Algebraic Topology II

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1 Singular Homology

1.1 Chain Complexes

We begin by defining a *chain complex*. A chain complex is a sequence of Abelian groups with homomorphisms between them:

$$
\begin{tikzcd}
\cdots \arrow[r, dashed] & C_{n+1} \arrow[r, "\partial_{n+1}"] & C_n \arrow[r, "\partial_n"] & C_{n-1} \arrow[r, dashed] & \cdots \arrow[r, dashed] & C_0 \arrow[r] & 0
\end{tikzcd}
$$

such that for every n, $\partial_n \circ \partial_{n+1} = 0$, in other words $\mathrm{Im}\partial_{n+1} \subseteq \ker \partial_n$. Define $Z_n = \ker \partial_n$, and its elements will be called *n-dimensional cycles*. And define $B_n = \mathrm{Im}\partial_{n+1}$, its elements will be called *boundaries*. Elements of the groups C_n will be called *n-dimensional chains*.

We now want to define a category of chain complexes. To do so we must define morphisms between chain complexes. So suppose we have two chain complexes $\mathscr{C} = \{C_n, \partial_n\}$ and $\mathscr{D} = \{D_n, \partial'_n\}$. We define a morphism from $\mathscr{C}$ to $\mathscr{D}$ to be a sequence of homomorphisms $f_n \colon C_n \longrightarrow D_n$ which preserves the structure of the chain. Meaning $\partial'_n \circ f_n = f_{n-1} \circ \partial_n$, in other words the following diagram commutes:

$$
\begin{tikzcd}
\cdots \arrow[r, dashed] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"'] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[r, dashed] \arrow[d, "f_{n-1}"'] & \cdots \arrow[r, dashed] & C_0 \arrow[r] \arrow[d, "f_0"'] & 0 \\
\cdots \arrow[r, dashed] & D_{n+1} \arrow[r, "\partial'_{n+1}"] & D_n \arrow[r, "\partial'_n"] & D_{n-1} \arrow[r, dashed] & \cdots \arrow[r, dashed] & D_0 \arrow[r] & 0
\end{tikzcd}
$$

To simplify writing, we will write $\partial \circ f = f \circ \partial$, which f and which ∂ is being referred to will be understood from context.

The composition of two morphisms $\{f_n\} \colon \mathscr{C} \longrightarrow \mathscr{D}$ and $\{g_n\} \colon \mathscr{D} \longrightarrow \mathscr{E}$ is defined to be $\{g_n \circ f_n\} \colon \mathscr{C} \longrightarrow \mathscr{E}$. This is indeed a morphism:

$$\partial \circ f \circ g = f \circ \partial \circ g = f \circ g \circ \partial$$

And then this implies that the identity morphism is just $\mathrm{Id}_{\mathscr{C}} = \{\mathrm{Id}_{\mathrm{C_n}}\} \colon \mathscr{C} \longrightarrow \mathscr{C}$, as if $\{f_n\} \colon \mathscr{C} \longrightarrow \mathscr{D}$ then

$$\{f_n\} \circ \mathrm{Id}_{\mathscr{C}} = \{f_n \circ \mathrm{Id}_{C_n}\} = \{f_n\}, \qquad \mathrm{Id}_{\mathscr{D}} \circ \{f_n\} = \{\mathrm{Id}_{D_n} \circ f_n\} = \{f_n\}$$

Associativity is clear, so **Comp**, the category of chain complexes, is indeed a category.

Now recall that by definition $\partial_n \circ \partial_{n+1} = 0$, meaning

$$B_n \subseteq Z_n \subseteq C_n$$

Since these groups are all Abelian, they are normal in one another, so let us define the *nth homology group* of a chain complex $\mathscr{C}$ as

$$H_n(\mathscr{C}) := {Z_n}\big/{B_n} = {\ker \partial_n}\big/{\mathrm{Im}\partial_{n+1}}$$

1.1 Proposition

A chain complex morphism $\{f_n\} \colon \mathscr{C} \longrightarrow \mathscr{D}$ maps cycles to cycles and boundaries to boundaries.

