Algebraic Topology II

Lectures by Tahl Nowik
Summary by Ari Feiglin (ari.feiglin@gmail.com)

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1 Chain Complexes

We begin by defining a *chain complex*. A chain complex is a sequence of Abelian groups with homomorphisms between them:

$$\begin{tikzcd} \cdots \arrow[r, dashed] & C_{n+1} \arrow[r, "\partial_{n+1}"] & C_n \arrow[r, "\partial_n"] & C_{n-1} \arrow[r, dashed] & \cdots \arrow[r, dashed] & C_0 \arrow[r] & 0 \end{tikzcd}$$

such that for every n, $\partial_n \circ \partial_{n+1} = 0$, in other words $\mathrm{Im}\partial_{n+1} \subseteq \ker \partial_n$. Define $Z_n = \ker \partial_n$, and its elements will be called *n-dimensional cycles*. And define $B_n = \mathrm{Im}\partial_{n+1}$, its elements will be called *boundaries*. Elements of the groups C_n will be called *n-dimensional chains*.

We now want to define a category of chain complexes. To do so we must define morphisms between chain complexes. So suppose we have two chain complexes $\mathscr{C} = \{C_n, \partial_n\}$ and $\mathscr{D} = \{D_n, \partial'_n\}$. We define a morphism from $\mathscr{C}$ to $\mathscr{D}$ to be a sequence of homomorphisms $f_n\colon C_n \longrightarrow D_n$ which preserves the structure of the chain. Meaning $\partial'_n \circ f_n = f_{n-1} \circ \partial_n$, in other words the following diagram commutes:

$$\begin{tikzcd} \cdots \arrow[r, dashed] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"'] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[r, dashed] & \cdots \arrow[r, dashed] & C_0 \arrow[r] \arrow[d, "f_0"'] & 0 \\ \cdots \arrow[r, dashed] & D_{n+1} \arrow[r, "\partial'_{n+1}"] & D_n \arrow[r, "\partial'_n"] & D_{n-1} \arrow[r, dashed] & \cdots \arrow[r, dashed] & D_0 \arrow[r] & 0 \end{tikzcd}$$

To simplify writing, we will write $\partial \circ f = f \circ \partial$, which f and which ∂ is being referred to will be understood from context.

The composition of two morphisms $\{f_n\}\colon \mathscr{C} \longrightarrow \mathscr{D}$ and $\{g_n\}\colon \mathscr{D} \longrightarrow \mathscr{E}$ is defined to be $\{g_n \circ f_n\}\colon \mathscr{C} \longrightarrow \mathscr{E}$. This is indeed a morphism:

$$\partial \circ f \circ g = f \circ \partial \circ g = f \circ g \circ \partial$$

And then this implies that the identity morphism is just $\mathrm{Id}_{\mathscr{C}} = \{\mathrm{Id}_{\mathrm{C_n}}\}\colon \mathscr{C} \longrightarrow \mathscr{C}$, as if $\{f_n\}\colon \mathscr{C} \longrightarrow \mathscr{D}$ then

$$\{f_n\} \circ \mathrm{Id}_{\mathscr{C}} = \{f_n \circ \mathrm{Id}_{C_n}\} = \{f_n\}, \qquad \mathrm{Id}_{\mathscr{D}} \circ \{f_n\} = \{\mathrm{Id}_{D_n} \circ f_n\} = \{f_n\}$$

Now recall that by definition $\partial_n \circ \partial_{n+1} = 0$, meaning

$$B_n \subseteq Z_n \subseteq C_n$$

Since these groups are all Abelian, they are normal in one another, so let us define the *nth homology group* of a chain complex $\mathscr{C}$ as

$$H_n(\mathscr{C}) := {}^{B_n}\big/_{Z_n} = {}^{\ker \partial_n}\big/_{\mathrm{Im}\partial_{n+1}}$$

1.1 Proposition

A chain complex morphism $\{f_n\}\colon \mathscr{C} \longrightarrow \mathscr{D}$ maps cycles to cycles and boundaries to boundaries.

