# Mathcord Mathematical Logic
## Problem Set 2 Solution

**Problem 1**

Prove the following:

$$\frac{X \vdash \alpha \to \beta}{X, \alpha \vdash \beta}, \qquad \frac{X, \alpha \vdash \beta}{X \vdash \alpha \to \beta}$$

For the first proof:

**(1)** $X, \alpha, \neg\beta \vdash \alpha \wedge \neg\beta$ by IS, MR, and $\wedge$1

**(2)** $X, \alpha, \neg\beta \vdash \neg(\alpha \wedge \neg\beta)$ by supposition and MR

**(3)** $X, \alpha, \neg\beta \vdash \beta$ by $\neg$1

**(4)** $X, \alpha \vdash \beta$ by $\neg$-elimination

For the second proof:

**(1)** $X, \alpha \wedge \neg\beta, \alpha \vdash \beta$ by supposition and MR

**(2)** $X, \alpha \wedge \neg\beta \vdash \alpha$ by IS, MR, and $\wedge$2

**(3)** $X, \alpha \wedge \neg\beta \vdash \beta$ by the cut rule

**(4)** $X, \alpha \wedge \neg\beta \vdash \neg\beta$ by IS, MR, and $\wedge$2

**(5)** $X, \alpha \wedge \neg\beta \vdash \alpha \to \beta$ by $\neg$1

**(6)** $X, \neg(\alpha \wedge \neg\beta) \vdash \alpha \to \beta$ by IS, MR

**(7)** $X \vdash \alpha \to \beta$ by $\neg$2

**Problem 2**

Complete section 2.4: prove claims 2.4.2 through 2.4.8.

See A Concise Introduction to Mathematical Logic, Wolfgang Rautenberg, Section 1.6.

**Problem 3**

A *substitution* is a mapping $\sigma\colon V \longrightarrow \mathcal{F}$, which we extend to $\sigma\colon \mathcal{F} \longrightarrow \mathcal{F}$ using recursion:

$$(\alpha \wedge \beta)^\sigma = \alpha^\sigma \wedge \beta^\sigma, \qquad (\neg\alpha)^\sigma = \neg\alpha^\sigma$$

For a set of formulas $X \subseteq \mathcal{F}$, define $X^\sigma = \{\varphi^\sigma \mid \varphi \in X\}$. Verify that $\vDash$ is *substitution invariant*:

$$X \vDash \alpha \implies X^\sigma \vDash \alpha^\sigma$$

Let $w$ be a valuation, we define $w^\sigma$ such that $w \vDash \alpha^\sigma$ iff $w^\sigma \vDash \alpha$. In order for this to hold for prime formulas, we must have $w^\sigma \vDash \pi$ iff $w \vDash \pi^\sigma$, this defines the valuation. Now we must verify that this identity holds for compound formulas, this is easy.

So

$$w \vDash X^\sigma \implies w^\sigma \vDash X \implies w^\sigma \vDash \alpha \implies w \vDash \alpha^\sigma$$

thus $X^\sigma \vDash \alpha^\sigma$, as required.

**Problem 4**

Let $\vdash \subseteq \mathcal{P}(\mathcal{F}) \times \mathcal{F}$ be a relation between sets of formulas and formulas (we write $X \vdash \varphi$). $\vdash$ is a *consequence relation* if it satisfies:

**(1)** Reflexivity: $\{\alpha\} \vdash \alpha$

**(2)** Monotonicity: $X \subseteq X'$ and $X \vdash \alpha$ implies $X' \vdash \alpha$

**(3)** Transitivity: $X \vdash Y$ ($X \vdash \varphi$ for all $\varphi \in Y$) and $Y \vdash \alpha$ implies $X \vdash \alpha$

**(4)** Substitution invariance: $X \vdash \alpha \implies X^\sigma \vdash \alpha^\sigma$ (see the previous question).

A consequence relation is called *finitary* if $X \vdash \alpha$ implies there exists a finite $X_0 \subseteq X$ such that $X_0 \vdash \alpha$.
Call a consequence relation $\vdash$ *inconsistent* if it is trivial: $\vdash \alpha$ for all $\alpha$ (equivalently $\vdash \bot$). Otherwise $\vdash$ is consistent.

**(1)** Let $\vdash$ be a consistent finitary consequence relation in $\mathcal{F}_{\{\wedge,\neg\}}$ which satisfies the properties ($\wedge$1) through ($\neg$2). Show that $\vdash$ is *maximally consistent* (meaning any consequence relation which contains $\vdash$ is inconsistent).

