Linear Reduction

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In this paper we will define the concept of linear reduction in the context of syntax parsing. We will progress through more and more complicated examples, beginning from the programming of a simple calculator until we ultimately have created an extensible programming language.

Table of Contents

0 Notation

$\mathbb{N}$ denotes the set of natural numbers, including 0.

$\overline{\mathbb{N}}$ is defined to be $\mathbb{N} \cup \{\infty\}$.

$f\colon A \longrightarrow B$ means that f is a partial function from A to B.

If X is a set and x is some symbol, then $\mathsf{X}_\mathsf{x} = \mathsf{X}^\mathsf{x} = \mathsf{X} \cup \{\mathsf{x}\}$.

1 Theoretical Background

1.1 Stateless Reduction

The idea of linear reduction is simple: given a string ξ the first character looks if it can bind with the second character to produce a new character, and the process repeats itself. There is of course, nuance. This nuance hides in the statement "if it can bind": we must define the rules for binding.

Let us define an *reducer* to be a tuple (Σ, β, π) where Σ is an alphabet; $\beta\colon \overline{\Sigma} \times \overline{\Sigma} \longrightarrow \overline{\Sigma}$ is a partial function called the *reduction function* where $\overline{\Sigma} = \Sigma \times \overline{\mathbb{N}}$; and π is the *initial priority function.* A *program* over an reducer is a string over $\overline{\Sigma}$. We write a program like $\sigma^1_{i_1} \cdots \sigma^n_{i_n}$ instead of as pairs $(\sigma^1, i_1) \dots (\sigma^n, i_n)$. In the character σ_i, we call i the *priority* of σ.

Then the rules of reduction are as follows, meaning we define $\beta(\xi)$ for a program: We do so in cases:

(**1**) If $\xi = \sigma_i$ then $\beta(\xi) = \sigma_0$.

(**2**) If $\xi = \sigma^1_i \sigma^2_j \xi'$ where $i \geq j$ and $\beta(\sigma^1_i, \sigma^2_j) = \sigma^3_k$ is defined then $\beta(\xi) = \sigma^3_k \xi'$.

(**3**) Otherwise, for $\xi = \sigma^1_i \sigma^2_j \xi'$, $\beta(\xi) = \sigma^1_i \beta(\sigma^2_j \xi')$.

A string ξ such that $\beta(\xi) = \xi$ is called *irreducible.* Notice that it is possible for a string of length more than 1 to be irreducible: for example if $\beta(\sigma^1, \sigma^2)$ is not defined then $\sigma^1_i \sigma^2_j$ is irreducible.

$$\beta(\sigma_1 \tau_2) \xrightarrow{(3)} \sigma_1 \beta(\tau_2) \xrightarrow{(1)} \sigma_1 \tau_2$$

But such strings are not desired, since in the end we'd like a string to give us a value. So an irreducible string which is not a single character is called *ill-written*, and a string which is not ill-written is *well-written.*

Now the initial priority function is $\pi\colon \Sigma \longrightarrow \overline{\mathbb{N}}$ which gives characters their initial priority. We can then canonically extend this to a function $\pi\colon \Sigma^* \longrightarrow (\Sigma \times \overline{\mathbb{N}})^*$ defined by $\pi(\sigma^1 \cdots \sigma^n) = \sigma^1_{\pi(\sigma^1)} \cdots \sigma^n_{\pi(\sigma^n)}$. Then a β-reduction of a string $\xi \in \Sigma^*$ is taken to mean a β-reduction of $\pi(\xi)$.

Notice that once again we require that π only be a partial function. This is since that we don't always need every character in Σ to have an initial priority; some symbols are only given their priority through the β-reduction of another pair of symbols. So we now provide a new definition of a *program*, which is a string $\xi = \sigma^1 \cdots \sigma^n \in \Sigma^*$ such that $\pi(\sigma^i)$ exists for all $1 \leq i \leq n$. We can only of course discuss the reductions of programs, as $\pi(\xi)$ is only defined if ξ is a program.

