Linear Reduction

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In this paper we will define the concept of linear reduction in the context of syntax parsing. We will progress through more and more complicated examples, beginning from the programming of a simple calculator until we ultimately have created an extensible programming language.

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0 Notation

$\mathbb{N}$ denotes the set of natural numbers, including 0.

$\overline{\mathbb{N}}$ is defined to be $\mathbb{N} \cup \{\infty\}$.

$f : A \longrightarrow B$ means that f is a partial function from A to B.

If X is a set and x is some symbol, then $\mathsf{X}_\mathsf{x} = \mathsf{X}^\mathsf{x} = \mathsf{X} \cup \{\mathsf{x}\}$.

1 Theoretical Background

1.1 Stateless Reduction

The idea of linear reduction is simple: given a string ξ the first character looks if it can bind with the second character to produce a new character, and the process repeats itself. There is of course, nuance. This nuance hides in the statement "if it can bind": we must define the rules for binding.

Let us define an *reducer* to be a tuple (Σ, β, π) where Σ is an alphabet; $\beta \colon \overline{\Sigma} \times \overline{\Sigma} \longrightarrow \overline{\Sigma}$ is a partial function called the *reduction function* where $\overline{\Sigma} = \Sigma \times \overline{\mathbb{N}}$; and π is the *initial priority function*. A *program* over an reducer is a string over $\overline{\Sigma}$. We write a program like $\sigma_{i_1}^1 \cdots \sigma_{i_n}^n$ instead of as pairs $(\sigma^1, i_1) \ldots (\sigma^n, i_n)$. In the character σ_i, we call i the *priority* of σ.

Then the rules of reduction are as follows, meaning we define $\beta(\xi)$ for a program: We do so in cases:

(**1**)   If $\xi = \sigma_i$ then $\beta(\xi) = \sigma_0$.

(**2**)   If $\xi = \sigma_i^1 \sigma_j^2 \xi'$ where $i \geq j$ and $\beta(\sigma_i^1, \sigma_j^2) = \sigma_k^3$ is defined then $\beta(\xi) = \sigma_k^3 \xi'$.

(**3**)   Otherwise, for $\xi = \sigma_i^1 \sigma_j^2 \xi'$, $\beta(\xi) = \sigma_i^1 \beta(\sigma_j^2 \xi')$.

A string ξ such that $\beta(\xi) = \xi$ is called *irreducible*. Notice that it is possible for a string of length more than 1 to be irreducible: for example if $\beta(\sigma^1, \sigma^2)$ is not defined then $\sigma_i^1 \sigma_j^2$ is irreducible.

$$\beta(\sigma_1 \tau_2) \xrightarrow{(3)} \sigma_1 \beta(\tau_2) \xrightarrow{(1)} \sigma_1 \tau_2$$

But such strings are not desired, since in the end we'd like a string to give us a value. So an irreducible string which is not a single character is called *ill-written*, and a string which is not ill-written is *well-written*.

Now the initial priority function is $\pi \colon \Sigma \longrightarrow \overline{\mathbb{N}}$ which gives characters their initial priority. We can then canonically extend this to a function $\pi \colon \Sigma^* \longrightarrow (\Sigma \times \overline{\mathbb{N}})^*$ defined by $\pi(\sigma^1 \cdots \sigma^n) = \sigma_{\pi(\sigma^1)}^1 \cdots \sigma_{\pi(\sigma^n)}^n$. Then a β-reduction of a string $\xi \in \Sigma^*$ is taken to mean a β-reduction of $\pi(\xi)$.

Notice that once again we require that π only be a partial function. This is since that we don't always need every character in Σ to have an initial priority; some symbols are only given their priority through the β-reduction of another pair of symbols. So we now provide a new definition of a *program*, which is a string $\xi = \sigma^1 \cdots \sigma^n \in \Sigma^*$ such that $\pi(\sigma^i)$ exists for all $1 \leq i \leq n$. We can only of course discuss the reductions of programs, as $\pi(\xi)$ is only defined if ξ is a program.

