Functional Analysis

Based on the book "Functional Analysis" by Walter Rudin

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1 Topological Vector Spaces

1.1 Separation Properties

Recall the definitions of vector, normed, and inner product spaces (or refer to the reference of your choice). Also recall the definitions of topological and metric spaces. A *complete* metric space is one where all Cauchy sequences converge.

1.1.1 Definition

A **topological field** is a field endowed with a topology which agrees with its operations. Explicitly, the maps $\alpha, \beta \mapsto \alpha + \beta$; $\alpha, \beta \mapsto \alpha\beta$; $\alpha \mapsto -\alpha$; and $\alpha \mapsto \alpha^{-1}$ are continuous.

We will focus mostly on the topological fields $\mathbb{R}$ and $\mathbb{C}$.

1.1.2 Definition

A **topological vector space** is a vector space endowed with a topology which behaves well with its existing vector space structure. Formally, it is a vector space $\mathbf{V}$ with a topology τ such that

(1) every singleton in $\mathbf{V}$ is closed;

(2) the vector space operations (vector addition and scalar multiplication) are continuous w.r.t. the topology.

We can explicitly state what it means for addition (an operation $\mathbf{V} \times \mathbf{V} \to \mathbf{V}$) and scalar multiplication (a mapping $F \times \mathbf{V} \to \mathbf{V}$) to be continuous. Addition is continuous if and only if for every neighborhood $\mathcal{U} \subseteq \mathbf{V}$ the set $\{(x_1, x_2) \mid x_1 + x_2 \in \mathcal{U}\}$ is open. This is equivalent to saying that there exists neighborhoods $\mathcal{U}_1, \mathcal{U}_2$ of x_1, x_2 such that $\mathcal{U}_1 + \mathcal{U}_2 \subseteq \mathcal{U}$, since then

$$\mathcal{U} = \bigcup_{x_1 + x_2 \in \mathcal{U}} \mathcal{U}_{x_1} + \mathcal{U}_{x_2} \Longrightarrow \{(x_1, x_2) \mid x_1 + x_2 \in \mathcal{U}\} = \bigcup \mathcal{U}_{x_1} \times \bigcup \mathcal{U}_{x_2}$$

Similarly scalar multiplication $\cdot \colon F \times \mathbf{V} \to \mathbf{V}$ is continuous if for every neighborhood $\mathcal{U}$ of αx, then there is some neighborhood O of α and $\mathcal{V}$ of x such that

$$O \cdot \mathcal{V} \subseteq \mathcal{U}$$

Let us define the mappings $T_a \colon x \mapsto a + x$ for $a \in X$ and $M_\lambda \colon x \mapsto \lambda x$ for $\lambda \in F$ nonzero. Then T_a and M_λ are homeomorphisms over X. They are both obviously bijective, with inverses T_{-a} and $M_{1/\lambda}$ respectively. Since T_a is just the restriction of addition to a, it is continuous (similar for M_λ). Thus they are homeomorphisms (since their inverses are also of the given form).

1.1.3 Proposition

The following are equivalent:

(1) $\mathcal{U}$ is open;

(2) $a + \mathcal{U}$ is open for some (all) a;

(3) $\lambda \mathcal{U}$ is open for some (all) λ.

This is due to T_a and M_λ being homeomorphisms.

Thus the topology τ is entirely determined by any local base (a local base of point x is a collection of neighborhoods of x such that any neighborhood of x contains some set in the base). In particular the local base around 0 (from now on, when we say local base we will mean around 0).

1.1.4 Definition

A metric on a vector space $\mathbf{V}$ is **invariant** if $d(x+z, y+z) = d(x,y)$ for all $x, y, z \in \mathbf{V}$.

1.1.5 Lemma

If $\mathcal{U}$ is a neighborhood of 0, then there exists a symmetric neighborhood of 0 $\mathcal{V}$ (i.e. $-\mathcal{V} = \mathcal{V}$) where $\mathcal{V} + \mathcal{V} \subseteq \mathcal{U}$.

