ENRICHED GROTHENDIECK TOPOLOGIES UNDER CHANGE OF BASE

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Abstract. In the presence of a monoidal adjunction between locally finitely presentable Bénabou cosmoi $\mathcal{U}$ and $\mathcal{V}$, we examine the behavior of enriched coverages on a $\mathcal{V}$-enriched category $\mathcal{U}$, and that of their constituent covering sieves, under the change of enriching category induced by the right adjoint $G : \mathcal{V} \to \mathcal{U}$ of the pair. We exhibit a construction of a $\mathcal{U}$-Grothendieck topology on $\mathcal{C}$ given a $\mathcal{V}$-Grothendieck topology, and prove in particular that when G is faithful and conservative, any upward-directed $\mathcal{V}$-coverage on $\mathcal{C}$ corresponds uniquely to an upward-directed $\mathcal{U}$-coverage on $G_*\mathcal{C}$. We show that when G is fully faithful, base change commutes with enriched sheafification in the sense of Borceux-Quinteiro.

Contents

1. Introduction

As outgrowths of the move to formalize algebraic geometry in terms of abelian categories, Grothendieck topologies and their accompanying categories of sheaves arose in the early 1960s as a framework for defining cohomology theories on schemes. Roughly speaking, a Grothendieck topology on a category $\mathcal{C}$ can be regarded as a way to specify, for all objects U of $\mathcal{C}$, which objects of $\mathcal{C}$ cover U. This is in exactly the same sense as, given a topological space X and an open set $U \subset X$, we might ask when $\bigcup_{i\in I} U_i = U$ for some family $\{U_i : i \in I\}$ of opens of X. Enriched categories, where the hom-sets of ordinary category theory are replaced, more generally, by objects of a closed monoidal category $\mathcal{V}$, were first introduced in the mid-1960s in the work of Maranda [22] and Bénabou [3], among others. Around the same time, Gabriel introduced in [13, V.2, p. 411] the notion of a (right) linear topology (*topologie linéaire à droite*) on a ring, an early example of an enriched Grothendieck topology in the particular case of a category with one object enriched over $\mathcal{V} = \mathsf{Ab}$.

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The definition of a Grothendieck topology admits a number of different formulations, but the definition in terms of sieves on objects $U \in \mathcal{C}$ (that is, subfunctors of $\mathcal{C}(-, U)$) is perhaps the most straightforwardly generalizable to the enriched setting. For a nice enough base category $\mathcal{V}$, enriched Grothendieck topologies on a $\mathcal{V}$-category $\mathcal{C}$ (now taken to be families of subfunctors of enriched hom-functors), their accompanying sheaves, and their correspondence with localizations of and universal closure operations on $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$, were introduced by Borceux and Quinteiro in 1996 with the publication of [5]. Their paper greatly inspires the current work. More recently, details of the theory of enriched sheaves in the case $\mathcal{V} = \mathsf{Ab}$ were established in the 2000s by Lowen in [19] and [20]; and in 2020 by Coulembier [8].

Given a category $\mathcal{C}$ enriched over $(\mathcal{V}, \otimes, I)$ and a lax monoidal functor $G : \mathcal{V} \to \mathcal{U}$, G canonically induces a 2-functor

$$G_* : \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{U}\text{-}\mathsf{Cat}$$

which acts via an operation called 'base change' or 'change of base,' changing $\mathcal{V}$-categories into $\mathcal{U}$-categories, $\mathcal{V}$-functors into $\mathcal{U}$-functors, and $\mathcal{V}$-natural transformations into $\mathcal{U}$-natural transformations. Base change first appeared in the literature around the same time as enriched categories themselves, with Eilenberg and Kelly's publication of [10], and is fundamental to the theory of enriched categories, in part because it allows one to view a $\mathcal{V}$-category $\mathcal{C}$ as an ordinary category by applying the functor

$$\mathrm{Hom}_{\mathcal{V}}(I, -) : \mathcal{V} \to \mathsf{Set}$$

to the hom-objects of $\mathcal{C}$. Many of the technical results in Section 3 of the current work rely heavily on the results and style of argument developed in Cruttwell's 2008 doctoral thesis [9], which, toward understanding normed spaces, addressed in detail the question of how base change interacts with the monoidal structures on $\mathcal{V}$ and $\mathcal{U}$.

A central theme of this work is the following: Changing base via a particular G may result in more or less loss of information about the hom-objects of $\mathcal{C}$. To illustrate, we consider the functors

$$\mathrm{Hom}_{\mathsf{Ab}}(\mathbb{Z}, -) : \mathsf{Ab} \to \mathsf{Set} \quad \text{and} \quad \mathrm{Hom}_{\mathsf{grMod}_k}(k, -) : \mathsf{grMod}_k \to \mathsf{Set},$$

where k is a field, and gradings are taken over $\mathbb{Z}$. Letting $\mathcal{V}$ be either of Ab or grMod_k, we define the hom-objects of the Set-category $G_*\mathcal{C}$ to be

$$G_*\mathcal{C}(x, y) := G(\mathcal{C}(x, y)).$$

In the former case, the hom-sets resulting from base change are (in bijection with) the underlying sets of the original hom-objects, and the $\mathcal{U}$-topology resulting from changing the base of a $\mathcal{V}$-topology is no coarser than the one we started with. In the latter case, however, for a graded k-module $M := \mathcal{C}(x, y)$, we only recover the set

$$\mathrm{Hom}_{\mathcal{V}}(k, M) \cong \mathrm{Hom}_k(k, M_0) \cong M_0$$

of degree-0 elements of M after changing base—in this case, the $\mathcal{U}$-topology resulting from a given $\mathcal{V}$-topology is much coarser. The key difference between these two examples lies in whether or not $\mathrm{Hom}_{\mathcal{V}}(I, -)$ is faithful; or equivalently, whether $\{I\}$ is a separating family for $\mathcal{V}$.

Below, we examine situations where this 'loss' is minimal, as in Theorems 4.16 and 4.6, and situations where changing base results in topologies which are radically coarser than the ones we started with, as in 6.9.

1.1. **Summary.**

- **§3**. In the presence of a monoidal right adjoint $G : \mathcal{V} \to \mathcal{U}$, we define the $\mathcal{U}$-sieve canonically induced by G given a $\mathcal{V}$-sieve (3.1).
- **§4**. We prove that the enriched Grothendieck topologies on a given category form a complete lattice (4.2), and that when G is faithful and conservative, there is an injection from the lattice of $\mathcal{V}$-coverages on $\mathcal{C}$ to the lattice of $\mathcal{U}$-coverages on $G_*\mathcal{C}$ (4.6). When G induces an order-reflecting map on sieves (a property defined in 4.15), there is an injective assignment from $\mathcal{V}$-sieves on $U \in \mathcal{C}$ to $\mathcal{U}$-sieves on $U \in G_*\mathcal{C}$ (4.16) and on upward-directed coverages (4.17).
- **§5**. We show that when G is fully faithful, change of base via G commutes with enriched sheafification in the sense of Borceux-Quinteiro (5.4).
- **§6**. We examine the special case of $\mathcal{V}$-sieves and $\mathcal{V}$-topologies on a monoid object in $\mathcal{V}$. Via an example, we show that when $\mathcal{V} = \mathsf{grMod}_k$, $\mathcal{C}$ is a graded k-algebra, and $G = \mathrm{Hom}_{\mathcal{V}}(k, -)$, the injectivity results of §3 and §4 do not hold (6.9). Generalizing the notions for $\mathcal{V} = \mathsf{Ab}$ and $\mathcal{V} = \mathsf{grMod}_k$, we propose a definition for a $\mathcal{V}$-Gabriel topology (6.3), and prove that $\mathcal{V}$-Gabriel topologies on monoid objects in $\mathcal{V}$ are exactly $\mathcal{V}$-Grothendieck topologies on one-object $\mathcal{V}$-categories (6.4).

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2. Preliminaries

We begin by addressing some questions of size in the categories at hand. By a **small** $\mathcal{V}$-category, we mean one which is equivalent to a $\mathcal{V}$-category with a small set of objects. $\mathcal{C}$ will always denote a small $\mathcal{V}$-category unless otherwise indicated.

For this work, we care only about those enriching categories $\mathcal{V}$ whose objects are built from 'finite' objects—for example, in the same sense that any object of Set is the union of its finite subsets, or that any object of R-Mod, for a commutative ring R, is the colimit of its finitely presented submodules. More precisely, we recall:

Definition 2.1. [1, 1.A, 1.1 and 1.9] An object x of a category $\mathcal{V}$ is called **finitely presentable** if the functor

$$\mathrm{Hom}_{\mathcal{V}}(x, -) : \mathcal{V} \to \mathsf{Set}$$

preserves filtered colimits. The category $\mathcal{V}$ is **locally finitely presentable** if it is cocomplete and has a set of objects

$$\mathcal{V}_{fp} = \{\text{finitely presentable objects of } \mathcal{V}\}$$

such that every object of $\mathcal{V}$ is a directed colimit of objects from $\mathcal{V}_{fp}$.

Borceux-Quinteiro [5] and Kelly [15] require $\mathcal{V}$ to be locally finitely presentable, among other conditions, to ensure that their results are sensibly analogous to more classical results. To ensure continuity of this work with theirs, we make the same assumptions on $\mathcal{V}$:

Hypothesis 2.2. Unless otherwise indicated,

(i) $\mathcal{V}$ is locally finitely presentable;
(ii) $(\mathcal{V}, \otimes, I)$ is closed symmetric monoidal;
(iii) $\mathrm{Hom}_{\mathcal{V}}(I, A)$ is a small set for all objects $A \in \mathcal{V}$ (in other words, the underlying category $\mathcal{V}_0$ of $\mathcal{V}$, defined below in 2.4, is locally small);
(iv) $\mathcal{V}$ admits all small conical limits and colimits, or equivalently, $\mathcal{V}_0$ is bicomplete (hence $\mathcal{V}$ as an enriched category is tensored and cotensored over itself);
(v) a finite tensor product of finitely presentable objects of $\mathcal{V}$ is again finitely presentable;
(vi) $\mathcal{V}$ is regular in the sense of [2].

Examples of categories which satisfy these conditions include

- Set, Ab, Mod_k for k a commutative ring, and the category grMod_k of $\mathbb{Z}$-graded k-modules;
- the category dgMod_k of differential graded k-modules, and by isomorphism, the category $\mathsf{Ch}_\bullet(\mathsf{Mod}_k)$ of chain complexes of k-modules;
- the category sSet of simplicial sets.

