Representation Theory
Homework 6

Ari Feiglin

1 Problem

(1) Let G be a finite group and k a field whose characteristic divides $|G|$. Show that $\sum_g g \in J(kG)$.

(2) Let G be a group of order p^n, and k have characteristic p. Show that kG has a unique maximal left ideal, and that it has codimension 1 over k.

(3) Let $f\colon R \to S$ be surjective, then show that $f(J(R)) \subseteq J(S)$.

(1) Let E be a simple kG-module, i.e. an irreducible G-representation, and let $v \in E$. Clearly we have that $\sum_g gv \in E^G$, and since E is simple, E^G is either zero or E. If $E^G = 0$ then $\sum_g gv = 0$, and we are finished. Otherwise, $E^G = E$, which means $gv = v$ for all $g \in G$. Thus $\sum_g gv = |G|v = 0$.

In either case $\sum_g gv = 0$, so $\sum_g g \in \mathrm{ann}_{kG}(E)$ for all simple kG-modules E. Thus, $\sum_g g \in J(kG)$ as required.

(2) We recall from homework 2 that irreducible representations of G over a field of characteristic p must be trivial. That is, there is a unique simple kG-module: k itself. Since simple kG-modules are in bijection with maximal left ideals of kG, this means kG has a unique maximal ideal, and it has codimension 1 (since $\dim_k k = 1$).

(3) Let E be a simple S-module, given by the ring morphism $\phi\colon S \to \mathrm{end}(E)$. Then by composition, $\phi \circ f$ gives E the structure of an R-module. It is also a simple R-module: let $F \leq E$ be an R-submodule, then for $x \in S$, let $f(r) = x$ and so $xF = f(r)F = rF \leq F$. This means that E must be a simple R-module.

So simple S-modules are simple R-modules. Now, suppose $x \in J(R)$ and let E be a simple S-module. Then $f(x)E = xE = 0$, and as such $f(x) \in J(S)$. Thus $f(J(R)) \subseteq J(S)$ as required.

2 Problem

(1) Show that a semisimple module is Artinian iff it is Noetherian.

(2) Show that if R is left Artinian, then each R-module $J(R)^k/J(R)^{k+1}$ is Artinian and semisimple.

(3) Show that a semisimple module M is Artinian iff it is Noetherian iff it is a direct sum of finitely many simple modules.

(4) Show that if R is a left Artinian ring, it is left Noetherian.

(1) Let M be semisimple. Suppose M is Artinian, and let $M_0 \subset M_1 \subset \cdots$ be infinitely increasing. Then we define $M_0' \supset M_1' \supset \cdots$ where M_i' is a complement of M_i as follows. First, let M_0' be a complement of M_0, i.e. $M = M_0 \oplus M_0'$. Now suppose we have defined M_i', then let M_{i+1}' to be $M_{i+1} \cap M_i'$'s complement in M_i' (since submodules of semisimple modules are semisimple). So we have

$$M_i' = (M_{i+1} \cap M_i') \oplus M_{i+1}'$$

And thus we have $M = M_i \oplus (M_{i+1} \cap M_i') \oplus M_{i+1}'$. Notice that $M_{i+1} = M_i \oplus (M_{i+1} \cap M_i')$ (right-to-left inclusion is trivial, for left-to-right, let $m \in M_{i+1}$, then $m = m' + m''$ for $m' \in M_i$ and $m'' \in M_i'$, since $M_i \subset M_{i+1}$, we have $m'' \in M_{i+1}$ as well). So we have $M = M_{i+1} \oplus M'_{i+1}$ as required, and $M'_{i+1} \subseteq M'_i$.

Notice that $M'_i = M'_{i+1}$ implies that $M_{i+1} \cap M'_i = 0$, but this would imply $M_{i+1} \subseteq M_i$, a contradiction. Thus we have an infinite decreasing chain of submodules, contradicting M being Artinian. So M must be Noetherian.

The other direction is proven much the same.

(2) Since $J(R)^k$ is an R-submodule of R, it too must be Artinian. Quotients of Artinian modules are Artinian, so $J(R)^k/J(R)^{k+1}$ is Artinian as well.

Recall that $J(R)^0/J(R)^1 = R/J(R)$ is a semisimple ring. The R-module structure of the quotient $J(R)^k/J(R)^{k+1}$ induces an $R/J(R)$-module structure, since $J(R) \cdot J(R)^k \subseteq J(R)^{k+1}$. Since modules over semisimple rings are semisimple, $J(R)^k/J(R)^{k+1}$ is semisimple.

