Complex Functions

Assignment 1
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Exercise 1.1:

Write the following in rectangular form:

(1) $\dfrac{(2+i)(3+2i)}{1-i}$

(2) $\left(-\dfrac{1}{2}+i\dfrac{\sqrt{3}}{2}\right)^4$

(1) We first expand the numerator to get $4+7i$ then we multiply the numerator and denominator by the conjugate of the denominator to get

$$\frac{(4+7i)(1+i)}{(1-i)(1+i)} = \frac{-3+11i}{2} = -\frac{3}{2}+\frac{11}{2}i$$

(2) We convert this to polar form, $r^2 = \frac{1}{4}+\frac{3}{4} = 1$ so $r = 1$ and $\theta = \frac{2}{3}\pi$. So $z = e^{i\frac{2}{3}\pi}$ and so

$$z^4 = e^{i\frac{8}{3}\pi} = \cos\left(\frac{8}{3}\pi\right) + i\sin\left(\frac{8}{3}\pi\right) = -\frac{1}{2}+\frac{\sqrt{3}}{2}i$$

Exercise 1.2:

Find the values of $\sqrt{-8+6i}$ (ie the solutions to $z^2 = -8+6i$).

Using some basic trigonometry we see that $|-8+6i| = 10$ and $2\theta = \arg(-8+6i) \approx \pi - \tan^{-1}\left(\frac{3}{4}\right)$. And so since

$$z^n = re^{i\theta} \iff z = \sqrt[n]{r}e^{i\frac{\theta}{n}+\frac{2\pi k}{n}}$$

we get that

$$\sqrt{-8+6i} \approx \sqrt{10}e^{\theta i}, \sqrt{10}e^{(\theta+\pi)i} = 1+3i, -1-3i$$

Exercise 1.3:

Find the solutions to $z^2 + \sqrt{32}iz - 6i = 0$.

Using the quadratic formula, we have to compute $\sqrt{(\sqrt{32}i)^2 - 4(-6i)} = \sqrt{-32+24i}$. If 2θ is the argument of $-32+24i$ then it equals $40e^{2\theta i}$ and so the square roots are $\pm 2\sqrt{10}e^{\theta i}$. And since $\theta = \frac{1}{2}(\pi - \tan^{-1}\left(\frac{3}{4}\right))$ we get that the square roots are

$$\pm 2\sqrt{10}\big(\cos(\theta) + i\sin(\theta)\big) = \pm(2+6i)$$

And so the solutions are

$$\frac{-\sqrt{32}i \pm (2+6i)}{2} = 1+(3-2\sqrt{2})i,\ -1-(3+2\sqrt{2})i$$

Exercise 1.4:

Prove the following identities:

(1) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(2) $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$

(3) If $p \in \mathbb{R}[x]$ then $\overline{p(z)} = p(\overline{z})$

(4) $\overline{\overline{z}} = z$

(1) Suppose $z_k = a_k + b_k i$ then $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$ and so

$$\overline{z_1 + z_2} = (a_1 + a_2) - (b_1 + b_2)i = (a_1 - b_1 i) + (a_2 - b_2 i)$$

And $\overline{z_i} = a_k - b_k i$ so

$$= \overline{z_1} + \overline{z_2}$$

as required.

(2) Recall that $\overline{re^{i\theta}} = re^{-i\theta}$. So if $z_k = r_k e^{i\theta_k}$ then

$$\overline{z_1 z_2} = \overline{r_1 r_2 e^{i(\theta_1 + \theta_2)}} = r_1 r_2 e^{-i(\theta_1 + \theta_2)}$$

And

$$\overline{z_1} \cdot \overline{z_2} = r_1 e^{-i\theta_1} \cdot r_2 e^{-i\theta_2} = r_1 r_2 e^{-i(\theta_1 + \theta_2)} = \overline{z_1 z_2}$$

As required.

(3) Notice that the above two identities can be generalized to any finite number of complex numbers by induction. ie $\overline{\sum z_k} = \sum \overline{z_k}$ and similar for products. Specifically $\overline{z^k} = \overline{z}^k$. Then if

$$p(x) = \sum_{k=0}^{n} a_k x^k$$

We have that

$$\overline{p(z)} = \overline{\sum_{k=0}^{n} a_k z^k} = \sum_{k=0}^{n} \overline{a_k z^k} = \sum_{k=0}^{n} \overline{a_k} \cdot \overline{z}^k$$

Since $a_k \in \mathbb{R}$ we have that $\overline{a_k} = a_k$ and so

$$= \sum_{k=0}^{n} a_k \cdot \overline{z}^k = p(\overline{z})$$

where the last equality comes from the definition of p.

