Complex Functions

Assignment 10
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Exercise 10.1:

Find the residues of each of the following functions at all of their singularities

(1) $\frac{1}{z^4+z^2}$

(2) $\cot(z)$

(3) $\csc(z)$

(4) $\frac{\exp(1/z^2)}{z-1}$

(5) $\frac{1}{z^2+3z+2}$

(6) $\sin\left(\frac{1}{z}\right)$

(7) $ze^{3/z}$

(8) $\frac{1}{az^2+bz+c}$

(1) This is equal to

$$\frac{1}{z^2}-\frac{1}{z^2+1}=\frac{1}{z^2}+\frac{i}{2}\frac{1}{z-i}-\frac{i}{2}\frac{1}{z+i}$$

Since the singularities of all these functions $(0, i, -i)$ are unique to each function, each function is analytic at the singularity of the other (ie. $\frac{1}{z^2}$ is analytic at $\pm i$, etc) and therefore does not contribute to the residue. Thus the only factor which contributes to the residue at 0 is $\frac{1}{z^2}$, whose residue is 0 (since all of these rational functions are already Laurent series). And the only factor which contributes to the residue at i is $\frac{1}{z-i}$, whose residue is $\frac{i}{2}$. And similarly the residue at $-i$ is $-\frac{i}{2}$. So

$$\operatorname{Res}(f,0)=0,\quad \operatorname{Res}(f,i)=\frac{i}{2},\quad \operatorname{Res}(f,-i)=-\frac{i}{2}$$

(2) Since $\cot(z)=\frac{\cos(z)}{\sin(z)}$, its singularities are when $\sin(z)=0$ ie. $z=\pi k$ for $k\in\mathbb{Z}$. Since these are not zeros of $\cos(z)$, these are simple poles and so if we let $z_k=\pi k$, we the residue of f at z_k is

$$\lim_{z\longrightarrow z_k}(z-z_k)\cot(z)=\lim_{z\longrightarrow z_k}\cos(z)\cdot\frac{z-z_k}{\sin(z)}=\cos(z_k)=\cos(\pi k)=(-1)^k$$

so we have

$$\operatorname{Res}(\cot(z),\pi k)=(-1)^k$$

(3) Similar to before $\csc(z)=\frac{1}{\sin(z)}$ has simple poles at $z_k=\pi k$ and

$$\operatorname{Res}(\csc(z),z_k)=\lim_{z\longrightarrow z_k}(z-z_k)\csc(z_k)=1$$

(4) Last week we showed that the Laurent series of this is

$$\sum_{n=-\infty}^{0}z^n\sum_{k=0}^{-n}a_k,\qquad a_{2k+1}=\frac{1}{k!},\quad a_{2k}=0$$

and so the coefficient of z^{-1} in this series is

$$a_0+a_1=1$$

meaning

$$\text{Res}(f, 0) = 1$$

And since $\exp(1/z^2)$ is analytic at the other singularity, 1, it is a simple pole. So

$$\text{Res}(f, 1) = \lim_{z \to 1} (z-1) \cdot \frac{\exp(1/z^2)}{z-1} = e$$

(5) The roots of the denominator are $z = -1, -2$ and using the solution to (8), we see that

$$\text{Res}(f, -1) = \frac{1}{(-1+2)} = 1, \qquad \text{Res}(f, -2) = \frac{1}{(-2+1)} = -1$$

(6) Using the Taylor expansion of $\sin(z)$ we see that

$$\sin(z) = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{z^{2k+1}}{(2k+1)!} \implies \sin\left(\frac{1}{z}\right) = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{z^{-2k-1}}{(2k+1)!}$$

and so the coefficient of z^{-1} is $(-1)^0 \cdot \frac{1}{1!} = 1$, so

$$\text{Res}(f, 0) = 1$$

(7) Using the Taylor expansion of $\exp(z)$ we see that

$$z \exp\left(\frac{3}{z}\right) = \sum_{k=0}^{\infty} z^{1-k} \frac{3^k}{k!}$$

and so

$$\text{Res}(f, 0) = \frac{9}{2!} = \frac{9}{2}$$

(8) If $az^2 + bz + c$ has two distinct roots $\alpha \neq \beta$ then $az^2 + bz + c = a(z-\alpha)(z-\beta)$ and so

