Probability and Statistics Homework #7

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Question 7.1:

People arrive at a certain store at a rate of 30 people per hour. Assuming the distribution in poisson, find the following probabilities:

(1) For 5 minutes no one comes.

(2) At least 4 people arrive within 5 minutes.

(3) 30 people arrive within the hour.

Answer:

Let us define the random variable X_n to be the number of people who arrive within n minutes. We know that:

$$\mathbb{E}\left[X_{\alpha n}\right] = \alpha\mathbb{E}\left[X_n\right]$$

As the expected number of people to arrive within $\alpha \cdot n$ minutes will be α times greater than the expected number of people to arrive within n minutes. (In double the amount of time, double the amount of people are expected to arrive, for example.)

Furthermore, we are told that:

$$\mathbb{E}\left[X_{60}\right] = 30$$

Which means:

$$\mathbb{E}\left[X_n\right] = \mathbb{E}\left[X_{\frac{n}{60}\cdot 60}\right] = \frac{n}{60}\cdot\mathbb{E}\left[X_{60}\right] = \frac{n}{2}$$

Furthermore, we can assume that the probability that a certain number of people arrive within n minutes has a poisson distribution as it represents a continuous binomial distribution. So:

$$X_n \sim \text{Poi}\left(\lambda_n\right)$$

But we know:

$$\mathbb{E}\left[X_n\right] = \lambda_n \implies \lambda_n = \frac{n}{2}$$

With this, we can answer the questions.

(1) This is the probability:

$$\mathbb{P}\left(X_5 = 0\right) = e^{-\lambda_5}\cdot\frac{\lambda_5^0}{0!} = \boxed{e^{-\frac{5}{2}}}$$

(2) This is the probability:

$$\mathbb{P}\left(X_5 \geq 4\right) = 1 - \mathbb{P}\left(X_5 < 4\right) = 1 - \sum_{k=0}^{3}\mathbb{P}\left(X_5 = k\right)$$

Which is equal to, by the definition of the poisson distribution:

$$1 - e^{-\frac{5}{2}}\cdot\sum_{k=0}^{3}\frac{\left(\frac{5}{2}\right)^k}{k!} \approx \boxed{0.24242}$$

(3) This is the probability:

$$\mathbb{P}\left(X_{60} = 30\right) = e^{-30}\cdot\frac{30^{30}}{30!} \approx \boxed{0.07263}$$

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Question 7.2:

In a box there are 20 parts, 4 of which are broken. 5 parts are removed randomly. Answer the following:

(1) What is the probability that at most 1 of the parts removed will be broken?

(2) It is known that at most 2 of the parts still in the box are broken. What now is the probability of the previous question?

(3) It is known that at least 2 of the parts still in the box are broken. What now is the probability of the previous question?

Answer:

Let X be the number of broken parts removed. We know that X has a hypergeometric distribution:

$$X \sim \mathrm{HG}\left(20, 4, 5\right)$$

(1) This is the probability:

$$\mathbb{P}\left(X \leq 1\right) = \mathbb{P}\left(X = 0\right) + \mathbb{P}\left(X = 1\right) = \frac{\binom{16}{5}}{\binom{20}{5}} + \frac{\binom{4}{1} \cdot \binom{16}{4}}{\binom{20}{5}} \approx \boxed{0.75129}$$

(2) The probability that at most 2 broken parts remain is the probability that at least $4 - 2 = 2$ broken parts were removed. So this is the probability:

$$\mathbb{P}\left(X \leq 1 \mid X \geq 2\right)$$

But these are disjoint events, so:

$$\mathbb{P}\left(X \leq 1 \mid X \geq 2\right) = \boxed{0}$$

(3) The probability that at least 2 broken parts remain is the probability that at most $4 - 2 = 2$ broken parts were removed. So this is the probability:

$$\mathbb{P}\left(X \leq 1 \mid X \leq 2\right) = \frac{\mathbb{P}\left(X \leq 1, X \leq 2\right)}{\mathbb{P}\left(X \leq 2\right)} = \frac{\mathbb{P}\left(X \leq 1\right)}{\mathbb{P}\left(X \leq 2\right)}$$

