Topology

Lecture 3, Sunday April 16, 2022
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3.1 Conditions for Compactness

Definition 3.1.1:

If $\{\mathcal{U}_\lambda\}_{\lambda\in\Lambda}$ is an open cover of X, then a **Lebesgue Number** of the open cover is a number $\varepsilon > 0$ such that for every $x \in X$ there exists a $\lambda \in \Lambda$ such that $B_\varepsilon(x) \subseteq \mathcal{U}_\lambda$.

Theorem 3.1.2:

A metric space is compact if and only if for every sequence there exists a convergent subsequence.

Proof:

Let us prove that if M is compact then every sequence of points has a convergent subsequence. Suppose there is not a convergent subsequence. Then for every $a \in M$, since x_n does not have a convergent subsequence to a, so there exists an $\varepsilon_a > 0$ such that the only element of x_n contained in $B_{\varepsilon_a}(a)$ is a itself if it is in x_n. This is true since otherwise for every $\varepsilon > 0$ there is an x_n in $B_\varepsilon(a)$ which is not a and so we could construct a subsequence x_{n_k} for $\varepsilon = \frac{1}{k}$. But notice that

$$M = \bigcup_{a\in M} B_{\varepsilon_a}(a)$$

And since M is compact, there exists a finite subcovering, ie. there exists $a_1, \dots, a_N \in M$ such that

$$M = \bigcup_{n=1}^{N} B_{\varepsilon_{a_n}}(a_n)$$

But since every ball contains at most once instance of x_n, this means there are at most N different values of x_n. So there must be some $a \in M$ where $x_n = a$ for an infinite number of as, and so we can take a subsequence of x_ns which are equal to a. Since this subsequence is constant, it is convergent (to a) in contradiction.
Now suppose that every sequence has a convergent subsequence, and let $\{\mathcal{U}_\lambda\}_{\lambda\in\Lambda}$ be an open cover of M. Then we claim that the open set has a Lebesgue number, let us assume the opposite. Then for every n there exists an $x_n \in M$ such that $B_{\frac{1}{n}}(x_n)$ is not contained in any $\mathcal{U}_\lambda$. We can assume x_n is convergent since it has a convergent subsequence so it can be chosen to converge, let its limit be x_0. Since $x_0 \in M$ there is a $\lambda \in \Lambda$ such that $x_0 \in \mathcal{U}_\lambda$ which is open so there exists an $\varepsilon > 0$ such that $B_\varepsilon(x_0) \subseteq \mathcal{U}_\lambda$. Then we can choose an x_n for which $\frac{1}{n}, \rho(x_n, x_0) < \frac{\varepsilon}{2}$ then if $\rho(x_n, y) < \frac{1}{n}$ then $\rho(x_0, y) < \rho(x_0, x_n) + \rho(x_n, y) < \varepsilon$ so $B_{\frac{1}{n}}(x_n) \subseteq B_\varepsilon(x_0) \subseteq \mathcal{U}_\lambda$ which contradicts the assumption that the balls around x_n are not contained in elements of the open cover.
Assume for the sake of a contradiction that there is no finite subcover. Let $\varepsilon > 0$ be the Lebesgue number of this open cover and so

$$M = \bigcup_{x\in X} B_\varepsilon(x)$$

If there is a finite subcover of this open covering of balls, then since $B_\varepsilon(x) \subseteq \mathcal{U}_\lambda$ for some $\lambda \in \Lambda$, this generates some finite subcovering of the original cover, so by our assumption there cannot be a finite subcover. Now we choose $x_n \in M$ inductively such that x_n is not in any $B_\varepsilon(x_m)$ for $x_m \neq x_n$. But since x_n has a convergent subsequence, it must be Cauchy so there must be some N for which $\rho(x_n, x_m) < \varepsilon$ which is a contradiction. ■

The idea of for every sequence there existing a convergent subsequence is called *sequential compactness*, and it is in general weaker than compactness (that is, it is implied by compactness).

Example 3.1.3:

Notice that if X is the discrete metric space, then every open cover has a Lebesgue number, namely $\varepsilon < 1$. But if X is infinite then the open cover $\{\{x\}\}_{x\in X}$ has no finite subcover. So the existence of a Lebesgue number is not sufficient for X to be compact.

Theorem 3.1.4:

$A \subseteq \mathbb{R}^n$ is compact if and only if A is closed and bounded.

