Representation Theory
Homework 1

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1 Group Actions

1.1 Problem

Let H be a subgroup of G, and X be a non-empty G-set. Show that there is an isomorphism $\hom_G(G/H, X) \cong X^H$ (where X^H is the set of points fixed by H).

A $f \in \hom_G(G/H, X)$ is a function $f\colon G/H \to X$ such that $f(g(aH)) = gf(aH)$ for all $g \in G$ and $aH \in G/H$. In particular, taking $g \in G$ and $H \in G/H$ we get that $f(gH) = gf(H)$. So each morphism is determined by uniquely its image of H. Furthermore, since for $h \in H$: $f(H) = f(hH) = hf(H)$, we get that $f(H)$ is a fixed point of H, i.e. $f(H) \in X^H$.

So mapping $f \in \hom_G(G/H, X)$ to $f(H)$ forms an injection into X^H (since each morphism is determined uniquely by its image of H, this is an injection). Now, for each $x \in X^H$, we can define $f_x(gH) = gx$, and this defines a morphism in $\hom_G(G/H, X)$ whose image under the bijection is clearly x. All that remains is to show that f_x is well-defined and a morphism of G-sets. Suppose $g_1H = g_2H$, meaning $g_1 = g_2h$ for some $h \in H$; now $f_x(g_1H) = g_1x = g_2hx = g_2x = f_x(g_2H)$ (where $hx = x$ precisely because $x \in X^H$), showing that f_x is indeed well-defined. And f_x is a morphism of G-sets: $f_x(g(aH)) = f_x((ga)H) = gax$, while $f_x(aH) = ax$ and so $f_x(g(aH)) = gf_x(aH)$.

1.2 Problem

Let $f\colon X \to Y$ be a morphism of G-sets, where Y is transitive. Show that f is surjective.

Let $x \in X$, and define $y = f(x)$. Then for any $y' \in Y$ by transitivity there is a $g \in G$ such that $gy = y'$. Meaning $y' = gy = gf(x) = f(gx)$. So y' is in the image of f, and therefore f is surjective.

1.3 Problem

Let X be a transitive finite G-set.

(1) Show that $\hom_G(X, X)$ is in bijection with N_H/H where H is the stabilizer of an element of X, and N_H is H's normalizer.

(2) Show that $\hom_G(X, X)$ has the structure of a group under composition, and the above bijection is an isomorphism.

Let $x_0 \in X$, and $H = \mathrm{Stab}_G(x_0) = \{g \in G \mid gx_0 = x_0\}$, then $N_H = \left\{g \in G \,\middle|\, gHg^{-1} = H\right\}$. Since X is transitive, every morphism out of X is determined by its image on x_0: for $y = gx_0$, $f(y) = gf(x_0)$.

Now, define a map $N_H \to \hom_G(X, X)$ by mapping gH to the unique morphism $f_g \in \hom_G(X, X)$ such that $f_g(x_0) = gx_0$ (i.e. $f_g(hx_0) = hgx_0$). Such a map, if it exists, is unique. We must show that f_g exists: if $ax_0 = bx_0$ then $agx_0 = bgx_0$. Now, if $ax_0 = bx_0$ then $a^{-1}b \in H$ as it keeps x_0 fixed, and since $g \in N_H$ we have that $g^{-1}a^{-1}bg \in H$, so $g^{-1}a^{-1}bgx_0 = x_0$ and so $bgx_0 = agx_0$ as required. So f_g does exist.

Now we claim that we can quotient this map out by H: if $aH = bH$ then $f_a = f_b$. Indeed, if $a = bh$ for $h \in H$ then $f_a(x_0) = ax_0 = bhx_0 = bx_0 = f_b(x_0)$ and since these maps are uniquely determined by their image of x_0, we have $f_a = f_b$. So we have defined a map $N_H/H \to \hom_G(X, X)$.

We claim that this map is injective: if $f_a = f_b$ then $ax_0 = bx_0$ and so $ab^{-1} \in H$ meaning $aH = bH$ as required. And this map is surjective: if $f \in \hom_G(X,X)$ then $f(x_0) = gx_0$ for some $g \in G$ since X is transitive, meaning $f = f_g$. Thus we have defined a bijection.

Furthermore, $\hom_G(X,X)$ forms a group: all we must show is that every $f \in \hom_G(X,X)$ is an isomorphism. By the previous problem, since X is transitive we know that all G-morphisms over X are surjective. Since X is finite, this means they are also injective and therefore $\hom_G(X,X)$ consists of only isomorphisms, as required.

