Introduction to Stochastic Processes

Assignment 6

Ari Feiglin

6.1 Exercise

A moneky is sitting at a typewritter randomly typing (by hitting the keys in a uniform and independent manner). Show that as time goes, the moneky will almost surely write the book "Treasure Island".

Suppose the text of the book is the string $s_1 \cdots s_N$. Then let us define X_i to be ith key pressed by the monkey, and let us define

$$A_i = \{X_{iN+1} = s_1, \dots, X_{(i+1)N} = s_N\}$$

which is the event that starting from the iNth keypress, the monkey writes Treasure Island. Then $\{A_i\}$ are independent as they all deal with distinct X_is, and $\mathbb{P}(A_i) = \frac{1}{c^N}$ where c is the number of characters on the typewriter. Then $\sum_{i=0}^{\infty} \mathbb{P}(A_i) = \infty$ so by Borel-Cantelli we have that $\mathbb{P}(A_i \text{ i.o.}) = 1$ and since A_i i.o. $\subseteq \bigcup_{i=0}^{\infty} A_i$ we have that $\mathbb{P}(\bigcup_{i=0}^{\infty} A_i) = 1$ meaning that the monkey will write Treasure Island with probability 1 (since the event that the monkey writes Treasure Island contains $\bigcup_{i=0}^{\infty} A_i$).

6.2 Exercise

Let S_n be a general random process, meaning $S_n = \sum_{j=0}^{n} X_j$ where X_j are all independent equal-distribution random variables. Show that one of the following must have probability 1:

(1) $S_n = 0$ for all n,

(2) $S_n \to \infty$,

(3) $S_n \to -\infty$,

(4) $\liminf S_n = -\infty$ and $\limsup S_n = \infty$.

Notice that for every $-\infty \leq a \leq \infty$, $\liminf S_n = \liminf \sum_{j=0}^{n} X_j \leq a$ is an event which occurs independent of a permutation of a finite number of indexes of X_j (since permuting a finite number of indexes does not alter S_n, eventually). Thus by the Hewitt-Savage zero-one law, $\mathbb{P}(\liminf S_n \leq a) \in \{0,1\}$. Let us define $\ell = \inf_a\{\mathbb{P}(\liminf S_n \leq a) = 1\}$, this infimum is on a nonempty set since for $a = \infty$ the probability is one. Then for every $a < \ell$, $\mathbb{P}(\liminf S_n \leq a) = 0$, and for every $a > \ell$, $\mathbb{P}(\liminf S_n \leq a) = 1$, thus

$$\mathbb{P}(\liminf S_n = \ell) = \mathbb{P}\left(\bigcap_{n=1}^{\infty} a - n^{-1} \leq \liminf S_n \leq a + n^{-1}\right) = \lim_n \mathbb{P}\big(a - n^{-1} \leq \liminf S_n \leq a + n^{-1}\big) = 1$$

The second equality is due to continuity of measures, and the final equality is since $a + n^{-1} \leq \liminf S_n \leq a + n^{-1}$ has a probability of 1 for each n. So there exists an ℓ such that $\liminf S_n \overset{as}{=} \ell$. With a similar proof we can show there exists an L such that $\limsup S_n \overset{as}{=} L$.

Now,

$$\ell + \liminf S_n = \liminf \sum_{j=0}^{n} X_j = X_0 + \liminf \sum_{j=1}^{n} X_j \overset{as}{=} X_0 + \ell$$

So either we have that $X_0 \overset{as}{=} 0$ or $\ell \overset{as}{=} \pm\infty$. If $X_0 \overset{as}{=} 0$ then $X_n \overset{as}{=} 0$ for all n since they have the same distribution, and so $S_n = 0$ for all n has probability 1 (since this contains the event that $X_n = 0$ for all n, which has probability 1 as the countable intersection of events with probability 1). Otherwise we similarly have that $L \overset{as}{=} \pm\infty$ and so since $\ell \leq L$, one of the following must have probability 1

$$\begin{gathered}\{\ell = L = -\infty\} = \{S_n \to -\infty\}, \\ \{\ell = -\infty,\, L = \infty\} = \{\liminf S_n = -\infty,\, \limsup S_n = \infty\}, \\ \{\ell = L = \infty\} = \{S - n \to \infty\}\end{gathered}$$

as required.

6.3 Exercise

Let $X_1, X_2, \ldots$ be independent random variables which all have the distribution $\mathrm{Exp}(1)$. Compute

(1) $\mathbb{P}(X_n > \log n \text{ i.o.})$,

(2) $\mathbb{P}(X_n > 2\log n \text{ i.o.})$,

(3) $\mathbb{P}((\exists N)(\forall n > N) \max\{X_{n^2+1}, \ldots, X_{n^2+2n}\} > 5)$,

(4) $\mathbb{P}((\exists N)(\forall n > N) \max\{X_{n^2+1}, \ldots, X_{n^2+30}\} > 5)$.

(1) Notice that the events $\{X_n > \log n\}$ are all independent and since X_n has an exponential distribution, we compute $\mathbb{P}(X_n > \log n) = e^{-\log n} = \frac{1}{n}$. Since $\sum_{n=1}^{\infty} \mathbb{P}(X_n > \log n) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$. So by Borel-Cantelli $\mathbb{P}(X_n > \log n \text{ i.o.}) = 1$.

(2) Similarly, $\mathbb{P}(X_n > 2\log n) = e^{-2\log n} = \frac{1}{n^2}$. And so $\sum_{n=1}^{\infty} \mathbb{P}(X_n > 2\log n) = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ and so again by Borel-Cantelli $\mathbb{P}(X + n > 2\log n \text{ i.o.}) = 0$.

(3) The complement of this event is $(\forall N)(\exists n > N)\, X_{n^2+1}, \ldots, X_{n^2+2n} \leq 5$ which is just $X_{n^2+1}, \ldots, X_{n^2+2n} \leq 5$ i.o.. Since $\mathbb{P}(X_{n^2+1}, \ldots, X_{n^2+2n} \leq 5) = \alpha^{2n}$ where $0 < \alpha < 1$ is some constant (what exactly it is can be easily computed but is not necessary). Then

$$\sum_{n=1}^{\infty} \mathbb{P}(X_{n^2+1}, \ldots, X_{n^2+2n} \leq 5) = \sum_{n=1}^{\infty} \alpha^n < \infty$$

So $\mathbb{P}(X_{n^2+1}, \ldots, X_{n^2+2n} \leq 5 \text{ i.o.}) = 0$ by Borel-Cantelli, so the original probability is 1.

(4) The complement of this event, similar to before, is $X_{n^2+1}, \ldots, X_{n^2+30} \leq 5$ i.o.. From $n = 6$ onward these events are all independent, and a finite number of events does not affect i.o.-ness, so we can just ignore these first few events. Since $\mathbb{P}(X_{n^2+1}, \ldots, X_{n^2+30} \leq 5) = \alpha$ for some constant $0 < \alpha < 1$ (this is constant for all n, it is equal to $\mathbb{P}(\mathrm{Exp}(1) \leq 5)^{30}$). Then $\sum_{n=1}^{\infty} \mathbb{P}(X_{n^2+1}, \ldots, X_{n^2+30} \leq 5) = \sum_{n=1}^{\infty} \alpha = \infty$ so by Borel-Cantelli since the events (from an index onward) are independent, $\mathbb{P}(X_{n^2+1}, \ldots, X_{n^2+30} \leq 5 \text{ i.o.}) = 1$ and so the original probability is 0.