Example: let $\Sigma = \mathbb{N} \cup \{+, \cdot\} \cup \{(n+), (n\cdot) \mid n \in \mathbb{N}\}$. β as follows:

σ_i^1, σ_j^2	$\beta(\sigma_i^1, \sigma_j^2)$
$n, +$	$(n+)$
$n, \cdot$	$(n\cdot)$
$(n+), m$	$n + m$
$(n\cdot), m$	$n \cdot m$
$(n\cdot), (m+)$	$(n \cdot m, +)$
$(n+), (m+)$	$(n + m, +)$
$(n\cdot), (m\cdot)$	$(n \cdot m, \cdot)$

Where n, m range over all values in $\mathbb{N}$. Here $\beta(\sigma_i, \sigma_j)$'s priority is j. We define the initial priorities

$$\pi(n) = \infty, \quad \pi(+) = 1, \quad \pi(\cdot) = 2$$

Now let us look at the string $1 + 2 \cdot 3 + 4$;. Here,

$$
\begin{aligned}
1_\infty +_1 2_\infty \cdot_2 3_\infty +_1 4_\infty &\longrightarrow (1+)_1 2_\infty \cdot_2 3_\infty +_1 4_\infty \\
&\longrightarrow (1+)_1 (2\cdot)_2 3_\infty +_1 4_\infty \\
&\longrightarrow (1+)_1 (2\cdot)_2 (3+)_1 4_\infty \\
&\longrightarrow (1+)_1 (6+)_1 4_\infty \\
&\longrightarrow (7+)_1 4_\infty \\
&\longrightarrow (7+)_1 4_0 \\
&\longrightarrow (11)_0
\end{aligned}
$$

So the rules for β we supplied seem to be sufficient for computing arithmetic expressions following the order of operations. $\diamond$

Example: We can also expand our language to include parentheses. So our alphabet becomes $\Sigma = \mathbb{N} \cup \{+, \cdot, (,)\} \cup \left\{ \underline{n+}, \underline{n\cdot}, \underline{n)} \mid n \in \mathbb{N} \right\}$. We distinguish between parentheses and bold parentheses for readability. We extend β as follows:

σ_i^1, σ_j^2	$\beta(\sigma_i^1, \sigma_j^2)$
$n, +$	$\underline{n+}_j$
$n, \cdot$	$\underline{n\cdot}_j$
$\underline{n+}, m$	$(n+m)_j$
$\underline{n\cdot}, m$	$(n \cdot m)_j$
$\underline{n\cdot}, m+$	$\underline{n \cdot m, +}_j$
$\underline{n+}, \underline{m+}$	$\underline{n + m, +}_j$
$\underline{n\cdot}, \underline{m\cdot}$	$\underline{n \cdot m, \cdot}_j$
$n,)$	$\underline{n)}_j$
$\underline{n+}, \underline{m)}$	$\underline{n + m)}_j$
$\underline{n\cdot}, \underline{m)}$	$\underline{n \cdot m)}_j$
$(, \underline{n)}$	n_i

$(n+m)_j$ means $n+m$ with a priority of j, not $\underline{n+m}_j$. And we define the initial priorities

$$\pi(n) = \infty, \quad \pi(+) = 1, \quad \pi(\cdot) = 2, \quad \pi(() = \infty, \quad \pi()) = 0$$

So for example reducing $2 \cdot ((1 + 2) \cdot 2) + 1$,

$$2_\infty *_2 (_\infty (_\infty 1_\infty +_1 2_\infty)_0 *_2 2_\infty)_0 +_1 1_\infty \longrightarrow \underline{2*}_2 (_\infty (_\infty 1_\infty +_1 2_\infty)_0 *_2 2_\infty)_0 +_1 1_\infty$$
$$\longrightarrow \underline{2*}_2 (_\infty (_\infty \underline{1+}_1 2_\infty)_0 *_2 2_\infty)_0 +_1 1_\infty$$
$$\longrightarrow \underline{2*}_2 (_\infty (_\infty \underline{1+_1 2)}_0 *_2 2_\infty)_0 +_1 1_\infty$$
$$\longrightarrow \underline{2*}_2 (_\infty (_\infty \underline{3)}_0 *_2 2_\infty)_0 +_1 1_\infty$$
$$\longrightarrow \underline{2*}_2 (_\infty 3_\infty *_2 2_\infty)_0 +_1 1_\infty$$
$$\longrightarrow \underline{2*}_2 (_\infty \underline{3*}_2 2_\infty)_0 +_1 1_\infty$$
$$\longrightarrow \underline{2*}_2 (_\infty \underline{3*_2 2)}_0 +_1 1_\infty$$
$$\longrightarrow \underline{2*}_2 (_\infty \underline{6)}_0 +_1 1_\infty$$
$$\longrightarrow \underline{2*}_2 6_\infty +_1 1_\infty$$
$$\longrightarrow \underline{2*}_2 \underline{6+}_1 1_\infty$$
$$\longrightarrow \underline{12+}_1 1_\infty$$
$$\longrightarrow \underline{12+}_1 1_0$$
$$\longrightarrow 13_0$$

$\Diamond$

1.2 Stateful Reduction

Suppose we'd like to reduce a program with variables in it. Then we cannot just use the previous definitions, as the actions of σ (which is to be understood as the function $\beta(\sigma, \bullet)$) are determined before any reduction occurs. We need a way to store the value of variables, a state.

