

Infinitesimal Calculus 3

Lecture 5, Sunday November 6, 2022
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4.1 Connected Spaces

Definition 4.1.1:

If X is a metric space, $E \subseteq X$ is **disconnected** if there exists two disjoint open sets \mathcal{O}_1 and \mathcal{O}_2 such that $E \subseteq \mathcal{O}_1 \cup \mathcal{O}_2$ and $E \cap \mathcal{O}_1, E \cap \mathcal{O}_2 \neq \emptyset$. A **connected** space is a space which is not disconnected.

This is equivalent to saying that if $E \subseteq \mathcal{O}_1 \cup \mathcal{O}_2$ then $S \subseteq \mathcal{O}_1$ or $S \subseteq \mathcal{O}_2$.

Proposition 4.1.2:

A metric space is connected if and only if the only clopen sets is the entire space and the empty set.

Proof:

Suppose X is disconnected, then there exists open sets \mathcal{O}_1 and \mathcal{O}_2 such that $X = \mathcal{O}_1 \cup \mathcal{O}_2$. Therefore the complement of \mathcal{O}_1 is \mathcal{O}_2 and so it is also closed. And since \mathcal{O}_1 and \mathcal{O}_2 are disjoint and have non-empty intersection with X , they are neither X nor the empty set. Therefore if X is disconnected there exists clopen sets other than X and \emptyset .

Suppose \mathcal{O}_1 is a clopen set in X not equal to X or \emptyset , then $\mathcal{O}_2 = \mathcal{O}_1^c$ is open and disjoint from \mathcal{O}_1 . Since $\mathcal{O}_1 \neq X, \emptyset$, $\mathcal{O}_2 \neq X, \emptyset$ since $X = \mathcal{O}_1 \cup \mathcal{O}_2$, X is disconnected. ■

So \mathbb{R} for example is connected. And $\mathbb{R} \setminus \{0\}$ is disconnected since $(-\infty, 0)$ and $(0, \infty)$ are open, disjoint, and cover $\mathbb{R} \setminus \{0\}$.

Definition 4.1.3:

A **line segment** between two vectors P and Q in a linear space X is

$$\overrightarrow{PQ} = \{P + t(Q - P) \mid t \in [0, 1]\}$$

Notice that $\overrightarrow{PQ} = \overleftarrow{QP}$, and both P and Q are in this segment. If the line segment in focus is understood, x_t is understood to be the point $P + t(Q - P)$.

Proposition 4.1.4:

A line segment in a normed linear space X is connected.

Proof:

Suppose $P, Q \in X$, suppose for the sake of a contradiction that A and B are open such that $\overrightarrow{PQ} \subseteq A \cup B$ as well as having non-empty intersections with the segment. We know $x_1 = Q$ and suppose $x_1 \in B$, we define:

$$K = \{t \in [0, 1] \mid x_t \in A\} \quad u = \sup K$$

K is non-empty since A has non-empty intersection with \overrightarrow{PQ} and K is bound by 1, so it has a supremum. We will show that $x_u \notin A$. If $u = 1$ then $x_u = Q \in B$ which is disjoint from A , so $x_u \notin A$. If $u < 1$ and $x_u \in A$, since A is open there is an $\varepsilon > 0$ such that $B_\varepsilon(x_u) \subseteq A$. Notice then that $x_{t'} = x_{t + \frac{\varepsilon}{2\|Q-P\|}}$ must be in A since $\|x_t - x_{t'}\| = \frac{\varepsilon}{2} < \varepsilon$. But $t' > t$ and $t' \in K$ which is a contradiction to t 's supremumness.

And $x_u \notin B$ since if it were, since B is open so there is a ball $B_\varepsilon(x_u) \subseteq B$. And so similar to before there must be a $\delta > 0$ such that for every $u - \delta < t < u + \delta$, $x_t \in B$. So $u - \delta$ is an upper bound to K , which is a contradiction.

So $x_u \notin A \cup B$, but it must be in \overrightarrow{PQ} since $u \in [0, 1]$ which is a contradiction. ■

Theorem 4.1.5:

If $\{S_\lambda\}_{\lambda \in \Lambda}$ are connected sets such that their intersection is non-empty, then

$$S = \bigcup_{\lambda \in \Lambda} S_\lambda$$

is also connected.

