Mathematical Logic

Lecture 4, Monday April 24, 2023

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Corollary 4.0.1:

Our system $\mathcal{L}$ is consistent: there is no well-formed formula φ such that φ and $\neg\varphi$ are both theorems.

Proof:

By the soundness theorem, if both φ and $\neg\varphi$ are theorems, they are both tautologies. This is a contradicition since they have opposite truth values. ■

Definition 4.0.2:

A well-formed formula φ is **independent** of a set of other well-formed formulas Γ if φ is not provable from (just) Γ.

Proposition 4.0.3:

Axioms **A1**, **A2**, and **A3** are all independent of one another.

Proof:

(1) First we show **A1** is independent of **A2** and **A3**. We construct the following "truth tables":

A	$\neg A$
0	1
1	1
2	0

A	B	$A \to B$
0	0	0
1	0	2
2	0	0
0	1	2
1	1	2
2	1	0
0	2	2
1	2	0
2	2	0

This gives us a method of evaluating well-formed formulas φ. If φ is always 0 then it is called *select*. We will show that modus ponens of two select formulas is itself select, and **A2** and **A3** are also select. Then these truth tables model the logical system without **A1**. We will then show that **A1** is not select (ie. it cannot be proven by **A2** and **A3**).

The proof that **A2** and **A3** are select is left as an exercise. If φ and $\varphi \to \psi$ is select then φ and $\varphi \to \psi$ are always 0. The only row where this is possible is when $\psi = 0$ (otherwise $\varphi \to \psi$ is 2). Thus ψ is also select and therefore modus ponens infers select values from select inputs, as required.

If $\varphi = 2$ and $\psi = 1$ then $\psi \to (\varphi \to \psi) = 1 \to (2 \to 1) = 1 \to 0 = 2$ so **A1** is not select, as required.

(2) For **A2** we take:

A	$\neg A$
0	1
1	0
2	1

A	B	$A \to B$
0	0	0
1	0	0
2	0	0
0	1	2
1	1	2
2	1	0
0	2	1
1	2	0
2	2	0

(3) We say that φ is "super" if $h(\varphi)$ is a tautology where $h(\varphi)$ is the same as φ except we remove all negations. Thus obviously **A1** and **A2** are super as they don't involve negation, but it can be shown that **A3** is not. Modus ponens holds since if φ and $\varphi \to \psi$ are super then $h(\varphi)$ and $h(\varphi \to \psi) = h(\varphi) \to h(\psi)$ are tautologies then by modus ponens, $h(\psi)$ is a tautology and ψ is "super" as required. ■

First we have Kleené's (1952):

(1) $\varphi \to (\psi \to \varphi)$

(2) $\big(\varphi \to (\psi \to \mu)\big) \to \big((\varphi \to \psi) \to (\varphi \to \mu)\big)$

(3) $(\varphi \wedge \psi) \to \varphi$

(4) $(\varphi \wedge \psi) \to \varphi$

(5) $\varphi \to \big(\psi \to (\varphi \wedge \psi)\big)$

(6) $\varphi \to (\varphi \vee \psi)$

(7) $\psi \to (\varphi \vee \psi)$

(8) $(\varphi \to \mu) \to \Big((\psi \to \mu) \to \big((\varphi \vee \psi) \to \mu\big)$

(9) $(\varphi \to \psi) \to \big((\varphi \to \neg\psi) \to \neg\varphi\big)$

(10) $\neg\neg\varphi \to \varphi$

Or Mezedith's (1953) which is just a single axiom:

$$\bigg(\Big(\big((\varphi \to \psi) \to (\neg\mu \to \neg\vartheta)\big) \to \mu\Big) \to \eta\bigg) \to \big((\eta \to \varphi) \to (\zeta \to \varphi)\big)$$

4.1 First Order Theories

Definition 4.1.1:

A **first order language** is a language consisting of the following characters:

(1) Connectives: $\neg$, $\to$, $\vee$, and $\wedge$.

(2) Punctuation: (and).

(3) The quantifier $\forall$ (for all, the universal quantifier).

(4) Individual variables: $x_1, x_2, \ldots$.

(5) Individual constants: $a_1, a_2, \ldots$.

(6) Predicate letters: A_k^n for $k, n \in \mathbb{N}$.

(7) Function letters: f_k^n for $k, n \in \mathbb{N}$.

(8) An equality symbol: $=$.

The set of individual constants, predicate letters, and function letters can be empty and an equality symbol is not necessary.

We know define what a **term** is:

(1) Any variable or constant is a term.

(2) If f_k^n is a function letter and $t_1, \ldots, t_n$ are terms then $f_k^n(t_1, \ldots, t_n)$ is a term.

And only strings which are recursively defined by the above two rules (using a finite number of steps) are terms.

Now we define **atomic formulas**: if A_k^n is a predicate letter then

(1) If $t_1, \ldots, t_n$ are terms then $A_k^n(t_1, \ldots, t_n)$ is an atomic formula.

(2) If there is an equality symbol, then $t_1 = t_2$ for any terms t_1 and t_2 is an atomic formula.

The well-formed formulas of the first order theory are defined by:

(1) Any atomic formula is well-formed.

(2) If φ and ψ are any well-formed formulas then so are $\neg\varphi$ and $\varphi \circ \psi$ for any connective $\circ$.

(3) If y is a variable letter and φ is a well-formed formula then $\forall y \varphi$ is also a well-formed formula.

Definition 4.1.2:

Given a well-formed formula $\forall y \varphi$ then φ is called a **scope** of the quantifier $\forall y$. An occurrence of a variable x in a well-formed formula is **bound** if the occurrence is in a substring $\forall x$, or if it is in the scope of some $\forall x$. And x is a **bound variable** in the well-formed formula if all of its occurrences are bound.

And an occurrence is **free** if it is not bound, and a variable is a **free variable** if it is not bound (equivalently, at least one of its occurrences is free).

Example 4.1.3:

For example given

$$\forall x_3 \Big(\big(\forall x_1 A_1(x_1, x_2) \big) \to A_2(x_3, a_1) \Big)$$

All of the occurrences of x_1 are bound, the only occurrence of x_2 is free, and all occurrences of x_3 are bound. So x_1 and x_3 are bound, and x_2 is free. Now look at a similar formula:

$$\Big(\forall x_3 \big(\forall x_1 A_1(x_1, x_2) \big) \Big) \to A_2(x_3, a_1)$$

The second occurrence of x_3 is free, so x_1 is bound and x_2 and x_3 are free.