Proof: let $z \in C_n$ be a cycle, i.e. $\partial z = 0$, but then $f(z)$ is a cycle since $\partial f(z) = f(\partial z) = f(0) = 0$. And let $b \in C_n$ be a boundary, so there exists an $a \in C_{n+1}$ such that $b = \partial a$. Then $f(b) = f\partial(a) = \partial f(a) = \partial b$, so $f(b)$ is a boundary as well. ■

This means that if $\{f_n\} \colon \mathscr{C} \longrightarrow \mathscr{D}$ is a morphism of chain complexes, $\{f_n\} \colon Z_n(\mathscr{C}) \longrightarrow Z_n(\mathscr{D})$ is well-defined, and so we have that

$$
\begin{tikzcd}
Z_n(\mathscr{C}) \arrow[r] \arrow[d] \arrow[dr, bend left, blue] & Z_n(\mathscr{D}) \arrow[d] \\
H_n(\mathscr{C}) & H_n(\mathscr{D})
\end{tikzcd}
$$

Where the blue arrow ψ is just the quotient map composed with f_n. This induces a group morphism

$$H_n(\{f_n\}) = f_* \colon H_n(\mathscr{C}) \longrightarrow H_n(\mathscr{D})$$

since we can define $f_*([z]) = \psi(z)$ since if $[z] = [z']$ then $z - z' \in B_n(\mathscr{C})$ and so $f(z - z') \in B_n(\mathscr{D})$ and thus the quotient of $f(z - z')$ is just 0, so $\psi(z) = \psi(z')$.

We now claim that H_n is a functor from the category of chain complexes **Comp** to the category of Abelian groups **Ab**. Now suppose $\{f_n\}\colon \mathscr{C} \longrightarrow \mathscr{D}$ and $\{g_n\}\colon \mathscr{D} \longrightarrow \mathscr{E}$ are chain complex morphisms, then the following diagram commutes

$$\begin{tikzcd}
Z_n(\mathscr{C}) \arrow[r, "f"] \arrow[d] & Z_n(\mathscr{D}) \arrow[r, "g"] \arrow[d] & Z_n(\mathscr{E}) \arrow[d] \\
H_n(\mathscr{C}) \arrow[r, "f_*"'] & H_n(\mathscr{D}) \arrow[r, "g_*"'] & H_n(\mathscr{E})
\end{tikzcd}$$

And so $(g \circ f)_* = g_* \circ f_*$, and it is easily verified that $\mathrm{id}_* = \mathrm{id}$ so H_n is a functor **Comp** $\longrightarrow$ **Ab** (the category of Abelian groups).

1.2 Singular Complex

We now define a functor from **Top** to **Comp**.

1.1 Definition

Let B be a set, then define the **free Abelian group** over B to be

$$\mathrm{FA}(B) = \bigoplus_{b \in B} \mathbb{Z} = \{\varphi\colon B \longrightarrow \mathbb{Z} \mid \varphi(b) \neq 0 \text{ for only finitely many } b \in B\}$$

Note then that there is a correspondence between B and $\mathrm{FA}(B)$: $b \leftrightarrow \varphi_b$ where

$$\varphi_b(x) = \begin{cases} 1 & x = b \\ 0 & x \neq b \end{cases}$$

so we can identify b with φ_b, and it is easy to see that every element of $\mathrm{FA}(B)$ can be written as $\sum_{i=1}^k n_i \varphi_{b_i}$, abusing notation $\sum_{i=1}^k n b_i$ and such a representation is unique.

Notice that if B is a set, G an Abelian group, and $g\colon B \longrightarrow G$ a function, then there exists a unique group homomorphism $L\colon \mathrm{FA}(B) \longrightarrow G$ which extends g. This is defined by

$$L\colon \sum_{i=1}^k n_i b_i \longmapsto \sum_{i=1}^k n_i g(b_i)$$

1.2 Definition

The **n-dimensional simplex** is defined to be

$$\Delta^n := \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; x_i \geq 0, \sum_{i=0}^n x_i = 1 \right\}$$

Δ^n has $n+1$ faces, and is homeomorphic to D^n.

1.3 Definition

Let X be a topological space, then an **n-dimensional singular simplex** in X is a morphism (in the category of topological spaces; a continuous map) $\Delta^n \longrightarrow X$. Define $S_n(x)$ to be the set of all n-dimensional singular simplexes in X, and define $C_n(X) = \mathrm{FA}(S_n(x))$.

