Complex Functions

Lecture 8, Wednesday June 7, 2023

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Theorem 8.1 (Morera's Theorem):

Let f be a continuous function in an open set D, if for every Γ which is a boundary of a rectangle in D, we have

$$\int_\Gamma f\,dz = 0$$

then f is analytic in D.

Proof:

Let $z_0 \in D$ and take an $\varepsilon > 0$ small enough such that $D_\varepsilon(z_0) \subseteq D$, then for every $z \in D_\varepsilon(z_0)$ define

$$F(z) = \int_{z_0}^{z} f(w)\,dw$$

where the path chosen is $[z_0, \mathrm{Re}(z) + i\,\mathrm{Im}(z_0)] + [\mathrm{Re}(z) + i\,\mathrm{Im}(z_0), z]$. We have shown already that $F'(z) = f(z)$ for every z in the disk, and thus F is analytic in the disk. Since the derivative of an analytic function is analytic, f is analytic. ■

Recall the following theorem from Calculus 3:

Theorem 8.2 (Fubini Toneli Theorem):

Suppose $f(x,y)$ is a real function continuous in the rectangle $R = [a_1, b_1] \times [a_2, b_2] \subseteq \overline{\mathbb{R}}^2$ ($\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$), then

$$\int_{a_1}^{b_1}\left(\int_{a_2}^{b_2} |f(x,y)|\,dy\right)dx = \int_{a_2}^{b_2}\left(\int_{a_1}^{b_1} |f(x,y)|\,dx\right)dy = \iint_R |f(x,y)|\,dxdy$$

and if these integrals are finite then

$$\int_{a_1}^{b_1}\left(\int_{a_2}^{b_2} f(x,y)\,dy\right)dx = \int_{a_2}^{b_2}\left(\int_{a_1}^{b_1} f(x,y)\,dx\right)dy = \iint_R f(x,y)\,dxdy$$

So for example let us define

$$f(z) = \int_0^\infty \frac{e^{zt}}{t+1}\,dt$$

for $z = x + iy$ where $x < 0$. It can be shown that f is continuous. Note that

$$|f(z)| \leq \int_0^\infty \frac{|e^{zt}|}{t+1}\,dt = \int_0^\infty \frac{e^{xt}}{t+1}\,dt \leq \int_0^\infty e^{xt}\,dt = \frac{e^{xt}}{x}\Big|_0^\infty = -\frac{1}{x}$$

thus the integral converges absolutely, and thus converges. So f is well-defined.
We will show that f is analytic in its domain D by **Morera's Theorem**. Let Γ be the boundary of a rectangle in D, then

$$\int_\Gamma |f(z)|\,dz = \int_\Gamma\left(\int_0^\infty \frac{e^{xt}}{t+1}\,dt\right)dz \leq \int_\Gamma -\frac{1}{\mathrm{Re}\,z}\,dz$$

which is finite, and thus by **Fubini Toneli Theorem** we can swap the order of integration and have

$$\int_\Gamma f(z)\,dz = \int_0^\infty \int_\Gamma \frac{e^{zt}}{t+1}\,dzdt = \int_0^\infty 0\,dt = 0$$

where the second equality is due to Cauchy's theorem since $\frac{e^{zt}}{t+1}$ is analytic within Γ.
Thus by **Morera's Theorem**, we have shown that f is analytic in its domain.

Proposition 8.3:

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of analytic functions in a domain D. And for every compact $K \subseteq D$,

$$\sup_{x\in K}|f_n - f| \longrightarrow 0$$

for some complex function f on D (ie. $f_n \rightrightarrows f$ on every compact subspace of D), then f is analytic on D.

Proof:

Let $z_0 \in D$ then there exists a compact set K_0 such that $z_0 \in K_0 \subseteq D$ and in K_0 f is the uniform limit of f_n, and since f_n are continuous at z_0, f is continuous at z_0. Thus f is continuous in D. Let Γ be the boundary of some rectangle R in D then since every rectangle is contained within some compact subspace of D, suppose $R \subseteq K_R$ for $K_R \subseteq D$ compact. Then since $f_n \rightrightarrows f$ in K_R, we have

$$\int_\Gamma f(z)\,dz = \int_\Gamma \lim f_n(z)\,dz = \lim \int_\Gamma f_n(z)\,dz = \lim 0 = 0$$

where the third equality is due to Cauchy. Thus by **Morera's Theorem**, f is analytic in D. ■

Proposition 8.4:

Suppose f is continuous in an open set D then suppose L is a line in D. If f is analytic in $D \setminus L$ then f is analytic in D.

Proof:

We will assume that $L \subseteq \mathbb{R}$ since L is of the form $z(t) = (x_0, y_0) + t(a, b)$ for $(a, b) = e^{i\theta}$. Let us look at $e^{-i\theta}(z(t) - x_0 - iy_0) = te^{-i\theta}(a+ib) = t$. Thus we can instead look at $\tilde{f}(z) = f(e^{-i\theta}(z-x_0-iy_0))$ which is just a shift and stretch of f, and thus the transform of L is $\tilde{L} = e^{-i\theta}(L - x_0 - iy_0) \subseteq \mathbb{R}$.
Let $z_0 \in L$. We will show that f is analytic in $D_r(z_0)$ for some $r > 0$. Note that $D_r(z_0) \cap L = (z_0 - r, z_0 + r)$. Let Γ be a rectangle R's boundary in $D_r(z_0)$. If $\varnothing = \overline{R} \cap L$ then $\overline{R} \subseteq D \setminus L$ and so f is analytic in $\overline{R}$ and so by Cauchy $\int_\Gamma f = 0$ by Cauchy.
Otherwise if one of R's edges in on L, let us define a new rectangle R_t with boundary Γ_t, which shares its edges with R other than the edge touching L, which we move away by t units. Since f is continuous on $\overline{R}$, which is compact, f is uniformly continuous on $\overline{R}$. Let

$$\Delta = \int_\Gamma f - \int_{\Gamma_t} f$$

this is the integral of f over the path which goes over the remaining edges of R not removed by R_t plus the reversed edge on R_t parallel to L,

