Infinitesimal Calculus 3

Lecture 18, Sunday January 8, 2023
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Recall the theorem from the previous lecture (to be typeset):

Theorem:

If f maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ and is in C^1 in some neighborhood of $x_0 \in \mathbb{R}^n$ then if $df\big|_{x_0}$ is invertible (as a linear transformation) then f maps a neighborhood of x_0 to a neighborhood of $f(x_0)$ bijectively.

Note that if we have a function $F(x, y(x))$ then $\frac{\partial}{\partial x}\big(F(x, y(x))\big) = \frac{\partial F}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = F_x + F_y \cdot \frac{dy}{dx}$. So if we require $F(x, y(x)) = 0$ then we get

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Proposition 18.1:

If $F(x, y) = 0$ and $F \in C^1$. Suppose $F(x_0, y_0) = 0$ and $F_y(x_0, y_0) \neq 0$ then y can be written as a C^1 function of x in a neighborhood of x_0.

Proof:

Let us focus on the following system:

$$\begin{cases} x = x \\ F(x, y) = 0 \end{cases}$$

We define $G(x, y) = \big(x, F(x, y)\big)$ then

$$J_G = \begin{pmatrix} 1 & F_x \\ 0 & F_y \end{pmatrix}$$

and since $\det J_G = F_y$ which is not equal to 0 at (x_0, y_0) we have that $J_G(x_0, y_0)$ is invertible and therefore G maps a neighborhood S of (x_0, y_0) to a neighborhood T of $(x_0, F(x_0, y_0))$ and has an inverse from T to S. We know that $G^{-1}(x, z) = (x, h(x, z))$ where $y = h(x, z)$ and $h \in C^1$ since G is. And since $F(x, y) = 0$ we have that $z = 0$ in the set that we are focusing on. And so $y = h(x, 0)$ which defines a C^1 single value function of x that is equal to y as required.

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Theorem 18.2:

Suppose $(F_1, \ldots, F_s) = F \colon \mathbb{R}^k \times \mathbb{R}^s \longrightarrow \mathbb{R}^s$ is in C^1 around $(x^0, y^0) = (x_1, \ldots, x_k, y_1, \ldots, y_s)$. Suppose $F(x^0, y^0) = 0$ and further suppose that

$$\det J_F = \frac{\partial(F_1, \ldots, F_S)}{\partial y_1, \ldots, y_s} \neq 0$$

Then there exists a neighborhood $I \subseteq \mathbb{R}^k$ of x^0 and a neighborhood $J \subseteq \mathbb{R}^s$ of y^0 such that for every $x \in I$ there is a unique $y = \varphi(x)$ which satisfies $F(x, y) = F(x, \varphi(x)) = 0$ and $\varphi \in C^1$.

Proof:

We focus on the system

$$\begin{cases} x_1 = x_1 \\ \vdots \\ x_k = x_k \\ F_1(x, y) = 0 \\ \vdots \\ F_s(x, y) = 0 \end{cases}$$

Which can be represented as

$$\begin{pmatrix} I & 0 \\ * & J_F \end{pmatrix}$$

which has determinant $J_f \neq 0$.