Differential and Analytic Geometry

Assignment 1
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Exercise 1.1:

(1) Suppose $a \geq b$, and we are given the formula for an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Let $c=\sqrt{a^2-b^2}$, and define $F_{1,2}=(\pm c,0)$ to be the foci of the ellipse. Prove that $A=(x,y)$ satisfies the ellipse equation if and only if $|AF_1|+|AF_2|=2a$.

(2) Show that the ellipse from the previous subquestion can be obtained by squashing a canonical circle. What is the radius of this circle, and how much was it squashed?

(3) Focus on the line $x=\frac{a^2}{c}$, and show that the ratio between the distance between a point A on the ellipse and the right focus, and the distance between A and the line is a constant which is less than one. What is its relationship with the constant found in the previous subquestion?

(1) Since

$$|AF_1|+|AF_2|=\sqrt{(x-c)^2+y^2}+\sqrt{(x+c)^2+y^2}$$

our goal is to show that

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1$$

and

$$\sqrt{(x-c)^2+y^2}+\sqrt{(x+c)^2+y^2}=2a$$

are equivalent.
Suppose the first equality holds, then

$$y^2=\frac{1}{a^2}(a^2-c^2)(a^2-x^2)=\frac{1}{a^2}(a^4-a^2x^2-c^2a^2+c^2x^2)=\left(\frac{c^2}{a^2}-1\right)x^2+a^2-c^2$$

And so

$$(x+c)^2+y^2=x^2+2xc+c^2+\left(\frac{c^2}{a^2}-1\right)x^2+a^2-c^2=\frac{c^2}{a^2}x^2+2xc+a^2=\left(\frac{c}{a}x+a\right)^2$$

And similarly

$$(x-c)^2+y^2=\frac{c^2}{a^2}x^2-2xc+a^2=\left(\frac{c}{a}x-a\right)^2$$

Since the first equality holds, we must have that $\frac{x^2}{a^2}\leq 1$, so $-a\leq x\leq a$, and so

$$0\leq -c+a\leq \frac{c}{a}x+a,\quad \frac{c}{a}x-a\leq c-a\leq 0$$

And therefore

$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=\frac{c}{a}x+a+a-\frac{c}{a}x=2a$$

as required. So the first equation implies the second.
Suppose the second equation holds, then if we fix $x\in(-a,a)$ then $\sqrt{(x-c)^2+y^2}+\sqrt{(x+c)^2+y^2}$ increases strictly as $|y|$ increases. So if it is equal to $2a$, it can only be equal at two distinct y values at most. And since we showed that the y values obtained from the first equation, $y^2=\frac{1}{a^2}(a^2-c^2)(a^2-x^2)$, satisfy the second equation, and there are two such y values, these must be the only y values which satisfy the second equation. Therefore if the second equation holds, then so too does the first.

(2) The circle is

$$x^2+y^2=a^2$$

and if we map $(x, y) \mapsto \left(x, \frac{b}{a}y\right)$ then we see that

$$x^2 + y^2 = a^2 \iff \frac{x^2}{a^2} + \frac{\left(\frac{b}{a}y\right)^2}{b^2} = \frac{x^2}{a^2} + \frac{y^2}{a^2} = \frac{a^2}{a^2} = 1$$

Meaning that this is a bijection between the points on the circle and the points on the ellipse. So the ellipse is obtained by scaling the circle of radius a on the y axis by $\frac{b}{a}$ (or squashing it by a factor of $\frac{a}{b}$).

(3) The maximum x value a point on the ellipse can be is a, and $a \leq \frac{a^2}{c}$ (since $\frac{a}{c} \geq 1$), so the line $x = \frac{a^2}{c}$ will always be to the right of the ellipse. And so if $A = (x, y)$ is on the ellipse, then its distance from the line is $\frac{a^2}{c} - x$. And we showed that the distance between A and the right focus is $a - \frac{c}{a}x$. So we need to find the constant

$$\frac{a - \frac{c}{a}x}{\frac{a^2}{c} - x} = \frac{c(a^2 - cx)}{a(a^2 - cx)} = \frac{c}{a}$$

And of course since $c < a$ (when $b \neq 0$), this constant is less than one. Since the ellipse is equal to the circle scaled by $\frac{b}{a}$, the relation between these two constants is that their quotient is $\frac{b}{c}$.

