Calculus Homework #9

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Question 9.1:

Determine the limits of the following series of functions, and determine if they converge uniformly or not.

(1) $f_n(x) = \cos(x)^{2n}$ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(2) $f_n(x) = \frac{\tan^{-1}(x)}{n}$ in $\mathbb{R}$.

(3) $f_n(x) = x^n - x^{2n}$ in $(-1, 1)$.

(4) $f_n(x) = \frac{1}{nx+1}$ in $(0, \infty)$.

(5) $f_n(x) = \sqrt{n+1} \cdot \sin(x)^n \cdot \cos(x)$ in $\mathbb{R}$.

(6) $f_n(x) = \frac{x}{n} \cdot \log\left|\frac{x}{n}\right|$ in $(0, 1)$.

(1) Notice that if $x \neq 0$, then $|\cos(x)| < 1$, which means that

$$\lim \cos(x)^{2n} = 0$$

And if $x = 0$, then $\cos(x) = 1$, so:

$$\lim \cos(x)^{2n} = \lim 1^{2n} = 1$$

Which means that f, the limit of f_n, is equal to:

$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$$

Notice that while f_n is continuous (as the composition of continuous functions), f is not. Since uniform convergence of continuous functions is continuous, this convergence is *not uniform*.

(2) Since:

$$-\frac{\pi}{2n} \leq f_n \leq \frac{\pi}{2n}$$

By the squeeze theorem, $f_n \longrightarrow 0 = f$.
Let:

$$\varepsilon_n \coloneqq \sup_{x\in\mathbb{R}} |f_n - f| = \sup_{x\in\mathbb{R}} \frac{\tan^{-1}(x)}{n}$$

We will prove that ε_n converges to 0.
Since $-\frac{\pi}{2n} \leq f_n \leq \frac{\pi}{2n}$, $|f_n| \leq \frac{\pi}{2n}$. So:

$$0 \leq \varepsilon_n \leq \frac{\pi}{2n} \longrightarrow 0$$

So by the squeeze theorem, ε_n converges to 0.
By the limit superior theorem for uniform convergence, this means that f_n *converges uniformly* to 0.

(3) Since $x \in (-1, 1)$, the limit of x^n and x^{2n} is 0, and:

$$f(x) = \lim f_n(x) = \lim x^n - x^{2n} = 0$$

So f_n converges to 0.
Let:

$$\varepsilon_n = \sup_{x\in(-1,1)} |f_n(x) - f(x)| = \sup_{x\in(-1,1)} \left|x^n - x^{2n}\right|$$

Differentiating $f_n(x)$ yields:

$$f_n'(x) = nx^{n-1} - 2nx^{2n-1} = nx^{n-1}(1 - 2x^n)$$

So f_n has a critical point when $1-2x^n = 0$, we'll take $x = \sqrt[n]{\frac{1}{2}}$. (It just so happens that this is a maximum, but that isn't necessary for this proof.)
Notice then that:

$$f_n(x) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

Which means that $\varepsilon_n \geq \frac{1}{4}$, so ε_n doesn't converge to 0, so f_n converges to 0 but *not uniformly*.

(4) In this case, the limit of f_n is 0 (since the limit of $nx + 1$ is ∞). We will once again perform the limit superior test. In this case:

$$\varepsilon_n = \sup_{x\in(0,\infty)} \left|\frac{1}{nx+1}\right|$$

Notice that when $x = \frac{1}{n}$, $f_n(x) = \frac{1}{2}$, so $\varepsilon_n \geq \frac{1}{2}$, so ε_n does not converge to 0. By the limit superior theorem, this means f_n converges to 0 *no uniformly*.

(5) Notice that when $x \neq \frac{\pi}{2} + \pi k$, $|\sin x| < 1$. Let $q = \sin(x)$. Notice that:

$$\lim \sqrt{n+1} \cdot q^n = \lim \frac{\sqrt{n+1}}{\left(\frac{1}{q}\right)^n}$$

Since $\left|\frac{1}{q}\right| > 1$, the denominator is exponential and since the numerator is a square root, the limit must be 0 (for $q < 0$, we split into subseries of even and odd n. Even n converge to 0 from the right and odd from the left). This means that:

$$\lim f_n(x) = \cos(x) \cdot \lim \sqrt{n+1} \sin(x)^n = 0$$

And for $x = \frac{\pi}{2} + \pi k$, $\cos(x) = 0$, so $f_n(x) = 0$, and therefore the limit equals 0 as well.
So:

$$f(x) = \lim f_n(x) = 0$$

We will once again use the limit superior test. Let:

$$\varepsilon_n = \sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \sup_{x\in\mathbb{R}} |f_n(x)|$$

We will differentiate $f_n(x)$:

$$f_n'(x) = \sqrt{n+1}\left(n\sin(x)^{n-1}\cos(x)^2 - \sin(x)^{n+1}\right) = \sqrt{n+1}\sin(x)^{n-1}\left(n\cos(x)^2 - \sin(x)^2\right)$$

So $f_n'(x)$ has a critical point when $n\cos(x)^2 - \sin(x)^2 = 0$. This has a solution, since if $\tan(x)^2 = n$, x satisfies this equation (so we can take $\tan^{-1}(\sqrt{n})$ for example, and has the benefit that sin and cos are then both positive).
Let x_0 satisfy this equation, then:

$$n(1 - \sin(x_0)^2) = \sin(x_0)^2 \implies \sin(x_0)^2 = \frac{n}{n+1}$$

Since $\sin(x_0)$ is positive:

$$\sin(x_0)^n = \frac{1}{\sqrt{\left(1+\frac{1}{n}\right)^n}}$$

And since:

$$n\cos(x_0)^2 = \sin(x_0)^2 \implies \cos(x_0)^2 = \frac{1}{n+1} \implies \cos(x_0)\,\frac{1}{\sqrt{n+1}}$$

So:

$$f_n(x_0) = \sqrt{n+1}\cdot\frac{1}{\sqrt{\left(1\frac{1}{n}\right)^n}}\cdot\frac{1}{\sqrt{n+1}} = \frac{1}{\sqrt{\left(1+\frac{1}{n}\right)^n}} \longrightarrow \frac{1}{\sqrt{e}}$$

And by definition:

$$\varepsilon_n \geq f_n(x_0) \longrightarrow \frac{1}{\sqrt{e}}$$

So ε_n does not converge to 0 and therefore f_n converges to 0 *not uniformly*.

