Mathematical Logic and Model Theory

Lectures by Slurp and Sharp

Contents

1 Lecture 1

Sources: A Concise Introduction to Mathematical Logic, Section 1, W. Rautenberg

We begin by defining *what* exactly mathematical logic is. Mathematical logic is a sort of metamathematical study of mathematics itself. It studies what sorts of logical statements we can make, how we can manipulate them, and what we can say about the mathematical objects which satisfy them. But the best way to understand what mathematical logic is, is to actually *do* it! So let's begin.

1.1 Propositional Logic

We begin our discussion with the simplest logic: propositional logic. This logic studies how we can connect propositions together, e.g. using *and*, *or*, *not*, etc. Suppose we wanted to say that "*if* it is cold outside *then* I will wear a coat", how could we go about this mathematically?

We begin with some definitions:

1.1.1 Definition

A **boolean function** is a function $\{0,1\}^n \longrightarrow \{0,1\}$ for some $n > 0$.

1.1.2 Definition

A **connective** is a symbol s with an associated boolean function (which will be named with s as well). A set of connectives is called a **logical signature**.

For boolean connectives $\circ$ (connectives whose function accepts two parameters), we can use a *truth table* to define it:

$$\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}$$

where a_{ij} is defined to be the value of $i \circ j$. So for example, we can define the following connective $\land$ as follows:

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

This connective takes two boolean values x and y and checks that both x *and* y are true. For this reason $\land$ is called *logical and* or a *conjunction*.

We can also define *logical or* or *disjunction* $\lor$:

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

And *logical negation* $\neg$, which is a unary connective: $\neg 0 = 1$ and $\neg 1 = 0$. Finally let us define *logical implication* $\to$:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Why should false imply false $(0 \to 0 = 1)$? Well suppose I said "if it rains then it is cloudy", but it is not raining. Is what I said false? Well, not necessarily! This is what is called a *vaccuous truth*.

Now suppose I wanted to string together connectives, like "if it rains then it is cloudy and I should wear a jacket".

1.1.3 Definition

Suppose we have a global set of variables V, whose elements are simply symbols which we will call *propositional variables*. Now suppose we have a logical signature ℓ, we define the set of **propositional formulas** $\mathcal{F}_\ell$ recursively as follows:

(1) If $p \in V$ is a propositional variable then it is a formula: $p \in \mathcal{F}_\ell$. Such formulas are called **prime formulas**.

(2) If $\mathsf{s} \in \ell$ is a connective of arity n and $\varphi_1, \ldots, \varphi_n \in \mathcal{F}_\ell$ are formulas, then so too is

$$\mathsf{s}\varphi_1 \cdots \varphi_n \in \mathcal{F}_\ell$$

So for example, if $\ell = \{\wedge, \vee, \neg\}$ and $V = \{p_1, p_2, \ldots\}$ then the following are formulas:

$$p_1, \qquad \wedge p_1 p_2, \qquad \wedge\vee p_1 p_2 \neg p_3, \qquad \neg\wedge\vee p_1 p_2 \wedge p_1 p_3$$

But using prefix notation like this is confusing, so we will adopt the custom that for binary connectives $\circ$, $\circ\varphi\psi$ is instead written as $(\varphi \circ \psi)$. So these formulas become

$$p_1, \qquad (p_1 \wedge p_2), \qquad ((p_1 \vee p_2) \wedge \neg p_3), \qquad \neg((p_1 \vee p_2) \wedge (p_1 \wedge p_3))$$

We will call the signature $\ell = \{\wedge, \vee, \neg\}$ the *standard signature*.

An important thing to keep in mind is that currently formulas are simply special strings. We haven't assigned to them any value yet.

Note that our definition of propositional formulas isn't really all that formal: how can we define a set using itself? Well formally, what we do is we look at the collection of all sets S of strings (over the alphabet $\ell \cup V$) with the properties that (1) $V \subseteq S$, (2) if $\mathsf{s} \in \ell$ has arity n and $\varphi_1, \ldots, \varphi_n \in S$ then $\mathsf{s}\varphi_1 \cdots \varphi_n \in S$. Then we simply define $\mathcal{F}_\ell$ to be the intersection of these sets.

From this definition the following is immediate:

1.1.4 Lemma (The Principle of Formula Induction)

Let ℓ be a logical signature. Suppose $\mathcal{E}$ is a property of strings (i.e. a subset of the set of all strings over $V \cup \ell$), with the following properties:

(1) For every $p \in V$, $\mathcal{E}p$.

(2) For every $\mathsf{s} \in \ell$ with arity n, if $\varphi_i \in \mathcal{F}_\ell$ for $i = 1, \ldots, n$ then $\mathcal{E}\mathsf{s}\varphi_1 \cdots \varphi_n$.

Then $\mathcal{E}\varphi$ holds for all formulas $\varphi \in \mathcal{F}_\ell$.

