Infintesimal Calculus 3

Assignment 7

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Exercise 7.1:

(1) Let $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$ and we define $g\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ by $g(x) = Ax + b$. Let $a \in \ker(f)$, show that $\ker(g) = a + \left\{R_1^T(A), \dots, R_m^T(A)\right\}^\perp$.

(2) Suppose $E \subseteq \mathbb{R}^n$ and $f\colon E \longrightarrow \mathbb{R}^m$ is continuously differentiable in E. Let $a \in \ker(f)$ and V be the affine space tangent to $\ker(f)$ at a. Prove that $V = a + \{\nabla f_1(a), \dots, \nabla f_m(a)\}^\perp$.

(1) First we will show that $\ker(A) = \left\{R_1^T(A), \dots, R_m^T(A)\right\}^\perp$. We know that $v \in \ker(A)$ if and only if $Av = 0$, that is if and only if for every $1 \le i \le m : R_i(A)v = 0$. Since $u^T v = u \cdot v$, this is equivalent to $R_i^T(A) \cdot v = 0$ for all $1 \le i \le m$, which is equivalent to $v \in \left\{R_1^T(A), \dots, R_m^T(A)\right\}$, as required.
We now claim that $\ker(g) = \ker(A) + a$. Suppose $v \in \ker(g)$ then $g(v) = Av + a = 0$, so $Av = -a$. Notice then that $A(v-a) = Av - Aa$, since $v, a \in \ker(g)$, $Av = Aa = -a$ so $A(v-a) = 0$ and so $v - a \in \ker(A)$ so $v \in \ker(A) + a$. And if $v \in \ker(A) + a$ then $A(v-a) = 0$ so $Av - Aa = Av + a = 0$ so $v \in \ker(g)$. So $\ker(g) = \ker(A) + a$ as required.
And since we showed that $\ker(A) = \left\{R_1^T(A), \dots, R_m^T(A)\right\}^\perp$ we have that

$$\ker(g) = \left\{R_1^T(A), \dots, R_m^T(A)\right\}^\perp + a$$

as required.

(2) Let $g(x) = df\big|_a(x-a) = J_f(a)\cdot(x-a)$ so $V = \ker(g)$. Notice that $g(x)$ is of the form $J_f(a)\cdot x + v$, so by the above subquestion $\ker(g) = v + \left\{R_1^T(J_f(a)), \dots, R_m^T(J_f(a))\right\}^\perp$ for any $v \in \ker(g)$. Since $g(a) = J_f(a)\cdot 0 = 0$, $a \in \ker(g)$ and since $R_i^T(J_f(a)) = \nabla f_i(a)$, we have that

$$\ker(g) = \{\nabla f_1(a), \dots, \nabla f_m(a)\}^\perp + a$$

Exercise 7.2:

At the point $a = (1,1,1)$ what is the direction where the function

$$f(x,y,z) = x \cdot \tan^{-1}(yz)$$

increases the most (as a unit vector)? Also compute the directional derivative of f at a in this point.

The largest rate of change is in the direction of $\nabla f(a)$ which is

$$\nabla f(a) = \begin{pmatrix} \tan^{-1}(yz) \\ \frac{xz}{1+z^2y^2} \\ \frac{xy}{1+z^2y^2} \end{pmatrix}(a) = \begin{pmatrix} \frac{\pi}{4} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

normalizing gives us the vector

$$u \approx \begin{pmatrix} 0.74317 \\ 0.47312 \\ 0.47312 \end{pmatrix}$$

which is the vector we were looking for.
We know that $D_u f(a) = u \cdot \nabla f(a)$ which in this case since u is the normalized vector of $\nabla f(a)$ is simply equal to $\|\nabla f(a)\| = \sqrt{\frac{\pi^2}{16} + \frac{1}{2}}$.

Exercise 7.3:

We define a surface in $\mathbb{R}^3$ by $z = x^2 + y^2$. Find a point on the surface such that the tangent plane at this point is perpendicular to $(1, 1, -2)^T$.

