Mathematical Logic and Model Theory

Lectures by Slurp and Sharp

Contents

1 Lecture 1

Sources: A Concise Introduction to Mathematical Logic, Section 1, W. Rautenberg

We begin by defining *what* exactly mathematical logic is. Mathematical logic is a sort of metamathematical study of mathematics itself. It studies what sorts of logical statements we can make, how we can manipulate them, and what we can say about the mathematical objects which satisfy them. But the best way to understand what mathematical logic is, is to actually *do* it! So let's begin.

1.1 Propositional Logic

We begin our discussion with the simplest logic: propositional logic. This logic studies how we can connect propositions together, e.g. using *and*, *or*, *not*, etc. Suppose we wanted to say that "*if* it is cold outside *then* I will wear a coat", how could we go about this mathematically?

We begin with some definitions:

1.1.1 Definition

A **boolean function** is a function $\{0,1\}^n \longrightarrow \{0,1\}$ for some $n > 0$.

1.1.2 Definition

A **connective** is a symbol s with an associated boolean function (which will be named with s as well). A set of connectives is called a **logical signature**.

For boolean connectives $\circ$ (connectives whose function accepts two parameters), we can use a *truth table* to define it:

$$\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}$$

where a_{ij} is defined to be the value of $i \circ j$. So for example, we can define the following connective $\wedge$ as follows:

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

This connective takes two boolean values x and y and checks that both x *and* y are true. For this reason $\wedge$ is called *logical and* or a *conjunction*.

We can also define *logical or* or *disjunction* $\vee$:

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

And *logical negation* $\neg$, which is a unary connective: $\neg 0 = 1$ and $\neg 1 = 0$. Finally let us define *logical implication* $\rightarrow$:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Why should false imply false $(0 \rightarrow 0 = 1)$? Well suppose I said "if it rains then it is cloudy", but it is not raining. Is what I said false? Well, not necessarily! This is what is called a *vaccuous truth*.

Now suppose I wanted to string together connectives, like "if it rains then it is cloudy and I should wear a jacket".

1.1.3 Definition

Suppose we have a global set of variables V, whose elements are simply symbols which we will call *propositional variables*. Now suppose we have a logical signature ℓ, we define the set of **propositional formulas** $\mathcal{F}_\ell$ recursively as follows:

(1) If $p \in V$ is a propositional variable then it is a formula: $p \in \mathcal{F}_\ell$. Such formulas are called **prime formulas**.

(2) If $\mathsf{s} \in \ell$ is a connective of arity n and $\varphi_1, \dots, \varphi_n \in \mathcal{F}_\ell$ are formulas, then so too is

$$\mathsf{s}\varphi_1 \cdots \varphi_n \in \mathcal{F}_\ell$$

So for example, if $\ell = \{\wedge, \vee, \neg\}$ and $V = \{p_1, p_2, \dots\}$ then the following are formulas:

$$p_1, \qquad \wedge p_1 p_2, \qquad \wedge\vee p_1 p_2 \neg p_3, \qquad \neg\wedge\vee p_1 p_2 \wedge p_1 p_3$$

But using prefix notation like this is confusing, so we will adopt the custom that for binary connectives $\circ$, $\circ\varphi\psi$ is instead written as $(\varphi \circ \psi)$. So these formulas become

$$p_1, \qquad (p_1 \wedge p_2), \qquad ((p_1 \vee p_2) \wedge \neg p_3), \qquad \neg((p_1 \vee p_2) \wedge (p_1 \wedge p_3))$$

We will call the signature $\ell = \{\wedge, \vee, \neg\}$ the *standard signature*.

An important thing to keep in mind is that currently formulas are simply special strings. We haven't assigned to them any value yet.

Note that our definition of propositional formulas isn't really all that formal: how can we define a set using itself? Well formally, what we do is we look at the collection of all sets S of strings (over the alphabet $\ell \cup V$) with the properties that (1) $V \subseteq S$, (2) if $\mathsf{s} \in \ell$ has arity n and $\varphi_1, \dots, \varphi_n \in S$ then $\mathsf{s}\varphi_1 \cdots \varphi_n \in S$. Then we simply define $\mathcal{F}_\ell$ to be the intersection of these sets.

From this definition the following is immediate:

1.1.4 Lemma (The Principle of Formula Induction)

Let ℓ be a logical signature. Suppose $\mathcal{E}$ is a property of strings (i.e. a subset of the set of all strings over $V \cup \ell$), with the following properties:

(1) For every $p \in V$, $\mathcal{E}p$.

(2) For every $\mathsf{s} \in \ell$ with arity n, if $\varphi_i \in \mathcal{F}_\ell$ for $i = 1, \dots, n$ then $\mathcal{E}\mathsf{s}\varphi_1 \cdots \varphi_n$.

Then $\mathcal{E}\varphi$ holds for all formulas $\varphi \in \mathcal{F}_\ell$.

Now how can we be sure that if we have a formula φ, suppose of the form $\wedge\alpha\beta$, it is not simultaneously of the form $\wedge\alpha'\beta'$ for some other $\alpha, \beta \in \mathcal{F}$? We can use formula induction to prove the following:

1.1.5 Lemma (The Unique Formula Reconstruction Property)

Every compound formula $\varphi \in \mathcal{F}_\ell$ is of the form $\mathsf{s}\varphi_1 \cdots \varphi_n$ for uniquely determined s and φ_i for $i = 1, \dots, n$.

Proof: s is obviously uniquely determined since it is the first character of φ. Now, we need to prove that if $\varphi_1 \cdots \varphi_n = \psi_1 \cdots \psi_m$ then $n = m$ and $\varphi_i = \psi_i$ for $i = 1, \dots, n$. To do so, we need to prove the claim that a proper prefix of a formula is not itself a formula. (Recall that a prefix of a string $\sigma_1 \cdots \sigma_n$ is a string $\sigma_1 \cdots \sigma_k$ for $k \leq n$. This prefix is *proper* if $k < n$).

This uses formula induction: for prime formulas this is trivial. Otherwise, let $\varphi = \mathsf{s}\varphi_1 \cdots \varphi_n$ then a proper prefix of φ is either s which is not a formula, or of the form $\mathsf{s}\varphi_1 \cdots \varphi_i'$ where φ_i' is a prefix of φ_i. In order for $\mathsf{s}\varphi_1 \cdots \varphi_i'$ to be a formula, $\varphi_1 \cdots \varphi_i'$ must be able to be split into n formulas. So suppose $\varphi_1 \cdots \varphi_i' = \psi_1 \cdots \psi_n$, but φ_1 and ψ_1 cannot be prefixes of one another, so $\varphi_1 = \psi_1$. And so on until $i-1$. Then we have $\varphi_i' = \psi_i \cdots \psi_n$, but then if $\varphi_i' \neq \varphi_i$ we have that ψ_i is a proper prefix of φ_i, a contradiction to our inductive assumption. So $\varphi_i = \psi_i \cdots \psi_n$, so we get that $i = n$, contradicting the assumption that the prefix is proper.

Note that in proving this claim, we have proven precisely the unique reconstruction property. ■

Note that the reconstruction property holds true even when our strings use the custom that binary connectives are written as $(\alpha \circ \beta)$.

Using the unique reconstruction property, we can define functions on formulas in a recursive manner. For example, we can define the *substring function*:

$$Sf\pi = \{\pi\} \text{ for prime } \pi, \qquad Sf\mathsf{s}\varphi_1 \cdots \varphi_n = \bigcup_{i=1}^{n} Sf\varphi_i \cup \{\mathsf{s}\varphi_1 \cdots \varphi_n\}$$

So $Sf\varphi$ is precisely all the subformulas of φ. Note that Sf is well-defined precisely because of the unique reconstruction property: a formula cannot satisfy multiple conditions in the definition of Sf at once.

More importantly than this example, we can assign truth to formulas:

1.1.6 Definition

Let $w\colon V \longrightarrow \{0,1\}$ be a **valuation** – a mapping of truth values to each propositional variable. Then we can extend w to a function $w\colon \mathcal{F}_\ell \longrightarrow \{0,1\}$ by recursion as follows:

(1) For π prime, $w\pi$ is already defined ($\pi \in V$).

(2) If $\mathsf{s} \in \ell$ and $\varphi_1, \dots, \varphi_n \in \mathcal{F}_\ell$ then

$$w\mathsf{s}\varphi_1 \cdots \varphi_n = \mathsf{s}(w\varphi_1, \dots, w\varphi_n)$$

So for example suppose $w(p) = 1$ and $w(q) = 0$ then

$$w(p \wedge q) = 0, \qquad w(p \wedge (q \vee \neg q)) = 1$$

Let $var\varphi$ denote all the variables in V occurring in φ. This can be defined recursively as follows:

$$var\pi = \{\pi\} \text{ for } \pi \text{ prime}, \qquad var\mathsf{s}\varphi_1 \cdots \varphi_n = \bigcup_{i=1}^{n} var\varphi_i$$

Then notice that $w\varphi$ is dependent only on w's values on $var\varphi$. That is, we have the following:

1.1.7 Lemma

If w, w' are two valuations such that $w(p) = w'(p)$ for all $p \in var\varphi$, then $w\varphi = w'\varphi$.

Proof: by formula induction. ■

Now, suppose φ is a formula with $var\varphi \subseteq \{p_1, \dots, p_n\}$, then we write $\varphi(\bar{p})$ for φ. Now suppose $\varphi = \varphi(\bar{p})$ and w is a valuation with $wp_i = x_i$, then instead of writing $w\varphi$, we can write $\varphi(\bar{x})$. This is well-defined by the above lemma.

Now, suppose that for every $n \in \mathbb{N}$ we have $p_n \in V$. Then we can define $\mathcal{F}_\ell^n = \{\varphi \in \mathcal{F}_\ell \mid var\varphi \subseteq \{p_1, \dots, p_n\}\}$, the set of all formulas whose variables are contained in $p_1, \dots, p_n$.

1.1.8 Definition

We say that a formula $\varphi \in \mathcal{F}_n^\ell$ **represents** a boolean function $f\colon \{0,1\}^n \longrightarrow \{0,1\}$ if for every valuation w,

$$w\varphi = f(wp_1, \dots, wp_n)$$

equivalently, for all boolean vectors $\bar{x} \in \{0,1\}^n$,

$$\varphi(\bar{x}) = f(\bar{x})$$

Since $w\varphi$ is dependent only $wp_1, \dots, wp_n$, f is uniquely determined: define $f(x_1, \dots, x_n)$ to be $w\varphi$ where $w(p_i) = x_i$ and $w(p)$ arbitrary for all other variables. So we can denote f by $\varphi^{(n)}$ since it is unique.

Notice that implication can be represented by $\neg(p_1 \wedge \neg p_2)$, thus in the standard signature, we can use $(\alpha \to \beta)$ as a standin for $\neg(\alpha \wedge \neg\beta)$. Similarly $\leftrightarrow$ can be represented by $(\alpha \to \beta) \wedge (\beta \to \alpha)$.

1.2 Logical Equivalence

1.2.1 Definition

Say two formulas α, β are **logically equivalent** if $w\alpha = w\beta$ for every valuation w. Denote this by $\alpha \equiv \beta$.

For example, $\alpha \equiv \neg\neg\alpha$. We can define $\top = p \vee \neg p$ and $\bot = p \wedge \neg p$ for $p \in V$. Notice that $w\top = 1$ for all w, and $w\bot = 0$, so $\top$ and $\bot$ represent truth and false respectively.

The following can be easily verified:

$$\begin{array}{ll}
\alpha \wedge (\beta \wedge \gamma) \equiv (\alpha \wedge \beta) \wedge \gamma & \alpha \vee (\beta \vee \gamma) \equiv (\alpha \vee \beta) \vee \gamma \\
\alpha \wedge \beta \equiv \beta \wedge \alpha & \alpha \vee \beta \equiv \beta \vee \alpha \\
\alpha \wedge \alpha \equiv \alpha & \alpha \vee \alpha \equiv \alpha \\
\alpha \wedge (\beta \vee \gamma) \equiv (\alpha \wedge \beta) \vee (\alpha \wedge \gamma) & \alpha \vee (\beta \wedge \gamma) \equiv (\alpha \vee \beta) \wedge (\alpha \vee \gamma) \\
\neg(\alpha \wedge \beta) \equiv \neg\alpha \vee \neg\beta & \neg(\alpha \vee \beta) \equiv \neg\alpha \wedge \neg\beta
\end{array}$$

Furthermore, $\alpha \vee \neg\alpha \equiv \top$ and $\alpha \wedge \neg\alpha \equiv \bot$ for all α. For implication, we adopt the custom that it is right associative: that is, $\alpha \to \beta \to \gamma$ means $\alpha \to (\beta \to \gamma)$. Then notice the interesting equivalence:

$$\alpha_1 \to \cdots \to \alpha_n \to \beta \equiv \bigwedge_{i=1}^{n} \alpha_i \to \beta$$

where $\bigwedge_{i=1}^{n} \alpha_i = \alpha_1 \wedge \cdots \wedge \alpha_n$.

