Differential and Analytic Geometry

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1 Conic sections

How do we define what a circle is? Historically, there are two approaches: Descartes defined it as the set of all points (x, y) which satisfy the equation

$$(x-a)^2 + (y-b)^2 = R^2$$

for some values a and b and $R > 0$. Euclid defined it as the set of all points whose distance from a specific point is some positive constant R.

We know that these two definitions are equivalent (given the standard norm/metric in $\mathbb{R}^2$), but Descartes's definition was introduced two thousand years after Euclid's. The idea of translating a visual or intuitive definition to an analytic one, as Descartes did, will be a motif of this course.

Now, recall the definition of an ellipse. Given two points, called the *foci* of the ellipse, F_1 and F_2 and a constant d, the ellipse defined is the set of all points A such that

$$|F_1A| + |F_2A| = d$$

We also must have that $|F_1F_2| < d$ as otherwise this just defines some line segment of F_1F_2. This is the Euclidean definition of an ellipse. Descartes's definition of an ellipse is the set of all points which satisfy

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

We must show that the cartesian definition satisfies the euclidean definition (and vice versa). Let us suppose that $a^2 > b^2$ (if we have an equality then this defines a circle), then we define $c = \sqrt{a^2 - b^2}$, and $F_1 = (-c, 0)$ and $F_2 = (c, 0)$. Then define $d = 2a$. Now we must show that given $A = (x, y)$, $|F_1A| + |F_2A| = d$ if and only if $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Now,

$$|F_1A| + |F_2A| = \sqrt{(x+c)^2 + y^2}, \qquad |F_2A| = \sqrt{(x-c)^2 + y^2}$$

And so we must show that

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a \iff \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Fortunately, we are not doing boring high school algebra, so we'll just assume that this is true. Thus the cartesian definition implies the eucliean definition.

Now suppose we have F_1, F_2, and d. Then we redefine the axes such that the x axis is parallel to F_1F_2 and the y axis is equidistant from F_1 and F_2. Define $a = \frac{d}{2}$, and $c = |F_1 0|$ (ie. half the distance between F_1 and F_2), and since $c = \sqrt{a^2 - b^2}$, this defines b. Now all that remains is to show that the points which satisfy $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are precisely the points which satisfy the euclidean definition of the ellipse defined by F_1, F_2, and d. Again, we won't be doing this.

Now, what about equations of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\,?$$

In the language of Euclid, this is defined by

$$\big||F_1A| - |F_2A|\big| = d$$

These are called hyperbolas.

And now for parabolas, Euclid defined them as the set of all points which satisfy

$$|A\ell| = |AF|$$

where ℓ is a line (called the directrix), and F is the focal point. $|A\ell|$ is defined as the metric between a point and a set is usually defined, by taking the infimum of all the distances between points on ℓ and A. This corresponds to the length of the line segment perpendicular to ℓ which intersects with A.

In cartesian terms, what we can do is define the x axis to be parallel to ℓ and halfway between it and F, and the y axis to pass through F. Let $F = (0, f)$ and $\ell\colon y = -f$. Then if $A = (x, y)$,

$$|AF| = \sqrt{x^2 + (y-f)^2}, \qquad |A\ell| = |y + f|$$

So

$$|AF| = |A\ell| \iff x^2 + (y-f)^2 = (y+f)^2 \iff x^2 = 4fy \iff y = \frac{1}{4f}x^2$$

Notice that all of these shapes are equivalent to the set of solutions of an equation of the form $Q(x, y) = 0$ where

$$Q(x, y) = ax^2 + by^2 + cxy + dx + ey + f$$

and two other forms of solutions are lines, or two lines (of the form $y = \pm\alpha x$).

Proposition 1.1.1:

The set of solutions to $Q(x,y)=0$ is either a line, two lines, an ellipse, a hyperbola, or a parabola.

Proof:

Notice that $Q(x,y)=0$ if and only if

$$\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} a & \frac{c}{2} \\ \frac{c}{2} & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} d & e \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + f = 0$$

Let A be the diagonal matrix in the equation above. Now recall that if a matrix is symmetric, it can be orthogonally diagonalized. Suppose that P is the orthogonal matrix which diagonalizes A, so

$$P^T A P = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$$

Now suppose

$$P^T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ s \end{pmatrix} \iff \begin{pmatrix} x \\ y \end{pmatrix} = P\begin{pmatrix} t \\ s \end{pmatrix}$$

Meaning that

$$\begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}^T = \begin{pmatrix} t & s \end{pmatrix} P^T$$

Thus $Q(x,y)=0$ if and only if

$$\begin{pmatrix} t & s \end{pmatrix} P^T A P \begin{pmatrix} t \\ s \end{pmatrix} + \begin{pmatrix} d & e \end{pmatrix} P \begin{pmatrix} t \\ s \end{pmatrix} + f = \begin{pmatrix} t & s \end{pmatrix}\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}\begin{pmatrix} t \\ s \end{pmatrix} + \begin{pmatrix} d & e \end{pmatrix} P \begin{pmatrix} t \\ s \end{pmatrix} + f = 0$$

if we denote $\begin{pmatrix} d & e \end{pmatrix} P = \begin{pmatrix} d' & e' \end{pmatrix}$ we get that this is if and only if

$$\lambda_1 t^2 + \lambda_2 s^2 + d't + e's + f = 0$$

Now utilizing this new equation, we will split into cases.

(1) If $\lambda_1, \lambda_2 \neq 0$, then we can complete the square, the equation is equivalent to

$$\lambda_1\left(t + \frac{d'}{2\lambda_1}\right)^2 + \lambda_2\left(s + \frac{e'}{2\lambda_2}\right)^2 + f - \frac{d'^2}{4\lambda_1} - \frac{e'^2}{4\lambda_2} = 0$$

This is equivalent to an equation of the form

$$\lambda_1 u^2 + \lambda_2 v^2 + f' = 0$$

If $f'=0$ then this is $\lambda_1 u^2 = -\lambda_2 v^2$, which defines two lines (with respect to u and v). Otherwise this defines an ellipse.
Note that these define shapes with respect to u and v, but since t and s are simply some (orthogonal) linear transformation of x and y, and u and v are shifts of t and s, the shape defined in x and y is some orthogonal linear transformation of this ellipse and a shift, which still defines two lines or an ellipse. This will be true of the other cases as well.

(2) If $\lambda_2 = 0$ and $\lambda_1 \neq 0$ then we get

$$\lambda_1 t^2 + d't + e's + f = 0$$

which defines a parabola (complete the square). Similar for if $\lambda_1 = 0$ and $\lambda_2 \neq 0$.

(3) If $\lambda_1 = \lambda_2 = 0$ then we get

$$d't + e's + f = 0$$

which defines a line. ■

Corollary 1.1.2:

The only bound set of the form $A = \{(x, y) \mid Q(x, y) = 0\}$ is an ellipse.

2 Curves

2.1 Isometries

Recall the following definition

Definition 2.1.1:

If (M, ρ) and (X, σ) are two metric spaces, a function

$$f\colon M \longrightarrow X$$

is an **isometry** if $\rho(x, y) = \sigma(f(x), f(y))$ for every $x, y \in M$. M and X are called **isometric**.

It is obvious that isometries are injective (if $f(x) = f(y)$ then $\rho(x, y) = 0$ so $x = y$).
If X is a normed vector space, and A is an orthogonal transformation then recall $\|Ax\| = \|x\|$, so

$$\|Ax - Ay\| = \|A(x - y)\| = \|x - y\|$$

so A is an isometry.

Definition 2.1.2:

If X is a normed vector space, and a is a unit vector then define

$$S_a(x) = x - 2\langle x, a\rangle \cdot a$$

This is the reflection about $\{a\}^\perp$.

Recall that $x - \langle x, a\rangle a \in \{a\}^\perp$, since

$$\langle x - \langle x, a\rangle a, a\rangle = \langle x, a\rangle - \langle\langle x, a\rangle a, a\rangle = \langle x, a\rangle - \langle x, a\rangle\langle a, a\rangle = \langle x, a\rangle - \langle x, a\rangle = 0$$

Now notice that

- If $x \in a^\perp$ then $S_a(x) = x$.
- $S_a(a) = -a$.
- $S_a^2(x) = S_a(x - 2\langle x, a\rangle a) = x - 2\langle x, a\rangle a - 2\langle x - 2\langle x, a\rangle a, a\rangle = x - 2\langle x, a\rangle a + 2\langle x, a\rangle = x$. So $S_a^2(x) = x$.
- $S_a(x + y) = S_a(x) + S_a(y)$ and $S_a(\lambda x) = \lambda S_a(x)$, so S_a is a linear transformation.

Also notice that $\langle x, a\rangle a = a\langle x, a\rangle = aa^T x$, thus

$$S_a(x) = (I - 2aa^T)x$$

this is another proof that S_a is a linear transformation, as $S_a(x) = Ax$ where $A = I - 2aa^T$. Now notice that $A^T = A$, we have that A is orthogonal, so S_a is an isometry.

Proposition 2.1.3:

If $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is an isometry which preserves the origin, ie. $f(0) = 0$, then f is an orthogonal linear transformation.

Proof:

Notice that f preserves norms, since $\|x\| = \|x - 0\| = \|f(x) - f(0)\| = \|f(x)\|$. And so f preserves the inner product since

$$\|x - y\|^2 = \langle x - y, x - y\rangle = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$$

And thus

$$2\langle x, y\rangle = \|x\|^2 - \|x - y\|^2 + \|y\|^2$$

So

$$2\langle x, y\rangle = \|f(x)\|^2 - \|f(x) - f(y)\|^2 + \|f(y)\|^2$$

But the equality is true for any x, y and so

$$2\langle f(x), f(y)\rangle = \|f(x)\|^2 - \|f(x) - f(y)\|^2 + \|f(y)\|^2$$

Thus $\langle x, y\rangle = \langle f(x), f(y)\rangle$ as required.
Let us define

$$A = \begin{pmatrix} | & & | \\ f(e_1) & \cdots & f(e_n) \\ | & & | \end{pmatrix}$$

Now recall that

$$\langle e_i, e_j\rangle = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

And so $\langle f(e_i), f(e_j)\rangle = \delta_{ij}$. Thus the rows of A form an orthogonal basis, meaning A is an orthogonal matrix.
Now let us define

$$g(x) = A^{-1} f(x)$$

and we will prove that $g(x) = x$, which means that $f(x) = Ax$. Notice that

$$g(e_i) = A^{-1} f(e_i) = A^{-1} C_i(A) = C_i(A^{-1}A) = e_i$$

Now, if g were a linear transformation, we could finish here. Since $g(0) = 0$, g is an isometry (as the composition of isometries) which preserves the origin, so it preserves inner products.
Now let $x \in \mathbb{R}^n$ have coefficients x_i, meaning $\langle x, e_i\rangle = x_i$, now let $g(x) = y$ with coefficients y_i. So

$$x_i = \langle x, e_i\rangle = \langle g(x), g(e_i)\rangle = \langle y, e_i\rangle = y_i$$

Thus $x = y$, so $g(x) = x$ and thus $f(x) = Ax$, so f is indeed an orthogonal transformation. ■

Thus if f is an isometry, let $g(x) = f(x) - f(0)$, then g is also an isometry which preserves the origin and so $g(x) = Ax$ where A is orthogonal. And so $f(x) = Ax + f(0)$.

Theorem 2.1.4 (Cartan-Dieudonne Theorem):

If $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is an isometry, then

$$f = T \circ S_1 \circ \cdots S_m$$

where T is a shift, and S_i are reflections, and $m \leq n$.

Proof:

We will prove this by induction on n. For $n = 1$, then we know that $f(x) = Ax + c$ where A is orthogonal, and in $\mathbb{R}$ that means that $A = \pm 1$. So $f(x) = \pm x + c$. The $+c$ is a shift, and $-x$ is a reflection about 1.
Now, for the inductive step let $g(x) = f(x) - f(0)$ so $g(x) = Ax$ where A is orthogonal. If $A = \mathrm{id}$, then $f(x) = x + c$ which is just a shift, and we have finished. Otherwise there exists an $a \in \mathbb{R}^n$ such that $g(a) \neq a$. Now, we want a $b \in \operatorname{span} a, g(a)$ such that $\|b\| = 1$ and $S_b(a) = g(a)$. Let

$$d = \frac{a}{\|a\|} + \frac{g(a)}{\|g(a)\|}$$

And let b be the unit normal to d in $\operatorname{span} a, g(a)$. Then $S_b(a)$ is the reflection of a about d, which gives $g(a)$.
Now let

$$h = S_b \circ g$$

then h is the composition of two orthogonal transformations, and is therefore also an orthogonal transformation. Let $\hat{a} = \frac{a}{\|a\|}$, and let us extend this to an orthogonal basis

$$B = \{\hat{a}, b_2, \cdots, b_n\}$$

And since h is orthogonal, $h(B)$ is also an orthogonal basis. And $h(a) = S_b(g(a)) = S_b(S_b(a)) = a$, and so $h(\hat{a}) = \hat{a}$.
Thus

$$h(\{b_2, \ldots, b_n\}) \perp \hat{a}$$

And so $h(\{b_2,\dots,b_n\})$ is an orthogonal basis of $V = \hat{a}^\perp$, which has a dimension of $n-1$. And so $h\big|_V: V \to V$ is an orthogonal transformation, since $\{b_2,\dots,b_n\}$ is an orthogonal basis of V, and so is its image. So by our inductive assumption,

$$h\big|_V = S_2 \circ \cdots \circ S_m$$

where S_i are reflections with respect to $u^\perp \subseteq V$, and $m \leq n$.
Let $\ell = \operatorname{span}\hat{a}$, and $h\big|_\ell = \mathrm{id}$, and since h is linear

$$h = S_2 \circ \cdots \circ S_m$$

where S_i is a reflection with respect to $u^\perp \subseteq \mathbb{R}^n$. And since $h = S_b \circ g$, and $f = T \circ g$, where T is a shift (adding $f(0)$), we have

$$T = T \circ S_b \circ S_2 \circ \cdots \circ S_m$$

where $m \leq n$ as required. ■

2.2 Curves and Reparameterization

Definition 2.2.1:

A **curve** is a continuous function

$$\gamma\colon [a,b] \longrightarrow \mathbb{R}^n$$

A curve is **smooth** if it is differentiable, and it is **regular** if its derivative is never zero. If $\gamma'(t) = 0$ then t is called a **singularity** of γ.

Definition 2.2.2:

Suppose $\alpha\colon [a,b] \longrightarrow \mathbb{R}^n$ is a curve, and $\varphi\colon [c,d] \longrightarrow [a,b]$ is differentiable and $\varphi' > 0$, then we define $\beta\colon [c,d] \longrightarrow \mathbb{R}^n$ by $\beta = \alpha \circ \varphi$. This is called a **reparametrization** of α.

Proposition 2.2.3:

"x is a reparametrization of y" is an equivalence relation.

Proof:

Obviously this is reflexive (take φ to be the identity function). And it is transitive since if $\beta = \alpha \circ \varphi$ and $\gamma = \beta \circ \psi$ then $\gamma = \alpha \circ (\varphi \circ \psi)$ (the derivative of the composition is still positive). restrict the definition, this still works). Now suppose $\beta = \alpha \circ \varphi$, then since $\varphi' > 0$, we know that φ is strictly increasing (and therefore injective). And so we can also assume that φ is surjective, since $\varphi([a,b]) = [\varphi(a), \varphi(b)]$. So φ is bijective and so $\alpha = \beta \circ \varphi^{-1}$, and $(\varphi^{-1})' > 0$ (since it is equal to the inverse of φ' of some point). ■

Definition 2.2.4:

Let $\alpha\colon [0,T] \to \mathbb{R}^n$ be a curve, let

$$s_\alpha(t) = \int_0^t \|\alpha'(f)\| = \int_a^T \left(\sum_{k=1}^n \alpha_k'(f)^2\right)^{1/2}$$

$s_\alpha(t)$ is the **arclength** of α.
α' is the componentwise derivative of α, which is equal to the Jacobian of α. We can continue with higher order componentwise derivatives.

The intuition behind the definition of $s(t)$ is that by the definition of integrals (using Riemman sums), we can partition $[0,T]$ into $t_0 = 0 < t_1 < \cdots < t_n = t$, and

$$\alpha'(f) \approx \frac{\alpha(t_{i+1}) - \alpha(t_i)}{\Delta_i} \implies \|\alpha'(f)\| \cdot \Delta_i \approx \|\alpha(t_{i+1}) - \alpha(t_i)\|$$

And $\|\alpha(t_{i+1}) - \alpha(t_i)\|$ approximates the length of α between t_i and t_{i+1}. And as we make the partition finer and finer, these approximations get more and more accurate.

Proposition 2.2.5:

Arclength is invariant under reparameterization. Meaning if $\alpha\colon [a,b] \longrightarrow \mathbb{R}^n$ and $\beta = \alpha \circ \varphi$ then

$$\int_a^b \|\alpha'(t)\| = \int_c^d \|\beta'(t)\|$$

Proof:

Notice that

$$\beta'(t) = \varphi'(t) \cdot \alpha'\big(\varphi(t)\big)$$

Since $\varphi'(t) > 0$ we have that

$$\int_c^d \|\beta'(t)\| = \int_c^d \|\alpha'(\varphi(t))\| \cdot \varphi'(t)\, dt$$

Let $u = \varphi(t)$ then $\varphi'(t)\, dt = du$ and since $\varphi(c) = a$ and $\varphi(d) = b$, so

$$= \int_a^b \|\alpha'(u)\|\, du$$

as required. ■

What we have shown is that $s_{\alpha\circ\varphi}(t) = s_\alpha(\varphi(t))$, ie

$$s_{\alpha\circ\varphi} = s_\alpha \circ \varphi$$

Notice that $s'_\alpha(t) = \|\alpha'(t)\|$. If α is regular then $\alpha'(t) \neq 0$ and so $s'_\alpha > 0$ so s_α is smooth and strictly increasing, meaning s_α is invertible.

Definition 2.2.6:

If α is a smooth regular curve, then let us define the curve β by

$$\beta(u) = \alpha \circ s_\alpha^{-1}(u) = \alpha(t)$$

β is called the **natural parameterization** of α.

Another way of thinking of the natural parameterization is realizing that $\beta(u)$ is equal to the value of α after traversing u units on the arc defined by α.
Notice that if β is a reparameterization of α, then they both have the same natural parameterizations, since if $\beta = \alpha \circ \varphi$ then

$$\beta \circ s_\beta^{-1} = \beta \circ s_{\alpha\circ\varphi}^{-1} = \beta \circ (s_\alpha \circ \varphi)^{-1} = \alpha \circ \varphi \circ \varphi^{-1} \circ s_\alpha^{-1} = \alpha \circ s_\alpha^{-1}$$

In other words:

Proposition 2.2.7:

The natural parameterization of a regular smooth curve is unique, up to reparameterization. Meaning if α and β are reparameterizations of one another, then they have the same natural parameterization.