(3) Simple modules are Artinian and Noetherian, so their finite direct sums are Artinian and Noetherian. Now let M be semisimple and Noetherian (equivalently Artinian by the first point). Then M is finitely generated (it is Noetherian), and as such it is the direct sum of finitely many simple modules (as we showed in class).

(4) If R is Artinian, then $J(R)$ is nilpotent and as such $J(R)^n = 0$ for some n. We induct on k to show that $R/J(R)^k$ is Noetherian. For $k = 1$, this is just $R/J(R)$ which is Artinian and semisimple, as such it is Noetherian. Inductively,

$$R/J(R)^{k+1} \cong \frac{R/J(R)^k}{J(R)^k/J(R)^{k+1}}$$

which is the quotient of a Noetherian ring, as such it is Noetherian. Thus we get $R \cong R/0 = R/J(R)^n$ is Noetherian, as required.

3 Problem

Let G, H be finite groups. Then any irreducible representation of $G \times H$ over an algebraically closed field k with characteristic not dividing $|G||H|$ has the form $V \boxtimes W$ for V, W irreducible representations of G, H respectively.

Note that if $V \boxtimes W \cong V' \boxtimes W$, then we must have $V \cong V'$ as G-representations. Indeed, we have by considering dimensions $\dim V \dim W = \dim V' \dim W$, so $V \cong V'$ as vector spaces. It then follows (somehow) that $V \cong V'$ as G-representations.

Now, suppose that $E_1, \ldots, E_n$ are the irreducible representations of G and $F_1, \ldots, F_m$ the irreducible representations of H. Notice then that $E_i \boxtimes F_j$ are all nonisomorphic irreducible representations of $G \times H$. Indeed, they are irreducible: suppose $A < E \boxtimes F$ is nonzero and proper, then consider the projection map $\pi\colon E \boxtimes F \to A$ (which exists due to Maschke), and compose this with the tensor map to give $\phi\colon E \times F \to A$ given by $\phi(e, f) = \pi(e \otimes f)$. Now consider, for $f \in F$, $\phi_f(e) = \phi(e, f)$. We cannot have that for all $f \in F$, $\phi_f = 0$ as then $\phi = 0$ (and so $\pi = 0$, so $A = 0$). So consider $F' = \{f \in F \mid \phi_f = 0\}$, this is a subrepresentation of F. But we have that $F' \neq F$, and so $F' = 0$. This means that $\ker\phi_f = 0$ for each $0 \neq f \in F$. In particular, we can see that A, ϕ forms a tensor product for E, F, and as such must be isomorphic to $E \boxtimes F$, a contradiction.

Now, notice that

$$\sum_{i,j} (\dim E_i \boxtimes F_j)^2 = \sum_{i,j} (\dim E_i)^2 (\dim F_j)^2 = \left(\sum_i (\dim E_i)^2\right)\left(\sum_j (\dim E_j)^2\right) = |G||H| = |G \times H|$$

So $E_i \boxtimes F_j$ must form an exhaustive list of irreducible $G \times H$-representations.

4 Problem

(1) Write explicitly n_i, D_i such that $\mathbb{R}[C_3] \cong \prod_i \mathrm{Mat}_{n_i}(D_i)$.

(2) Describe all irreducible representations of C_3 over $\mathbb{R}$.

(3) Find an irreducible representation V of C_3 such that $V \boxtimes V$ is not an irreducible representation of $C_3 \times C_3$.

(1) We note that since $\mathbb{R}[C_3]$ is commutative, each D_i must be a field, and moreso a field extension of $\mathbb{R}$. The dimensions of $\mathrm{Mat}_n(D)$ are thus at least n^2. Since the dimension of $\mathbb{R}[C_3]$ is 3, which contains no squares other than 1, we must have $n_i = 1$. So we only really have two possibilities: $\mathbb{R}[C_3] \cong \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ or $\mathbb{R}[C_3] \cong \mathbb{R} \times \mathbb{C}$. We know that the first one corresponds to $\mathbb{R}[C_3]$ having no non-trivial irreducible representations, we will attempt to show the latter.

Let us write $C_3 = \{1, x, x^2\}$, so $\mathbb{R}[C_3] = \{a + bx + cx^2 \mid a, b, c \in \mathbb{R}\}$. So our map $\mathbb{R}[C_3] \to \mathbb{R} \times \mathbb{C}$ is defined by its value on x (since 1 is mapped to $(1, 0)$). Let us map x to $(0, \omega_3)$ (the third root of unity). This defines a ring isomorphism.

