Linear Algebra 2, Homework 8 Solution

Exercise 1

Define an inner product on $\mathbb{R}^3$ by

$$\left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\rangle = x(a-b) + y(-a+2b-c) + z(-b+2c)$$

(1) Find an orthonormal basis for the space.

(2) Find the orthogonal complement of $(0,1,0)$ and compute the projection of $(1,2,3)$ onto it.

(1) To make the second part easier, we will start with the basis $\{(0,1,0),(1,0,0),(0,0,1)\}$. So we set $w_0 = (0,1,0)$, which has a squared norm of 2, so

$$w_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \pi_{w_0}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{2}\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

And w_1 has a squared norm of $1(1-0.5)+0.5(-1+1) = 0.5$, so

$$\begin{aligned} w_2 &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \pi_{w_0,w_1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 2\left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{1}{2} \\ 0 \end{pmatrix} \right\rangle \begin{pmatrix} 1 \\ \frac{1}{2} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ \frac{1}{2} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \end{aligned}$$

So we have an orthogonal basis

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

normalizing gives

$$\left\{ \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ 1/2\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

Alternatively, had we started with the standard basis in its canonical order then we would've gotten

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

(2) By the previous subquestion, we already have a basis for $(0,1,0)^\perp$, so

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^\perp = \operatorname{span}\left(\begin{pmatrix} 1 \\ \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right)$$

Notice that

$$\left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\rangle = 0$$

so it is already in the orthogonal complement, thus $\pi(1,2,3) = (1,2,3)$. ◊

Exercise 2

In recitation we defined

$$B_1 = \{u \in V \mid \|u\| = 1\}, \qquad \hat{v} = \frac{v}{\|v\|}$$

And we said that for all $v \neq 0$, $\min_{u \in B_1} \|u - v\|$ is obtained when $u = \hat{v}$. Now prove that this vector is unique, i.e.

$$\|v - \hat{v}\| = \|v - u\| \implies u = \hat{v}$$

By the recitation, we take $u \in B_1$ and we get that

$$\|v - u\|^2 = \|v\|^2 - 2\operatorname{Re}\langle v, u\rangle + 1$$

And

$$\|v - \hat{v}\|^2 = \|v\|^2 - 2\|v\| + 1$$

these are equal only when

$$\operatorname{Re}\langle v, u\rangle = \|v\|$$

Now, we have the chain of inequalities (from the recitation):

$$\operatorname{Re}\langle v, u\rangle \leq |\langle v, u\rangle| \leq \|v\|\|u\| = \|v\|$$

In order for this to be an equality, we must have $|\langle v, u\rangle| = \|v\|\|u\|$, which by Cauchy-Schwarz occurs only when v, u are linearly dependent. Thus $u = \alpha v$. Now we must also have

$$\operatorname{Re}\langle v, u\rangle = |\langle v, u\rangle| \iff \|v\|^2 \operatorname{Re}(\alpha) = \|v\|^2 |\alpha| \iff \operatorname{Re}(\alpha) = |\alpha|$$

This occurs only when $\alpha \in \mathbb{R}$ and $\alpha > 0$. Furthermore, since $\|u\| = 1$, we have $|\alpha| = \frac{1}{\|v\|}$, so $\alpha = \frac{1}{\|v\|}$. $\Diamond$

Exercise 3

Let V be an inner product space of dimension and T a linear operator over it such that for all $v \in V$: $\langle v, Tv\rangle = 0$. Prove or disprove:

(1) If T is invertible, it has no eigenvalues.

(2) If T's characteristic polynomial splits (into linear factors), then T is nilpotent.

(3) If T is not invertible, then T is nilpotent.

(4) If n is odd, then T is singular (not invertible).

(1) **True**: Suppose T did have an eigenvalue λ, so $Tv = \lambda v$ for some $v \neq 0$. Then $\langle Tv, v\rangle = \lambda\langle v, v\rangle = \lambda\|v\|^2$. This is zero, so $\lambda = 0$ in contradiction. Notice that this shows that the only eigenvalue of T is 0.

(2) **True**: Since T's only eigenvalue is 0, and its characteristic polynomial splits, its characteristic polynomial must be $p_T(x) = x^n$. By Cayley-Hamilton, we have $0 = p_T(T) = T^n$, so T is nilpotent.

(3) **False**: Take T over $\mathbb{R}$ with a characteristic polynomial of $x(x^2+1) = x^3 + x$. A matrix with this property is

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Notice that

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ z \\ -y \end{pmatrix}$$

which is orthogonal to (x, y, z) as required.

