so similar to before there must be a $\delta > 0$ such that for every $u - \delta < t < u + \delta$, $x_t \in B$. So $u - \delta$ is an upper bound to K, which is a contradiction.
So $x_u \notin A \uplus B$, but it must be in $\overleftrightarrow{PQ}$ since $u \in [0, 1]$ which is a contradiction.

■

Theorem 4.1.5:

If $\{S_\lambda\}_{\lambda\in\Lambda}$ are connected sets such that their intersection is non-empty, then

$$S = \bigcup_{\lambda\in\Lambda} S_\lambda$$

is also connected.

Proof:

Suppose B and C are open and disjoint such that $S \subseteq B \cup C$. Let $\lambda \in \Lambda$ and so $S_\lambda B \cup C$, so $S_\lambda \subseteq B$ or $S_\lambda \subseteq C$. Without loss of generality suppose it is a subset of B. Then take $\gamma \neq \lambda \in \Lambda$ then $S_\gamma \subseteq B$ or $S_\gamma \subseteq C$. Since there exists an $x \in S_\lambda \cap S_\gamma$ and $x \notin C$, $S_\lambda \subseteq B$. So for every $\lambda \in \Lambda$, $S_\lambda \subseteq B$, and therefore $S \subseteq B$, so S is connected.

■

Definition 4.1.6:

If X is a normed linear space and $P_1, \ldots, P_n \in X$, the **polygon chain** is

$$\overleftrightarrow{P_1P_2\cdots P_n} = \overleftrightarrow{P_1P_2} \cup \overleftrightarrow{P_2P_3} \cup \cdots \cup \overleftrightarrow{P_{n-1}P_n}$$

Proposition 4.1.7:

Polygonal chains are connected.

Proof:

We will prove so through induction on the number of vectors in the chain. For $n = 2$ this is simply a line segment, which we know is connected. We know that $\overleftrightarrow{P_1\cdots P_{n+1}} = \overleftrightarrow{P_1\cdots P_n} \cup \overleftrightarrow{P_nP_{n+1}}$. By our inductive hypothesis, these are both connected and have a non empty intersection (since P_n is in it), so by the above theorem, their union is connected as well.

■

Definition 4.1.8:

A **path** between two vectors P Q in a normed linear space X is a continuous function $f\colon [0,1] \longrightarrow X$ such that $f(0) = P$ and $f(1) = Q$.

Notice then that a line segment represents a path (let $f(t) = x_t$). And similarly so does a polygonal chain.

Definition 4.1.9:

A subset S of a normed linear space X is **connected pathwise** if for every $x, y \in S$ there is a path γ_{xy} between x and y whose image is contained in S.

When talking about paths it is often useful to focus on their image, so we when discussing paths we may use the function in place of its image.

Proposition 4.1.10:

Every path is connected.

A similar proof to showing line segments are connected can be used.

Proposition 4.1.11:

If a set is connected pathwise, then it is connected.

Proof:

Suppose S is connected pathwise, then let $x \in S$. We know that

$$S = \bigcup_{y \in S} \gamma_{xy}$$

Since $y \in \gamma_{xy}$ and $\gamma_{xy} \subseteq S$. And since paths are connected and the intersection of $\{\gamma_{xy} \mid y \in S\}$ contains x and is therefore non-empty, by the above theorem, S is also connected.

$\blacksquare$

Theorem 4.1.12:

An open set is connected if and only if it is connected pathwise.

Proof:

Suppose $\mathcal{O}$ is open, we know if it is connected pathwise then it is connected, so all that remains is to prove the converse. Take some $x \in S$ and define:

$$A = \{y \in S \mid \text{there exists a path between } x \text{ and } y \text{ contained entirely in } S\}$$

Let $B = S \setminus A$. We will show that A and B are open. We know that $S = A \cup B$ and $A \cap B = \varnothing$, so if we succeed in showing that they both are open, since S is connected, one must be empty. And since $x \in A$, it must be that $B = \varnothing$ so $S = A$ and therefore S is open.
First we will show that A is open. If $y \in A$ then since $\gamma_{xy} \subseteq S$, $y \in S$ which is open, so there exists an $r > 0$ such that $B_r(y) \subseteq S$. If $z \in B_r(y)$ then taking $\gamma_{xy} \cup \overleftrightarrow{yz}$ gives a path between x and y contained in S (we define the path of the union to squish γ_{xy} and then be $\overleftrightarrow{yz}$). So $z \in A$, and therefore $B_r(y) \subseteq A$, so A is open.
Let $y \in B$, since $y \in S$ then there exists an $r > 0$ such that $B_r(y) \subseteq S$. If there is a $z \in B_r(y) \cap A$ then there is a path γ_{zx} contained in S, and since $z \in B_r(y)$, $\overleftrightarrow{yz}$ is also contained in S, so the path $\overleftrightarrow{yz} \cup \gamma_{zx}$ is a path between y and x contained in S, so $y \in A$ in contradiction. So $B_r(y) \subseteq B$, so B is open.

$\blacksquare$