Complex Functions

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2.1 Complex Functions

Definition 2.1.1:

A complex series $\sum_{n=1}^{\infty} z_n$ **converges** to s if the sequence of partial sums:

$$s_n = \sum_{k=1}^{n} z_n$$

converges to s.
A complex series $\sum_{n=1}^{\infty} z_n$ **absolutely converges** if the real series $\sum_{n=1}^{\infty} |z_n|$ converges.

Proposition 2.1.2:

A complex series $\sum_{n=1}^{\infty} z_n$ converges to $a+bi$ if and only if $\sum_{n=1}^{\infty} \mathrm{Re}(z_n)$ converges to a and $\sum_{n=1}^{\infty} \mathrm{Im}(z_n)$ converges to b.
Specifically, a complex series converges if and only if its real and imaginary parts both converge.

Proof:

Notice that by linearity:

$$\mathrm{Re}(s_n) = \sum_{k=1}^{n} \mathrm{Re}(z_n), \qquad \mathrm{Im}(s_n) = \sum_{k=1}^{n} \mathrm{Im}(z_n)$$

and since s_n converges to $a+bi$ if and only if $\mathrm{Re}(s_n)$ converges to a and $\mathrm{Im}(s_n)$ converges to b, we have finished.

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Proposition 2.1.3:

If a complex series absolutely converges, it converges.

Proof:

Since

$$\sum_{n=1}^{\infty} |z_n| = \sum_{n=1}^{\infty} \sqrt{\mathrm{Re}(z_n)^2 + \mathrm{Im}(z_n)^2}$$

Since $|\mathrm{Re}(z_n)|, |\mathrm{Im}(z_n)| \leq \sqrt{\mathrm{Re}(z_n)^2 + \mathrm{Im}(z_n)^2}$, since these are nonnegative sequences, both $\sum_{n=1}^{\infty} |\mathrm{Re}(z_n)|$ and $\sum_{n=1}^{\infty} |\mathrm{Im}(z_n)|$ converge. Since absolute convergence implies convergence in $\mathbb{R}$, this means that the sums of $\mathrm{Re}(z_n)$ and $\mathrm{Im}(z_n)$ converge, and by above this means that the series converges.

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Note:

The topological definitions on $\mathbb{C}$ are equivalent to the topological definitions on $\mathbb{R}^2$. Eg. $B_r(z) = \{w \in \mathbb{C}\} |z-w| < r$, but balls are referred to as **disks** and denoted $D_r(z)$. The open sets is the topology defined by the open disks, and so on.
One final note is that an open connected set is called a **domain**. This is equivalent to being open and polygonal connected.

Notice that if we have a function $f\colon \mathbb{C} \longrightarrow \mathbb{C}$, we can define $u(x,y) = \mathrm{Re}\big(f(x+iy)\big)$ and $v(x,y) = \mathrm{Im}\big(f(x+iy)\big)$ then $f(x+iy) = u(x,y) + i\cdot v(x,y)$. So a function $\mathbb{C} \longrightarrow \mathbb{C}$ is equivalent in a sense to two functions $\mathbb{R}^2 \longrightarrow \mathbb{R}$, this shouldn't be surprising since we can generalize this to any function $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ as we have in infinitesimal calculus 3.
Notice then that f is continuous if and only if both u and v are. If u and v are, this is trivial by arithmetic of continuous functions. If f is continuous then this follows directly from the equivalence of complex and pointwise convergence of sequences (and thus functions).

Definition 2.1.4:

We say that $f \in C^n(E)$ for $E \subseteq \mathbb{C}$ if $u, v \in C^n(\tilde{E})$ where $\tilde{E} = \{(x,y) \mid x+iy \in E\} \subseteq \mathbb{R}^2$.

Definition 2.1.5:

A sequence of complex functions $\{f_n\}_{n=1}^\infty$ **converges uniformly** to a complex function f on a domain D if for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for every $n \geq N$ and $z \in D$

$$|f_n(z) - f(z)| < \varepsilon$$

Proposition 2.1.6:

f_n uniformly converges to f if and only if $\sup_{z\in D}(|f_n(z) - f(z)|) \xrightarrow[n\to\infty]{} 0$.

This is simple since for every $\varepsilon > 0$ there must be an N such that for every $n \geq N$ and for every $z \in D$: $|f_n(z) - f(z)| \leq \sup_{z\in D}(|f_n(z) - f(z)|) < \varepsilon$.

