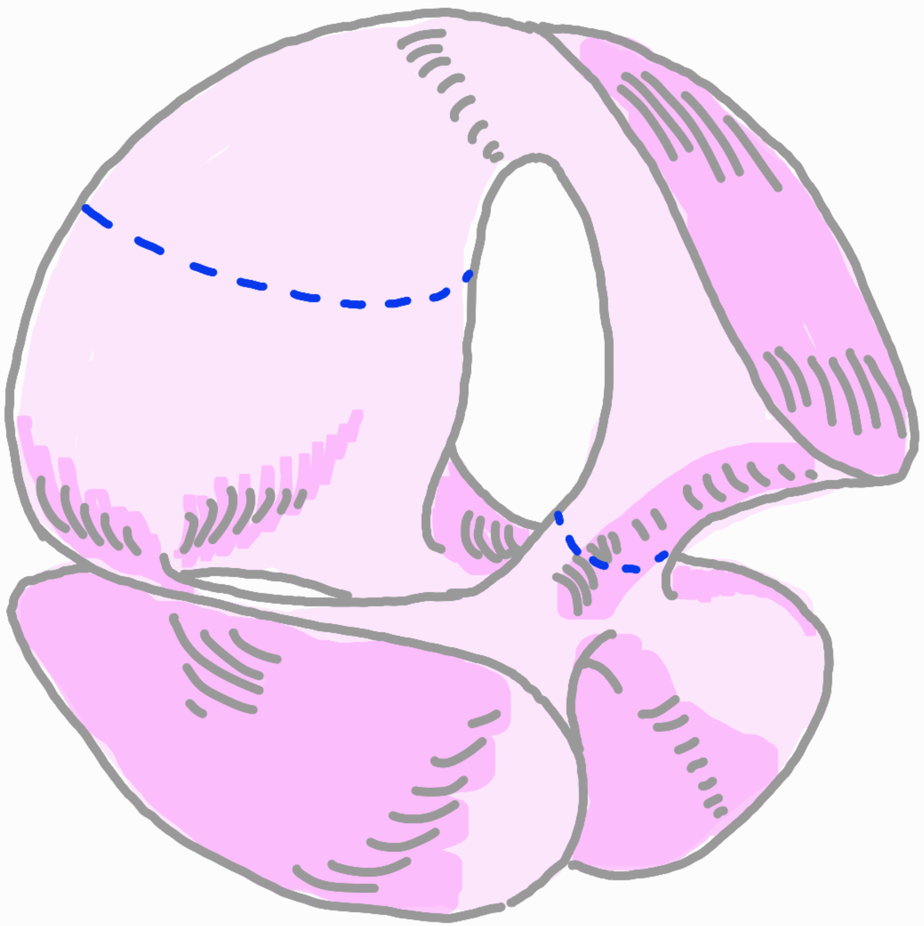

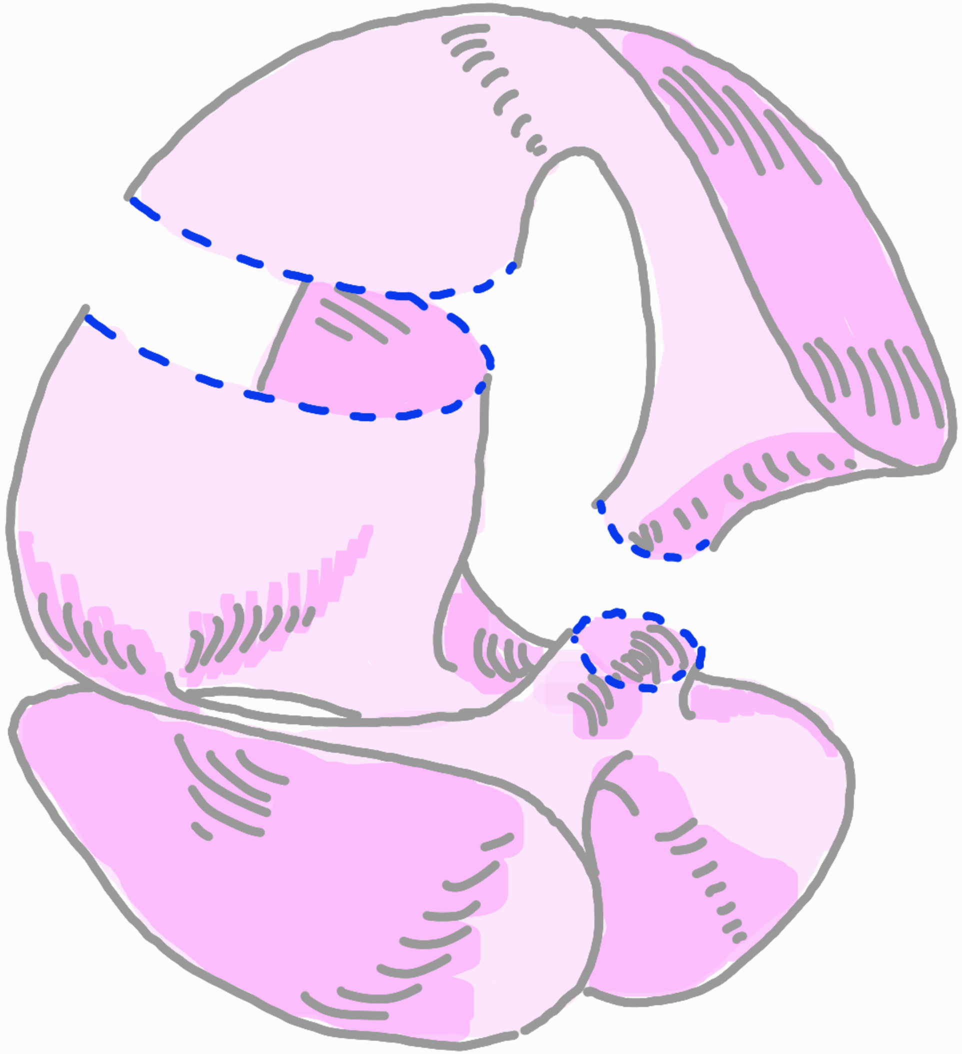

ON CATEGORIES OF ENRICHED SHEAVES

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OVERVIEW

TWO PROJECTS:

I. ENRICHED SHEAVES UNDER CHANGE OF ENRICHING BASE

II. (J/W ANA TENÓRIO) CLASSIFYING CATEGORIES OF ENRICHED SHEAVES

OVERVIEW

UNIFYING THEME:

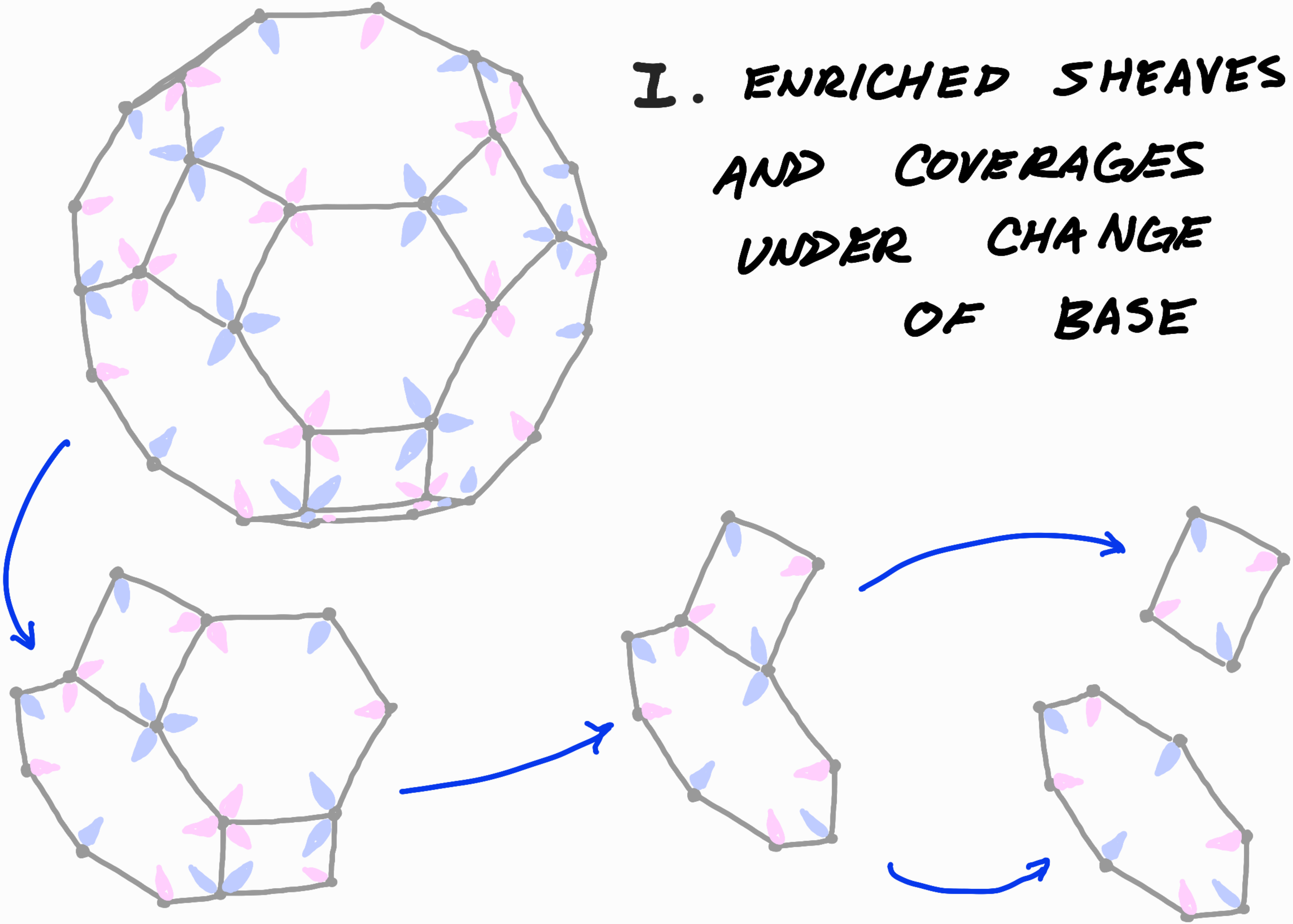
ENRICHED GROTHENDIECK TOPOSES.

OUR PERSPECTIVE

A **GROTHENDIECK TOPOS** IS A CATEGORY EQUIVALENT TO SOME CATEGORY OF SHEAVES.

AN **ENRICHED GROTHENDIECK TOPOS** IS AN ENRICHED CATEGORY EQUIVALENT TO SOME CATEGORY OF ENRICHED SHEAVES.

I. ENRICHED SHEAVES AND COVERAGES UNDER CHANGE OF BASE



arXiv: 2405.19529

I.A. INTRODUCTION

NOTATION AND CONVENTIONS

$(\mathcal{V}, \otimes, *_{\mathcal{V}})$ AND $(\mathcal{U}, \otimes, *_{\mathcal{U}})$ ARE ALWAYS SYMMETRIC MONOIDAL CLOSED AND LOCALLY SMALL.

$\mathcal{C}$ IS ALWAYS A SMALL (ESSENTIALLY SMALL) $\mathcal{V}$-ENRICHED CATEGORY.

ENRICHMENT

THINK OF AN ENRICHED CATEGORY AS A CATEGORY WHOSE HOMS HAVE MORE GENERAL STRUCTURE THAN THAT OF A SET.

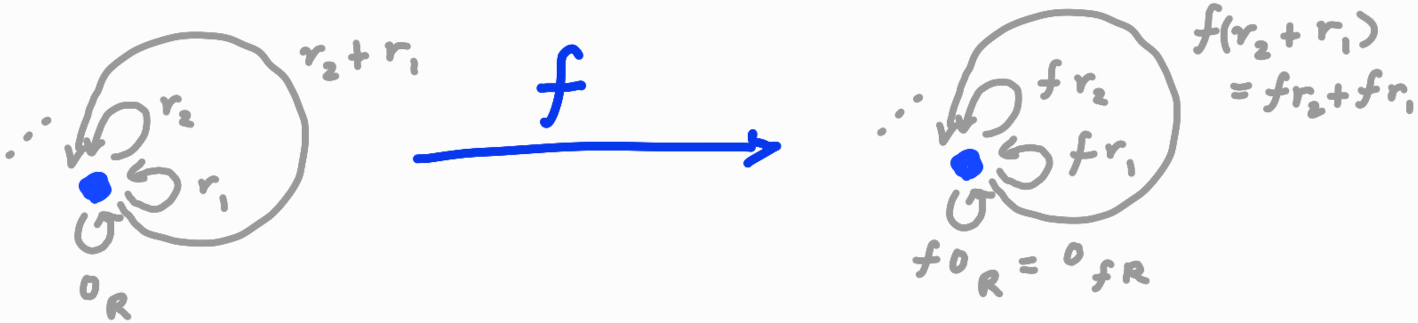
ENRICHED FUNCTORS, TRANSFORMATIONS RESPECT THIS STRUCTURE.

ENRICHMENT

e.g.,

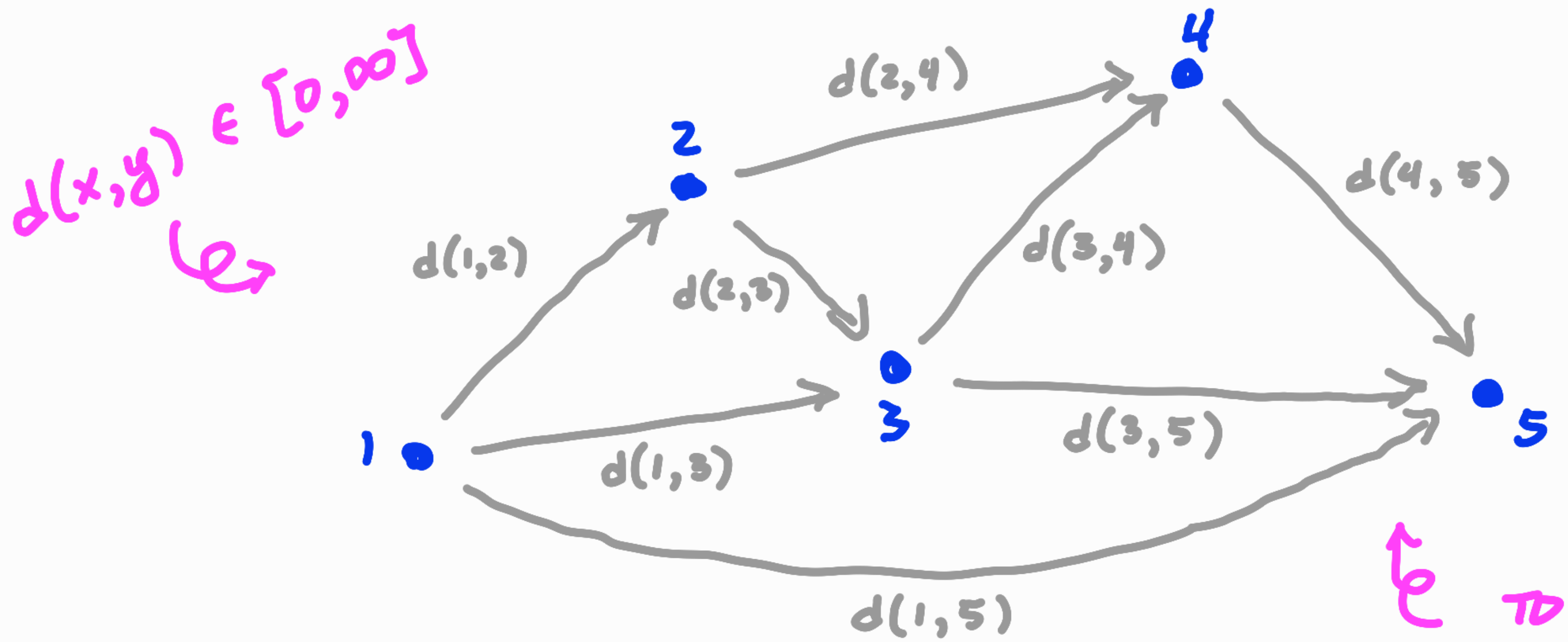
A RING IS A ONE-OBJECT Ab-CATEGORY.

