

Algebraic Topology II

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1 Singular Homology

1.1 Chain Complexes

We begin by defining a *chain complex*. A chain complex is a sequence of Abelian groups with homomorphisms between them:

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow 0$$

such that for every n , $\partial_n \circ \partial_{n+1} = 0$, in other words $\text{Im} \partial_{n+1} \subseteq \ker \partial_n$. Define $Z_n = \ker \partial_n$, and its elements will be called *n -dimensional cycles*. And define $B_n = \text{Im} \partial_{n+1}$, its elements will be called *boundaries*. Elements of the groups C_n will be called *n -dimensional chains*.

We now want to define a category of chain complexes. To do so we must define morphisms between chain complexes. So suppose we have two chain complexes $\mathcal{C} = \{C_n, \partial_n\}$ and $\mathcal{D} = \{D_n, \partial'_n\}$. We define a morphism from \mathcal{C} to \mathcal{D} to be a sequence of homomorphisms $f_n: C_n \longrightarrow D_n$ which preserves the structure of the chain. Meaning $\partial'_n \circ f_n = f_{n-1} \circ \partial_n$, in other words the following diagram commutes:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \longrightarrow \cdots & \longrightarrow & C_0 & \longrightarrow & 0 \\ & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} & & & \downarrow f_0 & & \\ \cdots & \longrightarrow & D_{n+1} & \xrightarrow{\partial'_{n+1}} & D_n & \xrightarrow{\partial'_n} & D_{n-1} & \longrightarrow \cdots & \longrightarrow & D_0 & \longrightarrow & 0 \end{array}$$

To simplify writing, we will write $\partial \circ f = f \circ \partial$, which f and which ∂ is being referred to will be understood from context.

The composition of two morphisms $\{f_n\}: \mathcal{C} \longrightarrow \mathcal{D}$ and $\{g_n\}: \mathcal{D} \longrightarrow \mathcal{E}$ is defined to be $\{g_n \circ f_n\}: \mathcal{C} \longrightarrow \mathcal{E}$. This is indeed a morphism:

$$\partial \circ f \circ g = f \circ \partial \circ g = f \circ g \circ \partial$$

And then this implies that the identity morphism is just $\text{Id}_{\mathcal{C}} = \{\text{Id}_{C_n}\}: \mathcal{C} \longrightarrow \mathcal{C}$, as if $\{f_n\}: \mathcal{C} \longrightarrow \mathcal{D}$ then

$$\{f_n\} \circ \text{Id}_{\mathcal{C}} = \{f_n \circ \text{Id}_{C_n}\} = \{f_n\}, \quad \text{Id}_{\mathcal{D}} \circ \{f_n\} = \{\text{Id}_{D_n} \circ f_n\} = \{f_n\}$$

Associativity is clear, so **Comp**, the category of chain complexes, is indeed a category.

Now recall that by definition $\partial_n \circ \partial_{n+1} = 0$, meaning

$$B_n \subseteq Z_n \subseteq C_n$$

Since these groups are all Abelian, they are normal in one another, so let us define the *n th homology group* of a chain complex \mathcal{C} as

$$H_n(\mathcal{C}) := Z_n / B_n = \ker \partial_n / \text{Im} \partial_{n+1}$$

1.1 Proposition

A chain complex morphism $\{f_n\}: \mathcal{C} \longrightarrow \mathcal{D}$ maps cycles to cycles and boundaries to boundaries.

Proof: let $z \in C_n$ be a cycle, i.e. $\partial z = 0$, but then $f(z)$ is a cycle since $\partial f(z) = f(\partial z) = f(0) = 0$. And let $b \in C_n$ be a boundary, so there exists an $a \in C_{n+1}$ such that $b = \partial a$. Then $f(b) = f(\partial a) = \partial f(a) = \partial b$, so $f(b)$ is a boundary as well. ■

This means that if $\{f_n\}: \mathcal{C} \longrightarrow \mathcal{D}$ is a morphism of chain complexes, $\{f_n\}: Z_n(\mathcal{C}) \longrightarrow Z_n(\mathcal{D})$ is well-defined, and so we have that

$$\begin{array}{ccc} Z_n(\mathcal{C}) & \xrightarrow{\quad} & Z_n(\mathcal{D}) \\ \downarrow & \searrow \text{blue arrow} & \downarrow \\ H_n(\mathcal{C}) & & H_n(\mathcal{D}) \end{array}$$

Where the blue arrow ψ is just the quotient map composed with f_n . This induces a group morphism

$$H_n(\{f_n\}) = f_*: H_n(\mathcal{C}) \longrightarrow H_n(\mathcal{D})$$

2 Singular Complex

since we can define $f_*([z]) = \psi(z)$ since if $[z] = [z']$ then $z - z' \in B_n(\mathcal{C})$ and so $f(z - z') \in B_n(\mathcal{D})$ and thus the quotient of $f(z - z')$ is just 0, so $\psi(z) = \psi(z')$. Explicitly,

$$f_*[z] = [f_n z]$$

We now claim that H_n is a functor from the category of chain complexes **Comp** to the category of Abelian groups **Ab**. Now suppose $\{f_n\}: \mathcal{C} \rightarrow \mathcal{D}$ and $\{g_n\}: \mathcal{D} \rightarrow \mathcal{E}$ are chain complex morphisms, then the following diagram commutes

$$\begin{array}{ccccc} Z_n(\mathcal{C}) & \xrightarrow{f} & Z_n(\mathcal{D}) & \xrightarrow{g} & Z_n(\mathcal{E}) \\ \downarrow & & \downarrow & & \downarrow \\ H_n(\mathcal{C}) & \xrightarrow{f_*} & H_n(\mathcal{D}) & \xrightarrow{g_*} & H_n(\mathcal{E}) \end{array}$$

And so $(g \circ f)_* = g_* \circ f_*$, and it is easily verified that $\text{id}_* = \text{id}$ so H_n is a functor **Comp** \rightarrow **Ab** (the category of Abelian groups).

1.2 Singular Complex

We now define a functor from **Top** to **Comp**.

1.1 Definition

Let B be a set, then define the **free Abelian group** over B to be

$$\text{FA}(B) = \bigoplus_{b \in B} \mathbb{Z} = \{ \varphi: B \rightarrow \mathbb{Z} \mid \varphi(b) \neq 0 \text{ for only finitely many } b \in B \}$$

Note then that there is a correspondence between B and $\text{FA}(B)$: $b \leftrightarrow \varphi_b$ where

$$\varphi_b(x) = \begin{cases} 1 & x = b \\ 0 & x \neq b \end{cases}$$

so we can identify b with φ_b , and it is easy to see that every element of $\text{FA}(B)$ can be written as $\sum_{i=1}^k n_i \varphi_{b_i}$, abusing notation $\sum_{i=1}^k n b_i$ and such a representation is unique.

Notice that if B is a set, G an Abelian group, and $g: B \rightarrow G$ a function, then there exists a unique group homomorphism $L: \text{FA}(B) \rightarrow G$ which extends g . This is defined by

$$L: \sum_{i=1}^k n_i b_i \mapsto \sum_{i=1}^k n_i g(b_i)$$

1.2 Definition

The n -**dimensional simplex** is defined to be

$$\Delta^n := \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0, \sum_{i=0}^n x_i = 1 \right\}$$

Δ^n has $n + 1$ faces, and is homeomorphic to D^n .

1.3 Definition

Let X be a topological space, then an n -**dimensional singular simplex** in X is a morphism (in the category of topological spaces; a continuous map) $\Delta^n \rightarrow X$. Define $S_n(x)$ to be the set of all n -dimensional

singular simplexes in X , and define $C_n(X) = \text{FA}(S_n(x))$.

We now want to define a chain complex on the sequence $C_n(X)$.

Let us define a set of maps $\tau_i^n: \Delta^{n-1} \rightarrow \Delta^n$ for $0 \leq i \leq n$ which maps

$$\tau_i^n: (x_0, \dots, x_{n-1}) \mapsto (x_0, \dots, x_{i-1}, 0, x_i, \dots, x_{n-1})$$

This is a well-defined continuous map, and geometrically it maps Δ^{n-1} to one of the faces of Δ^n .

Let $\sigma \in S_n(x)$, then let us define

$$\partial(\sigma) := \sum_{i=0}^n (-1)^i \sigma \circ \tau_i^n$$

Note that the composition is well-defined since $\Delta^{n-1} \xrightarrow{\tau_i^n} \Delta^n \xrightarrow{\sigma} X$, meaning $\sigma \circ \tau_i^n$ is an $n-1$ -dimensional singular simplex. Thus ∂ can be extended to a map $\partial: C_n(X) = \text{FA}(S_n(X)) \rightarrow \text{FA}(S_{n-1}(X)) = C_{n-1}(X)$. Notice that

$$\partial_{n-1}(\partial_n \sigma) = \partial_{n-1} \left(\sum_{i=0}^n (-1)^i \sigma \circ \tau_i^n \right) = \sum_{i=0}^n (-1)^i \partial_{n-1}(\sigma \circ \tau_i^n) = \sum_{i=0}^n \sum_{j=0}^{n-1} (-1)^{i+j} \sigma \circ \tau_i^n \circ \tau_j^{n-1}$$

Notice that $\tau_i^n \circ \tau_j^{n-1} = \tau_j^n \circ \tau_{i-1}^{n-1}$ which can be verified from its definition, but the first has a sign of $(-1)^{i+j}$ in the sum and the second has $-(-1)^{i+j}$. And so the sum is zero.

Thus we have defined a chain complex on $C_n(X)$, let us denote it by $\mathcal{C}(X)$, this is the first step in defining the functor. Next we must define the correspondence between morphisms.

Let $f: X \rightarrow Y$ be a continuous map between topological spaces. Let us define $f_\#: \mathcal{C}(X) \rightarrow \mathcal{C}(Y)$. First we define it for $\sigma \in S_n(X)$ by $f_\#(\sigma) = f \circ \sigma$. Since $\sigma: \Delta^n \rightarrow X$ is continuous, so is $f \circ \sigma: \Delta^n \rightarrow Y$ and so $f_\#$ is well-defined on the generators of $C_n(X)$. This can be extended by linearity to $f_\#: C_n(X) \rightarrow C_n(Y)$. Notice that we ignore the subscripts and superscripts $(f_\#)_n^X$ for brevity and readability.

