Complex Functions

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Ari Feiglin

Definition 9.1:

Let $z_0 \in \mathbb{C}$ and $r > 0$, then we call the set $D_r(z_0) \setminus \{z_0\}$ a pointed disk about (or around) z_0.

Definition 9.2:

Suppose f is a function defined in a pointed disk about z_0. z_0 is an **isolated singular point** of f, if there exists a pointed disk about z_0 in which f is analytic, but f is not analytic (or even not defined) at z_0.
There are three types of isolated singular points:

(1) If there exists $g(z)$ analytic in a neighborhood of z_0 where $f = g$ in this nighborhood without z_0, then z_0 is a **removable singularity**. Ie. if we can set $f(z_0) = w_0$ so that z_0 is no longer singular, it is removable.

(2) If there exists analytic functions at z_0, A and B, such that $A(z_0) \neq 0$ and $B(z_0) = 0$ where $f = \frac{A}{B}$ in a pointed neighborhood of z_0, then z_0 is a **pole** of f. If z_0 is a zero of degree k of B, then z_0 is a **pole of degree k** of f.

(3) If z_0 is not removable or a pole, then z_0 is an **essential singularity**.

Theorem 9.3 (Riemann's Criterion For Removable Singularities):

Let z_0 be an isolated singularity of f, where $\lim_{z\to z_0}(z - z_0)f(z) = 0$, then z_0 is removable.

Proof:

Let

$$h(z) = \begin{cases} (z - z_0)^2 f(z) & z \neq z_0 \\ 0 & z = z_0 \end{cases}$$

Then h is analytic on $D \setminus \{z_0\}$ and for $z = z_0$ then

$$h'(z_0) = \lim_{z\to z_0}(z - z_0)f(z) = 0$$

and so h is analytic on $z = z_0$ as well, so by the theorem from last lecture (take any line through D, then h is analytic on the line and on $D \setminus L$, so h is analytic). Note that $h(z_0) = 0$ and $h'(z_0) = 0$ and so h's taylor series is of the form

$$h(z) = c_2(z - z_0)^2 + c_3(z - z_0)^3 + \cdots$$

Since for $z \neq z_0$,

$$f(z) = \frac{h(z)}{(z - z_0)^2} = c_2 + c_3(z - z_0) + \cdots$$

which is analytic on D, and thus an analytic extension of f to D, and so z_0 is removable. ■

Corollary 9.4:

If f is bounded in a pointed neighborhood of an isolated singularity z_0 then z_0 is removable.

Proof:

Note that

$$|(z - z_0)f(z)| \leq M|z - z_0|$$

which goes to 0 as $z \to z_0$, and thus we have that $\lim_{z\to z_0}(z - z_0)f(z) = 0$ and thus the singularity is removable by the above theorem. ■

Theorem 9.5:

If f is analytic on a pointed neighborhood of z_0 such that there exists a $k \in \mathbb{N}$ where

$$\lim_{z \longrightarrow z_0})(z-z_0)^k f(z) \neq 0 \quad \text{and} \quad \lim_{z \longrightarrow z_0}(z-z_0)^{k+1} f(z) = 0$$

then z_0 is a pole.

Proof:

Let us define, similar to as before,

$$h(z) = \begin{cases} (z-z_0)^{k+2} f(z) & z \neq z_0 \\ 0 & z = z_0 \end{cases}$$

then h is analytic on $D \setminus \{z_0\}$ and for $z = z_0$ we have

$$h'(z_0) = \lim_{z \longrightarrow z_0}(z-z_0)^{k+1} f(z) = 0$$

and so h is analytic on D. Since $h(z_0) = 0$ and $h'(z_0) = 0$, its taylor series starts at $(z-z_0)^2$ and so if we define

$$A(z) = \frac{h(z)}{(z-z_0)^2} = c_2 + c_3(z-z_0) + \cdots$$

$A(z)$ is analytic. But $A(z) = (z-z_0)^k f(z)$ as well, and so $A(z_0) = \lim(z-z_0)^k f(z) \neq 0$ and so

$$f(z) = \frac{A(z)}{(z-z_0)^k}$$

and so z_0 is a pole singularity. ■

This is a generalization of Riemanns criterion.

