

# Algebraic Topology I

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# 1 Categories

## 1.0.1 Definition

A **category**  $\mathcal{C}$  is a mathematical object which contains the following

- (1) a class of objects  $\text{ob}(\mathcal{C})$  (the objects need not be sets),
- (2) for every two objects  $A, B \in \text{ob}(\mathcal{C})$  a class of **morphisms**  $\text{Mor}(A, B)$ ,
- (3) an operation on morphisms  $\circ$ , where for every  $f \in \text{Mor}(A, B)$  and  $g \in \text{Mor}(B, C)$ ,  $g \circ f \in \text{Mor}(A, C)$ ,
- (4) for every object  $A \in \text{ob}(\mathcal{C})$  there exists an identity morphism  $1_A \in \text{Mor}(A, A)$  where for every  $A, B \in \text{ob}(\mathcal{C})$  and  $f \in \text{Mor}(A, B)$ ,  $f \circ 1_A = 1_B \circ f = f$ ,
- (5) for every  $A, B, C, D \in \text{ob}(\mathcal{C})$  and  $f \in \text{Mor}(A, B), g \in \text{Mor}(B, C), h \in \text{Mor}(C, D)$ , there is associativity:  $(h \circ g) \circ f = h \circ (g \circ f)$ .

Although morphisms are not necessarily functions, we use similar notation: both  $f: A \longrightarrow B$  and  $A \xrightarrow{f} B$  are to be understood to mean  $f \in \text{Mor}(A, B)$ . And we write  $A \in \mathcal{C}$  to mean  $A \in \text{ob}(\mathcal{C})$ .

Notice that for every  $A \in \mathcal{C}$ ,  $1_A$  is unique: suppose  $1_A$  and  $1'_A$  are both identity morphisms then  $1_A \circ 1'_A = 1_A$  since  $1'_A$  is an identity, but  $1_A \circ 1'_A = 1'_A$  since  $1_A$  is an identity so  $1_A = 1'_A$ .

## 1.0.2 Definition

Suppose  $\mathcal{C}$  and  $\mathcal{D}$  are two categories, a **functor**  $F$  from  $\mathcal{C}$  to  $\mathcal{D}$  is a correspondence where for every  $A \in \mathcal{C}$  there is defined a single  $F(A) \in \mathcal{D}$ , and for every  $f \in \text{Mor}(A, B)$  there exists a unique  $F(f) \in \text{Mor}(F(A), F(B))$  such that for all  $A, B, C \in \mathcal{C}$  and  $f \in \text{Mor}(A, B)$  and  $g \in \text{Mor}(B, C)$  we have that  $F(g \circ f) = F(g) \circ F(f)$  and  $F(1_A) = 1_{F(A)}$ .

## 1.0.3 Example

The following are examples of categories:

- (1) The category of all groups, morphisms are taken to be homomorphisms between groups;
- (2) The category of all topological spaces, morphisms are taken to be homeomorphisms;
- (3) The category of all sets, the morphisms are taken to be set functions;
- (4) The category of pairs of topological spaces: the objects are of the form  $(X, A)$  where  $X$  is a topological space and  $A \subseteq X$ . Morphisms between  $(X, A)$  and  $(Y, B)$  of this category are continuous functions  $f$  between  $X$  and  $Y$  such that  $f(A) \subseteq B$ .
- (5) The category of pointed topological spaces: the objects are  $(X, a)$  where  $X$  is a topological space and  $a \in X$  and the morphisms between  $(X, a)$  and  $(Y, b)$  are continuous functions between  $X$  and  $Y$  such that  $a \mapsto b$ .

An example of a functor is the so-called *forgetful functor* from the category of topological spaces to the category of sets: map a topological space to itself as a pure set.

This course will focus on a specific functor between the category of pointed topological spaces to the category of groups.

## 1.0.4 Definition

Let  $\mathcal{C}$  be a category, and  $A, B \in \mathcal{C}$ . A morphism  $f: A \longrightarrow B$  is an **isomorphism** if there exists a morphism  $g: B \longrightarrow A$  such that  $g \circ f = 1_A$  and  $f \circ g = 1_B$ . Such a  $g$  is called the **inverse** of  $f$  and is denoted  $f^{-1}$ .

(notice that by symmetry the inverse is also an isomorphism). If there exists an isomorphism between  $A$  and  $B$ , we denote this by  $A \cong B$  and  $A$  and  $B$  are called **isomorphic**.

Inverses are unique: if  $g_1$  and  $g_2$  are inverses of  $f$  then  $(g_1 \circ f) \circ g_2 = 1_A \circ g_2 = g_2$  but  $g_1 \circ (f \circ g_2) = g_1 \circ 1_B = g_1$  and by associativity these are equal. Furthermore the composition of isomorphisms is an isomorphism: it is easily verified that  $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$ . Notice that  $1_A$  is an isomorphism and it is its own inverse.

