Complex Functions

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Theorem 11.1 (Rouche's Theorem):

Suppose f and g are analytic functions on and in a regular closed curve γ, such that $|g(z)| < |f(z)|$ for every $z \in \gamma$. Then $f + g$ has the same number of zeros as f in γ.

Proof:

Note that this means that on γ, $f(z) \neq 0$. So

$$f + g = f\left(1 + \frac{g}{f}\right)$$

In general

$$\frac{(A(z)B(z))'}{A(z)B(z)} = \frac{A'B + AB'}{AB} = \frac{A'}{A} + \frac{B'}{B}$$

This means that

$$\frac{(f+g)'}{f+g} = \frac{f'}{f} + \frac{\left(1 + \frac{g}{f}\right)'}{1 + \frac{g}{f}}$$

Since f and g are analytic and therefore have no poles,

$$Z(f+g) = \frac{1}{2\pi i}\int_\gamma \frac{f'}{f} + \frac{\left(1 + \frac{g}{f}\right)'}{1 + \frac{g}{f}}\,dz = Z(f) + \frac{1}{2\pi i}\int_\gamma \frac{\left(1 + \frac{g}{f}\right)'}{1 + \frac{g}{f}}\,dz$$

Let us define

$$w(z) = 1 + \frac{g(z)}{f(z)}$$

is analytic in a neighborhood of γ since $f(z) \neq 0$ on γ. Thus

$$\Gamma\colon [0,1] \longrightarrow \mathbb{C}, \quad t \mapsto w(\gamma(t))$$

is a closed smooth curve. So

$$\frac{1}{2\pi i}\int_\gamma \frac{w'(z)}{w(z)}\,dz = \frac{1}{2\pi i}\int_\Gamma \frac{dz}{z} = n(\Gamma, 0)$$

But notice that

$$|1 - \Gamma(t)| = |1 - w(\gamma(t))| = \left|\frac{g(\gamma(t))}{f(\gamma(t))}\right| < 1$$

And so $\Gamma \subseteq D_1(1)$, meaning 0 is in Γ's exterior, meaning $n(\Gamma, 0) = 0$.
So we have that

$$Z(f+g) = Z(f) + n(\Gamma, 0) = Z(f)$$

■

11.1 Computing real integrals with complex analysis

Example 11.1:

Let us compute

$$I = \int_0^{2\pi} \frac{\sin^2\theta}{5 - 4\cos\theta}\,d\theta$$

Let us define

$$z\colon [0,2\pi] \longrightarrow \mathbb{C}, \quad \theta \mapsto e^{i\theta}$$

Then $z' = ie^{i\theta} = iz$ so $d\theta = \frac{dz}{iz}$. Now we know that

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right), \quad \sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right)$$

So

$$\frac{\sin^2(\theta)}{5-4\cos\theta} = -\frac{1}{4}\frac{\left(z-\frac{1}{z}\right)^2}{5-2\left(z+\frac{1}{z}\right)} = -\frac{1}{4}\frac{(z^2-1)^2}{5z^2-2(z^3+z)} = \frac{1}{8}\frac{(z^2-1)}{z(z-2)\left(z-\frac{1}{2}\right)}$$

Since we have parameterized $|z|=1$, substituting is simply the same as integrating over $|z|=1$.

$$I = \int_{|z|=1} \frac{1}{8}\frac{(z^2-1)^2}{z(z-2)\left(z-\frac{1}{2}\right)}\frac{dz}{di} = \frac{-i}{8}\int_{|z|=1} \frac{(z^2-1)^2)}{z^2(z-2)\left(z-\frac{1}{2}\right)}\,dz$$

This is a rational function, and it has poles at $z = 0, 2, \frac{1}{2}$. 2 and $\frac{1}{2}$ are simple poles, and 0 has degree two. Since 2 is not in $|z|=1$, we ignore it. The residues are

$$\operatorname{Res}(0) = \lim_{z\to 0}(z^2 f(z))' = \lim_{z\to 0}\left(\frac{(z^2-1)^2}{(z-2)\left(z-\frac{1}{2}\right)}\right)' = \frac{5}{2}$$

and

$$\operatorname{Res}\left(\frac{1}{2}\right) = -\frac{3}{2}$$

So we get that

$$I = -\frac{i}{8}\cdot 2\pi i\left(\frac{5}{2}-\frac{3}{2}\right) = \frac{\pi}{4}$$

