Infinitesimal Calculus 3

Lecture 15, Sunday December 4, 2022
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Proposition 15.1:

Suppose $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is defined in a neighborhood and differentiable at $p \in \mathbb{R}^n$ let $q = f(p)$. Suppose $g\colon \mathbb{R}^m \longrightarrow \mathbb{R}^k$ which is defined in a neighborhood and differentiable at q. Then $g \circ f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^k$ is differentiable at p and satisfies:

$$dg \circ f\big|_p = dg\big|_q \cdot df\big|_p$$

Another way of thinking of this is if J_f is the Jacobian of f then:

$$J_{g\circ f}(p) = J_g\big(f(p)\big) \cdot J_f(p)$$

Proof:

Suppose $f(x+h) = f(x) + df\big|_x(h) + \varepsilon_1(h)$ and $g(x+h) = g(x) + dg\big|_x(h) + \varepsilon_2(h)$. Let $h \in \mathbb{R}^n$ then

$$g \circ f(p+h) - g \circ f(p) = g(f(p+h)) - g(q) = g\big(q + df\big|_p(h) + \varepsilon_1\big) - g(q) = dg\big|_q\big(df\big|_p(h) + \varepsilon_p(h)\big) + \varepsilon_q\big(f(p+h) - f(p)\big)$$

Now recall that by definition, differentials are linear, so this is equal to:

$$= dg\big|_q \circ df\big|_p(h) + dg\big|_q(\varepsilon_p) + \varepsilon_q\big(f(p+h) - f(p)\big)$$

So we will show that the right side of this is an ε function. We know that:

$$\lim_{h\to 0} \frac{dg\big|_q(\varepsilon_p(h))}{\|h\|} = \lim_{h\to 0} dg\big|_q\left(\frac{\varepsilon_p(h)}{h}\right) = dg\big|_q(0) = 0$$

Since differentials are continuous as finite linear transforms. And

$$\lim_{h\to 0} \frac{\varepsilon_q\big(\Delta f(p)\big)}{\|h\|} = \lim_{h\to 0} \frac{\varepsilon_q\big(\Delta f(p)\big)}{\Delta f(p)} \cdot \frac{\Delta f(p)}{\|h\|}$$

Since f is continuous at p, as h approaches 0, so too does $\Delta f(p)$. So

$$\lim_{h\to 0} \frac{\varepsilon_q\big(\Delta f(p)\big)}{\Delta f(p)} = 0$$

And $\frac{\Delta f(p)}{\|h\|}$ is bounded, as it is equal to

$$\frac{df\big|_p(h) + \varepsilon_1(h)}{\|h\|} = \frac{df\big|_p(h)}{\|h\|} + \frac{\varepsilon_1(h)}{\|h\|}$$

The left is bounded since differentials are linear transforms, and the right is bounded since it converges to 0. As required.

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Definition 15.2:

The **contour line** of a function $f\colon A \longrightarrow B$ is the set

$$\{a \in A \mid f(a) = b\}$$

for some $b \in B$. That is, it is the preimage of $\{b\}$.

For example, the contour lines of $f(x,y) = x^2 + y^2$ are circles. If γ is a contour line of f, if u is tangent to the contour, $D_u f = 0$. This is equivalent to saying that $\nabla f\big|_v$ is perpendicular to the contour line of $f(v)$ at the point v.

Definition 15.3:

A vector v is **perpendicular** to a set S at a point $p \in S$ if:

$$\lim_{S \ni u \to p} v \cdot \frac{p-u}{\|p-u\|} = 0$$

Proposition 15.4:

$\nabla f\big|_v$ is perpendicular to the contour line of $f(v)$ at the point v.

Proof:

We know that:

$$\lim_{u \to v} \nabla f\big|_v \cdot \frac{u-v}{\|u-v\|}$$

Recall that $f(v+h) = f(v) + \nabla f\big|_v \cdot h + \varepsilon(h)$, and so if $h = u - v$, then we have that $f(u) = f(v) + \nabla f\big|_v \cdot (u-v) + \varepsilon(u-v)$, and since $f(u) = f(v)$ since they're on the same contour, we have that $\nabla f\big|_v \cdot (u-v) = -\varepsilon(u-v)$, so the limit is equal to:

$$= \lim_{u \to v} -\frac{\varepsilon(u-v)}{\|u-v\|} = 0$$

by definition of an ε function.

$\blacksquare$.

Theorem 15.5 (Clairut-Schwarz Theorem):

Suppose $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ and its partial derivatives is defined in a neighborhood of $(x_1, \ldots, x_n)$. If its second order partial derivatives $\partial_j \partial_i f$ and $\partial_j \partial_i f$ exist in a neighborhood and are continuous at $(x_1, \ldots, x_n)$ then they are equal at $(x_1, \ldots, x_n)$.

Since this is identical to the 2 dimensional case, we will prove it only for the 2 dimensional case. Recall that an alternative notation for partial derivatives is:

$$f_{x_i} = \partial_{x_i} f$$

And:

$$f_{x_j x_i} = \partial_{x_j} \partial_{x_i} f$$

So this theorem states that $f_{x_i x_j} = f_{x_j x_i}$ under the specified conditions.

Proof:

We define the following auxillary functions:

$$\omega(h,k) = \frac{f(x_0+h, y_0+h) - f(x_0+h, y_0) - f(x_0, y_0+h) + f(x_0, y_0)}{hk}$$

$$\varphi(x) = \frac{f(x, y_0+k) - f(x, y_0)}{k}$$

Thus

$$\varphi'(x) = \frac{f_x(x, y_0+k) - f_x(x, y_0)}{k}$$

And

$$\omega(h,k) = \frac{\varphi(x_0+h) - \varphi(x_0)}{h}$$

By the mean value theorem, this means that $\omega(h,k) = \varphi'(x_0 + th)$ for some $0 < t < 1$, which is equal to

$$= \frac{f_x(x_0+th, y_0+k) - f_x(x_0+th, y_0}{k}$$

And by the mean value theorem this is equal to $f_{xy}(x_0 + t_1 h, y_0 + t_2 h)$ for $0 < t_1, t_2 < 1$ ($t_1 = t$). By symmetry (we can do the same thing but swapping x and y in φ), $\omega(h,k) = f_{yx}(x_0 + t_3 h, y_0 + t_4 k)$. That is:

$$\omega(h,k) = f_{xy}(x_0 + t_1 h, y_0 + t_2 k) = f_{yx}(x_0 + t_3 h, y_0 + t_4 k)$$

Since f_{xy} and f_{yx} are continuous at (x_0, y_0), as we take $(h,k) \to (0,0)$ we have that:

$$\lim_{(h,k)\to 0} f_{xy}(x_0 + t_1 h, y_0 + t_2 k) = f_{xy}(x_0, y_0)$$

But on the other hand:

$$\lim_{(h,k)\to 0} f_{xy}(x_0 + t_1 h, y_0 + t_2 k) = \lim_{(h,k)\to 0} f_{yx}(x_0 + t_3 h, y_0 + t_4 k) = f_{yx}(x_0, y_0)$$

And therefore

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$$

as required.

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Definition 15.6:

If $D \subseteq \mathbb{R}^n$ is open then $C^0(D)$ is the set of all continuous real-valued functions on D, and $C^n(D)$ is the set of functions which are differentiable n times (their partial derivatives have partial derivatives etc.) and the partial derivatives are continuous.