

Mathematical Logic

*A summary of “A Concise Introduction to Mathematical Logic”, W. Rautenberg
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1 Propositional Logic

1.1 Semantics of Propositional Logic

Propositional logic is the study of logic removed from interpretation of individual variables and context. I will assume that the reader already has experience with propositional logic, as this is something an undergraduate will cover in one of their first courses. While this subsection will focus mainly on the semantics of propositional logic, we will begin by defining its *syntax*,

1.1.1 Definition

Let PV be an arbitrary set of **propositional variables** (which are regarded as arbitrary symbols). **Propositional formulas** are formulas defined recursively by the following rules,

- (1) Propositional variables in PV are formulas, called **prime or atomic** formulas.
- (2) If α and β are formulas, then so are $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, and $\neg\alpha$. $(\alpha \wedge \beta)$ is called the **conjunction** of α and β , $(\alpha \vee \beta)$ their **disjunction**, and $\neg\alpha$ the **negation** of α .

The set of all the formulas constructed in this manner is denoted \mathcal{F} .

We can generalize this definition; instead of utilizing only the symbols \wedge and \vee , we can take a general *logical signature* σ consisting of logical connectives of differing arities. We then recursively define σ -formulas as following: if c is an n -ary logical connective in σ , and $\alpha_1, \dots, \alpha_n$ are formulas, then so is

$$(c\alpha_1, \dots, \alpha_n)$$

Alternatively, if we only consider binary and unary connectives, then if c is a unary connective, we define $c\alpha$ to be a formula, and if \circ is a binary connective, then $(\alpha \circ \beta)$ is a formula. But we don't have much need for such generalizations, as $\{\wedge, \vee, \neg\}$ is complete, in the sense that all connectives can be defined using them. This is a fact we will discuss soon.

We can define other connectives, for example \rightarrow and \leftrightarrow are used as shorthands:

$$(\alpha \rightarrow \beta) := \neg(\alpha \wedge \neg\beta), \quad (\alpha \leftrightarrow \beta) := ((\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha))$$

We similarly define symbols for false and true:

$$\perp := (p_1 \wedge \neg p_1), \quad \top = \neg\perp$$

For readability, we will use the following conventions when writing formulas (this is not a change to the definition of a formula, rather conventions for writing them in order to enhance readability)

- (1) We will omit the outermost parentheses when writing formulas, if there are any.
- (2) The order of operations for logical connectives is as follows, from first to last: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$.
- (3) We associate \rightarrow from the right, meaning $\alpha \rightarrow \beta \rightarrow \gamma$ is to be read as $\alpha \rightarrow (\beta \rightarrow \gamma)$. All other connectives associate from the left, for example $\alpha \wedge \beta \wedge \gamma$ is to be read as $(\alpha \wedge \beta) \wedge \gamma$.
- (4) Instead of writing $\alpha_0 \wedge \alpha_1 \wedge \dots \wedge \alpha_n$, we write $\bigwedge_{i=0}^n \alpha_i$, similar for \vee .

Since formulas are constructed in a recursive manner, most of our proofs about them are inductive.

1.1.2 Principle (Principle of Formula Induction)

Let \mathcal{E} be a property of strings which satisfies the following conditions:

- (1) $\mathcal{E}\pi$ for all prime formulas π ,
- (2) If $\mathcal{E}\alpha$ and $\mathcal{E}\beta$, then $\mathcal{E}(\alpha \wedge \beta)$, $\mathcal{E}(\alpha \vee \beta)$, and $\mathcal{E}\neg\alpha$ for all formulas $\alpha, \beta \in \mathcal{F}$.

Then $\mathcal{E}\varphi$ is true for all formulas φ .

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An example of this is that every formula $\varphi \in \mathcal{F}$ is either prime, or of one of the following forms

$$\varphi = \neg\alpha, \quad \varphi = (\alpha \wedge \beta), \quad \varphi = (\alpha \vee \beta)$$

The proof of this is straightforward: let \mathcal{E} be this property. Then trivially, $\mathcal{E}\pi$ for all prime formulas π . And if $\mathcal{E}\alpha$ and $\mathcal{E}\beta$, then of course we have

$$\mathcal{E}\neg\alpha, \quad \mathcal{E}(\alpha \wedge \beta), \quad \mathcal{E}(\alpha \vee \beta)$$

This is the first step in showing the *unique formula reconstruction property*. Let us prove a lemma before proving the property itself,

1.1.3 Lemma

Proper initial segments of formulas are not formulas. Equivalently (by contrapositive), if α and β are formulas and $\alpha\xi = \beta\eta$ for arbitrary strings ξ and η , then $\alpha = \beta$.

Let us prove this by induction on α . If α is a prime formula, suppose that β is not a prime formula, then its first character is either (or \neg , but then $\alpha = ($ or $\alpha = \neg$, in contradiction. Thus β is a prime formula and so $\alpha = \beta$ as they are both a single character. Now if $\alpha = (\alpha_1 \circ \alpha_2)$, then the first character of β must too be (, so β is of the form $(\beta_1 * \beta_2)\eta$. Thus

$$\alpha_1 \circ \alpha_2 \xi = \beta_1 * \beta_2 \eta$$

and so by our inductive assumption, $\alpha_1 = \beta_1$, and so $\circ = *$, and thus $\alpha_2 = \beta_2$ by our inductive assumption again. And so $\alpha = \beta$ as required. The proof for the case that $\alpha = \neg\alpha'$ is similar. ■

1.1.4 Proposition (Unique Formula Reconstruction Property)

Every compound formula $\varphi \in \mathcal{F}$ is of one of the following forms:

$$\varphi = \neg\alpha, \quad \varphi = (\alpha \wedge \beta), \quad \varphi = (\alpha \vee \beta)$$

For some formulas $\alpha, \beta \in \mathcal{F}$.

We have already shown existence. We will now show that this is unique, meaning that φ can be written uniquely as one of these strings. Using the lemma proven above, the proof for uniqueness of the reconstruction property is immediate. For example, if $\varphi = (\alpha_1 \wedge \beta_1)$ then obviously φ cannot be written as $\neg\alpha_2$ since $(\neq \neg$, and if $\varphi = (\alpha_2 \vee \beta_2)$ then by the lemma $\alpha_1 = \alpha_2$, and so we get that $\wedge = \vee$ in contradiction. And finally if $\varphi = (\alpha_2 \wedge \beta_2)$, then again by the lemma, $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$ as required. The proof for \neg and \vee are similar. ■

Utilizing formula recursion, we can define functions on formulas. For example,

1.1.5 Definition

For a formula φ , we define $Sf\varphi$ to be the set of all subformulas of φ . This is done recursively:

$$\begin{aligned} Sf\pi &= \{\pi\} \text{ for prime formulas } \pi, \\ Sf\neg\alpha &= Sf\alpha \cup \{\alpha\}, \quad Sf(\alpha \circ \beta) = Sf\alpha \cup Sf\beta \cup \{(\alpha \circ \beta)\} \text{ for a binary logical connective } \circ \end{aligned}$$

Similarly, we can define the **rank** of a formula φ ,

$$\begin{aligned} rank\pi &= 0 \text{ for prime formulas } \pi, \\ rank\neg\alpha &= rank\alpha + 1, \quad rank(\alpha \circ \beta) = \max\{rank\alpha, rank\beta\} + 1 \text{ for a binary logical connective } \circ \end{aligned}$$

And we can also define the set of variables in φ ,

$$\begin{aligned} var\pi &= \{\pi\} \text{ for prime formulas } \pi, \\ var\neg\alpha &= var\alpha, \quad var(\alpha \circ \beta) = var\alpha \cup var\beta \text{ for a binary logical connective } \circ \end{aligned}$$

In all definitions \circ is either \wedge or \vee .

So now that we have discussed the syntax of propositional logic, it is time to discuss its semantics; how we assign to formulas truth values. Recall the truth tables for \wedge , \vee , and \neg :

α	β	$\alpha \wedge \beta$	α	β	$\alpha \vee \beta$	α	$\neg\alpha$
1	1	1	1	1	1	1	0
1	0	0	1	0	1	0	1
0	1	0	0	1	1		
0	0	0	0	0	0		

These define how the logical connectives function as functions on $\{0, 1\}$.

1.1.6 Definition

A **propositional valuation**, or a **propositional model**, is a function

$$w: PV \longrightarrow \{0, 1\}$$

We can extend it to a function $w: PV \longrightarrow \mathcal{F}$ as follows:

$$w(\alpha \wedge \beta) = w\alpha \wedge w\beta, \quad w(\alpha \vee \beta) = w\alpha \vee w\beta, \quad w\neg\alpha = \neg w\alpha$$

Notice that we would need to define, for example, $w(\alpha \rightarrow \beta) = w\alpha \rightarrow w\beta$ had \rightarrow been an element of our logical signature. But since \rightarrow is defined using \wedge and \neg , we must prove this identity:

$$w(\alpha \rightarrow \beta) = w\neg(\alpha \wedge \neg\beta) = \neg w(\alpha \wedge \neg\beta) = \neg(w\alpha \wedge \neg w\beta) = w\alpha \rightarrow w\beta$$

This is of course not a coincidence, but a result of the fact that $\alpha \rightarrow \beta = \neg(\alpha \wedge \neg\beta)$ (where $\alpha, \beta \in \{0, 1\}$). Notice that furthermore,

$$w\top = 1, \quad w\perp = 0$$

1.1.7 Proposition

The valuation of a formula is dependent only on its variables. Meaning if φ is a formula and w and w' are two valuations where $w\pi = w'\pi$ for all $\pi \in \text{var}\varphi$, then $w\varphi = w'\varphi$.

We will prove this by induction on φ . For prime formulas, this is obvious as $\text{var}\varphi = \{\varphi\}$ and then $w\varphi = w'\varphi$ by the proposition's assumption. For $\varphi = \alpha \wedge \beta$, we have that

$$w\varphi = w\alpha \wedge w\beta = w'\alpha \wedge w'\beta = w'\varphi$$

where the second equality is our inductive assumption. The proof for $\varphi = \alpha \vee \beta$ and $\varphi = \neg\alpha$ is similar. ■

Let us suppose that $PV = \{p_1, p_2, \dots, p_n, \dots\}$, then we define \mathcal{F}_n to be the set of formulas φ such that $\text{var}\varphi \subseteq \{p_1, \dots, p_n\}$.

1.1.8 Definition

A **boolean function** is a function

$$f: \{0, 1\}^n \longrightarrow \{0, 1\}$$

for some $n \geq 0$. The set of boolean functions of arity n is denoted \mathbf{B}_n . A formula $\varphi \in \mathcal{F}_n$ **represents** a boolean function $f \in \mathbf{B}_n$ (similarly, f is represented by φ), if for all valuations w ,

$$w\varphi = f(w\vec{p}) \quad (w\vec{p} = (wp_1, \dots, wp_n))$$

So for example, $\alpha = p_1 \wedge p_2$ represents the function $f(p, q) = p \wedge q$. This is since

$$f(wp_1, wp_2) = wp_1 \wedge wp_2 = w(p_1 \wedge p_2) = w\alpha$$

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Since valuations of $\varphi \in \mathcal{F}_n$ are defined by their values on p_1, \dots, p_n , φ represents at most a single function f . In fact, it represents the function

$$\varphi^{(n)}(x_1, \dots, x_n) = w\varphi$$

where w is any valuation such that $w p_i = x_i$ (all of these valuations value φ the same). Now, notice that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ and $\mathbf{B}_n \subset \mathbf{B}_{n+1}$ and so $\varphi \in \mathcal{F}_n$ represents a function in \mathcal{B}_{n+1} as well. But this function is not essentially in \mathcal{B}_n in the sense that its last argument does not impact its value. Formally we say that given a function $f: M^n \rightarrow M$, we call its i th argument *fictional* if for all $x_1, \dots, x_i, \dots, x_n \in M$ and $x'_i \in M$:

$$f(x_1, \dots, x_i, \dots, x_n) = f(x_1, \dots, x'_i, \dots, x_n)$$

An *essentially n -ary* function is a function with no fictional arguments.

1.1.9 Definition

Two formulas α and β are **equivalent** if for every valuation w , $w\alpha = w\beta$. This is denoted $\alpha \equiv \beta$.

It is immediate that α and β are equivalent if and only if they represent the same function. A simple example of equivalence is $\alpha \equiv \neg\neg\alpha$. The following equivalences are easy to verify and the reader should already be familiar with them (α, β , and γ are formulas):

$$\begin{array}{lll} \alpha \wedge (\beta \wedge \gamma) \equiv \alpha \wedge \beta \wedge \gamma & \alpha \vee (\beta \vee \gamma) \equiv \alpha \vee \beta \vee \gamma & (\text{associativity}) \\ \alpha \wedge \beta \equiv \beta \wedge \alpha & \alpha \vee \beta \equiv \beta \vee \alpha & (\text{commutativity}) \\ \alpha \wedge \alpha \equiv \alpha & \alpha \vee \alpha \equiv \alpha & (\text{idempotency}) \\ \alpha \wedge (\alpha \vee \beta) \equiv \alpha & \alpha \vee \alpha \wedge \beta \equiv \alpha & (\text{absorption}) \\ \alpha \wedge (\beta \vee \gamma) \equiv \alpha \wedge \beta \vee \alpha \wedge \gamma & & (\wedge\text{-distributivity}) \\ \alpha \vee \beta \wedge \gamma \equiv (\alpha \vee \beta) \wedge (\alpha \vee \gamma) & & (\vee\text{-distributivity}) \\ \neg(\alpha \wedge \beta) \equiv \neg\alpha \vee \neg\beta & \neg(\alpha \vee \beta) \equiv \neg\alpha \wedge \neg\beta & (\text{de Morgan rules}) \end{array}$$

Furthermore,

$$\alpha \vee \neg\alpha \equiv \top, \quad \alpha \wedge \neg\alpha \equiv \perp, \quad \alpha \wedge \top \equiv \alpha \vee \perp \equiv \alpha$$

Since $\alpha \rightarrow \beta \equiv \neg(\alpha \wedge \neg\beta)$, by de Morgan rules, this is equivalent to

$$\equiv \neg\alpha \vee \neg\neg\beta \equiv \neg\alpha \vee \beta$$

Notice that

$$\alpha \rightarrow \beta \rightarrow \gamma \equiv \neg\alpha \vee (\beta \rightarrow \gamma) \equiv \neg\alpha \vee \neg\beta \vee \gamma \equiv \neg(\alpha \wedge \beta) \vee \gamma \equiv \alpha \wedge \beta \rightarrow \gamma$$

Inductively, we see that

$$\alpha_1 \rightarrow \dots \rightarrow \alpha_n \rightarrow \gamma \equiv \alpha_1 \wedge \dots \wedge \alpha_n \rightarrow \gamma$$

We could go on, but I assume you get the point.

\equiv is obviously reflexive, symmetric, and transitive: therefore it is an equivalence relation on \mathcal{F} . But moreso it is a *congruence relation*, meaning it respects connectives. Explicitly, for all formulas $\alpha, \beta, \alpha', \beta' \in \mathcal{F}$:

$$\alpha \equiv \alpha', \beta \equiv \beta' \implies \alpha \wedge \beta \equiv \alpha' \wedge \beta', \alpha \vee \beta \equiv \alpha' \vee \beta', \neg\alpha \equiv \neg\alpha'$$

Congruence relations will be discussed in more generality in later sections. Inductively, we can prove the following result:

1.1.10 Theorem (The Replacement Theorem)

Suppose α and α' are equivalent formulas. Let φ be some other formula, and define φ' to be the result of replacing all occurrences of α within φ by α' . Then $\varphi \equiv \varphi'$.

This will be proven more generally later.

1.1.11 Definition

Prime formulas and their negations are called **literals**. A formula of the form $\alpha_1 \vee \dots \vee \alpha_n$ where each

α_i is a conjunction of literals is called a **disjunctive normal form**. And similarly a formula of the form $\alpha_1 \wedge \cdots \wedge \alpha_n$ where each α_i is a disjunction of literals is called a **conjunctive normal form**. We will use the abbreviations DNF and CNF for disjunctive and conjunctive normal forms, respectively.

So a DNF is a formula of the form

$$\bigvee_{i=1}^n \bigwedge_{j=1}^{m_i} \ell_{i,j}$$

where for every i, j , $\ell_{i,j}$ is a literal: a formula of the form $p_{i,j}$ or $\neg p_{i,j}$ for some prime formula $p_{i,j}$. Similarly a CNF is a formula of the form

$$\bigwedge_{i=1}^n \bigvee_{j=1}^{m_i} \ell_{i,j}$$

Let us temporarily introduce the following notation: for a prime formula p , let

$$p^1 := p, \quad p^0 := \neg p$$

This allows us to more concisely state and prove the following theorem:

1.1.12 Theorem

Every boolean function $f \in \mathbf{B}_n$ for $n > 0$ is representable by the DNF

$$\alpha_f := \bigvee_{f(\vec{x})=1} p_1^{x_1} \wedge \cdots \wedge p_n^{x_n}$$

and a CNF

$$\beta_f := \bigwedge_{f(\vec{x})=0} p_1^{\neg x_1} \wedge \cdots \wedge p_n^{\neg x_n}$$

Let w be a valuation and $\vec{p} = (p_1, \dots, p_n)$ then

$$w\alpha_f = \bigvee_{f(\vec{x})=1} wp_1^{x_1} \wedge \cdots \wedge wp_n^{x_n}$$

Notice that wp^x is equal to 1 if and only if $wp = x$: suppose $x = 0$ then $wp^x = \neg wp$, which is equal to 1 if and only if $wp = 0 = x$, and similar for $x = 1$. Thus $w\alpha_f = 1$ if and only if there exists a \vec{x} such that $f(\vec{x}) = 1$ and $wp_1^{x_1} \wedge \cdots \wedge wp_n^{x_n} = 1$, meaning that for each i , $wp_i = x_i$. This means that $\vec{x} = w\vec{p}$, and so $f(w\vec{p}) = f(\vec{x}) = 1$. Similarly if $f(w\vec{p}) = 1$ then let $\vec{x} = w\vec{p}$, and then $wp_1^{x_1} \wedge \cdots \wedge wp_n^{x_n} = 1$ and $f(\vec{x}) = 1$, so $w\alpha_f = 1$. So $w\alpha_f = f(w\vec{p})$ for all valuations w , which means that f is represented by α_f , as required. The proof for β_f is similar. ■

Notice that since every formula represents a boolean function, which by above can be represented by a DNF and a CNF, we get that every formula is equivalent to a DNF and a CNF.

1.1.13 Corollary

Every formula is equivalent to a DNF and a CNF.

1.1.14 Definition

A logical signature σ is **functional complete** if every boolean function is representable by a formula in this signature.

By corollary 1.1.13, $\{\neg, \wedge, \vee\}$ is functional complete. Since

$$\alpha \vee \beta \equiv \neg(\neg\alpha \wedge \neg\beta), \quad \alpha \wedge \beta \equiv \neg(\neg\alpha \vee \neg\beta)$$

$\{\neg, \wedge\}$ and $\{\neg, \vee\}$ are both functional complete. Thus in order to show that a logical signature σ is functional complete, it is sufficient to show that \neg and \wedge or \neg and \vee can be represented by σ .

Note

If f is a function, instead of writing $f(x)$ or fx , many times we will instead write x^f . This is more concise and may reduce confusion in the case that x itself is a string wrapped in parentheses.

Let us define the function $\delta: \mathcal{F} \rightarrow \mathcal{F}$ on formulas recursively by $p^\delta = p$ for prime formulas p and

$$(\neg\alpha)^\delta = \neg\alpha^\delta, \quad (\alpha \wedge \beta)^\delta = \alpha^\delta \vee \beta^\delta, \quad (\alpha \vee \beta)^\delta = \alpha^\delta \wedge \beta^\delta$$

Alternatively, α^δ is simply the result of swapping all occurrences of \wedge with \vee , and all occurrences of \vee with \wedge . α^δ is called the *dual formula* of α . Notice that the dual formula of a DNF is a CNF, and vice versa.

Now, suppose $f \in \mathbf{B}_n$, then let us define the *dual* of f ,

$$f^\delta(\vec{x}) := \neg f(\neg\vec{x})$$

where $\neg\vec{x} = (\neg x_1, \dots, \neg x_n)$. Notice that δ is idempotent:

$$f^{\delta^2}(\vec{x}) = \neg f^\delta(\neg\vec{x}) = \neg\neg f(\neg\neg\vec{x}) = f(\vec{x})$$

1.1.15 Theorem (The Duality Principle for Two-Valued Logic)

If α represents the function f , then α^δ represents f^δ .

We will prove this by induction on α . If $\alpha = p$ is prime, then this is trivial. Now suppose that α and β represent f_1 and f_2 respectively. Then $\alpha \wedge \beta$ represents $f(\vec{x}) = f_1(\vec{x}) \wedge f_2(\vec{x})$, and $(\alpha \wedge \beta)^\delta = \alpha^\delta \vee \beta^\delta$ represents $g(\vec{x}) = f_1^\delta(\vec{x}) \vee f_2^\delta(\vec{x})$ by the induction hypothesis. Now,

$$f^\delta(\vec{x}) = \neg f(\neg\vec{x}) = \neg(f_1(\neg\vec{x}) \wedge f_2(\neg\vec{x})) = \neg f_1(\neg\vec{x}) \vee \neg f_2(\neg\vec{x}) = f_1^\delta(\vec{x}) \vee f_2^\delta(\vec{x}) = g(\vec{x})$$

So $f^\delta = g$, meaning that $(\alpha \wedge \beta)^\delta$ does indeed represent f^δ . The proof for $\alpha \vee \beta$ is similar. Now suppose α represents f , then $\neg\alpha$ represents $\neg f$, and α^δ represents f^δ by the induction hypothesis. And so $(\neg\alpha)^\delta = \neg\alpha^\delta$ represents $\neg f^\delta$, which is equal to $(\neg f)^\delta$ since

$$(\neg f)^\delta(\vec{x}) = (\neg\neg f)(\neg\vec{x}) = \neg(\neg f(\neg\vec{x})) = \neg f^\delta(\vec{x})$$

And so $(\neg\alpha)^\delta$ represents $(\neg f)^\delta$, as required. ■

1.1.16 Definition

Suppose α is a formula and w is a valuation. Instead of writing $w\alpha = 1$, we now write $w \models \alpha$, and this is read as “ w satisfies α ”. If X is a set of formulas, we write $w \models X$ if $w \models \alpha$ for all $\alpha \in X$, and w is called a **propositional model** for X . A formula α (respectively a set of formulas X) is **satisfiable** if there is some valuation w such that $w \models \alpha$ (respectively $w \models X$). \models is called the **satisfiability relation**.

\models has the following immediate properties:

$$\begin{aligned} w \models p &\iff wp = 1 \quad (p \in PV) & w \models \alpha &\iff w \not\models \neg\alpha \\ w \models \alpha \wedge \beta &\iff w \models \alpha \text{ and } w \models \beta & w \models \alpha \vee \beta &\iff w \models \alpha \text{ or } w \models \beta \end{aligned}$$

These properties uniquely define \models , meaning we could have defined \models recursively by these properties.

Notice that

$$w \models \alpha \rightarrow \beta \iff \text{if } w \models \alpha \text{ then } w \models \beta$$

This is due to the definition of \rightarrow coinciding with our common usage of implication. Had we not defined \rightarrow , but instead added it to our logical signature, this above equivalence would have to be taken in the definition of the satisfiability relation (when axiomized by the above properties).

1.1.17 Definition

α is **logically valid**, or a **tautology**, if $w \models \alpha$ for all valuations w . This is abbreviated by $\models \alpha$. A formula which cannot be satisfied; ie. for all valuations w , $w \not\models \alpha$; is called a **contradiction**.

For example, $\alpha \vee \neg\alpha$ is a tautology, while $\alpha \wedge \neg\alpha$ and $\alpha \leftrightarrow \neg\alpha$ are contradictions for all formulas α . Notice that the negation of a tautology is a contradiction and vice versa. \top is a tautology and \perp is a contradiction.

The following are important tautologies of implication (keep in mind how \rightarrow associates from the right):

$p \rightarrow p$	(self-implication)
$(p \rightarrow q) \rightarrow (q \rightarrow r) \rightarrow (p \rightarrow r)$	(chain rule)
$(p \rightarrow q \rightarrow r) \rightarrow (q \rightarrow p \rightarrow r)$	(exchange of premises)
$p \rightarrow q \rightarrow p$	(premise change)
$(p \rightarrow q \rightarrow r) \rightarrow (p \rightarrow q) \rightarrow (p \rightarrow r)$	(Frege's formula)
$((p \rightarrow q) \rightarrow p) \rightarrow p$	(Peirce's formula)

1.1.18 Definition

Suppose X is a set of formulas and α a formula, we say that α is a **logical consequence** if $w \models \alpha$ for every model w of X . In other words,

$$w \models X \implies w \models \alpha$$

This is denoted $X \models \alpha$.

Notice that \models here is used for logical consequence (the consequence relation), and we used it before as the symbol for the satisfiability relation. Context will make it clear as to its meaning. We use the notation $\alpha_1, \dots, \alpha_n \models \beta$ to mean $\{\alpha_1, \dots, \alpha_n\} \models \beta$. This justifies the notation for tautologies: α is a tautology if and only if $\emptyset \models \alpha$ (since every valuation models \emptyset), which is shortened by the above notation to $\models \alpha$.

And we also use $X \models \alpha, \beta$ to mean $X \models \alpha$ and $X \models \beta$. And $X, \alpha \models \beta$ to mean $X \cup \{\alpha\} \models \beta$.

The following are examples of logical consequences

$$\begin{aligned} \alpha, \beta \models \alpha \wedge \beta, \quad \alpha \wedge \beta \models \alpha, \beta \\ \alpha, \alpha \rightarrow \beta \models \beta \\ X \models \perp \implies X \models \alpha \quad \text{for all formulas } \alpha \\ X, \alpha \models \beta, X, \neg\alpha \models \beta \implies X \models \beta \end{aligned}$$

The final example is true because if $w \models X$ then either $w \models \alpha$ or $w \models \neg\alpha$, and in either case $w \models \beta$.

Let us now state some obvious properties of the consequence relation:

$$\begin{aligned} \alpha \in X \implies X \models \alpha & \quad (\text{reflexivity}) \\ X \models \alpha \text{ and } X \subseteq X' \implies X' \models \alpha & \quad (\text{monotonicity}) \\ X \models Y \text{ and } Y \models \alpha \implies X \models \alpha & \quad (\text{transitivity}) \end{aligned}$$

1.1.19 Definition

A **propositional substitution** is a mapping from prime formulas to formulas, $\sigma: PV \rightarrow \mathcal{F}$, which is extended to a mapping between formulas $\sigma: \mathcal{F} \rightarrow \mathcal{F}$ recursively:

$$(\alpha \wedge \beta)^\sigma = \alpha^\sigma \wedge \beta^\sigma, \quad (\alpha \vee \beta)^\sigma = \alpha^\sigma \vee \beta^\sigma, \quad (\neg\alpha)^\sigma = \neg\alpha^\sigma$$

If X is a set of formulas, we define

$$X^\sigma = \{\varphi^\sigma \mid \varphi \in X\}$$

Besides being intuitively important, the following proposition gives more insight into the usefulness of substitutions:

1.1.20 Proposition

Let X be a set of formulas, and α a formula. Then

$$X \models \alpha \implies X^\sigma \models \alpha^\sigma$$

Thus in a sense consequence is invariant under substitution.

Let w be a valuation, then we define w^σ as follows:

$$w^\sigma p = wp^\sigma$$

for prime formulas p . Now we claim that

$$w \models \alpha^\sigma \iff w^\sigma \models \alpha$$

We will prove this by induction on α . In the case that $\alpha = p$ is prime, then $w \models p^\sigma$ if and only if $wp^\sigma = w^\sigma p = 1$, and so this is if and only if $w^\sigma \models p$. Now by induction,

$$w \models (\alpha \wedge \beta)^\sigma \iff w \models \alpha^\sigma \text{ and } w \models \beta^\sigma \iff w^\sigma \models \alpha \text{ and } w^\sigma \models \beta \iff w^\sigma \models \alpha \wedge \beta$$

where the second equivalence is due to the induction hypothesis. The proof for formulas of the form $\alpha \vee \beta$ and $\neg\alpha$ proceed in a similar fashion.

Now, suppose $w \models X^\sigma$. This is if and only if $w \models \varphi^\sigma$ for all $\varphi \in X$, which is if and only if $w^\sigma \models \varphi$ by above. So $w \models X^\sigma$ if and only if $w^\sigma \models X$. And so if $X \models \alpha$ then let $w \models X^\sigma$, then $w^\sigma \models X$ meaning $w^\sigma \models \alpha$ and so $w \models \alpha^\sigma$ by above. So $X^\sigma \models \alpha^\sigma$ as required. ■

These four properties,

$$\begin{array}{ll} \alpha \in X \implies X \models \alpha & (\text{reflexivity}) \\ X \models \alpha \text{ and } X \subseteq X' \implies X' \models \alpha & (\text{monotonicity}) \\ X \models Y \text{ and } Y \models \alpha \implies X \models \alpha & (\text{transitivity}) \\ X \models \alpha \implies X^\sigma \models \alpha^\sigma & (\text{substitution invariance}) \end{array}$$

are what define general consequence relations, and form the basis for a general theory of logical systems. Another property is

$$X \models \alpha \implies X_0 \models \alpha \text{ for some finite } X_0 \subseteq X \quad (\text{finitary})$$

We will show in the next subsection that this is a property of our consequence relation.

Another property is the property

$$X, \alpha \models \beta \iff X \models \alpha \rightarrow \beta$$

termed the *semantic deduction theorem*. Let us prove the first direction: suppose w is a model of X , then if $w \models \alpha$ it is a model of $X \cup \{\alpha\}$ and so $w \models \beta$. So we have shown that if $w \models \alpha$, then $w \models \beta$, meaning $w \models \alpha \rightarrow \beta$ and so $X \models \alpha \rightarrow \beta$. end for the converse, if $w \models X, \alpha$ then it is a model of X and so $w \models \alpha, \alpha \rightarrow \beta$ and thus $w \models \beta$.

We can show by induction a generalization of this:

$$X, \alpha_1, \dots, \alpha_n \models \beta \iff X \models \alpha_1 \rightarrow \dots \rightarrow \alpha_n \rightarrow \beta \iff X \models \alpha_1 \wedge \dots \wedge \alpha_n \rightarrow \beta$$

The induction step is simple: take $X' = X \cup \{\alpha_1\}$ we get by our induction hypothesis,

$$\begin{aligned} X, \alpha_1, \dots, \alpha_n \models \beta &\iff X', \alpha_2, \dots, \alpha_n \models \beta \iff X, \alpha_1 \models \alpha_2 \rightarrow \dots \rightarrow \alpha_n \rightarrow \beta \\ &\iff X \models \alpha_1 \rightarrow \alpha_2 \rightarrow \dots \rightarrow \alpha_n \rightarrow \beta \end{aligned}$$

as required. The deduction theorem makes proving many tautologies relating to implication much easier.

1.2 Gentzen Calculi

To begin this subsection, we will define a derivability relation \vdash which axiomatizes the important properties of the consequence relation \models . Our goal is to show that by using these axioms, \vdash is equivalent to \models , and this will allow us to prove important facts about \models , namely its finitaryness.

1.2.1 Definition

We define **Gentzen style sequent calculus** of \vdash as follows: $X \vdash \alpha$ is to be read as “ α is derivable from X ” where α is a formula and X is a set of formulas. A pair $(X, \alpha) \in \mathcal{P}(\mathcal{F}) \times \mathcal{F}$, or more suggestively written $X \vdash \alpha$, is called a **sequent**. Gentzen-style rules have the form

$$\frac{X_1 \vdash \alpha_1 \mid \cdots \mid X_n \vdash \alpha_n}{X \vdash \alpha}$$

Which is to be understood as meaning that if for every i , $X_i \vdash \alpha_i$, then $X \vdash \alpha$.

Gentzen calculus has the following basic rules:

$$\begin{array}{ll} \text{(IS)} \quad \frac{}{\alpha \vdash \alpha} & \text{(MR)} \quad \frac{X \vdash \alpha}{X' \vdash \alpha} \quad (X \subseteq X') \\ \text{(\wedge 1)} \quad \frac{X \vdash \alpha, \beta}{X \vdash \alpha \wedge \beta} & \text{(\wedge 2)} \quad \frac{X \vdash \alpha \wedge \beta}{X \vdash \alpha, \beta} \\ \text{(\neg 1)} \quad \frac{X \vdash \alpha, \neg \alpha}{X \vdash \beta} & \text{(\neg 2)} \quad \frac{X, \alpha \vdash \beta \mid X, \neg \alpha \vdash \beta}{X \vdash \beta} \end{array}$$

(IS means “initial sequent”, MR means monotonicity rule.)

Now we say that α is derivable from X , in short $X \vdash \alpha$, if $S_n = X \vdash \alpha$ and there exists a sequence of sequents $(S_0; \dots; S_n)$ where for every S_i , S_i is either an initial sequent (IS) or derivable using the basic rules from previous sequents in the sequence.

