Infintesimal Calculus 3

Assignment 2
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Exercise 2.0.1:

Suppose (X, ρ) is a metric space, show that for any $x_1, \dots, x_n \in X$: $X \setminus \{x_1, \dots, x_n\}$ is open in X.

We will first show that for any $x \in X$, $\{x\}$ is closed. Suppose that $y \in \{x\}'$ then for any $\varepsilon > 0$, x must be in $B_\varepsilon(y)$ since it is the only point in the set. By the definition of a limit point, y cannot equal to x then. But that means $\rho(x, y) < \varepsilon$ for every $\varepsilon > 0$, and therefore $\rho(x, y) = 0$, so $x = y$. Which contradicts the definition of a limit point, so $\{x\}' = \varnothing \subseteq \{x\}$, so $\{x\}$ is closed.
Therefore $\{x_1, \dots, x_n\}$ is also closed as the finite union of closed sets $\{x_1\}, \dots, \{x_n\}$. And so $\{x_1, \dots, x_n\}^c$ is open and since X is open in X, $X \setminus \{x_1, \dots, x_n\} = X \cap \{x_1, \dots, x_n\}^c$ is open as the intersection of two open sets.

Exercise 2.0.2:

Suppose $A, B \subseteq \mathbb{R}^n$ are open under the standard euclidean metric, and $x \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$. Show that the following two sets are also open:

- $x + A = \{x + a \mid a \in A\}$
- $A + \alpha B = \{a + \alpha b \mid a \in A, b \in B\}$

- Suppose $y \in x + A$, then there exists an $a \in A$ such that $y = x + a$. Since A is open, there is an $r > 0$ such that $B_r(a) \subseteq A$. Notice then that if $z \in B_r(y)$, $\|z - y\| < r \implies \|z - x - a\| < r$, so $z - x \in B_r(a) \subseteq A$. So $B_r(y) - x \subseteq A$, so $B_r(y) \subseteq x + A$ and therefore $x + A$ is open.
- Notice that:
$$A + \alpha B = \bigcup_{b \in B} \alpha b + A$$
And as we showed above, for every $b \in B$, $\alpha b + A$ is open. So $A + \alpha B$ is the union of open sets, and is therefore itself open.

Exercise 2.0.3:

Suppose $x, y \in \mathbb{R}^2$ are linearly independent and $(a, b), (c, d) \subseteq \mathbb{R}$ are two open intervals. Show that the set $(a, b)x + (c, d)y = \{tx + sy \mid t \in (a, b), s \in (c, d)\}$ is open.

Notice that the linear transformation $T \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which is characterized by $e_1 \mapsto x$ and $e_2 \mapsto y$ (where $e_1 = (1, 0)$ and $e_2 = (0, 1)$) is bijective since x and y are linearly independent. And since

$$T\big((a, b)e_1 + (c, d)e_2\big) = (a, b)x + (c, d)y$$

if we can prove that for e_1 and e_2 the set is open, then by the following exercise, it must hold for any x and y linearly independent since T is bijective.
Suppose $v \in (a, b)e_1 + (c, d)e_2$ so $v = (v_1, v_2)$ where $v_1 \in (a, b)$ and $v_2 \in (c, d)$. Since (a, b) and (c, d) are open, there must be an $r > 0$ such that $B_r^{\mathbb{R}}(v_1) \subseteq (a, b)$ and $B_r^{\mathbb{R}}(v_2) \subseteq (c, d)$ (these are balls in $\mathbb{R}$). So if we take $u \in B_r^{\mathbb{R}^2}(v)$:

$$\|v - u\| < r \implies (v_1 - u_1)^2 + (v_2 - u_2)^2 < r^2$$

so $(v_1 - u_1)^2 < r^2$ and therefore $|v_1 - u_1| < r$ and so $u_1 \in B_r^{\mathbb{R}}(v_1) \subseteq (a, b)$. And similarly $u_1 \in (c, d)$, so by definition $u \in (a, b)e_1 + (c, d)e_2$, and therefore $B_r^{\mathbb{R}^2}(v)$ is contained in the set, so it is open.

Exercise 2.0.4:

If T is an invertible linear operator over $\mathbb{R}^n$ then T maps open sets to open sets.

This is equivalent with replacing T with an invertible matrix $A \in \mathbb{R}^{n\times n}$.

