Differential and Analytic Geometry

Assignment 5
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Exercise 5.1:

The unit sphere is parameterized by

$$x(\theta, \varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$

We are given a latitude line, γ given by $\varphi = \varphi_0$. Find a parallel unit vector field over γ.

Let us compute the tangent space to x,

$$x_1 = (-\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0), \qquad x_2 = (\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi)$$

We can form an orthonormal basis out of this,

$$E_1 = (-\sin\theta, \cos\theta, 0), \qquad E_2 = (\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi)$$

Now, since $\gamma(t) = x(t, \varphi_0)$, we have that

$$E_1(t) = (-\sin(t), \cos(t), 0), \qquad E_2(t) = (\cos(t)\cos(\varphi_0), \sin(t)\cos(\varphi_0), -\sin(\varphi_0))$$

Differentiating gives

$$E_1'(t) = (-\cos(t), -\sin(t), 0) = -\cos(\varphi_0)E_2(t) - \sin(\varphi_0)x(t, \varphi_0), \quad E_2'(t) = \cos(\varphi_0)\cdot(-\sin(t), \cos(t), 0) = \cos(\varphi_0)E_1(t)$$

And since $V(t) \in T_{\gamma(t)}\mathbb{S}^2$, and it is a unit

$$V(t) = \cos(\alpha(t))E_2(t, \varphi_0) + \sin(\alpha(t))E_2(t, \varphi_0)$$

This means

$$V'(t) = (\cos(\varphi_0) - \dot\alpha)\sin(\alpha)E_1 + (-\cos(\varphi_0) + \dot\alpha)\cos(\alpha)E_2 - \sin(\varphi_0)\cos(\alpha)n$$

(Since $x = n$ is the unit normal to $\mathbb{S}^2$). Thus we have that in order for $V(t)$ to be parallel, $V'(t) \perp T_{\gamma(t)}\mathbb{S}^2$ and so

$$\dot\alpha = \cos(\varphi_0)$$

Thus

$$\alpha(t) = \cos(\varphi_0)t + \alpha_0$$

And so the total change of angle is

$$\Delta\alpha = \alpha(2\pi) - \alpha(0) = \cos(\varphi_0)\cdot 2\pi$$

Which is what we got in recitation as well.

Exercise 5.2:

We are given the cylinder M parameterized by

$$x(u, v) = (\cos u, \sin u, v)$$

(1) For every $p \in M$, find the exponential $\exp_p\colon T_pM \longrightarrow M$ explicitly.

(2) For every curve $\gamma\colon I \to M$ and points $a, b \in I$, find the parallel transport $P_\gamma^{ab}\colon T_{\gamma(a)}M \to T_{\gamma(b)}M$

(1) We know that by definition $\exp_p(v)$ is equal to travelling a unit on the geodesic at p in the direction v. Since we are on a cylinder, the only geodesics are helixes ($t \mapsto (a(\cos(t) - 1), a\sin t, ct) + p$) and vertical lines ($t \mapsto (0, 0, ct) + p$). And let us recall the definition of the exponential map, for $v \in T_pM$ let γ_v be the geodesic where $\gamma_v(0) = p$ and $\gamma_v'(0) = v$ then $\exp_p(v) = \gamma_v(1)$. If $v = 0$ then $\exp_p(0) = p$.

So the two geodesics on cylinders have derivatives of the form $(-a\sin(t), a\cos(t), c)$ and $(0,0,c)$. So if $v=(0,0,c)$ then $\gamma_v(t)=(0,0,ct)+p$ and so $\exp_p(v)=(0,0,c)+p=v+p$. And if $v=(-a\sin(1), a\cos(1), c)$ then $\gamma_v(t)=(a(\cos(t)-1), a\sin t, ct)+p$ and so $\exp_p(v)=(a(\cos(1)-1), a\sin(1), c)+p$, this is equal to $R_{3\pi/2}v+(-a,0,0)+p$. And if $v=0$ then $\exp_p(v)=p$.

(2) Let $v\in T_{\gamma(a)}M$ then $P_\gamma^{ab}(v)=W(b)$ where $W(t)$ is the parallel transport on γ and $W(a)=v$. Then we know that

$$W(t)=\cos\left(\theta_0+\int_a^t\kappa_g\right)\gamma'(t)+\sin\left(\theta_0+\int_a^t\kappa_g\right)n\times\gamma'(t)$$

Now, since γ is smooth, let us focus on the surface whose boundary is given by γ (from a to b) and the geodesic from $\gamma(a)$ to $\gamma(b)$. Since the cylinder is isometric with the plane, both have a Gaussian curvature of zero. This means that (since the geodesic curvature of the geodesic is zero),

$$\int_a^t\kappa_g=\iint K\,ds=0$$

And so

$$W(t)=\cos(\theta_0)\gamma'(t)+\sin(\theta_0)n\times\gamma'(t)$$

So

$$P_\gamma^{ab}(v)=\cos(\theta_0)\gamma'(b)+\sin(\theta_0)n\times\gamma'(b)$$

Exercise 5.3:

Prove, without using Gauss-Bonnet, that given a geodesic triangle on the unit sphere whose interior angles are α, β, and γ, we have

$$\alpha+\beta+\gamma=\pi+T$$

where T is the area of the triangle.

Let S_x be the area between two geodesics at an angle x on the sphere. Since S_x covers $\frac{x}{\pi}$ of the total surface area of the sphere, we have that, since the area of the sphere is 4π,

$$S_x=4\pi\cdot\frac{x}{\pi}=4x$$

Now, notice that $S_\alpha+S_\beta+S_\gamma$ covers the entire sphere, but each S_x $(x=\alpha,\beta,\gamma)$ counts T twice (since the geodesics which define the triangle define a congruent triangle on the opposite side of the sphere). So this sum counts T six times, while it should only be counted twice (since the geodesics define two congruent triangles), meaning T is counted four more times than required. Thus (since the surface area of the sphere is 4π),

$$S_\alpha+S_\beta+S_\gamma=4\pi+4T$$

And so

$$4(\alpha+\beta+\gamma)=4(\pi+T)\implies\alpha+\beta+\gamma=\pi+T$$

as required.

