Representation Theory
Homework 5

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1 Problem

Let V be an $\mathbb{F}$-vector space, and define $\mathrm{end}_{\mathbb{F}}^{\mathsf{frk}}(V) \subseteq \mathrm{end}_{\mathbb{F}}(V)$ be the set of all finite-rank V-endomorphisms (endomorphisms whose image is finite-dimensional).

(1) Show that for $T \in \mathrm{end}_{\mathbb{F}}^{\mathsf{frk}}(V)$, TV is T-invariant and as such we can define $\mathrm{tr}_V(T) = \mathrm{tr}_{TV}(T)$. Show that $\mathrm{tr}\colon \mathrm{end}_{\mathbb{F}}^{\mathsf{frk}}(V) \to$ is linear.

(2) Show that $\mathrm{tr}\colon \mathrm{end}_{\mathbb{F}}^{\mathsf{frk}}(V) \to \mathbb{F}$ decomposes as the composition $\mathrm{end}_{\mathbb{F}}^{\mathsf{frk}}(V) \cong V^\vee \otimes_{\mathbb{F}} V \xrightarrow{\mathrm{ev}} \mathbb{F}$, where the first map is the standard isomorphism, and the second is induced by the bilinear map $V^\vee \times V \to \mathbb{F}, (\phi, v) \mapsto \phi(v)$.

(1) For any V-endomorphism, TV is T-invariant, as such tr is clearly well-defined. Note that if $TV \subseteq W$ is also finite-dimensional then $\mathrm{tr}_{TV}(T) = \mathrm{tr}_W(T)$. Indeed: take $B = \{v_1, \dots, v_n\}$ to be a basis for TV and extend it to a basis $B' = \{v_1, \dots, v_n, v_{n+1}, \dots, v_m\}$ of W. Then we have

$$\mathrm{tr}_{TV}(T) = \sum_{i=1}^{n} ([Tv_i]_B)_i$$

and

$$\mathrm{tr}_W(T) = \sum_{i=1}^{m} ([Tv_i]_{B'})_i$$

Now, for $1 \leq i \leq n$, notice that $Tv_i \in \mathrm{span}\{v_1, \dots, v_n\}$ and as such $[Tv_i]_{B'}$ is equal to $[Tv_i]_B$ but with $m-n$ padded zeroes. Thus, $([Tv_i]_{B'})_i = ([Tv_i]_B)_i$. And for $i > n$, $Tv_i \in TV = \mathrm{span}\{v_1, \dots, v_n\}$, so in the linear combination of basis vectors equalling Tv_i, the coefficients of $v_{n+1}, \dots, v_m$ are zero. In particular, $([Tv_i]_{B'})_i = 0$. So we have

$$\mathrm{tr}_W(T) = \sum_{i=1}^{n} ([Tv_i]_B)_i = \mathrm{tr}_{TV}(T)$$

Now, notice that for $T, S \in \mathrm{end}_{\mathbb{F}}^{\mathsf{frk}}(V)$, we have $(T+S)V, TV, SV \subseteq TV + SV$ (which is finite-dimensional) and so

$$\mathrm{tr}(T+S) = \mathrm{tr}_{(T+S)V}(T+S) = \mathrm{tr}_{TV+SV}(T+S) = \mathrm{tr}_{TV+SV}(T) + \mathrm{tr}_{TV+SV}(S) = \mathrm{tr}_{TV}(T) + \mathrm{tr}_{SV}(S) = \mathrm{tr}(T) + \mathrm{tr}(S)$$

And similarlt $(\alpha T)V \subseteq TV$ so

$$\mathrm{tr}(\alpha T) = \mathrm{tr}_{(\alpha T)V}(\alpha T) = \mathrm{tr}_{TV}(\alpha T) = \alpha \mathrm{tr}(T)$$

So tr is indeed linear.

