Algebraic Topology II

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1 Singular Homology

1.1 Chain Complexes

We begin by defining a *chain complex*. A chain complex is a sequence of Abelian groups with homomorphisms between them:

$$\begin{tikzcd} \cdots \arrow[r, dashed] & C_{n+1} \arrow[r, "\partial_{n+1}"] & C_n \arrow[r, "\partial_n"] & C_{n-1} \arrow[r, dashed] & \cdots \arrow[r, dashed] & C_0 \arrow[r] & 0 \end{tikzcd}$$

such that for every n, $\partial_n \circ \partial_{n+1} = 0$, in other words $\mathrm{Im}\partial_{n+1} \subseteq \ker \partial_n$. Define $Z_n = \ker \partial_n$, and its elements will be called *n-dimensional cycles*. And define $B_n = \mathrm{Im}\partial_{n+1}$, its elements will be called *boundaries*. Elements of the groups C_n will be called *n-dimensional chains*.

We now want to define a category of chain complexes. To do so we must define morphisms between chain complexes. So suppose we have two chain complexes $\mathscr{C} = \{C_n, \partial_n\}$ and $\mathscr{D} = \{D_n, \partial'_n\}$. We define a morphism from $\mathscr{C}$ to $\mathscr{D}$ to be a sequence of homomorphisms $f_n \colon C_n \longrightarrow D_n$ which preserves the structure of the chain. Meaning $\partial'_n \circ f_n = f_{n-1} \circ \partial_n$, in other words the following diagram commutes:

$$\begin{tikzcd} \cdots \arrow[r, dashed] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"'] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[r, dashed] \arrow[d, "f_{n-1}"'] & \cdots \arrow[r, dashed] & C_0 \arrow[r] \arrow[d, "f_0"'] & 0 \\ \cdots \arrow[r, dashed] & D_{n+1} \arrow[r, "\partial'_{n+1}"] & D_n \arrow[r, "\partial'_n"] & D_{n-1} \arrow[r, dashed] & \cdots \arrow[r, dashed] & D_0 \arrow[r] & 0 \end{tikzcd}$$

To simplify writing, we will write $\partial \circ f = f \circ \partial$, which f and which ∂ is being referred to will be understood from context.

The composition of two morphisms $\{f_n\} \colon \mathscr{C} \longrightarrow \mathscr{D}$ and $\{g_n\} \colon \mathscr{D} \longrightarrow \mathscr{E}$ is defined to be $\{g_n \circ f_n\} \colon \mathscr{C} \longrightarrow \mathscr{E}$. This is indeed a morphism:

$$\partial \circ f \circ g = f \circ \partial \circ g = f \circ g \circ \partial$$

And then this implies that the identity morphism is just $\mathrm{Id}_{\mathscr{C}} = \{\mathrm{Id}_{\mathrm{C_n}}\} \colon \mathscr{C} \longrightarrow \mathscr{C}$, as if $\{f_n\} \colon \mathscr{C} \longrightarrow \mathscr{D}$ then

$$\{f_n\} \circ \mathrm{Id}_{\mathscr{C}} = \{f_n \circ \mathrm{Id}_{C_n}\} = \{f_n\}, \qquad \mathrm{Id}_{\mathscr{D}} \circ \{f_n\} = \{\mathrm{Id}_{D_n} \circ f_n\} = \{f_n\}$$

Associativity is clear, so **Comp**, the category of chain complexes, is indeed a category.

Now recall that by definition $\partial_n \circ \partial_{n+1} = 0$, meaning

$$B_n \subseteq Z_n \subseteq C_n$$

Since these groups are all Abelian, they are normal in one another, so let us define the *nth homology group* of a chain complex $\mathscr{C}$ as

$$H_n(\mathscr{C}) := Z_n \big/ B_n = \ker \partial_n \big/ \mathrm{Im}\partial_{n+1}$$

1.1.1 Proposition

A chain complex morphism $\{f_n\} \colon \mathscr{C} \longrightarrow \mathscr{D}$ maps cycles to cycles and boundaries to boundaries.

Proof: let $z \in C_n$ be a cycle, i.e. $\partial z = 0$, but then $f(z)$ is a cycle since $\partial f(z) = f(\partial z) = f(0) = 0$. And let $b \in C_n$ be a boundary, so there exists an $a \in C_{n+1}$ such that $b = \partial a$. Then $f(b) = f\partial(a) = \partial f(a) = \partial b$, so $f(b)$ is a boundary as well. ■

This means that if $\{f_n\} \colon \mathscr{C} \longrightarrow \mathscr{D}$ is a morphism of chain complexes, $\{f_n\} \colon Z_n(\mathscr{C}) \longrightarrow Z_n(\mathscr{D})$ is well-defined, and so we have that

$$\begin{tikzcd} Z_n(\mathscr{C}) \arrow[r] \arrow[d] \arrow[dr, bend left] & Z_n(\mathscr{D}) \arrow[d] \\ H_n(\mathscr{C}) & H_n(\mathscr{D}) \end{tikzcd}$$

Where the blue arrow ψ is just the quotient map composed with f_n. This induces a group morphism

$$H_n(\{f_n\}) = f_* \colon H_n(\mathscr{C}) \longrightarrow H_n(\mathscr{D})$$

since we can define $f_*([z]) = \psi(z)$ since if $[z] = [z']$ then $z - z' \in B_n(\mathscr{C})$ and so $f(z - z') \in B_n(\mathscr{D})$ and thus the quotient of $f(z - z')$ is just 0, so $\psi(z) = \psi(z')$. Explicitly,

$$f_*[z] = [f_n z]$$

We now claim that H_n is a functor from the category of chain complexes **Comp** to the category of Abelian groups **Ab**. Now suppose $\{f_n\}: \mathscr{C} \longrightarrow \mathscr{D}$ and $\{g_n\}: \mathscr{D} \longrightarrow \mathscr{E}$ are chain complex morphisms, then the following diagram commutes

$$\begin{tikzcd}
Z_n(\mathscr{C}) \arrow[r, "f"] \arrow[d] & Z_n(\mathscr{D}) \arrow[r, "g"] \arrow[d] & Z_n(\mathscr{E}) \arrow[d] \\
H_n(\mathscr{C}) \arrow[r, "f_*"'] & H_n(\mathscr{D}) \arrow[r, "g_*"'] & H_n(\mathscr{E})
\end{tikzcd}$$

And so $(g \circ f)_* = g_* \circ f_*$, and it is easily verified that $\mathrm{id}_* = \mathrm{id}$ so H_n is a functor **Comp** $\longrightarrow$ **Ab** (the category of Abelian groups).

1.2 Singular Complex

We now define a functor from **Top** to **Comp**.

1.2.1 Definition

Let B be a set, then define the **free Abelian group** over B to be

$$\mathrm{FA}(B) = \bigoplus_{b \in B} \mathbb{Z} = \{\varphi : B \longrightarrow \mathbb{Z} \mid \varphi(b) \neq 0 \text{ for only finitely many } b \in B\}$$

Note then that there is a correspondence between B and $\mathrm{FA}(B)$: $b \leftrightarrow \varphi_b$ where

$$\varphi_b(x) = \begin{cases} 1 & x = b \\ 0 & x \neq b \end{cases}$$

so we can identify b with φ_b, and it is easy to see that every element of $\mathrm{FA}(B)$ can be written as $\sum_{i=1}^k n_i \varphi_{b_i}$, abusing notation $\sum_{i=1}^k n b_i$ and such a representation is unique.
Notice that if B is a set, G an Abelian group, and $g: B \longrightarrow G$ a function, then there exists a unique group homomorphism $L: \mathrm{FA}(B) \longrightarrow G$ which extends g. This is defined by

$$L: \sum_{i=1}^k n_i b_i \longmapsto \sum_{i=1}^k n_i g(b_i)$$

1.2.2 Definition

The **n-dimensional simplex** is defined to be

$$\Delta^n := \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; x_i \geq 0, \sum_{i=0}^n x_i = 1 \right\}$$

Δ^n has $n+1$ faces, and is homeomorphic to D^n.

1.2.3 Definition

Let X be a topological space, then an **n-dimensional singular simplex** in X is a morphism (in the category of topological spaces; a continuous map) $\Delta^n \longrightarrow X$. Define $S_n(x)$ to be the set of all n-dimensional singular simplexes in X, and define $C_n(X) = \mathrm{FA}(S_n(x))$.

We now want to define a chain complex on the sequence $C_n(X)$.
Let us define a set of maps $\tau_i^n\colon \Delta^{n-1} \longrightarrow \Delta^n$ for $0 \leq i \leq n$ which maps

$$\tau_i^n\colon (x_0, \ldots, x_{n-1}) \mapsto (x_0, \ldots, x_{i-1}, 0, x_i, \ldots, x_{n-1})$$

This is a well-defined continuous map, and geometrically it maps Δ^{n-1} to one of the faces of Δ^n.
Let $\sigma \in S_n(x)$, then let us define

$$\partial(\sigma) := \sum_{i=0}^{n} (-1)^i \sigma \circ \tau_i^n$$

Note that the composition is well-defined since $\Delta^{n-1} \xrightarrow{\tau_i^n} \Delta^n \xrightarrow{\sigma} X$, meaning $\sigma \circ \tau_i^n$ is an $n-1$-dimensional singular simplex. Thus ∂ can be extended to a map $\partial\colon C_n(X) = \mathrm{FA}(S_n(X)) \longrightarrow \mathrm{FA}(S_{n-1}(X)) = C_{n-1}(X)$ Notice that

$$\partial_{n-1}(\partial_n \sigma) = \partial_{n-1}\left(\sum_{i=0}^{n}(-1)^i \sigma \circ \tau_i^n\right) = \sum_{i=0}^{n}(-1)^i \partial_{n-1}(\sigma \circ \tau_i^n) = \sum_{i=0}^{n}\sum_{j=0}^{n-1}(-1)^{i+j}\sigma \circ \tau_i^n \circ \tau_j^{n-1}$$

Notice that $\tau_i^n \circ \tau_j^{n-1} = \tau_j^n \circ \tau_{i-1}^{n-1}$ which can be verified from its definition, but the first has a sign of $(-1)^{i+j}$ in the sum and the second has $-(-1)^{i+j}$. And so the sum is zero.
Thus we have defined a chain complex on $C_n(X)$, let us denote it by $\mathscr{C}(X)$, this is the first step in defining the functor. Next we must define the correspondence between morphisms.
Let $f\colon X \longrightarrow Y$ be a continuous map between topological spaces. Let us define $f_\sharp\colon \mathscr{C}(X) \longrightarrow \mathscr{C}(Y)$. First we define it for $\sigma \in S_n(X)$ by $f_\sharp(\sigma) = f \circ \sigma$. Since $\sigma\colon \Delta^n \longrightarrow X$ is continuous, so is $f \circ \sigma\colon \Delta^n \longrightarrow Y$ and so $f_\sharp$ is well-defined on the generators of $C_n(X)$. This can be extended by linearity to $f_\sharp\colon C_n(X) \longrightarrow C_n(Y)$. Notice that we ignore the subscripts and superscripts $(f_\sharp)_n^X$ for brevity and readability.
Now we must verify that this is a morphism of chain complexes, i.e. that $\partial f_\sharp = f_\sharp \partial$. So

$$f_\sharp \partial \sigma = f_\sharp\left(\sum_{i=0}^{n}(-1)^i \sigma \circ \tau_i^n\right) = \sum_{i=0}^{n}(-1)^i f_\sharp(\sigma \circ \tau_i^n) = \sum_{i=0}^{n}(-1)^i f \circ \sigma \circ \tau_i^n = \sum_{i=0}^{n}(-1)^i (f \circ \sigma) \circ \tau_i^n = \partial f_\sharp \sigma$$

and since this holds for generators, by linearity it holds for all $C_n(X)$. Thus $f_\sharp$ is indeed a morphism of chain complexes.
Thus we have defined a functor **Top** $\longrightarrow$ **Comp**.

