Mathcord Mathematical Logic
Problem Set 1

Problem 1

(1) Show that $\{\neg, \to\}$ is functional complete.

(2) Show that $\vee$ can be represented in the logical signature $\{\to\}$.

(3) Show that any formula φ over the signature $\to$ has $w\varphi = 1$ for the valuation w which evaluates every variable as 1.

(4) Show that $\wedge$ cannot be represented in the logical signature $\{\to\}$.

(5) Show that $\leftrightarrow$ cannot be represented in the logical signature $\{\to\}$.

Problem 2

We define the boolean *nand* connective, $\uparrow$, as follows:

$$0 \uparrow 0 = 0 \uparrow 1 = 1 \uparrow 0 = 1, \qquad 1 \uparrow 1 = 0$$

and *nor*, $\downarrow$:

$$0 \downarrow 0 = 1, \qquad 0 \downarrow 1 = 1 \downarrow 0 = 1 \downarrow 1 = 0$$

(1) Verify that $\alpha \uparrow \beta \equiv \neg(\alpha \wedge \beta)$ and $\alpha \downarrow \beta \equiv \neg(\alpha \vee \beta)$.

(2) Show that both $\{\uparrow\}$ and $\{\downarrow\}$ are functional complete.

(3) Show that if $\circ$ is a boolean connective and $\{\circ\}$ is functional complete, $\circ \in \{\uparrow, \downarrow\}$.

Problem 3

Let $\varphi \to \psi$ be a tautology, then show that there exists a formula θ which uses only variables common to both φ and ψ (i.e. $\mathit{var}\theta \subseteq \mathit{var}\varphi \cap \mathit{var}\psi$), such that both $\varphi \to \theta$ and $\theta \to \psi$ are tautologies.

Problem 4

Let φ be a formula, we define its *dual*, φ^δ, recursively as follows:

$$\pi^\delta = \pi \text{ for prime } \pi, \qquad (\neg\alpha)^\delta = \neg\alpha^\delta, \qquad (\alpha \wedge \beta)^\delta = (\alpha^\delta \vee \beta^\delta), \qquad (\alpha \vee \beta)^\delta = (\alpha^\delta \wedge \beta^\delta)$$

Equivalently, it just substitutes all $\wedge$ in φ with $\vee$ and all $\vee$ with $\wedge$.
Let $f\colon \{0,1\}^n \longrightarrow \{0,1\}$ be a boolean function, we define its *dual* to be

$$f^\delta \bar{x} = \neg f(\neg\bar{x})$$

(1) Show that if α represents f, then α^δ represents f^δ.

(2) Show that $f \mapsto f^\delta$ is a bijection on boolean functions.

(3) Using the fact that every boolean function can be represented by a DNF, show that every boolean function can be represented by a CNF.

Problem 5

Let $\bar{x} = (x_1, \dots, x_n)$ and $\bar{y} = (y_1, \dots, y_n)$ be boolean vectors. We write $\bar{x} \le \bar{y}$ to mean $x_i \le y_i$ for all $i = 1, \dots, n$. A boolean function f is said to be *increasing* if $\bar{x} \le \bar{y}$ implies $f\bar{x} \le f\bar{y}$.

Show that every increasing boolean function can be represented in the logical signature $\{\wedge, \vee\}$ and vice versa: every formula in this logical signature is increasing.

For the next two problems we will need the following definitions. Let X be a set of formulas, and φ another formula. We say φ is a *consequence* of X, written $X \vDash \varphi$, if $w \vDash X$ implies $w \vDash \varphi$ for all valuations w.

Furthermore, we say that a set of formulas X is *independent* if for every $\varphi \in X$, $X - \{\varphi\} \nvDash \varphi$. Equivalently, there is a valuation w such that $w \vDash \psi$ for all $\psi \in X - \{\varphi\}$ and $w \vDash \neg\varphi$.

Finally, we say that two sets of formulas, X and Y, are *equivalent* if $w \vDash X$ if and only if $w \vDash Y$.

Problem 6

Suppose ℓ is functional complete and countable, and V is countable (thus $\mathcal{F}$ is countable). Show that every countable set of formulas X is equivalent to an independent set.

Problem 7

Prove the same thing from the previous problem, without assuming X is countable.

Problem 6, while doable, should be quite challenging. Problem 7 should be extremely challenging.

Good Luck!