Notice that $\equiv$ is an equivalence relation. Furthermore it is a *congruence* (the precise definition of this will be given in a later lecture): for $\mathsf{s} \in \ell$ and $\varphi_1, \ldots, \varphi_n$ and $\psi_1, \ldots, \psi_n$, if $\varphi_i \equiv \psi_i$ for all i then $\mathsf{s}\varphi_1 \cdots \varphi_n \equiv \mathsf{s}\psi_1 \cdots \psi_n$. Thus we get the following:

1.2.2 Lemma (The Replacement Lemma)

Suppose $\alpha \equiv \alpha'$, and let $\varphi \in \mathcal{F}_\ell$. Let φ' result from φ by replacing one or more instances of α in φ with α'. Then $\varphi \equiv \varphi'$.

This will be proven in more generality in a later lecture.

1.3 Normal Forms

1.3.1 Definition

Prime formulas and their negations are called **literals**. If α_i are conjunctions of literals, then $\bigvee_{i=1}^{n} \alpha_i$ is called a **disjunctive normal form** (DNF). Similarly if β_i are disjunctions of literals, then $\bigwedge_{i=1}^{n} \beta_i$ is called a **conjunctive normal form** (CNF).

We get the following important theorem.

1.3.2 Theorem

Every boolean function $f \colon \{0,1\}^n \longrightarrow \{0,1\}$ can be represented by a DNF α_f and a CNF β_f.

Proof: define $p^1 = p$ and $p^0 = \neg p$, then define

$$\alpha_f = \bigvee_{f\bar{x}=1} \bigwedge_{i=1}^{n} p_i^{x_i}$$

Notice that if $w(p_i) = x_i$, then if $f\bar{x} = 1$ then $\bigwedge_{i=1}^{n} p_i^{x_i}$ is in the disjunction, and

$$w \bigwedge_{i=1}^{n} p_i^{x_i} = \bigwedge_{i=1}^{n} w p_i^{x_i} = 1$$

Otherwise if $f\bar{x} = 0$ then for every $f\bar{y} = 1$, there is a $y_i \neq x_i$ and so $wp_i^{y_i} = 0$, so every conjunction is not satisfied, and thus α_f is not.

A similar proof goes for

$$\beta_f = \bigwedge_{f\bar{x}=0} \bigvee_{i=1}^{n} p_i^{\neg x_i}$$

■

Note that by definition if two formulas represent the same boolean function then they are logically equivalent. Thus we have:

1.3.3 Corollary

Every formula is logically equivalent to a CNF and DNF.

1.3.4 Definition

A logical signature ℓ is **functional complete** if every boolean function can be represented by a formula in $\mathcal{F}_\ell$.

Since every boolean function can be represented by a DNF and CNF, this means that the standard signature is functional complete. Furtheremore, we know that $\wedge$ can be represented by $\vee$: $\alpha \wedge \beta \equiv \neg(\neg\alpha \vee \neg\beta)$, and vice versa. This means that $\{\wedge, \neg\}$ and $\{\vee, \neg\}$ are both functional complete.

2 Lecture 2

Sources: A Concise Introduction to Mathematical Logic, Section 1, W. Rautenberg

We will need Zorn's Lemma for this lecture. Recall that it is equivalent to the axiom of choice, and it can be phrased as follows:

2.0.1 Lemma (Zorn's Lemma)

Let $(X, <)$ be a poset, and let $\mathcal{H} \subseteq X$ be a set with the property that every chain $\mathcal{C} \subseteq \mathcal{H}$ has an upper bound. Then $\mathcal{H}$ has a maximal element.

A chain is a set $\mathcal{C}$ such that for every $x \neq y \in \mathcal{C}$, either $x < y$ or $y < x$.

2.1 Logical Consequence

From now on, we write $w \vDash \varphi$ to mean that $w\varphi = 1$, and we say w *satisfies* φ. For a set of formulas X, we write $w \vDash X$ if w satisfies every formula in X.

Notice that

$$w \vDash \alpha \wedge \beta \iff w \vDash \alpha \text{ and } w \vDash \beta, \qquad w \vDash \alpha \vee \beta \iff w \vDash \alpha \text{ or } w \vDash \beta, \qquad w \vDash \neg\alpha \iff w \nvDash \alpha$$

Furthermore, for defined connectives like $\to$, $w \vDash \alpha \to \beta$ iff if $w \vDash \alpha$ then $w \vDash \beta$, etc.

2.1.1 Definition

Let X be a set of formulas, and φ another formula (all over the same logical signature). Then φ is a **logical consequence** of X, denoted $X \vDash \varphi$, if $w \vDash X$ implies $w \vDash \varphi$.

Note that $\vDash$ is used both for the satisfaction and consequence relation. It will be understood from context which relation is meant.

2.1.2 Definition

If $\varnothing \vDash \varphi$ (meaning $w \vDash \varphi$ for all w), then φ is a **tautology**. This is also denoted by $\vDash \varphi$. If no valuation satisfies φ it is called a **contradiction** (equivalently, $\neg\varphi$ is a tautology).

For example $\alpha \vee \neg\alpha, \alpha \to \alpha, \alpha \leftrightarrow \alpha$ are tautologies. $\alpha \wedge \neg\alpha, \alpha \leftrightarrow \neg\alpha$ are contradictions.

Now, important properties of the consequence relation are as follows:

$$\begin{array}{ll} \alpha \vDash \alpha & (\textit{reflexivity}) \\ X \vDash \alpha \text{ and } X \subseteq Y \text{ then } Y \vDash \alpha & (\textit{monotonicity}) \\ X \vDash Y \text{ and } Y \vDash \alpha \text{ then } X \vDash \alpha & (\textit{transitivity}) \end{array}$$

Another interesting property is the *deduction theorem*: $X, \alpha \vDash \beta$ if and only if $X \vDash \alpha \to \beta$. Indeed: suppose $X, \alpha \vDash \beta$ then let $w \vDash X$, if $w \vDash \alpha$ then $w \vDash \beta$ since $X, \alpha \vDash \beta$. Thus $X \vDash \alpha \to \beta$. And conversely, suppose $X \vDash \alpha \to \beta$, then if $w \vDash X$, if $w \vDash \alpha$ then $w \vDash \beta$ so for every $w \vDash X, \alpha$ also $w \vDash \beta$.

2.2 Gentzen Calculi

For this section we work over the functional complete logical signature $\{\neg, \wedge\}$.

We now define what it means to *prove* something. This, in my opinion, is not the most natural method of defining this (we will get to a more natural method in the future), but it is quite useful.

2.2.1 Definition

A **Gentzen calculus** $\vdash$ is a syntactic (as opposed to semantic) relation between sets of formulas and formulas. The Gentzen calculus is dictated by a set of rules of the form

$$\frac{X_1 \vdash \alpha_1 \mid \cdots \mid X_n \vdash \alpha_n}{X \vdash \alpha}$$

which tells us that if $X_i \vdash \alpha_i$ for all $i = 1, \ldots, n$, then $X \vdash \alpha$. The top line is called the premises, and the bottom line is called the result.

Our Gentzen calculus for propositional logic has the following six basic rules:

$$\text{(IS)} \quad \frac{}{\alpha \vdash \alpha} \qquad\qquad \frac{X \vdash \alpha}{X' \vdash \alpha} \quad (X \subseteq X') \quad \text{(MR)}$$

$$(\wedge 1) \quad \frac{X \vdash \alpha, \beta}{X \vdash \alpha \wedge \beta} \qquad\qquad \frac{X \vdash \alpha \wedge \beta}{X \vdash \alpha, \beta} \quad (\wedge 2)$$

$$(\neg 1) \quad \frac{X \vdash \alpha, \neg\alpha}{X \vdash \beta} \qquad\qquad \frac{X, \alpha \vdash \beta \mid X, \neg\alpha \vdash \beta}{X \vdash \beta} \quad (\neg 2)$$

IS stands for *initial sequent*, and MR stands for *monotonicity rule*. $X \vdash \alpha$ is called a sequent. A **derivation** is a sequence of sequents $(S_1; \ldots; S_n)$ such that each S_i can be derived from a basic rule with no premises, or can be derived from previous sequents in the sequence by applications of any of the basic rules. We write $X \vdash \alpha$ if there exists a derivation whose last sequent is $X \vdash \alpha$.

For example, $\alpha, \beta \vdash \alpha \wedge \beta$ is derivable:

$$\begin{pmatrix} \alpha \vdash \alpha & \alpha, \beta \vdash \alpha & \beta \vdash \beta & \alpha, \beta \vdash \beta & \alpha, \beta \vdash \alpha \wedge \beta \\ \text{IS} & \text{MR} & \text{IS} & \text{MR} & \wedge 1 \end{pmatrix}$$

Let us prove some more useful rules

$$\frac{X, \neg\alpha \vdash \alpha}{X \vdash \alpha}$$
(¬-elimination)

1	$X, \alpha \vdash \alpha$	(IS), (MR)
2	$X, \neg\alpha \vdash \alpha$	supposition
3	$X \vdash \alpha$	$(\neg 2)$

$$\frac{X, \neg\alpha \vdash \beta, \neg\beta}{X \vdash \alpha}$$
(reductio ad absurdum)

1	$X, \neg\alpha \vdash \beta, \neg\beta$	supposition
2	$X, \neg\alpha \vdash \alpha$	$(\neg 1)$
3	$X \vdash \alpha$	¬-elimination

$$\frac{X \vdash \alpha \to \beta}{X, \alpha \vdash \beta}$$
(→-elimination)

1	$X, \alpha, \neg\beta \vdash \alpha, \neg\beta$	(IS), (MR)
2	$X, \alpha, \neg\beta \vdash \alpha \wedge \neg\beta$	$(\wedge 1)$
3	$X \vdash \neg(\alpha \wedge \neg\beta)$	supposition
4	$X, \alpha, \neg\beta \vdash \neg(\alpha \wedge \neg\beta)$	(MR)
5	$X, \alpha, \neg\beta \vdash \beta$	$(\neg 1)$ on 2 and 4
6	$X, \alpha \vdash \beta$	¬-elimination

$$\frac{X \vdash \alpha \mid X, \alpha \vdash \beta}{X \vdash \beta}$$
(cut rule)

1	$X, \neg\alpha \vdash \alpha$	supposition, (MR)
2	$X, \neg\alpha \vdash \neg\alpha$	(IS), (MR)
3	$X, \neg\alpha \vdash \beta$	$(\neg 1)$
4	$X, \alpha \vdash \beta$	supposition
5	$X \vdash \beta$	$(\neg 2)$ on 3 and 4

$$\frac{X, \alpha \vdash \beta}{X \vdash \alpha \to \beta}$$

($\to$-introduction)	1	$X, \alpha \wedge \neg\beta, \alpha \vdash \beta$	supposition, (MR)
	2	$X, \alpha \wedge \neg\beta \vdash \alpha$	(IS), (MR), ($\wedge$2)
	3	$X, \alpha \wedge \neg\beta \vdash \beta$	cut rule
	4	$X, \alpha \wedge \neg\beta \vdash \neg\beta$	(IS), (MR), ($\wedge$2)
	5	$X, \alpha \wedge \neg\beta \vdash \alpha \to \beta$	($\neg$1)
	6	$X, \neg(\alpha \wedge \neg\beta) \vdash \alpha \to \beta$	(IS), (MR)
	7	$X \vdash \alpha \to \beta$	($\neg$2) on 5 and 6

$$\frac{X \vdash \alpha, \alpha \to \beta}{X \vdash \beta}$$

(modus ponens)	1	$X \vdash \alpha \to \beta$	supposition
	2	$X, \alpha \to \beta$	$\to$-elimination
	3	$X \vdash \alpha$	supposition
	4	$X \vdash \beta$	cut rule

Notice that $\to$-elimination and introduction give us the *syntactic deduction theorem*: $X, \alpha \vdash \beta \iff X \vdash \alpha \to \beta$.

Now suppose we have a Gentzen rule:

$$R\colon \frac{X_1 \vdash \alpha_1 \mid \cdots \mid X_n \vdash \alpha_n}{X \vdash \alpha}$$

then we say that a property of sequents $\mathcal{E}$ is *closed under* R if when $\mathcal{E}(X_1, \alpha_1), \ldots, \mathcal{E}(X_n, \alpha_n)$ then $\mathcal{E}(X, \alpha)$. (Formally a sequent is (X, α), $X \vdash \alpha$ is anothe rway of writing it.)

2.2.2 Lemma (Principle of Rule Induction)

Let $\mathcal{E}$ be a property of sequents closed under all the basic rules of $\vdash$. Then $X \vdash \alpha$ implies $\mathcal{E}(X, \alpha)$.

