Differential and Analytic Geometry

Summer 2023 Summary

Ari Feiglin

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1 Conic sections

How do we define what a circle is? Historically, there are two approaches: Descartes defined it as the set of all points (x, y) which satisfy the equation

$$(x-a)^2+(y-b)^2=R^2$$

for some values a and b and $R>0$. Euclid defined it as the set of all points whose distance from a specific point is some positive constant R.
We know that these two definitions are equivalent (given the standard norm/metric in $\mathbb{R}^2$), but Descartes's definition was introduced two thousand years after Euclid's. The idea of translating a visual or intuitive definition to an analytic one, as Descartes did, will be a motif of this course.
Now, recall the definition of an ellipse. Given two points, called the *foci* of the ellipse, F_1 and F_2 and a constant d, the ellipse defined is the set of all points A such that

$$|F_1A|+|F_2A|=d$$

We also must have that $|F_1F_2|<d$ as otherwise this just defines some line segment of F_1F_2. This is the Euclidean definition of an ellipse. Descartes's definition of an ellipse is the set of all points which satisfy

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

We must show that the cartesian definition satisfies the euclidean definition (and vice versa). Let us suppose that $a^2>b^2$ (if we have an equality then this defines a circle), then we define $c=\sqrt{a^2-b^2}$, and $F_1=(-c,0)$ and $F_2=(c,0)$. Then define $d=2a$. Now we must show that given $A=(x,y)$, $|F_1A|+|F_2A|=d$ if and only if $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Now,

$$|F_1A|+|F_2A|=\sqrt{(x+c)^2+y^2}, \qquad |F_2A|=\sqrt{(x-c)^2+y^2}$$

And so we must show that

$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a \iff \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Fortunately, we are not doing boring high school algebra, so we'll just assume that this is true. Thus the cartesian definition implies the eucliean definition.
Now suppose we have F_1, F_2, and d. Then we redefine the axes such that the x axis is parallel to F_1F_2 and the y axis is equidistant from F_1 and F_2. Define $a=\frac{d}{2}$, and $c=|F_10|$ (ie. half the distance between F_1 and F_2), and since $c=\sqrt{a^2-b^2}$, this defines b. Now all that remains is to show that the points which satisfy $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are precisely the points which satisfy the euclidean definition of the ellipse defined by F_1, F_2, and d. Again, we won't be doing this.
Now, what about equations of the form

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1\,?$$

In the language of Euclid, this is defined by

$$\big||F_1A|-|F_2A|\big|=d$$

These are called hyperbolas.
And now for parabolas, Euclid defined them as the set of all points which satisfy

$$|A\ell|=|AF|$$

where ℓ is a line (called the directrix), and F is the focal point. $|A\ell|$ is defined as the metric between a point and a set is usually defined, by taking the infimum of all the distances between points on ℓ and A. This corresponds to the length of the line segment perpendicular to ℓ which intersects with A.
In cartesian terms, what we can do is define the x axis to be parallel to ℓ and halfway between it and F, and the y axis to pass through F. Let $F=(0,f)$ and $\ell\colon y=-f$. Then if $A=(x,y)$,

$$|AF|=\sqrt{x^2+(y-f)^2}, \qquad |A\ell|=|y+f|$$

So

$$|AF|=|A\ell| \iff x^2+(y-f)^2=(y+f)^2 \iff x^2=4fy \iff y=\frac{1}{4f}x^2$$

Notice that all of these shapes are equivalent to the set of solutions of an equation of the form $Q(x,y)=0$ where

$$Q(x,y)=ax^2+by^2+cxy+dx+ey+f$$

and two other forms of solutions are lines, or two lines (of the form $y=\pm\alpha x$).

Proposition 1.1.1:

The set of solutions to $Q(x,y)=0$ is either a line, two lines, an ellipse, a hyperbola, or a parabola.

Proof:

Notice that $Q(x,y)=0$ if and only if

$$\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} a & \frac{c}{2} \\ \frac{c}{2} & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} d & e \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + f = 0$$

Let A be the diagonal matrix in the equation above. Now recall that if a matrix is symmetric, it can be orthogonally diagonalized. Suppose that P is the orthogonal matrix which diagonalizes A, so

$$P^TAP = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$$

Now suppose

$$P^T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ s \end{pmatrix} \iff \begin{pmatrix} x \\ y \end{pmatrix} = P\begin{pmatrix} t \\ s \end{pmatrix}$$

Meaning that

$$\begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}^T = \begin{pmatrix} t & s \end{pmatrix}P^T$$

Thus $Q(x,y)=0$ if and only if

$$\begin{pmatrix} t & s \end{pmatrix}P^TAP\begin{pmatrix} t \\ s \end{pmatrix} + \begin{pmatrix} d & e \end{pmatrix}P\begin{pmatrix} t \\ s \end{pmatrix} + f = \begin{pmatrix} t & s \end{pmatrix}\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}\begin{pmatrix} t \\ s \end{pmatrix} + \begin{pmatrix} d & e \end{pmatrix}P\begin{pmatrix} t \\ s \end{pmatrix} + f = 0$$

if we denote $\begin{pmatrix} d & e \end{pmatrix}P = \begin{pmatrix} d' & e' \end{pmatrix}$ we get that this is if and only if

$$\lambda_1 t^2 + \lambda_2 s^2 + d't + e's + f = 0$$

Now utilizing this new equation, we will split into cases.

(1) If $\lambda_1, \lambda_2 \neq 0$, then we can complete the square, the equation is equivalent to

$$\lambda_1\left(t + \frac{d'}{2\lambda_1}\right)^2 + \lambda_2\left(s + \frac{e'}{2\lambda_2}\right)^2 + f - \frac{d'^2}{4\lambda_1} - \frac{e'^2}{4\lambda_2} = 0$$

This is equivalent to an equation of the form

$$\lambda_1 u^2 + \lambda_2 v^2 + f' = 0$$

If $f'=0$ then this is $\lambda_1 u^2 = -\lambda_2 v^2$, which defines two lines (with respect to u and v). Otherwise this defines an ellipse.
Note that these define shapes with respect to u and v, but since t and s are simply some (orthogonal) linear transformation of x and y, and u and v are shifts of t and s, the shape defined in x and y is some orthogonal linear transformation of this ellipse and a shift, which still defines two lines or an ellipse. This will be true of the other cases as well.

(2) If $\lambda_2 = 0$ and $\lambda_1 \neq 0$ then we get

$$\lambda_1 t^2 + d't + e's + f = 0$$

which defines a parabola (complete the square). Similar for if $\lambda_1 = 0$ and $\lambda_2 \neq 0$.

(3) If $\lambda_1 = \lambda_2 = 0$ then we get

$$d't + e's + f = 0$$

which defines a line. ■

Corollary 1.1.2:

The only bound set of the form $A = \{(x, y) \mid Q(x, y) = 0\}$ is an ellipse.

2 Curves

2.1 Isometries

Recall the following definition

Definition 2.1.1:

If (M,ρ) and (X,σ) are two metric spaces, a function

$$f\colon M \longrightarrow X$$

is an **isometry** if $\rho(x,y) = \sigma(f(x), f(y))$ for every $x,y \in M$. M and X are called **isometric**.

It is obvious that isometries are injective (if $f(x) = f(y)$ then $\rho(x,y) = 0$ so $x = y$).
If X is a normed vector space, and A is an orthogonal transformation then recall $\|Ax\| = \|x\|$, so

$$\|Ax - Ay\| = \|A(x-y)\| = \|x-y\|$$

so A is an isometry.

Definition 2.1.2:

If X is a normed vector space, and a is a unit vector then define

$$S_a(x) = x - 2\langle x, a\rangle \cdot a$$

This is the reflection about $\{a\}^\perp$.