**(2)** Conclude that $\vdash$ (our Gentzen calculus) is complete (is equal to $\vDash$).

**(1)** let $\vdash' \supset \vdash$ be a proper extension of $\vdash$, so there exist $X, \varphi$ such that $X \nvdash \varphi$ and $X \vdash' \varphi$. Let $Y$ be a maximal consistent extension of $X \cup \{\neg\varphi\}$ for $\vdash$. $Y$ exists since $X \cup \{\neg\varphi\}$ is consistent, and we can apply Zorn's lemma to $\mathcal{H} = \{Y \supseteq X \mid Y \text{ is consistent}\}$: let $\mathcal{C} \subseteq \mathcal{H}$ be a chain, then $\bigcup\mathcal{C}$ is consistent. For if $\mathcal{C} \vdash \bot$, then since $\vdash$ is finitary there exist $\varphi_1, \dots, \varphi_n \in \mathcal{C}$ such that $\varphi_1, \dots, \varphi_n \vdash \bot$. But since $\mathcal{C}$ is a chain, there exists a $C \in \mathcal{C}$ such that $\varphi_1, \dots, \varphi_n \in C$ and thus $C \in \mathcal{C} \subseteq \mathcal{H}$ is inconsistent, in contradiction.

Now let us define a substitution $\sigma$ where $\pi^\sigma = \top$ for $\pi \in Y$ and $\pi^\sigma = \bot$ otherwise. We claim

$$\alpha \in Y \implies \vdash \alpha^\sigma, \qquad \alpha \notin Y \implies \neg\alpha^\sigma$$

We prove this by induction. For prime $\pi$ this is trivial (since $\vdash$ satisfies $\wedge$1 through $\neg$2 we can show that $\vdash \top$ and $\vdash \neg\bot$, etc.).

Now for $\neg\alpha \in Y$, we have $\alpha \notin Y$ since $Y$ is consistent and so $\vdash \neg\alpha^\sigma$ as required. And for $\neg\alpha \notin Y$, we have $\alpha \in Y$ because it is maximally consistent, and so on.

For $\vdash'$ we have that $Y \vdash' \varphi, \neg\varphi$ and since $Y$ is maximally consistent, $\varphi, \neg\varphi \in Y$. Thus $\vdash' \varphi^\sigma, \neg\varphi^\sigma$. Thus $\vdash'$ is inconsistent.

**(2)** Let $\vdash$ be the smallest finitary consequence relation to satisfy $\wedge$1 through $\neg$2, this is our Gentzen calculus. Since $\vdash \subseteq \vDash$, and by the previous subquestion, $\vdash$ is maximal, $\vdash = \vDash$.

**(3)** This follows

**Problem 5**

A *positive formula* is a formula in $\mathcal{F}_{\{\wedge,\vee\}}$. Let $w\colon V \longrightarrow \{0,1\}$ be a valuation, we can also equivalently view it as a set $A \subseteq V$. Call a set of formulas $X$ *increasing* if $A \vDash X$ and $A \subseteq B$ implies $B \vDash X$. We say that $X$ is *equivalent* to $Y$ if $A \vDash X \iff A \vDash Y$.

Show that

**(1)** $A \subseteq B$ if and only if every positive sentence which holds in $A$ also holds in $B$.

**(2)** A consistent set of formulas $X$ is increasing iff it is equivalent to a set of positive formulas.

**(3)** A formula $\varphi$ is increasing (meaning $\{\varphi\}$ is) iff either $\varphi$ is equivalent to a positive formula, $\varphi$ is a tautology, or $\neg\varphi$ is a tautology.

**(1)** Let $A \subseteq B$, then by simple formula induction if $\varphi$ is positive then $A \vDash \varphi \implies B \vDash \varphi$. And conversely, if $A \vDash \varphi \implies B \vDash \varphi$ for all positive $\varphi$, in particular it holds for prime $\varphi = \pi$, and thus $\pi \in A \implies \pi \in B$.

(**2**) Suppose $X$ is increasing. Define $X^+ = \{\varphi \text{ positive} \mid X \vdash \varphi\}$, then $X$ and $X^+$ are equivalent. Obviously if $A \vDash X$ then $A \vDash X^+$. Conversely, let $A \vDash X^+$ and define

$$Y = \{\neg\varphi \mid \varphi \text{ positive},\ A \vDash \neg\varphi\}$$

then $X \cup Y$ is consistent, otherwise there exist $\varphi_1, \dots, \varphi_n$ positive such that $A \vDash \neg\varphi_i$ and $X, \neg\varphi_1, \dots, \neg\varphi_n$ is inconsistent. So

$$X \vdash \bigwedge_{i=1}^{n} \neg\varphi_i \to \bot \equiv \bigvee_{i=1}^{n} \varphi_i$$

So then $\bigvee_{i=1}^{n} \varphi_i \in X^+$, and so $A \vDash \bigvee_{i=1}^{n} \varphi_i$, and thus $A \vDash \varphi_i$ for some $i$. But $\neg\varphi_i \in Y$ and so $A \nvDash \varphi_i$ in contradiction.