Now how can we be sure that if we have a formula φ, suppose of the form $\wedge\alpha\beta$, it is not simultaneously of the form $\wedge\alpha'\beta'$ for some other $\alpha, \beta \in \mathcal{F}$? We can use formula induction to prove the following:

1.1.5 Lemma (The Unique Formula Reconstruction Property)

Every compound formula $\varphi \in \mathcal{F}_\ell$ is of the form $\mathsf{s}\varphi_1 \cdots \varphi_n$ for uniquely determined s and φ_i for $i = 1, \ldots, n$.

Proof: s is obviously uniquely determined since it is the first character of φ. Now, we need to prove that if $\varphi_1 \cdots \varphi_n = \psi_1 \cdots \psi_m$ then $n = m$ and $n = m$. To do so, we need to prove the claim that a proper prefix of a formula is not itself a formula.

This uses formula induction: for prime formulas this is trivial. Otherwise, let $\varphi = \mathsf{s}\varphi_1 \cdots \varphi_n$ then a proper prefix of φ is either s which is not a formula, or of the form $\mathsf{s}\varphi_1 \cdots \varphi_i'$ where φ_i' is a prefix of φ_i. In order for $\mathsf{s}\varphi_1 \cdots \varphi_i'$ to be a formula, $\varphi_1 \cdots \varphi_i'$ must be able to be split into n formulas. So suppose $\varphi_1 \cdots \varphi_i' = \psi_1 \cdots \psi_n$, but φ_1 and ψ_1 cannot be prefixes of one another, so $\varphi_1 = \psi_1$. And so on until $i-1$. Then we have $\varphi_i' = \psi_i \cdots \psi_n$, but then if $\varphi_i' \neq \varphi_i$ we have that ψ_i is a proper prefix of φ_i, a contradiction to our inductive assumption. So $\varphi_i = \psi_i \cdots \psi_n$, so we get that $i = n$, contradicting the assumption that the prefix is proper.

Note that in proving this claim, we have proven precisely the unique reconstruction property. ■

Note that the reconstruction property holds true even when our strings use the custom that binary connectives are written as $(\alpha \circ \beta)$.

Using the unique reconstruction property, we can define functions on formulas in a recursive manner. For example, we can define the *substring function*:

$$Sf\pi = \{\pi\} \text{ for prime } \pi, \qquad Sf\mathsf{s}\varphi_1 \cdots \varphi_n = \bigcup_{i=1}^{n} Sf\varphi_i \cup \{\mathsf{s}\varphi_1 \cdots \varphi_n\}$$

So $Sf\varphi$ is precisely all the subformulas of φ. Note that Sf is well-defined precisely because of the unique reconstruction property: a formula cannot satisfy multiple conditions in the definition of Sf at once.

More importantly than this example, we can assign truth to formulas:

1.1.6 Definition

Let $w\colon V \longrightarrow \{0,1\}$ be a **valuation** – a mapping of truth values to each propositional variable. Then we can extend w to a function $w\colon \mathcal{F}_\ell \longrightarrow \{0,1\}$ by recursion as follows:

(1) For π prime, $w\pi$ is already defined ($\pi \in V$).

(2) If $\mathsf{s} \in \ell$ and $\varphi_1, \dots, \varphi_n \in \mathcal{F}_\ell$ then

$$w\mathsf{s}\varphi_1 \cdots \varphi_n = \mathsf{s}(w\varphi_1, \dots, w\varphi_n)$$

So for example suppose $w(p) = 1$ and $w(q) = 0$ then

$$w(p \wedge q) = 0, \qquad w(p \wedge (q \vee \neg q)) = 1$$

Let $var\varphi$ denote all the variables in V occurring in φ. This can be defined recursively as follows:

$$var\pi = \{\pi\} \text{ for } \pi \text{ prime}, \qquad var\mathsf{s}\varphi_1 \cdots \varphi_n = \bigcup_{i=1}^{n} var\varphi_i$$

Then notice that $w\varphi$ is dependent only on w's values on $var\varphi$. That is, we have the following:

1.1.7 Lemma

If w, w' are two valuations such that $w(p) = w'(p)$ for all $p \in var\varphi$, then $w\varphi = w'\varphi$.

Proof: by formula induction. ■

Now, suppose φ is a formula with $var\varphi \subseteq \{p_1, \dots, p_n\}$, then we write $\varphi(\bar{p})$ for φ. Now suppose $\varphi = \varphi(\bar{x})$ and w is a valuation with $wp_i = x_i$, then instead of writing $w\varphi$, we can write $\varphi(\bar{x})$. This is well-defined by the above lemma.

Now, suppose that for every $n \in \mathbb{N}$ we have $p_n \in V$. Then we can define $\mathcal{F}_\ell^n = \{\varphi \in \mathcal{F}_\ell \mid var\varphi \subseteq \{1, \dots, n\}\}$, the set of all formulas whose variables are contained in $p_1, \dots, p_n$.

