Linear Algebra 2, Recitation 8

Final Part of Recitation 7

Definition 1

Let V be an inner product space, and $S \subseteq V$ a subset. Define S's **orthogonal complement** (**המרחב הניצב**) to be

$$S^\perp = \{v \in V \mid \forall u \in S\colon\ \langle u, v\rangle = 0\}$$

Exercise 2

Find the orthogonal complement of $S = \left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}\right\}$.

The orthogonal complement of S is

$$S^\perp = \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \in \mathbb{R}^3 \,\middle|\, \left\langle \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}x\\y\\z\end{pmatrix}\right\rangle = \left\langle \begin{pmatrix}0\\1\\2\end{pmatrix}, \begin{pmatrix}x\\y\\z\end{pmatrix}\right\rangle = 0\right\} = \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \in \mathbb{R}^3 \,\middle|\, \begin{cases} x + 2y + 3z = 0 \\ y + 2z = 0\end{cases}\right\}$$

This is just

$$N\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 2\end{pmatrix} = N\begin{pmatrix}1 & 0 & -1\\ 0 & 1 & 2\end{pmatrix} = \operatorname{span}\left(\begin{pmatrix}1\\-2\\1\end{pmatrix}\right)$$

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Exercise 3

Let U, W be subspaces of an inner product space V, prove

$$(U + W)^\perp = U^\perp \cap W^\perp$$

Suppose $v \in (U+W)^\perp$, let $u \in U$ and $w \in W$, then $u, w \in U + W$ so $\langle v, u\rangle = \langle v, w\rangle = 0$. Thus $v \in U^\perp \cap W^\perp$. Conversely, let $v \in U^\perp \cap W^\perp$, let $u + w \in U + W$ then $\langle v, u\rangle = \langle v, w\rangle = 0$ so $\langle v, u + w\rangle = 0$ by linearity. Thus $v \in (U + W)^\perp$ as required. ◊

Exercise 4

Let $A \in \mathbb{R}^{m\times n}$, find $C(A)^\perp$ and $C(A^\top)^\perp$.

We know that

$$C(A^\top) = \left\{A^\top w \,\middle|\, w \in \mathbb{R}^m\right\}$$

and so $C(A^\top)^\perp$ is the set of all vectors v such that for every $w \in \mathbb{R}^n$: $\left\langle A^\top w, v\right\rangle = (A^\top w)^\top v = w^\top A v$. Take in particular $w = e_i$, then this requires $e_i^\top A v = R_i(A)v = 0$. This precisely means that $v \in N(A)$. So we claim that $C(A^\top)^\perp = N(A)$, we have already shown one direction of the equality. Now suppose $Av = 0$ then for any w, $\left\langle A^\top w, v\right\rangle = w^\top A v = w^\top 0 = 0$. So we have shown equality.

Thus we also get

$$C(A)^\perp = N(A^\top)$$

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Note that $C(A^\top) = R(A)$, so we have in essence shown that $R(A)^\perp = N(A)$.

Recitation 8

Recall the following definition:

Definition 5

Let V be an inner product space, and $S = \{v_1, \ldots, v_n\} \subseteq V$, then S's **Gram Matrix** is

$$G_S := \begin{pmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & \cdots & \langle v_1, v_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle v_n, v_1 \rangle & \langle v_n, v_2 \rangle & \cdots & \langle v_n, v_n \rangle \end{pmatrix}$$

Last recitation we showed that G_S is invertible iff S is linearly independent. Now we show

Proposition 6

Let V be an inner product space and $B = \{v_1, \ldots, v_n\} \subseteq V$ a basis. Then for any $v, u \in V$:

$$\langle v, u \rangle = [v]_B^\top \cdot G_B \cdot \overline{[u]_B}$$

Proof: suppose $[v]_B = (\alpha_1, \ldots, \alpha_n)^\top$ and $[u]_B = (\beta_1, \ldots, \beta_n)^\top$. Then

$$\langle v, u \rangle = \left\langle \sum_i \alpha_i v_i,\ \sum_j \beta_j v_j \right\rangle = \sum_i \sum_j \alpha_i \overline{\beta}_j \langle v_i, v_j \rangle = \sum_i \sum_j \alpha_i \overline{\beta}_j (G_B)_{ij}$$

Now,

$$[v]_B^\top G_B \overline{[u]_B} = (\alpha_1, \ldots, \alpha_n) \begin{pmatrix} \sum_j \overline{\beta}_j (G_B)_{1j} \\ \vdots \\ \sum_j \overline{\beta}_j (G_B)_{nj} \end{pmatrix} = \sum_i \sum_j \alpha_i \overline{\beta}_j (G_B)_{ij}$$

as required. ■

Theorem 7 (Cauchy-Schwarz – קושי־שוורץ)

Let V be an inner product space, then for every $v, u \in V$:

$$|\langle v, u \rangle| \leq \|v\| \|u\|$$

and there is equality if and only if v and u are linearly dependent.