Now we must verify that this is a morphism of chain complexes, i.e. that $\partial f_\# = f_\# \partial$. So

$$f_\# \partial \sigma = f_\# \left(\sum_{i=0}^n (-1)^i \sigma \circ \tau_i^n \right) = \sum_{i=0}^n (-1)^i f_\#(\sigma \circ \tau_i^n) = \sum_{i=0}^n (-1)^i f \circ \sigma \circ \tau_i^n = \sum_{i=0}^n (-1)^i (f \circ \sigma) \circ \tau_i^n = \partial f_\# \sigma$$

and since this holds for generators, by linearity it holds for all $C_n(X)$. Thus $f_\#$ is indeed a morphism of chain complexes.

Thus we have defined a functor $\mathbf{Top} \rightarrow \mathbf{Comp}$.

1.3 Singular Homology

We have two functors $\mathbf{Top} \rightarrow \mathbf{Comp} \rightarrow \mathbf{Ab}$, and so composing them together gives us a functor $\mathbf{Top} \rightarrow \mathbf{Ab}$. For a topological space X , we will denote its image under this functor as $H_n(X)$, called the n th homological group of X . And for a continuous map f , we denote its image as f_* or $H_n(f)$.

Let us compute the homological groups of the trivial space: $X = \{p\}$. Notice that $S_n(X) = \{K_n\}$ where K_n is the constant map $\Delta^n \rightarrow \{p\}$, and so $C_n(X) = \mathbb{Z}$. We want to now compute what the boundary operators are, so

$$\partial K_n = \sum_{i=0}^n (-1)^i K_n \circ \tau_i^n$$

but $K_n \circ \tau_i^n$ is a morphism $\Delta^{n-1} \rightarrow \{p\}$ meaning it is equal to K_{n-1} , thus $\partial K_n = (\sum_{i=0}^n (-1)^i) K_{n-1}$. For n even this is then K_{n-1} (or 1), and 0 for n odd. This means that either $\ker \partial = 0$ or $\text{Im} \partial = \mathbb{Z}$, thus $H_n = 0$ for $n > 0$. For $n = 0$, we have that $\partial_0: \mathbb{Z} \rightarrow 0$ and so its kernel is \mathbb{Z} , but ∂_1 is trivial and so its image is 0. Thus $H_0 = \mathbb{Z}$.

So we have shown

1.1 Proposition

Let $X = \{p\}$ be the trivial topological space, then its homological groups are

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n > 0 \end{cases}$$

1.2 Proposition

Let X be path connected, then $H_0(X) \cong \mathbb{Z}$.

Proof: we are concerned with the chain:

$$C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

So first let us understand $C_0(X)$, this is generated by $S_0(X)$, all the maps $\Delta^0 \rightarrow X$ which are just all the points in X . And $S_1(X)$ is generated by all the maps $I \cong \Delta^1 \rightarrow X$, so all the paths in X . The boundary of a 1-simplex is then

$$\partial_1 \sigma = \sigma(1) - \sigma(0)$$

and thus $B_1(X) = \text{Im} \partial_1$ is generated by elements of the form $a - b$ where there exists a path between a and b . Since X is path-connected, this means that $B_1(X)$ is generated by $a - b$ for $a, b \in X$. Now, the subgroup generated by this is $\{\sum n_i p_i \mid p_i \in X, \sum n_i = 0\}$.

And now ∂_0 's kernel is just $C_0(X)$ which is simply the free group generated by X . Thus

$$H_0(X) = \left\{ \sum n_i p_i \right\} / \left\{ \sum n_i p_i \mid \sum n_i = 0 \right\}$$

This is isomorphic to \mathbb{Z} since we can define $\varphi: C_0(X) \rightarrow \mathbb{Z}$ by $\sum n_i p_i \mapsto \sum n_i$ and this is a group homomorphism whose image is \mathbb{Z} and whose kernel is all the points $\sum n_i p_i$ where $\sum n_i = 0$. Thus by the isomorphism theorem, $H_0(X) \cong \mathbb{Z}$. ■

1.3 Theorem

Let X be a topological space where $\{A_\alpha\}_{\alpha \in I}$ are its path connected components. Then for every n ,

$$H_n(X) \cong \bigoplus_{\alpha \in I} H_n(A_\alpha)$$

Proof: notice that if $\sigma: \Delta^n \rightarrow X$ is an n -simplex, then its image is contained within a path connected component. This is since Δ^n is path-connected, so $\sigma \Delta^n$ must be too. Thus for every $\gamma = \sum n_i \sigma_i \in S_n(X)$ we can write it as $\gamma = \sum \gamma_i$ for $\gamma_i \in S_n(A_i)$. And so $C_n(X) = \bigoplus_{\alpha \in I} C_n(A_\alpha)$.

Notice that γ is a cycle iff every γ_i is a cycle, since $\partial \gamma = \sum \partial \gamma_i$ and this is an element of a direct sum, so it is zero iff $\partial \gamma_i = 0$. Thus $Z_n(X) = \bigoplus_{\alpha \in I} Z_n(A_\alpha)$. And similarly we see that $B_n(X) = \bigoplus_{\alpha \in I} B_n(A_\alpha)$. Thus $H_n(X) = \bigoplus_{\alpha \in I} H_n(A_\alpha)$. ■

1.4 Corollary

If X is a topological space with $\{A_\alpha\}_{\alpha \in I}$ path connected components, $H_n(X) = \bigoplus_{\alpha \in I} \mathbb{Z}$.

1.5 Theorem

Let X be path-connected and $a \in X$, then $H_1(X) \cong \text{Ab } \pi_1(X, a)$.

For two chains, $a, b \in C_n(X)$ say that they are *homological* if $a - b$ is a boundary (i.e. $a - b \in B_n(X)$). Write this as $a \approx b$.

1.6 Lemma

Let σ, τ be 1-simplexes.

- (1) if σ is constant, then it is a boundary, i.e. $\sigma \approx 0$.
- (2) if $\sigma \stackrel{\partial I}{\sim} \tau$ (since they are maps from $I \cong \Delta^1 \longrightarrow X$), then $\sigma \approx \tau$.
- (3) if $\sigma(1) = \tau(0)$ then $\sigma * \tau \approx \sigma + \tau$.
- (4) $\sigma + \bar{\sigma} \approx 0$

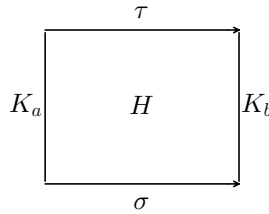
Proof:

- (1) If σ is constant, then it is K_p^1 for some $p \in X$. And as we have already computed

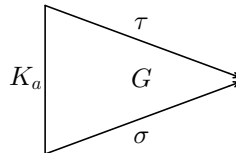
$$\partial K_p^n = \begin{cases} K_p^{n-1} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Thus $\partial K_p^2 = K_p^{n-1}$, meaning σ is a boundary.

- (2) Let us look at the homotopy



Since H is surjective, it induces a map on the quotient space $I \times I / I \times \{1\}$, the map G :



The quotient space can be viewed as a 2-simplex by assigning an order to its vertices. Then its boundary is

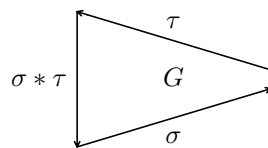
$$\partial G = K_a - \sigma + \tau$$

and since ∂G is a boundary, we have that

$$K_a - \sigma + \tau \approx 0$$

by (1) we have that $K_a \approx 0$ so $\sigma - \tau \approx 0$.

- (3) The idea is to define a simplex of the form



Notice that such a simplex is possible: each horizontal line in the domain can be made constant. And its boundary is

$$\partial G = \tau - \sigma * \tau + \sigma$$

so $\sigma * \tau \approx \sigma + \tau$ since $\partial G \approx 0$.

(4) This is direct from the previous three points:

$$\sigma + \bar{\sigma} \stackrel{(3)}{\approx} \sigma * \bar{\sigma} \stackrel{(2)}{\approx} K_b \stackrel{(1)}{\approx} 0$$

Proof (of theorem 1.5): let us define a homomorphism

$$F: \pi_1(X, a) \longrightarrow H_1(X)$$

Denote homotopy equivalence classes by $\langle \bullet \rangle$ and the equivalence classes of $H_1(X)$ by $[\bullet]$. Then we define

$$\langle \varphi \rangle \xrightarrow{F} [\varphi]$$

This is well-defined: if $\varphi \stackrel{\partial I}{\sim} \psi$ then $\varphi \approx \psi$ and so $[\varphi] = [\psi]$ (since $H_n(X)$ is the partition of $Z_n(X)$ relative to \approx). Notice that $\langle \varphi * \psi \rangle \mapsto [\varphi * \psi] = [\varphi + \psi] = [\varphi] + [\psi]$. So this is indeed a homomorphism. Since $H_1(X)$ is Abelian, this induces a homomorphism

$$\bar{F}: \text{Ab } \pi_1(X, a) \longrightarrow H_1(X)$$

Let us now define a homomorphism

$$G: C_1(X) \longrightarrow \text{Ab } \pi_1(X, a)$$

denote the equivalence classes of $\text{Ab } \pi_1(X, a)$ by $\langle \langle \bullet \rangle \rangle$. For every $x \in X$, choose a path γ_x from a to x , then for $\sigma \in S_1(X)$ define

$$\hat{\sigma} = \gamma_{\sigma(0)} * \sigma * \bar{\gamma}_{\sigma(1)} \text{ from } a \text{ to } a$$

And define

$$\sigma \xrightarrow{G} \langle \langle \hat{\sigma} \rangle \rangle$$

And extend by linearity to $G: C_1(X) \longrightarrow \text{Ab } \pi_1(X, a)$. We can then restrict G to $Z_1(X)$, and in order for this to induce a map on $Z_1(X)/B_1(X)$ we must have that $G|_{B_1(X)} = 0$. So let A be a 2-simplex, then we must show $G(\partial A) = 0$. We know

$$G(\partial A) = G(A \circ \tau_0 - A \circ \tau_1 + A \circ \tau_2) = \langle \langle \widehat{A \circ \tau_0} \rangle \rangle - \langle \langle \widehat{A \circ \tau_1} \rangle \rangle + \langle \langle \widehat{A \circ \tau_2} \rangle \rangle$$

