Complex Functions

Lecture 10, Wednesday June 21, 2023

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Notice that if $f(z)$ has a pole of degree m at $z = \alpha$, then

$$f(z) = \sum_{k=-m}^{\infty} c_k (z-\alpha)^k$$

and so

$$(z-\alpha)^m f(z) = \sum_{k=-m}^{\infty} c_k (z-\alpha)^{k+m} = \sum_{k=0}^{\infty} c_{k-m} (z-\alpha)^k$$

Meaning that c_{-1} is the coefficient of $(z-\alpha)^{m-1}$ in $(z-\alpha)^m f(z)$. So let $g(z) = (z-\alpha)^m f(z)$ be the analytic continuation of $(z-\alpha)^m f(z)$ to include α, and then we have that

$$g^{(m-1)}(\alpha) \cdot \frac{1}{(m-1)!} = c_{-1}$$

And since $(z-\alpha)^m f(z)$ is analytic about α, so are its derivatives and so we get the following

Proposition 10.1:

If $f(z)$ has a pole of degree m at $z = \alpha$ then

$$\mathrm{Res}(f(z), \alpha) = \frac{1}{(m-1)!} \cdot \lim_{z \to \alpha} \frac{d^{m-1}}{dz^{m-1}} \big((z-\alpha)^m f(z)\big)$$

In the case that α is a simple pole, $m = 1$ and so

$$\mathrm{Res}(f(z), \alpha) = \lim_{z \to \alpha} (z-\alpha) f(z)$$

Definition 10.2:

Let γ be a smooth closed curve and $\alpha \in \mathbb{C}$ is not on γ, then

$$n(\gamma, \alpha) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-\alpha}$$

is called the **winding number** of γ about α.

For example, if γ is the curve C_r if $|\alpha| < r$ it is in the interior of C_r, then by Cauchy's Integral Formula,

$$n(\gamma, \alpha) = \frac{1}{2\pi i} \int_{C_r} \frac{dz}{z-\alpha} = 1$$

and if $|\alpha| > r$ then $\frac{1}{z-\alpha}$ is analytic within C_r so the integral is 0. So

$$n(C_r, \alpha) = \begin{cases} 1 & |\alpha| < r \\ 0 & |\alpha| > r \end{cases}$$

Proposition 10.3:

For every smooth closed curve γ and $\alpha \notin \gamma$, $n(\gamma, \alpha)$ is an integer.

Proof:

We will assume that γ has a continuous derivative. Let us define $\Gamma\colon [0,1] \longrightarrow \mathbb{C}$ by

$$\Gamma(s) = \int_0^s \frac{\gamma'(t)}{\gamma(t) - \alpha}\, dt$$

Since the integrand is continuous, so is Γ, and Γ is differentiable everywhere

$$\Gamma'(s) = \frac{\gamma'(s)}{\gamma(s) - \alpha}$$

Then let us define

$$G(s) = (\gamma(s) - \alpha)e^{-\Gamma(s)}$$

And so

$$G'(s) = \gamma'(s)e^{-\Gamma(s)} - \gamma'(s)e^{-\Gamma(s)} = 0$$

So G is constant, and since

$$G(0) = (\gamma(0) - \alpha)e^0 = \gamma(0) - \alpha$$

So we have that

$$(\gamma(s) - \alpha)e^{-\Gamma(s)} = \gamma(0) - \alpha \implies e^{\Gamma(s)} = \frac{\gamma(s) - \alpha}{\gamma(0) - \alpha}$$

Therefore we have that

$$e^{\Gamma(1)} = \frac{\gamma(1) - \alpha}{\gamma(0) - \alpha} = 1$$

since γ is closed so $\gamma(0) = \gamma(1)$. Therefore there exists a $k \in \mathbb{Z}$ such that

$$\Gamma(1) = 2\pi i k$$

But notice that

$$\Gamma(1) = \int_0^1 \frac{1}{\gamma(t) - \alpha} \cdot \gamma'(t)\, dt = \int_\gamma \frac{1}{z - \alpha}\, dz = 2\pi i\, n(\gamma, \alpha)$$

So we have that

$$n(\gamma, \alpha) = k$$

as required. ■

By definition, $n(\gamma, \alpha) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - \alpha}$, so $n(\gamma, \alpha)$ is continuous in α, within the connected components of $\mathbb{C} \setminus \gamma$. (It is continuous everywhere it is defined, but these are connected.) But it is also an integer, and therefore is constant within the connected components.

Proposition 10.4:

$n(\gamma, \alpha)$ is constant within the connected components of $\mathbb{C} \setminus \gamma$.

Definition 10.5:

A curve is a **closed regular curve** if it is a simple closed curve and its winding numbers are either 0 or 1. In this case $\{\alpha \mid n(\gamma, \alpha) = 1\}$ is γ's interior and $\{\alpha \mid n(\gamma, \alpha) = 0\}$ is its exterior (since both these sets must be γ's connected components since it is a simple closed curve and therefore only has two, and the exterior has winding number zero in general).

