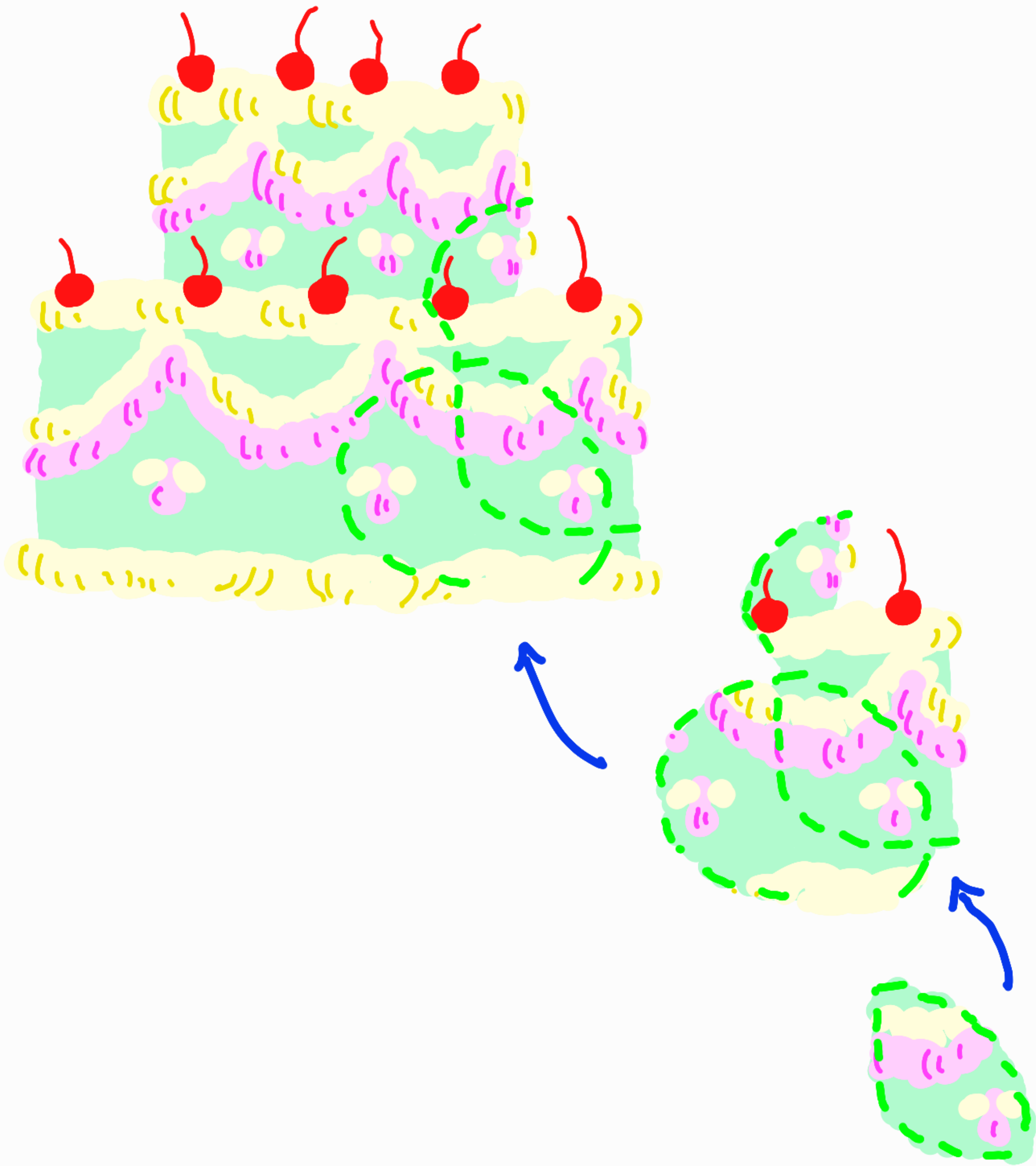


SHEAF THEORY FOR PASTRY CHEFS

ARI ROSENFIELD
UC - IRVINE

WHAT IS THIS?

IT'S AN EXPOSITORY TALK ABOUT SHEAVES, CATEGORIES OF SHEAVES, AND CLASSIFYING CATEGORIES OF SHEAVES.

WHAT'S "COMPASSIONATE" ABOUT IT?

OBSERVATION:

WHEN THINGS ARE VERY COMPLICATED, I (AND MY STUDENTS) TEND NOT TO GET MUCH FROM TALKS/LECTURES OTHER THAN MERE EXPOSURE.

WHAT'S "COMPASSIONATE" ABOUT IT?

IDEA:

ASSUME FROM THE START THAT THIS WILL HAPPEN, AND INSTEAD OF PRESENTING CONCEPTS IN PERFECT, RIGOROUS DETAIL, TRY TO PRESENT A MEMORABLE SKETCH.

WHAT'S "COMPASSIONATE" ABOUT IT?

