Topology

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Proposition 9.0.1:

If $X = \prod_{\lambda\in\Lambda} X_\lambda$ is a product topology, and $f_\lambda\colon X_\lambda \longrightarrow Y_\lambda$ be functions, then let $Y = \prod_{\lambda\in\Lambda} Y_\lambda$ and

$$f\colon X \longrightarrow Y, \qquad (x_\lambda)_\Lambda \mapsto (f_\lambda(x_\lambda))_\Lambda$$

then

(1) f is continuous if and only if each f_λ is continuous.

(2) If f is open then each f_λ is open.

(3) If f_λ are all surjective, or Λ is finite, then f is open if and only if each f_λ is open.

(4) f is a homeomorphism if and only if each f_λ is a homeomorphism.

Proof:

Suppose f_λ are continuous. Let $\mathcal{U} = \prod_{\lambda\in\Lambda} \mathcal{U}_\lambda$ be an element of the standard basis of the product topology Y, then

$$(x_\lambda)_\Lambda \in f^{-1}(\mathcal{U}) \iff (f_\lambda(x_\lambda))_\Lambda \in \prod_{\lambda\in\Lambda} \mathcal{U}_\lambda$$

which is if and only if $f_\lambda(x_\lambda) \in \mathcal{U}_\lambda$ for each $\lambda \in \Lambda$, and so

$$f^{-1}(\mathcal{U}) = \prod_{\lambda\in\Lambda} f_\lambda^{-1}(\mathcal{U}_\lambda)$$

and since $f_\lambda^{-1}(\mathcal{U}_\lambda)$ is open in X and since all but a finite number of $\mathcal{U}_\lambda \neq Y_\lambda$, so all but a finite number of $f_\lambda^{-1}(\mathcal{U}_\lambda) \neq X_\lambda$, meaning $f^{-1}(\mathcal{U})$ is an element of the basis of the product topology X, so it is open as required.
Now suppose f is continuous, then let $\mathcal{V}_\lambda$ be open in Y_λ, then we must show $f_\lambda^{-1}(\mathcal{V}_\lambda)$ is open in X_λ. If we take the open set $\mathcal{V}$ in Y which is the product of Y_γ with $\mathcal{V}_\lambda$ in the λth index then we get that $f^{-1}(\mathcal{V})$ is equal to the product of X_γ with $f_\lambda^{-1}(\mathcal{V}_\lambda)$. Since $f^{-1}(\mathcal{V})$ is open, $\pi_\lambda(f^{-1}(\mathcal{V})) = f_\lambda^{-1}(\mathcal{V}_\lambda)$ is open as required.
Now if f is open, let $\mathcal{U}_\lambda$ be open in X_λ, and let $\mathcal{U}$ be the product of X_γ with $\mathcal{U}_\lambda$ then $\mathcal{U}$ is open in X. So $f(\mathcal{U})$ is open and so $\pi_\lambda(f(\mathcal{U})) = f_\lambda(\mathcal{U}_\lambda)$ is open, so f_λ is open.
And if f_λ are all open and surjective or Λ is finite, then let $\mathcal{U} = \prod_\Lambda \mathcal{U}_\lambda$ be open in X then

$$f(\mathcal{U}) = \prod_\Lambda f_\lambda(\mathcal{U}_\lambda)$$

is open in Y (all but a finite number of $f_\lambda(\mathcal{U}_\lambda) \neq Y_\lambda$). So f is open.
Now suppose f is a homeomorphism, then f is necessarily bijective and so each f_λ must be bijective as well. If $f_\lambda(x_\lambda) = f_\lambda(y_\lambda)$ then if we take a $x \in X$ and $y \in X$ which are equal except at the λth coefficient, for which x's is x_λ and y's is y_λ, then we have by definition $f(x) = f(y)$ so $x = y$ meaning $x_\lambda = y_\lambda$. And if $y_\lambda \in Y_\lambda$, then the sequence y whose λth coefficient is y_λ has an origin in X, and so if x_λ is the λth coefficient in y's origin, then by definition $f_\lambda(x_\lambda) = y_\lambda$, so f_λ are all bijective. By above, f_λ are all continuous and open bijective mappings, meaning they are homeomorphisms.
And if f_λ are all homeomorphisms, then f is also bijective and open and continuous and is therefore also a homeomorphism. ∎