1.3 Singular Homology

We have two functors **Top** $\longrightarrow$ **Comp** $\longrightarrow$ **Ab**, and so composing them together gives us a functor **Top** $\longrightarrow$ **Ab**. For a topological space X, we will denote its image under this functor as $H_n(X)$, called the *nth homological group* of X. And for a continuous map f, we denote its image as f_* or $H_n(f)$.
Let us compute the homological groups of the trivial space: $X = \{p\}$. Notice that $S_n(X) = \{K_n\}$ where K_n is the constant map $\Delta^n \longrightarrow \{p\}$, and so $C_n(X) = \mathbb{Z}$. We want to now compute what the boundary operators are, so

$$\partial K_n = \sum_{i=0}^{n}(-1)^i K_n \circ \tau_i^n$$

but $K_n \circ \tau_i^n$ is a morphism $\Delta^{n-1} \longrightarrow \{p\}$ meaning it is equal to K_{n-1}, thus $\partial K_n = \left(\sum_{i=0}^{n}(-1)^i\right)K_{n-1}$. For n even this is then K_{n-1} (or 1), and 0 for n odd. This means that either $\ker\partial = 0$ or $\mathrm{Im}\partial = \mathbb{Z}$, thus $H_n = 0$ for $n > 0$. For $n = 0$, we have that $\partial_0\colon \mathbb{Z} \longrightarrow 0$ and so its kernel is $\mathbb{Z}$, but ∂_1 is trivial and so its image is 0. Thus $H_0 = \mathbb{Z}$.
So we have shown

1.3.1 Proposition

Let $X = \{p\}$ be the trivial topological space, then its homological groups are

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n > 0 \end{cases}$$

1.3.2 Proposition

Let X be path connected, then $H_0(X) \cong \mathbb{Z}$.

Proof: we are concerned with the chain:

$$C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

So first let us understand $C_0(X)$, this is generated by $S_0(X)$, all the maps $\Delta^0 \longrightarrow X$ which are just all the points in X. And $S_1(X)$ is generated by all the maps $I \cong \Delta^1 \longrightarrow X$, so all the paths in X. The boundary of a 1-simplex is then

$$\partial_1 \sigma = \sigma(1) - \sigma(0)$$

and thus $B_1(X) = \mathrm{Im}\partial_1$ is generated by elements of the form $a - b$ where there exists a path between a and b. Since X is path-connected, this means that $B_1(X)$ is generated by $a - b$ for $a, b \in X$. Now, the subgroup generated by this is $\{\sum n_i p_i \mid p_i \in X, \sum n_i = 0\}$.

And now ∂_0's kernel is just $C_0(X)$ which is simply the free group generated by X. Thus

$$H_0(X) = \left\{\sum n_i p_i\right\} \Big/ \left\{\sum n_i p_i \,\middle|\, \sum n_i = 0\right\}$$

This is isomorphic to $\mathbb{Z}$ since we can define $\varphi\colon C_0(X) \longrightarrow \mathbb{Z}$ by $\sum n_i p_i \mapsto \sum n_i$ and this is a group homomorphism whose image is $\mathbb{Z}$ and whose kernel is all the points $\sum n_i p_i$ where $\sum n_i = 0$. Thus by the isomorphism theorem, $H_0(X) \cong \mathbb{Z}$. ■

1.3.3 Theorem

Let X be a topological space where $\{A_\alpha\}_{\alpha \in I}$ are its path connected components. Then for every n,

$$H_n(X) \cong \bigoplus_{\alpha \in I} H_n(A_\alpha)$$

Proof: notice that if $\sigma\colon \Delta^n \longrightarrow X$ is an n-simplex, then its image is contained within a path connected component. This is since Δ^n is path-connected, so $\sigma\Delta^n$ must be too. Thus for every $\gamma = \sum n_i \sigma_i \in S_n(X)$ we can write it as $\gamma = \sum \gamma_i$ for $\gamma_i \in S_n(A_i)$. And so $C_n(X) = \bigoplus_{\alpha \in I} C_n(A_\alpha)$.

Notice that γ is a cycle iff every γ_i is a cycle, since $\partial\gamma = \sum \partial\gamma_i$ and this is an element of a direct sum, so it is zero iff $\partial\gamma_i = 0$. Thus $Z_n(X) = \bigoplus_{\alpha \in I} Z_n(A_\alpha)$. And similarly we see that $B_n(X) = \bigoplus_{\alpha \in I} B_n(A_\alpha)$. Thus $H_n(X) = \bigoplus_{\alpha \in I} H_n(A_\alpha)$. ■

1.3.4 Corollary

If X is a topological space with $\{A_\alpha\}_{\alpha \in I}$ path connected components, $H_n(X) = \bigoplus_{\alpha \in I} \mathbb{Z}$.

1.3.5 Theorem

Let X be path-connected and $a \in X$, then $H_1(X) \cong \mathrm{Ab}\, \pi_1(X, a)$.

For two chains, $a, b \in C_n(X)$ say that they are *homological* if $a - b$ is a boundary (i.e. $a - b \in B_n(X)$). Write this as $a \approx b$.

1.3.6 Lemma

Let σ, τ be 1-simplexes.

(1) if σ is constant, then it is a boundary, i.e. $\sigma \approx 0$.

(2) if $\sigma \overset{\partial I}{\sim} \tau$ (since they are maps from $I \cong \Delta^1 \longrightarrow X$), then $\sigma \approx \tau$.

(3) if $\sigma(1) = \tau(0)$ then $\sigma * \tau \approx \sigma + \tau$.

(4) $\sigma + \bar{\sigma} \approx 0$

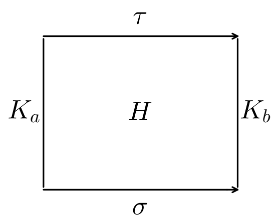
Proof:

(1) If σ is constant, then it is K_p^1 for some $p \in X$. And as we have already computed

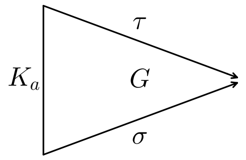
$$\partial K_p^n = \begin{cases} K_p^{n-1} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Thus $\partial K_p^2 = K_p^{n-1}$, meaning σ is a boundary.

(2) Let us look at the homotopy



Since H is surjective, it induces a map on the quotient space ${}^{I \times I}\!\big/_{I \times \{1\}}$, the map G:



The quotient space can be viewed as a 2-simplex by assigning an order to its vertices. Then its boundary is

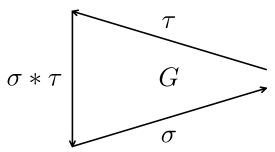
$$\partial G = K_a - \sigma + \tau$$

and since ∂G is a boundary, we have that

$$K_a - \sigma + \tau \approx 0$$

by (1) we have that $K_a \approx 0$ so $\sigma - \tau \approx 0$.

(3) The idea is to define a simplex of the form



Notice that such a simplex is possible: each horizontal line in the domain can be made constant. And its boundary is

$$\partial G = \tau - \sigma * \tau + \sigma$$

so $\sigma * \tau \approx \sigma + \tau$ since $\partial G \approx 0$.

(4) This is direct from the previous three points:

$$\sigma + \overline{\sigma} \overset{(3)}{\approx} \sigma * \overline{\sigma} \overset{(2)}{\approx} K_b \overset{(1)}{\approx} 0$$

Proof (of theorem 1.3.5): let us define a homomorphism

$$F\colon \pi_1(X, a) \longrightarrow H_1(X)$$

Denote homotopy equivalence classes by $\langle\bullet\rangle$ and the equivalence classes of $H_1(X)$ by $[\bullet]$. Then we define

$$\langle\varphi\rangle \xmapsto{F} [\varphi]$$

This is well-defined: if $\varphi \overset{\partial I}{\sim} \psi$ then $\varphi \approx \psi$ and so $[\varphi] = [\psi]$ (since $H_n(X)$ is the partition of $Z_n(X)$ relative to $\approx$). Notice that $\langle\varphi * \psi\rangle \mapsto [\varphi * \psi] = [\varphi + \psi] = [\varphi] + [\psi]$. So this is indeed a homomorphism. Since $H_1(X)$ is Abelian, this induces a homomorphism

$$\overline{F}\colon \mathrm{Ab}\,\pi_1(X, a) \longrightarrow H_1(X)$$

Let us now define a homomorphism

$$G\colon C_1(X) \longrightarrow \mathrm{Ab}\,\pi_1(X, a)$$

denote the equivalence classes of $\mathrm{Ab}\,\pi_1(X, a)$ by $\langle\langle\bullet\rangle\rangle$. For every $x \in X$, choose a path γ_x from a to x, then for $\sigma \in S_1(X)$ define

$$\hat{\sigma} = \gamma_{\sigma(0)} * \sigma * \overline{\gamma}_{\sigma(1)} \text{ from } a \text{ to } a$$

And define

$$\sigma \xmapsto{G} \langle\langle\hat{\sigma}\rangle\rangle$$

And extend by linearity to $G\colon C_1(X) \longrightarrow \mathrm{Ab}\,\pi_1(X, a)$. We can then restrict G to $Z_1(X)$, and in order for this to induce a map on $Z_1(X)\big/B_1(X)$ we must have that $G\big|_{B_1(X)} = 0$. So let A be a 2-simplex, then we must show $G(\partial A) = 0$. We know

$$G(\partial A) = G(A \circ \tau_0 - A \circ \tau_1 + A \circ \tau_2) = \left\langle\left\langle \widehat{A \circ \tau_0} \right\rangle\right\rangle - \left\langle\left\langle \widehat{A \circ \tau_1} \right\rangle\right\rangle + \left\langle\left\langle \widehat{A \circ \tau_2} \right\rangle\right\rangle$$

Now, $\langle\langle\sigma\rangle\rangle + \langle\langle\tau\rangle\rangle = \langle\langle\sigma\rangle\langle\tau\rangle\rangle$ and $-\langle\langle\sigma\rangle\rangle = \langle\sigma^{-1}\rangle$ by Abelianization, so this is equal to

$$\left\langle\left\langle \widehat{A \circ \tau_0} \right\rangle\left\langle \widehat{A \circ \tau_1} \right\rangle\left\langle \widehat{A \circ \tau_2} \right\rangle\right\rangle = \left\langle\left\langle \widehat{A \circ \tau_0} * \overline{\widehat{A \circ \tau_1}} * \widehat{A \circ \tau_2} \right\rangle\right\rangle$$

As is easily verified,

$$= \left\langle\left\langle \widehat{A \circ \tau_0} * \widehat{\overline{A \circ \tau_1}} * \widehat{A \circ \tau_2} \right\rangle\right\rangle = \left\langle\left\langle \overline{A \circ \tau_0 * \overline{A \circ \tau_1} * A \circ \tau_2} \right\rangle\right\rangle$$

Since $A\colon \Delta^2 \longrightarrow X$ is a simplex, $A \circ \tau_0 * \overline{A \circ \tau_1} * A \circ \tau_2$ is null-homotopic (the homotopy can condense the curve to a point through the image of A). Therefore its hat is as well, meaning this is all equal to zero, as required. So G induces a homomorphism

$$\overline{G}\colon H_1(X) \longrightarrow \mathrm{Ab}\,\pi_1(X, a)$$

Notice that

$$\overline{G} \circ \overline{F}\langle\langle\varphi\rangle\rangle = \overline{G}[\varphi] = \langle\langle\hat{\varphi}\rangle\rangle$$

