Topology

Lecture 8, Sunday June 11, 2022
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Definition 8.0.1:

Let (X, τ) be a topological space. $B \subseteq \tau$ is a **basis** for τ if every open set $\mathcal{U} \in \tau$ is the union of open sets in B.

Equivalently, B is a basis such that for every open set $\mathcal{U}$ and every $p \in \mathcal{U}$, there exists an $\mathcal{V} \in B$ such that $p \in \mathcal{V} \subseteq \mathcal{U}$. This is equivalent since if B satisfies this then for every open $\mathcal{U}$ we can take the union of $\mathcal{V}_p$ for $p \in \mathcal{U}$ and this gives $\mathcal{U}$, so B is a basis. And if B is a basis then for every $p \in \mathcal{U}$ then since B is a basis $\mathcal{U}$ is a union of open sets in B so p must be in the union, so $p \in \mathcal{V} \subseteq \mathcal{U}$ for $\mathcal{V} \in B$.

Example 8.0.2:

If (X, ρ) is a metric space, $B = \{B_r(x) \mid r > 0, x \in X\}$ the set of all open balls, is a basis for the topology of X. This is by definition.

Proposition 8.0.3:

Let X and Y be topological spaces and B a basis for Y. Then $f\colon X \longrightarrow Y$ is a continuous function if and only if for every $\mathcal{U} \in B$, $f^{-1}(\mathcal{U})$ is open in X.

Proof:

If f is continuous, this is trivial. Otherwise, let $\mathcal{V}$ be open in Y, then it is the union of elements of B:

$$\mathcal{V} = \bigcup_{\lambda \in \Lambda} \mathcal{U}_\lambda$$

and so we have that the preimage of $\mathcal{V}$ under f is

$$f^{-1}(\mathcal{V}) = f^{-1}\left(\bigcup_{\lambda \in \Lambda} \mathcal{U}_\lambda\right) = \bigcup_{\lambda \in \Lambda} f^{-1}(\mathcal{U}_\lambda)$$

and since $f^{-1}(\mathcal{U}_\lambda)$ is open for every $\lambda \in \Lambda$ we have that $f^{-1}(\mathcal{V})$ is open. Thus f is continuous. ■

Similarly we can show that

Proposition 8.0.4:

Let X and Y be topological spaces and B a basis for X. Then $f\colon X \longrightarrow Y$ is open if and only if for every $\mathcal{U} \in B$, $f(\mathcal{U})$ is open.

Proposition 8.0.5:

If X is a topological space and B is a basis and $A \subseteq X$ is a subspace, then $B_A = \{\mathcal{U} \cap A \mid \mathcal{U} \in B\}$ is a basis for A.

Proof:

Let $\mathcal{V}$ be open in A then $\mathcal{V} = \mathcal{V}' \cap A$ for $\mathcal{V}'$ open in A. Then since B is a basis, $\mathcal{V}' = \bigcup_{\lambda \in \Lambda} \mathcal{U}_\lambda$ and so $\mathcal{V} = \bigcup_{\lambda \in \Lambda} \mathcal{U}_\lambda \cap A$ which is a union of elements in B_A. ■

Proposition 8.0.6:

Let X be a topological space and B a basis. Then $A \subseteq X$ is dense in X if and only if for every $\varnothing \neq \mathcal{U} \in B, \mathcal{U} \cap A \neq \varnothing$.

Proof:

Since A is dense if and only if it intersects every open set, one direction is trivial. Otherwise, let $\mathcal{U}$ be open then since it is the union of open sets in B which have non-trivial intersection with A, so does $\mathcal{U}$, so A is dense.

■

Lemma 8.0.7:

If X is a topological space with a basis B, then there exists a dense set A with cardinality $|A| \leq |B|$.

