Mathematical Logic

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When we have a well-formed formula φ and we write $\varphi(x_1, \dots, x_n)$ this means that $x_1, \dots, x_n$ occur in φ as free occurrences.

Definition 5.0.1:

Given a term t, it is **collision-free** for the variable x in a formula φ, if no free occurrences of x in φ lies within the scope of any quantifier $\forall y$ where y is a variable of t.

What this means is that if we were to substitute x with t, we may have changed the meaning of the formula as we have swapped an independent occurrence of x in φ to some term dependent on y. So for example if $t = f(x, y)$ and we have $\varphi = \exists y P(x, y)$, then t is not free for x since the occurrence of x in φ is free, and within the domain of a quantifier on $y \in \operatorname{var} t$. But t is collision-free for y since every occurrence of y is bound. Equivalently if the substitution of a free occurrence of a variable x with t results in a new bounded occurrence of some variable y, then t is not free of x.

Note then that a variable-free term (a constant, or a function of constants) is collision-free of every variable in any formula. If a variable x is bound in φ (all of its occurrences are bound), then t is free of x in φ. And so term t is collision-free for any variable in φ if none of the variables of t are bound in φ. If φ contains no free occurrences of variables in t, then t is collision-free with every variable.

Definition 5.0.2:

Let $\mathcal{L}$ be a first order language, an **interpretation** of $\mathcal{L}$, $\mathcal{M}$, consists of

- A non-empty set $\mathcal{D}$ the **domain of interpretation**,
- For every predicate letter A_j^n of $\mathcal{L}$, an n-ary relation $(A_j^n)^{\mathcal{M}} \subseteq \mathcal{D}^n$,
- For every function letter f_j^n of $\mathcal{L}$, an n-ary function $(f_j^n)^{\mathcal{M}} \colon \mathcal{D}^n \longrightarrow \mathcal{D}$,
- For every constant letter c_j, some constant $(c_j)^{\mathcal{M}} \in \mathcal{D}$.

Definition 5.0.3:

A formula φ which has no free variables is a **sentence** or **closed formula**.

Definition 5.0.4:

Given an interpretation $\mathcal{M}$ and a valuation function $w \colon \mathrm{Var} \longrightarrow \mathcal{D}$ (or a sequence s in $\mathcal{D}^{\mathbb{N}}$ since Var is countable), we now define what the valuation of a term t is, t^w, recursively:

- If t is a variable $t = x$ then $t^w = x^w = w(x)$.
- If t is a constant $t = c$ then $t^w = c^{\mathcal{M}}$.
- Otherwise $t = f(t_1, \dots, t_n)$ for terms t_i so $t^w = f^{\mathcal{M}}(t_1^w, \dots, t_n^w)$.

Given an atomic formula φ, we say that $\mathcal{M}$ **satisfies** φ if:

- If $\varphi = A(t_1, \dots, t_n)$ for terms t_i, then $\mathcal{M}$ satisfies φ if $A^{\mathcal{M}}(t_1^w, \dots, t_n^w)$.
- If $\varphi = t = s$ for terms t and s, then $\mathcal{M}$ satisfies φ if $t^{\mathcal{M}} = s^{\mathcal{M}}$.

And given a general formula φ

- If $\varphi = \neg\alpha$ then $\mathcal{M}$ satisfies φ if it does not satisfy α.
- If $\varphi = \alpha \wedge \beta$ then $\mathcal{M}$ satisfies φ if $\mathcal{M}$ satisfies both α and β.
- If $\varphi = \forall x\alpha$ for variable x, $\mathcal{M}$ satisfies φ if for every $a \in \mathcal{D}$ when we define $\mathcal{M}_x^a$ to have the valuation function w' where
$$w'(v) = \begin{cases} w(v) & v \neq x \\ a & v = x \end{cases}$$
$\mathcal{M}_x^a$ satisfies α. That is $\mathcal{M}$ satisfies $\forall x\alpha$, if when we swap the value of $w(x)$ with any value in $\mathcal{D}$, α is satisfied.

w satisfies a formula φ is denoted by $(\mathcal{M}, w) \vDash \varphi$. And $\mathcal{M}$ satisfies a formula φ (alternatively φ is true for $\mathcal{M}$) if for every valuation function w, $(\mathcal{M}, w) \vDash \varphi$, this is denote $\mathcal{M} \vDash \varphi$. A formula φ is false for $\mathcal{M}$ if there is no valuation w which satisfies φ.

An interpretation $\mathcal{M}$ **models** a set Γ of formulas if every formula of Γ is true for $\mathcal{M}$.

Definition 5.0.5:

We now define what a **first order theory** is. Like any formal theory it has

(1) Axioms: these are split into **logical** and **proper** axioms. Logical axioms include all the axioms of predicate calculus, as well as
$$\big(\forall x\varphi(x)\big) \to \varphi(t)$$
Meaning that φ is a formula which contains x as a free variable, and it is true for every x, then it is true if we replace x with any term t. And the last logical axiom is
$$\big(\forall x(\varphi \to \psi)\big) \to \big(\varphi \to \forall x\psi\big)$$
if φ contains no free occurrences of x.

The second class of proper axioms are specific to each first order theory.

(2) Rules of inference:

(i) Modus ponens: if φ and $\varphi \to \psi$ then ψ.

(ii) Generalization: if φ then $\forall x\varphi$ for any variable x.

An interpretation models a first order theory if it satisfies all the axioms (logical and proper), and the rules of inference.

Example 5.0.6:

We define a **partial order theory**, which just has one predicate letter $A(x, y)$ which will be written as $x < y$. The proper axioms are:

(1) $(\forall x)\neg(x < x)$

(2) $(\forall x, y, z)(x < y \wedge y < z \to x < z)$

Any model of this theory is called a **partial order structure**.

Example 5.0.7:

The **group theory** has a binary predicate symbol $=$, a binary operation symbol $\cdot$, and a constant e. We take the predicate symbol $=$ instead of using the first order symbol $\boldsymbol{=}$, so we need extra axioms regarding $=$ as well as the normal axioms of group theory

(1) $(\forall x, y, z)\big((xy)z = x(yz)\big)$

(2) $(\forall x)(ex = xe = x)$

(3) $(\forall x)(\exists y)(xy = yx = e)$

(4) $(\forall x)(x = x)$

(5) $(\forall x)(\forall y)(x = y \to y = x)$

(6) $(\forall x, y, z)(x = y \land y = z \to x = z)$

(7) $(\forall x, y, z)\big(x = y \to (xz = yz \land zx = zy)\big)$

Definition 5.0.8:

If φ is a formula in a first order language, φ is **logically valid** if φ is true for any interpretation. φ is **satisfiable** if there exists an interpretation where φ is true. And φ is **contradictory** if it is false under any interpretation.

A set of formulas Γ is **satisfiable** if it has a model.