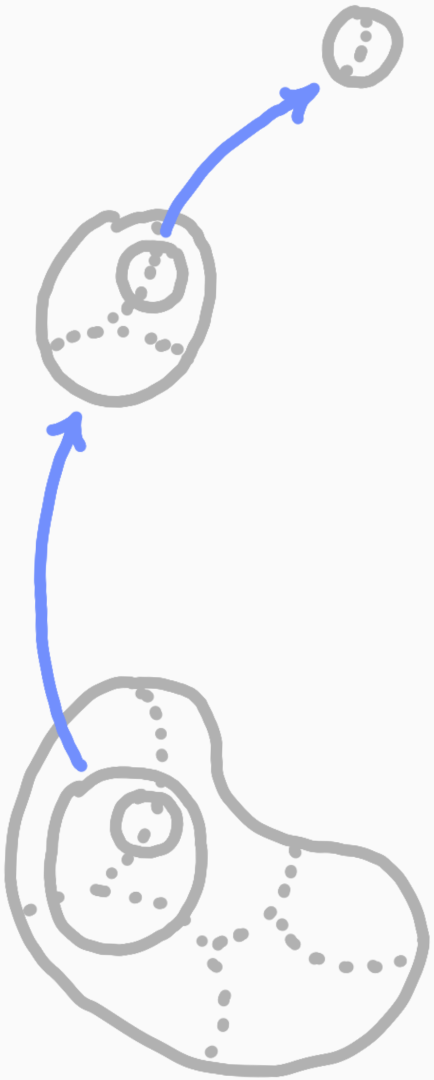

A FEW PUZZLES IN CATEGORICAL QUANTUM THEORY

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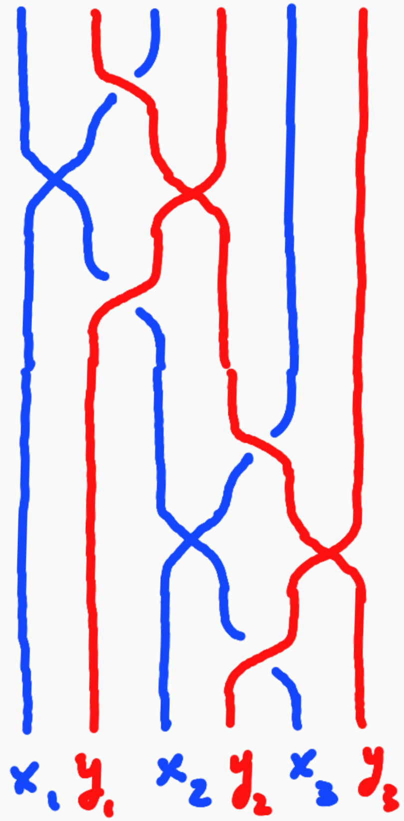


FIRST, WHAT IS A CATEGORY?

A CATEGORY $\mathcal{C}$ IS:

- A COLLECTION $\mathrm{Ob}(\mathcal{C})$ OF OBJECTS
- FOR EACH PAIR x, y OF OBJECTS, A SET $\mathcal{C}(x,y)$ OF MORPHISMS
- FOR EACH OBJECT $x \in \mathcal{C}$, AN IDENTITY MORPHISM $1_x : x \to x$
- FOR EACH TRIPLE x, y, z OF OBJECTS, A COMPOSITION FUNCTION

$\circ_{xyz} : \mathcal{C}(y,z) \times \mathcal{C}(x,y) \longrightarrow \mathcal{C}(x,z) \ldots$

FIRST, WHAT IS A CATEGORY?

...SUCH THAT THE OPERATION $\circ$ IS ASSOCIATIVE AND UNITAL (i.e., COMPATIBLE WITH IDENTITIES).

FIRST, WHAT IS A CATEGORY?

FOR EXAMPLE:

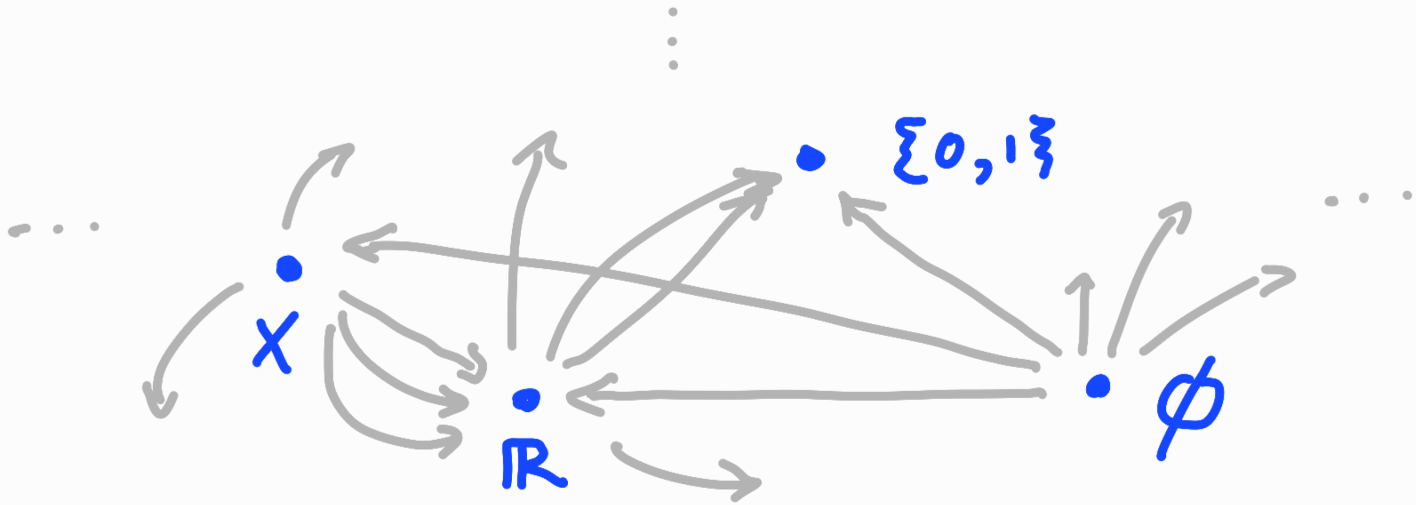
- ANY GROUP IS A CATEGORY WITH ONE OBJECT.

