Calculus Homework #4

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Question 4.1:

$f\colon [a,b] \longrightarrow \mathbb{R}$ is a continuous function such that for every function $g\colon [a,b] \longrightarrow \mathbb{R}$: if

$$\int_a^b g(x)\, dx = 0$$

Then

$$\int_a^b f(x) \cdot g(x)\, dx = 0$$

Prove that $f(x)$ is a constant function.

Answer:

For any $x_1 < x_2$ and $\alpha > x_2 - x_1$ we can construct the following function:

$$g(x) = \begin{cases} 1 & x \in [x_1, x_2] \\ -1 & x \in [x_1+\alpha, x_2+\alpha] \\ 0 & \text{else} \end{cases}$$

This function is well-defined as $[x_1, x_2] \cap [x_1+\alpha, x_2+\alpha] = \varnothing$, since $x_1 + \alpha > x_2$.
Furthermore, g is discontinuous only at $x_1, x_2, x_1+\alpha, x_2+\alpha$, so g is continuous almost everywhere in $[a,b]$. g is also bound, so g is integrable by Lebesgue's theorem.
We can then partition $[a,b]$ in such a way that the partitions of $[x_1, x_2]$ and $[x_1+\alpha, x_2+\alpha]$ are equivalent, that is we partition $[x_1, x_2]$ and then shift it over by α to partition $[x_1+\alpha, x_2+\alpha]$. We then partition the rest of $[a,b]$ arbitrarily.
The Riemman Sum of this partition will then be 0, as it will equal:

$$\sum_{i=1}^{n} \Delta_i \cdot f(d_i) = \sum_{d_i \in [x_1, x_2]} \Delta_i \cdot g(d_i) + \sum_{d_i \in [x_1+\alpha, x_2+\alpha]} \Delta_i \cdot g(d_i) + \sum_{d_i \notin [x_1, x_2], [x_1+\alpha, x_2+\alpha]} \Delta_i \cdot g(d_i)$$

Which equals, by the definition of g:

$$\sum_{d_i \in [x_1, x_2]} \Delta_i - \sum_{d_i \in [x_1+\alpha, x_2+\alpha]} \Delta_i$$

Since these Δ_is partition $[x_1, x_2]$ and $[x_1+\alpha, x_2+\alpha]$, this is equal to:

$$x_2 - x_1 - (x_2 + \alpha - x_1 - \alpha) = 0$$

So

$$\int_a^b g(x)\, dx = 0$$

As required.
Now notice that:

$$f(x) \cdot g(x) = \begin{cases} f(x) & x \in [x_1, x_2] \\ -f(x) & x \in [x_1+\alpha, x_2+\alpha] \\ 0 & \text{else} \end{cases}$$

We know this is integrable.
Furthermore, we know f is integrable over $[x_1, x_2]$ since $[x_1+\alpha, x_2+\alpha]$ since f is continuous over them.
We can then create a series of partitions of $[x_1, x_2]$ and $[x_1+\alpha, x_2+\alpha]$ and extend them to a partition of $[a,b]$.
The Riemman Sum of $f(x) \cdot g(x)$ will equal to the sum of the Riemman Sums of these two partitions. Let P_{1n}

be a series of partitions of $[x_1, x_2]$, and P_{2n} be a series of partitions of $[x_1+\alpha, x_2+\alpha]$, and P_n be the partition which is the extension of them onto $[a,b]$. So:

$$\sigma_{f\cdot g}(P_n) = \sigma_f(P_{1n}) + \sigma_{-f}(P_{2n})$$

Taking their limits, we get:

$$\int_a^b f(x)\cdot g(x)\,dx = \int_{x_1}^{x_2} f(x)\,dx + \int_{x_1+\alpha}^{x_2+\alpha} -f(x)\,dx$$

Which we know is equal to 0. So:

$$\int_{x_1}^{x_2} f(x)\,dx = \int_{x_1+\alpha}^{x_2+\alpha} f(x)\,dx$$

For every $x_1 < x_2$ and $\alpha > x_2 - x_1$.
Now, suppose for the sake of a contadiction that f is not continuous. This means that there exist $x_1 \neq x_2$ such that $f(x_1) < f(x_2)$.
Let $\varepsilon_1, \varepsilon_2 > 0$ such that $f(x_2) - f(x_1) \geq \varepsilon_1 + \varepsilon_2$. This means there exists $\delta_1, \delta_2 > 0$ such that for every $\tilde{x}_1 \in [x_1-\delta_1, x_1+\delta_1]$ and $\tilde{x}_2 \in [x_2-\delta_2, x_2+\delta_2]$:

$$|f(\tilde{x}_1) - f(x_1)| \leq \varepsilon_1 \quad |f(\tilde{x}_2) - f(x_2)| \leq \varepsilon_2$$