Proposition 11.2:

If $p(x)$ and $q(x)$ are real polynomials such that $d = \deg q - \deg p \geq 2$ and $q(x)$ has no real zeros, then

$$I = \int_{-\infty}^{\infty}\frac{p(x)}{q(x)}\,dx = 2\pi i\sum \operatorname{Res}\left(\frac{p}{q}, z_k\right)$$

Where the sum is over the singularities of $\frac{p(z)}{q(z)}$ for $0 < \operatorname{Im}(z)$.

Proof:

Recall that if $\lim_{x\to\infty}\frac{f(x)}{g(x)} = L \in [0,\infty]$ for non-negative f and g, if

(1) $L \in (0,\infty)$ then $\int_0^\infty f$ converges if and only if $\int_0^\infty g$ does.

(2) If $L = 0$ then if $\int_0^\infty g$ converges so does $\int_0^\infty f$.

(3) If $L = \infty$ then if $\int_0^\infty f$ converges so does $\int_0^\infty g$.

Since

$$\lim_{x\to\pm\infty}\frac{\left|\frac{p}{q}\right|}{\frac{1}{|x|^d}}$$

exists and is finite, and

$$\int_{\pm 1}^{\pm\infty}\frac{1}{|x|^d}$$

exists and is finite (since $d > 1$), then $\int_{\pm 1}^{\pm\infty} \left|\frac{p}{q}\right|$ also converges and so the integral of $\frac{p}{q}$ converges absolutely. Since

$$I = \int_{-\infty}^{-1} \frac{p}{q} + \int_{-1}^{1} \frac{p}{q} + \int_{1}^{\infty} \frac{p}{q}$$

I converges absolutely as well (since q has no real zeros, $\frac{p}{q}$ has no singularities and is therefore continuous in $(-1,1)$ so its integral exists).
Let us denote Δ_R as the top half of the circle C_R, and Γ_R be the top half of the circle with the line $[-R, R]$. Notice that $f(x) = \frac{p(x)}{q(x)}$ is meromorphic in $\mathbb{C}$. We can assume that Γ_R does not intersect any of its singularities. Notice that Γ_R is a closed regular curve and so

$$\int_{\Gamma_R} f(z)\, dz = 2\pi i \sum \operatorname{Res}(f, z_k)$$

where the sum is over the singularities of f within Γ_R.
Now notice that

$$\int_{\Gamma_R} f(z) = \int_{-R}^{R} f(x)\, dx + \int_{\Delta_R} f(z)\, dz$$

And so

$$\left|\int_{\Delta R} f(z)\, dz\right| \leq \sup_{z \in \Delta_R} |f(z)| \cdot \pi R$$

But since $z^2 \frac{p(z)}{q(z)}$ is bounded (since it is the division of two polynomials of equal degree), we get that there exists an A such that $\left|z^2\right| \cdot |f(z)| < A$. And so

$$\left|\int_{\Delta_R} f(z)\right| \leq \frac{A}{R^2} \cdot \pi R \xrightarrow[R \to \infty]{} 0$$

Meaning that

$$\lim_{R \to \infty} \int_{\Gamma_R} f(z)\, dz = \lim_{R \to \infty} \int_{-R}^{R} f(x)\, dx = \int_{-\infty}^{\infty} f(x)\, dx$$

And since

$$\int_{\Gamma_R} f(z) = 2\pi i \sum \operatorname{Res}(f, z_k)$$

where z_k are singularities in Γ_R. Since Γ_R is a half-circle with a positive imaginary part, for any singularity such that $\operatorname{Im}(z_k) > 0$, eventually z_k will be in Γ_R. So

$$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum \operatorname{Res}(f, z_k), \qquad z_k \text{ singularity, } \operatorname{Im}(z_k) > 0$$

Proposition 11.3:

Let $R(x) = \frac{p(x)}{q(x)}$ be a rational function (assume p and q are coprime in $\mathbb{C}[z]$; that they have no common root), and q is not zero on the real line. Suppose $\deg q > \deg p$ then

$$\int_{-\infty}^{\infty} R(x) \cos(x) = \operatorname{Re}\Big(2\pi i \sum \operatorname{Res}\big(R(z) e^{iz}, z_k\big)\Big)$$

and

$$\int_{-\infty}^{\infty} R(x) \sin(x) = \operatorname{Im}\Big(2\pi i \sum \operatorname{Res}\big(R(z) e^{iz}, z_k\big)\Big)$$

Where the sum is over singularities of $R(z)e^{iz}$ with positive imaginary parts.

