Mathematical Logic

A summary of "A Concise Introduction to Mathematical Logic", W. Rautenberg

Ari Feiglin

Contents

1 Propositional Logic

1.1 Semantics of Propositional Logic

Propositional logic is the study of logic removed from interpretation of individual variables and context. I will assume that the reader already has experience with propositional logic, as this is something an undergraduate will cover in one of their first courses. While this subsection will focus mainly on the semantics of propositional logic, we will begin by defining its *syntax*,

1.1.1 Definition

Let PV be an arbitrary set of **propositional variables** (which are regarded as arbitrary symbols). **Propositional formulas** are formulas defined recursively by the following rules,

(1) Propositional variables in PV are formulas, called **prime** or **atomic** formulas.

(2) If α and β are formulas, then so are $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, and $\neg\alpha$. $(\alpha \wedge \beta)$ is called the **conjunction** of α and β, $(\alpha \vee \beta)$ their **disjunction**, and $\neg\alpha$ the **negation** of α.

The set of all the formulas constructed in this matter is denoted $\mathcal{F}$.

We can generalize this definition; instead of utilizing only the symbols $\wedge$ and $\vee$, we can take a general *logical signature* σ consisting of logical connectives of differing arities. We then recursively define σ-formulas as following: if c is an n-ary logical connective in σ, and $\alpha_1, \dots, \alpha_n$ are formulas, then so is

$$(\mathsf{c}\alpha_1, \dots, \alpha_n)$$

Aleternatively, if we only consider binary and unary connectives, then if c is a unary connective, we define $\mathsf{c}\alpha$ to be a formula, and if $\circ$ is a binary connective, then $(\alpha \circ \beta)$ is a formula. But we don't have much need for such generalizations, as $\{\wedge, \vee, \neg\}$ is complete, in the sense that all connectives can be defined using them. This is a fact we will discuss soon.

We can define other connectives, for example $\to$ and $\leftrightarrow$ are used as shorthands:

$$(\alpha \to \beta) := \neg(\alpha \wedge \neg\beta), \qquad (\alpha \leftrightarrow \beta) := \big((\alpha \to \beta) \wedge (\beta \to \alpha)\big)$$

We similarly define symbols for false and true:

$$\bot := (p_1 \wedge \neg p_1), \qquad \top = \neg\bot$$

For readability, we will use the following conventions when writing formulas (this is not a change to the definition of a formula, rather conventions for writing them in order to enhance readability)

(1) We will omit the outermost parentheses when writing formulas, if there are any.

(2) The order of operations for logical connectives is as follows, from first to last: $\neg, \wedge, \vee, \to, \leftrightarrow$.

(3) We associate $\to$ from the right, meaning $\alpha \to \beta \to \gamma$ is to be read as $\alpha \to (\beta \to \gamma)$. All other connectives associate from the left, for example $\alpha \wedge \beta \wedge \gamma$ is to be read as $(\alpha \wedge \beta) \wedge \gamma$.

(4) Instead of writing $\alpha_0 \wedge \alpha_1 \wedge \cdots \alpha_n$, we write $\bigwedge_{i=0}^{n} \alpha_i$, similar for $\vee$.

Since formulas are constructed in a recursive manner, most of our proofs about them are inductive.

1.1.2 Principle (Principle of Formula Induction)

Let $\mathcal{E}$ be a property of strings which satisfies the following conditions:

(1) $\mathcal{E}\pi$ for all prime formulas π,

(2) If $\mathcal{E}\alpha$ and $\mathcal{E}\beta$, then $\mathcal{E}(\alpha \wedge \beta)$, $\mathcal{E}(\alpha \vee \beta)$, and $\mathcal{E}\neg\alpha$ for all formulas $\alpha, \beta \in \mathcal{F}$.

Then $\mathcal{E}\varphi$ is true for all formulas φ.

An example of this is that every formula $\varphi \in \mathcal{F}$ is either prime, or of one of the following forms

$$\varphi = \neg\alpha, \quad \varphi = (\alpha \wedge \beta), \quad \varphi = (\alpha \vee \beta)$$

The proof of this is straightforward: let $\mathcal{E}$ be this property. Then trivially, $\mathcal{E}\pi$ for all prime formulas π. And if $\mathcal{E}\alpha$ and $\mathcal{E}\beta$, then of course we have

$$\mathcal{E}\neg\alpha, \quad \mathcal{E}(\alpha \wedge \beta), \quad \mathcal{E}(\alpha \vee \beta)$$

This is the first step in showing the *unique formula reconstruction property*. Let us prove a lemma before proving the property itself,

1.1.3 Lemma
Proper initial segments of formulas are not formulas. Equivalently (by contrapositive), if α and β are formulas and $\alpha\xi = \beta\eta$ for arbitrary strings ξ and η, then $\alpha = \beta$.

Let us prove this by induction on α. If α is a prime formula, suppose that β is not a prime formula, then its first character is either (or $\neg$, but then $\alpha =$ (or $\alpha = \neg$, in contradiction. Thus β is a prime formula and so $\alpha = \beta$ as they are both a single character. Now if $\alpha = (\alpha_1 \circ \alpha_2)$, then the first character of β must too be (, so β is of the form $(\beta_1 * \beta_2)$. Thus

$$\alpha_1 \circ \alpha_2)\xi = \beta_1 * \beta_2)\eta$$

and so by our inductive assumption, $\alpha_1 = \beta_1$, and so $\circ = *$, and thus $\alpha_2 = \beta_2$ by our inductive assumption again. And so $\alpha = \beta$ as required. The proof for the case that $\alpha = \neg\alpha'$ is similar. ∎

1.1.4 Proposition (Unique Formula Reconstruction Property)
Every compound formula $\varphi \in \mathcal{F}$ is of one of the following forms:

$$\varphi = \neg\alpha, \quad \varphi = (\alpha \wedge \beta), \quad \varphi = (\alpha \vee \beta)$$

For some formulas $\alpha, \beta \in \mathcal{F}$.

We have already shown existence. We will now show that this is unique, meaning that φ can be written uniquely as one of these strings. Using the lemma proven above, the proof for uniqueness of the reconstruction property is immediate. For example, if $\varphi = (\alpha_1 \wedge \beta_1)$ then obviously φ cannot be written as $\neg\alpha_2$ since ($\neq \neg$, and if $\varphi = (\alpha_2 \vee \beta_2)$ then by the lemma $\alpha_1 = \alpha_2$, and so we get that $\wedge = \vee$ in contradiction. And finally if $\varphi = (\alpha_2 \wedge \beta_2)$, then again by the lemma, $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$ as required. The proof for $\neg$ and $\vee$ are similar. ∎

Utilizing formula recursion, we can define functions on formulas. For example,

1.1.5 Definition
For a formula φ, we define $\mathrm{Sf}\varphi$ to be the set of all subformulas of φ. This is done recursively:

$$\mathrm{Sf}\pi = \{\pi\} \text{ for prime formulas } \pi,$$
$$\mathrm{Sf}\neg\alpha = \mathrm{Sf}\alpha \cup \{\alpha\}, \quad \mathrm{Sf}(\alpha \circ \beta) = \mathrm{Sf}\alpha \cup \mathrm{Sf}\beta \cup \{(\alpha \circ \beta)\} \text{ for a binary logical connective } \circ$$

