Complex Functions

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Proposition 7.1:

The image of a non-constant analytic function is dense in $\mathbb{C}$.

Proof:

Suppose that $f(\mathbb{C})$ is not dense in $\mathbb{C}$, meaning that there is a $w \in \mathbb{C}$ and $r > 0$ such that $D_r(w) \cap f(\mathbb{C}) = \varnothing$. Let

$$g(z) = \frac{1}{f(z) - w}$$

this is defined on all of $\mathbb{C}$ as $f(z) \neq w$ so g is entire. Since $|f(z) - w| \geq r$ we have that

$$|g| \leq \frac{1}{r}$$

so by Liouville, this means g is constant and therefore f is constant. ■

Theorem 7.2:

If f is an entire function such that $\lim_{z\to\infty} f(z) = \infty$ then f is a polynomial.

$\lim_{z\to\infty} f(z) = \infty$ means that for any $M > 0$ there exists an $r > 0$ such that when $|z| > r$, $|f(z)| > M$.

Proof:

If there is a $z_0 \in \mathbb{C}$ where $f(z_0) = 0$ then take n such that $f^{(k)}(z_0) = 0$ for $k \leq n$ but $f^{(n+1)}(z_0) \neq 0$. This exists since otherwise, by the Taylor polynomial, f would be 0 (and so its limit would not be infinity). Thus

$$\frac{f(z)}{(z - z_0)^n} = \frac{1}{(z - z_0)^n}\left(c_n(z - z_0)^n + c_{n+1}(z - z_0)^{n+1} + \cdots\right)$$

where c_k are the Taylor coefficients, so $c_n \neq 0$. This does not equal to 0 at z_0 (it is equal to c_n). Since the limit of f is infinity, there exists an $R > 0$ such that for every $|z| \geq R$, $|f(z)| > 1$, so the zeros are contained within $D_R(0)$.
f is non-constant, so it has a finite number of roots. This is because if it had an infinite number of roots, then the roots form a sequence in $D_R(0)$ and since $\bar{D}_R(0)$ is compact, there is a convergent subsequence of roots. So we have a sequence of α_n where α_n are all distinct and the sequence is convergent and $f(\alpha_n) = 0$, which we showed means $f \equiv 0$ which is a contradiction.
So let $\alpha_1, \dots, \alpha_n$ be f's roots. Let the order of α_k be n_k (meaning $n_k + 1$ is the first where $f^{(n_k+1)}(\alpha_k) = 0$). Then we showed previously that we can define an entire function g such that for $z \neq \alpha_1, \dots, \alpha_n$

$$g(z) = \frac{f(z)}{(z - \alpha_1)^{n_1} \cdots (z - \alpha_n)^{n_n}}$$

And g has no roots. Let $h(z) = \frac{1}{g(z)}$.
We will show that there exist $A, B \geq 0$ such that

$$|h(z)| \leq A + B|z|^{n_1 + \cdots + n_n}$$

so h is a polynomial, but h does not have any roots, so h must be a constant polynomial, $h(z) = c$. Thus

$$f(z) = \frac{1}{c}(z - \alpha_1)^{n_1} \cdots (z - \alpha_n)^{n_n}$$

To show the existence of A and B notice that

$$|h(z)| = \left|\frac{(z - \alpha_1)^{n_1} \cdots (z - \alpha_n)^{n_n}}{f(z)}\right|$$

For $|z| \geq R$ (and we can assume $R \geq 1$), this is less than

$$< |(z-\alpha_1)^{n_1} \cdots (z-\alpha_n)^{n_n}| = \left| z^d + \sum_{k=0}^{d-1} a_k z^k \right| \leq |z|^d + (d-1)C|z|^{d-1}$$

for $d = n_1 + \cdots + n_n$ and $C = \max\{|a_0|, \ldots, |a_{d-1}|\}$, so

$$|h(z)| \leq ((d-1)C+1)|z|^d$$

So let $B = (d-1)C+1$, and for $|z| < R$, $h(z)$ is bound by some A, so all in all we have

$$|h(z)| \leq A + B|z|^d$$

as required. ■

Theorem 7.3 (Mean Value Theorem):

If f is analytic in D and $a \in D$, then

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta})\, d\theta$$

for every $r > 0$ where $D_r(a) \subseteq D$.

Proof:

By Cauchy we know

$$f(a) = \frac{1}{2\pi i} \int_{C_r(a)} \frac{f(z)}{z-a}\, dz$$

we parameterize $C_r(a)$ by $\theta \in [0, 2\pi) \mapsto a + re^{i\theta}$ and so this becomes

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}} rie^{i\theta}\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(a+re^{i\theta})\, d\theta$$

as required. ■

Definition 7.4:

Let f be a complex function and $z_0 \in \mathbb{C}$. If there exists a neighborhood $\mathcal{U}$ of z_0 such that $|f(z_0)| \geq |f(z)|$ for every $z \in \mathcal{U}$, then z is a **local maximum**. Similarly we define **local minimums**.

Theorem 7.5 (Maximal Modulus Principle):

If f is an analytic function in a domain D, then f has no local maxima in D.

This means that the local maxima must occur on the boundary of D.

