Summaries of Various Papers

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Contents

In this document I will summarize various interesting papers I have read, in no particular order. Enjoy!

1 Forcing

1.1 Set Theory 292B: Model-Theoretic Forcing and Its Applications

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In this paper Professor Ting Zhang discusses forcing and its applications to model theory.

In this paper, our logical signature consists of $\wedge, \vee, \neg, \exists$, all other logical symbols are considered abbreviations. The diagram of a structure $\mathcal{A}$ is denoted $\Delta[\mathcal{A}]$. A *basic sentence* is an atomic sentence or a negated atomic sentence (a “literal sentence”).

1.1.1 Definition (Forcing Condition)

Let T be an $\mathcal{L}$-theory, a **forcing condition** P is a set of basic sentences. of $\mathcal{L}[A]$ where A is a set of constant symbols, such that $T \cup P$ is consistent. For $\varphi \in \mathcal{L}[A]$, we define the **forcing relation** $P \Vdash \varphi$ on the recursive structure of φ as follows:

(1) if φ is atomic, then $P \Vdash \varphi \iff \varphi \in P$.

(2) $P \Vdash \varphi \vee \psi$ iff $P \Vdash \varphi$ or $P \Vdash \psi$.

(3) $P \Vdash \varphi \wedge \psi$ iff $P \Vdash \varphi$ and $P \Vdash \psi$.

(4) $P \Vdash \neg\varphi$ iff there exists no forcing condition $Q \supseteq P$ such that $Q \Vdash \varphi$.

(5) $P \Vdash \exists x\varphi$ iff there exists a closed term t of $\mathcal{L}[A]$ such that $P \Vdash \varphi(t)$.

If $P \Vdash \varphi$, we say P *forces* φ. We also define the **weak forcing relation**, where $P \Vdash^w \varphi \iff P \Vdash \neg\neg\varphi$.

Note that

$$P \Vdash^w \varphi \iff P \Vdash \neg\neg\varphi \iff (\forall Q \supseteq P)(Q \nVdash \neg\varphi) \iff (\forall Q \supseteq P)(\exists Q' \supseteq Q)(Q' \Vdash \varphi) \quad \textbf{(1)}$$

Note the following:

(1) P cannot force both φ and $\neg\varphi$, $P \Vdash \neg\varphi$ requires that for all $Q \supseteq P$ (including P), $Q \nVdash \varphi$.

(2) If $P \Vdash \varphi$ then $Q \Vdash \varphi$ for $Q \supseteq \varphi$ (proven by induction on φ).

(3) If $P \Vdash \varphi$ then $P \Vdash^w \varphi$, which is direct from equivalence (1) and point (2) by taking $Q' = Q$.

(4) If $P \Vdash^w \neg\varphi$ then $P \Vdash \neg\varphi$, since if $P \Vdash^w \neg\varphi$ then $(\forall Q \supseteq P)(\exists Q' \supseteq Q)(\forall Q'' \supseteq Q')(Q'' \nVdash \varphi)$. Now suppose P didn't force $\neg\varphi$, then that would mean there exists a $Q \supseteq P$ which forces φ. But this obviously contradicts the assumption.

(5) $P \Vdash^w \neg\neg\varphi \iff P \Vdash^w \varphi$. This can be seen quite easily by looking at equivalence (1).

(6) If $P \Vdash \varphi$ and φ is a literal, then $P \cup \{\varphi\}$ is a forcing condition for T. The only interesting case is with $\neg\varphi$ where φ is an atomic sentence. Since $P \Vdash \neg\varphi$, $P \cup \{\varphi\}$ cannot be a forcing condition and so it is not consistent with T. Since $T \cup P$ is consistent, this means that $T \cup P \cup \{\neg\varphi\}$ is consistent.

1.1.2 Definition

Let P be a forcing condition on T, then we denote $T[P]$ the set of all $\mathcal{L}$-sentences forced by P. Similarly $T^f[P]$ is the set of all sentences weakly forced by P. Denote $T^f = T^f[\varnothing]$ and call it the **forcing companion** of T. If $T \equiv T^f$, then T is considered **forcing complete**.

Write $P \Vdash_A \varphi$ if P forces φ in $\mathcal{L}[A]$. Furthermore write $P = P(\bar{a})$ and $\varphi = \varphi(\bar{a})$ to mean that P is a forcing condition in $\mathcal{L}[A]$ and φ is a $\mathcal{L}[A]$-sentence, where $\bar{a}$ denotes all the constants not in $\mathcal{L}$ but occur in either P or φ.

