Infintesimal Calculus 4

Lecture 4, Wednsday November 2, 2022
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Proposition 4.1.1:

If S is compact, S is closed.

Proof:

Suppose for the sake of a contradiction that S is not closed. Then there is a limit point x which is not in S. We will take a descending sequence of closed sets $F_n = \bar{B}_{\frac{1}{2^n}}(x)$, then let $\mathcal{O}_n = F_n^c$ which are open. Notice then that the intersection of F_n is $\{x\}$, since if y is in the intersection $\rho(x, y) \leq \frac{1}{2^n}$ for all n, so $\rho(x, y) = 0$, so $y = x$. And therefore the union of $\mathcal{O}_n$ is $X \setminus \{x\}$, and since x isn't in S, $\{\mathcal{O}_n\}_{n=1}^{\infty}$ is an open cover of S. Since S is compact, there is a finite subcover $\{\mathcal{O}_{m_n}\}_{n=1}^{N}$ which covers S. Now if we assume $m_n < m_{n+1}$, then since $\mathcal{O}_n$ is an increasing sequence:

$$S \subseteq \mathcal{O}_{m_N}$$

But then that means that F_{m_n} and S are disjoint, so there exists a ball around x which doesn't contain points of S. So x can't be a limit point, in contradiction. $\lightning$

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Definition 4.1.2:

If X is a metric space, $S \subseteq X$ is **bounded** if there is an x in X and an $r > 0$ such that $S \subseteq B_r(x)$.

Notice then that S is bounded if and only if for *every* $x \in X$ there is an $r > 0$ such that $S \subseteq B_r(x)$. If S is bounded, then suppose $S \subseteq B_r(x)$, then let $y \in X$, if we let $r' = r + \rho(x, y)$ $S \subseteq B_{r'}(y)$.

Proposition 4.1.3:

If S is compact, then S is bounded.

Proof:

Notice that for any $x \in X$, $\{B_n(x)\}_{n\in\mathbb{N}}$'s union is X since for any $y \in X$, there must be some $n \in \mathbb{N}$ such that $\rho(x, y) < n$. Since S is compact, there is a subcovering $\{B_{m_n}\}_{n=1}^{N}$ which covers S. But then since these balls form an increasing subsequence:

$$S \subseteq B_{m_N}(x)$$

and thus by definition S is bounded.

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Example:

Notice that not every bounded set is compact. Over a set X we can define the following **discrete metric**:

$$\rho(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

This is obviously a metric (and quite an interesting one as well). It has the interesting characteristic that every subset of X is open and thus it must also be closed (since its complement is open). This is because for $\varepsilon \leq 1$, $B_\varepsilon(x) = \{x\}$. So if we take $X = \mathbb{N}$ then we can create a covering $\{\{x\}\}_{x\in\mathbb{N}}$ (since every set is open), but for any finite subcovering, the union contains only a finite number of points and thus cannot cover X. So X is not compact. But X is bounded (this is true for every discrete metric space) since for every $x, y \in X$, then $\rho(x, y) \leq 1$, so $X \subseteq B_2(x)$.

Proposition 4.1.4:

If X is a metric space and $T \subseteq S \subseteq X$ where S is compact and T is closed, then T is also compact.

Proof:

Suppose $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$ is an open covering of T. Then we can add T^c to the cover which is open since T is closed, and that creates an open cover of X and therefore also of S. So there exists a finite subcovering of this which covers S:

$$T \subseteq S \subseteq \bigcup_{k=1}^{n} \mathcal{O}_k \cup T^c$$

Then since T and T^c are disjoint, we must have that

$$T \subseteq \bigcup_{k=1}^{n} \mathcal{O}_k$$

so $\{\mathcal{O}_k\})_{k=1}^n$ is a finite subcovering of T, so T is compact.

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Definition 4.1.5:

A **closed rectangle** in $\mathbb{R}^n$ is a set of the form:

$$[a_1, b_1] \times \cdots \times [a_n, b_n]$$

Where $a_k < b_k$. And the kth **vertex** of such a rectangle is $[a_k, b_k]$.

