Calculus Homework #3

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Question 3.1:

Compute the following definite integrals:

(1) $\int_0^2 \left(x^2 + x - 1\right) dx$

(2) $\int_0^1 x^3 dx$

(3) $\int_1^2 \frac{1}{x^2} dx$

Answer:

Firstly, notice that each of these functions are continuous over the the interval given, so they have a definite integral. Therefore we only need to compute the result for a specific series of partitions which approaches 0.

(1) Let P_n be given by:

$$x_i = \frac{2i}{n}$$

For $0 \leq i \leq n$. This obviously gives a partition as $x_0 = 0$, $x_n = 2$, and $x_i < x_{i+1}$. And we know:

$$\Delta_i = x_i - x_{i-1} = \frac{2i}{n} - \frac{2i-2}{n} = \frac{2}{n}$$

So $\lambda\left(P_n\right) = \sup \Delta_i = \frac{2}{n}$, whose limit is is 0, so this series of partitions satisfies everything we need.
Furthermore, let $d_i = x_i = \frac{2i}{n}$.
So the Riemman Sum of P_n is:

$$\sum_{i=1}^{n} \Delta_i \cdot f\left(d_i\right) = \sum_{i=1}^{n} \frac{2}{n} \cdot \left(\frac{4i^2}{n^2} + \frac{2i}{n} - 1\right) = \frac{8}{n^3} \sum_{i=1}^{n} i^2 + \frac{4}{n^2} \sum_{i=1}^{n} i - 2$$

Which, by the sums of the squares and algebraic series, is equal to:

$$= \frac{8}{n^3} \cdot \frac{n\left(n+1\right)\left(2n+1\right)}{6} + \frac{2\left(n-1\right)}{n} - 2 = \frac{8}{6} \cdot \frac{n\left(n+1\right)\left(2n+1\right)}{n^2} + 4 \cdot \frac{n-1}{n} - 2$$

Whose limit, as n approaches infinity, is

$$\frac{16}{6} + 2 - 2 = \boxed{2\frac{2}{3}}$$

(2) First, let's prove a simple lemma:

Statement 3.1.1:

$$\sum_{k=1}^{n} k^3 = \frac{n^2\left(n+1\right)^2}{4}$$

Proof:

We will prove this by induction on n:

Base case: $n = 1$

In this case, we need to prove:

$$1 = \frac{1 \cdot 2^2}{4} = \frac{4}{4} = 1$$

Which is true.

Inductive step:

Assume this is true for n, we need to prove this true for $n+1$:

$$\sum_{k=1}^{n+1} k^3 = \sum_{k=1}^{n} k^3 + (n+1)^3$$

By our inductive assumption, this is equal to:

$$= \frac{n^2 \cdot (n+1)^2}{4} + (n+1)^3 = \frac{(n+1)^2 \cdot \left(n^2 + 4(n+1)\right)}{4} = \frac{(n+1)^2 \cdot (n+2)^2}{4}$$

As required.

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So let P_n be defined as:

$$x_i = \frac{i}{n}$$

For $0 \leq i \leq n$. This obviously gives a partition for the same reason as the previous subquestion. And we know:

$$\Delta_i = \frac{i}{n} - \frac{i-1}{n} = \frac{1}{n}$$

And we define

$$d_i \coloneqq x_i = \frac{i}{n}$$

So:

$$\sigma(P_n) = \sum_{i=1}^{n} \Delta_i \cdot f(d_i) = \sum_{i=1}^{n} \frac{1}{n} \cdot \frac{i^3}{n^3} = \frac{1}{n^4} \cdot \sum_{i=1}^{n} i^3 = \frac{n^2 \cdot (n+1)^2}{4 \cdot n^4}$$

Whose limit is $\boxed{\frac{1}{4}}$.

