Programming Languages

Homework 2

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2.1 Exercise

(1) Prove the following semantic equivalence: $(S_1; S_2); S_3 \sim S_1; (S_2; S_3)$.

(2) Prove that the following is not an equivalence: $S_1; S_2 \sim S_2; S_1$.

(3) Prove the equivalence:

$$\text{if } b \text{ then } (\text{if } c \text{ then } S_1 \text{ else } S_2) \text{ else } S_3$$
$$\sim \text{if } b \text{ and } c \text{ then } S_1 \text{ else if } b \text{ and not } c \text{ then } S_2 \text{ else } S_3$$

(1) We have the following semantic trees:

$$\frac{\langle S_1; s\rangle \to s_1 \qquad \dfrac{\langle S_2, s_1\rangle \to s_2 \qquad \langle S_3, s_2\rangle \to s_3}{\langle S_2; S_3, s_1\rangle \to s_3}}{\langle S_1; (S_2; S_3), s\rangle \to s_3} \qquad \frac{\dfrac{\langle S_1, s\rangle \to s_1 \qquad \langle S_2, s_1\rangle \to s_2}{\langle S_1; S_2, s\rangle \to s_2} \qquad \langle S_3, s_2\rangle \to s_3}{\langle (S_1; S_2); S_3, s\rangle \to s_3}$$

So we see that both commands are equivalent as they both produce the same state.

(2) Let $S_1 = x := 1$ and $S_2 = x := 2$, let s be the default state and $s_1 = s[x \mapsto 1]$ and $s_2 = s[x \mapsto 2]$, so $s_1 \neq s_2$. Then

$$\frac{\langle S_1, s\rangle \to s_1 \qquad \langle S_2, s_1\rangle \to s_1[x \mapsto 2] = s_2}{\langle S_1; S_2, s\rangle \to s_2}$$

While

$$\frac{\langle S_2, s\rangle \to s_2 \qquad \langle S_1, s_2\rangle \to s_2[x \mapsto 1] = s_1}{\langle S_2; S_1, s\rangle \to s_1}$$

(3) We have the following semantic trees: If $\mathcal{B}[b]s = tt$ and $\mathcal{B}[c]s = tt$ then

$$\frac{\dfrac{\langle S_1, s\rangle \to s_1}{\langle \text{if } c \text{ then } S_1 \text{ else } S_2, s\rangle \to s_1}}{\langle \text{if } b \text{ then } (\text{if } c \text{ then } S_1 \text{ else } S_2) \text{ else } S_3, s\rangle \to s_1}$$

$$\frac{\langle S_1, s\rangle \to s_1}{\langle \text{if } b \text{ and } c \text{ then } S_1 \text{ else if } b \text{ and not } c \text{ then } S_2 \text{ else } S_3, s\rangle \to s_1}$$

If $\mathcal{B}[b]s = tt$ and $\mathcal{B}[c]s = ff$ then

$$\frac{\dfrac{\langle S_2, s\rangle \to s_2}{\langle \text{if } c \text{ then } S_1 \text{ else } S_2, s\rangle \to s_2}}{\langle \text{if } b \text{ then } (\text{if } c \text{ then } S_1 \text{ else } S_2) \text{ else } S_3, s\rangle \to s_2}$$

$$\frac{\dfrac{\langle S_2, s\rangle \to s_2}{\langle \text{if } b \text{ and not } c \text{ then } S_2 \text{ else } S_3, s\rangle \to s_2}}{\langle \text{if } b \text{ and } c \text{ then } S_1 \text{ else if } b \text{ and not } c \text{ then } S_2 \text{ else } S_3, s\rangle \to s_2}$$

If $\mathcal{B}[b]s = ff$ then

$$\frac{\langle S_3, s\rangle \to s_3}{\langle \text{if } b \text{ then } (\text{if } c \text{ then } S_1 \text{ else } S_2) \text{ else } S_3, s\rangle \to s_3}$$

$$\frac{\dfrac{\langle S_3, s\rangle \to s_3}{\langle \text{if } b \text{ and not } c \text{ then } S_2 \text{ else } S_3, s\rangle \to s_3}}{\langle \text{if } b \text{ and } c \text{ then } S_1 \text{ else if } b \text{ and not } c \text{ then } S_2 \text{ else } S_3, s\rangle \to s_3}$$

So in all cases, they are equivalent, as required.

2.2 Exercise

(1) Add semantics for a do...while loop, without utilizing the while loop.

(2) Prove

$$\text{do } S \text{ while } b \;\sim\; S; \text{if } b \text{ then (do } S \text{ while } b) \text{ else skip}$$

(1) Let $\langle S, s\rangle \to s'$ then

$$\text{if } B[b]s' = tt \qquad \frac{\langle S, s\rangle \to s' \qquad \langle \text{do } S \text{ while } b, s'\rangle \to s''}{\langle \text{do } S \text{ while } b, s\rangle \to s''}$$

$$\text{if } B[b]s' = ff \qquad \frac{\langle S, s\rangle \to s'}{\langle \text{do } S \text{ while } b, s\rangle \to s'}$$

(2) Let $\langle S, s\rangle \to s'$. If $B[b]s' = tt$ then

$$\frac{\langle s, s\rangle \to s' \qquad \dfrac{\langle \text{do } S \text{ while } b, s'\rangle \to s''}{\langle \text{if } b \text{ then (do } s \text{ while } b) \text{ else skip}, s'\rangle \to s''}}{\langle s; \text{if } b \text{ then (do } s \text{ while } b) \text{ else skip}, s\rangle \to s''}$$

And if $B[b]s' = ff$ then

$$\frac{\langle s, s\rangle \to s' \qquad \dfrac{\langle \text{skip}, s'\rangle \to s'}{\langle \text{if } b \text{ then (do } s \text{ while } b) \text{ else skip}, s'\rangle \to s'}}{\langle s; \text{if } b \text{ then (do } s \text{ while } b) \text{ else skip}, s\rangle \to s'}$$

Which are equivalent to the state results for do...while.

2.3 Exercise

Define syntax for numbers n by $n := 0 \,|\, 1 \,|\, n\,0 \,|\, n\,1$.

(1) Define semantics for these binary numbers.

(2) Show that this definition is total.

This is just we did in recitation: $\mathcal{N}[0] = 0$, $\mathcal{N}[1] = 1$, $\mathcal{N}[n\,0] = 2 \cdot \mathcal{N}[n]$, $\mathcal{N}[n\,1] = 2 \cdot \mathcal{N}[n] + 1$, which is clearly total by variant induction.