Algebraic Topology I

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1 A Brief Review of Select Topological Concepts

First, let us review a few concepts from point-set topology. Namely, we will review product and quotient topologies, since they are the most complicated constructions we had, the rest should not be too hard to recall.

Recall that if B is a set of subsets of X, then we define τ_B to be the set of all unions of sets from B:

$$\tau_B = \left\{ \bigcup_{A\in L} A \;\middle|\; L \subseteq B \right\}$$

then τ_B is a topology if and only if $X \in \tau_B$ and for every $\mathcal{U}, \mathcal{V} \in B$, $\mathcal{U} \cap \mathcal{V} \in \tau_B$. If τ_B is a topology, these hold because X and the intersection of open sets are open. Conversely, $\varnothing \in \tau_B$ vacuously, if $\{\mathcal{U}_\alpha\}_{\alpha\in I} \in \tau_B$ then suppose $\mathcal{U}_\alpha = \bigcup_{i\in J_\alpha} \mathcal{V}_i^\alpha$ then $\bigcup \mathcal{U}_\alpha = \bigcup_I \bigcup_{J_\alpha} \mathcal{V}_i^\alpha \in \tau_B$ so the union of open sets is open. And similarly if $\mathcal{U} = \bigcup_I \mathcal{U}_\alpha$ and $\mathcal{V} = \bigcup_I \mathcal{V}_\beta$ then $\mathcal{U} \cap \mathcal{V} = \bigcup_{I,J} \mathcal{U}_\alpha \cap \mathcal{V}_\beta \in \tau_B$, so the intersection of open sets is open.

1.1 Definition

Now if $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in I}$ is a collection of topological spaces then we define $X = \prod_{\alpha\in I} X_\alpha$

$$B = \left\{ \prod_{\alpha\in I} \mathcal{U}_\alpha \;\middle|\; \mathcal{U}_\alpha \in \tau_\alpha \text{ and all but a finite number of } \alpha \text{ satisfy } \mathcal{U}_\alpha = X_\alpha \right\}$$

then we define the **product topology** on X by $\tau = \tau_B$.

Firstly $X \in B$, and if $\mathcal{U} = \prod_{\alpha\in I} \mathcal{U}_\alpha \in B$ and $\mathcal{V} = \prod_{\alpha\in I} \mathcal{V}_\alpha \in B$ then $\mathcal{U} \cap \mathcal{V} = \prod_{\alpha\in I} (\mathcal{U}_\alpha \cap \mathcal{V}_\alpha)$ and all but a finite number of these are X_α, so $\mathcal{U} \cap \mathcal{V} \in B$. So τ_B is indeed a topology.

Let $X = \prod_{\alpha\in I} X_\alpha$ be equipped with the product topology. We now state a few facts of product topologies without proving them (proofs can be found in my topology summary):

(1) Let $\pi_\beta\colon X \longrightarrow X_\beta$ be the projection map $(x_\alpha)_{\alpha\in I} \mapsto x_\beta$. Then π_β is an open and continuous map.

(2) Let Y be an arbitrary topological space. Then $f\colon Y \longrightarrow X$ is continuous if and only if $\pi_\alpha \circ f$ is for every $\alpha \in I$.

(3) If X_α has a basis B_α then $B = \left\{ \prod_{\alpha\in I} \mathcal{U}_\alpha \;\middle|\; \mathcal{U}_\alpha \in B_\alpha \text{ and all but a finite number are } X_\alpha \right\}$ is a basis for X.

(4) If $Y = \prod_{\alpha\in I} Y_\alpha$ and $f_\alpha\colon X_\alpha \longrightarrow Y_\alpha$ are functions, define $f\colon X \longrightarrow Y$ by $(x_\alpha)_{\alpha\in I} \mapsto (f(x_\alpha))_{\alpha\in I}$. Then

  (1) f is continuous if and only if every f_α is,

  (2) if f is open then each f_α is, and this becomes if and only if if I is finite or f_α are all surjective,

  (3) f is a homeomorphism if and only if each f_α is.

(5) If $f_\alpha\colon Y \longrightarrow X_\alpha$ is continuous, so is $f\colon Y \longrightarrow X$ defined by $x \mapsto (f_\alpha(x))_{\alpha\in I}$.

(6) If every X_α is connected (respectively, path connected), so is X.

(7) X is compact if and only if every X_α is compact.

Now let us discuss quotient spaces, and in more depth than products.

1.2 Definition

If (X, τ) is a topological space and $\rho\colon X \longrightarrow Y$ is a surjection, then σ is a **quotient topology** of Y if

(1) $\rho\colon (X, \tau) \longrightarrow (Y, \sigma)$ is continuous,

(2) if γ is another topology on Y such that ρ is continuous with respect to it, then $\gamma \subseteq \sigma$.

In such a case, $\sigma = \left\{ \mathcal{U} \subseteq Y \;\middle|\; \rho^{-1}(\mathcal{U}) \in \tau \right\}$. To prove this, set $\sigma' = \left\{ \mathcal{U} \subseteq Y \;\middle|\; \rho^{-1}(\mathcal{U}) \in \tau \right\}$, our goal is to of course show that $\sigma = \sigma'$. First notice that σ' is a topology: $\rho^{-1}(Y) = X, \rho^{-1}(\varnothing) = \varnothing$ so both Y and the empty

set are in σ'. If $\rho^{-1}(\mathcal{U}), \rho^{-1}(\mathcal{V}) \in \tau$ then $\rho^{-1}(\mathcal{U}) \cap \rho^{-1}(\mathcal{V}) = \rho^{-1}(\mathcal{U} \cap \mathcal{V}) \in \tau$. σ' is also closed under unions since $\bigcup_{i\in I} \rho^{-1}(\mathcal{U}_i) = \rho^{-1}\big(\bigcup_{i\in I} \mathcal{U}_i\big)$.
σ' makes ρ continuous by definition so $\sigma' \subseteq \sigma$. And if $\mathcal{U} \in \sigma$ then $\rho^{-1}(\mathcal{U}) \in \tau$ since ρ is continuous with respect to σ, so $\mathcal{U} \in \sigma'$. Thus $\sigma = \sigma'$ as required.

And so we have shown that ρ is a quotient map if and only if it is a surjection and $\mathcal{U} \subseteq Y$ is open if and only if $\rho^{-1}(\mathcal{U})$ is.

1.3 Proposition

Suppose $\rho\colon X \longrightarrow Y$ is a quotient map, and $q\colon Y \longrightarrow Z$ is continuous. Then q is a quotient map if and only if $q \circ \rho$ is.

Proof: if q is a quotient map then $q \circ \rho$ is a surjection. And $\mathcal{U} \subseteq Z$ is open if and only if $q^{-1}(\mathcal{U})$ is, if and only if $\rho^{-1} \circ q^{-1}(\mathcal{U}) = (q \circ \rho)^{-1}(\mathcal{U})$ is. So $q \circ \rho$ is a quotient map. Conversely, if $q \circ \rho$ is a quotient map then it is a surjection and therefore so is q. Now if $\mathcal{U} \subseteq Y$ is open then so is $q^{-1}(\mathcal{U})$ since it is continuous. And if $q^{-1}(\mathcal{U})$ is open, then so is $\rho^{-1} \circ q^{-1}(\mathcal{U}) = (\rho \circ q)^{-1}(\mathcal{U})$ and therefore so is $\mathcal{U}$. ■

1.4 Proposition

If $\rho\colon X \longrightarrow Y$ is a quotient map, and $f\colon Y \longrightarrow Z$ is a function, then f is continuous if and only if $f \circ \rho$ is.

Proof: if f is continuous, then so is the composition of continuous functions $f \circ \rho$. If $f \circ \rho$ is continuous, then let $\mathcal{U} \subseteq Z$ open, then $(f \circ \rho)^{-1}(\mathcal{U}) = \rho^{-1} \circ f^{-1}(\mathcal{U})$ is open, and therefore so is $f^{-1}(\mathcal{U})$ since ρ is a quotient map. ■

This is an important result, as it allows us to prove that a function from a quotient space Y is continuous by proving that a function from a simpler space X is.

1.5 Proposition

If $\rho\colon X \longrightarrow Y$ is surjective, continuous, and open (or closed), then it is a quotient map.

Proof: if $\mathcal{U} \subseteq Y$ is open then $\rho^{-1}(\mathcal{U})$ is open since ρ is continuous. If $\rho^{-1}(\mathcal{U})$ is open then $\rho \circ \rho^{-1}(\mathcal{U}) = \mathcal{U}$ is open since ρ is. If ρ is closed, then $\rho^{-1}(\mathcal{U})$ is open if $\rho^{-1}(\mathcal{U})^c = \rho^{-1}(\mathcal{U}^c)$ is closed, which is implies $\rho \circ \rho^{-1}(\mathcal{U}^c) = \mathcal{U}^c$ is closed, and therefore $\mathcal{U}$ is open. ■

This shows us that the projection maps π_α are quotient maps.

1.6 Proposition

If $\rho\colon X \longrightarrow Y$ is a continuous bijection, it is a quotient map if and only if it is a homeomorphism.

Proof: if ρ is a homeomorphism then by above it is a quotient map. If ρ is a quotient map then $\rho(\mathcal{U})$ is open if and only if $\rho^{-1} \circ \rho(\mathcal{U}) = \mathcal{U}$ is, so ρ is an open map and thus a homeomorphism. ■

1.7 Proposition

If $\rho\colon X \longrightarrow Y$ is continuous and there exists an $A \subseteq X$ such that $\rho\big|_A\colon A \longrightarrow Y$ is a quotient map, then ρ itself is a quotient map.