Proposition 9.0.2:

Similarly if $f_\lambda\colon X \longrightarrow Y_\lambda$ is continuous, so is

$$f\colon X \longrightarrow \prod_\Lambda Y_\lambda, \qquad f(x) = (f_\lambda(x))_{\lambda\in\Lambda}$$

Proof:

This is because if $\prod_\Lambda \mathcal{U}_\lambda$ is in the basis of the product topology Y, then

$$x \in f^{-1}\left(\prod_\Lambda \mathcal{U}_\lambda\right) \iff (f_\lambda(x))_\Lambda \in \prod_\Lambda \mathcal{U}_\lambda \iff f_\lambda(x) \in \mathcal{U}_\lambda \iff x \in f^{-1}(\mathcal{U}_\lambda)$$

for each $\lambda \in \Lambda$. So

$$f^{-1}\Big(\prod_\Lambda \mathcal{U}_\lambda\Big) = \bigcap_\Lambda \mathcal{U}_\lambda$$

Since all but a finite of λ satisfy $f^{-1}(\mathcal{U}_\lambda) = X_\lambda$, this is a finite intersection, so it is open. ■

Proposition 9.0.3:

Let $X = \prod_{\lambda\in\Lambda} X_\lambda$ be a product topology, then if every X_λ is path connected then so is X.

Proof:

Let $(x_\lambda)_\Lambda, (y_\lambda)_\Lambda \in X$, then for every $\lambda \in \Lambda$ there exists a curve

$$\gamma_\lambda\colon [0,1] \longrightarrow X_\lambda, \quad \gamma_\lambda(0) = x_\lambda,\ \gamma_\lambda(1) = y_\lambda$$

Then let us define

$$\gamma\colon [0,1] \longrightarrow X, \qquad \gamma(t) = (\gamma_\lambda(t))_\Lambda$$

This is continuous since its components are. And it satisfies $\gamma(0) = (\gamma_\lambda(0)) = (x_\lambda)_\Lambda$ and similarly $\gamma(1) = (y_\lambda)_\Lambda$. ■

Proposition 9.0.4:

If X_λ are all connected, so is $X = \prod_\Lambda X_\lambda$.

Proof:

First we show this for the finite case, when $X = X_1 \times X_2$. Let $(a,b) \in X$ then $\{a\} \times Y \cong Y$ and $X \times \{b\} \cong X$ so these are both connected. And $(\{a\} \times Y) \cap (X \times \{b\}) = \{(a,b)\} \neq \varnothing$ and so $(\{a\} \times Y) \cup (X \times \{b\}) = X \times Y$ is connected as the non-disjoint union of two connected spaces. Therefore by induction $X_1 \times \cdots \times X_n$ is connected.

Let $(q_\lambda)_\Lambda \in X$ and let $F \subseteq \Lambda$ be finite, let

$$Q_F = \prod_\Lambda G_\lambda, \qquad G_\lambda = \begin{cases} X_\lambda & \lambda \in F \\ \{q_\lambda\} & \lambda \notin F \end{cases}$$

Then $Q_F \cong \prod_{f\in F} X_f$, so Q_F is connected. Let

$$Y = \bigcup_{F\subseteq I \text{ finite}} Q_F$$

and if $a, b \in Y$ then $a \in Q_{F_1}$ and $b \in Q_{F_2}$ so $a, b \in Q_{F_1\cup F_2}$ and so every two points in Y are contained within a connected subspace, and therefore Y is connected.

We now claim that Y is dense in X. Suppose $\mathcal{U} = \prod_\Lambda \mathcal{U}_\lambda$ is in the basis of X, suppose $F = \{\lambda_1, \ldots, \lambda_n\}$ is the set of indexes for which $\mathcal{U}_{\lambda_i} \neq X_{\lambda_i}$. Then we claim that $\mathcal{U} \cap Q_F \neq \varnothing$. This is equal to

$$\prod_\Lambda \mathcal{U}_\lambda \cap G_\lambda$$

So for $\lambda \in F$, $G_\lambda = X_\lambda$ otherwise $G_\lambda = \{q_\lambda\}$ and $\mathcal{U}_\lambda = X_\lambda$ so $\mathcal{U}_\lambda \cap G_\lambda$ is non-empty for every $\lambda \in \Lambda$ (either $\mathcal{U}_\lambda$ or $\{q_\lambda\}$). And so $\mathcal{U} \cap Q_F \neq \varnothing$ as required. ■