A RING HOMOMORPHISM IS AN Ab-FUNCTOR.



ENRICHMENT

e.g.) WE CAN ENRICH OVER MONOIDAL PREORDERS! A $[0,\infty]$ - CATEGORY IS A LAWVERE METRIC SPACE.



TO 'COMPOSE' IS TO ADD IN $[0,\infty]$.

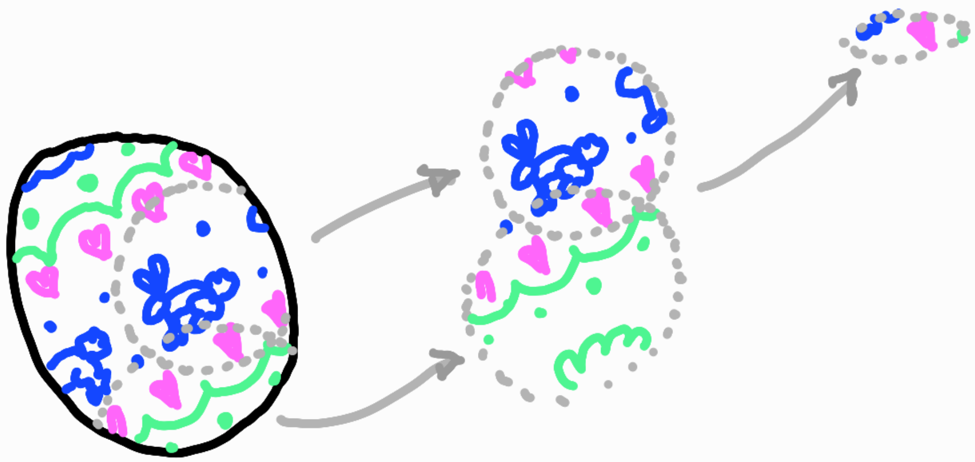
SHEAVES

A GROTHENDIECK TOPOLOGY IS A WAY TO ENDOW ANY CATEGORY (NOT JUST $\mathcal{O}(X)$) WITH A NOTION OF WHEN A GIVEN OBJECT IS "COVERED" BY OTHERS.

A COVERAGE ON A CATEGORY IS A "WEAK" GROTHENDIECK TOPOLOGY.

SHEAVES

WE WANT TO KNOW WHEN OBJECTS COVER ONE ANOTHER BECAUSE THEN WE CAN DEFINE SHEAVES: "COVER-RESPECTING" ASSIGNMENTS OF DATA TO THE OBJECTS OF OUR CAT'S.



ENRICHED SHEAVES!

WE HAVE ENRICHED GROTHENDIECK TOPOLOGIES AND ENRICHED SHEAVES, TOO (SAY, OVER $\mathcal{V}$).

e.g., A GABRIEL LOCALIZING SYSTEM ON A RING IS AN Ab-TOPOLOGY.
AN Ab-SHEAF ON A RING W/R/T SUCH A LOCALIZING SYSTEM IS ITS GABRIEL LOCALIZATION.

MORE ON THIS LATER.

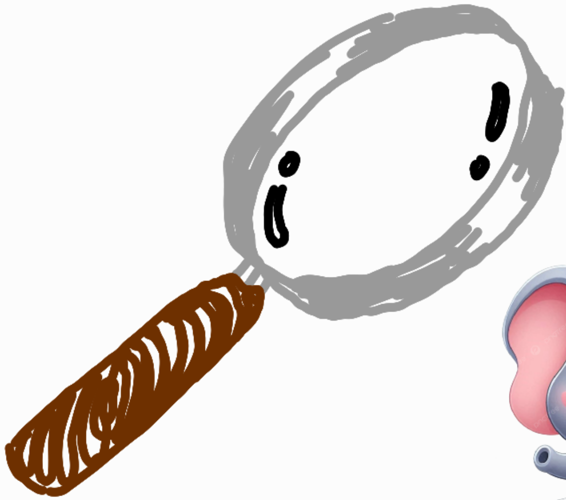
HISTORY

3 WAYS TO STUDY CAT'S OF $\mathcal{V}$-SHEAVES ON $\mathcal{C}$:

(1) "NICE" $\mathcal{V}$-SUBCATEGORIES (LOCALIZATIONS) OF $[\mathcal{C}^{op}, \mathcal{V}]$;

(2) $\mathcal{V}$-GROTHENDIECK TOPOLOGIES ON $\mathcal{C}$;

(3) UNIVERSAL CLOSURE OPERATIONS ON $[\mathcal{C}^{op}, \mathcal{V}]$.



HISTORY

BORCEUX - QUINTEIRO, 1996 —

FOR "NICE ENOUGH" $\mathcal{V}$, THERE ARE BIJECTIONS

$$(1) \Longleftrightarrow (2)$$

$$(2) \Longleftrightarrow (3)$$

$$(3) \Longleftrightarrow (1).$$

HISTORY

BORCEUX - QUINTEIRO, 1996 –

$\mathcal{V} = \mathbf{Set}$ IS "NICE ENOUGH", SO

WE RECOVER THE CLASSICAL

THEORY IN THAT CASE.

HISTORY

WE'LL FOCUS ON THE "NICE" SUBCATEGORY AND v-TOPOLOGY PERSPECTIVES.

ARCHAEOLOGY?

BY 1996, ENRICHED SHEAF THEORY HAD ALREADY BEEN DEVELOPED FOR $\mathcal{V} = Ab$ AND $\mathcal{V} = grMod_k$. (STENSTRÖM'S "RINGS OF QUOTIENTS" IS A TEXTBOOK ON $\mathcal{V} = Ab$!)

BORCEUX - QUINTEIRO'S WORK HASN'T BEEN WIDELY USED TO DATE OUTSIDE OF THESE CASES.

ARCHAEOLOGY?

OK, SO WHY DO YOU CARE ABOUT IT THEN?

WHY REVIVE THIS TOPIC?

NONCOMMUTATIVE SPECTRAL THEORY

CLASSICAL GEL'FAND DUALITY: AN EQUIVALENCE OF CATEGORIES

$$\begin{tikzcd} \mathrm{cpt\,Haus} \arrow[r, bend left, "\mathscr{C}_0(-)"] & \mathrm{comm}\,C^*\mathrm{Alg}^{op} \arrow[l, bend left, "\mathrm{Spec}"] \end{tikzcd}$$

STATE SPACES

OBSERVABLES ON CLASSICAL SYSTEMS

NONCOMMUTATIVE SPECTRAL THEORY

WHAT IF I WANT GEL'FAND DUALITY

FOR NON COMMUTATIVE C^*-ALGEBRAS?

(i.e., OBSERVABLES ON QUANTUM SYSTEMS?)

THAT WOULD BE SO COOL...

THEN I COULD DO

ACTUAL GEOMETRY...

I COULD TRY JUST APPLYING THE Spec

FUNCTOR I ALREADY HAVE??

NONCOMMUTATIVE SPECTRAL THEORY

PROBLEM! (REYES, 2012)

THEOREM: ANY FUNCTOR $F : C^*\mathrm{Alg}^{op} \longrightarrow \mathrm{Top}$

FOR WHICH

$$\begin{tikzcd} \mathrm{comm}\ C^*\mathrm{Alg}^{op} \arrow[r, "\mathrm{Spec}"] \arrow[d, hook, "\mathrm{inc}"'] & \mathrm{Top} \\ C^*\mathrm{Alg}^{op} \arrow[ur, "F"'] & \end{tikzcd}$$

COMMUTES MUST HAVE $F(M_n(\mathbb{C})) \cong \emptyset$

FOR $n \geq 3$.

NONCOMMUTATIVE SPECTRAL THEORY

IN PHYSICAL TERMS: GEL'FAND DUALITY SAYS ANY CLASSICAL SYSTEM (comm. C^*-Alg) IS DETERMINED BY A STATE SPACE (CPT. TOP. SPACE) IN A WAY THAT RESPECTS OPERATIONS ON THE SYSTEM (FUNCTORIALLY).