Proof: ρ is surjective since its restriction is. Now suppose $\mathcal{U} \subseteq Y$ such that $\rho^{-1}(\mathcal{U})$ is open. Then $\rho\big|_A^{-1}(\mathcal{U}) = \rho^{-1}(\mathcal{U}) \cap A$ is open with respect to A, and so $\mathcal{U}$ is open since $\rho\big|_A$ is a quotient map. ■

1.8 Definition

Let X be a topological space and $\sim$ an equivalence relation on X. Then we define $\rho\colon X \longrightarrow {}^{X}\!/\!_{\sim}$ by $\rho(x) = [x]_\sim$, then this is a quotient map with respect to the topology $\left\{\mathcal{U} \subseteq {}^{X}\!/\!_{\sim} \;\middle|\; \rho^{-1}(\mathcal{U}) \text{ is open in } X\right\}$. ${}^{X}\!/\!_{\sim}$ equipped with this topology is called the **quotient topology** of X with respect to $\sim$.

Since ρ is a quotient map, everything we proved above about quotient maps holds true for it.

Let us say that $f\colon X \longrightarrow Y$ *preserves* $\sim$ if $a \sim b$ implies $f(a) = f(b)$. And f *strongly preserves* $\sim$ if $a \sim b$ is equivalent to $f(a) = f(b)$. Then we have

1.9 Theorem

Let $f\colon X \longrightarrow Y$ be continuous. Then there exists a continuous $F\colon {}^{X}\!/\!_{\sim} \longrightarrow Y$ such that $f = F \circ \rho$ if and only if f preserves $\sim$. F is injective if and only if f strongly preserves $\sim$.

Proof: if $f = F \circ \rho$ then $a \sim b$ implies $\rho(a) = \rho(b)$ and so $f(a) = F\rho(a) = F\rho(b) = f(b)$, so f preserves $\sim$. And if f preserves $\sim$, then define $F[a] = f(a)$. This is well-defined since if $[a] = [b]$ then $a \sim b$ so $f(a) = f(b)$. And we showed that since ρ is a quotient map, F is continuous if and only if $F \circ \rho$ is already.

If F is injective, suppose $f(a) = f(b)$ then this means $F[a] = F[b]$, so $[a] = [b]$, meaning $a \sim b$, by injectivity. So f strongly preserves $\sim$. And if f strongly preserves $\sim$, then $F[a] = F[b]$ implies $f(a) = f(b)$ so $a \sim b$, and so $[a] = [b]$. Meaning F is injective. ∎

This theorem gives us the following commutative diagram:

$$\begin{tikzcd} X \arrow[d, "\rho"'] \arrow[dr, "f"] & \\ {}^{X}\!/\!_{\sim} \arrow[r, "F"'] & Y \end{tikzcd}$$

1.10 Proposition

If $f\colon X \longrightarrow Y$ is a quotient map that strongly preserves $\sim$ if and only if F is a homeomorphism.

Proof: if f is a quotient map and strongly preserves $\sim$, then F is injective. And since $f = F \circ \rho$ and ρ and f are quotient maps, so is F. So F is an injective quotient map, which is a homeomorphism.

And if F is a homeomorphism, then since $f = F \circ \rho$, f is a quotient map as well since homeomorphisms are quotient maps. Since F is injective, it trivially strongly preserves $\sim$ as well. ∎

2 Categories

2.1 Definition

A **category** $\mathcal{C}$ is a mathematical object which contains the following

(1) a class of objects $\mathrm{ob}(\mathcal{C})$ (the objects need not be sets),

(2) for every two objects $A, B \in \mathrm{ob}(\mathcal{C})$ a class of **morphisms** $\mathrm{Mor}(A, B)$,

(3) an operation on morphisms $\circ$, where for every $f \in \mathrm{Mor}(A, B)$ and $g \in \mathrm{Mor}(B, C)$, $g \circ f \in \mathrm{Mor}(A, C)$,

(4) for every object $A \in \mathrm{ob}(\mathcal{C})$ there exists an identity morphism $1_A \in \mathrm{Mor}(A, A)$ where for every $A, B \in \mathrm{ob}(\mathcal{C})$ and $f \in \mathrm{Mor}(A, B)$, $f \circ 1_A = 1_B \circ f = f$,

(5) for every $A, B, C, D \in \mathrm{ob}(\mathcal{C})$ and $f \in \mathrm{Mor}(A, B), g \in \mathrm{Mor}(B, C), h \in \mathrm{Mor}(C, D)$, there is associativity: $(h \circ g) \circ f = h \circ (g \circ f)$.

Although morphisms are not necessarily functions, we use similar notation: both $f\colon A \longrightarrow B$ and $A \xrightarrow{f} B$ are to be understood to mean $f \in \mathrm{Mor}(A, B)$. And we write $A \in \mathcal{C}$ to mean $A \in \mathrm{ob}(\mathcal{C})$.

Notice that for every $A \in \mathcal{C}$, 1_A is unique: suppose 1_A and $1'_A$ are both identity morphisms then $1_A \circ 1'_A = 1_A$ since $1'_A$ is an identity, but $1_A \circ 1'_A = 1'_A$ since 1_A is an identity so $1_A = 1'_A$.

2.2 Definition

Suppose $\mathcal{C}$ and $\mathcal{D}$ are two categories, a **functor** F from $\mathcal{C}$ to $\mathcal{D}$ is a correspondence where for every $A \in \mathcal{C}$ there is defined a single $F(A) \in \mathcal{D}$, and for every $f \in \mathrm{Mor}(A, B)$ there exists a unique $F(f) \in \mathrm{Mor}(F(A), F(B))$ such that for all $A, B, C \in \mathcal{C}$ and $f \in \mathrm{Mor}(A, B)$ and $g \in \mathrm{Mor}(B, C)$ we have that $F(g \circ f) = F(g) \circ F(f)$ and $F(1_A) = 1_{F(A)}$.

2.3 Example

The following are examples of categories:

(1) The category of all groups, morphisms are taken to be homomorphisms between groups;

(2) The category of all topological spaces, morphisms are taken to be homeomorphisms;

(3) The category of all sets, the morphisms are taken to be set functions;

(4) The category of pairs of topological spaces: the objects are of the form (X, A) where X is a topological space and $A \subseteq X$. Morphisms between (X, A) and (Y, B) of this category are continuous functions f between X and Y such that $f(A) \subseteq B$.

(5) The category of pointed topological spaces: the objects are (X, a) where X is a topological space and $a \in X$ and the morphisms between (X, a) and (Y, b) are continuous functions between X and Y such that $a \mapsto b$.

An example of a functor is the so-called *forgetful functor* from the category of topological spaces to the category of sets: map a topological to itself as a pure set.

This course will focus on a specific functor between the category of pointed topological spaces to the category of groups.

2.4 Definition

Let $\mathcal{C}$ be a category, and $A, B \in \mathcal{C}$. A morphism $f\colon A \longrightarrow B$ is an **isomorphism** if there exists a morphism $g\colon B \longrightarrow A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$. Such a g is called the **inverse** of f and is denoted f^{-1}

(notice that by symmetry the inverse is also an isomorphism). If there exists an isomorphism between A and B, we denote this by $A \cong B$ and A and B are called **isomorphic**.

Inverses are unique: if g_1 and g_2 are inverses of f then $(g_1 \circ f) \circ g_2 = 1_A \circ g_2 = g_2$ but $g_1 \circ (f \circ g_2) = g_1 \circ 1_B = g_1$ and by associativity these are equal. Furthermore the composition of isomorphisms is an isomorphism: it is easily verified that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$. Notice that 1_A is an isomorphism and it is its own inverse.

2.5 Proposition

A functor maps isomorphisms to isomorphisms, in particular $F(f^{-1}) = F(f)^{-1}$ if $f \colon A \longrightarrow B$ is an isomorphism.

Proof: notice that $F(f) \circ F(f^{-1}) = F(f \circ f^{-1}) = F(1_B) = 1_{F(B)}$ and $F(f^{-1}) \circ F(f) = F(f^{-1} \circ f) = F(1_A) = 1_{F(A)}$. So $F(f^{-1})$ is indeed the inverse of $F(f)$. ■

3 Homotopies

3.1 Definition

Let X and Y are topological spaces and $f, g: X \longrightarrow Y$ (meaning they are morphisms, thus continuous). We say that f is homotopic to g, denoted $f \sim g$, if there exists an $H: X \times I \longrightarrow Y$ ($I = [0,1]$, $X \times I$ is the product topology) such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. We denote $h_t(x) := H(x,t)$, and H is called a **homotopy** from f to g.

A homotopy is essentially a smooth mapping from one morphism f to another g. Homotopy is indeed an equivalence relation: firstly $f \sim f$ as we can define $H(x,t) = f(x)$ which is continuous as the composition of continuous functions ($H = f \circ \pi_1$), if $f \sim g$ then define $H'(x,t) = H(x, 1-t)$ which is also continuous (since $(x,t) \mapsto (x, 1-t)$ is continuous since its components are) and $H'(x,0) = g(x)$ and $H'(x,1) = f(x)$ so $g \sim f$, and if H_1 is a homotopy from f to g and H_2 is a homotopy from g to h, define

$$H(x,t) = \begin{cases} H_1(x, 2t) & 0 \leq t \leq \frac{1}{2} \\ H_2(x, 2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

$X \times [0, 1/2]$ and $X \times [1/2, 1]$ are closed (since $X \times [0, 1/2]$ is the preimage of $[0, 1/2]$ in the mapping $(x,t) \mapsto t$) and $H(x,t)$ is continuous on both of these (since $H_1(x, 2t)$ and $H_2(x, 2t-1)$ are continuous), so $H(x,t)$ is continuous.

