Algebraic Topology

Homework 4

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4.1 Exercise

Let X, Y be topological spaces, $a \in X, b \in Y$. Show that $\pi_1(X \times Y, (a,b)) \cong \pi_1(X,a) \times \pi_1(Y,b)$.

Define $f\colon \pi_1(X \times Y, (a,b)) \longrightarrow \pi_1(X,a) \times \pi_1(Y,b)$ by $f[\varphi] = ([p_1 \circ \varphi], [p_2 \circ \varphi])$. This is well-defined as the composition of continuous functions. It is a homomorphism since $p_i \circ (\varphi * \psi) = (p_i \circ \varphi) * (p_i \circ \psi)$ which is immediate, and so

$$f([\varphi\psi]) = \big([p_1 \circ (\varphi * \psi)], [p_2 \circ (\varphi * \psi)]\big) = \big([p_1 \circ \varphi], [p_2 \circ \varphi]\big)\big([p_1 \circ \psi], [p_2 \circ \psi]\big) = f[\varphi]f[\psi]$$

It is injective since if $f[\varphi] = f[\psi]$ then let H be a homotopy $p_1 \circ \varphi \overset{\partial I}{\sim} p_1 \circ \psi$, and K be a homotopy $p_2 \circ \varphi \overset{\partial I}{\sim} p_2 \circ \psi$. Then define $J\colon I \times I \longrightarrow X \times Y$ by $J(t,s) = (H(t,s), K(t,s))$ so that

$$J(t,0) = (H(t,0), K(t,0)) = (p_1 \circ \varphi(t), p_2 \circ \varphi(t)) = \varphi(t), \quad J(t,1) = \psi(t),$$
$$J(0,s) = (H(0,s), K(0,s)) = (p_1 \circ \varphi(0), p_2 \circ \varphi(0)) = \varphi(0), \quad J(1,s) = \varphi(1)$$

So $\varphi \overset{\partial I}{\sim} \psi$, meaning $[\varphi] = [\psi]$. It is surjective since if (φ_1, φ_2) are curves in $\Gamma_{aa} \times \Gamma_{bb}$ then $\varphi = (\varphi_1, \varphi_2)$ maps to $([\varphi_1], [\varphi_2])$. So f is a bijective homomorphism, an isomorphism.

4.2 Exercise

Show that $S^1 \times \{a\} \subseteq S^1 \times S^1$ is a retract, but not a deformation retract.

Define $r\colon S^1 \times S^1 \longrightarrow S^1 \times \{a\}$ by $r(p,q) = (p,a)$ which is continuous and holds $S^1 \times \{a\}$ constant, so is a retraction. But

$$\pi_1(S^1 \times \{a\}) \cong \pi_1(S^1) \times \pi_1(\{a\}) \cong \pi_1(S^1) \cong \mathbb{Z}$$

while

$$\pi_1(S^1 \times S^1) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z}^2$$

so these two fundamental groups are not isomorphic. But the fundamental groups of a space and a deformation retract are.

4.3 Exercise

Show that $S^1 \times \partial D^2 \subseteq S^1 \times D^2$ is not a retract.

We know $\partial D^2 \cong S^1$ and so $\pi_1(S^1 \times \partial D^2) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z}^2$ and $\pi_1(S^1 \times D^2) \cong \pi_1(S^1) \times \pi_1(D^2) \cong \mathbb{Z}$. So there is no embedding $\pi_1(S^1 \times \partial D^2) \hookrightarrow \pi_1(S^1 \times D^2)$, so it cannot be a retract.

4.4 Exercise

Let $h\colon \mathbb{Z} \longrightarrow \mathbb{Z}$ be the homomorphism $n \mapsto 2n$. Show that there does not exist a homomorphism $g\colon \mathbb{Z} \longrightarrow \mathbb{Z}$ such that $g \circ h = \mathrm{id}_{\mathbb{Z}}$.

Every homomorphism from $\mathbb{Z}$ is defined by its image on 1, so that $g(n) = an$ for some $a \in \mathbb{Z}$. Then $g \circ h(n) = 2an$ and this equals n if and only if $2a = 1$ but this cannot happen for any $a \in \mathbb{Z}$.

4.5 Exercise

Let M be a Möbius strip, and ∂M its boundary: $\rho(I \times \{0\} \cup I \times \{1\})$.

(**1**) prove that ∂M is a circle

(**2**) what is the induced homomorphism of the inclusion map $\iota\colon \partial M \longrightarrow M$?

(**3**) prove that ∂M is not a retract of M.