We know that $\hat{\varphi} = \gamma_a \varphi \overline{\gamma}_a$ which is conjugate to φ, so in the Abelianization they are equal. So $\overline{G} \circ \overline{F} = \mathrm{id}$. Now suppose $[z] \in H_1(X)$ where $z = \sum n_i \sigma_i$ then

$$\overline{F} \circ \overline{G}[z] = \overline{F}\Big(\sum n_i \langle\langle\hat{\sigma}_i\rangle\rangle\Big) = \sum n_i [\hat{\sigma}_i] = \Big[\sum n_i \hat{\sigma}_i\Big]$$

So we need to show that if $\sum n_i \sigma_i$ is a cycle then $\sum n_i \hat{\sigma}_i \approx \sum n_i \sigma_i$. Define $T\colon C_0(X) \longrightarrow C_1(X)$ by $T(p) = \gamma_p$, so

$$\hat{\sigma} = \gamma_{\sigma 0} * \sigma * \overline{\gamma}_{\sigma 1} \approx \gamma_{\sigma 0} + \sigma - \gamma_{\sigma 1} = \sigma - T\partial\sigma$$

And so

$$\sum n_i \hat{\sigma}_i \approx \sum n_i \sigma_i - T\partial \sum n_i \sigma_i = z - T\partial z$$

since z is a cycle, $\partial z = 0$ and so this is equal to z. Thus $\hat{z} \approx z$ as required.
So $\overline{F}, \overline{G}$ are inverse isomorphisms, meaning $H_1(X) \cong \mathrm{Ab}\,\pi_1(X, a)$. ■

1.3.7 Definition

Let $\mathscr{C}, \mathscr{D}$ be two categories and let $F, G\colon \mathscr{C} \longrightarrow \mathscr{D}$ be functors. Then a **natural transformation** η from F to G is a correspondence such that

(1) for every object $X \in \mathscr{C}$, η associates a morphism $\eta_X\colon F(X) \longrightarrow G(X)$ called the **component** of X.

(2) for every $f\colon X \longrightarrow Y$ morphism, $\eta_Y \circ F(f) = G(f) \circ \eta_X$, i.e. the following diagram commutes

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"'] & G(Y) \end{tikzcd}$$

So for every pointed topology (X, a) we defined a group homomorphism $F_{X,a}\colon \pi_1(X, a) \longrightarrow H_1(X)$. We claim that this is a natural transformation from π_1 to H_1.

Suppose there is a morphism $h\colon (X, a) \longrightarrow (Y, b)$, so we need the following diagram to commute:

$$\begin{tikzcd} \pi_1(X,a) \arrow[r, "F_{X,a}"] \arrow[d, "\pi_1(h)"'] & H_1(X) \arrow[d, "H_1(h)"] \\ \pi_1(Y,b) \arrow[r, "F_{Y,b}"'] & H_1(Y) \end{tikzcd}$$

This is indeed the case:

$$\langle\varphi\rangle \xrightarrow{F_{X,a}} [\varphi] \xrightarrow{H_1(h)} [h \circ \varphi], \qquad \langle\varphi\rangle \xrightarrow{\pi_1(h)} \langle h \circ \varphi\rangle \xrightarrow{F_{Y,b}} [h \circ \varphi]$$

1.3.8 Example

If we look at the identity functor (on the category of groups) and Abelianization, then $\rho_\bullet$, which is the quotient map $\bullet \longrightarrow \mathrm{Ab}\,\bullet$, is a natural transformation. Indeed

$$\begin{tikzcd} G \arrow[r, "\rho_G"] \arrow[d, "\varphi"'] & \mathrm{Ab}\,G \arrow[d, "\hat{\varphi}"] \\ H \arrow[r, "\rho_H"'] & \mathrm{Ab}\,H \end{tikzcd}$$

Where $\hat{\varphi}[g] = [\varphi(g)]$. This is indeed natural:

$$\rho_H \circ \varphi(g) = [\varphi(g)], \qquad \hat{\varphi} \circ \rho_G(g) = \hat{\varphi}[g] = [\varphi(g)]$$

1.3.9 Definition

The **simplified singular chain complex** of a topological space X is the chain complex

$$\cdots \dashrightarrow C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \dashrightarrow \cdots \dashrightarrow C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

Where we define ε as follows:

$$\varepsilon \sum n_i p_i = \sum n_i$$

i.e. $\varepsilon p = 1$ for every $p \in X$. And a morphism between two simplified singular chain complexes differ only from morphisms between normal singular chain complexes in that the map from $\mathbb{Z}$ to $\mathbb{Z}$ is the identity.

The homology induced by a simplified singular chain complex is called the **reduced homology** and denoted $\widetilde{H}_n(X)$.

Obviously for every $n \geq 1$, $\widetilde{H}_n(X) = H_n(X)$. Recall that if X is path-connected, then $B_0(X)$ is generated by $a - b$ for $a, b \in X$, so it is $\{\sum n_i p_i \mid \sum n_i = 0\}$. Now $\ker \varepsilon = \{\sum n_i p_i \mid \sum n_i = 0\}$ as well, and so we get that when X is path-connected, $\widetilde{H}_0(X) = 0$.

1.3.10 Definition

A chain of Abelian groups

$$\cdots \dashrightarrow A \xrightarrow{f} B \xrightarrow{g} C \dashrightarrow \cdots$$

is **exact** at B if $\mathrm{Im} f = \ker g$. If the sequence is exact at every group, then the sequence itself is called an **exact sequence**. (Recall that chain complexes require $\mathrm{Im} f \subseteq \ker g$.)

If we have an exact sequence in one of the following forms, then:

(1) $0 \longrightarrow A \xrightarrow{f} B$, then $0 = \ker f$ so f is injective.

(2) $A \xrightarrow{f} B \longrightarrow 0$, then $\mathrm{Im} f = B$ so f is surjective.

(3) $0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0$, then f is an isomorphism.

1.3.11 Definition

A **short exact sequence** is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

In a short exact sequence, by above f is injective and g is surjective, and furthermore $\mathrm{Im} f = \ker g$. In such a case, we can view A as being a subgroup of B (since f is an embedding) and since by the isomorphism theorem $C \cong B/\ker g = B/\mathrm{Im} f = B/A$, a short exact sequence can be viewed as

$$0 \longrightarrow A \xrightarrow{\text{inclusion}} B \xrightarrow{\text{quotient}} B/A \longrightarrow 0$$

1.3.12 Lemma (The Lemma of Five)

Suppose the chains $\{A_i\}_i, \{B_i\}_i$ are exact, and the following diagram commutes:

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "f_1"'] & A_2 \arrow[r] \arrow[d, "f_2"'] & A_3 \arrow[r] \arrow[d, "f_3"'] & A_4 \arrow[r] \arrow[d, "f_4"'] & A_5 \arrow[d, "f_5"'] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd}$$

(1) If f_2, f_4 are injective and f_1 is surjective, then f_3 is injective.

(2) If f_2, f_4 are surjective and f_5 is injective, then f_3 is surjective.

Proof: We write $x \overset{A}{\longmapsto} y$ to mean x maps to y in the exact sequence ($x \in A_i$).

(1) Suppose $f_3 a$, then $a \xmapsto{f_3} 0 \xmapsto{B} 0$, now suppose $a \xmapsto{A} b \xmapsto{f_4} c$. Since the diagram commutes, we must have that $c = 0$, but f_4 is injective so $b = 0$. This means $a \in \ker A$, so there exists some d such that $d \xmapsto{A} a$. Suppose $d \xmapsto{f_2} e$, then $e \xmapsto{B} 0$ by commutativity, so there exists an f such that $f \xmapsto{B} e$, and since f_1 is surjective there exists a $g \xmapsto{f_1} f$. Now suppose $g \xmapsto{A} h$. By commutativity, since $g \xmapsto{f_1} f \xmapsto{B} e$ we have $f_2 h = e$ and since f_2 is injective, $h = d$. So d is in the image of A, so it is in the kernel and so $a = 0$.

(2) is a little more complicated, but it's just chasing. ■

> **1.3.13 Definition**
>
> Suppose $\mathscr{C}$ and $\mathscr{D}$ are two chain complexes, with two morphisms $f, g\colon \mathscr{C} \longrightarrow \mathscr{D}$. Then a **chain homotopy** from f to g is a sequence of maps $T_n\colon C_n \longrightarrow D_{n+1}$ such that $\partial T + T\partial = f - g$. If there exists a chain homotopy between f and g, we write $f \overset{CH}{\sim} g$.

In a diagram, we have that T are the red arrows.

$$\begin{tikzcd}
\cdots \arrow[r, dashed] & C_{n+1} \arrow[r] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] \arrow[dl, dashed, red] & C_n \arrow[r] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] \arrow[dl, red] & C_{n-1} \arrow[r, dashed] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] \arrow[dl, red] & \cdots \\
\cdots \arrow[r, dashed] & D_{n+1} \arrow[r] & D_n \arrow[r] & D_{n-1} \arrow[r, dashed] & \cdots
\end{tikzcd}$$

Let $X \subseteq \mathbb{R}^k$ be convex. For $a \in X$ let us define the *cone construction* $C_a\colon C_n(X) \longrightarrow C_{n+1}(X)$ as follows: we start with generators of $C_n(X)$, i.e. we define $C_a\sigma$ for $\sigma\colon \Delta^n \longrightarrow X$ an n-simplex. Geometrically, $C_a\sigma$ will be a cone whose tip is a and whose base is σ. We define this by:

$$C_a\sigma(t_0, \dots, t_{n+1}) = t_0 b + (1 - t_0)\sigma\left(\frac{t_1}{1-t_0}, \dots, \frac{t_{n+1}}{1-t_0}\right)$$

Let us now compute the faces of $C_a\sigma$. For $i = 0$ then

$$(C_a\sigma)\tau_0^{n+1}(t_0, \dots, t_n) = C_a\sigma(0, t_0, \dots, t_n) = \sigma(t_0, \dots, t_n)$$

For $i > 0$ then

$$(C_a\sigma)\tau_i^{n+1}(t_0, \dots, t_n) = C_a\sigma(t_0, \dots, 0, \dots, t_n)$$

if $t_0 = 1$ as well, then this is just

$$C_a\sigma(1, 0, \dots, 0) = a$$

Otherwise,

$$\begin{aligned}
&= t_0 b + (1 - t_0)\sigma\left(\frac{t_1}{1-t_0}, \dots, 0, \dots, \frac{t_n}{1-t_0}\right) \\
&= t_0 b + (1 - t_0)\sigma\tau_{i-1}^n\left(\frac{t_1}{1-t_0}, \dots, \frac{t_n}{1-t_0}\right) \\
&= C_a^{n-1}(\sigma\tau_{i-1}^n)(t_0, \dots, t_n)
\end{aligned}$$

So we see that

$$(C_a\sigma)\tau_0^{n+1} = \sigma, \qquad (C_a\sigma)\tau_i^{n+1} = C_a^{n-1}(\sigma\tau_{i-1}^n)$$

So

$$\begin{aligned}
\partial_{n+1} C_a^n(\sigma) = \sum_{i=0}^{n+1} (-1)^i (C_a^n\sigma)\tau_i^{n+1} &= \sigma + \sum_{i=1}^{n+1} C_a^{n-1}(\sigma\tau_{i-1}^n) \\
&= \sigma - \sum_{i=0}^{n} (-1)^i C_a^{n-1}(\sigma\tau_i^n) \\
&= \sigma - C_a^{n-1}\left(\sum_{i=0}^{n} (-1)^i \sigma\tau_i^n\right) \\
&= \sigma - C_a^{n-1}\partial_n\sigma
\end{aligned}$$

So we see that

$$\partial C_a - C_a\partial = \mathrm{id}$$

so in other words, C_a is a chain homotopy from id to 0.