9.1 Tychonoff's Theorem

Lemma 9.1.1 (Tube Lemma):

Suppose X and Y are topological spaces, Y is compact, and $a \in X$. Then for every neighborhood of $\{a\} \times Y \subseteq \mathcal{O}$, there exists an open set $\mathcal{U} \subseteq X$ such that

$$\{a\} \times Y \subseteq \mathcal{U} \times Y \subseteq \mathcal{O}$$

Proof:

Recall that the basis of $X \times Y$ is the set of rectangles $\mathcal{U} \times \mathcal{V}$ for $\mathcal{U}$ and $\mathcal{V}$ open in $X \times Y$. So $\mathcal{O}$ is a union of sets of this form, and since for every $y \in Y$, $(a, y) \in \mathcal{O}$ and so there exists $a \in \mathcal{U}_y$ and $y \in \mathcal{V}_y$ open such that

$$(a, y) \in \mathcal{U}_y \times \mathcal{V}_y \subseteq \mathcal{O}$$

Then $\{\mathcal{V}_y\}_{y \in Y}$ is an open cover of Y and so there is a finite subcover $\{\mathcal{V}_{y_i}\}_{i=1}^n$, and so let us define

$$\mathcal{U} = \mathcal{U}_{y_1} \cap \cdots \cap \mathcal{U}_{y_n}$$

Then $a \in \mathcal{U}$ is an open neighborhood of a, and since

$$\mathcal{U} \times Y = \bigcup_{i=1}^n \mathcal{U} \times \mathcal{V}_{y_i}$$

and since $\mathcal{U} \times \mathcal{V}_{y_i} \subseteq \mathcal{U}_{y_i} \times \mathcal{V}_{y_i} \subseteq \mathcal{O}$ and so

$$\{a\} \times Y \subseteq \mathcal{U} \times Y \subseteq \mathcal{O}$$

as required. ■

Definition 9.1.2:

If X is a set and $B \subseteq \mathcal{P}(X)$, let τ_B be the smallest topology on X which contains B. This is well-defined since the arbitrary intersection of topologies is a topology, so we can take

$$\tau_B = \bigcap \{\tau \mid B \subseteq \tau \text{ is a topology on } X\}$$

τ_B is called the **topology generated by** B.

Notice that if $X \in B$ and B is closed under intersections then τ_B as defined previously is equal to the τ_B defined above (sicne τ_B is a topology and obviously any topology containing B must contain τ_B).
If we define $B^\cap$ to be the set of all finite unions of elements of B, then

$$\tau_B = \tau_{B^\cap \cup \{X\}}$$

this is because obviously any topology which contains B must contain $B^\cap \cup \{X\}$ and vice versa, and so the topology generated by B is equal to the topology generated by $B^\cap \cup \{X\}$. Since $B^\cap \cup \{X\}$ contains X and is closed under intersections, τ_B is equal to the union of finite intersections of elements in B and X. Thus if B is a subbasis, $B^\cap \cup \{X\}$ is a basis of the topology.

Definition 9.1.3:

If (X, τ) is a topological space, then $B \subseteq \tau$ is a **subasis** of τ if the topology generated by B is τ, and X is the union of elements in B.

This is equivalent to saying that $\tau_{B^\cap} = \tau$ in the sense of the previous lecture (every element of τ can be written as the union of elements in $B^\cap$). Or equivalently, $B^\cap$ is a basis of τ.
Note if B is a basis of τ, then $\tau_B = \tau$ and so B is a subasis.

Lemma 9.1.4 (Alexander Subbase Theorem):

If X is a topological space and B is a subasis, then X is compact if and only if for every open cover of X $\mathcal{C} = \{\mathcal{U}_\lambda\}_{\lambda\in\Lambda} \subseteq B$, there exists a finite subcover.

