Infintesimal Calculus 3

Assignment 1
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Exercise 1.0.1:

Suppose X and Y are normed vector spaces with $\left\|\cdot\right\|_X$ and $\left\|\cdot\right\|_Y$ as their respective norms. Determine which of the following defines a valid norm over the product space $X \times Y$:

- $\left\|(x, y)\right\| = \left\|x\right\|_X + \left\|y\right\|_Y$
- $\left\|(x, y)\right\| = \left\|x\right\|_X \cdot \left\|y\right\|_Y$
- $\left\|(x, y)\right\| = \max\left\{\left\|x\right\|_X, \left\|y\right\|_Y\right\}$

- This is a norm. We will show that it has all the necessary properties of a norm: It is positive, as the norm of X and Y are both positive. Since $\left\|x\right\|_X, \left\|y\right\|_Y \geq 0$, $\left\|(x, y)\right\| = 0$ if and only if $\left\|x\right\|_X = \left\|y\right\|_Y = 0$, which is if and only if $x, y = 0$ as required. It is absolutely homogeneus since

  $$\left\|\alpha(x, y)\right\| = \left\|(\alpha x, \alpha y)\right\| = \left\|\alpha x\right\|_X + \left\|\alpha y\right\|_Y = |\alpha| \left(\left\|x\right\|_X + \left\|y\right\|_Y \right)$$

  And it satisfies the triangle inequality since:

  $$\left\|(x, y) + (z, w)\right\| = \left\|x + z\right\|_X + \left\|y + w\right\|_Y \leq \left\|x\right\|_X + \left\|z\right\|_X + \left\|y\right\|_Y + \left\|w\right\|_Y = \left\|(x, y)\right\| + \left\|(x, w)\right\|$$

- This is not a norm. Let's take $X = Y = \mathbb{R}$. Notice that $\left\|(1, 0\right\| = \left\|1\right\|_X \cdot \left\|0\right\|_Y = 0$ even though $(1, 0) \neq (0, 0)$. In general this is not a norm over any non-trivial normed linear space.

- This is a norm. Notice that since the norm of X and Y are non-negative, so is this norm. And $\left\|(x, y)\right\| = 0$ if and only if $\left\|x\right\|_X = \left\|y\right\|_Y = 0$ since it is the maximum, which is if and only if $x = y = 0$. It satisfies absolute homogeneusness since:

  $$\left\|\alpha(x, y)\right\| = \max\left\{\left\|\alpha x\right\|_X, \left\|\alpha y\right\|_Y\right\} = |\alpha| \max\left\{\left\|x\right\|_X, \left\|y\right\|_Y\right\} = |\alpha| \left\|(x, y)\right\|$$

  It satisfies the triangle inequality since:

  $$\left\|(x, y) + (x', y')\right\| = \left\|(x + x', y + y')\right\| = \max\left\{\left\|x + x'\right\|_X, \left\|y + y'\right\|_Y\right\}$$

  Suppose the maximum is $\left\|x + x'\right\|_X$, then we know that

  $$\left\|x + x'\right\|_X \leq \left\|x\right\|_X + \left\|X'\right\|_X \leq \max\left\{\left\|x\right\|_X, \left\|y\right\|_Y\right\} + \max\left\{\left\|x'\right\|_X, \left\|y'\right\|_Y\right\} = \left\|(x, y)\right\| + \left\|(x', y')\right\|$$

  As required.

Exercise 1.0.2:

Suppose $1 \neq a \in \mathbb{N}$, we define a function $d_a \colon \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{R}$ by:

$$d_a(x, y) = \begin{cases} 0 & x = y \\ \frac{1}{a^{k(x,y)}} & \text{else} \end{cases}$$

Where for $x, y \in \mathbb{Z}$, $k(x, y)$ is defined as:

$$k(x, y) = \max\left\{n \in \mathbb{N}_{\geq 0} \mid a^n \mid (x - y)\right\}$$

Prove that d_a is a metric function over $\mathbb{Z}$.