We know that:

$$\mathbb{P}\left(X \leq 2\right) = \mathbb{P}\left(X \leq 1\right) + \mathbb{P}\left(X = 2\right)$$

Applying the formula for the hypergeometric distribution and plugging in our previous answer, we get:

$$\mathbb{P}\left(X \leq 1 \mid X \leq 2\right) = \boxed{\frac{52}{67}}$$

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Question 7.3:

Let $X \sim \mathrm{NB}\left(r, p\right)$. Compute the expected value of X.

Answer:

As proven in recitation (and my previous homework, albeit not stated explicitly), a random variable which has a negative binomial distribution is the sum of geometric distributions.
Let $\left\{X_i\right\}_{i=1}^{r}$ be a series of independent random variables such that

$$X_i \sim \mathrm{Geo}\left(p\right)$$

For all relevant i, thus

$$X \overset{d}{=} \sum_{i=1}^{r} X_i$$

By the linearity of the expected value, this means:

$$\mathbb{E}\left[X\right] = \sum_{i=1}^{r} \mathbb{E}\left[X_i\right]$$

Since X_i distributes geometrically, this means that $\mathbb{E}\left[X_i\right] = \frac{1}{p}$. Thus:

$$\mathbb{E}\left[X\right] = \sum_{i=1}^{r} \frac{1}{p} = \boxed{\frac{r}{p}}$$

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Question 7.4:

Let $\{X_n\}_{n\in\mathbb{N}}$ be a series of random variables defined by the following recurrence:

$$\mathbb{P}(X_{n+1} = k) = \begin{cases} p & k = X_n \\ 1-p & k = 1 - X_n \end{cases}$$

And $X_0 \coloneqq 1$. Determine the probability distribution of X_n.

Answer:

Let F_n be the number of sign swaps of X_k for $k \leq n$. That is:

$$F_n \coloneqq |\{m < n \in \mathbb{N}_0 \mid \operatorname{sign}(X_m) \neq \operatorname{sign}(X_{m+1})\}|$$

This means that:

$$X_n = (-1)^{F_n}$$

I will prove this through induction.

Base case: $n = 0$

We know that $F_0 = |\varnothing| = 0$ as there is no $m < 0 \in \mathbb{N}_0$. Which means:

$$(-1)^{F_0} = (-1)^0 = 1 = X_0$$

As required.

Inductive step:

We know X_{n+1} equals either X_n or $-X_n$.
If $X_{n+1} = X_n$, then $F_{n+1} = F_n$ as there is no sign swap between X_n and X_{n+1}. Thus:

$$(-1)^{F_{n+1}} = (-1)^{F_n} = X_n = X_{n+1}$$

As required.
If $X_{n+1} = -X_n$ then $F_{n+1} = F_n + 1$. So:

$$(-1)^{F_{n+1}} = (-1)^{F_n+1} = -(-1)^{F_n} = -X_n = X_{n+1}$$

As required.

Let us now find the probability distribution of F_n. Let W_m be the random variable that indicates whether or not a sign swap occurred between X_{m-1} and X_m. This means that $W_m \sim \operatorname{Ber}(1-p)$. Furthermore, we know that $\{W_m\}_{m\in\mathbb{N}}$ are independent. We also know that:

$$F_n = \sum_{m=1}^{n} W_m$$

As it counts the number of swaps between X_0 and X_n. Since W_m has a bernoulli distribution, this means:

$$F_n \sim \operatorname{Bin}(n, 1-p)$$

We know that $X_n = (-1)^{F_n} = \pm 1$. So let us compute $\mathbb{P}(X_n = 1)$, and then we know that $\mathbb{P}(X_n = -1) = 1 - \mathbb{P}(X_n = 1)$.
The event $X_n = 1$ is the event $(-1)^{F_n} = 1$, which is equivalent to the event that F_n is even. This is:

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \mathbb{P}(F_n = 2k)$$

Since we know $F_n \leq n$. This is equal to:

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \cdot p^{n-2k} \cdot (1-p)^{2k}$$

Lemma 7.4.1:

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} a^{n-2k} \cdot b^{2k} = \frac{1}{2}\left((a+b)^n + (a-b)^n\right)$$

Proof:

Using the binomial theorem:

$$\begin{aligned}
(a+b)^n + (a-b)^n = \\
&= \sum_{k=0}^{n}\left(\binom{n}{k}a^{n-k} \cdot b^k\right) + \sum_{k=0}^{n}\left(\binom{n}{k}a^{n-k} \cdot (-b)^k\right) \\
&= \sum_{k=0}^{n}\binom{n}{k} \cdot a^{n-k} \cdot \left(b^k + (-1)^k \cdot b^k\right)
\end{aligned}$$

For odd ks, $b^k + (-1)^k \cdot b^k = b^k - b^k = 0$. So we can sum over even ks between 0 and n. For even ks, $b^k + (-1)^k \cdot b^k = b^k + b^k = 2 \cdot b^k$. Summing over even ks is the same as summing from $k = 0$ to $\lfloor \frac{n}{2} \rfloor$, and using the index $2k+1$. So:

$$(a+b)^n + (a-b)^n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{2k} \cdot a^{n-2k} \cdot 2 \cdot b^{2k} = 2\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{2k} \cdot a^{n-2k} \cdot b^{2k}$$

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Using this lemma, we know:

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{2k} \cdot p^{n-2k} \cdot (1-p)^{2k} = \frac{1}{2} \cdot \left(1 + (2p-1)^n\right)$$

This means that:

$$\boxed{\mathbb{P}\left(X_n = 1\right) = \frac{1}{2} \cdot \left(1 + (2p-1)^n\right)}$$

And:

$$1 - \frac{1}{2} \cdot \left(1 + (2p-1)^n\right) = \frac{1}{2} \cdot \left(1 - (2p-1)^n\right)$$

So:

$$\boxed{\mathbb{P}\left(X_n = -1\right) = \frac{1}{2} \cdot \left(1 - (2p-1)^n\right)}$$

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Question 7.5:

Let X be a random variable such that $X \sim \mathrm{Poi}(\lambda)$. Let Y be a random variable defined as:

$$Y := \begin{cases} 0 & X_n \text{ is even} \\ 1 & X_n \text{ is odd} \end{cases}$$

Compute the probability distribution of Y.

Answer:

We know:

$$\mathbb{P}(Y=0) = \sum_{n=0}^{\infty} \mathbb{P}(X=2n) = \sum_{n=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{2n}}{(2n)!} = e^{-\lambda} \cdot \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{(2n)!}$$

We know:

$$e^{\lambda} + e^{-\lambda} = \sum_{n=0}^{\infty} \left(\frac{\lambda^n}{n!} + \frac{(-\lambda)^n}{n!} \right) = \sum_{n=0}^{\infty} \frac{\lambda^n + (-1)^n \lambda^n}{n!}$$

If n is odd, then $\lambda^n + (-1)^n \lambda^n = \lambda^n - \lambda^n = 0$, so we can sum over the even ns.
If n is even, then $\lambda^n + (-1)^n \lambda^n = 2\lambda^n$. So:

$$e^{\lambda} + e^{-\lambda} = \sum_{n=0}^{\infty} \frac{2\lambda^{2n}}{(2n)!} = 2 \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{(2n)!}$$

