

Group Theory

Lecture 5, Sunday November 20, 2022
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Notice that if we focus on $(\mathbb{Z}_n, +)$, then since the group is abelian, every subgroup is normal since $gH = Hg$ by abelianness. If we take $G = (\mathbb{Z}_{60}, +)$ and the subgroup $H = \langle 6 \rangle$ then:

$$G/H = \{H, 1 + H, 2 + H, 3 + H, 4 + H, 5 + H\}$$

And if we take $N = \langle 2 \rangle$, we have that $H < N < G$, so we can discuss $N/H = \{H, 2 + H, 4 + H\} \leq G/H$. And therefore we can go one step further and:

$$G/H \big/ N/H = \{\{H, 2 + H, 4 + H\}, \{1 + H, 3 + H, 5 + H\}\}$$

5.1 Homomorphisms

Definition 5.1.1:

Suppose (G, \cdot) and (H, \circ) are groups then a function $f: G \longrightarrow H$ is a **homomorphism** if for every $a, b \in G$: $f(a \cdot b) = f(a) \circ f(b)$.

Notice that $f(e_G) = f(e_G \cdot e_G) = f(e_G) \circ f(e_G)$, and so if we take the inverse of $f(e_G)$ we get $f(e_G) = e_H$. And $f(a) \circ f(a^{-1}) = f(a \cdot a^{-1}) = f(e) = e$ and similarly $f(a^{-1}) \circ f(a) = e$, so $f(a)^{-1} = f(a^{-1})$.

Definition 5.1.2:

If $f: G \longrightarrow H$ is a homomorphism, we define the **image** and **kernel** of f to be:

$$\text{Im } f = \{f(g) \mid g \in G\} \quad \text{Ker } f = \{g \in G \mid f(g) = e\}$$

Notice that $\text{Im } f \leq H$ and $\text{Ker } f \leq G$. The image is a subgroup since we have shown $e = f(e) \in \text{Im } f$, it is closed under inverses since if $f(a) \in \text{Im } f$ then $f(a)^{-1} = f(a^{-1}) \in \text{Im } f$, and if $f(a), f(b) \in \text{Im } f$ then so is $f(a)f(b) = f(ab) \in \text{Im } f$. And the kernel is a subgroup since $e \in \text{Ker } f$, if $f(a) = e$ then $f(a^{-1}) = f(a)^{-1} = e^{-1} = e$ so $a^{-1} \in \text{Ker } f$, and if $f(a) = f(b) = e$ then $f(ab) = f(a)f(b) = e$ so $ab \in \text{Ker } f$.

Moreso, the kernel of a subgroup is a normal subgroup.

Proposition 5.1.3:

If $f: G \longrightarrow H$ then $\text{Ker } f \trianglelefteq G$.

Proof:

Let $K = \text{Ker } f$ and let $g \in G, k \in K$. Then since $f(k) = e$:

$$f(gkg^{-1}) = f(g)f(k)f(g^{-1}) = f(g)f(g^{-1}) = e$$

And so $gkg^{-1} \in K$. So for every $g, gKg^{-1} \subseteq K$, and so K is normal. ■

Example:

The **trivial homomorphism** is a homomorphism $f: G \longrightarrow H$ such that for every $g \in G, f(g) = e$. Then $\text{Im } f = \{e\}$ and $\text{Ker } f = G$.

And the **identity homomorphism** over a group G is the homomorphism $f: G \longrightarrow G$ such that $f(g) = g$. Then $\text{Im } f = G$ and $\text{Ker } f = \{e\}$.

It is trivial to see why these functions are homomorphisms.

Lemma 5.1.4:

f a homomorphism is injective if and only if $\text{Ker } f = \{e\}$.

Proof:

If f is injective then since f is a homomorphism, $f(e) = e$, so the only element that can map to e is e (notice these may be in different sets) and therefore $\text{Ker } f = \{e\}$. Now to show the converse, suppose $f(g) = f(g')$ then $f(g)f(g')^{-1} = f(gg'^{-1}) = e$, and since the kernel is trivial, $gg'^{-1} = e$ and therefore $g = g'$. So f is injective. ■

Theorem 5.1.5 (The First Isomorphism Theorem):

If $f: G \longrightarrow H$ is an homomorphism then $G/\text{Ker } f \cong \text{Im } f$.

