Modern Analysis

Homework 2
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2.1 Exercise

Determine if the following collections are alegbras or σ-algebras:

(1) $X = \{1, 2, 3, 4\},\ S = \{\varnothing, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3, 4\}\}$

(2) $X = [0, 1],\ S = \{\varnothing\} \cup \{A \mid [0, 1/2] \subseteq A\}$

(3) $X = [0, 1],\ S = \{A \mid \{0, 1\} \subseteq A \text{ or } \{0, 1\} \cap A = \varnothing\}$

(4) $X = \{0, 1\}^{\mathbb{N}}$ and $A \in S$ if and only if every $(x_n) \in A$ satisfies that $\{(y_n) \mid y_1 = x_1\} \subseteq A$.

(1) This is not an algebra since $\{1\}^c = \{2, 3, 4\}$ is not in S.

(2) This is not an algebra since $[0, 1/2] \in S$ but $[0, 1/2]^c \notin S$.

(3) This is a σ-algebra: $\varnothing \in S$ since it has an empty intersection with $\{0, 1\}$, $A \cap \{0, 1\} = \varnothing$ if and only if $\{0, 1\} \subseteq A^c$ so S is closed under complements. Finally if $\{A_n\}_{n=1}^\infty \subseteq S$, if there exists any n such that $\{0, 1\} \subseteq A_n$ then $\{0, 1\} \subseteq \bigcup_n A_n$ so $\bigcup_n A_n \in S$. Otherwise $A_n \cap \{0, 1\} = \varnothing$ for every n, so $(\bigcup_n A_n) \cap \{0, 1\} = \bigcup_n (A_n \cap \{0, 1\}) = \varnothing$, so $\bigcup_n A_n \in S$ as well.

(4) This is a σ-algebra since $\varnothing \in S$ vaccuously. If $A \in S$, then let $(x_n) \in A^c$, if $y_1 = x_1$ then if $(y_n) \in A$, since $y_1 = x_1$ we'd have $(x_n) \in A$ in contradiction, so $(y_n) \in A^c$, meaning $A^c \in S$. Now suppose $\{A_n\}_{n=1}^\infty \subseteq S$, let $(x_k)_k \in \bigcup A_n$, then $(x_k)_k \in A_n$ for some n, and if $y_1 = x_1$ then $(y_k)_k \in A_n \subseteq \bigcup_n A_n$, meaning $\{(y_k)_k \mid y_1 = x_1\} \subseteq \bigcup_n A_n$, so $\bigcup_n A_n \in S$ as required.

2.2 Exercise

Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of continuous functions on $[0, 1]$. Prove that

$$A = \{x \in [0, 1] \mid f_n(x) \to 0\}$$

is a Borel set.

By definition $f_n(x) \to 0$ if and only if for every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $|f_n(x)| < \varepsilon$, meaning $x \in f_n^{-1}(-\varepsilon, \varepsilon)$. Since the rationals are dense, we can take only rational εs. Therefore

$$A = \bigcap_{\varepsilon\in\mathbb{Q}_+} \bigcup_{n_0\in\mathbb{N}} \bigcap_{n=n_0}^{\infty} f_n^{-1}(-\varepsilon, \varepsilon)$$

Since f_n is continuous, $f_n^{-1}(-\varepsilon, \varepsilon)$ is an open set for every $\varepsilon > 0$ and n. So this is a composition of intersections and unions of open sets and is therefore a Borel set.

2.3 Exercise

Let m be the Lebesgue measure and $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of measurable sets in $[0, 1]$. Define $F = \{x \mid \forall n \in \mathbb{N} \exists k > n \colon x \in A_k\}$. Prove

(1) $F = \bigcap_{n=1}^\infty \bigcup_{k=n}^\infty A_k$,

(2) if $m(A_n) \geq \delta > 0$ for all n then $m(F) \geq \delta$,

(3) if $\sum_{n=1}^{\infty} m(A_n) < \infty$ then $m(F) = 0$,

(4) there exists a sequence $\{A_n\}$ such that $\sum_{n=1}^{\infty} m(A_n) = \infty$ and $m(F) = 0$.

(1) $x \in F$ if and only if for every n there exists a $k > n$ such that $x \in A_k$. This is obviously contained in $\bigcap_n \bigcup_{k \geq n} A_k$, and if $x \in \bigcap_n \bigcup_{k \geq n} A_k$ then for every n there exists a $k \geq n$ such that $x \in A_k$. So for any n, there exists a $k \geq n+1 > n$ such that $x \in A_k$, so $x \in F$.