The first few requirements for metric functions are trivial: positivity, symmetry, and zero-ness. Positivity is true since for $a \in \mathbb{N}$, $a^x > 0$. This also gives us zero-ness since $a^x > 0$, so $d_a(x, y) = 0$ if and only if $x = y$. The symmetry of d_a is due to the symmetry of k: in general $\alpha \mid (x - y)$ if and only if $\alpha \mid (y - x)$, so the set $k(x, y)$ acts over is the same set $k(y, x)$ acts over, and therefore $k(x, y) = k(y, x)$.
The next step is to prove that d_a satisfies the triangle inequality. Let $x, y, z \in \mathbb{Z}$, then if any of them equal another the triangle inequality is trivial (this is true in general for a function which satisfies the other properties). Therefore, let's assume $x \neq y \neq z$. We must prove that:

$$d_a(x, y) \leq d_a(x, z) + d_a(z, y)$$

Using the definition of d_a we must show that:

$$\frac{1}{a^{k(x,y)}} \leq \frac{1}{a^{k(x,z)}} + \frac{1}{a^{k(z,y)}}$$

This is equivalent to showing

$$a^{-k(x,y)} \leq a^{-k(x,z)} + a^{-k(z,y)}$$

If we let $k = \min\left\{k(x, z), k(z, y)\right\}$, then:

$$a^{-k(x,z)} + a^{-k(z,y)} \geq a^{-k}$$

So we will show that $a^{-k(x,y)} \leq a^{-k}$, which is equivalent to $-k(x, y) \leq -k \iff k(x, y) \geq k$ since $a > 1$, and this will suffice. Notice then that since $k \leq k(x, z), k(z, y)$, $a^k \mid (x - z), (z - y)$, and therefore a^k divides $(x - z) + (z - y) = x - y$. Since $k(x, y)$ is the largest integer which does so, this means that $k \leq k(x, y)$, as required.
So d_a satisfies all the requirements for a metric, and therefore it is.

Exercise 1.0.3:

For $A \subseteq X$ a metric space A' is the set of limit points of A. We further inductively define $A^{(n)} = \left(A^{(n-1)}\right)'$.
Find a proper subset B of $\mathbb{R}$ such that $B^{(n)} \neq \varnothing$ for every $n \in \mathbb{N}$.

I didn't solve the question with $\varphi(x)$ since the definition wasn't understandable and no one I asked understood it either.

Let $B = \mathbb{Q}$ then $B' = \mathbb{R}$ since every $x \in \mathbb{R}$ has a limit of rationals (which don't equal x) whose limit is x. And since $\mathbb{R}' = \mathbb{R}$, $\mathbb{Q}^{(n)} = \mathbb{R} \neq \varnothing$. (Another example would be $[a, b]$, since every derived set is equal to it).

Exercise 1.0.4:

Determine which of the following sets are closed and which are open:

- $A = \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; y = 0, x \in (0, 1)\right\}$
- $B = \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; x^2 \leq y\right\}$
- $C = \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; x > 0, y < 0, x + y > -1\right\}$

- This set is neither open nor closed. It is not open since for every $x \in A$ (and A is non-empty!), for every $r > 0$ there is a point in $B_r(x)$ which is not in A. This is because the point $\left(x, \frac{r}{2}\right) \in B_r(x)$, and since $\frac{r}{2} \neq 0$, the point is not in A. It is not closed since the point $(0,0)$ is in the boundary of A, since for every $r > 0$, $\left(\min\left\{\frac{r}{2}, \frac{1}{2}\right\}, 0\right) \in B_r(x) \cap A$, and itself is in A^c, so every open ball around it contains points in both A and A^c. But $(0,0)$ itself is not in A, so A does not contain its boundary and is therefore not closed.

