Group Theory

Lecture 8, Sunday November 27, 2022

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Recall that in our discrete course, we proved the Cantor-Schroder-Bernstein theorem: if there exists injections $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$ then there exists a bijection between A and B. Does this result have a parallel in groups? That is, if there exists $f\colon A \hookrightarrow B$ and $g\colon B \hookrightarrow A$, are A and B isomorphic? The answer is no. Take for instance $\mathbb{F}_2$ and $\mathbb{F}_3$. $\mathbb{F}_2 \hookrightarrow \mathbb{F}_3$ trivially and $\mathbb{F}_3 \hookrightarrow \mathbb{F}_2$, but $\mathbb{F}_2$ and $\mathbb{F}_3$ are not isomorphic (prove this!).
Returning to chains, recall that the following are exact chains if and only if (we use 1 for the trivial group):

(1) $A \xrightarrow{f} B \longrightarrow 1$; $\operatorname{Im} f = \operatorname{Ker} g = B$ which is if and only if f is surjective (epimorphism).

(2) $1 \longrightarrow A \xrightarrow{f} B$; f is injective (monomorphism).

(3) $1 \longrightarrow A \xrightarrow{f} B \longrightarrow 1$; f is an isomorphism.

(4) $1 \longrightarrow A \hookrightarrow B \twoheadrightarrow C \longrightarrow 1$; $C \cong {}^{B}\!/_{K}$ and $K \cong A$, that is C "=" ${}^{B}\!/_{A}$.

Definition 8.0.1:

A diagram **commutes** if every possible path gives you the same output. That is if you have two paths $G \to \varphi_1, \dots, \varphi_n \to H$ and $G \to \psi_1, \dots, \psi_m \to H$ then

$$\varphi_n \circ \cdots \circ \varphi_1 = \psi_m \circ \cdots \circ \psi_1$$

(since composition is from right to left.)

So for example the following commutes

$$\begin{tikzcd}
G \arrow[r, "f"] \arrow[d, "h"'] & G \arrow[d, "g"] \\
G \arrow[r, "k"'] & G
\end{tikzcd}$$

if and only if $g \circ f = k \circ h$.

Example:

We define the **projective linear group** to be $\mathrm{PGL}_n(\mathbb{F}) = {}^{\mathrm{GL}_n(\mathbb{F})}\!/_{\{\alpha I\}}$. We further define the **projective special linear group** to be $\mathrm{PSL}_n(\mathbb{F}) = {}^{\mathrm{SL}_n(\mathbb{F})}\!/_{\{\alpha I\}}$. And one last definition, $\mu_n(\mathbb{F}) = \{a \in \mathbb{F} \mid a^n = 1\}$. Let us look at

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \mu_n(\mathbb{F}) \arrow[r, hook] \arrow[d, hook] & \mathbb{F}^\times \arrow[r, "\cdot n"] \arrow[d, hook] & \mathbb{F}^\times \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & \mathrm{SL}_n(\mathbb{F}) \arrow[r, hook] \arrow[d, two heads] & \mathrm{GL}_n(\mathbb{F}) \arrow[r, "\det"] \arrow[d, two heads] & \mathbb{F}^\times \arrow[r] \arrow[d, two heads] & 1 \\
1 \arrow[r] & \mathrm{PSL}_n(\mathbb{F}) \arrow[r, hook] \arrow[d] & \mathrm{PGL}_n(\mathbb{F}) \arrow[r, "\det"] \arrow[d] & {}^{\mathbb{F}^\times}\!/_{(\mathbb{F}^\times)^n} \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}$$

This diagram commutes.

8.1 Group actions

Definition 8.1.1:

Suppose X is any set, a **group action** of a group G on X is a function $\Phi\colon G \times X \longrightarrow X$. Which satisfies:

(1) $\Phi\big(g, \Phi(h, x)\big) = \Phi(gh, x)$.

(2) $\Phi(1, x) = x$.

Theorem 8.1.2:

An equivalent definition of a group action on X by G is a homomorphism $\varphi\colon G \longrightarrow S_X$.

