Infinitesimal Calculus 3

Lecture 24, Wednsday January 25, 2023
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Definition 24.1:

Alternative notation for the integral of a multivariable function $f(x_1, \dots, x_n)$ over $D \subseteq \mathbb{R}^n$ is

$$\int \cdots \int_D f(x_1, \dots, x_n)\, dx_1 \dots dx_n$$

Notice that if $D = [a,b] \times [c,d]$ and f is integrable over D then we can choose partions $a = x_0 < \cdots < x_n = b$ and $c = y_0 < \cdots < x_m = d$ we have that

$$s(f,P) = \sum_{i=1}^{n} \sum_{j=1}^{m} f(a_{ij})(x_i - x_{i-1})(y_{j-1} - y_j) = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} f(a_{ij})(y_{j-1} - y_j) \right)(x_i - x_{i-1})$$

we can take partitions which refine $[c,d]$ and get (let $a_{ij} = (a_i, a_j)$):

$$\sum_{i=1}^{n} \left(\int_c^d f(a_i, y) dy \right)(x_i - x_{i-1})$$

which is a Riemann sum of $A(x)$ over $[a,b]$ where A is the integral of $f(x,y)$ relative to y:

$$A(x) = \int_c^d f(x,y)\, dy$$

and so $s(f,P) = s(A, P_{a,b})$, and so $\int_D f = \int_a^b A\, dx$, ie

$$\int_D f = \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx$$

Using an identical proof, we can swap the order of integration. We summarize this in the following theorem:

Theorem 24.2:

Suppose f is integrable in $D = [a,b] \times [c,d]$ and for every $x \in [a,b]$ the following is defined

$$I(x) = \int_c^d f(x,y)\, dy$$

then

$$\iint_D f(x,y)\, dxdy = \int_a^b I(x)\, dx = \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx$$

Similarly if

$$I(y) = \int_a^b f(x,y)\, dx$$

is defined then the integral is equal to

$$\iint_D f(x,y)\, dxdy = \int_c^d I(x) = \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy$$

We can generalize this to $\mathbb{R}^3$, if the domain is a prism $[a,b] \times [c,d] \times [e,g]$ then

$$\iiint_D f = \int_a^b \int_c^d \int_e^g f$$

And in general we can extend this to n dimensions.

Definition 24.3:

A domain $D\mathbb{R}^2$ is a **normal domain** relative to x if there exists functions φ_1, φ_2 such that

$$D = \big\{(x, y) \in \mathbb{R}^2 \bigm| x \in [a, b\big\}, \varphi_1(x) \le y \le \varphi_2(x)]$$

Similar for normal domains relative to y.

Proposition 24.4:

Suppose D is a normal domain relative to x: $D = \{(x, y) \mid a \le x \le b, \varphi_1(x) \le y \le \varphi_2(x)\}$ and φ_i are continuous in $[a, b]$ and so is f. Then

$$\iint_D f(x, y) = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dydx$$

Proof:

Since φ_i are continuous, they have extrema: $c \le \varphi_1, \varphi_2 \le d$ and so if we define a function g on $R = [a, b] \times [c, d]$ by

$$g(x) = \begin{cases} f(x) & x \in D \\ 0 & x \notin D \end{cases}$$

then

$$\int_D f = \int_R g = \int_a^b \int_c^d g(x, y)\, dxdy$$

and since $g(x, y) = 0$ if y is outside $[\varphi_1(x), \varphi_2(x)]$ this is equal to

$$\int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dxdy$$

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