1.1.3 Lemma

Let $P = P(\bar{a})$ be a condition of T in $\mathcal{L}[A]$ and $\varphi = \varphi(\bar{a})$ be a sentence of $\mathcal{L}[A]$ weakly forced by P ($\bar{a}$ are constants not in $\mathcal{L}$). Then for any closed terms $\bar{t}$, if $P(\bar{t})$ is a condition for T, then $P(\bar{t}) \Vdash^w \varphi(\bar{t})$.

1.1.4 Lemma

Let φ be an $\mathcal{L}$-sentence, then $T^f[P] \vDash \varphi$ implies $\varphi \in T^f[P]$. In particular, both $T^f[P]$ and $T[P]$ are consistent.

Proof: by completeness, it is sufficient to prove this for $T^f[P] \vdash \varphi$. We induct on the length of the proof that if $T^f[P] \vdash \varphi(\bar{x})$ then $\varphi(\bar{t}) \in T^f[P]$ for all sequence of closed terms of $\mathcal{L}[A]$. Let us skip the proof for logical axioms, as these are routine and tedious.
Now consider modus ponens, so $T^f[P] \vdash \varphi(\bar{x}) \to \psi(\bar{x}), \varphi(\bar{x})$ and so $\varphi(\bar{t}) \to \psi(\bar{t}), \varphi(\bar{t}) \in T^f[P]$ (we can assume the same sequence of closed terms since we can extend the set of variables to be shared). Thus we have

$$(\forall Q \supseteq P)(\exists Q' \supseteq Q)\ (Q' \Vdash \neg\varphi(\bar{t}) \text{ or } Q' \Vdash \psi(\bar{t}))$$
$$(\forall Q \supseteq P)(\exists Q' \supseteq Q)\ (Q' \Vdash \varphi(\bar{t}))$$

We claim that $P \Vdash^w \psi(\bar{t})$, so let us suppose otherwise. Let $Q \supseteq P$ then by the first equivalence there exist $Q' \supseteq P$ such that $Q' \Vdash \neg\varphi(\bar{t})$, but then by the second equivalence there exists a $Q'' \supseteq Q'$ such that $Q'' \Vdash \varphi(\bar{t})$. By monotonocity, we also have $Q'' \Vdash \neg\varphi(\bar{t})$ in contradiction.
Now consider generalization, so $\varphi(\bar{t}) \in T^f[P]$ and suppose $P \nVdash^w \forall\bar{x}\varphi(\bar{x}) = \neg\exists\bar{x}\neg\varphi(\bar{x})$. By definition, we get that

$$(\exists Q \supseteq P)(\exists\bar{t})(\forall Q' \supseteq Q)(Q' \nVdash \varphi(\bar{t}))$$

and since $P \Vdash^w \varphi(\bar{t})$, we get

$$(\forall\bar{t})(\forall Q \supseteq P)(\exists Q' \supseteq Q)(Q' \Vdash \varphi(\bar{t}))$$

and these obviously form a contradiction. ■

Recall that for a structure $\mathcal{A}$, the following are equivalent:

(1) $\Delta[\mathcal{A}] \cup T$ is consistent,

(2) $\mathcal{A}$ can be embedded into a T-model,

(3) Every finite $P \subseteq \Delta[\mathcal{A}]$ is a condition for T.

(1) $\iff$ (3) is trivial.

1.1.5 Lemma

Let $\mathcal{A}$ be a structure such that $\Delta[\mathcal{A}] \cup T$ is consistent, then for all existential formulas $\varphi(\bar{x})$ and closed terms $\bar{t}$ of $\mathcal{L}[\mathcal{A}]$,

$$\mathcal{A} \vDash \varphi(\bar{t}) \iff P \Vdash \varphi(\bar{t}) \text{ for some finite } P \subseteq \Delta[\mathcal{A}]$$

By existential formula, we mean a formula in the form

$$\exists\bar{x}\varphi$$

where φ is a CNF (or DNF).
Proof: only $\implies$ is shown. We prove this by induction on the structure of φ:

(1) For $\varphi(\bar{x})$ literal, we let $P = \{\varphi(\bar{t})\}$ (since $\mathcal{A} \vDash \varphi(\bar{t})$, $P \subseteq \Delta[\mathcal{A}]$ and by definition $P \Vdash \varphi(\bar{t})$).

(2) For $\varphi(\bar{t}) = \varphi_1(\bar{t}) \vee \varphi_2(\bar{t})$, wlog $\mathcal{A} \vDash \varphi_1(\bar{t})$ and so $P \Vdash \varphi_1(\bar{t})$ and thus $P \Vdash \varphi(\bar{t})$.

