Calculus Homework #6

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Question 6.1:

Prove that $\frac{1}{\sin(x)+\cos(x)}$ is integrable in $\left[0,\frac{\pi}{2}\right]$, and prove the following inequality:

$$\frac{\pi}{2\sqrt{2}} \leq \int_0^{\frac{\pi}{2}} \frac{dx}{\sin(x)+\cos(x)} \leq \frac{\pi}{2}$$

The domain of the function is every $x \in \mathbb{R}$ such that $\sin(x)+\cos(x) \neq 0$, that is:

$$\sin\left(\frac{\pi}{2}-x\right) \neq \sin(-x) \implies x \neq -\frac{\pi}{4}+\pi k$$

So $\left[0,\frac{\pi}{2}\right]$ is totally within the domain of the function.
Furthermore, since $\sin(x)$ and $\cos(x)$ are continuous in $\left[0,\frac{\pi}{2}\right]$,so is $\frac{1}{\sin(x)+\cos(x)}$, which means it is integrable over $\left[0,\frac{\pi}{2}\right]$ (the integral exists).
Now I will prove that in this domain:

$$\frac{1}{\sqrt{2}} \leq \frac{1}{\sin(x)+\cos(x)} \leq 1$$

This is if and only if (since $\sin(x)+\cos(x) > 0$ in this domain, since for $x=0$, it is 1, and it can't be 0, so since its continuous it cannot be negative in this domain):

$$1 \leq \sin(x)+\cos(x) \leq \sqrt{2}$$

Notice that $f(x) := \sin(x)+\cos(x)$'s derivative is $f'(x) = \cos(x)-\sin(x)$, which equals 0 only once in this domain: at $x=\frac{\pi}{4}$.
Since $f(0)-1 = f\left(\frac{\pi}{2}\right)-1 = 0$, this means that $f(x)-1$ is either positive or negative, but not both (since if there's a place where it goes from negative to positive or vice versa, there must be a point a where $f(a)=0$ since f is continuous, but by Rolle's theorem then there must be two points where $f'-0$, in contradiction). And since $f\left(\frac{\pi}{2}\right)-1>0$, this means $f(x)-1 \geq 0$, so $f(x) \geq 1$, as required.
And similarly for $f(x)-\sqrt{2}$, there is one point where $f'=0$ at $x=\frac{\pi}{4}$, and at this point $f(x)=\sqrt{2}$. And since $f(0)-\sqrt{2}, f\left(\frac{\pi}{2}\right)-\sqrt{2}<0$, this point must be a maximum since the derivative before it is positive ($f'(0)=1$, and the deriavtive is continuous), and the derivative after it is negative ($f'\left(\frac{\pi}{2}\right)=-1$, and the derivative is continuous). So the maximum of $f(x)$ is $\sqrt{2}$.
So we've shown:

$$\frac{1}{\sqrt{2}} \leq \frac{1}{\sin(x)+\cos(x)} \leq 1$$

Which means their integrals follow this inequality as well:

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{2}} \leq \int_0^{\frac{\pi}{2}} \frac{dx}{\sin(x)+\cos(x)} \leq \int_0^{\frac{\pi}{2}} dx$$

These integrals are integrals of constants, so we get:

$$\frac{\pi}{2\sqrt{2}} \leq \int_0^{\frac{\pi}{2}} \frac{dx}{\sin(x)+\cos(x)} \leq \frac{\pi}{2}$$

As required.

Question 6.2:

Prove that:

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\sin(x) + \cos(x)} < \frac{\pi}{2}$$

We showed in the previous question that:

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\sin(x) + \cos(x)} \leq \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}$$

Suppose, for the sake of a contradiction, that:

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\sin(x) + \cos(x)} = \frac{\pi}{2}$$

This means that:

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\sin(x) + \cos(x)} = \int_0^{\frac{\pi}{2}} 1 \, dx$$

And since we proved that:

$$\frac{1}{\sin(x) + \cos(x)} \leq 1$$

This would mean that:

$$\frac{1}{\sin(x) + \cos(x)} = 1$$

Almost always in $\left[0, \frac{\pi}{2}\right]$.
But since since $f(x)$ is continuous, this would mean that $f(x) = 1$ (since the set of points where $f(x) = 1$ would be dense in $\left[0, \frac{\pi}{2}\right]$, so for every x, we could create a series of points x_n whose limit is x and where $f(x_n) = 1$, which would mean that $f(x) = 1$).
So this would mean that $f(x) = 1$, but this is not the case since $f\left(\frac{\pi}{4}\right) = \sqrt{2} \neq 1$, in contradiction.
So:

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\sin(x) + \cos(x)} < \frac{\pi}{2}$$

As required.