The bijection we defined $N_H/H \to \hom_G(X,X)$ is not a homomorphism: $f_a \circ f_b(x_0) = f_a(bx_0) = bf_a(x_0) = bax_0$, which is not equal to $f_{ab}(x_0)$. But if we instead map a to $f_{a^{-1}}$, then we get a homomorphism: $f_{a^{-1}} \circ f_{b^{-1}}(x_0) = b^{-1}a^{-1}x_0 = (ab)^{-1}x_0 = f_{(ab)^{-1}}(x_0)$.

If we denote our original bijection $\psi\colon N_H/H \to \hom_G(X,X)$ which maps gH to f_g, then our homomorphism is the composition of this with the inversion operator: $gH \mapsto g^{-1}H$. These are both bijections meaning our homomorphism is a bijection and thus an isomorphism, as required.

1.4 Problem

Let H, K be subgroups of G. Show that there is a bijection

$$\hom_G(G/H, G/K) \cong \{gK \in G/K \mid g^{-1}Hg \subseteq K\}$$

Furthermore show that G/H and G/K are isomorphic as G-sets iff H and K are conjugate.

The set on the right is well-defined: if $aK = bK$ then $a = bk$, so if $b^{-1}Hb \subseteq K$ then $a^{-1}Ha = k^{-1}b^{-1}Hbk \subseteq k^{-1}Kk = K$ as required. Now, a G-morphism $f\colon G/H \to G/K$ is uniquely determined by $f(H)$: $f(gH) = gf(H)$. So let us define the map which maps $f \in \hom_G(G/H, G/K)$ to $f(H)$. Suppose that $f(H) = gK$, then notice that for $h \in H$ we have $gK = f(H) = f(hH) = hf(H) = hgK$. So $gK = hgK$, meaning $g^{-1}hg \in K$, i.e. $g^{-1}Hg \subseteq K$, so this map is well-defined.

This map is clearly injective, since each G-morphism is uniquely determined by $f(H)$. And it is surjective: given gK such that $g^{-1}Hg \subseteq K$, define $f(aH) = agK$ (i.e. $f(H) = gK$). This is well-defined: if $a = bh$ then $f(aH) = agK = bhgK$ and $g^{-1}hg \in K$ so $hg = gk$, thus $f(aH) = bgkK = bgK = f(bH)$. So we have a bijection, as required.

Now, we know that G/K is transitive: (the single orbit is generated by $K \in G/K$). So $\hom_G(G/H, G/K)$ contains only surjections.

Now, if G/H and G/K are isomorphic, let f be an isomorphism: so $f(H) = gK$. Let $k \in K$, we want to show that $k \in g^{-1}Hg$ (so that since $g^{-1}Hg \subseteq K$, we have that they are conjugates). Indeed, notice that $f(g^{-1}H) = K$ and $f(kg^{-1}H) = K$, so since the isomorphism is injective we have $g^{-1}H = kg^{-1}H$, giving us the desired result, $k \in g^{-1}Hg$ as required.

Conversely, if $g^{-1}Hg = K$ then the unique map $f(H) = gH$ is an isomorphism (it is a G-morphism by our bijection: g is in our right-hand set and so $f(H) = gH$ defines a G-morphim). As already noted it must be surjective, and it is injective since if $f(aH) = f(bH)$ then $agK = bgK$, so $ag \in bgK$, so $ab^{-1} \in gKg^{-1} = H$ meaning $aH = bH$.

1.5 Problem

Show that the following G-sets are transitive, and choose an element from each and describe the stabilizer.

(1) For $1 \le k \le n$, $G = S_n$ and $X = \{(x_1, \ldots, x_k) \mid x_i \neq x_j \in [n]\}$ where G acts on X coordinate-wise.

(2) $G = D_n = \langle \rho, \epsilon \mid \rho^n, \epsilon^2, \epsilon\rho\epsilon\rho \rangle$ the dihedral group, and $X = \mathbb{Z}/n\mathbb{Z}$, where $\rho k = k+1 \bmod n$ and $\epsilon k = -k \bmod n$

(3) $G = \mathrm{GL}_n(\mathbb{F})$ and $X = \mathbb{F}^n - \{0\}$

(1) Let $\vec{x}, \vec{y} \in X$, then defining $\sigma(x_i) = y_i$ for $1 \leq i \leq k$ (and keeping it constant on elements not in $\vec{x}$) defines a well-defined bijection where $\sigma\vec{x} = \vec{y}$. It is well-defined since $\vec{x}$ has distinct components, and it is injective since $\vec{y}$ has distinct components. Surjectivity follows from the finiteness of $[n]$.

The stabilizer of any $\vec{x} \in X$ is the set of permutations $\sigma \in S_n$ whose support lies in $[n] - \vec{x}$.