9.1 Laurent Series

Definition 9.1:

Suppose $\{\mu_k\}_{k=-\infty}^{\infty} \in \mathbb{C}$ and $L \in \mathbb{C}$ then we say

$$\sum_{k=-\infty}^{\infty} \mu_k = L$$

if and only if the series

$$\sum_{k=0}^{\infty} \mu_k, \quad \sum_{k=1}^{\infty} \mu_{-k}$$

converge and their sum is L.

Theorem 9.2:

The function

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k$$

is convergent and analytic in the ring $D(R_1, R_2) = \{z \mid R_1 < |z| < R_2\}$ where

$$R_2 = \frac{1}{\limsup |a_k|^{\frac{1}{k}}}, \quad R_1 = \limsup |a_{-k}|^{\frac{1}{k}}$$

if $R_1 < R_2$.

Proof:

Let us define

$$f_1(z) = \sum_{k=1}^{\infty} a_{-k} z^{-k}$$

and

$$f_2(z) = \sum_{k=0}^{\infty} a_k z^k$$

Then if $f_1(z)$ and $f_2(z)$ converge, we have $f(z) = f_1(z) + f_2(z)$ (this is if and only if, too). Since f_2 is a (positive) powerseries, it has a radius of convergence which is precisely R_2, so f_2 converges (and is analytic) in $D_{R_2}(0)$.
Let us define for $z \neq 0$

$$g(z) = f\left(\frac{1}{z}\right) = \sum_{k=1}^{\infty} a_{-k} z^k$$

which has a radius of convergence of $\frac{1}{R_1}$, and so $g(z)$ converges when $z < \frac{1}{R_1}$. And for $z < \frac{1}{R_1}$, g is analytic. And since $f_1(z)$ converges if and only if $g\left(\frac{1}{z}\right)$ does, we have $f_1(z)$ converges when $R_1 < |z|$. And since $g(z)$ is analytic in $D_{R_1}(z) \setminus \{0\}$, f_1 is analytic in $\{z \mid |z| > R_1\}$.
So when $R_1 < z < R_2$, $f_1(z)$ and $f_2(z)$ are analytic and convergent and thus so is f. ■

Definition 9.3:

A series of the form

$$\sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

is called a **Laurent series** about z_0.

What we have shown is that a Laurent series about z_0 is convergent and analytic in

$$D(z_0, R_1, R_2) = \{|z| \mid R_1 < |z - z_0| < R_2\}$$

We get the general case for $z_0 \neq 0$ by shifting the function $f(z)$ to $f(z + z_0)$, and applying the above theorem.

Theorem 9.4:

If f is analytic in the ring $D(R_1, R_2)$ then f has a Laurent series in the same ring.

Proof:

Let $R_1 < r_1 < r_2 < R_2$, and $r_1 < |z| < r_2$, for every $w \in D$ (the ring) define

$$g(w) = \begin{cases} \frac{f(w)-f(z)}{w-z} & w \neq z \\ f'(z) & w = z \end{cases}$$

is analytic in D, and so by Cauchy

$$\int_{\partial D} g = 0$$

And by Cauchy we know

$$2\pi i = \int_{C_{r_2}} \frac{1}{w-z}\, dw$$

and so since $w \mapsto \frac{1}{w-z}$ is analytic in $D_{r_1}(w)$, we have

$$2\pi i f(z) = f(z)\int_{C_{r_2}} \frac{1}{w-z}\, dw + f(z)\int_{-C_{r_1}} \frac{1}{w-z}\, dw = \int_{\partial D} \frac{f(z)}{w-z}\, dw$$

And we know

$$0 = \int_{\partial D} g = \int_{\partial D} \frac{f(w)-f(z)}{w-z}\, dw$$

and so we have

$$\int_{\partial D} \frac{f(z)}{w-z}\, dw = \int_{\partial D} \frac{f(w)}{w-z}\, dw$$

and thus

$$2\pi i f(z) = \int_{\partial D} \frac{f(w)}{w-z}\, dw = \int_{C_{r_2}} \frac{f(w)}{w-z}\, dw + \int_{-C_{r_1}} \frac{f(w)}{w-z}\, dw$$