Example: let $\Sigma = \mathbb{N} \cup \{+, \cdot\} \cup \{(n+), (n\cdot) \mid n \in \mathbb{N}\}$. β as follows:

σ^1_i, σ^2_j	$\beta(\sigma^1_i, \sigma^2_j)$
$n, +$	$(n+)$
$n, \cdot$	$(n\cdot)$
$(n+), m$	$n + m$
$(n\cdot), m$	$n \cdot m$
$(n\cdot), (m+)$	$(n \cdot m, +)$
$(n+), (m+)$	$(n + m, +)$
$(n\cdot), (m\cdot)$	$(n \cdot m, \cdot)$

Where n, m range over all values in $\mathbb{N}$. Here $\beta(\sigma_i, \sigma_j)$'s priority is j. We define the initial priorities

$$\pi(n) = \infty, \quad \pi(+) = 1, \quad \pi(\cdot) = 2$$

Now let us look at the string $1 + 2 \cdot 3 + 4$;. Here,

$$\begin{aligned} 1_\infty +_1 2_\infty \cdot_2 3_\infty +_1 4_\infty &\longrightarrow (1+)_1 2_\infty \cdot_2 3_\infty +_1 4_\infty \\ &\longrightarrow (1+)_1 (2\cdot)_2 3_\infty +_1 4_\infty \\ &\longrightarrow (1+)_1 (2\cdot)_2 (3+)_1 4_\infty \\ &\longrightarrow (1+)_1 (6+)_1 4_\infty \\ &\longrightarrow (7+)_1 4_\infty \\ &\longrightarrow (7+)_1 4_0 \\ &\longrightarrow (11)_0 \end{aligned}$$

So the rules for β we supplied seem to be sufficient for computing arithmetic expressions following the order of operations. ◊

Example: We can also expand our language to include parentheses. So our alphabet becomes $\Sigma = \mathbb{N} \cup \{+, \cdot, (,)\} \cup \left\{ \underline{n+}, \underline{n\cdot}, \underline{n)} \;\middle|\; n \in \mathbb{N} \right\}$. We distinguish between parentheses and bold parentheses for readability. We extend β as follows:

σ_i^1, σ_j^2	$\beta(\sigma_i^1, \sigma_j^2)$
$n, +$	$\underline{n+}_j$
$n, \cdot$	$\underline{n\cdot}_j$
$\underline{n+}, m$	$(n+m)_j$
$\underline{n\cdot}, m$	$(n \cdot m)_j$
$\underline{n\cdot}, \underline{m+}$	$\underline{n \cdot m, +}_j$
$\underline{n+}, \underline{m+}$	$\underline{n + m, +}_j$
$\underline{n\cdot}, \underline{m\cdot}$	$\underline{n \cdot m, \cdot}_j$
$n,)$	$\underline{n)}_j$
$\underline{n+}, \underline{m)}$	$\underline{n+m)}_j$
$\underline{n\cdot}, \underline{m)}$	$\underline{n \cdot m)}_j$
$(, \underline{n)}$	n_i

$(n+m)_j$ means $n+m$ with a priority of j, not $\underline{n+m}_j$. And we define the initial priorities

$$\pi(n) = \infty, \quad \pi(+) = 1, \quad \pi(\cdot) = 2, \quad \pi(() = \infty, \quad \pi()) = 0$$