For example, we can derive $\alpha \wedge \beta$ from $\{\alpha, \beta\}$, meaning $\alpha, \beta \vdash \alpha \wedge \beta$. This can be done by the sequence:

$$\left(\begin{array}{l} \alpha \vdash \alpha ; \alpha, \beta \vdash \alpha ; \beta \vdash \beta ; \alpha, \beta \vdash \beta ; \alpha, \beta \vdash \alpha \wedge \beta \\ \text{IS} \quad ; \quad \text{MR} \quad ; \quad \text{IS} \quad ; \quad \text{MR} \quad ; \quad \wedge 1 \end{array} \right)$$

Let us prove some more useful rules

$$\frac{X, \neg \alpha \vdash \alpha}{X \vdash \alpha}$$

(\neg -elimination)

$$\begin{array}{ll} 1 & X, \alpha \vdash \alpha \quad \text{(IS), (MR)} \\ 2 & X, \neg \alpha \vdash \alpha \quad \text{supposition} \\ 3 & X \vdash \alpha \quad (\neg 2) \end{array}$$

$$\frac{X, \neg \alpha \vdash \beta, \neg \beta}{X \vdash \alpha}$$

(reductio ad absurdum)

$$\begin{array}{ll} 1 & X, \neg \alpha \vdash \beta, \neg \beta \quad \text{supposition} \\ 2 & X, \neg \alpha \vdash \alpha \quad (\neg 1) \\ 3 & X \vdash \alpha \quad \neg\text{-elimination} \end{array}$$

$$\frac{X \vdash \alpha \rightarrow \beta}{X, \alpha \vdash \beta}$$

(\rightarrow -elimination)

$$\begin{array}{ll} 1 & X, \alpha, \neg \beta \vdash \alpha, \neg \beta \quad \text{(IS), (MR)} \\ 2 & X, \alpha, \neg \beta \vdash \alpha \wedge \neg \beta \quad (\wedge 1) \\ 3 & X \vdash \neg(\alpha \wedge \neg \beta) \quad \text{supposition} \\ 4 & X, \alpha, \neg \beta \vdash \neg(\alpha \wedge \neg \beta) \quad \text{(MR)} \\ 5 & X, \alpha, \neg \beta \vdash \beta \quad (\neg 1) \text{ on 2 and 4} \\ 6 & X, \alpha \vdash \beta \quad \neg\text{-elimination} \end{array}$$

$$\frac{X \vdash \alpha \mid X, \alpha \vdash \beta}{X \vdash \beta}$$

(cut rule)

$$\begin{array}{ll} 1 & X, \neg \alpha \vdash \alpha \quad \text{supposition, (MR)} \\ 2 & X, \neg \alpha \vdash \neg \alpha \quad \text{(IS), (MR)} \\ 3 & X, \neg \alpha \vdash \beta \quad (\neg 1) \\ 4 & X, \alpha \vdash \beta \quad \text{supposition} \\ 5 & X \vdash \beta \quad (\neg 2) \text{ on 3 and 4} \end{array}$$

$$\frac{X, \alpha \vdash \beta}{X \vdash \alpha \rightarrow \beta}$$

(\rightarrow -introduction)

- | | | |
|---|--|---------------------------|
| 1 | $X, \alpha \wedge \neg\beta, \alpha \vdash \beta$ | supposition, (MR) |
| 2 | $X, \alpha \wedge \neg\beta \vdash \alpha$ | (IS), (MR), (\wedge 2) |
| 3 | $X, \alpha \wedge \neg\beta \vdash \beta$ | cut rule |
| 4 | $X, \alpha \wedge \neg\beta \vdash \neg\beta$ | (IS), (MR), (\wedge 2) |
| 5 | $X, \alpha \wedge \neg\beta \vdash \alpha \rightarrow \beta$ | (\neg 1) |
| 6 | $X, \neg(\alpha \wedge \neg\beta) \vdash \alpha \rightarrow \beta$ | (IS), (MR) |
| 7 | $X \vdash \alpha \rightarrow \beta$ | (\neg 2) on 5 and 6 |

$$\frac{X \vdash \alpha, \alpha \rightarrow \beta}{X \vdash \beta}$$

(modus ponens)

- | | | |
|---|-------------------------------------|----------------------------|
| 1 | $X \vdash \alpha \rightarrow \beta$ | supposition |
| 2 | $X, \alpha \rightarrow \beta$ | \rightarrow -elimination |
| 3 | $X \vdash \alpha$ | supposition |
| 4 | $X \vdash \beta$ | cut rule |

\rightarrow -elimination and \rightarrow -introduction give us the *syntactic deduction theorem*:

$$X, \alpha \vdash \beta \iff X \vdash \alpha \rightarrow \beta$$

Let R be a rule of the form

$$R: \frac{X_1 \vdash \alpha_1 \mid \cdots \mid X_n \vdash \alpha_n}{X \vdash \alpha}$$

Then we say that a property of sequents \mathcal{E} is *closed under R* if $\mathcal{E}(X_1, \alpha_1), \dots, \mathcal{E}(X_n, \alpha_n)$ implies $\mathcal{E}(X, \alpha)$.

1.2.2 Proposition (Principle of Rule Induction)

Let \mathcal{E} be a property of sequents which is closed under all the basic rules of \vdash . Then $X \vdash \alpha$ implies $\mathcal{E}(X, \alpha)$.

We will prove this by induction on the length of the derivation of $S = X \vdash \alpha$, n . If $n = 1$ then $X \vdash \alpha$ must be an initial sequent and so by assumption $\mathcal{E}(X, \alpha)$. For the induction step, suppose the derivation is $(S_0; \dots; S_n)$, so $S = S_n$. Then by our inductive hypothesis $\mathcal{E}S_i$ for all $i < n$. If S is an initial sequent then $\mathcal{E}S$ holds by assumption. Otherwise S is obtained by applying a basic rule on some of the sequents S_i for $i < n$. And since $\mathcal{E}S_i$ and \mathcal{E} is closed under basic rules, we have that $\mathcal{E}S$ as required. ■

1.2.3 Lemma (Soundness of \vdash)

If $X \vdash \alpha$ then $X \models \alpha$. More suggestively,

$$\vdash \subseteq \models$$

Using the principle of rule induction, let $\mathcal{E}(X, \alpha)$ mean $X \models \alpha$ (formally this means $\mathcal{E} = \{(X, \alpha) \mid X \models \alpha\}$). Then we must show that \mathcal{E} is closed under all the basic rules of \vdash . This means that we must show that

$$\begin{aligned} \alpha \models \alpha, \quad X \models \alpha \implies X' \models \alpha \text{ for } X \subseteq X', \quad X \models \alpha, \beta \iff X \models \alpha \wedge \beta, \\ X \models \alpha, \neg\alpha \implies X \models \beta, \quad X, \alpha \models \beta \text{ and } X, \neg\alpha \models \beta \implies X \models \beta \end{aligned}$$

These are all readily verifiable (and some we have already shown). So \mathcal{E} is indeed closed under all the basic rules of \vdash , and so $\mathcal{E}(X, \alpha)$ (meaning $X \models \alpha$) implies $X \vdash \alpha$. ■

The property above is called *soundness*, meaning \vdash does not derive anything “incorrect”.

1.2.4 Theorem

If $X \vdash \alpha$ then there exists a finite subset $X_0 \subseteq X$ such that $X_0 \vdash \alpha$.

Let $\mathcal{E}(X, \alpha)$ be the property that there exists a finite subset $X_0 \subseteq X$ such that $X_0 \vdash \alpha$. We will show that \mathcal{E} is closed under the basic rules of \vdash . Trivially, $\mathcal{E}(X, \alpha)$ holds for $X = \{\alpha\}$, meaning \mathcal{E} holds for (IS). And similarly

if $\mathcal{E}(X, \alpha)$ and $X \subseteq X'$, since there exists a finite $X_0 \subseteq X$ such that $X_0 \vdash \alpha$, this same X_0 is a subset of X' and so $\mathcal{E}(X', \alpha)$ so \mathcal{E} is closed under (MR).

Now if $\mathcal{E}(X, \alpha)$ and $\mathcal{E}(X, \beta)$ then suppose $X_1 \vdash \alpha$ and $X_2 \vdash \beta$ where $X_1, X_2 \subseteq X$ are finite. Then $X_0 = X_1 \cup X_2$ is finite, $X_0 \vdash \alpha, \beta$ and so $X_0 \vdash \alpha \wedge \beta$, and since $X_0 \subseteq X$ is finite, $\mathcal{E}(X, \alpha \wedge \beta)$ so \mathcal{E} is closed under $(\wedge 1)$. Closure under the rest of the basic rules can be shown similarly. ■

1.2.5 Definition

A set of formulas X is **inconsistent** if $X \vdash \alpha$ for every formula α . If X is not inconsistent, it is termed **consistent**. X is **maximally consistent** if X is consistent but for every proper superset $X \subset Y$, Y is inconsistent.

Notice that X is inconsistent if and only if $X \vdash \perp$. Obviously if X is inconsistent, $X \vdash \perp$. Conversely, if $X \vdash \perp$ then $X \vdash p_1 \wedge \neg p_1$ and so by $(\wedge 2)$, $X \vdash p_1, \neg p_1$ and thus by $(\neg 1)$ for all formulas α , $X \vdash \alpha$.

Furthermore, if X is consistent it is maximally consistent if and only if for every formula α , either $\alpha \in X$ or $\neg \alpha \in X$ exclusively. If neither α nor $\neg \alpha$ are in X , then since X is maximally consistent, $X, \alpha \vdash \perp$ and $X, \neg \alpha \vdash \perp$ and therefore by $(\neg 2)$, $X \vdash \perp$ contradicting X 's consistency. And if X contains α or $\neg \alpha$ for every formula α , then it is maximal: adding another formula α would mean that $\alpha, \neg \alpha \in X$ and so by (IS), (MR), and $(\neg 2)$, X would be inconsistent.

This means that maximally consistent sets X are *deductively closed*:

$$X \vdash \alpha \iff \alpha \in X$$

Obviously if $\alpha \in X$ then by (IS) and (MR), $X \vdash \alpha$. Now suppose that $X \vdash \alpha$, then since $\alpha \in X$ or $\neg \alpha \in X$, we cannot have $\neg \alpha \in X$ since X is consistent. Therefore $\alpha \in X$.

1.2.6 Lemma

The derivability relation has the following properties:

$$\mathcal{C}^+: X \vdash \alpha \iff X, \neg \alpha \vdash \perp, \quad \mathcal{C}^-: X \vdash \neg \alpha \iff X, \alpha \vdash \perp$$

Meaning α is derivable from X if and only if $X \cup \{\neg \alpha\}$ is inconsistent. And similarly $\neg \alpha$ is derivable from X if and only if $X \cup \{\alpha\}$ is inconsistent.

We will prove \mathcal{C}^+ . Suppose $X \vdash \alpha$, then $X, \neg \alpha \vdash \alpha$ by (MR) and $X, \neg \alpha \vdash \neg \alpha$ by (IS) and (MR). Thus by $(\neg 1)$, $X, \neg \alpha \vdash \perp$ for all formulas β by $(\neg 1)$ and in particular, $X, \neg \alpha \vdash \perp$. Now suppose $X, \neg \alpha \vdash \perp$ then by $(\wedge 2)$ and $(\neg 1)$, we have $X, \neg \alpha \vdash \alpha$ then by \neg -elimination, $X \vdash \alpha$. \mathcal{C}^- is proven similarly. ■

1.2.7 Lemma (Lindenbaum's Theorem)

Every consistent set of formulas $X \subseteq \mathcal{F}$ can be extended to a maximally consistent set of formulas $X \subseteq X' \subseteq \mathcal{F}$.

Let us define the set

$$H = \{Y \subseteq \mathcal{F} \mid Y \text{ is consistent and } X \subseteq Y\}$$

This is partially ordered with respect to \subseteq , and since $X \in H$, H is not empty. Let $C \subseteq H$ be a chain, meaning that for every $Z, Y \in C$, either $Z \subseteq Y$ or $Y \subseteq Z$. Now we claim that $U = \bigcup C$ is an upper bound for C . So we must show that $U \in H$. Suppose not, then U is not consistent meaning $U \vdash \perp$. But then there must exist a finite $U_0 = \{\alpha_1, \dots, \alpha_n\} \subseteq U$ such that $U_0 \vdash \perp$. Now suppose $\alpha_i \in Y_i \in C$ then since C is linearly ordered, we can assume that every Y_i is contained within Y_n . But then by (MR), $Y_n \vdash \perp$ which contradicts $Y_n \in H$ being consistent.

So U is consistent and so $U \in H$, and obviously for every $Y \in C$, $Y \subseteq U$. So U is an upper bound for C , meaning that every chain in H has an upper bound in H , and so by Zorn's Lemma, H has a maximal element. This maximal element, call it X' , is precisely a maximally consistent set containing X : it is consistent and contains X since it is in H , and it is maximal in H so for every $X \subseteq Y$, $Y \notin H$ so Y is inconsistent. ■

1.2.8 Lemma

A maximally consistent set of formulas X has the following property:

$$X \vdash \neg\alpha \iff X \not\vdash \alpha$$

for all formulas α .

If $X \vdash \neg\alpha$ then $X \not\vdash \alpha$ due to X 's consistency. If $X \not\vdash \alpha$ then $X \cup \{\neg\alpha\}$ is consistent in lieu of \mathcal{C}^+ . But since X is maximal, $X \cup \{\neg\alpha\} = X$ meaning $\neg\alpha \in X$ and so by (IS) and (MR), $X \vdash \neg\alpha$. ■

1.2.9 Lemma

Maximally consistent sets are satisfiable.

Suppose X is maximally consistent, then let us define the valuation w by $w \models p \iff X \vdash p$. Then we claim that

$$X \vdash \alpha \iff w \models \alpha$$

This is trivial for prime formulas. Now if $X \vdash \alpha \wedge \beta$:

$$X \vdash \alpha \wedge \beta \iff X \vdash \alpha, \beta \iff w \models \alpha, \beta \iff w \models \alpha \wedge \beta$$

where the second equivalence is due to the induction hypothesis. And if $X \vdash \neg\alpha$:

$$X \vdash \neg\alpha \iff X \not\vdash \alpha \iff w \not\models \alpha \iff w \models \neg\alpha$$

The first equivalence is due to the previous lemma, and the second is due to the induction hypothesis. And therefore $w \models X$, meaning X is satisfiable. ■

1.2.10 Theorem (The Completeness Theorem)

Let X and α be an arbitrary set of formulas and formula respectively. Then $X \vdash \alpha$ if and only if $X \models \alpha$. More suggestively,

$$\vdash = \models$$

We have already shown that $\vdash \subseteq \models$ and so all that remains is to show the converse. Suppose that $X \not\vdash \alpha$, then $X, \neg\alpha$ is consistent by \mathcal{C}^+ . Thus it can be extended to a maximally consistent set $X, \neg\alpha \subseteq X'$ which is satisfiable. Therefore so is $X, \neg\alpha$, which means that $X \not\models \alpha$. ■

We get the following theorem as an immediate result from The Completeness Theorem and theorem 1.2.4:

1.2.11 Theorem

$X \models \alpha$ if and only if $X_0 \models \alpha$ for a finite $X_0 \subseteq X$.

1.2.12 Theorem (The Compactness Theorem)

A set $X \subseteq \mathcal{F}$ is satisfiable if and only if every finite $X_0 \subseteq X$ is satisfiable.

Obviously if X is satisfiable, so is $X_0 \subseteq X$. Now if X is not satisfiable, then $X \vdash \perp$ and so there exists a finite $X_0 \subseteq X$ such that $X_0 \vdash \perp$ (and so $X_0 \models \perp$) by the previous theorem. And so if X is not satisfiable, there exists a finite $X_0 \subseteq X$ which is not satisfiable. ■

Let us now give some examples of applications of the compactness theorem.

1.2.13 Proposition

Every set M can be linearly (also known as totally) ordered.

If M is finite, this is trivial: if $M = \{m_1, \dots, m_n\}$ simply define $m_1 < \dots < m_n$. Now let M be any set, let us define the propositional variable (aka prime formula) p_{ab} for every $(a, b) \in M \times M$. This will represent $a < b$. So we define X to be the set of the following formulas, which represents M being linearly ordered,

$$\begin{aligned} \neg p_{aa} & \quad (a \in M), \\ p_{ab} \wedge p_{bc} & \rightarrow p_{ac} \quad (a, b, c \in M), \\ p_{ab} \vee p_{ba} & \quad (a \neq b \in M) \end{aligned}$$

If X is satisfiable, suppose $w \models X$, then we define the linear order $a < b$ if and only if $w \models p_{ab}$. Thus X is precisely the set of conditions necessary for $<$ to be a linear order: the first condition is irreflexivity, the second is transitivity, and the third totality (antisymmetry is gained through the combination of irreflexivity and transitivity).

So if X is satisfiable, then M can be linearly ordered. By the compactness theorem, we need only to show that every finite $X_0 \subseteq X$ is satisfiable. If $X_0 \subseteq X$ is finite, then let us define M_0 to be the set of all symbols in M which occur in formulas in X_0 . Since X_0 is finite, so is M_0 and therefore M_0 can be linearly ordered. Let us define $w_0 \models p_{ab} \iff a < b$ in M_0 , then $w_0 \models X_0$. So by the compactness theorem X is satisfiable, as required. ■

Recall that showing that every set can be well-ordered (the well-ordering theorem) is equivalent to the axiom of choice. Since the compactness theorem is actually weaker than the axiom of choice, the linear ordering theorem (what we just showed) is weaker than the well-ordering theorem. Which is not surprising.

1.2.14 Proposition

A graph is k -colorable if and only if every finite subgraph is k -colorable.

A *graph* is a pair $G = (V, E)$ where V is a set of *vertices* and E is a set of *edges*. E is a subset of $\{\{v, u\} \mid v \neq u \in V\}$. The graph G is k -colorable if V can be partitioned into k *color classes*: $V = C_1 \cup \dots \cup C_k$ such that if $a, b \in C_i$ then $\{a, b\} \notin E$, meaning two neighboring vertices do not have the same color.

Obviously if a graph is k -colorable, so is every subgraph. To show the converse, let $G = (V, E)$ be a graph, then let us define the set of formulas X , where prime formulas are of the form $p_{a,i}$ where $a \in V$ and $1 \leq i \leq k$:

$$\begin{aligned} p_{a,1} \vee \dots \vee p_{a,k} & \quad (a \in V) \\ \neg(p_{a,i} \wedge p_{a,j}) & \quad (a \in V, 1 \leq i < j \leq k) \\ \neg(p_{a,i} \wedge p_{b,i}) & \quad (\{a, b\} \in E, 1 \leq i \leq k) \end{aligned}$$

If X is satisfiable, $w \models X$, then we define $C_i = \{a \in V \mid w \models p_{a,i}\}$, ie. we color $a \in V$ with the color i if and only if $p_{a,i}$ is satisfied. Then $V = C_1 \cup \dots \cup C_k$ since for every $a \in V$, $w \models p_{a,1} \vee \dots \vee p_{a,k}$, so for every $a \in V$ there exists an $1 \leq i \leq k$ such that $w \models p_{a,i}$ so $a \in C_i$. And $C_i \cap C_j = \emptyset$ in lieu of $\neg(p_{a,i} \wedge p_{a,j})$. And if $\{a, b\} \in E$ then a and b cannot be in the same color class by $\neg(p_{a,i} \wedge p_{b,i})$. So the C_i s give a valid k -coloring of G .

Let $X_0 \subseteq X$ be finite, then let us define $G_0 = (V_0, E_0)$ where V_0 is the set of vertices appearing in formulas in X_0 , and E_0 be the edges connecting them. By assumption, G_0 is k -colorable since it is finite. Now we define the valuation w_0 such that $w_0 \models p_{a,i}$ if and only if a is in the i th color class for $a \in V_0$. This must model X_0 since X_0 includes only statements saying that G_0 can be k -colored. So by the compactness theorem, X is satisfiable, as required. ■

There are more examples of applications of the compactness theorem. For example, the ultrafilter theorem, which we will visit later on.

1.3 Hilbert Calculi

In this subsection we will define another form of sequent calculus.

1.3.1 Definition

A define $\Lambda \subseteq \mathcal{F}$ to be a set of axioms, called the **logical axiom scheme**. Now, let Γ be a set of **rules of inference**, predicates of the form $R \in \Lambda^n \times \Lambda$ for $n > 0$, where $R((\varphi_1, \dots, \varphi_n), \varphi)$ which is to be understood as “if $\varphi_1, \dots, \varphi_n$ then φ ”.

If $X \subseteq \mathcal{F}$ is a set of formulas, then a **proof** is a sequence $\Phi = (\varphi_0, \dots, \varphi_n)$ where for every i , φ_i is either in $X \cup \Lambda$ or there exists a rule of inference $R \in \Gamma$ and indexes $i_1, \dots, i_n < i$ such that $R((\varphi_{i_1}, \dots, \varphi_{i_n}), \varphi)$. In such a case, φ_n is termed **derivable** (or **provable**) from X , and is written $X \vdash \varphi_n$ (\vdash to differentiate it from the derivability relation \vdash from the previous subsection).

Hilbert-style calculi will use the following axiom scheme Λ :

$$\begin{array}{ll} \Lambda 1 & (\alpha \rightarrow \beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \gamma \\ \Lambda 2 & \alpha \rightarrow \beta \rightarrow \alpha \wedge \beta \\ \Lambda 3 & \alpha \wedge \beta \rightarrow \alpha, \quad \alpha \wedge \beta \rightarrow \beta \\ \Lambda 4 & (\alpha \rightarrow \neg \beta) \rightarrow \beta \rightarrow \neg \alpha \end{array}$$

And there is only a single rule of inference: $R((\alpha, \alpha \rightarrow \beta), \beta)$ called *modus ponens*, abbreviated MP. Essentially if α and $\alpha \rightarrow \beta$ then β .

The finiteness theorem for \vdash is immediate, since $X \vdash \alpha$ requires a *finite* proof from X . And notice that

$$X \vdash \alpha, \alpha \rightarrow \beta \implies X \vdash \beta$$

Since if $\Phi_1 = (\varphi_0, \dots, \varphi_n)$ is a proof of α , and $\Phi_2 = (\varphi'_0, \dots, \varphi'_m)$ is a proof of $\alpha \rightarrow \beta$, then

$$\Phi = (\varphi_0, \dots, \varphi_n, \varphi'_0, \dots, \varphi'_m, \beta)$$

is a proof of $\alpha \rightarrow \beta$.

1.3.2 Proposition (Principle of Induction for \vdash)

Let X be a set of formulas and \mathcal{E} a property of formulas. Then if

- (1) $\mathcal{E}\alpha$ is true for all $\alpha \in X \cup \Lambda$, and
- (2) $\mathcal{E}\alpha$ and $\mathcal{E}\alpha \rightarrow \beta$ implies $\mathcal{E}\beta$ for all formulas α, β .

Then $X \vdash \alpha$ implies $\mathcal{E}\alpha$.

We will prove this by induction on n , the length of the proof of α . If $n = 1$ then α is in $X \cup \Lambda$ and so by assumption $\mathcal{E}\alpha$. Now suppose $\Phi = (\varphi_0, \dots, \varphi_n)$ is a proof of $\alpha = \varphi_n$. If $\alpha \in X \cup \Lambda$ then by assumption $\mathcal{E}\alpha$. Otherwise Φ must contain formulas of the form α_i and $\alpha_i \rightarrow \alpha$. Since initial segments of proofs are themselves proofs, by our inductive hypothesis $\mathcal{E}\varphi_i$ for $i < n$. And thus $\mathcal{E}\alpha_i$ and $\mathcal{E}\alpha_i \rightarrow \alpha$ and so $\mathcal{E}\alpha$ as required. ■

This can obviously be generalized to a principle of induction for general rules of inferences, where the second condition is replaced with a general notion of closure under rules of inference.

Now we can show that $\vdash \subseteq \models$, meaning if $X \vdash \alpha$ then $X \models \alpha$ by defining the property $\mathcal{E}\alpha := X \models \alpha$. Since Λ contains only tautologies, for every $\alpha \in X \cup \Lambda$, $X \models \alpha$ meaning $\mathcal{E}\alpha$ for all $\alpha \in X \cup \Lambda$. And if $X \models \alpha$ and $X \models \alpha \rightarrow \beta$ then we know $X \models \beta$. So \mathcal{E} satisfies the inductive properties stated above, meaning $X \vdash \alpha$ implies $X \models \alpha$ as required.

Now, obviously \vdash is reflexive, monotonic, and transitive. Reflexivity follows directly from its definition. Monotonicity follows because a proof in X is also a proof in $X \subseteq X'$. And transitivity follows because if $X \vdash Y$ and $Y \vdash \alpha$, then by concatenating the proofs of $\varphi \in Y$ in X together with the proof of α in Y gives a proof of α in X .

Our goal for the remainder of this subsection is showing that $\vdash = \models$, we will do this by showing that $\vdash \subseteq \vdash$. As explained above, \vdash is reflexive and monotonic, meaning it satisfies (IS) and (MR) of the Gentzen-style calculus \vdash .

1.3.3 Lemma

- (1) If $X \vdash \alpha \rightarrow \neg\beta$ then $X \vdash \beta \rightarrow \neg\alpha$
- (2) $\vdash \alpha \rightarrow \beta \rightarrow \alpha$
- (3) $\vdash \alpha \rightarrow \alpha$
- (4) $\vdash \alpha \rightarrow \neg\neg\alpha$
- (5) $\vdash \beta \rightarrow \neg\beta \rightarrow \alpha$

(1) By $\Lambda 4$, we have $X \vdash (\alpha \rightarrow \neg\beta) \rightarrow \beta \rightarrow \neg\alpha$, and since $X \vdash \alpha \rightarrow \neg\beta$ by modus ponens we get $X \vdash \beta \rightarrow \neg\alpha$.

(2) By $\Lambda 3$, $\vdash \beta \wedge \neg\alpha \rightarrow \neg\alpha$, and so by (1) we have $\vdash \alpha \rightarrow \neg(\beta \wedge \neg\alpha) = \alpha \rightarrow \beta \rightarrow \alpha$.

(3) Let $\gamma = \alpha$ and $\beta = \alpha \rightarrow \alpha$, then $\Lambda 1$ gives

$$\vdash (\alpha \rightarrow (\alpha \rightarrow \alpha) \rightarrow \alpha) \rightarrow (\alpha \rightarrow \alpha \rightarrow \alpha) \rightarrow \alpha \rightarrow \alpha$$

We know by (2), $\vdash \alpha \rightarrow (\alpha \rightarrow \alpha) \rightarrow \alpha$ and $\vdash \alpha \rightarrow \alpha \rightarrow \alpha$, so by applying modus ponens twice we get $\vdash \alpha \rightarrow \alpha$.

(4) Since $\vdash \neg\alpha \rightarrow \neg\alpha$ by (3), and applying (1) gives $\vdash \alpha \rightarrow \neg\neg\alpha$.

(5) By $\Lambda 3$, $\vdash \neg\beta \wedge \neg\alpha \rightarrow \neg\beta$, applying (1) gives $\vdash \beta \rightarrow \neg(\neg\beta \rightarrow \neg\alpha) = \beta \rightarrow \neg\beta \rightarrow \alpha$. ■

Since $\Lambda 3$ gives $\alpha \wedge \beta \rightarrow \alpha, \beta$, \vdash satisfies $(\wedge 2)$ of \vdash . $\Lambda 2$ gives $\alpha \rightarrow \beta \rightarrow (\alpha \wedge \beta)$ and so by applying MP twice, we get $\alpha, \beta \vdash \alpha \wedge \beta$ and so \vdash satisfies $(\wedge 1)$ of \vdash . Now by (5) of the above lemma, since $\vdash \alpha \rightarrow \neg\alpha \rightarrow \beta$, by applying MP twice we get that $X, \alpha, \neg\alpha \vdash \beta$ for all formulas β . By transitivity, this means that $X \vdash \alpha, \neg\alpha$ implies $X \vdash \beta$. Thus \vdash satisfies (IS), (MR), $(\wedge 1)$, $(\wedge 2)$, and $(\neg 1)$ of \vdash . We will now do a bit more work to show that it also satisfies $(\neg 2)$.

1.3.4 Lemma (The Deduction Theorem)

$X, \alpha \vdash \gamma$ implies $X \vdash \alpha \rightarrow \gamma$.

We will prove this using the principle of induction for \vdash . Let $\mathcal{E}\gamma$ mean $X \vdash \alpha \rightarrow \gamma$, we will show that $X, \alpha \vdash \gamma$ implies $\mathcal{E}\gamma$ by showing \mathcal{E} is closed under the inductive properties stated in the Principle of Induction for \vdash . If $\gamma \in \Lambda \cup X \cup \{\alpha\}$, if $\gamma = \alpha$ then we showed above that $X \vdash \alpha \rightarrow \alpha$. Otherwise if $\gamma \in X \cup \Lambda$ then $X \vdash \gamma$ and $X \vdash \gamma \rightarrow \alpha \rightarrow \gamma$, so by MP $X \vdash \alpha \rightarrow \gamma$ meaning $\mathcal{E}\gamma$ as required.

Now, if $\mathcal{E}\beta$ and $\mathcal{E}\beta \rightarrow \gamma$, meaning $X \vdash \alpha \rightarrow \beta$ and $X \vdash \alpha \rightarrow \beta \rightarrow \gamma$. Then by $\Lambda 1$, applying MP twice gives $X \vdash \alpha \rightarrow \gamma$ as required. ■

1.3.5 Lemma

$\vdash \neg\neg\alpha \rightarrow \alpha$

By $\Lambda 3$ and MP, we have $\neg\neg\alpha \wedge \neg\alpha \vdash \neg\alpha, \neg\neg\alpha$. Let τ be any formula where $\vdash \tau$, then since we have already verified rule $(\neg 1)$, $\neg\neg\alpha \wedge \neg\alpha \vdash \neg\tau$. And so by the deduction theorem, $\vdash \neg\neg\alpha \wedge \neg\alpha \rightarrow \neg\tau$. We showed above that this means $\vdash \tau \rightarrow \neg(\neg\neg\alpha \wedge \neg\alpha)$, and since $\vdash \tau$ by MP we get $\vdash \neg(\neg\neg\alpha \wedge \neg\alpha) = \neg\neg\alpha \rightarrow \alpha$ as required. ■

1.3.6 Lemma

\vdash also satisfies rule $(\neg 2)$ of the Gentzen-style calculus \vdash .

This is the rule that $X, \alpha \vdash \beta$ and $X, \neg\alpha \vdash \beta$ implies $X \vdash \beta$. If $X, \alpha \vdash \beta$ and $X, \neg\alpha \vdash \beta$ then $X, \alpha \vdash \neg\neg\beta$ and $X, \neg\alpha \vdash \neg\neg\beta$. By the deduction theorem, this means $X \vdash \alpha \rightarrow \neg\neg\beta, \neg\alpha \rightarrow \neg\neg\beta$. And thus $X \vdash \neg\beta \rightarrow \neg\alpha, \neg\beta \rightarrow \neg\neg\alpha$. Thus MP yields $X, \neg\beta \vdash \neg\alpha, \neg\neg\alpha$. So let $\vdash \tau$ and so by $(\neg 1)$, we get $X, \neg\beta \vdash \neg\tau$, and so again by the deduction theorem, $X \vdash \neg\beta \rightarrow \neg\tau$, meaning $X \vdash \tau \rightarrow \neg\neg\beta$. Since $\vdash \tau$ by MP we get $X \vdash \neg\neg\beta$ and since $X \vdash \neg\neg\beta \rightarrow \beta$, by MP we get $X \vdash \beta$ as required. ■

1.3.7 Theorem (The Completeness Theorem)

$X \vdash \alpha$ if and only if $X \models \alpha$. More suggestively,

$$\vdash = \models$$

We have already shown $\vdash \subseteq \models$. Since \vdash satisfies all the basic rules of \vdash , $\vdash \subseteq \vdash$ (by \vdash 's principle of induction). Now since $\vdash = \models$, we get that $\models \subseteq \vdash \subseteq \models$, and so $\vdash = \models$. ■

It is important to note that Λ is sufficient to obtain all tautologies only because \rightarrow was defined via \neg and \wedge . Had it been taken as just another connective, we would've needed to add axioms to Λ stating the relation between \rightarrow and \neg and \wedge .

2 First Order Logic

2.1 Mathematical Structures

Our first step in studying first order logic (which will be defined later) is defining the general notion of a *mathematical structure*. Mathematical structures (also known as first order structures) give a useful generalization of many of the algebraic and relational objects mathematicians study.

2.1.1 Definition

An **extralogical signature** is a set σ of symbols of three types: function symbols, relational symbols, and constant symbols. Function symbols and relational symbols are also given an **arity**, a positive integer.

Formally, we can view σ as a tuple: $\sigma = (\sigma_f, \sigma_r, \sigma_c, \text{ar})$, where σ_f is a set of function symbols, σ_r is a set of relational symbols, and σ_c is a set of constant symbols (meaning that they are all just sets of symbols). Further assume that σ_f , σ_r , and σ_c are all disjoint. ar is a function mapping symbols in σ_f and σ_r to positive integers.