(2) We recall the isomorphism $V^\vee \otimes_{\mathbb{F}} V \cong \mathrm{end}_{\mathbb{F}}^{\mathsf{frk}}(V)$, induced by mapping $(\phi, v) \mapsto [u \mapsto \phi(u)v]$. Its inverse, recall is given as follows: for $T \in \mathrm{end}^{\mathsf{frk}}(V)$, let $\{e_1, \dots, e_n\}$ be a basis for TV. Then define $\pi_i\colon V \to \mathbb{F}$ by extending the ith projection of $TV \to \mathbb{F}$ to all of V. Then the image of T is

$$\sum_{i=1}^{n} (\pi_i \circ T) \otimes e_i$$

Now, we know that ev: $V^\vee \otimes_{\mathbb{F}} V \to \mathbb{F}$ by $\text{ev}(\phi \otimes v) \mapsto \phi(v)$. As such

$$\sum_{i=1}^{n} (\pi_i \circ T) \otimes e_i \mapsto \sum_{i=1}^{n} \pi_i(Te_i) = \text{tr}(T)$$

(Since $\pi_i(Te_i) = ([Te_i]_B)_i$).

2 Problem

Let V be a G-representation and W an H-representation. Show that there is an equivariant $G \times H$ injection $V^\vee \otimes_{\mathbb{F}} W^\vee \to (V \otimes_{\mathbb{F}} W)^\vee$. Show that if either V or W are finite-dimensional then this is an isomorphism.

We begin by defining a $G \times H$-equivariant bilinear map $V^\vee \times W^\vee \to (V \otimes_{\mathbb{F}} W)^\vee$, and this will induce a $G \times H$-equivariant map from the tensor product. Given $\lambda, \mu \in V^\vee \times W^\vee$, define $f(\lambda, \mu): V \times W \to \mathbb{F}$ by $f(\lambda, \mu)(v, w) = \lambda(v)\mu(w)$. This is clearly a bilinear map, and as such defines a map (abusing notation by recycling the name) $f(\lambda, \mu): V \otimes_{\mathbb{F}} W \to \mathbb{F}$, i.e. $f(\lambda, \mu) \in (V \otimes_{\mathbb{F}} W)^\vee$. This map $f: V^\vee \times W^\vee \to (V \otimes_{\mathbb{F}} W)^\vee$ is bilinear as well:

$$f(\lambda + \lambda', \mu)(v \otimes w) = (\lambda + \lambda')(v)\mu(w) = (f(\lambda, \mu) + f(\lambda', \mu))(v \otimes w)$$

etc. As such it defines a map $f: V^\vee \otimes W^\vee \to (V \otimes_{\mathbb{F}} W)^\vee$, explicitly given by

$$f(\lambda \otimes \mu)(v \otimes w) = \lambda(v)\mu(w)$$

(The work we have done thus far is to show that this is well-defined.)

Now we claim two things: that f is $G \times H$-equivariant, and that f is an injection. First, note that for $g, h \in G \times H$:

$$f((g, h)\lambda \otimes \mu)(v \otimes w) = f((g\lambda) \otimes (h\mu))(v \otimes w) = (g\lambda)(v) \cdot (h\mu)(w) = \lambda(g^{-1}v)\mu(h^{-1}w)$$

We now consider the action of $G \times H$ on $(V \otimes_{\mathbb{F}} W)^\vee$: given $\phi \in (V \otimes_{\mathbb{F}} W)^\vee$ and $g, h \in G \times H$:

$$(g, h) \star \phi: (v \otimes w) \mapsto \phi((g, h)^{-1}(v \otimes w)) = \phi((g^{-1}, h^{-1})(v \otimes w)) = \phi((g^{-1}v) \otimes (h^{-1}w))$$

As such,

$$(g, h)f(\lambda \otimes \mu): (v \otimes w) \mapsto f(\lambda \otimes \mu)(g^{-1}v \otimes h^{-1}w) = \lambda(g^{-1}v)\mu(h^{-1}w)$$

By linearity this holds for all $\phi \in V^\vee \otimes W^\vee$. Thus we have indeed that

$$f((g, h)\bullet) = (g, h)f(\bullet)$$

as required.