Now, $\langle \langle \sigma \rangle \rangle + \langle \langle \tau \rangle \rangle = \langle \langle \sigma \tau \rangle \rangle$ and $-\langle \langle \sigma \rangle \rangle = \langle \langle \sigma^{-1} \rangle \rangle$ by Abelianization, so this is equal to

$$\langle \langle \widehat{A \circ \tau_0} \rangle \rangle \langle \langle \widehat{A \circ \tau_1} \rangle \rangle \langle \langle \widehat{A \circ \tau_2} \rangle \rangle = \langle \langle \widehat{A \circ \tau_0 * \overline{A \circ \tau_1} * A \circ \tau_2} \rangle \rangle$$

As is easily verified,

$$= \langle \langle \widehat{A \circ \tau_0 * \overline{A \circ \tau_1} * A \circ \tau_2} \rangle \rangle = \langle \langle \overline{A \circ \tau_0 * \overline{A \circ \tau_1} * A \circ \tau_2} \rangle \rangle$$

Since $A: \Delta^2 \longrightarrow X$ is a simplex, $A \circ \tau_0 * \overline{A \circ \tau_1} * A \circ \tau_2$ is null-homotopic (the homotopy can condense the curve to a point through the image of A). Therefore its hat is as well, meaning this is all equal to zero, as required.

So G induces a homomorphism

$$\bar{G}: H_1(X) \longrightarrow \text{Ab } \pi_1(X, a)$$

Notice that

$$\bar{G} \circ \bar{F} \langle \langle \varphi \rangle \rangle = \bar{G}[\varphi] = \langle \langle \hat{\varphi} \rangle \rangle$$

We know that $\hat{\varphi} = \gamma_a \varphi \bar{\gamma}_a$ which is conjugate to φ , so in the Abelianization they are equal. So $\bar{G} \circ \bar{F} = \text{id}$. Now suppose $[z] \in H_1(X)$ where $z = \sum n_i \sigma_i$ then

$$\bar{F} \circ \bar{G}[z] = \bar{F} \left(\sum n_i \langle \langle \hat{\sigma}_i \rangle \rangle \right) = \sum n_i [\hat{\sigma}_i] = \left[\sum n_i \hat{\sigma}_i \right]$$

So we need to show that if $\sum n_i \sigma_i$ is a cycle then $\sum n_i \hat{\sigma}_i \approx \sum n_i \sigma_i$. Define $T: C_0(X) \longrightarrow C_1(X)$ by $T(p) = \gamma_p$, so

$$\hat{\sigma} = \gamma_{\sigma(0)} * \sigma * \bar{\gamma}_{\sigma(1)} \approx \gamma_{\sigma(0)} + \sigma - \gamma_{\sigma(1)} = \sigma - T\partial\sigma$$

And so

$$\sum n_i \hat{\sigma}_i \approx \sum n_i \sigma_i - T\partial \sum n_i \sigma_i = z - T\partial z$$

since z is a cycle, $\partial z = 0$ and so this is equal to z . Thus $\hat{z} \approx z$ as required.

So \bar{F}, \bar{G} are inverse isomorphisms, meaning $H_1(X) \cong \text{Ab } \pi_1(X, a)$. ■

1.7 Definition

Let \mathcal{C}, \mathcal{D} be two categories and let $F, G: \mathcal{C} \rightarrow \mathcal{D}$ be functors. Then a **natural transformation** η from F to G is a correspondence such that

- (1) for every object $X \in \mathcal{C}$, η associates a morphism $\eta_X: F(X) \rightarrow G(X)$ called the **component** of X .
- (2) for every $f: X \rightarrow Y$ morphism, $\eta_Y \circ F(f) = G(f) \circ \eta_X$, i.e. the following diagram commutes

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ F(f) \downarrow & & \downarrow G(f) \\ F(Y) & \xrightarrow{\eta_Y} & G(Y) \end{array}$$

So for every pointed topology (X, a) we defined a group homomorphism $F_{X,a}: \pi_1(X, a) \rightarrow H_1(X)$. We claim that this is a natural transformation from π_1 to H_1 .

Suppose there is a morphism $h: (X, a) \rightarrow (Y, b)$, so we need the following diagram to commute:

$$\begin{array}{ccc} \pi_1(X, a) & \xrightarrow{F_{X,a}} & H_1(X) \\ \pi_1(h) \downarrow & & \downarrow H_1(h) \\ \pi_1(Y, b) & \xrightarrow{F_{Y,b}} & H_1(Y) \end{array}$$

This is indeed the case:

$$\langle \varphi \rangle \xrightarrow{F_{X,a}} [\varphi] \xrightarrow{H_1(h)} [h \circ \varphi], \quad \langle \varphi \rangle \xrightarrow{\pi_1(h)} \langle h \circ \varphi \rangle \xrightarrow{F_{Y,b}} [h \circ \varphi]$$

1.8 Example

If we look at the identity functor (on the category of groups) and Abelianization, then ρ_\bullet , which is the quotient map $\bullet \rightarrow \text{Ab } \bullet$, is a natural transformation. Indeed

$$\begin{array}{ccc} G & \xrightarrow{\rho_G} & \text{Ab } G \\ \varphi \downarrow & & \downarrow \hat{\varphi} \\ H & \xrightarrow{\rho_H} & \text{Ab } H \end{array}$$

Where $\hat{\varphi}[g] = [\varphi(g)]$. This is indeed natural:

$$\rho_H \circ \varphi(g) = [\varphi(g)], \quad \hat{\varphi} \circ \rho_G(g) = \hat{\varphi}[g] = [\varphi(g)]$$

1.9 Definition

The **simplified singular chain complex** of a topological space X is the chain complex

$$\cdots \dashrightarrow C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \dashrightarrow \cdots \dashrightarrow C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \rightarrow 0$$

Where we define ε as follows:

$$\varepsilon \sum n_i p_i = \sum n_i$$

i.e. $\varepsilon p = 1$ for every $p \in X$. And a morphism between two simplified singular chain complexes differ only from morphisms between normal singular chain complexes in that the map from \mathbb{Z} to \mathbb{Z} is the identity.

The homology induced by a simplified singular chain complex is called the **reduced homology** and denoted $\tilde{H}_n(X)$.

Obviously for every $n \geq 1$, $\tilde{H}_n(X) = H_n(X)$. Recall that if X is path-connected, then $B_0(X)$ is generated by $a - b$ for $a, b \in X$, so it is $\{\sum n_i p_i \mid \sum n_i = 0\}$. Now $\ker \varepsilon = \{\sum n_i p_i \mid \sum n_i = 0\}$ as well, and so we get that when X is path-connected, $\tilde{H}_0(X) = 0$.

1.10 Definition

A chain of Abelian groups

$$\cdots \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \cdots$$

is **exact** at B if $\text{Im} f = \ker g$. If the sequence is exact at every group, then the sequence itself is called an **exact sequence**. (Recall that chain complexes require $\text{Im} f \subseteq \ker g$.)

If we have an exact sequence in one of the following forms, then:

- (1) $0 \longrightarrow A \xrightarrow{f} B$, then $0 = \ker f$ so f is injective.
- (2) $A \xrightarrow{f} B \longrightarrow 0$, then $\text{Im} f = B$ so f is surjective.
- (3) $0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0$, then f is an isomorphism.

1.11 Definition

A **short exact sequence** is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

In a short exact sequence, by above f is injective and g is surjective, and furthermore $\text{Im} f = \ker g$. In such a case, we can view A as being a subgroup of B (since f is an embedding) and since by the isomorphism theorem $C \cong B/\ker g = B/\text{Im} f = B/A$, a short exact sequence can be viewed as

$$0 \longrightarrow A \xrightarrow{\text{inclusion}} B \xrightarrow{\text{quotient}} B/A \longrightarrow 0$$

1.12 Lemma (The Lemma of Five)

Suppose the chains $\{A_i\}_i, \{B_i\}_i$ are exact, and the following diagram commutes:

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ f_1 \downarrow & & f_2 \downarrow & & f_3 \downarrow & & f_4 \downarrow & & f_5 \downarrow \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

- (1) If f_2, f_4 are injective and f_1 is surjective, then f_3 is injective.
- (2) If f_2, f_4 are surjective and f_5 is injective, then f_3 is surjective.

Proof: We write $x \xrightarrow{A} y$ to mean x maps to y in the exact sequence ($x \in A_i$).

- (1) Suppose $f_3 a$, then $a \xrightarrow{f_3} 0 \xrightarrow{B} 0$, now suppose $a \xrightarrow{A} b \xrightarrow{f_4} c$. Since the diagram commutes, we must have that $c = 0$, but f_4 is injective so $b = 0$. This means $a \in \ker A$, so there exists some d such that $d \xrightarrow{A} a$. Suppose $d \xrightarrow{f_2} e$, then $e \xrightarrow{B} 0$ by commutativity, so there exists an f such that $f \xrightarrow{B} e$, and since f_1 is surjective there exists a $g \xrightarrow{f_1} f$. Now suppose $g \xrightarrow{A} h$. By commutativity, since $g \xrightarrow{f_1} f \xrightarrow{B} e$ we have $f_2 h = e$ and since f_2 is injective, $h = d$. So d is in the image of A , so it is in the kernel and so $a = 0$.
- (2) is a little more complicated, but it's just chasing. ■

1.13 Definition

Suppose \mathcal{C} and \mathcal{D} are two chain complexes, with two morphisms $f, g: \mathcal{C} \rightarrow \mathcal{D}$. Then a **chain homotopy** from f to g is a sequence of maps $T_n: C_n \rightarrow D_{n+1}$ such that $\partial T + T\partial = f - g$. If there exists a chain homotopy between f and g , we write $f \stackrel{CH}{\sim} g$.

