Representation Theory
Homework 3

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1 Problem

Let $Q = \{\pm 1, \pm i, \pm j, \pm k\} \subseteq \mathbb{H}$ be the set of Quaternion units.

(1) Show that Q is a subgroup of $\mathbb{H}$ generated by i, j.

(2) Find 4 one-dimensional representations of Q over $\mathbb{C}$.

(3) Show that $\rho\colon Q \to \mathrm{GL}_2(\mathbb{C})$ given by

$$\rho(i) = \begin{pmatrix} i & \\ & -i \end{pmatrix}, \qquad \rho(j) = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

(4) Show that the irreducible representations found thus far comprise of all the irreducible complex representations of Q up to isomorphism.

(1) This is clear: since $i^2 = j^2 = -1 \in Q$ and $ij = k \in Q$ and $ji = -k \in Q$, we have that $\langle i, j\rangle \subseteq Q$. And clearly $Q \subseteq \langle i, j\rangle$ as well.

(2) A 1-dimensional representation of Q is a character $\chi\colon Q \to \mathbb{C}^\times$. Let $\chi(i) = x$ and $\chi(j) = y$, then since $Q = \langle i, j\rangle$, χ is uniquely determined by choice of x, y. We must have that $x^2 = y^2 = \chi(-1)$ and since $ijij = -1$, $\chi(-1) = x^2y^2$. So $x^2 = y^2 = x^2y^2$, so $x^2 = y^2 = 1$, meaning $x, y = \pm 1$. There are 4 choices of this, and they all the induced representations are non-isomorphic (since they determine the order of $\rho(i), \rho(j)$).

(3) Q has a known presentation $\langle i, j, k \mid i^2 = j^2 = k^2 = ijk\rangle$. So $k = ij$, and the relations can be simplified: $Q = \langle i, j \mid i^2 = j^2 = ijij\rangle$. Thus to show that ρ is a homomorphism, we just need to show that $\rho(i)^2 = \rho(j)^2 = (\rho(i)\rho(j))^2$. Clearly $\rho(i)^2 = \rho(j)^2 = -I$, and $\rho(i)\rho(j) = \mathrm{diag}(i, i)$, so $(\rho(i)\rho(j))^2 = -I$ as well, as required. This is an irreducible representation: a non-trivial subrepresentation must be one-dimensional and thus the span of an eigenvector of both $\rho(i), \rho(j)$. The eigenvectors of $\rho(i)$ are $(1, 0)$ and $(0, 1)$ (for distinct eigenvalues). But these are not eigenvectors of $\rho(j)$, so ρ must be irreducible.

(4) Since $\mathbb{C}$ is algebraically closed, $|Q| = 8$ is equal to the sum of the squares of the dimensions of its irreducible representations. Note that $8 = 1+1+1+1+2^2$, and so we have found all the irreducible representations (since the ones we have found are all pairwise non-isomorphic) up to isomorphism.

2 Problem

Let V be a G-representation over $\mathbb{F}$. Define the **coinvariants** of V:

$$V_G = V/\langle v - gv \mid g \in G, v \in V\rangle$$

(1) Show that the norm map $\mathrm{Nm}\colon V_G \to V^G$ defined by $\mathrm{Nm}([v]) = \sum_{g\in G} gv$ is well-defined.

(2) Let $V \to V_G$ be the canonical projection map, and $V^G \to V$ be the natural inclusion map. Show that the inclusions

$$V^G \to V \to V_G \xrightarrow{\text{Nm}} V^G, \qquad V_G \xrightarrow{\text{Nm}} V^G \to V \to V_G$$

are both multiplication by $|G|$ maps. Deduce that Nm is an isomorphism iff $|G| \in \mathbb{F}^\times$.

(1) Note that $\{v - gv\}_{v,g}$ is closed under scalar multiplication: $\alpha(v - gv) = (\alpha v) - g(\alpha v)$, and so $\langle v - gv\rangle_{g,v}$ is comprised of vectors of the form $\sum_i (w_i - g_i w_i)$. Let $\text{Nm}\colon V \to V^G$ be given by $\text{Nm}(v) = \sum_g gv$; this is clearly well-defined since $h\text{Nm}(v) = \sum_g hgv = \sum_g gv = \text{Nm}(v)$, so $\text{Nm}(v) \in V^G$. Now suppose $[v] = [u]$, that is $v - u \in \langle w - gw\rangle$, so $v - u = \sum_i (w_i - g_i w_i)$. Then

$$\begin{aligned}
\text{Nm}(v) - \text{Nm}(u) &= \sum_g gv - \sum_g gu = \sum_g g(v-u) = \sum_g g \sum_i (w_i - g_i w_i) \\
&= \sum_i \left(\sum_g g w_i \right) - \left(\sum_g g g_i w_i \right) = \sum_i \text{Nm}(w_i) - \text{Nm}(w_i) \\
&= 0
\end{aligned}$$

Thus Nm is constant on equivalence classes, and can thus be quotiented to give us $\text{Nm}\colon V_G \to V^G$.