Proof:

Let $K = \text{Ker } f$, then our goal is to construct an isomorphism (which recall is a bijective homomorphism) between G/K to $\text{Im } f$. We will denote this isomorphism as \tilde{f} . Given $gK \in G/K$ then the rational thing would be to map it to $f(g)$, that is $\tilde{f}(gK) = f(g)$. We must show 4 things: \tilde{f} is well defined, is a homomorphism, is injective, and is surjective. \tilde{f} is well defined since if $gK = g'K$ then $g' = gk$ for some $k \in K$, so $f(g') = f(g)f(k) = f(g)e = f(g)$. So no matter what representative we have for gK , their image in f is the same. \tilde{f} is a homomorphism since K is normal so $gK \cdot g'K = gg'K$ so:

$$\tilde{f}(gK \cdot g'K) = \tilde{f}(gg'K) = f(gg') = f(g)f(g') = \tilde{f}(gK)\tilde{f}(g'K)$$

To show \tilde{f} is injective, we will show that its kernel is trivial. Suppose $\tilde{f}(gK) = e$, then by definition $f(g) = e$, and therefore $g \in K$, which in turn means $gK = K$. So $\text{Ker } \tilde{f} = \{K\}$ (and K is the identity element of G/K), and therefore \tilde{f} is injective. \tilde{f} is surjective since if $f(g) \in \text{Im } f$ then $\tilde{f}(gK) = f(g)$. ■

Alternatively we could show that \tilde{f} is injective since if $\tilde{f}(gK) = \tilde{f}(g'K)$ then $f(g) = f(g')$, so $f(gg'^{-1}) = e$, so $gg'^{-1} \in K$, and therefore $g \in g'K$, so $gK = g'K$. And therefore \tilde{f} is injective. But this is less elegant and the lemma which gives a criterion for injectivity is an essential and important one.

Theorem 5.1.6 (The Second Isomorphism Theorem):

If $H \leq G$ and $N \trianglelefteq G$, then $HN \leq G$, $N \cap H \trianglelefteq H$, and $HN/N \cong H/(H \cap N)$.

Proof:

It is obvious that $HN \leq G$ since N is normal so $HN = NH$, and we showed that this is necessary and sufficient to show that HN is a subgroup. And $N \cap H$ is a normal subgroup of H since if $n \in N \cap H$ then $hN = Nh$ since N is normal, and if $hn \in h(N \cap H) \subseteq hN = Nh$ so $hn = n'h$, and so $n' = hnh^{-1}$. And since $n \in N \cap H$, $hnh^{-1} \in H$, so $n' \in H$ and therefore $hn = n'h$ for some $n' \in N \cap H$ and therefore $h(N \cap H) = (N \cap H)h$, and so $N \cap H$ is normal. We will define an isomorphism:

$$f: H \longrightarrow HN/N$$

By $f(h) = hN$. This is the only natural choice (recall that $H \leq HN$). It is simple to see why f is a homomorphism. Then if $h \in H$ then $hN = hnN$ for $n \in N$ so $hN \in HN/N$. And if $hnN \in HN/N$ then $hnN = hN = f(h)$, so $\text{Im } f = HN/N$. And $\text{Ker } f = N \cap H$ since $hN = N$ if and only if $h \in N$, and since $h \in H$ then h must be in $N \cap H$. So by the first isomorphism theorem:

$$H/\text{Ker } f = H/(N \cap H) \cong HN/N = \text{Im } f$$

Theorem 5.1.7 (The Third Isomorphism Theorem):

Suppose $K \leq H \leq G$ are groups such that $H, K \trianglelefteq G$. Then $K \trianglelefteq H$, $H/K \trianglelefteq G/K$, and:

$$G/K \Big/_{H/K} \cong G/H$$

Proof:

It is trivial to see that $K \trianglelefteq H$ since K is normal in G and cosets of K relative to H are cosets relative to G . It is also trivial to see why $H/K \leq G/K$ since $hK \in G/K$ since $h \in G$ and H/K is a group. It is a normal subgroup since if $g \in G$ then $gK \Big/_{H/K} = \{gKhK \mid h \in H\} = \{ghK \mid h \in H\}$ since H is normal. And since K is normal $ghK = Kgh$, and so $ghK = Kghg^{-1}g$. Since H is normal $ghg^{-1} \in H$, so $ghK = Kh'g = Kh'gK$, and therefore $gK \Big/_{H/K} = H/K \Big/_{H/K} gK$. So H/K is indeed normal.