For a locally small category $\mathcal{C}$, the collection of set functions $\{\bullet\} \to \mathcal{C}(x, y)$ encodes all available information about the structure of the hom-object $\mathcal{C}(x, y)$ as a set, in the sense that anytime we have $fg = hg$ for all $g : \{\bullet\} \to \mathcal{C}(x, y)$, we know that $f = h$. In a $\mathcal{V}$-category $\mathcal{C}$, it is no longer necessarily true that having $fg = hg$ for all $g : I \to \mathcal{C}(x, y)$ implies $f = h$ (for example, in the case where $\mathcal{V} = \mathsf{grMod}_k$ for k a field), so to capture all the information we want about hom-objects in our categories, we need a more general notion.

Definition 2.3. Let $\mathcal{V}$ be as in 2.2.

(i) By a **separating family** for $\mathcal{V}$, we mean a family $\mathcal{G}$ of objects of $\mathcal{V}$ such that if $fg = hg$ for any g with domain in $\mathcal{G}$, then $f = h$; or equivalently, that the family $\{\mathrm{Hom}_{\mathcal{V}}(G, -) : G \in \mathcal{G}\}$ is jointly faithful.
(ii) We say that $\mathcal{G}$ is an **extremal** separating family if for each object K of $\mathcal{V}$ and each proper subobject L of K there exists a morphism $G \to K$ with $G \in \mathcal{G}$ which does not factor through L.
(iii) We say that $\mathcal{G}$ is a **dense** separating family if every object of $\mathcal{V}$ is a filtered colimit of objects of $\mathcal{G}$.
(iv) If $\mathcal{G}$ is a separating family for $\mathcal{V}$, we denote
$$\mathrm{Hom}_{\mathcal{V}}(\mathcal{G}, X) = \{f \in \mathrm{Mor}(\mathcal{V}) : \mathrm{cod}(f) = X \text{ and } \mathrm{dom}(f) \in \mathcal{G}\}.$$
We will say $f \in_g X$ to mean $(f : g \to X) \in \mathrm{Hom}_{\mathcal{V}}(\mathcal{G}, X)$.

Recall that any dense separating family is an extremal separating family; and that in any locally finitely presentable category, the finitely presentable objects form a dense separating family.

2.1. **Change of base.** A very detailed treatment of this topic can be found in [9, 4], but for convenience, we recount the bare rudiments here. Let

$$(\mathcal{U}, \otimes, \mathbf{1}) \quad \text{and} \quad (\mathcal{V}, \times, *)$$

be closed symmetric monoidal categories, and let $\mathcal{C}$ be a $\mathcal{V}$-category. We denote an identity morphism in an enriched category $\mathcal{X}$ by $\mathrm{id}^{\mathcal{X}}$, and a composition morphism in $\mathcal{X}$ by $\circ^{\mathcal{X}}$. For visual simplicity, we will often omit subscripts which would ordinarily indicate the domain objects of the morphisms id and $\circ$.

We frequently refer to a special case of base change, namely the underlying category construction, in which the lax monoidal functor $\mathrm{Hom}_{\mathcal{V}}(*, -) : \mathcal{V} \to \mathsf{Set}$ is used to turn a $\mathcal{V}$-category into an ordinary one.

Definition 2.4. Given a $\mathcal{V}$-category $\mathcal{C}$, define an ordinary category $\mathcal{C}_0$ by setting $\mathrm{Ob}(\mathcal{C}_0) = \mathrm{Ob}(\mathcal{C})$ and $\mathcal{C}_0(x, y) = \mathrm{Hom}_{\mathcal{V}}(*, \mathcal{C}(x, y))$. Given morphisms $g : x \to y$ and $f : y \to z$ in $\mathcal{C}_0$, we define the composite $f \cdot g$ by

$$* \xrightarrow{\sim} * \times * \xrightarrow{f \times g} \mathcal{C}(y, z) \times \mathcal{C}(x, y) \xrightarrow{\circ^{\mathcal{C}}} \mathcal{C}(x, z) \ .$$

In light of the above, we note that having a morphism $* \to \mathcal{C}(x, y)$ in $\mathcal{V}$ no longer necessarily specifies an element of $\mathcal{C}(x, y)$ in the set-theoretic sense, and so referring to an 'arrow' in $\mathcal{C}$ is mildly nonsensical. Any diagrams in the work below should therefore be interpreted as living in the underlying category of the relevant enriched category.

In general, given a lax monoidal functor $G : \mathcal{V} \to \mathcal{U}$, we can form $\mathcal{U}$-categories, $\mathcal{U}$-functors, and $\mathcal{U}$-natural transformations in a canonical way.

Definition 2.5. Let $G : \mathcal{V} \to \mathcal{U}$ be a lax monoidal functor with coherence morphisms

$$u : \mathbf{1} \to G(*), \quad m_{xy} : G(x) \otimes G(y) \to G(x \times y).$$

(i) Form a $\mathcal{U}$-category $G_*\mathcal{C}$ by setting

$$\begin{aligned} \mathrm{Ob}(G_*\mathcal{C}) &:= \mathrm{Ob}(\mathcal{C}), \\ G_*\mathcal{C}(x, y) &:= G(\mathcal{C}(x, y)), \\ \mathrm{id}^{G_*\mathcal{C}} &:= G(\mathrm{id}^{\mathcal{C}}) \cdot u \\ \circ^{G_*\mathcal{C}} &:= G(\circ^{\mathcal{C}}) \cdot m. \end{aligned}$$

(ii) For a $\mathcal{V}$-functor $A : \mathcal{C} \to \mathcal{D}$, let

$$G_*A : G_*\mathcal{C} \to G_*\mathcal{D}$$

denote the $\mathcal{U}$-functor defined by

$$G_*Ax := Ax \ \text{ and } \ (G_*A)_{xy} := GA_{xy} : G(\mathcal{C}(x, y)) \to G(\mathcal{D}(Ax, Ay)).$$

(iii) For a $\mathcal{V}$-natural transformation

$$\{\alpha_x : * \to \mathcal{D}(Ax, Bx)\},$$

let $G_*\alpha$ denote the $\mathcal{U}$-natural transformation

$$\{G(\alpha_x) \cdot u : \mathbf{1} \to G(\mathcal{D}(Ax, Bx))\}.$$

We will often be concerned with the case where the functor $G : \mathcal{U} \to \mathcal{V}$ is half of a monoidal adjunction, rather than merely lax monoidal. To give a fully rigorous definition of a monoidal adjunction, we require a few elementary notions from the theory of 2-categories, which we recall in abbreviated form below.

Definition 2.6. [26, B.1.1] A (strict) **2-category** is a Cat-category. More explicitly, a 2-category $\mathbb{C}$ consists of

- a class of objects;
- for each pair a, b of objects, a category $\mathbb{C}(a, b)$, whose objects are called **1-cells**;
- for each pair $f, g : a \to b$ of 1-cells, a collection of arrows $f \Rightarrow g$ in $\mathbb{C}(a, b)$, called **2-cells**;

such that

- the objects and 1-cells form a 1-category;
- the objects and 2-cells form a 1-category;
- the composition laws in each of these 1-categories are compatible with one another, and with the category structure on $\mathbb{C}(a, b)$ for each pair of objects a, b.

Important examples include the 2-category MonCat_ℓ of monoidal categories with lax monoidal functors, as well as $\mathcal{V}$-Cat.

We omit the associated notions of 2-functors and 2-natural transformations, as knowledge of the definitions in full detail is not necessary for our discussion—the reader may consult [26, B.2.1, B.2.2], or simply think of them as Cat-enriched functors and natural transformations. The important fact is that, given a monoidal functor G as above, change of base as outlined in 2.5 defines a 2-functor

$$\mathcal{V}\text{-}\mathsf{Cat} \xrightarrow{G_*} \mathcal{U}\text{-}\mathsf{Cat} \ .$$

Moreover, we have an assignment

$$\mathsf{MonCat}_\ell \xrightarrow{(-)_*} 2\text{-}\mathsf{Cat}$$

which takes a monoidal category $\mathcal{V}$ to the 2-category $\mathcal{V}$-Cat, a monoidal functor G to the 2-functor G_*, et cetera. Proof that this assignment defines a 2-functor is [9, 4.3.2].

We note here, if only for the sake of the resulting nice algebraic expression, that given $\mathcal{V}$-categories $\mathcal{X}, \mathcal{Y}$,

$$G_* : [\mathcal{X}, \mathcal{Y}] \to [G_*\mathcal{X}, G_*\mathcal{Y}]$$

itself being a 1-functor means that for composable morphisms α, β in $[\mathcal{X}, \mathcal{Y}]$, we have

$$G_*(\alpha \cdot \beta) = G_*\alpha \cdot G_*\beta,$$

where the components of the natural transformations on both the left-hand and right-hand sides of the equality are simply composites in the underlying category.

Definition 2.7. [26, B.3] An **adjunction** internal to a 2-category $\mathbb{C}$ is

- a pair of objects a, b;
- a pair of 1-cells $u : a \to b$ and $f : b \to a$, called the right and left adjoint, respectively;

- a pair of 2-cells $\eta : 1_b \Rightarrow uf$, $\varepsilon : fu \Rightarrow 1_a$, called the unit and counit of the adjunction, respectively;

satisfying the triangle identities

$$(\varepsilon \cdot f)(f \cdot \eta) = \mathrm{id}_f,\ (u \cdot \varepsilon)(\eta \cdot u) = \mathrm{id}_u$$

in the hom-categories $\mathbb{C}(b, a)$ and $\mathbb{C}(a, b)$, respectively.

With the notions above in hand, we define a **monoidal adjunction** to be an adjunction internal to the 2-category MonCat_ℓ. For the remainder of this section, we suppose given a monoidal adjunction

$$\begin{tikzcd} \mathcal{U} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{V} \arrow[l, bend left, "G"] \end{tikzcd} . \tag{2.8}$$

The last two results we will need regarding monoidal adjunctions are the following:

Theorem 2.9. *Let $F, G, \mathcal{U}$, and $\mathcal{V}$ be as in 2.8.*

(i) [14, 1.4] The left adjoint of a monoidal adjunction is necessarily strong monoidal.

(ii) The monoidal adjunction 2.8 induces an adjunction

$$\begin{tikzcd} \mathcal{U}\text{-}\mathsf{Cat} \arrow[r, bend left, "F_*"] \arrow[r, phantom, "\perp"] & \mathcal{V}\text{-}\mathsf{Cat} \arrow[l, bend left, "G_*"] \end{tikzcd}$$

in 2-Cat *via the 2-functor $(-)_*$ mentioned above.*

Proof. (ii). Any 2-functor preserves adjunctions—this is [26, 2.1.3]. □

2.2. **$\mathcal{V}$-limits.** We will often need to deal with enriched limits. The cases we encounter in this work are as simple as possible, in that they behave for the most part like limits in an ordinary category.

