Differential and Analytic Geometry

Assignment 3
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Exercise 3.1:

Simplify the following expressions using Einstein notation. Assume that all vectors are in $\mathbb{R}^3$ and all matrices are in $M_3(\mathbb{R})$.

(1) $\delta^a_b g_{ca} g^{bd} \delta^c_d$

(2) $\delta^i_j g_{ik} \delta^k_m$

(3) $\delta_{ij} a^{ij}$

(4) $g^{1a} g_{a1}$

(5) $\delta^1_a \delta^a_b \delta^b_c \delta^c_d \delta^d_2$

(1) Here every index occurs both as a lower and upper index, and thus all indexes are being summed over. Since $\delta^a_b g_{ca} = g_{cb}$ and $g^{bd}\delta^c_d = g^{bc}$, we get that this is equal to $g_{cb}g^{bc}$, which is the product of a matrix and its inverse at the index c, c. So this is equal to δ^c_c, which again gets summed over and since $\delta^c_c = 1$ for a specific c, and the size of each matrix is 3×3, this is equal to

$$\sum_{c=1}^{3} \delta^c_c = 3$$

Thus the expression is equal to 3.

(2) Here, j and m only occur as lower indexes and so they are free. All other indexes are summed over. Since $\delta^i_j g_{ik} = g_{jk}$, this is equal to $g_{jk}\delta^k_m = g_{jm}$.

(3) Here all the indexes are summed over. And $\delta_{ij}a^{ij} = a^{ii}$, and so this is equal to $a^{11} + a^{22} + a^{33}$ which is equal to the trace of A^{-1} (the trace of the inverse of (a_{ij})).

(4) Here the only index is a and it is summed over (the other index is constant). $g^{1a}g_{a1} = \delta^1_1 = 1$ ($a^{ij}b_{jk}$ is equal to the product of AB at the index i, k).

(5) All indexes here are summed over (other than the constants). And so

$$\delta^1_a \delta^a_b \delta^b_c \delta^c_d \delta^d_2 = \delta^1_b \delta^b_c \delta^c_2 = \delta^1_c \delta^c_2 = \delta^1_2 = 0$$

Exercise 3.2:

Prove the following results using Einstein notation.

(1) Let A and B be matrices in $M_n(\mathbb{R})$, then $\mathrm{trace}(AB) = \mathrm{trace}(BA)$.

(2) Let $A \in M_{n\times m}(\mathbb{R})$, $B \in M_{m\times k}(\mathbb{R})$, and $C \in M_{k\times \ell}(\mathbb{R})$, then $A(BC) = (AB)C$.

Let $C = AB$ and $D = BA$, our goal is to show that $\mathrm{trace}(C) = \mathrm{trace}(D)$. Now,

$$\mathrm{trace}(C) = c^i_i = a^i_j b^j_i$$

and

$$\mathrm{trace}(D) = d^j_j = b^j_i a^i_j = a^i_j b^j_i = \mathrm{trace}(C)$$

as required.

(1) Let us define

$$D = AB, \quad E = (AB)C, \quad F = BC, \quad G = A(BC)$$

our goal is to show that $E = G$. Now, we know that $d^i_j = a^i_k b^k_j$ and $f^i_j = b^i_k c^k_j$ and so

$$e^i_j = d^i_k c^k_j = a^i_\ell b^\ell_k c^k_j$$

and

$$g^i_j = a^i_\ell f^\ell_j = a^i_\ell b^\ell_k c^k_j = e^i_j$$

Thus we have that $G = E$ as required.

Exercise 3.3:

Find a parameterization for a cone, as a rotational surface about z. And find the metric for this parameterization.

The curve which defines the surface is $\beta(t) = (t, 0, t)$. Then the surface is parameterized by

$$\sigma(\theta, t) = (t\cos\theta, t\sin\theta, t)$$

Then $\sigma_1 = (-t\sin\theta, t\cos\theta, 0)$ and $\sigma_2 = (\cos\theta, \sin\theta, 1)$. By definition $g_{ij} = \langle \sigma_i, \sigma_j \rangle$ and so

$$\begin{aligned} g_{11} &= t^2\sin^2\theta + t^2\cos^2\theta = t^2 \\ g_{12} = g_{21} &= -t\sin\theta\cos\theta + t\cos\theta\sin\theta = 0 \\ g_{22} &= \cos^2\theta + \sin^2\theta + 1 = 2 \end{aligned}$$

Thus the metric is

$$g = \begin{pmatrix} t^2 & 0 \\ 0 & 2 \end{pmatrix}$$

Exercise 3.4:

Find the metric of the surface $\sigma(u, v) = (u\cos v, u\sin v, v)$.