Recall that $x - \langle x,a\rangle a \in \{a\}^\perp$, since

$$\langle x - \langle x,a\rangle a, a\rangle = \langle x,a\rangle - \langle\langle x,a\rangle a, a\rangle = \langle x,a\rangle - \langle x,a\rangle\langle a,a\rangle = \langle x,a\rangle - \langle x,a\rangle = 0$$

Now notice that

- If $x \in a^\perp$ then $S_a(x) = x$.
- $S_a(a) = -a$.
- $S_a^2(x) = S_a(x - 2\langle x,a\rangle a) = x - 2\langle x,a\rangle a - 2\langle x - 2\langle x,a\rangle a, a\rangle = x - 2\langle x,a\rangle a + 2\langle x,a\rangle = x$. So $S_a^2(x) = x$.
- $S_a(x+y) = S_a(x) + S_a(y)$ and $S_a(\lambda x) = \lambda S_a(x)$, so S_a is a linear transformation.

Also notice that $\langle x,a\rangle a = a\langle x,a\rangle = aa^T x$, thus

$$S_a(x) = (I - 2aa^T)x$$

this is another proof that S_a is a linear transformation, as $S_a(x) = Ax$ where $A = I - 2aa^T$. Now notice that $A^T = A$, we have that A is orthogonal, so S_a is an isometry.

Proposition 2.1.3:

If $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is an isometry which preserves the origin, ie. $f(0) = 0$, then f is an orthogonal linear transformation.

Proof:

Notice that f preserves norms, since $\|x\| = \|x - 0\| = \|f(x) - f(0)\| = \|f(x)\|$. And so f preserves the inner product since

$$\|x-y\|^2 = \langle x-y, x-y\rangle = \|x\|^2 - 2\langle x,y\rangle + \|y\|^2$$

And thus

$$2\langle x,y\rangle = \|x\|^2 - \|x-y\|^2 + \|y\|^2$$

So

$$2\langle x,y\rangle = \|f(x)\|^2 - \|f(x) - f(y)\|^2 + \|f(y)\|^2$$

But the equality is true for any x, y and so

$$2\langle f(x), f(y)\rangle = \|f(x)\|^2 - \|f(x) - f(y)\|^2 + \|f(y)\|^2$$

Thus $\langle x, y\rangle = \langle f(x), f(y)\rangle$ as required.
Let us define

$$A = \begin{pmatrix} | & & | \\ f(e_1) & \cdots & f(e_n) \\ | & & | \end{pmatrix}$$

Now recall that

$$\langle e_i, e_j\rangle = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

And so $\langle f(e_i), f(e_j)\rangle = \delta_{ij}$. Thus the rows of A form an orthogonal basis, meaning A is an orthogonal matrix.
Now let us define

$$g(x) = A^{-1} f(x)$$

and we will prove that $g(x) = x$, which means that $f(x) = Ax$. Notice that

$$g(e_i) = A^{-1} f(e_i) = A^{-1} C_i(A) = C_i(A^{-1}A) = e_i$$

Now, if g were a linear transformation, we could finish here. Since $g(0) = 0$, g is an isometry (as the composition of isometries) which preserves the origin, so it preserves inner products.
Now let $x \in \mathbb{R}^n$ have coefficients x_i, meaning $\langle x, e_i\rangle = x_i$, now let $g(x) = y$ with coefficients y_i. So

$$x_i = \langle x, e_i\rangle = \langle g(x), g(e_i)\rangle = \langle y, e_i\rangle = y_i$$

Thus $x = y$, so $g(x) = x$ and thus $f(x) = Ax$, so f is indeed an orthogonal transformation. ■

Thus if f is an isometry, let $g(x) = f(x) - f(0)$, then g is also an isometry which preserves the origin and so $g(x) = Ax$ where A is orthogonal. And so $f(x) = Ax + f(0)$.

Theorem 2.1.4 (Cartan-Dieudonne Theorem):

If $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is an isometry, then

$$f = T \circ S_1 \circ \cdots S_m$$

where T is a shift, and S_i are reflections, and $m \leq n$.

Proof:

We will prove this by induction on n. For $n = 1$, then we know that $f(x) = Ax + c$ where A is orthogonal, and in $\mathbb{R}$ that means that $A = \pm 1$. So $f(x) = \pm x + c$. The $+c$ is a shift, and $-x$ is a reflection about 1.
Now, for the inductive step let $g(x) = f(x) - f(0)$ so $g(x) = Ax$ where A is orthogonal. If $A = \mathrm{id}$, then $f(x) = x + c$ which is just a shift, and we have finished. Otherwise there exists an $a \in \mathbb{R}^n$ such that $g(a) \neq a$. Now, we want a $b \in \operatorname{span} a, g(a)$ such that $\|b\| = 1$ and $S_b(a) = g(a)$. Let

$$d = \frac{a}{\|a\|} + \frac{g(a)}{\|g(a)\|}$$

And let b be the unit normal to d in $\operatorname{span} a, g(a)$. Then $S_b(a)$ is the reflection of a about d, which gives $g(a)$.
Now let

$$h = S_b \circ g$$

then h is the composition of two orthogonal transformations, and is therefore also an orthogonal transformation. Let $\hat{a} = \frac{a}{\|a\|}$, and let us extend this to an orthogonal basis

$$B = \{\hat{a}, b_2, \cdots, b_n\}$$

And since h is orthogonal, $h(B)$ is also an orthogonal basis. And $h(a) = S_b(g(a)) = S_b(S_b(a)) = a$, and so $h(\hat{a}) = \hat{a}$.
Thus

$$h(\{b_2, \ldots, b_n\}) \perp \hat{a}$$

And so $h(\{b_2, \dots, b_n\})$ is an orthogonal basis of $V = \hat{a}^{\perp}$, which has a dimension of $n-1$. And so $h\big|_V : V \to V$ is an orthogonal transformation, since $\{b_2, \dots, b_n\}$ is an orthogonal basis of V, and so is its image. So by our inductive assumption,

$$h\big|_V = S_2 \circ \cdots \circ S_m$$

where S_i are reflections with respect to $u^{\perp} \subseteq V$, and $m \leq n$.
Let $\ell = \operatorname{span} \hat{a}$, and $h\big|_{\ell} = \mathrm{id}$, and since h is linear

$$h = S_2 \circ \cdots \circ S_m$$

where S_i is a reflection with respect to $u^{\perp} \subseteq \mathbb{R}^n$. And since $h = S_b \circ g$, and $f = T \circ g$, where T is a shift (adding $f(0)$), we have

$$T = T \circ S_b \circ S_2 \circ \cdots \circ S_m$$

where $m \leq n$ as required. ■

2.2 Curves and Reparameterization

Definition 2.2.1:

A **curve** is a continuous function

$$\gamma \colon [a, b] \longrightarrow \mathbb{R}^n$$

A curve is **smooth** if it is differentiable, and it is **regular** if its derivative is never zero. If $\gamma'(t) = 0$ then t is called a **singularity** of γ.

Definition 2.2.2:

Suppose $\alpha \colon [a, b] \longrightarrow \mathbb{R}^n$ is a curve, and $\varphi \colon [c, d] \longrightarrow [a, b]$ is differentiable and $\varphi' > 0$, then we define $\beta \colon [c, d] \longrightarrow \mathbb{R}^n$ by $\beta = \alpha \circ \varphi$. This is called a **reparametrization** of α.

Proposition 2.2.3:

"x is a reparametrization of y" is an equivalence relation.