We will define a homomorphism:

$$f: G/K \longrightarrow G/H$$

By $f(gK) = gH$. This is well defined since if $g_1K = g_2K$ then $g_1H = g_1KH = g_2KH = g_2H$ ($H = KH$ since $K \leq H$). And it is a homomorphism since $f(g_1Kg_2K) = f(g_1g_2K) = g_1g_2H = g_1Hg_2H = f(g_1)f(g_2)$ since K and H are normal. And $\text{Im } f = G/H$ and $\text{Ker } f = \{gK \mid gH = H\} = \{gK \mid g \in H\} = H/K$. And so:

$$G/K \Big/_{\text{Ker } f} = G/K \Big/_{H/K} \cong \text{Im } f = G/H$$

As required.

Definition 5.1.8:

A **lattice** is a partially ordered set (Γ, \preceq) such that for every $\alpha, \beta \in \Gamma$ they have an **upper bound** and **lower bound** $\alpha \vee \beta$ and $\alpha \wedge \beta$ respectively such that $\alpha, \beta \preceq \gamma$ if and only if $\alpha \vee \beta \preceq \gamma$, and $\gamma \preceq \alpha, \beta$ if and only if $\gamma \preceq \alpha \wedge \beta$.

Notice that the upper and lower bounds are unique, since if x and y are both upper bounds to α and β then $\alpha, \beta \preceq x, y$ so $x \preceq y$ and $y \preceq x$ so $x = y$. A similar argument can be used for lower bounds.

Example:

If X is a set $(\mathcal{P} X, \subseteq)$ is a lattice where $A \vee B = A \cup B$ and $A \wedge B = A \cap B$.

Example:

If G is a group, let

$$\mathcal{L}(G) = \{H \leq G\}$$

Then (\mathcal{L}, \leq) is a lattice where $A \vee B = \langle A \cup B \rangle$ is the upper bound (since it is the smallest subgroup containing both A and B), and $A \wedge B = A \cap B$ is a lower bound.

If we define:

$$\mathcal{L}_N(G) = \{N \trianglelefteq G\}$$

Then $(\mathcal{L}_N(G), \leq)$ (the \leq can be replaced with \trianglelefteq) is a lattice where $A \vee B = AB$ and $A \wedge B = A \cap B$. AB is an upper bound since it is normal ($gABg^{-1} = gAg^{-1}B = AB$).

Definition 5.1.9:

If Γ and Π are lattices, a function $\varphi: \Gamma \longrightarrow \Pi$ is a **lattice homomorphism** if for every $\alpha, \beta \in \Gamma$:

$$\varphi(\alpha \vee \beta) = \varphi(\alpha) \vee \varphi(\beta) \text{ and } \varphi(\alpha \wedge \beta) = \varphi(\alpha) \wedge \varphi(\beta)$$

φ is an **lattice isomorphism** if it is bijective. Two lattices are **isomorphic** if there exists a lattice isomorphism

between them.

Lemma 5.1.10:

If $f: G \longrightarrow H$ is a homomorphism then for every $K \leq H$, $f^{-1}(K)$ is a subgroup of G .

Proof:

We know that since $e \in K$ and $f(e) = e$, we know that $e \in f^{-1}(K)$. And if $a \in f^{-1}(K)$, then $f(a) \in K$, so $f(a^{-1}) = f(a)^{-1} \in K$, so $a^{-1} \in f^{-1}(K)$. And if $a, b \in f^{-1}(K)$ then $f(a), f(b) \in K$ so $f(a)f(b) = f(ab) \in K$ so $ab \in f^{-1}(K)$, as required. ■

Lemma 5.1.11:

If $K \trianglelefteq G$ then every subgroup of G/K is of the form H/K for $K \leq H \leq G$ (and every set of that form is a subgroup).