(2) We see that C_3 has two irreducible real representations up to isomorphism: the trivial one, and one whose matrix algebra corresponds to $\mathbb{C}$. If we let it be E, then $D_E = \mathrm{end}_{C_3}(E)$ is isomorphic to $\mathbb{C}$, and $\mathrm{end}_{D_E}(E)$ must have dimension 1, so E is isomorphic to $D_E \cong \mathbb{C}$. So $x \in C_3$ has order 3 in D_E, we must have that it is ω_3. That is, the irreducible representation is given by $\rho(x)z = \omega_3 z$.

(3) So we claim that $\mathbb{C} \boxtimes_{\mathbb{R}} \mathbb{C}$ is not an irreducible representation of $C_3 \times C_3$. Indeed, $\mathbb{C} \cong \mathbb{C} \boxtimes_{\mathbb{R}} \mathbb{R}$ is clearly a subrepresentation.

5 Problem

Let k be a field.

(1) Let A be a k-algebra. Show that A can be given the structure of an $A \otimes_k A^{\mathsf{op}}$-module by

$$(a \otimes b^{\mathsf{op}}) \cdot x = axb$$

(2) Let D be a central division k-alegbra. Show that D is a simple $D \otimes_k D^{\mathsf{op}}$-module.

(3) Let C be a simple finite dimensional k-algebra, recall that $C \cong M_n(D)$ for a division algebra D, and show that C is a simple $C \otimes_k C^{\mathsf{op}}$-module.

(4) Let A be a k-algebra, show that there is a canonical map $Z(A) \to \mathrm{end}_{A \otimes A^{\mathsf{op}}}(A)$. Show that this is an isomorphism.

(5) Use the density theorem on the module C to show that if C is a finite dimensional simple k-algebra, then $C \otimes_k C^{\mathsf{op}} \cong \mathrm{end}_k(C)$ as k-algebras.

(1) Notice that we can define the map $A \times A^{\mathsf{op}} \to \mathrm{end}(A)$ by $(a, b^{\mathsf{op}})x = axb$. This is bilinear and k-balanced, and as such defines a map $A \otimes_k A^{\mathsf{op}} \to \mathrm{end}(A)$ which satisfies $(a \otimes b^{\mathsf{op}})x = axb$. That is, our structure is well-defined, and so it defines a module.

(2) Let $E \leq D$ be a nonzero submodule, so there exists a nonzero $e \in E$. But then for every $d \in D$, $(1/e \otimes d) \cdot e = d \in E$. Thus $E = D$, so D is simple.

(3) Note that if A is simple as a k-algebra, then it is simple as an $A \otimes_k A^{\mathsf{op}}$-module. Indeed, let $B \leq A$ be a submodule: then for every $a, b \in A$ we have $(a \otimes b)B = aBb \subseteq B$, i.e. B is a double-sided ideal of A, and as such is trivial.

(4) The map $Z(A) \to \mathrm{end}_{A \otimes_k A^{\mathsf{op}}}(A)$ is given by $\alpha \mapsto [x \mapsto \alpha x]$: this is well-defined since $\alpha(a \otimes b)x = \alpha a x b = a \alpha x b = (a \otimes b)\alpha x$. This is also clearly a ring morphism. It is also clearly injective: if $[x \mapsto \alpha x] = \mathrm{id}$, then $1 \mapsto \alpha = 1$. It is also surjective: let $f \in \mathrm{end}_{A \otimes_k A^{\mathsf{op}}}(A)$, then notice that $f(a) = f((1 \otimes a)1) = (1 \otimes a)f(1) = f(1)a$, so f is left-multiplication by $f(1)$. Notice that $f(a) = f((a \otimes 1)1) = (a \otimes 1)f(1) = af(1)$, so $f(1) \in Z(A)$, i.e. our morphism is surjective.

(5) As we showed before, C is a simple $C \otimes_k C^{\mathsf{op}}$-module. Further, let $c_1, \ldots, c_n$ form a basis for C. Now, let $S = \mathrm{end}_{C \otimes C^{\mathsf{op}}}(C) \cong Z(C) \cong k$. Density says that for every $T \in \mathrm{end}_S C \cong \mathrm{end}_k C$ there is an $r \in C \otimes_k C^{\mathsf{op}}$ such that $Tc_i = rc_i$. Since $c_1, \ldots, c_n$ form a k-basis for C, and T must be k-linear, we have that $T = r\mathrm{id}$.

Thus the map $C \otimes_k C^{\mathsf{op}} \mapsto \mathrm{end}_S C \cong \mathrm{end}_k C$ by $r \mapsto r\mathrm{id}$ is a surjection. It is also clearly a homomorphism and an injection, so it is an isomorphism as required.