1.1.8 Definition

We say that a formula $\varphi \in \mathcal{F}_n^\ell$ **represents** a boolean function $f\colon \{0,1\}^n \longrightarrow \{0,1\}$ if for every valuation w,

$$w\varphi = f(wp_1, \dots, wp_n)$$

Since $w\varphi$ is dependent only $wp_1, \dots, wp_n$, f is uniquely determined: define $f(x_1, \dots, x_n)$ to be $w\varphi$ where $w(p_i) = x_i$ and $w(p)$ arbitrary for all other variables. So we can denote f by $\varphi^{(n)}$ since it is unique.

Notice that implication can be represented by $\neg(p_1 \wedge \neg p_2)$, thus in the standard signature, we can use $(\alpha \to \beta)$ as a standin for $\neg(\alpha \wedge \neg\beta)$. Similarly $\leftrightarrow$ can be represented by $(\alpha \to \beta) \wedge (\beta \to \alpha)$.

1.2 Logical Equivalence

1.2.1 Definition

Say two formulas α, β are **logically equivalent** if $w\alpha = w\beta$ for every valuation w. Denote this by $\alpha \equiv \beta$.

For example, $\alpha \equiv \neg\neg\alpha$. We can define $\top = p \vee \neg p$ and $\bot = p \wedge \neg p$ for $p \in V$. Notice that $w\top = 1$ for all w, and $w\bot = 0$, so $\top$ and $\bot$ represent truth and false respectively.

The following can be easily verified:

$$\begin{array}{ll}
\alpha\wedge(\beta\wedge\gamma)\equiv(\alpha\wedge\beta)\wedge\gamma & \alpha\vee(\beta\vee\gamma)\equiv(\alpha\vee\beta)\vee\gamma \\
\alpha\wedge\beta\equiv\beta\wedge\alpha & \alpha\vee\beta\equiv\beta\vee\alpha \\
\alpha\wedge\alpha\equiv\alpha & \alpha\vee\alpha\equiv\alpha \\
\alpha\wedge(\beta\vee\gamma)\equiv(\alpha\wedge\beta)\vee(\alpha\wedge\gamma) & \alpha\vee(\beta\wedge\gamma)\equiv(\alpha\vee\beta)\wedge(\alpha\vee\gamma) \\
\neg(\alpha\wedge\beta)\equiv\neg\alpha\vee\neg\beta & \neg(\alpha\vee\beta)\equiv\neg\alpha\wedge\neg\beta
\end{array}$$

Furthermore, $\alpha\vee\neg\alpha\equiv\top$ and $\alpha\wedge\neg\alpha\equiv\bot$ for all α. For implication, we adopt the custom that it is right associative: that is, $\alpha\to\beta\to\gamma$ means $\alpha\to(\beta\to\gamma)$. Then notice the interesting equivalence:

$$\alpha_1\to\cdots\to\alpha_n\to\beta\equiv\bigwedge_{i=1}^n\alpha_i\to\beta$$

where $\bigwedge_{i=1}^n\alpha_i=\alpha_1\wedge\cdots\wedge\alpha_n$.

Notice that $\equiv$ is an equivalence relation. Furthermore it is a *congruence* (the precise definition of this will be given in a later lecture): for $\mathsf{s}\in\ell$ and $\varphi_1,\dots,\varphi_n$ and $\psi_1,\dots,\psi_n$, if $\varphi_i\equiv\psi_i$ for all i then $\mathsf{s}\varphi_1\cdots\varphi_n\equiv\mathsf{s}\psi_1\cdots\psi_n$. Thus we get the following:

1.2.2 Lemma (The Replacement Lemma)

Suppose $\alpha\equiv\alpha'$, and let $\varphi\in\mathcal{F}_\ell$. Let φ' result from φ by replacing one or more instances of α in φ with α'. Then $\varphi\equiv\varphi'$.

This will be proven in more generality in a later lecture.

1.3 Normal Forms

1.3.1 Definition

Prime formulas and their negations are called **literals**. If α_i are conjunctions of literals, then $\bigvee_{i=1}^n\alpha_i$ is called a **disjunctive normal form** (DNF). Similarly if β_i are disjunctions of literals, then $\bigwedge_{i=1}^n\beta_i$ is called a **conjunctive normal form** (CNF).

We get the following important theorem.

1.3.2 Theorem

Every boolean function $f\colon\{0,1\}^n\longrightarrow\{0,1\}$ can be represented by a DNF α_f and a CNF β_f.

Proof: define $p^1=p$ and $p^0=\neg p$, then define

$$\alpha_f=\bigvee_{f\bar{x}=1}\bigwedge_{i=1}^n p_i^{x_i}$$

Notice that if $w(p_i)=x_i$, then if $f\bar{x}=1$ then $\bigwedge_{i=1}^n p_i^{x_i}$ is in the disjunction, and

$$w\bigwedge_{i=1}^n p_i^{x_i}=\bigwedge_{i=1}^n wp_i^{x_i}=1$$

Otherwise if $f\bar{x}=0$ then for every $f\bar{y}=1$, there is a $y_i\neq x_i$ and so $wp_i^{y_i}=0$, so every conjunction is not satisfied, and thus α_f is not.