Proof:

Construct A by choosing a $p \in \mathcal{U}$ for every $\varnothing \neq \mathcal{U} \in B$. Then for every $\varnothing \neq \mathcal{U} \in B$, $A \cap \mathcal{U}$ is non-empty and so A is dense by above. By A's construction, $|A| \leq |B|$ as required. ■

Theorem 8.0.8:

Let M be a metric space, then M has a countable basis if and only if it is separable (has a countable dense set).

Proof:

If M has a countable basis then by the above lemma M has a basis which is countable. If M has a countable dense set A then we construct a basis B as follows:

$$B = \{B_p(x) \mid x \in A,\, p > 0,\, p \in \mathbb{Q}\}$$

This is countable since A and $\mathbb{Q}$ are countable. And for every open set $\mathcal{U}$ and for every $x \in \mathcal{U}$ there exists an $r > 0$ such that $B_r(x) \subseteq \mathcal{U}$. There exists a $p \in \mathbb{Q}$ such that $2p < r$ and an $a \in A$ such that $\rho(a, x) < p$ and so $B_p(a) \subseteq B_r(x)$ since if $\rho(a, y) < p$ then $\rho(y, x) < \rho(a, y) + \rho(a, x) < 2p = r$ as required.
So for every $x \in \mathcal{U}$ there exists a $B_p(a) \in B$ such that $x \in B_p(a) \subseteq \mathcal{U}$ so B is a basis. Note that in general the cardinality of this B will be $\leq \aleph_0 \cdot |A|$. ■

Example 8.0.9:

The **Sorgenfrey topology** on $\mathbb{R}$ is defined as the topology $\tau_{\mathbb{S}}$ consisting of all unions of sets of the form $[a, b)$. Thus we can take the basis

$$\mathbb{S} = \{[a, b) \mid a, b \in \mathbb{R}\}$$

$\mathbb{Q}$ is still dense in $\mathbb{R}$ under this topology since it intersects every set in the basis $\mathbb{S}$. We will show that $\tau_{\mathbb{S}}$ has no countable basis.
Let B be a basis for $\tau_{\mathbb{S}}$, then notice that if $x \in \mathbb{R}$ then since $x \in [x, x+1)$ there exists an $\mathcal{V} \in B$ such that $x \in \mathcal{V} \subseteq [x, x+1)$. So we define a mapping $f\colon \mathbb{R} \longrightarrow B$ by $f(x) = \mathcal{V}$. We claim that this is injective since if $f(x) = \mathcal{V}$ then since $\mathcal{V} \subseteq [x, x+1)$ for every $y < x$, $y \notin \mathcal{V}$ and so $f(y) \neq \mathcal{V}$. Since this is true for every x and $y < x$, this means that if $x \neq y$, $f(x) \neq f(y)$ so f is injective. Thus $|\mathbb{R}| \leq |B|$ meaning B is uncountable for any basis B.
We just showed that every metric space has a countable basis if and only if it is separable, but $\tau_{\mathbb{S}}$ is separable but does not have a countable basis. This means that $\tau_{\mathbb{S}}$ is not metricizable.

Proposition 8.0.10:

If X is a topological space with a countable basis then τ's cardinality is at most the cardinality of the continuum.

Proof:

Let B be the countable basis. Then we define $f\colon \mathcal{P}(B) \longrightarrow \tau$ by

$$f(L) = \bigcup_{\mathcal{U} \in L} \mathcal{U}$$

this is surjective since B is a basis, and so $|\tau| \leq |\mathcal{P}(B)| = 2^{|B|} = 2^{\aleph_0}$. ■

In general if τ has a basis B then

$$|\tau| \leq 2^{|B|}$$

Since $\mathbb{R}^n$ is separable, we know then that it has a countable basis and so the cardinality of its topology is at most the cardinality of the continuum. Since we can map x to $B_1(x)$, we know that the topology is actually equal to the cardinality of the continuum.

Proposition 8.0.11:

If X is a topological space and B a basis, then X is compact if and only if every open cover of X by open sets in B has an open subcover.