2.1.2 Definition

Let σ be an extralogical signature (for short, a signature), **mathematical structure** over σ (for short, a σ -structure) is a pair $\mathcal{A} = (A, \sigma^{\mathcal{A}})$ where A is some non-empty set, called the **domain** of the structure, and $\sigma^{\mathcal{A}}$ is an **interpretation** of σ . This means that for every function symbol $f \in \sigma$, $\sigma^{\mathcal{A}}$ consists of an operation $f^{\mathcal{A}}: A^{\text{ar}(f)} \rightarrow A$, for every relational symbol $r \in \sigma$, $\sigma^{\mathcal{A}}$ contains a relation $r^{\mathcal{A}} \subseteq A^{\text{ar}(r)}$, and for every constant symbol $c \in \sigma$, $\sigma^{\mathcal{A}}$ contains a constant $c^{\mathcal{A}} \in A$.

Constants may be viewed as 0-ary operations.

The domain of a mathematical structure \mathcal{A} will always be denoted by A .

We now define some general notions relating to structures.

Suppose $A \subseteq B$, and f is an n -ary operation on B . Then A is *closed under f* if $f(A^n) \subseteq A$, meaning that for every $\vec{a} \in A^n$, $f\vec{a} \in A$. If $n = 0$, ie. if f is a constant c , then this simply means that $c \in A$. It is obvious that the intersection of a family of sets closed under f is itself closed under f , and thus we can discuss the smallest set closed under f . For example, \mathbb{N} is closed under $+$ (when viewed as a binary operation of \mathbb{N} , \mathbb{Q} , etc.), but not under $-$.

Suppose $A \subseteq B$ again, and r^B is an n -ary relation on B . Then the *restriction* of r^B to A is the n -ary relation $r^A = r^B \cap A^n$. For example the restriction of $<^{\mathbb{Z}}$, the standard order of \mathbb{Z} , to \mathbb{N} is $<^{\mathbb{N}}$, the standard order of \mathbb{N} . If f^B is an n -ary operation on B and $A \subseteq B$ is closed under f^B , then we define f^B 's restriction to A to be the operation $f^A \vec{a} = f^B \vec{a}$.

So if \mathcal{B} is a σ -structure and $A \subseteq B$ is closed under all operations (including constants), then A can be given the structure of a σ -structure naturally: define $\mathcal{A} = (A, \sigma^{\mathcal{A}})$ where for $f \in \sigma_f$ take $f^{\mathcal{A}} = f^{\mathcal{B}}$ the restriction of $f^{\mathcal{B}}$ to A , for $r \in \sigma_r$ take $r^{\mathcal{A}} = r^{\mathcal{B}}$ the restriction of $r^{\mathcal{B}}$ to A , and for $c \in \sigma_c$ take $c^{\mathcal{A}} = c^{\mathcal{B}}$. \mathcal{A} is called a *substructure* of \mathcal{B} , denoted $\mathcal{A} \subseteq \mathcal{B}$.

Note that not every subset $A \subseteq B$ can be extended to a substructure of \mathcal{B} . For example, $\{1\} \subseteq \mathbb{Z}$ but if the signature σ is taken to include the constant 0, then since $\{1\}$ does not contain $0^{\mathbb{Z}} = 0$ it cannot be extended to a substructure. And similarly if σ includes $+$, then since $\{+\}$ is not closed under $+$, it cannot be extended to a substructure.

Suppose \mathcal{A} is a σ -structure and $\sigma_0 \subseteq \sigma$ is another extralogical signature (meaning $\sigma_{0_x} \subseteq \sigma_x$ for $x = f, r, c$ and $\text{ar}_0(\mathbf{s}) = \text{ar}(\mathbf{s})$ for all relational and function symbols $\mathbf{s} \in \sigma_0$). Then we define the σ_0 -structure \mathcal{A}_0 where the interpretation of each symbol $\mathbf{s} \in \sigma_0$ is $\mathbf{s}^{\mathcal{A}_0} = \mathbf{s}^{\mathcal{A}}$. \mathcal{A}_0 is called the σ_0 -*reduct* of \mathcal{A} , and conversely \mathcal{A} is called the σ -*expansion* of \mathcal{A}_0 .

Many times, if σ is a signature consisting of the symbols s_1, s_2, \dots , we will write a σ -structure as (A, s_1, s_2, \dots) instead of writing out the signature. And further, we will often write the signature as a set instead of as a tuple of sets and an arity function. What symbols are functions, relational, and constants, and their arities are to be understood from context.

Mathematical structures defined over a signature without relational symbols are termed *algebraic structures*, while structures defined over a signature without function or constant symbols are termed *relational structures*.

For example, mathematical structures of the form $\mathcal{A} = (A, \circ)$ where \circ is a binary operation are called *magmas*. If \circ is associative, \mathcal{A} is a *semigroup*, if it is invertible in each argument then it is a *group*, etc. These are

examples of very common algebraic structures. Another common algebraic structure are *rings* and *fields*: both are structures of the form $\mathcal{A} = (A, +, \cdot, 0, 1)$ which satisfy certain axioms. Notice that a structure of this form is not necessarily a ring, but all rings are structures of this form.

A *semilattice* is another type of algebraic structure, and is a special case of a magma where \circ is associative, commutative, and idempotent (meaning $a \circ a = a$ for all $a \in A$). For example $(\{0, 1\}, \wedge)$ is a semilattice. We can define the partial order \leq by $a \leq b \iff a \circ b = a$. This is reflexive since \circ is, anticommutative since \circ is commutative, and if $a \leq b$ and $b \leq c$ then $a = a \circ b = a \circ (b \circ c) = (a \circ b) \circ c = a \circ c$ so $a \leq c$. And a *lattice* is an algebraic structure of the form $\mathcal{A} = (A, \cap, \cup)$ where (A, \cap) and (A, \cup) are both semilattices and the following absorption laws hold: $a \cap (a \cup b) = a$ and $a \cup (a \cap b) = a$. A *distributive lattice* is a lattice which satisfies the distributive properties: $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$ and $x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$. For example if M is a set, then $(\mathcal{P}(M), \cap, \cup)$ is a lattice.

A *boolean algebra* is an algebraic structure $\mathcal{A} = (A, \cap, \cup, \neg)$ where the reduct (A, \cap, \cup) is a distributive lattice and

$$\neg \neg x = x, \quad \neg(x \cap y) = \neg x \cup \neg y, \quad x \cap \neg x = y \cap \neg y$$

The standard example is the boolean algebra $\mathcal{B} = (\{0, 1\}, \wedge, \vee, \neg)$.

A relational structure $\mathcal{A} = (A, \triangleleft)$ where \triangleleft is a binary relation is often called a *graph* (this coincides with the definition of a directed graph). If \triangleleft is irreflexive and transitive, this is a (*strict*) *partially ordered set*, or poset for short, and we generally write $<$ for \triangleleft . A *partially ordered set* is when \triangleleft is reflexive, transitive, and antisymmetric, then we usually write \leq for \triangleleft . Each partially ordered set gives rise to a strict partially ordered set and vice versa, by defining $a \leq b \iff a < b \vee a = b$

2.1.3 Definition

Let σ be some signature, and \mathcal{A} and \mathcal{B} be σ -structures. Then a map $h: A \longrightarrow B$ (though we will generally write $h: \mathcal{A} \longrightarrow \mathcal{B}$) is called a **homomorphism** provided that for every function symbol f , relational symbol r , and constant symbol c in σ , and $\vec{a} \in A^n$:

$$h(f^{\mathcal{A}}(\vec{a})) = f^{\mathcal{B}}(h(\vec{a})), \quad h(c^{\mathcal{A}}) = c^{\mathcal{B}}, \quad r^{\mathcal{A}}(\vec{a}) \implies r^{\mathcal{B}}(h(\vec{a}))$$

where $h(\vec{a}) = (h(a_1), \dots, h(a_n))$.

A **strong homomorphism** is a homomorphism where the third condition on relations is replaced by the stronger $r^{\mathcal{B}}(h(\vec{a}))$ if and only if there exists a $\vec{b} \in A^n$ such that $h(\vec{a}) = h(\vec{b})$ and $r^{\mathcal{A}}(\vec{b})$ (thus we need not require that every \vec{b} with the same image as \vec{a} under h satisfy $r^{\mathcal{A}}$, only that one does). In other words, the condition is replaced with

$$r^{\mathcal{B}}(h(\vec{a})) \iff (\exists \vec{b} \in A^n)(h(\vec{a}) = h(\vec{b}) \wedge r^{\mathcal{A}}(\vec{b}))$$

An injective strong homomorphism $\mathcal{A} \longrightarrow \mathcal{B}$ is called an **embedding** of \mathcal{A} into \mathcal{B} . If further the embedding is surjective, it is termed a **isomorphism**. If there exists an isomorphism between \mathcal{A} and \mathcal{B} , the two structures are called **isomorphic**, and this is denoted $\mathcal{A} \cong \mathcal{B}$. Similarly if $\mathcal{A} = \mathcal{B}$ then an isomorphism is called a **automorphism**.

We will sometimes dispense of parentheses and write $f\vec{a}$ instead of $f(\vec{a})$.

Notice that for algebraic structures, strong and “weak” homomorphisms are one and the same. Furthermore, if $h: \mathcal{A} \longrightarrow \mathcal{B}$ is an embedding, the condition on h being a strong isomorphism is simply

$$r^{\mathcal{A}}\vec{a} \iff r^{\mathcal{B}}h\vec{a}$$

as $(\exists \vec{b} \in A^n)(h\vec{a} = h\vec{b} \wedge r^{\mathcal{A}}\vec{a})$ is equivalent to $r^{\mathcal{A}}\vec{a}$ as $h\vec{a} = h\vec{b}$ implies $\vec{a} = \vec{b}$.

The composition of homomorphisms is itself a homomorphism: if $h_1: \mathcal{A} \longrightarrow \mathcal{B}$ and $h_2: \mathcal{B} \longrightarrow \mathcal{C}$ are homomorphisms then

$$\begin{aligned} h_2 \circ h_1(f^{\mathcal{A}}\vec{a}) &= h_2(f^{\mathcal{B}}h_1\vec{a}) = f^{\mathcal{C}}h_2 \circ h_1(\vec{a}) \\ h_2 \circ h_1(c^{\mathcal{A}}) &= h_2c^{\mathcal{B}} = c^{\mathcal{C}} \\ r^{\mathcal{A}}\vec{a} \implies r^{\mathcal{B}}h_1\vec{a} &\implies r^{\mathcal{C}}h_2h_1\vec{a} \end{aligned}$$

And if h_1 and h_2 are strong homomorphisms, and h_1 is surjective, then $h_2 \circ h_1$ is also a strong homomorphism:

$$r^{\mathcal{C}}h_2 \circ h_1\vec{a} \iff (\exists \vec{b} \in B^n)(h_2\vec{b} = h_2h_1\vec{a} \wedge r^{\mathcal{B}}\vec{b})$$

Since h_1 is surjective, suppose $h_1\vec{a}_0 = \vec{b}$ then

$$\iff (\exists \vec{a}_0 \in A^n)(h_2 h_1 \vec{a}_0 = h_2 h_1 \vec{a} \wedge r^B h_1 \vec{a}_0)$$

Since $r^B h_1 \vec{a}_0$ if and only if there exists an a_1 such that $h_1 \vec{a}_0 = h_1 \vec{a}_1$ and $r^A \vec{a}_1$, so this is equivalent to

$$\iff (\exists \vec{a}_1 \in A^n)(h_2 h_1 \vec{a}_1 = h_2 h_1 \vec{a} \wedge r^A \vec{a}_1)$$

As required.

2.1.4 Definition

Let σ be a signature and \mathcal{A} be a σ -structure. Then a **congruence** on \mathcal{A} is an equivalence relation on A , \approx , such that for all function symbols $f \in \sigma$ with arity $n > 0$,

$$\vec{a} \approx \vec{b} \implies f^A \vec{a} \approx f^A \vec{b}$$

where $\vec{a} \approx \vec{b}$ means $a_i \approx b_i$ for $i = 1, \dots, n$ where $\vec{a} = (a_1, \dots, a_n)$ and $\vec{b} = (b_1, \dots, b_n)$.

Let us denote a/\approx to be the equivalence class of a under \approx , and $\vec{a}/\approx = (a_1/\approx, \dots, a_n/\approx)$ for $\vec{a} \in A^n$. Let $f \in \sigma$ be a function symbol, $r \in \sigma$ be a relational symbol, and $c \in \sigma$ be a constant symbol, then let us define the σ -structure \mathcal{A}' over the domain partition A/\approx by

$$f^{\mathcal{A}'}(\vec{a}/\approx) := (f^A(\vec{a}))/\approx, \quad r^{\mathcal{A}'}(\vec{a}/\approx) \iff (\exists \vec{b} \approx \vec{a})(r^A \vec{b}), \quad c^{\mathcal{A}'} = (c^A)/\approx$$

These are well-defined as they are independent of the choice of representative from an equivalence class (only the first definition, for $f^{\mathcal{A}'}$, is not true for general equivalence relations). \mathcal{A}' is the **quotient structure** of \mathcal{A} modulo \approx , also denoted by \mathcal{A}/\approx (the use of \mathcal{A}' was to make it more readable in superscripts).

Let G be a group with the identity e and \approx be a congruence on G . Then let us define $N = \{g \in G \mid g \approx e\}$, and N is a normal subgroup: if $g \in N$ and $h \in G$ then $hgh^{-1} \approx heh^{-1} = e$, and so $hgh^{-1} \in N$. And if N is a normal subgroup, let us define $a \approx_N b$ if and only if $ab^{-1} \in N$, then if $a_1 \approx_N a_2$ and $b_1 \approx_N b_2$ then

$$a_1 b_1 \approx_N a_2 b_2 \iff a_1 b_1 b_2^{-1} a_2^{-1} \in N \iff a_1 (b_1 b_2^{-1} a_2^{-1} a_1) a_1^{-1} \in N$$

since $b_1 b_2^{-1} \in N$ and $a_2^{-1} a_1 \in N$, and since N is normal, this is indeed correct. So \approx_N is a congruence on G . This relation is deeper: recall that normal groups are simply kernels of group homomorphisms. So we can define the kernel of general homomorphisms:

2.1.5 Definition

Let $h: \mathcal{A} \longrightarrow \mathcal{B}$ be a homomorphism of σ -structures. Then h 's **kernel** is the congruence on \mathcal{A} defined by

$$a \approx_h b \iff h(a) = h(b)$$

This is indeed a congruence on \mathcal{A} : if $\vec{a} \approx_h \vec{b}$ and $f \in \sigma$ then

$$f^A \vec{a} \approx_h f^A \vec{b} \iff h f^A \vec{a} = h f^A \vec{b} \iff f^B h \vec{a} = f^B h \vec{b}$$

which is true since $h \vec{a} = h \vec{b}$ as $\vec{a} \approx_h \vec{b}$.

Let h be a group homomorphism, and K be its kernel (viewed as a normal subgroup) then $\approx_h = \approx_K$ where \approx_K is defined for groups as previously: $h(a) = h(b)$ if and only if $h(ab^{-1}) = e$ if and only if $ab^{-1} \in K$ if and only if $a \approx_K b$. So this definition of a kernel is natural, and generalizes much nicer than the group-theoretic definition.

2.1.6 Theorem (The Isomorphism Theorem)

(1) Let \mathcal{A} be a σ -structure, and \approx a congruence on \mathcal{A} . Then $k: a \mapsto a/\approx$ is a strong homomorphism from

\mathcal{A} onto \mathcal{A}/\approx .

- (2) Conversely, if $h: \mathcal{A} \rightarrow \mathcal{B}$ is a surjective strong homomorphism of σ -structures, then $\iota: \mathcal{A}/\approx_h \rightarrow \mathcal{B}$ is an isomorphism between \mathcal{A}/\approx_h and \mathcal{B} . Furthermore, $h = \iota \circ k$.

Let $f, r, c \in \sigma$ be function, relational, and constant symbols respectively. For readability, we will ignore superscripts.

- (1) We do this directly:

$$\begin{aligned} k(f\vec{a}) &= (f\vec{a})/\approx = f(\vec{a}/\approx) = f(k\vec{a}) \\ (\exists \vec{b} \in A^n)(k\vec{a} = k\vec{b} \wedge r\vec{b}) &\iff (\exists \vec{b} \approx \vec{a})(r\vec{b}) \iff r(\vec{a}/\approx) \iff rk\vec{a} \\ k(c) &= c/\approx = c^{\mathcal{A}/\approx} \end{aligned}$$

So k is indeed a strong homomorphism.

- (2) The definition of ι is obviously sound (ie. it is well-defined) and injective by the definition of \approx_h :

$$\iota(a/\approx_h) = \iota(b/\approx_h) \iff h(a) = h(b) \iff a \approx_h b \iff a/\approx_h = b/\approx_h$$

It is surjective since if $b \in \mathcal{B}$, since h is surjective there exists an $a \in \mathcal{A}$ such that $h(a) = b$ and so $\iota(a/\approx_h) = h(a) = b$. Now, ι is a strong homomorphism:

$$\begin{aligned} \iota f(\vec{a}/\approx_h) &= \iota(f\vec{a})/\approx_h = h(f\vec{a}) = f(h\vec{a}) = f\iota(\vec{a}/\approx_h) \\ r\iota(\vec{a}/\approx_h) &\iff rh(\vec{a}) \iff (\exists \vec{b} \approx_h \vec{a})(r(\vec{b})) \iff r(\vec{a}/\approx_h) \\ \iota c/\approx_h &= h(c) = c \end{aligned}$$

By the definitions of ι and k , $h = \iota \circ k$. ■

We need not require h be surjective: instead we alter the claim and ι becomes an isomorphism between \mathcal{A} and the image of \mathcal{A} under h (denoted $h\mathcal{A}$), which is a substructure of \mathcal{B} (this is easy to verify). This corollary is a direct result of the above theorem, as h is a strong homomorphism from \mathcal{A} to $h\mathcal{A}$.

2.1.7 Definition

Let $\{A_i\}_{i \in I}$ be a family of sets, then we define their **direct product** to be the set of function $I \rightarrow \bigcup_{i \in I} A_i$ such that for every $i \in I$, $i \mapsto a_i$ where $a_i \in A_i$. Such a function is denoted $(a_i)_{i \in I}$ (similar to how a sequence is denoted $(a_n)_{n=1}^\infty$ as it represents a function $\mathbb{N} \rightarrow \mathbb{R}$ which maps $n \mapsto a_n$). So the direct product is defined as, in set-theoretic terms:

$$\prod_{i \in I} A_i = \left\{ f: I \rightarrow \bigcup_{i \in I} A_i \mid (\forall i \in I)(f(i) \in A_i) \right\}$$

Where the function f is written as $(f(i))_{i \in I}$ (this is generally more readable).

If $\{\mathcal{A}_i\}_{i \in I}$ is a family of σ -structures, we define their **direct product** to be a σ -structure $\mathcal{B} = \prod_{i \in I} \mathcal{A}_i$ whose domain is the direct product of the domains of \mathcal{A}_i (so if A_i is the domain of \mathcal{A}_i , the domain is $B = \prod_{i \in I} A_i$) and for every function symbol f , relational symbol r , and constant symbol c in σ we define

$$f^{\mathcal{B}}\vec{a} = (f^{\mathcal{A}_i}\vec{a}_i)_{i \in I}, \quad r^{\mathcal{B}}\vec{a} \iff r^{\mathcal{A}_i}\vec{a}_i \text{ for all } i \in I, \quad c^{\mathcal{B}} = (c^{\mathcal{A}_i})_{i \in I}$$

Where $\vec{a} = ((a_i^1)_{i \in I}, \dots, (a_i^n)_{i \in I}) \in B^n$ and $\vec{a}_i = (a_i^1, \dots, a_i^n) \in A_i^n$ is obtained by looking at the components of \vec{a} at a specific $i \in I$ (take care, this is not the i th component of \vec{a}).

If all the structures are the same, $\mathcal{A}_i = \mathcal{A}$ for all $i \in I$, then $\prod_{i \in I} \mathcal{A}_i$ is called the **direct power** of \mathcal{A} and is denoted \mathcal{A}^I . If $I = \{1, \dots, n\}$ then $\prod_{i \in I} \mathcal{A}_i$ is also written $\mathcal{A}_1 \times \dots \times \mathcal{A}_n$ and $\prod_{i \in I} \mathcal{A}$ is written \mathcal{A}^n .

Notice that our concept of \mathbb{R}^n as an abelian group corresponds with the above definition. But here we have also defined the coordinate-wise product of vectors in \mathbb{R}^n .

We can define the *projection homomorphism* from a direct product to one of its components:

$$\pi_j: \prod_{i \in I} \mathcal{A}_i \longrightarrow \mathcal{A}_j, \quad (a_i)_{i \in I} \mapsto a_j$$

where $j \in I$. This is indeed a homomorphism, let $\vec{a} = ((a_i^1)_{i \in I}, \dots, (a_i^n)_{i \in I})$, then $f\vec{a} = ((fa_i^1)_{i \in I}, \dots, (fa_i^n)_{i \in I})$ and so

$$\pi_j f\vec{a} = (fa_j^1, \dots, fa_j^n) = f(a_j^1, \dots, a_j^n) = f\pi_j \vec{a}$$

As required, and $r\vec{a}$ if and only if $r(a_i^1, \dots, a_i^n)$ for all $i \in I$, which implies $r(a_j^1, \dots, a_j^n) = r\pi_j \vec{a}$. The case for constants is implied by the proof for functions.

But it is not necessarily strong: the condition for strongness is that $r\pi_j \vec{a}$ must be equivalent to

$$(\exists \vec{b})(\pi_j \vec{b} = \pi_j \vec{a} \wedge r\vec{b}) \iff (\exists \vec{b})(\pi_j \vec{b} = \pi_i \vec{a} \wedge (\forall i)(r\pi_i \vec{b}))$$

Since the definition of $r^B \vec{a}$ is literally $r^{A_i} \pi_i \vec{a}$ for all $i \in I$. So this is clearly stronger than $r\pi_j \vec{a}$, and unless we know that for every i , r^{A_i} can be satisfied, it is strictly stronger. But if we know that for all $i \in I$ (except for potentially j), there exists a $\vec{a}_i \in \mathcal{A}_i$ such that $r^{A_i} \vec{a}_i$, then this is equivalent.

2.2 Syntax of First-Order Languages

First-order logic allow us to discuss precise concepts relating to mathematical structures. Unlike propositional logic, first-order logic has the ability to discuss individual variables within a mathematical structure, and it can *quantify* them as well. Like propositional logic, we must first discuss the syntax of first-order logic, which is more involved.

Let us define a set of variables, which is taken to be a countably infinite set of distinct symbols: $\text{Var} = \{v_1, v_2, \dots\}$. Like any language, we must first define the *alphabet* over which we define the language of first-order logic. First-order logic over an extralogical signature σ has the alphabet consisting of: the extralogical symbols of σ ; the variables in Var ; the logical connectives \wedge and \neg ; the quantifier \forall (*for all*); the equality sign $=$ (in boldface to distinguish it from the metalogical symbol $=$); and parentheses (and). Other logical connectives, like \vee , \leftrightarrow , and \rightarrow can be defined via \wedge and \neg , as discussed before. Similarly other quantifiers like \exists (*there exists*) and $\exists!$ (*there exists a unique*) can be defined as well, which will be discussed later.

From the set of all strings over this alphabet are many meaningless ones, for example $)\forall\wedge$ would be a string over this alphabet, but it has no useful meaning. Like what we did in the previous section, we will recursively define meaningful strings from this alphabet.

2.2.1 Definition

We first define **terms** in this language. Terms are defined recursively as:

- (1) Variables and constant symbols in σ , are **prime terms**.
- (2) If $f \in \sigma$ is an n -ary function symbol, and t_1, \dots, t_n are terms, then $ft_1 \dots t_n$ is a term as well.

The set of all terms (ie. all strings constructed in this matter) is denoted \mathcal{T} .

Notice that we do not use parentheses with terms, as this simplifies syntax and parentheses turn out to be unnecessary. Despite this, when we actually need to write terms (ie. outside of proofs about terms), we may add parentheses for readability. Also note that all of these definitions (and all the coming definitions) are dependent on the choice of extralogical signature σ , so we may speak of *terms over σ* , or σ -*terms*.

Notice that we can view \mathcal{T} as a σ' -structure where σ' is the signature obtained from σ after removing all relational symbols. This is as for $f \in \sigma'$ we define $f^{\mathcal{T}}(t_1, \dots, t_n) = ft_1 \dots t_n$ (the right hand side is a term, a string, and is to be read literally) and for $c \in \sigma'$ we define $c^{\mathcal{T}} = c$. So \mathcal{T} is also sometimes called the *term algebra*.

2.2.2 Proposition (Principle of Term Induction)

Let \mathcal{E} be a property of strings (over this language) such that \mathcal{E} is true for all prime terms, and for all $n > 0$ and each n -ary function symbol $f \in \sigma$, $\mathcal{E}t_1, \dots, \mathcal{E}t_n$ implies $\mathcal{E}ft_1 \dots t_n$. Then \mathcal{E} holds for all terms.

This is true since \mathcal{T} is taken as the smallest set obtained by the two rules (that it contains all prime terms, and if t_1, \dots, t_n are terms then so is $ft_1 \dots t_n$). So \mathcal{E} must then contain \mathcal{T} .

2.2.3 Lemma

Let t be a term, then no proper initial segment of t is a term.

This follows from the principle of term induction: let $\mathcal{E}t$ be the property “no proper initial segment of t is a term, and t is not a proper initial segment of some other term”. Then \mathcal{E} holds for prime terms, as these are atomic characters from the alphabet and thus have no proper initial segments. And let p be a prime term since all other terms are either prime, or of the form $ft_1 \cdots t_n$, since $p \neq f$ are distinct symbols, p cannot be a proper initial segment of another term. Now if $\mathcal{E}t_1, \dots, \mathcal{E}t_n$ and $f \in \sigma$ is n -ary then any proper initial segment of $ft_1 \cdots t_n$ is of the form $ft_1 \cdots t_k \xi$, where ξ is a proper initial segment of t_{k+1} (it may also be empty). But in order for $ft_1 \cdots t_k \xi$ to be a term, it must be equal to $fs_1 \cdots s_n$ for other terms s_i , but by $\mathcal{E}t_1$, s_1 can not be an initial segment of t_1 nor can t_1 be an initial segment of s_1 , so $t_1 = s_1$. Continuing inductively, we have that $t_i = s_i$ for $i \leq k$, and so we get that $\xi = s_{k+1} \cdots s_n$, but this implies s_{k+1} is an initial segment of ξ , but then s_{k+1} is a proper initial segment of t_{k+1} , and so it cannot be a term by $\mathcal{E}t_{k+1}$ in contradiction.

So no proper initial segment of $ft_1 \cdots t_n$ is a term. And if $ft_1 \cdots t_n$ is the proper initial segment of some other term $fs_1 \cdots s_n$, then by induction we see that $t_i = s_i$ (since t_1 is either an initial segment of s_1 or vice versa) contradicting it being proper. ■

2.2.4 Proposition (Unique Term Concatenation Property)

Suppose t_i and s_j are terms, then if $t_1 \cdots t_n = s_1 \cdots s_m$ then $n = m$ and $t_i = s_i$ for all $1 \leq i \leq n$.

If $t_1 \cdots t_n = s_1 \cdots s_m$ then t_1 is either a initial segment of s_1 or vice versa, but by the lemma above, this cannot be proper, so $t_1 = s_1$. Thus $t_2 \cdots t_n = s_2 \cdots s_m$ and so inducting on n , we get $t_i = s_i$ and $n = m$ as required. ■

Using the unique term concatenation property, we can recursively define functions on terms without worrying about them being well-defined:

2.2.5 Definition

Let t be a term, then we define its **set of variables** recursively as follows:

$$\text{var } c = \emptyset \text{ for constant symbols } c, \quad \text{var } x = \{x\} \text{ for } x \in \text{Var}, \quad \text{var } ft_1 \cdots t_n = \text{var } t_1 \cup \cdots \cup \text{var } t_n$$

Alternatively we could simply define it as the set of all symbols in Var which occur in t .

2.2.6 Definition

We now define **formulas** in our language (again, these are defined with respect to a specific signature, and may be called *formulas over σ* or *σ -formulas*). These are strings defined recursively by the rules

- (1) If s and t are terms, then $s = t$ is a formula, called an **equation**.
- (2) If t_1, \dots, t_n are terms and $r \in \sigma$ is an n -ary relational symbol, then $rt_1 \cdots t_n$ is a formula.
- (3) If α and β are formulas and x is a variable, then $(\alpha \wedge \beta)$, $\neg \alpha$, and $\forall x \alpha$ are formulas.

Formulas defined by the first two rules are **prime formulas**. Formulas which do not contain any quantifiers (no occurrences of \forall , and since other quantifiers like \exists are defined using \forall , this includes all other quantifiers) are called **quantifier-free**.

The set of all formulas over a signature σ is denoted $\mathcal{L}\sigma$. In the case that σ contains only a single symbol, $\sigma = \{s\}$, we may write \mathcal{L}_s instead. And the set of all formulas over the signature \emptyset is denoted $\mathcal{L}_=$ and is called the **language of pure identity**.

If \circ is a binary operation, we will often write $t \circ s$ instead of $\circ ts$ as dictated by the definition of compound terms. Similarly if \triangleleft is a binary relation, we will often write $t \triangleleft s$ instead of $\triangleleft ts$. Formally these are abbreviations which refer to the correct form of writing the terms and formulas.

We define the following abbreviations:

$$(\alpha \vee \beta) := \neg(\neg\alpha \wedge \neg\beta), \quad (\alpha \rightarrow \beta) := \neg(\alpha \wedge \neg\beta), \quad (\alpha \leftrightarrow \beta) := ((\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha))$$

$$\exists x\alpha := \neg\forall x\neg\alpha$$

The first line of definitions should be familiar; they are the same as those defined in the previous section. The definition on the second line should make sense intuitively: “*there exists an x such that α* ” if and only if not all x s don’t satisfy α . The symbols \forall and \exists are called *quantifiers*. \forall is also called the *universal quantifier*, and \exists is the *existential quantifier*.

2.2.7 Proposition (Principle of Formula Induction)

If \mathcal{E} is a property of strings such that \mathcal{E} holds for all prime formulas and $\mathcal{E}\alpha$ and $\mathcal{E}\beta$ implies $\mathcal{E}(\alpha \wedge \beta)$, $\mathcal{E}\neg\alpha$, and $\mathcal{E}\forall x\alpha$, then \mathcal{E} holds for all formulas in \mathcal{L} .

Again, this is directly due to the definition of \mathcal{L} .

2.2.8 Proposition (Unique Formula Reconstruction Property)

Every formula $\alpha \in \mathcal{L}$ is either prime or can be written uniquely as $(\alpha \wedge \beta)$, $\neg\alpha$, or $\forall x\alpha$ for $\alpha, \beta \in \mathcal{L}$ and $x \in \text{Var}$.

The proof of this is similar to all similar previous propositions: first show that no proper initial segment of a formula is itself a formula, then this follows immediately.

Now, instead of discussing σ -structures and σ -terms, we will refer to them as \mathcal{L} -structures and \mathcal{L} -terms where \mathcal{L} is a first-order language (which is itself defined over a signature σ . This is simply accepted terminology.)

2.2.9 Definition

We define the set of variables in a formula φ recursively. First we define it on equations: $\text{vars} = t = \text{vars} \cup \text{var}t$, then we define it on prime formulas which are not equations: $\text{var}t_1 \cdots t_n = \text{var}t_1 \cup \cdots \cup \text{var}t_n$. Now for the recursive part:

$$\text{var}(\alpha \wedge \beta) = \text{var}\alpha \cup \text{var}\beta, \quad \text{var}\neg\alpha = \text{var}\alpha, \quad \text{var}\forall x\alpha = \text{var}\alpha \cup \{x\}$$

Alternatively we could define it as all the variables which occur in φ .

As before, we define the **rank** of a formula recursively as follows:

$$\text{rank}\pi = 0 \text{ for prime formulas } \pi, \quad \text{rank}(\alpha \wedge \beta) = \max\{\text{rank}\alpha, \text{rank}\beta\} + 1, \quad \text{rank}\neg\alpha = \text{rank}\alpha + 1,$$

$$\text{rank}\forall x\alpha = \text{rank}\alpha + 1$$

And we similarly define the **quantifier rank** of a formula, which measures the maximum nesting depth of a quantifier in the formula:

$$\text{qr}\pi = 0 \text{ for prime formulas } \pi, \quad \text{qr}(\alpha \wedge \beta) = \max\{\text{qr}\alpha, \text{qr}\beta\}, \quad \text{qr}\neg\alpha = \text{qr}\alpha,$$

$$\text{qr}\forall x\alpha = \text{qr}\alpha + 1$$

The set of **subformulas** of a formula is defined similar to before:

$$\text{Sf}\pi = \{\pi\} \text{ for prime formulas } \pi, \quad \text{Sf}(\alpha \wedge \beta) = \text{Sf}\alpha \cup \text{Sf}\beta \cup \{(\alpha \wedge \beta)\}, \quad \text{Sf}\neg\alpha = \text{Sf}\alpha \cup \{\alpha\},$$

$$\text{Sf}\forall x\alpha = \text{Sf}\alpha \cup \{\forall x\alpha\}$$

2.2.10 Definition

A string of the form $\forall x$ (and by extension $\exists x$) is called a **prefix**. And given a subformula of the form $\forall x\alpha$,

α is called the **scope** of $\forall x$. Occurrences of x within the scope of an occurrence of $\forall x$ are termed **bound occurrences** of x , all other occurrences of x are termed **free occurrences** of x . In general we say that a variable x **occurs bound** in a formula φ if the prefix $\forall x$ occurs in φ .

