Infintesimal Calculus 3

Assignment 5

Ari Feiglin

Exercise 5.1:

Suppose (X, ρ) is a metric space and $a \in X$.

(1) Prove that $f_a \colon X \longrightarrow \mathbb{R}$ defined by $f(x) = \rho(x, a)$ is continuous.

(2) Conclude that closed balls are closed.

(1) Suppose $x \in X$, then for any $x_n \longrightarrow x$ by definition $\rho(x_n, x) \longrightarrow 0$. Thus

$$|f_a(x_n) - f_a(x)| = |\rho(x_n, a) - \rho(x, a)| \leq \rho(x, x_n) \longrightarrow 0$$

And therefore $f_a(x_n) \longrightarrow f_a(x)$. So f is continuous.

(2) First we will prove the following lemma:

Lemma:

If $f \colon X \longrightarrow \mathbb{R}$ is continuous then for every $a \in \mathbb{R}$, $\{x \in X \mid f(x) \leq a\}$ is closed.

Proof:

Let $A = \{x \in X \mid f(x) \leq a\}$. Suppose $x \in \overline{A}$ then there exists a sequence $x_n \in A$ such that $x_n \longrightarrow x$. Since $x_n \in A$, $f(x_n) \leq a$, and since f is continuous $f(x_n) \longrightarrow f(x)$. Since limits preserve inequalities, this means $f(x) \leq a$ and therefore $x \in A$. So A contains all of its points of closure, and is therefore closed.

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Let $a \in X$, so since f_a is continuous and $\bar{B}_r(a) = \{x \in X \mid f_a(x) \leq r\}$, by the above lemma the closed ball is closed, as required.

Exercise 5.2:

We define a function $f \colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ as follows:

$$f(x, y) = \begin{cases} 8 & xy = 0 \\ \sqrt{2} & xy \neq 0 \end{cases}$$

find the set of points where f is continuous. Is it open or closed?

We claim that the set $S = \mathbb{R}^* \times \mathbb{R}^*$ is the set of all points where f is continuous. Suppose $(x, y) \in S$ then $x, y \neq 0$. So if we take $(x_n, y_n) \longrightarrow (x, y)$, at some point onward $x_n, y_n \neq 0$ so $x_n y_n \neq 0$ and so at some point onward $f(x_n, y_n) = \sqrt{2}$. So the limit of $f(x_n, y_n)$ is $\sqrt{2}$ which is equal to $f(x, y)$. And if $(x, y) \notin S$, then we can define the sequence $\vec{v}_n = \left(x + \frac{1}{n}, y + \frac{1}{n}\right)$ which converges to (x, y). Since at some point onward $x + \frac{1}{n}$ and $y + \frac{1}{n}$ are both non-zero (otherwise $\frac{1}{n} = \frac{1}{m}$ for $n \neq m$), $f(\vec{v}_n) = \sqrt{2}$ and so the limit of $f(\vec{v}_n)$ is $\sqrt{2}$ which doesn't equal $f(x, y) = 8$. So f is not continuous at (x, y) for every $(x, y) \notin S$ as required.
We claim that S is open. This is because if $(x, y) \in S$ by choosing $r = \min |x|, |y|$ then $B_r(x, y) \subseteq S$.

Exercise 5.3:

Are the following functions uniformly continuous?

(1) $f(x) = \sin(x^2)$ in $\mathbb{R}$.

(2) $f(x,y) = \sin^{-1}\left(\frac{x}{y}\right)$ in $D = \{(x,y) \mid |x| \leq |y|, y \neq 0\}$.

(1) Notice that the peaks get closer and closer together as x grows, and the distance gets arbitrarily small. So for $\varepsilon = 1$ for instance, if we take any $\delta > 0$ at some point the distance between two peaks is less than δ. So if we take x to be the x value of a peak, there exists a valley within a distance of δ, and thus the distance between the images of these two points is 2, which is greater than $\varepsilon = 1$. So $f(x)$ is not uniformly continuous.
To get numbers, let $x_n = \frac{\pi}{2} + 2\pi n$, the x values of the peaks. Then we need to find an h such that $(x_n+h)^2 = x_n^2+\pi$, and solving for a positive h gives $h = -x_n + \sqrt{x_n^2+\pi^2}$. So the distance between x_n and x_n+h (which is a valley) is $h = -x_n + \sqrt{x_n^2+\pi^2}$. The limit of h as n goes to infinity is 0 since we can rewrite h as

$$\frac{\pi^2}{x_n + \sqrt{x_n^2+\pi^2}}$$

So for every $\delta > 0$ there is some $h < \delta$ as required.

