Group Theory

Lecture 5, Sunday November 20, 2022

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Notice that if we focus on $(\mathbb{Z}_n, +)$, then since the group is abelian, every subgroup is normal since $gH = Hg$ by abelianness. If we take $G = (\mathbb{Z}_{60}, +)$ and the subgroup $H = \langle 6\rangle$ then:

$$G/H = \{H, 1+H, 2+H, 3+H, 4+H, 5+H\}$$

And if we take $N = \langle 2\rangle$, we have that $H < N < G$, so we can discuss $N/H = \{H, 2+H, 4+H\} \leq G/H$. And therefore we can go one step further and:

$$\left. {}^{G}\!/_{H} \middle/ {}_{N/H} \right. = \{\{H, 2+H, 4+H\}, \{1+H, 3+H, 5+H\}\}$$

5.1 Homomorphisms

Definition 5.1.1:

Suppose $(G, \cdot)$ and $(H, \circ)$ are groups then a function $f\colon G \longrightarrow H$ is a **homomorphism** if for every $a, b \in G$: $f(a \cdot b) = f(a) \circ f(b)$.

Notice that $f(e_G) = f(e_G \cdot e_G) = f(e_G) \circ f(e_G)$, and so if we take the inverse of $f(e_G)$ we get $f(e_G) = e_H$. And $f(a) \circ f(a^{-1}) = f(a \cdot a^{-1}) = f(e) = e$ and similarly $f(a^{-1}) \circ f(a) = e$, so $f(a)^{-1} = f(a^{-1})$.

Definition 5.1.2:

If $f\colon G \longrightarrow H$ is a homomorphism, we define the **image** and **kernel** of f to be:

$$\operatorname{Im} f = \{f(g) \mid g \in G\} \qquad \operatorname{Ker} f = \{g \in G \mid f(g) = e\}$$

Notice that $\operatorname{Im} f \leq H$ and $\operatorname{Ker} f \leq G$. The image is a subgroup since we have shown $e = f(e) \in \operatorname{Im} f$, it is closed under inverses since if $f(a) \in \operatorname{Im} f$ then $f(a)^{-1} = f(a^{-1}) \in \operatorname{Im} f$, and if $f(a), f(b) \in \operatorname{Im} f$ then so is $f(a)f(b) = f(ab) \in \operatorname{Im} f$. And the kernel is a subgroup since $e \in \operatorname{Ker} f$, if $f(a) = e$ then $f(a^{-1}) = f(a)^{-1} = e^{-1} = e$ so $a^{-1} \in \operatorname{Ker} f$, and if $f(a) = f(b) = e$ then $f(ab) = f(a)f(b) = e$ so $ab \in \operatorname{Ker} f$.
Moreso, the kernel of a subgroup is a normal subgroup.

Proposition 5.1.3:

If $f\colon G \longrightarrow H$ then $\operatorname{Ker} f \trianglelefteq G$.

Proof:

Let $K = \operatorname{Ker} f$ and let $g \in G$, $k \in K$. Then since $f(k) = e$:

$$f(gkg^{-1}) = f(g)f(k)f(g^{-1}) = f(g)f(g^{-1}) = e$$

And so $gkg^{-1} \in K$. So for every g, $gKg^{-1} \subseteq K$, and so K is normal.

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Example:

The **trivial homomorphism** is a homomorphism $f\colon G \longrightarrow H$ such that for every $g \in G$, $f(g) = e$. Then $\operatorname{Im} f = \{e\}$ and $\operatorname{Ker} f = G$.
And the **identity homomorphism** over a group G is the homomorphism $f\colon G \longrightarrow G$ such that $f(g) = g$. Then $\operatorname{Im} f = G$ and $\operatorname{Ker} f = \{e\}$.

It is trivial to see why these functions are homomorphisms.

Lemma 5.1.4:

f a homomorphism is injective if and only if $\operatorname{Ker} f = \{e\}$.

Proof:

If f is injective then since f is a homomorphism, $f(e) = e$, so the only element that can map to e is e (notice these may be in different sets) and therefore $\operatorname{Ker} f = \{e\}$. Now to show the converse, suppose $f(g) = f(g')$ then $f(g)f(g')^{-1} = f(gg'^{-1}) = e$, and since the kernel is trivial, $gg'^{-1} = e$ and therefore $g = g'$. So f is injective.

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Theorem 5.1.5 (The First Isomorphism Theorem):

If $f\colon G \longrightarrow H$ is an homomorphism then ${}^{G}\!/_{\operatorname{Ker} f} \cong \operatorname{Im} f$.

