Introduction to Stochastic Processes

Final Assignment

Ari Feiglin

1 Exercise

Given a finite-state Markov Chain, which of the following are necessarily true?

(1) If the chain has two different stationary distributions, it is reducible.

(2) If the chain is irreducible, then all of its states have a degree greater than one.

(3) If all the states have a degree of one, then there exists an initial distribution such that starting with it means the chain converges in distribution.

(4) There exists a recurrent state with degree one.

(5) There exists a state whose expected return time is infinite.

(6) If all the states have a degree of one, then there exists an initial state such that starting on it means the chain converges in distribution.

(1) This is true: we showed that if a finite chain is irreducible it has a unique stationary distribution.

(2) This is false: take for example $P = (1)$.

(3) This is true: suppose $C_1, \dots, C_n$ are the irreducible components of the chain and each has a unique stationary distribution π_i. Then if we let v_i be any initial distribution contained in C_i, we have that $v_i P^n = v_i P_i^n \xrightarrow{n\to\infty} \pi_i$ where P_i is the transition matrix of C_i (equality here is up to nonzero indexes), since each C_i can be seen as an irreducible aperiodic (finite, homogeneus) Markov chain. Thus the chain converges in distribution for any initial distribution contained in an irreducible component (and by extension any initial distribution taking values in the C_is will give convergence).

(4) This is false: take for example $P(1 \to 2) = 1$ and $P(2 \to 1) = 1$, then both states are recurrent and have degree two.

(5) This is false: take for example $P = (1)$, the expected return time to its only state is 1.

(6) This is true: as we showed in (3), for any initial distribution v_i contained in an irreducible component, the chain will converge in distribution. So take any recurrent state (which exist since the chain is finite-state) a, and take the initial distribution $\mathbb{P}(X_0 = a) = 1$.

2 Exercise

Given the following stochastic matrix:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{2}{5} & 0 & \frac{3}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

(1) What are the irreducible classes?

(2) Does there exist a reccurent state with a degree greater than 1?

(3) Find all the stationary distributions, does there exist only one?

(4) Given $X_0 = 4$, does the Markov chain converge in distribution?

(5) Compute $f_{2\to 4}$.

(1) 1 is an absorbing state so $\{1\}$ is an irreducible class. 2 is connected to 1, and so is transient. 3 is connected to 2 which is transient, so 3 is transient. $4, 5, 6$ are all connected and this set is closed so is an irreducible class. Thus

$$T = \{2, 3\}, \quad C_1 = \{1\}, \quad C_2 = \{4, 5, 6\}$$

(2) 1 is an absorbing state so its degree is 1. Since $P(5 \to 5) > 0$, 5 has a degree of 1 and since 4 and 6 are in the same irreducible class they have the same degree, so all the recurrent states have a degree of 1.

(3) The stationary distributions are simply eigenvectors of eigenvalue 1 of $P^\top$ (which are also normalized). But we also know that stationary distributions are those in the convex span of $\{\pi_1, \pi_2\}$ where π_i is the stationary distribution of the irreducible class. For C_1, $\pi_1 = (1)$. And for C_2 we have that the reduced transition matrix is

$$\tilde{P} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} \\ 1 & 0 & 0 \end{pmatrix}$$

and so we must find the eigenvectors of eigenvalue 1 of $\tilde{P}^\top$, ie. $N(I - \tilde{P}^\top)$. A quick computation yields $(1, 4/3, 1)$ as the basis for the eigenspace, and normalizing it gives $(0.3, 0.4, 0.3)$. Thus the stationary distributions of the Markov chain are of the form

$$\pi = (1 - \alpha, 0, 0, 0.3\alpha, 0.4\alpha, 0.3\alpha), \qquad 0 \leq \alpha \leq 1$$

so there are infinitely many stationary distributions.

(4) In question 1 I explained why the chain converges in distribution for initial recurrent states, and since 4 is recurrent it does converge in distribution.

