Probability and Statistics Homework #4

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Question 4.1:

Two fair independent dice are rolled. It is know that at least one of the dice rolled a 4. What is the probability that their sum is 8?

Answer:

The sample space is $\Omega = [6]^2$. Since both dice are fair, the probability space is uniform.
We want to find $\mathbb{P}(A \mid B)$ where the events A and B are defined as:

$$A \coloneqq \{(x_1, x_2) \mid x_1 + x_2 = 8\}$$
$$B \coloneqq \{(x_1, x_2) \mid x_1 = 4 \vee x_2 = 4\}$$

By the definition of conditional probability, and since the probability space is uniform:

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{|A \cap B|}{|B|}$$

We know that

$$B = \{(4, x)\} \cup \{(x, 4)\}$$

Each of these sets has a cardinality of 6 (as $1 \leq x \leq 6$), and their intersection is $\{(4, 4)\}$, so:

$$|B| = 6 + 6 - 1 = 11$$

We also know that

$$A \cap B = \{(x_1, x_2) \mid x_1 + x_2 = 8 \wedge (x_1 = 4 \vee x_2 = 4)\}$$

And if one x is 4, so must the oher x (as $8 - 4 = 4$), so:

$$A \cap B = \{(4, 4)\}$$

Which means:

$$\mathbb{P}(A \mid B) = \boxed{\frac{1}{11}}$$

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Question 4.2:

A student is taking a multiple-choice quiz where each question has m possible answers. Their strategy for answer is like so: if they know the answer to the question, they choose the correct answer. Otherwise, they guess one of the m possible answers. The probability that they know the answer to a question is p. Answer the following:

(a) What is the probability that the student knew the answer to a question if they answered it correctly?

(b) Compute and explain the result for $m = 1$ and $m \to \infty$.

Answer:

(a) Let A be the event that the student knew the answer, and let B be the event that the student got the answer correctly. Our goal then is to find $\mathbb{P}(A \mid B)$.
We know that $\mathbb{P}(A) = p$, and that $\mathbb{P}(B \mid A) = 1$ (as the probability that they got the answer right if they knew the answer is 1). Using Baye's Formula, we know:

$$\mathbb{P}(A \mid B) = \mathbb{P}(B \mid A) \cdot \frac{\mathbb{P}(A)}{\mathbb{P}(B)} = \frac{p}{\mathbb{P}(B)}$$

We know that:

$$\mathbb{P}(B) = \mathbb{P}(B \cap A) + \mathbb{P}(B \cap A^c)$$

But we also know that if the student knew the answer, they got the question correct, so $A \subseteq B$, meaning $A \cap B = A$. So

$$\mathbb{P}(B) = \mathbb{P}(A) + \mathbb{P}(B \cap A^c) = p + \mathbb{P}(B \mid A^c) \cdot \mathbb{P}(A^c)$$

We know that the probability of the student getting the answer right if they don't know the answer is $\frac{1}{m}$, so:

$$\mathbb{P}(B) = p + \frac{1-p}{m}$$

So:

$$\mathbb{P}(A \mid B) = \boxed{\frac{p \cdot m}{p \cdot m - p + 1}}$$

(b)

- If $m = 1$ then there is only one answer, so no matter what the student should get the answer right. Meaning $B = \Omega$, so $\mathbb{P}(B \mid A) = \mathbb{P}(A) = p$. This is in fact the case:

$$\mathbb{P}(B \mid A) = \frac{p}{p - p + 1} = p$$

- When $m \to \infty$, since the number of answers approaches ∞, the probability the student could get the answer right if they don't know the answer ($\mathbb{P}(B \mid A^c)$) should approach 0. So the probability of them getting them knowing the answer if they got it right should approach 1. This is true intuitively and because $\mathbb{P}(B \mid A^c) = 0$ so:

$$\mathbb{P}(B \mid A^c) = \mathbb{P}(A^c \mid B) \cdot \frac{\mathbb{P}(B)}{\mathbb{P}(A^c)} = 0 \implies \mathbb{P}(A^c \mid B) = 1 - \mathbb{P}(A \mid B) = 0 \implies \mathbb{P}(A \mid B) = 1$$

And this is indeed the case:

$$\lim_{m\to\infty} \mathbb{P}(A \mid B) = p \cdot \lim_{m\to\infty} \frac{m}{mp - p + 1} = p \cdot \lim_{m\to\infty} \frac{1}{p + \frac{-p+1}{m}} = p \cdot \frac{1}{p} = 1$$

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Question 4.3:

Two independent coins are flipped such that the probability of them landing on the same side that they started on is $\frac{2}{3}$. A die is rolled, if it lands on 1,2, or 3, then both coins start on heads. Otherwise, they start on tails. Let A be the event that the first coin lands on heads, and B be the event that the second coin lands on heads.

