Algebraic Topology I

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Contents

1 Categories

1.0.1 Definition

A **category** $\mathcal{C}$ is a mathematical object which contains the following

(1) a class of objects $\mathrm{ob}(\mathcal{C})$ (the objects need not be sets),

(2) for every two objects $A, B \in \mathrm{ob}(\mathcal{C})$ a class of **morphisms** $\mathrm{Mor}(A, B)$,

(3) an operation on morphisms $\circ$, where for every $f \in \mathrm{Mor}(A, B)$ and $g \in \mathrm{Mor}(B, C)$, $g \circ f \in \mathrm{Mor}(A, C)$,

(4) for every object $A \in \mathrm{ob}(\mathcal{C})$ there exists an identity morphism $1_A \in \mathrm{Mor}(A, A)$ where for every $A, B \in \mathrm{ob}(\mathcal{C})$ and $f \in \mathrm{Mor}(A, B)$, $f \circ 1_A = 1_B \circ f = f$,

(5) for every $A, B, C, D \in \mathrm{ob}(\mathcal{C})$ and $f \in \mathrm{Mor}(A, B), g \in \mathrm{Mor}(B, C), h \in \mathrm{Mor}(C, D)$, there is associativity: $(h \circ g) \circ f = h \circ (g \circ f)$.

Although morphisms are not necessarily functions, we use similar notation: both $f\colon A \longrightarrow B$ and $A \xrightarrow{f} B$ are to be understood to mean $f \in \mathrm{Mor}(A, B)$. And we write $A \in \mathcal{C}$ to mean $A \in \mathrm{ob}(\mathcal{C})$.

Notice that for every $A \in \mathcal{C}$, 1_A is unique: suppose 1_A and $1'_A$ are both identity morphisms then $1_A \circ 1'_A = 1_A$ since $1'_A$ is an identity, but $1_A \circ 1'_A = 1'_A$ since 1_A is an identity so $1_A = 1'_A$.

1.0.2 Definition

Suppose $\mathcal{C}$ and $\mathcal{D}$ are two categories, a **functor** F from $\mathcal{C}$ to $\mathcal{D}$ is a correspondence where for every $A \in \mathcal{C}$ there is defined a single $F(A) \in \mathcal{D}$, and for every $f \in \mathrm{Mor}(A, B)$ there exists a unique $F(f) \in \mathrm{Mor}(F(A), F(B))$ such that for all $A, B, C \in \mathcal{C}$ and $f \in \mathrm{Mor}(A, B)$ and $g \in \mathrm{Mor}(B, C)$ we have that $F(g \circ f) = F(g) \circ F(f)$ and $F(1_A) = 1_{F(A)}$.

1.0.3 Example

The following are examples of categories:

(1) The category of all groups, morphisms are taken to be homomorphisms between groups;

(2) The category of all topological spaces, morphisms are taken to be homeomorphisms;

(3) The category of all sets, the morphisms are taken to be set functions;

(4) The category of pairs of topological spaces: the objects are of the form (X, A) where X is a topological space and $A \subseteq X$. Morphisms between (X, A) and (Y, B) of this category are continuous functions f between X and Y such that $f(A) \subseteq B$.

(5) The category of pointed topological spaces: the objects are (X, a) where X is a topological space and $a \in X$ and the morphisms between (X, a) and (Y, b) are continuous functions between X and Y such that $a \mapsto b$.

An example of a functor is the so-called *forgetful functor* from the category of topological spaces to the category of sets: map a topological to itself as a pure set.

This course will focus on a specific functor between the category of pointed topological spaces to the category of groups.

1.0.4 Definition

Let $\mathcal{C}$ be a category, and $A, B \in \mathcal{C}$. A morphism $f\colon A \longrightarrow B$ is an **isomorphism** if there exists a morphism $g\colon B \longrightarrow A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$. Such a g is called the **inverse** of f and is denoted f^{-1}

(notice that by symmetry the inverse is also an isomorphism). If there exists an isomorphism between A and B, we denote this by $A \cong B$ and A and B are called **isomorphic**.

Inverses are unique: if g_1 and g_2 are inverses of f then $(g_1 \circ f) \circ g_2 = 1_A \circ g_2 = g_2$ but $g_1 \circ (f \circ g_2) = g_1 \circ 1_B = g_1$ and by associativity these are equal. Furthermore the composition of isomorphisms is an isomorphism: it is easily verified that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$. Notice that 1_A is an isomorphism and it is its own inverse.

1.0.5 Proposition

A functor maps isomorphisms to isomorphisms, in particular $F(f^{-1}) = F(f)^{-1}$ if $f\colon A \longrightarrow B$ is an isomorphism.

Proof: notice that $F(f) \circ F(f^{-1}) = F(f \circ f^{-1}) = F(1_B) = 1_{F(B)}$ and $F(f^{-1}) \circ F(f) = F(f^{-1} \circ f) = F(1_A) = 1_{F(A)}$. So $F(f^{-1})$ is indeed the inverse of $F(f)$. ∎

1.1 Homotopies

1.1.1 Definition

Let X and Y are topological spaces and $f, g\colon X \longrightarrow Y$ (meaning they are morphisms, thus continuous). We say that f is homotopic to g, denoted $f \sim g$, if there exists an $H\colon X \times I \longrightarrow Y$ ($I = [0,1]$, $X \times I$ is the product topology) such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. We denote $h_t(x) := H(x,t)$, and H is called a **homotopy** from f to g.

