Complex Functions

Lecture 6, Wednesday May 17, 2023
Ari Feiglin

Recall that for a function analytic in a rectangle, disk, or the entire complex plane:

(1) There exists an analytic antiderivative.

(2) The integral over every closed smooth curve is zero.

(3) The integral over a smooth curve is dependent only on its endpoints.

Theorem 6.1 (Cauchy's Integral Formula):

Suppose f is analytic in an open set $\mathcal{U}$, and let $a \in \mathcal{U}$. If C is a closed, positive-oriented curve contained in $\mathcal{U}$ and whose interior contains a, then

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$

Proof:

Since $\mathcal{U}$ is open and C is contained within $\mathcal{U}$, then there exists an $r > 0$ small enough such that

$$C_r(a) = \left\{\gamma(\theta) = a + re^{-i\theta} \;\middle|\; 0 \leq \theta \leq 2\pi\right\}$$

contained in the interior of C and $\mathcal{U}$.
The function $\frac{f}{z-a}$ is analytic in $\mathcal{U} \setminus \{a\}$. So we can take $\mathcal{O}$ to be the domain equal to the interior of C minus the interior of $C_r(a)$. Then $\partial\mathcal{O} = C \cup C_r(a)$, then f is analytic in $\mathcal{O}$. We can parameterize $C_r(a)$ by γ (in the definition of $C_r(a)$). Then since $\frac{f}{z-a}$ is analytic in $\mathcal{O}$, since $C_r(a)$ is negatively-oriented, by the last theorem from the previous lecture:

$$\int_C \frac{f}{z-a} + \int_{C_r} \frac{f}{z-a} = 0 \implies \int_C \frac{f}{z-a} = -\int_{C_r} \frac{f}{z-a} = \int_{-C_r} \frac{f}{z-a} = \int_{-C_r} \frac{f(z)-f(a)}{z-a} + \int_{-C_r} \frac{f(a)}{z-a}$$

$-C_r$ is parameterized by $\gamma_1(\theta) = \gamma(2\pi - \theta) = a + re^{i\theta}$, so

$$\int_{-C_r} \frac{f(a)}{z-a} = f(a) \cdot \int_0^{2\pi} \frac{rie^{i\theta}}{re^{i\theta}} = 2\pi i f(a)$$

Thus we have

$$\int_C \frac{f(z)}{z-a} = 2\pi i f(a) + \int_{-C_r} \frac{f(z)-f(a)}{z-a}$$

so we will show that the rightmost integral is zero.
Let $\varepsilon > 0$, then since f is continuous, there exists a $\delta > 0$ such that if $|z-a| < \delta$ then $|f(z) - f(a)| < \varepsilon$. We will choose $r > 0$ such that $0 < r < \delta$, then $\left|\frac{f(z)-f(a)}{z-a}\right| = \frac{|f(z)-f(a)|}{r} < \frac{\varepsilon}{r}$, and so

$$\left|\int_{-C_r} \frac{f(z)-f(a)}{z-a}\right| \leq \frac{\varepsilon}{r} \cdot 2\pi r = 2\pi\varepsilon$$

and since $\varepsilon > 0$ is arbitrary, this means that the integral is indeed zero.
So we have

$$\int_C \frac{f(z)}{z-a} = 2\pi i f(a)$$

or in other words

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a} = f(a)$$

■

We will use the notation $C_r(a) = \{z \in \mathbb{C} \mid |z-a| = r\}$, and we will give it the parameterization of $\gamma\colon [0, 2\pi] \longrightarrow \mathbb{C}$ by $\gamma(t) = a + re^{i\theta}$ (this was used as $-C_r$ in the previous proof). This is a positive-oriented Jordan curve.
Notice that for $k \in \mathbb{Z}$:

$$\int_{C_r(a)} \frac{dz}{(z-a)^k} = \int_0^{2\pi} \frac{rie^{i\theta}}{r^k e^{ki\theta}} = ir^{1-k} \int_0^{2\pi} e^{i\theta(1-k)}$$

