Complex Functions

Assignment 4

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Exercise 4.1:

Find the radius of convergence for

(1) $\sum_{n=0}^{\infty} z^{n!}$

(2) $\sum_{n=0}^{\infty}(n+2^n)z^n$

(1) The coefficient of z^n in the series is

$$a_n = \begin{cases} 1 & \exists k : n = k! \\ 0 & \text{else} \end{cases}$$

And so we can see that $\limsup \sqrt[n]{|a_n|} = 1$ since $|a_n| \leq 1$ and we can choose a subsequence where $a_n = 1$. And so the radius of convergence is

$$r = \frac{1}{\limsup \sqrt[n]{|a_n|}} = 1$$

(2) Notice that

$$2 = \sqrt[n]{2^n} \leq \sqrt[n]{n+2^n} \leq \sqrt[n]{n \cdot 2^n} = 2 \cdot \sqrt[n]{n}$$

And since $\sqrt[n]{n}$ converges to 1, by the squeeze theorem, $\lim \sqrt[n]{n+2^n} = 2$, and so the radius of convergence is

$$r = \frac{1}{\limsup \sqrt[n]{\sqrt[n]{n+2^n}}} = \frac{1}{2}$$

Exercise 4.2:

Suppose $\sum c_n z^n$ has a radius of convergence of R. Find the radius of convergence of

(1) $\sum_{n=0}^{\infty} n^p c_n z^n$

(2) $\sum_{n=0}^{\infty} |c_n| z^n$

(3) $\sum_{n=0}^{\infty} c_n^2 z^n$

Because the radius of convergence of $\sum c_n z^n$ is R, we know

$$\limsup_{n\to\infty} \sqrt[n]{|c_n|} = \frac{1}{R}$$

(1) We know

$$\limsup \sqrt[n]{|n^p \cdot c_n|} = \limsup \sqrt[n]{n}^p \cdot \limsup \sqrt[n]{|c_n|} = \frac{1}{R}$$

since the limit of $\sqrt[n]{n}$ is 1. So the radius of convergence is R.

(2) We know that $\sqrt[n]{||c_n||} = \sqrt[n]{|c_n|}$, and so the radius of convergence is also R.

(3) We know

$$\limsup \sqrt[n]{|c_n^2|} = \limsup \sqrt[n]{|c_n|}^2 = \left(\limsup \sqrt[n]{|c_n|}\right)^2 = \frac{1}{R^2}$$

and so the radius of convergence is R^2.

Exercise 4.3:

Given that

$$\sum_{n=0}^{\infty} a_n = A, \quad \sum_{n=0}^{\infty} b_n = B$$

absolutely, show that

$$\sum_{n=0}^{\infty} c_n = AB, \quad c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

Let us define

$$A' = \sum_{n=0}^{\infty} |a_n|$$

and A_N, B_N, and C_N to be the partial sums of a_n, b_n, and c_n respectively, ie.

$$X_N = \sum_{n=0}^{N} x_n$$

Let us notice that

$$C_N = \sum_{n=0}^{N} \sum_{k=0}^{n} a_k b_{n-k}$$

Let us rewrite this as the sum

$$C_N = \sum_{(a,b)\in S} ab$$

where $S = \{(a_k, b_{n-k}) \mid 0 \le n \le N, 0 \le k \le n\}$, but this is equal to $S' = \{(a_n, b_k) \mid 0 \le n \le N, 0 \le k \le N-n\}$. If $(a_k, b_{n-k}) \in S$ then $0 \le k \le n$ and so $0 \le n-k \le N-k$ and $0 \le k \le N$ so $(a_k, b_{n-k}) \in S'$. And if $(a_n, b_k) \in S'$ then we can rewrite it as $(a_n, b_{(n+k)-n})$ and $0 \le n+k \le N$ and $0 \le n \le n+k$ so $(a_n, b_k) \in S$ as required. So

$$C_N = \sum_{(a,b)\in S'} ab = \sum_{n=0}^{N} \sum_{k=0}^{N-n} a_n b_k = \sum_{n=0}^{N} a_n B_{N-n}$$

Then notice that

$$A_N \cdot B - C_N = B\sum_{n=0}^{N} a_n - \sum_{n=0}^{N} a_n B_{N-n} = \sum_{n=0}^{N} a_n (B - B_{N-n}) = \sum_{n=0}^{N} a_n \cdot R_{N-n}$$

where R_N is the remainder of the series of b_n, ie $R_N = \sum_{n=N+1}^{\infty} b_n$.