Note that for $w \in C_{r_2}(0)$, $|z| < |w|$ and so

$$\frac{1}{w-z} = \sum_{k=0}^{\infty} \frac{z^k}{w^{k+1}}$$

and for $w \in C_{r_1}(0)$, $|z| > |w|$ and so

$$\frac{1}{w-z} = -\sum_{k=0}^{\infty} \frac{w^k}{z^{k+1}}$$

And so

$$2\pi i f(z) = \int_{C_2} \sum_{k=0}^{\infty} \frac{f(w)z^k}{w^{k+1}} + \int_{-C_1} \sum_{k=0}^{\infty} \frac{f(w)w^k}{z^{k+1}}$$

since the series converge uniformly, we have that

$$= \sum_{k=0}^{\infty} z^k \int_{C_2} \frac{f(w)}{w^{k+1}} + \sum_{k=0}^{\infty} z^{-k-1} \int_{-C_1} f(w)w^k = \sum_{k=0}^{\infty} z^k \int_{C_2} \frac{f(w)}{w^{k+1}} + \sum_{k=-1}^{-\infty} z^k \int_{-C_1} \frac{f(w)}{w^{k+1}}$$

Since $\frac{f(w)}{w^{k+1}}$ is analytic on D, we can swap C_{r_1} and C_{r_2} with C_r for $R_1 < r < R_2$, and so we have

$$f(z) = \frac{1}{2\pi i} \sum_{k=-\infty}^{\infty} z^k \int_{C_r} \frac{f(w)}{w^{k+1}}\, dw$$

■

Proposition 9.5:

If f is analytic in $D(R_1, R_2)$ and

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k$$

then

$$a_k = \frac{1}{2\pi i} \int_{C_r} \frac{f(w)}{w^{k+1}}$$

Proof:

Note

$$\int_{C_r} \frac{f(z)}{z^{k+1}} = \int_{C_r} \sum_{n=-\infty}^{\infty} a_n z^{n-k-1} = \sum_{n=-\infty}^{\infty} a_n \int_{C_r} z^{n-k-1}$$

the integral

$$\int_{C_r} z^n = \begin{cases} 2\pi i & n = -1 \\ 0 & n \neq -1 \end{cases}$$

and thus

$$\int_{C_r} \frac{f(z)}{z^{k+1}} = 2\pi i a_k$$

as required. ■

So if z_0 is an isolated singularity, we can take $R_1 = 0$ and so there exists a Laurent series such that

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

Furthermore, if $a_k = 0$ for every $k < 0$ then this defines an Taylor series, and thus f is equal to an analytic function defined on z_0, so z_0 is removable (this is equivalent).
And if there exists a $k \geq 1$ such that $a_{-k} \neq 0$ but $a_t = 0$ for $t < -k$ then $(z - z_0)^k f(z) = a_{-k} \neq 0$ and $(z - z_0)^{k+1} = 0$ and so z_0 is a pole of degree k (this is equivalent).
Otherwise if there exist infinite $a_k \neq 0$ for $k < 0$ then z_0 is essential.

Definition 9.6:

For a Lauren series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

then $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ is the **analytic part** of f about z_0, and $\sum_{k=-1}^{-\infty} a_k (z - z_0)^k$ is the **essential part** of f about z_0.

Definition 9.7:

If $f(z) = \sum_{k=-\infty}^{\infty} c_k (z - z_0)^k$ in a pointed neighborhood of z_0 then c_{-1} is **f's residue at z_0**. This is denoted

$$c_{-1} = \operatorname{Res}(f, z_0)$$

Proposition 9.8:

If f has a simple pole (pole of degree 1) at z_0 then

(1) $c_{-1} = \lim_{z \to z_0} (z - z_0) f(z)$

(2) If $f(z) = \frac{A(z)}{B(z)}$ for A and B analytic about z_0 and $A(z_0) \neq 0$ and z_0 is a first order root of B, then

$$c_{-1} = \frac{A(z_0)}{B'(z_0)}$$

Proof:

Since z_0 is a simple pole, if $c_{-k} = 0$ for every $k > 1$. Thus

$$f(z) = c_{-1}(z - z_0)^{-1} + c_0 + c_1(z - z_0) + \cdots$$

Thus

$$(z-z_0)f(z) = c_{-1} + c_0(z-z_0) + \cdots$$

and so the limit is c_{-1}.
And since

$$c_{-1} = \lim_{z\to z_0}(z-z_0)\frac{A(z)}{B(z)} = \lim_{z\to z_0} A(z)\cdot\frac{z-z_0}{B(z)} = A(z_0)\cdot\frac{1}{B'(z_0)}$$

since the limits exist (the functions are analytic) and $B'(z_0) \neq 0$. ■