Mathematical Logic

Lecture 12, Monday June 26, 2023

Ari Feiglin

Proposition 12.0.1 (Tarski-Vaught Test):

Suppose $\mathcal{M}$ is a substructure of $\mathcal{N}$. Then $\mathcal{M}$ is an elementary substructure if and only if for any formula $\varphi(v, \vec{w})$ and $\vec{a} \in \mathcal{M}^n$, if there is a $b \in \mathcal{N}$ such that $\mathcal{N} \vDash \varphi(b, \vec{a})$, then there is a $c \in \mathcal{M}$ such that $\mathcal{N} \vDash \varphi(c, \vec{a})$.

Proof:

If $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$, then since

$$\mathcal{N} \vDash \exists x(\varphi(x, \vec{a}))$$

we have, by definition,

$$\mathcal{M} \vDash \exists x(\varphi(x, \vec{a}))$$

as required.

To show the converse, we must show that for every $\vec{a} \in \mathcal{M}^n$,

$$\mathcal{M} \vDash \varphi(\vec{a}) \iff \mathcal{N} \vDash \varphi(\vec{a})$$

we will prove this by formula induction. If φ is quantifier-free, then this is due to $\mathcal{M}$ being a substructure of $\mathcal{N}$. The induction step for boolean combinations is trivial. Now suppose

$$\mathcal{M} \vDash \exists x \varphi(x, \vec{a})$$

then there is a $b \in \mathcal{M}$ such that $\mathcal{M} \vDash \varphi(b, \vec{a})$ and since $b \in \mathcal{N}$, and so inductively $\mathcal{N} \vDash \varphi(b, \vec{a})$, which means that $\mathcal{N} \vDash \exists x \varphi(x, \vec{a})$, as required. And if

$$\mathcal{N} \vDash \exists x \varphi(x, \vec{a})$$

then there exists a $c \in \mathcal{N}$ such that $\mathcal{N} \vDash \varphi(c, \vec{a})$ which by our assumption means that $\mathcal{N} \vDash \varphi(b, \vec{a})$ for $b \in \mathcal{M}$ and thus $\mathcal{M} \vDash \varphi(b, \vec{a})$ so $\mathcal{M} \vDash \exists x \varphi(x, \vec{a})$ as required. ■

Definition 12.0.2:

An $\mathcal{L}$-theory T has **built-on Skolem functions** if for every $\mathcal{L}$-formula $\varphi(v, w_1, \ldots, w_n)$ there is a function symbol f such that

$$T \vdash \forall \vec{w}\big((\exists v \varphi(v, \vec{w})) \rightarrow \varphi(f(\vec{w}), \vec{w})\big)$$

Or in other words, if $\varphi(\cdot, \vec{w})$ can be witnessed, there is a function symbol f so that it can be witnessed by $f(\vec{w})$.

Lemma 12.0.3:

Let T be an $\mathcal{L}$-theory. Then there exists a signature $\mathcal{L} \subseteq \mathcal{L}^*$ and an $\mathcal{L}^*$-theory $T \subseteq T^*$ such that T^* has built-in Skolem functions. Furthermore, if $\mathcal{M} \vDash T$ then we can extend $\mathcal{M}$ to an $\mathcal{L}^*$-model $\mathcal{M}^*$ such that $\mathcal{M}^* \vDash T^*$. Even further, $\mathcal{L}^*$ can be chosen such that

$$|\mathcal{L}*| = |\mathcal{L}| + \aleph_0$$

Proof:

Let us construct an ascending sequence of languages $\{\mathcal{L}_i\}_{i=0}^{\infty}$, and an ascending sequence of theories $\{T_i\}_{i=0}^{\infty}$ where T_i is an $\mathcal{L}_i$-theory.

We define $\mathcal{L}_0 = \mathcal{L}$, and recursively

$$\mathcal{L}_{i+1} = \mathcal{L}_i \cup \{f_\varphi \mid \varphi(v, w_1, \ldots, w_n) \text{ is an } \mathcal{L}_i\text{-formula for } n = 1, 2, \ldots\}$$

where f_φ is a function symbol. Then for an $\mathcal{L}_i$-formula $\varphi(v, \vec{w})$, we define

$$\Phi_\varphi = \forall \vec{w}\big((\exists v \varphi(v, \vec{w})) \to \varphi(f_\varphi(\vec{w}), \vec{w})\big)$$

Then we define

$$T_{i+1} = T_i \cup \{\Phi_\varphi \mid \varphi \text{ is an } \mathcal{L}_i\text{-formula}\}$$

Now we claim that if $\mathcal{M} \vDash T_i$, it can be extended to an $\mathcal{L}_{i+1}$-model of T_{i+1}. Let $c \in \mathcal{M}$, then if $\varphi(v, w_1, \dots, w_n)$ is an $\mathcal{L}_i$-formula, we define a function $g \colon \mathcal{M}^n \longrightarrow \mathcal{M}$ such that for every $\vec{a} \in \mathcal{M}^n$ if

$$X_{\vec{a}} = \{b \in \mathcal{M} \mid \mathcal{M} \vDash \varphi(b, \vec{a})\}$$

is non-empty ($\varphi(\cdot, \vec{a})$ has a witness), then let $g(\vec{a}) \in X_{\vec{a}}$. Otherwise $g(\vec{a}) = c$. Such a function is guaranteed by the axiom of choice.

