Infintesimal Calculus 3

Assignment 4
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Exercise 4.0.1:

Determine if the following limits exist, and if they do, compute them:

(1) $\lim_{(x,y)\to(0,0)} \frac{-|x-y|}{e^{x^2-2xy-y^2}}$

(2) $\lim_{(x,y)\to(0,0)} \frac{x\cdot\sin\left(x^4+y^4\right)}{x^4+y^4}$

(3) $\lim_{(x,y)\to(0,0)} \frac{y^2}{x^4+y^2}$

(1) If we define $f(x,y)=|x-y|$ and $g(t)=\frac{-t}{e^t}$, then this limit is equal to:

$$\lim_{(x,y)\to(0,0)} g(f(x,y))$$

Since the limit of $f(x,y)$ as $(x,y)\longrightarrow(0,0)$ is 0, this is equal to:

$$\lim_{t\to 0} g(t)=\lim_{t\to} -\frac{t}{e^t}=0$$

So the limit exists and is equal to 0.

(2) We know that the limit of x as (x,y) approaches $(0,0)$ is 0, and we will prove that $\frac{\sin\left(x^4+y^4\right)}{x^4+y^4}$ converges. If we let $t=x^4+y^4$, since t approaches 0 this limit is equal to:

$$\lim_{t\to 0}\frac{\sin t}{t}=1$$

Since this converges, taking the limit of the product of this and x converges to 0 (since the limit of a product is the product of limits if the limits exist).

(3) This limit does not exist. If we focus on the points $(x,0)$ as x approaches 0, the limit under this family of points is

$$\lim_{x\to 0}\frac{0^2}{x^4+0^2}=0$$

And if we focus on $(0,y)$ as y approaches 0 the limit is:

$$\lim_{y\to 0}\frac{y^2}{y^2}=1$$

These two limits are not equal and therefore the limit does not exist.

(4) This limit does not exist. If we focus on the points $(x,0)$ as x approaches 0, the limit is:

$$\lim_{x\to 0}\frac{x^2\cdot 0^2}{x^2\cdot 0^2+x^2}=0$$

And if we focus on the points (x,x) as x approaches 0 we have:

$$\lim_{x\to 0}\frac{x^4}{x^4}=1$$

These two partial limits are not equal and therefore the limit does not exist.

Exercise 4.0.2:

Does there exist a ζ such that the following function is continuous?

$$f(x,y) = \begin{cases} x \cdot \log(x^2 + 3y^2) & (x,y) \neq (0,0) \\ \zeta & (x,y) = (0,0) \end{cases}$$

To find such a ζ we must first show that $\lim_{(x,y)\to(0,0)} f(x,y)$ exists and to find this limit. Notice that:

$$\left|x \cdot \log(x^2 + 3y^2)\right| \leq \sqrt{x^2 + 3y^2} \cdot \left|\log(x^2 + 3y^2)\right|$$

And so if we let $t = \sqrt{x^2 + 3y^2}$, then:

$$\lim_{(x,y)\to(0,0)} |f(x,y)| \leq \lim_{t\to 0} |t \log t| = 0$$

And so the limit is 0, therefore $\zeta = 0$ is the only solution.

Exercise 4.0.3:

Is the set

$$A = \{(x,y) \mid x \in \mathbb{Q} \vee y \in \mathbb{Q}\}$$

connected? Is it path connected?

