Introduction to Stochastic Processes

Assignment 8
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8.1 Exercise

Assuming $B^1(t)$ and $B^2(t)$ are two independent Brownian motions, show that the following are also Brownian motion:

(1) $X(t) = B^1(A+t) - B^1(A)$ for $A > 0$.

(2) $X(t) = \alpha B^1(t) + \sqrt{1-\alpha^2}B^2(t)$ for $0 < \alpha < 1$.

(1) $X(0) = 0$ and it is also trivially almost surely continuous. And

$$X(t+h) - X(t) = B^1(A+t+h) - B^1(A+t) \sim \mathcal{N}(0,h)$$

And in general $X(t_n) - X(t_{n-1}) = B(t_n) - B(t_{n-1})$ so differences are independent.

(2) We will show in general that if $\sum_{j=1}^k \alpha_j^2 = 1$ and $B^1,\dots,B^k$ are independent, then $X(t) = \sum_{j=1}^k \alpha_j B^j$ is Brownian motion. $X(0) \overset{as}{=} 0$ and $X(t)$ is also almost surely continuous as the sum of almost surely continuous functions.

$$X(t+h) - X(t) = \sum_{j=1}^k \alpha_j\big(B^j(t+h) - B^j(t)\big) \sim \sum_{j=1}^k \alpha^j \mathcal{N}(0,h) = \mathcal{N}\left(0, h\sum_{j=1}^k \alpha_j^2\right) = \mathcal{N}(0,h)$$

And since

$$\begin{aligned}\mathrm{Cov}(X(a)-X(b), X(c)-X(d)) &= \sum_{j=1}^k\sum_{i=1}^k \alpha_j\alpha_i \,\mathrm{Cov}\big(B^j(a)-B^j(b), B^i(c)-B^i(d)\big) \\ &= \sum_{j=1}^k \alpha_j^2 \,\mathrm{Cov}\big(B^j(a)-B^j(b), B^j(c)-B^j(d)\big) = 0\end{aligned}$$

Since for $i \neq j$, B^j and B^i are independent. Thus $X(t_n)-X(t_{n-1}),\dots,X(t_1)-X(t_0)$ are independent (since if coefficients of a Gaussian vector are pairwise uncorrelated, the coefficients are independent).

8.2 Exercise

Set $0 \le a < b$, show that Brownian motion is almost surely not monotonic on $[a,b]$. Then show that Brownian motion is almost surely not monotonic on any $[a,b]$.

Set a sequence $t_1 < \cdots < t_n < \cdots$ where $a \le t_i \le b$. Then $\mathbb{P}(B(t_i) - B(t_{i-1}) > 0) = \mathbb{P}(\mathcal{N}(0, t_i - t_{i-1}) > 0) = \frac{1}{2}$ and similarly $\mathbb{P}(B(t_i) - B(t_{i-1}) < 0) = \frac{1}{2}$. And we know that $\{B(t_i) - B(t_{i-1})\}$ is independent, and since

$$\sum_{i=1}^\infty \mathbb{P}(B(t_i) - B(t_{i-1}) > 0) = \sum_{i=1}^\infty \mathbb{P}(B(t_i) - B(t_{i-1}) < 0) = \infty$$

Thus by Borel-Cantelli, $\mathbb{P}(B(t_i) > B(t_{i-1}) \text{ i.o.}) = \mathbb{P}(B(t_i) < B(t_{i-1}) \text{ i.o.}) = 1$. So almost surely, there is a subsequence of t_n on which B is strictly increasing and another on which B is stricly decreasing. So $B(t)$ is almost surely not monotonic on t_n and therefore almost surely not monotonic on $[a,b]$.

Now, $B(t)$ is monotonic on some $[a,b]$ if and only if it is monotonic on some $[p,q]$ for $p,q \in \mathbb{Q}$ by the density of the rationals. Thus

$$\begin{aligned}\mathbb{P}(B(t) \text{ is monotonic on some } [a,b]) &= \mathbb{P}\left(\bigcup_{p<q\in\mathbb{Q}} B(t) \text{ is monotonic on } [p,q]\right) \\ &\le \sum_{p<q\in\mathbb{Q}} \mathbb{P}(B(t) \text{ is monotonic on } [p,q]) = 0\end{aligned}$$

The last equality is since $B(t)$ is almost surely not monotonic on any set $[a,b]$. Thus $B(t)$ is almost surely not monotonic on any $[a,b]$.

8.3 Exercise

Suppose f is a continuous function on $[0,1]$ such that $f(0)=0$. Let $\varepsilon>0$, show that

$$\mathbb{P}\left(\sup_{0\le t\le 1}|B(t)-f(t)|<\varepsilon\right)>0$$

Using Lèvy's construction, we have that $G_n \rightrightarrows B$ in $[0,1]$ where G_n are functions which are linearly interpolated through points in D_n. Since G_n converges to B uniformly, there exists an N such that for every $n\ge N$, $\|B-G_n\|_\infty<\frac{\varepsilon}{3}$. And since $[0,1]$ is compact and f is continuous, it is uniformly continuous. So let us define $f_n(d)=f(d)$ for $d\in D_n$ and interpolate f_n linearly through the points in D_n. Since f is uniformly continuous on $[0,1]$ there exists a $\delta>0$ such that if $|x-y|<\delta$ then $|f(x)-f(y)|<\frac{\varepsilon}{6}$. There must exist an n such that the difference in points in D_n ($=\frac{1}{2^n}$) is less than δ, and so if $x\in[d_1,d_2]$ for $d_1,d_2\in D_n$,

$$|f(x)-f_n(x)|\le|f(x)-f(d_1)|+|f(d_1)-f_n(d_1)|+|f_n(d_1)-f_n(x)|<\frac{\varepsilon}{3}$$

The final inequality is due to $f(d_1)=f_n(d_1)$ and $|f_n(d_1)-f_n(x)|\le|f_n(d_1)-f_n(d_2)|$ since f_n is linear in $[d_1,d_2]$. So there exists an N such that for every $n\ge N$ and $0\le t\le 1$, $|B(t)-G_n(t)|,|f(t)-f_n(t)|<\frac{\varepsilon}{3}$. Now,

$$|B(t)-f(t)|\le|B(t)-G_n(t)|+|G_n(t)-f_n(t)|+|f_n(t)-f(t)|<\frac{2\varepsilon}{3}+|G_n(t)-f_n(t)|$$

So we must show there is a non-zero probability that $|G_n(t)-f_n(t)|<\frac{\varepsilon}{3}$. Since f_n and G_n are both linearly interpolated through points on D_n, their maximum distance will be taken on D_n. For every $d\in D_n\setminus D_{n-1}$, by definition

$$G_n(t)=\frac{G_{n-1}(d-2^{-n})+G_{n-1}(d+2^{-n})}{2}+\frac{Z_d}{\sqrt{2^{n+1}}}$$

And so

$$\mathbb{P}\Big(|G_n(t)-f_n(t)|<\frac{\varepsilon}{3}\Big)=\mathbb{P}\Big((\forall d\in D_n)\,|G_n(d)-f_n(d)|<\frac{\varepsilon}{3}\Big)$$

Now since the values of $G_n(d)$ are determined by a finite number of independent normal distributions (which have full range), this probability must be nonzero.