Probability and Statistics Homework #6

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Question 6.1:

A student is attempting to call the university. The probability of their call being answered is 0.25 (and is independent of previous calls). The student attempts calling until their call is answered. Let X be a random variable which counts the number of calls the student makes. Answer the following:

(1) Determine the distribution of X.

(2) What is the probability the student is answered on their 7th attempt?

(3) What is the probability that the student will be answered after at least 4 attempts?

Answer:

(1) Let X_i be a random variable:

$$X_i(\omega) = \begin{cases} 1 & \text{the student is answered on their } i\text{th attempt} \\ 0 & \text{else} \end{cases}$$

We know that $X_i \sim \mathrm{Ber}\,(0.25)$, and:

$$X = \min\{i \in \mathbb{N} \mid X_i = 1\}$$

Which we know distributes geometrically, so:

$$\boxed{X \sim \mathrm{Geo}\,(0.25) \implies \mathbb{P}\,(X = n) = \begin{cases} 0.25 \cdot (0.75)^{n-1} & n \in \mathbb{N} \\ 0 & \text{else} \end{cases}}$$

(2) From our previous formula:

$$\mathbb{P}\,(X = 7) = 0.25 \cdot (0.75)^6 = \boxed{\frac{3^6}{4^7}}$$

(3) Our goal is to find:

$$\mathbb{P}\,(X \geq 4) = \mathbb{P}\,(X > 3)$$

Since X's image is $\mathbb{N}$. Since X distributes geometrically, this means:

$$\mathbb{P}\,(X > 3) = (1 - 0.25)^3 = \boxed{0.75^3}$$

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Question 6.2:

Roni gets allowance once per month and afterwards decides whether or not to buy a new book for himself. The probability that he buys a new book is 0.6. Roni starts off with no books. Answer the following:

(1) What is the probability that after 10 months, Roni has 6 books?

(2) What is the probability that after half a year, Roni has 4 books?

(3) What is the probability that Roni buys a fourth book after 6 months?

(4) What is the distribution of the number of books Roni will have after a year?

Answer:

Let X_n equal the number of books Roni has after n months.
Let:

$$B_i = \begin{cases} 1 & \text{Roni bought a book in the } i\text{th month} \\ 0 & \text{else} \end{cases}$$

We know that:

$$B_i \sim \text{Ber}\left(0.6\right)$$

Furthermore, we know that:

$$X_n = \sum_{i=1}^{n} B_i$$

And since B_i has a bernoulli distribution, this means that X_n has a binomial distribution:

$$X_n \sim \text{Bin}\left(n, 0.6\right)$$

Knowing this, we can easily answer the questions.

(1) This is asking:

$$\mathbb{P}\left(X_{10} = 6\right)$$

Which we know is equal to:

$$= \binom{10}{6} \cdot 0.6^6 \cdot 0.4^4 \approx \boxed{0.251}$$

(2) This is just:

$$\mathbb{P}\left(X_6 = 4\right) = \binom{6}{4} \cdot 0.6^4 \cdot 0.4^2 = \boxed{0.31104}$$

(3) This is the event that $B_6 = 1$ (as Roni bought a book in the 6th month) and $X_5 = 3$. These events are independent, so:

$$\mathbb{P}\left(B_6 = 1, X_5 = 3\right) = \mathbb{P}\left(B_6 = 1\right) \cdot \mathbb{P}\left(X_5 = 3\right) = 0.6 \cdot \binom{5}{3} \cdot 0.6^3 \cdot 0.4^2 = \boxed{0.20736}$$

(4) We know:

$$\boxed{X_{12} \sim \text{Bin}\left(12, 0.6\right)}$$

Or, equivalently:

$$\boxed{\mathbb{P}\left(X_{12} = n\right) = \begin{cases} \binom{12}{n} \cdot 0.6^n \cdot 0.4^{12-n} & 0 \leq n \leq 12 \in \mathbb{N} \\ 0 & \text{else} \end{cases}}$$

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Question 6.3:

Bob leaves his friend's house for his own after a night of drinking. Due to this, his sense of balance is slightly impaired; for every step, he has a probability of p of taking a step forward, and a probability of $1-p$ of stepping backwards. Let X_n be his position after n steps.