(AS A STUDENT, I'M ALWAYS MORE HUNGRY FOR INFORMAL MENTAL MODELS AND HEURISTICS FROM MY TEACHERS – I CAN GET RIGOR FROM BOOKS / PAPERS. MAYBE I'M IN FAVOR OF LESS FORMALITY IN FACE-TO-FACE MATH PEDAGOGY??)

WHAT'S "COMPASSIONATE" ABOUT IT?

MORE OBSERVATIONS:

- I'M ESPECIALLY BAD AT REMEMBERING NEW IDEAS WHEN THEY LACK CONTEXT.
- EVEN THE SILLIEST CONNECTION/BIT OF CONTEXT CAN HELP WITH RETENTION.
- SO CAN EVOKING EMOTION.

WHAT'S "COMPASSIONATE" ABOUT IT?

- EVEN IF THE SKETCH ISN'T MATHEMATICALLY PERFECT, HAVING SEEN THE CONCEPTS BEFORE AND MADE SOME CONNECTIONS TRICKS MY BRAIN INTO BELIEVING THEY'RE EASIER TO UNDERSTAND WHEN I SEE THEM AGAIN.

WHAT'S "COMPASSIONATE" ABOUT IT?

THE HOPE IS THAT A DISCUSSION LIKE THIS IS A STARTING POINT FOR STUDENTS TO LEARN THE TOPIC MORE RIGOROUSLY.

WHY PASTRY?

- FASHION
- B/C I CAN!

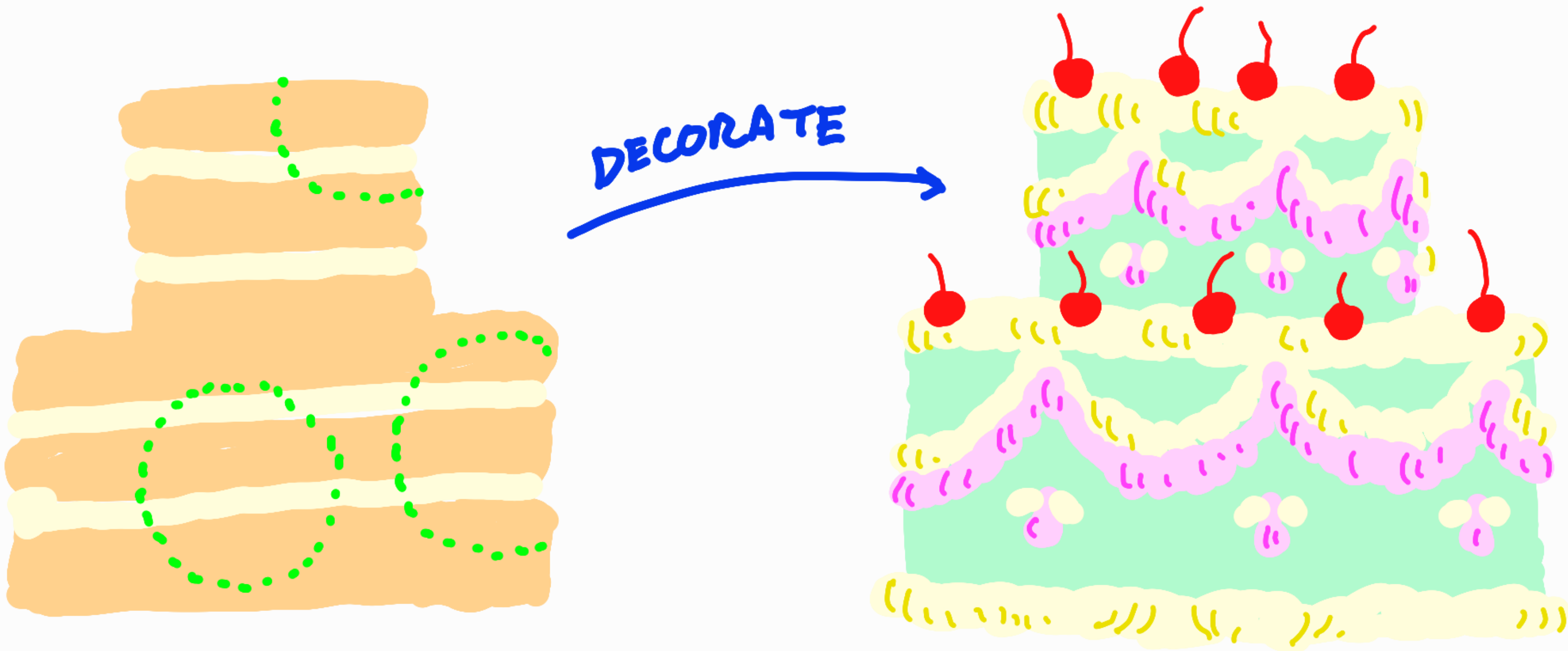
PRETEND...

