Representation Theory
Homework 2

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1 Problem

Let G be a group of order p^n, and V be a representation of G over a field $\mathbb{F}$ of characteristic p.

(1) Show that if $\mathbb{F}$ is finite and V is finite-dimensional, then there is a fixed nonzero vector.

(2) Show that if $\mathbb{F}$ is infinite, then we can view V as a $\mathbb{F}_p$ vector space and therefore a representation over $\mathbb{F}_p$. Show that V has a finite dimensional subrepresentation, and deduce that the requirements of finiteness in the previous point are redundant.

(3) Deduce that if V is irreducible then V is trivial and 1-dimensional.

(1) Let $0 \neq v \in V$, then by the orbit-stabilizer theorem we have that $|\mathrm{Orb}_G(v)| = |G|/|\mathrm{Stab}_G(v)|$. We claim that one of the stabilizers is all of G, equivalently one of the orbits is trivial. Note that the orbits partition V, and $\mathrm{Orb}_G(0) = 0$, furthermore since $|G| = p^n$, the orbits must have size of a power of p. V is a finite-dimensional vector space over a finite-dimensional field of characteristic p, and hence has cardinality p^m for some m.

Let $\mathrm{Orb}_G(v_1), \ldots, \mathrm{Orb}_G(v_t)$ partition $V - 0$. That is, $|V - 0| = p^m - 1 = \sum_i |\mathrm{Orb}_G(v_i)|$. If we assume that none of the orbits are trivial, then p divides all of their cardinalities, and thus the sum. But p does not divide $p^m - 1$, a contradiction.

(2) Let us consider $W = \mathrm{span}_{\mathbb{F}_p} \mathrm{Orb}_G(v)$ for $v \neq 0$. This is finite-dimensional, since G is finite, and is a subrepresentation of V: $g \sum \alpha_i g_i v = \sum \alpha_i g g_i v$. Since V must be infinite-dimensional as a $\mathbb{F}_p$-vector space, $V \neq W$. As per the previous point, W has a fixed non-zero vector. This vector is also fixed in the representation of V.

(3) If V is irreducible, then since V^G cannot be zero, $V^G = V$. V^G is trivial, and is irreducible iff it is one-dimensional.

Note: A G-set X is 2-transitive if $X \times X$ has two orbits. This is equivalent to saying that for every $x_1 \neq x_2, y_1 \neq y_2 \in X$ there is a $g \in G$ such that $gx_1 = y_1$ and $gx_2 = y_2$.

2 Problem

(1) Let X be a finite G-set. Show that $\mathbb{F}[X]$ has a non-trivial trivial subrepresentation. Furthermore show that

$$\mathbb{F}[X]_0 = \left\{ \sum_x a_x x \;\middle|\; \sum_x a_x = 0 \right\}$$

is a subrepresentation as well. Finally, show that if $|X|$ is invertible in $\mathbb{F}$ then $\mathbb{F}[X] = \mathrm{triv} \oplus \mathbb{F}[X]_0$ for some trivial subrepresentation triv.

(2) Let G be a finite group, and X be 2-transitive. Suppose that $|G|$ is invertible in $\mathbb{F}$. Show that $\mathbb{F}[X]_0$ is irreducible.

(3) Show that $\{1, \ldots, n\}$ is a 2-transitive S_n-set for $n \geq 2$.

(4) Deduce that the **standard representation** of S_n: $\mathbb{F}[\{1, \ldots, n\}]_0$, is irreducible for $n \geq 2$ if $n!$ is invertible in $\mathbb{F}$.

(1) Note that $\sum_x x \in \mathbb{F}[X]$ is a non-zero fixed point of G. This is because g acts bijectively on X, and so $g \sum_x x = \sum_x gx = \sum_x x$. Thus $\mathbb{F}[X]$ has a non-trivial trivial subrepresentation (i.e. $\mathbb{F}[X]^G \neq 0$).

$\mathbb{F}[X]_0$ is clearly a subrepresentation: $g \sum_x a_x x = \sum_x a_x gx = \sum_x a_{g^{-1}x} x$, and $\sum_x a_{g^{-1}x} = \sum_x a_x = 0$ as required. And clearly $\mathbb{F}[X]_0$ has dimension $|X| - 1$.