THIS THEOREM SAYS THAT NO COMMUTATIVE C^*-ALG DETERMINES A QUANTUM SYSTEM IN THIS WAY.

NONCOMMUTATIVE SPECTRAL THEORY

RELATED OBSTRUCTIONS HOLD FOR

- (v. d. BERG–HEUNEN, 2014) ZARISKI, STONE, AND PIERCE SPECTRA AS FUNCTORS INTO Set, Loc, Top, Scheme, Quantale, Topos;

- (REYES, 2014) SUBCANONICAL COVERAGES ON THE BIG ZARISKI SITE FOR cRing.

NONCOMMUTATIVE SPECTRAL THEORY

THE DREAM OF NONCOMMUTATIVE GEOMETRY:

$$\begin{tikzcd} \mathrm{comm}\, C^*\mathrm{Alg}^{op} \arrow[r, "\mathrm{Spec}"] \arrow[d, hook, "\mathrm{inc}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{cpt\,Haus} \arrow[d, hook, dashed, "??"] \\ C^*\mathrm{Alg}^{op} \arrow[r, dashed, "\text{``NCSpec''}"'] & \text{``NONCOMMUTATIVE CPT. HAUSDORFF SPACES''} \end{tikzcd}$$

NONCOMMUTATIVE SPECTRAL THEORY

TOPOLOGICAL SPACES ARE SETS OF POINTS, SO I MAY AS WELL THINK ABOUT A SLIGHTLY SIMPLER PROBLEM:

$$
\begin{tikzcd}
\mathrm{comm}\,\mathrm{Ring}^{op} \arrow[r, "\mathrm{Spec}"] \arrow[d, hook, "\mathrm{inc}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Set} \arrow[d, hook, dashed, "??"] \\
\mathrm{Ring}^{op} \arrow[r, dashed, "\text{``NCSpec''}"'] & \text{``}q\,\mathrm{Set}\text{''}
\end{tikzcd}
$$

FINDING A CAT. OF "QUANTUM / N.C." POINT - SETS.

NONCOMMUTATIVE SPECTRAL THEORY

WE EXPECT $q\mathrm{Set}$ TO BE A CLOSED SYMMETRIC MONOIDAL CATEGORY, AND WE EXPECT A FULLY FAITHFUL EMBEDDING

$$\mathrm{Set} \longrightarrow q\mathrm{Set}.$$

HOW TO TELL IF A GIVEN CANDIDATE FOR $q\mathrm{Set}$, SAY $\mathcal{V} \in \mathrm{MonCat}$, IS THE RIGHT ONE?

COOL, WHAT DOES IT HAVE TO DO WITH QUANTUM?

ONE WAY TO TEST WHETHER WE'VE FOUND A GOOD CANDIDATE FOR qSet IS TO MAKE SURE THERE ARE qSet - ENRICHED GROTH-ENDIECK TOPOLOGIES ON A GIVEN qSet - CATEGORY WHICH DON'T ARISE FROM UNENRICHED ONES.

COOL, WHAT DOES IT HAVE TO DO WITH QUANTUM?

SO, NEED TOOLS TO DETECT WHETHER A GIVEN ENRICHED GROTHENDIECK TOPOLOGY ARISES FROM AN UNENRICHED ONE.

THAT'S WHERE BASE CHANGE COMES IN...

BASE CHANGE

GIVEN A "NICE ENOUGH" FUNCTOR $G : \mathcal{V} \to \mathcal{U}$,
WE CAN CHANGE THE BASE OF $\mathcal{C}$ VIA G,
OBTAINING A $\mathcal{U}$-CATEGORY $G_* \mathcal{C}$.

THINK OF $\mathcal{U}$ & $\mathcal{V}$ AS DIFFERENT TYPES OF STRUCTURE THE HOMS IN $\mathcal{C}$ CAN HAVE.

BASE CHANGE

FOR EXAMPLE, WE CAN USE THE FUNCTOR

$$\mathrm{Hom}_{Ab}(\mathbb{Z}, -) : Ab \longrightarrow Set$$

TO TURN A 1-OBJECT Ab-CATEGORY (A RING) INTO A 1-OBJECT Set-CATEGORY (A MONOID).

NEW CONTRIBUTIONS

HOW BASE CHANGE INTERACTS WITH ENRICHED TOPOLOGIES OR ENRICHED LOCALIZATIONS HASN'T YET BEEN ADDRESSED, UNTIL NOW!

I.B. A CHANGE-OF-BASE CONSTRUCTION FOR PRESHEAVES

HYPOTHESIS

ENRICHING CATEGORIES $\mathcal{U}, \mathcal{V}$ ARE ASSUMED TO BE:

- SYMMETRIC MONOIDAL CLOSED
- LOCALLY FINITELY PRESENTABLE
- BARR REGULAR ($[\mathcal{C}^{op}, \mathcal{V}]_0$ IS REGULAR)
- FINITELY COMPLETE

WE'LL CALL SUCH A CATEGORY A COSMOS.

LFP CATEGORIES

A LOCALLY FINITELY PRESENTABLE ("LFP") CATEGORY IS ONE IN WHICH ANY OBJECT IS REALIZABLE AS A "WELL-BEHAVED" COLIMIT OF "FINITE" OBJECTS.

e.g., FOR $X \in \mathrm{Set}$, $\quad X \cong \bigcup_{\substack{S \subset X \\ \text{FINITE}}} S$.

WE SAY $\mathrm{Ob}(\mathrm{FinSet})$ IS A DENSE GENERATING FAMILY FOR Set.

LFP CATEGORIES

FACT: ANY LFP CATEGORY $\mathcal{W}$ ...

- ADMITS A DENSE GENERATING FAMILY $\mathcal{G}_\mathcal{W}$ MADE UP OF FINITELY PRESENTABLE OBJECTS.
- IS COCOMPLETE AND WELL-POWERED.

LFP CATEGORIES

WHO CARES ABOUT A GENERATING FAMILY?

IDEA. IF $\mathcal{C} \in \mathcal{V}\text{-Cat}$, SO $\mathcal{C}(x,y) \in Ob(\mathcal{V})$, AND $\mathcal{V} \neq Set$, SO $*_{\mathcal{V}} \neq \{\bullet\}$, WHAT'S AN "ELEMENT" OF $\mathcal{C}(x,y)$? WE HAVE TO CONSIDER GENERALIZED ELEMENTS INSTEAD:

MOR'S $g \to \mathcal{C}(x,y)$ FOR $g \in \mathcal{G}_{\mathcal{V}}$.

"COSMOI"

EXAMPLES OF COSMOI:

Set, **Ab**, R**Mod** (FOR COMM. RINGS R), **grMod**$_k$ (FOR FIELDS k), ANY PRESHEAF TOPOS.

"ALMOST EXAMPLES": THE MONOIDAL PREORDERS $[0,1]$ AND $[0,\infty]$ (NOT LFP), **Cat** (NOT REGULAR).

"COSMOI"

WHY CONSIDER THESE "ALMOST EXAMPLES?" IT WAS HARD TO FIND EXAMPLES OF BASE CHANGE THAT WEREN'T BASICALLY "FORGETFUL" FUNCTORS.

PLUS, NEITHER LFP NOR REGULARITY ARE NEEDED FOR THE DEFINITION OF ENRICHED TOPOLOGY.

GOAL! WANT TO FIGURE OUT HOW TO CHANGE THE BASE OF A $\mathcal{V}$-TOPOLOGY.

WHAT ARE $\mathcal{V}$-TOPOLOGIES & BASE CHANGE, REALLY?

WHAT IS A GROTHENDIECK TOPOLOGY?

