Model Theory
Homework 0
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> **0.1 Problem**
>
> Let L be the language over the signature $\{A\}$, where A is a unary predicate symbol. Define
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> $$\phi_n = \exists \vec{x}. \bigwedge_i A(x_i) \wedge \bigwedge_{i \neq j} x_i \neq x_j, \qquad \psi_n = \exists \vec{x}. \bigwedge_i \neg A(x_i) \wedge \bigwedge_{i \neq j} x_i \neq x_j$$
>
> and let T be the theory axiomatized by $\{\phi_n, \psi_n\}_n$. Show that T is $\aleph_0$-categorical.

We first note that ϕ_n claims the existence of n distinct elements which satisfy A, while ψ_n claims the existence of n distinct elements which satisfy $\neg A$. Thus T is the theory of sets $\mathcal{A}$ such that there are infinite elements satisfying A and not satisfying A. For a model $\mathcal{A} \vDash T$ define $\mathcal{A}_A = \{x \in A \mid Ax\}$ and $\mathcal{A}_{\neg A} = \{x \in A \mid \neg Ax\} = \mathcal{A} - \mathcal{A}_A$.

Note that for a countable model $\mathcal{A} \vDash T$, both $\mathcal{A}_A$ and $\mathcal{A}_{\neg A}$ must have cardinality $\aleph_0$ (as they are infinite subsets of $\mathcal{A}$). Therefore, given $\mathcal{A}, \mathcal{B} \vDash T$ countable models, $\mathcal{A}_A \cong \mathcal{B}_A$ and $\mathcal{A}_{\neg A} \cong \mathcal{B}_{\neg A}$ purely as sets. Let $f_A: \mathcal{A}_A \to \mathcal{B}_A$ and $f_{\neg A}: \mathcal{A}_{\neg A} \to \mathcal{B}_{\neg A}$ be bijections. Since $\mathcal{A}$ can be decomposed as the disjoint union $\mathcal{A}_A \cup \mathcal{A}_{\neg A}$, these bijections extend uniquely to a bijection $f: \mathcal{A} \to \mathcal{B}$. This bijection is clearly an isomorphism. Indeed, given $x \in \mathcal{A}$, $Ax \iff x \in \mathcal{A}_A$, which is iff $fx \in \mathcal{B}_A$ (since the bijection decomposes to $f_A \cup f_{\neg A}$) which is iff Afx.

And so all two countable models of T are isomorphic, meaning T is $\aleph_0$-categorical as required.

> **0.2 Problem**
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> Let L be the language over the constant signature $\{c_n\}_n$. Let T be the theory axiomatized by $\{c_i \neq c_j\}_{i \neq j}$. Show that while there are countably many non-isomorphic countable models of T, T is complete.

Given a model $\mathcal{A} \vDash T$, define $\mathcal{A}_C = \{c_n\}_n$ (i.e. $\mathcal{A}_C$ is the set of all the interpretations of the constant symbols in $\mathcal{A}$). We claim that the isomorphism class of $\mathcal{A}$ is determined purely by $\mathcal{A} - \mathcal{A}_C$. That is, if $\mathcal{B} \vDash T$ has that $|\mathcal{B} - \mathcal{B}_C| = |\mathcal{A} - \mathcal{A}_C|$ (bijection), then $\mathcal{A} \cong \mathcal{B}$, and clearly vice versa.

Indeed, let $f_-: \mathcal{B} - \mathcal{B}_C \to \mathcal{A} - \mathcal{A}_C$ be a bijection. This extends uniquely to $f: \mathcal{B} \to \mathcal{A}$ by mapping $f c_n^{\mathcal{B}} = c_n^{\mathcal{A}}$. This extension is well-defined since both model T: all interpretations of c_n are distinct. Since f_- is a bijection and the extension is also clearly a bijection between $\mathcal{B}_C$ and $\mathcal{A}_C$, f as a whole is a bijection. Furthermore, by construction $f c_n = c_n$ and so f is an isomorphism.

That is to say, the isomorphism class of a model of T is determined uniquely by the cardinality of its non-constant subset. Given a countable model $\mathcal{A} \vDash T$, we have that $|\mathcal{A} - \mathcal{A}_C| \in \{0, 1, \ldots, \aleph_0\}$. There are $\aleph_0$ many cardinalities, and thus $\aleph_0$ many isomorphism classes of countable models of T, as required.

Completeness follows from T's uncountable categoricity. Indeed, if $\mathcal{A} \vDash T$ is uncountable then since $\mathcal{A}_C$ is countable, $\mathcal{A} - \mathcal{A}_C$ must have the same cardinality as $\mathcal{A}$. Thus, there is a single isomorphism class of cardinality $|\mathcal{A}|$, meaning T is $|\mathcal{A}|$-categorical, and thus complete (since T is countable, and has no finite models).

> **0.3 Problem**

Note that both $<$ (less-than) and $|$ (divides) are definable in this signature. So are $0, 1$ as the identities of $+, \cdot$, and thus every number in $\mathbb{N}$ is definable. Let $S \subseteq \mathbb{N}$ be an infinite set of primes, and let us define the set of formulas

$$p_S(x) = \{p \mid x \mid p \in S\} \cup \{p \nmid x \mid p \notin S\}$$

Clearly by compactness, $p_S(x)$ is consistent with T.

Given a countable model $\mathcal{A} \vDash T$, $\mathcal{A}$ can only realize countably many $p_S(x)$s. This is because if $\mathcal{A} \vDash p_S(a)$, and $\mathcal{A} \vDash p_{S'}(b)$, then a and b must be distinct (by construction of $p_S(x)$). $\mathcal{A}$ is countable, so it only has room to realize countably many of these sets of formulas.