Proof: we will prove this for a general Gentzen calculus. We induct on the length of the derivation of $X \vdash \alpha$, which is $(S_1, \ldots, S_n)$. If $n = 1$ then $S_1 = X \vdash \alpha$ and there is a premise-less basic rule of $\vdash$ which gives us $X \vdash \alpha$ and so S_1 is a basic rule itself, so $\mathcal{E}(X, \alpha)$.

Now, suppose the derivation is $(S_1, \ldots, S_n)$ for $n > 1$. If S_n is premise-less then we are done. Otherwise we have $\mathcal{E}(S_1), \ldots, \mathcal{E}(S_{n-1})$ by induction (since a prefix of a derivation is a derivation itself). But $\mathcal{E}$ is closed under the basic rules, and S_n is obtained from $S_1, \ldots, S_{n-1}$ by the basic rules so $\mathcal{E}(S_n)$. ■

Notice that the property $\mathcal{E}(X, \alpha) =$ "$X \vDash \alpha$" is closed under all the basic rules of $\vdash$. Thus we have shown the *soundness* of $\vdash$: $X \vdash \alpha \implies X \vDash \alpha$.

2.2.3 Theorem (Finiteness of $\vdash$)

If $X \vdash \alpha$ then $X_0 \vdash \alpha$ for some finite $X_0 \subseteq X$.

Proof: we define $\mathcal{E}(X, \alpha) =$ "$X_0 \vdash \alpha$ for some finite $X_0 \subseteq X$" and show it is closed under the basic rules of $\vdash$.

(1) IS: for $X = \{\alpha\}$ take $X_0 = X$.

(2) MR: if $X_0 \vdash \alpha$ for finite $X_0 \subseteq X$, then for any $X' \supseteq X$, X_0 is still a finite subset of X'.

(3) $\wedge$1: if $X \vdash \alpha, \beta$ then there is $X_0, X_1 \subseteq X$ for which $X_0 \vdash \alpha$ and $X_1 \vdash \beta$, so by MR $X_2 = X_0 \cup X_1$ has $X_2 \vdash \alpha, \beta$ so $X_2 \vdash \alpha \wedge \beta$, and X_2 is finite.

(4) $\wedge$2: if $X_0 \vdash \alpha \wedge \beta$ is finite, then $X_0 \vdash \alpha, \beta$ is still finite.

(5) $\neg$1: similar to $\wedge$2.

(6) $\neg$2: if $X_0, \alpha \vdash \beta$ and $X_1, \neg\alpha \vdash \beta$ using our inductive hypothesis and MR, then using MR again we have $X_2, \alpha \vdash \beta$ and $X_2, \neg\alpha \vdash \beta$ for $X_2 = X_0 \cup X_1$. So then $X_2 \vdash \beta$. ■

2.2.4 Definition

A set of formulas is **inconsistent** (in $\vdash$) if $X \vdash \alpha$ for all formulas α. Otherwise X is **consistent**.

Note that X is inconsistent if and only if $X \vdash \bot$ by $\wedge 2$ and $\neg 1$.

Now, we are interested in *maximal consistency*, meaning X is consistent and any proper superset of X is inconsistent. If X is consistent, it is maximally consistent if and only if $\alpha \in X$ or $\neg\alpha \in X$ for all α. Indeed if X is maximally consistent and $\alpha, \neg\alpha \notin X$ then $X, \alpha \vdash \bot$ and $X, \neg\alpha \vdash \bot$ so $X \vdash \bot$ by $\neg 2$. And obviously if α or $\neg\alpha$ are in X for all α, it is maximal.

2.2.5 Lemma

$\vdash$ has the properties:

$$C^+ \colon X \vdash \alpha \iff X, \neg\alpha \vdash \bot, \qquad C^- \colon X \vdash \neg\alpha \iff X, \alpha \vdash \bot$$

Note that these can both be seen as a form of "proof by contradiction" where C^- is necessary since $\vdash$ is semantic.

Proof: suppose $X \vdash \alpha$ then $X, \neg\alpha \vdash \alpha$ by MR and since by MR as well we have $X, \neg\alpha \vdash \neg\alpha$, by $\neg 1$ we have $X, \neg\alpha \vdash \bot$. Conversely, if $X, \neg\alpha \vdash \bot$ then $X, \neg\alpha \vdash \alpha$ and so by $\neg$-elimination we have $X \vdash \alpha$. C^- is proven similarly. ∎

2.2.6 Lemma (Lindenbaum's Theorem)

Every consistent X can be extended to a maximally consistent $X' \supseteq X$.

Proof: let us define

$$\mathcal{H} = \{X' \mid X' \text{ is consistent and } X' \supseteq X\}$$

clearly, a maximal element of $\mathcal{H}$ is precisely a maximally consistent extension of X. So we must show that $\mathcal{H}$ has a maximal element, which is of course done with Zorn's Lemma. Let $\mathcal{C}$ be a chain in $\mathcal{H}$, and let us define $Y = \bigcup \mathcal{C}$, we claim that Y is an upper bound in $\mathcal{H}$ of $\mathcal{C}$. It is obviously an upper bound, all we need to show is that it is in $\mathcal{H}$, i.e. is consistent.

Suppose not, so $Y \vdash \bot$. By finiteness, there is a finite $Y_0 = \{\alpha_1, \dots, \alpha_n\} \subseteq Y$ such that $Y_0 \vdash \bot$. Suppose $\alpha_i \in X_i'$ for $X_i' \in \mathcal{C}$. Since $\mathcal{C}$ is a chain, there must be some X_i' which contains all other X_j's. Thus $Y_0 \subseteq X_i'$ and so by MR, $X_i' \vdash \bot$, contradicting X_i''s consistency. ∎

2.2.7 Lemma

A maximally consistent set X has the property $X \vdash \neg\alpha \iff X \nvdash \alpha$ for all α.

Proof: if $X \vdash \neg\alpha$ then $X \vdash \alpha$ cannot be due to X's consistency. Conversely if $X \nvdash \alpha$ then $\alpha \notin X$ so $\neg\alpha \in X$ so by IS and MR we have $X \vdash \neg\alpha$.

2.2.8 Lemma

A maximally consistent set X is satisfiable.

Proof: we define $w \vDash p$ if and only if $X \vdash p$. Then we will show that $X \vdash \alpha \iff w \vDash \alpha$. We proceed by formula induction. For prime formulas, this is trivial. And so:

$$X \vdash \alpha \wedge \beta \iff X \vdash \alpha, \beta \iff w \vDash \alpha, \beta \iff w \vDash \alpha \wedge \beta$$

and

$$X \vdash \neg\alpha \iff X \nvdash \alpha \iff w \nvDash \alpha \iff w \vDash \neg\alpha$$

Thus w models X, as required. ∎

2.2.9 Theorem (The Completeness Theorem)

$X \vdash \alpha$ if and only if $X \vDash \alpha$.

Proof: if $X \vdash \alpha$ by soundness, we have $X \vDash \alpha$. Now suppose $X \nvdash \alpha$ then $X, \neg\alpha$ is consistent (by C^+), and so it has a maximally consistent extension Y. This is satisfiable, so $X, \neg\alpha$ is satisfiable, meaning that $X \nvDash \alpha$. ■

Immediately, we get

2.2.10 Theorem (The Finiteness Theorem)

$X \vDash \alpha$ if and only if there is some finite $X_0 \subseteq X$ for which $X_0 \vDash \alpha$.

and

2.2.11 Theorem (The Compactness Theorem)

X is satisfiable if and only if every finite subset of X is satisfiable.

Proof: if X is satisfiable, so too is every subset. Conversely, if X is unsatisfiable then $X \vDash \bot$ and by finiteness, $X_0 \vDash \bot$ for some finite $X_0 \subseteq X$. ■

2.3 Applications of Compactness

We wish to prove the following lemma:

2.3.1 Lemma (König's Tree Lemma)

Let $(V, \lhd, r)$ be an infinite directed tree (so $\lhd$ is an irreflexive binary relation on V, V is infinite, and $r \in V$ is a special node in the tree called the *root* such that for every $a \in V$ there is precisely one path from r to a).

Now suppose that there exists arbitrarily long finite paths originating from r, and each $a \in V$ has finitely many successors, then there exists an infinite path originating from the root.

Proof: let us define "layers" of the tree inductively as follows: set $S_0 = \{c\}$ and

$$S_{k+1} = \{b \in V \mid a \lhd b \text{ for some } a \in S_k\}.$$

Since each node has finitely many successors, each S_k is finite.
Now for each $a \in V$, we define a propositional variable p_a. We will now define a set of formulas X which will say that there is an infinite path in V. This will be done by interpreting p_a as saying that the path traverses through a.

(1) for each k, $\bigvee_{a \in S_k} p_a$, since the path must traverse through some element in S_k.

(2) for each k and $a \neq b \in S_k$, $\neg(p_a \wedge p_b)$ since we want the path to traverse through only one node in each layer.

(3) $p_b \rightarrow p_a$ if $a \lhd b$ for $a, b \in V$, since if a node is in the path, so too must its parent.

Now since every finite subset $X_0 \subseteq X$ contains finitely many S_k's, it is satisfied by a finite path from the root through these layers. So X is satisfiable, suppose $w \vDash X$. Then we create a path $\{c_i\}_{i \in \mathbb{N}}$ where c_i is the node in S_i which is satisfied by w (i.e. c_i is the unique node in S_i for which $w \vDash p_{c_i}$).
Now notice that $c_0 = r$ and $c_i \lhd c_{i+1}$ for all i. This is because if a is the predecessor of $b = c_{i+1}$ then $w \vDash p_a$ in lieu of (3). So $\{c_i\}_{i \in \mathbb{N}}$ is an infinite path, as required. ■

2.4 Hilbert Calculi

What is the more natural calculus I mentioned earlier?

2.4.1 Definition

A **Hilbert calculus** $\vdash$ is one formed of

(1) **axioms**: a set of formulas Λ, and

(2) **rules of inference**: a set of relations Γ, where each $R \in \Lambda$ is a relation $R \subseteq \Gamma^n \times \Gamma$ for $n > 0$.

A **proof** of φ under a set of formulas X is a sequence of formulas $(\varphi_1, \ldots, \varphi_n = \varphi)$ such that each φ_i is either in $X \cup \Lambda$ or there exists a rule of inference $R \in \Gamma$ and for $\alpha_1, \ldots, \alpha_k$ occurring in the sequence before φ_i, $R(\alpha_1, \ldots, \alpha_k \implies \varphi_i)$. In such a case, we write $X \vdash \varphi$.

We can define a Hilbert calculus using the following axioms (which range over all α, β, γ):

$$\Lambda 1 \quad (\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma \qquad\qquad \Lambda 2 \quad \alpha \to \beta \to \alpha \wedge \beta$$

$$\Lambda 3 \quad \alpha \wedge \beta \to \alpha, \quad \alpha \wedge \beta \to \beta \qquad\qquad \Lambda 4 \quad (\alpha \to \neg\beta) \to \beta \to \neg\alpha$$

And the rule of inference of modus ponens: $(\alpha, \alpha \to \beta) \implies \beta$.

Now, we say that a property of formulas $\mathcal{E}$ is closed under a rule $R(\alpha_1, \ldots, \alpha_n \to \beta)$, if $\mathcal{E}\alpha_1, \ldots, \mathcal{E}\alpha_n$ implies $\mathcal{E}\beta$. So we can prove

2.4.2 Lemma (Principle of Induction for Hilbert Calculi)

Let X be a set of formulas and $\mathcal{E}$ a property of formulas $\mathcal{E}$. Then if

(1) $\mathcal{E}\alpha$ is true for all $\alpha \in X \cup \Lambda$ and

(2) $\mathcal{E}$ is closed under all the rules of inference of the Hilbert calculus,

$X \vdash \alpha$ implies $\mathcal{E}\alpha$.

Using this we can show the following:

2.4.3 Lemma

In our Hilbert calculus, if $X \vdash \alpha$ then $X \vDash \alpha$.

and

2.4.4 Lemma

A Hilbert calculus is reflexive ($\alpha \vdash \alpha$), monotonic ($X \vdash \alpha$ and $X \subseteq X'$ implies $X' \vdash \alpha$), and transitive (if $X \vdash Y$ and $Y \vdash \alpha$ then $X \vdash \alpha$).

And we can prove the following (in our Hilbert calculus):

2.4.5 Lemma

(1) $X \vdash \alpha \to \neg\beta$ implies $X \vdash \beta \to \neg\alpha$

(2) $\vdash \alpha \to \beta \to \alpha$

(3) $\vdash \alpha \to \alpha$

(4) $\vdash \alpha \to \neg\neg\alpha$

(5) $\vdash \beta \to \neg\beta \to \alpha$

2.4.6 Lemma (The Deduction Theorem)

$X, \alpha \vdash \beta$ iff $X \vdash \alpha \to \beta$.