Definition 2.10. Let $* : \mathcal{D} \to \mathcal{V}_0$ be an ordinary functor constant at the monoidal unit $*$ of $\mathcal{V}$, and let $F : \mathcal{D} \to \mathcal{C}$ be a $\mathcal{V}$-functor. The **conical limit** of F, if it exists, is an object $\lim^* F$ of $\mathcal{C}$ defined by the universal property

$$\mathcal{C}(m, \lim^* F) \cong [\mathcal{D}, \mathcal{V}](*, \mathcal{C}(m, F(-))).$$

Though we will not need the definition of the latter object in full detail, we note for the curious that it is in fact a $\mathcal{V}$-enriched end, as defined in [25, 7.3], so the isomorphisms above are truly isomorphisms as objects of $\mathcal{V}$.

In the setting of Remark 2.2, conical limits in $\mathcal{V}$ coincide with ordinary limits in $\mathcal{V}_0$, as observed in [17, p. 50]. We note here that conical limits are a special case of the more general notion of $\mathcal{V}$-limit, defined in [17, 3] and [25, 7.4], and that they do not encompass the full theory of limits in a $\mathcal{V}$-category.

In the presence of a monoidal adjunction 2.8 and a cotensored $\mathcal{V}$-category $\mathcal{C}$, change of base makes $G_*\mathcal{C}$ cotensored over $\mathcal{U}$ as follows:

Definition 2.11. Given a cotensored $\mathcal{V}$-category $\mathcal{C}$, $G_*\mathcal{C}$ is cotensored over $\mathcal{U}$ via

$$\{u, x\} := \{Fu, x\}$$

for $u \in \mathcal{U}$ and $x \in G_*\mathcal{C}$.

That the above object satisfies the appropriate universal property is a consequence of 2.9, (i).

2.3. **Enriched Grothendieck topologies.** We outline [5, 1.2] and a few of the notions surrounding it.

In order to define a sheaf on an ordinary category $\mathcal{X}$, we need to take each object x and specify a family $J(x)$ of 'sieves' on x, which we think of as being admissible coverings for that object. Precisely, in the unenriched case, a sieve is defined as a subobject of a representable presheaf of sets. To define the analogue in the $\mathcal{V}$-enriched case, we first need a good notion of monomorphism in the $\mathcal{V}$-category $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$—for this, we recall the definition used by Kelly in [16, p.7].

Definition 2.12. An arrow $f : x \to y$ in $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]_0$ is a **monomorphism** if

$$\begin{tikzcd} x \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\lrcorner", very near start] & x \arrow[d, "f"] \\ x \arrow[r, "f"'] & y \end{tikzcd}$$

is a pullback in $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$—equivalently, if every component of f is a monomorphism in $\mathcal{V}_0$ in the ordinary sense. As in the unenriched case, we denote a monomorphism by an arrow $f : x \rightarrowtail y$ with a forked tail.

Sieves in the enriched case are then defined as as follows:

Definition 2.13. Let $\mathcal{C}$ be a $\mathcal{V}$-category, and let $U \in \mathcal{C}$ be an object. A **sieve** on $U \in \mathcal{C}$ is a subobject of $\mathcal{C}(-, U) \in [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$.

There are several conditions we might require such an assignment J of admissible coverings to satisfy. Among the most fundamental of these is that each family $J(x)$ be closed under pullbacks, so we turn our attention to defining the pullbacks we will consider in the enriched setting.

The enriched functor category $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ has all small conical limits and colimits if $\mathcal{V}$ does, as explained in [17, 3.3]. Thus $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ is cotensored over $\mathcal{V}$:

Definition 2.14. The **cotensor** $\{v, A\}$ of $A \in [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ by $v \in \mathcal{V}$ is the $\mathcal{V}$-functor whose value at $x \in \mathcal{C}$ is $\{v, Ax\} \in \mathcal{V}$, together with $\mathcal{V}$-natural isomorphisms

$$[\mathcal{C}^{\mathrm{op}}, \mathcal{V}](B, \{v, A\}) \cong \mathcal{V}(v, [\mathcal{C}^{\mathrm{op}}, \mathcal{V}](A, B)).$$

Note that for any $v \in \mathcal{V}$, a monomorphism $R \rightarrowtail \mathcal{C}(-, U)$ of $\mathcal{V}$-functors induces, by naturality of cotensoring, a monomorphism

$$\{v, R\} \rightarrowtail \{v, \mathcal{C}(-, U)\},$$

which we denote by ι. Moreover, the enriched Yoneda lemma [25, 7.3.5] tells us that any $f : v \to \mathcal{C}(V, U)$ induces a map $v \to \mathsf{Nat}_{\mathcal{V}}(\mathcal{C}(-, V), \mathcal{C}(-, U))$, which in turn induces a $\mathcal{V}$-natural transformation $f : \mathcal{C}(-, V) \to \{v, \mathcal{C}(-, U)\}$.

The morphisms f and ι above, along with the fact that $\mathcal{V}$ is complete, allow us to define the pullback f^*R of a sieve R as follows:

Definition 2.15. The limit f^*R of the diagram

$$\mathcal{C}(-, V) \xrightarrow{\quad f \quad} \{v, \mathcal{C}(-, U)\} \xleftarrow{\quad \iota \quad} \{v, R\}$$

in $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]_0$ is defined pointwise as the functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{V}$ whose value f^*Rx at $x \in \mathcal{C}$ is the pullback of the diagram

$$\mathcal{C}(x, V) \xrightarrow{\quad f_x \quad} \{v, \mathcal{C}(x, U)\} \xleftarrow{\quad \iota_x \quad} \{v, R\}$$

in $\mathcal{V}_0$.

With 2.15 in hand, we list some conditions we might expect J to satisfy in the enriched case.

Hypothesis 2.16. Given a small $\mathcal{V}$-category $\mathcal{C}$, let $U \mapsto J(U)$ be an assignment to each object U in $\mathcal{C}$ of a family $J(U)$ of sieves on U, in the sense of 2.13.

(T1) $\mathcal{C}(-, U) \in J(U)$ for each object U;

(T2) For an arbitrary object V, for any $R \in J(U)$ and $f \in \mathrm{Hom}_{\mathcal{V}}(\mathcal{V}_{fp}, \mathcal{C}(V, U))$, we have $f^*R \in J(V)$, where f^*R is as in 2.15.

(T3) If $R \rightarrowtail \mathcal{C}(-, U)$ is an arbitrary sieve for which there exists $S \in J(U)$ such that for all objects V of $\mathcal{C}$,

$$f^*R \in J(V) \text{ for any } f \in \mathrm{Hom}_{\mathcal{V}}(\mathcal{V}_{fp}, S(V)),$$

then we have $R \in J(U)$.

We now recall Borceux's and Quinteiro's definition of a $\mathcal{V}$-Grothendieck topology on $\mathcal{C}$.

Definition 2.17. Given a small $\mathcal{V}$-category $\mathcal{C}$, let $U \mapsto J(U)$ be an assignment to each object U in $\mathcal{C}$ of a family $J(U)$ of sieves on U.

(i) We say J is a **$\mathcal{V}$-coverage** if it satisfies (T1) and (T2) of 2.16.

(ii) [5, 1.2] We say J is a **$\mathcal{V}$-Grothendieck topology**, or simply **$\mathcal{V}$-topology**, if it satisfies (T1)-(T3) of 2.16.

A very simple example of 2.17 occurs in the case where $\mathcal{V}$ is the monoidal preorder $([0, \infty], \geq, +, 0)$.

Example 2.18. Denote the monoidal preorder $([0, \infty], \geq, +, 0)$ by Cost. As described in [12, 2.51], we can view the real numbers $\mathbb{R}$ as a Cost-category whose hom-objects are defined by

$$\mathbb{R}(x, y) := |x - y|.$$

Cost-functors are exactly (1-)Lipschitz functions, and there is a unique Cost-sieve on each $U \in \mathbb{R}$, namely the maximal sieve $\mathbb{R}(-, U)$, which sends

$$x \longmapsto |x - U|.$$

There is thus a unique Cost-Grothendieck topology on $\mathbb{R}$, namely that with

$$J(U) = \{\mathbb{R}(-, U)\}$$

for each $U \in \mathbb{R}$. (In this case, since there is a unique subobject of $\mathbb{R}(-, U)$, the 'discrete' and 'indiscrete' topologies on $\mathbb{R}$, which we describe in more detail in §4.1, coincide.)

In the case where $\mathcal{C}$ is a poset—that is, a category enriched over the monoidal preorder $\mathcal{V} = \{0, 1\}$—a sieve on $p \in \mathcal{C}$ is exactly a downward-closed subset of $\downarrow p$, and the pullback of a sieve S on p along a morphism $q \leq p$ is exactly $S \cap \downarrow q$ (note that this set is again a sieve on q). We obtain the following example:

Example 2.19. [27, 247] A $\{0,1\}$-Grothendieck topology J on $\mathcal{C}$ is, to each $p \in \mathcal{C}$, a collection $J(p)$ of sieves satisfying

(P1) the maximal sieve $\downarrow p$ is in $J(p)$;
(P2) if $S \in J(p)$ and $q \leq p$, then $S \cap \downarrow q \in J(q)$;
(P3) if $S \in J(p)$ and R is a sieve on p such that $R \cap \downarrow q \in J(q)$ for all $q \in S$, then $R \in J(p)$.

Toward an algebraic example of 2.17, take an associative, unital, not-necessarily-commutative ring A, and think of it as a one-object Ab-category.

Example 2.20. Let A be a ring and let $\mathfrak{R}$ be a non-empty set of right ideals of A. $\mathfrak{R}$ is a **(right) Gabriel topology** on A if

(R1) $I \in \mathfrak{R}$ and $I \subset J$ implies $J \in \mathfrak{R}$;
(R2) if $I \in \mathfrak{R}$ and $x \in A$, then
$$(I : x) := \{r \in A : xr \in I\} \in \mathfrak{R};$$
(R3) if I is a right ideal and there exists $J \in \mathfrak{R}$ such that $(I : x) \in \mathfrak{R}$ for every $x \in J$, then $I \in \mathfrak{R}$.

Denoting the lone object of A by $\bullet$, an Ab-sieve on $\bullet$ is a right A-submodule of A, or in other words, a right ideal of A. The pullback f^*I of 2.17, (T2) is the right ideal $(I : f)$, where the group homomorphism $f : \mathbb{Z} \to A$ is identified with the element $f(1) \in A$, so (R2) is equivalent to (T2). Moreover (R1) and (R3) are respectively equivalent to (T1) and (T3). As remarked by Lowen in [19, 2.4], we see that a Gabriel topology on A is the same thing as an Ab-Grothendieck topology on A.

In light of 2.20, we can view Definition 2.17.ii as a generalization of what is alternately called a Gabriel topology [28, VI.5] or topologizing filter [11, p. 520] on A, to a setting where the category A might have many objects and be enriched over some general $\mathcal{V}$. In this light, we might also imagine (T2) of 2.16 as saying that a given family of ideals is 'closed under intersection,' and (T3) as saying (roughly) that a given family is 'upward-closed.'

