Infintesimal Calculus 3

Assignment 8
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Exercise 8.1:

Find the crtitical points of the following functions and determine their types:

(1) $f(x,y) = (x-1)^2 - 2y^2$

(2) $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

(1) The gradient here is

$$\nabla f = \begin{pmatrix} 2(x-1) \\ -4y \end{pmatrix}$$

which is equal to 0 only at $(1,0)$, so this is the only critical point. If we hold y constant at 0, then $x = 1$ is a minimum, ie $x = 1$ is the minimum of $f(x,0) = (x-1)^2$ as it is a positive parabola. But if we hold x constant at 1 then $y = 0$ is the maximum of $f(1,y) = -2y^2$ so $(1,0)$ is an inflection point (neither a maximum nor a minimum).

(2) The gradient here is

$$\nabla f = \begin{pmatrix} 4x^3 - 4x + 4y \\ 4y^3 + 4x - 4y \end{pmatrix}$$

which, if when 0, means that $4x^3 + 4y^3 = 0$ and so $y = -x$ and therefore $4x^3 - 8x = 0$ and so $x = 0$ or $x = \pm\sqrt{2}$. So the crtitical points are

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} \sqrt{2} \\ -\sqrt{2} \end{pmatrix} \quad \begin{pmatrix} -\sqrt{2} \\ \sqrt{2} \end{pmatrix}$$

We will now compute the hessian:

$$H_f(x,y) = \begin{pmatrix} 12x^2 - 4 & 4 \\ 4 & 12y^2 - 4 \end{pmatrix}$$

So for $\pm\sqrt{2}(1,-1)$, the hessian is the same:

$$\begin{pmatrix} 20 & 4 \\ 4 & 20 \end{pmatrix}$$

And therefore they are minima. For $(0,0)$, $x = 0$ is a maximum of $f(x,0) = x^4 - 2x^2$ but $x = 0$ is a minimum of $f(x,x) = 2x^4$, and so $(0,0)$ is an inflection point.

Exercise 8.2:

We define the following function:

$$f(x,y) = (y - 3x^2)(y - x^2)$$

(1) Show that $(0,0)$ is a critical point.

(2) Show that for every $a, b \in \mathbb{R}$, $f(at, bt)$ has a local minimum at $(0,0)$.

(3) Show that $(0,0)$ is not a minimum of f.

(1) The gradient is

$$\nabla f = \begin{pmatrix} -6x(y - x^2) - 2x(y - 3x^2) \\ y - x^2 + y - 3x^2 \end{pmatrix} = \begin{pmatrix} 12x^3 - 8xy \\ -4x^2 + 2y \end{pmatrix}$$

And since $\nabla f(0,0) = 0$, $(0,0)$ is indeed a critical point.

(2) We know that $d_t\big(f(at,bt)\big) = d_{x,y}f(at,bt) \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \nabla f(at,bt) \cdot \begin{pmatrix} a \\ b \end{pmatrix}$ by the chain rule. This is equal to $12a^4t^3 - 12a^2bt^2 + 2b^2t$. This is equal to 0 at $t = 0$ (as it should since $(0,0)$ is a critical point of f's), so $t = 0$ is a critical point. And its second derivative relative to t at $t = 0$ is $2b^2$ which is positive if $b \neq 0$, so if $b \neq 0$ then $t = 0$ is a minimum, as required. If $b = 0$ then

$$f(at, 0) = (-3a^2t^2)(-a^2t^2) = a^4t^4$$

which obviously has a minimum at $t = 0$ as required.

(3) Take $y = 2x^2$ then

$$f(x, 2x^2) = -x^2 \cdot x^2 = -x^4$$

and so $x = 0$ is a maximum here, so $(0,0)$ cannot be a minimum.

Exercise 8.3:

Find the critical points of the following function and categorize them:

$$f(x,y) = x^3y^2(1 - x - y)$$

We first find f's gradient:

$$\nabla f = \begin{pmatrix} x^2y^2(-4x - 3y + 3) \\ x^3y(-2x - 3y + 2) \end{pmatrix}$$

And so $\nabla f = 0$ if and only if $x = 0$ or $y = 0$ or at the point $\left(\frac{1}{2}, \frac{1}{3}\right)$. We will first deal with the case that $x = 0$ and $y \neq 0$.
Here, notice that if we consider y to be constant then $f(x,y)$ has an extrema (relative to x) whenever $x^3(1-x-y)$ does, and it is of the same type since $y^2 > 0$. So we now ask a more general question: when does a function $g(x) = -x^4 + \alpha x^3$ have an extrema at $x = 0$? Notice that $g'(x) = -4x^3 + 3\alpha x^2$ and so $g(x)$ has critical points at $x = 0$ and $x = \frac{3\alpha}{4}$. Computing $g'\left(\frac{\alpha}{2}\right)$ gives $\frac{\alpha^3}{4}$. So if $\alpha < 0$ then $g'(x) < 0$ for $x > 0$ and it is also negative at $\frac{\alpha}{2}$. So around $x = 0$, the derivative of g is negative and therefore $x = 0$ is an inflection point. If $\alpha > 0$ then $g'(x) > 0$ for $x < 0$ and it is also positive at $\frac{\alpha}{2}$ and therefore $x = 0$ is an inflection point. If $\alpha = 0$ then $g(x) = -x^4$ and therefore $x = 0$ is a maximum. In our case $\alpha = 1 - y$ so unless $y = 1$, $(0, y)$ is an inflection point.
We now deal with the case that $x \neq 0$ and $y = 0$. We split it up into cases:

- If $x > 1$ then $(x,0)$ is above the line $y = 1 - x$ and therefore there exists a ball around $(x,0)$ such that for every $(a,b) \in B$, $b > 1 - a$ and $a > 0$ so $1 - a - b < 0$ and therefore $f(a,b) < 0 = f(x,0)$. So $(x,0)$ is a local maximum for $x > 1$.

- If $x < 0$ then $(x,0)$ is below the line $y = 1 - x$ so there exists a ball around $(x,0)$ which is also underneath $y = 1 - x$ and its x values are negative. So for every $(a,b) \in B$, $f(a,b) < 0 = f(x,0)$ and so $(x,0)$ is a local maximum for $0 < x < 1$.

- If $0 < x < 1$ then $(x,0)$ is below the line $y = 1 - x$, so there exists a ball around $(x,0)$ which is underneath the line and has positive x values, and so for every $(a,b) \in B$, $f(a,b) > 0$, so $(x,0)$ is a local minimum for $0 < x < 1$.

- If $x = 1$, then it lies on the line $y = 1 - x$, so every ball around $(x,0)$ has values above and below this line, and for radii small enough, the x values are positive, so there are elements in the ball whose image is positive and some whose image is negative, so $(1,0)$ is an inflection point.

- For reasons nearly identical to the ones given above, $(0,1)$ and $(0,0)$ are also an inflection points.

Lastly, for the point $\left(\frac{1}{2}, \frac{1}{3}\right)$ we find the Hessian of the function:

$$H_f = \begin{pmatrix} y^2\big(2x(-4x - 3y + 3) - 4x^2\big) & x^2\big(2y(-4x - 3y + 3) - 3y^2\big) \\ x^2\big(2y(-4x - 3y + 3) - 3y^2\big) & x^3\big(-2x - 3y + 2 - 3y\big) \end{pmatrix}$$

Specifically at this point

$$H_f\left(\frac{1}{2},\frac{1}{3}\right) = -\begin{pmatrix} \frac{1}{9} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{8} \end{pmatrix}$$

which is a negative-definite matrix (the negative of a positive-definite matrix), and therefore $\left(\frac{1}{2},\frac{1}{3}\right)$ is a maximum. I summarize the findings below:

- Maxima: $(x,0)$ for $x>1$ or $x<0$, and $\left(\frac{1}{2},\frac{1}{3}\right)$.
- Minima: $(x,0)$ for $0<x<1$.
- Inflection points: $(0,y)$ for $y\in\mathbb{R}$ and $(1,0)$.

Exercise 8.4:

Show that the following equations define z as a function of x and y in a neighborhood of the given point, and further find z_x and z_y in this neighborhood.

(1) $F(x,y,z) = y^2+xy+z^2-e^z-4 = 0$ around $(0,e,2)$, further compute z_{yy}.

(2) $F(x,y,z) = xz + y\ln z + x^2 = 0$ around $(-2,0,2)$, further compute z_{xy}.

Both functions here are functions $F\colon \mathbb{R}^{2\times 1}\longrightarrow \mathbb{R}^1$, so by using the implicit function theorem at the given point, if the restricted Jacobian is invertible, then z is indeed a function of x and y. Note that the reduced Jacobian is simply $J_{f,z} = (f_z)$, so all we need to show is that at this point, $f_z\neq 0$. Further, it satisfies

$$\begin{pmatrix} z_x & z_y \end{pmatrix} = J_z = -J_{f,z}^{-1}\cdot J_{f,\binom{x}{y}} = -f_z^{-1}\cdot J_{f,\binom{x}{y}}$$

(1) Here

$$f_z = 2z - e^z$$

so at this point $f_z = 4-e^2\neq 0$, as required. And so

$$\begin{pmatrix} z_x & z_y \end{pmatrix} = -\frac{1}{2z-e^z}\cdot\begin{pmatrix} y & 2y+x \end{pmatrix} = \begin{pmatrix} \frac{y}{e^z-2z} & \frac{2y+x}{e^z-2z} \end{pmatrix}$$

and by further differentiating z_y by y, we find that:

$$z_{yy} = \frac{2(e^z-2z)-(2y+x)\cdot z_y(e^z-2)}{(e^z-2z)^2}$$

And so

$$\begin{aligned} z_x(0,e) &= \frac{e}{e^2-4} \\ z_y(0,e) &= \frac{2e}{e^2-4} \\ z_{yy}(0,e) &= \frac{2(e^2-4)-2e\cdot\frac{2e}{e^2-4}(e^2-2)}{(e^2-4)^2} \end{aligned}$$

