# Complex Functions

*Assignment 6*

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**Exercise 6.1:**

Find the Taylor expansion of $f(z)=\frac{1}{z}$ about $z=1+i$.

Since $f(z)=z^{-1}$, we have that $f^{(k)}(z)=(-1)^k\cdot k!\cdot z^{-k-1}$. This is true by induction: for $k=0$ this is $f^{(0)}(z)=(-1)^0\cdot 0!\cdot z^{-1}=z^{-1}=f(z)$, and

$$f^{(k+1)}(z)=(-1)^k k!\cdot(-k-1)z^{-k-2}=(-1)^{k+1}(k+1)!\cdot z^{-(k+1)-1}$$

Since the components of the Taylor series are $\frac{f^{(k)}(z_0)}{k!}$, we have that the components are $(-1)^k\cdot(1+i)^{-k-1}$. Since $1+i=\sqrt{2}e^{\frac{\pi}{4}i}$, we have

$$f(z)=\sum_{k=0}^{\infty}(-1)^k 2^{-\frac{k+1}{2}}\cdot e^{-\frac{\pi}{4}(k+1)i}\cdot(z-1-i)^k$$

**Exercise 6.2:**

Find the Taylor expansion of $f(z)=\frac{1}{1-z-2z^2}$ about 0.

Since $1-z-2z^2=(1+z)(1-2z)$, by partial fraction decomposition

$$\frac{1}{1-z-2z^2}=\frac{A}{1+z}+\frac{B}{1-2z}$$

and so

$$A+B=1,\quad B-2A=0$$

thus $B=2A$ and so $A=\frac{1}{3}$ and $B=\frac{2}{3}$.
Furthermore, we know for $|w|<1$,

$$\sum_{k=0}^{\infty}w^k=\frac{1}{1-w}$$

and thus

$$\frac{1}{1+z}=\sum_{k=0}^{\infty}(-1)^k z^k,\qquad \frac{1}{1-2z}=\sum_{k=0}^{\infty}2^k z^k$$

So we have that

$$\frac{1}{1-z-2z^2}=\frac{1}{3}\cdot\frac{1}{1+z}+\frac{2}{3}\cdot\frac{1}{1-2z}=\sum_{k=0}^{\infty}\frac{1}{3}\Big((-1)^k+2^{k+1}\Big)z^k$$

**Exercise 6.3:**

Show that if $f$ is analytic in the closed disk $|z|\leq 1$, then there exists an $n\in\mathbb{N}$ such that

$$f\left(\frac{1}{n}\right)\neq\frac{1}{n+1}$$

Suppose the contrary, that

$$f\left(\frac{1}{n}\right)=\frac{1}{n+1}$$

then define

$$g(z) = \frac{z}{z+1}$$

which is analytic in $D_1(0)$. Notice that for $z_n = \frac{1}{n}$,

$$g(z_n) = \frac{\frac{1}{n}}{\frac{1}{n}+1} = \frac{1}{n+1} = f(z_n)$$

And since $z_n \to 0$, by the uniqueness theorem this means that $f(z) = g(z)$ on $D_1(0)$.
But then

$$\lim_{z\to -1} f(z) = \lim_{z\to -1} \frac{z}{z+1}$$

is undefined, which contradicts $f$ being analytic and thus continuous on the *closed* disk $|z| \leq 1$.

**Exercise 6.4:**

Show that if an analytic function $f$ agrees with $\tan x$ for $0 \leq x \leq 1$, then there is no solution to $f(z) = i$. Can $f$ be entire?

Let us define $z_n = \frac{1}{n}$, then since $f(z_n) = \tan(z_n)$ this means that by the uniqueness theorem, $f(z) = \tan(z)$ whenever they are defined. Since $\tan(z)$ is defined whenever $\cos(z) \neq 0$, which is only when $z = \frac{\pi}{2} + \pi k$, all singularities are isolated. Thus if $f(z) = i$ then we can take $z_n \to z$ and $\tan(z_n) = f(z_n)$ and since tan is continuous, $\tan(z) = f(z) = i$. But thus would mean $\sin(z) = i\cos(z)$, or $-i\sin(z) = \cos(z)$. Thus

$$-\frac{e^{iz} - e^{-iz}}{2} = \frac{e^{iz} + e^{-iz}}{2} \implies e^{iz} = 0$$

which is impossible.
And if $f$ were entire then let $z_n \to \frac{\pi}{2}$. $f(z_n) = \tan(z_n)$, and so $f(z_n)$ would not converge, which contradicts $f$ being entire.

**Exercise 6.5:**

Suppose $f$ is an entire function where $|f(z)| \geq |z|^N$ when $z$ is large enough. Show that $f$ must be a polynomial of degree at least $N$.

Let us notice that

$$\lim_{z\to\infty} f(z) = \infty$$

Since eventually $|f(z)| \geq |z|^N$ and the limit of $|z|^N$ is infinity. Thus we showed in lecture that $f$ is a polynomial. Suppose

$$f(z) = \sum_{k=0}^{M} a_k z^k$$

Thus we have

$$\left|\frac{f(z)}{z^N}\right| \leq \sum_{k=0}^{M} |a_k| z^{k-N}$$

Now suppose $M < N$, then for each $k$, $k - N < 0$ and so $\left|z^{k-N}\right| \xrightarrow[z\to\infty]{} 0$, meaning

$$\lim_{z\to\infty} \left|\frac{f(z)}{z^N}\right| = 0$$

but

$$\left|\frac{f(z)}{z^N}\right| \geq 1$$

for sufficiently large $z$, so the limit either would not exist or would be greater than 1 (inclusive), in contradiction. Thus $M \geq N$, ie. $f$ is a polynomial whose degree is at least $N$.

**Exercise 6.6:**

Suppose $P_n(z) = a_0 + a_1 z + \cdots + a_n z^n$ is bounded by 1 in the disk $|z| \leq 1$. Show that $|P(z)| \leq |z|^n$ when $1 < |z|$.

By question 5 from the previous homework, since $P_n$ is bounded by 1 on $D_1(0)$, we have $|a_k| \leq 1$ for each $k$. Let us define

$$Q(z) = \frac{P_n(z)}{z^n}$$

for $|z| > 1$. $Q(z)$ is obviously analytic on its domain. Now let us focus on $Q(z)$ in the ring $1 < |z| < R$. Since $Q(z)$ is analytic, it takes its maxima on the boundary of this ring, ie. when $|z| = 1$ or $|z| = R$. When $|z| = 1$ we have

$$|Q(z)| \leq |P_n(z)| \leq 1$$

since $P_n$ is bounded by 1 on the closed disk $|z| \leq 1$ (including $|z| = 1$). And when $|z| = R$ then

$$|Q(z)| = \frac{|P_n(z)|}{|R|^n}$$

But notice that

$$|P_n(z)| \leq \sum_{k=0}^{n} |a_k||z|^k \leq \sum_{k=0}^{n} R^k = \frac{R^{n+1} - 1}{R - 1}$$

and thus

$$|Q(z)| \leq \frac{R^{n+1} - 1}{R^{n+1} - R^n} \leq \frac{R^{n+1}}{R^{n+1} - R^n} = \frac{R}{R - 1}$$

Thus we have that for $1 < |z| < R$,

$$|P_n(z)| \leq |z|^n \cdot \frac{R}{R - 1}$$

Now let $|z| > 1$, then for every $R > |z|$ we have the above equality, so let us take $R \to \infty$ and we have that

$$|P_n(z)| \leq |z|^n$$

as required.