Mathematical Logic

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Ari Feiglin

6.1 Zermelo-Frankel Set Theory

In this lecture we present *Zermelo-Frankel Set Theory*. As a first order language, it contains no functional symbols, no constants, and just the predicate symbol $\in$ ($x \in y$ can be thought of as "x is an element of y"). This language is an identity language, meaning it also has the equality symbol, or it can be thought of as its own predicate symbol $=$. We now present the axioms of the theory:

(1) Axiom of extensionality: $\forall x \forall y\big(x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y)\big)$ (two-sided inclusion)

(2) Axiom of the empty set: $\exists x \forall y \neg(y \in x)$ (there exists an empty set) By the axiom of extensionality, the empty set is unique. We denote the empty set by the constant $\varnothing$ (or 0).

(3) Axiom of pairing: $\forall x \forall y \exists z \forall u\big(u \in z \leftrightarrow (x = u \vee y = u)\big)$ (z is taken to be thought of as the set $\{x, y\}$).

If $x = y$, by this axiom you get a set $z = \{x\}$, meaning a set consisting of just x. So $\varnothing$ is a set by the axiom of the empty set, $\{\varnothing\}$ is a set by the axiom of pairing (for $x = y = \varnothing$), and $\{\varnothing, \{\varnothing\}\}$ is a set by the axiom of pairing yet again. These sets are denoted 0, 1, and 2 respectively.

If we have two elements, x and y, we define $(x, y) = \{\{x\}, \{x, y\}\}$ which exists by the axiom of pairing. And if $(x, y) = (a, b)$ then $\{\{x\}, \{x, y\}\} = \{\{a\}, \{a, b\}\}$ which means that $x = a$ and $y = b$.

(4) Axiom of union: $\forall x \exists y \forall z\big(z \in y \leftrightarrow \exists w(z \in w \wedge w \in x)\big)$ (z is in y if and only if there exists a set $w \in x$ such that $z \in w$). y is denoted by $\cup x$.

By the axiom of union, we can recursively define n by $n + 1 = \{0, 1, \dots, n\}$. We have that $\cup(n + 1) = n$ and $\cup 0 = 0$.

(5) Axiom of power: $\forall x \exists y \forall z\big(z \in y \leftrightarrow \forall w(w \in z \to w \in x)\big)$ (z is in y if and only if when $w \in z$ then $w \in x$). If we define the predicate $\subseteq$ by $x \subseteq y$ if and only if $\forall z(z \in x \to z \in y)$, then the axiom of power can be written as $\forall x \exists y \forall z\big(z \in y \leftrightarrow z \subseteq x\big)$. y is denoted as $\mathcal{P}(x)$, or 2^x.

Notice that $x \in \mathcal{P}(x)$ since $w \in x \to w \in x$ is a tautology. And $\mathcal{P}(\varnothing) = \{\varnothing\}$, $\mathcal{P}(\{\varnothing\}) = \{\{\varnothing\}, \varnothing\}$. So $2^0 = 1$ and $2^1 = 2$ (in general this is true for all n).

Notice that a possible model of the above axioms are the natural numbers where $\in$ is taken to mean $<$. This is is not desirable, as natural numbers do not have some desired properties.

(6) Axiom of infinity: $\exists x\Big(\exists y(y \in x) \wedge \forall y\big(y \in x \to \exists z(y \in z \wedge z \in x)\big)\Big)$. This means that x is non-empty ($\exists y(y \in x)$), and if $y \in x$ then there exists a $z \in x$ such that $y \in z$. For example, if we define $\omega = \{0, 1, 2, \dots, n, \dots\}$, ω satisfies the axiom of infinity.

(7) Axiom of regularity: $\forall x\Big(\exists y(y \in x) \to \exists y\big(y \in x \wedge \forall z(z \notin y \vee z \notin x)\big)\Big)$ (if x is non empty, then there exists a $y \in x$ disjoint from x). Despite not defining intersections yet, we can reformulate this informally as follows: $\forall x \neq \varnothing \exists y \in x(x \cap y \neq \varnothing)$.

Lemma 6.1.1:

$\forall x(x \notin x)$

Proof:

Suppose $x \in x$, then let $y = \{x\}$, then y is non-empty and x intersects with y since $x \in x$ and $x \in y$. Thus y does not satisfy the axiom of regularity. ∎

This proof uses just the axiom of regularity and pairing.

Lemma 6.1.2:

There exists no cycle where $x_1 \in x_2 \in \cdots \in x_n \in x_1$.