1.3.14 Theorem

Let X be a convex set in $\mathbb{R}^k$, then for all $n > 0$, $H_n(X) = 0$.

Proof: let $\gamma \in C_n(X)$, then $\gamma = \partial C_a\gamma + C_a\partial\gamma$. If $\gamma \in Z_n(X)$, i.e. it is a cycle, then $\partial\gamma = 0$ and so $\gamma = \partial C_a\gamma$. Thus $\gamma \in B_n(X)$, so $Z_n(X) = B_n(X)$, and then $H_n(X) = 0$. ■

1.3.15 Lemma

If $f, g\colon X \longrightarrow Y$ are two homotopic continuous maps, then $f_\sharp$ and $g_\sharp$ are chain homotopic.

Proof: let us define $i, j\colon X \longrightarrow X \times I$ where $i(x) = (x, 0)$ and $j(x) = (x, 1)$. If $H\colon X \times I \longrightarrow Y$ is a homotopy from f to g, then $f = H \circ i$ and $g = H \circ j$. Also $i \sim j$, so if we can show that $i_\sharp \overset{CH}{\sim} j_\sharp$ then we have that

$$f_\sharp = H_\sharp \circ i_\sharp \overset{CH}{\sim} H_\sharp \circ j_\sharp = g_\sharp$$

so it is sufficient to show that $i_\sharp \overset{CH}{\sim} j_\sharp$.
So we need to define a sequence of morphisms $T_n^X\colon C_n(X) \longrightarrow C_{n+1}(X \times I)$ such that $\partial T^X + T^X\partial = i_\sharp^X - j_\sharp^X$. We will define T_n^X by induction on n, such that T^X is natural. Natural between what two functors? The first functor maps topological spaces X to their chain complexes $\mathscr{C}(X)$ and maps morphisms $X \xrightarrow{f} Y$ to $f_\sharp\colon \mathscr{C}(X) \longrightarrow \mathscr{C}(Y)$. The second functor maps topological spaces X to the chain complex $C_{n+1}(X \times I)$ and morphisms $X \xrightarrow{f} Y$ to $(f \times \mathrm{id})_\sharp\colon C_{n+1}(X) \longrightarrow C_{n+1}(Y)$.
T^X being natural means that the diagram commutes for all $f\colon X \longrightarrow Y$:

$$\begin{tikzcd}
C_n(X) \arrow[r, "T^X"] \arrow[d, "f_\sharp"'] & C_{n+1}(X \times I) \arrow[d, "(f \times \mathrm{id})_\sharp"] \\
C_n(Y) \arrow[r, "T^Y"'] & C_{n+1}(Y \times I)
\end{tikzcd}$$

So $T_Y \circ f_\sharp = (f \times \mathrm{id})_\sharp \circ T_X$.
Let $I_n\colon \Delta^n \longrightarrow \Delta^n$ be the identity n-dimensional simplex. If we determine $T^{\Delta^n}(I_n)$, then we have determined $T^X(\sigma)$ for all $\sigma \in C_n(X)$ for all X. This is because $\sigma = \sigma \circ I_n = \sigma_\sharp(I_n)$, since we can view σ as a continuous map $X \longrightarrow \Delta^n$ and so $\sigma_\sharp$ is defined. Thus

$$T^X(\sigma) = T^X \circ \sigma_\sharp(I_n) = (\sigma \times \mathrm{id})_\sharp \circ T^{\Delta^n}(I_n)$$

And so determining $T^{\Delta^n}(I_n)$ determines $T^X(\sigma)$. So if we define $A = T^{\Delta^n}(I_n)$, then

$$T^X(\sigma) = (\sigma \times \mathrm{id})_\sharp(A)$$

A is some simplex in $C_{n+1}(\Delta^n \times I)$, and we claim that for any choice of A, this defines a natural transformation. This is because

$$T^Y \circ f_\sharp(\sigma) = T^Y(f \circ \sigma) = \big((f \circ \sigma) \times \mathrm{id}\big)_\sharp(A) = (f \times \mathrm{id})_\sharp \circ (\sigma \times \mathrm{id})_\sharp(A)$$

And

$$(f \times \mathrm{id})_\sharp \circ T^X(\sigma) = (f \times \mathrm{id})_\sharp \circ (\sigma \times \mathrm{id})_\sharp(A)$$

so these are indeed equal, as required.
Now we claim that

$$(\partial T^X + T^X\partial)(\sigma) = (i_\sharp^X - j_\sharp^X)(\sigma)$$

for all X, σ. It is sufficient to show this for $X = \Delta^n$ and $\sigma = I_n$, since if

$$(\partial T^{\Delta^n} + T^{\Delta^n}\partial)(I_n) = (i_\sharp^{\Delta^n} - j_\sharp^{\Delta^n})(I_n)$$

if we compose it on the left with $(\sigma \times \mathrm{id})_\sharp$, the LHS gives

$$(\partial(\sigma \times \mathrm{id})_\sharp T^{\Delta^n} + (\sigma \times \mathrm{id})_\sharp T^{\Delta^n}\partial)(I_n) = (\partial T^X\sigma_\sharp + T^X\partial\sigma_\sharp)(I_n) = \partial T^X\sigma + T^X\partial\sigma$$

since T is natural, $T^Y \circ f_\sharp = (f \times \mathrm{id})_\sharp \circ T^X$ and $\partial f_\sharp = f_\sharp \partial$. The RHS is

$$\left((\sigma \times \mathrm{id})_\sharp \circ i_\sharp^{\Delta^n} - (\sigma \circ \mathrm{id}) \circ j_\sharp^{\Delta^n}\right)(I_n)$$

Now notice that

$$\begin{array}{ccccc} \Delta^n & \xrightarrow{i^{\Delta^n}} & \Delta^n \times I & \xrightarrow{\sigma \times \mathrm{id}} & X \times I \\ s & \longmapsto & (s,0) & \longmapsto & (\sigma(s),0) \end{array}$$

So $(\sigma \times \mathrm{id}) \circ i^{\Delta^n} = i^X \circ \sigma$, and similar for j. So the RHS is just

$$i_\sharp^X \circ \sigma_\sharp(I_n) - j_\sharp^X \circ \sigma_\sharp(I_n) = i_\sharp^X(\sigma) - j_\sharp^X(\sigma)$$

So we get

$$\partial T^X(\sigma) + T^X \partial\sigma = i_\sharp^X(\sigma) - j_\sharp^X(\sigma)$$

as required.

So we must show that

$$\partial T I_n + T\partial I_n = i_\sharp I_n - j_\sharp I_n$$

in order to get this for every $\sigma \in C_n(\Delta^n)$. So we must show $\partial T I_n = -T\partial I_n + i_\sharp - j_\sharp I_n$, since $\partial T I_n \in C_n(\Delta^n \times I)$, and $\Delta^n \times I$ is a convex set in $\mathbb{R}^{n+2}$. In a convex set so a simplex is a boundary if and only if it is a cycle. We want $-T\partial I_n + i_\sharp I_n - j_\sharp I_n$ to be a boundary, and so it is sufficient to check that it is a cycle:

$$-\partial T\partial I_n + \partial i_\sharp I_n - \partial j_\sharp I_n$$

Since $\partial I_n \in C_{n-1}(\Delta^n)$, we have that

$$\partial T\partial I_n + T\partial\partial I_n = i_\sharp \partial I_n - j_\sharp \partial I_n$$

and thus we must have that the following is zero:

$$T\partial\partial I_n - i_\sharp \partial I_n + j_\sharp \partial I_n + \partial i_\sharp I_n - \partial j_\sharp I_n$$

Since $\partial\partial = 0$, and $i_\sharp, j_\sharp$ are chain homomorphisms, this is indeed zero. So $-T\partial I_n + i_\sharp I_n - j_\sharp I_n$ is a cycle and thus a boundary since the universe is convex. So let us take A to be a chain such that ∂A is this element. ■

So notice now that if $f \sim g$, then $f_\sharp \sim g_\sharp$ are chain homotopic, and so $f_* = g_*$.

1.3.16 Corollary

If $f\colon X \longrightarrow Y$ is a homotopy equivalence, then $f_*\colon H_n(X) \longrightarrow H_n(Y)$ is an isomorphism.

Proof: there exists a $g\colon Y \longrightarrow X$ such that $fg \sim \mathrm{id}_Y$ and $gf \sim \mathrm{id}_X$. Thus

$$g_* \circ f_* = (g \circ f)_* = (\mathrm{id}_X)_* = \mathrm{id}_{H_n(X)}$$

and similarly $f_* \circ g_* = \mathrm{id}_{H_n(Y)}$, so f_* is an isomorphism. ■

1.4 Mayer-Vietoris

1.4.1 Definition

Let $p_1, \ldots, p_n$ be vectors in a vector space, then their **affine hull** is

$$\mathrm{CH}(p_1, \ldots, p_n) = \left\{ \sum_{i=1}^n \alpha_i p_i \;\middle|\; \sum_{i=1}^n \alpha_i = 1 \right\}$$

Elements of the affine hull are called **affine combinations**. We similarly define the **convex hull**:

$$\mathrm{CH}(p_1,\dots,p_n) = \left\{ \sum_{i=1}^n \alpha_i p_i \;\middle|\; \sum_{i=1}^n \alpha_i = 1,\ \alpha_i \geq 0 \right\}$$

And its elements are called **convex combinations**.

1.4.2 Definition

$p_1,\dots,p_n$ are **affine independent** if $\sum_{i=1}^n \alpha_i v_i = 0$ and $\sum_{i=1}^n \alpha_i = 0$ implies every α_i is 0.

1.4.3 Definition

$A \subseteq \mathbb{R}^k$ is an n**-simplex** if it is the convex hull of a set of $n+1$ affine independent set of vectors.

1.4.4 Definition

Let $\Sigma = \mathrm{CH}(p_0,\dots,p_n)$ be an n-simplex, then its ith **face** is $\mathrm{CH}(p_0,\dots,p_{i-1},p_i,\dots,p_n)$. And its **barycenter** is

$$b = \frac{1}{n+1}\sum_{i=0}^n p_i$$

We define the **barycentric subdivision** of Σ, denoted $\mathrm{Sd}\,\Sigma$, to be a set of n-simplices which we define inductively on n as follows:

(1) For a 0-simplex, $\mathrm{Sd}\,\Sigma = \Sigma$.

(2) If Σ is an n-simplex, then let $\varphi_0,\dots,\varphi_n$ be its faces (which are $n-1$-simplices) and b its barycenter. Then define $\mathrm{Sd}\,\Sigma$ to be the n-simplices spanned by b and the simplices in the barycentric subdivisions of φ_i. I.e.

$$\mathrm{Sd}\,\Sigma = \left\{ \mathrm{CH}\big(b,\Sigma^{n-1}\big) \;\middle|\; \Sigma^{n-1} \in \mathrm{Sd}\,\varphi_i, 0 \leq i \leq n \right\}$$

Inductively, $\Sigma = \bigcup \mathrm{Sd}\,\Sigma$ and $\#\,\mathrm{Sd}\,\Sigma = (n+1)!$.

