Complex Functions

Assignment 2
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Exercise 2.1:

Find the Taylor series of $f(z) = z^2$ around $z = 2$.

We know that $f'(z) = 2z$ and $f''(z) = 2$ and $f^{(k)}(z) = 0$ for $k \geq 3$. Thus

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(2)}{k!}(z-2)^k = 4 + 4(z-2) + (z-2)^2$$

Exercise 2.2:

Find the Taylor series of $f(z) = e^z$ about $a \in \mathbb{C}$.

We know that $f'(z) = e^z$ and so inductively $f^{(k)}(z) = e^z$. So

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(z-a)^k = \sum_{k=0}^{\infty} \frac{e^a}{k!}(z-a)^k = e^a \cdot \sum_{k=0}^{\infty} \frac{1}{k!}(z-a)^k$$

Exercise 2.3:

An *odd* function is a function such that $f(-z) = -f(z)$, and an *even* function is a function such that $f(-z) = f(z)$. Let f be an entire odd function. Prove that f's Taylor polynomial has only odd powers. Prove a similar result for entire even functions.

If f is an entire odd function then

$$f(z) = \frac{f(z) + f(z)}{2} = \frac{f(z) - f(-z)}{2}$$

And so

$$f'(z) = \frac{f'(z) + f'(-z)}{2}$$

And this is also entire (as the derivative of an entire function is entire), and it is also even since

$$f'(-z) = \frac{f'(-z) + f(z)}{2} = f'(z)$$

And if f is an entire even function then

$$f(z) = \frac{f(z) + f(z)}{2} = \frac{f(z) + f(-z)}{2}$$

And so

$$f'(z) = \frac{f'(z) - f'(-z)}{2}$$

And this is an entire odd function.
So if f is odd, inductively we see that for k even $f^{(k)}$ is odd and if k is odd $f^{(k)}$ is even. This is true for $k = 0$ and $k = 1$ as we showed above. And if it is true for k even then $f^{(k)}$ is odd and so $f^{(k+1)}$ is even and $k + 1$ is odd, and

similar for k odd. Notice that if g is an odd function then $g(0) = -g(0)$ so $g(0) = 0$. Thus for k even, $f^{(k)}(0) = 0$ and so all of the coefficients with even indexes in the Taylor series are zero (since $c_k = \frac{f^{(k)}(0)}{k!}$), as required.
And if f is even then inductively we see that for k even $f^{(k)}$ is even and for k odd $f^{(k)}$ is odd. And the odd coefficients in the Taylor series are zero, as required.

Exercise 2.4:

Prove that if f is an entire function and C is a circle which contains a then for $k \in \mathbb{N}$:

$$f^{(k)}(a) = \frac{k!}{2\pi i} \int_C \frac{f(w)}{(w-a)^{k+1}}\, dw$$

We prove this inductively on k. For $k = 0$ this reduces to Cauchy's integral theorem which we proved in lecture. Let $\tilde{C}$ be the open circle whose boundary is C, thus $a \in C$. Now suppose it is true for $k-1$ then for small enough h, $a + h \in \tilde{C}$ as it is open and so by our induction hypothesis

$$\frac{f^{(k-1)}(a+h) - f^{(k-1)}(a)}{h} = \frac{(k-1)!}{2\pi i} \int_C f(w) \cdot \frac{1}{h} \left(\frac{1}{(w-a-h)^k} - \frac{1}{(w-a)^k} \right) dw$$

Our goal is to show that this converges to the target integral (in the statement of the exercise) as $h \longrightarrow 0$.
Let $h_n \longrightarrow 0$, and we can assume without loss of generality that $a + h_n \in \tilde{C}$ for every n. Then let $g_n \colon C \longrightarrow \mathbb{C}$ by

$$g_n(w) = f(w) \cdot \frac{1}{h_n} \left(\frac{1}{(w-a-h_n)^k} - \frac{1}{(w-a)^k} \right) dw$$

Our goal is to show that g_n converges uniformly to g on C where

$$g(w) = \frac{k \cdot f(w)}{(w-a)^{k+1}}$$

We can expand out the fraction in g_n:

$$\frac{1}{h_n} \cdot \frac{1}{(w-a-h_n)^k} - \frac{1}{(w-a)^k} = \frac{(w-a)^k - (w-a-h_n)^k}{(w-a)^k (w-a-h_n)^k}$$

Using the binomial theorem, we can see that

$$\frac{1}{h_n} \cdot \left((w-a)^k - (w-a-h_n)^k \right) = \frac{1}{h_n} \left((w-a)^k - \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^\ell (w-a)^{k-\ell} h_n^\ell \right) = \sum_{\ell=1}^{k} \binom{k}{\ell} (-1)^{\ell-1} (w-a)^{k-\ell} h_n^{\ell-1}$$

