

# Group Theory

Lecture 8, Sunday November 27, 2022  
Ari Feigin

Recall that in our discrete course, we proved the Cantor-Schroder-Bernstein theorem: if there exists injections  $f: A \rightarrow B$  and  $g: B \rightarrow A$  then there exists a bijection between  $A$  and  $B$ . Does this result have a parallel in groups? That is, if there exists  $f: A \rightarrow B$  and  $g: B \rightarrow A$ , are  $A$  and  $B$  isomorphic? The answer is no. Take for instance  $\mathbb{F}_2$  and  $\mathbb{F}_3$ .  $\mathbb{F}_2 \rightarrow \mathbb{F}_3$  trivially and  $\mathbb{F}_3 \rightarrow \mathbb{F}_2$ , but  $\mathbb{F}_2$  and  $\mathbb{F}_3$  are not isomorphic (prove this!).

Returning to chains, recall that the following are exact chains if and only if (we use 1 for the trivial group):

- (1)  $A \xrightarrow{f} B \rightarrow 1$ ;  $\text{Im } f = \text{Ker } g = B$  which is if and only if  $f$  is surjective (epimorphism).
- (2)  $1 \rightarrow A \xrightarrow{f} B$ ;  $f$  is injective (monomorphism).
- (3)  $1 \rightarrow A \xrightarrow{f} B \rightarrow 1$ ;  $f$  is an isomorphism.
- (4)  $1 \rightarrow A \rightarrow B \twoheadrightarrow C \rightarrow 1$ ;  $C \cong B/K$  and  $K \cong A$ , that is  $C \cong B/A$ .

## Definition 8.0.1:

A diagram **commutes** if every possible path gives you the same output. That is if you have two paths  $G \rightarrow \varphi_1, \dots, \varphi_n \rightarrow H$  and  $G \rightarrow \psi_1, \dots, \psi_m \rightarrow H$  then

$$\varphi_n \circ \dots \circ \varphi_1 = \psi_m \circ \dots \circ \psi_1$$

(since composition is from right to left.)

So for example the following commutes

$$\begin{array}{ccc} G & \xrightarrow{f} & G \\ h \downarrow & & \downarrow g \\ G & \xrightarrow{k} & G \end{array}$$

if and only if  $g \circ f = k \circ h$ .

## Example:

We define the **projective linear group** to be  $\text{PGL}_n(\mathbb{F}) = \text{GL}_n(\mathbb{F}) / \{\alpha I\}$ . We further define the **projective special linear group** to be  $\text{PSL}_n(\mathbb{F}) = \text{SL}_n(\mathbb{F}) / \{\alpha I\}$ . And one last definition,  $\mu_n(\mathbb{F}) = \{a \in \mathbb{F} \mid a^n = 1\}$ . Let us look at

$$\begin{array}{ccccccc} & & 1 & & 1 & & 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mu_n(\mathbb{F}) & \hookrightarrow & \mathbb{F}^\times & \xrightarrow{\cdot n} & \mathbb{F}^\times \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \text{SL}_n(\mathbb{F}) & \hookrightarrow & \text{GL}_n(\mathbb{F}) & \xrightarrow{\det} & \mathbb{F}^\times \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \text{PSL}_n(\mathbb{F}) & \hookrightarrow & \text{PGL}_n(\mathbb{F}) & \xrightarrow{\det} & \mathbb{F}^\times / (\mathbb{F}^\times)^n \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 1 & & 1 & & 1 \end{array}$$

This diagram commutes.

## 8.1 Group actions

### Definition 8.1.1:

Suppose  $X$  is any set, a **group action** of a group  $G$  on  $X$  is a function  $\Phi: G \times X \longrightarrow X$ . Which satisfies:

- (1)  $\Phi(g, \Phi(h, x)) = \Phi(gh, x)$ .
- (2)  $\Phi(1, x) = x$ .

### Theorem 8.1.2:

An equivalent definition of a group action on  $X$  by  $G$  is a homomorphism  $\varphi: G \longrightarrow S_X$ .