Proof

Since addition is continuous, the preimage of $\mathcal{U}$ is open and therefore contains $\mathcal{V}_1 \times \mathcal{V}_2$; i.e. $\mathcal{V}_1 + \mathcal{V}_2 \subseteq \mathcal{U}$ for $\mathcal{V}_i$ open. Let

$$\mathcal{V} = \mathcal{V}_1 \cap \mathcal{V}_2 \cap (-\mathcal{V}_1) \cap (-\mathcal{V}_2),$$

it is easy to see that $\mathcal{V}$ has the desired properties. ■

Note that there then exists a symmetric neighborhood $\mathcal{V}'$ of $\mathcal{V}$ such that $\mathcal{V}' + \mathcal{V}' \subseteq \mathcal{V}$ and thus $\sum_4 \mathcal{V}' \subseteq \mathcal{U}$. We can induct, and we get that for any n there exists a symmetric neighborhood $\mathcal{V}$ of 0 such that

$$\sum_n \mathcal{V} \subseteq \mathcal{U}.$$

1.1.6 Theorem

Suppose $K, C \subseteq \mathbf{V}$ where K compact and C closed and both sets are disjoint. Then there exists a neighborhood $\mathcal{U}$ of 0 such that $K + \mathcal{U}$ and $C + \mathcal{U}$ are disjoint.

Proof

If K is empty then this is trivial (since $K + \mathcal{U}$ is empty). Otherwise let $x \in K$, since $x \notin C$ this means there exists an open set $\mathcal{V}_x'$ such that $\mathcal{V}_x'$ is disjoint from C. Then $\mathcal{V}_x' - x$ is a neighborhood of 0 and thus contains $\mathcal{V}_x + \mathcal{V}_x + \mathcal{V}_x$ for $\mathcal{V}_x$ symmetric. And so $x + \mathcal{V}_x + \mathcal{V}_x + \mathcal{V}_x$ is disjoint from C (as a subset of $\mathcal{V}_x'$). Thus $x + \mathcal{V}_x + \mathcal{V}_x$ is disjoint from $C - \mathcal{V}_x = C + \mathcal{V}_x$.
Since K is compact, we can find finitely many points such that $K \subseteq \bigcup x_i + \mathcal{V}_{x_i}$. Let $\mathcal{U} = \bigcap \mathcal{V}_{x_i}$, then

$$K + \mathcal{U} \subseteq \bigcup (x_i + \mathcal{V}_{x_i} + \mathcal{U}) \subseteq \bigcup (x_i + \mathcal{V}_{x_i} + \mathcal{V}_{x_i}),$$

each of which is disjoint from $C + \mathcal{V}_{x_i}$ and thus $C + \mathcal{U}$. ■

Notice that

$$K + \mathcal{U} = \bigcup_{k \in K} k + \mathcal{U},$$

and is therefore open. This means that compact and closed sets can be separated by open sets. Since singletons are closed and compact, this means that topological vector spaces are Hausdorff.

1.1.7 Theorem

A topological vector space is Hausdorff.

1.1.8 Theorem

If $\mathbf{B}$ is a local base for $\mathbf{V}$, then every member of $\mathbf{B}$ contains the closure of some member of $\mathbf{B}$.

Proof

Let $\mathcal{U} \in \mathbf{B}$; by the theorem above, there exists a neighborhood $\mathcal{V}$ (we can assume to be in $\mathbf{B}$) of 0 such that $\mathcal{U}^c + \mathcal{V}$ and $\{0\} + \mathcal{V} = \mathcal{V}$ are disjoint. Recall that if X, Y are open and disjoint then $\overline{X}, Y$ are disjoint. Thus $\mathcal{U}^c \subseteq \mathcal{U}^c + \mathcal{V}$ and $\overline{\mathcal{V}}$ are disjoint, meaning $\overline{\mathcal{V}} \subseteq \mathcal{U}$. ∎

1.1.9 Proposition

Let $\mathbf{V}$ be a topological vector space.

