Mathcord Mathematical Logic
Problem Set 1 Solution

Problem 1

(1) Show that $\{\neg, \to\}$ is functional complete.

(2) Show that $\vee$ can be represented in the logical signature $\{\to\}$.

(3) Show that any formula φ over the signature $\to$ has $w\varphi = 1$ for the valuation w which evaluates every variable as 1.

(4) Show that $\wedge$ cannot be represented in the logical signature $\{\to\}$.

(5) Show that $\leftrightarrow$ cannot be represented in the logical signature $\{\to\}$.

(1) Clearly $\neg$ is representable in this signature, and so is $\vee$, by $\varphi \vee \psi = \neg\varphi \to \psi$. Since $\{\neg, \vee\}$ is functional complete, so too is this signature.

(2) Suppose $\varphi(x, y) \to \psi(x, y)$ represents $\vee$, so we must have $\varphi(0, 0) = 1, \psi(0, 0) = 0$. So let us take $\varphi(x, y) = x \to y$ and $\psi(x, y) = (x \to y) \to y$. We claim then that $\vee$ is represented by

$$\theta(x, y) = (x \to y) \to ((x \to y) \to y)$$

We must verify this: $\theta(0, 0) = 0, \theta(0, 1) = 1, \theta(1, 0) = 1, \theta(1, 1) = 1$ as required.

(3) This is clear by formula induction. For $\varphi = \pi$ prime, $w\pi = 1$ by definition. And otherwise $\varphi = \psi \to \theta$ we have $w\varphi = {\to}(w\psi, w\theta)$, which by our inductive assumption is ${\to}(1, 1) = 1$.

(4) Suppose $\wedge$ was representable by $\to$, so by a formula of the form $\varphi(x, y) \to \psi(x, y)$. Thus we must have that

$$\varphi(0, 0) = \varphi(0, 1) = \varphi(1, 0) = 1, \qquad \psi(0, 0) = \psi(0, 1) = \psi(1, 0) = 0$$

We can assume that $\varphi(x, y) \to \psi(x, y)$ is the shortest representation. Thus we must have $\varphi(1, 1) = 0$ since otherwise φ is a tautology, and $\varphi \to \psi \equiv \psi$, contradicting minimality. But as showed in the previous subquestion, we cannot have $\varphi(1, 1) = 0$.

(5) Similarly suppose it is representable by a minimal formula $\varphi(x, y) \to \psi(x, y)$. Then we know $\varphi(1, 1) = 1$ and $\psi(1, 1) = 1$ for the same reason as above. Furthermore

$$\varphi(0, 1) = \varphi(1, 0) = 1, \qquad \psi(0, 1) = \psi(1, 0) = 0$$

Now, $\psi(0, 0) = 0$ since otherwise ψ would represent $\leftrightarrow$, contradicting minimality. But then ψ represents $\wedge$, which we just showed cannot be.

Problem 2

We define the boolean *nand* connective, $\uparrow$, as follows:

$$0 \uparrow 0 = 0 \uparrow 1 = 1 \uparrow 0 = 1, \qquad 1 \uparrow 1 = 0$$

and *nor*, $\downarrow$:

$$0 \downarrow 0 = 1, \qquad 0 \downarrow 1 = 1 \downarrow 0 = 1 \downarrow 1 = 0$$

(1) Verify that $\alpha \uparrow \beta \equiv \neg(\alpha \wedge \beta)$ and $\alpha \downarrow \beta \equiv \neg(\alpha \vee \beta)$.

(2) Show that both $\{\uparrow\}$ and $\{\downarrow\}$ are functional complete.

(3) Show that if $\circ$ is a boolean connective and $\{\circ\}$ is functional complete, $\circ \in \{\uparrow, \downarrow\}$.

(1) This is easily verified with a truth table.

(2) For $\uparrow$ we will represent $\neg$ and $\wedge$. For $\neg$, notice that it can be represented by $x \uparrow x \equiv \neg(x \wedge x) \equiv \neg x$. And then since we can represent $\neg$, we can represent $\wedge$ by $\neg(x \uparrow y)$.

Similarly for $\downarrow$, we will represent $\neg$ and $\vee$. $\neg$ can be represented by $x \downarrow x$, and $\vee$ by $\neg(x \downarrow y)$.