Proof:

Obviously this is reflexive (take φ to be the identity function). And it is transitive since if $\beta = \alpha \circ \varphi$ and $\gamma = \beta \circ \psi$ then $\gamma = \alpha \circ (\varphi \circ \psi)$ (the derivative of the composition is still positive). restrict the definition, this still works). Now suppose $\beta = \alpha \circ \varphi$, then since $\varphi' > 0$, we know that φ is strictly increasing (and therefore injective). And so we can also assume that φ is surjective, since $\varphi([a, b]) = [\varphi(a), \varphi(b)]$. So φ is bijective and so $\alpha = \beta \circ \varphi^{-1}$, and $(\varphi^{-1})' > 0$ (since it is equal to the inverse of φ' of some point). ■

Definition 2.2.4:

Let $\alpha \colon [0, T] \to \mathbb{R}^n$ be a curve, let

$$s_\alpha(t) = \int_0^t \|\alpha'(f)\| = \int_a^T \left(\sum_{k=1}^n \alpha_k'(f)^2 \right)^{1/2}$$

$s_\alpha(t)$ is the **arclength** of α.
α' is the componentwise derivative of α, which is equal to the Jacobian of α. We can continue with higher order componentwise derivatives.

The intuition behind the definition of $s(t)$ is that by the definition of integrals (using Riemman sums), we can partition $[0, T]$ into $t_0 = 0 < t_1 < \cdots < t_n = t$, and

$$\alpha'(f) \approx \frac{\alpha(t_{i+1}) - \alpha(t_i)}{\Delta_i} \implies \|\alpha'(f)\| \cdot \Delta_i \approx \|\alpha(t_{i+1}) - \alpha(t_i)\|$$

And $\|\alpha(t_{i+1}) - \alpha(t_i)\|$ approximates the length of α between t_i and t_{i+1}. And as we make the partition finer and finer, these approximations get more and more accurate.

Proposition 2.2.5:

Arclength is invariant under reparameterization. Meaning if $\alpha\colon [a,b] \longrightarrow \mathbb{R}^n$ and $\beta = \alpha \circ \varphi$ then

$$\int_a^b \|\alpha'(t)\| = \int_c^d \|\beta'(t)\|$$

Proof:

Notice that

$$\beta'(t) = \varphi'(t) \cdot \alpha'\big(\varphi(t)\big)$$

Since $\varphi'(t) > 0$ we have that

$$\int_c^d \|\beta'(t)\| = \int_c^d \|\alpha'(\varphi(t))\| \cdot \varphi'(t)\,dt$$

Let $u = \varphi(t)$ then $\varphi'(t)\,dt = du$ and since $\varphi(c) = a$ and $\varphi(d) = b$, so

$$= \int_a^b \|\alpha'(u)\|\,du$$

as required. ∎

What we have shown is that $s_{\alpha\circ\varphi}(t) = s_\alpha(\varphi(t))$, ie

$$s_{\alpha\circ\varphi} = s_\alpha \circ \varphi$$

Notice that $s'_\alpha(t) = \|\alpha'(t)\|$. If α is regular then $\alpha'(t) \neq 0$ and so $s'_\alpha > 0$ so s_α is smooth and strictly increasing, meaning s_α is invertible.

Definition 2.2.6:

If α is a smooth regular curve, then let us define the curve β by

$$\beta(u) = \alpha \circ s_\alpha^{-1}(u) = \alpha(t)$$

β is called the **natural parameterization** of α.

Another way of thinking of the natural parameterization is realizing that $\beta(u)$ is equal to the value of α after traversing u units on the arc defined by α.
Notice that if β is a reparameterization of α, then they both have the same natural parameterizations, since if $\beta = \alpha \circ \varphi$ then

$$\beta \circ s_\beta^{-1} = \beta \circ s_{\alpha\circ\varphi}^{-1} = \beta \circ (s_\alpha \circ \varphi)^{-1} = \alpha \circ \varphi \circ \varphi^{-1} \circ s_\alpha^{-1} = \alpha \circ s_\alpha^{-1}$$

In other words:

Proposition 2.2.7:

The natural parameterization of a regular smooth curve is unique, up to reparameterization. Meaning if α and β are reparameterizations of one another, then they have the same natural parameterization.

Notice that α is a natural parameterization if and only if $s_\alpha = \mathrm{id}$. If α is a natural parameterization, then $\alpha = \alpha \circ s_\alpha^{-1}$, and so $s_\alpha = \mathrm{id}$. And if $s_\alpha = \mathrm{id}$, then $\alpha \circ s_\alpha^{-1} = \alpha$.

Proposition 2.2.8:

If α is a curve, it is a natural parameterization if and only if $\|\alpha'\| = 1$.

Proof:

Since

$$s_\alpha(t) = \int_0^t \|\alpha'(u)\|$$

so $s'_\alpha = \|\alpha'\|$, so if $s_\alpha = \text{id}$ then $s'_\alpha = \|\alpha'\| = 1$. And if $\|\alpha'\| = 1$ then $s'_\alpha = 1$ so $s_\alpha(t) = t + c$ and since $s_\alpha(0) = 0$, $c = 0$ as required. ■

2.3 Curvature

Definition 2.3.1:

Let α be a natural parameterization. We define $T_\alpha(s) = \alpha'(s)$, and in the case that we are in 2 dimensions, we define $N_\alpha(s) = R_{\frac{\pi}{2}} \cdot T(s)$. R_θ is the rotation matrix

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Since α is a natural parameterization and R_θ is orthogonal, $\|T_\alpha\| = \|N_\alpha\| = 1$ and thus $\{T(s), N(s)\}$ forms an orthonormal basis, called the **Frenet-Serret Frame**.

We can think of T_α as the direction of motion, or the velocity, of α, and T'_α as its acceleration. Since T_α is constant, its derivative is perpendicular to itself, meaning the acceleration of α is orthogonal to its velocity. We will prove this formally:

Proposition 2.3.2:

Suppose $V : \mathbb{R} \longrightarrow \mathbb{R}^n$ (ie. V is a vector field over $\mathbb{R}$), if $\|V\| = c$ then $V' \perp V$ whenever V is differentiable.

Proof:

Since $\langle V, V\rangle = c^2$ is constant, we have that the function

$$f(t) = \langle V(t), V(t)\rangle = \sum_{k=1}^n V_i(t)V_i(t)$$

Is constant and therefore if V is differentiable at t, then so must V_i be, and therefore $f(t)$ is. And since f is constant, $f'(t) = 0$. Therefore

$$f'(t) = \sum_{k=1}^n V'_i(t)V_i(t) + V_i(t)V'_i(t) = \langle V'(t), V(t)\rangle + \langle V(t), V'(t)\rangle = 0$$

And since this inner product is over $\mathbb{R}$, this means $\langle V, V'\rangle = 0$ so $V' \perp V$ as required. ■

So when $n = 2$, this means that T'_α is parallel with N_α and so

$$T'_\alpha(s) = \kappa(s) \cdot N_\alpha(s)$$

For some function $k : \mathbb{R} \longrightarrow \mathbb{R}$. In fact, since $\{T_\alpha, N_\alpha\}$ is an orthonormal basis,

$$T' = \langle T', T\rangle T + \langle T', N\rangle N = \langle T', N\rangle N$$

So $\kappa(s) = \langle T'(s), N(s)\rangle$.
Let us look at this function κ.

(1) When $\kappa(s) = 0$, then $T'(s) = 0$ and so there is no acceleration, and we are moving in a straight line.

(2) When $\kappa(s) > 0$, then the curve α is accelerating away from T "upward" (toward N), and this creates a steep curve.

(3) When $\kappa(s) < 0$, the curve is accelerating away from T "downward", also creating a steep curve.

Thus κ can be seen as a measure of curvature.

Definition 2.3.3:

The **curvature** of a regular two-dimensional curve α at point s is defined to be

$$\kappa(s) = \langle T'_\alpha(s), N_\alpha(s)\rangle$$

Where T_α and N_α are taken as their values for the natural reparameterization of α.