### 1.0.5 Proposition

A functor maps isomorphisms to isomorphisms, in particular  $F(f^{-1}) = F(f)^{-1}$  if  $f: A \longrightarrow B$  is an isomorphism.

**Proof:** notice that  $F(f) \circ F(f^{-1}) = F(f \circ f^{-1}) = F(1_B) = 1_{F(B)}$  and  $F(f^{-1}) \circ F(f) = F(f^{-1} \circ f) = F(1_A) = 1_{F(A)}$ . So  $F(f^{-1})$  is indeed the inverse of  $F(f)$ . ■

## 1.1 Homotopies

### 1.1.1 Definition

Let  $X$  and  $Y$  be topological spaces and  $f, g: X \longrightarrow Y$  (meaning they are morphisms, thus continuous). We say that  $f$  is homotopic to  $g$ , denoted  $f \sim g$ , if there exists an  $H: X \times I \longrightarrow Y$  ( $I = [0, 1]$ ,  $X \times I$  is the product topology) such that  $H(x, 0) = f(x)$  and  $H(x, 1) = g(x)$  for all  $x \in X$ . We denote  $h_t(x) := H(x, t)$ , and  $H$  is called a **homotopy** from  $f$  to  $g$ .

A homotopy is essentially a smooth mapping from one morphism  $f$  to another  $g$ . Homotopy is indeed an equivalence relation: firstly  $f \sim f$  as we can define  $H(x, t) = f(x)$  which is continuous as the composition of continuous functions ( $H = f \circ \pi_1$ ), if  $f \sim g$  then define  $H'(x, t) = H(x, 1 - t)$  which is also continuous (since  $(x, t) \mapsto (x, 1 - t)$  is continuous since its components are) and  $H'(x, 0) = g(x)$  and  $H'(x, 1) = f(x)$  so  $g \sim f$ , and if  $H_1$  is a homotopy from  $f$  to  $g$  and  $H_2$  is a homotopy from  $g$  to  $h$ , define

$$H(x, t) = \begin{cases} H_1(x, 2t) & 0 \leq t \leq \frac{1}{2} \\ H_2(x, 2t - 1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

$X \times [0, 1/2]$  and  $X \times [1/2, 1]$  are closed (since  $X \times [0, 1/2]$  is the preimage of  $[0, 1/2]$  in the mapping  $(x, t) \mapsto t$  and  $H(x, t)$  is continuous on both of these (since  $H_1(x, 2t)$  and  $H_2(x, 2t - 1)$  are continuous), so  $H(x, t)$  is continuous.

### 1.1.2 Proposition

For every topological space  $X$  and every two morphisms  $f, g: X \longrightarrow \mathbb{R}^n$ ,  $f$  and  $g$  are homotopic.

**Proof:** define  $H(x, t) = (1 - t)f(x) + tg(x)$  (addition and scalar multiplication are continuous). ■

### 1.1.3 Definition

A topological space  $X$  is **contractible** if the identity map  $\text{id}_X$  is homotopic to some constant map.

Notice that all two constant maps are homotopic if and only if the space is path connected. If all two constant maps are homotopic, for  $x_1, x_2 \in X$  let  $H(x, t)$  be a homotopy from  $x_1$  to  $x_2$  and define  $\gamma(t) = H(x_0, t)$  for any  $x_0 \in X$ , this is a continuous path from  $x_1$  to  $x_2$ . And if  $X$  is path connected, for  $x_1$  and  $x_2$  and  $\gamma$  connecting them, define  $H(x, t) = \gamma(t)$ .

### 1.1.4 Proposition

Let  $X, Y, Z$  be topological spaces,  $f, f': X \rightarrow Y$  and  $g, g': Y \rightarrow Z$  such that  $f \sim f'$  and  $g \sim g'$ , then  $g \circ f \sim g' \circ f'$ .

**Proof:** let  $H$  be a homotopy from  $f$  to  $f'$  and  $K$  a homotopy from  $g$  to  $g'$ . Then define  $J(x, t) = K(H(x, t), t)$  which is a composition of continuous functions (map  $(x, t)$  to  $(H(x, t), t)$  to  $K(H(x, t), t)$ ). ■

We call the equivalence classes of morphisms under  $\sim$  *homotopy classes*, and the homotopy class of a morphism  $f$  is denoted  $[f]$ . So by above,  $[f] \circ [g] := [f \circ g]$  is a well-defined operation. This gives us a new category whose objects are topological spaces and morphisms are homotopy classes. What are the isomorphisms in this category? Well the identities are obviously  $[1_X]$  since  $[f] \circ [1_X] = [f \circ 1_X] = [f]$  and  $[1_X] \circ [g] = [1_X \circ g] = [g]$ . So an isomorphism  $X \xrightarrow{[f]} Y$  is a homotopy class such that there exists a  $Y \xrightarrow{[g]} X$  such that  $[f] \circ [g] = [f \circ g] = [1_X]$  and  $[g \circ f] = [1_Y]$ . We give these isomorphisms a different name:

### 1.1.5 Definition

Let  $X$  and  $Y$  be topological spaces, then  $f: X \rightarrow Y$  is a **homotopic equivalence** if there exists a  $g: Y \rightarrow X$  such that  $g \circ f \sim \text{id}_X$  and  $f \circ g \sim \text{id}_Y$ . If a homotopy equivalence exists between  $X$  and  $Y$ , then  $X$  and  $Y$  are said to be **homotopy equivalent**, denoted  $X \simeq Y$ .