A PICTURE OF $\mathbb{Z}/3\mathbb{Z}$ AS A CATEGORY.

FIRST, WHAT IS A CATEGORY?

FOR EXAMPLE:

- SETS WITH FUNCTIONS BETWEEN THEM FORM A CATEGORY, CALLED **Set**.



IT'S MUCH HARDER TO DRAW THAN $\mathbb{Z}/3\mathbb{Z}$.

MORE BASICS

- A FUNCTOR IS A "MORPHISM BETWEEN CATEGORIES" – AN ASSIGNMENT ON OBJECTS AND ON MORPHISMS, COMPATIBLE WITH COMPOSITION & IDENTITIES.

  e.g., ANY GROUP HOMOMORPHISM.

MORE BASICS

- A MONOIDAL CATEGORY $\mathcal{X}$ IS ONE EQUIPPED WITH A RULE $\otimes$ FOR COMBINING OBJECTS AND MORPHISMS "IN PARALLEL."

  (THINK OF COMPOSITION AS "IN SEQUENCE.")

- IT IS SYMMETRIC IF $A \otimes B \cong B \otimes A$ FOR $A, B \in Ob(\mathcal{X})$ AND CLOSED IF $\mathcal{X}(A,B) \in Ob(\mathcal{X})$.

MORE BASICS

- e.g., THE CATEGORY Hilb OF HILBERT SPACES WITH BOUNDED LINEAR MAPS.

WHAT IS CATEGORY THEORY FOR?

SYSTEMATICALLY ABSTRACTING PROPERTIES OF A GIVEN MATHEMATICAL OBJECT IN ORDER TO MAKE ANALOGIES BETWEEN DIFFERENT FIELDS OF MATH.

e.g.,

$$\begin{tikzcd} P \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & A \end{tikzcd}$$

PULLBACK

$\mathcal{P}(A)$

Top

$$\begin{tikzcd} X \cap Y \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, hook] \\ Y \arrow[r, hook] & A \end{tikzcd}$$

SET INTERSECTION

$$\begin{tikzcd} X \times_A Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ Y \arrow[r] & A \end{tikzcd}$$

FIBER PRODUCT

BROADLY, I STUDY

CATEGORICAL QUANTUM THEORY.

SPECIFICALLY:

PUZZLE 1 :

ENRICHED GROTHENDIECK TOPOLOGIES.

(arXiv: 2405.19529)

BACKGROUND

- CLASSICAL GEL'FAND DUALITY: AN EQUIVALENCE OF CATEGORIES

$$\begin{tikzcd} \mathrm{cpt\,Haus} \arrow[r, bend left, "\mathcal{C}_0(-)"] & \mathrm{comm}\, C^*\mathrm{Alg}^{op} \arrow[l, bend left, "\mathrm{Spec}"] \end{tikzcd}$$

STATE SPACES

OBSERVABLES ON CLASSICAL SYSTEMS

NONCOMMUTATIVE SPECTRAL THEORY

WHAT IF I WANT GEL'FAND DUALITY
FOR NON COMMUTATIVE C^*-ALGEBRAS?
(i.e., OBSERVABLES ON QUANTUM SYSTEMS?)

THAT WOULD BE SO COOL...

THEN I COULD DO
ACTUAL GEOMETRY...

I COULD TRY JUST APPLYING THE Spec
FUNCTOR I ALREADY HAVE??

NONCOMMUTATIVE SPECTRAL THEORY

PROBLEM! (REYES, 2012 ; v. d. BERGH – HEUNEN, 2014)

THEOREM: ANY FUNCTOR $F : C^*\mathrm{Alg}^{op} \longrightarrow \mathrm{Top}$

FOR WHICH

$$\begin{tikzcd} \mathrm{comm}\, C^*\mathrm{Alg}^{op} \arrow[r, "\mathrm{Spec}"] \arrow[d, hook, "\mathrm{inc}"'] & \mathrm{Top} \\ C^*\mathrm{Alg}^{op} \arrow[ur, "F"'] & \end{tikzcd}$$

COMMUTES MUST HAVE $F(M_n(\mathbb{C})) \cong \emptyset$

FOR $n \geq 3$.

