Mathematical Logic

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Definition 13.0.1:

Let I be a non-empty set, a **filter** over I is a set $D \subseteq \mathcal{P}(I)$ such that

(1) $I \in D$

(2) If $X, Y \in D$ then $X \cap Y \in D$

(3) If $X \in D$ and $X \subseteq Z \subseteq I$, then $Z \in D$ (D is upwards closed)

Example 13.0.2:

(1) The filter $\{I\}$ is the **trivial filter**.

(2) The filter $\mathcal{P}(I)$ is the **improper filter**. Filters which are not the improper filter are called **proper filters**.

(3) For each $Y \subseteq I$, $D = \{X \subseteq I \mid Y \subseteq X\}$ is the **principal filter generated by Y**.

(4) The **Frechét** filter $D = \{X \subseteq I \mid I \setminus X \text{ is finite}\}$.

Notice that if $\{D_\lambda\}_{\lambda\in\Lambda}$ is a family of filters over I, then

$$D = \bigcap_{\lambda\in\Lambda} D_\lambda$$

is also a filter. Obviously $I \in D$, and if $X, Y \in D$ then $X, Y \in D_\lambda$ for every $\lambda \in \Lambda$ and so $X \cap Y \in D_\lambda$ for every $\lambda \in \Lambda$ and so $X \cap Y$. And if $X \in D$ and $X \subseteq Z \subseteq I$ then $Z \in D_\lambda$ for every $\lambda \in \Lambda$ and so $Z \in D$.

Definition 13.0.3:

If $E \subseteq \mathcal{P}(I)$, then the **filter generated by E** is the smallest filter over I which contains E. Since the intersection of arbitrary non-empty families of filters is also a filter, the filter generated by E is equal to

$$\bigcap_{\substack{F \text{ is a filter} \\ E\subseteq F}} F$$

since this intersection is non-empty as the improper filter is in it.

Definition 13.0.4:

If $E \subseteq \mathcal{P}(I)$, it is said to have the **finite intersection property** if the intersection of any finite number of sets in E is non-empty.

Proposition 13.0.5:

Let $E \subseteq \mathcal{P}(I)$, and let D be the filter generated by E, then

(1) D is a filter over I

(2) D is the set of all $X \in \mathcal{P}(I)$ such that $X = I$ or for some $Y_1, \dots, Y_n \in E$

$$Y_1 \cap \cdots \cap Y_n \subseteq X$$

(3) D is a proper filter if and only if E has the finite intersection property

Proof:

(1) We have shown this, as it is the intersection of a non-empty family of filters which is itself a filter.

(2) Let D' be the set of all X such that there exist $Y_1, \dots, Y_n \in E$ such that $Y_1 \cap \dots \cap Y_n \subseteq X$, or $X = I$, ie

$$D' = \{X \in \mathcal{P}(I) \mid X = I \text{ or } \exists Y_1, \dots, Y_n \in E \colon Y_1 \cap \dots \cap Y_n \subseteq X\}$$

we will show $D = D'$.

Firstly we will show that D' is a filter containing E. Obviously $I \in D'$. If $X, X' \in D'$ then let $Y_1, \dots, Y_n, Y_1', \dots, Y_m' \in E$ such that

$$Y_1 \cap \dots \cap Y_n \subseteq X, \quad Y_1' \cap \dots \cap Y_m' \subseteq X'$$

then

$$Y_1 \cap \dots \cap Y_n \cap Y_1' \cap \dots \cap Y_m' \subseteq X \cap X'$$

and so $X \cap X' \in D'$. And if $X \in D$ and $X \subseteq Z \subseteq I$ then if $Y_1 \cap \dots \cap Y_n \subseteq X$, $Y_1 \cap \dots \cap Y_n \subseteq Z$ and so $Z \in D'$. Therefore D' is a filter. And if $Y \in E$, then $Y \subseteq Y$ and so $E \subseteq D'$, meaning D' is a filter containing E as required.