Notice that α is a natural parameterization if and only if $s_\alpha = \mathrm{id}$. If α is a natural parameterization, then $\alpha = \alpha \circ s_\alpha^{-1}$, and so $s_\alpha = \mathrm{id}$. And if $s_\alpha = \mathrm{id}$, then $\alpha \circ s_\alpha^{-1} = \alpha$.

Proposition 2.2.8:

If α is a curve, it is a natural parameterization if and only if $\|\alpha'\| = 1$.

Proof:

Since

$$s_\alpha(t) = \int_0^t \|\alpha'(u)\|$$

so $s'_\alpha = \|\alpha'\|$, so if $s_\alpha = \text{id}$ then $s'_\alpha = \|\alpha'\| = 1$. And if $\|\alpha'\| = 1$ then $s'_\alpha = 1$ so $s_\alpha(t) = t + c$ and since $s_\alpha(0) = 0$, $c = 0$ as required. ∎

2.3 Curvature

Definition 2.3.1:

Let α be a natural parameterization. We define $T_\alpha(s) = \alpha'(s)$, and in the case that we are in 2 dimensions, we define $N_\alpha(s) = R_{\frac{\pi}{2}} \cdot T(s)$. R_θ is the rotation matrix

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Since α is a natural parameterization and R_θ is orthogonal, $\|T_\alpha\| = \|N_\alpha\| = 1$ and thus $\{T(s), N(s)\}$ forms an orthonormal basis, called the **Frenet-Serret Frame**.

We can think of T_α as the direction of motion, or the velocity, of α, and T'_α as its acceleration. Since T_α is constant, its derivative is perpendicular to itself, meaning the acceleration of α is orthogonal to its velocity. We will prove this formally:

Proposition 2.3.2:

Suppose $V: \mathbb{R} \longrightarrow \mathbb{R}^n$ (ie. V is a vector field over $\mathbb{R}$), if $\|V\| = c$ then $V' \perp V$ whenever V is differentiable.

Proof:

Since $\langle V, V\rangle = c^2$ is constant, we have that the function

$$f(t) = \langle V(t), V(t)\rangle = \sum_{k=1}^n V_i(t)V_i(t)$$

Is constant and therefore if V is differentiable at t, then so must V_i be, and therefore $f(t)$ is. And since f is constant, $f'(t) = 0$. Therefore

$$f'(t) = \sum_{k=1}^n V'_i(t)V_i(t) + V_i(t)V'_i(t) = \langle V'(t), V(t)\rangle + \langle V(t), V'(t)\rangle = 0$$

And since this inner product is over $\mathbb{R}$, this means $\langle V, V'\rangle = 0$ so $V' \perp V$ as required. ∎

So when $n = 2$, this means that T'_α is parallel with N_α and so

$$T'_\alpha(s) = \kappa(s) \cdot N_\alpha(s)$$

For some function $k: \mathbb{R} \longrightarrow \mathbb{R}$. In fact, since $\{T_\alpha, N_\alpha\}$ is an orthonormal basis,

$$T' = \langle T', T\rangle T + \langle T', N\rangle N = \langle T', N\rangle N$$

So $\kappa(s) = \langle T'(s), N(s)\rangle$.
Let us look at this function κ.

(1) When $\kappa(s) = 0$, then $T'(s) = 0$ and so there is no acceleration, and we are moving in a straight line.

(2) When $\kappa(s) > 0$, then the curve α is accelerating away from T "upward" (toward N), and this creates a steep curve.

(3) When $\kappa(s) < 0$, the curve is accelerating away from T "downward", also creating a steep curve.

Thus κ can be seen as a measure of curvature.

Definition 2.3.3:

The **curvature** of a regular two-dimensional curve α at point s is defined to be

$$\kappa(s) = \langle T'_\alpha(s), N_\alpha(s)\rangle$$

Where T_α and N_α are taken as their values for the natural reparameterization of α.

Notice that

$$N' = \left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}T\right)' = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}T' = \kappa\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}N = \kappa\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}^2 T = \kappa\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}T = -\kappa T$$

Therefore T and N are solutions to the ODE,

$$T' = kappaN, \quad N' = -\kappa T$$

Thus by the uniqueness theorem for ODEs, if we are given the function $\kappa(s)$, and $N(0)$ and $T(0)$, then we can solve for N and T. Since N is determined by T, we need only $T(0)$ and $\kappa(s)$. And since $T = \alpha'$,

$$\alpha(s) - \alpha(0) = \int_0^s T$$

for all s, so if we are given T and $\alpha(0)$, we can find $\alpha(s)$. Thus given $\kappa(s)$, $\alpha(0)$, and $T(0)$ we can determine α.

Theorem 2.3.4 (The Fundamental Theorem of Curves):

Every regular curve is uniquely determined by its curvature, initial position, and $T(0)$.

Now, recall that

$$\kappa(s) = \langle T'(s), N(s)\rangle = \left\langle \alpha''(s), \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\alpha'(s)\right\rangle = \left\langle \begin{pmatrix}\alpha_1''(s)\\ \alpha_2''(s)\end{pmatrix}, \begin{pmatrix}-\alpha_2'(s)\\ \alpha_1'(s)\end{pmatrix}\right\rangle = \alpha_2''(s)\alpha_1'(s) - \alpha_2'(s)\alpha_1''(s)$$

And so

$$\kappa(s) = \alpha_2''\alpha_1' - \alpha_2'\alpha_1''$$

Where α is the natural parameterization.

Example 2.3.5:

Suppose α is the curve in $\mathbb{R}^2$ connecting x and y, ie.

$$\alpha\colon [0,1] \longrightarrow \mathbb{R}^2, \quad s \mapsto x\cdot\frac{s}{L} + y\cdot\frac{1-s}{L}$$

where $L = \|x-y\|$. Thus

$$\alpha'(s) = \frac{x}{L} - \frac{y}{L}$$

And so $\alpha''(s) = 0$, meaning $\kappa(s) = 0$.

Example 2.3.6:

Suppose α is the curve which parameterizes the circle of radius R,

$$\alpha\colon [0, 2\pi R] \longrightarrow \mathbb{R}^2, \quad s \mapsto R\left(\cos\frac{s}{R}, \sin\frac{s}{R}\right)$$

Thus

$$\alpha'(s) = \left(-\sin\frac{s}{R}, \cos\frac{s}{R}\right), \qquad \alpha''(s) = -\frac{1}{R}\left(\cos\frac{s}{R}, \sin\frac{s}{R}\right)$$

$\|\alpha\| = 1$, so α is the natural parameterization. And thus

$$\kappa(s) = -\frac{1}{R}\left(-\sin^2\frac{s}{R} + \cos^2\frac{s}{R}\right) = \frac{1}{R}$$

So the curvature of a circle of radius R is $\frac{1}{R}$.

Since the curves are determined by $\alpha(0)$, $T(0)$, and their curvature, by the above two examples, if

(1) $\kappa(s) = c \neq 0$ then α is a circle. If $\kappa(s) > 0$ then the curve is drawn counterclockwise, and if $\kappa(s) < 0$ the curve is parameterized clockwise (the proof above means that $\alpha(-s)$ is a circle of radius $-R$).

(2) $\kappa = 0$ then α is a line.

Notice that if γ is a natural parameterization then

$$\gamma'(s) = T(s) = \begin{pmatrix} \cos(\alpha(s)) \\ \sin(\alpha(s)) \end{pmatrix}$$

This means that

$$\alpha(s) = \operatorname{atan2}(\cos\alpha(s), \sin\alpha(s))$$

Now we claim that $\kappa(s) = \alpha'(s)$. Since

$$T(s) = \begin{pmatrix} \cos(\alpha(s)) \\ \sin(\alpha(s)) \end{pmatrix} \implies T'(s) = \begin{pmatrix} -\sin(\alpha(s)) \\ \cos(\alpha(s)) \end{pmatrix} \cdot \alpha'(s) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} T \cdot \alpha'(s) = \alpha'(s)N$$

And since $T'(s) = \kappa(s)N$ this means that $\alpha'(s) = \kappa(s)$ as required.
So if we are given $\gamma' = T$, then we can compute α based on T and then taking its derivative gives $\kappa(s)$.
But what if we aren't given the natural parameterization of the curve? Let β be any regular smooth curve, and γ its natural parameterization. Then recall that $\gamma = \beta \circ s_\beta^{-1}$ and so $\beta = \gamma \circ s_\beta$. Thus

$$\beta'(t) = s'_\beta(t) \cdot \gamma'(s_\beta(t))$$

(This is a bit confusing, since s_β is a scalar, and γ is a vector). We know that there exists an α such that

$$\alpha = \operatorname{atan}\left(\frac{\gamma'_2}{\gamma'_2}\right)$$

And since

$$\frac{\gamma'_2(s)}{\gamma'_1(s)} = \frac{\beta'_2(t)}{\beta'_1(t)}$$

And thus

$$\alpha(s) = \operatorname{atan}\left(\frac{\beta'_2(t)}{\beta'_1(t)}\right)$$

Recall that the derivative of $\operatorname{atan}(x) = \frac{1}{1+x^2}$, and since

$$\frac{d}{ds}\operatorname{atan}\left(\frac{\beta'_2(t)}{\beta'_1(t)}\right) = \frac{d}{dt}\operatorname{atan}\left(\frac{\beta'_2(t)}{\beta'_1(t)}\right) \cdot \frac{dt}{ds}$$

We have that

$$\kappa(s) = \alpha'(s) = \frac{1}{1 + \left(\frac{\beta'_2(t)}{\beta'_1(t)}\right)^2} \cdot \frac{\beta''_2(s)\beta'_1(s) - \beta'_2(s)\beta''_1(s)}{\beta'_1(s)^2} \cdot \frac{dt}{ds} = \frac{\beta''_2(s)\beta'_1(s) - \beta'_2(s)\beta''_1(s)}{\beta'_1(t)^2 + \beta'_2(t)^2} \cdot \frac{dt}{ds}$$

By definition,

$$s(t) = \int_0^t \|\beta'(u)\|\, du \implies s'(t) = \|\beta'(t)\|$$

So

$$\frac{dt}{ds} = \frac{1}{\|\beta'(t)\|} = \frac{1}{\sqrt{\beta'_1(s)^2 + \beta'_2(s)^2}}$$

And so all in all we have that

$$\kappa(s) = \frac{\beta''_2(s)\beta'_1(s) - \beta'_2(s)\beta''_1(s)}{\left(\beta'_1(t)^2 + \beta'_2(t)^2\right)^{1.5}}$$

So we have proven the following proposition:

Proposition 2.3.7:

If β is a regular smooth curve, then its curvature is given by

$$\kappa(s) = \frac{\beta_2''(s)\beta_1'(s) - \beta_2'(s)\beta_1''(s)}{\left(\beta_1'(t)^2 + \beta_2'(t)^2\right)^{1.5}}$$

Example 2.3.8:

So if $\beta(t) = (t, f(t))$ then

$$\kappa_\beta(t) = \frac{f''(t)}{(1 + f'(t))^{1.5}}$$

Thus if we have a function $f\colon \mathbb{R} \to \mathbb{R}$, then we can discuss its curvature as the parameterization of its graph.

Suppose we have a regular smooth curve α which is a natural parameterization. Our goal is to find the circle tangent to α at the point s_0.

(1) First, we can write α as a second order Taylor series

$$\alpha(s_0 + h) = \alpha(s_0) + h\alpha'(s_0) + \frac{h^2}{2}\alpha''(s_0) + \varepsilon(h)$$

where $\varepsilon(h) \in o(h^2)$ (meaning $\frac{\|\varepsilon(h)\|}{h^2} \xrightarrow[h \to 0]{} 0$).

(2) Now, we know that $T = \alpha'$ and $\alpha'' = T' = \kappa(s)N$ and thus

$$\alpha(s_0 + h) - \alpha(s_0) = hT(s_0) + \kappa(s_0)\frac{h^2}{2}N(s_0) + \varepsilon(h)$$

Let us define

$$\Delta(h) = \alpha(s_0 + h) - \alpha(s_0), \quad x(h) = \langle \Delta(h), T(s_0)\rangle, \quad y(h) = \langle \Delta(h), N(s_0)\rangle$$

Thus $\Delta(h) = x(h)T(s_0) + y(h)N(s_0)$, and so

$$x(h) = \left\langle hT(s_0) + \kappa(s_0)\frac{h^2}{2}N(s_0) + \varepsilon(h), T(s_0)\right\rangle = h + \langle \varepsilon(h), T(s_0)\rangle$$

Since $\|T\| = 1$ and T and N are orthogonal. Now since by Cauchy-Schwarz, $|\langle u, v\rangle| \leq \|u\|\|v\|$, we have that $\langle \varepsilon(h), T(s_0)\rangle = \varepsilon_1(h) \in o(h^2)$. Similarly

$$y(h) = \kappa(s_0) \cdot \frac{h^2}{2} + \varepsilon_2(h)$$

where $\varepsilon_1(h) \in o(h^2)$.

(3) Now, let us define the axis system $(T(s_0), N(s_0))$ centered at $\alpha(s_0)$, then since in this axis system $\alpha(s_0) = 0$, we will denote $\alpha(s_0 + h)$ by $\alpha(h)$, and $T(s_0)$ and $N(s_0)$ by T and N, and $\kappa(s_0)$ by k. Thus

$$\alpha(h) = x(h)T + y(h)N = hT + k \cdot \frac{h^2}{2}N + \big(\varepsilon_1(h)T + \varepsilon_2(h)N\big)$$

So given an h, we will define a circle through $(0, 0)$, $\left(\pm h, k \cdot \frac{h^2}{2}\right)$. Such a circle would have the form $(x-a)^2 + (y-b)^2 = R^2$. Let us assume $a = 0$ (the reason for assuming this is by symmetry). Thus we must have

$$b^2 = R^2, \quad h^2 + \left(k \cdot \frac{h^2}{2} - b\right)^2 = R^2$$

So

$$h^2 + \kappa^2 \cdot \frac{h^4}{4} - b\kappa h^2 + b^2 = R^2 \implies \kappa^2 \cdot \frac{h^4}{4} = b\kappa h^2 - h^2 \implies \kappa^2 \cdot \frac{h^2}{2} = b\kappa - 1$$

So as $h \to 0$ we get that

$$bk = 1 \implies b = \frac{1}{\kappa} \implies R = |b| = \left|\frac{1}{\kappa}\right|$$

And the center of the circle is $\left(0, \frac{1}{\kappa}\right)$.

(4) Now, we know that (x, y) in this axis system corresponds to $xT(s_0) + yN(s_0) + \alpha(s_0)$ in $\mathbb{R}^2$, and so the circle we got is the set

$$\left\{ \alpha(s_0) + xT(s_0) + yT(s_0) \;\middle|\; x^2 + \left(y - \frac{1}{\kappa(s_0)}\right)^2 = \frac{1}{\kappa(s_0)^2} \right\}$$

We can also see this because the center of the circle is at $\frac{1}{k}$ in the new axis system, which is the point

$$c(s_0) = \alpha(s_0) + \frac{1}{\kappa(s_0)} N$$

And the radius of the circle is still $\frac{1}{\kappa(s_0)}$ (since the new axis system is simply an isometry).

Newton was originally the person who came up with this formula (for the center of the circle and its radius). The way he approached it was by taking the points $\alpha(s_0)$ and $\alpha(s_0 + h)$ and looking at the intersection of the normal lines at these points, $o(h)$. Then we will show that $o(h) \to c(s_0)$. Let $\ell_1(t)$ and $\ell_2(t)$ be the normal lines at $\alpha(s_0)$ and $\alpha(s_0 - h)$ respectively. We know that

$$\ell_1(t) = \alpha(s_0) + tN(s_0), \qquad \ell_2(t) = \alpha(s_0 + h) + tN(s_0 + h)$$

And since we know that

$$\alpha(s_0 + h) = \alpha(s_0) + h\alpha'(s_0) + o(h) = \alpha(s_0) + hT(s_0) + o(h)$$

And

$$N(s_0 + h) = N(s_0) + hN'(h) + o(h)$$

And since $N' = -\kappa(s_0)T$, we have

$$N(s_0 + h) = N(s_0) - h\kappa(s_0)T(h) + o(h)$$

Then $\ell_1(t) = \ell_2(p)$ if and only if

$$\alpha(s_0) + tN(s_0) = \alpha(s_0) + hT(s_0) + o(h) + p(N(s_0) - h\kappa(s_0)T(s_0) + o(h)) \iff (t - p)N(s_0) = h(1 - p\kappa(s_0))T(s_0) + o(h)$$

Meaning that

$$(p - t)N(s_0) + h(1 - p\kappa(s_0))T(s_0) \in o(h)$$

Thus

$$\frac{p - t}{h} N(s_0) + (1 - p\kappa(s_0))T(s_0) \xrightarrow[h \to 0]{} 0$$

Since N and T are orthonormal, this means that $p - t = 0$ and $1 - p\kappa(s_0) = 0$. So $t = p = \frac{1}{\kappa(s_0)}$. And so the center point is

$$c(s_0) = \alpha(s_0) + \frac{1}{\kappa(s_0)} N(s_0)$$

as we showed before. Let us summarize this in the following definition:

Definition 2.3.9:

If α is a regular smooth planar curve, then the **osculating circle** of α at the point s_0 (this is the input, we could also think of it as the point $\alpha(s_0)$) is the circle centered at

$$c_\alpha(s_0) = \alpha(s_0) + \frac{1}{\kappa_\alpha(s_0)} N_\alpha(s_0)$$

and whose radius is $\frac{1}{\kappa_\alpha(s_0)}$. The curve c_α is called the **evolute** of α.

Suppose α is a natural parameterization, and $\varphi\colon v \mapsto Ax + c$ is an isometry (and so A is orthonormal). Then let $\beta = \varphi \circ \alpha$, so

$$\beta(s) = A\alpha(s) + c$$

Then $\beta'(s) = A\alpha'(s)$, and since A is orthonormal, $\|\beta'\| = \|\alpha'\| = 1$ since α is natural. Thus β is also a natural parameterization. And so

$$\kappa_\beta(s) = \left\langle \beta''(s), R_{\frac{\pi}{2}} \beta'(s) \right\rangle = \left\langle A\alpha''(s), R_{\frac{\pi}{2}} A\alpha'(s) \right\rangle$$

Now, rotations and A commute up to sign. If $\det(A) = 1$ then they commute, and if $\det(A) = -1$ then $R_\theta A = -AR_\theta$. So this is equal to $\det(A)\left\langle A\alpha''(s), AR_{\frac{\pi}{2}}\alpha'(s)\right\rangle$, since A is orthogonal this is equal to

$$= \det(A)\left\langle \alpha''(s), R_{\frac{\pi}{2}}\alpha'(s)\right\rangle = \pm\kappa_\alpha(s)$$

So we have proven the following:

Proposition 2.3.10:

If A is an orthogonal matrix, and c a vector then $\varphi\colon x \mapsto Ax + c$ is an isometry, and if α is a natural parameterization, then so is $\beta = \varphi \circ \alpha$, and $\kappa_\alpha = \kappa_\beta$.

