Linear Algebra 2, Homework 7 Solution

Exercise 1

Let V be an inner product space over $\mathbb{R}$, show that for all $u, v \in V$:

$$(u - v) \perp (u + v) \iff \|v\| = \|u\|$$

We have that

$$(u - v) \perp (u + v) \iff \langle u - v, u + v\rangle = \|u\|^2 + \langle u, v\rangle - \langle v, u\rangle - \|v\|^2 = 0 \iff \|u\|^2 = \|v\|^2 \iff \|u\| = \|v\|$$

$\Diamond$

Exercise 2

Find for which $\alpha \in \mathbb{R}$ the following is an inner product over $\mathbb{R}$:

$$\langle (x_0, x_1), (y_0, y_1)\rangle = x_0y_0 - 3x_0y_1 - 3x_1y_0 + \alpha x_1y_1$$

Firstly, notice that

$$\langle (x_0, x_1), (y_0, y_1)\rangle = (x_0, x_1)\begin{pmatrix} 1 & -3 \\ -3 & \alpha \end{pmatrix}\begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$$

So the function is obviously linear, and since the matrix is symmetric, so is the function. So all we need to confirm is that $\langle (x_0, x_1), (x_0, x_1)\rangle \geq 0$ and equal to zero iff $x_0, x_1 = 0$. Computing gives

$$\langle (x_0, x_1), (x_0, x_1)\rangle = x_0^2 - 6x_0x_1 + \alpha x_1^2 = (x_0 - 3x_1)^2 + (\alpha - 9)x_1^2$$

So we must have that $\alpha \geq 9$ for nonnegativity. And in such a case, this is equal to zero if and only if $x_1 = 0$ and $x_0 = 3x_1 = 0$, as required. $\Diamond$

Exercise 3

Let V be a complex inner product space, and $T : V \longrightarrow V$ a linear operator such that for every $v \in V$, $\langle Tv, v\rangle = 0$. Show that $T = 0$.

Let $u, w \in V$ and $\alpha \in \mathbb{C}$, then by setting $v = u + \alpha w$, we get

$$0 = \langle Tv, v\rangle = \langle Tu + \alpha Tw, u + \alpha w\rangle = \langle Tu, u\rangle + \overline{\alpha}\langle Tu, w\rangle + \alpha\langle Tw, u\rangle + |\alpha|^2\langle Tw, w\rangle = \overline{\alpha}\langle Tu, w\rangle + \alpha\langle Tw, u\rangle$$

Thus we get, by choosing $\alpha = 1, i$,

$$\langle Tu, w\rangle + \langle Tw, u\rangle = -\langle Tu, w\rangle + \langle Tw, u\rangle = 0$$

Thus for all $u, w \in V$ we have $\langle Tw, u\rangle = 0$. Choosing $u = Tw$ we have $\langle Tw, Tw\rangle = 0$ for all $w \in V$ and so $Tw = 0$ for all $w \in V$, so $T = 0$. $\Diamond$

Exercise 4

Let V be an n-dimensional inner product space and let $B \subseteq V$ be a set of vectors. Show that B is an orthonormal basis if and it is an orthonormal set of n vectors.

If B is an orthonormal basis, it must be a set of $\dim V = n$ orthonormal vectors. If B is a set of n orthonormal vectors, then it must be linearly independent since a set of orthogonal vectors not containing zero is linearly independent. So it is a set of n linearly independent vectors, and thus a basis. $\Diamond$

Exercise 5

Let V be a complex inner product space of dimension n.

(1) Let $B \subseteq V$ be a basis. Define an inner product on V such that B is orthnormal with respect to this inner product.

(2) Let $V = \mathbb{C}^2$ and $B = \left\{ \begin{pmatrix} i \\ i \end{pmatrix}, \begin{pmatrix} 1+i \\ -2+i \end{pmatrix} \right\}$, find the corresponding inner product,

(1) We know that every inner product can be written as

$$\langle v, u \rangle = [v]_B^\top G_B \overline{[u]_B}$$

so in order for B to be an orthnormal basis, we must have $G_B = I$ and so

$$\langle v, u \rangle = [v]_B^\top \overline{[u]_B}$$

We now prove that this is an inner product and that B is orthnormal with respect to it:

(i) Linearity in the first component:

$$\langle v + \alpha w, u \rangle = [v + \alpha w]_B^\top \overline{[u]_B} = [v]_B^\top \overline{[u]_B} + \alpha [w]_B^\top \overline{[u]_B} = \langle v, u \rangle + \alpha \langle w, u \rangle$$

(ii) Hermitianess:

$$\overline{\langle v, u \rangle} = \overline{[v]_B^\top \overline{[u]_B}} = \overline{[v]_B}^\top [u]_B$$

since this is a scalar, it is equal to its transpose:

$$= [u]_B^\top \overline{[v]_B} = \langle u, v \rangle$$

(iii) Nonnegativity: suppose $[v]_B = (\alpha_1, \dots, \alpha_n)^\top$, then

$$\langle v, v \rangle = [v]_B^\top \overline{[v]_B} = \sum_i |\alpha_i|^2$$

this is nonnegative and zero iff $\alpha_i = 0$ for all i, i.e. iff $[v]_B = 0$, which is iff $v = 0$.

Now suppose $B = (b_1, \dots, b_n)$ then

$$\langle b_i, b_j \rangle = [b_i]_B^\top \overline{[b_j]_B} = e_i^\top e_j = \delta_{ij}$$

as required.

