Infinitesimal Calculus 3

Lecture 12, Wednsday November 23, 2022
Ari Feiglin

We will use $\varepsilon\colon X \longrightarrow X$ where X is some linear space as a function such that:

$$\lim_{x\to 0} \frac{\varepsilon(x)}{\|x\|} = 0$$

Then $\lim_{x\to 0} \varepsilon(x) = 0$. So if $X = \mathbb{R}^n$ then ε must satisfy:

$$\lim_{(x_1,\dots,x_n)\to 0} \frac{\varepsilon(x_1,\dots,x_n)}{\sqrt{x_1^2+\cdots+x_n^2}} = 0$$

Notice then that f is continuous if and only if $f(x) = f(x_0) + \frac{\varepsilon(\Delta x)}{\Delta x}$ where $\Delta x = x - x_0$. This is direct since the limit of $\frac{\varepsilon(\Delta x)}{\Delta x}$ is $f(x) = f(x_0+\delta x) - f(x_0)$ which is 0 if and only if f is continuous at x_0.

Proposition 12.0.1:

Suppose $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ defined in some neighborhood of x_0, then f is differentiable at x_0 if and only if there is some $A \in \mathbb{R}$ and $\varepsilon(\Delta x)$ such that

$$f(x) = f(x_0) + A\Delta x + \varepsilon(\Delta x)$$

where $\Delta x = x - x_0$.

Proof:

If f is differentiable then define $\varepsilon(\Delta) = f(x) - f(x_0) - A\Delta x$:

$$\lim_{\Delta x\to 0} \frac{\varepsilon(\Delta)}{\Delta x} = \lim_{\Delta x\to 0} \frac{f(x_0+\Delta x) - f(x_0) - A\Delta x}{\Delta x} = f'(x_0) - A$$

so if $A = f'(x_0)$, then ε and A satisfy the condition.
Suppose there exists some A and ε where this occurs, then:

$$\lim_{\Delta x\to 0} \frac{f(x_0+x) - f(x_0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{A\Delta x + \varepsilon(\Delta x)}{\Delta x} = A + \lim_{\Delta x\to} \frac{\varepsilon(\Delta x)}{\Delta x} = A$$

As required.

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Definition 12.0.2:

A function $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ is differentiable at (x_0, y_0) if it is defined in some neighborhood of it and there exist $A, B \in \mathbb{R}$ and $\varepsilon(\Delta x, \Delta y)$ such that

$$f(x,y) = f(x_0,y_0) + A\Delta x + B\Delta y + \varepsilon(\Delta x, \Delta y)$$

Proposition 12.0.3:

If $f(x,y)$ is defined in some neighborhood of (x_0,y_0) and differentiable at (x_0,y_0) then

(1) f is continuous at (x_0,y_0)

(2) $\partial_x f$ and $\partial_y f$ exist at (x_0,y_0) and are equal to A and B

Proof:

(1) We know that:

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0) + \lim_{(\Delta x,\Delta y)\to 0} \varepsilon(\Delta x,\Delta y) = f(x_0,y_0)$$

Δx and Δy approach 0 since convergence in $\mathbb{R}^n$ is pointwise.

(2) We know

$$\partial_x f(x_0,y_0) = \lim_{\Delta x\to 0} \frac{f(x_0+\Delta x,y_0) - f(x_0,y_0)}{\Delta x} = A + \lim_{\Delta x\to 0} \frac{B\Delta y + \varepsilon(\Delta x,\Delta y)}{\Delta x}$$

In this case $\Delta y = 0$ since $y = y_0$ so this is equal to:

$$A + \lim_{\Delta x\to 0} \frac{\varepsilon(\Delta x,0)}{\Delta x} = A$$

And similarly for $\partial_y f$.

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Proposition 12.0.4:

Suppose $f(x,y)$ is a function which has partial derivatives in a neighborhood of (x_0,y_0) and $\partial_x f$ and $\partial_y f$ are continuous at (x_0,y_0). Then f is differentiable at (x_0,y_0).

Proof:

Notice that

$$f(x_0+\Delta x,y_0+\Delta y) - f(x_0,y_0) = (f(x_0+\Delta x,y_0+\Delta y) - f(x_0,y_0+\Delta y)) + (f(x_0,y_0+\Delta y) - f(x_0,y_0))$$

So by the mean value theorem on the "single value representations" of the partial derivatives, there exists $0 \leq t,s \leq 1$ such that

$$f(x_0+\Delta x,y_0+\Delta y) - f(x_0,y_0) = \partial f_x(x_0+t\Delta x,y_0+\Delta y)\Delta x + \partial f_y(x_0+\Delta x,y_0+s\Delta y)\Delta y$$

Since $\partial_x f$ and $\partial_y f$ are continuous at (x_0,y_0) this is equal to:

$$= (\partial_x f(x_0,y_0) + \varepsilon_1(t\Delta x,\Delta y))\Delta x + (\partial_y f(x_0,y_0) + \varepsilon_2(\Delta x,s\Delta y))$$

And so if we define $A = \partial_x f(x_0,y_0)$ and $B = \partial_y f(x_0,y_0)$ and $\varepsilon(\Delta x,\Delta y) = \varepsilon_1\Delta x + \varepsilon_2\Delta y$ we have:

$$f(x,y) - f(x_0,y_0) = A\Delta x + B\Delta x + \varepsilon(\Delta x,\Delta y)$$

And we know that:

$$\lim \frac{\varepsilon(\Delta x,\Delta y)}{\sqrt{\Delta x^2+\Delta y^2}} = 0$$

Since $\frac{\Delta x}{\sqrt{\Delta x^2+\Delta y^2}}$ is bounded by 1, and similiar for Δy and the limit of this for ε_1 and ε_2 is 0 (by pointwise convergence), so this is indeed 0 as required.

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This is not an equivalent condition though.

Example:

Let

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & (x,y) \neq 0 \\ 0 & (x,y) = 0 \end{cases}$$

Then $f(x,0) = x^2\sin\left(\frac{1}{x}\right)$ the derivative of this is $\partial_x f(x,0) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$, which does not exist for $x = 0$, but it is continuous. Similarly for $\partial_y f$. But since the limit of f as (x,y) approaches 0 is 0, f is a valid ε, so it is necessarily differentiable.