Topology Recitation

Recitation 1, Sunday March 19, 2022
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Definition 1.0.1:

If M is a metric space, then the **diameter** of a set $A \subseteq M$ is:

$$\operatorname{diam}(A) = \sup\{\rho(x,y) \mid x,y \in A\}$$

and A is **bounded** if $\operatorname{diam}(A) < \infty$.

Proposition 1.0.2:

The union of two bounded sets is itself bounded.

Proof:

Suppose A and B are bounded, let $a \in A$ and $b \in B$, then $M = \operatorname{diam}(A) + \operatorname{diam}(B) + \rho(a,b)$. We claim $\operatorname{diam}(A \cup B) \leq M$. Let $x,y \in A \cup B$. If both x and y are in the same set then $\rho(x,y) \leq \operatorname{diam}(A)$ or $\operatorname{diam}(B)$ depending on which set they're in, and necessarily $\rho(x,y) \leq \operatorname{diam}(A) + \operatorname{diam}(B) \leq M$. Otherwise suppose $x \in A$ and $y \in B$ then $\rho(x,y) \leq \rho(x,a) + \rho(a,b) + \rho(b,y) \leq \operatorname{diam}(A) + \rho(a,b) + \operatorname{diam}(B) = M$ as required.

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Definition 1.0.3:

An **ultrametric space** is a set M equipped with an **ultrametric**, a function $\rho\colon M \times M \longrightarrow \mathbb{R}_{\geq 0}$ where

(1) $\rho(x,y) = 0$ if and only if $x = y$

(2) $\rho(x,y) = \rho(y,x)$

(3) $\rho(x,y) \leq \max\{\rho(x,z), \rho(z,y)\}$

An ultrametric is trivially also a metric.

Let X be the set of infinite sequences above $\{0,\dots,n-1\}$, ie $X = \{0,\dots,n-1\}^{\mathbb{N}}$. If $w \in X$ is a sequence we understand that w_i is the ith index of w. Then for $w,v \in X$ we define:

$$k(w,v) = \begin{cases} \infty & w = v \\ \min\{i \in \mathbb{N} \mid w_i \neq v_i\} & \end{cases}$$

and we define:

$$\rho(w,v) = \begin{cases} 0 & u = v \\ \frac{1}{p^{k(w,v)}} & u \neq v \end{cases}$$

for $1 < p$. We claim that ρ is an ultrametric. It is obvious that it is both nonnegative and symmetric, as well as that it is 0 only when $w = v$. So we must prove the triangle inequality. Suppose $w,u,v \in X$ then if any one of them are equal, this is trivial. Otherwise we must show that

$$\rho(w,v) \leq \max\{\rho(w,u), \rho(u,v)\} \iff \frac{1}{p^{k(w,v)}} \leq \max\left\{\frac{1}{p^{k(w,u)}}, \frac{1}{p^{k(u,v)}}\right\}$$

This is equivalent to

$$p^{-k(w,v)} \leq \max\left\{p^{-k(w,u)}, p^{-k(u,v)}\right\}$$

and since $p > 1$ this is only if

$$-k(w,v) \leq \max\{-k(w,u), -k(u,v)\} \iff k(w,v) \geq \min\{k(w,u), k(u,v)\}$$

Suppose for the sake of a contradiction that this is false, ie $k(w,v) < \min\{k(w,u), k(u,v)\}$. Let $i = k(w,v)$ and so $w_i \neq v_i$, and we furthermore know that $i < k(w,u)$, so $u_i = w_i$ and $i < k(u,v)$ so $u_i = v_i$ so $w_i = v_i$ in contradiction to $w_i \neq v_i$. So ρ is in fact a metric.

Definition 1.0.4:

Let p be prime, then we define the metric d_p over the integers $\mathbb{Z}$ by:

$$d_p(x,y) = \begin{cases} 0 & x = y \\ \frac{1}{p^{k(x,y)}} & x \neq y \end{cases}$$

where

$$k(x,y) = \max\{n \in \mathbb{N}_{\geq 0} \mid p^n \mid (x-y)\}$$

This is called the **p-adic** metric.

We claim that d_p is an ultrametric:

$$d_p\, x, y \leq \max\{d_p\, x, z, d_p\, z, y\} \iff k(x,y) \geq \min\{k(x,z), k(z,y)\}$$

Suppose $i = \min\{k(x,z), k(z,y)\}$ then $p^i \mid (x-z), (z-y)$ and so $p^i \mid \big((x-z)+(z-y)\big) \implies p^i \mid (x-y) \implies i \leq k(x,y)$ as required.
There is a connection between the two metric spaces we defined above.

Definition 1.0.5:

A **iosmetry** between two metric spaces (M,ρ) and (N,σ) is a function $f\colon M \longrightarrow N$ such that for every $x, y \in M$:

$$\rho(x,y) = \sigma(f(x), f(y))$$

Every isometry is an injection: if $f(x) = f(y)$ then $\rho(x,y) = \sigma(f(x),f(y)) = 0$ so $x = y$.
Recall that every $z \in Z$ has a unique representation in base p, that is there is a unique sequence $\{a_i\}_{i=0}^{k} \in \{0,\dots,p-1\}$ such that:

$$z = \pm\sum_{i=0}^{k} z_i p^i$$

So if we let C_p be the set of all sequences above $\{0,\dots,p-1\}$ with the metric defined above, we define an isometry:

$$\varphi\colon \mathbb{Z} \longrightarrow C_p$$

By

$$\varphi(z) = \begin{cases} \{z_0, z_1, \dots, a_k, 0, \dots\} & z \geq 0 \\ \{z_0, \dots, z_k, (p-1), \dots\} & z < 0 \end{cases}$$

We claim that φ is indeed an isometry. So we must prove that:

$$d_p(z,w) = p^{-k(\{z_i\},\{w_i\})} = \rho(\{z_i\},\{w_i\})$$

where k is the function defined on C_p. Let $t = k(\{z_i\},\{w_i\})$, so t is the first index where $z_i \neq w_i$, that is the first index where $z_i - w_i \neq 0$. So we must show that $p^t \mid (z-w)$ and p^{t+1} does not. We know that p^t divides $z-w$ since $z - w = (z_i - w_i)p^t + \dots$ so p^t divides $z-w$, and since $z_i - w_i \neq 0$ p^{t+1} does not. So we have shown that $k(z,w)$ in $\mathbb{Z}$ is equal to $k(\{z_i\},\{w_i\}) = k(f(z), f(w))$ in C_p, and so $d_p(z,w) = \rho(f(z), f(w))$ as required.

Proposition 1.0.6:

If M is an ultrametric space, then if $y \in B_r(x)$ then $B_r(y) = B_r(x)$, that is every point in a ball is its center.

Proof:

We will show that $B_r(y) \subseteq B_r(x)$. Suppose $z \in B_r(y)$, so $\rho(z,x) \leq \max\{\rho(z,y), \rho(x,y)\} < \max\{r,r\} = r$. So $z \in B_r(x)$. Then by symmetry (since $x \in B_r(y)$), $B_r(x) \subseteq B_r(y)$.

$\blacksquare$

Notice that p^n converges to 0 in the p-adics: $d_p\, p^n, 0 = p^{-n}$ which converges to 0 (in $\mathbb{R}$), so p^n converges to 0 in the p-adics.