A homotopy is essentially a smooth mapping from one morphism f to another g. Homotopy is indeed an equivalence relation: firstly $f \sim f$ as we can define $H(x,t) = f(x)$ which is continuous as the composition of continuous functions ($H = f \circ \pi_1$), if $f \sim g$ then define $H'(x,t) = H(x, 1-t)$ which is also continuous (since $(x,t) \mapsto (x, 1-t)$ is continuous since its components are) and $H'(x,0) = g(x)$ and $H'(x,1) = f(x)$ so $g \sim f$, and if H_1 is a homotopy from f to g and H_2 is a homotopy from g to h, define

$$H(x,t) = \begin{cases} H_1(x,2t) & 0 \le t \le \frac{1}{2} \\ H_2(x,2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

$X \times [0,1/2]$ and $X \times [1/2,1]$ are closed (since $X \times [0,1/2]$ is the preimage of $[0,1/2]$ in the mapping $(x,t) \mapsto t$) and $H(x,t)$ is continuous on both of these (since $H_1(x,2t)$ and $H_2(x,2t-1)$ are continuous), so $H(x,t)$ is continuous.

1.1.2 Proposition

For every topological space X and every two morphisms $f, g\colon X \longrightarrow \mathbb{R}^n$, f and g are homotopic.

Proof: define $H(x,t) = (1-t)f(x) + tg(x)$ (addition and scalar multiplication are continuous). ∎

1.1.3 Definition

A topological space X is **contractible** if the identity map id_X is homotopic to some constant map.

Notice that all two constant maps are homotopic if and only if the space is path connected. If all two constant maps are homotopic, for $x_1, x_2 \in X$ let $H(x,t)$ be a homotopy from x_1 to x_2 and define $\gamma(t) = H(x_0,t)$ for any $x_0 \in X$, this is a continuous path from x_1 to x_2. And if X is path connected, for x_1 and x_2 and γ connecting them, define $H(x,t) = \gamma(t)$.

1.1.4 Proposition

Let X, Y, Z be topological spaces, $f, f' \colon X \longrightarrow Y$ and $g, g' \colon Y \to Z$ such that $f \sim f'$ and $g \sim g'$, then $g \circ f \sim g' \circ f'$.

Proof: let H be a homotopy from f to f' and K a homotopy from g to g'. Then define $J(x,t) = K(H(x,t),t)$ which is a composition of continuous functions (map (x,t) to $((x,t),t)$ to $(H(x,t),t)$ to $K(H(x,t),t)$). ■

We call the equivalence classes of morphisms under $\sim$ *homotopy classes*, and the homotopy class of a morphism f is denoted $[f]$. So by above, $[f] \circ [g] := [f \circ g]$ is a well-defined operation. This gives us a new category whose objects are topological spaces and morphisms are homotopy classes. What are the isomorphisms in this category? Well the identities are obviously $[1_X]$ since $[f] \circ [1_X] = [f \circ 1_X] = [f]$ and $[1_X] \circ [g] = [1_X \circ g] = [g]$. So an isomorphism $X \xrightarrow{[f]} Y$ is a homotopy class such that there exists a $Y \xrightarrow{[g]} X$ such that $[f] \circ [g] = [f \circ g] = [1_X]$ and $[g \circ f] = [1_Y]$. We give these isomorphisms a different name:

1.1.5 Definition

Let X and Y be topological spaces, then $f \colon X \longrightarrow Y$ is a **homotopic equivalence** if there exists a $g \colon Y \longrightarrow X$ such that $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$. If a homotopy equivalence exists between X and Y, then X and Y are said to be **homotopy equivalent**, denoted $X \simeq Y$.

Notice that homeomorphisms are homotopic equivalences, since $f^{-1} \circ f = \mathrm{id}_X$ and $f \circ f^{-1} = \mathrm{id}_Y$.

1.1.6 Definition

Let X and Y be topological spaces, $A \subseteq X$, and $f, g \colon X \longrightarrow Y$. We say that f and g are homotopic relative to A, denoted $f \overset{A}{\sim} g$, if there exists a homotopy H from f to g such that $H(a,t) = f(a)$ for all $a \in A$ and $t \in I$. In such a case we must have $f\big|_A = g\big|_A$.

It is not enough for $f \sim g$ and $f\big|_A = g\big|_A$ for f and g to be homotopic relative to A. For example take I and S^1 and the points 0 and 1 on I. Then we can continuously deform I so that it maps onto the bottom or top of the circle. These are two continuous mappings which are homotopic, but no homotopy between them which keeps the image of 0 and 1 constant.

Notice that $\overset{A}{\sim}$ is an equivalence relation, the proof of this is analogous to the proof that homotpy is an equivalence relation. It also preserves composition, if $f, f' \colon (X, A) \longrightarrow (Y, B)$ (meaning they are morphisms from X to Y and $f(A), f'(A) \subseteq B$) and $g, g' \colon (Y, B) \longrightarrow (Z, C)$ such that $f \overset{A}{\sim} f'$ and $g \overset{B}{\sim} g'$, then $g \circ f \overset{A}{\sim} g' \circ f'$.

1.1.7 Definition

Let X be a topological space. $A \subseteq X$ is called a **retract** if there exists an $r \colon X \longrightarrow A$ such that $r \circ \iota = \mathrm{id}_A$ where $\iota \colon A \longrightarrow X$ is the inclusion map. In other words $r(a) = a$ for all $a \in A$. r is called a **retraction**.

