Calculus Homework #5

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Question 5.1:

Compute the following integral:

$$\int e^{2\cdot\sin^{-1}(x)}\,dx$$

Answer:

Let:

$$x = \sin\theta$$

For $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so:

$$dx = \cos\theta\,dx$$

Then the integral becomes:

$$I = \int e^{2\theta}\cdot\cos\theta\,dx$$

By integration by parts, this is equal to:

$$= e^{2\theta}\cdot\sin\theta - 2\int e^{2\theta}\sin\theta\,d\theta$$

Integrating by parts again, we get:

$$= e^{2\theta}\cdot\sin\theta - 2\left(-\cos\theta\cdot e^{2\theta} + 2\int e^{2\theta}\cos\theta\,d\theta\right)$$

The rightmost integral is just I, so:

$$I = \frac{e^{2\theta}}{5}\cdot(\sin\theta + 2\cos\theta)$$

We know in this domain:

$$\theta = \sin^{-1}(x)$$

And $\cos\theta \geq 0$ in this domain, so:

$$\cos\theta = \sqrt{1-x^2}$$

Which means:

$$I = \boxed{\frac{e^{2\sin^{-1}(x)}}{5}\cdot\left(x + 2\sqrt{1-x^2}\right) + C}$$

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Question 5.2:

Compute the following integral:

$$\int x \log \left|x^2+1\right| \, dx$$

Answer:

We can substitute:

$$u := x^2 + 1$$

Which means:

$$du = 2x \, dx$$

So the integral becomes:

$$\frac{1}{2} \int \log |u| \, du$$

And as we computed in the lecture, the integral of the natural logarithm is:

$$= \frac{1}{2} \cdot (u \log |u| - u)$$

Thus the integral is equal to:

$$= \frac{x^2+1}{2} \cdot \log \left|x^2+1\right| - \frac{x^2}{2} - \frac{1}{2}$$

We can ignore that trailing constant as the cosests of the antiderivative is equal to the antiderivative itself. So the integral is equal to:

$$\boxed{\frac{x^2}{2} \cdot \log \left|x^2+1\right| - \frac{x^2}{2} + C}$$

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Question 5.3:

Compute the following integral:

$$\int x^3 \sin\left(x^3\right)\, dx$$

Answer:

We can use integration by parts:

$$\begin{array}{rclcrcl} u & = & x^2 & \qquad & dv & = & x\sin\left(x^2\right)\, dx \\ du & = & 2x\, dx & & v & = & -\frac{1}{2}\cos\left(x^2\right) \end{array}$$

And the integral becomes:

$$-\frac{x^2}{2}\cdot\cos\left(x^2\right) + \int x\cos\left(x^2\right)\, dx$$

By substituting $u \coloneqq x^2$ into the right integral, we see it is equal to:

$$\int x\cos\left(x^2\right)\, dx = \frac{1}{2}\int \cos\left(u\right)\, du = \frac{\sin\left(u\right)}{2} = \frac{\sin\left(x^2\right)}{2}$$

So all in all, the integral is equal to:

$$\boxed{\frac{1}{2}\cdot\left(\sin\left(x^2\right) - x^2\cdot\cos\left(x^2\right)\right) + C}$$

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Question 5.4:

Compute the following integral:

$$\int x^2 \tan^{-1}(x)\ dx$$

Answer:

We can use integration by parts:

$$\begin{array}{rclcrcl} u & = & \tan^{-1}(x) & \quad & dv & = & x^2\,dx \\ du & = & \frac{dx}{1+x^2} & & v & = & \frac{x^3}{3} \end{array}$$

The integral becomes:

$$\frac{x^3}{3} \cdot \tan^{-1}(x) - \frac{1}{3} \cdot \int \frac{x^3}{1+x^2}\,dx$$

We know:

$$\frac{x^3}{1+x^2} = x - \frac{x}{1+x^2}$$

And so its integral is:

$$\int \frac{x^3}{1+x^2}\,dx = \frac{x^2}{2} - \int \frac{x}{1+x^2}\,dx$$

Substituting $u = 1 + x^2$, we get that the rightmost integral above is equal to:

$$= \frac{1}{2} \cdot \int \frac{du}{u} = \frac{1}{2} \cdot \log|u| = \frac{1}{2} \cdot \log\left|1 + x^2\right|$$

So the original integral is equal to:

$$\boxed{\frac{x^3}{3} \cdot \tan^{-1}(x) - \frac{x^2}{6} + \frac{\log\left|1+x^2\right|}{6} + C}$$

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Question 5.5:

Compute the following integral:

$$\int e^{\sin(x)} \cdot \sin(2x)\ dx$$

Answer:

We know $\sin(2x) = 2\sin(x) \cdot \cos(x)$, so the integral is equal to:

$$2 \cdot \int e^{\sin(x)} \cdot \sin(x)\cos(x)\ dx$$

Substituting $u \coloneqq \sin(x)$, we get $du = \cos(x)\ dx$, so the integral is equal to:

$$2 \cdot \int e^{u} \cdot u\, du$$

Using integration by parts

$$\begin{array}{rclcrcl} t & = & u & & dv & = & e^u\, du \\ dt & = & du & & v & = & e^u \end{array}$$

the integral becomes:

$$= 2 \cdot \left(ue^u - \int e^u\, du \right) = 2 \cdot (ue^u - e^u) = 2e^u \cdot (u-1)$$

Resubstituting the definition of u, we find that the integral is equal to:

$$\boxed{2e^{\sin(x)} \cdot (\sin(x) - 1) + C}$$

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Question 5.6:

Compute the following integral:

$$\int e^{2x+e^x}\,dx$$

Answer:

The integral is equal to:

$$\int e^{2x} \cdot e^{e^x}\,dx$$

Substituting $x = \log|u|$ for $u > 0$ and thus $dx = \frac{du}{u}$ and $u = e^x$, we get that the integral is equal to:

$$\int u^2 \cdot e^u \cdot \frac{1}{u}\,du = \int u \cdot e^u\,du$$

As computed above, this is equal to:

$$= e^u \cdot (u-1) = e^{e^x} \cdot (e^x - 1)$$

Thus the integral is equal to:

$$\boxed{e^{e^x} \cdot (e^x - 1) + C}$$

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Question 5.7:

Compute the following integral:

$$\int \frac{e^{\tan(x)} \cdot \sin(x)}{\cos^3(x)}\, dx$$

Answer:

Moving around some terms, we can see that the integral is equal to:

$$\int e^{\tan(x)} \cdot \tan(x) \cdot \frac{1}{\cos^2(x)}\, dx$$

Let $u \coloneqq \tan(x)$, thus $du = \frac{1}{\cos^2(x)}\, dx$. So the integral is equal to:

$$= \int e^u \cdot u\, du$$

Which, as computed above, is equal to:

$$= e^u \cdot (u - 1) = e^{\tan(x)} \cdot (\tan(x) - 1)$$

Thus the integral is equal to:

$$\boxed{e^{\tan(x)} \cdot (\tan(x) - 1) + C}$$

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Question 5.8:

Compute the following integral:

$$\int \frac{x^4}{x^4 + 5x^2 + 4}\, dx$$

Answer:

Using partial fraction decomposition, we want to find two polynomials of degree < 2 such that:

$$\frac{x^4}{x^4 + 5x^2 + 4} = \frac{p_1}{x^2 + 1} + \frac{p_2}{x^2 + 4}$$

Letting $p_i = \alpha_i x + \beta_i$ where $\alpha, \beta \in \mathbb{R}$, we find that:

$$\frac{x^4}{x^4 + 5x^2 + 4} = \frac{1}{3} \cdot \frac{1}{x^2 + 1} - 5\frac{1}{3} \cdot \frac{1}{x^2 + 4}$$

So we just need to integrate these two fractions.

Lemma 5.8.1:

$$\int \frac{1}{x^2 + \alpha^2}\, dx = \frac{1}{\alpha} \cdot \tan^{-1}\left(\frac{x}{\alpha}\right) + C$$

Proof:

We can substitute $x = \alpha \cdot \tan(\theta)$ for $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, and thus $dx = \frac{\alpha}{\cos^2 \theta}\, d\theta$, so the integral becomes:

$$\int \frac{1}{\alpha \tan^2 \theta + \alpha^2} \cdot \frac{\alpha}{\cos^2 \theta}\, d\theta$$

And we know that $\tan^2 \theta + 1 = \frac{1}{\cos^2 \theta}$, so this becomes:

$$\int \frac{1}{\alpha^2 \cdot \frac{1}{\cos^2 \theta}} \cdot \frac{1}{\cos^2 \theta}\, d\theta = \int \frac{1}{\alpha}\, d\theta = \frac{1}{\alpha} \cdot \theta$$

And by the definition of θ, we know $\theta = \tan^{-1}\left(\frac{x}{\alpha}\right)$. So the integral is equal to:

$$\frac{1}{\alpha} \cdot \tan^{-1}\left(\frac{x}{\alpha}\right) + C$$

As required.