Now, to show that f is injective. Let $\{\lambda_i\}_{i \in I}$ be a basis for $V^\vee$, then elements of $V^\vee \otimes W^\vee$ are of the form $\sum_{i \in I} \lambda_i \otimes \phi_i$ for $\phi_i \in W^\vee$, all but finitely many being zero. Now, if $x = \sum_{i \in I} \lambda_i \otimes \phi_i \neq 0$, we claim that its image is also nonzero (and so f has a trivial kernel, making it injective). Indeed, since it is nonzero there is some $i_0 \in I$ with $\phi_{i_0} \neq 0$. As such, suppose $\phi_{i_0}(w) \neq 0$. Let us define $\psi \in V^\vee$ by $\psi(v) = \sum_{i \in I} \lambda_i(v)\phi_i(w)$, i.e. $\psi(v) = f(x)(v, w)$. Now, we have

$$\psi = \sum_{i \in I} \lambda_i \cdot \phi_i(w)$$

i.e. ψ is a linear combination of λ_is. Since λ_i forms a basis for $V^\vee$, and a coefficient ($\phi_{i_0}(w)$) is nonzero, we must have that $\psi \neq 0$. In particular, this means that $f(x) \neq 0$, as desired.

If V or W are finite-dimensional, we claim that this is a surjection as well. Without loss of generality, assume V is finitely generated, with a basis $\{e_1, \ldots, e_n\}$ and dual basis $\{e_1^*, \ldots, e_n^*\}$. Now let $\phi \in (V \otimes_{\mathbb{F}} W)^\vee$. Let us define $\phi_j \in W^\vee$ by $\phi_j(w) = \phi(e_j \otimes w)$. This is clearly linear. Furthermore, notice that

$$\phi(v \otimes w) = \phi\left(\sum_{i=1}^n e_i^*(v) e_i \otimes w\right) = \sum_{i=1}^n e_i^*(v)\phi(e_i \otimes w) = \sum_{i=1}^n e_i^*(v)\phi_i(w)$$

Thus

$$\phi(v \otimes w) = f\left(\sum_{i=1}^n e_i^* \otimes \phi_i\right)(v \otimes w) \implies \phi = f\left(\sum_{i=1}^n e_i^* \otimes \phi_i\right)$$

as required.

3 Problem

Let V be a vector space over a field of characteristic $\neq 2$. Define

$$\mathrm{Sym}^2 V = \mathrm{span}\left\{v_1 \otimes v_2 + v_2 \otimes v_1\right\}, \qquad \Lambda^2 V = \mathrm{span}\left\{v_1 \otimes v_2 - v_2 \otimes v_1\right\} \subseteq V \otimes_{\mathbb{F}} V$$

(1) Show that $V \otimes_{\mathbb{F}} V = \mathrm{Sym}^2 V \oplus \Lambda^2 V$;

(2) Given $n = \dim V$, compute $\dim \mathrm{Sym}^2 V, \dim \Lambda^2 V$;

(3) S_2 acts naturally on $V \otimes_{\mathbb{F}} V$. Show that $\mathrm{Sym}^2 V, \Lambda^2 V$ are subrepresentations. How does S_2 act on each?

(4) Assume that V is a G-representations. Show that $\mathrm{Sym}^2 V, \Lambda^2 V \subseteq V \otimes_{\mathbb{F}} V$ are subrepresentations. Deduce that if $\dim V > 1$, $V \otimes_{\mathbb{F}} V$ is not irreducible.