Proof:

One direction is trivial. Suppose $\{\mathcal{U}_\lambda\}_{\lambda \in \Lambda}$ is an open cover, then every $\mathcal{U}_\lambda$ can be written as a union of sets in the basis, so

$$X = \bigcup_{\lambda \in \Lambda'} \mathcal{V}_\lambda$$

where $\mathcal{V}_\lambda \in B$, and so there exists a finite subcover

$$X = \bigcup_{n=1}^{N} \mathcal{V}_n$$

And since $\mathcal{V}_n \subseteq \mathcal{U}_n$ for some $\mathcal{U}_n$ in the open cover, we get

$$X = \bigcup_{n=1}^{N} \mathcal{U}_n$$

so X is compact. ■

Definition 8.0.12:

Let X be a set and B be a set of subsets of X. We define τ_B as the set of all unions of sets from B:

$$\tau_B = \left\{ \bigcup_{A \in L} A \;\middle|\; L \subseteq B \right\}$$

Note $B \subseteq \tau_B$.

Theorem 8.0.13:

τ_B is a topology on X if and only if:

(1) $X \in \tau_B$.

(2) For every $\mathcal{U}, \mathcal{V} \in B$, $\mathcal{U} \cap \mathcal{V} \in \tau_B$ (the intersection of sets in B is equal to some union of sets in B).

Proof:

If τ_B is a topology then these conditions hold trivially. In general τ_B is closed under arbitrary unions and contains

the empty set. Furthermore given this condition, we know that $X \in \tau_B$. All that remains is to show that τ_B is closed under finite intersections.
Let $\mathcal{U}, \mathcal{V} \in \tau_B$ suppose $\mathcal{U} = \bigcup_{\lambda\in\Lambda} \mathcal{U}_\lambda$ and $\mathcal{V} = \bigcup_{\gamma\in\Gamma} \mathcal{V}_\gamma$ and so

$$\mathcal{U} \cap \mathcal{V} = \bigcup_{\lambda\in\Lambda} \bigcup_{\gamma\in\Gamma} \mathcal{U}_\lambda \cap \mathcal{V}_\gamma$$

by the conditions given, $\mathcal{U}_\lambda \cap \mathcal{V}_\gamma \in \tau_B$ and so the union, $\mathcal{U} \cap \mathcal{V}$, is in τ_B as required. ■

Thus if B is closed under intersections, then by the above theorem τ_B is necessarily a topology.
The conditions are also equivalent to that for every $a \in X$ there is a $\mathcal{U} \in B$ where $x \in \mathcal{U}$, and for every $\mathcal{U}, \mathcal{V} \in B$ and every $a \in \mathcal{U} \cap \mathcal{V}$ there is a $\mathcal{W} \in B$ where $a \in \mathcal{W} \subseteq \mathcal{U} \cap \mathcal{V}$. This is trivial and left as an exercise.

Definition 8.0.14:

If $\{(X_\lambda, \tau_\lambda)\}_{\lambda\in\Lambda}$ is a collection of topological spaces, we define $X = \prod_{\lambda\in\Lambda} X_\lambda$ and

$$B = \left\{ \prod_{\lambda\in\Lambda} \mathcal{U}_\lambda \;\middle|\; \mathcal{U}_\lambda \in \tau_\lambda \text{ and all but a finite number of } \mathcal{U}_\lambda = X_\lambda \right\}$$

then we define a topology on X by τ_B. This is called the **product topology**.

