Group Theory

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13.1 Subgroups

Definition 13.1.1:

A group is **solvable** if it has a subnormal series with abelian factors.

Note that this means that an abelian group is necessarily solvable since the series $\{e\} \triangleleft A$ is a subnormal series whose factor is isomorphic to A. We can further prove inductively that a finite abelian group has cyclic factors, we start by taking a non-trivial element $a \in A$ and constructing $\{e\} \triangleleft \langle a\rangle \triangleleft A$. Then the factor ${}^{A}\!/_{\langle a\rangle}$ is abelian and so inductively has a subnormal series with cyclic factors:

$$\{\langle a\rangle\} = {}^{A_n}\!/_{\langle a\rangle} \triangleleft \cdots \triangleleft {}^{A_0}\!/_{\langle a\rangle} = \frac{A}{\langle a\rangle}$$

note that these factors are

$${}^{A_i/\langle a\rangle}\Big/_{A_{i+1}/\langle a\rangle} \cong {}^{A_i}\!/_{A_{i+1}}$$

thus the quotient of

$$\{e\} \triangleleft \langle a\rangle = A_n \triangleleft \cdots \triangleleft A_0 = A$$

are cyclic, and this is a subnormal series of cyclic factors, we can further refine this series to a series of cyclic groups of prime order (since these are the only simple cyclic groups). Thus, if a finite group is solvable, it has a subnormal series of abelian factors, which in turn have subnormal series with factors of cyclic groups of prime order. This condition is obviously sufficient, and so we summarize this result in the following proposition:

Proposition 13.1.2:

A finite group is solvable if and only if it has a subnormal series with factors which are cyclic groups of prime order.

Lemma 13.1.3:

Suppose $A, B, C \leq G$ where C is normal and $A \subseteq B$, then

$$(B \cap C)A = B \cap (CA)$$

Note that this multiplication is a group because C is normal.

Proof:

We already know $(B \cap C)A \subseteq B \cap (CA)$ since $B \cap C \subseteq B, CA$ and $A \subseteq B, CA$. Now suppose $b \in B \cap (CA)$, then $b = ca$ for some $c \in C$ and $a \in A$. Then $c = ba^{-1} \in B$, so $c \in B \cap C$ so $ca \in (B \cap C)A$ as required.

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Proposition 13.1.4:

Suppose N is a normal subgroup of G, then G is solvable if and only if both N and ${}^{G}\!/_{N}$ are solvable.

Proof:

First, suppose G is solvable, so there is a subnormal series

$$\{e\} = G_t \triangleleft G_{t-1} \triangleleft \cdots \triangleleft G_1 \triangleleft G_0 = G$$

where ${}^{G_i}\!/_{G_{i+1}}$ are abelian. Then we can take the two series: $G_{i+1} \cap N \triangleleft G_i \cap N$, and ${}^{G_{i+1}\cdot N}\!/_{N} \triangleleft {}^{G_i \cdot N}\!/_{N}$. Note that by the lemma above:

$$(G_i \cap N)G_{i+1} = G_i \cap (N \cdot G_{i+1})$$

And since $G_{i+1} \cap N = G_{i+1} \cap (G_i \cap N)$ (since $G_{i+1} \subseteq G_i$) so:

$$G_i \cap N / G_{i+1} \cap N = G_i \cap N / G_{i+1} \cap (G_i \cap N) \cong (G_i \cap N)G_{i+1} / G_{i+1} = G_i \cap (N \cdot G_{i+1} / G_{i+1} \leq G_i / G_{i+1}$$

and so the factors of $G_i \cap N$ are abelian, as the subgroups of abelian groups. This defines a subnormal series with abelian factors:

$$\{e\} = G_t \cap N \triangleleft G_{t-1} \cap N \triangleleft \cdots \triangleleft G_1 \cap N \triangleleft G_0 \cap N = G \cap N = N$$

so N is solvable, as required.
And:

$$G_iN/N \Big/ G_{i+1}N/N \cong G_iN / Gi+1N \cong G_i / G_i \cap (G_{i+1}N) = G_i / (G_i \cap N)G_{i+1}$$

and this can be embedded in G_i / G_{i+1}, so it is also abelian. Thus the subnormal series:

$$\{G\} = G_tN/N \triangleleft G_{t-1}N/N \triangleleft \cdots \triangleleft G_1N/N \triangleleft G_0N/N = G/N$$

has abelian factors and so G/N is solvable.
To show the converse, suppose that there are subnormal series

$$\{e\} = N_s \triangleleft \cdots \triangleleft N_1 \triangleleft N_0 = N$$
$$\{N\} = B_r \triangleleft \cdots \triangleleft B_1 \triangleleft B_0 = G/N$$

with abelian factors. Since B_i is a subgroup of G/N there is some subgroup $N \leq H_i \leq G$ such that $B_i = H_i/N$. And so

$$\{N\} = H_r/N \triangleleft \cdots \triangleleft H_1/N \triangleleft H_0/N = G/N$$