So for example reducing $2 \cdot ((1+2) \cdot 2) + 1$,

$$\begin{aligned}
2_\infty *_2 (_\infty (_\infty 1_\infty +_1 2_\infty)_0 *_2 2_\infty)_0 +_1 1_\infty &\longrightarrow \underline{2*}_2 (_\infty (_\infty 1_\infty +_1 2_\infty)_0 *_2 2_\infty)_0 +_1 1_\infty \\
&\longrightarrow \underline{2*}_2 (_\infty (_\infty \underline{1+}_1 2_\infty)_0 *_2 2_\infty)_0 +_1 1_\infty \\
&\longrightarrow \underline{2*}_2 (_\infty (_\infty \underline{1+}_1 \underline{2)}_0 *_2 2_\infty)_0 +_1 1_\infty \\
&\longrightarrow \underline{2*}_2 (_\infty (_\infty \underline{3)}_0 *_2 2_\infty)_0 +_1 1_\infty \\
&\longrightarrow \underline{2*}_2 (_\infty 3_\infty *_2 2_\infty)_0 +_1 1_\infty \\
&\longrightarrow \underline{2*}_2 (_\infty \underline{3*}_2 2_\infty)_0 +_1 1_\infty \\
&\longrightarrow \underline{2*}_2 (_\infty \underline{3*}_2 \underline{2)}_0 +_1 1_\infty \\
&\longrightarrow \underline{2*}_2 (_\infty \underline{6)}_0 +_1 1_\infty \\
&\longrightarrow \underline{2*}_2 6_\infty +_1 1_\infty \\
&\longrightarrow \underline{2*}_2 \underline{6+}_1 1_\infty \\
&\longrightarrow \underline{12+}_1 1_\infty \\
&\longrightarrow \underline{12+}_1 1_0 \\
&\longrightarrow 13_0
\end{aligned}$$

$\diamondsuit$

1.2 Stateful Reduction

Suppose we'd like to reduce a program with variables in it. Then we cannot just use the previous definitions, as the actions of σ (which is to be understood as the function $\beta(\sigma, \bullet)$) are determined before any reduction occurs. We need a way to store the value of variables, a state.

This leads us to the following definition: let Σ_P and Σ_A be two disjoint sets of symbols: Σ_P the set of *printable symbols* and Σ_A the set of *abstract symbols*. Σ_P will generally be a set consisting of the string representations of abstract symbols, be it operators like $+$ and $\cdot$ or variable names. Σ_A are the actual objects which can "execute something". Let us further define $\Sigma = \Sigma_P \cup \Sigma_A$.

Now a state is a mapping from printable symbols to strings. So for example, if x is a printable symbol a line like $\mathsf{let} x = 1$ should change the state so that x maps to the abstract symbol representing 1.

A *point state* is a partial function $s\colon \Sigma_P \longrightarrow \Sigma_A$. If s_1, s_2 are point states, define their composition to be a point state $s_1 s_2$ such that

$$s_1 s_2(\sigma) = \begin{cases} s_2(\sigma) & \sigma \in \mathrm{dom}(s_2) \\ s_1(\sigma) & \sigma \in \mathrm{dom}(s_1) \end{cases}$$

A *state* is a sequence of point states: $\overline{s} = (s_1, \ldots, s_n)$. Let us define

$$\mathsf{State} = \{\Sigma_P \longrightarrow \Sigma_A\}^+$$

the set of all states.

Let $\overline{s} = (s_1 \cdots s_n) \in \mathsf{State}$ be a state, then define

- for $\sigma \in \Sigma_P$ we define $s(\sigma) = s_1 \cdots s_n(\sigma)$ (the composition of states),
- define *pop* $\overline{s} = (s_1, \ldots, s_{n-1})$,
- define *push* $\overline{s} = (s_1, \ldots, s_n, \varnothing)$ ($\varnothing$ is the empty state),
- if s is a point state, $\overline{s}s = (s_1, \ldots, s_{n-1}, s_n s)$,
- if s is a point state, $\overline{s} + s = (s_1, \ldots, s_n, s)$ (so *push* $\overline{s} = \overline{s} + \varnothing$).

So if we'd like to revert to a previous state, we simply pop from the current state. And substituting the current state only alters the current (topmost) point state.