And so

$$g_n - g = f(w) \cdot \left(\frac{\sum_{\ell=1}^{k} \binom{k}{\ell} (-1)^{\ell-1} (w-a)^{k-\ell} h_n^{\ell-1}}{(w-a)^k (w-a-h_n)^k} - \frac{k}{(w-a)^{k+1}} \right) =$$

$$= f(w) \cdot \frac{1}{(w-a)^{k+1} (w-a-h_n)^k} \cdot \left(\sum_{\ell=1}^{k} \binom{k}{\ell} (-1)^{\ell-1} (w-a)^{k-\ell+1} h_n^{\ell-1} - k(w-a-h_n)^k \right)$$

The subformula in the parentheses can be rewritten like so

$$\sum_{\ell=0}^{k-1} \binom{k}{\ell+1} (-1)^\ell (w-a)^{k-\ell} h_n^\ell - k \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^\ell (w-a)^{k-\ell} h_n^\ell = \sum_{\ell=0}^{k-1} (-1)^\ell (w-a)^{k-\ell} h_n^\ell \cdot \left(\binom{k}{\ell+1} - k \binom{k}{\ell} \right) - (-1)^k h_n^k$$

Since when $\ell = 0$ $\binom{k}{1} - k\binom{k}{0} = k - k = 0$, we can start indexing the sum at $\ell = 1$. Thus we get that the modulus of this is less than

$$|\ldots| \leq M \sum_{\ell=1}^{k-1} |w-a|^{k-\ell} |h_n|^\ell + |h_n|^k$$

where M is the maximum value of $\binom{k}{\ell+1} - k\binom{k}{\ell}$. Let N be the maximum value of $|w-a|^{k-\ell}$ iterating over all $w \in C$ and $\ell = 1, \dots, k-1$, and we can assume that $|h_n| < 1$ so $|h_n|^\ell \leq |h_n|$. Thus this in turn is less than

$$\leq MN(k-2)|h_n| + |h_n|$$

Let us simply denote this as h'_n, and it is clear that $h'_n \longrightarrow 0$ since h_n does.
And so

$$|g_n - g| \leq |f(w)| \cdot \frac{1}{|w-a|^{k+1}|w-a-h_n|^k} \cdot h'_n$$

We can bound $|f(w)|$ since $|f|$ is continuous and C is closed and bounded, and therefore compact, so $|f|$ takes a maximum value on C, let it be B. And since $a \in \tilde{C}$ which is open and $w \in C$ which is the boundary of $\tilde{C}$, $|w-a|$ must take a minimum value $\alpha > 0$ over all $w \in C$ (take a ball of radius $\alpha > 0$ about a contained in C, which must exist as it is open). And since

$$|w-a-h_n| \geq |w-a| - |h_n| \geq \alpha - |h_n|$$

we can assume $|h_n| \leq \frac{\alpha}{2}$ and so $|w-a-h_n| \geq \frac{\alpha}{2}$. And so we have a bound inpdenent of w:

$$|g_n - g| \leq B \cdot \frac{2^k}{\alpha^{2k+1}} \cdot h'_n$$

And this bound converges to 0 as $n \longrightarrow \infty$, so $g_n \rightrightarrows g$ as required.
This means that

$$\frac{(k-1)!}{2\pi i}\int_C g_n\, dz \longrightarrow \frac{(k-1)!}{2\pi i}\int_C g\, dz$$

And we recall that

$$\frac{(k-1)!}{2\pi i}\int_C g_n\, dz = \frac{f^{(k-1)}(a+h_n) - f^{(k-1)}(a)}{h_n}, \qquad \frac{(k-1)!}{2\pi i}\int_C g\, dz = \frac{k!}{2\pi i}\int_C \frac{f(w)}{(w-a)^{k+1}}\, dz$$

Since this is true for every $h_n \longrightarrow 0$, this means that

$$f^{(k)}(a) = \lim_{h\to 0}\frac{f^{(k-1)}(a+h) - f^{(k-1)}(a)}{h} = \frac{k!}{2\pi i}\int_C \frac{f(w)}{(w-a)^{k+1}}\, dz$$

as required.

Exercise 2.5:

Let f be an entire function bound by M on the circle $|z| = R$.

(1) Show that the coefficients c_k in f's Taylor series about 0 satisfy

$$|C_k| \leq \frac{M}{R^k}$$

(2) Show that for the polynomial $p(z) = \sum_{k=0}^n c_k z^k$ bound by 1 on the open disk $D_1(0)$, every coefficient c_k is bound by 1.