For example $\partial I = \{0, 1\}$ is not a retraction of I since every continuous image of I must be connected, and ∂I is not. But if we take X to be an eight shape, and A its bottom circle, then we can map the top circle to the middle point and A to itself and this is a retraction.

1.1.8 Definition

$A \subseteq X$ is called a **deformation retract** if there exists a retraction r such that $\iota \circ r \overset{A}{\sim} \mathrm{id}_X$.

Instead of requiring r be a retraction, we can require only that $r(X) \subseteq A$. Since then if $\iota \circ r \overset{A}{\sim} \mathrm{id}_X$, this means that $r(a) = \mathrm{id}_X(a) = a$ for all $a \in A$ so it is already a retraction. Explicitly, this is equivalent to saying that there exists a homotopy $H \colon X \times I \longrightarrow X$ such that $H(x, 0) = x$ for all $x \in X$, $H(a, t) = a$ for all $a \in A, t \in I$, $H(x, 1) \in A$ for all $x \in X$.

Notice that if $A \subseteq X$ is a deformation retract then $\iota \colon A \longrightarrow X$ is a homotopy equivalence, since $r \circ \iota = \mathrm{id}_A$ and $\iota \circ r \sim \mathrm{id}_X$.

1.1.9 Example

Let $X = \mathbb{R}^n \setminus \{0\}$ and $A = S^{n-1} \subseteq \mathbb{R}^n \setminus \{0\}$. Then $r(x) := \frac{x}{\|x\|}$ is a retraction with the homotopy $H(x,t) = (1-t)x + t\frac{x}{\|x\|}$. This is the homotopy we used to show that all morphisms to $\mathbb{R}^n$ are homotopic.

A morphism f is called *null-homotopic* if it is homotopic to a constant morphism.

1.1.10 Proposition

Let X be a topological space and $f\colon S^1 \longrightarrow X$, then the following are equivalent

(1) f is null-homotopic,

(2) f is null-homotopic relative to any point on S^1,

(3) f can be expanded to a morphism on D^2 (the disk in $\mathbb{R}^2$), meaning there exists an $F\colon D^2 \longrightarrow X$ such that $F\big|_{S'} = f$.

(2) $\implies$ (1) is trivial since a null-homotopy relative to a point is still a null-homotopy. (3) $\implies$ (2): let $\iota\colon S^1 \longrightarrow D^2$ be the inclusion map, and let $a \in S^1$, define the homotopy $H\colon S^1 \longrightarrow I \longrightarrow D^2$ by $H(x,t) = (1-t)\iota(x) + ta$, which is a homotopy from ι to the constant map K_a. Then $F \circ H$ is a null-homotopy between f and $K_{f(a)}$ (since $F \circ H(x,0) = F(x) = f(x)$ and $F \circ H(x,1) = F(a)$) relative to a since $F \circ H(a,t) = F(a)$. (1) $\implies$ (3): so there exists a homotopy $H\colon S^1 \times I \longrightarrow X$ such that $H(x,0) = f(x)$ for every $x \in S^1$ and there exists a $p \in X$ such that $H(x,1) = p$ for all $x \in S^1$. Let us define $\rho\colon S^1 \times I \longrightarrow D^2$ by $\rho(x,t) = (1-t)x$, this is a continuous map from a compact (since S^1 and I are compact and therefore so is their product) to a Hausdorff space, and so it is closed. And it is surjective, so it is a quotient map. So D^2 is the quotient space of $S^1 \times I$ with respect to ρ, and H respects ρ, since $\rho(x,t) = \rho(y,s)$ implies $(1-t)x = (1-s)y$ and this means that either $(x,t) = (y,s)$ or $t = s = 1$. But in both cases $H(x,t) = H(y,s)$, and so there exists an $F\colon D^2 \longrightarrow X$ which is continuous such that $H = F \circ \rho$, meaning $F(x) = H(x,0) = f(x)$ as required. ∎

This proof uses the fact that if ρ is a quotient map, and $f\colon X \longrightarrow Y$ is continuous then there exists a $F\colon \overline{X} \longrightarrow Y$ such that $f = F \circ \rho$ if and only if $\rho(a) = \rho(b)$ implies $f(a) = f(b)$.

1.1.11 Definition

Let X be a topological space, and for every $a, b \in X$ define Γ_{ab} to be the set of all paths from a to b, which are continuous maps $I \longrightarrow X$. On Γ_{ab} we take the equivalence relation of homotopy relative to $\partial I = \{0,1\}$. Take $\hat{\Gamma}_{ab}$ to be the partition defined by this relation, ie. $\hat{\Gamma}_{ab} = {}^{\Gamma_{ab}}\big/_{\overset{\partial I}{\sim}}$.

If $[\gamma] \in \hat{\Gamma}_{ab}$ and $[\delta] \in \hat{\Gamma}_{bc}$ then we define $[\gamma][\delta] := [\gamma * \delta]$ (their concatenation).