Proof:

Let $K = \operatorname{Ker} f$, then our goal is to construct an isomorphism (which recall is a bijective homomorphism) between ${}^{G}\!/_{K}$ to $\operatorname{Im} f$. We will denote this isomorphism as $\tilde{f}$. Given $gK \in {}^{G}\!/_{K}$ then the rational thing would be to map it to $f(g)$, that is $\tilde{f}(gK) = f(g)$. We must show 4 things: $\tilde{f}$ is well defined, is a homomorphism, is injective, and is surjective.
$\tilde{f}$ is well defined since if $gK = g'K$ then $g' = gk$ for some $k \in K$, so $f(g') = f(g)f(k) = f(g)e = f(g)$. So no matter what representative we have for gK, their image in f is the same.
$\tilde{f}$ is a homomorphism since K is normal so $gK \cdot g'K = gg'K$ so:

$$\tilde{f}(gK \cdot g'K) = \tilde{f}(gg'K) = f(gg') = f(g)f(g') = \tilde{f}(gK)\tilde{f}(g'K)$$

To show $\tilde{f}$ is injective, we will show that its kernel is trivial. Suppose $\tilde{f}(gK) = e$, then by definition $f(g) = e$, and therefore $g \in K$, which in turn means $gK = K$. So $\operatorname{Ker} \tilde{f} = \{K\}$ (and K is the identity element of ${}^{G}\!/_{K}$), and therefore $\tilde{f}$ is injective.
$\tilde{f}$ is surjective since if $f(g) \in \operatorname{Im} f$ then $\tilde{f}(gK) = f(g)$.

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Alternatively we could show that $\tilde{f}$ is injective since if $\tilde{f}(gK) = \tilde{f}(g'K)$ then $f(g) = f(g')$, so $f(gg'^{-1}) = e$, so $gg'^{-1} \in K$, and therefore $g \in g'K$, so $gK = g'K$. And therefore $\tilde{f}$ is injective. But this is less elegant and the lemma which gives a criterion for injectivity is an essential and important one.

Theorem 5.1.6 (The Second Isomorphism Theorem):

If $H \leq G$ and $N \trianglelefteq G$, then $HN \leq G$, $N \cap H \trianglelefteq H$, and ${}^{HN}\!/_{N} \cong {}^{H}\!/_{H \cap N}$.

Proof:

It is obvious that $HN \leq G$ since N is normal so $HN = NH$, and we showed that this is necessary and sufficient to show that HN is a subgroup. And $N \cap H$ is a normal subgroup of H since if $n \in N \cap H$ then $hN = Nh$ since N is normal, and if $hn \in h(N \cap H) \subseteq hN = Nh$ so $hn = n'h$, and so $n' = hnh^{-1}$. And since $n \in N \cap H$, $hnh^{-1} \in H$, so $n' \in H$ and therefore $hn = n'h$ for some $n' \in N \cap H$ and therefore $h(N \cap H) = (N \cap H)h$, and so $N \cap H$ is normal.
We will define an isomorphism:

$$f\colon H \longrightarrow {}^{HN}\!/_{N}$$

By $f(h) = hN$. This is the only natural choice (recall that $H \leq HN$). It is simple to see why f is a homomorphism. Then if $h \in H$ then $hN = hnN$ for $n \in N$ so $hN \in {}^{HN}\!/_{N}$. And if $hnN \in {}^{HN}\!/_{N}$ then $hnN = hN = f(h)$, so $\operatorname{Im} f = {}^{HN}\!/_{N}$. And $\operatorname{Ker} f = N \cap H$ since $hN = N$ if and only if $h \in N$, and since $h \in H$ then h must be in $N \cap H$.
So by the first isomorphism theorem:

$${}^{H}\!/_{\operatorname{Ker} f} = {}^{H}\!/_{H \cap N} \cong {}^{NH}\!/_{N} = \operatorname{Im} f$$

Theorem 5.1.7 (The Third Isomorphism Theorem):

Suppose $K \leq H \leq G$ are groups such that $H, K \trianglelefteq G$. Then $K \trianglelefteq H$, ${}^{H}/_{K} \trianglelefteq {}^{G}/_{K}$, and:

$$\left. {}^{G}/_{K} \middle/ {}^{H}/_{K} \right. \cong {}^{G}/_{H}$$

Proof:

It is trivial to see that $K \trianglelefteq H$ since K is normal in G and cosets of K relative to H are cosest relative to G. It is also trivial to see why ${}^{H}/_{K} \leq {}^{G}/_{K}$ since $hK \in {}^{G}/_{K}$ since $h \in G$ and ${}^{H}/_{K}$ is a group. It is a normal subgroup since if $g \in G$ then $gK\, {}^{H}/_{K} = \{gKhK \mid h \in H\} = \{ghK \mid h \in H\}$ since H is normal. And since K is normal $ghK = Kgh$, and so $ghK = Kghg^{-1}g$. Since H is normal $ghg^{-1} \in H$, so $ghK = Kh'g = Kh'gK$, and therefore $gK\, {}^{H}/_{K} = {}^{H}/_{K}\, gK$. So ${}^{H}/_{K}$ is indeed normal.
We will dfefine a homomorphism:

$$f\colon\ {}^{G}/_{K} \longrightarrow {}^{G}/_{H}$$

By $f(gK) = gH$. This is well defined since if $g_1K = g_2K$ then $g_1H = g_1KH = g_2KH = g_2H$ ($H = KH$ since $K \leq H$). And it is a homomorphism since $f(g_1Kg_2K) = f(g_1g_2K) = g_1g_2H = g_1Hg_2H = f(g_1)f(g_2)$ since K and H are normal. And $\operatorname{Im} f = {}^{G}/_{H}$ and $\operatorname{Ker} f = \{gK \mid gH = H\} = \{gK \mid g \in H\} = {}^{H}/_{K}$. And so:

$$\left. {}^{G}/_{K} \middle/ \operatorname{Ker} f \right. = \left. {}^{G}/_{K} \middle/ {}^{H}/_{K} \right. \cong \operatorname{Im} f = {}^{G}/_{H}$$

As required.

Definition 5.1.8:

A **lattice** is a partially ordered set $(\Gamma, \preccurlyeq)$ such that for every $\alpha, \beta \in \Gamma$ they have an **upper bound** and **lower bound** $\alpha \vee \beta$ and $\alpha \wedge \beta$ respectively such that $\alpha, \beta \preccurlyeq \gamma$ if and only if $\alpha \vee \beta \preccurlyeq \gamma$, and $\gamma \preccurlyeq \alpha, \beta$ if and only if $\gamma \preccurlyeq \alpha \wedge \beta$.

Notice that the upper and lower bounds are unique, since if x and y are both upper bounds to α and β then $\alpha, \beta \preccurlyeq x, y$ so $x \preccurlyeq y$ and $y \preccurlyeq x$ so $x = y$. A similar argument can be used for lower bounds.

Example:

If X is a set $(\mathcal{P}\, X, \subseteq)$ is a lattice where $A \vee B = A \cup B$ and $A \wedge B = A \cap B$.

Example:

If G is a group, let

$$\mathcal{L}(G) = \{H \leq G\}$$

Then $(\mathcal{L}, \leq)$ is a lattice where $A \vee B = \langle A \cup B \rangle$ is the upper bound (since it is the smallest subgroup containing both A and B), and $A \wedge B = A \cap B$ is a lower bound.
If we define:

$$\mathcal{L}_N(G) = \{N \trianglelefteq G\}$$

Then $(\mathcal{L}_N(G), \leq)$ (the $\leq$ can be replaced with $\trianglelefteq$) is a lattice where $A \vee B = AB$ and $A \wedge B = A \cap B$. AB is an upper bound since it is normal ($gABg^{-1} = gAg^{-1}B = AB$).

Definition 5.1.9:

If Γ and Π are lattices, a function $\varphi\colon \Gamma \longrightarrow \Pi$ is a **lattice homomorphism** if for every $\alpha, \beta \in \Gamma$:

$$\varphi(\alpha \vee \beta) = \varphi(\alpha) \vee \varphi(\beta) \text{ and } \varphi(\alpha \wedge \beta) = \varphi(\alpha) \wedge \varphi(\beta)$$

φ is an **lattice isomorphism** if it is bijective. Two lattices are **isomorphic** if there exists a lattice isomorphism between them.

Lemma 5.1.10:

If $f\colon G \longrightarrow H$ is a homomorphism then for every $K \leq H$, $f^{-1}(K)$ is a subgroup of G.

Proof:

We know that since $e \in K$ and $f(e) = e$, we know that $e \in f^{-1}(K)$. And if $a \in f^{-1}(K)$, then $f(a) \in K$, so $f(a^{-1}) = f(a)^{-1} \in K$, so $a^{-1} \in f^{-1}(K)$. And if $a, b \in f^{-1}(K)$ then $f(a), f(b) \in K$ so $f(a)f(b) = f(ab) \in K$ so $ab \in f^{-1}(K)$, as required.

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Lemma 5.1.11:

If $K \trianglelefteq G$ then every subgroup of ${}^{G}/_{K}$ is of the form ${}^{H}/_{K}$ for $K \leq H \leq G$ (and every set of that form is a subgroup).

Proof:

Let us focus on the function:

$$\pi\colon G \longrightarrow {}^{G}/_{K}, \qquad g \mapsto gK$$

This function is a surjective homomorphism. Then if B is a subgroup of ${}^{G}/_{K}$, then by above $H = \pi^{-1}(B)$ is a subgroup of G. We now argue that $B = {}^{H}/_{K}$. This is true since π is surjective so $\pi(H) = \pi^{-1}(\pi(B)) = B$, and by definition $\pi(H) = \{hK \mid h \in H\} = {}^{H}/_{K}$, so $B = {}^{H}/_{K}$.