Since D is the smallest filter containing E, $D \subseteq D'$.

Now let F be any filter over I which includes E, then if $Y_1, \dots, Y_n \in E$ we must have that $Y_1 \cap \dots \cap Y_n \in F$. Moreso, since filters are upwards-closed, we must have that for every $X \in \mathcal{P}(I)$ such that $Y_1 \cap \dots \cap Y_n \subseteq X$, $X \in F$. Meaning that $D' \subseteq F$ and in particular $D' \subseteq D$. Thus $D = D'$ as required.

(3) Note that a filter F is a proper filter if and only if $\varnothing \notin F$. If $\varnothing \in F$ then for every $\varnothing \subseteq X$, $X \in F$ meaning $F = \mathcal{P}(I)$ so it is the improper filter. And if $\varnothing \notin F$ then it is obviously is a proper filter.

So D is a proper filter if and only if $\varnothing \notin D$, which is if and only if for every $Y_1, \dots, Y_n \in E$, $Y_1 \cap \dots \cap Y_n \neq \varnothing$, which is precisely what it means for E to have the finite intersection property. ∎

Let us give an example of a particularly important filter. Let J be an infinite set and let $I = \mathcal{P}_\omega(J)$ be the set of all finite subsets of J. For each $j \in J$ let

$$\mathcal{J}_j = \{i \in I \mid j \in i\} \subseteq I$$

be the set of all finite subsets of J which contain j. And let

$$E = \{\mathcal{J}_j \mid j \in J\} \subseteq \mathcal{P}(I)$$

Then let D be the filter over I generated by E. E has the finite intersection property since if $j_1, \dots, j_n \in J$ then $\{j_1, \dots, j_n\} \in \mathcal{J}_{j_k}$ for every $1 \leq k \leq n$ and so $\{j_1, \dots, j_n\} \in \mathcal{J}_{j_1} \cap \dots \cap \mathcal{J}_{j_n}$.

Definition 13.0.6:

D is said to be an **ultrafilter** over I if D is a filter over I and for every $X \subseteq I$, $X \in D$ if and only if $I \setminus X \notin D$. Meaning that for every $X \in \mathcal{P}(I)$, D contains either X or $I \setminus X$.

Notice that if D is a filter over I then $\bigcup_{X \in D} X = I$ since $I \in D$. Thus if we say D is a filter, we do not need to state over what.

Proposition 13.0.7:

The following are equivalent:

(1) D is an ultrafilter over I

(2) D is a maximal proper filter over I (if F is a proper filter over I such that $D \subseteq F$ then $F = D$)

Proof:

Suppose D is an ultrafilter over I, then D is a proper filter since $I \in D$ so $I \setminus I = \varnothing \notin D$. Let F be a proper filter which includes D. If $X \in F$ and $X \notin D$ then $I \setminus X \in D$ which means $I \setminus X \in F$ but then $\varnothing = X \cap (I \setminus X)$ and so $\varnothing \in F$ which contradicts F being proper. And so $F \subseteq D$, meaning $F = D$ as required.

Now suppose D is a maximal proper filter over I, then let $X \in \mathcal{P}(I)$. We cannot have both $X \in D$ and $I \setminus X \in D$ since then $\varnothing \in D$ but D is proper. So we will show that if $I \setminus X \notin D$ then $X \in D$. Let $E = D \cup \{X\}$, then let F be the filter generated by E. Let $Y_1, \dots, Y_n \in E$ and let $Z = Y_1 \cap \cdots \cap Y_n$, then since D is closed under finite intersections, $Z = Y$ or $Z = Y \cap X$ for $Y \in D$. In the first case $Z \in D$ and so $Z \neq \varnothing$. For the second case, if $Z = \varnothing$ then $Y \cap X = \varnothing$ meaning that $Y \subseteq I \setminus X$ and so $I \setminus X \in D$, which is a contradiction. So we have in both cases that $Z \neq \varnothing$ and so E has the finite intersection property, and therefore F is a proper filter. Since D is maximal, this means $F = D$ and so $X \in E \subseteq F = D$ as required. ■

Lemma 13.0.8:

If C is a chain of proper filters over I, then $D = \bigcup_{F \in C} F$ is a proper filter over I.