Exercise 1.2:

(1) Given the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, let $c = \sqrt{a^2 + b^2}$, and $F_{1,2} = (\pm c, 0)$. Prove that a point $A = (x, y)$ satisfies the equation if and only if $||AF_1| - |AF_2|| = 2a$.

(2) Focus on the line $x = \frac{a^2}{c}$. Show that the ratio between the distance of a point A on the hyperbola and the right focus, and A and the line, is a constant larger than one.

(1) If the hyperbolic equation is true, then we have

$$y^2 = b^2\left(\frac{x^2}{a^2} - 1\right) = (c^2 - a^2)\left(\frac{x^2}{a^2} - 1\right) = x^2\left(\frac{c^2}{a^2} - 1\right) - c^2 + a^2$$

And so

$$(x + c)^2 + y^2 = x^2 + 2xc + c^2 + x^2\left(\frac{c^2}{a^2} - 1\right) - c^2 + a^2 = \frac{c^2}{a^2}x^2 + 2xc + a^2 = \left(\frac{c}{a}x + a\right)^2$$

and similarly

$$(x - c)^2 + y^2 = \left(\frac{c}{a}x - a\right)^2$$

Now, for the equation to hold, we must have

$$\frac{x^2}{a^2} \geq 1 \implies x^2 \geq a^2 \implies x \geq a \vee x \leq -a$$

So if $x \geq a$, then $\frac{c}{a}x - a \geq c - a \geq 0$ and $\frac{c}{a}x + a \geq c + a \geq 0$ and therefore

$$\sqrt{(x + c)^2 + y^2} - \sqrt{(x - c)^2 + y^2} = \frac{c}{a}x + a - \frac{c}{a}x + a = 2a$$

And if $x \leq -a$ then $\frac{c}{a}x - a, \frac{c}{a}x + a \leq 0$ and so

$$\sqrt{(x + c)^2 + y^2} - \sqrt{(x - c)^2 + y^2} = -a - \frac{c}{a}x - a + \frac{c}{a}x = -2a$$

And so in both cases the absolute value is $2a$, as required.
We will show that when x is held constant ($x \leq -a$ or $x \geq a$), then there are at most two solutions to $||AF_1| - |AF_2|| = 2a$, and since the hyperbolic equation gives two, if the distance equation holds, so too must

the hyperbolic. Now, if $x \geq a$, then $(x-c)^2 > (x+c)^2$ and so $\sqrt{(x-c)^2+y^2} - \sqrt{(x+c)^2+y^2} > 0$. This is a strictly decreasing function in terms of $|y|$, since the derivative of $\sqrt{\alpha+u}$ is $(\alpha+u)^{-1/2}$, so if $\beta < \alpha$ then

$$\left(\sqrt{\alpha+u} - \sqrt{\beta+u}\right)' = (\alpha+u)^{-1/2} - (\beta+u)^{-1/2} < 0$$

since $\beta < \alpha$. And so the function is decreasing in terms of u. Taking $u = y^2$ and α and β as $(x-c)^2$ and $(x+c)^2$ respectively, this means that the distance function is decreasing in terms of y^2, ie. in terms of $|y|$. Thus it can only equal $2a$ for at most one $|y|$ value, meaning for two y values at most. Similar for when $x \leq -a$, but now $(x-c)^2 < (x+c)^2$.

(2) Let $A = (x, y)$ be on the hyperbola. We showed that its distance from A to $(c, 0)$ is $\left|\frac{c}{a}x - a\right|$. So the ratio is equal to

$$\frac{\left|\frac{c}{a}x - a\right|}{\left|x - \frac{a^2}{c}\right|} = \frac{c\left|cx - a^2\right|}{a\left|cx - a^2\right|} = \frac{c}{a}$$

And since $a < c$, this constant is less than one as required.