(a) Prove $\mathbb{P}(A) = \frac{1}{2}$

(b) Compute $\mathbb{P}(B \mid A)$

(c) Explain how A and B are dependent, but independent given the result of the die roll.

Answer:

(a) Let H be the event that the coins started on heads. This means that the die rolled a 1,2, or 3, so $\mathbb{P}(H) = \frac{1}{2}$.So $\mathbb{P}(H^c) = \frac{1}{2}$ as well. We know:

$$\mathbb{P}(A) = \mathbb{P}(A \cap H) + \mathbb{P}(A \cap H^c) = \mathbb{P}(A \mid H) \cdot \mathbb{P}(H) + \mathbb{P}(A \mid H^c) \cdot \mathbb{P}(H^c)$$

We know that $\mathbb{P}(A \mid H) = \frac{2}{3}$ and $\mathbb{P}(A \mid H^c) = 1 - \mathbb{P}(A^c \mid H^c) = 1 - \frac{2}{3} = \frac{1}{3}$. So:

$$\mathbb{P}(A) = \frac{1}{2} \cdot \left(\frac{2}{3} + \frac{1}{3}\right) = \frac{1}{2}$$

As required.

(b) Since the two coins are independent, the event that one lands on a side and another lands on a side given that they both started at the same state (same side) are independent. That means that:

$$\mathbb{P}(A \cap B \mid H) = \mathbb{P}(A \mid H) \cdot \mathbb{P}(B \mid H)$$

And we know $\mathbb{P}(A \mid H) = \mathbb{P}(B \mid H) = \frac{2}{3}$ so:

$$\mathbb{P}(A \cap B \mid H) = \left(\frac{2}{3}\right)^2$$

And similarly:

$$\mathbb{P}(A \cap B \mid H^c) = \mathbb{P}(A \mid H^c) \cdot \mathbb{P}(B \mid H^c)$$

And as we showed before, $\mathbb{P}(A \mid H^c) = \mathbb{P}(B \mid H^c) = \frac{1}{3}$ so

$$\mathbb{P}(A \cap B \mid H^c) = \left(\frac{1}{3}\right)^2$$

We know

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = 2\mathbb{P}(A \cap B)$$

And we know:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A \cap B \mid H) \cdot \mathbb{P}(H) + \mathbb{P}(A \cap B \mid H^c) \cdot \mathbb{P}(H^c) = \frac{1}{2} \cdot \left(\frac{4}{9} + \frac{1}{9}\right)$$

So:

$$\mathbb{P}(B \mid A) = \boxed{\frac{5}{9}}$$

(c) We can see that A and B are indeed dependent as $\mathbb{P}(B \mid A) \neq \mathbb{P}(B)$ (by symmetry $\mathbb{P}(B) = \mathbb{P}(A)$). They are dependent because they are both dependent on a third event, the die roll.
On the other hand, they are independent once their initial state is known (which is dependent only on the die roll), as nothing affects both of them afterwards.

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Question 4.4:

3 boxes are set upon a table, each with 2 black balls and a red ball. A ball is randomly taken from the first box and put in the second box. Then a ball is moved from the second to the third box, then from the third to the first. At the end a ball is removed from the first box. What is the probability that the ball is red?

Answer:

We will first generalize this question. Let the number of boxes be m, the number of balls per box be n, and the number of red balls per box be k (so $k \leq n$).
Let R_t be the event that a red ball is moved from box t to box $t+1$ mod m (the first box is considered the 0-th box and the last box is box $m-1$, for ease of indexing).
Let A be the event that a red ball is removed from the the first box at the last step. We know:

$$\mathbb{P}(A) = \mathbb{P}(A \mid R_0) \cdot \mathbb{P}(R_0) + \mathbb{P}(A \mid R_0^c) \cdot \mathbb{P}(R_0^c)$$