This means:

$$\mathbb{P}(Y=0) = e^{-\lambda} \cdot \frac{1}{2} \cdot \left(e^{\lambda} + e^{-\lambda}\right) = \frac{1}{2} \cdot \left(1 + e^{-2\lambda}\right)$$

And we know that $Y \in \{0, 1\}$, so

$$\mathbb{P}(Y=1) = 1 - \mathbb{P}(Y=0) = 1 - \frac{1}{2} \cdot \left(1 + e^{-2\lambda}\right) = \frac{1}{2} \cdot \left(1 - e^{-2\lambda}\right)$$

So all in all:

$$\mathbb{P}(Y=0) = \frac{1}{2} \cdot \left(1 + e^{-2\lambda}\right)$$
$$\mathbb{P}(Y=1) = \frac{1}{2} \cdot \left(1 - e^{-2\lambda}\right)$$

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Question 7.6:

Suppose X is a random variable with a poisson distribution over λ. Prove that for every natural k:

$$\mathbb{E}\left[X \cdot (X-1) \cdots (X-k)\right] = \lambda^{k+1}$$

Answer:

We know that:

$$\mathbb{E}\left[X \cdot (X-1) \cdots (X-k)\right] = \sum_{x \in \mathbb{R}} \left(x \cdots (x-k) \cdot \mathbb{P}\left(X \cdots (X-k) = x \cdots (x-k)\right)\right)$$

Since $X \in \mathbb{N}_0$, we can sum for xs in $\mathbb{N}_0$. And we then know that if $x - k \leq 0$, then $x \leq k$, which means that $x - x = 0$ is between x and $x - k$. This means that the product $x \cdots (x-k) = 0$. So we can sum over $x > k$. Since $k \in \mathbb{N}_0$, this means $x \in \mathbb{N}$. So:

$$\mathbb{E}\left[X \cdots (X-k)\right] = \sum_{x > k \in \mathbb{N}} \left(x \cdots (x-k) \cdot \mathbb{P}\left(X \cdots (X-k) = x \cdots (x-k)\right)\right)$$

Since $x \cdots (x-k) \neq 0$ (as $x, x-k > 0$), $X \cdots (X-k) = x \cdots (x-k)$ if and only if $X = x$. Furthermore, we can express $x \cdots (x-k)$ as $\frac{x!}{(x-k-1)!}$

$$= \sum_{x > k \in \mathbb{N}} \frac{x!}{(x-k-1)!} \cdot \mathbb{P}\left(X = x\right) = \sum_{x=k+1}^{\infty} \frac{x!}{(x-k-1)!} \cdot e^{-\lambda} \cdot \frac{\lambda^x}{x!} = e^{-\lambda} \cdot \sum_{x=k+1}^{\infty} \frac{\lambda^x}{(x-k-1)!} =$$

$$= e^{-\lambda} \cdot \sum_{x=0}^{\infty} \frac{\lambda^{x+k+1}}{x!} = \lambda^{k+1} \cdot e^{-\lambda} \cdot \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = \lambda^{k+1} \cdot e^{-\lambda} \cdot e^{\lambda} = \lambda^{k+1}$$

So $\mathbb{E}\left[X \cdots (X-k)\right] = \lambda^{k+1}$, as required.

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Question 7.7:

In an urn there are n black balls, m white balls, and k red balls, such that $n, m, k \geq 2$. We remove balls one after another from the urn with replacement. What is the expected number of times that we'll need to balls until we get the same color ball as the ball we removed originally? What would the answer be without replacement?