Exercise 1.3:

Let $x^2 = 4py$ be a parabola, and let $F = (0, p)$ be the focus. Prove that every point on the parabola is equidistant from F and $y = -p$.

Let $A = (x, y)$ be on the parabola, then its distance from F is

$$\sqrt{x^2 + (y-p)^2} = \sqrt{4py + y^2 - 2py + p^2} = \sqrt{y^2 + 2py + p^2} = \sqrt{(y+p)^2} = |y+p|$$

And the distance from $A = (x, y)$ to $y = -p$ is also $|y+p|$, as required.

Exercise 1.4:

Characterize the following curves

(1) $x^2 + 8xy + y^2 + 4x + 6y + 2 = 0$

(2) $12x^2 + 12xy + 12y^2 + 6x + 6y + 1 = 0$

(3) $x^2 - 3xy - 3y^2 - 4x + 6y + 4 = 0$

(4) $-x^2 + 4xy + 2y^2 + 4y + 2 = 0$

(5) $9x^2 - 4xy + 9y^2 + 2x - 2y + 1 = 0$

(6) $x^2 - xy + y^2 + 2x - 2y + 1 = 0$

(7) $x^2 + xy + y^2 - x - y - 1 = 0$

(8) $2x^2 + 4xy + 2y^2 - x - 2y - 1 = 0$

(9) $2x^2 + 4xy + 2y^2 - x - y - 1 = 0$

(1) In order to reduce this to a form we can deal with more easily, we define the matrix $A = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$, so

$$A = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$$

Now we will attempt to unitarily diagonalize A, which we can since A is symmetric. Let us first find the eigenvalues of A:

$$p_A(x) = (x-1)^2 - 16 = x^2 - 2x - 15$$

So the eigenvalues of A are the roots of this polynomial, $5, -3$. Now for the eigenvalue 5, the eigenspace is

$$N(5I - A) = N\begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

And for -3, the eigenspace is

$$N(A + 3I) = N\begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

This forms an orthogonal basis, and we can reduce it to an orthonormal basis by dividing the vectors by their norm. So the matrix

$$P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

diagonalizes A. Recalling the process we did in the proof during the lecture, we get new values for d and e:

$$(d', e') = (d, e)P = \frac{1}{\sqrt{2}}(4, 6)\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(10, -2)$$

And so we have the new equation (which we obtain by transforming the vector space with respect to $P^T = P^{-1}$),

$$\lambda_1 t^2 + \lambda_2 s^2 + d't + e's + f = 5t^2 - 3t^2 + 5\sqrt{2}t - \sqrt{2}s + 2 = 5\left(t + \frac{1}{\sqrt{2}}\right)^2 - 3\left(s + \frac{1}{3\sqrt{2}}\right)^2 - \frac{1}{3} = 0$$

Which defines an *hyperbola*.

(2) We will sort of just go through the steps without explicitly explaining each one, since the steps were explained in (1). We have

$$A = \begin{pmatrix} 12 & 6 \\ 6 & 12 \end{pmatrix} \implies p_A(x) = (x - 12)^2 - 36 = x^2 - 24x + 108 \implies \operatorname{spec}(A) = \{18, 6\}$$

$$V_{18} = N(18I - A) = N\begin{pmatrix} 6 & -6 \\ -6 & 6 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

$$V_6 = N(A - 6I) = N\begin{pmatrix} 6 & 6 \\ 6 & 6 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

And so our orthonormal basis is $\left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$, and so

$$P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \implies (d', e') = \frac{1}{\sqrt{2}}(6, 6)\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(12, 0)$$

Thus we get the equation

$$18t^2 + 6s^2 + 6\sqrt{2}t + 1 = 0 \iff 18\left(t + \frac{\sqrt{2}}{6}\right)^2 + 6s^2 = 0$$

Which defines *two lines*.