Taking $\operatorname{triv} = \operatorname{span} \sum_x x$, we get that triv is a non-trivial subrepresentation of dimension 1. Since $|X| \neq 0$ by assumption, triv is disjoint from $\mathbb{F}[X]_0$ and thus they direct sum to $\mathbb{F}[X]$.

(2) By corollary 1.5 in the previous recitation notes, $\dim \hom_G(\mathbb{F}[X], \mathbb{F}[X]) = |(X \times X)/G| = 2$. Note that X must be finite (orbits have order dividing $|G|$, so a finite number of orbits means a finite set). Therefore by Maschke, we can view $\hom_G(\mathbb{F}[X]_0, \mathbb{F}[X]_0)$ as a subgroup of $\hom_G(\mathbb{F}[X], \mathbb{F}[X])$ (choose a complementary subrepresentation to $\mathbb{F}[X]_0$, W; then endomorphisms over $\mathbb{F}[X]_0$ can be viewed as endomorphisms over all of $\mathbb{F}[X]$ by making them identically zero on W). This is a non-trivial subgroup of $\hom_G(\mathbb{F}[X], \mathbb{F}[X])$, and thus must have dimension 1. By the last problem set, this means $\mathbb{F}[X]_0$ is irreducible.

(3) In the previous problem set we showed that $X = \{(x_1, x_2) \mid x_1 \neq x_2\} \subseteq [n]^2$ is transitive. Clearly $\Delta = \{(x, x)\} \subseteq [n]^2$ is also transitive (it is isomorphic as a G-set to $[n]$), and since $X \cup \Delta = [n]^2$, we get that $[n] \times [n]$ has two orbits (X and Δ), so $[n]$ is 2-transitive.

(4) This is a direct result of the previous two points: $[n]$ is 2-transitive and $|S_n| = n!$ is invertible.

3 Problem

Let A be a finite Abelian group.

(1) Show that if V is a complex irreducible representation of G, then there exists a character $\xi \colon Z(G) \to \mathbb{C}^\times$ such that for every $z \in Z(G)$ and $v \in V$, $\rho(z)(v) = \xi(z)v$.

(2) Deduce that any irreducible complex representation of A is 1-dimensional.

(3) Let $\rho \colon A \to \mathrm{GL}_n(\mathbb{R})$ be an irreducible representation of A. Let $\rho_{\mathbb{C}}$ be the composition of ρ with the inclusion $\mathrm{GL}_n(\mathbb{R}) \subseteq \mathrm{GL}_n(\mathbb{C})$. Use Maschke and the previous point to show that $\rho_{\mathbb{C}}$ decomposes into a sum of n 1-dimensional subrepresentations.

(4) Use the previous point to show that, when viewing $\mathbb{C}^n$ as a real vector space, $\rho_{\mathbb{C}}$ decomposes into a sum of at least n 1-dimensional subrepresentations.

(5) Show that $(\mathbb{C}^n, \rho_{\mathbb{C}})$ is isomorphic over $\mathbb{R}$ to $(\mathbb{R}^n, \rho) \oplus (\mathbb{R}^n, \rho)$.

(6) Use the previous two points to show that any irreducible representation of A over $\mathbb{R}$ is 1 or 2-dimensional.

(1) Since V is irreducible, it is finite dimensional. For $z \in Z(G)$, note that $\rho(z) \colon V \to V$ is a G-morphism: $\rho(z)(gv) = \rho(zg)v = g\rho(z)v$. Therefore $\rho(z) \in \operatorname{end}_G(V)$, which by Schur is equal to $\mathbb{C} \cdot \mathrm{id}_V$, so $\rho(z) = \xi(z) \cdot \mathrm{id}_V$ for some unique $\xi(z) \in \mathbb{C}$. Since $\rho(z)$ is invertible, $\xi(z) \in \mathbb{C}^\times$.