We define $\text{bnd}\varphi$ to be the set of all variables which occur bound in φ , and $\text{free}\varphi$ to be the set of all variables which have free occurrences in φ .

$\text{bnd}\varphi$ and $\text{free}\varphi$ can also be defined recursively:

$$\begin{aligned} \text{bnd}\pi &= \emptyset \text{ for prime formulas } \pi, & \text{bnd}(\alpha \wedge \beta) &= \text{bnd}\alpha \cup \text{bnd}\beta, & \text{bnd}\neg\alpha &= \text{bnd}\alpha, \\ \text{bnd}\forall x\alpha &= \text{bnd}\alpha \cup \{x\} \\ \text{free}\pi &= \text{var}\pi \text{ for prime formulas } \pi, & \text{free}(\alpha \wedge \beta) &= \text{free}\alpha \cup \text{free}\beta, & \text{free}\neg\alpha &= \text{free}\alpha, \\ \text{free}\forall x\alpha &= \text{free}\alpha \setminus \{x\} \end{aligned}$$

Notice that a variable can occur both free and bound in a formula, for example in the below formula x occurs both free and bound

$$\forall x(x = y) \wedge (x = y)$$

This will generally be avoided, but it can happen. We could strengthen our definitions of formulas to ensure that this does not occur, but there is no need to do so.

2.2.11 Definition

Let us define \mathcal{L}^k to be the set of all formulas φ such that $\text{free}\varphi \subseteq \{v_0, \dots, v_{k-1}\}$. Thus $\mathcal{L}^0 \subseteq \mathcal{L}^1 \subseteq \dots$ and $\mathcal{L} = \bigcup_{k=0}^{\infty} \mathcal{L}^k$. \mathcal{L}^0 is the set of all formulas which contain no free variables, formulas belonging to \mathcal{L}^0 are called **sentences** or **closed formulas**.

Note

If φ is a formula, we write $\varphi(\vec{x})$ to mean that $\text{free}\varphi \subseteq \{x_1, \dots, x_n\}$, where $\vec{x} = (x_1, \dots, x_n)$ and x_i are all arbitrary and distinct. Similarly if t is a term, $t(\vec{x})$ means $\text{var}t \subseteq \{x_1, \dots, x_n\}$.

And we write $f\vec{t}$ to mean the compound term $ft_1 \cdots t_n$ where $\vec{t} = (t_1, \dots, t_n)$ where t_i are terms. Similarly we write $r\vec{t}$ to mean the prime formula $rt_1 \cdots t_n$.

Now we would like to define a notion of substitution, which is a natural concept to have. But importantly we'd only like to substitute variables at their free occurrences, why? Suppose you have the formula $\varphi(y) = \exists x(x + x = y)$ (in the context of integers, this means y is even). We could substitute y for 2 and get $\exists x(x + x = 2)$. But now say we wanted to substitute x for 2, then should we get $\exists 2(2 + 2 = y)$ (which is not a valid formula), or $\exists x(2 + 2 = y)$? Well, neither, because neither really makes sense. This is since bound occurrences already have meaning associated with them by their quantifier, so it makes little sense to substitute them.

2.2.12 Definition

A **substitution** (also called a *global* substitution) is a function which assigns to every variable a term, meaning it is a function $\sigma: \text{Var} \rightarrow \mathcal{T}$, where $x \mapsto x^\sigma$. We first extend it to a substitution of terms, ie. a function $\sigma: \mathcal{T} \rightarrow \mathcal{T}$ as follows:

$$c^\sigma = c \text{ for constant symbols } c, \quad (ft_1 \cdots t_n)^\sigma = ft_1^\sigma \cdots t_n^\sigma$$

and now we extend it to a substitution of formulas, ie. a function $\sigma: \mathcal{L} \rightarrow \mathcal{L}$ as follows:

$$(s = t)^\sigma = s^\sigma = t^\sigma, \quad (rt_1 \cdots t_n)^\sigma = rt_1^\sigma \cdots t_n^\sigma, \quad (\alpha \wedge \beta)^\sigma = \alpha^\sigma \wedge \beta^\sigma, \quad (\neg\alpha)^\sigma = \neg\alpha^\sigma, \quad (\forall x\alpha)^\sigma = \forall x\alpha^\tau$$

where τ is the substitution $y^\tau = y^\sigma$ for variables y distinct from x , and $x^\tau = x$ as we'd like to substitute only at free occurrences of variables.

A **simultaneous substitution** is a substitution σ such that there exist variables $x_1, \dots, x_n \in \text{Var}$ and terms $t_1, \dots, t_n \in \mathcal{T}$ such that

$$x^\sigma = \begin{cases} t_i & x = x_i \\ x & \text{else} \end{cases}$$

So we substitute only x_i s with t_i s, and leave all other variables the same. Instead of φ^σ we write instead $\varphi_{x_1 \dots x_n}^{t_1 \dots t_n}$. In the case that $n = 1$ (we only substitute a single variable), this is called a **simple substitution**.

Notice that while by definition there is no significance in the order of writing the variables and their substitutions in a simultaneous substitution (meaning there is no difference between $\varphi_{x_1 \dots x_n}^{t_1 \dots t_n}$ and $\varphi_{x_{\sigma 1} \dots x_{\sigma n}}^{t_{\sigma 1} \dots t_{\sigma n}}$ where σ is a permutation), it is not true in general that

$$\varphi_{x_1 x_2}^{t_1 t_2} = \varphi_{x_1}^{t_1} \varphi_{x_2}^{t_2} \left(= \left(\varphi_{x_1}^{t_1} \right) \varphi_{x_2}^{t_2} \right)$$

For example, let $\varphi = x_1 < x_2$, then $\varphi_{x_1 x_2}^{x_2 x_1} = x_2 < x_1$, while $\varphi_{x_1}^{x_2} \varphi_{x_2}^{x_1} = x_2 < x_2 \frac{x_1}{x_2} = x_1 < x_1$. Though it is the case that

$$\varphi_{\vec{x}}^{\vec{t}} = \varphi_{x_n}^{t_n} \varphi_{x_1 \dots x_{n-1}}^{t_1 \dots t_{n-1}} \varphi_{y_n}^{t_n}$$

where y is a variable not in $\text{var}\varphi \cup \text{var}\vec{x} \cup \text{var}\vec{t}$. This should make sense, as we substitute x_n first with y , which remains unchanged by the next simultaneous substitution, and then substitute y for t_n . Thus inductively we see that every simultaneous substitution can be written as a composition of simple substitutions.

2.3 Semantics of First-Order Languages

Similar to how in propositional logic we defined models to give meaning to propositional formulas, we do the same for first-order logic.

2.3.1 Definition

Suppose \mathcal{L} is a first-order language, then an \mathcal{L} -**model** (or an \mathcal{L} -**interpretation**) is a pair $\mathcal{M} = (\mathcal{A}, w)$ where \mathcal{A} is an \mathcal{L} -structure and w is a **valuation function**, $w: \text{Var} \longrightarrow A$, $x \mapsto x^w$. We denote $f^{\mathcal{A}}$, $r^{\mathcal{A}}$, $c^{\mathcal{A}}$, and x^w also by $f^{\mathcal{M}}$, $r^{\mathcal{M}}$, $c^{\mathcal{M}}$, and $x^{\mathcal{M}}$ respectively.

We can extend valuations to \mathcal{T} in an obvious manner:

$$c^{\mathcal{M}} = c \text{ for constant symbols } c, \quad (ft_1 \dots t_n)^{\mathcal{M}} = f^{\mathcal{M}} t_1^{\mathcal{M}} \dots t_n^{\mathcal{M}}$$

In place of $t^{\mathcal{M}}$ we may write $t^{\mathcal{A}, w}$ or simply t^w if the structure is understood. But we will usually stick with $t^{\mathcal{M}}$. Notice that the valuation of a term t depends only on the valuation of the variables and extralogical symbols occurring in t :

2.3.2 Proposition

Suppose t is an \mathcal{L} -term, and \mathcal{M} and \mathcal{M}' are two \mathcal{L} -models. Let V be a set of variables where $\text{var}t \subseteq V$. Now suppose that \mathcal{M} and \mathcal{M}' agree on their valuations of V and extralogical symbols in t : for every $x \in V$, $x^{\mathcal{M}} = x^{\mathcal{M}'}$ and for every extralogical symbol s occurring in t , $s^{\mathcal{M}} = s^{\mathcal{M}'}$. Then $t^{\mathcal{M}} = t^{\mathcal{M}'}$.

This is done by term induction. If t is a prime term, then $t = c$ for some constant or $t = x$ for some variable. In either case the proposition is satisfied by its assumption (that \mathcal{M} and \mathcal{M}' agree on variables and extralogical symbols occurring in t). Now suppose $t = ft_1 \dots t_n$ then by the assumption of the proposition, $f^{\mathcal{M}} = f^{\mathcal{M}'}$ and by the induction hypothesis $t_i^{\mathcal{M}} = t_i^{\mathcal{M}'}$. Thus

$$t^{\mathcal{M}} = f^{\mathcal{M}} t_1^{\mathcal{M}} \dots t_n^{\mathcal{M}} = f^{\mathcal{M}'} t_1^{\mathcal{M}'} \dots t_n^{\mathcal{M}'} = t^{\mathcal{M}'}$$

as required. ■

2.3.3 Definition

We now define the **satisfiability relation** for first-order models. Let $\mathcal{M} = (\mathcal{A}, w)$ be a model. For every $a \in \mathcal{A}$ and $x \in \text{Var}$ let us define the model $\mathcal{M}_x^a = (\mathcal{A}, w')$ where $y^{w'} = y^w$ for variables y distinct from x , and $x^{w'} = a$. Meaning

$$y^{\mathcal{M}_x^a} = \begin{cases} a & y = x \\ y^w & \text{else} \end{cases}$$

So now we define the satisfiability relation \models recursively as follows:

$$\begin{aligned} \mathcal{M} \models s = t &\iff s^{\mathcal{M}} = t^{\mathcal{M}}, & \mathcal{M} \models r\vec{t} &\iff r^{\mathcal{M}}\vec{t}^{\mathcal{M}}, \\ \mathcal{M} \models (\alpha \wedge \beta) &\iff \mathcal{M} \models \alpha \text{ and } \mathcal{M} \models \beta, & \mathcal{M} \models \neg\alpha &\iff \mathcal{M} \not\models \alpha, \\ & \mathcal{M} \models \forall x\alpha &\iff \mathcal{M}_x^a \models \alpha \text{ for all } a \in \mathcal{A} \end{aligned}$$

If $\mathcal{M} \models \varphi$, then \mathcal{M} is said to model φ . And if $X \subseteq \mathcal{L}$ is a set of formulas, we write $\mathcal{M} \models X$ if for all $\varphi \in X$, $\mathcal{M} \models \varphi$, and we similarly say \mathcal{M} models X .

We can generalize \mathcal{M}_x^a to $\mathcal{M}_{\vec{x}}^{\vec{a}}$ where the underlying structure remains the same and

$$y^{\mathcal{M}_{\vec{x}}^{\vec{a}}} = \begin{cases} a_i & y = x_i \\ y^{\mathcal{M}} & \text{else} \end{cases}$$

Notice that $\mathcal{M}_{\vec{x}}^{\vec{a}} = (\mathcal{M}_{x_1}^{a_1})_{x_2}^{a_2} \dots$. It follows immediately that if we use $\forall \vec{x}$ as an abbreviation for $\forall x_1 \forall x_2 \dots \forall x_n$, then we get

$$\mathcal{M} \models \forall \vec{x}\alpha \iff \mathcal{M}_{\vec{x}}^{\vec{a}} \models \alpha \text{ for all } \vec{a} \in \mathcal{A}^n$$

It is easily verifiable that

$$\begin{aligned} \mathcal{M} \models (\alpha \vee \beta) &\iff \mathcal{M} \models \alpha \text{ or } \mathcal{M} \models \beta & \mathcal{M} \models (\alpha \rightarrow \beta) &\iff \text{if } \mathcal{M} \models \alpha \text{ then } \mathcal{M} \models \beta \\ \mathcal{M} \models (\alpha \leftrightarrow \beta) &\iff \mathcal{M} \models \alpha \text{ if and only if } \mathcal{M} \models \beta \end{aligned}$$

And also $\mathcal{M} \models \exists x\alpha = \neg \forall x \neg \alpha$ if and only if $\mathcal{M} \not\models \forall x \neg \alpha$ so there exists an $a \in \mathcal{A}$ such that $\mathcal{M}_x^a \not\models \neg \alpha$, meaning there exists an $a \in \mathcal{A}$ such that $\mathcal{M}_x^a \models \alpha$. This chain of reasoning is readily seen to be reversible. So we have shown

$$\mathcal{M} \models \exists x\alpha \iff \text{there exists an } a \in \mathcal{A} \text{ such that } \mathcal{M}_x^a \models \alpha$$

2.3.4 Definition

A formula or set of formulas is said to be **satisfiable** if it has a model. $\varphi \in \mathcal{L}$ is called a **tautology** (or **generally/logically valid**), denoted $\models \varphi$, if $\mathcal{M} \models \varphi$ for every model \mathcal{M} . Two formulas α and β are said to be **logically equivalent**, denoted $\alpha \equiv \beta$, if for every model \mathcal{M} ,

$$\mathcal{M} \models \alpha \iff \mathcal{M} \models \beta$$

Now, say \mathcal{A} is an \mathcal{L} -structure, then we write $\mathcal{A} \models \varphi$ for a formula φ if $(\mathcal{A}, w) \models \varphi$ for all valuations $w: \text{Var} \rightarrow \mathcal{A}$. Similarly one writes $\mathcal{A} \models X$ for a set of formulas X if $\mathcal{A} \models \varphi$ for all $\varphi \in X$.

2.3.5 Definition

Finally we define the **consequence relation** for first-order logic. Suppose X is a set of formulas and φ is a formula, then we write $X \models \varphi$ if every model of X models φ . Meaning $\mathcal{M} \models X \implies \mathcal{M} \models \varphi$.

Again, \models is used to denote both the satisfaction and consequence relations. The meaning of the notation is to be understood from context. Moreover, \models is also used for the satisfaction relation of structures. And again we write $\varphi_1, \dots, \varphi_n \models \varphi$ in place of $\{\varphi_1, \dots, \varphi_n\} \models \varphi$ and all the usual shorthands.

Notice that while by definition if \mathcal{M} is a model then $\mathcal{M} \models \varphi$ or $\mathcal{M} \models \neg\varphi$ for all formulas φ . But if \mathcal{A} is a structure, then it is possible for \mathcal{A} to satisfy neither φ nor $\neg\varphi$ (but if it does satisfy one, it cannot satisfy the other obviously). Take for example the formula $x = y$, then if \mathcal{A} is a structure with at least two elements, suppose $a \neq b \in \mathcal{A}$, then we can define a valuation which satisfies $x = y$ and one which does not. And so \mathcal{A} satisfies neither $x = y$ nor $\neg x = y = x \neq y$.

Now suppose φ is a formula and let x_1, \dots, x_n be an enumeration of $\text{free}\varphi$ (according to some accepted total order of Var , for example by index), then we define the *generalized* of φ or its *universal closure* to be the sentence

$$\varphi^g := \forall x_1 \cdots \forall x_n \varphi$$

From the definitions provided above, it is immediate that if \mathcal{A} is a structure then

$$\mathcal{A} \models \varphi \iff \mathcal{A} \models \varphi^g$$

And in general $\mathcal{A} \models X \iff \mathcal{A} \models X^g := \{\varphi^g \mid \varphi \in X\}$.

2.3.6 Theorem (The Coincidence Theorem)

Let φ be a formula, and V be a set of variables such that $\text{free}\varphi \subseteq V$. Let \mathcal{M} and \mathcal{M}' be two models over the same domain A such that $x^{\mathcal{M}} = x^{\mathcal{M}'}$ for all variables $x \in V$, and $s^{\mathcal{M}} = s^{\mathcal{M}'}$ for all extralogical symbols s occurring in φ . Then $\mathcal{M} \models \varphi$ if and only if $\mathcal{M}' \models \varphi$.

We prove this by induction on φ . If φ is a prime formula of the form $rt_1 \cdots t_n$, by the assumptions of the theorem and proposition 2.3.2, $t_i^{\mathcal{M}} = t_i^{\mathcal{M}'}$ for all $1 \leq i \leq n$, and $r^{\mathcal{M}} = r^{\mathcal{M}'}$, so $r^{\mathcal{M}} \vec{t}^{\mathcal{M}} \iff r^{\mathcal{M}'} \vec{t}^{\mathcal{M}'}$ as required. This proof holds for equations as well. Now by the inductive hypothesis we get

$$\mathcal{M} \models \alpha \wedge \beta \iff \mathcal{M} \models \alpha \text{ and } \mathcal{M} \models \beta \iff \mathcal{M}' \models \alpha \text{ and } \mathcal{M}' \models \beta \iff \mathcal{M}' \models \alpha \wedge \beta$$

Similar for formulas of the form $\neg\alpha$.

Now, let $a \in \mathcal{A}$ and suppose $\mathcal{M}_x^a \models \varphi$. Then let $V' = V \cup \{x\}$ then $\text{free}\varphi \subseteq V'$ (since $\text{free}\varphi \subseteq \text{free}\forall x\varphi \cup \{x\} \subseteq V \cup \{x\}$) and \mathcal{M}_x^a and \mathcal{M}'_x^a coincide for all $y \in V'$ (though it is possible that $x^{\mathcal{M}} \neq x^{\mathcal{M}'}$). Thus by our inductive hypothesis $\mathcal{M}_x^a \models \varphi$ if and only if $\mathcal{M}'_x^a \models \varphi$. Thus

$$\mathcal{M} \models \forall x\varphi \iff \mathcal{M}_x^a \models \varphi \text{ for all } a \in \mathcal{A} \iff \mathcal{M}'_x^a \models \varphi \text{ for all } a \in \mathcal{A} \iff \mathcal{M}' \models \forall x\varphi$$

as required. ■

Let $\sigma \subseteq \sigma'$ be two signatures, and $\mathcal{L} \subseteq \mathcal{L}'$ be their respective first-order languages. Now, if $\mathcal{M} = (\mathcal{A}, w)$ is an \mathcal{L} -model, it can be arbitrarily extended to an \mathcal{L}' -model $\mathcal{M}' = (\mathcal{A}', w)$, where \mathcal{A}' is the σ' -expansion of \mathcal{A} , by arbitrarily setting $s^{\mathcal{M}'}$ for $s \in \sigma' \setminus \sigma$. Now, let us set $V = \text{Var}$ and by the coincidence theorem we get that for every $\varphi \in \mathcal{L}$ since \mathcal{M} and \mathcal{M}' agree on the extralogical symbols (as \mathcal{A}' is an expansion of \mathcal{A}) and variables in V (since the valuation remains the same), we get that

$$\mathcal{M} \models \varphi \iff \mathcal{M}' \models \varphi$$

If we denote the consequence relation of \mathcal{L} by $\models_{\mathcal{L}}$, then it follows that if $\mathcal{L} \subseteq \mathcal{L}'$, $\models_{\mathcal{L}'}$ is a *conservative* extension of $\models_{\mathcal{L}}$: for every $\varphi \in \mathcal{L}$ and $X \subseteq \mathcal{L}$, $X \models_{\mathcal{L}'} \varphi$ if and only if $X \models_{\mathcal{L}} \varphi$. Indeed: if \mathcal{M}' is an \mathcal{L}' -model then let \mathcal{M} be the \mathcal{L} -reduct of \mathcal{M}' and so $\mathcal{M} \models_{\mathcal{L}} X$ if and only if $\mathcal{M}' \models_{\mathcal{L}'} X$, and same for φ .

So the satisfiability of φ depends only on the symbols occurring in φ , we need not the subscripts in \models .

Another consequence of the coincidence theorem is the *omission of superfluous quantifiers*:

$$\forall x\varphi \equiv \varphi \equiv \exists x\varphi \text{ if } x \notin \text{free}\varphi$$

To see this, let \mathcal{M} be a model and $a \in \mathcal{A}$ be arbitrary. Then let $V = \text{free}\varphi$ and $\mathcal{M}' = \mathcal{M}_x^a$, and by the coincidence theorem since $y^{\mathcal{M}} = y^{\mathcal{M}'}$ for all $y \in V$ (since $x \notin \text{free}\varphi$) we have that $\mathcal{M} \models \varphi$ if and only if $\mathcal{M}_x^a \models \varphi$. So $\mathcal{M} \models \forall x\varphi$ if and only if $\mathcal{M}_x^a \models \varphi$ for all $a \in \mathcal{A}$, which is if and only if $\mathcal{M} \models \varphi$, which is if and only if $\mathcal{M}_x^a \models \varphi$ for some $a \in \mathcal{A}$, which is by definition $\mathcal{M} \models \exists x\varphi$.

This fact should be intuitive, for example $\forall x\exists x(x > 0)$ is the same as $\exists x(x > 0)$ and $\exists x\exists x(x > 0)$ since the outermost quantifier is superfluous.

Another thing which is simple to show is that if \mathcal{A} is a substructure of \mathcal{B} , and $\mathcal{M} = (\mathcal{A}, w)$ and $\mathcal{M}' = (\mathcal{B}, w)$ are models with the same valuation $w: \text{Var} \rightarrow A$, then for every term t , $t^{\mathcal{M}} = t^{\mathcal{M}'}$. For prime terms this is obvious (for terms of the form x , this is since the same valuation is used, and for constant terms this is by definition of substructures). Now inductively, if $t_i^{\mathcal{M}} = t_i^{\mathcal{M}'}$ then since $t_i^{\mathcal{M}} \in \mathcal{A}$, and $f^{\mathcal{M}'} \vec{a} = f^{\mathcal{M}} \vec{a}$ for values $\vec{a} \in \mathcal{A}^n$, we have that

$$(f\vec{t})^{\mathcal{M}'} = f^{\mathcal{M}'} \vec{t}^{\mathcal{M}'} = f^{\mathcal{M}} \vec{t}^{\mathcal{M}} = (f\vec{t})^{\mathcal{M}}$$

Now by the coincidence theorem, we know that the satisfaction of a formula φ is dependent only on the valuations of variables occurring in φ . So let $\varphi = \varphi(\vec{x})$ (meaning $\text{free}\varphi \subseteq \{x_1, \dots, x_n\}$), then saying

$$(\mathcal{A}, w) \models \varphi \text{ for a valuation } w \text{ where } x_1^w = a_1, \dots, x_n^w = a_n$$

can be written more suggestively by

$$(\mathcal{A}, \vec{a}) \models \varphi, \text{ or } \mathcal{A} \models \varphi[a_1, \dots, a_n], \text{ or } \mathcal{A} \models \varphi[\vec{a}]$$

So there is no need to mention the global valuation w .

2.3.7 Theorem (The Substructure Theorem)

For \mathcal{L} -structures \mathcal{A} and \mathcal{B} such that $A \subseteq B$ (where A represents the domain of \mathcal{A} , and so on), the following conditions are equivalent:

- (1) \mathcal{A} is a substructure of \mathcal{B} : $\mathcal{A} \subseteq \mathcal{B}$,
- (2) $\mathcal{A} \models \varphi[\vec{a}]$ if and only if $\mathcal{B} \models \varphi[\vec{a}]$ for all quantifier-free $\varphi = \varphi(\vec{x})$ and $\vec{a} \in A^n$,
- (3) $\mathcal{A} \models \varphi[\vec{a}]$ if and only if $\mathcal{B} \models \varphi[\vec{a}]$ for all prime $\varphi = \varphi(\vec{x})$ and $\vec{a} \in A^n$.

First we prove (1) \implies (2): so we must show that for models $\mathcal{M} = (\mathcal{A}, w)$ and $\mathcal{M}' = (\mathcal{B}, w)$, $\mathcal{M} \models \varphi$ if and only if $\mathcal{M}' \models \varphi$. For prime formulas this is obvious as we showed that $\vec{t}^{\mathcal{M}} = \vec{t}^{\mathcal{M}'}$ for all vectors of terms \vec{t} . The inductive step for \wedge and \neg are simple. (2) \implies (3) is trivial as prime formulas are quantifier-free. Now to show that (3) \implies (1) we must show that $f^{\mathcal{A}}\vec{a} = f^{\mathcal{B}}\vec{a}$ and $r^{\mathcal{A}}\vec{a} \iff r^{\mathcal{B}}\vec{a}$ for $\vec{a} \in A^n$. By (3), we know that

$$r^{\mathcal{A}}\vec{a} \iff \mathcal{A} \models r\vec{x}[\vec{a}] \iff \mathcal{B} \models r\vec{x}[\vec{a}] \iff r^{\mathcal{B}}\vec{a}$$

($\vec{x} = (x_1, \dots, x_n)$ is any vector of n distinct variables.) Similarly

$$f^{\mathcal{A}}\vec{a} = b \iff \mathcal{A} \models f\vec{x} = y[\vec{a}, b] \iff \mathcal{B} \models f\vec{x} = y[\vec{a}, b] \iff f^{\mathcal{B}}\vec{a} = b$$

and so $f^{\mathcal{A}}\vec{a} = f^{\mathcal{B}}\vec{a}$, meaning $\mathcal{A} \subseteq \mathcal{B}$ as required (constants are viewed as 0-ary functions; the proof is the same). \blacksquare

2.3.8 Definition

A **universal formula** (or a **\forall -formula**) is a formula α of the form $\forall \vec{x}\beta$ where β is quantifier-free. If α is a sentence, it is also called a **universal sentence** or **\forall -sentence**. Similarly, an **existential formula** (or a **\exists -formula**) is a formula α of the form $\exists \vec{x}\beta$ where β is quantifier-free. And again if α is a sentence, it is also called an **existential sentence** or **\exists -sentence**.

For example, we can define the following existential sentences:

$$\exists_1 := \exists v_0 v_0 = v_0, \quad \exists_n := \exists v_0 \cdots \exists v_{n-1} \bigwedge_{0 \leq i < j < n} v_i \neq v_j$$

So \exists_n states “there exists at least n (distinct) elements”. This means that $\neg \exists_{n+1}$ means “there exists at most n elements”, and so we can define $\exists_{=n} := \exists_n \wedge \neg \exists_{n+1}$ which means “there exists exactly n elements”.

Since structures are non-empty by definition, \exists_1 is a tautology, and so we define $\top := \exists_1$, and $\perp := \exists_0 := \neg \top$. From the substructure theorem we get the following corollary:

2.3.9 Corollary

Let $\mathcal{A} \subseteq \mathcal{B}$, then every \forall -sentence which is valid in \mathcal{B} is also valid in \mathcal{A} . Conversely, every \exists -sentence which is valid in \mathcal{A} is also valid in \mathcal{B} .

Suppose $\mathcal{B} \models \forall \vec{x}\beta$ where β is quantifier-free. Then let $\vec{a} \in A^n$, and so $\mathcal{B} \models \beta[\vec{a}]$, and by the substructure theorem since β is quantifier-free, we get $\mathcal{A} \models \beta[\vec{a}]$. Since \vec{a} is arbitrary, this means $\mathcal{A} \models \forall \vec{x}\beta$ as required. And similarly, if $\mathcal{A} \models \exists \vec{x}\beta$ then there exists a $\vec{a} \in A^n$ such that $\mathcal{A} \models \beta[\vec{a}]$ and so again by the substructure theorem we get $\mathcal{B} \models \beta[\vec{a}]$, so $\mathcal{B} \models \exists \vec{x}\beta$. \blacksquare

2.3.10 Theorem (The Invariance Theorem)

Suppose \mathcal{A} and \mathcal{B} are isomorphic \mathcal{L} -structures with an isomorphism $\iota: \mathcal{A} \longrightarrow \mathcal{B}$. Then for every formula

$\varphi = \varphi(\vec{x})$, and all $\vec{a} \in A^n$:

$$\mathcal{A} \models \varphi[\vec{a}] \iff \mathcal{B} \models \varphi[\iota\vec{a}]$$

where again, $\iota\vec{a} = (\iota a_1, \dots, \iota a_n)$.

It is sufficient to show that for models $\mathcal{M} = (\mathcal{A}, w)$ and $\mathcal{M}' = (\mathcal{B}, w')$ where $w' = \iota \circ w$, $\mathcal{M} \models \varphi \iff \mathcal{M}' \models \varphi$. We will first prove by term induction that for every term t , $\iota(t^{\mathcal{M}}) = t^{\mathcal{M}'}$. For prime terms, this is obvious:

$$x^{\mathcal{M}'} = x^{w'} = \iota x^w = \iota(x^{\mathcal{M}}), \quad c^{\mathcal{M}'} = c^{\mathcal{B}} = \iota(c^{\mathcal{A}}) = \iota(c^{\mathcal{M}})$$

For compound terms, suppose $\iota(t_i^{\mathcal{M}}) = t_i^{\mathcal{M}'}$ for $1 \leq i \leq n$, then we get $\bar{t}^{\mathcal{M}'} = \iota \bar{t}^{\mathcal{M}}$ and so

$$(f\bar{t})^{\mathcal{M}'} = f^{\mathcal{M}'} \bar{t}^{\mathcal{M}'} = f^{\mathcal{B}} \iota \bar{t}^{\mathcal{M}} = \iota(f^{\mathcal{A}} \bar{t}^{\mathcal{M}}) = \iota(f\bar{t})^{\mathcal{M}}$$

the third equality is due to ι being a homomorphism.

Now we will prove the goal: $\mathcal{M} \models \varphi \iff \mathcal{M}' \models \varphi$, by formula induction on φ . For prime formulas φ :

$$\begin{aligned} \mathcal{M}' \models t = s &\iff t^{\mathcal{M}'} = s^{\mathcal{M}'} \iff \iota t^{\mathcal{M}} = \iota s^{\mathcal{M}} \iff t^{\mathcal{M}} = s^{\mathcal{M}} \iff \mathcal{M} \models t = s \\ \mathcal{M}' \models r\bar{t} &\iff r^{\mathcal{B}} \bar{t}^{\mathcal{M}'} \iff r^{\mathcal{B}} \iota \bar{t}^{\mathcal{M}} \iff r^{\mathcal{A}} \bar{t}^{\mathcal{M}} \iff \mathcal{M} \models r\bar{t} \end{aligned}$$

For compound formulas we use the inductive hypothesis:

$$\begin{aligned} \mathcal{M}' \models \alpha \wedge \beta &\iff \mathcal{M}' \models \alpha \text{ and } \mathcal{M}' \models \beta \iff \mathcal{M} \models \alpha \text{ and } \mathcal{M} \models \beta \iff \mathcal{M} \models \alpha \wedge \beta \\ \mathcal{M}' \models \neg \alpha &\iff \mathcal{M}' \not\models \alpha \iff \mathcal{M} \not\models \alpha \iff \mathcal{M} \models \neg \alpha \end{aligned}$$

And for quantified formulas, $\mathcal{M}' \models \forall x \varphi$ if and only if $\mathcal{M}'_x^b \models \varphi$ for all $b \in \mathcal{B}$. Since ι is surjective, $b = \iota a$ for some $a \in A$ and so if $\mathcal{M}_x^a = (\mathcal{A}, w_0)$ then $\mathcal{M}'_x^b = (\mathcal{B}, \iota \circ w_0)$ and so by our inductive hypothesis (as \mathcal{M} and \mathcal{M}' are arbitrary), this is if and only if $\mathcal{M}_x^a \models \varphi$. Since b is arbitrary, and therefore so is a , $\mathcal{M} \models \forall x \varphi$. This chain of logic is reversible, which can also be seen since ι^{-1} is also an isomorphism. So we have indeed shown that for all formulas, $\mathcal{M} \models \varphi \iff \mathcal{M}' \models \varphi$ as required. ■

This proof obviously covers sentences $\varphi \in \mathcal{L}^0$. So for example, if G is a group and ι is an isomorphism to some other o-structure, then $\iota(G)$ is also a group. This is as we can have φ run over the axioms of group theory (which are sentences) and $G \models \varphi \implies \iota(G) \models \varphi$.

2.3.11 Definition

Two \mathcal{L} -structures are **elementarily equivalent** if $\mathcal{A} \models \varphi \iff \mathcal{B} \models \varphi$ for all sentences $\varphi \in \mathcal{L}^0$. This is denoted $\mathcal{A} \equiv \mathcal{B}$.

So by the invariance theorem isomorphic structures are elementarily equivalent.

Notice that substitutions do not pay attention to *collision of variables*, for example if $\varphi = \exists y(x \neq y)$ then certainly $\forall x \varphi$ is true when the domain has at least two elements. But $\varphi_x^y = \exists y(y \neq y)$ is false. So we have shown that $\forall x \varphi$ does not necessarily imply φ_x^t , as one may expect. But the issue is that we have changed the context of the variable x from a free to a bound variable, and so the meaning of φ_x^y is different from φ .

2.3.12 Definition

A simple substitution $\varphi, \frac{t}{x}$ is called **collision-free** if every $y \in \text{var } t$ distinct from x does not occur bound in φ (meaning $\text{bnd} \varphi \cap \text{var } t \subseteq \{x\}$, or $y \in \text{var } t \setminus \{x\} \implies y \notin \text{bnd} \varphi$). This means that a variable being inserted into φ does not get potentially inserted into a scope of its own quantifier.

