Real Analysis

Real Analysis Modern Techniques and Their Applications, Gerald B. Folland

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1 Measures

1.1 σ-Algebras

1.1.1 Definition

Let X be a nonempty set, then an **algebra** of sets on X is a nonempty collection $\mathcal{A} \subseteq \mathcal{P}(X)$ which is closed under finite unions and complements. Meaning if $E_1, \dots, E_n \in \mathcal{A}$ then $\bigcup_{i=1}^n E_i \in \mathcal{A}$ and if $E \in \mathcal{A}$ then $E^c \in \mathcal{A}$. If $\mathcal{A}$ is closed under countable unions, then it is called a **σ-algebra**.

Notice that since $\bigcap_{i\in I} E_i = \left(\bigcup_{i\in I} E_i^c\right)^c$, algebras (respectively σ-algebras) are closed under finite (respectively countable) intersections. And if $\mathcal{A}$ is an algebra then since it is non-empty, there exists some $E \in \mathcal{A}$ and so $E \cap E^c = \varnothing \in \mathcal{A}$ and $\varnothing^c = X \in \mathcal{A}$.

Further notice that if $\mathcal{A}$ is an algebra, it is sufficient for it to be closed under countable *disjoint* unions in order for it to be a σ-algebra. Suppose $\{E_i\}_{i=1}^\infty \subseteq \mathcal{A}$ then let us define

$$F_k = E_k \cap \left(\bigcup_{i=1}^{k-1} E_j\right)^c$$

then $\{F_k\}_{k=1}^\infty$ are disjoint and and $\bigcup_{i=1}^\infty E_i = \dot\bigcup_{i=1}^\infty F_i$ and since $F_k \in \mathcal{A}$ since it is an algebra, and $\mathcal{A}$ is closed under countable disjoint unions, the union is in $\mathcal{A}$. So $\mathcal{A}$ is a σ-algebra.

Some trivial examples of σ-algebras are $\mathcal{P}(X)$ and $\{\varnothing, X\}$. If X is uncountable then

$$\mathcal{A} = \{E \subseteq X \mid E \text{ is countable or cocountable}\}$$

(cocountable meaning its complement is countable.) $\mathcal{A}$ is obviously closed under complements and is nonempty. If $\{E_i\}_{i=1}^\infty \subseteq \mathcal{A}$ then if all E_i are countable then $\bigcup_{i=1}^\infty E_i$ is also countable and in $\mathcal{A}$. Otherwise if any E_i is cocountable, so is the union.

Notice that if $\{\mathcal{A}_i\}_{i\in I}$ is an arbitrary family of σ-algebras on X, then so is $\bigcap_{i\in I} \mathcal{A}_i$. This is nonempty since it contains $\varnothing$; if $E \in \bigcap_{i\in I} \mathcal{A}_i$ then $E \in \mathcal{A}_i$ and so $E^c \in \mathcal{A}_i$ for every $i \in I$, meaning $E^c \in \bigcap_{i\in I} \mathcal{A}_i$; and similarly if $\{E_j\}_{j=1}^\infty \subseteq \bigcap_{i\in I} \mathcal{A}_i$ then $\{E_j\}_{j=1}^\infty \subseteq \mathcal{A}_i$ and so $\bigcup_{j=1}^\infty E_j \in \mathcal{A}_i$ for every $i \in I$, and so $\bigcup_{j=1}^\infty E_j \in \bigcap_{i\in I} \mathcal{A}_i$ as required. Thus if $\mathcal{E}$ is an arbitrary family of subsets of X, we can discuss the smallest σ-algebra containing $\mathcal{E}$:

$$\mathcal{M}(\mathcal{E}) := \bigcap\{\mathcal{A} \mid \mathcal{A} \text{ is a } \sigma\text{-algebra containing } \mathcal{E}\}$$

We will often use the following argument:

1.1.2 Lemma

If $\mathcal{E} \subseteq \mathcal{M}(\mathcal{F})$ then $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{F})$.

