Mathcord Mathematical Logic
Problem Set 3

Submission to mathcord.pset.submissions@gmail.com

Problem 1

Let $\mathcal{A}$ be a σ-structure and $\{\mathcal{B}_\lambda\}_{\lambda\in\Lambda}$ a non-empty family of substructures, then prove that $\mathcal{B} = \bigcap_{\lambda\in\Lambda}\mathcal{B}_\lambda$ is a substructure of $\mathcal{A}$ as well.

Thus if $S \subseteq A$, we can look at the *substructure generated by* S:

$$\langle S\rangle = \bigcap\{\mathcal{B} \subseteq \mathcal{A} \mid S \subseteq B\}$$

this is the smallest substructure of $\mathcal{A}$ containing S. Show that

$$\langle S\rangle = \left\{t^{\mathcal{A}}(a_1,\dots,a_n) \;\middle|\; t \in \mathcal{T},\ a_1,\dots,a_n \in S\right\}$$

Find lower and upper bounds on the cardinality of $|\langle S\rangle|$ in terms of $|S|$ and $|\mathcal{L}|$ (notice that $|\mathcal{L}|$ is the first infinite cardinality at least as large as $|\sigma|$).

Problem 2

The following are the four isomorphism theorems for groups:

(**1**) If $h\colon G \longrightarrow H$ is a homomorphism $G/\ker h \cong hG$,

(**2**) If $H \leq G$ is a subgroup and $N \trianglelefteq G$ a normal subgroup, then $HN \leq G$ and $N \cap H \trianglelefteq H$ and ${}^{HN}\!/_{N} \cong {}^{H}\!/_{H\cap N}$,

(**3**) If $N \leq K$ are normal subgroups of G, then $K/N \trianglelefteq G/N$ and

$${}^{G/N}\Big/_{K/N} \cong {}^{G}\!/_{K}$$

(**4**) If $N \trianglelefteq G$ is a normal subgroup of G, then there is a bijection of subgroups of G/N and intermediate subgroups $N \leq H \leq G$ given by

$$H \mapsto H/N$$

that is, every subgroup of G/N is of the form H/N for $N \leq H \leq G$, and all such Hs form a subgroup. Furthermore this bijection has the property:

- (**1**) $H_1 \leq H_2$ if and only if $H_1/N \leq H_2/N$,
- (**2**) if $H_1 \leq H_2$ then the indices are equal: $[H_2 : H_1] = [H_2/N : H_1/N]$ (where $[A : B] = |A/B|$),
- (**3**) $\langle H_1, H_2\rangle/N = \langle H_1/N, H_2/N\rangle$,
- (**4**) ${}^{H_1\cap H_2}\!/_{N} = {}^{H_1}\!/_{N} \cap {}^{H_2}\!/_{N}$,
- (**5**) $H \trianglelefteq G$ if and only if $H/N \trianglelefteq G/N$.

Formulate and prove analogous results for general σ-structures, where σ is an algebraic signature. (An analogous result for the first isomorphism theorem was proven in the lecture already.)

We say that φ and ψ are *(elementarily) equivalent* if for all models $\mathcal{M}$, $\mathcal{M} \vDash \varphi \iff \mathcal{M} \vDash \psi$. Denote this by $\varphi \equiv \psi$.

Problem 3

Show that

(**1**) A conjunction of $\exists_i$s and their negations is equivalent to $\exists_n \wedge \neg\exists_m$ for suitable n, m. (Note that $\exists_n \wedge \neg\exists_0 \equiv \exists_n$, and $\exists_1 \wedge \neg\exists_m \equiv \neg\exists_m$).

(**2**) A boolean combination of $\exists_i$ is equivalent to either $\bigvee_{i=0}^n \exists_{=k_i}$ or $\exists_k \vee \bigvee_{i=0}^n \exists_{=k_i}$ for $k_0 < \cdots < k_n$. (Note that $\bigvee_{i=0}^n \exists_{=k_i}$ is equal to $\exists_{=0} \equiv \bot$ for $n = k_0 = 0$, and $\neg\exists_n \equiv \bigvee_{i=0}^{n-1} \exists_{=i}$ for $n > 0$.)

Problem 4

Show that isomorphisms and elementary equivalence coincide for finite structures. That is, if $\mathcal{A}$ and $\mathcal{B}$ are finite structures, they are isomorphic if and only if they are elementarily equivalent. (Hint: why is it okay to assume that $\mathcal{L}$ is finite?)

Problem 5

Let σ be a finite signature, and κ an infinite cardinal.

(**1**) Show that there are at most 2^κ non-isomorphic σ-structures of cardinality κ.

(**2**) Find a finite signature σ such that there are exactly 2^κ non-isomorphic σ-structures of cardinality κ.

(**3**) Suppose σ consists only of k unary relation symbols. Let φ be a formula of length n (literally, its length as a string). Show that if φ has a model, it has a model of size $\leq n \cdot 2^k$.

Problem 6

Call a $\mathcal{L}$-formula a *literal* if it is atomic or the negation of an atomic formula. If C is a set of constants, let $\mathcal{L}C$ be the language obtained by adjoining constant symbols in C to the signature of $\mathcal{L}$. In particular if $\mathcal{A}$ is an $\mathcal{L}$-structure, let $\mathcal{L}\mathcal{A}$ be the language obtained by adding constant symbols for every $a \in A$ to $\mathcal{L}$. $\mathcal{A}$ can be canonically extended to a $\mathcal{L}\mathcal{A}$-structure in the natural way.