Notice that

$$N' = \left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}T\right)' = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}T' = \kappa\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}N = \kappa\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}^2 T = \kappa\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}T = -\kappa T$$

Therefore T and N are solutions to the ODE,

$$T' = kappaN, \quad N' = -\kappa T$$

Thus by the uniqueness theorem for ODEs, if we are given the function $\kappa(s)$, and $N(0)$ and $T(0)$, then we can solve for N and T. Since N is determined by T, we need only $T(0)$ and $\kappa(s)$. And since $T = \alpha'$,

$$\alpha(s) - \alpha(0) = \int_0^s T$$

for all s, so if we are given T and $\alpha(0)$, we can find $\alpha(s)$. Thus given $\kappa(s)$, $\alpha(0)$, and $T(0)$ we can determine α.

Theorem 2.3.4 (The Fundamental Theorem of Curves):

Every regular curve is uniquely determined by its curvature, initial position, and $T(0)$.

Now, recall that

$$\kappa(s) = \langle T'(s), N(s)\rangle = \left\langle \alpha''(s), \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\alpha'(s)\right\rangle = \left\langle \begin{pmatrix}\alpha_1''(s)\\ \alpha_2''(s)\end{pmatrix}, \begin{pmatrix}-\alpha_2'(s)\\ \alpha_1'(s)\end{pmatrix}\right\rangle = \alpha_2''(s)\alpha_1'(s) - \alpha_2'(s)\alpha_1''(s)$$

And so

$$\kappa(s) = \alpha_2''\alpha_1' - \alpha_2'\alpha_1''$$

Where α is the natural parameterization.

Example 2.3.5:

Suppose α is the curve in $\mathbb{R}^2$ connecting x and y, ie.

$$\alpha\colon [0,1] \longrightarrow \mathbb{R}^2, \quad s \mapsto x\cdot\frac{s}{L} + y\cdot\frac{1-s}{L}$$

where $L = \|x-y\|$. Thus

$$\alpha'(s) = \frac{x}{L} - \frac{y}{L}$$

And so $\alpha''(s) = 0$, meaning $\kappa(s) = 0$.

Example 2.3.6:

Suppose α is the curve which parameterizes the circle of radius R,

$$\alpha\colon [0, 2\pi R] \longrightarrow \mathbb{R}^2, \quad s \mapsto R\left(\cos\frac{s}{R}, \sin\frac{s}{R}\right)$$

Thus

$$\alpha'(s) = \left(-\sin\frac{s}{R}, \cos\frac{s}{R}\right), \qquad \alpha''(s) = -\frac{1}{R}\left(\cos\frac{s}{R}, \sin\frac{s}{R}\right)$$

$\|\alpha\| = 1$, so α is the natural parameterization. And thus

$$\kappa(s) = -\frac{1}{R}\left(-\sin^2\frac{s}{R} + \cos^2\frac{s}{R}\right) = \frac{1}{R}$$

So the curvature of a circle of radius R is $\frac{1}{R}$.

Since the curves are determined by $\alpha(0)$, $T(0)$, and their curvature, by the above two examples, if

(1) $\kappa(s) = c \neq 0$ then α is a circle. If $\kappa(s) > 0$ then the curve is drawn counterclockwise, and if $\kappa(s) < 0$ the curve is parameterized clockwise (the proof above means that $\alpha(-s)$ is a circle of radius $-R$).

(2) $\kappa = 0$ then α is a line.

Notice that if γ is a natural parameterization then

$$\gamma'(s) = T(s) = \begin{pmatrix} \cos(\alpha(s)) \\ \sin(\alpha(s)) \end{pmatrix}$$

This means that

$$\alpha(s) = \operatorname{atan2}(\cos\alpha(s), \sin\alpha(s))$$

Now we claim that $\kappa(s) = \alpha'(s)$. Since

$$T(s) = \begin{pmatrix} \cos(\alpha(s)) \\ \sin(\alpha(s)) \end{pmatrix} \implies T'(s) = \begin{pmatrix} -\sin(\alpha(s)) \\ \cos(\alpha(s)) \end{pmatrix} \cdot \alpha'(s) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} T \cdot \alpha'(s) = \alpha'(s) N$$

And since $T'(s) = \kappa(s) N$ this means that $\alpha'(s) = \kappa(s)$ as required.
So if we are given $\gamma' = T$, then we can compute α based on T and then taking its derivative gives $\kappa(s)$.
But what if we aren't given the natural parameterization of the curve? Let β be any regular smooth curve, and γ its natural parameterization. Then recall that $\gamma = \beta \circ s_\beta^{-1}$ and so $\beta = \gamma \circ s_\beta$. Thus

$$\beta'(t) = s_\beta'(t) \cdot \gamma'(s_\beta(t))$$

(This is a bit confusing, since s_β is a scalar, and γ is a vector). We know that there exists an α such that

$$\alpha = \operatorname{atan}\left(\frac{\gamma_2'}{\gamma_2'}\right)$$

And since

$$\frac{\gamma_2'(s)}{\gamma_1'(s)} = \frac{\beta_2'(t)}{\beta_1'(t)}$$

And thus

$$\alpha(s) = \operatorname{atan}\left(\frac{\beta_2'(t)}{\beta_1'(t)}\right)$$

Recall that the derivative of $\operatorname{atan}(x) = \frac{1}{1+x^2}$, and since

$$\frac{d}{ds} \operatorname{atan}\left(\frac{\beta_2'(t)}{\beta_1'(t)}\right) = \frac{d}{dt} \operatorname{atan}\left(\frac{\beta_2'(t)}{\beta_1'(t)}\right) \cdot \frac{dt}{ds}$$

We have that

$$\kappa(s) = \alpha'(s) = \frac{1}{1 + \left(\frac{\beta_2'(t)}{\beta_1'(t)}\right)^2} \cdot \frac{\beta_2''(s)\beta_1'(s) - \beta_2'(s)\beta_1''(s)}{\beta_1'(s)^2} \cdot \frac{dt}{ds} = \frac{\beta_2''(s)\beta_1'(s) - \beta_2'(s)\beta_1''(s)}{\beta_1'(t)^2 + \beta_2'(t)^2} \cdot \frac{dt}{ds}$$

By definition,

$$s(t) = \int_0^t \|\beta'(u)\|\, du \implies s'(t) = \|\beta'(t)\|$$

So

$$\frac{dt}{ds} = \frac{1}{\|\beta'(t)\|} = \frac{1}{\sqrt{\beta_1'(s)^2 + \beta_2'(s)^2}}$$

And so all in all we have that

$$\kappa(s) = \frac{\beta_2''(s)\beta_1'(s) - \beta_2'(s)\beta_1''(s)}{\left(\beta_1'(t)^2 + \beta_2'(t)^2\right)^{1.5}}$$

So we have proven the following proposition:

Proposition 2.3.7:

If β is a regular smooth curve, then its curvature is given by

$$\kappa(s) = \frac{\beta_2''(s)\beta_1'(s) - \beta_2'(s)\beta_1''(s)}{\left(\beta_1'(t)^2 + \beta_2'(t)^2\right)^{1.5}}$$

Example 2.3.8:

So if $\beta(t) = (t, f(t))$ then

$$\kappa_\beta(t) = \frac{f''(t)}{(1 + f'(t))^{1.5}}$$

Thus if we have a function $f\colon \mathbb{R} \to \mathbb{R}$, then we can discuss its curvature as the parameterization of its graph.

Suppose we have a regular smooth curve α which is a natural parameterization. Our goal is to find the circle tangent to α at the point s_0.

(1) First, we can write α as a second order Taylor series

$$\alpha(s_0 + h) = \alpha(s_0) + h\alpha'(s_0) + \frac{h^2}{2}\alpha''(s_0) + \varepsilon(h)$$

where $\varepsilon(h) \in o(h^2)$ (meaning $\frac{\|\varepsilon(h)\|}{h^2} \xrightarrow[h \to 0]{} 0$).