(1) if $A \subseteq \mathbf{V}$ then $\overline{A} = \bigcap(A + \mathcal{U})$ (where $\mathcal{U}$ runs over all neighborhoods of 0);

(2) if $A, B \subseteq \mathbf{V}$ then $\overline{A} + \overline{B} \subseteq \overline{A + B}$

(3) if $Y \leq \mathbf{V}$ (i.e. is a subspace) then $\overline{Y} \leq \mathbf{V}$;

(4) if C is a convex subset of $\mathbf{V}$ then so are $\overline{C}$ and C°;

Proof

(1) $x \in \overline{A}$ iff $x + \mathcal{U} \cap A \neq \emptyset$ for all $\mathcal{U}$ neighborhoods of 0. This is iff $x \in A - \mathcal{U}$ for all such $\mathcal{U}$, and $-\mathcal{U}$ is a neighborhood of 0 iff $\mathcal{U}$ is one, completing the proof.

(2) Let $x \in \overline{A}, y \in \overline{B}$ and $\mathcal{U}$ a neighborhood of $x + y$. Then there exists $x \in \mathcal{U}_1, y \in \mathcal{U}_2$ such that $\mathcal{U}_1 + \mathcal{U}_2 \subseteq \mathcal{U}$. Since $x \in \overline{A}$ and $y \in \overline{B}$, there are $a \in \mathcal{U}_1 \cap A, b \in \mathcal{U}_2 \cap B$, and so $a + b \in \mathcal{U} \cap (A + B)$. Thus $\mathcal{U} \cap (A + B) \neq \emptyset$. This means that $x + y \in \overline{A + B}$.

(3) Since M_a is a homeomorphism for $a \neq 0$ we have that $a\overline{Y} = \overline{aY}$ (and for $a = 0$ this is true trivially). We then have by the previous point

$$a\overline{Y} + b\overline{Y} = \overline{aY} + \overline{bY} \subseteq \overline{aY + bY} = \overline{Y}$$

as required.

(4) Notice that

$$t\overline{C} + (1-t)\overline{C} \subseteq \overline{tC + (1-t)C} \subseteq \overline{C}$$

and since $C^\circ \subseteq C$ and C is convex, we have that $tC^\circ + (1-t)C^\circ \subseteq C$. The sum of two open sets is open, we have that $tC^\circ + (1-t)C^\circ \subseteq C^\circ$ as required. ∎

1.1.10 Definition

When $F = \mathbb{R}$ or $\mathbb{C}$ (with the usual topology) and $\mathbf{V}$ a F-topological vector space, a **balanced subset** of $\mathbf{V}$ is a set B such that $\alpha B \subseteq B$ for every $|\alpha| \leq 1$. And E is **bounded** if for every neighborhood of 0 $\mathcal{U}$ there is a number $s_{\mathcal{U}}$ such that $E \subseteq t\mathcal{U}$ for every $t > s_{\mathcal{U}}$.

1.1.11 Proposition

Let $\mathbf{V}$ be a real or complex topological vector space.

(1) If B is balanced, so is $\overline{B}$. If $0 \in B^\circ$ then B° is also balanced.

(2) If E is bounded, so is $\overline{E}$.

Proof

(1) If $0 < |a| < 1$ then M_a is a homeomorphism so $aB^\circ = (aB)^\circ$. And so $aB^\circ = (aB)^\circ \subseteq B^\circ$ since $aB \subseteq B$. And if $0 \in B^\circ$ then $0B^\circ = \{0\} \subseteq B^\circ$.

(2) Let $\mathcal{U}$ be a neighborhood of 0; by the previous theorem, $\overline{\mathcal{V}} \subseteq \mathcal{U}$ for some neighborhood $\mathcal{V}$ of 0. Then $E \subseteq t\mathcal{V}$ for all $t > s_{\mathcal{V}}$ and then $\overline{E} \subseteq t\overline{\mathcal{V}} \subseteq t\mathcal{U}$. ■

Note:
From now on, all vector spaces under consideration are either real or complex.

1.1.12 Theorem
In a topological vector space $\mathbf{V}$,

(1) every neighborhood of 0 contains a balanced neighborhood of 0;

(2) every convex neighborhood of 0 contains a balanced convex neighborhood of 0.