This means that

$$N = H_r \triangleleft \cdots \triangleleft H_1 \triangleleft H_0 = G$$

where the factors are abelian. Appending this to N's series gives

$$\{e\} = N_s \triangleleft \cdots \triangleleft N_1 \triangleleft N_0 = N = H_r \triangleleft \cdots \triangleleft H_1 \triangleleft H_0 = G$$

which has abelian factors (since it is concatenation of two series with abelian factors) and so G is solvable, as required. ■

We could use this as an alternative definition for the solvability of a finite group.

Example:

We can construct the following subnormal series:

$$1 \overset{\mathbb{Z}_3}{\triangleleft} A_3 \overset{\mathbb{Z}_2}{\triangleleft} S_3$$

so S_3 is solvable, and

$$1 \overset{\mathbb{Z}_2 \times \mathbb{Z}_2}{\triangleleft} K_4 \overset{\mathbb{Z}_3}{\triangleleft} A_4 \overset{\mathbb{Z}_2}{\triangleleft} S_4$$

so is S_4. But for $n \geq 5$ then since A_n is simple and the only normal subgroup of S_n's then the only non-trivial subnormal series is

$$1 \overset{A_n}{\triangleleft} A_n \overset{\mathbb{Z}_2}{\triangleleft} S_n$$

which doesn't have abelian factors, so S_n is not solvable for $n \geq 5$.

13.2 Commutator Subgroups

Definition 13.2.1:

For $a, b \in G$, their **commutator** is

$$[a,b] = aba^{-1}b^{-1}$$

We define the commutator of two subgroups A and B to be

$$[A,B] = \langle [a,b] \mid a \in A, b \in B \rangle$$

The group generated by all commutators of G is defined to be

$$G' = [G,G] = \langle [a,b] \mid a,b \in G \rangle$$

this is the **commutator subgroup** of G.

Note that $[a,b]$ is trivial if and only if a and b commute. Further note that

$$[a,b] \cdot [b,a] = aba^{-1}b^{-1}bab^{-1}a^{-1} = e$$

so $[a,b]^{-1} = [b,a]$, so commutators are closed under inversions, so G' is simply the group of all finite products of commutators. We will quickly prove some basic characteristics of the commutator subgroup

(1) G' is normal: suppose $c \in G$ then

$$c[a,b]c^{-1} = caba^{-1}b^{-1}c^{-1} = (cac^{-1})(cbc^{-1})(cac^{-1})^{-1}(cbc^{-1})^{-1} = \left[cac^{-1}, cbc^{-1}\right]$$

and since elements of G' are products of commutators, and this holds for products of commutators, G' is normal.

(2) $G' = \{e\}$ if and only if G is abelian, this is trivial.

(3) If $K \trianglelefteq G$ then

$$\left({}^{G}\!/_{K}\right)' = \langle [aK, bK] \mid a,b \in G \rangle = \langle [a,b]K \mid a,b \in G \rangle = {}^{G'K}\!\Big/_{K}$$

(4) And so

$$\left({}^{G}\!/_{G'}\right)' = {}^{G'G'}\!\Big/_{G'} = {}^{G'}\!\Big/_{G'} = 1$$

So ${}^{G}\!/_{G'}$ is abelian.

(5) And ${}^{G}\!/_{K}$ is abelian if and only if ${}^{G'K}\!\Big/_{K} = 1$ which is if and only if $G'K = K$ which occurs if and only if $G' \subseteq K$. That is, ${}^{G}\!/_{K}$ is abelian if and only if $G' \subseteq K$. So G' is the minimal normal subgroup whose quotient is abelian, and so ${}^{G}\!/_{G'}$ is the maximal abelian quotient, this is called the *abelization*.

(6) If $A, B \trianglelefteq G$ then $A \ni a(ba^{-1}b^{-1}) = (aba^{-1})b^{-1} \in B$, so $[A,B] \leq A \cap B$. Furthermore, $c[a,b]c^{-1} = \left[cac^{-1}, cbc^{-1}\right] \in [A,B]$, so $[A,B]$ is normal.

Example:

Notice that $S_n' \leq A_n$ since the sign of a commutator must be even since it is multiplicative. And since $\left[\begin{pmatrix} i & j \end{pmatrix}, \begin{pmatrix} i & k \end{pmatrix}\right] = \begin{pmatrix} i & j & k \end{pmatrix}$, so all the three-cycles are commutators. A_n is generated by the three-cycles, and so every element of A_n is a product of commutators and therefore in S_n'.
So $S_n' = A_n$. And so ${}^{S_n}\!/_{A_n} \cong \mathbb{Z}_2$ is the maximal abelian quotient.