(2) Let $\phi\colon V^G \to V \to V_G \xrightarrow{\text{Nm}} V^G$, then

$$\phi(v) = \text{Nm}([v]) = \sum_g gv = \sum_g v = |G|v$$

and let $\psi\colon V_G \xrightarrow{\text{Nm}} V^G \to V \to V_G$, then

$$\psi([v]) = \left[\sum_g gv\right] = \sum_g [gv]$$

Note that $[gv] = [v]$ since $v - gv \in \langle w - gw\rangle$ clearly, so $\psi([v]) = |G|[v]$, as required.

Now, if $|G|$ is invertible, then ϕ, ψ are isomorphisms (as they are invertible), and thus Nm must be too (since if $f \circ g, h \circ f$ are bijective, so too must f be).

3 Problem

Let V be a finite-dimensional irreducible representation of G over an algebraically closed $\mathbb{F}$.

(1) Show that V has at most one nonzero G-invariant bilinear form up to scalar multiplication. Furthermore show that every nonzero bilinear form is nondegenerate.

(2) Show that if $\text{char}\mathbb{F} \neq 2$ and $(\bullet, \bullet)$ is an invariant nondegenerate bilinear form then it is either symmetric or antisymmetric.

(3) Conclude that exactly one of the following is correct:

(i) V has no nonzero invariant bilinear forms;

(ii) V has a nondegenerate symmetric bilinear form;

(iii) V has a nondegenerate antisymmetric bilinear form.

(1) Suppose $(\bullet,\bullet)$ is a nonzero G-invariant bilinear form on V. Define the map $V \to V^\vee$ defined by $v \mapsto (\bullet, v)$. Recall the structure of the G-representation on $\hom(U,W)$: $(g \star f)(u) = gf(g^{-1}u)$. Now our map is indeed a G-morphism: $(\bullet, gv) = (g^{-1}\bullet, v) = g \star (\bullet, v)$ (since $\mathbb{F}$ is trivial).

Note that if V is reducible, so too is $V^\vee$. Indeed let $V = U \oplus W$, then $V^\vee = U^\vee \oplus W^\vee$, where $U^\vee = \{\phi\colon V \to \mathbb{F} \mid \phi(W) = 0\}$ (which is isomorphic to the dual of U). These are clearly G-subrepresentations: for $\phi \in U^\vee$ and $g \in G$, $g \star \phi(w) = \phi(g^{-1}w) = 0$ since W is a subrepresentation of V. And since $U^\vee$ is isomorphic (as a vector space) to U, it is non-trivial iff U is.

And V is isomorphic to $V^{\vee\vee}$ as G-representations. Indeed the natural isomorphism $v \mapsto (v, \bullet)$ where $(v, \phi) = \phi(v)$ is a G-morphism. See, $g \star (v, \phi) = (v, g^{-1} \star \phi) = (g^{-1} \star \phi)(v) = \phi(gv) = (gv, \phi)$.

So if $V^\vee$ is reducible, so too is $V^{\vee\vee}$, and therefore so too is V. Thus V is irreducible iff $V^\vee$ is.

Since our V is irreducible, $V^\vee$ is irreducible, and thus by Schur $\hom_G(V, V^\vee)$ is at most one-dimensional. So any two elements in $\hom_G(V, V^\vee)$ are linearly dependent. Thus if $(\bullet,\bullet)_1$ and $(\bullet,\bullet)_2$ are two nonzero G-invariant bilinear forms, then $(v \mapsto (\bullet, v)_2) = \alpha(v \mapsto (\bullet, v)_1)$. That is, they are scalar multiples of one another.

Now, if $(\bullet,\bullet)$ is a degenerate bilinear form, then there is a $0 \neq u \in V$ such that $(\bullet, u) \in V^\vee$ is zero. But then that means the map $v \mapsto (\bullet, v)$ has a nonzero kernel, and the kernel cannot be all of V since the bilinear form is nonzero. Thus the kernel gives a nontrivial subrepresentation of V, a contradiction.

