Mathematical Logic

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Let $\mathcal{L} = \{<\}$ and le DLO be the theory of dense linear orders without endpoints. That is DLO consists of all the axioms of linear (total) orders, as well as the two axioms:

$$\forall x \forall y \big((x < y) \to \exists z(x < z < y)\big)$$
$$\forall x \exists y \exists z (y < x < z)$$

Proposition 11.0.1:

DLO is $\aleph_0$-categorical and complete.

Proof:

Let $(A, <)$ and $(B, <)$ be two countable models of DLO, let us enumerate them by $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$. We construct a sequence of bijections $f_i\colon A_i \to B_i$ for $A_i \subseteq A$ and $B_i \subseteq B$ recursively. Our goal is to construct such bijections such that $A_i \subseteq A$ and $B_i \subseteq B$ and $f_i \subseteq f_{i+1}$ (ie $f_{i+1}\big|_{A_i} = f_i$). We also want $\bigcup_{i=0}^{\infty} A_i = A$ and $\bigcup_{i=0}^{\infty} B_i = B$. In order for our bijections to have meaning we finally desire that $x < y \in A_i$ if and only if $f_i(x) < f_i(y)$.

Such functions are called *partial embeddings*.

For $n = 0$, let $A_n = B_n = \varnothing$, and f_n the empty function (formally $f_n = \varnothing$).

When $n = 2m+1$, we will construct A_n such that $a_m \in A_n$, as this is a surefire way of ensuring the union of A_ns is A. If $a_m \in A_{n-1}$ then we simply take $A_n = A_{n-1}$, $B_n = B_{n-1}$, $f_n = f_{n-1}$. Otherwise, if $a_m \notin A_{n-1}$ then we know that for every $a \in A_{n-1}$ if $a < a_m$, then $f_{n-1}(a) = f_n(a) < f_n(a_m)$. So we must find a $b \in B \setminus B_{n-1}$ such that $b > f_{n-1}(a)$ for all such $a \in A_{n-1}$.

If a_m is greater than every $a \in A_{n-1}$ then this means that $b > f_{n-1}(a)$ for every $a \in A_{n-1}$, and such a b exists since the theory requires the model have no endpoints. If a_m is less than every $a \in A_{n-1}$ then we can take $b \in B \setminus B_{n-1}$ less than every other element in B_{n-1}. Otherwise $a_i < a_m < a_j$ and so we must have a b such that $f_{n-1}(a_i) < b < f_{n-1}(a_j)$ which exists since the theory requires the model be dense. Thus in any case such a b exists.

Now we define $A_n = A_{n-1} \cup \{a_m\}$, $B_n = A_{n-1} \cup \{b\}$ and $f_n = f_{n-1} \cup \{(a_m, b)\}$.

When $n = 2m+2$, we will construct B_n such that $b_m \in B_n$, as this ensures the union of B_ns is B. If $b_m \in B_{n-1}$, we change nothing. Otherwise, we do the same process as above but on f_{n-1}^{-1} to generate f_n^{-1} (since all these functions are bijections).

As explained above, we have that the unions of A_n and B_n are A and B respectively. So for every $a \in A$ we can take an n such that $a \in A_n$ and define $f(a) = f_n(a)$. The choice of n is immaterial since $f_n \subseteq f_{n+1}$, and since the f_ns are partial embeddings (bijective), f is a bijection. f is an isomorphism since $a < a'$ if and only if $f(a) < f(a')$ and thus the two models are isomorphic.

Therefore DLO is $\aleph_0$-categorical. Since there are no finite dense linear orders, by Vaught's test, DLO is complete, as required. ∎

Note that DLO is satisfiable as $(\mathbb{Q}, <)$ models it.

Definition 11.0.2:

A theory T has **quantifier elimination** if for every formula φ there exists a quantifier-free formula ψ such that

$$T \vDash \varphi \leftrightarrow \psi$$

(meaning every model of T satisfies $\varphi \leftrightarrow \psi$).

Lemma 11.0.3:

If $(A, <)$ and $(B, <)$ are countable dense linear orders, and $a_1 < \cdots < a_n \in A$ and $b_1 < \cdots < b_n \in B$ then there exists an isomorphism $f\colon A \to B$ such that $f(a_i) = f(b_i)$ for every relevant i.