Proof:

Let $z \in D$ and $r > 0$ such that $D_r(z) \subseteq D$. By the **Mean Value Theorem**

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta})\, d\theta$$

so

$$|f(z)| \leq \frac{1}{2\pi} \max\left|f(z+re^{i\theta})\right| d\theta = \max\left|f(z+re^{i\theta})\right|$$

The maximum exists since $C_r(z)$ is compact. Suppose the maximum is induced by w. If we assume that z is a local maxima, then for $r > 0$ small enough $|f(z)| = |f(w)|$.

Thus if u is on $C_r(z)$ then $|f(u)| \leq |f(z)|$ but the integral from 0 to 2π of both of these is equal which means that $|f(u)| = |f(z)|$ for every u on $C_r(z)$. So $|f|$ is constant on every $C_r(z)$ and therefore on $D_r(z)$, which means that f is constant on $D_r(z)$, which means f is constant on D (since the derivatives of f at z are all zero, so the taylor expansion is constant). This is a contradiction. ∎

Theorem 7.6 (Minimal Modulus Principle):

Suppose f is a non-constant analytic function in a domain D. Then z is a local minimum if and only if $f(z) = 0$.

Proof:

If $f(z) = 0$ then it is obvious that z is a local minimum. If $f(z) \neq 0$ then there exists an $r > 0$ such that for $w \in D_r(z) \subseteq D$, $f(w) \neq 0$ by continuity. So $\frac{1}{f}$ is defined in $D_r(z)$, is analytic, and is non-constant, so it therefore has no local maxima. This means that z is not a local minimum of f, since if it were, it would be a local maximum of $\frac{1}{f}$. ∎

Theorem 7.7 (Open Mapping Theorem):

A non-constant analytic function in a domain is an open mapping (maps open sets to open sets).

Proof:

Let $\alpha \in D$, we can assume $f(\alpha) = 0$, by simply defining a new function $\tilde{f}(z) = f(z) - f(\alpha)$ which is a shift of f and therefore open if and only if f is open. There must be a neighborhood of α not containing any other roots of f, as otherwise we could take a sequence of roots $z_n \to \alpha$ which means that $f \equiv 0$ which is a contradiction. So suppose for $r > 0$, $\bar{D}_r(\alpha)$ has no other roots of f.
Let $\varepsilon = \frac{1}{2} \min_{z \in C_r(\alpha)} |f(z)|$. $\varepsilon > 0$ since $C_r(\alpha)$ is compact so its minimum exists, and is in $\bar{D}_r(\alpha)$. We will show that $D_\varepsilon(0) \subseteq f(D_r(\alpha))$.
Let $w \in D_\varepsilon(0)$ and $z \in C_r(\alpha)$ then

$$|f(z) - w| \geq |f(z)| - |w| \geq 2\varepsilon - \varepsilon = \varepsilon$$

Let us define $h(z) = f(z) - w$, which is analytic. Notice that $|f(\alpha) - w| = |w| < \varepsilon$. We know that $|h|$ must have a minimum in $\bar{D}_r(\alpha)$, and since $|h(\alpha)| < |h(z)|$ for $z \in C_r(\alpha)$, this minimum must be in $D_r(\alpha)$. This means that h has a local minimum in $D_r(\alpha)$, so there is a point where $h(z) = 0$ so $f(z) = w$ for $z \in D_r(\alpha)$, meaning $w \in f(D_r(\alpha))$ as required.

Lemma 7.8 (Schwarz Lemma):

Suppose f is analytic on $D_1(0)$ such that $f(0) = 0$ and $|f(z)| \leq 1$. Then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. If there exists a non-zero z_0 where $|f(z_0)| = |z_0|$ or $|f'(0)| = 1$ then $f(z) = re^{i\theta}z$ (f is a rotation).

Proof:

Let us define

$$g(z) = \begin{cases} \frac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases}$$

which is analytic. Take $0 < r < 1$ and $z \in C_r(0)$, so

$$|g(z)| = \left| \frac{f(z)}{z} \right| \leq \frac{1}{r}$$

By the maximal modulus principle, the local maxima of g on $\bar{D}_r(0)$ are on $C_r(0)$, and the value of the modulus of the maximum is $\leq \frac{1}{r}$. So if we take $w \in D_r(0)$ where $|w| = \rho < r < 1$, then since maxima are found on the boundary

$$|g(w)| \leq \max_{|z|=r} |g(z)| \leq \frac{1}{r}$$

since r is arbitrarily between ρ and 1, this means that

$$|g(w)| \leq 1 \implies \left|\frac{f(z)}{z}\right| \leq 1 \implies |f(z)| \leq |z|$$

And taking the limit of $|g(w)|$ as $w \to 0$ gives $|f'(0)|$ so we also get $|f'(0)| \leq 1$ as required.
If $|f(z_0)| = |z_0|$ for $z_0 \neq 0$ then $|g(z_0)| = 1$ which is maximal. Or if $z_0 = 0$ and $|f'(0)| = 1$ then $|g(z_0)| = 1$ is maximal as well. Since z_0 is an interior point, this means that g must be constant (non-constant analytic functions have maxima only on their boundaries). So $f(z) = az$ where $g(z) = a$, since $|g(z)| = 1$ this means $|a| = 1$ so $f(z) = e^{i\theta} z$. ■