Infintesimal Calculus 3

Assignment 6

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Exercise 6.1:

Compute the partial derivatives of each of the following functions:

(1) $f(x,y) = x^3 + 3y^2 - \frac{x}{y}$

(2) $f(x,y) = e^{\cos(xy)}$

(3) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$

(4) $f(x,y,z) = \log(x^3+y^3-z^3)$

Partial derivatives are simply derivatives of functions where all other variables are held constant, we will use this fact to compute the partial derivatives:

(1) $f_x(x,y) = 3x^2 - \frac{1}{y}$ and $f_y(x,y) = 6y + \frac{x}{y^2}$. This is defined for $\mathbb{R}^2 \setminus \{(x,0)\}$ which is the domain of the function as well, so the partial derivatives cannot be defined anywhere else.

(2) $f_x(x,y) = -y\sin(xy)\cdot e^{\cos(xy)}$ and by symmetry $f_y(x,y) = -x\sin(xy)\cdot e^{\cos(x,y)}$. This is defined on all of $\mathbb{R}^2$.

(3) In general we can write this as a specific case of the function

$$f(x_1,\dots,x_n) = \sqrt{\sum_{i=1}^n x_i^2}$$

Whose partial derivatives are

$$f_{x_i}(x_1,\dots,x_n) = \frac{2x_i}{\sqrt{\sum_{i=1}^n x_i^2}} = \frac{x_i}{\sqrt{\sum_{i=1}^n x_i^2}}$$

This is defined only when the denominator is non-zero which can only be when $(x_1,\dots,x_n) = 0$ since $x_i^2 \geq 0$. If $(x_1,\dots,x_n) = 0$ then the ith partial derivative is (since $f(0) = 0$):

$$f_{x_i}(0,\dots,0) = \lim \frac{f(0,\dots,\Delta,\dots,0)}{\Delta} = \lim_{\Delta\to 0} \frac{|\Delta|}{\Delta}$$

Which doesn't converge since for $\Delta < 0$ it is -1 and for $\Delta > 0$ it is 1. So the partial derivatives are defined over $\mathbb{R}^n \setminus \{(0,\dots,0)\}$.

(4) In this case:

$$f_x(x,y,z) = \frac{3x^2}{x^3+y^3-z^3} \quad f_y(x,y,z) = \frac{3y^2}{x^3+y^3-z^3} \quad f_z(x,y,z) = \frac{-3z^2}{x^3+y^3-z^3}$$

Which are defined except for when $x^3+y^3-z^3 = 0$, but this is outside the domain of the function, which is the set of all points where $x^3+y^3-z^3 > 0$ so the partial derivatives are defined where ever the function is.

Exercise 6.2:

Determine the differentiability of the following functions:

(1) $f(x,y) = \begin{cases} \frac{x^3+y^4}{x^2+y^2} & (x,y) \neq 0 \\ 0 & (x,y) = 0 \end{cases}$

(2) $f(x,y) = \begin{cases} \frac{x^3-y^2}{\sqrt{x^2+y^2}} & (x,y) \neq 0 \\ 0 & (x,y) = 0 \end{cases}$

(3) $f(x,y) = \begin{cases} x\sin\left(\frac{y^2}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$

(1) Since for when $(x,y) \neq 0$ this is simply the composition of elementary functions which are defined in this domain, it is differentiable there. For when $(x,y) = 0$ we have that:

$$f_x(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x} = \lim \frac{x}{x} = 1$$

And

$$f_y(0,0) = \lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y} = \lim \frac{y^2}{y} = 0$$

So $\nabla f\big|_{(0,0)} = (1,0)^T$ and to determine differentiability we must show that the following limit converges to 0:

$$\lim_{(x,y)\to 0} \frac{\varepsilon}{\|(x,y)\|} = \lim \frac{f(x,y) - f(0,0) - \nabla f\big|_{(0,0)} \cdot (x,y)}{\sqrt{x^2+y^2}} = \lim \frac{x^3 + y^4 - x}{(x^2+y^2)^{1.5}}$$

For $y = 0$ this is the tail

$$\lim \frac{x^3 - x}{x^3} = \lim 1 - x^{-2}$$

Which does not converge to 0, so the function is not differentiable at $(0,0)$. So the domain of differentiability for the function is $\mathbb{R}^2 \setminus \{(0,0)\}$.

