Modern Analysis

Homework 4
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4.1 Exercise

Give an example of a function f which isn't Lebesgue measurable, such that $|f|$ is.

Let S be non-measurable, then define $f = \chi_S - \chi_{S^c}$. Since $f^{-1}\{1\} = S$ which isn't measurable, f is not measurable. But $|f| = 1$ is measurable.

4.2 Exercise

Let $\{A_i\}_{i=1}^\infty$ be a sequence of disjoint measurable sets in (X, Σ).

(1) let $\{g_i\}_1^\infty$ be a sequence of S-measurable functions $g_i\colon X \longrightarrow \mathbb{R}$. Show that $g(x) = \sum_{i=1}^\infty \chi_{A_i} g_i$ converges and is measurable.

(2) suppose $\dot\bigcup_{i=1}^\infty A_i = X$, let $\mathcal{F} = \sigma\{A_i\}_1^\infty$. Show that h is $\mathcal{F}$-measurable if and only if h is constant on each A_i.

(1) Let $x \in X$ then either $x \notin A_i$ for all i in which case $g(x) = 0$, or $x \in A_i$ for a single i, in which case $g(x) = g_i(x)$. So g converges. Let $\alpha \geq 0$ and $x \in X$, then $g(x) > \alpha$ if $x \in A_i$ for some i and $g_i(x) > \alpha$. So $g^{-1}(\alpha, \infty) = \bigcup_i g_i^{-1}(\alpha, \infty) \cap A_i$, which is measurable since g_i is. If $\alpha < 0$ then $g(x) > \alpha$ if $g(x) = 0$, meaning $x \in \bigcap_i A_i^c$ or $x \in \bigcup_i g_i^{-1}(\alpha, \infty) \cap A_i$. Both sets are measurable.

(2) If h is constant on $\mathcal{A}_i$ then $h = \sum_i a_i \chi_{A_i}$ which is measurable by above (constant functions are measurable). Conversely, let $\sigma = \left\{\bigcup_{n\in M} A_n \;\middle|\; M \subseteq \mathbb{N}\right\}$, this is a σ-algebra: $X, \varnothing \in \sigma$ since the union of all A_i is X, it is obviously closed under arbitrary unions, and it is closed under complements:

$$\left(\bigcup_{n\in M} A_n\right)^c = \bigcup_{n\in\mathbb{N}} A_n \setminus \bigcup_{n\in M} A_n = \bigcup_{n\in M^c} A_n$$

Since for all i, $A_i \in \sigma$ and so $\mathcal{F} \subseteq \sigma$, and obviously the converse holds as well so $\mathcal{F} = \sigma$.

So let $x \in A_i$, then $h^{-1}\{h(x)\}$ is measurable, so it is a union of A_js. But it includes x, so A_i must be in this union meaning $A_i \subseteq h^{-1}\{h(x)\}$, so $h(A_i) = h(x)$, meaning h is constant over A_i.

4.3 Exercise

Let (X, Σ) be a measurable space, $f_1, f_2, f_3\colon X \longrightarrow \mathbb{R}$ measurable. For $x \in X$ define the polynomial $p_x(t) = f_1(x)t^2 + f_2(x)t + f_3(x)$. Show that the set of $x \in X$ for which p_x has two roots is measurable.

Recall that the polynomial has two roots if and only if $f_1(x) \neq 0$ and $f_2(x)^2 - 4\cdot f_1(x)f_3(x) \geq 0$. Define $g(x) = f_2(x)^2 - 4 \cdot f_1(x)f_3(x)$ which is measurable since arithmetic operations on measurable functions produce measurable functions. Then the set is $f_1^{-1}(\mathbb{R}\setminus\{0\}) \cap g^{-1}[0, \infty)$ which is measurable (since $f_1^{-1}(\mathbb{R}\setminus\{0\}) = f_1^{-1}\{0\}^c$ is measurable).

4.4 Exercise

Let (X, Σ) be a measurable space, let $f, g\colon X \longrightarrow \mathbb{R}$ be measurable. Show that $h(x) = \frac{f(x)}{g(x)}\chi_{\{x \mid g(x)\neq 0\}}$ is measurable.

It is sufficient to prove this for $f = 1$ since then we can multiply h by f. For $\alpha \geq 0$, $h(x) > \alpha$ if and only if $\frac{1}{g(x)} > \alpha$, if and only if $0 < g(x) < \frac{1}{\alpha}$, so $h^{-1}(\alpha, \infty) = g^{-1}(0, \alpha^{-1})$ $(0^{-1} = \infty)$ is measurable. And for $\alpha < 0$, $h(x) > \alpha$ if and only if $g(x) = 0$ or $\frac{1}{g(x)} > \alpha$, which is if and only if $g(x) < \frac{1}{\alpha}$. So $h^{-1}(\alpha, \infty) = g^{-1}\{0\} \cup g^{-1}(-\infty, \alpha^{-1})$ is measurable.

4.5 Exercise

Define $\mathbb{A} = \{E \in \mathfrak{B}(\mathbb{R}) \mid E = -E\}$.

(1) show that $\mathbb{A}$ is a σ-algebra.

(2) show that f is $\mathbb{A}$-measurable if and only if it is Borel measurable and even.

(1) $\mathbb{R}, \varnothing \in \mathbb{A}$ obviously. If $E \in \mathbb{A}$ then $x \in -E^c \iff -x \notin E \iff x \notin E \iff x \in E^c$ so $E^c = -E^c$ as well, meaning $E^c \in \mathbb{A}$. And if $\{E_i\} \subseteq \mathbb{A}$ then $-\bigcup E_i = \bigcup -E_i = \bigcup E_i$ so $\bigcup E_i \in \mathbb{A}$.

(2) If f is $\mathbb{A}$ measurable then it is also Borel measurable by definition. And $x \in X$ then $x \in f^{-1}\{f(x)\}$ is measurable so $-x \in f^{-1}\{f(x)\}$ meaning $f(x) = f(-x)$, so an $\mathbb{A}$-measurable function is Borel-measurable and even. If f is Borel measurable and even, then let $\alpha \in \mathbb{R}$ then if $x \in f^{-1}(\alpha, \infty]$ then $f(x) \in (\alpha, \infty]$ so $f(-x) \in (\alpha, \infty]$. So $x \in f^{-1}(\alpha, \infty] \implies -x \in f^{-1}(\alpha, \infty]$, meaning $f^{-1}(\alpha, \infty] \in \mathbb{A}$, so f is $\mathbb{A}$-measurable.