(4) $\overline{\overline{a + bi}} = \overline{a - bi} = a + bi = z$.

Exercise 1.5:

Prove that $\left|z^2\right| = |z|^2$ both in polar and rectangular coordinates.

We will show that for every $z, w \in \mathbb{C} : |zw| = |z||w|$. Thus when $z = w$ this proves what we have been asked to prove. Suppose $z = a + bi$ and $w = c + di$ then $zw = (ac - bd) + i(ad + cb)$ so

$$|zw| = \sqrt{(ac - bd)^2 + (ad + bc)^2} = \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2 - 2acbd + 2adbc} = \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2}$$

And

$$|z||w| = \sqrt{a^2 + b^2} \cdot \sqrt{c^2 + d^2} = \sqrt{a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2} = |zw|$$

As required.
And if $z = re^{i\alpha}$ and $w = se^{i\beta}$ we have that $|z| = r$, $|w| = s$ and $zw = rse^{i(\alpha+\beta)}$ so $|zw| = rs = |z||w|$.

Exercise 1.6:

Prove the following:

(1) $|z^n| = |z|^n$

(2) $|z|^2 = z\overline{z}$

(3) $|\mathrm{Re}(z)|, |\mathrm{Im}(z)| \leq |z| \leq |\mathrm{Re}(z)| + |\mathrm{Im}(z)|$.

(1) Inductively, by my proof of the previous problem $|z_1 \cdots z_n| = |z_1| \cdots |z_n|$. The base case where $n = 2$ was shown in the proof of the previous problem. Then $|z_1 \cdot z_2 \cdots z_n| = |z_1| \cdot |z_2 \cdots z_n|$ by the $n = 2$ case, and inductively $|z_2 \cdots z_n| = |z_2| \cdots |z_n|$ and so all in all we get that $|z_1 \cdots z_n| = |z_1| \cdots |z_n|$.
So when all the $z_k = z$ we get that $|z^n| = |z \cdots z| = |z| \cdots |z| = |z|^n$ as required.

(2) Suppose $z = re^{i\theta}$ then $z\overline{z} = re^{i\theta}re^{-i\theta} = r^2e^{i(\theta-\theta)} = r^2$, and since $|z|^2 = r^2$ we have finished.

(3) Note that $\mathrm{Im}(z)^2, \mathrm{Re}(z)^2 \leq \mathrm{Re}(z)^2 + \mathrm{Im}(z)^2$, and since the square root function is monotonic (these are nonnegative real numbers) we get that (since $\sqrt{a^2} = |a|$):

$$|\mathrm{Re}(z)|, |\mathrm{Im}(z)| \leq \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2} = |z|$$

And since
$$\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2 \leq \mathrm{Re}(z)^2 + 2|\mathrm{Re}(z)||\mathrm{Im}(z)| + \mathrm{Im}(z)^2 = \big(|\mathrm{Re}(z)| + |\mathrm{Im}(z)|\big)^2$$
taking the square root of both sides we get

$$|z| \leq |\mathrm{Re}(z)| + |\mathrm{Im}(z)|$$

as required.
Note that by our proof, the left inequality is an equality when $\mathrm{Re}(z)^2, \mathrm{Im}(z)^2 = \mathrm{Re}(z)^2 + \mathrm{Im}(z)^2$. So in order to get both equalities we get $\mathrm{Re}(z) = \mathrm{Im}(z) = 0$ (ie. $z = 0$) and for just one, $z \in \mathbb{R}$ or $z \in i\mathbb{R}$.
And the right inequality is an equality when

$$\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2 = \mathrm{Re}(z)^2 + 2|\mathrm{Re}(z)||\mathrm{Im}(z)| + \mathrm{Im}(z)^2$$

so $|\mathrm{Re}(z)||\mathrm{Im}(z)| = 0$, and so $\mathrm{Re}(z) = 0$ or $\mathrm{Im}(z) = 0$.
That is, in order to get all inequalities ($|\mathrm{Re}(z)| = |\mathrm{Im}(z)| = \dots$), z must be 0. And to get just one set of inequalities ($|\mathrm{Re}(z)| = |z| = \dots$ or $|\mathrm{Im}(z)| = |z| = \dots$), $z \in \mathbb{R}$ or $z \in i\mathbb{R}$ respectively.