(1) Let us define an endomorphism of $V \otimes V$, namely

$$f(v \otimes w) = \frac{v \otimes w + w \otimes v}{2}$$

This is induced by the bilinear function $f(v, w) = \frac{v \otimes w + w \otimes v}{2}$, and as such is well-defined. Clearly, the image of f is $\mathrm{Sym}^2 V$: it is clear that $f(v \otimes w) \in \mathrm{Sym}^2 V$, and as such $\mathrm{im} f \subseteq \mathrm{Sym}^2 V$. Conversely, $\mathrm{Sym}^2 V$ is spanned by $\frac{v \otimes w + w \otimes v}{2}$, i.e. elements of the form $f(v \otimes w)$. Also, f is idempotent: since $f(v \otimes w) = f(w \otimes v)$,

$$f^2(v \otimes w) = \frac{f(v \otimes w) + f(w \otimes v)}{2} = f(v \otimes w)$$

Thus f is a projection operator onto $\mathrm{Sym}^2 V$. So we have $V \otimes_{\mathbb{F}} V = \mathrm{Sym}^2 V \oplus \ker f$. Notice that since $f(v \otimes w) = f(w \otimes v)$, we have $\Lambda^2 V \subseteq \ker f$.

Let us also define

$$g(v \otimes w) = \frac{v \otimes w - w \otimes v}{2}$$

Which is again well-defined, and a projection onto $\Lambda^2 V$. But notice that $f + g = \mathrm{id}$.

As such, if $v \in \ker f$ then $v = fv + gv = gv \in \Lambda^2 V$. Thus, $\ker f = \Lambda^2 V$. So we have shown

$$V \otimes_{\mathbb{F}} V = \mathrm{Sym}^2 V \oplus \Lambda^2 V$$

as required.

(2) Let $B = \{e_i\}_{i=1}^n$ be a basis for V. Then $B_2 = \{e_i \otimes e_j\}$ is a basis for $V \otimes_{\mathbb{F}} V$, and in particular we have $\dim V \otimes_{\mathbb{F}} V = n^2$. Since f, g are projection operators into $\mathrm{Sym}^2 V$ and $\Lambda^2 V$ respectively, fB_2, gB_2 form bases for $\mathrm{Sym}^2, \Lambda^2 V$ respectively. Now, notice that since $f(e_i \otimes e_j) = f(e_j \otimes e_i)$,

$$\mathrm{Sym}^2 V = \mathrm{span} f(B_2) = \mathrm{span}\{f(e_i \otimes e_j)\}_{i,j} = \mathrm{span}\{f(e_i \otimes e_j)\}_{i \leq j}$$

and similarly since $g(e_i \otimes e_j) = -g(e_j \otimes e_i)$ (and in particular $g(v \otimes v) = 0$),

$$\Lambda^2 V = \mathrm{span} g(B_2) = \mathrm{span}\{f(e_i \otimes e_j)\}_{i<j}$$

In particular, we have $\dim \mathrm{Sym}^2 V \leq \binom{n}{2} + n = \frac{n(n+1)}{2}$ and $\dim \Lambda^2 V \leq \binom{n}{2} = \frac{n(n-1)}{2}$ (as these are the number of ways of choosing $i \leq j$ and $i < j$ respectively). Now, notice that since $V \otimes V = \mathrm{Sym}^2 V \oplus \Lambda^2 V$, we have

$$\dim V \otimes V = n^2 = \dim \mathrm{Sym}^2 V + \dim \Lambda^2 \leq \frac{n(n+1)}{2} + \frac{n(n-1)}{2} = n^2$$

As such, these inequalities must be equalities, and we have

$$\dim \mathrm{Sym}^2 V = \frac{n(n+1)}{2}, \qquad \dim \Lambda^2 V = \frac{n(n-1)}{2}$$

(3) S_2 acts on $V \otimes V$ by $\sigma(v_1 \otimes v_2) = v_{\sigma 1} \otimes v_{\sigma 2}$. The only non-trivial element of S_2 is $\sigma = (1\ 2)$, so this can be simplified to $(1\ 2)(v \otimes u) = u \otimes v$.

On $\mathrm{Sym}^2 V$, note that for $v \otimes u + u \otimes v$, $\sigma(v \otimes u + u \otimes v) = u \otimes v + v \otimes u$. So σ acts trivially on the generators of $\mathrm{Sym}^2 V$ (as such it is a subrepresentation), and thus on all of $\mathrm{Sym}^2 V$. So S_2 acts trivially on $\mathrm{Sym}^2 V$.