Which means:

$$f(\tilde{x}_1) \leq f(x_1) + \varepsilon_1 \quad f(\tilde{x}_2) \geq f(x_2) - \varepsilon_2$$

We know:

$$f(x_2) - \varepsilon_2 \geq f(x_1) + \varepsilon_1$$

So:

$$f(\tilde{x}_1) \leq f(\tilde{x}_2)$$

For every $\tilde{x}_1 \in [x_1-\delta_1, x_1+\delta_1]$ and $\tilde{x}_2 \in [x_2-\delta_2, x_2+\delta_2]$.
We want to find a common δ that also makes the intervals disjoint. So we want $x_1+\delta < x_2-\delta$ or $x_2+\delta < x_1-\delta$ depending on which x_i is larger. So:

$$2\delta < |x_1 - x_2|$$

So we can define $\delta \coloneqq \min\left\{\delta_1, \delta_2, \left|\frac{x_1-x_2}{3}\right|\right\}$. So we get:

$$\int_{x_1-\delta}^{x_1+\delta} f(x)\,dx = \int_{x_2-\delta}^{x_2+\delta} f(x)\,dx$$

Statement 4.1.1:

$$\int_a^b f(x)\,dx = \int_{a+\alpha}^{b+\alpha} f(x-\alpha)\,dx$$

Proof:

Let P be a pointed partition of $[a,b]$:

$$P\colon a = x_0^P < \cdots < x_n^P = b$$

I'll denote $P+\alpha$ to be the partition where $x_i^{P+\alpha} = x_i^P + \alpha$ and $d_i^{P+\alpha} = d_i^P + \alpha$. Every partition of $[a+\alpha, b+\alpha]$ can be constructed this way, and so can every partition of $[a,b]$ (by subtracting α from a partition of $a+\alpha, b+\alpha]$).
Notice that the Δ_is of the two partitions are equal.
So:

$$\sigma_{f(x-\alpha)}(P+\alpha) = \sum_{i=1}^{n} \Delta_i \cdot f\left(d^{P+\alpha} - \alpha\right) = \sum_{i=1}^{n} \Delta_i \cdot f(d_i + \alpha - \alpha) = \sum_{i=1}^{n} \Delta_i \cdot f(d_i) = \sigma_{f(x)}(P)$$

So their limits are equal as well, so:

$$\int_{a}^{b} f(x)\,dx = \int_{a+\alpha}^{b+\alpha} f(x-\alpha)\,dx$$

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So we can define:

$$h(x) \coloneqq f(x + x_2 - x_1)$$

So:

$$\int_{x_2-\delta}^{x_2+\delta} f(x)\,dx = \int_{x_1-\delta}^{x_1+\delta} f(x + x_2 - x_1)\,dx = \int_{x_1-\delta}^{x_1+\delta} h(x)\,dx$$

We also know that for every $x \in [x_1 - \delta, x_1 + \delta]$: $x + x_2 - x_1 \in [x_2 - \delta, x_2 + \delta]$ so $f(x + x_2 - x_1) \geq f(x)$, which means $h(x) \geq f(x)$.
Which means that $h(x) = f(x)$ almost always.
Let $x_0 \in [x_1 - \delta, x_1 + \delta]$. We can construct a series of points $x'_n \longrightarrow x_0$ such that $f(x'_n) = h(x'_n)$ since the points where this is true is dense in $[x_1 - \delta, x_1 + \delta]$ (the complement of a null set is dense). Which means:

$$\lim f(x'_n) = \lim h(x'_n)$$

And since f and h are continuous (since h is a shift of f), this means that:

$$f(x_0) = h(x_0)$$

This is true for $x_0 \in [x_1 = \delta, x_1 + \delta]$, so it must also be true for x_1.
So:

$$f(x_1) = h(x_1) = f(x_1 - x_1 + x_2) = f(x_2)$$

In contradiction.
So for every two points $x_1 \neq x_2$, $f(x_1) = f(x_2)$, which means $f(x)$ is constant.

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Question 4.2:

f is a function over $[a, b]$. Dis/Prove the following:

(1) If f is integrable, there exists a partition P such that:

$$\overline{s}(P) = \int_a^b f(x)\,dx$$

(2) If f is continuous over (a, b) then f is integrable over $[a, b]$.