(5) By definition $f_{2\to 4} = \mathbb{P}(T_4 < \infty \mid X_0 = 2)$. To go from 2 to 4 we must take a path of the form $2 \to 3 \to 2$ for n times then $2 \to 3 \to 4$. This has a probability of

$$f_{2\to 4} = P(2 \to 3) \cdot \sum_{n=0}^{\infty} \big(P(3 \to 2)P(2 \to 3)\big)^n \cdot P(3 \to 4) = \frac{3}{10} \sum_{n=0}^{\infty} \frac{1}{5^n} = \frac{3}{10} \cdot \frac{5}{4} = \frac{3}{8}$$

3 Exercise

Every day in the morning Naomi goes to work and every evening she returns home. She has 3 elephants, which she can leave by home or by her work. If there is an elephant available when she needs to move between home and work she will decide with probability $\frac{1}{5}$ and will walk on foot with probability $\frac{4}{5}$. If there are no elephants available she will simply walk on foot.

(1) After many years, what is the probability Naomi will be at a place (either home or work) without elephants?

(2) Naomi is happy to see that by her there are 3 elephants. What is the expected time until this will happen again?

(3) This morning Naomi sees that she has exactly 1 elephant by her, what is the expected time it will take until she has 0 elephants by her?

(1) Let us define X_n to be the number of elephants by Naomi. Then $P(0 \to 3) = 1$ as if she has no elephants, she must walk to the other place which has three. $P(3 \to 1) = P(1 \to 3) = \frac{1}{5}$ as these are the events that she rides an elephant if there are 3 or 1 available. $P(3 \to 0) = \frac{4}{5}$ as this is the event that she walks on foot if she has three elephants available. $P(1 \to 2) = P(2 \to 1) = \frac{4}{5}$ as these are the events she walks on foot if she has a single or two elephants available. And $P(2 \to 2) = \frac{1}{5}$ as this is the event she rides an elephant if there are two available. So we have that the transition matrix is

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{4}{5} & \frac{1}{5} \\ 0 & \frac{4}{5} & \frac{1}{5} & 0 \\ \frac{4}{5} & \frac{1}{5} & 0 & 0 \end{pmatrix}$$

This Markov chain is also irreducible and aperiodic: it is irreducible since all the states are connected, and it is aperiodic since $P(2 \to 2) > 0$ so 2 has a degree of 1 and since the chain is irreducible all the other states have the same degree. Thus $vP^n \xrightarrow{n\to\infty} \pi$ where π is P's unique stationary distribution. Let us compute π, this is simply the normalized eigenvector of eigenvalue 1 of $P^\top$. A simple row reduction of $I - P^\top$ gives us $\pi = \left(\frac{4}{19}, \frac{5}{19}, \frac{5}{19}, \frac{5}{19}\right)$. Thus

$$\mathbb{P}_v(X_n = 0) \xrightarrow{n\to\infty}= \pi(0) = \frac{4}{19}$$

(2) We would like to compute $\mathbb{E}[T_3 \mid X_0 = 3]$. Let us define $\mu_k = \mathbb{E}[T_3 \mid X_0 = k]$. So then we have that

$$\mu_3 = \frac{4}{5}\,\mathbb{E}[T_3 \mid X_1 = 0] + \frac{1}{5}\,\mathbb{E}[T_3 \mid X_1 = 1] = 1 + \frac{4}{5}\mu_0 + \frac{1}{5}\mu_1 = \frac{9}{5} + \frac{1}{5}\mu_1$$

since $\mu_0 = 1$. And similarly

$$\mu_1 = \frac{1}{5}\,\mathbb{E}[T_3 \mid X_1 = 3] + \frac{4}{5}\,\mathbb{E}[T_3 \mid X_1 = 2] = 1 + \frac{4}{5}\mu_2, \qquad \mu_2 = 1 + \frac{1}{5}\mu_2 + \frac{4}{5}\mu_1$$

So then $\frac{4}{5}\mu_2 = 1 + \frac{4}{5}\mu_1$, thus $\mu_1 = 2 + \frac{4}{5}\mu_1$ so $\mu_1 = 10$ and thus $\mu_3 = 3.8$.

(3) We would like to compute $\mathbb{E}[T_0 \mid X_0 = 1]$. So similar to before, let us define $\mu_k = \mathbb{E}[T_0 \mid X_0 = k]$. Then

$$\mu_1 = 1 + \frac{1}{5}\mu_3 + \frac{4}{5}\mu_2, \quad \mu_2 = 1 + \frac{1}{5}\mu_2 + \frac{4}{5}\mu_1, \quad \mu_3 = \frac{9}{5} + \frac{1}{5}\mu_1$$

Solving this yields $\mu_1 = 14.75$.