Now we begin with an initial β function which is a partial function

$$\beta \colon \overline{\Sigma}_A \times (\overline{\Sigma} \cup \{\varepsilon\}) \times \mathsf{State} \longrightarrow \overline{\Sigma}^* \times \mathsf{State}$$

Recall that $\overline{\Sigma}$ is $\Sigma \times \mathbb{N}$. We will denote tuples in $X \times \mathsf{State}$ by $\langle x, \mid s\rangle$ for $x \in X$ and $s \in \mathsf{State}$ for the sake of readability. So we now wish to extend to a β function

$$\beta \colon \overline{\Sigma}^* \times \mathsf{State} \longrightarrow \overline{\Sigma}^* \times \mathsf{State}$$

We do this as follows: given $\xi \in (\Sigma \times \overline{\mathbb{N}})^*$ and $s \in \mathsf{State}$ we define $\beta\langle \xi \mid s\rangle$ as follows:

(1) if $\xi = \sigma_i \xi'$ for $\sigma \in \Sigma_P$ then $\beta\langle \xi \mid s\rangle = \langle s(\sigma)_i \xi' \mid s\rangle$,

(2) if $\xi = \sigma_i \xi'$ such that $\beta\langle \sigma_i \varepsilon \mid s\rangle = \langle \xi'' \mid s'\rangle$ is defined then $\beta\langle \xi \mid s\rangle = \langle \xi'' \xi' \mid s'\rangle$,

(3) if $\xi = \sigma^1_i \sigma^2_j \xi'$ for $\sigma^1 \in \Sigma_A$, $i \geq j$, such that $\beta\big\langle \sigma^1_i \sigma^2_j \bigm| s\big\rangle = \langle \xi'' \mid s'\rangle$ is defined, then $\beta\langle \xi \mid s\rangle = \langle \xi'' \xi' \mid s'\rangle$,

(4) otherwise for $\xi = \sigma^1_i \sigma^2_j \xi'$, if $\beta\big\langle \sigma^2_j \xi' \bigm| s\big\rangle = \langle \xi'' \mid s'\rangle$ then $\beta\langle \xi \mid s\rangle = \big\langle \sigma^1_i \xi'' \bigm| s'\big\rangle$.

Notice that **(2)** cares not about the priority of σ, and neither if $\beta(\sigma_i, \tau_j)$ is defined for some $\tau \neq \varepsilon$.

We also define the *initial priority function* to be a map $\pi \colon \Sigma_P \longrightarrow \overline{\mathbb{N}}$ (this is not a partial function: every printable symbol must be given a priority). This is once again canonically extended to a function $\pi \colon \Sigma_P^* \longrightarrow (\Sigma_P \times \overline{\mathbb{N}})^*$. And an *initial state* s_0 which is a point state. The quintuple $(\Sigma_P, \Sigma_A, \beta, \pi, s_0)$ is called an *reducer*. The reduction of a string $\xi \in S$ is the process of iteratively applying β to $\langle \pi(\xi) \mid s_0\rangle$.

Example: let

$$\Sigma_P = \mathbb{N} \cup \{+, \cdot, =, ;\} \cup \{\mathtt{let}\} \cup \big\{x^i \bigm| i \in \mathbb{N}\big\},$$
$$\Sigma_A = \mathbb{N} \cup \{(n+), (n\cdot) \mid n \in \mathbb{N}\} \cup \{\mathtt{let}\} \cup \big\{(\mathtt{let}x^i), (\mathtt{let}x^i =) \bigm| i \in \mathbb{N}\big\}$$

where the natural numbers in Σ_A are not the same as the natural numbers in Σ_P since they must be disjoint, same for `let`. But they both essentially represent the same thing: s_0 maps $n \mapsto n$ for $n \in \mathbb{N}$ (the left-hand n is in Σ_p, the right-hand n is in Σ_A) and $\mathtt{let} \mapsto \mathtt{let}$. All other printable symbols are mapped to ε.