Notice that homeomorphisms are homotopic equivalences, since  $f^{-1} \circ f = \text{id}_X$  and  $f \circ f^{-1} = \text{id}_Y$ .

### 1.1.6 Definition

Let  $X$  and  $Y$  be topological spaces,  $A \subseteq X$ , and  $f, g: X \rightarrow Y$ . We say that  $f$  and  $g$  are homotopic relative to  $A$ , denoted  $f \stackrel{A}{\sim} g$ , if there exists a homotopy  $H$  from  $f$  to  $g$  such that  $H(a, t) = f(a)$  for all  $a \in A$  and  $t \in I$ . In such a case we must have  $f|_A = g|_A$ .

It is not enough for  $f \sim g$  and  $f|_A = g|_A$  for  $f$  and  $g$  to be homotopic relative to  $A$ . For example take  $I$  and  $S^1$  and the points 0 and 1 on  $I$ . Then we can continuously deform  $I$  so that it maps onto the bottom or top of the circle. These are two continuous mappings which are homotopic, but no homotopy between them which keeps the image of 0 and 1 constant.

Notice that  $\stackrel{A}{\sim}$  is an equivalence relation, the proof of this is analogous to the proof that homotopy is an equivalence relation. It also preserves composition, if  $f, f': (X, A) \rightarrow (Y, B)$  (meaning they are morphisms from  $X$  to  $Y$  and  $f(A), f'(A) \subseteq B$ ) and  $g, g': (Y, B) \rightarrow (Z, C)$  such that  $f \stackrel{A}{\sim} f'$  and  $g \stackrel{B}{\sim} g'$ , then  $g \circ f \stackrel{A}{\sim} g' \circ f'$ .

### 1.1.7 Definition

Let  $X$  be a topological space.  $A \subseteq X$  is called a **retract** if there exists an  $r: X \rightarrow A$  such that  $r \circ \iota = \text{id}_A$  where  $\iota: A \rightarrow X$  is the inclusion map. In other words  $r(a) = a$  for all  $a \in A$ .  $r$  is called a **retraction**.

For example  $\partial I = \{0, 1\}$  is not a retraction of  $I$  since every continuous image of  $I$  must be connected, and  $\partial I$  is not. But if we take  $X$  to be an eight shape, and  $A$  its bottom circle, then we can map the top circle to the middle point and  $A$  to itself and this is a retraction.

### 1.1.8 Definition

$A \subseteq X$  is called a **deformation retract** if there exists a retraction  $r$  such that  $\iota \circ r \stackrel{A}{\sim} \text{id}_X$ .

Instead of requiring  $r$  be a retraction, we can require only that  $r(X) \subseteq A$ . Since then if  $\iota \circ r \stackrel{A}{\sim} \text{id}_X$ , this means that  $r(a) = \text{id}_X(a) = a$  for all  $a \in A$  so it is already a retraction. Explicitly, this is equivalent to saying that there exists a homotopy  $H: X \times I \rightarrow X$  such that  $H(x, 0) = x$  for all  $x \in X$ ,  $H(a, t) = a$  for all  $a \in A, t \in I$ ,  $H(x, 1) \in A$  for all  $x \in X$ .

Notice that if  $A \subseteq X$  is a deformation retract then  $\iota: A \rightarrow X$  is a homotopy equivalence, since  $r \circ \iota = \text{id}_A$  and  $\iota \circ r \sim \text{id}_X$ .

## 1.1.9 Example

Let  $X = \mathbb{R}^n \setminus \{0\}$  and  $A = S^{n-1} \subseteq \mathbb{R}^n \setminus \{0\}$ . Then  $r(x) := \frac{x}{\|x\|}$  is a retraction with the homotopy  $H(x, t) = (1-t)x + t \frac{x}{\|x\|}$ . This is the homotopy we used to show that all morphisms to  $\mathbb{R}^n$  are homotopic.

A morphism  $f$  is called *null-homotopic* if it is homotopic to a constant morphism.

