Linear Algebra 2, Recitation 7

Definition 1

An **inner product space** (מרחב מכפלה פנימי) is a vector space V along with an **inner product** (פונקצית מכפלה פנימי) which is

$$\langle \bullet, \bullet \rangle \colon V \times V \longrightarrow \mathbb{F}$$

where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and $\langle \bullet, \bullet \rangle$ satisfies the following three axioms:

(1) Linearity (in the first component): $\langle v + \alpha w, u \rangle = \langle v, u \rangle + \alpha \langle w, u \rangle$.

(2) Hermitianness (הרמיטיות): $\langle v, u \rangle = \overline{\langle u, v \rangle}$.

(3) Nonnegativity: $\langle v, v \rangle \geq 0$ where equality occurs iff $v = 0$.

Exercise 2

Let $T \colon V \longrightarrow V$ be a linear operator on the inner product space V. Show that $\langle v, u \rangle_T = \langle Tv, Tu \rangle$ is an inner product iff T is injective (חחע)

First, suppose $\langle \bullet, \bullet \rangle_T$ is an inner product, then $Tv = 0$ means that $\langle v, v \rangle_T = \langle Tv, Tv \rangle = \langle 0, 0 \rangle = 0$ so $v = 0$. Thus T is injective.

Conversely, we must check the three axioms of inner products:

(1) Linearity:

$$\langle v + \alpha w, u \rangle_T = \langle T(v + \alpha w), Tu \rangle = \langle Tv + \alpha Tw, Tu \rangle = \langle Tv, Tu \rangle + \alpha \langle Tw, Tu \rangle = \langle v, u \rangle_T + \alpha \langle w, u \rangle_T$$

(2) Hermitianness:

$$\langle v, u \rangle_T = \langle Tv, Tu \rangle = \overline{\langle Tu, Tv \rangle} = \overline{\langle u, v \rangle_T}$$

(3) Nonnegativity: $\langle v, v \rangle_T = \langle Tv, Tv \rangle \geq 0$, and $\langle v, v \rangle_T = 0$ iff $Tv = 0$ iff $v = 0$ since T is injective.

$\Diamond$

Exercise 3

Prove or disprove: for an inner product space V and any $v_1, \ldots, v_n \in V$:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \langle v_i, v_j \rangle \geq 0$$

We will prove this. We can change the order of summation:

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} \langle v_i, v_j \rangle$$

By linearity:

$$= \sum_{j=1}^{n} \left\langle \sum_{i=1}^{n} v_i, v_j \right\rangle$$

And by linearity again + Hermitianness:

$$= \left\langle \sum_{i=1}^{n} v_i, \sum_{j=1}^{n} v_j \right\rangle$$

So defining $u = \sum_{i=1}^{n} v_i$, this is just equal to $\langle u, u \rangle \geq 0$. $\Diamond$

Exercise 4

Let V be an inner product space.

(1) Take $v \in V$, prove that for all $u \in V$: $\langle v, u\rangle = 0$ iff $v = 0$.

(2) Take $B = (v_1, \ldots, v_n)$ a basis of V, and $v, u \in V$. Suppose for all $1 \leq i \leq n$: $\langle v, v_i\rangle = \langle u, v_i\rangle$. Show that $v = u$.

(1) Take $v = u$ then $\langle v, v\rangle = 0$ so $v = 0$. The converse was shown in lecture.

(2) We have that $\langle v - u, v_i\rangle = 0$ for all $1 \leq i \leq n$. Now take any $w \in V$, there must exist α_is such that $w = \sum_i \alpha_i v_i$, then

$$\langle v-u, w\rangle = \left\langle v-u, \sum_i \alpha_i v_i \right\rangle = \sum_i \overline{\alpha}_i \langle v-u, v_i\rangle = \sum_i \overline{\alpha}_i \cdot 0 = 0$$

So for every $w \in V$, $\langle v-u, w\rangle = 0$ and so by (1), $v - u = 0 \implies v = u$.

Definition 5

Let V be an inner product space, and $S = (v_1, \ldots, v_n) \subseteq V$ a subset. Define S's **Gram matrix** by $(G_S)_{ij} = \langle v_i, v_j\rangle$.

Exercise 6

Let $S \subseteq V$, show that G_S is singular (not invertible) iff S is linearly dependent.