(Hint: one direction uses induction.)

2.4.7 Lemma

$\vdash$ satisfies all the rules of our Gentzen calculus.

(Hint: only $\neg 2$ is complicated.)

2.4.8 Theorem (The Completeness Theorem)

$X \vdash \alpha \iff X \vDash \alpha$

3 Lecture 3

3.1 Signatures and Structures

Let us recall a few definitions that should be familiar with you all.

(1) A *group* is a mathematical structure $(G, \circ)$ which satisfies certain axioms.

(2) A *field* is a mathematical structure $(F, \cdot, +, 0, 1)$ which satisfies certain axioms.

(3) A *poset* is a mathematical structure $(X, <)$ which satisfies certain axioms.

We see that in all these examples, our structures are of the form $(A, \ldots)$ where $\ldots$ is a collection of operators, constants, and relations. Let us generalize this notion:

3.1.1 Definition

An **(extralogical) signature** is a set σ consisting of three types of symbols: operator symbols, constant symbols, and relation symbols. For each operator symbol and relation symbol, we assign it a value $\operatorname{ar}\mathsf{s} \in \mathbb{N}_{\geq 1}$.

Note that in general, we can think of constants as 0-ary functions (ones which take no inputs and give an output).

A signature is called *algebraic* if it has no relation symbols.

3.1.2 Definition

Let σ be a signature, then a **σ-structure** is an object $\mathcal{A}$ consisting of the following:

(1) A set A, called the **domain** of $\mathcal{A}$ (also denoted $\operatorname{dom}\mathcal{A}$).

(2) For every operator symbol $f \in \sigma$, an operator $f^{\mathcal{A}}\colon A^{\operatorname{ar} f} \longrightarrow A$.

(3) For every constant symbol $c \in \sigma$, a constant $c^{\mathcal{A}} \in A$.

(4) For every relation symbol $r \in \sigma$, a relation $r^{\mathcal{A}} \subseteq A^{\operatorname{ar} r}$.

These objects $\mathsf{s}^{\mathcal{A}}$ are called the **interpretations** of the symbols s.

For a σ-structure with domain A, we often write $\mathcal{A} = (A, \sigma^{\mathcal{A}})$. For example, given the signature $\{+, \cdot, 0, 1\}$, we can define the structure $\mathcal{N} = (\mathbb{N}, +^{\mathcal{N}}, \cdot^{\mathcal{N}}, 0^{\mathcal{N}}, 1^{\mathcal{N}})$, where all the symbols are interpreted canonically. We will also often be lazy and lose the exponent and just write the symbol.

3.1.3 Definition

Let $\mathcal{A}$ be a σ-structure, and $B \subseteq A$ such that

(1) For every $c \in \sigma$, $c^{\mathcal{A}} \in B$,

(2) For every $f \in \sigma$ and $\bar{b} \in B$, $f\bar{b} \in B$.

Then we can define the **substructure** of $\mathcal{A}$ whose domain is B and whose interpretations of the symbols are as follows:

$$c^{\mathcal{B}} = c^{\mathcal{A}}, \qquad f^{\mathcal{B}} = f^{\mathcal{A}}\big|_{B}, \qquad r^{\mathcal{B}} = r^{\mathcal{A}} \cap B^{\operatorname{ar} r}$$

Notice something interesting though: $\mathcal{Z} = (\mathbb{Z}, +)$ is a group, but $\mathcal{N} = (\mathbb{N}, +)$ is a substructure of $\mathcal{Z}$ and is not a group. But on the other hand all the substructures of $\mathcal{Z} = (\mathbb{Z}, +, -, 0)$ are groups. So the signature matters!

3.1.4 Definition

Let $\sigma_0 \subseteq \sigma$ be signatures, and $\mathcal{A}$ a σ-structure. We define the **σ_0-reduct** of $\mathcal{A}$ to be the σ_0-structure $\mathcal{A}_0 = \mathcal{A}\big|_{\sigma_0}$ where for every $\mathsf{s} \in \sigma$, $\mathsf{s}^{\mathcal{A}_0} = \mathsf{s}^{\mathcal{A}}$.

3.2 Homomorphisms and Isomorphisms

3.2.1 Definition

Let $\mathcal{A}, \mathcal{B}$ be σ-structures. Then $h: \mathcal{A} \longrightarrow \mathcal{B}$ is a **homomorphism** if for all $f, c, r \in \sigma$ and $\bar{a} \in A^n$:

$$hf^{\mathcal{A}}\bar{a} = f^{\mathcal{A}}h\bar{a}, \qquad hc^{\mathcal{A}} = c^{\mathcal{B}}, \qquad r^{\mathcal{A}}\bar{a} \implies r^{\mathcal{B}}h\bar{a}, \qquad \text{where } h\bar{a} = (ha_1, \dots, ha_n)$$

If we replace the third condition with $r^{\mathcal{B}}h\bar{a}$ if and only if there exists a $\bar{a}_0 \in A^n$ such that $h\bar{a} = h\bar{a}_0$ and $r^{\mathcal{A}}\bar{a}_0$:

$$r^{\mathcal{B}}h\bar{a} \iff (\exists \bar{a}_0 \in A^n)(h\bar{a} = h\bar{a}_0 \text{ and } r^{\mathcal{A}}\bar{a}_0$$

then h is a **strong homomorphism**. An injective strong homomorphism is an **embedding**, and a surjective embedding is an **isomorphism**.

Note that for an embedding, we have that

$$r^{\mathcal{B}}h\bar{a} \iff (\exists \bar{a}_0 \in A^n)(h\bar{a} = h\bar{a}_0 \text{ and } r^{\mathcal{A}}\bar{a}_0 \iff r^{\mathcal{A}}\bar{a}$$

since $h\bar{a} = h\bar{a}_0 \iff \bar{a} = \bar{a}_0$.

If there is an isomorphism $\iota: \mathcal{A} \longrightarrow \mathcal{B}$ then ι^{-1} is also an isomorphism. We write $\mathcal{A} \cong \mathcal{B}$ to say that the two structures are isomorphic. This is an equivalence relation.

Furthermore, if $h: \mathcal{A} \longrightarrow \mathcal{B}$ then $f\mathcal{A} \subseteq \mathcal{B}$ is a substructure.

3.2.2 Definition

A **congruence** on a σ-structure $\mathcal{A}$ is an equivalence relation θ such that for all $f \in \sigma$ and $\bar{a}, \bar{b} \in A^n$:

$$\bar{a}\ \theta\ \bar{b} \implies f^{\mathcal{A}}\bar{a}\ \theta\ f^{\mathcal{A}}\bar{b}$$

where $\bar{a}\ \theta\ \bar{b}$ means $a_i\ \theta\ b_i$ for $i = 1, \dots, n$.

Notice that if θ is a congruence on $\mathcal{A}$, then we can define a σ-structure on the quotient A/θ. We will denote this structure by $\mathcal{A}/\theta$, and we interpret the symbols of σ as

$$c^{\mathcal{A}/\theta} = c^{\mathcal{A}}/\theta, \qquad f^{\mathcal{A}/\theta}(\bar{a}/\theta) = (f^{\mathcal{A}}\bar{a})/\theta, \qquad r^{\mathcal{A}/\theta}(\bar{a}/\theta) \iff (\exists \bar{a}_0\ \theta\ \bar{a})r^{\mathcal{A}}\bar{a}_0$$

where a/θ is the equivalence class of a, and $\bar{a}/\theta = (a_1/\theta, \dots, a_n/\theta)$. Since θ is a congruence, these are all well-defined.

Notice that if $h: \mathcal{A} \longrightarrow \mathcal{B}$ is a homomorphism, then we can define its *kernel* to be the congruence $\ker h$:

$$a \ker h\ b \iff ha = hb$$

Verifying that this is a congruence is easy.

3.2.3 Theorem (The First Isomorphism Theorem)

(1) Let $\mathcal{A}$ be a σ-structure and θ a congruence on $\mathcal{A}$. Then $\pi: \mathcal{A} \longrightarrow \mathcal{A}/\theta$ defined by $a \mapsto a/\theta$ is a surjective strong homomorphism.

(2) Let $h: \mathcal{A} \longrightarrow \mathcal{B}$ be a strong homomorphism, then $\tilde{h}: \mathcal{A}/\ker h \longrightarrow h\mathcal{A}$ defined by $a/\ker h \mapsto ha$ is an isomorphism and $h = \tilde{h} \circ \pi$. In other words, $\mathcal{A}/\ker h \cong h\mathcal{A}$.

Proof: verify yourself. ■

So we get the classic commutative diagram (where h is surjective):

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\pi"] \arrow[dr, "h"'] & \mathcal{A}/\ker h \arrow[d, "\tilde{h}"] \\ & \mathcal{B} \end{tikzcd}$$

3.3 The Syntax of First-Order Logic

In order to discuss logic, we need a language to talk about. Namely, we want to define first order logic. This language will be built in two steps.

First, let us define $Var = \{v_1, v_2, \ldots\}$. Unlike in propositional logic, our set of variables is kept constant and global.

3.3.1 Definition

Let σ be an extralogical signature, we define σ-terms recursively as follows:

(1) $c \in \sigma$ and $x \in Var$ are both terms.

(2) if $f \in \sigma$ and $t_1, \ldots, t_n$ are terms, then so is $ft_1 \cdots t_n$.

Let the set of terms be denoted $\mathcal{T}_\sigma$ (we omit the subscript generally).

$\mathcal{T}_\sigma$ is an algebraic structure over the reduct of σ to only function and constant symbols. For $c \in \sigma$, $c^{\mathcal{T}_\sigma} = c$ (the string). And for $f \in \sigma$ and $t_1, \ldots, t_n \in \mathcal{T}_\sigma$, $f(t_1, \ldots, t_n) = ft_1 \cdots t_n$. Thus we call $\mathcal{T}_\sigma$ the *term algebra* over σ.

There is of course a notion of term induction: let $\mathcal{E}$ be a property of strings such that

(1) for $c \in \sigma$ and $x \in \mathrm{Var}$, both $\mathcal{E}c$ and $\mathcal{E}x$,

(2) for $f \in \sigma$ and $t_1, \ldots, t_n \in \mathcal{T}_\sigma$, $\mathcal{E}t_1, \ldots, \mathcal{E}t_n$ implies $\mathcal{E}ft_1 \cdot t_n$

then $\mathcal{E}t$ for all terms t.

There is also a unique reconstruction property for terms:

$$ft_1 \cdots t_n = fs_1 \cdots s_n \implies t_1 = s_1, \ldots, t_n = s_n$$

Thus we can define functions on terms by term recursion. Namely, we wish to define *vart*, the set of variables in t. This is done by

$$\mathrm{var}c = \varnothing, \qquad \mathrm{var}x = \{x\}, \qquad \mathrm{var}ft_1 \cdots t_n = \bigcup_{i=1}^{n} \mathrm{var}t_i$$

3.3.2 Definition

Let σ be an extralogical signature, we define σ-formulas recursively as follows:

(1) if $s, t \in \mathcal{T}_\sigma$ then $s \boldsymbol{=} t$ is a formula. We use the boldface $\boldsymbol{=}$ to distinguish the equality symbol and the metatheoretical equality.

(2) if $r \in \sigma$ and $t_1, \ldots, t_n \in \mathcal{T}_\sigma$ then $rt_1 \cdots t_n$ is a formula.

(3) If α, β are formulas and x is a variable, then $(\alpha \wedge \beta), \neg\alpha, \forall x\alpha$ are formulas.

Formulas constructed by (1) and (2) are called **atomic formulas**. Let us denote the set of σ-formulas by $\mathcal{L}_\sigma$. We will identify σ with $\mathcal{L}_\sigma$, and just write $\mathcal{L}$ in place of σ.

$\forall$ is called the *universal quantifier*.

Note that we take only the logical signature $\ell = \{\neg, \wedge\}$. We do not lose any generality since it is complete. We can abbreviate other logical symbols as before: $(\alpha \vee \beta) = \neg(\neg\alpha \wedge \neg\beta)$, etc.