In §5 we take this perspective, and address $\mathcal{V}$-Grothendieck topologies on one-object $\mathcal{V}$-categories in greater detail.

3. Sieves under change of base

Below, we consider categories $\mathcal{U}$ and $\mathcal{V}$ satisfying the hypotheses in 2.2. We denote the unit objects in $\mathcal{U}, \mathcal{V}$ by $*_\mathcal{U}, *_\mathcal{V}$, the monoidal operation on both categories by $\otimes$, and refer to a fixed lax monoidal functor $G : \mathcal{V} \to \mathcal{U}$, whose coherence morphisms we denote by
$$u : *_\mathcal{U} \to G(*_\mathcal{V}), \quad m_{ab} : G(a) \otimes G(b) \to G(a \otimes b).$$
For an enriched category $\mathcal{X}$ (over either $\mathcal{U}$ or $\mathcal{V}$), we will continue to denote composition in $\mathcal{X}_0$, as defined in 2.4, by $\cdot$. Toward answering the question of how base change affects $\mathcal{V}$-Grothendieck topologies, we first address the behavior of sieves, as defined in 2.13, under the change of base induced by G.

Our main examples of interest occur when G is half of a monoidal adjunction 2.8, whose unit and counit we denote respectively by $\eta : \mathbb{1}_\mathcal{U} \to GF$ and $\varepsilon : FG \to \mathbb{1}_\mathcal{V}$, since we will require the existence of the counit ε to make sense of a '$\mathcal{U}$-sieve induced by a $\mathcal{V}$-sieve.' In this setting, we denote the induced 2-adjunction by

$$\begin{tikzcd} \mathcal{U}\text{-}\mathsf{Cat} \arrow[r, bend left, "F_*"] \arrow[r, phantom, "\perp"] & \mathcal{V}\text{-}\mathsf{Cat} \arrow[l, bend left, "G_*"] \end{tikzcd} .$$

The unenriched adjunction 2.8 induces a $\mathcal{U}$-adjunction: For $x, y \in \mathcal{U}$, we have (unenriched) natural isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathcal{U}}(-, \mathcal{U}(x, Gy)) &\cong \mathrm{Hom}_{\mathcal{U}}(- \otimes x, Gy) \\ &\cong \mathrm{Hom}_{\mathcal{V}}(F(- \otimes x), y) \\ &\cong \mathrm{Hom}_{\mathcal{V}}(F(-) \otimes Fx, y) \\ &\cong \mathrm{Hom}_{\mathcal{V}}(F(-), \mathcal{V}(Fx, y)) \\ &\cong \mathrm{Hom}_{\mathcal{U}}(-, G(\mathcal{V}(Fx, y))), \end{aligned}$$

whence $\mathcal{U}(x, Gy) \cong G(\mathcal{V}(Fx, y))$ as objects of $\mathcal{U}$ by Yoneda's lemma, and naturally in x and y. Denote the components (in $\mathcal{U}_0$) of this natural isomorphism by

$$\Phi_{xy} : G(\mathcal{V}(Fx, y)) \to \mathcal{U}(x, Gy).$$

Following the discussion in [17, 1.11], the family Φ corresponds uniquely to an adjunction in $\mathcal{U}$-Cat in the sense of 2.7.

Definition 3.1. The right adjoint of the adjunction above necessarily has the following form:

(i) The right adjoint of the $\mathcal{U}$-adjunction
$$\mathcal{U} \leftrightarrows G_*\mathcal{V}$$
induced by $(F \dashv G, \varepsilon, \eta)$ is the $\mathcal{U}$-functor
$$G^{\mathcal{U}} : G_*\mathcal{V} \to \mathcal{U}$$
defined on objects by $G^{\mathcal{U}}x = Gx$ and with hom-components
$$G^{\mathcal{U}}_{xy} : G(\mathcal{V}(x, y)) \to \mathcal{U}(Gx, Gy) := \Phi_{(Gx)y} \cdot G(\varepsilon^*_x).$$

(ii) Let $\widetilde{G}$ be the functor defined as the composite
$$[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]_0 \xrightarrow{G_*} [G_*\mathcal{C}^{\mathrm{op}}, G_*\mathcal{V}]_0 \xrightarrow{G^{\mathcal{U}} \circ -} [G_*\mathcal{C}^{\mathrm{op}}, \mathcal{U}]_0 ,$$
whose effect on a $\mathcal{V}$-subfunctor $R \rightarrowtail \mathcal{C}(-, U)$ is
$$\widetilde{G}R := G^{\mathcal{U}} \circ (G_*R) : G_*\mathcal{C}^{\mathrm{op}} \to \mathcal{U}.$$

The following lemma will allow us to pass between conical $\mathcal{U}$-limits and conical $\mathcal{V}$-limits.

Lemma 3.2. *Let $\mathcal{C}$ be a $\mathcal{V}$-category, and suppose G is faithful and conservative.*

(i) G_* preserves pointwise limits in $[\mathcal{C}^{op}, \mathcal{V}]_0$
(ii) G_* reflects pointwise limits in the category $[\mathcal{C}^{op}, \mathcal{V}]_0$.

Thus $\widetilde{G}(-) = G^{\mathcal{U}} \circ G_*(-)$ preserves and reflects conical limits in $[\mathcal{C}^{op}, \mathcal{V}]$.

Proof. (i). Let $\mathcal{L}$ be a locally small category and $T : \mathcal{L} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]_0$ be an ordinary functor such that

$$\lim T(-)(x) := \lim_{\ell \in \mathcal{L}} (T(\ell)(x))$$

exists in $\mathcal{V}$ for every $x \in \mathcal{C}$. We have $\mathcal{V}$-natural isomorphisms

$$\mathcal{V}(v, \lim(T(-)(x))) \cong \lim \mathcal{V}(v, T(-)(x))$$

in $\mathcal{V}_0$. Since G is a right adjoint, we then have $\mathcal{U}$-natural isomorphisms

$$G\left(\mathcal{V}(v, \lim(T(-)(x)))\right) \cong G\left(\lim \mathcal{V}(v, T(-)(x))\right) \tag{3.3}$$

$$\cong \lim G\left(\mathcal{V}(v, T(-)(x))\right) \tag{3.4}$$

in $\mathcal{U}_0$. Thus $\lim T$ exists pointwise in $[G_*\mathcal{C}, G_*\mathcal{V}]_0$.

(ii). With $\mathcal{L}$ and T as above, suppose that 3.4 holds for each $x \in \mathcal{C}$, so that 3.3 holds. Since G is conservative, we have $\mathcal{V}$-natural isomorphisms

$$\mathcal{V}(v, \lim(T(-)(x))) \cong \lim \mathcal{V}(v, T(-)(x)),$$

whence $\lim T$ exists pointwise in $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]_0$.

Since conical limits in the $\mathcal{V}$-category $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ coincide with ordinary limits in the category $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]_0$ as long as $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ is tensored over $\mathcal{V}$ (as noted in [17, §3.8]), (i) implies that G_* preserves pointwise conical limits in $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$. Since $G^{\mathcal{U}}$ is a right $\mathcal{U}$-adjoint, $G^{\mathcal{U}} \circ (-)$ preserves $\mathcal{U}$-limits, and thus the composite $\widetilde{G}(-) = G^{\mathcal{U}} \circ G_*(-)$ preserves conical limits.

To see that

$$G^{\mathcal{U}} \circ - : [G_*\mathcal{C}^{\mathrm{op}}, G_*\mathcal{V}] \to [G_*\mathcal{C}^{\mathrm{op}}, \mathcal{U}]$$

reflects pointwise conical limits, observe that if $G^{\mathcal{U}} \circ k$ is the limit of

$$G^{\mathcal{U}} \circ T : \mathcal{L} \to [G_*\mathcal{C}^{\mathrm{op}}, G_*\mathcal{V}]_0 \to [G_*\mathcal{C}^{\mathrm{op}}, \mathcal{U}]_0,$$

so that

$$G^{\mathcal{U}} \circ k(x) \cong \lim_{\ell \in \mathcal{L}} G^{\mathcal{U}} \circ T(\ell)(x),$$

then we have

$$G(k(x)) \cong \lim_{\ell \in \mathcal{L}} G(T(\ell)(x))$$

by definition of $G^{\mathcal{U}}$. Since G preserves limits, we have

$$\lim_{\ell \in \mathcal{L}} G(T(\ell)(x)) \cong G\left(\lim_{\ell \in \mathcal{L}} T(\ell)(x)\right).$$

Since G is conservative, we then have $k(x) \cong \lim_{\ell \in \mathcal{L}} T(\ell)(x)$. $\square$

Corollary 3.5. *Suppose G is the right adjoint of the pair 2.8. For $y \in \mathcal{U}_{fp}$ and $R \in [\mathcal{C}^{op}, \mathcal{V}]$,*

$$\widetilde{G}\{Fy, R\} := G^{\mathcal{U}} \circ G_*\{Fy, R\} \cong \{y, \widetilde{G}R\}.$$

Proof. Cotensors in enriched functor categories can be realized as pointwise conical limits - see [25, 7.4.3]. $\square$

To shorten the statements of the results below, we collect all of the conditions we might require G to satisfy.

Hypothesis 3.6. With $G : \mathcal{V} \to \mathcal{U}$ as in the beginning of the section,

(i) G is faithful;

(ii) G is conservative;
(iii) G is the right adjoint of the pair 2.8.

The prototypical G to keep in mind is the forgetful functor

$$\mathrm{Hom}_{\mathsf{Ab}}(\mathbb{Z}, -) : \mathsf{Ab} \longrightarrow \mathsf{Set}.$$

Our goal for this section is to prove that the assignment $R \mapsto \widetilde{G}R$ is injective on subfunctors of representable presheaves (Theorem 3.9), for which we need a handful of technical results. The following proposition, which will ensure that we can sensibly pass between $\mathcal{U}$-sieves and $\mathcal{V}$-sieves, is a generalization of the observation made in [17, 1.3] that if $V = \mathrm{Hom}_{\mathcal{V}}(I, -)$ is faithful, $\mathcal{V}$-naturality of a family

$$\{\alpha_x : *_{\mathcal{V}} \to \mathcal{D}(Ax, Bx)\}$$

is equivalent to ordinary naturality of

$$\{(\alpha_x)_0 : A_0x \to B_0x\}.$$

Proposition 3.7. *Let $\mathcal{U}$ and $\mathcal{V}$ be as above, and suppose G is faithful.*

(i) For $\mathcal{V}$-functors $A, B : \mathcal{C} \to \mathcal{D}$, the family

$$\{\alpha_x : *_{\mathcal{V}} \to \mathcal{D}(Ax, Bx)\}$$

is $\mathcal{V}$-natural if and only if the family

$$\{(G_*\alpha)_x : *_{\mathcal{U}} \to G_*\mathcal{D}(Ax, Bx)\}$$

is $\mathcal{U}$-natural.