Proof:

If X is compact, this is obvious. To show the converse, suppose X is not compact. Let S be the set of all open covers of X which have no finite subcover, and so by assumption $S \neq \varnothing$. So S is partially ordered by inclusion, we will use Zorn's Lemma to show that S contains a maximal element.
Let $\{\mathcal{C}^\gamma\}_{\gamma\in\Gamma}$ be a chain of covers in S, then we claim that $\mathcal{C} = \bigcup_{\gamma\in\Gamma} \mathcal{C}^\gamma$ is in S. $\mathcal{C}$ obviously covers X (since it is a superset of an open cover). But if $\mathcal{C}$ had a finite subcover, then since $\{\mathcal{C}^\gamma\}_{\gamma\in\Gamma}$ forms a chain, this finite subcover is contained entirely within some $\mathcal{C}^\gamma$ and so $\mathcal{C}^\gamma$ has a finite subcover, which contradicts it being in S. Thus every chain has an upper bound in S and therefore S has a maximal element.
Suppose $\mathcal{C} \in S$ is a maximal element. Since $\mathcal{C}$ is maximal, if $\mathcal{U} \notin \mathcal{C}$ then $\mathcal{C} \cup \{\mathcal{U}\} \notin S$ and so $\mathcal{C} \cup \{\mathcal{U}\}$ has a finite subcover, which is of the form $\mathcal{C}_\mathcal{U} \cup \{\mathcal{U}\}$ for some finite subset $\mathcal{C}_\mathcal{U}$ of $\mathcal{C}$. But $\mathcal{C} \cap B$ cannot cover X as if it did, since $\mathcal{C} \cap B \subseteq B$, by our assumption in the lemma, $\mathcal{C} \cap B$ and in particular $\mathcal{C}$ would have a finite subcover. Thus there exists a $x \in X$ which is not covered by $\mathcal{C} \cap B$, but there exists a $\mathcal{V} \in \mathcal{C}$ such that $x \in \mathcal{V}$, and s
Since $B^\cap$ is a basis there exists a $\mathcal{O} \in B^\cap$ such that $x \in \mathcal{O} \subseteq \mathcal{V}$, and $\mathcal{O} = \mathcal{O}_1 \cap \cdots \mathcal{O}_n$ for $\mathcal{O}_i \in B$, so

$$x \in \mathcal{O}_1 \cap \cdots \cap \mathcal{O}_n \subseteq \mathcal{V}$$

But $\mathcal{O}_i \notin \mathcal{C}$, since then $\mathcal{C}$ would cover x, and so by above, there exist $\mathcal{C}_i = \mathcal{C}_{\mathcal{O}_i} \subset \mathcal{C}$ finite such that $\mathcal{C}_i \cup \{\mathcal{O}_i\}$ is a finite cover of X. Thus if we denote $\mathcal{U}_i = \bigcup \mathcal{C}_i$, then $\mathcal{U}_i \cup \mathcal{O}_i = X$ and so

$$X = \bigcap_{i=1}^n \mathcal{U}_i \cup \mathcal{O}_i \subseteq \bigcup_{i=1}^n \mathcal{U}_i \cup \bigcap_{i=1}^n \mathcal{O}_i \subseteq \bigcup_{i=1}^n \mathcal{U}_i \cup \mathcal{V}$$

But this means X is equal to a finite union of elements of $\mathcal{C}$ ($\mathcal{U}_i$ is the union of $\mathcal{C}_i$ which is finite), in contradiction to $\mathcal{C}$ not having a finite subcover. ■

Theorem 9.1.5 (Tychonoff Theorem):

If $\{X_\lambda\}_{\lambda\in\Lambda}$ is a family of compact topological spaces, then $X = \prod_{\lambda\in\Lambda} X_\lambda$ is compact if and only if every X_λ is compact.

Proof:

If X is compact, then since π_λ is continuous, $\pi_\lambda(X) = X_\lambda$ is compact as well. To show the converse, let

$$B = \left\{\pi_\lambda^{-1}(\mathcal{U}_\lambda) \,\middle|\, \mathcal{U}_\lambda \in \tau_\lambda,\, \lambda \in \Lambda\right\}$$

