Algebraic Topology II

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1 Singular Homology

1.1 Chain Complexes

We begin by defining a *chain complex*. A chain complex is a sequence of Abelian groups with homomorphisms between them:

$$\cdots \dashrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \dashrightarrow \cdots \dashrightarrow C_0 \longrightarrow 0$$

such that for every n, $\partial_n \circ \partial_{n+1} = 0$, in other words $\mathrm{Im}\partial_{n+1} \subseteq \ker \partial_n$. Define $Z_n = \ker \partial_n$, and its elements will be called *n-dimensional cycles*. And define $B_n = \mathrm{Im}\partial_{n+1}$, its elements will be called *boundaries*. Elements of the groups C_n will be called *n-dimensional chains*.

We now want to define a category of chain complexes. To do so we must define morphisms between chain complexes. So suppose we have two chain complexes $\mathscr{C} = \{C_n, \partial_n\}$ and $\mathscr{D} = \{D_n, \partial'_n\}$. We define a morphism from $\mathscr{C}$ to $\mathscr{D}$ to be a sequence of homomorphisms $f_n\colon C_n \longrightarrow D_n$ which preserves the structure of the chain. Meaning $\partial'_n \circ f_n = f_{n-1} \circ \partial_n$, in other words the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r, dashed] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"'] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[r, dashed] \arrow[d, "f_{n-1}"'] & \cdots \arrow[r, dashed] & C_0 \arrow[r] \arrow[d, "f_0"'] & 0 \\
\cdots \arrow[r, dashed] & D_{n+1} \arrow[r, "\partial'_{n+1}"] & D_n \arrow[r, "\partial'_n"] & D_{n-1} \arrow[r, dashed] & \cdots \arrow[r, dashed] & D_0 \arrow[r] & 0
\end{tikzcd}$$

To simplify writing, we will write $\partial \circ f = f \circ \partial$, which f and which ∂ is being referred to will be understood from context.

The composition of two morphisms $\{f_n\}\colon \mathscr{C} \longrightarrow \mathscr{D}$ and $\{g_n\}\colon \mathscr{D} \longrightarrow \mathscr{E}$ is defined to be $\{g_n \circ f_n\}\colon \mathscr{C} \longrightarrow \mathscr{E}$. This is indeed a morphism:

$$\partial \circ f \circ g = f \circ \partial \circ g = f \circ g \circ \partial$$

And then this implies that the identity morphism is just $\mathrm{Id}_{\mathscr{C}} = \{\mathrm{Id}_{\mathrm{C_n}}\}\colon \mathscr{C} \longrightarrow \mathscr{C}$, as if $\{f_n\}\colon \mathscr{C} \longrightarrow \mathscr{D}$ then

$$\{f_n\} \circ \mathrm{Id}_{\mathscr{C}} = \{f_n \circ \mathrm{Id}_{C_n}\} = \{f_n\}, \qquad \mathrm{Id}_{\mathscr{D}} \circ \{f_n\} = \{\mathrm{Id}_{D_n} \circ f_n\} = \{f_n\}$$

Associativity is clear, so **Comp**, the category of chain complexes, is indeed a category.

Now recall that by definition $\partial_n \circ \partial_{n+1} = 0$, meaning

$$B_n \subseteq Z_n \subseteq C_n$$

Since these groups are all Abelian, they are normal in one another, so let us define the *nth homology group* of a chain complex $\mathscr{C}$ as

$$H_n(\mathscr{C}) := Z_n \big/ B_n = \ker \partial_n \big/ \mathrm{Im}\partial_{n+1}$$

1.1 Proposition

A chain complex morphism $\{f_n\}\colon \mathscr{C} \longrightarrow \mathscr{D}$ maps cycles to cycles and boundaries to boundaries.

Proof: let $z \in C_n$ be a cycle, i.e. $\partial z = 0$, but then $f(z)$ is a cycle since $\partial f(z) = f(\partial z) = f(0) = 0$. And let $b \in C_n$ be a boundary, so there exists an $a \in C_{n+1}$ such that $b = \partial a$. Then $f(b) = f\partial(a) = \partial f(a) = \partial b$, so $f(b)$ is a boundary as well. ■

This means that if $\{f_n\}\colon \mathscr{C} \longrightarrow \mathscr{D}$ is a morphism of chain complexes, $\{f_n\}\colon Z_n(\mathscr{C}) \longrightarrow Z_n(\mathscr{D})$ is well-defined, and so we have that

$$\begin{tikzcd}
Z_n(\mathscr{C}) \arrow[r] \arrow[d] \arrow[dr, bend left] & Z_n(\mathscr{D}) \arrow[d] \\
H_n(\mathscr{C}) & H_n(\mathscr{D})
\end{tikzcd}$$

Where the blue arrow ψ is just the quotient map composed with f_n. This induces a group morphism

$$H_n(\{f_n\}) = f_*\colon H_n(\mathscr{C}) \longrightarrow H_n(\mathscr{D})$$

since we can define $f_*([z]) = \psi(z)$ since if $[z] = [z']$ then $z - z' \in B_n(\mathscr{C})$ and so $f(z - z') \in B_n(\mathscr{D})$ and thus the quotient of $f(z - z')$ is just 0, so $\psi(z) = \psi(z')$. Explicitly,

$$f_*[z] = [f_n z]$$

We now claim that H_n is a functor from the category of chain complexes **Comp** to the category of Abelian groups **Ab**. Now suppose $\{f_n\}: \mathscr{C} \longrightarrow \mathscr{D}$ and $\{g_n\}: \mathscr{D} \longrightarrow \mathscr{E}$ are chain complex morphisms, then the following diagram commutes

$$\begin{tikzcd}
Z_n(\mathscr{C}) \arrow[r, "f"] \arrow[d] & Z_n(\mathscr{D}) \arrow[r, "g"] \arrow[d] & Z_n(\mathscr{E}) \arrow[d] \\
H_n(\mathscr{C}) \arrow[r, "f_*"'] & H_n(\mathscr{D}) \arrow[r, "g_*"'] & H_n(\mathscr{E})
\end{tikzcd}$$

And so $(g \circ f)_* = g_* \circ f_*$, and it is easily verified that $\mathrm{id}_* = \mathrm{id}$ so H_n is a functor **Comp** $\longrightarrow$ **Ab** (the category of Abelian groups).