(ii) Suppose G is the right adjoint of the pair 2.8. For $\mathcal{V}$-presheaves $A, B : \mathcal{C}^{op} \to \mathcal{V}$, the family

$$\{\iota_x : *_{\mathcal{U}} \to G(\mathcal{V}(Ax, Bx))\}$$

is $\mathcal{U}$-natural if and only if the family

$$\{\Phi_{(GAx)(Bx)} \cdot G_*(\varepsilon^*)_{Ax} \cdot \iota_x : *_{\mathcal{U}} \to \mathcal{U}(GAx, GBx)\}$$

is $\mathcal{U}$-natural.

Proof. (i) Denote the left and right unitors in a monoidal category $\mathcal{X}$ by $\lambda_{\mathcal{X}}, \rho_{\mathcal{X}}$. If $\{\alpha_x\}$ is $\mathcal{V}$-natural, $\mathcal{U}$-naturality of $\{(G_*\alpha)_x\}$ follows from [9, 4.1.1]. Conversely, suppose $\{(G_*\alpha)_x\}$ is $\mathcal{U}$-natural, so that

$$\begin{tikzcd}
 & G\mathcal{C}(x,y) \arrow[dl, "GB_{xy}"'] \arrow[dr, "GA_{xy}"] & \\
G\mathcal{D}(Bx,By) \arrow[d, "(\mathrm{id}\otimes u)\cdot\rho_{\mathcal{U}}^{-1}"'] \arrow[dddr, bend left, "(G_*\alpha)_x^*"] & & G\mathcal{D}(Ax,Ay) \arrow[d, "(u\otimes\mathrm{id})\cdot\lambda_{\mathcal{U}}^{-1}"] \arrow[dddl, bend right, "{[(G_*\alpha)_y]_*}"'] \\
G\mathcal{D}(Bx,By)\otimes G(*_{\mathcal{V}}) \arrow[d, "m\cdot(\mathrm{id}\otimes G\alpha_x)"'] \arrow[r, phantom, "="] & & G(*_{\mathcal{V}})\otimes G\mathcal{D}(Ax,Ay) \arrow[d, "m\cdot(G\alpha_y\otimes\mathrm{id})"] \arrow[l, phantom, "="] \\
G(\mathcal{D}(Bx,By)\otimes\mathcal{D}(Ax,Bx)) \arrow[dr, "G\circ"'] & & G(\mathcal{D}(Ay,By)\otimes\mathcal{D}(Ax,Ay)) \arrow[dl, "G\circ"] \\
 & G\mathcal{D}(Ax,By) &
\end{tikzcd}$$

commutes. Suppressing subscripts, naturality of m implies that

$$m \cdot (G\alpha \otimes \mathrm{id}) = G(\alpha \otimes \mathrm{id}) \cdot m,$$

so the above diagram becomes

$$\begin{tikzcd}
 & & G\mathcal{C}(x,y) \arrow[dll, "GB_{xy}"'] \arrow[drr, "GA_{xy}"] & & \\
G\mathcal{D}(Bx,By) \arrow[d, "m\cdot(\mathrm{id}\otimes u)\cdot\rho_{\mathcal{U}}^{-1}"'] \arrow[ddrr, bend left, "(G_*\alpha)^*_x"] \arrow[dr, phantom, "="] & & & & G\mathcal{D}(Ax,Ay) \arrow[d, "m\cdot(u\otimes\mathrm{id})\cdot\lambda_{\mathcal{U}}^{-1}"] \arrow[ddll, bend right, "{[(G_*\alpha)_y]_*}"'] \arrow[dl, phantom, "="] \\
G(\mathcal{D}(Bx,By) \otimes *_{\mathcal{V}}) \arrow[drr, "G(\circ\cdot(\mathrm{id}\otimes\alpha_x))"'] & & & & G(*_{\mathcal{V}} \otimes \mathcal{D}(Ax,Ay)) \arrow[dll, "G(\circ\cdot(\alpha_y\otimes\mathrm{id}))"] \\
 & & G\mathcal{D}(Ax,By) & &
\end{tikzcd}$$

Finally, coherence of the monoidal functor G means that we have

$$m \cdot (u \otimes \mathrm{id}) \cdot \lambda_{\mathcal{U}}^{-1} = G\lambda_{\mathcal{V}}^{-1} \quad \text{and} \quad m \cdot (\mathrm{id} \otimes u) \cdot \rho_{\mathcal{U}}^{-1} = G\rho_{\mathcal{V}}^{-1},$$

so in fact both of the composites

$$[(G_*\alpha)_x]_* \cdot GA_{xy} \qquad \text{and} \qquad (G_*\alpha)^*_y \cdot GB_{xy}$$

are of the form $G(f)$ for some morphism f. We can therefore apply faithfulness of G, obtaining a commuting diagram

$$\begin{tikzcd}
\mathcal{C}(x,y) \arrow[r, "A_{xy}"] \arrow[d, "B_{xy}"'] & \mathcal{D}(Ax,Ay) \arrow[d, "\lambda_{\mathcal{V}}^{-1}"] \\
\mathcal{D}(Bx,By) \arrow[d, "\rho_{\mathcal{V}}^{-1}"'] & *_{\mathcal{V}} \otimes \mathcal{D}(Ax,Ay) \arrow[d, "\circ\cdot(\alpha_y\otimes\mathrm{id})"] \\
\mathcal{D}(Bx,By) \otimes *_{\mathcal{V}} \arrow[r, "\circ\cdot(\mathrm{id}\otimes\alpha_x)"'] & \mathcal{D}(Ax,By)
\end{tikzcd},$$

which says exactly that $\{\alpha_x\}$ is $\mathcal{V}$-natural.

(ii) For visual simplicity, we omit alphanumeric subscripts. Naturality of the counit ε for $F \dashv G$ implies that the top-right square in the diagram

$$\begin{tikzcd}
G(\mathcal{C}(x,y)) \arrow[r, "G_*B"] \arrow[d, "G_*A"'] & G(\mathcal{V}(Bx,By)) \arrow[r, "G_*(\varepsilon^*)"] \arrow[d, "\iota"'] & G(\mathcal{V}(FGBx,By)) \arrow[d, "\iota"'] \\
G(\mathcal{V}(Ax,Ay)) \arrow[r, "\iota"] \arrow[d, "G_*(\varepsilon^*)"'] & G(\mathcal{V}(Ax,By)) \arrow[r, "G_*(\varepsilon^*)"] \arrow[d, "G_*(\varepsilon^*)"'] & G(\mathcal{V}(FGAx,By)) \arrow[d, equal] \\
G(\mathcal{V}(FGAx,Ay)) \arrow[r, "\iota"] & G(\mathcal{V}(FGAx,By)) \arrow[r, equal] & G(\mathcal{V}(FGAx,By))
\end{tikzcd}$$

commutes for any x, y, while commutativity of the bottom-left square follows from associativity of composition in $\mathcal{V}$. Thus commutativity of the outer square, expressing $\mathcal{U}$-naturality of $G_*(\varepsilon^*)\cdot\iota$, is equivalent to commutativity of the upper-left square, expressing $\mathcal{U}$-naturality of ι. Postcomposing each instance of $G_*(\varepsilon^*)$ above with the appropriate component of Φ yields squares which trivially commute (they are of the form $(\Phi\cdot\iota\cdot\Phi^{-1})\cdot\Phi = \Phi\cdot\iota$), so commutativity of the diagram above is sufficient.

$\square$

Proposition 3.7 shows that $\mathcal{V}$-naturality of $\alpha : A \to B$ is equivalent to $\mathcal{U}$-naturality of

$$\Phi \cdot (G_*\varepsilon^*) \cdot (G_*\alpha)$$

as long as G is faithful and a right adjoint, so we define the following:

Definition 3.8. Suppose G is faithful and satisfies 2.8. If $\alpha : A \to B$ is a $\mathcal{V}$-natural transformation between sieves $A, B \rightarrowtail \mathcal{C}(-,U)$, denote the induced $\mathcal{U}$-natural transformation $\widetilde{G}A \to \widetilde{G}B$, as in 3.7, by $\widetilde{G}\alpha$, with components

$$(\widetilde{G}\alpha)_x := \Phi_{(GAx)(Bx)} \cdot (G_*\varepsilon^*)_{Ax} \cdot (G_*\alpha)_x : *_{\mathcal{U}} \to \mathcal{U}(GAx, GBx).$$

Denote

$$\mathcal{M}_{\mathcal{V}}(\mathcal{C}(-,x)) := \{\alpha \in \mathrm{Mor}([\mathcal{C}^{\mathrm{op}}, \mathcal{V}]_0) : \alpha \text{ monic and } \mathrm{cod}(\alpha) = \mathcal{C}(-,x)\}$$

and

$$\mathcal{M}_{\mathcal{U}}(\widetilde{G}\mathcal{C}(-,x)) := \{\beta \in \mathrm{Mor}([G_*\mathcal{C}^{\mathrm{op}}, \mathcal{U}]_0) : \beta \text{ monic and } \mathrm{cod}(\beta) = \widetilde{G}\mathcal{C}(-,x)\}.$$

Note that these sets may in general be large, and that each is a preorder under the relation

$$(\alpha : A \to \mathcal{C}(-,x)) \le (\beta : B \to \mathcal{C}(-,x)) \iff \exists(\sigma : B \to A) \text{ such that } \beta = \alpha\sigma.$$

We note that $\mathcal{V}$-sieves on x and $\mathcal{U}$-sieves on x stand in the following relationship:

Theorem 3.9. *If G satisfies (i)-(iii) in 3.6, then the assignment*

$$\mathcal{M}_{\mathcal{V}}(\mathcal{C}(-,x)) \xrightarrow{\widetilde{G}(-)} \mathcal{M}_{\mathcal{U}}(\widetilde{G}\mathcal{C}(-,x))$$

is an injective morphism of preorders.

Proof. To see that the assignment is well-defined, first note that by 3.7, its outputs are indeed $\mathcal{U}$-natural transformations. To see that they are monic in the sense of 2.12, suppose that $\alpha \in [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]_0(A,B)$ is monic, so that

$$\begin{tikzcd} A \arrow[r, Rightarrow, "\mathrm{id}"] \arrow[d, Rightarrow, "\mathrm{id}"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, Rightarrow, "\alpha"] \\ A \arrow[r, Rightarrow, "\alpha"'] & B \end{tikzcd} \tag{3.10}$$

is a pullback in $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$. By 3.2, we have a pullback square

$$\begin{tikzcd} \widetilde{G}A \arrow[r, Rightarrow, "\mathrm{id}"] \arrow[d, Rightarrow, "\mathrm{id}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{G}A \arrow[d, Rightarrow, "\widetilde{G}\alpha"] \\ \widetilde{G}A \arrow[r, Rightarrow, "\widetilde{G}\alpha"'] & \widetilde{G}B \end{tikzcd} \tag{3.11}$$

in $[G_*\mathcal{C}^{\mathrm{op}}, \mathcal{U}]$, so $\widetilde{G}\alpha \in [G_*\mathcal{C}^{\mathrm{op}}, \mathcal{U}]_0(\widetilde{G}A, \widetilde{G}B)$ is monic. Functoriality of $\widetilde{G}$ ensures that the assignment is monotone.

