Modern Analysis

Homework 1
Ari Feiglin

1.1 Exercise

Let $E \subseteq \mathbb{R}$ and $a, b \in \mathbb{R}$. Show that $m^*(aE+b) = |a|m^*(E)$.

Note that there is a one-to-one correspondence between open covers of $aE+b$ and E: if $\{I_n\}$ is a cover of open intervals of E, $\{aI_n+b\}$ is a cover of $aE+b$, and every cover of $aE+b$ is of this form (since if $\{J_n\}$ is a cover of $aE+b$, $\{\frac{J_n-b}{a}\}$ is a cover of E). And since $|aI+b| = |a||I|$,

$$m^*(aE+b) = \inf\left\{\sum |aI_n+b| \;\middle|\; E \subseteq \bigcup I_n\right\} = |a| \inf\left\{\sum |I_n| \;\middle|\; E \subseteq \bigcup I_n\right\} = |a|m^*(E)$$

1.2 Exercise

Prove or disprove:

(1) if $A \subseteq \mathbb{R}$ is bound, then $m^*(A) < \infty$,

(2) if $A \subseteq \mathbb{R}$ has finite measure, A is bound.

(1) This is true: if A is bound then $A \subseteq [a,b]$ for some $a, b \in \mathbb{R}$ and by monotonicity $m^*(A) \leq m^*([a,b]) = b-a < \infty$.

(2) This is false: $m^*(\mathbb{Q}) = 0$ but $\mathbb{Q}$ is unbound.

1.3 Exercise

We say that a set S is G_δ if it is the countable intersection of open sets. Show that for every $E \subseteq \mathbb{R}$, there exists a G_δ set S such that $m^*(E) = m^*(S)$.

For every n, let $\{I_{n,k}\}_k$ be an open cover of E such that $\sum_k |I_{n,k}| \leq m^*(E) + \frac{1}{n}$. Then define

$$S := \bigcap_n \bigcup_k I_{n,k}$$

which is a G_δ set since $\bigcup_k I_{n,k}$ are open as the unions of open sets. Furthermore, $S \subseteq \bigcup_k I_{n,k}$ for every n and so by subadditivity for every n,

$$m^*(S) \leq \sum_k |I_{n,k}| \leq m^*(E) + \frac{1}{n}$$

thus $m^*(S) \subseteq m^*(E)$. And since $E \subseteq \bigcup_k I_{n,k}$ for every n, $E \subseteq S$ so $m^*(E) \leq m^*(S)$ and therefore $m^*(E) = m^*(S)$ as required.

1.4 Exercise

Suppose $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, show that $f^{-1}\{0\}$ is G_δ.

We claim that

$$f^{-1}\{0\} = \bigcap_{n=1}^{\infty} f^{-1}\left(-\frac{1}{n}, \frac{1}{n}\right)$$

since $f^{-1}\{0\} \subseteq f^{-1}\left(-\frac{1}{n}, \frac{1}{n}\right)$ for every n we have $\subseteq$. And if $-\frac{1}{n} < f(x) < \frac{1}{n}$ for all n then $f(x) = 0$ so we have $\supseteq$ and therefore equality. Since f is continuous and $\left(-\frac{1}{n}, \frac{1}{n}\right)$ is open, $f^{-1}\left(-\frac{1}{n}, \frac{1}{n}\right)$ is open and therefore $f^{-1}\{0\}$ is G_δ.