- This set is closed, but not open. Intuitively, it is closed since its boundary is the parabola $y = x^2$ which is contained in B. But we will prove this more rigorously by showing that $B' \subseteq B$. Suppose (x, y) is a limit point of B', then for every $\varepsilon > 0$, $B_r(x, y) \cap B \setminus \{(x, y)\} \neq \varnothing$. So we can construct a sequence of points (x_n, y_n) which converge to (x, y) (as in their distance approaches), for example by taking (x_n, y_n) to be contained in $B_{1/n}(x, y) \cap B)$. Then:

$$\|(x - x_n, y - y_n)\|_2 \longrightarrow 0 \implies (x - x_n)^2 + (y - y_n)^2 \longrightarrow 0 \implies x_n \longrightarrow x \quad y_n \longrightarrow y$$

  And so $y - x^2 = \lim y_n - x_n^2$, and since $(x_n, y_n) \in B$, $y_n - x_n^2 \geq 0$, so $y - x^2 \geq 0$.
  Notice that this set has a boundary: $(0,0)$ is one point in this boundary: for every $r > 0$, $\left(0, -\frac{r}{2}\right)$ is in the ball $B_r(0,0)$ but it is not in B. And B is closed so $(0,0)$ is in its boundary. And since the boundary exists and is contained in B, B cannot be open (since open sets don't contain their boundaries).

- This set is open, but not closed. Notice that

$$C = \{x > 0\} \cap \{y < 0\} \cap \{x + y > -1\}$$

  It is obvious that $\{x > 0\}$ and $\{y < 0\}$ are open: if $x > 0$ then take $r = \frac{x}{2}$ and then for every $(x', y') \in B_r(x)$, $x' > x - r = \frac{x}{2} > 0$, so this ball is contained in $\{x > 0\}$ so it is open. A nearly identical proof can be constructed for showing that $\{y < 0\}$ is open. Suppose $x + y > -1$, then we can take r to be half the distance from (x, y) to the line $y = -x - 1$. Then $B_r(x, y)$ is contained in $\{x + y > -1\}$ since the distance between (x, y) and the line is the length of the shortest path to the line, and since the distance between (x, y) and each point in the ball is less than this distance, none of the points can cross the line. So C is the intersection of a finite number of open sets, and therefore C is open.
  And since this set has a boundary (for example $(0,0)$ is in the boundary, since it is not contained in the set but for every $r > 0$, $\left(\frac{r}{2}, 0\right)$). And since the set is open, it doesn't contain this boundary, so it cannot be closed.

Exercise 1.0.5:

Suppose $a_1, \dots, a_n$ and b are real numbers. Show that the following set is closed:

$$P = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \;\middle|\; \sum_{k=1}^{n} a_k x_k = b \right\}$$

We will show that P contains all of its limit points. Let $\vec{x} = (x_1, \dots, x_n) \in P'$, then for every $r > 0$ there is a point in $B_r(\vec{x}) \cap P$ which is not $\vec{x}$. Then take $\vec{x}_k = (x_1^k, \dots, x_n^k)$ to be this point for $r = \frac{1}{k}$. Then $\|\vec{x} - \vec{x}_k\|_2$ converges to 0 (since it is less than $\frac{1}{k}$) so:

$$\sum_{i=1}^{n} \left(x_i - x_i^k\right)^2 \longrightarrow 0$$

Since squares are positive, it must be that for every $1 \leq i \leq n$ $x_i^k \to x_i$. By limit arithmetic:

$$\sum_{i=1}^{n} a_i x_i = \lim_{k \to \infty} \sum_{i=1}^{n} a_i x_i^k$$

And since $\vec{x}_k \in P$, the right sum must be equal to b so:

$$\sum_{i=1}^{n} a_i x_i = \lim_{k \to \infty} b = b$$

So $\vec{x}$ is in P, as required.
Notice that this proof is valid no matter what p-norm is chosen for $\mathbb{R}^n$, it is not necessary for p to be 2.