Complex Functions

Lecture 2, Monday July 17, 2023
Ari Feiglin

Recall the following definition

Definition 2.1:

If (M,ρ) and (X,σ) are two metric spaces, a function

$$f\colon M \longrightarrow X$$

is an **isometry** if $\rho(x,y) = \sigma(f(x),f(y))$ for every $x,y \in M$. M and X are called **isometric**.

It is obvious that isometries are injective (if $f(x) = f(y)$ then $\rho(x,y) = 0$ so $x = y$).
If X is a normed vector space, and A is an orthogonal transformation then recall $\|Ax\| = \|x\|$, so

$$\|Ax - Ay\| = \|A(x-y)\| = \|x-y\|$$

so A is an isometry.

Definition 2.2:

If X is a normed vector space, and a is a unit vector then define

$$S_a(x) = x - 2\langle x, a\rangle \cdot a$$

This is the reflection about $\{a\}^\perp$.

Recall that $x - \langle x,a\rangle a \in \{a\}^\perp$, since

$$\langle x - \langle x,a\rangle a, a\rangle = \langle x,a\rangle - \langle\langle x,a\rangle a, a\rangle = \langle x,a\rangle - \langle x,a\rangle\langle a,a\rangle = \langle x,a\rangle - \langle x,a\rangle = 0$$

Now notice that

- If $x \in a^\perp$ then $S_a(x) = x$.
- $S_a(a) = -a$.
- $S_a^2(x) = S_a(x - 2\langle x,a\rangle a) = x - 2\langle x,a\rangle a - 2\langle x - 2\langle x,a\rangle a, a\rangle = x - 2\langle x,a\rangle a + 2\langle x,a\rangle = x$. So $S_a^2(x) = x$.
- $S_a(x+y) = S_a(x) + S_a(y)$ and $S_a(\lambda x) = \lambda S_a(x)$, so S_a is a linear transformation.

Also notice that $\langle x,a\rangle a = a\langle x,a\rangle = aa^T x$, thus

$$S_a(x) = (I - 2aa^T)x$$

this is another proof that S_a is a linear transformation, as $S_a(x) = Ax$ where $A = I - 2aa^T$. Now notice that $A^T = A$, we have that A is orthogonal, so S_a is an isometry.

Proposition 2.3:

If $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is an isometry which preserves the origin, ie. $f(0) = 0$, then f is an orthogonal linear transformation.

Proof:

Notice that f preserves norms, since $\|x\| = \|x - 0\| = \|f(x) - f(0)\| = \|f(x)\|$. And so f preserves the inner product since

$$\|x-y\|^2 = \langle x-y, x-y\rangle = \|x\|^2 - 2\langle x,y\rangle + \|y\|^2$$

And thus

$$2\langle x,y\rangle = \|x\|^2 - \|x-y\|^2 + \|y\|^2$$

So

$$2\langle x, y\rangle = \|f(x)\|^2 - \|f(x)-f(y)\|^2 + \|f(y)\|^2$$

But the equality is true for any x, y and so

$$2\langle f(x), f(y)\rangle = \|f(x)\|^2 - \|f(x)-f(y)\|^2 + \|f(y)\|^2$$

Thus $\langle x, y\rangle = \langle f(x), f(y)\rangle$ as required.
Let us define

$$A = \begin{pmatrix} | & & | \\ f(e_1) & \cdots & f(e_n) \\ | & & | \end{pmatrix}$$

Now recall that

$$\langle e_i, e_j\rangle = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

And so $\langle f(e_i), f(e_j)\rangle = \delta_{ij}$. Thus the rows of A form an orthogonal basis, meaning A is an orthogonal matrix.
Now let us define

$$g(x) = A^{-1}f(x)$$

and we will prove that $g(x) = x$, which means that $f(x) = Ax$. Notice that

$$g(e_i) = A^{-1}f(e_i) = A^{-1}C_i(A) = C_i(A^{-1}A) = e_i$$

Now, if g were a linear transformation, we could finish here. Since $g(0) = 0$, g is an isometry (as the composition of isometries) which preserves the origin, so it preserves inner products.
Now let $x \in \mathbb{R}^n$ have coefficients x_i, meaning $\langle x, e_i\rangle = x_i$, now let $g(x) = y$ with coefficients y_i. So

$$x_i = \langle x, e_i\rangle = \langle g(x), g(e_i)\rangle = \langle y, e_i\rangle = y_i$$

Thus $x = y$, so $g(x) = x$ and thus $f(x) = Ax$, so f is indeed an orthogonal transformation. ■

Thus if f is an isometry, let $g(x) = f(x) - f(0)$, then g is also an isometry which preserves the origin and so $g(x) = Ax$ where A is orthogonal. And so $f(x) = Ax + f(0)$.