Thus if $\mathcal{M} \vDash \exists v \varphi(v, \vec{a})$ then $X_{\vec{a}}$ is non-empty and so $\mathcal{M} \vDash \varphi(g(\vec{a}), \vec{a})$. So we interpret f_φ as g. And thus $\mathcal{M} \vDash \Phi_\varphi$.

Let us define

$$\mathcal{L}^* = \bigcup_{i=0}^{\infty} \mathcal{L}_i, \qquad T^* = \bigcup_{i=0}^{\infty} T_i$$

And $\mathcal{M}^*$ is the extension of $\mathcal{M}$ we have defined above. And if we have $\Phi \in T^*$, either $\Phi \in T$ in which case $\mathcal{M}^* \vDash \Phi$ as it extends $\mathcal{M}$, and otherwise it is equal to Φ_φ for some $\mathcal{L}^*$-formula φ, which must be an $\mathcal{L}_i$-formula for some i, and we showed that $\mathcal{M} \vDash \Phi_\varphi$. Thus $\mathcal{M}^* \vDash T^*$.

Then if $\varphi(v, \vec{w})$ is an $\mathcal{L}^*$-formula, it is a $\mathcal{L}_i$-formula for some i and so $\Phi_\varphi \in T_{i+1} \subseteq T^*$, and this states exactly the property of φ having a built-in Skolem function. (Note that φ may be Φ_ψ for some other $\mathcal{L}^*$-formula ψ).

Furthermore, note that since we've added function symbols to $\mathcal{L}_{i+1}$ for every formula of $\mathcal{L}_i$, we have that $|\mathcal{L}_{i+1}| = |\mathcal{L}_i| + \aleph_0$ (if $\mathcal{L}_i$ is uncountable then the number of functions added is $|\mathcal{L}_i| = |\mathcal{L}_i| + \aleph_0$, so this still holds). And so every $\mathcal{L}_i$ has the same cardinality for each $i > 0$, which is $|\mathcal{L}| + \aleph_0$. Thus their union, as a countable union, also has this cardinality. ■

Definition 12.0.4:

The T^* defined in the proof above is called the **skolemization** of T.

Theorem 12.0.5 (Downward Lowenheim-Skolem Theorem):

Suppose $\mathcal{M}$ is an $\mathcal{L}$-structure and $X \subseteq \mathcal{M}$. Then there exists an elementary substructure $\mathcal{N}$ of $\mathcal{M}$ such that $X \subseteq \mathcal{N}$ and $|\mathcal{N}| \leq |X| + |\mathcal{L}| + \aleph_0$.

Proof:

By the above lemma, we can assume $\mathrm{Th}(\mathcal{M})$ has built-in Skolem functions. Let $X_0 = X$, then we recursively define

$$X_{i+1} = X_i \cup \left\{ f^{\mathcal{M}}(\vec{a}) \;\middle|\; f \text{ is an } n\text{-ary function symbol and } \vec{a} \in X_i^n \text{ for } n = 0, 1, 2, \dots \right\}$$

Then let

$$\mathcal{N} = \bigcup_{i=0}^{\infty} X_i$$

Notice that $|X_{i+1}| \leq |X_i| + |\mathcal{L}| \cdot \varkappa_{X_i}$ where

$$\varkappa_X = \left| \bigcup_{n \in \mathbb{N}} X^n \right|$$

We can split this into cases, but it is not hard to show that we get $|\mathcal{N}| \leq |X| + |\mathcal{L}| + \aleph_0$.

If f is an n-ary function symbol of $\mathcal{L}$ and $\vec{a} \in \mathcal{N}^n$, then there exists some i such that $\vec{a} \in X_i^n$, and so $f^{\mathcal{M}}(\vec{a}) \in X_{i+1} \subseteq \mathcal{N}$. Thus $f^{\mathcal{M}}$ can be restricted on $\mathcal{N}^n$, ie. $\mathcal{N}$ is a substructure of $\mathcal{M}$.

If $\varphi(v, \vec{w})$ is an $\mathcal{L}$-structure and $\mathcal{M} \vDash \varphi(b, \vec{a})$ then since $\mathcal{M}$ has built-in skolem functions, there exists some function f such that $\mathcal{M} \vDash \varphi(f(\vec{a}), \vec{a})$. But since $f^{\mathcal{M}}(\vec{a}) \in \mathcal{N}$, thus by **Tarski-Vaught Test**, $\mathcal{N}$ is an elementary substructure of $\mathcal{M}$. ■