Complex Functions

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Proposition 5.1:

Let $\{f_n\}$ be a sequence of complex functions which are continuous on a curve C and converge uniformly to f. Then

$$\int_C f(z)\,dz = \lim_{n\to\infty}\int_C f_n(z)\,dz$$

Proof:

Let $\varepsilon > 0$ then there exists an N such that for every $n \geq N$, $|f_n(z) - f(z)| < \varepsilon$ on the curve. Then we know that

$$\left|\int_C f_n - f\,dz\right| < \varepsilon\ell$$

where ℓ is the length of C. Thus

$$\int_C f_n \longrightarrow \int_C f$$

as required. ■

Proposition 5.2:

Let F be a continuously differentiable complex function. Let C be the smooth curve given by $z\colon [a,b] \longrightarrow \mathbb{C}$ then

$$\int_C F'(z)\,dz = F(z(b)) - F(z(a))$$

Proof:

Let us define $\gamma\colon [a,b] \longrightarrow \mathbb{C}$ by $\gamma(t) = F(z(t))$ then

$$\gamma'(t) = \lim_{h\to 0}\frac{F\big(z(t+h)\big) - F\big(z(t)\big)}{h}$$

Since the curve is differentiable except for at a finite number of points, and is smooth:

$$0 \neq z'(t) = \lim_{\mathbb{R}\ni h\to 0}\frac{z(t+h) - z(t)}{h}$$

And so there exists a $\delta > 0$ such that for every $|h| < \delta$, $z(t+h) - z(t) \neq 0$. So then we can divide by $z(t+h) - z(t)$ in:

$$\gamma'(t) = \lim_{h\to 0}\frac{F(z(t+h)) - F(z(t))}{z(t+h) - z(t)}\cdot\frac{z(t+h) - z(h)}{h} = \lim_{s\to z(t)}\frac{F(s) - F(z(t))}{s - z(t)}\cdot\lim_{h\to 0}\frac{z(t+h) - z(t)}{h}$$

Where the final equality is due to z being continuous. Since F is differentiable, this is equal to

$$= F'(z(t))\cdot z'(t)$$

So we have that

$$\int_C F'(z)\,dz = \int_a^b F'\big(z(t)\big)z'(t)\,dt = \int_a^b \gamma'(t)\,dt$$

Since γ has real domain, this is equal to:

$$= \gamma(b) - \gamma(a) = F(z(a)) - F(z(b))$$

As required. ■

Definition 5.3:

A curve C given by $z\colon [a,b] \longrightarrow \mathbb{C}$ is **closed** if $z(a) = z(b)$. A closed curve which is also injective is called a **Jordan curve**.

The following theorem is stated without proof:

Theorem 5.4 (Jordan's Theorem):

If C is a Jordan curve then $\mathbb{C} \setminus C$ has exactly two connected components. One is bound and is called the interior of C and the other is unbound and is the exterior.

Definition 5.5:

A **complex closed rectangle** is a set of the form $R = \{z \in \mathbb{C} \mid (\operatorname{Re} z, \operatorname{Im} z) \in [a,b] \times [c,d]\}$ We say that a complex function is closed in a closed rectangle R if there exists a neighborhood of R, $R \subseteq \mathcal{U}$ such that f is analytic in $\mathcal{U}$.

When we study the parameterization of a boundary as a smooth curve, we consider its *positive direction* to be counter-clockwise.

Theorem 5.6 (Cauchy's Theorem For Rectangles):

Suppose f is analytic in a rectangle R. Let Γ be the positive boundary of R. Then

$$\int_\Gamma f(z)\,dz = 0$$

Proof:

