Topology

Lecture 6, Sunday April 7, 2022
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Definition 6.0.1:

If X is a topological space, a **curve** in X is a continuous function $\gamma\colon [0,1] \longrightarrow X$. It is called a curve from $\gamma(0)$ to $\gamma(1)$.

If γ is a curve we define its reverse as $\overline{\gamma}(t) = \gamma(1-t)$ which is a curve (continuous) as the composition of continuous functions. Note that $\overline{\gamma}(0) = \gamma(1)$ and $\overline{\gamma}(1) = \gamma(0)$.
If γ_1 and γ_2 are two curves such that $\gamma_1(1) = \gamma_2(0)$ then we define their *concatenation* as

$$(\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_1(2t) & 0 \leq t \leq \frac{1}{2} \\ \gamma_2(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

This is continuous since its restrictions under the finite closed cover $\left[0, \frac{1}{2}\right] \cup \left[\frac{1}{2}, 1\right]$ are continuous as the composition of continuous functions.

Definition 6.0.2:

A topological space X is **pathwise connected** if for every $a, b \in X$ there exists a curve from a to b in X.

Proposition 6.0.3:

If X is pathwise connected, X is connected.

Proof:

Suppose for the sake of a contradiction that $X = \mathcal{U} \cupdot \mathcal{V}$ which are non-empty closed sets. Then there exists $a \in \mathcal{U}$ and $b \in \mathcal{V}$, and so there exists a curve γ from a to b. But since $\mathcal{U}$ is closed in X, $\gamma^{-1}(\mathcal{U})$ is closed in $[0,1]$ and it is open, so it is also open in $[0,1]$. The same is true for $\mathcal{V}$, so $\gamma^{-1}(\mathcal{V})$ is also clopen in $[0,1]$. But $[0,1]$ is connected so the only clopen sets are $\varnothing$ and $[0,1]$. Since $a \in \mathcal{U}$, $0 \in \gamma^{-1}(\mathcal{U})$ and $1 \in \gamma^{-1}(\mathcal{V})$ so they cannot be empty and so are both equal to $[0,1]$. But then they cannot be disjoint, as then their preimages would have to be disjoint, which is a contradiction.

$\blacksquare$

Alternatively, for every $a, b \in X$ there exists a path γ between them and so $a, b \in \gamma([0,1]) = A$. Since γ is continuous and $[0,1]$ is connected, A is connected. So every two points in X are contained in a connected set, and so X is connected.

Proposition 6.0.4:

If X and Y are topological spaces where X is pathwise connected, if there exists a surjective continuous function $f\colon X \longrightarrow Y$, then Y is also pathwise connected.

Proof:

If $a, b \in Y$ then there exists $x, y \in X$ such that $f(x) = a$ and $f(y) = b$ since f is surjective. There exists a path γ from x to y, so $f \circ \gamma$ is continuous and $f \circ \gamma(0) = f(x) = a$ and $f \circ \gamma(1) = f(y) = b$ so $f \circ \gamma$ is a curve from a to b as required. $\blacksquare$

Thus if $f\colon X \longrightarrow Y$ is a continuous function from a pathwise connected space X, then $f(X)$ is pathwise connected. This is because $f\colon X \longrightarrow f(X)$ is surjective and continuous.
We define an equivalence relation on X where $a \sim b$ if and only if there exists a curve from a to b. This is reflexive (constant curve), symmetric (reverse curve), and transitive (concatenation of curves), so it is indeed an equivalence relation. This defines a partition of X, $X/\sim$. The equivalence classes under this relation are called *pathwise connected components*.
Note that if $a \sim b$ then $\gamma([0,1])$ is connected and contains a and b, so if a and b are in a pathwise connected relation, they are in a connected relation (the relation from last lecture).

Proposition 6.0.5:

(1) If $A \subseteq X$ is pathwise connected, it is contained within one of pathwise connected components.

(2) Pathwise connected components are pathwise connected.

Proof:

(1) For every $a, b \in A$ then there exists a path $\gamma\colon [0,1] \longrightarrow A$ from a to b. But expanding the codomain of γ to X (composing with the inclusion function) gives a curve in X from a to b, so $a \sim b$. So for every $b \in A$, $a \sim b$, so $A \subseteq {}^{a}/_{\sim}$ as required.

(2) Let C be a pathwise connected component. Let $a, b \in C$ then $a \sim b$ so there exists a curve $\gamma\colon [0,1] \longrightarrow X$ from a to b. Note that $[0,1]$ is pathwise connected (this is trivial), so $\gamma([0,1])$ is pathwise connected and so $\gamma([0,1]) \subseteq C'$ for some connected component C' by above. Since $a, b \in \gamma([0,1])$, $C' = C$ so restricting the codomain to get $\gamma\colon [0,1] \longrightarrow C$ gives a continuous curve from a to b in C. So for every $a, b \in C$ there exists a curve in C from a to b, so C is pathwise connected. ∎

Thus the partition of X into pathwise connected components partitions X into maximal pathwise connected components. Thus X is pathwise connected if and only if there exists a single pathwise connected component.
Since every pathwise connected component is pathwise connected and therefore connected, every pathwise connected component is contained within a connected component.

