Complex Functions

Assignment 2
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Exercise 2.1:

(1) Suppose $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ is a complex function such that for every real z, f is differentiable at z and $f(z) \in \mathbb{R}$. Prove that for every $z \in \mathbb{R}$, $f'(z) \in \mathbb{R}$.

(2) Suppose $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ is a complex function such that for every imaginary z, f is differentiable at z and $f(z) \in \mathbb{R}$. Prove that for every imaginary z, $f'(z) \in i\mathbb{R}$.

(1) Since $f'(z)$ exists for $z \in \mathbb{R}$, it is equal to (since we can take any path to 0):

$$f'(z) = \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{f(z+h) - f(z)}{h}$$

Since $z + h \in \mathbb{R}$, $f(z+h) - f(z) \in \mathbb{R}$ and so $\frac{f(z+h)-f(z)}{h} \in \mathbb{R}$ and so $f'(z)$ is the limit of a real sequence, and therefore $f'(z) \in \mathbb{R}$ as required.

(2) Since $f'(z)$ exists for $z \in i\mathbb{R}$, it is equal to:

$$f'(z) = \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{f(z+ih) - f(z)}{ih} = -i \cdot \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{f(z+ih) - f(z)}{h}$$

Since $ih \to 0$. Since $z + ih \in i\mathbb{R}$, $f(z+ih) - f(z) \in \mathbb{R}$ so $\frac{f(z+ih)-f(z)}{h} \in \mathbb{R}$, so the limit is real and therefore $f'(z) \in i\mathbb{R}$ (since the limit is multiplied by $-i$) as required.

Exercise 2.2:

Suppose f and g are two complex functions which are differentiable at $z \in \mathbb{C}$, then

(1) $f + g$ is differentiable at z and $(f+g)'(z) = f'(z) + g'(z)$.

(2) $f \cdot g$ is differentiable at z and $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$.

(3) $\frac{f}{g}$ is differentiable at z and $g \neq 0$ in a neighborhood of z, then $\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$.

(1) Notice that

$$(f+g)'(z) = \lim_{h\to 0} \frac{f(z+h) + g(z+h) - (f(z) + g(z))}{h} = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} + \lim_{h\to 0} \frac{g(z+h) - g(z)}{h} = f'(z) + g'(z)$$

since the two limits on the right exist. So the limit defining $(f+g)'(z)$ exists and is equal to $f'(z) + g'(z)$ as required.

(2) Notice that

$$(fg)'(z) = \lim_{h\to 0} \frac{f(z+h)g(z+h) - f(z)g(z)}{h} = \lim_{h\to 0} \frac{f(z+h)\big(g(z+h) - g(z)\big) + g(z)\big(f(z+h) - f(z)\big)}{h}$$

$$= g(z) \cdot \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} + \lim_{h\to 0} f(z+h) \cdot \frac{g(z+h) - g(z)}{h} = f'(z)g(z) + f(z)g'(z)$$

where the right limit equals $f(z)g'(z)$ as the product of two convergent limits. So the limit defining $(fg)'(z)$ exists and is equal to the desired result, as required.

(3) Notice that

$$(g^{-1})'(z) = \lim_{h\to 0} \frac{\frac{1}{g(z+h)} - \frac{1}{g(z)}}{h} = \lim_{h\to 0} \frac{1}{g(z)\cdot g(z+h)} \cdot \frac{g(z)-g(z+h)}{h} =$$

$$= \frac{1}{g(z)} \cdot \lim_{h\to 0} \frac{1}{g(z+h)} \cdot \lim_{h\to 0} \frac{g(z)-g(z+h)}{h} = -\frac{g'(z)}{g(z)^2}$$

We can take this limit since $g \neq 0$ in a neighborhood of z, so for any sequence $h_n \to 0$, eventually $g(z+h_n) \neq 0$. Thus by above:

$$\left(\frac{f}{g}\right)'(z) = \left(f \cdot \frac{1}{g}\right)'(z) = \frac{f'(z)}{g(z)} - \frac{f(z)g'(z)}{g(z)^2} = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$$

Exercise 2.3:

Show that $f(z) = x^2 + iy^2$ is differentiable at z if and only if $x = y$, and thus show why f is not analytic.

