Algebraic Topology

Homework 3

Ari Feiglin

3.1 Exercise

Let $a, b \in X$. Show that $\pi_1(X, a)$ is abelian if and only if for every two paths γ, δ from a to b, $F_\gamma = F_\delta$.

We can generalize this slightly to ease up on notation: let $\mathcal{G}$ be a groupoid, then for $A \in \mathcal{G}$, $\mathrm{Mor}(A, A)$ is abelian if and only if for every $\gamma, \delta \in \mathrm{Mor}(A, B)$, $F_\gamma = F_\delta\colon \mathrm{Mor}(A, A) \longrightarrow \mathrm{Mor}(B, B)$. Notice that $F_\gamma = F_\delta$ if and only if for every $a \in \mathrm{Mor}(A, A)$, $F_\gamma(a) = \gamma^{-1}a\gamma = \delta^{-1}a\delta = F_\delta(a)$, which is if and only if $\delta\gamma^{-1}a\gamma\delta^{-1} = a$. Now notice that $\delta\gamma^{-1}, \gamma\delta^{-1} \in \mathrm{Mor}(A, A)$, so if $\mathrm{Mor}(A, A)$ is abelian, this holds since $(\delta\gamma^{-1})^{-1} = \gamma\delta^{-1}$.

And if this holds, then let $b \in \mathrm{Mor}(A, A)$ and define $\delta = b\gamma$, so $F_\gamma = F_\delta$ and thus $\delta\gamma^{-1}a\gamma\delta^{-1} = bab^{-1} = a$, ie. $ba = ab$. So for every $a, b \in \mathrm{Mor}(A, A)$, a and b commute, meaning $\mathrm{Mor}(A, A)$.

3.2 Exercise

Let $f, g\colon X \longrightarrow Y$ such that $f \sim g$ and let $a \in X$. Let $f_*\colon \pi_1(X, a) \longrightarrow \pi_1(X, f(a))$ and $g_*\colon \pi_1(X, a) \longrightarrow \pi_1(X, g(a))$ be the induced homomorphisms.

(1) Show that f_* is trivial if and only if g_* is.

(2) Conclude that if $f\colon X \longrightarrow Y$ is null-homotopic then f_* is trivial.

(1) Since f and g are homotopic, $g_* = F_\gamma \circ f_*$. So if f_* is trivial then $g_*[\varphi] = F_\gamma \circ f_*[\varphi] = F_\gamma(1) = 1$, so g_* is trivial. The converse holds by symmetry.

(2) Suppose $f \sim K_p$ then since $(K_p)_*[\varphi] = [K_p \circ \varphi] = [K_p] = 1$, $(K_p)_*$ is trivial and so f_* is trivial as well.

3.3 Exercise

Suppose X is path connected, show that the following are equivalent:

(1) X is simply connected,

(2) for every $a \in X$, $\pi_1(X, a)$ is trivial,

(3) for every two $a, b \in X$, every two paths from a to b are homotopic relative to ∂I,

(4) there exist two $a, b \in X$, every two paths from a to b are homotopic relative to ∂I,

(1) $\Longrightarrow$ (2): let $[\varphi] \in \pi_1(X, a)$ then φ can be viewed as a map $S^1 \longrightarrow X$ since it has the same endpoints (formally φ respects the equivalence relation and so $\varphi = \varphi' \circ \rho$ for some φ'). Thus φ is null-homotopic relative to any point by simple connectivity, so $\varphi \overset{\partial I}{\sim} K_a$ (since the endpoints of I are mapped to the same point), so $[\varphi] = [K_a] = 1$, meaning $\pi_1(X, a)$ is trivial. (2) $\Longrightarrow$ (3): let $\gamma, \delta \in \Gamma_{ab}$ then $\gamma\overline{\delta} \in \Gamma_{aa}$ so $[\gamma\overline{\delta}] = 1$ by (2), so $[\gamma] = [\delta]$ ie. $\gamma \overset{\partial I}{\sim} \delta$. (3) $\Longrightarrow$ (4): trivial. (4) $\Longrightarrow$ (1): let $\gamma \in \Gamma_{cc}$, then let $\delta_1 \in \Gamma_{ac}$ and $\delta_2 \in \Gamma_{cb}$. Then $\delta_1\delta_2, \delta_1\gamma\delta_2 \in \Gamma_{ab}$ so $[\delta_1\delta_2] = [\delta_1\gamma\delta_2]$, meaning $[\gamma] = 1$, ie. $\pi_1(X, c)$ is trivial for every $c \in X$.