## 1.1.10 Proposition

Let  $X$  be a topological space and  $f: S^1 \rightarrow X$ , then the following are equivalent

- (1)  $f$  is null-homotopic,
- (2)  $f$  is null-homotopic relative to any point on  $S^1$ ,
- (3)  $f$  can be expanded to a morphism on  $D^2$  (the disk in  $\mathbb{R}^2$ ), meaning there exists an  $F: D^2 \rightarrow X$  such that  $F|_{S^1} = f$ .

(2)  $\Rightarrow$  (1) is trivial since a null-homotopy relative to a point is still a null-homotopy. (3)  $\Rightarrow$  (2): let  $\iota: S^1 \rightarrow D^2$  be the inclusion map, and let  $a \in S^1$ , define the homotopy  $H: S^1 \times I \rightarrow D^2$  by  $H(x, t) = (1-t)\iota(x) + ta$ , which is a homotopy from  $\iota$  to the constant map  $K_a$ . Then  $F \circ H$  is a null-homotopy between  $f$  and  $K_{f(a)}$  (since  $F \circ H(x, 0) = F(x) = f(x)$  and  $F \circ H(x, 1) = F(a)$ ) relative to  $a$  since  $F \circ H(a, t) = F(a)$ . (1)  $\Rightarrow$  (3): so there exists a homotopy  $H: S^1 \times I \rightarrow X$  such that  $H(x, 0) = f(x)$  for every  $x \in S^1$  and there exists a  $p \in X$  such that  $H(x, 1) = p$  for all  $x \in S^1$ . Let us define  $\rho: S^1 \times I \rightarrow D^2$  by  $\rho(x, t) = (1-t)x$ , this is a continuous map from a compact (since  $S^1$  and  $I$  are compact and therefore so is their product) to a Hausdorff space, and so it is closed. And it is surjective, so it is a quotient map. So  $D^2$  is the quotient space of  $S^1 \times I$  with respect to  $\rho$ , and  $H$  respects  $\rho$ , since  $\rho(x, t) = \rho(y, s)$  implies  $(1-t)x = (1-s)y$  and this means that either  $(x, t) = (y, s)$  or  $t = s = 1$ . But in both cases  $H(x, t) = H(y, s)$ , and so there exists an  $F: D^2 \rightarrow X$  which is continuous such that  $H = F \circ \rho$ , meaning  $F(x) = H(x, 0) = f(x)$  as required. ■

This proof uses the fact that if  $\rho$  is a quotient map, and  $f: X \rightarrow Y$  is continuous then there exists a  $F: \bar{X} \rightarrow Y$  such that  $f = F \circ \rho$  if and only if  $\rho(a) = \rho(b)$  implies  $f(a) = f(b)$ .

## 1.1.11 Definition

Let  $X$  be a topological space, and for every  $a, b \in X$  define  $\Gamma_{ab}$  to be the set of all paths from  $a$  to  $b$ , which are continuous maps  $I \rightarrow X$ . On  $\Gamma_{ab}$  we take the equivalence relation of homotopy relative to  $\partial I = \{0, 1\}$ . Take  $\hat{\Gamma}_{ab}$  to be the partition defined by this relation, ie.  $\hat{\Gamma}_{ab} = \Gamma_{ab} / \sim_{\partial I}$ .

If  $[\gamma] \in \hat{\Gamma}_{ab}$  and  $[\delta] \in \hat{\Gamma}_{bc}$  then we define  $[\gamma][\delta] := [\gamma * \delta]$  (their concatenation).

We must show that this is well-defined, meaning we must show that if  $\gamma \stackrel{\partial I}{\sim} \gamma'$  and  $\delta \stackrel{\partial I}{\sim} \delta'$  then  $\gamma * \delta \stackrel{\partial I}{\sim} \gamma' * \delta'$ . So let  $H: I \times I \rightarrow X$  be a homotopy relative to  $\partial I$  between  $\gamma$  and  $\gamma'$ , and  $G: I \times I \rightarrow X$  between  $\delta$  and  $\delta'$ . Then define

$$K(s, t) := \begin{cases} H(2s, t) & 0 \leq s \leq \frac{1}{2} \\ G(2s - 1, t) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

this is continuous,  $K(0, t) = H(0, t) = 0$  and  $K(1, t) = G(1, t) = 1$  so it is a homotopy between the concatenations relative to  $\partial I$ .

Notice that concatenation is not necessarily associative, since in  $(\gamma * \delta) * \varepsilon$ , the speed of  $\gamma$  and  $\delta$  is quadrupled while in  $\gamma * (\delta * \varepsilon)$ ,  $\gamma$ 's speed is only doubled. But it is the case that  $[\gamma]([\delta][\varepsilon]) = ([\gamma][\delta])[\varepsilon]$ , so in homotopy concatenation is associative. So we need to prove  $\gamma(\delta\varepsilon) \stackrel{\partial I}{\sim} (\gamma\delta)\varepsilon$ , the idea behind this is that for every  $x$  and  $y$  where  $\gamma(\delta\varepsilon)(x) = (\gamma\delta)\varepsilon(y)$ , define in  $I \times I$  the line between  $(x, 0)$  and  $(y, 1)$ . These lines cover  $I \times I$  and for every point  $(t, s)$  which is on the line from  $(x, 0)$  map it to  $\gamma(\delta\varepsilon)(x)$ .