We can also define the *existential quantifier* $\exists$ by $\exists x\alpha = \neg\forall x\neg\alpha$.
So for example, over the signature $\{+,<,1\}$, we can write the formulas

$$\forall x(x < x+1), \qquad \forall x\exists y(y = x+x), \qquad \forall x\exists y(x = 2y)$$

There is of course also formula induction and the unique formula reconstruction property. Thus we can define formulas by recursion on formulas. For example $\mathit{var}\varphi$:

$$\mathit{var}s = t = \mathit{var}s \cup \mathit{var}t, \qquad \mathit{var}rt_1\cdots t_n = \bigcup_{i=1}^{n} \mathit{var}t_i, \quad \mathit{var}\forall x\alpha = \{x\} \cup \mathit{var}\alpha$$

Let us examine the following formula (over the necessary signature):

$$\forall x\, x+y=0$$

here we quantify over x, but y remains unquantified. Similarly, we can examine

$$(\forall x\exists y\, x+y=0) \wedge (\forall x\, x<y)$$

which does have a quantifier which quantifies over y, but it also contains a y which is not within the scope of a quantifier of y. We call such variables *free*.
We define $\mathit{free}\varphi$ to be the set of free variables of φ:

$$\mathit{free}\alpha = \mathit{var}\alpha \text{ for atomic } \alpha, \qquad \mathit{free}(\alpha\wedge\beta) = \mathit{free}\alpha \cup \mathit{free}\beta, \qquad \mathit{free}\neg\alpha = \mathit{free}\alpha, \qquad \mathit{free}\forall x\alpha = \mathit{free}\alpha - \{\alpha\}$$

We write $\varphi = \varphi(\bar{x})$ to mean that φ is a variable with free variables in $\{x_1,\dots,x_n\}$, i.e. $\mathit{free}\varphi \subseteq \{x_1,\dots,x_n\}$. Similarly for a term t, we write $t = t(\bar{x})$ to mean $\mathit{var}t \subseteq \{x_1,\dots,x_n\}$.

3.3.3 Definition

An $\mathcal{L}$**-sentence** is an $\mathcal{L}$-formula with no free variables.

3.4 The Semantics of First-Order Logic

Just like we evaluated propositional formulas using a valuation, we will define a *first-order model* which will be used to evaluate formulas.

3.4.1 Definition

A **model** is a pair $(\mathcal{A}, w)$ consisting of an $\mathcal{L}$-structure $\mathcal{A}$ and a valuation $w\colon \mathit{Var} \longrightarrow A$ (we denote the image of x under w by x^w). We denote $r^{\mathcal{A}}, f^{\mathcal{A}}, c^{\mathcal{A}}, x^w$ by $r^{\mathcal{M}}, f^{\mathcal{M}}, c^{\mathcal{M}}, x^{\mathcal{M}}$.

Let $\mathcal{M}$ be a model and $t \in \mathcal{T}$ a term. Then we can define the value of t in $\mathcal{M}$, which we denote $t^{\mathcal{M}}$, recursively:

(1) $c^{\mathcal{M}}$ and $x^{\mathcal{M}}$ are already defined,

(2) $(ft_1\cdots t_n)^{\mathcal{M}} = f^{\mathcal{M}}(t_1^{\mathcal{M}},\dots,t_n^{\mathcal{M}})$.

We define the satisfaction relation as follows: for a model $\mathcal{M}$ and formula φ we write $\mathcal{M} \vDash \varphi$ to mean that φ is valid under $\mathcal{M}$. Formally, we do so recursively:

$$\begin{array}{lcl}
\mathcal{M} \vDash s = t & \iff & s^{\mathcal{M}} = t^{\mathcal{M}} \\
\mathcal{M} \vDash r\bar{t} & \iff & r^{\mathcal{M}}\bar{t}^{\mathcal{M}} \\
\mathcal{M} \vDash (\alpha\wedge\beta) & \iff & \mathcal{M} \vDash \alpha \text{ and } \mathcal{M} \vDash \beta \\
\mathcal{M} \vDash \neg\alpha & \iff & \mathcal{M} \nvDash \alpha \\
\mathcal{M} \vDash \forall x\alpha & \iff & \mathcal{M}_x^a \vDash \alpha \text{ for all } a \in A
\end{array}$$

For the last clause, we define $\mathcal{M}_x^a$ to be the model whose valuation is w_x^a which agrees with w except on x, which it defines to be a. We can similarly define $\mathcal{M}_{\bar{x}}^{\bar{a}}$ which maps $\bar{x}$ to $\bar{a}$. Then writing $\forall\bar{x}\varphi$ for $\forall x_1\cdots\forall x_n\varphi$,

$$\mathcal{M} \vDash \forall\bar{x}\varphi \iff \mathcal{M}_{\bar{x}}^{\bar{a}} \vDash \varphi \text{ for all } \bar{a} \in A$$

We can also see easily that $\mathcal{M} \vDash \alpha\vee\beta$ iff $\mathcal{M} \vDash \alpha$ or $\mathcal{M} \vDash \beta$, etc. And $\mathcal{M} \vDash \exists x\varphi \iff \mathcal{M}_x^a \vDash \varphi$ for some $a \in A$.

3.4.2 Definition

(1) A formula or set of formulas is **satisfiable** if it has a $\mathcal{L}$-model.

(2) A formula is a **tautology** if it is satisfied by every $\mathcal{L}$-model.

(3) $\alpha, \beta \in \mathcal{L}$ are **equivalent**, $\alpha \equiv \beta$, if for all $\mathcal{L}$-models $\mathcal{M}$, $\mathcal{M} \vDash \alpha \iff \mathcal{M} \vDash \beta$.

(4) If $\mathcal{A}$ is an $\mathcal{L}$-structure, then $\mathcal{A} \vDash \varphi$ iff $\mathcal{M} \vDash \varphi$ for every model over $\mathcal{A}$.

(5) Let X be a set of formulas, then $X \vDash \varphi$ iff $\mathcal{M} \vDash \varphi$ for all $\mathcal{M} \vDash X$ (this defines the consequence relation).

3.4.3 Lemma (The Coincidence Lemma)

Let $\varphi \in \mathcal{L}$ be an $\mathcal{L}$-formula, $\mathit{free}\varphi \subseteq U \subseteq \mathit{Var}$, and $\mathcal{M}, \mathcal{M}'$ be two $\mathcal{L}$-models over the same domain A such that $x^{\mathcal{M}} = x^{\mathcal{M}'}$ for all $x \in U$ and $\mathsf{s}^{\mathcal{M}} = \mathsf{s}^{\mathcal{M}'}$ for all extralogical symbols occurring in φ. Then $\mathcal{M} \vDash \varphi \iff \mathcal{M}' \vDash \varphi$.

Proof: first we prove by term induction that for any term $t(\bar{x})$ with $\bar{x} \subseteq U$, $t^{\mathcal{M}} = t^{\mathcal{M}'}$. For prime terms this is by assumption. Suppose $t = ft_1 \cdots t_n$, then by assumption $f^{\mathcal{M}} = f^{\mathcal{M}'}$ and by induction $t_i^{\mathcal{M}} = t_i^{\mathcal{M}'}$ for $i = 1, \dots, n$. So

$$t^{\mathcal{M}} = f^{\mathcal{M}}(t_1^{\mathcal{M}}, \dots, t_n^{\mathcal{M}}) = f^{\mathcal{M}'}(t_1^{\mathcal{M}'}, \dots, t_n^{\mathcal{M}'}) = t^{\mathcal{M}'}$$

as required.

Now we proceed by formula induction to show $\mathcal{M} \vDash \varphi \iff \mathcal{M}' \vDash \varphi$. For atomic formulas this is due to what we just proved:

$$\mathcal{M} \vDash t = s \iff t^{\mathcal{M}} = s^{\mathcal{M}} \iff t^{\mathcal{M}'} = s^{\mathcal{M}'} \iff \mathcal{M}' \vDash t = s, \qquad \mathcal{M} \vDash r\bar{t} \iff r^{\mathcal{M}}\bar{t}^{\mathcal{M}} \iff r^{\mathcal{M}'}\bar{t}^{\mathcal{M}'} \iff \mathcal{M}' \vDash r\bar{t}$$

For compound formulas this is trivial:

$$\mathcal{M} \vDash \alpha \wedge \beta \iff \mathcal{M} \vDash \alpha \text{ and } \mathcal{M} \vDash \beta \iff \mathcal{M}' \vDash \alpha \text{ and } \mathcal{M}' \vDash \beta \iff \mathcal{M}' \vDash \alpha \wedge \beta,$$
$$\mathcal{M} \vDash \neg\alpha \iff \mathcal{M} \nvDash \alpha \iff \mathcal{M}' \nvDash \alpha \iff \mathcal{M}' \vDash \neg\alpha$$

Now, notice that for $\varphi = \forall x\alpha$, let $a \in A$ then $\mathcal{M}_x^a$ and $(\mathcal{M}')_x^a$ agree on $U \cup \{x\}$. Since $\mathit{free}\varphi \subseteq U$, we must have that $\mathit{free}\alpha \subseteq U \cup \{x\}$. And thus we have $\mathcal{M}_x^a \vDash \alpha \iff (\mathcal{M}')_x^a \vDash \alpha$ for all $a \in A$, thus $\mathcal{M} \vDash \forall x\alpha \iff \mathcal{M}' \vDash \forall x\alpha$ as required. ■

We now adopt the notation that for $\varphi = \varphi(\bar{x})$ and $\bar{a} \in A$, $\mathcal{A} \vDash \varphi[\bar{a}]$ if $(\mathcal{A}, w) \vDash \varphi$ for $\bar{x}^w = \bar{a}$. In particular if φ is a sentence, then we can write $\mathcal{A} \vDash \varphi$. Similarly if $t = t(\bar{x})$ then we write $t^{\mathcal{A}}(\bar{a})$ for $t^{(\mathcal{A},w)}$ where $\bar{x}^w = \bar{a}$.

A corollary of this lemma is that if $\mathcal{L} \subseteq \mathcal{L}'$ and $\mathcal{M}$ is an $\mathcal{L}'$-model, then it can be expanded arbitrarily to $\mathcal{M}'$ an $\mathcal{L}'$-model by setting $\mathsf{s}^{\mathcal{M}'}$ arbitrarily for $\mathsf{s} \in \mathcal{L}' - \mathcal{L}$. Then for every $\varphi \in \mathcal{L}$, $\mathcal{M} \vDash_{\mathcal{L}} \varphi \iff \mathcal{M}' \vDash_{\mathcal{L}'} \varphi$. Thus for $X \subseteq \mathcal{L}$ and $\varphi \in \mathcal{L}$, $X \vDash_{\mathcal{L}} \varphi \iff X \vDash_{\mathcal{L}'} \varphi$.

Another corollary is the "omission of superfluous quantifiers": that is if $x \notin \mathit{free}\varphi$:

$$\forall x\varphi \equiv \varphi \equiv \exists x\varphi$$

Indeed if $x \notin \mathit{free}\varphi$ then $\mathcal{M}_x^a$ and $\mathcal{M}$ agree on $\mathit{free}\varphi$, and thus $\mathcal{M} \vDash \varphi \iff \mathcal{M}_x^a \vDash \varphi$.

Now, if $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{M} = (\mathcal{A}, w)$ and $\mathcal{M}' = (\mathcal{B}, w)$ then for every $t \in \mathcal{T}$, $t^{\mathcal{M}} = t^{\mathcal{M}'}$. This is proven easily using term induction.

3.4.4 Lemma (The Substructure Lemma)

For $\mathcal{L}$-structures $\mathcal{A}, \mathcal{B}$ with $A \subseteq B$, the following are equivalent:

(1) $\mathcal{A} \subseteq \mathcal{B}$,

(2) $\mathcal{A} \vDash \varphi[\bar{a}] \iff \mathcal{B} \vDash \varphi[\bar{a}]$ for all atomic $\varphi(\bar{x})$ and $\bar{a} \in A$,

(3) $\mathcal{A} \vDash \varphi[\bar{a}] \iff \mathcal{B} \vDash \varphi[\bar{a}]$ for all quantifier free $\varphi(\bar{x})$ and $\bar{a} \in A$.

Proof: (1) $\implies$ (2): it is sufficient to prove this for $\mathcal{M} = (\mathcal{A}, w)$ and $\mathcal{M}' = (\mathcal{B}, w)$ that $\mathcal{M} \vDash \varphi \iff \mathcal{M}' \vDash \varphi$ for quantifier free φ. Recall that $t^{\mathcal{M}} = t^{\mathcal{M}'}$, so we obtain the result immediately for atomic formulas.
(2) $\implies$ (3): we prove this by formula induction. The base case is our assumption and the two steps: conjunction and negation, are trivial.
(3) $\implies$ (2) is trivial, and we show (2) $\implies$ (1): we know that

$$r^{\mathcal{A}}\bar{a} \iff \mathcal{A} \vDash r\bar{x}[\bar{a}] \iff \mathcal{B} \vDash r\bar{x}[\bar{a}] \iff r^{\mathcal{B}}\bar{a}$$

and

$$f^{\mathcal{A}}\bar{a} = b \iff \mathcal{A} \vDash (f\bar{x} = y)[\bar{a}, b] \iff \mathcal{B} \vDash (f\bar{x} = y)[\bar{a}, b] \iff f^{\mathcal{B}}\bar{a} = b$$

thus $f^{\mathcal{A}}$ and $f^{\mathcal{B}}$ agree on A. Thus $\mathcal{A} \subseteq \mathcal{B}$ as required (we can view constants as 0-ary functions). ■

A formula of the form $\forall\bar{x}\varphi$ with φ quantifier free is called a *universal formula* or a $\forall$*-formula*. Similarly a formula of the form $\exists\bar{x}\varphi$ with φ quantifier free is a *existential formula* or a $\exists$*-formula*. For example, we can define the existential sentences:

$$\exists_1 = \exists v_1\, v_1 = v_1, \qquad \exists_n = \exists v_1 \cdots \exists v_n \bigwedge_{1 \le i < j \le n} v_i \neq v_j$$

so $\mathcal{A} \vDash \exists_n$ if and only if there exists at least n elements in $\mathcal{A}$. We can similarly define $\exists_{=n} = \exists_n \wedge \neg\exists_{n+1}$. Since structures must be nonempty, $\exists_1$ is a tautology, and thus we can define $\top = \exists_1$, and $\exists_0 = \bot = \neg\top$.