Proof

(1) Let $\mathcal{U}$ be a neighborhood of 0, since scalar multiplication is continuous the preimage of $\mathcal{U}$ under multiplication contains $B_r(0) \times \mathcal{V}$ where $\mathcal{V}$ is a neighborhood of 0. This means that if $|a| < r$ then $a\mathcal{V} \subseteq \mathcal{U}$. Let $\mathcal{W}$ be the union of all such $a\mathcal{V}$, then $\mathcal{W}$ is a balanced neighborhood of 0 contained in $\mathcal{U}$.

(2) Let $A = \bigcap_{|a|=1} a\mathcal{U}$ and $\mathcal{W}$ be as in the previous point. Since $\mathcal{W}$ is balanced, when $|a| = 1$ we have $a^{-1}\mathcal{W} \subseteq \mathcal{W}$ and so $\mathcal{W} \subseteq a\mathcal{U}$. Therefore $\mathcal{W} \subseteq A$. Since $\mathcal{W}$ is a neighborhood of 0, this means that A° is a neighborhood of 0 as well, clearly contained in $\mathcal{U}$. Since $a\mathcal{U}$ is convex, A is an intersection of convex sets and is therefore convex, and therefore so is A°.

All that remains is to show that A° is balanced. It is sufficient to show that A is balanced. Let $0 \le r \le 1$ and $|b| = 1$ then,

$$rbA = \bigcap_{|a|=1} rba\mathcal{U} = \bigcap_{|a|=1} ra\mathcal{U}.$$

Since $a\mathcal{U}$ is a convex set containing 0, we have $ra\mathcal{U} \subseteq aU$ (since $ra\mathcal{U} \subseteq ra\mathcal{U} + (1-r)a\mathcal{U}$). Thus $rbA \subseteq A$ as required. ■

Let us say that a local base $\mathbf{B}$ is *balanced* if its members are balanced, and *convex* if its members are convex.

1.1.13 Corollary

(1) Every topological vector space has a balanced local base.

(2) Every topological vector space has a convex local base.

1.1.14 Theorem

Let $\mathcal{U}$ be a neighborhood of 0 in $\mathbf{V}$.

(1) If $\{r_i\}_i$ is an increasing sequence to ∞, then

$$V = \bigcup_i r_i\mathcal{U}.$$

(2) Every compact subset of $\mathbf{V}$ is bounded.

(3) If $\{\delta_i\}_i$ is a decreasing sequence to 0 and $\mathcal{U}$ is bounded, then $\{\delta_i\mathcal{U}\}_i$ is a local base for $\mathbf{V}$.

Proof

(1) Let $x \in \mathbf{V}$, then $\mathcal{V}_x = \{a \in F \mid ax \in \mathcal{U}\}$ is open (as the projection of the preimage of $\mathcal{U}$ under scalar multiplication). Furthermore, it contains 0 and therefore $1/r_n$ for large enough n (as it is open). Therefore $1/r_n x \in \mathcal{U}$ and so $x \in r_n\mathcal{U}$ as required.

(2) Let $\mathcal{U}$ be a balanced neighborhood of 0. By the above point, $\{n\mathcal{U}\}$ is an open cover of K, and thus there exist $n_1, \dots, n_k$ such that $K \subseteq \bigcup_1^k n_i\mathcal{U}$. Since $\mathcal{U}$ is balanced, for $a < b$ we have $a\mathcal{U} \subseteq b\mathcal{U}$ since $a/b\mathcal{U} \subseteq \mathcal{U}$. Thus $K \subseteq n_k\mathcal{U}$, and we can take $s = n_k$. And for $\mathcal{U}$ arbitrary, it contains a balanced neighborhood.

(3) Let $\mathcal{W}$ be a neighborhood of 0. Since $\mathcal{U}$ is bounded $\mathcal{U} \subseteq t\mathcal{W}$ for $t > s$ and therefore $1/t\mathcal{U} \subseteq \mathcal{W}$. Since δ_i is decreasing, there is a δ_n such that $\delta_n\mathcal{U} \subseteq \mathcal{W}$ and therefore $\{\delta_i\mathcal{U}\}_i$ is a local base.

■

1.2 Linear Mappings

Note that if Λ is a linear map, then it and its preimage preserve subspaces, convex sets, and balanced sets.