(3) Again,

$$A = \begin{pmatrix} 1 & -1.5 \\ -1.5 & -3 \end{pmatrix} \implies p_A(x) = (x - 1)(x + 3) - 2.25 \implies \operatorname{spec}(A) = \{1.5, -3.5\}$$

$$V_{1.5} = N(1.5I - A) = N\begin{pmatrix} 0.5 & 1.5 \\ 1.5 & 4.5 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} 3 \\ -1 \end{pmatrix} \right\}$$

$$V_{-3.5} = N(A + 3.5I) = N\begin{pmatrix} 4.5 & -1.5 \\ -1.5 & 0.5 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$$

And thus we define

$$P = \frac{1}{\sqrt{10}}\begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix} \implies (d', e') = \frac{1}{\sqrt{10}}(-4, 6)\begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix} = \frac{1}{\sqrt{10}}(-18, 14)$$

Thus we get

$$1.5t^2 - 3.5s^2 - \frac{18}{\sqrt{10}}t + \frac{14}{\sqrt{10}}s + 4 = 0 \iff 1.5\left(t - \frac{6}{\sqrt{10}}\right)^2 - 3.5\left(s - \frac{2}{\sqrt{10}}\right)^2 = 0$$

Which defines *two lines*.

(4)

$$A = \begin{pmatrix} -1 & 2 \\ 2 & 2 \end{pmatrix} \implies p_A(x) = (x+1)(x-2) - 4 \implies \operatorname{spec}(A) = \{3, -2\}$$

$$V_3 = N\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right\}, \qquad V_{-2} = N\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$$

And thus

$$P = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \implies (d', e') = \frac{1}{\sqrt{5}}(0, 4)\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} = \frac{1}{\sqrt{5}}(-8, 4)$$

And so we get

$$3t^2 - 2s^2 - \frac{8}{\sqrt{5}}t + \frac{4}{\sqrt{5}}s + 2 = 0 \iff 3\left(t - \frac{4}{3\sqrt{5}}\right)^2 - 2\left(s - \frac{1}{\sqrt{5}}\right)^2 + \frac{4}{3} = 0$$

Which defines a *hyperbola*.

(5)

$$A = \begin{pmatrix} 9 & -2 \\ -2 & 9 \end{pmatrix} \implies p_A(x) = (x-9)^2 - 4 \implies \operatorname{spec}(A) = \{11, 7\}$$

$$V_{11} = N\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}, \qquad V_7 = N\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$$

And thus

$$P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \implies (d', e') = \frac{1}{\sqrt{2}}(2, -2)\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(4, 0)$$

And so we get

$$11t^2 + 7s^2 + 2\sqrt{2}t + 1 = 0 \iff 11\left(t + \frac{\sqrt{2}}{11}\right)^2 + 7s^2 + \frac{9}{11} = 0$$

Which defines *the empty set*.

(6)

$$A = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix} \implies p_A(x) = (x-1)^2 - \frac{1}{4} \implies \operatorname{spec}(A) = \{1.5, 0.5\}$$

$$V_{1.5} = N\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}, \qquad V_{0.5} = N\begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$$

And thus

$$P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \implies (d', e') = \frac{1}{\sqrt{2}}(2, -2)\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(4, 0)$$

And so we get

$$1.5t^2 + 0.5s^2 + \frac{4}{\sqrt{2}}t + 1 = 0 \iff 1.5\left(t + \frac{2\sqrt{2}}{3}\right)^2 + \frac{1}{2}s^2 - \frac{1}{3} = 0$$

Which defines an *ellipse*.

(7)

$$A = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} \implies p_A(x) = (x-1)^2 - \frac{1}{4} \implies \operatorname{spec}(A) = \{1.5, 0.5\}$$

Similarly

$$V_{1.5} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}, \qquad V_{0.5} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$$

And thus

$$P = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \implies (d', e') = \frac{1}{\sqrt{2}}(-2, 0)$$

And so we get

$$1.5t^2 + 0.5s^2 - \sqrt{2}t - 1 = 0 \iff 1.5\left(t - \frac{\sqrt{2}}{3}\right)^2 + 0.5s^2 - \frac{4}{3} = 0$$

Which defines an *ellipse*.