(2) By the previous subquestion, all we must find is $[\bullet]_B$. We can do this by computing $[I]_B^S$, which is just the inverse of $[I]_S^B$:

$$[I]_B^S = \left([I]_S^B\right)^{-1} = \begin{pmatrix} i & 1+i \\ i & -2+i \end{pmatrix}^{-1} = \frac{1}{3} \begin{pmatrix} -1-2i & 1-i \\ 1 & -1 \end{pmatrix} = A$$

So $[v]_B = Av$ and so

$$\langle v, u \rangle = (Av)^\top \overline{Au} = v^\top A^\top \overline{A} \overline{u}$$

So we must compute $A^\top \overline{A}$:

$$= \frac{1}{9} \begin{pmatrix} -1+2i & 1+i \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -1-2i & 1-i \\ 1 & -1 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 6+i & 2i \\ -2-2i & 2-i \end{pmatrix}$$

Thus we get that

$$\langle v, u \rangle = \frac{1}{9} v^\top \begin{pmatrix} 6+i & 2i \\ -2-2i & 2-i \end{pmatrix} \overline{u} = \frac{1}{9} \big((6+i) v_1 \overline{u}_1 + 2i v_1 \overline{u}_2 - (2+2i) v_2 \overline{u}_1 + (2-i) v_2 \overline{u}_2 \big)$$

$\Diamond$

Exercise 6

Let V be a **pseudonorm space**, i.e. it is equipped with a function

$$\|\bullet\|: V \longrightarrow \mathbb{R}_{\geq 0}$$

which satisfies $\|\alpha v\| = |\alpha|\|v\|$ and the triangle inequality (but not necessarily $\|v\| = 0 \implies v = 0$). Let us define a relation $\sim$ on V by

$$u \sim v \iff \|u - v\| = 0$$

(1) Show that this is indeed an equivalence relation.

(2) Show that one can define a vector-space structure on the quotient set $V/\sim$ by

$$\alpha[v] = [\alpha v], \qquad [v] + [u] = [v + u]$$

(3) Define a norm over $V/\sim$ and show that it is indeed a norm.

(1) We need to show the following three properties:

- **(i)** Reflexivity: $\|v - v\| = \|0\|$ and $\|0\| = norm 0 \cdot 0 = 0\|0\| = 0$, so $v \sim v$.
- **(ii)** Symmetry:

$$u \sim v \iff \|u - v\| = 0 \iff |-1|\|u - v\| = \|v - u\| = 0 \iff v \sim u$$

- **(iii)** Transitivity: suppose $u \sim v$ and $v \sim w$, then

$$\|u - w\| = \|(u - v) + (v - w)\| \leq \|u - v\| + \|v - w\| = 0$$

so $u \sim w$.

(2) First let us show that these operations are well-defined:

- **(i)** Suppose $v_1 \sim v_2$, we must then show that $\alpha v_1 \sim \alpha v_2$. This is because $\|\alpha v_1 - \alpha v_2\| = |\alpha|\|v_1 - v_2\| = 0$.
- **(ii)** Now suppose $v_1 \sim v_2, u_1 \sim u_2$, we must show that $v_1 + u_1 \sim v_2 + u_2$. This is because

$$\|(v_1 + u_1) - (v_2 + u_2)\| = \|(v_1 - v_2) + (u_1 - u_2)\| \leq \|v_1 - v_2\| + \|u_1 - u_2\| = 0$$

so indeed $v_1 + u_1 \sim v_2 + u_2$.

Now we show that they form a vector space:

- **(i)** Associativity:

$$[v] + ([u] + [w]) = [v] + [u + w] = [v + u + w] = [v + u] + [w] = ([v] + [u]) + [w]$$

- **(ii)** Commutativity:

$$[v] + [u] = [v + u] = [u + v] = [u] + [v]$$

- **(iii)** Additive identity: we will show that $[0]$ is the additive identity:

$$[0] + [v] = [0 + v] = [v]$$

- **(iv)** Additive inverses: we will show $-[v] = [-v]$ satisfies this:

$$[v] + (-[v]) = [v] + [-v] = [v - v] = [0]$$

- **(v)** Compatibility:

$$\alpha(\beta[v]) = \alpha[\beta v] = [\alpha\beta v] = (\alpha\beta)[v]$$

(vi) Identity:

$$1[v] = [1v] = [v]$$

(vii) Scalar distributivity:

$$\alpha([v] + [u]) = \alpha[v + u] = [\alpha v + \alpha u] = [\alpha v] + [\alpha u] = \alpha[v] + \alpha[u]$$

(viii) Vector distributivity:

$$(\alpha + \beta)[v] = [(\alpha + \beta)v] = [\alpha v] + [\beta v] = \alpha[v] + \beta[v]$$

Phew.

(3) Let us define the norm $\|\bullet\|_\sim$ on $V/\sim$ by

$$\|[v]\|_\sim = \|v\|$$

This is well-defined: if $v \sim u$ then $\|v - u\| = 0$ and so

$$\|v\| = \|v - u + u\| = \|v - u\| + \|u\| = \|u\|$$

and similarly $\|u\| \leq \|v\|$, thus $v \sim u \implies \|v\| = \|u\|$, as required.
And this is a norm:

(i) Nonnegativity: $\|[v]\|_\sim = \|v\| \geq 0$. And this is equal to zero if and only if $\|v\| = 0$, which is iff $\|v - 0\| = 0$, iff $v \sim 0$, iff $[v] = [0]$.

(ii) Scalar multiplication:

$$\|[\alpha v]\|_\sim = \|\alpha v\| = |\alpha|\|v\| = |\alpha|\|[v]\|_\sim$$

(iii) Triangle inequality:

$$\|[v] + [u]\|_\sim = \|[v + u]\|_\sim = \|v + u\| \leq \|v\| + \|u\| = \|[v]\|_\sim + \|[u]\|_\sim$$

as required. ◊