First we compute the partial derivatives of σ.

$$\begin{aligned} \sigma_1(u, v) &= (\cos v, \sin v, 0) \\ \sigma_2(u, v) &= (-u\sin v, u\cos v, 1) \end{aligned}$$

And since $g_{ij} = \langle g_i, g_j \rangle$ we get

$$\begin{aligned} g_{11} &= \cos^2 v + \sin^2 v = 1 \\ g_{12} = g_{21} &= -u\cos v\sin v + u\cos v\sin v = 0 \\ g_{22} &= u^2\sin^2 v + u^2\cos^2 v + 1 = u^2 + 1 \end{aligned}$$

And so the metric is

$$g = \begin{pmatrix} 1 & 0 \\ 0 & u^2 + 1 \end{pmatrix}$$

Exercise 3.5:

(1) Find the parameterization for the catenoid defined as the rotational surface of $(\cosh\varphi, 0, \varphi)$ about the z axis.

(2) Find the metric for the catenoid.

(1) Rotational surfaces of curves of the form $(r(\varphi), 0, z(\varphi))$ are $\sigma(\theta, \varphi) = (r(\varphi)\cos\theta, r(\varphi)\sin\theta, z(\varphi))$, as this is the result of multiplying R_θ by them, where

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So the parameterization is

$$\sigma(\theta, \varphi) = (\cosh(\varphi)\cos\theta, \cosh(\varphi)\sin\theta, \varphi)$$

(2) Once again, we find σ's partial derivatives:

$$\sigma_1(\theta, \varphi) = (-\cosh(\varphi)\sin\theta, \cosh(\varphi)\cos\theta, 0)$$
$$\sigma_2(\theta, \varphi) = (\sinh(\varphi)\cos\theta, \sinh(\varphi)\sin\theta, 1)$$

And since $g_{ij} = \langle\sigma_1, \sigma_2\rangle$ we get

$$g_{11} = \cosh(\varphi)^2$$
$$g_{12} = g_{21} = 0$$
$$g_{22} = \sinh(\varphi)^2 + 1$$

And so

$$g = \begin{pmatrix} \cosh(\varphi)^2 & 0 \\ 0 & \sinh(\varphi)^2 + 1 \end{pmatrix}$$

Exercise 3.6:

(1) Find a parameterization for the rotation of the ellipse

$$2x^2 + 3(z-4)^2 = 5$$

about the z axis.

(2) Find the metric for this parameterization.

(1) The ellipse itself can be parameterized by

$$\beta(\varphi) = \left(\sqrt{\frac{5}{2}}\cos\varphi, 0, \sqrt{\frac{5}{3}}\sin\varphi + 4\right)$$

And so the rotation is

$$\sigma(\theta, \varphi) = \left(\sqrt{\frac{5}{2}}\cos\varphi\cos\theta,\ \sqrt{\frac{5}{2}}\cos\varphi\sin\theta,\ \sqrt{\frac{5}{3}}\sin\varphi + 4\right)$$

(2) Again, we find the partial derivatives of σ,

$$\sigma_1 = \left(-\sqrt{\frac{5}{2}}\cos\varphi\sin\theta,\ \sqrt{\frac{5}{2}}\cos\varphi\cos\theta,\ 0\right)$$
$$\sigma_2 = \left(-\sqrt{\frac{5}{2}}\sin\varphi\cos\theta,\ -\sqrt{\frac{5}{2}}\sin\varphi\sin\theta,\ \sqrt{\frac{5}{3}}\cos\varphi\right)$$

And using $g_{ij} = \langle \sigma_i, \sigma_j \rangle$ we get

$$\begin{aligned} g_{11} &= \frac{5}{2}\cos^2\varphi \\ g_{12} = g_{21} &= 0 \\ g_{22} &= \frac{5}{3} + \frac{5}{6}\sin^2\varphi \end{aligned}$$

So the metric is

$$g = \begin{pmatrix} \frac{5}{2}\cos^2\varphi & 0 \\ 0 & \frac{5}{3} + \frac{5}{6}\sin^2\varphi \end{pmatrix}$$

Exercise 3.7:

Let us focus on the rotation surface of the parabola $x = z^2 + \frac{1}{4}$ around the z axis.

(1) Find the first and second fundamental forms of the surface.

(2) Find the shape operator of the surface.

(3) Find the Gaussian and mean curvature of the surface.