Definition 13.2.2:

The **derived series** of a group G is

$$\cdots \triangleleft G''' \triangleleft G'' \triangleleft G' \triangleleft G$$

we further recursively define $G^{(n+1)} = \left(G^{(n)}\right)'$, where $G^{(0)} = G$.

Theorem 13.2.3:

A group G is solvable if and only if $G^{(n)} = 1$ for some n.

Proof:

If $G^{(n)} = 1$ then the derived series

$$1 = G^{(n)} \triangleleft G^{(n-1)} \triangleleft \cdots \triangleleft G^{(1)} \triangleleft G^{(0)} = G$$

is a subnormal series whose quotients are abelian (recall that ${}^{G}\!/_{G'}$ is abelian), so G is solvable.
To show the converse, suppose G is solvable, so it has a subnormal series

$$1 = G_t \triangleleft \cdots \triangleleft G_1 \triangleleft G_0 = G$$

where ${}^{G_i}\!/_{G_{i+1}}$ is abelian, so $G_i' \subseteq G_{i+1}$, thus $G' \subseteq G_1$, and so $G'' \subseteq G_1' \subseteq G_2$, and inductively $G^{(i)} \subseteq G_i$. So $G^{(t)} \subseteq G_t = 1$, as required.

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Notice that we showed in this proof that the derived series is the quickest subnormal series with abelian factors.

Corollary 13.2.4:

If G is solvable, so are all of its subgroups.

This is because if $H \leq G$, $H^{(n)} \subseteq G^{(n)} = 1$, so H is solvable by the theorem above.
Note that $(A \times B)' = A' \times B'$ since the group operation is done elementwise. And more general:

$$\left(\prod_{\lambda\in\Lambda} A_\lambda\right) = \prod_{\lambda\in\Lambda} A_\lambda'$$

Thus $\prod A_\lambda$ is solvable if and only each A_λ is solvable in nth class for a constant n (meaning $A_\lambda^{(n)} = 1$).

13.3 Nilpotent Groups

Definition 13.3.1:

A **central series** is a normal series (every element is normal in G) where for every i:

$${}^{G_i}\!/_{G_{i+1}} \leq Z\left({}^{G}\!/_{G_{i+1}}\right)$$

In order to give an example of such a series, we define the *lower central series* where $G_{(1)} = G$ and recursively $G_{(n)} = [G, G_{(n-1)}] \subseteq G_{(n-1)}$. This is inductively a normal series (since $[A, B]$ is normal when A and B are). Now notice that ${}^{B}\!/_{A} \in Z\left({}^{G}\!/_{A}\right)$ if and only if for every $g \in G$ and $b \in B$, bA and gA commute, that is $[bA, gA] = [b, g]A = 1$. So ${}^{B}\!/_{A} \in Z\left({}^{G}\!/_{A}\right)$ if and only if $1 = \left[{}^{B}\!/_{A}, {}^{G}\!/_{A}\right] = {}^{[B,G]A}\!/_{A}$ which is if and only if $[B, G] \subseteq A$ (the order of B and G doesn't matter here). So a normal series is central if and only if for every i:

$$[G, G_i] \subseteq G_{i+1}$$

Thus by definition the lower central series is in fact a central series. We define this inductively: $\zeta_0 = 1$ and $\zeta_{i+1} = \{x \in G \mid \forall y \in G : [x, y] \in \zeta_i\}$, notice here that inductively ζ_i are all normal, and so if $x \in \zeta_i$ then for any $y \in G$, $[x, y] \in \zeta_i$ so $\zeta_i \leq \zeta_{i+1}$. So here we have an ascending series

$$1 = \zeta_0 \triangleleft \zeta_1 \triangleleft \zeta_2 \triangleleft \cdots$$

This is central since

$$[G, \zeta_{i+1}] \subseteq \zeta_i$$

by definition (we reverse i and $i + 1$ since the series is descending). But notice that if $x\zeta_i \in Z\left({}^{G}\!/_{\zeta_i}\right)$ then for every $y\zeta_i \in {}^{G}\!/_{\zeta_i}$ we have that

$$\zeta_i = [x\zeta_i, y\zeta_i] = [x, y]\zeta_i$$

so $[x, y] \in \zeta_i$ and then by definition $x \in \zeta_{i+1}$, so $x\zeta_i \in {}^{\zeta_{i+1}}\!/_{\zeta_i}$, and therefore we have that

$${}^{\zeta_{i+1}}\!/_{\zeta_i} = Z\left({}^{G}\!/_{\zeta_i}\right)$$

Proposition 13.3.2:

If $\cdots \triangleleft H_2 \triangleleft H_1 = G$ is a central series then $G_{(n)} \subseteq H_n$. And if $1 = K_0 \triangleleft K_1 \triangleleft K_2 \triangleleft \cdots$ then $K_n \subseteq \zeta_n$.