If $k = 1$ then this is equal to $2\pi i$. Otherwise

$$\int_0^{2\pi} e^{i\theta(1-k)} = \frac{i(1-k)}{e}^{i\theta(1-k)} \Big|_0^{2\pi} = 0$$

So

$$\int_{C_r(a)} \frac{dz}{(z-a)^k} = \begin{cases} 2\pi i & k = 1 \\ 0 & k \neq 1 \end{cases}$$

Lemma 6.2:

If $a \in D_r(x)$ then

$$\int_{C_r(x)} \frac{dz}{z-a} = 2\pi i$$

Proof:

For $w = \frac{a-x}{z-x}$ we have

$$\frac{1}{z-a} = \frac{1}{z-x} \cdot \frac{1}{1-w}$$

And for $z \in C_r(x)$ we have that

$$|w| = \frac{|a-x|}{|z-a|} = \frac{|a-x|}{r} < 1$$

since $a \in D_r(x)$ so $|a-x| < r$. Thus we have that

$$\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n = \sum_{n=0}^{\infty} \left(\frac{a-x}{z-x}\right)^n$$

Thus for every $z \in C_r(x)$ and $a \in D_r(x)$

$$\frac{1}{z-a} = \sum_{k=0}^{\infty} \frac{(a-x)^n}{(z-x)^{n+1}}$$

Thus

$$\int_{C_r(x)} \frac{dz}{z-a} = \int_{C_r(x)} \frac{dz}{z-x} + \sum_{n=1}^{\infty} \int_{C_r(x)} \frac{(a-x)^n}{(z-x)^{n+1}}$$

Since we showed that

$$\int_{C_r(a)} \frac{dz}{(z-a)^n} = 0$$

for $n \neq 1$, this means that the right sum is zero, and we also showed that

$$\int_{C_r(x)} \frac{dz}{z-x} = 2\pi i$$

as required. ■

Theorem 6.3:

Suppose f is analytic in $D_R(x)$ then there exist constants $c_k \in \mathbb{C}$ such that

$$f(z) = \sum_{k=0}^{\infty} c_k (z-x)^k$$

for every $z \in D_R(x)$.

Proof:

Since $C_R(x)$ is not contained in $D_R(x)$, we first focus on $r > 0$ such that $0 < r < R$. Recall that

$$f(a) = \frac{1}{2\pi} \int_{C_r(a)} \frac{f(z)}{z-a}$$

and we showed that

$$\frac{1}{z-a} = \sum_{k=0}^{\infty} \frac{(a-x)^k}{(z-x)^{k+1}}$$

uniformly for $a \in D_r(x)$ and $z \in C_r(x)$. Thus

$$f(a) = \frac{1}{2\pi} \int_{C_r(a)} f(z) \cdot \sum_{k=0}^{\infty} \frac{(a-x)^k}{(z-x)^{k+1}} = \sum_{k=0}^{\infty} \left(\frac{1}{2\pi i} \int_{C_r(x)} \frac{f(z)}{(z-x)^{k+1}} \right) (a-x)^k$$

where the final equality is due to the uniform convergence. Thus if we set

$$c_k(r) = \frac{1}{2\pi i} \int_{C_r(x)} \frac{f(z)}{(z-x)^{k+1}}$$

we have found the constants satisfying our condition in $D_r(x)$.
If $r_1, r_2 > 0$, f can be written as a powerseries with $c_k(r_1)$ and $c_k(r_2)$, but since $D_{r_1}(x) \subseteq D_{r_2}(x)$ and powerseries are unique, this means $c_k(r_1) = c_k(r_2)$. So $c_k(r)$ is independent of r, we can write them as c_k. And for every $z \in D_R(x)$ there is a $0 < r < R$ where $z \in D_r(x)$, we have that

$$f(z) = \sum_{k=0}^{\infty} c_k (z-x)^k$$

as required. ∎

As we know, power series are infinitely differentiable, and we can differentiate and find that

$$\frac{f^{(k)}}{k!} = c_k = \frac{1}{2\pi i} \int_{C_r(x)} \frac{f(z)}{(x-z)^{k+1}}$$