Exercise 1.7:

(1) Prove that $|z_1 + z_2| \leq |z_1| + |z_2|$ (follow the proof given).

(2) When does the inequality become an equality?

(3) Prove that $|z_1| - |z_2| \leq |z_1 - z_2|$.

(1) We know that $|z_1 + z_2|^2 = (z_1 + z_2) \cdot \overline{(z_1 + z_2)}$ by the identity we showed before, and by linearity of the conjugate we get that this is equal to $(z_1 + z_2) \cdot (\overline{z_1} + \overline{z_2})$ and distributing we get that this is equal to $z_1\overline{z_1} + z_2\overline{z_1} + z_1\overline{z_2} + z_1\overline{z_2}$ which is equal, again by that same identity, to $|z_1|^2 + |z_2|^2 + z_1\overline{z_2} + z_2\overline{z_1}$. Notice that $\overline{z_1\overline{z_2}} = \overline{z_1} \cdot \overline{\overline{z_2}} = \overline{z_1} \cdot z_2$. And

since $z+\overline{z}=2\,\mathrm{Re}(z)$, this is equal to $|z_1|^2+|z_2|^2+2\,\mathrm{Re}(z_1\overline{z_2})$. And by above, this is

$$\leq |z_1|^2+|z_2|^2+2|z_1\overline{z_2}| = |z_1|^2+|z_2|^2+2|z_1||\overline{z_2}| = |z_1|^2+|z_2|^2+2|z_1||z_2| = \big(|z_1|+|z_2|\big)^2$$

And so we have that

$$|z_1+z_2|^2 \leq \big(|z_1|+|z_2|\big)^2 \implies |z_1+z_2| \leq |z_1|+|z_2|$$

(2) This is an inequality when our inequality in the proof, $2\,\mathrm{Re}(z_1\overline{z_2}) \leq 2|z_1\overline{z_2}|$ is an equality. Recall that from before, $\mathrm{Re}(z)=|z|$ only when $z\in\mathbb{R}$ and so this is an inequality only when $z_1\overline{z_2}\in\mathbb{R}$. So if $z_1=r_1e^{i\theta_1}$ and $z_2=r_2e^{i\theta_2}$, $z_1\overline{z_2}=r_1r_2e^{i(\theta_1-\theta_2)}$, and this is real only when $\theta_1-\theta_2=0$, that is this is an equality only when $\arg(z_1)=\arg(z_2)$.

(3) We know that

$$|z_1| = |(z_1-z_2)+z_2| \leq |z_1-z_2|+|z_2|$$

And so we have that, after subtracting $|z_2|$ from both sides:

$$|z_1|-|z_2| \leq |z_1-z_2|$$

as required.

Exercise 1.8:

Suppose $z=a+bi$. Explain the connection between $\arg(z)$ and $\tan^{-1}\left(\frac{b}{a}\right)$.

Let $\theta=\arg(z)$. Then we know that the line connecting (a,b) and $(0,0)$ creates an angle of θ with the x axis. This means that $\tan(\theta)=\frac{b}{a}$, and in general

- If (a,b) is in the first quadrant $\theta=\tan^{-1}\left(\frac{b}{a}\right)$.
- If (a,b) is in the second quadrant $\theta=\pi+\tan^{-1}\left(\frac{b}{a}\right)$.
- If (a,b) is in the third quadrant $\theta=\pi+\tan^{-1}\left(\frac{b}{a}\right)$.
- If (a,b) is in the fourth quadrant $\theta=2\pi+\tan^{-1}\left(\frac{b}{a}\right)$.

Exercise 1.9:

Solve the equation $z^4=-1+\sqrt{3}i$.

In polar coordinates $-1+\sqrt{3}i=2e^{i\frac{2}{3}\pi}$ and so the solutions are

$$z_k=\sqrt[4]{2}e^{i(16+\frac{k}{2})\pi}$$

And so

$$z_0=\sqrt[4]{2}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right),\quad z_1=\sqrt[4]{2}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right),\quad z_2=\sqrt[4]{2}\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}\right),\quad z_3=\sqrt[4]{2}\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$$

Exercise 1.10:

Describe the following complex sets. Which of them are domains?