NONCOMMUTATIVE SPECTRAL THEORY

IN PHYSICAL TERMS: (I'LL DO MY BEST!)

- ANY CLASSICAL SYSTEM (comm. C^*-Alg) IS DETERMINED BY A STATE SPACE (CPT. TOP. SPACE) IN A WAY THAT RESPECTS OPERATIONS ON THE SYSTEM (FUNCTORIALLY).

- NO COMMUTATIVE C^*-ALG DETERMINES A QUANTUM SYSTEM IN THIS WAY.

NONCOMMUTATIVE SPECTRAL THEORY

- THE DREAM OF NONCOMMUTATIVE GEOMETRY:

$$
\begin{tikzcd}
\mathrm{comm}\,C^*\mathrm{Alg}^{op} \arrow[r, "\mathrm{Spec}"] \arrow[d, hook, "\mathrm{inc}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{cpt\ Haus} \arrow[d, hook, dashed, "??"] \\
C^*\mathrm{Alg}^{op} \arrow[r, dashed, "\text{``NCSpec''}"'] & \text{``NONCOMMUTATIVE CPT. HAUSDORFF SPACES''}
\end{tikzcd}
$$

OBSERVABLES ON QUANTUM SYSTEMS: THESE ARE WHAT N.C. GEOMETRY STUDIES.

NONCOMMUTATIVE SPECTRAL THEORY

- TOPOLOGICAL SPACES ARE SETS OF POINTS, SO I COULD JUST AS WELL THINK ABOUT A SLIGHTLY SIMPLER PROBLEM:

$$\begin{tikzcd}
\mathrm{commRing}^{op} \arrow[r, "\mathrm{Spec}"] \arrow[d, hook, "\mathrm{inc}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Set} \arrow[d, hook, dashed, "??"] \\
\mathrm{Ring}^{op} \arrow[r, dashed, "\text{``NCSpec''}"'] & \text{``}q\mathrm{Set}\text{''}
\end{tikzcd}$$

FINDING A CAT. OF "QUANTUM / N.C." SETS.

NONCOMMUTATIVE SPECTRAL THEORY

WE EXPECT "qSet" TO BE A CLOSED SYMMETRIC MONOIDAL CATEGORY, AND WE EXPECT IT TO "CONTAIN" Set.

HOW TO TELL IF A GIVEN CANDIDATE, SAY $\mathcal{V} \in$ MonCat, IS THE RIGHT ONE?

ENRICHED GROTHENDIECK TOPOLOGIES

- GIVEN A CLOSED SYMMETRIC MON. CAT. $\mathcal{V}$, A $\mathcal{V}$-ENRICHED CATEGORY IS "A CAT. WITH $\mathcal{V}(x,y) \in Ob(\mathcal{V})$."

  ENRICHED FUNCTORS RESPECT THE MNDL. STRUCTURE OF $\mathcal{V}$.

e.g., A RING IS A ONE-OBJECT Ab-CATEGORY. A RING HOMOMORPHISM IS AN Ab-FUNCTOR.

- GIVEN ENRICHING CAT'S $\mathcal{U}$, $\mathcal{V}$, A $\mathcal{V}$-CAT. $\mathcal{C}$, AND A FUNCTOR

$$G : \mathcal{V} \longrightarrow \mathcal{U},$$

WE CAN CHANGE BASE VIA G, OBTAINING A $\mathcal{U}$-CAT, DENOTED $G_* \mathcal{C}$.

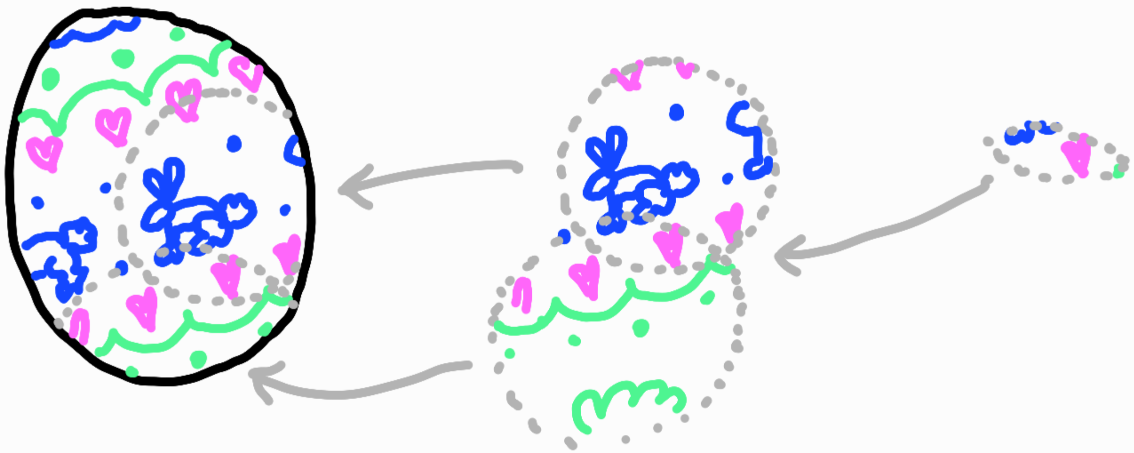
e.g., USING $\mathrm{Hom}_{Ab}(\mathbb{Z}, -) : Ab \longrightarrow Set$, WE CAN "FORGET" THE ADDITIVE STRUCTURE ON A RING TO OBTAIN A MERE MULTIPLICATIVE MONOID.