Firstly, the curve (parabola) is given by

$$\beta(\varphi) = \left(\varphi^2 + \frac{1}{4}, 0, \varphi\right)$$

and so the rotation surface is

$$\sigma(\theta, \varphi) = \left(\left(\varphi^2 + \frac{1}{4}\right)\cos\theta,\, \left(\varphi^2 + \frac{1}{4}\right)\sin\theta,\, \varphi\right)$$

And so the first and second partial derivatives of σ are:

$$\begin{aligned} \sigma_1 &= \left(-\left(\varphi^2 + \frac{1}{4}\right)\sin\theta,\, \left(\varphi^2 + \frac{1}{4}\right)\cos\theta,\, 0\right) \\ \sigma_2 &= (2\varphi\cos\theta,\, 2\varphi\sin\theta,\, 1) \\ \sigma_{11} &= \left(-\left(\varphi^2 + \frac{1}{4}\right)\cos\theta,\, -\left(\varphi^2 + \frac{1}{4}\right)\sin\theta,\, 0\right) \\ \sigma_{22} &= (2\cos\theta,\, 2\sin\theta,\, 0) \\ \sigma_{12} &= (-2\varphi\sin\theta,\, 2\varphi\cos\theta,\, 0) \end{aligned}$$

(1) We know that the first fundamental form is given by $g_{ij} = \langle \sigma_i, \sigma_j \rangle$. So

$$\begin{aligned} g_{11} &= \left(\varphi^2 + \frac{1}{4}\right)^2 \\ g_{12} = g_{21} &= 0 \\ g_{22} &= 4\varphi^2 + 1 \end{aligned}$$

And so the first fundamental form is

$$g = \begin{pmatrix} \left(\varphi^2 + \frac{1}{4}\right)^2 & 0 \\ 0 & 4\varphi^2 + 1 \end{pmatrix}$$

Now let us find the unit normal vector to the surface. Recall that this is given by

$$\rho = \frac{\sigma_1 \times \sigma_2}{\|\sigma_1 \times \sigma_2\|}$$

And so

$$\sigma_1 \times \sigma_2 = \det\begin{pmatrix} e_1 & e_2 & e_3 \\ -(\varphi^2+1/4)\sin\theta & (\varphi^2+1/4)\cos\theta & 0 \\ 2\varphi\cos\theta & 2\varphi\sin\theta & 1 \end{pmatrix} = \begin{pmatrix} (\varphi^2+1/4)\cos\theta \\ (\varphi^2+1/4)\sin\theta \\ -2\varphi(\varphi^2+1/4) \end{pmatrix}$$

Normalizing this gives

$$\rho = \frac{\begin{pmatrix} \cos\theta & \sin\theta & -2\varphi \end{pmatrix}}{\sqrt{1+4\varphi^2}}$$

Since the representation of second fundamental form, B, satisfies $b_{ij} = \langle \rho, \sigma_{ij} \rangle$ we have

$$\begin{aligned} b_{11} &= -\frac{\sqrt{\varphi^2+\frac{1}{4}}}{2} \\ b_{12} = b_{21} &= 0 \\ b_{22} &= \frac{1}{\sqrt{\varphi^2+\frac{1}{4}}} \end{aligned}$$

So the second fundamental form is

$$B = \begin{pmatrix} -\frac{\sqrt{\varphi^2+\frac{1}{4}}}{2} & 0 \\ 0 & \frac{1}{\sqrt{\varphi^2+\frac{1}{4}}} \end{pmatrix}$$

(2) Recall that $S = g^{-1}B$ and so

$$S = g^{-1}B = \begin{pmatrix} \left(\varphi^2+\frac{1}{4}\right)^{-2} & 0 \\ 0 & \frac{1}{4}\left(\varphi^2+\frac{1}{4}\right)^{-1} \end{pmatrix} \cdot \begin{pmatrix} -\frac{\sqrt{\varphi^2+\frac{1}{4}}}{2} & 0 \\ 0 & \frac{1}{\sqrt{\varphi^2+\frac{1}{4}}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\left(\varphi^2+\frac{1}{4}\right)^{-1.5} & 0 \\ 0 & \frac{1}{4}\left(\varphi^2+\frac{1}{4}\right)^{-1.5} \end{pmatrix}$$

(3) We know that the Guassian curvature is given by $K = \det(S)$, so

$$K = \det(S) = -\frac{1}{8}\left(\varphi^2+\frac{1}{4}\right)^{-3}$$

And the mean curvature is given by $H = \frac{1}{2}\operatorname{trace}(S)$, so

$$H = \frac{1}{2}\operatorname{trace}(S) = -\frac{1}{8}\left(\varphi^2+\frac{1}{4}\right)^{-1.5}$$

Exercise 3.8:

Given the surface

$$M = \left\{(x,y,z) \;\middle|\; x^2+y^2=4\right\}$$

find the shape operator of M.

