Complex Functions

Assignment 3

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Exercise 3.1:

Compute the integral $\int_C f$ where $f(z) = x^2 + iy^2$ and $C\colon z(t) = t^2 + it^2$ for $0 \leq t \leq 1$.

We do this by definition, the integral is equal to

$$\int_0^1 f(z(t))z'(t) = \int_0^1 (t^4 + it^4)2t(1+i) = 2(1+i)^2 \int_0^1 t^5 = 4i \cdot \frac{t^6}{6}\bigg|_0^1 = \frac{2i}{3}$$

Exercise 3.2:

Compute the integral $\int_C f$ where $f(z) = \frac{1}{z}$ and $C\colon z(t) = \sin t + i\cos t$ for $0 \leq t \leq 2\pi$.

We again do this by definition, noting that $z(t) = \cos\left(\frac{\pi}{2} - t\right) + i\sin\left(\frac{\pi}{2} - t\right) = e^{i\left(\frac{\pi}{2} - t\right)}$:

$$\int_0^{2\pi} e^{-i\left(\frac{\pi}{2}-t\right)} \cdot (-1) \cdot e^{i\left(\frac{\pi}{2}-t\right)} = \int_0^{2\pi} (-1) = -2\pi i$$

Exercise 3.3:

Prove that if F' is identically 0 on a domain D then F is constant on D.

Let $a, b \in D$, then since D is a connected domain there exists a smooth curve connecting them, so we can integrate F' from a to b Then

$$\int_a^b F' = F(b) - F(a)$$

But since $F' = 0$, we have

$$\int_a^b F' = \int_a^b 0 = 0$$

So $F(a) = F(b)$ for every two points in D, as required.

Exercise 3.4:

Show that if f is a continuous real function where $|f| \leq 1$ on all of $\mathbb{C}$ then

$$\left|\int_{|z|=1} f\right| \leq 4$$

Let

$$I = \int_{|z|=1} f$$

If $I = 0$, we have finished. Otherwise, let

$$z_0 = \frac{\overline{I}}{|I|}$$

Then $|z_0| = 1$, so $z_0 = e^{i\theta}$ and $z_0 I = |I|$. So we have that

$$\left| \int_{|z|=1} f \right| = e^{i\theta} \int_{|z|=1} f = \int_0^{2\pi} f\left(e^{it}\right) \cdot i e^{i(t+\theta)}\, dt$$

Since the left hand side is real, so must the right hand side be. And since f is real, the real part of the right hand side is

$$= \int_0^{2\pi} -f\left(e^{it}\right) \sin(t+\theta)\, dt$$

And this is less than

$$\leq \int_0^{2\pi} \left| -f\left(e^{it}\right) \sin(t+\theta) \right| dt \leq \int_0^{2\pi} |\sin(t+\theta)|\, dt = \int_0^{2\pi} |\sin t|\, dt$$

since sin has a period of 2π. And this is equal to

$$\int_0^{\pi} \sin t\, dt - \int_{\pi}^{2\pi} \sin t\, dt = -\cos t \Big|_0^{\pi} + \cos t \Big|_{\pi}^{2\pi} = 4$$

So all in all we have

$$\left| \int_{|z|=1} f \right| \leq 4$$

as required.

Exercise 3.5:

Show that $\int_C z^k = 0$ for every $-1 \neq k \in \mathbb{Z}$ where $C = Re^{i\theta}$ for $0 \leq \theta \leq 2\pi$ and constant $R > 0$, in two ways:

(1) Representing z^k as the derivative of an analytic function.

(2) Directly.

(1) Let $f(z) = \frac{z^{k+1}}{k+1}$, then $f'(z) = z^k$ (since $k+1 \neq 0$ this is well-defined). So

$$\int_C z^k = \int_C f' = f(C(2\pi)) - f(C(0)) = f(R) - f(R) = 0$$

(2) Directly we have

$$\int_C z^k = \int_0^{2\pi} R^k e^{ik\theta} Rie^{i\theta} = R^{k+1} \int_0^{2\pi} e^{(k+1)i\theta} = \frac{R^{k+1}}{k+1} e^{(k+1)i\theta} \Big|_0^{2\pi} = 0$$

Since $e^0 = e^{(k+1)2\pi} = 1$.

Exercise 3.6:

Compute $\int_C z - i$ where $C\colon z(t) = t + it^2$ for $-1 \leq t \leq 1$, by

(1) Finding an antiderivative.

(2) By computing the integral on the line from $-1+i$ to $1+i$ and using Cauchy's theorem.

(1) We can find the antiderivative of $z-i$, which is $\frac{z^2}{2}-iz$. The curve C is from $z(-1)=-1+i$ to $z(1)=1+i$, and so

$$\int_C z-i=\frac{z^2}{2}-iz\Big|_{-1+i}^{1+i}=i-i(1+i)-(-i-i(-1+i))=1-1=0$$

(2) By Cauchy's theorem we know that the integral from $-1+i$ to $1+i$ is equal no matter which curve you choose since $z-i$ is analytic. Then we can take the line $[-1+i,1+i]$, which is parameterized by $z(t)=-1+i+2t$ for $0\leq t\leq 1$. This gives

$$\int_0^1(-1+i+2t-i)2\,dt=\int_0^1 4t-2=2t^2-2t\Big|_0^1=2-2=0$$

Exercise 3.7:

Compute the following integrals:

(1) $\int_0^i e^z$

(2) $\int_{\frac{\pi}{2}}^{\frac{\pi}{2}+i}\cos(2z)$

(1) Since the antiderivative of e^z is e^z, this is equal to $e^i-e^0=\cos(1)-1+i\sin(1)$.