Proof:

Suppose we have a homomorphism $\varphi\colon G \longrightarrow S_X$. We define Φ by

$$\Phi(g, x) = \big(\varphi(g)\big)(x)$$

We claim this is a group action:

$$\Phi\big(g, \Phi(h, x)\big) = \Phi\big(g, (\varphi(h)(x))\big) = \varphi(g)\big(\varphi(h)(x)\big) = \big(\varphi(g) \circ \varphi(h)\big)(x) = \varphi(gh)(x) = \Phi(gh, x)$$

which proves the first axiom, and

$$\Phi(1, x) = \varphi(1)(x) = \mathrm{id}(x) = x$$

which proves Φ is indeed a group action.
Now suppose Φ is a group action, we must find a homomorphism φ. Given $g \in G$ we define $\sigma = \varphi(g)$ by $\sigma(x) = \Phi(g, x)$. This is just a longer way of saying φ is defined by:

$$\big(\varphi(g)\big)(x) = \Phi(g, x)$$

It is not immediately clear why φ is well defined (as a function), but we will first show that it has the homomorphism property.

$$\varphi(gh)(x) = \Phi(gh, x) = \Phi\big(g, \Phi(h, x)\big) = \Phi\big(g, \varphi(h)(x)\big) = \varphi(g)\big(\varphi(h)(x)\big) = \varphi(g) \circ \varphi(h)(x)$$

And therefore $\varphi(gh) = \varphi(g) \circ \varphi(h)$ as required.
Now we will show that $\varphi(g)$ is indeed a permutation. Notice that since φ has the homomorphism property:

$$\varphi(g) \circ \varphi(g^{-1}) = \varphi(gg^{-1}) = \varphi(1)$$

And $\varphi(1) = \mathrm{id}$, so $\varphi(g)$ has an inverse, namely $\varphi(g^{-1})$ so it is a bijection as required.

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Notice that this theorem gives us a simple proof that if G acts on X and $H \leq G$, then H acts on X (in the same way). This is because we can take the same homomorphism from G to S_X an restrict it to H.
We use a compact notation for group actions: instead of writing $\Phi(g, x)$ we instead write gx. This means that it must satisfy

(1) $g\big(hx\big) = \big(gh\big)x$. (Note that on the right side gh is not a group action rather it is the *group's* action, its operation).

(2) $ex = x$.

Example:

(1) S_n acts on $\{1, \ldots, n\}$ by $\Phi(\sigma, k) = \sigma(k)$ or with the compact notation: $\sigma k = \sigma(k)$. This is a group action since $\sigma \cdot (\tau \cdot x) = \sigma(\tau(x)) = (\sigma\tau)(x) = (\sigma\tau)x$.

(2) If G is a graph, $\mathrm{Aut}(G)$ acts on V.

(3) $\mathrm{GL}[n]\mathbb{F}$ acts on $\mathbb{F}^n$ by $\Phi(A, v) = Av$ (matrix multiplication).

Definition 8.1.3:

A group action of G on X is **faithful** if the homomorphism $\varphi\colon G \longrightarrow S_X$ is injective.

Proposition 8.1.4:

A group action is faithful if and only if for every $e \neq g \in G$, there is a $x \in X$ such that $gx \neq x$.

Proof:

Suppose a group action is faithful, then if $gx = x$ for every x, $\varphi(g)(x) = x$ for every x, so $\varphi(g) = \mathrm{id}$ and therefore $g = e$. To show the converse, suppose $\varphi(g) = \mathrm{id}$ then $gx = \varphi(g)(x) = x$ for every $x \in X$, so $g = e$ and therefore φ is injective.

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Theorem 8.1.5 (Cayley's Theorem):

If G is a group $G \hookrightarrow S_G$. Specifically G is isomorphic to a subgroup of S_G.

Proof:

We will show this by showing that there is a faithful group action of G on G. We define this group action by $\Phi(g,h) = gh$. We claim this is a group action:

(1) $\Phi(g, \Phi(h,k)) = g(hk) = (gh)k = \Phi(gh,k)$.

(2) $\Phi(e,g) = eg = g$.

