Differential and Analytic Geometry

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Ari Feiglin

Let $\gamma\colon [0,L] \longrightarrow \mathbb{R}^2$ be a natural parameterization, then $T = \gamma'$ and $k(s) = \langle T', N\rangle$. Suppose $T(0)$ has an angle of θ_0 then let us define

$$\theta(s) = \int_0^s k(p)\,dp + \theta_0$$

And we define the curve

$$\beta(s) = \gamma(0) + \begin{pmatrix} \int_0^s \cos(\theta(s))\,dp \\ \int_0^s \sin(\theta(s))\,dp \end{pmatrix}$$

Now, notice that

$$\beta'(s) = \begin{pmatrix} \cos(\theta(s)) \\ \sin(\theta(s)) \end{pmatrix}$$

And since $\|\beta'\| = 1$, β is a natural parameterization. And further

$$\beta''(s) = \theta'(s) \cdot \begin{pmatrix} -\sin(\theta(s)) \\ \cos(\theta(s)) \end{pmatrix} = \theta'(s) \cdot R_{\frac{\pi}{2}}\beta'(s)$$

Which means that

$$k_\beta(s) = \langle \beta''(s), N_\beta(s)\rangle = \left\langle \theta'(s) \cdot R_{\frac{\pi}{2}}\beta'(s), R_{\frac{\pi}{2}}\beta'(s)\right\rangle = \theta'(s)\langle \beta'(s), \beta'(s)\rangle = \theta'(s) = k(s)$$

(The third equality is since $R_{\frac{\pi}{2}}$ is orthogonal.) So the curvature of β is equal to that of γ.
Now,

$$T_\beta(0) = \beta'(0) = \begin{pmatrix} \cos(\theta(0)) \\ \sin(\theta(0)) \end{pmatrix} = \begin{pmatrix} \cos(\theta_0) \\ \sin(\theta_0) \end{pmatrix} = T(0)$$

And $\beta(0) = \gamma(0)$.
So by the fundamental theorem of curves, since $k_\beta = k_\gamma$, $\beta(0) = \gamma(0)$, and $T_\beta(0) = T_\gamma(0)$, we have that $\beta = \gamma$. This means that

$$T_\gamma(s) = T_\beta(s) = \beta'(s) = \begin{pmatrix} \cos(\theta(s)) \\ \sin(\theta(s)) \end{pmatrix}$$

So θ is the angle function of γ (ie. it gives the angle of γ). So we have proven the following proposition:

Proposition 5.1:

If $\gamma\colon [0,L] \longrightarrow \mathbb{R}^2$ is a regular smooth curve, then its angle is given by

$$\theta_\gamma(s) = \int_0^s k_\gamma(p)\,dp + \theta_0$$

where θ_0 is the angle of $T_\gamma(0)$.

Definition 5.2:

If $\gamma\colon [0,L] \longrightarrow \mathbb{R}^2$ is a natural parameterization, then we define

$$K_\gamma = \int_0^L k_\gamma(s)\,ds$$

to be the **total curvature** of γ.

So by the above definitions,

$$K_\gamma = \theta_\gamma(L) - \theta_\gamma(0)$$

So K_γ can also be thought of the total difference in the angle of γ.

Example 5.3:

If γ is a circle, then intuitively $K_\gamma = 2\pi$ since the total difference in the angle of the curve is 2π. And since the natural parameterization is given by a curve from $[0, 2\pi R]$ whose curvature is $\frac{1}{R}$ and thus

$$K_\gamma = \int_0^{2\pi R} \frac{1}{R} = 2\pi$$

as expected.

Definition 5.4:

A smooth curve $\gamma\colon [a,b] \longrightarrow \mathbb{R}^n$ is **n-closed** if $\gamma^{(k)}(a) = \gamma^{(k)}(b)$ for every $0 \leq k \leq n$. If γ is n-closed for every n, then γ is called **closed**.

Proposition 5.5:

If γ is a 1-closed regular smooth curve then $K_\gamma = 2\pi n$ for some $n \in \mathbb{Z}$.

Proof:

Since γ is 1-closed, $\gamma'(0) = \gamma'(L)$. But recall that

$$\gamma'(s) = \begin{pmatrix} \cos(\theta(s)) \\ \sin(\theta(s)) \end{pmatrix}$$

So we have that

$$\begin{pmatrix} \cos(\theta(0)) \\ \sin(\theta(0)) \end{pmatrix} = \begin{pmatrix} \cos(\theta(L)) \\ \sin(\theta(L)) \end{pmatrix}$$

Which is if and only if $\theta(L) = \theta(0) + 2\pi n$ for some $n \in \mathbb{Z}$, and so $K_\gamma = 2\pi n$ as required. ■

Definition 5.6:

If γ is a 1-closed regular smooth curve, then $\frac{1}{2\pi} K_\gamma$ is called γ's **winding number** (about 0).

Theorem 5.7 (Hopf's Theorem):

If $\gamma\colon [0, L] \longrightarrow \mathbb{R}^2$ is a closed natural parameterization, then γ is injective (other than at the points 0 and L).

We will not be proving this theorem.
This means that if γ is closed, then $K_\gamma = \pm 2\pi$. This is because the winding number is ± 1, as otherwise γ would have to intersect with itself. The sign of K_γ correlates with its orientation. We will prove this formally:

Proof:

Suppose $\gamma(0) = 0$, and $T(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $0 \leq \gamma_1(s)$ for every $s \neq 0, T$ (we can get to this via an isometry). Let $B = \{(x,y) \mid 0 \leq x \leq y \leq T\}$ and we define a function $g\colon B \longrightarrow [-1, 1]$ by

$$g(s,t) = \begin{cases} \frac{\gamma(t) - \gamma(s)}{\|\gamma(t) - \gamma(s)\|} & s \neq t \text{ and } s \neq 0, t \neq T \\ \gamma'(s) & s = t \\ -\gamma'(0) & s = 0 \text{ and } t = T \end{cases}$$

g is therefore continuous. Let us define $\alpha_0(t)$ to be the line which connects $(0,0)$ to (T,T), ie. $\alpha_0(t) = t(T,T)$. Thus α_0 is contained within B. Then

$$g(\alpha_0(s)) = \gamma'(s) = \begin{pmatrix} \cos(\theta_0(s)) \\ \sin(\theta_0(s)) \end{pmatrix}$$

Where θ_0 is $g \circ \alpha_0$'s angle function. Thus

$$K = \theta_0(T) - \theta_0(0)$$