(1) What is X_n's distribution?

(2) What is the probability that after n steps, Bob is back at his friend's house?

Answer:

(1) Let Y_n represent Bob's nth step:

$$Y_n = \begin{cases} 1 & \text{Bob steps forward} \\ -1 & \text{Bob steps backwards} \end{cases}$$

And we know Y_n has the distribution:

$$\mathbb{P}\left(Y_n = x\right) = \begin{cases} p & x = 1 \\ 1-p & x = -1 \\ 0 & \text{else} \end{cases}$$

Which means that:

$$X_n = \sum_{i=1}^{n} Y_i$$

Now, let's define the random variable:

$$Z_n := \frac{Y_n + 1}{2}$$

This means that if $Y_n = 1$, $Z_n = 1$, and if $Y_n = -1$, then $Z_n = 0$. This is essentially "norming" Y_n, ie. transforming it into a bernoulli distribution.
Z_n has the distribution:

$$\mathbb{P}\left(Z_n = x\right) = \begin{cases} p & x = 1 \\ 1-p & x = 0 \\ 0 & \text{else} \end{cases}$$

Which means

$$Z_n \sim \text{Ber}\left(p\right)$$

And since $Y_n = 2 \cdot Z_n - 1$:

$$X_n = \sum_{i=1}^{n} \left(2 \cdot Z_i - 1\right) = 2 \cdot \sum_{i=1}^{n} Z_i - n$$

Which means that:

$$\mathbb{P}\left(X_n = x\right) = \mathbb{P}\left(\sum_{i=1}^{n} Z_i = \frac{x+n}{2}\right)$$

Now, we know that Z_i has a bernoulli distribution, which means that $\sum_{i=1}^{n} Z_i$ has a binomial distribution, meaning:

$$\mathbb{P}\left(\sum_{i=1}^{n} Z_i = z\right) = \binom{n}{z} \cdot p^z \cdot \left(1-p\right)^{n-z}$$

So, the distribution o X_n is:

$$\mathbb{P}\left(X_n = x\right) = \boxed{\begin{cases} \binom{n}{(x+n)/2} \cdot p^{(x+n)/2} \cdot \left(1-p\right)^{(n-x)/2} & \frac{x+n}{2} \leq n \in \mathbb{N}_0 \\ 0 & \text{else} \end{cases}}$$

(2) This is asking the probability that $X_n = 0$:

$$\boxed{\mathbb{P}\left(X_n = 0\right) = \begin{cases} \binom{n}{n/2} \cdot (p - p^2)^{n/2} & 2 \mid n \\ 0 & \text{else} \end{cases}}$$

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Question 6.4:

X and Y are two independent random variables which distribute geometrically. Prove that $\min(X,Y)$ also distributes geometrically.

Answer:

Suppose:

$$X \sim \text{Geo}(p) \quad Y \sim \text{Geo}(r)$$

Furthermore, we know:

$$\min(X,Y) = n \iff (X = n, Y > n) \vee (X > n, Y = n) \vee (X = n, Y = n)$$

Since these are all disjoint events, it follows that

$$\mathbb{P}(\min(X,Y) = n) = \mathbb{P}(X = n, Y > n) + \mathbb{P}(X > n, Y = n) + \mathbb{P}(X = n, Y = n)$$

Since X and Y are independent:

$$\mathbb{P}(X = n, Y > n) = \mathbb{P}(X = n) \cdot \mathbb{P}(Y > n)$$

We know that

$$\mathbb{P}(X = n) = p(1-p)^{n-1}$$

And

$$\mathbb{P}(Y > n) = (1-r)^n$$

So:

$$\mathbb{P}(X = n, Y > n) = p(1-p)^{n-1} \cdot (1-r)^n$$

It follows that:

$$\mathbb{P}(X > n, Y = n) = (1-p)^n \cdot r(1-r)^{n-1}$$

And we know that:

$$\mathbb{P}(X = n, Y = n) = \mathbb{P}(X = n) \cdot \mathbb{P}(Y = n) = p(1-p)^{n-1} \cdot r(1-r)^{n-1}$$