(2) Here

$$f_z = x + \frac{y}{z}$$

and at this point $f_z = -2\neq 0$, as required. And so

$$\begin{pmatrix} z_x & z_y \end{pmatrix} = -\frac{1}{x+\frac{y}{z}}\cdot\begin{pmatrix} z+2x & \ln(z) \end{pmatrix} = \begin{pmatrix} -\frac{z^2+2xz}{xz+y} & \frac{-z\ln(z)}{xz+y} \end{pmatrix}$$

and by further differentiating z_x by y:

$$z_{xy} = -z_y\frac{2(z+x)(xz+y)^2 - xz(z+2x)}{(zx+y)^2}$$

And so

$$z_x(-2,0) = -1$$
$$z_y(-2,0) = \frac{\ln(2)}{2}$$
$$z_{xy}(-2,0) = \frac{\ln(2)}{4}$$

Exercise 8.5:

Does the following equation define z as a function of x and y around $(-1,0,0)$? Does it define a y as a function of x and z? x as a function of y and z?

- It does not. Since z^4 and $\cos(z)$ are even, if z is a solution to this equation then so is $-z$, and so there can be no function which maps from values of x and y to values of z unless it is the constant zero function. But $z = 0$ is a solution if and only if $x^2 + y^5 = 1$, but this does not define an open set.
- It does, and we can find the function explicitly:
$$y = \sqrt[5]{\left(z^4+1\right)^2 - \cos(z) - x^2 + 1}$$
Since this is defined on all of $\mathbb{R}^2$, it is defined over every neighborhood of $(-1,0,0)$.
- Using the implicit function theorem, since
$$\frac{\partial f}{\partial x}(-1,0,0) = \frac{x}{\sqrt{x^2+y^5+\cos(z)-1}}(-1,0,0) = -1$$
and therefore the differential of x is invertible, so by the implicit function theorem, x can be written as a function of y and z in some environment of the point, as required.

Exercise 8.6:

Prove that there exists a ball $B \subseteq \mathbb{R}^4$ whose center is at $(2,1,-1,-2)$ and functions $f,g\colon B \longrightarrow \mathbb{R}$ continuously differentiable such that

$$f(2,1,-1,-2) = 4 \quad g(2,1,-1,2) = 3$$

and for every $(x,y,z,a) \in B$:

$$f^2 + g^2 + a^2 = 29 \quad \frac{f^2}{x^2} + \frac{g^2}{y^2} + \frac{a^2}{z^2} = 17$$

We can use the implicit function theorem to prove this. We first define the function $h\colon \mathbb{R}^{4\times 2} \longrightarrow \mathbb{R}^2$ by

$$h(x,y,z,a,f,g) = \left(f^2+g^2+a^2-29,\ \frac{f^2}{x^2} + \frac{g^2}{y^2} + \frac{a^2}{z^2} - 17\right)$$

Notice then that

$$J_{h,\binom{f}{g}}(x,y,z,a,f,g) = \begin{pmatrix} 2f & 2g \\ \frac{2f}{x^2} & \frac{2g}{y^2} \end{pmatrix}$$

so at $(2,1,-1,2,4,3)$:

$$J_{h,\binom{f}{g}} = \begin{pmatrix} 8 & 6 \\ \frac{1}{2} & 6 \end{pmatrix}$$

which is invertible as it has a non-zero determinant. Thus by the implicit function theorem, in a neighborhood of $(2,1,-1,2)$ (which contains a ball centered at this point B, so we'll just take the ball B) such that f and g are indeed continuously differentiable functions of x, y, z, a where $h\big(x, y, z, a, f(x, y, z, a), g(x, y, z, a)\big) = 0$ (which exactly defines the equations given in the question) and $f(2, 1, -1, 2) = 4$ and $g(2, 1, -1, 2) = 3$ (which are the initial conditions given in the question), as required.

Exercise 8.7:

Show that the function $f(x, y) = \big(e^x \cos(y),\ e^x \sin(y)\big)$ is not invertible but is locally invertible in a neighborhood of every point in $\mathbb{R}^n$.

Firstly we know that f is continuously differentiable. And

$$J_f = \begin{pmatrix} e^x \cos(y) & -e^x \sin(y) \\ e^x \sin(y) & e^x \cos(y) \end{pmatrix} = e^x \begin{pmatrix} \cos(y) & -\sin(y) \\ \sin(y) & \cos(y) \end{pmatrix}$$

which has a determinant of

$$e^{2x}\big(\cos^2(y) + \sin^2(y)\big) + e^{2x} \neq 0$$

so J_f is always invertible, and since f is continuously differentiable, for every point there is a neighborhood of it in which f is locally invertible.
f itself is not globally invertible since $f(0, 0) = f(0, 2\pi) = (1, 0)$ so it is not injective.