(3) For $\varphi(\bar{t}) = \varphi_1(\bar{t}) \wedge \varphi_2(\bar{t})$, we have $\mathcal{A} \vDash \varphi_1(\bar{t}), \varphi_2(\bar{t})$. Thus there exists finite $P, Q \subseteq \Delta[\mathcal{A}]$ such that $P \Vdash \varphi_1(\bar{t})$ and $Q \Vdash \varphi_2(\bar{t})$. Then $P \cup Q \subseteq \Delta[\mathcal{A}]$ is also finite and by monotonicity, $P \cup Q \Vdash \varphi_1(\bar{t}), \varphi_2(\bar{t})$ and so $P \cup Q \Vdash \varphi(\bar{t})$.

(4) For $\varphi(\bar{t}) = \exists\bar{x}\psi(\bar{x}, \bar{t})$, then $\mathcal{A} \vDash \psi(\bar{s}, \bar{t})$ for some closed terms $\bar{s}$ of $\mathcal{L}[\mathcal{A}]$. Thus $P \Vdash \psi(\bar{s}, \bar{t})$ for some finite $P \subseteq \Delta[\mathcal{A}]$, and so $P \Vdash \exists\bar{x}\psi(\bar{x}, \bar{t})$. ■

1.1.6 Lemma

Let φ be a universal sentence of $\mathcal{L}[\mathcal{A}]$, then for all conditions P, $P \Vdash \varphi$ if and only if $T \cup P \vDash \varphi$.

Proof: φ is universal iff it is of the form $\varphi = \neg\psi$ for ψ existential. So it is sufficient to show that $P \Vdash \neg\varphi$ iff $T \cup P \vDash \neg\varphi$ for φ existential. Let $\mathcal{A}$ model $T \cup P$ and so surely $P \subseteq \Delta[\mathcal{A}]$. If $P \Vdash \neg\varphi$, for every $Q \supseteq P$ we have $Q \nVdash \varphi$ and in particular for every $Q \subseteq \Delta[\mathcal{A}]$ it cannot be that $Q \Vdash \varphi$ (as then $Q \cup P \Vdash \varphi$). So by the above lemma, we have that $\mathcal{A} \nvDash \varphi$ so $\mathcal{A} \vDash \neg\varphi$, meaning $T \cup P \vDash \varphi$ as required. And if $\mathcal{A} \vDash \neg\varphi$ then $\mathcal{A} \nvDash \varphi$, so for any finite $Q \subseteq \Delta[\mathcal{A}]$, $Q \nVdash \varphi$. In particular for every $Q \supseteq P$, $Q \nVdash \varphi$ and so $P \Vdash \neg\varphi$. ■

Let T be a theory, its *universal part* denoted $T_\forall$ is all universal sentences consequent of T.

1.1.7 Lemma

Let P be a finite set of basic sentences of $\mathcal{L}[\mathcal{A}]$. P is a forcing condition for T iff P is a forcing condition for $T_\forall$. And for all $\mathcal{L}[\mathcal{A}]$-sentences φ, $P \Vdash \varphi$ in T if and only if $P \Vdash \varphi$ in $T_\forall$. That is, $T[P] = T_\forall[P]$ and moreso $T^f[P] = T^f_\forall[P]$.

Proof: if P is consistent with T, it must be consistent with $T_\forall$. So suppose P is consistent with $T_\forall$, and suppose it is not consistent with T. Let $\varphi(\bar{a}) = \bigwedge P$ where $\bar{a}$ are constants not in $\mathcal{L}$, then $T \vDash \neg\varphi(\bar{a})$ and since $\bar{a}$ doesn't occur in T, $T \vDash \forall\bar{x}\neg\varphi(\bar{x})$. Thus $\forall\bar{x}\neg\varphi(\bar{x}) \in T_\forall$ so $T_\forall \vDash \neg\varphi(\bar{a})$ which contradicts $T_\forall$ being consistent with P.

Since the forcing relation is totally determined by the forcing conditions, the rest follows immediately. ■

1.1.8 Lemma

Let $P(\bar{a})$ be a finite forcing condition in $\mathcal{L}[\mathcal{A}]$, and $\varphi(\bar{a})$ a sentence of $\mathcal{L}[\mathcal{A}]$ (where $\bar{a}$ lists constants not in $\mathcal{L}$). Let $P(\bar{x}), \varphi(\bar{x})$ be the results of substituting $\bar{a}$ with the variables $\bar{x}$. If $P(\bar{a}) \Vdash \varphi(\bar{a})$, then

$$\forall\bar{x}\Big(\bigwedge P(\bar{x}) \to \varphi(\bar{x})\Big) \in T^f$$

Proof: we need to show that

$$\varnothing \Vdash^w \neg\exists\bar{x}\neg\Big(\neg\bigwedge P(\bar{x}) \vee \varphi(\bar{x})\Big)$$