In a diagram, we have that T are the red arrows.

$$\begin{array}{ccccccc}
 \cdots & \xrightarrow{\quad} & C_{n+1} & \xrightarrow{\quad} & C_n & \xrightarrow{\quad} & C_{n-1} & \xrightarrow{\quad} & \cdots \\
 & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} & & \\
 & & D_{n+1} & \xrightarrow{\quad} & D_n & \xrightarrow{\quad} & D_{n-1} & \xrightarrow{\quad} & \cdots
 \end{array}$$

(Red arrows represent $T_n: C_n \rightarrow D_{n+1}$)

Let $X \subseteq \mathbb{R}^k$ be convex. For $a \in X$ let us define the *cone construction* $C_a: C_n(X) \rightarrow C_{n+1}(X)$ as follows: we start with generators of $C_n(X)$, i.e. we define $C_a\sigma$ for $\sigma: \Delta^n \rightarrow X$ an n -simplex. Geometrically, $C_a\sigma$ will be a cone whose tip is a and whose base is σ . We define this by:

$$C_a\sigma(t_0, \dots, t_{n+1}) = t_0 b + (1 - t_0)\sigma\left(\frac{t_1}{1 - t_0}, \dots, \frac{t_{n+1}}{1 - t_0}\right)$$

Let us now compute the faces of $C_a\sigma$. For $i = 0$ then

$$(C_a\sigma)\tau_0^{n+1}(t_0, \dots, t_n) = C_a\sigma(0, t_0, \dots, t_n) = \sigma(t_0, \dots, t_n)$$

For $i > 0$ then

$$(C_a\sigma)\tau_i^{n+1}(t_0, \dots, t_n) = C_a\sigma(t_0, \dots, 0, \dots, t_n)$$

if $t_0 = 1$ as well, then this is just

$$C_a\sigma(1, 0, \dots, 0) = a$$

Otherwise,

$$\begin{aligned}
 &= t_0 b + (1 - t_0)\sigma\left(\frac{t_1}{1 - t_0}, \dots, 0, \dots, \frac{t_n}{1 - t_0}\right) \\
 &= t_0 b + (1 - t_0)\sigma\tau_{i-1}^n\left(\frac{t_1}{1 - t_0}, \dots, \frac{t_n}{1 - t_0}\right) \\
 &= C_a^{n-1}(\sigma\tau_{i-1}^n)(t_0, \dots, t_n)
 \end{aligned}$$

So we see that

$$(C_a\sigma)\tau_0^{n+1} = \sigma, \quad (C_a\sigma)\tau_i^{n+1} = C_a^{n-1}(\sigma\tau_{i-1}^n)$$

So

$$\begin{aligned}
 \partial_{n+1} C_a^n(\sigma) &= \sum_{i=0}^{n+1} (-1)^i (C_a^n \sigma) \tau_i^{n+1} = \sigma + \sum_{i=1}^{n+1} C_a^{n-1}(\sigma \tau_{i-1}^n) \\
 &= \sigma - \sum_{i=0}^n (-1)^i C_a^{n-1}(\sigma \tau_i^n) \\
 &= \sigma - C_a^{n-1} \left(\sum_{i=0}^n (-1)^i \sigma \tau_i^n \right) \\
 &= \sigma - C_a^{n-1} \partial_n \sigma
 \end{aligned}$$

So we see that

$$\partial C_a - C_a \partial = \text{id}$$

so in other words, C_a is a chain homotopy from id to 0.

1.14 Theorem

Let X be a convex set in \mathbb{R}^k , then for all $n > 0$, $H_n(X) = 0$.

Proof: let $\gamma \in C_n(X)$, then $\gamma = \partial C_a \gamma + C_a \partial \gamma$. If $\gamma \in Z_n(X)$, i.e. it is a cycle, then $\partial \gamma = 0$ and so $\gamma = \partial C_a \gamma$. Thus $\gamma \in B_n(X)$, so $Z_n(X) = B_n(X)$, and then $H_n(X) = 0$. ■

1.15 Lemma

If $f, g: X \rightarrow Y$ are two homotopic continuous maps, then $f_\#$ and $g_\#$ are chain homotopic.

Proof: let us define $i, j: X \rightarrow X \times I$ where $i(x) = (x, 0)$ and $j(x) = (x, 1)$. If $H: X \times I \rightarrow Y$ is a homotopy from f to g , then $f = H \circ i$ and $g = H \circ j$. Also $i \sim j$, so if we can show that $i_\# \stackrel{CH}{\sim} j_\#$ then we have that

$$f_\# = H_\# \circ i_\# \stackrel{CH}{\sim} H_\# \circ j_\# = g_\#$$

so it is sufficient to show that $i_\# \stackrel{CH}{\sim} j_\#$.

So we need to define a sequence of morphisms $T_n^X: C_n(X) \rightarrow C_{n+1}(X \times I)$ such that $\partial T^X + T^X \partial = i_\#^X - j_\#^X$. We will define T_n^X by induction on n , such that T^X is natural. Natural between what two functors? The first functor maps topological spaces X to their chain complexes $\mathcal{C}(X)$ and maps morphisms $X \xrightarrow{f} Y$ to $f_\#: \mathcal{C}(X) \rightarrow \mathcal{C}(Y)$. The second functor maps topological spaces X to the chain complex $C_{n+1}(X \times I)$ and morphisms $X \xrightarrow{f} Y$ to $(f \times \text{id})_\#: C_{n+1}(X) \rightarrow C_{n+1}(Y)$.

T^X being natural means that the diagram commutes for all $f: X \rightarrow Y$:

$$\begin{array}{ccc} C_n(X) & \xrightarrow{T^X} & C_{n+1}(X \times I) \\ f_\# \downarrow & & \downarrow (f \times \text{id})_\# \\ C_n(Y) & \xrightarrow{T^Y} & C_{n+1}(Y \times I) \end{array}$$

So $T_Y \circ f_\# = (f \times \text{id})_\# \circ T_X$.

Let $I_n: \Delta^n \rightarrow \Delta^n$ be the identity n -dimensional simplex. If we determine $T^{\Delta^n}(I_n)$, then we have determined $T^X(\sigma)$ for all $\sigma \in C_n(X)$ for all X . This is because $\sigma = \sigma \circ I_n = \sigma_\#(I_n)$, since we can view σ as a continuous map $X \rightarrow \Delta^n$ and so $\sigma_\#$ is defined. Thus

$$T^X(\sigma) = T^X \circ \sigma_\#(I_n) = (\sigma \times \text{id})_\# \circ T^{\Delta^n}(I_n)$$

And so determining $T^{\Delta^n}(I_n)$ determines $T^X(\sigma)$. So if we define $A = T^{\Delta^n}(I_n)$, then

$$T^X(\sigma) = (\sigma \times \text{id})_\#(A)$$

A is some simplex in $C_{n+1}(\Delta^n \times I)$, and we claim that for any choice of A , this defines a natural transformation. This is because

$$T^Y \circ f_\#(\sigma) = T^Y(f \circ \sigma) = ((f \circ \sigma) \times \text{id})_\#(A) = (f \times \text{id})_\# \circ (\sigma \times \text{id})_\#(A)$$

And

$$(f \times \text{id})_\# \circ T^X(\sigma) = (f \times \text{id})_\# \circ (\sigma \times \text{id})_\#(A)$$

so these are indeed equal, as required.

Now we claim that

$$(\partial T^X + T^X \partial)(\sigma) = (i_\#^X - j_\#^X)(\sigma)$$

for all X, σ . It is sufficient to show this for $X = \Delta^n$ and $\sigma = I_n$, since if

$$(\partial T^{\Delta^n} + T^{\Delta^n} \partial)(I_n) = (i_\#^{\Delta^n} - j_\#^{\Delta^n})(I_n)$$

if we compose it on the left with $(\sigma \times \text{id})_\#$, the LHS gives

$$(\partial(\sigma \times \text{id})_\# T^{\Delta^n} + (\sigma \times \text{id})_\# T^{\Delta^n} \partial)(I_n) = (\partial T^X \sigma_\# + T^X \partial \sigma_\#)(I_n) = \partial T^X \sigma + T^X \partial \sigma$$

since T is natural, $T^Y \circ f_{\#} = (f \times \text{id})_{\#} \circ T^X$ and $\partial f_{\#} = f_{\#} \partial$. The RHS is

$$((\sigma \times \text{id})_{\#} \circ i_{\#}^{\Delta^n} - (\sigma \circ \text{id}) \circ j_{\#}^{\Delta^n})(I_n)$$

Now notice that

$$\begin{aligned} \Delta^n &\xrightarrow{i^{\Delta^n}} \Delta^n \times I \xrightarrow{\sigma \times \text{id}} X \times I \\ s &\longmapsto (s, 0) \longmapsto (\sigma(s), 0) \end{aligned}$$

So $(\sigma \times \text{id}) \circ i^{\Delta^n} = i^X \circ \sigma$, and similar for j . So the RHS is just

$$i_{\#}^X \circ \sigma_{\#}(I_n) - j_{\#}^X \circ \sigma_{\#}(I_n) = i_{\#}^X(\sigma) - j_{\#}^X(\sigma)$$

So we get

$$\partial T^X(\sigma) + T^X \partial \sigma = i_{\#}^X(\sigma) - j_{\#}^X(\sigma)$$

as required.

So we must show that

$$\partial T I_n + T \partial I_n = i_{\#} I_n - j_{\#} I_n$$

in order to get this for every $\sigma \in C_n(\Delta^n)$. So we must show $\partial T I_n = -T \partial I_n + i_{\#} I_n - j_{\#} I_n$, since $\partial T I_n \in C_n(\Delta^n \times I)$, and $\Delta^n \times I$ is a convex set in \mathbb{R}^{n+2} . In a convex set so a simplex is a boundary if and only if it is a cycle. We want $-T \partial I_n + i_{\#} I_n - j_{\#} I_n$ to be a boundary, and so it is sufficient to check that it is a cycle:

$$-\partial T \partial I_n + \partial i_{\#} I_n - \partial j_{\#} I_n$$

Since $\partial I_n \in C_{n-1}(\Delta^n)$, we have that

$$\partial T \partial I_n + T \partial \partial I_n = i_{\#} \partial I_n - j_{\#} \partial I_n$$

and thus we must have that the following is zero:

$$T \partial \partial I_n - i_{\#} \partial I_n + j_{\#} \partial I_n + \partial i_{\#} I_n - \partial j_{\#} I_n$$

Since $\partial \partial = 0$, and $i_{\#}, j_{\#}$ are chain homomorphisms, this is indeed zero. So $-T \partial I_n + i_{\#} I_n - j_{\#} I_n$ is a cycle and thus a boundary since the universe is convex. So let us take A to be a chain such that ∂A is this element. ■

So notice now that if $f \sim g$, then $f_{\#} \sim g_{\#}$ are chain homotopic, and so $f_* = g_*$.