This leads us to the following definition: let Σ_P and Σ_A be two disjoint sets of symbols: Σ_P the set of *printable symbols* and Σ_A the set of *abstract symbols*. Σ_P will generally be a set consisting of the string representations of abstract symbols, be it operators like $+$ and $\cdot$ or variable names. Σ_A are the actual objects which can "execute something". Let us further define $\Sigma = \Sigma_P \cup \Sigma_A$.

Now a state is a mapping from printable symbols to strings. So for example, if x is a printable symbol a line like `let` $x = 1$ should change the state so that x maps to the abstract symbol representing 1.

A *point state* is a partial function $s \colon \Sigma_P \longrightarrow \Sigma_A$. If s_1, s_2 are point states, define their composition to be a point state $s_1 s_2$ such that

$$s_1 s_2(\sigma) = \begin{cases} s_2(\sigma) & \sigma \in \mathrm{dom}(s_2) \\ s_1(\sigma) & \sigma \in \mathrm{dom}(s_1) \end{cases}$$

A *state* is a sequence of point states: $\bar{s} = (s_1, \ldots, s_n)$. Let us define

$$\mathsf{State} = \{\Sigma_P \longrightarrow \Sigma_A\}^+$$

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the set of all states.

Let $\overline{s} = (s_1 \cdots s_n) \in \mathsf{State}$ be a state, then define

- for $\sigma \in \Sigma_P$ we define $s(\sigma) = s_1 \cdots s_n(\sigma)$ (the composition of states),
- define $pop\ \overline{s} = (s_1, \ldots, s_{n-1})$,
- define $push\ \overline{s} = (s_1, \ldots, s_n, \varnothing)$ ($\varnothing$ is the empty state),
- if s is a point state, $\overline{s}s = (s_1, \ldots, s_{n-1}, s_n s)$,
- if s is a point state, $\overline{s} + s = (s_1, \ldots, s_n, s)$ (so $push\ \overline{s} = \overline{s} + \varnothing$).

So if we'd like to revert to a previous state, we simply pop from the current state. And substituting the current state only alters the current (topmost) point state.

Now we begin with an initial β function which is a partial function

$$\beta \colon \overline{\Sigma}_A \times (\overline{\Sigma} \cup \{\varepsilon\}) \times \mathsf{State} \longrightarrow \overline{\Sigma}^* \times \mathsf{State}$$

Recall that $\overline{\Sigma}$ is $\Sigma \times \mathbb{N}$. We will denote tuples in $X \times \mathsf{State}$ by $\langle x, \mid s \rangle$ for $x \in X$ and $s \in \mathsf{State}$ for the sake of readability. So we now wish to extend to a β function

$$\beta \colon \overline{\Sigma}^* \times \mathsf{State} \longrightarrow \overline{\Sigma}^* \times \mathsf{State}$$

We do this as follows: given $\xi \in (\Sigma \times \overline{\mathbb{N}})^*$ and $s \in \mathsf{State}$ we define $\beta \langle \xi \mid s \rangle$ as follows:

(**1**) if $\xi = \sigma_i \xi'$ for $\sigma \in \Sigma_P$ then $\beta \langle \xi \mid s \rangle = \langle s(\sigma)_i \xi' \mid s \rangle$,

(**2**) if $\xi = \sigma_i \xi'$ such that $\beta \langle \sigma_i \varepsilon \mid s \rangle = \langle \xi'' \mid s' \rangle$ is defined then $\beta \langle \xi \mid s \rangle = \langle \xi'' \xi' \mid s' \rangle$,

(**3**) if $\xi = \sigma_i^1 \sigma_j^2 \xi'$ for $\sigma^1 \in \Sigma_A$, $i \geq j$, such that $\beta \langle \sigma_i^1 \sigma_j^2 \mid s \rangle = \langle \xi'' \mid s' \rangle$ is defined, then $\beta \langle \xi \mid s \rangle = \langle \xi'' \xi' \mid s' \rangle$,

(**4**) otherwise for $\xi = \sigma_i^1 \sigma_j^2 \xi'$, if $\beta \langle \sigma_j^2 \xi' \mid s \rangle = \langle \xi'' \mid s' \rangle$ then $\beta \langle \xi \mid s \rangle = \langle \sigma_i^1 \xi'' \mid s' \rangle$.