8.1 Branches of the logarithm

We'd like to find a continuous inverse to e^z. But e^z is not surjective, its image is $\mathbb{C} \setminus \{0\}$, and it is periodic: $e^{z+2\pi ik} = e^z$. Thus we'd like to find a function such that

$$e^{\log z} = z$$

for every $0 \neq z \in \mathbb{C}$.
Suppose $\log = u + iv$ then suppose $re^{i\theta}$

$$e^{u(z)+iv(z)} = e^{u(z)}e^{iv(z)} = re^{i\theta}$$

if and only if $e^{u(z)} = r$ and $v(z) - \theta = 2\pi k$. Thus we can define $u(z) = \log|z| = \log r$ and $v(z) = \arg(z) + 2\pi k = \theta + 2\pi k$. If $z \neq 0$ then $r > 0$ so this is well-defined for any choice of k. So all in all we have

$$\log(z) = \log(re^{i\theta}) = \log r + i\theta + i2\pi ik$$

Suppose we have a circle $|z| = R$, parameterized by $\gamma(\theta) = Re^{i\theta}$ then

$$\log(\gamma(0)) = \log R + 2\pi ik$$

Despite this

$$\lim_{\theta\to 2\pi^-} \log(\gamma(\theta)) = \lim(\log R + i\theta + i2\pi ik) = \log R + i2\pi(k+1) \neq \log(\gamma(0))$$

Thus the function we've defined is not continuous on any canonical circle.

Definition 8.1:

We say that $f(z)$ is an **analytic branch** of $\log(z)$ on a domain D if f is analytic on D and for every $z \in D$, $e^{f(z)} = z$.

Lemma 8.2:

If f iz an analytic branch of log on D then for every $k \in \mathbb{Z}$, so is $g(z) = f(z) + 2\pi ik$.

This is trivial.

Theorem 8.3:

Let $D \subseteq \mathbb{C}$ be a simple domain where $0 \notin D$. For every choice of $z_0 \in D$ and choice of $\log(z_0)$ define

$$f(z) = \int_{z_0}^{z} \frac{1}{w}\,dw + \log(z_0)$$

where the integral is integrated over a smooth curve in D, is an analytic branch of log in D.

Proof:

Since $0 \notin D$, $\frac{1}{w}$ is analytic. Since the choice of smooth curve does not matter in simple domains f is well-defined. We have shown that

$$F(z) = \int_{z_0}^{z} \frac{1}{w}\,dw$$

is an antiderivative of $\frac{1}{z}$ in D and thus so is $f(z)$, so f is analytic in D.
Let us define

$$g(z) = ze^{-f(z)}$$

then g's derivative is

$$g'(z) = e^{-f(z)}(1 + z(-f'(z))) = e^{-f(z)}\left(1 - z \cdot \frac{1}{z}\right) = 0$$

and so g is constant, suppose $g \equiv c$. For $z = z_0$ we get that

$$g(z_0) = z_0 e^{-f(z_0)} = z_0 e^{-\log(z_0)} = c$$

if and only if $z_0 = ce^{\log(z_0)}$ if and only if $z_0 = cz_0$ so $c = 1$. Thus $g \equiv 1$ and so

$$ze^{-f(z)} = 1 \implies e^{f(z)} = z$$

for every $z \in D$ as required.

Lemma 8.4:

If f_1 and $f-2$ are two analytic branches of log in D then $f_1(z) - f_2(z) = 2\pi ik$ for every $z \in D$ for some $k \in \mathbb{Z}$.

Proof:

Notice that

$$e^{f_1(z)-f_2(z)} = \frac{e^{f_1(z)}}{e^{f_2(z)}} = \frac{z}{z} = 1$$

Recall $e^u = 1$ if and only if $u = 2\pi ik$ for some $k \in \mathbb{Z}$. ∎

Definition 8.5:

For an analytic function $G\colon V \longrightarrow V$ surjective, we define an analytic branch of G^{-1} on a domain $V_1 \subseteq V$ as an analytic function $g\colon V_1 \longrightarrow V$ such that for every $z \in V_1$, $G(g(z)) = z$.

For example take $G(z) = z^2$ for $V = \mathbb{C}$ and then for an analytic branch of log on a domain $0 \notin D$, f, we can define

$$g(z) = \exp\left(\frac{1}{2}f(z)\right)$$

and this defines a branch of G^{-1}:

$$G(g(z)) = \exp(f(z)) = z$$

Proposition 8.6:

For every simply connected domain D where $0 \notin D$, $\sqrt{z}$ has two branches given by the formula above.

Proof:

Let D be such a domain, and f a branch of log. Since every other branch of log is of the form $f_k = f + 2\pi i k$, we have

$$g_k(z) = \exp\left(\frac{1}{2} f_k(z)\right) = \exp\left(\frac{1}{2} f(z) + \pi i k\right) = \exp\left(\frac{1}{2} f(z)\right) \exp(\pi i k)$$

so all k even give the same g_k and all k odd give the same g_k. ■