Theorem 6.4:

If f is analytic on an open set $\mathcal{U}$ then for every $k \in \mathbb{N}_0$ and every $x \in \mathcal{U}$, and every positive-oriented Jordan curve C contained in $\mathcal{U}$, if x is in the interior of C:

$$f^{(k)}(x) = \frac{k!}{2\pi i} \int_C \frac{f(z)}{(x-z)^{k+1}}$$

Corollary 6.5:

If $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ is analytic at $\alpha \in \mathbb{C}$, then so is $f^{(k)}$ for $k \in \mathbb{N}_0$.

Proof:

If f is analytic at α then there exists a disk $D_R(\alpha)$ in which f is differentiable. So there exist $c_k \in \mathbb{C}$ such that

$$f(z) = \sum_{k=0}^{\infty} c_k (z-\alpha)^k$$

for every $z \in D_R(\alpha)$. Inductively we see that

$$f^{(m)}(z) = \sum_{k=0}^{\infty} k(k-1)\cdots(k-m+1)\cdot c_k \cdot (z-\alpha)^{k-m}$$

All of these powerseries have the same radius of convergence (using limsup), and thus define analytic functions in $D_R(\alpha)$. ■

If $\mathcal{U}$ is open and f is analytic in it, then for every $x \in \mathcal{U}$ there exists a $D_r(x) \subseteq \mathcal{U}$, and so f can be written as a powerseries in $D_r(x)$. But in general f may not be a powerseries in all of $D_r(x)$.

Proposition 6.6:

If f is analytic in a domain D such that there exists a sequence $z_n \in D$ such that $z_n \longrightarrow z_0 \in D$ where all z_n are distinct from z_0, and $f(z_n) = 0$, then f is identically zero on D.

Proof:

There exists an $R > 0$ such that $D_R(z_0) \subseteq D$. And there exists an $N > 0$ such that for every $n \geq N$, $z_n \in D_R(z_0)$. Since f can be written as a powerseries in $D_R(z_0)$, we use the similar theorem for powerseries to conclude that $f = 0$ on $D_R(z_0)$.
Let us define

$$A = \{z \in D \mid \exists \{x_n\}_{n=1}^{\infty} \in D, x_n \longrightarrow z, f(x_n) = 0\}$$

A is non-empty by assumption, and it is open since if $z \in A$ then by above there exists a $r > 0$ such that f is zero in $D_r(z)$, and so $D_r(z) \subseteq A$. We define $B = D \setminus A$, and if $b \in B$ then there must be an $r > 0$ such that in $D_r(b)$, $f \neq 0$ as otherwise you could take a sequence to b of zeros of f. And $D_r(b) \subseteq B$ for this same reason, so B is open.
So $D = A \cup B$, but A and B are open and D is connected, so A or B must be empty. Since A is non-empty, $B = \varnothing$ and so $A = D$. And since for every $z \in A$, $f(z) = 0$, $f = 0$ in D. ■

Thus if f and g are two analytic functions in a domain D which agree on a convergent sequence of distinct numbers, then $f = g$ in D ($f - g$ is equal to zero in this sequence). So analytic functions are defined uniquely by their values on convergent sequences.

Proposition 6.7:

Let $f(z)$ be an entire function, then for any $a \in \mathbb{C}$

$$g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a} & z \neq a \\ f'(a) & z = a \end{cases}$$

$g(z)$ is entire.

Proof:

Since

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} \cdot (z-a)^k$$

for all $z \neq a$ we have

$$g(z) = \sum_{k=1}^{\infty} \frac{f^{(k)}(a)}{k!} \cdot (z-a)^{k-1}$$

And for $z = a$, we have that this powerseries is equal to $\frac{f^{(1)}(a)}{1!} = f'(a)$, so the above equation holds for all $z \in \mathbb{C}$. Thus g is a powerseries which is convergent on all of $\mathbb{C}$ and is therefore entire. ■

If a is a root of f, the definition of g simplifies to $\frac{f(z)}{z-a}$ for $z \neq a$. We can continue this inductively on g if $\alpha_1, \ldots, \alpha_n$ are roots of f and define

$$g(z) = \frac{f(z)}{(z-\alpha_1)\cdots(z-\alpha_n)}$$

for $z \neq \alpha_i$, and if $z = \alpha_i$ then the limit of $g(z)$ as z approaches α_i exists (and is equal to $f'(\alpha_i)$ divided by the product of $\alpha_i - \alpha_j$ for $j \neq i$). This is done inductively and at every step the function is entire.