- A GROTHENDIECK TOPOLOGY IS A WAY TO ENDOW ANY CATEGORY (NOT JUST $\mathcal{O}(X)$) WITH A NOTION OF "CLOSE-TOGETHER-NESS" OF ITS OBJECTS.

"A FAMILY OF PRESHEAVES SATISFYING SOME CLOSURE CONDITIONS."

"A TOPOLOGY ON A CATEGORY."

- WE WANT TO KNOW WHEN OBJECTS ARE CLOSE TOGETHER BECAUSE THEN WE CAN DEFINE SHEAVES: "CLOSENESS-RESPECTING" ASSIGNMENTS OF DATA TO THE "LOCATIONS" IN (i.e., OBJECTS OF) OUR CAT'S.



- THE "NO-GO" THEOREMS ABOVE ALSO EXTEND TO SIMILAR RESULTS ABOUT WHAT GROTHENDIECK TOPOLOGIES CAN BE DEFINED ON, e.g., Ring.

WE HAVE ENRICHED GROTHENDIECK TOPOLOGIES AND ENRICHED SHEAVES, TOO.

e.g., A GABRIEL LOCALIZING SYSTEM ON A RING IS AN Ab-TOPOLOGY. AN Ab-SHEAF ON A RING w/r/t SUCH A LOCALIZING SYSTEM IS ITS GABRIEL LOCALIZATION.

WHAT I WANTED TO KNOW:

- HOW CAN I USE BASE CHANGE TO TURN A $\mathcal{V}$-TOPOLOGY INTO A $\mathcal{U}$-TOPOLOGY?

- UNDER WHAT CONDITIONS IS SUCH AN ASSIGNMENT INJECTIVE?

COOL, WHAT DOES IT HAVE TO DO WITH QUANTUM?

- ONE WAY TO TEST WHETHER WE'VE FOUND A GOOD CANDIDATE FOR "qSet" — THUS A GOOD CAT. TO MODEL QUANTUM SYSTEMS WITH — IS TO MAKE SURE THERE ARE qSet-ENRICHED GROTH. TOPOLOGIES WHICH DON'T ARISE FROM UNENRICHED ONES.

COOL, WHAT DOES IT HAVE TO DO WITH QUANTUM?

- SO, NEED TOOLS TO DETECT WHETHER A GIVEN ENRICHED GROTHENDIECK TOPOLOGY ARISES FROM AN UNENRICHED ONE.

Some Key Results ...

CONDITIONS

- ENRICHING CAT'S $\mathcal{U}, \mathcal{V}$ ARE ALWAYS CLOSED SYMMETRIC MONOIDAL AND LOCALLY FINITELY PRESENTABLE.
- $\mathcal{C}$ DENOTES A $\mathcal{V}$-CAT.
- $G : \mathcal{V} \longrightarrow \mathcal{U}$ IS A LAX MONOIDAL FUNCTOR WHICH MAY BE

$$\left.\begin{array}{ll} \text{i.} & \text{FAITHFUL} \\ \text{ii.} & \text{CONSERVATIVE} \\ \text{iii.} & \text{A RIGHT ADJOINT (IN MonCat)} \end{array}\right\} (\blacktriangle)$$

MISCELLANY

- A COVERAGE IS A TOPOLOGY BUT MINUS ONE "SATURATION" CONDITION.

  (A MILD WEAKENING.)

- DENOTE

$$\Sigma(\mathscr{C}, \mathcal{V}) = \{\text{ALL } \mathcal{V}\text{-COVERAGES ON } \mathscr{C}\}$$

$$\tau(\mathscr{C}, \mathcal{V}) = \{\text{ALL } \mathcal{V}\text{-TOPOLOGIES ON } \mathscr{C}\}$$

FIX $\mathcal{U}, \mathcal{V}$, AND $G : \mathcal{V} \to \mathcal{U}$.
SUPPOSE $\mathcal{C}$ IS SMALL.

PROPOSITION. GIVEN A $\mathcal{V}$-COVERAGE J ON $\mathcal{C}$, G INDUCES A $\mathcal{U}$-COVERAGE $\widetilde{GJ}$ ON $G_* \mathcal{C}$.

PROPOSITION. GIVEN A $\mathcal{V}$-TOPOLOGY J ON $\mathcal{C}$, G INDUCES A $\mathcal{U}$-TOPOLOGY $\overline{GJ}$ ON $G_* \mathcal{C}$.

THEOREM. $\Sigma(\mathcal{C}, \mathcal{V})$ AND $\mathcal{T}(\mathcal{C}, \mathcal{V})$ ARE COMPLETE LATTICES.