Now, let $\mathcal{A}_S$ be a model of $T \cup p_S(x)$. We quotient out the set of $\mathcal{A}_S$s by the equivalence relation of realizing the same $p_S(x)$s. That is, $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent iff they realize the same set of $p_S(x)$s. An equivalence class thus gives a unique set of $S \subseteq \mathbb{N}$ for which all of its structures realize the $p_S(x)$s. But this set of $S \subseteq \mathbb{N}$s must be countable, and therefore therefore we must have $2^{\aleph_0}$ equivalence classes. This is because the original set of models has cardinality $2^{\aleph_0}$ and every equivalence class is countable.

Notice now that two isomorphic models must be in the same equivalence class. Indeed, if $\mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic, then under the isomorphism f we have $\mathcal{A}_1 \vDash p_S(a) \implies \mathcal{A}_2 \vDash p_S(fa)$. This means we have at least $2^{\aleph_0}$ isomorphism classes.

Finally, notice that given a countably infinite carrier set X, there are $2^{\aleph_0}$ ways of turning it into a $(+, \cdot)$-structure. This is because, the number of binary functions on X is $2^{\aleph_0}$. Thus there can only be at most $2^{\aleph_0}$ isomorphism classes, concluding the proof.

0.4 Problem

Let κ be an infinite cardinal, and L a language (possibly of cardinality $> \kappa$). Let T be an L-theory which is κ-categorical and has no finite models. Is T necessarily complete?

Let us take the $\aleph_0$-categorical theory $T = \mathrm{DLO}$. We extend the signature by adding uncountably many constant symbols $\{c_x\}_{x \in X}$ (X is uncountable). Now we add to T the axioms $(c_x \neq c_y) \rightarrow (c_x \neq c_z)$ for all $x, y, z \in X$. These axioms say that if any of the c_xs are distinct, then all of the c_xs are distinct.

Now, if $\mathcal{A} \vDash T$ is countable, it must satisfy $\mathcal{A} \vDash c_x = c_y$ for all $x, y \in X$, as otherwise all the c_xs are distinct from one another, but then $\mathcal{A}$ would have an uncountable amount of elements. So there is a unique element $c^{\mathcal{A}} \in \mathcal{A}$ which is the interpretation of these constants.

Given any other countable $\mathcal{B} \vDash T$, there is an order-preserving isomorphism from $\mathcal{A}$ to $\mathcal{B}$ which carries $c^{\mathcal{A}}$ to $c^{\mathcal{B}}$. That is, it is an isomorphism over our extended language. So T is $\aleph_0$-categorical.

But T is not complete: the sentence $c_x = c_y$ (for $x \neq y$) is independent of T. Suppose that $X = \mathbb{R}$, then interpret c_x as x in the model $\mathbb{R} \vDash T$. In this model $c_x = c_y$ is not valid, but we can just as easily interpret $c_x = 0$ for all $x \in X$ in $\mathbb{R}$.

0.5 Problem

Show that $T = \mathrm{Th}(\mathbb{Q}, \leq)$ is not uncountably categorical.

We know that $T = \mathrm{Th}(\mathbb{Q}, \leq) = \mathrm{DLO}_{00}$, the theory of dense linear orders without endpoints. Thus, we know that $(\mathbb{R}, \leq) \vDash T$ (since it is a dense linear order without endpoints). Let us construct $\mathbb{R}_2 = \mathbb{R} \amalg \mathbb{R}$, the disjoint union of two copies of $\mathbb{R}$. The order on $\mathbb{R}_2$ is identical to $\mathbb{R}$'s on both of its copies, and one copy is taken to be greater than the other. (So if $\mathbb{R}_2 = \mathbb{R}^0 \cup \mathbb{R}^1$, we have $\mathbb{R}^0 < \mathbb{R}^1$.) Clearly $\mathbb{R}_2$ is a dnese linear order without endpoints, and thus is a model of T.

Now, we claim that $\mathbb{R}$ and $\mathbb{R}_2$ are not isomorphic. For ease of the argument, assume $\mathbb{R}_2$ consists of two copies $\mathbb{R}^0 < \mathbb{R}^1$. Assume the contrary; that there exists $f\colon \mathbb{R} \to \mathbb{R}_2$ an isomorphism. Then $\left\{ x \in \mathbb{R} \mid fx \in \mathbb{R}^0 \right\}$ is a bounded set: there must be a $y \in \mathbb{R}$ such that $fy \in \mathbb{R}^1$ and for every $fx \in \mathbb{R}^0$, $fx < fy$ and so $x < y$. It is also nonempty as the preimage of $\mathbb{R}^0$ under an isomorphism. Therefore it has a supremum, s.

We consider two cases: $fs \in \mathbb{R}^0$ and $fs \in \mathbb{R}^1$.

(1) If $fs \in \mathbb{R}^0$ then for every $fx \in \mathbb{R}^0$, we have that since $x \leq s$, $fx \leq fs$. And so the image of f in $\mathbb{R}^0$ is bound by fs, and thus cannot be surjective, a contradiction.

(2) If $fs \in \mathbb{R}^1$, then for every $fx \in \mathbb{R}^1$, we must have that $s \leq x$ (since if $s > x$ then there is an x' such that $fx' \in \mathbb{R}^0$ but $x < x' < s$, in contradiction). Thus $fs \leq fx$ for every $fx \in \mathbb{R}^1$. But this means that the image of f in $\mathbb{R}^1$ is bound from below by fs, and therefore f cannot be surjective, a contradiction.

So $\mathbb{R}$ and $\mathbb{R}_2$ are non-isomorphic models of T, both having the cardinality of the continuum.