Similarly, we can define the **rank** of a formula φ,

$$\mathrm{rank}\pi = 0 \text{ for prime formulas } \pi,$$
$$\mathrm{rank}\neg\alpha = \mathrm{rank}\alpha + 1, \quad \mathrm{rank}(\alpha \circ \beta) = \max\{\mathrm{rank}\alpha, \mathrm{rank}\beta\} + 1 \text{ for a binary logical connective } \circ$$

And we can also define the set of variables in φ,

$$\mathrm{Var}\pi = \{\pi\} \text{ for prime formulas } \pi,$$
$$\mathrm{Var}\neg\alpha = \mathrm{Var}\alpha, \quad \mathrm{Var}(\alpha \circ \beta) = \mathrm{Var}\alpha \cup \mathrm{Var}\beta \text{ for a binary logical connective } \circ$$

In all definitions $\circ$ is either $\wedge$ or $\vee$.

So now that we have discussed the syntax of propositional logic, it is time to discuss its semantics; how we assign to formulas truth values. Recall the truth tables for $\wedge$, $\vee$, and $\neg$:

α	β	$\alpha \wedge \beta$
1	1	1
1	0	0
0	1	0
0	0	0

α	β	$\alpha \vee \beta$
1	1	1
1	0	1
0	1	1
0	0	0

α	$\neg\alpha$
1	0
0	1

These define how the logical connectives function as functions on $\{0, 1\}$.

1.1.6 Definition
A **propositional valuation**, or a **propositional model**, is a function

$$w\colon PV \longrightarrow \{0,1\}$$

We can extend it to a function $w\colon PV \longrightarrow \mathcal{F}$ as follows:

$$w(\alpha\wedge\beta) = w\alpha\wedge w\beta, \quad w(\alpha\vee\beta) = w\alpha\vee w\beta, \quad w\neg\alpha = \neg w\alpha$$

Notice that we would need to define, for example, $w(\alpha\to\beta) = w\alpha\to w\beta$ had $\to$ been an element of our logical signature. But since $\to$ is defined using $\wedge$ and $\neg$, we must prove this identity:

$$w(\alpha\to\beta) = w\neg(\alpha\wedge\neg\beta) = \neg w(\alpha\wedge\neg\beta) = \neg(w\alpha\wedge\neg w\beta) = w\alpha\to w\beta$$

This is of course not a coincidence, but a result of the fact that $\alpha\to\beta = \neg(\alpha\wedge\neg\beta)$ (where $\alpha,\beta\in\{0,1\}$). Notice that furthermore,

$$w\top = 1, \quad w\bot = 0$$

1.1.7 Proposition
The valuation of a formula is dependent only on its variables. Meaning if φ is a formula and w and w' are two valuations where $w\pi = w'\pi$ for all $\pi\in\mathrm{Var}\varphi$, then $w\varphi = w'\varphi$.

We will prove this by induction on φ. For prime formulas, this is obvious as $\mathrm{Var}\varphi = \{\varphi\}$ and then $w\varphi = w'\varphi$ by the proposition's assumption. For $\varphi = \alpha\wedge\beta$, we have that

$$w\varphi = w\alpha\wedge w\beta = w'\alpha\wedge w'\beta = w'\varphi$$

where the second equality is our inductive assumption. The proof for $\varphi = \alpha\vee\beta$ and $\varphi = \neg\alpha$ is similar.

$\blacksquare$

Let us suppose that $PV = \{p_1, p_2, \dots, p_n, \dots\}$, then we define $\mathcal{F}_n$ to be the set of formulas φ such that $\mathrm{Var}\varphi\subseteq\{p_1,\dots,p_n\}$.

1.1.8 Definition
A **boolean function** is a function

$$f\colon \{0,1\}^n \longrightarrow \{0,1\}$$

for some $n\geq 0$. The set of boolean functions of arity n is denoted $\mathbf{B}_n$. A formula $\varphi\in\mathcal{F}_n$ **represents** a boolean function $f\in\mathbf{B}_n$ (similarly, f is represented by φ), if for all valuations w,

$$w\varphi = f(w\vec{p}) \qquad (w\vec{p} = (wp_1,\dots,wp_n))$$

So for example, $\alpha = p_1\wedge p_2$ represents the function $f(p,q) = p\wedge q$. This is since

$$f(wp_1, wp_2) = wp_1\wedge wp_2 = w(p_1\wedge p_2) = w\alpha$$

Since valuations of $\varphi\in\mathcal{F}_n$ are defined by their values on $p_1,\dots,p_n$, φ represents at most a single function f. In fact, it represents the function

$$\varphi^{(n)}(x_1,\dots,x_n) = w\varphi$$

where w is any valuation such that $wp_i = x_i$ (all of these valuations valuate φ the same). Now, notice that $\mathcal{F}_n\subset\mathcal{F}_{n+1}$ and $\mathbf{B}_n\subset\mathbf{B}_{n+1}$ and so $\varphi\in\mathcal{F}_n$ represents a function in $\mathcal{B}_{n+1}$ as well. But this function is not essentially in $\mathcal{B}_n$ in the sense that its last argument does not impact its value. Formally we say that given a function $f\colon M^n\longrightarrow M$, we call its ith argument *fictional* if for all $x_1,\dots,x_i,\dots,x_n\in M$ and $x_i'\in M$:

$$f(x_1,\dots,x_i,\dots,x_n) = f(x_1,\dots,x_i',\dots,x_n)$$

An *essentially n-ary* function is a function with no fictional arguments.

1.1.9 Definition
Two formulas α and β are **equivalent** if for every valuation w, $w\alpha = w\beta$. This is denoted $\alpha \equiv \beta$.

It is immediate that α and β are equivalent if and only if they represent the same function. A simple example of equivalence is $\alpha \equiv \neg\neg\alpha$. The following equivalences are easy to verify and the reader should already be familiar with them (α, β, and γ are formulas):

$$\begin{array}{ccl}
\alpha \wedge (\beta \wedge \gamma) \equiv \alpha \wedge \beta \wedge \gamma & \alpha \vee (\beta \vee \gamma) \equiv \alpha \vee \beta\gamma & \text{(associativity)} \\
\alpha \wedge \beta \equiv \beta \wedge \alpha & \alpha \vee \beta \equiv \beta \vee \alpha & \text{(commutativity)} \\
\alpha \wedge \alpha \equiv \alpha & \alpha \vee \alpha \equiv \alpha & \text{(idempotency)} \\
\alpha \wedge (\alpha \vee \beta) \equiv \alpha & \alpha \vee \alpha \wedge \beta \equiv \alpha & \text{(absorption)} \\
\multicolumn{2}{c}{\alpha \wedge (\beta \vee \gamma) \equiv \alpha \wedge \beta \vee \alpha \wedge \gamma} & (\wedge\text{-distributivity}) \\
\multicolumn{2}{c}{\alpha \vee \beta \wedge \gamma \equiv (\alpha \vee \beta) \wedge (\alpha \vee \gamma)} & (\vee\text{-distributivity}) \\
\neg(\alpha \wedge \beta) \equiv \neg\alpha \vee \neg\beta & \neg(\alpha \vee \beta) \equiv \neg\alpha \wedge \neg\beta & \text{(de Morgan rules)}
\end{array}$$

