Complex Functions

Assignment 8

Ari Feiglin

Exercise 8.1:

Suppose f is analytic and not constant on a compact domain D. Show that $\operatorname{Re} f$ and $\operatorname{Im} f$ obtain their maxima and minima on the boundary of D.

I will prove this using the result of the next exercise. Suppose z_0 induces a maximum or minimum for $\operatorname{Re} f$ or $\operatorname{Im} f$, then $f(z_0)$ is on the boundary of $f(D)$. This is because either $f(z_0) \pm \frac{\varepsilon}{2}$ or $f(z_0) \pm i\frac{\varepsilon}{2}$ is not in $f(D)$ for any $\varepsilon > 0$ (depending on whether $f(z_0)$ is a maximum or minimum, and for which function). And since $f(z_0)$ is necessarily in $f(D)$, we have that for every $\varepsilon > 0$, $D_\varepsilon(z_0)$ is not disjoint with $f(D)$ or $f(D)^c$, which is precisely what $f(z_0)$ being a boundary point of $f(D)$ means.
Thus by the result of the next question, $z_0 \in \partial D$ as required.

Exercise 8.2:

(1) Show that if f is analytic and not constant on S, and $f(S) = T$ then if $f(z)$ is a boundary point of T then z is a boundary point of S.

(2) Let $f(z) = z^2$, and let S be the union of S_1 and S_2 where

$$S_1 = \{z \mid |z| \leq 2, \operatorname{Re} z \leq 0\}, \qquad S_2 = \{z \mid |z| \leq 1, \operatorname{Re} z \geq 0\}$$

Show that there exists a boundary point of S, z, such that $f(z)$ is an interior point of $f(S)$.

(1) Since we know non-constant analytic functions are open maps, $f(\operatorname{int} S)$ is open in T, and since the interior of a set is the largest open set contained within said set, we have that $f(\operatorname{int} S) \subseteq \operatorname{int} T$. Since $f(z) \in \partial T$, this means that $f(z) \notin \operatorname{int} T$ and thus $z \notin \operatorname{int} S$. So $z \in S \setminus \operatorname{int} S \subseteq \partial S$ as required.

(2) Notice that $f(S_1) = \bar{D}_4(0)$ and $f(S_2) = \bar{D}_1(0)$. This is because if $z = re^{i\theta} \in S_2$ then $r \leq 2$ and $\frac{\pi}{2} \leq \theta \leq \frac{3}{2}\pi$. Thus $f(re^{i\theta}) = r^2 e^{2i\theta}$ and since $r^2 \leq 4$, $f(z) \in \bar{D}_4(0)$. And if $re^{i\theta} \in \bar{D}_4(0)$, let $z = \sqrt{r}e^{i\alpha}$ where $\alpha = \frac{\theta}{2}$ if $\pi \leq \theta$, and $\alpha = \frac{\theta}{2} + \pi$ otherwise. In any case, we have that $z \in S_1$, and $f(z) = re^{i\theta}$. Thus $f(S_1) = \bar{D}_4(0)$ as required. A nearly identical proof holds for S_2.
Thus $f(S) = \bar{D}_4(0)$. So let us take $z = 1$, which is on the boundary of S (for any $\varepsilon > 0$, $z + \frac{\varepsilon}{2}$ is not in S), but $f(z) = 1$ which is in the interior of $f(S) = \bar{D}_4(0)$ ($D_1(1)$ is contained within $f(S)$).

Exercise 8.3:

Suppose f is an analytic function strictlu bounded by 1 on the unit disk. Further suppose that there exists an α on the unit disk where $f(\alpha) \neq 0$. Show that there exists an analytic function g which is also strictly bounded by 1 on the unit disk where $|f'(\alpha)| < |g'(\alpha)|$.

Let us define

$$g(z) = \frac{f(z) - f(\alpha)}{1 - f(z) \cdot \overline{f(\alpha)}}$$

We note that this is defined over all of $D_1(0)$, since it is only undefined when

$$f(z) = \frac{1}{\overline{f(\alpha)}} \implies |f(z)| = \frac{1}{|f(\alpha)|} > 1$$

since $|f| < 1$, this is a contradiction. And since $g(z)$ is the quotient of two analytic functions, it itself is analytic in $D_1(0)$.
$g(z)$ is also strictly bounded by 1 in $D_1(0)$ since

$$|g(z)| < 1 \iff |f(z) - f(\alpha)| < \left|1 - f(z) \cdot \overline{f(\alpha)}\right| \iff |f(z) - f(\alpha)|^2 < \left|1 - f(z) \cdot \overline{f(\alpha)}\right|^2$$