So all in all:

$$\mathbb{P}(\min(X,Y) = n) = p(1-p)^{n-1} \cdot (1-r)^n + (1-p)^n \cdot r(1-r)^{n-1} + p(1-p)^{n-1} \cdot r(1-r)^{n-1} =$$

$$= (1-p)^{n-1} \cdot (1-r)^{n-1} \cdot (p(1-r) + r(1-p) + pr)$$

We know that:

$$p(1-r) + r(1-p) + pr = p + r - pr = 1 - (1-p)(1-r)$$

So let:

$$q \coloneqq 1 - (1-p)(1-r)$$

Which means:

$$\mathbb{P}(\min(X,Y) = n) = q \cdot (1-q)^{n-1}$$

Which means that:

$$\min(X,y) \sim \text{Geo}(q)$$

As required.

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Question 6.5:

X and Y are two independent random variables which distribute geometrically over p. Prove that the distribution of $X \mid X+Y=n$ is uniform over $\{1,\dots,n-1\}$.

Answer:

We know that

$$\mathbb{P}\left(X=m \mid X+Y=n\right)=\frac{\mathbb{P}\left(X=m,Y=n-m\right)}{\mathbb{P}\left(X+Y=n\right)}$$

Since X and Y are independent:

$$\mathbb{P}\left(X=m,Y=n-m\right)=\mathbb{P}\left(X=n\right)\cdot\mathbb{P}\left(Y=n-m\right)=p(1-p)^{m-1}\cdot p(1-p)^{n-m-1}=p^2(1-p)^{n-2}$$

And:

$$\mathbb{P}\left(X+Y=n\right)=\sum_{i=1}^{n-1}\mathbb{P}\left(X=i,Y=n-i\right)=\sum_{i=1}^{n-1}p^2(1-p)^{n-2}=(n-1)\cdot p^2(1-p)^{n-2}$$

So all in all:

$$\mathbb{P}\left(X=m \mid X+Y=n\right)=\frac{1}{n-1}$$

Which means that

$$X \mid X+Y=n \sim \text{Unif}\left\{1,\dots,n-1\right\}$$

As required.

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Question 6.6:

Suppose $\{G_i\}_{i\in\mathbb{N}}$ is a series of independent events which distribute geometrically over p. For every natural n, we define the random variable:

$$Z_n := \max\left\{m \geq 0 \,\middle|\, \sum_{i=1}^{m} G_i \leq n\right\}$$

Prove that $Z \sim \operatorname{Bin}(n, p)$

Answer:

Lemma 6.6.1:

$$\sum_{i=m}^{n} \binom{i-1}{m-1} = \binom{n}{m}$$

Proof:

I will prove this through induction on n.

Base case: $n = m$

In this case, the left hand side becomes:

$$\sum_{i=m}^{m} \binom{i-1}{m-1} = \binom{m-1}{m-1} = 1$$

And the right hand side becomes:

$$\binom{m}{m} = 1$$

So the left and right hand sides are equal.

Inductive step:

Suppose this is true for n, I will prove this for $n+1$.

$$\sum_{i=m}^{n+1} \binom{i-1}{m-1} = \sum_{i=m}^{n} \binom{i-1}{m-1} + \binom{n}{m-1}$$

Which by our inductive hypotheses is equal to

$$= \binom{n}{m} + \binom{n}{m-1}$$

Which equals $\binom{n+1}{m}$ by Pascal's identity.

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Lemma 6.6.2:

If $\{G_i\}_{i=1}^{m}$ are independent random variables which distribute geometrically over p, then

$$\mathbb{P}\left(\sum_{i=1}^{m} G_i = n\right) = \binom{n-1}{m-1} \cdot p^m \cdot (1-p)^{n-m}$$

Proof:

Let A_n be the set:

$$A_n := \left\{\{a_i\}_{i=1}^{m} \in \mathbb{N}_0 \,\middle|\, \sum_{i=1}^{m} a_i = n\right\}$$

