Complex Functions

Assignment 9

Ari Feiglin

Exercise 9.1:

Suppose f is analytic in a punctured disk about z_0 which is an isolated singularity. Further suppose that $f(z) \to \infty$ as $z \to z_0$. Show that z_0 is a pole.

Let $g(z) = \frac{1}{f(z)}$, then $g(z) \to 0$ as $z \to z_0$. Thus $g(z)$ is analytic in some disk about z_0 (since zeros of $f(z)$ are isolated since it is not consantly zero, so it is well-defined on some disk about z_0). So suppose

$$g(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

Then since $g(z_0) = 0 = a_0$, we have that $g(z) = (z - z_0)^k \cdot h(z)$ for some $k \geq 1$ and $h(z)$ analytic where $h(z_0) \neq 0$ (we take $h(z)$ to be the taylor series where $b_k = a_{k+n}$ where z^n a_n is the first non-zero coefficient). Since $h(z)$ is analytic and $h(z_0) \neq 0$, it is non-zero in some disk about z_0, so in this disk $f(z) = \frac{1}{(z-z_0)^k} \frac{1}{h(z)}$ and so

$$\lim_{z \to z_0} (z - z_0)^k f(z) = \lim_{z \to z_0} \frac{1}{h(z)} = \frac{1}{h(z_0)} \neq 0$$

and

$$\lim_{z \to z_0} (z - z_0)^{k+1} f(z) = \lim_{z \to z_0} \frac{z}{h(z)} = 0$$

so z_0 is a pole of f.

Exercise 9.2:

Suppose f is analytic when $z \neq 0$ and

$$|f(z)| \leq \sqrt{|z|} + \frac{1}{\sqrt{|z|}}$$

show that f is constant.

Notice that

$$|z \cdot f(z)| \leq |z|^{1.5} + \sqrt{|z|} \xrightarrow[z \to 0]{} 0$$

and so

$$\lim_{z \to 0} z \cdot f(z) = 0$$

so by Riemman's criterion for removable singularities, $z = 0$ is a removable singularity of f's. Thus we can analytically extend f to all of $\mathbb{C}$, ie. let us just assume f is entire.
Let $g(z) = z \cdot f(z^2)$ then we have

$$|g(z)| \leq |z|^2 + 1$$

and so by the general Liouville theorem since g is entire, $g(z) = az^2 + bz + c$. Since $g(0) = 0$, $c = 0$ and so

$$f(z^2) = az + b$$

But then since $f(1^2) = f((-1)^2)$, $a + b = -a + b$ so $a = 0$. Thus $f(z^2) = b$ and so $f(z) = b$ (since z^2 is surjective).

Exercise 9.3:

Let $f(z) = e^{1/z}$. Show that $\mathrm{Im}(f)$ takes on every value on the ring $0 < |z| < 1$, other than one. What is the value which it doesn't take on?

Suppose that $w \in \mathbb{C}$, then if $w \neq 0$ we claim there is a solution to $f(z) = w$. Suppose $w = re^{i\theta}$, so this is if and only if

$$e^{1/z} = w \iff \frac{1}{z} \in \mathrm{Log}(w) \iff \frac{1}{z} = \log(r) + i(\theta + 2\pi k)$$

for some $k \in \mathbb{Z}$. So now we must show there exists a $k \in \mathbb{Z}$ such that

$$\frac{1}{|\log(r) + i(\theta + 2\pi k)|} < 1$$

note that

$$|\log(r) + i(\theta + 2\pi k)|^2 = \log(r)^2 + (\theta + 2\pi k)^2$$

Since the right hand side diverges to infinity as $k \to \infty$, if we take a large enough k this will be greater than 1 and thus $z = \frac{1}{\log(r)+i(\theta+2\pi k)} < 1$ will have $|z| < 1$ as required. Thus $f(z)$ takes on every complex value other than zero, and therefore in particular $\mathrm{Im}(f(z))$ takes on every value.

Exercise 9.4:

Suppose f and g both have poles at z_0 of degree n and m respectively. What is the degree of the pole z_0 for

(1) $f + g$

(2) $f \cdot g$

(3) $\frac{f}{g}$

(1) Suppose

$$f(z) = \sum_{k=-n}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=-m}^{\infty} b_k z^k$$

then let $N = \max\{n, m\}$ so we have

$$f(z) + g(z) = \sum_{k=-N}^{\infty} (a_k + b_k) z^k$$

then we have that the degree of z_0 is $\leq N$ (we may not have equality, if $a_{-N} = -b_{-N}$. In general suppose $f = -g$ then the point would be removable; not even a pole).