(2) For the same reason as above when $(x,y) \neq 0$ the function is differentiable as the composition of elementary functions. And for $(0,0)$:

$$f_x(0,0) = \lim_{x\to 0} \frac{x^3}{|x| \cdot x} = \lim_{x\to 0} \frac{x^2}{|x|} = \lim_{x\to 0} |x| = 0$$

And

$$f_y(0,0) = \lim_{y\to 0} \frac{-y^2}{|y| \cdot y} = \lim_{y\to 0} \frac{-y}{|y|}$$

Which does not converge, so f does not have a y partial derivative at $(0,0)$ and is therefore not differentiable there. So the domain of differentiability is $\mathbb{R}^2 \setminus \{(0,0)\}$.

(3) Since when $x \neq 0$, this is the composition of elementary functions, and $\{x \neq 0\}$ is open, f is differentiable when $x \neq 0$. When $x = 0$:

$$f_x(0,y) = \lim_{x\to 0} \frac{x\sin\left(\frac{y^2}{x}\right)}{x} = \lim_{x\to 0} \sin\left(\frac{y^2}{x}\right)$$

This does not converge unless $y = 0$ as well (since $\sin\left(\frac{1}{x}\right)$ is discontinuous at $x = 0$). And if $y = 0$ then it is equal to 0. And

$$f_y(0,0) = \lim_{y\to 0} \frac{0}{y} = 0$$

So $\nabla f\big|_{(0,0)} = (0,0)^T$. So in order to show differentiability we must show that the following limit exists and is equal to 0:

$$\lim_{(x,y)\to 0} \frac{f(x,y) - f(0,0) - \nabla f\big|_{(0,0)} \cdot (x,y)}{\|(x,y)\|} = \lim_{(x,y)\to 0} \frac{x\sin\left(\frac{y^2}{x}\right)}{\sqrt{x^2+y^2}}$$

since for $x = 0$ this limit is trivially equal to 0. We can rewrite this as

$$\frac{\sin\left(\frac{y^2}{x}\right)}{\sqrt{1 + \frac{y^2}{x^2}}}$$

Now suppose we have a sequence $(x_n, y_n) \in \mathbb{R}^2$, we can assume that $\frac{y_n^2}{x_n}$ is a convergent sequence. If it is not a convergent sequence there must be a convergent subsequence since $\mathbb{R}$ is sequentially compact, and we will show that if this sequence converges then the limit above converges to 0. So every convergent subsequence of any sequence which converges to 0 has the limit converge to 0, so the limit converges to 0. If $\frac{y_n^2}{x_n}$ converges to 0 then so does the limit, since the numerator converges to 0 and the denominator is larger than 1. Otherwise if it doesn't converge to 0 then since $\left|\frac{1}{x_n}\right|$ converges to infinity (since x_n converges to 0), $\frac{y_n^2}{x_n^2}$ converges to infinity as well. So in this case the denominator converges to infinity and the numerator is bounded (since sin is bounded), so the limit also converges to 0.

So the limit does indeed converge to 0 as required, so f is differentiable at $(0,0)$. Thus the domain of differentiability is $\mathbb{R}^2 \setminus \{(0,y) \mid y \neq 0\}$.

Exercise 6.3:

Suppose f is differentiable at $(0,0)$, then we define:

$$h(x,y) = \begin{cases} f(x,y) & xy > 0 \\ 0 & xy \leq 0 \end{cases}$$

Show that if $f(0,0) = f_x(0,0) = f_y(0,0) = 0$ then h is differentiable at $(0,0)$.

We know that

$$h_x(0,0) = \lim_{x\to 0} \frac{h(x,0) - h(0,0)}{x} = \lim_{x\to 0} \frac{0}{x} = 0$$

And similarly $h_y(0,0) = 0$ so $\nabla h\big|_{(0,0)} = (0,0)^T$. And so all that remains to show is that the following limit converges to 0:

$$\lim_{(x,y)\to 0} \frac{h(x,y) - h(0,0) - \nabla h\big|_{(0,0)}}{\|(x,y)\|} = \lim_{(x,y)\to 0} \frac{h(x,y)}{\sqrt{x^2+y^2}}$$

If $xy \leq 0$ then $h(x,y) = 0$ so the limit holds. Otherwise if $xy > 0$ then the limit is equal to the limit of

$$\frac{f(x,y)}{\sqrt{x^2+y^2}} = \frac{f(0,0) + \nabla f\big|_{(0,0)} \cdot (x,y) + \varepsilon(x,y)}{\sqrt{x^2+y^2}} = \frac{\varepsilon(x,y)}{\sqrt{x^2+y^2}}$$

since f is differentiable at $(0,0)$. And since ε is an ε function, this limit is indeed 0 as required.