(8)

$$A = \begin{pmatrix}2 & 2\\ 2 & 2\end{pmatrix} \implies \operatorname{spec}(A) = \{4, 0\}$$

$$V_4 = \operatorname{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}, \qquad V_0 = \operatorname{span}\left\{\begin{pmatrix}1\\-1\end{pmatrix}\right\}$$

Thus

$$P = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \implies (d', e') = \frac{1}{\sqrt{2}}(1, -3)$$

And so we get

$$4t^2 + \frac{1}{\sqrt{2}}t - \frac{3}{\sqrt{2}}s - 1 = 0 \iff 4\left(t + \frac{\sqrt{2}}{16}\right)^2 - \frac{3}{\sqrt{2}}s - \frac{33}{32} = 0$$

Which defines a *parabola*.

(9) The matrix A here is the same as the previous subquestion, since all the values are the same (save e). So we have the same eigenvalues and P as well, and so

$$(d', e') = \frac{1}{\sqrt{2}}(-1, -1)\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \frac{1}{\sqrt{2}}(-2, 0)$$

So we get

$$4t^2 - \sqrt{2}t - 1 = 0$$

This has two solutions, and thus defines *two parallel lines*.

Exercise 1.5:

Determine what surfaces are defined by the following equations

(1) $x^2 + y^2 + z^2 + 2xz + 2y - 3 = 0$

(2) $\frac{2}{5}x^2 - x + \frac{3}{5}y^2 + y + 5z^2 + z = 0$

(3) $x^2 + y^2 + 6z^2 - 2x - 4y + 6 = 0$

(4) $2x^2 - 3y^2 - 6y - 6z - z^2 = 0$

(5) $5x^2 + 5z^2 + 12xy - 9z + \frac{101}{20} = 0$

(6) $32x^2 + 16xy + 2y^2 + 2z^2 - 17x + 2 = 0$

(7) $168x^2 + 192xz + 24z^2 + 144y^2 + 168y + 49 = 0$

(8) $4x^2 + 4xz - 3y^2 + z^2 + 15x - 12y - 3 = 0$

(9) $25x^2 + 60yz - 25z^2 + 60x + 36 = 0$

(10) $16x^2 + 8xy + y^2 + z^2 - 256z = 0$

(1) We must first transform this into a form without coefficients like xy etc. To do so we define the matrix

$$A = \begin{pmatrix}a & b/2 & d/2\\ b/2 & c & f/2\\ d/2 & f/d & e\end{pmatrix} = \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 1\end{pmatrix}$$

now we will unitarily diagonalize A, but first we must find its eigenvalues, and an orthonormal basis of eigenvectors.

$$p_A(x) = \det(xI - A) = \det\begin{pmatrix} x-1 & 0 & -1 \\ 0 & x-1 & 0 \\ -1 & 0 & x-1 \end{pmatrix} = (x-1)\det\begin{pmatrix} x-1 & -1 \\ -1 & x-1 \end{pmatrix} =$$

$$(x-1)(x^2 - 2x + 1 - 1) = x(x-1)(x-2)$$

Now we find the eigenspaces,

$$V_0 = N(A) = N\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}, \quad V_1 = N(I - A) = N\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$V_2 = N(2I - A) = N\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Thus the unitary diagonalizer of A is

$$P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

And now we transform the coefficients g, h, and i to get

$$(g', h', i') = (g, h, i)P = \frac{1}{\sqrt{2}}(0, 2, 0)\begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix} = (0, 2, 0)$$

Thus the new transformed equation is

$$s^2 + 2r^2 + 2s - 3 = 0$$

Completing the square gives

$$(s+1)^2 + 2r^2 = 4$$

Which defines an *elliptical cylinder*.

(2) Here we can simply complete a few squares,

$$\frac{2}{5}\left(x - \frac{5}{4}\right)^2 + \frac{3}{5}\left(y + \frac{5}{6}\right)^2 + 5\left(z + \frac{1}{10}\right)^2 - c = 0$$

where $c > 0$, and this defines an *ellipsoid*.

(3) Again, we can simply complete the squares

$$(x-1)^2 + (y-2)^2 + 6z^2 + 1 = 0$$

which defines the *empty set*.

(4) Once again, we complete the squares

$$2x^2 - 3(y+1)^2 - (z+3)^2 + 12 = 0$$

Which defines a *hyperboloid*.