Since $\mathcal{M}(\mathcal{F})$ is a σ-algebra containing $\mathcal{E}$, it must contain $\mathcal{M}(\mathcal{E})$. ■

1.1.3 Definition

If X is a topological space (in particular a metric space), then the σ-algebra generated by the set of open sets in X (the topology) is called the **Borel σ-algebra** on X, and is denoted $\mathcal{B}_X$. Members of $\mathcal{B}_X$ are called **Borel sets**.

Examples of Borel sets are open and closed sets, countable intersections of open sets, countable unions of closed sets, etc. In general a countable intersection of open sets is called a G_δ set, a countable union of closed sets is a F_σ set, a countable union of G_δ sets is a $G_{\delta\sigma}$ set, a countable intersection of F_σ sets is a $F_{\sigma\delta}$ set, and so on. This is called the *Borel hierarchy.*

The Borel σ-algebra on $\mathbb{R}$ plays a foundational role in what is to come.

1.1.4 Proposition

$\mathcal{B}_{\mathbb{R}}$ can be generated by each of the following:

(1) the open intervals: $\mathcal{E}_1 = \{(a,b) \mid a < b\}$,

(2) the closed intervals: $\mathcal{E}_2 = \{[a,b] \mid a < b\}$,

(3) the half open intervals: $\mathcal{E}_3 = \{(a,b] \mid a < b\}$ or $\mathcal{E}_4 = \{[b,a) \mid a < b\}$,

(4) the open rays: $\mathcal{E}_5 = \{(a,\infty)\}$ or $\mathcal{E}_6 = \{(-\infty,a)\}$,

(5) the closed rays: $\mathcal{E}_7 = \{[a,\infty)\}$ or $\mathcal{E}_8 = \{(-\infty,a]\}$.

$\mathcal{E}_1$ generates $\mathcal{B}_{\mathbb{R}}$ since every open set is the countable union of open intervals, and so $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}(\mathcal{E}_1) \subseteq \mathcal{B}_{\mathbb{R}}$: the first inclusion is due to lemma 1.1.2 and the second is since $\mathcal{E}_1$ contains only open sets. Elements of $\mathcal{E}_j$ for all j are either G_δ or F_δ sets, for example $(a,b] = \bigcap_{n=1}^\infty (a, b+n^{-1})$, and so $\mathcal{M}(\mathcal{E}_j) \subseteq \mathcal{B}_{\mathbb{R}}$ by lemma 1.1.2. It is readily verifiable that open intervals can be generated by any $\mathcal{E}_j$ and so $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{E}_1) \subseteq \mathcal{M}(\mathcal{E}_j)$ for every other j, and so all $\mathcal{E}_j$ generate $\mathcal{B}_{\mathbb{R}}$. For example, $(a,b) = \bigcup_{n=1}^\infty [a+n^{-1}, b-n^{-1}]$. ■

1.1.5 Definition

If $\{X_\alpha\}_{\alpha\in A}$ is a collection of nonempty sets, let $X = \prod_{\alpha\in A} X_\alpha$ be their direct product and $\pi_\alpha \colon X \longrightarrow X_\alpha$ be the coordinate maps: $(x_a)_{a\in A} \mapsto x_\alpha$. If $\mathcal{M}_\alpha$ is a σ-algebra on X_α for each $\alpha \in A$, then we define their **product σ-algebra** to be the σ-algebra generated by

$$\left\{\pi_\alpha^{-1}(E_\alpha) \;\middle|\; E_\alpha \in \mathcal{M}_\alpha, \alpha \in A\right\}$$

This is denoted by $\bigotimes_{\alpha\in A} \mathcal{M}_\alpha$.