(3) Suppose $\{\circ\}$ is functional complete. Then we must have that $\circ(1,1) = 0$ and $\circ(0,0) = 1$ as an easy formula induction will show that otherwise no formula will map $(1,1)$ to 0 or $(0,0)$ to 1. We now run down possibilities:

- **(1)** If $\circ(1,0) = \circ(0,1) = 0$ then $\circ$ is $\downarrow$.
- **(2)** If $\circ(1,0) = \circ(0,1) = 1$ then $\circ$ is $\uparrow$.
- **(3)** If $\circ(1,0) = 1$ and $\circ(0,1) = 0$ then $\circ$ is just $\circ(x,y) = \neg y$, so it will always be constant in the first input, and thus cannot be functional complete.
- **(4)** If $\circ(1,0) = 0$ and $\circ(0,1) = 1$ similar as before.

Problem 3

Let $\varphi \to \psi$ be a tautology, then show that there exists a formula θ which uses only variables common to both φ and ψ (i.e. $var\theta \subseteq var\varphi \cap var\psi$), such that both $\varphi \to \theta$ and $\theta \to \psi$ are tautologies.

Suppose that

$$\mathrm{var}\varphi = \{x_1, \dots, x_n, y_1, \dots, y_k\}, \qquad \mathrm{var}\psi = \{x_1, \dots, x_n, z_1, \dots, z_\ell\}$$

we then represent φ and ψ in DNF:

$$\varphi \equiv \bigvee_{j=1}^{n_\varphi} \left(\bigwedge_{i=1}^{n} x_i^{\varepsilon_{ij}^\varphi} \wedge \bigwedge_{i=1}^{k} y_i^{\delta_{ij}^\varphi} \right), \qquad \psi \equiv \bigvee_{j=1}^{n_\psi} \left(\bigwedge_{i=1}^{n} x_i^{\varepsilon_{ij}^\psi} \wedge \bigwedge_{i=1}^{\ell} z_i^{\delta_{ij}^\psi} \right)$$

Then we define

$$\theta = \bigvee_{j=1}^{n_\varphi} \bigwedge_{i=1}^{n} x_i^{\varepsilon_{ij}^\varphi}$$

It is trivial to see that $\varphi \to \theta$. Nowe we want to show that $\theta \to \psi$, suppose not then there is a valuation w such that $w\theta = 1$ and $w\psi = 0$. Since $\varphi \to \psi$, we must have $w\varphi = 0$.

Now, since $w\theta = 1$, there is a $1 \leq j \leq n_\varphi$ such that $w \vDash \bigwedge_{i=1}^{n} x_i^{\varepsilon_{ij}^\varphi}$. But $w\varphi = 0$ so there must be some is with $1 \leq i \leq k$ such that $wy_i^{\delta_{ij}^\varphi} = 0$. So let us simply alter w to w' such that $w'y_i^{\delta_{ij}^\varphi} = 1$ for these is.
Since all we've done is alter y_is, we still have $w'\psi = 0$ and $w'\theta = 1$, but now $w'\varphi = 1$. But this contradicts $\varphi \to \psi$ being a tautology.

Problem 4

Let φ be a formula, we define its *dual*, φ^δ, recursively as follows:

$$\pi^\delta = \pi \text{ for prime } \pi, \qquad (\neg\alpha)^\delta = \neg\alpha^\delta, \qquad (\alpha \wedge \beta)^\delta = (\alpha^\delta \vee \beta^\delta), \qquad (\alpha \vee \beta)^\delta = (\alpha^\delta \wedge \beta^\delta)$$

Equivalently, it just substitutes all $\wedge$ in φ with $\vee$ and all $\vee$ with $\wedge$.
Let $f\colon \{0,1\}^n \longrightarrow \{0,1\}$ be a boolean function, we define its *dual* to be

$$f^\delta \bar{x} = \neg f(\neg \bar{x})$$

- **(1)** Show that if α represents f, then α^δ represents f^δ.
- **(2)** Show that $f \mapsto f^\delta$ is a bijection on boolean functions.
- **(3)** Using the fact that every boolean function can be represented by a DNF, show that every boolean function can be represented by a CNF.