A similar proof goes for

$$\beta_f=\bigwedge_{f\bar{x}=0}\bigvee_{i=1}^n p_i^{\neg x_i} \qquad \blacksquare$$

Note that by definition if two formulas represent the same boolean function then they are logically equivalent. Thus we have:

1.3.3 Corollary

Every formula is logically equivalent to a CNF and DNF.

1.3.4 Definition

A logical signature ℓ is **functional complete** if every boolean function can be represented by a formula in $\mathcal{F}_\ell$.

Since every boolean function can be represented by a DNF and CNF, this means that the standard signature is functional complete. Furtheremore, we know that $\wedge$ can be represented by $\vee$: $\alpha \wedge \beta \equiv \neg(\neg\alpha \vee \neg\beta)$, and vice versa. This means that $\{\wedge, \neg\}$ and $\{\vee, \neg\}$ are both functional complete.

2 Lecture 2

Sources: A Concise Introduction to Mathematical Logic, Section 1, W. Rautenberg

We will need Zorn's Lemma for this lecture. Recall that it is equivalent to the axiom of choice, and it can be phrased as follows:

2.0.1 Lemma (Zorn's Lemma)

Let $(X, <)$ be a poset, and let $\mathcal{H} \subseteq X$ be a set with the property that every chain $\mathcal{C} \subseteq \mathcal{H}$ has an upper bound. Then $\mathcal{H}$ has a maximal element.

A chain is a set $\mathcal{C}$ such that for every $x \neq y \in \mathcal{C}$, either $x < y$ or $y < x$.

2.1 Logical Consequence

From now on, we write $w \vDash \varphi$ to mean that $w\varphi = 1$, and we say w *satisfies* φ. For a set of formulas X, we write $w \vDash X$ if w satisfies every formula in X.

Notice that

$$w \vDash \alpha \wedge \beta \iff w \vDash \alpha \text{ and } w \vDash \beta, \qquad w \vDash \alpha \vee \beta \iff w \vDash \alpha \text{ or } w \vDash \beta, \qquad w \vDash \neg\alpha \iff w \nvDash \alpha$$

Furthermore, for defined connectives like $\to$, $w \vDash \alpha \to \beta$ iff if $w \vDash \alpha$ then $w \vDash \beta$, etc.

2.1.1 Definition

Let X be a set of formulas, and φ another formula (all over the same logical signature). Then φ is a **logical consequence** of X, denoted $X \vDash \varphi$, if $w \vDash X$ implies $w \vDash \varphi$.

Note that $\vDash$ is used both for the satisfaction and consequence relation. It will be understood from context which relation is meant.

2.1.2 Definition

If $\varnothing \vDash \varphi$ (meaning $w \vDash \varphi$ for all w), then φ is a **tautology**. This is also denoted by $\vDash \varphi$. If no valuation satisfies φ it is called a **contradiction** (equivalently, $\neg\varphi$ is a tautology).

For example $\alpha \vee \neg\alpha, \alpha \to \alpha, \alpha \leftrightarrow \alpha$ are tautologies. $\alpha \wedge \neg\alpha, \alpha \leftrightarrow \neg\alpha$ are contradictions.

Now, important properties of the consequence relation are as follows:

$$\begin{array}{ll} \alpha \vDash \alpha & (\textit{reflexivity}) \\ X \vDash \alpha \text{ and } X \subseteq Y \text{ then } Y \vDash \alpha & (\textit{monotonicity}) \\ X \vDash Y \text{ and } Y \vDash \alpha \text{ then } X \vDash \alpha & (\textit{transitivity}) \end{array}$$

Another interesting property is the *deduction theorem*: $X, \alpha \vDash \beta$ if and only if $X \vDash \alpha \to \beta$. Indeed: suppose $X, \alpha \vDash \beta$ then let $w \vDash X$, if $w \vDash \alpha$ then $w \vDash \beta$ since $X, \alpha \vDash \beta$. Thus $X \vDash \alpha \to \beta$. And conversely, suppose $X \vDash \alpha \to \beta$, then if $w \vDash X$, if $w \vDash \alpha$ then $w \vDash \beta$ so for every $w \vDash X, \alpha$ also $w \vDash \beta$.

2.2 Gentzen Calculi

For this section we work over the functional complete logical signature $\{\neg, \wedge\}$.

We now define what it means to *prove* something. This, in my opinion, is not the most natural method of defining this (we will get to a more natural method in the future), but it is quite useful.