Let $\mathcal{A}$ be a $\mathcal{L}$-structure, define its *diagram* to be:

$$\Delta\mathcal{A} = \{\varphi \in \mathcal{L}\mathcal{A} \mid \varphi \text{ is a literal sentence and } \mathcal{A} \vDash \varphi\}$$

and its *positive diagram* to be:

$$\Delta^+\mathcal{A} = \{\varphi \in \mathcal{L}\mathcal{A} \mid \varphi \text{ is an atomic sentence and } \mathcal{A} \vDash \varphi\}$$

(a literal sentence is a literal which is a sentence, i.e. it has no variables. Atomic sentences are defined analogously.)

Let $\mathcal{B}_\mathcal{A}$ be an $\mathcal{L}\mathcal{A}$-structure, and let $\mathcal{B}$ be its $\mathcal{L}$-reduct. Show the following:

(**1**) $\mathcal{B}_\mathcal{A} \vDash \Delta^+\mathcal{A}$ if and only if there is a homomorphism $\mathcal{A} \longrightarrow \mathcal{B}$.

(**2**) $\mathcal{B}_\mathcal{A} \vDash \Delta\mathcal{A}$ if and only if there is an embedding $\mathcal{A} \longrightarrow \mathcal{B}$.

Hints on next page

Hint for Problem 2

Recall that the analog of a normal subgroup for general structures is a congruence. Thus for the second isomorphism theorem you need to define what the product of a substructure by a congruence is. Notice that for groups

$$g \in HN \iff \exists h \in H\colon gh^{-1} \in N \iff \exists h \in H\colon g\ \theta_N\ h$$

where θ_N is the congruence derived from N. Consider how to generalize this to a general congruence θ. You also need to generalize $H \cap N \trianglelefteq H$, notice that $\theta_{H\cap N} = \theta_N \cap H^2 = \theta_N\big|_H$.

For the third isomorphism theorem you need to generalize the congruence $\theta_{K/N}$ on G/N. That is, given congruences $\theta_2 \subseteq \theta_1$ on $\mathcal{A}$, define what a *quotient congruence* θ_1/θ_2 is on $\mathcal{A}/\theta_1$. Notice that

$$aN\ \theta_{K/N}\ bN \iff aN \cdot K/N = bN \cdot K/N \iff ab^{-1}N \in K/N \iff ab^{-1} \in K \iff a\ \theta_K\ b$$

Hint for Problem 5

(1) Let $|X| = \kappa$, how many distinct n-ary relations and functions are there on X? How many constants?

(2) Consider $\sigma = \{\leq, R\}$ where $\leq$ is a binary relation and R a unary relation. Consider only the σ-structures which are well-ordered (recall that isomorphic well-ordered sets have a unique isomorphism). Show that if R defines different subsets relative to $\leq$, then the structures are non-isomorphic.

(3) Let $\sigma = \{r_1, \ldots, r_k\}$. For a σ-structure $\mathcal{A}$ and a vector $\bar{\varepsilon} \in \{0,1\}^k$ define

$$r^{\mathcal{A}}_{\bar{\varepsilon}} = \bigcap_{i=1}^{k} \varepsilon_i r_i^{\mathcal{A}}$$

where $\varepsilon r^{\mathcal{A}}$ is $r^{\mathcal{A}}$ when $\varepsilon = 1$ and its complement otherwise. For $s \in \mathbb{N}_{>0}$, say that $\mathcal{A}$ and $\mathcal{B}$ are *s-close* if

$$\min\{s, |r^{\mathcal{A}}_{\bar{\varepsilon}}|\} = \min\{s, |r^{\mathcal{B}}_{\bar{\varepsilon}}|\}, \qquad \text{for all } \bar{\varepsilon} \in \{0,1\}^k$$

Call $\bar{a} \in A^n$ and $\bar{b} \in B^n$ *similar* if

$$\mathcal{A} \vDash \varphi[\bar{a}] \iff \mathcal{B} \vDash \varphi[\bar{b}], \qquad \text{for all atomic } \varphi$$

Finally define the *weight* of a formula φ to be the sum of the number of quantifiers in φ and the number of free variables.

Show the following:

(1) Suppose $\mathcal{A}, \mathcal{B}$ are s-close, $\bar{a} \in A^n, \bar{b} \in B^n$ are similar, and $n < s$. Then for every $a' \in A$ there exists a $b' \in B$ such that $(\bar{a}, a')$ and $(\bar{b}, b')$ are similar.

(2) Suppose $\mathcal{A}, \mathcal{B}$ are s-close, the weight of $\varphi(\bar{x})$ is less than s, and $\bar{a}, \bar{b}$ are similar. Then for all formulas φ, $\mathcal{A} \vDash \varphi[\bar{a}] \iff \mathcal{B} \vDash \varphi[\bar{b}]$.

(3) For any σ-structure $\mathcal{A}$, there is a $\mathcal{B}$ which is s-close to $\mathcal{A}$ and has at most $s \cdot 2^k$ elements. Conclude the desired result.

Hint for Problem 6

Suppose $\mathcal{B}_{\mathcal{A}} \vDash \Delta^+\mathcal{A}$, show that $h(a) = a^{\mathcal{B}_{\mathcal{A}}}$ is a homomorphism. If $\mathcal{B}_{\mathcal{A}} \vDash \Delta\mathcal{A}$ show that this homomorphism is an embedding.