Notice that (**2**) cares not about the priority of σ, and neither if $\beta(\sigma_i, \tau_j)$ is defined for some $\tau \neq \varepsilon$.

We also define the *initial priority function* to be a map $\pi \colon \Sigma_P \longrightarrow \overline{\mathbb{N}}$ (this is not a partial function: every printable symbol must be given a priority). This is once again canonically extended to a function $\pi \colon \Sigma_P^* \longrightarrow (\Sigma_P \times \overline{\mathbb{N}})^*$. And an *initial state* s_0 which is a point state. The quintuple $(\Sigma_P, \Sigma_A, \beta, \pi, s_0)$ is called an *reducer*. The reduction of a string $\xi \in S$ is the process of iteratively applying β to $\langle \pi(\xi) \mid s_0 \rangle$.

Example: let

$$\Sigma_P = \mathbb{N} \cup \{+, \cdot, =, ;\} \cup \{\texttt{let}\} \cup \{x^i \mid i \in \mathbb{N}\},$$
$$\Sigma_A = \mathbb{N} \cup \{(n+), (n\cdot) \mid n \in \mathbb{N}\} \cup \{\texttt{let}\} \cup \{(\texttt{let}x^i), (\texttt{let}x^i =) \mid i \in \mathbb{N}\}$$

where the natural numbers in Σ_A are not the same as the natural numbers in Σ_P since they must be disjoint, same for $\texttt{let}$. But they both essentially represent the same thing: s_0 maps $n \mapsto n$ for $n \in \mathbb{N}$ (the left-hand n is in Σ_p, the right-hand n is in Σ_A) and $\texttt{let} \mapsto \texttt{let}$. All other printable symbols are mapped to ε.

And similar to the previous example we define $\pi(n) = \infty$, $\pi(+) = 1$, and $\pi(\cdot) = 2$. We extend this to $\pi(;) = 0$, $\pi(=) = 0$, $\pi(\texttt{let}) = \infty$, and $\pi(x^i) = \infty$.

Let us take the same transitions as the example in the previous section for $n, (n+), (n\cdot)$ (we have to add the condition that the state doesnt change). We further add the transitions

$$\frac{\langle \sigma_i^1 \sigma_j^2 \mid s \rangle \qquad\qquad \beta \langle \sigma^1 \sigma^2 \mid s \rangle}{\begin{array}{cc} \langle \sigma; \mid s \rangle & \langle \sigma_j \mid s \rangle \\ \langle \texttt{let}\ x^i \mid s \rangle & \langle (\texttt{let}\ x^i)_j \mid s \rangle \\ \langle (\texttt{let}\ x^i) = \mid s \rangle & \langle (\texttt{let}\ x^i =)_j \mid s \rangle \\ \langle (\texttt{let}\ x^i =)\sigma \mid s \rangle & \langle \varepsilon \mid s[x^i \mapsto \sigma] \rangle \end{array}}$$

In the final transition, $n \in \Sigma_A$. Then for example (we will be skipping trivial reductions):

$$\texttt{let}\ x^1 = 1 + 2;\ \texttt{let}\ x^2 = 2;\ x^1 \cdot x^2; \longrightarrow \texttt{let}_\infty x_\infty^1 =_0 1_\infty +_1 2_{\infty;0}\ \texttt{let}_\infty x_\infty^2 =_0 2_{\infty;0}\ x_\infty^1 \cdot_2 x_{\infty;0}^2$$

$$s_0 \longrightarrow (\texttt{let}x^1 =)_0 1_\infty +_1 2_{\infty;0}\ \texttt{let}_\infty x_\infty^2 =_0 2_{\infty;0}\ x_\infty^1 \cdot_2 x_{\infty;0}^2$$

$$s_0 \longrightarrow (\texttt{let}x^1 =)_0 3_0\ \texttt{let}_\infty x_\infty^2 =_0 2_{\infty;0}\ x_\infty^1 \cdot_2 x_{\infty;0}^2$$

$$s_0[x^1 \mapsto 3] \longrightarrow \texttt{let}_\infty x_\infty^2 =_0 2_{\infty;0}\ x_\infty^1 \cdot_2 x_{\infty;0}^2$$

$$s_0[x^1 \mapsto 3,\ x^2 \mapsto 2] \longrightarrow x_\infty^1 \cdot_2 x_{\infty;0}^2$$

