Infinitesimal Calculus 3

Lecture 6, Wednsday November 9, 2022
Ari Feiglin

6.1 Sequences and Limits in Metric Spaces

Definition 6.1.1:

If (X,ρ) is a metric space, a sequence $\{x_n\}_{n=1}^{\infty}$ in X **converges** to $x \in X$ if

$$\lim_{n\to\infty} \rho\left(x_n, x\right) = 0$$

We denote this by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$.

Proposition 6.1.2:

- Limits, if they exist, are unique.
- Constant sequences $\{x\}_{n=1}^{\infty}$ converge to x.
- If $x_n \to x$ and $y_n \to y$ then $\rho\left(x_n, y_n\right) \to \rho\left(x, y\right)$

Proof:

- Suppose $x_n \to x, y$ so by the triangle inequality for every $n \in \mathbb{N}$:

  $$\rho\left(x,y\right) \leq \rho\left(x,x_n\right) + \rho\left(x_n,y\right)$$

  Taking the limit of the right side gives 0 by definition, so $\rho\left(x,y\right) = 0$ and therefore $x = y$.
- This is trivial since $\rho\left(x,x\right) = 0$.
- Since:

  $$\rho\left(x,y\right) \leq \rho\left(x,x_n\right) + \rho\left(x_n,y_n\right) + \rho\left(y_n,y\right)$$

  The limit of the right side is $\lim\rho\left(x_n,y_n\right)$, so $\rho\left(x,y\right) \leq \lim\rho\left(x_n,y_n\right)$. And:

  $$\rho\left(x_n,y_n\right) \leq \rho\left(x_n,x\right) + \rho\left(x,y\right) + \rho\left(y,y_n\right)$$

  The limit of the right side is $\rho\left(x,y\right)$ so

  $$\lim\rho\left(x_n,y_n\right) \leq \rho\left(x,y\right) \leq \lim\rho\left(x_n,y_n\right)$$

  And therefore $\rho\left(x_n,y_n\right) \to \rho\left(x,y\right)$.

$\blacksquare$

Proposition 6.1.3:

Suppose X is a normed linear space and $x_n \to x$ and $y_n \to y$ and let $c \in \mathbb{R}$.

- $x_n + y_n \to x + y$
- $cx_n \to cx$
- $\|x_n\| \to \|x\|$
- If $X = \mathbb{R}^n$ and if $x^{(m)} = \left(x_1^{(m)}, \dots, x_n^{(m)}\right)$ and $x = (x_1, \dots, x_n)$ then $x^{(m)} \to x$ if and only if $x_k^{(m)} \to x_k$ for every relevant k.

- If $X = \mathbb{R}^n$ and $x^{(m)} \to x$ and $y^{(m)} \to y$ then $x^{(m)} \cdot y^{(m)} \to x \cdot y$ (dot product).

Proof:

- We know $\|x_n + y_n - x - y\| \leq \|x_n - x\| + \|y_n - y\| \to 0$ so $x_n + y_n \to x + y$ as required.
- We know $\|cx_n - cx\| = |c| \|x_n - x\| \to 0$ so $cx_n \to cx$ as required.
- Since $|\|x_n\| - \|x\|| \leq \|x_n - x\| \to 0$, it must be that $|\|x_n\| - \|x\|| \to 0$ so $\|x_n\| \to \|x\|$ as required.
- Notice that
$$\left\|x^{(m)} - x\right\|^2 = \sum_{k=1}^{n} \left(x_k^{(m)} - x_k\right)^2 \to 0$$
So the left converges to 0 if and only if every $\left(x_k^{(m)} - x_k\right)^2$ converges to 0 since squares are non-negative. And this in turn is equivalent to $x_k^{(m)} \to x_k$. It is easy to see how this is actually true for any p-norm, not just for $p = 2$.
- By above we know that $x_k^{(m)} \to x_k$ and $y_k^{(m)} \to y_k$, and since:
$$x^{(m)} \cdot y^{(m)} = \sum_{k=1}^{n} x_k^{(m)} \cdot y_k^{(m)}$$
And by limit arithmetic, we know this converges to
$$\sum_{k=1}^{n} x_k \cdot y_k = x \cdot y$$
As required.