Exercise 6.4:

Write the second order Taylor polynomial for $f(x,y) = \sin(xe^y)$ about the point $\left(\frac{\pi}{2}, 0\right)$.

First we compute the partial derivatives:

$$f_x = e^y \cos(xe^y) \qquad f_y = xe^y \cos(xe^y)$$
$$f_{xx} = -e^{2y} \sin(xe^y) \qquad f_{yy} = xe^y\big(\cos(xe^y) - xe^y \sin(xe^y)\big)$$
$$f_{xy} = e^y\big(\cos(xe^y) - xe^y \sin(xe^y)\big)$$

And so:

$$f(x,y) \approx 1 + \left(x - \frac{\pi}{2}\right) \cdot 0 + y \cdot 0 + \frac{1}{2}\left(-\left(x - \frac{\pi}{2}\right)^2 - \pi\left(x - \frac{\pi}{2}\right)y - \frac{\pi^2 y^2}{4}\right)$$

Simplifying gives:

$$f(x,y) \approx -\frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 - \frac{\pi}{2}\left(x - \frac{\pi}{2}\right)y - \frac{\pi^2 y^2}{8} + 1$$

Exercise 6.5:

Find the Taylor series expansions of the following functions about $(1, 1)$ of order 2:

(1) $f(x, y) = x^y$

(2) $f(x, y) = \frac{x}{y}$

(1) We will find partial derivatives:

$$f_x = yx^{y-1} \qquad f_y = x^y \log x$$
$$f_{xx} = y(y-1)x^{y-2} \qquad f_{yy} = x^y \left(\log x\right)^2$$
$$f_{xy} = yx^{y-1}\log x + x^{y-1}$$

And so:

$$f(x, y) \approx 1 + (x-1) + (y-1)\cdot 0 + \frac{1}{2}\left((x-1)^2 \cdot 0 + 2(x-1)(y-1) + (y-1)^2 \cdot 0\right) = x + (x-1)(y-1)$$

(2) We will find partial derivatives:

$$f_x = \frac{1}{y} \qquad f_y = -\frac{x}{y^2}$$
$$f_{xx} = 0 \qquad f_{yy} = \frac{2x}{y^3}$$
$$f_{xy} = -\frac{1}{y^2}$$

And so:

$$\begin{aligned} f(x, y) &\approx 1 + (x-1) - (y-1) + \frac{1}{2}\left((x-1)^2 \cdot 0 - 2(x-1)(y-1) + 2(y-1)^2\right) \\ &= 1 + x - y - (x-1)(y-1) + (y-1)^2 \end{aligned}$$

Exercise 6.6:

Find the Taylor polynomial of $f(x, y) = \frac{1}{1-x^2y}$ about $(0, 0)$ and using it find $\partial_{x^4y^2} f(0, 0)$.

If we let $t = x^2y$ then we have that $f(t) = \frac{1}{1-t}$ and this has a well-known Taylor series:

$$f(t) = \sum_{n=0}^{\infty} t^n$$

This is the taylor series around $t = 0$, which is given by $x = y = 0$. And so

$$f(x, y) = \sum_{n=0}^{\infty} x^{2n} y^n$$

this is the taylor series of f around $(0, 0)$ since the terms are of the form $(x-0)^n(y-0)^k$, as required.
To find $\partial_{x^4y^2} f(0, 0)$ recall that

$$f(x, y) = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\ell=0}^{m} \binom{m}{\ell} \cdot \partial_{x^\ell y^{m-\ell}} f(0, 0) x^\ell y^{m-\ell}$$

And the summand which has $\partial_{x^4y^2} f(0, 0)$ is given when $\ell = 4$ and $m = 6$, which is the only summand with the term

x^4y^2, which has a coefficient of 1 in f's Taylor series found above (it is given by $n = 2$). So:

$$\frac{1}{m!}\binom{m}{\ell} \cdot \partial_{x^\ell y^{m-\ell}} f(0,0) = 1 \implies \frac{1}{6!}\binom{6}{4} \cdot \partial_{x^4 y^2} f(0,0) = 1$$

And so

$$\partial_{x^4 y^2} f(0,0) = 48$$