this is the standard subasis of the product topology (since $B^\cap$ is the standard basis of the product topology). Let us assume that X is not compact, then there exists $\mathcal{C} \subseteq B$, an open cover of X without a finite subcover.
For $\lambda \in \Lambda$, let $\mathcal{C}_\lambda$ be the set of all $\pi_\lambda^{-1}(\mathcal{U}_\lambda) \in \mathcal{C}$ (the set of all elementary prisms of the coefficient X_λ in $\mathcal{C}$). So $\mathcal{C} = \bigcup_{\lambda\in\Lambda} \mathcal{C}_\lambda$. Then for every $\lambda \in \Lambda$, $\pi_\lambda(\mathcal{C}_\lambda)$ contains no finite subcover of X_λ, since if $\{\mathcal{U}_n\}_{n=1}^N \subseteq \pi_\lambda(\mathcal{C}_\lambda)$ is a finite subcover of X_λ then

$$\bigcup_{n=1}^N \pi_\lambda^{-1}(\mathcal{U}_n) = X$$

(since $\pi_\lambda^{-1}(\mathcal{U}_n)$ is the vector whose λth coefficient is $\mathcal{U}_n$ and all other are X_γ). And so $\mathcal{C}_\lambda \subseteq \mathcal{C}$ would have a finite subcover.
But since X_λ is compact, $\pi_\lambda(\mathcal{C}_\lambda)$ cannot cover X, and so there exists a $x_\lambda \in X_\lambda$ not covered by $\pi_\lambda(\mathcal{C}_\lambda)$. Then $x = (x_\lambda)_{\lambda\in\Lambda} \in X$ is not covered by $\mathcal{C}$ in contradiction. ■

Since

$$\prod_{\lambda\in\Lambda} X = X^\Lambda$$

we have that X^Λ is compact if and only if X is compact for any set Λ. For example $[0,1]^S$ are called *Tychonoff cubes*, and if $S = \mathbb{N}$ it is called a *Hilbert cube*.

9.2 Disjoint Unions

Definition 9.2.1:

Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of topological spaces, then we can ensure they are disjoint by replacing them with $X_\lambda \times \{\lambda\}$ which is homeomorphic with X_λ, then we define

$$\coprod_{\lambda\in\Lambda} X_\lambda = \bigcup_{\lambda\in\Lambda} X_\lambda$$

with the topology

$$\tau = \left\{ \bigcup_{\lambda\in\Lambda} \mathcal{U}_\lambda \,\middle|\, \mathcal{U}_\lambda \text{ is open in } X_\lambda \right\}$$

This is a topology since obviously $\varnothing, \coprod_\Lambda X_\lambda \in \tau$ and

$$\left(\bigcup \mathcal{U}_\lambda\right) \cap \left(\bigcup \mathcal{V}_\lambda\right) = \bigcup \mathcal{U}_\lambda \cap \mathcal{V}_\lambda$$

and

$$\bigcup_{\gamma\in\Gamma} \bigcup_{\lambda\in\Lambda} \mathcal{U}_\lambda^\gamma = \bigcup_{\lambda\in\Lambda} \bigcup_{\gamma\in\Gamma} \mathcal{U}_\lambda^\gamma$$

Since X_λ are all disjoint, X_λ's topology is equal to its subspace topology as a subspace of $\coprod_{\lambda\in\Lambda} X_\lambda$.
Notice that if we define ι_λ as the inclusion function from X_λ to $\coprod_\Lambda X_\lambda$ (which is continuous as the inclusion function from a subspace), then a function

$$f\colon \coprod_\Lambda X_\lambda \longrightarrow Y$$

is continuous if and only if $f \circ \iota_\lambda \colon X_\lambda \longrightarrow Y$ is continuous. If f is continuous, this is obvious. For the converse, this is because then f is continuous over every X_λ which form an open cover of the disjoint union.

9.3 Quotient Spaces

Definition 9.3.1:

If (X,τ) is a topological space and $q\colon X \longrightarrow Y$ is a surjective function, then σ is a **quotient topology of Y** if

(1) $q\colon (X,\tau) \longrightarrow (Y,\sigma)$ is continuous

(2) If $q\colon (X,\tau) \longrightarrow (Y,\gamma)$ is continuous then $\gamma \subseteq \sigma$ (σ is the finest topology which makes q surjective).

We call q the **quotient mapping**.