Proof:

Notice that

$$R'(x) = \frac{p'q - pq'}{q^2}$$

and since $\deg(p'q - pq') > 0$, and eventually polynomials are either always negative or positive (since otherwise they'd have infinite roots), $R'(x)$ is either always negative or positive in some ray $[a, \infty)$ and $(-\infty, b]$ meaning R is eventually monotonic.
Now note that if we define

$$F_+(x) = \int_b^x \cos(t)\,dt, \qquad F_-(x) = \int_x^b \cos(t)\,dt$$

Then F_+ and F_- are bounded in $[b, \infty)$ and $(-\infty, b]$ respectively, and so by Dirichlet's test

$$\int_{-\infty}^{\infty} R(x)\cos(x)\,dx$$

exists. Similar for $\sin(x)$.
Let us use the same Γ_R and Δ_R as last proof, and for $0 < h < R$ let us define

$$\Delta_{R,+} = \{z \in \Delta_R \mid \operatorname{Im}(z) \geq h\}, \quad \Gamma_{R,-} = \{z \in \Delta_R \mid \operatorname{Im}(z) < h\}$$

Since $\deg q > \deg p$, there exists a $k > 0$ such that for $|z|$ sufficiently large

$$|R(z)| < \frac{k}{|z|}$$

Then

$$\sup_{z\in\Delta_{R,+}} \left|R(z)e^{iz}\right| = \sup_{z\in\Delta_{R,+}} |R(z)| \cdot e^{-\operatorname{Im}(z)} \leq \frac{k}{R} \cdot e^{-h}$$

So

$$\left|\int_{\Delta_{R,+}} R(z)e^{iz}\right| \leq \frac{k}{R}e^{-h}\pi R = k\pi e^{-h}$$

And since the perimiter of $\Delta_{R,-} \leq 4h$ (this involves some trigonometry), we have

$$\left|\int_{\Delta_{R,-}} R(z)e^{iz}\right| \leq \frac{k}{R}4h$$

So

$$\left|\int_{\Delta_R} R(z)e^{iz}\right| \leq k\pi e^{-h} + \frac{4k}{R}h$$

Let $h = \sqrt{R}$ and we get

$$\left|\int_{\Delta_R} R(z)e^{iz}\right| \leq k\pi e^{-\sqrt{R}} + \frac{4k}{\sqrt{R}} \xrightarrow[R\to\infty]{} 0$$

So we get

$$\lim \int_{\Gamma_R} R(z)e^{iz} = \lim \int_{-R}^{R} R(z)e^{iz} = \int_{-\infty}^{\infty} R(x)e^{ix}$$

And since

$$\int_{\Gamma_R} R(z)e^{iz} = 2\pi i \sum \operatorname{Res}\big(R(z)e^{iz}, z_k\big)$$

where z_k are singularities in Γ_R we get

$$\int_{-\infty}^{\infty} R(x)e^{ix}\,dx = 2\pi i \sum \operatorname{Res}\big(R(z)e^{iz}, z_k\big)$$

where $\operatorname{Im}(z_k) > 0$.
The imaginary and real parts of this integral correspond with the integrals of $R(x)\cos(x)$ and $R(x)\sin(x)$ respectively, as required. ■

Definition 11.4:

The **principal value** of the integral $\int_{-\infty}^{\infty} f(x)\,dx$ is defined as

$$\mathrm{pv}\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx$$

The principal value of an integral may exist without the integral existing, but if the integral exists then they are equal. For example if $f(x) = x$ then $\mathrm{pv}\int_{-\infty}^{\infty} x\,dx = 0$ but the indefinite integral does not exist.