And similar to the previous example we define $\pi(n) = \infty$, $\pi(+) = 1$, and $\pi(\cdot) = 2$. We extend this to $\pi(;) = 0$, $\pi(=) = 0$, $\pi(\mathtt{let}) = \infty$, and $\pi(x^i) = \infty$.

Let us take the same transitions as the example in the previous section for $n, (n+), (n\cdot)$ (we have to add the condition that the state doesnt change). We further add the transitions

$\big\langle \sigma^1_i \sigma^2_j \bigm| s\big\rangle$	$\beta\big\langle \sigma^1 \sigma^2 \bigm| s\big\rangle$
$\langle \sigma; \mid s\rangle$	$\langle \sigma_j \mid s\rangle$
$\big\langle \mathsf{let}x^i \bigm| s\big\rangle$	$\big\langle (\mathsf{let}x^i)_j \bigm| s\big\rangle$
$\big\langle (\mathsf{let}x^i) = \bigm| s\big\rangle$	$\big\langle (\mathsf{let}x^i =)_j \bigm| s\big\rangle$
$\big\langle (\mathsf{let}x^i =)\sigma \bigm| s\big\rangle$	$\big\langle \varepsilon \bigm| s[x^i \mapsto \sigma]\big\rangle$

In the final transition, $n \in \Sigma_A$. Then for example (we will be skipping trivial reductions):

$$\begin{aligned}
\mathsf{let}x^1 = 1 + 2;\ \mathsf{let}x^2 = 2;\ x^1 \cdot x^2; &\longrightarrow \mathtt{let}_\infty x^1_\infty =_0 1_\infty +_1 2_\infty ;_0\ \mathtt{let}_\infty x^2_\infty =_0 2_\infty ;_0\ x^1_\infty \cdot_2 x^2_\infty ;_0 \\
s_0 &\longrightarrow (\mathtt{let}x^1 =)_0 1_\infty +_1 2_\infty ;_0\ \mathtt{let}_\infty x^2_\infty =_0 2_\infty ;_0\ x^1_\infty \cdot_2 x^2_\infty ;_0 \\
s_0 &\longrightarrow (\mathtt{let}x^1 =)_0 3_0\ \mathtt{let}_\infty x^2_\infty =_0 2_\infty ;_0\ x^1_\infty \cdot_2 x^2_\infty ;_0 \\
s_0[x^1 \mapsto 3] &\longrightarrow \mathtt{let}_\infty x^2_\infty =_0 2_\infty ;_0\ x^1_\infty \cdot_2 x^2_\infty ;_0 \\
s_0[x^1 \mapsto 3,\ x^2 \mapsto 2] &\longrightarrow x^1_\infty \cdot_2 x^2_\infty ;_0 \\
s_0[x^1 \mapsto 3,\ x^2 \mapsto 2] &\longrightarrow 3_\infty \cdot_2 x^2_\infty ;_0 \\
s_0[x^1 \mapsto 3,\ x^2 \mapsto 2] &\longrightarrow (3\cdot)_2 x^2_\infty ;_0 \\
s_0[x^1 \mapsto 3,\ x^2 \mapsto 2] &\longrightarrow (3\cdot)_2 2_\infty ;_0 \\
s_0[x^1 \mapsto 3,\ x^2 \mapsto 2] &\longrightarrow 6_0
\end{aligned}$$

$\Diamond$

1.3 Valued Reduction

We define the following four base sets:

(1) $\mathcal{U}$ the universe of *values*, these are all the internal values an object may have.

(2) $\mathcal{T}_\mathcal{P}$ the set of *printable terms*, these are the tokens which a programmer may pass to the reducer.

(3) $\mathcal{T}_\Sigma$ the set of *type terms*.

(4) $\mathcal{T}_\mathcal{A}$ the set of *abstract terms*.