R_0 is the event that a red ball was removed from the first box in the first step, so $\mathbb{P}(R_0) = \frac{k}{n}$ and $\mathbb{P}(R_0^c) = \frac{n-k}{n}$. Now we need to determine $\mathbb{P}(A \mid R_0)$ and $\mathbb{P}(A \mid R_0^c)$. We know that

$$\mathbb{P}(A \mid R_0) = \mathbb{P}_{R_0}(A \mid R_{m-1}) \cdot \mathbb{P}_{R_0}(R_{m-1}) + \mathbb{P}_{R_0}\left(A \mid R_{m-1}^c\right) \cdot \mathbb{P}_{R_0}\left(R_{m-1}^c\right)$$

We know that $\mathbb{P}_{R_0}(A \mid R_{m-1}) = \frac{k}{n}$ since a red ball was originally removed from the first box (R_0), and then replaced again (R_{m-1}). And similarly $\mathbb{P}_{R_0}\left(A \mid R_{m-1}^c\right) = \frac{k-1}{n}$ since a red ball was originally removed (R_0) and replaced with a black ball (R_{m-1}^c). So:

$$\mathbb{P}(A \mid R_0) = \left(\frac{k}{n} - \frac{k-1}{n}\right) \cdot \mathbb{P}_{R_0}(R_{m-1}) + \frac{k-1}{n} = \frac{1}{n} \cdot \mathbb{P}_{R_0}(R_{m-1}) + \frac{k-1}{n}$$

Similarly, we need to compute $\mathbb{P}(A \mid R_0^c)$:

$$\mathbb{P}(A \mid R_0^c) = \mathbb{P}_{R_0^c}(A \mid R_{m-1}) \cdot \mathbb{P}_{R_0^c}(R_{m-1}) + \mathbb{P}_{R_0^c}\left(A \mid R_{m-1}^c\right) \cdot \mathbb{P}_{R_0^c}(R_{m-1})$$

And we know that $\mathbb{P}_{R_0^c}(A \mid R_{m-1}) = \frac{k+1}{n}$ since a black ball was taken out originally (R_0^c) and replaced with a red ball (R_{m-1}). And $\mathbb{P}_{R_0^c}\left(A \mid R_{m-1}^c\right) = \frac{k}{n}$ since a black ball was removed (R_0^c) and replaced with a black ball (R_{m-1}^c). So:

$$\mathbb{P}(A \mid R_0^c) = \left(\frac{k+1}{n} - \frac{k}{n}\right) \cdot \mathbb{P}_{R_0^c}(R_{m-1}) + \frac{k}{n} = \frac{1}{n} \cdot \mathbb{P}_{R_0^c}(R_{m-1}) + \frac{k}{n}$$

So, by plugging these results back into the original formula:

$$\mathbb{P}(A) = \mathbb{P}(R_0) \cdot \left(\frac{1}{n} \cdot \mathbb{P}_{R_0}(R_{m-1}) + \frac{k-1}{n}\right) + \mathbb{P}(R_0^c) \cdot \left(\frac{1}{n} \cdot \mathbb{P}_{R_0^c}(R_{m-1}) + \frac{k}{n}\right)$$

Since $\mathbb{P}_{R_0}(R_{m-1}) \cdot \mathbb{P}(R_0) = \mathbb{P}(R_{m-1} \cap R_0)$ and $\mathbb{P}(R_0^c) \cdot \mathbb{P}_{R_0^c}(R_{m-1}) = \mathbb{P}(R_{m-1} \cap R_0^c)$:

$$\mathbb{P}(A) = \frac{1}{n}\mathbb{P}(R_{m-1} \cap R_0) + \frac{1}{n}\mathbb{P}(R_{m-1} \cap R_0^c) + \frac{k-1}{n} \cdot \mathbb{P}(R_0) + \frac{k}{n} \cdot \mathbb{P}(R_0^c)$$

And we know that $\mathbb{P}(R_{m-1} \cap R_0) + \mathbb{P}(R_{m-1} \cap R_0^c) = \mathbb{P}(R_{m-1})$, and we'll plug in the probability of R_0 determined above:

$$\mathbb{P}(A) = \frac{1}{n} \cdot \mathbb{P}(R_{m-1}) + \frac{k^2-k}{n^2} + \frac{nk-k^2}{n^2} = \frac{1}{n} \cdot \mathbb{P}(R_{m-1}) + \frac{nk-k}{n^2}$$

We can now create a recursive function for $\mathbb{P}(R_{m-1})$ in order to compute it.