2.4 Total Curvature

Let $\gamma\colon [0, L] \longrightarrow \mathbb{R}^2$ be a natural parameterization, then $T = \gamma'$ and $\kappa(s) = \langle T', N\rangle$. Suppose $T(0)$ has an angle of θ_0 then let us define

$$\theta(s) = \int_0^s \kappa(p)\,dp + \theta_0$$

And we define the curve

$$\beta(s) = \gamma(0) + \begin{pmatrix} \int_0^s \cos(\theta(s))\,dp \\ \int_0^s \sin(\theta(s))\,dp \end{pmatrix}$$

Now, notice that

$$\beta'(s) = \begin{pmatrix} \cos(\theta(s)) \\ \sin(\theta(s)) \end{pmatrix}$$

And since $\|\beta'\| = 1$, β is a natural parameterization. And further

$$\beta''(s) = \theta'(s) \cdot \begin{pmatrix} -\sin(\theta(s)) \\ \cos(\theta(s)) \end{pmatrix} = \theta'(s) \cdot R_{\frac{\pi}{2}}\beta'(s)$$

Which means that

$$\kappa_\beta(s) = \langle \beta''(s), N_\beta(s)\rangle = \left\langle \theta'(s) \cdot R_{\frac{\pi}{2}}\beta'(s), R_{\frac{\pi}{2}}\beta'(s)\right\rangle = \theta'(s)\langle \beta'(s), \beta'(s)\rangle = \theta'(s) = \kappa(s)$$

(The third equality is since $R_{\frac{\pi}{2}}$ is orthogonal.) So the curvature of β is equal to that of γ.
Now,

$$T_\beta(0) = \beta'(0) = \begin{pmatrix} \cos(\theta(0)) \\ \sin(\theta(0)) \end{pmatrix} = \begin{pmatrix} \cos(\theta_0) \\ \sin(\theta_0) \end{pmatrix} = T(0)$$

And $\beta(0) = \gamma(0)$.
So by the **The Fundamental Theorem of Curves**, since $\kappa_\beta = \kappa_\gamma$, $\beta(0) = \gamma(0)$, and $T_\beta(0) = T_\gamma(0)$, we have that $\beta = \gamma$.
This means that

$$T_\gamma(s) = T_\beta(s) = \beta'(s) = \begin{pmatrix} \cos(\theta(s)) \\ \sin(\theta(s)) \end{pmatrix}$$

So θ is the angle function of γ (ie. it gives the angle of γ). So we have proven the following proposition:

Proposition 2.4.1:

If $\gamma\colon [0, L] \longrightarrow \mathbb{R}^2$ is a regular smooth curve, then its angle is given by

$$\theta_\gamma(s) = \int_0^s \kappa_\gamma(p)\,dp + \theta_0$$

where θ_0 is the angle of $T_\gamma(0)$.

Definition 2.4.2:

If $\gamma\colon [0, L] \longrightarrow \mathbb{R}^2$ is a natural parameterization, then we define

$$K_\gamma = \int_0^L \kappa_\gamma(s)\,ds$$

to be the **total curvature** of γ.

So by the above definitions,

$$K_\gamma = \theta_\gamma(L) - \theta_\gamma(0)$$

So K_γ can also be thought of the total difference in the angle of γ.

Example 2.4.3:

If γ is a circle, then intuitively $K_\gamma = 2\pi$ since the total difference in the angle of the curve is 2π. And since the natural parameterization is given by a curve from $[0, 2\pi R]$ whose curvature is $\frac{1}{R}$ and thus

$$K_\gamma = \int_0^{2\pi R} \frac{1}{R} = 2\pi$$

as expected.

Definition 2.4.4:

A smooth curve $\gamma\colon [a,b] \longrightarrow \mathbb{R}^n$ is **n-closed** if $\gamma^{(k)}(a) = \gamma^{(k)}(b)$ for every $0 \leq k \leq n$. If γ is n-closed for every n, then γ is called **closed**.

Proposition 2.4.5:

If γ is a 1-closed regular smooth curve then $K_\gamma = 2\pi n$ for some $n \in \mathbb{Z}$.

Proof:

Since γ is 1-closed, $\gamma'(0) = \gamma'(L)$. But recall that

$$\gamma'(s) = \begin{pmatrix} \cos(\theta(s)) \\ \sin(\theta(s)) \end{pmatrix}$$

So we have that

$$\begin{pmatrix} \cos(\theta(0)) \\ \sin(\theta(0)) \end{pmatrix} = \begin{pmatrix} \cos(\theta(L)) \\ \sin(\theta(L)) \end{pmatrix}$$

Which is if and only if $\theta(L) = \theta(0) + 2\pi n$ for some $n \in \mathbb{Z}$, and so $K_\gamma = 2\pi n$ as required. ■

Definition 2.4.6:

If γ is a 1-closed regular smooth curve, then $\frac{1}{2\pi} K_\gamma$ is called γ's **winding number** (about 0).

Theorem 2.4.7 (Hopf's Theorem):

If $\gamma\colon [0,L] \longrightarrow \mathbb{R}^2$ is a closed natural parameterization, then γ is injective (other than at the points 0 and L).

We will not be proving this theorem.
This means that if γ is closed, then $K_\gamma = \pm 2\pi$. This is because the winding number is ± 1, as otherwise γ would have to intersect with itself. The sign of K_γ correlates with its orientation. We will prove this formally:

Proposition 2.4.8:

If γ is a closed curve then $K_\gamma = \pm 2\pi$.

Proof:

We assume that $\gamma\colon [0,T] \to \mathbb{R}^2$ is the natural parameterization of the curve. Suppose $\gamma(0) = 0$, and $T(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $0 \leq \gamma_1(s)$ for every $s \neq 0, T$ (we can get to this via an isometry). Let $B = \{(x,y) \mid 0 \leq x \leq y \leq T\}$ and we define a function $g\colon B \longrightarrow S^1$ (S^1 is the unit circle) by

$$g(s,t) = \begin{cases} \frac{\gamma(t) - \gamma(s)}{\|\gamma(t) - \gamma(s)\|} & s \neq t \text{ and } s \neq 0, t \neq T \\ \gamma'(s) & s = t \\ -\gamma'(0) & s = 0 \text{ and } t = T \end{cases}$$

g is therefore continuous.
Let us define $\alpha_0(t)$ (from $[0,T] \to B$) to be the line which connects $(0,0)$ to (T,T), ie. $\alpha_0(t) = t(1,1)$. And let us define $\alpha_1(t)$ to be the concatenation of the line from $(0,0)$ to $(0,T)$ with the line from $(0,T)$ to (T,T). α_0 and α_1 are both contained within B. And for $0 \leq \lambda \leq 1$, let us define $\alpha_\lambda = (1-\lambda)\alpha_0 + \lambda\alpha_1$.
Since $g \circ \alpha_\lambda(t)$ is a unit vector (since $g(t)$ always is), there exists a function θ_λ such that

$$g \circ \alpha_\lambda = \begin{pmatrix} \cos(\theta_\lambda(t)) \\ \sin(\theta_\lambda(t)) \end{pmatrix}$$

Since g and α_λ are continuous (though α_λ is not differentiable for $\lambda > 0$ as α_1 is not), so is θ_λ. Let us define

$$D(\lambda) = \theta_\lambda(T) - \theta_\lambda(0)$$

Since $g \circ \alpha_\lambda(T) = \gamma'(T)$ which is equal to $\gamma'(0) = g \circ \alpha_\lambda(0)$ since γ is closed, we have that

$$\begin{pmatrix} \cos\theta_\lambda(T) \\ \sin\theta_\lambda(T) \end{pmatrix} = \begin{pmatrix} \cos\theta_\lambda(0) \\ \sin\theta_\lambda(0) \end{pmatrix}$$

and therefore $D(\lambda) = \theta_\lambda(T) - \theta_\lambda(0)$ is a multiple of 2π.
Now, notice that $g \circ \alpha_0(t) = g(t,t) = \gamma'(t)$ and so θ_0 is the angle of γ, so

$$D(0) = \theta_0(T) - \theta_0(0) = K_\gamma$$

Notice that $g \circ \alpha_1(0) = g(0,0) = \gamma'(0) = (1,0)$ and $g \circ \alpha_1(T/2) = g \circ (0,T) = -\gamma'(0) = (-1,0)$, $g \circ \alpha_1$ rotated π radians on its path from $(0,0)$ to $(0,T)$. And similarly $g \circ \alpha_1(T) = g(T,T) = \gamma'(T) = \gamma'(0) = (1,0)$. And so $g \circ \alpha_1$ rotated another π radians on its path from $(0,T)$ to (T,T). Thus all in all $D(1) = 2\pi$ (or -2π if we were to change our orientation).
We will now prove that D is continuous. And since D is always a multiple of 2π this would mean that it is constant, and so $K_\gamma = \pm 2\pi$.
Suppose that λ is a point of discontinuity for D, then for h small enough $D(\lambda) \neq D(\lambda+h)$ and so let $\delta = \theta_\lambda - \theta_{\lambda+h}$. Then

$$|\delta(T) - \delta(0)| = |\theta_\lambda(T) - \theta_{\lambda+h}(T) - \theta_\lambda(0) + \theta_{\lambda+h}(0)| = |D(\lambda+h) - D(\lambda)|$$

and since D is always a multiple of 2π and $D(\lambda+h) \neq D(\lambda)$, this means that

$$|\delta(T) - \delta(0)| \geq 2\pi$$

Since δ is continuous, and the difference between the endpoints $\delta(0)$ and $\delta(T)$ is greater than 2π, there must exists some $0 \leq t_0 \leq T$ and n natural such that $\delta(t_0) = \pm\pi(2n+1)$ (ie. there must be a point where δ is an odd multiple of π). And so $\theta_\lambda(t_0) - \theta_{\lambda+h}(t_0) = \pm\pi(2n+1)$ and therefore

$$g \circ \alpha_\lambda(t_0) = \begin{pmatrix} \cos\theta_\lambda(t_0) \\ \sin\theta_\lambda(t_0) \end{pmatrix} = \begin{pmatrix} \cos(\theta_{\lambda+h}(t_0) \pm \pi(2n+1)) \\ \sin(\theta_{\lambda+h}(t_0) \pm \pi(2n+1)) \end{pmatrix} = -\begin{pmatrix} \cos\theta_{\lambda+h}(t_0) \\ \sin\theta_{\lambda+h}(t_0) \end{pmatrix} = -g \circ \alpha_{\lambda+h}(t_0)$$

But we can make h small enough so that α_λ and $\alpha_{\lambda+h}$ are arbitrarily close, and since g is continuous $g \circ \alpha_\lambda(t_0)$ and $g \circ \alpha_{\lambda+h}(t_0)$ must be arbitrarily close. But they are on opposite ends of the unit circle, in contradiction. ■

2.5 Three Dimensional Curves

Let α be the natural parameterization of some curve. In two dimensions, recall that we define N_α by rotating $T_\alpha = \alpha'$ ninety degrees. But rotation by ninety degrees has less meaning in three dimensions, as there are an infinite number of planes on which we can rotate ninety degrees. But recall by **proposition 2.3.2** that if $\|T_\alpha\|$ is constant, then T'_α is orthogonal to T_α. Thus we can define N_α to be the unit vector in the direction of T'_α, ie $N_\alpha = \frac{T'_\alpha}{\|T'_\alpha\|}$. And recall that we defined curvature as the scalar function κ_α such that

$$T'_\alpha = \kappa_\alpha N_\alpha$$

in three dimensions this becomes

$$T'_\alpha = \kappa_\alpha \frac{T'_\alpha}{\|T'_\alpha\|} \implies \kappa_\alpha = \|T'_\alpha\|$$

So in three dimensions, curvature is always positive, while in two dimensions it may be signed.

But $\{T_\alpha, N_\alpha\}$ is not yet an orthonormal basis, we require one more vector. We can obtain it by simply defining

$$B_\alpha = T_\alpha \times N_\alpha$$

this is orthogonal to T_α and N_α and since $\|T_\alpha\| = \|N_\alpha\| = 1$, $\|B_\alpha\| = 1$. So $\{T_\alpha, N_\alpha, B_\alpha\}$ is an orthonormal basis. Let us summarize the definitions:

Definition 2.5.1:

Let α be a natural parameterization, then we define

(1) $T_\alpha(s)$ as $\alpha'(s)$.

(2) $N_\alpha(s) = \frac{T'_\alpha(s)}{\|T'_\alpha(s)\|}$.

(3) $B_\alpha(s) = T_\alpha(s) \times N_\alpha(s)$.

And the **curvature** of α is defined to be $\kappa_\alpha(s) = \|T'_\alpha(s)\|$. Or alternatively

$$\kappa_\alpha(s) = \langle T'_\alpha(s), N_\alpha(s)\rangle$$

Now, since $\|B\| = 1$, B' is orthogonal to B and so

$$B' = \langle B', T\rangle T + \langle B', N\rangle N + \langle B', B\rangle B = \langle B', T\rangle T + \langle B', N\rangle N$$

And since we know that $\langle B, T\rangle = \langle B, N\rangle = 0$, we get that by differentiating

$$\langle B', T\rangle = -\langle B, T'\rangle, \qquad \langle B', N\rangle = -\langle B, N'\rangle$$

And since

$$\langle B, T'\rangle = \langle B, \kappa N\rangle = 0$$

and so $\langle B', T\rangle = 0$. Therefore

$$B' = \langle B', N\rangle N$$

Definition 2.5.2:

Let α be the natural parameterization of a curve, we define the **torsion** of α to be

$$\tau_\alpha(s) = -\langle B'_\alpha(s), N_\alpha(s)\rangle = \langle B_\alpha(s), N'_\alpha(s)\rangle$$

Now, we know that since T's norm is constant, $T' \perp T$ and so

$$N' = \langle N', T\rangle T + \langle N', N\rangle N + \langle N', B\rangle B = \langle N', T\rangle T + \langle N', B\rangle B = -\langle N, T'\rangle T - \langle N, B'\rangle B = -\kappa T + \tau B$$

Thus we have the system of ODEs:

$$\begin{aligned} T'_\alpha(s) &= \kappa_\alpha(s) N_\alpha(s) \\ N'_\alpha(s) &= -\kappa(s) T_\alpha(s) + \tau(s) B_\alpha(s) \\ B'_\alpha(s) &= -\tau_\alpha(s) N_\alpha(s) \end{aligned}$$

Or using matrices,

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

Thus if we are given $\kappa_\alpha(s)$ and $\tau_\alpha(s)$, and $T_\alpha(0)$, $N_\alpha(0)$, and $B_\alpha(0)$ then we can solve the ODE for T_α and integrate to get α. In fact, we need only two out of $T_\alpha(0)$, $N_\alpha(0)$, and $B_\alpha(0)$, since that will determine the third. This proves the fundamental theorem of curves for three dimensions,

Theorem 2.5.3 (The Fundamental Theorem of Curves):

Every natural parameterization is determined uniquely by its curvature, torsion, and initial conditions for T, N, and B.

Example 2.5.4:

Suppose we have a natural parameterization

$$\gamma(s) = \begin{pmatrix} \gamma_1(s) \\ \gamma_2(s) \\ 0 \end{pmatrix}$$

which is a two dimensional curve embedded onto the $[xy]$ plane in $\mathbb{R}^3$. Then

$$T(s) = \begin{pmatrix} \gamma_1'(s) \\ \gamma_2'(s) \\ 0 \end{pmatrix}, \qquad T'(s) = \begin{pmatrix} \gamma_1''(s) \\ \gamma_2''(s) \\ 0 \end{pmatrix}$$

If $T'(s) = 0$ then $\kappa(s) = \|T'(s)\| = 0$ and so $N(s)$ is undefined, and therefore so is $B(s)$ and $\tau(s)$. Otherwise since T' is on the $[xy]$ plane, so is N. Thus since $B = T \times N$, $B = (0, 0, \pm 1)$. Using the right-hand rule, we can see that B's sign is $+1$ when the curve is turning left, and -1 when turning right, and so B's sign encodes the sign of the curvature of the curve when viewed as a planar curve.
And since for any neighborhood in which B is defined, B is constant, and so $B' = 0$ and thus $\tau(s) = 0$ when defined.

Lemma 2.5.5:

Suppose f_{ij} are all differential at x_0 then let us define

$$D = \det \begin{pmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{pmatrix}$$

then

$$D'(x_0) = \sum_{i=1}^{n} \det \begin{pmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{i1}' & \cdots & f_{in}' \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{pmatrix}$$

Proof:

By definition

$$D = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \cdot \prod_{i=1}^{n} f_{i\sigma(i)}$$

and thus

$$D' = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \cdot \left(\prod_{i=1}^{n} f_{i\sigma(i)} \right)' = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \cdot \sum_{i=1}^{n} f_{i\sigma(i)}' \cdot \prod_{i \neq j = 1}^{n} f_{j\sigma(j)} = \sum_{i=1}^{n} \sum_{\sigma \in S_n} f_{i\sigma(i)}' \cdot \prod_{i \neq j = 1}^{n} f_{j\sigma(j)}$$

Each component in this sum is the determinant of D if instead we swapped the ith row with $f_{i1}', \ldots, f_{in}'$, thus we get the equality required by the lemma.

This means that by using the determinant formula for cross products, we have that

$$(f_1 \times f_2)' = f_1' \times f_2 + f_1 \times f_2'$$

Proposition 2.5.6:

Curvature and torsion are invariant under isometries.