Furthermore,

$$\alpha \vee \neg\alpha \equiv \top, \quad \alpha \wedge \neg\alpha \equiv \bot, \quad \alpha \wedge \top \equiv \alpha \vee \bot \equiv \alpha$$

Since $\alpha \to \beta = \neg(\alpha \wedge \neg\beta)$, by de Morgan rules, this is equivalent to

$$\equiv \neg\alpha \vee \neg\neg\beta \equiv \neg\alpha \vee \beta$$

Notice that

$$\alpha \to \beta \to \gamma \equiv \neg\alpha \vee (\beta \to \gamma) \equiv \neg\alpha \vee \neg\beta \vee \gamma \equiv \neg(\alpha \wedge \beta) \vee \gamma \equiv \alpha \wedge \beta \to \gamma$$

Inductively, we see that

$$\alpha_1 \to \cdots \to \alpha_n \to \gamma \equiv \alpha_1 \wedge \cdots \wedge \alpha_n \to \gamma$$

We could go on, but I assume you get the point.
$\equiv$ is obviously reflexive, symmetric, and transitive: therefore it is an equivalence relation on $\mathcal{F}$. But moreso it is a *congruence relation*, meaning it respects connectives. Explicitly, for all formulas $\alpha, \beta, \alpha', \beta' \in \mathcal{F}$:

$$\alpha \equiv \alpha',\, \beta \equiv \beta' \implies \alpha \wedge \beta \equiv \alpha' \wedge \beta',\, \alpha \vee \beta \equiv \alpha' \vee \beta',\, \neg\alpha \equiv \neg\alpha'$$

Congruence relations will be discussed in more generality in later sections. Inductively, we can prove the following result:

1.1.10 Theorem (The Replacement Theorem)

Suppose α and α' are equivalent formulas. Let φ be some other formula, and define φ' to be the result of replacing all occurences of α within φ by α'. Then $\varphi \equiv \varphi'$.

This will be proven more generally later.

1.1.11 Definition
Prime formulas and their negations are called **literals**. A formula of the form $\alpha_1 \vee \cdots \vee \alpha_n$ where each α_i is a conjunction of literals is called a **disjunctive normal form**. And similarly a formula of the form $\alpha_1 \wedge \cdots \wedge \alpha_n$ where each α_i is a disjunction of literals is called a **conjunctive normal form**. We will use the abreviations DNF and CNF for disjunctive and conjunctive normal forms, respectively.

So a DNF is a formula of the form

$$\bigvee_{i=1}^{n} \bigwedge_{j=1}^{m_i} \ell_{i,j}$$

where for every i, j, $\ell_{i,j}$ is a literal: a formula of the form $p_{i,j}$ or $\neg p_{i,j}$ for some prime formula $p_{i,j}$. Similarly a CNF is a formula of the form

$$\bigwedge_{i=1}^{n} \bigvee_{j=1}^{m_i} \ell_{i,j}$$

Let us temporarily introduce the following notation: for a prime formula p, let

$$p^1 := p, \qquad p^0 := \neg p$$

This allows us to more concisely state and prove the following theorem:

1.1.12 Theorem
Every boolean function $f \in \mathbf{B}_n$ for $n > 0$ is representable by the DNF

$$\alpha_f := \bigvee_{f(\vec{x})=1} p_1^{x_1} \wedge \cdots \wedge p_n^{x_n}$$

and a CNF

$$\beta_f := \bigwedge_{f(\vec{x})=0} p_1^{\neg x_1} \wedge \cdots \wedge p_n^{\neg x_n}$$

Let w be a valuation and $\vec{p} = (p_1, \ldots, p_n)$ then

$$w\alpha_f = \bigvee_{f(\vec{x})=1} wp_1^{x_1} \wedge \cdots \wedge wp_n^{x_n}$$

Notice that wp^x is equal to 1 if and only if $wp = x$: suppose $x = 0$ then $wp^x = \neg wp$, which is equal to 1 if and only if $wp = 0 = x$, and similar for $x = 1$. Thus $w\alpha_f = 1$ if and only if there exists a $\vec{x}$ such that $f(\vec{x}) = 1$ and $wp_1^{x_1} \wedge \cdots \wedge wp_n^{x_n} = 1$, meaning that for each i, $wp_i = x_i$. This means that $\vec{x} = w\vec{p}$, and so $f(w\vec{p}) = f(\vec{x}) = 1$. Similarly if $f(w\vec{p}) = 1$ then let $\vec{x} = w\vec{p}$, and then $wp_1^{x_1} \wedge \cdots \wedge wp_n^{x_n} = 1$ and $f(\vec{x}) = 1$, so $w\alpha_f = 1$. So $w\alpha_f = f(w\vec{p})$ for all valuations w, which means that f is represented by α_f, as required. The proof for β_f is similar. ∎

Notice that since every formula represents a boolean function, which by above can be represented by a DNF and a CNF, we get that every formula is equivalent to a DNF and a CNF.

1.1.13 Corollary
Every formula is equivalent to a DNF and a CNF.

1.1.14 Definition
A logical signature σ is **functional complete** if every boolean function is representable by a formula in this signature.

By corollary 1.1.13, $\{\neg, \wedge, \vee\}$ is functional complete. Since

$$\alpha \vee \beta \equiv \neg(\neg\alpha \wedge \neg\beta), \quad \alpha \wedge \beta \equiv \neg(\neg\alpha \vee \neg\beta)$$

$\{\neg, \wedge\}$ and $\{\neg, \vee\}$ are both functional complete. Thus in order to show that a logical signature σ is functional complete, it is sufficient to show that $\neg$ and $\wedge$ or $\neg$ and $\vee$ can be represented by σ.

Note
If f is a function, instead of writing $f(x)$ or fx, many times we will instead write x^f. This is more concise and may reduce confusion in the case that x itself is a string wrapped in parentheses.

