Introduction to Stochastic Processes

Assignment 3
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3.1 Exercise

Given an undirected graph $G = (V, E)$, what are the requirements for the simple random walk on it to

(1) be irreducible?

(2) contain a transient state?

(3) be 2-periodic?

(4) be 3-periodic?

(5) be aperiodic?

(1) A Markov chain is irreducible if and only if it is connected (as S is by definition closed). Since for every $a, b \in V$, $a \to b$ if and only if a and b are connected in G (as if there exists a path of length n then $P^n(a \to b) > 0$). Thus in order for the walk to be irreducible, G must be connected (these are equivalent).

(2) Suppose G contains a transient state $a \in V$. Then define $C = \{b \in V \mid a \to b\}$, and since $C \subseteq V$, C is finite. And C must be closed: if $b \in C$ and $b \to b'$ then $a \to b \to b'$ and so $a \to b'$ meaning $b' \in C$. Therefore this means C contains a recurrent state as a finite closed set. But if $b \in C$ is recurrent, since $a \to b$ and therefore $b \to a$ since G is undirected, we must have that a is recurrent, in contradiction. So G cannot have a transient state.

(3) We claim that G is 2-periodic if and only if it contains no odd cycles. Any cycle containing a of length n means that $n \in \tau(a) = \{t \geq 1 \mid P^t(a \to a) > 0\}$, and so $d(a) | n$. So if G is 2-periodic, it cannot contain odd cycles as 2 must divide the length of any cycle. And if G contains no odd cycles, then for every $a \in V$, $\{a, a\}$ cannot be an edge as it is a 1-cycle, and so a must be connected to some $b \neq a$. But then a, b, a is a 2-cycle and so $P^2(a \to a) > 0$ and therefore $2 \in \tau(a)$. And since all cycles are even, this means that $\tau(a)$ contains only even numbers, so $d(a) = 2$.

(4) G must contain two vertices a, b such that $\{a, b\} \in E$ (a and b need not be distinct) then a, b, a is a path and so $2 \in \tau(a)$. Thus $d(a) \neq 3$ as it doesn't divide 2.

(5) We already showed that $2 \in d(a)$ for every $a \in V$. So for every $a \in V$, either $d(a) = 2$ or $d(a) = 1$. So G is aperiodic if and only if there exists a vertex a whose period is 1 (for the converse, if $d(a) = 1$ then either all vertices have degree one or some vertices have different periods, and in either case the graph is then aperiodic). This is if and only if there exists a path of odd length (since $2 \in \tau(a)$, $d(a) = 2$ if and only if there exists an odd number in $\tau(a)$, which corresponds to an odd cycle). So G is aperiodic if and only if there exists an odd cycle.

3.2 Exercise

Cookie Monster is collecting cookies. At every step, he either buys a new cookie or eats all the cookies he has collected, each with the same probability (when he has no cookies, he buys one with probability 1).

(1) What is the probability that within a finite amount of time, he will have half a million cookies?

(2) If Cookie Monster starts with 3 cookies, what is the expected time that it will take him to return to having 3 cookies?

(3) One day Cookie Monster decides that he will limit himself to only a million cookies, and once he gets to a million cookies, he eats them all. How will your previuous answers change?

(1) Let us define X_n to be the number of cookies Cookie Monster has on the nth step. So

$$\mathbb{P}(X_n = i+1 \mid X_{n-1} = i) = \frac{1}{2}, \quad \mathbb{P}(X_n = 0 \mid X_{n-1} = i) = \frac{1}{2}, \quad \mathbb{P}(X_n = 1 \mid X_{n-1} = 0) = 1 \qquad (i > 0)$$

This is asking for $\mathbb{P}\left(T_{\frac{1}{2}\text{mil}} \,\middle|\, X_0 = 0\right)$. We claim that all states (which are $\mathbb{N}_{\geq 0}$) are recurrent, meaning that this probability is 1. Since 0 is connected to all the states, it is sufficient to show that 0 is recurrent. Notice that the probability that $T_0 > n$ is $\frac{1}{2}^n$, and by the continuity of probability, $\mathbb{P}(T_0 = \infty \mid X_0 = 0) = 0$. Thus $\mathbb{P}(T_0 < \infty \mid X_0 = 0) = 1$ and so 0 is recurrent, therefore so is half a million, so the probability is one.

