Infintesimal Calculus 3

Assignment 9
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Exercise 9.1:

Find the minimal distance from $(0,0)$ to the hyperbola:

$$7x^2 + 8xy + y^2 = 45$$

We define the lagrangian:

$$\mathcal{L}(x,y,\lambda) = x^2 + y^2 + \lambda(7x^2 + 8xy + y^2 - 45)$$

Whose gradient is

$$\begin{pmatrix} 2x + 14\lambda x + 8\lambda y \\ 2y + 2\lambda y + 8\lambda x \\ 7x^2 + 8xy + y^2 - 45 \end{pmatrix}$$

Solving for zero gives the system

$$\begin{pmatrix} 2+14\lambda & 8\lambda \\ 8\lambda & 2+2\lambda \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

So the determinant must be 0, meaning λ is $\frac{1}{9}$ or -1. If $\lambda = \frac{1}{9}$ then this gives us the solution $x = 2y$, plugging this into the hyperbola we get $y = \pm 1$ and so the point is $\pm(2,1)$. $\lambda = 1$ gives $y = 2x$ and $x = \pm\frac{\sqrt{15}}{3}$.
$\pm(2,1)$ gives the minimum distance of $\sqrt{5}$.

Exercise 9.2:

Find the maximum and minimum of the function

$$f(x,y,z) = \sqrt{2}x + \sqrt{2}y + \sqrt{3}z$$

within the ball

$$B = \left\{(x,y,z) \in \mathbb{R}^3 \;\middle|\; x^2 + y^2 + z^2 \le 2\right\}$$

First we look for critical points within the ball by comparing the gradient of f to 0, but the gradient of f is

$$\nabla f = \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \\ \sqrt{3} \end{pmatrix}$$

So we define the lagrangian:

$$\mathcal{L}(x,y,z,\lambda) = f(x,y,z) + \lambda(x^2 + y^2 + z^2 - 2)$$

And we find critical points relative to it

$$\nabla\mathcal{L} = \begin{pmatrix} \sqrt{2} + 2\lambda x \\ \sqrt{2} + 2\lambda y \\ \sqrt{3} + 2\lambda z \\ x^2 + y^2 + z^2 - 2 \end{pmatrix} = 0$$

So

$$x = \frac{\sqrt{2}}{2\lambda} \quad y = \frac{\sqrt{2}}{2\lambda} \quad z = \frac{\sqrt{3}}{2\lambda}$$

Plugging these into the constraint function we get that

$$x^2+y^2+z^2-2=\frac{7}{4\lambda^2}-2=0 \implies \lambda=\pm\frac{\sqrt{14}}{4}$$

And so we have the points $\pm\left(\frac{2}{\sqrt{7}},\frac{2}{\sqrt{7}},\sqrt{\frac{6}{7}}\right)$, and so the maximum is $\sqrt{14}$ (when the point is positive) and the minimum is $-\sqrt{14}$, as the maximum and minimum must be one of these two points.

Exercise 9.3:

Find the maximum distance between the origin and the set $\left\{(x,y,z)\in\mathbb{R}^3 \,\middle|\, x^2+y^2=1, z=x+y\right\}$.

We are trying to maximize the function $f(x,y,z)=x^2+y^2+z^2$ under the constraints $h_1(x,y,z)=x^2+y^2-1=0$ and $h_2(x,y,z)=x+y-z=0$. The Lagrangian of this is:

$$\mathcal{L}(x,y,z,\lambda_1,\lambda_2)=x^2+y^2+z^2+\lambda_1(x^2+y^2-1)+\lambda_2(x+y-z)$$

whose gradient is

$$\nabla\mathcal{L}=\begin{pmatrix}2x+2\lambda_1x+\lambda_2\\2y+2\lambda_1y+\lambda_2\\2z-\lambda_2\\x^2+y^2-1\\x+y-z\end{pmatrix}$$

So

$$2x(\lambda_1+1)=-\lambda_2 \quad 2y(\lambda_1+1)=-\lambda_2 \quad z=\frac{\lambda_2}{2}$$

If $\lambda_1=-1$ then $\lambda_2=0$ and so $z=0$ and so we're left with $x^2+y^2=1$ and $y=-x$, since there is a solution to this this gives us a distance of $f(x,y,0)=x^2+y^2=1$.
If $\lambda_1\neq-1$ then

$$x,y=-\frac{\lambda_2}{2(\lambda_1+1)} \quad z=\frac{\lambda_2}{2}$$

So $x=y$ and $x^2+y^2=1$ meaning $x^2=\frac{1}{2}$ so $x=y=\pm\frac{1}{\sqrt{2}}$ and $z=x+y=\pm\frac{2}{\sqrt{2}}$. These are obviously points in the set, so we don't even need to find the values of λ_1 and λ_2 since they are inconsequential. For these values of x, y, and z we get

$$f(x,y,z)=1+z^2=3$$

So the maximum value of f is 3 and so the maximum distance is $\sqrt{3}$.

Exercise 9.4:

Compute

$$\iiint_D y\,dxdydz$$

where D is the volume bounded by $y=1-x^2$, $z=0$, and $z=y$.

