Mathcord Mathematical Logic
Problem Set 2

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Problem 1

Prove the following:

$$\frac{X \vdash \alpha \to \beta}{X, \alpha \vdash \beta}, \qquad \frac{X, \alpha \vdash \beta}{X \vdash \alpha \to \beta}$$

Problem 2

Complete section 2.4: prove claims 2.4.2 through 2.4.8.

Problem 3

A *substitution* is a mapping $\sigma\colon V \longrightarrow \mathcal{F}$, which we extend to $\sigma\colon \mathcal{F} \longrightarrow \mathcal{F}$ using recursion:

$$(\alpha \wedge \beta)^\sigma = \alpha^\sigma \wedge \beta^\sigma, \qquad (\neg\alpha)^\sigma = \neg\alpha^\sigma$$

For a set of formulas $X \subseteq \mathcal{F}$, define $X^\sigma = \{\varphi^\sigma \mid \varphi \in X\}$. Verify that $\vDash$ is *substitution invariant*:

$$X \vDash \alpha \implies X^\sigma \vDash \alpha^\sigma$$

Problem 4

Let $\vdash \subseteq \mathcal{P}(\mathcal{F}) \times \mathcal{F}$ be a relation between sets of formulas and formulas (we write $X \vdash \varphi$). $\vdash$ is a *consequence relation* if it satisfies:

- **(1)** Reflexivity: $\{\alpha\} \vdash \alpha$
- **(2)** Monotonicity: $X \subseteq X'$ and $X \vdash \alpha$ implies $X' \vdash \alpha$
- **(3)** Transitivity: $X \vdash Y$ ($X \vdash \varphi$ for all $\varphi \in Y$) and $Y \vdash \alpha$ implies $X \vdash \alpha$
- **(4)** Substitution invariance: $X \vdash \alpha \implies X^\sigma \vdash \alpha^\sigma$ (see the previous question).

A consequence relation is called *finitary* if $X \vdash \alpha$ implies there exists a finite $X_0 \subseteq X$ such that $X_0 \vdash \alpha$. Call a consequence relation $\vdash$ *inconsistent* if it is trivial: $\vdash \alpha$ for all α (equivalently $\vdash \bot$). Otherwise $\vdash$ is consistent.

- **(1)** Let $\vdash$ be a consistent finitary consequence relation in $\mathcal{F}_{\{\wedge,\neg\}}$ which satisfies the properties ($\wedge$1) through ($\neg$2). Show that $\vdash$ is *maximally consistent* (meaning any consequence relation which contains $\vdash$ is inconsistent).
- **(2)** Show that there is exactly one consistent consequence relation in $\mathcal{F}_{\{\wedge,\neg\}}$ which satisfies ($\wedge$1) through ($\neg$2).
- **(3)** Conclude that $\vdash$ (our Gentzen calculus) is complete.

Problem 5

A *positive formula* is a formula in $\mathcal{F}_{\{\wedge,\vee\}}$. Let $w\colon V \longrightarrow \{0,1\}$ be a valuation, we can also equivalently view it as a set $A \subseteq V$. Call a set of formulas X *increasing* if $A \vDash X$ and $A \subseteq B$ implies $B \vDash X$. We say that

X is *equivalent* to Y if $A \vDash X \iff A \vDash Y$.

Show that

(1) $A \subseteq B$ if and only if every positive sentence which holds in A also holds in B.

(2) A consistent set of formulas X is increasing iff it is equivalent to a set of positive formulas.

(3) A formula φ is increasing (meaning $\{\varphi\}$ is) iff either φ is equivalent to a positive formula, φ is a tautology, or $\neg\varphi$ is a tautology.

Hints on next page

Hint for Problem 3

For valuation w, define w^σ such that $w \vDash \alpha^\sigma \iff w^\sigma \vDash \alpha$.

Hint for Problem 4

Let $\vdash' \supset \vdash$ be a proper extension of $\vdash$, so there exists X, φ where $X \nvdash \varphi$ and $X \vdash' \varphi$. Let Y be a maximal consistent extension of $X \cup \{\neg\varphi\}$ for $\vdash$ (why does this exist?). Define a substitution σ such that $p^\sigma = \top$ for $p \in Y$ and $p^\sigma = \bot$ for $p \notin Y$. Then show

$$\alpha \in Y \implies \vdash \alpha^\sigma, \qquad \alpha \notin Y \implies \vdash \neg\alpha^\sigma$$

Hint for Problem 5

(1) One direction uses formula induction, the other is trivial (since prime formulas are positive).

(2) Define $X^+ = \{\varphi \text{ positive} \mid X \vdash \varphi\}$, we claim that X and X^+ are equivalent. Obviously $A \vDash X \implies A \vDash X^+$, so let $A \vDash X^+$, define

$$Y = \{\neg\varphi \mid \varphi \text{ positive}, A \vDash \neg\varphi\}$$

Then show that $X \cup Y$ is consistent, and conclude from that that $A \vDash X$.

(3) Suppose φ is satisfiable, then let $X = \{\psi \mid \varphi \vdash \psi\}$. Then X is equivalent to a set of positive sentences X^+. Show then that there are formulas $\psi_1, \ldots, \psi_n \in X^+$ such that $\{\psi_1, \ldots, \psi_n\} \vDash \varphi$. Conclude that φ is equivalent to their conjunction.