Introduction to Stochastic Processes

Assignment 4
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4.1 Exercise

Given the following transition matrix,

$$P = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

(1) Find the general form of a stationary distribution of P.

(2) Compute $\mathbb{E}[T_5 \mid X_0 = 5]$ and $\mathbb{E}[T_6 \mid X_0 = 6]$.

(1) A stationary distribution of P is just an eigenvector of $P^\top$ whose coefficients are all nonnegative and sum to 1. The eigenspace of eigenvalue 1 of $P^\top$ can be readily computed through row reduction, and it gives $\mathrm{span}\{(0,0,0,0,1,1),\ (0,0,1,0,0,0)\}$. So the stationary distributions of P are

$$\left(0, 0, 1-p, 0, \frac{p}{2}, \frac{p}{2}\right) \qquad (0 \leq p \leq 1)$$

(2) Since $\{5,6\}$ is an irreducible component, there exists a unique stationary distribution π which is zero on all coefficients other than 5 and 6. And for $k = 5, 6$, $\pi_k = \frac{1}{\mathbb{E}_k[T_k]}$. This stationary distribution is $\left(0,0,0,0,\frac{1}{2},\frac{1}{2}\right)$, and so $\mathbb{E}[T_6 \mid X_0 = 6] = \mathbb{E}[T_5 \mid X_0 = 5] = 2$. And

$$\begin{aligned} \mathbb{E}[T_6 \mid X_0 = 5] = \mathbb{E}[T_6 \mid X_1 = 6] \cdot \mathbb{P}(X_1 = 6 \mid X_0 = 5) + \mathbb{E}[T_6 \mid X_1 = 5] \cdot \mathbb{P}(X_1 = 5 \mid X_0 = 5) \\ = \frac{1}{3} + \left(1 + \mathbb{E}[T_6 \mid X_0 = 5]\right) \cdot \frac{2}{3} \end{aligned}$$

Solving this gives $\mathbb{E}[T_6 \mid X_0 = 5] = 3$.

4.2 Exercise

You are given a deck of 52 cards, enumerated from 1 to 52, to shuffle. In order to shuffle them, you take the top card and arbitrarily place it back into the deck.

(1) Is there a limit to the probability that the cards are ordered from 1 to 52 after n shuffles, as n goes to infinity? If so, what is it?

(2) If the deck begins ordered, what is the expected amount of shuffles it will take for the deck to become ordered again?

(1) Let $S = S_{52}$ be the state space of all possible orderings of the cards. Notice that each state is connected with equal probability to 52 other states, as the top card can be inserted into 52 different spots in the deck with equal probability. So this Markov chain, which we will denote by $\{X_n\}$, corresponds to a random walk on the (directed) graph S (the edges of this graph aren't important). This graph is obviously connected, and it is also aperiodic since if a is any state it is possible to reach a again in a single move by placing the card back on top. (Note that if this move isn't allowed, then a can be reached in two moves by placing the card below the new top card, and then placing the new top card beneath that. And it can similarly be reached in 3 moves, and since these are coprime, $d(a) = 1$ still.)

So by the convergence theorem for finite state, irreducible, aperiodic Markov chains $\mathbb{P}(X_n = \text{id} \mid X_0 \sim v) \xrightarrow{n\to\infty} \pi(\text{id})$ where π is the chain's unique stationary distribution. So it does indeed converge. And since the Markov chain is irreducible, we have that $\pi_a = \frac{1}{\mathbb{E}_a[T_a]}$. By symmetry, we have that $\mathbb{E}_a[T_a] = \mathbb{E}_b[T_b]$ for all two states a, b (since there is a symmetry between starting at a and b since the way we can shuffle cards is not affected by the current permutation). This means that $\pi = \frac{1}{|S|}\mathbf{1}$ (where $\mathbf{1}$ is the vector of all 1s), and so $\pi(\text{id}) = \frac{1}{|S|} = \frac{1}{52!}$, so the probability converges to $\frac{1}{52!}$.

Instead of relying on symmetry, we can just show that $\mathbf{1}$ is a row eigenvector of P (since $P(x \to y) = \frac{1}{52}$ if x is connected to y and 0 otherwise),

$$(\mathbf{1}P)_y = \sum_{x\in S} P_{xy} = \sum_{x\in S} \frac{1}{52}\delta(x \to y) = \frac{1}{52}\sum_{x\in S}\delta(x\to y) = \frac{\text{indeg}(y)}{52}$$

The in-degree of y is 52 as there are exactly possible 52 previous states, depending which card was previously at the top. So $(\mathbf{1}P)_y = 1$ meaning $\mathbf{1}P = \mathbf{1}$, so $\pi = \frac{1}{|S|}\pi$ as required.

(2) As said before, $\pi_a = \frac{1}{\mathbb{E}_a[T_a]}$ and since $\pi_a = \frac{1}{|S|} = \frac{1}{52!}$ we have that $\mathbb{E}_a[T_a] = 52!$ for all states a, and in particular for $a = \text{id}$. So the expected hitting time is 52!.

4.3 Exercise

N balls are divided between two containers, one white and one black. At every step a ball is randomly chosen and moved to the other container. Let X_k be the number of balls in the white container after k steps.

(1) Show that $\{X_k\}$ is a Markov chain over the state space $\{0, \ldots, N\}$, and compute its transistion matrix.

