Probability and Statistics Homework #11

Ari Feiglin

Question 11.1:

Let X be a random variable with the following probability density function:

$$f_X(x) = \begin{cases} \frac{k}{\sqrt{X}} & 0 < x < 1 \\ 0 & \text{else} \end{cases}$$

(1) What is the value k?

(2) Find and sketch the comulative probability function of X.

(3) What is $\mathbb{P}\left(\frac{1}{4} < X \leq 2\right)$?

(1) We know that the integral of f_X over $\mathbb{R}$ must be 1 by definition, so:

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^1 \frac{k}{\sqrt{x}}\,dx = 2k \cdot \sqrt{x}\Big|_0^1 = 2k$$

So $\boxed{k = \frac{1}{2}}$.

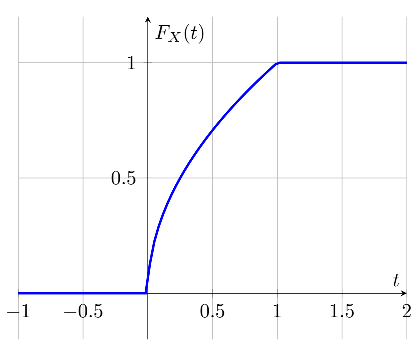
(2) In this case, if $0 < t < 1$:

$$F_X(t) = \int_{-\infty}^{t} \frac{k}{\sqrt{x}}\,dx = \int_0^t \frac{1}{2\sqrt{x}}\,dx = \sqrt{x}\Big|_0^t = \sqrt{t}$$

And if $t \leq 0$, then $F_X(t) = 0$, since the integral over $x \leq 0$ is 0 as $f_X(x) = 0$.
And if $t \geq 1$, then $F_X(t) = 1$, since the integral of $f_X(x)$ over $x < 1 \geq t$ is 1 (by definition of density functions). So:

$$F_X(t) = \begin{cases} \sqrt{t} & 0 < t < 1 \\ 0 & t \leq 0 \\ 1 & t \geq 1 \end{cases}$$

The following is a graph of the comulative distribution of X:



(3) We know that:

$$\mathbb{P}\left(\frac{1}{4} < X \leq 2\right) = \int_{\frac{1}{4}}^{2} f_X(x)\,dx = \int_{\frac{1}{4}}^{1} \frac{1}{2\sqrt{x}}\,dx = \sqrt{x}\Big|_{\frac{1}{4}}^{1} = 1 - \frac{1}{2} = \frac{1}{2}$$

So:

$$\boxed{\mathbb{P}\left(\frac{1}{4} < X \leq 2\right) = \frac{1}{2}}$$

Question 11.2:

Let $a > 1$ and X be a random variable with a probability density function:

$$f_X(x) = \begin{cases} C \cdot x^{-a} & x > 1 \\ 0 & \text{else} \end{cases}$$

(1) Find C and the comulative probability function of X by a.

(2) For which values of a does X have an expected value? And for which does it have variance?

(1) We know that:

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_1^{\infty} C \cdot x^{-a}\,dx = C \cdot \frac{x^{1-a}}{1-a}\bigg|_1^{\infty}$$

Since $1 - a < 0$, this means that $\lim_{x\to\infty} x^{1-a} = 0$, so:

$$= -C \cdot \frac{1}{1-a} \implies \boxed{C = a - 1}$$

And if $t \geq 1$, then:

$$F_X(t) = \int_1^t C \cdot x^{-a}\,dx = -x^{1-a}\bigg|_1^t = 1 - t^{1-a}$$

If $t < 1$, then $F_X(t) = 0$, since the integral of $f_X(x)$ for $x < 1$ is 0 as it is equal to 0 there.
So:

$$F_X(x) = \begin{cases} 1 - t^{1-a} & t > 1 \\ 0 & \text{else} \end{cases}$$

(2) We know X has an expected value if the following converges:

$$\int_{-\infty}^{\infty} |x| \cdot f_X(x)\,dx$$

Since f_X is defined only over $x > 1$, this is equal to:

$$= \int_1^{\infty} x \cdot C \cdot x^{-a}\,dx = C \cdot \int_1^{\infty} x^{1-a}$$

And since the integral of x^{-b} converges if and only if $b > 1$, this converges if and only if

$$a - 1 > 1 \iff a > 2$$

And X has variance if and only if both X^2 and X have expected values. X^2 has an expected value if and only if the following converges (by LOTUS):

$$\int_{-\infty}^{\infty} x^2 \cdot f_X(x)\,dx = C \cdot \int_1^{\infty} x^{2-a}\,dx$$

Which converges if and only if $a - 2 > 1 \iff a > 3$. Since if $a > 3$, then $a > 2$ so X has an expected value. So X has variance only if $a > 3$.
So X has an expected value only if $a > 2$, and it has variance only if $a > 3$.

