Infintesimal Calculus 3

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Just like how the focus of Calculus 1 and 2 were of single value functions:

$$f\colon X \subseteq \mathbb{R} \longrightarrow \mathbb{R}$$

The focus of this course will be on functions:

$$f\colon X \subseteq \mathbb{R} \longrightarrow \mathbb{R}^m$$

1.1 A Soft Review of Linear Algebra

Definition 1.1.1:

We will recall that a **normed vector space** V is a vector space equipped with a norm function:

$$\|\cdot\| : W \longrightarrow \mathbb{R}$$

Which satisfies the following axioms:

- Positivity: $\|v\| \geq 0$ for every $v \in V$.
- Homogeneity: $\|\alpha v\| = |\alpha| \cdot \|v\|$ for every $\alpha \in \mathbb{R}$ and $v \in V$.
- The Triangle Inequality: $\|v + u\| \leq \|v\| + \|u\|$ for every $v, u \in V$.

Notice that by the triangle inequality:

$$\|v - u\| + \|u\| \geq \|v\| \qquad \|v - u\| + \|v\| \geq \|u\|$$

So we have that

$$\|v - u\| \geq |\|v\| - \|u\||$$

Example:

Some examples of norms over $\mathbb{R}^n$ are:

- $\|x\|_\infty = \max |x_1|, \ldots, |x_n|$
- $\|x\|_p = \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}}$ for $1 \leq p < \infty$.

These are actually special cases of the more general ℓ^p norm, which itself can be seen as a special case of the $L^p(\mathbb{R})$ norm.

Example:

The set:

$$C([a, b]) \coloneqq \{f\colon [a, b] \longrightarrow \mathbb{R} \mid f \text{ is continuous}\}$$

is also a vector space, and we can norm it by the $\|\cdot\|_{\max}$ norm:

$$\|f\|_{\max} \coloneqq \max_{x \in [a,b]} f(x)$$

We can also define the $\|\cdot\|_p$ norm:

$$\|f\|_p \coloneqq \left(\int_a^b |f(x)|^p \, dx\right)^{\frac{1}{p}}$$

This too is a specific case of the $L^p[a, b]$ norm.

Note that $C[a,b]$ is infinite-dimensional. We can show this by showing that the set $\{x^n\}_{n\in\mathbb{N}} \subseteq C[a,b]$ is linearly independent. Suppose then that there is a finite sum of x^ns which equals 0:

$$\sum_{k=1}^{n} a_k x^{k_n} = 0$$

Notice that the sum above is a polynomial, we will let it equal $p(x)$, so $p \equiv 0$. But we know that a polynomial is identically 0 if and only if its coefficients are all 0, meaning that $a_k = 0$. So $\{x^n\}_{n\in\mathbb{N}}$ is indeed linearly independent, so $C[a,b]$ is infinite dimensional.

Definition 1.1.2:

Again recall that given a vector space V, an **inner product** on V is a function:

$$\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{R}$$

Such that:

- $\langle v, v \rangle \geq 0$ and is 0 if and only if $v = 0$.
- $\langle \alpha v, u \rangle = \alpha \langle v, u \rangle$
- $\langle v, u \rangle = \overline{\langle u, v \rangle}$. Since we are working over $\mathbb{R}$ we can simplify this to $\langle v, u \rangle = \langle u, v \rangle$.

If V has an inner product, it is called an **inner product space**.

An inner product also generates a norm, we can define:

$$\|v\| = \sqrt{\langle v, v \rangle}$$

Example:

Over $C[a,b]$ we can define:

$$\langle f, g \rangle = \int_a^b f(x) \cdot g(x)\, dx$$

This inner product actually generates the $\|\cdot\|_2$ norm.

Example:

The ℓ^p space is the space of all infinite sequences $\{a_n\}_{n=1}^{\infty}$ such that:

$$\sum_{n=1}^{\infty} |a_n|^p < \infty$$

And we define the ℓ^p norm, also denoted $\|\cdot\|_p$ to be the pth root of this.
Over ℓ^2 we can define an inner product:

$$\langle \{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \rangle = \sum_{n=1}^{\infty} a_n \cdot b_n$$

It can be seen that this is well-defined and it is trivial to see that this generates the ℓ^2 norm.

Definition 1.1.3:

Given a normed vector space V, we can define the **distance** metric to be:

$$d(v,u) = \|v - u\|$$

Definition 1.1.4:

A set M equipped with a function:

$$d(\cdot,\cdot)\colon M \times M \longrightarrow \mathbb{R}_{\geq 0}$$

Which satisfies the following:

- Positivity: $d(v,u) \geq 0$ and is 0 if and only if $v = u$.
- Symmetry: $d(v,u) = d(u,v)$.
- The Triangle Inequality: $d(v,u) \leq d(v,w) + d(w,u)$.

Notice then that a normed vector space is a metric space since:

$$d(v,w) + d(w,u) = \|v - w\| + \|w - u\| \geq \|v - u\| = d(v,u)$$

And the other two requirements are simple to prove.

Example:

Not every metric space is a vector space. We can define the $\mathbb{S}^n$ space as:

$$\mathbb{S}^n \coloneqq \left\{ v \in \mathbb{R}^{n+1} \;\middle|\; \|v\| = 1 \right\} \subset \mathbb{R}^{n+1}$$

We can define a metric on $\mathbb{S}^n$ to be the length of the smallest arc between two points. This is obviously not a vector space, but it *is* a metric space.