$$f(z) = \frac{1}{a(z-\alpha)(z-\beta)}$$

and so the singularities, α and β, are simple poles and so

$$\text{Res}(f, \alpha) = \lim_{z \to \alpha} (z-\alpha) \cdot f(z) = \lim_{z \to \alpha} \frac{1}{a(z-\beta)} = \frac{1}{a(\alpha-\beta)}$$

and similarly

$$\text{Res}(f, \beta) = \frac{1}{a(\beta-\alpha)}$$

And if the polynomial only has a single root α, then $f(z) = \frac{1}{a(z-\alpha)^2}$ whose residue at α is 0.

Exercise 10.2:

Compute the following integrals

(1) $\int_{|z|=1} \cot(z)\, dz$

(2) $\int_{|z|=2} \frac{dz}{(z-4)(z^3-1)}$

(3) $\int_{|z|=1} \sin\left(\frac{1}{z}\right) dz$

(4) $\int_{|z|=2} ze^{3/z}\, dz$

(1) We know that the singularities of $\cot(z)$ are when $z = \pi k$. Of which there is only $z = 0$ within $|z| < 1$, and so by the residue theorem

$$\int_{|z|=1} \cot(z) = 2\pi i \operatorname{Res}(\cot(z), 0) = 2\pi i$$

(2) The only singularities of this function when $|z| < 4$ are ω_3^k for $k = 0, 1, 2$ ($\omega_3 = \exp\bigl(i \cdot \frac{2\pi}{3}\bigr)$). And since the denominator is equal to

$$(z-4)(z-\omega_3^0)(z-\omega_3^1)(z-\omega_3^2)$$

we get that

$$\operatorname{Res}(f, 1) = -9, \quad \operatorname{Res}\bigl(f, \omega_3^1\bigr) = 9 + 6\sqrt{3}i, \quad \operatorname{Res}\bigl(f, \omega_3^2\bigr) = 9 - 6\sqrt{3}i$$

and so we get that by the residue theorem

$$\int_{|z|=2} f(z)\, dz = 2\pi i(-9 + 9 + 6\sqrt{3}i + 9 - 6\sqrt{3}i) = 18\pi i$$

(3) We saw before that $\operatorname{Res}(f, 0) = 1$ and so the integral is equal to, by the residue theorem,

$$\int_{|z|=1} \sin\left(\frac{1}{z}\right) dz = 2\pi i$$

(4) And here $\operatorname{Res}(f, 0) = 4.5$ and so

$$\int_{|z|=2} z e^{3/z}\, dz = 9\pi i$$

Exercise 10.3:

Suppose f is an entire function and $f(z)$ is real if and only if z is real. Show that f has at most one zero.

Let C be any circle centered about the origin, and let γ be the differentiable function which parameterizes it ($\theta \mapsto re^{i\theta}$). Then we know that

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}$$

is equal to the number of times which $f(z)$ winds around the origin while z traverses C. Now since $f \circ \gamma$ is smooth, if it winds once around C it must cross over the real axis twice. But $f(\gamma(\theta))$ is real only when $\gamma(\theta)$ is real, which is only when $\theta = 0, \pi, 2\pi$. Thus f only crosses the real axis three times, and therefore must winds at most once around C. Thus

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} \leq 1$$

And since this integral is also equal to the number of zeros of f, we have that the number of zeros of f is at most 1.

Exercise 10.4:

Find the number of zeros of f in the domain

(1) $3e^z - z$ in $|z| \leq 1$

(2) $\frac{1}{3}e^z - z$ in $|z| \leq 1$

(3) $z^4 - 5z + 1$ in $1 \leq |z| \leq 2$

(4) $z^6 - 5z^4 + 3z^2 - 1$ in $|z| \leq 1$

(1) Notice that

$$|3e^z| = 3e^x \geq 3e^{-1} > 1 = |-z|$$

and so by Rouché's theorem, $3e^z - z$ has the same number of zeros as $3e^z$ does in the domain, which is no zeros.