BORCEUX - QUINTEIRO, 1996:

$\mathcal{V}$- $\mathcal{V}$-

Definition 1.2 *Let $\mathcal{C}$ be a small* $\mathcal{V}$- *category. A* $\mathcal{V}$- *Grothendieck topology on $\mathcal{C}$ is the choice, for every object $C \in \mathcal{C}$, of a family $\mathcal{T}(C)$ of subobjects of the representable* $\mathcal{V}$- *presheaf $\mathcal{C}(-,C)$. Those data must satisfy the following axioms:*

$f : G \to \mathcal{C}(D,C)$ FOR $G \in \mathcal{G}_{\mathcal{V}}$

(T1) $\mathcal{C}(-,C) \in \mathcal{T}(C)$ for every object $C \in \mathcal{C}$;

(T2) given $R \in \mathcal{T}(C)$ and $f \in_G \mathcal{C}(D,C)$, one has $f^{-1}(R) \in \mathcal{T}(D)$, where $f^{-1}(R)$ is defined by the following pullback:

$$\begin{tikzcd}
f^{-1}(R) \arrow[r] \arrow[d, tail] & \{G,R\} \arrow[d, tail] \\
\mathcal{C}(-,D) \arrow[r, "f"'] & \{G,\mathcal{C}(-,C)\};
\end{tikzcd}$$

COTENSOR IN $[\mathcal{C}^{op}, \mathcal{V}]$

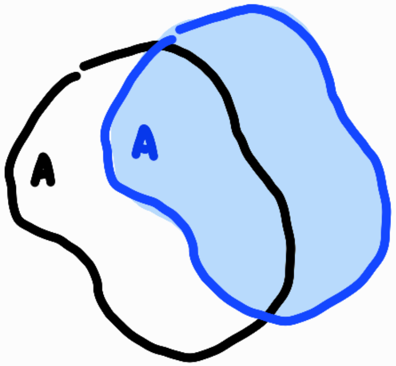
(T3) given $S \in \mathcal{T}(C)$ and a subobject $R \rightarrowtail \mathcal{C}(-,C)$ such that $f^{-1}(R) \in \mathcal{T}(D)$ for all $f \in_G S(D)$, one has $R \in \mathcal{T}(C)$.

WHAT IS A GROTHENDIECK TOPOLOGY?

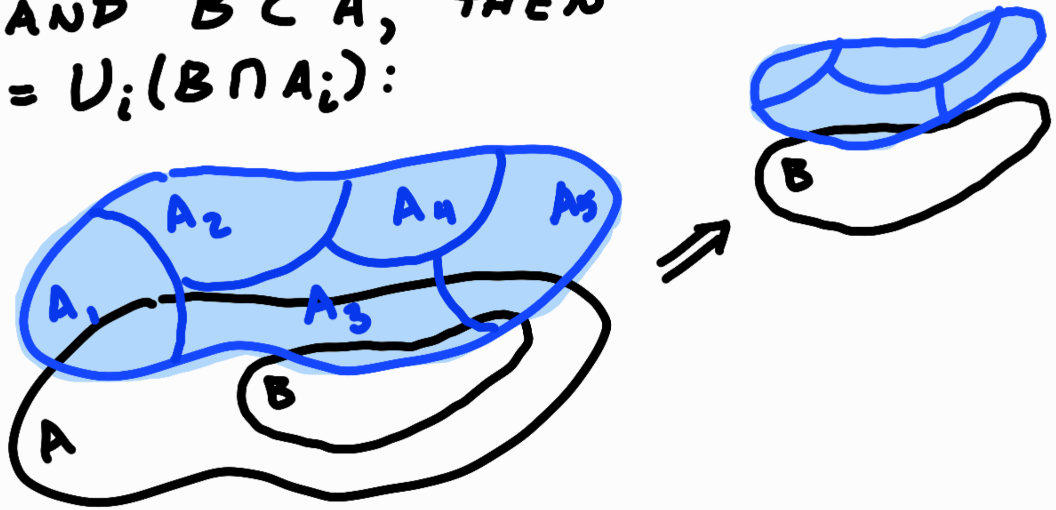
(T1) – (T3) ENCODE SOME FAMILIAR PROPERTIES OF OPEN COVERS IN TOPOLOGICAL SPACES X:

WHAT IS A GROTHENDIECK TOPOLOGY?

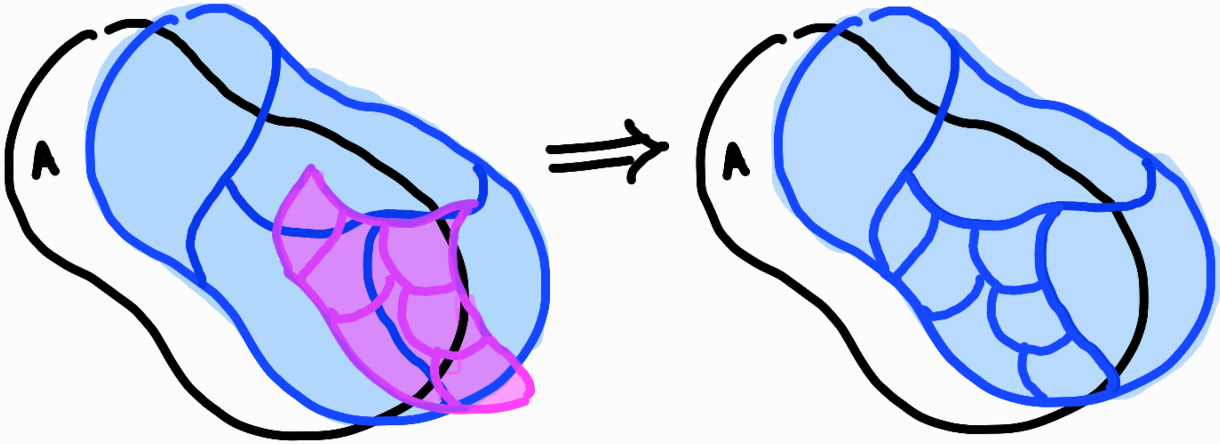
(T1) $A \in \mathcal{O}(X)$ IS AN OPEN COVER OF ITSELF:



(T2) IF $A = \bigcup_i A_i$ AND $B \subset A$, THEN $B = \bigcup_i (B \cap A_i)$:



(T3) IF $A = \bigcup_i A_i$ AND $A_i = \bigcup_j B_{ij}$ $\forall i$, THEN $A = \bigcup_{ij} B_{ij}$.



WHAT IS A GROTHENDIECK TOPOLOGY?

ABUSE!

A SIEVE ON $x \in Ob(\mathcal{C})$ IS A SUB-OBJECT $R \rightarrowtail \mathcal{C}(-,x)$ – i.e., AN EQUIVALENCE CLASS OF MONOMORPHISMS INTO $\mathcal{C}(-,x)$. THESE FORM A COMPLETE LATTICE, $\mathrm{Sub}(\mathcal{C}(-,x))$.

"AN OPEN COVER OF x."

WHAT IS A GROTHENDIECK TOPOLOGY?

SAY THAT A FAMILY J OF SIEVES IS A **COVERAGE** ON $\mathcal{C}$ IF IT SATISFIES (T1) AND (T2).

DENOTE

$$\Sigma(\mathcal{C}, \mathcal{V}) = \{ \text{ALL } \mathcal{V}\text{-COVERAGES ON } \mathcal{C} \}$$

$$\tau(\mathcal{C}, \mathcal{V}) = \{ \text{ALL } \mathcal{V}\text{-TOPOLOGIES ON } \mathcal{C} \}$$

COVERAGES VS. TOPOLOGIES

CONDITION (T3) WAS TECHNICALLY DIFFICULT TO WORK WITH UNDER BASE CHANGE.

CONJECTURE. ANY $\mathcal{V}$-COVERAGE UNIQUELY DETERMINES A $\mathcal{V}$-TOPOLOGY WITH THE SAME SHEAVES.

WHAT IS A GROTHENDIECK TOPOLOGY?

E.G., FOR A RING A, AN Ab-SIEVE ON A'S ONE OBJECT IS A RIGHT IDEAL OF A. AN Ab-COVERAGE J ON A IS A FAMILY OF RIGHT IDEALS S.T.

(T1) $A \in J$

(T2) $\forall a \in A, \forall I \in J,$

$$(I : a) := \{x \in A : xa \in I\} \in J$$

"CLOSURE UNDER IDEAL QUOTIENT"

WHAT IS A GROTHENDIECK TOPOLOGY?

e.g., LET $\mathcal{X}$ BE A LAWVERE METRIC SPACE WITH DISTANCE FUNCTION $d(-,-)$.