So we have $u = x^2$ and $v = y^2$ so $u_x = 2x, u_y = 0, v_x = 0, v_y = 2y$. In order to satisfy the Cauchy-Riemann equations we must have $u_x = v_y$ and $u_y = -v_x$, so $2x = 2y$ and $0 = 0$. So it is necessary and sufficient for $x = y$ in order to satisfy the Cauchy-Riemann equations. Since f is differentiable when u and v are and satisfy the Cauchy-Riemann equations, and since u and v are differentiable everywhere, f is differentiable if and only if $x = y$.
Notice that in order for f to be analytic at $z \in \mathbb{C}$, it must be differentiable in a domain D of z's. So $z = x + ix$, but since D is open, there must be an element $w \in D$ which is not on the line $x = y$ and so f is not differentiable at w and hence not in D. So f is nowhere analytic.

Exercise 2.4:

Prove the chain rule for complex derivatives.

Note that a function f is differentiable at z_0 if and only if there exists a function $\varepsilon\colon \mathbb{C} \longrightarrow \mathbb{C}$ and a value $f'(z_0)$ such that:

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + \varepsilon(z - z_0)$$

where $\frac{\varepsilon(h)}{h} \xrightarrow[h\to 0]{} 0$. This is trivial and is very reminiscent of infinitesimal calculus 3.
And so we have ε_1 and ε_2 where:

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + \varepsilon_1(z - z_0), \qquad g(z) = g\big(f(z_0)\big) + \big(z - f(z_0)\big)g'\big(f(z_0)\big) + \varepsilon_2\big(z - f(z_0)\big)$$

And we need to find an ε_3 such that

$$g \circ f(z) = g \circ f(z_0) + (z - z_0)\Big(f'(z_0) \cdot g'\big(f(z_0)\big)\Big) + \varepsilon_3(z - z_0)$$

So then:

$$\begin{aligned} g \circ f(z) &= g\big(f(z_0)\big) + \big(f(z) - f(z_0)\big)g'\big(f(z_0)\big) + \varepsilon_2\big(f(z) - f(z_0)\big) \\ &= g \circ f(z_0) + (z - z_0)\Big(f'(z_0) \cdot g'\big(f(z_0)\big)\Big) + \varepsilon_1(z - z_0)g'\big(f(z_0)\big) + \varepsilon_2\big((z - z_0)f'(z_0) + \varepsilon_1(z - z_0)\big) \end{aligned}$$

So we define

$$\varepsilon_3(h) = \varepsilon_1(h) \cdot g'\big(f(z_0)\big) + \varepsilon_2\big(hf'(z_0) + \varepsilon_1(h)\big)$$

And we claim that $\frac{\varepsilon_3(h)}{h}$ converges to 0 as h approaches 0. This is simple for the $\varepsilon_1 \ldots$ part, let us look at the ε_2 part:

$$\frac{\varepsilon_2\big(hf'(z_0) + \varepsilon_1(h)\big)}{h} = \frac{\varepsilon_2\Big(h\Big(f'(z_0) + \frac{\varepsilon_1(h)}{h}\Big)\Big)}{h\Big(f'(z_0) + \frac{\varepsilon_1(h)}{h}\Big)}\left(f'(z_0) + \frac{\varepsilon_1(h)}{h}\right)$$

Which converges to 0 (the left converges to 0 by the characteristic of ε_2 and the right converges to $f'(z_0)$), as required.

Exercise 2.5:

Show that a non-constant analytic function cannot map a domain onto a line or curve.

Suppose f is a non-constant analytic function. Then there exists $z \in \mathbb{C}$ such that $f'(z) \neq 0$ and so if we view f as a function $f\colon \mathbb{R}^2 \longrightarrow \mathbb{C}^2$, by the Cauchy-Riemann equations $|J_f(z)| = u_x(z)^2 + u_y(z)^2 = v_x(z)^2 + v_y(z)^2$ which must be non-zero, otherwise $f'(z) = 0$. So by the inverse function theorem, there is a neighborhood $\mathcal{U}$ of z and $\mathcal{V}$ of $f(z)$ such that $f\colon \mathcal{U} \longrightarrow \mathcal{V}$ is bijective. So the curve contains an open set, but that it means its interior is non-empty which is a contradiction since (injective) curves are hollow.

Exercise 2.6:

Prove that there are no analytic functions $f = u + iv$ where $u(x, y) = x^2 + y^2$.