Proof:

Obviously $I \in D$, and if $X, Y \in D$ then there exist $F_1, F_2 \in C$ such that $X \in F_1$ and $Y \in F_2$, we can assume that $F_1 \subseteq F_2$ in which case $X, Y \in F_2$ and so $X \cap Y \in F_2 \subseteq D$. And finally if $X \in D$ and $X \subseteq Z$, then $X \in F \in C$, and so $Z \in F$ meaning $F \in D$, so D is indeed a filter. If $\varnothing \in D$, then $\varnothing \in F$ for some $F \in C$, but C is a chain of proper filters so this cannot be. Thus $\varnothing \notin D$ meaning D is proper. ■

Theorem 13.0.9:

If $E \subseteq \mathcal{P}(I)$ and E has the finite intersection property, then there exists an ultrafilter D over I such that $E \subseteq D$.

Proof:

Let F be the filter generated by E, it is proper since E has the finite intersection property. Let

$$\mathcal{S} = \{F \subset \mathcal{P}(I) \mid F \text{ is a proper filter and } E \subseteq F\}$$

then let C be a chain in $\mathcal{S}$, then by the above lemma $\bigcup_{F \in C} F$ is a proper filter over I, and it obviously contains E. Thus every chain in $\mathcal{S}$ has an upper bound in $\mathcal{S}$ and so by Zorn's Lemma $\mathcal{S}$ has a maximal element, D. Therefore $E \subseteq D$ and D is a maximal proper filter over I meaning D is an ultrafilter containing E. ■

Corollary 13.0.10:

Any proper filter over I can be extended to an ultrafilter over I.

This is because every proper filter has the finite intersection property.

Definition 13.0.11:

If I is a non-empty set and $\{A_i\}_{i \in I}$ is a family of sets, recall that

$$C = \prod_{i \in I} A_i$$

is the set of all function $f\colon I \longrightarrow \bigcup_{i \in I} A_i$ such that for every $i \in I$, $f(i) \in A_i$.

Now suppose D is a proper filter over I, we say that $f, g \in C$ are **D-equivalent** if the set of all $i \in I$ such that $f(i) = g(i)$ is an element of D, ie

$$\{i \in I \mid f(i) = g(i)\} \in D]$$

and we denote this by $f \equiv_D g$.

Proposition 13.0.12:

The relation $\equiv_D$ is an equivalence relation over C.

Proof:

Since $\{i \in I \mid f(i) = f(i)\} = I \in D$ since D is a filter, we have that $f \equiv_D f$, so the relation is reflexive. Obviously the relation is symmetric. Now suppose $f \equiv_D g$ and $g \equiv_D h$, then

$$\{i \in I \mid f(i) = h(i)\} \supseteq \{i \in I \mid f(i) = g(i)\} \cap \{i \in I \mid g(i) = h(i)\}$$

and since both of the sets on the right hand side are in D, so is their intersection and so $\{i \in I \mid f(i) = h(i)\}$ contains a set in D and thus is itself contained in D since D is a filter. So $f \equiv_D h$, so $\equiv_D$ is transitive as required. ■

Definition 13.0.13:

If $f \in C$, let f_D be the equivalence class of f under $\equiv_D$:

$$f_D = \{g \in C \mid f \equiv_D g\}$$

We then define the **reduced product of A_i modulo D** to be the set of all equivalence classes of $\equiv_D$, it is denoted by $\prod_D A_i$:

$$\prod_D A_i = \left\{ f_D \,\middle|\, f \in \prod_{i \in I} A_i \right\}$$

or in other words, $\prod_D A_i$ is the partition of $\prod_{i \in I} A_i$ under $\equiv_D$.

