Introduction to Rings and Modules

Lecture 13, Wednesday June 7 2023
Ari Feiglin

Definition 13.0.1:

An R-module M is **finitely generated** if there exists a finite set $\mathscr{S} \subseteq M$ such that $\langle S \rangle = M$. Equivalently, M is finitely generated if and only if there exists $m_1, \dots, m_n \in M$ such that every element $m \in M$ can be written as a linear combination of m_is.

Definition 13.0.2:

If M and N are both R-modules, a **module homomorphism** form M to N is a function

$$f \colon M \longrightarrow N$$

such that $f(m_1 + m_2) = f(m_1) + f(m_2)$ for every $m_1, m_2 \in M$ and $f(rm) = rf(m)$ for all $r \in R$ and $m \in M$. If f is injective, surjective, or bijective then f is also called a monomorphism, epimorphism, or isomorphism respectively. And the **kernel** of a module homomorphism is

$$\mathrm{Ker}(f) = f^{-1}(0_N) = \{m \in M \mid f(m) = 0_N\}$$

Note that since module homomorphisms are group homomorphisms, a module homomorphism is injective if and only if its kernel is trivial.
Note that if R is a field, module homomorphisms are exactly linear transformations. And if $R = \mathbb{Z}$ then the condition that $f(rm) = rf(m)$ is redundant as $f(rm) = f(m + \cdots + m) = f(m) + \cdots + f(m) = rf(m)$, so if M and N are $\mathbb{Z}$-modules (abelian groups) then module homomorphism are simply abelian group homomorphisms.

Proposition 13.0.3:

If $f \colon M \longrightarrow N$ is a homomorphism of R-modules then

(1) $M' \subseteq M$ is a submodule then $f(M') \subseteq N$ is also a submodule.

(2) If $N' \subseteq N$ is a submodule then so is $f^{-1}(N')$.

Proof:

(1) Let $f(m_1), f(m_2) \in f(M')$ then $f(m_1) + f(m_2) = f(m_1 + m_2)$ and since M' is a submodule, $m_1 + m_2 \in M'$ so $f(m_1) + f(m_2) \in f(M')$ as required. And if $f(m) \in f(M')$ and $r \in R$ then $rf(m) = f(rm) \in f(M')$ since $rm \in M'$ since it is a submodule.

(2) If $m_1, m_2 \in f^{-1}(N')$ then $f(m_1), f(m_2) \in N'$ so $f(m_1 + m_2) = f(m_1) + f(m_2) \in N'$ so $m_1 + m_2 \in f^{-1}(N')$ as required. And if $m \in f^{-1}(N')$ and $r \in R$ then $f(rm) = rf(m) \in N'$ since $f(m) \in N'$ which is a submodule, so $rm \in f^{-1}(N')$ as required. ■

Note that since $\{0_N\} \subseteq N$ is a submodule, $\mathrm{Ker}(f) = f^{-1}\{0_N\}$ is a submodule of M.
If M is an R-module and $N \subseteq M$ is a submodule, in particular it is a (normal, as the group is abelian) subgroup. Then we can discuss the quotient group ${}^{M}\!/_{N}$, which already has addition defined by

$$(m_1 + N) + (m_2 + N) = (m_1 + m_2) + N$$

and we define scalar multiplication by

$$r(m + N) = rm + N$$

This is well defined since if $m_1 + N = m_2 + N$ then $m_1 - m_2 \in N$, and so $r(m_1 - m_2) \in N$ (since it is a submodule), so $rm_1 - rm_2 \in N$ so $rm_1 + N = rm_2 + N$ as required.

Definition 13.0.4:

If M is an R-module and N is a submodule of M, then ${}^{M}\!/_{N}$ obtains a module structure where

$$(m_1 + N) + (m_2 + N) = (m_1 + m - 2) + N$$

and

$$r(m + N) = rm + N$$

Theorem 13.0.5 (First Isomorphism Theorem for Modules):

If $f\colon M \longrightarrow N$ is a homomorphism of R-modules, then

$${}^{M}\!\Big/_{\mathrm{Ker}(f)} \cong \mathrm{Im}(f)$$

Recall that $\mathrm{Im}(f) = f(M)$ is a submodule of N.

