Algebraic Topology

Homework 4

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4.1 Exercise

Let X be a topological space, $\mathcal{U}, \mathcal{V} \subseteq X$ open such that $X = \mathcal{U} \cup \mathcal{V}$, $\mathcal{U}, \mathcal{V}$ are simply connected, $\mathcal{U} \cap \mathcal{V}$ is nonempty and path connected. Prove that X is simply connected.

Let $f \in \pi_1(X, a)$ for some $a \in \mathcal{U} \cap \mathcal{V}$. Then $f^{-1}\mathcal{U}, f^{-1}\mathcal{V}$ forms an open cover of I, which is compact and thus has a Lebesgue number. So we can partition I into closed intervals I_i whose length is at most the Lebesgue number so that $f(I_i) \subseteq \mathcal{U}$ or $\mathcal{V}$. Define f_i to be the curve obtained by restricting f to I_i, so up to homotopy $f = f_1 \cdots f_n$. Thus f_i is a curve from $f_i(0)$ to $f_i(1)$ in $\mathcal{U}$ or $\mathcal{V}$. Since $\mathcal{U} \cap \mathcal{V}$ is path connected, let γ_i be a path from $f_i(1) = f_{i+1}(0)$ to a, then

$$[f] = [\overline{\gamma}_0 f_1 \gamma_1 \overline{\gamma}_1 f_2 \gamma_2 \overline{\gamma}_2 \cdots \overline{\gamma}_{n-1} f_n \gamma_n] = [\overline{\gamma}_0 f_1 \gamma_1][\overline{\gamma}_1 f_2 \gamma_2] \cdots [\overline{\gamma}_{n-1} f_n \gamma_n]$$

where γ_0, γ_n are just the constant loops on a. Now, $\overline{\gamma}_{i-1} f_i \gamma_i$ is a loop on a contained within $\mathcal{U}$ or $\mathcal{V}$, which are simply connected meaning these are loop-homotopic to K_a. Thus $[f] = 1$ as required.

4.2 Exercise

(1) Show that for $n \geq 2$, S^n is simply-connected.

(2) How does your proof fail for $n = 1$?

(1) Define $\mathcal{U} = \{(x_1, \dots, x_{n+1}) \in S^n \mid x_{n+1} > -\varepsilon\}$ and $\mathcal{V} = \{(x_1, \dots, x_{n+1}) \in S^n \mid x_{n+1} < \varepsilon\}$. These are both hemispheres of S^n, which are homeomorphic to D^{n+1} and thus simply connected. And $\mathcal{U} \cap \mathcal{V}$ is the band of points $\{(x_1, \dots, x_{n+1}) \in S^n \mid -\varepsilon < x_{n+1} < \varepsilon\}$ which is path-connected and non-empty. So by the first exercise, $S^1 = \mathcal{U} \cup \mathcal{V}$ is simply-connected.

(2) For $n = 1$, $\mathcal{U} \cap \mathcal{V}$ is two segments on the side of S^1 and is not path connected.

4.3 Exercise

Let G, H be two nontrivial groups. Show that $G * H$ is infinite.

Let $g \in G, h \in H$. Then we define $f\colon \mathbb{N} \longrightarrow G * H$ recursively:

$$f(n) = \begin{cases} (g, f(n-1)) & n \text{ even} \\ (h, f(n-1)) & n \text{ odd} \end{cases}$$

so that $f(0) = g$, $f(1) = hg$, $f(2) = ghg$, etc. Then $f(n) \neq f(m)$ for $n \neq m$ since $f(n)$ is irreducible, so f is an injection from $\mathbb{N}$ to $G * H$ meaning the free product is infinite.

4.4 Exercise

Show that the center of the free product of two groups is trivial.

Let $g_1 h_1 \cdots g_n h_n \in G * H$, then this doesn't commute with g_1 since $g_1 g_1 h_1 \cdots g_n h_n$ ends with an element of H while $g_1 h_1 \cdots g_n h_n g_1$ ends with an element of G. And for the case $h_1 g_1 \cdots h_n g_n$ similar. For the case $g_1 h_1 \cdots g_n h_n g_{n_1} \in G * H$, this doesn't commute with $h \in H$. Similar for words which begin and end with $G * H$. So no nontrivial words commute with every other word.

4.5 Exercise

Let A, B be abelian groups, show that $A \times B$ satisfies the same property as the free product in the category of Abelian groups.