So we have defined $\xi \colon Z(G) \to \mathbb{C}^\times$. We claim that this defines a group morphism. Indeed: $\rho(zw)(v) = \rho(z)(\rho(w)(v)) = \xi(z)\xi(w)v$, so $\xi(zw) = \xi(z)\xi(w)$.

(2) Given a representation ρ of A and let ξ be as before, since $Z(A) = A$, we have that $\rho(z)(v) = \xi(z)v$ for all $z \in A$. Now let $0 \neq v \in V$, then the span of v is a subrepresentation. Indeed:

$\rho(z)(v) = \xi(z)v \in \operatorname{span} v$. Since every 1-dimensional subspace is a subrepresentation, all irreducible subrepresentations are 1-dimensional.

(3) By Maschke, there exists a decomposition $\mathbb{C}^n = E_1 \oplus \cdots \oplus E_k$ where E_i are irreducible subrepresentations. By the above point, these are all 1-dimensional, and so $k = n$.

(4) Let $\mathbb{C}^n = E_1 \oplus \cdots \oplus E_n$ be a decomposition into complex irreducible subrepresentations. These are all also real subrepresentations, and so a decomposition of $\mathbb{C}^n$ further into real irreducible subrepresentations must use more than n summands.

(5) Define $\phi\colon \mathbb{R}^n \oplus \mathbb{R}^n \to \mathbb{C}^n$ by $\phi(\vec{a}, \vec{b}) = \vec{a} + i\vec{b}$. This is clearly isomorphic and linear, it is a real A-morphism as for every $z \in A$:

$$\phi(\rho(z)(\vec{a}, \vec{b})) = \phi(\rho(z)\vec{a}, \rho(z)\vec{b}) = (\rho(z)\vec{a}) + i(\rho(z)\vec{b}) = (\rho_{\mathbb{C}}(z)\vec{a}) + i(\rho_{\mathbb{C}}(z)\vec{b}) = \rho_{\mathbb{C}}(z)(\vec{a} + i\vec{b})$$

(6) Let $(\mathbb{R}^n, \rho)$ (wlog $V = \mathbb{R}^n$) be an irreducible representation of A. Then $(\mathbb{C}^n, \rho_{\mathbb{C}}) \cong (\mathbb{R}^n, \rho) \oplus (\mathbb{R}^n, \rho)$. Now, $\mathbb{C}^n$ has at least n irreducible components in its decomposition: $\mathbb{C}^n = E_1 \oplus \cdots \oplus E_k$ for $k \geq n$. But $\mathbb{C}^n = \mathbb{R}^n \oplus \mathbb{R}^n$ is another irreducible decomposition, so $2 \geq n$, i.e. $n = 1, 2$.

4 Problem

(1) Show that the irreducible representations of D_n over $\mathbb{C}$ are at most 2-dimensional.

(2) Calculate the number of 1-dimensional representations of D_n and write them all out explicitly.

(3) Using the previous two points, show that if n is even (odd), there are precisely $(n-2)/2$ ($(n-1)/2$) 2-dimensional irreducible representations of D_n.

(4) Let $V = \mathbb{C}^n$ and $\xi = \exp(2\pi i/n)$. Show that for $0 < k < n/2$, the following defines a representation (V, ρ_k):

$$\rho_k(\sigma) = \begin{pmatrix} \xi^k & \\ & \xi^k \end{pmatrix}, \qquad \rho_k(\tau) = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

(5) Show that if $0 < k \neq j < n/2$, then ρ_k and ρ_j are not isomorphic.

(6) Deduce that we have found all the irreducible representations of D_n up to isomorphism.

(1) This was proven as-is in recitation... Anyway this is direct from lemma 4.1.

(2) Given a one-dimensional representation $(\mathbb{C}, \rho)$, we note that ρ is entirely determined based on the image of D_n's generators σ, τ. Furthermore, $\rho(g) \in \mathrm{GL}(\mathbb{C}) \cong \mathbb{C}^\times$, as such $\rho(\sigma), \rho(\tau)$ may be viewed as nonzero complex numbers. And even further, we have $\rho(\sigma)^n = 1$ and $\rho(\tau)^2 = 1$ and $\rho(\sigma\tau\sigma\tau) = \rho(\sigma)^2\rho(\tau)^2 = 1$, and hence $\rho(\sigma)^2 = 1$. So to every representation ρ corresponds a pair of complex numbers x, y such that $x^n = 1$ and $x^2 = 1$ and $y^2 = 1$.