Definition 4.1.6:

If X is a metric space and $S \subseteq X$, the **diameter** of S is:

$$\operatorname{diam} S = \sup_{x,y\in S} \rho(x, y)$$

A **contracting sequence** of sets $\{E_n\}_{n=1}^\infty$ in X is a descending sequence of sets whose diameter approaches 0, that is:

$$E_{n+1} \subseteq E_n \text{ and } \lim_{n\to\infty} \operatorname{diam}(E_n) = 0$$

Theorem 4.1.7 (Cantor's Lemma):

If $\{T_k\}_{k\in\mathbb{N}}$ is a contracting sequence of rectangles in $\mathbb{R}^n$ then there exists an $x \in \mathbb{R}^n$ such that:

$$\bigcap_{k\in\mathbb{N}} T_k = \{x\}$$

Proof:

For every m we can take the kth vertex of T_k: $[a_k^{(m)}, b_k^{(m)}]$. And since T_k is decreasing:

$$[a_k^{(1)}, b_k^{(1)}] \supseteq \cdots \supseteq [a_k^{(m)}, b_k^{(m)}] \supseteq \cdots$$

Then by Cantor's Lemma in $\mathbb{R}$, the intersection of these intervals is non-empty, so there exists an x_k in the intersection. Then $(x_1, \dots, x_n)$ is in the intersection of T_k. This must be the only point in the intersection since if there is another y in the intersection, since $x, y \in T_k$ for every k, $\operatorname{diam} T_k \geq \rho(x, y)$, and then the limit of the diameters wouldn't be 0.

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This theorem can be generalized to any complete metric space (which we will learn about later). In fact this trait is actually equivalent to completeness.

Theorem 4.1.8:

A rectangle $T = [a_1, b_1] \times \cdots \times [a_n, b_n]$ in $\mathbb{R}^n$ is compact.

Proof:

Let $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$ be an open covering of T. Suppose for the sake of a contradiction that T has no finite open subcovering. For every vertex, we note that $[a_k, b_k] = [a_k, c_k] \times [c_k, b_k]$ where $c_k = \frac{1}{2}(a_k + b_k)$. Then we now have 2^n smaller rectangles. And one of these smaller rectangles must also not have a finite open subcovering (if they all did, we could take the union of these finite open subcoverings which would also be a finite open subcovering of T which is a contradiction). Let T_1 be this subrectangle which doesn't have a finite open subcovering. We can find a subrectangle T_2 of T_1 which also doesn't have a finite open subcovering, and thus we can recursively define a sequence $\{T_n\}_{n\in\mathbb{N}}$. And since the diameter of T_{n+1} is half that of T_n, $\operatorname{diam} T_n \longrightarrow 0$. Since T_n is closed, by Cantor's lemma:

$$\bigcap_{n\in\mathbb{N}} T_n = \{x\}$$

for some $x \in \mathbb{R}^n$. Then there must be some $\lambda \in \Lambda$ such that $x \in \mathcal{O}_\lambda$, and so there must be an $r > 0$ such that $x \in B_r(x) \subseteq \mathcal{O}_\lambda$. Since $\operatorname{diam} T_n \longrightarrow 0$, at some point $\operatorname{diam} T_n < r$, so therefore

$$T_n \subseteq B_r(x) \subseteq \mathcal{O}_\lambda$$

So T_n *does* have a finite open subcovering, in contradiction. $\lightning$

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Theorem 4.1.9 (Heine-Borel Theorem):

$T \subseteq \mathbb{R}^n$ is compact if and only if T is closed and bounded.

Proof:

We have already shown that compactness implies closedness and boundedness, so all that remains is to prove the converse. Since T is bound, it must be contained inside of a rectangle U (we can take $x \in T$, and then vertices around x_k whose lengths are the diameter of T). By above, U is compact. And since $T \subseteq U$ and T is closed, T is compact.

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This result does not hold in general metric spaces.

Example:

Recall the definition of ℓ^2:

$$\ell^2 = \left\{ \{a_n\}_{n\in\mathbb{N}} \;\middle|\; \sum_{n=1}^{\infty} a_n^2 < \infty \right\}$$

Let e_k be the sequence whose kth element is 1 and the rest are 0. Let $T = \{e_1, \dots, e_n, \dots\}$. T is bounded since $\rho(e_n, e_m) = \sqrt{2}$ and it is closed since if x is a limit point of T, then it must be equal to some e_k, because the distance between each e_k is constant. But T is not compact since if we focus on the cover $\{B_1(e_k)\}_{k=1}^{\infty}$, each ball contains only one e_k and thus there can't be a finite subcover since T is infinite.