Calculus Homework #8

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Lemma:

(1) The integral $\displaystyle\int_1^\infty \frac{\sin x}{x^\alpha}\,dx$ converges if and only if $\alpha > 0$.

(2) The integral $\displaystyle\int_0^1 \frac{\sin x}{x^\alpha}\,dx$ converges if and only if $\alpha < 2$.

Proof:

(1) If $\alpha > 0$ then $\frac{1}{x^\alpha}$ is decreasing and the integral of $\sin x$ is bound so by Dirichlet, the integral converges. And if $\alpha = 0$ the integral diverges since the integral of $\sin x$ diverges.
If $\alpha < 0$, suppose that the definite integral converges. That means that the integral of $\frac{\sin x}{x^\alpha}$ is bound. And we know then that x^α monotonically decreases to 0, so by Dirichlet, that would mean that the integral

$$\int_1^\infty \sin x\,dx$$

converges, in contradiction.
So the integral converges if and only if $\alpha > 0$.

(2) Using integration by parts the integral is equal to:

$$= x^\alpha(1-\cos x)\Big|_0^1 + \alpha\int_0^1 \frac{1-\cos x}{x^{\alpha+1}}\,dx$$

The leftmost element converges if and only if the following limit converges:

$$\lim_{x\to 0^+} \frac{1-\cos x}{x^\alpha}$$

By L'Hopital:

$$= \lim \frac{\sin x}{\alpha x^{\alpha-1}}$$

If $\alpha - 1 \leq 0 \iff \alpha \leq 1$ then this limit converges to 0, otherwise:

$$= \lim \frac{\cos x}{\alpha(\alpha-1)x^{\alpha-2}}$$

Which converges if and only if $\alpha - 2 \leq 0 \iff \alpha \leq 2$.
Notice that:

$$\frac{\frac{1-\cos x}{x^{\alpha+1}}}{x^{1-\alpha}} = \frac{1-\cos x}{x^2} \longrightarrow \frac{1}{2}$$

So the right integral converges if and only if the integral $\int_0^1 x^{1-\alpha}\,dx$ converges, which is if and only if $\alpha - 1 < 1 \iff \alpha < 2$.
So the integral converges if and only if $\alpha < 2$.

Question 8.1:

Determine for which values of p the following integrals converge:

(1) $\displaystyle\int_1^\infty \frac{e^{\sin x}\sin(2x)}{x^p}\,dx$

(2) $\displaystyle\int_1^\infty \frac{\log|x|^p \sin x}{x}\,dx$

(1) We know that the integral of $\int e^{\sin x}\sin(2x)\,dx$ is bound since a substitution of $u = \sin x$ yields:

$$= 2\int ue^u\,du \ni 2(ue^u - e^u) = 2e^u(u-1) = 2e^{\sin x}(\sin x - 1)$$

Which is bound between 0 and $-2e$.
And if $p > 0$ then $\frac{1}{x^p}$ is monotonically decreasing to 0, so by Dirichlet the integral converges.
If $p = 0$, the integral is equal to

$$2e^{\sin x}(\sin x - 1)\Big|_1^\infty$$

Which doesn't converge (we can choose xs where $\sin x = 1$ and others where $\sin x = 0$).
If $p < 0$ then suppose the integral converges. That means that:

$$\int \frac{e^{\sin x}\sin(2x)}{x^p}\,dx$$

Is bound, and since $p < 0$ x^p decreases to 0. So by dirichlet, that would mean the integral

$$\int_1^\infty e^{\sin x}\sin(2x)\,dx$$

Converges. But this is a contradiction since we showed that this doesn't converge.
So the integral converges only for $p > 0$.

(2) We know that at some point $\frac{\log|x|^p}{x}$ is decreasing, as its derivative is equal to:

$$\frac{\log|x|^{p-1}(p - \log|x|)}{x^2}$$

And for $x > e^p$ this is negative.
The integral converges if and only if:

$$\int_{e^p}^\infty \frac{\log|x|^p \sin x}{x}\,dx$$

converges. And since $\frac{\log|x|^p}{x}$ is decreasing to 0 and the integral of $\sin x$ is bound, by Dirichlet this integral for every p.