The sets $\mathcal{T}_\mathcal{P}, \mathcal{T}_\Sigma, \mathcal{T}_\mathcal{A}$ are all disjoint, we place no such restriction on $\mathcal{U}$ as the purpose it serves is different. Let $\mathcal{A}$ be a set of *atomic abstract terms*, then the construction of abstract terms is

$$\mathcal{T}_\mathcal{A} ::= \mathcal{A} \mid \mathcal{A}\mathcal{T}_\Sigma$$

And let Σ be a set of *atomic types*, each with an associated arity, which may be ∞. Let Σ^n be the set of atomic types of arity n, then the construction of type terms is

$$\mathcal{T}_\Sigma ::= \Sigma^0 \mid \Sigma^n \mathcal{T}_\Sigma^1 \cdots \mathcal{T}_\Sigma^n \mid \Sigma^\infty \mathcal{T}_\Sigma^1 \cdots \mathcal{T}_\Sigma^n$$

as n ranges over all $\mathbb{N}_{>0}$.

Define

(1) $\mathcal{T} := \mathcal{T}_\mathcal{P} \cup \mathcal{T}_\Sigma \cup \mathcal{T}_\mathcal{A}$ the set of *basic terms*.

(2) $\mathcal{T}_\mathcal{I} := \mathcal{T}_\Sigma \cup \mathcal{T}_\mathcal{A}$ the set of *internal terms*.

(3) $\Pi_\mathcal{I} := \mathcal{T}_\mathcal{I} \times \mathcal{U}$ the set of *termed values*.

(4) $\Pi := \Pi_\mathcal{I} \cup \mathcal{T}_\mathcal{P}$ the set of *atomic expressions*.

Elements of $\overline{\Pi}$ will be written like $\sigma_n(v)$ where σ is the term, n the priority, and v the value (nothing for printable terms).

In valued reduction, we abstract away some inputs to the initial beta-reducer in order to allow for easier implementation. An initial beta-reducer is a partial function

$$\widehat{\beta}\colon \mathcal{T}_\mathcal{I} \times \mathcal{T}^\varepsilon \longrightarrow \mathcal{T}_\mathcal{I}^\varepsilon \times (\overline{\mathbb{Z}} \times \overline{\mathbb{Z}} \to \overline{\mathbb{Z}}) \times (\mathcal{U} \times \mathcal{U} \times \text{State} \rightharpoonup \mathcal{U} \times \mathcal{T}_\mathcal{P}^* \times \text{State})$$

We extend this to a derived β-reducer,

$$\beta\colon \overline{\Pi}^* \times \text{State} \longrightarrow \overline{\Pi}^* \times \text{State}$$

with the following rules: given an input $\langle \xi \mid s \rangle$ its image is

(1) If $\xi = \sigma_n \xi'$ for $\sigma \in \mathcal{T}_\mathcal{P}$ then
$$\beta\langle \xi \mid s \rangle = \langle s(\sigma)_n \xi' \mid s \rangle.$$

(2) If $\xi = \sigma_i(v)\xi'$ and $\widehat{\beta}(\sigma, \varepsilon) = (\alpha, \rho, f)$ is defined, then if $f(v, \_, s) = (w, \zeta, s')$ and $\rho(i) = k$ then
$$\beta\langle \xi \mid s \rangle = \langle \alpha_k(w)\pi(\zeta)\xi' \mid s' \rangle.$$

(3) If $\xi = \sigma_i(v)\tau_j(u)\xi'$ and $i \geq j$ and $\widehat{\beta}(\sigma, \tau) = (\alpha, \rho, f)$ is defined, then if $f(v, u, s) = (w, \zeta, s')$ and $\rho(i, j) = k$ then
$$\beta\langle \xi \mid s \rangle = \langle \alpha_k(w)\pi(\zeta)\xi' \mid s \rangle.$$

