

Infinitesimal Calculus 3

Lecture 15, Sunday December 4, 2022
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Proposition 15.1:

Suppose $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is defined in a neighborhood and differentiable at $p \in \mathbb{R}^n$ let $q = f(p)$. Suppose $g: \mathbb{R}^m \rightarrow \mathbb{R}^k$ which is defined in a neighborhood and differentiable at q . Then $g \circ f: \mathbb{R}^n \rightarrow \mathbb{R}^k$ is differentiable at p and satisfies:

$$dg \circ f|_p = dg|_q \cdot df|_p$$

Another way of thinking of this is if J_f is the Jacobian of f then:

$$J_{g \circ f}(p) = J_g(f(p)) \cdot J_f(p)$$

Proof:

Suppose $f(x+h) = f(x) + df|_x(h) + \varepsilon_1(h)$ and $g(x+h) = g(x) + dg|_x(h) + \varepsilon_2(h)$. Let $h \in \mathbb{R}^n$ then

$$g \circ f(p+h) - g \circ f(p) = g(f(p+h)) - g(q) = g(q + df|_p(h) + \varepsilon_1) - g(q) = dg|_q(df|_p(h) + \varepsilon_p(h)) + \varepsilon_q(f(p+h) - f(p))$$

Now recall that by definition, differentials are linear, so this is equal to:

$$= dg|_q \circ df|_p(h) + dg|_q(\varepsilon_p) + \varepsilon_q(f(p+h) - f(p))$$

So we will show that the right side of this is an ε function. We know that:

$$\lim_{h \rightarrow 0} \frac{dg|_q(\varepsilon_p(h))}{\|h\|} = \lim_{h \rightarrow 0} dg|_q\left(\frac{\varepsilon_p(h)}{h}\right) = dg|_q(0) = 0$$

Since differentials are continuous as finite linear transforms. And

$$\lim_{h \rightarrow 0} \frac{\varepsilon_q(\Delta f(p))}{\|h\|} = \lim_{h \rightarrow 0} \frac{\varepsilon_q(\Delta f(p))}{\Delta f(p)} \cdot \frac{\Delta f(p)}{\|h\|}$$

Since f is continuous at p , as h approaches 0, so too does $\Delta f(p)$. So

$$\lim_{h \rightarrow 0} \frac{\varepsilon_q(\Delta f(p))}{\Delta f(p)} = 0$$

And $\frac{\Delta f(p)}{\|h\|}$ is bounded, as it is equal to

$$\frac{df|_p(h) + \varepsilon_1(h)}{\|h\|} = \frac{df|_p(h)}{\|h\|} + \frac{\varepsilon_1(h)}{\|h\|}$$

The left is bounded since differentials are linear transforms, and the right is bounded since it converges to 0. As required. ■

Definition 15.2:

The **contour line** of a function $f: A \rightarrow B$ is the set

$$\{a \in A \mid f(a) = b\}$$

for some $b \in B$. That is, it is the preimage of $\{b\}$.

For example, the contour lines of $f(x, y) = x^2 + y^2$ are circles. If γ is a contour line of f , if u is tangent to the contour, $D_u f = 0$.

Theorem 15.3 (Clairut-Schwarz Theorem):

Suppose $f: \mathbb{R}^n \rightarrow \mathbb{R}$ and its partial derivatives is defined in a neighborhood of (x_1, \dots, x_n) . If its second order partial derivatives $\partial_j \partial_i f$ and $\partial_i \partial_j f$ exist in a neighborhood and are continuous at (x_1, \dots, x_n) then they are equal at (x_1, \dots, x_n) .

Since this is identical to the 2 dimensional case, we will prove it only for the 2 dimensional case. Recall that an alternative notation for partial derivatives is:

$$f_{x_i} = \partial_{x_i} f$$

And:

$$f_{x_j x_i} = \partial_{x_j} \partial_{x_i} f$$

So this theorem states that $f_{x_i x_j} = f_{x_j x_i}$ under the specified conditions.

Proof:

We define the following auxillary functions:

$$\omega(h, k) = \frac{f(x_0 + h, y_0 + h) - f(x_0 + h, y_0) - f(x_0, y_0 + h) + f(x_0, y_0)}{hk}$$

$$\varphi(x) = \frac{f(x, y_0 + k) - f(x, y_0)}{k}$$

Thus

$$\varphi'(x) = \frac{f_x(x, y_0 + k) - f_x(x, y_0)}{k}$$

And

$$\omega(h, k) = \frac{\varphi(x_0 + h) - \varphi(x_0)}{h}$$

By the mean value theorem, this means that $\omega(h, k) = \varphi'(x_0 + th)$ for some $0 < t < 1$, which is equal to

$$= \frac{f_x(x_0 + th, y_0 + k) - f_x(x_0 + th, y_0)}{k}$$

And by the mean value theorem this is equal to $f_{xy}(x_0 + t_1 h, y_0 + t_2 k)$ for $0 < t_1, t_2 < 1$ ($t_1 = t$). By symmetry (we can do the same thing but swapping x and y in φ), $\omega(h, k) = f_{yx}(x_0 + t_3 h, y_0 + t_4 k)$. That is:

$$\omega(h, k) = f_{xy}(x_0 + t_1 h, y_0 + t_2 k) = f_{yx}(x_0 + t_3 h, y_0 + t_4 k)$$

Since f_{xy} and f_{yx} are continuous at (x_0, y_0) , as we take $(h, k) \rightarrow (0, 0)$ we have that:

$$\lim_{(h,k) \rightarrow 0} f_{xy}(x_0 + t_1 h, y_0 + t_2 k) = f_{xy}(x_0, y_0)$$

But on the other hand:

$$\lim_{(h,k) \rightarrow 0} f_{xy}(x_0 + t_1 h, y_0 + t_2 k) = \lim_{(h,k) \rightarrow 0} f_{yx}(x_0 + t_3 h, y_0 + t_4 k) = f_{yx}(x_0, y_0)$$

And therefore

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$$

as required. ■