Let us define the function $\delta\colon \mathcal{F} \longrightarrow \mathcal{F}$ on formulas recursively by $p^\delta = p$ for prime formulas p and

$$(\neg\alpha)^\delta = \neg\alpha^\delta, \quad (\alpha \wedge \beta)^\delta = \alpha^\delta \vee \beta^\delta, \quad (\alpha \vee \beta)^\delta = \alpha^\delta \wedge \beta^\delta$$

Alternatively, α^δ is simply the result of swapping all occurences of $\wedge$ with $\vee$, and all occurences of $\vee$ with $\wedge$. α^δ is called the *dual formula* of α. Notice that the dual formula of a DNF is a CNF, and vice versa.
Now, suppose $f \in \mathbf{B}_n$, then let us define the *dual* of f,

$$f^\delta(\vec{x}) := \neg f(\neg\vec{x})$$

where $\neg\vec{x} = (\neg x_1, \ldots, \neg x_n)$. Notice that δ is idempotent:

$$f^{\delta^2}(\vec{x}) = \neg f^\delta(\neg\vec{x}) = \neg\neg f(\neg\neg\vec{x}) = f(\vec{x})$$

1.1.15 Theorem (The Duality Principle for Two-Valued Logic)
If α represents the function f, then α^δ represents f^δ.

We will prove this by induction on α. If $\alpha = p$ is prime, then this is trivial. Now suppose that α and β represent f_1 and f_2 respectively. Then $\alpha \wedge \beta$ represents $f(\vec{x}) = f_1(\vec{x}) \wedge f_2(\vec{x})$, and $(\alpha \wedge \beta)^\delta = \alpha^\delta \vee \beta^\delta$ represents $g(\vec{x}) = f_1^\delta(\vec{x}) \vee f_2^\delta(\vec{x})$ by the induction hypothesis. Now,

$$f^\delta(\vec{x}) = \neg f(\neg\vec{x}) = \neg(f_1(\neg\vec{x}) \wedge f_2(\neg\vec{x})) = \neg f_1(\neg\vec{x}) \vee \neg f_2(\neg\vec{x}) = f_1^\delta(\vec{x}) \vee f_2^\delta(\vec{x}) = g(\vec{x})$$

So $f^\delta = g$, meaning that $(\alpha \wedge \beta)^\delta$ does indeed represent f^δ. The proof for $\alpha \vee \beta$ is similar. Now suppose α represents f, then $\neg\alpha$ represents $\neg f$, and α^δ represents f^δ by the induction hypothesis. And so $(\neg\alpha)^\delta = \neg\alpha^\delta$ represents $\neg f^\delta$, which is equal to $(\neg f)^\delta$ since

$$(\neg f)^\delta(\vec{x}) = (\neg\neg f)(\neg\vec{x}) = \neg(\neg f(\neg\vec{x})) = \neg f^\delta(\vec{x})$$

And so $(\neg\alpha)^\delta$ represents $(\neg f)^\delta$, as required. ∎

1.1.16 Definition
Suppose α is a formula and w is a valuation. Instead of writing $w\alpha = 1$, we now write $w \vDash \alpha$, and this is read as "w *satisfies* α". If X is a set of formulas, we write $w \vDash X$ if $w \vDash \alpha$ for all $\alpha \in X$, and w is called a **propositional model** for X. A formula α (respectively a set of formulas X) is **satisfiable** if there is some valuation w such that $w \vDash \alpha$ (respectively $w \vDash X$). $\vDash$ is called the **satisfiability relation**.

$\vDash$ has the following immediate properties:

$$\begin{array}{rclcrcl} w \vDash p & \iff & wp = 1\ (p \in PV) & \qquad & w \vDash \alpha & \iff & w \nvDash \alpha \\ w \vDash \alpha \wedge \beta & \iff & w \vDash \alpha \text{ and } w \vDash \beta & & w \vDash \alpha \vee \beta & \iff & w \vDash \alpha \text{ or } w \vDash \beta \end{array}$$

These properties uniquely define $\vDash$, meaning we could have defined $\vDash$ recursively by these properties.
Notice that

$$w \vDash \alpha \to \beta \iff \text{if } w \vDash \alpha \text{ then } w \vDash \beta$$

This is due to the definition of $\to$ coinciding with our common usage of implication. Had we not defined $\to$, but instead added it to our logical signature, this above equivalence would have to be taken in the definition of the satisfiability relation (when axiomized by the above properties).

1.1.17 Definition
α is **logically valid**, or a **tautology**, if $w \vDash \alpha$ for all valuations w. This is abbreviated by $\vDash \alpha$. A formula which cannot be satisfied; ie. for all valuations w, $w \nvDash \alpha$; is called a **contradiction**.

For example, $\alpha \vee \neg\alpha$ is a tautology, while $\alpha \wedge \neg\alpha$ and $\alpha \leftrightarrow \neg\alpha$ are contradictions for all formulas α. Notice that the negation of a tautology is a contradiction and vice versa. $\top$ is a tautology and $\bot$ is a contradiction. The following are important tautologies of implication (keep in mind how $\to$ associates from the right):

$p \to p$	(self-implication)
$(p \to q) \to (q \to r) \to (p \to r)$	(chain rule)
$(p \to q \to r) \to (q \to p \to r)$	(exchange of premises)
$p \to q \to p$	(premise change)
$(p \to q \to r) \to (p \to q) \to (p \to r)$	(Frege's formula)
$((p \to q) \to p) \to p$	(Peirce's formula)

1.1.18 Definition

Suppose X is a set of formulas and α a formula, we say that α is a **logical consequence** if $w \vDash \alpha$ for every model w of X. In other words,

$$w \vDash X \implies w \vDash \alpha$$

This is denoted $X \vDash \alpha$.

Notice that $\vDash$ here is used for logical consequence (the consequence relation), and we used it before as the symbol for the satisfiability relation. Context will make it clear as to its meaning. We use the notation $\alpha_1, \dots, \alpha_n \vDash \beta$ to mean $\{\alpha_1, \dots, \alpha_n\} \vDash \beta$. This justifies the notation for tautologies: α is a tautology if and only if $\varnothing \vDash \alpha$ (since every valuation models $\varnothing$), which is shortened by the above notation to $\vDash \alpha$.
And we also use $X \vDash \alpha, \beta$ to mean $X \vDash \alpha$ and $X \vDash \beta$. And $X, \alpha \vDash \beta$ to mean $X \cup \{\alpha\} \vDash \beta$.
The following are examples of logical consequences

$$\alpha, \beta \vDash \alpha \wedge \beta, \quad \alpha \wedge \beta \vDash \alpha, \beta$$
$$\alpha, \alpha \to \beta \vDash \beta$$
$$X \vDash \bot \implies X \vDash \alpha \quad \text{for all formulas } \alpha$$
$$X, \alpha \vDash \beta,\ X, \neg\alpha \vDash \beta \implies X \vDash \beta$$

The final example is true because if $w \vDash X$ then either $w \vDash \alpha$ or $w \vDash \neg\alpha$, and in either case $w \vDash \beta$.
Let us now state some obvious properties of the consequence relation:

$$\begin{array}{ll} \alpha \in X \implies X \vDash \alpha & (\textit{reflexivity}) \\ X \vDash \alpha \text{ and } X \subseteq X' \implies X' \vDash \alpha & (\textit{monotonicity}) \\ X \vDash Y \text{ and } Y \vDash \alpha \implies X \vDash \alpha & (\textit{transitivity}) \end{array}$$

1.1.19 Definition
A **propositional substitution** is a mapping from prime formulas to formulas, $\sigma\colon PV \longrightarrow \mathcal{F}$, which is extended to a mapping between formulas $\sigma\colon \mathcal{F} \longrightarrow \mathcal{F}$ recursively:

$$(\alpha \wedge \beta)^\sigma = \alpha^\sigma \wedge \beta^\sigma, \quad (\alpha \vee \beta)^\sigma = \alpha^\sigma \vee \beta^\sigma, \quad (\neg\alpha)^\sigma = \neg\alpha^\sigma$$

If X is a set of formulas, we define
$$X^\sigma = \{\varphi^\sigma \mid \varphi \in X\}$$

Besides being intuitively important, the following proposition gives more insight into the usefulness of substitutions:

1.1.20 Proposition
Let X be a set of formulas, and α a formula. Then

$$X \vDash \alpha \implies X^\sigma \vDash \alpha^\sigma$$

Thus in a sense consequence is invariant under substitution.