On $\Lambda^2 V$ we have $\sigma(v \otimes u - u \otimes v) = u \otimes v - v \otimes u$. So σ acts as multiplication by -1 on $\Lambda^2 V$ (and as such it is a subrepresentation).

(4) We will show that f, g (defined above) are G-equivariant. Indeed, let us define $s \colon V \otimes V \to V \otimes V$ by $v \otimes u \mapsto u \otimes v$. Then

$$f = \frac{\mathrm{id} + s}{2}, \qquad g = \frac{\mathrm{id} - s}{2}$$

So we just need to show that s is G-equivariant, and then by linearity so to are f, g.

Indeed, let $h \in G$, then

$$s(h(v \otimes u)) = s((hv) \otimes (hu)) = (hu) \otimes (hv) = h(u \otimes v) = h(s(v \otimes u))$$

as required.

Since f, g are G-equivariant, their images are G-subrepresentations. These images are the subspaces $\mathrm{Sym}^2 V, \Lambda^2 V$, and so we have shown the desired result.

Now, if $n = \dim V > 1$, we have that $\frac{n(n+1)}{2}, \frac{n(n-1)}{2} < n^2$. As such, by considering dimensions, $\mathrm{Sym}^2 V, \Lambda^2 V$ are proper (nontrivial) subspaces of $V \otimes V$. Therefore $V \otimes V$ cannot be simple.

4 Problem

Let V be a G-representation over a field with characteristic $\neq 2$.

(1) Identify bilinear forms on V with elements in $\hom_{\mathbb{F}}(V \otimes V, \mathbb{F})$. Show that a form is G-invariant iff it is in $\hom_G(V \otimes V, \mathbb{F})$.

(2) Show that a map $V \otimes V \to \mathbb{F}$ factors as a composition $V \otimes V \xrightarrow{p_1} \mathrm{Sym}^2 V \to \mathbb{F}$ iff it corresponds to a symmetric form, and $V \otimes V \xrightarrow{p_2} \Lambda^2 V \to \mathbb{F}$ iff it corresponds to an antisymmetric form.

(1) Let $(\bullet, \bullet)$ be a bilinear form. Then it corresponds to the map $f \in (V \otimes V)^\vee$ given by $f(v \otimes w) = (v, w)$. We claim that $(\bullet, \bullet)$ is G-invariant iff f is G-equivariant (note that $\mathbb{F}$ is trivial, so f being equivariant means $f(gx) = f(x)$). Indeed:

$$f(g(v \otimes w)) = f((gv) \otimes (gw)) = (gv, gw)$$

So $(gv, gw) = (v, w)$ iff $f(g(v \otimes w)) = f(v \otimes w)$. This means that if $(\bullet, \bullet)$ is G-invariant, then f is G-equivariant (since it satisfies $f(gx) = f(x)$ for generators x, by linearity this extends to all x). And conversely if f is G-equivariant, then the bilinear form is G-invariant.

Thus $(\bullet, \bullet)$ is G-invariant iff $f \in \hom_G(V \otimes V, \mathbb{F})$ (i.e. it corresponds to an element in $\hom_G(V \otimes V, \mathbb{F})$).

(2) Notice that $p_1(v \otimes u) = p_1(u \otimes v)$ (as $v \otimes u - u \otimes v \in \Lambda^2 V$, which is the complement of $\mathrm{Sym}^2 V$). As such a map $f \colon V \otimes V \xrightarrow{p_1} \mathrm{Sym}^2 V \xrightarrow{q_1} \mathbb{F}$ satisfies $f(v \otimes u) = q_1 p_1(v \otimes u) = q_1 p_1(u \otimes v) = f(u \otimes v)$. Thus f corresponds to a symmetric bilinear form.

Similarly, $p_2(v \otimes u) = -p_2(u \otimes v)$. Thus $f(v \otimes u) = q_2 p_2(v \otimes u) = -q_2 p_2(u \otimes v) = -f(u \otimes v)$, so f corresponds to an antisymmetric bilinear form.

