# Infinitesimal Calculus 3

*Lecture 11, Wednesday November 23, 2022*
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## 11.1 Funky Functions

**Proposition 11.1.1:**

If $f\colon X \longrightarrow Y$ is continuous and $E \subseteq X$ is compact, $f(E)$ is also compact.

**Proof:**

Suppose $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$ is an open cover of $f(E)$, that is:

$$f(E) \subseteq \bigcup_{\lambda\in\Lambda} \mathcal{O}_\lambda \implies E \subseteq \bigcup_{\lambda\in\Lambda} f^{-1}(\mathcal{O}_\lambda)$$

Since $f$ is continuous, $f^{-1}(\mathcal{O}_\lambda)$ is open so $\left\{f^{-1}(\mathcal{O}_\lambda)\right\}_{\lambda\in\Lambda}$ is an open cover of $E$, which is compact so there is an open subcover of $E$, let it be $\left\{f^{-1}(\mathcal{O}_k)\right\}_{k=1}^n$, so:

$$E \subseteq \bigcup_{k=1}^n f^{-1}(\mathcal{O}_k) \implies f(E) \subseteq \bigcup_{k=1}^n \mathcal{O}_k$$

And so $\{\mathcal{O}_k\}_{k=1}^n$ is an open subcovering of $f(E)$, and therefore $f(E)$ is compact.

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**Proposition 11.1.2:**

If $E \subseteq X$ is compact and $f\colon E \longrightarrow Y$ is continuous, $f$ takes a minimum and maximum in $E$.

**Proof:**

Since $f(E)$ itself is compact, it is bounded. And so there exists $s = \sup f(E)$, which means that there exists $x_n \in E$ such that $f(x_n) \longrightarrow s$. And since $E$ is compact there exists a subsequence $x_{n_k}$ which converges to some value $x \in E$, and so $f\left(x_{n_k}\right) \longrightarrow s$, but at the same time since $f$ is continuous so $f\left(x_{n_k}\right) \longrightarrow f(x)$, and so $f(x) = s$. So $f$ takes a maximum in $E$. A similar proof can be used to show it takes a minimum.

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**Example:**

The continuous preimage of a compact set is not necessarily compact. Take for example $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ where $f(x) = c$. Then even though $\{c\}$ is compact, $f^{-1}(\{c\}) = \mathbb{R}$ is not compact.

**Proposition 11.1.3:**

If $E \subseteq X$ is compact and $f\colon E \longrightarrow Y$ is continuous and injective, then $f^{-1}\colon f(E) \longrightarrow E$ is continuous.

**Proof:**

Let $g = f^{-1}$ on $f(E)$. We will show that for every $K \subseteq E$ closed, $g^{-1}(K)$ is also closed. Since $E$ is compact and $K$ is closed in $E$, $K$ is compact in $X$. And $g^{-1} = f$, so $g^{-1}(K) = f(K)$ which is compact since $f$ is continuous, and so it is closed. So the preimage under $g$ of any closed set is also closed, therefore $g$ is continuous.

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**Definition 11.1.4:**

A mapping $f$ between two metric spaces $X$ and $Y$ is **uniformly continuous** if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $x, y \in X$ if $\rho(x, y) < \delta$ then $\sigma(f(x), f(y)) < \varepsilon$.

**Theorem 11.1.5:**

If $E \subseteq X$ is compact and $f\colon E \longrightarrow X$ is continuous, then it is uniformly continuous.