$$\mathbb{P}(R_{m-1}) = \mathbb{P}(R_{m-1} \mid R_{m-2}) \cdot \mathbb{P}(R_{m-2}) + \mathbb{P}\left(R_{m-1} \mid R_{m-2}^c\right) \cdot \mathbb{P}\left(R_{m-2}^c\right)$$

We know that the event R_{m-2} corresponds to the event that a red ball was moved to the box, so there will be $k+1$ red balls and $n+1$ balls total in the box, so in this case the probability of moving a red ball out is $\frac{k+1}{n+1}$. That is to say $\mathbb{P}(R_{m-1} \mid R_{m-2}) = \frac{k+1}{n+1}$.
Similarly, the event R_{m-2}^c means that a black ball was moved. So the number of red balls is k while the total number of balls is $n+1$. So $\mathbb{P}\left(R_{m-1} \mid R_{m-2}^c\right) = \frac{k}{n+1}$. Meaning:

$$\mathbb{P}(R_{m-1}) = \frac{k+1}{n+1} \cdot \mathbb{P}(R_{m-2}) + \frac{k}{n+1} \cdot (1 - \mathbb{P}(R_{m-2})) = \frac{1}{n+1} \cdot \mathbb{P}(R_{m-2}) + \frac{k}{n+1}$$

If we continue to plug in this formula (replacing $\mathbb{P}(R_{m-2})$ with this formula), we get that for every $i \leq m-1$:

$$\mathbb{P}(R_{m-1}) = \frac{1}{(n+1)^i} \cdot \mathbb{P}(R_{m-1-i}) + k \cdot \sum_{j=1}^{i} \frac{1}{(n+1)^j}$$

The right sum is geometric, so we can compute it:

$$\mathbb{P}(R_{m-1}) = \frac{1}{(n+1)^i} \cdot \mathbb{P}(R_{m-1-i}) + k \cdot \frac{(n+1)^i - 1}{n(n+1)^i}$$

But before we take the last step to solve the question, let's first prove that this formula is true. We will prove this through induction on i.

Base case: $i = 0$

For $i = 0$ the formula is:

$$\mathbb{P}(R_{m-1}) = \frac{1}{(n+1)^0} \cdot \mathbb{P}(R_{m-1}) + k \cdot \sum_{j=1}^{0} \frac{1}{(n+1)^j} = \mathbb{P}(R_{m-1})$$

Which is obviously true.

Inductive step:

Assume that this holds for i, we will prove it for $i+1$ (under the assumption $i+1 \leq m-1$). We know:

$$\mathbb{P}(R_{m-1-i}) = \mathbb{P}(R_{m-1-i} \mid R_{m-2-i}) \cdot \mathbb{P}(R_{m-2-i}) + \mathbb{P}\left(R_{m-1-i} \mid R^c_{m-2-i}\right) \cdot \mathbb{P}\left(R^c_{m-2-i}\right)$$

And for reasons similar to those explained before,

$$\mathbb{P}(R_{m-1-i} \mid R_{m-2-i}) = \frac{k+1}{n+1} \text{ and } \mathbb{P}(R_{m-1-i} \mid R_{m-2-i}) = \frac{k}{n+1}$$

So:

$$\mathbb{P}(R_{m-1-i}) = \frac{1}{n+1} \cdot \mathbb{P}(R_{m-2-i}) + \frac{k}{n+1}$$

By our inductive hypothesis:

$$\mathbb{P}(R_{m-1}) = \frac{1}{(n+1)^i} \cdot \mathbb{P}(R_{m-1-i}) + k \cdot \sum_{j=1}^{i} \frac{1}{(n+1)^j}$$

Which we now know is equal to:

$$= \frac{1}{(n+1)^i} \cdot \left(\frac{1}{n+1} \cdot \mathbb{P}(R_{m-2-i}) + \frac{k}{n+1}\right) + k \sum_{j=1}^{i} \frac{1}{(n+1)^j} =$$

$$= \frac{1}{(n+1)^{i+1}} \cdot \mathbb{P}(R_{m-2-i}) + k \left(\sum_{j=1}^{i} \left(\frac{1}{(n+1)^j}\right) + \frac{1}{(n+1)^{i+1}}\right) =$$

$$= \frac{1}{(n+1)^{i+1}} \cdot \mathbb{P}\left(R_{m-1-(i+1)}\right) + k \cdot \sum_{j=1}^{i+1} \frac{1}{(n+1)^j}$$

As required.

