Programming Languages

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1 Some Model Theory

Let $\Sigma \subseteq \mathcal{L}$ be a set of formulas in free variables $x_1, \ldots, x_n$. Then if $\mathcal{A}$ is an $\mathcal{L}$-structure and $a_1, \ldots, a_n \in A$, we write $\mathcal{A} \vDash \Sigma[a_1, \ldots, a_n]$ to mean that $\mathcal{A} \vDash \varphi[a_1, \ldots, a_n]$ for every $\varphi \in \Sigma$. We then say that $a_1, \ldots, a_n$ *realizes* or *satisfies* Σ.

1.1 Definition

An **n-type** $\Gamma(x_1, \ldots, x_n)$ is a maximally consistent set of $\mathcal{L}$-formulas in free variables $x_1, \ldots, x_n$. Let $\mathcal{A}$ be an $\mathcal{L}$-structure and $a_1, \ldots, a_n \in A$ then the **type** of $a_1, \ldots, a_n$ is all the formulas $\varphi(x_1, \ldots, x_n) \in \mathcal{L}$ satisfied in $\mathcal{A}$ by the sequence $a_1, \ldots, a_n$.

This is indeed a type, as it is consistent $(\mathcal{A}, a_1, \ldots, a_n)$ models it, and it is maximal since for every $\varphi(\vec{x}) \in \mathcal{L}$ either φ or $\neg\varphi$ are in the type.

1.2 Example

$(\mathbb{R}, +, \cdot, 0, 1 <)$ is the ordered field of real numbers. Then for every $a < b$, a and b have distinct 1-types, as there exists a rational number $a < \frac{n}{m} < b$ and so $x < \frac{n}{m}$ is in a's type but not b's ($\frac{n}{m}$ is definable in the signature).

1.3 Definition

Let $\Sigma(\vec{x})$ be a set of $\mathcal{L}$-formulas in $\vec{x}$ and $\mathcal{A}$ a $\mathcal{L}$-structure. $\mathcal{A}$ **realizes** Σ if Σ is satisfied by some sequence $\vec{a} \in A^n$. Otherwise $\mathcal{A}$ **omits** Σ.

1.4 Definition

Let $\Sigma(\vec{x})$ be a set of $\mathcal{L}$-formulas and T a $\mathcal{L}$-theory. Then Σ is **compatible** with T if T has a model realizing Σ.

1.5 Definition

Let $\varkappa$ be a cardinal, then a model $\mathcal{A}$ is called **$\varkappa$-saturated** if for every $X \subset A$ of cardinality strictly less than $\varkappa$, $\mathcal{A}_X$ realizes every 1-type $\Sigma(v)$ in the language $\mathcal{L}X$ compatible with $Th\mathcal{A}_X$. And $\mathcal{A}$ is **saturated** if it is $|A|$-saturated.

The reason we consider $\mathcal{A}$ $\varkappa$-saturated using sets of cardinality *strictly* less than $\varkappa$ is because otherwise no model would be saturated: since the type $\{x \neq a \mid a \in \mathcal{A}\}$ is finitely satisfiable by $\mathcal{A}$ and thus is compatible with $Th\mathcal{A}_X$, but it is not realized by $\mathcal{A}$.

1.6 Theorem (Craig's Interpolation Theorem)

Let φ, ψ be $\mathcal{L}$-formulas such that $\varphi \vDash \psi$. Then there exists a $\mathcal{L}$-formula θ such that $\varphi \vDash \theta$ and $\theta \vDash \psi$ such that all extralogical symbols occurring in θ occur in both φ and ψ. θ is called a *Craig interpolate* of φ and ψ.

Proof: suppose φ and ψ have no Craig interpolate, then we will show that $\{\varphi, \neg\psi\}$ is satisfiable by constructing a model for it. Without loss of generality, we can assume that $\mathcal{L}$'s signature contains only the extralogical symbols occurring in φ or ψ, and in particular it is then countable. Define $\mathcal{L}_1$ to be the language whose signature consists of only symbols in φ and $\mathcal{L}_2$ for ψ. Define $\mathcal{L}_0 = \mathcal{L}_1 \cap \mathcal{L}_2$.

Let C be a countably infinite set of new constant symbols and define $\mathcal{L}'_i = \mathcal{L}_i C$.

Let T be a $\mathcal{L}'_1$-theory and S a $\mathcal{L}'_2$-theory. Say that $\theta \in \mathcal{L}'_0$ *separates* them if $T \vDash \theta$ and $S \vDash \neg\theta$. Call T and S *inseparable* if no $\mathcal{L}'_0$-formula separates them.

First notice that $\{\varphi\}$ and $\{\neg\psi\}$ are inseparable: as if $\theta(c_1, \ldots, c_n)$ separated them ($c_i \in C$) then $\exists\vec{x}\theta(\vec{x})$ would be a Craig interpolate of φ and ψ.