Let $\varepsilon > 0$, since $\sum a_n$ converges absolutely, its own absolute tail must converge to 0, and so there exists an M such that

$$\sum_{n=M+1}^{\infty} |a_n| < \varepsilon$$

and since $\sum b_n$ converges, there must be an M' such that for every $m \ge M'$, $|R_m| < \varepsilon$. By taking the maximum between M and M', we can assume without loss of generality that $M = M'$. So we have that

$$|A_N \cdot B - C_N| \le \overbrace{\left|\sum_{n=0}^{M} a_n R_{N-n}\right|}^{(1)} + \overbrace{\left|\sum_{n=M+1}^{N} a_n R_{N-n}\right|}^{(2)}$$

Let us focus briefly on (1). Notice that if $N \geq 2M$ then $N - n \geq N - M \geq M$ and so $|R_{N-n}| < \varepsilon$. So if $N \geq 2M$ then

$$(1) \leq \sum_{n=0}^{M} |a_n| \cdot \varepsilon \leq A' \cdot \varepsilon$$

Now for (2), since $\sum b_n$ converges, R_m must converge to 0, and therefore R_m is bound, suppose $|R_m| \leq R$. Thus

$$(2) \leq \sum_{n=M+1}^{N} |a_n| \cdot |R_{N-n}| \leq R \cdot \sum_{n=M+1}^{N} |a_n| \leq R \cdot \varepsilon$$

And so all in all we have that for every $\varepsilon > 0$ there exists an M' $(= 2M)$ such that for every $N \geq M'$,

$$|A_N \cdot B - C_N| \leq \varepsilon(R + A')$$

and so $A_N \cdot B - C_N \xrightarrow[N\to\infty]{} 0$. But since $A_N \xrightarrow[N\to\infty]{} A$ by definition, this means that

$$C_N \xrightarrow[N\to\infty 0]{} A \cdot B$$

as required.

Exercise 4.4:

Let $\sum a_n z^n$ and $\sum b_n z^n$ be two powerseries with radii of convergence R_1 and R_2 respectively. Show that the Cauchy product (defined in the previous exercise) $\sum c_n z^n$ converges for $|z| < \min\{R_1, R_2\}$.

If $|z| < \min\{R_1, R_2\}$ then $\sum a_n z^n$ and $\sum b_n z^n$ converge absolutely, and therefore does $\sum c'_n$ where

$$c'_n = \sum_{k=0}^{n} a_k z^k b_{n-k} z^{n-k} = \left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n = c_n z^n$$

that is, $\sum c_n z^n$ converges, as required.

Exercise 4.5:

Show that there does not exist a powerseries $f(z) = \sum c_n z^n$ such that $f(z)$ for $z = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ and also $f'(0) > 0$.

Let $z_n = \frac{1}{n}$, then $f(z_n) = 1$ for every n. But since 1 is itself a powerseries:

$$1 = 1 \cdot z^0 + \sum_{n=1}^{\infty} 0 \cdot z^n$$

So if we let $g(z) = 1$ (so g is a powerseries), then $f(z_n) = g(z_n)$ and $z_n \longrightarrow 0$ and $z_n \neq 0$. We showed in lecture that this means $f(z) = g(z)$ for every $z \in \mathbb{C}$. This means that $f(z) = 1$ for every $z \in \mathbb{C}$, meaning f is constant and therefore $f'(z) = 0$ for every $z \in \mathbb{C}$ and specifically for $z = 0$.

Exercise 4.6:

Show that if $\limsup |c_n|^{\frac{1}{n}} < \infty$ then if we define

$$f(z) = \sum_{n=0}^{\infty} c_n (z - \alpha)^n$$

then

$$c_n = \frac{f^{(n)}(\alpha)}{n!}$$

Let R be the radius of convergence of the series, since $\limsup|c_n|^{\frac{1}{n}} < \infty$, $R > 0$. We showed in lecture that if we define the partial sum to be f_n:

$$f_n(z) = \sum_{k=0}^{n} c_k(z-\alpha)^k$$

then $f_n \rightrightarrows f$ in $D_R(\alpha)$. (We showed this in the case $\alpha = 0$, so we can define $g(z) = f(z+\alpha)$ and then $g_n(z) = f(z+\alpha)$, and then we have that $g_n \rightrightarrows g$ in $D_R(0)$ as g is centered about $z = 0$. And from this it follows directly that $f_n \rightrightarrows f$ in $D_R(0) + \alpha = D_R(\alpha)$.) Thus $f_n'(z) \longrightarrow f'(z)$ for $z \in D_R(\alpha)$. And since

$$f_n'(z) = \sum_{k=0}^{n} k \cdot c_k(z-\alpha)^{k-1} = \sum_{k=1}^{n} k \cdot c_k(z-\alpha)^{k-1}$$