THEOREM. SUPPOSE G SATISFIES ALL CONDITIONS IN $(\blacktriangle)$. THE ASSIGNMENT

$$\Sigma(\mathcal{C},\mathcal{V}) \xrightarrow{\tilde{G}(-)} \Sigma(G_*\mathcal{C},\mathcal{U})$$

IS AN INJECTIVE LATTICE MORPHISM.

"DISTINCT $\mathcal{V}$-COVERAGES GIVE RISE TO DISTINCT $\mathcal{U}$-COVERAGES."

CONJECTURE. SUPPOSE G SATISFIES ALL CONDITIONS IN (▲), PLUS ONE MORE TECHNICAL CONDITION. THE ASSIGNMENT

$$\tau(\mathcal{C}, \mathcal{V}) \xrightarrow{\overline{G(-)}} \tau(G_*\mathcal{C}, \mathcal{U})$$

IS AN INJECTIVE LATTICE MORPHISM.

ONE MORE THING

THEOREM. IN CASE $G : \mathcal{V} \to \mathcal{U}$ IS FULLY FAITHFUL, BASE CHANGE "COMMUTES" WITH ENRICHED SHEAFIFICATION.

"CANONICAL WAY TO TURN A PRESHEAF INTO A SHEAF."

PUZZLE 2 : CONCURRENCY IN MONOIDAL CATEGORIES

(J/W CARMEN CONSTANTIN, CHRIS HEUNEN)

BACKGROUND

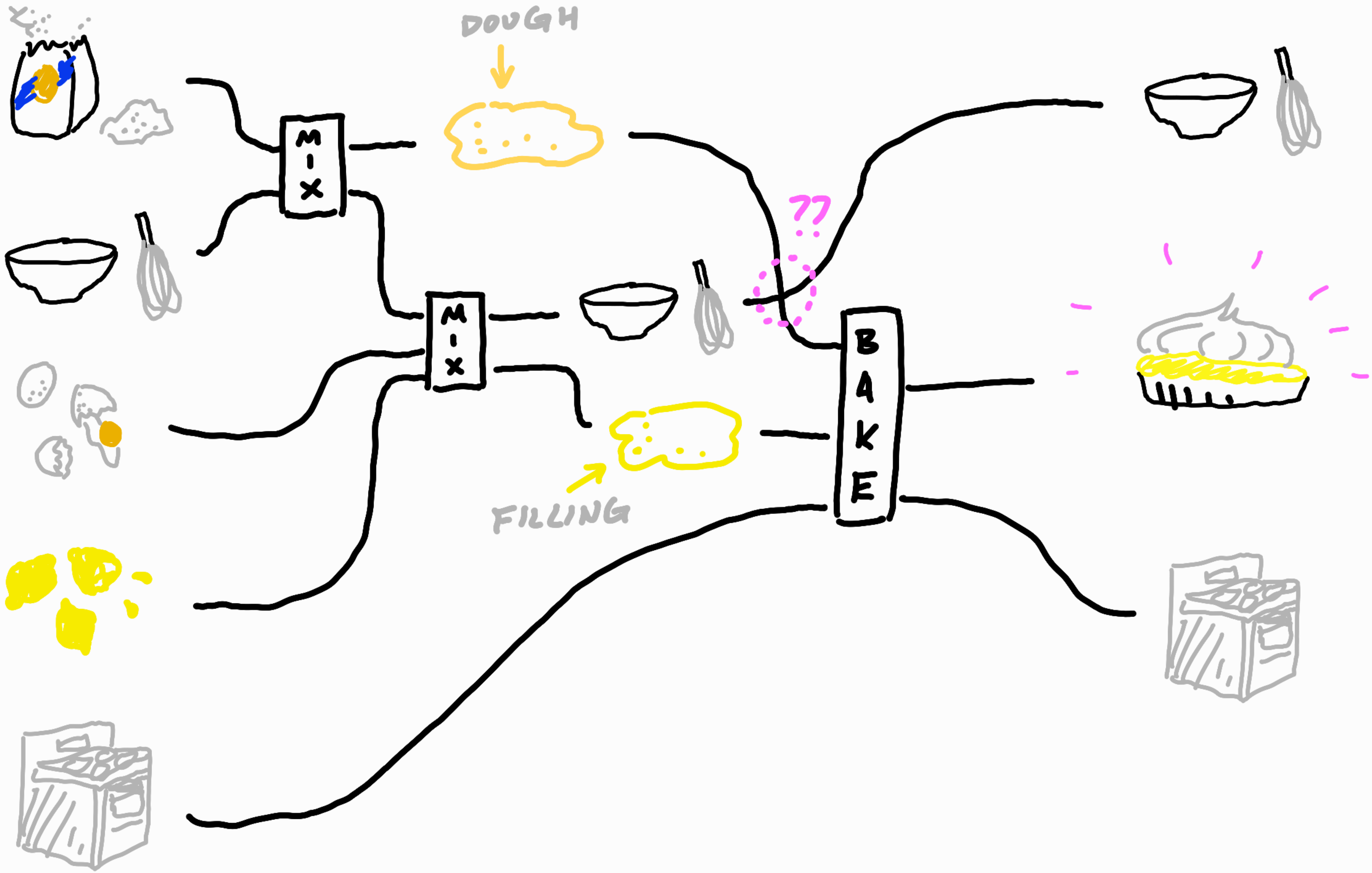
MODELING CONCURRENT PROCESSES IS AN IMPORTANT PROBLEM IN DEVELOPING QUANTUM COMPUTERS & PROGRAMS.