3.2 Proposition

For every topological space X and every two morphisms $f, g: X \longrightarrow \mathbb{R}^n$, f and g are homotopic.

Proof: define $H(x,t) = (1-t)f(x) + tg(x)$ (addition and scalar multiplication are continuous). ∎

3.3 Definition

A topological space X is **contractible** if the identity map id_X is homotopic to some constant map.

Notice that all two constant maps are homotopic if and only if the space is path connected. If all two constant maps are homotopic, for $x_1, x_2 \in X$ let $H(x,t)$ be a homotopy from x_1 to x_2 and define $\gamma(t) = H(x_0, t)$ for any $x_0 \in X$, this is a continuous path from x_1 to x_2. And if X is path connected, for x_1 and x_2 and γ connecting them, define $H(x,t) = \gamma(t)$.

3.4 Proposition

Let X, Y, Z be topological spaces, $f, f': X \longrightarrow Y$ and $g, g': Y \to Z$ such that $f \sim f'$ and $g \sim g'$, then $g \circ f \sim g' \circ f'$.

Proof: let H be a homotopy from f to f' and K a homotopy from g to g'. Then define $J(x,t) = K(H(x,t), t)$ which is a composition of continuous functions (map (x,t) to $((x,t),t)$ to $(H(x,t),t)$ to $K(H(x,t),t)$). ∎

We call the equivalence classes of morphisms under $\sim$ *homotopy classes*, and the homotopy class of a morphism f is denoted $[f]$. So by above, $[f] \circ [g] := [f \circ g]$ is a well-defined operation. This gives us a new category whose objects are topological spaces and morphisms are homotopy classes. What are the isomorphisms in this category? Well the identities are obviously $[1_X]$ since $[f] \circ [1_X] = [f \circ 1_X] = [f]$ and $[1_X] \circ [g] = [1_X \circ g] = [g]$. So an isomorphism $X \xrightarrow{[f]} Y$ is a homotopy class such that there exists a $Y \xrightarrow{[g]} X$ such that $[f] \circ [g] = [f \circ g] = [1_X]$ and $[g \circ f] = [1_Y]$. We give these isomorphisms a different name:

3.5 Definition

Let X and Y be topological spaces, then $f: X \longrightarrow Y$ is a **homotopic equivalence** if there exists a $g: Y \longrightarrow X$ such that $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$. If a homotopy equivalence exists between X and Y, then X and Y are said to be **homotopy equivalent**, denoted $X \simeq Y$.

Notice that homeomorphisms are homotopic equivalences, since $f^{-1} \circ f = \mathrm{id}_X$ and $f \circ f^{-1} = \mathrm{id}_Y$.

3.6 Definition

Let X and Y be topological spaces, $A \subseteq X$, and $f, g\colon X \longrightarrow Y$. We say that f and g are homotopic relative to A, denoted $f \overset{A}{\sim} g$, if there exists a homotopy H from f to g such that $H(a,t) = f(a)$ for all $a \in A$ and $t \in I$. In such a case we must have $f\big|_A = g\big|_A$.

It is not enough for $f \sim g$ and $f\big|_A = g\big|_A$ for f and g to be homotopic relative to A. For example take I and S^1 and the points 0 and 1 on I. Then we can continuously deform I so that it maps onto the bottom or top of the circle. These are two continuous mappings which are homotopic, but no homotopy between them which keeps the image of 0 and 1 constant.

Notice that $\overset{A}{\sim}$ is an equivalence relation, the proof of this is analogous to the proof that homotpy is an equivalence relation. It also preserves composition, if $f, f'\colon (X,A) \longrightarrow (Y,B)$ (meaning they are morphisms from X to Y and $f(A), f'(A) \subseteq B$) and $g, g'\colon (Y,B) \longrightarrow (Z,C)$ such that $f \overset{A}{\sim} f'$ and $g \overset{B}{\sim} g'$, then $g \circ f \overset{A}{\sim} g' \circ f'$.

3.7 Definition

Let X be a topological space. $A \subseteq X$ is called a **retract** if there exists an $r\colon X \longrightarrow A$ such that $r \circ \iota = \mathrm{id}_A$ where $\iota\colon A \longrightarrow X$ is the inclusion map. In other words $r(a) = a$ for all $a \in A$. r is called a **retraction**.

For example $\partial I = \{0,1\}$ is not a retraction of I since every continuous image of I must be connected, and ∂I is not. But if we take X to be an eight shape, and A its bottom circle, then we can map the top circle to the middle point and A to itself and this is a retraction.

3.8 Definition

$A \subseteq X$ is called a **deformation retract** if there exists a retraction r such that $\iota \circ r \overset{A}{\sim} \mathrm{id}_X$.

Instead of requiring r be a retraction, we can require only that $r(X) \subseteq A$. Since then if $\iota \circ r \overset{A}{\sim} \mathrm{id}_X$, this means that $r(a) = \mathrm{id}_X(a) = a$ for all $a \in A$ so it is already a retraction. Explicitly, this is equivalent to saying that there exists a homotopy $H\colon X \times I \longrightarrow X$ such that $H(x,0) = x$ for all $x \in X$, $H(a,t) = a$ for all $a \in A, t \in I$, $H(x,1) \in A$ for all $x \in X$.

Notice that if $A \subseteq X$ is a deformation retract then $\iota\colon A \longrightarrow X$ is a homotopy equivalence, since $r \circ \iota = \mathrm{id}_A$ and $\iota \circ r \sim \mathrm{id}_X$.

3.9 Example

Let $X = \mathbb{R}^n \setminus \{0\}$ and $A = S^{n-1} \subseteq \mathbb{R}^n \setminus \{0\}$. Then $r(x) := \frac{x}{\|x\|}$ is a retraction with the homotopy $H(x,t) = (1-t)x + t\frac{x}{\|x\|}$. This is the homotopy we used to show that all morphisms to $\mathbb{R}^n$ are homotopic.

A morphism f is called *null-homotopic* if it is homotopic to a constant morphism.

3.10 Proposition

Let X be a topological space and $f\colon S^1 \longrightarrow X$, then the following are equivalent

(1) f is null-homotopic,

(2) f is null-homotopic relative to any point on S^1,

(3) f can be expanded to a morphism on D^2 (the disk in $\mathbb{R}^2$), meaning there exists an $F\colon D^2 \longrightarrow X$ such that $F\big|_{S'} = f$.

(2) $\implies$ (1) is trivial since a null-homotopy relative to a point is still a null-homotopy. (3) $\implies$ (2): let $\iota\colon S^1 \longrightarrow D^2$ be the inclusion map, and let $a \in S^1$, define the homotopy $H\colon S^1 \longrightarrow I \longrightarrow D^2$ by $H(x,t) = (1-t)\iota(x) + ta$, which is a homotopy from ι to the constant map K_a. Then $F \circ H$ is a null-homotopy between f and $K_{f(a)}$ (since $F \circ H(x,0) = F(x) = f(x)$ and $F \circ H(x,1) = F(a)$) relative to a since $F \circ H(a,t) = F(a)$. (1) $\implies$ (3): so there

exists a homotopy $H: S^1 \times I \longrightarrow X$ such that $H(x, 0) = f(x)$ for every $x \in S^1$ and there exists a $p \in X$ such that $H(x, 1) = p$ for all $x \in S^1$. Let us define $\rho: S^1 \times I \longrightarrow D^2$ by $\rho(x, t) = (1 - t)x$, this is a continuous map from a compact (since S^1 and I are compact and therefore so is their product) to a Hausdorff space, and so it is closed. And it is surjective, so it is a quotient map. So D^2 is the quotient space of $S^1 \times I$ with respect to ρ, and H respects ρ, since $\rho(x, t) = \rho(y, s)$ implies $(1 - t)x = (1 - s)y$ and this means that either $(x, t) = (y, s)$ or $t = s = 1$. But in both cases $H(x, t) = H(y, s)$, and so there exists an $F: D^2 \longrightarrow X$ which is continuous such that $H = F \circ \rho$, meaning $F(x) = H(x, 0) = f(x)$ as required. ■

This proof uses the fact that if ρ is a quotient map, and $f: X \longrightarrow Y$ is continuous then there exists a $F: \overline{X} \longrightarrow Y$ such that $f = F \circ \rho$ if and only if $\rho(a) = \rho(b)$ implies $f(a) = f(b)$.

4 The Fundamental Group

4.1 Definition

Let X be a topological space, and for every $a, b \in X$ define Γ_{ab} to be the set of all paths from a to b, which are continuous maps $I \longrightarrow X$. On Γ_{ab} we take the equivalence relation of homotopy relative to $\partial I = \{0, 1\}$. Take $\hat{\Gamma}_{ab}$ to be the partition defined by this relation, ie. $\hat{\Gamma}_{ab} = {}^{\Gamma_{ab}}\big/_{\overset{\partial I}{\sim}}$.

If $[\gamma] \in \hat{\Gamma}_{ab}$ and $[\delta] \in \hat{\Gamma}_{bc}$ then we define $[\gamma][\delta] := [\gamma * \delta]$ (their concatenation).