(5) Here we define

$$A = \begin{pmatrix} 5 & 6 & 0 \\ 6 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix} \implies p_A(x) = (x-5)(x-9)(x+4)$$

And so the eigenspaces are

$$V_5 = N\begin{pmatrix} 0 & 6 & 0 \\ 6 & -5 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \quad V_9 = N\begin{pmatrix} 4 & -6 & 0 \\ -6 & 9 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} \right\}$$

$$V_{-4} = N\begin{pmatrix}9 & 6 & 0\\ 6 & 4 & 0\\ 0 & 0 & 9\end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix}2\\ -3\\ 0\end{pmatrix}\right\}$$

Thus the unitary diagonalizer of A is

$$P = \frac{1}{\sqrt{13}}\begin{pmatrix}0 & 3 & 2\\ 0 & 2 & -3\\ \sqrt{13} & 0 & 0\end{pmatrix} \implies (g', h', i') = \frac{1}{\sqrt{13}}(0,0,-9)\begin{pmatrix}0 & 3 & 2\\ 0 & 2 & -3\\ \sqrt{13} & 0 & 0\end{pmatrix} = (-9,0,0)$$

And so we get the equation

$$5t^2 + 9s^2 - 4r^2 - 9t + \frac{101}{20} = 5\left(t - \frac{9}{10}\right)^2 + 9s^2 - 4r^2 + \frac{101}{20} - \frac{81}{100} = 0$$

which defines a *hyperboloid*.

(6) The method for solving the rest of the questions is the same,

$$A = \begin{pmatrix}32 & 8 & 0\\ 8 & 2 & 0\\ 0 & 0 & 2\end{pmatrix} \implies \operatorname{spec}(A) = \{0, 2, 34\}$$

And the eigenspaces are

$$V_0 = \operatorname{span}\left\{\begin{pmatrix}1\\ -4\\ 0\end{pmatrix}\right\}, \quad V_2 = \operatorname{span}\left\{\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}\right\}, \quad V_{34} = \operatorname{span}\left\{\begin{pmatrix}4\\ 1\\ 0\end{pmatrix}\right\}$$

And so the unitary diagonalizer is

$$P = \frac{1}{\sqrt{17}}\begin{pmatrix}1 & 0 & 4\\ -4 & 0 & 1\\ 0 & \sqrt{17} & 0\end{pmatrix} \implies (g', h', i') = \frac{1}{\sqrt{17}}(-17,0,0)\begin{pmatrix}1 & 0 & 4\\ -4 & 0 & 1\\ 0 & \sqrt{17} & 0\end{pmatrix} = (-\sqrt{17}, 0, -4\sqrt{17})$$

Thus the transformed equation is

$$2s^2 + 34r^2 - \sqrt{17}t - 4\sqrt{17}r + 2 = 2s^2 + 34\left(r - \frac{1}{\sqrt{17}}\right)^2 - \sqrt{17}t = 0$$

Which defines an *elliptical cone.*

(7)

$$A = \begin{pmatrix}168 & 0 & 96\\ 0 & 144 & 0\\ 96 & 0 & 24\end{pmatrix} \implies \operatorname{spec}(A) = \{144, 216, -24\}$$

And the eigenspaces are

$$V_{144} = \operatorname{span}\left\{\begin{pmatrix}0\\ 1\\ 0\end{pmatrix}\right\}, \quad V_{216} = \operatorname{span}\left\{\begin{pmatrix}2\\ 0\\ 1\end{pmatrix}\right\}, \quad V_{-24} = \operatorname{span}\left\{\begin{pmatrix}1\\ 0\\ 2\end{pmatrix}\right\}$$

Thus the unitary diagonalizer is

$$P = \frac{1}{\sqrt{5}}\begin{pmatrix}0 & 2 & 1\\ \sqrt{5} & 0 & 0\\ 0 & 1 & -2\end{pmatrix} \implies (g', h', i') = \frac{1}{\sqrt{5}}(0,168,0)\begin{pmatrix}0 & 2 & 1\\ \sqrt{5} & 0 & 0\\ 0 & 1 & -2\end{pmatrix} = (0,168,0)$$

Thus the transformed equation is

$$144t^2 + 216s^2 - 24r^2 + 168s + 49 = 144t^2 + 216\left(s + \frac{7}{18}\right)^2 - 24r^2 + \frac{49}{3} = 0$$

Which defines a *hyperboloid*.