2.2.1 Definition

A **Gentzen calculus** $\vdash$ is a syntactic (as opposed to semantic) relation between sets of formulas and formulas. The Gentzen calculus is dictated by a set of rules of the form

$$\frac{X_1 \vdash \alpha_1 \mid \cdots \mid X_n \vdash \alpha_n}{X \vdash \alpha}$$

which tells us that if $X_i \vdash \alpha_i$ for all $i = 1, \dots, n$, then $X \vdash \alpha$. The top line is called the premises, and the bottom line is called the result.

Our Gentzen calculus for propositional logic has the following six basic rules:

$$\text{(IS)} \quad \frac{}{\alpha \vdash \alpha} \qquad\qquad \frac{X \vdash \alpha}{X' \vdash \alpha} \quad (X \subseteq X') \quad \text{(MR)}$$

$$(\wedge 1) \quad \frac{X \vdash \alpha, \beta}{X \vdash \alpha \wedge \beta} \qquad\qquad \frac{X \vdash \alpha \wedge \beta}{X \vdash \alpha, \beta} \quad (\wedge 2)$$

$$(\neg 1) \quad \frac{X \vdash \alpha, \neg\alpha}{X \vdash \beta} \qquad\qquad \frac{X, \alpha \vdash \beta \mid X, \neg\alpha \vdash \beta}{X \vdash \beta} \quad (\neg 2)$$

IS stands for *initial sequent*, and MR stands for *monotonicity rule*. $X \vdash \alpha$ is called a sequent. A **derivation** is a sequence of sequents $(S_1; \dots; S_n)$ such that each S_i can be derived from a basic rule with no premises, or can be derived from previous sequents in the sequence by applications of any of the basic rules. We write $X \vdash \alpha$ if there exists a derivation whose last sequent is $X \vdash \alpha$.

For example, $\alpha, \beta \vdash \alpha \wedge \beta$ is derivable:

$$\begin{pmatrix} \alpha \vdash \alpha & \alpha, \beta \vdash \alpha & \beta \vdash \beta & \alpha, \beta \vdash \beta & \alpha, \beta \vdash \alpha \wedge \beta \\ \text{IS} & \text{MR} & \text{IS} & \text{MR} & \wedge 1 \end{pmatrix}$$

Let us prove some more useful rules

$$\frac{X, \neg\alpha \vdash \alpha}{X \vdash \alpha}$$
($\neg$-elimination)

1	$X, \alpha \vdash \alpha$	(IS), (MR)
2	$X, \neg\alpha \vdash \alpha$	supposition
3	$X \vdash \alpha$	($\neg 2$)

$$\frac{X, \neg\alpha \vdash \beta, \neg\beta}{X \vdash \alpha}$$
(reductio ad absurdum)

1	$X, \neg\alpha \vdash \beta, \neg\beta$	supposition
2	$X, \neg\alpha \vdash \alpha$	($\neg 1$)
3	$X \vdash \alpha$	$\neg$-elimination

$$\frac{X \vdash \alpha \to \beta}{X, \alpha \vdash \beta}$$
($\to$-elimination)

1	$X, \alpha, \neg\beta \vdash \alpha, \neg\beta$	(IS), (MR)
2	$X, \alpha, \neg\beta \vdash \alpha \wedge \neg\beta$	($\wedge 1$)
3	$X \vdash \neg(\alpha \wedge \neg\beta)$	supposition
4	$X, \alpha, \neg\beta \vdash \neg(\alpha \wedge \neg\beta)$	(MR)
5	$X, \alpha, \neg\beta \vdash \beta$	($\neg 1$) on 2 and 4
6	$X, \alpha \vdash \beta$	$\neg$-elimination

$$\frac{X \vdash \alpha \mid X, \alpha \vdash \beta}{X \vdash \beta}$$
(cut rule)

1	$X, \neg\alpha \vdash \alpha$	supposition, (MR)
2	$X, \neg\alpha \vdash \neg\alpha$	(IS), (MR)
3	$X, \neg\alpha \vdash \beta$	($\neg 1$)
4	$X, \alpha \vdash \beta$	supposition
5	$X \vdash \beta$	($\neg 2$) on 3 and 4

$$\frac{X, \alpha \vdash \beta}{X \vdash \alpha \to \beta}$$

(→-introduction)

1	$X, \alpha \wedge \neg\beta, \alpha \vdash \beta$	supposition, (MR)
2	$X, \alpha \wedge \neg\beta \vdash \alpha$	(IS), (MR), ($\wedge$2)
3	$X, \alpha \wedge \neg\beta \vdash \beta$	cut rule
4	$X, \alpha \wedge \neg\beta \vdash \neg\beta$	(IS), (MR), ($\wedge$2)
5	$X, \alpha \wedge \neg\beta \vdash \alpha \to \beta$	($\neg$1)
6	$X, \neg(\alpha \wedge \neg\beta) \vdash \alpha \to \beta$	(IS), (MR)
7	$X \vdash \alpha \to \beta$	($\neg$2) on 5 and 6

$$\frac{X \vdash \alpha, \alpha \to \beta}{X \vdash \beta}$$

(modus ponens)

1	$X \vdash \alpha \to \beta$	supposition
2	$X, \alpha \to \beta$	→-elimination
3	$X \vdash \alpha$	supposition
4	$X \vdash \beta$	cut rule

Notice that →-elimination and introduction give us the *syntactic deduction theorem*: $X, \alpha \vdash \beta \iff X \vdash \alpha \to \beta$.