Now, if $(\bullet, \bullet)$ is a symmetric bilinear form, then let it correspond to $f \colon V \otimes V \to \mathbb{F}$. We need to show that if $p_1(v \otimes u) = p_1(z \otimes w)$ then $(v, u) = (z, w)$ (since then $q_1(p_1(v \otimes u)) = (v, u)$ will be well-defined, and extends by linearity to satisfy our demand). Indeed, $v \otimes u - z \otimes w \in \ker p_1 = \Lambda^2 V$. Since $(\bullet, \bullet)$ is symmetric, we must have $f(v \otimes u - u \otimes v) = (v, u) - (u, v) = 0$. As such f is the zero map on $\Lambda^2 V$, so $f(v \otimes u) = f(z \otimes w)$, meaning $(v, u) = (z, w)$ as required.

Similarly if $(\bullet, \bullet)$ is antisymmetric, then if $p_2(v \otimes u) = p_2(z \otimes w)$, then $v \otimes u - z \otimes w \in \ker p_2 = \mathrm{Sym}^2 V$. Since $(\bullet, \bullet)$ is antisymmetric we have $f(v \otimes u + u \otimes v) = (v, u) + (u, v) = 0$, so f is the zero map on $\mathrm{Sym}^2 V$ and thus $(v, u) = f(v \otimes u) = f(z \otimes w) = (z, w)$ as required.

5 Problem

Let G be a finite group and V be an irreducible G-representation over $\mathbb{C}$. Let $\sigma \colon \mathbb{C} \to \mathbb{C}$ denote complex conjugation. Suppose V admits a nondegenerate invariant bilinear form $(\bullet, \bullet)$. Let $\langle \bullet, \bullet \rangle$ be an invariant inner product (which always exists) on V.

(1) Let J be the composition $V \to V^\vee \to V$ of the function $v \mapsto (\bullet, v)$ composed with the inverse of $v \mapsto \langle \bullet, v \rangle$. Show that J is G-equivariant and $\mathbb{R}$-linear, but not $\mathbb{C}$-linear. Show that J^2 is multiplication by a scalar $0 \neq \lambda \in \mathbb{C}$.

(2) Show that $J^2 = \lambda \in \mathbb{R}$, and that if $(\bullet, \bullet)$ is symmetric then $\lambda > 0$, and if it is antisymmetric $\lambda < 0$.

(3) Replacing $(\bullet, \bullet)$ with $\frac{1}{\sqrt{|\lambda|}}(\bullet, \bullet)$, show that we may assume $\lambda = \pm 1$.

(4) Consider V a real representation, and as such an $\mathbb{R}[G]$-module. Show that if $(\bullet, \bullet)$ is antisymmetric, there is a homomorphism $\mathbb{H} \to \mathrm{end}_{\mathbb{R}[G]}(V)$ which sends i to multiplication by i and j to J.

(1) Recall that the inverse of $v \mapsto \langle \bullet, v \rangle$ is $\phi \mapsto \sum_{i=1}^n \sigma\phi(e_i) e_i$, where $\{e_i\}_{i=1}^n$ an orthonormal basis of V. As such, J is defined by

$$v \mapsto \sum_{i=1}^n \sigma(e_i, v) e_i$$

Now, let $g \in G$, then

$$gv \mapsto \sum_{i=1}^{n} \sigma(e_i, gv)e_i = \sum_i \sigma(g^{-1}e_i, v)e_i = \sum_i \sigma(e_i, v)ge_i = gJv$$

So J is indeed G-equivariant.

Note that

$$J = \sum_{i=1}^{n} \sigma(e_i, \bullet)e_i$$

so it is the sum of maps of the form $v \mapsto \sigma(e_i, \bullet)e_i$. All of which are $\mathbb{R}$-linear (since $(\bullet, \bullet)$ is bilinear and σ is $\mathbb{R}$-linear). Thus J is $\mathbb{R}$-linear.

