Infinitesimal Calculus 3

Lecture 3, Sunday October 30, 2022
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Proposition 3.1.1:

Any arbitrary union of open sets is itself open, and a finite intersection of open sets is open as well. Similarly, any intersection of closed sets is closed, and a finite union of closed sets is closed as well.

Proof:

This is trivial. Suppose $\{\mathcal{O}_i\}_{i\in I}$ is a set of open sets, then if:

$$x \in \bigcup_{i\in I} \mathcal{O}_i$$

there must be some $i \in I$ such that $x \in \mathcal{O}_i$. Since $\mathcal{O}_i$ is open, there is a $r > 0$ such that $B_r(x) \subseteq \mathcal{O}_i$, and therefore $B_r(x) \subseteq \bigcup \mathcal{O}_i$, so the union is open.
Now suppose $\{\mathcal{O}_n\}_{n=1}^N$ is a finite set of open sets. Then if

$$x \in \bigcap_{n=1}^N \mathcal{O}_n$$

x must be in $\mathcal{O}_n$ for every $n = 1 \dots N$. Then for every $n = 1 \dots N$, there must be a $r_n > 0$ such that $B_{r_n}(x) \subseteq \mathcal{O}_n$. If we let $r = \min\{r_1, \dots, r_N\} > 0$, $B_r(x) \subseteq B_{r_n}(x) \subseteq \mathcal{O}_n$ for every n, so:

$$B_r(x) \subseteq \bigcap_{n=1}^N \mathcal{O}_n$$

as required.
Now suppose $\{F_i\}_{i\in I}$ is a set of closed sets, then:

$$\left(\bigcap_{i\in I} F_i\right)^c = \bigcup_{i\in I} F_i^c$$

which is a union of open sets since F_i^c is open, which we just proved is open. So the complement of the intersection is open, and therefore the intersection is closed.
Similarly for a finite set of closed sets:

$$\left(\bigcup_{n=1}^N F_n\right)^c = \bigcap_{n=1}^N F_n^c$$

which is open, and therefore the union is closed.

$\blacksquare$

Notice that an arbitrary intersection of open sets, and thus an arbitrary intersection of closed sets, is not necessarily open (or closed). Take for instance $\mathcal{O}_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$, then:

$$\bigcap_{n\in\mathbb{N}} \mathcal{O}_n = \{0\}$$

And $\{0\}$ is not open.

Definition 3.1.2:

For a subset S of a metric space X, we denote its **interior**, the set of all interior points of S as S° or $\operatorname{int} S$. Similarly, the **exterior**, the set of all exterior points is $\operatorname{ext} S$ (or $S^{c\circ}$), and the **boundary** of S is denoted ∂S.

Notice that $\operatorname{int} S \subseteq S$. Furthermore, $S \cup \partial S = \operatorname{int} S \uplus \partial S$. These sets are obviously disjoint by definition, and if $x \in S$, x is either in the interior of S or the boundary of S.

Lemma 3.1.3:

Every limit point of S is in $\operatorname{int} S \uplus \partial S = S \cup \partial S$.

Proof:

Suppose x is a limit point of S and $x \notin \operatorname{int} S$. Since x is a limit point, for every $r > 0$ there is a $x \neq y \in S$ such that $y \in B_r(x)$, so $B_r(x)$ is not a subset of S^c. And since x is not an interior point of S, for every $r > 0$, $B_r(x)$ is not a subset of S. So for every $r > 0$, $B_r(x) \cap S$ and $B_r(x) \cap S^c$ are not empty and therefore x is a boundary point of S.

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Theorem 3.1.4:

Suppose $S \subseteq X$ for X metric space. Then the following are equivalent:

- S is closed.
- S contains all of its boundary points.
- S contains all of its limit points.

Proof:

We will prove the first relation. Suppose $x \in \partial S$, then if $x \notin S$, $x \in S^c$, and since $\partial (S^c) = \partial S$, $x \in \partial (S^c) \cap S^c$. But S^c is open and thus doesn't contain its boundary points, in contradiction, so $x \in S$. Therefore $\partial S \subseteq S$.
Now suppose $\partial S \subseteq S$. Then since every limit point is in $\partial S \cup S = S$, every limit point is in S.
Now suppose S contains all of its limit points. Suppose $x \in S^c$, then x is not a limit point of S, so there must be a $r > 0$ such that $B_r(x)$ contains no points in S other than x. Since x is not in S^c, $B_r(x) \subseteq S^c$. So S^c is open and therefore S is closed.

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Definition 3.1.5:

For $S \subseteq X$ a metric space, S' is the set of all limit points of S, and is also called the **limit set** of S.

Lemma 3.1.6:

Let $S \subseteq X$ a metric space and $x \in S^c$. Then x is a boundary point of S if and only if x is a limit point of S.

This proof is trivial, if x is a boundary point of S, for every $r > 0$ there must be a $y \in B_r(x) \cap S$, and since $x \notin S$, $y \neq x$, so x is a limit point. And if x is a limit point, then since x is in $B_r(x) \cap S^c$, it is not open, and since there must be a $y \in B_r(x) \cap S$ since it's a limit point, x is a boundary point of S.
Notice then that

$$S' \subseteq S \cup \partial S = \operatorname{int} S \uplus \partial S$$

Since if $x \in S'$ and $x \notin S$, then we just showed $x \in \partial S$.

Definition 3.1.7:

If S is a subset of a metric space X, then the closure of S, denoted $\overline{S}$, is the smallest possible closed set containing S:

$$\overline{S} = \bigcap_{\substack{F \text{ closed} \\ S \subseteq F}} F$$

Proposition 3.1.8:

If S is a subset of a metric space X, then $\overline{S} = \operatorname{int} S \cup \partial S = \operatorname{int} S \cup \partial (S^c)$.

Proof:

Since $X = \operatorname{int} S \cup \partial S \cup \operatorname{int}(S^c)$, $\operatorname{int}(S^c)$ is open so its complement, $\operatorname{int} S \cup \partial S$ is closed. This contains S so $\overline{S} \subseteq \operatorname{int} S \cup \partial S$. And if F is a closed set which contains S, it must contain S's interior (since it is a subset of S) and its boundary, so the sets are equal.

$\blacksquare$.

Theorem 3.1.9:

If $S \subseteq X$ a metric space, the following are equal:

- $\overline{S}$
- $S \cup S'$
- $S \cup \partial S$

Proof:

The first equality was proven above. We know that $S' \subseteq S \cup \partial S$, so $S \cup S' \subseteq S \cup \partial S$. Suppose $x \in \partial S$ and $x \notin S$. Then we know that x is a limit point, so $S \cup \partial S \subseteq S \cup S'$. Therefore $S \cup S' = S \cup \partial S$.

$\blacksquare$

Definition 3.1.10:

A set $S \subseteq X$ is **compact** if for every set of open sets $\{\mathcal{O}_i\}_{i \in I}$ such that

$$S \subseteq \bigcup_{i \in I} \mathcal{O}_i$$

there is a finite subcovering, that is a finite indexing set $I' \subseteq I$ such that:

$$S \subseteq \bigcup_{i \in I'} \mathcal{O}_i$$