Proof:

If A is compact, then take

$$A \subseteq \bigcup_{a\in A} B_1(a)$$

Then since A is compact there is a subcover where

$$A \subseteq \bigcup_{n=1}^{n} B_1(a_n)$$

this is bounded since we can take the maximum distance between a_ns and add 2 to get a bound for the diameter of A. And we will show later on in more generality that compact sets are bounded in general.
Suppose A is closed and bounded, then let $\{x_m\}_{m=1}^{\infty}$ be a sequence in A then we can show that every element of the vectors of x_n form a bounded sequence and therefore by the completeness of $\mathbb{R}$, for every k there is a convergent subsequence $x^{(k)}_{m_j^{(k)}}$. We can construct these subsequences as subsequences of $m_j^{(k-1)}$, and so we finally get a subsequence $m_j = m_j^{(n)}$ where $x^{(k)}_{m_j}$ converges for every k to some $x^{(k)}$. We know that pointwise convergence in $\mathbb{R}^n$ is equivalent to convergence, so x_{m_j} converges to x (where the indexes of x are $x^{(k)}$). Since A is closed, all its limit points are in A, so $x \in A$ and thus x_n has a convergent subsequence in A, so A is compact. ■

3.2 Topologies

The following definition is a generalization of the concept of open sets generated by metrics:

Definition 3.2.1:

A **topological space** is a pair (X, τ) where X is a set and τ is a family of subsets of X where

(1) $\varnothing, X \in \tau$

(2) If $\{\mathcal{U}_\lambda\}_{\lambda\in\Lambda}$ is an arbitrary family of sets in τ then $\bigcup_{\lambda\in\Lambda} \mathcal{U}_\lambda \in \tau$

(3) If $\{\mathcal{U}_n\}_{n=1}^{N}$ is a finite family of sets in τ then $\bigcap_{n=1}^{N} \mathcal{U}_n \in \tau$

τ is called a **topology** on X, and a set $\mathcal{U} \in \tau$ is called a **open set**.

Example 3.2.2:

If M is a metric space, then the set of all open sets generated by its metric is a topology on M.

Example 3.2.3:

Take an arbitrary set X and the topology $\tau = \{\varnothing, X\}$. This is called the **trivial topology** on X. No metric can generate this if X is non-empty and not a singleton since this would imply that for every $x \in X$, every open ball around x is X. But if we take another $y \in X$ and take $r < \rho(x, y)$ then $B_r(x)$ does not contain y and therefore is not X.
Thus not every topology can be generated by a metric, but every metric generates a topology.

Definition 3.2.4:

A topological space (X, τ) is called **metrizable** if there exists a metric ρ on X which generates τ.

By the example above, not every topology is metrizable.

Example 3.2.5:

Given an arbitrary set X, the topology $\tau = \mathcal{P}(X)$ is called the **discrete topology** on X. This topology is metrizable, as it is generated by the discrete metric on X.

Definition 3.2.6:

If (X, τ_1) and (Y, τ_2) are two topological spaces, then a function $f: X \longrightarrow Y$ is **continuous** if for every $\mathcal{U} \in \tau_2$, $f^{-1}(\mathcal{U}) \in \tau_1$. That is, the preimage of every open set is open.

Definition 3.2.7:

If (X, τ_1) and (Y, τ_2) are two topological spaces, then a function $f: X \longrightarrow Y$ is **continuous** at $a \in X$ if and only if for every neighborhood $\mathcal{U} \in \tau_2$ of $f(a)$, there exists a neighborhood $\mathcal{O} \in \tau_1$ of a such that $f(\mathcal{O}) \subseteq \mathcal{U}$.

Notice that if f is continuous at every $a \in X$ and if $\mathcal{U} \in \tau_2$ then for every $a \in f^{-1}(\mathcal{U})$, there is a neighborhood $\mathcal{O}_a \in \tau_1$ such that $f(\mathcal{O}_a) \subseteq \mathcal{U}$. And since

$$f^{-1}(\mathcal{U}) = \bigcup_{a \in f^{-1}(\mathcal{U})} \mathcal{O}_a$$

so the preimage of $\mathcal{U}$ is open as the union of open sets.
And if f is continuous and $a \in X$ let $\mathcal{U}$ be a neighborhood of $f(a)$ then $\mathcal{O} = f^{-1}(\mathcal{U})$ is a neighborhood of a and $f(\mathcal{O}) \subseteq \mathcal{U}$. So f is continuous if and only if it is continuous at every $a \in X$.

Proposition 3.2.8:

Every constant function is continuous.