Let w be a valuation, then we define w^σ as follows:

$$w^\sigma p = wp^\sigma$$

for prime formulas p. Now we claim that
$$w \vDash \alpha^\sigma \iff w^\sigma \vDash \alpha$$

We will prove this by induction on α. In the case that $\alpha = p$ is prime, then $w \vDash p^\sigma$ if and only if $wp^\sigma = w^\sigma p = 1$, and so this is if and only if $w^\sigma \vDash p$. Now by induction,

$$w \vDash (\alpha \wedge \beta)^\sigma \iff w \vDash \alpha^\sigma \text{ and } w \vDash \beta^\sigma \iff w^\sigma \vDash \alpha \text{ and } w^\sigma \vDash \beta \iff w^\sigma \vDash \alpha \wedge \beta$$

where the second equivalence is due to the induction hypothesis. The proof for formulas of the form $\alpha \vee \beta$ and $\neg\alpha$ proceed in a similar fashion.
Now, suppose $w \vDash X^\sigma$. This is if and only if $w \vDash \varphi^\sigma$ for all $\varphi \in X$, which is if and only if $w^\sigma \vDash \varphi$ by above. So $w \vDash X^\sigma$ if and only if $w^\sigma \vDash X$. And so if $X \vDash \alpha$ then let $w \vDash X^\sigma$, then $w^\sigma \vDash X$ meaning $w^\sigma \vDash \alpha$ and so $w \vDash \alpha^\sigma$ by above. So $X^\sigma \vDash \alpha^\sigma$ as required. ■

These four properties,

$$\begin{array}{ll} \alpha \in X \implies X \vDash \alpha & (\textit{reflexivity}) \\ X \vDash \alpha \text{ and } X \subseteq X' \implies X' \vDash \alpha & (\textit{monotonicity}) \\ X \vDash Y \text{ and } Y \vDash \alpha \implies X \vDash \alpha & (\textit{transitivity}) \\ X \vDash \alpha \implies X^\sigma \vDash \alpha^\sigma & (\textit{substitution invariance}) \end{array}$$

are what define general consequence relations, and form the basis for a general theory of logical systems. Another property is

$$X \vDash \alpha \implies X_0 \vDash \alpha \text{ for some finite } X_0 \subseteq X \quad (\textit{finitary})$$

We will show in the next subsection that this is a property of our consequence relation.
Another property is the property

$$X, \alpha \vDash \beta \iff X \vDash \alpha \to \beta$$

termed the *semantic deduction theorem*. Let us prove the first direction: suppose w is a model of X, then if $w \vDash \alpha$ it is a model of $X \cup \{\alpha\}$ and so $w \vDash \beta$. So we have shown that if $w \vDash \alpha$, then $w \vDash \beta$, meaning $w \vDash \alpha \to \beta$ and so $X \vDash \alpha \to \beta$. end for the converse, if $w \vDash X, \alpha$ then it is a model of X and so $w \vDash \alpha, \alpha \to \beta$ and thus $w \vDash \beta$.
We can show by induction a generalization of this:

$$X, \alpha_1, \dots, \alpha_n \vDash \beta \iff X \vDash \alpha_1 \to \cdots \to \alpha_n \to \beta \iff X \vDash \alpha_1 \wedge \cdots \wedge \alpha_n \to \beta$$

The induction step is simple: take $X' = X \cup \{\alpha_1\}$ we get by our induction hypothesis,

$$X, \alpha_1, \dots, \alpha_n \vDash \beta \iff X', \alpha_2, \dots, \alpha_n \vDash \beta \iff X, \alpha_1 \vDash \alpha_2 \to \cdots \to \alpha_n \to \beta$$
$$\iff X \vDash \alpha_1 \to \alpha_2 \to \cdots \to \alpha_n \to \beta$$

as required. The deduction theorem makes proving many tautologies relating to implication much easier.

1.2 Gentzen Calculi

To begin this subsection, we will define a derivability relation $\vdash$ which axiomatizes the important properties of the consequence relation $\vDash$. Our goal is to show that by using these axioms, $\vdash$ is equivalent to $\vDash$, and this will allow us to prove important facts about $\vDash$, namely its finitaryness.

1.2.1 Definition
We define **Gentzen style sequent calculus** of $\vdash$ as follows: $X \vdash \alpha$ is to be read as "α is derivable from X" where α is a formula and X is a set of formulas. A pair $(X, \alpha) \in \mathcal{P}(\mathcal{F}) \times \mathcal{F}$, or more suggestively written $X \vdash \alpha$, is called a **sequent**. Gentzen-style rules have the form

$$\frac{X_1 \vdash \alpha_1 \mid \cdots \mid X_n \vdash \alpha_n}{X \vdash \alpha}$$

Which is to be understood as meaning that if for every i, $X_i \vdash \alpha_i$, then $X \vdash \alpha$.
Gentzen calculus has the following basic rules:

$$\begin{array}{llll} (\text{IS}) & \dfrac{}{\alpha \vdash \alpha} & (\text{MR}) & \dfrac{X \vdash \alpha}{X' \vDash \alpha} \quad (X \subseteq X') \\ (\wedge 1) & \dfrac{X \vdash \alpha, \beta}{X \vdash \alpha \wedge \beta} & (\wedge 2) & \dfrac{X \vdash \alpha \wedge \beta}{X \vdash \alpha, \beta} \\ (\neg 1) & \dfrac{X \vdash \alpha, \neg\alpha}{X \vdash \beta} & (\neg 2) & \dfrac{X, \alpha \vdash \beta \mid X, \neg\alpha \vdash \beta}{X \vdash \beta} \end{array}$$

(IS means "initial sequent", MR means monotonacity rule.)
Now we say that α is derivable from X, in short $X \vdash \alpha$, if $S_n = X \vdash \alpha$ and there exists a sequence of sequents $(S_0; \dots; S_n)$ where for every S_i, S_i is either an initial sequent (IS) or derivable using the basic rules from previous sequents in the sequence.