- YOU ARE MY UNDERGRADUATE STUDENT.
- YOU HAVE JUST LEARNED WHAT CATEGORIES, FUNCTORS, NATURAL TRANSFORMATIONS, AND LIMITS ARE.
- YOU HAVE JUST ASKED ME "WHAT IS YOUR RESEARCH ABOUT?"

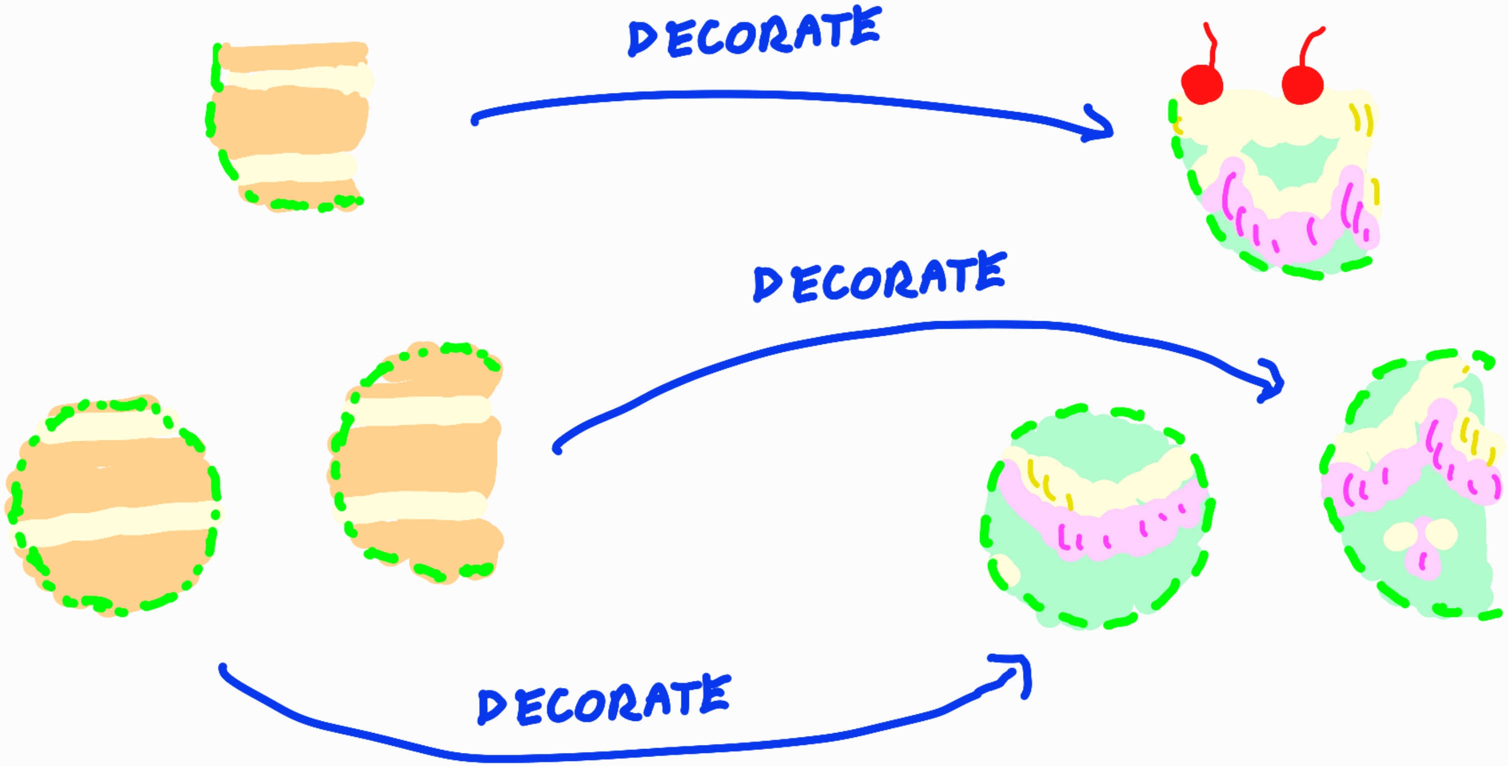
PRETEND...

I SAY: "SHEAVES AND CATEGORIES OF SHEAVES!"

WHAT IS A SHEAF?