Question 11.3:

Show that if $X \sim \text{Exp}(\lambda)$, then $\lceil X \rceil \sim \text{Geo}\left(1 - e^{-\lambda}\right)$.

We know that $\lceil X \rceil = k$ if and only if $k - 1 < X \leq k$, so:

$$\mathbb{P}(\lceil X \rceil = k) = \mathbb{P}(k - 1 < X \leq k)$$

And since X is continuous, this is equal to:

$$\int_{k-1}^{k} f_X(x)\, dx$$

So if $k < 1$ (ie. $k \leq 0$), this is equal to 0 since $f_X(x) = 0$ for $x < 0$, so the integral for $x \leq 0$ is 0.
If $k \geq 1$, then this is equal to:

$$= \int_{k-1}^{k} \lambda e^{-\lambda x}\, dx = -e^{-\lambda x}\Big|_{k-1}^{k} = e^{-\lambda(k-1)} - e^{-\lambda \cdot k} = e^{-\lambda(k-1)} \cdot \left(1 - e^{-\lambda}\right)$$

If we let $p = 1 - e^{-\lambda}$, we get that this is equal to $p \cdot (1-p)^{k-1}$, that is:

$$\mathbb{P}(\lceil X \rceil = k) = p \cdot (1-p)^{k-1}$$

This means that $\lceil X \rceil \sim \text{Geo}(p) = \text{Geo}\left(1 - e^{-\lambda}\right)$, as required.

Question 11.4:

In an experiment, a dart is thrown at a circular target with radius r. Let X be the distance of the dart from the center of the target. Assume that the dart always hits the target, and the probability it hits a point is uniform.

(1) What is the range of X?

(2) What is the probability $\mathbb{P}(X < a)$?

(3) What is the probability $\mathbb{P}(a < X < b)$?

(1) The distance of the dart from the center must be nonnegative (since distance is nonnegative), and since the dart always hits the circle, the distance must be less than or equal to r. So the range of X is $\boxed{[0, r]}$.

(2) The probability the dart lands in any part of the target is equal to the area of that part divided by the total area of the target since the probability is uniform. This means that X must be continuous, since $\mathbb{P}(X = a) = 0$, as the part of the target that gives $X = a$ has no area (since it forms a circle, not disk, around the center of radius a, and a circle has 0 area as it doesn't include its interior).
So $\mathbb{P}(X < a) = \mathbb{P}(X \leq a)$, and the part which gives $X \leq a$ is the disk around the center of radius a (since this is defined as all the points with distance $\leq a$). The area of such a disk is $\pi \cdot a^2$. The total area of the target is $\pi \cdot r^2$, so:

$$\mathbb{P}(X \leq a) = \frac{\pi \cdot a^2}{\pi \cdot r^2} = \left(\frac{a}{r}\right)^2$$

That is:

$$\boxed{\mathbb{P}(X < a) = \left(\frac{a}{r}\right)^2}$$

(3) As explained above, $\mathbb{P}(a < X < b) = \mathbb{P}(a \leq X \leq b)$, and the part of the target where $a \leq X \leq b$ is the annulus (ring) with outer radius b and inner radius a (as this is defined as all the points with distance greater than the inner radius, a, and less than the radius of the outer radius, b).
The area of such an annulus is $\pi \cdot (b^2 - a^2)$, so the probability is equal to:

$$\mathbb{P}(a \leq X \leq b) = \frac{\pi \cdot (b^2 - a^2)}{r^2} = \frac{b^2 - a^2}{r^2}$$

That is:

$$\boxed{\mathbb{P}(a < X < b) = \frac{b^2 - a^2}{r^2}}$$

Question 11.5:

X is a random variable with a probability density function:

$$f_X(x) = \begin{cases} 1.5 \cdot x^2 & -1 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

Find the probability density functions of:

(1) $U = X^2$

(2) $V = \sqrt{|X|}$

(3) $W = \frac{1}{|X|}$

We will find the probability density functions by finding the cumulative probability distributions, and since the density function is the derivative of the cumulative distribution, from that we can find the density.