Thus $X \cup Y$ is consistent, and has a model $B \vDash X \cup Y$. Since $B \cup Y$, $A \nvDash \varphi \implies B \nvDash \varphi$ for positive $\varphi$, thus $B \vDash \varphi \implies A \vDash \varphi$ for positive $\varphi$. By the previous subquestion, this means $B \subseteq A$, and since $X$ is increasing this means $A \vDash X$ as required.

Conversely, we showed that positive formulas are increasing and thus so is a set of positive formulas, and surely then a set equivalent to a set of positive formulas.

(**3**) Suppose neither $\varphi$ nor $\neg\varphi$ are tautologies. Then define $X = \{\psi \mid \varphi \vdash \psi\}$, then $X$ is increasing since $\varphi \in X$. So $A \vDash X$ means $A \vDash \varphi$ and so if $A \subseteq B$ then $B \vDash \varphi$ as well, and thus $B \vDash \psi$ for all $\psi \in X$. So by the previous subquestion, $X \equiv X^+$ for some set of positive formulas $X^+$.

Now, $X \vDash \varphi$ (since $\varphi \in X$), and thus $X^+ \vDash \varphi$, and by finitness, there exist $\psi_1, \dots, \psi_n \in X^+$ such that $\psi_1, \dots, \psi_n \vDash \varphi$. Now we claim that $\varphi \equiv \bigwedge_{i=1}^{n} \varphi_i$, we clearly have $\bigwedge_{i=1}^{n} \varphi_i \vDash \varphi$. ANd $\bigwedge_{i=1}^{n} \varphi_i \in X^+$ so if $A \vDash \varphi$ then $A \vDash X^+$ and so $A \vDash \bigwedge_{i=1}^{n} \varphi_i$. So we have $\varphi \vDash \bigwedge_{i=1}^{n} \varphi_i$ as well, as required.

**Problem 6**

A *graph* is a pair $G = (V, E)$ where $E$ is an irreflexive binary relation on $V$. Elements of $V$ are called *vertices* and pairs $(u, v) \in E$ are called *edges*. If $G$ is a graph, an *$n$-coloring* is a map $\pi\colon V \longrightarrow \{1, \dots, n\}$ such that for every edge $(u, v) \in E$ the vertices $u$ and $v$ are not given the same color:

$$(u, v) \in E \implies \pi(u) \neq \pi(v)$$

We say that $G$ is *$n$-colorable* if there exists an $n$-coloring on it. A *subgraph* of $G = (V, E)$ is a graph $G' = (V', E')$ where $V' \subseteq V$ and $E' = E \cap (V')^2$.

Show that an infinite graph $G$ (meaning $V$ is infinite) is $n$-colorable iff every finite subgraph is $n$-colorable.

Obviously if $G$ is $n$-colorable, so is every subgraph of $G$. Now, suppose every finite subgraph of $G$ is $n$-colorable. We will define a set of propositional formulas as follows: for every $v \in V$ and $i \in \{1, \dots, n\}$ define a propositional variable $p_{v,i}$. Now define $X$ to consist of all the formulas of the form

$$(1)\quad \bigvee_{i=1}^{n} p_{v,i} \text{ for all } v \in V, \qquad (2)\quad p_{v,i} \to \neg p_{v,j} \text{ for } v \in V, i \neq j, \qquad (3)\quad p_{u,i} \to \neg p_{v,i} \text{ for } (u, v) \in E$$

Now, if $w \vDash X$ then define $\pi\colon V \longrightarrow \{1, \dots, n\}$ where $\pi(v) = i$ if and only if $w \vDash p_{v,i}$. By formulas (1), every $v \in V$ has an image, and by (2) this image is unique. Thus $\pi$ is well-defined. By (3) it is a coloring.

So all we need to show is that $X$ is satisfiable. Let $X' \subseteq X$ be a finite subset, then we can define a finite subgraph $G'$ to consist of all vertices whose symbols occur in $X'$. Then we can extend $X'$ to $X''$ which contains all the formulas of type $(1), (2), (3)$ for vertices occurring in $G'$. This is satisfiable because $G'$ is $n$-colorable (take an $n$-coloring of $G'$, $\pi$, and define $w \vDash p_{v,i}$ if and only if $\pi(v) = i$).

Thus $X$ is finitely satisfiable, and thus satisfiable by the compactness theorem.