Complex Functions

Lecture 1, Monday July 17, 2023
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How do we define what a circle is? Historically, there are two approaches: Descartes defined it as the set of all points (x, y) which satisfy the equation

$$(x-a)^2 + (y-b)^2 = R^2$$

for some values a and b and $R > 0$. Euclid defined it as the set of all points whose distance from a specific point is some positive constant R.
We know that these two definitions are equivalent (given the standard norm/metric in $\mathbb{R}^2$), but Descartes's definition was introduced two thousand years after Euclid's. The idea of translating a visual or intuitive definition to an analytic one, as Descartes did, will be a motif of this course.
Now, recall the definition of an ellipse. Given two points, called the *foci* of the ellipse, F_1 and F_2 and a constant d, the ellipse defined is the set of all points A such that

$$|F_1A| + |F_2A| = d$$

We also must have that $|F_1F_2| < d$ as otherwise this just defines some line segment of F_1F_2. This is the Euclidean definition of an ellipse. Descartes's definition of an ellipse is the set of all points which satisfy

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

We must show that the cartesian definition satisfies the euclidean definition (and vice versa). Let us suppose that $a^2 > b^2$ (if we have an equality then this defines a circle), then we define $c = \sqrt{a^2 - b^2}$, and $F_1 = (-c, 0)$ and $F_2 = (c, 0)$. Then define $d = 2a$. Now we must show that given $A = (x, y)$, $|F_1A| + |F_2A| = d$ if and only if $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Now,

$$|F_1A| + |F_2A| = \sqrt{(x+c)^2 + y^2}, \qquad |F_2A| = \sqrt{(x-c)^2 + y^2}$$

And so we must show that

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a \iff \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Fortunately, we are not doing boring high school algebra, so we'll just assume that this is true. Thus the cartesian definition implies the eucliean definition.
Now suppose we have F_1, F_2, and d. Then we redefine the axes such that the x axis is parallel to F_1F_2 and the y axis is equidistant from F_1 and F_2. Define $a = \frac{d}{2}$, and $c = |F_1 0|$ (ie. half the distance between F_1 and F_2), and since $c = \sqrt{a^2 - b^2}$, this defines b. Now all that remains is to show that the points which satisfy $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are precisely the points which satisfy the euclidean definition of the ellipse defined by F_1, F_2, and d. Again, we won't be doing this.
Now, what about equations of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\,?$$

In the language of Euclid, this is defined by

$$\big||F_1A| - |F_2A|\big| = d$$

These are called hyperbolas.
And now for parabolas, Euclid defined them as the set of all points which satisfy

$$|A\ell| = |AF|$$

where ℓ is a line (called the directrix), and F is the focal point. $|A\ell|$ is defined as the metric between a point and a set is usually defined, by taking the infimum of all the distances between points on ℓ and A. This corresponds to the length of the line segment perpendicular to ℓ which intersects with A.
In cartesian terms, what we can do is define the x axis to be parallel to ℓ and halfway between it and F, and the y axis to pass through F. Let $F = (0, f)$ and $\ell\colon y = -f$. Then if $A = (x, y)$,

$$|AF| = \sqrt{x^2 + (y-f)^2}, \qquad |A\ell| = |y + f|$$

So

$$|AF| = |A\ell| \iff x^2 + (y-f)^2 = (y+f)^2 \iff x^2 = 4fy \iff y = \frac{1}{4f}x^2$$

Notice that all of these shapes are equivalent to the set of solutions of an equation of the form $Q(x,y)=0$ where

$$Q(x,y) = ax^2 + by^2 + cxy + dx + ey + f$$

and two other forms of solutions are lines, or two lines (of the form $y = \pm\alpha x$).

Proposition 1.1:

The set of solutions to $Q(x,y)=0$ is either a line, two lines, an ellipse, a hyperbola, or a parabola.

Proof:

Notice that $Q(x,y)=0$ if and only if

$$\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} a & \frac{c}{2} \\ \frac{c}{2} & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} d & e \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + f = 0$$

Let A be the diagonal matrix in the equation above. Now recall that if a matrix is symmetric, it can be orthogonally diagonalized. Suppose that P is the orthogonal matrix which diagonalizes A, so

$$P^TAP = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$$

Now suppose

$$P^T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ s \end{pmatrix} \iff \begin{pmatrix} x \\ y \end{pmatrix} = P\begin{pmatrix} t \\ s \end{pmatrix}$$

Meaning that

$$\begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}^T = \begin{pmatrix} t & s \end{pmatrix}P^T$$

Thus $Q(x,y)=0$ if and only if

$$\begin{pmatrix} t & s \end{pmatrix}P^TAP\begin{pmatrix} t \\ s \end{pmatrix} + \begin{pmatrix} d & e \end{pmatrix}P\begin{pmatrix} t \\ s \end{pmatrix} + f = \begin{pmatrix} t & s \end{pmatrix}\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}\begin{pmatrix} t \\ s \end{pmatrix} + \begin{pmatrix} d & e \end{pmatrix}P\begin{pmatrix} t \\ s \end{pmatrix} + f = 0$$

if we denote $\begin{pmatrix} d & e \end{pmatrix}P = \begin{pmatrix} d' & e' \end{pmatrix}$ we get that this is if and only if

$$\lambda_1 t^2 + \lambda_2 s^2 + d't + e's + f = 0$$

Now utilizing this new equation, we will split into cases.

(1) If $\lambda_1, \lambda_2 \neq 0$, then we can complete the square, the equation is equivalent to

$$\lambda_1\left(t + \frac{d'}{2\lambda_1}\right)^2 + \lambda_2\left(s + \frac{e'}{2\lambda_2}\right)^2 + f - \frac{d'^2}{4\lambda_1} - \frac{e'^2}{4\lambda_2} = 0$$

This is equivalent to an equation of the form

$$\lambda_1 u^2 + \lambda_2 v^2 + f' = 0$$

If $f'=0$ then this is $\lambda_1 u^2 = -\lambda_2 v^2$, which defines two lines (with respect to u and v). Otherwise this defines an ellipse.

Note that these define shapes with respect to u and v, but since t and s are simply some (orthogonal) linear transformation of x and y, and u and v are shifts of t and s, the shape defined in x and y is some orthogonal linear transformation of this ellipse and a shift, which still defines two lines or an ellipse. This will be true of the other cases as well.

(2) If $\lambda_2 = 0$ and $\lambda_1 \neq 0$ then we get

$$\lambda_1 t^2 + d't + e's + f = 0$$

which defines a parabola (complete the square). Similar for if $\lambda_1 = 0$ and $\lambda_2 \neq 0$.

(3) If $\lambda_1 = \lambda_2 = 0$ then we get

$$d't + e's + f = 0$$

which defines a line. ∎

Corollary 1.2:

The only bound set of the form $A = \{(x, y) \mid Q(x, y) = 0\}$ is an ellipse.