Infinitesimal Calculus 3

Lecture 9, Sunday November 20, 2022

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9.1 Continuity Continued

Notice that f is continuous at x if and only if for every $x_n \longrightarrow x$ (and there exists such a sequence, since we can take $x_n = x$), $f(x_n) \longrightarrow f(x)$. Suppose f is continuous at x, let $\varepsilon > 0$ and $\delta > 0$ satisfy continuity. Then since $\rho(x_n, x) \longrightarrow 0$, so for some N for every $n \geq N$, $\rho(x_n, x) < \delta$ and so $\rho(f(x_n), f(x)) < \varepsilon$. And therefore $f(x_n) \longrightarrow f(x)$. To show the converse, suppose that f isn't continuois at x. Then there is an $\varepsilon > 0$ such that every $\delta > 0$ doesn't satisfy continuity. So for $\frac{1}{n}$ there is a x_n such that $\rho(x_n, x) < \frac{1}{n}$ but $\rho(f(x_n), f(x)) \geq \varepsilon$. So while $x_n \longrightarrow x$, $f(x_n)$ doesn't converge to $f(x)$, in contradiction.

Proposition 9.1.1:

We define the function $\chi_k\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ by $f_k(x_1, \ldots, x_n) = x_k$. Then f_k is continuous.

Proof:

Suppose $x^m \longrightarrow x = (x_1, \ldots, x_n)$. Then we must have that $x_k^m \longrightarrow x_k$ since pointwise convergence in $\mathbb{R}^n$ is equivalent to convergence. And so $f_k(x^m) = x_k^m \longrightarrow x_k = f_k(x)$, and so f_k is continuous.

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Example:

The function $f(x, y, z) = z^3 e^{\sin z}$ is continuous since if we define $g(z) = z^3 e^{\sin z}$, we have that $f(x, y, z) = g(f_3(x, y, z))$. And g and f_3 are continuous, and the composition of continuous functions is continuous, so f is continuous.

Proposition 9.1.2:

If $f, g\colon E \longrightarrow \mathbb{R}$ are continuous at $p \in E$ and if $c \in \mathbb{R}$ then the following are also continuous at p:

$$\begin{gathered} f + g \\ f + cg \\ f \cdot g \\ \frac{f}{g} \quad \text{If } g(p) \neq 0 \end{gathered}$$

This is trivial to prove.
Also notice that $f\colon X \longrightarrow \mathbb{R}^n$ is continuous if and only if $\chi_k \circ f$ is continuous for every $1 \leq k \leq n$. This is due to a proposition we proved in the previous lecture.

Example:

Circles are continuous. We define the function:

$$f\colon [0, 2\pi) \longrightarrow \mathbb{R}, \qquad t \mapsto (\cos(t), \sin(t))$$

which is the parametric representation of the unit circle. Since both of the coordinate functions, $\chi_1 \circ f$ and $\chi_2 \circ f$ ($\cos t$ and $\sin t$ respectively) are continuous, so is f. Notice that this parametric representation is the *counter-clockwise* representation. The clockwise representation, $(\cos t, -\sin t)$ is also continuous.

9.2 Surfaces

Definition 9.2.1:

If $\vec{v}, \vec{n} \in \mathbb{R}^k$ if $\vec{n} \neq 0$, the **hyperplane** normal to $\vec{n}$ which contains $\vec{v}$ is defined as:

$$H_{v,n} = \left\{\vec{u} \in \mathbb{R}^k \;\middle|\; n \cdot (u - v) = 0\right\} = \left\{\vec{u} \in \mathbb{R}^k \;\middle|\; n \cdot u = n \cdot v\right\}$$

Notice that if $u \in H_{v,n}$ then $H_{v,n} = H_{u,n}$ and for every $\alpha \neq 0$, $H_{v,\alpha n} = H_{v,n}$. Further notice that the set $\left\{\vec{u} \in \mathbb{R}^k \;\middle|\; n \cdot u = d\right\}$ for any $d \in \mathbb{R}$ defines a hyperplane. And if $u \in \mathbb{R}^n$ then $H_{v,n}+u = \left\{u + w \in \mathbb{R}^k \;\middle|\; n \cdot (w - v) = 0\right\} = \left\{w \in \mathbb{R}^k \;\middle|\; n \cdot (w - v - u) = 0\right\} = H_{v+u,n}$. And if $0 \in H_{v,n}$ then $H_{v,n}$ represents a subspace of $\mathbb{R}^k$ whose dimension is $k-1$, and so $H_{v,n} - v = H_{0,n}$ is a $k-1$ dimension subspace of $\mathbb{R}^k$.

Definition 9.2.2:

If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a real-valued function, then its rotation around an axis, for example the z axis is defined as the set:

$$\left\{(x, y, f(\sqrt{x^2 + y^2})) \;\middle|\; x, y \in \mathbb{R}\right\}$$

If we intersect a rotation with some plane $z = c$ then we get the set of points (x, y, c) where $f(\sqrt{x^2 + y^2}) = c$. This is either empty or contains circles whose radii are in $f^{-1}(c)$.

Example:

If $f(x) = \alpha x$, then rotating it around z creates the set: $\left\{(x, y, \alpha\sqrt{x^2 + y^2}) \;\middle|\; x, y \in \mathbb{R}\right\}$ which is called a **cone**. And for $f(x) = \alpha x^2$, rotating it gives $z = \alpha(x^2 + y^2)$, this is called an elliptical parabola.

Example:

If we look at $f(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2}$, $z = f(x, y)$. Cutting this at $z = c$ yields an ellipse, and thus this is not a rotation. Cutting this at $y = 0$ gives $z = \frac{x^2}{a^2}$ which is a parabola.