Calculus Homework #10

Ari Feiglin

Question 10.1:

Do the following sums converge, converge uniformly, or diverge in the given domains?

(1) $\displaystyle\sum_{n=2}^{\infty}\log\left(1+\frac{x^2}{n\log^2 n}\right)$ in $(-a,a)$.

(2) $\displaystyle\sum_{n=1}^{\infty}\frac{x^2}{e^{nx}}$ in $[0,\infty)$.

(3) $\displaystyle\sum_{n=1}^{\infty}\frac{x}{(1+x^2)^n}$ in $[0,\infty)$.

(1) Notice that each element (as every element in positive) in the sum is less than

$$\log\left(1+\frac{a^2}{n\log^2 n}\right)$$

since log is a monotonically increasing function. Notice that if we multiply and divide by $n\log^2 n$ we get:

$$\frac{1}{n\log^2 n}\cdot\left(n\log^2 n\cdot\log\left(1+\frac{a^2}{n\log^2 n}\right)\right)=\frac{1}{n\log^2 n}\cdot\log\left(\left(1+\frac{a^2}{n\log^2 n}\right)^{n\log^2 n}\right)$$

Notice that the logarithm has a limit of:

$$\log\left(a^{x^2}\right)=a^2$$

So all in all this acts like the function $\dfrac{a^2}{n\log^2 n}$. This is true since dividing this by that fraction gives a limit of 1, as explained above. And we know that this sum converges by the condensation test (it becomes the sum of $\frac{1}{n^2\cdot\log^2 2}$). So by the Weirstrass M-test, the sum *uniformly converges*.

(2) Notice that $\frac{1}{e^{nx}}$ is a geometric series whose sum converges if and only if $\frac{1}{e^x}<1 \iff x>0$. And notice that if $x=0$, then the sum is just a sum of 0, which also converges. So the sum converges everywhere in the given domain. Let us now try and find the maximum of $f_n(x)$ so we can create a series $f_n\leq M_n$. Differentiating f_n yields:

$$f_n'(x)=\frac{2ne^{nx}-ne^{nx}x^2}{e^{2nx}}=\frac{xe^{nx}\cdot(2-nx)}{e^{2nx}}$$

This means that f_n has a maximum when $2-nx=0$ so $x=\frac{2}{n}$ (this is true since before this, f_n' is negative, and afterward it is positive). So f_n has a maximum (its only maximum as it is increasing beforehand and decreasing afterward) of:

$$M_n\coloneqq f_n\left(\frac{2}{n}\right)=\frac{\frac{4}{n^2}}{e^2}=\frac{4}{e^2}\cdot\frac{1}{n^2}$$

And so $\sum M_n$ converges, so by the Weirstrass M-test, the sum *uniformly converges*.

(3) For $x>0$ this is also a geometric series whose sum converges since the quotient is less than 1. The sum is equal to:

$$x\cdot\frac{\frac{1}{1+x^2}}{1-\frac{1}{1+x^2}}=\frac{x}{x^2}=\frac{1}{x}$$

And if $x = 0$ then it is the sum of 0, so the sum is just 0. Let $f(x)$ be this sum, so:

$$f(x) = \begin{cases} \frac{1}{x} & x > 0 \\ 0 & x = 0 \end{cases}$$

Notice that while $f(x)$ is discontinuous at $x = 0$, $f_n(x)$ is continuous. Therefore the convergence is not uniform. So the convergence is *pointwise*.

Question 10.2:

Find the sum:

$$\sum_{n=0}^{\infty} n^2 x^n$$

In $(-1, 1)$.

Recall that:

$$\frac{1}{(1-x)^{k+1}} = \sum_{n=0}^{\infty} \binom{n+k}{k} x^n$$

So specifically

$$\frac{1}{(1-x)^3} = \frac{1}{2}\sum_{n=0}^{\infty}(n+2)(n+1)x^n = \frac{1}{2}\left(\sum n^2x^n + 3\sum(n+1)x^n - \sum x^n\right)$$