(1) $|z-i|\leq 1$

(2) $\left|\dfrac{z-1}{z+1}\right|=1$

(3) $|z-2| > |z-3|$

(4) $|z| < 1$ and $\text{Im}(z) > 0$

(5) $\dfrac{1}{z} = \overline{z}$

(6) $|z|^2 = \text{Im}(z)$

(7) $\left|z^2 - 1\right| < 1$

(1) This is by definition $\bar{D}_1(i)$ (the closed disk around i) which is closed. Since $\mathbb{C}$ is connected, it can't be open (the only clopen sets in a connected space is the space itself and $\varnothing$). So it's not open and therefore not a domain.

(2) This is equivalent to $|z-1| = |z+1|$. So if $z = a+bi$ this is equivalent to

$$(a-1)^2 + b^2 = (a+1)^2 + b^2 \iff (a-1)^2 = (a+1)^2 \iff 1-a = a+1 \iff a = 0$$

And so this is equivalent to $z \in i\mathbb{R}$, so the set is $i\mathbb{R}$ (since $z+1=0 \iff z=-1$ is not in the domain, this is still true). This too is closed and therefore not a domain.

(3) Suppose $z = a+bi$, this is equivalent to

$$(a-2)^2 + b^2 > (a-3)^2 + b^2 \iff (a-2)^2 > (a-3)^2$$

There is equality when $a = 2.5$ and for $a \leq 2.5$ this is false, and for $a > 2.5$ this is true (it is sufficient to check one such a for each case since the functions are continuous). So the set is $\{\text{Re}(z) > 2.5\}$ which is open and (line) connected and therefore a domain.

(4) This is $D_1(0) \cap \{\text{Im}(z) > 0\}$, which is the open unit half circle above the real axis, and is (line) connected and open. So it's a domain.

(5) This is equivalent $1 = z\overline{z} = |z|^2$ and equivalent to $|z| = 1$. So it is the unit circle, or $\partial D_1(0)$, which is closed and therefore not a domain.

(6) Let $z = a+bi$ then this is

$$a^2 + b^2 = b \iff a^2 + b^2 - b + \frac{1}{4} = \frac{1}{4} \iff a^2 + \left(b - \frac{1}{2}\right)^2 = \frac{1}{4}$$

which is the circle around $i\frac{1}{2}$ of radius $\frac{1}{2}$, ie $\partial D_{\frac{1}{2}}\left(i\frac{1}{2}\right)$ which is closed and therefore not a domain.

(7) Let $z = re^{i\theta}$ then we are looking for $\left|r^2\cos(2\theta) - 1 + ir^2\sin(2\theta)\right| < 1$, this is equivalent to (squaring both sides):

$$\left(r^2\cos(2\theta) - 1\right)^2 + r^4\sin(2\theta)^2 = r^4 - 2\cos(2\theta) + 1 < 1$$

which is equivalent to $r^2 < 2\cos(2\theta)$.
We claim that this set, let it be S, is not connected. We claim that for every $z = ix \in i\mathbb{R}$, ie. $\text{Re}(z) = 0$, $z \notin S$. This is true since $\text{Re}(z) = 0$ which means that (if $z \neq 0$) $\theta = \arg(z) = \pm\frac{\pi}{2}$, and so $\cos(2\theta) = \cos(\pm\pi) = -1$, and since $r^2 \geq 0 > -2 = 2\cos(2\theta)$ this means that $z \notin S$ and $0 \notin S$ either since $|0-1| = 1 \not< 1$.
And so $S \subseteq \{z \in \mathbb{C} \mid \text{Re}(z) > 0\} \cup \{z \in \mathbb{C} \mid \text{Re}(z) < 0\}$ and the intersections between S and these sets are non-empty (for example $\pm 1 \in S$), and these sets are open so by definition S is not connected and therefore is not a domain.

Exercise 1.11:

Describe the complex sets which satisfy:

(1) $|z| = \text{Re}(z) + 1$

(2) $|z-1| + |z+1| = 4$

(3) $z^{n-1} = \bar{z}$

(1) If we let $z = a + bi$ then this becomes $a^2 + b^2 = (a+1)^2 \iff b^2 = 2a + 1$, so $a = \frac{b^2-1}{2}$. Since we must further require $a > -1$, this becomes $b^2 > -1$, which is true, so this is true for all b. So the set is $\left\{ \frac{b^2-1}{2} + bi \,\middle|\, b \in \mathbb{R} \right\}$, which is a rotated parabola.