If we plug $i = m-1$ into the formula, we get:

$$\mathbb{P}(R_{m-1}) = \frac{1}{(n+1)^{m-1}} \cdot \mathbb{P}(R_0) + k \cdot \frac{(n+1)^{m-1} - 1}{n(n+1)^{m-1}}$$

And since we know $\mathbb{P}(R_0) = \frac{k}{n}$, this is equal to:

$$\frac{k}{n(n+1)^{m-1}} + k \cdot \frac{(n+1)^{m-1} - 1}{n(n+1)^{m-1}} = \frac{k}{n(n+1)^{m-1}} \cdot \left(1 + (n+1)^{m-1} - 1\right) = \frac{k}{n}$$

Plugging this in, we arrive at:

$$\mathbb{P}(A) = \frac{1}{n} \cdot \frac{k}{n} + \frac{nk-k}{n^2} = \frac{k}{n^2} + \frac{nk-k}{n^2} = \frac{nk}{n^2} = \frac{k}{n}$$

So

$$\mathbb{P}(A) = \frac{k}{n}$$

Which is also equal to $\mathbb{P}(R_{m-1})$, which is pretty neat. It's also neat because it's independent of the number of boxes.
Plugging in our values for n and k ($n = 3$ and $k = 1$), we get:

$$\mathbb{P}(A) = \boxed{\frac{1}{3}}$$

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Question 4.5:

Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. Suppose $B_1, B_2 \in \mathcal{F}$ events with non-zero probability, such that $\mathbb{P}(B_1 \triangle B_2) = 0$. Prove that for every event A, $\mathbb{P}(A \mid B_1) = \mathbb{P}(A \mid B_2)$.

Answer:

Firstly, we know that $B_1 \triangle B_2 = (B_1 \setminus B_2) \uplus (B_2 \triangle B_1)$, so:

$$\mathbb{P}(B_1 \setminus B_2) + \mathbb{P}(B_2 \setminus B_1) = 0$$

And since the probability of events are all non-negative, this means

$$\mathbb{P}(B_1 \setminus B_2) = \mathbb{P}(B_1 \setminus B_2) = 0$$

We also know that $\mathbb{P}(B_1 \setminus B_2) = \mathbb{P}(B_1) - \mathbb{P}(B_1 \cap B_2)$, so:

$$\mathbb{P}(B_1) - \mathbb{P}(B_1 \cap B_2) = \mathbb{P}(B_2) - \mathbb{P}(B_1 \cap B_2) = 0 \implies \mathbb{P}(B_1 \cap B_2) = \mathbb{P}(B_1) = \mathbb{P}(B_2)$$

Now, suppose A is an event. Then:

$$\mathbb{P}(A \mid B_1) = \frac{\mathbb{P}(A \cap B_1)}{\mathbb{P}(B_1)}$$

And

$$\mathbb{P}(A \mid B_2) = \frac{\mathbb{P}(A \cap B_2)}{\mathbb{P}(B_2)} = \frac{\mathbb{P}(A \cap B_2)}{\mathbb{P}(B_1)}$$

So it is sufficient (and necessary) to prove that $\mathbb{P}(A \cap B_1) = \mathbb{P}(A \cap B_2)$.
We know that

$$\mathbb{P}(A \cap B_1) = \mathbb{P}(A \cap B_1 \cap B_2) + \mathbb{P}(A \cap B_1 \cap B_2^c) = \mathbb{P}(A \cap B_1 \cap B_2) + \mathbb{P}(A \cap (B_1 \setminus B_2)$$

And since $A \cap (B_1 \setminus B_2) \subseteq B_1 \setminus B_2$ which has probability zero, so does $A \cap (B_1 \setminus B_2)$, so:

$$\mathbb{P}(A \cap B_1) = \mathbb{P}(A \cap B_1 \cap B_2)$$

And similarly

$$\mathbb{P}(A \cap B_2) = \mathbb{P}(A \cap B_1 \cap B_2) + \mathbb{P}(A \cap (B_2 \setminus B_1))$$

And $A \cap (B_2 \setminus B_1) \subseteq B_2 \setminus B_1$, which also has zero probability, so:

$$\mathbb{P}(A \cap B_2) = \mathbb{P}(A \cap B_1 \cap B_2) = \mathbb{P}(A \cap B_1)$$

As required.