3.4 Exercise

Let $f\colon X \longrightarrow Y$.

(1) Show that f defines a function $\tilde{f}$ from the path connected components of X to the path connected components of Y.

(2) Show that if f is a homotopy equivalence then $\tilde{f}$ is bijective.

(1) For $a \in X$, define $\langle a \rangle$ to be a's path connected component. Then define $f\langle a \rangle := \langle fa \rangle$. This is well defined: if $\langle a \rangle = \langle b \rangle$ then $f(\langle a \rangle) = f(\langle b \rangle)$, and since $a \in \langle a \rangle$, $f(\langle a \rangle) \subseteq \langle fa \rangle$ (since the image of a path connected space is path connected). And so $fb \in f(\langle b \rangle) \subseteq \langle fa \rangle$, meaning $fb \in \langle fa \rangle, \langle fb \rangle$ but since path connected components are disjoint, $\langle fa \rangle = \langle fb \rangle$.

(2) Suppose $g\colon Y \longrightarrow X$ is f's homotopy inverse: $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$. Then by above $\tilde{g}$ is well-defined, and so if $\tilde{f}\langle a \rangle = \tilde{f}\langle b \rangle$ then $\langle fa \rangle = \langle fb \rangle$, so $\tilde{g}\langle fa \rangle = \langle gfa \rangle = \langle gfb \rangle = \tilde{g}\langle fb \rangle$. Thus gfa and gfb are connected.

In general notice that if $h \sim \mathrm{id}$ then a and ha are path connected for every $a \in X$: if H is a homotopy from h to id, define $\gamma(t) := H(a, t)$. Then $\gamma(0) = H(a, 0) = ha$ and $\gamma(1) = H(a, 1) = a$.

So a and gfa are connected, and b and gfb are connected, meaning a and b are connected so $\langle a \rangle = \langle b \rangle$. Thus $\tilde{f}$ is injective.

And for $y \in Y$, $\tilde{f}\langle gy \rangle = \langle fgy \rangle = \langle y \rangle$ since fgy and y are connected (by the same proof above: $fg \sim \mathrm{id}_Y$).

3.5 Exercise

Let M be a Möbius strip, meaning it is the quotient space of $I \times I$ under $(0, t) \sim (1, 1-t)$ for $t \in I$. If $\rho\colon I \times I \longrightarrow M$ is its quotient map, define $S := \rho(I \times \{1/2\})$.

(1) Show that S is a circle,

(2) Show that S is a deformation retract.

(1) Let us define $f\colon I \longrightarrow S$ by $f(t) = \rho(t, 1/2)$. Now $f(t) = f(s)$ if and only if $\rho(t, 1/2) = \rho(s, 1/2)$, which is if and only if $(t, 1/2) \sim (s, 1/2)$ which is if and only if $t = s$ or $t = 0$ and $s = 1$ or vice versa. Thus f defines a homeomorphism from S^1 to S.

(2) Define $r[t, s] = [t, 1/2]$. r is continuous if and only if $r \circ \rho\colon (t, s) \mapsto \rho(t, 1/2)$ is continuous, which it is. So r is a retraction. Now define the homotopy $H\colon M \times I \longrightarrow M$ by $H([t, s], x) := \left[t, \frac{1}{2} + x\left(s - \frac{1}{2}\right)\right]$. Notice that $H([t, s], 0) = [t, 1/2] = r[t, s]$ and $H([t, s], 1) = [t, s]$, so H is a homotopy from $\iota \circ r$ to id_M as required.