BACKGROUND

- WE CAN MODEL A SINGLE PROCESSOR AS A MONOIDAL CAT., THINKING OF OBJECTS AS RESOURCE TYPES AND MORPHISMS AS PROGRAMS/PROCESSES TURNING ONE TYPE INTO ANOTHER.

- WE CAN REPRESENT MORPHISMS IN MONOIDAL CAT'S BY STRING DIAGRAMS.

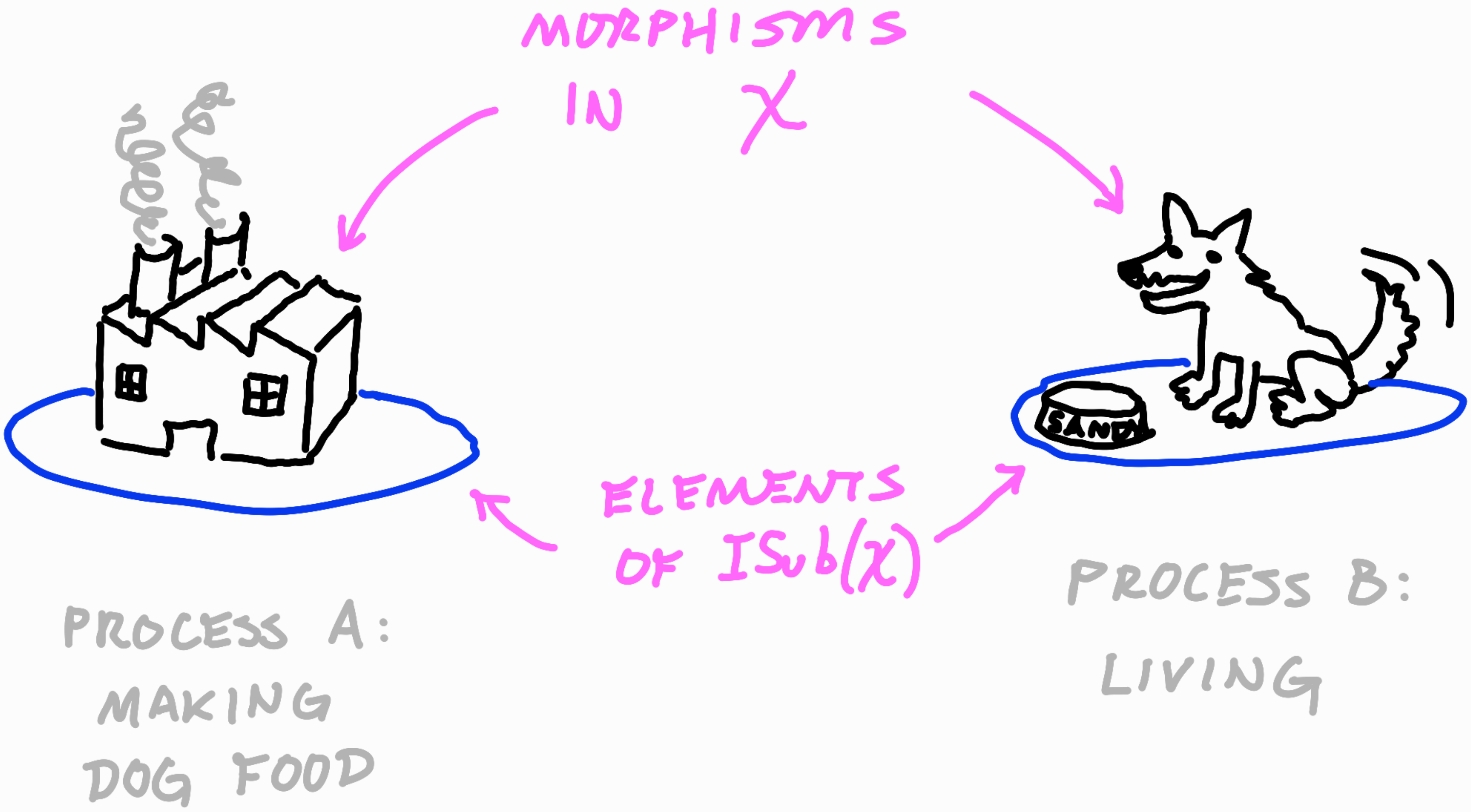