For example, we can derive $\alpha \wedge \beta$ from $\{\alpha, \beta\}$, meaning $\alpha, \beta \vdash \alpha \wedge \beta$. This can be done by the sequence:

$$\begin{pmatrix} \alpha \vdash \alpha \;;\; \alpha, \beta \vdash \alpha \;;\; \beta \vdash \beta \;;\; \alpha, \beta \vdash \beta \;;\; \alpha, \beta \vdash \alpha \wedge \beta \\ \text{IS} \;;\; \text{MR} \;;\; \text{IS} \;;\; \text{MR} \;;\; \wedge 1 \end{pmatrix}$$

Let us prove some more useful rules

$$\frac{X, \neg\alpha \vdash \alpha}{X \vdash \alpha}$$

(¬-elimination)

1	$X, \alpha \vdash \alpha$	(IS), (MR)
2	$X, \neg\alpha \vdash \alpha$	supposition
3	$X \vdash \alpha$	(¬2)

$$\frac{X, \neg\alpha \vdash \beta, \neg\beta}{X \vdash \alpha}$$

(reductio ad absurdum)

1	$X, \neg\alpha \vdash \beta, \neg\beta$	supposition
2	$X, \neg\alpha \vdash \alpha$	(¬1)
3	$X \vdash \alpha$	¬-elimination

$$\frac{X \vdash \alpha \to \beta}{X, \alpha \vdash \beta}$$

(→-elimination)

1	$X, \alpha, \neg\beta \vdash \alpha, \neg\beta$	(IS), (MR)
2	$X, \alpha, \neg\beta \vdash \alpha \wedge \neg\beta$	(∧1)
3	$X \vdash \neg(\alpha \wedge \neg\beta)$	supposition
4	$X, \alpha, \neg\beta \vdash \neg(\alpha \wedge \neg\beta)$	(MR)
5	$X, \alpha, \neg\beta \vdash \beta$	(¬1) on 2 and 4
6	$X, \alpha \vdash \beta$	¬-elimination

$$\frac{X \vdash \alpha \mid X, \alpha \vdash \beta}{X \vdash \beta}$$

(cut rule)

1	$X, \neg\alpha \vdash \alpha$	supposition, (MR)
2	$X, \neg\alpha \vdash \neg\alpha$	(IS), (MR)
3	$X, \neg\alpha \vdash \beta$	(¬1)
4	$X, \alpha \vdash \beta$	supposition
5	$X \vdash \beta$	(¬2) on 3 and 4

$$\frac{X, \alpha \vdash \beta}{X \vdash \alpha \to \beta}$$

(→-introduction)

1	$X, \alpha \wedge \neg\beta, \alpha \vdash \beta$	supposition, (MR)
2	$X, \alpha \wedge \neg\beta \vdash \alpha$	(IS), (MR), (∧2)
3	$X, \alpha \wedge \neg\beta \vdash \beta$	cut rule
4	$X, \alpha \wedge \neg\beta \vdash \neg\beta$	(IS), (MR), (∧2)
5	$X, \alpha \wedge \neg\beta \vdash \alpha \to \beta$	(¬1)
6	$X, \neg(\alpha \wedge \neg\beta) \vdash \alpha \to \beta$	(IS), (MR)
7	$X \vdash \alpha \to \beta$	(¬2) on 5 and 6

$$\frac{X \vdash \alpha, \alpha \to \beta}{X \vdash \beta}$$

(modus ponens)

1	$X \vdash \alpha \to \beta$	supposition
2	$X, \alpha \to \beta$	→-elimination
3	$X \vdash \alpha$	supposition
4	$X \vdash \beta$	cut rule

→-elimination and →-introduction give us the *syntactic deduction theorem*:

$$X, \alpha \vdash \beta \iff X \vdash \alpha \to \beta$$

Let R be a rule of the form

$$R\colon \frac{X_1 \vdash \alpha_1 \mid \cdots \mid X_n \vdash \alpha_n}{X \vdash \alpha}$$

Then we say that a property of sequents $\mathcal{E}$ is *closed under* R if $\mathcal{E}(X_1, \alpha_1), \ldots, \mathcal{E}(X_n, \alpha_n)$ implies $\mathcal{E}(X, \alpha)$.

1.2.2 Proposition (Principle of Rule Induction)

Let $\mathcal{E}$ be a property of sequents which is closed under all the basic rules of $\vdash$. Then $X \vdash \alpha$ implies $\mathcal{E}(X, \alpha)$.

We will prove this by induction on the length of the derivation of $S = X \vdash \alpha$, n. If $n = 1$ then $X \vdash \alpha$ must be an initial sequent and so by assumption $\mathcal{E}(X, \alpha)$. For the induction step, suppose the derivation is $(S_0; \ldots; S_n)$, so $S = S_n$. Then by our inductive hypothesis $\mathcal{E}S_i$ for all $i < n$. If S is an initial sequent then $\mathcal{E}S$ holds by assumption. Otherwise S is obtained by applying a basic rule on some of the sequents S_i for $i < n$. And since $\mathcal{E}S_i$ and $\mathcal{E}$ is closed under basic rules, we have that $\mathcal{E}S$ as required. ■

1.2.3 Lemma
If $X \vdash \alpha$ then $X \vDash \alpha$. More suggestively,

$$\vdash \subseteq \vDash$$

Using the principle of rule induction, let $\mathcal{E}(X, \alpha)$ mean $X \vDash \alpha$ (formally this means $\mathcal{E} = \{(X, \alpha) \mid X \vDash \alpha\}$). Then we must show that $\mathcal{E}$ is closed under all the basic rules of $\vdash$. This means that we must show that

$$\alpha \vDash \alpha, \quad X \vDash \alpha \implies X' \vDash \alpha \text{ for } X \subseteq X', \quad X \vDash \alpha, \beta \iff X \vDash \alpha \wedge \beta,$$
$$X \vDash \alpha, \neg\alpha \implies X \vDash \beta, \quad X, \alpha \vDash \beta \text{ and } X, \neg\alpha \vDash \beta \implies X \vDash \beta$$

These are all readily verifiable (and some we have already shown). So $\mathcal{E}$ is indeed closed under all the basic rules of $\vdash$, and so $\mathcal{E}(X, \alpha)$ (meaning $X \vDash \alpha$) implies $X \vdash \alpha$. ■

1.2.4 Theorem
If $X \vdash \alpha$ then there exists a finite subset $X_0 \subseteq X$ such that $X_0 \vdash \alpha$.

Let $\mathcal{E}(X, \alpha)$ be the property that there exists a finite subset $X_0 \subseteq X$ such that $X_0 \vdash \alpha$. We will show that $\mathcal{E}$ is closed under the basic rules of $\vdash$. Trivially, $\mathcal{E}(X, \alpha)$ holds for $X = \{\alpha\}$, meaning $\mathcal{E}$ holds for (IS). And similarly if $\mathcal{E}(X, \alpha)$ and $X \subseteq X'$, since there exists a finite $X_0 \subseteq X$ such that $X_0 \vdash \alpha$, this same X_0 is a subset of X' and so $\mathcal{E}(X', \alpha)$ so $\mathcal{E}$ is closed under (MR).
Now if $\mathcal{E}(X, \alpha)$ and $\mathcal{E}(X, \beta)$ then suppose $X_1 \vdash \alpha$ and $X_2 \vdash \beta$ where $X_1, X_2 \subseteq X$ are finite. Then $X_0 = X_1 \cup X_2$ is finite, $X_0 \vdash \alpha, \beta$ and so $X_0 \vdash \alpha \wedge \beta$, and since $X_0 \subseteq X$ is finite, $\mathcal{E}(X, \alpha \wedge \beta)$ so $\mathcal{E}$ is closed under ($\wedge$1). Closure under the rest of the basic rules can be shown similarly. ■

1.2.5 Definition
A set of formulas X is **inconsistent** if $X \vdash \alpha$ for every formula α. If X is not inconsistent, it is termed **consistent**. X is **maximally consistent** if X is consistent but for every proper superset $X \subset Y$, Y is inconsistent.

