Introduction to Rings and Modules

Lecture 19, Wednesday June 21 2023

Ari Feiglin

Proposition 19.0.1:

Every $\mathcal{O}_d$ is a Dedekind domain.

Proof:

Since $\mathcal{O}_d \subseteq \mathbb{C}$, it is obvious that it is an integral domain. $\mathcal{O}_d$ is integrally closed as the integral closure of $\mathbb{Z}$. Since $\mathcal{O}_d = \mathbb{Z}[\alpha]$, then we define $f\colon \mathbb{Z}[x] \longrightarrow \mathbb{Z}[\alpha]$ where $f(x) = \alpha$ and $f(1) = 1$, so in general

$$f(a_n x^n + \cdots + a_0) = a_n \alpha^n + \cdots + a_0$$

this is a homomorphism, as it is a restriction of the evaluation homomorphism. Since every element of $\mathbb{Z}[\alpha]$ is of the form $a + b\alpha$ (for the αs we are studying), so $a + b\alpha = f(a + bx)$. Thus by the first isomorphism theorem

$$\mathbb{Z}[x] \big/ \mathrm{Ker}\, f \cong \mathcal{O}_d$$

Since $\mathbb{Z}$ is a PID, and therefore noetherian, by Hilbert's basis theorem so is $\mathbb{Z}[x]$. Since the quotient of a noetherian ring is noetherian, we have that $\mathcal{O}_d$ is noetherian.
The final criterion is that $\dim \mathcal{O}_d = 1$. We must show that there exist non-zero prime ideals, and that every such ideal is maximal. Assume P is a non-zero prime ideal, let $I = P \cap \mathbb{Z}$. Then I is a prime ideal over $\mathbb{Z}$, it is an ideal since it is a group (intersection of groups), and if $n \in \mathbb{Z}$ and $i \in I \subseteq \mathbb{Z}$ then $ni \in P$ and $\mathbb{Z}$. Now suppose $ab \in I$ for $a, b \in \mathbb{Z}$ then $ab \in P$ so either a or b is in P (and $\mathbb{Z}$) and thus I.
We also claim that $I \neq (0)$. Suppose $0 \neq y \in P$ then let us look at the isomorphism from the previous lecture

$$\varphi\colon \mathbb{Q}(\sqrt{d}) \longrightarrow \mathbb{Q}(\sqrt{d}), \qquad a + b\sqrt{d} \mapsto a - b\sqrt{d}$$

we showed that $\mathcal{O}_d$ is φ-invariant, and $\alpha\varphi(\alpha) \in \mathbb{Z}$ for every $\alpha \in \mathbb{Q}(\sqrt{d})$. Thus since $y \neq 0$, $\varphi(y) \neq 0$ and $y\varphi(y) \in \mathbb{Z}$ and in P (since $y \in \mathcal{O}_d$ so $\varphi(y) \in \mathcal{O}_d$ and P is an ideal), so $0 \neq y\varphi(y) \in I$.
Thus $I = p\mathbb{Z}$ where p is prime, and since $p\mathbb{Z} = I = P \cap \mathbb{Z} \subseteq P$, thus $p \in P$ and so we have $p\mathcal{O}_d \subseteq P$. So we can look at the natural homomorphism

$$\mathcal{O}_d \big/ p\mathcal{O}_d \longrightarrow \mathcal{O}_d \big/ P, \qquad w + p\mathcal{O}_d \mapsto w + P$$

But note that

$$\mathcal{O}_d \big/ p\mathcal{O}_d = \left\{ [a] + [b]\gamma \;\middle|\; [a], [b] \in \mathbb{Z} \big/ p\mathbb{Z} \right\}$$

where $\mathcal{O}_d = \mathbb{Z}[\gamma]$, and thus the order of this quotient is p^2, and since the homomorphism is surjective, $\mathcal{O}_d/P$ is a finite integral domain (since P is prime), meaning it is a field. Thus P is maximal, as required. ■

Note:

Material from the end of this lecture has been moved to the next lecture's file.