# Infintesimal Calculus 3

Lecture 3, Sunday October 30, 2022 Ari Feiglin

# Proposition 3.1.1:

Any arbitrary union of open sets is itself open, and a finite intersection of open sets is open as well. Similarly, any intersection of closed sets is closed, and a finite union of closed sets is closed as well.

#### **Proof:**

This is trivial. Suppose  $\{\mathcal{O}_i\}_{i\in I}$  is a set of open sets, then if:

$$x \in \bigcup_{i \in I} \mathcal{O}_i$$

there must be some  $i \in I$  such that  $x \in \mathcal{O}_i$ . Since  $\mathcal{O}_i$  is open, there is a r > 0 such that  $B_r(x) \subseteq \mathcal{O}_i$ , and therefore  $B_r(x) \subseteq \bigcup \mathcal{O}_i$ , so the union is open.

Now suppose  $\{\mathcal{O}_n\}_{n=1}^N$  is a finite set of open sets. Then if

$$x \in \bigcap_{n=1}^{N} \mathcal{O}_n$$

x must be in  $\mathcal{O}_n$  for every  $n=1\ldots N$ . Then for every  $n=1\ldots N$ , there must be a  $r_n>0$  such that  $B_{r_n}(x)\subseteq \mathcal{O}_n$ . If we let  $r=\min\{r_1,\ldots,r_N\}>0$ ,  $B_r(x)\subseteq B_{r_n}(x)\subseteq \mathcal{O}_n$  for every n, so:

$$B_r(x) \subseteq \bigcap_{n=1}^N \mathcal{O}_n$$

as required.

Now suppose  $\{F_i\}_{i\in I}$  is a set of closed sets, then:

$$\left(\bigcap_{i\in I} F_i\right)^c = \bigcup_{i\in I} F_i^c$$

which is a union of open sets since  $F_i^c$  is open, which we just proved is open. So the complement of the intersection is open, and therefore the intersection is closed.

Similarly for a finite set of closed sets:

$$\left(\bigcup_{n=1}^{N} F_n\right)^c = \bigcap_{n=1}^{N} F_n^c$$

which is open, and therefore the union is closed.

Notice that an arbitrary intersection of open sets, and thus an arbitrary intersection of closed sets, is not necessarily open (or closed). Take for instance  $\mathcal{O}_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ , then:

$$\bigcap_{n\in\mathbb{N}}\mathcal{O}_n=\{0\}$$

And  $\{0\}$  is not open.

## Definition 3.1.2:

For a subset S of a metric space X, we denote its interior, the set of all interior points of S as  $S^{\circ}$  or int S. Similarly,

the exterior, the set of all exterior points is ext S (or  $S^{c\circ}$ ), and the boundary of S is denoted  $\partial S$ .

Notice that int  $S \subseteq S$ . Furthermore,  $S \cup \partial S = \text{int } S \cup \partial S$ . These sets are obviously disjoint by definition, and if  $x \in S$ , x is either in the interior of S or the boundary of S.

#### Lemma 3.1.3:

Every limit point of S is in int  $S \cup \partial S = S \cup \partial S$ .

#### **Proof:**

Suppose x is a limit point of S and  $x \notin \text{int } S$ . Since x is a limit point, for every r > 0 there is a  $x \neq y \in S$  such that  $y \in B_r(x)$ , so  $B_r(x)$  is not a subset of  $S^c$ . And since x is not an interior point of S, for every r > 0,  $B_r(x)$  is not a subset of S. So for every r > 0,  $B_r(x) \cap S$  and  $B_r(x) \cap S^c$  are not empty and therefore x is a boundary point of S.

#### Theorem 3.1.4:

Suppose  $S \subseteq X$  for X metric space. Then the following are equivalent:

- S is closed.
- S contains all of its boundary points.
- S contains all of its limit points.

#### **Proof:**

We will prove the first relation. Suppose  $x \in \partial S$ , then if  $x \notin S$ ,  $x \in S^c$ , and since  $\partial (S^c) = \partial S$ ,  $x \in \partial (S^c) \cap S^c$ . But  $S^c$  is open and thus doesn't contain its boundary points, in contradiction, so  $x \in S$ . Therefore  $\partial S \subseteq S$ .

Now suppose  $\partial S \subseteq S$ . Then since every limit point is in  $\partial S \cup S = S$ , every limit point is in S.

Now suppose S contains all of its limit points. Suppose  $x \in S^c$ , then x is not a limit point of S, so there must be a r > 0 such that  $B_r(x)$  contains no points in S other than x. Since x is not in  $S^c$ ,  $B_r(x) \subseteq S^c$ . So  $S^c$  is open and therefore S is closed.

# Definition 3.1.5:

For  $S \subseteq X$  a metric space, S' is the set of all limit points of S, and is also called the limit set of S.

# Lemma 3.1.6:

Let  $S \subseteq X$  a metric space and  $x \in S^c$ . Then x is a boundary point of S if and only if x is a limit point of S.

This proof is trivial, if x is a boundary point of S, for every r > 0 there must be a  $y \in B_r(x) \cap S$ , and since  $x \notin S$ ,  $y \neq x$ , so x is a limit point. And if x is a limit point, then since x is in  $B_r(x) \cap S^c$ , it is not open, and since there must be a  $y \in B_r(x) \cap S$  since it's a limit point, x is a boundary point of S. Notice then that

$$S' \subseteq S \cup \partial S = \operatorname{int} S \cup \partial S$$

Since if  $x \in S'$  and  $x \notin S$ , then we just showed  $x \in \partial S$ .

## Definition 3.1.7:

If S is a subset of a metric space X, then the closure of S, denoted  $\overline{S}$ , is the smallest possible closed set containing S:

$$\overline{S} = \bigcap_{\substack{F \text{ closed} \\ S \subset F}} F$$

# Proposition 3.1.8:

If S is a subset of a metric space X, then  $\overline{S} = \operatorname{int} S \cup \partial S = \operatorname{int} S \cup \partial (S^c)$ .

#### **Proof:**

Since  $X = \text{int } S \cup \partial S \cup \text{int } (S^c)$ , int  $(S^c)$  is open so its complement, int  $S \cup \partial S$  is closed. This contains S so  $\overline{S} \subseteq \text{int } S \cup \partial S$ . And if F is a closed set which contains S, it must contain S's interior (since it is a subset of S) and its boundary, so the sets are equal.

Theorem 3.1.9:

If  $S \subseteq X$  a metric space, the following are equal:

- $\bullet$   $\overline{S}$
- $S \cup S'$
- $S \cup \partial S$

#### **Proof:**

The first equality was proven above. We know that  $S' \subseteq S \cup \partial S$ , so  $S \cup S' \subseteq S \cup \partial S$ . Suppose  $x \in \partial S$  and  $x \notin S$ . Then we know that x is a limit point, so  $S \cup \partial S \subseteq S \cup S'$ . Therefore  $S \cup S' = S \cup \partial S$ .

Definition 3.1.10:

A set  $S \subseteq X$  is compact if for every set of open sets  $\{\mathcal{O}_i\}_{i \in I}$  such that

$$S\subseteq\bigcup_{i\in I}\mathcal{O}_i$$

there is a finite subcovering, that is a finite indexing set  $I' \subseteq I$  such that:

$$S\subseteq\bigcup_{i\in I'}\mathcal{O}_i$$

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