Infinitesimal Calculus 3

Lecture 23, Sunday January 22, 2023
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By the previous theorem, f is integrable if and only if for every $\varepsilon > 0$ there is a partition such that $\bar{s}(f,P) - \underset{\sim}{s}(f,P) < \varepsilon$, and this is simply

$$\sum_{j=1}^{n}(M_j - m_j)|D_j| < \varepsilon$$

where $P = \{D_j\}_{j=1}^{n}$.

Theorem 23.1:

Let $D \subseteq \mathbb{R}^n$ compact and content and $f\colon D \longrightarrow \mathbb{R}$ continuous, then f is integrable over D.

Proof:

By Weirstrauss, f is bounded, which is a necessary condition for integrability. Let $\varepsilon > 0$, since D is compact, f is uniformly continuous over it, so there exist a $\delta > 0$ such that for every $\|x-y\| < \delta$ in D, $|f(x) - f(y)| < \frac{\varepsilon}{2|D|}$. Now we take a partition $P = \{D_j\}_{j=1}^{k}$ where $\lambda(P) < \delta$. So for every $x, y \in D$ in the same D_j, $\|x-y\| < \delta$ so $|f(x) - f(y)| < \frac{\varepsilon}{2|D|}$ and so $|M_j - m_j| \leq \frac{\varepsilon}{2|D|} < \frac{\varepsilon}{|D|}$, so

$$\bar{s}(f,P) - \underset{\sim}{s}(f,P) = \sum_{j=1}^{k}(M_j - m_j)|D_j| < \sum_{j=1}^{k}\frac{\varepsilon}{|D|}|D_j| = \varepsilon$$

so f is integrable as required.

$\blacksquare$

Note that we showed that if f is continuous in D contented compact then

$$\lim_{\lambda(P)\to 0} \bar{s}(f,P) - \underset{\sim}{s}(f,P) = 0$$

since it is less than every $\varepsilon > 0$, and since

$$\underset{\sim}{s}(f,P) \leq \int_D f \leq \bar{s}(f,P) \implies 0 \leq \bar{s}(f,P) - \int_D f \leq \bar{s}(f,P) - \underset{\sim}{s}(f,P) \longrightarrow 0$$

and so

$$\int_D f = \lim_{\lambda(P)\to 0} \bar{s}(f,P) = \lim_{\lambda(P)\to 0} \underset{\sim}{s}(f,P)$$

Definition 23.2:

Suppose $P = \{D_j\}_{j=1}^{k}$ is a partition of D and $f\colon D \longrightarrow \mathbb{R}$ is bounded. Then a **Riemann sum** of f over P is a sum of the form

$$\sum_{j=1}^{k} f(x_j)|D_j|$$

where $x_j \in D_j$.

Notice then that by definition $\bar{s}(f,P)$ is the supremum of all Riemann sums over P and $\underset{\sim}{s}(f,P)$ is the infimum, and so if s is a Riemman sum:

$$\underset{\sim}{s}(f,P) \leq s \leq \bar{s}(f,P)$$

Definition 23.3:

A function is **Riemann Integrable** in D if when $\lambda(P) \longrightarrow 0$ all the Riemann sums over P converge to the same limit L. Then we say

$$\int_D f = L$$

We say that for our previous definition of integrability, f is *Darboux Integrable*. But we will prove that it doesn't matter, both definitions are equivalent.

Theorem 23.4:

Let $D \in \mathbb{R}^n$ compact and $f\colon D \longrightarrow \mathbb{R}$ bounded, then f is Riemann integrable if and only if it is Darboux integrable.

Proof:

Since for every Riemman sum over P we have

$$\underline{s}(f, P) \leq s(P) \leq \bar{s}(f, P)$$

and if f is Darboux integrable then the limits on both sides are the same, say L. So the limit of $s(P)$ as $\lambda(P) \longrightarrow 0$ is equal to L by the squeeze theorem.
Now suppose f is Riemann integrable, then for every Riemman sum over the partition P, we have that $\lim\limits_{\lambda(P)\to 0} s(P) = L$.
And since the upper and lower Darboux sums are the supremum and infimum of the Riemman sums, this means that they too must converge to L.