Since $P \Vdash \neg\varphi \iff P \Vdash^w \neg\varphi$, we shall show this for $\Vdash$ in place of $\Vdash^w$. Suppose not, then there exists a condition Q and a set of closed terms $\bar{t}$ such that

$$Q \Vdash \neg\Big(\neg\bigwedge P(\bar{t}) \vee \varphi(\bar{t})\Big)$$

This means that for all $Q' \supseteq Q$:

$$Q' \nVdash \neg\bigwedge P(\bar{t}) \text{ and } Q' \nVdash \varphi(\bar{t})$$

This means there exists a $Q'' \supseteq Q'$ such that $Q'' \Vdash \bigwedge P(\bar{t})$. Since $\bigwedge P(\bar{t})$ is universal (as it is quantifier-free), by a previous lemma this means $T \cup Q'' \vDash \bigwedge P(\bar{t})$. In particular $P(\bar{t}) \cup Q''$ is a condition for T. Since $P(\bar{a}) \Vdash \varphi(\bar{a})$ we have $P(\bar{t}) \Vdash \varphi(\bar{t})$ by a previous lemma (this seems like a mistake, it should be $\Vdash^w$, but the proof requires normal forcing...). Thus $Q \subseteq P(\bar{t}) \cup Q'' \Vdash \varphi(\bar{t})$, which is a contradiction. ■

1.1.9 Definition

Let $\mathcal{L}[A]$ be a countable language, T a $\mathcal{L}$-theory, and $\mathbb{P} = \{P_i\}_{i<\omega}$ a sequence of finite forcing conditions on T. $\mathbb{P}$ is called ***T*-generic** if

(1) for any atomic sentence φ, exactly one of $\varphi, \neg\varphi$ is in $\bigcup\mathbb{P}$,

(2) for any $\mathcal{L}[\mathcal{A}]$-sentence φ, exactly one of $\varphi, \neg\varphi$ is forced by some $P \in \mathbb{P}$.

1.1.10 Theorem

If $\mathcal{L}[A]$ is countable, then for every consistent $\mathcal{L}$-theory T there exists a T-generic sequence $\mathbb{P}$.

Proof: since $\mathcal{L}[A]$ is countable, we can enumerate its sentences by $\varphi_0, \varphi_1, \ldots$. Let $P_0 = \varnothing$ and assuming P_i is constructed we define P_i as follows: if $P_i \Vdash \neg\varphi_i$, then define

$$P_{i+1} = \begin{cases} P_i \cup \{\neg\varphi_i\} & \text{if } \varphi_i \text{ is atomic} \\ P_i & \text{else} \end{cases}$$

Otherwise $P_i \nVdash \neg\varphi_i$ so there exists a $Q \supseteq P_i$ such that $Q \vDash \varphi_i$, so set $P_{i+1} = Q$. Then let $\mathbb{P} = \{P_i\}_{i<\omega}$, this is an increasing sequence of conditions and so it is T-generic (since if $P_i \Vdash \varphi$ and $P_j \Vdash \neg\varphi$ then since $\mathbb{P}$ is increasing, one is contained in the other, so one forces both φ and $\neg\varphi$ in contradiction). ■

1.1.11 Definition

Let $\mathcal{A}$ be a $\mathcal{L}$-structure, then $\mathcal{A}$ is ***T*-generic** if the following two conditions hold:

(**1**) $T \cup \Delta[\mathcal{A}]$ is consistent,

(**2**) For every $\mathcal{L}[\mathcal{A}]$-sentence φ, $\mathcal{A} \vDash \varphi$ iff there exists a finite $P \subseteq \Delta[\mathcal{A}]$ which forces φ.

1.1.12 Theorem

Let $\mathcal{L}$ be countable, then for every T-generic sequence $\mathbb{P}$, there exists a countable T-generic structure $\mathcal{A}$ such that $\Delta[\mathcal{A}] = \bigcup\mathbb{P}$.

Proof: let $\mathcal{A}$ be the canonical term model over $\bigcup\mathbb{P}$, then we have that $\Delta[\mathcal{A}] = \bigcup\mathbb{P}$. Now, $T \cup \Delta[\mathcal{A}]$ is consistent because every finite $P \subseteq \Delta[\mathcal{A}]$ is a subset of some $P' \in \mathbb{P}$ and $T \cup P'$ is consistent. Furthermore to show that $\mathcal{A} \vDash \varphi \iff P \Vdash \varphi$ for some finite $P \subseteq \Delta[\mathcal{A}]$ it is sufficient to take some $P_i \in \mathbb{P}$. We prove this by induction.

