Complex Functions

Lecture 4, Wednesday April 19, 2023
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4.1 Power Series

Definition 4.1.1:

A sequence of functions $\{f_n\}_{n=1}^\infty$ **converges uniformly** to f in a set X if for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for every $n \geq N$, $|f - f_n| < \varepsilon$. Equivalently, $\sup_{z \in X} |f(z) - f_n(z)|$ converges to 0.
A power series is uniformly convergent if its partial sums converge uniformly to it.

Proposition 4.1.2:

A power series converges uniformly if and only if for every $\varepsilon > 0$ there is an N such that for every $N \leq n < m$ such that

$$\left| \sum_{k=n}^{m} c_k z^k \right| < \varepsilon$$

Theorem 4.1.3:

If $\sum c_k z^k$ is a power series and R is its radius of convergence and D is its domain of convergence, then if $|z| < R$ the power series converges and if $|z| > R$ the power series diverges. Specifically if $R < \infty$ then

$$D_R(0) \subseteq D \subseteq \bar{D}_R(0)$$

Furthermore, for every $0 < r < R$, the convergence of the power series in $D_r(0)$ is uniform.

Proof:

Recall the definition of R:

$$R = \sup\left\{ |w| \;\middle|\; \sum c_k w^k \text{ converges} \right\}$$

Thus if $|z| > R$, the power series does not converge for z. If $|z| < R$ then there is a $w \in D$ such that $|z| < |w|$. Since $w \in D$, we must have that $c_k w^k \longrightarrow 0$ and so $\left|c_k w^k\right| \leq M$. Let $\rho = \frac{|z|}{|w|} < 1$ then

$$\sum_{k=0}^{\infty} \left|c_k z^k\right| = \sum_{k=0}^{\infty} \left|c_k \rho^k w^k\right| \leq M \sum_{k=0}^{\infty} \rho^k$$

which converges since $0 \leq \rho < 1$ and so the series converges absolutely for z as required.
Let $r < \rho < R$, from above we know that $\sum_{k=0}^{\infty} c_k \rho^k$ converges so $c_k \rho^k$ converges to 0. Thus it is bound by some M: $\left|c_k \rho^k\right| < M$. Let $z \in D_r(0)$ then

$$\left|c_k z^k\right| = \left|c_k \rho^k\right| \cdot \left|\frac{z}{\rho}\right|^k < M \cdot \left|\frac{z}{\rho}\right|^k < M \cdot \left|\frac{r}{\rho}\right|^k$$

And $\left|\frac{z}{\rho}\right| < 1$ so:

$$\sum_{k=0}^{\infty} M \cdot \left|\frac{r}{\rho}\right|^k = M \cdot \frac{1}{1 - \left|\frac{r}{\rho}\right|} < \infty$$

Thus by the Weirestrauss M test, this convergence is uniform. And it is also absolute, as we can see in our proof.
■

Since the power series is the uniform convergence of continuous functions, the power series itself is continuous in $D_r(0)$ for $r < R$.

Note:

The border of the domain of convergence is problematic. The inclusion chain may be proper or D may be equal to one of the disks.

Theorem 4.1.4:

If $f(z) = \sum_{k=0}^{\infty} c_k z^k$ then for every $|z| < R$:

$$f'(z) = \sum_{k=0}^{\infty} k c_k z^{k-1}$$

Proof:

Let $|z| < r < R$ then since the partial sums converge uniformly to $f(z)$ in $D_r(0)$ and the partial sums are analytic, their derivatives exist and converge to $f'(z)$. ■

Notice then:

(1) A power series with radius of convergence $0 < R$ is differentiable an infinite number of times in $D_R(0)$, and thus it is also analytic.

(2) A power series $f(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k$ with radius of convergence $0 < R$ satisfies:

$$c_k = \frac{f^{(k)}(z_0)}{k!}$$

this stems from plugging in z_0 to $f^{(k)}$ which we obtain by the above theorem.

Lemma 4.1.5:

Let $\{z_k\}_{k=1}^{\infty}$ be a sequence of complex points such that $0 \neq z_k \longrightarrow 0$, if the power series $f(z) = \sum_{k=0}^{\infty} c_k z^k$ with positive radius of convergence is equal to zero for every z_k, then for every k, $c_k = 0$.

Proof:

We will show this inductively. Notice that

$$c_0 = f(0) = \lim_{k \to \infty} f(z_k) = 0$$

since f is continuous in $D_R(0)$.
Now suppose it is true for $n - 1$, then:

$$f(z) = \sum_{k=n}^{\infty} c_k z^k = \sum_{k=0}^{\infty} c_{k+n} z^{k+n}$$

then we have that

$$c_n = \lim_{z \to 0} \frac{f(z)}{z^n} = \lim_{k \to \infty} \frac{f(z_k)}{z_k^n} = 0$$

as required. ■

Theorem 4.1.6:

If you have two power series $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ with positive radii of convergence such that there is a sequence $z_n \longrightarrow 0$ where $f(z_n) = g(z_n)$ then $a_k = b_k$ for all k.

This proof is quite trivial using the above lemma, look at the power series $f - g$, since $f(z_n) - g(z_n) = 0$ by the lemma $a_k - b_k = 0$ for all k.

4.2 Complex Integrals

Definition 4.2.1:

Let $f\colon [a,b] \longrightarrow \mathbb{C}$ where $a < b \in \mathbb{R}$ be a complex function $f = u + iv$, then we define

$$\int_a^b f\,dt = \int_a^b u\,dt + i\int_a^b v\,dt$$

when the right hand side is defined (u and v are integrable; notice that u and v are real functions here). It is also common to leave out the dt.