A $[0,\infty]$-SIEVE ON $x \in \mathcal{X}$ IS A LIPSCHITZ FUNCTION $f : \mathcal{X} \to [0,\infty]$ SUCH THAT $\forall y, z$,

$$\text{(i)} \quad fz \geq d(z,x)$$

$$\text{(ii)} \quad d(y,z) \geq \underbrace{\max\{0, d(y,x) - fz\}}_{=: [0,\infty](fz,\, d(y,x))}.$$

THESE SAY THAT $f \rightarrowtail d(-,x)$ IS $[0,\infty]$-NATURAL.

WHAT IS A GROTHENDIECK TOPOLOGY?

A $[0,\infty]$-COVERAGE ON $\mathcal{X}$ IS, $\forall x$, A COLLECTION $J(x)$ OF SIEVES S.T.

(T1) $d(-,x) \in J(x)$

(T2) $\forall y \in \mathcal{X}$, $\forall q \in [0,\infty]$ WITH $q \geq \max\{0, y - x\}$, AND $\forall f \in J(x)$, THE FUNCTION

$$f_q(z) := \max\{fz, \max\{0, d(z,y) - q\}\}$$

IS IN $J(y)$.

WHAT IS A GROTHENDIECK TOPOLOGY?

IN SHORT:

"A FAMILY OF SIEVES SATISFYING SOME CLOSURE CONDITIONS."

BASE CHANGE

FIX $\mathcal{U}, \mathcal{V}$, AND LAX MONOIDAL $G : \mathcal{V} \to \mathcal{U}$.

G DETERMINES A 2-FUNCTOR

$$\mathcal{V}\text{-}\mathbf{Cat} \xrightarrow{G_*} \mathcal{U}\text{-}\mathbf{Cat}.$$

GIVEN A $\mathcal{V}$-CATEGORY $\mathcal{C}$, THE $\mathcal{U}$-CATEGORY

$G_*\mathcal{C}$ HAS

$$\mathrm{Ob}(G_*\mathcal{C}) := \mathrm{Ob}(\mathcal{C}),$$

$$G_*\mathcal{C}(x,y) := G(\mathcal{C}(x,y)).$$

BASE CHANGE

GIVEN A $\mathcal{V}$-FUNCTOR $A : \mathcal{X} \longrightarrow \mathcal{Y}$, DENOTE THE CORR. $\mathcal{U}$-FUNCTOR BY

$$G_* A : G_* \mathcal{X} \longrightarrow G_* \mathcal{Y}$$

(G_* ALSO ACTS ON $\mathcal{V}$-NATURAL TRANSFORMATIONS).

HYPOTHESIS

WE CONSIDER LAX MONOIDAL FUNCTORS $G : \mathcal{V} \longrightarrow \mathcal{U}$ WHICH ARE

$$\left.\begin{array}{l} \text{i. FAITHFUL} \\ \text{ii. CONSERVATIVE} \\ \text{iii. RIGHT ADJOINTS (IN MonCat)} \end{array}\right\} \quad (\blacktriangle)$$

BASE CHANGE

FORGETFUL FUNCTORS ON $(G\circ F)$-Mod FOR MONADS $G\circ F$

EXAMPLES!

(1) MONADIC FUNCTORS. AMONG THEM:

- FOR $(f: R \to S) \in c\mathrm{Ring}$, RESTRICTION OF SCALARS $S\mathrm{Mod} \xrightarrow{f^*} R\mathrm{Mod}$.

- FOR $\mathcal{W} \in \mathrm{Cosmos}$, THE FORGETFUL FTR. $\mathrm{Mon}(\mathcal{W}) \longrightarrow \mathcal{W}$

- REFLECTIVE SUBCATEGORY INCLUSIONS.

BASE CHANGE

ALMOST EXAMPLES!

(2) THE INCLUSION $\{0,1\} \hookrightarrow [0,1]$. A $\{0,1\}$-CATEGORY IS JUST A PREORDER. ENRICHING OVER $[0,1]$ IS SIMILAR TO $[0,\infty]$.

(3) THE FUNCTIONS

$$-\log(*) : [0,1] \rightleftarrows [0,\infty] : e^{-(*)}.$$

(IN FACT, THIS IS AN ISOMORPHISM OF CATEGORIES.)

BASE CHANGE

ALMOST EXAMPLES!

(4) THE NERVE CONSTRUCTION $\mathcal{C}at \longrightarrow s\mathcal{S}et$.

IT'S MONADIC, BUT THESE AREN'T COSMOI.

BASE CHANGE FOR SIEVES

GOAL! WANT TO FIGURE OUT HOW TO CHANGE THE BASE OF A COVERAGE.

A COVERAGE IS MADE UP OF SIEVES, SO LET'S FIGURE OUT BASE CHANGE FOR SIEVES.

BASE CHANGE FOR SIEVES

IF $R : \mathcal{C}^{op} \longrightarrow \mathcal{V}$ IS A $\mathcal{V}$-PRESHEAF, BASE CHANGE GIVES US A $\mathcal{U}$-FUNCTOR

$$G_* R : G_* \mathcal{C}^{op} \longrightarrow G_* \mathcal{V}$$

IT DOESN'T TAKE VALUES IN $\mathcal{U}$.

– NOT QUITE A $\mathcal{U}$-PRESHEAF!

BASE CHANGE FOR SIEVES

MONOIDAL

IN CASE G IS A MONOIDAL RIGHT ADJOINT, WE HAVE AN INDUCED ADJUNCTION

$$\begin{tikzcd} G_*\mathcal{V} \arrow[r, bend left, "G^{\mathcal{U}}"] \arrow[r, phantom, "\top"] & \mathcal{U} \arrow[l, bend left, "F^{\mathcal{U}}"] \end{tikzcd}$$

IN $\mathcal{U}$-Cat:

BASE CHANGE FOR SIEVES

SAY $(F \dashv G, \varepsilon, \eta)$ IS OUR ADJUNCTION AND x, y, z ARE OBJECTS:

$$\mathrm{Hom}_{\mathcal{U}}(x, \mathcal{U}(y, Gz)) \cong \mathrm{Hom}_{\mathcal{U}}(x \otimes y, Gz)$$

HOM - TENSOR IN $\mathcal{U}$

$$\cong \mathrm{Hom}_{\mathcal{V}}(Fx \otimes Fy, z)$$

$F \dashv G$, STRENGTH OF F

$$\cong \mathrm{Hom}_{\mathcal{U}}(x, G\mathcal{V}(Fy, z))$$

HOM - TENSOR IN $\mathcal{V}$, $F \dashv G$

よ LEMMA $\Longrightarrow$

$$\phi : \mathcal{U}(y, Gz) \cong G\mathcal{V}(Fy, z) \quad \text{IN} \quad \mathcal{U}.$$

BASE CHANGE FOR SIEVES

SETTING

$$G^{\mathcal{U}}_{yz} := G\mathcal{V}(y,z) \xrightarrow{G(\varepsilon^*)} G\mathcal{V}(FGy,z) \xrightarrow{\Phi} \mathcal{U}(Gy,Gz)$$

DEFINES A $\mathcal{U}$-FUNCTOR WITH $G^{\mathcal{U}}(x) = Gx$ FOR $x \in Ob(\mathcal{V})$.

BASE CHANGE FOR SIEVES

WE CAN THEN DEFINE

$$\tilde{G}R := G^{\mathcal{U}} \circ G_* R : G_* \mathcal{C}^{op} \longrightarrow \mathcal{U}.$$

A PERFECTLY GOOD $\mathcal{U}$-PRESHEAF.

BASE CHANGE FOR SIEVES

PROPOSITION. SUPPOSE G SATISFIES ($\blacktriangle$). THE OPERATION $R \longmapsto \tilde{G}R$ EXTENDS TO A FAITHFUL, CONSERVATIVE FUNCTOR

$$[\mathcal{C}^{op}, \mathcal{V}] \xrightarrow{\tilde{G}} [G_* \mathcal{C}^{op}, \mathcal{U}]$$

WHICH PRESERVES AND REFLECTS LIMITS.