1.4.5 Theorem

For every n, there exists a constant $c < 1$ such that for every n-simplex Σ then for every $\Sigma' \in \mathrm{Sd}\,\Sigma$:

$$\mathrm{diam}(\Sigma') \leq c\,\mathrm{diam}(\Sigma)$$

1.4.6 Definition

We define $\mathrm{Sd}_n\colon C_n(\Delta^n) \longrightarrow C_n(\Delta^n)$ by induction on n. Let $\sigma\colon \Delta^n \longrightarrow \Delta^n$ be a generator, then

(1) $\mathrm{Sd}_0(\sigma) = \sigma$

(2) $\mathrm{Sd}_n(\sigma) = C_{\sigma(b)}(\mathrm{Sd}_{n-1}(\partial\sigma))$ where b is the barycenter of Δ^n.

Let X be a topological space, then let $\mathrm{Sd}_n\colon C_n(X) \longrightarrow C_n(X)$ be defined on generators $\sigma\colon \Delta^n \longrightarrow X$ by $\mathrm{Sd}\,\sigma = \sigma_\sharp\,\mathrm{Sd}_n\,\mathrm{id}_{\Delta^n}$.

1.4.7 Theorem

Sd is a chain map ($\mathrm{Sd} = \{\mathrm{Sd}_n\}_{n=0}^{\infty}$) and is natural (between the chain functor **Top** $\to$ **Comp** and itself).

Sd being natural means the following diagram commutes

$$\begin{tikzcd} C_n(X) \arrow[r, "\mathrm{Sd}_n"] \arrow[d, "f_\sharp"'] & C_n(X) \arrow[d, "f_\sharp"] \\ C_n(Y) \arrow[r, "\mathrm{Sd}_n"'] & C_n(Y) \end{tikzcd}$$

1.4.8 Theorem

Sd is chain homotopic to $\mathrm{id}_{\mathscr{C}(X)}$.

1.4.9 Definition

Let X be a topological space, and $\mathcal{U} = \{\mathcal{U}_\alpha\}_{\alpha\in I}$ a collection of subsets of X such that $\bigcup \mathring{\mathcal{U}}_\alpha = X$ (where $\mathring{\mathcal{U}}$ is the interior of $\mathcal{U}$). Such a collection will be called a **good cover** of X.

We will say that $\sigma\colon \Delta^n \longrightarrow X$ **preserves** the cover if there exists an $\alpha \in I$ such that $\sigma(\Delta^n) \subseteq \mathcal{U}_\alpha$. And we will say that $\sum_i n_i\sigma_i \in C_n(X)$ **preserves** the cover if each σ_i preserves the cover.

Let us define

$$C_n^{\mathcal{U}}(X) = \{\sigma \in C_n(X) \mid \sigma \text{ preserves } \mathcal{U}\}$$

$C_n^{\mathcal{U}}(X)$ is a subgroup of $C_n(X)$, as can be easily verified. Notice that if $\sigma(\Delta^n) \subseteq \mathcal{U}_\alpha$ then $\sigma\tau_i(\Delta^{n-1}) = \sigma(\tau_i\Delta^{n-1}) \subseteq \mathcal{U}_\alpha$ so that $\sigma\tau_i \in C_{n-1}^{\mathcal{U}}(X)$. Thus $\partial\sigma \in C_{n-1}^{\mathcal{U}}(X)$, so we can define a subcomplex of $\mathscr{C}(X)$, $\mathscr{C}^{\mathcal{U}}(X)$ whose coefficients are $C_n^{\mathcal{U}}(X)$. So we can define $H_n^{\mathcal{U}}(X)$ to be the nth homology group of $\mathscr{C}^{\mathcal{U}}(X)$.

The inclusion map $\iota\colon C_n^{\mathcal{U}}(X) \longrightarrow C_n(X)$ is a chain morphism, so this induces a $\iota_*\colon H_n^{\mathcal{U}}(X) \longrightarrow H_n(X)$.

1.4.10 Theorem

This ι_* is an isomorphism.

This is not a trivial proof, and it relies on the following observations. But from here on, I will only be putting in the simpler/enlightening proofs so that I can finish this summary. Notice that

$$\mathrm{Sd}_n\colon C_n^{\mathcal{U}}(X) \longrightarrow C_n^{\mathcal{U}}(X)$$

is defined, since if $\sigma \in C_n^{\mathcal{U}}(X)$ then that means for some $\alpha \in I$ $\sigma(\Delta^n) \subseteq \mathcal{U}_\alpha$, and $\mathrm{Sd}_n\sigma = \sigma_\sharp \mathrm{Sd}_n \mathrm{id}_n$. Thus the image of $\mathrm{Sd}_n\sigma$ is contained in the image of σ, which in turn is contained in $\mathcal{U}_\alpha$. Now, the chain homotopy between Sd and $\mathrm{id}_{\mathscr{C}(X)}$ can also be restricted to $\mathscr{C}^{\mathcal{U}}(X) \longrightarrow \mathscr{C}^{\mathcal{U}}(X)$. Thus Sd is chain homotopic to $\mathrm{id}_{\mathscr{C}^{\mathcal{U}}(X)}$.

1.4.11 Definition

A **short exact sequence** of chain complexes is a chain of chain morphisms $\mathscr{C} \xrightarrow{f} \mathscr{D} \xrightarrow{g} E$ such that for every n, $0 \to C_n \xrightarrow{f_n} D_n \xrightarrow{g_n} E_n \to 0$ is a short exact sequence.

1.4.12 Lemma

A short exact sequence of chain complexes $\mathscr{C} \xrightarrow{f} \mathscr{D} \xrightarrow{g} \mathscr{E}$ induces a long exact sequence on the homology groups:

$$\begin{tikzcd}
{} \arrow[rr, dashed] & & H_{n+1}\mathscr{E} \arrow[dll] \\
H_n\mathscr{C} \arrow[r] & H_n\mathscr{D} \arrow[r] & H_n\mathscr{E} \arrow[dll] \\
H_{n-1}\mathscr{C} \arrow[rr, dashed] & & {}
\end{tikzcd}$$

Proof: a diagram chase. ■

1.4.13 Definition

If $\mathscr{C}, \mathscr{D}$ are chain complexes then their **direct sum** is the chain complex $\mathscr{C} \oplus \mathscr{D}$ whose terms are $C_n \oplus D_n$ and whose boundary operator is $\partial_{\mathscr{C}} \oplus \partial_{\mathscr{D}}$ (i.e. $(a, b) \mapsto (\partial a, \partial b)$).

1.4.14 Lemma

If X is a topological space, $\mathcal{U}, V \subseteq X$ such that $\mathring{\mathcal{U}} \cup \mathring{\mathcal{V}} = X$, then there exists a short exact sequence of chain complexes

$$0 \longrightarrow \mathscr{C}(\mathcal{U} \cap \mathcal{V}) \longrightarrow \mathscr{C}(\mathcal{U}) \oplus \mathscr{C}(\mathcal{V}) \longrightarrow \mathscr{C}^{\mathcal{U},\mathcal{V}}(X) \longrightarrow 0$$

where $\mathscr{C}^{\mathcal{U},\mathcal{V}}(X)$ is the chain complex modulo the cover $\{\mathcal{U}, \mathcal{V}\}$.

Proof: we have the inclusions, which commute:

$$\begin{tikzcd}
 & \mathcal{U} \arrow[dr, "k"] & \\
\mathcal{U} \cap \mathcal{V} \arrow[ur, "i"] \arrow[dr, "j"'] & & X \\
 & \mathcal{V} \arrow[ur, "\ell"'] &
\end{tikzcd}$$

And from them we build:

$$\begin{tikzcd}
0 \arrow[r] & C_n(\mathcal{U} \cap \mathcal{V}) \arrow[r] & C_n(\mathcal{U}) \oplus C_n(\mathcal{V}) \arrow[r] & C_n^{\mathcal{U},\mathcal{V}}(X) \arrow[r] & 0 \\
 & a \arrow[r, mapsto] & (i_\sharp a, -j_\sharp a) & & \\
 & & (a, b) \arrow[r, mapsto] & k_\sharp a + \ell_\sharp b &
\end{tikzcd}$$

This is exact because composing the two maps gives $k_\sharp i_\sharp a - \ell_\sharp j_\sharp a = (ki)_\sharp a - (\ell j)_\sharp a$, and since $ki = \ell j$, this is zero. So the image of the first is contained within the kernel of the second. And if $k_\sharp a = -\ell_\sharp b$, then a, b must be chains in $\mathcal{U} \cap \mathcal{V}$ (since k maps chains of $\mathcal{U}$ to X, and ℓ maps chains of $\mathcal{V}$), so they must be in the image of the first map. It can be verified that these are chain morphisms. ■

Notice that the homology group of $C_n(X) \oplus C_n(Y)$ is just $H_n(X) \oplus H_n(Y)$ since the image of $\partial \oplus \partial$ is just $\operatorname{Im} \partial \oplus \operatorname{Im} \partial$, and similar for kernel. From the previous two lemmas, the following is immediate (recall that $H_n^{\mathcal{U}}(X) \cong H_n(X)$):

1.4.15 Theorem (Mayer-Vietoris)

If $\mathcal{U}, \mathcal{V} \subseteq X$ such that $\mathring{\mathcal{U}} \cup \mathring{\mathcal{V}} = X$, then there is an exact sequence

$$\begin{tikzcd}
{} \arrow[rr, dashed] & & H_{n+1}(X) \arrow[dll] \\
H_n(\mathcal{U} \cap \mathcal{V}) \arrow[r] & H_n(\mathcal{U}) \oplus H_n(\mathcal{V}) \arrow[r] & H_n(X) \arrow[dll] \\
H_{n-1}(\mathcal{U} \cap \mathcal{V}) \arrow[rr, dashed] & & {}
\end{tikzcd}$$

Notice that at $n = 0$ for the reduced homology if $\mathcal{U} \cap \mathcal{V} \neq \varnothing$, then we get the same exact sequence but with the reduced homology.

1.4.16 Theorem

$$\widetilde{H}_i(S^n) = \begin{cases} \mathbb{Z} & i = n \\ 0 & \text{else} \end{cases}$$

Proof: by induction on n. For $n = 0$, we have that S^0 is just the space of two points, so $H_0(S^0) = \mathbb{Z} \oplus \mathbb{Z}$ and for $i > 0$ it is zero since the homology of the one-point space is zero. The reduced homology removes a factor of $\mathbb{Z}$ and so $\widetilde{H}_0(S^0) = \mathbb{Z}$ and for $n > 0$ $\widetilde{H}_i(S^0) = 0$. Now inductively, we can choose contractible $\mathcal{U}, \mathcal{V}$ such that $\mathcal{U} \cap \mathcal{V}$ are homotopic to S^{n-1} (by choosing hemispheres which overlap), and so $H_i(\mathcal{U} \cap \mathcal{V}) \cong H_i(S^{n-1})$. We have an exact sequence by Mayer-Vietoris:

$$\widetilde{H}_i(\mathcal{U}) \oplus \widetilde{H}_i(\mathcal{V}) \longrightarrow \widetilde{H}_i(S^n) \longrightarrow \widetilde{H}_{i-1}(S^{n-1}) \longrightarrow \widetilde{H}_{i-1}(\mathcal{U}) \oplus \widetilde{H}_{i-1}(\mathcal{V})$$

since $\mathcal{U}, \mathcal{V}$ are contractible, $\widetilde{H}_i(\mathcal{U}) = \widetilde{H}_i(\mathcal{V}) = 0$ for all i and so we get the exact sequence

$$0 \longrightarrow \widetilde{H}_i(S^n) \longrightarrow \widetilde{H}_{i-1}(S^{n-1}) \longrightarrow 0$$

which means that $\widetilde{H}_i(S^n) \cong \widetilde{H}_{i-1}(S^{n-1})$, and so inductively we have our result. ■

Since their homology groups differ, we immediately get

1.4.17 Theorem

If $n \neq m$ then S^n is not homotopic to S^m.