Proposition 9.3.2:

$\sigma = \left\{\mathcal{U} \subseteq Y \,\middle|\, q^{-1}(\mathcal{U}) \in \tau\right\}$

Proof:

Let $\sigma' = \left\{\mathcal{U} \subseteq Y \,\middle|\, q^{-1}(\mathcal{U}) \in \tau\right\}$. Obviously since σ makes q continuous, if $\mathcal{U}$ is open then $q^{-1}(\mathcal{U}) \in \tau$ so $\sigma \subseteq \sigma'$. So now we show that σ' is a topology, and thus $\sigma' \subseteq \sigma$, meaning $\sigma = \sigma'$. Firstly, $q^{-1}(Y) = X$ and $q^{-1}(\varnothing) = \varnothing$ and so $Y, \varnothing \in \sigma'$. If $\mathcal{U}, \mathcal{V} \in \sigma'$ then

$$q^{-1}(\mathcal{U} \cap \mathcal{V}) = q^{-1}(\mathcal{U}) \cap q^{-1}(\mathcal{V}) \in \tau$$

so $\mathcal{U} \cap \mathcal{V} \in \sigma'$. And if $\{\mathcal{U}_\lambda\}_\Lambda \subseteq \sigma'$ then

$$q^{-1}\left(\bigcup_{\lambda\in\Lambda} \mathcal{U}_\lambda\right) = \bigcup_{\lambda\in\Lambda} q^{-1}(\mathcal{U}_\lambda) \in \tau$$

so $\bigcup_\Lambda \mathcal{U}_\lambda \in \tau$, thus σ' is a topology as required. ■

Thus q is a quotient map if and only if q is surjective and for every $\mathcal{U} \subseteq Y$, $\mathcal{U}$ is open if and only if $q^{-1}(\mathcal{U})$ is open in X.

Proposition 9.3.3:

Suppose $q \circ X \longrightarrow Y$ is a quotient map and $f\colon Y \longrightarrow Z$, then f is a quotient map if and only if $f \circ q$ is a quotient map.

Proof:

If f is a quotient map then $f \circ q$ is surjective as the composition of surjective functions. We must show that $\mathcal{U} \subseteq Y$ is open if and only if $(f \circ q)^{-1}(\mathcal{U})$ is open. $\mathcal{U}$ is open if and only if $f^{-1}(\mathcal{U})$ is open, and since q is a quotient map this is if and only if $q^{-1}(f^{-1}(\mathcal{U})) = (f \circ q)^{-1}(\mathcal{U})$ is open as required.
And if $f \circ q$ is surjective, so is f. And so we must show that $\mathcal{U} \subseteq Y$ is open if and only if $f^{-1}(\mathcal{U})$ is. $\mathcal{U}$ is open if and only if $(f \circ q)^{-1}(\mathcal{U}) = q^{-1}(f^{-1}(\mathcal{U}))$ is open and $q^{-1}(\mathcal{V})$ is open if and only if $\mathcal{V}$ is open, so this is open if and only if $f^{-1}(\mathcal{U})$ is open.

Proposition 9.3.4:

Suppose $q \circ X \longrightarrow Y$ is a quotient map and $f\colon Y \longrightarrow Z$, then f is a continuous function if and only if $f \circ q$ is continuous.

Proof:

Obviously if f and q are continuous, so is $f \circ q$. To show the converse, let $\mathcal{U} \subseteq Z$ is open then we must show $f^{-1}(\mathcal{U})$ is open in Y, but this is equivalent to $q^{-1}(f^{-1}(\mathcal{U}))$ being open in X, and since

$$q^{-1}(f^{-1}(\mathcal{U})) = (q \circ f)^{-1}(\mathcal{U})$$

which is open, this is indeed true. ■

Proposition 9.3.5:

If $q\colon X \longrightarrow Y$ is surjective, continuous, and open (or closed) then it is a quotient map.

Proof:

We must show $\mathcal{U} \subseteq Y$ is open if and only if $q^{-1}(\mathcal{U})$ is open. If $\mathcal{U}$ is open then since q is continuous, $q^{-1}(\mathcal{U})$ is open. And if $q^{-1}(\mathcal{U})$ is open then $q(q^{-1}(\mathcal{U})) = \mathcal{U}$ since q is surjective, and since q is open it is open as well. ■

Thus every projective function $\pi_\lambda\colon \prod_\Lambda X_\lambda$ is a quotient map.

Proposition 9.3.6:

If $f\colon X \longrightarrow Y$ is bijective and continuous, then f is a quotient map if and only if f is a homeomorphism.

