Infintesimal Calculus 3

Assignment 3
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Exercise 3.0.1:

For every $n \in \mathbb{N}$ the set $A_n \subseteq \mathbb{R}^d$ is compact. Prove or disprove the following:

- $\bigcup_{n=1}^m A_n$ is compact.
- $\bigcap_{n=1}^m A_n$ is compact.
- $A_m \setminus A_n$ is compact.
- $\bigcup_{n\in\mathbb{N}} A_n$ is compact.
- $\bigcap_{n\in\mathbb{N}} A_n$ is compact.

- This is true since if $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$ is an open cover of the union then:

$$\bigcup_{n=1}^m A_n \subseteq \bigcup_{\lambda\in\Lambda} \mathcal{O}_\lambda \implies A_n \subseteq \bigcup_{\lambda\in\Lambda} \mathcal{O}_\lambda$$

  So $\{\mathcal{O}_\lambda\}$ is an open cover of every A_n. So for every n there exists a finite subcover $\{\mathcal{O}_{n_k}\}_{k=1}^{k_n}$ of A_n. And therefore

$$\bigcup_{n=1}^m A_n \subseteq \bigcup_{n=1}^m \bigcup_{k=1}^{k_n} \mathcal{O}_{n_k}$$

  And so the union of these finite subcovers of A_n is itself a finite subcover of the union. And so the union is compact.
- This is true. An arbitrary intersection of compact sets in $\mathbb{R}^d$ is compact: suppose $\{A_\lambda\}_{\lambda\in\Lambda}$ are compact. Then each of the sets are closed, and therefore $\bigcap_{\lambda\in\Lambda} A_\lambda$ is closed. And each of the sets are bounded since they are compact, and therefore the intersection is also bounded (it is a subset of the ball containing any A_λ). And therefore the intersection is compact. This also shows that 5 is true.
- This is false. The sets $[0,2]$ and $[1,2]$ are compact since they are closed and bounded, but $[0,2] \setminus [1,2] = [0,1)$ is not closed and is therefore not compact.
- This is false. For every n let $A_n = [-n, n]$, then A_n is compact since it is closed and bounded. But the union of A_n gives $\mathbb{R}$ which is not compact since it is not bounded.
- This is true, refer to the proof of 2.

Exercise 3.0.2:

Suppose $(X, \|\cdot\|)$ is a normed linear space. Show that every closed ball in X is path connected.

Suppose $\bar{B}_r(x)$ is the offending ball. If $z, y \in \bar{B}_r(x)$ then the line segment between them, denoted $\overleftrightarrow{zy}$ is a path between the two points contained in $\bar{B}_r(x)$. It is contained in $\bar{B}_r(x)$ since for every $0 \leq t \leq 1$:

$$\|ty + (1-t)z - x\| \leq \|t(y-x)\| + \|(1-t)(z-x)\| = t\,\|y-x\| + (1-t)\,\|z-x\|$$

and since z, y are in the ball, this is less than or equal to $tr + (1-t)r = r$. So every point on the line segment is in the ball.

Exercise 3.0.3:

Suppose $A, B \subseteq \mathbb{R}^n$ are connected. Which of the following are also connected?

- $A \cap B$
- If $A \cap B \neq \varnothing$, $\operatorname{int}(A \cup B)$.
- $A \setminus B$

- This is not necessarily connected.
  We will first show that rectangles are connected. Suppose we have the rectangle $R = \prod(a_n, b_n)$, and $x, y \in R$. Then the line $\overleftrightarrow{xy}$ is contained in R, so it is path connected so R is connected. Then if x_k and y_k are the kth components of x and y respectively, notice that for every relevant k:

  $$a_k = ta_k + (1-t)a_k < tx_k + (1-t)y_k < tb_k + (1-t)b_k = b_k$$

  So the line segment is conntained in R as required.
  So if we take the rectangle: $A = (0,3) \times (0,1)$ and the union of non-disjoint rectangles $B = \big((0,1) \times (0,2)\big) \cup \big((0,3) \times (1,2)\big) \cup \big((2,3) \times (0,2)\big)$. Then these are both connected and $A \cap B = \big((0,1) \times (0,1)\big) \mathbin{\dot\cup} \big((2,3) \times (0,1)\big)$, which is the disjoint union of two open sets. So $A \cap B$ is disconnected.
- This is not necessarily connected.
  So if we define $A = \bar{B}_1(0,0)$ and $B = \bar{B}_1(2,0)$ then by the previous question both of these sets are path connected and therefore connected. But $\operatorname{int}(A \cup B) = B_1(0,0) \cup B_2(2,0)$ which is the disjoint union of two open sets which is therefore not connected.
- This is not necessarily connected. If $A = (-2,2)$ and $B = [-1,1]$ these two sets are connected since they are path connected, but $A \setminus B = (-2,-1) \cup (1,2)$ which is the disjoint union of two open sets and is therefore not connected.