1.4.18 Corollary

If $n \neq m$ then $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$.

Proof: suppose $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a homeomorphism, then it is a homeomorphism $f\colon \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^m \setminus \{f(0)\}$. So we have

$$S^n \simeq \mathbb{R}^n \setminus \{0\} \cong \mathbb{R}^m \setminus \{f(0)\} \simeq S^m$$

in contradiction. ■

1.4.19 Theorem

∂D^n is not a retract of D^n.

Proof: suppose $r\colon D^n \longrightarrow \partial D^n$ is a retraction, then $r\iota = \mathrm{id}_{\partial D^n}$ where ι is the inclusion $\partial D^n \longrightarrow D$. Thus $r_* \iota_* = \mathrm{id}_{H_i(\partial D^n)}$. This implies that ι_* is injective, in particular for $i = n-1$ and so $i_*\colon \widetilde{H}_{n-1}(\partial D^n) \longrightarrow \widetilde{H}_{n-1}(D^n)$. Since $\partial D^n \cong S^{n-1}$ and D^n is contractible, we have an injective map $\mathbb{Z} \longrightarrow 0$ in contradiction. ■

1.4.20 Lemma

Let us define $R\colon S^n \longrightarrow S^n$ by $R(x_1, \ldots, x_n) = (-x_1, x_2, \ldots, x_n)$. Then $R_*\colon \widetilde{H}_n(S^n) \longrightarrow \widetilde{H}_n(S^n)$ satisfies $R_* = -\mathrm{id}_{\widetilde{H}_n(S^n)}$.

Proof: by induction on n. For $n = 0$, $R(1) = -1$ and $R(-1) = 1$, and $\widetilde{H}_0(S^0) = \mathbb{Z}$. Now, ε must map the generator of the reduced homology to zero, so the generator must be $kp_1 - kp_2$, and composing R_* on this gives $kp_2 - kp_1$ which is the inverse of the generator, so R_* is indeed minus the identity.

Now for $n > 0$, let us split the sphere S^n into two hemispheres $\mathcal{U}$ and $\mathcal{V}$ whose intersection is homotopic to S^{n-1}. By Mayer-Vietoris, we have

$$\begin{tikzcd} 0 \arrow[r] & H_n(S^n) \arrow[r] \arrow[d, "R_*"'] & H_{n-1}(S^{n-1}) \arrow[r] \arrow[d, "R_* = -\mathrm{id}"] & 0 \\ 0 \arrow[r] & H_n(S^n) \arrow[r] & H_{n-1}(S^{n-1}) \arrow[r] & 0 \end{tikzcd}$$

This diagram commutes by naturality, so $R_* = -\mathrm{id}$ for S^n. ∎

By symmetry, we can define $R_i\colon (x_1, \ldots, x_i, \ldots, x_{n+1}) \mapsto (x_1, \ldots, -x_i, \ldots, x_{n+1})$ and we have that $R_{i,*} = -\mathrm{id}$. Let us define

$$A\colon S^n \longrightarrow S^n, \qquad x \mapsto -x$$

the *antipodal map*. Since $A = R_1 \circ \cdots \circ R_{n+1}$, we have that $A_* = (-\mathrm{id})^{n+1} = (-1)^{n+1}\mathrm{id}$.

1.4.21 Corollary

If n is even, then the antipodal map is not homotopic to the identity.

Note that for $n = 2k-1$, we can view S^n as the unit sphere in $\mathbb{C}^k$ and take the homotopy $H(z,t) = e^{\pi i t} z$ which is a homotopy from id to A.

1.4.22 Lemma

Let $n \geq 0$, and let $f, g\colon S^n \longrightarrow S^n$ such that for all $x \in S^n$, $f(x) \neq -g(x)$. Then $f \sim g$.

Proof: we define the homotopy

$$H(x,t) = \frac{(1-t)f(x) + tg(x)}{\|(1-t)f(x) + tg(x)\|}$$

this cannot be zero, since the line $(1-t)f(x)+tg(x)$ connects $f(x)$ and $g(x)$, and it can only be zero when $f(x)$ and $g(x)$ are antipodal points on the sphere. ∎

1.4.23 Theorem

Let n be even and $f\colon S^n \longrightarrow S^n$, then there exists an $x \in S^n$ such that either $f(x) = x$ or $f(x) = -x$.

Proof: suppose not. Then for all x, $f(x) \neq x$, so $f(x) \neq -A(x)$ so $f \sim A$. And for all x, $f(x) \neq -x$, i.e. $f(x) \neq -\mathrm{id}(x)$ so $f \sim \mathrm{id}$. Thus $A \sim \mathrm{id}$, which contradicts n being even. ∎

1.4.24 Definition

A **vector field** of S^n is a continuous map $f\colon S^n \longrightarrow \mathbb{R}^{n+1}$ such that for all $x \in S^n$, $\langle f(x), x\rangle = 0$.

1.4.25 Theorem (Hairy Ball Theorem)

Let n be even. Then for every vector field on S^n, there is an $x \in S^n$ such that $f(x) = 0$.

Proof: suppose not, then we can define a continuous map $x \mapsto \frac{f(x)}{\|f(x)\|}$ which is a map $S^n \longrightarrow S^n$. These points are still tangent to x, in particular they cannot be x or antipodal to x, in contradiction to n being even.

∎

Note that in general if $\mathcal{U}, \mathcal{V}$ is a good cover of X and $\mathcal{U} \cap \mathcal{V}$ is contractible, then by Mayer-Vietoris we have

$$0 = \widetilde{H}_i(\mathcal{U} \cap \mathcal{V}) \longrightarrow \widetilde{H}_i(\mathcal{U}) \oplus \widetilde{H}_i(\mathcal{V}) \longrightarrow \widetilde{H}_i(X) \longrightarrow \widetilde{H}_{i-1}(\mathcal{U} \cap \mathcal{V}) = 0$$

so $\widetilde{H}_i(X) \cong \widetilde{H}_i(\mathcal{U}) \oplus \widetilde{H}_i(\mathcal{V})$. In particular let us look at $S^n \vee S^m$, we can take $\mathcal{U}$ to be S^n with a bit of S^m and $\mathcal{V}$ to be S^m with a bit of S^n, then $\mathcal{U} \cap \mathcal{V}$ is contractible and $\mathcal{U}$ is homotopic to S^n and $\mathcal{V}$ to S^m so

$$\widetilde{H}_i(S^n \vee S^m) \cong \widetilde{H}_i(S^n) \oplus \widetilde{H}_i(S^m)$$

and similarly by induction

$$\widetilde{H}_i\left(\bigvee_{j=1}^{k} S^{n_j}\right) \cong \bigoplus_{j=1}^{k} \widetilde{H}_i(S^{n_j})$$

Now let us look at $X = nT$. Let us take $\mathcal{U}$ to be a disk in X, and $\mathcal{V}$ to be the rest of X with a bit of $\mathcal{U}$. Then $\mathcal{U}$ is homotopic to a point, $\mathcal{U} \cap \mathcal{V} \simeq S^1$ and we showed last semester that $\mathcal{V} \simeq \bigvee_{2n} S^1$. Mayer-Vietoris gives us

$$\widetilde{H}_i(\mathcal{U} \cap \mathcal{V}) \longrightarrow \widetilde{H}_i(\mathcal{U}) \oplus \widetilde{H}_i(\mathcal{V}) \longrightarrow \widetilde{H}_i(X) \longrightarrow \widetilde{H}_{i-1}(\mathcal{U} \cap \mathcal{V})$$

when $i \geq 2$ $\widetilde{H}_i(\mathcal{U} \cap \mathcal{V}) = \widetilde{H}_i(\mathcal{U}) \oplus \widetilde{H}_i(\mathcal{V}) = 0$, but we require $i \geq 3$ for $\widetilde{H}_{i-1}(\mathcal{U} \cap \mathcal{V}) = 0$. So when $i \geq 3$, $\widetilde{H}_i(nT) = 0$. So let us look at $i = 2$:

$$\begin{aligned} H_2(\mathcal{U} \cap \mathcal{V}) \longrightarrow H_2\mathcal{U} \oplus H_2\mathcal{V} \longrightarrow H_2X \longrightarrow H_1(\mathcal{U} \cap \mathcal{V}) \longrightarrow H_1\mathcal{U} \oplus H_1\mathcal{V} \longrightarrow H_1X \longrightarrow \\ \longrightarrow \widetilde{H}_0(\mathcal{U} \cap \mathcal{V}) \longrightarrow \widetilde{H}_0\mathcal{U} \oplus \widetilde{H}_0\mathcal{V} \longrightarrow \widetilde{H}_0X \longrightarrow 0 \end{aligned}$$

We get from this

$$0 \longrightarrow 0 \longrightarrow H_2X \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^{2n} \longrightarrow H_1X \longrightarrow 0 \longrightarrow 0 \longrightarrow \widetilde{H}_0X \longrightarrow 0$$

So we get that $\widetilde{H}_0X = 0$. Let us focus on the map $\mathbb{Z} \longrightarrow \mathbb{Z}^{2n}$ here, that is we need to understance $H_1(\mathcal{U} \cap \mathcal{V}) \longrightarrow H_1(\mathcal{U}) \oplus H_1(\mathcal{V}) = H_1(\mathcal{V})$. Visually, this can be shown to just be zero (using abelianization of π_1). We can then just insert 0 into the sequence where the zero morphism was:

$$0 \longrightarrow 0 \longrightarrow H_2X \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z}^{2n} \longrightarrow H_1X \longrightarrow 0 \longrightarrow 0 \longrightarrow \widetilde{H}_0X \longrightarrow 0$$

So we get that $H_2X \cong \mathbb{Z}$ and $H_1X \cong \mathbb{Z}^{2n}$.

1.4.26 Theorem

If $f\colon D^k \longrightarrow S^n$ is injective, then $\widetilde{H}_i(S^n - f(D^k)) = 0$ for all i.