The proof of this is the same as the proof of the proposition above, but we have $A_0 = \{a_1, \ldots, a_n\}$ and $B_0 = \{b_1, \ldots, b_n\}$ and $f_0(a_i) = f_0(b_i)$.

Theorem 11.0.4:

DLO has quantifier elimination.

Proof:

We begin by assuming that φ is a sentence (a closed formula, ie. it has no free variables). We know that $DLO \vdash \varphi$ if and only if $\mathbb{Q} \vDash \varphi$ since *DLO* is complete. So if $\mathbb{Q} \vDash \varphi$ then $DLO \vdash \varphi$ and so

$$DLO \vdash \varphi \leftrightarrow x = x$$

where x is some variable. And if $\mathbb{Q} \vDash \neg\varphi$ then $DLO \vdash \neg\varphi$ and so

$$DLO \vdash \varphi \leftrightarrow x \neq x$$

and thus *DLO* has quantifier elimination for sentences (what we have shown is that this is true for complete theories, not just *DLO*).

Suppose that φ has free variables $x_1, \ldots, x_n$ where $n \geq 1$. We will denote $\vec{x} = (x_1, \ldots, x_n)$.

For every function

$$\sigma\colon \{(i, j \mid 1 \leq i < j \leq n\} \longrightarrow \{0, 1, 2\}$$

we define the formula

$$\chi_\sigma(\vec{x})\colon \quad \bigwedge_{\sigma(i,j)=0} (x_i = x_j) \wedge \bigwedge_{\sigma(i,j)=1} (x_i < x_j) \wedge \bigwedge_{\sigma(i,j)=2} (x_i > x_j)$$

thus χ_σ only evaluates as true if $\sigma(i, j) = 0$ when $i = j$, $\sigma(i, j) = 1$ when $x_i < x_j$, and $\sigma(i, j) = 2$ when $x_i > x_j$. We call such σ and χ_σ *sign conditions.*

Let Λ_φ be the set of sign conditions such that there is a $\vec{a} \in \mathbb{Q}^n$ where

$$\mathbb{Q} \vDash \chi_\sigma(\vec{a}) \wedge \varphi(\vec{a})$$

If Λ_φ is empty, then for every $\vec{a} \in \mathbb{Q}^n$ since $\chi_\sigma(\vec{a})$ is true for some sign condition, we have that $\mathbb{Q} \nvDash \varphi(\vec{a})$ and so

$$\mathbb{Q} \vDash \forall\vec{x}\neg\varphi(\vec{x})$$

and so

$$DLO \vDash \varphi \leftrightarrow x_1 \neq x_1$$

for some variable x_1.

Otherwise, Λ_φ is non-empty (and by definition finite), so let

$$\psi_\varphi(\vec{x}) = \bigvee_{\sigma \in \Lambda_\varphi} \chi_\sigma(\vec{x})$$

If $\mathbb{Q} \vDash \varphi(\vec{x})$ since there is a sign condition such that $\mathbb{Q} \vDash \chi_\sigma(\vec{x})$, we have $\mathbb{Q} \vDash \chi_\sigma(\vec{x}) \wedge \varphi(\vec{x})$. Thus $\mathbb{Q} \vDash \varphi(\vec{x}) \to \psi_\varphi(\vec{x})$.

To show the converse, suppose that $\vec{b} \in \mathbb{Q}^n$ and $\mathbb{Q} \vDash \psi_\varphi(\vec{b})$. Let $\sigma \in \Lambda_\varphi$ such that $\mathbb{Q} \vDash \chi_\sigma(\vec{b})$ then by definition there exists an $\vec{a} \in \mathbb{Q}^n$ such that $\mathbb{Q} \vDash \varphi(\vec{a}) \wedge \chi_\sigma(\vec{a})$. By the lemma above there exists an automorphism of $(\mathbb{Q}, <)$ such that $f(\vec{a}) = \vec{b}$ and so $\mathbb{Q} \vDash \varphi(\vec{b}) \wedge \chi_\sigma(\vec{b})$, in particular $\mathbb{Q} \vDash \varphi(\vec{b})$. Thus we also have $\mathbb{Q} \vDash \psi_\varphi(\vec{b}) \to \varphi(\vec{b})$, as required. ■

Let us take the signature $\mathcal{L} = \{=, 0, S\}$ where S is a unary function (the successor function, in $\mathbb{Z}$ taken to mean $x \mapsto x+1$).