And we know claim it is faithful: if $gh = h$ then $g = e$, so if $\Phi(g,h) = h$ then $g = e$. (Notice that this is a stronger claim than the group action being faithful, for any $g \neq e$ than $gh \neq h$ for *any* $h \in G$, such an action is called *free*). Since the action is faithful, its induced homomorphism is injective.

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Note that the monomorphism $G \hookrightarrow S_G$ is given by $\big(\varphi(g)\big)(h) = gh$. Thus if $G \subseteq \{1,\dots,n\}$ then if $\varphi(k) = \sigma_k$, $\sigma_k(j) = k \circ j$. So if $G = \mathbb{Z}_n$ then $\sigma_k(j) = k + j$, etc.

Example:

We will define a monomorphism $\mathrm{Euler}(9) \hookrightarrow S_9$ by:

$$\begin{aligned}
1 &\mapsto \mathrm{id}\\
2 &\mapsto \begin{pmatrix}1 & 2 & 4 & 8 & 7 & 5\end{pmatrix}\\
4 &\mapsto \begin{pmatrix}1 & 4 & 7\end{pmatrix}\begin{pmatrix}2 & 8 & 5\end{pmatrix}\\
5 &\mapsto \begin{pmatrix}1 & 5 & 7 & 8 & 4 & 2\end{pmatrix}\\
6 &\mapsto \begin{pmatrix}1 & 5 & 7 & 8 & 4 & 2\end{pmatrix}\\
7 &\mapsto \begin{pmatrix}1 & 7 & 4\end{pmatrix}\begin{pmatrix}2 & 5 & 8\end{pmatrix}\\
8 &\mapsto \begin{pmatrix}1 & 8\end{pmatrix}\begin{pmatrix}2 & 7\end{pmatrix}\begin{pmatrix}4 & 5\end{pmatrix}
\end{aligned}$$

Definition 8.1.6:

Suppose G acts on X, then the **orbit** of $x_0 \in X$ is:

$$G \cdot x_0 = \{gx_0 \mid g \in G\}$$

The **stabilizer** of x_0 is:

$$G_{x_0} = \{g \in G \mid g \cdot x_0 = x_0\} \leq G$$

This is a subgroup since if gx_0 and $hx_0 = x_0$ then $gh(x_0) = g(x_0) = x_0$ so $gh \in G_{x_0}$, and if $g \in G_{x_0}$ then $(g^{-1}g)x_0 = ex_0 = x_0$, but $(gg^{-1})x_0 = g^{-1}(gx_0) = g^{-1}x_0 = x_0$.

Proposition 8.1.7:

The set of orbits partition X.

Proof:

Suppose $y \in G \cdot x$ then $y = gx$ so if $g'y \in Gy$ then $g'y = g'gx \in Gx$, so $Gy \subseteq Gx$, and by symmetry $Gx = Gy$. And since $x \in Gx$, the orbits partition X.

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Proposition 8.1.8:

The stabilizer of an element $x \in X$ is a subgroup of G.

Proof:

Firstly, by definition $e \in G_x$. If $g, h \in G_x$ then $(gh) \cdot x = g \cdot (h \cdot x) = g \cdot x = x$ and so $gh \in G_x$. And finally if $g \in G_x$, then $g^{-1} \cdot x = g^{-1} \cdot (g \cdot x) = (g^{-1}g) \cdot x = x$ so $g^{-1} \in G_x$ and so G_x is a subgroup of G.

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Proposition 8.1.9:

$|G \cdot x_0| = \left[G : G_{x_0}\right]$

Proof:

We will map $g \cdot G_{x_0}$ to $g \cdot x_0$. This is well defined: if $g \cdot G_{x_0} = g' \cdot G_{x_0}$ then $g^{-1}g' \in G_{x_0}$ so $g^{-1}g'x_0 = x_0$ so $g'x_0 = gx_0$. This is surjective since every point in the orbit is of the form $g \cdot x_0$ and the image of $g \cdot G_{x_0}$ is gx_0. This is injective since if $g \cdot x_0 = g' \cdot x_0$ then $g^{-1}g' \in G_{x_0}$ so $g' \in g \cdot G_{x_0}$ and therefore since cosets partition G, $g' \cdot G_{x_0} = g \cdot G_{x_0}$ as required.

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