Mathematical Logic

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2.1 Normal Forms

Recall that every boolean function/formula can be written in *conjunctive normal form*, that is it can be written in the form

$$(\delta_1^1 A_1 \wedge \cdots \wedge \delta_n^1 A_n) \vee \cdots \vee (\delta_1^m A_1 \wedge \cdots \wedge \delta_n^m A_n) = \bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} \delta_j^i A_j$$

where δ_i^j is either negation or nothing. Now notice that if we denote this formula as φ then we know that $\neg\varphi$ has its own conjuctive normal form:

$$\neg\varphi = \bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} \varepsilon_j^i A_j$$

and so if we negate both sides above recalling that $\neg(A \vee B) = \neg A \wedge \neg B$ and $\neg(A \wedge B) = \neg A \vee \neg B$, we get

$$\varphi = \neg\left(\bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} \varepsilon_j^i A_j\right) = \bigwedge_{i=1}^{m} \bigvee_{j=1}^{n} \neg\varepsilon_j^i A_j$$

Thus if we define $\delta_j^i = \neg\varepsilon_j^i$ we get that

$$\varphi = \bigwedge_{i=1}^{m} \bigvee_{j=1}^{n} \delta_j^i A_j$$

This is called the *disjunctive normal form of* φ.

Since every formula can be written in disjunctive normal form, so can $A \to B$. Specifically its disjunctive normal form is $\neg A \vee B$.

Notice that not every formula can be written using just conjunction, disjunction, and implication. This is because the value of any of these connectives when both input values are true is true. Thus any arbitrary composition of these connectives must have an output value of true when both inputs are true, and so these connectives cannot compose to create formulas like negation.

Definition 2.1.1:

nor is a connective, denoted $\downarrow$ with the following truth table

		$\downarrow$
true	true	false
true	false	false
false	true	false
false	false	true

and **nand** is a connective denoted $\uparrow$:

		$\uparrow$
true	true	true
true	false	true
false	true	true
false	false	false

Notice that $\neg A \Leftrightarrow (A \downarrow A)$ and $A \wedge B \Leftrightarrow (A \downarrow A) \downarrow (B \downarrow B)$, and since every formula can be written using just conjunctions and negations, every formula can be written with nor. Similarly $\neg A \Leftrightarrow (A \uparrow A)$ and $A \wedge B \Leftrightarrow (A \uparrow A) \uparrow (B \uparrow B)$, so every formula can also be written using nand.

Proposition 2.1.2:

Nand and nor are the only connectives which are sufficient for constructing any formula.

Proof:

Suppose $\star$ is a connective which can construct any formula. Notice that $\mathsf{true} \star \mathsf{true}$ must be false since otherwise any formula which maps two true values to a false value cannot be written as a composition of $\star$. Similarly we must have $\mathsf{false} \star \mathsf{false} = \mathsf{true}$. Now if $\star$ isn't $\uparrow$ or $\downarrow$ then $\mathsf{true} \star \mathsf{false} = \mathsf{true}$ and $\mathsf{false} \star \mathsf{true} = \mathsf{false}$ or $\mathsf{true} \star \mathsf{false} = \mathsf{false}$ and $\mathsf{false} \star \mathsf{true} = \mathsf{true}$. But then notice that $p \star q = \neg q$ or $p \star q = \neg p$ and these cannot construct any formula dependent on two variables. ■

Example 2.1.3:

Suppose you are wandering in a town inhabited by truth-tellers (people who always tell the truth) and liars (people who always lie). As a person with big aspirations, you are uninterested in remaining in this backwater town and wish to make your way to the capital. But being a shortsighted person, you forgot your map and are at a fork in the road and you don't know whether to turn right or left. Unfortunately the townsfolk are all inside due to a tornado warning and you are only able to find a single person to ask for directions, not knowing whether they are a truth-teller or liar. You only have time to ask one question, what can you ask them in order to ensure that you know which way to turn?

Think of your question as a logical connective, and we have two formulas, truth and left which are true only when the person is a truth-teller and when the capital is on the left respectively. We want a formula φ which is independent of truth and is true only when left is true. We can also think of two basic questions A "are you a truth-teller" and B "is the capital on the left", these essentially generate two more formulas. Thus we have the truth table:

truth	left	A	B	φ
true	true	true	true	true
true	false	true	false	false
false	true	true	false	true
false	false	true	true	false

Note that if Q is any question you ask you

2.2 Formal Theories

Definition 2.2.1:

Given a countble set of symbols $\mathcal{L}$, any finite string composed of characters in $\mathcal{L}$ (the elements of $\mathcal{L}^*$) is called a **experssion**.

A **formal language** is a subset of $\mathcal{L}^*$, its elements are called **well-formed formulas**.

A **formal theory** is a formal language in which there is a subset of well-formed formulas called **axioms**. If there exists an algorithm to determine if a well-formed formula is an axiom, then the theory is called **axiomatic**. Furthermore a formal theory must be equipped with a finite set of relations between well-formed formulas $R_1, \dots, R_n$ called **rules of inference** such that for every i there is a unique j where every set of j well-formed formulas and every well-formed formula φ, we can determine whether or not the j well-formed formulas are in relation R_i with φ.