Theorem 11.0.5:

$(\mathbb{Z}, =, 0, S)$ has quantifier elimination.

Proof:

We will show this by formula induction. Instead of using the universal quantifier, we will use the existence quantifier.

If φ is a prime formula, then this is true since it by definition has no quantifiers. If φ is the logical connective of two other formulas, it is sufficient to induct over those and replace them with their quantifier-free equivalents.

Otherwise φ is of the form $\exists x\psi$. We can assume by induction that ψ is quantifier-free. Since ψ can have other free variables, we will write $\psi(x, x_1, \ldots, x_n)$.

Since the atomic formulas of this theory are formulas of the form

$$S(\ldots(S(u))\ldots) = S(\ldots(S(v))\ldots)$$

where u and v are either variables or 0. ψ is some boolean combination of these atomic formulas (eg. using conjunction and negation). Let us focus on the atomic formulas in ψ which contain x. If in such an atomic formula, $u = v = x$ then this atomic formula is either always true or always false (as it essentially says $x + n = x + m$), and so it can be replaced with an equivalent formula which is always true/always false which does not contain x.

The other atomic formula in ψ which contain x are of the form $S^n(x) = S^m(x_i)$ meaning they are all of the form $x = t_j$ where $t_j = x_i + c$ for some i and integer c. And so let us define

$$\psi' = \vee_j \psi(t_j, x_1, \ldots, x_n)$$

For some $x_1, \ldots, x_n$, if ψ' is true then obviously φ is true. If $\varphi(x_1, \ldots, x_n)$ is true then suppose $\psi'(x_1, \ldots, x_n)$ is false, then this means that there is an x where $\psi(x, x_1, \ldots, x_n)$ is true, and so $x \neq t_j$ for any j. But as we discussed above, all such x must make all the atomic formula in ψ false.

So let ψ'' be the formula obtained by replacing all atomic formula in ψ with identically false formulas. Thus $\psi' \vee \psi'$ is equivalent to φ. ■

The structure $(\mathbb{Z}, =, <, S)$ also has quantifier elimination, for a similar reason. Instead of just having atomic formulas of the form $x = t_i$ we also have of the form $x < t_i$. But the last step when considering when $x \neq t_i$ fails here.

Definition 11.0.6:

Two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ are **elementarily equivalent** if for every $\mathcal{L}$-sentence φ, $\mathcal{M} \vDash \varphi$ if and only if $\mathcal{N} \vDash \psi$. This is denoted $\mathcal{M} \equiv \mathcal{N}$.

Thus $\mathcal{M} \equiv \mathcal{N}$ if and only if $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$.

Theorem 11.0.7 (Invariance Theorem):

If $\mathcal{M}$ and $\mathcal{N}$ are isomorphic $\mathcal{L}$-structures with an isomorphism ι, and $\varphi(\vec{x})$ is an $\mathcal{L}$-formula, then for every $\vec{a} \in \mathcal{M}^n$

$$\mathcal{M} \vDash \varphi(\vec{a}) \iff \mathcal{N} \vDash \varphi(\iota(\vec{a}))$$

Proof:

We first prove that for terms $t(\vec{x})$, $t(\iota(\vec{a})) = \iota(t(\vec{a}))$. This is done by term induction, if t is a variable then this is obviously true. Otherwise suppose $t = f(t_1, \ldots, t_n)$ then

$$t(\iota(\vec{a})) = f\big(t_1(\iota(\vec{a})), \ldots, t_n(\iota(\vec{a}))\big) = f\big(\iota(t_1(\vec{a})), \ldots, \iota(t_n(\vec{a}))\big) = \iota\big(f(t_1(\vec{a}), \ldots, t_n(\vec{a}))\big) = \iota(t(\vec{a}))$$

where the second equality is due to term induction, and the third is due to ι being an isomorphism.

We now prove the theorem using formula induction. For prime formulas, suppose $\varphi = R(\vec{t})$ we have that $\mathcal{N} \vDash \varphi(\iota(\vec{a}))$ if and only if $R(\vec{t}(\iota(\vec{a})))$ which is if and only if $R\big(\iota(\vec{t}(\vec{a}))\big)$, which is if and only if $R(\vec{t}(\vec{a}))$ since ι is a isomorphism which is if and only if $\mathcal{M} \vDash \varphi(\vec{a})$ as required.