1.1.6 Proposition

If A is countable then $\bigotimes_{\alpha\in A} \mathcal{M}_\alpha$ is the σ-algebra generated by $\mathcal{E} = \left\{\prod_{\alpha\in A} E_\alpha \;\middle|\; E_\alpha \in \mathcal{M}_\alpha\right\}$.

If $E_\alpha \in \mathcal{M}_\alpha$ then $\pi_\alpha^{-1}(E_\alpha) = \prod_{\beta\in A} E_\beta$ where $E_\beta = X_\beta$ for $\beta \neq \alpha$, and so elements of the generating set of the product algebra are in $\mathcal{E}$ so $\bigotimes_{\alpha\in A} \mathcal{M}_\alpha \subseteq \mathcal{M}(\mathcal{E})$. Conversely $\prod_{\alpha\in A} E_\alpha = \bigcap_{\alpha\in A} \pi_\alpha^{-1}(E_\alpha)$ which is a countable union and is therefore in $\bigotimes_{\alpha\in A} \mathcal{M}_\alpha$. So by lemma 1.1.2, $\mathcal{M}(\mathcal{E}) \subseteq \bigotimes_{\alpha\in A} \mathcal{M}_\alpha$. ■

1.1.7 Proposition

If $\mathcal{M}_\alpha$ is generated by $\mathcal{E}_\alpha$ for every $\alpha \in A$ then $\bigotimes_{\alpha\in A} \mathcal{M}_\alpha$ is generated by $\mathcal{F}_1 = \left\{\pi_\alpha^{-1}(E_\alpha) \;\middle|\; E_\alpha \in \mathcal{E}_\alpha, \alpha \in A\right\}$. If A is countable and $X_\alpha \in \mathcal{E}_\alpha$ for all $\alpha \in A$ then $\bigotimes_{\alpha\in A} \mathcal{M}_\alpha$ is generated by $\mathcal{F}_2 = \left\{\prod_{\alpha\in A} E_\alpha \;\middle|\; E_\alpha \in \mathcal{E}_\alpha, \alpha \in A\right\}$.

Obviously $\mathcal{M}(\mathcal{F}_1) \subseteq \bigotimes_{\alpha\in A} \mathcal{M}_\alpha$. Conversely, $\left\{E \subseteq X_\alpha \;\middle|\; \pi_\alpha^{-1}(E) \in \mathcal{M}(\mathcal{F}_1)\right\}$ is easily seen to be a σ-algebra on X_α which contains $\mathcal{E}_\alpha$ and therefore $\mathcal{M}(\mathcal{E}_\alpha) = \mathcal{M}_\alpha$. Thus $\pi_\alpha^{-1}(E) \in \mathcal{M}(\mathcal{F}_1)$ for all $E \in \mathcal{M}_\alpha$, which means that $\bigotimes_{\alpha\in A} \mathcal{M}_\alpha \subseteq \mathcal{M}(\mathcal{F}_1)$ as required. The second assertion follows from the first. ■

1.1.8 Proposition

Let $X_1, \ldots, X_n$ be metric spaces and let $X = \prod_{i=1}^n X_i$ be equipped with the product metric (maximum). Then $\bigotimes_{i=1}^n \mathcal{B}_{X_i} \subseteq \mathcal{B}_X$. If the X_is are separable then $\bigotimes_{i=1}^n \mathcal{B}_{X_i} = \mathcal{B}_X$.

By the above proposition, $\bigotimes_{i=1}^n \mathcal{B}_{X_i}$ is generated by the sets $\pi_i^{-1}(U_i)$ for $1 \leq i \leq n$ where U_i is open in X_i. Since these sets are open X, $\bigotimes_{i=1}^n \mathcal{B}_{X_i} \subseteq \mathcal{B}_X$. Now suppose C_i is countably dense in X_i and let $\mathcal{E}_i$ be the collection of balls in X_i centered around points in C_i with rational radii. Every open set in X_i is a union of elements of $\mathcal{E}_i$, a countable union since $\mathcal{E}_i$ is countable, so $\mathcal{B}_{X_i}$ is generated by $\mathcal{E}_i$. Furthermore, the set of points in X whose ith coordinate is in C_i for all i is a countable dense subset of X. Balls of radius r in X are simply products of balls of radius r in the X_i so X is generated by $\{\prod_{i=1}^n E_i \mid E_i \in \mathcal{E}_i\}$ which also generated $\bigotimes_{i=1}^n \mathcal{B}_{X_i}$ by the above proposition. ■