Now suppose we have a Gentzen rule:

$$R\colon \frac{X_1 \vdash \alpha_1 \mid \cdots \mid X_n \vdash \alpha_n}{X \vdash \alpha}$$

then we say that a property of sequents $\mathcal{E}$ is *closed under* R if when $\mathcal{E}(X_1, \alpha_1), \ldots, \mathcal{E}(X_n, \alpha_n)$ then $\mathcal{E}(X, \alpha)$. (Formally a sequent is (X, α), $X \vdash \alpha$ is anothe rway of writing it.)

2.2.2 Lemma (Principle of Rule Induction)

Let $\mathcal{E}$ be a property of sequents closed under all the basic rules of $\vdash$. Then $X \vdash \alpha$ implies $\mathcal{E}(X, \alpha)$.

Proof: we will prove this for a general Gentzen calculus. We induct on the length of the derivation of $X \vdash \alpha$, which is $(S_1, \ldots, S_n)$. If $n = 1$ then $S_1 = X \vdash \alpha$ and there is a premise-less basic rule of $\vdash$ which gives us $X \vdash \alpha$ and so S_1 is a basic rule itself, so $\mathcal{E}(X, \alpha)$.

Now, suppose the derivation is $(S_1, \ldots, S_n)$ for $n > 1$. If S_n is premise-less then we are done. Otherwise we have $\mathcal{E}(S_1), \ldots, \mathcal{E}(S_{n-1})$ by induction (since a prefix of a derivation is a derivation itself). But $\mathcal{E}$ is closed under the basic rules, and S_n is obtained from $S_1, \ldots, S_{n-1}$ by the basic rules so $\mathcal{E}(S_n)$. ■

Notice that the property $\mathcal{E}(X, \alpha) =$ "$X \vDash \alpha$" is closed under all the basic rules of $\vdash$. Thus we have shown the *soundness* of $\vdash$: $X \vdash \alpha \implies X \vDash \alpha$.

2.2.3 Theorem (Finiteness of $\vdash$)

If $X \vdash \alpha$ then $X_0 \vdash \alpha$ for some finite $X_0 \subseteq X$.

Proof: we define $\mathcal{E}(X, \alpha) =$ "$X_0 \vdash \alpha$ for some finite $X_0 \subseteq X$" and show it is closed under the basic rules of $\vdash$.

(1) IS: for $X = \{\alpha\}$ take $X_0 = X$.

(2) MR: if $X_0 \vdash \alpha$ for finite $X_0 \subseteq X$, then for any $X' \supseteq X$, X_0 is still a finite subset of X'.

(3) $\wedge$1: if $X \vdash \alpha, \beta$ then there is $X_0, X_1 \subseteq X$ for which $X_0 \vdash \alpha$ and $X_1 \vdash \beta$, so by MR $X_2 = X_0 \cup X_1$ has $X_2 \vdash \alpha, \beta$ so $X_2 \vdash \alpha \wedge \beta$, and X_2 is finite.

(4) $\wedge$2: if $X_0 \vdash \alpha \wedge \beta$ is finite, then $X_0 \vdash \alpha, \beta$ is still finite.

(5) $\neg$1: similar to $\wedge$2.

(6) $\neg$2: if $X_0, \alpha \vdash \beta$ and $X_1, \neg\alpha \vdash \beta$ using our inductive hypothesis and MR, then using MR again we have $X_2, \alpha \vdash \beta$ and $X_2, \neg\alpha \vdash \beta$ for $X_2 = X_0 \cup X_1$. So then $X_2 \vdash \beta$. ■

2.2.4 Definition

A set of formulas is **inconsistent** (in $\vdash$) if $X \vdash \alpha$ for all formulas α. Otherwise X is **consistent**.

Note that X is inconsistent if and only if $X \vdash \bot$ by $\wedge 2$ and $\neg 1$.

Now, we are interested in *maximal consistency*, meaning X is consistent and any proper superset of X is inconsistent. If X is consistent, it is maximally consistent if and only if $\alpha \in X$ or $\neg\alpha \in X$ for all α. Indeed if X is maximally consistent and $\alpha, \neg\alpha \notin X$ then $X, \alpha \vdash \bot$ and $X, \neg\alpha \vdash \bot$ so $X \vdash \bot$ by $\neg 2$. And obviously if α or $\neg\alpha$ are in X for all α, it is maximal.