3.4.5 Corollary

Let $\mathcal{A} \subseteq \mathcal{B}$, then every $\forall$-sentence valid in $\mathcal{B}$ is valid in $\mathcal{A}$. Dually, every $\exists$-sentence valid in $\mathcal{A}$ is valid in $\mathcal{B}$.

Proof: if $\mathcal{B} \vDash \forall\bar{x}\varphi(\bar{x})$ then let $\bar{a} \in A$, so $\mathcal{B} \vDash \varphi[\bar{a}]$. By the lemma, since φ is quantifier-free, $\mathcal{A} \vDash \varphi[\bar{a}]$. Thus $\mathcal{A} \vDash \forall\bar{x}\varphi$. Similarly for the dual. ■

3.4.6 Theorem (The Invariance Theorem)

Let $\mathcal{A} \cong \mathcal{B}$ with an isomorphism $\iota\colon \mathcal{A} \longrightarrow \mathcal{B}$. Then for all $\varphi = \varphi(\bar{x})$ and $\bar{a} \in A$,

$$\mathcal{A} \vDash \varphi[\bar{a}] \iff \mathcal{B} \vDash \varphi[\iota\bar{a}]$$

Notice that in the case that φ is a sentence, this theorem tells us $\mathcal{A} \vDash \varphi \iff \mathcal{B} \vDash \varphi$. This theorem generalizes to higher-order logics as well, and it formalizes why we care about isomorphisms so much: isomorphic structures satisfy the same sentences and for all intents and purposes, the same.
Proof: it is convenient to reformulate this as $\mathcal{M} \vDash \varphi \iff \mathcal{M}' \vDash \varphi$ for $\mathcal{M} = (\mathcal{A}, w)$ and $\mathcal{M} = (\mathcal{B}, \iota \circ w)$. First we prove that $\iota(t^{\mathcal{M}}) = t^{\mathcal{M}'}$ for terms t: for prime terms we have $\iota(x^{\mathcal{M}}) = \iota \circ w(x) = x^{\mathcal{M}'}$ and $\iota(c^{\mathcal{M}}) = \iota(c^{\mathcal{A}}) = c^{\mathcal{B}} = c^{\mathcal{M}'}$. Inductively since ι is a homomorphism:

$$\iota((f\bar{t})^{\mathcal{M}}) = \iota(f^{\mathcal{M}}\bar{t}^{\mathcal{M}}) = f^{\mathcal{M}'}\iota\bar{t}^{\mathcal{M}} = f^{\mathcal{M}'}\bar{t}^{\mathcal{M}'} = (f\bar{t})^{\mathcal{M}'}$$

Notice that we only relied on ι being a homomorphism: this claim holds for all homomorphisms, not just isomorphisms.
We proceed by formula induction. For atomic formulas this is simple:

$$\mathcal{M} \vDash t = s \iff t^{\mathcal{M}} = s^{\mathcal{M}} \iff \iota(t^{\mathcal{M}}) = \iota(s^{\mathcal{M}}) \iff t^{\mathcal{M}'} = s^{\mathcal{M}'} \iff \mathcal{M} \vDash t = s$$

similar for relations. Conjunction and negation are clear, and quantification is due to ι's surjectivity. ■

3.4.7 Definition

Two $\mathcal{L}$-structures are **(elementarily) equivalent**, denoted $\mathcal{A} \equiv \mathcal{B}$, if $\mathcal{A} \vDash \varphi \iff \mathcal{B} \vDash \varphi$ for all sentences φ.

By the previous theorem, isomorphic structures are elementarily equivalent.

4 Lecture 4

4.1 Substitutions

A *(global) substitution* is a function $\sigma\colon Var \longrightarrow \mathcal{T}$ which substitutes variables with terms. We can then expand this to a function $\sigma\colon \mathcal{T} \longrightarrow \mathcal{T}$ recursively as follows:

$$c^\sigma = c, \qquad x^\sigma = \sigma(x), \qquad (f\bar{t})^\sigma = ft_1^\sigma \cdots t_n^\sigma$$

and finally to a function $\sigma\colon \mathcal{L} \longrightarrow \mathcal{L}$ recursively by

$$(t = s)^\sigma = t^\sigma = s^\sigma, \qquad (r\bar{t})^\sigma = r\bar{t}^\sigma, \qquad (\alpha \wedge \beta)^\sigma = \alpha^\sigma \wedge \beta^\sigma, \qquad (\neg\alpha)^\sigma = \neg\alpha^\sigma$$

finally for $\varphi = \forall x\alpha$ we define $\varphi^\sigma = \forall x\alpha^{\sigma'}$ where σ' agrees with σ for $y \in Var - \{x\}$ and $x^{\sigma'} = x$.

If σ maps only $x_1 \mapsto t_1, \ldots, x_n \mapsto t_n$ and keeps all other variables constant, we write $\varphi\frac{t_1,\ldots,t_n}{x_1,\ldots,x_n}$ for φ^σ. Such a substitution is called a *simultaneous substitution*, and if $n = 1$ a *simple substitution*. Notice that we can compose substitutions, but be careful – this notation can be dangerous! Note that in general we don't have

$$\varphi\frac{t_1,\ldots,t_n}{x_1,\ldots,x_n} = \varphi\frac{t_1}{x_1}\cdots\frac{t_n}{x_n}$$

take for example $t_1 = x_2$ and $t_2 = x_1$. Then $\frac{t_1,t_2}{x_1,x_2}$ swaps x_1 and x_2, but $\frac{t_1}{x_1}\frac{t_2}{x_2}$ will just swap all x_2 with x_1. We need a condition: indeed

$$\varphi\frac{t_1,\ldots,t_n}{x_1,\ldots,x_n} = \varphi\frac{t_1}{x_1}\cdots\frac{t_n}{x_n}, \qquad \text{if } x_i \notin \mathrm{var}t_j \text{ for } i \neq j$$

Now, notice that $\mathcal{M} \vDash \forall x\varphi$ does not imply $\mathcal{M} \vDash \varphi\frac{t}{x}$ for all $t \in \mathcal{T}$, as one might hope. Indeed take $\varphi = \exists y\, x \neq y$, then $\mathcal{M} \vDash \forall x\varphi = \forall x\exists y\, x \neq y$ whenever $\mathcal{M}$ has at least two elements. But $\mathcal{M} \nvDash \varphi\frac{y}{x} = \exists y\, y \neq y$. The issue here is that we substituted a variable within its scope with a term which includes it.

We would like to define a condition which allows us to avoid this.

4.1.1 Definition

Call $\varphi, \frac{t}{x}$ **collision-free** if the following hold recursively:

(1) if φ is prime,

(2) for $\varphi = \alpha \wedge \beta$ if $\alpha, \frac{t}{x}$ and $\beta, \frac{t}{x}$ are collision-free,

(3) for $\varphi = \neg\alpha$ if $\alpha, \frac{t}{x}$ is collision-free,

(4) if $\varphi = \forall x\alpha$,

(5) for $\varphi = \forall y\alpha$ and $x \neq y$, if $x \notin free\alpha$ or $y \notin vart$.

We then say that φ, σ is **collision-free** if $\varphi, \frac{x^\sigma}{x}$ is for every $x \in Var$.

This is a necessary and sufficient condition for everything we want, but it is a bit too complicated for our taste. So instead we use a more crude definition: we say that $\varphi, \frac{t}{x}$ is **collision-free** if $y \notin bnd\varphi$ for all $y \in vart - \{x\}$. Where $bnd\varphi$ is all the variables y such that $\forall y$ occurs in φ.

For $\mathcal{M} = (\mathcal{A}, w)$ and σ a substitution, define $\mathcal{M}^\sigma = (\mathcal{A}, w^\sigma)$ where $x^{w^\sigma} = \sigma(x)^\mathcal{M}$. In other words $x^{\mathcal{M}^\sigma} = (x^\sigma)^\mathcal{M}$. By term induction, we have $t^{\mathcal{M}^\sigma} = t^{\sigma\mathcal{M}}$. Notice that $\mathcal{M}^\sigma$ coincides with $\mathcal{M}^{\bar{t}^\mathcal{M}}_{\bar{x}}$ for $\sigma = \frac{\bar{t}}{\bar{x}}$.

4.1.2 Lemma (The Substitution Lemma)

Let $\mathcal{M}$ be a model and σ a substitution. Then for $\varphi \in \mathcal{L}$ such that φ, σ is collision-free:

$$\mathcal{M} \vDash \varphi^\sigma \iff \mathcal{M}^\sigma \vDash \varphi$$

Proof: for prime formulas $t = s$ we have

$$\mathcal{M} \vDash (t_1 = t_2)^\sigma \iff t_1^{\sigma\mathcal{M}} = t_2^{\sigma\mathcal{M}} \iff t_1^{\mathcal{M}^\sigma} = t_2^{\mathcal{M}^\sigma} \iff \mathcal{M}^\sigma \vDash t_1 = t_2$$

and prime formulas $r\bar{t}$ are proven similarly. Conjunction and negation are clear, all that remains is to show for $\varphi = \forall x\alpha$.

We have $\mathcal{M} \vDash (\forall x\alpha)^\sigma \iff \mathcal{M} \vDash \forall x\alpha^\tau$ for $x^\tau = x$ and $y^\tau = y^\sigma$ for $x \neq y$. This is equivalent to $\mathcal{M}_x^a \vDash \alpha^\tau$ for all $a \in A$. By the induction hypothesis, this is equivalent to $(\mathcal{M}_x^a)^\tau \vDash \alpha$. Now we claim that $\mathcal{M}_1 = (\mathcal{M}_x^a)^\tau = (\mathcal{M}^\sigma)_x^a = \mathcal{M}_2$. This is true since $x^{\mathcal{M}_1} = x^{\tau\mathcal{M}_x^a} = x^{\mathcal{M}_x^a} = a$ and $x^{\mathcal{M}_2} = a$. And for $x \neq y$, $y^{\mathcal{M}_1} = y^{\tau\mathcal{M}_x^a} = y^{\sigma\mathcal{M}_x^a}$, since $\forall x\alpha, \frac{y^\sigma}{y}$ is collision-free this means that $x \notin vary^\sigma$, so this is just equal to $y^{\sigma\mathcal{M}} = y^{\mathcal{M}^\sigma}$. And $y^{\mathcal{M}_2} = y^{\mathcal{M}^\sigma}$ as required.

Thus we have $(\mathcal{M}^\sigma)_x^a \vDash \varphi$ for all $a \in A$ and so $\mathcal{M}^\sigma \vDash \forall x\alpha$. ■

4.1.3 Corollary

If $\varphi, \frac{\bar{t}}{\bar{x}}$ is collision-free then

(1) $\forall\bar{x}\varphi \vDash \varphi\frac{\bar{t}}{\bar{x}}$.

(2) $\varphi\frac{\bar{t}}{\bar{x}} \vDash \exists\bar{x}\varphi$

(3) $\varphi\frac{s}{x}, s = t \vDash \varphi\frac{t}{x}$ provided $\varphi, \frac{s}{x}$ is collision-free.

Proof: (1): let $\mathcal{M} \vDash \forall\bar{x}\varphi$ then $\mathcal{M}_{\bar{x}}^{\bar{a}} \vDash \varphi$ for all $\bar{a} \in A$, in particular for $\bar{t}^{\mathcal{M}}$, so $\mathcal{M}_{\bar{x}}^{\bar{t}^{\mathcal{M}}} \vDash \varphi$. By the previous lemma, this means $\mathcal{M} \vDash \varphi\frac{\bar{t}}{\bar{x}}$.

(2) is obtained from (1) since $\varphi \vDash \psi$ implies $\neg\psi \vDash \neg\varphi$.