Notice that X is inconsistent if and only if $X \vdash \bot$. Obviously if X is inconsistent, $X \vdash \bot$. Conversely, if $X \vdash \bot$ then $X \vdash p_1 \wedge \neg p_1$ and so by ($\wedge$2), $X \vdash p_1, \neg p_2$ and thus by ($\neg$1) for all formulas α, $X \vdash \alpha$.
Furthermore, if X is consistent it is maximally consistent if and only if for every formula α, either $\alpha \in X$ or $\neg\alpha \in X$ exclusively. If neither α nor $\neg\alpha$ are in X, then since X is maximally consistent, $X, \alpha \vdash \bot$ and $X, \neg\alpha \vdash \bot$ and therefore by ($\neg$2), $X \vdash \bot$ contradicting X's consistency. And if X contains α or $\neg\alpha$ for every formula α, then it is maximal: adding another formula α would mean that $\alpha, \neg\alpha \in X$ and so by (IS), (MR), and ($\neg$2), X would be inconsistent.
This means that maximally consistent sets X are *deductively closed*:

$$X \vdash \alpha \iff \alpha \in X$$

Obviously if $\alpha \in X$ then by (IS) and (MR), $X \vdash \alpha$. Now suppose that $X \vdash \alpha$, then since $\alpha \in X$ or $\neg\alpha \in X$, we cannot have $\neg\alpha \in X$ since X is consistent. Therefore $\alpha \in X$.

1.2.6 Lemma
The derivability relation has the following properties:

$$\texttt{C}^+\!: \quad X \vdash \alpha \iff X, \neg\alpha \vdash \bot, \qquad \texttt{C}^-\!: \quad X \vdash \neg\alpha \iff X, \alpha \vdash \bot$$

Meaning α is derivable from X if and only if $X \cup \{\neg\alpha\}$ is inconsistent. And similarly $\neg\alpha$ is derivable from X if and only if $X \cup \{\alpha\}$ is inconsistent.

We will prove $\texttt{C}^+$. Suppose $X \vdash \alpha$, then $X, \neg\alpha \vdash \alpha$ by (MR) and $X, \neg\alpha \vdash \neg\alpha$ by (IS) and (MR). Thus by ($\neg$1), $X, \neg\alpha \vdash \beta$ for all formulas β by ($\neg$1) and in particular, $X, \neg\alpha \vdash \bot$. Now suppose $X, \neg\alpha \vdash \bot$ then by ($\wedge$2) and ($\neg$1), we have $X, \neg\alpha \vdash \alpha$ then by $\neg$-elimination, $X \vdash \alpha$. $\texttt{C}^-$ is proven similarly. ■

1.2.7 Lemma (Lindenbaum's Theorem)
Every consistent set of formulas $X \subseteq \mathcal{F}$ can be extended to a maximally consistent set of formulas $X \subseteq X' \subseteq \mathcal{F}$.

Let us define the set

$$H = \{Y \subseteq \mathcal{F} \mid Y \text{ is consistent and } X \subseteq Y\}$$

This is partially ordered with respect to $\subseteq$, and since $X \in H$, H is not empty. Let $C \subseteq H$ be a chain, meaning that for every $Z, Y \in C$, either $Z \subseteq Y$ or $Y \subseteq Z$. Now we claim that $U = \bigcup C$ is an upper bound for C. So we must show that $U \in H$. Suppose not, then U is not consistent meaning $U \vdash \bot$. But then there must exist a finite $U_0 = \{\alpha_1, \ldots, \alpha_n\} \subseteq U$ such that $U_0 \vdash \bot$. Now suppose $\alpha_i \in Y_i \in C$ then since C is linearly ordered, we can assume that every Y_i is contained within Y_n. But then by (MR), $Y_n \vdash \bot$ which conradicts $Y_n \in H$ being consistent.
So U is consistent and so $U \in H$, and obviously for every $Y \in C$, $Y \subseteq U$. So U is an upper bound for C, meaning that every chain in H has an upper bound in H, and so by Zorn's Lemma, H has a maximal element. This maximal element, call it X', is precisely a maximally consistent set containing X: it is consistent and contains X since it is in H, and it is maximal in H so for every $X \subseteq Y$, $Y \notin H$ so Y is inconsistent. ■

1.2.8 Lemma
A maximally consistent set of formulas X has the following property:

$$X \vdash \neg\alpha \iff X \nvdash \alpha$$

for all formulas α.

If $X \vdash \neg\alpha$ then $X \nvdash \alpha$ due to X's consistency. If $X \nvdash \alpha$ then $X \cup \{\neg\alpha\}$ is consistent in lieu of $\mathtt{C}^+$. But since X is maximal, $X \cup \{\neg\alpha\} = X$ meaning $\neg\alpha \in X$ and so by (IS) and (MR), $X \vdash \neg\alpha$. ■

1.2.9 Lemma
Maximally consistent sets are satisfiable.

Suppose X is maximally consistent, then let us define the valuation w by $w \vDash p \iff X \vdash p$. Then we claim that

$$X \vdash \alpha \iff w \vDash \alpha$$

This is trival for prime formulas. Now if $X \vdash \alpha \wedge \beta$:

$$X \vdash \alpha \wedge \beta \iff X \vdash \alpha, \beta \iff w \vDash \alpha, \beta \iff w \vDash \alpha \wedge \beta$$

where the second equivalence is due to the induction hypothesis. And if $X \vdash \neg\alpha$:

$$X \vdash \neg\alpha \iff X \nvdash \alpha \iff w \nvDash \alpha \iff w \vDash \neg\alpha$$

The first equivalence is due to the previous lemma, and the second is due to the induction hypothesis. And therefore $w \vDash X$, meaning X is satisfiable. ■

1.2.10 Theorem (The Completeness Theorem)
Let X and α be an arbitrary set of formulas and formula respectively. Then $X \vdash \alpha$ if and only if $X \vDash \alpha$. More suggestively,

$$\vdash = \vDash$$

We have already shown that $\vdash \subseteq \vDash$ and so all that remains is to show the converse. Suppose that $X \nvdash \alpha$, then $X, \neg\alpha$ is consistent by $\mathtt{C}^+$. Thus it can be extended to a maximally consistent set $X, \neg\alpha \subseteq X'$ which is satisfiable. Therefore so is $X, \neg\alpha$, which means that $X \nvDash \alpha$. ■

We get the following theorem as an immediate result from The Completeness Theorem and theorem 1.2.4:

1.2.11 Theorem
$X \vDash \alpha$ if and only if $X_0 \vDash \alpha$ for a finite $X_0 \subseteq X$.

1.2.12 Theorem (The Compactness Theorem)
A set $X \subseteq \mathcal{F}$ is satisfiable if and only if every finite $X_0 \subseteq X$ is satisfiable.