Notice then that:

(1) The integral is a linear functional:

$$\int_a^b f + g = \int_a^b f + \int_a^b g \text{ and } \int_a^b \alpha f = \alpha\int_a^b f$$

these come directly from the same properties for real integrals and the definition of the complex integral.

(2) If $f' = u' + iv'$ exists and is continuous (this does not require f be complex analytic since f is not a function whose domain is complex) then

$$\int_a^b f'\,dt = f(b) - f(a)$$

this comes from the fundamental theorem of (real) calculus.

Proposition 4.2.2:

If $f\colon [a,b] \longrightarrow \mathbb{C}$ is integrable then

$$\left|\int_a^b f\,dt\right| \leq \int_a^b |f|\,dt$$

Proof:

Suppose

$$\int_a^b f\,dt = re^{i\theta}$$

then we have that

$$\left|\int_a^b f\,dt\right| = r = \int_a^b e^{-i\theta} f\,dt = \int_a^b \mathrm{Re}\big(e^{-i\theta} f\big)\,dt + i\int_a^b \mathrm{Im}\big(e^{-i\theta} f\big)\,dt$$

since r is real, the imaginary part of this integral must be 0 so we have that

$$= \int_a^b \mathrm{Re}\big(e^{-i\theta} f\big)\,dt \leq \int_a^b \big|e^{-i\theta} f\big|\,dt$$

since $\mathrm{Re}(z) \leq |z|$, and since $\left|e^{-i\theta}\right| = 1$ we have that

$$= \int_a^b |f|\,dt$$

as required. ■

Definition 4.2.3:

The **length** of a differentiable function $f\colon [a,b] \longrightarrow \mathbb{C}$ is

$$L = \int_a^b |f'(t)|\,dt$$

Definition 4.2.4:

A **complex curve** is a continuous function $z\colon [a,b] \longrightarrow \mathbb{C}$. A complex curve $z(t) = x(t) + iy(t)$ $(x, y\colon [a,b] \longrightarrow \mathbb{R})$ is **piecewise differentiable** if for every point $t \in [a,b]$ the derivative $z'(t) = x'(t) + iy'(t)$ exists except possibly at a finite number of points where only one of the one-sided derivatives of z exists. If furthermore $z'(t) \neq 0$ except for possibly at a finite number of points, then the curve is **smoothe**.
Sometimes we call the *image* of a complex curve a curve.

Another way to think of piecewise differentiability is that there is a finite partition of $[a,b]$, $a = x_0 < \cdots < x_n = b$, where x and y are differentiable over (x_i, x_{i+1}) for every relevant i, and for every i, x and y have a one-sided derivative at x_i (and the one sided derivatives are on the same side).

Definition 4.2.5:

Given a smoothe complex curve $z\colon [a,b] \longrightarrow \mathbb{C}$, we denote $C = z([a,b])$. For a complex function f which is continuous and defined over C we define

$$\int_C f(z)\,dz = \int_a^b f\big(z(t)\big)z'(t)\,dt$$

Proposition 4.2.6:

If $z\colon [a,b] \longrightarrow \mathbb{C}$ and $w\colon [c,d] \longrightarrow \mathbb{C}$ are two smoothe curves such that there is a differentiable bijection

$$\lambda\colon [c,d] \longrightarrow [a,b]$$

such that

(1) λ is (almost everywhere) continuously differentiable.

(2) $\lambda(c) = a$ and $\lambda(d) = b$.

(3) $w(t) = z\big(\lambda(t)\big)$

then

$$\int_z f\,dz = \int_w f\,dw$$

Proof:

Notice that $w'(t) = \lambda'(t) \cdot z'\big(\lambda(t)\big)$ and so

$$\int_w f\,dw = \int_c^d f \cdot \lambda'(t) \cdot z'\big(\lambda(t)\big)\,dt$$

then by substituting $u = \lambda(t)$ then we get that $du = \lambda'(t)\,dt$ (this is just change of variables) so

$$= \int_a^b f \cdot z'(u)\,du = \int_z f(z)\,dz$$

■

Proposition 4.2.7:

Let $z\colon [a,b] \longrightarrow \mathbb{C}$ be a smoothe curve with $C = z([a,b])$, we define $-C = w([a,b])$ where $w(t) = z(b+a-t)$ then

$$\int_{-C} f(w)\,dw = -\int_C f(z)\,dz$$

Proof:

Let $\lambda\colon [c,d] \longrightarrow [a,b]$ where $\lambda(t) = a+b-t$ then $\lambda'(t) = -1$ and $w = z \circ \lambda$ so λ satisfies the conditions for the proposition above and we get our desired result. ■

Proposition 4.2.8:

Suppose C is a curve with length L and f is a continuous function bounded by M, then

$$\left|\int_C f\right| \leq M \cdot L$$

Proof:

Suppose $z\colon [a,b] \longrightarrow \mathbb{C}$ is the curve whose image is C. Then

$$\left|\int_C f\right| = \left|\int_a^b f\big(z(t)\big) \cdot z'(t)\,dt\right| \leq \int_a^b \big|f\big(z(t)\big)\big| \cdot |z'(t)|\,dt \leq M \cdot \int_a^b |z'(t)|\,dt = M \cdot L$$

as required. ■