Definition 2.2.2:

A **proof** in a formal theory $\mathcal{T}$ is a sequence of $\varphi_1, \dots, \varphi_n$ of well-formed formulas such that for every i either φ_i is an axiom or φ_i follows from some $\varphi_{i_1}, \dots, \varphi_{i_\ell}$ by the rules of inference of the theory for $i_1, \dots, i_\ell < i$. If a well-formed formula φ can be proven then we write $\vdash \varphi$.

A **theorem** is a well-formed formula which is used in a proof (that is, it can be proven by the theory).

A theory is **decidable** if given any well-formed formula, it can be determined if it is a theorem (can be proven) or not. Otherwise the theory is **undecidable**.

We can create a formal theory over the language $\mathcal{L} = \{(,), \neg, \to, A_1, \dots, A_n, \dots\}$ where well-formed formulas are constructed recursively:

(1) All statement letters A_i are well-formed.

(2) If φ and ψ are well-formed, then so are $(\neg\varphi)$ and $(\varphi \to \psi)$.

Furthermore the axioms of the theory are

(1) $(\psi \to (\varphi \to \psi))$

(2) $\Big((\varphi \to (\psi \to \mu)) \to \big((\varphi \to \psi) \to (\varphi \to \mu)\big)\Big)$

(3) $\Big(((\neg\psi) \to (\neg\varphi)) \to \big(((\neg\psi) \to \varphi) \to \psi\big)\Big)$

Note that this actually defined countably many axioms, as φ, ψ, and μ may be any well-formed formulas.

Finally, the only rule of inference is that the well-formed formulas φ and $(\varphi \to \psi)$ infer ψ (inference is denoted by $\Rightarrow$ as well). This rule of inference is famously called modus ponens.

Lemma 2.2.3:

$\vdash (\varphi \to \varphi)$

Proof:

By the second axiom where we have replaced ψ by $(\varphi \to \varphi)$ and μ with φ we have:

$$\big((\varphi \to ((\varphi \to \varphi) \to \varphi)\big) \to \big((\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi)\big)$$

and by the first axiom we have

$$\varphi \to \big((\varphi \to \varphi) \to \varphi\big)$$

By modus ponens we have then that

$$(\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi)$$

And by the first axiom we have $\varphi \to (\varphi \to \varphi)$ so again by modus ponens we have $\varphi \to \varphi$, as required. ■

Definition 2.2.4:

If Γ is a set of well-formed formulas, we say that $\Gamma \vdash \varphi$ if there exists a sequence of well-formed formulas $\varphi_1, \dots, \varphi_n = \varphi$ where every φ_i is either an axiom or in Γ or is inferred from previous φ_js by the rules of inference.

Theorem 2.2.5 (The Deduction Theorem):

If Γ is a set of well-formed formulas and φ and ψ are well-formed formulas where $\Gamma, \varphi \vdash \psi$ then $\Gamma \vdash (\varphi \to \psi)$. In particular $\varphi \vdash \psi$ means that $\vdash (\varphi \to \psi)$.

Proof:

We will show inductively on the length of the proof $\psi_1, \dots, \psi_n$.

For the base case, notice that either $\psi_1 \in \Gamma$, $\psi_1 = \varphi$, or ψ_1 is an axiom.

(1) If $\psi_1 \in \Gamma$: since $\psi_1 \to (\varphi \to \psi_1)$ and $\psi_1 \in \Gamma$ so when proving with Γ by modus ponens we have $\varphi \to \psi_1$, so $\Gamma \vdash (\varphi \to \psi_1)$.

(2) If $\psi_1 = \varphi$: by our lemma above, $\vdash (\varphi \to \varphi)$ and thus it is also true when proving with Γ.

(3) If ψ_1 is an axiom: similarly we have $\psi_1 \to (\varphi \to \psi_1)$ and since ψ_1 is an axiom, by modus ponens we have $\varphi \to \psi_1$.

Now inductively, we know that ψ_i is either in Γ, equal to φ, is an axiom, or is inferred by previous φ_js. The first three cases are identical by above. Otherwise, since the only rule of inference is modus ponens, we must must show that there is some $j < i$ such that ψ_j and $\psi_j \to \psi_i$ are proven. We know that $\Gamma \vdash (\varphi \to \psi_j)$ for $j < i$ by induction since the proof of $\psi_j \to \psi_i$ is fewer than i steps (since it is used to prove ψ_i), we have that $\Gamma \vdash \big(\varphi \to (\psi_j \to \psi_i)\big)$ also by induction. By the second axiom we have:

$$\Gamma \vdash \big(\varphi \to (\psi_j \to \psi_i)\big) \to \big((\varphi \to \psi_j) \to (\varphi \to \psi_i)\big)$$

Since we know that $\Gamma \vdash \big(\varphi \to (\psi_j \to \psi_i)\big)$ we have

$$\Gamma \vdash (\varphi \to \psi_j) \to (\varphi \to \psi_i)$$

and since $\Gamma \vdash (\varphi \to \psi_j)$ this means $\Gamma \vdash (\varphi \to \psi_i)$ as required. ■