Proof:

Define a module isomorphism by

$$\varphi(m + \mathrm{Ker}(f)) = f(m)$$

this is well-defined since this can be viewed as a group homomorphism and we know this is well-defined for groups. Or we can show it directly: if $m + \mathrm{Ker}(f) = n + \mathrm{Ker}(f)$ then $m - n \in \mathrm{Ker}(f)$ so $f(m - n) = f(m) - f(n) = 0$ so $f(m) = f(n)$.

Again, since this is a group homomorphism we know that $\varphi((m+\mathrm{Ker}(f))+(n+\mathrm{Ker}(f))) = \varphi(m+\mathrm{Ker}(f))+\varphi(n+\mathrm{Ker}(f))$ (this is also trivial to show). And it satisfies the condition for scalar multiplication as

$$\varphi(r(m + \mathrm{Ker}(f))) = \varphi(rm + \mathrm{Ker}(f)) = f(rm) = rf(m) = r\varphi(m + \mathrm{Ker}(f))$$

as required.

Again, since this is a group homomorphism which we know is an isomorphism, it is an isomorphism as a module homomorphism. ∎

Proposition 13.0.6:

If M is an R-module and N is a submodule where both N and ${}^{M}\!/_{N}$ are finitely generated, then so is M.

Proof:

Let $n_1, \ldots, n_k$ be a set of generators for N, and let $m_1 + N, \ldots, m_s + N$ be a set of generators for ${}^{M}\!/_{N}$. Then let $m \in M$, so there exist $r_1, \ldots, r_s \in R$ where

$$m + N = \sum_{i=1}^{s} r_i(m_i + N) = \sum_{i=1}^{s} r_i m_i + N$$

This means that

$$m - \sum_{i=1}^{s} r_i m_i \in N$$

and so there exist $s_1, \ldots, s_k \in R$ such that

$$m - \sum_{i=1}^{s} r_i m_i = \sum_{i=1}^{k} s_i n_i \implies m = \sum_{i=1}^{s} r_i m_i + \sum_{i=1}^{k} s_i n_i$$

so $\{n_1, \ldots, n_k, m_1, \ldots, m_s\}$ generates M. ∎

Recall the following definitions

Definition 13.0.7:

If R is a ring, then R is (left/right/two-sided) **Noetherian** if every ascending chain of (left/right/two-sided) ideals stabilizes, and R is (left/right/twosided) **Artinian** if every descending chain of (left/right/twosided) ideals stabilizes.

Theorem 13.0.8 (Hopkins-Levitzki Theorem):

Every (left/right/two-sided) Artinian ring is (left/right/two-sided) Noetherian.

We prove this for the case that R is commutative.

Proof:

Recall that R is Noetherian if and only if every ideal is finitely generated. Suppose that R is not Noetherian, so there exists an ideal which is not finitely generated. Let

$$\mathscr{S} = \{I \trianglelefteq R \mid I \text{ is not finitely generated}\}$$

There exists an $I \in \mathscr{S}$ which is minimal. Otherwise we could create an infinite chain of ideals $I_1 \supset I_2 \supset \ldots$, which contradicts R being Artinian. Thus I is not finitely generated but every ideal $J \subset I$ is.
Note that for $r \in R$, if J is an ideal of R then so is rJ. We know $0_R = r0_R \in rJ$. And $ra + rb = r(a+b) \in rJ$, so rI is closed under addition. And $-ra = r(-a) \in rJ$ so rJ is closed under inverses. And if $s \in R$ and $ra \in rJ$ then $s(ra) = r(sa) \in rJ$ so rI is closed under multiplication by R.
We claim that if $r \in R$ then $rI = \{ra \mid a \in I\}$ $rI = I$ or $rI = (0)$. This is true since $rI \subseteq I$, so if $rI \neq I$, then rI is finitely generated. Since I and rI are ideals of R and therefore R-modules, we can look at the module homomorphism

$$f\colon I \longrightarrow rI, \quad f(a) = ra$$

This is obviously a homomorphism: $f(a+b) = r(a+b) = ra + rb = f(a) + f(b)$ and $f(sa) = r(sa) = s(ra) = sf(a)$. This is also obviously surjective. If $\operatorname{Ker} f = I$ then $rI = (0)$ (since $rI = f(I)$) and we are finished. Otherwise, $\operatorname{Ker} f \subset I$ and by the minimality of I this means that $\operatorname{Ker} f$ is finite generated (since $\operatorname{Ker} f$ is an R-module, it is an ideal of R). Since

$$rI = f(I) \cong {}^{I}\big/_{\operatorname{Ker} f}$$

and since rI is finitely generated, this means that so is ${}^{I}/_{\operatorname{Ker} f}$. So ${}^{I}/_{\operatorname{Ker} f}$ and $\operatorname{Ker} f$ are both finitely generated and therefore so is I, in contradiction.
We take a break from this proof to define more objects and prove results.