(2) Take $k, m \in \mathbb{Z}/n\mathbb{Z}$, then $\rho^{k-m}m = k$, and so the action is transitive. For any $k \in \mathbb{Z}/n\mathbb{Z}$ its stabilizer is the set $\left\{1, \rho^{2k \bmod n}\epsilon\right\}$. This is because all elements of D_n can be written as $\rho^m\epsilon$ or ρ^m. ρ^m is in the stabilizer for $m = 0$. $\rho^m\epsilon k = m - k \bmod n$, and $m - k \equiv k \pmod{n}$ if and only if $m \equiv 2k \pmod{n}$.

(3) Transitivity of this action is a trivial consequence from linear alegbra. Given $\vec{x}, \vec{y} \in \mathbb{F}^n - \{0\}$, extend $\vec{x}$ to a basis B of $\mathbb{F}^n$ and similarly $\vec{y}$ to B'. We know that there exists a linear transformation which maps B to B', and so $\vec{x}$ to $\vec{y}$. Since this linear transformation maps a base to a base, it is an isomorphism and thus in $\mathrm{GL}_n(\mathbb{F})$.

2 Representations and Equivariant Maps

2.1 Problem

Let V be a representation of a group G, where $V = V_1 \oplus V_2$ for subrepresentations V_1, V_2. Show that the inclusions $\iota_i : V_i \to V$ and projections $\pi_i : V \to V_i$ are equivariant maps.

We simply need to show that for all $g \in G, v \in V_i$, $\iota_i(gv) = g\iota_i(v)$. This reduces to $gv = gv$, since as a subrepresentation V_i inherits the same action on it as V. And for the projections, since V_1, V_2 are subrepresentations for $v_i \in V_i$, $gv_i \in V_i$ for $g \in G$. So $\pi_i(g(v_1+v_2)) = \pi_i(gv_1+gv_2) = gv_i$, while $g\pi_i(v_1+v_2) = gv_i$ as well. (The first equality is since $gv_1 + gv_2$ splits in $V_1 \oplus V_2$.)

2.2 Problem

Let V_1, V_2 be irreducible representations of G. Consider their direct sum $V = V_1 \oplus V_2$. Show that V_1 and V_2 are isomorphic if and only if V has a non-trivial subrepresentations other than V_1 and V_2.

Suppose V_1 and V_2 are isomorphic, with $\iota : V_1 \to V_2$ an isomorphism. Consider $W = \{(v, \iota v) \mid v \in V_1\} \subseteq V$. This is clearly a subspace: $\alpha(v, \iota v) + \beta(u, \iota u) = (\alpha v + \beta u, \iota(\alpha v + \beta u))$. It is also a subrepresentation: $g(v, \iota v) = (gv, g\iota v) = (gv, \iota gv)$, so W is closed under G. Furthermore W is not V_1 or V_2, as its coordinates are both non-zero.

Now suppose that W is a non-trivial subrepresentation of V distinct from V_1, V_2. Then let us define $f_i : W \to V_i$ by $f_i = \pi_i \circ \iota$ ($\pi_i : V \to V_i$ the projection operator and $\iota : W \to V$ the inclusion operator). Now, $\mathrm{im} f_i$ must be trivial: either V_i or 0, since V_i is irreducible. Notice that $\ker f_i = W \cap V_{\neg i}$ (as a projection operator), and so $\ker f_i$ is a subrepresentation of $V_{\neg i}$: it too must be trivial then.

If $\mathrm{im} f_i = 0$ then $\ker f_i = W$, which is non-trivial and so $W = V_{\neg i}$, contradicting W being distinct from V_1, V_2. So $\mathrm{im} f_i = V_i$. Now, if $\ker f_i = 0$ then we are finished: f_i forms an isomorphism between W and V_i, so $V_1 \cong W \cong V_2$ as required. Otherwise, $\ker f_i = V_i$ and so $V_i \subseteq W$, but then $W = V$ (we will take $i = 1$ here): indeed, take $v_1 + v_2 \in W$ then since f_i is surjective there is an $u_1 + v_2 \in W$. But $V_1 \subseteq W$, so $u_1 \in W$, meaning $v_2 \in W$. So $V_1, V_2 \subseteq W$, meaning $W = V$ as required. So $\ker f_i = 0$ and we have the desired result.

2.3 Problem

Let G be a finite group whose order is not dividable by $\mathbb{F}$'s characteristic. Let V be a representation of G over $\mathbb{F}$.

(1) Show that if $\hom_G(V, V)$ is 1-dimensional then V is irreducible.

(2) Show that if $\mathbb{F}$ is algebraically closed, then $\hom_G(V, V)$ is 1-dimensional iff V is irreducible.