Suppose G_S is singular, then there exist $\alpha_1, \ldots, \alpha_n$ such that at least one is not zero and

$$\sum_{i=1}^n \alpha_i C_i(A) = \sum_{i=1}^n \alpha_i \begin{pmatrix} \langle v_1, v_i\rangle \\ \vdots \\ \langle v_n, v_i\rangle \end{pmatrix}$$

So for every j, $0 = \sum_{i=1}^n \alpha_i \langle v_j, v_i\rangle = \langle v_j, \sum_{i=1}^n \overline{\alpha}_i v_i\rangle$. Let us define $u = \sum_{i=1}^n \overline{\alpha}_i v_i$. Now, we have that

$$0 = \sum_{j=1}^n \overline{\alpha}_j \langle v_j, u\rangle = \langle u, u\rangle$$

Thus $u = 0$ and since at least one α_i is nonzero, S is linearly dependent.
The converse is shown similarly. $\Diamond$

Definition 7

A **normed vector space** (מרחב נורמי) is a $\mathbb{R}$- or $\mathbb{C}$-vector space V equipped with a **norm function** (פונקצית נורמה)

$$\|\bullet\| \colon V \longrightarrow \mathbb{R}$$

Which satisfies the axioms:

(1) Nonnegativity: $\|v\| \geq 0$ and equality occurs iff $v = 0$.

(2) Homogeneity (הומוגניות): for all $\alpha \in \mathbb{F}$ and $v \in V$: $\|\alpha v\| = |\alpha|\|v\|$.

(3) The triangle inequality: $\|v + u\| \leq \|v\| + \|u\|$.

Intuitively, the norm measures the "length" of a vector.

Theorem 8

Let V be an inner product space, then

$$\|v\| = \sqrt{\langle v, v\rangle}$$

defines a norm function on V.

For example, the standard inner product $\langle v, u\rangle = \sum_i \overline{v}_i u_i$ on $\mathbb{C}^n$ induces the norm $\|v\| = \sqrt{\sum_i |v_i|^2}$.

Exercise 9

Let V be an inner product space, show that

$$|\|v\| - \|u\|| \leq \|v - u\|$$

for all $v, u \in V$.

We have that

$$\|v\| = \|(v - u) + u\| \leq \|v - u\| + \|u\|$$

and so

$$\|v\| - \|u\| \leq \|v - u\|$$

and similarly

$$\|u\| - \|v\| \leq \|u - v\|$$

so we get the desired result. $\Diamond$

Definition 10

Let V be a normed vector space, then $v \in V$ is called **normal** (נורמלי) if $\|v\| = 1$. Every vector $v \in V$, save zero, can be **normalized** (ניתנת לנירמול) by $v \mapsto \frac{v}{\|v\|}$.

Definition 11

Let V be an inner product space, then two vectors $v, u \in V$ are **orthogonal** (אורתוגונליים) if $\langle v, u\rangle = 0$. A set of vectors $\{v_1, \ldots, v_n\}$ is said to be orthogonal if it is pairwise orthogonal (i.e. $\langle v_i, v_j\rangle = 0$ for every $i \neq j$). A set of vectors $\{v_1, \ldots, v_n\}$ is said to be **orthonormal** (אורתונורמלי) if it is orthogonal and every vector is normal.

Theorem 12

Every orthogonal set which doesn't contain zero is linearly independent.

Exercise 13

Find an orthonormal basis to $\mathbb{C}^{n\times n}$ wrt the inner product $\langle A, B\rangle = \mathrm{tr}(AB^*)$.

Let E_{ij} be the elementary matrix where $(E_{ij})_{k\ell} = \delta_{ik}\delta_{j\ell}$ ($\delta_{xy} = 1$ when $x = y$ and zero otherwise). We claim that $\{E_{ij}\}_{i,j=1}^n$ is an orthonormal basis. Indeed:

$$\langle E_{ij}, E_{k\ell}\rangle = \mathrm{tr}(E_{ij}E_{k\ell}^*) = \mathrm{tr}(E_{ij}E_{\ell k})$$

Now

$$(E_{ab}E_{cd})_{xy} = \sum_t (E_{ab})_{xt}(E_{cd})_{ty}$$

A coefficient of this sum is zero unless $a = x$, $b = t$, $c = t$, and $d = y$. So $E_{ab}E_{cd} = 0$ if $b \neq c$ and $E_{ab}E_{cd} = E_{ad}$ if $b = c$. So in the case $(i, j) = (k, \ell)$ the inner product is

$$\langle E_{ij}, E_{k\ell}\rangle = \operatorname{tr}(E_{ij}E_{ji}) = \operatorname{tr}(E_{ii}) = 1$$

and otherwise we either have $j \neq k$ in which case the product of the two elementary matrices is zero, or $j = k$ and then

$$\langle E_{ij}, E_{k\ell}\rangle = \operatorname{tr}(E_{ij}E_{jk}) = \operatorname{tr}(E_{ik}) = 0$$

since $i \neq k$ so the diagonal is zero. $\lozenge$

Exercise 14

Prove the generalized Pythagorean theorem: if V is an inner product space, $\{v_1, \dots, v_n\}$ an orthogonal set, then

$$\left\|\sum_{i=1}^n v_i\right\|^2 = \sum_{i=1}^n \|v_i\|^2$$

We know that

$$\left\|\sum_{i=1}^n v_i\right\|^2 = \left\langle \sum_{i=1}^n v_i, \sum_{j=1}^n v_j \right\rangle = \sum_{i,j=1}^n \langle v_i, v_j\rangle$$