Since $\mathcal{L}$ and C are countable, so is $\mathcal{L}C$ and therefore every $\mathcal{L}'_i$. So enumerate the sentences of $\mathcal{L}'_1$ by $\{\varphi_i\}_{i=0}^{\infty}$ and $\mathcal{L}'_2$ by $\{\psi_i\}_{i=0}^{\infty}$. Let us define two sequences of theories

$$\{\varphi\} = T_0 \subseteq T_1 \subseteq \cdots, \qquad \{\neg\psi\} = S_0 \subseteq S_1 \subseteq \cdots$$

inductively as follows:

(1) if $T_m \cup \{\varphi_m\}$ and S_m are inseparable, then put $\varphi_m \in T_{m+1}$;

(2) if $S_m \cup \{\psi_m\}$ and T_{m+1} are inseparable, then put $\psi_m \in S_{m+1}$;

(3) if $\varphi_m = \exists x \sigma(x)$ and $\varphi_m \in T_{m+1}$ then put $\sigma(c) \in T_{m+1}$ for some unused $c \in C$. Similar for ψ_m.

After steps 1 and 2, if T_m and S_m were inseparable, so is T_{m+1} and S_{m+1}. Notice that 3 still preserves inseparability (why?) Then let us define

$$T_\omega = \bigcup_{n=0}^{\infty} T_n, \qquad S_\omega = \bigcup_{n=0}^{\infty} S_n$$

these are inseparable theories, if $T_\omega \vDash \theta$ then $T_n \vDash \theta$ for some n by compactness, and so then we cannot have that $S_n \vDash \neg\theta$ and therefore $S_\omega \nvDash \neg\theta$. Both of these theories are then consistent, as otherwise $\bot$ would separate them.

Now we claim that T_ω is maximally consistent in $\mathcal{L}'_1$ and S_ω is maximally consistent in $\mathcal{L}'_2$. Suppose not: that $\varphi_m, \neg\varphi_m \notin T_\omega$. That means then that $T_m \cup \{\varphi_m\}$ is separable from S_m, so there exists an $\mathcal{L}'_0$-sentence θ such that

$$T_\omega \vDash \varphi_m \to \theta, \qquad S_\omega \vDash \neg\theta$$

Similarly there exists θ' such that

$$T_\omega \vDash \neg\varphi_m \to \theta', \qquad S_\omega \vDash \neg\theta'$$

But then we'd have that

$$T_\omega \vDash \theta \vee \theta', \qquad S_\omega \vDash \neg(\theta \vee \theta')$$

and so T_ω and S_ω are separable, in contradiction.

We now claim that $T_\omega \cap S_\omega$ is a maximally consistent $\mathcal{L}'_0$-theory. Let σ be a $\mathcal{L}'_0$-sentence, so either $\sigma \in T_\omega$ or $\neg\sigma \in T_\omega$ and $\sigma \in S_\omega$ or $\neg\sigma \in S_\omega$. But T_ω and S_ω are inseparable, so we have that $\sigma \in T_\omega \cap S_\omega$ or $\neg\sigma \in T_\omega \cap S_\omega$ as required.

Finally let us construct a model for $T_\omega \cup S_\omega$, which contains both φ and $\neg\psi$. Let $\mathcal{B}'_1 = (\mathcal{B}_1, b_i)_{i \in C}$ be a model for T_ω. By (3) in the construction of T_ω, this means that if we take the substructure $\mathcal{A}'_1 = (\mathcal{A}_1, b_i)_{i \in C}$ whose domain is $\{b_i\}_{i \in C}$, $\mathcal{A}'_1$ also satisfies T_ω. Similarly we can take $\mathcal{A}'_2 = (\mathcal{A}_2, d_i)_{i \in C}$ a structure whose domain is $\{d_i\}_{i \in C}$ which satisfies S_ω. Then the map $b_i \mapsto d_i$ is an isomorphism since both structures model the complete theory $T_\omega \cap S_\omega$ (so it contains all formulas of the form $fc_1 \cdots c_n = c$ and $rc_1 \cdots c_n$ and their negations). So without loss of generality, $b_i = d_i$, meaning $\mathcal{A}'_1$ and $\mathcal{A}'_2$ have the same $\mathcal{L}_0$-reduct. Thus we can take an $\mathcal{L}$-structure $\mathcal{A}$ whose $\mathcal{L}_1$-reduct is $\mathcal{A}_1$ and $\mathcal{L}_2$-reduct is $\mathcal{A}_2$, and so it models both T_ω and S_ω, meaning it models $\varphi \wedge \neg\psi$ in contradiction. ∎

1.7 Theorem (Robinson's Theorem)

Let $\mathcal{L}_1, \mathcal{L}_2$ be two first-order languages and define $\mathcal{L} = \mathcal{L}_1 \cap \mathcal{L}_2$. If T is a complete $\mathcal{L}$-theory, $T_1 \supseteq T$ and $T_2 \supseteq T$ are consistent $\mathcal{L}_1$- and $\mathcal{L}_2$-theories respectively, then $T_1 \cup T_2$ is a consistent $\mathcal{L}_1 \cup \mathcal{L}_2$-theory.

