Infinitesimal Calculus 3

Lecture 8, Wednesday November 16, 2022
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8.1 Continuous Limits

Proposition 8.1.1:

If (X, ρ) is a metric space and p is a limit point of some $E \subseteq X$, then for every mapping between E and $\mathbb{R}$ (or a normed linear space):

- $\lim\limits_{x \to p} f(x) + g(x) = \lim\limits_{x \to p} f(x) + \lim\limits_{x \to p} g(x)$
- If $c \in \mathbb{R}$ then $\lim\limits_{x \to p} cf(x) = c \lim\limits_{x \to p} f(x)$
- If f and g map to $\mathbb{R}$ then $\lim\limits_{x \to p} f(x) \cdot g(x) = \lim\limits_{x \to p} f(x) \cdot \lim\limits_{x \to p} g(x)$
- If f and g map to $\mathbb{R}$ and $\lim\limits_{x \to p} g(x) \neq 0$ then $\lim\limits_{x \to p} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to p} f(x)}{\lim_{x \to p} g(x)}$

The proof of this is trivial using that the limit $\lim_{x \to p} f(x)$ is equal to the discrete limit $\lim f(x_n)$ if $x_n \longrightarrow p$.

Proposition 8.1.2 (The Squeeze Theorem):

If p is a limit point of E and $f, g, h \colon E \longrightarrow \mathbb{R}$ such that $f(x) \leq g(x) \leq h(x)$ and $\lim\limits_{x \to p} f(x) = \lim\limits_{x \to p} h(x) = c$. Then $\lim\limits_{x \to p} g(x) = c$ as well.

The proof of this is again trivial using discrete limits.

Proposition 8.1.3:

If f is a mapping from a metric space E to a normed linear space X then $\lim\limits_{x \to p} f(x) = 0$ if and only if $\lim\limits_{x \to p} \|f(x)\| = 0$.

Notice that the 0 in $\lim\limits_{x \to p} f(x) = 0$ is the 0 of X, and the 0 in the norm's limit is $0 \in \mathbb{R}$.

Proof:

Suppose the limit is 0, then for any $x_n \longrightarrow p$ we have that $f(x_n) \longrightarrow 0$ which by definition means $\|f(x_n)\| \longrightarrow 0$ as required. If the limit of the norm is 0 then for any $x_n \longrightarrow p$ we have $\|f(x_n)\| \longrightarrow 0$ which means $f(x_n) \longrightarrow 0$ so the limit is 0 as required.

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Proposition 8.1.4:

If $\lim\limits_{x \to p} f(x) = 0$ and $g(x)$ is bounded then $\lim\limits_{x \to p} g(x) \cdot f(x) = 0$ as well.

Notice that $\lim\limits_{x \to p} g(x)$ doesn't need to exist. If it did, then this proof would be immediate.

Proof:

Suppose $|g(x)| < M$ so $|g(x)f(x)| < M|f(x)|$ which converges to 0 and so by the above proposition so does $g(x) \cdot f(x)$.

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Theorem 8.1.5:

If (X, ρ_1), (Y, ρ_2), and (Z, ρ_3) are metric spaces, and f is a mapping between X and Y and g is a mapping between Y and Z. If

$$\lim_{x \to p} f(x) = q \in Y \qquad \lim_{x \to q} g(x) = r \in Z$$

then

$$\lim_{x \to p} g(f(x)) = r$$

Proof:

Suppose $x_n \longrightarrow p$ then $f(x_n) \longrightarrow \lim_{x \to p} f(x) = q$ by the definition of limits. Then since $f(x_n)$ is a sequence converging to q, $g(f(x_n)) \longrightarrow \lim_{x \to q} g(x) = r$. And so for every $x_n \longrightarrow p$ we have that $g(f(x_n)) \longrightarrow r$ so the limit of $g(f(x))$ is r as required.