We can parameterize this by

$$\sigma(\theta, r) = (2\cos\theta, 2\sin\theta, r)$$

Now, σ's partial derivatives:

$$\begin{aligned} \sigma_1 &= (-2\sin\theta, 2\cos\theta, 0) \\ \sigma_2 &= (0,0,1) \\ \sigma_{11} &= (-2\cos\theta, -2\sin\theta, 0) \\ \sigma_{12} &= (0,0,0) \\ \sigma_{22} &= (0,0,0) \end{aligned}$$

And so

$$g_{11} = 2$$
$$g_{12} = 0$$
$$g_{22} = 1$$

So the metric is

$$g = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

And now we will compute the unit normal to M. To do so we must compute $\sigma_1 \times \sigma_2$:

$$\sigma_1 \times \sigma_2 = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2\cos\theta \\ 2\sin\theta \\ 0 \end{pmatrix}$$

Normalizing this gives

$$\rho = (\cos\theta, \sin\theta, 0)$$

Since $b_{ij} = \langle \rho, \sigma_{ij} \rangle$ we have

$$b_{11} = -2$$
$$b_{12} = b_{21} = 0$$
$$b_{22} = 0$$

So

$$B = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$$

Now, $S = g^{-1}B$ and so the shape operator is

$$S = g^{-1}B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

Exercise 3.9:

Let

$$f(x, y) = 6x^2 + 8xy + 2y^2$$

(1) Compute the shape operator of the graph of f at $p = (0, 0, 0)$.

(2) What does the surface look like at $p = (0, 0, 0)$?

(1) We showed in lecture that for the graph of the function, which is $\sigma(x, y) = (x, y, f(x, y))$, the first and second fundamental forms are

$$g = \begin{pmatrix} 1 + f_x^2 & f_x f_y \\ f_x f_y & 1 + f_y^2 \end{pmatrix}, \qquad B = \frac{1}{\sqrt{1 + f_x^2 + f_y^2}} \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}$$

So we will compute the partial derivatives of f.

$$f_x = 12x + 8y$$
$$f_y = 8x + 4y$$
$$f_{xx} = 12$$
$$f_{xy} = 8$$
$$f_{yy} = 4$$

So at $p = (0,0,0)$ (and so its origin, $q = (0,0)$), we have

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 12 & 8 \\ 8 & 4 \end{pmatrix}$$

And since $S = g^{-1}B$ we get

$$S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 12 & 8 \\ 8 & 4 \end{pmatrix} = \begin{pmatrix} 12 & 8 \\ 8 & 4 \end{pmatrix}$$

(2) At $p = (0,0,0)$ notice that $\nabla f(0,0) = 0$ and so $p = (0,0,0)$ is a critical point on the graph. The Gaussian curvature is $K = \det S = 48 - 64 = -16$ which is negative, and so around p the graph curves both up and down, creating a saddle point.

Exercise 3.10:

Let $f(x,y) = y^4$, and focus on the graph of f.

(1) Find a parameterization for the graph, and its unit normal.

(2) What is the Gaussian curvature at each point on the surface?

(3) At the critical points of f, verify the solution to the previous question by computing the Hessian matrix.

(1) A parameterization is

$$\sigma(x,y) = (x, y, y^4)$$

Now, to compute the unit normal we must compute $\sigma_x \times \sigma_y$.

$$\sigma_x = (1,0,0)$$
$$\sigma_y = (0,1,4y^3)$$

And so

$$\sigma_x \times \sigma_y = \det\begin{pmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & 0 \\ 0 & 1 & 4y^3 \end{pmatrix} = \begin{pmatrix} 0 \\ -4y^3 \\ 1 \end{pmatrix}$$

Normalizing gives

$$\rho = \frac{1}{\sqrt{1+16y^6}}(0, -4y^3, 1)$$

(2) We showed in lecture formulas for computing g and B for the graph of functions, but instead of utilizing those we will instead compute them directly. We know $g_{ij} = \langle \sigma_i, \sigma_j \rangle$ and so

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 1+16y^6 \end{pmatrix}$$

And to compute $b_{ij} = \langle \rho, \sigma_{ij} \rangle$ we will compute σ's second partial derivatives:

$$\sigma_{xx} = (0,0,0)$$
$$\sigma_{xy} = (0,0,0)$$
$$\sigma_{yy} = (0,0,12y^2)$$

And thus

$$B = \begin{pmatrix} 0 & 0 \\ 0 & \frac{12y^2}{\sqrt{1+16y^6}} \end{pmatrix}$$

Since the Gaussian curvature is given by $K = \frac{\det(B)}{\det(G)}$, and $\det(B) = 0$ we have that $K = 0$.

(3) At critical points (which are points of the form $(x, 0)$ for f), $K = \det(H_f)$ where H_f is the Hessian matrix. And

$$H_f = \begin{pmatrix} 0 & 0 \\ 0 & 12y^6 \end{pmatrix}$$

And so $\det(H_f) = 0 = K$ as required.

Exercise 3.11:

Find the Gaussian curvature of the rotation surface of the curve $t \mapsto (\cosh(t), 0, t)$ about the z axis, by

(1) a direct computation of the shape operator.

(2) computing the second fundamental form.

Finally, draw a sketch of the curve and the rotation surface.