LIKE 6 TECHNICAL LEMMATA REQUIRED TO PROVE THIS

BASE CHANGE FOR SIEVES

IN PARTICULAR, SINCE $\tilde{G}$ PRESERVES LIMITS, IT INDUCES A FUNCTION

$$\mathrm{Sub}_{\mathcal{V}}(\mathcal{C}(-,x)) \xrightarrow{\tilde{G}} \mathrm{Sub}_{\mathcal{U}}(\tilde{G}\mathcal{C}(-,x)).$$

BASE CHANGE FOR SIEVES

MORE IS TRUE WHEN G SATISFIES ($\blacktriangle$):

THEOREM: THE FUNCTION

$$\mathrm{Sub}(\mathcal{C}(-,x)) \xrightarrow{\tilde{G}} \mathrm{Sub}(\tilde{G}\mathcal{C}(-,x))$$

IS AN INJECTIVE LATTICE MORPHISM.

"G PRESERVES COARSENESS"

BASE CHANGE FOR COVERAGES

WE CAN EXTEND THIS RESULT TO COVERAGES AFTER ADDRESSING SOME TECHNICALITIES:

BASE CHANGE FOR COVERAGES

THEOREM. $\Sigma(\mathcal{C}, \mathcal{V})$ AND $\mathcal{T}(\mathcal{C}, \mathcal{V})$ ARE COMPLETE LATTICES. ← SAME FOR $G_* \mathcal{C}$, $\mathcal{U}$, ETC.

PROPOSITION. GIVEN A $\mathcal{V}$-COVERAGE J ON $\mathcal{C}$, G INDUCES A $\mathcal{U}$-COVERAGE $\tilde{G}J$ ON $G_* \mathcal{C}$ BY SETTING $\tilde{G}J(x) := \{ \tilde{G}R : R \in J(x) \}$.

BASE CHANGE FOR COVERAGES

THEOREM. SUPPOSE G SATISFIES $(\blacktriangle)$.

THE ASSIGNMENT

$$\Sigma(\mathcal{C}, \mathcal{V}) \xrightarrow{\tilde{G}} \Sigma(G_*\mathcal{C}, \mathcal{U})$$

IS AN INJECTIVE LATTICE MORPHISM.

BASE CHANGE FOR COVERAGES

IT APPEARS THAT FAITHFULNESS IS NECESSARY!

THEOREM. FOR A FIELD k, $\mathcal{V} = \mathrm{grMod}_k$, $\mathcal{U} = \mathrm{Set}$, AND

NOT FAITHFUL!

$$G = \mathrm{Hom}_{\mathcal{V}}(k, -) : \mathcal{V} \to \mathcal{U},$$

THERE EXIST DISTINCT $\mathcal{V}$-COVERAGES ON $k[x,y]$ WHICH GIVE RISE TO THE SAME $\mathcal{U}$-COVERAGE.

BASE CHANGE FOR TOPOLOGIES??

CAN WE STRENGTHEN THAT TO HOLD FOR TOPOLOGIES? IDK, MAYBE!

PROPOSITION. GIVEN A $\mathcal{V}$-TOPOLOGY J ON $\mathcal{C}$, G INDUCES A $\mathcal{U}$-TOPOLOGY $\overline{GJ}$ ON $G_*\mathcal{C}$.

↑ SMALLEST TOPOLOGY CONTAINING $\tilde{G}J$.

IDEA: FORM $\tilde{G}J$, THEN "CLOSE UNDER (T3)" USING A TRANSFINITE CONSTRUCTION.

BASE CHANGE FOR TOPOLOGIES??

IT'S STILL ONLY A CONJECTURE.

CONJECTURE. SUPPOSE G SATISFIES ($\blacktriangle$).

THE ASSIGNMENT

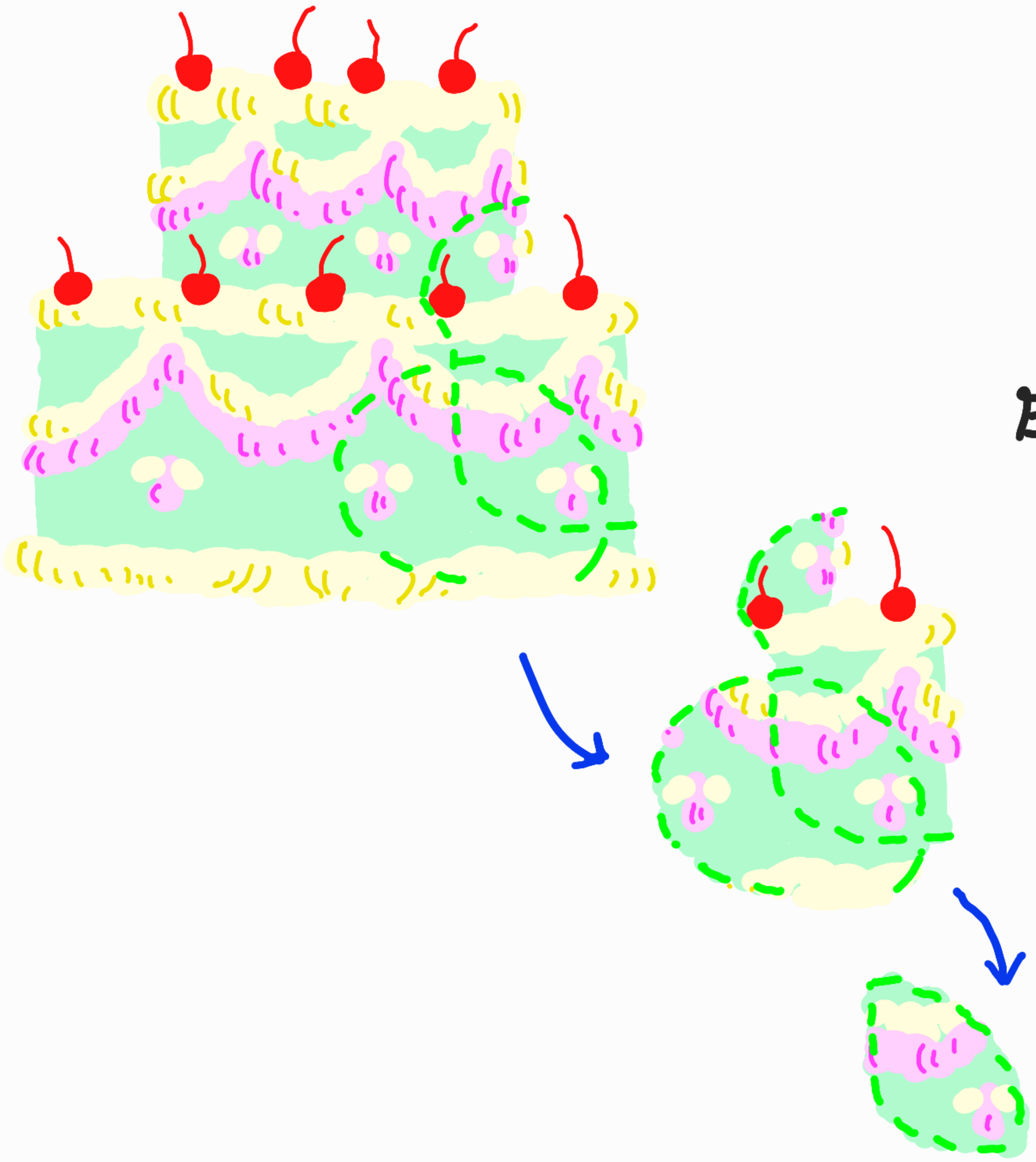
$$\mathcal{T}(\mathcal{C}, \mathcal{V}) \xrightarrow{\overline{G(-)}} \mathcal{T}(G_*\mathcal{C}, \mathcal{U})$$

IS AN INJECTIVE LATTICE MORPHISM.

BASE CHANGE FOR TOPOLOGIES??

THIS WOULD LET US MAKE USE OF BORCEUX-QUINTEIRO'S CORRESPONDENCE THEOREM, SINCE IT APPLIES TO TOPOLOGIES, NOT COVERAGES.

PROBLEM: $\overline{G(-)}$ IS HARD TO WORK WITH – NEED AN EASIER WAY TO CONSTRUCT.