1.2 Singular Complex

We now define a functor from **Top** to **Comp**.

1.1 Definition

Let B be a set, then define the **free Abelian group** over B to be

$$\mathrm{FA}(B) = \bigoplus_{b \in B} \mathbb{Z} = \{\varphi: B \longrightarrow \mathbb{Z} \mid \varphi(b) \neq 0 \text{ for only finitely many } b \in B\}$$

Note then that there is a correspondence between B and $\mathrm{FA}(B)$: $b \leftrightarrow \varphi_b$ where

$$\varphi_b(x) = \begin{cases} 1 & x = b \\ 0 & x \neq b \end{cases}$$

so we can identify b with φ_b, and it is easy to see that every element of $\mathrm{FA}(B)$ can be written as $\sum_{i=1}^k n_i \varphi_{b_i}$, abusing notation $\sum_{i=1}^k n b_i$ and such a representation is unique.
Notice that if B is a set, G an Abelian group, and $g: B \longrightarrow G$ a function, then there exists a unique group homomorphism $L: \mathrm{FA}(B) \longrightarrow G$ which extends g. This is defined by

$$L: \sum_{i=1}^k n_i b_i \longmapsto \sum_{i=1}^k n_i g(b_i)$$

1.2 Definition

The **n-dimensional simplex** is defined to be

$$\Delta^n := \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; x_i \geq 0, \sum_{i=0}^n x_i = 1 \right\}$$

Δ^n has $n + 1$ faces, and is homeomorphic to D^n.

1.3 Definition

Let X be a topological space, then an **n-dimensional singular simplex** in X is a morphism (in the category of topological spaces; a continuous map) $\Delta^n \longrightarrow X$. Define $S_n(x)$ to be the set of all n-dimensional singular simplexes in X, and define $C_n(X) = \mathrm{FA}(S_n(x))$.

We now want to define a chain complex on the sequence $C_n(X)$.

Let us define a set of maps $\tau_i^n\colon \Delta^{n-1} \longrightarrow \Delta^n$ for $0 \le i \le n$ which maps

$$\tau_i^n\colon (x_0, \dots, x_{n-1}) \mapsto (x_0, \dots, x_{i-1}, 0, x_i, \dots, x_{n-1})$$

This is a well-defined continuous map, and geometrically it maps Δ^{n-1} to one of the faces of Δ^n.

Let $\sigma \in S_n(x)$, then let us define

$$\partial(\sigma) := \sum_{i=0}^{n} (-1)^i \sigma \circ \tau_i^n$$

Note that the composition is well-defined since $\Delta^{n-1} \xrightarrow{\tau_i^n} \Delta^n \xrightarrow{\sigma} X$, meaning $\sigma \circ \tau_i^n$ is an $n-1$-dimensional singular simplex. Thus ∂ can be extended to a map $\partial\colon C_n(X) = \mathrm{FA}(S_n(X)) \longrightarrow \mathrm{FA}(S_{n-1}(X)) = C_{n-1}(X)$ Notice that

$$\partial_{n-1}(\partial_n \sigma) = \partial_{n-1}\left(\sum_{i=0}^{n} (-1)^i \sigma \circ \tau_i^n\right) = \sum_{i=0}^{n} (-1)^i \partial_{n-1}(\sigma \circ \tau_i^n) = \sum_{i=0}^{n} \sum_{j=0}^{n-1} (-1)^{i+j} \sigma \circ \tau_i^n \circ \tau_j^{n-1}$$

Notice that $\tau_i^n \circ \tau_j^{n-1} = \tau_j^n \circ \tau_{i-1}^{n-1}$ which can be verified from its definition, but the first has a sign of $(-1)^{i+j}$ in the sum and the second has $-(-1)^{i+j}$. And so the sum is zero.

Thus we have defined a chain complex on $C_n(X)$, let us denote it by $\mathscr{C}(X)$, this is the first step in defining the functor. Next we must define the correspondence between morphisms.

Let $f\colon X \longrightarrow Y$ be a continuous map between topological spaces. Let us define $f_\sharp\colon \mathscr{C}(X) \longrightarrow \mathscr{C}(Y)$. First we define it for $\sigma \in S_n(X)$ by $f_\sharp(\sigma) = f \circ \sigma$. Since $\sigma\colon \Delta^n \longrightarrow X$ is continuous, so is $f \circ \sigma\colon \Delta^n \longrightarrow Y$ and so $f_\sharp$ is well-defined on the generators of $C_n(X)$. This can be extended by linearity to $f_\sharp\colon C_n(X) \longrightarrow C_n(Y)$. Notice that we ignore the subscripts and superscripts $(f_\sharp)_n^X$ for brevity and readability.

Now we must verify that this is a morphism of chain complexes, i.e. that $\partial f_\sharp = f_\sharp \partial$. So

$$f_\sharp \partial \sigma = f_\sharp\left(\sum_{i=0}^{n} (-1)^i \sigma \circ \tau_i^n\right) = \sum_{i=0}^{n} (-1)^i f_\sharp(\sigma \circ \tau_i^n) = \sum_{i=0}^{n} (-1)^i f \circ \sigma \circ \tau_i^n = \sum_{i=0}^{n} (-1)^i (f \circ \sigma) \circ \tau_i^n = \partial f_\sharp \sigma$$

and since this holds for generators, by linearity it holds for all $C_n(X)$. Thus $f_\sharp$ is indeed a morphism of chain complexes.

Thus we have defined a functor **Top** $\longrightarrow$ **Comp**.

1.3 Singular Homology

We have two functors **Top** $\longrightarrow$ **Comp** $\longrightarrow$ **Ab**, and so composing them together gives us a functor **Top** $\longrightarrow$ **Ab**. For a topological space X, we will denote its image under this functor as $H_n(X)$, called the *nth homological group* of X. And for a continuous map f, we denote its image as f_* or $H_n(f)$.

