Group Theory

Lecture 10, Sunday January 1, 2023
Ari Feiglin

10.1 Group Actions, continued

Recall that we have the following general group actions:

- G acts on itself via its own operation.
- G acts on itself via conjugation.
- G acts on normal subgroups via conjugation.
- G acts on $\mathcal{L}(G)$ (the lattice of all subgroups) via conjugation.
- $\text{Aut}(G)$ acts on G via evaluation.
- G acts on ${}^{G}\!/_{H}$ via left multiplication.

One more action is the action of conjugation on $H \leq G$ (not necessarily normal) by $N_G(H)$, the normalizer of H.

Definition 10.1.1:

The **kernel** of a group action is the kernel of the homomorphism enduced by it.

Lemma 10.1.2:

The kernel of a group action is $K = \{g \in G \mid \forall x \in X : g \cdot x = x\}$.

Proof:

Let φ be the homomorphism induced by the group action and suppose $g \in \operatorname{Ker}\varphi$ then if $x \in X$ then $g \cdot x = \varphi(g)(x) = \text{id}(x) = x$, so $g \in K$. And if $g \in K$ then $\varphi(g)(x) = g \cdot x = x$, so $\varphi(g) = \text{id}$ and therefore $g \in \operatorname{Ker}\varphi$. So $K = \operatorname{Ker}\varphi$ as required.

■

Theorem 10.1.3:

For every subgroup H of G there is a monomorphism

$$ {}^{N_G(H)}\!\Big/_{C_G(H)} \hookrightarrow \text{Aut}(H) $$

Recall that $N_G(H) = \left\{g \in G \;\middle|\; gHg^{-1} = H\right\}$, and $C_G(H) = \{g \in G \mid \forall h \in H : gh = hg\} \subseteq N_G(H)$.

Proof:

If $H = G$ then $N_G(H) = G$ and $C_G(H) = Z(G)$ and so ${}^{N_G(H)}\!/_{C_G(H)} = {}^{G}\!/_{Z(G)}$ which we know is isomorphic to $\text{Inn}(G)$ which defines a monomorphism to $\text{Aut}(G)$.
Firstly, we know that $H \trianglelefteq N_G(H)$, and so $N_G(H)$ acts on H via conjugation. The kernel of this is

$$ \left\{g \in N_G(H) \;\middle|\; \forall h \in H : ghg^{-1} = h\right\} $$

which is a subset of $C_G(H)$, and since if $C_G(H) \subseteq N_G(H)$, this is equal to $C_G(H)$. So the kernel is $C_G(H)$, and we know by the first isomorphism theorem that

$$ {}^{N_G(H)}\!\Big/_{C_G(H)} \hookrightarrow S_H $$

Now suppose $\sigma = \varphi(g)$ for $g \in N_G(H)$. Then $\sigma(h_1h_2) = gh_1h_2g^{-1} = gh_1g^{-1}gh_2g^{-1} = \sigma(h_1)\sigma(h_2)$ so the image of the monomorphism has only homomorphism, and thus only automorphisms (by definition of S_H), so this is an embedding into $\mathrm{Aut}(H)$.

■

10.2 Simple Groups

Definition 10.2.1:

A group is **simple** if it has no non-trivial normal subgroups.

Notice that an abelian group is simple if and only if it has no subgroups at all, in other words it is isomorphic to $\mathbb{Z}_p$ for some prime p.

Lemma 10.2.2:

If D is a conjugacy class of S_n contained in A_n, then A_n is a conjugacy class of A_n if $C_{S_n}(x) \nsubseteq A_n$ for every $x \in D$, and if $C_{S_n}(x) \subseteq A_n$ then D is the disjoint union of two conjugacy classes of the same class.

Proof:

We know

$$|[x]_{S_n}| = [S_n : C_{S_n}(x)] \qquad |[x]_{A_n}| = [A_n : C_{A_n}(x)]$$

And $C_{A_n}(x) = A_n \cap C_{S_n}(x)$ by the second isomorphism theorem:

$$C_{S_n}(x) \big/ C_{A_n}(x) = C_{S_n} \big/ A_n \cap C_{S_n} = C_{S_n}A_n \big/ A_n$$

Which has a cardinality of either 1 or 2 (since the index of A_n is 2). And recall that this is equal to:

$$\frac{2[A_n : C_{A_n}]}{[S_n : C_{S_n}]} = \frac{2|[x]_{A_n}|}{|[x]_{S_n}|}$$

Suppose this is equal to 1, that is $C_{S_n} \subseteq A_n$ then $2|[x]_{A_n}| = |[x]_{S_n}|$, so $[x]_{A_n}$ is not a conjugacy class in S_n. And if we take an element y in $[x]_{S_n}$ not in $[x]_{A_n}$, then $y \in A_n$ (since the conjugacy class in S_n preserves signs) so $[y]_{A_n}$ is disjoint from $[x]_{A_n}$ since orbits are disjoint, and they have the same size since otherwise $[y]_{S_n}$ would have the same size as $[y]_{A_n} = [y]_{S_n} = [x]_{S_n}$, and since this size is half that of $[x]_{S_n}$, $[x]_{S_n}$ is the disjoint union of these. And if it is equal to 2, that is it is not a subset, then $|[x]_{A_n}| = |[x]_{S_n}|$, so $[x]_{A_n} = [x]_{S_n}$ (since it is a subset).

■

Using this lemma, it can be shown through somewhat length computations that A_5 is simple.

Lemma 10.2.3:

For $5 \geq n$ all the cycles of length 3 are conjugates in A_n.