D can be written as

$$D=\left\{(x,y,z)\in\mathbb{R}^3 \,\middle|\, 0\leq y\leq z,\ y\leq 1-x^2\right\}$$

So $0\leq y\leq 1-x^2$ meaning $-1\leq x\leq 1$, so the integral is equal to

$$\int_{-1}^{1}\int_{0}^{1-x^2}\int_{0}^{y} y\,dzdydx=\int_{-1}^{1}\int_{0}^{1-x^2} y^2\,dydx=\frac{1}{3}\int_{-1}^{1}(1-x^2)^3\,dx$$

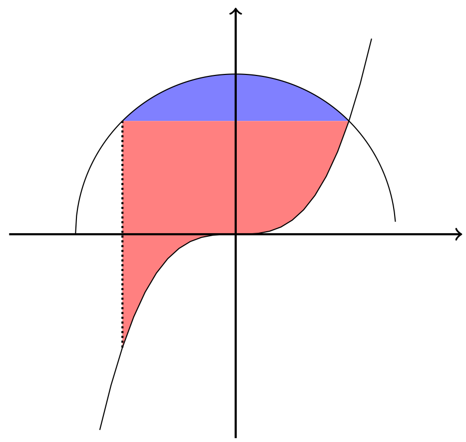
This is a polynomial, and integrating gives $\frac{32}{105}$.

Exercise 9.5:

Reverse the order of integration of

$$\int_{-1}^{1}\int_{x^3}^{\sqrt{2-x^2}} f(x,y)\,dydx$$

Let's take a look at the graph of the domain:



The blue region is given by $1 \le y \le \sqrt{2}$ and $-\sqrt{2-y^2} \le x \le \sqrt{2-y^2}$, and the red region is $-1 \le y \le 1$ and $-1 \le x \le \sqrt[3]{y}$. So the integral is

$$\int_{-1}^{1}\int_{-1}^{\sqrt[3]{y}} f(x,y)\,dxdy + \int_{1}^{\sqrt{2}}\int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} f(x,y)\,dxdy$$

Exercise 9.6:

Compute

$$\iint_D \frac{1}{\sqrt{x^2+y^2}}\,dxdy$$

where $D = \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; \left(x-\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 \le \frac{1}{2}\right\}$.

Note that

$$\left(x-\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 = x^2+y^2-x-y+\frac{1}{2}$$

So $(x,y) \in D$ if and only if $x^2+y^2-x-y \le 0$. Let us transform to polar coordinates, this means that we're in the domain if and only if

$$r^2 \le r(\cos\theta + \sin\theta) \iff r \le \cos\theta + \sin\theta$$

And $0 \le \cos\theta + \sin\theta$ if and only if $-\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$ so the integral is

$$\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_{0}^{\cos\theta+\sin\theta} 1\,drd\theta = \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \cos\theta + \sin\theta\,d\theta$$

This is equal to $2\sqrt{2}$.

Exercise 9.7:

Compute

$$\iint_D e^{\frac{x-y}{x+y}}\,dxdy$$

where $D = \{1 \le x + y \le 2, x \ge 0, y \ge 0\}$.

We transform $u = x + y$ and $v = x - y$, or $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$. Thus the Jacobian is

$$\left\| \begin{matrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{matrix} \right\| = \frac{1}{2}$$

And the domain becomes $1 \le u \le 2$, $u + v \ge 0$, and $u - v \ge 0$, so $\{1 \le u \le 2, -u \le v \le u\}$, so:

$$\frac{1}{2}\int_1^2 \int_{-u}^{u} e^{\frac{v}{u}}\, dvdu = \frac{1}{2}\int_1^2 u\left(e - \frac{1}{e}\right) dy = \frac{3}{4}\left(e - \frac{1}{e}\right)$$

Exercise 9.8:

Compute

$$\iiint_D (yz + zx)\, dxdydz$$

where D is the domain contained within the first octant, $x = 0$, $z = 0$, $y = x$, $x^2 + y^2 + z^2 = R^2$.

Here

$$D = \left\{z \ge 0, 0 \le y \le x, x^2 + y^2 + z^2 \le R^2\right\}$$

Let us use spherical coordinates (ρ, φ, θ) where

$$x = \rho \sin\varphi \cos\theta, \quad y = \rho \sin\varphi \sin\theta, \quad z = \rho\cos\varphi$$

subject to

$$\rho \ge 0, \quad 0 \le \varphi \le \pi, \quad -\pi \le \theta \le \pi$$

The Jacobian is well-known $\varphi^2 \sin\varphi$. And the domain requires $\varphi \le R$, $\cos\varphi \ge 0$ meaning $0 \le \varphi \le \frac{\pi}{2}$. This means that $\sin\varphi \ge 0$, and the domain requires $0 \le \sin\varphi\cos\theta \le \sin\varphi\sin\theta$, so $0 \le \cos\theta \le \sin\theta$. $\cos\theta \ge 0$ so $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $\sin\theta \ge 0$ so $0 \le \theta \le \frac{\pi}{2}$ and finally $\tan\theta \ge 1$ so $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$.
And the integrand is

$$\rho^2 \cos\varphi \sin\varphi(\cos\theta + \sin\theta)\rho^2 \sin\varphi = \rho^4 \cos\varphi \sin^2\varphi(\cos\theta + \sin\theta)$$

So the integral is

$$\int_0^R \int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \rho^4 \cos\varphi \sin^2\varphi(\cos\theta + \sin\theta)\, d\theta d\varphi d\rho$$

This is simply the product

$$\int_0^R \rho^4\, d\rho \cdot \int_0^{\frac{\pi}{2}} \sin^2\varphi \cos\varphi\, d\varphi \cdot \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos\theta + \sin\theta\, d\theta$$

The first integral is equal to $\frac{R^5}{5}$, and the last integral is

$$\sin\theta - \cos\theta\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = 1$$

The second integral is

$$\int_0^{\frac{\pi}{2}} \left(\sin\varphi\right)^2 d\left(\sin\varphi\right) = \frac{1}{3}$$

So the integral is equal to

$$\frac{R^5}{15}$$