1.1.9 Corollary

$\mathcal{B}_{\mathbb{R}^n} = \bigotimes_{i=1}^n \mathcal{B}_{\mathbb{R}}$.

1.1.10 Definition

An **elementary family** on X is a collection $\mathcal{E}$ of subsets of X such that

(1) $\varnothing \in \mathcal{E}$,

(2) if $E, F \in \mathcal{E}$ then $E \cap F \in \mathcal{E}$,

(3) if $E \in \mathcal{E}$ then E^c is a finite disjoint union of members of $\mathcal{E}$.

1.1.11 Proposition

If $\mathcal{E}$ is an elementary family then the collection $\mathcal{A}$ of finite disjoint unions of members of $\mathcal{E}$ is an algebra.

If $A, B \in \mathcal{E}$ and $B^c = \dot\bigcup_{i=1}^I C_i$ where $C_i \in \mathcal{E}$ are disjoint, then $A \setminus B = \dot\bigcup_{i=1}^I (A \cap C_i) \in \mathcal{E}$ and $A \cup B = (A \setminus B) \mathbin{\dot\cup} B$. Thus $A \setminus B, A \cup B \in \mathcal{A}$. By induction if $A_1, \dots, A_n \in \mathcal{E}$, $\bigcup_{i=1}^n A_i \in \mathcal{A}$: we can assume that $A_1, \dots, A_{n-1}$ are disjoint (since their union is in $\mathcal{A}$ which is the set of disjoint unions), and then $\bigcup_{i=1}^n A_i = A_n \mathbin{\dot\cup} \dot\bigcup_{i=1}^{n-1} (A_i \setminus A_n)$ which is a disjoint union (of disjoint unions of elements in $\mathcal{E}$) and so is in $\mathcal{A}$.
To show that $\mathcal{A}$ is closed under complements, suppose $A_1, \dots, A_n \in \mathcal{E}$ are disjoint and $A_m^c = \dot\bigcup_{j=1}^{J_m} B_m^j$ then

$$\left(\dot\bigcup_{m=1}^n A_m \right)^c = \bigcap_{m=1}^n \dot\bigcup_{j=1}^{J_m} B_m^j = \dot\bigcup \left\{ B_1^{j_1} \cap \dots \cap B_n^{j_n} \;\middle|\; 1 \le j_m \le J_m, 1 \le m \le n \right\}$$

which is a disjoint union of elements in $\mathcal{E}$, and so is in $\mathcal{A}$. ∎

Exercise

A family of sets $\mathcal{R} \subseteq \mathcal{P}(X)$ is called an **ring** if it is closed under finite unions and differences (meaning if $E, F \in \mathcal{R}$ then $E \setminus F \in \mathcal{R}$). A ring closed under countable unions is called a **σ-ring**. Show that

(1) Rings (respectively σ-rings) are closed under finite (respectively countable) intersections,

(2) If $\mathcal{R}$ is a ring (respectively σ-ring), then $\mathcal{R}$ is an algebra (respectively σ-algebra) if and only if $X \in \mathcal{R}$,

(3) If $\mathcal{R}$ is a σ-ring then $\mathcal{F}_1 = \{E \subseteq X \mid E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$ is a σ-algebra,

(4) If $\mathcal{R}$ is a σ-ring then $\mathcal{F}_2 = \{E \subseteq X \mid E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$.