Proof:

Suppose it is possible, then let $y = \{x_1, \dots, x_n\}$, which exists as the recursive union of $\{x_i\}$s. Then for every $x_i \in y$, x_i intersects with y since $x_{i-1} \in x_i$ and $x_{i-1} \in y$. ■

(8) Axiom schema of separation: if $\varphi(x, y)$ is a first order formula, then

$$\forall x \forall u \exists z \forall y \big(y \in z \leftrightarrow (y \in u \land \varphi(x, y))\big)$$

meaning that there exists a z such that $y \in z$ if and only if $y \in u$ and $\varphi(x, y)$ is true. z is denoted $\{y \in u \mid \varphi(x, y)\}$.

(9) Axiom schema of replacement: if $\varphi(x, y, u, \dots)$ is a first order formula, then

$$\forall u \Big(\big(\forall x \exists! z \varphi(x, z, u, \dots)\big) \to \Big(\exists y \forall z \big(z \in y \leftrightarrow \exists x (x \in u \land \varphi(x, z, u, \dots))\big)\Big)\Big)$$

in words, if φ is a "mapping", as in for every x there is a unique "image" ($\varphi(x, z, u, \dots)$ is true), then there exists a y such that $z \in y$ if and only if there exists an $x \in u$ such that z is the "image" of x. This y can be thought of as the image of u, and specifically if you have a function definable using this first order theory, the image of any set under this function is also a set.

We haven't yet defined $\exists!$, it means "there exists a unique", and we can define satisfability in the obvious way.

Lemma 6.1.3:

Any chain $x_1 \ni x_2 \ni \cdots \ni x_n \ni x_{n+1} \ni \cdots$ is finite. Meaning there exists an N such that for every $M \geq N$, $x_N = x_M$.

What this lemma implies, informally, is that every set can be thought of being constructed by pairing with empty sets.

Proof:

If there does exist such a chain, let $y = \{x_1, x_2, \dots\}$ then for any x_i, $x_{i+1} \in y$ and $x_{i+1} \in x_i$. This contradicts regularity. ■

This lemma relies on the definability of y. Assuming one can define the natural numbers (which is possible), one can use the axiom schema of replacement in order to define y (mapping $n \mapsto x_n$).

6.2 Ordinals

If X is a set, then X is partially ordered by $\subseteq$ (recall: all sets are sets of sets). X is called an *chain* if it is linearly (or totally) ordered by $\subseteq$ (meaning for every $x_1, x_2 \in X$, $x_1 \subseteq x_2$ or $x_2 \subseteq x_1$). X is called a *well-ordered chain* if it well-ordered by $\subseteq$. All of these notioncs can be formulated in a first-order manner, and it does not pay to repeat it here.

Definition 6.2.1:

An **ordinal** is a set α such that

(1) $\forall \beta (\beta \in \alpha \to \beta \subseteq \alpha)$

(2) α is well-ordered by $\in$

For example, natural numbers are ordinals.

Lemma 6.2.2:

Every element of an ordinal is itself an ordinal.

Proof:

Suppose α be an ordinal, and $x \in \alpha$. Then $x \subseteq \alpha$, so x is well-ordered (since every subset of x is a subset of α and therefore must have a minimum element). If $y \in x$, we must show $y \subseteq x$, meaning if $z \in y$ then $z \in x$. So $y \in x \subseteq \alpha$ so $y \in \alpha$ and therefore $y \subseteq \alpha$, and so $z \in \alpha$. So $z \in y$ and $y \in x$ and $x, y, z \in \alpha$. Since $\in$ is a well-order and therefore a partial order on α, it is transitive, and so $z \in x$ as required (so $y \subseteq x$). ■

Lemma 6.2.3:

If α and β are ordinals, then $\alpha \subseteq \beta$ if and only if $\alpha \in \beta$ or $\alpha = \beta$.

Proof:

If $\alpha \in \beta$ or $\alpha = \beta$ this is trivial, by definition (or by triviallity). To show the converse, suppose $\alpha \subseteq \beta$ and $\alpha \neq \beta$. We can look at the set $\beta \setminus \alpha$ (which exists by the axiom of separation), let $\gamma \in \beta \setminus \alpha$ be the minimum in this set. We will attempt to show that $\gamma = \alpha$. Let $\delta \in \gamma$, then $\delta \in \beta$ since β is an ordinal, and since γ is a minimum, $\delta \notin \beta \setminus \alpha$ so $\delta \in \alpha$. So we have inclusion in one direction. Suppose now that $\delta \in \alpha$, so $\delta \in \beta$ and since $\gamma \in \beta$ and β is well-ordered. So either $\gamma \in \delta$, $\gamma = \delta$, or $\delta \in \gamma$. If either of the first two are true, then since $\delta \in \alpha$, we have $\gamma \in \alpha$, which is a contradiction to its definition. So $\delta \in \gamma$ and therefore $\alpha = \gamma$, and specifically $\alpha \in \beta$. ■

Lemma 6.2.4:

Every ordinal α is a well-ordered chain by $\subseteq$.