Let $I = \int_\Gamma f(z)\,dz$. We then define $\Gamma_1, \ldots, \Gamma_4$ be the positive smooth curves obtained by splitting R into four equal subrectangles (and halving the domains). Then we have

$$\int_\Gamma f(z)\,dz = \sum_{i=1}^{4} \int_{\Gamma_i} f(z)\,dz$$

As the "internal" components of the integrals (the values of $f(z)$ taken on Γ_i not on Γ) cancel out as the directions differ on different Γ_is. So you can create an orientation-reversing bijection between the curves on these vertices, and you get the negative curve. There exists some k such that

$$\left| \int_{\Gamma_k} f(z)\,dz \right| \geq \frac{|I|}{4}$$

Let this rectangle be $R^{(1)}$ and its boundary $\Gamma^{(1)}$. Recurively we get

$$\begin{array}{ccccccc} R & \supset & R^{(1)} & \supset & R^{(2)} & \supset & \cdots \\ \Gamma, & & \Gamma^{(1)}, & & \Gamma^{(2)}, & & \ldots \end{array}$$

Where

$$\operatorname{diam} R^{(k)} = \frac{\operatorname{diam} R^{(k-1)}}{2} = \frac{\operatorname{diam} R}{2^k}$$

And

$$\left| \int_{\Gamma^{(k)}} f(z)\,dz \right| \geq \frac{|I|}{4^k}$$

By Cantor's lemma, since $R^{(k)}$ is a contracting sequence of closed sets

$$\bigcap_{k=0}^{\infty} R^{(k)} = \{z_0\}$$

Since f is analytic:

$$\lim_{z \longrightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) \implies f(z) = f(z_0) + f'(z_0)(z - z_0) + \alpha(z - z_0)(z - z_0)$$

Where $\frac{\varepsilon(h)}{h} \xrightarrow[h\to 0]{} 0$. So

$$\int_{\Gamma^{(k)}} f(z)\,dz = \int_{\Gamma^{(k)}} f(z_0)\,dz + \int_{\Gamma^{(k)}} f'(z_0)(z-z_0)\,dz + \int_{\Gamma^{(k)}} \alpha(z-z_0)(z-z_0)\,dz$$

The first two integrals are equal to zero. The first is zero since the integral $\int_{\Gamma^{(k)}} dz = 0$ since opposite sides of the rectangle have opposite directions and therefore cancel out. And the first integral is a scalar multiple of this. The second integral is equal to a scalar multiple of

$$\int_a^b z(t)\cdot z'(t)\,dt = \int_{z(a)}^{z(b)} z(t)\,dz = 0$$

since $z(a) = z(b)$.
Thus

$$\int_{\Gamma^{(k)}} f(z)\,dz = \int_{\Gamma^{(k)}} \alpha(z-z_0)(z-z_0)\,dz$$

Suppose the longer edge of R has length S then

$$\int_{\Gamma^{(k)}} |dz| \leq \frac{4S}{2^k}$$

as the left side is equal to the perimeter of $\Gamma^{(k)}$. Since for every $z \in \Gamma^{(k)}$

$$|z-z_0| \leq \operatorname{diam}\Gamma^{(k)} \leq \frac{\operatorname{diam} R}{2^n} \leq \frac{\sqrt{2}S}{2^n}$$

Let $\varepsilon > 0$, by the continuity of α at 0, there exists an N such that whenever

$$|z-z_0| \leq \frac{\sqrt{2}S}{2^N} \implies |\alpha(z-z_0)| < \varepsilon$$

Thus for every $n \geq N$:

$$\frac{|I|}{4^n} \leq \left|\int_{\Gamma^{(n)}} f(z)\,dz\right| = \left|\int_{\Gamma^{(n)}} \alpha(z-z_0)(z-z_0)\right| \leq \sup_{z\in\Gamma^{(n)}}|\alpha(z-z_0)||z-z_0|\cdot\ell_n$$

where ℓ_n is the perimiter of $\Gamma^{(n)}$. And this is less than

$$\leq \varepsilon\cdot\frac{\sqrt{2}S}{2^n}\cdot\frac{4S}{2^n} = \varepsilon\cdot\frac{4\sqrt{2}S^2}{4^n}$$

So

$$|I| \leq \varepsilon\cdot(4\sqrt{2}S^2)$$

And since $\varepsilon > 0$ is arbitrary, this means $I = 0$ as required. ∎

Notice then that if we have two straight-edged paths from z to w, then integrating a function over either of these two paths will result in the same result. This is because if we reverse the direction of one of the curves we have a closed path, and so the integral will be 0. But this integral is also equal to the difference of the integral of the function over the curves, so they must be equal.