We can prove in a similar manner that for  $\gamma \in \Gamma_{ab}$ ,  $[K_a][\gamma] = [\gamma][K_b] = [\gamma]$ .

And so we have defined a category. The objects of this category are the points  $a \in X$  and the morphisms between  $a$  and  $b$  are  $\hat{\Gamma}_{ab}$  (notice that  $[\gamma] \in \hat{\Gamma}_{ab}$  can be composed with elements from  $\hat{\Gamma}_{bc}$ , so the order of composition is reversed). Here the identity morphisms are  $[K_a]$ .

Notice that every morphism in this category is an isomorphism. This is since for every  $\gamma \in \Gamma_{ab}$  we defined its reverse  $\bar{\gamma} \in \Gamma_{ba}$  by  $\bar{\gamma}(t) := \gamma(1 - t)$ .

### 1.1.12 Proposition

$$[\gamma][\bar{\gamma}] = [K_a] \text{ and } [\bar{\gamma}][\gamma] = [K_b].$$

The idea is that at time  $t$  we take the path  $\gamma$  but not all the way, then wait, then take the reverse path  $\bar{\gamma}$ . So

$$H(x, t) = \begin{cases} \gamma(2x) & 0 \leq x \leq \frac{1-t}{2} \\ \gamma(1-t) & \frac{1-t}{2} \leq x \leq \frac{1+t}{2} \\ \gamma(2-2x) & \frac{1+t}{2} \leq x \leq 1 \end{cases}$$

is a homotopy from  $\gamma * \bar{\gamma}$  to  $K_a$ . ■

### 1.1.13 Definition

A **groupoid** is a small category (a category whose objects form a set, not a pure class) such that every morphism is an isomorphism. If  $\mathcal{C}$  is a groupoid, then  $\text{Mor}(A, A)$  is then a group for every  $A \in \mathcal{C}$ .

### 1.1.14 Definition

Given a pointed topological space  $(X, a)$  (call  $a$  the basis point), define the **first homotopy group**  $\pi_1(X, a) := \hat{\Gamma}_{aa}$ . And given a morphism  $f: (X, a) \rightarrow (Y, b)$  (meaning  $f$  is continuous and  $f(a) = b$ ), then we define a group homomorphism  $f_*: \pi_1(X, a) \rightarrow \pi_1(Y, b)$  by  $f_*([\gamma]) = [f \circ \gamma]$ . The correspondence  $(X, a) \mapsto \pi_1(X, a)$  and  $f \mapsto f_*$  is a functor.

We need to show that  $f_*$  is well-defined and also a group homomorphism. To show that it is well-defined, suppose  $\gamma \stackrel{\partial I}{\sim} \delta$ , then we must show  $f \circ \gamma \stackrel{\partial I}{\sim} f \circ \delta$ . Now we showed that if  $f \stackrel{A}{\sim} f'$  and  $g \stackrel{B}{\sim} g'$  such that  $f(A), f'(A) \subseteq B$  then  $g \circ f \stackrel{A}{\sim} g' \circ f'$ . And we have that  $\gamma \stackrel{\partial I}{\sim} \delta$  and  $f \stackrel{\{a\}}{\sim} f$  so  $f \circ \gamma \stackrel{\partial I}{\sim} f \circ \delta$  as required. Now we must show that  $f_*$  is a homomorphism, ie.

$$f_*([\gamma][\delta]) = [f \circ (\gamma * \delta)] = [(f \circ \gamma) * (f \circ \delta)] = f_*([\gamma])f_*([\delta])$$

Actually a stronger result holds,  $f \circ (\gamma * \delta) = (f \circ \gamma) * (f \circ \delta)$ , as both are given by

$$\begin{cases} f \circ \gamma(2t) & 0 \leq t \leq \frac{1}{2} \\ f \circ \delta(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

To finish the proof that the correspondence is a functor, we need to show that  $(g \circ f)_* = g_* \circ f_*$  and  $(\text{id}_X)_* = \text{id}_{\pi_1(X, a)}$ . We do so directly:

$$(g \circ f)_*([\gamma]) = [(g \circ f) \circ \gamma] = [g \circ (f \circ \gamma)] = g_*([f \circ \gamma]) = g_* \circ f_*([\gamma])$$

and

$$(\text{id}_X)_*([\varphi]) = [\text{id}_X \varphi] = [\varphi]$$

so  $(\text{id}_X)_* = \text{id}_{\pi_1(X, a)}$  as required. Thus we have defined a functor from the category of pointed topological spaces to the category of groups.