1.2.1 Theorem

The following are equivalent of a linear map $\Lambda\colon X \to Y$:

(1) Λ is continuous at 0;

(2) Λ is continuous everywhere;

(3) Λ is uniformly continuous: for every $0 \in \mathcal{V} \subseteq Y$ open there is a $0 \in \mathcal{U} \subseteq X$ open where

$$x - y \in \mathcal{U} \iff \Lambda x - \Lambda y \in \mathcal{V}$$

Proof

Obviously (2) $\implies$ (1), so we show that (1) $\implies$ (2), (3) and (3) $\implies$ (1).
Let $\mathcal{U}$ be a neighborhood of $\Lambda(x) \in Y$, then $\mathcal{U} - \Lambda(x)$ is a neighborhood of 0 and so $\Lambda^{-1}(\mathcal{U} - \Lambda(x))$ is open. But this is equal to $\{a \in X \mid \Lambda a \in \mathcal{U} - \Lambda(x)\} = \{a \in X \mid \Lambda(a + x) \in \mathcal{U}\} = \Lambda^{-1}\mathcal{U} - x$. So $\Lambda^{-1}\mathcal{U}$ is open, and therefore Λ is continuous.
For (1) $\implies$ (3), let $\mathcal{U} = \Lambda^{-1}\mathcal{V}$ which is open as Λ is continuous at 0 (and $\Lambda 0 = 0$). Now,

$$x - y \in \mathcal{U} \iff \Lambda x - \Lambda y \in \mathcal{V}$$

This also shows (3) $\implies$ (1). ■

1.2.2 Theorem

Let Λ be a non-constant linear functional $X \to F$. Then the following are equivalent:

(1) Λ is continuous;

(2) The kernel $\ker \Lambda$ is closed;

(3) $\ker \Lambda$ is not dense in X;

(4) Λ is bounded in some neighborhood $\mathcal{U}$ of 0.

Proof

(1) $\implies$ (2) follows since $\ker \Lambda = \Lambda^{-1}\{0\}$ and $\{0\}$ is closed.
(2) $\implies$ (3): since Λ is non-constant, $\ker \Lambda \neq X$ and is closed. Since it is closed and non-trivial, it cannot be dense.
(3) $\implies$ (4): so $\ker \Lambda^c$ has non-empty interior. We know then that there exists a balanced neighborhood $\mathcal{U}$ of 0 and $x \in X$ such that $x + \mathcal{U}$ and $\ker \Lambda$ are disjoint. (This is because we can have a local basis of balanced neighborhoods.) Now, $\Lambda\mathcal{U}$ is a balanced subset of F, which means that if it is unbounded $\Lambda\mathcal{U} = F$. But then there exists a $y \in \mathcal{U}$ such that $\Lambda y = -\Lambda x$ and so $x + y \in x + \mathcal{U}$ and $\ker \Lambda$, in contradiction. ■

1.3 Finite-Dimension Vector Spaces

In this chapter we focus on finite-dimension vector spaces. We endow upon $\mathbb{R}^\ltimes$ and $\mathbb{C}^\ltimes$ the topologies induced by their euclidean norms (which is equivalent to the product topology).

1.3.1 Lemma

If X is a complex topological vector space, and $f\colon \mathbb{C}^\ltimes \to \mathbb{X}$ is linear, then it is continuous.

Proof

Let $\{e_1, \ldots, e_n\}$ form the standard basis for $\mathbb{C}^\ltimes$. Note that

$$f(z) = \sum_{i=1}^{n} \pi_i(z) f(e_i),$$

where π_i are the projections $\mathbb{C}^\ltimes \to \mathbb{C}$. Projections are continuous, and thus f is the linear combination of continuous functions and is therefore continuous. ■

1.3.2 Theorem

If Y is an n-dimensional subspace of a complex topological vector space X, then

(1) every vector space isomorphism of $\mathbb{C}^\ltimes$ and Y is a homeomorphism;

(2) Y is closed.