1.16 Corollary

If $f: X \rightarrow Y$ is a homotopy equivalence, then $f_*: H_n(X) \rightarrow H_n(Y)$ is an isomorphism.

Proof: there exists a $g: Y \rightarrow X$ such that $fg \sim \text{id}_Y$ and $gf \sim \text{id}_X$. Thus

$$g_* \circ f_* = (g \circ f)_* = (\text{id}_X)_* = \text{id}_{H_n(X)}$$

and similarly $f_* \circ g_* = \text{id}_{H_n(Y)}$, so f_* is an isomorphism. ■

1.4 Mayer-Vietoris

1.1 Definition

Let p_1, \dots, p_n be vectors in a vector space, then their **affine hull** is

$$\text{CH}(p_1, \dots, p_n) = \left\{ \sum_{i=1}^n \alpha_i p_i \mid \sum_{i=1}^n \alpha_i = 1 \right\}$$

Elements of the affine hull are called **affine combinations**. We similarly define the **convex hull**:

$$\text{CH}(p_1, \dots, p_n) = \left\{ \sum_{i=1}^n \alpha_i p_i \mid \sum_{i=1}^n \alpha_i = 1, \alpha_i \geq 0 \right\}$$

And its elements are called **convex combinations**.

1.2 Definition

p_1, \dots, p_n are **affine independent** if $\sum_{i=1}^n \alpha_i v_i = 0$ and $\sum_{i=1}^n \alpha_i = 0$ implies every α_i is 0.

1.3 Definition

$A \subseteq \mathbb{R}^k$ is an **n -simplex** if it is the convex hull of a set of $n + 1$ affine independent set of vectors.

1.4 Definition

Let $\Sigma = \text{CH}(p_0, \dots, p_n)$ be an n -simplex, then its i th **face** is $\text{CH}(p_0, \dots, p_{i-1}, p_i, \dots, p_n)$. And its **barycenter** is

$$b = \frac{1}{n+1} \sum_{i=0}^n p_i$$

We define the **barycentric subdivision** of Σ , denoted $\text{Sd } \Sigma$, to be a set of n -simplices which we define inductively on n as follows:

- (1) For a 0-simplex, $\text{Sd } \Sigma = \Sigma$.
- (2) If Σ is an n -simplex, then let $\varphi_0, \dots, \varphi_n$ be its faces (which are $n - 1$ -simplices) and b its barycenter. Then define $\text{Sd } \Sigma$ to be the n -simplices spanned by b and the simplices in the barycentric subdivisions of φ_i . I.e.

$$\text{Sd } \Sigma = \{ \text{CH}(b, \Sigma^{n-1}) \mid \Sigma^{n-1} \in \text{Sd } \varphi_i, 0 \leq i \leq n \}$$

Inductively, $\Sigma = \bigcup \text{Sd } \Sigma$ and $\# \text{Sd } \Sigma = (n + 1)!$.

1.5 Theorem

For every n , there exists a constant $c < 1$ such that for every n -simplex Σ then for every $\Sigma' \in \text{Sd } \Sigma$:

$$\text{diam}(\Sigma') \leq c \text{diam}(\Sigma)$$

1.6 Definition

We define $\text{Sd}_n: C_n(\Delta^n) \longrightarrow C_n(\Delta^n)$ by induction on n . Let $\sigma: \Delta^n \longrightarrow \Delta^n$ be a generator, then

- (1) $\text{Sd}_0(\sigma) = \sigma$
- (2) $\text{Sd}_n(\sigma) = C_{\sigma(b)}(\text{Sd}_{n-1}(\partial\sigma))$ where b is the barycenter of Δ^n .

Let X be a topological space, then let $\text{Sd}_n: C_n(X) \longrightarrow C_n(X)$ be defined on generators $\sigma: \Delta^n \longrightarrow X$ by $\text{Sd } \sigma = \sigma_{\sharp} \text{Sd}_n \text{id}_{\Delta^n}$.

1.7 Theorem

Sd is a chain map ($\text{Sd} = \{\text{Sd}_n\}_{n=0}^\infty$) and is natural (between the chain functor $\mathbf{Top} \rightarrow \mathbf{Comp}$ and itself).

Sd being natural means the following diagram commutes

$$\begin{array}{ccc} C_n(X) & \xrightarrow{\text{Sd}_n} & C_n(X) \\ f_\# \downarrow & & \downarrow f_\# \\ C_n(Y) & \xrightarrow{\text{Sd}_n} & C_n(Y) \end{array}$$

1.8 Theorem

Sd is chain homotopic to $\text{id}_{\mathcal{C}(X)}$.

1.9 Definition

Let X be a topological space, and $\mathcal{U} = \{\mathcal{U}_\alpha\}_{\alpha \in I}$ a collection of subsets of X such that $\bigcup \mathring{\mathcal{U}}_\alpha = X$ (where $\mathring{\mathcal{U}}$ is the interior of \mathcal{U}). Such a collection will be called a **good cover** of X .

We will say that $\sigma: \Delta^n \rightarrow X$ **preserves** the cover if there exists an $\alpha \in I$ such that $\sigma(\Delta^n) \subseteq \mathcal{U}_\alpha$. And we will say that $\sum_i n_i \sigma_i \in C_n(X)$ **preserves** the cover if each σ_i preserves the cover.

Let us define

$$C_n^{\mathcal{U}}(X) = \{\sigma \in C_n(X) \mid \sigma \text{ preserves } \mathcal{U}\}$$

$C_n^{\mathcal{U}}(X)$ is a subgroup of $C_n(X)$, as can be easily verified. Notice that if $\sigma(\Delta^n) \subseteq \mathcal{U}_\alpha$ then $\sigma\tau_i(\Delta^{n-1}) = \sigma(\tau_i\Delta^{n-1}) \subseteq \mathcal{U}_\alpha$ so that $\sigma\tau_i \in C_{n-1}^{\mathcal{U}}(X)$. Thus $\partial\sigma \in C_{n-1}^{\mathcal{U}}(X)$, so we can define a subcomplex of $\mathcal{C}(X)$, $\mathcal{C}^{\mathcal{U}}(X)$ whose coefficients are $C_n^{\mathcal{U}}(X)$. So we can define $H_n^{\mathcal{U}}(X)$ to be the n th homology group of $\mathcal{C}^{\mathcal{U}}(X)$.

The inclusion map $\iota: C_n^{\mathcal{U}}(X) \rightarrow C_n(X)$ is a chain morphism, so this induces a $\iota_*: H_n^{\mathcal{U}}(X) \rightarrow H_n(X)$.

1.10 Theorem

This ι_* is an isomorphism.

This is not a trivial proof, and it relies on the following observations. But from here on, I will only be putting in the simpler/enlightening proofs so that I can finish this summary. Notice that

$$\text{Sd}_n: C_n^{\mathcal{U}}(X) \rightarrow C_n^{\mathcal{U}}(X)$$

is defined, since if $\sigma \in C_n^{\mathcal{U}}(X)$ then that means for some $\alpha \in I$ $\sigma(\Delta^n) \subseteq \mathcal{U}_\alpha$, and $\text{Sd}_n \sigma = \sigma_\# \text{Sd}_n \text{id}_n$. Thus the image of $\text{Sd}_n \sigma$ is contained in the image of σ , which in turn is contained in \mathcal{U}_α . Now, the chain homotopy between Sd and $\text{id}_{\mathcal{C}(X)}$ can also be restricted to $\mathcal{C}^{\mathcal{U}}(X) \rightarrow \mathcal{C}^{\mathcal{U}}(X)$. Thus Sd is chain homotopic to $\text{id}_{\mathcal{C}^{\mathcal{U}}(X)}$.

1.11 Definition

A **short exact sequence** of chain complexes is a chain of chain morphisms $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{g} \mathcal{E}$ such that for every n , $0 \rightarrow C_n \xrightarrow{f_n} D_n \xrightarrow{g_n} E_n \rightarrow 0$ is a short exact sequence.

1.12 Lemma

A short exact sequence of chain complexes $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{g} \mathcal{E}$ induces a long exact sequence on the homology groups:

$$\begin{array}{ccccc}
& & \cdots & \xrightarrow{\quad} & H_{n+1}\mathcal{C} \\
& \swarrow & & \nearrow & \\
H_n\mathcal{C} & \xrightarrow{\quad} & H_n\mathcal{D} & \xrightarrow{\quad} & H_n\mathcal{E} \\
& \nwarrow & & \nwarrow & \\
H_{n-1}\mathcal{C} & \cdots & \xrightarrow{\quad} & &
\end{array}$$

Proof: a diagram chase. ■

1.13 Definition

If \mathcal{C}, \mathcal{D} are chain complexes then their **direct sum** is the chain complex $\mathcal{C} \oplus \mathcal{D}$ whose terms are $C_n \oplus D_n$ and whose boundary operator is $\partial_{\mathcal{C}} \oplus \partial_{\mathcal{D}}$ (i.e. $(a, b) \mapsto (\partial a, \partial b)$).

1.14 Lemma

If X is a topological space, $\mathcal{U}, \mathcal{V} \subseteq X$ such that $\mathring{\mathcal{U}} \cup \mathring{\mathcal{V}} = X$, then there exists a short exact sequence of chain complexes

$$0 \longrightarrow \mathcal{C}(\mathcal{U} \cap \mathcal{V}) \longrightarrow \mathcal{C}(\mathcal{U}) \oplus \mathcal{C}(\mathcal{V}) \longrightarrow \mathcal{C}^{\mathcal{U}, \mathcal{V}}(X) \longrightarrow 0$$

where $\mathcal{C}^{\mathcal{U}, \mathcal{V}}(X)$ is the chain complex modulo the cover $\{\mathcal{U}, \mathcal{V}\}$.