Proof: suppose that $T_1 \cup T_2$ is inconsistent, then take $\Sigma_1 \subseteq T_1$ and $\Sigma_2 \subseteq T_2$ finite such that $\Sigma_1 \cap \Sigma_2$ is inconsistent. Define $\sigma_1 = \bigwedge \Sigma_1$ and $\sigma_2 = \bigwedge \Sigma_2$, and so we have that $\sigma_1 \vDash \neg\sigma_2$. By Craig's Interpolation Theorem, there is a Craig interpolate θ where $\sigma_1 \vDash \theta$ and $\theta \vDash \neg\sigma_2$ and θ contains only extralogical symbols contained in both σ_1 and σ_2. So θ is a $\mathcal{L}_1 \cap \mathcal{L}_2$-sentence. Since T_1 is consistent, $T_1 \nvDash \neg\theta$ meaning $T \nvDash \neg\theta$, but $T_2 \vDash \sigma_2 \vDash \neg\theta$ so by consistency $T_2 \nvDash \theta$ meaning $T \nvDash \neg\theta$. But this contradicts T's completeness. ∎

1.8 Lemma

Let $\mathcal{L}$ be a first-order language of cardinality $\leq \alpha$ and $\mathcal{A}$ be an $\mathcal{L}$-structure whose cardinality is $\omega \leq |\mathcal{A}| \leq$

2^α. Then there exists an elementary extension $\mathcal{A} \preceq \mathcal{B}$ of cardinality 2^α such that for every $X \subseteq A$ of cardinality α, $(\mathcal{B}, a)_{a\in X}$ realizes all types consistent with $(\mathcal{A}, a)_{a\in X}$.

Proof: since $|A| \leq 2^\alpha$, we have that $|\{X \subseteq A \mid |X| = \alpha\}| \leq 2^\alpha$, meaning there are at most 2^α subsets X of cardinality α. Furthermore $\mathcal{L}X$ is of cardinality $\leq \alpha$ and so there are at most 2^α 1-types over $\mathcal{L}X$. So for every $X \subseteq A$ of cardinality α and every 1-type $\Sigma(v)$ define a new constant symbol $c_{X\Sigma}$. Let us define

$$T = D_{el}\mathcal{A} \cup \bigcup_{X,\Sigma} \Sigma[c_{X\Sigma}]$$

Notice that $D_{el}\mathcal{A}$ is a complete theory consistent with $\Sigma(v)$ by definition, and so consistent with $\Sigma[c_{X\Sigma}]$ and thus $D_{el}\mathcal{A} \cup \Sigma[c_{X\Sigma}]$ is a consistent extension of $D_{el}\mathcal{A}$. So by Robinson's Theorem, every finite subset of T is consistent and therefore T is consistent.

Since the language of T contains at most 2^α symbols, it has a model of cardinality 2^α. Since this model models $D_{el}\mathcal{A}$, it is an elementary extension of $\mathcal{A}$. ■

1.9 Theorem

Let $\mathcal{A}$ be an $\mathcal{L}$-structure where $|\mathcal{L}| \leq \alpha$ and $\omega \leq |\mathcal{A}| \leq 2^\alpha$. Then there exists an α^+-saturated elementary extension $\mathcal{B} \succeq \mathcal{A}$ of cardinality 2^α.

Proof: we will construct an elementary chain $\{\mathcal{B}_\xi\}_{\xi<2^\alpha}$ such that every $\mathcal{B}_\xi$ is an elementary extension of $\mathcal{A}$ of cardinality 2^α, for every subset $X \subseteq \mathcal{B}_\varepsilon$ of cardinality α, $(\mathcal{B}_{\xi+1}, a)_{a\in X}$ realizes every type over $(\mathcal{B}_\xi, a)_{a\in X}$. For $\mathcal{B}_0$ we take the structure created in the previous lemma. If η is a limit ordinal, define $\mathcal{B}_\eta = \bigcup_{\xi<\eta} \mathcal{B}_\xi$. Otherwise if $\eta = \xi + 1$, then take $\mathcal{B}_\eta$ to be the structure created in the previous lemma, with $\mathcal{B}_\xi$ instead of $\mathcal{A}$. Then define

$$\mathcal{B} = \bigcup_{\xi<2^\alpha} \mathcal{B}_\xi$$

Clearly $\{\mathcal{B}_\xi\}$ is an elementary chain and so $\mathcal{B}$ is an elementary extension of $\mathcal{A}$. Now let $X \subseteq \mathcal{B}$ of cardinality α and $\Sigma(v)$ a type over $(\mathcal{B}, a)_{a\in X}$. Since 2^α has larger cofinality than α there must exist $\xi < 2^\alpha$ such that $X \subseteq \mathcal{B}_\xi$. But since $\mathcal{B}_\xi$ is an elementary substructure of $\mathcal{B}$, $\Sigma(v)$ is also a type over $(\mathcal{B}_\xi, a)_{a\in X}$ and so is realized by $\mathcal{B}_{\xi+1}$ and thus by $\mathcal{B}$ as an elementary extension. ■