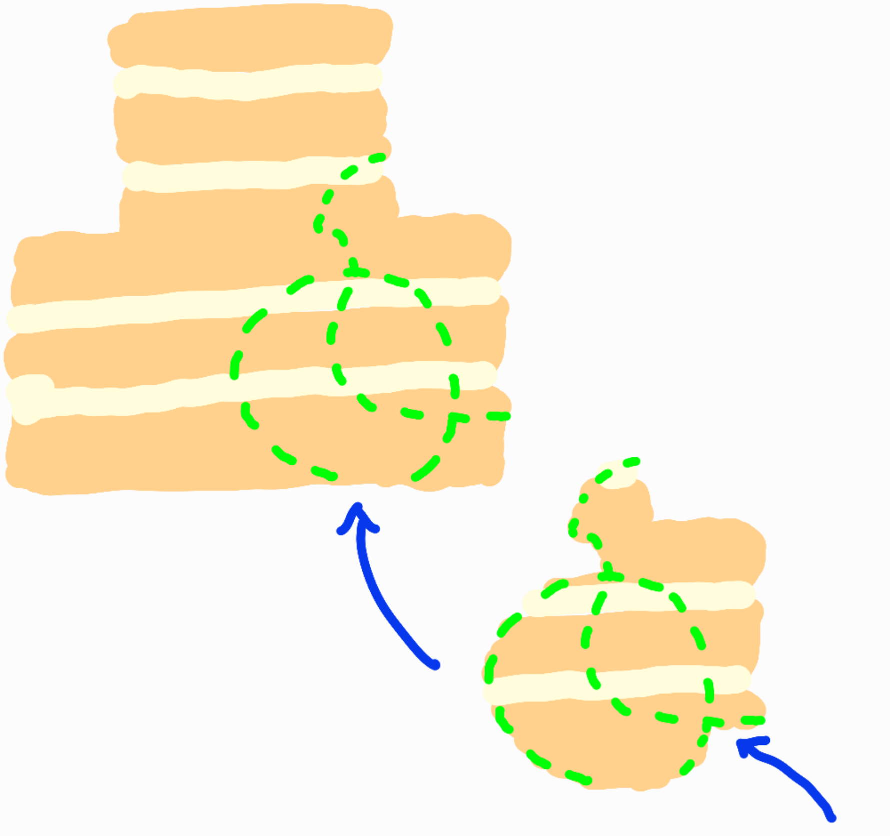


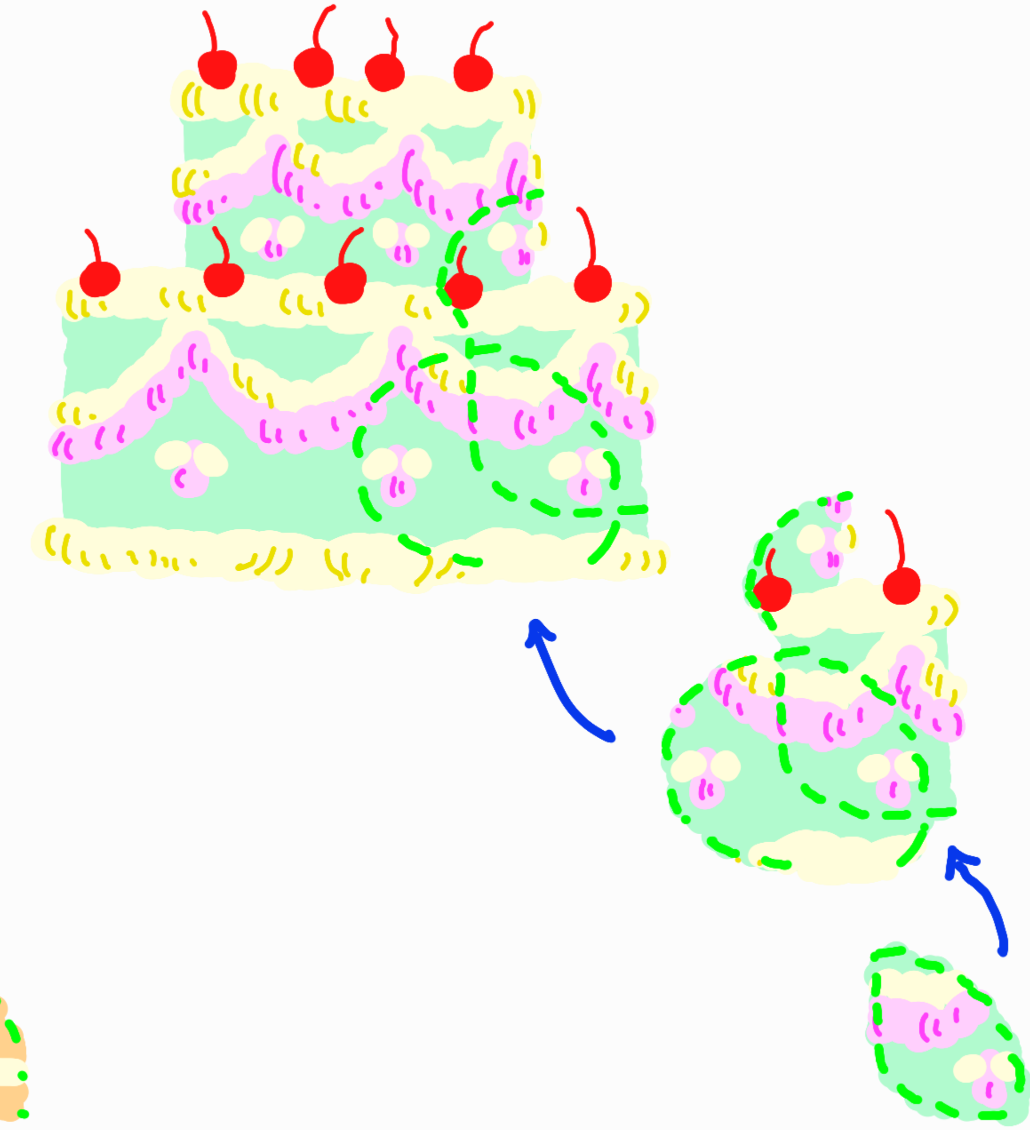
WHAT IS A SHEAF ?



