Calculus 2 Homework #2

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Question 2.1:

Compute $\int \frac{x^5}{(x^3+1)(x^3+8)}dx$

Answer:

First, we will split up the integrand into the sum of smaller rational functions. We know that there exists polynomials p and q such that:

$$\frac{x^5}{(x^3+1)(x^3+8)} = \frac{p}{x^3+1} + \frac{q}{x^3+8}$$

And the degrees of both p and q are less than 3. This requires that:

$$p(x^3+8) + q(x^3+1) = x^5 \implies (p+q)x^3 + 8p + q = x^5$$

Since the degree of $p+q$ is less than 3 and the degree of $(p+q)x^3$ is more than or equal to 3 (or it's 0), this means $8p+q=0$. Otherwise, some part of $(p+q)x^3$ must be equal to $-8p-q$ and must therefore have a degree of less than 3, which is a contradiction since it is multiplied by x^3.
So $q = -8p$. Which means:

$$-7p \cdot x^3 = x^5 \implies p = -\frac{x^2}{7}$$

And

$$q = \frac{8x^2}{7}$$

Which means the integral is equal to:

$$-\frac{1}{7} \cdot \int \frac{x^2}{x^3+1}dx + \frac{8}{7} \cdot \int \frac{x^2}{x^3+8}dx$$

So we just need to figure out how to compute integrals of the form:

$$\int \frac{x^2}{x^3+a}dx$$

We can substitute $u = x^3 + a$ which means $du = 3x^2dx$, so

$$= \frac{1}{3}\int \frac{du}{u} = \frac{1}{3} \cdot \log|u| = \frac{1}{3} \cdot \log\left|x^3+a\right|$$

So the integral is equal to:

$$-\frac{1}{21} \cdot \log\left|x^3+1\right| + \frac{8}{21} \cdot \log\left|x^3+8\right| = \frac{1}{21}\left(\log\left|\frac{(x^3+8)^8}{x^3+1}\right|\right)$$

For simplicity, I did not add C, so the definite integral is actually:

$$\boxed{\frac{1}{21}\left(\log\left|\frac{(x^3+8)^8}{x^3+1}\right|\right)}$$

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Question 2.2:

Compute the integral $\int \frac{x}{\left(\sqrt{-x^2+7x-10}\right)^3} dx$

Answer:

We can complete the square to get:

$$-x^2+7x-10 = -(x-\frac{7}{2})^2 + 2\frac{1}{4} = \frac{3}{2}^2 - (x-\frac{7}{2})^2$$

So we will substitute:

$$x - \frac{7}{2} = \frac{3}{2}\sin\theta$$

For $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, when sin is bijective and differentiable. This means that:

$$\left(\sqrt{-x^2+7x-10}\right)^3 = \left(\sqrt{\frac{3}{2}^2 - \frac{3}{2}^2 \sin^2\theta}\right)^3 = \left(\frac{3}{2}\sqrt{1-\sin^2\theta}\right)^3$$

Since in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\cos\theta \geq 0$, this means that this is equal to:

$$\left(\frac{3}{2}\cos\theta\right)^3 = \frac{3}{2}^3 \cos^3\theta$$

And:

$$dx = \frac{3}{2}\cos\theta d\theta$$

So the integral is equal to:

$$\int \frac{\frac{3}{2}\sin\theta + \frac{7}{2}}{\frac{3}{2}^3 \cos^3\theta} \cdot \frac{3}{2}\cos\theta d\theta = \int \frac{\frac{3}{2}\sin\theta + \frac{7}{2}}{\frac{3}{2}^2 \cos^2\theta} d\theta$$

We can split this fraction up:

$$= \frac{2}{3} \cdot \int \frac{\sin\theta}{\cos^2\theta} d\theta + \frac{14}{9} \cdot \int \frac{d\theta}{\cos^2\theta}$$