(4) **True**: Suppose T is invertible, then it has no eigenvalues by (1). So all of the roots of $p_T(x)$ must be non-real. But non-real roots come in pairs, so the degree of $p_T(x)$ must be even (alternatively, all odd-degree polynomials have a real root). $\Diamond$

Exercise 4

Let $\langle \bullet, \bullet \rangle_1$ and $\langle \bullet, \bullet \rangle_2$ be two inner products over V. Let $\|\bullet\|_1, \|\bullet\|_2$ be their respective induced norms. Show that there exists a $c > 0$ such that for all $v \in V$:

$$\|v\|_1 \leq c\|v\|_2$$

Let $E = \{e_1, \ldots, e_n\}$ be an orthonormal basis with respect to $\langle \bullet, \bullet \rangle_2$. Now take $v \in V$, suppose $v = \sum_i \alpha_i e_i$, then by Pythagoras:

$$\|v\|_2^2 = \sum_i |\alpha_i|^2$$

and

$$\|v\|_1^2 = \left\| \sum_i \alpha_i e_i \right\|^2 \leq \left(\sum_i |\alpha_i| \|e_i\|_1 \right)^2$$

Let $M = \max_{1 \leq i \leq n} \|e_i\|_1$ so that

$$\|v\|_1^2 \leq M^2 \left(\sum_i |\alpha_i| \right)^2$$

We know by Cauchy-Schwarz (we showed this in recitation) that $(a_1 + \cdots + a_n)^2 \leq n(a_1^2 + \cdots + a_n^2)$, so

$$\|v\|_1^2 \leq M^2 n \sum_i |\alpha_i|^2 = M^2 n \|v\|_2^2$$

So we take $c = M\sqrt{n}$ and we have the desired result. $\Diamond$

Exercise 5

Moshe wants to find a correspondence between the number of hours he studies for a test (P_1), the amount of homework he solved (P_2), and the number of books he read (P_3). He put the data in a table:

	P_1	P_2	P_3	Grade
1	4	2	3	7
2	2	3	3	4
3	4	4	5	8
4	2	5	5	6

Moshe wanted to find values x_1, x_2, x_3 which satisfy

$$x_1 P_1 + x_2 P_2 + x_3 P_3 = \text{Grade}$$

unfortunately no such solution exists, so instead he decided to find a solution to

$$\begin{aligned} 4x_1 + 2x_2 + 3x_3 &= b_1' \\ 2x_1 + 3x_2 + 3x_3 &= b_2' \\ 4x_1 + 4x_2 + 5x_3 &= b_3' \\ 2x_1 + 5x_2 + 5x_3 &= b_4' \end{aligned}$$

which is closest (in norm) to $b = (7, 4, 8, 6)$.

We need $b' \in C(A)$ to be closest to b, where

$$A = \begin{pmatrix} 4 & 2 & 3 \\ 2 & 3 & 3 \\ 4 & 4 & 5 \\ 2 & 5 & 5 \end{pmatrix}$$

Notice that indeed $Ax = b$ has no solution (using row reduction). So we just want to find $b' = \pi_{C(A)}(b)$. So we must find an orthogonal basis for $C(A)$. Using row reduction we can see that A's columns are linearly independent, so we start by applying the Gram-Schmidt process to A's columns. Let v_1, v_2, v_3 be the columns of A, then

$$w_1 = v_1 = \begin{pmatrix} 4 \\ 2 \\ 4 \\ 2 \end{pmatrix}$$

$$w_2 = v_2 - \pi_{w_1}(v_2) = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 3 \end{pmatrix}$$

$$w_3 = v_3 - \pi_{w_1, w_2}(v_3) = v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 = \frac{1}{35} \begin{pmatrix} -3 \\ -9 \\ 7 \\ 1 \end{pmatrix}$$

Now, all we need to do is compute $b' = \pi_{C(A)}(b) = \pi_{w_1, w_2, w_3}(b)$. This is just

$$b' = \frac{\langle b, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 + \frac{\langle b, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 + \frac{\langle b, w_3 \rangle}{\langle w_3, w_3 \rangle} = \frac{1}{4} \begin{pmatrix} 27 \\ 17 \\ 33 \\ 23 \end{pmatrix}$$

And solving $Ax = b'$ gives

$$x = \begin{pmatrix} 1 \\ -1/2 \\ 5/4 \end{pmatrix}$$

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Exercise 6

Let $H \in \mathbb{R}^{m \times n}$ whose columns are linearly independent.

(1) Prove that $H^\top H$ is invertible.

(2) Given the linear system $Hx = b$, $\tilde{x} = (H^\top H)^{-1} H^\top b$ is called the **least squares** (LSQ). Show that

$$\min_{x \in \mathbb{R}^n} \left\{ \sum_{i=1}^{m} \left(b_i - (Hx)_i \right)^2 \right\}$$

is obtained at $\tilde{x}$.

(1) We will show that $N(H^\top H) = N(H)$. Obviously if $Hx = 0$ then $H^\top H x = 0$. Conversely, if $H^\top H x = 0$ then $x^\top H^\top H x = (Hx)^\top Hx = \|Hx\|^2 = 0$, thus $Hx = 0$. Thus $r(H^\top H) = r(H) = n$ since H's columns are linearly independent, thus $H^\top H$ has full rank and is therefore invertible.

(2) Notice that we are trying to minimize $\|b - Hx\|^2$, which is obtained when $Hx = \pi_{C(H)}(b)$. So we want to show that $H\tilde{x} = \pi_{C(H)}(b)$, that is:

$$H(H^\top H)^{-1} H^\top b = \pi_{C(H)}(b)$$

Let us define $\tilde{b} = H(H^\top H)^{-1} H^\top b$, and we can see that $\tilde{b} \in C(H)$. Now all that remains to show is that $b - \tilde{b} \in C(H)^\perp = N(H^\top)$ (we showed this in recitation). So

$$H^\top \tilde{b} = H^\top H (H^\top H)^{-1} H^\top b = H^\top b$$

so $H^\top (b - \tilde{b}) = 0$ as required.