We now want to define a chain complex on the sequence $C_n(X)$.

Let us define a set of maps $\tau_i^n\colon \Delta^{n-1} \longrightarrow \Delta^n$ for $0 \leq i \leq n$ which maps

$$\tau_i^n\colon (x_0, \ldots, x_{n-1}) \mapsto (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{n-1})$$

This is a well-defined continuous map, and geometrically it maps Δ^{n-1} to one of the faces of Δ^n.

Let $\sigma \in S_n(x)$, then let us define

$$\partial(\sigma) := \sum_{i=0}^{n}(-1)^i \sigma \circ \tau_i^n$$

Note that the composition is well-defined since $\Delta^{n-1} \xrightarrow{\tau_i^n} \Delta^n \xrightarrow{\sigma} X$, meaning $\sigma \circ \tau_i^n$ is an $n-1$-dimensional singular simplex. Thus ∂ can be extended to a map $\partial\colon C_n(X) = \mathrm{FA}(S_n(X)) \longrightarrow \mathrm{FA}(S_{n-1}(X)) = C_{n-1}(X)$ Notice that

$$\partial_{n-1}(\partial_n\sigma) = \partial_{n-1}\left(\sum_{i=0}^{n}(-1)^i \sigma \circ \tau_i^n\right) = \sum_{i=0}^{n}(-1)^i \partial_{n-1}(\sigma \circ \tau_i^n) = \sum_{i=0}^{n}\sum_{j=0}^{n-1}(-1)^{i+j}\sigma \circ \tau_i^n \circ \tau_j^{n-1}$$

Notice that $\tau_i^n \circ \tau_j^{n-1} = \tau_j^n \circ \tau_{i-1}^{n-1}$ which can be verified from its definition, but the first has a sign of $(-1)^{i+j}$ in the sum and the second has $-(-1)^{i+j}$. And so the sum is zero.

Thus we have defined a chain complex on $C_n(X)$, let us denote it by $\mathscr{C}(X)$, this is the first step in defining the functor. Next we must define the correspondence between morphisms.

Let $f\colon X \longrightarrow Y$ be a continuous map between topological spaces. Let us define $f_\sharp\colon \mathscr{C}(X) \longrightarrow \mathscr{C}(Y)$. First we define it for $\sigma \in S_n(X)$ by $f_\sharp(\sigma) = f \circ \sigma$. Since $\sigma\colon \Delta^n \longrightarrow X$ is continuous, so is $f \circ \sigma\colon \Delta^n \longrightarrow Y$ and so $f_\sharp$ is well-defined on the generators of $C_n(X)$. This can be extended by linearity to $f_\sharp\colon C_n(X) \longrightarrow C_n(Y)$. Notice that we ignore the subscripts and superscripts $(f_\sharp)_n^X$ for brevity and readability.

Now we must verify that this is a morphism of chain complexes, i.e. that $\partial f_\sharp = f_\sharp \partial$. So

$$f_\sharp\partial\sigma = f_\sharp\left(\sum_{i=0}^{n}(-1)^i \sigma \circ \tau_i^n\right) = \sum_{i=0}^{n}(-1)^i f_\sharp(\sigma \circ \tau_i^n) = \sum_{i=0}^{n}(-1)^i f \circ \sigma \circ \tau_i^n = \sum_{i=0}^{n}(-1)^i (f \circ \sigma) \circ \tau_i^n = \partial f_\sharp \sigma$$

and since this holds for generators, by linearity it holds for all $C_n(X)$. Thus $f_\sharp$ is indeed a morphism of chain complexes.

Thus we have defined a functor **Top** $\longrightarrow$ **Comp**.

1.3 Singular Homology

We have two functors **Top** $\longrightarrow$ **Comp** $\longrightarrow$ **Ab**, and so composing them together gives us a functor **Top** $\longrightarrow$ **Ab**. For a topological space X, we will denote its image under this functor as $H_n(X)$, called the *nth homological group* of X. And for a continuous map f, we denote its image as f_* or $H_n(f)$.