Exercise 8

Let $a_1, \ldots, a_n \in \mathbb{R}$, show that

$$(a_1 + \cdots + a_n)^2 \leq n(a_1^2 + \cdots + a_n^2)$$

Proof: let $V = \mathbb{R}^n$, and define $v = (a_1, \ldots, a_n)^\top$ and $\mathbf{1} = (1, \ldots, 1)^\top$. Then $\langle v, \mathbf{1} \rangle = a_1 + \cdots + a_n$, $\|v\| = \sqrt{a_1^2 + \cdots + a_n^2}$, and $\|\mathbf{1}\| = \sqrt{n}$. So by Cauchy-Schwarz,

$$|a_1 + \cdots + a_n| \leq \sqrt{n}\sqrt{a_1^2 + \cdots + a_n^2} \implies (a_1 + \cdots + a_n)^2 \leq n(a_1^2 + \cdots + a_n^2) \qquad \Diamond$$

Exercise 9

Let V be an inner product space and define $B_1 = \{u \in V \mid \|u\| = 1\}$. Show that for every $0 \neq v \in V$, $\min\{\|v - u\| \mid u \in B_1\}$ is given by v's normalization: $\hat{v} = \frac{v}{\|v\|}$.

Proof: note that minimizing $\|v-u\|$ is the same as minimizing $\|v-u\|^2$ since norms are nonnegative. For $\hat{v}$:

$$\|v-\hat{v}\|^2 = \left\|v\left(1-\frac{1}{\|v\|}\right)\right\|^2 = \left|1-\frac{1}{\|v\|}\right|^2 \|v\|^2 = \|v\|^2 - 2\|v\| + 1$$

And for a general $u \in B_1$:

$$\|v-u\|^2 = \langle v-u, v-u\rangle = \|v\|^2 - \langle v,u\rangle - \langle u,v\rangle + \|u\|^2$$

notice that $\langle v,u\rangle + \langle u,v\rangle = \langle v,u\rangle + \overline{\langle v,u\rangle} = 2\operatorname{Re}\langle v,u\rangle$. So we need to show that

$$\|v\|^2 - 2\|v\| + 1 \leq \|v\|^2 - 2\operatorname{Re}\langle v,u\rangle + 1 \iff \operatorname{Re}\langle v,u\rangle \leq \|v\|$$

This is indeed true, as by Cauchy-Schwarz:

$$\operatorname{Re}\langle v,u\rangle \leq |\langle v,u\rangle| \leq \|v\|\|u\| = \|v\| \qquad \Diamond$$

Theorem 10

Let $U \leq V$ be a subspace, then $V = U \oplus U^\perp$.

Exercise 11

Let $U \leq V$ be a subspace, show that $(U^\perp)^\perp = U$.

Proof: firstly, it is obvious that $U \subseteq (U^\perp)^\perp$ since for every $u \in U$, u is orthogonal to everything in $U^\perp$. Now, suppose $\dim V = n$ and $\dim U = k$, then

$$\dim V = \dim U + \dim U^\perp \implies \dim U^\perp = n-k$$

and

$$\dim V = \dim U^\perp + \dim(U^\perp)^\perp \implies \dim(U^\perp)^\perp = k = \dim U$$

so by considering dimensions, we have that $U = (U^\perp)^\perp$. $\Diamond$

Definition 12

Let V be an inner product space, and $W \leq V$ a subspace with an orthogonal basis $B = \{w_1, \ldots, w_n\}$. Then define the **projection map on** W (**היטל**) by

$$\pi_W \colon V \longrightarrow W, \qquad \pi_W(v) = \sum_{i=1}^{n} \frac{\langle v, w_i\rangle}{\|w_i\|^2} w_i$$

Theorem 13

The following hold for projection maps:

(1) $\pi_W(v) \in W$ for all $v \in V$ (i.e. π_W is well-defined).

(2) $\pi_W(v) = v \iff v \in W$.

(3) $\pi_W(v) = 0 \iff v \in W^\perp$.

(4) $v - \pi_W(v) \in W^\perp$.

(5) π_W is independent on choice of orthogonal basis.

Exercise 14

Let $V = \mathbb{R}^3$ and $W = \operatorname{span}\left(\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}2\\2\\-3\end{pmatrix}\right)$. Find the projection of $v = \begin{pmatrix}1\\1\\0\end{pmatrix}$ onto W.