Notice that this does not guarantee the existence of a *saturated* elementary extension, as this requires the generalized continuum hypothesis (GCH): that $\alpha^+ = 2^\alpha$ which is independent of ZFC. If it were true, then $\mathcal{B}$ would be $\alpha^+ = 2^\alpha$-saturated and of cardinality 2^α, as required.

1.10 Lemma (Shuttle Lemma)

Let α be an infinite cardinal, $\mathcal{A}, \mathcal{B}$ be α-saturated and elementary equivalent. Let $a\colon \alpha \longrightarrow A$, $b\colon \alpha \longrightarrow B$ be injective, then there exists $a'\colon \alpha \longrightarrow A$, $b'\colon \alpha \longrightarrow B$ such that

$$\mathrm{Im} a \subseteq \mathrm{Im} a', \qquad \mathrm{Im} b \subseteq \mathrm{Im} b', \qquad (\mathcal{A}, a'_\xi)_{\xi<\alpha} \equiv (\mathcal{B}, b'_\xi)_{\xi<\alpha}$$

Proof: every ordinal ξ has a unique representation as $\xi = \lambda + \eta$ where λ is a limit ordinal and $\eta \in \omega$. Call ξ *even* if η is even, otherwise odd. We will define two injective functions $a'\colon \alpha \longrightarrow A$ and $b'\colon \alpha \longrightarrow B$ such that for all ordinals $\xi < \alpha$:

(1) if $\xi = \lambda + 2n$ is even, then $a'_\xi = a_{\lambda+n}$,

(2) if $\xi = \lambda + 2n + 1$ is odd, then $b'_\xi = b_{\lambda+n}$,

(3) $(\mathcal{A}, a'_\eta)_{\eta\leq\xi} \equiv (\mathcal{B}, b'_\eta)_{\eta\leq\xi}$

Notice that (3) can indeed be satisfied: first suppose $(\mathcal{A}, a'_\eta)_{\eta<\xi} \vDash \varphi$, we must have that $(\mathcal{A}, a'_\eta)_{\eta<\xi'} \vDash \varphi$ for some $\xi' < \xi$ by compactness (look at the theory of the model). And so $(\mathcal{B}, b'_\eta)_{\eta<\xi} \vDash \varphi$, meaning $(\mathcal{A}, a'_\eta)_{\eta<\xi} \equiv (\mathcal{B}, b'_\eta)_{\eta<\xi}$.

So let us assume that ξ is even, then let us define the 1-type

$$\Sigma(x) = \left\{ \varphi(b'_\eta, x)_{\eta<\xi} \in \mathcal{L}(b'_\eta)_{\eta<\xi} \;\middle|\; \mathcal{A} \vDash \varphi(a'_\eta)_{\eta\leq\xi} \right\}$$

$\Sigma(x)$ is consistent with the theory of $\mathcal{B}(b'_\eta)_{\eta<\xi}$ since (since $\mathcal{A}$ is a deductively closed theory we can consider single formulas) for $\varphi(\bar{b}'_\eta, x) \in \Sigma(x)$, we have that $\mathcal{A} \vDash \exists x \varphi(\bar{a}'_\eta, x)$ and so $\mathcal{B} \vDash \exists x \varphi(\bar{b}'_\eta, x)$. Since $\mathcal{B}$ is α-saturated, we have that there must exist a b'_ξ which realizes $\Sigma(x)$, and thus satisfies (3).

If we have a sequence which satisfies $(1), (2), (3)$, then we must have the required results. ■

1.11 Theorem (Uniqueness of Saturated Models)

If $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent saturated models of the same cardinality, then they are isomorphic.

Proof: suppose $|A| = |B| = \alpha$, then there exist enumerations $a\colon \alpha \longrightarrow A, b\colon \alpha \longrightarrow B$. By the Shuttle Lemma, there exists a', b' whose images contain A and B repspectively such that $(\mathcal{A}, a'_\xi)_{\xi<\alpha} \equiv (\mathcal{B}, b'_\xi)_{\xi<\alpha}$. But then $a'_\xi \mapsto b'_\xi$ is an isomorphism. ■