We must show that this is well-defined, meaning we must show that if $\gamma \overset{\partial I}{\sim} \gamma'$ and $\delta \overset{\partial I}{\sim} \delta'$ then $\gamma * \delta \overset{\partial I}{\sim} \gamma' * \delta'$. So let $H\colon I \times I \longrightarrow X$ be a homotopy relative to ∂I between γ and γ', and $G\colon I \times I \longrightarrow X$ between δ and δ'. Then define

$$K(s,t) := \begin{cases} H(2s,t) & 0 \leq s \leq \frac{1}{2} \\ G(2s-1,t) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

this is continuous, $K(0,t) = H(0,t) = 0$ and $K(1,t) = G(1,t) = 1$ so it is a homotopy between the concatenations relative to ∂I.

Notice that concatenation is not necessarily associative, since in $(\gamma * \delta) * \varepsilon$, the speed of γ and δ is quadrupled while in $\gamma * (\delta * \varepsilon)$, γ's speed is only doubled. But it is the case that $[\gamma]\big([\delta][\varepsilon]\big) = \big([\gamma][\delta]\big)[\varepsilon]$, so in homotopy concatenation is associative. So we need to prove $\gamma(\delta\varepsilon) \overset{\partial I}{\sim} (\gamma\delta)\varepsilon$, the idea behind this is that for every x and y where $\gamma(\delta\varepsilon)(x) = (\gamma\delta)\varepsilon(y)$, define in $I \times I$ the line between $(x,0)$ and $(y,1)$. These lines cover $I \times I$ and for every point (t,s) which is on the line from $(x,0)$ map it to $\gamma(\delta\varepsilon)(x)$.

We can prove in a similar manner that for $\gamma \in \Gamma_{ab}$, $[K_a][\gamma] = [\gamma][K_b] = [\gamma]$.

And so we have defined a category. The objects of this category are the points $a \in X$ and the morphisms between a and b are $\hat{\Gamma}_{ab}$ (notice that $[\gamma] \in \hat{\Gamma}_{ab}$ can be composed with elements from $\hat{\Gamma}_{bc}$, so the order of composition is reversed). Here the identity morphisms are $[K_a]$.

Notice that every morphism in this category is an isomorphism. This is since for every $\gamma \in \Gamma_{ab}$ we defined its reverse $\overline{\gamma} \in \Gamma_{ba}$ by $\overline{\gamma}(t) := \gamma(1-t)$.

1.1.12 Proposition

$[\gamma][\overline{\gamma}] = [K_a]$ and $[\overline{\gamma}][\gamma] = [K_b]$.

The idea is that at time t we take the path γ but not all the way, then wait, then take the reverse path $\overline{\gamma}$. So

$$H(x,t) = \begin{cases} \gamma(2x) & 0 \le x \le \frac{1-t}{2} \\ \gamma(1-t) & \frac{1-t}{2} \le x \le \frac{1+t}{2} \\ \gamma(2-2x) & \frac{1+t}{2} \le x \le 1 \end{cases}$$

is a homotopy from $\gamma * \overline{\gamma}$ to K_a. ■

1.1.13 Definition

A **groupoid** is a small category (a category whose objects form a set, not a pure class) such that every morphism is an isomorphism. If $\mathcal{C}$ is a groupoid, then $\mathrm{Mor}(A,A)$ is then a group for every $A \in \mathcal{C}$.

1.1.14 Definition

Given a pointed topological space (X,a) (call a the basis point), define the **first homotopy group** (or the **fundamental group**) $\pi_1(X,a) := \hat{\Gamma}_{aa}$. And given a morphism $f\colon (X,a) \longrightarrow (Y,b)$ (meaning f is continuous and $f(a)=b$), then we define a group homomorphism $f_*\colon \pi_1(X,a) \longrightarrow \pi_1(Y,b)$ by $f_*([\gamma]) = [f\circ\gamma]$. The correspondence $(X,a) \mapsto \pi_1(X,a)$ and $f \mapsto f_*$ is a functor.

We need to show that f_* is well-defined and also a group homomorphism. To show that it is well-defined, suppose $\gamma \overset{\partial I}{\sim} \delta$, then we must show $f\circ\gamma \overset{\partial I}{\sim} f\circ\delta$. Now we showed that if $f \overset{A}{\sim} f'$ and $g \overset{B}{\sim} g'$ such that $f(A), f'(A) \subseteq B$ then $g \circ f \overset{A}{\sim} g' \circ f'$. And we have that $\gamma \overset{\partial I}{\sim} \delta$ and $f \overset{\{a\}}{\sim} f$ so $f \circ \gamma \overset{\partial I}{\sim} f \circ \delta$ as required. Now we must show that f_* is a homomorphism, ie.

$$f_*([\gamma][\delta]) = [f\circ(\gamma*\delta)] = [(f\circ\gamma)*(f\circ\delta)] = f_*([\gamma])f_*([\delta])$$

Actually a stronger result holds, $f \circ *(\gamma * \delta) = (f\circ\gamma)*(f\circ\delta)$, as both are given by

$$\begin{cases} f\circ\gamma(2t) & 0 \le t \le \frac{1}{2} \\ f\circ\delta(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

To finish the proof that the correspondence is a functor, we need to show that $(g\circ f)_* = g_* \circ f_*$ and $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X,a)}$. We do so directly:

$$(g\circ f)_*([\gamma]) = [(g\circ f)\circ\gamma] = [g\circ(f\circ\gamma)] = g_*([f\circ\gamma]) = g_*\circ f_*([\gamma])$$

and

$$(\mathrm{id}_X)_*([\varphi]) = [\mathrm{id}_X\varphi] = [\varphi]$$

so $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X,a)}$ as required. Thus we have defined a functor from the category of pointed topological spaces to the category of groups.