Firstly we will show that T is bounded in the sense that there is a constant $M > 0$ such that for every $v \in V$:

$$\|Tv\| \leq M \|v\|$$

Suppose $\{e_1, \ldots, e_n\}$ is the standard basis for $\mathbb{R}^n$, let:

$$N = \max_{1\leq i,j\leq n} \{|\langle Te_i, Te_j\rangle|\}$$

Let $v \in \mathbb{R}^n$ and suppose $v = \sum_{i=1}^n a_i e_i$, then

$$\|Tv\| = \left\|\sum_{i=1}^n a_i Te_i\right\| \leq \sum_{i=1}^n |a_i| \|Te_i\| \leq N \cdot \sum_{i=1}^n |a_i|$$

And notice that if we define $\vec{x} = (|a_1|, \ldots, |a_n|)$ and $\vec{y} = (1, \ldots, 1)$, then $\|\vec{x}\| = \|v\|$ and $\|\vec{y}\| = \sqrt{n}$, so by the Cauchy-Schwarz inequality:

$$|\langle x, y\rangle| = \sum_{i=1}^n |a_i| \leq \|\vec{x}\| \cdot \|\vec{y}\| = \sqrt{n}\|v\|$$

So we have that:

$$\|Tv\|_W \leq N\sqrt{n} \cdot \|v\|$$

So $M = N\sqrt{n}$ satisfies our requirement.
Now we will show that if $\mathcal{O} \subseteq W$ is open, then $T^{-1}(\mathcal{O})$ is as well. Suppose $x \in T^{-1}(\mathcal{O})$, then $Tx \in \mathcal{O}$ so there is an $r' > 0$ such that $B_{r'}(Tx) \subseteq \mathcal{O}$. If we let $r = \frac{r'}{M}$, then if $y \in B_r(x)$ then $\|Tx - Ty\| = \|T(x-y)\| < M\|x-y\| < Mr = r'$. So $Ty \in B_{r'}(Tx)$, and so Ty is in $\mathcal{O}$, and therefore $y \in T^{-1}(\mathcal{O})$. Therefore $B_r(x) \subseteq T^{-1}(\mathcal{O})$, and so $T^{-1}(\mathcal{O})$ is open as required.
Notice that these two above claims don't rely on anything about T, not even T's invertibility. But T^{-1} is a linear transformation, so if $\mathcal{O}$ is open then $\left(T^{-1}\right)^{-1}(\mathcal{O}) = T(\mathcal{O})$ is open. So $T(\mathcal{O})$ is open for every open set $\mathcal{O}$ as required.

Exercise 2.0.5:

Suppose X is a metric space and $K \subseteq X$ is compact. $\{F_\lambda\}_{\lambda\in\Lambda}$ is a collection of closed sets whose union is K, and every finite intersection of theirs is non-empty. Then show that:

$$\bigcap_{\lambda\in\Lambda} F_\lambda \neq \varnothing$$

The idea is to think of K itself as a metric space. Then F_λ is closed in K, and so $K \setminus F_\lambda$ is open in K. We will prove this: suppose $k \in K \setminus F_\lambda$. So $k \in F_\lambda^c$ and therefore there is an $r > 0$ such that $B_r^X(x) \subseteq F_\lambda^c$. And since $B_r^K(x) = B_r^X(x) \cap K$ (as a direct result of the definition), we have that $B_r^K(x) \subseteq K \setminus F_\lambda$, so $K \setminus F_\lambda$ is open in K.
So if we assume that the intersection of F_λ is empty, then:

$$K \setminus \bigcap_{\lambda\in\Lambda} F_\lambda = \bigcup_{\lambda\in\Lambda} K \setminus F_\lambda = K$$

So $\{K \setminus F_\lambda\}_{\lambda\in\Lambda}$ is an open cover of K. Since K is compact, there must be a finite subcover of this:

$$\bigcup_{k=1}^n K \setminus F_k = K$$

But we know that

$$\bigcup_{k=1}^n K \setminus F_k = K \setminus \bigcap_{k=1}^n F_k = K$$

And so (since F_λ is a subset of K), it must be that:

$$\bigcap_{k=1}^{n} F_k = \varnothing$$

which contradicts what we're told about every finite intersection of F_λs being non-empty. So the intersection of F_λ is non-empty.

Exercise 2.0.6:

Suppose (X, ρ) is a metric space and $A, B \subseteq X$. Prove or disprove:

- $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$
- $\overline{A \cap B} \supseteq \overline{A} \cap \overline{B}$
- $\operatorname{int}(A \cup B) \subseteq \operatorname{int} A \cup \operatorname{int} B$
- $\operatorname{int}(A \cup B) \supseteq \operatorname{int} A \cup \operatorname{int} B$

