Introduction to Stochastic Processes

Assignment 5
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5.1 Exercise

Given a fair walk on $\mathbb{Z}$, compute $\mathbb{E}[\min\{T_0, T_N\} \mid S_0 = k]$ for all $k \in \mathbb{Z}$.

Let us define $\tau = \min\{T_0, T_N\}$ and $\mu(k) = \mathbb{E}[\tau \mid S_0 = k]$. Then $\mu(0) = \mu(N) = 0$. Using first step analysis,

$$\begin{aligned}\mu(k) = \mathbb{E}[\tau \mid S_1 = k+1] \cdot \mathbb{P}(S_1 = k+1 \mid S_0 = k) + \mathbb{E}[\tau \mid S_1 = k-1] \cdot \mathbb{P}(S_1 = k-1 \mid S_0 = k) \\ = \frac{1}{2}(1 + \mu(k+1)) + \frac{1}{2}(1 + \mu(k-1))\end{aligned}$$

and so we get the linear recurrence

$$\mu(k) = 1 + \frac{1}{2}\mu(k+1) + \frac{1}{2}\mu(k-1) \implies \mu(k) = 2\mu(k-1) - \mu(k-2) - 2$$

Let us define $\Delta(k) = \mu(k) - \mu(k-1)$ and so we have that $\Delta(k) - \Delta(k-1) = \mu(k) - 2\mu(k-1) + \mu(k-2) = -2$. This means that $\Delta(k)$ is an arithmetic sequence and so $\Delta(k) = c - 2k$. Now, $\sum_{k=1}^{N} \Delta(k) = \mu(N) - \mu(0) = 0$ and so $0 = \frac{N}{2}(\Delta(1) + \Delta(N)) = \frac{N}{2}(c - 2 + c - 2N)$ thus $c = N + 1$, meaning $\Delta(k) = N + 1 - 2k$. And in general

$$\mu(n) = \mu(n) - \mu(0) = \sum_{k=1}^{n} \Delta(k) = \frac{n}{2}(\Delta(1) + \Delta(n)) = \frac{n}{2}(N - 1 + N + 1 - 2n) = \frac{n}{2}(2N - 2n) = n(N-n)$$

And so

$$\mathbb{E}[\min\{T_0, T_N\} \mid S_0 = n] = n(N-n)$$

5.2 Exercise

Suppose $p \neq \frac{1}{2}$, then let S_n be a p-weighted walk on $\mathbb{Z}$, meaning $P(i \to i+1) = p$ and $P(i \to i-1) = 1-p$ for every $i \in \mathbb{Z}$. Find a formula for $p(k) = \mathbb{P}(T_N < T_0 \mid S_0 = k)$. Show that if $p < \frac{1}{2}$ then there exists an $\alpha \in (0,1)$ and a $c > 0$ such that $p(k) < c\alpha^{N-k}$.

Using first step analysis,

$$\begin{aligned}p(k) = \mathbb{P}(T_N < T_0 \mid S_1 = k+1) \cdot \mathbb{P}(S_1 = k+1 \mid S_0 = k) + \mathbb{P}(T_N < T_0 \mid S_1 = k-1) \cdot \mathbb{P}(S_1 = k-1 \mid S_0 = k) \\ = p \cdot p(k+1) + (1-p) \cdot p(k-1)\end{aligned}$$

and so we get the recurrence

$$p(k) = \frac{1}{p}p(k-1) + \frac{p-1}{p}p(k-2)$$

The characteristic polynomial of this recurrence is $x^2 - \frac{1}{p}x + \frac{1-p}{p}$ which has the same roots as $px^2 - x + (1-p)$, the roots being

$$\frac{1 \pm \sqrt{1 - 4p(1-p)}}{2p} = \frac{1 \pm \sqrt{4p^2 - 4p + 1}}{2p} = \frac{1 \pm (2p-1)}{2p}$$

so the roots are 1 and $\frac{1}{p} - 1$, thus the general solution to this recurrence is

$$p(k) = c_1 + c_2\left(\frac{1}{p} - 1\right)^k$$

The initial conditions are $p(0) = 0$ and $p(N) = 1$, so

$$0 = c_1 + c_2, \qquad 1 = c_1 + c_2\left(\frac{1}{p} - 1\right)^N$$

solving this gives

$$p(k) = a\left(\frac{1}{p} - 1\right)^k - a, \qquad a = \left(\left(\frac{1}{p} - 1\right)^N - 1\right)^{-1}$$

Now if $p < \frac{1}{2}$ then $\frac{1}{p} - 1 > 1$ and so if we let $\alpha = \left(\frac{1}{p} - 1\right)^{-1}$ then $\alpha < 1$ and we get that

$$p(k) = \frac{\alpha^{-k}}{\alpha^{-N} - 1} - \frac{1}{\alpha^{-N} - 1} = \frac{\alpha^{N-k}}{1 - \alpha^N} - \frac{\alpha^N}{1 - \alpha^N}$$

So if we set $c = \frac{1}{1-\alpha^N} > 0$ then we have that $p(k) < c\alpha^{N-k}$ as required.