Proof:

Suppose φ is an isometry, then it is of the form $v \mapsto Av + c$ for some orthogonal matrix A and vector c. Then if α is a natural parameterization, so is $\varphi \circ \alpha$ and

$$(\varphi \circ \alpha(t))' = J_\varphi(\alpha(t)) \cdot \alpha'(t) = A\alpha'(t)$$

And thus $T_{\varphi\circ\alpha} = AT_\alpha$ and so

$$\kappa_{\varphi\circ\alpha} = \left\|T'_{\varphi\circ\alpha}\right\| = \|AT'_\alpha\| = \|T'_\alpha\| = \kappa_\alpha$$

so curvature is indeed invariant.
And

$$B_{\varphi\circ\alpha} = T_{\varphi\circ\alpha} \times N_{\varphi\circ\alpha} = \frac{1}{\left\|T'_{\varphi\circ\alpha}\right\|} \cdot (AT_\alpha) \times (AT'_\alpha) = \frac{1}{\kappa_\alpha}(AT_\alpha) \times (AT'_\alpha)$$

And so

$$B'_{\varphi\circ\alpha} = \frac{1}{\kappa}(AT'_\alpha \times AT'_\alpha + AT_\alpha \times AT''_\alpha) = \frac{1}{\kappa}(AT_\alpha \times AT''_\alpha)$$

And since $N_{\varphi\circ\alpha} = \frac{T'_{\varphi\circ\alpha}}{\kappa} = \frac{AT'_\alpha}{\kappa}$,

$$\tau_{\varphi\circ\alpha} = \frac{1}{\kappa^2}\langle AT_\alpha \times AT''_\alpha, AT'_\alpha\rangle = \frac{1}{\kappa^2}\langle T_\alpha \times T''_\alpha, T'_\alpha\rangle = \tau_\alpha \qquad \blacksquare$$

Proposition 2.5.7:

Let γ be a regular smooth curve such that $\kappa \neq 0$ in the entire domain. Then the image of γ is contained within a plane if and only if $\tau = 0$ everywhere.

Proof:

If γ is contained within a plane, we can compose it with an isometry to move the plane to $[xy]$. Then we showed in the above example that the torsion of the transformed curve is zero. And since torsion is preserved under isometries, γ's torsion is zero.
Now, if $\tau = 0$ then $B' = 0$ so $B = w$ is constant. Since T is orthogonal to B, $\langle T, w\rangle = 0$ and so

$$(\langle \gamma, w\rangle') = \langle \gamma', w\rangle + \langle \gamma, 0\rangle = \langle T, w\rangle = 0$$

and thus $\langle \gamma, w\rangle = c$ is constant. Thus γ is contained within the plane $\{x \mid \langle x, w\rangle = c\}$. $\blacksquare$

In two dimensions, the osculating circle is the circle centered at

$$c(s_0) = \alpha(s_0) + \frac{1}{\kappa(s_0)}N(s_0)$$

whose radius is $\frac{1}{\kappa(s_0)}$. Using the same derivation for the two dimensional case, we get the same result in three dimensions. The plane spanned by $T(s_0)$ and $N(s_0)$ which contains $\alpha(s_0)$ is the *tangent plane* to the curve at s_0.
How do we compute κ and τ for arbitrary regular curves? Let β be an arbitrary regular curve and γ its natural parameterization:

$$\gamma = \beta \circ s^{-1} \implies \beta = \gamma \circ s$$

then

$$\begin{aligned}
\beta'(t) &= s'(t)\gamma'(s(t)), \\
\beta''(t) &= s''(t)\gamma'(s(t)) + s'(t)^2\gamma''(s(t)), \\
\beta'''(t) &= s'''(t)\gamma'(s(t)) + s''(t)s'(t)\gamma''(s(t)) + 2s'(t)s''(t)\gamma''(s(t)) + s'(t)^3\gamma'''(t)
\end{aligned}$$

Now, $\gamma' = T$ and $\gamma'' = T' = \kappa N$ and

$$\gamma''' = \kappa' N + \kappa N' = \kappa' N + \kappa(-\kappa T + \tau B) = \kappa' N - \kappa^2 T + \kappa\tau B$$

And so

$$\beta' \times \beta'' = (s'T) \times (s''T + (s')^2 \kappa N) = (s')^3 \kappa B$$

Thus

$$\|\beta' \times \beta''\| = \|\beta'\|^3 \kappa \implies \kappa = \frac{\|\beta' \times \beta''\|}{\|\beta'\|^3}$$

And

$$\langle \beta''', B \rangle = \big\langle (s')^3 \gamma''', B \big\rangle = (s')^3 \kappa\tau$$

Now, $s' = \|\beta'\|$, and so

$$(s')^3 \kappa = \|\beta'\|^3 \frac{\|\beta' \times \beta''\|}{\|\beta'\|^3} = \|\beta' \times \beta''\|$$

And since

$$\beta' \times \beta'' = (s')^3 \kappa B = \|\beta' \times \beta''\| B$$

And so

$$\tau = \frac{\langle \beta''', B \rangle}{\|\beta' \times \beta''\|} = \frac{\langle \beta''', \beta' \times \beta'' \rangle}{\|\beta' \times \beta''\|^2}$$

And since

$$\langle \beta''', \beta' \times \beta'' \rangle = \langle \beta' \times \beta'', \beta''' \rangle = \det(\beta', \beta'', \beta''')$$

we get

$$\tau = \frac{\det(\beta', \beta'', \beta''')}{\|\beta' \times \beta''\|^2}$$

Let us summarize this in the following proposition:

Proposition 2.5.8:

If β is an arbitrary regular smooth curve in $\mathbb{R}^3$, then its curvature and torsion are given by

$$\kappa_\beta(s) = \frac{\|\beta'(s) \times \beta''(s)\|}{\|\beta'(s)\|^3}, \qquad \tau_\beta(s) = \frac{\det\big(\beta'(s), \beta''(s), \beta'''(s)\big)}{\|\beta'(s) \times \beta''(s)\|^2}$$

3 Surfaces

3.1 Surfaces

Definition 3.1.1:

A **chart** for a topological space M is a homeomorphism from an open set $\mathcal{U} \subseteq M$ to an open subset of $\mathbb{R}^n$.

Definition 3.1.2:

A **surface** is a set $M \subseteq \mathbb{R}^3$ such that for every $p \in M$ there exists a chart $\sigma_p : \mathcal{U} \longrightarrow A$ where

(1) $\mathcal{U}$ is an open subset of $\mathbb{R}^2$,

(2) $A \subseteq M$,

(3) $p \in A$

A surface is **regular** if for every $p \in M$, $\sigma_p : \mathcal{U} \longrightarrow A$ is a **diffeomorphism**, that is J_{σ_p} has rank two (note that J_{σ_p} is a 3×2 matrix).

Definition 3.1.3:

Suppose M is a regular surface and $p \in M$, then we define the **tangent space** of M at p to be

$$T_pM = \{\gamma'(0) \mid \gamma\colon (-\varepsilon, \varepsilon) \longrightarrow M \text{ is a regular curve, and } \gamma(0) = p\}$$

Proposition 3.1.4:

If M is a regular surface, and $p \in M$, then let $\sigma(u, v)\colon \mathcal{U} \to M$ be a regular chart, and p is in its image. Suppose that $\sigma(q) = p$, then $T_pM = \operatorname{span}\{\sigma_1, \sigma_2\}$ where

$$\sigma_1 = \frac{d\sigma}{du}(q), \quad \sigma_2 = \frac{d\sigma}{dv}(q)$$

Proof:

First we will prove that $\sigma_1, \sigma_2 \in T_pM$. Suppose that $q = (u_0, v_0)$, then let us define the curve

$$\beta_1(t) = \begin{pmatrix} u_0 + t \\ v_0 \end{pmatrix}$$

and

$$\gamma_1(t) = \sigma \circ \beta_1(t) = \sigma(u_0 + t, v_0)$$

Now,

$$\gamma_1' = \sigma_u(\beta_1)\beta_{1u} + \sigma_v(\beta_1)\beta_{1v}$$

And we know that $\beta_{1u} = 1$ and $\beta_{1v} = 0$ so we have that

$$\gamma_1'(t) = \sigma_u(u_0 + t, v_0)$$

And therefore $\gamma_1'(0) = \sigma_u(u_0, v_0) = \sigma_1$. Therefore $\sigma_1 \in T_pM$. Similarly, $\sigma_2 \in T_pM$.
Now, let A and B be constants, we will prove that $w = A\sigma_1 + B\sigma_2$ is in T_pM. Let us define

$$\gamma(t) = \sigma(u_0 + At, v_0 + Bt)$$

Then

$$\gamma'(t) = \sigma_u(u_0 + At, v_0 + Bt) \cdot A + \sigma_v(u_0 + At, v_0 + Bt) \cdot B$$

and therefore

$$\gamma'(0) = \sigma_u(u_0, v_0)A + \sigma_v(u_0, v_0)B = \sigma_1 A + \sigma_2 B = w$$

and so $w \in T_pM$ as required. Therefore

$$\operatorname{span}\{\sigma_1, \sigma_2\} \subseteq T_pM$$

Now suppose that $\gamma'(0) \in T_pM$. Let us define the curve

$$\beta(t) = \sigma^{-1} \circ \gamma(t)$$

Notice that

$$\beta(0) = \sigma^{-1}(\gamma(0)) = \sigma^{-1}(p) = q$$

Since σ is a diffeomorphism, its inverse is differentiable. And since γ is differentiable, we have that β is differentiable.
Since $\gamma = \sigma \circ \beta$, we have that

$$\gamma' = \sigma_u(\beta) \cdot \beta_u + \sigma_v(\beta) \cdot \beta_v$$

And so

$$\gamma'(0) = \sigma_u(\beta(0))\beta_u(0) + \sigma_v(\beta(0))\beta_v(0)$$

Since $\sigma_u(\beta(0)) = \sigma_u(q) = \sigma_1$ and similarly $\sigma_v(\beta(0)) = \sigma_2$, let us define $A = \beta_u(0)$ and $B = \beta_v(0)$, and so we have that

$$\gamma'(0) = A\sigma_1 + B\sigma_2$$

and therefore $T_pM \subseteq \operatorname{span}\{\sigma_1, \sigma_2\}$. ■

Definition 3.1.5:

Suppose M is a regular surface, and let $p \in M$. Then there exists a chart $\sigma_p \colon \mathcal{U} \longrightarrow M$, where $\sigma_p(q) = p$. We define its **differential** to be a function

$$d\sigma_p \colon T_q\mathcal{U} \longrightarrow T_pM$$

Since $\mathcal{U} \subseteq \mathbb{R}^2$, $T_q\mathcal{U}$ is a line in $\mathbb{R}^2$, and so we define $d\sigma_p$ by its Jacobian,

$$d\sigma_p \begin{pmatrix} \delta u \\ \delta v \end{pmatrix} = J_{\sigma_p}(q) \begin{pmatrix} \delta u \\ \delta v \end{pmatrix} = \sigma_1 \delta u + \sigma_2 \delta v$$

where $\sigma_1 = \frac{d\sigma_p}{du}(q)$ and $\sigma_2 = \frac{d\sigma_p}{dv}(q)$.
Since $J_{\sigma_p}(q)$ has rank two, as M is regular, $d\sigma_p$ is invertible.

Example 3.1.6:

Suppose f is a function $X \to \mathbb{R}$, where $X \subseteq \mathbb{R}^2$. Then let us define the surface

$$M = \{(x, y, f(x, y)) \mid x, y \in X\}$$

Then let us define the chart

$$\sigma \colon \mathcal{U} \longrightarrow M, \qquad \sigma(u, v) = (u, v, f(u, v))$$

Then

$$\sigma_1 = \begin{pmatrix} 1 \\ 0 \\ f_u \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 \\ 1 \\ f_v \end{pmatrix}$$

These are linearly independent and so J_σ does indeed have a rank of two, therefore σ is indeed a regular chart. Thus M is regular.
And we can look at M's tangent space:

$$T_pM = \operatorname{span}\{\sigma_1, \sigma_2\} = \left\{ \begin{pmatrix} \delta u \\ \delta v \\ f_u \delta u + f_v \delta v \end{pmatrix} \,\middle|\, \delta u, \delta v \in \mathbb{R} \right\}$$

Suppose that $\gamma \colon [a, b] \longrightarrow \mathbb{R}^3$ is a curve such that $\gamma([a, b])$ is contained within a surface M. Let us define $\beta \colon [a, b] \longrightarrow \mathbb{R}^2$ by

$$\beta = \sigma^{-1} \circ \gamma \implies \gamma = \sigma \circ \beta$$

Then the arclength of γ is

$$L = \int_a^b \|\gamma'\| = \int_a^b \|\sigma_u(\beta) \cdot \beta_1' + \sigma_v(\beta) \cdot \beta_2'\|$$

Now recall that, as this is a norm generated by an inner product,

$$\|v + u\| = \sqrt{\langle v, v\rangle + 2\langle v, u\rangle + \langle u, u\rangle}$$

and thus

$$L = \int_a^b \sqrt{(\beta_1')^2 \langle \sigma_u(\beta), \sigma_u(\beta)\rangle + 2\langle \sigma_u, \sigma_v\rangle \beta_1' \beta_2' + (\beta_2')^2 \langle \sigma_v(\beta), \sigma_v(\beta)\rangle}$$

Let us define g to be a matrix

$$g_{11} = \langle \sigma_u, \sigma_u\rangle, \quad g_{12} = g_{21} = \langle \sigma_u, \sigma_v\rangle, \quad g_{22} = \langle \sigma_v, \sigma_v\rangle$$

And thus

$$L = \int_a^b \left(\begin{pmatrix} \beta_1' & \beta_2' \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} \beta_1' \\ \beta_2' \end{pmatrix} \right)^{1/2}$$

Notice that

$$g = \begin{pmatrix} \text{---} & \sigma_u^\top & \text{---} \\ \text{---} & \sigma_v^\top & \text{---} \end{pmatrix} \begin{pmatrix} | & | \\ \sigma_u & \sigma_v \\ | & | \end{pmatrix} = J_\sigma^\top J_\sigma$$

Proposition 3.1.7:

Suppose that $\sigma(q) = p$, then if $w, r \in T_q\mathcal{U}$ then

$$w^\top gr = \langle d\sigma_p(w), d\sigma_p(r)\rangle$$

where g is evaluated at $\sigma^{-1}(p) = q$.

Proof:

We know that

$$w^\top gr = \langle \sigma_u, \sigma_u\rangle w_1 r_2 + \langle \sigma_u, \sigma_v\rangle w_1 r_2 + \langle \sigma_u, \sigma_v\rangle w_2 r_1 + \langle \sigma_v, \sigma_v\rangle w_2 r_2 = \langle \sigma_u w_1 + \sigma_v w_2,\, \sigma_u r_1 + \sigma_v r_2\rangle$$

Now, recall that since

$$J_\sigma = \begin{pmatrix} | & | \\ \sigma_u & \sigma_v \\ | & | \end{pmatrix}$$

and so this is equal to

$$w^\top gr = \langle J_\sigma(q)w, J_\sigma(q)r\rangle = \langle d\sigma_p(w), d\sigma_p(r)\rangle$$

■

And so g defines an inner product on $T_q\mathcal{U}$, by

$$\langle w, r\rangle_g = w^\top gr = \langle d\sigma_p(w), d\sigma_p(r)\rangle$$

This is an inner product as the inner product on $\mathbb{R}^3$ and $d\sigma_p$ is an invertible linear transform.
g is called the *metric*, and $\langle \cdot, \cdot\rangle_g$ is called the *pullback* inner product of $\langle \cdot, \cdot\rangle_{\mathbb{R}^3}$.
Now, recall that we define the angle between two vectors by

$$\theta = \cos^{-1}\left(\frac{\langle v, u\rangle}{\|v\|\|u\|}\right)$$

And so using the pullback inner product, if we have two tangent vectors $\gamma'(0)$ and $\delta'(0)$, then let

$$\beta = \sigma^{-1} \circ \gamma, \quad \gamma = \sigma^{-1} \circ \delta$$

This means that

$$\gamma'(t) = J_\sigma(\beta(t)) \cdot \beta'(t)$$

and since $\beta(0) = \sigma^{-1}(\gamma(0)) = \sigma^{-1}(p)$, so

$$\gamma'(0) = J_\sigma(\sigma^{-1}(p))\beta'(0) = d\sigma(\beta'(0))$$

And therefore

$$\langle \gamma'(0), \delta'(0)\rangle = \langle d\sigma(\beta'(0)), d\sigma(\lambda'(0))\rangle = \langle \beta'(0), \lambda'(0)\rangle_g$$

So the angle between $\gamma'(0)$ and $\delta'(0)$ is

$$\theta = \cos^{-1}\left(\frac{\langle \gamma'(0), \delta'(0)\rangle}{\|\gamma'(0)\|\|\delta'(0)\|}\right) = \cos^{-1}\left(\frac{\langle \beta'(0), \lambda'(0)\rangle_g}{\|\beta'(0)\|_g\|\lambda'(0)\|_g}\right)$$

Using the definition of the pullback inner product,

$$\theta = \cos^{-1}\left(\frac{\beta'(0)^\top g\lambda'(0)}{\left(\beta'(0)^\top g\beta'(0) \cdot \lambda'(0)^\top g\lambda'(0)\right)^{1/2}}\right)$$

Suppose we have a surface M and a subset $A \subseteq M$. We can approximate the area of A by looking at the area of the parallelogram defined by $\delta u\sigma_u$ and $\delta v\sigma_v$ at every point $p \in A$ as δu and δv become smaller and smaller. The area of this parallelogram is

$$\|\delta u\sigma_u \times \delta v\sigma_v\| = \delta u\delta v\|\sigma_u \times \sigma_v\|$$

Now we can show that the determinant of g is equal to the square of this same volume, ie $\det(g) = \|\sigma_u \times \sigma_v\|^2$.
So we can define area as follows:

Definition 3.1.8:

So then the area of A can be defined to be

$$S(A) = \int_{\sigma^{-1}(A)} \|\sigma_u \times \sigma_v\| \, dudv = \int_{\sigma^{-1}(A)} \sqrt{\det(g)} \, dudv$$

Example 3.1.9:

We can define a chart for polar coordinates:

$$\sigma(r, \theta) = (r\cos\theta,\, r\sin\theta,\, 0)$$

where $-\pi < \theta < \pi$ and $r > 0$. Then

$$\sigma_r = (\cos\theta, \sin\theta, 0), \quad \sigma_\theta = (-r\sin\theta,\, r\cos\theta, 0)$$

And so

$$g = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

Then we can define the curve

$$\beta(t) = (R, t)$$

And so $\sigma \circ \beta$ corresponds to a circle of radius R. And so

$$L(\sigma\circ\beta) = \int_{-\pi}^{\pi} \Big((\beta')^\top \cdot g(\beta) \cdot \beta'\Big)^{1/2} = \int_{-\pi}^{\pi} \left(\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & R^2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)^{1/2} = \int_{-\pi}^{\pi} R = 2\pi R$$

So the area (but the surface σ is a plane, so the area is really the length) of $\sigma\circ\beta$ (the circle of radius R) is $2\pi R$.