(1) We prove this by formula induction. For $\alpha = x$ then it represents $f(\bar{x}) = x$, and so $\alpha^\delta = x = \alpha$ and $f^\delta(\bar{x}) = x$ so $f^\delta = f$ as required.

For $\alpha = \varphi \wedge \psi$, then $\alpha^\delta = \varphi^\delta \vee \psi^\delta$. Suppose α represents f and φ, ψ represent g_1, g_2 respectively, this means $f(\bar{x}) = g_1(\bar{x}) \wedge g_2(\bar{x})$. Then

$$w\alpha^\delta = (w\varphi^\delta) \vee (w\psi^\delta) = g_1^\delta(w\bar{x}) \vee g_2^\delta(w\bar{x})$$

while

$$f^\delta(w\bar{x}) = \neg f(\neg w\bar{x}) = \neg(g_1(\neg w\bar{x}) \wedge g_2(\neg w\bar{x})) = g_1^\delta(w\bar{x}) \vee g_2^\delta(w\bar{x})$$

as required.

For $\alpha = \neg\varphi$, then $\alpha^\delta = \neg\varphi^\delta$. Suppose α represents f and φ represents g, then $f(\bar{x}) = \neg g(\bar{x})$. Then

$$w\alpha^\delta = \neg w\varphi^\delta = \neg g^\delta(w\bar{x}) = g(\neg w\bar{x})$$

and

$$f^\delta(w\bar{x}) = \neg f(\neg w\bar{x}) = \neg(\neg g(\neg w\bar{x}) = g(\neg w\bar{x})$$

as required.

(2) This map is an involution: $(f^\delta)^\delta = f$, so it is its own inverse and thus a bijection.

(3) Note that the dual of a DNF is a CNF. Let f be a boolean function, it can be represented by a DNF φ. Then f^δ is represented by φ^δ, a CNF. Since every boolean function can be written as f^δ, every boolean function can be represented by a CNF.

Problem 5

Let $\bar{x} = (x_1, \ldots, x_n)$ and $\bar{y} = (y_1, \ldots, y_n)$ be boolean vectors. We write $\bar{x} \leq \bar{y}$ to mean $x_i \leq y_i$ for all $i = 1, \ldots, n$. A boolean function f is said to be *increasing* if $\bar{x} \leq \bar{y}$ implies $f\bar{x} \leq f\bar{y}$.

Show that every increasing boolean function can be represented in the logical signature $\{\wedge, \vee\}$ and vice versa: every formula in this logical signature is increasing.

First we will show that every formula in this signature is increasing. We do so by formula induction. For prime formulas, these are just projections which are trivially increasing.

Now notice that if $(x_1, x_2) \leq (y_1, y_2)$ then $x_1 \wedge x_2 \leq y_1 \wedge y_2$ and $x_1 \vee x_2 \leq y_1 \vee y_2$. This is easily verifiable. Now suppose $\alpha = \varphi \wedge \psi$, then if $\bar{x} \leq \bar{y}$ then $\alpha(\bar{x}) = \varphi(\bar{x}) \wedge \psi(\bar{x})$ and $\varphi(\bar{x}) \leq \varphi(\bar{y})$ and $\psi(\bar{x}) \leq \psi(\bar{y})$ by induction. So we have that

$$\alpha(\bar{x}) = \varphi(\bar{x}) \wedge \psi(\bar{x}) \leq \varphi(\bar{y}) \wedge \psi(\bar{y})$$

as required. Similar for $\vee$.

Let f be an increasing boolean function, then it can be represented by

$$\varphi = \bigvee_{f\bar{x}=1} \bigwedge_{x_i=1} p_i$$

If $f(\bar{x}) = 1$ then trivially $\varphi(\bar{x}) = 1$.

And if $\varphi(\bar{x}) = 1$ then there is some $\bar{y}$ such that $f(\bar{y}) = 1$ and $\bigwedge_{i=1, y_i=1}^n x_i = 1$. Thus $y_i = 1$ implies $x_i = 1$. This means that $\bar{y} \leq \bar{x}$ and $f(\bar{y}) = 1$ so $f(\bar{x}) = 1$, as required.