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Example:

Suppose we'd "like" to compute:

$$\lim_{(x,y) \to (2,1)} \left(x^2y^2 - 4xy\right) \cdot \frac{\arcsin(xy - 2)}{\arctan(3xy - 6)}$$

Let $t = xy - 2$ then we have that $t^2 = x^2y^2 - 4xy + 4$ so the limit is equal to:

$$= \lim_{t \to 0} \left(t^2 - 4\right) \cdot \frac{\arcsin(t)}{\arctan(3t)}$$

Essentially what we have done is defined a function $f(x, y) = xy - 2$ and $g(x) = \left(t^2 - 4\right) \cdot \frac{\arcsin(t)}{\arctan(3t)}$ and the original limit is simply the limit of $g(f(x))$ as (x, y) approaches $(2, 1)$. Now since the limit of $f(x, y)$ as (x, y) approaches to $(2, 1)$ is 0. So the original limit is equal to the limit of $g(x)$ as x approaches 0, which is what we wrote above. Now this is the limit of a single variable function which we know how to compute. The limit is the product of two limits, the polynomial and the rational trignometric function, whose limit is (after applying L'hôpital's rule) is $-\frac{4}{3}$.

Example:

It is *not* always the case that

$$\lim_{(x,y) \to (a,b)} f(x, y) = \lim_{x \to a} \lim_{y \to b} f(x, y)$$

For example take

$$\lim_{(x,y) \to (0,1)} x \cdot \sin\left(\frac{1}{1 - y}\right)$$

Then since sin is bounded and x converges to 0, this limit is 0. But the limit of this as y approaches 1 does not exist, that is:

$$\lim_{x \to 0} \lim_{y \to} x \cdot \sin\left(\frac{1}{1 - y}\right)$$

does not exist, but the original limit does.

Proposition 8.1.6:

Suppose a function $f \colon E \longrightarrow \mathbb{R}^n$ is defined by $f = (f_1, \ldots, f_n)$. If $p = (p_1, \ldots, p_n)$ is a limit point of E then $\lim_{x \to p} f(x)$ exists if and only if for every relevant k, the limit $\lim_{x \to p_k} f(x_k)$ exists and is equal to q_k. Then

$$\lim_{x \to p} f(x) = q = (q_1, \ldots, q_n)$$

This is true since convergence in $\mathbb{R}^n$ is equivalent to pointwise convergence. So $f(x_m) = (f_1(x_m), \ldots, f_n(x_m))$ converges to $(q_1, \ldots, q_n)$ if and only if every $f_k(x_k)$ converges to q_k.

8.2 Continuous Functions

Definition 8.2.1:

If X and Y are metric spaces and $E \subseteq X$ is a subset then $f\colon E \longrightarrow Y$ is **continuous** at $p \in E$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $\rho(x,p) < \delta$ then $\sigma(f(x), f(p)) < \varepsilon$.
And f is **continuous** in E if it is continuous for every $p \in E$.

This is equivalent to saying that for every $\varepsilon > 0$ there is a $\delta > 0$ such that:

$$f\big(B_\delta^E(p)\big) \subseteq B_\varepsilon^Y(f(p))$$

If p is not an isolated point then this is equivalent to saying

$$\lim_{x \to p} f(x) = f(p)$$

if p is an isolated point though, then the limit doesn't exist but f *is* continuous at p. This is because for any $\varepsilon > 0$ we can take a $\delta > 0$ such that $B_\delta^E(p) = \{p\}$ and then for every x in this ball (which is only p), we have that $\sigma(f(x), f(p)) = \sigma(f(p), f(p)) = 0 < \varepsilon$.
So an equivalent definition would be to say that f is continuous at p if and only if p is either an isolated point or $\lim\limits_{x \to p} f(x) = f(p)$.

Proposition 8.2.2:

If X, Y, and Z are metric spaces and $f\colon X \supseteq E \longrightarrow Y$ is continuous at $x \in E$ and $g\colon f(E) \longrightarrow Z$ is continuous at $f(x) \in f(E)$ then $g \circ f$ is continuous at x.

Proof:

If x is an isolated point of E, by above $g \circ f$ is continuous at x. Otherwise take $E \ni x_n \longrightarrow x$, so $f(x_n) \longrightarrow f(x)$. Since g is continuous at $f(x)$, we know that $g(f(x_n)) \longrightarrow g(f(x))$ (since if $f(x_n) = f(x)$ this is true trivially, and otherwise it is true since $\lim_{y \to f(x)} g(y) = f(x)$). And so all in all for every $x_n \longrightarrow x$, $g(f(x_n)) \longrightarrow g(f(x))$, so $g \circ f$ is continuous at x.

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This shows that if f is continuous in E and g is continuous in $f(E)$, $g \circ f$ is continuous in E.