Suppose there does exist such an analytic function. By the Cauchy-Riemann equations, $v_x = -u_y$ and $v_y = u_x$ so $v_x = -2y$ and $v_y = 2x$ and so $v_{xy} = -2$ and $v_{yx} = 2$. But these second order derivatives are constant, and therefore by Clairut's theorem, $v_{xy} = v_{yx}$ in contradiction.

Exercise 2.7:

Show that if $f = u + iv$ is differentiable at $z \in \mathbb{C}$ then u and v are differentiable at $(x, y) = z$ and satisfy the Cauchy-Riemann equations.

Notice that since f is differentiable, for its differentiation we can take any path of $h \to 0$ and get the same result. Specifically, we will take a look at what happens when $h \in \mathbb{R}$ and $h \in i\mathbb{R}$. So for $h \in \mathbb{R}$:

$$f'(z) = \lim_{\mathbb{R} \ni h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \frac{u(x+h, y) + iv(x+h, y) - u(x, y) - iv(x, y)}{h} = \lim_{h \to 0} \frac{u(x+h, y) - u(x, y)}{h} + i \lim_{h \to 0} \frac{v(x+h, y}{h}$$

And similarly for $ih \in i\mathbb{R}$:

$$f'(z) = \lim_{h \to 0} \frac{f(z+ih) - f(z)}{ih} = -i \lim_{h \to 0} \frac{u(x, y+h) + iv(x, y+h) - u(x, y) - iv(x, y)}{h} = -i\big(u_y(x, y) + iv_y(x, y)\big) = v_y(x, y) - iu_y(x,$$

And so we get that $u_x + iv_x = v_y - iu_y$ so $u_x(x, y) = v_y(x, y)$ and $v_x(x, y) = -u_y(x, y)$ as required.
Notice that since f is differentiable at z there exists α and β such that

$$f(z+h) = f(z) + hf'(z) + \alpha(h) + i\beta(h)$$

where $\frac{\alpha(h)}{h}, \frac{\beta(h)}{h} \longrightarrow 0$ as $h \to 0$. We want to show that there exists an ε such that

$$u(z) = u(z+h) + u_x(z)h_1 + u_y(z)h_2 + \varepsilon(h)$$

where $\frac{\varepsilon(h)}{\sqrt{h_1^2 + h_2^2}} \longrightarrow 0$ as $h_1, h_2 \to 0$. Notice that by differentiability of f and the Cauchy-Riemann equations, we can take the real part of the equation above and get:

$$u(x + h_1, y + h_2) = \mathrm{Re}\Big(u(x) + (h_1 + ih_2)\big(u_x(x, y) - iu_y(x, y)\big) + \alpha(h)\Big) = u(x) + u_x(x, y)h_1 + u_y(x, y)h_2 + \alpha(h_1, h_2)$$

So all we need to show is that $\frac{\alpha(h_1, h_2)}{\sqrt{h_1^2 + h_2^2}} = \frac{\alpha(h)}{|h|} \longrightarrow 0$. This is true since $\left|\frac{\alpha(h)}{|h|}\right| = \frac{|\alpha(h)|}{|h|}$, which must converge to 0 since $\frac{\alpha(h)}{h}$ does and convergence in $\mathbb{C}$ is convergence in modulus, which for that same reason implies $\frac{\alpha(h)}{h}$ converges to 0.
The proof is very similar for v.

Exercise 2.8:

(1) Show that $e^z = e^x\cos(y) + ie^x\sin(y)$ is analytic over all of $\mathbb{C}$ (entire).

(2) Prove that $e^{z_1+z_2} = e^{z_1}e^{z_2}$.

(1) Notice that $u(x,y) = e^x\cos(y)$ and $v(x,y) = e^x\sin(y)$ which are both differentiable as the product of elementary functions. And

$$u_x(x,y) = e^x\cos(y), \quad u_y(x,y) = -e^x\sin(y), \quad v_x(x,y) = e^x\sin(u), \quad v_y(x,y) = e^x\cos(y)$$

So we have that

$$u_x = v_y, \quad u_y = -v_x$$

So f satisfies the Cauchy-Riemann equations for every $z\in\mathbb{C}$ and u and v are differentiable for every $z\in\mathbb{C}$, so f is differentiable over all of $\mathbb{C}$ and is therefore entire. Furthermore, notice that

$$f'(z) = u_x(z) + iv_x(z) = u(z) + iv(z) = f(z)$$

(2) Suppose $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ so $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$ so:

$$e^{z_1+z_2} = e^{x_1+x_2}\big(\cos(y_1+y_2) + i\sin(y_1+y_2)\big) = e^{x_1}e^{x_2}\big(\cos(y_1) + i\sin(y_1)\big)\big(\cos(y_2) + i\sin(y_2)\big) = e^{z_1}\cdot e^{z_2}$$

as required.