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Question 4.6:

$(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. $A, B \in \mathcal{F}$.Prove that the following are equivalent:

(a) $A \perp\!\!\!\perp B$

(b) $A^c \perp\!\!\!\perp B$

(c) $A \perp\!\!\!\perp B^c$

(d) $A^c \perp\!\!\!\perp B^c$

Answer:

(a) $\implies$ ***(b)*** We know that

$$\mathbb{P}(B \cap A^c) = \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

Since A and B are independent, this is equal to

$$\mathbb{P}(B) - \mathbb{P}(A) \cdot \mathbb{P}(B) = (1 - \mathbb{P}(A)) \cdot \mathbb{P}(B) = \mathbb{P}(A^c) \cdot \mathbb{P}(B)$$

So $\mathbb{P}(B \cap A^c) = \mathbb{P}(B) \cdot \mathbb{P}(A^c)$, so A^c and B are independent, as required.

(b) $\implies$ ***(b)*** By our first proof, we know that $A \perp\!\!\!\perp B \implies A^c \perp\!\!\!\perp B$, if we replace A with B and B with A^c, we get

$$A^c \perp\!\!\!\perp B \equiv B \perp\!\!\!\perp A^c \implies B^c \perp\!\!\!\perp A \equiv A \perp\!\!\!\perp B^c$$

As required.

(c) $\implies$ ***(d)*** If we replace B with B^c in the first proof, we get:

$$A \perp\!\!\!\perp B^c \implies A^c \perp\!\!\!\perp B^c$$

As required.

(d) $\implies$ ***(a)*** If we replace A with A^c and B with B^c in the first proof, we get:

$$A^c \perp\!\!\!\perp B^c \implies A \perp\!\!\!\perp B^c$$

And from the second proof, we know:

$$A \perp\!\!\!\perp B^c \equiv B^c \perp\!\!\!\perp A \implies B \perp\!\!\!\perp A \equiv A \perp\!\!\!\perp B$$

So all in all

$$A^c \perp\!\!\!\perp B^c \implies A \perp\!\!\!\perp B$$

As required.

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Question 4.7:

$(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. Given two events A and B, A *confirms* B if $\mathbb{P}(B) < \mathbb{P}(B \mid A)$.
Give an example of a probability space, and three events A, B, and C, such that A confirms B, B confirms C, but A doesn't confirm C.

Answer:

Let's first solve this generally before giving an explicit example.
Suppose A and C are two disjoint events with non-zero probability such that their union has probability of less than 1. We can define:

$$B := A \cupdot C$$

This means that $A \cap B = A$ and $C \cap B = C$. Also since B is the union of two non-zero probability events and by the given, $0 < \mathbb{P}(B) < 1$.
Then:

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A)}{\mathbb{P}(A)} = 1 > \mathbb{P}(B)$$

So A confirms B.
And:

$$\mathbb{P}(C \mid B) = \frac{\mathbb{P}(B \cap C)}{\mathbb{P}(B)} = \frac{\mathbb{P}(C)}{\mathbb{P}(B)}$$

And since $\mathbb{P}(B) < 1 \implies \frac{1}{\mathbb{P}(B)} > 1$. Which means $\frac{\mathbb{P}(C)}{\mathbb{P}(B)} > \mathbb{P}(C)$, so

$$\mathbb{P}(C \mid B) > \mathbb{P}(C)$$

Which means B confirms C.
But:

$$\mathbb{P}(C \mid A) = \frac{\mathbb{P}(A \cap C)}{\mathbb{P}(A)} = \frac{\mathbb{P}(\varnothing)}{\mathbb{P}(A)} = 0$$

Which is less than $\mathbb{P}(C)$, so A does not confirm C, as required.
Now let's find an example of a probability space with events that satisfy these properties.
Let $\Omega = [3]$, and let $A = \{1\}$, $C = \{2\}$, $B = A \cupdot C = \{1, 2\}$. And let $\mathbb{P}$ be a uniform probability function, ie:

$$\mathbb{P}(X) = \frac{|X|}{|\Omega|}$$

This satisfies the properties above since $\mathbb{P}(A), \mathbb{P}(C) \neq 0$ and $B = A \cupdot C$ and $\mathbb{P}(B) < 1$. So these events and probability space satisfies the requirements of the question.

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Question 4.8:

A cup holds a black ball and a white ball. We repeatedly choose a random ball from the cup and add another ball of the same color. Prove that for every $1 \le n \le t+1$, the probability of there being n white balls after t steps is $\frac{1}{t+1}$.