The right integral is just $\tan\theta$.
For the left integral, we substitute $u = \cos\theta$, so $du = -\sin\theta d\theta$, so it is equal to:

$$-\int \frac{du}{u^2} = \frac{1}{u} = \frac{1}{\cos\theta}$$

So the integral as a whole is equal to:

$$= \frac{2}{3} \cdot \frac{1}{\cos\theta} + \frac{14}{9} \cdot \tan\theta$$

We know that in this interval (as explained above):

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1 - \frac{2}{3}^2 \left(x - \frac{7}{2}\right)^2} = \frac{2}{3} \cdot \sqrt{\frac{3}{2}^2 - (x-\frac{7}{2})^2} = \frac{2}{3} \cdot \sqrt{-x^2+7x-10}$$

So

$$\tan\theta = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} = \frac{x - \frac{7}{2}}{\sqrt{-x^2+7x-10}}$$

Which means the integral is equal to:

$$\frac{1}{\sqrt{-x^2+7x-10}} + \frac{14}{9} \cdot \frac{x-\frac{7}{2}}{\sqrt{-x^2+7x-10}} = \boxed{\frac{\frac{14}{9} \cdot x - 4\frac{4}{9}}{\sqrt{-x^2+7x-10}} + C}$$

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Question 2.3:

Compute the integral $\int \frac{\cos(2x)}{\cos^2 x \sin^2 x} dx$

Answer:

We know $\cos(2x) = \cos^2 x - \sin^2 x$, so this is equal to:

$$\int \frac{\cos^2 x - \sin^2 x}{\cos^2 x \sin^2 x} dx = \int \frac{1}{\sin^2 x} dx - \int \frac{1}{\cos^2 x} dx = -\cot x - \tan x$$

And we know:

$$\cot x + \tan x = \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} = \frac{1}{\sin x \cos x} = \frac{2}{\sin(2x)}$$

So the integral is equal to

$$\boxed{-\frac{2}{\sin(2x)} + C}$$

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Question 2.4:

Compute the integral $\int x^3\sqrt{9-x^2}dx$

Answer:

We'll substitute $x = 3\sin\theta$ for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. So

$$\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = 3\sqrt{1-\sin^2\theta}$$

And as explained in the previous question, this is equal to $3\cos\theta$.
Also:

$$dx = 3\cos\theta d\theta$$

So the integral is equal to

$$\int 27\sin^3\theta \cdot 3\cos\theta \cdot 3\cos\theta d\theta = 3^5 \cdot \int \sin^3\theta \cdot \cos^2\theta d\theta = 3^5 \cdot \int \sin\theta \left(\cos^2\theta - \cos^4\theta\right) d\theta$$

We can substitute $u = \cos\theta$, so $du = -\sin\theta d\theta$, so the integral is equal to:

$$= -3^5 \cdot \int u^2 - u^4 du = 3^5\left(\frac{u^5}{5} - \frac{u^3}{3}\right) = 3^5\left(\frac{\cos^5\theta}{5} - \frac{\cos^3\theta}{3}\right)$$

We know that in this interval:

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{x^2}{9}} = \frac{1}{3}\cdot\sqrt{9-x^2}$$

So the integral is equal to:

$$3^5\left(\frac{\left(\sqrt{9-x^2}\right)^5}{3^5\cdot 5} - \frac{\left(\sqrt{9-x^2}\right)^3}{3^4}\right) = \left(9-x^2\right)^{3/2}\cdot\left(\frac{9-x^2}{5} - 3\right) = -\frac{1}{5}\left(9-x^2\right)^{3/2}\cdot\left(x^2+6\right)$$

So all in all, the integral is equal to:

$$\boxed{-\frac{1}{5}\left(9-x^2\right)^{3/2}\cdot\left(x^2+6\right)+C}$$

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Question 2.5:

Compute the integral $\int \frac{x^2}{\sqrt{a^2-x^2}}dx$

Answer:

We will substitute $x = a \cdot \sin\theta$ for $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Which means

$$dx = a \cdot \cos\theta d\theta$$

And:

$$\sqrt{a^2-x^2} = \sqrt{a^2 - a^2 \cdot \sin^2\theta} = a\cos\theta$$

So the integral is equal to:

$$\int \frac{a^2\sin^2\theta}{a\cos\theta} \cdot a\cos\theta d\theta = a^2 \cdot \int \sin^2\theta d\theta$$

We know that

$$1 - 2\sin^2\theta = \cos(2\theta) \implies \sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

So the integral is equal to

$$\frac{a^2}{2} \cdot \int 1 - \cos(2\theta)d\theta = \frac{a^2}{2} \cdot \left(\theta - \frac{\sin(2\theta)}{2}\right)$$

We know $\theta = \sin^{-1}\left(\frac{x}{a}\right)$, and in this interval:

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1 - \frac{x^2}{a^2}} = \frac{1}{a} \cdot \sqrt{a^2-x^2}$$

Which means

$$\sin(2\theta) = 2\sin\theta\cos\theta = \frac{2}{a^2} \cdot x \cdot \sqrt{a^2-x^2}$$

So the integral is equal to

$$\frac{a^2}{2} \cdot \left(\sin^{-1}\left(\frac{x}{a}\right) - \frac{2}{2 \cdot a^2} \cdot x \cdot \sqrt{a^2-x^2}\right) = \frac{a^2 \cdot \sin^{-1}\left(\frac{x}{a}\right)}{2} - \frac{1}{2} \cdot x \cdot \sqrt{a^2-x^2}$$

So all in all, the answer is:

$$\boxed{\frac{a^2 \cdot \sin^{-1}\left(\frac{x}{a}\right)}{2} - \frac{1}{2} \cdot x \cdot \sqrt{a^2-x^2} + C}$$

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Question 2.6:

Compute the integral $\int \frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x\left(1+\sqrt[3]{x}\right)}dx$

Answer:

We will substitute $u=\sqrt[6]{x}$, so $x=u^6 \implies dx=6u^5du$, which means the integral is equal to:

$$\int \frac{u^6+u^4+u}{u^6\left(1+u^2\right)}\cdot 6u^5du = 6\cdot\int \frac{u^6+u^4+u}{u\left(1+u^2\right)}du = 6\cdot\int \frac{u^5+u^3+1}{1+u^2}du$$

Using polynomial division, we can compute that:

$$u^5+u^3+1 = u^3\cdot\left(u^2+1\right)+1$$

So the integral is equal to:

$$6\cdot\int u^3+\frac{1}{u^2+1}du = \frac{3u^4}{2}+6\cdot\int\frac{1}{u^2+1}du$$

And we know the right integral is $\tan^{-1}(u)$, so:

$$= \frac{3u^4}{2}+6\tan^{-1}(u) = \frac{3\sqrt[3]{x^2}}{2}+6\tan^{-1}\left(\sqrt[6]{x}\right)$$

So the integral is equal to

$$\boxed{\frac{3\sqrt[3]{x^2}}{2}+6\tan^{-1}\left(\sqrt[6]{x}\right)+C}$$

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Question 2.7:

Compute the integral $\int \frac{dx}{1+\sin x+\cos x}$

Answer:

We will substitute $u=\tan\frac{x}{2}$.

Lemma 1.1.1:

If $u=\tan\frac{x}{2}$ then:

- $dx=\frac{2}{1+u^2}du$
- $\sin x=\frac{2u}{1+u^2}$
- $\cos x=\frac{1-u^2}{1+u^2}$

Proof:

- We know that $x=2\tan^{-1}u$, and we can differentiate both sides to get:

$$dx=\frac{2}{1+u^2}du$$

As required.