(1) We know that U is nonnegative, so $F_U(t) = 0$ for $t < 0$. Otherwise:

$$F_U(t) = \mathbb{P}(U \le t) = \mathbb{P}\left(X^2 \le t\right) = \mathbb{P}\left(-\sqrt{t} \le X \le \sqrt{t}\right) = \int_{-\sqrt{t}}^{\sqrt{t}} f_X(x)\, dx$$

So if $t < 1$, this is equal to:

$$= \int_{-\sqrt{t}}^{\sqrt{t}} 1.5x^2\, dx = \frac{1}{2}x^3\bigg|_{-\sqrt{t}}^{\sqrt{t}} = t^{\frac{3}{2}}$$

And if $t \ge 1$, this is equal to:

$$= \int_{-1}^{1} 1.5x^2\, dx = 1$$

Since $f_X(x)$ is defined as 0 for xs outside of $[-1, 1]$, and so we can remove from the bounds of the integral all points where x is outside of $[-1, 1]$.
So:

$$F_U(t) = \begin{cases} t^{\frac{3}{2}} & 0 < t < 1 \\ 0 & t \le 0 \\ 1 & t \ge 1 \end{cases}$$

Thus f_U is equal to, as the derivative of F_U:

$$\boxed{f_U(t) = \begin{cases} \frac{3}{2}\sqrt{t} & 0 < t < 1 \\ 0 & \text{else} \end{cases}}$$

(We can ignore ts where F_U is not differentiable since a countable number of ts do not affect f_U's integral.)

(2) Again, we know V is nonnegative, so $F_V(t) = 0$ for $t < 0$. Otherwise:

$$F_V(t) = \mathbb{P}\left(\sqrt{|X|} \le t\right) = \mathbb{P}\left(|X| \le t^2\right) = \mathbb{P}\left(-t^2 \le X \le t^2\right) = F_U(t^4)$$

This means that:

$$f_V(t) = 4t^3 \cdot f_U(t^4)$$

For $0 < t < 1$, this is equal to $4t^3 \cdot \frac{3}{2} \cdot \sqrt{t^4} = 6t^3 \cdot t^2 = 6t^5$, and otherwise this is equal to 0.
So:

$$\boxed{f_V(t) = \begin{cases} 6t^5 & 0 < t < 1 \\ 0 & \text{else} \end{cases}}$$

(3) Notice that:

$$F_W(t) = \mathbb{P}\left(\frac{1}{|X|} \leq t\right) = \mathbb{P}\left(|X| \geq \frac{1}{t}\right) = \mathbb{P}\left(X \geq \frac{1}{t}\right) + \mathbb{P}\left(X \leq -\frac{1}{t}\right)$$

So if $\frac{1}{t} \leq 1 \iff t \geq 1$, then this is equal to:

$$\int_{\frac{1}{t}}^{1} 1.5x^2\,dx + \int_{-1}^{-\frac{1}{t}} 1.5x^2\,dx = \frac{1}{2}x^3\bigg|_{\frac{1}{t}}^{1} + \frac{1}{2}x^3\bigg|_{-1}^{-\frac{1}{t}} = 1 - \frac{1}{t^3}$$

Otherwise, if $t < 1$, then $1 < \frac{1}{t}$, so $F_W(t) = \mathbb{P}\left(|X| \geq \frac{1}{t}\right) \leq \mathbb{P}\left(|X| \geq 1\right)$, which must be 0 since $-1 \leq X \leq 1$ almost always, so $|X| \leq 1$ almost always.
So:

$$F_W(t) = \begin{cases} 1 - \frac{1}{t^3} & t \geq 1 \\ 0 & t < 1 \end{cases}$$

Which means that:

$$\boxed{f_W(t) = \begin{cases} \frac{3}{t^4} & t \geq 1 \\ 0 & t < 1 \end{cases}}$$

Question 11.6:

Prove that Markov's and Chebyshev's Inequalities hold for Absolutely Continuous random variables.

Note:

The proof given in lecture while covering discrete random variables makes no assumptions about the random variable, and is thus valid for all random variables. Nevertheless, I will provide a proof for the specific case of absolutely continuous random variables.

Suppose X is a random variable with expected value and $X \overset{as}{\geq} 0$, and let $a \in \mathbb{R}_{>0}$. Then since $X \overset{as}{\geq} 0$:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_{0}^{\infty} x f_X(x)\, dx$$

Since $f_X(x)$ is nonnegative, this is greater than:

$$\geq \int_{a}^{\infty} x f_X(x)\, dx$$

And since $f_X(x)$ is nonnegative, for every $x \geq a$, $x f_X(x) \geq a \cdot f_X(x)$, so this is greater then:

$$\geq a \cdot \int_{a}^{\infty} f_X(x)\, dx = a \cdot \mathbb{P}(X \geq a)$$

Thus:

$$\mathbb{E}[X] \geq a \cdot \mathbb{P}(X \geq a) \implies \mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X]}{a}$$

As required.
Notice that $Y = (X - \mathbb{E}[X])^2 \geq 0$, and $\mathbb{E}[Y] = \mathrm{Var}(X)$ by definition. So by Markov, if $a \in \mathbb{R}_{>0}$:

$$\mathbb{P}\left(Y \geq a^2\right) \leq \frac{\mathbb{E}[Y]}{a^2} = \frac{\mathrm{Var}(X)}{a^2}$$

And $Y \geq a^2 \iff (X - \mathbb{E}[X])^2 \geq a^2 \iff |X - \mathbb{E}[X]| \geq a$, so:

$$\mathbb{P}(|X - \mathbb{E}[X]| \geq a) = \mathbb{P}\left(Y \geq a^2\right) \leq \frac{\mathrm{Var}(X)}{a^2}$$

As required.