(2) Show that $\text{Bin}\left(N, \frac{1}{2}\right)$ is a stationary distribution of the Markov chain, meaning

$$\pi(j) = \binom{N}{j}2^{-N}$$

(3) Assuming $X_0 = 0$, does X_n converge in distribution? Does X_{2n} converge in distribution? If so, to what limit?

(4) Assuming that N is even, show that the expected return time to 0 is at least 2^N, and that the expected return time to $\frac{N}{2}$ is at most $10\sqrt{N}$.

(1) It is obviously a Markov chain as the number of balls in the white container is dependent only on how many balls was in the container previously.

$$P(i \to j) = \begin{cases} \frac{i}{N} & j = i-1 \\ 1 - \frac{i}{N} & j = i+1 \\ 0 & \text{else} \end{cases}$$

since the probability of going from i balls to $i-1$ balls is the probability of choosing one of the i balls in the white container, and the probability of going from i balls to $i+1$ balls is the probability of choosing one of the balls in the black container.

(2) We compute πP, for $0 < j < N$:

$$(\pi P)_j = \sum_{i=0}^{N} \pi_i P(i \to j) = \pi(j-1)P(j-1\to j) + \pi(j+1)P(j+1\to j)$$
$$= 2^{-N}\left(\binom{N}{j-1}\cdot\frac{N-j+1}{N} + \binom{N}{j+1}\cdot\frac{j-1}{N}\right)$$

And

$$\begin{aligned}\binom{N}{j-1}\cdot\frac{N-j+1}{N} + \binom{N}{j+1}\cdot\frac{j+1}{N} &= \frac{1}{N}\cdot\left(\frac{N!\cdot(N-j+1)}{(j-1)!(N-j+1)!} + \frac{N!(j+1)}{(j+1)!(N-j-1)!}\right)\\ &= \frac{1}{N}\cdot\left(\frac{N!}{(j-1)!(N-j)!} + \frac{N!}{j!(N-j-1)!}\right)\\ &= \frac{N!}{N}\cdot\frac{N}{j!(N-j)!} = \binom{N}{j}\end{aligned}$$

So we have that $(\pi P)_j = \binom{N}{j}2^{-N} = \pi_j$ for $0 < j < N$. And for $j = 0, N$:

$$(\pi P)_0 = \pi(1)P(1 \to 0) = 2^{-N} \cdot \binom{N}{1} \cdot \frac{1}{N} = 2^{-N}\binom{N}{0} = \pi_0$$
$$(\pi P)_N = \pi(N-1)P(N-1 \to N) = 2^{-N} \cdot \binom{N}{N-1}\frac{1}{N} = 2^{-N}\binom{N}{N} = \pi_N$$

So $\pi P = \pi$ as required.

(3) Firstly notice that X_n is periodic since for every state $a \in S$, its period is 2. This is since in order to get back to a, an even amount of steps must be taken, as the same amount of balls must be moved from the white container to the black and vice versa. And furthermore $P^2(a \to a) > 0$ since it is possible to move a ball from white to black, then black to white. So the convergence theorem does not hold here. And in general the probability that $X_n = a$ if $n \not\equiv i \pmod 2$ is zero, and otherwise it is non-zero. So $\mathbb{P}_0(X_n = a)$ cannot converge, meaning X_n does not converge in probability.

But X_{2n} is indeed aperiodic since $P^2(a \to a) > 0$ (this corresponds to $n = 1$, since the transition matrix for X_{2n} is P^2). And since the period of X_n is 2, X_{2n} is irreducible, so by the convergence theorem X_{2n} converges in probability to its stationary distribution. Since the stationary distribution of P is the same as P^2 ($\pi P^2 = \pi P = \pi$), X_{2n} converges in distribution to $\pi = \mathrm{Bin}\big(N, \frac{1}{2}\big)$. In other words

$$\mathbb{P}_0(X_{2n} = i) \xrightarrow{n \to \infty} \binom{N}{i}2^{-N}$$

(4) We want to compute $\mathbb{E}_0[T_0]$ and $\mathbb{E}_{N/2}\big[T_{N/2}\big]$. Since X_n is irreducible, $\pi(i) = \frac{1}{\mathbb{E}_i[T_i]}$ for all $0 \le i \le N$ and so $\mathbb{E}_0[T_0] = \frac{1}{\pi(0)} = 2^N$. And $\mathbb{E}_{N/2}\big[T_{N/2}\big] = \frac{1}{\pi(N/2)} = \frac{2^N}{\binom{N}{N/2}} = \frac{2^N (N/2)!(N/2)!}{N!}$. Using the inequalities: $\sqrt{2\pi N}\big(\frac{N}{e}\big)^N \exp\big(\frac{1}{12N+1}\big) \le N! \le \sqrt{2\pi N}\big(\frac{N}{e}\big)^N \exp\big(\frac{1}{12N}\big)$, we get

$$\mathbb{E}_{N/2}\big[T_{N/2}\big] \le \frac{\exp\big(\frac{1}{3N}\big)}{\exp\big(\frac{1}{12N+1}\big)} \cdot \sqrt{2\pi} \cdot \sqrt{N} = \exp\left(\frac{1}{3N} - \frac{1}{12N+1}\right) \cdot \sqrt{2\pi} \cdot \sqrt{N}$$

Since $\frac{1}{3N} - \frac{1}{12N+1}$ is decreasing, this takes a maximum at $N = 1$ and so we can bound this by $\exp\big(\frac{1}{3} - \frac{1}{13}\big) \le 1.3$. So we get that the expected value is $\le 1.3\sqrt{2\pi}\sqrt{N} \le 3.3\sqrt{N}$.