Question 6.3:

Find the following sum:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{4n^2-k^2}}$$

Notice that:

$$\frac{1}{\sqrt{4n^2-k^2}}=\frac{1}{n}\cdot\frac{1}{\sqrt{4-\left(\frac{k}{n}\right)^2}}$$

So the sum is equal to:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\cdot\frac{1}{\sqrt{4-\left(\frac{k}{n}\right)^2}}$$

Let:

$$f(x)=\frac{1}{\sqrt{4-x^2}}$$

Since $f(x)$ is continuous over $[0,1]$ (as it is defined over $(-2,2)$ and is the composition of continuous functions), it is integrable over it as well.
And let $\{P_n\}_{n=1}^{\infty}$ be a series of partitions over $[0,1]$ defined by:

$$P_n : 0=x_0<\cdots<x_n<1$$

Where:

$$x_k \coloneqq \frac{k}{n}$$

Then if we define $d_i=x_i$, we get that $\Delta_i=\frac{1}{n}$. Notice that $\lambda(P_n)=\frac{1}{n}$, since all the Δ_is are equal to $\frac{1}{n}$, so $P_n\longrightarrow 0$. Furthermore, notice that:

$$\sigma(P_n)=\sum_{k=1}^{n}\Delta_i f(d_i)=\sum_{k=1}^{n}\frac{1}{n}\cdot\frac{1}{\sqrt{4-\left(\frac{k}{n}\right)^2}}$$

So:

$$\lim_{n\to\infty}\sigma(P_n)=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\cdot\frac{1}{\sqrt{4-\left(\frac{k}{n}\right)^2}}$$

This is the sum we want to find!
And since $f(x)$ is integrable over $[0,1]$, this means that:

$$\lim_{n\to\infty}\sigma(P_n)=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\cdot\frac{1}{\sqrt{4-\left(\frac{k}{n}\right)^2}}=\int_0^1\frac{dx}{\sqrt{4-x^2}}$$

So all that remains is to find this integral.
Notice that the definite integral is equal to:

$$\frac{1}{2}\int\frac{dx}{\sqrt{1-\left(\frac{x}{2}\right)^2}}$$

Let $u=\frac{x}{2}$, so $dx=2du$, we get that this is equal to:

$$=\int\frac{du}{\sqrt{1-u^2}}=\sin^{-1}(u)+C=\sin^{-1}\left(\frac{x}{2}\right)+C$$

So:

$$\int_0^1 \frac{dx}{\sqrt{4-x^2}} = \sin^{-1}\left(\frac{x}{2}\right)\Big|_0^1 = \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0) = \frac{\pi}{6}$$

So all in all we get:

$$\boxed{\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{4n^2-k^2}} = \frac{\pi}{6}}$$

Question 6.4:

A function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is **convex** if for every $a, b \in \mathbb{R}$ and for every $\lambda \in [0,1]$:

$$f\left(\lambda a + (1-\lambda)b\right) \leq \lambda f\left(a\right) + (1-\lambda) f\left(b\right)$$

In the following questions, $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is convex.

(1) Suppose f is differentiable and $g\colon \mathbb{R} \longrightarrow \mathbb{R}$ is continuous. Prove:

$$f\left(\int_0^1 g\, dx\right) \leq \int_0^1 f(g)\, dx$$

(2) Suppose f is non-negative and $f(1) = 1$, prove:

$$\int_0^2 f\, dx \geq 1$$

Lemma:

If f is **convex**, then for every $\{\Delta_i\}_{i=1}^n \in [0,1]$ such that $\sum\limits_{i=1}^n \Delta_i = 1$, the following is true:

$$f\left(\sum_{i=1}^n \Delta_i x_i\right) \leq \sum_{i=1}^n \Delta_i \cdot f\left(x_i\right)$$

Proof:

This is a simple proof by induction on n.
Base case: This is trivial for $n = 1$, and is true by the definition of convexity for $n = 2$ (since $\Delta_2 = 1 - \Delta_1$).
Inductive step: Suppose this is true for n, then let $\{\Delta_i\}_{i=1}^{n+1} \in [0,1]$ satisfy:

$$\sum_{i=1}^n \Delta_i = 1$$

So:

$$f\left(\sum_{i=1}^{n+1} \Delta_i x_i\right) = f\left(\sum_{i=1}^n \Delta_i x_i + \Delta_{n+1} x_{n+1}\right) = f\left((1-\Delta_{n+1})\left(\sum_{i=1}^n \frac{\Delta_i}{1-\Delta_{n+1}} x_i\right) + \Delta_{n+1} x_{n+1}\right)$$

Which, by the definition of convexity, is less than:

$$\leq (1-\Delta_{n+1}) f\left(\sum_{i=1}^n \frac{\Delta_i}{1-\Delta_{n+1}} x_i\right) + \Delta_{n+1} f\left(x_{n+1}\right)$$