This means that the lower central series is the quickest descending central series and the upper central series is the quickest ascending central series.

Proof:

We show this inductively (the base being $H_1 = G_{(1)} = G$):

$$G_{(n+1)} = [G, G_{(n)}] \subseteq [G, H_n] \subseteq H_{n+1}$$

where the last equality is because H_n is a central series.
Again we show this inductively: let $x \in K_{n+1}$ and $y \in G$ then $[x, g] \in K_n \subseteq \zeta_n$, so $[x\zeta_n, g\zeta_n] = 1$ so $x\zeta_n \in Z\left({}^{G}/_{\zeta_n}\right) = {}^{\zeta_{n+1}}/_{\zeta_n}$, so $x \in \zeta_{n+1}$.

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Definition 13.3.3:

A group is **nilpotent** if it has a central sequence from G to 1.

So G is nilpotent if there is a series

$$1 = H_n \triangleleft H_{n-1} \triangleleft \cdots \triangleleft H_2 \triangleleft H_1 = G$$

and we know that $G_{(i)} \subseteq H_i \subseteq \zeta_{n-i}(G)$ (we need to reindex ζ since it is defined as an increasing series).

Proposition 13.3.4:

G is nilpotent if and only if there exists an n and m such that $G_{(n)} = 1$ and $\zeta_m(G) = G$ (even one is sufficient).

Proof:

Suppose G is nilpotent then $G_{(n)} \subseteq H_n = 1$ and $G = H_1 \subseteq \zeta_{n-1}$ For the converse notice that the series (either $G_{(n)}$ or ζ_{m-n}) defines a central series from 1 to G, and thus G is nilpotent by definition.

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Notice that every abelian group is nilpotent since $A_{(2)} = 1$ and $\zeta_1(A) = A$. Furthermore, since the factors of a central group are subgroups of the center of another group, they are necessarily abelian, and therefore every nilpotent group is solvable.
It isn't hard to see that $\zeta_i(G \times H) = \zeta_i(G) \times \zeta_i(H)$ and so the finite direct product of nilpotent groups is also nilpotent.
Note that by the definition of ζ_i, G is nilpotent if and only if $G = 1$ or $Z(G) \neq 1$ and ${}^{G}/_{Z(G)}$ is nilpotent.

Lemma 13.3.5:

For every proper subgroup of a nilpotent group $H < G$, H is a proper subgroup of its normalizer.

Proof:

Take the maximal n such that $\zeta_n(G) \subseteq H$ (meaning $\zeta_{n+1} \nsubseteq H$). We will show that $\zeta_{n+1}(G) \subseteq N_G(H)$. Let $x \in \zeta_{n+1}(G)$ and $h \in H$ then $xhx^{-1} = [x, h]h \in \left[\zeta_{n+1}, G\right] \cdot H \subseteq \zeta_n \cdot H = \zeta_n$, where the final inequality comes from the necessary and sufficient condition for central series, and the final equality since $\zeta_n \subseteq H$. So $x \in N_G(H)$, and since ζ_{n+1} is not a subset of H, H must be a proper subset of $N_G(H)$.

Theorem 13.3.6:

Let G be a group, then the following are equivalent:

(1) G is nilpotent

(2) For every proper subgroup of H of G, H is a proper subgroup of its normalizer

(3) Every maximal subgroup is normal

(4) Every p-Sylow subgroup is normal

(5) G is the direct product of p-groups

Proof:

- $(1 \implies 2)$ We just proved this.
- $(2 \implies 3)$ Since $H \subset N_G(H)$, and H is maximal, this means that $N_G(H) = G$, so H is normal.
- $(3 \implies 4)$ If P is a p-Sylow subgroup, suppose P is not normal. Then $P \subseteq N_G(P) < G$, then let $N_G(P) \leq M < G$ be maximal. Then we know $N_G(M) = M$ so M isn't normal but this is a contradiction.
- $(4 \implies 5)$ We proved this last lecture.
- $(5 \implies 1)$ We first show that a p-group is nilpotent. Since the center of a p-group is non-trivial, and ${}^{P}\!/_{Z(P)}$ is itself a p-group, this is proven inductively. So G is the product of p-groups, specifically G is the product of nilpotent groups, and is therefore itself nilpotent.

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