Answer:

(1) This is false. Let's take a look at Thomae's Function:

$$f(x) = \begin{cases} \frac{1}{m} & x = \frac{n}{m},\, n \in \mathbb{Z}, m \in \mathbb{N},\, \gcd(n, m) = 1 \\ 0 & x \notin \mathbb{Q} \end{cases}$$

We know that Thomae's function is continuous over the irrationals (and therefore discontinuous only over the rationals, which is countable and therefore a null set), and bount by 1. So by Lebesgue's Theorem, f is integrable.
Furthermore, we know:

$$\int_a^b f(x)\,dx = 0$$

As we can choose a partition where $d_i \notin \mathbb{Q}$, which gives a Riemman Sum of 0.
But for every partition P, there exists some rational $q_i \in [x_{i-1}, x_i]$ for every i. Which means $f(q_i) > 0$. And we know $\beta_i \geq f(q_i) > 0$, and so $\Delta_i \cdot \beta_i > 0$. We know

$$\overline{s}(P) = \sum_{i=1}^{n} \Delta_i \cdot \beta_i$$

Which is the finite sum of positive integers, which means that $\overline{s}(P) > 0$ for every partiton P.
Which means that for every partition P:

$$\int_a^b f(x)\,dx \neq \overline{s}(P)$$

(2) This is also false. Let's look at the function:

$$f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Defined over $[0, 1]$.
f is continuous over $(0, 1)$ as it is equal to $\frac{1}{x}$ in this case, which is the composition of elementary functions.
But f isn't bound, so f is not integrable over $[0, 1]$.

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Question 4.3:

Another form of a Cantor Set is:

$$\left\{x \in [0,1] \;\middle|\; x = \sum_{n=1}^{\infty} a_n \cdot 3^{-n},\ \forall 1 \le n \in \mathbb{N} : a_n \in \{0,2\}\right\}$$

Prove that for every $x \in C$ there exists a series $x \neq \{x_n\}_{n=1}^{\infty} \in C$ such that $\lim_{n\to\infty} x_n = x$.

Answer:

Let $x \in C$, so there exists $\{a_n\}_{n=1}^{\infty} \in \{0,2\}$ such that:

$$x = 0.a_1a_2a_3\dots$$

In trinary.
We need to find a series $x \neq \{x_n\}_{n=1}^{\infty} \in C$ whose limit is x. Since $x_n \in C$, there exists $\{a_i^n\}_{i=1}^{\infty}$ such that:

$$x_n = 0.a_1^n a_2^n a_3^n \dots$$

Case 1: there exists some n_0 such that for every $n > n_0$: $a_n = 0$

In this case:

$$x = \sum_{i=1}^{n_0} a_i \cdot 3^{-i}$$

We can define:

$$a_i^n := \begin{cases} a_i & i \le n_0 \\ 2 & i = n_0 + n \\ 0 & \text{else} \end{cases}$$

Which means:

$$x_n = \sum_{i=1}^{n_0} a_i \cdot 3^{-i} + 2 \cdot 3^{-n_0-n} = x + 2 \cdot 3^{-n_0-n}$$

($x_n \in C$ since $a_i^n \in \{0,2\}$)
Which doesn't equal x as $2 \cdot 3^{-n_0-n} \neq 0$. But:

$$\lim 2 \cdot 3^{-n_0-n} = 0$$

Which means:

$$\lim x_n = \lim x + 2 \cdot 3^{-n_0-n} = x$$

As required.

Case 2: For every n_0 there exists some $n > n_0$ such that $a_n = 2$

In this case:

$$x = \sum_{i=1}^{\infty} a_i \cdot 3^{-i}$$

We can also define:

$$a_i^n := \begin{cases} a_i & i \le n \\ 0 & i > n \end{cases}$$

So:

$$x_n = \sum_{i=1}^{n} a_i \cdot 3^{-i}$$

($x_n \in C$ since $a_i^n \in \{0,2\}$)
We know that:

$$x_n = x - \sum_{i=n+1}^{\infty} a_i \cdot 3^{-n}$$

And since for every n there exists some $i > n$ such that $a_i \neq 0$, that means that the sum:

$$\sum_{i=n+1}^{\infty} a_i \cdot 3^{-n} > 0$$

Since it is the sum of non-negative integers, some of which are positive. This means $x_n \neq x$. Furthermore:

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i \cdot 3^{-i} = \sum_{i=1}^{\infty} a_i \cdot 3^{-i} = x$$

As required.

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