(2) Now, we know that $T = \alpha'$ and $\alpha'' = T' = \kappa(s)N$ and thus

$$\alpha(s_0 + h) - \alpha(s_0) = hT(s_0) + \kappa(s_0)\frac{h^2}{2}N(s_0) + \varepsilon(h)$$

Let us define

$$\Delta(h) = \alpha(s_0 + h) - \alpha(s_0), \quad x(h) = \langle \Delta(h), T(s_0)\rangle, \quad y(h) = \langle \Delta(h), N(s_0)\rangle$$

Thus $\Delta(h) = x(h)T(s_0) + y(h)N(s_0)$, and so

$$x(h) = \left\langle hT(s_0) + \kappa(s_0)\frac{h^2}{2}N(s_0) + \varepsilon(h), T(s_0)\right\rangle = h + \langle \varepsilon(h), T(s_0)\rangle$$

Since $\|T\| = 1$ and T and N are orthogonal. Now since by Cauchy-Schwarz, $|\langle u, v\rangle| \leq \|u\|\|v\|$, we have that $\langle \varepsilon(h), T(s_0)\rangle = \varepsilon_1(h) \in o(h^2)$. Similarly

$$y(h) = \kappa(s_0) \cdot \frac{h^2}{2} + \varepsilon_2(h)$$

where $\varepsilon_1(h) \in o(h^2)$.

(3) Now, let us define the axis system $(T(s_0), N(s_0))$ centered at $\alpha(s_0)$, then since in this axis system $\alpha(s_0) = 0$, we will denote $\alpha(s_0 + h)$ by $\alpha(h)$, and $T(s_0)$ and $N(s_0)$ by T and N, and $\kappa(s_0)$ by k. Thus

$$\alpha(h) = x(h)T + y(h)N = hT + k \cdot \frac{h^2}{2}N + \left(\varepsilon_1(h)T + \varepsilon_2(h)N\right)$$

So given an h, we will define a circle through $(0,0)$, $\left(\pm h, k \cdot \frac{h^2}{2}\right)$. Such a circle would have the form $(x-a)^2 + (y-b)^2 = R^2$. Let us assume $a = 0$ (the reason for assuming this is by symmetry). Thus we must have

$$b^2 = R^2, \quad h^2 + \left(k \cdot \frac{h^2}{2} - b\right)^2 = R^2$$

So

$$h^2 + \kappa^2 \cdot \frac{h^4}{4} - b\kappa h^2 + b^2 = R^2 \implies \kappa^2 \cdot \frac{h^4}{4} = b\kappa h^2 - h^2 \implies \kappa^2 \cdot \frac{h^2}{2} = b\kappa - 1$$

So as $h \to 0$ we get that

$$bk = 1 \implies b = \frac{1}{\kappa} \implies R = |b| = \left|\frac{1}{\kappa}\right|$$

And the center of the circle is $\left(0, \frac{1}{\kappa}\right)$.

(4) Now, we know that (x, y) in this axis system corresponds to $xT(s_0) + yN(s_0) + \alpha(s_0)$ in $\mathbb{R}^2$, and so the circle we got is the set

$$\left\{ \alpha(s_0) + xT(s_0) + yT(s_0) \;\middle|\; x^2 + \left(y - \frac{1}{\kappa(s_0)}\right)^2 = \frac{1}{\kappa(s_0)^2} \right\}$$

We can also see this because the center of the circle is at $\frac{1}{k}$ in the new axis system, which is the point

$$c(s_0) = \alpha(s_0) + \frac{1}{\kappa(s_0)} N$$

And the radius of the circle is still $\frac{1}{\kappa(s_0)}$ (since the new axis system is simply an isometry).

Newton was originally the person who came up with this formula (for the center of the circle and its radius). The way he approached it was by taking the points $\alpha(s_0)$ and $\alpha(s_0 + h)$ and looking at the intersection of the normal lines at these points, $o(h)$. Then we will show that $o(h) \to c(s_0)$. Let $\ell_1(t)$ and $\ell_2(t)$ be the normal lines at $\alpha(s_0)$ and $\alpha(s_0 - h)$ respectively. We know that

$$\ell_1(t) = \alpha(s_0) + tN(s_0), \qquad \ell_2(t) = \alpha(s_0 + h) + tN(s_0 + h)$$

And since we know that

$$\alpha(s_0 + h) = \alpha(s_0) + h\alpha'(s_0) + o(h) = \alpha(s_0) + hT(s_0) + o(h)$$

And

$$N(s_0 + h) = N(s_0) + hN'(h) + o(h)$$

And since $N' = -\kappa(s_0)T$, we have

$$N(s_0 + h) = N(s_0) - h\kappa(s_0)T(h) + o(h)$$

Then $\ell_1(t) = \ell_2(p)$ if and only if

$$\alpha(s_0) + tN(s_0) = \alpha(s_0) + hT(s_0) + o(h) + p(N(s_0) - h\kappa(s_0)T(s_0) + o(h)) \iff (t - p)N(s_0) = h(1 - p\kappa(s_0))T(s_0) + o(h)$$

Meaning that

$$(p - t)N(s_0) + h(1 - p\kappa(s_0))T(s_0) \in o(h)$$

Thus

$$\frac{p - t}{h} N(s_0) + (1 - p\kappa(s_0))T(s_0) \xrightarrow[h \to 0]{} 0$$

Since N and T are orthonormal, this means that $p - t = 0$ and $1 - p\kappa(s_0) = 0$. So $t = p = \frac{1}{\kappa(s_0)}$. And so the center point is

$$c(s_0) = \alpha(s_0) + \frac{1}{\kappa(s_0)} N(s_0)$$

as we showed before. Let us summarize this in the following definition:

Definition 2.3.9:

If α is a regular smooth planar curve, then the **osculating circle** of α at the point s_0 (this is the input, we could also think of it as the point $\alpha(s_0)$) is the circle centered at

$$c_\alpha(s_0) = \alpha(s_0) + \frac{1}{\kappa_\alpha(s_0)} N_\alpha(s_0)$$

and whose radius is $\frac{1}{\kappa_\alpha(s_0)}$. The curve c_α is called the **evolute** of α.

Suppose α is a natural parameterization, and $\varphi\colon v \mapsto Ax + c$ is an isometry (and so A is orthonormal). Then let $\beta = \varphi \circ \alpha$, so

$$\beta(s) = A\alpha(s) + c$$

Then $\beta'(s) = A\alpha'(s)$, and since A is orthonormal, $\|\beta'\| = \|\alpha'\| = 1$ since α is natural. Thus β is also a natural parameterization. And so

$$\kappa_\beta(s) = \left\langle \beta''(s), R_{\frac{\pi}{2}}\beta'(s) \right\rangle = \left\langle A\alpha''(s), R_{\frac{\pi}{2}}A\alpha'(s) \right\rangle$$

Now, rotations and A commute up to sign. If $\det(A) = 1$ then they commute, and if $\det(A) = -1$ then $R_\theta A = -AR_\theta$. So this is equal to $\det(A)\left\langle A\alpha''(s), AR_{\frac{\pi}{2}}\alpha'(s) \right\rangle$, since A is orthogonal this is equal to

$$= \det(A)\left\langle \alpha''(s), R_{\frac{\pi}{2}}\alpha'(s) \right\rangle = \pm\kappa_\alpha(s)$$

So we have proven the following:

Proposition 2.3.10:

If A is an orthogonal matrix, and c a vector then $\varphi\colon x \mapsto Ax + c$ is an isometry, and if α is a natural parameterization, then so is $\beta = \varphi \circ \alpha$, and $\kappa_\alpha = \kappa_\beta$.