(3) Give an example where V is irreducible and $\hom_G(V, V)$ is not 1-dimensional.

(1) We assume V is finite-dimensional. Suppose $W \subseteq V$ is a subrepresentation. By Maschke's theorem, W has a complementary subrepresentation: $V = W \oplus W'$ for a subrepresentation W'. Let $\pi\colon V \to V$ be the projection operator on W: $\pi(w + w') = w$. By the previous question this is a G-morphism, i.e. it is in $\hom_G(V, V)$. Since this space is 1-dimensional, it must be in the span of the identity. But this can only happen if W is trivial: if not, then π has two eigenvalues.

(2) V is irreducible, and let $T \in \hom_G(V, V)$. We want to show that T is scalar multiplication. Since V is irreducible, it must have finite dimension and therefore T has an eigenvalue (since all linear operators over a finite-dimension vector space over an algebraically closed field have an eigenvalue). So suppose $Tv = \lambda v$. Since T is a G-morphism, this means that for all $g \in G$, $T(gv) = \lambda gv$.

Let $W = \operatorname{span}\{gv\}_{g\in G}$, this is clearly a subrepresentation, and since V is irreducible it must be equal to V. So $\{gv\}_{g\in G}$ forms a spanning set, and since T is scalar multiplication by λ on this spanning set, it is scalar multiplication on all of V.

(3) Take $V = \mathbb{R}^2$ (over $\mathbb{R}$) and $G = \mathbb{Z}/3\mathbb{Z}$, with the representation given by $n \mapsto R^n_{2\pi/3}$ (R_θ is the rotation matrix of angle θ, we will write it simply as R). This is clearly a representation (all that needs to be shown is well-definedness, which is simple since $R^3 = I$, the rest follows).

Now, we want to find an element of $\hom_G(V, V)$ which is not scalar multiplication. Let us take $Tv = R_{\pi/4}v$ (rotation by $90°$). Since $R_\theta \circ R_\phi = R_{\theta+\phi}$, we get that all rotation matrices commute, meaning T is a G-morphism: $T(nv) = TRv = RTv$ (since T, R commute as rotation matrices). So $T \in \hom_G(V, V)$.

Furthermore, T is not a multiple of the identity, and is therefore linearly independent of $\mathrm{id} \in \hom_G(V, V)$. So $\dim \hom_G(V, V) \geq 2$.

2.4 Problem

Let $G = \mathbb{Z}/p\mathbb{Z}$ for prime p, and let $\mathbb{F} = \mathbb{F}_p$. Define the representation

$$\rho\colon n \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

(1) Show that ρ defines a representation on $\mathbb{F}^2$.

(2) Find a one-dimensional subrepresentation of ρ that does not have an invariant complement. Conclude that ρ is not semisimple.

(3) Use the ideas of this exercise to also show that if $\mathbb{F}$ has characteristic 0 but G has infinite order, Maschke's theorem may still fail.

(1) Firstly, this is well-defined since the base sets of G and $\mathbb{F}$ are equal. $\rho(n) \in \mathrm{GL}_2(\mathbb{F})$ since the determinant of $\rho(n)$ is 1. And

$$\rho(n)\rho(m) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n+m \\ 0 & 1 \end{pmatrix} = \rho(n+m)$$

So $\rho\colon G \to \mathrm{GL}_2(\mathbb{F})$ is a homomorphism, as required.

Note that this only works because G and $\mathbb{F}$ have the same carrier set ($\mathbb{Z}/p\mathbb{Z}$), this is the key to the counterexample.

(2) Let $v = (a, b) \in \mathbb{F}^2$ then span $\{v\}$ is a subrepresentation iff for each n,

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a + bn \\ b \end{pmatrix}$$

is in the span of (a, b). That is, we must have

$$(a + bn,\, b) = \lambda(a, b)$$

This requires that $\lambda = 1$, and so $bn = 0$. Since this holds for all n, in particular 1, this means that $b = 0$. So our subrepresentation must be the span of $(a, 0)$, which is just the span of $(1, 0)$.

In summary, the *only* subrepresentation of $\mathbb{F}^2$ is span $\{(1, 0)\}$, and thus it cannot have a complementary subrepresentation.

(3) Take $G = \mathbb{Z}$ and $\mathbb{F} = \mathbb{R}$, and define the representation on $\mathbb{R}^2$:

$$\rho\colon n \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

The proof that this is a representation is the same as in the first point. And the proof of the second point still holds, it did not rely on any special characteristic of the fields: the only subrepresentation is span $\{(1, 0)\}$, and so the representation is not semisimple.