"AN ASSIGNMENT OF DATA TO EACH LOCAL REGION ..."

WHAT IS A SHEAF?





"... SUCH THAT DATA ATTACHED TO SMALLER REGIONS IS COMPATIBLE WITH DATA ATTACHED TO LARGER REGIONS."

WHAT IS A SHEAF?

MATHEMATICALLY:

ON A CATEGORY $\mathcal{C}$, A FUNCTOR

$$F : \mathcal{C}^{op} \longrightarrow \mathbf{Set}$$

← "LOCAL ASSIGNMENT"

WHICH IS CONTINUOUS, i.e., SUCH THAT

$$F(\lim d_i) \cong \lim F(d_i)$$

"GLOBAL COHERENCE"

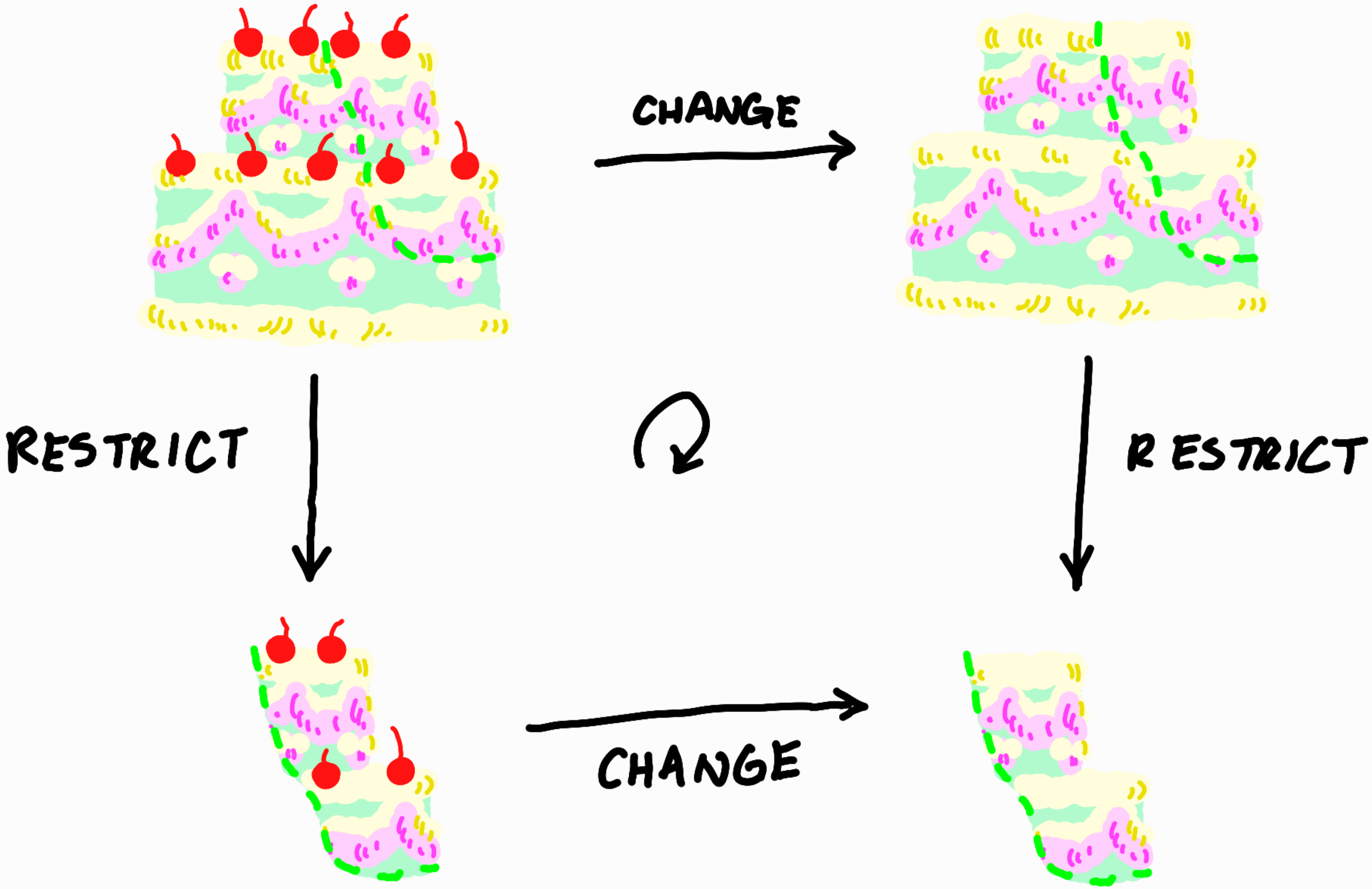
WHAT IS A SHEAF?