Furthermore, let $\{\mathcal{G}_i\}_{i=1}^{m}$ be random variables defined like so:

$$\mathcal{G}_i := G_i - 1$$

Which means that the event $\sum_{i=1}^{m} G_i = n$ is the same event as $\sum_{i=1}^{m} (\mathcal{G}_i + 1) = n \iff \sum_{i=1}^{m} \mathcal{G}_i = n - m$. This, in turn, is the same event as:

$$\{\mathcal{G}_i\}_{i=1}^m \in A_{n-m}$$

Furthermore, since $\{G_i\}_{i=1}^m$ are independent, so are $\{\mathcal{G}_i\}_{i=1}^m$. This means:

$$\mathbb{P}\left(\sum_{i=1}^{m} G_i = n\right) = \mathbb{P}\left(\sum_{i=1}^{m} \mathcal{G}_i = n - m\right) = \mathbb{P}\left(\forall i : \mathcal{G}_i = a_i \mid \{a_i\}_{i=1}^m \in A_{n-m}\right) =$$

$$= \sum_{\{a_i\}_{i=1}^m \in A_{n-m}} \mathbb{P}\left(\forall i : \mathcal{G}_i = a_i\right)$$

Suppose $\{a_i\}_{i=1}^m \in A$, then:

$$\mathbb{P}\left(\forall i : \mathcal{G}_i = a_i\right) = \prod_{i=1}^{m} \mathbb{P}\left(\mathcal{G}_i = a_i\right)$$

Since $\{\mathcal{G}_i\}_{i=1}^m$ is independent. Note that:

$$\mathcal{G}_i = a_i \iff G_i = a_i + 1$$

Since G_i distributes geometrically:

$$\mathbb{P}\left(\mathcal{G}_i = a_i\right) = \mathbb{P}\left(G_i = a_i + 1\right) = p(1-p)^{a_i}$$

So:

$$\prod_{i=1}^{m} \mathbb{P}\left(\mathcal{G}_i = a_i\right) = \prod_{i=1}^{m} p \cdot (1-p)^{a_i} = p^m \cdot (1-p)^{\sum_{i=1}^m a_i} = p^m \cdot (1-p)^{n-m}$$

So all in all:

$$\mathbb{P}\left(\sum_{i=1}^{m} G_i = n\right) = \sum_{\{a_i\}_{i=1}^m \in A_{n-m}} p^m (1-p)^{n-m} = |A_{n-m}| \cdot p^m \cdot (1-p)^{n-m}$$

So what is $|A_{n-m}|$? The idea is we can take $n-m$ 1s and partition them m ways. We can then uniquely determine each a_i by its respective partition.
How many ways are there to construct such a partition? What we do is allocate $n-m$ spots for the ones, and an extra $m-1$ spots for "dividers". So all in all there are $n-m+m-1 = n-1$ spots, of which $m-1$ are dividers. All we need to do is count the number of unique ways there are to place these $m-1$ dividers amongst these $n-1$ spots. But we already know how to do this, this is just

$$\binom{n-1}{m-1}$$

(This is analogous to ordering $m-1$ black balls amongst $n-1$ balls total.)
That is,

$$|A_{n-m}| = \binom{n-1}{m-1}$$

So

$$\mathbb{P}\left(\sum_{i=1}^{m} G_i = n\right) = \binom{n-1}{m-1} \cdot p^m \cdot (1-p)^{n-m}$$

As required.

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Now, for the final step. Notice that the event:

$$Z_n = m$$

Is equivalent to the event:

$$\sum_{i=1}^{m} G_i \leq n,\, G_{m+1} > n - \sum_{i=1}^{m} G_i$$

This is obviously necessary, and less obviously (but not surprisingly) sufficient.
As we know since $G_i > 0$, this means:

$$\sum_{i=1}^{m} G_i < \sum_{i=1}^{m+1} G_i$$

So

$$G_{m+1} > n - \sum_{i=1}^{m} G_i \iff \sum_{i=1}^{m+1} G_i > n \implies \forall k > m : \sum_{i=1}^{k} G_i > n$$

And since $\sum_{i=1}^{m} G_i \leq n$ this means $Z_n = m$.
So we know that:

$$\mathbb{P}\left(Z_n = m\right) = \mathbb{P}\left(\sum_{i=1}^{m} G_i \leq n\,,\, G_{m+1} > n - \sum_{i=1}^{m} G_i\right)$$

Which is equal to:

$$\sum_{j=m}^{n} \mathbb{P}\left(\sum_{i=1}^{m} G_i = j\,,\, G_{m+1} > n - j\right) = \sum_{j=m}^{n}\left(\mathbb{P}\left(\sum_{i=1}^{m} G_i = j\right) \cdot \mathbb{P}\left(G_{m+1} > n - j\right)\right)$$

