Infinitesimal Calculus 3

Lecture 16, Wednsday December 14, 2022
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We would like to generalize the Taylor-Maclaurin Theorem to functions of multiple variables. Suppose we have a function $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ which is differentiable $n+1$ times (as in it has $n+1$th order partial derivatives). Then if h, k are constants we can define

$$g(t) = f(x_0 + th, y_0 + tk)$$

where $0 \leq t \leq 1$. Then by the chain rule:

$$g'(t) = f_x(x_0 + th, y_0 + tk)h + f_y(x_0 + th, y_0 + tk)k$$

And so differentiating again we have (by Clairut-Schwarz):

$$g''(t) = f_{xx}(x_0 + th, y_0 + tk)h^2 + 2f_{xy}(x_0 + th, y_0 + tk)hk + f_{yy}(x_0 + th, y_0 + tk)k^2$$

And so if $t = 0$ we have that

$$g''(0) = f_{xx}(x_0, y_0)h^2 + 2f_{xy}(x_0, y_0)hk + f_{yy}(x_0, y_0)k^2$$

And differentiating again gives:

$$g'''(0) = f_{xxx}(x_0, y_0)h^3 + 3f_{xxy}(x_0, y_0)h^2k + 3f_{xyy}(x_0, y_0)hk^2 + f_{yyy}(x_0, y_0)k^3$$

So we can see that:

$$g^{(m)}(t) = \sum_{\ell=0}^{m} \binom{m}{\ell} \cdot \frac{\partial^\ell f}{\partial x^\ell \partial y^{m-\ell}}(x_0 + th, y_0 + tk) \cdot h^\ell \cdot k^{m-\ell}$$

And since g is differentiable $n+1$ times, we can use a taylor series:

$$g(t) = \sum_{m=0}^{n} \frac{g^{(m)}(0)}{m!} t^m + \frac{g^{(n+1)}(c)}{(n+1)!} t^{n+1}$$

So if $t = 1$ then

$$g(1) = \sum_{m=0}^{n} \frac{g^{(m)}(0)}{m!} + \frac{g^{(n+1)}(\theta)}{(n+1)!}$$

where $0 < \theta < 1$. So then we have that

$$\begin{aligned} f(x_0 + h, y_0 + k) = g(1) = \sum_{m=0}^{n} \frac{1}{m!} \sum_{\ell=0}^{m} \binom{m}{\ell} \cdot \frac{\partial^\ell f}{\partial x^\ell \partial y^{m-\ell}}(x_0 + h, y_0 + k) \cdot h^\ell \cdot k^{m-\ell} \\ + \sum_{\ell=0}^{n+1} \binom{n+1}{\ell} \cdot \frac{\partial^\ell f}{\partial x^\ell \partial y^{n+1-\ell}}(x_0 + \theta h, y_0 + \theta k) \cdot h^\ell \cdot k^{n+1-\ell} \end{aligned}$$

If we use the following notation:

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}$$

And we define the exponent of this as if we actually multiplied it out (but composing derivatives):

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^m f = \sum_{\ell=0}^{m} \binom{m}{\ell} \cdot \frac{\partial^m f}{\partial x^\ell \partial y^{m-\ell}} \cdot h^\ell k^{m-\ell}$$

And so we have that

$$f(x, y) = f(x_0 + h, y_0 + k) = \sum_{m=0}^{n} \frac{1}{m!} \left(\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^m f\right)(x_0, y_0) + \frac{1}{(n+1)!} \left(\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f\right)(x_0 + \theta h, y_0 + \theta k)$$

So for $n = 2$ we have that:

$$f(x,y)=f(x_0,y_0)+\left(h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}\right)(x_0,y_0)+\frac{1}{2}\left(h^2\frac{\partial^2 f}{\partial x^2}+2hk\frac{\partial^2 f}{\partial xy}+k^2\frac{\partial^2 f}{\partial x^2}\right)+\frac{1}{6}\left(\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^3 f\right)(x_0+\theta h,y_0+\theta k)$$

Example:

Suppose we have the function

$$f(x,y)=x^2\log y$$

We would like to compute its taylor sequence about $(2,1)$, notice that $f(2,1)=0$. So:

$$\begin{aligned}f(x,y)=f(2,1)+\big(hf_x(2,1)+kf_y(2,1)\big)&+\frac{1}{2}\big(h^2f_{xx}+2hkf_{xy}+k^2f_{yy}\big)(2,1)\\&+\frac{1}{6}\big(h^3f_{xxx}+3h^2kf_{xxy}+3hk^2f_{xyy}+k^3f_{yyy}\big)(2+\theta h,1+\theta k)\end{aligned}$$

Now, we know that:

$$\begin{aligned}f_x&=2x\log y\Big|_{(2,1)}=0 & f_y&=x^2y^{-1}\Big|_{(2,1)}=4\\ f_{xx}&=2\log y\Big|_{(2,1)}=0 & f_{yy}&=-x^2y^{-1}\Big|_{(2,1)}=-4\\ f_{xy}&=\frac{2x}{y}\Big|_{(2,1)}=4\end{aligned}$$

So we have that:

$$f(2+h,1+k)=-4k+\frac{1}{2}\big(8hk-4k^2\big)+R_2$$

Where R_2 is the error, and is equal to:

$$R_2=\frac{1}{6}\Big(0h^3+3\frac{2}{y}h^2k+3\frac{-2x}{y}^2hk^2+\frac{2x^2}{y^3}k^3\Big)(x,y)$$

If we attempt to compute $f(2.2,0.8)$ we have $h=0.2$ and $k=-0.2$ and we get that

$$f(2.2,0.8)=-1.04+R_2(x,y)$$

Where $2\le x\le 2.2$ and $0.8\le y\le 1$. So in this case:

$$|R_2|\le\frac{1}{6}\Big(0+\frac{2}{0.8}(0.2)^3+3\cdot\frac{2\cdot 2.2}{0.8^2}0.2^3+\frac{2\cdot 2.2^2}{0.8^3}0.2^3\Big)\le 0.0558$$

So $f(2.2,0.8)\approx -1.04$ is a good approximation.

Notice that in the Taylor expansion, if we can't sufficiently bound R_n then it is pretty much useless. The Taylor expansion is only useful for functions whose error can be bounded sufficiently.