(1) Firstly, a parameterization of the surface is

$$\sigma(\theta, t) = (\cosh(t)\cos\theta, \cosh(t)\sin\theta, t)$$

And so its partial derivatives

$$\begin{aligned} \sigma_1 &= (-\cosh(t)\sin\theta, \cosh(t)\cos\theta, 0) \\ \sigma_2 &= (\sinh(t)\cos\theta, \sinh(t)\sin\theta, 1) \end{aligned}$$

Now, recall that $S(\sigma_i) = -\rho_i$. So we must compute the unit normal and its derivatives.

$$\sigma_1 \times \sigma_2 = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ -\cosh(t)\sin\theta & \cosh(t)\cos\theta & 0 \\ \sinh(t)\cos\theta & \sinh(t)\sin\theta & 1 \end{pmatrix} = \begin{pmatrix} \cosh(t)\cos\theta \\ \cosh(t)\sin\theta \\ -\cosh(t)\sinh(t) \end{pmatrix}$$

Normalizing gives (since $1 + \sinh(t)^2 = \cosh(t)^2$),

$$\rho = \frac{1}{\cosh(t)} \begin{pmatrix} \cos\theta \\ \sin\theta \\ -\sinh(t) \end{pmatrix}$$

Thus

$$\begin{aligned} \rho_1 &= \frac{1}{\cosh(t)}(-\sin\theta, \cos\theta, 0) \\ \rho_2 &= \left(-\frac{\sinh(t)\cos\theta}{\cosh(t)^2}, -\frac{\sinh(t)\sin\theta}{\cosh(t)^2}, -\frac{1}{\cosh(t)^2}\right) \end{aligned}$$

Now, notice that σ_1 and σ_2 are orthogonal, so we can find the components of a vector in $\operatorname{span}\{\sigma_1, \sigma_2\}$ via the inner product. Ie. if $v = a_1\sigma_1 + a_2\sigma_2$ then

$$\langle v, \sigma_i \rangle = a_i \langle \sigma_i, \sigma_i \rangle \implies a_i = \frac{\langle v, \sigma_i \rangle}{\langle \sigma_i, \sigma_i \rangle}$$

So we must compute $\langle \rho_i, \sigma_j \rangle$.

$$\begin{aligned} \langle \rho_1, \sigma_1 \rangle &= 1 \\ \langle \rho_1, \sigma_2 \rangle &= 0 \\ \langle \rho_2, \sigma_1 \rangle &= 0 \\ \langle \rho_2, \sigma_2 \rangle &= -1 \end{aligned}$$

Now,

$$\langle \sigma_1, \sigma_1 \rangle = \cosh(t)^2$$
$$\langle \sigma_2, \sigma_2 \rangle = \cosh(t)^2$$

Thus, since $S(\sigma_i) = -\rho_i$ and

$$\rho_i = \frac{\langle \rho_i, \sigma_1 \rangle}{\langle \sigma_1, \sigma_1 \rangle}\sigma_1 + \frac{\langle \rho_i, \sigma_2 \rangle}{\langle \sigma_2, \sigma_2 \rangle}\sigma_2$$

we have that

$$S = \frac{1}{\cosh(t)^2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

So $K = \det(S) = -\frac{1}{\cosh(t)^4}$.

(2) Here we have that (since we have already computed $g_{ij} = \langle \sigma_i, \sigma_j \rangle$),

$$g = \cosh(t)^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Now, to compute the second fundamental form, we must find its coefficients $b_{ij} = \langle \rho, \sigma_{ij} \rangle$ (we've actually already computed this as it is equal to $-\langle \rho_i, \sigma_j \rangle$, but whatever). Now,

$$\sigma_{11} = (-\cosh(t)\cos\theta, -\cosh(t)\sin\theta, 0)$$
$$\sigma_{12} = \sigma_{21} = (-\sinh(t)\sin\theta, \sinh(t)\cos\theta, 0)$$
$$\sigma_{22} = (\cosh(t)\cos\theta, \cosh(t)\sin\theta, 0)$$

Thus

$$b_{11} = -1$$
$$b_{12} = b_{21} = 0$$
$$b_{22} = 1$$

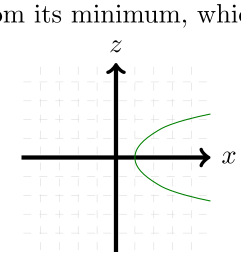
Thus

$$S = g^{-1}B = \frac{1}{\cosh(t)^2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

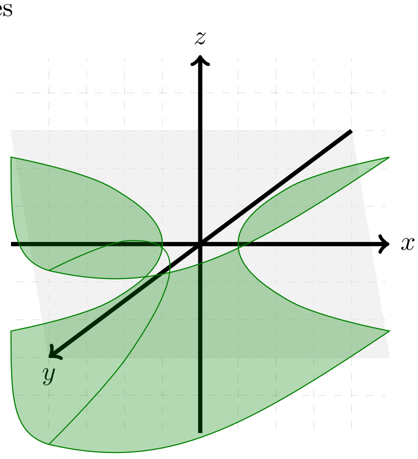
And so as before

$$K = \det(S) = \frac{1}{\cosh(t)^4}$$

(3) $x = \cosh(z)$ is a curve which curves up from its minimum, which is at 0. So the curve looks like