"A TOOL FOR KEEPING TRACK OF LOCAL DATA AND HOW TO BUILD GLOBAL DATA FROM IT."

I/T/O PASTRY:

"ONE POSSIBLE WAY TO DECORATE A GIVEN CAKE."

WHAT IS A MORPHISM OF SHEAVES, THEN?



WHAT IS A SHEAF TOPOS?

A CATEGORY EQUIVALENT TO $Sh(X)$ FOR SOME $X \in \mathbf{Top}$.

ADVANCED: TO $Sh(\mathcal{C}, J)$ FOR SOME <u>SITE</u> $(\mathcal{C}, J)$!

"CATEGORY THAT IS SPACE-LIKE."

WHAT IS A SHEAF TOPOS ?

1/T/0 PASTRY :

" THE COLLECTION OF ALL POSSIBLE WAYS OF DECORATING A GIVEN CAKE AND ALL WAYS OF TURNING ONE INTO ANOTHER."

CLASSIFICATION

SHEAF TOPOSES HAVE A TON OF NICE CATEGORY-THEORETIC PROPERTIES, SO IT'S IMPORTANT TO HAVE METHODS FOR DETECTING THEM.

CLASSIFICATION

GIVEN $X \in \mathbf{Top}$, I KNOW

$Sh(X)$

IS A SHEAF TOPOS.

CLASSIFICATION

WHAT IF YOU HAND ME SOME CATEGORY $\mathcal{E}$ AND ASK ME WHETHER IT'S A SHEAF TOPOS?

HOW COULD I TELL?

CLASSIFICATION

TAKE EVERY SINGLE $X \in \mathrm{Top}$, COMPUTE $\mathrm{Sh}(X)$, AND TRY TO FIND AN EQUIVALENCE $\mathcal{E} \cong \mathrm{Sh}(X)$?!?

THAT'S WAY TOO HARD...

IMPOSSIBLE, EVEN.

CLASSIFICATION

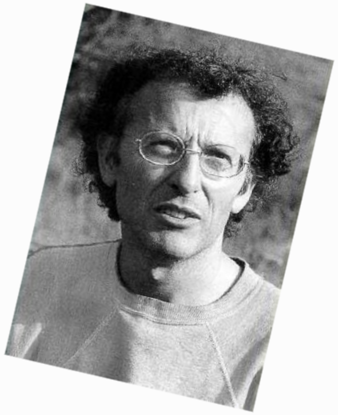
INSTEAD, LET'S DECIDE ON THE ESSENTIAL PROPERTIES A SHEAF TOPOS SHOULD HAVE,

HARD, BUT NOT IMPOSSIBLE...

THEN PROVE THAT FOR ANY $\mathcal{E}$ WITH THESE PROPERTIES, WE CAN EXTRACT AN X FOR WHICH

$$\mathcal{E} \simeq Sh(X).$$

Giraud's Theorem



(AN AXIOMATIC CLASSIFICATION OF SHEAF TOPOSES)

THEOREM:

A CATEGORY $\mathcal{E}$ IS EQUIVALENT TO $Sh(X)$ FOR SOME X EXACTLY WHEN IT HAS THE FOLLOWING PROPERTIES 1 – 4 :