Let us compute both sides with the identity $|z|^2 = z \cdot \overline{z}$:

$$|f(z) - f(\alpha)|^2 = (f(z) - f(\alpha))(\overline{f(z)} - \overline{f(\alpha)}) = |f(z)|^2 + |f(\alpha)|^2 - f(z)\overline{f(\alpha)} - f(\alpha)\overline{f(z)}$$

and

$$\left|1 - f(z) \cdot \overline{f(\alpha)}\right|^2 = (1 - f(z) \cdot \overline{f(\alpha)})(1 - \overline{f(z)}f(\alpha)) = 1 + |f(z)|^2|f(\alpha)|^2 - f(z)\overline{f(\alpha)} - f(\alpha)\overline{f(z)}$$

Thus the inequality holds if and only if

$$|f(z)|^2 + |f(\alpha)|^2 < 1 + |f(z)|^2|f(\alpha)|^2$$

Which is if and only if

$$|f(\alpha)|^2\left(1 - |f(z)|^2\right) < 1 - |f(z)|^2$$

and since $|f(z)| < 1$, we can divide both sides by $1 - |f(z)|^2$ and preserve the inequality, meaning this is if and only if

$$|f(\alpha)|^2 < 1$$

Thus we have shown that $|g(z)| < 1$ in $D_1(0)$ as required.
Now notice that $g(\alpha) = 0$ and so

$$g'(\alpha) = \lim_{z \to \alpha} \frac{g(z)}{z - \alpha} = \lim_{z \to \alpha} \frac{f(z) - f(\alpha)}{z - \alpha} \cdot \frac{1}{1 - f(z) \cdot \overline{f(\alpha)}} = f'(\alpha) \cdot \frac{1}{1 - f(\alpha) \cdot \overline{f(\alpha)}} = f'(\alpha) \cdot \frac{1}{1 - |f(\alpha)|^2}$$

Since $0 < |f(\alpha)| < 1$ we have that $\frac{1}{1 - |f(\alpha)|^2} > 1$ and so

$$|g'(\alpha)| = |f'(\alpha)| \cdot \frac{1}{1 - |f(\alpha)|^2} > |f'(\alpha)|$$

as required.

Exercise 8.4:

Suppose f is an entire function such that $|f(z)| \leq \frac{1}{\operatorname{Re}(z)^2}$. Prove that f is identically zero.

Let $R > 0$ and let us define for $|z| < R$

$$g(z) = (z^2 + R^2)^4 f(z)$$

Let $|z| = R$, and notice that for such a z, suppose $z = a + bi$ then $R^2 = a^2 + b^2$ and so

$$z^2 + R^2 = a^2 - b^2 + 2abi + a^2 + b^2 = 2a^2 + 2abi = 2a(a + bi) = 2z \operatorname{Re} z$$

and so

$$|g(z)| = \left|z^2 + R^2\right|^4 \cdot |f(z)| \leq |2z \operatorname{Re} z|^4 \cdot \frac{1}{\operatorname{Re}(z)^2} \leq 16R^4 \operatorname{Re} z^4 \cdot \frac{1}{\operatorname{Re}(z)^2} = 16R^4 \operatorname{Re}(z)^2 \leq 16R^6$$

since $\operatorname{Re}(z) \leq R$.
So for every $z \in \partial D_R(0)$ we have $|g(z)| \leq 16R^6$. But by the maximum modulus principal, for the maximum of $g(z)$ on $D_R(0)$ is obtained on its boundary, ie when $|z| = R$. Thus for every $|z| \leq R$, $|g(z)| \leq 16R^6$.
So let $z \in \mathbb{C}$, then for every $R > 0$ such that $|z| \leq R$, we have

$$|f(z)| \cdot \left|z^2 + R^2\right|^4 \leq 16R^6 \implies |f(z)| \leq \frac{16R^6}{|z^2 + R^2|^4}$$

And by letting $R \to \infty$, we get that $\frac{16R^6}{|z^2+R^2|^4} \to 0$ and so $|f(z)| \leq 0$ meaning $f(z) = 0$ for every $z \in \mathbb{C}$ as required.

Exercise 8.5:

Show that

$$f(z) = \int_0^1 \frac{\sin(zt)}{t}\, dt$$

is an entire function by

(1) Morera's theorem

(2) Finding a powerseries for f

(1) Let Γ be the boundary of a complex rectangle, we must show that

$$\int_\Gamma f(z)\, dz = 0$$

and then by Morera's theorem, f is analytic.
Notice that

$$\int_\Gamma \int_0^1 \left|\frac{\sin(zt)}{t}\right| dt\, dz$$

converges since the inner integral converges (we showed this in calculus 2), and is bounded.
Thus by Fubini-Tonelli, we have that

$$\int_\Gamma f(z)\, dz = \int_\Gamma \int_0^1 \frac{\sin(zt)}{t}\, dtdz = \int_0^1 \int_\Gamma \frac{\sin(zt)}{t}\, dzdt$$

by Cauchy's theorem, since $z \mapsto \frac{\sin(zt)}{t}$ is analytic in Γ's interior, the inner integral is 0, and thus the integral as a whole is zero, as required.

