Modern Analysis

Homework 3

Ari Feiglin

3.1 Exercise

Let E_1, E_2 be two measurable sets and μ a positive measure. Show that if $\mu(E_1 \triangle E_2) = 0$ then $\mu(E_1) = \mu(E_2)$.

We know $E_1 \triangle E_2 = (E_1 \cup E_2) \setminus (E_1 \cap E_2)$. If $\mu(E_1 \cap E_2) < \infty$ then this means $0 = \mu(E_1 \cup E_2) - \mu(E_1 \cap E_2)$ and so $\mu(E_1 \cap E_2) = \mu(E_1 \cup E_2)$. And if $\mu(E_1 \cap E_2) = \infty$ then since $E_1 \cap E_2 \subseteq E_1 \cup E_2$, $\mu(E_1 \cup E_2) = \infty$ and so we obtain the equality as well. Finally we have $E_1 \cap E_2 \subseteq E_1, E_2 \subseteq E_1 \cup E_2$ so

$$\mu(E_1 \cap E_2) \leq \mu(E_1), \mu(E_2) \leq \mu(E_1 \cup E_2) \implies \mu(E_1) = \mu(E_2) = \mu(E_1 \cap E_2) = \mu(E_1 \cup E_2)$$

3.2 Exercise

Let (X, Σ) be a measurable space and $\mu_1, \mu_2, \dots$ a sequence of positive measures on X, such that for every $A \in \Sigma$, $\mu_n(A)$ is increasing. Prove or disprove that $\mu\colon \Sigma \longrightarrow \overline{\mathbb{R}}$ defined by $\mu(A) := \lim_{n\to\infty} \mu_n(A)$ is a measure.

We will prove this. Firstly note that μ is well-defined since $\mu_n(A)$ is increasing and therefore has a limit. And,

$$\mu(\varnothing) = \lim_{n\to\infty} \mu_n(\varnothing) = \lim_{n\to\infty} 0 = 0$$

And μ is also finitely additive:

$$\mu(A \uplus B) = \lim_{n\to\infty} \mu_n(A \uplus B) = \lim_{n\to\infty} \mu_n(A) + \mu_n(B) = \lim_{n\to\infty} \mu_n(A) + \lim_{n\to\infty} \mu_n(B) = \mu(A) + \mu(B)$$

Now it is sufficient to show that if $\{A_k\}_k$ is increasing then $\mu(\bigcup_k A_k) = \lim_k \mu(A_k)$. Notice three things: first that since $\mu_n(A)$ is increasing, $\mu(A) = \sup_n \mu_n(A)$; second that since $\mu(A_k)$ is increasing (since $\mu(A_k) = \lim_n \mu_n(A_k) \leq \lim_n \mu_n(A_{k+1})$), $\lim_k \mu(A_k) = \sup_k \mu(A_k)$; and third that $\mu_n(A_k)$ is increasing so $\mu_n(\bigcup_k A_k) = \lim_k \mu_n(A_k) = \sup_k \mu_n(A_k)$. Thus

$$\lim_k \mu(A_k) = \sup_k \sup_n \mu_n(A_k) = \sup_{n,k} \mu_n(A_k) = \sup_n \sup_k \mu_n(A_k) = \sup_n \mu_n\left(\bigcup_k A_k\right) = \mu\left(\bigcup_k A_k\right)$$

as required.

3.3 Exercise

Let X be a metric space with its Borel σ-algebra, and let μ be a measure on X. μ's **support** is the smallest closed set F such that $\mu(F^c) = 0$.

Let $F \subseteq [0,1]$, prove there exists a measure on $[0,1]$ such that F is its support.

Define μ to be the counting measure relative to F: $\mu(A) := |A \cap F|$ (where $|S|$ is the number of elements in S, or ∞ if S is infinite). Then μ is a measure (the counting measure defined by $S \mapsto |S|$ is trivially a measure, and this is just the counting measure taken relative to F, so it is still a measure). Now if $\mu(F_0^c) = 0$ then $|F_0^c \cap F| = 0$ so $F_0^c \cap F = \varnothing$ meaning $F_0^c \subseteq F^c$ and so $F \subseteq F_0$. Thus F is the smallest set (not just closed) such that $\mu(F^c) = 0$.