2.4 Total Curvature

Let $\gamma\colon [0, L] \longrightarrow \mathbb{R}^2$ be a natural parameterization, then $T = \gamma'$ and $\kappa(s) = \langle T', N \rangle$. Suppose $T(0)$ has an angle of θ_0 then let us define

$$\theta(s) = \int_0^s \kappa(p)\, dp + \theta_0$$

And we define the curve

$$\beta(s) = \gamma(0) + \begin{pmatrix} \int_0^s \cos(\theta(s))\, dp \\ \int_0^s \sin(\theta(s))\, dp \end{pmatrix}$$

Now, notice that

$$\beta'(s) = \begin{pmatrix} \cos(\theta(s)) \\ \sin(\theta(s)) \end{pmatrix}$$

And since $\|\beta'\| = 1$, β is a natural parameterization. And further

$$\beta''(s) = \theta'(s) \cdot \begin{pmatrix} -\sin(\theta(s)) \\ \cos(\theta(s)) \end{pmatrix} = \theta'(s) \cdot R_{\frac{\pi}{2}} \beta'(s)$$

Which means that

$$\kappa_\beta(s) = \langle \beta''(s), N_\beta(s) \rangle = \left\langle \theta'(s) \cdot R_{\frac{\pi}{2}} \beta'(s), R_{\frac{\pi}{2}} \beta'(s) \right\rangle = \theta'(s) \langle \beta'(s), \beta'(s) \rangle = \theta'(s) = \kappa(s)$$

(The third equality is since $R_{\frac{\pi}{2}}$ is orthogonal.) So the curvature of β is equal to that of γ.
Now,

$$T_\beta(0) = \beta'(0) = \begin{pmatrix} \cos(\theta(0)) \\ \sin(\theta(0)) \end{pmatrix} = \begin{pmatrix} \cos(\theta_0) \\ \sin(\theta_0) \end{pmatrix} = T(0)$$

And $\beta(0) = \gamma(0)$.
So by the **The Fundamental Theorem of Curves**, since $\kappa_\beta = \kappa_\gamma$, $\beta(0) = \gamma(0)$, and $T_\beta(0) = T_\gamma(0)$, we have that $\beta = \gamma$. This means that

$$T_\gamma(s) = T_\beta(s) = \beta'(s) = \begin{pmatrix} \cos(\theta(s)) \\ \sin(\theta(s)) \end{pmatrix}$$

So θ is the angle function of γ (ie. it gives the angle of γ). So we have proven the following proposition:

Proposition 2.4.1:

If $\gamma\colon [0, L] \longrightarrow \mathbb{R}^2$ is a regular smooth curve, then its angle is given by

$$\theta_\gamma(s) = \int_0^s \kappa_\gamma(p)\, dp + \theta_0$$

where θ_0 is the angle of $T_\gamma(0)$.

Definition 2.4.2:

If $\gamma\colon [0, L] \longrightarrow \mathbb{R}^2$ is a natural parameterization, then we define

$$K_\gamma = \int_0^L \kappa_\gamma(s)\, ds$$

to be the **total curvature** of γ.

So by the above definitions,

$$K_\gamma = \theta_\gamma(L) - \theta_\gamma(0)$$

So K_γ can also be thought of the total difference in the angle of γ.

Example 2.4.3:

If γ is a circle, then intuitively $K_\gamma = 2\pi$ since the total difference in the angle of the curve is 2π. And since the natural parameterization is given by a curve from $[0, 2\pi R]$ whose curvature is $\frac{1}{R}$ and thus

$$K_\gamma = \int_0^{2\pi R} \frac{1}{R} = 2\pi$$

as expected.

Definition 2.4.4:

A smooth curve $\gamma\colon [a,b] \longrightarrow \mathbb{R}^n$ is **n-closed** if $\gamma^{(k)}(a) = \gamma^{(k)}(b)$ for every $0 \leq k \leq n$. If γ is n-closed for every n, then γ is called **closed**.

Proposition 2.4.5:

If γ is a 1-closed regular smooth curve then $K_\gamma = 2\pi n$ for some $n \in \mathbb{Z}$.

Proof:

Since γ is 1-closed, $\gamma'(0) = \gamma'(L)$. But recall that

$$\gamma'(s) = \begin{pmatrix} \cos(\theta(s)) \\ \sin(\theta(s)) \end{pmatrix}$$

So we have that

$$\begin{pmatrix} \cos(\theta(0)) \\ \sin(\theta(0)) \end{pmatrix} = \begin{pmatrix} \cos(\theta(L)) \\ \sin(\theta(L)) \end{pmatrix}$$

Which is if and only if $\theta(L) = \theta(0) + 2\pi n$ for some $n \in \mathbb{Z}$, and so $K_\gamma = 2\pi n$ as required. ■

Definition 2.4.6:

If γ is a 1-closed regular smooth curve, then $\frac{1}{2\pi}K_\gamma$ is called γ's **winding number** (about 0).

Theorem 2.4.7 (Hopf's Theorem):

If $\gamma\colon [0,L] \longrightarrow \mathbb{R}^2$ is a closed natural parameterization, then γ is injective (other than at the points 0 and L).

We will not be proving this theorem.
This means that if γ is closed, then $K_\gamma = \pm 2\pi$. This is because the winding number is ± 1, as otherwise γ would have to intersect with itself. The sign of K_γ correlates with its orientation. We will prove this formally:

Proposition 2.4.8:

If γ is a closed curve then $K_\gamma = \pm 2\pi$.

Proof:

We assume that $\gamma\colon [0,T] \to \mathbb{R}^2$ is the natural parameterization of the curve. Suppose $\gamma(0) = 0$, and $T(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $0 \leq \gamma_1(s)$ for every $s \neq 0, T$ (we can get to this via an isometry). Let $B = \{(x,y) \mid 0 \leq x \leq y \leq T\}$ and we define a function $g\colon B \longrightarrow S^1$ (S^1 is the unit circle) by

$$g(s,t) = \begin{cases} \frac{\gamma(t)-\gamma(s)}{\|\gamma(t)-\gamma(s)\|} & s \neq t \text{ and } s \neq 0, t \neq T \\ \gamma'(s) & s = t \\ -\gamma'(0) & s = 0 \text{ and } t = T \end{cases}$$

g is therefore continuous.
Let us define $\alpha_0(t)$ (from $[0,T] \to B$) to be the line which connects $(0,0)$ to (T,T), ie. $\alpha_0(t) = t(1,1)$. And let us define $\alpha_1(t)$ to be the concatenation of the line from $(0,0)$ to $(0,T)$ with the line from $(0,T)$ to (T,T). α_0 and α_1 are both contained within B. And for $0 \leq \lambda \leq 1$, let us define $\alpha_\lambda = (1-\lambda)\alpha_0 + \lambda\alpha_1$.
Since $g \circ \alpha_\lambda(t)$ is a unit vector (since $g(t)$ always is), there exists a function θ_λ such that

$$g \circ \alpha_\lambda = \begin{pmatrix} \cos(\theta_\lambda(t)) \\ \sin(\theta_\lambda(t)) \end{pmatrix}$$

