Mathcord Mathematical Logic
Problem Set 2

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Problem 1

Prove the following:

$$\frac{X \vdash \alpha \to \beta}{X, \alpha \vdash \beta}, \qquad \frac{X, \alpha \vdash \beta}{X \vdash \alpha \to \beta}$$

Problem 2

Complete section 2.4: prove claims 2.4.2 through 2.4.8.

Problem 3

A *substitution* is a mapping $\sigma\colon V \longrightarrow \mathcal{F}$, which we extend to $\sigma\colon \mathcal{F} \longrightarrow \mathcal{F}$ using recursion:

$$(\alpha \wedge \beta)^\sigma = \alpha^\sigma \wedge \beta^\sigma, \qquad (\neg\alpha)^\sigma = \neg\alpha^\sigma$$

For a set of formulas $X \subseteq \mathcal{F}$, define $X^\sigma = \{\varphi^\sigma \mid \varphi \in X\}$. Verify that $\vDash$ is *substitution invariant*:

$$X \vDash \alpha \implies X^\sigma \vDash \alpha^\sigma$$

(Hint: for valuation w, define w^σ such that $w \vDash \alpha^\sigma \iff w^\sigma \vDash \alpha$)

Problem 4

Let $\vdash \subseteq \mathcal{P}(\mathcal{F}) \times \mathcal{F}$ be a relation between sets of formulas and formulas (we write $X \vdash \varphi$). $\vdash$ is a *consequence relation* if it satisfies:

(1) Reflexivity: $\{\alpha\} \vdash \alpha$

(2) Monotonicity: $X \subseteq X'$ and $X \vdash \alpha$ implies $X' \vdash \alpha$

(3) Transitivity: $X \vdash Y$ ($X \vdash \varphi$ for all $\varphi \in Y$) and $Y \vdash \alpha$ implies $X \vdash \alpha$

(4) Substitution invariance: $X \vdash \alpha \implies X^\sigma \vdash \alpha^\sigma$ (see the previous question).

A consequence relation is called *finitary* if $X \vdash \alpha$ implies there exists a finite $X_0 \subseteq X$ such that $X_0 \vdash \alpha$. Call a consequence relation $\vdash$ *inconsistent* if it is trivial: $\vdash \alpha$ for all α. Otherwise $\vdash$ is consistent.

(1) Let $\vdash$ be a consistent finitary consequence relation in $\mathcal{F}_{\{\wedge,\neg\}}$ which satisfies the properties ($\wedge$1) through ($\neg$2). Show that $\vdash$ is *maximally consistent* (meaning any consequence relation which contains $\vdash$ is inconsistent).

(2) Show that there is exactly one consistent consequence relation in $\mathcal{F}_{\{\wedge,\neg\}}$ which satisfies ($\wedge$1) through ($\neg$2).

(3) Conclude that $\vdash$ (our Gentzen calculus) is complete.

Problem 5

A *Horn formula* is one which is of one of the following forms:

$$\alpha_0 \vee \neg\alpha_1 \vee \cdots \vee \alpha_n \quad \text{or} \quad \neg\alpha_0 \vee \neg\alpha_1 \vee \cdots \vee \alpha_n \qquad \text{for } n \geq 0,\ \alpha_i \text{ prime}$$

Let $w\colon V \longrightarrow \{0,1\}$ be a valuation, we can also equivalently view it as a set $A \subseteq V$. We say that a set of formulas X is *preserved under intersections* if $A \vDash X$ and $B \vDash X$ implies $A \cap B \vDash X$. And X is *preserved under arbitrary intersections* if $\{A_i\}_{i \in I}$ all model X then so too does $B = \bigcap_{i \in I} A_i$.
Show that

(1) X is preserved under finite intersections if and only if it is closed under arbitrary intersections.

(2) X is preserved under intersections if and only if it can be axiomatized (meaning it has the same models as) by a set of Horn formulas.

(3) A formula φ is preserved under intersections if and only if it is equivalent to a Horn formula.