(**1**) For basic sentences φ, $\mathcal{A} \vDash \varphi$ iff $\varphi \in \Delta[\mathcal{A}] = \bigcup\mathbb{P}$ iff there exists some $P_i \in \mathbb{P}$ which contains, and thus forces, φ.

(**2**) For $\vee, \wedge$ the step is trivial.

(**3**) $\mathcal{A} \vDash \exists\bar{x}\varphi(\bar{x}, \bar{a})$ if and only if there exists closed $\mathcal{L}[A]$-terms $\bar{t}$ such that $\mathcal{A} \vDash \varphi(\bar{t}, \bar{a})$ which by induction means $P_i \Vdash \varphi(\bar{t}, \bar{a})$ for some $P_i \in \mathbb{P}$ and then $P_i \Vdash \exists\bar{x}\varphi(\bar{x}, \bar{a})$.

(**4**) $\mathcal{A} \vDash \neg\varphi$ iff $\mathcal{A} \nvDash \varphi$ iff $P_i \nVdash \varphi$ for all $P_i \in \mathbb{P}$ iff $P_i \Vdash \neg\varphi$ for some $P_i \in \mathbb{P}$ (since $\mathbb{P}$ is generic). ■

Conversely, if $\mathcal{A}$ is a countable T-generic structure, then the set of finite subsets of $\Delta[\mathcal{A}]$ is a T-generic sequence. The following is immediate from the previous proof and this comment:

1.1.13 Corollary

A countable $\mathcal{L}$-structure $\mathcal{A}$ is T-generic iff there exists a T-generic sequence $\mathbb{P}$ such that $\Delta[\mathcal{A}] = \bigcup\mathbb{P}$.

1.1.14 Lemma

Any T-generic structure is a model of the forcing companion of T, T^f.

Proof: suppose $\varphi \in T^f$, then $\varnothing \Vdash \neg\neg\varphi$, and $\varnothing \subseteq \Delta[\mathcal{A}]$ is finite, so $\mathcal{A} \vDash \varphi$. ■

1.1.15 Definition

A first-order theory T is **model-complete** if every embedding betwee T-models is elementary.

1.1.16 Theorem

T is model-complete iff every formula is equivalent to an existential formula modulo T.

1.1.17 Lemma

T is model-complete iff for any T-model $\mathcal{A}$, $T \cup \Delta[\mathcal{A}]$ is complete in $\mathcal{L}[\mathcal{A}]$.

Proof: if T is model complete, let $\mathcal{A} \vDash T$. Now suppose $\mathcal{B} \vDash T \cup \Delta[\mathcal{A}]$, so $\mathcal{B}$ is a T-model which $\mathcal{A}$ is embeddable in, meaning $\mathcal{A} \equiv \mathcal{B}$ in $\mathcal{L}[\mathcal{A}]$. Thus all models of $T \cup \Delta[\mathcal{A}]$ are elementarily equivalent, meaning $T \cup \Delta[\mathcal{A}]$ is complete. And conversely if $T \cup \Delta[\mathcal{A}]$ is complete, suppose $\mathcal{A}$ is embeddable into a T-model $\mathcal{B}$, so $\mathcal{B} \vDash T \cup \Delta[\mathcal{A}]$, then $\mathcal{A} \equiv \mathcal{B}$ in $\mathcal{L}[\mathcal{A}]$. And this means that the embedding from $\mathcal{A}$ to $\mathcal{B}$ is elementary. ■

1.1.18 Definition

Let T, T^* be two $\mathcal{L}$-theories with $T \subseteq T^*$. We say that T^* is the **model completion** of T if for any T-model $\mathcal{A}$, $T^* \cup \Delta[\mathcal{A}]$ is complete in $\mathcal{L}[\mathcal{A}]$.

Note the following:

(1) If T^* is the model completion of T, then any model of T can be embedded into a model of T^*. This is because $T^* \cup \Delta[\mathcal{A}]$ is consistent, and thus there is a T^*-model in which $\mathcal{A}$ is embeddable.

(2) If T^* is the model completion of T, T^* is model-complete. This is because a model of T^* is a model of T (since $T \subseteq T^*$) and so $T^* \cup \Delta[\mathcal{A}]$ is complete.

(3) If T is model complete, then T is its own model completion. This is direct from the previous lemma.

(4) If T_1^*, T_2^* are two model completions of T then they are logically equivalent. This will be proven in more depth later.

