Computability and Complexity

Recitation 2, Thursday August 3, 2023

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Exercise 2.1:

Let us define the decision problem

$$\mathsf{DoubleIS} = \{(G, k) \mid G \text{ has two distinct independent sets whose size is at least } k\}$$

Define a Karp reduction from IS (the decision problem of G having an independent set of size k, check recitation 1) to $\mathsf{DoubleIS}$.

Let us define a function $f\colon \mathsf{IS} \longrightarrow \mathsf{DoubleIS}$ where $f(G, k) = (G', k)$ where we define G' as follows: if $G = (V, E)$ then we define $G_i = (V_i, E_i)$ to be two distinct copies of G and

$$G' = \big(V_1 \cup V_2, E_1 \cup E_2 \cup \{(v, u) \mid v \in V_1,\, u \in V_2\}\big)$$

Then if $(G, k) \in \mathsf{IS}$ then $(G', k) \in \mathsf{DoubleIS}$ since the independent set in G is copied twice into G'.

And if $(G', k) \in \mathsf{DoubleIS}$ then (G', k) has an independent set S, and S is either contained entirely within G_1 or entirely within G_2, as otherwise there would be an edge connecting two nodes of S (since all nodes in G_1 are connected with all nodes in G_2). Thus S corresponds with an independent set in G.

Thus $f(G, k) \in \mathsf{DoubleIS}$ if and only if $(G, k) \in \mathsf{IS}$, so f is a Karp reduction as required.

Definition:

A set of nodes S in a graph are **almost independent** if there is at most one pair of nodes in S with an edge between them.

Exercise 2.2:

We define the problem $\mathsf{AlmostIS}$ by

$$\mathsf{AlmostIS} = \{(G, k) \mid G \text{ has an almost independent set whose size is at least } k\}$$

Define a Karp reduction from IS to $\mathsf{AlmostIS}$.

So we need to define a function $f\colon \mathsf{IS} \longrightarrow \mathsf{AlmostIS}$. Let $G = (V, E)$ be a graph, then we define $G' = (V', E')$ where $V' = V \cup \{u_1, u_2\}$ where $u_1, u_2 \notin V$, and $E' = E \cup \{(u_1, u_2)\}$, and set $f(G, k) = (G', k + 2)$.

If $(G, k) \in \mathsf{IS}$ then $(G', k + 2) \in \mathsf{AlmostIS}$ as if S is independent in G then we can take the almost independent set $S \cup \{u_1, u_2\}$ in G'. Since $|k| \geq k$, $|S \cup \{u_1, u_2\}| \geq k + 2$, and so $(G', k + 2) \in \mathsf{AlmostIS}$ as required.

Now, if $(G', k + 2) \in \mathsf{AlmostIS}$, then let S be the almost independent set of size $\geq k + 2$ in G'. We split into cases:

(1) If $u_1, u_2 \in S$ then $S \setminus \{u_1, u_2\}$ is an independent set in G of size $\geq k$ (since otherwise S would have two edges), and so $(G, k) \in \mathsf{IS}$ as required.

(2) Otherwise, let $S' = S \setminus u_1, u_2$ and so $|S'| \geq k + 1$ (since not both u_1 and u_2 are in S'), and $S' \subseteq G$, and we split into two subcases:

   (i) If S' is independent, then $(G, k) \in \mathsf{IS}$ as required.

   (ii) Otherwise, since S' is still almost independent, there exist $u, v \in S$ such that $(u, v) \in E$ and so $S'' = S \setminus \{u\}$ is independent and of size $|S''| \geq k$, and so $(G, k) \in \mathsf{IS}$ as required.

Exercise 2.3:

Define a Karp reduction from SAT to IS.

So we need a function $f\colon \mathsf{SAT} \longrightarrow \mathsf{IS}$ which satisfies the conditions for a Karp reduction. Let $(\varphi, \tau) \in \mathsf{SAT}$, and suppose

$$\varphi = \bigwedge_{i=1}^{m} \bigvee_{j=1}^{n} \varepsilon_{ij} x_j$$

where ε_{ij} is either $\neg$ or nothing. Let us define a graph $G = (V, E)$ where $V = \{(i, j) \mid 1 \le i \le m,\, 1 \le j \le n\}$. The vertex (i, j) represents $\varepsilon_{ij} x_j$, and we define edges

$$E = \left\{ \big((i, j_1),\, (i, j_2)\big) \;\middle|\; 1 \le i \le n,\, 1 \le j_1, j_2 \le n \right\} \cup \left\{ \big((i_1, j),\, (i_2, j)\big) \;\middle|\; \varepsilon_{i_1 j} \ne \varepsilon_{i_2 j} \right\}$$

And let us define $k = m$.

So we must show that $\varphi \in \mathsf{SAT}$ if and only if $(G, m) \in \mathsf{IS}$.

If $\varphi \in \mathsf{SAT}$ then there exists a boolean vector τ which satisfies φ. Then for every disjunction $(\bigvee_{j=1}^{n} \varepsilon_{ij} x_j)$, there exists at least one variable which is satisfies by τ. So from the ith disjunction, suppose $\varepsilon_{ij} x_j$ is satisfied by τ, then we add (i, j) to S. Since there are m disjunctions, $|S| = m$, and S is also independent as if (i_1, j_1) and (i_2, j_2) are neighbors and in S, then since we only choose one node from each disjunction, $i_1 \ne i_2$. Then $j_1 = j_2 = j$ and $\varepsilon_{i_1 j} \ne \varepsilon_{i_2 j}$, but then $\varepsilon_{i_1 j} x_j$ and $\varepsilon_{i_2 j} x_j$ can't both be satisfied by τ, in contradiction.

So S is independent and of size m, meaning $(G, n) \in \mathsf{IS}$.

Suppose that $(G, n) \in \mathsf{IS}$, so there exists an independent set of size n. If $(i, j) \in S$ then if $\varepsilon_{ij} = \neg$ set $\tau_j = 0$ and otherwise if ε_{ij} is empty set $\tau_j = 1$. The rest of the indexes of τ can be set as wanted.

τ is well-defined since if $(i_1, j), (i_2, j) \in S$ then they are not neighbors so $\varepsilon_{i_1 j} = \varepsilon_{i_2 j}$, so τ_j is set to one value. Since S is of size n and independent, for every $1 \le i \le m$ there exists a $1 \le j \le n$ such that $(i, j) \in S$ (as otherwise there would be an i such that $(i, j_1), (i, j_2) \in S$ but these are neighbors). And so every disjunction has a variable which is satisfied, and thus every disjunction is satisfied, and so φ is satisfied.

Thus $\varphi \in \mathsf{SAT}$ if and only if $(G, n) \in \mathsf{IS}$.