(2) Notice that

$$\lim_{z\to\infty} (z - z_0)^{n+m} f(z) \cdot g(z) = \lim_{z\to z_0} (z - z_0)^n f(z) \cdot \lim_{z\to z_0} (z - z_0)^m g(z) \neq 0$$

since both of the limits are non-zero, but

$$\lim_{z\to\infty} (z - z_0)^{n+m+1} f(z) \cdot g(z) = \lim_{z\to z_0} (z - z_0)^{n+1} f(z) \cdot \lim_{z\to z_0} (z - z_0)^m g(z) = 0$$

since the left limit is zero and the right is convergent. So z_0 is a pole of degree $n + m$.

(3) We have that

$$f(z) = \frac{A(z)}{(z - z_0)^n}, \qquad g(z) = \frac{B(z)}{(z - z_0)^m}$$

for analytic functions A and B such that $A(z_0), B(z_0) \neq 0$. Then

$$\frac{f(z)}{g(z)} = (z - z_0)^{m-n} \cdot \frac{A(z)}{B(z)}$$

since $B(z_0) \neq 0$, $\frac{A}{B}$ is analytic in a disk around z_0, and since $A(z_0) \neq 0$ we get that by definition if $m < n$ then the singularity is a pole of degree $m - n$. And if $m \geq n$ then the singularity is removable (the quotient is equal to an analytic function defined at z_0).

Classify the singularities of

(1) $f(z) = \frac{1}{z^4+z^2}$

(2) $f(z) = \cot z$

(3) $f(z) = \csc z$

(4) $f(z) = \exp\left(\frac{1/z^2}{z-1}\right)$

(1) The singularities of the function are the "problematic" points, here $z^4 + z^2 = z^2(z^2+1)$ so the problematic points are $z = 0, \pm i$. Let $g(z) = z^4 + z^2$, then $f(z) = \frac{1}{g(z)}$ and since $g(z)$ is analytic about the singularities (not at them), the singularities are poles. Since $g''(z) = 12z^2 + 2$, we have that $g''(0) \neq 0$ but $g'''(z) = 24z$ and so $g'''(0) = 0$. And $g^{(4)} = 24$ and so $g^{(4)}(\pm i) \neq 0$ but $g^{(5)} = 0$. So 0 is a second order pole, and $\pm i$ are fourth order poles.

(2) Since $\cot(z) = \frac{\cos(z)}{\sin(z)}$, the singularities are when $\sin(z) = 0$ (since $\cos(z)$ is entire). This is when $z = 2\pi k$ for $k \in \mathbb{Z}$. At these points $\cos(z) \neq 0$ (since $\sin(z)^2 + \cos(z)^2 = 1$, so they cannot both be zero), and so these singularities are poles (the function is of the form $\frac{A}{B}$ where $A(z_0) \neq 0$ and $B(z_0) = 0$ where A and B are analytic at z_0).
Since $\sin'(z) = \cos(z)$, $\sin'(z_0) \neq 0$ and so these are first degree poles.

(3) Since $f(z) = \frac{1}{\sin(z)}$, the singularities are when $\sin(z) = 0$, so when $z = 2\pi k$. For the same exact reason as above (except now $A = 1$ which is also entire and non-zero, in particular at these singularities), these are first degree poles.

(4) The poles here are at $z = 0$ and $z = 1$. But for any $k \in \mathbb{N}$, and $z_0 = 0, 1$,

$$(z - z_0)^k \cdot \exp\left(\frac{1}{z^2(z-1)}\right) \xrightarrow[z \to z_0]{} \infty$$

since exponential growth is much faster than polynomial growth.
So these are both essential singularities.

Exercise 9.5:

Find the Laurent expansions of

(1) $f(z) = \frac{1}{z^4+z^2}$ about $z = 0$

(2) $f(z) = \frac{\exp(1/z^2)}{z-1}$ about $z = 0$

(3) $f(z) = \frac{1}{z^2-4}$ about $z = 2$

(1) Notice that

$$f(z) = \frac{1}{z^2} - \frac{1}{z^2+1}$$

and since

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n \implies \frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n}$$

and so

$$f(z) = \sum_{n=-1}^{\infty} (-1)^{n+1} z^{2n}$$

(2) Let $g(z) = f(1/z)$ then

$$g(z) = \frac{\exp\left(z^2\right)}{\frac{1}{z} - 1} = \frac{z \cdot \exp\left(z^2\right)}{1-z}$$

