Introduction to Rings and Modules

Lecture 14, Friday June 9 2023
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Theorem 14.0.1 (Hilbert's Basis Theorem):

If R is a left (right) noetherian ring, then $R[x]$ is a left (right) noetherian ring as well.

It will be shown in recitation that if R is a UFD then so is $R[x]$.

Proof:

Recall that R is left noetherian if and only if every left ideal is finitely generated. Let I be a left ideal of $R[x]$. For every $n \geq 0$ let us define

$$I_n = \left\{ a \in R \;\middle|\; \exists b_0, \dots, b_{n-1} \in R \colon ax^n + b_{n-1}x^{n-1} + \cdots + b_0 \in I \right\}$$

or in other words, I_n is the set of all leading coefficients on polynomials of degree $\leq n$ in I (if $a \neq 0$ the degree is n). We claim that I_n is a left ideal of R. If $a \in I_n$ and $a' \in I_n$ then suppose $ax^n + b_{n-1}x^{n-1} + \cdots + b_0, a'x^n + b'_{n-1}x^{n-1} + \cdots + b'_0 \in I$ and so their difference is in I, and since the leading coefficient of their difference is $a - a'$, we have that $a - a' \in I_n$. So I_n is a group under addition. If $a \in I_n$ and $b \in R$ then there exists $ax^n + b_{n-1}x^{n-1} + \cdots + b_0 \in I$ and thus $bax^n + bb_{n-1}x^{n-1} + \cdots + b_0 \in I$ since $b \in R[x]$ and I is a left ideal, thus $ab \in I_n$. So I_n is closed under left multiplication of R. Thus I_n is an ideal of R.

Notice that if $a \in I_n$ then we can multiply the polynomial in I whose leading coefficient is a by x to get a polynomial of degree $n+1$ in I, whose leading coefficient is also a. Thus $a \in I_{n+1}$ and in particular I_n is an ascending chain of ideals of R. Thus since R is noetherian, at some point $I_n = I_{n+1}$ for every $n \geq N$. Furthermore since R is noetherian, I_n is finitely generated for every n. For every $n \leq N$ suppose I_n is generated by $a_{n,1}, \dots, a_{n,t_n}$, and for every $a_{n,k}$ we will choose the polynomial

$$f_{n,k} = a_{n,k}x^n + b_{n,k,n-1}x^{n-1} + \cdots + b_{n,k,0} \in I$$

We know claim that $\{f_{n,k} \mid n \leq N,\, k \leq t_n\}$ generates I. Since this set is finite, if we prove this then we have shown that I is finitely generated and therefore R is noetherian.

Let $g \in I$ then

$$g = c_m x^m + \cdots + c_0$$

Suppose g has degree 0, meaning g is constant: $g = c_0$, thus $c_0 \in I_0$. We claim that g is a linear combination of $f_{0,1}, \dots, f_{0,t_0}$. We know that since $f_{n,k} \in I_n$ is of degree $\leq n$, thus $f_{0,i}$ has degree 0 and is therefore constant, in other words $f_{0,i} = a_{0,i}$. Since $c_0 \in I_0$ and $a_{0,i}$ generate I_0, c_0 is a linear combination of $a_{0,i}$s, and since $g = c_0$ and $f_{0,i} = a_{0,i}$, g is a linear combination of $f_{0,i}$s as required.

We make one final subclaim: If $g \in I$ then g is a linear combination of $f_{n,k}$ (with coefficients in $R[x]$). We do this inductively on $m = \deg g$. For $m = 0$, this is simply what we proved above. If $1 \leq m \leq N$ then $c_m \in I_m$ and so

$$c_m = r_{m,1}a_{m,1} + \cdots + r_{m,t_m}a_{m,t_m}$$

for $r_{m,i} \in R$. Since $f_{m,i}$ are polynomials in $I \subseteq R[x]$ with leading coefficients of $a_{m,i}$, the polynomial

$$g' = r_{m,1}f_{m,1} + \cdots + r_{m,t_m}f_{m,t_m}$$

has a leading coefficient of c_m, a degree of m, and is a linear combination of elements of I and thus $g' \in I$. So defining

$$h = g - g'$$

gives a polynomial of degree strictly less than m and is in I. Thus by induction, h is a linear combination of $f_{k,i}$s and since g' is a linear combination of $f_{m,i}$s, we have $g = h + g'$ is a linear combination of $f_{k,i}$s as required.

If $m > N$, then $c_m \in I_m = J_N$. So we have that

$$c_m = r_{N,1}a_{N,1} + \cdots + r_{N,t_N}a_{N,t_N}$$

and so

$$g' = r_{N,1}f_{N,1} + \cdots + r_{N,t_n}f_{N,t_N} \in I$$

and has a leading coefficient of c_m, but a degree of N. Since $m > N$ we can define

$$g'' = x^{m-N} g' = r_{N,1} x^{m-N} f_{N,1} + \cdots + r_{N,t_n} x^{m-N} f_{N,t_N}$$

which has a leading coefficient of c_m and is of degree N, since $g' \in I$ we have $g'' \in I$. We continue as before and define

$$h = g - g''$$

which has degree strictly less than m, and inductively is a linear combination of $f_{k,i}$s and so $g = h + g''$ is a linear combination of $f_{k,i}$s, as required.
This proves that I is finitely generated, and thus $R[x]$ is noetherian, as required. ■

Notice that if R is noetherian, then $R[x_1]$ is noetherian, and so $R[x_1, x_2] = (R[x_1])[x_2]$ is noetherian, and so on for $R[x_1, \dots, x_n]$.
Notice that if F is a field, then the only ideals of F are itself and (0), so it is obviously noetherian. Thus by the above theorem, $F[x_1, \dots, x_n]$ is noetherian as well.
Suppose we define $V \subseteq F^n$ as follows: let $\{f_\lambda\}_{\lambda \in \Lambda}$ be a set of polynomials in $F[x_1, \dots, x_n]$ then we define

$$V = \{(a_1, \dots, a_n) \in F^n \mid \forall \lambda \in \Lambda \colon f_\lambda(a_1, \dots, a_n) = 0\}$$

and let

$$I(V) = \{f \in F[x_1, \dots, x_n] \mid \forall (a_1, \dots, a_n) \in V \colon f(a_1, \dots, a_n) = 0\}$$

This is an ideal of $F[x_1, \dots, x_n]$ since if you multiply $f \in I(V)$ by some other polynomial, it will still be zero when f is. Since F is noetherian, $F[x_1, \dots, x_n]$ is as well and so $I(V)$ is finitely generated, suppose by $g_1, \dots, g_k$. Thus

$$V = \{\vec{a} = (a_1, \dots, a_n) \in F^n \mid g_1(\vec{a}) = \cdots = g_k(\vec{a}) = 0\}$$

since $\vec{a} \in V$ if and only if $f(\vec{a}) = 0$ for all $f \in I(V)$, since if $\vec{a} \in V$ then for every $f \in I(V)$ by definition $f(\vec{a}) = 0$. And if $f(\vec{a}) = 0$ for all $f \in I(V)$ then since $f_\lambda \in I(V)$ for all λ, we have $f_\lambda(\vec{a}) = 0$ and so $\vec{a} = 0$. Since $I(V) = (g_1, \dots, g_k)$, $f(\vec{a}) = 0$ for $f \in I(V)$ if and only if $g_i(\vec{a}) = 0$ for all i.
So every such V can be defined by finitely many polynomials (since recall that Λ may not be finite).