If D is an ultrafilter $\prod_D A_i$ is called the **ultraproduct of A_i modulo D**.

If all of the sets A_i are equal to $A_i = A$, then $\prod_D A_i$ is called the **reduced power of A modulo D** and is written $\prod_D A$. If D is an untrafilter then $\prod_D A$ is called the **ultrapower of A modulo D**.

Definition 13.0.14:

Suppose I is a non-empty set and $\mathcal{L}$ a signature and for every $i \in I$ let $\mathcal{A}_i$ be an $\mathcal{L}$-structure, we use the convention that the domain of $\mathcal{A}_i$ is understood to be A_i. Let D be a proper filter over I, we define the **reduced (filtered) product $\mathcal{A} = \prod_D \mathcal{A}_i$** to be an $\mathcal{L}$-interpretation whose domain is $\prod_D A_i$ and

- If P is an n-ary relation in $\mathcal{L}$ then

  $$P^{\mathcal{A}}(f^1_D, \dots, f^n_D) \text{ if and only if } \left\{i \in I \,\middle|\, P^{\mathcal{A}_i}(f^1(i), \dots, f^n(i))\right\} \in D$$

  this is well defined since if $f^k \equiv_D g^k$ for each k then if $\left\{i \in I \,\middle|\, P^{\mathcal{A}_i}(f^1(i), \dots, f^n(i))\right\} \in D$ then

  $$\left\{i \in I \,\middle|\, P^{\mathcal{A}_i}(f^1(i), \dots, f^n(i))\right\} \cap \left\{i \in I \,\middle|\, \forall 1 \leq k \leq n\colon f^k(i) = g^k(i)\right\} \in D$$

  as the intersection of sets in D. And this is a subset of $\left\{i \in I \,\middle|\, P^{\mathcal{A}_i}(g^1(i), \dots, g^n(i))\right\}$, meaning that $\left\{i \in I \,\middle|\, P^{\mathcal{A}_i}(g^1(i), \dots, g^n(i))\right\} \in D$ as required.

- Using the notation that $(a_i)_{i \in I}$ is the function $f \in \prod_{i \in I} A_i$ where $f(i) = a_i$, if F is an n-ary function in $\mathcal{L}$ then

  $$F^{\mathcal{A}}(f^1_D, \dots, f^n_D) = \left[\left(F^{\mathcal{A}_i}(f^1(i), \dots, f^n(i))\right)_{i \in I}\right]_D$$

  the equivalence class under the relation $\equiv_D$.

  This is well defined since if $f^k \equiv_D g^k$ for all $1 \leq k \leq n$ then we must show that

  $$\left(F^{\mathcal{A}_i}(f^1(i), \dots, f^n(i))\right)_{i \in I} \equiv_D \left(F^{\mathcal{A}_i}(g^1(i), \dots, g^n(i))\right)_{i \in I}$$

  this is true because the set where these two functions are equivalent is a superset of $\left\{i \in I \,\middle|\, \forall 1 \leq k \leq n\colon f^k(i) = g^k(i)\right\}$ which is in D as the intersection of sets in D. So this definition is also well-defined.

- Since constants are just 0-ary functions, the interpretation of constants inherits from the definition above:

  $$c^{\mathcal{A}} = \left(c^{\mathcal{A}_i}\right)_{i \in I}$$

Theorem 13.0.15 (The Expansion Theorem):

Let $\mathcal{L}'$ be an extension of the signature $\mathcal{L}$. Let I be a non-empty set, and let $\mathcal{A}_i$ be an $\mathcal{L}$-structure for each $i \in I$ and $\mathcal{B}_i$ be an extension of $\mathcal{A}_i$ to an $\mathcal{L}'$-structure. Let D be a proper filter over I then $\prod_D \mathcal{B}_i$ is an extension of $\prod_D \mathcal{A}_i$.