Proof: we have the inclusions, which commute:

$$\begin{array}{ccccc}
& & \mathcal{U} & & \\
& \nearrow i & & \searrow k & \\
\mathcal{U} \cap \mathcal{V} & & & & X \\
& \searrow j & & \nearrow \ell & \\
& & \mathcal{V} & &
\end{array}$$

And from them we build:

$$0 \longrightarrow C_n(\mathcal{U} \cap \mathcal{V}) \longrightarrow C_n(\mathcal{U}) \oplus C_n(\mathcal{V}) \longrightarrow C_n^{\mathcal{U}, \mathcal{V}}(X) \longrightarrow 0$$

$$a \longmapsto (i_{\#}a, -j_{\#}a)$$

$$(a, b) \longmapsto k_{\#}a + \ell_{\#}b$$

This is exact because composing the two maps gives $k_{\#}i_{\#}a - \ell_{\#}j_{\#}a = (ki)_{\#}a - (\ell j)_{\#}a$, and since $ki = \ell j$, this is zero. So the image of the first is contained within the kernel of the second. And if $k_{\#}a = -\ell_{\#}b$, then a, b must be chains in $\mathcal{U} \cap \mathcal{V}$ (since k maps chains of \mathcal{U} to X , and ℓ maps chains of \mathcal{V}), so they must be in the image of the first map. It can be verified that these are chain morphisms. ■

Notice that the homology group of $C_n(X) \oplus C_n(Y)$ is just $H_n(X) \oplus H_n(Y)$ since the image of $\partial \oplus \partial$ is just $\text{Im } \partial \oplus \text{Im } \partial$, and similar for kernel. From the previous two lemmas, the following is immediate (recall that $H_n^{\mathcal{U}}(X) \cong H_n(X)$):

1.15 Theorem (Mayer-Vietoris)

If $\mathcal{U}, \mathcal{V} \subseteq X$ such that $\mathring{\mathcal{U}} \cup \mathring{\mathcal{V}} = X$, then there is an exact sequence

$$\begin{array}{ccccc}
& & \cdots & \xrightarrow{\quad} & H_{n+1}(X) \\
& \swarrow & & \nearrow & \\
H_n(\mathcal{U} \cap \mathcal{V}) & \xrightarrow{\quad} & H_n(\mathcal{U}) \oplus H_n(\mathcal{V}) & \xrightarrow{\quad} & H_n(X) \\
& \nwarrow & & \nwarrow & \\
H_{n-1}(\mathcal{U} \cap \mathcal{V}) & \cdots & \xrightarrow{\quad} & &
\end{array}$$

Notice that at $n = 0$ for the reduced homology if $\mathcal{U} \cap \mathcal{V} \neq \emptyset$, then we get the same exact sequence but with the reduced homology.

1.16 Theorem

$$\tilde{H}_i(S^n) = \begin{cases} \mathbb{Z} & i = n \\ 0 & \text{else} \end{cases}$$

Proof: by induction on n . For $n = 0$, we have that S^0 is just the space of two points, so $H_0(S^0) = \mathbb{Z} \oplus \mathbb{Z}$ and for $i > 0$ it is zero since the homology of the one-point space is zero. The reduced homology removes a factor of \mathbb{Z} and so $\tilde{H}_0(S^0) = \mathbb{Z}$ and for $n > 0$ $\tilde{H}_i(S^0) = 0$. Now inductively, we can choose contractible \mathcal{U}, \mathcal{V} such that $\mathcal{U} \cap \mathcal{V}$ are homotopic to S^{n-1} (by choosing hemispheres which overlap), and so $H_i(\mathcal{U} \cap \mathcal{V}) \cong H_i(S^{n-1})$. We have an exact sequence by Mayer-Vietoris:

$$\tilde{H}_i(\mathcal{U}) \oplus \tilde{H}_i(\mathcal{V}) \longrightarrow \tilde{H}_i(S^n) \longrightarrow \tilde{H}_{i-1}(S^{n-1}) \longrightarrow \tilde{H}_{i-1}(\mathcal{U}) \oplus \tilde{H}_{i-1}(\mathcal{V})$$

since \mathcal{U}, \mathcal{V} are contractible, $\tilde{H}_i(\mathcal{U}) = \tilde{H}_i(\mathcal{V}) = 0$ for all i and so we get the exact sequence

$$0 \longrightarrow \tilde{H}_i(S^n) \longrightarrow \tilde{H}_{i-1}(S^{n-1}) \longrightarrow 0$$

which means that $\tilde{H}_i(S^n) \cong \tilde{H}_{i-1}(S^{n-1})$, and so inductively we have our result. ■

Since their homology groups differ, we immediately get

1.17 Theorem

If $n \neq m$ then S^n is not homotopic to S^m .

1.18 Corollary

If $n \neq m$ then \mathbb{R}^n is not homeomorphic to \mathbb{R}^m .

Proof: suppose $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a homeomorphism, then it is a homeomorphism $f: \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^m \setminus \{f(0)\}$. So we have

$$S^n \simeq \mathbb{R}^n \setminus \{0\} \cong \mathbb{R}^m \setminus \{f(0)\} \simeq S^m$$

in contradiction. ■

1.19 Theorem

∂D^n is not a retract of D^n .

Proof: suppose $r: D^n \longrightarrow \partial D^n$ is a retraction, then $r\iota = \text{id}_{\partial D^n}$ where ι is the inclusion $\partial D^n \longrightarrow D$. Thus $r_*\iota_* = \text{id}_{H_i(\partial D^n)}$. This implies that ι_* is injective, in particular for $i = n-1$ and so $i_*: \tilde{H}_{n-1}(\partial D^n) \longrightarrow \tilde{H}_{n-1}(D^n)$. Since $\partial D^n \cong S^{n-1}$ and D^n is contractible, we have an injective map $\mathbb{Z} \longrightarrow 0$ in contradiction. ■

1.20 Lemma

Let us define $R: S^n \longrightarrow S^n$ by $R(x_1, \dots, x_n) = (-x_1, x_2, \dots, x_n)$. Then $R_*: \tilde{H}_n(S^n) \longrightarrow \tilde{H}_n(S^n)$ satisfies $R_* = -\text{id}_{\tilde{H}_n(S^n)}$.

Proof: by induction on n . For $n = 0$, $R(1) = -1$ and $R(-1) = 1$, and $\tilde{H}_0(S^0) = \mathbb{Z}$. Now, ε must map the generator of the reduced homology to zero, so the generator must be $kp_1 - kp_2$, and composing R_* on this gives $kp_2 - kp_1$ which is the inverse of the generator, so R_* is indeed minus the identity.

Now for $n > 0$, let us split the sphere S^n into two hemispheres \mathcal{U} and \mathcal{V} whose intersection is homotopic to S^{n-1} . By Mayer-Vietoris, we have

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_n(S^n) & \longrightarrow & H_{n-1}(S^{n-1}) & \longrightarrow & 0 \\ & & R_* \downarrow & & \downarrow R_* = -\text{id} & & \\ 0 & \longrightarrow & H_n(S^n) & \longrightarrow & H_{n-1}(S^{n-1}) & \longrightarrow & 0 \end{array}$$

This diagram commutes by naturality, so $R_* = -\text{id}$ for S^n . ■

By symmetry, we can define $R_i: (x_1, \dots, x_i, \dots, x_{n+1}) \mapsto (x_1, \dots, -x_i, \dots, x_{n+1})$ and we have that $R_{i,*} = -\text{id}$. Let us define

$$A: S^n \longrightarrow S^n, \quad x \mapsto -x$$

the *antipodal map*. Since $A = R_1 \circ \dots \circ R_{n+1}$, we have that $A_* = (-\text{id})^{n+1} = (-1)^{n+1} \text{id}$.

1.21 Corollary

If n is even, then the antipodal map is not homotopic to the identity.

Note that for $n = 2k - 1$, we can view S^n as the unit sphere in \mathbb{C}^k and take the homotopy $H(z, t) = e^{\pi i t} z$ which is a homotopy from id to A .

1.22 Lemma

Let $n \geq 0$, and let $f, g: S^n \longrightarrow S^n$ such that for all $x \in S^n$, $f(x) \neq -g(x)$. Then $f \sim g$.

Proof: we define the homotopy

$$H(x, t) = \frac{(1-t)f(x) + tg(x)}{\|(1-t)f(x) + tg(x)\|}$$

this cannot be zero, since the line $(1-t)f(x) + tg(x)$ connects $f(x)$ and $g(x)$, and it can only be zero when $f(x)$ and $g(x)$ are antipodal points on the sphere. ■

1.23 Theorem

Let n be even and $f: S^n \longrightarrow S^n$, then there exists an $x \in S^n$ such that either $f(x) = x$ or $f(x) = -x$.

Proof: suppose not. Then for all x , $f(x) \neq x$, so $f(x) \neq -A(x)$ so $f \sim A$. And for all x , $f(x) \neq -x$, i.e. $f(x) \neq -\text{id}(x)$ so $f \sim \text{id}$. Thus $A \sim \text{id}$, which contradicts n being even. ■

1.24 Definition

A **vector field** of S^n is a continuous map $f: S^n \longrightarrow \mathbb{R}^{n+1}$ such that for all $x \in S^n$, $\langle f(x), x \rangle = 0$.

1.25 Theorem (Hairy Ball Theorem)

Let n be even. Then for every vector field on S^n , there is an $x \in S^n$ such that $f(x) = 0$.