(1) If $\mathcal{R}$ is a ring then let $A, B \in \mathcal{R}$ and so $A \setminus (A \setminus B) = A \cap (A \cap B^c)^c = A \cap (A^c \cup B) = A \cap B \in \mathcal{R}$ as required. If $\mathcal{R}$ is a σ-ring and $\{A_n\}_{n=1}^\infty \subseteq \mathcal{R}$ then

$$A_1 \setminus \left(\bigcup_{n=1}^\infty A_1 \setminus A_n \right) = \bigcap_{n=1}^\infty A_1 \cap A_n = \bigcap_{n=1}^\infty A_n$$

so it is closed under countable intersections.

(2) Obviously if $\mathcal{R}$ is a ring then $X \in \mathcal{R}$. Conversely then $\mathcal{R}$ is nonempty and closed under unions and complements (since $A^c = X \setminus A$), and is thus a algebra. And if it is a σ-ring it is further closed under countable unions and is thus a σ-algebra.

(3) A ring is nonempty and so $\mathcal{F}_1$ is nonempty. $\mathcal{F}_1$ is also obviously closed under complements. And if $\{A_n\}_{n=1}^\infty \subseteq \mathcal{F}_1$ then if for every n, $A_n \in \mathcal{R}$ so is their union. Otherwise let $I = \{i \mid A_i \in \mathcal{R}\}$ and $J = \left\{ j \;\middle|\; A_j^c \in \mathcal{R} \right\}$, then

$$\left(\bigcup_{n=1}^\infty A_n \right)^c = \bigcap_{j \in J} A_j^c \setminus \bigcup_{i \in I} A_i$$

since $\bigcap_{j\in J} A_j^c \in \mathcal{R}$ since σ-rings are closed under countable intersections by (1), and $\bigcup_{i\in I} A_i \in \mathcal{R}$, and rings are closed under differences, this means that $\left(\bigcup_{n=1}^\infty A_n\right)^c \in \mathcal{R}$ so $\bigcup_{n=1}^\infty A_n \in \mathcal{F}_1$ as required.

(4) Since $X \in \mathcal{F}_2$, $\mathcal{F}_2$ is nonempty. And if $E \in \mathcal{F}_2$ then $E \cap F \in \mathcal{R}$ for every $F \in \mathcal{R}$, since $E^c \cap F = F \setminus (E \cap F) \in \mathcal{R}$, this means that $E^c \in \mathcal{F}_1$. And if $\{A_n\}_{n=1}^\infty \in \mathcal{F}_2$ then for every $F \in \mathcal{R}$, $F \cap \bigcup_{n=1}^\infty A_n = \bigcup_{n=1}^\infty A_n \cap F \in \mathcal{R}$ as required.

Exercise

Let $\mathcal{M}$ be an infinite σ-algebra, then

(1) $\mathcal{M}$ contains an infinite sequence of nonempty disjoint sets,

(2) $\operatorname{card}(\mathcal{M}) \geq \mathfrak{c}$.

(1) Let $A_1 \in \mathcal{M}$ be a nonempty set such that $\{B \setminus A_1 \mid B \in \mathcal{M}\}$ is infinite. Otherwise for every $A \in \mathcal{M}$, we'd have that $\{B \setminus A \mid B \in \mathcal{M}\}$ and $\{B \cap A \mid B \in \mathcal{M}\}$ are finite (the second is for A^c), but $\mathcal{M}$ is just

$$\mathcal{M} = \{(B \setminus A) \cup (B \cap A) \mid B \in \mathcal{M}\} \subseteq \{B \setminus A \mid B \in \mathcal{M}\} \cup \{B \cap A \mid B \in \mathcal{M}\}$$

and so $\mathcal{M}$ would be finite, in contradiction.