Proof: let $z \in C_n$ be a cycle, i.e. $\partial z = 0$, but then $f(z)$ is a cycle since $\partial f(z) = f(\partial z) = f(0) = 0$. And let $b \in C_n$ be a boundary, so there exists an $a \in C_{n+1}$ such that $b = \partial a$. Then $f(b) = f\partial(a) = \partial f(a) = \partial b$, so $f(b)$ is a boundary as well. ■

This means that if $\{f_n\}\colon \mathscr{C} \longrightarrow \mathscr{D}$ is a morphism of chain complexes, $\{f_n\}\colon Z_n(\mathscr{C}) \longrightarrow Z_n(\mathscr{D})$ is well-defined, and so we have that

$$\begin{tikzcd} Z_n(\mathscr{C}) \arrow[r] \arrow[d] \arrow[dr, bend left, blue] & Z_n(\mathscr{D}) \arrow[d] \\ H_n(\mathscr{C}) & H_n(\mathscr{D}) \end{tikzcd}$$

Where the blue arrow ψ is just the quotient map composed with f_n. This induces a group morphism

$$f_*\colon H_n(\mathscr{C}) \longrightarrow H_n(\mathscr{D})$$

since we can define $f_*([z]) = \psi(z)$ since if $[z] = [z']$ then $z - z' \in B_n(\mathscr{C})$ and so $f(z - z') \in B_n(\mathscr{D})$ and thus the quotient of $f(z - z')$ is just 0, so $\psi(z) = \psi(z')$.

So we have shown that H_n is a functor between the category of chain complexes and the category of Abelian groups.

1.2 Definition

Let B be a set, then define the **free Abelian group** over B to be

$$\mathrm{FA}(B) = \bigoplus_{b\in B}\mathbb{Z} = \{\varphi\colon B \longrightarrow \mathbb{Z} \mid \varphi(b) \neq 0 \text{ for only finitely many } b \in B\}$$

Note then that there is a correspondence between B and $\mathrm{FA}(B)$: $b \leftrightarrow \varphi_b$ where

$$\varphi_b(x) = \begin{cases} 1 & x = b \\ 0 & x \neq b \end{cases}$$

so we can identify b with φ_b, and it is easy to see that every element of $\mathrm{FA}(B)$ can be written as $\sum_{i=1}^k n_i\varphi_{b_i}$, abusing notation $\sum_{i=1}^k nb_i$ and such a representation is unique.
Notice that if B is a set, G an Abelian group, and $g\colon B \longrightarrow G$ a function, then there exists a unique group homomorphism $L\colon \mathrm{FA}(B) \longrightarrow G$ which extends g. This is defined by

$$L\colon \sum_{i=1}^k n_i b_i \longmapsto \sum_{i=1}^k n_i g(b_i)$$

1.3 Definition

The n**-dimensional simplex** is defined to be

$$\Delta^n := \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; x_i \geq 0, \sum_{i=0}^n x_i = 1 \right\}$$

Δ^n has $n+1$ faces, and is homeomorphic to D^n.

1.4 Definition

Let X be a topological space, then an n**-dimensional singular simplex** in X is a morphism (in the category of topological spaces; a continuous map) $\Delta^n \longrightarrow X$. Define $S_n(x)$ to be the set of all n-dimensional singular simplexes in X, and define $C_n(X) = \mathrm{FA}(S_n(x))$.

We now want to define a chain complex on the sequence $C_n(X)$.
Let us define a set of maps $\tau_i^n\colon \Delta^{n-1} \longrightarrow \Delta^n$ for $0 \leq i \leq n$ which maps

$$\tau_i^n\colon (x_0, \dots, x_{n-1}) \mapsto (x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_{n-1})$$

This is a well-defined continuous map, and geometrically it maps Δ^{n-1} to one of the faces of Δ^n.
Let $\sigma \in S_n(x)$, then let us define

$$\partial(\sigma) := \sum_{i=0}^n (-1)^i \sigma \circ \tau_i^n$$

Note that the composition is well-defined since $\Delta^{n-1} \xrightarrow{\tau_i^n} \Delta^n \xrightarrow{\sigma} X$, meaning $\sigma \circ \tau_i^n$ is an $n-1$-dimensional singular simplex. Thus ∂ can be extended to a map $\partial\colon C_n(X) = \mathrm{FA}(S_n(X)) \longrightarrow \mathrm{FA}(S_{n-1}(X)) = C_{n-1}(X)$
Notice that

$$\partial_{n-1}(\partial_n\sigma) = \partial_{n-1}\left(\sum_{i=0}^n (-1)^i \sigma \circ \tau_i^n\right) = \sum_{i=0}^n (-1)^i \partial_{n-1}(\sigma \circ \tau_i^n) = \sum_{i=0}^n \sum_{j=0}^{n-1} (-1)^{i+j} \sigma \circ \tau_i^n \circ \tau_j^{n-1}$$

Notice that $\tau_i^n \circ \tau_j^{n-1} = \tau_j^n \circ \tau_{i-1}^{n-1}$ which can be verified from its definition, but the first has a sign of $(-1)^{i+j}$ in the sum and the second has $-(-1)^{i+j}$. And so the sum is zero.
Thus we have defined a chain complex on $C_n(X)$, let us denote it by $\mathscr{C}(X)$.