Let us compute the homological groups of the trivial space: $X = \{p\}$. Notice that $S_n(X) = \{K_n\}$ where K_n is the constant map $\Delta^n \longrightarrow \{p\}$, and so $C_n(X) = \mathbb{Z}$. We want to now compute what the boundary operators are, so

$$\partial K_n = \sum_{i=0}^{n} (-1)^i K_n \circ \tau_i^n$$

but $K_n \circ \tau_i^n$ is a morphism $\Delta^{n-1} \longrightarrow \{p\}$ meaning it is equal to K_{n-1}, thus $\partial K_n = \left(\sum_{i=0}^{n} (-1)^i\right) K_{n-1}$. For n even this is then K_{n-1} (or 1), and 0 for n odd. This means that either $\ker \partial = 0$ or $\mathrm{Im}\partial = \mathbb{Z}$, thus $H_n = 0$ for $n > 0$. For $n = 0$, we have that $\partial_0\colon \mathbb{Z} \longrightarrow 0$ and so its kernel is $\mathbb{Z}$, but ∂_1 is trivial and so its image is 0. Thus $H_0 = \mathbb{Z}$.

So we have shown

1.1 Proposition

Let $X = \{p\}$ be the trivial topological space, then its homological groups are

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n > 0 \end{cases}$$

1.2 Proposition

Let X be path connected, then $H_0(X) \cong \mathbb{Z}$.

Proof: we are concerned with the chain:

$$C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

So first let us understand $C_0(X)$, this is generated by $S_0(X)$, all the maps $\Delta^0 \longrightarrow X$ which are just all the points in X. And $S_1(X)$ is generated by all the maps $I \cong \Delta^1 \longrightarrow X$, so all the paths in X. The boundary of a 1-simplex is then

$$\partial_1\sigma = \sigma(1) - \sigma(0)$$

and thus $B_1(X) = \mathrm{Im}\partial_1$ is generated by elements of the form $a - b$ where there exists a path between a and b. Since X is path-connected, this means that $B_1(X)$ is generated by $a - b$ for $a, b \in X$. Now, the subgroup generated by this is $\{\sum n_i p_i \mid p_i \in X, \sum n_i = 0\}$.

And now ∂_0's kernel is just $C_0(X)$ which is simply the free group generated by X. Thus

$$H_0(X) = {\left\{\sum n_i p_i\right\}}\Big/{\left\{\sum n_i p_i \,\Big|\, \sum n_i = 0\right\}}$$

This is isomorphic to $\mathbb{Z}$ since we can define $\varphi\colon C_0(X) \longrightarrow \mathbb{Z}$ by $\sum n_i p_i \mapsto \sum n_i$ and this is a group homomorphism whose image is $\mathbb{Z}$ and whose kernel is all the points $\sum n_i p_i$ where $\sum n_i = 0$. Thus by the isomorphism theorem, $H_0(X) \cong \mathbb{Z}$. ∎

1.3 Theorem

Let X be a topological space where $\{A_\alpha\}_{\alpha\in I}$ are its path connected components. Then for every n,

$$H_n(X) \cong \bigoplus_{\alpha\in I} H_n(A_\alpha)$$

Proof: notice that if $\sigma\colon \Delta^n \longrightarrow X$ is an n-simplex, then its image is contained within a path connected component. This is since Δ^n is path-connected, so $\sigma\Delta^n$ must be too. Thus for every $\gamma = \sum n_i\sigma_i \in S_n(X)$ we can write it as $\gamma = \sum \gamma_i$ for $\gamma_i \in S_n(A_i)$. And so $C_n(X) = \bigoplus_{\alpha\in I} C_n(A_\alpha)$.

Notice that γ is a cycle iff every γ_i is a cycle, since $\partial\gamma = \sum \partial\gamma_i$ and this is an element of a direct sum, so it is zero iff $\partial\gamma_i = 0$. Thus $Z_n(X) = \bigoplus_{\alpha\in I} Z_n(A_\alpha)$. And similarly we see that $B_n(X) = \bigoplus_{\alpha\in I} B_n(A_\alpha)$. Thus $H_n(X) = \bigoplus_{\alpha\in I} H_n(A_\alpha)$. ∎

1.4 Corollary

If X is a topological space with $\{A_\alpha\}_{\alpha\in I}$ path connected components, $H_n(X) = \bigoplus_{\alpha\in I} \mathbb{Z}$.

1.5 Theorem

Let X be path-connected and $a \in X$, then $H_1(X) \cong \operatorname{Ab} \pi_1(X, a)$.

For two chains, $a, b \in C_n(X)$ say that they are *homological* if $a - b$ is a boundary (i.e. $a - b \in B_n(X)$). Write this as $a \approx b$.

1.6 Lemma

Let σ, τ be 1-simplexes.

(1) if σ is constant, then it is a boundary, i.e. $\sigma \approx 0$.

(2) if $\sigma \overset{\partial I}{\sim} \tau$ (since they are maps from $I \cong \Delta^1 \longrightarrow X$), then $\sigma \approx \tau$.

(3) if $\sigma(1) = \tau(0)$ then $\sigma * \tau \approx \sigma + \tau$.

(4) $\sigma + \bar{\sigma} \approx 0$

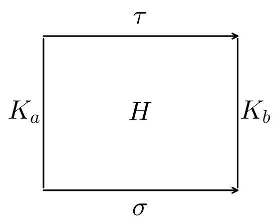
Proof:

(1) If σ is constant, then it is K_p^1 for some $p \in X$. And as we have already computed

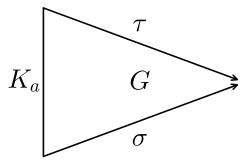
$$\partial K_p^n = \begin{cases} K_p^{n-1} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Thus $\partial K_p^2 = K_p^{n-1}$, meaning σ is a boundary.

(2) Let us look at the homotopy



Since H is surjective, it induces a map on the quotient space ${}^{I \times I}/_{I \times \{1\}}$, the map G:



The quotient space can be viewed as a 2-simplex by assigning an order to its vertices. Then its boundary is

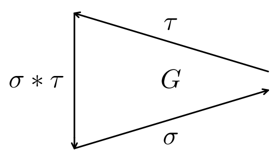
$$\partial G = K_a - \sigma + \tau$$

and since ∂G is a boundary, we have that

$$K_a - \sigma + \tau \approx 0$$

by (1) we have that $K_a \approx 0$ so $\sigma - \tau \approx 0$.