1.1.19 Theorem

A T^f-model $\mathcal{A}$ is T-generic iff $T^f \cup \Delta[\mathcal{A}]$ is complete.

Proof: ($\Longrightarrow$) suppose $\mathcal{A} \vDash T^f$ is T-generic. Then let $\mathcal{A} \vDash \varphi$ for $\varphi \in \mathcal{L}[\mathcal{A}]$, then by definition there exists a finite $P \subseteq \Delta[\mathcal{A}]$ such that $P \Vdash \varphi$. Let $P = P(\bar{a})$ and $\varphi = \varphi(\bar{a})$ where $\bar{a}$ lists constants not in $\mathcal{L}$, then by lemma 1.1.8 we have that $\forall \bar{x}(\bigwedge P(\bar{x}) \to \varphi(\bar{x}))$ is in T^f. Thus $T^f \vDash \bigwedge P(\bar{a}) \to \varphi(\bar{a})$. Since $\Delta[\mathcal{A}] \vDash \bigwedge P(\bar{a})$, we have $T^f \cup \Delta[\mathcal{A}] \vDash \varphi(\bar{a})$, so it is complete (since every φ is either satisfied or its negation is by $\mathcal{A}$).

($\Longleftarrow$) suppose that $T^f \cup \Delta[\mathcal{A}]$ is complete, then we must show the following:

(1) $T \cup \Delta[\mathcal{A}]$ is consistent: this will be proven in more generality later.

(2) $\mathcal{A} \vDash \varphi$ iff $P \Vdash \varphi$ for some finite $P \subseteq \Delta[\mathcal{A}]$. Since we have already proven this for existential φ by induction, it is sufficient to induct on the case that φ is negated: $\varphi = \neg\psi$. Suppose $\mathcal{A} \vDash \neg\psi$, then by completeness $T^f \cup \Delta[\mathcal{A}] \vDash \neg\psi$. Then by compactness there exists a finite $P \subseteq \Delta[\mathcal{A}]$ such that $T^f \cup P \vDash \neg\psi$, we claim that $P \Vdash \neg\psi$. Otherwise, there exists a $Q \supseteq P$ such that $Q \Vdash \psi$, and so

$$T^f \vDash \forall \bar{x}\Big(\bigwedge Q(\bar{x}) \to \psi(\bar{x})\Big)$$

Then

$$T^f[Q] \vDash \forall \bar{x}\Big(\bigwedge Q(\bar{x}) \to \psi(\bar{x})\Big)$$

and since $T^f[Q] \vDash \bigwedge Q(\bar{a})$, we have that $T^f[Q] \vDash \psi(\bar{a})$. But $T^f \cup P \subseteq T^f[P] \subseteq T^f[Q]$ and so by completeness, $T^f \cup P \vDash \psi(\bar{a})$, in contradiction.

Now conversely, suppose $P \Vdash \varphi$ for some finite $P \subseteq \Delta[\mathcal{A}]$, then for any $Q \supseteq P$, $Q \nVdash \psi$. We then claim there is no $Q \subseteq \Delta[\mathcal{A}]$ which forces ψ. As otherwise by montonocity $P \cup Q$ would force ψ. So by induction $\mathcal{A} \nvDash \psi$, meaning $\mathcal{A} \vDash \neg\psi$ as required. ■

An immediate consequence of this theorem is

1.1.20 Corollary

T^f is model-complete iff every one of its models is T-generic.

1.1.21 Definition

Call a class of structures ***K* inductive** if it is closed under union of chains.

1.1.22 Theorem

The class of T-generic structures is inductive.

Proof: let $\{\mathcal{A}_\alpha\}_{\alpha<\lambda}$ be an increasing chain of T-generic structures, where λ is a limit ordinal. We know that every T-generic structure is a model of T^f, and so $T^f \cup \Delta[\mathcal{A}_\alpha]$ is complete in $\mathcal{L}[\mathcal{A}_\alpha]$. Since $\mathcal{A}_{\alpha+1} \vDash T^f \cup \Delta[\mathcal{A}_\alpha]$ as $\mathcal{A}_\alpha \subseteq \mathcal{A}_{\alpha+1}$, we have that $\mathcal{A}_\alpha \equiv \mathcal{A}_{\alpha+1}$ in $\mathcal{L}[\mathcal{A}_\alpha]$, meaning $\mathcal{A}_\alpha \preceq \mathcal{A}_{\alpha+1}$. By Tarski's elementary chain lemma, this means that $\mathcal{A}_\alpha \preceq \bigcup_{\alpha<\lambda} \mathcal{A}_\alpha = \mathcal{A}$. Since every finite subset of $\Delta[\mathcal{A}]$ contains symbols entirely from some $\mathcal{A}_\alpha$, $T \cup \Delta[\mathcal{A}]$ is consistent (take a finite subset, it is contained in $T \cup \Delta[\mathcal{A}_\alpha]$ which is consistent). Let φ be a sentence of $\mathcal{L}[\mathcal{A}]$, then since there are only finitely many constant symbols in φ, it is a sentence of some $\mathcal{L}[\mathcal{A}_\alpha]$. Then