Since g and α_λ are continuous (though α_λ is not differentiable for $\lambda > 0$ as α_1 is not), so is θ_λ. Let us define

$$D(\lambda) = \theta_\lambda(T) - \theta_\lambda(0)$$

Since $g \circ \alpha_\lambda(T) = \gamma'(T)$ which is equal to $\gamma'(0) = g \circ \alpha_\lambda(0)$ since γ is closed, we have that

$$\begin{pmatrix} \cos\theta_\lambda(T) \\ \sin\theta_\lambda(T) \end{pmatrix} = \begin{pmatrix} \cos\theta_\lambda(0) \\ \sin\theta_\lambda(0) \end{pmatrix}$$

and therefore $D(\lambda) = \theta_\lambda(T) - \theta_\lambda(0)$ is a multiple of 2π.
Now, notice that $g \circ \alpha_0(t) = g(t,t) = \gamma'(t)$ and so θ_0 is the angle of γ, so

$$D(0) = \theta_0(T) - \theta_0(0) = K_\gamma$$

Notice that $g \circ \alpha_1(0) = g(0,0) = \gamma'(0) = (1,0)$ and $g \circ \alpha_1(T/2) = g \circ (0,T) = -\gamma'(0) = (-1,0)$, $g \circ \alpha_1$ rotated π radians on its path from $(0,0)$ to $(0,T)$. And similarly $g \circ \alpha_1(T) = g(T,T) = \gamma'(T) = \gamma'(0) = (1,0)$. And so $g \circ \alpha_1$ rotated another π radians on its path from $(0,T)$ to (T,T). Thus all in all $D(1) = 2\pi$ (or -2π if we were to change our orientation).
We will now prove that D is continuous. And since D is always a multiple of 2π this would mean that it is constant, and so $K_\gamma = \pm 2\pi$.
Suppose that λ is a point of discontinuity for D, then for h small enough $D(\lambda) \neq D(\lambda+h)$ and so let $\delta = \theta_\lambda - \theta_{\lambda+h}$. Then

$$|\delta(T) - \delta(0)| = |\theta_\lambda(T) - \theta_{\lambda+h}(T) - \theta_\lambda(0) + \theta_{\lambda+h}(0)| = |D(\lambda+h) - D(\lambda)|$$

and since D is always a multiple of 2π and $D(\lambda+h) \neq D(\lambda)$, this means that

$$|\delta(T) - \delta(0)| \geq 2\pi$$

Since δ is continuous, and the difference between the endpoints $\delta(0)$ and $\delta(T)$ is greater than 2π, there must exists some $0 \leq t_0 \leq T$ and n natural such that $\delta(t_0) = \pm\pi(2n+1)$ (ie. there must be a point where δ is an odd multiple of π). And so $\theta_\lambda(t_0) - \theta_{\lambda+h}(t_0) = \pm\pi(2n+1)$ and therefore

$$g \circ \alpha_\lambda(t_0) = \begin{pmatrix} \cos\theta_\lambda(t_0) \\ \sin\theta_\lambda(t_0) \end{pmatrix} = \begin{pmatrix} \cos(\theta_{\lambda+h}(t_0) \pm \pi(2n+1)) \\ \sin(\theta_{\lambda+h}(t_0) \pm \pi(2n+1)) \end{pmatrix} = -\begin{pmatrix} \cos\theta_{\lambda+h}(t_0) \\ \sin\theta_{\lambda+h}(t_0) \end{pmatrix} = -g \circ \alpha_{\lambda+h}(t_0)$$

But we can make h small enough so that α_λ and $\alpha_{\lambda+h}$ are arbitrarily close, and since g is continuous $g \circ \alpha_\lambda(t_0)$ and $g \circ \alpha_{\lambda+h}(t_0)$ must be arbitrarily close. But they are on opposite ends of the unit circle, in contradiction. ■

2.5 Three Dimensional Curves

Let α be the natural parameterization of some curve. In two dimensions, recall that we define N_α by rotating $T_\alpha = \alpha'$ ninety degrees. But rotation by ninety degrees has less meaning in three dimensions, as there are an infinite number of planes on which we can rotate ninety degrees. But recall by **proposition 2.3.2** that if $\|T_\alpha\|$ is constant, then T'_α is orthogonal to T_α. Thus we can define N_α to be the unit vector in the direction of T'_α, ie $N_\alpha = \frac{T'_\alpha}{\|T'_\alpha\|}$. And recall that we defined curvature as the scalar function κ_α such that

$$T'_\alpha = \kappa_\alpha N_\alpha$$

in three dimensions this becomes

$$T'_\alpha = \kappa_\alpha \frac{T'_\alpha}{\|T'_\alpha\|} \implies \kappa_\alpha = \|T'_\alpha\|$$

So in three dimensions, curvature is always positive, while in two dimensions it may be signed.

But $\{T_\alpha, N_\alpha\}$ is not yet an orthonormal basis, we require one more vector. We can obtain it by simply defining

$$B_\alpha = T_\alpha \times N_\alpha$$

this is orthogonal to T_α and N_α and since $\|T_\alpha\| = \|N_\alpha\| = 1$, $\|B_\alpha\| = 1$. So $\{T_\alpha, N_\alpha, B_\alpha\}$ is an orthonormal basis. Let us summarize the definitions:

Definition 2.5.1:

Let α be a natural parameterization, then we define

(1) $T_\alpha(s)$ as $\alpha'(s)$.

(2) $N_\alpha(s) = \frac{T'_\alpha(s)}{\|T'_\alpha(s)\|}$.

(3) $B_\alpha(s) = T_\alpha(s) \times N_\alpha(s)$.

And the **curvature** of α is defined to be $\kappa_\alpha(s) = \|T'_\alpha(s)\|$. Or alternatively

$$\kappa_\alpha(s) = \langle T'_\alpha(s), N_\alpha(s)\rangle$$

Now, since $\|B\| = 1$, B' is orthogonal to B and so

$$B' = \langle B', T\rangle T + \langle B', N\rangle N + \langle B', B\rangle B = \langle B', T\rangle T + \langle B', N\rangle N$$

And since we know that $\langle B, T\rangle = \langle B, N\rangle = 0$, we get that by differentiating

$$\langle B', T\rangle = -\langle B, T'\rangle, \qquad \langle B', N\rangle = -\langle B, N'\rangle$$

And since

$$\langle B, T'\rangle = \langle B, \kappa N\rangle = 0$$

and so $\langle B', T\rangle = 0$. Therefore

$$B' = \langle B', N\rangle N$$

Definition 2.5.2:

Let α be the natural parameterization of a curve, we define the **torsion** of α to be

$$\tau_\alpha(s) = -\langle B'_\alpha(s), N_\alpha(s)\rangle = \langle B_\alpha(s), N'_\alpha(s)\rangle$$

Now, we know that since T's norm is constant, $T' \perp T$ and so

$$N' = \langle N', T\rangle T + \langle N', N\rangle N + \langle N', B\rangle B = \langle N', T\rangle T + \langle N', B\rangle B = -\langle N, T'\rangle T - \langle N, B'\rangle B = -\kappa T + \tau B$$

Thus we have the system of ODEs:

$$\begin{aligned} T'_\alpha(s) &= \kappa_\alpha(s) N_\alpha(s) \\ N'_\alpha(s) &= -\kappa(s) T_\alpha(s) + \tau(s) B_\alpha(s) \\ B'_\alpha(s) &= -\tau_\alpha(s) N_\alpha(s) \end{aligned}$$

Or using matrices,

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

Thus if we are given $\kappa_\alpha(s)$ and $\tau_\alpha(s)$, and $T_\alpha(0)$, $N_\alpha(0)$, and $B_\alpha(0)$ then we can solve the ODE for T_α and integrate to get α. In fact, we need only two out of $T_\alpha(0)$, $N_\alpha(0)$, and $B_\alpha(0)$, since that will determine the third. This proves the fundamental theorem of curves for three dimensions,

Theorem 2.5.3 (The Fundamental Theorem of Curves):

Every natural parameterization is determined uniquely by its curvature, torsion, and initial conditions for T, N, and B.

Example 2.5.4:

Suppose we have a natural parameterization

$$\gamma(s) = \begin{pmatrix} \gamma_1(s) \\ \gamma_2(s) \\ 0 \end{pmatrix}$$

which is a two dimensional curve embedded onto the $[xy]$ plane in $\mathbb{R}^3$. Then

$$T(s) = \begin{pmatrix} \gamma_1'(s) \\ \gamma_2'(s) \\ 0 \end{pmatrix}, \qquad T'(s) = \begin{pmatrix} \gamma_1''(s) \\ \gamma_2''(s) \\ 0 \end{pmatrix}$$

If $T'(s) = 0$ then $\kappa(s) = \|T'(s)\| = 0$ and so $N(s)$ is undefined, and therefore so is $B(s)$ and $\tau(s)$. Otherwise since T' is on the $[xy]$ plane, so is N. Thus since $B = T \times N$, $B = (0, 0, \pm 1)$. Using the right-hand rule, we can see that B's sign is $+1$ when the curve is turning left, and -1 when turning right, and so B's sign encodes the sign of the curvature of the curve when viewed as a planar curve.
And since for any neighborhood in which B is defined, B is constant, and so $B' = 0$ and thus $\tau(s) = 0$ when defined.