(3) The idea is to define a simplex of the form



Notice that such a simplex is possible: each horizontal line in the domain can be made constant. And its boundary is

$$\partial G = \tau - \sigma * \tau + \sigma$$

so $\sigma * \tau \approx \sigma + \tau$ since $\partial G \approx 0$.

(4) This is direct from the previous three points:

$$\sigma + \overline{\sigma} \overset{(3)}{\approx} \sigma * \overline{\sigma} \overset{(2)}{\approx} K_b \overset{(1)}{\approx} 0$$

Proof (of theorem 1.5): let us define a homomorphism

$$F\colon \pi_1(X,a) \longrightarrow H_1(X)$$

Denote homotopy equivalence classes by $\langle\bullet\rangle$ and the equivalence classes of $H_1(X)$ by $[\bullet]$. Then we define

$$\langle\varphi\rangle \overset{F}{\longmapsto} [\varphi]$$

This is well-defined: if $\varphi \overset{\partial I}{\sim} \psi$ then $\varphi \approx \psi$ and so $[\varphi] = [\psi]$ (since $H_n(X)$ is the partition of $Z_n(X)$ relative to $\approx$). Notice that $\langle\varphi * \psi\rangle \mapsto [\varphi * \psi] = [\varphi + \psi] = [\varphi] + [\psi]$. So this is indeed a homomorphism. Since $H_1(X)$ is Abelian, this induces a homomorphism

$$\overline{F}\colon \mathrm{Ab}\,\pi_1(X,a) \longrightarrow H_1(X)$$

Let us now define a homomorphism

$$G\colon C_1(X) \longrightarrow \mathrm{Ab}\,\pi_1(X,a)$$

denote the equivalence classes of $\mathrm{Ab}\,\pi_1(X,a)$ by $\langle\langle\bullet\rangle\rangle$. For every $x \in X$, choose a path γ_x from a to x, then for $\sigma \in S_1(X)$ define

$$\hat{\sigma} = \gamma_{\sigma(0)} * \sigma * \overline{\gamma}_{\sigma(1)} \text{ from } a \text{ to } a$$

And define

$$\sigma \overset{G}{\longmapsto} \langle\langle\hat{\sigma}\rangle\rangle$$

And extend by linearity to $G\colon C_1(X) \longrightarrow \mathrm{Ab}\,\pi_1(X,a)$. We can then restrict G to $Z_1(X)$, and in order for this to induce a map on ${Z_1(X)}/{B_1(X)}$ we must have that $G\big|_{B_1(X)} = 0$. So let A be a 2-simplex, then we must show $G(\partial A) = 0$. We know

$$G(\partial A) = G(A\circ\tau_0 - A\circ\tau_1 + A\circ\tau_2) = \left\langle\!\left\langle \widehat{A\circ\tau_0} \right\rangle\!\right\rangle - \left\langle\!\left\langle \widehat{A\circ\tau_1} \right\rangle\!\right\rangle + \left\langle\!\left\langle \widehat{A\circ\tau_2} \right\rangle\!\right\rangle$$

Now, $\langle\langle\sigma\rangle\rangle + \langle\langle\tau\rangle\rangle = \langle\langle\sigma\rangle\langle\tau\rangle\rangle$ and $-\langle\langle\sigma\rangle\rangle = \langle\sigma^{-1}\rangle$ by Abelianization, so this is equal to

$$\left\langle\!\left\langle \widehat{A\circ\tau_0} \right\rangle \left\langle \widehat{A\circ\tau_1} \right\rangle \left\langle \widehat{A\circ\tau_2} \right\rangle\!\right\rangle = \left\langle\!\left\langle \widehat{A\circ\tau_0} * \overline{\widehat{A\circ\tau_1}} * \widehat{A\circ\tau_2} \right\rangle\!\right\rangle$$

As is easily verified,

$$= \left\langle\!\left\langle \widehat{A\circ\tau_0} * \widehat{\overline{A\circ\tau_1}} * \widehat{A\circ\tau_2} \right\rangle\!\right\rangle = \left\langle\!\left\langle \widehat{A\circ\tau_0 * \overline{A\circ\tau_1} * A\circ\tau_2} \right\rangle\!\right\rangle$$

Since $A\colon \Delta^2 \longrightarrow X$ is a simplex, $A\circ\tau_0 * \overline{A\circ\tau_1} * A\circ\tau_2$ is null-homotopic (the homotopy can condense the curve to a point through the image of A). Therefore its hat is as well, meaning this is all equal to zero, as required. So G induces a homomorphism

$$\overline{G}\colon H_1(X) \longrightarrow \mathrm{Ab}\,\pi_1(X,a)$$

Notice that

$$\overline{G}\circ\overline{F}\langle\langle\varphi\rangle\rangle = \overline{G}[\varphi] = \langle\langle\hat{\varphi}\rangle\rangle$$

We know that $\hat{\varphi} = \gamma_a\varphi\overline{\gamma}_a$ which is conjugate to φ, so in the Abelianization they are equal. So $\overline{G}\circ\overline{F} = \mathrm{id}$. Now suppose $[z] \in H_1(X)$ where $z = \sum n_i\sigma_i$ then

$$\overline{F}\circ\overline{G}[z] = \overline{F}\Big(\sum n_i\langle\langle\hat{\sigma}_i\rangle\rangle\Big) = \sum n_i[\hat{\sigma}_i] = \Big[\sum n_i\hat{\sigma}_i\Big]$$

So we need to show that if $\sum n_i\sigma_i$ is a cycle then $\sum n_i\hat{\sigma}_i \approx \sum n_i\sigma_i$. Define $T\colon C_0(X) \longrightarrow C_1(X)$ by $T(p) = \gamma_p$, so

$$\hat{\sigma} = \gamma_{\sigma 0} * \sigma * \overline{\gamma}_{\sigma 1} \approx \gamma_{\sigma 0} + \sigma - \gamma_{\sigma 1} = \sigma - T\partial\sigma$$