Proof:

Since the domain of $\mathcal{A}_i$ and $\mathcal{B}_i$ are the same, $A_i = B_i$ the domain of the reduced products are the same. Since $\mathcal{B}_i$ is an extension of $\mathcal{A}_i$, each symbol in $\mathcal{L}$ has the same interpretation in $\mathcal{A}_i$ as it does in $\mathcal{B}_i$. Since the interpretations of symbols in $\mathcal{L}$ by $\prod_D \mathcal{B}_i$ depends only on the interpretations of those symbols by the $\mathcal{B}_i$s, which is the same as the interpretations of the symbols in the $\mathcal{A}_i$s, it follows that the interpretations of the symbols in $\prod_D \mathcal{B}_i$ is the same as the interpretations in $\prod_D \mathcal{A}_i$. ∎

Theorem 13.0.16 (The Fundamental Theorem of Ultraproducts):

Let $\mathcal{A}$ be the ultraproduct $\prod_D \mathcal{A}_i$ and let I be the indexing set. Then

(1) For any $\mathcal{L}$-term $t(x_1, \dots, x_n)$ and $f_D^1, \dots, f_D^n \in \mathcal{A}$,

$$t^{\mathcal{A}}(f_D^1, \dots, f_D^n) = \Big[\big(t^{\mathcal{A}_i}(f^1(i), \dots, f^n(i))\big)_{i \in I}\Big]_D$$

(2) For any $\mathcal{L}$-formula $\varphi(x_1, \dots, x_n)$, and $f_D^1, \dots, f_D^n \in \mathcal{A}$,

$$\mathcal{A} \vDash \varphi(f_D^1, \dots, f_D^n) \text{ if and only if } \big\{i \in I \bigm| \mathcal{A}_i \vDash \varphi(f_D^1(i), \dots, f^n(i)) \in D\big\}$$

(3) For any $\mathcal{L}$-sentence φ,

$$\mathcal{A} \vDash \varphi \iff \{i \in I \mid \mathcal{A}_i \vDash \varphi\} \in D$$

Proof:

(1) We will do this by term induction. If $t = x$ is a variable then all we must show is that

$$f_D = \Big[\big(f^1(i)\big)_{i \in I}\Big]_D$$

which is true because $(f^1(i))_{i \in I}$ is precisely f. And if $t = c$ is a constant then this is a direct result of the definition of $c^{\mathcal{A}}$.

Now if

$$t(x_1, \dots, x_n) = F(t_1(x_1, \dots, x_n), \dots, x_m(x_1, \dots, x_n))$$

then

$$t^{\mathcal{A}}(f_D^1, \dots, f_D^n) = F^{\mathcal{A}}(t_1^{\mathcal{A}}(f_D^1, \dots, f_D^n), \dots, t_m^{\mathcal{A}}(f_D^1, \dots, f_D^n))$$

By our inductive assumption

$$t_k^{\mathcal{A}}(f_D^1, \dots, f_D^n) = g_D^k$$

where

$$g^k = \big(t_k^{\mathcal{A}_i}(f^1(i), \dots, f^n(i))\big)_{i \in I}$$

and so

$$t^{\mathcal{A}}(f_D^1, \dots, f_D^n) = F^{\mathcal{A}}(g_D^1, \dots, g_D^n)$$

And by definition

$$F^{\mathcal{A}}(g_D^1, \dots, g_D^n) = \Big[\big(F^{\mathcal{A}_i}(g^1(i), \dots, g^n(i))\big)_{i \in I}\Big]_D$$

And again by definition

$$t^{\mathcal{A}_i}(f_D^1(i), \dots, f_D^n(i)) = F^{\mathcal{A}_i}(t_1^{\mathcal{A}_i}(f_D^1(i), \dots, f_D^n(i)), \dots, t_m^{\mathcal{A}_i}(f_D^1(i), \dots, f_D^n(i))) = F^{\mathcal{A}_i}(g^1(i), \dots, g^n(i))$$

And so we get that

$$t^{\mathcal{A}}(f_D^1, \dots, f_D^n) = \Big[\big(t^{\mathcal{A}_i}(f_D^1(i), \dots, f_D^n(i))\big)_{i \in I}\Big]_D$$

as required.