We must show that this is well-defined, meaning we must show that if $\gamma \overset{\partial I}{\sim} \gamma'$ and $\delta \overset{\partial I}{\sim} \delta'$ then $\gamma * \delta \overset{\partial I}{\sim} \gamma' * \delta'$. So let $H: I \times I \longrightarrow X$ be a homotopy relative to ∂I between γ and γ', and $G: I \times I \longrightarrow X$ between δ and δ'. Then define

$$K(s,t) := \begin{cases} H(2s,t) & 0 \leq s \leq \frac{1}{2} \\ G(2s-1,t) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

this is continuous, $K(0,t) = H(0,t) = 0$ and $K(1,t) = G(1,t) = 1$ so it is a homotopy between the concatenations relative to ∂I.

Notice that concatenation is not necessarily associative, since in $(\gamma * \delta) * \varepsilon$, the speed of γ and δ is quadrupled while in $\gamma * (\delta * \varepsilon)$, γ's speed is only doubled. But it is the case that $[\gamma]\big([\delta][\varepsilon]\big) = \big([\gamma][\delta]\big)[\varepsilon]$, so in homotopy concatenation is associative. So we need to prove $\gamma(\delta\varepsilon) \overset{\partial I}{\sim} (\gamma\delta)\varepsilon$, the idea behind this is that for every x and y where $\gamma(\delta\varepsilon)(x) = (\gamma\delta)\varepsilon(y)$, define in $I \times I$ the line between $(x,0)$ and $(y,1)$. These lines cover $I \times I$ and for every point (t,s) which is on the line from $(x,0)$ map it to $\gamma(\delta\varepsilon)(x)$.

We can prove in a similar manner that for $\gamma \in \Gamma_{ab}$, $[K_a][\gamma] = [\gamma][K_b] = [\gamma]$.

And so we have defined a category. The objects of this category are the points $a \in X$ and the morphisms between a and b are $\hat{\Gamma}_{ab}$ (notice that $[\gamma] \in \hat{\Gamma}_{ab}$ can be composed with elements from $\hat{\Gamma}_{bc}$, so the order of composition is reversed). Here the identity morphisms are $[K_a]$.

Notice that every morphism in this category is an isomorphism. This is since for every $\gamma \in \Gamma_{ab}$ we defined its reverse $\overline{\gamma} \in \Gamma_{ba}$ by $\overline{\gamma}(t) := \gamma(1-t)$.

4.2 Proposition

$[\gamma][\overline{\gamma}] = [K_a]$ and $[\overline{\gamma}][\gamma] = [K_b]$.

The idea is that at time t we take the path γ but not all the way, then wait, then take the reverse path $\overline{\gamma}$. So

$$H(x,t) = \begin{cases} \gamma(2x) & 0 \leq x \leq \frac{1-t}{2} \\ \gamma(1-t) & \frac{1-t}{2} \leq x \leq \frac{1+t}{2} \\ \gamma(2-2x) & \frac{1+t}{2} \leq x \leq 1 \end{cases}$$

is a homotopy from $\gamma * \overline{\gamma}$ to K_a. ■

4.3 Definition

A **groupoid** is a small category (a category whose objects form a set, not a pure class) such that every morphism is an isomorphism. If $\mathcal{C}$ is a groupoid, then $\mathrm{Mor}(A,A)$ is then a group for every $A \in \mathcal{C}$.

4.4 Definition

Given a pointed topological space (X,a) (call a the basis point), define the **first homotopy group** (or the **fundamental group**) $\pi_1(X,a) := \hat{\Gamma}_{aa}$. And given a morphism $f: (X,a) \longrightarrow (Y,b)$ (meaning f is continuous and $f(a) = b$), then we define a group homomorphism $f_*: \pi_1(X,a) \longrightarrow \pi_1(Y,b)$ by $f_*([\gamma]) = [f \circ \gamma]$. The correspondence $(X,a) \mapsto \pi_1(X,a)$ and $f \mapsto f_*$ is a functor.

We need to show that f_* is well-defined and also a group homomorphism. To show that it is well-defined, suppose $\gamma \overset{\partial I}{\sim} \delta$, then we must show $f \circ \gamma \overset{\partial I}{\sim} f \circ \delta$. Now we showed that if $f \overset{A}{\sim} f'$ and $g \overset{B}{\sim} g'$ such that $f(A), f'(A) \subseteq B$

then $g \circ f \overset{A}{\sim} g' \circ f'$. And we have that $\gamma \overset{\partial I}{\sim} \delta$ and $f \overset{\{a\}}{\sim} f$ so $f \circ \gamma \overset{\partial I}{\sim} f \circ \delta$ as required. Now we must show that f_* is a homomorphism, ie.

$$f_*([\gamma][\delta]) = [f \circ (\gamma * \delta)] = [(f \circ \gamma) * (f \circ \delta)] = f_*([\gamma]) f_*([\delta])$$

Actually a stronger result holds, $f \circ *(\gamma * \delta) = (f \circ \gamma) * (f \circ \delta)$, as both are given by

$$\begin{cases} f \circ \gamma(2t) & 0 \leq t \leq \frac{1}{2} \\ f \circ \delta(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

To finish the proof that the correspondence is a functor, we need to show that $(g \circ f)_* = g_* \circ f_*$ and $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X,a)}$. We do so directly:

$$(g \circ f)_*([\gamma]) = [(g \circ f) \circ \gamma] = [g \circ (f \circ \gamma)] = g_*([f \circ \gamma]) = g_* \circ f_*([\gamma])$$

and

$$(\mathrm{id}_X)_*([\varphi]) = [\mathrm{id}_X \varphi] = [\varphi]$$

so $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X,a)}$ as required. Thus we have defined a functor from the category of pointed topological spaces to the category of groups.

4.5 Proposition

Let X be a topological space, $a \in X$, and A be a's connected component. Let $\iota\colon A \longrightarrow X$ be the inclusion map, then $\iota_*\colon \pi_1(A,a) \longrightarrow \pi_1(X,a)$ is an isomorphism.

Proof: ι_* is injective: $\iota_*([\gamma]) = [K_a]$ if and only if $[\iota \circ \gamma] = [K_a]$, which means $\iota \circ \gamma \overset{\partial I}{\sim} K_a$, let H be such a homotopy. Then $H\colon I \times I \longrightarrow X$, but we claim that H's image is contained in A so that it is also a homotopy $\gamma \overset{\partial I}{\sim} K_a$, meaning $[\gamma] = [K_a]$. Suppose not, that $H(t_0, s_0) \notin A$, then define $\delta(t) = H(t_0 \cdot t, s_0)$. Then $\delta(0) = H(0, s_0) = a$ since H is a homotopy relative to ∂I, and $\delta(1) = H(t_0, s_0) \notin A$, so a is connected to a value not in A, in contradiction. So ι_* is indeed injective.

Now suppose $[\gamma] \in \pi_1(X,a)$, meaning γ is a path connecting a to itself in X. But every point in γ's image must also be connected to a, meaning γ is a path connecting a to itself contained within A. So there exists a $\gamma' \in \Gamma_{aa}^A$ such that $\gamma = \iota \circ \gamma'$ and in particular $\iota_*([\gamma']) = [\gamma]$ as required. So ι_* is a bijective homomorphism, an isomorphism. ■

Suppose $a, b \in X$ such that there exists a path between them, $\gamma\colon I \longrightarrow X, \gamma(0) = a, \gamma(1) = b$. Let us define

$$F_\gamma\colon \pi_1(X,a) \longrightarrow \pi_1(X,b), \qquad F_\gamma[\varphi] = [\overline{\gamma} * \varphi * \gamma] = [\overline{\gamma}][\varphi][\gamma]$$

so $F_\gamma[\varphi]$ is the class of curves homotopic to the curve obtained by walking from b along γ backward to a, then going back along γ to b. Notice that $F_\gamma[\varphi] = [\gamma]^{-1}[\varphi][\gamma]$, so F_γ is simply conjugation by $[\gamma]$.

In general, suppose $\mathcal{G}$ is a groupoid and let $A, B \in \mathcal{G}$ such that $\mathrm{Mor}(A,B) \neq \varnothing$. Then $\mathrm{Mor}(A,A)$ and $\mathrm{Mor}(B,B)$ are isomorphic groups. Let $\varphi \in \mathrm{Mor}(A,B)$ and define $F_\varphi\colon \mathrm{Mor}(A,A) \longrightarrow \mathrm{Mor}(B,B)$ by $F_\varphi(\varkappa) = \varphi \circ \varkappa \circ \varphi^{-1}$. This is a group homomorphism:

$$F_\varphi(\varkappa_1) \circ F_\varphi(\varkappa_2) = \varphi \circ \varkappa_1 \circ \varphi^{-1} \circ \varphi \circ \varkappa_2 \circ \varphi^{-1} = \varphi \circ (\varkappa_1 \circ \varkappa_2) \circ \varphi^{-1} = F_\varphi(\varkappa_1 \circ \varkappa_2)$$

and it is injective:

$$F_\varphi(\varkappa) = 1_B \iff \varphi \circ \varkappa \circ \varphi^{-1} = 1_B \iff \varphi \circ \varkappa = \varphi \iff \varkappa = \varphi^{-1} \circ \varphi = 1_A$$

and it is surjective: let $\varkappa' \in \mathrm{Mor}(B,B)$ and define $\varkappa := \varphi^{-1} \circ \varkappa' \circ \varphi$, it is clear $F_\varphi(\varkappa) = \varkappa'$. It is clear that $F_\varphi^{-1} = F_{\varphi^{-1}}$ by this.

Our F_γ above is precisely this $F_{[\gamma]}$ defined in the groupoid of first homotopy groups above X, meaning it is an isomorphism between $\pi_1(X,a)$ and $\pi_1(X,b)$. And so $F_\gamma^{-1} = F_{\overline{\gamma}}$. Let us summarize this:

4.6 Proposition

Let $a, b \in X$ be two points in X connected by a path γ. Then $\pi_1(X, a)$ and $\pi_1(X, b)$ are isomorphic.