#### Proof:

Suppose we have a homomorphism  $\varphi: G \longrightarrow S_X$ . We define  $\Phi$  by

$$\Phi(g, x) = (\varphi(g))(x)$$

We claim this is a group action:

$$\Phi(g, \Phi(h, x)) = \Phi(g, (\varphi(h)(x))) = \varphi(g)(\varphi(h)(x)) = (\varphi(g) \circ \varphi(h))(x) = \varphi(gh)(x) = \Phi(gh, x)$$

which proves the first axiom, and

$$\Phi(1, x) = \varphi(1)(x) = \text{id}(x) = x$$

which proves  $\Phi$  is indeed a group action.

Now suppose  $\Phi$  is a group action, we must find a homomorphism  $\varphi$ . Given  $g \in G$  we define  $\sigma = \varphi(g)$  by  $\sigma(x) = \Phi(g, x)$ . This is just a longer way of saying  $\varphi$  is defined by:

$$(\varphi(g))(x) = \Phi(g, x)$$

It is not immediately clear why  $\varphi$  is well defined (as a function), but we will first show that it has the homomorphism property.

$$\varphi(gh)(x) = \Phi(gh, x) = \Phi(g, \Phi(h, x)) = \Phi(g, \varphi(h)(x)) = \varphi(g)(\varphi(h)(x)) = \varphi(g) \circ \varphi(h)(x)$$

And therefore  $\varphi(gh) = \varphi(g) \circ \varphi(h)$  as required.

Now we will show that  $\varphi(g)$  is indeed a permutation. Notice that since  $\varphi$  has the homomorphism property:

$$\varphi(g) \circ \varphi(g^{-1}) = \varphi(gg^{-1}) = \varphi(1)$$

And  $\varphi(1) = \text{id}$ , so  $\varphi(g)$  has an inverse, namely  $\varphi(g^{-1})$  so it is a bijection as required. ■

Notice that this theorem gives us a simple proof that if  $G$  acts on  $X$  and  $H \leq G$ , then  $H$  acts on  $X$  (in the same way). This is because we can take the same homomorphism from  $G$  to  $S_X$  and restrict it to  $H$ .

We use a compact notation for group actions: instead of writing  $\Phi(g, x)$  we instead write  $gx$ . This means that it must satisfy

- (1)  $g(hx) = (gh)x$ . (Note that on the right side  $gh$  is not a group action rather it is the *group's* action, its operation).
- (2)  $ex = x$ .

#### Example:

- (1)  $S_n$  acts on  $\{1, \dots, n\}$  by  $\Phi(\sigma, k) = \sigma(k)$  or with the compact notation:  $\sigma k = \sigma(k)$ . This is a group action since  $\sigma \cdot (\tau \cdot x) = \sigma(\tau(x)) = (\sigma\tau)(x) = (\sigma\tau)x$ .
- (2) If  $G$  is a graph,  $\text{Aut}(G)$  acts on  $V$ .
- (3)  $\text{GL}[n]\mathbb{F}$  acts on  $\mathbb{F}^n$  by  $\Phi(A, v) = Av$  (matrix multiplication).

**Definition 8.1.3:**

A group action of  $G$  on  $X$  is **faithful** if the homomorphism  $\varphi: G \longrightarrow S_X$  is injective.

**Proposition 8.1.4:**

A group action is faithful if and only if for every  $e \neq g \in G$ , there is a  $x \in X$  such that  $gx \neq x$ .

**Proof:**

Suppose a group action is faithful, then if  $gx = x$  for every  $x$ ,  $\varphi(g)(x) = x$  for every  $x$ , so  $\varphi(g) = \text{id}$  and therefore  $g = e$ . To show the converse, suppose  $\varphi(g) = \text{id}$  then  $gx = \varphi(g)(x) = x$  for every  $x \in X$ , so  $g = e$  and therefore  $\varphi$  is injective. ■

**Theorem 8.1.5 (Cayley's Theorem):**

If  $G$  is a group  $G \longrightarrow S_G$ . Specifically  $G$  is isomorphic to a subgroup of  $S_G$ .