(2) Using sin's powerseries

$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

and thus

$$\frac{\sin(zt)}{t} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1} t^{2n}}{(2n+1)!}$$

This still has a radius of convergence of infinity (both as a powerseries for z and t, since we are taking a powerseries defined everywhere and dividing it by t, and this still results in a powerseries). Thus since powerseries converge uniformly

$$\int_0^1 \frac{\sin(zt}{t}\, dt = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \cdot \int_0^1 t^{2n}\, dt = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)! \cdot (2n+1)}$$

And this has a radius of convergence of infinity, as is obviously apparent by the ratio test. Thus $f(z)$ has a powerseires which is convergent everywhere, meaning it is entire.

Exercise 8.6:

Show that the function f from the previous exercise satisfies

$$f'(z) = \int_0^1 \cos(zt)\, dt$$

by

(1) Using the change of order of integration.

(2) Using the powerseries from the previous exercise.

(1) Notice that

$$\int_0^z \cos(wt)\,dw = \frac{\sin(wt)}{w}\bigg|_0^z = \frac{\sin(zt)}{z}$$

and thus

$$f(z) = \int_0^1 \int_0^z \cos(wt)\,dwdt$$

Now for any $z \in \mathbb{C}$, since

$$\int_0^z \int_0^1 |\cos(wt)|\,dt|dw| \le \int_0^z |dw|$$

is convergent, by Fubini-Tonelli we have

$$f(z) = \int_0^1 \int_0^z \cos(wt)\,dwdt = \int_0^z \int_0^1 \cos(wt)\,dtdw$$

and by the Fundamental theorem of calculus, this means

$$f'(z) = \int_0^1 \cos(zt)\,dt$$

as required.

(2) Using the powerseries of cos,

$$\cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

we have that

$$\int_0^1 \cos(zt)\,dt = \int_0^1 \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}t^{2n}}{(2n)!}\,dt = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \int_0^1 t^{2n}\,dt = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!}$$

And we know that by the previous exercise

$$f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)! \cdot (2n+1)}$$

So

$$f'(z) = \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)z^{2n}}{(2n+1)! \cdot (2n+1)} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!} = \int_0^1 \cos(zt)\,dt$$

as required.

Exercise 8.7:

Show that

$$L(z) = \pi i + \int_{-1}^z \frac{dw}{w}$$

is a branch of the complex logarithm in $D = \{z \in \mathbb{C} \mid z \in \mathbb{R} \implies z < 0\}$. And further show that

$$0 < \operatorname{Im} L(z) = \arg(z) < 2\pi$$

We showed in lecture that if D is a simply connected domain where $0 \notin D$, and $e^{L_0} = z_0$ then

$$L(z) = L_0 + \int_{z_0}^z \frac{dw}{w}$$

is an analytic branch of the complex logarithm in D.
Since the domain D defined in the question is simply connected, and $e^{i\pi} = -1$, we have that the L defined in the question is indeed an analytic branch of the complex logarithm.
For $z \in D$ let us define the smooth curve Γ as the concatenation of the curve from -1 to $-|z|$ (contained within $\mathbb{R}$), and then the arc from $-|z|$ to z (this is part of the circle around 0 of radius $|z|$). Let us denote the first part of this curve by Γ_1, and the second (the arc) by Γ_2. So we have that

$$L(z) = i\pi + \int_\Gamma \frac{dw}{w} = i\pi + \int_{\Gamma_1} \frac{dw}{w} + \int_{\Gamma_2} \frac{dw}{w}$$

Since Γ_1 is contained entirely within $\mathbb{R}$, the integral over Γ_1 does not contribute to $\mathrm{Im}(L(z))$.
Suppose $z = re^{i\alpha}$, we can parameterize Γ_2 by

$$[\pi, \alpha] \longrightarrow \Gamma_2, \quad \theta \mapsto re^{i\theta}$$

and thus we have that

$$\int_{\Gamma_2} \frac{dw}{w} = \int_{\Gamma_2} \frac{\overline{w}}{|w|^2}\, dw = \int_\pi^\alpha \frac{re^{-i\theta}}{r^2} \cdot rie^{i\theta}\, d\theta = i \int_\pi^\alpha d\theta = i(\alpha - \pi)$$

So we have

$$\mathrm{Im}(L(z)) = \mathrm{Im}\left(i\pi + \int_{\Gamma_1} \frac{dw}{w} + \int_{\Gamma_2} \frac{dw}{w} \right) = \pi + \mathrm{Im}(i(\alpha - \pi)) = \pi + \alpha - \pi = \alpha = \arg(z)$$

as required. ($\arg(z) > 0$ since if $\arg(z) = 0$ then $z \in \mathbb{R}$ and $z \geq 0$, so $z \notin D$).