Theorem 5.7:

Let f be an analytic function in the rectangle $R = R\big([a,b]\times[c,d]\big)$. Then there exists a function F in R such that F is analytic in R and $F'(z) = f(z)$ for every $z \in R$.

Proof:

We can assume that $R = R\big([0,b]\times[0,d]\big)$ by shifting f. For every $z \in R$ we define:

$$F(z) = \int_0^z f(w)\,dw$$

where w is the curve which is the concatenation of the curves $[0, \operatorname{Re} z]$ and $[\operatorname{Re} z, z]$ ($[a, b]$ is the line from a to b). If we let $\Gamma_1 = [0, \operatorname{Re} z + \operatorname{Re} h] \cup [\operatorname{Re} z + \operatorname{Re} h, z + h]$ and $\Gamma_1 = [0, \operatorname{Re} z] \cup [\operatorname{Re} z, z] \cup [z, z + \operatorname{Re} h] \cup [z + \operatorname{Re} h, z + h]$, then by definition

$$F(z+h) = \int_{\Gamma_1} f(w)\, dw$$

But by **Cauchy's Theorem For Rectangles** (specifically the note following it), this is equal to

$$F(z+h) = \int_{\Gamma_2} f(w)\, dw$$

And so

$$F(z+h) - F(z) = \int_z^{z+h} f(w)\, dw$$

where the curve is $[z, z + \operatorname{Re} h] \cup [z + \operatorname{Re} h, z + h]$. So

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h} \int_z^{z+h} f(w)\, dw$$

And we know that

$$\frac{1}{h} \int_z^{z+h} dz = \frac{1}{h}(z + h - z) = 1$$

as the antiderivative of 1 is z. Thus

$$\frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h} \int_z^{z+h} f(w) - f(z)\, dw$$

Since f is continuous at z, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that when $|z - w| < \delta$, $|f(z) - f(w)| < \varepsilon$. So if we let $|h| < \delta$ then for every point w on the curve from z to $z + h$, $|f(w) - f(z)| < \varepsilon$. And so we have that

$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| = \left| \frac{1}{h} \int_z^{z+h} f(w) - f(z)\, dw \right| \leq \frac{1}{|h|} \varepsilon \big(|\operatorname{Re} h| + |\operatorname{Im} h| \big) \leq \frac{2|h|}{|h|} \varepsilon = 2\varepsilon$$

So

$$F'(z) = \lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z)$$

as required.
F is obviously analytic since its derivative is f. ■

By reducing our restrictions on f we can generalize to the following theorem, using the same proof:

Theorem 5.8:

If f is a continuous function on a rectangle R such that for every rectangular path Γ

$$\int_\Gamma f(z)\, dz = 0$$

then f has an antiderivative on the rectangle.

Corollary 5.9:

If f is analytic on all of $\mathbb{C}$ then f has an antiderivative on all of $\mathbb{C}$.

Proof:

Let F_n be f's antiderivative on $R_n = R\big([-n, n]^2\big)$, and we further require $F_{n+1} = F_n$ on R_n. We define $F\colon \mathbb{C} \longrightarrow \mathbb{C}$ by $F(z) = F_n(z)$ if $z \in R_n$ (this is well defined since $F_{n+1} = F_n$ on R_n). It is obvious that F is continuous and $F'(z) = f(z)$ by looking at a neighborhood of z enclosed in a R_n. ■

We can use the same proof as before to show that analytic functions on disks $D_r(a)$ have antiderivatives on these disks.

Proposition 5.10:

Let $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ be a countable collection of open sets such that for every $k \in \mathbb{N}$, $\mathcal{U}_{k+1} \cap \bigcup_{n=1}^{k} \mathcal{U}_n$ is connected. Let f be an analytic function on $\mathcal{U} = \bigcup_{n\in\mathbb{N}} \mathcal{U}_n$. If for every $n \in \mathbb{N}$, f has an antiderivative on $\mathcal{U}_n$ then f has an antiderivative on $\mathcal{U}$.