Recall the Taylor expansion of $\exp(z)$:

$$\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot z^n \implies z\exp(z^2) = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot z^{2n+1}$$

and

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

So let $a_n = \frac{1}{k!}$ when $n = 2k+1$ and $a_n = 0$ when n is even. Then

$$z\exp(z^2) = \sum_{n=0}^{\infty} a_n z^n$$

and so multiplying this by the Taylor series of $\frac{1}{1-z}$ gives

$$g(z) = \sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} a_k$$

And so, since $f(z) = g(1/z)$,

$$f(z) = \sum_{n=-\infty}^{0} z^n \sum_{k=0}^{-n} a_k$$

(3) Let $g(z) = f(z+2)$, then a Laurent expansion of $f(z)$ about $z = 2$ is equivalent to an expansion of $g(z)$ about $z = 0$. Since

$$g(z) = \frac{1}{(z-2)(z+2)} \implies f(z) = \frac{1}{z(z+4)}$$

doing partial fraction decomposition gives

$$g(z) = \frac{1}{3} \cdot \frac{1}{z} - \frac{1}{3} \cdot \frac{1}{z+4}$$

Now, in general

$$\frac{1}{z+c} = -\frac{1}{c} \cdot \frac{1}{\left(-\frac{z}{c}\right) - 1} = -\frac{1}{c} \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{c}\right)^n \cdot z^n$$

and so in particular

$$\frac{1}{z+4} = -\frac{1}{4} \sum_{n=0}^{\infty} (-1)^n \frac{1}{4^n} \cdot z^n$$

And so

$$g(z) = \frac{1}{3} \cdot \frac{1}{z} + \frac{1}{3} \cdot \sum_{n=0}^{\infty} (-1)^n \frac{1}{4^{n+1}} z^n$$

and thus

$$f(z) = g(z-2) = \frac{1}{3} \cdot (z-2)^{-1} + \frac{1}{3} \cdot \sum_{n=0}^{\infty} (-1)^n \frac{1}{4^{n+1}} (z-2)^n$$

Exercise 9.6:

Show that if f is analytic on the complex plane where $z \neq 0$ and f is odd, then all of the even coefficients in f's Laurent expansion about 0 are zero.

Suppose

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

is f's Laurent expansion about $z = 0$. Then $f(z) + f(-z) = 0$ so

$$0 = \sum_{n=-\infty}^{\infty} a_n z^n + \sum_{n=-\infty}^{\infty} (-1)^n a_n z^n = \sum_{n=-\infty}^{\infty} (a_n + (-1)^n a_n) z^n$$

and since Laurent expansions are unique, this means that $a_n + (-1)^n a_n = 0$ for all $n \in \mathbb{Z}$. For odd ns this is already zero, and for even n this is equal to $2a_n = 0$ so $a_n = 0$ for all even n.

Exercise 9.7:

Show that if f is analytic in a punctured domain D of z_0, and $z_n \to z_0$ are poles of f (f is also of course not analytic at z_n), then $f(D)$ is dense in $\mathbb{C}$.

Suppose not, then there exists a $w \in \mathbb{C}$ and a $\delta > 0$ such that for every $z \in D$, $|w - f(z)| > \delta$. Then let us define

$$g(z) = \frac{1}{f(z) - w}$$

Which is defined for all $z \in D$ where $z \neq z_n$ for $n \geq 0$. Since f is analytic, so is g and $|g(z)| < \frac{1}{\delta}$ for $z_n \neq z \in D$. Furthermore, since z_n are all poles of f, and so $\lim_{z \to z_n} \frac{1}{f(z)-w} = 0$ since the limit of $f(z)$ is ∞. And so g can be analytically continued to z_n for all $n > 0$ (these are the zeros of g).
Now, since $|g(z)| < \frac{1}{\delta}$ for all $z \in D$, we have that

$$\lim_{z \to z_0} |zg(z)| \leq \lim_{z \to z_0} |z| \cdot \frac{1}{\delta} = 0$$

and so

$$\lim_{z \to z_0} z \cdot g(z) = 0$$

and therefore by Riemman's critierion, z_0 is a removable singularity of g. Thus we can extend g analytically to $D \cup \{z_0\}$, but $z_n \to z_0$ is a convergent sequence of zeroes of $g(z)$ in this domain. Therefore g is identically equal to zero in D. But this is a contradiction, as $\frac{1}{f(z)-w} \neq 0$ whenever $f(z)$ is defined (which it is).

Exercise 9.8:

Show that the image of an entire non-constant function is dense in $\mathbb{C}$.

Similarly to before, suppose not. Then there exists a $w \in \mathbb{C}$ and a $\delta > 0$ such that for every $z \in \mathbb{C}$, $|f(z) - w| > \delta$. And so let us define

$$g(z) = \frac{1}{f(z) - w}$$

which is entire since $f(z)$ is, and $f(z) \neq w$. And

$$|g(z)| = \frac{1}{|f(z) - w|} < \frac{1}{\delta}$$

so g is a bounded entire function and therefore by Liouville's theorem, it is constant. So suppose $g(z) = c$, and therefore

$$f(z) = \frac{1}{c} + w$$

but then $f(z)$ is constant, in contradiction.