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Lemma 5.1.12:

If $K \trianglelefteq G$ and $K \leq H_1, H_2 \leq G$ then:

$${}^{H_1}/_{K} \cap {}^{H_2}/_{K} = {}^{H_1 \cap H_2}/_{K} \qquad \left\langle {}^{H_1}/_{K}, {}^{H_2}/_{K} \right\rangle = {}^{\langle H_1, H_2 \rangle}/_{K}$$

Proof:

We know that if $h \in H_1 \cap H_2$ then hK is in both ${}^{H_1}/_{K}$ and ${}^{H_2}/_{K}$. And if A is in both quotient groups, then $A = h_1K = h_2K$ for $h_1 \in H_1$, $h_2 \in H_2$. Then $h_1 = h_2k \in H_2$, so $h_1 \in H_1 \cap H_2$ and therefore $A \in {}^{H_1 \cap H_2}/_{K}$ as required. That proves the first equality.
We know that ${}^{H_1}/_{K}$, ${}^{H_2}/_{K}$ are subgroups of ${}^{\langle H_1, H_2 \rangle}/_{K}$, so the cycle of these quotient groups is also a subgroup (as the smallest group to contain them both). And if $h \in \langle H_1, H_2 \rangle$ then hK is in the cycle $\left\langle {}^{H_1}/_{K}, {}^{H_2}/_{K} \right\rangle$ since h can be written as a product of elements in H_1 and H_2 and thus hK is equal to that product times K. That is equivalent to taking every element in that product and multiplying it by K, which is in that cycle. This proves the second equality.

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Theorem 5.1.13:

Let G be a group and $K \trianglelefteq G$ a normal subgroup, let:

$$\mathcal{L}(G, K) = \{H \leq G \mid K \subseteq H\} \subseteq \mathcal{L}(G)$$

Then $\mathcal{L}\left({}^{G}/_{K}\right)$ and $\mathcal{L}(G, K)$ are isomorphic.

Proof:

Notice that by one of the above lemmas, $\mathcal{L}\left({}^{G}/_{K}\right) = \left\{ {}^{H}/_{K} \;\middle|\; H \in \mathcal{L}(G, K) \right\}$.

We must define a lattice isomorphism $\varphi\colon \mathcal{L}(G,K) \longrightarrow \mathcal{L}\left({}^{G}\!/\!_{K}\right)$. We define $\varphi(H) = {}^{H}\!/\!_{K}$ (this is well defined since ${}^{H}\!/\!_{K} \subseteq {}^{G}\!/\!_{K}$). φ is a lattice homomorphism since $\varphi(H_1 \cap H_2) = {}^{H_1 \cap H_2}\!/\!_{K} = {}^{H_1}\!/\!_{K} \cap {}^{H_2}\!/\!_{K} = \varphi(H_1) \cap \varphi(H_2)$. And $\varphi(\langle H_1, H_2\rangle) = {}^{\langle H_1, H_2\rangle}\!/\!_{K} = \left\langle {}^{H_1}\!/\!_{K}, {}^{H_2}\!/\!_{K} \right\rangle$. This is true since ${}^{\langle H_1, H_2\rangle}\!/\!_{K}$ contains both ${}^{H_{1/2}}\!/\!_{K}$s, and it is the smallest group to do so.
And we define another lattice ψ in the other direction:

$$\psi\colon \mathcal{L}\left({}^{G}\!/\!_{K}\right) \longrightarrow \mathcal{L}(G,K) \qquad \psi\left({}^{H}\!/\!_{K}\right) = H$$

And this is a lattice homomorphism since:

$$\psi\left({}^{H_1}\!/\!_{K} \cap {}^{H_2}\!/\!_{K}\right) = \psi\left({}^{H_1 \cap H_2}\!/\!_{K}\right) = H_1 \cap H_2$$

And

$$\psi\left(\left\langle {}^{H_1}\!/\!_{K}, {}^{H_2}\!/\!_{K} \right\rangle\right) = \psi\left({}^{\langle H_1, H_2\rangle}\!/\!_{K}\right) = \langle H_1, H_2\rangle$$

Notice that φ and ψ are inverses: if $H \in \mathcal{L}(G,K)$ then:

$$\psi(\varphi(H)) = \psi\left({}^{H}\!/\!_{K}\right) = H$$

and if ${}^{H}\!/\!_{K} \leq {}^{G}\!/\!_{K}$ then:

$$\varphi\left(\psi\left({}^{H}\!/\!_{K}\right)\right) = \varphi(H) = {}^{H}\!/\!_{K}$$

So the lattics are isomorphisms.

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These isomorphisms preserve indexes, normalness, and subgrouping.