$$\mathcal{A} \vDash \varphi \iff \mathcal{A}_\alpha \vDash \varphi \iff P \Vdash \varphi \text{ for some finite } P \subseteq \Delta[\mathcal{A}_\alpha]$$

now, $P \subseteq \Delta[\mathcal{A}]$ as well. And if such a P exists, then it has only finitely many symbols in $\mathcal{A}$, so it is in $\Delta[\mathcal{A}_\alpha]$. ■

1.1.23 Definition

Let T_1, T_2 be first-order theories. T_1 is **model-consistent** with T_2 if every T_2-model can be embedded into a T_1-model. If T_1 is model-consistent with T_2 and T_2 is model-consistent with T_1, then T_1 and T_2 are **mutually model-consistent**.

1.1.24 Lemma

T_1 is model-consistent with T_2 iff $(T_1)_\forall \subseteq (T_2)_\forall$. In particular they are mututally model-consistent if $(T_1)_\forall = (T_2)_\forall$.

Proof: recall that $\mathcal{A} \vDash T_\forall$ iff $\mathcal{A}$ can be embedded into a T-model. Thus T_1 is model-consistent with T_2 if $\mathcal{A} \vDash T_2 \implies \mathcal{A} \vDash (T_1)_\forall$, which is equivalent to $T_2 \vDash (T_1)_\forall$. In turn this is equivalent to $(T_1)_\forall \subseteq (T_2)_\forall$, as required. ■

1.1.25 Lemma

T^f is model-consistent with T for any theory.

Proof: let $\varphi \in (T^f)_\forall$, by lemma 1.1.4, we have that $\varphi \in T^f$, i.e. $\varnothing \Vdash \varphi$, and by lemma 1.1.6 $T \vDash \varphi$ and so $\varphi \in T_\forall$ as required. ■

1.1.26 Lemma

For any theory T, $T_{\forall\exists} \subseteq T^f$, in particular $T_\forall \subseteq (T^f)_\forall$.

Proof: let $\varphi \in T_{\forall\exists}$, so $\varphi = \neg\exists\bar{x}\neg\psi(\bar{x})$ where $\psi(\bar{x})$ is existential. Now suppose $\varphi \notin T^f$, meaning $\varnothing \nVdash \varphi$ so there exists a condition $P \Vdash \exists\bar{x}\neg\psi(\bar{x})$. So there are closed $\mathcal{L}[A]$-terms $\bar{t}$ such that $P \Vdash \neg\psi(\bar{t})$. Since $\neg\psi(\bar{t})$ is universal, by lemma 1.1.6, $T \cup P \vDash \neg\psi(\bar{t})$. But this contradicts $T \vDash \forall\bar{x}\psi(\bar{x})$.
Note that a $\forall$-formula is a $\forall\exists$-formula, so $T_\forall \subseteq T^f$ and thus $T_\forall \subseteq (T^f)_\forall$. ■

As a direct consequence of the previous lemma:

1.1.27 Corollary

If T is a $\forall\exists$-theory, then $T \subseteq T^f$ (so T^f is the forcing completion of T).

And as a direct consequence of the previous two lemmas, we have

1.1.28 Theorem

T and T^f are mutually model-consistent.

1.1.29 Corollary

Let T_1, T_2 be two first-order theories. Then they are mutually model-consistent if and only if $T_1^f = T_2^f$.

Proof: ($\Longrightarrow$) if T_1 and T_2 are mutually model-consistent then $(T_1)_\forall = (T_2)_\forall$. By lemma 1.1.7 $T_\forall^f = T^f$ so

$$T_1^f = ((T_1)_\forall)^f = ((T_2)_\forall)^f = T_2^f$$

as required. ($\Longleftarrow$) by the above theorem we have $T_\forall = (T^f)_\forall$. So

$$(T_1)_\forall = (T_1^f)_\forall = (T_2^f)_\forall = (T_2)_\forall$$

as required. ■

1.1.30 Corollary

Let T_1 and T_2 be two mutually model-consistent theories. Then a structure $\mathcal{A}$ is T_1-generic iff it is T_2-generic.

