Infinitesimal Calculus 3

Lecture 22, Wednsday January 18, 2023
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Theorem 22.1:

If $f, h_1, \dots, h_k \colon \mathbb{R}^{n+k} \longrightarrow \mathbb{R}$ functions in C^1. We define

$$S = \left\{x \in \mathbb{R}^{n+k} \;\middle|\; h_1(x) = \cdots = h_k(x) = 0\right\}$$

If $f\big|_S$ has a critical point $p \in S$ and $\{\nabla h_i(p)\}_{i=1,\dots,k}$ are linearly independent then there exists $\lambda_1, \dots, \lambda_k$ such that

$$\nabla f(p) = \sum_{i=1}^{k} \lambda_i \cdot \nabla h_i(p)$$

Proof:

Since $\{\nabla h_i\}$ are independent, then the function $h = (h_1, \dots, h_k)$ has an invertible Jacobian at p and therefore by the implicit function theorem, S is the graph of an implicit function around p in C^1. Let TS be the tangent space to S. We know that by the previous lemma $\nabla f(p) \in \mathrm{TS}$, and since S is defined via k variables, TS has a dimension of n so $\mathrm{TS} = \mathrm{span}\{\nabla h_1(p), \dots, \nabla h_k(p)\}^\top$, and so $\nabla f(p)$ is in this span.

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22.1 Integrals

Definition 22.1:

A set $D \subseteq \mathbb{R}^n$ is a **open domain** if it is an open and connected set. And $\bar{D}$ is called a **closed domain**.

A prism $T = \{(x_1, \dots, x_n) \mid a_i < x_j < b_j\} = \prod_{i=1}^n (a_i, b_i)$ has volume

$$|T| = \prod_{i=1}^{n} (b_i - a_i)$$

For a set D we define its *internal volume* to be

$$|D|_{\text{int}} = \sup \sum_{i=1}^{k} |T_i|$$

where the supremum is taken over all sets of prisms $\{T_i\}_{i=1}^k$ where $\cup T_i \subseteq D$. The *external volume* is defined to be

$$|D|_{\text{ext}} = \inf \sum_{i=1}^{k} |T_i|$$

where the infimum is taken over all sets of prisms where $\cup T_i \supseteq D$.

Definition 22.2:

A set D is **contented** if its internal and external volumes are equal.

It can be shown that D is contented if and only if $|\partial D|_{\text{ext}} = 0$.

Definition 22.3:

A **partition** of the domain $\bar{D}$ is a set $\left\{\bar{D}_i \;\middle|\; 1 \le i \le k\right\}$ of contented closed domains which are pairwise disjoint and

$$\dot{\bigcup_{i=1}^{k}} \bar{D}_i = \bar{D}$$

Definition 22.4:

Given a partition $P = \left\{\bar{D}_i\right\}_{i=1}^{k}$ we define

$$M_i = \sup\left\{f(x) \;\middle|\; x \in \bar{D}_i\right\} \qquad m_i = \inf\left\{f(x) \;\middle|\; x \in \bar{D}_i\right\}$$

The **upper sum** of P is

$$\bar{S}(f, P) = \sum_{i=1}^{k} M_j \cdot \left|\bar{D}_j\right|$$

and the **lower sum** of P

$$\underline{S}(f, P) = \sum_{i=1}^{k} m_j \cdot \left|\bar{D}_j\right|$$

Definition 22.5:

The **width** of a partition P is $\lambda(P) = \max \operatorname{diam}(D_j)$.

Suppose D is contented, and so is every D_j. Further suppose $f\colon \bar{D} \longrightarrow \mathbb{R}$ is bounded, then

$$m\left|\bar{D}\right| = m\sum_{i=1}^{k}|D_j| = \sum_{i=1}^{k} m|D_j| \le \sum_{i=1}^{k} m_j|D_j| \le \sum_{i=1}^{k} M_j|D_j| \le M\sum_{i=1}^{k}|D_j| \le M\left|\bar{D}_j\right|$$

And specifically

$$m\left|\bar{D}\right| \le \underline{S}(f, P) \le \bar{S}(f, P) \le M|D|$$

Thus all the lower and upper sums are bounded by constants and thus the following is well-defined:

Definition 22.6:

The **upper integral** and **lower integral** are defined as followed, respectively:

$$\overline{\int_D} f = \inf_P \bar{S}(f, P) \qquad \underline{\int_D} f = \sup_P \underline{S}(f, P)$$

And f is **integrable** over D if

$$\int_D f(x_1, \ldots, x_n) dx_1 \ldots dx_n = S_D = \underline{\int_D} f = \overline{\int_D} f$$

Definition 22.7:

A partition Q is a **refinement** of a partition $P = \{P_i\}_{i=1}^{k}$ if it is obtained from P by further partitioning each P_i.

Proposition 22.8:

If Q is a refinement of P then

$$\underline{S}(f, P) \le \underline{S}(f, Q) \le \bar{S}(f, Q) \le \bar{S}(f, P)$$

Proof:

We can write Q as the "union" of partitions Q_j (partitions, not domains) which partition P_j. It is then trivial to see that

$$\bar{S}(f,Q) = \sum_{j=1}^{k} \bar{S}(f,Q_j)$$

and we know by the previous proposition that

$$\bar{S}(f,Q_j) \leq M_j |P_j|$$

where M_j is taken from P_j, since Q_j is a partition of the domain P_j. Thus

$$\bar{S}(f,Q) \leq \sum_{j=1}^{k} M_j |P_j| = \bar{S}(f,P)$$

as required. Similar for lower sums.

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Proposition 22.9:

If P and Q are *any* two partitions then

$$\underline{S}(f,P) \leq \bar{S}(f,Q)$$

Proof:

We define a new partition T which is a refinement of both P and Q, this can be done by taking all possible intersections of P_i and Q_j, that is:

$$T = \{T_{ij} \mid T_{ij} = P_i \cap Q_j \neq \varnothing\}$$

This is a contented partition since $\partial(P_i \cap Q_j) \subseteq \partial P_i \cup \partial Q_j$ and it can be shown that $|A \cup B| \leq |A| + |B|$, and so $\partial P_i \cap Q_j = 0$. Thus

$$\underline{S}(f,P) \leq \underline{S}(f,T) \leq \bar{S}(f,T) \leq \bar{S}(f,Q)$$

as required.

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Proposition 22.10:

$$\underline{\int_D} f \leq \overline{\int_D} f$$

This is true because the set of lower integrals is less than the set of upper integrals.

Proposition 22.11:

f is integrable in D if and only if for every $\varepsilon > 0$ there is a partition P such that

$$\bar{S}(f,P) - \underline{S}(f,P) < \varepsilon$$

Proof:

Suppose this is true for every $\varepsilon > 0$, then

$$\overline{\int_D} f - \underline{\int_D} f \leq \bar{S}(f,P) - \underline{S}(f,P) < \varepsilon$$

so the upper and lower integrals are equal, and therefore f is integrable.
And if f is integrable, take $\varepsilon > 0$ then there are partitions P and Q such that

$$0 \leq \bar{s}(f, P) - \overline{\int} f < \frac{\varepsilon}{2}$$

and

$$0 \leq \underline{\int} f - \underset{\sim}{s}(f, Q) < \frac{\varepsilon}{2}$$

If we take a common refinement K of P and Q then we get the same inequalities. Then adding both inequalities gives

$$0 \leq \bar{s}(f, K) - \underline{S}(f, K) < \varepsilon$$

since the upper and lower integrals are equal.

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