And so

$$\sum n_i\hat{\sigma}_i \approx \sum n_i\sigma_i - T\partial\sum n_i\sigma_i = z - T\partial z$$

since z is a cycle, $\partial z = 0$ and so this is equal to z. Thus $\hat{z} \approx z$ as required.
So $\overline{F}, \overline{G}$ are inverse isomorphisms, meaning $H_1(X) \cong \mathrm{Ab}\,\pi_1(X,a)$. ■

1.7 Definition

Let $\mathscr{C}, \mathscr{D}$ be two categories and let $F, G \colon \mathscr{C} \longrightarrow \mathscr{D}$ be functors. Then a **natural transformation** η from F to G is a correspondence such that

(**1**) for every object $X \in \mathscr{C}$, η associates a morphism $\eta_X \colon F(X) \longrightarrow G(X)$ called the **component** of X.

(**2**) for every $f \colon X \longrightarrow Y$ morphism, $\eta_Y \circ F(f) = G(f) \circ \eta_X$, i.e. the following diagram commutes

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"'] & G(Y) \end{tikzcd}$$

So for every pointed topology (X, a) we defined a group homomorphism $F_{X,a} \colon \pi_1(X, a) \longrightarrow H_1(X)$. We claim that this is a natural transformation from π_1 to H_1.

Suppose there is a morphism $h \colon (X, a) \longrightarrow (Y, b)$, so we need the following diagram to commute:

$$\begin{tikzcd} \pi_1(X,a) \arrow[r, "F_{X,a}"] \arrow[d, "\pi_1(h)"'] & H_1(X) \arrow[d, "H_1(h)"] \\ \pi_1(Y,b) \arrow[r, "F_{Y,b}"'] & H_1(Y) \end{tikzcd}$$

This is indeed the case:

$$\langle \varphi \rangle \xrightarrow{F_{X,a}} [\varphi] \xrightarrow{H_1(h)} [h \circ \varphi], \qquad \langle \varphi \rangle \xrightarrow{\pi_1(h)} \langle h \circ \varphi \rangle \xrightarrow{F_{Y,b}} [h \circ \varphi]$$

1.8 Example

If we look at the identity functor (on the category of groups) and Abelianization, then $\rho_\bullet$, which is the quotient map $\bullet \longrightarrow \mathrm{Ab}\,\bullet$, is a natural transformation. Indeed

$$\begin{tikzcd} G \arrow[r, "\rho_G"] \arrow[d, "\varphi"'] & \mathrm{Ab}\,G \arrow[d, "\hat{\varphi}"] \\ H \arrow[r, "\rho_H"'] & \mathrm{Ab}\,H \end{tikzcd}$$

Where $\hat{\varphi}[g] = [\varphi(g)]$. This is indeed natural:

$$\rho_H \circ \varphi(g) = [\varphi(g)], \qquad \hat{\varphi} \circ \rho_G(g) = \hat{\varphi}[g] = [\varphi(g)]$$

1.9 Definition

The **simplified singular chain complex** of a topological space X is the chain complex

$$\cdots \dashrightarrow C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \dashrightarrow \cdots \dashrightarrow C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

Where we define ε as follows:

$$\varepsilon \sum n_i p_i = \sum n_i$$

i.e. $\varepsilon p = 1$ for every $p \in X$. And a morphism between two simplified singular chain complexes differ only from morphisms between normal singular chain complexes in that the map from $\mathbb{Z}$ to $\mathbb{Z}$ is the identity.

The homology induced by a simplified singular chain complex is called the **simplified homology** and denoted $\tilde{H}_n(X)$.

Obviously for every $n \geq 1$, $\tilde{H}_n(X) = H_n(X)$.

1.10 Definition

A chain of Abelian groups

$$\cdots \dashrightarrow A \xrightarrow{f} B \xrightarrow{g} C \dashrightarrow \cdots$$

is **exact** at B if $\mathrm{Im} f = \ker g$. If the sequence is exact at every group, then the sequence itself is called an **exact sequence**. (Recall that chain complexes require $\mathrm{Im} f \subseteq \ker g$.)

If we have an exact sequence in one of the following forms, then:

(1) $0 \longrightarrow A \xrightarrow{f} B$, then $0 = \ker f$ so f is injective.

(2) $A \xrightarrow{f} B \longrightarrow 0$, then $\mathrm{Im} f = B$ so f is surjective.

(3) $0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0$, then f is an isomorphism.

1.11 Definition

A **short exact sequence** is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

In a short exact sequence, by above f is injective and g is surjective, and furthermore $\mathrm{Im} f = \ker g$. In such a case, we can view A as being a subgroup of B (since f is an embedding) and since by the isomorphism theorem $C \cong {}^{B}\!/_{\ker g} = {}^{B}\!/_{\mathrm{Im} f} = {}^{B}\!/_{A}$, a short exact sequence can be viewed as

$$0 \longrightarrow A \xrightarrow{\textit{inclusion}} B \xrightarrow{\textit{quotient}} {}^{B}\!/_{A} \longrightarrow 0$$

1.12 Lemma (The Lemma of Five)

Suppose the chains $\{A_i\}_i, \{B_i\}_i$ are exact, and the following diagram commutes:

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "f_1"'] & A_2 \arrow[r] \arrow[d, "f_2"'] & A_3 \arrow[r] \arrow[d, "f_3"'] & A_4 \arrow[r] \arrow[d, "f_4"'] & A_5 \arrow[d, "f_5"'] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd}$$

(1) If f_2, f_4 are injective and f_1 is surjective, then f_3 is injective.

(2) If f_2, f_4 are surjective and f_5 is injective, then f_3 is surjective.

Proof: We write $x \overset{A}{\longmapsto} y$ to mean x maps to y in the exact sequence ($x \in A_i$).