A global substitution σ is collision-free if $\varphi, \frac{x^\sigma}{x}$ is collision-free for every variable $x \in \text{Var}$. So for a simultaneous substitution $\varphi, \frac{\vec{t}}{\vec{x}}$ it is only necessary to verify that $\varphi, \frac{t_i}{x_i}$ is collision-free.

The reason for not requiring $x \notin \text{bnd} \varphi$ in a substitution $\frac{t}{x}$ is since t is only substituted at free occurrences of x , so whether or not x occurs bound in φ is immaterial. (Though such a restriction wouldn't matter practically: it is bad practice to have a formula where the same variable occurs both free and bound. In some texts, different symbols are used for free and bound variables even.)

This is a crude definition, for example if $\varphi = \forall x(x = x) \wedge \forall y(y = y)$, then $\varphi_{\frac{y}{x}}$ is not collision-free since y occurs bound in φ . But such a substitution would not alter φ . We could refine the definition of collision-free substitutions to mean that x does not occur within the scope of any variable in *var* t , other than potentially x itself. Some texts refer to this definition as *t is free for x in φ*. Such a refinement is unnecessary for our purposes though.

For a model $\mathcal{M} = (\mathcal{A}, w)$ and a global substitution σ , we define $\mathcal{M}^\sigma = (\mathcal{A}, w^\sigma)$ where $w^\sigma = (x^\sigma)^\mathcal{M}$ (first substitute x to get a term, and then valuate it via \mathcal{M}) for every $x \in \text{Var}$. In other words $x^{\mathcal{M}^\sigma} = x^{\sigma\mathcal{M}} = (x^\sigma)^\mathcal{M}$. This extends to terms: $t^{\mathcal{M}^\sigma} = t^{\sigma\mathcal{M}}$. This is obvious for prime terms (variables and constants), and for compound terms:

$$(ft_1 \dots t_n)^{\mathcal{M}^\sigma} = f^{\mathcal{M}} t_1^{\mathcal{M}^\sigma} \dots t_n^{\mathcal{M}^\sigma} = f^{\mathcal{M}} t_1^{\sigma\mathcal{M}} \dots t_n^{\sigma\mathcal{M}} = (ft_1 \dots t_n)^{\sigma\mathcal{M}}$$

Notice that in the case $\sigma = \frac{\vec{t}}{\vec{x}}$,

$$x^{\mathcal{M}^\sigma} = x^{\sigma\mathcal{M}} = \left(x \frac{\vec{t}}{\vec{x}} \right)^\mathcal{M}$$

and so \mathcal{M}^σ coincides with $\mathcal{M}_{\vec{x}}^{\vec{t}\mathcal{M}}$ ($\vec{t}^\mathcal{M}$ since the terms must be interpreted, $\mathcal{M}_{\vec{x}}^{\vec{t}}$ makes no sense since \vec{t} is not a vector in the domain).

2.3.13 Theorem (The Substitution Theorem)

Let \mathcal{M} be a model and σ be a global substitution. If φ, σ are collision-free then

$$\mathcal{M} \models \varphi^\sigma \iff \mathcal{M}^\sigma \models \varphi$$

In particular, $\mathcal{M} \models \varphi_{\frac{\vec{t}}{\vec{x}}}$ if and only if $\mathcal{M}_{\vec{x}}^{\vec{t}\mathcal{M}} \models \varphi$, provided $\varphi, \frac{\vec{t}}{\vec{x}}$ is collision-free.

This will be proven by formula induction. In the case that φ is an equation $t_1 = t_2$,

$$\mathcal{M} \models (t_1 = t_2)^\sigma \iff t_1^{\sigma\mathcal{M}} = t_2^{\sigma\mathcal{M}} \iff t_1^{\mathcal{M}^\sigma} = t_2^{\mathcal{M}^\sigma} \iff \mathcal{M}^\sigma \models t_1 = t_2$$

Similarly, for prime formulas of the form rt :

$$\mathcal{M} \models (rt)^\sigma \iff r^{\mathcal{M}} t^{\sigma\mathcal{M}} \iff r^{\mathcal{M}^\sigma} t^{\mathcal{M}^\sigma} \iff \mathcal{M}^\sigma \models rt$$

The induction steps for \wedge and \neg are obvious and simple. Now, recall that $\mathcal{M} \models \forall x\varphi$ if and only if $\mathcal{M} \models \forall x\varphi^\tau$ where $x^\tau = x$ and $y^\tau = y^\sigma$ for $x \neq y$,

$$\iff \mathcal{M}_x^a \models \varphi^\tau \text{ for every } a \in A \iff (\mathcal{M}_x^a)^\tau \models \varphi$$

now we claim that $(\mathcal{M}_x^a)^\tau = (\mathcal{M}^\sigma)_x^a$. Since $\forall x\varphi, \sigma$ is collision-free, meaning $\forall x\varphi, \frac{y^\tau}{y}$ for every y is collision-free so $x \notin \text{var } y^\sigma$ for every $y \neq x$. And since $y^\tau = y^\sigma$, we get $y^{(\mathcal{M}_x^a)^\tau} = y^{\tau\mathcal{M}_x^a} = y^{\sigma\mathcal{M}_x^a} = y^{\sigma\mathcal{M}} = y^{\mathcal{M}^\sigma} = y^{(\mathcal{M}^\sigma)_x^a}$. And $x^{(\mathcal{M}_x^a)^\tau} = x^{\mathcal{M}_x^a} = a = x^{(\mathcal{M}^\sigma)_x^a}$. Thus

$$\iff (\mathcal{M}^\sigma)_x^a \models \varphi \text{ for every } a \in A \iff \mathcal{M}^\sigma \models \forall x\varphi$$

as required. ■

This proof also obviously works for the refinement of the concept of collision-free substitutions (what we termed “... free for x in φ ”).

2.3.14 Corollary

For all formulas φ and $\frac{\vec{t}}{\vec{x}}$ such that $\varphi, \frac{\vec{t}}{\vec{x}}$ is collision-free, the following are true:

- (1) $\forall \vec{x}\varphi \models \varphi_{\frac{\vec{t}}{\vec{x}}}$,
- (2) $\varphi_{\frac{\vec{t}}{\vec{x}}} \models \exists \vec{x}\varphi$,
- (3) $\varphi_{\frac{s}{x}}, s = t \models \varphi_{\frac{t}{x}}$, provided $\varphi, \frac{s}{x}$ and $\varphi, \frac{t}{x}$ are collision-free.

- (1) Let $\mathcal{M} \models \forall \vec{\varphi}$, so $\mathcal{M}_{\vec{x}}^{\vec{a}} \models \varphi$ for all $\vec{a} \in A^n$. This means that $\mathcal{M}_{\vec{x}}^{\vec{t}} \models \varphi$, and so by the substitution theorem this is equivalent to $\mathcal{M} \models \varphi_{\vec{x}}^{\vec{t}}$.
- (2) Since $\neg \exists \vec{x} \varphi = \forall \vec{x} \neg \varphi$, by (1) $\neg \exists \vec{x} \varphi \models \neg \varphi_{\vec{x}}^{\vec{t}}$. And so $\varphi_{\vec{x}}^{\vec{t}} \models \exists \vec{\varphi}$ (since $\neg X \models \neg Y$ if and only if $Y \models X$).
- (3) Suppose $\mathcal{M} \models \varphi_{\vec{x}}^s$, $s = t$ so $s^{\mathcal{M}} = t^{\mathcal{M}}$ and $\mathcal{M}_{\vec{x}}^{s^{\mathcal{M}}} \models \varphi$ by the substitution theorem, and so $\mathcal{M}_{\vec{x}}^{t^{\mathcal{M}}} \models \varphi$, and by applying the substitution theorem again we get $\mathcal{M} \models \varphi_{\vec{x}}^t$. ■

This shows that collision-free substitutions act as substitutions should: if φ is true for all \vec{x} , then we should be able to substitute any terms \vec{t} in place of \vec{x} and φ should hold. What we need to be careful of is that this substitution doesn't mess with the syntax of φ , which is why it must be collision-free.

Let us define the following quantifier, “there exists exactly one”:

$$\exists! x \varphi := \exists x \varphi \wedge \forall x \forall y (\varphi \wedge \varphi_x^y \rightarrow y = x)$$

where y is any variable not in $\text{var} \varphi$. So if $\mathcal{M} \models \forall x \forall y (\varphi \wedge \varphi_x^y \rightarrow y = x)$ means that $\mathcal{M}_x^a \models \varphi$ and $\mathcal{M}_x^b \models \varphi_x^y$ implies $a = b$ (since $y \notin \text{var} \varphi$). Putting it all together, $\mathcal{M} \models \exists! x \varphi$ if and only if there exists a single a such that $\mathcal{M}_x^a \models \varphi$. An example of a tautology is $\exists! x x = t$ for any term t .

An equivalent definition of $\exists!$ is

$$\exists! x \varphi := \exists x \forall y (\varphi_x^y \leftrightarrow y = x)$$

Exercise

Suppose $X \models \varphi$ and $x \notin \text{free} \varphi$, then show that $X \models \forall x \varphi$.

Suppose $\mathcal{M} \models X$ and let $a \in A$, then notice that \mathcal{M}_x^a and \mathcal{M} agree on all variables in $\text{free} \varphi$ and extralogical symbols (as they are defined over the same structure). So by The Substitution Theorem since $\mathcal{M} \models \varphi$, $\mathcal{M}_x^a \models \varphi$. Since a is arbitrary, this means $\mathcal{M} \models \forall x \varphi$.

Exercise

Show that $\forall x (\alpha \rightarrow \beta) \models \forall x \alpha \rightarrow \forall x \beta$.

Let $\mathcal{M} \models \forall x (\alpha \rightarrow \beta)$. Then if $\mathcal{M} \models \forall x \alpha$ then let $a \in A$, so $\mathcal{M}_x^a \models \alpha, \alpha \rightarrow \beta$ so $\mathcal{M}_x^a \models \beta$ by modus ponens. Since a is arbitrary, this means $\mathcal{M} \models \forall x \beta$. Thus $\mathcal{M} \models \forall x \alpha \rightarrow \forall x \beta$.

Exercise

Suppose \mathcal{A} is an \mathcal{L} -structure and \mathcal{A}' is obtained by adjoining a constant symbol \mathbf{a} for some $a \in A$ (meaning $\mathbf{a}^{\mathcal{A}'} = a$). Show that $\mathcal{A} \models \alpha[a] \iff \mathcal{A}' \models \alpha_{\mathbf{a}}^{\mathbf{a}}$ for a formula $\alpha = \alpha(x)$.

We will first show that $t(x)^{\mathcal{A},a} = (t_{\mathbf{a}}^{\mathbf{a}})^{\mathcal{A}'}$. For prime terms this is obvious. For compound terms this is simple by induction.

It is sufficient to show that for arbitrary $\mathcal{M} = (\mathcal{A}, w)$ and $\mathcal{M}' = (\mathcal{A}', w)$ such that $x^w = a$, $\mathcal{M} \models \alpha \iff \mathcal{M}' \models \alpha_{\mathbf{a}}^{\mathbf{a}}$. By formula induction on α : in the case that $\alpha = t_1 = t_2$,

$$\mathcal{M}' \models t_1_{\mathbf{a}}^{\mathbf{a}} = t_2_{\mathbf{a}}^{\mathbf{a}} \iff (t_1_{\mathbf{a}}^{\mathbf{a}})^{\mathcal{M}'} = (t_2_{\mathbf{a}}^{\mathbf{a}})^{\mathcal{M}'} \iff t_1^{\mathcal{M}} = t_2^{\mathcal{M}} \iff \mathcal{M} \models t_1 = t_2$$

Similar for prime formulas of the form $r\vec{t}$ (there is of course a similarity between equations and prime formulas of the form $r\vec{t}$, $=$ is a “primitive” binary relation; it represents the identity relation). For compound formulas using connectives \wedge and \neg the inductive step is clear. And for formulas $\alpha = \forall y \beta$, if $y \neq x$ then this is clear. If $y = x$ then this is clear by the substitution theorem, as \mathcal{A} and \mathcal{A}' act the same on α as it is a sentence.

Exercise

Prove the following:

- (1) A conjunction of formulas \exists_i and their negations is logically equivalent to a formula of the form $\exists_n \wedge \neg \exists_m$ for suitable n and m .
- (2) A boolean combination of \exists_i is equivalent to a formula of the form $\bigvee_{i \leq n} \exists_{=k_i}$ or $\exists_k \vee \bigvee_{i \leq n} \exists_{=k_i}$ where $k_0 < \dots < k_n < k$.

- (1) Every such conjunction is equivalent, by commutativity, to a formula of the form

$$\bigwedge_{1 \leq i \leq n} \exists_{k_i} \wedge \bigwedge_{1 \leq j \leq m} \neg \exists_{\ell_j}$$

Where $k_1 < \dots < k_n$ and $\ell_1 < \dots < \ell_m$. Since \exists_i represents there existing at least i elements, and $\neg \exists_i$ means there exists $< i$ elements, this is equivalent to $\exists_{k_n} \wedge \neg \exists_{\ell_1}$. If $n = 0$ then take $k_0 = 1$ (since \exists_1 is a tautology) and if $m = 0$ then take $\ell_1 = 0$ (since $\neg \exists_0$ is a tautology).

- (2) Let us prove this by induction on the rank of the boolean combination, r . If $r = 0$ then the formula is \exists_n , which is of the second form. Otherwise, there are three cases:

- (1) The formula is of the form $\neg \alpha$ where $\alpha \equiv \bigvee_{i \leq n} \exists_{=k_i}$ or $\exists_k \vee \bigvee_{i \leq n} \exists_{=k_i}$. In the first case,

$$\neg \alpha \equiv \bigwedge_{i \leq n} \neg \exists_{=k_i} \equiv \exists_{k_n+1} \vee \bigwedge_{i \leq k_n, i \neq k_j} \exists_{=i}$$

This is as the negation of α refers to there not existing precisely k_i elements, so this means there is either $\geq k_n + 1$ elements or precisely i elements for $i \leq k_n$ where i differs from any k_j . In the second case,

$$\neg \alpha \equiv \neg \exists_k \wedge \bigwedge_{i \leq n} \neg \exists_{=k_i}$$

By above, both parts of the conjunction is equivalent to one of the forms required by the exercise, so this case is covered by case (3).

- (2) The formula is of the form $\alpha \wedge \beta$ where α and β are boolean combinations. It is obvious how this formula is equivalent to a formula of one of the forms required.
- (3) If the formula is of the form $\alpha \vee \beta$, then it is already (after α and β are converted to the correct form) in a required form.

2.4 General Validity and Logical Equivalence

2.4.1 Example

You may recall from set theory that the so-called “Russellian set” of sets not containing themselves is a proper class (not a set). This is reflective of the more general logical result, $\models \neg \exists u \forall x (x \in u \leftrightarrow x \notin x)$. This sentence states that there does not exist a u such that $x \in u$ if and only if $x \notin x$ for all x . But note that we require no conditions on the binary relation \in , so this result is purely logical one.

We will prove this in steps. Firstly, $\forall x (x \in u \leftrightarrow x \notin x) \models u \in u \leftrightarrow u \notin u$ which is due to corollary 2.3.14, using the substitution $\frac{u}{x}$. Clearly $u \in u \leftrightarrow u \notin u$ is not satisfiable, and thus neither is $\forall x (x \in u \leftrightarrow x \notin x)$, and therefore neither is $\exists u \forall x (x \in u \leftrightarrow x \notin x)$. Therefore $\models \neg \exists u \forall x (x \in u \leftrightarrow x \notin x)$.

This proof probably parallels the proof you are familiar with from an introductory set theory/discrete math course: one considers the Russellian set u itself.

It is obvious that $X \models \alpha \rightarrow \beta$ if and only if $X, \alpha \models \beta$. Suppose $X \models \alpha \rightarrow \beta$, and so suppose $\mathcal{M} \models X, \alpha$ then $\mathcal{M} \models \alpha \rightarrow \beta, \alpha$ and so $\mathcal{M} \models \beta$. Thus $X, \alpha \models \beta$. And conversely if $X, \alpha \models \beta$, suppose $\mathcal{M} \models X$. If $\mathcal{M} \models \alpha$ then $\mathcal{M} \models \beta$, thus $X \models \alpha \rightarrow \beta$ as required.

Thus we immediately get, for example

$$\forall \vec{x} \alpha \rightarrow \alpha \frac{\vec{t}}{\vec{x}}$$

for collision-free $\alpha, \frac{\bar{x}}{x}$.

Now, \mathcal{L} is a structure over the signature $\{\wedge, \neg\} \cup \{\forall x \mid x \in \text{Var}\}$. And so we can talk about *congruences in \mathcal{L}* . These are equivalence relations \approx such that for all formulas $\alpha, \alpha', \beta, \beta' \in \mathcal{L}$ and $x \in \text{Var}$:

$$\alpha \approx \alpha', \beta \approx \beta' \implies \alpha \wedge \beta \approx \alpha' \wedge \beta', \neg\alpha \approx \neg\alpha', \forall x\alpha \approx \forall x\alpha'$$

(As a technicality the symbol \wedge represents the function $(\alpha, \beta) \mapsto (\alpha \wedge \beta)$ and not $(\alpha, \beta) \mapsto \alpha \wedge \beta$ but I digress.)

2.4.2 Theorem (The Replacement Theorem)

Let \approx be a congruence in \mathcal{L} and $\alpha \approx \alpha'$. Let φ be an arbitrary formula. If φ' is obtained by replacing some of the occurrences (meaning not necessarily all the occurrences are replaced) of α with α' , then $\varphi \approx \varphi'$.

We will prove this by induction on φ . If φ is a prime formula, then either $\varphi = \alpha$ or $\varphi \neq \alpha'$, but in any case $\varphi' = \alpha'$ or $\varphi' = \varphi = \alpha$ (since prime formulas are the smallest units of formulas). And so $\varphi \approx \varphi'$ since $\alpha \approx \alpha$ and $\alpha \approx \alpha'$. In general, the case where $\varphi = \alpha$ is trivial.

Now, if $\varphi = \varphi_1 \wedge \varphi_2$ then $\varphi' = \varphi'_1 \wedge \varphi'_2$ (since we don't consider $\varphi = \alpha$) and inductively $\varphi_i \approx \varphi'_i$. Since \approx is a congruence, $\varphi' \approx \varphi$. The steps for \neg and $\forall x$ are analogous. ■

So for example, since \equiv is a congruence in \mathcal{L} , the replacement theorem tells us that if we replace subformulas of φ with equivalent subformulas, the resulting formula remains equivalent to φ . This is not surprising, but provides a rigorous proof of our intuition.

By this theorem, the “congruence-ness” holds for all defined connectives and quantifiers (meaning $\alpha \approx \alpha', \beta \approx \beta'$ implies $\alpha \rightarrow \alpha' \equiv \beta \rightarrow \beta'$ and $\exists x\alpha \approx \exists x\alpha'$ and so on).

2.4.3 Definition

Suppose \mathcal{A} is an \mathcal{L} -structure. Two \mathcal{L} -formulas α and β are **equivalent in \mathcal{A}** (or **equivalent modulo \mathcal{A}**) if $\mathcal{A}, w \models \alpha$ if and only if $\mathcal{A}, w \models \beta$ for all valuations $w: \mathcal{A} \rightarrow \text{Var}$. This is denoted $\alpha \equiv_{\mathcal{A}} \beta$.

Obviously $\alpha \equiv_{\mathcal{A}} \beta$ is equivalent to $\mathcal{A} \models \alpha \leftrightarrow \beta$. And if $\alpha \equiv \beta$ then $\alpha \equiv_{\mathcal{A}} \beta$, or more suggestively: $\equiv \subseteq \equiv_{\mathcal{A}}$.

2.4.4 Proposition

Suppose $\{\approx_i\}_{i \in I}$ is a non-empty family of congruences in \mathcal{L} , then so is their intersection, $\approx = \bigcap_{i \in I} \approx_i$.

This is trivial. ■

Thus if $\mathbf{K} \neq \emptyset$ is a class of \mathcal{L} -structures, $\cong_{\mathbf{K}} := \bigcap_{\mathcal{A} \in \mathbf{K}} \equiv_{\mathcal{A}}$ is also a congruence in \mathcal{L} . If \mathbf{K} is the class of *all* \mathcal{L} -structures, then $\equiv_{\mathbf{K}}$ is equal to \equiv .

We will state some basic properties of \equiv below:

- | | |
|---|---|
| (1) $\forall x(\alpha \wedge \beta) \equiv \forall x\alpha \wedge \forall x\beta$ | (2) $\exists x(\alpha \vee \beta) \equiv \exists x\alpha \vee \exists x\beta$ |
| (3) $\forall x\forall y\alpha \equiv \forall y\forall x\alpha$ | (4) $\exists x\exists y\alpha \equiv \exists y\exists x\alpha$ |

If x does not occur free in the formula β , then

- | | |
|--|---|
| (5) $\forall x(\alpha \vee \beta) \equiv \forall x\alpha \vee \beta$ | (6) $\exists x(\alpha \wedge \beta) \equiv \exists x\alpha \wedge \beta$ |
| (7) $\forall x\beta \equiv \beta$ | (8) $\exists x\beta \equiv \beta$ |
| (9) $\forall x(\alpha \rightarrow \beta) \equiv \exists x\alpha \rightarrow \beta$ | (10) $\exists x(\alpha \rightarrow \beta) \equiv \forall x\alpha \rightarrow \beta$ |

These proofs are relatively simple. Proofs of (5) through (10) will of course rely on The Substitution Theorem. Another important property is the *renaming of bound variables* (*bound renaming* for short): let α be a formula and $x \neq y$ be distinct variables where $y \notin \text{var}\alpha$. Then

$$\forall x\alpha \equiv \forall y(\alpha \frac{y}{x}), \quad \exists x\alpha \equiv \exists y(\alpha \frac{y}{x})$$

The second equivalence results from the first due to the replacement theorem (its necessity can easily be directly verified). So we prove the first equivalence (\mathcal{M}_x^y means $\mathcal{M}_x^{y^M}$):

$$\begin{aligned} \mathcal{M} \models \forall x\alpha &\iff \mathcal{M}_x^a \models \alpha \quad \text{for all } a && \text{(definition)} \\ &\iff (\mathcal{M}_y^a)_x^a \models \alpha \quad \text{for all } a && \text{(by The Coincidence Theorem)} \end{aligned}$$

$$\begin{aligned}
&\iff (\mathcal{M}_y^a)_x \models \alpha \text{ for all } a && ((\mathcal{M}_y^a)_x = (\mathcal{M}_y^a)_x) \\
&\iff \mathcal{M}_y^a \models \alpha_x^y \text{ for all } a && (\text{by The Substitution Theorem}) \\
&\iff \mathcal{M} \models \forall y (\alpha_x^y). && (\text{definition})
\end{aligned}$$

We also have the following properties: (assuming $\alpha, \frac{t}{x}$ are collision-free)

$$\begin{aligned}
\forall x(x = t \rightarrow \alpha) &\equiv \alpha_x^t \equiv \exists x(x = t \wedge \alpha) && (x \notin \text{var } t) \\
\forall y(y = t \rightarrow \alpha_x^y) &\equiv \alpha_x^t \equiv \exists y(y = t \wedge \alpha_x^y) && (y \notin \text{var } \alpha, \text{var } t)
\end{aligned}$$

We will prove the first line of equivalences. $\forall x(x = t \rightarrow \alpha) \models (x = t \rightarrow \alpha)_x^t = t = t \rightarrow \alpha_x^t \models \alpha_x^t$, by corollary 2.3.14. Conversely, let $\mathcal{M} \models \alpha_x^t$, so by The Substitution Theorem $\mathcal{M}_x^t \models \alpha$. If $\mathcal{M}_x^a \models x = t$, then obviously $a = t^{\mathcal{M}}$, and so $\mathcal{M}_x^a \models \alpha$. So for all $a \in A$, $\mathcal{M}_x^a \models x = t \rightarrow \alpha$, thus $\mathcal{M} \models \forall x(x = t \rightarrow \alpha)$ as required. Now, for the rightmost formula by what we just proved,

$$\exists x(x = t \wedge \alpha) = \neg \forall x \neg(x = t \wedge \alpha) \equiv \neg \forall x(x = t \rightarrow \neg \alpha) \equiv \neg \neg \alpha_x^t \equiv \alpha_x^t$$

The second line is proven similarly.

2.4.5 Definition

A formula of the form $\alpha = Q_1 x_1 \cdots Q_n x_n \beta$ where Q_i are either universal or existential quantifiers and β is quantifier-free is a **prenex normal form**, for short PNF.

In PNFs, we may assume that all the variables x_i are distinct, as we can drop superfluous quantifiers as stated earlier. For example, \forall - and \exists -formulas are prenex normal forms.

2.4.6 Theorem

Every formula is equivalent to a prenex normal form.

Let φ be a formula. Without loss of generality, we can assume that φ contains only $\neg, \wedge, \forall, \exists$ and not any other connectives or quantifiers (formally φ would not have \exists , but I digress). For each prefix Qx in φ , consider the number of occurrences of the symbols \neg and \wedge to the left of the prefix, and let $s\varphi$ be the sum of these numbers, summed over all prefixes in φ . Obviously φ is a PNF if and only if $s\varphi = 0$. So we will induct on $s\varphi$. In the case that $s\varphi = 0$, then φ is already a PNF. Otherwise, there must be some prefix Qx such that immediately to its left there is an occurrence of \wedge or \neg . Applying the appropriate equivalence from the following

$$\neg \forall x \alpha \equiv \exists x \neg \alpha, \quad \neg \exists x \alpha \equiv \forall x \neg \alpha, \quad \beta \wedge Qx \alpha \equiv Qy(\beta \wedge \alpha_x^y) \quad (y \notin \text{var } \alpha, \text{var } \beta)$$

(the third equivalence is the result of applying the appropriate basic properties (1), (2), (5), (6), (7), (8) as well as renaming bound variables) will reduce $s\varphi$, and the proof (and construction of a PNF for φ) proceeds inductively. ■

As a shorthand, if \triangleleft is a binary relation and t is a term, then we will write $(\forall x \triangleleft t) \alpha$ in place of $\forall x(x \triangleleft t \rightarrow \alpha)$. Similarly we will write $(\exists x \triangleleft t) \alpha$ in place of $\exists x(x \triangleleft t \wedge \alpha)$. So for example, in fields we may write $(\forall x \neq 0) \exists y x \cdot y = 1$.

Exercise

Suppose $\alpha \equiv \beta$ are equivalent formulas, show that $\alpha_x^{\frac{t}{x}} \equiv \beta_x^{\frac{t}{x}}$ where the substitutions are collision-free.

We will use the notation $\mathcal{M}_x^{\frac{t}{x}}$ as an abbreviation for $\mathcal{M}_x^{t^{\mathcal{M}}}$. Then we have by The Substitution Theorem:

$$\mathcal{M} \models \alpha_x^{\frac{t}{x}} \iff \mathcal{M}_x^{\frac{t}{x}} \models \alpha \iff \mathcal{M}_x^{\frac{t}{x}} \models \beta \iff \mathcal{M} \models \beta_x^{\frac{t}{x}}$$

as required.

Exercise

Show that $\neg(\forall x \triangleleft t)\alpha \equiv (\exists x \triangleleft t)\neg\alpha$ and $\neg(\exists x \triangleleft t)\alpha \equiv (\forall x \triangleleft t)\neg\alpha$.

We know that

$$\neg(\forall x \triangleleft t)\alpha = \neg\forall x(x \triangleleft t \rightarrow \alpha) \equiv \exists x(x \triangleleft t \wedge \neg\alpha) = (\exists x \triangleleft t)\neg\alpha$$

This means that

$$\neg(\exists x \triangleleft t)\alpha \equiv \neg\neg(\forall x \triangleleft t)\neg\alpha \equiv (\forall x \triangleleft t)\neg\alpha$$

Exercise

Show that the conjunction and disjunction of \forall -formulas is equivalent to a \forall -formula. Show the same for \exists -formulas.

Suppose $\forall \vec{x}\alpha$ and $\forall \vec{y}\beta$ are \forall -formulas (α, β are quantifier-free). By bound renaming, we can assume that \vec{x} is distinct from $\text{free}\beta$ and \vec{y} , and \vec{y} is distinct from $\text{free}\alpha$ and \vec{x} . Thus by $\forall \vec{x}\alpha \wedge \beta \equiv \forall \vec{x}(\alpha \wedge \beta)$,

$$\forall \vec{x}\alpha \wedge \forall \vec{y}\beta \equiv \forall \vec{x}(\alpha \wedge \forall \vec{y}\beta) \equiv \forall \vec{x}\forall \vec{y}(\alpha \wedge \beta)$$

similar for disjunction and \exists -formulas.

Exercise

Show that every formula $\varphi \in \mathcal{L}$ is equivalent to some formula $\varphi' \in \mathcal{L}$ built up from literals using only \wedge , \vee , and \exists .

We will prove this by induction on φ . If φ is a prime formula, this is trivial. Now suppose $\varphi = \neg\varphi_1$, then by induction φ_1 is equivalent to some formula built up from literals using only conjunction, disjunction, and existential quantifiers. Let us assume for simplicity that φ_1 is (replace φ_1 with its equivalent formula). Then φ_1 is either a literal, a conjunction, a disjunction, or a combination quantified using an existential quantifier. If φ_1 is a literal, then $\neg\varphi_1$ is equivalent to a literal. If $\varphi_1 = \varphi_2 \vee \varphi_3$, then $\varphi \equiv \neg\varphi_2 \wedge \neg\varphi_3$ and by induction $\neg\varphi_2$ and $\neg\varphi_3$ are equivalent to formulas of the required form, and so therefore so is their conjunction, φ . Similar for $\varphi_1 = \varphi_2 \wedge \varphi_3$. If $\varphi_1 = \exists x\varphi_2$ then by induction φ_2 is equivalent to a formula of the required form and therefore so is φ_1 .

Exercise

Let P be a unary relational symbol, show that $\exists x(Px \rightarrow \forall yPy)$ is a tautology.

The negation of this is equivalent to $\forall x(Px \wedge \neg\forall yPy) \equiv \forall x(Px \wedge \exists y(\neg Py)) \equiv \forall x Px \wedge \exists y(\neg Py)$ (since $\forall x(\alpha \wedge \beta) \equiv \forall x\alpha \wedge \forall x\beta$, and $\forall x\exists y(\neg Py) \equiv \exists y(\neg Py)$ by removing superfluous quantifiers). This is obviously not satisfiable.

2.5 Logical Consequence and Theories

Similar to the expansion and reduct of structures, if $\mathcal{L} \subseteq \mathcal{L}'$, \mathcal{L} is called the *reduct* of \mathcal{L}' and \mathcal{L}' is called the *expansion* of \mathcal{L} . Since the satisfaction requirements for \wedge and \neg are the same as in propositional logic, the basic rules for \models in propositional logic carry over to first order logic. These are

$$\begin{array}{ll} \text{(IS)} & \frac{}{\alpha \models \alpha} \\ \text{(\wedge 1)} & \frac{X \models \alpha, \beta}{X \models \alpha \wedge \beta} \\ \text{(\neg 1)} & \frac{X \models \alpha, \neg\alpha}{X \models \beta} \\ \text{(MR)} & \frac{X \models \alpha}{X' \models \alpha} \quad (X \subseteq X') \\ \text{(\wedge 2)} & \frac{X \models \alpha \wedge \beta}{X \models \alpha, \beta} \\ \text{(\neg 2)} & \frac{X, \alpha \models \beta \mid X, \neg\alpha \models \beta}{X \models \beta} \end{array}$$

But we also have new properties:

$$(1) \quad \frac{X \models \forall x\alpha}{X \models \alpha \frac{t}{x}} \quad (\alpha, \frac{t}{x} \text{ collision-free})$$

- $$\begin{array}{ll}
(2) \quad \frac{X \models \alpha \frac{s}{x}, s = t}{X \models \alpha \frac{t}{x}} & (\alpha, \frac{s}{x} \text{ and } \alpha, \frac{t}{x} \text{ collision-free}) \\
(3) \quad \frac{X, \beta \models \alpha}{X, \forall x \beta \models \alpha} & (\text{anterior generalization}) \\
(4) \quad \frac{X \models \alpha}{X \models \forall x \alpha} & (x \notin \text{free}X, \text{posterior generalization}) \\
(5) \quad \frac{X, \beta \models \alpha}{X, \exists x \beta \models \alpha} & (x \notin \text{free}X, \text{free}\alpha, \text{anterior particularization}) \\
(6) \quad \frac{X \models \alpha \frac{t}{x}}{X \models \exists x \alpha} & (\alpha, \frac{t}{x} \text{ collision-free, posterior particularization})
\end{array}$$

(1) is due to $\forall x \alpha \models \alpha \frac{t}{x}$ and the transitivity of \models . Similarly (2) is due to $\alpha \frac{s}{x}, s = t \models \alpha \frac{t}{x}$. And (3) is due to $\forall x \beta \models \beta$. We now prove (4). Suppose $X \models \alpha$, $\mathcal{M} \models X$, and $x \notin \text{free}X$. Then by The Coincidence Theorem, for every $a \in A$, $\mathcal{M}_x^a \models X$ and so $\mathcal{M}_x^a \models \alpha$ meaning $\mathcal{M} \models \forall x \alpha$ as required. For (5), notice that by (4):

$$X, \beta \models \alpha \implies X, \neg \alpha \models \neg \beta \implies X, \neg \alpha \models \forall x \neg \beta \implies X, \neg \forall x \neg \beta \models \alpha \implies X, \exists x \beta \models \alpha$$

And by corollary 2.3.14, $\alpha \frac{t}{x} \models \exists x \alpha$ proving (6).

