Computability and Complexity

Recitation 4, Thursday August 10, 2023

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Definition:

A **Hamiltonian cycle** is a Hamiltonian path which at the end returns to the first vertex.

Exercise 4.1:

Show that the decision problem

$$\mathsf{DHC} = \{G \mid G \text{ is a directed graph which has a Hamiltonian cycle}\}$$

is **NP**-complete.

DHC is obviously in **NP**. We will create a reduction from DHP to DHC. Let $G = (V, E)$ be some graph, let us define $G' = (V', E')$ by

$$V' = V \cup \{u\}, \quad E' = E \cup \{(v, u), (v, u) \mid v \in V\}$$

If G is in DHP then suppose P is the Hamiltonian path in G. Then $u \to P \to u$ is a Hamiltonian cycle in G'.

And if G' has a Hamiltonian cycle, then it is of the form $u \to P \to u$ (since it is a cycle, it has no start point). Then P is a Hamiltonian path in G, as it must visit every vertex in G (since $u \to P \to u$ is a Hamiltonian cycle), and visits them only once and does not visit u so it is indeed a path in G (since each vertex is visited only once in a Hamiltonian cycle).

Exercise 4.2:

Show that the decision problem

$$\mathsf{HC} = \{G \mid G \text{ is an undirected graph which has a Hamiltonian cycle}\}$$

is **NP**-complete.

We will define a reduction from DHC to HC. Suppose $G = (V, E)$ is a directed graph, then we define $G' = (V', E')$ by

$$V' = \{v_{\text{in}}, v_{\text{out}}, v_{\text{mid}} \mid v \in V\}, \quad E' = \{(u_{\text{out}}, v_{\text{in}}) \mid (u, v) \in E\} \cup \{(v_{\text{in}}, v_{\text{mid}}), (v_{\text{mid}}, v_{\text{out}}) \mid v \in V\}$$

Now suppose G has a Hamiltonian cycle $v_1 \to v_2 \to \cdots \to v_n \to v_{n+1} = v_1$. Then

$$v_{1,\text{in}} \to v_{1,\text{mid}} \to v_{1,\text{out}} \to v_{2,\text{in}} \to v_{2,\text{mid}} \to v_{2,\text{out}} \to \cdots \to v_{n,\text{in}} \to v_{n,\text{mid}} \to v_{n,\text{out}} \to v_{1,\text{in}}$$

is a Hamiltonian cycle in G'.

Now suppose G' has a Hamiltonian cycle. Suppose it starts at $v_{1,\text{mid}}$, then the next vertex must be $v_{1,\text{out}}$ or $v_{1,\text{in}}$, since these are the only neighbors of $v_{1,\text{mid}}$, the cycle must contain both of these edges. And so we can choose whatever one we'd like to "start" with, suppose $v_{1,\text{mid}} \to v_{1,\text{out}}$. Then since $v_{1,\text{out}}$ is connected only to in-nodes and $v_{1,\text{in}}$. Since we've already visited $v_{1,\text{mid}}$, we must go to $v_{2,\text{in}}$. Now suppose we go to $v_{3,\text{out}}$, then at some point later we must go to $v_{2,\text{mid}}$ and the only way to get there is via $v_{2,\text{out}} \to v_{2,\text{mid}}$. But then from $v_{2,\text{mid}}$ we cannot move (since we've visited both of its neighbors), but this is a cycle so it must end at $v_{1,\text{mid}}$. So we must go from $v_{2,\text{in}}$ to $v_{2,\text{mid}}$.

The same argument can be used recursively to state that in the Hamiltonian cycle, we move from $v_{i,\text{out}} \to v_{i+1,\text{in}} \to v_{i+1,\text{mid}} \to v_{i+1,\text{out}}$. Thus the Hamiltonian cycle has the form

$$v_{1,\text{mid}} \to v_{1,\text{out}} \to v_{2,\text{in}} \to v_{2,\text{mid}} \to v_{2,\text{out}} \to \cdots \to v_{n,\text{in}} \to v_{n,\text{mid}} \to v_{n,\text{out}} \to v_{1,\text{in}} \to v_{1,\text{mid}}$$

And so

$$v_1 \to v_2 \to \cdots \to v_n \to v_1$$

is a Hamiltonian cycle in G.

Thus G has a Hamiltonian cycle if and only if G' does. Thus $G \mapsto G'$ is a Karp reduction from DHC to HC as required.

Exercise 4.3:

We define the **travelling salesman problem**,

$$\mathsf{TSP} = \left\{ (X, d, D) \;\middle|\; \begin{array}{l} X \text{ is a set of targets, } d \text{ is a (symmetric) distance function } X \times X \longrightarrow \mathbb{N}\text{, and } D \text{ is} \\ \text{a natural number such that there exists a cycle which visits every target whose total} \\ \text{distance/weight is at most } D. \end{array} \right\}$$

Show that TSP is **NP**-complete.

We define a reduction from HC to TSP. Given a graph G, we define $X = V$ and $d(u, v) = 1$ if $(u, v) \in E$ and otherwise $d(u, v) = |V| + 1$, and $D = |V|$. Then if G has a Hamiltonian cycle, then this cycle is also a cycle which visits every node in X, which is V, and its length is $|V|$, so $(X, d, D) \in \mathsf{TSP}$. And if X has a cycle which visits every node and has a distance $\leq |V|$, then it can only use $(u, v) \in E$. Thus it is a cycle which visits every vertex in G, and it cannot visit a vertex twice as then its distance would be greater than $|V|$. So this cycle is a Hamiltoniian cycle in G.