But

$$J(iv) = \sum_j \sigma(e_j, iv)e_i = \sum_j -i\sigma(e_j, v)e_i = -iJ(v)$$

Now, if $J(v) \neq 0$ then we have $-iJ(v) \neq iJ(v)$. Indeed, since the bilinear form is nondegenerate, there is a v such that $(e_i, v) \neq 0$. As such $Jv \neq 0$ and we have shown that J is not $\mathbb{C}$-linear.

Finally, let us consider J^2:

$$J^2v = \sum_i \sigma\left(e_i, \sum_j \sigma(e_j, v)e_j\right)e_i = \sum_i \sum_j (e_j, v)\sigma(e_i, e_j)e_i$$

Now, notice that J^v is $\mathbb{C}$-linear. Indeed:

$$J^2(zv) = \sum_{i,j} (e_j, zv)\sigma(e_i, e_j)e_i = z\sum_{i,j} (e_j, v)\sigma(e_i, e_j)e_i = zJ^2v$$

So since J^2 is G-equivariant, $J^2 \in \mathrm{end}_G(V) = \mathbb{F} \cdot \mathrm{id}$, since V is irreducible.

Now, $J^2 \neq 0$. After all, if we focus on J^2v's ith coordinate, it is equal to

$$\sum_j (e_j, v)\sigma(e_i, e_j) = \left(\sum_j \sigma(e_i, e_j)e_j, v\right)$$

Now, since $\{e_i\}$ forms a basis, we cannot have $(e_i, e_j) = 0$ (since the bilinear form is nondegenerate). As such the vector $\sum_j \sigma(e_i, e_j)e_j$ is nonzero. Again by non-degeneracy this means there exists a v such that $J^2v \neq 0$. So indeed we have $J^2 = \lambda\mathrm{id}$ for $\lambda \neq 0$.

(2) Notice that

$$\langle v, Jw\rangle = \left\langle v, \sum_i \sigma(e_i, w)e_i \right\rangle = \sum_i (e_i, w)\langle v, e_i\rangle = (sum_i \langle v, e_i\rangle e_i, w) = (v, w)$$

And so $\langle w, Jv\rangle = (w, v)$. So if $(v, w) = \epsilon(w, v)$ then $\langle v, Jw\rangle = \epsilon\langle w, Jv\rangle$. In particular,

$$\bar{\lambda}\langle v, v\rangle = \langle v, J^2v\rangle = \epsilon\langle Jv, Jv\rangle$$

So let $v \in V$ where $Jv \neq 0$, then we have $\bar{\lambda}$ must be real and have the same sign as ϵ. In particular, if the bilinear form is symmetric, $\lambda > 0$, and if it is antisymmetric $\lambda < 0$.

(3) Note that if we replace $(\bullet, \bullet)$ with $\alpha(\bullet, \bullet)$ then we scale the map $v \mapsto (\bullet, v)$ by α, while the other map doesn't change. In particular, this means that J is scaled by α and J^2 by α^2, i.e. $J^2 = \alpha^2\lambda\mathrm{id}$. Specifically, we can take $\alpha = 1/\sqrt{|\lambda|}$, and we get that $J^2 = \lambda/|\lambda|\mathrm{id}$, so $J^2 = \pm\mathrm{id}$.

(4) To define a morphism $f\colon \mathbb{H} \to \mathrm{end}_{\mathbb{R}[G]}(V)$, we define the image on i, j and extend by linearity and multiplicity. By the presentation of $Q_8 \subseteq \mathbb{H}$ (which is $Q_8 = \left\langle i, j \mid i^4 = 1, i^2 = j^2, iji = j\right\rangle$), it is sufficient to have $f(i), f(j)$ satisfy the following relations:

$$f(i)^4 = 1, \qquad f(i)^2 = f(j)^2, \qquad f(i)f(j)f(i) = f(j)$$

Indeed, since the bilinear form is antisymmetric, we have $\lambda = -1$ and so $J^2 = -\mathrm{id}$. Thus we have

$$i^4\mathrm{id} = \mathrm{id}, \qquad i^2\mathrm{id} = -\mathrm{id} = J^2, \qquad i^2J = -J \neq J$$

so no such homomorphism can exist.