(3) Notice the integral is equal to:

$$\frac{1}{3}\int_1^\infty \frac{\sin(x^3)\cdot 3x^2}{x^{p+2}}\,dx = \frac{1}{3}\int_1^\infty \frac{\sin u}{u^{\frac{p+2}{3}}}\,du$$

For $u = x^3$. By **lemma 8.1**, this converges if and only if $p > -2$.

(4) Notice that if $p < 0$ then:

$$\frac{|\tan(x)|^p}{x^p} = \left|\frac{\cot(x)}{\frac{1}{x}}\right|^{-p}$$

And the limit inside the absolute value is equal to:

$$\left|\frac{x}{\sin x}\right|^{-2p} \longrightarrow 1$$

So the integral of $|\tan(x)|^p$ converges if and only if the integral of x^p converges, which is if and only if $p > -1$.
And if $p > 0$:

$$\int_0^{\frac{\pi}{2}} |\tan x|^p \, dx = \int_0^{\frac{\pi}{2}} |\cot x|^p \, dx$$

By substituting $u = \frac{\pi}{2} - x$, which converges if and only if $p < 1$ as proven above.
If $p = 0$, the integrand is 1 so it is trivial.
So the integral converges if and only if $-1 < p < 1$.

Question 8.2:

Determine if the following converge or diverge:

(1) $\displaystyle\int_1^\infty \frac{\sqrt{x}}{\sqrt{1+5x^4}}\,dx$

(2) $\displaystyle\int_0^1 \frac{\log x}{1-x}\,dx$

(3) $\displaystyle\int_1^\infty \sin\left(\frac{1}{\sqrt{x^2+1}}\right)dx$

(4) $\displaystyle\int_0^\infty \frac{\tan^{-1}\left(\frac{1}{x}\right)}{\sqrt{x}}\,dx$

(5) $\displaystyle\int_2^\infty \frac{x\sqrt{x}\sin\left(\frac{1}{x}\right)}{\sqrt{x^2-x}}\,dx$

(1) Notice that:

$$\frac{\sqrt{\frac{x}{5x^4+1}}}{\sqrt{\frac{1}{x^3}}} = \sqrt{\frac{x^4}{5x^4+1}} \longrightarrow \sqrt{\frac{1}{5}}$$

So this integral and $\int_1^\infty \frac{1}{x^{\frac{3}{2}}}\,dx$ converge together. Since $\frac{3}{2} > 1$, the integral *converges*.

(2) Notice that:

$$\lim_{x\to1}\frac{\log x}{1-x} = \lim_{x\to1} -\frac{1}{x} = -1$$

So the integral:

$$\int_{0.5}^1 \frac{\log x}{1-x}\,dx$$

Exists, since the function is bound and continuous.
Let $u = \log x$, the rest of the integral becomes:

$$\int_{-\infty}^{-\log 2} \frac{u}{e^{-u}-1}\,du$$

And notice:

$$\frac{\frac{u}{e^{-u}-1}}{-ue^u} = -\frac{1}{1-e^u} \xrightarrow{x\to-\infty} -1$$

So the integral and the integral of $-ue^u$ converge together, and:

$$\int_{-\infty}^{-\log 2} -ue^u\,du = -(ue^u - e^u)\Big|_{-\infty}^{-\log 2} = 1$$

So the integral *converges*.

(3) Notice that:

$$0 \le \frac{1}{\sqrt{x^2+1}} \le \frac{1}{\sqrt{2}} \implies \sin\left(\frac{1}{\sqrt{x^2+1}}\right) \ge 0$$

And:

$$\lim_{x\to\infty}\frac{\sin\left(\frac{1}{\sqrt{x^2+1}}\right)}{\frac{1}{sqrtx^2+1}} = \lim_{\theta\to0^+}\frac{\sin\theta}{\theta} = 1$$

So the integral converges if and only if the integral of $\frac{1}{\sqrt{x^2+1}}$ converges, and:

$$\frac{\sqrt{x}}{\sqrt{x^2+1}} \longrightarrow 1$$

So the integral converges if and only if the integral of $\frac{1}{\sqrt{x}}$ converges. Since it diveres, the integral *diverges*.