■

Definition 6.1.4:

Suppose (X, ρ) is a metric space and $\{x_n\}_{n=1}^{\infty}$ is a sequence in it. If $\{n_k\}$ is a strictly increasing sequence ($n_k < n_{k+1}$), then $\{x_{n_k}\}_{k=1}^{\infty}$ is a **subsequence** of $\{x_n\}_{n=1}^{\infty}$. If x_{n_k} converges to x, then x is a **partial limit** of $\{x_n\}_{n=1}^{\infty}$.

Proposition 6.1.5:

If $x_n \to x$ then every subsequence of $\{x_n\}$ converges to x.

This is trivial.

Theorem 6.1.6:

If $S \subseteq X$ is compact, then every sequence $\{x_n\}_{n=1}^{\infty}$ in S has a convergent subsequence.

Proof:

If there is an element x which is in the sequence an infinite amount of times, we can construct a subsequence of all of its instances, and this subsequence converges to x. Otherwise there are an infinite number of different elements in $\{x_n\}$. Let $x \in S$, then if for every $\varepsilon > 0$ there is an element $x \neq x_n \in B_\varepsilon(x)$, we can taken a sequence $\varepsilon_n \to 0$ such and the associated x_{n_k}s converge to x, and thus we have a convergent subsequence. Otherwise, there must be some ε_x such that there is no $x \neq x_n \in B_{\varepsilon_x}(x)$. So we can take an open cover of S by $\{B_{\varepsilon_x}(x)\}_{x \in S}$, and since every one of these balls contains at most one element in x_n, every finite subcover contains only a finite number of x_ns, so it can't cover S. This contradicts the compactness of S. And therefore $\{x_n\}$ must have a convergent subsequence.

■

Theorem 6.1.7 (Bolzano-Weierstrauss Theorem):

Suppose $\{x_m\}$ is bounded in $\mathbb{R}^n$ then there exists a convergent subsequence of it.

Proof:

Since $x_m \in B_M(0)$ for some $M > 0$, so $x_m \in \bar{B}_M(0)$. And since this ball is closed and bounded (by $B_{M+1}(0)$) for example, then by Heine-Borel, it is compact. So x_m is contained inside a compact space and therefore by the above theorem it has a convergent subsequence (moreso, its limit is in $\bar{B}_M(0)$).

■

Example:

Let $e_n = \{0, \ldots, 1, \ldots\}$ be the sequence in ℓ^2 which is 0 except for at its nth position. Then $\{e_n\}$ is bounded since it is contained in the closed unit ball. But no subsequence of it is convergent: let $x \in \ell^2$ then:

$$\|x - e_{n_k}\|^2 \geq (x_{n_k} - 1)^2$$

And since $x \in \ell^2$, this converges to 1, so e_{n_k} is not convergent to x.

Definition 6.1.8:

Suppose (X, ρ) is a metric space. Then a sequence $\{x_n\}_{n=1}^{\infty}$ in X is a **cauchy sequence** if for every $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that for every $n, m \geq N$:

$$\rho(x_n, x_m) < \varepsilon$$

Proposition 6.1.9:

Every convergent sequence is also Cauchy.

Proof:

Suppose $x_n \to x$, then let $\varepsilon > 0$ then there exists an N such that for every $n \geq N$:

$$\rho(x_n, x) < \frac{\varepsilon}{2}$$

And so if $n, m \geq N$:

$$\rho(x_n, x_m) \leq \rho(x_n, x) + \rho(x, x_m) < \varepsilon$$

And so $\{x_n\}$ is a cauchy sequence, as required.

■

The reverse of this proposition is not true. Take $x \in \mathbb{R} \setminus \mathbb{Q}$ and take a sequence q_n of rationals which converge to $\mathbb{Q}$. Then q_n is cauchy in $\mathbb{Q}$ (since it is cauchy in $\mathbb{R}$), but it is not convergent in $\mathbb{Q}$.