(1) Suppose $f_3 a$, then $a \xmapsto{f_3} 0 \xmapsto{B} 0$, now suppose $a \xmapsto{A} b \xmapsto{f_4} c$. Since the diagram commutes, we must have that $c = 0$, but f_4 is injective so $b = 0$. This means $a \in \ker A$, so there exists some d such that $d \xmapsto{A} a$. Suppose $d \xmapsto{f_2} e$, then $e \xmapsto{B} 0$ by commutativity, so there exists an f such that $f \xmapsto{B} e$, and since f_1 is surjective there exists a $g \xmapsto{f_1} f$. Now suppose $g \xmapsto{A} h$. By commutativity, since $g \xmapsto{f_1} f \xmapsto{B} e$ we have $f_2 h = e$ and since f_2 is injective, $h = d$. So d is in the image of A, so it is in the kernel and so $a = 0$.

(2) is a little more complicated, but it's just chasing. ■

1.13 Definition

Suppose $\mathscr{C}$ and $\mathscr{D}$ are two chain complexes, with two morphisms $f, g\colon \mathscr{C} \longrightarrow \mathscr{D}$. Then a **chain homotopy** from f to g is a sequence of maps $T_n\colon C_n \longrightarrow D_{n+1}$ such that $\partial T + T\partial = f - g$. If there exists a chain homotopy between f and g, we write $f \overset{CH}{\sim} g$.

In a diagram, we have that T are the red arrows.

$$
\begin{tikzcd}
\cdots \arrow[r, dashed] & C_{n+1} \arrow[r] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] \arrow[dl, dashed, red] & C_n \arrow[r] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] \arrow[dl, red] & C_{n-1} \arrow[r, dashed] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] \arrow[dl, red] & \cdots \\
\cdots \arrow[r, dashed] & D_{n+1} \arrow[r] & D_n \arrow[r] & D_{n-1} \arrow[r, dashed] & \cdots
\end{tikzcd}
$$

Let $X \subseteq \mathbb{R}^k$ be convex. For $a \in X$ let us define the *cone construction* $C_a\colon C_n(X) \longrightarrow C_{n+1}(X)$ as follows: we start with generators of $C_n(X)$, i.e. we define $C_a\sigma$ for $\sigma\colon \Delta^n \longrightarrow X$ an n-simplex. Geometrically, $C_a\sigma$ will be a cone whose tip is a and whose base is σ. We define this by:

$$C_a\sigma(t_0, \dots, t_{n+1}) = t_0 b + (1 - t_0)\sigma\left(\frac{t_1}{1 - t_0}, \dots, \frac{t_{n+1}}{1 - t_0}\right)$$

Let us now compute the faces of $C_a\sigma$. For $i = 0$ then

$$(C_a\sigma)\tau_0^{n+1}(t_0, \dots, t_n) = C_a\sigma(0, t_0, \dots, t_n) = \sigma(t_0, \dots, t_n)$$

For $i > 0$ then

$$(C_a\sigma)\tau_i^{n+1}(t_0, \dots, t_n) = C_a\sigma(t_0, \dots, 0, \dots, t_n)$$

if $t_0 = 1$ as well, then this is just

$$C_a\sigma(1, 0, \dots, 0) = a$$

Otherwise,

$$
\begin{aligned}
&= t_0 b + (1 - t_0)\sigma\left(\frac{t_1}{1 - t_0}, \dots, 0, \dots, \frac{t_n}{1 - t_0}\right) \\
&= t_0 b + (1 - t_0)\sigma\tau_{i-1}^n\left(\frac{t_1}{1 - t_0}, \dots, \frac{t_n}{1 - t_0}\right) \\
&= C_a^{n-1}(\sigma\tau_{i-1}^n)(t_0, \dots, t_n)
\end{aligned}
$$

So we see that

$$(C_a\sigma)\tau_0^{n+1} = \sigma, \qquad (C_a\sigma)\tau_i^{n+1} = C_a^{n-1}(\sigma\tau_{i-1}^n)$$

So

$$
\begin{aligned}
\partial_{n+1} C_a^n(\sigma) = \sum_{i=0}^{n+1} (-1)^i (C_a^n\sigma)\tau_i^{n+1} &= \sigma + \sum_{i=1}^{n+1} C_a^{n-1}(\sigma\tau_{i-1}^n) \\
&= \sigma - \sum_{i=0}^{n} (-1)^i C_a^{n-1}(\sigma\tau_i^n) \\
&= \sigma - C_a^{n-1}\left(\sum_{i=0}^{n} (-1)^i \sigma\tau_i^n\right) \\
&= \sigma - C_a^{n-1}\partial_n\sigma
\end{aligned}
$$

So we see that

$$\partial C_a - C_a\partial = \mathrm{id}$$

so in other words, C_a is a chain homotopy from id to 0.

1.14 Theorem

Let X be a convex set in $\mathbb{R}^k$, then for all $n > 0$, $H_n(X) = 0$.

Proof: let $\gamma \in C_n(X)$, then $\gamma = \partial C_a\gamma + C_a\partial\gamma$. If $\gamma \in Z_n(X)$, i.e. it is a cycle, then $\partial\gamma = 0$ and so $\gamma = \partial C_a\gamma$. Thus $\gamma \in B_n(X)$, so $Z_n(X) = B_n(X)$, and then $H_n(X) = 0$. ■

1.15 Lemma

If $f, g\colon X \longrightarrow Y$ are two homotopic continuous maps, then $f_\sharp$ and $g_\sharp$ are chain homotopic.

