Complex Functions

Lecture 1, Wednesday March 15, 2023

Ari Feiglin

This lecture was largely a review of the basics of the complex field $\mathbb{C}$, and so I did not transcribe the first two hours or so, as everyone taking this course should already be familiar with it.

Definition 1.1:

Suppose $\{z_n\}_{n=1}^{\infty}$ is a complex sequence, then it converges to $z \in \mathbb{C}$ if:

$$|z_n - z| \xrightarrow[n\to\infty]{} 0$$

this is denoted

$$z_n \xrightarrow[n\to\infty]{} z \text{ or } \lim z_n = z$$

Note that since $|z_n - z|$ is equal to the norm of $z_n - z$ when viewed as a vector, a sequence converges to at most one value. And since convergence in $\mathbb{R}^n$ is equivalent to pointwise convergence, z_n converges to z if and only if $\mathrm{Re}(z_n)$ converges to $\mathrm{Re}(z)$ and $\mathrm{Im}(z_n)$ converges to $\mathrm{Im}(z)$.

The arithmetic of sequences is the same in $\mathbb{C}$ as it is in $\mathbb{R}$ since for addition this is simply the addition of two vector sequences, scaling a sequence by $w \in \mathbb{C}$ has that

$$|wz_n - wz| = |w| \cdot |z_n - z| \xrightarrow[n\to\infty]{} 0$$

and since $\mathbb{C}$ is a field, we can also multiply two sequences: suppose $\{z_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ are two complex sequences which converge to z and w respectively. Then $\{z_n w_n\}_{n=1}^{\infty}$ converges to zw:

$$|z_n w_n - zw| = |z_n(w_n - w) + w(z_n - z)| \leq |z_n||w_n - w| + |w||z_n - z|$$

which converges to 0 since $|z_n|$ must be bounded (since $|z_n| \leq |z_n - z| + |z|$).

The definition of a complex series is analogous to a real one, and similarly if $\sum z_n$ converges, then z_n converges to 0 (the proof is simple using sequence arithmetic).