2.2.5 Lemma

$\vdash$ has the properties:

$$C^+ \colon X \vdash \alpha \iff X, \neg\alpha \vdash \bot, \qquad C^- \colon X \vdash \neg\alpha \iff X, \alpha \vdash \bot$$

Note that these can both be seen as a form of "proof by contradiction" where C^- is necessary since $\vdash$ is semantic.

Proof: suppose $X \vdash \alpha$ then $X, \neg\alpha \vdash \alpha$ by MR and since by MR as well we have $X, \neg\alpha \vdash \neg\alpha$, by $\neg 1$ we have $X, \neg\alpha \vdash \bot$. Conversely, if $X, \neg\alpha \vdash \bot$ then $X, \neg\alpha \vdash \alpha$ and so by $\neg$-elimination we have $X \vdash \alpha$. C^- is proven similarly. ■

2.2.6 Lemma (Lindenbaum's Theorem)

Every consistent X can be extended to a maximally consistent $X' \supseteq X$.

Proof: let us define

$$\mathcal{H} = \{X' \mid X' \text{ is consistent and } X' \supseteq X\}$$

clearly, a maximal element of $\mathcal{H}$ is precisely a maximally consistent extension of X. So we must show that $\mathcal{H}$ has a maximal element, which is of course done with Zorn's Lemma. Let $\mathcal{C}$ be a chain in $\mathcal{H}$, and let us define $Y = \bigcup \mathcal{C}$, we claim that Y is an upper bound in $\mathcal{H}$ of $\mathcal{C}$. It is obviously an upper bound, all we need to show is that it is in $\mathcal{H}$, i.e. is consistent.

Suppose not, so $Y \vdash \bot$. By finiteness, there is a finite $Y_0 = \{\alpha_1, \dots, \alpha_n\} \subseteq Y$ such that $Y_0 \vdash \bot$. Suppose $\alpha_i \in X'_i$ for $X'_i \in \mathcal{C}$. Since $\mathcal{C}$ is a chain, there must be some X'_i which contains all other X'_js. Thus $Y_0 \subseteq X'_i$ and so by MR, $X'_i \vdash \bot$, contradicting X'_i's consistency. ■

2.2.7 Lemma

A maximally consistent set X has the property $X \vdash \neg\alpha \iff X \nvdash \alpha$ for all α.

Proof: if $X \vdash \neg\alpha$ then $X \vdash \alpha$ cannot be due to X's consistency. Conversely if $X \nvdash \alpha$ then $\alpha \notin X$ so $\neg\alpha \in X$ so by IS and MR we have $X \vdash \neg\alpha$.

2.2.8 Lemma

A maximally consistent set X is satisfiable.

Proof: we define $w \vDash p$ if and only if $X \vdash p$. Then we will show that $X \vdash \alpha \iff w \vDash \alpha$. We proceed by formula induction. For prime formulas, this is trivial. And so:

$$X \vdash \alpha \wedge \beta \iff X \vdash \alpha, \beta \iff w \vDash \alpha, \beta \iff w \vDash \alpha \wedge \beta$$

and

$$X \vdash \neg\alpha \iff X \nvdash \alpha \iff w \nvDash \alpha \iff w \vDash \neg\alpha$$

Thus w models X, as required. ■

2.2.9 Theorem (The Completeness Theorem)

$X \vdash \alpha$ if and only if $X \vDash \alpha$.

Proof: if $X \vdash \alpha$ by soundness, we have $X \vDash \alpha$. Now suppose $X \nvdash \alpha$ then $X, \neg\alpha$ is consistent (by C^+), and so it has a maximally consistent extension Y. This is satisfiable, so $X, \neg\alpha$ is satisfiable, meaning that $X \nvDash \alpha$. ■

Immediately, we get

2.2.10 Theorem (The Finiteness Theorem)

$X \vDash \alpha$ if and only if there is some finite $X_0 \subseteq X$ for which $X_0 \vDash \alpha$.

and

2.2.11 Theorem (The Compactness Theorem)

X is satisfiable if and only if every finite subset of X is satisfiable.

Proof: if X is satisfiable, so too is every subset. Conversely, if X is unsatisfiable then $X \vDash \bot$ and by finiteness, $X_0 \vDash \bot$ for some finite $X_0 \subseteq X$. ■

2.3 Applications of Compactness

We wish to prove the following lemma:

2.3.1 Lemma (König's Tree Lemma)

Let $(V, \lhd, r)$ be an infinite directed tree (so $\lhd$ is an irreflexive binary relation on V, V is infinite, and $r \in V$ is a special node in the tree called the *root* such that for every $a \in V$ there is precisely one path from r to a).

Now suppose that there exists arbitrarily long finite paths originating from r, and each $a \in V$ has finitely many successors, then there exists an infinite path originating from the root.

Proof: let us define "layers" of the tree inductively as follows: set $S_0 = \{c\}$ and

$$S_{k+1} = \{b \in V \mid a \lhd b \text{ for some } a \in S_k\}.$$

Since each node has finitely many successors, each S_k is finite.