Now notice that:

$$\frac{\Delta_i}{1-\Delta_{n+1}} \leq 1 \iff \Delta_i \leq 1 - \Delta_{n+1} \iff \Delta_i + \Delta_{n+1} \leq 1$$

Which is true, since the Δ_is are non-negative and their sum is 1.
Furthermore:

$$\sum_{i=1}^n \frac{\Delta_i}{1-\Delta_{n+1}} = \frac{\sum_{i=1}^n \Delta_i}{1-\Delta_{n+1}}$$

And recall that:

$$\sum_{i=1}^{n+1} \Delta_i = 1 \implies \sum_{i=1}^{n} \Delta_i = 1 - \Delta_{n+1}$$

So:

$$\sum_{i=1}^{n} \frac{\Delta_i}{1-\Delta_{n+1}} = \frac{1-\Delta_{n+1}}{1-\Delta_{n+1}} = 1$$

So $\left\{\frac{\Delta_i}{1-\Delta_{n+1}}\right\}_{i=1}^{n}$ satisfy the restrictions for the inductive hypothesis:

$$(1-\Delta_{n+1})f\left(\sum_{i=1}^{n} \frac{\Delta_i}{1-\Delta_{n+1}} x_i\right) \leq (1-\Delta_{n+1}) \cdot \sum_{i=1}^{n} \frac{\Delta_i}{1-\Delta_{n_1}} f(x_i) = \sum_{i=1}^{n} \Delta_i f(x_i)$$

Which means that:

$$f\left(\sum_{i=1}^{n+1} \Delta_i x_i\right) \leq \sum_{i=1}^{n} \Delta_i f(x_i) + \Delta_{n+1} f(x_{n+1}) = \sum_{i=1}^{n+1} \Delta_i f(x_i)$$

As required.

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(1) Since g is continuous, it has an integral over $[0,1]$. And since f is differentiable, it is continous, and therefore so is $f \circ g$, which means it has an integral over $[0,1]$ as well.
Let $\{P_n\}_{n=1}^{\infty}$ be a set of pointed partitions such that $P_n \longrightarrow 0$. Which means that:

$$\sigma(P_n) = \sum_{i=1}^{n} \Delta_i g(d_i) \longrightarrow \int_0^1 g\ dx$$

Notice that:

$$f(\sigma_g(P_n)) = f\left(\sum_{i=1}^{n} \Delta_i g(d_i)\right)$$

And since P_n is a partition of $[0,1]$, this means that $\sum\limits_{i=1}^{n} \Delta_i = 1$, so by our lemma above:

$$\leq \sum_{i=1}^{n} \Delta_i f(g(d_i)) = \sigma_{f \circ g}(P_n)$$

So we have that:

$$f(\sigma_g(P_n)) \leq \sigma_{f \circ g}(P_n)$$

Since f is continuous, the limit of the left side is:

$$\lim_{n\to\infty} f(\sigma_g(P_n)) = f\left(\lim_{n\to\infty} \sigma_g(P_n)\right) = f\left(\int_0^1 g\ dx\right)$$

And the limit of the right side is:

$$\lim_{n\to\infty} \sigma_{f \circ g}(P_n) = \int_0^1 f(g)\ dx$$

(This is true since the integrals exist and $P_n \longrightarrow 0$.)
And since limits preserve weak inequalities, we have:

$$f\left(\int_0^1 g\ dx\right) \leq \int_0^1 f(g)\ dx$$

As required.

(2) Let's try and make this as simple as possible. We can start by letting $\lambda = \frac{1}{2}$, so $\lambda = 1 - \lambda = \frac{1}{2}$. So we have:

$$f\left(\frac{1}{2}(a+b)\right) \leq \frac{1}{2}(f(a) + f(b))$$

And let's require that:

$$\frac{1}{2}(a+b) = 1 \implies b = 2 - a$$

Since we know what $f(1)$ is. And *something* has to be x, so let $a = x$, which means $b = 2 - x$.
So we get:

$$f(1) \leq \frac{1}{2}(f(x) + f(2-x)) \implies 2 \leq f(x) + f(2-x)$$

Integrating the left side between 0 and 2 yields $2 \cdot (2 - 0) = 4$, and integrating the right side gives:

$$\int_0^2 f(x)\, dx + \int_0^2 f(2-x)\, dx$$

Now, notice that both of these integrals are the same, the right one just is f flipped in $[0, 2]$. This can be proven by letting $u = 2 - x$, so we get $du = -dx$, so the right integral is equal to:

$$-\int_{u(0)}^{u(2)} f(u)\, du = -\int_2^0 f(u)\, du = \int_0^2 f(u)\, du = \int_0^2 f(x)\, dx$$

So we have:

$$4 \leq 2 \cdot \int_0^2 f(x)\, dx \implies 2 \leq \int_0^2 f(x)\, dx$$

And if the integral is greater than 2, it is greater than 1, as required.