Theorem 2.4 (Cartan-Dieudonne Theorem):

If $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is an isometry, then

$$f = T \circ S_1 \circ \cdots S_m$$

where T is a shift, and S_i are reflections, and $m \leq n$.

Proof:

We will prove this by induction on n. For $n = 1$, then we know that $f(x) = Ax + c$ where A is orthogonal, and in $\mathbb{R}$ that means that $A = \pm 1$. So $f(x) = \pm x + c$. The $+c$ is a shift, and $-x$ is a reflection about 1.
Now, for the inductive step let $g(x) = f(x) - f(0)$ so $g(x) = Ax$ where A is orthogonal. If $A = \mathrm{id}$, then $f(x) = x + c$ which is just a shift, and we have finished. Otherwise there exists an $a \in \mathbb{R}^n$ such that $g(a) \neq a$. Now, we want a $b \in \operatorname{span} a, g(a)$ such that $\|b\| = 1$ and $S_b(a) = g(a)$. Let

$$d = \frac{a}{\|a\|} + \frac{g(a)}{\|g(a)\|}$$

And let b be the unit normal to d in $\operatorname{span} a, g(a)$. Then $S_b(a)$ is the reflection of a about d, which gives $g(a)$.
Now let

$$h = S_b \circ g$$

then h is the composition of two orthogonal transformations, and is therefore also an orthogonal transformation. Let $\hat{a} = \frac{a}{\|a\|}$, and let us extend this to an orthogonal basis

$$B = \{\hat{a}, b_2, \cdots, b_n\}$$

And since h is orthogonal, $h(B)$ is also an orthogonal basis. And $h(a) = S_b(g(a)) = S_b(S_b(a)) = a$, and so $h(\hat{a}) = \hat{a}$. Thus

$$h(\{b_2, \dots, b_n\}) \perp \hat{a}$$

And so $h(\{b_2, \dots, b_n\})$ is an orthogonal basis of $V = \hat{a}^\perp$, which has a dimension of $n-1$. And so $h\big|_V : V \to V$ is an orthogonal transformation, since $\{b_2, \dots, b_n\}$ is an orthogonal basis of V, and so is its image. So by our inductive assumption,

$$h\big|_V = S_2 \circ \cdots \circ S_m$$

where S_i are reflections with respect to $u^\perp \subseteq V$, and $m \leq n$.
Let $\ell = \operatorname{span} \hat{a}$, and $h\big|_\ell = \mathrm{id}$, and since h is linear

$$h = S_2 \circ \cdots \circ S_m$$

where S_i is a reflection with respect to $u^\perp \subseteq \mathbb{R}^n$. And since $h = S_b \circ g$, and $f = T \circ g$, where T is a shift (adding $f(0)$), we have

$$T = T \circ S_b \circ S_2 \circ \cdots \circ S_m$$

where $m \leq n$ as required. ∎

Definition 2.5:

A **curve** is a continuous function

$$\gamma \colon [a, b] \longrightarrow \mathbb{R}^n$$

A curve is **smooth** if it is differentiable, and it is **regular** if its derivative is never zero. If $\gamma'(t) = 0$ then t is called a **singularity** of γ.

Definition 2.6:

Suppose $\alpha \colon [a, b] \longrightarrow \mathbb{R}^n$ is a curve, and $\varphi \colon [c, d] \longrightarrow [a, b]$ is differentiable and $\varphi' > 0$, then we define $\beta \colon [c, d] \longrightarrow \mathbb{R}^n$ by $\beta = \alpha \circ \varphi$. This is called a **reparametrization** of α.

Proposition 2.7:

"x is a reparametrization of y" is an equivalence relation.