4 Exercise

Karen the giraffe is walking on a $\mathbb{Z}^2$ plane which has a strong east-west wind. She leaves her equipment at $(0,0)$ and begins a random walk on the plane. She moves right with a probability of $\frac{2}{5}$, left with a probability of $\frac{1}{10}$, and up and down with a probability of $\frac{1}{4}$.

(1) Does Karen have a positive probability of returning to her equipment within a finite number of steps?

(2) Does Karen have a positive probability of never returning to her equipment?

(3) Will Karen almost surely enter the domain $\{(x,y) \mid x > 3\}$?

(1) This is true, for example Karen could move up and then down which has a probability of $\frac{1}{16} > 0$. Thus $\mathbb{P}(\text{Karen returns to her equipment}) \geq \frac{1}{16} > 0$.

(2) We can view Karen's walk on $\mathbb{Z}^2$ as being composed of two walks on $\mathbb{Z}$: X_n^1 and X_n^2. X_n^1 is associated with Karen's walk projected onto the x axis, and thus $\mathbb{P}\big(X_n^1 = X_{n-1}^1 + 1\big) = \frac{4}{5}$ and $\mathbb{P}\big(X_n^1 = X_{n-1}^1 - 1\big) = \frac{1}{5}$. X_n^2 is associated with Karen's walk projected onto the y axis, and thus $\mathbb{P}\big(X_n^2 = X_{n-1}^2 + 1\big) = \mathbb{P}\big(X_n^2 = X_{n-1}^2 - 1\big) = \frac{1}{2}$. At each step we choose $\{X_n^1\}$ or $\{X_n^2\}$ randomly, and take a step using that walk.

Notice that X_n^1 is not a fair walk on $\mathbb{Z}$, and so it is transient. This means that $\mathbb{P}_0\Big(T_0^{X_1} = \infty\Big) > 0$, and so surely $\mathbb{P}_{\vec{0}}(T_{\vec{0}} = \infty) > 0$ for Karen's walk on $\mathbb{Z}^2$, as if X^1 never returns to 0, so too won't Karen.

(3) Now, recall that almost surely one of the following must occur for a random walk $X_n = I_1 + \cdots + I_n$ where $\{I_i\}_i$ are all independent and have the same distribution:

$$(1)\ (\forall n)\, X_n = 0, \qquad (2)\ \lim_{n\to\infty} X_n = \infty, \qquad (3)\ \lim_{n\to\infty} X_n = -\infty, \qquad (4)\ \limsup_{n\to\infty} X_n = \infty, \liminf_{n\to\infty} X_n = -\infty$$

This was stated in class, and proven in our homework, so I will prove it again here (using my same proof from my homework): Notice that for every $-\infty \leq a \leq \infty$, $\liminf X_n = \liminf \sum_{j=0}^n I_j \leq a$ is an event which occurs independent of a permutation of a finite number of indexes of I_j (since permuting a finite number of indexes does not alter X_n, eventually). Thus by the Hewitt-Savage zero-one law, $\mathbb{P}(\liminf X_n \leq a) \in \{0,1\}$. Let us define $\ell = \inf_a\{\mathbb{P}(\liminf X_n \leq a) = 1\}$, this infimum is on a nonempty set since for $a = \infty$ the probability is one. Then for every $a < \ell$, $\mathbb{P}(\liminf X_n \leq a) = 0$, and for every $a > \ell$, $\mathbb{P}(\liminf X_n \leq a) = 1$, thus

$$\mathbb{P}(\liminf X_n = \ell) = \mathbb{P}\left(\bigcap_{n=1}^{\infty} a - n^{-1} \leq \liminf X_n \leq a + n^{-1}\right) = \lim_n \mathbb{P}\big(a - n^{-1} \leq \liminf X_n \leq a + n^{-1}\big) = 1$$

The second equality is due to continuity of measures, and the final equality is since $a+n^{-1} \leq \liminf X_n \leq a+n^{-1}$ has a probability of 1 for each n. So there exists an ℓ such that $\liminf X_n \overset{as}{=} \ell$. With a similar proof we can show there exists an L such that $\limsup X_n \overset{as}{=} L$.

