Infinitesimal Calculus 3

Lecture 18, Sunday January 1, 2023

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Definition 18.1:

A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is

(1) **Positive** if $k^tAk > 0$ for all $0 \neq k \in \mathbb{R}^n$.

(2) **Negative** if $k^tAk < 0$.

(3) **Nonnegative** if $k^tAk \geq 0$.

(4) **Nonpositive** if $k^tAk \leq 0$.

(5) **Nonsigned** if there are vectors k_1 and k_2 such that $k_1^tAk_1 > 0$ and $k_2^tAk_2 < 0$.

Proposition 18.2:

A symmetric matrix A is positive if and only if for every $1 \leq k \leq n$, M_k is positive where M_k is defined as

$$M_k = \det\begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix}$$

This proof is pretty lengthy, so we will not prove it.

Proposition 18.3:

A symmetric matrix A is negative if and only if $-A$ is positive.

The proof of this is trivial. But this means that A is negative if and only if $(-1)^kM_k > 0$ for all k.

Proposition 18.4:

Suppose $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ is defined and in C^2 in a neighborhood of $x^0 \in \mathbb{R}^n$. Further suppose x^0 is a critical point of f. Let $H(x)$ be the Hessian matrix of f, if

(1) If $H(x^0)$ is positive, then x^0 is a local minimum.

(2) If $H(x^0)$ is negative, then x^0 is a local maximum.

(3) If $H(x^0)$ is nonsigned, then x^0 is not a local maximum nor minimum.

(4) Otherwise, it is unknown.

Proof:

By Taylor's expansion we have that:

$$f(x^0+k) = f(x^0) + \nabla f\big|_{x_0}\cdot k + k^tH(x^0+\theta k)k = f(x^0) + k^tH(x^0+\theta k)k$$

Since f is in C^2, its second order derivatives are continuous, if $k^tH(x^0)k > 0$ then it is positive in a neighborhood of x^0, and so $f(x^0+k) > f(x^0)$ in this neighborhood, and so x^0 is a local minimum. And similarly if H is negative. If $H(x^0)$ is nonsigned, we can take k such that $k^tHk > 0$ since we can scale k we can assume it has any norm, that is we can find such a k in any neighborhood of x^0. So for any neighborhood of x^0 we can find a k such that $f(x^0+k) > f(x^0)$ and similarly we can find a k where $f(x^0+k) < f(x^0)$ so x^0 is not a minimum nor a maximum.

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Definition 18.5:

Suppose (X, ρ) is a metric space and $T\colon X \longrightarrow X$ is a **contraction mapping** if there exists a $0 < k < 1$ such that for every x_1, x_2:

$$\rho(T(x_1), T(x_2)) \leq k\,\rho(x_1, x_2)$$

Theorem 18.6:

If T is a contraction mapping over a complete metric space then T has a unique fixed point.

Proof:

Let $x_0 \in X$, then we define $x_{i+1} = T(x_i)$, that is $x_n = T^n(x_0)$. Then notice that

$$\rho(x_i, x_{i+1}) = \rho(T(x_{i-1}), T(x_i)) \leq k\,\rho(x_{i-1}, x_i)$$

And so on we have that

$$\rho(x_i, x_{i+1}) \leq k^i \cdot \rho(x_0, T(x_0)) = k^i \cdot c$$

Notice then that if $n < m$:

$$\rho(x_n, x_m) \leq \sum_{j=n}^{m-1} \rho(x_j, x_{j+1}) \leq c \sum_{j=n}^{\infty} k^j$$

since $0 < k < 1$, the infinite series converges and thus the sum on the right converges to 0 as the tail of a convergent series. That is, for any $\varepsilon > 0$ there is an N such that

$$c \sum_{j=n}^{\infty} k^j < \varepsilon$$

and threfore for any $N \leq n, m$, $\rho(x_n, x_m) < \varepsilon$, so $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence, which converges to $x \in X$ since X is complete.
Since T is a contraction it is continuous, so:

$$0 = \lim \rho(T(x), T(x_n)) = \lim \rho(T(x), x_n)$$

And:

$$\rho(T(x), x) \leq \rho(T(x), x_n) + \rho(x_n, x)$$

which converges to 0 so $\rho(T(x), x) = 0$ so $T(x) = x$ as required.
Now suppose $T(x_1) = x_1$ and $T(x_2) = x_2$ then

$$\rho(x_1, x_2) = \rho(T(x_1), T(x_2)) \leq k\,\rho(x_1, x_2)$$

which means that $k \geq 1$ which is a contradiction, or $\rho(x_1, x_2) = 0$, ie $x_1 = x_2$. So the fixed point is unique.

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Lemma 18.7:

Suppose $S \subseteq \mathbb{R}^n$ is a neighborhood of $x_0 \in \mathbb{R}^n$, and $f\colon S \longrightarrow \mathbb{R}^n$. Suppose f's components are in C^1 and $df\big|_{x_0} = 0$ then for every $\varepsilon > 0$ there is a $r > 0$ such that for every $x_1, x_2 \in B_r(x_0)$, $\|f(x_1) - f(x_2)\| < \varepsilon\|x_1 - x_2\|$.

Proof:

Suppose $f(x) = (f_1(x), \ldots, f_n(x))$ where $x \in S$. Then for some $0 \leq \theta \leq 1$, $f_k(x) = f_k(x_0) + \nabla f_k(x_0 + \theta h)$. Since $\nabla f_k(0)$ and $f_k \in C^1$, so there exists a radius $r > 0$ such that forall $x \in B_r(x_0)$:

$$\|\nabla f_k(x)\| < \frac{\varepsilon}{\sqrt{n}}$$

And by Cauchy-Schwarz:

$$\|f_k(x) - f_k(x_0)\| = \|\nabla f_k \cdot (\theta(x - x_0))\| \leq \|\nabla f_k\| \|x - x_0\| \leq \frac{\varepsilon}{\sqrt{n}} \cdot \|x - x_0\|$$

And so:

$$\|f(x) - f(x_0)\|^2 = \sum_{k=1}^{n} \|f_k(x) - f_k(x_0)\|^2 \leq \sum_{k=1}^{n} \frac{\varepsilon^2}{n} \|x - x_0\|^2 = \varepsilon^2 \|x - x_0\|^2$$

so $\|f(x) - f(x_0)\|^2 \leq \varepsilon \|x - x_0\|$ as required. ■