Topology

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Definition 7.0.1:

Given a family of sets $\mathcal{F} = \{A_\lambda\}_{\lambda\in\Lambda}$, $\mathcal{F}$ has the **finite intersection property** if every finite intersection of sets in $\mathcal{F}$ is non-empty.

Proposition 7.0.2:

A space X is compact if and only if every family of closed sets $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ with the finite intersection property has non-empty intersection.

Proof:

Suppose X is compact and $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ has the finite intersection property but an empty intersection. Then

$$X = \left(\bigcap_{\lambda\in\Lambda} \mathcal{F}_\lambda\right)^c = \bigcup_{\lambda\in\Lambda} \mathcal{F}_\lambda^c$$

so this forms an open cover, and so there is a finite subcover

$$X = \bigcup_{n=1}^{N} \mathcal{F}_n^c$$

and so the intersections of these $\mathcal{F}_n$s is empty, in contradiction.
If this condition is true, let

$$X = \bigcup_{\lambda\in\Lambda} \mathcal{U}_\lambda$$

is an open cover. Then suppose there is no finite subcover, which means for every finite subset:

$$X \neq \bigcup_{n=1}^{N} \mathcal{U}_n \implies \bigcap_{n=1}^{N} \mathcal{U}_n^c \neq \varnothing$$

and so this family has the finite intersectiion property, and so its intersection is non-empty

$$\bigcap_{\lambda\in\Lambda} \mathcal{U}_\lambda^c \neq \varnothing \implies \bigcup_{\lambda\in\Lambda} \mathcal{U}_\lambda \neq X$$

in contradiction. ■

Proposition 7.0.3:

Let X be a compact space, and Y is another topological space. If $f\colon X \longrightarrow Y$ is continuous and surjective, then Y is compact.

Proof:

Let

$$Y = \bigcup_{\lambda\in\Lambda} \mathcal{U}_\lambda$$

be an open cover of Y. Then we have that

$$X = f^{-1}(Y) = \bigcup_{\lambda\in\Lambda} f^{-1}(\mathcal{U}_\lambda)$$

and since f is continuous, this is an open cover of X, and so there exists a finite subcover:

$$X = \bigcup_{n=1}^{N} f^{-1}(\mathcal{U}_n)$$

and since f is surjective we have

$$Y = f(X) = f\left(\bigcup_{n=1}^{N} f^{-1}(\mathcal{U}_n)\right) = \bigcup_{n=1}^{N} f\big(f^{-1}(\mathcal{U}_n)\big) \subseteq \bigcup_{n=1}^{N} \mathcal{U}_n$$

(the last inclusion is actually an equality) and so there is a finite subcover, so Y is compact. ■

Thus if X is a compact space and Y is a topological space, $f\colon X \longrightarrow Y$ is continuous, $f(X)$ is compact. This is because the restriction of f on its codomain is still continuous.

Proposition 7.0.4:

If X is a compact topological space, and $S \subseteq X$ is closed, S is also compact.

Proof:

We show that every family of closed sets with the finite intersection property in S has non-empty intersection. Let $\{Q_\lambda\}_{\lambda\in\Lambda}$ be a family of closed sets in S, and since S is closed in X, Q_λ is closed in X. Thus if this family has the finite intersection property, since X is compact, the intersection over all of the sets is also non-empty, and so S is compact. ■

Definition 7.0.5:

Suppose X is a topological space and $A \subseteq X$. A **open cover** of A is a family $\{\mathcal{U}_\lambda\}_{\lambda\in\Lambda}$ of open sets in X such that

$$A \subseteq \bigcup_{\lambda\in\Lambda} \mathcal{U}_\lambda$$

Notice that every open cover of A (not relative to X) induces an open cover of A relative to X, and vice versa. This is because open sets in A are of the form $A \cap \mathcal{U}$ for $\mathcal{U}$ open in X.
Thus all the statements we have formulated about compact spaces are true for compact subspaces with this "new" definition of open covers for subspaces.

Definition 7.0.6:

A topological space X satisfies the **first separation axiom** (denoted T_1) if for every two points $a \neq b \in X$ there exists a neighborhood $\mathcal{U}$ of a such that $b \notin \mathcal{U}$ and a neighborhood $\mathcal{V}$ of b such that $a \notin \mathcal{V}$.
A topological space X satisfies the **second separation axiom** (denoted T_2) if for every $a \neq b \in X$ there exists a neighborhood $\mathcal{U}$ of a and a neighborhood $\mathcal{V}$ of b such that $\mathcal{U} \cap \mathcal{V} = \varnothing$. A T_2 space is also called a **Hausdorff space**.

It is trivial to see that if X is a T_2 space, it is also a T_1 space.

Example 7.0.7:

All metric spaces are Hausdorff spaces: let $a \neq b \in X$, and take $r = \frac{1}{2}\rho(a,b)$. Then $B_r(a) \cap B_r(b) = \varnothing$, and these are neighborhoods of a and b.

Proposition 7.0.8:

X is a T_1 space if and onlt if for every $a \in X$, $\{a\}$ is closed.

Proof:

If X is T_1 then for every $a \neq b \in X$, let $\mathcal{U}_b$ be a neighborhood of b such that $a \neq \mathcal{U}_b$. Then

$$\{a\}^c = \bigcup_{a \neq b \in X} \mathcal{U}_b$$

since $a \notin \mathcal{U}_b$ for every b, and $b \in \mathcal{U}_b$ for every $a \neq b$. So $\{a\}^c$ is open as the union of open sets, and so $\{a\}$ is closed.
Let $a \neq b \in X$, then $\mathcal{U} = \{b\}^c$ is a neighborhood of a which doesn't contain b, and $\mathcal{V} = \{a\}^c$ is a neighborhood of b which doesn't contain a, so X is T_1. ■

Thus in a Hausdorff space, every singleton is closed.