Now,

$$\ell + \liminf X_n = \liminf \sum_{j=0}^{n} I_j = I_0 + \liminf \sum_{j=1}^{n} I_j \overset{as}{=} I_0 + \ell$$

So either we have that $I_0 \overset{as}{=} 0$ or $\ell \overset{as}{=} \pm\infty$. If $I_0 \overset{as}{=} 0$ then $I_n \overset{as}{=} 0$ for all n since they have the same distribution, and so $X_n = 0$ for all n has probability 1 (since this contains the event that $I_n = 0$ for all n, which has probability 1 as the countable intersection of events with probability 1). Otherwise we similarly have that $L \overset{as}{=} \pm\infty$ and so since $\ell \leq L$, one of the following must have probability 1

$$\begin{aligned} \{\ell = L = -\infty\} &= \{X_n \to -\infty\}, \\ \{\ell = -\infty,\, L = \infty\} &= \{\liminf X_n = -\infty,\, \limsup X_n = \infty\}, \\ \{\ell = L = \infty\} &= \{X - n \to \infty\} \end{aligned}$$

as required.

Let us define $\{X_n\}$ to be the projection of Karen's walk onto the x-axis, thus $X_n = I_1 + \cdots + I_n$ where $\{I_i\}_i$ are independent and $\mathbb{P}(I_i = 1) = \frac{2}{5}, \mathbb{P}(I_i = -1) = \frac{1}{10}, \mathbb{P}(I_i = 0) = \frac{1}{2}$. Then since I_i is not almost surely zero, X_n is not almost surely always zero. Notice then that $\mathbb{P}(\limsup X_n = -\infty) \leq \mathbb{P}(\limsup X_n = \infty)$ since X_n is biased towards the right, and since one of these events must have probability 1 this means that $\limsup X_n \overset{as}{=} \infty$. And in particular this means that X_n almost surely is eventually greater than 3, meaning Karen does almost surely eventually enter the domain.

5 Exercise

Let $\{X_n\}_{n=0}^{\infty}$ be a sequence of independent random variables which all distribute $\mathcal{N}(0,1)$. Compute the following probabilities:

(1) $\mathbb{P}\big(X_n > 0.5\sqrt{\log n} \text{ io}\big)$,

(2) $\mathbb{P}\big(X_n > 2\sqrt{\log n} \text{ io}\big)$,

(3) $\mathbb{P}((\exists N)(\forall n > N) \max\{X_{n^2+1}, \ldots, X_{n^2+10}\} > 2)$

(4) $\mathbb{P}((\exists N)(\forall n > N) \max\{X_{n^2+1}, \ldots, X_{n^2+n}\} > 2)$

(1) Let us define $A_n = \big\{X_n > 0.5\sqrt{\log n}\big\}$. We know that for $Z \sim \mathcal{N}(0,1)$,

$$\frac{1}{\sqrt{2\pi}}\left(\frac{1}{t} - \frac{1}{t^3}\right)e^{-t^2/2} \leq \mathbb{P}(Z > t) \leq \frac{1}{\sqrt{2\pi}}\frac{1}{t}e^{-t^2/2} \implies \mathbb{P}(Z > t) \sim \frac{1}{\sqrt{2\pi}}\frac{1}{t}e^{-t^2/2}$$

This means that

$$\mathbb{P}(A_n) \sim \frac{1}{\sqrt{2\pi}}\frac{2}{\sqrt{\log n}}e^{-\log n/8} = \frac{1}{\sqrt{2\pi}}\frac{2}{\sqrt{\log n} \cdot \sqrt[8]{n}}$$

this means that $\sum \mathbb{P}(A_n) = \infty$, and since $\{A_n\}$ are independent by Borel-Cantelli we get that $\mathbb{P}(A_n \text{ io}) = 1$.

(2) Similarly let us define $A_n = \big\{X_n > 2\sqrt{\log n}\big\}$. And so we get that

$$\mathbb{P}(A_n) \sim \frac{1}{\sqrt{2\pi}}\frac{1}{2\sqrt{\log n}}e^{-2n} = \frac{1}{\sqrt{2\pi}}\frac{1}{2\sqrt{\log n}} \cdot \frac{1}{n^2}$$

so $\sum \mathbb{P}(A_n) < \infty$ and thus by Borel-Cantelli we have that $\mathbb{P}(A_n \text{io}) = 0$.