### 1.1.15 Proposition

Let  $X$  be a topological space,  $a \in X$ , and  $A$  be  $a$ 's connected component. Let  $\iota: A \rightarrow X$  be the inclusion map, then  $\iota_*: \pi_1(A, a) \rightarrow \pi_1(X, a)$  is an isomorphism.

**Proof:**  $\iota_*$  is injective:  $\iota_*([\gamma]) = [K_a]$  if and only if  $[\iota \circ \gamma] = [K_a]$ , which means  $\iota \circ \gamma \stackrel{\partial I}{\sim} K_a$ , let  $H$  be such a homotopy. Then  $H: I \times I \rightarrow X$ , but we claim that  $H$ 's image is contained in  $A$  so that it is also a homotopy  $\gamma \stackrel{\partial I}{\sim} K_a$ , meaning  $[\gamma] = [K_a]$ . Suppose not, that  $H(t_0, s_0) \notin A$ , then define  $\delta(t) = H(t_0 \cdot t, s_0)$ . Then

$\delta(0) = H(0, s_0) = a$  since  $H$  is a homotopy relative to  $\partial I$ , and  $\delta(1) = H(t_0, s_0) \notin A$ , so  $a$  is connected to a value not in  $A$ , in contradiction. So  $\iota_*$  is indeed injective.

Now suppose  $[\gamma] \in \pi_1(X, a)$ , meaning  $\gamma$  is a path connecting  $a$  to itself in  $X$ . But every point in  $\gamma$ 's image must also be connected to  $a$ , meaning  $\gamma$  is a path connecting  $a$  to itself contained within  $A$ . So there exists a  $\gamma' \in \Gamma_{aa}^A$  such that  $\gamma = \iota \circ \gamma'$  and in particular  $\iota_*([\gamma']) = [\gamma]$  as required. So  $\iota_*$  is a bijective homomorphism, an isomorphism. ■

Suppose  $a, b \in X$  such that there exists a path between them,  $\gamma: I \rightarrow X, \gamma(0) = a, \gamma(1) = b$ . Let us define

$$F_\gamma: \pi_1(X, a) \rightarrow \pi_1(X, b), \quad F_\gamma[\varphi] = [\bar{\gamma} * \varphi * \gamma] = [\bar{\gamma}][\varphi][\gamma]$$

so  $F_\gamma[\varphi]$  is the class of curves homotopic to the curve obtained by walking from  $b$  along  $\gamma$  backward to  $a$ , then going back along  $\gamma$  to  $b$ . Notice that  $F_\gamma[\varphi] = [\gamma]^{-1}[\varphi][\gamma]$ , so  $F_\gamma$  is simply conjugation by  $[\gamma]$ .

In general, suppose  $\mathcal{G}$  is a groupoid and let  $A, B \in \mathcal{G}$  such that  $\text{Mor}(A, B) \neq \emptyset$ . Then  $\text{Mor}(A, A)$  and  $\text{Mor}(B, B)$  are isomorphic groups. Let  $\varphi \in \text{Mor}(A, B)$  and define  $F_\varphi: \text{Mor}(A, A) \rightarrow \text{Mor}(B, B)$  by  $F_\varphi(\kappa) = \varphi \circ \kappa \circ \varphi^{-1}$ . This is a group homomorphism:

$$F_\varphi(\kappa_1) \circ F_\varphi(\kappa_2) = \varphi \circ \kappa_1 \circ \varphi^{-1} \circ \varphi \circ \kappa_2 \circ \varphi^{-1} = \varphi \circ (\kappa_1 \circ \kappa_2) \circ \varphi^{-1} = F_\varphi(\kappa_1 \circ \kappa_2)$$

and it is injective:

$$F_\varphi(\kappa) = 1_B \iff \varphi \circ \kappa \circ \varphi^{-1} = 1_B \iff \varphi \circ \kappa = \varphi \iff \kappa = \varphi^{-1} \circ \varphi = 1_A$$

and it is surjective: let  $\kappa' \in \text{Mor}(B, B)$  and define  $\kappa := \varphi^{-1} \circ \kappa' \circ \varphi$ , it is clear  $F_\varphi(\kappa) = \kappa'$ . It is clear that  $F_\varphi^{-1} = F_{\varphi^{-1}}$  by this.

Our  $F_\gamma$  above is precisely this  $F_{[\gamma]}$  defined in the groupoid of first homotopy groups above  $X$ , meaning it is an isomorphism between  $\pi_1(X, a)$  and  $\pi_1(X, b)$ . And so  $F_\gamma^{-1} = F_{\bar{\gamma}}$ . Let us summarize this:

#### 1.1.16 Proposition

Let  $a, b \in X$  be two points in  $X$  connected by a path  $\gamma$ . Then  $\pi_1(X, a)$  and  $\pi_1(X, b)$  are isomorphic.