Some texts define a stricter consequence relation, which we call the *global consequence relation*, denoted here by \models^g . For a set of formulas $X \subseteq \mathcal{L}$ and a formula $\varphi \in \mathcal{L}$, we define $X \models^g \varphi$ if and only if $\mathcal{A} \models X$ implies $\mathcal{A} \models \varphi$ for all *structures* \mathcal{A} . (The difference here is subtle: the “local” consequence relation deals in models, while the global consequence relation deals in structures.)

Obviously $X \models \varphi$ implies $X \models^g \varphi$: let $\mathcal{A} \models X$ then since for every valuation $\mathcal{A}, w \models X \implies \mathcal{A}, w \models \varphi$ so $\mathcal{A} \models \varphi$. The converse is not true though: for example $x = y \models^g \forall x \forall y x = y$ since if $\mathcal{A} \models x = y$ then \mathcal{A} has a single element. But $x = y \not\models \forall x \forall y x = y$, since any model over a structure with two or more elements whose valuations of x and y are equal is a counterexample. By posterior generalization, $X \models \varphi \implies X \models \varphi^g$ is true in general only if the free variables of φ do not occur free in X . But on the other hand $X \models^g \varphi \implies X \models \varphi^g$ is always true (since $\mathcal{A} \models \varphi \iff \mathcal{A} \models \varphi^g$).

Now, since for every structure \mathcal{A} , valuation w , and set of formulas X : $\mathcal{A}, w \models X^g \iff \mathcal{A} \models X^g$ by the coincidence theorem ($\text{free}X^g = \emptyset$). Thus

$$X \models^g \varphi \iff X^g \models \varphi$$

If $X \models^g \varphi$, let $\mathcal{A}, w \models X^g$ then $\mathcal{A} \models X^g$ (explained above) and so $\mathcal{A} \models X$ meaning $\mathcal{A} \models \varphi$ and in particular $\mathcal{A}, w \models \varphi$. Conversely, if $X^g \models \varphi$ suppose $\mathcal{A} \models X$ then $\mathcal{A} \models X^g$ and so $\mathcal{A} \models \varphi$. If S is a set of sentences, then $S^g = S$ and so we get that

$$S \models^g \varphi \iff S \models \varphi$$

so for sets of sentences, there is no difference. But otherwise the two consequence relations act differently, for example neither the rule of case distinction nor the deduction theorem hold. These are the rules:

$$\frac{X, \alpha \models^g \beta \mid X, \neg \alpha \models^g \beta}{X \models^g \beta} \quad (\text{rule of case distinction}), \quad \frac{X, \alpha \models^g \beta}{X \models^g \alpha \rightarrow \beta} \quad (\text{deduction theorem})$$

For example, $x = y \models^g \forall x \forall y x = y$ but it is not necessarily true that $\models^g x = y \rightarrow \forall x \forall y x = y$.

2.5.1 Definition

A **first-order theory** over a language \mathcal{L} (also a **\mathcal{L} -theory**), is a set of sentences $T \subseteq \mathcal{L}^0$ which is **deductively closed** in \mathcal{L} , meaning $T \models \varphi \iff \varphi \in T$ for all sentences $\varphi \in \mathcal{L}^0$. If $\varphi \in T$ then we say that φ is a **theorem** of T (or holds or is true in T). If $T \subseteq T'$ are both \mathcal{L} -theories, then T is called a **subtheory** of T' and T' is called an **extension** of T . An \mathcal{L} -structure \mathcal{A} such that $\mathcal{A} \models T$ is called an **model** of T , we denote the class of all models of T by $\text{Md } T$.

Different authors define theories slightly differently, for example some may not require the theory be deductively closed, etc.

For example, if $X \subseteq \mathcal{L}$ is a set of formulas we define its *deductive closure* to be $\{\alpha \in \mathcal{L}^0 \mid X \models \alpha\}$. Since \models is transitive, the deductive closure of a set of formulas is indeed a theory. When we discuss the *theory of* S , where S is a set of sentences, we mean the deductive closure of S . A set $X \subseteq \mathcal{L}$ is called an *axiom system* for a theory T if $T = \{\alpha \in \mathcal{L}^0 \mid X^g \models \alpha\}$. Notice how we tacitly generalize the free variables in axioms.

Furthermore, if T is a theory then $T \models \varphi \iff \mathcal{A} \models \varphi$ for all $\mathcal{A} \models T$ since T is a set of sentences (in general we require that $\mathcal{M} \models \varphi$ for all $\mathcal{M} \models T$ but as discussed previously, there is no distinction between global and local consequence). And $T \models \varphi \iff T \models \varphi^g$.

The smallest theory in \mathcal{L} is the theory containing only tautologies, which we denote $Taut_{\mathcal{L}}$. An axiom system for $Taut_{\mathcal{L}}$ is the empty set. The set of all sentences, \mathcal{L}^0 , is the largest theory in \mathcal{L} which is also called the *inconsistent theory*, and all other theories are called or *satisfiable*.

Notice that if $\{T_i\}_{i \in I}$ is a family of \mathcal{L} -theories, $T = \bigcap_{i \in I} T_i$ is also an \mathcal{L} -theory: if $T \models \alpha$ then $T_i \models \alpha$ for all $i \in I$ (by (MR)) and so $\alpha \in T$.

2.5.2 Definition

Let T be an \mathcal{L} -theory, and $\varphi \in \mathcal{L}^0$ be a sentence. Then we define $T + \varphi$ to be the smallest theory which extends T and contains φ . Similarly if $S \subseteq \mathcal{L}^0$ a set of sentences, then $T + S$ is the smallest theory which extends T and contains S . Alternatively,

$$T + S = \bigcap \{T' \mid T' \text{ is a theory which extends } T \text{ and contains } S\}$$

If $T + \varphi$ is satisfiable, then φ is said to be **compatible with T** . If $\neg\varphi$ is compatible with T , then φ is said to be **refutable in T** . If φ is both compatible and refutable (meaning $\neg\varphi$ is also compatible) with T , then φ is said to be **independent of T** .

2.5.3 Definition

If T is a \mathcal{L} -theory and $\alpha, \beta \in \mathcal{L}$ are two formulas, we say α and β are **equivalent modulo T** , denoted $\alpha \equiv_T \beta$, if $\alpha \equiv_{\mathcal{A}} \beta$ for all $\mathcal{A} \models T$. Meaning

$$\equiv_T := \bigcap_{\mathcal{A} \in \text{Md } T} \equiv_{\mathcal{A}}$$

So \equiv_T is an intersection of congruences, and is therefore also a congruence. Obviously

$$\alpha \equiv_T \beta \iff T \models \alpha \leftrightarrow \beta \iff T \models (\alpha \leftrightarrow \beta)^g$$

Similarly if t and s are \mathcal{L} -terms, we call them **equivalent modulo T** , denoted $t \approx_T s$, if $T \models s = t$ (meaning for all $\mathcal{A} \models T$ and valuations $w: \text{Var} \longrightarrow A$, $\mathcal{A}, w \models s = t$).

For example, let T_G^- be the theory of groups in the signature $\{\circ, e, ^{-1}\}$ (an equivalent formulation can be defined over the signature $\{\circ, e\}$ and gives us the equivalent theory T_G), then $(x \circ y)^{-1} \approx_{T_G^-} y^{-1} \circ x^{-1}$.

Let \mathcal{A} be an \mathcal{L} -structure, then we define its *theory* to be:

$$Th\mathcal{A} := \{\alpha \in \mathcal{L}^0 \mid \mathcal{A} \models \alpha\}$$

this is indeed a theory, as if $Th\mathcal{A} \models \alpha$ then since $\mathcal{A} \models Th\mathcal{A}$ by definition, we have $\mathcal{A} \models \alpha$ so $\alpha \in Th\mathcal{A}$. And if \mathbf{K} is a class of \mathcal{L} -structures, then its theory is defined to be

$$Th\mathbf{K} := \bigcap_{\mathcal{A} \in \mathbf{K}} Th\mathcal{A}$$

which is a theory as the intersection of theories.

Exercise

Suppose $x \notin \text{free}X$ and c is not in X or α . Prove

$$X \models \alpha \iff X \models \forall x \alpha \iff X \models \alpha^c_x$$

Since $x \notin \text{free}X$, we have $X \models \alpha \implies \forall x \alpha$. And similarly $X \models \forall x \alpha \implies X \models \alpha$ since $\alpha, \frac{x}{x}$ is collision-free. And since $\alpha, \frac{c}{x}$ is collision-free, we have $X \models \forall x \alpha \implies X \models \alpha^c_x$.

Now suppose $X \models \alpha^c_x$, then suppose $\mathcal{M} \models X$ then $\mathcal{M}_x^c \models \alpha$ by the substitution theorem. Let $a \in A$, since c is not in X , if we set $c^{\mathcal{M}} = a$ then \mathcal{M} still satisfies X , and so $\mathcal{M}_x^a \models \alpha$, thus $\mathcal{M} \models \forall x \alpha$. Thus $X \models \forall x \alpha$ as required.

Exercise

Let S be a set of sentences, α and β be formulas, $x \notin \text{free}\beta$, and c be a constant not occurring in S , α , or β . Show that

$$S \models \alpha_x^c \rightarrow \beta \iff S \models \exists x \alpha \rightarrow \beta$$

If $S \models \alpha_x^c \rightarrow \beta$, then if $\mathcal{M} \models S$, if $\mathcal{M} \models \exists x \alpha$ then there exists an $a \in A$ such that $\mathcal{M}_x^a \models \alpha$. Since we can set $c^{\mathcal{M}} = a$ as c does not occur in S, α, β (again, the coincidence theorem), so $\mathcal{M} \models \alpha_x^c$ by the substitution theorem. Thus $\mathcal{M} \models \beta$, meaning $\mathcal{M} \models \exists x \alpha \rightarrow \beta$ so $S \models \exists x \alpha \rightarrow \beta$.

And if $S \models \exists x \alpha \rightarrow \beta$ then since $\models \alpha_x^c \rightarrow \exists x \alpha$ we have $S \models \alpha_x^c \rightarrow \beta$ as required.

Exercise

Show that for all sentences $\alpha, \beta \in \mathcal{L}^0$, if T is an \mathcal{L} -theory then

$$\beta \in T + \alpha \iff \alpha \rightarrow \beta \in T$$

If $\beta \in T + \alpha$ then $T, \alpha \models \beta$ so $T \models \alpha \rightarrow \beta$ by the deduction theorem, thus $\alpha \rightarrow \beta \in T$ since T is deductively closed. Conversely if $\alpha \rightarrow \beta \in T$ then $T, \alpha \models \alpha \rightarrow \beta, \alpha$ and so $T, \alpha \models \beta$ thus $\beta \in T + \alpha$.

Exercise

Let $T \subseteq \mathcal{L}$ be an \mathcal{L} -theory and $\mathcal{L}_0 \subseteq \mathcal{L}$ a reduction of the language \mathcal{L} . Show that $T_0 := T \cap \mathcal{L}_0$ is also a (\mathcal{L}_0) -theory.

Suppose $T_0 \models \alpha$ for a \mathcal{L}_0 -sentence α , then $T \models \alpha$ and so $\alpha \in T$ and $\alpha \in \mathcal{L}_0$ so $\alpha \in T_0$. (Note that T_0 is not necessarily a \mathcal{L} -theory. For example if c is a constant symbol not occurring in \mathcal{L}_0 then $T_0 \models c = c$ but $c = c$ is a sentence not in \mathcal{L}_0 , so $c = c \notin T_0$).

2.6 Explicit Definitions—Language Expansions

It is often very useful to define notions within a theory on top of the language of the theory. For example, in rings the concept of divisibility is often useful, where divisibility is defined by $x|y \iff \exists z x \cdot z = y$.

In this section, if $\varphi(\vec{x})$ is a formula (meaning $\text{free}\varphi \subseteq \{x_1, \dots, x_n\}$), we use the notation $\varphi(\vec{t})$ in place of $\varphi_{\vec{x}}^{\vec{t}}$. Substitutions here are tacitly assumed to be collision-free.

2.6.1 Definition

Let r be an n -ary relational symbol not occurring in \mathcal{L} . An **explicit definition of r in \mathcal{L}** is a formula of the form

$$\eta_r: \quad r\vec{x} \leftrightarrow \delta(\vec{x})$$

where $\delta(\vec{x}) \in \mathcal{L}$ and \vec{x} is a vector of distinct variables. δ is called the **defining formula of r** . If T is an \mathcal{L} -theory, then $T_r := T + \eta_r^g$ is the **definitorial extension of T by r** .

Similarly if f is an n -ary ($n \geq 0$) function (or constant, in the case $n = 0$) symbol not occurring in \mathcal{L} its explicit definition is of the form

$$\eta_f: \quad y = f\vec{x} \leftrightarrow \delta(\vec{x}, y)$$

where $\delta \in \mathcal{L}$ is the defining formula, and \vec{x}, y are distinct. η_f is called **legitimate** in a theory T if $T \models \forall \vec{x} \exists! y \delta$ (so f is indeed a function), and if η_f is legitimate then $T_f := T + \eta_f^g$ is the definitorial extension of T by f .

2.6.2 Definition

Let s be a (function, relational, or constant) symbol and \mathcal{L} be a first-order language. Then we define the

language extension $\mathcal{L}[s]$ to be the first-order language obtained by adding s to the signature.

If r is an n -ary relational symbol not occurring in \mathcal{L} defined by η_r , then for every $\varphi \in \mathcal{L}[r]$ we define the **reduced formula** φ^{rd} by replacing every prime formula $r\vec{t}$ in φ by $\delta_r(\vec{t})$.

Now, if f is an n -ary function symbol legitimately defined by η_f in T , then for every $\varphi \in \mathcal{L}[f]$ define φ^{rd} in steps: for the first occurrence of f in φ on the left, we may write $\varphi = \varphi_0 \frac{f\vec{t}}{y}$ for appropriate φ_0 , \vec{t} , and $y \notin \text{var}\varphi$. So $\varphi \equiv_{T_f} \exists y(\varphi_0 \wedge y = f\vec{t}) \equiv_{T_f} \varphi_1$ where $\varphi_1 = \exists y(\varphi_0 \wedge \delta_f(\vec{t}, y))$. Continue inductively on φ_1 until getting a formula φ_m in which f does not occur, set $\varphi^{rd} = \varphi_m$.

Obviously if $\varphi \in \mathcal{L}$ (ie. they do not include the defined symbols), $\varphi^{rd} = \varphi$. For example, in T_G we can define $^{-1}$ by $y = x^{-1} \leftrightarrow x \circ y = e$, and this is legitimate in T_G .

2.6.3 Theorem (The Elimination Theorem)

Let T be an \mathcal{L} -theory, and T_s be the definitorial extension of T by some η_s . Then for all $\varphi \in \mathcal{L}[s]$,

$$T_s \models \varphi \iff T \models \varphi^{rd}$$

We will prove this in the case that $s = r$ is an n -ary relational symbol. If $\mathcal{A} \models T$ then \mathcal{A} can be expanded to a model $\mathcal{A}' \models T_r$ by setting $r\mathcal{A}'\vec{a} \iff \mathcal{A} \models \delta[\vec{a}]$. Since $r\vec{t} \equiv_{T_r} \delta(\vec{t})$ for any \vec{t} , by The Replacement Theorem $\varphi \equiv_{T_r} \varphi^{rd}$ for all formulas $\varphi \in \mathcal{L}[r]$. Thus

$$\begin{aligned} T_r \models \varphi &\iff \mathcal{A}' \models \varphi && \text{for all } \mathcal{A}' \models T && (\text{Md } T_r = \{\mathcal{A}' \mid \mathcal{A}' \models T\}) \\ &\iff \mathcal{A}' \models \varphi^{rd} && \text{for all } \mathcal{A}' \models T && (\varphi \equiv_{T_r} \varphi^{rd}) \\ &\iff \mathcal{A} \models \varphi^{rd} && \text{for all } \mathcal{A} \models T && (\text{The Coincidence Theorem}) \\ &\iff T \models \varphi^{rd} \end{aligned}$$

Sometimes function symbols are defined similar to

$$f\vec{x} := t(\vec{x})$$

where t is a term. This is simply equivalent to

$$\eta_f: \quad y = f\vec{x} \leftrightarrow y = t(\vec{x})$$

and so it is a special case of explicit definitions. And all definitions of this form are legitimate: $\forall \vec{x} \exists! y y = t(\vec{x})$ is a tautology.

2.6.4 Definition

Suppose T is an \mathcal{L} -theory, and Δ is a set of explicit definitions, then $T' = T + \Delta$ is a **definitorial extension** of T . Let \mathcal{L}' be the language expansion of \mathcal{L} obtained by adjoining all the new symbols defined in Δ to \mathcal{L} , then for $\varphi \in \mathcal{L}'$, we define φ^{rd} as above where the procedure is done stepwise for every new symbol in φ .

From The Elimination Theorem it is immediate

2.6.5 Theorem (The General Elimination Theorem)

Let T' be a definitorial extension of T , then $\alpha \in T' \iff \alpha^{rd} \in T$ for all $\alpha \in \mathcal{L}'$. In particular if $\alpha \in \mathcal{L}$, $\alpha \in T' \iff \alpha \in T$, thus T' is a conservative extension of T .

2.6.6 Definition

Let T be an \mathcal{L} -theory, and s be a symbol in \mathcal{L} . Then s is **explicitly definable** in T if T contains an explicit definition of s where s 's defining formula is in \mathcal{L}_0 , the language obtained by removing s from \mathcal{L} .

For example, the constant e is explicitly definable in the theory of groups T_G , since $x = e \leftrightarrow x \circ x = x$ is an explicit definition of e which is in T_G (more specifically, its generalized is).

2.6.7 Proposition

Suppose s is explicitly definable in $Th\mathcal{A}$ and legitimate, suppose we extend \mathcal{A} to \mathcal{A}' by adding the symbol s to its signature. Then automorphisms of \mathcal{A} are automorphisms of \mathcal{A}' (the converse is trivial).

Let σ be an automorphism of \mathcal{A} . If $s = r$ is a relational symbol, we must show that $r\vec{x} \iff r\sigma\vec{x}$. Now suppose that δ is the defining formula of r , so this is equivalent to showing $\mathcal{A} \models \delta[\vec{a}]$ if and only if $\mathcal{A} \models \delta[\sigma\vec{a}]$ for all $\vec{a} \in A^n$. This is true by The Invariance Theorem. Now suppose that $s = f$ is a function symbol, we must show that $f\sigma\vec{x} = \sigma f\vec{x}$. Since $y = f\vec{x}$ if and only if $\delta[\vec{x}, y]$, we must show that $\delta[\vec{x}, y]$ if and only if $\delta[\sigma\vec{x}, \sigma y]$. This is as then if $y = f\vec{x} \implies \sigma y = \sigma f\vec{x}$ so $\sigma f\vec{x} = f\sigma\vec{x}$. But this is again true by The Invariance Theorem. ■

Let us now turn our attention to another type of normal forms. We call two formulas α and β *satisfiably equivalent* if they are both satisfiable (not necessarily by the same model) or they are both not satisfiable. Now suppose α is a formula, we assume without loss of generality that it is a PNF $\alpha = Q_1x_1 \cdots Q_nx_n\beta$ (β quantifier-free). We will construct a satisfiably equivalent \forall -formula $\hat{\alpha}$ such that α and $\hat{\alpha}$ are satisfiably equivalent. We do this inductively/stepwise: let $\alpha_0 = \alpha$ and suppose we have already constructed α_i . If α_i is already a \forall -formula, set $\hat{\alpha} = \alpha_i$, otherwise α_i has the form $\forall x_1 \cdots \forall x_k \exists y \beta_i$ for $k \geq 0$. Let f be some k -ary (if $k = 0$ this is a constant) function symbol not yet used (not in \mathcal{L} , not used in the previous i steps) and let $\alpha_{i+1} = \forall x_1 \cdots \forall x_k \beta_i \frac{f\vec{x}}{y}$. $\hat{\alpha}$ is called the *Skolem Normal Form* (SNF for short) of α .

Essentially what is done here is that variables which are existentially quantified are viewed instead as functions of the universally quantified variables (which precede them on the left).

If m is the number of \exists quantifiers in Q_1, \dots, Q_n , then after m steps we will have $\hat{\alpha} = \alpha_m$ and $free\hat{\alpha} = free\alpha$ (since $free\alpha_i = free\alpha$ for all i). And $\hat{\alpha}$ is a \forall -formula.

For example, let α be the formula $\forall x \exists y x < y$ then $\hat{\alpha}$ is $\forall x x < fx$. And for $\alpha = \exists x \forall y x \cdot y = y$, we have $\hat{\alpha} = \forall y c \cdot y = y$. And for $\alpha = \forall x \forall y \exists z (x < z \wedge y < z)$ then $\hat{\alpha} = \forall x \forall y (x < fxy \wedge y < fxy)$.

2.6.8 Theorem

Let $\hat{\alpha}$ be the Skolem normal form of the formula α . Then

- (1) $\hat{\alpha} \models \alpha$ (2) $\hat{\alpha}$ and α are satisfiably equivalent

To prove (1), it is sufficient to show that $\alpha_{i+1} \models \alpha_i$ for each of the construction steps of $\hat{\alpha}$ (this is sufficient due to the transitivity of \models). Since $\beta_i \frac{f\vec{x}}{y}$ is collision-free (β_i is a PNF not containing quantifiers of \vec{x}) we have that $\beta_i \frac{f\vec{x}}{y} \models \exists y \beta_i$. Then by anterior and posterior generalization, $\alpha_{i+1} = \forall \vec{x} \beta_i \frac{f\vec{x}}{y} \models \forall \vec{x} \exists y \beta_i = \alpha_i$ as required.

By (1), if $\hat{\alpha}$ is satisfiable, so is α . Conversely, if $\mathcal{A} \models \alpha_i[\vec{c}] = \forall \vec{x} \exists y \beta_i(\vec{x}, y, \vec{z})[\vec{c}]$. For each $\vec{a} \in A^n$, choose a $b \in A$ such that $\mathcal{A} \models \beta_i[\vec{a}, b, \vec{c}]$ (choosing such a b is possible due to the axiom of choice), and expand \mathcal{A} to \mathcal{A}' by setting $f\vec{a} = b$ where f is the new function symbol. Then $\mathcal{A}' \models \alpha_{i+1}[\vec{c}]$. So if α is satisfiable, so is every α_i , and in particular $\hat{\alpha}$ (and the model which satisfies $\hat{\alpha}$ is an expansion of the one which satisfies α). ■

Notice that if α is a formula then $\neg\alpha$ has a SNF, and so its negation is an \exists -formula and is satisfiably equivalent to α . We will denote this \exists -formula by $\check{\alpha}$: if $\beta = \neg\alpha$ then $\check{\alpha} = \neg\hat{\beta}$.

For example, if $\alpha = \exists x \forall y (ry \rightarrow rx)$ then $\neg\alpha \equiv \beta = \forall x \exists y (ry \wedge \neg rx)$ and then $\hat{\beta} = \forall x (rfx \wedge \neg rx)$ and so $\check{\alpha} = \exists x (rfx \rightarrow rx)$. This is actually a tautology. Skolem normal forms are used in model theory and logic programming.

Exercise

Let T_f result from adjoining an explicit definition η_f for f to a theory T . Show that T_f is a conservative extension of T if and only if η_f is legitimate.

Suppose δ_f is the defining formula for f , meaning $\eta_f = y = f\vec{x} \leftrightarrow \delta(\vec{x}, y)$. We have already shown that if η_f is legitimate then T_f is a conservative extension of T , we will now show that if T_f is conservative then η_f is legitimate. We know that $T_f \models \forall \vec{x} \exists! y y = f\vec{x}$ and since $y = f\vec{x} \equiv_{T_f} \delta(\vec{x}, y)$ we have that $T_f \models \forall \vec{x} \exists! y \delta(\vec{x}, y)$ and since T_f is conservative this means $T \models \forall \vec{x} \exists! y \delta(\vec{x}, y)$ so η_f is legitimate.

Exercise

Let $S: n \mapsto n+1$ be the successor function in $\mathcal{N} = (\mathbb{N}, 0, S, +, \cdot)$. Show that $Th\mathcal{N}$ is a definitorial extension of $Th(\mathbb{N}, S, \cdot)$; ie. 0 and + are explicitly definable by S and \cdot in \mathbb{N} .

0 is the only number which is not the successor of some other number, so we could define 0 by $\delta_0(x) = \forall y Sy \neq x$. Alternatively 0 is the only number whose product with every number is itself, so we could also define $\delta_0(x) = \forall y x \cdot y = x$. Addition requires more consideration.

Exercise

$<$ is explicitly definable in $(\mathbb{N}, 0, +)$ by $x < y \leftrightarrow (\exists z \neq 0)x + z = y$. Show that $<$ is not explicitly definable in $(\mathbb{Z}, 0, +)$.

We showed that explicitly definable relations are invariant under automorphisms. But $n \mapsto -n$ is an automorphism of $(\mathbb{Z}, 0, +)$ (since $-0 = 0$ and $(-x) + (-y) = -(x + y)$) but $<$ is certainly not invariant under it.

3 Complete Logical Calculi

3.1 A Calculus of Natural Deduction

Let \mathcal{L} be an arbitrary first-order language in the logical signature $\neg, \wedge, \forall, =$. We define a calculus \vdash using the following basic rules (the definitions for sequents and proofs remain the same as they were in propositional logic):

$$\begin{array}{ll}
(\text{IR}) \quad \frac{}{X \vdash \alpha} \quad (\alpha \in X \cup \{t = t\}) & (\text{MR}) \quad \frac{X \vdash \alpha}{X' \vdash \alpha} \quad (X \subseteq X') \\
(\wedge 1) \quad \frac{X \vdash \alpha, \beta}{X \vdash \alpha \wedge \beta} & (\wedge 2) \quad \frac{X \vdash \alpha \wedge \beta}{X \vdash \alpha, \beta} \\
(\neg 1) \quad \frac{X \vdash \alpha, \neg \alpha}{X \vdash \beta} & (\neg 2) \quad \frac{X, \alpha \vdash \beta \mid X, \neg \alpha \vdash \beta}{X \vdash \beta} \\
(\forall 1) \quad \frac{X \vdash \forall x \alpha}{X \vdash \alpha_{\frac{t}{x}}} \quad (\alpha, \frac{t}{x} \text{ collision-free}) & (\forall 2) \quad \frac{X \vdash \alpha_{\frac{y}{x}}}{X \vdash \forall x \alpha} \quad (y \notin \text{free} X \cup \text{var} \alpha) \\
(=) \quad \frac{X \vdash s = t, \alpha_{\frac{s}{x}}}{X \vdash \alpha_{\frac{t}{x}}} \quad (\alpha \text{ prime}) &
\end{array}$$

Note that $(\forall 2)$ is the result of posterior generalization: if $X \vdash \alpha$ then $X \vdash \forall x \alpha$ for $x \notin \text{free} X$. Set $\alpha = \alpha_{\frac{y}{x}}$ then then $X \vdash \alpha_{\frac{y}{x}}$ implies $X \vdash \forall y \alpha_{\frac{y}{x}} \equiv \forall x \alpha$ for $y \notin \text{free} X \cup \text{var} \alpha$ by renaming bound variables. Thus all of these rules are sound in the sense that they are true for \models . The rule $(=)$ could be strengthened to require $\alpha, \frac{t}{x}, \frac{s}{x}$ be collision-free, but requiring α be prime instead is sufficient.

Since this calculus is an extension of the Gentzen-style calculus of propositional logic, all of the rules we proved in propositional logic carry over here.

Again, let R be a rule of the form

$$R: \frac{X_1 \vdash \alpha_1 \mid \cdots \mid X_n \vdash \alpha_n}{X \vdash \alpha}$$

Then we say that a property of sequents (which are again pairs (X, α) where $X \subseteq \mathcal{L}$ and $\alpha \in \mathcal{L}$) \mathcal{E} is *closed under R* if $\mathcal{E}(X_1, \alpha_1), \dots, \mathcal{E}(X_n, \alpha_n)$ implies $\mathcal{E}(X, \alpha)$. The same principle from propositional calculus holds here, and the proof is the same.

3.1.1 Proposition (Principle of Rule Induction)

Let \mathcal{E} be a property of sequents (X, α) such that \mathcal{E} is closed under all of the basic rules defined above, then $X \vdash \alpha$ implies $\mathcal{E}(X, \alpha)$.

Thus if we let $\mathcal{E}(X, \alpha)$ be the property that $X \vdash \alpha$ then we get that \mathcal{E} is closed under all of the basic rules, and so $X \vdash \alpha \implies X \vdash \alpha$, or more suggestively $\vdash \subseteq \vdash$. Similarly if we have $\mathcal{L} \subseteq \mathcal{L}'$ and we let $\mathcal{E}(X, \alpha)$ be the property $X \vdash_{\mathcal{L}'} \alpha$ then we get $X \vdash_{\mathcal{L}} \alpha \implies X \vdash_{\mathcal{L}'} \alpha$, or $\vdash_{\mathcal{L}} \subseteq \vdash_{\mathcal{L}'}$.

Similarly let $\mathcal{E}(X, \alpha)$ be the property that there exists a finite $X_0 \subseteq X$ and a finite $\mathcal{L}_0 \subseteq \mathcal{L}$ (meaning the signature of \mathcal{L}_0 is finite, not \mathcal{L}_0 itself, which is always infinite) where $X_0 \vdash_{\mathcal{L}_0} \alpha$. Notice that this requires that $X_0 \cup \{\alpha\} \subseteq \mathcal{L}_0$. We will first show that (IR) is closed under \mathcal{E} , so if $\alpha \in X \cup \{t = t\}$ then let $X_0 = \{\alpha\}$ and let \mathcal{L}_0 be constructed from the symbols in α which is finite. And then we get $X_0 \vdash_{\mathcal{L}_0} \alpha$. For (MR) , it is trivial. For $(\wedge 1)$ suppose $X_1 \vdash_{\mathcal{L}_1} \alpha_1$ and $X_2 \vdash_{\mathcal{L}_2} \alpha_2$ where $X_i \subseteq X$ and the signatures of \mathcal{L}_i are finite. Then let \mathcal{L}_0 be constructed by taking the union of the signatures of \mathcal{L}_1 and \mathcal{L}_2 , and $X_0 = X_1 \cup X_2$ then we get that $X_0 \vdash_{\mathcal{L}} \alpha, \beta$ and so $X_0 \vdash_{\mathcal{L}} \alpha \wedge \beta$ and $X_0 \subseteq X$ is finite and \mathcal{L} has a finite signature, as required. The proofs for the rest of the rules are similar.

3.1.2 Theorem

If $X \vdash_{\mathcal{L}} \alpha$ then there exists a finite $X_0 \subseteq X$ and a language $\mathcal{L}_0 \subseteq \mathcal{L}$ with finitely many symbols such that $X_0 \vdash_{\mathcal{L}_0} \alpha$.

Let us now prove the following

$$(1) \quad \frac{X \vdash s = t, s = t'}{X \vdash t = t'} \quad (2) \quad \frac{X \vdash s = t}{X \vdash t = s} \quad (3) \quad \frac{X \vdash t = s, s = t'}{X \vdash t = t'}$$

To show (1), let $x \notin \text{var} t'$ and let $\alpha = x = t'$ then the premise of (1) can be rewritten as $X \vdash s = t, \alpha \frac{s}{x}$. Rule (=) yields $X \vdash \alpha \frac{t}{x} = t = t'$ as required. For (2), by (IR) $X \vdash s = t, s = s$ so by (1) we get $X \vdash t = s$. And so follows (3), as the premise gives $X \vdash s = t, s = t'$ by (2) which gives $X \vdash t = t'$ by (1).

Now we show (f and r are function and relation symbols of arity n),

$$\begin{array}{ll} (1) \frac{X \vdash t_i = t}{X \vdash f\vec{t} = ft_1 \cdots t_{i-1} t t_{i+1} \cdots t_n} & (2) \frac{X \vdash \vec{t} = \vec{t}'}{X \vdash f\vec{t} = f\vec{t}'} \\ (3) \frac{X \vdash t_i = t, r\vec{t}}{X \vdash r t_1 \cdots t_{i-1} t t_{i+1} \cdots t_n} & (4) \frac{X \vdash \vec{t} = \vec{t}', r\vec{t}}{X \vdash r\vec{t}'} \end{array}$$

We prove (1): let us define $\alpha = f\vec{t} = ft_1 \cdots x \cdots t_n$ where x does not occur in any t_j . Then if $X \vdash t_i = t$ we also have $X \vdash \alpha \frac{t_i}{x}$ by (IR), as $\alpha \frac{t_i}{x} = f\vec{t} = f\vec{t}$. So by (=) we get $X \vdash \alpha \frac{t}{x} = f\vec{t} = ft_1 \cdots t \cdots t_n$, as required. for (2), apply (1) n times (since $\vec{t}' = \vec{t}$ means $t'_i = t_i$ for each i). Rule (3) is proven similarly to rule (1), and (4) is proven similar to (2).