(3): let $\mathcal{M} \vDash \varphi\frac{s}{x}, s = t$, so $s^{\mathcal{M}} = t^{\mathcal{M}}$ and $\mathcal{M}_x^{s^{\mathcal{M}}} \vDash \varphi$ and so $\mathcal{M}_x^{t^{\mathcal{M}}} \vDash \varphi \implies \mathcal{M} \vDash \varphi\frac{t}{x}$. ■

Notice that we can define the *unique existential quantifier*: $\exists!$ by

$$\exists! x\alpha = \exists x\alpha \land \forall x\forall y\Big(\alpha \land \alpha\frac{y}{x} \to x = y\Big), \qquad \text{for } y \notin var\alpha$$

we can also define it by (again for $y \notin var\alpha$):

$$\exists! x\alpha = \exists x\forall y\Big(\varphi\frac{y}{x} \leftrightarrow x = y\Big)$$

4.2 Elementary Equivalence

4.2.1 Definition

Two $\mathcal{L}$-formulas, α, β are **elementarily equivalent** if $\mathcal{M} \vDash \alpha \iff \mathcal{M} \vDash \beta$. This is written $\alpha \equiv \beta$.

There are a few equivalent formulations of equivalence: $\vDash \alpha \leftrightarrow \beta$, $\alpha \vDash \beta$ and $\beta \vDash \alpha$, etc.

Notice that $\mathcal{L}$ forms an algebra over the signature $\{\land, \neg, \forall x \mid x \in Var\}$. So we can talk about congruences over $\mathcal{L}$, those are relations $\approx$ such that

$$\alpha \approx \alpha',\ \beta \approx \beta' \implies \alpha \land \beta \approx \alpha' \land \beta',\ \neg\alpha \approx \neg\alpha',\ \forall x\alpha \approx \forall x\alpha'$$

As is easily verified, $\equiv$ is a congruence.

4.2.2 Lemma (The Replacement Lemma)

Let $\approx$ be a congruence on $\mathcal{L}$, and $\alpha \approx \alpha'$. Let $\varphi \in \mathcal{L}$ and φ' be obtained by substituting one or more occurances of α with α' in φ. Then $\varphi \approx \varphi'$.

Proof: by formula induction. ■

4.2.3 Definition

Let $\mathcal{A}$ be an $\mathcal{L}$-structure. Then $\alpha, \beta \in \mathcal{L}$ are **equivalent in** $\mathcal{A}$ if $\mathcal{A}, w \vDash \alpha \iff \mathcal{A}, w \vDash \beta$ for all valuations w. We denote this $\alpha \equiv_{\mathcal{A}} \beta$.

This too is a congruence, is equivalent to $\mathcal{A} \vDash \alpha \leftrightarrow \beta$, and $\equiv \subseteq \equiv_{\mathcal{A}}$.
Trivially, an arbitrary intersection of congruences is a congruence. So if $\mathbf{K}$ is a class of $\mathcal{L}$-structures, $\equiv_{\mathbf{K}} = \bigcap_{\mathcal{A}\in\mathbf{K}} \equiv_{\mathcal{A}}$ is also a congruence.
The following is a list of simple equivalences:

(1) $\forall x(\alpha \wedge \beta) \equiv (\forall x\alpha) \wedge (\forall x\beta)$, $\exists x(\alpha \vee \beta) \equiv (\exists x\alpha) \vee (\exists x\beta)$,

(2) $\forall x \forall y \alpha \equiv \forall y \forall x \alpha$, $\exists x \exists y \alpha \equiv \exists y \exists x \alpha$,

If $x \notin \mathit{free}\beta$ then

(3) $\forall x(\alpha \vee \beta) \equiv (\forall x\alpha) \vee \beta$, $\exists x(\alpha \wedge \beta) \equiv (\exists x\alpha) \wedge \beta$,

(4) $\forall x\beta \equiv \beta \equiv \exists x\beta$,

(5) $\forall x(\alpha \to \beta) \equiv (\exists x\alpha) \to \beta$, $\exists x(\alpha \to \beta) \equiv (\forall x\alpha) \to \beta$.

A non-trivial equivalence is that of *renaming bound variables*: if $y \notin \mathit{var}\alpha$ then

$$\forall x\alpha \equiv \forall y\Big(\alpha\frac{y}{x}\Big), \qquad \exists x\alpha \equiv \exists y\Big(\alpha\frac{y}{x}\Big)$$

Indeed

$$\mathcal{M} \vDash \forall y\Big(\alpha\frac{y}{x}\Big) \iff \mathcal{M}^a_y \vDash \alpha\frac{y}{x} \text{ for all } a \iff (\mathcal{M}^a_y)^{y^{\mathcal{M}^a_y}}_x \vDash \alpha \iff (\mathcal{M}^a_y)^a_x \vDash \alpha$$

since $y \notin \mathit{var}\alpha$, its valuation has no effect on its satisfaction and thus

$$\iff \mathcal{M}^a_x \vDash \alpha \iff \mathcal{M} \vDash \forall x\alpha$$

4.2.4 Definition

A **prenex normal form** (PNF) is a formula of the form $\mathtt{Q}_1 x_1 \cdots \mathtt{Q}_n x_n \varphi$ where $\mathtt{Q}_i \in \{\forall, \exists\}$ are quantifiers and φ is quantifier-free.

4.2.5 Theorem

Every formula φ is equivalent to a formula in prenex normal form.

Proof: for each $\mathtt{Q}x$ let us consider the number of symbols $\neg, \wedge$ occurring to the left of $\mathtt{Q}x$, and let $s\varphi$ be the sum of these numbers. φ is clearly a PNF iff $s\varphi = 0$. We can iteratively decrement $s\varphi$ while remaining elementarily equivalent, and thus conclude that φ is equivalent to a PNF. So suppose $s\varphi > 0$, then there exists some $\mathtt{Q}x$ with a symbol $\neg$ or $\wedge$ before it. Then apply one of the following:

$$\neg\forall x \equiv \exists x\neg\alpha, \quad \neg\exists x\alpha \equiv \forall x\neg\alpha, \quad \beta \wedge \mathtt{Q}x\alpha \equiv \mathtt{Q}y\Big(\beta \wedge \alpha\frac{y}{x}\Big)$$

for $y \notin \mathit{var}\alpha, \mathit{var}\beta$. ∎

4.3 Logical Consequence

4.3.1 Definition

As before, for a set of $\mathcal{L}$-formuals X and a formula φ, we write $X \vDash \varphi$ to mean $\mathcal{M} \vDash X \implies \mathcal{M} \vDash \varphi$ for all $\mathcal{L}$-models $\mathcal{M}$. This is called the **consequence relation**.

We can state some properties about logical consequence:

(1) $\dfrac{X \vDash \forall x\alpha}{X \vDash \alpha\frac{t}{x}}$ for $\alpha, \frac{t}{x}$ collision-free.

(2) $\dfrac{X \vDash \alpha\frac{s}{x}, s = t}{X \vDash \alpha\frac{t}{x}}$ for $\alpha, \frac{t}{x}$, $\alpha, \frac{s}{x}$ collision-free.

(3) $\dfrac{X, \beta \vDash}{X, \forall x\beta \vDash \alpha}$ (anterior generalization).

(4) $\dfrac{X \vDash \alpha}{X \vDash \forall x\alpha}$ for $x \notin \mathit{free}X$ (posterior generalization).

(5) $\dfrac{X, \beta \vDash \alpha}{X, \exists x\beta \vDash \alpha}$ for $x \notin \mathit{free}X, \mathit{free}\alpha$ (anterior particularization).

(6) $\dfrac{X \vDash \alpha\frac{t}{x}}{X \vDash \exists x\alpha}$ for $\alpha, \frac{t}{x}$ collision-free (posterior particularization).

4.4 A Gentzen Calculus for FOL

We define a Gentzen calculus for first-order logic, which has the following basic rules:

(IS) $\dfrac{}{\alpha \vdash \alpha}$ $\qquad$ $\dfrac{}{t = t}$ (ES)

$$\frac{X \vdash \alpha}{X' \vdash \alpha} \quad (X \subseteq X') \qquad \text{(MR)}$$

($\wedge$1) $\dfrac{X \vdash \alpha, \beta}{X \vdash \alpha \wedge \beta}$ $\qquad$ $\dfrac{X \vdash \alpha \wedge \beta}{X \vdash \alpha, \beta}$ ($\wedge$2)

($\neg$1) $\dfrac{X \vdash \alpha, \neg\alpha}{X \vdash \beta}$ $\qquad$ $\dfrac{X, \beta \vdash \alpha \mid X, \neg\beta \vdash \alpha}{X \vdash \alpha}$ ($\neg$2)

($\forall$1) $\dfrac{X \vdash \forall x\alpha}{X \vdash \alpha\frac{t}{x}}$ $\left(\alpha, \dfrac{t}{x}\text{ collision-free}\right)$ $\qquad$ $\dfrac{X \vdash \alpha\frac{y}{x}}{X \vdash \forall x\alpha}$ $(y \notin \mathit{free}X \cup \mathit{var}\alpha)$ ($\forall$2)

(=) $\dfrac{X \vdash s = t, \alpha\frac{s}{x}}{X \vdash \alpha\frac{t}{x}}$ (α prime)

Note that every first-order language $\mathcal{L}$ defines a calculus, which we can denote with a subscript: $\vdash_{\mathcal{L}}$. This is an extension of our propositional logic Gentzen calculus, and thus all the rules we proved there hold here as well.

As before, since this is a Gentzen calculus, we can induct on it. Doing so, we can prove

4.4.1 Proposition

Suppose $\mathcal{L}$'s signature is σ. If $X \vdash_{\mathcal{L}} \alpha$, then there exists a finite $X_0 \subseteq X$ and a finite signature $\sigma_0 \subseteq \sigma$ such that $X_0 \vdash_{\mathcal{L}_0} \alpha$ (where $\mathcal{L}_0 = \mathcal{L}_{\sigma_0}$).

Of course we must have that $X_0, \alpha \subseteq \mathcal{L}_0$.

We can prove the following results:

$$\frac{X \vdash s = t, s = t'}{X \vdash t = t'}, \qquad \frac{X \vdash s = t}{X \vdash t = s}, \qquad \frac{X \vdash t = s, s = t'}{X \vdash t = t'}$$

To prove the first, let α be the formula $x = t'$ for $x \notin \mathit{var}t'$. Then the premise is $X \vdash s = t, \alpha\frac{s}{x}$. By (=) we have $X \vdash \alpha\frac{t}{x} = t = t'$. The second follows from the first by $t' = s$, and the third follows.

We can also prove

$$\frac{X \vdash t_i = t_i'}{X \vdash f\bar{t} = ft_1 \cdots t_i' \cdots t_n}, \qquad \frac{X \vdash t_i = t_i', r\bar{t}}{X \vdash t_1 \cdots t_i' \cdots t_n}$$

To prove the first, let $X \vdash t_i = t_i'$ and α be the formula $f\bar{t} = ft_1 \cdots x \cdots t_n$ where $x \notin vart_j$. Then by $(=)$, since $X \vdash \alpha\frac{t_i}{x}$, we have $X \vdash \alpha\frac{t_i'}{x}$ as required. Similarly for the second.
By applying these n times we get

$$\frac{X \vdash \bar{t} = \bar{t}'}{X \vdash f\bar{t} = f\bar{t}'}, \qquad \frac{X \vdash \bar{t} = \bar{t}', r\bar{t}}{X \vdash r\bar{t}'}$$

We can show $\vdash \exists x\, t = x$ for all terms t with $x \notin vart$. Since $(\forall 1)$ gives $\forall x\, t \neq x \vdash (t \neq x)\frac{t}{x} = t \neq t$, and certainly $\forall x\, t \neq x \vdash t = t$, so by $(\neg 1)$ we have $\forall x\, t \neq x \vdash \exists x\, t = x$. And trivially $\neg\forall x\, t \neq x \vdash \exists x\, t = x$, so by $(\neg 2)$, $\vdash \exists x\, t = x$. Similarly we can show $\exists x\, x = x = \top$.

4.4.2 Definition

A set of formulas $X \subseteq \mathcal{L}$ is **inconsistent** if $X \vdash \alpha$ for all $\alpha \in \mathcal{L}$. Otherwise X is consistent.

Inconsistency, as before, is equivalent to $X \vdash \bot$. This is since all $X \vdash \top = \exists x\, x = x$. And again, we can show the C^+, C^- properties:

$$C^+: \quad X \vdash \alpha \iff X, \neg\alpha \vdash \bot, \qquad C^-: \quad X \vdash \neg\alpha \iff X, \alpha \vdash \bot$$

4.5 Completeness

4.5.1 Definition

Let $\mathcal{L}$ be a language and C a set of constants, then $\mathcal{L}C$ is the language obtained by adding the constants in C to the language.

Let c be a constant, and x a variable, then let $\alpha\frac{z}{c}$ be the formula which results from substituting all occurrences of c with x.

4.5.2 Lemma

Suppose $X \vdash_{\mathcal{L}c} \alpha$, then $X\frac{z}{c} \vdash_{\mathcal{L}} \alpha\frac{z}{c}$ for all but a finite number of variables x.