(2) The proof for atomic formulas is similar to the proof for (1). We proceed inductively, suppose $\varphi = \neg\psi$ then

$$\mathcal{A} \vDash \varphi(f^1_D, \dots, f^n_D) \iff \mathcal{A} \nvDash \psi(f^1_D, \dots, f^n_D) \iff \left\{i \in I \,\middle|\, \mathcal{A}_i \vDash \psi(f^1(i), \dots, f^n(i))\right\} \notin D$$

and since D is an ultrafilter this is if and only if

$$\iff \left\{i \in I \,\middle|\, \mathcal{A}_i \nvDash \psi(f^1(i), \dots, f^n(i))\right\} \in D \iff \left\{i \in I \,\middle|\, \mathcal{A}_i \vDash \varphi(f^1(i), \dots, f^n(i))\right\}$$

as required.

The step for formulas of the form $\varphi \wedge \psi$ is simple, knowing that $X \cap Y \in D$ if and only if $X \in D$ and $Y \in D$ (this is true for filters in general).

Now suppose $\varphi(x_1, \dots, x_n) = \exists x_0 \psi(x_0, x_1, \dots, x_n)$, then $\mathcal{A} \vDash \varphi(f^1_D, \dots, f^n_D)$ if and only if there exists an $f^0_D \in \mathcal{A}$ such that $\mathcal{A} \vDash \psi(f^0_D, \dots, f^n_D)$ which inductively is if and only if $\left\{i \in I \,\middle|\, \mathcal{A}_i \vDash \psi(f^0(i), \dots, f^n(i))\right\} \in D$. And so if this holds then we get that since $\mathcal{A}_i \vDash \psi(f^0(i), \dots, f^n(i)$, we have $\mathcal{A}_i \vDash \exists x_0 \psi(x_0, f^1(i), \dots, f^n(i))$ and so $\mathcal{A}_i \vDash \varphi(f^1(i), \dots, f^n(i))$ and so

$$\left\{i \in I \,\middle|\, \mathcal{A}_i \vDash \varphi(f^1(i), \dots, f^n(i))\right\} \in D$$

as required.

And if

$$\left\{i \in I \,\middle|\, \mathcal{A}_i \vDash \varphi(f^1(i), \dots, f^n(i))\right\} \in D$$

then for each $i \in I$ we can choose $a_i \in \mathcal{A}_i$ such that $\mathcal{A}_i \vDash \psi(a_i, f^1(i), \dots, f^n(i))$ and define $f^0(i) = a)_i$ and so we have

$$\left\{i \in I \,\middle|\, \mathcal{A}_i \vDash \psi(f^0(i), \dots, f^n(i))\right\}$$

is a superset of the above set and is therefore also in D. Thus as shown above this means $\mathcal{A} \vDash \varphi(f^1_D, \dots, f^n_D)$ as required.

(3) This is a particular result of the previous part. ■

Corollary 13.0.17:

For any structure $\mathcal{A}$ and ultrafilter D, $\prod_D \mathcal{A} \equiv \mathcal{A}$.

This is because if φ is an $\mathcal{L}$-sentence then

$$\prod_D \mathcal{A} \vDash \varphi \iff \{i \in I \mid \mathcal{A} \vDash \varphi\} \in D$$

which is if and only if $\mathcal{A} \vDash \varphi$ (since if the set is in D it cannot be empty so $\mathcal{A} \vDash \varphi$ and if $\mathcal{A} \vDash \varphi$ then the set above is equal to I).

We can use the **The Fundamental Theorem of Ultraproducts** to provide an alternative proof of the compactness theorem:

Corollary 13.0.18:

Let Σ be a set of sentences of $\mathcal{L}$, and let $I = \mathcal{P}_\omega(\Sigma)$, the set of all finite subsets of Σ. Then if every for every $i \in I$, i is satisfiable by $\mathcal{A}_i$ then there exists an ultrafilter D over I such that $\prod_D \mathcal{A}_i$ models Σ.