Our goal is to define the curvature of a surface, similar to how we defined the curvature of a curve. Suppose we have a point p on the surface M, then we can define the unit normal to M at p to be a unit vector orthogonal to T_pM. Suppose $\sigma_1 = \frac{d\sigma}{du}(q)$ and $\sigma_2 = \frac{d\sigma}{dv}(q)$ where $q = \sigma^{-1}(p)$ then we showed $T_pM = \operatorname{span}\{\sigma_1, \sigma_2\}$, so we could define

$$N(p) = \frac{\sigma_1 \times \sigma_2}{\|\sigma_1 \times \sigma_2\|}$$

And since $\sqrt{g} = \|\sigma_1 \times \sigma_2\|$, this is equal to

$$N(p) = \frac{\sigma_1 \times \sigma_2}{\sqrt{g}}$$

So N is a function $M \longrightarrow S^2$, and we can define

$$\rho\colon \mathcal{U} \longrightarrow S^2, \quad \rho = N \circ \sigma$$

Definition 3.1.10:

The **second fundamental form** of a surface M gives the curvature of the surface at $p \in M$ in the direction of $v \in T_pM$. For every $v \in T_pM$ there exists some curve $\gamma\colon (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$ such that $\gamma'(0) = v$ and $\gamma(0) = p$. So we define the second fundamental form by

$$\mathrm{II}_p\colon T_pM \longrightarrow \mathbb{R}, \quad \mathrm{II}_p(v) = \langle -(N\circ\gamma)'(0), v\rangle$$

This should be reminiscient of the curvature of a planar curve being defined by $\kappa = \langle -N', T\rangle$.
So if $v \in T_pM$ then there exists some curve $\gamma\colon (-\varepsilon, \varepsilon) \longrightarrow M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$ then let us define a curve $\beta\colon (-\varepsilon, \varepsilon) \longrightarrow \mathcal{U}$ where

$$\gamma = \sigma \circ \beta$$

Then

$$\gamma'(t) = d\sigma(\beta(t)) \cdot \beta'(t)$$

and so

$$p = \gamma(0) = \sigma(\beta(0)) \implies \beta(0) = q$$

and

$$v = \gamma'(0) = d\sigma(q) \cdot \beta'(0) = d\sigma_p(\beta'(0))$$

thus

$$\beta'(0) = d\sigma_p^{-1}(v)$$

Now let us show that the definition of the second form is independent on choice of γ. This is since

$$(N \circ \gamma)'(t) = (N \circ \sigma \circ \beta)'(t) = (\rho \circ \beta)'(t) = \rho_u(\beta(t)) \cdot \beta_1'(t) + \rho_v(\beta(t)) \cdot \beta_2'(t) = J_\rho(\beta(t)) \cdot \beta'(t)$$

Thus we get that

$$(N \circ \gamma)'(0) = J_\rho(p) \cdot \beta'(0) = J_\rho(p) \cdot d\sigma_p^{-1} v$$

And so we get that $(N \circ \gamma)'(0)$ is dependent only on ρ and v, and thus the second fundamental form $\mathrm{II}_p(v)$ is independent on choice of $\gamma \in T_pM$ where $\gamma'(0) = v$.
Furthermore, since

$$\gamma'(0) = d\sigma_p(\beta'(0)) = \begin{pmatrix} | & | \\ \sigma_u & \sigma_v \\ | & | \end{pmatrix} \begin{pmatrix} \beta_1' \\ \beta_2' \end{pmatrix} = \sigma_u \beta_1' + \sigma_v \beta_2'$$

we get that

$$\mathrm{II}_p(v) = -\langle J_\rho(p)\beta'(0), \gamma'(0) \rangle = -\left\langle \begin{pmatrix} | & | \\ \rho_u & \rho_v \\ | & | \end{pmatrix} \begin{pmatrix} \beta_1' \\ \beta_2' \end{pmatrix}, \begin{pmatrix} \gamma_1' \\ \gamma_2' \end{pmatrix} \right\rangle = -\langle \rho_u \beta_1' + \rho_v \beta_2', \sigma_u \beta_1' + \sigma_v \beta_2' \rangle = -\sum_{i,j=1}^{2} \beta_i' \beta_j' \langle \rho_i, \sigma_j \rangle$$

Thus if we compute $\langle \rho_i(q), \sigma_j(q) \rangle$ for each $1 \le i, j \le 2$ then we can compute $\mathrm{II}_p(v)$ for each $v \in T_pM$. Let us summarize this in the following proposition

Proposition 3.1.11:

Suppose M is a surface parameterized by the chart σ, and p is a point in M such that $\sigma(q) = p$. Then for every $v \in T_pM$ given by the curve γ, if we define β to be $\sigma^{-1} \circ \gamma$, then

$$\beta'(0) = d\sigma_p^{-1}(v)$$

and

$$\mathrm{II}_p(v) = \sum_{i,j=1}^{2} \beta_i' \beta_j' \langle -\rho_i, \sigma_j \rangle$$

Let us denote

$$b_{ij} = \langle -\rho_i, \sigma_j \rangle, \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

and so we get

$$\mathrm{II}_p(v) = \beta'(0)^\top B \beta'(0)$$

For this reason, some people refer to the function

$$\mathrm{II}(x, y) = x^\top B y$$

as the *second fundamental form* instead of how we defined it.
Now, notice that

$$\langle \rho(q), \sigma_i(q) \rangle = 0$$

since $\sigma_i(q) \in T_pM$ and $\rho(q)$ is its unit normal. Thus

$$0 = \frac{d}{du_i} \langle \rho, \sigma_j \rangle = \langle \rho_i, \sigma_j \rangle + \langle \rho, \sigma_{ij} \rangle \implies \langle -\rho_i, \sigma_j \rangle = \langle \rho, \sigma_{ij} \rangle$$

And therefore we get that

$$b_{ij} = \langle \rho, \sigma_{ij} \rangle = \left\langle \frac{\sigma_1 \times \sigma_2}{\sqrt{\det g}}, \sigma_{ij} \right\rangle = \frac{1}{\sqrt{\det g}} \det \begin{pmatrix} — & \sigma_1 & — \\ — & \sigma_2 & — \\ — & \sigma_{ij} & — \end{pmatrix}$$

Which makes computing b_{ij} much simpler, as we only need to differentiate σ twice. But also notice that this means $b_{ij} = b_{ji}$, so B is a symmetric matrix.
Suppose n is some unit vector, then $\langle v, n\rangle$ is the distance between v and $n^\perp$, as the projection of v onto $n^\perp$ is $v - \langle v, n\rangle n$. So if p is on some surface M and v is in its tangent space T_pM suppose that $q = \sigma^{-1}(p)$ and $x = \sigma_p^{-1}(v)$ then

$$D_x(t) = \langle \sigma(q + tx) - \sigma(q),\, \rho(q)\rangle$$

tells us the distance of $\sigma(q + tx) - \sigma(q)$ from $\rho(q)^\perp = T_pM$. Thus $D_x(t)$ tells us the distance between the surface and T_pM after walking t units in the direction of x starting from p.

Proposition 3.1.12:

$$D_x(t) = \frac{t^2}{2}\,\mathrm{I\!I}_p(v) + o(t^2)$$

Proof:

We find the Taylor series of $\sigma(q + tx) - \sigma(q)$ centered at q,

$$\sigma(q + tx) - \sigma(q) = \sigma_1(q) \cdot tx_1 + \sigma_2(q) \cdot tx^2 + \frac{1}{2}\sum_{i,j=1}^{2} \sigma_{ij}(q) \cdot (tx_i)(tx_j) + o(t^2)$$

Now since $\sigma_i(q)$ are in T_pM, their inner product with $\rho(q)$ is zero. Thus

$$D_x(t) = \frac{1}{2}\sum_{i,j=1}^{2} (tx_i)(tx_j)\langle \sigma_{ij}, \rho\rangle + \big\langle o(t^2), \rho\big\rangle = \frac{t^2}{2} x^\top B x + o(t^2)$$

Now recall that

$$\mathrm{I\!I}_p(v) = \beta'(0)^\top B \beta'(0)$$

where $\beta'(0) = d\sigma_p^{-1}(v) = x$ and so

$$D_x(t) = \frac{t^2}{2}\,\mathrm{I\!I}_p(v) + o(t^2)$$

as required. ■

Proposition 3.1.13:

Let $\gamma\colon [0, T] \longrightarrow M$ be the natural parameterization of a curve on the surface M. Then

$$\langle \gamma''(s),\, N(\gamma(s))\rangle = \mathrm{I\!I}_{\gamma(s)}(\gamma'(s))$$

where $N(\gamma(s))$ is the unit normal to $\gamma(s)$ (not necessarily the Frenet-Serret frame).

Proof:

Since $\gamma'(s) \in T_{\gamma(s)}M$ (define $\beta = \sigma^{-1} \circ \gamma$, then $\gamma' = \sigma_u(\beta)\beta_1' + \sigma_v(\beta)\beta_2'$ which is in $T_{\gamma(s)}M$), we have that it is orthogonal to $N(\gamma(s))$. So

$$\langle \gamma'(s),\, N(\gamma(s))\rangle = 0 \implies \langle \gamma''(s), N(\gamma(s))\rangle + \langle \gamma'(s),\, (N \circ \gamma)'(s)\rangle = 0$$

And so

$$\langle \gamma''(s), N(\gamma(s))\rangle = \langle -(N \circ \gamma)'(s), \gamma'(s)\rangle$$

and so

$$\mathrm{I\!I}_{\gamma(s)}(\gamma'(s)) = \langle -(N \circ \gamma)'(s), \gamma'(s)\rangle = \langle \gamma''(s), N(\gamma(s)\rangle$$

The first equality is not trivial here, let us define $\lambda(t) = \gamma(t + s)$ and so $\lambda'(t) = \gamma'(t + s)$. Then $\lambda'(0) = \gamma'(s) \in T_{\lambda(0)}M = T_{\gamma(s)}M$ and so

$$\mathrm{I\!I}_{\gamma(s)}(\gamma'(s)) = \mathrm{I\!I}_{\lambda(0)}(\lambda'(0)) = \langle -(N \circ \lambda)'(0), \lambda'(0)\rangle = \langle -(N \circ \gamma)'(s), \gamma'(s)\rangle$$ ■

This means that $\langle \gamma'', N\rangle$ is dependent only on γ'. And also since the curvature of a three-dimensional curve is defined by $\kappa(s) = \|\gamma''(s)\|$ we have

$$\left|\mathrm{II}_{\gamma(s)}(\gamma'(s))\right| = |\langle \gamma''(s), N(\gamma(s))\rangle| \leq \|\gamma''(s)\| \cdot \|N(\gamma(s)\| = \kappa(s)$$

Definition 3.1.14:

Suppose M is a surface, p is a point on the surface, and v is a unit vector in p's tangent space. Then the **normal curvature of M at p in the direction of v** is defined to be $\mathrm{II}_p(v)$.

Notice that since

$$D_x(t) = \frac{t^2}{2}\,\mathrm{II}_p(v) + o(t^2)$$

if the curvature in the direction v is negative, then the curve is curving away from the normal at p. And if it is positive then the curve curves towards the normal.

Now, suppose γ is the natural parameterization of some curve. Then $N(\gamma(s))$, $\gamma'(s)$, and $R_{\frac{\pi}{2}}\gamma'(s)$ forms an orthonormal basis (where $R_{\frac{\pi}{2}}$ is the linear operator on $T_{\gamma(s)}M$). So

$$\gamma'' = \langle \gamma'', N\rangle N + \langle \gamma'', \gamma'\rangle \gamma' + \left\langle \gamma'', R_{\frac{\pi}{2}}\gamma'\right\rangle R_{\frac{\pi}{2}}\gamma'$$

We know that $\langle \gamma'', N\rangle = \mathrm{II}_\gamma(\gamma')$ and since $\|\gamma'\| = 1$ is constant, $\gamma'' \perp \gamma'$. Thus

$$\gamma'' = \mathrm{II}_\gamma(\gamma')N + \left\langle \gamma'', R_{\frac{\pi}{2}}\gamma'\right\rangle R_{\frac{\pi}{2}}\gamma'$$

These components of γ'' are significant:

Definition 3.1.15:

If γ is the natural parameterization of a curve on a surface M, then we define the **normal curvature** of γ to be

$$\kappa_n(s) = \mathrm{II}_{\gamma(s)}(\gamma'(s))$$

and the **geodesic curvature** of γ to be

$$\kappa_g(s) = \left\langle \gamma''(s), R_{\frac{\pi}{2}}\gamma'(s)\right\rangle$$

The normal curvature of a curve is the curvature of the surface in the direction of the curve (the direction of its derivative). The geodesic curvature corresponds to how non-straight the curve is (how orthogonal γ'' is to γ'), relative to a straight line on M (since M itself may be curved, a straight line on M may not correspond to a straight line in $\mathbb{R}^3$).

And the curvature of γ is given by

$$\kappa(s) = \|\gamma''(s)\| = \sqrt{\kappa_n(s)^2 + \kappa_g(s)^2}$$

Definition 3.1.16:

A **geodesic** is a a curve whose geodesic curvature is zero.

So a geodesic has as little curvature as possible on M; its only curvature is curving in the direction of M. Thus a geodesic corresponds to a straight line on M.

Recall that we showed that if γ is a smooth curve where $\gamma(0) = p$ and $\gamma'(0) = w$ then

$$(N \circ \gamma)'(0) = J_\rho(p) \cdot d\sigma_p^{-1}(w)$$

now since $\|N \circ \gamma\| = 1$, $(N \circ \gamma)'$ is orthogonal to $N \circ \gamma$. So $(N \circ \gamma)'(0)$ is orthogonal to $N \circ \gamma(0) = N(p)$, and so $(N \circ \gamma)'(0)$ is on the tangent space of p. Thus we can define

Definition 3.1.17:

Let M be a surface. We define a linear transformation

$$S\colon T_pM \longrightarrow T_{N(p)}S^2$$

For every $w \in T_pM$, there exists a curve γ such that $\gamma(0) = p$ and $\gamma'(0) = w$, and we define

$$S(w) = -(N \circ \gamma)'(0)$$

S is called the **shape operator**.

Since p is (a unit) normal to T_pS^2, we have that $T_pS^2 = p^\perp$. And since $\|N \circ \gamma\| = 1$ is constant, $(N \circ \gamma)'$ is orthogonal to $N \circ \gamma$, and therefore $(N \circ \gamma)'(0)$ is on $T_{N\circ\gamma(0)}S^2 = T_{N(p)}S^2$ as required. Notice that v is in $T_{N(p)}S^2$ if and only if v is tangent to S^2 at $N(p)$, which is if and only if v is orthogonal to $N(p)$, which is if and only if $v \in T_pM$. Thus $T_{N(p)}S^2 = T_pM$, and $(N \circ \gamma)'(0)$ is orthogonal to $N \circ \gamma(0) = N(p)$ (since the norm of N is constant), and so $(N \circ \gamma)'(0)$ is in $T_pM = T_{N(p)}S^2$, so S's codomain is indeed $T_{N(p)}S^2 = T_pM$. As we showed,

$$S(w) = -J_\rho(p) \cdot d\sigma_p^{-1}(w)$$

which is the product of linear transformations, and independent of the choice of γ, and therefore S is therefore well-defined and a linear transformation.
By definition,

$$\mathrm{I\!I}_p(v) = \langle S(v), v\rangle$$

Example 3.1.18:

(1) If M is a plane, then N is some constant vector. Thus $(-N \circ \gamma)' = 0$ and so $S = 0$ and $\mathrm{I\!I}_p(v) = 0$ for every p and v.

(2) On a sphere of radius R, $N(p) = \frac{p}{R}$ and so

$$(N \circ \gamma)' = \left(\frac{\gamma}{R}\right)' = \frac{\gamma'}{R}$$

and thus

$$(N \circ \gamma)'(0) = \frac{\gamma'(0)}{R} = \frac{v}{R}$$

And so $S(v) = -\frac{v}{R}$, and therefore

$$\mathrm{I\!I}_p(v) = \left\langle -\frac{v}{R}, v\right\rangle = -\frac{\|v\|^2}{R} = -\frac{1}{R}$$

since $v \in S^2$ so $\|v\| = 1$.

In general it is hard to compute S. But since it is a linear transformation, we can compute it based on its image of a basis. Since $\{\sigma_1, \sigma_2\}$ is a basis for T_pM where $\sigma_i = \frac{\sigma}{du_i}(q)$ ($\sigma(q) = p$), we get

$$S(\sigma_1) = S\left(d\sigma_p\begin{pmatrix}1\\0\end{pmatrix}\right) = -J_\rho(p)\cdot\begin{pmatrix}1\\0\end{pmatrix} = -\frac{d\rho}{du}(q) = \rho_1$$

and similarly

$$S(\sigma_2) = \rho_2$$

Now, we said that the codomain of S is $T_{N(p)}S^2 = T_pM$, so let us find $-\rho_i$ in terms of the basis $\{\sigma_1, \sigma_2\}$. Suppose

$$-\rho_1 = c_1\sigma_1 + c_2\sigma_2$$

Now recall the following definitions

$$B = \begin{pmatrix} b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}, \quad b_{ij} = \langle -\rho_i, \sigma_j\rangle$$

and

$$g = \begin{pmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix}, \quad g_{ij} = \langle \sigma_i, \sigma_j\rangle$$

And so

$$b_{11} = \langle -\rho_1, \sigma_1\rangle = c_1\langle\sigma_1, \sigma_1\rangle + c_2\langle\sigma_1,\sigma_2\rangle = c_1g_{11} + c_2g_{12}$$

And similarly

$$b_{12} = \langle -\rho_1, \sigma_2 \rangle = c_1 g_{12} + c_2 g_{22}$$

And since $b_{12} = b_{21}$ we get

$$g \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} b_{11} \\ b_{12} \end{pmatrix} = R_1(B)$$

Similarly we get that if $-\rho_2 = c_3\sigma_1 + c_4\sigma_2$, then

$$g \begin{pmatrix} c_3 \\ c_4 \end{pmatrix} = R_2(B)$$

Thus we get

$$g \begin{pmatrix} c_1 & c_3 \\ c_2 & c_4 \end{pmatrix} = B \implies \begin{pmatrix} c_1 & c_3 \\ c_2 & c_4 \end{pmatrix} = g^{-1}B$$

(g is invertible since it is the product $J_\sigma^\top J_\sigma$ which both have a rank of two.) This means that if we define the basis $B = (\sigma_1, \sigma_2)$, then

$$[S]_B = g^{-1}B$$

So we have proven the following

Proposition 3.1.19:

If S is the shape operator of a surface M at the point p, g is the metric at p, and B is the second fundamental form at p then if we take $B = (\sigma_1, \sigma_2)$ as a basis for T_pM, then

$$[S]_B = g^{-1}B$$

Proposition 3.1.20:

The shape operator is self-adjoint, ie. for every $x, y \in T_pM$:

$$\langle x, S(y) \rangle = \langle S(x), y \rangle$$

Proof:

Notice first that

$$\langle S(\sigma_1), \sigma_2 \rangle = \langle -\rho_1, \sigma_2 \rangle = \langle \rho, \sigma_{12} \rangle = b_{12}$$

the last equality was proven before. Now

$$\langle \sigma_1, S(\sigma_2) \rangle = \langle \sigma_1, -\rho_2 \rangle = b_{12}$$

So $\langle S(\sigma_1), \sigma_2 \rangle = \langle \sigma_1, S(\sigma_2) \rangle$. The rest follows as S is a linear transformation (operator) and (σ_1, σ_2) is a basis for T_pM. ■

Since S is self-adjoint, it is diagonalizable. The eigenvalues and eigenvectors have an important geometric meaning.