(3) Notice that the complement of this event is $(\forall N)(\exists n > N)\, X_{n^3+1}, \ldots, X_{n^3+10} \leq 2$ which is just $X_{n^3+1}, \ldots, X_{n^3+10} \leq 2$ io. So let $A_n = \{X_{n^3+1}, \ldots, X_{n^3+10} \leq 2\}$. Then we have that since $\{X_n\}$ are independent

$$\mathbb{P}(A_n) = \mathbb{P}(X_{n^3+1} \leq 2) \cdots \mathbb{P}(X_{n^3+10} \leq 2)$$

so let $p = \mathbb{P}(\mathcal{N}(0,1) \leq 2)$, so $\mathbb{P}(A_n) = p^{10}$. This means that $\sum \mathbb{P}(A_n) = \infty$ and since eventually (for $n \geq 3$) $\{A_n\}$ is independent and io-ness is not affected by a finite amount of events, we get that by Borel-Cantelli $\mathbb{P}(A_n \text{ io}) = 1$, and since this is the complement of the event we get that the probability of the original event is zero.

(4) Let us define $A_n = \{X_{n^3+1}, \ldots, X_{n^3+n} \leq 2\}$, and so as before the complement of the event is A_n io. Let us again define $p = \mathbb{P}(\mathcal{N}(0,1) \leq 2)$ and so

$$\mathbb{P}(A_n) = \mathbb{P}(X_{n^3+1} \leq 2) \cdots \mathbb{P}(X_{n^3+n} \leq 2) = p^n$$

since $0 < p < 1$ as normal distributions have full range, we get that $\sum \mathbb{P}(A_n) < \infty$ and thus by Borel-Cantelli $\mathbb{P}(A_n \text{ io}) = 0$. This is the complement of the original event, and so that has a probability of one.

6 Exercise

Let $B(t)$ be Brownian motion which starts at 0. Which of the following are necessarily true?

(1) Almost surely, there exists an open interval in which $B(t)$ is monotonic.

(2) Almost surely, for every zero of B, s, there exists a sequence of points s_n which converge to s from above such that $B(s_n) = 0$.

(3) Almost surely, there exists an $x > 0$ such that $B(x + 2) - B(x) > 2$.

(4) $B(2)$ is independent of $B(1)$.

(5) Almost surely, $B(t)$ is continuous yet not differentiable in $[0, 10]$.

(6) Almost surely, if $B(3) = 0$ then there exists a sequence of points s_n which converge to 3 from below such that $B(s_n) = 0$.

(7) $B(t + 1) - B(t)$ is also Brownian motion.

(1) Every open interval contains a closed interval, so this would mean that there exists a closed interval $[a, b]$ in which $B(t)$ is monotonic. Since the rationals are dense, we can assume that a and b are rational. In any case, this would mean that $\max_{[a,b]}\{B\}(t) = B(a)$ or $\max_{[a,b]}\{B\}(t) = B(b)$ depending on whether $B(t)$ is increasing or not on $[a, b]$. Both of these events have a probability of zero, and thus so does their union. Thus $B(t)$ is almost surely not monotonic on $[a, b]$ for any set $a < b$. And so

$$\mathbb{P}(B(t) \text{ is monotonic on some open interval}) = \mathbb{P}\left(\bigcup_{a<b\in\mathbb{Q}} B(t) \text{ is monotonic on } [a,b]\right)$$
$$\leq \sum_{a<b\in\mathbb{Q}} \mathbb{P}(B(t) \text{ is monotonic on } [a,b]) = 0$$

So in fact the opposite is true: almost surely there does not exist an open interval in which $B(t)$ is monotonic.

(2) We showed that $\inf\{t > 0 \mid B(t) = 0\} \overset{as}{=} 0$, and since $B(t+s)-B(s)$ is Brownian motion, we get that by applying this to $\{B(t + s) - B(s)\}$, $\inf\{t > s \mid B(t) = B(s)\} = s$. Thus there must almost surely exist a sequence $s_n \downarrow s$ such that $B(s_n) = B(s)$ as required.

(3) Let us define $A_n = \{B(2n + 2) - B(2n) > 10\}$. Then $\{A_n\}$ is independent since differences in Brownian motion are independent. And we have that since $B(2n + 2) - B(2n) \sim \mathcal{N}(0, 2)$, $\mathbb{P}(A_n) = \mathbb{P}(\mathcal{N}(0, 2) > 10) =: p$ where $p > 0$ since normal distributions have full range. Thus $\sum \mathbb{P}(A_n) = \infty$ and so by Borel-Cantelli $\mathbb{P}(A_n \text{ io}) = 1$ thus there almost surely exist an $n > 0$ such that $B(2n + 2) - B(2n) > 10$ as required.