Proof:

Let us focus on the function:

$$\pi: G \longrightarrow G/K, \quad g \mapsto gK$$

This function is a surjective homomorphism. Then if B is a subgroup of G/K , then by above $H = \pi^{-1}(B)$ is a subgroup of G . We now argue that $B = H/K$. This is true since π is surjective so $\pi(H) = \pi^{-1}(\pi(B)) = B$, and by definition $\pi(H) = \{hK \mid h \in H\} = H/K$, so $B = H/K$. ■

Lemma 5.1.12:

If $K \trianglelefteq G$ and $K \leq H_1, H_2 \leq G$ then:

$$H_1/K \cap H_2/K = H_1 \cap H_2 / K \quad \langle H_1/K, H_2/K \rangle = \langle H_1, H_2 \rangle / K$$

Proof:

We know that if $h \in H_1 \cap H_2$ then hK is in both H_1/K and H_2/K . And if A is in both quotient groups, then $A = h_1K = h_2K$ for $h_1 \in H_1$, $h_2 \in H_2$. Then $h_1 = h_2k \in H_2$, so $h_1 \in H_1 \cap H_2$ and therefore $A \in H_1 \cap H_2 / K$ as required. That proves the first equality.

We know that $H_1/K, H_2/K$ are subgroups of $\langle H_1, H_2 \rangle / K$, so the cycle of these quotient groups is also a subgroup (as the smallest group to contain them both). And if $h \in \langle H_1, H_2 \rangle$ then hK is in the cycle $\langle H_1/K, H_2/K \rangle$ since h can be written as a product of elements in H_1 and H_2 and thus hK is equal to that product times K . That is equivalent to taking every element in that product and multiplying it by K , which is in that cycle. This proves the second equality. ■

Theorem 5.1.13:

Let G be a group and $K \trianglelefteq G$ a normal subgroup, let:

$$\mathcal{L}(G, K) = \{H \leq G \mid K \subseteq H\} \subseteq \mathcal{L}(G)$$

Then $\mathcal{L}(G/K)$ and $\mathcal{L}(G, K)$ are isomorphic.

Proof:

Notice that by one of the above lemmas, $\mathcal{L}(G/K) = \{H/K \mid H \in \mathcal{L}(G, K)\}$.

We must define a lattice isomorphism $\varphi: \mathcal{L}(G, K) \longrightarrow \mathcal{L}(G/K)$. We define $\varphi(H) = H/K$ (this is well defined since $H/K \subseteq G/K$). φ is a lattice homomorphism since $\varphi(H_1 \cap H_2) = (H_1 \cap H_2)/K = H_1/K \cap H_2/K = \varphi(H_1) \cap \varphi(H_2)$. And $\varphi(\langle H_1, H_2 \rangle) = \langle H_1, H_2 \rangle / K = \langle H_1/K, H_2/K \rangle$. This is true since $\langle H_1, H_2 \rangle / K$ contains both H_1/K and H_2/K , and it is the smallest group to do so.

And we define another lattice ψ in the other direction:

$$\psi: \mathcal{L}(G/K) \longrightarrow \mathcal{L}(G, K) \quad \psi(H/K) = H$$

And this is a lattice homomorphism since:

$$\psi(H_1/K \cap H_2/K) = \psi((H_1 \cap H_2)/K) = H_1 \cap H_2$$

And

$$\psi(\langle H_1/K, H_2/K \rangle) = \psi(\langle H_1, H_2 \rangle / K) = \langle H_1, H_2 \rangle$$

Notice that φ and ψ are inverses: if $H \in \mathcal{L}(G, K)$ then:

$$\psi(\varphi(H)) = \psi(H/K) = H$$

and if $H/K \in \mathcal{L}(G/K)$ then:

$$\varphi(\psi(H/K)) = \varphi(H) = H/K$$

So the lattices are isomorphisms. ■

These isomorphisms preserve indexes, normalness, and subgrouping.