1.1.15 Proposition

Let X be a topological space, $a \in X$, and A be a's connected component. Let $\iota\colon A \longrightarrow X$ be the inclusion map, then $\iota_*\colon \pi_1(A,a) \longrightarrow \pi_1(X,a)$ is an isomorphism.

Proof: ι_* is injective: $\iota_*([\gamma]) = [K_a]$ if and only if $[\iota\circ\gamma] = [K_a]$, which means $\iota\circ\gamma \overset{\partial I}{\sim} K_a$, let H be such a homotopy. Then $H\colon I\times I \longrightarrow X$, but we claim that H's image is contained in A so that it is also a homotopy $\gamma \overset{\partial I}{\sim} K_a$, meaning $[\gamma] = [K_a]$. Suppose not, that $H(t_0,s_0) \notin A$, then define $\delta(t) = H(t_0\cdot t, s_0)$. Then

$\delta(0) = H(0, s_0) = a$ since H is a homotopy relative to ∂I, and $\delta(1) = H(t_0, s_0) \notin A$, so a is connected to a value not in A, in contradiction. So ι_* is indeed injective.

Now suppose $[\gamma] \in \pi_1(X, a)$, meaning γ is a path connecting a to itself in X. But every point in γ's image must also be connected to a, meaning γ is a path connecting a to itself contained within A. So there exists a $\gamma' \in \Gamma^A_{aa}$ such that $\gamma = \iota \circ \gamma'$ and in particular $\iota_*([\gamma']) = [\gamma]$ as required. So ι_* is a bijective homomorphism, an isomorphism. ∎

Suppose $a, b \in X$ such that there exists a path between them, $\gamma\colon I \longrightarrow X, \gamma(0) = a, \gamma(1) = b$. Let us define

$$F_\gamma\colon \pi_1(X, a) \longrightarrow \pi_1(X, b), \qquad F_\gamma[\varphi] = [\overline{\gamma} * \varphi * \gamma] = [\overline{\gamma}][\varphi][\gamma]$$

so $F_\gamma[\varphi]$ is the class of curves homotopic to the curve obtained by walking from b along γ backward to a, then going back along γ to b. Notice that $F_\gamma[\varphi] = [\gamma]^{-1}[\varphi][\gamma]$, so F_γ is simply conjugation by $[\gamma]$.

In general, suppose $\mathcal{G}$ is a groupoid and let $A, B \in \mathcal{G}$ such that $\mathrm{Mor}(A, B) \neq \varnothing$. Then $\mathrm{Mor}(A, A)$ and $\mathrm{Mor}(B, B)$ are isomorphic groups. Let $\varphi \in \mathrm{Mor}(A, B)$ and define $F_\varphi\colon \mathrm{Mor}(A, A) \longrightarrow \mathrm{Mor}(B, B)$ by $F_\varphi(\varkappa) = \varphi \circ \varkappa \circ \varphi^{-1}$. This is a group homomorphism:

$$F_\varphi(\varkappa_1) \circ F_\varphi(\varkappa_2) = \varphi \circ \varkappa_1 \circ \varphi^{-1} \circ \varphi \circ \varkappa_2 \circ \varphi^{-1} = \varphi \circ (\varkappa_1 \circ \varkappa_2) \circ \varphi^{-1} = F_\varphi(\varkappa_1 \circ \varkappa_2)$$

and it is injective:

$$F_\varphi(\varkappa) = 1_B \iff \varphi \circ \varkappa \circ \varphi^{-1} = 1_B \iff \varphi \circ \varkappa = \varphi \iff \varkappa = \varphi^{-1} \circ \varphi = 1_A$$

and it is surjective: let $\varkappa' \in \mathrm{Mor}(B, B)$ and define $\varkappa := \varphi^{-1} \circ \varkappa' \circ \varphi$, it is clear $F_\varphi(\varkappa) = \varkappa'$. It is clear that $F_\varphi^{-1} = F_{\varphi^{-1}}$ by this.

Our F_γ above is precisely this $F_{[\gamma]}$ defined in the groupoid of first homotopy groups above X, meaning it is an isomorphism between $\pi_1(X, a)$ and $\pi_1(X, b)$. And so $F_\gamma^{-1} = F_{\overline{\gamma}}$. Let us summarize this:

1.1.16 Proposition

Let $a, b \in X$ be two points in X connected by a path γ. Then $\pi_1(X, a)$ and $\pi_1(X, b)$ are isomorphic.

1.1.17 Proposition

Suppose $H\colon I \times I \longrightarrow X$ is a homotopy from the closed loop φ to the closed loop ψ such that for all $t \in I$, $H(0, t) = H(1, t)$. Define the path $\gamma(t) = H(0, t) = H(1, t)$, then

$$[\psi] = [\overline{\gamma}][\varphi][\gamma]$$

Proof: this is equivalent to saying

$$[\overline{\psi}][\overline{\gamma}][\varphi][\gamma] = [K_p] \iff \overline{\psi} * \overline{\gamma} * \varphi * \gamma \overset{\partial I}{\sim} K_p$$

Now, $\overline{\psi} * \overline{\gamma} * \varphi * \gamma$ is a curve $I \longrightarrow X$ whose endpoints are the same, so it can be viewed as a curve $S^1 \longrightarrow X$. And it can be extended to the curve H on D^2 (since the unit disc and unit square are homeomorphic), which we showed above is equivalent to $\overline{\psi} * \overline{\gamma} * \varphi * \gamma$ being null-homotopic relative to any point on S^1, so we can choose the point which corresponds to 0 and 1. ∎

1.1.18 Theorem

Let $f, g\colon X \longrightarrow Y$ be homotopic with homotopy H. Define $\gamma(t) := H(a, t)$, so $\gamma(0) = f(a)$ and $\gamma(1) = g(a)$. Then $g_* = F_\gamma \circ f_*$ (recall that $f_*\colon \pi_1(X, a) \longrightarrow \pi_1(Y, f(a))$ and $g_*\colon \pi_1(X, a) \longrightarrow \pi_1(Y, g(a))$).