Answer:

(1) Let us generalize this question. Suppose instead we have an urn with n colors, each of which has a_i balls. Let X be the number of balls we removed until we get the first same color as the first ball (not including the first ball). And let C_i be the event that the first ball had color i. Let $A \coloneqq \sum_{i=1}^{n} a_i$. We know:

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} k \cdot \mathbb{P}(X = k) = \sum_{k=1}^{\infty} \left(k \cdot \sum_{i=1}^{n} \mathbb{P}(X = k \mid C_i) \cdot \mathbb{P}(C_i) \right)$$

$$= \sum_{i=1}^{n} \left(\mathbb{P}(C_i) \cdot \sum_{k=1}^{\infty} k \cdot \mathbb{P}(X = k \mid C_i) \right) = \sum_{i=1}^{n} \mathbb{P}(C_i) \cdot \mathbb{E}[X \mid C_i]$$

We know $\mathbb{P}(C_i) = \frac{a_i}{A}$ as it is the probability we choose one the of a_i balls out of A total.
And we know $\mathbb{P}(X = k \mid C_i)$ is the probability that we $k-1$ balls not colored i and then a ball colored i. Since the balls are replaced, this is equal to:

$$\frac{A - a_i}{A}^{k-1} \cdot \frac{a_i}{A} = \frac{a_i}{A} \cdot \left(1 - \frac{a_i}{A}\right)^{k-1}$$

Which means:

$$X \mid C_i \sim \text{Geo}\left(\frac{a_i}{A}\right)$$

This means that $\mathbb{E}[X \mid C_i] = \frac{A}{a_i}$. So:

$$\mathbb{E}[X] = \sum_{i=1}^{n} \frac{a_i}{A} \cdot \frac{A}{a_i} = \sum_{i=1}^{n} 1 = n$$

Since in our case there are only 3 colors ($n = 3$), this means:

$$\boxed{\mathbb{E}[X] = 3}$$

(2) Let us first answer a simpler question.

Subquestion 7.7.1:

Given an urn with n red balls and m blue balls, what is the expected number of balls that'll be removed until we remove a blue ball?

Answer:

Let Ω be the sample space consisting of ways to remove all of the balls in the urn. We know that:

$$P(\omega_1) = P(\omega_2)$$

For every $\omega_1, \omega_2 \in \Omega$, since the order in which we remove the balls does not affect the probability. (This has been proven explicitly in previous homeworks. It is because the probability of each of these ωs are the same albeit with permuted products in the numerators and denominators.)
Let $\{S_i\}_{i=1}^{m+1}$ be the number of red balls chosen between the $i-1$st blue ball chosen and the ith. We know that:

$$\sum_{i=1}^{n+1} S_i = n$$

As there are n red balls in the urn.

Furthermore, we can construct a bijection between:

$$\{\omega \in \Omega \mid S_t(\omega) = k\} \longleftrightarrow \{\omega \in \Omega \mid S_\ell(\omega) = k\}$$

By simply taking ω and shifting the indexes of when the blue balls were taken in such a way to satisfy the condition.
That is, suppose $\{a_i\}_{i=0}^{m+1}$ are the indexes of when the blue balls were removed ($0 = a_0 < a_1 < \cdots < a_n < a_{n+1} = n+m+1$). We know:

$$S_i = a_i - a_{i-1} - 1$$

We can then define a new indexing series where we shift the a_is in such a way that $a_t - a_{t-1}$ is swapped with $a_\ell - a_{\ell-1}$. This is just a shift, so it is a bijection.
Since the probability is uniform (as the mass probability function is independent of ω), this means that:

$$\mathbb{P}(S_t = k) = \mathbb{P}(\{\omega \mid S_t(\omega) = k\}) = \mathbb{P}(\{\omega \mid S_\ell(\omega) = k\}) = \mathbb{P}(S_\ell = k)$$

This means that $S_t \overset{d}{=} S_\ell$, so $\mathbb{E}[S_t] = \mathbb{E}[S_\ell]$. And since we know:

$$\sum_{i=1}^{m+1} S_i = n \implies \sum_{i=1}^{m+1} \mathbb{E}[S_i] = n \implies (m+1)\cdot\mathbb{E}[S_1] = n$$

Which means that:

$$\mathbb{E}[S_1] = \frac{n}{m+1}$$

So the expected number of red balls removed until the first blue ball is removed is $\frac{n}{m+1}$, which means the expected number of balls removed until the first blu ball is removed is $1 + \frac{n}{m+1}$ (since we're also remvoning the blue ball).