If we define $f(x, y, z) = x^2 + y^2 - z$ then the surface is defined by $\ker(f)$. Let v be a point on this surface then $v \in \ker(f)$, and we know that the tangent to the plane at v is given by $v + \{\nabla f(v)\}^\top$. And so the space of vectors perpendicular to this plane is $\operatorname{span}(\nabla f(v))$. So we need a v such that $(1, 1, -2) \in \operatorname{span}(\nabla f(v))$. We know that

$$\nabla f = \begin{pmatrix} 2x \\ 2y \\ -1 \end{pmatrix}$$

So we need to find x, y, and α such that $(1, 1, -2) = (2\alpha x, 2\alpha y, -\alpha)$. So we have that $\alpha = 2$ and $x = y = \frac{1}{4}$, and so $z = x^2 + y^2 = \frac{1}{8}$, thus the point is

$$\begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{8} \end{pmatrix}$$

Exercise 7.4:

Find the directional derivative of f at a in the direction h:

(1) $f(x, y) = x \sin(x + y)$, $a = \left(\frac{\pi}{4}, \frac{\pi}{4}\right)$, $h = (-1, 0)$.

(2) $f(x, y, z) = xy^2z^3$, $a = (3, 2, 1)$, $h = (4, 3, 0)$.

We first notice that all these functions are differentiable as the composition of standard functions. Thus $D_h f(a) = h \cdot \nabla f(a)$ for h unit vector.

(1) We have

$$\nabla f = \begin{pmatrix} \sin(x + y) + x\cos(x + y) \\ x\cos(x + y) \end{pmatrix}$$

Thus

$$D_h(a) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -1$$

(2) We have

$$\nabla f = \begin{pmatrix} y^2z^3 \\ 2xyz^3 \\ 3xy^2z^2 \end{pmatrix}$$

We must normalize h to get $\frac{1}{5}h$ and we have

$$D_h(a) = \frac{1}{5}\begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 12 \\ 36 \end{pmatrix} = 10.4$$

Exercise 7.5:

Find $dg\big|_a(h)$ where $g = \varphi \circ f$ where $f(x, y) = (x^2 + xy + 1, y^2 + 2)$ and $\varphi(x, y) = (x + y, 2x, y^2)$.

We know that $dg\big|_a = d\varphi\big|_{f(a)} \circ df\big|_a$, and the representation of the differentials is their Jacobian:

$$J_f = \begin{pmatrix} 2x + y & x \\ 0 & 2y \end{pmatrix} \qquad J_\varphi = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 2y \end{pmatrix}$$

And since $f(a) = (3,3)$:

$$dg\big|_a = J_g(a) = J_\varphi(f(a)) \cdot J_f(a) = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 6 \end{pmatrix} \cdot \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 6 & 2 \\ 0 & 12 \end{pmatrix}$$

And so:

$$dg\big|_a(h) = J_g(a) \cdot h = \begin{pmatrix} 10.5 \\ 19 \\ 6 \end{pmatrix}$$

Exercise 7.6:

Suppose $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ is a function such that there is a constant $M > 0$ such that for every $x, y \in \mathbb{R}^n$:

$$|f(x) - f(y)| \le M\|x - y\|$$

Such a function is called **Lipschitz**. Prove or disprove:

(1) f is continuous on $\mathbb{R}^n$.

(2) f is differentiable on all of $\mathbb{R}^n$.

(3) If $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuously differentiable on the closed unit ball around 0 then it is Lipschitz-continuous.

(1) This is true, suppose $x \in \mathbb{R}^n$ and $x_n \longrightarrow x$ then

$$|f(x_n) - f(x)| \le M\|x_n - x\| \longrightarrow 0$$

So $f(x_n) \longrightarrow f(x)$ and therefore f is continuous at x for all $x \in \mathbb{R}^n$ as required.

(2) This is false, take $f(x) = |x|$ in $\mathbb{R}$. This function is Lipschitz-continuous:

$$|f(x) - f(y)| = ||x| - |y|| \le |x - y|$$

but it is not differentiable at $x = 0$.

(3) We know that since $f \in C^1$, for every x and y in the closed unit ball:

$$f(y) - f(x) = \nabla f(x + t(y - x)) \cdot (y - x)$$

where $0 \le t \le 1$ by f's 0th order Taylor series expansion. Let us define

$$g(x) = \|\nabla f(x)\|$$

for every x in the closed unit ball. Since $f \in C^1$, ∇f is continuous in the closed unit ball, and therefore so is g as its norm (if a function is continuous, so is its norm). And since the closed unit ball is compact, g must be bounded, so $g(x) \le M$, that is $\|\nabla f\| \le M$ for some N. So then by the Cauchy-Schwarz inequality:

$$|f(y) - f(x)| = \big|\nabla f\big(x + t(y - x)\big) \cdot (y - x)\big| \le \big\|\nabla f\big(x + t(y - x)bigr)\big\| \cdot \|y - x\| \le M\|y - x\|$$

since $x + t(y - x)$ is in the closed unit ball. So f is Lipschitz continuous as required.