Proof: let $[\varphi] \in \pi_1(X, a)$ then

$$F_\gamma(f_*[\varphi]) = [\overline{\gamma}][f \circ \gamma][\gamma], \qquad g_*[\varphi] = [g \circ \varphi]$$

Define $K(s,t) := H(\varphi(s),t)$ which is continuous and

$$K(s,0) = H(\varphi(s),0) = f \circ \varphi(s), \quad K(s,1) = H(\varphi(s),1) = g \circ \varphi(s), \quad K(0,t) = H(\varphi(0),t) = H(a,t) = K(1,t)$$

so by the above proposition, since K is a homotopy from the closed loop $f \circ \varphi$ to the closed loop $g \circ \varphi$,

$$[g \circ \varphi] = [\overline{\gamma}][f \circ \varphi][\gamma]$$

as required. ■

Notice then that $g_* = f_*$ if and only if $F_\gamma = \mathrm{id}$ (requiring $f(a) = g(a) = b$). This happens when $[\gamma] = [K_b]$ for example, which can happen when $\gamma = K_b$. Ie. if $H(a,t) = b$ for all $t \in I$, then $f_* = g_*$. But notice that this is simply the condition for H to be a homotopy relative to $\{a\}$, so

1.1.19 Proposition

If $f \overset{\{a\}}{\sim} g$ then $f_* = g_*$.

1.1.20 Theorem

If $f\colon X \longrightarrow Y$ is a homotopy equivalence, then $f_*\colon \pi_1(X,a) \longrightarrow \pi_1(Y,f(a))$ is an isomorphism of groups.

Proof: there exists a $g\colon Y \longrightarrow X$ such that $g \circ f \sim \mathrm{id}_X$ and $g \circ f \sim \mathrm{id}_Y$. Now since $g \circ f \sim \mathrm{id}_X$, then by above $(g \circ f)_* = F_\gamma \circ (\mathrm{id}_X)_* = F_\gamma$, since $(\cdot)_*$ is a functor, $g_* \circ f_* = F_\gamma$. Now recall that $F_\gamma = g_* \circ f_*\colon \pi_1(X,a) \longrightarrow \pi_1(X,g(f(a)))$ is an isomorphism. And similarly $f \circ g \sim \mathrm{id}_Y$, so there exists an F_δ such that $(f \circ g)_* = f_* \circ g_* = F_\delta\colon \pi_1(Y,f(a)) \longrightarrow \pi_1(X,f(g(f(a))))$ is an isomorphism.

Recall that if $f \circ g$ is injective, then g is injective, and if $f \circ g$ is surjective, f is surjective. Thus f_* is injective and surjective, meaning it is a bijective homomorphism, an isomorphism. ■

Thus if X is contractible, then it is homotopic to the singleton space (an exercise in homework), $\{b\}$. Thus $\pi_1(X,a) \cong \pi_1(\{b\},b)$, and $\pi_1(\{b\},b)$ has a single point, meaning $\pi_1(X,a)$ is the trivial group.

1.1.21 Corollary

If X is contractible, then $\pi_1(X,a)$ is trivial.

1.1.22 Definition

Let E, B be topological spaces, then a map $p\colon E \longrightarrow B$ is a **covering map** (or just a *covering*) if

(1) p is surjective,

(2) for every $x \in B$, there exists a neighborhood $x \in \mathcal{U}$ such that $p^{-1}(\mathcal{U})$ is the disjoint union of open sets $\{\mathcal{V}_\alpha\}_{\alpha\in I}$ such that for every $\alpha \in I$, $p\big|_{\mathcal{V}_\alpha}\colon \mathcal{V}_\alpha \longrightarrow \mathcal{U}$ is a homeomorphism.

For example, take $B = S^1$ and $E = \mathbb{R}$. Define $p(t) := (\cos(2\pi t), \sin(2\pi t))$, or if we identify S^1 with $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, then $p(t) = e^{2\pi i t}$. This is like spiraling the real line so that all the integers are all on a vertical line (they all map to $1 \in S^1$), and projecting this spiral onto a plane to form the circle S^1. p is obviously a surjective continuous map. For every $e^{2\pi i t}$, let $\varepsilon < 2\pi$ then the preimage of the open set $\mathcal{U}_t := \left\{e^{2\pi i s} \;\middle|\; t-\varepsilon < s < t+\varepsilon\right\}$ is $\bigcup_{n\in\mathbb{Z}}\big(n + (t-\varepsilon, t+\varepsilon)\big)$ and every $n + (t-\varepsilon, t+\varepsilon)$ is homeomorphic (via p) to this $\mathcal{U}_t$.