(2) Since

$$\left|\frac{1}{3}e^z\right| = \frac{1}{3}e^x \leq \frac{1}{3}e < 1 = |-z|$$

by Rouché's theorem, $\frac{1}{3}e^z - z$ has the same number of zeros as $-z$ does in the domain, which is one.

(3) For $|z| = 2$,

$$|-5z+1| \leq 5|z| + 1 = 11, \qquad \left|z^4\right| = 16$$

and so $|-5z+1| < \left|z^4\right|$, so by Rouché's theorem, in $|z| \leq 2$, $z^4 - 4z + 1$ has the same number of zeros as z^4, which is one.
And for $|z| = r$,

$$|-5z+1| \geq |-5z| - 1 = 5r - 1, \qquad \left|z^4\right| = r^4$$

Since for $r = 1$, $5r - 1 = 4 > 1 = r^4$, since these functions are continuous there exists an $0 < r < 1$ such that $5r - 1 < r^4$. So $\left|z^4\right| < |-5z+1|$ and therefore by Rouché's theorem, in $|z| \leq r < 1$, $z^4 - 4z + 1$ has the same number of zeros as $-5z + 1$ which is one (we can assume $r > \frac{1}{5}$). Thus all the zeros of f in $|z| \leq 2$ are in $|z| \leq r$, meaning they are in $|z| < 1$. So there are no zeros in $1 \leq |z| \leq 2$.

(4) Since

$$\left|z^6 - 5z^4\right| = \left|z^4\right| \cdot \left|z^2 - 5\right| = \left|z^2 - 5\right| \geq 5 - |z|^2 = 4$$

and

$$\left|3z^2 - 1\right| \leq 3|z|^2 + 1 = 4$$

We have that $\left|3z^2 - 1\right| \leq \left|z^6 - 5z^4\right|$ on $|z| = 1$. Notice that $\left|3z^2 - 1\right| = 4$ only if z^2 has the same direction as -1, meaning $z = \pm i$. In this case, $z^2 - 5 = -6$ and so $\left|3z^2 - 1\right| < \left|z^2 - 5z^4\right|$. Thus we have that $\left|3z^2 - 1\right| < \left|z^6 - 5z^4\right|$ on $|z| = 1$, and so by applying Rouché's theorem we get $z^6 - 5z^4 + 3z^2 - 1$ has the same number of zeros in $|z| \leq 1$ as $z^6 - 5z^4$ does. Since $z^6 - 5z^4 = z^4(z^2 - 5)$, and $\pm\sqrt{5}$ is not in $|z| \leq 1$, the function has four zeros (since 0 has multiplicity 4).

Exercise 10.5:

Suppose f is analytic on and within a regular smooth closed contour γ, without zeros in f. Show that if m is a non-negative integer then

$$\frac{1}{2\pi i}\int_\gamma z^m \frac{f'(z)}{f(z)}\,dz = \sum_k z_k^m$$

where the sum is done over the zeros of f.

We will show that for $g(z)$ entire,

$$\frac{1}{2\pi i}\int_\gamma g(z) \cdot \frac{f'(z)}{f(z)}\,dz = \sum_k g(z_k)$$

and so if $g(z) = z^m$ (which is entire), we get the desired result.
Let us denote $F(z) = g(z) \cdot \frac{f'(z)}{f(z)}$. By the residue theorem, we have that

$$\frac{1}{2\pi i}\int_\gamma g(z) \cdot \frac{f'(z)}{f(z)} = \sum_k \operatorname{Res}(F(z), z_k)$$