(4) We know that $B(2) - B(1)$ and $B(1) - B(0) = B(1)$ are independent and so $0 = \mathrm{Cov}(B(2) - B(1), B(1)) = \mathrm{Cov}(B(2), B(1)) - \mathrm{Var}(B(1))$. Thus $\mathrm{Cov}(B(2), B(1)) = \mathrm{Var}(B(1)) = \mathrm{Var}(\mathcal{N}(0, 1)) = 1$. Thus $B(2)$ and $B(1)$ are correlated and thus cannot be independent.

(5) We showed that for any set t_0, $B(t)$ is almost surely not differentiable at t_0. So yes, almost surely $B(t)$ is not differentiable in $[0, 10]$ (take for example $t_0 = 5$), and by definition it is also almost surely continuous.

(6) Since $\{B(A - t) - B(A)\}_{0\leq t\leq A}$ is Brownian motion: it is almost surely continuous and $B(A-t-h)-B(A-t) \sim -\mathcal{N}(0, h) = \mathcal{N}(0, h)$ and differences are just differences in $B(t)$ so are independent. And so we have that $\inf\{t > 0 \mid B(A - t) = B(A)\} = 0$, thus $\sup\{t < A \mid B(t) = B(A)\} = A$, meaning there exists a sequence $s_n \uparrow A$ such that $B(s_n) = B(A)$ (take $A = 3$).

(7) Yes, we showed that $\{B(A + t) - B(A)\}_{t\geq 0}$ is Brownian motion for any $A \geq 0$.

7 Exercise

Let $B(t)$ be Brownian motion starting at 0.

(1) Does $\mathbb{P}\big(\max_{[0,1]} B(t) < 2\big) > \frac{3}{4}$?

(2) What is $\mathbb{E}\big[B(3)^2 B(8)\big]$?

(3) Is it true that $\mathbb{E}[T_3] < \mathbb{E}[T_8]$?

(1) By the reflection principle,

$$\mathbb{P}\left(\max_{[0,1]} B(t) < 2\right) = 1 - \mathbb{P}\left(\max_{[0,1]} B(t) \geq 2\right) = 1 - 2\,\mathbb{P}(B(1) \geq 2) = 2\Phi(2) - 1$$

Since $B(1) \sim \mathcal{N}(0,1)$. Since $\Phi(2) \approx 0.9772$, we get that the probability is ≈ 0.9544 which is greater than $\frac{3}{4}$.

(2) Firstly, we claim that $(B(3), B(3), B(8))$ is a Gaussian vector. In general if $X = (X_1, \ldots, X_n)$ is a Gaussian vector, then we claim that $X' = (X_1, X_1, \ldots, X_n)$ is. This is as if $X = AZ + \mu$ then

$$X' = \begin{pmatrix} \text{---} \; e_1 A \; \text{---} \\ A \end{pmatrix} \cdot Z + \begin{pmatrix} \mu_1 \\ \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}$$

Now, if $X \sim \mathcal{N}(0, \Sigma)$ then $-X \sim \mathcal{N}(0, \Sigma)$ as well (if $X = AZ$ then $-X = A(-Z)$ and $-Z \sim \mathcal{N}(0, I)$), thus $X \overset{d}{=} -X$ and therefore for every measurable function f, $f(X) \overset{d}{=} f(-X)$. So let $f(X) = X_1 \cdots X_n$ and so we get that $X_1 \cdots X_n \overset{d}{=} (-1)^n X_1 \cdots X_n$ and in particular $\mathbb{E}[X_1 \cdots X_n] = (-1)^n\, \mathbb{E}[X_1 \cdots X_n]$ so if n is odd we get that $\mathbb{E}[X_1 \cdots X_n] = 0$. And in our case, $n = 3$ is odd so we have that $\mathbb{E}\big[B(3)^2 B(8)\big] = 0$.

(3) We showed that for any $a > 0$,

$$f_{T_a}(t) = \frac{a}{\sqrt{2\pi}} t^{-3/2} e^{-a^2/2t}$$

and so this means that $t f_{T_a}(t) = \frac{a}{\sqrt{2\pi}} t^{-1/2} e^{-a^2/2t} \sim \frac{a}{\sqrt{2\pi}} t^{-1/2}$. Thus we have that $\mathbb{E}[T_a] = \int_0^\infty t f_{T_a}(t)\, dt = \infty$, and so $\mathbb{E}[T_3] = \mathbb{E}[T_8] = \infty$.