Proof: let us define $i, j\colon X \longrightarrow X \times I$ where $i(x) = (x, 0)$ and $j(x) = (x, 1)$. If $H\colon X \times I \longrightarrow Y$ is a homotopy from f to g, then $f = H \circ i$ and $g = H \circ j$. Also $i \sim j$, so if we can show that $i_\sharp \overset{CH}{\sim} j_\sharp$ then we have that

$$f_\sharp = H_\sharp \circ i_\sharp \overset{CH}{\sim} H_\sharp \circ j_\sharp = g_\sharp$$

so it is sufficient to show that $i_\sharp \overset{CH}{\sim} j_\sharp$.
So we need to define a sequence of morphisms $T_n^X\colon C_n(X) \longrightarrow C_{n+1}(X \times I)$ such that $\partial T^X + T^X\partial = i_\sharp^X - j_\sharp^X$. We will define T_n^X by induction on n, such that T^X is natural. Natural between what two functors? The first functor maps topological spaces X to their chain complexes $\mathscr{C}(X)$ and maps morphisms $X \xrightarrow{f} Y$ to $f_\sharp\colon \mathscr{C}(X) \longrightarrow \mathscr{C}(Y)$. The second functor maps topological spaces X to the chain complex $C_{n+1}(X \times I)$ and morphisms $X \xrightarrow{f} Y$ to $(f \times \mathrm{id})_\sharp\colon C_{n+1}(X) \longrightarrow C_{n+1}(Y)$.
T^X being natural means that the diagram commutes for all $f\colon X \longrightarrow Y$:

$$\begin{tikzcd} C_n(X) \arrow[r, "T^X"] \arrow[d, "f_\sharp"'] & C_{n+1}(X \times I) \arrow[d, "(f \times \mathrm{id})_\sharp"] \\ C_n(Y) \arrow[r, "T^Y"'] & C_{n+1}(Y \times I) \end{tikzcd}$$

So $T_Y \circ f_\sharp = (f \times \mathrm{id})_\sharp \circ T_X$.
Let $I_n\colon \Delta^n \longrightarrow \Delta^n$ be the identity n-dimensional simplex. If we determine $T^{\Delta^n}(I_n)$, then we have determined $T^X(\sigma)$ for all $\sigma \in C_n(X)$ for all X. This is because $\sigma = \sigma \circ I_n = \sigma_\sharp(I_n)$, since we can view σ as a continuous map $X \longrightarrow \Delta^n$ and so $\sigma_\sharp$ is defined. Thus

$$T^X(\sigma) = T^X \circ \sigma_\sharp(I_n) = (\sigma \times \mathrm{id})_\sharp \circ T^{\Delta^n}(I_n)$$

And so determining $T^{\Delta^n}(I_n)$ determines $T^X(\sigma)$. So if we define $A = T^{\Delta^n}(I_n)$, then

$$T^X(\sigma) = (\sigma \times \mathrm{id})_\sharp(A)$$

A is some simplex in $C_{n+1}(\Delta^n \times I)$, and we claim that for any choice of A, this defines a natural transformation. This is because

$$T^Y \circ f_\sharp(\sigma) = T^Y(f \circ \sigma) = \big((f \circ \sigma) \times \mathrm{id}\big)_\sharp(A) = (f \times \mathrm{id})_\sharp \circ (\sigma \times \mathrm{id})_\sharp(A)$$

And

$$(f \times \mathrm{id})_\sharp \circ T^X(\sigma) = (f \times \mathrm{id})_\sharp \circ (\sigma \times \mathrm{id})_\sharp(A)$$

so these are indeed equal, as required.
Now we claim that

$$(\partial T^X + T^X\partial)(\sigma) = (i_\sharp^X - j_\sharp^X)(\sigma)$$

for all X, σ. It is sufficient to show this for $X = \Delta^n$ and $\sigma = I_n$, since if

$$(\partial T^{\Delta^n} + T^{\Delta^n}\partial)(I_n) = (i_\sharp^{\Delta^n} - j_\sharp^{\Delta^n})(I_n)$$

if we compose it on the left with $(\sigma \times \mathrm{id})_\sharp$, the LHS gives

$$(\partial(\sigma \times \mathrm{id})_\sharp T^{\Delta^n} + (\sigma \times \mathrm{id})_\sharp T^{\Delta^n}\partial)(I_n) = (\partial T^X\sigma_\sharp + T^X\partial\sigma_\sharp)(I_n) = \partial T^X\sigma + T^X\partial\sigma$$

since T is natural, $T^Y \circ f_\sharp = (f \times \mathrm{id})_\sharp \circ T^X$ and $\partial f_\sharp = f_\sharp \partial$. The RHS is

$$\big((\sigma \times \mathrm{id})_\sharp \circ i_\sharp^{\Delta^n} - (\sigma \circ \mathrm{id}) \circ j_\sharp^{\Delta^n}\big)(I_n)$$

Now notice that

$$\begin{array}{ccccc} \Delta^n & \xrightarrow{i^{\Delta^n}} & \Delta^n \times I & \xrightarrow{\sigma \times \mathrm{id}} & X \times I \\ s & \longmapsto & (s, 0) & \longmapsto & (\sigma(s), 0) \end{array}$$

So $(\sigma \times \mathrm{id}) \circ i^{\Delta^n} = i^X \circ \sigma$, and similar for j. So the RHS is just

$$i_\sharp^X \circ \sigma_\sharp(I_n) - j_\sharp^X \circ \sigma_\sharp(I_n) = i_\sharp^X(\sigma) - j_\sharp^X(\sigma)$$

So we get

$$\partial T^X(\sigma) + T^X \partial\sigma = i_\sharp^X(\sigma) - j_\sharp^X(\sigma)$$

as required.

So we must show that

$$\partial T I_n + T\partial I_n = i_\sharp I_n - j_\sharp I_n$$

in order to get this for every $\sigma \in C_n(\Delta^n)$. So we must show $\partial T I_n = -T\partial I_n + i_\sharp - j_\sharp I_n$, since $\partial T I_n \in C_n(\Delta^n \times I)$, and $\Delta^n \times I$ is a convex set in $\mathbb{R}^{n+2}$. In a convex set so a simplex is a boundary if and only if it is a cycle. We want $-T\partial I_n + i_\sharp I_n - j_\sharp I_n$ to be a boundary, and so it is sufficient to check that it is a cycle:

$$-\partial T\partial I_n + \partial i_\sharp I_n - \partial j_\sharp I_n$$

Since $\partial I_n \in C_{n-1}(\Delta^n)$, we have that

$$\partial T\partial I_n + T\partial\partial I_n = i_\sharp \partial I_n - j_\sharp \partial I_n$$

and thus we must have that the following is zero:

$$T\partial\partial I_n - i_\sharp \partial I_n + j_\sharp \partial I_n + \partial i_\sharp I_n - \partial j_\sharp I_n$$

Since $\partial\partial = 0$, and $i_\sharp, j_\sharp$ are chain homomorphisms, this is indeed zero. So $-T\partial I_n + i_\sharp I_n - j_\sharp I_n$ is a cycle and thus a boundary since the universe is convex. So let us take A to be a chain such that ∂A is this element. ■

So notice now that if $f \sim g$, then $f_\sharp \sim g_\sharp$ are chain homotopic, and so $f_* = g_*$.