Definition 13.0.9:

Suppose R is a ring, and M an R-module. The **annihilator** of M is defined as

$$\operatorname{Ann}_R(M) = \{r \in R \mid \forall m \in M \colon rm = 0_M\}$$

Proposition 13.0.10:

$\operatorname{Ann}_R(M)$ is a two-sided ideal of R.

Proof:

Firstly it is obvious that $0_R \in \operatorname{Ann}_R(M)$, if $r, s \in \operatorname{Ann}_R(M)$ then let $m \in M$ we get

$$(r+s)m = rm + sm = 0_M$$

so $r + s \in \operatorname{Ann}_R(M)$ and if $r \in \operatorname{Ann}_R(M)$ then

$$(-r)m = (-1_R)rm = 0_M$$

so $-r \in \operatorname{Ann}_R(M)$ so $\operatorname{Ann}_R(M)$ is a subgroup of R. And if $r \in \operatorname{Ann}_R(M)$ and $s \in R$ then for any $m \in M$ it is obvious that $(sr)m = s(rm) = 0_M$ so $sr \in \mathrm{Ann}_R(M)$ and $(rs)m = r(sm) = 0_M$ since $sm \in M$ so $rs \in \mathrm{Ann}_R(M)$ meaning $\mathrm{Ann}_R(M)$ is closed under multiplication by R on both sides, and is therefore a two-sided ideal. ■

So if M is an R-module, there is a natural extension of it to a ${}^{R}\!/_{\mathrm{Ann}_R(M)}$-module by

$$(r + \mathrm{Ann}_R(M)) \cdot m = rm$$

This is well-defined since if $r + \mathrm{Ann}_R(M) = s + \mathrm{Ann}_R(M)$ then $r - s \in \mathrm{Ann}_R(M)$ so $(r - s)m = 0_M$ meaning $rm = sm$.
Let us return to our proof of **Hopkins-Levitzki Theorem**:

So we have a commutative Artinian ring R, and an ideal $I \trianglelefteq R$ which is not finitely generated, but any ideal $J \subset I$ is. We showed that for any $r \in R$, rI is either I or (0).
We now claim that $\mathrm{Ann}_R(I) \trianglelefteq R$ is a prime ideal. We showed above that it is an ideal. Suppose that $rs \in \mathrm{Ann}_R(I)$ and suppose $r \notin \mathrm{Ann}_R(I)$ then $rI \neq (0)$ since $\mathrm{Ann}_R(I) = \{r \in R \mid rI = (0)\}$. So $rI = I$. And so we get that

$$sI = s(rI) = (sr)I = (rs)I = (0)$$

since $rs \in \mathrm{Ann}_R(I)$ and so $s \in \mathrm{Ann}_R(I)$ meaning $\mathrm{Ann}_R(I)$ is prime.
We showed that quotients of Artinian rings are Artinian, and since $\mathrm{Ann}_R(I)$ is prime we get that ${}^{R}\!/_{\mathrm{Ann}_R(I)}$ is an Artinian integral domain. We showed that this means that $F = {}^{R}\!/_{\mathrm{Ann}_R(I)}$ is a field, and since I is an R-module, it is also a $F = {}^{R}\!/_{\mathrm{Ann}_R(I)}$, ie. a linear space over F. Since I is not finitely generated in R, it is not finitely generated in F. This is because $(r + \mathrm{Ann}_R(I))i = ri$, so any generating set in F induces a generating set of the same cardinality in R. Thus I's dimension is infinite.
Let B be a basis of I, and let $b \in B$. Then $B \setminus \{b\}$ is a basis for a subspace $V \subset I$. V must be an ideal in R since for every $r \in R$, $rv = (r + \mathrm{Ann}_R(I))v \in V$ Since $V \subset I$ by I's minimality, V must be finitely generated and so has finite dimension. But $B \setminus \{b\}$ is infinite and is a basis for V so V has infinite dimension, in contradiction. ■