Proof: the set $\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}2\\2\\-3\end{pmatrix}\right\}$ is orthogonal so

$$\pi_W(v) = \frac{\left\langle \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix} \right\rangle}{\left\| \begin{pmatrix}1\\2\\3\end{pmatrix} \right\|^2} \begin{pmatrix}1\\2\\3\end{pmatrix} + \frac{\left\langle \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\2\\-3\end{pmatrix} \right\rangle}{\left\| \begin{pmatrix}2\\2\\-3\end{pmatrix} \right\|^2} \begin{pmatrix}2\\2\\-3\end{pmatrix} = \frac{3}{14}\begin{pmatrix}1\\2\\3\end{pmatrix} + \frac{4}{12}\begin{pmatrix}2\\2\\-3\end{pmatrix} \qquad \Diamond$$

Definition 15

The **Gram-Schmidt process** (תהליך גראם־שמידט) is a process of converting a basis to an orthogonal basis. Suppose $B = \{v_1, \ldots, v_n\}$ is a basis, then we define $C = \{w_1, \ldots, w_n\}$ recursively as follows:

$$w_1 = v_1, \qquad w_{i+1} = v_{k+1} - \pi_{\operatorname{span}(w_1,\ldots,w_k)}(v_{k+1}) = v_{i+1} - \sum_{i=1}^{k} \frac{\langle v_{k+1}, w_i \rangle}{\|w_i\|^2} w_i$$

Using the Gram-Schmidt process, we see that every inner product space has an orthogonal, and thus orthonormal, basis.

Exercise 16

Let us define an inner product on $\mathbb{R}^3$ by

$$\left\langle \begin{pmatrix}x\\y\\z\end{pmatrix}, \begin{pmatrix}a\\b\\c\end{pmatrix} \right\rangle = xa + xb + ya + 2yb + zc$$

Find an orthogonal basis relative to this inner product.

Proof: let us take the standard basis as our initial basis. We will then perform the Gram-Schmidt process on it to convert it to an orthogonal basis. So $w_1 = (1, 0, 0)^\top$ and

$$w_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} - \frac{\left\langle \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix} \right\rangle}{\left\| \begin{pmatrix}1\\0\\0\end{pmatrix} \right\|^2} \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\1\\0\end{pmatrix}$$

and

$$w_3 = \begin{pmatrix}0\\0\\1\end{pmatrix} - \frac{\left\langle \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix} \right\rangle}{\left\| \begin{pmatrix}1\\0\\0\end{pmatrix} \right\|^2} \begin{pmatrix}1\\0\\0\end{pmatrix} - \frac{\left\langle \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\0\end{pmatrix} \right\rangle}{\left\| \begin{pmatrix}-1\\1\\0\end{pmatrix} \right\|^2} \begin{pmatrix}-1\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix} \qquad \Diamond$$

Exercise 17

Let V be an inner product space, and W a subspace, show that for every $v \in V$ and $w \in W$:

$$\|v - \pi_W(v)\| \leq \|v - w\|$$

and there is equality iff $w = \pi_W(v)$.

Proof: take an orthonormal basis of W (so that π_W looks nicer), $E_W = \{e_1, \dots, e_k\}$, and then extend it to an orthonormal basis of $E = \{e_1, \dots, e_k, \dots, e_n\}$ of V (why can we do this? Don't explain; it'll be in the homework.) Suppose $v = \sum_{i=1}^n \alpha_i e_i$, then

$$\pi_W(v) = \sum_{i=1}^k \langle v, e_i \rangle e_i = \sum_{i=1}^k \alpha_i e_i$$

let $w \in W$ and suppose $w = \sum_{i=1}^k \beta_i e_i$. Then by Pythagoras:

$$\|v - \pi_W(v)\|^2 = \left\| \sum_{i=k+1}^n \alpha_i e_i \right\| = \sum_{i=k+1}^n \alpha_i^2$$

and

$$\|v - w\|^2 = \left\| \sum_{i=1}^k (\alpha_i - \beta_i) e_i + \sum_{i=k+1}^n \alpha_i e_i \right\|^2 = \sum_{i=1}^k (\alpha_i - \beta_i)^2 + \sum_{i=k+1}^n \alpha_i^2$$

So we get that

$$\|v - \pi_W(v)\|^2 \leq \|v - 2\|^2$$

as required, and there is equality iff $\sum_{i=1}^k (\alpha_i - \beta_i)^2 = 0$ which is iff $\alpha_i = \beta_i$ for $1 \leq i \leq k$, which just means $w = \pi_W(v)$ as required. $\Diamond$