(2) Given a G-invariant bilinear form $(\bullet,\bullet)$, define $(\bullet,\bullet)^{\mathsf{op}}$ by $(v,u)^{\mathsf{op}} = (u,v)^{\mathsf{op}}$. This is clearly a G-invariant bilinear form, and by the previous point $(\bullet,\bullet) = \epsilon(\bullet,\bullet)^{\mathsf{op}}$ for some $\epsilon \in \mathbb{F}$. That is, $(v,u) = \epsilon(u,v)$ for all $v, u \in V$. From this we see that $(v,u) = \epsilon^2(v,u)$, and since the bilinear form is nonzero, $\epsilon^2 = 1$. This means that $\epsilon = \pm 1$, so the bilinear form is either symmetric ($\epsilon = 1$), or antisymmetric ($\epsilon = -1$).

(3) If V has a nonzero invariant bilinear form, then by above it must be symmetric or anti symmetric (and nondegenerate). So one of the points must hold. V cannot have both a symmetric and an antisymmetric nondegenerate bilinear form, as then they would not be scalar multiples of one another. Indeed, if $(v,u)_1 = \alpha(v,u)_2$ then $-\alpha(v,u)_2 = \alpha(u,v)_2 = (u,v)_1 = (v,u)_1 = \alpha(v,u)_2$. And then $\alpha = -\alpha$ and so $\alpha = 0$, a contradiction to the nonzero-ness of the bilinear forms.

4 Problem

Let A be a finite abelian group, and $\mathbb{F}$ be algebraically closed with $|A| \in \mathbb{F}^\times$.

(1) Show that $A^\vee$ is closed under pointwise multiplication, and such multiplication gives $A^\vee$ the structure of a group.

(2) Show that there is a natural isomorphism of groups $A \cong A^{\vee\vee}$ sending $a \mapsto [\chi \mapsto \chi(a^{-1})]$.

(3) Consider the composition of isomorphisms $\mathcal{G}\colon \mathrm{Set}(A^\vee, \mathbb{F}) \to \mathrm{Set}(A^{\vee\vee}, \mathbb{F}) \to \mathrm{Set}(A, \mathbb{F})$, where the first isomorphism is the Fourier transform of $A^\vee$ and the second is given by the previous point. Show that there is a $\lambda \in \mathbb{F}^\times$ such that $\mathcal{G} = \lambda\mathcal{F}^{-1}$, where $\mathcal{F}$ is the Fourier transform of A.

(1) Let $\chi, \mu \in A^\vee$, then $\chi\mu$ is a group homomorphism $A \to \mathbb{F}^\times$. Indeed, $\chi\mu(ab) = \chi(ab)\mu(ab) = \chi(a)\chi(b)\mu(a)\mu(b) = (\chi\mu)(a)(\chi\mu)(b)$. And clearly pointwise multiplication is associative with identity

$1\colon a \mapsto 1$ (this is clearly in $\hom(A, \mathbb{F}^\times)$). Every $\chi \in A^\vee$ has an inverse given by $\chi^{-1}(a) = \chi(a)^{-1}$. Note that $A^\vee$ is abelian, since $\chi(a)\mu(a) = \mu(a)\chi(a)$.

(2) Let $[\bullet]\colon A \to A^{\vee\vee}$ be defined by $[a](\chi) = \chi(a^{-1})$. This is a group homomorphism: $[ab](\chi) = \chi(b^{-1}a^{-1}) = \chi(b^{-1})\chi(a^{-1}) = [b](\chi)[a](\chi)$. So we get $[ab] = [b][a]$, and since $A^{\vee\vee}$ is abelian, this is indeed a homomorphism.

Furthermore, this is an injection: if $[a] = 1$, then $\chi(a^{-1}) = 1$ and so $\chi(a) = 1$ for all $\chi\colon A \to \mathbb{F}^\times$. Suppose that $a \neq 1$, then we know that $A \cong \langle a \rangle \times B$ where $\langle a \rangle$ is the subgroup generated by a. Let $n = |a|$ be the order of a, then $x^n = 1$ has a solution in $\mathbb{F} - 1$ as an alegbraically closed field; if the only root of $x^n - 1$ is 1, then $x^n - 1 = (x-1)^n$, and so the characteristic of $\mathbb{F}$ divides n, which divides $|A|$, a contradiction. Then $\phi(a, 1) = x$ and $\phi(1, b) = 1$ for $b \in B$ is a homomorphism for which $\phi(a) \neq 1$, a contradiction. So this map is indeed an injection.