I. C.

ENRICHED SHEAVES AND BASE CHANGE

ENRICHED SHEAVES + BASE CHANGE

BORCEUX - QUINTEIRO ALSO GIVE DEFINITIONS FOR

(i) ENRICHED SHEAF

(ii) ENRICHED SHEAFIFICATION.

WHEN G IS A FULLY FAITHFUL RIGHT ADJOINT, IT INTERACTS NICELY WITH THESE.

i.e., $\mathcal{V}$ IS A REFLECTIVE SUBCAT. OF $\mathcal{U}$

QUICKLY, DOCTOR! THE DEFINITIONS!

SAY $\mathcal{W} \in$ Cosmos AND $\mathcal{X}$ IS A SMALL $\mathcal{W}$-CATEGORY.

Definition 5.1. A $\mathcal{W}$-functor $P \in [\mathcal{X}^{\mathrm{op}}, \mathcal{W}]$ is a **sheaf** for a $\mathcal{W}$-coverage J when, given any object $x \in \mathcal{X}$, any $R \in J(x)$, any $g \in \mathcal{G}_{\mathcal{W}}$, and α such that

$$\begin{tikzcd} R \arrow[r, tail, "r"] \arrow[d, "\alpha"'] & \mathcal{X}(-,x) \arrow[dl, dotted, "\exists! \beta"] \\ \{g, P\} & \end{tikzcd},$$

there exists a unique β for which the diagram commutes.

QUICKLY, DOCTOR! THE DEFINITIONS!

THIS DEFINITION ENCODES THE MORE FAMILIAR ONE FROM GEOMETRY:

"GIVEN A FAMILY OF COMPATIBLE LOCAL SECTIONS, $\exists!$ GLOBAL SECTION WHICH RESTRICTS TO THEM."

QUICKLY, DOCTOR! THE DEFINITIONS!

FOR A COVERAGE J, DENOTE THE CAT. OF SHEAVES ON $\mathcal{X}$ BY

$$Sh(\mathcal{X}, J).$$

QUICKLY, DOCTOR! THE DEFINITIONS!

FACT. WE CAN DEFINE A $\mathcal{W}$-FUNCTOR

$$[\mathcal{X}^{op}, \mathcal{W}] \xrightarrow{\Sigma} [\mathcal{X}^{op}, \mathcal{W}]$$

WHICH SENDS PRESHEAVES TO "NICER" ONES AND "NICER" PRESHEAVES TO SHEAVES W/R/T A COVERAGE J.

QUICKLY, DOCTOR! THE DEFINITIONS!

FACT. THE COMPOSITE $\Sigma\Sigma$ EXHIBITS $Sh(\mathcal{X}, J)$ AS A REFLECTIVE SUBCAT. OF $[\mathcal{X}^{op}, \mathcal{W}]$.

WE CALL

$$[\mathcal{X}^{op}, \mathcal{W}] \xrightarrow{\Sigma\Sigma_{\mathcal{W}}} Sh(\mathcal{X}, J)$$

SHEAFIFICATION (W/R/T J).

ENRICHED SHEAVES + BASE CHANGE

FOR A FULLY FAITHFUL RIGHT ADJOINT G:

PROPOSITION. FOR $J \in \Sigma(\mathcal{C}, \mathcal{V})$ AND A SHEAF P ON $\mathcal{C}$ WITH RESPECT TO J, $\tilde{G}P$ IS A SHEAF FOR $\tilde{G}J$.

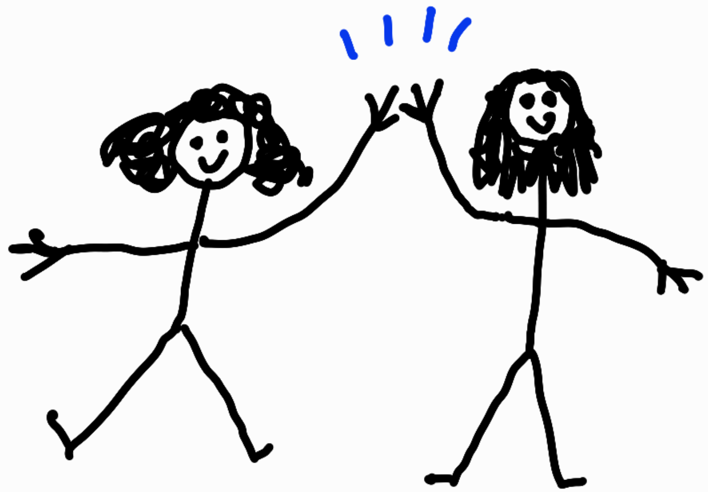
THEOREM. FOR $P \in [\mathcal{C}^{op}, \mathcal{V}]$, WE HAVE

$$\tilde{G}(\Sigma\Sigma_{\mathcal{V}} P) \cong \Sigma\Sigma_{\mathcal{U}}(\tilde{G}P).$$

END OF I.

II. TOWARD A CLASSIFICATION OF ENRICHED GROTHENDIECK TOPOSES

(J/W ANA TENÓRIO)



THE UNENRICHED SETTING

WHAT IF I WANT TO KNOW WHETHER A GIVEN CATEGORY $\mathcal{E}$ (ENRICHED OR NOT) IS A SHEAF CATEGORY?

IN THE UNENRICHED CASE, WE CAN CHECK GIRAUD'S AXIOMS.

THE UNENRICHED SETTING

UNFORTUNATELY, GENERALIZING THEM TO $\mathcal{V}$ IS HARD:

THE UNENRICHED SETTING

(GIRAUD, c. 1960)

A CATEGORY $\mathcal{E}$ IS EQUIVALENT TO $Sh(X, J)$ FOR SOME (Set-) SITE (X, J) EXACTLY WHEN IT

1. IS LOCALLY PRESENTABLE
2. HAS PULLBACK-STABLE COLIMITS
3. HAS DISJOINT COPRODUCTS
4. HAS EFFECTIVE QUOTIENTS

THE UNENRICHED SETTING

THE CLASSIFICATION AS REFLECTIVE SUBCAT'S OF PRESHEAF CATEGORIES HOLDS BOTH FOR Set AND GENERAL $\mathcal{V}$.

BUT THERE'S ANOTHER CLASSIFICATION FOR Set WHICH WE CAN TRY TO EXTEND TO $\mathcal{V}$...

THE UNENRICHED SETTING

THEOREM (STREET, 1981) A LOCALLY SMALL CATEGORY $\mathcal{E}$ IS A GROTHENDIECK TOPOS IFF

$$\mathcal{L}_{\mathcal{E}} : \mathcal{E} \longrightarrow [\mathcal{E}^{op}, \mathbf{Set}]$$

HAS A LEFT EXACT LEFT ADJOINT.

" TOPOS $\Leftrightarrow$ LEX-TOTAL "

THE ENRICHED SETTING

WELL, THAT'S NICE! I KNOW WHAT A LEFT $\mathcal{V}$-ADJOINT IS, I HAVE A GOOD NOTION OF ENRICHED YONEDA EMBEDDING, AND (LACK, TENDAS 2019) ALREADY DEFINED $\mathcal{V}$-LEFT EXACTNESS OF $\mathcal{V}$-FUNCTORS.

(PRESERVE FINITE WEIGHTED $\mathcal{V}$-LIMITS)

THE ENRICHED SETTING

PROVED SO FAR:

THEOREM. IF $\mathcal{E}$ IS A $\mathcal{V}$-GROTHENDIECK TOPOS, THEN

$$\text{よ}_{\mathcal{E}} : \mathcal{E} \longrightarrow [\mathcal{E}^{op}, \mathcal{V}]$$

HAS A $\mathcal{V}$-LEFT EXACT $\mathcal{V}$-LEFT ADJOINT.

" $\mathcal{V}$-TOPOS $\Longrightarrow$ $\mathcal{V}$-LEX TOTAL. "

THE ENRICHED SETTING

IDEA. SHOW THAT THE "STANDARD" YONEDA STRUCTURE ON $\mathcal{V}$-Cat RESTRICTS TO ONE ON $\mathcal{V}$-Lex, THEN APPLY ANOTHER RESULT OF STREET THAT ALLOWS US TO DETECT "TOTAL" 0-CELLS.

THANKS FOR LISTENING!