Proof: suppose not, then we can define a continuous map $x \mapsto \frac{f(x)}{\|f(x)\|}$ which is a map $S^n \longrightarrow S^n$. These points are still tangent to x , in particular they cannot be x or antipodal to x , in contradiction to n being even. ■

Note that in general if \mathcal{U}, \mathcal{V} is a good cover of X and $\mathcal{U} \cap \mathcal{V}$ is contractible, then by Mayer-Vietoris we have

$$0 = \tilde{H}_i(\mathcal{U} \cap \mathcal{V}) \longrightarrow \tilde{H}_i(\mathcal{U}) \oplus \tilde{H}_i(\mathcal{V}) \longrightarrow \tilde{H}_i(X) \longrightarrow \tilde{H}_{i-1}(\mathcal{U} \cap \mathcal{V}) = 0$$

so $\tilde{H}_i(X) \cong \tilde{H}_i(\mathcal{U}) \oplus \tilde{H}_i(\mathcal{V})$. In particular let us look at $S^n \vee S^m$, we can take \mathcal{U} to be S^n with a bit of S^m and \mathcal{V} to be S^m with a bit of S^n , then $\mathcal{U} \cap \mathcal{V}$ is contractible and \mathcal{U} is homotopic to S^n and \mathcal{V} to S^m so

$$\tilde{H}_i(S^n \vee S^m) \cong \tilde{H}_i(S^n) \oplus \tilde{H}_i(S^m)$$

and similarly by induction

$$\tilde{H}_i\left(\bigvee_{j=1}^k S^{n_j}\right) \cong \bigoplus_{j=1}^k \tilde{H}_i(S^{n_j})$$

Now let us look at $X = nT$. Let us take \mathcal{U} to be a disk in X , and \mathcal{V} to be the rest of X with a bit of \mathcal{U} . Then \mathcal{U} is homotopic to a point, $\mathcal{U} \cap \mathcal{V} \simeq S^1$ and we showed last semester that $\mathcal{V} \simeq \bigvee_{2n} S^1$. Mayer-Vietoris gives us

$$\tilde{H}_i(\mathcal{U} \cap \mathcal{V}) \longrightarrow \tilde{H}_i(\mathcal{U}) \oplus \tilde{H}_i(\mathcal{V}) \longrightarrow \tilde{H}_i(X) \longrightarrow \tilde{H}_{i-1}(\mathcal{U} \cap \mathcal{V})$$

when $i \geq 2$ $\tilde{H}_i(\mathcal{U} \cap \mathcal{V}) = \tilde{H}_i(\mathcal{U}) \oplus \tilde{H}_i(\mathcal{V}) = 0$, but we require $i \geq 3$ for $\tilde{H}_{i-1}(\mathcal{U} \cap \mathcal{V}) = 0$. So when $i \geq 3$, $\tilde{H}_i(nT) = 0$. So let us look at $i = 2$:

$$\begin{aligned} H_2(\mathcal{U} \cap \mathcal{V}) &\longrightarrow H_2\mathcal{U} \oplus H_2\mathcal{V} \longrightarrow H_2X \longrightarrow H_1(\mathcal{U} \cap \mathcal{V}) \longrightarrow H_1\mathcal{U} \oplus H_1\mathcal{V} \longrightarrow H_1X \longrightarrow \\ &\longrightarrow \tilde{H}_0(\mathcal{U} \cap \mathcal{V}) \longrightarrow \tilde{H}_0\mathcal{U} \oplus \tilde{H}_0\mathcal{V} \longrightarrow \tilde{H}_0X \longrightarrow 0 \end{aligned}$$

We get from this

$$0 \longrightarrow 0 \longrightarrow H_2X \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^{2n} \longrightarrow H_1X \longrightarrow 0 \longrightarrow 0 \longrightarrow \tilde{H}_0X \longrightarrow 0$$

So we get that $\tilde{H}_0X = 0$. Let us focus on the map $\mathbb{Z} \longrightarrow \mathbb{Z}^{2n}$ here, that is we need to understand $H_1(\mathcal{U} \cap \mathcal{V}) \longrightarrow H_1\mathcal{U} \oplus H_1\mathcal{V} = H_1(\mathcal{V})$. Visually, this can be shown to just be zero (using abelianization of π_1). We can then just insert 0 into the sequence where the zero morphism was:

$$0 \longrightarrow 0 \longrightarrow H_2X \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z}^{2n} \longrightarrow H_1X \longrightarrow 0 \longrightarrow 0 \longrightarrow \tilde{H}_0X \longrightarrow 0$$

So we get that $H_2X \cong \mathbb{Z}$ and $H_1X \cong \mathbb{Z}^{2n}$.

1.26 Theorem

If $f: D^k \longrightarrow S^n$ is injective, then $\tilde{H}_i(S^n - f(D^k)) = 0$ for all i .

Proof: by induction on k . For $k = 0$, $S^n - \{\cdot\} \cong \mathbb{R}^n$ which is contractible and thus has a homotopy group of zero. We will be working with the k -dimensional cube $I^k \cong D^k$. So $f: I^k \times I \longrightarrow S^n$ is injective. Define $A_1 = I^k \times [0, 1/2]$ and $B_1 = I^k \times [1/2, 1]$, and let $\mathcal{U} = S^n - f(A_1) = f(A_1)^c$ and $\mathcal{V} = S^n - f(B_1) = f(B_1)^c$. So $\mathcal{U} \cup \mathcal{V} = f(A_1 \cap B_1)^c = f(I^k \times \{1/2\})^c$. So inductively, $\tilde{H}_i(\mathcal{U} \cup \mathcal{V}) = 0$ for all i . And $\mathcal{U} \cap \mathcal{V} = f(A_1 \cup B_1)^c = f(I_{k+1})^c$ which is the space we want to compute the homology groups of. By Mayer-Vietoris:

$$0 = \tilde{H}_{i+1}(\mathcal{U} \cup \mathcal{V}) \longrightarrow \tilde{H}_i(\mathcal{U} \cap \mathcal{V}) \longrightarrow \tilde{H}_i(\mathcal{U}) \oplus \tilde{H}_i(\mathcal{V}) \longrightarrow \tilde{H}_i(\mathcal{U} \cup \mathcal{V}) = 0$$

So $\tilde{H}_i(\mathcal{U} \cap \mathcal{V}) \cong \tilde{H}_i\mathcal{U} \oplus \tilde{H}_i\mathcal{V}$. So suppose that $\tilde{H}_i(\mathcal{U} \cap \mathcal{V}) \neq 0$, then take $[z] \neq 0$ in $\tilde{H}_i(\mathcal{U} \cap \mathcal{V})$. Taking the inclusion maps i, j we have that one of $i_*[z]$ and $-j_*[z]$ is nonzero. Continue. ■

1.27 Theorem

Let $f: S^k \longrightarrow S^n$ be injective, then

$$\tilde{H}_i(S^n - f(S^k)) = \begin{cases} \mathbb{Z} & i = n - k - 1 \\ 0 & \text{else} \end{cases}$$

Proof: similarly by induction on k . ■

1.28 Theorem (Jordan's Theorem)

Let $f: S^{n-1} \longrightarrow S^n$ injective then $S^n - f(S^{n-1})$ has two path-connected components and they are open.

1.29 Theorem (Invariance of Domain)

Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, and $f: \mathcal{U} \longrightarrow \mathbb{R}^n$ injective. Then $f(\mathcal{U})$ is open.

Proof: let $x \in \mathcal{U}$ then let us look at the restriction of f to a closed ball around x , and show that the image of its interior is open. If we choose this closed ball to lie in \mathcal{U} , looking at the union of these balls, we see that \mathcal{U} 's image is open. So we must show that if $f: D^n \longrightarrow \mathbb{R}^n$ is injective, then $f(\hat{D}^n)$ is open.

Now, $\mathbb{R}^n - f(\partial D^n) = f(\dot{D}^n) \sqcup (\mathbb{R}^n - f(D^n))$. This is the union of two disjoint path-connected spaces (the left is the continuous map of a path-connected space, and the right is because $\tilde{H}_i(\mathbb{R}^n - f(D^n)) = 0$ as an exercise). By Jordan's theorem, there are two path-connected components and they are open. So these are the two open path-connected components, in particular $f(\dot{D}^n)$ is open. ■

Note then that if $\mathcal{U} \subseteq \mathbb{R}^n$ is open, then it is not homeomorphic to any subspace $A \subseteq \mathbb{R}^n$ which is not open. This is because the homeomorphism $\mathcal{U} \rightarrow A$ would mean by the invariance of domain that A is open. In particular, an open set in \mathbb{R}^n is not homeomorphic to any open set in \mathbb{R}^m for $n \neq m$. This is because $\mathbb{R}^m \subset \mathbb{R}^n$ assuming $m < n$, and so $\mathcal{U} \rightarrow A$ means that A is open, but the last coordinates of A are all zero and so it cannot be open.

1.30 Definition

An n -dimensional **manifold** is a Hausdorff topological space M with a countable basis such that for every $x \in M$ there exists a neighborhood homeomorphic to an open ball in \mathbb{R}^n . An n -dimensional **manifold with boundary** is a Hausdorff topological space M with a countable basis such that every $x \in M$ has a neighborhood homeomorphic either to an open ball or to the half-open ball (which is defined to be $\{(x_1, \dots, x_n) \mid \|(x_1, \dots, x_n)\| < 1, x_1 \geq 0\}$). A **closed manifold** is a compact manifold (without a boundary).

1.5 Excision

Let $A \subseteq X$ be a subspace, then $\mathcal{C}(A) \subseteq \mathcal{C}(X)$ is a subcomplex, so we can define the *quotient complex* $\mathcal{C}(X, A) = \mathcal{C}(X)/\mathcal{C}(A)$. Explicitly, $C_n(X, A) = C_n(X)/C_n(A)$. The boundary operator ∂ maps from $C_n(A)$ to $C_{n-1}(A)$, so we can simply take $\partial[z] = [\partial z]$ in the quotient complex. Thus we can define the *relative homology groups* of X with respect to A to be

$$H_n(X, A) := H_n(\mathcal{C}(X, A))$$

Now, suppose $f: (X, A) \rightarrow (Y, B)$ is a map, then we have $f_\# : C_n(X) \rightarrow C_n(Y)$. Do we have that this induces a map $f_\# : C_n(X, A) \rightarrow C_n(Y, B)$? In order for this to occur we must have $f_\#(C_n(A)) \subseteq C_n(B)$, which is indeed the case (since $f: A \rightarrow B$). Thus we have a function $f_\# : C_n(X, A) \rightarrow C_n(Y, B)$ and we can see that this is a chain morphism. So we have defined a functor $\mathbf{Top}^2 \rightarrow \mathbf{Comp}$. And in particular we can compose this with our functor $\mathbf{Comp} \rightarrow \mathbf{Ab}$ to get $\mathbf{Top}^2 \rightarrow \mathbf{Ab}$.