Proof: by induction on k. For $k = 0$, $S^n - \{\cdot\} \cong \mathbb{R}^n$ which is contractible and thus has a homotopy group of zero. We will be working with the k-dimensional cube $I^k \cong D^k$. So $f\colon I^k \times I \longrightarrow S^n$ is injective. Define $A_1 = I^k \times [0, 1/2]$ and $B_1 = I^k \times [1/2, 1]$, and let $\mathcal{U} = S^n - f(A_1) = f(A_1)^c$ and $\mathcal{V} = S^n - f(B_1) = f(B_1)^c$. So $\mathcal{U} \cup \mathcal{V} = f(A_1 \cap B_1)^c = f(I^k \times \{1/2\})^c$. So inductively, $\widetilde{H}_i(\mathcal{U} \cup \mathcal{V}) = 0$ for all i. And $\mathcal{U} \cap \mathcal{V} = f(A_1 \cup B_1)^c = f(I_{k+1})^c$ which is the space we want to compute the homology groups of. By Mayer-Vietoris:

$$0 = \widetilde{H}_{i+1}(\mathcal{U} \cup \mathcal{V}) \longrightarrow \widetilde{H}_i(\mathcal{U} \cap \mathcal{V}) \longrightarrow \widetilde{H}_i(\mathcal{U}) \oplus \widetilde{H}_i(\mathcal{V}) \longrightarrow \widetilde{H}_i(\mathcal{U} \cup \mathcal{V}) = 0$$

So $\widetilde{H}_i(\mathcal{U} \cap \mathcal{V}) \cong \widetilde{H}_i\mathcal{U} \oplus \widetilde{H}_i\mathcal{V}$. So suppose that $\widetilde{H}_i(\mathcal{U} \cap \mathcal{V}) \neq 0$, then take $[z] \neq 0$ in $\widetilde{H}_i(\mathcal{U} \cap \mathcal{V})$. Taking the inclusion maps i, j we have that one of $i_*[z]$ and $-j_*[z]$ is nonzero. Continue. ■

1.4.27 Theorem

Let $f\colon S^k \longrightarrow S^n$ be injective, then

$$\widetilde{H}_i(S^n - f(S^k)) = \begin{cases} \mathbb{Z} & i = n - k - 1 \\ 0 & \text{else} \end{cases}$$

Proof: similarly by induction on k. ■

1.4.28 Theorem (Jordan's Theorem)

Let $f\colon S^{n-1} \longrightarrow S^n$ injective then $S^n - f(S^{n-1})$ has two path-connected components and they are open.

1.4.29 Theorem (Invariance of Domain)

Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, and $f\colon \mathcal{U} \longrightarrow \mathbb{R}^n$ injective. Then $f(\mathcal{U})$ is open.

Proof: let $x \in \mathcal{U}$ then let us look at the restriction of f to a closed ball around x, and show that the image of its interior is open. If we choose this closed ball to lie in $\mathcal{U}$, looking at the union of these balls, we see that $\mathcal{U}$'s image is open. So we must show that if $f\colon D^n \longrightarrow \mathbb{R}^n$ is injective, then $f(\mathring{D}^n)$ is open.

Now, $\mathbb{R}^n - f(\partial D^n) = f(\mathring{D}^n) \sqcup (\mathbb{R}^n - f(D^n))$. This is the union of two disjoint path-connected spaces (the left is the continuous map of a path-connectedt space, and the right is because $\widetilde{H}_i(\mathbb{R}^n - f(D^n)) = 0$ as an exercise). By Jordan's theorem, there are two path-connected components and they are open. So these are the two open path-connected components, in particular $f(\mathring{D}^n)$ is open. ■

Note then that if $\mathcal{U} \subseteq \mathbb{R}^n$ is open, then it is not homeomorphic to any subspace $A \subseteq \mathbb{R}^n$ which is not open. This is because the homeomorphism $\mathcal{U} \longrightarrow A$ would mean by the invariance of domain that A is open. In particular, an open set in $\mathbb{R}^n$ is not homeomorphic to any open set in $\mathbb{R}^m$ for $n \neq m$. This is because $\mathbb{R}^m \subset \mathbb{R}^n$ assuming $m < n$, and so $\mathcal{U} \longrightarrow A$ means that A is open, but the last coordinates of A are all zero and so it cannot be open.

1.4.30 Definition

An n-dimensional **manifold** is a Hausdorff topological space M with a countable basis such that for every $x \in M$ there exists a neighborhood homeomorphic to an open ball in $\mathbb{R}^n$. An n-dimensional **manifold with boundary** is a Hausdorff topological space M with a countable basis such that every $x \in M$ has a neighborhood homeomorphic either to an open ball or to the half-open ball (which is defined to be $\{(x_1, \dots, x_n) \mid \|(x_1, \dots, x_n)\| < 1, x_1 \geq 0\}$). A **closed manifold** is a compact manifold (without a boundary).

1.5 Excision

Let $A \subseteq X$ be a subspace, then $\mathscr{C}(A) \subseteq \mathscr{C}(X)$ is a subcomplex, so we can define the *quotient complex* $\mathscr{C}(X, A) = \mathscr{C}(X)/\mathscr{C}(A)$. Explicitly, $C_n(X, A) = C_n(X)/C_n(A)$. The boundary operator ∂ maps from $C_n(A)$ to $C_{n-1}(A)$, so we can simply take $\partial[z] = [\partial z]$ in the quotient complex. Thus we can define the *relative homology* groups of X with respect to A to be

$$H_n(X, A) := H_n(\mathscr{C}(X, A))$$

Now, suppose $f\colon (X, A) \longrightarrow (Y, B)$ is a map, then we have $f_\sharp\colon C_n(X) \longrightarrow C_n(Y)$. Do we have that this induces a map $f_\sharp\colon C_n(X, A) \longrightarrow C_n(Y, B)$? In order for this to occur we must have $f_\sharp(C_n(A)) \subseteq C_n(B)$, which is indeed the case (since $f\colon A \longrightarrow B$). Thus we have a function $f_\sharp\colon C_n(X, A) \longrightarrow C_n(Y, B)$ and we can see that this is a chain morphism. So we have defined a functor $\mathbf{Top}^2 \longrightarrow \mathbf{Comp}$. And in particular we can compose this with our functor $\mathbf{Comp} \longrightarrow \mathbf{Ab}$ to get $\mathbf{Top}^2 \longrightarrow \mathbf{Ab}$.

Now, we have a short exact sequence of chain complexes

$$0 \longrightarrow C_n(A) \longrightarrow C_n(X) \longrightarrow C_n(X, A) \longrightarrow 0$$

since this is precisely an inclusion-quotient chain, and the boundary operators are defined in such a way so that the diagram commutes. Thus we have an exact sequence of homology groups:

$$\begin{tikzcd}
{} \arrow[rr, dashed] & & H_{n+1}(X,A) \arrow[dll] \\
H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X,A) \arrow[dll] \\
H_{n-1}(A) \arrow[rr, dashed] & & {}
\end{tikzcd}$$

And this short exact sequence of chain complexes is natural, so this exact sequence is natural as well.

1.5.1 Theorem

$$H_i(D^n, \partial D^n) = \begin{cases} \mathbb{Z} & i = n \\ 0 & \text{else} \end{cases}$$

Proof: by the exact sequence of homology groups we have

$$0 = \widetilde{H}_i(D^n) \longrightarrow H_i(D^n, \partial D^n) \longrightarrow \widetilde{H}_{i-1}(\partial D^n) \longrightarrow \widetilde{H}_{i-1}(D^n) = 0$$

so $H_i(D^n, \partial D^n) \cong \widetilde{H}_{i-1}(\partial D^n) = \widetilde{H}_{i-1}(S^{n-1})$ which is exactly what we want. ■

We can generalize this:

1.5.2 Lemma

Let $A \subseteq X$ then

(1) if A is contractible, then $\widetilde{H}_i(X) \cong H_i(X, A)$;

(2) if X is contractible, then $\widetilde{H}_{i-1}(A) \cong H_i(X, A)$.

Proof: again we use the exact sequence:

$$0 = \widetilde{H}_i(A) \longrightarrow \widetilde{H}_i(X) \longrightarrow H_i(X, A) \longrightarrow \widetilde{H}_{i-1}(A) = 0$$

so $H_i(X, A) \cong \widetilde{H}_i(X)$. Similar for the second case. ■

1.5.3 Proposition

Note if we have $f\colon (X, A) \longrightarrow (Y, B)$ then we have

$$f_*\colon H_n(X) \longrightarrow H_n(Y), \quad f_*\colon H_n(A) \longrightarrow H_n(Y), \quad f_*\colon H_n(X, A) \longrightarrow H_n(Y, B)$$

If any two of these are isomorphisms, so is the third.

Proof: immediate from the naturality of the exact sequence of homology groups, and the lemma of five. ■

In particular we have that

1.5.4 Corollary

If $f\colon X \longrightarrow Y$ and $f\colon A \longrightarrow B$ are both homotopic equivalences, then $f_*\colon H_n(X, A) \longrightarrow H_n(Y, B)$ is an isomorphism.

In particular, the inclusion map $(D^n, \partial D^n) \subseteq (D^n, D^n - \{0\})$ is a homotopic equivalence, and so $H_i(D^n, \partial D^n) \cong H_i(D^n, D^n - \{0\})$. Thus

$$H_i(D^n, D^n - \{0\}) = \begin{cases} \mathbb{Z} & i = n \\ 0 & \text{else} \end{cases}$$

And we can look at our good friend $R\colon (x_1, \dots, x_n) \mapsto (-x_1, x_2, \dots, x_n)$ which can be viewed as $R\colon (D^n, \partial D^n) \longrightarrow (D^n, \partial D^n)$ and we have the commutative diagram

$$\begin{tikzcd} H_n(D^n, \partial D^n) \arrow[r, "\cong"] \arrow[d, "R_*"'] & \widetilde{H}_{n-1}(\partial D^n) \arrow[d, "R_* = -\mathrm{id}"] \\ H_n(D^n, \partial D^n) \arrow[r, "\cong"'] & \widetilde{H}_{n-1}(\partial D^n) \end{tikzcd}$$

And so $R_* = -\mathrm{id}$ for the map over $H_n(D^n, \partial D^n)$.

1.5.5 Definition

Let $\mathcal{U} = \{\mathcal{U}_\alpha\}_{\alpha \in I}$ be a good covering of X, and let $A \subseteq X$, then we define

$$C_n^{\mathcal{U}}(X, A) = C_n^{\mathcal{U}}(X) \Big/ C_n^{\mathcal{U}}(X) \cap C_n(A)$$

This is indeed a chain complex, since if a chain preserves $\mathcal{U}$ and is contained in A, then its boundary preserves $\mathcal{U}$ and is contained in A.

Similar to before, the inclusion map $\iota\colon C_n^{\mathcal{U}}(X, A) \longrightarrow C_n(X, A)$ induces an isomorphism $\iota_*\colon H_n^{\mathcal{U}}(X, A) \longrightarrow H_n(X, A)$.

1.5.6 Theorem (Excision)

Let $K \subseteq A \subseteq X$ such that $\overline{K} \subseteq \mathring{A}$, then the inclusion $(X-K, A-K) \longrightarrow (X, A)$ induces an isomorphism of all homology groups $H_n(X-K, A-K) \longrightarrow H_n(X, A)$.

Proof: note that $\overline{K} \subseteq \mathring{A}$ is equivalent to $\mathcal{U} = \{A, K^c\}$ being a good cover. So $C_n(A), C_n(X-K) \subseteq C_n^{\mathcal{U}}(X)$. So we can compose the inclusion map with the quotient map to get $C_n(X-K) \longrightarrow C_n^{\mathcal{U}}(X)/C_n(A) \cap C_n^{\mathcal{U}}(X) = C_n^{\mathcal{U}}(X)/C_n(A)$. We claim that this is a surjective map, as chains in $C_n^{\mathcal{U}}(X)/C_n(A)$ are classes of chains which respect $\{X-K, A\}$, but the simplexes which respect A are identified with zero, so we are left with formal sums of simplexes which respect $X-K$. The kernel is just $C_n(X-K) \cap C_n(A) = C_n((X-K) \cap A) = C_n(A-K)$. Thus by the first isomorphism theorem

$$C_n(X-K, A-K) = {C_n(X-K)}\Big/{C_n(A-K)} \cong {C_n^{\mathcal{U}}(X)}\Big/{C_n(A)} = C_n^{\mathcal{U}}(X, A)$$

Thus we get that

$$H_n(X-K, A-K) \cong H_n^{\mathcal{U}}(X, A) \cong H_n(X, A)$$

■

1.5.7 Theorem

Let M be an n-dimensional manifold with or without a boundary, and $p \in M$ be a point in its interior. Then

$$H_i(M, M-\{p\}) = \begin{cases} \mathbb{Z} & i = n \\ 0 & \text{else} \end{cases}$$

Proof: let $j \colon D^n \longrightarrow M$ be an embedding of D^n into M which maps 0 to p and which maps $\mathring{D}^n$ to a neighborhood of p. Let us identify D^n with its image in M and p with 0. Then $(D^n, D^n - \{0\}) \subseteq (M, M - \{0\})$ is an excision: take $A = M - \{0\}$ and $K = M - D^n$. Thus $H_i(M, M-\{0\}) \cong H_i(D^n, D^n - \{0\})$ which is precisely what we want. ■

1.5.8 Corollary

The dimension of a manifold M is a topological property of M (i.e. it is unique).