1.1.23 Theorem

Suppose $p\colon E \longrightarrow B$ is a covering, $b \in B$ and $e \in E$ such that $p(e) = b$. Now suppose $\gamma\colon I \longrightarrow B$ is a curve starting at b, $\gamma(0) = b$. Then there exists a unique curve $\delta\colon I \longrightarrow E$ starting at e such that $p \circ \delta = \gamma$.

Proof: let $\{\mathcal{U}_\beta\}_{\beta\in J}$ be an open covering of B, then $\left\{\gamma^{-1}(\mathcal{U}_\beta)\right\}_{\beta\in J}$ is an open covering of I. Since I is a compact metric space, it has a Lebesgue number, and so we can take n small enough such that for every k, $I_k := \left[\frac{k-1}{n}, \frac{k}{n}\right]$ is contained within a single $\gamma^{-1}(\mathcal{U}_\beta)$. Meaning for every $1 \leq k \leq n$ there exists a $\beta_k \in J$ such that $I_k \subseteq \gamma^{-1}(\mathcal{U}_{\beta_k})$, so $\gamma(I_k) \subseteq \mathcal{U}_{\beta_k}$.

Now since p is a covering, there exists a neighborhood $b \in \mathcal{U}$ such that $p^{-1}(\mathcal{U}) = \bigcupdot_{\alpha\in I} \mathcal{V}_\alpha$. Since this is a disjoint union and $p(e) \in \mathcal{U}$, there exists a single $\alpha \in I$ such that $e \in \mathcal{V}_\alpha$, and let us denote this neighborhood by $\mathcal{V}$. And so $p\big|_{\mathcal{V}}$ is a homeomorphism from $\mathcal{V}$ to $\mathcal{U}$. So define

$$\delta\big|_{I_1} := \left(p\big|_{\mathcal{V}}\right)^{-1} \circ \gamma\big|_{I_1}$$

We have then that $p \circ \left(\delta\big|_{I_1}\right) = \gamma\big|_{I_1}$ and $\delta(0) = \left(p\big|_{\mathcal{V}}\right)^{-1} \circ \gamma(0) = e$. This is the only such transform of $\gamma\big|_{I_1}$ which satisfies $\delta\big|_{I_1}(0) = e$ since every transform $\delta\colon I_1 \longrightarrow E$ where $p \circ \delta = \gamma$, $\delta(I_1) = p^{-1}\gamma(I_1) \subseteq p^{-1}(\mathcal{U}) = \bigcupdot_{\alpha\in I} \mathcal{V}_\alpha$. Since $\delta(I_1)$ is connected, it must be contained within a single $\mathcal{V}_\alpha$, this being $\mathcal{V}$ since it contains e.

We continue with $\gamma\big|_{I_2}$ which starts at $\frac{1}{n}$ to get $\delta\big|_{I_2}$, and so on. Together these form δ, which is continuous as $\{I_k\}_{1\leq k\leq n}$ is a finite partition of I into closed sets and δ is continuous over each of these closed sets. ■

Since this curve is unique, we can denote it by $\hat{\gamma}^e$ (since there is a unique curve for every $e \in p^{-1}(b)$).

1.1.24 Proposition

Let $p\colon E \longrightarrow B$ be a covering, $\gamma, \delta\colon I \longrightarrow B$ are curves such that $\gamma \overset{\partial I}{\sim} \delta$. Let $a = \gamma(0) = \delta(0)$, and let $e \in E$ such that $p(e) = a$. Then $\hat{\gamma}^e \overset{\partial I}{\sim} \hat{\delta}^e$, and in particular $\hat{\gamma}^e(1) = \hat{\delta}^e(1)$; both of the curves finish on the same point.

Proof: let $H\colon I \times I \longrightarrow B$ be a homotopy relative to ∂I from γ to δ. Let $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ be an open cover of B by good open sets, then $\left\{H^{-1}(\mathcal{U}_\alpha)\right\}_{\alpha\in I}$ is an open cover of $I \times I$. Since $I \times I$ is a compact metric space, it has a Lebesgue number and so there is an n large enough so that we can partition $I \times I$ into squares $S_{ij} = \left[\frac{i}{n}, \frac{i+1}{n}\right] \times \left[\frac{j}{n}, \frac{j+1}{n}\right]$ such that each S_{ij} is contained within some $H^{-1}(\mathcal{U}_\alpha)$.

Continuing similar to the previous proof, we get a homotopy $\hat{H}\colon I \times I \longrightarrow E$ such that $p \circ \hat{H} = H$ and $\hat{H}(0,0) = e$. So $p \circ \hat{H}(t,0) = H(t,0) = \gamma(t)$ and since $\hat{\gamma}^e$ is unique, this means $\hat{H}(t,0) = \hat{\gamma}^e(t)$. And similarly $p \circ \hat{H}(t,1) = H(t,1) = \delta(t)$ so $\hat{H}(t,1) = \hat{\delta}^e(t)$. And $p \circ \hat{H}(0,s) = H(0,s) = a$, we can view this as the curve K_a from a to a, and so by uniqueness we have again that $\hat{H}(0,s) = \hat{K}_a^e(s) = e$. Similar for $\hat{H}(1,s)$. Thus $\hat{H}$ is a homotopy from $\hat{\gamma}^e$ to $\hat{\delta}^e$ relative to ∂I. ■

Let $p\colon E \longrightarrow B$ be a covering, $a \in B$, and $p(e) = a$. Then we define a function $F\colon \pi_1(B,a) \longrightarrow p^{-1}(a)$ by $F([\gamma]) := \hat{\gamma}^e(1)$. This is well-defined by the previous proposition (it is in $p^{-1}(a)$ since $p \circ \hat{\gamma}^e = \gamma$, so $p \circ \hat{\gamma}^e(1) = a$).