4.7 Proposition

Suppose $H: I \times I \longrightarrow X$ is a homotopy from the closed loop φ to the closed loop ψ such that for all $t \in I$, $H(0, t) = H(1, t)$. Define the path $\gamma(t) = H(0, t) = H(1, t)$, then

$$[\psi] = [\overline{\gamma}][\varphi][\gamma]$$

Proof: this is equivalent to saying

$$[\overline{\psi}][\overline{\gamma}][\varphi][\gamma] = [K_p] \iff \overline{\psi} * \overline{\gamma} * \varphi * \gamma \overset{\partial I}{\sim} K_p$$

Now, $\overline{\psi} * \overline{\gamma} * \varphi * \gamma$ is a curve $I \longrightarrow X$ whose endpoints are the same, so it can be viewed as a curve $S^1 \longrightarrow X$. And it can be extended to the curve H on D^2 (since the unit disc and unit square are homeomorphic), which we showed above is equivalent to $\overline{\psi} * \overline{\gamma} * \varphi * \gamma$ being null-homotopic relative to any point on S^1, so we can choose the point which corresponds to 0 and 1. ∎

4.8 Theorem

Let $f, g: X \longrightarrow Y$ be homotopic with homotopy H. Define $\gamma(t) := H(a, t)$, so $\gamma(0) = f(a)$ and $\gamma(1) = g(a)$. Then $g_* = F_\gamma \circ f_*$ (recall that $f_*: \pi_1(X, a) \longrightarrow \pi_1(Y, f(a))$ and $g_*: \pi_1(X, a) \longrightarrow \pi_1(Y, g(a))$).

Proof: let $[\varphi] \in \pi_1(X, a)$ then

$$F_\gamma(f_*[\varphi]) = [\overline{\gamma}][f \circ \gamma][\gamma], \qquad g_*[\varphi] = [g \circ \varphi]$$

Define $K(s, t) := H(\varphi(s), t)$ which is continuous and

$$K(s, 0) = H(\varphi(s), 0) = f \circ \varphi(s), \quad K(s, 1) = H(\varphi(s), 1) = g \circ \varphi(s), \quad K(0, t) = H(\varphi(0), t) = H(a, t) = K(1, t)$$

so by the above proposition, since K is a homotopy from the closed loop $f \circ \varphi$ to the closed loop $g \circ \varphi$,

$$[g \circ \varphi] = [\overline{\gamma}][f \circ \varphi][\gamma]$$

as required. ∎

Notice then that $g_* = f_*$ if and only if $F_\gamma = \mathrm{id}$ (requiring $f(a) = g(a) = b$). This happens when $[\gamma] = [K_b]$ for example, which can happen when $\gamma = K_b$. Ie. if $H(a, t) = b$ for all $t \in I$, then $f_* = g_*$. But notice that this is simply the condition for H to be a homotopy relative to $\{a\}$, so

4.9 Proposition

If $f \overset{\{a\}}{\sim} g$ then $f_* = g_*$.

4.10 Theorem

If $f: X \longrightarrow Y$ is a homotopy equivalence, then $f_*: \pi_1(X, a) \longrightarrow \pi_1(Y, f(a))$ is an isomorphism of groups.

Proof: there exists a $g: Y \longrightarrow X$ such that $g \circ f \sim \mathrm{id}_X$ and $g \circ f \sim \mathrm{id}_Y$. Now since $g \circ f \sim \mathrm{id}_X$, then by above $(g \circ f)_* = F_\gamma \circ (\mathrm{id}_X)_* = F_\gamma$, since $(\cdot)_*$ is a functor, $g_* \circ f_* = F_\gamma$. Now recall that $F_\gamma = g_* \circ f_*: \pi_1(X, a) \longrightarrow \pi_1(X, g(f(a)))$ is an isomorphism. And similarly $f \circ g \sim \mathrm{id}_Y$, so there exists an F_δ such that $(f \circ g)_* = f_* \circ g_* = F_\delta: \pi_1(Y, f(a)) \longrightarrow \pi_1(X, f(g(f(a))))$ is an isomorphism.

Recall that if $f \circ g$ is injective, then g is injective, and if $f \circ g$ is surjective, f is surjective. Thus f_* is injective and surjective, meaning it is a bijective homomorphism, an isomorphism. ∎

Thus if X is contractible, then it is homotopic to the singleton space (an exercise in homework), $\{b\}$. Thus $\pi_1(X, a) \cong \pi_1(\{b\}, b)$, and $\pi_1(\{b\}, b)$ has a single point, meaning $\pi_1(X, a)$ is the trivial group.

4.11 Corollary

If X is contractible, then $\pi_1(X, a)$ is trivial.

4.12 Definition

Let E, B be topological spaces, then a map $p: E \longrightarrow B$ is a **covering map** (or just a *covering*) if

(1) p is surjective,

(2) There exists an open cover $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ of B such that for every $\alpha \in I$, $p^{-1}\mathcal{U}_\alpha \subseteq E$ is equal to the disjoint union of open sets $\dot\bigcup_{\beta\in J}\tilde{\mathcal{U}}_\beta$ such that p forms a homeomorphism from $\tilde{\mathcal{U}}_\beta$ to $\mathcal{U}_\alpha$. For the sake of conciseness, such an open cover will be called an **open cover for** p.

For example, take $B = S^1$ and $E = \mathbb{R}$. Define $p(t) := (\cos(2\pi t), \sin(2\pi t))$, or if we identify S^1 with $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, then $p(t) = e^{2\pi it}$. This is like spiraling the real line so that all the integers are all on a vertical line (they all map to $1 \in S^1$), and projecting this spiral onto a plane to form the circle S^1. p is obviously a surjective continuous map. For every $e^{2\pi it}$, let $\varepsilon < 2\pi$ then the preimage of the open set $\mathcal{U}_t := \left\{e^{2\pi is} \;\middle|\; t-\varepsilon < s < t+\varepsilon\right\}$ is $\dot\bigcup_{n\in\mathbb{Z}}\big(n + (t-\varepsilon, t+\varepsilon)\big)$ and every $n + (t-\varepsilon, t+\varepsilon)$ is homeomorphic (via p) to this $\mathcal{U}_t$.

4.13 Definition

If $p: E \longrightarrow B$ is a covering, $F: X \longrightarrow B$ a map, a **lift** of F is a function $\tilde{F}: X \longrightarrow E$ such that $F = p \circ \tilde{F}$.

4.14 Lemma

Let $p: E \longrightarrow B$ be a covering, $F: X \times I \longrightarrow B$ be a map such that there exists an initial lift of $F\big|_{X\times\{0\}}$, $\tilde{F}: X \times \{0\} \longrightarrow E$, then there is a unique extension of $\tilde{F}$ to a lift $X \times I \longrightarrow E$.

Proof: let $\{\mathcal{U}_\alpha\}_{\alpha\in J}$ be a covering of B for p, and $x_0 \in X$ and $t \in I$. Then $(x_0, t) \in F^{-1}(\mathcal{U}_\alpha)$ for some $\alpha \in J$, and since $X \times I$ is a product space, there exists an open neighborhood N_t of x_0 and (a_t, b_t) a neighborhood of t such that $N_t \times (a_t, b_t) \subseteq F^{-1}(\mathcal{U}_\alpha)$ so $F(N_t \times (a_t, b_t)) \subseteq \mathcal{U}_\alpha$. Now $\{N_t \times (a_t, b_t)\}_{t\in I}$ forms an open cover of $\{x_0\} \times I$, and therefore it has a finite subcover and $\{(a_t, b_t)\}_{t\in I}$ has a Lebesgue number. Let N be the (finite) intersection of the N_t in the subcover, and $0 = t_0 < \cdots < t_n = 1$ be a partition of I where the difference between subsequent t_is is less than the Lebesgue number. This means that $N \times [t_i, t_{i+1}] \subseteq N_t \times (a_t, b_t)$ and thus is mapped into a single $\mathcal{U}_\alpha$, let it be denoted by $\mathcal{U}_i$.

Let us assume inductively that $\tilde{F}$ has been constructed on $X \times [0, t_i]$, we will extend it to $X \times [0, t_{i+1}]$. Since p is a covering, there exists a $\tilde{\mathcal{U}}_i$ homeomorphic by p to $\mathcal{U}_i$ which contains $\tilde{F}(x_0, t_i)$ (which is in $p^{-1}F(x_0, t_i) \in p^{-1}\mathcal{U}_i$). We can assume that $\tilde{F}(N \times \{t_i\})$ is contained within $\tilde{\mathcal{U}}_i$ by replacing $N \times \{t_i\}$ with its intersection with $\tilde{F}\big|^{-1}_{N\times\{t_i\}}(\tilde{\mathcal{U}}_i)$. Then since $p \circ \tilde{F} = F$, we must have that on $N \times [t_i, t_{i+1}]$, $\tilde{F} = p^{-1} \circ F$ for the restriction $p: \tilde{\mathcal{U}}_i \longrightarrow \mathcal{U}_i$. So define $\tilde{F}\big|_{N\times[t_i,t_{i+1}]} = p^{-1} \circ F\big|_{N\times[t_i,t_{i+1}]}$ where p is the restriction. Since $\tilde{F}(N \times \{t_i\})$ is contained within $\tilde{\mathcal{U}}_i$, its definition in $[t_{i-1}, t_i]$ must agree with this definition, so $\tilde{F}$ is well-defined and continuous.