Proof: we proceed by rule induction on $\vdash_{\mathcal{L}c}$. If $\alpha \in X$, then $\alpha\frac{z}{c} \in X\frac{z}{c}$; and if α is of the form $t = t$, so is $\alpha\frac{z}{c}$. So all rules but $(\forall 1), (\forall 2), (=)$ are clear. We prove the step for $(\forall 1)$. Let $\alpha, \frac{t}{x}$ be collision-free and $X \vdash_{\mathcal{L}c} \forall x\alpha$. Then we can assume that $X\frac{z}{c} \vdash_{\mathcal{L}} (\forall x\alpha)\frac{z}{c}$ for almost all z. So we can assume that $z \notin var(\forall x\alpha, t)$ (since there are only finitely many variables here). We can verify that $\alpha\frac{t}{x}\frac{z}{c} = \alpha'\frac{t'}{x}$ where $\alpha' = \alpha\frac{z}{c}$ and $t' = t\frac{z}{c}$. Since $\alpha', \frac{t'}{x}$ are collision-free, we have $X\frac{z}{c} \vdash_{\mathcal{L}} (\forall x\alpha)\frac{z}{c} = \forall x\alpha'$. By rule $(\forall 1)$ we have $X\frac{z}{c} \vdash_{\mathcal{L}} \alpha'\frac{t'}{x} = \alpha\frac{t}{x}\frac{z}{c}$ for almost all z. ■

Thus we get

$$(\forall 3)\ \frac{X \vdash \alpha\frac{c}{x}}{X \vdash \forall x\alpha} \qquad c \text{ not in } X, \alpha$$

Suppose $X \vdash \alpha\frac{c}{x}$, we can assume that X is finite and by the previous lemma (where $\mathcal{L}c = \mathcal{L}$), we take some y not in X, α such that $X = X\frac{y}{c} \vdash \alpha\frac{c}{x}\frac{y}{c} = \alpha\frac{y}{x}$. So by $(\forall 2)$ we have $X \vdash \forall x\alpha$.
We can also show

4.5.3 Lemma

Let C be a set of constants, then for $X \subseteq \mathcal{L}$ and $\alpha \in \mathcal{L}$, $X \vdash_{\mathcal{L}} \alpha \iff X \vdash_{\mathcal{L}C} \alpha$.

We now wish to prove that a consistent model is satisfiable (the converse is due to soundness). For every $x \in Var$ and $\alpha \in \mathcal{L}$, define a constants $c_{x,\alpha} \notin \mathcal{L}$. Define

$$\alpha^x = \neg\forall x\alpha \wedge \alpha\frac{c}{x} \qquad c = c_{x,\alpha}$$

Notice that $\neg\alpha^x = \exists x\neg\alpha \to \neg\alpha\frac{c}{x}$, that is if α doesn't hold for all x, $c_{x,\alpha}$ exists as a counterexample for it. Notice that $\alpha^x \equiv \bot$ whenever $x \notin free\alpha$.

4.5.4 Lemma

Let $\Gamma_{\mathcal{L}} = \{\neg\alpha^x \mid \alpha \in \mathcal{L}, x \in Var\}$, and let X be consistent. Then $X \cup \Gamma_{\mathcal{L}}$ is consistent as well.

Proof: suppose not; so $X \cup \Gamma_{\mathcal{L}} \vdash \bot$. By finiteness, we have $X \cup \{\neg\alpha_i^{x_i}\}_{i=1}^n \vdash \bot$. Since X is consistent, we can assume that $n \geq 1$ is minimal. Let $X' = X \cup \{\neg\alpha_i^{x_i}\}_{i=1}^{n-1}$ and $x = x_n, \alpha = \alpha_n, c = c_{x,\alpha}$. Then $X' \vdash \alpha^x$, so $X' \vdash \neg\forall x\alpha, \alpha\frac{c}{x}$. By ($\forall$3) we have $X' \vdash \forall x\alpha$, meaning $X' \vdash \bot$, contradicting n's minimality. ■

4.5.5 Definition

$X \subseteq \mathcal{L}$ is a **Henkin set** if it satisfies the following two conditions:

$$\text{(H1)} \quad X \vdash \neg\alpha \iff X \nvdash \alpha, \qquad \text{(H2)} \quad X \vdash \forall x\alpha \iff X \vdash \alpha\frac{c}{x} \text{ for all constants } c$$

(H1) and (H2) imply (H3): for each t there is a c such that $X \vdash t = c$. Indeed: $X \vdash \neg\forall x t \neq x$ for $x \notin vart$, and so $X \nvdash \forall x\, t \neq x$ by (H1) so $X \nvdash t \neq c$ for some c by (H2), and thus $X \vdash t = c$ by (H1).

4.5.6 Lemma

Let $X \subseteq \mathcal{L}$ be consistent. Then there exists a Henkin set $Y \supseteq X$ (in a suitable constant expansion of $\mathcal{L}$).

Proof: define $\mathcal{L}_0 = \mathcal{L}$ and $X_0 = X$. Define $\mathcal{L}_{n+1}$ and X_n as follows: let $\mathcal{L}_{n+1}$ be obtained from $\mathcal{L}_n$ by adding new constant symbols $c_{x,\alpha,n}$ for all $x \in Var, \alpha \in \mathcal{L}_n$. Then let $X_{n+1} = X_n \cup \Gamma_{\mathcal{L}_n}$ (where $c_{x,\alpha}$ is chosen to be $c_{x,\alpha,n}$). By the previous lemma, X_{n+1} is consistent. Then define $X' = \bigcup X_n \subseteq \mathcal{L}' = \bigcup \mathcal{L}_n$. By finiteness, and sonce $\{X_n\}$ form a chain, X' is also consistent. Let $\mathcal{H}$ be the set of consistent extensions of X', then $\mathcal{H}$ has a maximal element Y by Zorn's lemma (same proof as in propositional logic). We claim that Y is a Henkin set.
Let $\alpha \in \mathcal{L}_n$ where n is minimal, and α^x be taken with respect to $\mathcal{L}_n$, so $\neg\alpha^x \in X_{n+1} \subseteq X' \subseteq Y$. So we have that $Y \vdash \neg\alpha^x$ for all $\alpha \in \mathcal{L}'$.
For (H1): $Y \vdash \neg\alpha$ implies $Y \nvdash \alpha$ due to the consistency of Y. And conversely if $Y \nvdash \alpha$ then $\alpha \notin Y$ and so $Y, \alpha \vdash \bot$ since Y is maximally consistent and thus $Y \vdash \neg\alpha$ in lieu of C^-.
For (H2): if $X \vdash \forall x\alpha$ then for all constants c $X \vdash \alpha\frac{c}{x}$ by ($\forall$1). Now suppose $Y \vdash \alpha\frac{c}{x}$ for all c, in particular $c = c_{x,\alpha,n}$ where n is minimal with $\alpha \in \mathcal{L}_n$. Suppose $Y \nvdash \forall x\alpha$, then by (H1) $Y \vdash \neg\forall x\alpha, \alpha\frac{c}{x}$ so $Y \vdash \alpha^x$. This contradicts Y's consistency and $Y \vdash \neg\alpha^x$. ■

4.5.7 Lemma

Every Henkin set $Y \subseteq \mathcal{L}$ is satisfiable.

Proof: define a congruence on terms by $t \approx s$ whenever $Y \vdash t = s$. Let $A = Y/\approx$ be the resulting quotient algebra, let us denote the congruence class of t by $\bar{t}$. This will be the domain of our model $\mathcal{M} = (\mathcal{A}, w)$ for Y. Let C be the set of constants in $\mathcal{L}$, so by (H3) for each term $t \in \mathcal{T}$ there is some $c \in C$ such that $c \approx t$. So we can write $A = \{\bar{c} \mid c \in C\}$. Since $\approx$ is a congruence on the term algebra, $\mathcal{T}/\approx$ forms a quotient algebra. Specifically, we can define $x^{\mathcal{M}} = \bar{x}$ and $c^{\mathcal{M}} = \bar{c}$ for variables and constants, and

$$f^{\mathcal{M}}(\bar{t}_1, \dots, \bar{t}_n) = \overline{ft_1 \cdots t_n}$$

And for relations

$$r^{\mathcal{M}}\bar{t}_1 \cdots \bar{t}_n \iff Y \vdash r\vec{t}$$

This is because we showed $Y \vdash \vec{t} = \vec{t}', r\vec{t}$ then $Y \vdash t\vec{t}'$.
By term induction we have $t^{\mathcal{M}} = \bar{t}$ for all terms.
And we wish to prove $\mathcal{M} \vDash \alpha \iff Y \vdash \alpha$. For prime formulas, negations, and conjunctions this is clear. Now, we have

$$\mathcal{M} \vDash \forall x\alpha \iff \mathcal{M}_x^{\bar{c}} \vDash \alpha \text{ for all } c \in C \iff \mathcal{M}_x^{c^{\mathcal{M}}} \vDash \alpha \iff \mathcal{M} \vDash \alpha\frac{c}{x} \text{ (substitution theorem)}$$

$$\iff Y \vDash \alpha\frac{c}{x} \iff Y \vdash \forall x\alpha \text{ (H2)}$$

As in propositional logic, we immediately have the following:

4.5.8 Theorem (Completeness)

For any $X \subseteq \mathcal{L}$ and $\alpha \in \mathcal{L}$, $X \vdash \alpha \iff X \vDash \alpha$.

Proof: one direction follows by soundness. Conversely, suppose $X \nvdash \alpha$ then $X, \neg\alpha$ is consistent and has a Henkin expansion which has a model. Thus $\mathcal{M} \vDash X, \neg\alpha$ and so $X \nvDash \alpha$. ■

4.5.9 Theorem (Finiteness)

$X \vDash \alpha$ if and only if $X_0 \vDash \alpha$ for some finite $X_0 \subseteq X$.

4.5.10 Theorem (Compactness)

$X \subseteq \mathcal{L}$ is satisfiable if and only if every finite $X_0 \subseteq X$ is satisfiable.

4.5.11 Definition

A **theory** is a set of sentences T, such that $T \vDash \varphi \iff \varphi \in T$ for all sentences φ (i.e. T is deductively closed in the set of senetnces).

4.6 Theories and Applying Compactness

For example, if $\mathcal{A}$ is a structure, then $Th\mathcal{A} = \{\varphi \mid \varphi \text{ is a sentence and } \mathcal{A} \vDash \varphi\}$ is a theory. Indeed: if $Th\mathcal{A} \vDash \psi$ then since $\mathcal{A} \vDash Th\mathcal{A}$, we have $\mathcal{A} \vDash \psi$ so $\psi \in Th\mathcal{A}$. And if $X \subseteq \mathcal{L}$ is a set of formulas, we can look at its *deductive closure*: $X^{\vDash} = \{\alpha \text{ sentence} \mid X \vDash \alpha\}$ is a theory.

In particular, when S is a set of sentences and we say "the theory S", we are referring to its deductive closure $S^{\vDash}$. S is then an *axiom system* for this theory.

If T is a theory and α a sentence, then $T+\alpha$ is the smallest theory containing both T and α (namely $(T \cup \{\alpha\})^{\vDash}$). Similar if S is a set of sentences.

4.6.1 Definition

Let T be a theory, then two formulas α, β are **equivalent modulo** T, $\alpha \equiv_T \beta$ if $T \vDash \alpha \leftrightarrow \beta$ (equivalently $\equiv_T = \bigcap_{\mathcal{A} \vDash T} \equiv_{\mathcal{A}}$ so $\equiv_T$ is a congruence).

We now give an example of the application of the compactness theorem. Let $\mathbb{N}$ the natural numbers in any signature which contains $<$, $+$, and 1. We can then form a non-standard model of $Th\mathbb{N}$ (i.e. a model of it which is non-isomorphic to $\mathbb{N}$) as follows. Define

$$X = Th\mathbb{N} \cup \{x > n\}_{n \in \mathbb{N}} \qquad \text{where } n \text{ is } 1 + \cdots + 1$$

We claim that X has a model $\mathcal{M}$ (and thus $\mathbb{N}$ and $\mathcal{M}$ are non-isomorphic). Indeed, let $\Delta \subseteq X$ be a finite subset, then $\mathbb{N}$ models Δ by taking a sufficiently large interpretation of x (namely, if $x > n_1, \ldots, x > n_k$ are the formulas in X, then define x to be $\max\{n_i\}_{i=1}^{k} + 1$).

We can prove a similar result for $\mathbb{R}$.

Also notice:

4.6.2 Theorem (Intermediate Löwenheim-Skolem)

A consistent theory T in a countable language $\mathcal{L}$ has a countable model.

Proof: in our proof of the satisfiability of a Henkin set, the domain of the model is a quotient of the set of terms in $\mathcal{L}C$ where C is a set of constants obtained by expanding T to a Henkin set. In our expansion of T to a Henkin set, C is countable, and thus so is $\mathcal{L}C$. Since $\mathcal{L}C$ is countable, so is $\mathcal{T}C$, and therefore so is the constructed model of T. ■