Modern Analysis

Homework 5

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5.1 Exercise

Let f, f_n be nonnegative measurable functions on X such that $f_n \to f$ and $f_n \leq f$. Prove that

$$\lim \int f_n = \int f$$

Define $g_n = \max\{f_1, \dots, f_n\}$, then for every n, $f_n \leq g_n \leq f$ so g_n is monotonically increasing to f, meaning $\lim \int f_n \leq \lim \int g_n = \int f$. By Fatou,

$$\int f = \int \liminf f_n \leq \liminf \int f_n = \lim \int f_n$$

So we have both directions of the inequality, meaning $\lim \int f_n = \int f$ as required.

5.2 Exercise

Let f be a measurable nonnegative function, then define $F(x) = \int_{-\infty}^{x} f\,dx$. Show that F is continuous.

Let $a \in \mathbb{R}$ and take $x_n \nearrow a$, then $F(x_n) = \int_{-\infty}^{x_n} f\,d\mu = \int_{\mathbb{R}} f\chi_{(-\infty,x_n)}$. Since $f \cdot \chi_{(-\infty,x_n)}$ is increasing to $f \cdot \chi_{(-\infty,a)}$ we get that $F(x_n) \longrightarrow \int_{-\infty}^{a} f = F(a)$ by the monotone convergence theorem. Note that the monotone convergence theorem works for f_n decreasing if f_1 is integrable: define $g_n = f_1 - f_n$ which is nonnegative and increasing and $g_n \nearrow f_1 - f$ so

$$\lim \int g_n = \int f_1 - f \implies \lim \int f_1 - f_n = \int f_1 - \int f \implies \lim \int f_n = \int f$$

So for $x_n \searrow a$, $f \cdot \chi_{(-\infty,x_n)}$ is decreasing to $f \cdot \chi_{(-\infty,a)}$, and since these are all bound by f which is integrable they are integrable, so the monotone convergence theore gives us again that $F(x_n) \longrightarrow \int_{-\infty}^{a} f = F(a)$ again.

5.3 Exercise

Let (Ω, Σ, μ) be a measure space such that $\mu(X) = 1$. Then suppose $A, B, C \in \Sigma$ are measurable such that $\mu(A) + \mu(B) + \mu(C) \geq 2.5$. Is it possible for $A \cap B \cap C = \varnothing$?

Suppose so, then $A^c \cup B^c \cup C^c = X$ so $\mu(A^c \cup B^c \cup C^c) = 1$, but by subadditivity

$$\mu(A^c \cup B^c \cup C^c) \leq \mu(A^c) + \mu(B^c) + \mu(C^c) = 3 - \mu(A) - \mu(B) - \mu(C) \leq 3 - 2.5 = 0.5$$

So we have that $1 \leq 0.5$ in contradiction.