Lemma 2.5.5:

Suppose f_{ij} are all differential at x_0 then let us define

$$D = \det \begin{pmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{pmatrix}$$

then

$$D'(x_0) = \sum_{i=1}^{n} \det \begin{pmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{i1}' & \cdots & f_{in}' \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{pmatrix}$$

Proof:

By definition

$$D = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \cdot \prod_{i=1}^{n} f_{i\sigma(i)}$$

and thus

$$D' = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \cdot \left(\prod_{i=1}^{n} f_{i\sigma(i)} \right)' = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \cdot \sum_{i=1}^{n} f_{i\sigma(i)}' \cdot \prod_{i \neq j=1}^{n} f_{j\sigma(j)} = \sum_{i=1}^{n} \sum_{\sigma \in S_n} f_{i\sigma(i)}' \cdot \prod_{i \neq j=1}^{n} f_{j\sigma(j)}$$

Each component in this sum is the determinant of D if instead we swapped the ith row with $f_{i1}', \dots, f_{in}'$, thus we get the equality required by the lemma.

This means that by using the determinant formula for cross products, we have that

$$(f_1 \times f_2)' = f_1' \times f_2 + f_1 \times f_2'$$

Proposition 2.5.6:

Curvature and torsion are invariant under isometries.

Proof:

Suppose φ is an isometry, then it is of the form $v \mapsto Av + c$ for some orthogonal matrix A and vector c. Then if α is a natural parameterization, so is $\varphi \circ \alpha$ and

$$(\varphi \circ \alpha(t))' = J_\varphi(\alpha(t)) \cdot \alpha'(t) = A\alpha'(t)$$

And thus $T_{\varphi\circ\alpha} = AT_\alpha$ and so

$$\kappa_{\varphi\circ\alpha} = \left\|T'_{\varphi\circ\alpha}\right\| = \|AT'_\alpha\| = \|T'_\alpha\| = \kappa_\alpha$$

so curvature is indeed invariant.
And

$$B_{\varphi\circ\alpha} = T_{\varphi\circ\alpha} \times N_{\varphi\circ\alpha} = \frac{1}{\left\|T'_{\varphi\circ\alpha}\right\|} \cdot (AT_\alpha) \times (AT'_\alpha) = \frac{1}{\kappa_\alpha}(AT_\alpha) \times (AT'_\alpha)$$

And so

$$B'_{\varphi\circ\alpha} = \frac{1}{\kappa}(AT'_\alpha \times AT'_\alpha + AT_\alpha \times AT''_\alpha) = \frac{1}{\kappa}(AT_\alpha \times AT''_\alpha)$$

And since $N_{\varphi\circ\alpha} = \frac{T'_{\varphi\circ\alpha}}{\kappa} = \frac{AT'_\alpha}{\kappa}$,

$$\tau_{\varphi\circ\alpha} = \frac{1}{\kappa^2}\langle AT_\alpha \times AT''_\alpha, AT'_\alpha\rangle = \frac{1}{\kappa^2}\langle T_\alpha \times T''_\alpha, T'_\alpha\rangle = \tau_\alpha \qquad \blacksquare$$

Proposition 2.5.7:

Let γ be a regular smooth curve such that $\kappa \neq 0$ in the entire domain. Then the image of γ is contained within a plane if and only if $\tau = 0$ everywhere.

Proof:

If γ is contained within a plane, we can compose it with an isometry to move the plane to $[xy]$. Then we showed in the above example that the torsion of the transformed curve is zero. And since torsion is preserved under isometries, γ's torsion is zero.
Now, if $\tau = 0$ then $B' = 0$ so $B = w$ is constant. Since T is orthogonal to B, $\langle T, w\rangle = 0$ and so

$$(\langle \gamma, w\rangle') = \langle \gamma', w\rangle + \langle \gamma, 0\rangle = \langle T, w\rangle = 0$$

and thus $\langle \gamma, w\rangle = c$ is constant. Thus γ is contained within the plane $\{x \mid \langle x, w\rangle = c\}$. $\blacksquare$

In two dimensions, the osculating circle is the circle centered at

$$c(s_0) = \alpha(s_0) + \frac{1}{\kappa(s_0)}N(s_0)$$

whose radius is $\frac{1}{\kappa(s_0)}$. Using the same derivation for the two dimensional case, we get the same result in three dimensions.
The plane spanned by $T(s_0)$ and $N(s_0)$ which contains $\alpha(s_0)$ is the *tangent plane* to the curve at s_0.
How do we compute κ and τ for arbitrary regular curves? Let β be an arbitrary regular curve and γ its natural parameterization:

$$\gamma = \beta \circ s^{-1} \implies \beta = \gamma \circ s$$

then

$$\begin{aligned}
&\beta'(t) = s'(t)\gamma'(s(t)), \\
&\beta''(t) = s''(t)\gamma'(s(t)) + s'(t)^2\gamma''(s(t)), \\
&\beta'''(t) = s'''(t)\gamma'(s(t)) + s''(t)s'(t)\gamma''(s(t)) + 2s'(t)s''(t)\gamma''(s(t)) + s'(t)^3\gamma'''(t)
\end{aligned}$$

Now, $\gamma' = T$ and $\gamma'' = T' = \kappa N$ and

$$\gamma''' = \kappa' N + \kappa N' = \kappa' N + \kappa(-\kappa T + \tau B) = \kappa' N - \kappa^2 T + \kappa\tau B$$

And so

$$\beta' \times \beta'' = (s'T) \times (s''T + (s')^2 \kappa N) = (s')^3 \kappa B$$

Thus

$$\|\beta' \times \beta''\| = \|\beta'\|^3 \kappa \implies \kappa = \frac{\|\beta' \times \beta''\|}{\|\beta'\|^3}$$

And

$$\langle \beta''', B \rangle = \big\langle (s')^3 \gamma''', B \big\rangle = (s')^3 \kappa\tau$$

Now, $s' = \|\beta'\|$, and so

$$(s')^3 \kappa = \|\beta'\|^3 \frac{\|\beta' \times \beta''\|}{\|\beta'\|^3} = \|\beta' \times \beta''\|$$

And since

$$\beta' \times \beta'' = (s')^3 \kappa B = \|\beta' \times \beta''\| B$$

And so

$$\tau = \frac{\langle \beta''', B \rangle}{\|\beta' \times \beta''\|} = \frac{\langle \beta''', \beta' \times \beta'' \rangle}{\|\beta' \times \beta''\|^2}$$

And since

$$\langle \beta''', \beta' \times \beta'' \rangle = \langle \beta' \times \beta'', \beta''' \rangle = \det(\beta', \beta'', \beta''')$$

we get

$$\tau = \frac{\det(\beta', \beta'', \beta''')}{\|\beta' \times \beta''\|^2}$$

Let us summarize this in the following proposition:

Proposition 2.5.8:

If β is an arbitrary regular smooth curve in $\mathbb{R}^3$, then its curvature and torsion are given by

$$\kappa_\beta(s) = \frac{\|\beta'(s) \times \beta''(s)\|}{\|\beta'(s)\|^3}, \qquad \tau_\beta(s) = \frac{\det\big(\beta'(s), \beta''(s), \beta'''(s)\big)}{\|\beta'(s) \times \beta''(s)\|^2}$$