Rotating this about the z axis gives



(This isn't perfect, the fill should fill into the rotated hyperbolic cosine graph, but I'm not really sure how to do that using Ti*k*Z. Note that I did use Ti*k*Z, not pgfplots. I computed coordinates on the curve, and then using Ti*k*Z interpolated a curve through them.)

Exercise 3.12:

Find the mean curvature at every point on the surface from the previous questions in two ways: via direct computation and via the Laplacian.

(1) We know the mean curvature is $H = \frac{1}{2}\operatorname{trace}(S) = 0$.

(2) Our parameterization is an isomothermic parameterization, as g is a scalar matrix ($f = \cosh$). Thus $\Delta\sigma = -2\cosh^2(t)H\rho$, and since

$$\Delta\sigma = \sigma_{11} + \sigma_{22} = 0$$

this means that $H = 0$.

Exercise 3.13:

Give an example for a surface which satisfies:

(1) There is no point which has negative Gaussian curvature.

(2) Some points have negative Gaussian curvature, but not all of them.

(3) Every point has negative Gaussian curvature, and the curvature is not constant.

(1) Let the surface be a paraboloid,

$$\sigma(u, v) = (u, v, u^2 + v^2)$$

this is the graph of the function $f(u, v) = u^2 + v^2$. Thus it has Gaussian curvature:

$$K = \frac{f_{uu}f_{vv} - f_{uv}^2}{(1 + f_u^2 + f_v^2)^2} = \frac{4}{(1 + 4u^2 + 4v^2)^2}$$

and this is always positive.

(2) Let the surface be given by the graph of $f(u, v) = u^3 + v^3$,

$$\sigma(u, v) = (u, v, u^3 + v^3)$$

Then its Gaussian curvature is

$$K = \frac{f_{uu}f_{vv} - f_{uv}^2}{(1 + f_u^2 + f_v^2)^2} = \frac{36uv}{(1 + 9u^4 + 9v^4)^2}$$

This is not always negative or positive (when u and v have the same sign, it is positive, when they have differing signs the curvature is negative).

(3) Let us alter the example we gave for the first question, we want $f_{uu}f_{vv}$ to be negative, so let $f(u, v) = u^2 - v^2$. And σ is the graph of f,

$$\sigma(u, v) = (u, v, u^2 - v^2)$$

This has the Gaussian curvature

$$K = \frac{f_{uu}f_{vv} - f_{uv}^2}{(1 + f_u^2 + f_v^2)^2} = -\frac{4}{(1 + 4u^2 + 4v^2)^2}$$

This is always negative, and not constant.

Exercise 3.14:

Show that the Enneper surface is a minimal surface,

$$\sigma(u,v) = \left(u - \frac{1}{3}u^3 + uv^2, -v + \frac{1}{3}v^3 - vu^2, u^2 - v^2\right)$$

Let us compute the partial derivatives of σ:

$$\begin{aligned}
\sigma_1 &= \big(1 - u^2 + v^2, -2uv, 2u\big) \\
\sigma_2 &= \big(2uv, -1 + v^2 - u^2, -2v\big) \\
\sigma_{11} &= (-2u, -2v, 2) \\
\sigma_{12} = \sigma_{21} &= (2v, -2u, 0) \\
\sigma_{22} &= (2u, 2v, -2)
\end{aligned}$$

Then

$$\begin{aligned}
g_{11} &= (1 + u^2 + v^2)^2 \\
g_{12} = g_{21} &= 0 \\
g_{22} &= (1 + u^2 + v^2)
\end{aligned}$$

So the metric is equal to

$$g = (1 + u^2 + v^2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This means that σ is an isothermic parameterization. So in order to show that the surface is minimal, ie. that $H = 0$, it is sufficient to show that the Laplacian of σ is zero. And since

$$\Delta\sigma = \sigma_{11} + \sigma_{22} = (0,0,0)$$

this is indeed the case. So the surface is indeed minimal.