#### 1.1.17 Proposition

Suppose  $H: I \times I \rightarrow X$  is a homotopy from the closed loop  $\varphi$  to the closed loop  $\psi$  such that for all  $t \in I$ ,  $H(0, t) = H(1, t)$ . Define the path  $\gamma(t) = H(0, t) = H(1, t)$ , then

$$[\psi] = [\bar{\gamma}][\varphi][\gamma]$$

**Proof:** this is equivalent to saying

$$[\bar{\psi}][\bar{\gamma}][\varphi][\gamma] = [K_p] \iff \bar{\psi} * \bar{\gamma} * \varphi * \gamma \stackrel{\partial I}{\sim} K_p$$

Now,  $\bar{\psi} * \bar{\gamma} * \varphi * \gamma$  is a curve  $I \rightarrow X$  whose endpoints are the same, so it can be viewed as a curve  $S^1 \rightarrow X$ . And it can be extended to the curve  $H$  on  $D^2$  (since the unit disc and unit square are homeomorphic), which we showed above is equivalent to  $\bar{\psi} * \bar{\gamma} * \varphi * \gamma$  being null-homotopic relative to any point on  $S^1$ , so we can choose the point which corresponds to 0 and 1. ■

#### 1.1.18 Theorem

Let  $f, g: X \rightarrow Y$  be homotopic with homotopy  $H$ . Define  $\gamma(t) := H(a, t)$ , so  $\gamma(0) = f(a)$  and  $\gamma(1) = g(a)$ . Then  $g_* = F_\gamma \circ f_*$  (recall that  $f_*: \pi_1(X, a) \rightarrow \pi_1(Y, f(a))$  and  $g_*: \pi_1(X, a) \rightarrow \pi_1(Y, g(a))$ ).

**Proof:** let  $[\varphi] \in \pi_1(X, a)$  then

$$F_\gamma(f_*[\varphi]) = [\bar{\gamma}][f \circ \gamma][\gamma], \quad g_*[\varphi] = [g \circ \varphi]$$

Define  $K(s, t) := H(\varphi(s), t)$  which is continuous and

$$K(s, 0) = H(\varphi(s), 0) = f \circ \varphi(s), \quad K(s, 1) = H(\varphi(s), 1) = g \circ \varphi(s), \quad K(0, t) = H(\varphi(0), t) = H(a, t) = K(1, t)$$

so by the above proposition, since  $K$  is a homotopy from the closed loop  $f \circ \varphi$  to the closed loop  $g \circ \varphi$ ,

$$[g \circ \varphi] = [\bar{\gamma}][f \circ \varphi][\gamma]$$

as required. ■

Notice then that  $g_* = f_*$  if and only if  $F_\gamma = \text{id}$  (requiring  $f(a) = g(a) = b$ ). This happens when  $[\gamma] = [K_b]$  for example, which can happen when  $\gamma = K_b$ . I.e. if  $H(a, t) = b$  for all  $t \in I$ , then  $f_* = g_*$ . But notice that this is simply the condition for  $H$  to be a homotopy relative to  $\{a\}$ , so

### 1.1.19 Proposition

If  $f \stackrel{\{a\}}{\sim} g$  then  $f_* = g_*$ .

### 1.1.20 Theorem

If  $f: X \rightarrow Y$  is a homotopy equivalence, then  $f_*: \pi_1(X, a) \rightarrow \pi_1(Y, f(a))$  is an isomorphism of groups.

**Proof:** there exists a  $g: Y \rightarrow X$  such that  $g \circ f \sim \text{id}_X$  and  $f \circ g \sim \text{id}_Y$ . Now since  $g \circ f \sim \text{id}_X$ , then by above  $(g \circ f)_* = F_\gamma \circ (\text{id}_X)_* = F_\gamma$ , since  $(\cdot)_*$  is a functor,  $g_* \circ f_* = F_\gamma$ . Now recall that  $F_\gamma = g_* \circ f_*: \pi_1(X, a) \rightarrow \pi_1(X, g(f(a)))$  is an isomorphism. And similarly  $f \circ g \sim \text{id}_Y$ , so there exists an  $F_\delta$  such that  $(f \circ g)_* = f_* \circ g_* = F_\delta: \pi_1(Y, f(a)) \rightarrow \pi_1(X, f(g(f(a))))$  is an isomorphism.

Recall that if  $f \circ g$  is injective, then  $g$  is injective, and if  $f \circ g$  is surjective,  $f$  is surjective. Thus  $f_*$  is injective and surjective, meaning it is a bijective homomorphism, an isomorphism. ■

Thus if  $X$  is contractible, then it is homotopic to the singleton space (an exercise in homework),  $\{b\}$ . Thus  $\pi_1(X, a) \cong \pi_1(\{b\}, b)$ , and  $\pi_1(\{b\}, b)$  has a single point, meaning  $\pi_1(X, a)$  is the trivial group.