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So the integral becomes:

$$\frac{1}{3} \cdot \tan^{-1}(x) - 5\frac{1}{3} \cdot \frac{1}{2} \cdot \tan^{-1}\left(\frac{x}{2}\right)$$

Which simplifies to:

$$\boxed{\frac{1}{3} \cdot \tan^{-1}(x) - 2\frac{2}{3} \cdot \tan^{-1}\left(\frac{x}{2}\right)}$$

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Question 5.9:

Compute the following integral:

$$\int \frac{x^2+5x+4}{x^4+5x^2+4}\,dx$$

Answer:

Using partial fraction decomposition, we find that:

$$\frac{x^2+5x+4}{x^4+5x^2+4} = \frac{\frac{5}{3}x+1}{x^2+1} - \frac{\frac{5}{3}x}{x^2+4}$$

The first fraction can be written as:

$$\frac{\frac{5}{3}x}{x^2+1} + \frac{1}{x^2+1}$$

And we can generalize:

$$\int \frac{\alpha x}{x^2+\beta}\,dx = \int \frac{\frac{\alpha}{2}\cdot d\left(x^2+\beta\right)}{x^2+\beta} = \frac{\alpha}{2}\cdot \log\left|x^2+\beta\right|$$

And we know:

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}(x)$$

So the integral becomes (integrating each fraction):

$$\frac{5}{6}\cdot \log\left|x^2+1\right| + \tan^{-1}(x) - \frac{5}{6}\cdot \log\left|x^2+4\right|$$

Which simplifies to:

$$\boxed{\frac{5}{6}\cdot \log\left|\frac{x^2+1}{x^2+4}\right| + \tan^{-1}(x) + C}$$

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Question 5.10:

Compute the following integral:

$$\int \frac{dx}{x^3+1}$$

Answer:

Using partial fraction decomposition, we get:

$$\frac{1}{x^3+1} = \frac{1}{3} \cdot \left(\frac{1}{x+1} - \frac{x-2}{x^2-x+1} \right)$$

We know:

$$\int \frac{1}{x+1}\, dx = \log|x+1|$$

And:

$$\frac{x-2}{x^2-x+1} = \frac{x-\frac{1}{2}}{x^2-x+1} - \frac{1\frac{1}{2}}{x^2-x+1}$$

We know:

$$\int \frac{x-\frac{1}{2}}{x^2-x+1}\, dx = \int \frac{\frac{1}{2}\, d\left(x^2-x+1\right)}{x^2-x+1} = \frac{1}{2} \cdot \log\left|x^2-x+1\right|$$

And we can reorder the denominator in:

$$\frac{1}{x^2-x+1} = \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}$$

So:

$$\int \frac{dx}{x^2-x+1} = \int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}$$

As per lemma 5.8.1, this is equal to:

$$\frac{2}{\sqrt{3}} \cdot \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)$$

So:

$$\int \frac{-1\frac{1}{2}}{x^2-x+1}\, dx = -\frac{3}{\sqrt{3}} \cdot \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)$$

So all in all, the integral is equal to:

$$\boxed{\frac{1}{3}\log|x+1| - \frac{1}{6} \cdot \log\left|x^2-x+1\right| + \frac{1}{\sqrt{3}} \cdot \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C}$$

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Question 5.11:

Compute the following integral:

$$\int \frac{dx}{(x^2+1)(x^2-1)}$$

Answer:

Using partial fraction decomposition, we know there must be polynomials of the relevant degree such that:

$$\frac{1}{x^4-1} = \frac{\alpha_1}{x-1} + \frac{\alpha_2}{x+1} + \frac{\beta_1 x + \beta_2}{x^2+1}$$

Before we find what these values are, we can compute the integral. We know that:

$$\int \frac{\beta_1 x + \beta_2}{x^2+1}\, dx = \frac{\beta_1}{2} \cdot \int \frac{d\left(x^2+1\right)}{x^2+1} + \beta_2 \int \frac{1}{x^2+1} = \frac{\beta_1}{2} \cdot \log\left|x^2+1\right| + \beta_2 \cdot \tan^{-1}(x)$$

And the left integrals are $\alpha_1 \cdot \log|x-1|$ and $\alpha_2 \cdot \log|x+1|$ respectively.
Now to find the values of these paremeters. Multiplying each side by x^4-1, we get:

$$\alpha_1(x^3+x^2+x+1) + \alpha_2(x^3-x^2+x-1) + \beta_1(x^3-x) + \beta_2(x^2-1) = 1$$

So:

$$\begin{cases} \alpha_1+\alpha_2+\beta_1 &= 0 \\ \alpha_1-\alpha_2+\beta_2 &= 0 \\ \alpha_1+\alpha_2-\beta_1 &= 0 \\ \alpha_1-\alpha_2-\beta_2 &= 1 \end{cases}$$

Which gives us the solution:

$$\alpha_1 = \frac{1}{4} \quad \alpha_2 = -\frac{1}{4} \quad \beta_1 = 0 \quad \beta_2 = -\frac{1}{2}$$