Proof:

For each $\sigma \in \Sigma$ let $\hat{\sigma}$ be the set of all $i \in I$ such that $\sigma \in I$:

$$\hat{\sigma} = \{i \in I \mid \sigma \in i\}$$

ie. $\hat{\sigma}$ is the set of all finite subsets of Σ for which σ is an element of. Then let

$$E = \{\hat{\sigma} \mid \sigma \in \Sigma\}$$

E has the finite intersection property since if $\sigma_1, \dots, \sigma_n \in \Sigma$ then $\{\sigma_1, \dots, \sigma_n\} \in \hat{\sigma}_k$ for each $1 \leq k \leq n$.

Thus E can be extended to an ultrafilter D (since it generates a proper filter which can be extended to an ultrafilter). If $i \in \hat{\sigma}$ then $\sigma \in i$ meaning that $\mathcal{A}_i \vDash \sigma$. Thus

$$\hat{\sigma} \subseteq \{i \in I \mid \mathcal{A}_i \vDash \sigma\}$$

and since $\hat{\sigma} \in E \subseteq D$, we have that $\{i \in I \mid \mathcal{A}_i \vDash \sigma\} \in D$. By **The Fundamental Theorem of Ultraproducts**, this means that $\prod_D \mathcal{A}_i \vDash \sigma$ for all $\sigma \in \Sigma$, and thus $\prod_D \mathcal{A}_i \vDash \Sigma$ as required. ■

We now discuss classes of structures, these are many times proper classes.

Definition 13.0.19:

Suppose $\mathcal{K}$ is a class of $\mathcal{L}$-structures, then

- $\mathcal{K}$ is an **elementary class** if there exists an $\mathcal{L}$-theory T such that $\mathcal{K}$ is precisely all the models of T.
- $\mathcal{K}$ is a **basic elementary class** if there exists an $\mathcal{L}$-sentence φ such that $\mathcal{K}$ is precisely all the models which satisfy φ.
- $\mathcal{K}$ is **closed under elementary equivalence** if $\mathcal{A} \in \mathcal{K}$ and $\mathcal{A} \equiv \mathcal{B}$ then $\mathcal{B} \in \mathcal{K}$.
- $\mathcal{K}$ is **closed under ultraproducts** if for every family of structures in $\mathcal{K}$, $\{\mathcal{A}_i\}_{i \in I}$, and ultrafilter D over I, $\prod_D \mathcal{A}_i \in \mathcal{K}$.

Theorem 13.0.20:

Let $\mathcal{K}$ be a class of $\mathcal{L}$-structures. Then

(1) $\mathcal{K}$ is an elementary class if and only if $\mathcal{K}$ is closed under ultraproducts and elementary equivalence.

(2) $\mathcal{K}$ is a basic elementary class if and only if both $\mathcal{K}$ and the complement of $\mathcal{K}$ are closed under ultraproducts and elementary equivalence.

Proof:

(1) If $\mathcal{K}$ is an elementary class, then it is obviously closed under elementary equivalence. And if $\prod_D \mathcal{A}_i$ is an ultraproduct of structures in $\mathcal{K}$, then since $\{i \in I \mid \mathcal{A}_i \vDash \varphi\} = I \in D$, $\prod_D \mathcal{A}_i \vDash \varphi$. And so if $\mathcal{A}_i \vDash T$ for all $i \in I$ then $\prod_D \mathcal{A}_i \vDash T$ and so since $\mathcal{K}$ is an elementary class this means that the ultraproduct is in $\mathcal{K}$.