We will also show

$$(1) \vdash \exists x t = x \ (x \notin \text{var} t) \quad (2) \vdash \exists x x = x$$

These are trivial to show semantically, but require a bit more work to show derive them using our calculus. ($\forall 1$) gives $\forall x t \neq x \vdash (t \neq x) \frac{t}{x} = t \neq t$. And by (IR), $\forall x t \neq x \vdash t = t$, and so by ($\neg 1$) we get $\forall x t \neq x \vdash \exists x t = x$. And so $\neg \forall x t \neq x \vdash \exists x t = x$ (as by definition these are equal, so this is by (IR)). So by ($\neg 2$), we get $\vdash \exists x t = x$ as required. (2) is verified similarly, starting with $\forall x x \neq x \vdash x \neq x, x = x$.

3.1.3 Definition

A set $X \subseteq \mathcal{L}$ is called **inconsistent** if $X \vdash \alpha$ for all formulas $\alpha \in \mathcal{L}$. A non-inconsistent set is called **consistent**.

If a set is satisfiable, it is obviously consistent (since $\vdash \subseteq \models$). Also by ($\neg 1$), the inconsistency of X is equivalent to $X \vdash \alpha, \neg \alpha$ for any α , which is equivalent to $X \vdash \perp$ (since $\perp = \neg \top = \exists \mathbf{v}_0 \mathbf{v}_0 = \mathbf{v}_0$ and $X \vdash \top$ as we showed above).

And as before (the proof remains valid), we have the following two equivalences:

$$\mathbf{C}^+: X \vdash \alpha \iff X, \neg \alpha \vdash \perp \quad \mathbf{C}^-: X \vdash \neg \alpha \iff X, \alpha \vdash \perp$$

Similarly again to propositional logic, a set X is *maximally consistent* if X is consistent but every $X \subset X'$ is inconsistent.

Exercise

Derive the rule $\frac{X \vdash \alpha \frac{t}{x}}{X \vdash \exists x \alpha}$ where $\alpha, \frac{t}{x}$ are collision-free.

By ($\forall 1$) we have $X, \forall x \neg \alpha \vdash \neg \alpha \frac{t}{x}$, then by the premise and (MR) we have $X, \forall x \neg \alpha \vdash \alpha \frac{t}{x}$. Thus by ($\neg 1$) we have $X, \forall x \neg \alpha \vdash \neg \forall x \neg \alpha$. And by (IR) and (MR) we also have $X, \neg \forall x \neg \alpha \vdash \neg \forall x \neg \alpha$, and so $X \vdash \neg \forall x \neg \alpha = \exists x \alpha$ as required.

Exercise

Prove $\forall x \alpha \vdash \forall y (\alpha \frac{y}{x})$ and $\forall y (\alpha \frac{y}{x}) \vdash \forall x \alpha$ provided $y \notin \text{var} \alpha$.

Since let $z \notin \text{var} \alpha$ and $z \neq y$ then $\alpha \frac{z}{x} = \alpha \frac{y}{x} \frac{z}{y}$, and so by ($\forall 1$), $\forall x \alpha \vdash \alpha \frac{y}{x} \frac{z}{y}$. Then by ($\forall 2$) we get $\forall x \alpha \vdash \forall y (\alpha \frac{y}{x})$.

Exercise

Prove the rule $\frac{X \vdash \forall y (\alpha \frac{y}{x})}{X \vdash \forall z (\alpha \frac{z}{x})}$ where $y, z \notin \text{var} \alpha$.

We have that $\forall y (\alpha \frac{y}{x}) \vdash \forall z (\alpha \frac{y}{x} \frac{z}{y}) = \forall z (\alpha \frac{z}{x})$ by the previous problem. So we have $X \vdash \forall y (\alpha \frac{y}{x})$ and $X, \forall y (\alpha \frac{y}{x}) \vdash \forall z (\alpha \frac{z}{x})$ so by the cut rule, we get $X \vdash \forall z (\alpha \frac{z}{x})$, as required.

Exercise

Show that a consistent set $X \subseteq \mathcal{L}$ is maximally consistent if and only if for all $\varphi \in \mathcal{L}$, $\varphi \in X$ or $\neg\varphi \in X$. This means that maximally consistent sets are deductively closed, ie. $X \vdash \varphi \implies \varphi \in X$.

Let $\varphi \in \mathcal{L}$ then suppose $\varphi, \neg\varphi \notin X$ then $X, \varphi \vdash \perp$ and $X, \neg\varphi \vdash \perp$ since X is maximally consistent, so any extension is inconsistent. Thus by $(\neg 2)$ we have $X \vdash \perp$ contradicting X being consistent. Now, if $X \vdash \varphi$ then if $\varphi \notin X$ we have $\neg\varphi \in X$ and so $X \vdash \varphi, \neg\varphi$ contradicting X 's consistency.

3.2 The Completeness Proof

We previously used the notation $\mathcal{L}[s]$ for the language obtained by adjoining the symbol s to the signature of \mathcal{L} . For a set of symbols S we will similarly use the notation $\mathcal{L}[S]$ for the language obtained by adjoining the symbols in S to the signature of \mathcal{L} . We will also sometimes omit the brackets and instead write $\mathcal{L}s$ or $\mathcal{L}S$.

If C is a set of constants, then $\mathcal{L}C$ is a *constant expansion* of \mathcal{L} . If $\alpha \in \mathcal{L}c$ and $z \in \text{Var}$ then we write α_c^z to be the formula in \mathcal{L} obtained by replacing every occurrence of c in α with the variable z . Similarly if $X \subseteq \mathcal{L}c$, then $X_c^z := \{\alpha_c^z \mid \alpha \in X\}$.

3.2.1 Lemma

Suppose $X \vdash_{\mathcal{L}c} \alpha$ then $X_c^z \vdash_{\mathcal{L}} \alpha_c^z$ for almost all (meaning all but a finite number) of variables z .

We do this by rule induction on $\vdash_{\mathcal{L}c}$. We start with (IR): if $\alpha \in X$ then $\alpha_c^z \in X_c^z$ as well, so $X_c^z \vdash_{\mathcal{L}} \alpha_c^z$ as required. If $\alpha = t = t$ then α_c^z is also of the form $t' = t'$ and so $X_c^z \vdash_{\mathcal{L}} \alpha_c^z$ as well. (MR) through $(\neg 2)$ are similarly obvious and impose no restrictions on z ; only the steps for $(\forall 1)$, $(\forall 2)$, and $(=)$ aren't immediately obvious. We will show only $(\forall 1)$ as the other two are proven in a similar manner. Let $\alpha, \frac{t}{x}$ be collision-free and $X \vdash_{\mathcal{L}c} \forall x \alpha$, then by induction we have $X_c^z \vdash_{\mathcal{L}} (\forall x \alpha)_c^z$ for almost all z . We may assume $z \notin \text{var} \forall x \alpha, \text{var} t$. We can verify through formula induction that $\alpha_c^{\frac{t}{x} z} = \alpha'_{\frac{t'}{x}}$ where $\alpha' = \alpha_c^z$ and $t' = t_c^z$. Then $\alpha', \frac{t'}{x}$ is collision-free (the bound variables of α' are the bound variables of α , and t' contains potentially only one more variable than t , z which does not occur in α). So we have that $X_c^z \vdash_{\mathcal{L}} (\forall x \alpha)_c^z = \forall x \alpha'$, thus by $(\forall 1)$ we have $X_c^z \vdash_{\mathcal{L}} \alpha'_{\frac{t'}{x}} = (\alpha_c^{\frac{t}{x} z})_c^z$, and this is true for almost all variables z , as required. ■

This lemma gives rise to the following rule:

$$(\forall 3) \quad \frac{X \vdash \alpha_c^{\frac{c}{x}}}{X \vdash \forall x \alpha} \quad (c \text{ not in } X, \alpha)$$

Suppose $X \vdash \alpha_c^{\frac{c}{x}}$, by the finiteness of \vdash we may assume that X is finite. By the above lemma, where $\mathcal{L}c = \mathcal{L}$ (ie. we adjoin a constant already in \mathcal{L}), we can find a y not occurring in $X \cup \{\alpha\}$ (as X is finite) such that $X_c^y \vdash \alpha_c^{\frac{c}{x} y} = \alpha_c^{\frac{y}{x}}$ (since c does not occur in α). Since c does not occur in X , we have $X_c^y = X$ and thus we get $X \vdash \alpha_c^{\frac{y}{x}}$ and so $X \vdash \forall x \alpha$ by $(\forall 2)$.

3.2.2 Lemma

Let C be a set of constant symbols and $\mathcal{L}' = \mathcal{L}C$. Then $X \vdash_{\mathcal{L}} \alpha \iff X \vdash_{\mathcal{L}'} \alpha$ for all $X \subseteq \mathcal{L}$ and $\alpha \in \mathcal{L}$. Thus $\vdash_{\mathcal{L}'}$ is a conservative expansion of $\vdash_{\mathcal{L}}$.

By the monotonicity of \vdash , we have $X \vdash_{\mathcal{L}} \alpha \implies X \vdash_{\mathcal{L}'} \alpha$. Now conversely, suppose $X \vdash_{\mathcal{L}'} \alpha$ then we may assume by finiteness that C is finite, and since adjoining finitely many constants can be done stepwise, we may assume that $\mathcal{L}' = \mathcal{L}c$ for a single constant c not occurring in \mathcal{L} (if c occurs in \mathcal{L} this is trivial). Then lemma 3.2.1 tells us that $X_c^z \vdash_{\mathcal{L}} \alpha_c^z$ for some variable z , but since $X \subseteq \mathcal{L}$ and $\alpha \in \mathcal{L}$ so c occurs in neither X nor α so this is equivalent to $X \vdash_{\mathcal{L}} \alpha$ as required. ■

By this lemma, we may write \vdash for the derivability relation \mathcal{L} and every constant expansion \mathcal{L}' and there will be no misunderstandings as the expansions are conservative. This also means that X is consistent in \mathcal{L} if and only if it is consistent in \mathcal{L}' as $X \vdash_{\mathcal{L}} \perp \iff X \vdash_{\mathcal{L}'} \perp$.

For every variable x and formula $\alpha \in \mathcal{L}$ let us define a distinct constant symbol $c_{x,\alpha}$ which does not occur in \mathcal{L} . Then we define

$$\alpha^x := \neg \forall x \alpha \wedge \alpha \frac{c_{x,\alpha}}{x}$$

Notice that $\neg \alpha^x \equiv \exists x \neg \alpha \rightarrow \neg \alpha_c^{\frac{c}{x}}$, meaning $\neg \alpha^x$ says that “if α is not true for all x , c provides a counterexample for α ”. Thus $\neg \alpha^x \equiv \top$ whenever $x \notin \text{free} \alpha$ (since then $\neg \alpha^x \equiv \neg \alpha \rightarrow \neg \alpha$).

3.2.3 Lemma

Let $\Gamma_{\mathcal{L}} := \{\neg\alpha^x \mid \alpha \in \mathcal{L}, x \in \text{Var}\}$, and let $X \subseteq \mathcal{L}$ be consistent. Then $X \cup \Gamma_{\mathcal{L}}$ is also consistent.

Suppose $X \cup \Gamma_{\mathcal{L}} \vdash \perp$, then by the finiteness of \vdash , there exist $\neg\alpha_0^{x_0}, \dots, \neg\alpha_n^{x_n} \in \Gamma_{\mathcal{L}}$ such that $X \cup \{\neg\alpha_i^{x_i} \mid i \leq n\} \vdash \perp$. Since X is consistent and so $X \not\vdash \perp$, we can assume that n is minimal so that $X' = X \cup \{\neg\alpha_i^{x_i} \mid i < n\} \not\vdash \perp$. Let $x = x_n$, $\alpha = \alpha_n$, and $c = c_{x,\alpha}$ then $X' \cup \{\neg\alpha^x\} \vdash \perp$ and so by \mathbf{C}^+ we have $X' \vdash \alpha^x = \neg\forall x\alpha \wedge \alpha_x^c$. Thus by $(\wedge 2)$ we have $X' \vdash \neg\forall x\alpha, \alpha_x^c$. By $(\forall 3)$, we get that $X' \vdash \alpha_x^c$ means $X' \vdash \forall x\alpha$, and this means that X' is inconsistent, in contradiction. ■

3.2.4 Definition

A set $X \subseteq \mathcal{L}$ is called a **Henkin set** if X satisfies the following two conditions:

- (H1) $X \vdash \neg\alpha \iff X \not\vdash \alpha$, (equivalently $X \vdash \alpha \iff X \not\vdash \neg\alpha$)
- (H2) $X \vdash \forall x\alpha \iff X \vdash \alpha_x^c$ for all constants c in \mathcal{L} .

(H1) and (H2) imply a third property:

- (H3) For each term t there is a constant c such that $X \vdash t = c$

We showed $X \vdash \neg\forall x t \neq x$ ($= \exists x t = x$), and so $X \not\vdash \forall x t \neq x$ by (H1) and so $X \not\vdash t \neq c$ for some c by (H2) and thus $X \vdash t = c$ by (H1).

3.2.5 Lemma

Let $X \subseteq \mathcal{L}$ be consistent. Then there exists a Henkin set $Y \supseteq X$ in a suitable constant expansion $\mathcal{L}C$ of \mathcal{L} .

Let $\mathcal{L}_0 := \mathcal{L}$ and $X_0 := X$, and assume that \mathcal{L}_n and X_n have already be defined. Then let \mathcal{L}_{n+1} be obtained by adding new constants $c_{x,\alpha,n}$ for all $x \in \text{Var}$ and $\alpha \in \mathcal{L}_n$, meaning $\mathcal{L}_{n+1} = \mathcal{L}_n C_n$ where C_n is the set of constants $c_{x,\alpha,n}$. Furthermore, let $X_{n+1} = X_n \cup \Gamma_{\mathcal{L}_n}$ (where $\Gamma_{\mathcal{L}_n}$ is defined as in the previous lemma), and so $X_{n+1} \subseteq \mathcal{L}_{n+1}$. Inductively we see that X_n is consistent for each n by the previous lemma. Let $X' = \bigcup_{n=0}^{\infty} X_n$ and $\mathcal{L}' = \bigcup_{n=0}^{\infty} \mathcal{L}_n = \mathcal{L}C$ where $C = \bigcup_{n=0}^{\infty} C_n$. Thus $X' \subseteq \mathcal{L}'$, and X' must be consistent: if $X' \vdash \perp$ then by the finiteness theorem we get that some $\varphi_1, \dots, \varphi_n \vdash \perp$. Since X' is the union of a chain of sets, we have that $\varphi_1, \dots, \varphi_n \in X_m$ for some m and thus $X_m \vdash \perp$ contradicting its consistency.

Let $\alpha \in \mathcal{L}'$ and $x \in \text{Var}$, then say $\alpha \in \mathcal{L}_n$ where n is minimal, and take α^x defined with respect to \mathcal{L}^n , and so $\neg\alpha^x \in X_{n+1} \subseteq X'$. Let us define

$$H = \{Y \subseteq \mathcal{L}' \mid X' \subseteq Y \text{ and } Y \text{ is consistent}\}$$

the set of all consistent extensions of X' in \mathcal{L}' . This is partially ordered by \subseteq . Let K be a chain in H , then $\bigcup K$ is an upper bound of the chain since the union of a chain of consistent sets is consistent (we showed this earlier in the proof). And $H \neq \emptyset$ as $X \in H$, so by Zorn's lemma H contains a maximal element Y ; Y is a maximally consistent extension of X . Since $\neg\alpha^x \in X' \subseteq Y$, we have that $Y \vdash \neg\alpha^x$ for all $\alpha \in \mathcal{L}'$.

Y is also Henkin: for (H1), $Y \vdash \neg\alpha$ implies $Y \not\vdash \alpha$ by Y 's consistency. And if $Y \not\vdash \alpha$ then $\alpha \notin Y$ and so $Y, \alpha \vdash \perp$ since Y is maximally consistent and so by \mathbf{C}^- we get $Y \vdash \neg\alpha$. For (H2), $Y \vdash \forall x\alpha \implies Y \vdash \alpha_x^c$ is true in general by $(\forall 1)$. Now, if $Y \vdash \alpha_x^c$ for all c in \mathcal{L}' and thus also for $c = c_{x,\alpha,n}$ where n is minimal such that $\alpha \in \mathcal{L}_n$. Assume $Y \not\vdash \forall x\alpha$ then $Y \vdash \neg\forall x\alpha$ by (H1) and so $Y \vdash \neg\forall x\alpha, \alpha_x^c$ implies $Y \vdash \neg\forall x\alpha \wedge \alpha_x^c = \alpha^x$ by $(\wedge 1)$. But $Y \vdash \neg\alpha^x$ and so this contradicts Y 's consistency. Therefore $Y \vdash \forall x\alpha$ as required.

Thus Y is indeed a Henkin extension of X . ■

This lemma is not true if we require Y be Henkin in \mathcal{L} . For example, let \mathcal{L} consist of constants c_i for $i \in I$ where I is infinite and let $X = \{v_0 \neq c_i \mid i \in I\}$. Then X is consistent, but in no consistent extension of X in \mathcal{L} can $v_0 = c_i$ be derived, contradicting (H3).

3.2.6 Lemma

Every Henkin set $Y \subseteq \mathcal{L}$ possess a model.

We will construct a model for Y called a *term model*. Let us define $t \approx t'$ whenever $Y \vdash t = t'$, this is a congruence relation on the term algebra \mathcal{T} of \mathcal{L} . This means that \approx is an equivalence relation and $\vec{t} \approx \vec{t}'$ implies $f\vec{t} \approx f\vec{t}'$ for all vectors of terms \vec{t}, \vec{t}' and function symbols f of \mathcal{L} (these were both proven explicitly in the previous subsection). Let us denote the equivalence class of t under \approx by \bar{t} , and let us denote the quotient algebra of \mathcal{T} by A ; meaning $A = \{\bar{t} \mid t \in \mathcal{T}\}$. A will be the domain of the model \mathcal{M} of Y .

Let C be the set of constants in \mathcal{L} , then by (H3) for each term $t \in \mathcal{T}$ there exists a $c \in C$ such that $c \approx t$. This means that $A = \{\bar{c} \mid c \in C\}$. For every variable x , let us define $x^{\mathcal{M}} = \bar{x}$ and for constants $c \in C$, define $c^{\mathcal{M}} = \bar{c}$. For a function symbol f in \mathcal{L} we define

$$f^{\mathcal{M}}(\bar{t}_1, \dots, \bar{t}_n) = \overline{ft_1 \cdots t_n}$$

This is well-defined since \approx is a congruence: so if $t'_i \approx t_i$ then $ft'_1 \cdots t'_n \approx ft_1 \cdots t_n$. And if r is a relation symbol, then define

$$r^{\mathcal{M}}\bar{t}_1 \cdots \bar{t}_n \iff Y \vdash rt_1 \cdots t_n$$

this too is well-defined since we showed that $Y \vdash \vec{t} = \vec{t}'$, $Y \vdash r\vec{t}$ implies $Y \vdash r\vec{t}'$.

We will now show that

$$(1) t^{\mathcal{M}} = \bar{t}, \quad (2) \mathcal{M} \models \alpha \iff Y \vdash \alpha$$

(2) is certainly sufficient to show that \mathcal{M} models Y : if $\alpha \in Y$ then $Y \vdash \alpha$ and so $\mathcal{M} \models \alpha$, meaning $\mathcal{M} \models Y$. We prove (1) by term induction. This is obvious by definition for prime terms. For compound terms

$$(ft_1 \cdots t_n)^{\mathcal{M}} = f^{\mathcal{M}}t_1^{\mathcal{M}} \cdots t_n^{\mathcal{M}} = f^{\mathcal{M}}\bar{t}_1 \cdots \bar{t}_n = \overline{ft_1 \cdots t_n}$$

by definition, as required. We prove (2) by formula induction.

$$\begin{aligned} \mathcal{M} \models t = s &\iff t^{\mathcal{M}} = s^{\mathcal{M}} \iff \bar{t} = \bar{s} && (\text{since } t^{\mathcal{M}} = \bar{t}) \\ &\iff Y \models t = s && (\text{definition}) \\ \mathcal{M} \models r\vec{t} &\iff r^{\mathcal{M}}\vec{t}^{\mathcal{M}} \iff r^{\mathcal{M}}\vec{\bar{t}} \iff Y \vdash r\vec{t} \\ \mathcal{M} \models \alpha \wedge \beta &\iff \mathcal{M} \models \alpha, \beta \iff Y \vdash \alpha, \beta && (\text{induction hypothesis}) \\ &\iff Y \vdash \alpha \wedge \beta && (\text{by } (\wedge 1) \text{ and } (\wedge 2)) \\ \mathcal{M} \models \neg \alpha &\iff \mathcal{M} \not\models \alpha \iff Y \not\vdash \alpha && (\text{induction hypothesis}) \\ &\iff Y \vdash \neg \alpha && (\text{by (H1)}) \\ \mathcal{M} \models \forall x \alpha &\iff \mathcal{M}_{\bar{c}}^{\bar{c}} \models \alpha \text{ for all } c \in C && (\text{since } A = \{\bar{c} \mid c \in C\}) \\ &\iff \mathcal{M}_x^{c^{\mathcal{M}}} \models \alpha \text{ for all } c \in C && (\text{since } c^{\mathcal{M}} = \bar{c}) \\ &\iff \mathcal{M} \models \alpha_x^c \text{ for all } c \in C && (\text{by The Substitution Theorem}) \\ &\iff Y \vdash \alpha_x^c \text{ for all } c \in C && (\text{induction hypothesis}) \\ &\iff Y \vdash \forall x \alpha && (\text{by (H2)}) \end{aligned}$$

So we have shown (2), and as explained (2) is sufficient to show the premise of the lemma. ■

3.2.7 Theorem (The Model Existence Theorem)

Every consistent set $X \subseteq \mathcal{L}$ has a model.

Let $Y \supseteq X$ be a Henkin expansion of X in a suitable constant expansion \mathcal{LC} of \mathcal{L} . By the above lemma, Y has a model \mathcal{M}' in \mathcal{LC} . Let \mathcal{M} be the \mathcal{L} -reduct of \mathcal{M}' . Then since $\mathcal{M}' \models X$, by The Coincidence Theorem we get $\mathcal{M} \models X$. ■

3.2.8 Theorem (The Completeness Theorem)

Let \mathcal{L} be a first-order language, and $X \subseteq \mathcal{L}$ and $\alpha \in \mathcal{L}$. Then $X \vdash \alpha \iff X \models \alpha$. More suggestively, $\vdash = \models$.

The soundness of \vdash states $X \vdash \alpha \implies X \models \alpha$. If $X \not\models \alpha$ then $X, \neg\alpha$ is consistent and therefore $X \cup \{\neg\alpha\}$ has a model and therefore $X \not\models \alpha$. ■

So we can freely alternate between using \vdash and \models , and we will often prove $X \vdash \alpha$ by showing $X \models \alpha$ in a semi-formal manner as is common in mathematics. In particular for theories T , $T \models \alpha$ is equivalent to $T \vdash \alpha$ and we will often write $\vdash_T \alpha$ in place of it. For sentences α , $\vdash_T \alpha$ means the same as $\alpha \in T$. More generally, we write $X \vdash_T \alpha$ to mean $X \cup T \vdash \alpha$. Notice that

$$\alpha \vdash_T \beta \iff \vdash_T \alpha \rightarrow \beta \iff \vdash_{T+\alpha} \beta, \quad \vdash_T \alpha \iff \vdash_T \alpha^g$$

are all immediate due to $\vdash = \models$.

Exercise

Let $K \neq \emptyset$ be a chain of theories in \mathcal{L} . Show that $T_K = \bigcup K$ is a theory that is consistent if and only if every $T \in K$ is consistent.

If every $T \in K$ is consistent, then suppose $T_K \vdash \perp$ then by finiteness, we get $\varphi_1, \dots, \varphi_n \vdash \perp$ for $\varphi_i \in T_K$. Since T_K is the union of a chain, $\varphi_1, \dots, \varphi_n \in T$ for some $T \in K$ and thus $T \vdash \perp$, contradicting T 's consistency. If T_K is consistent, since $T \subseteq T_K$ for every $T \in K$, T is consistent as well.

Exercise

Suppose T is consistent and $Y \subseteq \mathcal{L}$. Prove that the following are equivalent:

- (1) $Y \vdash_T \perp$,
- (2) $\vdash_T \neg\alpha$ for some conjunction α of formulas in Y .

$Y \vdash_T \perp$ if and only if $Y \cup T \vdash \perp$ and since T is consistent and by finiteness, this is equivalent to $T, \alpha_1, \dots, \alpha_n \vdash \perp$. By \mathcal{C}^- , this is if and only if $T, \alpha_1, \dots, \alpha_{n-1} \vdash \neg\alpha_n$, which is equivalent to

$$\vdash_T \alpha_1 \rightarrow \dots \rightarrow \alpha_{n-1} \rightarrow \neg\alpha_n \equiv \alpha_1 \wedge \dots \wedge \alpha_{n-1} \rightarrow \neg\alpha_n \equiv \neg(\alpha_1 \wedge \dots \wedge \alpha_n)$$

so setting $\alpha = \alpha_1 \wedge \dots \wedge \alpha_n$ gives the desired result. This chain is obviously reversible.

Exercise

Let $x \notin \text{var } t$ and $\alpha, \frac{t}{x}$ collision-free. Verify the following equivalence chain:

$$(1) \vdash_T \alpha \frac{t}{x} \iff (2) x = t \vdash_T \alpha \iff (3) \vdash_T x = t \rightarrow \alpha \\ \iff (4) \vdash_T \forall x(x = t \rightarrow \alpha) \iff (5) \vdash_T \exists x(x = t \wedge \alpha)$$

If $T \vdash \alpha \frac{t}{x}$ then $T, x = t \vdash x = t, \alpha \frac{t}{x}$ so by $(=)$ (but which is true, due to the completeness theorem for all formulas α where $\alpha, \frac{t}{x}$ is collision-free) we get $T, x = t \vdash \alpha$ as required. $(2) \implies (3)$ is due to the deduction theorem. As explained above (it is due to the completeness theorem), $\vdash_T \alpha \iff \vdash_T \alpha^g$ and in particular $\vdash_T \alpha \iff \vdash_T \forall x \alpha$, which proves $(3) \implies (4)$. $(4) \implies (5)$ is due to $(\forall x = t)\alpha \models \exists(x = t)\alpha$ which we showed in a previous exercise. For $(5) \implies (1)$: suppose $\mathcal{M} \models \exists x(x = t \wedge \alpha)$ then $\mathcal{M}_x^a \models x = t \wedge \alpha$ for some $a \in A$ and thus $t^{\mathcal{M}} = a$ and $\mathcal{M}_x^t \models \alpha$ thus $\mathcal{M} \models \alpha \frac{t}{x}$ by the substitution theorem, so $\exists x(x = t \wedge \alpha) \models \alpha \frac{t}{x}$ and this is sufficient by the completeness theorem.

3.3 First Applications: Nonstandard Models

By the completeness theorem, $\vdash = \models$ and so immediately we get

3.3.1 Theorem (The Finiteness Theorem)

$X \models \alpha$ implies $X_0 \models \alpha$ for a finite $X_0 \subseteq X$.

Let us define the sentence $\text{char}_p = 1 + \cdots + 1 = 0$ (1 is added p times) in the language of rings $\mathcal{L}\{+, \cdot, 0, 1\}$. A *field of characteristic p* is a field F such that $F \models \text{char}_p$ and $F \not\models \text{char}_q$ for $q < p$ (it is simple to show that if a field is of characteristic p , then p is prime). We define the first-order theory of fields of characteristic 0 to be the theory axiomatized by X : the set of axioms for the theory of fields, as well as $\neg \text{char}_p$ for all prime p .

3.3.2 Proposition

A sentence α which is valid in all fields of characteristic 0 is also valid in all fields of sufficiently high prime characteristic p (p is dependent on α).

A sentence α is valid in all fields of characteristic 0 if and only if $X \models \alpha$. By the finiteness theorem this means $X_0 \models \alpha$ for some finite $X_0 \subseteq X$. Since X_0 is finite, this means it can only contain finitely many sentences of the form $\neg \text{char}_q$, thus there exists some p such that for all $\neg \text{char}_q \in X_0$, $q < p$. Thus for all fields F of characteristic $\geq p$, we have $F \models X_0$ and so $F \models \alpha$. Meaning α is valid in all fields whose characteristic is at least p , as required. ■

3.3.3 Definition

A set of strings Z is **decidable** if there is an algorithm (this will be formally defined later) which after finitely many steps provides an answer to whether a string of symbols ξ belongs to Z . Otherwise Z is **undecidable**. For example, it is intuitively clear that it is decidable whether ξ is a formula (or a sentence). A theory T is **recursively axiomatizable** (for short **axiomatizable**) if it possess a decidable axiom system. For example, T is recursively axiomatizable if it is finitely axiomatizable: if it has a finite axiom system.

So by the above proposition, the theory of fields of characteristic zero is not finitely axiomatizable. For if F were a finite set of axioms, then their conjunction $\alpha = \bigwedge F$ would be valid in fields of sufficiently large characteristic. But then these fields would be of characteristic zero, in contradiction.

Here is another example: an abelian group \mathcal{G} is *n -divisible* if $\mathcal{G} \models \vartheta_n$ where $\vartheta_n = \forall x \exists y x = ny$ where $ny = y + \cdots + y$ (n times). And \mathcal{G} is *divisible* if it is n -divisible for all $n \geq 1$. The theory of divisible abelian groups is denoted DAG and is axiomatized by the axioms of abelian groups and the sentences ϑ_n for $n \geq 1$.

3.3.4 Proposition

A sentence $\alpha \in \mathcal{L}\{+, 0\}$ valid in all divisible abelian groups is also valid in at least one nondivisible abelian group.

If $\text{DAG} \models \alpha$ then $X_0 \models \alpha$ for a finite subset of the axioms X_0 . In particular there exists some m such that for every $\vartheta_n \in X_0$, $n < m$, let p be a prime larger than m . Now let \mathbb{Z}_p be the cyclic group of order p , then $\mathbb{Z}_p \models \vartheta_n$ for $0 < n < p$ (since n is coprime with p , and thus is divisible in \mathbb{Z}_p thus take $y = n^{-1}x$). Thus $\mathbb{Z}_p \models X_0$, but \mathbb{Z}_p is not divisible, as $\mathbb{Z}_p \not\models \vartheta_p$ (since $py = 0$). ■

So DAG is not finitely axiomatizable, for the same reason that the theory of fields of characteristic zero is not finitely axiomatizable.

3.3.5 Theorem (The Compactness Theorem)

A set of formulas X is satisfiable if and only if every finite subset $X_0 \subseteq X$ is satisfiable.

If X is satisfiable, then so is every $X_0 \subseteq X$. If X is not satisfiable, then by the model existence theorem this is equivalent to X being inconsistent so $X \vdash \perp$ and thus $X_0 \vdash \perp$ meaning $X_0 \models \perp$ for some finite subset $X_0 \subseteq X$. ■

3.3.6 Definition

An \mathcal{L} -theory T is **complete** if it is consistent and has no consistent proper extension in \mathcal{L}^0 .

If T is complete and if $\not\vdash_T \alpha, \neg\alpha$ then $T + \alpha$ and $T + \neg\alpha$ are consistent by \mathbf{C}^+ and \mathbf{C}^- and are therefore consistent proper extensions in contradiction. So if T is complete then $\vdash_T \alpha$ or $\vdash_T \neg\alpha$ (but not both, for then T would not be consistent) for every $\alpha \in \mathcal{L}^0$. Conversely if this is true then T is consistent as if $\vdash_T \perp$ then $\vdash_T \alpha, \neg\alpha$ contradicting the “not both” part. And it is maximally consistent as then if $\alpha \notin T$, $\neg\alpha \in T$ and so $T + \alpha \vdash \alpha, \neg\alpha$ and is therefore inconsistent. So T is complete if and only if $\vdash_T \alpha$ or $\vdash_T \neg\alpha$ but not both, or equivalently $\vdash_T \alpha \iff \not\vdash_T \neg\alpha$ for all sentences α .

So for example, if \mathcal{A} is an \mathcal{L} -structure, then since for every sentence α , either $\mathcal{A} \models \alpha$ or $\mathcal{A} \not\models \alpha$, which is equivalent to $\mathcal{A} \models \neg\alpha$ (since α this is true for specific models over \mathcal{A} , and since α is a sentence, all the models agree on the validity of α). Thus for every sentence α , $\mathcal{A} \models \alpha$ or $\mathcal{A} \models \neg\alpha$, and it obviously cannot satisfy both. Thus $Th\mathcal{A}$ is a complete theory for every structure \mathcal{A} .

Now, a frequently occurring theory is the theory $Th\mathcal{N}$, which is the theory of the structure $\mathcal{N} = (\mathbb{N}, 0, \mathbf{S}, +, \cdot)$ where \mathbf{S} is the *successor function*: $\mathbf{S}: n \mapsto n + 1$. \mathcal{N} is the standard structure of the arithmetical language $\mathcal{L}_{ar} := \mathcal{L}\{0, \mathbf{S}, +, \cdot\}$. We can define the relationship \leq in \mathcal{L}_{ar} by $x \leq y \iff \exists z \, x + z = y$ and the relationship $<$ by $x < y \iff x \leq y \wedge x \neq y$.