(4) Notice that:

$$0 \leq \int_0^1 \frac{\tan^{-1}\left(\frac{1}{x}\right)}{\sqrt{x}}\,dx \leq \frac{\pi}{2}\int_0^1 \frac{1}{\sqrt{x}}\,dx$$

And the right integral converges, so so does the left.
And:

$$\lim_{x\to\infty} \frac{\frac{\tan^{-1}\left(\frac{1}{x}\right)}{\sqrt{x}}}{\frac{1}{x^{1.1}}} = \lim \frac{\tan^{-1}\left(\frac{1}{x}\right)}{x^{-0.6}} = \lim \frac{-\frac{1}{x^2}\cdot\frac{1}{1+\frac{1}{x^2}}}{-0.6x^{-1.6}} = \lim \frac{1}{0.6}\frac{x^{1.6}}{x^2+1} = 0$$

Since

$$\int_1^\infty \frac{1}{x^{1.1}}\,dx$$

Converges, so does the integral.
Both parts of the integral converge, so the integral *converges*.

(5) Notice that:

$$\lim_{x\to\infty} \frac{\frac{x\sqrt{x}\sin\left(\frac{1}{x}\right)}{\sqrt{x^2-x}}}{\frac{\sqrt{x}}{\sqrt{x^2-x}}} = \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to 0} \frac{\sin x}{x} = 1$$

So the integral converges if and only if:

$$\int_2^\infty \frac{1}{\sqrt{x-1}}\,dx = \int_1^\infty \frac{1}{\sqrt{x}}\,dx$$

Converges.
It diverges, so the integral *diverges*.

Question 8.3:

Determine whether the integrals diverge, conditionally converge, or converge absolutely:

(1) $\displaystyle\int_2^{\infty} \frac{\sin x}{\sqrt{x}-1}\,dx$

(2) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{x\sin(\tan x)}{\cos x}\,dx$

(1) Since the integral of $\sin x$ is bound and $\frac{1}{\sqrt{x}-1}$ monotonically decreases to 0, by Dirichlet, the integral converges.
We know that $|\sin x| \geq \frac{1-\cos(2x)}{2}$, so the absolute value integral diverges if the following diverges:

$$\int_2^{\infty} \frac{1-\cos(2x)}{\sqrt{x}-1}\,dx = \int_1^{\infty} \frac{dx}{\sqrt{x}} - \int_2^{\infty} \frac{\cos(2x)}{\sqrt{x}-1}\,dx$$

The left integral diverges since $0.5 < 1$ and the right integral converges by Dirichlet, so the integral itself diverges. So the integral does not converge absolutely.
The integral *converges conditionally*.

(2) If we let $u = \tan x$, we get $\cos x = \frac{1}{\sqrt{1+u^2}}$, and the integral is equal to:

$$\int_0^{\frac{\pi}{2}} \frac{x\cos x\sin(\tan x)}{\cos^2 x}\,dx = \int_0^{\infty} \frac{\tan^{-1}(u)\sin u}{\sqrt{1+u^2}}\,du$$

Since the integral of $\sin u$ is bound and at some point $\frac{\tan^{-1}(u)}{\sqrt{1+u^2}}$ is monotonically decreasing to 0 (Since its derivative is $\frac{1-u\tan^{-1}(u)}{(1+u^2)^{1.5}}$, which is negative for large enough us), by Dirichlet, the integral converges.
At some point $\tan^{-1}(u) \geq 1$, so from some point a the absolute integral greater than:

$$\geq \int_a^{\infty} \frac{|\sin u|}{\sqrt{1+u^2}}\,du \geq \frac{1}{\sqrt{2}}\int_a^{\infty} \frac{|\sin u|}{u}\,du$$

And we have shown that this diverges. So the integral does not absolutely converge.
The integral *converges conditionally*.