Proof: this is since it is determined by its homology groups. ■

1.5.9 Theorem

Let M be a manifold with a boundary and p a point on its boundary. Then $H_i(M, M-\{p\}) = 0$ for all i.

Proof: take $j \colon C \longrightarrow M$ an embedding of the half-open ball into M. Then as before $(C, C-\{0\}) \subseteq (M, M-\{0\})$ is an excision and both C and $C - \{0\}$ are contractible. We have the exact sequence

$$0 = \widetilde{H}_i(C) \longrightarrow H_i(C, C-\{0\}) \longrightarrow \widetilde{H}_i(C - \{0\}) = 0$$

so $H_i(M, M-\{p\}) \cong H_i(C, C-\{0\}) = 0$. ■

1.5.10 Corollary

The boundary of a manifold is a topological property of M (a point cannot be both in its boundary and interior).

Note that if $p \in M$ is a boundary point, then it has a neighborhood homeomorphic to $\{\vec{x} \in B_1^n(0) \mid x_n \geq 0\}$. Thus it has a neighborhood homeomorphic to $B_1^{n-1}(0)$ (taking the last coordinate equal to 0), and all the points in this neighborhood must also be boundary points. Thus the boundary of an n-dimensional manifold is an $n-1$-dimensional manifold.

1.5.11 Theorem

$[\mathrm{id}_n] \in H_n(\Delta^n, \partial\Delta^n)$ generates the homological group.

1.5.12 Theorem

Let $A \subseteq X$ be closed and suppose that there exists an open $\mathcal{U}$ such that $A \subseteq \mathcal{U} \subseteq X$ and A is a deformation retract of $\mathcal{U}$. Then

$$H_n(X, A) = \widetilde{H}_n(X/A)$$

Proof: since A is a deformation retract, the inclusion $(X, A) \longrightarrow (X, \mathcal{U})$ induces an isomorphism $H_n(X, A) \cong H_n(X, \mathcal{U})$. Furthermore by excision, $H_n(X - A, \mathcal{U} - A) \cong H_n(X, \mathcal{U})$. Let a be the point which represents A in X/A. Then $H_n(X/A, \{a\}) \cong H_n(X/A, \mathcal{U}/A)$ similar to above. And by excision $H_n(X/A - \{a\}, \mathcal{U}/A - \{a\}) \cong H_n(X/A, \mathcal{U}/A)$. Note though that $(X - A, \mathcal{U} - A)$ is homeomomorphic to $(X/A - \{a\}, \mathcal{U}/A - \{a\})$ (both just remove A). Thus

$$H_n(X, A) \cong H_n(X, \mathcal{U}) \cong H_n(X - A, \mathcal{U} - A) \cong H_n(X/A - \{a\}, \mathcal{U}/A - \{a\}) \cong H_n(X/A, \mathcal{U}/A) \cong H_n(X/A, \{a\})$$

But since $\{a\}$ is contractible, by our exact sequence we see that this isomorphic to $H_n(X/A)$.

1.5.13 Definition

An **orientation** on an n-dimensional manifold M is a choice of a generator of $a_p \in H_n(M, M - \{p\})$ such that for every $p \in M$ there is a euclidean neighborhood $\mathcal{U}$ and a choice of generator $a \in H_n(M, M - \mathcal{U})$ such that for every $q \in \mathcal{U}$ with the inclusion $i_q\colon (M, M - \mathcal{U}) \longrightarrow (M, M - \{q\})$ we have $i_{q,*}(a) = a_q$.

If we can choose an orientation of a manifold, call it **orientable**.

Note that since $H_n(M, M - \{p\}) \cong \mathbb{Z}$, there are two choices of orientation for each $p \in M$.

Further note that if we have a path on a manifold, $\gamma\colon I \longrightarrow M$, we can choose an orientation for $p = \gamma(0)$. Then by covering the path with open balls, we can ensure that this orientation is consistent in each open ball. This will give us an orientation for $q = \gamma(1)$. This is independent on the choice of covering of the path. If M is orientable then the orientation of q is also independent on the choice of the path (and is dependent only on p's orientation). Notice then that a closed loop in M must start and end with the same orientation (i.e. it is *orientation-preserving*), in fact this is equivalent to M being orientable.

1.5.14 Theorem

M is orientable if and only if every closed loop in M is orientation-preserving.

So for example, since a loop on the center of the Möbius strip is not orientation-preserving, the Möbius strip is not orientable.

Furthermore, if we have two paths γ, δ which are homotopic relative to their endpoints, then they have the same orientation (i.e. if $p = \gamma(0) = \delta(0)$ is given an orientation, both paths give the same orientation to $q = \gamma(1) = \delta(1)$). Further note that if γ preserves orientation and δ flips orientation then $\gamma * \delta$ flips orientation, and so on for all combinations. So we can assign to orientation-preserving loops the value 0, and to orientation-flipping loops the value 1. For example if γ, δ are both orientation-flipping, the value of $\gamma * \delta$ is 0. By these two facts, we can define a homomorphism

$$\varphi\colon \pi_1(M, b) \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

which assigns to each closed loop on b 0 if it preserves orientation and 1 if it flips orientation. M is orientable if and only if this is the trivial homomorphism for all b (all closed loops preserve orientation).

The issue is to check if M is orientable we must check this homomorphism for every path-connected component of M. But since $\mathbb{Z}/2\mathbb{Z}$ is Abelian and $H_1(M) = \mathrm{Ab}\pi_1(M, b)$, there is an induced homomorphism $H_1(M) \longrightarrow \mathbb{Z}/2\mathbb{Z}$. And M is orientable if and only if this homomorphism is trivial. And this homomorphism is trivial if it is trivial on the generators of $H_1(M)$.

So M is orientable if and only if the generators of $H_1(M)$ preserve orientation.

For example take $M = nT$. All of the generators of $H_1(M)$ (which are the center circles of the torii) preserve orientatiom, so M is orientable.

Note that if M is not orientable, there exists a closed loop on M which flips orientation. This loop can be blown up (since the orientation is taken in a neighborhood) to a quotient of $D^{n-1} \times I$ where $D^{n-1} \times \{0\}$ and $D^{n-1} \times \{1\}$ are identified but with the orientation swapped. Such a space is called a *full Klein bottle*. So M is not orientable if and only if a full Klein bottle can be embedded into it.

1.6 Homology of CW Complexes

Let a CW complex K be constructed out of skeletons $K^0 \subseteq K^1 \subseteq \cdots \subseteq K^m = K$. We would like to compute $H_i(K^n, K^{n-1})$. We claim that there exists an open $\mathcal{U}$ such that $K^{n-1} \subseteq \mathcal{U} \subseteq K^n$ and K^{n-1} is a deformation retract of $\mathcal{U}$. This $\mathcal{U}$ can be taken to include part of the cells added to K^{n-1} (in particular, something like $D^n - \{0\}$). Thus we have that $H_i(K^n, K^{n-1}) = \widetilde{H}_i(K^n/K^{n-1})$.

Recall that K^n is obtained by adding disks to K^{n-1}. So if we contract K^{n-1} to a point, we have essentially just added these disks to a point. And we know that contracting the disk D^n at its boundary to a point is just S^n, so we have that $K^{n-1}/K^n = \bigvee_{f_n} S^n$, and thus

$$H_i(K^n, K^{n-1}) = H_i\left(\bigvee_{f_n} S^n\right) = \begin{cases} \mathbb{Z}^{f_n} & i = n \\ 0 & \text{else} \end{cases}$$

We have an exact sequence

$$H_{i+1}(K^n, K^{n-1}) \longrightarrow H_i(K^{n-1}) \longrightarrow H_i(K^n) \longrightarrow H_i(K^n, K^{n-1})$$

If $n \neq i, i+1$ then $H_i(K^n) = H_i(K^{n-1})$. In particular for $n < i$ we have $H_i(K^n) = 0$ (since $H_i(K^n) = H_i(K^0)$ and the homology group of a set of points is 0). We have a sequence of homomorphisms (not necessarily exact, it is induced by the inclusion maps):

$$0 = H_i(K^{i-1}) \longrightarrow H_i(K^i) \longrightarrow H_i(K^{i+1})$$

So let $A = H_i(K^i)$ and $B = H_i(K^{i+1})$, then we know that for $n > i+1$ we have $H_i(K^n) = H_i(K^{i+1}) = B$. In particular $H_i(K) = H_i(K^{i+1})$. Thus we get the following

1.6.1 Theorem

Let $K^0 \subseteq \cdots \subseteq K^m = K$ be a CW complex. Then

(1)
$$H_i(K^n, K^{n-1}) = \begin{cases} \mathbb{Z}^{f_n} & i = n \\ 0 & \text{else} \end{cases}.$$

(2) $H_i(K^n) = 0$ for $i < n$.

(3) $H_i(K^n) = H_i(K)$ for $n > i$.

Let us define $E_n = H_n(K^n, K^{n-1}) = \mathbb{Z}^{f_n}$. Now, recall that we have two exact sequences:

$$E_n = H_n(K^n, K^{n-1}) \longrightarrow H_{n-1}(K^{n-1})$$

and

$$E_{n-1} = H_{n-1}(K^{n-1}) \longrightarrow H_{n-1}(K^{n-1}, K^{n-2})$$

composing them gives a sequence (not necessarily exact):

$$E_n \xrightarrow{\Delta} H_{n-1}(K^{n-1}) \xrightarrow{i_*} E_{n-1}$$

If we now look at the composition of these maps, we get a sequence

$$\cdots \longrightarrow E_n \longrightarrow E_{n-1} \longrightarrow \cdots \longrightarrow E_0$$

This is a chain complex, as if we look at

$$E_n \xrightarrow{\Delta} H_{n-1}(K^{n-1}) \xrightarrow{i_*} E_{n-1} \xrightarrow{\Delta} H_{n-2}(K^{n-2}) \xrightarrow{i_*} E_{n-2}$$

$$\to \to$$

Let us look at the K^n skeleton of a CW complex, it is of the form $K^{n-1} \coprod_{i=1,\varphi_i}^{f_n} D^n$, meaning

$$K^n = {K^{n-1} \coprod_{i=1}^{f_n} D_i^n} \Big/ {x \sim \varphi_i(x) \text{ for } x \in \partial D_i^n}$$

where $\varphi_i : \partial D_i^n \longrightarrow K^{n-1}$ are the attaching maps. We can look at the sequence

$$D^n \amalg \cdots \amalg D^n \xrightarrow{i} K^{n-1} \amalg D^n \amalg \cdots \amalg D^n \xrightarrow{\rho} K^n$$

where i is the inclusion map and ρ is the quotient map. Note that $\rho \circ i$ restricted to ∂D_i^n s the attaching map, i.e. φ_i.