1.16 Corollary

If $f\colon X \longrightarrow Y$ is a homotopy equivalence, then $f_*\colon H_n(X) \longrightarrow H_n(Y)$ is an isomorphism.

Proof: there exists a $g\colon Y \longrightarrow X$ such that $fg \sim \mathrm{id}_Y$ and $gf \sim \mathrm{id}_X$. Thus

$$g_* \circ f_* = (g \circ f)_* = (\mathrm{id}_X)_* = \mathrm{id}_{H_n(X)}$$

and similarly $f_* \circ g_* = \mathrm{id}_{H_n(Y)}$, so f_* is an isomorphism. ■

1.4 Affine Spaces

1.1 Definition

Let $p_1, \ldots, p_n$ be vectors in a vector space, then their **affine hull** is

$$\mathrm{CH}(p_1, \ldots, p_n) = \left\{ \sum_{i=1}^n \alpha_i p_i \;\middle|\; \sum_{i=1}^n \alpha_i = 1 \right\}$$

Elements of the affine hull are called **affine combinations**. We similarly define the **convex hull**:

$$\mathrm{CH}(p_1,\dots,p_n)=\left\{\sum_{i=1}^n \alpha_i p_i \,\middle|\, \sum_{i=1}^n \alpha_i=1,\ \alpha_i\geq 0\right\}$$

And its elements are called **convex combinations**.

1.2 Definition

$p_1,\dots,p_n$ are **affine independent** if $\sum_{i=1}^n \alpha_i v_i=0$ and $\sum_{i=1}^n \alpha_i=0$ implies every α_i is 0.

1.3 Definition

$A\subseteq\mathbb{R}^k$ is an n**-simplex** if it is the convex hull of a set of $n+1$ affine independent set of vectors.

1.4 Definition

Let $\Sigma=\mathrm{CH}(p_0,\dots,p_n)$ be an n-simplex, then its ith **face** is $\mathrm{CH}(p_0,\dots,p_{i-1},p_i,\dots,p_n)$. And its **barycenter** is

$$b=\frac{1}{n+1}\sum_{i=0}^n p_i$$

We define the **barycentric subdivision** of Σ, denoted $\operatorname{Sd}\Sigma$, to be a set of n-simplices which we define inductively on n as follows:

(1) For a 0-simplex, $\operatorname{Sd}\Sigma=\Sigma$.

(2) If Σ is an n-simplex, then let $\varphi_0,\dots,\varphi_n$ be its faces (which are $n-1$-simplices) and b its barycenter. Then define $\operatorname{Sd}\Sigma$ to be the n-simplices spanned by b and the simplices in the barycentric subdivisions of φ_i. I.e.

$$\operatorname{Sd}\Sigma=\left\{\mathrm{CH}\big(b,\Sigma^{n-1}\big)\,\middle|\,\Sigma^{n-1}\in\operatorname{Sd}\varphi_i, 0\leq i\leq n\right\}$$

Inductively, $\Sigma=\bigcup\operatorname{Sd}\Sigma$ and $\#\operatorname{Sd}\Sigma=(n+1)!$.

1.5 Theorem

For every n, there exists a constant $c<1$ such that for every n-simplex Σ then for every $\Sigma'\in\operatorname{Sd}\Sigma$:

$$\operatorname{diam}(\Sigma')\leq c\operatorname{diam}(\Sigma)$$

1.6 Definition

We define $\operatorname{Sd}_n\colon C_n(\Delta^n)\longrightarrow C_n(\Delta^n)$ by induction on n. Let $\sigma\colon\Delta^n\longrightarrow\Delta^n$ be a generator, then

(1) $\operatorname{Sd}_0(\sigma)=\sigma$

(2) $\operatorname{Sd}_n(\sigma)=C_{\sigma(b)}(\operatorname{Sd}_{n-1}(\partial\sigma))$ where b is the barycenter of Δ^n.

Let X be a topological space, then let $\operatorname{Sd}\colon C_n(X)\longrightarrow C_n(X)$ be defined on generators $\sigma\colon\Delta^n\longrightarrow X$ by $\operatorname{Sd}\sigma=\sigma_\sharp\operatorname{Sd}_n\mathrm{id}_{\Delta^n}$.

1.7 Theorem

Sd is a chain map ($\mathrm{Sd} = \{\mathrm{Sd}_n\}_{n=0}^{\infty}$) and is natural (between the chain functor **Top** $\to$ **Comp** and itself).

Sd being natural means the following diagram commutes

$$\begin{tikzcd} C_n(X) \arrow[r, "\mathrm{Sd}_n"] \arrow[d, "f_\sharp"'] & C_n(X) \arrow[d, "f_\sharp"] \\ C_n(Y) \arrow[r, "\mathrm{Sd}_n"'] & C_n(Y) \end{tikzcd}$$

1.8 Definition

Let X be a topological space, and $\mathcal{U} = \{\mathcal{U}_\alpha\}_{\alpha \in I}$ a collection of subsets of X such that $\bigcup \mathring{\mathcal{U}}_\alpha = X$ (where $\mathring{\mathcal{U}}$ is the interior of $\mathcal{U}$). Such a collection will be called a **good cover** of X.

We will say that $\sigma\colon \Delta^n \longrightarrow X$ **preserves** the cover if there exists an $\alpha \in I$ such that $\sigma(\Delta^n) \subseteq \mathcal{U}_\alpha$. And we will say that $\sum_i n_i \sigma_i \in C_n(X)$ **preserves** the cover if each σ_i preserves the cover.

Let us define

$$C_n^{\mathcal{U}}(X) = \{\sigma \in C_n(X) \mid \sigma \text{ preserves } \mathcal{U}\}$$

$C_n^{\mathcal{U}}(X)$ is a