The proof here is not important.
Notice that since if $f' = 0$ then $u_x = u_y = v_x = v_y = 0$ and since these are real functions, u and v are constant, so $f = u + iv$ is constant. Thus if $F' = G'$ then $(F - G)' = 0$ so $F - G$ is constant. Thus if F is an antiderivative of f then all antiderivatives of f are of the form $F + \alpha$ (and all functions of this form are antiderivatives of f).

Lemma 5.11:

Let f be an analytic function with an antiderivative F on a set W (meaning there exists an open set containing W where this is true), then for any closed smooth curve C in W:

$$\int_C f(z)\,dz = 0$$

Proof:

Suppose C is given by $\gamma\colon [a,b] \longrightarrow \mathbb{C}$. Then

$$\int_C f(z)\,dz = \int_C F'(z)\,dz = F(\gamma(b)) - F(\gamma(a)) = 0$$

as $\gamma(a) = \gamma(b)$. ∎

Theorem 5.12 (Cauchy's Integral Theorem):

Let f be an analytic function on the set S where S is a disk, rectangle, or all of $\mathbb{C}$. Let C be a closed smooth curve contained in S then

$$\int_C f(z)\,dz = 0$$

Since in all of these cases, f has an antiderivative, this result follows from the previous lemma.
This result is also true for open simply connected sets S.

Theorem 5.13:

Suppose S is a rectangle, disk, or all of $\mathbb{C}$ and f is analytic in S. Then for any $z_0, z_1 \in S$ and every smooth curve C from z_0 to z_1 contained in S

$$\int_C f(z)\,dz$$

gives the same value.

Proof:

Suppose $\gamma_1\colon [a_1,b_1] \longrightarrow \mathbb{C}$ and $\gamma_2\colon [a_2,b_2] \longrightarrow \mathbb{C}$ are two smooth curves such that $\gamma_i(a_i) = z_0$ and $\gamma_i(b_i) = z_1$ for $i = 1,2$. We can assume that $a_2 = b_1$, then we define

$$\gamma\colon [a_1,b_2] \longrightarrow \mathbb{C}, \qquad \gamma(t) = \begin{cases} \gamma_1(t) & t \in [a_1,b_1] \\ \gamma_2(a_2+b_2-t) & t \in [a_2,b_2] \end{cases}$$

This is also a smooth closed curve and so

$$\int_\gamma f(z)\,dz = 0$$

But at the same time

$$0 = \int_\gamma f(z)\,dz = \int_{a_1}^{b_2} f(\gamma(t))\gamma'(t)\,dt = \int_{a_1}^{b_1} f(\gamma_1(t))\gamma_1'(t)\,dt - \int_{a_2}^{b_2} f(\gamma_2(a_2+b_2-t))\gamma_2'(a_2+b_2-t)\,dt =$$

$$= \int_{a_1}^{b_1} f(\gamma_1(t))\gamma_1'(t)\,dt - \int_{a_2}^{b_2} f(\gamma_2(t))\gamma_2'(t)\,dt = \int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz$$

And so we have

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$$

as required ∎

We say that a curve obtained from the boundary of a domain D is positive-oriented relative to D if as we proceed along the curve, D is to the left.

Theorem 5.14:

Suppose C is a closed simple smooth curve which is positive-oriented, and let $C_1, \ldots, C_k$ be closed simple smooth curves contained inside the domain whose boundary is C and are negative-oriented, and the interior of C_i does not intersect the interior of C_j for $i \neq j$. Let D be the domain contained in C and D_i the domain contained in C_i.
Then if f is an analytic function in $\overline{D} \setminus \{D_1 \uplus \cdots \uplus D_k\}$, then

$$\int_C f(z)\,dz + \sum_{i=1}^{k} \int_{C_i} f(z)\,dz = 0$$