So we claim inductively that for every $m \in \mathbb{N}_0$ and $z \in D_R(\alpha)$.

$$f^{(m)}(z) = \sum_{k=m}^{\infty} k \cdots (k-m+1) \cdot c_k(z-\alpha)^{k-m} = \sum_{k=0}^{\infty}(k+m)\cdots(k+1)\cdot c_{k+m}(z-\alpha)^k$$

The base cases for $m = 0$ and $m = 1$ were shown above.
Suppose that this is true for m, then notice that the radius of convergence of $f^{(m)}$ is

$$\frac{1}{\limsup \sqrt[k]{\left|c_k^{(m)}\right|}}$$

where $c_k^{(m)} = (k+m)\cdots(k+1)\cdot c_{k+m}$ and so

$$\sqrt[k]{\left|c_k^{(m)}\right|} = \sqrt[k]{k+m}\cdots\sqrt[k]{k+1}\cdot\sqrt[k]{|c_{k+m}|}$$

Since the number of elements in the product is constant (m), and the limit of $\sqrt[k]{k+i}$ is 1, the limit superior of this is

$$\limsup \sqrt[k]{|c_{k+m}|} = \limsup \sqrt[k]{|c_k|} = \frac{1}{R}$$

And so we have that $f_n^{(m)} \rightrightarrows f^{(m)}$ in $D_R(\alpha)$ and so this means that $\left(f_n^{(m)}\right)' \longrightarrow f^{(m+1)}$. So by taking the derivatives of $f_n^{(m)}$ we get

$$f^{(m+1)}(z) = \sum_{k=0}^{\infty}(k+m)\cdots(k+1)\cdot k\cdot c_{k+m}(z-\alpha)^{k-1} = \sum_{k=m+1}^{\infty}(k+m)\cdots k\cdot c_{k+m+1}(z-\alpha)^k$$

as required.
And so we have that

$$f^{(m)}(\alpha) = \sum_{k=0}^{\infty}(k+m)\cdots(k+1)\cdot c_{k+m}(z-\alpha)^k$$

since $(z-\alpha)^k = 0^k$, this is zero when $k \neq 0$ and 1 when $k = 0$. So we have

$$f^{(m)}(\alpha) = m\cdots 1\cdot c_m = m!\cdot c_m \implies c_m = \frac{f^{(m)}(\alpha)}{m!}$$

as required.

Exercise 4.7:

Find the radius of convergence of

(1) $\displaystyle\sum_{n=0}^{\infty} n(z-1)^n$

(2) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot (z+1)^n$

(3) $\displaystyle\sum_{n=0}^{\infty} n^2(2z-1)^n$

(1) Here we have $c_n = n$ and $\lim \sqrt[n]{n} = 1$, so the radius of convergence is $\frac{1}{\limsup \sqrt[n]{n}} = \frac{1}{1} = 1$.

(2) Here we have $c_n = \frac{(-1)^n}{n!}$ and so $|c_n| = \frac{1}{n!}$. Notice that if n is even, $n! \geq \left(\frac{n}{2}\right)^{n/2}$ as

$$n! = \prod_{k=1}^{n} k \geq \prod_{k=n/2}^{n} k \geq \prod_{k=n/2} \frac{n}{2} = \left(\frac{n}{2}\right)^{n/2}$$

So if we take a subsequence of even ns, we get that

$$\limsup \frac{1}{n!} \leq \limsup \frac{1}{(n/2)^{n/2}} = 0$$

since the limit of $\left(\frac{n}{2}\right)^{n/2}$ is infinity. And so the radius of convergence is ∞.

(3) We can rewrite $n^2(2z-1)^n$ as $2^n n^2\left(z - \frac{1}{2}\right)^n$. So $c_n = 2^n n^2$, and

$$\sqrt[n]{c_n} = 2 \cdot \sqrt[n]{n}^2$$

since the limit of $\sqrt[n]{n}$ is 1, we get that the limit of $\sqrt[n]{c_n}$ is 2 and so the radius of convergence is $\frac{1}{2}$.