(2) If we let $z = a + bi$ then this becomes

$$\begin{aligned}
\sqrt{(a-1)^2 + b^2} + \sqrt{(a+1)^2 + b^2} &= 4 \\
(a-1)^2 + b^2 &= 16 - 8\sqrt{(a+1)^2 + b^2} + (a+1)^2 + b^2 \\
-4a - 16 &= -8\sqrt{(a+1)^2 + b^2} \\
a^2 + 8a + 16 &= 4\big(a^2 + 2a + 1 + b^2\big) \\
3a^2 + 4b^2 &= 12 \\
\frac{a^2}{4} + \frac{b^2}{3} &= 1
\end{aligned}$$

This is the canonical ellipse with width 2 and height $\sqrt{3}$.

(3) $z = 0$ satisfies this, otherwise this is equivalent to the solution given by multiplying both sides by z:

$$z^n = |z|^2$$

And so z is any of the n-degree roots of unity multiplied $|z|^{\frac{2}{n}}$. This means that $|z| = |z|^{\frac{2}{n}}$ and so $|z|^{\frac{2}{n}-1} = 1$ which means that $|z| = 1$ or $\frac{2}{n} = 1 \iff n = 2$.
If $n \neq 2$ then $|z| = 1$ so this set is $\Omega_n = \left\{ e^{i\frac{2\pi k}{n}} \,\middle|\, k = 0, \ldots, n-1 \right\}$. Otherwise $n = 2$ and so this is $z^2 = |z|^2$ and so $z = \pm|z|$ meaning that $z \in \mathbb{R}$, so the set for $n = 2$ is $\mathbb{R}$.

Exercise 1.12:

Prove the following:

(1) $f(z) = \displaystyle\sum_{k=0}^{\infty} kz^k$ is continuous for $|z| < 1$.

(2) $g(z) = \displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2 + z}$ is continuous for $\operatorname{Re}(z) > 0$.

(1) Let $0 < r < 1$ then for any $|z| < r$ we have that $\left|kz^k\right| < kr^k$ and it is well-known that this series converges. This is because using the ratio test we have $\frac{(k+1)r^{k+1}}{kr^k} = \frac{k+1}{k} \cdot r$ which converges to $r < 1$. And so $f_k(z) = kz^k$ satisfies $|f_k(z)| \leq M_k = kr^k$ where $\sum M_k$ is convergent, so by the Weierstrass M test, $\sum f_k$ converges uniformly for $|z| < r$ and since f_k is continuous, so is $\sum f_k$ for $|z| < r$. Since this is true for all $r < 1$, taking any $|z| < 1$ then there is an r satisfying $|z| < r < 1$ and so $\sum f_k$ is continuous at z, as required.

(2) Notice that

$$\frac{1}{k^2 + z} = \frac{k^2 + \bar{z}}{\left|k^2 + z\right|^2}$$

And since $\operatorname{Re}(z) > 0$, $\left|k^2 + z\right| \geq k^2$ since the former has a factor of $(k^2 + \operatorname{Re}(z))^2 \geq k^2$. And $\left|k^2 + \bar{z}\right| \leq \left|k^2\right| + |\bar{z}| = \left|k^2\right| + |z|$.

$$\left|\frac{1}{k^2 + z}\right| = \frac{\left|k^2 + \bar{z}\right|}{\left|k^2 + z\right|^2} \leq \frac{1}{k^2} + \frac{|z|}{k^2}$$

So if we let $r > 0$ then for any $|z| < r$ with $\mathrm{Re}(z) > 0$ we have that

$$|f_k(z)| = \left|\frac{1}{k^2+z}\right| \leq \frac{1}{k^2} + \frac{r}{k^2} = M_k$$

and since $\sum M_k$ converges (since $\sum \frac{1}{k^2}$ converges), by Weierstrass $\sum f_k$ converges to a continuous function for $|z| < r$ and $\mathrm{Re}(z) > 0$ (note f_k is defined for all such z obviously) since f_k itself are continuous. And since this is true for every $r > 0$, this is true for every $\mathrm{Re}(z) > 0$ as required.

Exercise 1.13:

Let $T \subseteq \mathbb{C}$ and Σ be the sphere used for stereographic projection onto the plane. Suppose that $S \subseteq \Sigma$ is the preimage of T under the stereographic projection, then

(1) S is a circle if T is a circle.

(2) S is a circle without the point $(0,0,1)$ if T is a line.