(6) First lets find the limit of $f_n(x)$:

$$\lim \frac{x}{n}\cdot\log\left|\frac{x}{n}\right| = \lim \frac{\log\left|\frac{x}{n}\right|}{\frac{n}{x}}$$

The numerator approaches $-\infty$ and the denominator approaches ∞, so we can apply L'Hopital (differentiating relative to n):

$$= \frac{-\frac{x}{n^2}\cdot\frac{n}{x}}{\frac{1}{x}} = -\lim\frac{x}{n} = 0$$

So $f_n \longrightarrow 0 = f$, so f is continuous.
Notice though that:

$$\lim_{x\to\infty}\lim_{n\to\infty} f_n(x) = \lim_{x\to\infty} 0 = 0$$

But:

$$\lim_{x\to\infty} f_n(x) = \lim_{x\to\infty}\frac{x}{n}\cdot\log\left|\frac{x}{n}\right|$$

And since the limit of both $\frac{x}{n}$ and $\log\left|\frac{x}{n}\right|$ is ∞, this limit is ∞. So:

$$\lim_{n\to\infty}\lim_{x\to\infty} f_n(x) = \lim_{n\to\infty}\infty = \infty$$

This means that:

$$\lim_{n\to\infty}\lim_{x\to\infty} f_n(x) \neq \lim_{x\to\infty}\lim_{n\to\infty} f_n(x)$$

Even though f, the limit of f_n, is continuous.
So f_n converges to 0 but *not uniformly*.

Question 9.2:

Dis/Prove the following:

(1) If $f_n(x)$ and $g_n(x)$ converge uniformly in I, then so does $f_n(x) + g_n(x)$.

(2) If $f_n(x)$ converges uniformly to $f(x)$ in I, then $g(x) \cdot f_n(x)$ converges uniformly to $g(x) \cdot f(x)$ in I.

(3) If $f_n(x)$ converges uniformly to $f(x)$ in I, and every one of $f_n(x)$ are uniformly continuous, then so is $f(x)$.

(1) This is true (in fact we proved a stronger proposition in lecture).
Suppose f_n and g_n converge to f and g respectively. Let $\varepsilon > 0$, then there exists some n_1 such that for every $n \geq n_1$: $|f_n(x) - f(x)| \leq \varepsilon$. And there also exists some n_2 such that for every $n \geq n_2$: $|g_n(x) - g(x)| \leq \varepsilon$. So let $n_0 \coloneqq \max\{n_1, n_2\}$. Then for every $n \geq n_0$, the two above inequalities still hold.
And by the triangle inequality:

$$|f_n + g_n - (f + g)| \leq |f_n - f| + |g_n - g| \leq 2\varepsilon$$

So $f_n + g_n \rightrightarrows f + g$.

(2) This is *false*.
Let $f_n(x) = \frac{1}{nx}$ and $g(x) = e^x$ in $I = \mathbb{R}_{\geq 1}$. The limit of f_n is 0, and since:

$$|f_n(x) - f(x)| = |f_n(x)| = \left|\frac{1}{nx}\right| = \frac{1}{nx} \leq \frac{1}{n} \longrightarrow 0$$

(Since $x \geq 1$.)
This means that $f_n \rightrightarrows 0 = f$.
So $f \cdot g = 0$. But:

$$\varepsilon_n = \sup_{x \geq 1} |f_n g - fg| = \sup_{x \geq 1} \frac{e^x}{nx} \geq \frac{e^n}{n^2} \longrightarrow \infty$$

So $\varepsilon_n \nrightarrow 0$, so $f_n g$ does not converge uniformly to fg.

(3) This *true*.
Let $\varepsilon > 0$, then there exists some n such that from it and onward:

$$|f_n(x) - f(x)| \leq \varepsilon$$

And since f_n is uniformly continuous, there exists some $\delta > 0$ such that for every $|x_1 - x_2| < \delta$:

$$|f_n(x_1) - f_n(x_2)| \leq \varepsilon$$

So for every $|x_1 - x_2| < \delta$:

$$|f(x_1) - f(x_2)| \leq |f(x_1) - f_n(x_1)| + |f_n(x_1) - f_n(x_2)| + |f_n(x_2) - f(x_2)| \leq 3\varepsilon$$

This means that f is uniformly continuous, as required.

Question 9.3:

Suppose $f_n(x)$ is a series of functions which converges to $f(x)$ in $[a,b]$.
Prove that if $f_n(x)$ does not converge to f uniformly in $[a,b]$ then it does not converge uniformly in (a,b).

I will prove the contrapositive: if f_n converges uniformly in (a,b) then it converges uniformly in $[a,b]$.
Since f_n converges to f in $[a,b]$, the limit of $f_n(a)$ and $f_n(b)$ must exist. And so f_n converges uniformly in $\{a,b\}$ since it is countable and the limit exists.
Since f_n converges uniformly in (a,b) and $\{a,b\}$, it converges uniformly in $(a,b) \cup \{a,b\} = [a,b]$, as required.