Now suppose $\mathcal{K}$ is closed under elementary equivalence and ultraproducts. Let T be the theory of all $\mathcal{L}$-sentences which hold in every $\mathcal{A} \in \mathcal{K}$. Then $\mathcal{K}$ is a class of models of T. Now suppose $\mathcal{B}$ models T, let Σ be the set of $\mathcal{L}$-sentences true in $\mathcal{B}$ and let $I = \mathcal{P}_\omega(\Sigma)$. Then for every $i = \{\sigma_1, \dots, \sigma_n\} \in I$, there exists a structure $\mathcal{A}_i \in \mathcal{K}$ which models i, as otherwise every $\mathcal{A} \in \mathcal{K}$ satisfies $\varphi = \neg(\sigma_1 \wedge \cdots \wedge \sigma_n)$. And thus by definition $\varphi \in T$ and so $\mathcal{B} \vDash \varphi$, which is a contradiction since φ is false in $\mathcal{B}$.

And so by above, there exists an ultrafilter D such that $\prod_D \mathcal{A}_i \vDash \Sigma$, and since $\mathcal{K}$ is closed under ultraproducts, $\prod_D \mathcal{A}_i \in \Sigma$. And since the ultraproduct models Σ, it is elementarily equivalent to $\mathcal{B}$, and since $\mathcal{K}$ is closed under elementary equivalence, $\mathcal{B} \in \mathcal{K}$. So $\mathcal{K}$ is the class of all models of T, and is therefore an elementary class as required.

(2) If $\mathcal{K}$ is a basic elementary class then $\mathcal{K}$ and $\mathcal{K}^c$ are elementary classes ($\mathcal{K}^c$ is all the models which satisfy $\neg\varphi$), and so by above they are both closed under ultraproducts and elementary equivalence.

Suppose T_1 is the theory of $\mathcal{K}$ and T_2 that of $\mathcal{K}^c$. Then let $T = T_1 \cup T_2$, if $\mathcal{A} \vDash T$ then $\mathcal{A} \vDash T_1$ and $\mathcal{A} \vDash T_2$ which means $\mathcal{A} \in \mathcal{K}$ and $\mathcal{A} \in \mathcal{K}^c$ which is a contradiction. Thus T is unsatisfiable and therefore there exists a $\varphi \in T_1$ and $\neg\varphi \in T_2$. Let us define $T' = \{\varphi\}$, and so if $\mathcal{A} \in \mathcal{K}$ then it obviously satisfies $\varphi \in T_1$, and thus T'. And if $\mathcal{A} \vDash T'$ then $\mathcal{A} \nvDash \neg\varphi$ which means $\mathcal{A} \notin \mathcal{K}^c$, so $\mathcal{A} \in \mathcal{K}$ meaning $T' = \{\varphi\}$ is the theory of $\mathcal{K}$, so $\mathcal{K}$ is a basic elementary class. ■

Let $\mathcal{A}$ be a structure and $\prod_D \mathcal{A}$ an ultrapower where D is an ultrafilter over I. The *natural embedding* of $\mathcal{A}$ into $\prod_D \mathcal{A}$

is defined by

$$d(a) = \left[(a)_{i\in I}\right]_D$$

Corollary 13.0.21:

The natural embedding is an elementary embedding.

Proof:

Let $\varphi(x_1, \ldots, x_n)$ be an $\mathcal{L}$-formula and $a_1, \ldots, a_n \in \mathcal{A}$ then

$$\prod_D \mathcal{A} \vDash \varphi(d(a_1), \ldots, d(a_n)) \iff \{i \in I \mid \mathcal{A} \vDash \varphi(a_1, \ldots, a_n)\} \in D \iff \mathcal{A} \vDash \varphi(a_1, \ldots, a_n)$$

where the last equivalence is because if not then the set is empty, and if so then the set is equal to $I \in D$. ■

Theorem 13.0.22 (Keisler-Shelah Isomorphism Theorem):

If $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{L}$-structures then $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent if and only if there exists an ultrafilter D such that

$$\prod_D \mathcal{A} \cong \prod_D \mathcal{B}$$