(4) Otherwise, if $\xi = \sigma_i(v)\xi'$ and $\beta\langle \xi' \mid s \rangle = \langle \xi'' \mid s' \rangle$,
$$\beta\langle \xi \mid s \rangle = \langle \sigma_i(v)\xi'' \mid s' \rangle.$$

1.3.1 States

Similar to before, we define point-states as partial maps $\mathcal{T}_\mathcal{P} \longrightarrow \Pi_\mathcal{I}$. And if s_1, s_2 are two point-states and $\sigma \in \mathcal{T}_\mathcal{P}$ then

$$s_1 s_2(\sigma) = \begin{cases} s_2(\sigma) & \sigma \in \mathrm{dom} s_2 \\ s_1(\sigma) & \sigma \in \mathrm{dom} s_1 \end{cases}$$

We will denote finite point states as $[\sigma_1 \mapsto \varkappa_1, \dots, \sigma_n \mapsto \varkappa_n]$, and this denotes the point-state which maps σ_i to $\varkappa_i$.

A state will now have two fields: a sequence of point-states, as well as a sequence of indexes. For a state $\bar{s} = \big[(s_1, \dots, s_n), I = (i_1, \dots, i_k)\big]$, let us define

(1) $\bar{s} + s = \big[(s_1, \dots, s_n, s), I\big]$

(2) $\bar{s} +_c s = \big[(s_1, \dots, s_n, s), (i_1, \dots, i_k, n+1)\big]$

(3) $pop\ \bar{s} = \big[(s_1, \dots, s_{n-1}), I\big]$ if $i_k < n$ otherwise, $\big[(s_1, \dots, s_{n-1}), (i_1, \dots, i_{k-1})\big]$

(4) $\bar{s}s = \big[(s_1, \dots, s_n s), I\big]$

(5) $\bar{s}(\sigma) = s_1 \cdots s_n(\sigma)$ for $\sigma \in \Sigma_P$

Furthermore, if $\sigma \in \mathcal{T}_\mathcal{P}$ and $\varkappa \in \Pi_\mathcal{I}$ let us define $\bar{s}\{\sigma \mapsto \varkappa\}$ as $(s_1, \dots, s_i[\sigma \mapsto \varkappa], \dots, s_n)$ where i is the maximum index such that $\sigma \in \mathrm{dom} s_i$.

1.3.2 The Initial Beta Reducer

We now describe the initial beta reducer. By convention, atomic abstract terms will be red, type terms will be green, internal terms will be blue.

End:

- σ end $\longrightarrow$ σ minfty $(u, \_, s \to u, \varepsilon, s)$

Arithmetic:

- σ op $\longrightarrow$ opσ snd $(u, f, s \to (u, f), \varepsilon, s)$
- opσ opσ $\longrightarrow$ opσ snd $\big((u, f), (v, g), s \to (f(u, v), g), \varepsilon, s\big)$
- opσ σ $\longrightarrow$ σ snd $\big((u, f), v, s \to f(u, v), \varepsilon, s\big)$
- σ rparen $\longrightarrow$ rparenσ snd $(u, \_, s \to u, \varepsilon, s)$
- opσ rparenσ $\longrightarrow$ rparenσ snd $\big((f, u), v, s \to f(u, v), \varepsilon, s\big)$
- lparen rparenσ $\longrightarrow$ σ fst $(\_, u, s \to u, \varepsilon, s)$

Lists:

- lbrack σ $\longrightarrow$ lbrackσ fst $(\_, u, s \to (u), \varepsilon, s)$
- lbrackσ σ $\longrightarrow$ lbrackσ fst $(\ell, u, s \to (\ell, u), \varepsilon, s)$
- lbrackσ rbrack $\longrightarrow$ listσ infty $(\ell, \_, s \to \ell, \varepsilon, s)$
- period num $\longrightarrow$ index zero $(\_, n, s \to n, \varepsilon, s)$
- listσ index $\longrightarrow$ σ fst $(\ell, i, s \to \ell_i, \varepsilon, s)$