Exercise 2.9:

Find all the solutions to:

(1) $e^z = 1$

(2) $e^z = i$

(3) $e^z = -3$

(4) $e^z = 1 + i$

Lemma:

$e^z = e^y$ if and only if $z = y + 2\pi ik$ for some $k\in\mathbb{Z}$.

Proof:

If $z = a+bi$ and $y = c+di$ then $e^z = e^a\big(\cos(b)+i\sin(b)\big)$ and $e^y = e^c\big(\cos(d)+i\sin(d)\big)$, and so in polar coordinates, $e^z = e^a\angle b$ and $e^y = e^c\angle d$, so $e^z = e^y$ if and only if $e^a = e^c$ and $b = d$ as angles, so $a = c$ by the injectivity of exponentials and $b = d + 2\pi k$ for some $k\in\mathbb{Z}$. Thus $z = a + bi = c + i(d + 2\pi k) = y + 2\pi ik$ as required. ∎

To solve this problem, we transform w into polar form $|w|\angle\theta$, and from that we know $w = |w|\cdot e^{i\theta}$ by definition of the complex exponential, and so this is equal to $e^{\log|w|+i\theta}$. So the set of solutions to $e^z = w$ is $\{\log|w| + i\theta + i2\pi k \mid k\in\mathbb{Z}\}$.

(1) Since $1 = e^0$ by our lemma above, $e^z = 1$ if and only if $z = 2\pi ik$ for any $k\in\mathbb{Z}$, ie $\{2\pi ik \mid k\in\mathbb{Z}\}$ is the set of solutions.

(2) Since $i = e^{\frac{\pi}{2}i}$ by our lemma above, $e^z = i$ if and only if $z\in\left\{\frac{\pi}{2}i + 2\pi ik \;\middle|\; k\in\mathbb{Z}\right\}$.

(3) Since $-3 = 3e^{\pi i} = e^{\log 3 + i\pi}$, the solutions are $\{\log 3 + i\pi(2k+1) \mid k\in\mathbb{Z}\}$.

(4) Since $1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$ so the solutions are $\left\{\frac{1}{2}\log 2 + i\pi\left(\frac{1}{4} + 2k\right) \;\middle|\; k\in\mathbb{Z}\right\}$.

Exercise 2.10:

Find the derivative of $\cos(z)$ for $z \in \mathbb{C}$.

Recall the definition of the complex cosine function:

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

Thus by linearity of the derivative and the chain rule (the derivative of $f(\alpha x)$ is $\alpha \cdot f'(\alpha x)$) we get that the complex cosine function is also entire (since the exponential is) and since $\left(e^z\right)' = e^z$:

$$\cos'(z) = \frac{ie^{iz} - ie^{-iz}}{2} = \frac{-e^{iz} + e^{-iz}}{2i} = -\sin(z)$$

So for every $z \in \mathbb{C}$, $\cos'(z) = -\sin(z)$ as we'd expect.

Exercise 2.11:

Show that

$$\sin(x+iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

where

$$\cosh(y) = \frac{e^y + e^{-y}}{2}, \quad \sinh(y) = \frac{e^y - e^{-y}}{2}$$

We know that

$$\sin(x+iy) = -\frac{i}{2}\left(e^{-y+ix} - e^{y-ix}\right) = -\frac{i}{2}\left(e^{-y}\operatorname{cis}(x) - e^{y}\operatorname{cis}(-x)\right) = -\frac{i}{2}\left(\cos(x)\left(e^{-y} - e^{y}\right) + i\sin(x)\left(e^{-y} + e^{y}\right)\right)$$

$$= \sin(x) \cdot \frac{e^y + e^{-y}}{2} + i\cos(x) \cdot \frac{e^y - e^{-y}}{2} = \sin(x) \cdot \cosh(y) + i\cos(x) \cdot \sinh(y)$$

as required