Even more frequent than $Th\mathcal{N}$ is one of its subtheories: *Peano arithmetic* denoted PA. It is axiomatized by

$$\begin{array}{lll} \forall x \, \mathbf{S}x \neq 0 & \forall x \, x + 0 = x & \forall x \, x \cdot x = 0 \\ \forall x \forall y \, (\mathbf{S}x = \mathbf{S}y \rightarrow x = y) & \forall x \forall y \, x + \mathbf{S}y = \mathbf{S}(x + y) & \forall x \forall y \, x \cdot \mathbf{S}y = x \cdot y + x \\ \text{IS: } \varphi_x^0 \wedge \forall x (\varphi \rightarrow \varphi_{\mathbf{S}x}^{\mathbf{S}x}) \rightarrow \forall x \varphi & & \end{array}$$

IS is the *induction schema* and is a *schema* of rules: it runs over all formulas φ in \mathcal{L}_{ar} . Since φ may not be a sentence, and axioms are, by our convention IS must be generalized, so the correct axiom is $(\varphi_x^0 \wedge \forall x (\varphi \rightarrow \varphi_{\mathbf{S}x}^{\mathbf{S}x}) \rightarrow \forall x \varphi)^g$. The purpose of IS is to prove $\forall x \varphi$ by *induction on x* : first one proves φ for when $x = 0$ (hence showing φ_x^0), and then one shows that if φ then $\varphi_{\mathbf{S}x}^{\mathbf{S}x}$. Proving $\vdash_{PA} \forall x (\varphi \rightarrow \varphi_{\mathbf{S}x}^{\mathbf{S}x})$ is equivalent to showing $\varphi \vdash_{PA} \varphi_{\mathbf{S}x}^{\mathbf{S}x}$.

For example, let $\varphi = x = 0 \vee \exists v \, \mathbf{S}v = x$. We will prove $\vdash_{PA} \forall x \varphi$, meaning every $x \neq 0$ has a predecessor. Obviously φ_x^0 ($= 0 = 0 \vee \exists v \, \mathbf{S}v = x$). Now we must show that $\varphi \vdash_{PA} \varphi_{\mathbf{S}x}^{\mathbf{S}x}$. In general we have $\mathbf{S}v = x \vdash_{PA} \mathbf{S}\mathbf{S}v = \mathbf{S}x$, and so by particularization we get $\exists v \, \mathbf{S}v = x \vdash_{PA} \exists v \, \mathbf{S}\mathbf{S}v = \mathbf{S}x$. And since $x = 0 \vdash_{PA} \exists v \, \mathbf{S}v = \mathbf{S}x$, we get $\varphi \vdash_{PA} \exists v \, \mathbf{S}v = \mathbf{S}x \vdash_{PA} \mathbf{S}x = 0 \vee \exists v \, \mathbf{S}v = \mathbf{S}x = \varphi_{\mathbf{S}x}^{\mathbf{S}x}$ as required.

We will now show that $Th\mathcal{N}$ and PA have what are called *nonstandard models*, models which are not isomorphic to the standard model of the theory, which is \mathcal{N} in this case. Let us define $\underline{n} := \mathbf{S}^n 0 := \mathbf{S} \cdots \mathbf{S} 0$ where \mathbf{S} is composed with itself n times. So for example $\underline{1} = \mathbf{S}0$, $\underline{2} = \mathbf{S}1$ and in general $\underline{\mathbf{S}n} = \underline{\mathbf{S}n}$. Let $x \in \text{Var}$ and we define $X := Th\mathcal{N} \cup \{\underline{n} < x \mid n \in \mathbb{N}\}$.

Let $X_0 \subseteq X$ be finite, then X_0 is satisfiable: since X_0 is finite, there must be some m such that $X_0 \subseteq Th\mathcal{N} \cup \{\underline{n} < x \mid n < m\}$. Thus, we can simply assign x the value of m and then (\mathcal{N}, m) (meaning the valuation function maps x to m) is a model of X_0 . So by The Compactness Theorem, X has a model (\mathcal{N}', c) where the domain of the model is \mathbb{N}' and $c \in \mathbb{N}'$ is the valuation of x . Since $Th\mathcal{N}$ is a subset of X and is complete, \mathbb{N}' satisfies precisely the same sentences as \mathcal{N} . In particular the following sentences are valid in \mathcal{N}' : $\underline{\mathbf{S}n} = \underline{\mathbf{S}n}$, $\underline{n + m} = \underline{n} + \underline{m}$, and $\underline{n \cdot m} = \underline{n} \cdot \underline{m}$. Thus $n \mapsto \underline{n}^{\mathcal{N}'}$ is an embedding of \mathcal{N} into \mathcal{N}' . We can identify the image of this embedding with \mathcal{N} , meaning it is legitimate to assume $\underline{n}^{\mathcal{N}'} = n$ so $\mathcal{N} \subseteq \mathcal{N}'$.

Since $\mathcal{N}' \models X$, \mathcal{N} and \mathcal{N}' are elementarily equivalent as $Th\mathcal{N}$ is complete. But on the other hand, $n < a$ for any $a \in \mathbb{N}' \setminus \mathbb{N}$, since in \mathcal{N} and thus in \mathcal{N}' we have $(\forall x \leq \underline{n}) \bigvee_{i \leq \underline{n}} x = \underline{i}$. So if $a \leq n$ then we'd have $a = \underline{i}$ and so $a \in \mathbb{N}$ in contradiction. And since $\mathbb{N}' \setminus \mathbb{N}$ is nonempty by the formulas added to X , \mathbb{N} is a proper initial segment of \mathbb{N}' . The elements of $\mathbb{N}' \setminus \mathbb{N}$ are called *nonstandard numbers* (the existence of nonstandard numbers are precisely what make \mathbb{N}' and \mathbb{N} non-isomorphic). c is an example of a nonstandard number, as well as $c + c$ and so on.

Since $(\forall x \neq 0) \exists! y \, \mathbf{S}y = x$ is a theorem of \mathcal{N} it too must be valid in \mathcal{N}' and so c must have an immediate successor. But if we were to chase the successors of c , we would never find one which is in \mathbb{N} (ie. there is no natural x and n such that $\mathbf{S}^n x = c$ as then c would be natural).

In no nonstandard model \mathcal{N}' of $Th\mathcal{N}$ is \mathbb{N} definable (for the same reason as above, \mathbb{N} can be embedded in all models of $Th\mathcal{N}$, so we really mean that the image of this embedding is not definable). In fact it is not even *parameter definable*, meaning there is no $\alpha = \alpha(x, \vec{y})$ and $b_1, \dots, b_n \in \mathbb{N}'$ such that $\mathbb{N} = \{a \in \mathbb{N}' \mid \mathcal{N}' \models \alpha[a, \vec{b}]\}$.

Otherwise, we'd have $\mathcal{N}' \models \alpha \frac{0}{x}[\vec{b}]$ since $0 \in \mathbb{N}$ and $\mathcal{N}' \models \forall x(\alpha \rightarrow \alpha \frac{Sx}{x})[\vec{b}]$ since \mathbb{N} is closed under S . Thus by IS we have $\mathcal{N}' \models \forall x \alpha[\vec{b}]$, which contradicts $\mathbb{N}' \setminus \mathbb{N}$ being nonempty.

Similar to \mathcal{N} we can find nonstandard models for the theory of real numbers, $\mathcal{R} = (\mathbb{R}, +, \cdot, <, \{a \mid a \in \mathbb{R}\})$ which contains a constant symbol a for every number $a \in \mathbb{R}$. Consider $X = Th\mathcal{R} \cup \{a < x \mid a \in \mathbb{R}\}$, which is finitely satisfiable and therefore by The Compactness Theorem X has a model \mathcal{R}^* which is a *nonstandard model of analysis*. \mathcal{R}^* models $Th\mathcal{R}$ and therefore every theorem valid in \mathcal{R} is valid in \mathcal{R}^* and vice versa. We could also add functions like \exp, \ln, \sin, \cos to the signature of \mathcal{R} and obtain a model \mathcal{R}^* where the properties of these functions which can be formulated in first-order language are preserved, like

$$\forall x, y \exp(x + y) = \exp x \cdot \exp y, \quad (\forall x > 0) \exp \ln x = x, \quad \forall x \sin^2 x + \cos^2 x = 1$$

Since continuity and differentiability can be formulated in first-order language (via ε - δ shenanigans), these functions remain continuous and repeatedly differentiable. Though topological properties like Bolzano-Weirestrass are not generally true, but can be replaced by infinitesimal arguments.

For a nonstandard model \mathcal{R}^* of $Th\mathcal{R}$, with $\mathcal{R} \subseteq \mathcal{R}^*$ contains infinitely large numbers (meaning numbers c where $r < c$ for all $r \in \mathbb{R}$), but also infinitely small numbers (meaning numbers ε where $0 < \varepsilon < r$ for all $r \in \mathbb{R}$, as we can take $\varepsilon := \frac{1}{c}$). These infinitely small numbers are termed *infinitesimal numbers*, and utilizing their properties one can study *nonstandard analysis* which is a branch of analysis utilizing these nonstandard models of analysis and their infinitesimal numbers. This branch gives arguably more intuitive proofs for analytic results, which can more closely follow the arguments of the original analytical researchers like Leibniz.

While every bounded set in \mathcal{R} possesses a supremum, this is not true in \mathcal{R}^* . Rather we can only say that every bounded definable set in \mathcal{R}^* possesses a supremum, as this is first-order formulatable,

$$\exists x \varphi \wedge \exists y \forall x (\varphi \rightarrow x \leq y) \rightarrow \exists z \forall x ((\varphi \rightarrow x \leq z) \wedge \forall y ((\varphi \rightarrow x \leq y) \rightarrow z \leq y))$$

(the set is defined by φ , ie. it is the set $\{x \mid \mathcal{R}^* \models \varphi\}$.)

Exercise

Prove in PA the associativity, commutativity, and distributivity of $+$ and \cdot .

We will first derive $Sx + y = x + Sy$ by induction on y , ie. we define $\varphi = Sx + y = x + Sy$. The base case $\varphi \frac{0}{y} = Sx + 0 = x + S0$: since $Sx + 0 = Sx$ and $x + S0 = S(x + 0) = Sx$, this is true. Now we show $\varphi \vdash \varphi \frac{Sy}{y} = Sx + Sy = x + SSy$. Now, $Sx + Sy = S(Sx + y) = S(x + Sy) = x + SSy$ as required. So we have shown $\forall y \varphi$ as required.

Now, we will also show that $0 + y = y$ by induction on y : let $\varphi = 0 + y = y$. Then $\varphi \frac{0}{y} = 0 + 0 = 0$ which is valid in PA. Now we must show $0 + y = y \vdash 0 + Sy = Sy$. Since $0 + Sy = S(0 + y) = Sy$ we have finished.

Now we will show associativity, by inducting $\varphi = (x + y) + z = x + (y + z)$ on y . $\varphi \frac{0}{y} = (x + 0) + z = x + (0 + z)$ and since $x + 0 = x$ and $0 + z = z$ this is true. For the inductive step, $(x + Sy) + z = S(x + y) + z = S((x + y) + z) = S(x + (y + z)) = x + S(y + z) = x + (y + Sz) = x + (Sy + z)$ as required.

For commutativity, we induct $x + y = y + x$ on y (we could also on x , but why change?). This is obviously true when substituting 0 for y , as $x + 0 = x = x + 0$. Now $x + Sy = S(x + y) = S(y + x) = Sy + x$, as required. Proofs for \cdot and distributivity are similar.

Exercise

Prove the antisymmetry of \leq in PA.

So we must show that $x \leq y$ and $y \leq x$ implies $x = y$. Now,

$$x \leq y \leftrightarrow \exists z x + z = y, \quad y \leq x \leftrightarrow \exists u y + u = x$$

This means that $x \leq y \wedge y \leq x \rightarrow \exists u, z y + u + z = y$. Now simple induction on x will yield $y + x = y \rightarrow x = 0$, and so $u + z = 0$. And induction on z will yield $u + z = 0 \rightarrow z = 0, u = 0$ (the base case is trivial, the inductive step holds vacuously). And so $u + z = 0$ and thus $y = x$.

Exercise

Prove $x < y \equiv_{\text{PA}} Sx \leq y$. Use this to prove $\vdash_{\text{PA}} x \leq y \vee y \leq x$ by induction on x .

We will first prove $x < y \vdash_{\text{PA}} Sx \leq y$. By definition if $x < y$ then $\exists z x + z = y$ and since $x \neq y$ we have that $z \neq 0$ and so z has a predecessor, u . Thus $x < y$ implies $\exists u x + Su = y$ and since $x + Su = Sx + u$ this means $Sx \leq y$ as required. Conversely, since $x < Sx$ by induction on x , we have that if $Sx \leq y$ by transitivity $x < y$.

Let $\varphi = x \leq y \vee y \leq x$, then substituting 0 for x , we get $0 \leq y \vee y \leq 0$ and since we know that $\forall y 0 \leq y$ (by induction on y), this is true. Now for the inductive step we must show $Sx \leq y \vee y \leq Sx \equiv x < y \vee y \leq Sx$. We know that $x \leq y \vee y \leq x$, so either $x < y$ in which case this is true, or $y \leq x$ in which case $y \leq Sx$.

Exercise

Let $\alpha, \beta, \gamma \in \mathcal{L}_{ar}$ and $y \notin \text{var}\{\alpha, \beta\}$ and $z \notin \text{var}\gamma$. Verify the following:

- (1) $\vdash_{\text{PA}} \forall x ((\forall y < x) \alpha \frac{y}{x} \rightarrow \alpha) \rightarrow \forall x \alpha$, the *schema of $<$ -induction* (or *strong induction*).
- (2) $\vdash_{\text{PA}} \exists x \beta \rightarrow \exists x (\beta \wedge (\forall y < x) \neg \beta \frac{y}{x})$, the *minimum schema* (or *well-ordering principle*).
- (3) $\vdash_{\text{PA}} (\forall x < v) \exists y \gamma \rightarrow \exists z (\forall x < v) (\exists y < z) \gamma$, the *schema of collection*.

- (1) Let $\varphi = (\forall y < x) \alpha \frac{y}{x}$, we will prove $\forall x (\varphi \rightarrow \alpha) \vdash_{\text{PA}} \varphi \frac{0}{x}, \varphi \rightarrow \varphi \frac{Sx}{x}$ which proves $\forall x \varphi$ by IS. This means that in particular $\forall x \varphi \frac{Sx}{x}$, and so for every $y < Sx$, $\alpha \frac{y}{x}$ and since $x < Sx$ we get α as required. Since $\varphi \frac{0}{x} = (\forall y < 0) \alpha \frac{y}{x}$ which holds vacuously. And since $\forall x (\varphi \rightarrow \alpha) = \forall x ((\forall y < x) \alpha \frac{y}{x} \rightarrow \alpha)$, we have that if $\varphi = (\forall y < x) \alpha \frac{y}{x}$ then α , meaning $(\forall y \leq x) \alpha \frac{y}{x}$, which is equivalent to $\varphi \frac{Sx}{x}$. Thus $\forall x (\varphi \rightarrow \alpha) \vdash_{\text{PA}} \varphi \rightarrow \varphi \frac{Sx}{x}$, as required.
- (2) Note that the contrapositive of this is $\forall x ((\forall y < x) \neg \beta \frac{y}{x} \rightarrow \neg \beta) \rightarrow \forall x (\neg \beta)$, which follows from (1).
- (3) Let $\varphi = (\forall x < v) \exists y \gamma \rightarrow \exists z (\forall x < v) (\exists y < z) \gamma$, we will prove $\forall v \varphi$ by induction on v . $\vdash_{\text{PA}} \varphi \frac{0}{v}$ holds vacuously. Now we must prove $\varphi \vdash_{\text{PA}} \varphi \frac{Sv}{v} \equiv_{\text{PA}} (\forall x \leq v) \exists y \gamma \rightarrow \exists z (\forall x \leq v) (\exists y < z) \gamma$. By φ , if $(\forall x \leq v) \exists y \gamma$, then we have $\exists z (\forall x < v) (\exists y < z) \gamma$. So the only “issue” is when $x = v$, but in which case we know there exists a y_x such that γ , so let $z' = \max\{z, Sy_x\}$ and we get that for every $x < v$ there exists a $y < z \leq z'$ such that γ and for $x = v$ there exists a $y_x < Sy_x \leq z'$ such that γ . And so we get $\exists z (\forall x < v) (\exists y < z) \gamma$ (where the z is z') as required.

3.4 ZFC and Skolem's Paradox

We briefly review fundamentals of set theory. Recall that a set M is *countable* if there exists a surjection $f: \mathbb{N} \rightarrow M$ (assuming the axiom of choice, this is equivalent to there being an injection $M \rightarrow \mathbb{N}$). A set which is not countable is called *uncountable*. Two sets M and N are *equipotent* if there exists a bijection between M and N , we denote this as $M \sim N$. If $M \sim \mathbb{N}$, M is *countably infinite*. Recall that \mathbb{R} is uncountable and equipotent to $\mathcal{P}(\mathbb{N})$. In general due to Cantor's theorem which states that the powerset of a set, $\mathcal{P}(M)$, is of a higher cardinality than M (meaning there is an injection $M \rightarrow \mathcal{P}(M)$ but no surjection).

And if M and N are countable, then so is $M \times N$. And if $\{M_i\}_{i \in I}$ is a countable family of countable sets (meaning I is countable and M_i is countable for every $i \in I$), then their union $\bigcup_{i \in I} M_i$ is also countable. Notice then that if M is countable, the set of all finite sequences of symbols in M is $\bigcup_{n \in \mathbb{N}} M^n$, which is countable. Thus if \mathcal{L} is a first-order language over a countable signature then \mathcal{L} is countable as well.

A *countable theory* is a theory formalized in a countable first-order language.

3.4.1 Theorem (Löwenheim-Skolem Theorem)

A countable consistent theory T has a countable model.

By The Model Existence Theorem, T has a model whose domain A consists of the equivalence classes \bar{c} for $c \in C$, where $C = \bigcup_{n \in \mathbb{N}} C_n$ is a new set of constants. By definition, C_0 is equipotent to $\mathcal{L} \times \text{Var}$ which is countable, and so $\mathcal{L}_1 := \mathcal{L}C_0$ is equipotent. And in general, C_n is equipotent to $\mathcal{L}_n \times \text{Var}$ so C_n and \mathcal{L}_{n+1} are countable for every n , by induction. Thus C is also countable as the countable union of countable sets. The map $c \mapsto \bar{c}$ is a surjection from C to A , and therefore A is also countable, as required. ■

This theorem will later be generalized, but this theorem already leads to interesting consequences. For example, there must be countable ordered fields which are nonstandard models of the first-order theory $Th(\mathbb{R}, 0, 1, +, \cdot, <, \exp, \sin, \dots)$. Even more surprising (for reasons explained shortly) is that this theorem implies the existence of countable models of set theory.

To explain why, we must first define set theory. The most common and thus most important formalization of set theory is ZFC (short for Zermelo-Fraenkel+Choice, after the two mathematicians which first formalized it as well as the axiom of choice AC. Excluding AC from the axioms of the theory, gives rise to a proper (this is not a trivial fact) subtheory, ZF.).

The language upon which ZFC is defined is $\mathcal{L}\{\in\}$, it contains only the extralogical symbols \in (and of course $=$). We denote this set-theoretic language by \mathcal{L}_\in . We use a boldface \in symbol in order to differentiate it from the metalogical \in symbol, much like $=$. Importantly, in ZFC all objects are sets, there are no other types of objects like in other formulations of set theory.

We define the relation \subseteq by $x \subseteq y \leftrightarrow \forall z(z \in x \rightarrow z \in y)$.

We begin to list the axioms of ZFC (recall that these axioms must be generalized):

$$\begin{aligned} \text{AE : } & \quad \forall z(x \in x \leftrightarrow z \in y) \rightarrow x = y \quad (\text{axiom of extensionality}) \\ \text{AS : } & \quad \exists y \forall z(z \in y \leftrightarrow \varphi \wedge z \in x) \quad (\text{axiom of separation}) \end{aligned}$$

AS is an axiom schema: φ runs over all \mathcal{L}_\in -formulas with $y \notin \text{free}\varphi$. Now, let $\varphi = \varphi(x, z, \vec{a})$, then we can derive $\forall x \exists! y \forall z(z \in y \leftrightarrow \varphi \wedge z \in x)$, meaning that $y = \{z \in x \mid \varphi\} \leftrightarrow \forall z(z \in y \leftrightarrow \varphi \wedge z \in x)$ is a legitimate definition of the *set term* $\{z \in x \mid \varphi\}$ (the set of all $z \in x$ such that $\varphi(x, z, \vec{a})$ holds). The set term $\{z \in x \mid \varphi\}$ is simply a suggestive way of writing a function term $f_{\vec{a}}x$ (which depends on the “parameter” term \vec{a}). To derive the required formula for legitimacy, let $y, y' \notin \text{free}\varphi$ then it is obvious to see

$$(x \in y \leftrightarrow \varphi \wedge z \in x) \wedge (z \in y' \leftrightarrow \varphi \wedge z \in x) \rightarrow (z \in y \leftrightarrow z \in y')$$

This implies that $\forall z(x \in y \leftrightarrow \varphi \wedge z \in x) \wedge \forall z(z \in y' \leftrightarrow \varphi \wedge z \in x) \rightarrow y = y'$, and so we have proven the claim of legitimacy.

We can explicitly define the empty set by $y = \emptyset \leftrightarrow \forall z z \notin y$. But we must show that this is legitimate. According to AS, $\exists y \forall z(z \in y \leftrightarrow z \notin x \wedge z \in x)$. But since $z \in y \leftrightarrow z \in x \wedge z \in x \equiv z \notin y$, so this is equivalent to $\exists y \forall z(z \notin y)$ (proving the existence part of legitimacy; we must still prove uniqueness). Now by AE, $\forall z z \notin y \wedge \forall z z \notin y' \rightarrow y = y'$, and since we just showed $\exists y \forall z(z \notin y)$ gives the required $\exists! y \forall z(z \notin y)$.

We also have

$$\text{AU : } \quad \forall x \exists y \forall z(z \in y \leftrightarrow (\exists u \in x) z \in u) \quad (\text{axiom of union})$$

y here is the *union* of x , $z \in y$ if and only if there exists a set $u \in x$ such that $z \in u$. By AE, the $\exists y$ may be replaced with $\exists! y$. So we can define an operator (we use the word operator instead of function as the word function has a special meaning in ZFC), the union operator $x \mapsto \bigcup x$. In view of AS, the axiom of union could be weakened to $\forall x \exists y \forall z((\exists u \in x) z \in u \rightarrow z \in y)$ as then we can use AS to get the union from this set y . Similarly the following axiom could be weakened:

$$\text{AP : } \quad \forall x \exists y \forall z(z \in y \leftrightarrow z \subseteq x) \quad (\text{power set axiom})$$

And again, $\exists y$ can be replaced with $\exists! y$ due to AE. We define the *powerset* of a set x to be this unique y , which we denote $\mathcal{P}x$, which contains all the subsets of x . Now, since $x \subseteq \emptyset \leftrightarrow \forall y \in x y \in \emptyset$, and $y \in \emptyset \equiv_{\text{ZFC}} \perp$, we have that $x \subseteq \emptyset \leftrightarrow \perp$, so $\forall x(x \in \mathcal{P}\emptyset \leftrightarrow x = \emptyset)$. Similarly $\forall x(x \in \mathcal{P}\mathcal{P}\emptyset \leftrightarrow x = \emptyset \vee x = \mathcal{P}\emptyset)$. Since $\mathcal{P}\emptyset \neq \emptyset$ as it is not empty, $\mathcal{P}\mathcal{P}\emptyset$ has precisely two elements.

The following axiom was added by Fraenkel to the axioms formulated by Zermelo:

$$\text{AR : } \quad \forall x \exists! y \varphi \rightarrow \forall u \exists v \forall y(y \in v \leftrightarrow (\exists x \in u) \varphi) \quad (\text{axiom of replacement})$$

Here $\varphi = \varphi(x, y, \vec{a})$ is viewed as a function which maps x to y , where y is the unique value satisfying φ (hence the $\forall x \exists! y \varphi$). If φ does represent such a function, AR says that for every u we can define the image of u under this function. More suggestively, we can define an operator $x \mapsto Fx$ where $y = Fx \leftrightarrow \varphi$ (since F actually is dependent on \vec{a} , we should instead write $F_{\vec{a}}$ for F). Then AR states that for every u we can define its image under u which is denoted $\{Fx \mid x \in u\}$.

For example, let us define $\varphi = \varphi(x, y, a, b) := x = \emptyset \wedge y = a \vee x \neq \emptyset \wedge y = b$. $\forall x \exists! y \varphi$ is obvious, and so this defines an operator $F = F_{a,b}$ which satisfies $F\emptyset = a$ and $Fx = b$ for all $x \neq \emptyset$ (and such an x exists, for example $\mathcal{P}\emptyset$). We then define

$$\{a, b\} := \{F_{a,b}x \mid x \in \mathcal{P}\mathcal{P}\emptyset\}$$

(since $\mathcal{PP}\emptyset$ contains $\mathcal{P}\emptyset$ and \emptyset , two distinct elements.) This is called the *pair set* of a, b . Then we can define the union of two sets (which would not necessarily exist, as we have only defined the union of a set) to be $a \cup b := \bigcup \{a, b\}$. The intersection of two sets already exists by AS: $a \cap b := \{z \in a \mid z \in b\}$. But we can also define the intersection of a set as $\bigcap x := \{z \in \bigcup x \mid (\forall y \in x) z \in y\}$. Notice that $\bigcap \{a, b\} = \{z \in a \cup b \mid z \in a \wedge z \in b\}$ which by AE is equal to $a \cap b$. We further define $\{a\} := \{a, a\}$ and inductively $\{a_1, \dots, a_{n+1}\} := \{a_1, \dots, a_n\} \cup \{a_{n+1}\}$. So for example, we can prove $\mathcal{P}\emptyset = \{\emptyset\}$, $\mathcal{PP}\emptyset = \{\emptyset, \{\emptyset\}\}$. We define the *ordered pair* of a, b to be $(a, b) := \{\{a\}, \{a, b\}\}$. This definition is useful as it has the basic property $(a, b) = (c, d) \leftrightarrow a = c \wedge b = d$.

So far our theory has many features, but it lacks an important one: there does not necessarily exist an infinite set! Let us define the successor operator, defined by $S: x \mapsto x \cup \{x\}$. So for example $S\emptyset = \{\emptyset\}$, $SS\emptyset = \{\emptyset, \{\emptyset\}\}$, ... Notice that Sx contains precisely one more element than x (assuming that $x \notin x$, which requires another axiom which will be stated next). So there is a natural correspondence between this successor operator and the one from PA (in fact, in some definitions of \mathbb{N} , these two operators are one and the same). Now we state the following axiom which asserts the existence of an infinite set

$$\text{AI} : \quad \exists u (\emptyset \in u \wedge \forall (x \in u) Sx \in u) \quad (\text{axiom of infinity})$$

This set u is a so-called *inductive set*: it is closed under successors and contains \emptyset .

We also have

$$\text{AF} : \quad (\forall x \neq \emptyset)(\exists y \in x) x \cap y = \emptyset \quad (\text{axiom of foundation})$$

This means that every nonempty set x has a \in -minimal element: an element y such that no other $y' \in x$ is in y . The axiom of foundation implies that there cannot be a circular chain of inclusion, ie. $x_0 \in \dots \in x_n \in x_0$, as then $y = \{x_0, \dots, x_n\}$ would have no \in -minimal element.

These previously mentioned axioms axiomatize the theory ZF. If we adjoin to ZF the *axiom of choice*, we get the theory ZFC.

$$\text{AC} : \quad \forall u (\emptyset \notin u \wedge (\forall x \in u)(\forall y \in u)(x \neq y \rightarrow x \cap y = \emptyset) \rightarrow \exists z (\forall x \in u) \exists! y (y \in x \cap z)) \quad (\text{axiom of choice})$$

The axiom of choice states that if u is a set of nonempty disjoint sets, then there exists a *choice set* z which contains precisely one element from each $x \in u$. Equivalent to AC (among many other important equivalences) is that $\prod_{i \in I} A_i \neq \emptyset$ for any index set I .

ZFC is often viewed as the purest of all first-order theories, as it is sophisticated and thought to be able to formalize (almost) all proofs in mathematics (the *almost* here is technical: ZFC is not complete, and so some mathematicians may study results independent of it. One such example is the *continuum hypothesis*). But whether or not this is true is not of too much significance, as most mathematics does not care whether or not it can be formalized in ZFC.

If ZFC is consistent (which no one really doubts, although we cannot prove), then by the Löwenheim-Skolem Theorem, it must contain a countable model $\mathcal{V} = (V, \in^{\mathcal{V}})$. But certainly the existence of an uncountable set is provable within ZFC: for every inductive set u , $\mathcal{P}u$ is uncountable. But at the same time, $(\mathcal{P}u)^{\mathcal{V}} \subseteq V$ and so $\mathcal{P}u$ must be countable. The fallacy here is that our notion of *countable* “outside” of \mathcal{V} is different from the notion “within” \mathcal{V} . This “paradox” is known as *Skolem's paradox*.

The explanation of Skolem's paradox is that the countable model \mathcal{V} contains fewer sets and functions than expected, and in particular there do not exist enough to satisfy a bijection between $u^{\mathcal{V}}$ and $(\mathcal{P}u)^{\mathcal{V}}$ although “outside” of \mathcal{V} such a bijection exists.

Another “paradox” is that while V is by definition a set, but $\vdash_{\text{ZFC}} \neg \exists u \forall z z \in u$, ie. there does not exist a universal set. This is derived as follows: $\exists u \forall z z \in u$ implies, using AE and AS, the existence of a Russelian set $v = \{x \in u \mid x \notin x\}$. Thus $\exists u \forall z z \in u \vdash_{\text{ZFC}} \exists u \forall x (x \in u \leftrightarrow x \notin x)$. Though we have previously shown that the right-hand formula is logically invalid, and thus equivalent to \perp . So $\exists u \forall z z \in u \vdash_{\text{ZFC}} \perp$ and so $\vdash_{\text{ZFC}} \neg \exists u \forall z z \in u$. So similar to how there is no absolute concept of countability, there is no absolute concept of a set: a set “outside” a model of ZFC is not necessarily a set within ZFC.

Exercise

Let T be an elementary theory with arbitrarily large finite models. Prove that T also has an infinite model.

Since T has arbitrarily large finite models, \exists_n does not contradict T for every n (since $T + \exists_n$ is not unsatisfiable). Thus let $X = \{\exists_n \mid n \in \mathbb{N}\}$, then $T + X$ is satisfiable: let $X_0 \subseteq T \cup X$ be finite then $X_0 \subseteq T \cup \{\exists_n \mid n \leq m\}$ for some m . Then X_0 is satisfiable as $T + \exists_m$ is satisfiable, and therefore so is $T \cup \{\exists_n \mid n \leq m\}$ (as $\exists_m \rightarrow \exists_n$ is a tautology) and thus so is X_0 . So every finite subset of $T \cup X$ is satisfiable, and so by The Compactness Theorem, so is $T \cup X$.

Now, a model of $T + X$ must satisfy \exists_n for every n , and so it must have at least n elements for every n . So it cannot be finite, and thus must be infinite. Furthermore, by definition a model of $T + X$ models T as well.

Exercise

Suppose $\mathcal{A} = (A, <)$ is an infinite well-ordered set. Show that there is a not well-ordered set elementarily equivalent to \mathcal{A} . Thus being well-ordered is not a first-order property (at least in the language $\mathcal{L}\{<\}$).

Let us define $X = Th\mathcal{A} \cup \{v_{n+1} < v_n \mid n \in \mathbb{N}\}$. X is finitely satisfiable (as finite subsets of X can be modeled over \mathcal{A}), but every model of X has an infinitely descending chain (and such a chain therefore has no minimum element) and therefore models of X are not well-ordered.

Exercise

Prove that a consistent theory T is equal to the intersection of all its complete extensions. Meaning $T = \bigcap \{T' \supseteq T \mid T' \text{ complete}\}$.

Firstly, T does have a complete extension as there exists a $\mathcal{A} \models T$ and $Th\mathcal{A}$ is a complete extension of T . Obviously then $T \subseteq \bigcap \{T' \supseteq T \mid T' \text{ complete}\}$. Now suppose $\varphi \in T'$ for every complete extension of T , then if $\varphi \notin T$ then $\neg\varphi$ doesn't contradict T since $T, \neg\varphi \not\vdash \perp$ (in general if $X \not\vdash \varphi$ then by \mathcal{C}^+ , $X, \neg\varphi \not\vdash \perp$). But then $T, \neg\varphi$ is consistent and therefore has a complete extension, which contradicts φ being in every complete extension of T .

Exercise

Derive AS from AR.

Let $\varphi = \varphi(x, z, \vec{a})$ be a formula and let $y, b \notin \text{free}\varphi$.

$$\psi = \left(\varphi_z^b \wedge ((y = z \wedge \varphi) \vee (y = b \wedge \neg\varphi)) \right) \vee \left(\neg\varphi_x^b \wedge y = \emptyset \right)$$

Then $\forall z \exists! y \psi$, as either φ or $\neg\varphi$ is true. This operator defined by ψ maps z to z if φ and z to b otherwise. If $\{z \in x \mid \varphi\}$ is nonempty then there exists a b such that φ_z^b and for this b , the operator defined by ψ when parameterized with b and x gives this set. If φ is invalid, then $\{z \in x \mid \varphi\}$ is the empty set which exists by definition.

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