Question 8.4:

Determine for which values of p the following integrals converge conditionally and for which they converge absolutely:

(1) $\displaystyle\int_0^\infty \frac{\sin\left(x^2\right)}{x^p}\,dx$

(2) $\displaystyle\int_0^\infty \sin(x)\cdot\frac{\log|x+1|}{x^p}\,dx$

(1) Let $u = x^2$, we get that the integral is equal to:

$$\frac{1}{2}\int_0^\infty \frac{\sin\left(x^2\right)2x}{x^{p+1}}\,dx = \frac{1}{2}\int_0^\infty \frac{\sin u}{\sqrt{u^{p+1}}}\,du$$

By **lemma 8.1**, this converges if and only if $0 < \frac{p+1}{2} < 2 \iff -1 < p < 3$.
For $0 \le x \le 1$ the absolute integral is just equal to the integral, which we know by **lemma 8.1**, converges if and only if $p < 3$.
For $x \ge 1$, the integral is eqyal to:

$$\frac{1}{2}\int_1^\infty \frac{|\sin u|}{\sqrt{u^{p+1}}}\,du$$

If $\frac{p+1}{2} > 1 \iff p > 1$, then then this integral converges, since it is less than the integral of $\frac{1}{x^{\frac{p+1}{2}}}$ which converges.
If $p = 1$ then it is equal to:

$$\frac{1}{2}\int_1^\infty \frac{|\sin u|}{u}\,du$$

Which we showed diverges.
And for every $p < 1$, its integral is greater than if $p = 1$, so it too diverges.
This means that the integral absolutely converges for $1 < p < 3$.
And so the integral conditionally converges for $-1 \le p \le 1$ (since it converges for $-1 < p < 3$).

(2) For $p > 0$, $\frac{\log|x+1|}{x^p}$ monotonically decreases to 0 at some point since its derivative is

$$x^{p-1}\left(\frac{x}{x+1} - p\log|x+1|\right)$$

Whose limit is $-\infty$.
And since the integral of $\sin x$ is bound, the integral:

$$\int_1^\infty \sin(x)\,\frac{\log|x+1|}{x^p}\,dx$$

Converges.
Suppose for $p \le 0$ the integral converges. Then it is bound, and since $\frac{1}{\log|x+1|}$ decreases to 0, by Dirichlet that would mean:

$$\int_1^\infty \frac{\sin x}{x^p}\,dx$$

Converges, which is contradictory to **lemma 8.1**.
So the integral from 1 to ∞ converges if and only if $p > 0$.
Notice that:

$$\lim_{x\to 0}\frac{\sin(x)\cdot\frac{\log|x+1|}{x^p}}{\frac{1}{x^{p-2}}} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\log|x+1|}{x}$$

And since the limit of $\frac{\sin x}{x}$ is 1 and:

$$\lim \frac{\log|x+1|}{x} = \lim \frac{1}{x+1} = 1$$

This limit is equal to 1, so the integral between 0 and 1 converges if and only if the integral of $\frac{1}{x^{p-2}}$ converges, which is if and only if $p-2<1 \iff p<3$.
So the integral converges if and only if $0<p<3$.
For the absolute integral, notice that:

$$\int_0^\infty |\sin(x)| \cdot \frac{\log|x+1|}{x^p}\,dx \geq \frac{1}{2}\left(\int_0^\infty \frac{\log|x+1|}{x^p}\,dx - \int_0^\infty \cos(2x)\frac{\log|x+1|}{x^p}\,dx\right)$$

The rightmost integral converges by Dirichlet, and the left integral is greater than:

$$\geq \int_0^\infty \frac{1}{x^p}\,dx$$

Which diverges for all p.
So for all p the absolute integral diverges.
So the integral converges conditionally for $0<p<3$, and does not converge absolutely.