Proposition 2.1.7:

If f_n are all continuous and uniformly converge to f, then f is also continuous.

Theorem 2.1.8 (Weierstrass M Test):

Suppose $\{f_n\}_{n=1}^\infty$ are complex functions such that there exists numbers M_n such that for every n, $|f_n(z)| \leq M_n$ for every $z \in D$ and $\sum_{n=1}^\infty M_n$ converges, then $\sum_{n=1}^\infty f_n$ converges absolutely and uniformly on D.

2.2 Stereographical Projection

We define the boundary of the ball $B_{\frac{1}{2}}\big(0,0,\frac{1}{2}\big)$ by Σ, ie

$$\Sigma = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4} \right\}$$

And we define Σ_0 to be Σ without its "northern point" $(0,0,1)$. And we define a projection from Σ_0 to $\mathbb{C}$

$$\pi\colon \Sigma_0 \longrightarrow \mathbb{C}$$

where $\pi(u,v,w)$ is defined to be the (unique) point on $\mathbb{C} \cong \big\{(x,y,0) \in \mathbb{R}^3\big\}$ which is also on the line which passes through $(0,0,1)$ and (u,v,w). This line is given by

$$(0,0,1) + t\big((u,v,w) - (0,0,1)\big)$$

and so this is equal to $(x,y,0)$ when $1 + t(w-1) = 0$ and so $t = \frac{1}{1-w}$, thus

$$\pi(u,v,w) = \frac{u}{1-w} + i\frac{v}{1-w}$$

this a bijection, it is obviously surjective and we can see why geometrically this is injective. If two lines starting from the same point intersect then they must be the same line:

$$v + t(v-u) = v + t'(v-u') \implies t(v-u) = t'(v-u') \implies v - u' = \alpha(v-u)$$

So the lines are equal and since Σ_0 is on a sphere, it this would mean $v - u' = v - u$ (this is not a formal proof).
We can extend the projection to $\pi\colon \Sigma \longrightarrow \mathbb{C} \cup \{\infty\}$ where $\pi(0,0,1) = \infty$, and this is still a bijection, so there is an inverse projection π^{-1}.

Definition 2.2.1:

A sequence $\{z_n\}_{n=1}^{\infty} \in \mathbb{C}$ converges/diverges to ∞ if $|z_n| \xrightarrow[n\to\infty]{} \infty$.

A neighborhood of $(0,0,1)$ in Σ is an intersection of a neighborhood of $(0,0,1)$ in $\mathbb{R}^3$ with Σ. And a neighborhood of ∞ in $\mathbb{C} \cup \{\infty\}$ is an image of a neighborhood of $(0,0,1)$ in Σ under π. And a *circle* in Σ is an intersection of a hyperplane in $\mathbb{R}^3$ $(Ax + By + Cz = D)$ with Σ.

Proposition 2.2.2:

If S is a circle in Σ, then if $(0,0,1) \in S$, $\pi(0,0,1)$ is a plane. Otherwise $\pi(0,0,1)$ is a circle.

The stereographical projection is useful for some reason.

2.3 Complex Derivatives

Definition 2.3.1:

Suppose $f = u + iv$ is a complex function then its **partial derivatives** are:

$$f_x = u_x + iv_x, \qquad f_y = u_y + iv_y$$

The alternative notations used in Infinitesimal Calculus 3 are used as well.
And its **complex derivative** at $z \in \mathbb{C}$ is:

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

If this limit exists, then f is called **differentiable** at z.

It is simple to see why the usual results of differentiation hold (derivatives of sums and products and scalings) with complex derivatives as well.
Notice that if f is differentiable at $z = x + iy$ then taking the path $h \to$ where $h \in \mathbb{R}$ then we get that $f'(z) = \lim_{h\to 0} \frac{f(x+h+iy)-f(x+iy)}{h}$, but:

$$f'(z) = \lim_{h\to 0} \frac{f(x+h+iy) - f(x+iy)}{h} = \lim_{h\to 0} \frac{u(x+h,y) + iv(x+h,y) - u(x,y) - iv(x+h,y)}{h} = u_x(x,y) + iv_x(x,y)$$