(2) Notice that if $x = y$ the $f(x,y) = \frac{\pi}{2}$ while $f(x,2y) = \frac{\pi}{6}$. So if we choose a sequence (a_n, a_n) and $(b_n, 2b_n)$ such that $|a_n - b_n|$ and $|a_n - 2b_n|$ converge to 0 then we will have shown that the function is not uniformly continuous. Let $a_n = b_n = \frac{1}{n}$. Then notice that $f\left(\frac{1}{n}, \frac{1}{n}\right) = \frac{\pi}{2}$ and $f\left(\frac{1}{n}, \frac{2}{n}\right) = \frac{\pi}{6}$ but $\left|\frac{2}{n} - \frac{1}{n}\right| = \frac{1}{n}$ converges to 0, so $\left\|\left(\frac{1}{n}, \frac{1}{n}\right) - \left(\frac{1}{n}, \frac{2}{n}\right)\right\|$ converges to 0 while the difference in their image is a non-zero constant and consequently doesn't converge to 0. Therefore the function is not uniformly continuous.

Exercise 5.4:

Suppose $f(x,y)$ is defined on D and is continuous in the x variable and lipschitz continuous in its second variable y. Is f continuous?

Suppose $(x,y) \in D$, then we will show for any $D \ni (x_n, y_n) \longrightarrow (x,y)$, $f(x_n, y_n) \longrightarrow f(x,y)$. We know that:

$$|f(x_n,y_n) - f(x,y)| \leq |f(x_n,y_n) - f(x_n,y)| + |f(x_n,y) - f(x,y)|$$

Since f is continuous in x, $|f(x_n,y) - f(x,y)| \longrightarrow 0$, and since f is Lipschitz in y then $|f(x_n,y_n) - f(x_n,y)| \leq K \cdot |y_n - y| \longrightarrow 0$. So $|f(x_n,y_n) - f(x,y)|$ converges to 0. Therefore f is indeed continuous in D.

Exercise 5.5:

Compute the following limit:

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{|xzyw|^{\frac{n}{3}}}{|x|^n + |y|^n + |z|^n + |w|^n}$$

For any $n > 0$.

Firstly, let

$$f(x,y,z,w) = \frac{|xzyw|^{\frac{n}{3}}}{|x|^n + |y|^n + |z|^n + |w|^n}$$

Then let us define:

$$a = \max\{|x|,|y|,|z|,|w|\} \qquad b = \min\{|x|,|y|,|z|,|w|\}$$

Then notice that a and b are both equal to one of the values, $|xzyw| \leq a^3 \cdot b$ and $|x|^n + |y|^n + |z|^n + |w|^n \geq 3b^n + a^n$. So all in all:

$$0 \leq f(x,y,z,w) \leq \frac{a^n \cdot b^{\frac{n}{3}}}{a^n + 3b^n} = b^{\frac{n}{3}} \cdot \frac{1}{1 + 3\left(\frac{a}{b}\right)^n}$$

We can divide by a^n since it is not equal to 0, since if it were, $x = y = z = w = 0$ which is not the case since limits don't include the point they converge to. Notice that since $1 + 3\left(\frac{a}{b}\right)^n \geq 1$, $\frac{1}{1+3\left(\frac{a}{b}\right)^n}$ is bounded (between 0 and 1).

And since (x, y, z, w) converges to 0, since convergence in $\mathbb{R}^n$ is pointwise, b converges to 0 as well, since it is less than $|x|$ etcetra. And therefore so does $b^{\frac{n}{3}}$. So the limit of the right hand side is 0, and so by the squeeze theorem, the limit of $f(x, y, z, w)$ is 0 as well. That is:

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{|xzyw|^{\frac{n}{3}}}{|x|^n + |y|^n + |z|^n + |w|^n} = 0$$