We need to show that there exists a unique $L\colon A \times B \longrightarrow K$ to make the following diagram commute (where $\rho_A\colon a \mapsto (a, 0)$ and $\rho_B\colon b \mapsto (0, b)$. All other objects and morphisms are given):

$$\begin{tikzcd}
A \arrow[dr, "\rho_A"] \arrow[drr, bend left, "\varphi"] & & \\
& A \times B \arrow[r, "\exists! L"] & K \\
B \arrow[ur, "\rho_B"'] \arrow[urr, bend right, "\psi"'] & &
\end{tikzcd}$$

To satisfy this, we must have that $L \circ \rho_A = L\varphi$ and $L \circ \rho_B = \psi$ so that $L(a, 0) = \varphi(a)$ and $L(0, b) = \psi(b)$. So we must define $L(a, b) = \varphi(a) + \psi(b)$ and this is well-defined and a unique Abelian group homomorphism.

4.6 Exercise

Show that $\mathrm{Ab}(G * H) \cong \mathrm{Ab}(G) \times \mathrm{Ab}(H)$.

Let K be a group such that there exists a morphism $G * H \longrightarrow K$, then we will show that $\mathrm{Ab}(G) \times \mathrm{Ab}(H)$ has the Abelianization property: there exists a unique morphism L which makes the appropriate diagram commute. But first we need to find the cannonical morphism $G * H \longrightarrow \mathrm{Ab}(G) \times \mathrm{Ab}(H)$.

Because $G \longrightarrow \mathrm{Ab}(G) \longrightarrow \mathrm{Ab}(G) \times \mathrm{Ab}(H)$ and $H \longrightarrow \mathrm{Ab}(H) \longrightarrow \mathrm{Ab}(G) \times \mathrm{Ab}(H)$ are both morphisms from G and H to $\mathrm{Ab}(G) \times \mathrm{Ab}(H)$, by the universal property of the free product $G * H$, there exists a unique morphism which makes the following commute:

$$\begin{tikzcd}
G * H \arrow[r, "\exists!"] & \mathrm{Ab}(G) \times \mathrm{Ab}(H) \\
G \arrow[u] \arrow[r] & \mathrm{Ab}(G) \arrow[u] \\
H \arrow[uu, bend left] \arrow[r] & \mathrm{Ab}(H) \arrow[uu, bend right]
\end{tikzcd}$$

Now let us suppose there is an Abelian group K group and a morphism $G * H \longrightarrow K$. We want to prove there exists a unique morphism $\mathrm{Ab}(G) \times \mathrm{Ab}(H) \longrightarrow K$ which makes the following diagram commute:

$$\begin{tikzcd}
& K & \\
G * H \arrow[ur] \arrow[rr, "\exists!"] & & \mathrm{Ab}(G) \times \mathrm{Ab}(H) \\
G \arrow[u] \arrow[rr] & & \mathrm{Ab}(G) \arrow[u] \\
H \arrow[uu, bend left] \arrow[rr] & & \mathrm{Ab}(H) \arrow[uu, bend right]
\end{tikzcd}$$

Composing $G \longrightarrow G * H \longrightarrow K$ and $H \longrightarrow G * H \longrightarrow K$, by the Abelianization property there exist unique morphisms which make the following commute

$$
\begin{tikzcd}
 & K & \\
G * H \arrow[ur] \arrow[rr, "\exists!"] & & \mathrm{Ab}(G) \times \mathrm{Ab}(H) \\
G \arrow[u] \arrow[rr] & & \mathrm{Ab}(G) \arrow[u] \arrow[uul, bend right, "\exists!"'] \\
H \arrow[uu, bend left] \arrow[rr] & & \mathrm{Ab}(H) \arrow[uu, bend right] \arrow[uuul, bend left, "\exists!"]
\end{tikzcd}
$$

Now, as we showed above $A \times B$ has the free group universal property for Abelian groups, and thus there exists a unique morphism $\mathrm{Ab}(G) \times \mathrm{Ab}(H) \longrightarrow K$ which makes the diagram commute

$$
\begin{tikzcd}
 & K & \\
G * H \arrow[ur] \arrow[rr, "\exists!"] & & \mathrm{Ab}(G) \times \mathrm{Ab}(H) \arrow[ul, "\exists!"'] \\
G \arrow[u] \arrow[rr] & & \mathrm{Ab}(G) \arrow[u] \arrow[uul, bend right, "\exists!"'] \\
H \arrow[uu, bend left] \arrow[rr] & & \mathrm{Ab}(H) \arrow[uu, bend right] \arrow[uuul, bend left, "\exists!"]
\end{tikzcd}
$$