Obviously if X is satisfiable, so is $X_0 \subseteq X$. Now if X is not satisfiable, then $X \vdash \bot$ and so there exists a finite $X_0 \subseteq X$ such that $X_0 \vdash \bot$ (and so $X_0 \vDash \bot$) by the previous theorem. And so if X is not satisfiable, there exists a finite $X_0 \subseteq X$ which is not satisfiable. ∎

Let us now give some examples of applications of the compactness theorem.

1.2.13 Proposition
Every set M can be linearly (also known as totally) ordered.

If M is finite, this is trivial: if $M = \{m_1, \dots, m_n\}$ simply define $m_1 < \cdots < m_n$. Now let M be any set, let us define the propositional variable (aka prime formula) p_{ab} for every $(a, b) \in M \times M$. This will represent $a < b$. So we define X to be the set of the following formulas, which represents M being linearly ordered,

$$\begin{aligned} \neg p_{aa} \quad & (a \in M), \\ p_{ab} \wedge p_{bc} \to p_{ac} \quad & (a, b, c \in M), \\ p_{ab} \vee p_{ba} \quad & (a \neq b \in M) \end{aligned}$$

If X is satisfiable, suppose $w \vDash X$, then we define the linear order $a < b$ if and only if $w \vDash p_{ab}$. Thus X is precisely the set of conditions necessary for $<$ to be a linear order: the first condition is irreflexivity, the second is transitivity, and the third totality (antisymmetry is gained through the combination of irreflexitivity and transitivity).
So if X is satisfiable, then M can be linearly ordered. By the compactness theorem, we need only to show that every finite $X_0 \subseteq X$ is satisfiable. If $X_0 \subseteq X$ is finite, then let us define M_0 to be the set of all symbols in M which occur in formulas in X_0. Since X_0 is finite, so is M_0 and therefore M_0 can be linearly ordered. Let us define $w_0 \vDash p_{ab} \iff a < b$ in M_0, then $w_0 \vDash X_0$. So by the compactness theorem X is satisfiable, as required. ∎

Recall that showing that every set can be well-ordered (the well-ordering theorem) is equivalent to the axiom of choice. Since the compactness theorem is actually weaker than the axiom of choice, the linear ordering theorem (what we just showed) is weaker than the well-ordering theorem. Which is not surprising.

1.2.14 Proposition
A graph is k-colorable if and only if every finite subgraph is k-colorable.

A *graph* is a pair $G = (V, E)$ where V is a set of *vertices* and E is a set of *edges*. E is a subset of $\{\{v, u\} \mid v \neq u \in V\}$. The graph G is k-colorable if V can be partitioned into k *color classes*: $V = C_1 \cup \cdots \cup C_k$ such that if $a, b \in C_i$ then $\{a, b\} \notin E$, meaning two neighboring vertices do not have the same color.
Obviously if a graph is k-colorable, so is every subgraph. To show the converse, let $G = (V, E)$ be a graph, then let us define the set of formulas X, where prime formulas are of the form $p_{a,i}$ where $a \in V$ and $1 \leq i \leq k$:

$$\begin{aligned} & p_{a,1} \vee \cdots \vee p_{a,k} (a \in V) \\ & \neg(p_{a,i} \wedge p_{a,j})(a \in V,\ 1 \leq i < j \leq k) \\ & \neg(p_{a,i} \wedge p_{b,i})(\{a, b\} \in E,\ 1 \leq i \leq k) \end{aligned}$$

If X is satisfiable, $w \vDash X$, then we define $C_i = \{a \in V \mid w \vDash p_{a,i}\}$, ie. we color $a \in V$ with the color i if and only if $p_{a,i}$ is satisfied. Then $V = C_1 \cup \cdots \cup C_k$ since for every $a \in V$, $w \vDash p_{a,1} \vee \cdots \vee p_{a,k}$, so for every $a \in V$ there exists an $1 \leq i \leq k$ such that $w \vDash p_{a,i}$ so $a \in C_i$. And $C_i \cap C_j = \varnothing$ in lieu of $\neg(p_{a,i} \wedge p_{a,j})$. And if $\{a, b\} \in E$ then a and b cannot be in the same color class by $\neg(p_{a,i} \wedge p_{b,i})$. So the C_is give a valid k-coloring of G.
Let $X_0 \subseteq X$ be finite, then let us define $G_0 = (V_0, E_0)$ where V_0 is the set of vertices appearing in formulas in X_0, and E_0 be the edges connecting them. By assumption, G_0 is k-colorable since it is finite. Now we define the

valuation w_0 such that $w_0 \vDash p_{a,i}$ if and only if a is in the ith color class for $a \in V_0$. This must model X_0 since X_0 includes only statements saying that G_0 can be k-colored. So by the compactness theorem, X is satisfiable, as required. ■

There are more examples of applications of the compactness theorem. For example, the ultrafilter theorem, which we will visit later on.

1.3 Hilbert Calculi

In this subsection we will define another form of sequent calculus.

1.3.1 Definition

A define $\Lambda \subseteq \mathcal{F}$ to be a set of axioms, called the **logical axiom scheme**. Now, let Γ be a set of **rules of inference**, predicates of the form $R \in \Lambda^n \times \Lambda$ for $n > 0$, where $R((\varphi_1, \dots, \varphi_n), \varphi)$ which is to be understood as "*if* $\varphi_1, \dots, \varphi_n$ *then* φ".
If $X \subseteq \mathcal{F}$ is a set of formulas, then a **proof** is a sequence $\Phi = (\varphi_0, \dots, \varphi_n)$ where for every i, φ_i is either in $X \cup \Gamma$ or there exists a rule of inference $R \in \Gamma$ and indexes $i_1, \dots, i_n < i$ such that $R((\varphi_{i_1}, \dots, \varphi_{i_n}), \varphi)$.
In such a case, φ_n is termed **derivable** (or **provable**) from X, and is written $X \vdash \varphi_n$ ($\vdash$ to differentiate it from the derivability relation $\vdash$ from the previous subsection).

Hilbert-style calculi will use the following axiom scheme Λ:

$\Lambda1$	$(\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \beta$	$\Lambda2$	$\alpha \to \beta \to \alpha \land \beta$
$\Lambda3$	$\alpha \land \beta \to \alpha, \quad \alpha \land \beta \to \beta$	$\Lambda3$	$(\alpha \to \neg\beta) \to \beta \to \neg\alpha$

And there is only a single rule of inference: $R((\alpha, \alpha \to \beta), \beta)$ called *modus ponens*, abbreviated MP. Essentially if α and $\alpha \to \beta$ then β.
The finiteness theorem for $\vdash$ is immediate, since $X \vdash \alpha$ requires a *finite* proof from X. And notice that

$$X \vdash \alpha, \alpha \to \beta \implies X \vdash \beta$$

Since if $\Phi_1 = (\varphi_0, \dots, \varphi_n)$ is a proof of α, and $\Phi_2 = (\varphi'_0, \dots, \varphi'_m)$ is a proof of $\alpha \to \beta$, then

$$\Phi = (\varphi_0, \dots, \varphi_n, \varphi'_0, \dots, \varphi'_m, \beta)$$

is a proof of $\alpha \to \beta$.

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