Proof:

Obviously this is reflexive (take φ to be the identity function). And it is transitive since if $\beta = \alpha \circ \varphi$ and $\gamma = \beta \circ \psi$ then $\gamma = \alpha \circ (\varphi \circ \psi)$ (the derivative of the composition is still positive). restrict the definition, this still works). Now suppose $\beta = \alpha \circ \varphi$, then since $\varphi' > 0$, we know that φ is strictly increasing (and therefore injective). And so we can also assume that φ is surjective, since $\varphi([a, b]) = [\varphi(a), \varphi(b)]$. So φ is bijective and so $\alpha = \beta \circ \varphi^{-1}$, and $(\varphi^{-1})' > 0$ (since it is equal to the inverse of φ' of some point). ∎

Definition 2.8:

Let $\alpha \colon [0, T] \to \mathbb{R}^n$ be a curve, let

$$s_\alpha(t) = \int_0^t \|\alpha'(f)\| = \int_a^T \left(\sum_{k=1}^n \alpha_k'(f)^2 \right)^{1/2}$$

$s_\alpha(t)$ is the **arclength** of α.
α' is the componentwise derivative of α, which is equal to the Jacobian of α. We can continue with higher order componentwise derivatives.

The intuition behind the definition of $s(t)$ is that by the definition of integrals (using Riemman sums), we can partition $[0, T]$ into $t_0 = 0 < t_1 < \cdots < t_n = t$, and

$$\alpha'(f) \approx \frac{\alpha(t_{i+1}) - \alpha(t_i)}{\Delta_i} \implies \|\alpha'(f)\| \cdot \Delta_i \approx \|\alpha(t_{i+1}) - \alpha(t_i)\|$$

And $\|\alpha(t_{i+1}) - \alpha(t_i)\|$ approximates the length of α between t_i and t_{i+1}. And as we make the partition finer and finer, these approximations get more and more accurate.

Proposition 2.9:

Arclength is invariant under reparameterization. Meaning if $\alpha\colon [a,b] \longrightarrow \mathbb{R}^n$ and $\beta = \alpha \circ \varphi$ then

$$\int_a^b \|\alpha'(t)\| = \int_c^d \|\beta'(t)\|$$

Proof:

Notice that

$$\beta'(t) = \varphi'(t) \cdot \alpha'\big(\varphi(t)\big)$$

Since $\varphi'(t) > 0$ we have that

$$\int_c^d \|\beta'(t)\| = \int_c^d \|\alpha'(\varphi(t))\| \cdot \varphi'(t)\, dt$$

Let $u = \varphi(t)$ then $\varphi'(t)\, dt = du$ and since $\varphi(c) = a$ and $\varphi(d) = b$, so

$$= \int_a^b \|\alpha'(u)\|\, du$$

as required. ■

What we have shown is that $s_{\alpha\circ\varphi}(t) = s_\alpha(\varphi(t))$, ie

$$s_{\alpha\circ\varphi} = s_\alpha \circ \varphi$$

Notice that $s'_\alpha(t) = \|\alpha'(t)\|$. If α is regular then $\alpha'(t) \neq 0$ and so $s'_\alpha > 0$ so s_α is smooth and strictly increasing, meaning s_α is invertible. Now let us define

$$\beta(u) = \alpha \circ s_\alpha^{-1}(u) = \alpha(t)$$

where

$$u = \int_0^t \|\alpha'(x)\|$$

So $\beta(u)$ is equal to the value of α after walking u units on the arc α. β is called the *natural parameterization* of α. Notice that if β is a reparameterization of α, then they both have the same natural parameterizations, since if $\beta = \alpha \circ \varphi$ then

$$\beta \circ s_\beta^{-1} = \beta \circ s_{\alpha\circ\varphi}^{-1} = \beta \circ (s_\alpha \circ \varphi)^{-1} = \alpha \circ \varphi \circ \varphi^{-1} \circ s_\alpha^{-1} = \alpha \circ s_\alpha^{-1}$$

So the natural parameterization of a regular curve is unique.
Notice that α is a natural parameterization if and only if $s_\alpha = \mathrm{id}$. If α is a natural parameterization, then $\alpha = \alpha \circ s_\alpha^{-1}$, and so $s_\alpha = \mathrm{id}$. And if $s_\alpha = \mathrm{id}$, then $\alpha \circ s_\alpha^{-1} = \alpha$.

Proposition 2.10:

If α is a curve, it is a natural parameterization if and only if $\|\alpha'\| = 1$.

Proof:

Since

$$s_\alpha(t) = \int_0^t \|\alpha'(u)\|$$

so $s'_\alpha = \|\alpha'\|$, so if $s_\alpha = \mathrm{id}$ then $s'_\alpha = \|\alpha'\| = 1$. And if $\|\alpha'\| = 1$ then $s'_\alpha = 1$ so $s_\alpha(t) = t + c$ and since $s_\alpha(0) = 0$, $c = 0$ as required. ■