Theorem 3.1.21 (Euler's Theorem):

Let w be a unit vector in T_pM, then for every angle θ we define R_θ to be the rotation operator on T_pM which rotates by an angle of θ. For each θ we define the plane $p + \operatorname{span}\{N, R_\theta w\}$ which intersects with M to give some curve n_θ. Then we define $\kappa_n(\theta)$ as the function

$$\kappa_n \colon [0, 2\pi] \longrightarrow \mathbb{R}, \quad \kappa_n(\theta) = \mathrm{II}_p(R_\theta w)$$

Since σ and N are smooth, so is II_p and κ_n is continuous and therefore obtains a minimum and maximum κ_1 and κ_2 by θ_1 and θ_2 respectively. Let us define $w_i = R_{\theta_i} w$, then

(1) w_1 and w_2 are orthogonal.

(2) Since w_1 and w_2 are orthogonal, there exists an angle α such that $w = w_1 \cos\alpha + w_2 \sin\alpha$ then

$$\mathrm{II}_p(w) = \kappa_1 \cos^2\alpha + \kappa_2 \sin^2\alpha$$

First let us prove a lemma:

Lemma 3.1.22:

If $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a self-adjoint linear operator then

(1) There exists an orthonormal basis of eigenvectors of A.

(2) If $\lambda_1 \leq \lambda_2$ are T's eigenvalues, then

$$\lambda_1 = \min_{\|v\|=1} \langle Tv, v\rangle, \quad \lambda_2 = \max_{\|v\|=1} \langle Tv, v\rangle$$

(3) For every $v = v_1 \cos\alpha + v_2 \sin\alpha$,

$$\langle Tv, v\rangle = \lambda_1 \cos^2\alpha + \lambda_2 \sin^2\alpha$$

Proof:

Since the function $f(v) = \langle Tv, v\rangle$ over $\{v \mid \|v\| = 1\}$ is a continuous function over a compact set, f obtains a minimum λ_1 at v_1. Let v_2 be any unit vector orthogonal to v_1, and let $\lambda_2 = \langle Tv_2, v_2\rangle$. Since (v_1, v_2) forms an orthonormal basis,

$$Tv_1 = \langle Tv_1, v_1\rangle v_1 + \langle Tv_2, v_2\rangle v_2$$

and so on. Since $b = \langle Tv_1, v_2\rangle = \langle v_1, Tv_2\rangle$ as T is self-adjoint, the matrix which represents T relative to the basis (v_1, v_2) is

$$A = \begin{pmatrix} \langle Tv_1, v_1\rangle & \langle Tv_2, v_1\rangle \\ \langle Tv_1, v_2\rangle & \langle Tv_2, v_2\rangle \end{pmatrix} = \begin{pmatrix} \lambda_1 & b \\ b & \lambda_2 \end{pmatrix}$$

Now suppose $v = v_1 \cos\alpha + v_2 \sin\alpha$ then

$$Tv = Tv_1 \cos\alpha + Tv_2 \sin\alpha$$

and so

$$f(v) = \langle Tv, v\rangle = \langle Tv_1 \cos\alpha + Tv_2 \sin\alpha,\, v_1 \cos\alpha + v_2 \sin\alpha\rangle = \cos^2\alpha \langle Tv_1, v_1\rangle + 2\cos\alpha \sin\alpha \langle Tv_1, v_2\rangle + \sin^2\alpha \langle Tv_2, v_2\rangle$$

which is equal to, by definition,

$$= \lambda_1 \cos^2\alpha + 2b \cos\alpha \sin\alpha + \lambda_2 \sin^2\alpha$$

Since if we were to define $g(\alpha) = f(v_1 \cos\alpha + v_2 \sin\alpha)$ we'd get that $f(v) = g(\alpha)$. And since $g(0) = f(v_1)$ is a minimum, $g'(0) = 0$. Thus

$$g'(\alpha) = -2\lambda_1 \sin(2\alpha) + 2b \cos(2\alpha) + \lambda_2 \sin(2\alpha)$$

and so

$$g'(0) = 2b = 0 \implies b = 0$$

And so the representation of T is

$$A = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$$

thus v_1 and v_2 indeed form an orthonormal basis of eigenvectors (as they are orthogonal and induce A). And

$$f(v) = \langle Tv, v\rangle = \lambda_1 \cos^2\alpha + \lambda_2 \sin^2\alpha$$

And since λ_1 is the minimum value of f, we have that $\lambda_1 \leq \lambda_2$ and so

$$f(v) \leq \lambda_2 \cos^2\alpha + \lambda_2 \sin^2\alpha = \lambda_2$$

And so $f(v_2) = \lambda_2$ is the maximum value of f, as required. ■

So let us now prove Euler's theorem:

Proof (Euler's Theorem):

We know that

$$\mathrm{II}_p(w) = \langle S(w), w\rangle$$

and S is self-adjoint, and so there exists w_1 and w_2 orthonormal such that

$$\kappa_1 = \mathrm{II}_p(w_1), \quad \kappa_2 = \mathrm{II}_p(w_2)$$

where κ_1 and κ_2 are the minimum and maximum of II_p respectively. Since $w = w_1\cos\alpha + w_2\sin\alpha$, by the above lemma

$$\mathrm{II}_p(w) = \kappa_1\cos^2\alpha + \kappa_2\sin^2\alpha$$

■

Definition 3.1.23:

The unit vectors w_1 and w_2 are called the **principal directions** of M at p. And κ_1 and κ_2 are the **principal curvatures** of M at p.

Now, notice that $\det S = \kappa_1\kappa_2$. And so if $\det S > 0$ then κ_1 and κ_2 have the same parity and since κ_1 is the minimum and κ_2 the maximum normal curvatures of M at p, all the curvatures of M at p have the same parity. And thus M curves in the same direction in every direction from p. And similarly if $\det S < 0$ then M curves in different directions at p.
For this reason we define

Definition 3.1.24:

If M is a surface then its **Gaussian curvature** is defined to be $K(p) = \det S = \kappa_1\kappa_2$.

Now, recall that the representation of S by the basis (σ_1, σ_2) is equal to $g^{-1}B$ and thus

$$K = \det S = \frac{\det B}{\det g}$$

Example 3.1.25:

Let f be a smooth function, and let us define the chart

$$\sigma(u,v) = (u, v, f(u,v))$$

Then

$$\sigma_1 = (1, 0, f_u), \quad \sigma_2 = (0, 1, f_v), \quad \sigma_{12} = (0, 0, f_{uv})$$

Thus

$$T_pM = \left\{ \begin{pmatrix} \delta u \\ \delta v \\ \nabla f \begin{pmatrix} \delta u \\ \delta v \end{pmatrix} \end{pmatrix} \middle| \; \delta u, \delta v \in \mathbb{R} \right\}$$

And so

$$g = \begin{pmatrix} 1 + f_u^2 & f_u f_v \\ f_u f_v & 1 + f_v^2 \end{pmatrix} \implies \det(g) = 1 + f_u^2 + f_v^2$$

And since

$$b_{ij} = \langle \rho, \sigma_{ij}\rangle = \frac{1}{\sqrt{\det g}} \det \begin{pmatrix} 1 & 0 & f_u \\ 0 & 1 & f_v \\ 0 & 0 & f_{ij} \end{pmatrix} = \frac{f_{ij}}{\sqrt{1 + f_u^2 + f_v^2}}$$

Thus

$$B = \frac{1}{\sqrt{1 + f_u^2 + f_v^2}} \begin{pmatrix} f_{uu} & f_{uv} \\ f_{uv} & f_{vv} \end{pmatrix}$$

which is equal to $\frac{1}{\sqrt{1+f_u^2+f_v^2}}$ times the Hessian matrix of f, H_f. And so

$$\det B = \frac{f_{uu}f_{vv} - f_{uv}^2}{1 + f_u^2 + f_v^2}$$

And so

$$K = \frac{\det B}{\det g} = \frac{f_{uu}f_{vv} - f_{uv}^2}{(1 + f_u^2 + f_v^2)^2}$$

Notice then that at critical points, $\nabla f = 0$ and so

$$K = \det(H_f)$$

This tells us the significance of the Hessian, as if at a critical point $\det(H_f) = K < 0$ then the surface (and so f), curves both up and down, and so the critical point is an inflection point. Otherwise if $K = \det(H_f) > 0$, the point is a maximum or a minimum (if the surface is locally above T_pM then it is a maximum, and if the surface is locally below T_pM then it is a minimum).

3.2 Geodesics

Recall that for every curve γ on a surface M, we defined its *normal* and *geodesic* curvature by

$$\kappa_n(s) = \mathrm{I\!I}_{\gamma(s)}(\gamma'(s)), \qquad \kappa_g(s) = \left\langle \gamma''(s), R_{\frac{\pi}{2}}\gamma'(s) \right\rangle$$

where $R_{\frac{\pi}{2}}$ is the rotation matrix on $T_{\gamma(s)}M$. The normal curvature, $\kappa_n(s)$, is equal to the curvature of M at $\gamma(s)$ in the direction of its velocity ($\gamma'(s)$). The geodesic curvature measures how orthogonal $\gamma''(s)$ is from $\gamma'(s)$, or indeed how orthogonal it is from $\gamma(s)$'s tangent space.

Proposition 3.2.1:

The geodesic curvature of γ is zero if and only if $\gamma''(s)$ is orthogonal to $T_{\gamma(s)}M$.

Proof:

If $\kappa_g(s) = 0$ then $\left\langle \gamma''(s), R_{\frac{\pi}{2}}\gamma'(s) \right\rangle = 0$. And since $\|\gamma'(s)\| = 1$ is constant, $\gamma''(s)$ and $\gamma'(s)$ are orthogonal. Now, $\left\{ R_{\frac{\pi}{2}}\gamma'(s), \gamma'(s) \right\}$ form a basis for $T_{\gamma(s)}M$ and so $\gamma''(s)$ is orthogonal to $T_{\gamma(s)}M$ as required.
And if $\gamma''(s)$ is orthogonal to $T_{\gamma(s)}M$, then it is necessarily orthogonal to $R\frac{\pi}{2}\gamma'(s)$ and thus $\kappa_g(s) = 0$ as required. ■

This allows us to define what it means for a non-natural parameterization to be a geodesic:

Definition 3.2.2:

Let M be a surface and $\gamma\colon [a,b] \longrightarrow M$ an arbitrary curve on M. γ is a **geodesic** if and only if $\gamma''(t)$ is orthogonal to $T_{\gamma(t)}M$ for every $t \in [a,b]$.

Note that the parameterization of a curve affects if it is a geodesic or not. Suppose that γ is a geodesic and $\beta = \gamma \circ \varphi$ is a reparameterization of γ. Then

$$\beta' = \gamma'(\varphi) \cdot \varphi' \implies \beta'' = \gamma''(\varphi) \cdot (\varphi')^2 + \gamma'(\varphi) \cdot \varphi''$$

Let $t = \varphi(s)$, then $\gamma''(t)$ is orthogonal to $T_{\gamma(t)}M = T_{\beta(s)}M$. And so

$$\beta''(s) = \gamma''(t) \cdot (\varphi'(s))^2 + \gamma'(t) \cdot \varphi''(s)$$

and this is orthogonal to $T_{\beta(s)}M = T_{\gamma(t)}M$ if and only if $\gamma'(t) \cdot \varphi''(s)$ is (since $\gamma''(t)$ is, since γ is a geodesic). But $\gamma'(t) \in T_{\gamma(t)}M$, and so $\beta(s)$ is orthogonal to $T_{\beta(s)}M$ if and only if $\varphi''(s) = 0$. So β is a geodesic if and only if $\varphi'' = 0$, meaning $\varphi = at + b$. Let us summarize this in the following proposition:

Proposition 3.2.3:

If γ is a geodesic, the only reparameterizations of γ which are also geodesics are of the form $\gamma(at + b)$ (and all reparameterizations of this form are geodesics).

This is pretty significant, as generally up until now we've discussed properties of curves which are independent of the parameterization.

Proposition 3.2.4:

If γ is a geodesic, then $\|\gamma'\|$ is constant.

Proof:

Since γ'' is orthogonal to $T_\gamma M$ and γ is in $T_\gamma M$, we have that γ'' is orthogonal to γ'. Thus $\langle\gamma'',\gamma'\rangle = 0$. And since

$$\langle\gamma',\gamma'\rangle' = 2\langle\gamma'',\gamma'\rangle = 0$$

and so $\|\gamma'\|^2 = \langle\gamma',\gamma'\rangle$ is constant, as required. ■

Example 3.2.5:

(1) If M is a plane, then γ is a planar curve. Then γ'' is on the plane M shifted to the origin. This is equal to T_pM, and so $\gamma'' \in T_\gamma M$ and thus γ'' is orthogonal to γ's tangent space if and only if $\gamma'' = 0$. This is if and only if $\gamma(t) = c_0 t + c_1$ for some vectors c_0, c_1. So the only geodesics on a plane are lines.

(2) Let M be the sphere S^2. Then let u, v be orthonormal vectors then $\gamma(t) = u\cos(t) + v\sin(t)$ is a geodesic. This is as

$$\gamma''(t) = -u\cos(t) - v\sin(t) = -\gamma(t)$$

and $\gamma(t)$ is orthogonal to $T_{\gamma(t)}S^2$, and therefore so is $\gamma''(t)$. Thus γ is indeed a geodesic.
γ corresponds to a circle around S^2 which passes through opposite ends on the sphere (great circle).

(3) Let $M = \left\{(x,y,z) \,\middle|\, x^2+y^2=1\right\}$ be a cylinder. There are two types of geodesics:

(1) Helixes: $(\cos t, \sin t, ct)$

(2) Lines: $(\cos\alpha, \sin\alpha, ct)$ for some constant α.

We can chart M by $(u,v) \mapsto (\cos u, \sin u, v)$ and so the tangent space

$$T_{(\cos u, \sin u, v)}M = \operatorname{span}\left\{\begin{pmatrix}-\sin u\\ \cos u\\ 0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$

A normal to this tangent space is $(\cos u, \sin u, 0)$.
For helixes,

$$\gamma''(t) = -(\cos t, \sin t, 0)$$

which is orthogonal to the tangent space at $\gamma(t) = (\cos t, \sin t, ct)$, as we showed above.
And for lines, the second derivative is zero which is trivially orthogonal to the tangent space.

So how do we find geodesics in general? Let σ be a regular chart of the surface M, then $\{\sigma_1, \sigma_2, N\}$ is a basis of $\mathbb{R}^3$ at $p \in M$. But σ_1 and σ_2 need not be orthogonal, and their norm need not be 1. So this basis is not the simplest to deal with.
Now, since this is a basis we have that there exist Γ_{11}^1 and Γ_{11}^2 where

$$\sigma_{11} = \Gamma_{11}^1\sigma_1 + \Gamma_{11}^2\sigma_2 + b_{11}N$$

as N is orthogonal to σ_i and has a norm of one, and $\langle N, \sigma_{11}\rangle = b_{11}$. Similarly

$$\sigma_{12} = \Gamma_{12}^1\sigma_1 + \Gamma_{12}^2\sigma_2 + b_{12}N, \quad \sigma_{21} = \Gamma_{21}^1\sigma_1 + \Gamma_{21}^2\sigma_2 + b_{21}N$$

since $\sigma_{12} = \sigma_{21}$, $\Gamma_{12} = \Gamma_{21}$. And

$$\sigma_{22} = \Gamma_{22}^1\sigma_1 + \Gamma_{22}^2\sigma_2 + b_{22}N$$

So in general

$$\sigma_{ij} = \Gamma_{ij}^1\sigma_1 + \Gamma_{ij}^2\sigma_2 + b_{ij}N$$

The coefficients Γ_{ij}^k are called *Christoffel symbols*.

Theorem 3.2.6:

Let M be a surface and $\gamma = \sigma \circ \beta$ be a regular curve on M. Then γ is a geodesic if and only if

$$0 = \beta_1'' + (\beta_1')^2\Gamma_{11}^1 + 2\beta_1'\beta_2'\Gamma_{12}^1 + (\beta_2')^2\Gamma_{22}^1$$

and

$$0 = \beta_2'' + (\beta_1')^2\Gamma_{11}^2 + 2\beta_1'\beta_2'\Gamma_{12}^2 + (\beta_2')^2\Gamma_{22}^2$$

Or in other words for $i = 1, 2$:

$$\beta_i'' + \sum_{k,j=1}^{2} \beta_k'\beta_j'\Gamma_{kj}^i = 0$$

These equations are called the *geodesic equations*.