■

4.15 Theorem

Suppose $p: E \longrightarrow B$ is a covering, $b \in B$ and $e \in E$ such that $p(e) = b$. Now suppose $\gamma: I \longrightarrow B$ is a curve starting at b, $\gamma(0) = b$. Then there exists a unique lift of γ which starts at e, ie. a curve $\hat{\gamma}^e: I \longrightarrow E$ such that $\hat{\gamma}^e(0) = e$.

Proof: using the lemma above, take X to be the trivial space of a single point so we can ignore it in the proof and result of the lemma above. So the initial $\hat{\gamma}^e$ ($\tilde{F}$ in the proof) is a single point, which we can define to be e (which is an initial lift since $p(e) = b$). And by the lemma, γ has a unique lift which extends this. ■

4.16 Proposition

Let $p\colon E \longrightarrow B$ be a covering, $\gamma, \delta\colon I \longrightarrow B$ are curves such that $\gamma \overset{\partial I}{\sim} \delta$. Let $a = \gamma(0) = \delta(0)$, and let $e \in E$ such that $p(e) = a$. Then $\hat{\gamma}^e \overset{\partial I}{\sim} \hat{\delta}^e$, and in particular $\hat{\gamma}^e(1) = \hat{\delta}^e(1)$; both of the curves finish on the same point.

Proof: let $H\colon I \times I \longrightarrow B$ be a homotopy relative to ∂I from γ to δ. We can then lift H to a homotopy $\hat{H}\colon I \times I \longrightarrow E$, using the lemma above with $X = I$ we must have that the initial lift satisfies $p \circ \hat{H}(t,0) = H(t,0) = \gamma(t)$ so $\hat{H}(t,0) = \hat{\gamma}^e(t)$ is the initial lift. And similarly $p \circ \hat{H}(t,1) = H(t,1) = \delta(t)$ so $\hat{H}(t,1) = \hat{\delta}^e(t)$ by the uniqueness of lifts (since $t \mapsto \hat{H}(t,1)$ forms a lift of δ). And $p \circ \hat{H}(0,s) = H(0,s) = a$, we can view this as the curve K_a from a to a, and so by uniqueness we have again that $\hat{H}(0,s) = \hat{K}_a^e(s) = e$. Similar for $\hat{H}(1,s)$. Thus $\hat{H}$ is a homotopy from $\hat{\gamma}^e$ to $\hat{\delta}^e$ relative to ∂I. ■

Let $p\colon E \longrightarrow B$ be a covering, $a \in B$, and $p(e) = a$. Then we define a function $F\colon \pi_1(B,a) \longrightarrow p^{-1}(a)$ by $F([\gamma]) := \hat{\gamma}^e(1)$. This is well-defined by the previous proposition (it is in $p^{-1}(a)$ since $p \circ \hat{\gamma}^e = \gamma$, so $p \circ \hat{\gamma}^e(1) = a$).

4.17 Proposition

(1) If E is path connected, then F is surjective.

(2) If E is simply connected (see homework 2+3, for every two $a, b \in E$ every two paths between them are homotopic relative to ∂I), then it is also injective (it is bijective).

Proof:

(1) Let $p(x) = a$ then x and e are connected since E is path connected, so let δ be a path e to x, then $\widehat{p \circ \delta}^e = \delta$ by uniqueness. And so $F[p \circ \delta] = \delta(1) = x$. So F is surjective.

(2) Suppose $F[\gamma] = F[\delta]$, then $\hat{\gamma}^e(1) = \hat{\delta}^e(1)$. Of course $\hat{\gamma}^e(0) = \hat{\delta}^e(0) = e$, and so by simple connectivity, $\hat{\gamma}^e \overset{\partial I}{\sim} \hat{\delta}^e$ since they begin at the same point and end at the same point. Now recall that $\gamma = p \circ \hat{\gamma}^e$ and since homotopy respects the composition of homotopic functions, $\gamma \overset{\partial I}{\sim} \delta$. ■

So for example, the covering $p\colon \mathbb{R} \longrightarrow S^1$ by $t \mapsto e^{2\pi it}$ is a covering from a simply connected space (since contractible implies simply connected, homework 2) to S^1, this means that $F\colon \pi_1(S^1, e^{2\pi it}) \longrightarrow \{t + n \mid n \in \mathbb{Z}\} \cong \mathbb{Z}$ is a bijection. Since the fundamental group of a simply connected space is trivial, S^1 is not simply connected.

Notice that F^{-1} can be viewed as mapping a curve γ which starts at e and ends at $b \in p^{-1}(a)$ to $[p \circ \gamma]$. So for example, $\pi_1(S^1, 1)$ is $\{[\text{the curve which winds around the circle } n \text{ times}] \mid n \in \mathbb{Z}\}$. This is since a curve from 0 to n in $\mathbb{R}$ is mapped to a curve which winds around the circle n times (for negative n this winds in the opposite direction). Notice that F is a group isomorphism here, since concatenating two curves which wind around the circle n and m times gives a curve which winds around the circle $n + m$ times. So $\pi_1(S^1) \cong \mathbb{Z}$.

Now, since S^1 is a deformation retract of $\mathbb{R}^2 \setminus \{0\}$, so the inclusion map is a homotopic equivalence and thus defines an isomorphism of their fundamental groups. So $\pi_1(\mathbb{R}^2 \setminus \{0\}) \cong \pi_1(S^1) \cong \mathbb{Z}$ as well.

Now suppose $A \subseteq X$ is a retract, meaning there exists a retraction $r\colon X \longrightarrow A$ such that $r \circ \iota = \mathrm{id}_A$. So $\iota_*\colon \pi_1(A,a) \longrightarrow \pi_1(X,a)$ and $r_*\colon \pi_1(X,a) \longrightarrow \pi_1(A,a)$ and since this is a functor, $r_* \circ \iota_* = (r \circ \iota)_* = (\mathrm{id}_A)_* = \mathrm{id}_{\pi_1(A,a)}$. So ι_* is injective. Let us summarize this:

4.18 Proposition

If $A \subseteq X$ is a retract, then $\iota_*\colon \pi_1(A,a) \longrightarrow \pi_1(X,a)$ is a monomorphism (injective).

Now, for example $S^1 = \partial D^2 \subseteq D^2$ is not a retract: if it were then ι_* would be a monomorphism $\pi_1(S^1,1) \longrightarrow \pi_1(D^2,1)$. But $\pi_1(S^1,1) \cong \mathbb{Z}$ and $\pi_1(D^2,1) = 1$ since D^2 is contractible ($H(x,t) = (1-t)x$). And there is no monomorphism $\mathbb{Z} \longrightarrow 1$.

4.19 Theorem (Brouwer Fixed-Point Theorem (for D^2))

If $f\colon D^2 \longrightarrow D^2$ is continuous, then it has a fixed point: a point $x \in D^2$ such that $f(x) = x$.

Proof: suppose not, then $f(x) \neq x$ for all x. Using this f we will construct an $r\colon D^2 \longrightarrow S^1$ such that $r \circ \iota = \mathrm{id}_{S^1}$, meaning then S^1 would be a retract of D^2, which we just showed it is not. For $x \in D^2$, look at the segment which begins at $f(x)$ to x, and set $r(x)$ to be the intersection of this line with the segment. The segment is $f(x) + t(x - f(x))$ and so we want to solve (setting $f(x) = (f_1, f_2)$ and $x = (x_1, x_2)$)

$$\big(f_1 + t(x_1 - f_1)\big)^2 + \big(f_2 + t(x_2 - f_2)\big)^2 = 1$$

which can be solved, and gives a t which is continuous in f and x. Notice that if $x \in S^1$ then by definition $r(x) = x$ since the intersection of the segment with the boundary of the circle is x. So r is indeed a retract, in contradiction. ∎

4.20 Lemma

Let $f, g\colon X \longrightarrow \mathbb{C} \setminus \{0\}$ such that $|f(x) - g(x)| < |g(x)|$ for all $x \in X$. Then f and g are homotopic.

Proof: define $H(x,t) = (1-t)g(x) + tf(x)$, then $|H(x,t)| = |(1-t)g(x) + tf(x)| \geq |g(x)| - t|f(x) - g(x)| > 0$ so $H(x,t)$ is indeed a homotopy to $\mathbb{C} \setminus \{0\}$. ∎

4.21 Theorem (The Fundamental Theorem of Algebra)

If $p(z)$ is a non-constant polynomial over $\mathbb{C}$, then it has a root.

Proof: we can assume $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ for $n > 0$. Suppose that $p(z) \neq 0$ for all $z \in \mathbb{C}$, so p is a map $\mathbb{C} \longrightarrow \mathbb{C} \setminus \{0\}$. For $r > 0$ define $S_r = \{z \mid |z| = r\}$ and $D_r = \{z \mid |z| \leq r\}$ so that $\partial D_r = S_r$. Then $p\big|_{S_r}\colon S_r \longrightarrow \mathbb{C} \setminus \{0\}$, and $p\big|_{D_r}$ is an extension of $p\big|_{S_r}$ to D_r, so $p\big|_{S_r}$ is null-homotopic. Define $g(z) = z^n$, then $p(z) - g(z) = a_{n-1}z^{n-1} + \cdots + a_0$, then

$$\frac{|p(z) - g(z)|}{|g(z)|} \leq \frac{|a_{n-1}|\left|z^{n-1}\right| + \cdots + |a_0|}{|z^n|} = |a_{n-1}|\frac{1}{|z|} + \cdots + |a_0|\frac{1}{|z^n|}$$

let $r = |z|$, then for r large enough this becomes less than 1, so $|p(z) - g(z)| < |g(z)|$ on S_r, so $p\big|_{S_r} \sim g\big|_{S_r}$.