### 1.1.21 Corollary

If  $X$  is contractible, then  $\pi_1(X, a)$  is trivial.

### 1.1.22 Definition

Let  $E, B$  be topological spaces, then a map  $p: E \rightarrow B$  is a **covering map** (or just a *covering*) if

- (1)  $p$  is surjective,
- (2) for every  $x \in B$ , there exists a neighborhood  $x \in \mathcal{U}$  such that  $p^{-1}(\mathcal{U})$  is the disjoint union of open sets  $\{\mathcal{V}_\alpha\}_{\alpha \in I}$  such that for every  $\alpha \in I$ ,  $p|_{\mathcal{V}_\alpha}: \mathcal{V}_\alpha \rightarrow \mathcal{U}$  is a homeomorphism.

For example, take  $B = S^1$  and  $E = \mathbb{R}$ . Define  $p(t) := (\cos(2\pi t), \sin(2\pi t))$ , or if we identify  $S^1$  with  $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ , then  $p(t) = e^{2\pi i t}$ . This is like spiraling the real line so that all the integers are all on a vertical line (they all map to  $1 \in S^1$ ), and projecting this spiral onto a plane to form the circle  $S^1$ .  $p$  is obviously a surjective continuous map. For every  $e^{2\pi i s}$ , let  $\varepsilon < 2\pi$  then the preimage of the open set  $\mathcal{U}_t := \{e^{2\pi i s} \mid t - \varepsilon < s < t + \varepsilon\}$  is  $\bigcup_{n \in \mathbb{Z}} (n + (t - \varepsilon, t + \varepsilon))$  and every  $n + (t - \varepsilon, t + \varepsilon)$  is homeomorphic (via  $p$ ) to this  $\mathcal{U}_t$ .

### 1.1.23 Theorem

Suppose  $p: E \rightarrow B$  is a covering,  $b \in B$  and  $e \in E$  such that  $p(e) = b$ . Now suppose  $\gamma: I \rightarrow B$  is a curve starting at  $b$ ,  $\gamma(0) = b$ . Then there exists a unique curve  $\delta: I \rightarrow E$  starting at  $e$  such that  $p \circ \delta = \gamma$ .



## 8 Homotopies

**Proof:** let  $\{\mathcal{U}_\beta\}_{\beta \in J}$  be an open covering of  $B$ , then  $\{\gamma^{-1}(\mathcal{U}_\beta)\}_{\beta \in J}$  is an open covering of  $I$ . Since  $I$  is a compact metric space, it has a Lebesgue number, and so we can take  $n$  small enough such that for every  $k$ ,  $I_k := [\frac{k-1}{n}, \frac{k}{n}]$  is contained within a single  $\gamma^{-1}(\mathcal{U}_\beta)$ . Meaning for every  $1 \leq k \leq n$  there exists a  $\beta_k \in J$  such that  $I_k \subseteq \gamma^{-1}(\mathcal{U}_{\beta_k})$ , so  $\gamma(I_k) \subseteq \mathcal{U}_{\beta_k}$ .

Now since  $p$  is a covering, there exists a neighborhood  $b \in \mathcal{U}$  such that  $p^{-1}(\mathcal{U}) = \bigcup_{\alpha \in I} \mathcal{V}_\alpha$ . Since this is a disjoint union and  $p(e) \in \mathcal{U}$ , there exists a single  $\alpha \in I$  such that  $e \in \mathcal{V}_\alpha$ , and let us denote this neighborhood by  $\mathcal{V}$ . And so  $p|_{\mathcal{V}}$  is a homeomorphism from  $\mathcal{V}$  to  $\mathcal{U}$ . So define

$$\delta|_{I_1} := (p|_{\mathcal{V}})^{-1} \circ \gamma|_{I_1}$$

We have then that  $p \circ (\delta|_{I_1}) = \gamma|_{I_1}$  and  $\delta(0) = (p|_{\mathcal{V}})^{-1} \circ \gamma(0) = e$ . This is the only such transform of  $\gamma|_{I_1}$  which satisfies  $\delta|_{I_1}(0) = e$  since every transform  $\delta: I_1 \rightarrow E$  where  $p \circ \delta = \gamma$ ,  $\delta(I_1) = p^{-1}(\gamma(I_1)) \subseteq p^{-1}(\mathcal{U}) = \bigcup_{\alpha \in I} \mathcal{V}_\alpha$ . Since  $\delta(I_1)$  is connected, it must be contained within a single  $\mathcal{V}_\alpha$ , this being  $\mathcal{V}$  since it contains  $e$ .

We continue with  $\gamma|_{I_2}$  which starts at  $\frac{1}{n}$  to get  $\delta|_{I_2}$ , and so on. Together these form  $\delta$ , which is continuous as  $\{I_k\}_{1 \leq k \leq n}$  is a finite partition of  $I$  into closed sets and  $\delta$  is continuous over each of these closed sets. ■