① BEING LOCALLY PRESENTABLE

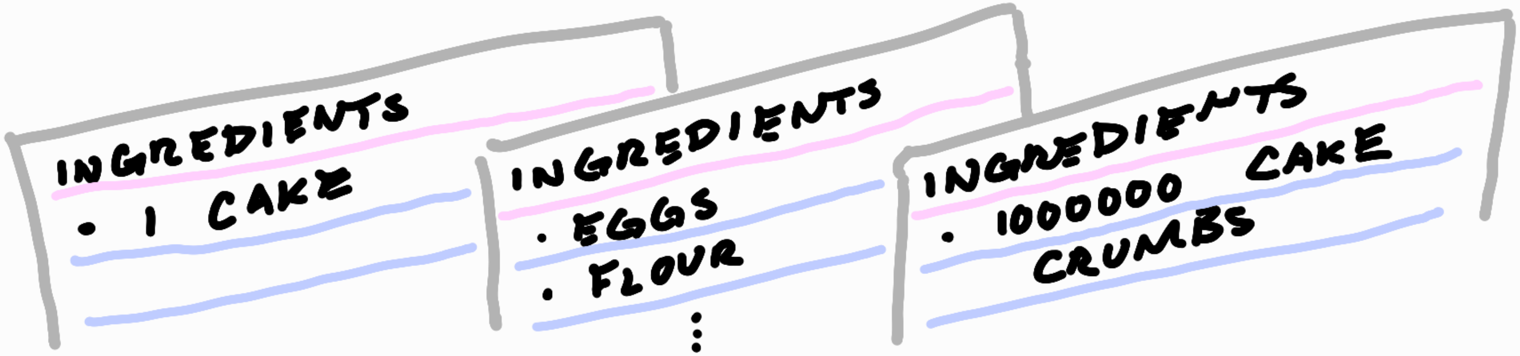
$\mathcal{E}_{\mathrm{fp}}$

THERE'S A SET OF ______ OBJECTS OF $\mathcal{E}$ WHICH WE CAN USE TO BUILD ANY OTHER OBJECT BY SMUSHING TOGETHER / AGGREGATING. (TAKING FILTERED COLIMITS.)

'SMALL' 'ELEMENTAL' 'FINITE'

① BEING LOCALLY PRESENTABLE

SAY OBJECTS OF Cake ARE TYPES OF POSSIBLE INGREDIENTS FOR CAKES (AND ANY-SIZED BITS THEREOF).



① BEING LOCALLY PRESENTABLE

(NOTE: IN Cake, WE ALLOW "COMPOUND" TYPES LIKE [EGGS AND SUGAR] AS WELL AS [JUST EGGS] OR [JUST SUGAR])

① BEING LOCALLY PRESENTABLE

(SMUSHED-TOGETHER CAKE CRUMBS $\cong$ CAKE, ETC.)

ONE POSSIBILITY FOR Cake_{pr} IS THE SET

$\{$ MOLECULES OF FAT, PROTEIN, CARBOHYDRATE, ETC., FOUND IN CAKE INGREDIENTS $\}$.

IMAGINE I CAME UP WITH A COMPLETE LIST.

(2) PULLBACK-STABLE COLIMITS

SAY $G : \mathcal{D} \to \mathcal{E}$ IS A DIAGRAM IN $\mathcal{E}$

$$\begin{tikzcd} & G(d_2) \arrow[dr] & & \\ G(d_1) \arrow[ur] \arrow[rr, bend right] & & G(d_3) \arrow[r] & G(d_4) \end{tikzcd}$$

WHICH HAS A COLIMIT IN $\mathcal{E}$. FOR ANY PULLBACK

$$\begin{tikzcd} (\operatorname{Colim}_d G) \times_Z Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Colim}_d G \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd} ,$$

$$(\operatorname{Colim}_d G(d)) \times_Z Y \cong \operatorname{Colim}_d (G(d) \times_Z Y).$$

2 PULLBACK-STABLE COLIMITS

IN Cake, MORPHISMS ARE WAYS TO TURN ONE TYPE OF INGREDIENT INTO ANOTHER.

② PULLBACK-STABLE COLIMITS

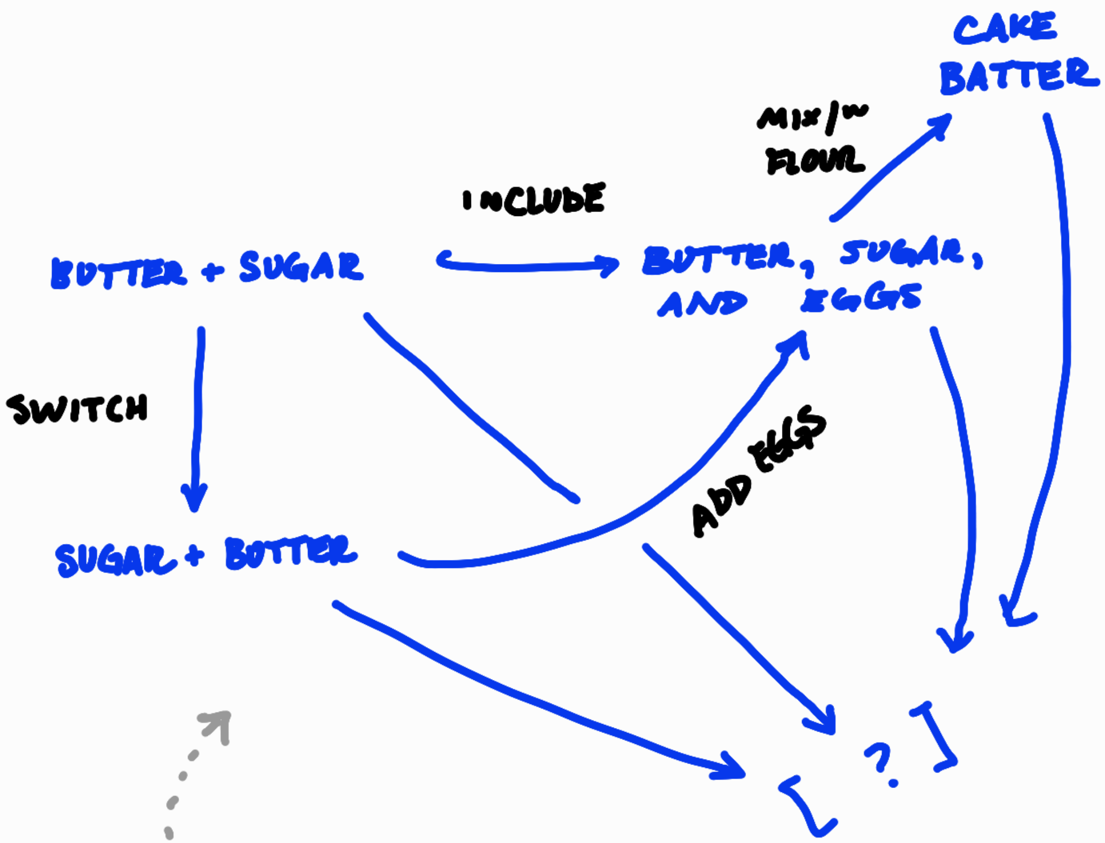
$$\begin{tikzcd} ? \arrow[r, "?"] \arrow[d, "?"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, "f"] \\ A \arrow[r, "g"'] & X \end{tikzcd}$$