To see that the assignment is bijective onto its image (and hence injective), note that by 3.2, if we are given $\widetilde{G}\alpha \in \mathcal{M}_{\mathcal{U}}(\widetilde{G}\mathcal{C}(-,x))$ the square 3.11 is a pullback only if 3.10 is a pullback. Thus the corestriction of $\widetilde{G}$ to its image is invertible. $\square$

4. Enriched Grothendieck topologies under change of base

Here we prove the main theorem of this work: A result analogous to Theorem 4.16, namely Theorem 4.6, holds for $\mathcal{V}$-coverages. Below, we refer to the monoidal adjunction 2.8 of the previous sections.

4.1. **Lattices of enriched coverages.** Given a category $\mathcal{W}$ satisfying Hypothesis 2.2 and a $\mathcal{W}$-category $\mathcal{X}$, we establish some properties of the collection of $\mathcal{W}$-coverages on $\mathcal{X}$. If $\mathcal{X}$ is small and $\mathcal{W}$ is both complete and well-powered, as is true in the case where $\mathcal{W}$ satisfies 2.2, then $[\mathcal{X}^{\mathrm{op}}, \mathcal{W}]$ is well-powered, as proven in [7, 4.15]. It follows that the collection of $\mathcal{W}$-coverages on $\mathcal{X}$, which we will denote by $\Sigma(\mathcal{X}, \mathcal{W})$, is a small set.

Exactly as for ordinary topologies on a set of points, as in [18], and Grothendieck topologies on an ordinary category, as in [4, V3, 3.2.13], $\mathcal{W}$-coverages form a complete lattice:

Definition 4.1. Let J, K be two $\mathcal{W}$-coverages on $\mathcal{X}$. K is a **refinement** of J (and J is **coarser** than K) if

$$J(U) \subseteq K(U)$$

for all objects U. Say $J = K$ if $J(U) = K(U)$ for all U.

It is routine to check that $\Sigma(\mathcal{X}, \mathcal{W})$ is partially ordered under refinement, with top element the discrete topology

$$D(U) := \mathrm{Sub}(\mathcal{X}(-,U))$$

and bottom element the indiscrete topology

$$I(U) := \{\mathcal{X}(-,U)\}.$$

Moreover, given a family $\{J_\alpha\}_{\alpha \in A} \subset \Sigma(\mathcal{X}, \mathcal{W})$, the assignment

$$S(U) := \bigcap_{\alpha} J_\alpha(U)$$

defines a $\mathcal{W}$-coverage, which is easily seen to be the finest one which is coarser than any of the J_α. Using the fact that the greatest lower bound property implies the least upper bound property on a small set proves the following:

Proposition 4.2. *For $\mathcal{X}$ small and $\mathcal{W}$ satisfying 2.2, the set $\Sigma(\mathcal{X}, \mathcal{W})$ of $\mathcal{W}$-coverages is a complete lattice.*

Note that the subset

$$\mathcal{T}(\mathcal{X}, \mathcal{W}) \subset \Sigma(\mathcal{X}, \mathcal{W})$$

of $\mathcal{W}$-topologies on $\mathcal{X}$ is a complete sublattice, since an arbitrary intersection of families satisfying (T1)-(T3) of 2.16 is easily seen to satisfy (T1)-(T3). With this, we recall the following lemma, stated for the unenriched case as [4, V3, 3.2.6], and whose proof in the enriched case is exactly the same.

Lemma 4.3. *Let $\mathcal{W}$ be a category satisfying 2.2, and let $\mathcal{X}$ be a $\mathcal{W}$-category. For any $\mathcal{W}$-coverage $J \in \Sigma(\mathcal{X}, \mathcal{W})$, there exists a smallest $\mathcal{W}$-topology $\overline{J}$ containing J.*

4.2. **Change of base for $\mathcal{V}$-coverages.** Since $\mathcal{U}$ and $\mathcal{V}$ are locally finitely presentable, the collections of finitely presentable objects in each category, denoted respectively by $\mathcal{U}_{fp}$ and $\mathcal{V}_{fp}$, are extremally separating. In this situation, we want to be able to say that the left adjoint F preserves extremally separating families, a property already ensured by the requirements 3.6 imposed on G.

Lemma 4.4. *[6, 2.2.1] The following are equivalent:*

(a) G is faithful and conservative;

(b) the family

$$\{Fx : x \in \mathcal{H}\}$$

is (extremally) separating in $\mathcal{V}$ whenever $\mathcal{H}$ is (extremally) separating in $\mathcal{U}$.

Since F is a left adjoint functor between locally finitely presentable (hence $\aleph_0$-accessible) categories, F is $\aleph_0$-accessible. By [1, 2.19], F preserves finitely presentable objects, and thus

$$F\mathcal{U}_{fp} := \{Fx : x \in \mathcal{U}_{fp}\}$$

is an extremally separating family of finitely presentable objects in $\mathcal{V}$.

Since $\mathcal{V}$ is finitely complete, the notions of extremally and strongly separating families coincide as a consequence of [4, V1, 4.3.7]. Thus, by [5, 1.6], in proving the results of this section, when considering generalized elements of the hom-objects of $\mathcal{C}$, it suffices to restrict our attention to those of shape g, for some $g \in F\mathcal{U}_{fp}$.

Proposition 4.5. *Suppose G is as in* 3.6*. For a $\mathcal{V}$-coverage J on $\mathcal{C}$, the assignment to each object $U \in \mathcal{C}$ of the family*

$$\widetilde{G}J(U) = \{\widetilde{G}r \mid (r : R \rightarrowtail \mathcal{C}(-,U)) \in J(U)\}$$

defines a $\mathcal{U}$-coverage on $G_*\mathcal{C}$.

Proof. Recalling Definition 2.15, we show that $\widetilde{G}J$ satisfies (T1) and (T2) of 2.16.

(T1) Immediate from the definition of $\widetilde{G}J$.

(T2) Take any $\mathcal{V}$-sieve $r : R \rightarrowtail \mathcal{C}(-,U)$ in $J(U)$ (so that $\widetilde{G}R$ is an arbitrary element of $\widetilde{G}J(U)$), any $y \in \mathcal{U}_{fp}$, and any $a : y \to G(\mathcal{C}(V,U))$. We first show that the pullback $a^*(\widetilde{G}R)$ defined by

$$\begin{tikzcd} a^*(\widetilde{G}R) \arrow[r] \arrow[d] & \{y, \widetilde{G}R\} \arrow[d, "r"] \\ \widetilde{G}\mathcal{C}(-,V) \arrow[r, "a"] & \{y, \widetilde{G}\mathcal{C}(-,U)\} \end{tikzcd}$$

is in $\widetilde{G}J(U)$. Take the transpose $a^\flat : Fy \to \mathcal{C}(V,U)$ of a (recalling that we have $Fy \in \mathcal{V}_{fp}$, as discussed above). Forming the pullback

$$\begin{tikzcd} (a^\flat)^*R \arrow[r] \arrow[d] & \{Fy, R\} \arrow[d, "r"] \\ \mathcal{C}(-,V) \arrow[r, "b"] & \{Fy, \mathcal{C}(-,U)\} \end{tikzcd}$$

in $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$, we have $(a^\flat)^*R \in J(V)$, since J is a $\mathcal{V}$-topology. Applying $\widetilde{G}$ to the diagram above, Prop. 3.2 and Corollary 3.5 imply that the resulting square

$$\begin{array}{ccc} \widetilde{G}((a^\flat)^*R) & \longrightarrow & \{y, \widetilde{G}R\} \\ \big\downarrow & & \big\downarrow \\ \widetilde{G}\mathcal{C}(-,V) & \longrightarrow & \{y, \widetilde{G}\mathcal{C}(-,U)\} \end{array}$$

is a pullback, whence $\widetilde{G}((a^\flat)^*R) \cong a^*(\widetilde{G}R)$, since they are pullbacks of the same diagram. Since J is a $\mathcal{V}$-topology, we have $(a^\flat)^*R \in J$. By definition, we thus have $\widetilde{G}((a^\flat)^*R) \in \widetilde{G}J$, so that $a^*(\widetilde{G}R) \in \widetilde{G}J(U)$. We see that $\widetilde{G}J$ satisfies (T2) in 2.17.

□

With $\mathcal{U}, \mathcal{V}$, and $\mathcal{C}$ as in Section 3, we know that since $\mathcal{C}$ is small, $G_*\mathcal{C}$ is small. Thus, since both $\mathcal{U}$ and $\mathcal{V}$ satisfy 2.2, we know that both $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ and $[G_*\mathcal{C}^{\mathrm{op}}, \mathcal{U}]$ are well-powered. It follows that $\Sigma(\mathcal{C}, \mathcal{V})$ and $\Sigma(G_*\mathcal{C}, \mathcal{U})$ are small sets.

Theorem 4.6. *Suppose G satisfies all conditions in 3.6. The assignment*

$$\Sigma(\mathcal{C}, \mathcal{V}) \xrightarrow{\widetilde{G}(-)} \Sigma(G_*\mathcal{C}, \mathcal{U})$$

is an injective morphism of lattices.

Proof. Monotonicity follows immediately from the definition of $\widetilde{G}J$. To see that meets are preserved, observe that

$$\begin{aligned} \widetilde{G}\left[\bigcap_\alpha J_\alpha\right](x) &= \{\widetilde{G}r \mid (r : R \rightarrowtail \mathcal{C}(-,x)) \in J_\alpha(x) \text{ for all } \alpha\} \\ &= \bigcap_\alpha \{\widetilde{G}r : r \in J_\alpha(x)\} \\ &= \bigcap_\alpha \widetilde{G}J_\alpha(x). \end{aligned}$$

To prove injectivity, suppose J, K are $\mathcal{V}$-coverages such that $\widetilde{G}J = \widetilde{G}K$. For all x, we thus have that (i) for each $\widetilde{G}r \in \widetilde{G}J(x)$, there exists an $s \in K(x)$ such that $\widetilde{G}r = \widetilde{G}s$; (ii) for each $\widetilde{G}s \in \widetilde{G}K(x)$, there exists an $r \in J(x)$ such that $\widetilde{G}s = \widetilde{G}r$. By 3.9, (i) implies that $J(x) \subset K(x)$, and (ii) implies that $K(x) \subset J(x)$, whence $J = K$. □

4.3. **Change of base for $\mathcal{V}$-topologies.** We will frequently refer to an arbitrary category $\mathcal{W}$ satisfying 2.2, and an arbitrary small $\mathcal{W}$-category $\mathcal{X}$.

