Infinitesimal Calculus 3

Lecture 2, Wednesday October 26, 2022
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Notice that if (X, ρ) is a metric space, if $Y \subseteq X$, then by restricting ρ onto $Y \times Y$, we define another metric space, (Y, ρ') where ρ' is the restriction of ρ. This new metric space is called a *metric subspace* of X.

Definition 2.1.1:

Suppose (X, ρ) is a metric space and $r > 0$ is a positive real number. If $x \in X$, then we define $B_r(x)$ to be the **open ball** centered at x with radius r:

$$B_r(x) \coloneqq \{y \in X \mid \rho(x, y) < r\}$$

And the **closed ball** is defined similarly:

$$\bar{B}_r(x) \coloneqq \{y \in X \mid \rho(x, y) \leq r\}$$

These balls are called the *basic neighborhoods*.

Definition 2.1.2:

If X is a metric space, $\mathcal{O} \subseteq X$ is **open** if for every $x \in S$, there is a $r > 0$ such that $B_r(x) \subseteq \mathcal{O}$. A set $F \subseteq X$ is **closed** if F^c is open.

Example:

Every open ball $B_r(x)$ is indeed open. This is because if $y \in B_r(x)$ then if we let $s = r - \rho(x, y)$ then $B_s(y) \subseteq B_r(x)$, since if:

$$\rho(z, y) < s \implies \rho(z, y) < r - \rho(x, y) \implies \rho(z, y) + \rho(x, y) < r$$

By the triangle inequality, this means $\rho(x, z) < r$, so $z \in B_r(x)$, as required.

Example:

The closed ball $\bar{B}_r(x)$ is indeed closed. To prove this, we need to show that $\left(\bar{B}_r(x)\right)^c$ is open. Suppose that $y \notin \bar{B}_r(x)$. That means that $\rho(y, x) > r$, so take $\varepsilon > 0$ such that $r < r + \varepsilon < \rho(x, y)$. Then for every $z \in B_\varepsilon y$, $\rho(y, z) < \varepsilon$, so $\rho(z, x) \geq \rho(y, x) - \rho(z, y) > r + \varepsilon - \varepsilon = r$. So $z \notin \bar{B}_r(x)$, and therefore $B_\varepsilon(y) \subseteq \left(\bar{B}_r(x)\right)^c$ as required.

Example:

X and $\varnothing$ are both open and closed. $\varnothing$ is open vaccuously. If $x \in X$, then for any $r > 0$, $B_r(x) \subseteq X$ so X is open. Since $\varnothing^c = X$, X and $\varnothing$ are also closed.

Example:

If $X = \mathbb{R}^+ \cup \mathbb{R}^-$, then $\mathbb{R}^+$ is open since if $x \in \mathbb{R}^+$ since $B_x(x) \in \mathbb{R}^+$. Similarly so is $\mathbb{R}^-$. So both $\mathbb{R}^+$ and $\mathbb{R}^-$ are closed and open in X.

Such sets which are both open and closed are sometimes called *clopen* sets.

Example:

If $X = \mathbb{R}$ let $S = [0, 1)$. Then S is neither open nor closed. S is not open since no ball around 0 is contained entirely in S. And since $S^c = (-\infty, 0) \cup [1, \infty)$ so no ball around 1 is contained entirely in S^c, so S^c is not open, and therefore S is not closed. So S is neither closed nor open.

Definition 2.1.3:

Suppose X is a metric space and $S \subseteq X$.

- $x \in S$ is an **interior point** of S if there is an $r > 0$ such that $B_r(x) \subseteq S$.
- $x \in X$ is an **exterior point** of S if there is an $r > 0$ such that $B_r(x) \subseteq S^c$.
- $x \in X$ is a **boundary point** of S if every open ball containing x intersects with both S and S^c.
- $x \in X$ is a **isolated point** of S if there is an open ball containing x which does not contain any other point of S. That is, there is an $r > 0$ such that $B_r(x) \cap S = \{x\}$.
- $x \in X$ is a **limit point** of S if every open ball containing x contains another element of S. That is, for all $r > 0$ $\exists x \neq s \in B_r(x) \cap S$.

Proposition 2.1.4:

If X is a metric space and $S \subseteq X$, then the following are equivalent:

- S is open.
- Every $x \in S$ is an interior point.
- S does not contain any of its boundary points.

Proof:

The equivalence of the first two points is a direct consequence of the definition of open sets and interior points. Now, suppose S is open and x is a boundary point. Then for every $r > 0$, $B_r(x) \cap S^c \neq \varnothing$, so $B_r(x)$ is not a subset of S, so x is not in S. Therefore if S is open, it does not contain any of its boundary points.

Now suppose that S doesn't contain any of its boundary points. So if $x \in S$, there is an $r > 0$ such that $B_r(x)$ doesn't intersect both S and S^c. Since $x \in B_r(x)$, it must intersect S, so $B_r(x)$ cannot intersect S^c. Therefore $B_r(x) \subseteq S$. So for every $x \in S$, there is a $r > 0$ such that $B_r(x) \subseteq S$, and therefore S is open.

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