(8)

$$A=\begin{pmatrix}4&0&2\\0&-3&0\\2&0&1\end{pmatrix}\implies \operatorname{spec}(A)=\{0,-3,5\}$$

And so the eigenspaces are

$$V_0=\operatorname{span}\left\{\begin{pmatrix}1\\0\\-2\end{pmatrix}\right\},\quad V_{-3}=\operatorname{span}\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}\right\},\quad V_5=\operatorname{span}\left\{\begin{pmatrix}2\\0\\1\end{pmatrix}\right\}$$

And so the unitary diagonalizer is

$$P=\frac{1}{\sqrt{5}}\begin{pmatrix}1&0&2\\0&\sqrt{5}&0\\-2&0&1\end{pmatrix}\implies (g',h',i')=\frac{1}{\sqrt{5}}(15,-12,0)\begin{pmatrix}1&0&2\\0&\sqrt{5}&0\\-2&0&1\end{pmatrix}=\frac{1}{\sqrt{5}}(15,-12\sqrt{5},30)$$

And so we get the equation

$$-3s^2+5r^2+3\sqrt{5}t-12s+6\sqrt{5}r-3=-(s+2)^2+5\left(r+\frac{3}{5\sqrt{5}}\right)^2+3\sqrt{5}t+\frac{16}{25}=0$$

This defines a *hyperbolic paraboloid*.

(9)

$$A=\begin{pmatrix}25&0&0\\0&0&30\\0&30&-25\end{pmatrix}\implies \operatorname{spec}(A)=\{25,20,-45\}$$

And so the eigenspaces are

$$V_{25}=\operatorname{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\},\quad V_{20}=\operatorname{span}\left\{\begin{pmatrix}0\\3\\-2\end{pmatrix}\right\},\quad V_{-45}=\operatorname{span}\left\{\begin{pmatrix}0\\2\\-3\end{pmatrix}\right\}$$

And so the unitary diagonalizer is

$$P=\frac{1}{\sqrt{13}}\begin{pmatrix}\sqrt{13}&0&0\\0&3&2\\0&-2&-3\end{pmatrix}\implies (g',h',i')=\frac{1}{\sqrt{13}}(60,0,0)\begin{pmatrix}\sqrt{13}&0&0\\0&3&2\\0&-2&-3\end{pmatrix}=(60,0,0)$$

And so we get the transformed equation

$$25t^2+20s^2-45r^2+60t+36=25(t+1.2)^2+20s^2-45r^2=0$$

This defines a *elliptical paraboloid*.

(10)

$$A=\begin{pmatrix}16&4&0\\4&1&0\\0&0&1\end{pmatrix}\implies \operatorname{spec}(A)=\{0,1,17\}$$

And so the eigenspaces are

$$V_0=\operatorname{span}\left\{\begin{pmatrix}1\\-4\\0\end{pmatrix}\right\},\quad V_1=\operatorname{span}\left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\},\quad V_{17}=\operatorname{span}\left\{\begin{pmatrix}4\\1\\0\end{pmatrix}\right\}$$

And so the unitary diagonalizer is

$$P=\frac{1}{\sqrt{13}}\begin{pmatrix}1&0&4\\-4&0&1\\0&\sqrt{13}&0\end{pmatrix}\implies (g',h',i')=\frac{1}{\sqrt{13}}(0,0,-256)\frac{1}{\sqrt{13}}\begin{pmatrix}1&0&4\\-4&0&1\\0&\sqrt{13}&0\end{pmatrix}=(0,0,-256)$$

And so we get the transformed equation

$$s^2 + 17r^2 - 256r = s^2 + 17\left(r - \frac{128}{17}\right)^2 + \frac{128^2}{17} = 0$$

This defines the *empty set*.