And we know that:

$$\sum(n+1)x^n = \frac{1}{(1-x)^2} \qquad \sum x^n = \frac{1}{1-x}$$

So:

$$\frac{2}{(1-x)^3} = \sum n^2x^n + \frac{3}{(1-x)^2} - \frac{1}{1-x}$$

So we get that:

$$\sum n^2x^n = \frac{2-3(1-x)+(1-x)^2}{(1-x)^3} = \frac{x(x+1)}{(1-x)^3}$$

Question 10.3:

Compute the following sum:

$$\sum_{n=0}^{\infty} \frac{n}{(n+1) \cdot 2^n}$$

Recall that for $x \in (-1, 1)$:

$$\log(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -x \sum \frac{x^n}{n+1}$$

So

$$-\frac{\log(1-x)}{x} = \sum \frac{x^n}{n+1}$$

Notice that this is almost the sum we want. We will differentiate both sides (which we can do elementwise to the sum since it is a powerseries). This will not affect the radius of convergence of the sum so this will strill be true for all $x \in (-1, 1)$. We get:

$$-\frac{-\frac{x}{1-x} - \log(1-x)}{x^2} = \sum \frac{n}{n+1} x^{n-1}$$

We now multiply both sides by x and we get:

$$\sum \frac{n}{n+1} x^n = \frac{\frac{x}{1-x} + \log(1-x)}{x}$$

This is the sum we want. Let $x = \frac{1}{2} \in (-1, 1)$ and we get:

$$\sum_{n=0}^{\infty} \frac{n}{(n+1) \cdot 2^n} = \frac{1 + \log\left(\frac{1}{2}\right)}{\frac{1}{2}} = 2 - 2\log(2)$$

Question 10.4:

Find the radius of convergence for each of the following powerseries:

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^p}$ where $p \in \mathbb{R}$.

(2) $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} x^n$

(1) We will use the root test for radii of convergence. So we need to compute:

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\left|\frac{1}{n^p}\right|} = \overline{\lim} \frac{1}{\sqrt[n]{n}^p}$$

We know that the limit of $\sqrt[n]{n}$ is 1, so this is equal to 1. This means that the radius of convergence is $\frac{1}{1} = 1$.

(2) We will use the quotient test for radii.

$$\left|\frac{a_n}{a_{n+1}}\right| = \frac{(n!)^2 \cdot (2n+2)!}{((n+1)!)^2 \cdot (2n)!} = \frac{(2n+2)(2n+1)}{(n+1)^2}$$

The limit of this is 4 since the numerator is a degree 2 polynomial with a leading coefficient 4 and the denominator is also a degree 2 polynomial but with a leading coefficient of 1. So the radius is 4.

Question 10.5:

Find the domain of convergence for each of the following powerseries:

(1) $\sum_{n=0}^{\infty} n^3 x^n$

(2) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{(2n)!}$

(3) $\sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{3}\right) \cdot x^n$

(4) $\sum_{n=1}^{\infty} n! \cdot x^n$

(5) $\sum_{n=1}^{\infty} \frac{\log(x)^n}{n^{\log(n)}} x^n$

(1) We see that:

$$\overline{\lim} \sqrt[n]{|a_n|} = \overline{\lim} \sqrt[n]{n^3} = \left(\overline{\lim} \sqrt[n]{n}\right)^3 = 1^3 = 1$$

So the radius of convergence is $\frac{1}{1} = 1$. If $x = \pm 1$, $n^3 x^n$ does not converge to 0 so the sum diverges. So the domain of convergence is $(-1, 1)$.

(2) We see that:

$$\left|\frac{a_n}{a_{n+1}}\right| = \frac{(2n+2)!}{(2n)!} = (2n+2)(2n+1)$$

the limit of this is infinity, so the radius of convergence is infinity and thus the domain convergence is all of $\mathbb{R}$.

(3) Notice that $\left|\cos\left(\frac{n\pi}{3}\right)\right| \leq 1$ so the limit superior of the nth root of this is less than or equal to 1 as well. We can take $m_n = 6n$ and then $a_{m_n} = \cos(2n\pi) = 1$, so the limit of $\sqrt[n]{a_{m_n}} = \sqrt[n]{1}$ is 1. Therefore the limit superior of $\sqrt[n]{|a_n|}$ is 1, and therefore the radius of convergence is 1. If $x = \pm 1$, $a_n x^n$ doesn't converge to 0 so the sum does not converge. So the domain of convergence is $(-1, 1)$.