Theorem 6.8 (Liouville's Theorem):

Any bound entire function is constant.

Proof:

Let $a, b \in \mathbb{C}$ and let $R \geq 1 + \max\{|a|, |b|\}$. Then by **Cauchy's Integral Formula**, since a and b are contained within the interior of $C_R(0)$

$$f(b) - f(a) = \frac{1}{2\pi i}\left(\int_{C_R(0)} \frac{f(z)}{z-a} - \frac{f(z)}{z-b}\,dz\right) = \frac{1}{2\pi i}\left(\int_{C_R(0)} \frac{f(z)(b-a)}{(z-a)(z-b)}\,dz\right)$$

Since $f(z)$ is bound, suppose by M, and since $|z-a| \geq |R - |a||$, we have that

$$|f(b) - f(a)| \leq \frac{M|b-a| \cdot 2\pi R}{2\pi \cdot |R-|a|| \cdot |R-|b||} = \frac{M|b-a| \cdot R}{|R-|a|| \cdot |R-|b||}$$

As we let $R \longrightarrow \infty$, this converges to 0 and so $|f(b) - f(a)| = 0$, meaning $f(b) = f(a)$ for any $a, b \in \mathbb{C}$, meaning f is constant. ■

Theorem 6.9 (Generalized Liouville's Theorem):

Suppose f is an entire function such that there exist $A, B \in \mathbb{C}$ and $k \in \mathbb{N}_0$ where $|f| \leq A + B|z|^k$, then f is a polynomial of degree at most k.

Proof:

For $k = 0$ this is simply **Liouville's Theorem**. If we assume that this is true for $k \leq n$, we will show that this is true for $k = n+1$. Suppose $|f| \leq A + B|z|^{n+1}$ (we can assume that $A, B \geq 0$, which is true in any case), then we define

$$g(z) = \begin{cases} \frac{f(z)-f(0)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases}$$

we have shown above that such a g is entire. If $z \neq 0$ then

$$|g(z)| = \frac{|f(z) - f(0)|}{|z|} \leq \frac{|f(z)| + |f(0)|}{|z|} = \frac{2A + B|z|^{n+1}}{|z|} = \frac{2A}{|z|} + B|z|^n$$

if $|z| > 1$ then this is less than

$$\leq 2A + B|z|^n$$

and if $|z| \leq 1$ then since g is analytic and therefore continuous, it is bound by some M, which we can assume without loss of generality is equal to $2A$ (by taking the maximum between M and $2A$). So we have that

$$|g(z)| \leq 2A + B|z|^n$$

and therefore g is a polynomial of degree $\leq n$. And since $f(z) = zg(z) + f(0)$ for every $z \in \mathbb{C}$ (for $z = 0$ this is trivial), and so f is a polynomial of degree one more than g's, which is less than or equal to $n+1$. ■

Theorem 6.10 (Fundamental Theorem of Algebra):

Every non-constant polynomial $p(z) \in \mathbb{C}[z]$ has a root in $\mathbb{C}$.

Proof:

If we assume that $p(z) \neq 0$ for every $z \in \mathbb{C}$ then $f(z) = \frac{1}{p(z)}$ is an entire function in $\mathbb{C}$. Since p is non-constant, $|p(z)| \xrightarrow[z \to \infty]{} \infty$ and so $|f(z)| \xrightarrow[z \to \infty]{} 0$, and so f is bound. But then by **Liouville's Theorem**, f is constant and therefore p is as well, in contradiction. ■