- This is true. It is quite trivial to see that if $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$. Since $\overline{B}$ is the smallest closed set containing B, $\overline{B}$ is a closed set containing A and thus is larger than $\overline{A}$ (as in, it contains it). Since $A \cap B \subseteq A, B$, $\overline{A \cap B} \subseteq \overline{A}, \overline{B}$ so $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.
- This is false. Take $A = (0, 1)$ and $B = (1, 2)$. Then $A \cap B = \varnothing$ so $\overline{A \cap B} = \varnothing$ (since the empty set is closed). But $\overline{A} = [0, 1]$ and $\overline{B} = [1, 2]$, so $\overline{A} \cap \overline{B} = \{1\}$.
- This is false. Let $A = (-1, 1) \setminus \{0\}$ and $B = \{0\}$. Then since $\{0\}$ is closed (as we showed earlier), so $(-1, 1)$ and $\{0\}^c$ are open, A is open as the intersection of two open sets, so $\operatorname{int} A = A$. And $\operatorname{int} B = \varnothing$ since 0 can't be in the interior since for every $r > 0$, there exists a non-zero element in $(-r, r)$. So while $\operatorname{int} A \cup \operatorname{int} B = A = (-1, 1) \setminus \{0\}$, $A \cup B = (-1, 1)$ which is open so $\operatorname{int}(A \cup B) = (-1, 1)$.
- This is true. Again, we will show that if $A \subseteq B$ then $\operatorname{int} A \subseteq \operatorname{int} B$. Suppose $x \in \operatorname{int} A$ then there is an $r > 0$ such that $B_r(x) \subseteq A \subseteq B$, so $x \in \operatorname{int} B$. So since $A \cup B \supseteq A, B$, $\operatorname{int}(A \cup B) \supseteq \operatorname{int} A \cup \operatorname{int} B$ as required.

Exercise 2.0.7:

Let $A = \{(0, x_2, \ldots, x_n) \in \mathbb{R}^n\}$, prove:

- A is closed.
- $\operatorname{int} A = \varnothing$

- We will show that A^c is open. Suppose $\vec{x} = (x_1, \ldots, x_n) \in A^c$, so $x_1 \neq 0$. Let $r = |x_1|$, then for any $\vec{y} = (y_1, \ldots, y_n) \in B_r(x_1, \ldots, x_n)$, we have that:

  $$\|(x_1 - y_1, \ldots, x_n - y_n)\|_2^2 = (x_1 - y_1)^2 + \cdots + (x_n - y_n)^2 < r^2 = x_1^2$$

  If we suppose for the sake of a contradiction that $(y_1, \ldots, y_n) \in A$, then $y_1 = 0$, so:

  $$\|\vec{x} - \vec{y}\| = x_1^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2 < x_1^2$$

  which is a contradiction since that implies that

  $$(x_1 - y_2)^2 + \cdots (x_n - y_n)^2 < 0$$

  which is impossible since squares are non-negative. Notice that this is only for $n \geq 2$, if $n = 1$, then $A = \{0\}$ which have already proved is closed. So $\vec{y} \in A^c$ so $B_r(\vec{x}) \subseteq A^c$ and therefore A^c is closed.
- Suppose $\vec{x} = (0, x_2, \ldots, x_n) \in A$ and $r > 0$. Then $\vec{y} = \left(\frac{r}{2}, x_2, \ldots, x_n\right) \in B_r(\vec{x})$ since:

  $$\|\vec{x} - \vec{y}\| = \frac{r}{2} < r$$

but $\vec{y} \notin A$, so for any $\vec{x} \in A$ and $r > 0$, $B_r(\vec{x}) \nsubseteq A$, and therefore $\vec{x} \notin \operatorname{int} A$, so $\operatorname{int} A = \varnothing$.

Exercise 2.0.8:

Definition:

A subset A of a linear space X is **convex** if for every $a, b \in A$ and $\lambda \in [0,1]$, $\lambda a + (1-\lambda)b \in A$. And A is **symmetric** around A if for any $a \in A$, $-a \in A$

Show that if $\varnothing \neq A \subseteq \mathbb{R}^n$ is open, convex, bounded, and symmetric around 0 then the function:

$$\|v\|_A = \inf\left\{k > 0 \;\middle|\; \frac{v}{k} \in A\right\}$$

is a norm over $\mathbb{R}^n$ and $A = B_1(0)$ relative to $\|\cdot\|_A$.

For $v \in \mathbb{R}^n$, let $S_v = \left\{k > 0 \mid \frac{v}{k} \in A\right\}$.
The first step is to show that $\|\cdot\|_A$ is well-defined. To do so we will first show that $0 \in A$, this is true since if $a \in A$ then $-a \in A$, and by convexity for $\lambda = \frac{1}{2}$ we have that:

$$\frac{1}{2} \cdot a - \frac{1}{2} \cdot a = 0 \in A$$

Then we know that there is some $r > 0$ such that $B_r(x) \subseteq A$ since A is open. Then if $0 \neq v \in A$, we know that:

$$\left\|\frac{r}{2\|v\|} \cdot v\right\| = \frac{r}{2}$$

So for $k = \frac{2\|v\|}{r}$, $\frac{v}{k} \in B_r(0) \subseteq A$, and therefore S_v is non-empty and bound from below by 0, and therefore its infimum (and by extension $\|v\|_A$) exists. And for $v = 0$, then $S_v = \mathbb{R}_{<0}$ since $\frac{0}{k} = 0$, so the infimum exists and is 0.
Since $S_v > 0$, $\|v\|_A = \inf S_v \geq 0$, and $\|v\|_A = 0$ if and only if we can construct a sequence of $k_n \in S_v$ such that $k_n \to 0$. But we know that $\frac{v}{k_n} \in A$, and $\left\|\frac{v}{k_n}\right\|_2 = \frac{1}{k_n}\|v\|_2$ is bounded. Since $\frac{1}{k_n} \to \infty$, this must mean that $\|v\|_2 = 0$, and since this is a norm, $v = 0$. And if $v = 0$, for any $k > 0$ we know that $\frac{v}{k} \in A$, so $\inf S_v = 0$. And therefore $\|v\|_A \geq 0$ and $\|v\|_A = 0$ if and only if $v = 0$.
Now we will show that $\|\alpha v\|_A = |\alpha| \cdot \|v\|_A$. This is trivial if $\alpha = 0$ since then $\alpha v = 0$. If $\alpha > 0$ then notice that $S_{\alpha v} = \alpha S_v$ since $\frac{\alpha v}{k} \in A$, then $k' = \frac{k}{\alpha}$ satisfies $\frac{v}{k'} = \frac{\alpha v}{k} \in A$ so $k' \in S_v$, so if $k \in S_{\alpha v}$ then $k = \alpha k'$ for some $k' \in S_v$. And if $k \in S_v$ then $\frac{\alpha v}{\alpha k} = \frac{v}{k} \in A$, so $\alpha k \in S_{\alpha v}$. So $S_{\alpha v} = \alpha S_v$ as required, and therefore

$$\|\alpha v\|_A = \inf S_{\alpha v} = \inf \alpha S_v = \alpha \inf S_v = \alpha \|v\|_A$$

And $S_{-v} = S_v$ since $\frac{-v}{k} \in A$ if and only if $\frac{v}{k} \in A$. So $\|-v\|_A = \|v\|_A$, and therefore if $\alpha < v$ then $\|\alpha v\|_A = \|-\alpha v\|_A = -\alpha\|v\|_A$ as required.
Now we will prove the triangle inequality: suppose $a, b \in \mathbb{R}^n$. We will show that $S_{a+b} \supseteq S_a + S_b$. Suppose $k_1 \in S_a$ and $k_2 \in S_b$, then we will show that $k_1 + k_2 \in S_{a+b}$. This is because:

$$\frac{a+b}{k_1+k_2} = \frac{k_1}{k_1+k_2} \cdot \frac{a}{k_1} + \frac{k_2}{k_1+k_2} \cdot \frac{b}{k_2}$$

And since:

$$\frac{a}{k_1}, \frac{b}{k_2} \in A, \quad \frac{k_1}{k_1+k_2} \in [0,1], \quad 1 - \frac{k_1}{k_1+k-2} = \frac{k-2}{k_1+k_2}$$

by A's convexity, $\frac{a+b}{k_1+k_2} \in A$, and so $k_1 + k_2 \in S_{a+b}$. So for any $k \in S_a + S_b$, $k \in S_{a+b}$ so $S_{a+b} \supseteq S_a + S_b$. And therefore

$$\|a+b\|_A = \inf S_{a+b} \leq \inf S_a + \inf S_b = \|a\|_A + \|b\|_A$$

As required.

Now all that's left is to show that $A = B_1(0)$. Let $v \in A$, if $v = 0$ then $v \in B_1(0)$, otherwise we know that A is open, so there is some $r > 0$ such that $B_r(v) \subseteq A$ (relative to $\left\|\cdot\right\|_2$), and we know that if

$$u = v\left(1 + \frac{r}{2\left\|v\right\|_2}v\right)$$

Then:

$$\left\|v - u\right\|_2 = \left\|\frac{r}{2\left\|v\right\|_2}v\right\|_2 = \frac{r}{2}$$

So $u \in B_r(v) \subseteq A$, and therefore $\dfrac{1}{1 + \frac{r}{2\left\|v\right\|_2}} \geq \left\|v\right\|_A$, and since this is strictly less than 1, $\left\|v\right\|_A < 1$ for any $v \in A$, and therefore $A \subseteq B_1(0)$.
Suppose $v \in B_1(0)$, then $\inf S_V < 1$ so there must be some $k \in S_v$ such that $0 < k < 1$. So $\frac{v}{k} \in A$, so $v = k \cdot a$ for some $a \in A$. And since A is convex and $k \in [0, 1]$:

$$k \cdot a + (1 - k)0 = ka = v \in A$$

(recall that $0 \in A$.) So $B_1(0) \subseteq A$, and therefore $A = B_1(0)$ as required.