Modern Analysis

Homework 6

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6.1 Exercise

Let (X, Σ, μ) be a measure space and f a real integrable function in $L^1(\mu)$. For every $t \in \mathbb{R}$, define

$$F(t) := \int_X f(x)\cos(e^t f(x))\,d\mu(x)$$

show that F is defined and continuous in $\mathbb{R}$.

Define $g_t(x) = f(x)\cos(e^t f(x))$ so that $F(t) = \int g_t\,d\mu(x)$. Then $|g_t| \leq |f|$ and so g_t is integrable and has a finite integral since $|f|$ does (since it is in L^1). Furthermore, if t_n is a sequence converging to some $t_0 \in \mathbb{R}$ then g_{t_n} converges to g_{t_0} pointwise and since $|g_{t_n}| \leq |f|$, by the dominated convergence theorem,

$$F(t_n) = \int_X g_{t_n}\,d\mu(x) \longrightarrow \int_X g_{t_0}\,d\mu(x) = F(t_0)$$

so F is continuous.

6.2 Exercise

Let (X, Σ, μ) be a measure space and f an extended real integrable function. Let $c = \int_X f$ and suppose $0 < c < \infty$. Show that

$$\lim_{n\to\infty} \int_X n\log\left(1 + \left(\frac{f}{n}\right)^a\right)d\mu = \begin{cases} c & a = 1 \\ \infty & 0 < a < 1 \\ 0 & 1 < a < \infty \end{cases}$$

Notice that

$$n\log\left(1 + \frac{f^a}{n^a}\right) = \frac{1}{n^{a-1}}\log\left[\left(1 + \frac{f^a}{n^a}\right)^{n^a}\right]$$

Now, $\log\left[\left(1 + \frac{f^a}{n^a}\right)^{n^a}\right]$ increases to $\log e^{f^a} = f^a$, and when $0 < a < 1$, n^{1-a} also increases to ∞. So by the monotone convergence theorem, when $0 < a < 1$ the integral increases to $\int \infty = \infty$ (since $\mu(X) > 0$ as $c > 0$). When $a > 1$, n^{1-a} converges to zero, and so the function converges to zero. Furthermore,

$$n\log\left(1 + \left(\frac{f}{n}\right)^a\right) \leq n\log\left(e^{a\frac{f}{n}}\right) = af$$

which is integrable, and since the left-hand side is nonnegative by the dominated convergence theorem the limit of its integral is the integral of its limit, which is zero. When $a = 1$ we have the above inequality so we can still utilize the dominated convergence theorem, but this time n^{1-a} is 1, and $f^a = f$, so its limit is f. Thus the limit of the integral is the integral of f, which is c.

6.3 Exercise

Let (X, Σ, μ) be a finite measure space. Show that a nonnegative measurable function f is integrable if and only if

$$\sum_{n=1}^{\infty} \mu\{x \mid f(x) \geq n\} < \infty$$

Let us define

$$g(x) := \sum_{n=1}^{\infty} \chi\{x \mid f(x) \geq n\}$$

Then $g(x) = n$ if and only if $n \leq f(x) < n+1$ (since $g(x)$ is the number of times $f(x) \geq n$). Thus $g(x) \leq f(x) < g(x) + 1$ (if $f(x) = \infty$ then $g(x) = \infty$ as well, so $g(x) = f(x)$). This means that

$$\int_X g(x) \leq \int_X f(x) \leq \int_X g(x) + \mu(X)$$

And we know that $\mu(X) < \infty$, so $\int_X f(x)$ is finite if and only if $\int_X g(x)$ is, and $\int_X g(x) = \sum_{n=1}^{\infty} \mu\{x \mid f(x) \geq n\}$ as it is the countable sum of nonnegative functions (we showed that $\int \sum f_n = \sum \int f_n$ when f_n are nonnegative).

6.4 Exercise

Let (X, Σ, μ) be a finite measure space, and $f \in L^1(\mu)$ nonnegative. Show that

$$\lim_{\alpha \to 1^-} \int_X f^\alpha = \int_X f$$

Let $\alpha_n \nearrow 1$, and define $E = \{x \mid f(x) \geq 1\}$. So for $x \in E$, $f(x)^{\alpha_n}$ increases to $f(x)$, and for $x \notin E$ $f(x)^{\alpha_n}$ decreases to $f(x)$. Thus by the monotone convergence theorem,

$$\int_E f^{\alpha_n} \longrightarrow \int_E f$$

And for $x \in E^c$, $f(x)^{\alpha_n}, f(x) \leq 1$ which is integrable as the space is finite ($\int 1 = \mu(X) < \infty$), so by the dominated convergence theorem

$$\int_{E^c} f^{\alpha_n} \longrightarrow \int_{E^c} f$$

Thus we have that

$$\int_X f^{\alpha_n} = \int_E f^{\alpha_n} + \int_{E^c} f^{\alpha_n} \longrightarrow \int_E f + \int_{E^c} f = \int_X f$$

as required.

6.5 Exercise

Let (X, Σ, μ) be a σ-finite space. Let $f \colon X \longrightarrow [0, \infty)$ be nonnegative and integrable. Show that for every $\varepsilon > 0$, there exists an $A \in \Sigma$ with finite measure such that

$$\int_X f < \int_A f + \varepsilon$$

X is σ-finite so there exists $\{X_i\}$ with finite measure so that $X = \bigcup X_i$. Define $A_n = \bigcup_{i=1}^n X_i$, so $X = \bigcup A_n$ and $\mu(A_n)$ is still finite (as the finite union of finite measured sets). Now, $f \cdot \chi_{A_n}$ is an increasing sequence of functions which increases to f (since A_n is increasing to X). So by the monotone convergence theorem, $\int_{A_n} f \nearrow \int f$. Thus there exists an n such that $\int_{A_n} f > \int f - \varepsilon$, so choose $A = A_n$ and we have the desired result.