BACKGROUND



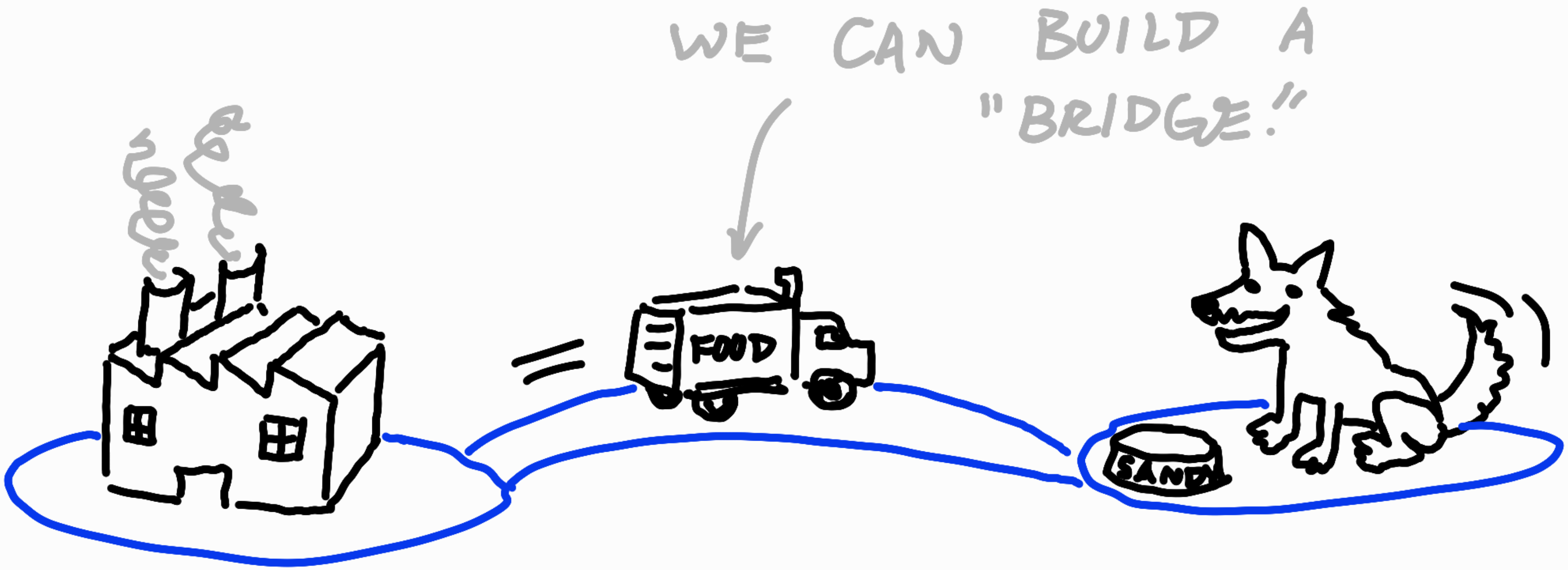
BACKGROUND

- ANY MONOIDAL CATEGORY $\mathcal{X}$ IS EQUIPPED WITH A COLLECTION OF OBJECTS WHICH WE THINK OF AS ADMISSIBLE "LOCATIONS" FOR PROCESSES TO OCCUR AT, DENOTED $\mathrm{ISub}(\mathcal{X})$.