Proof:

By definition γ is a geodesic if and only if $\gamma''(s)$ is orthogonal to $T_{\gamma(s)}M$. Now, we know

$$\gamma' = \sigma'(\beta) \cdot \beta' = \sigma_1(\beta)\beta_1' + \sigma_2(\beta)\beta_2' = \sum_i \sigma_i(\beta) \cdot \beta_i'$$

And so

$$\gamma'' = \sum_i \left(\beta_i' \sum_j \sigma_{ij}(\beta) \cdot \beta_j' \right) + \sum_i \sigma_i(\beta) \cdot \beta_i''$$

And since $\sigma_{ij} = \Gamma_{ij}^1\sigma_1 + \Gamma_{ij}^2\sigma_2 + b_{ij}N$, we have

$$\gamma'' = \sum_i \sigma_i\beta_i'' + \sum_i \left(\beta_i' \sum_j \beta_j' \left(\Gamma_{ij}^1\sigma_1 + \Gamma_{ij}^2\sigma_2 + b_{ij}N \right) \right) =$$

$$\sigma_1 \left(\beta_1'' + \sum_{i,j} \beta_i'\beta_j'\Gamma_{ij}^1 \right) + \sigma_2 \left(\beta_2'' + \sum_{i,j} \beta_i'\beta_j'\Gamma_{ij}^2 \right) + N \left(\sum_{ij} \beta_i'\beta_j' b_{ij} \right)$$

Since N is orthogonal to $T_\gamma M$, and σ_1 and σ_2 are on it, γ'' is orthogonal to $T_\gamma M$ if and only if its coefficients for σ_1 and σ_2 are zero. Meaning

$$\beta_1'' + \sum_{i,j} \beta_i'\beta_j'\Gamma_{ij}^1 = \beta_2'' + \sum_{i,j} \beta_i'\beta_j'\Gamma_{ij}^2 = 0$$

which are precisely the geodesic equations. ■

We know that $g_{11} = \langle \sigma_1, \sigma_1 \rangle$ and differentiating this relative to u gives

$$g_{11,1} = 2\langle \sigma_{11}, \sigma_1 \rangle$$

The notation $g_{ij,k}$ is used instead of $\frac{d}{u_k} g_{ij}$. Similarly

$$g_{12,1} = \langle \sigma_{11}, \sigma_2 \rangle + \langle \sigma_1, \sigma_{12} \rangle$$
$$g_{11,2} = 2\langle \sigma_{12}, \sigma_1 \rangle$$

Thus

$$\langle \sigma_{11}, \sigma_2 \rangle = g_{12,1} - \frac{1}{2} g_{11,2}$$

On the other hand, using Christoffel symbols:

$$\langle \sigma_{11}, \sigma_1 \rangle = \Gamma_{11}^1 \langle \sigma_1, \sigma_1 \rangle + \Gamma_{11}^2 \langle \sigma_2, \sigma_1 \rangle = \Gamma_{11}^1 g_{11} + \Gamma_{11}^2 g_{12}$$

and similarly

$$\langle \sigma_{11}, \sigma_2 \rangle = \Gamma_{11}^1 g_{12} + \Gamma_{11}^2 g_{22}$$

Thus we have

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} \begin{pmatrix} \Gamma_{11}^1 \\ \Gamma_{11}^2 \end{pmatrix} = \begin{pmatrix} \langle \sigma_{11}, \sigma_1 \rangle \\ \langle \sigma_{11}, \sigma_2 \rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} g_{11,1} \\ g_{12,1} - \frac{1}{2} g_{11,2} \end{pmatrix}$$

Thus

$$\begin{pmatrix} \Gamma_{11}^1 \\ \Gamma_{11}^2 \end{pmatrix} = g^{-1} \begin{pmatrix} \frac{1}{2} g_{11,1} \\ g_{12,1} - \frac{1}{2} g_{11,2} \end{pmatrix}$$

Using the same process but with σ_{12}, we get

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} \begin{pmatrix} \Gamma_{12}^1 \\ \Gamma_{12}^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} g_{11,2} \\ \frac{1}{2} g_{22,1} \end{pmatrix}$$

and

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} \begin{pmatrix} \Gamma_{22}^1 \\ \Gamma_{22}^2 \end{pmatrix} = \begin{pmatrix} g_{12,2} - \frac{1}{2} g_{22,1} \\ \frac{1}{2} g_{22,2} \end{pmatrix}$$

Using Einstein notation, we denote the inverse of g by

$$g^{-1} = \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix}$$

and so by the equations above we get

$$\Gamma_{ij}^k = \frac{1}{2} g^{k1} \big(g_{1i,j} + g_{j1,i} - g_{ij,1} \big) + \frac{1}{2} g^{k2} \big(g_{2i,j} + g_{j2,i} - g_{ij,2} \big)$$

Using Einstein notation

$$\Gamma_{ij}^k = \frac{1}{2} g^{km} \big(g_{mi,j} + g_{jm,i} - g_{ij,m} \big)$$

This means that the Christoffel symbols are dependent only on g, and not on a choice of N.
Furthermore, notice that this means the geodesic equations give a system of second order ODEs (in terms of β), and so by the uniqueness and existence theorem, there exists an $\varepsilon > 0$ such that for every initial point p and velocity v, there exists a unique curve $\beta\colon (-\varepsilon, \varepsilon) \longrightarrow \mathcal{U}$ where $\beta(0) = \sigma^{-1}(p)$ and $\beta'(0) = d\sigma_p^{-1}(v)$. This means that $\gamma(0) = p$ and $\gamma'(0) = v$. So we have shown the following proposition

Proposition 3.2.7:

For any intial point p and initial velocity $v \in T_pM$, there exists a geodesic γ such that $\gamma(0) = p$ and $\gamma'(0) = v$.

Note that γ's domain $(-\varepsilon, \varepsilon)$ may be arbitrarily small, so the geodesic exists within an arbitrarily small neighborhood of p.
Suppose $p \in M$ and $v \in T_pM$ is a unit vector. Then let us define $\gamma_v\colon (-\varepsilon, \varepsilon) \longrightarrow M$ to be the geodesic starting at p in the direction v. This means that $\gamma'(0) = v$, and so $\|\gamma'(0)\| = 1$. Since we showed that the norm of the derivative of a geodesic is constant, this means that $\|\gamma'\| = 1$ and so γ is a natural parameterization.
Let us define the function

$$\exp_p(v) = \begin{cases} 0 & v = 0 \\ \gamma_{\frac{v}{\|v\|}}(\|v\|) & v \neq 0 \end{cases}$$

$\exp_p$ is a function $V \longrightarrow M$ where $V = \{v \in T_pM \mid \|v\| < \varepsilon\}$ (since $\gamma_{\frac{v}{\|v\|}}$ is defined on the domain $(-\varepsilon, \varepsilon)$). Geometrically, $\exp_p(v)$ is the point you'd end up after walking on the geodesic starting from p in the direction of v for $\|v\|$ units. This is geometrically similar to what $\exp(it)$ does (it gives you the point you'd end up after walking t units from 1 on the unit circle).
Using $\exp_p$ we will reparameterize M.

(1) Choose an orthonormal basis $\{w_1, w_2\}$ for T_pM.

(2) Define the map $y(x) = w_1x_1 + w_2x_2$ (this will become the differential operator of the new parameterization).

(3) We define $\sigma\colon \mathcal{U} \longrightarrow M$ by

$$\sigma(u, v) = \exp_p(y(u, v)) = \exp_p(uw_1 + vw_2)$$

where

$$\mathcal{U} = \big\{ (u, v) \bigm| u^2 + v^2 < \varepsilon^2 \big\}$$

Proposition 3.2.8:

For some $\varepsilon > 0$, the restriction of σ to $B_\varepsilon(0)$ is a diffeomorphism (a smooth bijection whose inverse is also smooth).

Proof:

Taking the partial derivative of σ at $q = (0, 0)$ relative to u gives

$$\sigma_1(q) = \frac{d}{du}\exp_p(uw_1)\Big|_{u=0} = \frac{d}{du}\gamma_{w_1}(u)\Big|_{u=0} = \gamma'_{w_1}(0) = w_1$$

similarly

$$\sigma_2(q) = w_2$$

This means that $\sigma_1(0)$ and $\sigma_2(0)$ are linearly independent (since w_1 and w_2 forms an orthonormal basis). Since σ is smooth, so are σ_1 and σ_2, which means that in a neighborhood of 0, σ_1 and σ_2 are still linearly independent. ■

This means that σ is a reparameterization of M.
Now let us define $\tilde{\sigma}$ to be σ in polar coordinates:

$$\tilde{\sigma}(r, \theta) = \sigma(r\cos\theta, r\sin\theta) = \exp_p(r(w_1\cos\theta + w_2\sin\theta))$$

So $\tilde{\sigma} = \sigma \circ (r\cos\theta, r\sin\theta)$ and since σ and $(r, \theta) \mapsto (r\cos\theta, r\sin\theta)$ are both diffeomorphisms, so is $\tilde{\sigma}$. Thus $\tilde{\sigma}$ is another reparameterization of M.

Definition 3.2.9:

Let M be a surface, then the reparameterization σ defined above is called a **normal coordinate chart** of M. And the reparameterization $\tilde{\sigma}$ defined above is called a **normal polar coordinate chart** of M.

Of course there exist many normal coordinate charts, as we could choose different vectors for w_1 and w_2.

Lemma 3.2.10 (Gauss's Lemma):

The metric induced by a polar coordinate chart $\tilde{\sigma}$ is equal to

$$g_{\tilde{\sigma}} = \begin{pmatrix} 1 & 0 \\ 0 & G(r,\theta) \end{pmatrix}$$

where $G(r, \theta)$ is some function.

Proof:

So

$$\tilde{\sigma}_r = \frac{\partial}{\partial r}\exp_p\big(r(\cos\theta w_1 + \sin\theta w_2)\big)$$

let $w = \cos\theta w_1 + \sin\theta w_2$, this is a unit vector and

$$= \frac{d}{dr}\exp_p(rw) = \frac{d}{dr}\gamma_w(r) = \gamma'_w$$

So $\tilde{\sigma}_r$ is equal to the derivative of a geodesic in its natural parameterization (since w is a unit vector). Thus $\|\tilde{\sigma}_r\| = 1$ and so $\langle\tilde{\sigma}_r, \tilde{\sigma}_r\rangle = 1$.
And $\tilde{\sigma}_{rr} = \gamma''_w$, and so σ_{rr} is orthogonal to T_pM since γ is a geodesic. Therefore

$$\frac{\partial}{\partial r}\langle\tilde{\sigma}_r, \tilde{\sigma}_\theta\rangle = \langle\tilde{\sigma}_{rr}, \tilde{\sigma}_\theta\rangle + \langle\tilde{\sigma}_r, \tilde{\sigma}_{\theta r}\rangle$$

Since $\tilde{\sigma}_\theta$ is on T_pM, $\tilde{\sigma}_{rr}$ is orthogonal to it and so

$$= \langle\tilde{\sigma}_r, \tilde{\sigma}_{\theta r}\rangle$$

Now since $\langle\sigma_r, \sigma_r\rangle$ is constant,

$$0 = \frac{\partial}{\partial\theta}\langle\tilde{\sigma}_r, \sigma_r\rangle = 2\langle\tilde{\sigma}_{r\theta}, \tilde{\sigma}_r\rangle$$

and so $\langle \tilde{\sigma}_r, \tilde{\sigma}_{\theta r}\rangle = 0$. Thus

$$\frac{\partial}{\partial r}\langle \tilde{\sigma}_r, \tilde{\sigma}_\theta\rangle = 0$$

so for a constant θ_0, $g_{12}(r, \theta_0) = \langle \tilde{\sigma}_r, \tilde{\sigma}_\theta\rangle$ is equal to some constant c, in other words g_{12} is a function of θ.
Let us define

$$f(r, \theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

thus $\tilde{\sigma} = \sigma \circ f$. And

$$\frac{\partial \tilde{\sigma}}{\partial \theta} = \sigma_1 \cdot \frac{\partial f_1}{\partial \theta} + \sigma_2 \cdot \frac{\partial f_2}{\partial \theta} = -\sigma_1 r\sin\theta + \sigma_2 r\cos\theta$$

As we let $r \to 0$, this converges to zero. And so $g_{12} = \langle \tilde{\sigma}_\theta, \tilde{\sigma}_r\rangle$ and as we let $r \to 0$, we get that $g_{12} \to \langle 0, \sigma_r\rangle = 0$. But since g_{12} is constant in r, this means that $g_{12} = 0$.
And so

$$g_{11} = \langle \tilde{\sigma}_r, \tilde{\sigma}_r\rangle = 1, \quad g_{12} = g_{21} = 0$$

and so

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \langle \tilde{\sigma}_\theta, \tilde{\sigma}_\theta\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & G(r, \theta) \end{pmatrix}$$

as required. ■

We will now prove the following remarkable theorem (its name is literally Latin for "Remarkable Theorem").

Theorem 3.2.11 (Theorema Egregium):

The Gaussian curvature of a surface can be computed solely using the metric g and its derivatives.

Proof:

Now we will show later that there exists a reparameterization of the surface M, $\sigma(r, \theta) \colon \mathcal{U} \longrightarrow M$ such that its metric is

$$g = \begin{pmatrix} 1 & 0 \\ 0 & G(r, \theta) \end{pmatrix} \implies g^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{G(r,\theta)} \end{pmatrix}$$

This means that

$$\begin{pmatrix} \Gamma^1_{11} \\ \Gamma^2_{11} \end{pmatrix} = g^{-1}\begin{pmatrix} \frac{1}{2}g_{11,1} \\ g_{12,1} - \frac{1}{2}g_{11,2} \end{pmatrix} = g^{-1}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$$

thus $\Gamma^1_{11} = \Gamma^2_{11} = 0$. And

$$\begin{pmatrix} \Gamma^1_{12} \\ \Gamma^2_{12} \end{pmatrix} = g^{-1}\begin{pmatrix} 0 \\ \frac{1}{2}g_{22,1} \end{pmatrix} = g^{-1}\begin{pmatrix} 0 \\ \frac{1}{2}G_r \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2}\frac{G_r}{G} \end{pmatrix}$$

$$\begin{pmatrix} \Gamma^1_{22} \\ \Gamma^2_{22} \end{pmatrix} = g^{-1}\begin{pmatrix} -\frac{1}{2}g_{22,1} \\ \frac{1}{2}g_{22,2} \end{pmatrix} = g^{-1}\begin{pmatrix} -\frac{1}{2}G_r \\ \frac{1}{2}G_\theta \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}G_r \\ \frac{1}{2}\frac{G_\theta}{G} \end{pmatrix}$$

Thus

$$\Gamma^1_{11} = \Gamma^2_{11} = \Gamma^1_{12} = 0, \quad \Gamma^2_{12} = \frac{1}{2}\frac{G_r}{G}, \quad \Gamma^1_{22} = -\frac{1}{2}G_r, \quad \Gamma^2_{22} = \frac{1}{2}\frac{G_\theta}{G}$$

So we have

$$\sigma_{12} = \Gamma^1_{12}\sigma_1 + \Gamma^2_{12}\sigma_2 + b_{12}N = \Gamma^2_{12}\sigma_2 + b_{12}N$$

And so

$$\sigma_{122} = \Gamma^2_{122}\sigma_2 + \Gamma^2_{12}\sigma_{22} + b_{122}N + b_{12}N_2$$

The coefficients of σ_1 in σ_{122} are found in $\Gamma^2_{12}\sigma_{22}$ which gives $\Gamma^2_{12}\Gamma^1_{22}$, and from $b_{12}N_2$. Recall that $S(\sigma_2) = -\rho_2$ (recall that $\rho = N \circ \sigma$, which is really what we mean by N above), and recall that the representation of the shape operator by $\{\sigma_1, \sigma_2\}$ is

$$g^{-1}B = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$

meaning that the coefficient of σ_1 in $-\rho_2$ is S_{12}. Thus the coefficient of σ_1 in $b_{12}N_2$ is $-b_{12}S_{12}$. Thus the coefficient of σ_1 in σ_{122} is

$$\Gamma^2_{12}\Gamma^1_{22} - b_{12}S_{12}$$

Similarly,

$$\sigma_{22} = \Gamma^1_{22}\sigma_1 + \Gamma^2_{22}\sigma_2 + b_{22}N$$

so

$$\sigma_{221} = \Gamma^1_{22,1}\sigma_1 + \Gamma^1_{22}\sigma_{11} + \Gamma^2_{221}\sigma_2 + \Gamma^2_{22}\sigma_{21} + b_{221}N + b_{22}N_1$$

Recall that

$$\begin{aligned}
\sigma_{11} &= \Gamma^1_{11}\sigma_1 + \Gamma^2_{11}\sigma_2 + b_{11}N = b_{11}N \\
\sigma_{21} &= \Gamma^1_{21}\sigma_1 + \Gamma^2_{21}\sigma_2 + b_{21}N = \frac{1}{2}\frac{G_r}{G}\sigma_2 + b_{21}N \\
N_1 &= -S_{11}\sigma_1 - S_{21}\sigma_2
\end{aligned}$$

Thus the coefficient of σ_1 in σ_{221} is

$$\Gamma^1_{22,1} - b_{22}S_{11}$$

But since $\sigma_{122} = \sigma_{221}$ we get that

$$\Gamma^2_{12}\Gamma^1_{22} - b_{12}S_{12} = \Gamma^1_{22,1} - b_{22}S_{11} \implies \Gamma^1_{22,1} - \Gamma^2_{12}\Gamma^1_{22} = b_{22}S_{11} - b_{12}S_{12}$$

Now,

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = g^{-1}B = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{G} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ \frac{b_{21}}{G} & \frac{b_{22}}{G} \end{pmatrix}$$

Thus

$$S_{11} = b_{11}, \quad S_{12} = b_{12}, \quad S_{21} = \frac{b_{12}}{G}, \quad S_{22} = \frac{b_{22}}{G}$$

This means that

$$b_{22}S_{11} - b_{12}b_{12} = b_{22}b_{11} - b_{12}b_{12} = \det(B)$$

meaning that

$$\det(B) = \Gamma^1_{22,1} - \Gamma^2_{12}\Gamma^1_{22}$$

And so the Gaussian curvature of M is

$$K = \det(S) = \frac{\det(B)}{\det(g)} = \frac{1}{G}\left(\Gamma^1_{22,1} - \Gamma^2_{12}\Gamma^1_{22}\right)$$

Recall that Γ^k_{ij} is dependent only on g and its derivatives. Thus the Gaussian curvature of M is dependent only on g and its derivatives, as required. ■

3.3 Tangent Vector Fields

Definition 3.3.1:

Let M be a surface, then a **tangent vector field** on M is an assignment of each point on M to a vector in its tangent space, ie it is a function

$$V : M \longrightarrow \bigcup_{p\in M} T_pM$$

which satisfies the condition that for every $p \in M$, $V(p) \in T_pM$.

Usually it is unnecessary for us to require a vector field on all of M, rather generally we need a tangent vector field to a curve on M.

Definition 3.3.2:

Let M be a surface and $\gamma\colon [a,b] \longrightarrow M$ be a curve on M, then a **tangent vector field** on γ is a function

$$V\colon [a,b] \longrightarrow \bigcup_{p\in M} T_pM$$

where for every $t \in [a,b]$, $V(t) \in T_{\gamma(t)}M$.

Definition 3.3.3:

Suppose V is a tangent vector field on the curve γ, then we define its **covariant derivative** to be

$$\big(\nabla_\gamma V\big)(t) = V'(t) - \langle V'(t), N(\gamma(t))\rangle N(\gamma(t))$$

Where N is the unit normal vector to M.

The covariant derivative is the projection of V' onto $N^\perp = T_\gamma M$ the tangent space of M. Thus $\nabla_\gamma V \in N^\perp = T_\gamma M$.

Proposition 3.3.4:

The covariant derivative satisfies the following properties:

(1) $\nabla_\gamma(V_1 + V_2) = \nabla_\gamma V_1 + \nabla_\gamma V_2$

(2) $\nabla_\gamma(f(t)\cdot V) = f'\cdot V + f\cdot \nabla_\gamma V$, where f is a real-valued function.