Now similarly, for every n, we claim that if $A_1, \dots, A_n \in \mathcal{M}$ such that $\{B \setminus \bigcup_{k=1}^n A_k \mid B \in \mathcal{M}\}$ is infinite, then there exists an A_{n+1} disjoint from $A_1, \dots A_n$ such that $\left\{B \setminus \bigcup_{k=1}^{n+1} A_k \;\middle|\; B \in \mathcal{M}\right\}$ is infinite. There must exist such an A_{n+1} as otherwise for every A,

$$\left\{B \setminus \bigcup_{k=1}^n A_k \;\middle|\; B \in \mathcal{M}\right\} \subseteq \left\{B \setminus \left(\bigcup_{k=1}^n A_k \cup A\right) \;\middle|\; B \in \mathcal{M}\right\} \cup \left\{B \setminus \left(\bigcup_{k=1}^n A_k \cup A'\right) \;\middle|\; B \in \mathcal{M}\right\}$$

($A' = A^c \setminus \bigcup_{k=1}^n A_k$) which is finite, in contradiction. And so we have inductively created an infinite sequence of disjoint sets, as required.

(2) Let $\{A_n\}_{n=1}^\infty$ be an infinite sequence of nonempty disjoint sets, then we can define an injection $\mathcal{P}(\mathbb{N}) \to \mathcal{M}$ by $I \mapsto \bigcup_{i\in I} A_i$. Since all A_n are disjoint, this is indeed an injection, and since $\operatorname{card}\big(\mathcal{P}(\mathbb{N})\big) = \mathfrak{c}$, this means that $\operatorname{card}(\mathcal{M}) \geq \mathfrak{c}$.

Exercise

Show that an algebra $\mathcal{A}$ is a σ-algebra if and only if it is closed under countable increasing unions.

If $\mathcal{A}$ is a σ-algebra it is necessarily closed under countable increasing unions. Conversely suppose $\{A_n\}_{n=1}^\infty \subseteq \mathcal{A}$ then let us define $B_n = \bigcup_{k=1}^n A_k \in \mathcal{A}$ and $B_n \subseteq B_{n+1}$ so

$$\bigcup_{n=1}^\infty A_n = \bigcup_{n=1}^\infty B_n \in \mathcal{A}$$

Exercise

If $\mathcal{M}$ is the σ-algebra generated by $\mathcal{E}$ then it is the union of σ-algebras generated by $\mathcal{F}$ as $\mathcal{F}$ ranges over all countable subsets of $\mathcal{E}$.

Let us define

$$\mathcal{M}' = \bigcup \{\mathcal{M}(\mathcal{F}) \mid \mathcal{F} \subseteq \mathcal{E} \text{ is countable}\}$$

We will prove that $\mathcal{M}'$ is a σ-algebra. Firstly obviously $\mathcal{M}'$ is nonempty. If $E \in \mathcal{M}'$ then it is in some $\mathcal{M}(\mathcal{F})$ and therefore so is E^c. And if $\{A_n\}_{n=1}^\infty \subseteq \mathcal{M}'$ then suppose $A_n \in \mathcal{M}(\mathcal{F}_n)$ and then $\mathcal{F} = \bigcup_{n=1}^\infty \mathcal{F}_n$ is a countable subset of $\mathcal{E}$ and $A_n \in \mathcal{M}(\mathcal{F}_n) \subseteq \mathcal{M}(\mathcal{F})$ for every n so $\bigcup_{n=1}^\infty A_n \in \mathcal{M}(\mathcal{F}) \subseteq \mathcal{M}'$. So $\mathcal{M}'$ is indeed a σ-algebra. Now suppose $A \in \mathcal{E}$ then it is certainly in $\mathcal{M}(\{A\}) \subseteq \mathcal{M}'$, thus $\mathcal{M} = \mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}'$. And if $A \in \mathcal{M}'$ then it is in some $\mathcal{M}(\mathcal{F}) \subseteq \mathcal{M}(\mathcal{E}) = \mathcal{M}$. So $\mathcal{M} = \mathcal{M}'$ as required.

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