Infintesimal Calculus 3

Lecture 9, Sunday November 20, 2022 Ari Feiglin

9.1 Continuity Continued

Notice that f is continuous at x if and only if for every $x_n \longrightarrow x$ (and there exists such a sequence, since we can take $x_n = x$), $f(x_n) \longrightarrow f(x)$. Suppose f is continuous at x, let $\varepsilon > 0$ and $\delta > 0$ satisfy continuity. Then since $\rho(x_n, x) \longrightarrow 0$, so for some N for every $n \ge N$, $\rho(x_n, x) < \delta$ and so $\rho(f(x_n), f(x)) < \varepsilon$. And therefore $f(x_n) \longrightarrow f(x)$. To show the converse, suppose that f isn't continuous at x. Then there is an $\varepsilon > 0$ such that every $\delta > 0$ doesn't satisfy continuity. So for $\frac{1}{n}$ there is a x_n such that $\rho(x_n, x) < \frac{1}{n}$ but $\rho(f(x_n), f(x)) \ge \varepsilon$. So while $x_n \longrightarrow x$, $f(x_n)$ doesn't converge to f(x), in contradiction.

Proposition 9.1.1:

We define the function $\chi_k \colon \mathbb{R}^n \longrightarrow \mathbb{R}$ by $f_k(x_1, \dots, x_n) = x_k$. Then f_k is continuous.

Proof:

Suppose $x^m \longrightarrow x = (x_1, \dots, x_n)$. Then we must have that $x_k^m \longrightarrow x_k$ since pointwise convergence in \mathbb{R}^n is equivalent to convergence. And so $f_k(x^m) = x_k^m \longrightarrow x_k = f_k(x)$, and so f_k is continuous.

Example:

The function $f(x, y, z) = z^3 e^{\sin z}$ is continuous since if we define $g(z) = z^3 e^{\sin z}$, we have that $f(x, y, z) = g(f_3(x, y, z))$. And g and f_3 are continuous, and the composition of continuous functions is continuous, so f is continuous.

Proposition 9.1.2:

If $f, g: E \longrightarrow \mathbb{R}$ are continuous at $p \in E$ and if $c \in \mathbb{R}$ then the following are also continuous at p:

$$f + g$$

$$f + cg$$

$$f \cdot g$$

$$\frac{f}{g} \quad \text{If } g(p) \neq 0$$

This is trivial to prove.

Also notice that $f: X \longrightarrow \mathbb{R}^n$ is continuous if and only if $\chi_k \circ f$ is continuous for every $1 \leq k \leq n$. This is due to a proposition we proved in the previous lecture.

Example:

Circles are continuous. We define the function:

$$f: [0, 2\pi) \longrightarrow \mathbb{R}, \qquad t \mapsto (\cos(t), \sin(t))$$

which is the parametric representation of the unit circle. Since both of the coordinate functions, $\chi_1 \circ f$ and $\chi_2 \circ f$ (cos t and sin t respectively) are continuous, so is f. Notice that this parametric representation is the *counter-clockwise* representation. The clockwise representation, (cos t, $-\sin t$) is also continuous.

9.2 Surfaces

Definition 9.2.1:

If $\vec{v}, \vec{n} \in \mathbb{R}^k$ if $\vec{n} \neq 0$, the hyperplane normal to \vec{n} which contains \vec{v} is defined as:

$$H_{v,n} = \left\{ \vec{u} \in \mathbb{R}^k \mid n \cdot (u - v) = 0 \right\} = \left\{ \vec{u} \in \mathbb{R}^k \mid n \cdot u = n \cdot v \right\}$$

Notice that if $u \in H_{v,n}$ then $H_{v,n} = H_{u,n}$ and for every $\alpha \neq 0$, $H_{v,\alpha n} = H_{v,n}$. Further notice that the set $\{\vec{u} \in \mathbb{R}^k \mid n \cdot u = d\}$ for any $d \in \mathbb{R}$ defines a hyperplane. And if $u \in \mathbb{R}^n$ then $H_{v,n} + u = \{u + w \in \mathbb{R}^k \mid n \cdot (w - v) = 0\} = \{w \in \mathbb{R}^k \mid n \cdot (w - v - u) = 0\} = H_{v+u,n}$. And if $0 \in H_{v,n}$ then $H_{v,n}$ represents a subspace of \mathbb{R}^k whose dimension is k-1, and so $H_{v,n} - v = H_{0,n}$ is a k-1 dimension subspace of \mathbb{R}^k .

Definition 9.2.2:

If $f: \mathbb{R} \longrightarrow \mathbb{R}$ is a real-valued function, then its rotation around an axis, for example the z axis is defined as the set:

$$\left\{ (x, y, f(\sqrt{x^2 + y^2})) \mid x, y \in \mathbb{R} \right\}$$

If we intersect a rotation with some plane z = c then we get the set of points (x, y, c) where $f(\sqrt{x^2 + y^2}) = c$. This is either empty or contains circles whose radii are in $f^{-1}(c)$.

Example:

If $f(x) = \alpha x$, then rotating it around z creates the set: $\{(x, y, \alpha \sqrt{x^2 + y^2}) \mid x, y \in \mathbb{R}\}$ which is called a cone. And for $f(x) = \alpha x^2$, rotating it gives $z = \alpha (x^2 + y^2)$, this is called an elliptical parabola.

Example:

If we look at $f(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2}$, z = f(x,y). Cutting this at z = c yields an ellipse, and thus this is not a rotation. Cutting this at y = 0 gives $z = \frac{x^2}{a^2}$ which is a parabola.