# Automorphisms and Logical Equivalences of Linear Groups

*Lectures by Elena Bunina*
*Summary by Ari Feiglin* (ari.feiglin@gmail.com)

## Contents

# 1 Linear Groups

Let $R$ be a ring and $V$ a free $R$-module with rank $n$ (meaning it has a basis, and all its basis have cardinality $n$). We define the *general linear group* of $V$ to be $\mathrm{GL}(V)$, the group of invertible $R$-linear endomorphisms over $V$. This is contained within the group $\mathrm{End}(V)$, the group of all $R$-linear endomorphisms over $V$.

By fixing a basis of $V$, we can identify $\mathrm{GL}(V)$ with $\mathrm{GL}_n(R)$, the group of invertible $n \times n$ $R$-matrices. We define the *special linear group* $\mathrm{SL}_n(R)$ to be the group of all invertible $n \times n$ $R$-matrices with a determinant of 1.

**1.1 Definition**

Let $I = I_n$ be the $n \times n$ identity matrix, $E_ij$ to be the standard unit matrix (i.e. $(E_{ij})_{\ell k} = 1$ iff $i = \ell, j = k$ otherwise zero). Then define the **elementary transvection matrix** to be $t_{ij}(\lambda) = I + \lambda E_{ij}$ for $i \neq j$ and $\lambda \in R$.

Notice that

$$t_{ij}(\lambda)t_{ij}(\delta) = I + \lambda E_{ij} + \delta E_{ij} + \lambda\delta E_{ij}^2 = I + (\lambda + \delta)E_{ij} = t_{ij}(\lambda + \delta)$$

If we define $X_{ij} = \{t_{ij}(\lambda) \mid \lambda \in R\}$ then $X_{ij}$ is an Abelian subgroup of $\mathrm{SL}_n(R)$.

Define $E_n(R)$ to be the group generated by all elementary transvection matrices, called the *elementary linear group*. $E_n(R)$ contains the following set of automorphisms which are called *standard*:

**(1)** Let $S/R$ be a (suitable; i.e. the following definition is well-defined) ring extension, and $g \in \mathrm{GL}_n(S)$, then define $\iota_g$ to be the inner automorphism generated by $g$: $a \mapsto g^{-1}a$ This is of course an *inner automorphism*, if $g \in \mathrm{GL}_n(R)$ then this is a *strict inner automorphism*.

**(2)** If $\delta\colon R \longrightarrow R$ is an $R$-automorphism, then $\bar{\delta}$ defined by

$$\bar{\delta}\colon (a_{ij}) \mapsto (\delta(a_{ij}))$$

is a an automorphism in $E_n(R)$. In the case that $A = t_{ij}(\lambda)$ notice that $\bar{\delta}(t_{ij}(\lambda)) = t_{ij}(\delta\lambda)$.

**(3)** If $e \in R$ is idempotent, meaning $e^2 = e$, then

$$\Lambda_e\colon A \mapsto (A^\top)^{-1}e + A(1 - e)$$

is also in $E_n(R)$. In the case that $R$ has no idempotents other than the identity, then we simply write $\Lambda\colon A \mapsto (A^\top)^{-1}$.

Compositions of automorphisms of the above forms $(1) - (3)$ are called *standard* in $E_n(R)$. Beyone these automorphisms, $\mathrm{GL}_n(R)$ and $\mathrm{SL}_n(R)$ have another form of automorphism:

**(4)** If $\gamma$ is some homomorphism from $\mathrm{SL}_n(R)$ or $\mathrm{GL}_n(R)$ to the center of the group, then

$$\Gamma_\gamma\colon A \mapsto \gamma(A)A$$

is an automorphism.

A composition of automorphisms of the form $(1) - (4)$ is called *standard* in $\mathrm{GL}_n(R)$ or $\mathrm{SL}_n(R)$.

**1.2 Theorem**

All automorphisms of $E_n(R)$ for $n \geq 4$ and $R$ commutative are standard. If $2 \in R^\times$ then all the automorphisms of $E_3(R)$ are also standard.

**1.3 Theorem**

All automorphisms of $\mathrm{GL}_n(R)$ and $\mathrm{SL}_n(R)$ for $n \geq 4$ and $R$ commutative are standard. If $2 \in R^\times$ then all automorphisms of $\mathrm{GL}_3(R)$ and $\mathrm{SL}_3(R)$ are also standard.