Answer:

Let the event that there are n white balls after t steps be denoted by A_n^t. We need to prove

$$\mathbb{P}\left(A_n^t\right) = \frac{1}{1+t}$$

We will prove this by induction on t.

Base case: $t = 0$

In this case, no balls have been doubled, and we are in the inital state. We need to prove that for $n = 1$ the probability of there being n balls is $\frac{1}{t+1} = 1$. This is obviously true, because we know that initially there is only **one** white ball and one black ball.

Inductive step:

Assume this is true for t, we must prove it holds true for $t+1$. Let $1 \le n \le t+2$. We know that $\{A_n^t\}_{n=1}^{t+1}$ is a set of disjoint unions that covers the sample space, so:

$$\mathbb{P}\left(A_n^{t+1}\right) = \sum_{i=1}^{t+1} \mathbb{P}\left(A_n^{t+1} \mid A_i^t\right) \cdot \mathbb{P}\left(A_i^t\right)$$

But we know that for every $i \neq n, n-1$, $A_n^{t+1} \cap A_i^t = \varnothing$ as if there aren't n or $n-1$ white balls at step $t-1$, there cannot be n white balls at step $t+1$ (as we're only adding 1 or no white balls each step). So:

$$\mathbb{P}\left(A_n^{t+1}\right) = \mathbb{P}\left(A_n^{t+1} \mid A_n^t\right) \cdot \mathbb{P}\left(A_n^t\right) + \mathbb{P}\left(A_n^{t+1} \mid A_{n-1}^t\right) \cdot \mathbb{P}\left(A_{n-1}^t\right)$$

Case 1: $n = t+2$

If this is the case, then $A_n^t = \varnothing$ because you can't have $t+2$ white balls after t steps, since after t steps there are $t+2$ balls (since we're adding one ball each time), and there must be at least one black ball. So:

$$\mathbb{P}\left(A_{t+2}^{t+1}\right) = \mathbb{P}\left(A_{t+2}^{t+1} \mid A_{t+1}^t\right) \cdot \mathbb{P}\left(A_{t+1}^t\right)$$

The probability of there being $t+2$ white balls after $t+1$ steps given that there were $t+1$ white balls after t steps means that a white ball must have been chosen during the $t+1$-th step. We know that there were $t+1$ white balls before the $t+1$-th step and $t+2$ balls in total, so the probability of choosing a white ball is $\mathbb{P}\left(A_{t+2}^{t+1} \mid A_{t+1}^t\right) = \frac{t+1}{t+2}$.

By our inductive assumption, we also know that $\mathbb{P}\left(A_{t+1}^t\right) = \frac{1}{t+1}$. So:

$$\mathbb{P}\left(A_{t+2}^{t+1}\right) = \frac{t+1}{t+2} \cdot \frac{1}{t+1} = \frac{1}{t+2}$$

As required.

Case 2: $n < t+2$

If there are n white balls after $t+1$ steps if there were n after t steps, that means that during the $t+1$-th step, a black ball must have been chosen. There are $t+2-n$ black balls after t steps, and $t+2$ in total, so the probability of this happening is:

$$\mathbb{P}\left(A_n^{t+1} \mid A_n^t\right) = \frac{t+2-n}{t+2}$$

And if there are n white balls after $t+1$ steps if there were $n-1$ after t steps, that means during the $t+1$-th step, a white ball must've been taken. Since there were $n-1$ white balls and $t+2$ balls in total, the probability of this happening is:

$$\mathbb{P}\left(A_n^{t+1} \mid A_{n-1}^t\right) = \frac{n-1}{t+2}$$

And by our inductive assumption:

$$\mathbb{P}\left(A_{n-1}^{t}\right) = \mathbb{P}\left(A_{n}^{t}\right) = \frac{1}{t+1}$$

So all in all:

$$\mathbb{P}\left(A_{n}^{t+1}\right) = \frac{1}{t+1} \cdot \left(\frac{t+2-n}{t+2} + \frac{n-1}{t+2}\right) = \frac{1}{t+1} \cdot \frac{t+1}{t+2} = \frac{1}{t+2}$$

As required.

So for any $1 \leq n \leq t+2$:

$$\mathbb{P}\left(A_{n}^{t+1}\right) = \frac{1}{t+2}$$

As required.

■