Proof:

We know that this is true in S_n since they have the same periodic structure. Since $\begin{pmatrix}1 & 2 & 3\end{pmatrix}$ commutes with $\begin{pmatrix}4 & 5\end{pmatrix}$, we have that $C_{S_n}\begin{pmatrix}1 & 2 & 3\end{pmatrix} \nsubseteq A_n$, so the conjugacy class of all cycles of length 2 which is $[\begin{pmatrix}1 & 2 & 3\end{pmatrix}]_{S_n}$, which by the lemma above is equal to $[\begin{pmatrix}1 & 2 & 3\end{pmatrix}]_{A_n}$, as required.

■

Notice that in S_4, the conjugacy class of $\begin{pmatrix}1 & 2 & 3\end{pmatrix}$ has a size of 8 ($\frac{4\cdot 3\cdot 2}{3} = 8$), which does not divide 12, the size of A_4, so it cannot be a subgroup, and therefore a conjugacy class, in A_n.

Theorem 10.2.4:

For $n \geq 5$, A_n is simple.

Proof:

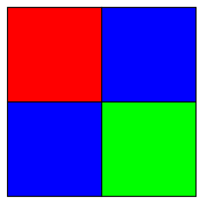
Suppose A_n has a non-trivial subgroup N. Let $\text{id} \neq \sigma \in N$, then let $i \in [n]$ such that $\sigma(i) = j \neq i$. Let $k \in [n]$ which is distinct from i, j, and $\sigma(j)$ (note that it is possible for $\sigma(j) = i$). Notice that σ does not commute with (i, j, k):

$$\sigma(i,j,k)(i) = \sigma(j) \neq k \qquad (i,j,k)\sigma(i) = (i,j,k)(j) = k$$

Let us focus on $\sigma(i,j,k)\sigma^{-1}(i,j,k)^{-1} = \sigma\big((i,j,k)\sigma^{-1}(i,j,k)^{-1}\big)^{-1}$ which is in N as the composition of two elements of N. This is equal to $\begin{pmatrix} j & \sigma(j) & \sigma(k) \end{pmatrix}\begin{pmatrix} i & k & j \end{pmatrix} \in A_{\{i,j,k,\sigma(j),\sigma(k)\}}$ which is normal in the symmetric group S_5 (up to isomorphism). Since N is normal in A_n which is a supergroup of A_5 so $N \cap A_5$ is normal in A_5. The element we were looking at before is in A_5 and N so $A_5 \cap N \neq \{e\}$, so it must be A_5 since A_5 is simple, so $A_5 \subseteq N$. So N contains a cycle of length 3, as A_5 does, and therefore it contains them all since they are all conjugates. But A_n is generated by cycles of length 3 so $N = A_n$, which contradicts that N is a *non-trivial* normal subgroup of A_n ↯

■

Suppose we want to color a square with 4 regions with 3 colors, for example red, green, and blue. The following would be a valid coloring:



A rotation of such a coloring is considered to be the same. So we'd like to count the number of distinct colorings. Let Ω be the set of all colorings, then $G = \mathbb{Z}_4 \subseteq D_4$ acts on Ω as rotations, we'd like to count the number of orbits of this action. For a group action of G on Ω let $\mathit{fp}(\sigma) = |\{x \in \Omega \mid \sigma x = x\}|$ for $\sigma \in G$.

Lemma 10.2.5 (Frobenius's Lemma):

Suppose G is a finite group which acts on Ω, then the number of orbits is:

$$\left|{}^{\Omega}/_{G}\right| = \sum_{\sigma \in G} \mathit{fp}(\sigma)$$

Proof:

Firstly we know that

$$\sum_{\sigma \in G} \mathit{fp}(\sigma) = |\{(\sigma, x) \in G \times \Omega \mid \sigma x = x\}| = \sum_{x \in \Omega} |G_x|$$

And by the orbit-stabilizer theorem this is equal to:

$$= \sum_{x \in \Omega} \frac{|G|}{|G \cdot x|} = |G| \cdot \sum_{x \in \Omega} \frac{1}{|G \cdot x|}$$

Now notice that since orbits partition Ω:

$$\sum_{x \in \Omega} \frac{1}{|G \cdot x|} = \sum_{A \in {}^{\Omega}/_{G}} \sum_{x \in A} \frac{1}{|A|} = \sum_{A \in {}^{\Omega}/_{G}} 1 = \left|{}^{\Omega}/_{G}\right|$$

as required.

Notice that if σ and σ' are conjugates, $\sigma' = g\sigma g^{-1}$:

$$\sigma'(x) = x \iff \sigma(g^{-1}(x)) = g^{-1}x$$

so $\mathit{fp}(\sigma) = \mathit{fp}(\sigma')$ and so by Frobenius's lemma:

$$\left|{}^{\Omega}/_{G}\right| = \frac{1}{|G|} \sum_{c \subseteq G} \mathit{fp}(\sigma \in c)|c|$$

where c is a conjugacy class.
Using this lemma, we can answer our above question. Since $fp(\mathrm{id}) = |\Omega| = 3^4 = 81$ and $fp(\sigma) = fp\left(\sigma^3\right) = 3$ (since a coloring is fixed after a single rotation if the square is colored with the same color) and $fp\left(\sigma^2\right) = 9$ (since the coloring is preserved after two rotations if the diagonals are the same color) so:

$$\left| {}^{\Omega}/_{\mathbb{Z}_4} \right| = \frac{1}{4} \cdot 96 = 24$$