5.3 Exercise

Show that for every $d > 3$, the fair walk on $\mathbb{Z}^d$ is transient.

The transition probabilities for a fair walk on $\mathbb{Z}^d$ is $P(v \to v \pm e_i) = \frac{1}{2d}$ (where e_i is the standard vector). And since all states in the walk are connected, it is sufficient to show that 0 is transient. We will progress with a proof similar to showing that 0 is recurrent for $d = 1$. Notice that if $X_0 = 0$ then we can only reach 0 again on even steps, ie. $X_{2n} = 0$. And in order for us to reach 0 again we must take t_i steps in the direction of e_i and t_i steps in the direction of $-e_i$. And so the total number of steps is $2t_1 + \cdots + 2t_d = 2n$, thus

$$\mathbb{P}(X_{2n} = 0) = \sum_{t_1+\cdots+t_d=n} \binom{2n}{t_1, t_1, \ldots, t_d, t_d} \frac{1}{(2d)^{2n}}$$

this is since of the $2n$ steps we must choose t_1 steps in e_1, t_1 steps in $-e_1$, etc. and the number of ways to choose these steps is $\binom{2n}{t_1,t_1,\ldots,t_d,t_d}$. And for each choice of $t_1, \ldots, t_d$ the probability of taking this specific path is $\frac{1}{2d} \cdots \frac{1}{2d} = \frac{1}{(2d)^{2n}}$. Now,

$$\begin{aligned}\binom{2n}{t_1, t_1, \ldots, t_d, t_d} = \frac{(2n)!}{(t_1! \cdots t_d!)^2} = \binom{n}{t_1, \ldots, t_d} \cdot \frac{(n+1)\cdots(2n)}{t_1! \cdots t_d!} &= \binom{n}{t_1, \ldots, t_d}^2 \cdot \frac{(n+1)\cdots(2n)}{n!} \\ &= \binom{n}{t_1, \ldots, t_d}^2 \cdot \binom{2n}{n}\end{aligned}$$

Thus

$$\mathbb{P}(X_{2n} = 0) = \binom{2n}{n} \cdot \frac{1}{2^{2n}} \cdot \frac{1}{d^n} \sum_{t_1+\cdots+t_d=n} \binom{n}{t_1, \ldots, t_d}^2 \cdot \frac{1}{d^n}$$

Now if we let $m = \frac{n}{d}$ (n not being divisible by d is not really a concern since we will use Stirling's approximation which holds for non-integers), then we have that $\binom{n}{t_1,\ldots,t_d} \leq \binom{n}{m,\ldots,m}$ and so

$$\mathbb{P}(X_{2n} = 0) \leq \binom{2n}{n} \cdot \frac{1}{2^{2n}} \cdot \frac{1}{d^n} \cdot \binom{n}{m, \ldots, m} \cdot \sum_{t_1+\cdots+t_d=n} \binom{n}{t_1, \ldots, t_d} \cdot \frac{1}{d^n}$$

By the multinomial theorem, we have that

$$\sum_{t_1+\cdots+t_d=n} \binom{n}{t_1, \ldots, t_d} \frac{1}{d^n} = \left(\sum_{i=1}^{d} \frac{1}{d}\right)^n = 1$$

And by Stirling's approximation

$$\binom{2n}{n} \sim \frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}}{2\pi n\left(\frac{n}{e}\right)^{2n}} = \frac{2^{2n}}{\sqrt{\pi}\sqrt{n}}$$

$$\binom{n}{m, \ldots, m} \sim \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{\sqrt{2\pi m}^d\left(\frac{m}{e}\right)^{md}} = \frac{\sqrt{2\pi n}}{\sqrt{\frac{2\pi n}{d}}^d} \cdot d^n$$

And so

$$\mathbb{P}(X_{2n} = 0) \leq c\frac{1}{n^{d/2}}$$

And so

$$N_0(0) = \sum_{n=1}^{\infty} \chi\{X_{2n} = 0\} \implies \mathbb{E}[N_0(0)] = \sum_{n=1}^{\infty} \mathbb{P}(X_{2n} = 0) \leq c\sum_{n=1}^{\infty} \frac{1}{n^{d/2}}$$

And this converges if $d/2 > 1$, meaning if $d > 2$. So for $d \geq 3$ then $\mathbb{E}[N_0(0)] < \infty$, meaning 0 is transient and therefore the whole walk is transient, as required.