BACKGROUND

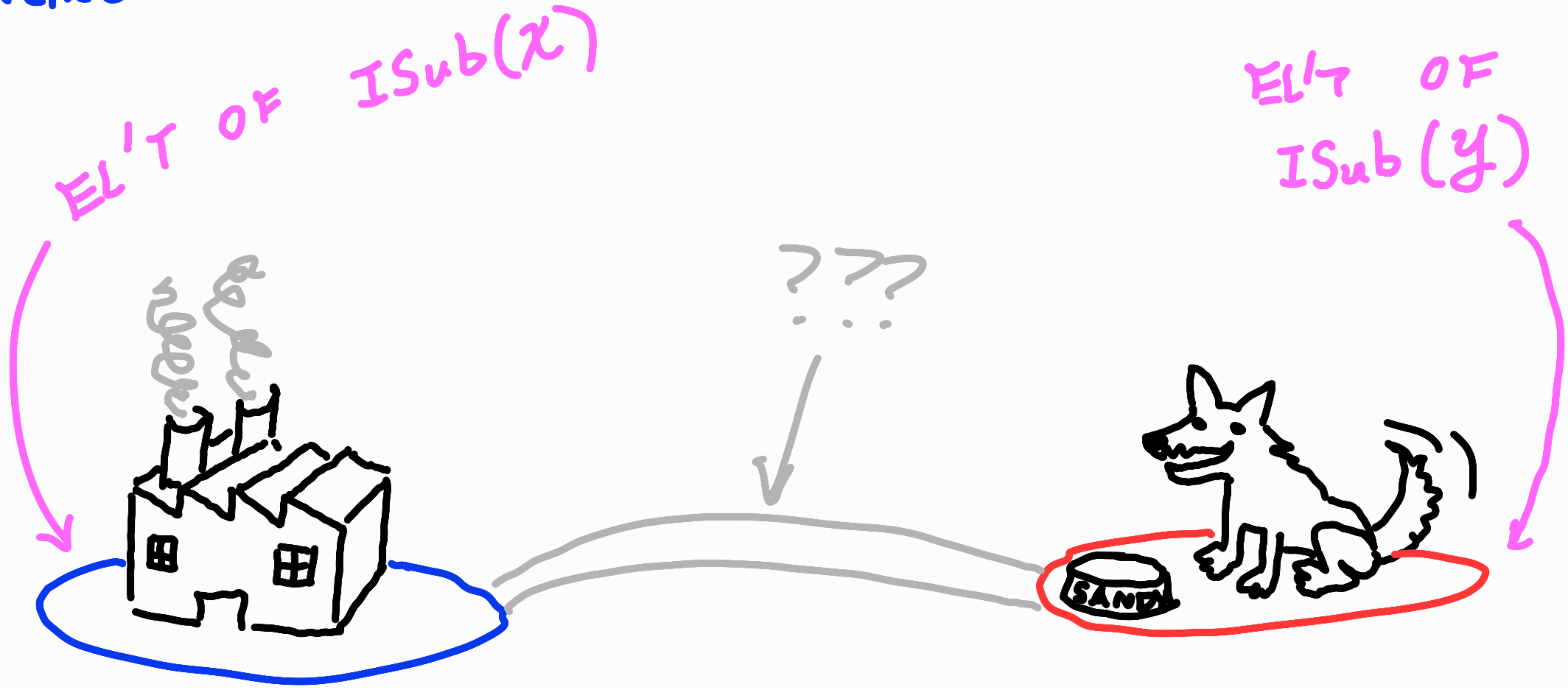

MORPHISMS IN $\mathcal{X}$
SANDY
ELEMENTS OF $\mathrm{ISub}(\mathcal{X})$
PROCESS A: MAKING DOG FOOD
PROCESS B: LIVING

BACKGROUND



A & B CAN EXCHANGE RESOURCES IN A GIVEN MONOIDAL CATEGORY.

BACKGROUND



WHAT IF A & B ARE OCCURRING IN DIFFERENT MONOIDAL CAT.'S ?
(e.g., DIFFERENT PROCESSORS ?)

KEY RESULT

GIVEN $\mathcal{X}, \mathcal{Y}, \mathcal{A} \in \mathsf{symMonCat}$, WE CONSTRUCTED A SYM. MONOIDAL CAT. WHICH SHOULD BE THE PUSHOUT OF $\mathcal{X} \div \mathcal{Y}$ ALONG $\mathcal{A}$:

$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[d] \arrow[dr, phantom, very near end, "\ulcorner"] & \mathcal{X} \arrow[d] \\
\mathcal{Y} \arrow[r] & \mathcal{X} *_{\mathcal{A}} \mathcal{Y}
\end{tikzcd}$$

ALGEBRAICALLY, MORPHISMS ARE MATRICES W/ ENTRIES IN A CERTAIN RIG CATEGORY.

TO-DO

(WITH UNDERGRADS OR EARLY-YEAR GRADS!)

1. VERIFY THAT $\mathcal{X} *_{\Delta} \mathcal{Y}$ IS INDEED THE PUSHOUT!

2. VERIFY THAT $\mathrm{ISub}(\mathcal{X} *_{\Delta} \mathcal{Y})$ CONTAINS "ENOUGH BRIDGES."

3. DEFINE A "NORMAL FORM" FOR MOR'S IN $\mathcal{X} *_{\Delta} \mathcal{Y}$.

4. PROVE THAT IT'S A CATEGORY OF TRACES.
   (i.e., IS "FREE" IN SOME SENSE.)

PUZZLES 3 – ∞ :

PROPERTIES OF DAGGER CATEGORIES

BACKGROUND

QUANTUM THEORY TRADITIONALLY TAKES PLACE IN THE CAT. **Hilb** OF HILBERT SPACES & BOUNDED LINEAR OPERATORS.

Hilb IS SYMMETRIC MONOIDAL, AND IS EQUIPPED WITH AN ADJOINT OPERATION: FOR OPERATORS f, g ON H, WE HAVE

$$(g \circ f)^\dagger = f^\dagger \circ g^\dagger, \quad id_H^\dagger = id_H,$$

$$(f^\dagger)^\dagger = f.$$

BACKGROUND

THE ADJOINT ON **Hilb** IS AN EXAMPLE OF A DAGGER STRUCTURE ON A CATEGORY: A CONTRAVARIANT INVOLUTIVE FUNCTOR

$$(-)^\dagger : \mathcal{C} \longrightarrow \mathcal{C}$$

WHICH IS THE IDENTITY ON OBJECTS.

MANY PROPERTIES OF DAGGER CATEGORIES HAVE YET TO BE INVESTIGATED...

TO-DO

JUST TWO SUCH PROPERTIES I'M MOST INTERESTED IN:

- IS THERE A YONEDA STRUCTURE ON DagCat? (NONEXISTENCE WOULD MAKE DEFINING "DAGGER SHEAVES" IMPOSSIBLE...)
- IS THERE A MONADICITY THEOREM FOR DAGGER MONADS?

MANY MORE, SOME PROBABLY ACCESSIBLE TO EARLY-YEAR GRADS!

THANKS FOR LISTENING!

SLIDES: ari-rosenfield.github.io