(2) Since

$$\cos(2z)=\frac{e^{2z}+e^{-2z}}{2}$$

So its antiderivative is

$$e^{2z}-e^{-2z}$$

And so the integral is equal to

$$e^{\pi+2i}-e^{-\pi-2i}-e^{\pi}+e^{-\pi}$$

Exercise 3.8:

Suppose f is analytic in a convex domain D such that $|f'|\leq 1$. Prove that $|f(b)-f(a)|\leq|b-a|$ for every $a,b\in D$.

Let C be a curve from a to b, this can be the line $t\mapsto a+t(b-a)$. Then we know that since f is analytic

$$|f(b)-f(a)|=\left|\int_a^b f'\,dz\right|\leq\max|f'(z)|\cdot|b-a|\leq|b-a|$$

as required.

Exercise 3.9:

Let $a,b\in\{z\in\mathbb{C}\mid\operatorname{Re}z<0\}$, prove that $\left|e^b-e^a\right|<|b-a|$. Is this true for $a,b\in\{z\in\mathbb{C}\mid\operatorname{Re}z\leq 0\}$.

We know that

$$\left|e^b-e^a\right|=\left|\int_a^b e^z\right|\leq\max|e^z|\cdot|b-a|$$

from the proposition proven in lecture. We know that $|e^z|=e^{\operatorname{Re}z}$, and so if we take the curve as the line connecting

a to b, then $|e^z| \le \max\{|e^a|, |e^b|\}$, depending on whose real value is larger. This is since for every $z \in [a,b]$ (the line connecting the points), $\operatorname{Re} z$ is between $\operatorname{Re} a$ and $\operatorname{Re} b$. So

$$\left|e^b - e^a\right| \le \max\{e^{\operatorname{Re} a}, e^{\operatorname{Re} b}\} \cdot |b-a|$$

since $e^{\operatorname{Re} a}, e^{\operatorname{Re} b} < e^0 = 1$ and this inequality is strict, we have that

$$\left|e^b - e^a\right| < |b-a|$$

as required.
We know that for $\operatorname{Re} z \le 0$, $|e^z| \le 1$, so we get that

$$\left|e^b - e^a\right| \le |b-a|$$

from the first inequality above. If there exists a and b where this inequality is an equality, let ℓ be the line connecting a and b so we have that

$$\left|e^b - e^a\right| = \left|\int_\ell e^z \, dz\right| = \left|\int_0^1 e^{\ell(t)}\ell'(t)\, dz\right| \le \int_0^1 \left|e^{\ell(t)}\right| \cdot |\ell'(t)|\, dz \le \int_0^1 |\ell'(t)|\, dz = |b-a|$$

where the last inequality is because $\left|e^{\ell(t)}\right| \le 1$ since $\operatorname{Re}(\ell(t)) \le 0$. So if this equality holds, we must have that

$$\int_0^1 \left|e^{\ell(t)} \cdot \ell'(t)\right| dz = \int_0^1 |\ell'(t)|\, dz$$

so $\left|e^{\ell(t)}\right| \cdot |\ell'(t)| = |\ell'(t)|$ almost everywhere, and since these are continuous functions this is equality everywhere. If $a \ne b$ then $\ell'(t) \ne 0$ anywhere (it is constant as a line), and so $\left|e^{\ell(t)}\right| = 1$ for every $t \in [0,1]$. This means that $e^{\operatorname{Re}(\ell(t))} = 1$ so $\operatorname{Re}(\ell(t)) = 0$, and so a and b are both imaginary numbers.
So we need to solve for when

$$\begin{aligned}
\left|e^{ai} - e^{bi}\right| = |a-b| &\iff (\cos a - \cos b)^2 + (\sin a - \sin b)^2 = (a-b)^2 \\
&\iff \cos^2 a - 2\cos a \cos b + \cos^2 b + \sin^2 a - 2 \sin a \sin b + \sin^2 b = (a-b)^2 \\
&\iff 2\big(1 - \cos a \cos b - \sin a \sin b\big) = (a-b)^2 \\
&\iff 2\big(1 - \cos(a-b)\big) = (a-b)^2
\end{aligned}$$

Let $t = a - b$, so we must find a solution to $f(t) = 0$ where $f(t) = 2(1-\cos t) - t^2$ Our goal is to show that this inequality does hold, and this means that we have equality if and only if $a = b$, and so $f(t) = 0$ if and only if $t = 0$. For $t = 0$ it is the case that $f(0) = 0$ (which would have to be the case). Now let us compute its derivatives:

$$f'(t) = 2\sin t - 2t, \qquad f''(t) = 2\cos t - 2$$

Notice that $f''(t) \le 0$ so f' is decreasing, and it is never constant since f'' is only zero at points (not intervals), so f' is injective. Thus since $f'(0) = 0$, $t = 0$ is the only critical point of f. And this is a maximum since f' is decreasing and $f'(0) = 0$ so f is increasing for $t \le 0$ (since $f'(t) \ge 0$ then) and decreasing afterward.
So $f(t) \ge 0$ with equality only when $t = 0$ (at the maximum). So we have equality only when $a = b$.
So the inequality does hold.