Calculus Homework #7

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Question 7.1:

Determine the convergance of the following improper integrals:

(1) $\displaystyle\int_1^\infty e^{-\log^2 x}\,dx$

(2) $\displaystyle\int_0^\infty x^2\sin\left(x^4\right)\,dx$

(3) $\displaystyle\int_1^\infty \frac{\cos x}{x}\,dx$

(4) $\displaystyle\int_1^\infty \frac{\cos\left(x^2\right)}{x}\,dx$

(5) $\displaystyle\int_1^\infty \frac{|\cos x|}{x}\,dx$

(6) $\displaystyle\int_1^\infty \frac{x-\tan^{-1}x}{x(1+x^2)\tan^{-1}x}\,dx$

(7) $\displaystyle\int_1^\infty \frac{\tan^{-1}x}{\sqrt{x^3+x}}\,dx$

(1) This integral *converges*. This is because we can rewrite it as:

$$e^{-\log^2 x} = x^{-\log x}$$

At some point, $\log x \geq 2$, which means that $-\log x \leq -2$, and since $x \geq 1$:

$$0 \leq x^{-\log x} \leq x^{-2}$$

And since the integral of x^{-2} converges and both functions are nonnegative, the integral converges.

(2) This integral *converges*. By substituting $u = x^4$, we get that $du = 4x^3\,dx$, so the integral is equal to:

$$\int_0^1 x^2\sin\left(x^4\right)\,dx + \frac{1}{4}\int_1^\infty \frac{\sin u}{\sqrt[4]{u}}\,du$$

The left integral converges since the function is continuous in $[0,1]$, and the right integral converges since the integral of $\sin u$ is bound and $\frac{1}{\sqrt[4]{u}}$ monotonically converges to 0, so by Dirichlet, the integral converges.

(3) This integral *converges*. Since at every point:

$$\int_1^x \cos t\,dt = \sin x - \sin 1$$

Which is bound, and since $\frac{1}{x}$ monotonically converges to 0, by Dirichlet, the integral

$$\int_1^\infty \frac{\cos x}{x}\,dx$$

Converges.

(4) This integral *converges*. Doing a wee bit of manipulation we get:

$$\int_1^\infty \frac{\cos\left(x^2\right)}{x}\,dx = \int_1^\infty \frac{x\cdot\cos\left(x^2\right)}{x^2}\,dx$$

We can substitute $u = x^2$, so $du = 2x\,dx$, and we get:

$$= \frac{1}{2}\cdot\int_1^\infty \frac{\cos u}{u}\,du$$

And we showed above that this integral converges.

(5) This integral *diverges*. Since $0 \le |\cos x| \le 1$, $|\cos x| \ge \cos\left(x\right)^2$, and therefore:

$$\frac{|\cos x|}{x} \le \frac{\cos\left(x\right)^2}{x}$$

So I will prove that the improper integral of $\frac{\cos(x)^2}{x}$ from 1 to ∞ diverges, and by comparison, that would mean the integral of $\frac{|\cos x|}{x}$ diverges.
We know:

$$\cos\left(x\right)^2 = \frac{\cos\left(2x\right) - 1}{2}$$

So we get:

$$\int_1^\infty \frac{\cos\left(x\right)^2}{x}\,dx = \int_1^\infty \frac{\cos\left(2x\right)-1}{2x}\,dx$$

By linearity of the integral, this is equal to:

$$= \frac{1}{2}\int_1^\infty \frac{\cos\left(2x\right)}{x}\,dx - \frac{1}{2}\int_1^\infty \frac{dx}{x}$$

The left integral converges, (a substitution of $u = 2x$ yields the integral of $\frac{\cos u}{u}$, which converges as shown above). But the right integral is equal to:

$$\frac{1}{2}\cdot\log x\Big|_1^\infty$$

Which diverges.
So the integral is the sum of a divergent and a convergent integral, and therefore diverges.

(6) This integral *converges*. We can substitute $u = \tan^{-1} x$, and we get that the integral is equal to:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\tan u - u}{u\tan u}\,du$$

This is equal to by linearity:

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{du}{u} - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{du}{\tan u}$$

The left integral converges since its $\frac{1}{u}$ is bound and continuous in the domain.
The right integral is equal to (since the integral is an improper integral of type two, it is the integral between $\frac{\pi}{4}$ and a as a approaches $\frac{\pi}{2}$, and in this domain, the function is equal to $\cot u$).

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot u\,du$$

The function is bound and continuous in the domain and therefore the integral exists.

(7) This integral *converges*. This is because the function satisfies:

$$0 \leq \frac{\tan^{-1}(x)}{\sqrt{x^3+x}} \leq \frac{\pi}{2} \cdot \frac{1}{\sqrt{x^3}}$$

Since $\tan^{-1}$ is positive in this domain. The right integral converges since

$$\int_1^\infty \frac{dx}{\sqrt{x^3}}$$

Converges, as shown in lecture. Therefore the integral converges.

Question 7.2:

For which αs does the following integral converge?

$$\int_1^\infty \frac{\sin(x)^2}{x^\alpha}\,dx$$

If $\alpha > 1$, let $\varepsilon \coloneqq \alpha - 1$. So $\varepsilon > 0$.
We can compare limits between this function and $\frac{1}{x^{1+\frac{\varepsilon}{2}}}$. We get:

$$\frac{\frac{\sin(x)^2}{x^\alpha}}{\frac{1}{x^{1+\frac{\varepsilon}{2}}}} = \frac{\sin(x)^2}{x^{\frac{\varepsilon}{2}}} \longrightarrow 0$$

And since $1 + \frac{\varepsilon}{2} > 1$, the improper integral

$$\int_1^\infty \frac{dx}{x^{1+\frac{\varepsilon}{2}}}$$

Converges.
By the limit comparison test, since this converges, so does our integral.