Now, we have a short exact sequence of chain complexes

$$0 \rightarrow C_n(A) \rightarrow C_n(X) \rightarrow C_n(X, A) \rightarrow 0$$

since this is precisely an inclusion-quotient chain, and the boundary operators are defined in such a way so that the diagram commutes. Thus we have an exact sequence of homology groups:

$$\begin{array}{ccccccc} & & & & \xrightarrow{\quad\quad\quad} & H_{n+1}(X, A) & \\ & & & \swarrow & & & \\ H_n(A) & \xrightarrow{\quad\quad} & H_n(X) & \xrightarrow{\quad\quad} & H_n(X, A) & & \\ & & \swarrow & & & & \\ H_{n-1}(A) & \xrightarrow{\quad\quad\quad} & & & & & \end{array}$$

And this short exact sequence of chain complexes is natural, so this exact sequence is natural as well.

1.1 Theorem

$$H_i(D^n, \partial D^n) = \begin{cases} \mathbb{Z} & i = n \\ 0 & \text{else} \end{cases}$$

Proof: by the exact sequence of homology groups we have

$$0 = \tilde{H}_i(D^n) \rightarrow H_i(D^n, \partial D^n) \rightarrow \tilde{H}_{i-1}(\partial D^n) \rightarrow \tilde{H}_{i-1}(D^n) = 0$$

so $H_i(D^n, \partial D^n) \cong \tilde{H}_{i-1}(\partial D^n) = \tilde{H}_{i-1}(S^{n-1})$ which is exactly what we want. ■

We can generalize this:

1.2 Lemma

Let $A \subseteq X$ then

- (1) if A is contractible, then $\tilde{H}_i(X) \cong H_i(X, A)$;
- (2) if X is contractible, then $\tilde{H}_{i-1}(A) \cong H_i(X, A)$.

Proof: again we use the exact sequence:

$$0 = \tilde{H}_i(A) \longrightarrow \tilde{H}_i(X) \longrightarrow H_i(X, A) \longrightarrow \tilde{H}_{i-1}(A) = 0$$

so $H_i(X, A) \cong \tilde{H}_i(X)$. Similar for the second case. ■

1.3 Proposition

Note if we have $f: (X, A) \longrightarrow (Y, B)$ then we have

$$f_*: H_n(X) \longrightarrow H_n(Y), \quad f_*: H_n(A) \longrightarrow H_n(Y), \quad f_*: H_n(X, A) \longrightarrow H_n(Y, B)$$

If any two of these are isomorphisms, so is the third.

Proof: immediate from the naturality of the exact sequence of homology groups, and the lemma of five. ■

In particular we have that

1.4 Corollary

If $f: X \longrightarrow Y$ and $f: A \longrightarrow B$ are both homotopic equivalences, then $f_*: H_n(X, A) \longrightarrow H_n(Y, B)$ is an isomorphism.

In particular, the inclusion map $(D^n, \partial D^n) \subseteq (D^n, D^n - \{0\})$ is a homotopic equivalence, and so $H_i(D^n, \partial D^n) \cong H_i(D^n, D^n - \{0\})$. Thus

$$H_i(D^n, D^n - \{0\}) = \begin{cases} \mathbb{Z} & i = n \\ 0 & \text{else} \end{cases}$$

And we can look at our good friend $R: (x_1, \dots, x_n) \mapsto (-x_1, x_2, \dots, x_n)$ which can be viewed as $R: (D^n, \partial D^n) \longrightarrow (D^n, \partial D^n)$ and we have the commutative diagram

$$\begin{array}{ccc} H_n(D^n, \partial D^n) & \xrightarrow{\cong} & \tilde{H}_{n-1}(\partial D^n) \\ R_* \downarrow & & \downarrow R_* = -\text{id} \\ H_n(D^n, \partial D^n) & \xrightarrow{\cong} & \tilde{H}_{n-1}(\partial D^n) \end{array}$$

And so $R_* = -\text{id}$ for the map over $H_n(D^n, \partial D^n)$.

1.5 Definition

Let $\mathcal{U} = \{\mathcal{U}_\alpha\}_{\alpha \in I}$ be a good covering of X , and let $A \subseteq X$, then we define

$$C_n^{\mathcal{U}}(X, A) = C_n^{\mathcal{U}}(X) / C_n^{\mathcal{U}}(X) \cap C_n(A)$$

This is indeed a chain complex, since if a chain preserves \mathcal{U} and is contained in A , then its boundary preserves \mathcal{U} and is contained in A .

Similar to before, the inclusion map $\iota: C_n^{\mathcal{U}}(X, A) \longrightarrow C_n(X, A)$ induces an isomorphism $\iota_*: H_n^{\mathcal{U}}(X, A) \longrightarrow H_n(X, A)$.

1.6 Theorem (Excision)

Let $K \subseteq A \subseteq X$ such that $\bar{K} \subseteq \mathring{A}$, then the inclusion $(X - K, A - K) \longrightarrow (X, A)$ induces an isomorphism of all homology groups $H_n(X - K, A - K) \longrightarrow H_n(X, A)$.

Proof: note that $\bar{K} \subseteq \mathring{A}$ is equivalent to $\mathcal{U} = \{A, K^c\}$ being a good cover. So $C_n(A), C_n(X - K) \subseteq C_n^{\mathcal{U}}(X)$. So we can compose the inclusion map with the quotient map to get $C_n(X - K) \longrightarrow C_n^{\mathcal{U}}(X)/C_n(A) \cap C_n^{\mathcal{U}}(X) = C_n^{\mathcal{U}}(X)/C_n(A)$. We claim that this is a surjective map, as chains in $C_n^{\mathcal{U}}(X)/C_n(A)$ are classes of chains which respect $\{X - K, A\}$, but the simplexes which respect A are identified with zero, so we are left with formal sums of simplexes which respect $X - K$. The kernel is just $C_n(X - K) \cap C_n(A) = C_n((X - K) \cap A) = C_n(A - K)$. Thus by the first isomorphism theorem

$$C_n(X - K, A - K) = C_n(X - K) / C_n(A - K) \cong C_n^{\mathcal{U}}(X) / C_n(A) = C_n^{\mathcal{U}}(X, A)$$

Thus we get that

$$H_n(X - K, A - K) \cong H_n^{\mathcal{U}}(X, A) \cong H_n(X, A) \quad \blacksquare$$

1.7 Theorem

Let M be an n -dimensional manifold with or without a boundary, and $p \in M$ be a point in its interior. Then

$$H_i(M, M - \{p\}) = \begin{cases} \mathbb{Z} & i = n \\ 0 & \text{else} \end{cases}$$

Proof: let $j: D^n \longrightarrow M$ be an embedding of D^n into M which maps 0 to p and which maps \dot{D}^n to a neighborhood of p . Let us identify D^n with its image in M and p with 0. Then $(D^n, D^n - \{0\}) \subseteq (M, M - \{0\})$ is an excision: take $A = M - \{0\}$ and $K = M - D^n$. Thus $H_i(M, M - \{0\}) \cong H_i(D^n, D^n - \{0\})$ which is precisely what we want. \blacksquare

1.8 Corollary

The dimension of a manifold M is a topological property of M (i.e. it is unique).

Proof: this is since it is determined by its homology groups. \blacksquare

1.9 Theorem

Let M be a manifold with a boundary and p a point on its boundary. Then $H_i(M, M - \{p\}) = 0$ for all i .

Proof: take $j: C \longrightarrow M$ an embedding of the half-open ball into M . Then as before $(C, C - \{0\}) \subseteq (M, M - \{0\})$ is an excision and both C and $C - \{0\}$ are contractible. We have the exact sequence

$$0 = \tilde{H}_i(C) \longrightarrow H_i(C, C - \{0\}) \longrightarrow \tilde{H}_i(C - \{0\}) = 0$$

so $H_i(M, M - \{p\}) \cong H_i(C, C - \{0\}) = 0$. \blacksquare

1.10 Corollary

The boundary of a manifold is a topological property of M (a point cannot be both in its boundary and interior).

Note that if $p \in M$ is a boundary point, then it has a neighborhood homeomorphic to $\{\vec{x} \in B_1^n(0) \mid x_n \geq 0\}$. Thus it has a neighborhood homeomorphic to $B_1^{n-1}(0)$ (taking the last coordinate equal to 0), and all the points in this neighborhood must also be boundary points. Thus the boundary of an n -dimensional manifold is an $n - 1$ -dimensional manifold.

1.11 Theorem

$[\text{id}_n] \in H_n(\Delta^n, \partial\Delta^n)$ generates the homological group.

1.12 Theorem

Let $A \subseteq X$ be closed and suppose that there exists an open \mathcal{U} such that $A \subseteq \mathcal{U} \subseteq X$ and A is a deformation retract of \mathcal{U} . Then

$$H_n(X, A) = \tilde{H}_n(X/A)$$

Proof: since A is a deformation retract, the inclusion $(X, A) \longrightarrow (X, \mathcal{U})$ induces an isomorphism $H_n(X, A) \cong H_n(X, \mathcal{U})$. Furthermore by excision, $H_n(X - A, \mathcal{U} - A) \cong H_n(X, \mathcal{U})$. Let a be the point which represents A in X/A . Then $H_n(X/A, \{a\}) \cong H_n(X/A, \mathcal{U}/A)$ similar to above. And by excision $H_n(X/A - \{a\}, \mathcal{U}/A - \{a\}) \cong H_n(X/A, \mathcal{U}/A)$. Note though that $(X - A, \mathcal{U} - A)$ is homeomorphic to $(X/A - \{a\}, \mathcal{U}/A - \{a\})$ (both just remove A). Thus

$$H_n(X, A) \cong H_n(X, \mathcal{U}) \cong H_n(X - A, \mathcal{U} - A) \cong H_n(X/A - \{a\}, \mathcal{U}/A - \{a\}) \cong H_n(X/A, \mathcal{U}/A) \cong H_n(X/A, \{a\})$$

But since $\{a\}$ is contractible, by our exact sequence we see that this is isomorphic to $H_n(X/A)$.