Computability and Complexity

Lecture 5, Thursday August 15, 2023

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Exercise 5.1:

We define the following decision problem

$$\mathsf{Partition} = \left\{ A \;\middle|\; \begin{array}{l} A \text{ is a set of natural numbers which can be partitioned into two subsets which have} \\ \text{the same sum} \end{array} \right\}$$

show that $\mathsf{Partition}$ is **NP**-complete.

Note:

Recall that a partition is a set of disjoint subsets of A whose union is A. So the statement "A can be partitioned into two subsets which have the same sum" means that there exist $A_1, A_2 \subseteq A$ where $A_1 \cupdot A_2 = A$ and $\sum A_1 = \sum A_2$.

Showing that $\mathsf{Partition}$ is in **NP** is simple. We will define a reduction from $\mathsf{SubsetSum}$ to $\mathsf{Partition}$. So given an input (A, b) for $\mathsf{SubsetSum}$, let $S = \sum A$, and we define a set B which is an input for $\mathsf{Partition}$ by

$$B = A \cup \{2S - b, S + b\}$$

Notice that $\sum B = \sum A + 3S = 4S$.

Now, if $(A, b) \in \mathsf{SubsetSum}$ then there exists a subset (let us view A as a multiset) $A' \subseteq A$ where $\sum A' = b$. Then if we define $B_1 = A' \cup \{2S - b\}$ and $B_2 = B \setminus B = A \setminus A' \cup \{S + b\}$, we have

$$\sum B_1 = \sum A' + 2S - b = 2S, \qquad \sum B_2 = \sum A \setminus A' + S + b = S - b + S + b = 2S$$

and so B_1, B_2 forms a partition of B and both sets have the same sum. Thus $B \in \mathsf{Partition}$.

And if $B \in \mathsf{Partition}$, then suppose $B = B_1 \cupdot B_2$ and $\sum B_1 = \sum B_2$. Now, $2S - b$ and $S + b$ cannot both be in the same B_i, as then $\sum B_i \geq 3S$ and $B_j \subseteq A$ and so $\sum B_j \leq S$ and thus the sums are not the same, in contradiction. Suppose $2S - b \in B_1$ and $S + b \in B_2$, then let $A_1 = B_1 \setminus \{2S - b\}$ and $A_2 = B_2 \setminus \{S + b\}$, and so

$$\sum B_1 + \sum B_2 = \sum B = 4S$$

and so $\sum B_1 = \sum B_2 = 2S$. This means that

$$\sum A_1 = \sum B_1 - (2S - b) = b$$

and so $(A, b) \in \mathsf{SubsetSum}$ as A_1 is a subset whose sum is b.

Exercise 5.2:

We define the following decision problem

$$\mathsf{BinPacking} = \left\{ (X, \omega, k) \;\middle|\; \begin{array}{l} X \text{ is a set of items, and } \omega\colon X \longrightarrow [0,1] \text{ is a weight function on } X. \ k \text{ is a natural} \\ \text{number, where we can pack all the elements in } X \text{ into } k \text{ boxes where the weight of} \\ \text{each box is at most } 1 \end{array} \right\}$$

Show $\mathsf{BinPacking}$ is **NP**-complete.

The verifier for $\mathsf{BinPacking}$ is the list of elements in X in each box, so $\mathsf{BinPacking}$ is in **NP**. We will define a reduction from $\mathsf{Partition}$ to $\mathsf{BinPacking}$. Suppose A is an input for $\mathsf{Partition}$ ie a set of natural numbers. Let us define (X, ω, k) where

(1) $X = \{x_a \mid a \in A\}$

(2) Let us denote $S = \sum A$, and $\omega(x_a) = \frac{2a}{S}$.

(3) We define $k = 2$.

If $A \in \mathsf{Partition}$, then there exists $A = A_1 \cup A_2$ where $\sum A_1 = \sum A_2 = \frac{S}{2}$. Let us define

$$X_1 = \{x_a \mid a \in A_1\}, \quad X_2 = \{x_a \mid a \in A_2\}$$

then

$$\omega(X_1) = \sum_{a \in A_1} \omega(x_a) = \frac{2\sum A_1}{S} = 1$$

and similarly $\omega(X_2) = 1$, and so packing the elements of X into X_1 and X_2 satisfies the constraints, so $(X, \omega, k) \in \mathsf{BinPacking}$.

Now, if $(X, \omega, k) \in \mathsf{BinPacking}$ there exists a partition of X into X_1 and X_2 where $\omega(X_1), \omega(X_2) \leq 1$. Let

$$A_1 = \{a \in A \mid x_a \in X_1\}, \quad A_2 = \{a \in A \mid x_a \in X_2\}$$

this is a partition of A. And

$$\omega(X_1) = \sum_{a \in A_1} \frac{2a}{S} = \frac{2}{S} \sum A_1, \qquad \omega(X_2) = \frac{2}{S} \sum A_2$$

And so

$$\omega(X_1) + \omega(X_2) = \frac{2}{S}\left(\sum A_1 + \sum A_2\right) = 2$$

and since $\omega(X_1), \omega(X_2) \leq 1$, which means $\omega(X_1) = \omega(X_2) = 1$, and thus

$$\sum A_1 = \sum A_2 = \frac{S}{2}$$

so $A \in \mathsf{Partition}$ as required.

Exercise 5.3:

We define the following decision problem

$$\mathsf{PartitionIntoIS} = \{(G, k) \mid G \text{ is an undirected graph which can be partitioned into } k \text{ independent sets}\}$$

show that $\mathsf{PartitionIntoIS}$ is **NP**-complete.

This is sort of a trick, since

$$\mathsf{PartitionIntoIS} = \mathsf{Color}$$

as G has k independent sets if and only if it can be k-colored (the colors define the independent sets, and vice versa).

Exercise 5.4:

Show that for every search problem $R \in \mathbf{PC}$, if S_R is **NP**-complete, then R has a self-reduction.

We showed that for every search problem R, there exists a Cook reduction from R to S'_R:

$$S'_R = \{(x, u) \mid \exists w \colon (x, uw) \in R\}$$

(proof in lecture 2) Since $R \in \mathbf{PC}$, S'_R is in $\mathbf{NP}$ as let A be the polynomial-time verifier for R, then we define $V((x,u), w) = A(x, uw)$ is a polynomial verifier for S'_R. Thus $S'_R \in \mathbf{NP}$. And since S_R is $\mathbf{NP}$-complete, there exists a reduction from S'_R to S_R, and so there exists a Cook reduction from R to S_R.