(3) We want P_n's intervals to form a geometric series. So:

$$x_i = x_0 \cdot q^i$$

Since $x_0 = 1$, this means:

$$x_i = q^i$$

And we require:

$$2 = x_n = q^n \implies q = 2^{\frac{1}{n}}$$

So we define:

$$x_i \coloneqq 2^{\frac{i}{n}}$$

Which means that:

$$\Delta_i = x_i - x_{i-1} = 2^{\frac{i}{n}} - 2^{\frac{i-1}{n}} = 2^{\frac{i-1}{n}} \cdot \left(2^{\frac{1}{n}} - 1\right)$$

And let $d_i \coloneqq x_i = 2^{\frac{i}{n}}$. So:

$$\sigma(P_n) = \sum_{i=1}^{n} \frac{2^{\frac{i-1}{n}} \cdot \left(2^{\frac{1}{n}} - 1\right)}{2^{\frac{2i}{n}}} = \frac{2^{\frac{1}{n}} - 1}{2^{\frac{1}{n}}} \cdot \sum_{i=1}^{n} \frac{2^{\frac{i}{n}}}{4^{\frac{i}{n}}} = \frac{2^{\frac{1}{n}} - 1}{2^{\frac{1}{n}}} \cdot \sum_{i=1}^{n} \left(\frac{1}{2}^{\frac{1}{n}}\right)^i$$

This is a geometric sum, which we can compute:

$$= \frac{2^{\frac{1}{n}} - 1}{2^{\frac{1}{n}}} \cdot \frac{\frac{1}{2} \cdot \frac{1}{2^{\frac{1}{n}}}}{1 - \frac{1}{2^{\frac{1}{n}}}} = \frac{1}{2} \cdot \frac{1 - \frac{1}{2^{\frac{1}{n}}}}{2^{\frac{1}{n}} - 1} = \frac{1}{2} \cdot \frac{2^{\frac{1}{n}} - 1}{2^{\frac{2}{n}} - 2^{\frac{1}{n}}} = \frac{1}{2} \cdot \frac{1}{2^{\frac{1}{n}}}$$

Whose limit is $\boxed{\frac{1}{2}}$.

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Question 3.2:

Let C be a constant and $f(x)$ an integrable function over $[a,b]$, such that for every rational x in $[a,b]$, $f(x) = C$. Prove that:

$$\int_a^b f(x)\,dx = C(b-a)$$

Answer:

Since f is integrable, we only need to show that for a specific series of pointed partitions P_n such that $P_n \longrightarrow 0$, $\lim \sigma(P_n) = C(b-a)$.
Let $\{P_n\}$ be a set of arbitrary partitions:

$$P_n\colon a = x_0 < \cdots < x_n = b$$

For every $1 \le i \le n$, we define $d_i \in [x_{i-1}, x_i]$ to be a rational number in $[x_{i-1}, x_i]$. There exists such a rational number because the rationals are dense in $\mathbb{R}$. Because $d_i \in \mathbb{Q}$, we know that $f(d_i) = C$.
So for every P_n:

$$\sigma(P_n) = \sum_{i=1}^{n} \Delta_i \cdot f(d_i) = \sum_{i=1}^{n} \Delta_i \cdot C = C \cdot \sum_{i=1}^{n} \Delta_i$$

And since P_n is a partition, we know that $\sum_{i=1}^{n} \Delta_i = b - a$. So:

$$\sigma(P_n) = C \cdot (b-a)$$

Which means that:

$$\int_a^b f(x)\,dx = \lim \sigma(P_n) = C \cdot (b-a)$$

As required.

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Question 3.3:

Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a continuous function. For every $c \leq d \in [a,b]$ we know:

$$\int_c^d f(x)\,dx = 0$$

Prove that $f(x) = 0$.

Answer:

Suppose, for the sake of a contradiction, that there exists some $x_0 \in [a,b]$ such that $f(x_0) \neq 0$. Because f is continuous:

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall |x - x_0| \leq \delta : |f(x) - f(x_0)| \leq \varepsilon$$

So let $\varepsilon := \frac{f(x_0)}{2}$, so for every x in the neighborhood:

$$\frac{f(x_0)}{2} \leq f(x) \leq \frac{3 \cdot f(x_0)}{2}$$

So take $c < d \in [x_0 - \delta, x_0 + \delta] \cap [a,b]$. This means that:

$$\int_c^d f(x)\,dx \leq \int_c^d \frac{3 \cdot f(x_0)}{2} dx = \frac{3 \cdot f(x_0)}{2} \cdot (d - c)$$

And:

$$\int_c^d f(x)\,dx \geq \int_c^d \frac{f(x_0)}{2} dx = \frac{f(x_0)}{2} \cdot (d - c)$$

Since the sign of these bounds is the same (and non-zero, as $f(x_0) \neq 0$, and $d - c \neq 0$), this means that

$$\int_c^d f(x)\,dx \neq 0$$

In contradiction.

