Introduction to Rings and Modules

Lecture 20, Thursday June 22 2023
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Recall the product of ideals is given by

$$IJ = \left\{ \sum_{i=1}^n a_i b_i \;\middle|\; a_i \in I,\, b_i \in J \right\}$$

and thus inductively

$$I_1 \cdots I_k = \left\{ \sum_{i=1}^n a_i^1 \cdots a_i^k \;\middle|\; a_i^j \in I_j \right\}$$

And suppose M, N are sub-R-modules (of some larger R-module), then so is

$$MN = \left\{ \sum_{i=1}^k m_i n_i \;\middle|\; m_i \in M,\, n_i \in N \right\}$$

Lemma 20.0.1:

Suppose R is left/right noetherian and $\mathscr{S}$ is a set of left/right ideals. Then R contains a maximal ideal.

Proof:

Since every chain in $\mathscr{S}$ has a maximal element by R being noetherian, Zorn's lemma tells us $\mathscr{S}$ has a maximal element. ■

Lemma 20.0.2:

Suppose R is a noetherian ring, and $0 \neq I \trianglelefteq R$. Then I contains a product of non-zero prime ideals.

Proof:

Let

$$\mathscr{S} = \{0 \neq I \trianglelefteq R \mid I \text{ does not contain a product of non-zero prime ideals}\}$$

we must prove $\mathscr{S} = \varnothing$.
Suppose not, then by the previous lemma $\mathscr{S}$ has a maximal element, J. J cannot be prime as then it would contain a product of primes (itself). Since J is not prime there exist ideals I_1 and I_2 which are not subsets of J, but $I_1 I_2 \subseteq J$. Thus J is a proper subset of $I_1 + J$ and $I_2 + J$, and so by J's maximality $I_1 + J, I_2 + J \notin \mathscr{S}$. Thus

$$P_1 \cdots P_n \subseteq I_1 + J, \quad Q_1 \cdots Q_m \subseteq I_2 + J$$

where P_i and Q_i are prime. But $(I_1 + J)(I_2 + J) = I_1 I_2 + I_1 J + J I_2 + J^2$, and since $I_1 I_2 \subseteq J$ by our definition of I_1 and I_2, and $I_1 J, J I_2, j^2 \subseteq J$ since J is an ideal, we have that $(I_1 + J)(I_2 + J) \subseteq J$ and so

$$P_1 \cdots P_n Q_1 \cdots Q_m \subseteq J$$

meaning $J \notin \mathscr{S}$ in contradiction. ■

Definition 20.0.3:

Suppose R is an integral domain, and $(0) \neq P \triangleleft R$ is a prime ideal. We define

$$P^{-1} = \{z \in \mathrm{Frac}(R) \mid zP \subseteq R\}$$

P^{-1} is closed under R-scalar multiplication and is thus an R-submodule of $\mathrm{Frac}(R)$. Since P is an ideal, $R \subseteq P^{-1}$.

Lemma 20.0.4:

Suppose R is a noetherian integral domain, and $\dim R = 1$. Suppose $(0) \neq P \triangleleft R$ a prime ideal, then $R \subset P^{-1}$.

Proof:

Let $0 \neq y \in P$ and $(0) \neq I = (y) = Ry$. By the previous lemma there exist P_i prime such that

$$P_1 \cdots P_n \subseteq I \subseteq P$$

we assume n is minimal. Since P is prime, there exists an i such that $P_i \subseteq P$, without loss of generality $i = n$. But P_n and P are non-zero ideals and since $\dim R = 1$ this means they are maximal so $P_n = P$. By the minimality of n,

$$P_1 \cdots P_{n-1} \nsubseteq I$$

and so there exists an $x \in P_1 \cdots P_{n-1}$ where $x \notin I$. Let $z = \frac{x}{y} \in \operatorname{Frac}(R)$. Note that $z \notin R$ since if so $x = zy \in (y) = I$ in contradiction.
We will show $z \in P^{-1}$, meaning $zP \subseteq R$. Let $a \in P$, so

$$za = \frac{xa}{y}$$

since $x \in P_1 \cdots P_{n-1}$ and $a \in P = P_n$, we have $xa \in P_1 \cdots P_n \subseteq I = Ry$. So there exists an $r \in R$ such that $xa = ry$ and so $\frac{xa}{y} = r \in R$ meaning $za \in P$ as required. ■

Lemma 20.0.5:

Suppose R is a Dedekind domain and $(0) \neq I \trianglelefteq R$. Suppose $I \subseteq P$ for P prime ideal. Then $I \subset IP^{-1}$.