Now recall that for $a \in X$, if $f \sim g$ then $f_* = 0$ if and only if $g_* = 0$, so since $p\big|_{S_r}$ is null-homotopic this means $\left(g\big|_{S_r}\right)_*$ is trivial. But notice that $g_*\colon \pi_1(S_r) \longrightarrow \pi_1(S_{r^n})$ maps the generator of S_r (which is the loop which wraps around S_r once) to the loop which wraps around S_{r^n} n times. So g_* (the restriction) maps 1 to n, which is not a trivial homomorphism. ∎

5 Universal Properties

Direct products: Suppose we have two groups G and H, then we'd like some universal group U along with two morphisms $g\colon U \longrightarrow G$ and $f\colon U \longrightarrow H$ such that for every K and $\varphi\colon K \longrightarrow G$, $\psi\colon K \longrightarrow H$, there exists a unique $L\colon K \longrightarrow U$ such that the following diagram commutes

$$\begin{tikzcd}
& & G \\
K \arrow[r, "\exists!L"] \arrow[urr, bend left, "\varphi"] \arrow[drr, bend right, "\psi"'] & U \arrow[ur, "g"] \arrow[dr, "f"] & \\
& & H
\end{tikzcd}$$

Such a group would be unique up to isomorphism: if U' is another, then our diagram becomes

$$\begin{tikzcd}
& & G \\
U' \arrow[r, shift left, "\exists!L"] \arrow[urr, bend left, "f'"] \arrow[drr, bend right, "g'"'] & U \arrow[l, shift left, "\exists!L'"] \arrow[ur, "g"] \arrow[dr, "f"] & \\
& & H
\end{tikzcd}$$

Taking $U = U'$ then we see that $L = \mathrm{id}$ satisfies the conditions for f, g, so it must be unique. And so now when $U \neq U'$, $L \circ L'$ also satisfies the conditions, so $L \circ L' = \mathrm{id}$. And similarly $L' \circ L = \mathrm{id}$, meaning L is an isomorphism between U and U'.

One such universal construction is $U = G \times H$ with the projection maps p_G and p_H, since if $\varphi\colon K \longrightarrow G$ and $\psi\colon K \longrightarrow H$, then $p_G \circ L = \varphi$ and $p_H \circ L = \psi$ if and only if $L = (\varphi, \psi)$. So such a map exists, and it is unique, meaning $G \times H$ indeed has this universal property. Any universal construction satisfying this is thus called a *product* of G and H.

Abelianization: Similarly let G be a group, then we want an abelian group U and a morphism $\rho\colon G \longrightarrow U$ such that for every other abelian group K and $\varphi\colon G \longrightarrow K$, there exists a unique $L\colon U \longrightarrow K$ such that $\varphi = L \circ \rho$. In other words, the following diagram commutes

$$\begin{tikzcd}
G \arrow[rr, "\varphi"] \arrow[dr, "\rho"'] & & K \\
& U \arrow[ur, "\exists!L"'] &
\end{tikzcd}$$

Such a U is once again unique, since by setting $K = U$ and $\varphi = \rho$ we get that id is the unique morphism which makes the diagram commute. And if we add another universal construction U', we get

$$\begin{tikzcd}
G \arrow[rr, "\rho'"] \arrow[dr, "\rho"'] & & U' \arrow[dl, shift right, "\exists!L'"'] \\
& U \arrow[ur, shift right, "\exists!L"'] &
\end{tikzcd}$$

So $L' \circ L$ makes the previous diagram commute, since $L' \circ L \circ \rho = L' \circ \rho' = \rho$. But since this is unique, that means $L' \circ L = \mathrm{id}$, and similarly $L \circ L' = \mathrm{id}$, meaning L is an isomorphism between U and U'. Such a U is called the *abelianization* of G.

To construct this, define the commutator of $g, h \in G$ by $[g,h] = g^{-1}h^{-1}gh$. Notice that $[g,h]^{-1} = [h,g]$ and $s[g,h]s^{-1} = [sgs^{-1}, sh^{-1}s^{-1}]$. Then define the *commutator subgroup* $[G,G]$ to be the subgroup of G generated by the set of its commutators. Since commutators are closed under conjugation, it follows that $[G,G]$ is normal. Then this means that elements of $[G,G]$ are of the form $[g_1,h_1]\cdots[g_n,h_n]$. Notice that ${}^{G}\!/_{[G,G]}$ is abelian: $ghg^{-1}h^{-1} \in [G,G]$ so $gh[G,G] = hg[G,G]$. In fact, this is what we define to be the abelianization of G: $\mathrm{Ab}(G) := {}^{G}\!/_{[G,G]}$.

Define ρ naturally, $\rho(g) = g[G,G]$. Then if $\varphi\colon G \longrightarrow K$ is a homomorphism to an abelian group, then $L \circ \rho = \varphi$ if $L(g[G,G]) = \varphi(g)$. This is well-defined since if $g_1[G,G] = g_2[G,G]$ then $g_1 g_2^{-1} \in [G,G]$ and every commutator is mapped to 1 by φ (since K is abelian), we get that $\varphi(g_1) = \varphi(g_2)$. And L is of course a homomorphism.

Coproduct (free product): We now define another universal construct, where G, H are groups. Then we want another group U and $g: G \longrightarrow U$ and $h: H \longrightarrow U$ such that for every other group K and $\varphi: G \longrightarrow K$ and $\psi: H \longrightarrow K$, there exists a unique $L: U \longrightarrow K$ such that the following diagram commutes:

$$\begin{tikzcd}
G \arrow[dr, "g"] \arrow[drr, bend left, "\varphi"] & & \\
& U \arrow[r, "\exists! L"] & K \\
H \arrow[ur, "f"'] \arrow[urr, bend right, "\psi"'] & &
\end{tikzcd}$$

Again, U is unique. Let us define M to be the set of all words which utilize letters in G and H (which we assume to be disjoint). We define an equivalence relation on M where consecutive letters of the same group in a word are merged together, meaning for example

$$(\cdots, g, g', \cdots) \sim (\cdots, gg', \cdots)$$

where $g, g' \in G$. And we can remove any identity from any word,

$$(\cdots, 1, \cdots) \sim (\cdots, \cdots)$$

Define the partition of M by $\sim$ as $G * H$ (called the *free product* of G and H). $G * H$ is a group under concatenation: $[\omega_1][\omega_2] = [\omega_1\omega_2]$ where $\omega_1\omega_2$ is the concatenation of the two words in M. So for example $[(g_1, h_1)][(h_2, g_3, h_3)] = [(g_1, h_1, h_2, g_3, h_3)] = [(g_1, h_1h_2, g_3, h_3)]$. This is well-defined, since if $\omega_1 \sim \omega_1'$ and $\omega_2 \sim \omega_2'$ then $\omega_1\omega_1' \sim \omega_2\omega_2'$. This is since if ω_1' results in ω_1 by combining consecutive elements or removing an identity and similar for ω_2', then $\omega_2\omega_2'$ results in $\omega_1\omega_1'$ by the combined results of these operations. Then continue inductively. Since concatenation in M is associative, so is $G * H$'s operation. The identity is $[\varepsilon]$ (the equivalence class of the empty word). The inverse of $[(x_1, \dots, x_n)]$ is $[(x_n^{-1}, \dots, x_1^{-1})]$. So indeed $G * H$ is a group.

We claim that the free product satisfies this universal construction. We can define $\iota_G: G \longrightarrow G * H$ by $\iota_G(g) = [(g)]$ and $\iota_H: H \longrightarrow G*H$ by $\iota_H(h) = [(h)]$ which are obviously embeddings (so we can view G and H as subgroups of their free product). Now if $\varphi: G \longrightarrow K$ and $\psi: H \longrightarrow K$ then $L \circ \iota_G = \varphi$ and $L \circ \iota_H = \psi$ if and only if $L[g] = \varphi(g)$ and $L[h] = \psi(h)$ which uniquely determines the homomorphism L. Namely, $L[x_1, \dots, x_n] = y_1 \cdots y_n$ where $y_i = \varphi(x_i)$ if $x_i \in G$ and $y_i = \psi(x_i)$ if $x_i \in H$.