(2) By continuity from above, since $\left\{\bigcup_{k \geq n} A_k\right\}_n$ is a decreasing sequence,

$$m(F) = \lim_{n \to \infty} m\left(\bigcup_{k \geq n} A_k\right) \geq \lim_{n \to \infty} m(A_n) \geq \lim_{n \to \infty} \delta = \delta$$

(3) Again by continuity from above,

$$m(F) = \lim_{n \to \infty} m\left(\bigcup_{k \geq n} A_k\right) \leq \lim_{n \to \infty} \sum_{k \geq n} m(A_k)$$

since $\sum_{k=1}^{\infty} m(A_k)$ converges, its tail must converge to zero, meaning $\lim_{n \to \infty} \sum_{k \geq n} m(A_k) = 0$, so $m(F) = 0$.

(4) Define $A_n = (0, 1/n)$, then $m(A_n) = \frac{1}{n}$ so its sum diverges. But $F = \varnothing$ since if $x \in F$, then eventually $\frac{1}{n} < x$ so for every $k \geq n$, $x \notin A_k$. So $m(F) = 0$ as required.

2.4 Exercise

Let $\mathcal{C}$ be the Cantor set, prove it

(1) is compact,

(2) does not contain any interval of positive measure,

(3) is not the countable union of closed intervals.

(1) Recall the definition of the Cantor set, defining $C_0 := [0,1]$ and $C_{n+1} := \frac{1}{3}C_n \bigcup \left(\frac{2}{3} + \frac{1}{3}C_n\right)$,

$$\mathcal{C} := \bigcap_{n=1}^{\infty} C_n$$

Since all C_n are closed (by induction: C_0 is closed, then C_{n+1} is the union of two closed sets which is also closed), $\mathcal{C}$ is the countable intersection of closed sets, and is therefore also closed. Therefore $\mathcal{C}$ is closed and bound, meaning it is compact.

(2) By induction, $m(C_n) = \left(\frac{2}{3}\right)^n$, since

$$m(C_{n+1}) = \frac{1}{3}m(C_n) + \frac{1}{3}m(2 + C_n) = \frac{2}{3}m(C_n) = \left(\frac{2}{3}\right)^{n+1}$$

And so by continuity from above,

$$m(\mathcal{C}) = \lim_{n \to \infty} m\left(\bigcap_{k=1}^{n} C_k\right) \leq \lim_{n \to \infty} m(C_n) = \lim_{n \to \infty} \left(\frac{2}{3}\right)^n = 0$$

so $\mathcal{C}$ has zero measure, and therefore cannot contain any set of positive measure.

(3) Since $\mathcal{C}$ contains no set of positive measure, it cannot contain any interval and in particular cannot be the union of intervals.

2.5 Exercise

Let X be a set and $\mathcal{E} \subseteq \mathcal{P}(X)$. Show that for every $F \in \sigma(\mathcal{E})$, there exists a countable subset $\mathcal{E}' \subseteq \mathcal{E}$ such that $F \in \sigma(\mathcal{E}')$.

This is equivalent to showing that

$$\sigma(\mathcal{E}) = \bigcup_{\mathcal{E}' \subseteq \mathcal{E} \text{ countable}} \sigma(\mathcal{E}')$$

Since $\sigma(\mathcal{E}') \subseteq \sigma(\mathcal{E})$ for every $\mathcal{E}' \subseteq \mathcal{E}$, $\supseteq$ is trivial. Now we claim that the right hand side (denote it $\mathcal{U}$) is a σ-algebra: it obviously contains $\varnothing$ and if $A \in \mathcal{U}$ then $A \in \sigma(\mathcal{E}')$ for some countable $\mathcal{E}' \subseteq \mathcal{E}$ and so $A^c \in \sigma(\mathcal{E}') \subseteq \mathcal{U}$. And if $\{A_n\}_n \in \mathcal{U}$, then for every n there exists a countable $\mathcal{E}_n \subseteq \mathcal{E}$ such that $A_n \in \sigma(\mathcal{E}_n)$. Then define $\mathcal{E}' := \bigcup_n \mathcal{E}_n$ which is the countable union of countable sets, so $\mathcal{E}'$ is countable. And for every n, $A_n \in \sigma(\mathcal{E}_n) \subseteq \sigma(\mathcal{E}')$, so $\bigcup_n A_n \in \sigma(\mathcal{E}') \subseteq \mathcal{U}$. Thus $\mathcal{U}$ is a σ-algebra containing $\mathcal{E}$, since for every $A \in \mathcal{E}$, $A \in \sigma(\{A\})$, thus $\sigma(\mathcal{E}) \subseteq \mathcal{U}$ and so $\sigma(\mathcal{E}) = \mathcal{U}$ as required.