Since $\{G_i\}$ is independent. By our previous lemma, this is equal to:

$$= \sum_{j=m}^{n}\left(\binom{j-1}{m-1} \cdot p^m \cdot (1-p)^{j-m} \cdot (1-p)^{n-j}\right) = \sum_{j=m}^{n}\left(\binom{j-1}{m-1} \cdot p^m \cdot (1-p)^{n-m}\right) =$$

$$= p^m \cdot (1-p)^{n-m} \cdot \sum_{j=m}^{n} \binom{j-1}{m-1}$$

Which, by our first lemma, is equal to:

$$\binom{n}{m} \cdot p^m \cdot (1-p)^{n-m}$$

So, all in all:

$$\mathbb{P}\left(Z_n = m\right) = \binom{n}{m} \cdot p^m \cdot (1-p)^{n-m}$$

As required.

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Question 6.7:

Alice is in a room with 3 doors: one door goes to Wonderland, one door to England, and one door goes back to the room itself. During each visit to the room, Alice chooses the door to England with probability p_0 and the door to Wonderland with probability p_1 such that p_0 and p_1 are parameters where $p_0 + p_1 < 1$.

(1) What is the probability that in the end Alice will go to Wonderland?

(2) How does the number of times Alice visits the room distribute, if we know that in the end Alice gets to Wonderland?

Answer:

Let X_i be the random variable which represents the choice Alice makes on her ith visit to the room. That is:

$$X_i = \begin{cases} 0 & \text{Alice chooses the door that brings her back to the room} \\ 1 & \text{Alice chooses the door to England} \\ 2 & \text{Alice chooses the door to Wonderland} \end{cases}$$

We know that:

$$\mathbb{P}(X_i = 1) = p_0 \quad \mathbb{P}(X_i = 2) = p_1 \quad \mathbb{P}(X_i = 0) = 1 - p_0 - p_1$$

(1) The event that Alice eventually chooses the door to Wonderland is the event that there exists some n such that $X_n = 2$ and for every $i < n$, $X_i = 0$ (Alice never chooses the door to England). That is we must compute:

$$\mathbb{P}(\exists n \forall i < n : X_n = 2, X_i = 0) = \sum_{n=1}^{\infty} \mathbb{P}(\forall i < n : X_n = 2, X_i = 0)$$

Since $\{X_i\}_{i=1}^{\infty}$ are independent, this is equal to:

$$\sum_{n=1}^{\infty} \left(\mathbb{P}(X_n = 2) \cdot \prod_{i=1}^{n-1} \mathbb{P}(X_i = 0) \right) = \sum_{n=1}^{\infty} p_1 \cdot (1 - p_1 - p_0)^{n-1} = p_1 \sum_{n=0}^{\infty} (1 - p_1 - p_0)^n =$$

$$= \boxed{\frac{p_1}{p_1 + p_0}}$$

(2) Let A be the event that in the end Alice goes to Wonderland, by the previous subquestion, we know:

$$\mathbb{P}(A) = \frac{p_1}{p_1 + p_0}$$

Let X be the random variable which is the number of times Alice visits the room.
We want to compute:

$$\mathbb{P}(X = n \mid A) = \frac{\mathbb{P}(X = n \wedge A)}{\mathbb{P}(A)}$$

The event $X = n \wedge A$ is the event that after n visits, Alice goes to Wonderland. This is the event:

$$\forall i < n : X_n = 2, X_i = 0$$

Which we know has a probability of:

$$\mathbb{P}(X_n = 2) \cdot \prod_{i=1}^{n-1} \mathbb{P}(X_i = 0) = p_1 \cdot (1 - p_1 - p_0)^{n-1}$$

So:

$$\mathbb{P}(X = n \mid A) = \boxed{(p_0 + p_1)(1 - p_0 - p_1)^{n-1}}$$

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Question 6.8:

We define a series of random variables like so: $X_0 \coloneqq 0$ and for every natural $n > 0$ we define X_n to be the number of successes in $X_{n-1} + 1$ independent trials which have a bernoulli distribution over p.

(1) Determine the probability of $X_2 = 0$.

