Complex Functions

Assignment 7

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Exercise 7.1:

Show directly that on a compact domain D, the maximal and minimal modulus of e^z are attained on the boundary.

Firstly, $|e^z|$ does attain a maximum value since it is continuous and D is compact. We know that $|e^z| = e^{\operatorname{Re} z}$, thus the maximum of e^z is when $\operatorname{Re} z$ is maximal, and the minimum is when $\operatorname{Re} z$ is minimal. But if $\operatorname{Re} z$ is maximal, then for every $\varepsilon > 0$, $\operatorname{Re}(z+\varepsilon) = \operatorname{Re} z + \varepsilon > \operatorname{Re} z$ and since $\operatorname{Re} z$ is maximal, $z + \varepsilon$ is not in D, thus $D_\varepsilon(z) \cap D, D_\varepsilon(z) \neq \varnothing$ meaning z is on the boundary of D. Similar for when z induces a minimum $(z - \varepsilon)$.

Exercise 7.2:

Find the minimum and maximum modulus of $z^2 - z$ on the closed disk $|z| \leq 1$.

Since $z^2 - z$ is a polynomial and thus entire, it attains maximum and minimum on the boundary of the disk, ie. when $|z| = 1$, or for the minimum when $\left|z^2 - z\right| = 0$. So

$$\left|z^2 - z\right| = |z||z-1| = |z-1|$$

This is the distance from the point $(1,0)$ on the circle of radius 1 about 0, and thus the maximum distance is attained at $(-1,0)$, for $\left|(-1)^2 + 1\right| = |2| = 2$. And the minimum is attained whenever $\left|z^2 - z\right| = 0$, which can be attained on the boundary at $z = 1$ or on the interior at $z = 0$.

Exercise 7.3:

Suppose $\{f_i\}_{i=1}^n$ are analytic on a compact domain D. Show that the maximum of $f(z) = \sum_{i=1}^n |f_i(z)|$ is obtained on the boundary of D.

Firstly, such a maximum is obtained in D since the function is continuous and D is compact. Now, suppose it is obtained at $z_0 \in D$. Then suppose that for each $1 \leq j \leq n$,

$$f_j(z_0) = |f_j(z_0)| \cdot e^{i\theta_j} \implies |f_j(z_0)| = f_j(z_0) \cdot e^{-i\theta_j}$$

let $\omega_j = e^{-i\theta_j}$, thus we have

$$|f_j(z_0)| = f_j(z_0) \cdot \omega_j$$

And so let us define

$$g(z) = \sum_{i=1}^n f_i(z) \cdot \omega_i$$

Notice that

$$|g(z)| = \left|\sum_{i=1}^n f_i(z) \cdot \omega_i\right| \leq \sum_{i=1}^n |f_i(z)| \cdot |\omega_i| = \sum_{i=1}^n |f_i(z)| = f(z) \leq f(z_0)$$

So $|g|$ is bounded by $f(z_0)$. But at the same time,

$$g(z_0) = \sum_{i=1}^n f_i(z_0) \cdot \omega_i = \sum_{i=1}^n |f_i(z_0)| = f(z_0)$$

and thus $|g(z_0)| = f(z_0)$, so z_0 induces a maximum of g. And since g is analytic on D as well, as the linear combination of analytic functions, this means that z_0 is on the boundary of D.