If $i \neq j$ then $\langle v_i, v_j\rangle = 0$, and so this is equal to

$$= \sum_{i=1}^n \langle v_i, v_i\rangle = \sum_{i=1}^n \|v_i\|^2$$

$\lozenge$

Exercise 15

Let $e_1, \dots, e_n$ be an orthonormal basis of V. Show that if $v_1, \dots, v_n$ are vectors of V such that for every i,

$$\|e_i - v_i\| < \frac{1}{\sqrt{n}}$$

then $v_1, \dots, v_n$ is a basis of V.

Since there are n v_is, we need only prove that $v_1, \dots, v_n$ is linearly independent. Suppose $\sum_i \alpha_i v_i = 0$, then

$$\sum_i (\alpha_i v_i - \alpha_i e_i) = -\sum_i \alpha_i e_i$$

The norm of the left-hand side can be bound by

$$\left\|\sum_i \alpha_i(v_i - e_i)\right\| \leq \sum_i |\alpha_i|\|v_i - e_i\| < \sum_i |\alpha_i| \frac{1}{\sqrt{n}}$$

And by the Pythagorean theorem, the right hand-side's norm is

$$\left\|-\sum_i \alpha_i e_i\right\| = \sqrt{\sum_{i=1}^n |\alpha_i|^2}$$

Squaring these both, we want to prove

$$\left(\sum_i |\alpha_i|\right)^2 = \sum_{i,j} |\alpha_i||\alpha_j| \leq n\sum_{i=1}^n |\alpha_i|^2 = \sum_{i,j} |\alpha_i||\alpha_i|$$

That is, we want to show that $\sum_{i,j} |\alpha_i|(|\alpha_i| - |\alpha_j|) \geq 0$. This is just equal to

$$\sum_{i<j} \big(|\alpha_i|(|\alpha_i| - |\alpha_j|) + |\alpha_j|(|\alpha_j| - |\alpha_i|)\big) = \sum_{i<j} \big(|\alpha_i| - |\alpha_j|\big)^2 \geq 0$$

as required. $\lozenge$

Definition 16

Let V be an inner product space, and $S \subseteq V$ a subset. Define S's **orthogonal complement** (**המרחב הניצב**) to be

$$S^{\perp} = \{v \in V \mid \forall u \in S \colon \langle u, v \rangle = 0\}$$

Exercise 17

Find the orthogonal complement of $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$.

The orthogonal complement of S is

$$S^{\perp} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \left\langle \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\rangle = 0 \right\} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{cases} x + 2y + 3z = 0 \\ y + 2z = 0 \end{cases} \right\}$$

This is just

$$N\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix} = N\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix} = \operatorname{span}\left(\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right)$$ ◊

Exercise 18

Let U, W be subspaces of an inner product space V, prove

$$(U + W)^{\perp} = U^{\perp} \cap W^{\perp}$$

Suppose $v \in (U + W)^{\perp}$, let $u \in U$ and $w \in W$, then $u, w \in U + W$ so $\langle v, u \rangle = \langle v, w \rangle = 0$. Thus $v \in U^{\perp} \cap W^{\perp}$. Conversely, let $v \in U^{\perp} \cap W^{\perp}$, let $u + w \in U + W$ then $\langle v, u \rangle = \langle v, w \rangle = 0$ so $\langle v, u + w \rangle = 0$ by linearity. Thus $v \in (U + W)^{\perp}$ as required. ◊

Exercise 19

Let $A \in \mathbb{R}^{m \times n}$, find $C(A)^{\perp}$ and $C(A^{\top})^{\perp}$.

We know that

$$C(A^{\top}) = \left\{ A^{\top} w \;\middle|\; w \in \mathbb{R}^m \right\}$$

and so $C(A^{\top})^{\perp}$ is the set of all vectors v such that for every $w \in \mathbb{R}^n$: $\left\langle A^{\top} w, v \right\rangle = (A^{\top} w)^{\top} v = w^{\top} A v$. Take in particular $w = e_i$, then this requires $e_i^{\top} A v = R_i(A) v = 0$. This precisely means that $v \in N(A)$. So we claim that $C(A^{\top})^{\perp} = N(A)$, we have already shown one direction of the equality. Now suppose $Av = 0$ then for any w, $\left\langle A^{\top} w, v \right\rangle = w^{\top} A v = w^{\top} 0 = 0$. So we have shown equality.

Thus we also get

$$C(A)^{\perp} = N(A^{\top})$$