Since $X_\lambda \in \tau_\lambda$, we have $X \in B$. And if $\mathcal{U} = \prod_{\lambda\in\Lambda} \mathcal{U}_\lambda, \mathcal{V} = \prod_{\lambda\in\Lambda} \mathcal{V}_\lambda \in B$ then

$$\mathcal{U} \cap \mathcal{V} = \left(\prod_{\lambda\in\Lambda} \mathcal{U}_\lambda \right) \cap \left(\prod_{\lambda\in\Lambda} \mathcal{V}_\lambda \right) = \prod_{\lambda\in\Lambda} \mathcal{U}_\lambda \cap \mathcal{V}_\lambda$$

and since $\mathcal{U}_\lambda \cap \mathcal{V}_\lambda \in \tau_\lambda$ and the only $\mathcal{U}_\lambda \cap \mathcal{V}_\lambda \neq X_\lambda$ is when either is not X_λ which is finite, so we have that $\mathcal{U} \cap \mathcal{V} \in B$ so B is indeed a basis and this definition is well-defined.

Proposition 8.0.15:

Let π_γ denote the function $\pi_\gamma \colon \prod_{\lambda\in\Lambda} X_\lambda \longrightarrow X_\gamma$ which maps $(x_\lambda)_{\lambda\in\Lambda} \mapsto x_\gamma$. Then π_γ is open and continuous.

Proof:

Let $\mathcal{U}_\gamma \subseteq X_\gamma$ be open then $\pi_\gamma^{-1}(\mathcal{U}_\gamma) = (\mathcal{V}_\lambda)_{\lambda\in\Lambda}$ where $\mathcal{V}_\gamma = \mathcal{U}_\gamma$ and $\mathcal{V}_\lambda = X_\lambda$ for $\lambda \neq \gamma$. This is in B and so is open. Thus π_γ is continuous.
And if $\prod_{\lambda\in\Lambda} \mathcal{U}_\lambda$ is in B then its image in π_γ is by definition $\mathcal{U}_\gamma$ which is open (by the definition of B). Thus π_γ is open as well. ■

Proposition 8.0.16:

Let $X = \prod_{\lambda\in\Lambda} X_\lambda$ be the product topology, and let Y be another topological space. Then a function $f \colon Y \longrightarrow X$ is continuous if and only if $\pi_\gamma \circ f$ is for every $\gamma \in \Gamma$.

Proof:

If f is continuous then $\pi_\gamma \circ f$ is as the composition of continuous functions. Notice that $f(y) = (f_\lambda(y))_{\lambda\in\Lambda}$ where $\pi_\gamma \circ f = f_\gamma$. It is sufficient to show that the preimage of $\prod_{\lambda\in\Lambda} \mathcal{U}_\lambda$ where all but a finite number of $\mathcal{U}_\lambda = X_\lambda$ under f is open. Notice that

$$f(y) \in \prod_{\lambda\in\Gamma} \mathcal{U}_\lambda \iff f_\lambda(y) \in \mathcal{U}_\lambda \iff y \in f_\lambda^{-1}(\mathcal{U}_\lambda)$$

for every $\lambda \in \Gamma$. So

$$f^{-1}\left(\prod_{\lambda\in\Gamma} \mathcal{U}_\lambda \right) = \bigcap_{\lambda\in\Lambda} f_\lambda^{-1}(\mathcal{U}_\lambda)$$

Since $f_\lambda^{-1}(\mathcal{U}_\lambda)$ is open as f_λ is continuous, and since only a finite number of $\mathcal{U}_\lambda \neq X_\lambda$ and thus only a finite number of $f_\lambda^{-1}(\mathcal{U}_\lambda) \neq Y$, this intersection is finite, and so we get that the preimage of every open set is open. Therefore f is continuous. ■

This shows the significance of requiring that all but a finite number of $\mathcal{U}_\lambda = X_\lambda$.
If $A \subseteq X$ and $B \subseteq Y$ where X and Y are topological spaces, there are two ways of defining a topology on $A \times B$. We can view $A \times B$ as a subspace of $X \times Y$, or we can view A and B as subspaces and take their product topology. But notice that if B defines $X \times Y$ then

$$B' = \{\mathcal{U} \times \mathcal{V} \cap A \times B \mid \mathcal{U} \times \mathcal{V} \in B\} = \{\mathcal{U} \times \mathcal{V} \cap A \times B \mid \mathcal{U} \in \tau_X,\, \mathcal{V} \in \tau_Y\}$$