Question 11.7:

Suppose X is an random variable, whose cumulative probability function is F.

(1) Show that F is constant over an interval $[a, b]$ if and only if $\mathbb{P}(X \in (a, b]) = 0$.

(2) Show that F has a jump discontinuity at a if and only if $\mathbb{P}(X = a)$ is positive.

(1) We know that:

$$\mathbb{P}(X \in (a, b]) = \mathbb{P}(X \leq b) - \mathbb{P}(X \leq a) = F(b) - F(a)$$

So if F is continuous over $[a, b]$, then $F(b) = F(a)$, so $\mathbb{P}(X \in (a, b]) = 0$, as required.
And if $\mathbb{P}(X \in (a, b]) = 0$, then $F(b) = F(a)$. And since F is monotonic increasing, for every $c \in [a, b]$:

$$F(a) \leq F(c) \leq F(b) \implies F(c) = F(a)$$

So F is constant, as required.

Note:

If X is also absolutely continuous, then $\mathbb{P}(X \in (a, b]) = \mathbb{P}(X \in [a, b])$, so this becomes:
"F is constant over $[a, b]$ if and only if $\mathbb{P}(X \in [a, b]) = 0$."

(2) We know that F is right-continuous, and has a limit from the left, so F has a jump at a if and only if:

$$\lim_{x \to a^-} F(x) \neq F(a)$$

And since $F(x) \leq F(a)$ for $x \leq a$, this can only happen if (still biconditional):

$$\lim_{x \to a^-} F(x) < F(a)$$

And since we know F has a limit from the left, let $x_n \nearrow a$, this limit is equal to:

$$\lim_{x \to a^-} F(x) = \lim F(x_n) = \lim \mathbb{P}(X \leq x_n)$$

And since $x_n \leq x_{n+1}$, the set $\{\omega \in \Omega \mid X(\omega) \leq x_n\}$ is increasing, so this is equal to:

$$= \mathbb{P}\left(\bigcup_{n \in \mathbb{N}} \{\omega \in \Omega \mid X(\omega) \leq x_n\}\right)$$

By the continuity of probability. And this is equal to:

$$= \mathbb{P}(\{\omega \in \Omega \mid X(\omega) < a\}) = \mathbb{P}(X < a) = F(a) - \mathbb{P}(X = a)$$

Since the sets are equal (the set of $\{X < a\}$ is obviously a superset of the other, and if $X(\omega) < a$, then it must be less than some x_n, so it is in the union of $X < x_n$).
So we have that F has a jump if and only if:

$$F(a) - \mathbb{P}(X = a) < F(a)$$

Which is if and only if $\mathbb{P}(X = a) > 0$, as required.

Question 11.8:

Suppose $X \sim \mathrm{Exp}(\lambda)$.

(1) Prove that $\mathbb{E}[X] = \frac{1}{\lambda}$ and $\mathrm{Var}(X) = \frac{1}{\lambda^2}$.

(2) Find an upper bound for $\mathbb{P}$ using Chebyshev's inequality.

(1) By the definition of expected value:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_0^{\infty} \lambda \cdot x e^{-\lambda x}\, dx$$

And we know by integration by parts:

$$\int u e^{-u}\, du = -ue^{-u} + \int e^{-u}\, du = -ue^{-u} - e^{-u} + C$$

Thus:

$$\int_0^{\infty} e^{-u}\, du = 1$$

If we let $u = \lambda x$, then $dx = \frac{1}{\lambda}\, du$, so:

$$\mathbb{E}[X] = \frac{1}{\lambda} \int_0^{\infty} u e^{-u}\, du = \frac{1}{\lambda} \cdot 1 = \frac{1}{\lambda}$$

As required.
And:

$$\mathbb{E}\left[X^2\right] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_0^{\infty} \lambda x^2 e^{-\lambda x}\, dx = \frac{1}{\lambda} \int_0^{\infty} \lambda^2 x^2 e^{-\lambda x}\, dx$$

And by integration by parts:

$$\int u^2 e^{-u}\, du = -u^2 e^{-u} + 2 \int u e^{-u}\, du$$

So:

$$\int_0^{\infty} u^2 e^{-u}\, du = 2 \int_0^{\infty} u e^{-u}\, du = 2$$

If we let $u = \lambda x$, then:

$$\mathbb{E}\left[X^2\right] = \frac{1}{\lambda^2} \int_0^{\infty} u^2 e^{-u}\, du = \frac{2}{\lambda^2}$$

And:

$$\mathrm{Var}(X) = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

As required.