If $\alpha = 1$, then notice that:

$$\int_1^\infty \frac{\sin(x)^2}{x}\,dx = \frac{1}{2}\int_1^\infty \frac{1}{x}\,dx - \frac{1}{2}\int_1^\infty \frac{\cos(2x)}{x}\,dx$$

Now, note that the right integral converges: a substitution of $u = 2x$ yields:

$$\frac{1}{2}\int_2^\infty \frac{\cos u}{u}\,du$$

Which we proved converges in the previous question.
But the left integral diverges (as proven in the lecture).
So the integral is a combination of a divergent and convergent integral, and therefore diverges.

If $\alpha < 1$, then:

$$x^\alpha \leq x \implies \frac{\sin(x)^2}{x^\alpha} \geq \frac{\sin(x)^2}{x}$$

Since $\sin(x)^2$ is positive.
And since we showed above that the integral of $\frac{\sin(x)^2}{x}$ diverges, so does the integral of $\frac{\sin(x)^2}{x^\alpha}$.

So all in all, the integral converges only if $\alpha > 1$.

Question 7.3:

$f(x)$ is a decreasing function such that

$$\int_0^\infty f(x)\,dx$$

Converges. Prove that

$$\lim_{x\to\infty} f(x) = 0$$

Suppose there exists some x_0 such that $f(x_0) < 0$, then let $\varepsilon = f(x_0) < 0$, and for every $x \geq x_0$ we know $f(x) \leq \varepsilon$ since f is decreasing. And since improper integrals converge if and only if their tails converge, the integral converges if and only if

$$\int_{x_0}^\infty f(x)\,dx \leq \int_{x_0}^\infty \varepsilon$$

And the right integral diverges to $-\infty$ since $\varepsilon < 0$, so so does the left integral, and therefore so does the whole integral. This is a contadiction, so $f(x) \geq 0$.
Since $f(x)$ is decreasing and greater than 0, it must have a limit at infinity. Let

$$\varepsilon \coloneqq \lim_{x\to\infty} f(x) \geq 0$$

Since f is decreasing $f(x) \geq \varepsilon$. So:

$$\int_0^\infty f(x)\,dx \geq \int_0^\infty \varepsilon\,dx$$

If $\varepsilon > 0$ then the right integral diverges (to infinity), and therefore the left integral would diverge. Since we know the integral converges, this means that $\varepsilon = 0$, as required.

Question 7.4:

(1) f is a continuous positive function, and we know that

$$\int_0^\infty f(x)\,dx = \infty$$

Prove that:

$$\int_1^\infty \frac{f(x)}{\int_0^x f(t)\,dt}\,dx = \infty$$

(2) Show that this is not true if

$$\int_0^\infty f(x)\,dx \neq \infty$$

(1) First, we define:

$$\varphi(x) := \int_0^x f(t)\,dt$$

By the fundamental theorem of calculus, this means that $\varphi' = f$ (since f is continuous everywhere). So the integral is equal to:

$$\int_1^\infty \frac{f(x)}{\int_0^x f(t)\,dt}\,dx = \int_1^\infty \frac{\varphi'(x)}{\varphi(x)}\,dx$$

And a substitution of $u = \varphi(x)$ gives $du = \varphi'(x)\,dx$, so this integral is equal to:

$$\int_{\varphi(1)}^{\varphi(\infty)} \frac{1}{u}\,du$$

($\varphi(\infty)$ is the limit of φ at infinity).
Thus the integral is equal to:

$$\log|u|\Big|_{\varphi(1)}^{\varphi(\infty)} = \log|\varphi(\infty)| - \log|\varphi(1)|$$

Now, notice that:

$$\varphi(\infty) = \int_0^\infty f(x)\,dx = \infty$$

So $\log|\varphi(\infty)| = \infty$.
And:

$$\varphi(1) = \int_0^1 f(x)\,dx \geq 0$$

Since f is positive, and so if $\varphi(1) = 0$, then the integral is equal to:

$$\infty - \log|0| = \infty + \infty = \infty$$

And if $\varphi(1) > 0$, then $\log|\varphi(1)|$ is a real number, and the integral is equal to:

$$\infty - \log|\varphi(1)| = \infty$$

(By $\log|\varphi(1)|$, I mean the limit of the logarithm of $\varphi(x)$ as x approaches 1 from the right. This doesn't really matter since log and φ are continuous).
So in any case, the integral is equal to ∞ as required.

(2) So we know from above that the integral is equal to:

$$\log|\varphi(\infty)| - \log|\varphi(1)|$$

So if

$$\int_0^\infty f(x) < \infty$$

Then $\varphi(\infty) < \infty$, so $\log|\varphi(\infty)| < \infty$. But if $\varphi(1) = 0$, then $-\log|\varphi(1)| = \infty$, so this integral can still be ∞. But in any other case (where $\varphi(1) \neq 0$), then the integral cannot be infinite.
Let's look at this integral when $f(x) = \frac{1}{(x+1)^2}$. We know that

$$\int_0^\infty \frac{1}{(x+1)^2}\,dx = \int_1^\infty \frac{1}{x^2}\,dx = 1$$

So $\varphi(\infty) = 1$.
And:

$$\varphi(1) = \int_0^1 \frac{1}{(x+1)^2}\,dx = \int_1^2 \frac{1}{x^2}\,dx = 1 - \frac{1}{2} = \frac{1}{2}$$

So the integral is equal to:

$$\log|\varphi(\infty)| - \log|\varphi(1)| = \log(1) - \log\left(\frac{1}{2}\right) = \log(2)$$

Which is not infinity, as required.