Differential and Analytic Geometry

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Recall that if $\alpha\colon [a,b] \longrightarrow \mathbb{R}^n$ is a regular smooth curve, then we define its *natural parameterization* as the curve

$$\beta\colon [0,L] \longrightarrow \mathbb{R}^n$$

where $L = s_\alpha(b)$ is the arclength of α by

$$\beta(u) = \alpha \circ s_\alpha^{-1}(u)$$

And this is unique (up to reparameterization). A curve from $[0,L]$ is a natural parameterization if and only if $\|\alpha'\| = 1$.

Definition 3.1:

Let α be a natural parameterization. We define $T_\alpha(s) = \alpha'(s)$, and in the case that we are in 2 dimensions, we define $N_\alpha(s) = R_{\frac{\pi}{2}} \cdot T(s)$. R_θ is the rotation matrix

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Since α is a natural parameterization and R_θ is orthogonal, $\|T_\alpha\| = \|N_\alpha\| = 1$ and thus $\{T(s), N(s)\}$ forms an orthonormal basis, called the **Frenet-Serret Frame**.

We can think of T_α as the direction of motion, or the velocity, of α, and T'_α as its acceleration. Since T_α is constant, its derivative is perpendicular to itself, meaning the acceleration of α is orthogonal to its velocity. We will prove this formally:

Proposition 3.2:

Suppose $V\colon \mathbb{R} \longrightarrow \mathbb{R}^n$ (ie. V is a vector field over $\mathbb{R}$), if $\|V\| = c$ then $V' \perp V$ whenever V is differentiable.

Proof:

Since $\langle V, V\rangle = c^2$ is constant, we have that the function

$$f(t) = \langle V(t), V(t)\rangle = \sum_{k=1}^n V_i(t)V_i(t)$$

Is constant and therefore if V is differentiable at t, then so must V_i be, and therefore $f(t)$ is. And since f is constant, $f'(t) = 0$. Therefore

$$f'(t) = \sum_{k=1}^n V_i'(t)V_i(t) + V_i(t)V_i'(t) = \langle V'(t), V(t)\rangle + \langle V(t), V'(t)\rangle = 0$$

And since this inner product is over $\mathbb{R}$, this means $\langle V, V'\rangle = 0$ so $V' \perp V$ as required. ■

So when $n = 2$, this means that T'_α is parallel with N_α and so

$$T'_\alpha(s) = k(s) \cdot N_\alpha(s)$$

For some function $k\colon \mathbb{R} \longrightarrow \mathbb{R}$. In fact, since $\{T_\alpha, N_\alpha\}$ is an orthonormal basis,

$$T' = \langle T', T\rangle T + \langle T', N\rangle N = \langle T', N\rangle N$$

So $k(s) = \langle T'(s), N(s)\rangle$.
Let us look at this function k.

(1) When $k(s) = 0$, then $T'(s) = 0$ and so there is no acceleration, and we are moving in a straight line.

(2) When $k(s) > 0$, then the curve α is accelerating away from T "upward" (toward N), and this creates a steep curve.

(3) When $k(s) < 0$, the curve is accelerating away from T "downward", also creating a steep curve.

Thus k can be seen as a measure of curvature.

Definition 3.3:

The **curvature** of a regular two-dimensional curve α at point s is defined to be

$$k(s) = \langle T'_\alpha(s), N_\alpha(s) \rangle$$

Where T_α and N_α are taken as their values for the natural reparameterization of α.

Notice that

$$N' = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} T \right)' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} T' = k \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} N = k \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 T = k \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} T = -kT$$

Therefore T and N are solutions to the ODE,

$$T' = kN, \quad N' = -kT$$

Thus by the uniqueness theorem for ODEs, if we are given the function $k(s)$, and $N(0)$ and $T(0)$, then we can solve for N and T. Since N is determined by T, we need only $T(0)$ and $k(s)$. And since $T = \alpha'$,

$$\alpha(s) - \alpha(0) = \int_0^s T$$

for all s, so if we are given T and $\alpha(0)$, we can find $\alpha(s)$. Thus given $k(s)$, $\alpha(0)$, and $T(0)$ we can determine α.

Theorem 3.4 (The Fundamenta Theorem of Curves):

Every regular curve is uniquely determined by its curvature, initial position, and $T(0)$.

Now, recall that

$$k(s) = \langle T'(s), N(s) \rangle = \left\langle \alpha''(s), \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \alpha'(s) \right\rangle = \left\langle \begin{pmatrix} \alpha''_1(s) \\ \alpha''_2(s) \end{pmatrix}, \begin{pmatrix} -\alpha'_2(s) \\ \alpha'_1(s) \end{pmatrix} \right\rangle = \alpha''_2(s)\alpha'_1(s) - \alpha'_2(s)\alpha''_1(s)$$

And so

$$k(s) = \alpha''_2\alpha'_1 - \alpha'_2\alpha''_1$$

Where α is the natural parameterization.

Example 3.5:

Suppose α is the curve in $\mathbb{R}^2$ connecting x and y, ie.

$$\alpha\colon [0,1] \longrightarrow \mathbb{R}^2, \quad s \mapsto x \cdot \frac{s}{L} + y \cdot \frac{1-s}{L}$$

where $L = \|x - y\|$. Thus

$$\alpha'(s) = \frac{x}{L} - \frac{y}{L}$$

And so $\alpha''(s) = 0$, meaning $k(s) = 0$.

Example 3.6:

Suppose α is the curve which parameterizes the circle of radius R,

$$\alpha\colon [0, 2\pi R] \longrightarrow \mathbb{R}^2, \quad s \mapsto R\left(\cos\frac{s}{R}, \sin\frac{s}{R}\right)$$

Thus

$$\alpha'(s) = \left(-\sin\frac{s}{R}, \cos\frac{s}{R}\right), \qquad \alpha''(s) = -\frac{1}{R}\left(\cos\frac{s}{R}, \sin\frac{s}{R}\right)$$

$\|\alpha\| = 1$, so α is the natural parameterization. And thus

$$k(s) = -\frac{1}{R}\left(-\sin^2\frac{s}{R} + \cos^2\frac{s}{R}\right) = \frac{1}{R}$$

So the curvature of a circle of radius R is $\frac{1}{R}$.

Since the curves are determined by $\alpha(0)$, $T(0)$, and their curvature, by the above two examples, if

(1) $k(s) = c \neq 0$ then α is a circle. If $k(s) > 0$ then the curve is drawn counterclockwise, and if $k(s) < 0$ the curve is parameterized clockwise (the proof above means that $\alpha(-s)$ is a circle of radius $-R$).

(2) $k = 0$ then α is a line.

Notice that if γ is a natural parameterization then

$$\gamma'(s) = T(s) = \begin{pmatrix} \cos(\alpha(s)) \\ \sin(\alpha(s)) \end{pmatrix}$$

This means that

$$\alpha(s) = \operatorname{atan2}(\cos\alpha(s), \sin\alpha(s))$$

Now we claim that $k(s) = \alpha'(s)$. Since

$$T(s) = \begin{pmatrix} \cos(\alpha(s)) \\ \sin(\alpha(s)) \end{pmatrix} \implies T'(s) = \begin{pmatrix} -\sin(\alpha(s)) \\ \cos(\alpha(s)) \end{pmatrix} \cdot \alpha'(s) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} T \cdot \alpha'(s) = \alpha'(s) N$$

And since $T'(s) = k(s)N$ this means that $\alpha'(s) = k(s)$ as required.
So if we are given $\gamma' = T$, then we can compute α based on T and then taking its derivative gives $k(s)$.