(2) Let us define $e_k = \mathbb{E}[T_k \mid X_0 = 0]$. Let us denote $t := T_k$, and so ($\mathbb{E}_t[T_{k+1}] := \mathbb{E}[T_{k+1} \mid T_k]$, and all probability is implicitly under the assumption that $X_0 = 0$),

$$\mathbb{E}_t[T_{k+1}] = \frac{1}{2}\,\mathbb{E}_t[T_{k+1} \mid X_{t+1} = k_1] + \frac{1}{2}\,\mathbb{E}_t[T_{k+1} \mid X_{t+1} = 0] = \frac{1}{2}(t+1) + \frac{1}{2}\,\mathbb{E}_t[T_{k+1} \mid X_{t+1} = 0]$$

Now, $\mathbb{E}_t[T_{k+1} \mid X_{t+1} = 0] = \mathbb{E}_t[T_{k+1}] + t + 1$ since this is like starting over but having "wasted" $t+1$ steps (due to homogeneity). Thus

$$\mathbb{E}_t[T_{k+1}] = \frac{1}{2}\,\mathbb{E}_t[T_{k+1}] + t + 1 \implies \mathbb{E}_t[T_{k+1}] = 2t + 2$$

By the law of total expectation, we then get $\mathbb{E}[T_{k+1}] = 2\,\mathbb{E}[T_k] + 2$, so $e_{k+1} = 2e_k + 2$. Solving this recurrence we get $e_k = 2^k + 2^{k-1} - 2$.

Now, let us define $\tilde{e}_k = \mathbb{E}[T_k \mid X_0 = 3]$, so we are attempting to comput $\tilde{e}_3$. Then we get that, now with probability being conditioned under $X_0 = 3$,

$$\mathbb{E}_t[T_{k+1}] = \frac{1}{2}\,\mathbb{E}_t[T_{k+1} \mid X_{t+1} = k_1] + \frac{1}{2}\,\mathbb{E}_t[T_{k+1} \mid X_{t+1} = 0] = \frac{1}{2}(t+1) + \frac{1}{2}(e_{k+1} + t + 1) = t + 1 + \frac{1}{2}e_{k+1}$$

Thus again by the law of total expectation, $\tilde{e}_{k+1} = \tilde{e}_{k+1} + 2^k + 2^{k-1}$. Now, notice that $\mathbb{P}_3(T_0 = k) = \frac{1}{2^k}$ since in order for $T_0 = k$, Cookie Monster must buy $k-1$ cookies and then eat them all, so $\frac{1}{2^{k-1}} \cdot \frac{1}{2}$. Thus $T_0 \mid X_0 = 3 \sim \mathrm{Geo}\left(\frac{1}{2}\right)$ and so $\tilde{e}_0 = 2$. And so we get from our recurrence that $\tilde{e}_1 = 3, \tilde{e}_2 = 6, \tilde{e}_3 = 12$. So the answer is 12.

(3) For the first subquestion, the answer will not change as Cookie Monster will reach half a million cookies before one million, and so all the probabilities will remain the same. For the second subquestion, all the computations remain valid until computing $\mathbb{P}_3(T_0 = k)$. While for $k \leq 1\text{mil} - 2$ this remains the same, for $k = 1\text{mil} - 1$ this is just $\mathbb{P}_3(T_0 = k) = \frac{1}{2^{k-1}}$, as the probability of returning to zero is one. And for larger k it is zero. Thus the difference between this new $\mathbb{E}_3[T_0]$ by

$$\frac{a}{2^{a-1}} + \sum_{k=a+1}^{\infty} \frac{k}{2^k} \qquad (a = 1\text{mil} - 1)$$

this is very very small, so the change in the answer is negligible.

3.3 Exercise

Let $p \in [0,1]$ and X_n be a random walk on $\mathbb{Z}$ where the transition probabilities are

$$P(i,j) = \begin{cases} p & j = i+1, i > 0 \\ 1-p & j = i-1, i > 0 \\ \frac{1}{2} & j = i+1, i \leq 0 \\ \frac{1}{2} & j = i-1, i \leq 0 \\ 0 & |i-j| \neq 1 \end{cases}$$

Determine, for each p, whether or not 0 is transient or recurrent.

If $p > \frac{1}{2}$, then 0 must be transient. This is as $0 \to 1$ and 1 is transient:

$$\mathbb{P}(T_1 < \infty \mid X_0 = 1) = p\,\mathbb{P}(T_1 < \infty \mid X_1 = 2) + (1-p)\,\mathbb{P}(T_1 < \infty \mid X_1 = 0)$$

The probability being summed on the left is the same if we replaced X_n with a random walk on $\mathbb{Z}$ of probability p: all the probabilities until it hits 1 again are the same. Now since $p > \frac{1}{2}$, if we were to replace this with a random walk of probability p, $\mathbb{P}(T_1 < \infty \mid X_1 = 2) \leq \mathbb{P}(T_1 < \infty \mid X_1 = 1)$. And since all states in a random walk of probability p are transient, this means that this is less than 1, and so $\mathbb{P}(T_1 < \infty \mid X_0 = 1) < 1$ so 1 is transient and therefore so is 0.

If $p \leq \frac{1}{2}$, then the probability of returning to zero if we go to the right is more than it would be on a fair walk. And the left is a fair walk. We showed last assignment that in a fair walk, all states are recurrent, so 0 must be a recurrent state.