Proof:

Since $1 \in R \subset P^{-1}$, for every $x \in I$, $x = x \cdot 1 \in IP^{-1}$ as required. Now suppose $I = IP^{-1}$. Let $z \in P^{-1}$ such that $z \notin R$, and since I is closed under multiplication by z, it is closed under multiplication by any exponent of z. Since I is closed under multiplication by R, it has the structure of an $R[z]$ module.
Since R is noetherian, I is finitely generated as an ideal, meaning an R-module. We will show that I is a faithful $R[z]$-module. Suppose

$$b \in \operatorname{Ann}_{R[z]}(I)$$

then $ba = 0$ for every $i \in I$. But since $R[z], I \subseteq \operatorname{Frac}(R)$ which is a field and thus has no zero divisors and $I \neq (0)$ so $b = 0$, meaning

$$\operatorname{Ann}_{R[z]}(I) = (0)$$

Thus I is a faithful finitely generated $R[z]$-module, and thus z is integral over R. But $z \in \operatorname{Frac}(R)$, and R is a Dedekind domain so it is integrally closed, and so $z \in R$. Thus $\operatorname{Frac}(R) = R$, and so $z \in R$. But $z \notin R$ in contradiction.

■

Lemma 20.0.6:

Suppose R is a Dedekind domain and P a non-trivial prime ideal. Then $PP^{-1} = R$.

Proof:

By definition $PP^{-1} \subseteq R$. Since P and P^{-1} are sub-R-modules of $\operatorname{Frac}(R)$, we have PP^{-1} is an R module, and thus an ideal. But we know $P \subset PP^{-1}$, but since P is non-trivial and $\dim R = 1$ so P is maximal, thus $PP^{-1} = R$.

Theorem 20.0.7:

Suppose R is a Dedekind domain and $(0) \neq I \trianglelefteq R$ is a non-zero ideal. Then I factorizes uniquely as a product of non-trivial prime ideals:

$$I = P_1 \cdot P_2 \cdots P_n$$

Proof:

Let us first prove the existence of such a factorization. Let us define

$$\mathscr{S} = \{(0) \neq I \trianglelefteq R \mid I \text{ does not factorize as a product of non-trivial prime ideals}\}$$

suppose that $\mathscr{S} \neq \varnothing$. Thus since R is noetherian, $\mathscr{S}$ has a maximal ideal I, and I is contained in a maximal (and thus prime) ideal $I \subseteq P$. Since $I \subset IP^{-1}$, but $IP^{-1} \subseteq PP^{-1} = R$, thus IP^{-1} is an ideal of R. Since $I \subset IP^{-1}$ and I is maximal, $IP^{-1} \notin \mathscr{S}$ so IP^{-1} factorizes, suppose

$$IP^{-1} = P_1 \cdots P_n$$

but then

$$P_1 \cdots P_n P = IP^{-1}P = IR = I$$

meaning $I \notin \mathscr{S}$ in contradiction.
Thus $\mathscr{S} = \varnothing$ and so every non-zero ideal factorizes as a product of non-trivial prime ideals.
Now suppose that

$$I = P_1 \cdots P_n = Q_1 \cdots Q_m$$

we will assume without loss of generality that $n \leq m$. We will prove on induction of n that this factorization is unique.
If $n = 0$ then suppose $m > 0$, so $I = R = Q_1 \cdots Q_m$ but $I \subseteq Q_1 \subset R$ which is a contradiction.
Now suppose that the factorization is unique for n, then suppose

$$P_1 \cdots P_n \cdot P_{n+1} = Q_1 \cdots Q_m$$

Thus $Q_1 \cdots Q_m \subseteq P_{n+1}$, but since P_{n+1} is prime there exists an i such that $Q_i \subseteq P_{n+1}$. But $\dim R = 1$ and Q_i and P_{n+1} are non-trivial so $Q_i = P_{n+1}$. Thus

$$P_1 \cdots P_{n+1} = Q_1 \cdots Q_m = (Q_1 \cdots Q_{i-1} \cdot Q_i \cdots Q_m)P_{n+1}$$

multiplying both sides by P_{n+1}^{-1} we get

$$P_1 \cdots P_n = Q_1 \cdots Q_{i-1} \cdot Q_i \cdots Q_m$$

and thus by our inductive assumption, $m = n + 1$ and the above products have the same coefficients. ■

Proposition 20.0.8:

Suppose R is a Dedekind domain, then R is a UFD if and only if R is a PID.

Proof:

If R is a PID, then R is a UFD. For the converse, suppose R is a Unique factorization Dedekind domain. Let P be a prime ideal of R, if $P = (0)$ then P is of course principal. Otherwise suppose $0 \neq y \in P$ then since R is a UFD, there exist p_i irreducible such that

$$y = p_1 \cdots p_n$$

but P is prime, and so there exists a $p_i \in P$. Since in a UFD every irreducible element is prime, p_i is prime and so $(p_i) \subseteq P$ is prime. But since $0 \neq p_i$, $(0) \neq (p_i) \subseteq P$ and $\dim R = 1$ so $P = (p_i)$.
Thus every prime ideal is principal. If I is any ideal, then since it is the product of prime ideals, which are of the form (p_i), we have

$$I = (p_1) \cdots (p_n) = (p_1 \cdots p_n)$$

and so R is a PID. ■