$$\begin{tikzcd} ? \arrow[r, "?"] \arrow[d, "?"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \text{PLAIN SPONGE} \arrow[d, "\text{DECORATE}"] \\ \text{DOUGH} \arrow[r, "\text{BAKE + DECORATE}"'] & \text{CAKE} \end{tikzcd}$$

TO TAKE THE PULLBACK IS TO ASK

"WHAT'S THE MOST GENERAL TYPE OF INGREDIENT WHICH CAN BE TURNED INTO BOTH A AND B, COMPATIBLY WITH f & g?"

② PULLBACK-STABLE COLIMITS



OK, DON'T TAKE THIS TOO LITERALLY...

SIMILARLY, TAKING THE COLIMIT IS TO ASK:

"WHAT TYPE OF INGREDIENT DO I GET IF I COMBINE ALL THE INGREDIENTS IN THIS DIAGRAM, COMPATIBLY WITH ALL THE MORPHISMS IN IT?"

② PULLBACK-STABLE COLIMITS

AXIOM ② SAYS THAT WE SHOULD BE ABLE TO DO THESE TWO OPERATIONS IN EITHER ORDER AND GET THE SAME RESULT.

③ DISJOINT COPRODUCTS

$\mathcal{E}$ HAS AN INITIAL ("EMPTY") OBJECT $\emptyset$, AND COPRODUCTS BEHAVE LIKE THE DISJOINT UNION OF SETS:

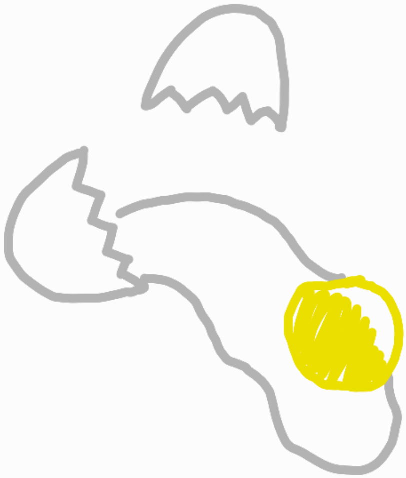
$$
\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & b \arrow[d, hook] \\
a \arrow[r, hook] & a+b
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & a \arrow[d, hook] \\
a \arrow[r, hook] & a+b
\end{tikzcd}
\qquad
\begin{tikzcd}
b \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & b \arrow[d, hook] \\
b \arrow[r, hook] & a+b
\end{tikzcd}
$$

③ DISJOINT COPRODUCTS

IN *Cake*, THIS SAYS SOMETHING LIKE:

"DISTINCT TYPES OF INGREDIENT DON'T HAVE ANYTHING IN COMMON"

" $\cap$ "



" $=$ " $\emptyset$

④ EFFECTIVE QUOTIENTS

IF $X \in \mathrm{Ob}(\mathcal{E})$ ADMITS A CONGRUENCE ("IS INTERNALLY SORTABLE"), SAY R, THEN X/R IS ITSELF AN OBJECT OF $\mathcal{E}$.

④ EFFECTIVE QUOTIENTS

$$X := \{\text{ALL BAGS OF SUGAR AT RALPHS}\}$$

④ EFFECTIVE QUOTIENTS

$$X/R = \{\text{CANE SUGAR},\ \text{POWDERED SUGAR},\ \text{BROWN SUGAR},\ \dots\}$$

THEOREM:

A CATEGORY $\mathcal{E}$ IS EQUIVALENT TO $Sh(X)$ FOR SOME X EXACTLY WHEN IT

1. IS LOCALLY PRESENTABLE
2. HAS PULLBACK-STABLE COLIMITS
3. HAS DISJOINT COPRODUCTS
4. HAS EFFECTIVE QUOTIENTS

THANKS FOR LISTENING!

IT'S UNDERBAKED...

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