Recall that

$$\pi^{-1}(u,v) = \left(\frac{u}{u^2+v^2+1}, \frac{v}{u^2+v^2+1}, \frac{u^2+v^2}{u^2+v^2+1}\right)$$

Or in terms of complex numbers:

$$\pi^{-1}(z) = \left(\frac{\mathrm{Re}(z)}{|z|^2+1}, \frac{\mathrm{Im}(z)}{|z|^2+1}, \frac{|z|^2}{|z|^2+1}\right)$$

So we will show that if P is a plane $Ax + By + Cz = D$ then $\pi(P)$ is a circle if $(0,0,1) \notin P$ ($C \neq D$) and a line if $(0,0,1) \in P$ ($C = D$).
Let $z = x + yi$. Notice that $\pi^{-1}(z) \in P$ if and only if

$$\frac{1}{x^2+y^2+1}(x, y, x^2+y^2) \in P \iff Ax + By + C(x^2+y^2) = D(x^2+y^2+1)$$

If $C = D$ this becomes $Ax + By = D$ which is the equation of a line, so $z \in \pi(P)$ if and only if z is on some line. So the image of a circle with (without) the north pole is a line. And notice that for any line we can write it as $Ax + By = 1$ and this line is the image of the intersection of the plane $Ax + By + z = 1$ with the sphere (and this intersection is not trivial since A and B can't both be zero). And so every line is the image of the circle with (without) the north pole.
Otherwise we end up with (after completing the square):

$$\left(x - \frac{A}{2(D-C)}\right)^2 + \left(y - \frac{B}{2(D-C)}\right)^2 = \frac{D}{C-D} + \frac{A^2+B^2}{4(D-C)^2}$$

This is the formula for a circle, so if the circle doesn't contain the north pole, its projection is the circle. And suppose we have a circle around $a + bi$ with radius r then we need to find A, B, C, and D such that $\frac{A}{2(D-C)} = a$ and $\frac{B}{2(D-C)} = b$ and $r^2 = \frac{D}{C-D} + \frac{A^2+B^2}{4(D-C)^2}$. This means solving

$$a = \frac{A}{2(D-C)}$$
$$b = \frac{B}{2(D-C)}$$
$$r^2 - a^2 - b^2 = \frac{D}{C-D}$$

This has a solution since for any choice of D and C which satisfies the bottom equation we can find A and B to satisfy the top two. So the circle around $a + bi$ with radius r is the image of the intersection of a plane which doesn't pass through the point $(0,0,1)$ with the sphere (ie. a circle).

Exercise 1.14:

We are given that z is the stereographic projection of (u, v, w) onto $\mathbb{C}$ and $\frac{1}{z}$ is the projection of (u', v', w'). Prove that $(u', v', w') = (u, -v, 1-w)$.

So we have that

$$\pi^{-1}(z) = (u, v, w) = \left(\frac{\text{Re}(z)}{|z|^2+1}, \frac{\text{Im}(z)}{|z|^2+1}, \frac{|z|^2}{|z|^2+1}\right)$$

And since $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$ we have that $\text{Re}\left(\frac{1}{z}\right) = \frac{\text{Re}(z)}{|z|^2}$ and $\text{Im}\left(\frac{1}{z}\right) = -\frac{\text{Im}(z)}{|z|^2}$ and $\left|\frac{1}{z}\right| = \frac{1}{|z|}$. And so

$$\frac{\text{Re}\left(\frac{1}{z}\right)}{\left|\frac{1}{z}\right|^2+1} = \frac{\text{Re}(z)}{\frac{|z|^2}{|z|^2}+|z|^2} = \frac{\text{Re}(z)}{|z|^2+1} = u$$

$$\frac{\text{Im}\left(\frac{1}{z}\right)}{\left|\frac{1}{z}\right|^2+1} = -\frac{\text{Im}(z)}{\frac{|z|^2}{|z|^2}+|z|^2} = -\frac{\text{Im}(z)}{|z|^2+1} = -v$$

$$\frac{\left|\frac{1}{z}\right|^2}{\left|\frac{1}{z}\right|^2+1} = \frac{1}{\frac{|z|^2}{|z|^2}+|z|^2} = \frac{1}{|z|^2+1} = 1-w$$

So we have that

$$\pi^{-1}\left(\frac{1}{z}\right) = (u', v', w') = \left(\frac{\text{Re}\left(\frac{1}{z}\right)}{\left|\frac{1}{z}\right|^2+1}, \frac{\text{Im}\left(\frac{1}{z}\right)}{\left|\frac{1}{z}\right|^2+1}, \frac{\left|\frac{1}{z}\right|^2}{\left|\frac{1}{z}\right|^2+1}\right) = (u, -v, 1-w)$$

as required.