is a basis for $A \times B$. But since $\mathcal{U} \times \mathcal{V} \cap A \times B = (\mathcal{U} \cap A) \times (\mathcal{V} \cap B)$, and the basis which defines $A \times B$ as the product of two subspace topologies is

$$B'' = \{(\mathcal{U} \cap A) \times (\mathcal{V} \cap B) \mid \mathcal{U} \in \tau_x,\, \mathcal{V} \in \tau_Y\}$$

we have $B'' = B'$ and so the topology defined for $A \times B$ is the same with both methods.

Proposition 8.0.17:

If $X_1, \ldots, X_n$ are topological spaces with respective basis B_i then

$$C = \{\mathcal{V}_1 \times \cdots \times \mathcal{V}_n \mid \mathcal{V}_i \in B_i\}$$

is a basis for $X = X_1 \times \cdots \times X_n$.

Proof:

By definition, C is a set of open sets in X. We must show that every set in B (the basis for X) can be written as a union of elements in C. Let $\mathcal{U}_1 \times \cdots \times \mathcal{U}_n \in B$ then $\mathcal{U}_i$ is open in X_i and thus a union of elements in B_i, so let

$$\mathcal{U}_i = \bigcup_{\lambda_i \in \Lambda_i} \mathcal{V}_{\lambda_i}$$

thus

$$\mathcal{U}_1 \times \cdots \times \mathcal{U}_n = \prod_{i=1}^{n} \left(\bigcup_{\lambda_i \in \Lambda_i} \mathcal{V}_{\lambda_i} \right) = \bigcup_{\lambda_1 \in \Lambda_1} \cdots \bigcup_{\lambda_n \in \Lambda_n} \prod_{i=1}^{n} \mathcal{V}_{\lambda_i}$$

which is a union of elements in C, as required. ■

Proposition 8.0.18:

If $\{X_\lambda\}_{\lambda \in \Lambda}$ is a collection of topological spaces with respective bases B_λ then

$$C = \left\{ \prod_{\lambda \in \Lambda} \mathcal{V}_\lambda \;\middle|\; \mathcal{V}_\lambda \in B_\lambda \text{ and all but a finite number of } \mathcal{V}_\lambda = X_\lambda \right\}$$

is a basis for $X = \prod_{\lambda \in \Lambda} X_\lambda$.

Proof:

By definition all elements of C are open in X. Let $\mathcal{U} \in B$ then $\mathcal{U}$ is of the form $\prod_{\lambda \in \Lambda} \mathcal{U}_\lambda$ where all but a finite number of $\mathcal{U}_\lambda = X_\lambda$. Since every $\mathcal{U}_\lambda$ can be written as a union of sets in B_λ, for instance

$$\mathcal{U}_\lambda = \bigcup_{\gamma_\lambda \in \Gamma_\lambda} \mathcal{V}_{\gamma_\lambda}$$

then we get that

$$\mathcal{U} = \prod_{\lambda \in \Lambda} \mathcal{U}_\lambda = \prod_{\lambda \in \Lambda} \bigcup_{\gamma_\lambda \in \Gamma_\lambda} \mathcal{V}_{\gamma_\lambda} = \bigcup_{\substack{\lambda \in \Lambda \\ \gamma_\lambda \in \Gamma_\lambda}} \prod_{\gamma \in \Gamma} \mathcal{V}_{\gamma_\lambda}$$

Now, note that since all but a finite number of $\mathcal{U}_\lambda = X_\lambda$, we can assume that if $\mathcal{U}_\lambda = X_\lambda$ then we just take $\mathcal{U}_\lambda = X_\lambda$ as $\mathcal{V}_\gamma$, and so every product in the above union has all but a finite amount of $\mathcal{V}_{\gamma_\lambda} = X_\lambda$, and so the product is in C.

■