Proof: by lemma 1.1.7, a condition for T is a condition for $T_\forall$. So P is a condition for T_1 iff it is a condition for $(T_1)_\forall = (T_2)_\forall$ iff it is a condition for T_2. So $\Delta[\mathcal{A}]$ is consistent with T_1 (i.e. a condition) iff it is consistent with T_2, this proves the first condition. And for any condition P, $P \Vdash \varphi$ for T_1 iff $P \Vdash \varphi$ for $(T_1)_\forall = (T_2)_\forall$ iff $P \Vdash \varphi$ for T_2. So if $\mathcal{A}$ is T_1-generic then $\mathcal{A} \vDash \varphi$ iff $P \Vdash \varphi$ for T_1 for some finite $P \subseteq \Delta[\mathcal{A}]$ iff $P \Vdash \varphi$ for T_2. Meaning $\mathcal{A}$ is T_2-generic. ■

1.1.31 Definition

If T is an $\mathcal{L}$-theory, a $\mathcal{L}$-theory T' is T's **model companion** if

(1) T and T' are mutually model-consistent.

(2) T' is model complete.

Recall that if T^* is the model completion of T, then it is model complete and every model of T can be embedded into a model of T^*. Since T^* is an extension of T, every model of T^* is a model of T and thus can be trivially embedded into itself. So T and T^* are mutually model-consistent. Thus a model completion is a model companion.

1.1.32 Theorem

If T_1, T_2 are model companions of T then they are logically equivalent. In particular two model completions of T are logically equivalent.

Proof: we will show that a model of T_1 is a model of T_2. Let $\mathcal{A}_0 \vDash T_1$ be a T_1-model. Then since T_1, T_2 ar mutually model-consistent, it can be embedded into a T_2-model, which can be embedded into a T_1-model, and so on. So we get a chain

$$\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots$$

where $\mathcal{A}_{2i} \vDash T_1$ and $\mathcal{A}_{2i+1} \vDash T_2$. Since T_1 and T_2 are both model complete,

$$\mathcal{A}_0 \subseteq \mathcal{A}_2 \subseteq \mathcal{A}_4 \subseteq \cdots, \qquad \mathcal{A}_1 \subseteq \mathcal{A}_3 \subseteq \mathcal{A}_5 \subseteq \cdots$$

are both elementary chains, and so

$$\mathcal{A} = \bigcup_{i<\omega} \mathcal{A}_i = \bigcup_{2i<\omega} \mathcal{A}_{2i} = \bigcup_{2i+1<\omega} \mathcal{A}_{2i+1}$$

is an elementary extension of $\mathcal{A}_i$ by the Tarski chain theorem. So $\mathcal{A} \vDash T_2$ since $\mathcal{A}_1 \vDash T_2$, and since $\mathcal{A}_0 \equiv \mathcal{A}$, we have $\mathcal{A}_0 \vDash T_2$ as required. ■

1.1.33 Corollary

If a theory T has a model companion T', then T' and T^f are logically equivalent.

Proof: since T' is model-complete, by theorem 1.1.16 it is equivalent to a set of existential sentences. Thus it is a $\forall\exists$-theory, and so $T' \subseteq (T')^f$ by corollary 1.1.27. Since T and T' are by definition mutually model-consistent, $(T')^f = T^f$, so $T' \subseteq T^f$. Since T^f is a superset of a model-complete theory, it too is therefore model-complete. Thus T and T^f are mutually model-complete and T^f is model-complete, so T^f is T's model companion. By the previous theorem, T' and T^f are therefore logically equivalent. ■

1.1.34 Theorem

A theory T is model-complete iff every model of T is T-generic.

Proof: ($\Longrightarrow$) if T is model-complete it is its own model companion. So T and T^f are logically equivalent, so by corollary 1.1.20 every model of T is T-generic.

($\Longleftarrow$) if every T-model is T-generic, then every model of T models T^f by lemma 1.1.14. So by theorem 1.1.19 for every T-model $\mathcal{A}$, it is a T^f-model and so $T^f \cup \Delta[\mathcal{A}]$ and thus $T \cup \Delta[\mathcal{A}]$ is complete. Thus T is model-complete. ■

1.1.35 Corollary

Every model-complete theory is forcing-complete.

Proof: this is because every T-model is T-generic, and so every model of T models T^f. Thus $T \vDash T^f$, meaning $T^f \subseteq T$. Since T is a $\forall\exists$-theory, we showed that $T \subseteq T^f$. So $T = T^f$, and T is forcing-complete. ■