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Proposition 23.5:

Suppose $D \subseteq \mathbb{R}^n$ is a compact domain and $f, g\colon D \longrightarrow \mathbb{R}$ integrable and $c \in \mathbb{R}$ constant, then

(1) $f + cg$ is integrable over D and satisfies

$$\int_D f + cg = \int_D f + c\int_D g$$

(2) If $f \leq g$ in D then

$$\int_D f \leq \int_D g$$

(3) If $\{D_i\}_{i=1}^k$ is a partition of D then f is integrable over each D_i and

$$\int_D = \sum_{i=1}^{k} \int_{D_i} f$$

(4)

$$\int_D 1 = |D|$$

(5) If $m \leq f \leq M$ in D then

$$m|D| \leq \int_D f \leq M|D|$$

(6) If f is continuous in D then there is an $x_0 \in D$ such that

$$\int_D f = f(x_0)|D|$$

(7) $|f|$ is integrable and

$$\left|\int_D f\right| \leq \int_D |f|$$

Proof:

(1) For any partition P:

$$s(f+cg,P)=\sum_{i=1}^{k}\big(f(x_i)+cg(x_i)\big)|D_i|=\sum_{i=1}^{k}f(x_i)|D_i|+c\sum_{i=1}^{k}g(x_i)|D_i|=s(f,P)+c\cdot s(g,P)$$

and thus taking limits as $\lambda(P)$ approaches 0 yields

$$\int_D f+cg=\int_D f+c\int_D g$$

(2) Since $|D_i|\geq 0$ we have that

$$s(f,P)=\sum_{i=1}^{k}f(x_i)|D_i|\leq\sum_{i=1}^{k}g(x_i)|D_i|=s(g,P)$$

so taking the limit gives the inequality.

(3) Take P_j to be a partition of D_j then $P=\bigcup P_j$ is a partition of D then it is simple to show that

$$\bar{s}(f,P)=\sum_{j=1}^{k}\bar{s}(f,P_j)\qquad \underline{s}(f,P)=\sum_{j=1}^{k}\underline{s}(f,P_j)$$

And so

$$\bar{s}(f,P)-\underline{s}(f,P)=\sum_{j=1}^{k}\bar{s}(f,P_j)-\underline{s}(f,P_j)$$

since the right side is a sum of non-negatives it follows that for every j

$$0\leq\bar{s}(f,P_j)-\underline{s}(f,P_j)\leq\bar{s}(f,P)-\underline{s}(f,P)$$

thus taking the limit of the right side (since for every $\lambda(P)$ we can create a partition of P_js as needed) gives that the difference between the upper and lower sums converges to 0 and therefore f is integrable over D_j. And so

$$\int_D=\lim_{\lambda(P)\to 0}\bar{s}(f,P)=\lim\sum_{j=1}^{k}\bar{s}(f,P_j)=\sum_{j=1}^{k}\int_{D_j}f$$

(4) This is simple: for every partition P:

$$s(1,P)=\sum_{i=1}^{k}|D_j|=|D|$$

and so the integral is also $|D|$.

(5) By above we know that $\int m=m|D|$ and $\int M=M|D|$ and since $m\leq f\leq M$:

$$m|D|=\int_D m\leq\int_D f\leq\int_D M=M|D|$$

(6) Since D is compact, f has a maximum and minimum so $m|D|\leq\int f\leq M|D|$, so

$$m\leq\frac{1}{|D|}\int_D f\leq M$$

And since f is continuous, for every value between m and M, there is an element $x_0\in D$ such that $f(x_0)$ is equal to it. And specifically there is an $x_0\in D$ such that

$$f(x_0)=\frac{1}{|D|}\int_D f$$

as required.

(7) For any partition P of D, we know that $M_j(|f|) - m_j(|f|) \leq M_j(f) - m_j(f)$. And so

$$0 \leq \bar{s}(|f|, P) - \underline{s}(|f|, P) \leq \bar{s}(f, P) - \underline{s}(f, P)$$

and since f is integrable, this means that so is $|f|$. And since both f and $-f$ are both less than or equal to $|f|$, their integrals, which are $\int f$ and $-\int f$ are less than $\int |f|$, so

$$\left| \int_D f \right| = \pm \int_D f \leq \int_D |f|$$

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Proposition 23.6:

Suppose D is a domain which can be separated into $D = \bigcup_{i=1}^k D_i$. If f is integrable over each D_i then it is integrable over D and

$$\int_D f = \sum_{i=1}^k \int_{D_i} f$$

The proof of this is similar to the proof of the converse we showed above. Since

$$\bar{s}(f, P) - \underline{s}(f, P) = \sum_{i=1}^k \bar{s}(f, P_j) - \underline{s}(f, P_j)$$

choose P_js such that the right difference is less than $\frac{\varepsilon}{k}$ and then we get that $\bar{s}(f, P) - \underline{s}(f, P) < \varepsilon$ as required. Then by above

$$\int_D f = \sum_{i=1}^k \int_{D_i} f$$