Infinitesimal Calculus 3

Lecture 7, Sunday November 13, 2022
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7.1 Complete Metric Spaces

Definition 7.1.1:

A metric space (X, ρ) is **complete** if every cauchy sequence in X is convergent.

For example, $\mathbb{R}$ is complete but $\mathbb{Q}$ is not.

Proposition 7.1.2:

If $\{x_n\}_{n=1}^{\infty}$ is a cauchy sequence, it is bounded. And if $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence, it itself is convergent.

Proof:

Let $\varepsilon > 0$ then there exists a N such that for every $n, m \geq N$: $\rho(x_n, x_m) < \varepsilon$. Let $x = X_N$, and we define

$$M = \max_{1 \leq n < N} \{\rho(x_n, x), \varepsilon\}$$

then for every $n \in \mathbb{N}$ we have that $\rho(x_n, x) \leq M$ since if $n < N$ then by definition $\rho(x_n, x) \leq M$ since M is the maximum distance. And if $n \geq N$ then $\rho(x_n, x) < \varepsilon \leq M$. So $\{x_n\}_{n=1}^{\infty}$ is bounded.
Suppose x_{n_k} is a convergent subsequence which converges to $x \in X$. Then let $\varepsilon > 0$, and so there exists an N which satsifes the definition of cauchy sequences for ε. Let $n \geq N$, and there must be a k such that $n_k \geq N$ and $\rho(x_{n_k}, x) < \varepsilon$ (since it converges to x). So for every $n \geq N$ by the definition of a cauchy sequence:

$$\rho(x_n, x) \leq \rho(x_n, x_{n_k}) + \rho(x_{n_k}, x) < 2\varepsilon$$

And so for every $\varepsilon > 0$ there is an N such that for every $n \geq N$: $\rho(x_n, x) < 2\varepsilon$, so x_n converges to x as required.

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Proposition 7.1.3:

$\mathbb{R}^n$ is complete.

Proof:

Suppose $\{x_n\}_{n=1}^{\infty}$ is a cauchy sequence in $\mathbb{R}^n$. Then it is bound, and by Weierstrauss it has a convergent subsequence. So by above since it is a cauchy sequence with a convergent subsequence, it itself is convergent. So $\mathbb{R}^n$ is complete.

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Proposition 7.1.4:

If (X, ρ) is a compact metric space, it is complete.

Proof:

Suppose $\{x_n\}_{n=1}^{\infty}$ is a cauchy sequence in X, then since X is a compact space x_n has a convergent subsequence. And since it is cauchy, x_n is therefore convergent. So X is complete.

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Proposition 7.1.5:

If (X, ρ) is a complete metric space and $S \subseteq X$, then S is closed if and only if (S, ρ) is complete (we restrict ρ to $S \times S$).

Proof:

Suppose S is closed and $\{x_n\}_{n=1}^{\infty}$ is a cauchy sequence in S, then it is cauchy in X and therefore converges to some $x \in X$. Therefore since $x_n \longrightarrow x$ and $x_n \in S$, $x \in \overline{S}$. Because S is closed, $\overline{S} = S$ and therefore $x \in S$. So x_n converges to a value in S, that is it is convergent in S. So every cauchy sequence in S is convergent, and therefore S is complete.
Suppose S is complete and $x \in S'$ then there exists a sequence $x_n \in S$ such that $x_n \longrightarrow x$. Since $\{x_n\}$ is convergent in X, it is cauchy in S, and therefore converges to a value in S. Therefore $x \in S$, that is $S' \subseteq S$, so S is closed.

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7.2 Continuous Mappings Between Metric Spaces

Definition 7.2.1:

If (X, ρ) and (Y, σ) are metric spaces, a **mapping** (or a function) f between them is a function:

$$f\colon X \longrightarrow Y$$

And a **restriction** of f onto $E \subseteq X$ is a mapping $f|_E$ between (E, ρ) and (Y, σ) such that for every $x \in E : f|_E(x) = f(x)$.

Definition 7.2.2:

If f is a mapping between X and Y and p is a limit point of X, we say

$$\lim_{x \to p} f(x) = q$$

if for every sequence $p \neq x_n \longrightarrow p$ in X, $f(x_n) \longrightarrow q$.

Theorem 7.2.3:

Suppose f is a mapping between metric space, then the following are equivalent:

- $\lim_{x \to p} f(x) = q$
- For every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $p \neq x \in B_\delta(p)$, $f(x) \in B_\varepsilon(q)$.
- For every $K \subseteq X$ where p is a limit point of K: $\lim_{x \to p} f|_K(x) = q$ in K.

Proof:

Suppose $\lim f(x) = q$ and assume for the sake of a contradiction that there is an $\varepsilon > 0$ such that for every $\delta > 0$ there is a $p \neq x \in B_\delta(p)$ such that $f(x) \notin B_\varepsilon(q)$. Take $\delta_n = \frac{1}{n}$ and x_n to be the x_n which satisfies the above for δ_n. Then $p \neq x_n \longrightarrow p$, but $\rho(x_n, q) \geq \varepsilon$, so x_n doesn't converge to q in contradiction.
To prove the converse, suppose $p \neq x_n \longrightarrow p$. Then let $\varepsilon > 0$, so there exists a $\delta > 0$ which satisfies the $\varepsilon - \delta$ criterion, and since $x_n \longrightarrow p$, there exists an N such that for every $n \geq N$ we have that $x_n \in B_\delta(p)$, so $f(x_n) \in B_\varepsilon(q)$. So for every $\varepsilon > 0$ there is an N such that for every $n \geq N$ we have $\rho(f(x_n), q) < \varepsilon$ so $f(x_n)$ converges to q.
We will now show that 1 is equivalent to 3. If we assume 1 then 3 is trivial. Now assume 3, suppose $p \neq x_n \to p$, then p is a limit point of $K = \{x_n \mid n \in \mathbb{N}\}$, and so:

$$\lim_{x \to p} f|_K(x) = q$$

and since $\{x_n\}$ is a sequence in K which converges to p in X and isn't equal to p:

$$p = \lim_{x \to p} f|_K(x) = \lim f|_K(x_n) = \lim f(x_n)$$

as required. ∎

Example:

We define the function $f\colon \mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow \mathbb{R}$ by:

$$f(x,y) = \frac{xy}{x^2+y^2}$$

and we'd like to compute the limit

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

If we take $K_k = \{(x,xk) \mid x \in \mathbb{R}\}$ then $(0,0)$ is a limit point of every K_k. But the limit in K is equal to:

$$\lim_{(x,xk)\to(0,0)} f(x) = \lim_{x\to 0} \frac{kx^2}{x^2(1+k^2)} = \lim_{x\to 0} \frac{k}{1+k^2} = \frac{k}{1+k^2}$$

which is different for every k, and therefore the limit doesn't exist.

Example:

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$$

What happens on $y = kx$? We get:

$$\lim_{x\to 0} \frac{kx^3}{x^2(x^2+k^2)} = \lim_{x\to 0} \frac{kx}{x^2+k^2} = 0$$

So if the limit exists, it is 0. But if we take $y = kx^2$ then the limit:

$$\lim_{x\to 0} f(x,kx^2) = \lim_{x\to 0} \frac{kx^4}{x^4(1+k^2)} = \frac{k}{1+k^2}$$

which is not equal to 0 if $k \neq 0$, and therefore the limit doesn't exist.