**Proof:**

Let $\varepsilon > 0$ then for every $x \in E$ let $\delta_x > 0$ satisfy continuity for $\varepsilon$, that is $f\big(B_{\delta_x}(x)\big) \subseteq B_\varepsilon\big(f(x)\big)$. And so:

$$E \subseteq \bigcup_{x \in E} B_{\frac{1}{2}\delta_x}(x)$$

This is an open cover so there is an open subcover $\left\{B_{\frac{1}{2}\delta_k}(x_k)\right\}_{k=1}^{n}$. We can take $\delta = \frac{1}{2} \min_{1 \leq k \leq n} \delta_k > 0$. Let $y, z \in E$ such that $\rho(x, y) < \delta$, then there exists some $x_1$ and $x_2$ in the open subcover such that $y \in B_1(x_1)$ and $z \in B_2(x_2)$ so

$$\rho(z, x_1) \leq \rho(z, y) + \rho(y, x_1) < \frac{\delta_1}{2} + \delta \leq \delta_1$$

So $z \in B_1(x_1)$ and therefore $\sigma(f(z), f(x_1)) < \varepsilon$ and so $\sigma(f(y), f(z)) < 2\varepsilon$. And so $f$ is uniformly continuous.

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**Proposition 11.1.6:**

Suppose $f\colon X \longrightarrow Y$ is continuous and $E \subseteq X$ is connected, then $f(E)$ is connected in $Y$.

**Proof:**

Suppose $f(E) \subseteq \mathcal{O}_1 \cup \mathcal{O}_2$, so $E \subseteq f^{-1}(\mathcal{O}_1) \cup f^{-1}(\mathcal{O}_2)$, which are open since $f$ is continuous. Since $E$ is connected, $E \subseteq f^{-1}(\mathcal{O}_1)$ or $f^{-1}(\mathcal{O}_2)$ and therefore $f(E) \subseteq \mathcal{O}_1$ or $\mathcal{O}_2$. So $f(E)$ is connected as required.

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Note that if $E$ is connected in $\mathbb{R}$ and $a < b \in E$ than $(a, b) \subseteq E$. Suppose $c \in (a, b) \setminus E$ then $E \subseteq \mathbb{R}_{<c} \cup \mathbb{R}_{>c}$ which are both open and have non-empty intersections with $E$ (namely $a$ and $b$), in contradiction to $E$'s connectedness. This helps prove our next corollary.

**Corollary 11.1.7:**

Suppose $X$ is a metric space and $f\colon X \longrightarrow \mathbb{R}$ is continuous. Suppose $E \subseteq X$ is connected, then if $a, b \in X$ for $f(a) < f(b)$ then for every $f(a) < y < f(b)$, there is an $x \in E$ such that $f(x) = y$.

**Proof:**

Since $f$ is continuous and $E$ is connected, $f(E)$ is connected in $\mathbb{R}$. Since it contains $f(a)$ and $f(b)$, it must contain everything between them (since it is a connected set in $\mathbb{R}$), ie $\big(f(a), f(b)\big) \subseteq f(E)$, as required.

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Thus concludes the topological portion of our course.

## 11.2 Partial Derivatives

**Definition 11.2.1:**

If $f$ is a real valued function defined at $(x_0, y_0)$ then the **partial derivative** of $f$ relative to $x$ is defined as:

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

And the partial derivative relative to $y$ is:

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

Other notations include:

$$\frac{\partial f}{\partial x}(x_0, y_0) = f_x(x_0, y_0) = \partial_x f(x_0, y_0)$$

**Lemma 11.2.2:**

If we define $g(x) = f(x, y_0)$ then $\partial_x f(x_0, y_0) = g'(x_0)$.

This lemma is trivial, but helps us understand how to compute partial derivatives. We simply keep the other variable constant and diffeentiate.

**Proof:**

The proof is trivial:

$$g'(x_0) = \lim_{h \to 0} \frac{g(x_0 + h) - g(x_0)}{h} = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} = \partial_x f(x_0, y_0)$$

**Example:**

$$f(x, y) = \sin\left(x^2 y + y^3\right)$$

We will compute $\partial_x f(1, 2)$:

$$\partial_x f(1, 2) = \left(\sin\left(2x^2 + 8\right)\right)'(1) = \left(4x \cdot \cos\left(2x^2 + 8\right)\right)(1) = 4\cos(10)$$