**Proof:**

We will show this by showing that there is a faithful group action of  $G$  on  $G$ . We define this group action by  $\Phi(g, h) = gh$ . We claim this is a group action:

$$(1) \quad \Phi(g, \Phi(h, k)) = g(hk) = (gh)k = \Phi(gh, k).$$

$$(2) \quad \Phi(e, g) = eg = g.$$

And we know claim it is faithful: if  $gh = h$  then  $g = e$ , so if  $\Phi(g, h) = h$  then  $g = e$ . (Notice that this is a stronger claim than the group action being faithful, for any  $g \neq e$  than  $gh \neq h$  for *any*  $h \in G$ , such an action is called *free*).

Since the action is faithful, its induced homomorphism is injective. ■

Note that the monomorphism  $G \longrightarrow S_G$  is given by  $(\varphi(g))(h) = gh$ . Thus if  $G \subseteq \{1, \dots, n\}$  then if  $\varphi(k) = \sigma_k$ ,  $\sigma_k(j) = k \circ j$ . So if  $G = \mathbb{Z}_n$  then  $\sigma_k(j) = k + j$ , etc.

**Example:**

We will define a monomorphism  $\text{Euler}(9) \longrightarrow S_9$  by:

$$\begin{aligned} 1 &\mapsto \text{id} \\ 2 &\mapsto (1 \ 2 \ 4 \ 8 \ 7 \ 5) \\ 4 &\mapsto (1 \ 4 \ 7)(2 \ 8 \ 5) \\ 5 &\mapsto (1 \ 5 \ 7 \ 8 \ 4 \ 2) \\ 6 &\mapsto (1 \ 5 \ 7 \ 8 \ 4 \ 2) \\ 7 &\mapsto (1 \ 7 \ 4)(2 \ 5 \ 8) \\ 8 &\mapsto (1 \ 8)(2 \ 7)(4 \ 5) \end{aligned}$$

**Definition 8.1.6:**

Suppose  $G$  acts on  $X$ , then the **orbit** of  $x_0 \in X$  is:

$$G \cdot x_0 = \{gx_0 \mid g \in G\}$$

The **stabilizer** of  $x_0$  is:

$$G_{x_0} = \{g \in G \mid g \cdot x_0 = x_0\} \leq G$$

This is a subgroup since if  $gx_0$  and  $hx_0 = x_0$  then  $gh(x_0) = g(x_0) = x_0$  so  $gh \in G_{x_0}$ , and if  $g \in G_{x_0}$  then  $(g^{-1}g)x_0 = ex_0 = x_0$ , but  $(gg^{-1})x_0 = g^{-1}(gx_0) = g^{-1}x_0 = x_0$ .

**Proposition 8.1.7:**

The set of orbits partition  $X$ .

**Proof:**

Suppose  $y \in G \cdot x$  then  $y = gx$  so if  $g'y \in Gy$  then  $g'y = g'gx \in Gx$ , so  $Gy \subseteq Gx$ , and by symmetry  $Gx = Gy$ . And since  $x \in Gx$ , the orbits partition  $X$ . ■

**Proposition 8.1.8:**

$$|G \cdot x_0| = [G : G_{x_0}]$$

**Proof:**

We will map  $g \cdot G_{x_0}$  to  $g \cdot x_0$ . This is well defined: if  $g \cdot G_{x_0} = g' \cdot G_{x_0}$  then  $g^{-1}g' \in G_{x_0}$  so  $g^{-1}g'x_0 = x_0$  so  $g'x_0 = gx_0$ . This is surjective since every point in the orbit is of the form  $g \cdot x_0$  and the image of  $g \cdot G_{x_0}$  is  $gx_0$ . This is injective since if  $g \cdot x_0 = g' \cdot x_0$  then  $g^{-1}g' \in G_{x_0}$  so  $g' \in g \cdot G_{x_0}$  and therefore since cosets partition  $G$ ,  $g' \cdot G_{x_0} = g \cdot G_{x_0}$  as required. ■