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Suppose, again, that there are instead n colors, of which there are each a_i balls.
Now, let X and C_i be defined as before. Similar to the previous question, we know:

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{P}(C_i)\cdot\mathbb{E}[X \mid C_i]$$

$\mathbb{E}[X \mid C_i]$ is the expected number of balls not of color i removed until a ball of color i is removed, from the $a_i - 1$ balls of color i remaining. This is equivalent to the expected number of balls removed until we get a blue ball if there are $a_i - 1$ blue balls and $A - a_i$ red balls (as there are $A - a_i$ non-i-colored balls). From the subquestion above, we know this to be:

$$\mathbb{E}[X \mid C_i] = 1 + \frac{A - a_i}{a_i} = \frac{A}{a_i}$$

The probability of C_i hasn't changed, since we're still removing one of the a_i balls from A total. So:

$$\mathbb{E}[X] = \sum_{i=1}^{n} \frac{a_i}{A}\cdot\frac{A}{a_i} = \sum_{i=1}^{n} 1 = n$$

Since there are 3 colors in our case ($n = 3$), this means:

$$\boxed{\mathbb{E}[X] = 3}$$

Note 7.7:

Note that the expected value computed here in both subquestions is the expected number of balls removed *after* the first ball. If we inlude the first ball, then in both subquestions the expected value is $\boxed{4}$.

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Question 7.8:

X and Y are random variables with an expected value. Prove:

$$\mathbb{E}\left[|X+Y|\right] \leq \mathbb{E}\left[|X|\right] + \mathbb{E}\left[|Y|\right]$$

Answer:

Firstly, we know that $|X|, |Y|$ have an expected value, as:

$$\mathbb{E}\left[|X|\right] = \sum_{x\in\mathbb{R}} |x| \cdot \mathbb{P}\left(X = x\right)$$

Which must be finite since X has an absolute value. Similar for Y.
So if we can prove that $\mathbb{E}\left[|X+Y|\right] \leq \mathbb{E}\left[|X|\right] + \mathbb{E}\left[|Y|\right]$, that will be sufficient (as in it will also prove that $|X+Y|$ has an expected value).
We know:

$$\mathbb{E}\left[|X+Y|\right] = \sum_{x,y\in\mathbb{R}} |x+y| \cdot \mathbb{P}\left(X = x, Y = y\right)$$

Since $|x+y| \leq |x| + |y|$, we know:

$$\leq \sum_{x,y\in\mathbb{R}} |x| \cdot \mathbb{P}\left(X = x, Y = y\right) + \sum_{x,y\in\mathbb{R}} |y| \cdot \mathbb{P}\left(X = x, Y = y\right)$$

We can expand:

$$\sum_{x,y\in\mathbb{R}} |x|\, \mathbb{P}\left(X = x, Y = y\right) = \sum_{x\in\mathbb{R}} \left(|x| \cdot \sum_{y\in\mathbb{R}} \mathbb{P}\left(X = x, Y = y\right)\right)$$

We can reorder the summation like this since the series is non-negative. We know:

$$\sum_{y\in\mathbb{R}} \mathbb{P}\left(X = x, Y = y\right) = \mathbb{P}\left(X = x\right)$$

So:

$$\sum_{x,y\in\mathbb{R}} \left(|x|\, \mathbb{P}\left(X = x, Y = y\right)\right) = \sum_{x\in\mathbb{R}} |x|\, \mathbb{P}\left(X = x\right) = \mathbb{E}\left[|X|\right]$$

And we can do an identical process for the second sum, so we have:

$$\mathbb{E}\left[|X+Y|\right] \leq \mathbb{E}\left[|X|\right] + \mathbb{E}\left[|Y|\right]$$

As required.

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