Exercise 3.15:

Show that the Scherk surface is minimal,

$$\sigma(u,v) = \left(u, v, \log\left(\frac{\cos v}{\cos u}\right)\right)$$

We once again compute the partial derivatives of σ.

$$\begin{aligned}
\sigma_1 &= (1, 0, -\tan u) \\
\sigma_2 &= (0, 1, \tan v) \\
\sigma_{11} &= \left(0, 0, -\frac{1}{\cos^2(u)}\right) \\
\sigma_{12} = \sigma_{21} &= (0,0,0) \\
\sigma_{22} &= \left(0, 0, \frac{1}{\cos^2(v)}\right)
\end{aligned}$$

Thus we have that using $g_{ij} = \langle \sigma_i, \sigma_j \rangle$,

$$g = \begin{pmatrix} 1 + \tan^2(u) & -\tan(u)\tan(v) \\ -\tan(u)\tan(v) & 1 + \tan^2(v) \end{pmatrix}$$

Notice that $\det(g) = 1 + \tan^2(u) + \tan^2(v)$. Now, we will compute the unit normal, but first $\sigma_1 \times \sigma_2$:

$$\sigma_1 \times \sigma_2 = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & -\tan(u) \\ 0 & 1 & \tan(v) \end{pmatrix} = \begin{pmatrix} \tan(u) \\ -\tan(v) \\ 1 \end{pmatrix}$$

Normalizing gives

$$\rho = \frac{1}{\sqrt{1 + \tan^2(u) + \tan^2(v)}} \begin{pmatrix} \tan(u) \\ -\tan(v) \\ 1 \end{pmatrix}$$

Now we will compute the second fundamental form, B, utilizing $b_{ij} = \langle \rho, \sigma_{ij} \rangle$.

$$\begin{aligned} \langle \rho, \sigma_{11} \rangle &= \frac{1}{\sqrt{1 + \tan^2(u) + \tan^2(v)}} \cdot \left(-\frac{1}{\cos^2(u)} \right) \\ \langle \rho, \sigma_{12} \rangle = \langle \rho, \sigma_{21} \rangle &= 0 \\ \langle \rho, \sigma_{22} \rangle &= \frac{1}{\sqrt{1 + \tan^2(u) + \tan^2(v)}} \cdot \frac{1}{\cos^2(v)} \end{aligned}$$

Thus

$$B = \frac{1}{\sqrt{\det(g)}} \begin{pmatrix} -\cos^{-2}(u) & 0 \\ 0 & \cos^{-2}(v) \end{pmatrix}$$

And computing the adjugate and from that the inverse, we have that

$$g^{-1} = \frac{1}{\det g} \begin{pmatrix} 1 + \tan^2(v) & \tan(u)\tan(v) \\ \tan(u)\tan(v) & 1 + \tan^2(u) \end{pmatrix}$$

And since $S = g^{-1}B$ we have

$$S = \frac{1}{\det(g)^{1.5}} \begin{pmatrix} -\cos^{-2}(u)\big(1 + \tan^2(v)\big) & * \\ * & \cos^{-2}(v)\big(1 + \tan^2(u)\big) \end{pmatrix}$$

(We can ignore the coefficients which aren't on the diagonal since we are interested in $H = \frac{1}{2}\operatorname{trace}(S)$.) So we can now compute the mean curvature by $H = \frac{1}{2}\operatorname{trace}(S)$,

$$\begin{aligned} \operatorname{trace}(S) &= -\frac{1}{\cos^2(u)} - \frac{\sin^2(v)}{\cos^2(u)\cos^(v)} + \frac{1}{\cos^2(v)} + \frac{\sin^2(u)}{\cos^2(u)\cos^2(v)} \\ &= \frac{-\cos^2(v) - \sin^2(v) + \cos^2(u) + \sin^2(u)}{\cos^2(u)\cos^2(v)} = \frac{-1+1}{\cos^2(u)\cos^2(v)} = 0 \end{aligned}$$

Thus $H = 0$, so the surface is minimal, as required.

Exercise 3.16:

Compute the total curvature of

$$\alpha(t) = \big(\cos(2t), \sin(2t)\big)$$

where $t \in [0, 4\pi]$.

Let $\beta(t) = \alpha\left(\frac{t}{2}\right) = (\cos(t), \sin(t))$ be a reparameterization of α. Since $\alpha\colon [0, 4\pi] \longrightarrow \mathbb{R}^2$, and the preimage of $[0, 4\pi]$ under $t \mapsto \frac{t}{2}$ is $[0, 8\pi]$, we have that β's domain is $[0, 8\pi]$. Further notice that $\|\beta'\| = 1$ and so β is α's natural parameterization. β is the natural parameterization of the unit circle, and so its curvature is $\frac{1}{R} = 1$. So

$$K = \int_0^{8\pi} 1 = 8\pi$$

Exercise 3.17:

Let us focus on the *Leminscate of Gerono*, the curve

$$\alpha(t) = \left(\sin(t), \frac{1}{2}\sin(2t)\right), \quad t \in [0, 2\pi]$$

(1) Sketch a graph of α.

(2) Using the sketch estimate α's total curvature.

(3) Find the total curvature of α.