Let us compute the homological groups of the trivial space: $X = \{p\}$. Notice that $S_n(X) = \{K_n\}$ where K_n is the constant map $\Delta^n \longrightarrow \{p\}$, and so $C_n(X) = \mathbb{Z}$. We want to now compute what the boundary operators are, so

$$\partial K_n = \sum_{i=0}^{n}(-1)^i K_n \circ \tau_i^n$$

but $K_n \circ \tau_i^n$ is a morphism $\Delta^{n-1} \longrightarrow \{p\}$ meaning it is equal to K_{n-1}, thus $\partial K_n = \left(\sum_{i=0}^{n}(-1)^i\right)K_{n-1}$. For n even this is then K_{n-1} (or 1), and 0 for n odd. This means that either $\ker\partial = 0$ or $\mathrm{Im}\partial = \mathbb{Z}$, thus $H_n = 0$ for $n > 0$. For $n = 0$, we have that $\partial_0\colon \mathbb{Z} \longrightarrow 0$ and so its kernel is $\mathbb{Z}$, but ∂_1 is trivial and so its image is 0. Thus $H_0 = \mathbb{Z}$.

So we have shown

1.1 Proposition

Let $X = \{p\}$ be the trivial topological space, then its homological groups are

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n > 0 \end{cases}$$

1.2 Proposition

Let X be path connected, then $H_0(X) \cong \mathbb{Z}$.

Proof: we are concerned with the chain:

$$C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

So first let us understand $C_0(X)$, this is generated by $S_0(X)$, all the maps $\Delta^0 \longrightarrow X$ which are just all the points in X. And $S_1(X)$ is generated by all the maps $I \cong \Delta^1 \longrightarrow X$, so all the paths in X. The boundary of a 1-simplex is then

$$\partial_1 \sigma = \sigma(1) - \sigma(0)$$

and thus $B_1(X) = \mathrm{Im}\partial_1$ is generated by elements of the form $a - b$ where there exists a path between a and b. Since X is path-connected, this means that $B_1(X)$ is generated by $a - b$ for $a, b \in X$. Now, the subgroup generated by this is $\{\sum n_i p_i \mid p_i \in X, \sum n_i = 0\}$.

And now ∂_0's kernel is just $C_0(X)$ which is simply the free group generated by X. Thus

$$H_0(X) = \left\{\sum n_i p_i\right\} \Big/ \left\{\sum n_i p_i \,\middle|\, \sum n_i = 0\right\}$$

This is isomorphic to $\mathbb{Z}$ since we can define $\varphi \colon C_0(X) \longrightarrow \mathbb{Z}$ by $\sum n_i p_i \mapsto \sum n_i$ and this is a group homomorphism whose image is $\mathbb{Z}$ and whose kernel is all the points $\sum n_i p_i$ where $\sum n_i = 0$. Thus by the isomorphism theorem, $H_0(X) \cong \mathbb{Z}$. ■

1.3 Theorem

Let X be a topological space where $\{A_\alpha\}_{\alpha \in I}$ are its path connected components. Then for every n,

$$H_n(X) \cong \bigoplus_{\alpha \in I} H_n(A_\alpha)$$

Proof: notice that if $\sigma \colon \Delta^n \longrightarrow X$ is an n-simplex, then its image is contained within a path connected component. This is since Δ^n is path-connected, so $\sigma\Delta^n$ must be too. Thus for every $\gamma = \sum n_i \sigma_i \in S_n(X)$ we can write it as $\gamma = \sum \gamma_i$ for $\gamma_i \in S_n(A_i)$. And so $C_n(X) = \bigoplus_{\alpha \in I} C_n(A_\alpha)$.

Notice that γ is a cycle iff every γ_i is a cycle, since $\partial\gamma = \sum \partial\gamma_i$ and this is an element of a direct sum, so it is zero iff $\partial\gamma_i = 0$. Thus $Z_n(X) = \bigoplus_{\alpha \in I} Z_n(A_\alpha)$. And similarly we see that $B_n(X) = \bigoplus_{\alpha \in I} B_n(A_\alpha)$. Thus $H_n(X) = \bigoplus_{\alpha \in I} H_n(A_\alpha)$. ■

1.4 Corollary

If X is a topological space with $\{A_\alpha\}_{\alpha \in I}$ path connected components, $H_n(X) = \bigoplus_{\alpha \in I} \mathbb{Z}$.