Since the singularities of $F(z) = g(z) \cdot \frac{f'(z)}{f(z)}$ are the zeros of $f(z)$ since it is analytic. Suppose α is a zero of degree k, then there exists a function which is analytic in γ and non-zero at α, h, such that

$$f(z) = (z-\alpha)^k h(z)$$

then

$$f'(z) = k(z-\alpha)^{k-1}h(z) + (z-\alpha)^k h'(z)$$

and so

$$\frac{f'(z)}{f(z)} = \frac{k}{z-\alpha} + \frac{h'(z)}{h(z)}$$

And so

$$F(z) = g(z) \cdot \frac{k}{z-\alpha} + g(z) \cdot \frac{h'(z)}{h(z)}$$

Since $h(\alpha) \neq 0$, $g(z) \cdot \frac{h'(z)}{h(z)}$ is analytic about α and therefore does not contribute to the residue of $F(z)$ at α. So

$$\text{Res}(F, \alpha) = \text{Res}\left(g(z) \cdot \frac{k}{z-\alpha}, \alpha\right)$$

Now we make the general claim that if g is analytic at α then

$$\text{Res}\left(\frac{g(z)z-\alpha}{,}\alpha\right) = g(\alpha)$$

this is since g has a Taylor series about α,

$$g(z) = \sum_{k=0}^{\infty} c_k (z-\alpha)^k$$

and so

$$\frac{g(z)}{z-\alpha} = \sum_{k=0}^{\infty} c_k (z-\alpha)^{k-1}$$

and so $\text{Res}\left(\frac{g(z)}{z-\alpha}\right) = c_0$, and recall that $c_0 = g(\alpha)$ as required.
Thus we have that

$$\text{Res}(F, \alpha) = k \cdot g(\alpha)$$

Where k is the multiplicity of α.
Thus

$$\frac{1}{2\pi i} \int_\gamma F(z)\, dz = \sum_\alpha \text{Res}(F, \alpha) = \sum_\alpha k \cdot g(\alpha)$$

as required (since the sum of $g(z_k)$ will sum z_k as per its multiplicity).

Exercise 10.6:

Show that for every $R > 0$, there exists an n large enough such that

$$P_n(z) = \sum_{k=0}^{n} \frac{z^k}{k!}$$

has no zeros in $|z| \leq R$.

Since P_n is the partial sum of the powerseries of $\exp(z)$, we have $P_n(z) \xrightarrow[n\to\infty]{} \exp(z)$. Thus there exists an n where $|\exp(z) - P_n(z)| < e^{-R}$ for all $|z| \leq R$. Since $|\exp(z)| = e^x \geq e^{-R}$ we have that

$$|P_n(z) - \exp(z)| < e^{-R} \leq |\exp(z)|$$

and so $P_n(z) - \exp(z) + \exp(z) = P_n(z)$ has the same number of zeros on $|z| \leq R$ as $\exp(z)$, which is none. Meaning $P_n(z)$ has no zeros on $|z| \leq R$.

Exercise 10.7:

Prove the fundamental theorem of algebra.

Let

$$p(z) = \sum_{n=0}^{N} a_n z^n$$

For some $N \geq 1$, our goal is to prove $p(z)$ has a zero. Then let

$$g(z) = \sum_{n=0}^{N-1} a_n z^n$$

Then notice that

$$\left|\frac{g(z)}{z^N}\right| = \left|\sum_{n=0}^{N-1} a_n z^{n-N}\right| \leq \sum_{n=0}^{N-1} |a_n||z|^{n-N}$$

and since for every $0 \leq n < N$, $|z|^{n-N} \xrightarrow[z\to\infty]{} 0$, we have that $\left|\frac{g(z)}{z^N}\right| \xrightarrow[z\to\infty]{} 0$. So we can take an $R > 0$ arbitrarily large such that when $|z| = R$,

$$\left|\frac{g(z)}{z^N}\right| < 1$$

and so

$$|g(z)| < \left|z^N\right|$$

on $|z| = R$. So by Rouché's theorem, we have that $P_n(z) = g(z) + z^N$ has the same number of zeros in $|z| \leq R$ as z^N does. Since z^N has N zeros (0 with multiplicity N), that means $P_n(z)$ has $N \geq 1$ zeros in $|z| \leq R$, as required.