Introduction to Stochastic Processes

Assignment 9
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9.1 Exercise

Suppose $B(t)$ is Brownian motion and let $t_0 > 0$. Show that almost surely, t_0 is not a local critical point of $B(t)$.

We know that almost surely, $\inf\{t > 0 \mid B(t) > 0\} = 0$ and $\inf\{t > 0 \mid B(t) < 0\} = 0$. By the Markov property, this means that $\inf\{t > t_0 \mid B(t) > B(t_0)\} = t_0$ and $\inf\{t > t_0 \mid B(t) < B(t_0)\} = t_0$, as we can focus on the Brownian motion $X(t) = B(t + t_0) - B(t_0)$, since

$$0 = \inf\{t > 0 \mid X(t) > 0\} = \inf\{t > 0 \mid B(t + t_0) > B(t_0)\} = \inf\{t > t_0 \mid B(t) > B(t_0)\} - t_0$$

This means that for every $\varepsilon > 0$, there exists $t_1, t_2 \in (t_0 - \varepsilon, t_0 + \varepsilon)$ such that $B(t_1) < B(t_0) < B(t_2)$, so t_0 cannot be a local maximum or minimum of $B(t)$.

9.2 Exercise

Let $B(t)$ be Brownian motion, and define $\mathcal{F}_\infty = \bigcap_{t\geq 0} \sigma\{B(s)\}_{s\geq t}$.

(1) Give an example of an event in $\mathcal{F}_\infty$.

(2) Show that for every $A \in \mathcal{F}_\infty$, $\mathbb{P}(A) \in \{0, 1\}$.

(3) Show that for every $A \in \mathcal{F}_\infty$ and every initial state x, $\mathbb{P}_x(A) = \mathbb{P}_0(A)$.

(1) Events in $\mathcal{F}_\infty$ are events which are dependent only on the tails $\{B(s)\}_{s\geq t}$. For example, $\{\limsup_{t\to\infty} B(t) = \infty\}$ is dependent only on these tails, as in it is in $\sigma\{B(s)\}_{s\geq t}$ for every $t \geq 0$ and so it is surely therefore in the intersection.

(2) Let us define $X(t) = tB(1/t)$ for $t > 0$ and $X(0) = 0$, this is Brownian motion as we showed in lecture. Then

$$\mathcal{F}_0^{X,+} = \bigcap_{\varepsilon>0} \sigma\{X(t)\}_{0\leq t\leq\varepsilon} = \bigcap_{\varepsilon>0} \sigma\{B(1/t)\}_{0<t\leq\varepsilon} = \bigcap_{t>0} \sigma\{B(s)\}_{s\geq t} = \mathcal{F}_\infty^B$$

By Blumenthal's zero-one law, events in $\mathcal{F}_0^{X,+}$ have trivial probability and therefore so do events in $\mathcal{F}_\infty^B$.

(3) Similar to before, if we define $X(t) = tB(1/t) + B(0)$ and $X(0) = B(0)$ then $\mathcal{F}_0^{X,+} = \mathcal{F}_\infty^B$. So the sequence $\{X(t) - X(0)\}_{t\geq 0} = \{tB(1/t)\}_{t>0}$ is independent of $\mathcal{F}_\infty^B$, meaning $\{B(t)\}_{0<t}$ is independent of $\mathcal{F}_\infty^B$. Now, since $B(t)$ is almost surely continuous, this means that $\{B(0) = x\} \in \sigma\{B(t)\}_{0<t}$ so it is also indepedent of $\mathcal{F}_\infty^B$ as required.

9.3 Exercise

Show that almost surely, $B(t)$ has uncountably many zeroes.

Firstly, $B(t)$ is almost surely continuous and so $B^{-1}\{0\}$ is closed almost surely. Let $F = B^{-1}\{0\}$, and so we claim that almost surely, F has no isolated points. This is since $\inf\{t > 0 \mid B(t) = 0\} = 0$ and so by the Markov property (similar to in question 1), almost surely $\inf\{t > t_0 \mid B(t) = B(t_0)\} = t_0$ and in particular for every $t_0 \in F$, $\inf\{t > t_0 \mid B(t) = 0\} = t_0$. Thus for every zero t_0 and every $\varepsilon > 0$, there exists another zero in $(t_0 - \varepsilon, t_0 + \varepsilon)$ and so t_0 is almost surely not isolated. So F is almost surely closed and has no isolated points, and so it must be uncountable as required.