1.1.25 Proposition

(1) If E is path connected, then F is surjective.

(2) If E is simply connected (see homework 2+3, for every two $a, b \in E$ every two paths between them are homotopic relative to ∂I), then it is also injective (it is bijective).

Proof:

(1) Let $p(x) = a$ then x and e are connected since E is path connected, so let δ be a path e to x, then $\widehat{p \circ \delta}^e = \delta$ by uniqueness. And so $F[p \circ \delta] = \delta(1) = x$. So F is surjective.

(2) Suppose $F[\gamma] = F[\delta]$, then $\hat{\gamma}^e(1) = \hat{\delta}^e(1)$. Of course $\hat{\gamma}^e(0) = \hat{\delta}^e(0) = e$, and so by simple connectivity, $\hat{\gamma}^e \overset{\partial I}{\sim} \hat{\delta}^e$ since they begin at the same point and end at the same point. Now recall that $\gamma = p \circ \hat{\gamma}^e$ and since homotopy respects the composition of homotopic functions, $\gamma \overset{\partial I}{\sim} \delta$. ■

So for example, the covering $p\colon \mathbb{R} \longrightarrow S^1$ by $t \mapsto e^{2\pi it}$ is a covering from a simply connected space (since contractible implies simply connected, homework 2) to S^1, this means that $F\colon \pi_1(S^1, e^{2\pi it}) \longrightarrow \{t + n \mid n \in \mathbb{Z}\} \cong \mathbb{Z}$ is a bijection. Since the fundamental group of a simply connected space is trivial, S^1 is not simply connected.

Notice that F^{-1} can be viewed as mapping a curve γ which starts at e and ends at $b \in p^{-1}(a)$ to $[p \circ \gamma]$. So for example, $\pi_1(S^1, 1)$ is $\{$[the curve which winds around the circle n times] $\mid n \in \mathbb{Z}\}$. This is since a curve from 0 to n in $\mathbb{R}$ is mapped to a curve which winds around the circle n times (for negative n this winds in the opposite direction). Notice that F is a group isomorphism here, since concatenating two curves which wind around the circle n and m times gives a curve which winds around the circle $n + m$ times. So $\pi_1(S^1) \cong \mathbb{Z}$.

Now, since S^1 is a deformation retract of $\mathbb{R}^2 \setminus \{0\}$, so the inclusion map is a homotopic equivalence and thus defines an isomorphism of their fundamental groups. So $\pi_1(\mathbb{R}^2 \setminus \{0\}) \cong \pi_1(S^1) \cong \mathbb{Z}$ as well.

Now suppose $A \subseteq X$ is a retract, meaning there exists a retraction $r\colon X \longrightarrow A$ such that $r \circ \iota = \mathrm{id}_A$. So $\iota_*\colon \pi_1(A, a) \longrightarrow \pi_1(X, a)$ and $r_*\colon \pi_1(X, a) \longrightarrow \pi_1(A, a)$ and since this is a functor, $r_* \circ \iota_* = (r \circ \iota)_* = (\mathrm{id}_A)_* = \mathrm{id}_{\pi_1(A,a)}$. So ι_* is injective. Let us summarize this:

1.1.26 Proposition

If $A \subseteq X$ is a retract, then $\iota_*\colon \pi_1(A, a) \longrightarrow \pi_1(X, a)$ is a monomorphism (injective).

Now, for example $S^1 = \partial D^2 \subseteq D^2$ is not a retract: if it were then ι_* would be a monomorphism $\pi_1(S^1, 1) \longrightarrow \pi_1(D^2, 1)$. But $\pi_1(S^1, 1) \cong \mathbb{Z}$ and $\pi_1(D^2, 1) = 1$ since D^2 is contractible ($H(x, t) = (1 - t)x$). And there is no monomorphism $\mathbb{Z} \longrightarrow 1$.

1.1.27 Theorem (Brouwer Fixed-Point Theorem (for D^2))

If $f\colon D^2 \longrightarrow D^2$ is continuous, then it has a fixed point: a point $x \in D^2$ such that $f(x) = x$.

Proof: suppose not, then $f(x) \neq x$ for all x. Using this f we will construct an $r\colon D^2 \longrightarrow S^1$ such that $r \circ \iota = \mathrm{id}_{S^1}$, meaning then S^1 would be a retract of D^2, which we just showed it is not. For $x \in D^2$, look at the segment which begins at $f(x)$ to x, and set $r(x)$ to be the intersection of this line with the segment. The segment is $f(x) + t(x - f(x))$ and so we want to solve (setting $f(x) = (f_1, f_2)$ and $x = (x_1, x_2)$)

$$\big(f_1 + t(x_1 - f_1)\big)^2 + \big(f_2 + t(x_2 - f_2)\big)^2 = 1$$

which can be solved, and gives a t which is continuous in f and x. Notice that if $x \in S^1$ then by definition $r(x) = x$ since the intersection of the segment with the boundary of the circle is x. So r is indeed a retract, in contradiction. ∎