Definition 4.7. Given $\mathcal{W}$-sieves $r : R \rightarrowtail \mathcal{X}(-,x)$ and $s : S \rightarrowtail \mathcal{X}(-,x)$, say that $R \geq S$ if $s = rh$ for some $h : S \to R$. We say a coverage $J \in \Sigma(\mathcal{X}, \mathcal{W})$ is **upward-directed** if whenever a sieve R on x is such that $R \geq S$ for some $S \in J(x)$, then we have $R \in J(x)$.

With the above definition, it is straightforward to check that the following lemma, stated in the unenriched case as [4, V3, 3.2.7], holds in the enriched case.

Lemma 4.8. *Given $J \in \Sigma(\mathcal{X}, \mathcal{W})$, the assignment*

$$J_\uparrow(x) := \{R \rightarrowtail \mathcal{X}(-,x) \mid R \geq S\}$$

defines an upward-directed $\mathcal{W}$-coverage on $\mathcal{X}$, and is the smallest upward-directed coverage containing J.

The following lemmas also appear for the unenriched case in [4]. Their proofs are reproducible without subtlety in the enriched case if, for $\mathcal{W}$-sieves R, S and objects y, one considers elements of $\mathrm{Hom}_{\mathcal{W}}(\mathcal{W}_{fp}, R(y))$ and monomorphisms $R \rightarrowtail S$ (rather than elements of a set $R(y)$ and subset inclusions $R(y) \subset S(y)$).

Lemma 4.9. *[4, V3, 3.2.9] Given an upward-directed coverage $J \in \Sigma(\mathcal{X}, \mathcal{W})$ and an object $x \in \mathcal{X}$, let $J^+(x)$ denote the collection of all sieves $R \rightarrowtail \mathcal{X}(-, x)$ satisfying*

$$\exists\, S \in J(x) \text{ such that } \forall\, f \in Hom_{\mathcal{W}}(\mathcal{W}_{fp}, S(y)),\ \text{we have } f^*R \in J(y)$$

for all $y \in \mathcal{X}$. The assignment $J^+(x)$ defines an upward-directed $\mathcal{W}$-coverage on $\mathcal{X}$ containing J.

Lemma 4.10. *[4, V3, 3.2.5] Any $\mathcal{W}$-topology $J \in \mathcal{T}(\mathcal{X}, \mathcal{W})$ is an upward-directed coverage.*

In the particular case of $J \in \Sigma(\mathcal{C}, \mathcal{V})$ and $\widetilde{G}J \in \Sigma(G_*\mathcal{C}, \mathcal{U})$, we follow the method given in [4, V3, 3.2.11], and exhibit the smallest $\mathcal{U}$-topology on $G_*\mathcal{C}$ containing $\widetilde{G}J$.

Construction 4.11. *Consider the transfinite sequence $((\widetilde{G}J)_\lambda)_\lambda$ defined by:*

- $(\widetilde{G}J)_\uparrow(x) := \{R \leq \mathcal{C}(-, x) : R \geq \widetilde{G}S$ *for some* $S \in J(x)\}$*;*
- $(\widetilde{G}J)_{\lambda+1} := [(\widetilde{G}J)_\lambda]^+$ *for each ordinal* λ*;*
- $(\widetilde{G}J)_\lambda := \bigcup_{\mu<\lambda} (\widetilde{G}J)_\mu$ *for each limit ordinal* λ*.*

Define $\overline{GJ} := \bigcup_\lambda (\widetilde{G}J)_\lambda$.

For notational purposes, we have given the construction of $\overline{GJ}$. However, note that for $J \in \Sigma(\mathcal{C}, \mathcal{V})$, a similar construction yields the smallest $\mathcal{V}$-topology $\overline{J}$ containing J, proved in the unenriched case in [4, V3, 3.2.11].

Since $\Sigma(G_*\mathcal{C}, \mathcal{U})$ is a small set, the proof that 4.11 defines a $\mathcal{U}$-topology proceeds along exactly the same argument as that given in [4, V3, 3.2.11] for the unenriched case, provided that we check two technicalities.

Lemma 4.12. *For $J \in \Sigma(\mathcal{C}, \mathcal{V})$ and any ordinal λ, $\widetilde{G}J_\lambda \subset (\widetilde{G}J)_\lambda$.*

Proof. For $\lambda = 0$, observe that

$$\begin{aligned}
(\widetilde{G}J_\uparrow)(x) &= \{\widetilde{G}R \leq \widetilde{G}\mathcal{C}(-, x) : \exists S \in J(x) \text{ such that } R \geq S\} \\
&\subset \{T \leq \widetilde{G}\mathcal{C}(-, x) : \exists \widetilde{G}S \in \widetilde{G}J(x) \text{ such that } T \geq \widetilde{G}S\} \\
&= (\widetilde{G}J)_\uparrow(x)
\end{aligned}$$

for any x, whence $\widetilde{G}J_\uparrow \subset (\widetilde{G}J)_\uparrow$. Suppose that for some fixed λ, we have

$$\widetilde{G}J_\lambda \subset (\widetilde{G}J)_\lambda.$$

Say $R \in J_{\lambda+1}(x)$, so that for some $S \in J_\lambda(x)$, we have

$$\forall\, f \in \mathrm{Hom}_{\mathcal{V}}(F\mathcal{U}_{fp}, S(y)),\ f^*R \in J_\lambda(y).$$

Then $\widetilde{G}S \in \widetilde{G}J_\lambda(x) \subset (\widetilde{G}J_\lambda)(x)$ is such that

$$\forall\, g \in \mathrm{Hom}_{\mathcal{U}}(\mathcal{U}_{fp}, GS(y)),\ g^*(\widetilde{G}R) \in \widetilde{G}J_\lambda(y) \subset (\widetilde{G}J_\lambda)(y),$$

whence $\widetilde{G}R \in (\widetilde{G}J)_{\lambda+1}(x)$. The case where λ is a limit ordinal follows immediately. $\square$

Observe that given $J \in \Sigma(\mathcal{C}, \mathcal{V})$, there is a relationship between the ordinal at which the sequence $(J_\lambda)_\lambda$ stabilizes and that at which the corresponding sequence $((\widetilde{G}J)_\lambda)_\lambda$ stabilizes.

Lemma 4.13. *If $J \in \Sigma(\mathcal{C}, \mathcal{V})$ is such that the transfinite sequence $(J_\lambda)_\lambda$ stabilizes at an ordinal λ, then the sequence $((\widetilde{G}J)_\mu)_\mu$ stabilizes at an ordinal μ with $\mu \leq \lambda+1$.*

Proof. Suppose λ is such that $J_\lambda = J_{\lambda+1}$. Then

$$\widetilde{G}(J_\lambda) = \widetilde{G}(J_{\lambda+1}),$$

so

$$(\widetilde{G}J)_{\lambda+1} = [(\widetilde{G}J_\lambda)_\uparrow]^+ = [(\widetilde{G}J_{\lambda+1})_\uparrow]^+ = (\widetilde{G}J)_{\lambda+2}.$$

If μ is the least ordinal for which

$$(\widetilde{G}J)_\mu = (\widetilde{G}J)_{\mu+1},$$

we must have $\mu \leq \lambda + 1$. $\square$

Together with Lemmas 4.3, 4.8-4.10, and 4.12, this proves:

Theorem 4.14. *Given $J \in \Sigma(\mathcal{C}, \mathcal{V})$ and G as in 3.6, the family $\overline{GJ}$ of 4.11 is the smallest $\mathcal{U}$-topology on $G_*\mathcal{C}$ containing the coverage $\widetilde{G}J$.*

4.4. **Injectivity on $\mathcal{V}$-topologies.** We can prove a stronger version of 4.6 if we require a mild condition on $\widetilde{G}$.

Definition 4.15. Say that a functor $H : \mathcal{X} \to \mathcal{Y}$ is **order-reflecting** if it preserves monomorphisms, and if f, g are morphisms with common codomain such that

$$H(f) = H(g) \circ \sigma,$$

then $\sigma = H(\sigma_0)$ for some σ_0, and we have $f = g \circ \sigma_0$.

Any monadic functor satisfies this condition; in particular, our prototype functor

$$\mathrm{Hom}_{\mathsf{Ab}}(\mathbb{Z}, -) : \mathsf{Ab} \longrightarrow \mathsf{Set}$$

does. In case $\mathcal{R}$ is a ring and G is the functor $\mathrm{Hom}_{\mathsf{Ab}}(\mathbb{Z}, -)$, we moreover have that the functor

$$\{\text{right ideals of } \mathcal{R}\} \xrightarrow{\widetilde{G}(-)} \{\text{right ideals of } \mathrm{Hom}_{\mathsf{Ab}}(\mathbb{Z}, \mathcal{R})\}$$

is order-reflecting. We do not know of an example where G is order-reflecting and $\widetilde{G}$ is not, but proof that the former implies the latter appears to require some unknown condition on G. For the moment, we treat the two conditions as independent.

In the case of a general G, whenever $\widetilde{G}$ is order-reflecting, we immediately have the following:

Theorem 4.16. *If G satisfies (i)-(iii) in 3.6, and $\widetilde{G}$ is additionally order-reflecting, then the assignment*

$$\mathsf{Sub}_{\mathcal{V}}(\mathcal{C}(-, x)) \xrightarrow{\widetilde{G}(-)} \mathsf{Sub}_{\mathcal{U}}(\widetilde{G}\mathcal{C}(-, x))$$

is an injective morphism of posets.

Denote the collection of upward-directed $\mathcal{V}$-coverages on $\mathcal{C}$ by

$$\Delta(\mathcal{C},\mathcal{V}) \subset \Sigma(\mathcal{C},\mathcal{V}).$$

In the case where $\widetilde{G}$ is order-reflecting, the closure operations $(-)_{\uparrow}$ and $(-)^{+}$ of Lemmas 4.8 and 4.9 yield an injective mapping

$$\mathcal{T}(\mathcal{C},\mathcal{V}) \longrightarrow \Delta(G_*\mathcal{C},\mathcal{U}) \ .$$

Proposition 4.17. *Suppose G is as in 3.6, and moreover that $\widetilde{G}$ is order-reflecting. Suppose $J, K \in \Sigma(\mathcal{C},\mathcal{V})$ are such that*

$$[(\widetilde{G}J)_{\uparrow}]^{+} = [(\widetilde{G}K)_{\uparrow}]^{+}.$$

Then $(J_{\uparrow})^{+} = (K_{\uparrow})^{+}$. In particular, if J, K are $\mathcal{V}$-topologies, we have $J = K$.