$$s_0[x^1 \mapsto 3,\ x^2 \mapsto 2] \longrightarrow 3_\infty \cdot_2 x_{\infty;0}^2$$

$$s_0[x^1 \mapsto 3,\ x^2 \mapsto 2] \longrightarrow (3\cdot)_2 x_{\infty;0}^2$$

$$s_0[x^1 \mapsto 3,\ x^2 \mapsto 2] \longrightarrow (3\cdot)_2 2_{\infty;0}$$

$$s_0[x^1 \mapsto 3,\ x^2 \mapsto 2] \longrightarrow 6_0$$

1.3 Valued Reduction

Notice that in the previous examples, many symbols had values, be it a number, a list, etc. This leads us to the next variation of linear reduction: *valued* linear reduction. Here we start with the sets: $\mathcal{U}$ the universe of values, Σ_P the set of printable types, and Σ_A the set of abstract types.

Here $\mathcal{U}$ is some arbitrary set, it will contain of all possible values (numbers, lists, strings, functions, etc.) in our program. Σ_P and Σ_A are sets of symbols which we call *types*.

Let us further define $\Pi_A := \Sigma_A \times \mathcal{U}$ to be the set of *abstract values*, $\Sigma := \Sigma_P \cup \Sigma_A$ the set of types, and $\Pi = \Sigma_P \cup \Pi_A$ the set of *values* (printable types are also values).

Let us define a point-state, similar to stateful reductions, but now the state of a printable type can also be a sequence of printable types. This is so we can define functions: we can allow a symbol to expand to the code of a function. So a point-state is a partial function $\Sigma_P \longrightarrow (\Sigma_A \cup \Sigma_P^*)$, and a state is a sequence of point-states. Define State to be the set of states. *push* , *pop* ,etc. is defined as before.

Now we define a *initial β-reducer* to be

$$\hat{\beta} \colon \overline{\Sigma}_A \times \overline{\Sigma}_\varepsilon \times \text{State} \longrightarrow \overline{\Sigma}^* \times (\mathcal{U} \times \mathcal{U}_\varnothing \longrightarrow \mathcal{U}_\varnothing^*) \times \text{State}$$

The idea is that given two types, you create a new sequence of types, a value function f, and a new state. If the values of the original two types are v, u then the values of the new sequence of types will be $f(v, u)$. Here we consider the value of a printable type to be $\varnothing$.

For readability, an element of $\overline{\Pi}$ will be written as $\sigma_n(v)$ for a type of σ, a priority of n, and a value of v ($v = \varnothing$ if $\sigma \in \Sigma_P$). And for $\xi = \sigma_{n_1}^1 \cdots \sigma_{n_k}^k \in \overline{\Sigma}^*$ and $\vec{v} = v_1 \cdots v_k \in \mathcal{U}_\varnothing^*$, define $\xi(\vec{v})$ to be $\sigma_{n_1}^1(v_1) \cdots \sigma_{n_k}^k(v_k)$.

A *β-reducer* is a partial function

$$\beta \colon \overline{\Pi}^* \times \text{State} \longrightarrow \overline{\Pi}^* \times \text{State}$$

The β-reducer *derived* from an initial β-reducer $\hat{\beta}$ is defined recursively as follows: for $\langle \xi \mid s \rangle \in \overline{\Pi}^* \times \text{State}$,

(**1**) If $\xi = \sigma_n \xi'$ where $\sigma \in \Sigma_P$ is a printable symbol, then:

  (**1**) if $s(\sigma) = \tau \in \Sigma_A$ then $\langle \tau_n \xi' \mid s \rangle$,

  (**2**) otherwise, $s(\sigma) = \xi''$ then $\langle \pi(\xi'') \xi' \mid s \rangle$.

(**2**) If $\xi = \sigma_n(v) \xi'$ where $\hat{\beta}(\sigma, \varepsilon, s) = (\xi'', f, s')$ then $\langle \xi''(f(v)) \xi' \mid s \rangle$.

(**3**) If $\xi = \sigma_n(v) \tau_m(u) \xi'$ where $n \geq m$ and $\hat{\beta}(\sigma, \tau, s) = (\xi'', f, s')$ then $\langle \xi''(f(v)) \xi' \mid s \rangle$.

(**4**) Otherwise $\xi = \sigma_n(v) \tau_m(u) \xi'$ and $n < m$ or $\hat{\beta}(\sigma, \tau, s)$ is undefined, then if $\beta \langle \tau_m(u) \xi' \mid s \rangle = \langle \xi'', s' \rangle$, then $\langle \sigma_n(v) \xi'' \mid s' \rangle$.