Proof:

If f is a homeomorphism, then by above it is a quotient map. Otherwise, f is a quotient map, let us show that f is an open mapping. Suppose $\mathcal{U}$ is open in X then $f(\mathcal{U})$ is an open in Y if and only if $f^{-1}(f(\mathcal{U})) = \mathcal{U}$ is open in X, which is true. So f is an open, continuous, bijective map and is therefore a homeomorphism. ■

Proposition 9.3.7:

If $q\colon X \longrightarrow Y$ is continuous and $A \subseteq X$ such that $q\big|_A\colon A \longrightarrow Y$ is a quotient map, then q is a quotient map.

Proof:

q is surjective since its restriction is. We must show that $\mathcal{U} \subseteq Y$ is open if and only if $q^{-1}(\mathcal{U})$ is. If $\mathcal{U}$ is open, then

$q^{-1}(\mathcal{U})$ is since it is continuous. If $q^{-1}(\mathcal{U})$ is open then $q\big|_A^{-1}(\mathcal{U}) = q^{-1}(\mathcal{U}) \cap A$ is open as well and so $\mathcal{U}$ is open. ■

Definition 9.3.8:

Let X be a topological space and $\sim$ an equivalence relation on X. Let us denote $\overline{X} = {}^{X}\!/\!_{\sim}$ be the partition of X with respect to $\sim$, then let us define

$$\rho\colon X \longrightarrow \overline{X}, \qquad \rho(x) = [x]_\sim$$

and we define the **quotient topology** on $\overline{X}$ by

$$\left\{\mathcal{U} \subseteq \overline{X} \;\middle|\; \rho^{-1}(\mathcal{U}) \text{ is open in } X\right\}$$

This is indeed a topology, since final topologies are topologies. Thus ρ is a quotient map for $\overline{X}$.

Proposition 9.3.9:

Suppose $f\colon X \longrightarrow Y$ is continuous, then there exists a continuous function $\bar{f}\colon \overline{X} \longrightarrow Y$ such that $f = \bar{f} \circ \rho$ if and only if $a \sim b$ implies $f(a) = f(b)$.
$\bar{f}$ is injective if and only if $a \sim b \iff f(a) = f(b)$.

Proof:

If $f = \bar{f} \circ \rho$ then if $a \sim b$ then $\rho(a) = \rho(b)$ and so $f(a) = \bar{f}(\rho(a)) = \bar{f}(\rho(b)) = f(b)$. And if the condition holds then let us define $\bar{f}([a]) = f(a)$. This is well-defined since if $a \sim b$ then $f(a) = f(b)$, and we showed that $\bar{f}$ is continuous if and only if $\bar{f} \circ \rho = f$ is.
Now suppose $\bar{f}$ is injective then we already know that $a \sim b$ implies $f(a) = f(b)$ so it remains to be shown that $f(a) = f(b)$ implies $a \sim b$. If $f(a) = f(b)$ then $\bar{f}([a]) = \bar{f}([b])$ which means $[a] = [b]$ so $a \sim b$ as required. To show the converse, suppose $\bar{f}([a]) = \bar{f}([b])$ then $f(a) = f(b)$ which means $a \sim b$ so $[a] = [b]$. ■

Definition 9.3.10:

A function $f\colon X \longrightarrow Y$ **preserves** $\sim$ if $a \sim b \implies f(a) = f(b)$. And f **strongly preserves** $\sim$ if $a \sim b \iff f(a) = f(b)$.

Thus we can rephrase the result above as there exists a continuous function $\bar{f}$ such that $f = \bar{f} \circ \rho$ if and only if f preserves $\sim$, and $\bar{f}$ is injective if and only if f strongly preserves $\sim$.

Proposition 9.3.11:

If $f\colon X \longrightarrow Y$ is a quotient map then f strongly preserves $\sim$ if and only if $\bar{f}$ is a homeomorphism.

Proof:

If f strongly preserves $\sim$ then $\bar{f}$ is injective, and since $f = \bar{f} \circ q$ and f and q are quotient maps, $\bar{f}$ is a quotient map as well. So $\bar{f}$ is an injective quotient map, and is therefore a homeomorphism.
And if $\bar{f}$ is a homeomorphism, then since $f = \bar{f} \circ q$, f is a quotient map. And since $\bar{f}$ is injective, f strongly preserves $\sim$. ■