Introduction to Stochastic Processes

Assignment 7
Ari Feiglin

7.1 Exercise

Suppose $(X_1, X_2, X_3) \sim \mathcal{N}(0, \Sigma)$ where

$$\Sigma = \begin{pmatrix} 14 & -4 & -3 \\ -4 & 2 & 1 \\ -3 & 1 & 1 \end{pmatrix}$$

(1) Find a triangle matrix U such that $U^\top U = \Sigma$.

(2) Compute $\mathbb{E}[X_1^2 X_2 X_3]$.

(1) We will look for a lower triangle matrix, so we are trying to solve

$$\begin{pmatrix} 14 & -4 & -3 \\ -4 & 2 & 1 \\ -3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ & a_4 & a_5 \\ & & a_6 \end{pmatrix} \begin{pmatrix} a_1 & & \\ a_2 & a_4 & \\ a_3 & a_5 & a_6 \end{pmatrix} = \begin{pmatrix} a_1^2 + a_2^2 + a_3^2 & a_2 a_4 + a_3 a_5 & a_3 a_6 \\ a_2 a_4 + a_3 a_5 & a_4^2 + a_5^2 & a_5 a_6 \\ a_3 a_6 & a_5 a_6 & a_6^2 \end{pmatrix}$$

So we can assume that $a_6 = 1$ and then $a_5 = 1$ and $a_3 = -3$. We can then further assume $a_4 = -1$ and then $a_2 = -1$ and $a_1 = 2$. So we get

$$U = \begin{pmatrix} 2 & & \\ -1 & 1 & \\ -3 & 1 & 1 \end{pmatrix}$$

(2) Since $\Sigma = AA^\top$ where A is a transition matrix, $U^\top$ is a transition matrix for (X_1, X_2, X_3) so there exist $(Z_1, Z_2, Z_3) \sim \mathcal{N}(0, I)$ such that

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 2 & -1 & -3 \\ & 1 & 1 \\ & & 1 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix} = \begin{pmatrix} 2Z_1 - Z_2 - 3Z_3 \\ Z_2 + Z_3 \\ Z_3 \end{pmatrix}$$

So we get that

$$\begin{aligned} X_1^2 X_2 X_3 = (2Z_1 - Z_2 - 3Z_3)^2 (Z_2 + Z_3) Z_3 = 4Z_1^2 Z_2 Z_3 - 4Z_1 Z_2^2 Z_3 - 12 Z_1 Z_2 Z_3^2 + Z_2^3 Z_3 - 6 Z_2^2 Z_3^2 + 9 Z_3^2 Z_2 \\ + 4Z_1^2 Z_3^2 - 4Z_1 Z_2 Z_3^2 - 12 Z_1 Z_3^3 + Z_2^2 Z_3^2 - 6 Z_2 Z_3^3 + 9 Z_3^4 \end{aligned}$$

Now since Z_1, Z_2, Z_3 are independent, the expected value of their products is the product of their expected values. And since their expected values are all zero, we focus only on the terms which have powers of Z_i greater than 1. So

$$\mathbb{E}[X_1^2 X_2 X_3] = 4\,\mathbb{E}[Z_1^2]\,\mathbb{E}[Z_3^2] + \mathbb{E}[Z_2^2]\,\mathbb{E}[Z_3^2] + 9\,\mathbb{E}[Z_3^4]$$

For $Z \sim \mathcal{N}(0, 1)$, $\mathbb{E}[Z^2] = 1$ as this is its variance, and $\mathbb{E}[Z^4] = 3$ (this can be computed using its moment generating function, $e^{-x^2/2}$). So

$$\mathbb{E}[X_1^2 X_2 X_3] = 32$$

7.2 Exercise

Suppose $B(t)$ is Brownian motion starting from 0 and $t_1 < t_2 < t_3$, then compute

(1) $\mathbb{E}[B(t_1)B(t_2)]$,

(2) $\mathbb{E}[B(t_1)B(t_2)B(t_3)]$.

(1) We know that $B(t_2) - B(t_1) \sim \mathcal{N}(0, t_2 - t_1)$ and $B(t_i) \sim \mathcal{N}(0, t_i)$ so

$$\mathbb{E}[B(t_1)B(t_2)] = \mathbb{E}\left[-\frac{(B(t_2) - B(t_1))^2 - B(t_2)^2 - B(t_1)^2}{2}\right] = -\frac{(t_2 - t_1) - t_2 - t_1}{=} t_1$$

(2) We will prove something more general: if $X = (X_1, \ldots, X_n)$ is a Gaussian vector and n is odd, then $\mathbb{E}[X_1 \cdots X_n] = 0$. This is since the covariance matrix of X is the same as that of $-X$, since $\mathrm{Cov}(-X_i, -X_j) = \mathrm{Cov}(X_i, X_j)$ by bilinearity. And so the joint probability of X and $-X$ are the same, which means that $\mathbb{E}[X_1 \cdots X_n] = \mathbb{E}[(-X_1) \cdots (-X_n)] = -\mathbb{E}[X_1 \cdots X_n]$, so $\mathbb{E}[X_1 \cdots X_n] = 0$ as required. And since $\big(B(t_1), B(t_2), B(t_3)\big)$ is a Gaussian vector of odd length, we get the desired result.

7.3 Exercise

Suppose $B(t)$ is Brownian motion, then show that $\limsup \frac{|B(t)|}{\sqrt{t}} \geq 1$ almost surely.

Notice that

$$\begin{aligned}\mathbb{P}\big(|B(n)| \geq \sqrt{n} \text{ i.o.}\big) = \mathbb{P}\big((\forall n)(\exists m \geq n)\, |B(m)| \geq \sqrt{m}\big) &= \lim_{n\to\infty} \mathbb{P}\big((\exists m \geq n)\, |B(m)| \geq \sqrt{m}\big) \\ &\geq \lim_{n\to\infty} \mathbb{P}\big(|B(n)| \geq \sqrt{n}\big) = \mathbb{P}\big(|\mathcal{N}(0, n)| \geq \sqrt{n}\big) > 0\end{aligned}$$

Now

$$X(n) = \frac{|B(n)|}{\sqrt{n}} = \frac{\left|\sum_{k=1}^{n} \big(B(k) - B(k-1)\big)\right|}{\sqrt{n}}$$

and $\{B(k) - B(k-1)\}_{k=1}^{\infty}$ are by definition independent, and since $\limsup X(n)$ is invariant under finite permutations of $\{B(k) - B(k-1)\}$, by Hewitt-Savage the event $\limsup X(n) \geq 1$ has trivial probability. But if $|B(n)| \geq \sqrt{n}$ i.o. then certainly $\limsup X(n) \geq 1$ since we can form a sequence of all the indexes m_n where $|B(m_n)| \geq \sqrt{m_n}$ and in this case we would have $\lim X(m_n) \geq 1$. And we showed that $|B(n)| \geq \sqrt{n}$ i.o. has nonzero probability, and so $\limsup X(n) \geq 1$ also has nonzero probability, meaning almost surely $\limsup X(n) \geq 1$, as required.