So by adding up all the smaller integrals we computed, we find that the original integral is equal to:

$$\frac{1}{4} \cdot (\log|x-1| - \log|x+1|) - \frac{1}{2} \cdot \tan^{-1}(x)$$

Simplified, the integral is equal to:

$$\boxed{\frac{1}{4}\log\left|\frac{x-1}{x+1}\right| - \frac{1}{2} \cdot \tan^{-1}(x) + C}$$

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Question 5.12:

Compute the following integral:

$$\int \frac{2x^3+3x^2-x-14}{x^3-8}\,dx$$

Answer:

Firstly, we know:

$$\frac{2x^3+3x^2-x-14}{x^3-8} = 2 + \frac{3x^2-x+2}{x^3-8}$$

And we can split up the fraction:

$$= 2 + \frac{3x^2}{x^3-8} - \frac{x-2}{x^3-8}$$

And $3x^2$ is the derivative of x^3-8, so:

$$\int \frac{3x^2}{x^3-8}\,dx = \log\left|x^3-8\right|$$

And:

$$\frac{x-2}{x^3-8} = \frac{1}{x^2+2x+4} = \frac{1}{(x+1)^2+2}$$

And by lemma 5.8.1, we know:

$$\int \frac{dx}{(x+1)^2+2} = \frac{1}{\sqrt{2}} \cdot \tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$$

So in total, the integral is equal to:

$$\boxed{2x + \log\left|x^3-8\right| - \frac{1}{\sqrt{2}} \cdot \tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + C}$$

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Question 5.13:

Compute the following integral:

$$\int \frac{\cos(x) - \sin(x) + 1}{\cos(x) + \sin(x) + 1} \, dx$$

Answer:

Substituting $t \coloneqq \tan\left(\frac{x}{2}\right)$, the integral becomes (as per what we were taught in recitation):

$$\int \frac{1 - t^2 - 2t + 1 + t^2}{1 - t^2 + 2t + 1 + t^2} \cdot \frac{2}{1 + t^2} \, dt = \int \frac{1 - t}{1 + t} \cdot \frac{2}{1 + t^2} \, dt$$

Using partial fraction decomposition, we find that:

$$\frac{1 - t}{(1 + t)(1 + t^2)} = \frac{1}{1 + t} - \frac{t}{1 + t^2}$$

This integrates to

$$\log|1 + t| - \frac{1}{2} \log\left|1 + t^2\right|$$

So the integral becomes:

$$2 \log|1 + t| - \log\left|1 + t^2\right| = \log\left|\frac{(1 + t)^2}{1 + t^2}\right|$$

Now recall that $t = \tan\left(\frac{x}{2}\right)$, let $\alpha = \frac{x}{2}$. Notice:

$$\frac{(1 + \tan\alpha)^2}{1 + \tan^2\alpha} = \frac{(\cos\alpha + \sin\alpha)^2}{\cos^2\alpha + \sin^2\alpha} = (\cos\alpha + \sin\alpha)^2$$

So the integral is equal to:

$$\boxed{2 \log\left|\cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)\right| + C}$$

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Question 5.14:

Compute the following integral:

$$\int \frac{1}{x^4+1}\,dx$$

Answer:

Notice that this is equal to:

$$\frac{1}{2}\cdot\left(\frac{x^2+1}{x^4+1}-\frac{x^2-1}{x^4+1}\right)$$

And:

$$\int \frac{x^2+1}{x^4+1}\,dx = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx = \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}\,dx$$

Let $u = x - \frac{1}{x}$, this is equal to:

$$\int \frac{du}{u^2+2} = \frac{1}{\sqrt{2}}\cdot\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\cdot\tan^{-1}\left(\frac{x^2-1}{x\sqrt{2}}\right)$$

Similarly:

$$\int \frac{x^2-1}{x^4+1}\,dx = \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx = \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-2}$$

Substituting $u = x + \frac{1}{x}$ yields:

$$\int \frac{du}{u^2-2}$$

Using partial fraction decomposition, this is equal to:

$$\frac{1}{2\sqrt{2}}\int \frac{1}{u-\sqrt{2}}-\frac{1}{u+\sqrt{2}}\,du = \frac{1}{2\sqrt{2}}\cdot\log\left|\frac{u-\sqrt{2}}{u+\sqrt{2}}\right| = \frac{1}{2\sqrt{2}}\log\left|\frac{x^2-x\sqrt{2}+1}{x^2+x\sqrt{2}+1}\right|$$

Adding these together and dividing by two, we get:

$$\boxed{\frac{1}{4\sqrt{2}}\cdot\left(2\tan^{-1}\left(\frac{x^2-1}{x\sqrt{2}}\right)+\log\left|\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}\right|\right)+C}$$

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