(1) Let us notice that

$$\alpha(\theta+\pi) = \left(\sin(\theta+\pi), \frac{1}{2}\sin(2\theta+2\pi)\right) = \left(-\sin(\theta), \frac{1}{2}\sin(2\theta)\right)$$

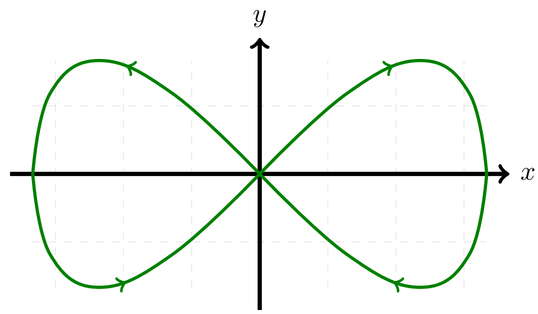
and

$$\alpha(\pi-\theta) = \left(\sin(\pi-\theta), \frac{1}{2}\sin(2\pi-2\theta)\right) = \left(\sin(\theta), -\frac{1}{2}\sin(2\theta)\right)$$

Meaning there is an antisymmetry around π, as $\alpha(\pi-\theta)$ reflects $\alpha(\theta)$ across the x axis, and $\alpha(\pi+\theta)$ reflexts $\alpha(\theta)$ across the y axis. This means that the curve is symmetric across the x and y axis, so it is sufficient to draw one quarter of the curve (ie for $0 \le t \le \frac{\pi}{2}$). So let us sample the points where $t = \frac{\pi}{8}a$ where $a = 0, \dots, 4$.

$$\alpha(0) = 0, \quad \alpha\left(\frac{\pi}{8}\right) = (0.383, 0.354), \quad \alpha\left(\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}, 0.5\right), \quad \alpha\left(\frac{3\pi}{8}\right) = (0.924, 0.354), \quad \alpha\left(\frac{\pi}{2}\right) = (1, 0)$$

So interpolating a curve through these points and reflecting it across the x and y axes gives the curve



(2) Notice that the left half of the curve is reflected onto the right half. But while the left half moves clockwise, the right half moves counterclockwise. So the total curvature of the left half is equal to -1 times the total curvature of the right half, as they are symmetric but move in different directions. So the total curvature of the curve as a whole should be zero.

(3) Now, since

$$\alpha'(t) = (\cos t, \cos(2t)), \quad \alpha''(t) = -(\sin t, 2\sin(2t))$$

And the curvature is given by

$$\kappa = \frac{\alpha_2''\alpha_1' - \alpha_2'\alpha_1''}{\left((\alpha_1')^2 + (\alpha_2')^2\right)^{1.5}} = \frac{-2\sin(2t)\cos(t) - \cos(2t)\sin(t)}{(\cos^2(t) + \cos^2(2t))^{1.5}}$$

Thus the total curvature is given by

$$K = \int_0^{2\pi} \kappa(t) = \int_0^{2\pi} \frac{-2\sin(2t)\cos(t) - \cos(2t)\sin(t)}{(\cos^2(t) + \cos^2(2t))^{1.5}}$$

But notice that

$$\kappa(\pi+t) = \frac{2\sin(2t)\cos(t) + \cos(2t)\sin(t)}{(\cos^2(t) + \cos^2(2t))^{1.5}} = -\kappa(t)$$

And

$$\kappa(\pi - t) = \frac{-2\sin(2t)\cos(t) - \cos(2t)\sin(t)}{(\cos^2(t) + \cos^2(2t))^{1.5}} = \kappa(t)$$

So κ is antisymmetric around π, thus integrating on a symmetric domain around π, for example $[0, 2\pi]$ will equal zero. Meaning

$$K = \int_0^{2\pi} \kappa = 0$$

as required.

Exercise 3.18:

Show that if γ is a regular Jordan curve then the radial vector is orthogonal to the tangent vector of γ at its closest point to the origin.

The distance between the radius vector ($\gamma(t)$) and the origin is given by $\|\gamma(t)\|$, and since this is positive, it has a minimum if and only if $\|\gamma(t)\| = \langle\gamma(t), \gamma(t)\rangle$ has a minimum. Let $f(t) = \langle\gamma(t), \gamma(t)\rangle$, so we are trying to show that at a minimum point of f, $\gamma(t)$ (the radial vector) and $\gamma'(t)$ (the tangent vector) are orthogonal. At a minimum, the derivative of f becomes zero, and

$$f'(t) = \langle\gamma'(t), \gamma(t)\rangle + \langle\gamma(t), \gamma'(t)\rangle = 2\langle\gamma(t), \gamma'(t)\rangle$$

So if $p = f(t)$ is the closest point to the origin then $f'(t) = 0$ and so $\langle\gamma(t), \gamma'(t)\rangle = 0$, meaning $\gamma(t)$ and $\gamma'(t)$ are orthogonal, as required.