Note that (1) is not a trivial consequence of the above lemma: an isomorphism $\mathbb{C}^\ltimes \to \mathbb{Y}$ is continuous, but it is not obvious that its inverse $Y \to \mathbb{C}^\ltimes$ is!

Proof

Let $S = \partial B_1(0) \subseteq \mathbb{C}^\ltimes$ (i.e. $z \in S \iff \sum_i |z|_i^2 = 1$). Now suppose $f\colon \mathbb{C}^\ltimes \to \mathbb{Y}$ is an isomorphism. By the above lemma is continuous, and therefore since S is compact so is $K = fS$. Since f is injective and $0 \notin S$, this means that $0 \notin K$, and so there is a balanced neighborhood of 0 which does not intersect K, let this be $\mathcal{U}$. Define

$$E = f^{-1}(\mathcal{U} \cap Y),$$

this is disjoint from S (since $x \in E \cap S$ would imply $fx \in \mathcal{U}$ and $fx \in K$). Since f is linear, and $\mathcal{U} \cap Y$ is balanced, so is E. Since balanced subsets of $\mathbb{C}^\ltimes$ are connected, E is connected. Since $0 \in E$, $E \subseteq B_1(0)$ (since otherwise fE would have an element of K).

Now, f^{-1} is a tuple of linear functionals, each of which is bounded around 0 and therefore are continuous. Therefore f is a homeomorphism.

For (2), let $p \in \overline{Y}$, and have f and $\mathcal{U}$ has above. Then for some $t > 0$, $p \in t\mathcal{U}$, so p is in the closure of $Y \cap t\mathcal{U}$. Since $\mathcal{U} \subseteq fB$ we have that $\overline{Y \cap t\mathcal{U}} \subseteq \overline{f(tB)} \subseteq \overline{f(t\overline{B})}$. But $\overline{B}$ is compact and therefore so is $f(t\overline{B})$. Thus $p \in f(t\overline{B})$. And $f(t\overline{B}) \subseteq Y$, as required. ∎

1.3.3 Theorem

Every locally compact topological vector space has finite dimension.

Proof

Since X is locally compact, $0 \in X$ has a neighborhood $\mathcal{U}$ whose closure is compact. Thus $\mathcal{U}$ is bounded and therefore $2^{-n}\mathcal{U}$ form a local basis for X. Compactness of $\overline{\mathcal{U}}$ shows that there exist $x_1, \ldots, x_n \in X$ such that $\overline{\mathcal{U}} \subseteq \bigcup_i x_i + 1/2\mathcal{U}$. Let $Y = \langle x_1, \ldots, x_n \rangle$, and so Y has dimension $\leq n$ and by the previous theorem is closed. So we have $\overline{\mathcal{U}} \subseteq \bigcup_i x_i + 1/2\mathcal{U} \subseteq Y + 1/2\mathcal{U}$. This is open so $\mathcal{U} \subseteq Y + 1/2\mathcal{U}$. Since $\lambda Y = Y$ for $\lambda \neq 0$ we see that $\frac{1}{2}\mathcal{U} \subseteq Y + 1/4\mathcal{U}$ and so $\mathcal{U} \subseteq Y + 1/4\mathcal{U}$. Continuing we get

$$\mathcal{U} \subseteq \bigcap_{n=1}^{\infty} (Y + 2^{-n}\mathcal{U}).$$

Now, $\{2^{-n}\mathcal{U}\}$ is a local base, and so $\mathcal{U} \subseteq \overline{Y} = Y$ (since Y is closed). But then $n\mathcal{U} \subseteq Y$ for all $n > 0$, and since $\bigcup_n n\mathcal{U} = X$ we see that $X = Y$, and so X is finite-dimensional. ∎

1.3.4 Definition

A topological vector space has the **Heine-Borel property** if compactness is equivalent to being closed and bounded.

1.3.5 Theorem

If X is a locally bounded topological vector space with the Heine-Borel property, then X is finite-dimensional.

Proof

By assumption, $0 \in X$ has a bounded neighborhood $\mathcal{U}$. Therefore $\overline{\mathcal{U}}$ is bounded too, and closed, and therefore compact. So 0 has a compact neighborhood, and therefore X is locally compact so we can apply the above theorem. ∎