For conjugate formulas, this is trivial. Suppose $\varphi = \forall x\psi$ then we will take $\mathcal{M}$ as a model with any valuation function w, and we view the valuation function of $\mathcal{N}$ as $\iota \circ w$. $\mathcal{M} \vDash \varphi$ if and only if every b, $\mathcal{M}^b_x \vDash \psi$, since the inductive result is true for any valuation function, this is if and only $\mathcal{N}_x^{\iota(b)} \vDash \psi$ for every $b \in \mathcal{N}$. But since ι is surjective, this is if and only if $\mathcal{N} \vDash \varphi$ as required. ■

Thus we get the following corollary easily

Corollary 11.0.8:

Isomorphic structures are elementarily equivalent.

This is since $\mathcal{M} \vDash \varphi$ if and only if $\mathcal{N} \vdash \varphi$ if φ is a sentence (so it has no free variables to substitute).

Proposition 11.0.9:

Any two models of a complete theory are are elementarily equivalent.

Proof:

Suppose $\mathcal{M}, \mathcal{N} \vDash T$, then let φ be a sentence. If $\mathcal{M} \vDash \varphi$ then T cannot prove $\neg\varphi$ so $T \vdash \varphi$ and so $\mathcal{N} \vDash \varphi$. By symmetry, $\mathcal{M} \vDash \varphi$ if and only if $\mathcal{N} \vDash \varphi$, so $\mathcal{M} \equiv \mathcal{N}$. ■

Thus for example, $(\mathbb{R}, <)$ and $(\mathbb{Q}, <)$ are elementarily equivalent (but not isomorphic) models of DLO. And any two algebraically closed fields of the same characteristic are equivalent.

Definition 11.0.10:

Suppose $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures and $\mu\colon \mathcal{M} \longrightarrow \mathcal{N}$ is an $\mathcal{L}$-embedding (injection preserving constants, functions, and relations). Then μ is an **elementary embedding** if

$$\mathcal{M} \vDash \varphi(a_1, \dots, a_n) \iff \mathcal{N} \vDash \varphi(\mu(a_1), \dots, \mu(a_n))$$

for every $\mathcal{L}$-formula $\varphi(x_1, \dots, x_n)$ and $a_1, \dots, a_n \in \mathcal{M}$.

Suppose $\mathcal{M} \subseteq \mathcal{N}$ is a substructure (meaning it contains all of $\mathcal{N}$'s constants, the functions are closed under $\mathcal{M}$, and the relations $R^{\mathcal{M}} = R^{\mathcal{N}} \cap \mathcal{M}^n$). Then $\mathcal{M}$ is called an **elementary substructure** of $\mathcal{N}$ and $\mathcal{N}$ an **elementary extension** of $\mathcal{M}$, denoted $\mathcal{M} \prec \mathcal{N}$, if the inclusion mapping is an elementary embedding.

Definition 11.0.11:

If $\mathcal{M}$ is an $\mathcal{L}$-structure, let $\mathcal{L}_{\mathcal{M}}$ be the language where a constant symbol c_m is added for every $m \in \mathcal{M}$. We extend $\mathcal{M}$ to an $\mathcal{L}_{\mathcal{M}}$-structure by interpreting each such constructed constant symbol as its actual value in $\mathcal{M}$ (ie. interpreting c_m as m).

The **atomic diagram** of $\mathcal{M}$ is the set of atomic $\mathcal{L}_{\mathcal{M}}$-formulas or negation of atomic $\mathcal{L}_{\mathcal{M}}$-formulas which are satisfied by $\mathcal{M}$. Or in other words:

$$\mathrm{Diag}(\mathcal{M}) = \{\varphi(c_{m_1}, \dots, c_{m_n}) \mid \varphi \text{ is an atomic } \mathcal{L} \text{ formula, or the negation of one, and } \mathcal{M} \vDash \varphi(m_1, \dots, m_n)\}$$

Note that it is necessary to define $\mathcal{L}_{\mathcal{M}}$ in order to talk about $\varphi(m_1, \dots, m_n)$.