- We know:

$$u=\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}=\frac{\sin^2\frac{x}{2}}{\cos\frac{x}{2}\cdot\sin\frac{x}{2}}=\frac{2\sin^2\frac{x}{2}}{\sin x}=\frac{1-\cos x}{\sin x}$$

And we know that

$$\cos x=\pm\sqrt{1-\sin^2 x}$$

So this is equal to:

$$=\frac{1\mp\sqrt{1-\sin^2 x}}{\sin x}$$

Which means

$$u\cdot\sin x=1\mp\sqrt{1-\sin^2 x}$$

Subtracting 1 from both sides and squaring results in:

$$u^2\sin^2 x-2u\sin x+1=1-\sin^2 x\implies\sin x\left(u^2\sin x+\sin x-2u\right)=0$$

Which means that

$$u^2\sin x+\sin x-2u=0\implies\sin x\left(u^2+1\right)=2u\implies\sin x=\frac{2u}{u^2+1}$$

As required.

- From before, we know:

$$u\sin x=1-\cos x\implies\cos x=1-u\sin x=1-\frac{2u^2}{u^2+1}=\frac{1-u^2}{1+u^2}$$

As required.

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So by the above lemma the integral is equal to:

$$\int\frac{\frac{2du}{1+u^2}}{1+\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}}=2\cdot\int\frac{du}{1+u^2+2u+1-u^2}=2\cdot\int\frac{du}{2u+2}=\int\frac{du}{u+1}=\log|u+1|=$$

$$=\log\left|\tan\left(\frac{x}{2}\right)+1\right|$$

So the answer is

$$\boxed{\log\left|\tan\left(\frac{x}{2}\right)+1\right|+C}$$

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Question 2.8:

Compute the integral $\int \frac{dx}{x\sqrt{x^2-7x+6}}$

Answer:

By completing the square, we find:

$$x^2-7x+6=\left(x-\frac{7}{2}\right)^2-6\frac{1}{4}=\frac{1}{4}\cdot\left((2x-7)^2-25\right)$$

So the integral is equal to:

$$\int\frac{2dx}{x\sqrt{(2x-7)^2-25}}$$

So we will substitute:

$$2x-7=\frac{5}{\cos\theta}$$

For $\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. Which means:

$$\sqrt{(2x-7)^2-25}=\sqrt{\frac{25}{\cos^2\theta}-25}=5\sqrt{\tan^2\theta}=5\tan\theta$$

And:

$$dx=\frac{5\sin\theta}{2\cos^2\theta}d\theta$$

So the integral is equal to

$$\int\frac{\frac{5\sin\theta}{\cos^2\theta}}{\frac{\frac{5}{\cos\theta}+7}{2}\cdot 5\tan\theta}d\theta=2\int\frac{\sin\theta}{(5+7\cos\theta)\sin\theta}d\theta=2\int\frac{d\theta}{5+7\cos\theta}$$

We can then substitute $u=\tan\left(\frac{\theta}{2}\right)$, which by lemma 1.1.1, means the integral is equal to:

$$4\int\frac{\frac{1}{1+u^2}}{5+7\cdot\frac{1-u^2}{1+u^2}}du=4\int\frac{1}{12-2u^2}du=2\int\frac{du}{6-u^2}$$

We know there exist $\alpha,\beta\in\mathbb{R}$ such that

$$\frac{1}{6-u^2}=\frac{\alpha}{\sqrt{6}-u}+\frac{\beta}{\sqrt{6}+u}$$

Which means

$$\alpha(\sqrt{6}+u)+\beta(\sqrt{6}-u)=1\implies\left\{\begin{array}{lcl}\alpha-\beta&=&0\\\alpha+\beta&=&\frac{1}{\sqrt{6}}\end{array}\right.$$

So $\alpha=\beta$ and $\alpha=\beta=\frac{1}{2\sqrt{6}}$.
So the integral is:

$$\frac{1}{\sqrt{6}}\left(\log\left|\frac{\sqrt{6}+u}{\sqrt{6}-u}\right|\right)$$