Exercise 5.4:

Let M be a compact orientable surface with a positive genus. Show that the surface has ellipitic, hyperbolic, and parabolic points (points where the Gaussian curvature is positive, zero, and negative).

Let $O\in\mathbb{R}^3$ be a point within M, since M is compact there exist radii $R>0$ such that M is contained within the ball $B_R(O)$. Let r be the infimum of all such radii, and then M must intersect with $B_r(O)$ at at least one point p, as otherwise we could reduce the radius r. In fact, they must be tangent at p as otherwise we could increase r and get a ball not containing M. So if we look at a normal section at p, we must have that its normal curvature with M is greater than its normal curvature with $B_r(O)$ (since it must curve away from p quicker, as it is contained within $B_r(O)$). Since the Gaussian curvature is equal to the product of the maximum and minimum of these values, we conclude that the Gaussian curvature of M at p is greater than that with $B_r(O)$ at p. Since the Gaussian curvature of $B_r(O)$ is $\frac{1}{r}>0$, we have that the Gaussian curvature of M at p is positive, and so we have an elliptic point.
Since M has a positive genus, it must have a hole somewhere. Let $O\in\mathbb{R}^3$ contained within this hole, and let r be the supremum of all the radii R such that $B_r(O)$ does not intersect M. Then again, $B_r(O)$ must be tangent to M at some point p. If we look at the normal curvatures, not all can curve inward or all outward for every choice of O, and so there must be some point p where it has normal curvature both inward and outward. And so its Gaussian curvature is negative. And since Gaussian curvature is continuous, there must be a point with zero Gaussian curvature.

Exercise 5.5:

Let us look at the rotation of the curve $\alpha(\psi) = (\cosh\varphi, 0, \psi)$. Find the total Gaussian curvature of the surface.

We have computed that the Gaussian curvature of this surface is $K(t,\varphi) = -\frac{1}{\cosh(t)^4}$. And the metric has a determinant of $\cosh(t)^4$, meaning that the total curvature is

$$\iint_M K\,ds = -\int_{-\infty}^{\infty}\int_0^{2\pi} \frac{1}{\cosh(t)^2}\,d\varphi dt = -2\pi\int_{-\infty}^{\infty} \frac{1}{\cosh(t)}^2\,dt = -2\pi\tanh(t)\Big|_{-\infty}^{\infty} = -4\pi$$

Exercise 5.6:

Find the total Gaussian curvature of the rotation of $\alpha(\varphi) = (\varphi, 0, \varphi^2)$ where $\varphi \geq 0$.

This forms a parabaloid, which we can parameterize by $x(u,v) = (u, v, u^2+v^2)$ (Gaussian curvature is intrinsic and does not depend on parameterization). This is the graph of $f(u,v) = u^2+v^2$, and we have shown that the Gaussian curvature of a graph is given by

$$K = \frac{f_{uu}f_{vv} - f_{uv}^2}{(1+f_u^2+f_v^2)^2}$$

Here that is equal to

$$\frac{4}{(1+4u^2+4v^2)^2}$$

And so the total Gaussian curvature is

$$\iint_M K\,ds = 4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1+4u^2+4v^2)^{-2}\,dudv = \pi\int_{-\infty}^{\infty}(1+4v^2)^{-3/2}\,dv = \pi$$

Exercise 5.7:

Suppose M_1 and M_2 are two disjoint compact orientable surfaces, show that

$$\chi(M_1 \uplus M_2) = \chi(M_1) + \chi(M_2)$$

We can show this in two ways. Firstly, since $M_1 \uplus M_2$ is compact and orientable, by Gauss-Bonnet:

$$\iint_{M_1\uplus M_2} K\,ds = 2\pi\chi(M_1 \uplus M_2)$$

But at the same time

$$\iint_{M_1\uplus M_2} K\,ds = \iint_{M_1} K\,ds + \iint_{M_2} K\,ds = 2\pi\chi(M_1) + 2\pi\chi(M_2)$$

And so

$$\chi(M_1 \uplus M_2) = \chi(M_1) + \chi(M_2)$$

Alternatively, any triangularization of M_1 and M_2 defines a triangularization of $M_1 \uplus M_2$. So if V^i, F^i, and E^i are the number of vertices, faces, and edges of the triangularization of M_i then V^1+V^2, F^1+F^2, and E^1+E^2 are the number of vertices, faces, and edges of a triangularization of $M_1 \uplus M_2$. And so

$$\chi(M_1 \uplus M_2) = V^1 + V^2 - (E^1+E^2) + (F^1+F^2) = V^i - E^i + F^i = \chi(M_1) + \chi(M_2)$$

Exercise 5.8:

Given the surface M defined by

$$(x^2+y^2+2z^2-1)((x-10)^2+y^2+2z^2-1) = 0$$

compute its total curvature.

This surface is the union of $x^2+y^2+2z^2-1=0$ and $(x-10)^2+y^2+2z^2-1=0$. These are two disjoint ellipsoids, let them be M_1 and M_2. Since ellipsoids are compact and orientable surfaces of genus zero, we have

$$\iint_M K\,ds = \iint_{M_1\uplus M_2} K\,ds = \iint_{M_1} K\,ds + \iint_{M_2} K\,ds = 4\pi + 4\pi = 8\pi$$