Now, recall that the number of irreducible representations of A is equal to $|A|$ (in general it is equal to the number of conjugacy classes, since A is abelian this is just $|A|$). And there is an isomorphism between $A^\vee$ and the set of irreducible representations of A. Thus $|A^\vee| = |A|$. Since these are finite sets, an injection between them is a bijection, so $[\bullet]$ is an isomorphism, as required.

(3) Let $\mathcal{F}_\vee$ be the Fourier transform of $A^\vee$. Given the previous isomorphism $[\bullet]\colon A \to A^{\vee\vee}$, consider the induced isomorphism $\langle \bullet \rangle : \mathrm{Set}(A^{\vee\vee}, \mathbb{F}) \to \mathrm{Set}(A, \mathbb{F})$ defined by $\langle f \rangle(a) = f([a])$. Now, we note that for $\chi \in A^\vee$,

$$\mathcal{G}(\delta_\chi)(a) = \langle \mathcal{F}_\vee(\delta_\chi) \rangle (a) = \mathcal{F}_\vee(\delta_\chi)[a] = [a]\chi = \chi(a^{-1})$$

And by recitation,

$$\mathcal{F}^{-1}(\delta_\chi)(a) = \frac{1}{|A|}\sum_{b \in A} \chi(b^{-1})\delta_b(a) = \frac{1}{|A|}\chi(a^{-1}) = \frac{1}{|A|}\mathcal{G}(\delta_\chi)(a)$$

And so we indeed have that $\mathcal{G} = |A|\mathcal{F}^{-1}$, as required.

5 Problem

Let A be a finite abelian group, and let $\mathbb{F}$ be algebraically closed with $|A| \in \mathbb{F}^\times$.

(1) Show that if A is cyclic, then there exists an isomorphism of Abelian groups $A \cong A^\vee$.

(2) Deduce that if A is any finite Abelian group, $A \cong A^\vee$.

(1) Given $a \in A$, map $\chi \in A^\vee$ to $\chi(a) \in \mathbb{F}^\times$. This evaluation map $\mathrm{ev}_a\colon A^\vee \to \mathbb{F}^\times$ is clearly a homomorphism. Now, if $a \in A$ generates A, then ev_a is an injection: $\chi \in A^\vee$ is determined by its image on a. Thus $A^\vee \cong \mathrm{im}\,\mathrm{ev}_a \leq \mathbb{F}^\times$. We know that finite subgroups of the multiplicative group of $\mathbb{F}$ is cyclic, and therefore $A^\vee$ is cyclic. We further know that (from the previous question) $|A| = |A^\vee|$, so A and $A^\vee$ are isomorphic as cyclic groups of the same order.

(2) Suppose A, B are abelian groups, then $(A \times B)^\vee \cong A^\vee \times B^\vee$. Indeed, map $\phi\colon f\colon A \times B \to \mathbb{F}^\times$ to $(f \circ \iota_A, f \circ \iota_B)$ (where $\iota_A \circ A \to A \times B$ maps $a \mapsto (a, 1)$). And map $\psi\colon (f, g) \in A^\vee \times B^\vee$ to $(a, b) \mapsto f(a)g(b)$. These are clearly homomorphisms, and they are inverses:

$$\phi \circ \psi(f, g) = (\psi(f, g) \circ \iota_A, \psi(f, g) \circ \iota_B) = (f, g)$$

(Since $\psi(f, g) \circ \iota_A(a) = \psi(f, g)(a, 1) = f(a)$ so $\psi(f, g) \circ \iota_A = f$.) And

$$\psi \circ \phi(f) = \psi(f \circ \iota_A, f \circ \iota_B) \colon (a, b) \mapsto f(a, 1)f(1, b) = f(a, b)$$

so $\psi \circ \phi(f) = f$ as well.

Inductively, $(A_1 \times \cdots \times A_n)^\vee \cong A_1^\vee \times \cdots \times A_n^\vee$ for Abelian groups $A_1, \ldots, A_n$.

If A is a finite Abelian group, then $A \cong A_1 \times \cdots \times A_n$ where A_i are finite cyclic groups. Then $A_i^\vee \cong A_i$ by the previous point, and we just showed

$$A^\vee \cong A_1^\vee \times \cdots \times A_n^\vee \cong A_1 \times \cdots \times A_n \cong A$$

as required.