Conversely, suppose x, y satisfy $x^n = x^2 = y^2 = 1$, then $\rho(\sigma) = x$ and $\rho(\tau) = y$ defines a representation. Indeed, by the presentation of D_n this is well-defined.

So the one-dimensional representations are in bijection with $x, y \in \mathbb{C}$ such that $x^n = x^2 = y^2 = 1$. If n is even, then this requirement is simply $x^2 = y^2 = 1$. Thus there are two choices for both x and y, in total 4 (these representations are given $\rho_{ij}(\sigma) = i$ and $\rho_{ij}(\tau) = j$ for $i, j \in \{\pm 1\}$). And if n is odd, then $(n, 2) = 1$ and hence $x^{(n,2)} = x = 1$, so there are 2 (given by $\rho_i(\sigma) = 1$ and $\rho_i(\tau) = i$ for $i = \pm 1$).

(3) Let $E_1, \ldots, E_k$ be all the 2-dimensional irreducible representations of D_n. We know that $|G| = \sum_i (\dim E_i)^2$, so

$$2n = (2 \text{ or } 4) + 4k$$

If n is even then we get $4k = 2n - 4$ and so $k = (n-2)/2$. If n is odd then we get $4k = 2n - 2$ and so $k = (n-1)/2$, as required.

(4) By the presentation of D_n, we must show that $\rho_k(\sigma)^n = \rho_k(\tau)^2 = \rho_k(\sigma)\rho(\tau)\rho(\sigma)\rho(\tau) = 1$. Indeed $\rho_k(\tau)^2 = I$ clearly, and since $\exp(2\pi ik) = 1$, we have that $\xi^{kn} = 1$ so

$$\rho_k(\sigma)^n = \begin{pmatrix} \xi^{kn} & \\ & \xi^{-kn} \end{pmatrix} = I$$

and finally

$$\rho_k(\sigma)\rho_k(\tau)\rho_k(\sigma)\rho_k(\tau) = \begin{pmatrix} & \xi^{k} \\ \xi^{-k} & \end{pmatrix}^2 = I$$

as required.

(5) Let $0 < k, j < n/2$, then if $\rho_k \cong \rho_j$ then there is a isomorphism $M \in \mathrm{GL}_2(\mathbb{C})$ such that $M\rho_k(g) = \rho_j(g)M$. Note that since M must commute with $\rho_k(\tau)$, its diagonals must be equal. Then

$$M\rho_k(\sigma) = \begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} \xi^{k} & \\ & \xi^{-k} \end{pmatrix} = \begin{pmatrix} a\xi^{k} & b\xi^{-k} \\ b\xi^{k} & a\xi^{-k} \end{pmatrix}$$

and

$$\rho_j(\sigma)M = \begin{pmatrix} \xi^{j} & \\ & \xi^{-j} \end{pmatrix}\begin{pmatrix} a & b \\ b & a \end{pmatrix} = \begin{pmatrix} a\xi^{j} & b\xi^{j} \\ b\xi^{-j} & a\xi^{-j} \end{pmatrix}$$

One of a, b must be nonzero, in which case we get $\xi^k = \xi^j$ or $\xi^k = \xi^{-j}$. In the first case, since the order of ξ is n, we get $k = j$. In the second case, we get that $k = n - j$ but both are less than $n/2$, a contradiction.

So ρ_k, ρ_j are in isomorphism iff $k = j$.

(6) If n is even, then there are $n/2 - 1$ ks between 0 and $n/2$ (exclusive). So by our previous point of there being $n/2 - 1$ 2-dimensional irreducible representations, ρ_k form all of these.

If n is odd, then there are $(n+1)/2 - 1 = (n-1)/2$ ks between 0 and $n/2$ (exclusive). And hence ρ_k are all the 2-dimensional irreducible representations of D_n, as required.