Amalgation: Now let us work upon this construction. Suppose we have the following commutative diagram:

$$\begin{tikzcd}
& G \arrow[dr, "\varphi"] & \\
P \arrow[ur, "\eta_G"] \arrow[dr, "\eta_H"'] & & K \\
& H \arrow[ur, "\psi"'] &
\end{tikzcd}$$

Meaning $\varphi \circ \eta_G = \psi \circ \eta_H$. Then we can add in $G * H$,

$$\begin{tikzcd}
& G \arrow[dr, "\varphi"] \arrow[d, "\delta_G"] & \\
P \arrow[ur, "\eta_G"] \arrow[dr, "\eta_H"'] & G * H \arrow[r, "\exists! L'"] & K \\
& H \arrow[ur, "\psi"'] \arrow[u, "\delta_H"'] &
\end{tikzcd}$$

But we no longer have that this diagram commutes, since $\delta_G \circ \eta_G$ and $\delta_H \circ \eta_H$ are not necessarily equal. So let us define N to be the normal subgroup of $G * H$ generated by $\left\{ \big(\delta_G\eta_G(p)\big)\big(\delta_H\eta_H(p)\big)^{-1} \;\middle|\; p \in P \right\}$. Now, $N \subseteq \ker(L')$ since

$$L'\big(\delta_G\eta_G(p)\big)L'\big(\delta_H\eta_H(p)\big)^{-1} = \varphi\eta_G(p)\psi\eta_H(p)^{-1}$$

since $\varphi\eta_G = \psi\eta_H$, this is just 1. Thus L' generates a homomorphism $L: {}^{G*H}/_{N} \longrightarrow K$ by $\omega N \mapsto L'(\omega)$, so we get the following diagram

$$\begin{tikzcd}
& G \arrow[d, "\delta_G"] \arrow[dr, "\varepsilon_G"] \arrow[drr, "\varphi"] & & \\
P \arrow[ur, "\eta_G"] \arrow[dr, "\eta_H"'] & G * H \arrow[r, "\rho"] \arrow[rr, bend left, "L'"] & {}^{G*H}\!/_{N} \arrow[r, "L"'] & K \\
& H \arrow[u, "\delta_H"] \arrow[ur, "\varepsilon_H"] \arrow[urr, "\psi"'] & &
\end{tikzcd}$$

Now we claim that the following diagram, obtained by removing $G * H$, commutes:

$$\begin{tikzcd}
& G \arrow[d, "\varepsilon_G"] \arrow[dr, "\varphi"] & \\
P \arrow[ur, "\eta_G"] \arrow[dr, "\eta_H"'] & {}^{G*H}\!/_{N} \arrow[r, "L"'] & K \\
& H \arrow[u, "\varepsilon_H"] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

Indeed, let $p \in P$, then $\varepsilon_G \circ \eta_G(p) = \rho \circ \delta_G \circ \eta_G(p) = \delta_G \circ \eta_G(p)N$ and $\varepsilon_H \circ \eta_H(p) = \delta_H \circ \eta_H(p)N$. By definition of N, these are equal. And $L \circ \varepsilon_G = L \circ \rho \circ \delta_G = L' \circ \delta_G = \varphi$ and similarly $L \circ \varepsilon_H = \psi$. So this diagram does indeed commute. ${}^{G*H}\!/_{N}$ is called the *amalgated product* of G and H relative to P, denoted $G *_P H$.

Free groups: Let X be a set, then the *free group* over X is a group U with $i\colon X \longrightarrow U$ set function, such that for every group K and set function $f\colon X \longrightarrow K$, there exists a unique homomorphism $L\colon U \longrightarrow K$ such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[rr, "i"] \arrow[dr, "f"'] & & K \\
& U \arrow[ur, "\exists! L"'] &
\end{tikzcd}$$

Let $W = W(X)$ be the set of all words over $X \coprod X$. For elements $a \in X$, let us denote $(a, 0)$ by a and $(a, 1)$ by a^{-1}, meaning we denote elements $a \in X$ in the left X by a and in the right X by a^{-1}. We define an equivalence relation over W as follows: a word U is equivalent to a word V if U and V can be brought to a similar word using a finite number of the following operations:

(1) A word of the form $(\underline{\ A\ }, a, a^{-1}, \underline{\ B\ })$ is brought to $(\underline{\ A\ }, \underline{\ B\ })$.

(2) Similarly a word of the form $(\underline{\ A\ }, a^{-1}, a, \underline{\ B\ })$ is also brought to $(\underline{\ A\ }, \underline{\ B\ })$.

We define the group operation over this set by $[U][V] = [UV]$, meaning the product of (the equivalence classes) of two words is the (equivalence class) of the concatenation of the two words. This is indeed a group operation as it is associative, the identity is the empty word, and the inverse of U is U^{-1} (the word obtained by reversing the order of U and swapping a with a^{-1} and a^{-1} with a). The set W equipped with this operation is called the *free group* over X, denoted $F(X)$.

The free group satisfies the above universal property, with $i\colon a \mapsto [a]$. Let $f\colon X \longrightarrow K$ be a set function, then $f = L \circ i$ if and only if $f(a) = L[a]$, so define $L[a] = f(a)$ for $a \in X$. And similarly define $L[a^{-1}] = f(a)^{-1}$, so that this extends to a homomorphism:

$$L[a_1^{x_1} \cdots a_n^{x_n}] = f(a_1)^{x_1} \cdots f(a_n)^{x_n}, \qquad x_i = 1, -1$$

Call a word in W *reduced* if there are no adjacent occurrences of a and a^{-1} in it.

5.1 Proposition

For every $g \in F(X)$, there exists a unique representative of g which is a reduced word.

Proof: let $\omega \in W$ be a word, and define $\#\omega$ be the number of times an element $a \in X$ occurs adjacent to a^{-1}. We claim that ω is equivalent to a reduced word. If $\#\omega = 0$ then ω is reduced. Otherwise we induct on ω: if $\#\omega = n+1$ then we can apply one of the operations to get an ω' equivalent to ω such that $\#\omega' = n$. But then by induction, ω' is equivalent to a reduced word so by transitivity so is ω. The reduced word is unique, since a reduced word cannot have one of the operations applied to it, so two reduced words cannot be equivalent. ■

Let $R \subseteq W(X)$ be a set of words over $X \coprod X$. Then let N be the normal subgroup of $F(X)$ generated by the equivalence classes of words in R (ie. N is generated by $\{[\omega] \mid \omega \in R\}$). Let $f\colon X \longrightarrow K$ be a function to a group K, notice that it can be extended to $W(X) \longrightarrow K$ which maps $a_1^{x_1} \cdots a_n^{x_n}$ to $f(a_1)^{x_1} \cdots f(a_n)^{x_n}$. If $f(\omega) = 1$ for all $\omega \in R$. Since $F(X)$ is the free group, by its construction it induces a unique homomorphism $L'\colon F(X) \longrightarrow K$ such that $f = L' \circ i'$ (where i' is the inclusion $X \longrightarrow F(X)$). Now, $N \subseteq \ker(L')$ since $L'[\omega] = f(\omega) = 1$ for $\omega \in R$. Thus L' induces a homomorphism $L\colon {}^{F(X)}\!/_{N} \longrightarrow K$ such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "i'"] \arrow[ddr, "i"'] & & K \\
& F(X) \arrow[ur, "L'"] \arrow[d, "\rho"] & \\
& {}^{F(X)}\!/_{N} \arrow[uur, "L"'] &
\end{tikzcd}$$

We denote ${}^{F(X)}\!/_{N}$ by $\langle X \mid R \rangle$, called the free group generated by X modulo the relations R. Notice that $F(X) = \langle X \mid \varnothing \rangle$, also denoted $\langle X \rangle$.

Every group is isomorphic to such a free group. Let G be a group with generators X (which could just be $X = G$), then there exists a homomorphism $\varphi\colon F(X) \longrightarrow G$ given by the set function $X \longrightarrow G$ which is simply the inclusion. φ is surjective since X generates G, so $G \cong {}^{F(X)}\!/_{\ker\varphi}$. So if we let R be the set of generators of $\ker\varphi$ as a normal subgroup (in the worst case, $R = \ker\varphi$, or more correctly representatives of elements in $\ker\varphi$), then $G \cong {}^{F(X)}\!/_{\ker\varphi} = \langle X \mid R \rangle$ as required.

6 The Van-Kampen Theorem

6.1 Theorem (Van Kampen)

Let X be a topological space, and $\mathcal{U}, \mathcal{V}$ be open such that $X = \mathcal{U} \cup \mathcal{V}$ and $\mathcal{U} \cap \mathcal{V}$ is non-empty and path connected. Choose a basepoint $a \in \mathcal{U} \cap \mathcal{V}$, we have the diagram of topological spaces

$$\begin{tikzcd}
 & (\mathcal{U}, a) \arrow[dr, "k"] & \\
(\mathcal{U} \cap \mathcal{V}, a) \arrow[ur, "i"] \arrow[dr, "j"'] & & (X, a) \\
 & (\mathcal{V}, a) \arrow[ur, "\ell"'] &
\end{tikzcd}$$

Where i, j, k, ℓ are inclusion maps. This then induces the following commutative diagram

$$\begin{tikzcd}
 & \pi_1(\mathcal{U}) \arrow[d] \arrow[drr, "k_*"] & & \\
\pi_1(\mathcal{U} \cap \mathcal{V}) \arrow[ur, "i_*"] \arrow[dr, "j_*"'] & \pi_1(\mathcal{U}) *_{\pi_1(\mathcal{U} \cap \mathcal{V})} \pi_1(\mathcal{V}) \arrow[rr, "L"'] & & \pi_1(X) \\
 & \pi_1(\mathcal{V}) \arrow[u] \arrow[urr, "\ell_*"'] & &
\end{tikzcd}$$

Van Kampen's theorem then states that L is an isomorphism.

Proof: in our proof we will assume $\mathcal{U}$ and $\mathcal{V}$ are path connected. Let $x \in X$, then take γ_x to be a path from a to x such that

(**1**) γ_a is constant,

(**2**) if $x \in \mathcal{U}$ then γ_x is contained within $\mathcal{U}$,

(**3**) if $x \in \mathcal{V}$ then γ_x is contained within $\mathcal{V}$,

Thus if $x \in \mathcal{U} \cap \mathcal{V}$, γ_x is contained within $\mathcal{U} \cap \mathcal{V}$. If φ is a path in X from x to y, define $\hat{\varphi} := \gamma_x * \varphi * \overline{\gamma}_y$ a loop on a. Notice that if φ is contained within $\mathcal{U}$, so is $\hat{\varphi}$ and similar for $\mathcal{V}$. Then if $[\varphi] \in \pi_1(X, a)$ then $\hat{\varphi} = \gamma_a * \varphi * \overline{\gamma}_a = K_a * \varphi * K_a$ so $[\varphi] = [\hat{\varphi}]$.