And the **elementary diagram** of $\mathcal{M}$ is the set of all $\mathcal{L}_{\mathcal{M}}$-formulae which are satisfied by $\mathcal{M}$:

$$\mathrm{Diag}_{\mathrm{el}}(\mathcal{M}) = \{\varphi(c_{m_1}, \dots, c_{m_n}) \mid \varphi \text{ is a } \mathcal{L}\text{-formula, and } \mathcal{M} \vDash \varphi(m_1, \dots, m_n)\}$$

Lemma 11.0.12:

(1) Suppose $\mathcal{N}$ is an $\mathcal{L}_{\mathcal{M}}$-structure such that $\mathcal{N} \vDash \mathrm{Diag}(\mathcal{M})$ then viewing $\mathcal{N}$ as an $\mathcal{L}$-structure, there is an $\mathcal{L}$-embedding of $\mathcal{M}$ into $\mathcal{N}$.

(2) If $\mathcal{N} \vDash \mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$, then there exists an elementary $\mathcal{L}$-embedding of $\mathcal{M}$ into $\mathcal{N}$.

Proof:

(1) Let us define $\mu\colon \mathcal{M} \longrightarrow \mathcal{N}$ by $\mu(m) = c_m^{\mathcal{N}}$, $\mathcal{N}$'s interpretation of the constant symbol c_m. Since if $m_1 \neq m_2 \in \mathcal{M}$ then

$$c_{m_1} \neq c_{m_2} \in \mathrm{Diag}(\mathcal{M})$$

and so $c_{m_1}^{\mathcal{N}} \neq c_{m_2}^{\mathcal{N}}$ and so μ is injective.

If f is a function symbol of $\mathcal{L}$ and $f^{\mathcal{M}}(m_1, \dots, m_n) = m$, then $f(c_{m,1}, \dots, c_{m,n}) = c_m$ is a formula in $\mathrm{Diag}(\mathcal{M})$, and so

$$f^{\mathcal{N}}(c_{m,1}^{\mathcal{N}}, \dots, c_{m,n}^{\mathcal{N}}) = c_m^{\mathcal{N}}$$

meaning

$$f^{\mathcal{N}}(\mu(m_1), \dots, \mu(m_n)) = \mu(f^{\mathcal{M}}(m_1, \dots, m_n))$$

so μ preserves functions.

Similarly if R is a relation symbol and $R^{\mathcal{M}}(m_1, \dots, m_n)$, then $R(c_{m,1}, \dots, c_{m,n}) \in \mathrm{Diag}(\mathcal{M})$ and so

$$R^{\mathcal{N}}(c_{m,1}^{\mathcal{N}}, \dots, c_{m,n}^{\mathcal{N}}) \iff R^{\mathcal{N}}(\mu(m_1), \dots, \mu(m_n))$$

thus if $R^{\mathcal{M}}(m_1, \dots, m_n)$ then $R^{\mathcal{N}}(\mu(m_1), \dots, \mu(m_n))$ so μ is an $\mathcal{L}$-embedding.

(2) If $\mathcal{N} \vDash \mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$ then this embedding is elementary. This is because $\mathcal{M} \vDash \varphi(m_1, \dots, m_n)$ if and only if $\mathcal{N} \vDash \varphi(c_{m,1}, \dots, c_{m,n})$ which is if and only if

$$\mathcal{N} \vDash \varphi(c_{m,1}^{\mathcal{N}}, \dots, c_{m,n}^{\mathcal{N}}) = \varphi(\mu(c_{m,1}), \dots, \mu(c_{m,n}))$$

as required. ■

Theorem 11.0.13 (Upward Lowenheim-Skolem Theorem):

Let $\mathcal{M}$ be an infinite $\mathcal{L}$-structure and $\varkappa$ be an infinite cardinal $\varkappa \geq |\mathcal{M}| + |\mathcal{L}|$. Then there exists an $\mathcal{L}$-structure $\mathcal{N}$ of cardinality $\varkappa$ and $\mu\colon \mathcal{M} \longrightarrow \mathcal{N}$ elementary.

Proof:

Since $\mathcal{M} \vDash \mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$, the theory $\mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$ is satisfiable and thus since $|\mathcal{L}_{\mathcal{M}}| = |\mathcal{L}| + |\mathcal{M}| \leq \varkappa$, there exists a model of $\mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$ of cardinality $\varkappa$. Let this model be $\mathcal{N}$, so $\mathcal{N} \vDash \mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$. By the above lemma, there exists an elementary embedding of $\mathcal{M}$ into $\mathcal{N}$. ■

Since an elementary embedding can be viewed as an elementary extension (swap out $\mu(m)$ with m), the Upward L owenheim-Skolem Theorem tells us that structures have arbitrarily large elementary extensions.