(1) We know that by Cauchy's integral theorem:

$$f^{(k)}(0) = \frac{k!}{2\pi i}\int_{|z|=R} \frac{f(z)}{z^{k+1}}\, dz$$

since the function $\frac{f(z)}{z^{k+1}}$ is bound by $\frac{M}{R^{k+1}}$ on $|z| = R$, we get that

$$\left|f^{(k)}(0)\right| \leq \frac{k!}{2\pi} \cdot \frac{M}{R^{k+1}} \cdot \int_{|z|=R} |dz| = k! \cdot \frac{M}{R^k}$$

since the length of the curve $|z| = R$ is $2\pi R$. And since

$$c_k = \frac{f^{(k)}(0)}{k!} \implies |c_k| \leq \frac{M}{R^k}$$

as required.

(2) For every $R < 1$, p is bound by 1 on the circle $|z| = R$, and since the coefficients of the Taylor series are precisely c_k, we get

$$|c_k| \leq \frac{1}{R^k}$$

for every $R < 1$. And since $R < 1$ is arbitrary, this means that $|c_k| \leq 1$ as required.

Exercise 2.6:

f is an entire function such that $|f(z)| \leq A + B|z|^{3/2}$ for every z. Show that f is a linear polynomial.

Let $z_0 \in \mathbb{C}$ and $R > 0$, set $C_R = \{z \in \mathbb{C} \mid |z - z_0| = R\}$. Then

$$f''(z_0) = \frac{1}{\pi i} \int_{C_R} \frac{f(z)}{(z - z_0)^3}\, dz$$

Thus

$$|f''(z_0)| \leq \frac{1}{\pi} \max_{z \in C_R} \frac{|f(z)|}{|z - z_0|^3} \cdot 2\pi R \leq \frac{2}{R^2} \cdot \max_{z \in C_R} A + B|z|^{3/2}$$

Since $R = |z - z_0| \geq |z| - |z_0|$, we have $|z| \leq |z_0| + R$ and so

$$|f''(z_0)| \leq \frac{2\big(A + B(|z_0| + R)^{3/2}\big)}{R^2}$$

As $R \longrightarrow \infty$ this goes to 0, so $f'' = 0$ and therefore $f(z) = a + bz$ as required.

Exercise 2.7:

Let f be an entire function where $|f'(z)| \leq |z|$ for every z. Show that $f(z) = a + bz^2$ where $|b| \leq \frac{1}{2}$.

Let $z_0 \in \mathbb{C}$ and let $R > 0$ and $C_R = \{z \in \mathbb{C} \mid |z - z_0| = R\}$. Then

$$f''(z_0) = \frac{1}{2\pi i} \int_{C_R} \frac{f'(x)}{(z - z_0)^2}\, dz$$

Thus

$$|f''(z_0)| \leq \frac{1}{2\pi} \max_{z \in C_R} \frac{|f'(x)|}{|z - z_0|^2} \cdot 2\pi R \leq \max_{z \in C_R} \frac{|z|}{R}$$

Since $R = |z - z_0| \geq |z| - |z_0|$, we have $|z| \leq R + |z_0|$ for every $z \in C_R$ so

$$|f''(z_0)| \leq \frac{|z_0| + R}{R} = 1 + \frac{|z_0|}{R}$$

And as $R \longrightarrow \infty$ this approaches 1, so $f''(z_0)$ is bounded and entire and therefore constant.
So $f''(z) = 2b$ so $f'(z) = c + 2bx$, and since $|f'(0)| = 0$ this means that $c = 0$. And so $f(z) = a + bx^2$. Since $|f'(1)| \leq 1$, this means that $|2b| \leq 1$ so $|b| \leq \frac{b}{2}$ as required.

Exercise 2.8:

Prove that no non-constant entire function can satisfy that for all $z \in \mathbb{C}$, $f(z + 1) = f(z)$ and $f(z + i) = f(z)$.

Suppose f is an entire function which satisfies these conditions.
Let $m, n \in \mathbb{Z}$ and $z \in \mathbb{C}$, then inductively we can see that $f(z+m) = f(z)$ for positive integers m, and for negative integers $f(z) = f(z+m-m) = f(z+m)$. And similarly $f(z+in) = f(z)$. Thus $f(z+m+in) = f(z)$.
This means that $f(\mathbb{C}) = f\big(\{z \in \mathbb{C} \mid 0 \leq \operatorname{Re} z, \operatorname{Im} z \leq 1\}\big)$ since for every $z \in \mathbb{C}$, $z - \lfloor \operatorname{Re} z \rfloor - i\lfloor \operatorname{Im} z \rfloor$ is in the unit square and has the same image as z. But the unit square is closed and bounded, so it is compact. And since f is continuous, it is therefore bounded on the unit square and therefore on all of $\mathbb{C}$. Therefore f is constant.