Theorem 10.6 (The Residue Theorem):

Suppose f is analytic in D except for a finite number of isolated singularities $z_1, \dots, z_n$. Let γ be a closed curve in D which doesn't intersect any of the singularities, then

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \sum_{k=1}^{n} n(\gamma, z_k)\cdot \mathrm{Res}(f, z_k)$$

Proof:

Let $P_i\left(\frac{1}{z-z_i}\right)$ be the essential part of the Laurent series of f in the ring $0 < |z - z_i| < R_i$, suppose

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z - z_i)^k$$

then

$$P_i\left(\frac{1}{z-z_i}\right) = \sum_{k=-\infty}^{-1} c_k (z - z_i)^k$$

And so

$$P_i(z) = \sum_{k=1}^{\infty} c_{-k} z^k$$

So $P_i(z)$ defines an analytic function. Now, P_i must be defined on all of $\mathbb{C}$ since if $z = \frac{1}{w-z_i}$ and so $w - z_i = \frac{1}{z}$, so if $\left|\frac{1}{z}\right| < R_i$ then plugging w into the essential part of f must converge, and so $P_i(z)$ must converge. Since P_i is analytic this means it has an infinite radius of convergence, and so $P_i\left(\frac{1}{z-z_i}\right)$ is analytic in $\mathbb{C} \setminus \{z_i\}$.
So let

$$g(z) = f(z) - P_1\left(\frac{1}{z-z_1}\right) - \cdots - P_n\left(\frac{1}{z-z_n}\right)$$

And so g is analytic in D, and so since γ is closed,

$$\int_\gamma g(z)\,dz = 0 \implies \int_\gamma f(z)\,dz = \sum_{i=1}^{m}\int_\gamma P_i\left(\frac{1}{z-z_i}\right)$$

Now,

$$\int_\gamma P_i\left(\frac{1}{z-z_i}\right) = \int_\gamma \sum_{k=1}^{\infty} c_{-k}(z-z_i)^{-k}$$

which is equal to, by uniform convergence,

$$= \sum_{k=1}^{\infty} c_{-k}\int_\gamma (z-z_i)^{-k}$$

Note that for $k \neq 1$,

$$(z-z_i)^{-k} = \frac{d}{dz}\left(\frac{(z-z_i)^{-k+1}}{-k+1}\right)$$

So the integral $\int_\gamma (z-z_i)^{-k} = 0$ (since it has an antiderivative, and the curve is closed). Thus

$$\int_\gamma P_i\left(\frac{1}{z-z_i}\right) = c_{-1}\int_\gamma \frac{1}{z-z_i} = 2\pi i \cdot c_{-1}\, n(\gamma, z_i) = 2\pi i\,\mathrm{Res}(f(z), z_i)\cdot n(\gamma, z_i)$$

So all in all

$$\frac{1}{2\pi i}\int_\gamma f(z) = \sum_{i=1}^{m}\mathrm{Res}(f(z), z_i)\cdot n(\gamma, z_i)$$

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Corollary 10.7:

If f is analytic in D except for a finite number of singularities, and if γ is a closed regular curve in D which doesn't intersect any singularity, then

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \sum_{k=1}^{m} \mathrm{Res}(f, z_k)$$

where the sum is over singularities contained within z_k.

This is because the residue of all the other singularities are on the exterior of γ and thus have a winding number of zero. And the winding number of all the singularities in γ's interior is one.

Definition 10.8:

We say that f is **meromorphic** in a domain D if it is analytic except for isolated poles.

Theorem 10.9 (The Argument Principle):

Suppose γ is a closed regular curve and f is meromorphic in and on γ, and it has no poles on γ then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} = Z(f) - P(f)$$

where $Z(f)$ is the number of zeros of f within γ, and $P(f)$ is the number of poles of within γ (counted with multiplicity).

Proof:

The only poles of $\frac{f'(z)}{f(z)}$ are at the zeros of $f(z)$ and the poles of $f(z)$.
If $z = \alpha$ is a zero of f of order k then

$$f(z) = (z-\alpha)^k g(z)$$

where $g(\alpha) \neq 0$, then

$$f'(z) = k(z-\alpha)^{k-1}g(z) + (z-\alpha)^k g'(z)$$

and so

$$\frac{f'(z)}{f(z)} = \frac{k}{z-\alpha} + \frac{g'(z)}{g(z)}$$

since $g(z)$ is non-zero and analytic at α, $\frac{g'(z)}{g(z)}$ is analytic at α and therefore does not contribute to the residue. So

$$\mathrm{Res}\left(\frac{f'}{f}, \alpha\right) = k \cdot \mathrm{Res}(1z-\alpha, \alpha) = k$$

And if $z = \alpha$ is a pole of order k then

$$f(z) = (z-\alpha)^{-k} g(z)$$

where $g(z)$ is analytic at α and $g(\alpha) \neq 0$. Then similarly

$$f'(z) = -k(z-\alpha)^{k-1}g(z) + (z-\alpha)^{-k} g'(z)$$

so

$$\frac{f'(z)}{f(z)} = -\frac{k}{z-\alpha} + \frac{g'(z)}{g(z)}$$

and so

$$\mathrm{Res}\left(\frac{f'}{f}, \alpha\right) = -k$$

Thus we have that by **The Residue Theorem**,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_\alpha \mathrm{Res}\left(\frac{f'}{f}, \alpha\right) = Z(f) - P(f)$$

since the sum over α zeros gives $Z(f)$ and the sum over α poles gives $-P(f)$. ∎