Example 7.0.9:

In the **cofinite** topology:

$$\tau = \{\varnothing\} \cup \{A \subseteq X \mid X \setminus A \text{ is finite}\}$$

(This is quite simple to verify as a topology). Since every finite set F is closed (since $X \setminus (X \setminus F) = F$ so $X \setminus F$ is closed), and in fact all closed sets are finite, the cofinite topology is T_1 (since singletons are finite).
But if X is infinite, the cofinite topology is not Hausdorff. Let $\mathcal{U}$ and $\mathcal{V}$ be open sets, then if $\mathcal{U} \cap \mathcal{V} = \varnothing$ then $\mathcal{U}^c \cup \mathcal{V}^c = X$, but the closed sets are finite so $\mathcal{U}^c \cup \mathcal{V}^c$ must be finite, and since X is infinite, this is a contradiction. So every two open sets have non-empty intersection, and so X cannot be Hausdorff (for any $a \neq b$, every neighborhood of a and every neighborhood of b must have non-empty intersection).
So for infinite X, the cofinite topology is T_1 but not T_2. When X is finite, the cofinite topology is simply the discrete topology and therefore is Hausdorff (take the singletons as neighborhoods).

It is obvious that both T_1 and T_2 are inherited by subspaces: if X satisfies one of these axioms, so does every $A \subseteq X$.

Proposition 7.0.10:

Let X be a Hausdorff space, and $A, B \subseteq X$ be two disjoint compact subspaces. Then there exist $\mathcal{U}, \mathcal{V} \subseteq X$ disjoint open sets such that $A \subseteq \mathcal{U}$ and $B \subseteq \mathcal{V}$.

Proof:

If B is a singleton $\{p\}$, and $x \in A$ then since X is Hausdorff, there exist disjoint open sets $x \in \mathcal{U}_x$ and $p \in \mathcal{V}_x$. Then

$$A \subseteq \bigcup_{x \in A} \mathcal{U}_x$$

is an open cover, and since A is compact there exists a finite subcover

$$A \subseteq \bigcup_{n=1}^{N} \mathcal{U}_{x_n} = \mathcal{U}$$

and so

$$\mathcal{V} = \bigcap_{n=1}^{N} \mathcal{V}_{x_n}$$

is an open set containing p, and is disjoint from $\mathcal{U}$ since if $a \in \mathcal{U} \cap \mathcal{V}$ then $a \in \mathcal{U}_{x_n}$ for some n, and $a \in \mathcal{V}_{x_n}$ for every n. But $\mathcal{U}_{x_n}$ and $\mathcal{V}_{x_n}$ are disjoint.
Now in general, if $x \in A$ then $x \notin B$ so there exists $\mathcal{U}_x, \mathcal{V}_x$ open in X such that $x \in \mathcal{U}_x$ and $B \subseteq \mathcal{V}_x$ and these are disjoint (take the union of all $\mathcal{V}_x$ found before where $p \in B$). The family $\{\mathcal{U}_x\}_{x \in A}$ is an open cover of A and so there is a finite open subcover $\{\mathcal{U}_{x_n}\}_{n=1}^{N}$. And taking

$$\mathcal{U} = \bigcup_{n=1}^{N} \mathcal{U}_{x_n}, \qquad \mathcal{V} = \bigcap_{n=1}^{N} \mathcal{V}_{x_n}$$

which are disjoint, since if $x \in \mathcal{U} \cap \mathcal{V}$ then $x \in \mathcal{U}_{x_n} \cap \mathcal{V}_{x_n}$ for some n, which is impossible. ■

Theorem 7.0.11:

If X is a Hausdorff space, and $A \subseteq X$ is compact, then A is closed.

Proof:

If $A = X$ this is trivial. Otherwise, let $p \notin A$, then there exists $\mathcal{U}_p, \mathcal{V}_p \subseteq X$ open and disjoint such that $A \subseteq \mathcal{U}_p$ and $p \in \mathcal{V}_p$. We can do this for every $p \in A^c$, and since $\mathcal{V}_p \cap A = \varnothing$, we have that

$$A^c = \bigcup_{p \in A^c} \mathcal{V}_p$$

so A^c is open and therefore A is closed. ■

Proposition 7.0.12:

If X is a compact space, and Y is Hausdorff. If $f\colon X \longrightarrow Y$ is continuous, it is also a closed mapping.

Proof:

Suppose $S \subseteq X$ is closed, and therefore $S \subseteq X$ is compact. Then $f(S)$ is compact (the continuous image of a compact space is compact), and therefore $f(S)$ is closed since Y is Hausdorff. ■

Thus if f is also a bijection, then f is a homeomorphism. Thus if there exists a continuous bijection between a compact and Hausdorff space, the bijection is also a homeomorphism.

Definition 7.0.13:

A continuous mapping between topological spaces $f\colon X \longrightarrow Y$ is called an **embedding** if the induced mapping $f\colon X \longrightarrow f(X)$ is a homeomorphism.

Thus if f is an embedding, it is a continuous injection. The converse is not true (f^{-1} from $f(X)$ to X must also be continuous).
Thus if $f\colon X \longrightarrow Y$ is continuous and injective, and X is compact and Y is Hausdorff, $f\colon X \longrightarrow f(X)$ is a bijection and thus a homeomorphism. So f is an embedding.