Example 6.0.6:

We define $C \subseteq \mathbb{R}^2$ as:

$$C = [0,1] \times \{0\} \cup \bigcup_{n \in \mathbb{N}} \left\{\tfrac{1}{n}\right\} \times [0,1]$$

C is a line on the x axis from 0 to 1 with lines sticking up at every $x = \frac{1}{n}$ from $y = 0$ to $y = 1$. Let S be th point $(0,1)$, then define

$$X = C \cup \{S\}$$

We now claim that X is connected, but not pathwise.
C is path connected, since for every $p, q \in C$ we can take the composition of lines from $p = (p_x, p_y)$ to $(p_x, 0)$ to $(q_x, 0)$ to $q = (q_x, q_y)$.
C is also dense in X, since for every $x \in X$ if $x \in C$ then $x \in \overline{C}$. Otherwise $x = S$ and for every $B_r(S)$, there is a $\frac{1}{n} < r$ and so $\left(0, \frac{1}{n}\right)$ is in C and in $B_r(S)$ as required. So for every open neighborhood of S, it has non-trivial intersection with C and therefore $S \in \overline{C}$. Therefore $\overline{C} = X$ as required.
Since C is dense in X and is connected, then X is connected (proven last lecture).
But X is not path connected. We show this by defining:

$$\mathcal{U}_1 = \left\{ (x,y) \in X \;\middle|\; y > \frac{1}{3} \right\}, \qquad \mathcal{U}_2 = \left\{ (x,y) \in X \;\middle|\; y < \frac{2}{3} \right\}$$

this is an open cover of X. Let $\gamma\colon [0,1] \longrightarrow X$ be a curve such that $\gamma(0) = S$, we claim that γ is constant. Since $\gamma^{-1}(\mathcal{U}_1) \cup \gamma^{-1}(\mathcal{U}_2)$ is an open cover of $[0,1]$ Since $[0,1]$ is compact, this open cover has a Lebesgue number $\varepsilon > 0$. Let m be a number such that $\frac{1}{m} < 2\varepsilon$, then we define

$$I_k = \left[\frac{k-1}{m}, \frac{k}{m} \right]$$

for $1 \leq k \leq m$. This is a finite closed cover of $[0,1]$. And since every I_k is contained within a ball of radius ε, every I_k is contained within one of $\gamma^{-1}(\mathcal{U}_i)$.
Since $0 \in I_1$, and $S \notin \mathcal{U}_2$, we have that $0 \notin \gamma^{-1}(\mathcal{U}_2)$ so $I_1 \subseteq \gamma^{-1}(\mathcal{U}_1)$.
Now for every k we define

$$w_k = \left\{ (x,y) \in \mathcal{U}_1 \;\middle|\; x < \frac{1}{k + \frac{1}{2}} \right\}, \qquad w_k' = \left\{ (x,y) \in \mathcal{U}_1 \;\middle|\; x > \frac{1}{k + \frac{1}{2}} \right\}$$

Since any point with x value $\frac{1}{k+\frac{1}{2}}$ cannot be in $\mathcal{U}_1$ (it can only have y value 0 if it is in X), w_k and w'_k form an open cover of $\mathcal{U}_1$. Since $S \in w_k$, and we can restrict $\gamma\colon I_1 \longrightarrow \mathcal{U}_1$. $\gamma(I_1)$ is connected, and so $\gamma(I_1) \subseteq w_k$ since $S \in \gamma(I_1) \cap w_k$, and $\gamma(I_1) \subseteq w_k \cup w'_k$ and is connected. So then

$$\gamma(I_1) \subseteq \bigcap_{k=1}^{\infty} w_k = \{S\}$$

Thus $\gamma\big|_{I_1}$ is constant.
Since the right endpoint of I_1 is the left endpoint of I_2, this proof continues the same on I_2 and so on until I_k. So γ is constant on all I_k to S. So if $x \in X$ is not equal to S then S cannot be connected to x by a continuous curve, so X is not pathwise connected.
The path connected components here are C and $\{S\}$.

Definition 6.0.7:

Suppose V is a real linear space and $X \subseteq V$. X is called **convex** if for every $p, q \in V$ and every $0 \leq t \leq 1$, $p + t(q-p) \in X$.

Proposition 6.0.8:

If $X \subseteq \mathbb{R}^n$ is convex, it is path connected.

Proof:

Let $p, q \in X$ then $\gamma\colon [0,1] \longrightarrow \mathbb{R}^n$ by $\gamma(t) = p + t(q-p)$ is a continuous function from p to q. Since scalar multiplication is continuous, this is indeed a curve from p to q. And since X is convex, $\gamma(t) \in X$ so we can restrict the codomain of γ to X and get a continuous curve $\gamma\colon [0,1] \longrightarrow X$ as required. ■