(3) $\langle V_1', V_2\rangle = \langle \nabla_\gamma V_1, V_2\rangle$

(4) $\frac{d}{dt}\langle V_1, V_2\rangle = \langle \nabla_\gamma V_1, V_2\rangle + \langle V_1, \nabla_\gamma V_2\rangle$

Proof:

The first two properties can be easily verified by applying the definition of the covariant derivative. We can also directly verify the third property:

$$\langle \nabla_\gamma V_1, V_2\rangle = \langle V_1' - \langle V_1', N\rangle N,\, V_2\rangle$$

now, since V_2 is on $T_\gamma M$ and N is orthogonal to this tangent space, we get

$$= \langle V_1', V_2\rangle$$

as required. And this implies the fourth property, as $\langle V_1, V_2\rangle' = \langle V_1', V_2\rangle + \langle V_1, V_2'\rangle$. ■

Definition 3.3.5:

We say that a tangent vector field V is a **parallel vector field** if at every point, V' is orthogonal to $T_\gamma M$.

Notice that if V is parallel then V' and V are orthogonal, which means $\|V\|$ is constant. (This is since the derivative of $\langle V, V\rangle$ is zero, we discussed this before.)

Proposition 3.3.6:

The norm of a parallel vector field is constant.

This means that V is a parallel vector field if and only if its covariant derivative is always zero. This is since the covariant derivative is the component of V' on $T_\gamma M$, and V' is orthogonal to $T_\gamma M$ if and only if its component in $T_\gamma M$ is zero. Formally,

$$V' = \nabla_\gamma V + \langle V', N\rangle N$$

And since N is orthogonal to $T_\gamma M$, V' is orthogonal to $T_\gamma M$ if and only if $\nabla_\gamma V = 0$. Let us summarize this in the following proposition,

Proposition 3.3.7:

A tangent vector field is parallel if and only if its covariant derivative is always zero.

Now, how does one go about computing the covariant derivative? Since $V \in T_\gamma M$,

$$V = x^1\sigma_1 + x^2\sigma_2 = x^i\sigma_i$$

now recall that the derivative of σ is being taken at $\gamma(t)$ on the surface, which is $\beta(t) = \sigma^{-1} \circ \gamma(t)$ in its origin. So we mean the derivative of σ evaluated at $\beta(t)$, ie $\sigma_i(\beta(t))$. Thus

$$\frac{d}{dt}\sigma_i = \frac{d}{dt}\big(\sigma_i(\beta(t))\big) = \sigma_{i1}(\beta(t))\dot{\beta}^1 + \sigma_{i2}(\beta(t))\dot{\beta}^2$$

And so

$$V' = \dot{x}^i\sigma_i + x^i(\sigma_{i1}\dot{\beta}^1 + \sigma_{i2}\dot{\beta}^2) = \dot{x}^i\sigma_i + x^i\dot{\beta}^j\sigma_{ij}$$

Now, since σ_i is orthogonal to N, we get that

$$\langle V', N\rangle = x^i\dot{\beta}^j\langle\sigma_{ij}, N\rangle = x^i\dot{\beta}^j b_{ij}$$

So we get that

$$\nabla_\gamma V = \dot{x}^i\sigma_i + x^i\dot{\beta}^j\sigma_{ij} - x^i\dot{\beta}^j b_{ij}N$$

Now let us recall that

$$\sigma_{ij} = \Gamma^k_{ij}\sigma_k + b_{ij}N$$

And so

$$\nabla_\gamma V = \dot{x}^i\sigma_i + x^i\dot{\beta}^j\Gamma^k_{ij}\sigma_k + \big(x^i\dot{\beta}^j b_{ij} - x^i\dot{\beta}^j\big)N = \dot{x}^i\sigma_i + x^i\dot{\beta}^j\Gamma^k_{ij}\sigma_k$$

Now, we can determine x^i uniquely by the metric g, so this can be uniquely determined by g. We have proven

Proposition 3.3.8:

The covariant derivative is equal to

$$\nabla_\gamma V = \dot{x}^i\sigma_i + x^i\dot{\beta}^j\Gamma^k_{ij}\sigma_k$$

Note:

I have decided to use Einstein summation notation as well as physics notation for derivatives from here onward, since that is what is used in the course and it's important to use and understand it for the sake of the course.

Example 3.3.9:

Recall that

$$\Gamma^k_{ij} = \frac{1}{2}g^{km}(g_{mi,j} + g_{jm,i} - g_{ij,m})$$

and so if $g = I$ (eg. on a plane), then g's derivatives are zero and so $\Gamma^k_{ij} = 0$ and thus

$$\nabla_\gamma V = \dot{x}^i\sigma_i = \dot{x}^1\sigma_1 + \dot{x}^2\sigma_2$$

Now, we know that V is parallel if and only if $\nabla_\gamma V = 0$ which is if and only if

$$\dot{x}^k\sigma_k + x^i\dot{\beta}^j\Gamma^k_{ij}\sigma_k = (\dot{x}^k + x^i\dot{\beta}^j\Gamma^k_{ij})\sigma_k = 0$$

This means that we get the system of first order ODEs (the variables are $\dot{x}^k$),

$$\dot{x}^1 + x^i\dot{\beta}^j\Gamma^1_{ij} = 0$$
$$\dot{x}^2 + x^i\dot{\beta}^j\Gamma^2_{ij} = 0$$

So given an initial condition on x (ie. initial conditions on x^1 and x^2), then there exists a unique solution. So given some initial condition $V(t_0) \in T_{\gamma(t_0)}M$, then there exists a unique parallel vector field which satisfies this.

Proposition 3.3.10:

Let M be a surface and γ be a curve on M. Then for an initial condition $V_0 \in T_{\gamma(t_0)}M$, there exists a unique parallel vector field V on γ which satisfies $V(t_0) = V_0$.

Now we also claim that this parallel vector field is independent of the parameterization of γ.

Definition 3.3.11:

If V is a tangent vector field to the curve γ, and $\gamma \circ \varphi$ is a reparameterization, then we define the reparameterization of V to simply be $V(\varphi(t))$. And we define $\nabla_{\gamma\circ\varphi}V$ to be the covariant derivative of the reparameterization of V (otherwise this would not be well-defined).

Notice that if $V \circ \varphi$ is a reparameterization of V, then

$$\begin{aligned}\nabla_{\gamma\circ\varphi}V = \nabla_{\gamma\circ\varphi}(V\circ\varphi) = (V\circ\varphi)' - \langle (V\circ\varphi)', N(\gamma\circ\varphi)\rangle N(\gamma\circ\varphi) &= \varphi' V'(\varphi) - \langle \varphi' V'(\varphi), N(\gamma\circ\varphi)\rangle N(\gamma\circ\varphi) \\ &= \varphi'(t)\cdot \big(\nabla_\gamma V\big)(\varphi(t))\end{aligned}$$

So we have that

$$\nabla_{\gamma\circ\varphi}V = \varphi' \cdot \nabla_\gamma V(\varphi(t))$$

Since $\gamma\circ\varphi$ is a reparameterization, $\varphi' > 0$ and is bijective, so $\nabla_{\gamma\circ\varphi}V = 0$ if and only if $\nabla_\gamma V = 0$. So we have shown that being parallel is independent of the parameterization of the curve.

Proposition 3.3.12:

Let γ_1 and γ_2 be reparameterizations of the same curve, and V be a tangent vector field for this curve, then V is parallel for γ_1 if and only if it is parallel for γ_2.

Example 3.3.13:

Suppose γ is the natural parameterization of a curve on M, then we know $\gamma' \in T_\gamma M$, so γ' is a tangent vector field to γ so we can ask what $\nabla_\gamma \gamma'$ is equal to. We know that

$$\gamma'' = \kappa_n N + \kappa_g R_{\frac{\pi}{2}}\gamma'$$

And so

$$\nabla_\gamma\gamma' = \kappa_n N + \kappa_g R_{\frac{\pi}{2}}\gamma' - \big\langle \kappa_n N + \kappa_g R_{\frac{\pi}{2}}\gamma', N\big\rangle N = \kappa_g R_{\frac{\pi}{2}}\gamma'$$

Let us denote

$$V = R_{\frac{\pi}{2}}\gamma'$$

And so

$$\nabla_\gamma\gamma' = \kappa_g V$$

Now, V is also a tangent vector field to γ (since $R_{\frac{\pi}{2}}$ operates on $T_\gamma M$). Furthermore, we know that (V, γ') is a basis for $T_\gamma M$ and so $\langle V, \gamma'\rangle = 0$ and furthermore

$$0 = \frac{d}{dt}\langle V, \gamma'\rangle = \langle \nabla_\gamma V, \gamma'\rangle + \langle V, \nabla_\gamma\gamma'\rangle$$

Thus

$$\langle \nabla_\gamma V, \gamma'\rangle = -\langle V, \nabla_\gamma\gamma'\rangle$$

Since γ is a natural parameterization, $\|\gamma'\| = 1$ and so $\|V\| = 1$ as well, thus

$$= -\langle V, \kappa_g V\rangle = -\kappa_g$$

Now, since $\|V\|$ is constant, V is orthogonal with its derivative and so

$$\langle \nabla_\gamma V, V\rangle = \langle V', V\rangle = 0$$

Thus we get the system

$$\nabla_\gamma \gamma' = \kappa_g V$$
$$\nabla_\gamma V = -\kappa_g \gamma'$$

This is similar to the Frenet-Serret frame for a plane, and when M is a plane these are the same equations.

Let γ be some curve on M, and let us choose some initial unit vector $W(t_0) \in T_{\gamma(t_0)}M$. We can then extend this to a unique parallel vector field on γ, let us denote it $W(t)$. Since parallel vector fields have a constant norm, $\|W(t)\| = \|W(t_0)\| = 1$. Furthermore, since $W(t) \in T_{\gamma(t)}M$ which has a basis of $\{\gamma'(t), V(t)\}$ there exists a $\theta(t)$ such that

$$W(t) = \cos(\theta(t))\gamma'(t) + \sin(\theta(t))V(t)$$

We get that

$$\nabla_\gamma W = -\theta' \sin(\theta)\gamma' + \cos(\theta)\nabla_\gamma \gamma' + \theta' \cos(\theta)V + \sin(\theta)\nabla_\gamma V$$

Applying the equations from the above example we get

$$= \theta' \cdot \big(\cos(\theta)V - \sin(\theta)\gamma'\big) + \kappa_g \cdot \big(\cos(\theta)V - \sin(\theta)\gamma'\big) = (\theta' + \kappa_g)\big(\cos(\theta)V - \sin(\theta)\gamma'\big)$$

Now, we know that $W(t)$ is parallel, and so $\nabla_\gamma W = 0$. Since V and γ' are linearly independent, this is if and only if

$$\theta'(t) = -\kappa_g(t)$$

So if we define

$$\theta(t) = \theta_0 - \int_{t_0}^t \kappa_g(s)\, ds$$

Then we get the desired function,

$$W(t) = \cos(\theta(t))\gamma'(t) + \sin(\theta(t))V(t)$$

If $W(t_0)$ isn't a unit vector, let $R = \|W(t_0)\|$ then we instead get that $\frac{W(t)}{R}$ is still a parallel vector field, whose norm is one. Thus we get the result above for $\frac{W(t)}{R}$, and so

$$W(t) = R\cos(\theta(t))\gamma'(t) + R\sin(\theta(t))V(t), \qquad \theta(t) = \theta_0 - \int_{t_0}^t \kappa_g(s)\, ds$$

3.4 The Gauss-Bonnet Theorem

Definition 3.4.1:

Suppose M is a surface and $\gamma\colon [a,b] \longrightarrow M$ is a curve on M. We define the **parallel movement operator** to be the function

$$P_\gamma\colon T_{\gamma(a)}M \longrightarrow T_{\gamma(b)}M$$

where for every $v_0 \in T_{\gamma(a)}M$, let us denote W_{v_0} to be the parallel vector field on γ where $W_{v_0}(a) = v_0$ then we define

$$P_\gamma(v_0) = W_{v_0}(b)$$

In other words,

$$P_\gamma(W_v(a)) = W_v(b)$$

Now, notice that $W_{\alpha v+\beta u} = \alpha W_v + \beta W_u$, this is since $\alpha W_v + \beta W_u$ is a parallel vector field:

$$\nabla_\gamma(\alpha W_v + \beta W_u) = \alpha\nabla_\gamma W_v + \beta\nabla_\gamma W_u = 0$$

And also

$$\big(\alpha W_v + \beta W_u\big)(a) = \alpha v + \beta u$$

So $\alpha W_v + \beta W_u$ is the parallel vector field which starts at $\alpha v + \beta u$, ie. it is $W_{\alpha v+\beta u}$.

Proposition 3.4.2:

P_γ is an orthonormal linear transformation.

Proof:

Firstly, suppose we will show that P_γ is a linear transformation. So let v and u be tangent vectors, and α and β be scalars, then

$$P_\gamma(\alpha v + \beta u) = W_{\alpha v+\beta u}(b) = \alpha W_v(b) + \beta W_u(b) = \alpha P_\gamma(v) + \beta P_\gamma(u)$$

as required. Now, since W_v is a parallel vector field, $\|W_v(t)\| = \|W_v(a)\| = \|v\|$ for every $a \leq t \leq b$, and in particular b. Therefore

$$\|P_\gamma(v)\| = \|W_v(b)\| = \|W_v(a)\| = \|v\|$$

So P_γ preserves the norm, and is therefore orthonormal. ∎

Let us denote W_v^γ as the parallel vector field which begins at v on γ. Now, further notice that if $\gamma\circ\varphi$ is a reparameterization where $\varphi\colon [c,d] \to [a,b]$, then

$$W_v^{\gamma\circ\varphi}(t) = W_v^\gamma(\varphi(t))$$

Since $W_v^\gamma \circ \varphi$ is a reparameterization of a parallel vector field, it too remains parallel. And

$$W_v^\gamma(\varphi(c)) = W_v^\gamma(a) = v$$

So by the uniqueness of $W_v^{\gamma\circ\varphi}$, it is equal to $W_v^\gamma \circ \varphi$. Notice then that

$$P_{\gamma\circ\varphi}(v) = W_v^{\gamma\circ\varphi}(d) = W_v^\gamma \circ \varphi(d) = W_v^\gamma(b) = P_\gamma(v)$$

Thus $P_{\gamma\circ\varphi} = P_\gamma$.
But if $\gamma\circ\varphi$ is an *anti*-reparameterization, ie $\varphi' < 0$ then $\varphi(d) = a$ and $\varphi(c) = b$, so if $P_\gamma(v) = W_v^\gamma(b) = u$ then

$$W_u^{\gamma\circ\varphi} = W_v^\gamma \circ \varphi$$

This is as $W_v^\gamma \circ\varphi$ still remains a parallel vector field (our proof for reparameterizations works for anti-reparameterizations), and

$$W_u^\gamma \circ \varphi(c) = W_v^\gamma(b) = u$$

So again by the uniqueness of $W_u^{\gamma\circ\varphi}$, it is equal to $W_v^\gamma \circ \varphi$. And then

$$P_{\gamma\circ\varphi}(u) = W_u^{\gamma\circ\varphi}(d) = W_v^\gamma \circ \varphi(d) = W_v^\gamma(a) = v$$

Thus

$$P_{\gamma\circ\varphi}(P_\gamma(v)) = P_{\gamma\circ\varphi}(u) = v$$

and by symmetry (since γ is an anti-reparameterization of $\gamma\circ\varphi$), we get that

$$P_\gamma(P_{\gamma\circ\varphi}(u)) = u$$

So they are inverses, thus we have proven

Proposition 3.4.3:

Let P_γ be the parallel movement operator of γ. Then if $\gamma\circ\varphi$ is a reparameterization of γ,

$$P_{\gamma\circ\varphi} = P_\gamma$$

and if $\gamma\circ\varphi$ is an anti-reparameterization of γ,

$$P_{\gamma\circ\varphi} = P_\gamma^{-1}$$

This should make sense as a reparameterization of the curve should not alter where moving v goes to, but if we reverse the direction of the curve we move the vector to the opposite end.
Now, recall that we showed if W is a parallel vector field on a curve γ, then the angle between W and γ' is given by

$$\theta(t) = \theta_0 - \int_{t_0}^t \kappa_g(s)\,ds$$

Let us denote

$$\Delta\theta = \theta(T) - \theta(t_0) = -\int_{t_0}^{T} \kappa_g(s)\,ds$$

(Where γ's domain is $[t_0, T]$.) But if we instead view W as rotating around γ', then we get that the angle between W and γ' is equal to

$$2\pi - \theta(t) = (2\pi - \theta_0) + \int_{t_0}^{t} \kappa_g(s)\,ds$$

This is why sometimes we actually define $\theta(t)$ as above,

$$\theta(t) = \theta_0 + \int_{t_0}^{t} \kappa_g(s)$$

This is the angle between γ' and W (instead of between W and γ').

Definition 3.4.4:

If γ is a closed curve $\gamma\colon [a,b] \longrightarrow M$ then P_γ forms a linear operator

$$P_\gamma\colon T_pM \longrightarrow T_pM$$

where $p = \gamma(a) = \gamma(b)$. In this case, P_γ is called the **holonomy operator**.

So if P_γ is a holonomy operator, then $P_\gamma(v)$ will equal a rotation of v on T_pM (since T_pM is two-dimensional). We know that W_v rotates around γ' with at angle of $\theta(t)$, and the total rotation is $\Delta\theta = \theta(T) - \theta(t_0)$, so $P_\gamma(v)$ is equal to the rotation of v with an angle of $\Delta\theta$ (or $2\pi - \Delta\theta$ if we utilize the angle between γ' and W).

Example 3.4.5:

Suppose we'd like to focus on the parallel movement operator on a great circle on a sphere. This is a geodesic connecting two poles on the sphere, let them be N and S. Let this geodesic be γ, then by definition γ'' is orthogonal to $T_\gamma M$ at every point, which makes γ' a parallel movement operator on γ. Let us divide γ into the first geodesic, γ_1, from N to S and then γ_2 which is from S to N. So

$$\begin{aligned} \gamma\colon [0,2T] &\longrightarrow M \\ \gamma_1\colon [0,T] &\longrightarrow M \\ \gamma_2\colon [T,2T] &\longrightarrow M \end{aligned}$$

To move $\gamma_1'(0)$ over γ in parallel, first we move it parallel over γ_1, and since γ_1' is parallel to γ_1 we get that this goes to $\gamma_1'(T)$. Since γ_1 and γ_2 are both parts of the smooth geodesic γ, we have that $\gamma_2'(T) = \gamma_1'(T)$. And so we now move $\gamma_1'(T)$ over γ_2 in parallel, since γ_2' is parallel to γ_2 and starts at $\gamma_1'(T)$, this is equal to $\gamma_2'(2T)$.