Proof:

Suppose B and C are open and disjoint such that $S \subseteq B \cup C$. Let $\lambda \in \Lambda$ and so $S_\lambda \subseteq B \cup C$, so $S_\lambda \subseteq B$ or $S_\lambda \subseteq C$. Without loss of generality suppose it is a subset of B . Then take $\gamma \neq \lambda \in \Lambda$ then $S_\gamma \subseteq B$ or $S_\gamma \subseteq C$. Since there exists an $x \in S_\lambda \cap S_\gamma$ and $x \notin C$, $S_\gamma \subseteq B$. So for every $\lambda \in \Lambda$, $S_\lambda \subseteq B$, and therefore $S \subseteq B$, so S is connected. ■

Definition 4.1.6:

If X is a normed linear space and $P_1, \dots, P_n \in X$, the **polygon chain** is

$$\overleftrightarrow{P_1 P_2} \cdots \overleftrightarrow{P_n} = \overleftrightarrow{P_1 P_2} \cup \overleftrightarrow{P_2 P_3} \cup \cdots \cup \overleftrightarrow{P_{n-1} P_n}$$

Proposition 4.1.7:

Polygonal chains are connected.

Proof:

We will prove so through induction on the number of vectors in the chain. For $n = 2$ this is simply a line segment, which we know is connected. We know that $\overleftrightarrow{P_1 \cdots P_{n+1}} = \overleftrightarrow{P_1 \cdots P_n} \cup \overleftrightarrow{P_n P_{n+1}}$. By our inductive hypothesis, these are both connected and have a non empty intersection (since P_n is in it), so by the above theorem, their union is connected as well. ■

Definition 4.1.8:

A **path** between two vectors P, Q in a normed linear space X is a continuous function $f: [0, 1] \rightarrow X$ such that $f(0) = P$ and $f(1) = Q$.

Notice then that a line segment represents a path (let $f(t) = x_t$). And similarly so does a polygonal chain.

Definition 4.1.9:

A subset S of a normed linear space X is **connected pathwise** if for every $x, y \in S$ there is a path γ_{xy} between x and y whose image is contained in S .

When talking about paths it is often useful to focus on their image, so we when discussing paths we may use the function in place of its image.

Proposition 4.1.10:

Every path is connected.

A similar proof to showing line segments are connected can be used.

Proposition 4.1.11:

If a set is connected pathwise, then it is connected.

Proof:

Suppose S is connected pathwise, then let $x \in S$. We know that

$$S = \bigcup_{y \in S} \gamma_{xy}$$

Since $y \in \gamma_{xy}$ and $\gamma_{xy} \subseteq S$. And since paths are connected and the intersection of $\{\gamma_{xy} \mid y \in S\}$ contains x and is therefore non-empty, by the above theorem, S is also connected. ■

Theorem 4.1.12:

An open set is connected if and only if it is connected pathwise.

Proof:

Suppose \mathcal{O} is open, we know if it is connected pathwise then it is connected, so all that remains is to prove the converse. Take some $x \in S$ and define:

$$A = \{y \in S \mid \text{there exists a path between } x \text{ and } y \text{ contained entirely in } S\}$$

Let $B = S \setminus A$. We will show that A and B are open. We know that $S = A \cup B$ and $A \cap B = \emptyset$, so if we succeed in showing that they both are open, since S is connected, one must be empty. And since $x \in A$, it must be that $B = \emptyset$ so $S = A$ and therefore S is open.

First we will show that A is open. If $y \in A$ then since $\gamma_{xy} \subseteq S$, $y \in S$ which is open, so there exists an $r > 0$ such that $B_r(y) \subseteq S$. If $z \in B_r(y)$ then taking $\gamma_{xy} \cup \vec{yz}$ gives a path between x and y contained in S (we define the path of the union to squish γ_{xy} and then be \vec{yz}). So $z \in A$, and therefore $B_r(y) \subseteq A$, so A is open.

Let $y \in B$, since $y \in S$ then there exists an $r > 0$ such that $B_r(y) \subseteq S$. If there is a $z \in B_r(y) \cap A$ then there is a path γ_{zx} contained in S , and since $z \in B_r(y)$, \vec{yz} is also contained in S , so the path $\vec{yz} \cup \gamma_{zx}$ is a path between y and x contained in S , so $y \in A$ in contradiction. So $B_r(y) \subseteq B$, so B is open. ■