Proof:

Let $f(x) = a \in Y$ be a constant function. Let $\mathcal{U}$ be an open set in the codomain Y. If $a \in \mathcal{U}$ then $f^{-1}(\mathcal{U}) = X$ which is open, and if $a \notin \mathcal{U}$ then $f^{-1}(\mathcal{U}) = \varnothing$ which is open. So the preimage of every open set (and in fact every set) is open, so f is continuous. ■

Proposition 3.2.9:

Suppose X, Y, and Z are topological spaces where $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$ are continuous, then so is $g \circ f: X \longrightarrow Z$.

Proof:

Suppose $\mathcal{U} \subseteq Z$ is open, then $(g \circ f)^{-1}(\mathcal{U}) = f^{-1}\big(g^{-1}(\mathcal{U})\big)$ (recall that these are preimages) is open since $g^{-1}(\mathcal{U})$ is open. So the preimage of every open set is open as required. ■

Definition 3.2.10:

Suppose (X, τ) is a topological space, and $A \subseteq X$ is an arbitrary subset. We denote

$$\tau_A = \{\mathcal{U} \cap A \mid \mathcal{U} \in \tau\}$$

then (A, τ_A) is a topological space and τ_A is called the **subspace topology** of A.

τ_A is indeed a topology on A:

(1) It contains $\varnothing = \varnothing \cap A$ and $A = X \cap A$.

(2) If $\{\mathcal{U}_\lambda \cap A\}_{\lambda\in\Lambda}$ are open sets in τ_A then

$$\bigcup_{\lambda\in\Lambda} \mathcal{U}_\lambda \cap A = \left(\bigcup_{\lambda\in\Lambda} \mathcal{U}\right) \cap A$$

is also in τ_A since arbitrary unions of open sets are open.

(3) If $\{\mathcal{U}_n \cap A\}_{n=1}^N$ are open sets in τ_A then

$$\bigcap_{n=1}^{N} \mathcal{U}_n \cap A = \left(\bigcap_{n=1}^{N} \mathcal{U}_n\right) \cap A$$

is also in τ_A as finite intersections of open sets are open.

Notice that if $A \subseteq B \subseteq X$, τ_A generated from τ_X is equal to τ'_A generated from τ_B since

$$\tau'_A = \{\mathcal{U} \cap A \mid \mathcal{U} \in \tau_B\} = \{\mathcal{U} \cap B \cap A \mid \mathcal{U} \in \tau_X\} = \{\mathcal{U} \cap A \mid \mathcal{U} \in \tau_X\} = \tau_A$$

Example 3.2.11:

Suppose $A \subseteq X$ then the **inclusion map** $\iota\colon A \longrightarrow X$ is defined by $\iota(x) = x$.
If (X, τ) is a topology, then ι is a continuous mapping from (A, τ_A) to (X, τ). Let $\mathcal{U} \in \tau$ be open, then

$$\iota^{-1}(\mathcal{U}) = \{x \in A \mid \iota(x) \in \mathcal{U}\} = \{x \in A \mid x \in \mathcal{U}\} = \mathcal{U} \cap A$$

which is in τ_A. Thus ι is continuous.

Notice that if τ' is a topology on A such that ι is continuous, then we showed that $\mathcal{U} \cap A \in \tau'$ by above. Thus $\tau_A \subseteq \tau'$, and so τ_A is the minimal topology on A for which ι is continuous. It may not be the only topology since the discrete topology $\tau' = \mathcal{P}(A)$ also creates a topology where ι is continuous (in general, every mapping where the domain has the discrete topology is continuous).

Proposition 3.2.12:

If $f\colon X \longrightarrow Y$ is continuous and $A \subseteq X$ then $f\big|_A\colon A \longrightarrow Y$ is continuous as well.

This is trivial since $f\big|_A = f \circ \iota_A$ (ι_A is the inclusion mapping from A to X) and the composition of continuous functions is continuous.

Proposition 3.2.13:

Suppose $f\colon X \longrightarrow Y$ is continuous and $f(X) \subseteq B \subseteq Y$, and let $\tilde{f}\colon X \longrightarrow B$ be the generated function, then $\tilde{f}$ is continuous.

The proof is simple, let $\mathcal{U} \cap B \in \tau_B$ then $\tilde{f}^{-1}(\mathcal{U} \cap B) = f^{-1}(\mathcal{U} \cap B) = f^{-1}(\mathcal{U}) \cap f^{-1}(B) = f^{-1}(\mathcal{U})$ since the preimage of B is X. Since f is continuous, this is open. So the preimage of every open set in τ_B is open and so $\tilde{f}$ is continuous as required.