The final equality is due to convergence in $\mathbb{C}$ being equivalent to pointwise convergence (of the real and complex parts). So $u_x(x,y)$ and $v_x(x,y)$ exist and $f'(z) = u_x(x,y) + iv_x(x,y)$. And if we take $h \in \mathbb{R}$ then notice that $f(z + ih) = f(x + i(y+h)) = u(x, y+h) + iv(x, y+h)$ and so:

$$f'(z) = \lim_{h\to 0} \frac{u(x,y+h) - u(x,y) + i\big(v(x,y+h) - v(x,y)\big)}{ih} = -i(u_y(x,y) + iv_y(x,y)) = v_y(x,y) - iu_y(x,y)$$

So if $f'(z)$ exists then so does $u_y(x,y)$ and $u_y(x,y)$ and satisfies $f'(z) = v_y(x,y) - iu_y(x,y)$. Thus we get the following result:

Proposition 2.3.2:

If f is differentiable at $z \in \mathbb{C}$ then its derivative satisfies:

$$f'(z) = u_x(x,y) + iv_x(x,y) = v_y(x,y) - iu_y(x,y)$$

and specifically

$$u_x(x,y) = v_y(x,y), \qquad v_x(x,y) = -u_y(x,y)$$

Example 2.3.3:

The derivative of $f(z) = \bar{z}$ does not exist at any $z \in \mathbb{C}$. The derivative at z is equal to:

$$\lim_{h\to} \frac{\overline{z+h} - \overline{z}}{h} = \lim \frac{\overline{h}}{h}$$

This limit does not exist, since if we take $h \in \mathbb{R}$ it equals 1 but if we take $h \in i\mathbb{R}$ this equals -1. And in general if $h = re^{i\theta}$ then $\frac{\overline{h}}{h} = e^{i2\theta} = \cos(2\theta) + i\sin(2\theta)$, so this isn't even dependent on r and the limit doesn't exist.

Proposition 2.3.4:

If f is differentiable at z_0 and g is differentiable at $f(z_0)$ then $h = g \circ f$ is differentiable at z_0 and satisfies:

$$h'(z_0) = g\big(f(z_0)\big) \cdot f(z_0)$$

Proof:

Note that a function f is differentiable at z_0 if and only if there exists a function $\varepsilon\colon \mathbb{C} \longrightarrow \mathbb{C}$ and a value $f'(z_0)$ such that:

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + \varepsilon(z - z_0)$$

where $\frac{\varepsilon(h)}{h} \xrightarrow[h\to 0]{} 0$. This is trivial and is very reminiscent of infinitesimal calculus 3.
And so we have ε_1 and ε_2 where:

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + \varepsilon_1(z - z_0), \qquad g(z) = g\big(f(z_0)\big) + \big(z - f(z_0)\big)g'\big(f(z_0)\big) + \varepsilon_2\big(z - f(z_0)\big)$$

And we need to find an ε_3 such that

$$g \circ f(z) = g \circ f(z_0) + (z - z_0)\Bigg(f'(z_0) \cdot g'\big(f(z_0)\big)\Bigg) + \varepsilon_3(z - z_0)$$

So then:

$$\begin{aligned} g \circ f(z) &= g\big(f(z_0)\big) + \big(f(z) - f(z_0)\big)g'\big(f(z_0)\big) + \varepsilon_2\big(f(z) - f(z_0)\big) \\ &= g \circ f(z_0) + (z - z_0)\Bigg(f'(z_0) \cdot g'\big(f(z_0)\big)\Bigg) + \varepsilon_1(z - z_0)g'\big(f(z_0)\big) + \varepsilon_2\big((z - z_0)f'(z_0) + \varepsilon_1(z - z_0)\big) \end{aligned}$$

So we define

$$\varepsilon_3(h) = \varepsilon_1(h) \cdot g'\big(f(z_0)\big) + \varepsilon_2\big(hf'(z_0) + \varepsilon_1(h)\big)$$

And we claim that $\frac{\varepsilon_3(h)}{h}$ converges to 0 as h approaches 0. This is simple for the $\varepsilon_1 \ldots$ part, let us look at the ε_2 part:

$$\frac{\varepsilon_2\big(hf'(z_0) + \varepsilon_1(h)\big)}{h} = \frac{\varepsilon_2\Bigg(h\Big(f'(z_0) + \frac{\varepsilon_1(h)}{h}\Big)\Bigg)}{h\Big(f'(z_0) + \frac{\varepsilon_1(h)}{h}\Big)}\Bigg(f'(z_0) + \frac{\varepsilon_1(h)}{h}\Bigg)$$

Which converges to 0 (the left converges to 0 by the characteristic of ε_2 and the right converges to $f'(z_0)$), as required.

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