Notice that the set is equal to:

$$A = \mathbb{Q} \times \mathbb{R} \cup \mathbb{R} \times \mathbb{Q}$$

We can think of this as:

$$A = \left(\bigcup_{q\in\mathbb{Q}} \{q\} \times \mathbb{R}\right) \cup \left(\bigcup_{q\in\mathbb{Q}} \mathbb{R} \times \{q\}\right)$$

This is essentially a grid of intersecting (perpendicular as well) lines, which is intuitively path connected.
Suppose we have points $u, v \in A$. Suppose $u = (p, x)$ and $v = (q, y)$ where $p, q \in \mathbb{Q}$ and $x, y \in \mathbb{R}$. Without loss of generality suppose $x < y$ then there exists a $r \in \mathbb{Q}$ such that $x < r < y$. So if we define $u' = (p, r)$ and $v' = (q, r)$, then since r is rational the line segment $\overleftrightarrow{u'v'}$ is contained in A. So the polygonal chain:

$$\overleftrightarrow{uu'} \cup \overleftrightarrow{u'v'} \cup \overleftrightarrow{v'v}$$

is a path contained in A which connects u and v. An identical proof can be constructed if $u = (x, p)$ and $v = (y, q)$.
Similarly if $u = (p, x)$ and $v = (y, q)$ then if we define $u' = (p, q) \in A$ then the polygonal chain:

$$\overleftrightarrow{uu'} \cup \overleftrightarrow{u'v}$$

connects u and v and is contained in A.
So A is path connected and therefore also connected.

Exercise 4.0.4:

Prove or disprove: if $A \subseteq \mathbb{R}^2$ is countable then $\mathbb{R}^2 \setminus A$ is path connected.

This is true. Suppose for the sake of a contradiction that it is not. Then there exists two points $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2 \setminus A$ where there is no path between them in $\mathbb{R}^2 \setminus A$. Let ℓ be the line perpendicular to $\overleftrightarrow{x_1x_2}$ (suppose we only take one side of it, where the sides are divided by $\overleftrightarrow{x_1x_2}$). Then for every $a \in \ell$ there is a unqiue circle centered at a which intersects x_1 and x_2 since ℓ is perpendicular to $\overleftrightarrow{x_1x_2}$ so $\triangle x_1 a x_2$ is an isosceles triangle (so take the radius to be the distance between a and x_1). And these circles are disjoint other than at x_1 and x_2. We will show this last point for $x_1 = (0,1)$ and $x_2 = (0,-1)$ and $\ell = \mathbb{R}_{>0} \times \{0\}$ The circle around $(a,0)$ is given by

$$(x-a)^2 + y^2 = a^2 + 1 \equiv x^2 + y^2 - 2ax = 1$$

And so for two different values, they intersect only when:

$$\begin{cases} x^2 + y^2 - 2ax = 1 \\ x^2 + y^2 - 2bx = 1 \end{cases}$$

And so $2x(a-b) = 0$, so $x = 0$ and therefore the point is x_1 or x_2. And since all circles are just scales and shifts of another circle, this holds for all circles.
So if we define γ_a to be the arc on the circle around a between x_1 and x_2, this is a path between x_1 and x_2 in $\mathbb{R}^2$. Since we assumed $\mathbb{R}^2 \setminus A$ is not path connected, for every $a \in \ell$ there is a point in $\gamma_a \cap A$. So we can define a function $f\colon A \longrightarrow \ell$ where $f(x) = a$ such that $x \in \gamma_a$. As explained above, this must be injective since the circles are disjoint other than for x_1 and x_2 which are not in A. And so it must also be surjective since for every $a \in \ell$ there is a point $x \in A$ such that $x \in \gamma_a$ and this point cannot be sent to any other point other than a, so $f(x) = a$. So f is a bijection. But A is countable and ℓ is a line in $\mathbb{R}^2$ so it is uncountable, so there cannot be a bijection between them, in contradiction.

Exercise 4.0.5:

Suppose $A \subseteq \mathbb{R}^2$, prove or disprove: $\overline{\operatorname{int} A} = \operatorname{int}\left(\overline{A}\right)$.

Proof:

Let $A = B_1(0)$.

$$\overline{\operatorname{int} A} = \overline{A} = \bar{B}_1(0) \qquad \operatorname{int}\left(\overline{A}\right) = \operatorname{int}\left(\bar{B}_1(0)\right) = B_1(0)$$

And these are not equal.