Now, we know that $u=\tan\frac{\theta}{2}$, and from our proof of lemma 1.1.1, we know:

$$\tan\frac{\theta}{2}=\frac{1-\cos\theta}{\sin\theta}=\frac{1-\cos\theta}{\sqrt{1-\cos^2\theta}}$$

And we know that

$$\cos\theta=\frac{5}{2x-7}$$

So

$$\tan\frac{\theta}{2}=\frac{1-\frac{5}{2x-7}}{\sqrt{1-\frac{25}{(2x-7)^2}}}=\frac{2x-7-5}{\sqrt{(2x-7)^2-25}}=\frac{2x-12}{\sqrt{4x^2-28x+24}}=\frac{x-6}{\sqrt{x^2-7x+6}}$$

Which means that the integral is equal to:

$$\frac{1}{\sqrt{6}}\cdot\log\left|\frac{\sqrt{6}+\frac{x-6}{\sqrt{x^2-7x+6}}}{\sqrt{6}-\frac{x-6}{\sqrt{x^2-7x+6}}}\right|=\frac{1}{\sqrt{6}}\cdot\log\left|\frac{\sqrt{6x^2-42x+36}+x-6}{\sqrt{6x^2-42x+36}-x+6}\right|$$

So all in all the integral is:

$$\boxed{\frac{1}{\sqrt{6}}\cdot\log\left|\frac{\sqrt{6x^2-42x+36}+x-6}{\sqrt{6x^2-42x+36}-x+6}\right|+C}$$

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Question 2.9:

Compute the integral $\displaystyle\int \frac{dx}{2+2\sin x}$

Answer:

We will substitute $u = \tan\frac{x}{2}$, and according to lemma 1.1.1, the integral is equal to:

$$\int \frac{\frac{2du}{1+u^2}}{2+\frac{4u}{1+u^2}} = \int \frac{2du}{2+2u^2+4u} = \int \frac{du}{u^2+2u+1} = \int \frac{du}{(u+1)^2} = -\frac{1}{u+1}$$

So the integral is equal to:

$$\boxed{-\frac{1}{\tan\left(\frac{x}{2}\right)+1}+C}$$

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Question 2.10:

Find a recursive formula for $I_n = \int \sin^n x dx$

Answer:

Boundary conditions

$\boldsymbol{n = 0}$: If $n = 0$, then

$$I_n = I_0 = \int \sin^0 x dx = \int dx = x + C$$

So $I_0 \coloneqq x + C$

$\boldsymbol{n = 1}$: If $n = 1$, then:

$$I_n = I_1 = \int \sin x dx = -\cos x + C$$

So $I_n = -\cos x + C$

Recursion

Notice that for $n \geq 1$:

$$\sin^n x = \sin^{n-1}(x) \cdot \sin(x)$$

So the integral

$$I_n = \int \sin^{n-1}(x) \cdot \sin(x) dx$$

Using integration by parts

$$\begin{array}{rclcrcl} u & = & \sin^{n-1}(x) & \qquad & dv & = & \sin(x)dx \\ du & = & (n-1)\sin^{n-2}(x) \cdot \cos(x)dx & \qquad & v & = & -\cos(x) \end{array}$$

The integral is equal to:

$$I_n = -\sin^{n-1}(x) \cdot \cos(x) + (n-1) \int \sin^{n-2}(x) \cos^2(x) dx$$

Now, let's focus on the right integral:

$$\int \sin^{n-2}(x) \cos^2(x) dx = \int \sin^{n-2}(x) - \sin^n(x) dx = I_{n-2} - I_n$$

So:

$$I_n = -\sin^{n-1}(x) \cdot \cos(x) + (n-1)(I_{n-2} - I_n) \implies n \cdot I_n = -\sin^{n-1}(x) \cdot \cos(x) + (n-1) I_{n-2}$$

Which means:

$$\boxed{I_n = -\frac{1}{n} \cdot \sin^{n-1}(x) \cdot \cos(x) + \frac{n-1}{n} \cdot I_{n-2}}$$

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