(**1**) Define $I_i = I \times \{i\}$, then we utilize the following commutative diagram:

$$\begin{tikzcd}
I_0 \cup I_1 \arrow[r, "\rho"] \arrow[d, "f"'] & \partial M \arrow[ddl, "F"] \\
I \arrow[d, "q"'] & \\
S^1 &
\end{tikzcd}$$

We define $f\colon I_0 \uplus I_1 \longrightarrow I$ such that $q \circ f$ strongly preserves $\sim$, then this defines an injective $F\colon \partial M \longrightarrow S^1$. Which we then claim is our homeomorphism. First, define

$$f(t,0) = \frac{1}{2}t, \qquad f(t,1) = \frac{1}{2} + \frac{1}{2}t$$

This is continuous on each I_i which form a finite closed cover of the domain, and thus f is continuous. The only two similar elements in the domain are $(0,0)$ and $(1,1)$, in $q \circ f$, $(0,0)$ maps to $[0]$ and $(1,1)$ maps to $[1]$, which are equal. So $q \circ f$ preserves $\sim$. Notice that if $fa = fb$ occurs only when $a = (1,0)$ and $b = (0,1)$ or vice versa, and so $a \sim b$. And if $q \circ fa = q \circ fb$ then $[fa] = [fb]$, so $fa = fb$, or $fa = 0$ and $fb = 1$, or vice versa. For the first case we already showed $a \sim b$, if $fa = 0$ and $fb = 1$ then $a = (0,0)$ and $b = (1,1)$ so $a \sim b$. So $q \circ fa = q \circ fb$ implies $a \sim b$, meaning $q \circ f$ strongly preserves $\sim$ and therefore F is injective.

q and f are surjective, so $q \circ f$ is surjective. We claim that $q \circ f$ is a quotient map, and so this means that F is a homeomorphism. All that remains is to show that $q \circ f$ is closed. Notice that q is closed: if $\mathcal{F} \subseteq I$ is closed then $q^{-1} \circ q\mathcal{F}$ is of one of the following forms: $\mathcal{F}, \mathcal{F} \cup \{0\}, \mathcal{F} \cup \{1\}, \mathcal{F} \cup \{0,1\}$. All of these are closed, so $q^{-1} \circ q\mathcal{F}$ is closed, and thus $q\mathcal{F}$ is closed since q is a quotient map. f is also closed, since a closed subset of $I_0 \uplus I_1$ is of the form $\mathcal{F}_0 \uplus \mathcal{F}_1$ and its image is $\frac{1}{2}\mathcal{F}_0 \cup \big(|frac12 + \frac{1}{2}\mathcal{F}_1\big)$ which is closed. So $q \circ f$ is closed, as required.

(**2**) Since ∂M is homeomorphic to the circle, $\pi_1(\partial M, a) \cong \mathbb{Z}$. Now, we showed previously that $\rho\big(I \times \{\frac{1}{2}\}\big) \cong S^1 \subseteq M$ is a deformation retract, meaning $\pi_1(M,a) \cong \pi_1(S^1,a) \cong \mathbb{Z}$.

In general suppose $A \subseteq X$ is a deformation retract with r being the retraction, and $B \subseteq X$. Then r_* is an isomorphism as it is a homotopy equivalence. Now from the previous homework, $A = \rho\big(I \times \{\frac{1}{2}\}\big)$ is a deformation retract with $r[t,s] = [t, 1/2]$. This is then an isomorphism over $\mathbb{Z}$, so we can view it as the identity (since the precise isomorphism is unimportant). Now, $(r \circ \iota)_* = \iota_*$ then, meaning ι_* is equal to the induced homomorphism of the restriction of r to ∂M.

Since $\partial M \simeq S^1$ by the above isomorphism $F[t,0] = \frac{1}{2}t$ and $F[t,1] = \frac{1}{2} + \frac{1}{2}t$, the generator of $\pi_1(M) \cong \mathbb{Z}$ is $F^{-1} \circ \varphi$ where φ is a generator of $\pi_1(S^1)$ which is just $\varphi(t) = [t]$ (the curves are taken as their homotopy class). Then the generator of $\pi_1(\partial M)$ is

$$\psi(t) = \begin{cases} [2t, 0] & t \leq \frac{1}{2} \\ [2t-1, 1] & t \geq \frac{1}{2} \end{cases}$$

By definition $r_*[\psi] = [r \circ \psi]$ and

$$r \circ \psi(t) = \begin{cases} [2t, 1/2] & t \leq \frac{1}{2} \\ [2t-1, 1/2] & t \geq \frac{1}{2} \end{cases}$$

which is equal to $\varphi * \varphi$, meaning that viewed as a $\mathbb{Z}$-homomorphism, $r_*(1) = 2$. And thus $\iota_*\colon n \mapsto 2n$.

(**3**) Recall that if r is a retraction then $r_* \circ \iota_* = \mathrm{id}$, but we showed that $\iota_*\colon n \mapsto 2n$ and we also showed that then there is no homomorphism which satisfies this.