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Question 3.4:

Dis/Prove the following:

(1) If $|f|$ is integrable over the interval $[a, b]$, then f is as well.

(2) If f is integrable over the interval $[a, b]$, then $|f|$ is as well.

Answer:

(1) This is false. Let's look at the Dirichlet function:

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \notin \mathbb{Q} \end{cases}$$

We proved in the lecture that this is not integrable (over any interval). But I will quickly prove it (in two ways) here.
The simplest way to prove this is to recall that f isn't continuous at any point. So f isn't continuous almost everywhere, so by Lebesgue's Theorem, f isn't integrable.
Another way to prove it is to look at any partition P. We can take d_is which are rational (as $\mathbb{Q}$ is dense in $\mathbb{R}$) so the riemman sum of this pointed P is $b - a$. But we can also take d_is which are irrational (as $\mathbb{Q}^c$ is dense in $\mathbb{R}$), so the riemman sum of this pointed P is $a - b$. Since we can do this for *any* partition, especially for partitions whose norm approaches 0, it follows that the limit of the riemman sums does not exist. Therefore f has no integral.
But on the other hand:

$$|f(x)| = \begin{cases} 1 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases} = 1$$

So $|f(x)|$ is a constant function, which *is* is integrable.
So $|f(x)|$ is integrable, but $f(x)$ isn't.

(2) This is true. Firstly, because f is integrable, f is bound. Therefore $|f|$ is also bound.
Since f is integrable, for every $\varepsilon > 0$, there exists a partition P such that:

$$\sum_{i=1}^{n} \Delta_i \cdot \omega_i^f \leq \varepsilon$$

And we know that:

$$\omega_i^f = \sup_{x \in [x_{i-1}, x_i]} f(x) - \inf_{x \in [x_{i-1}, x_i]} f(x) = \sup_{x, y \in [x_{i-1}, x_i]} f(x) - f(y)$$

Now let's focus on the oscillation of $|f(x)|$ on the same interval:

$$\omega_i^{|f|} = \sup_{x, y \in [x_{i-1}, x_i]} |f(x)| - |f(y)|$$

By the triangle inequality, we know:

$$|f(x)| - |f(y)| \leq |f(x) - f(y)|$$

So:

$$\omega_i^{|f|} \leq \sup_{x, y \in [x_{i-1}, x_i]} |f(x) - f(y)|$$

Now, we know that:

$$\{f(x) - f(y)\} = \{|f(x) - f(y)|\} \cup \{-|f(x) - f(y)|\}$$

Since $f(x) - f(y) = \pm |f(x) - f(y)|$. And we also know:

$$\{-|f(x) - f(y)|\} \leq 0 \leq \{|f(x) - f(y)|\}$$

Which means that:

$$\sup\left\{\left|f\left(x\right)-f\left(y\right)\right|\right\}=\sup\left\{f\left(x\right)-f\left(y\right)\right\}$$

As an upper bound of the set is an upper bound of the absolute value of the set (since it is a subset), and an upper bound of the absolute value of the set must be an upper bound of the set (as otherwise, there must be an element $-\left|f\left(x\right)-f\left(y\right)\right|$ which is greater than the upper bound, which is a contradiction as they are less than or equal to 0, and therefore less than or equal to the absolute values, and by extension their upper bounds).
So a number is an upper bound of $\left\{f\left(x\right)-f\left(y\right)\right\}$ if and only if it is an upper bound of $\left\{\left|f\left(x\right)-f\left(y\right)\right|\right\}$. Therefore their supremums are equal.
Now recall that:

$$\omega_i^{|f|}\leq\sup_{x,y\in[x_{i-1},x_i]}\left|f\left(x\right)-f\left(y\right)\right|=\sup_{x,y\in[x_{i-1},x_i]}f\left(x\right)-f\left(y\right)=\omega_i^f$$

Which means that:

$$\sum_{i=1}^{n}\Delta_i\omega_i^{|f|}\leq\sum_{i=1}^{n}\Delta_i\omega_i^f\leq\varepsilon$$

So for every $\varepsilon>0$, there exists a partition P (which happens to be the same partition that works for f) such that:

$$\sum_{i=1}^{n}\Delta_i\cdot\omega_i^{|f|}\leq\varepsilon$$

And since $|f|$ is bound, by Riemman's criteria for integrability, $|f|$ is integrable.

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