Proof:

Let $\beta, \gamma \in \alpha$ then β and γ are ordinals and $\beta, \gamma \subseteq \alpha$. We must show that $\beta \subseteq \gamma$ or $\gamma \subseteq \beta$. Since α is well-ordered by $\in$, either $\beta \in \gamma$ or $\gamma \in \beta$, and therefore $\beta \subseteq \gamma$ or $\gamma \subseteq \beta$.

And if $x \subseteq \alpha$, then there is a minimum $\beta \in x$ relative to $\in$. And so for every $\gamma \in x$, $\beta \in \gamma$ and so $\beta \subseteq \gamma$, so β is also a minimum relative to $\subseteq$. ■

Lemma 6.2.5:

(1) Every set of ordinals is well-ordered by $\in$.

(2) Every set of ordinals is a well-ordered chain.

Proof:

(1) Suppose $\varnothing \neq Y \subseteq X$. Let $\alpha = \bigcap_{\beta \in Y} \beta$, then this is an ordinal since if $\gamma \in \alpha$ then $\gamma \in \beta$ for $\beta \in Y$ and so $\gamma \subseteq \beta$, so $\gamma \subseteq \alpha$. And if $Z \subseteq \alpha$, $Z \subseteq \beta$ for every $\beta \in Y$ and so there exists a minimum element.

So α is an ordinal, and we claim it is the minimum element of Y. First we must show that $\alpha \in Y$, let $\alpha^+ = \alpha \cup \{\alpha\}$. So for every $\beta \in Y$, either $\beta \in \alpha^+$ or $\alpha^+ \in \beta$. If $\alpha^+ \in \beta$ then $\alpha^+ \subseteq \beta$ for every $\beta \in Y$ and so $\alpha^+ \subseteq \alpha$ since α is the intersection, which is a contradiction. So there exists a $\beta \in \alpha^+$, so $\beta \in \alpha$ or $\beta = \alpha$. If $\beta \in \alpha$ then $\beta \subseteq \alpha$, but $\alpha \subseteq \beta$ so $\alpha = \beta$ (which is a contradiction), so $\alpha = \beta$ and therefore $\alpha \in Y$.

Now suppose $\beta \in Y$, since $\alpha \subseteq \beta$, either $\alpha = \beta$ or $\alpha \in \beta$, as required.

(2) Let $\alpha \neq \beta \in X$ and suppose that β is not a subset of α, we will show $\alpha \subseteq \beta$. Let γ be the minimum of $\beta \setminus \alpha$, then if $\delta \in \gamma$, $\delta \in \beta$ and since it is smaller than γ, $\delta \in \alpha$. So $\gamma \subseteq \alpha$ and so $\gamma \in \alpha$ or $\gamma = \alpha$, but $\gamma \in \beta \setminus \alpha$, so $\gamma = \alpha$ and therefore $\alpha \in \beta$. So X is totally ordered.

Now we show that X is well-ordered, suppose $\varnothing \neq Y \subseteq X$. Let $\alpha \in Y$. If $\alpha \cap Y = \varnothing$ then let $\beta \in Y$, then $\beta \notin \alpha$ since α and Y are disjoint, but since X is totally-ordered, $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$. But if $\beta \subseteq \alpha$ then $\beta \in \alpha$ or $\beta = \alpha$, so $\beta = \alpha$ or $\alpha \subseteq \beta$, and so α is the minimum of Y. Otherwise, if $\alpha \cap Y$ is a subset of an ordinal (α), and so it has a smallest element γ with respect to $\in$. Then if $\beta \in \gamma \cap Y$ then $\beta \in \alpha \cap Y$ and $\beta \in \gamma$ which contradicts γ's minimumness. So $\gamma \cap Y = \varnothing$ and so from above, γ is the minimum. ■

Definition 6.2.6:

If α is an ordinal, we define the **successor** of α to be $\alpha + 1 = \alpha \cup \{\alpha\}$. If α is not a successor for any ordinal β, it is called a **limit ordinal**.

0 and ω are examples of limit ordinals.