(4) We will use the quotient test:

$$\lim \left|\frac{a_n}{a_{n+1}}\right| = \lim \frac{n!}{(n+1)!} = \lim \frac{1}{n+1} = 0$$

So the domain of convergence is $\{0\}$.

(5) Using the root test we have that:

$$\overline{\lim} \sqrt[n]{|a_n|} = \overline{\lim} \frac{\log(n)}{n^{\frac{\log(n)}{n}}}$$

Now notice that

$$n^{\frac{\log(n)}{n}} = e^{\frac{\log(n)^2}{n}}$$

And the limit of $\frac{\log(n)^2}{n}$ is 0, so $n^{\frac{\log(n)}{n}}$'s limit is 1. Since $\log(n)$ diverges to infinity and the numerator converges to 1:

$$\overline{\lim} \sqrt[n]{|a_n|} = \infty$$

So the radius of convergence is ∞ and therefore the domain of convergence is $\{0\}$.

Question 10.6:

Does the powerseries $\sum\limits_{n=1}^{\infty} \frac{x^n}{n!}$ converge uniformly in:

(1) $(-100, 100)$?

(2) All of $\mathbb{R}$?

(1) Let's compute the radius of convergence:

$$\lim \left| \frac{a_n}{a_{n+1}} \right| = \lim \frac{(n+1)!}{n!} = \lim n+1 = \infty$$

So the domain of convergence is $\mathbb{R}$. Since a powerseries uniformly converges in every closed interval of its domain, this one converges uniformly in $[-100, 100]$ and therefore also converges uniformly in $(-100, 100)$.

(2) This is not true. Firstly note that this is the powerseries of $e^x - 1$, Now notice that for every n:

$$\lim_{n \to \infty} \frac{1 + \sum_{k=1}^{n} \frac{x^k}{k!}}{e^x - 1} = 0$$

Since exponential growth is faster than polynomial growth. So at some point e^x is greater than the power series, and so at some point:

$$e^x - 1 > 1 + \sum_{k=1}^{n} \frac{x^k}{k!}$$

So:

$$\left| e^x - 1 - \sum_{k=1}^{n} \frac{x^k}{k!} \right| > 1$$

For great enough values of x. Therefore $\varepsilon_n > 1$ (ε_n is the supremum of the difference between the powerseries and $e^x - 1$ in $\mathbb{R}$), and thus doesn't converge to 0. So the convergence is not uniform.

Question 10.7:

Suppose a_n is a monotonically decreasing series to 0 such that $\sum a_n$ diverges. Find the domain of convergence of $\sum a_n x^n$.

Notice that if $x = 1$, then:

$$\sum_{n=0}^{\infty} a_n x^n = \sum a_n = \infty$$

And if $x = -1$, the sum is $\sum (-1)^n a_n$ which converges by Leibinz's alternating series test. Let the radius of convergence be r. r cannot be greater than 1 since the series diverges for 1 and for every $|x| < r$ (and 1 would be less than r if this were the case) the powerseries converges. And r cannot be less than 1 because the series converges for -1, and for every $|x| > r$ (which -1 would be in this case) the powerseries diverges. So the radius must be $r = 1$. And as already shown above, it diverges for $x = 1$ and converges for $x = -1$ so the domain of convergence is $[-1, 1)$.

Question 10.8:

Compute $\displaystyle\int_0^1 e^{-t^2}\,dt$ with an error of at most 0.01.

Recall that:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

So:

$$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}$$

Using elementwise integration (since powerseries converge uniformly):

$$\int_0^1 e^{-t^2}\,dt = \sum_{n=0}^{\infty} \int_0^1 \frac{(-1)^n t^{2n}}{n!}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{n! \cdot (2n+1)}\Bigg|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \cdot (2n+1)}$$

Note that if we sum n terms here, by Leibinz's rule for alternate sums, the error will be

$$\leq \frac{1}{n! \cdot (2n+1)}$$

So summing 4 terms gives an error less than one one hundredth. So the integral is approximately:

$$\sum_{n=0}^{3} \frac{(-1)^n}{n! \cdot (2n+1)} = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} = 0.743$$