Now for each $a \in V$, we define a propositional variable p_a. We will now define a set of formulas X which will say that there is an infinite path in V. This will be done by interpreting p_a as saying that the path traverses through a.

(1) for each k, $\bigvee_{a \in S_k} p_a$, since the path must traverse through some element in S_k.

(2) for each k and $a \neq b \in S_k$, $\neg(p_a \wedge p_b)$ since we want the path to traverse through only one node in each layer.

(3) $p_b \to p_a$ if $a \lhd b$ for $a, b \in V$, since if a node is in the path, so too must its parent.

Now since every finite subset $X_0 \subseteq X$ contains finitely many S_k's, it is satisfied by a finite path from the root through these layers. So X is satisfiable, suppose $w \vDash X$. Then we create a path $\{c_i\}_{i \in \mathbb{N}}$ where c_i is the node in S_i which is satisfied by w (i.e. c_i is the unique node in S_i for which $w \vDash p_{c_i}$).

Now notice that $c_0 = r$ and $c_i \lhd c_{i+1}$ for all i. This is because if a is the predecessor of $b = c_{i+1}$ then $w \vDash p_a$ in lieu of (3). So $\{c_i\}_{i \in \mathbb{N}}$ is an infinite path, as required. ■

2.4 Hilbert Calculi

What is the more natural calculus I mentioned earlier?

2.4.1 Definition

A **Hilbert calculus** $\vdash$ is one formed of

(1) **axioms**: a set of formulas Λ, and

(2) **rules of inference**: a set of relations Γ, where each $R \in \Lambda$ is a relation $R \subseteq \Gamma^n \times \Gamma$ for $n > 0$.

A **proof** of φ under a set of formulas X is a sequence of formulas $(\varphi_1, \ldots, \varphi_n = \varphi)$ such that each φ_i is either in $X \cup \Lambda$ or there exists a rule of inference $R \in \Gamma$ and for $\alpha_1, \ldots, \alpha_k$ occurring in the sequence before φ_i, $R(\alpha_1, \ldots, \alpha_k \implies \varphi_i)$. In such a case, we write $X \vdash \varphi$.

We can define a Hilbert calculus using the following axioms (which range over all α, β, γ):

$$\Lambda 1 \quad (\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma \qquad \Lambda 2 \quad \alpha \to \beta \to \alpha \wedge \beta$$

$$\Lambda 3 \quad \alpha \wedge \beta \to \alpha, \quad \alpha \wedge \beta \to \beta \qquad \Lambda 4 \quad (\alpha \to \neg\beta) \to \beta \to \neg\alpha$$

And the rule of inference of modus ponens: $(\alpha, \alpha \to \beta) \implies \beta$.

Now, we say that a property of formulas $\mathcal{E}$ is closed under a rule $R(\alpha_1, \ldots, \alpha_n \to \beta)$, if $\mathcal{E}\alpha_1, \ldots, \mathcal{E}\alpha_n$ implies $\mathcal{E}\beta$. So we can prove

2.4.2 Lemma (Principle of Induction for Hilbert Calculi)

Let X be a set of formulas and $\mathcal{E}$ a property of formulas $\mathcal{E}$. Then if

(1) $\mathcal{E}\alpha$ is true for all $\alpha \in X \cup \Lambda$ and

(2) $\mathcal{E}$ is closed under all the rules of inference of the Hilbert calculus,

$X \vdash \alpha$ implies $\mathcal{E}\alpha$.

Using this we can show the following:

2.4.3 Lemma

In our Hilbert calculus, if $X \vdash \alpha$ then $X \vDash \alpha$.

and

2.4.4 Lemma

A Hilbert calculus is reflexive ($\alpha \vdash \alpha$), monotonic ($X \vdash \alpha$ and $X \subseteq X'$ implies $X' \vdash \alpha$), and transitive (if $X \vdash Y$ and $Y \vdash \alpha$ then $X \vdash \alpha$).

And we can prove the following (in our Hilbert calculus):

2.4.5 Lemma

(1) $X \vdash \alpha \to \neg\beta$ implies $X \vdash \beta \to \neg\alpha$

(2) $\vdash \alpha \to \beta \to \alpha$

(3) $\vdash \alpha \to \alpha$

(4) $\vdash \alpha \to \neg\neg\alpha$

(5) $\vdash \beta \to \neg\beta \to \alpha$

2.4.6 Lemma (The Deduction Theorem)

$X, \alpha \vdash \beta$ iff $X \vdash \alpha \to \beta$.

(Hint: one direction uses induction.)

2.4.7 Lemma

$\vdash$ satisfies all the rules of our Gentzen calculus.

(Hint: only $\neg 2$ is complicated.)

2.4.8 Theorem (The Completeness Theorem)

$X \vdash \alpha \iff X \vDash \alpha$