(2) Determine the probability of the event $A_r \coloneqq \left\{\omega \in \Omega \;\middle|\; \lim_{n\to\infty} X_n(\omega) = r\right\}$ for a parameter $r \in \mathbb{R}$.

Answer:

(1) We know that $X_1 \in \{1, 0\}$ as it is the number of successes in $0 + 1 = 1$ trials. So

$$\mathbb{P}(X_2 = 0) = \mathbb{P}(X_2 = 0 \mid X_1 = 1) \cdot \mathbb{P}(X_1 = 1) + \mathbb{P}(X_2 = 0 \mid X_1 = 0) \cdot \mathbb{P}(X_1 = 0)$$

Since X_1 is the number of successes in 1 trial, it has a bernoulli distribution over p. Furthermore, we know:

$$X_{i+1} \mid X_i = x \sim \mathrm{Bin}(x + 1, p)$$

As it is the number of successes in $x + 1$ independent trials which have a bernoulli distribution. This means that:

$$\mathbb{P}(X_2 = 0) = (1-p)^2 \cdot p + (1-p)^2 = \boxed{(1-p)^2 \cdot (1+p)}$$

(2) Firstly, notice that if $\{a_i\}_{i=1}^{\infty} \in \mathbb{N}_0$ then:

$$\lim_{n\to\infty} a_n = r \iff \exists n_0\ \forall n \geq n_0 : a_n = r$$

It is obvious why this is sufficient. The proof why this is necessary is also simple.
Firstly, it is obvious that $r \in \mathbb{N}_0$, as otherwise we could choose an epsilon equal to $|\mathrm{round}(r) - r|$ and a_n would never get within that epsilon of r.
Otherwise let $\varepsilon \coloneqq \frac{1}{3}$, this means there must be some n_0 such that for every $n \geq n_0$:

$$|a_n - r| \leq \varepsilon = \frac{1}{3} \implies r - \frac{1}{3} \leq a_n \leq r + \frac{1}{2}$$

The only integer between these two values is r, which means that $a_n = r$, as required.
Since $X_n(\omega) \in \mathbb{N}_0$, this means that:

$$\lim_{n\to\infty} X_n(\omega) = r \iff \exists n_0\ \forall n \geq n_0 : X_n(\omega) = r$$

This means that:

$$\mathbb{P}(A_r) = \mathbb{P}(\forall n \geq n_0 : X_n(\omega) = r) \leq \prod_{n=n_0+1}^{\infty} \mathbb{P}(X_n(\omega) = r \mid \forall m < n : X_m(\omega) = r)$$

And we know that since only X_{n-1} affects X_n, this is equal to:

$$\prod_{n=n_0+1}^{\infty} \mathbb{P}(X_n(\omega) = r \mid X_{n-1}(\omega) = r)$$

And we know that:

$$\forall n > n_0 : X_n \mid X_{n-1} = r \sim \mathrm{Bin}(r + 1, p)$$

So $\mathbb{P}(X_n = r \mid X_{n-1} = r) = (r+1) \cdot p^r \cdot (1-p)$.
This equals 1 only if p and r are both 0, otherwise it is less than 1. (If $p = 0$ and $r \neq 0$, it equals $(r+1) \cdot 0 \cdot 1 = 0$. If $p \neq 0$ and $r = 0$, it equals $1 - p < 1$.) So:

$$\prod_{n=n_0+1}^{\infty} \mathbb{P}(X_n(\omega) = r) = \lim_{n\to\infty} ((r+1) \cdot p^r(1-p))^n = 0$$

Which means that $\mathbb{P}\left(A_r\right) = 0$.
But if $p = 0$ and $r = 0$, then no matter how many trials are done, none will succeed (as $p = 0$), so $X_i = 0$ for every i. This means that:

$$\lim_{n\to\infty} X_n(\omega) = \lim_{n\to\infty} 0 = 0$$

Which means that:

$$A_r = \Omega \implies \mathbb{P}\left(A_r\right) = 1$$

So all in all:

$$\boxed{\mathbb{P}\left(A_r\right) = \begin{cases} 0 & p \neq 0 \vee r \neq 0 \\ 1 & p = 0, r = 0 \end{cases}}$$

$\blacksquare$