Proof. Suppose J and K are such that

$$((\widetilde{G}J)_{\uparrow})^{+} = ((\widetilde{G}K)_{\uparrow})^{+},$$

and suppose $R \in (J_{\uparrow})^{+}(x)$. We have $\widetilde{G}R \in \widetilde{G}((J_{\uparrow})^{+})(x) \subset ((\widetilde{G}J)_{\uparrow})^{+}(x)$, so $\widetilde{G}R \in ((\widetilde{G}K)_{\uparrow})^{+}(x)$. By definition of the latter set, there is an $S \in (\widetilde{G}K)_{\uparrow}(x)$ such that

$$\forall f \in \mathrm{Hom}_{\mathcal{U}}(\mathcal{U}_{fp}, S(y)),\ f^*(\widetilde{G}R) \in (\widetilde{G}K)_{\uparrow}(y). \tag{4.18}$$

Having $S \in (\widetilde{G}K)_{\uparrow}(x)$ is true exactly when there is some $\widetilde{G}T \in \widetilde{G}K(x)$ such that $S \geq \widetilde{G}T$. Thus, in particular, 4.18 holds for all those $f \in \mathrm{Hom}_{\mathcal{U}}(\mathcal{U}_{fp}, S(y))$ factoring through the monomorphism $GT(y) \rightarrowtail S(y)$. We see that

$$f^*(\widetilde{G}R) \cong \widetilde{G}((f^\flat)^*R) \in (\widetilde{G}K)_{\uparrow}(y)$$

for all $f \in \mathrm{Hom}_{\mathcal{U}}(\mathcal{U}_{fp}, GT(y))$. Denote $h = f^\flat$.

Having $\widetilde{G}(h^*R) \in (\widetilde{G}K)_{\uparrow}(y)$ is true exactly when there is some $\widetilde{G}Q \in \widetilde{G}K(y)$ such that $\widetilde{G}(h^*R) \geq \widetilde{G}Q$. Since $\widetilde{G}$ is order-reflecting, we see that $h^*R \geq Q$, whence $h^*R \in K_{\uparrow}(y)$.

Thus $T \in K(x)$ is such that for all $h \in \mathrm{Hom}_{\mathcal{V}}(F\mathcal{U}_{fp}, T(y))$, we have $h^*R \in K_{\uparrow}(y)$, which proves $R \in (K_{\uparrow})^{+}(x)$. A symmetric argument proves that $(K_{\uparrow})^{+} \subset (J_{\uparrow})^{+}$. If J and K happen to be upward-directed, we immediately obtain $J^{+} = K^{+}$ as a consequence of 4.8. $\square$

We might naturally ask if Proposition 4.17 can be extended to the analogous result for topologies; that is, given $\mathcal{V}$-topologies J and K for which

$$\overline{GJ} = \overline{GK},$$

is it always true that $J = K$? The complicated nature of the definitions involved makes it difficult to spot a proof of the analogue of 4.17, even in the case of

$$(\widetilde{G}J)_2 = [[\widetilde{G}J)_{\uparrow}]^{+}]^{+},$$

and equally difficult to spot a counterexample to the claim. We therefore defer the answer to a later work, hoping that perhaps using an alternative construction of $\overline{GJ}$ will make the problem more tractable.

5. Enriched sheaves under change of base

We make a few observations on how change of base interacts with enriched sheaves in the sense of [5]. Throughout this section, $\mathcal{C}$ denotes a small $\mathcal{V}$-category equipped with a $\mathcal{V}$-topology J.

Definition 5.1. [5, 1.3] A presheaf $P \in [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ is a **sheaf** for J when, given R and α as in

$$\begin{tikzcd} R \arrow[r, tail, "r"] \arrow[d, "\alpha"'] & \mathcal{C}(-,U) \arrow[dl, dotted, "\exists!\beta"] \\ \{g, P\} & \end{tikzcd}$$

with $g \in \mathcal{V}_{fp}$ and $R \in J(U)$, there exists a unique β for which the diagram commutes.

Definition 5.2. [5, 4.1, 4.4] Given a presheaf $P \in [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$, define a new presheaf ΣP on objects by

$$\Sigma P(x) = \mathrm{colim}_{R \in J(x)}[R, P],$$

where square brackets denote the internal $\mathcal{V}$-hom in $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$. The **sheafification** or **associated sheaf** of P with respect to J is $\Sigma\Sigma P$. We will refer to the right adjoint

$$\ell : [\mathcal{C}^{\mathrm{op}}, \mathcal{V}] \longrightarrow \mathrm{Sh}_{\mathcal{V}}(\mathcal{C}, J)$$

to the inclusion functor $i : \mathrm{Sh}_{\mathcal{V}}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$, where $\ell(P) = \Sigma\Sigma P$.

A classical example is the case where $\mathcal{V} = \mathsf{Ab}$ and J is a $\mathcal{V}$-topology as in 2.20.

Example 5.3. [28, IX.1] Given a commutative ring A equipped with an Ab-topology (that is, Gabriel topology) $\mathfrak{R}$, and viewing A as a right A-module, the module

$$A_{\mathfrak{R}} := \mathrm{colim}_{I \in \mathfrak{R}} \mathrm{Hom}_A(I, A/t(A)),$$

where

$$t(A) := \{a \in A : aJ = 0 \text{ for some } J \in \mathfrak{R}\},$$

is the sheafification of A with respect to $\mathfrak{R}$. In particular, if S is a multiplicatively closed subset of A containing no zero divisors and such that for $s \in S$ and $a \in A$, there exist $t \in S$ and $b \in A$ such that $sb = at$, the family

$$\mathfrak{R} := \{I \triangleleft A : I \cap S \neq \varnothing\}$$

(where $I \triangleleft A$ means that I is an ideal of A) defines a Gabriel topology on A, and $A_{\mathfrak{R}}$ is isomorphic to the ring of fractions $A[S^{-1}]$.

When G satisfies Hypothesis 3.6, we can also sheafify $\mathcal{U}$-presheaves on $G_*\mathcal{C}$ with respect to the $\mathcal{U}$-topology $\overline{GJ}$ of 4.11. We will use the notation

$$\ell_G \dashv i_G : \mathrm{Sh}_{\mathcal{U}}(G_*\mathcal{C}, \overline{GJ}) \leftrightarrows [G_*\mathcal{C}^{\mathrm{op}}, \mathcal{U}]$$

for the resulting localization, and denote the units of both adjunctions $i \dashv \ell$ and $i_G \dashv \ell_G$ by η.

It seems natural to ask whether sheafification 'commutes' with change of base, in the sense that $\widetilde{G}(i\ell P) \cong i_G\ell_G(\widetilde{G}P)$ as sheaves. We will see that in the case where G is only faithful, we at least obtain a distinguished morphism $\widetilde{G}(i\ell P) \to i_G\ell_G(\widetilde{G}P)$; but when G is also full, the isomorphism is guaranteed.

Lemma 5.4. *Let J be a $\mathcal{V}$-topology on $\mathcal{C}$ and $P \in [\mathcal{C}^{op}, \mathcal{V}]$ be a sheaf for J. If G is fully faithful and a right adjoint, then $\widetilde{G}P$ is a sheaf for $\overline{GJ}$.*

Proof. Say $P \in \mathrm{Sh}_{\mathcal{V}}(\mathcal{C}, J)$, and suppose that $\gamma : \widetilde{G}\mathcal{C}(-,U) \to \{y, \widetilde{G}P\}$, $r : R \rightarrowtail \mathcal{C}(-,U)$, $g \in \mathcal{V}_{fp}$ and $\alpha : R \to \{g, P\}$ are such that

$$\widetilde{G}\alpha = \gamma \circ \widetilde{G}r.$$

By Definition 5.2, there exists a unique $\beta : \mathcal{C}(-,U) \to \{g, P\}$ for which

$$\gamma_x Gr_x = G\beta_x \circ Gr_x = G\alpha_x$$

in $\mathcal{U}_0$ for each object $x \in \mathcal{C}$. Since G is full, γ_x has the form $G\delta_x$ for some $\delta_x : \mathcal{C}(x,U) \to \{Fy, Px\}$. Since G is faithful, uniqueness of β implies that $\delta_x = \beta_x$, whence $\gamma = \widetilde{G}\beta$. □

Given $S \in [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ and $r : R \rightarrowtail S$, define $\widehat{R}$ to be the pullback

$$\begin{tikzcd} \widehat{R} \arrow[r] \arrow[d] & i\ell(R) \arrow[d, "i\ell(r)"] \\ S \arrow[r, "\eta_S"] & i\ell(S) \end{tikzcd}.$$

The operation $R \mapsto \widehat{R}$ is a universal closure operation on $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ in the sense of [5, 1.4]. A presheaf R is called **dense** if $\widehat{R} = S$.

For visual simplicity, we define

$$\widetilde{G}(\eta_Q) := \widetilde{\eta_Q} \quad \text{and} \quad \eta_{\widetilde{Q}} := \eta_{\widetilde{G}Q}.$$

Theorem 5.5. *Suppose G satisfies all conditions in 3.6. For $P \in [\mathcal{C}^{op}, \mathcal{V}]$, the unit*

$$\eta_{\widetilde{P}} : \widetilde{G}P \to i_G\ell_G(\widetilde{G}P)$$

factors uniquely through $\widetilde{G}(i\ell P)$; and if G is full, $\widetilde{G}(i\ell P) \cong i_G\ell_G(\widetilde{G}P)$.

Proof. Since i is fully faithful, we have for any $Q \in [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ that the unit $\eta_Q : Q \to i\ell Q$ is an isomorphism. Then $i\ell(\eta_Q)$ is an isomorphism, and since isomorphisms are pullback stable, we have $\widehat{Q} \cong i\ell Q$; in other words, η_Q is dense. Since $\widetilde{G}$ preserves conical limits, we have

$$\widetilde{G}\widehat{Q} \cong \widehat{\widetilde{G}Q} \cong \widetilde{G}(i\ell Q),$$

so that $\widetilde{\eta_Q}$ is dense.

The result [5, 2.2] says that P is (isomorphic to) a sheaf for J exactly when, for every dense monomorphism $r : R \rightarrowtail Q$ and morphism $s : R \to P$, there is a unique $t : Q \to P$ for which $r = ts$. In particular, since $i_G\ell_G(\widetilde{G}P)$ is a sheaf for $\widetilde{G}J$ and $\widetilde{\eta_P} : \widetilde{G}P \to \widetilde{G}(i\ell P)$ is dense, there is a unique morphism τ for which

$$\begin{tikzcd} \widetilde{G}P \arrow[r, "\eta_{\widetilde{P}}"] \arrow[d, tail, "\widetilde{\eta_P}"'] & i_G\ell_G(\widetilde{G}P) \\ \widetilde{G}(i\ell P) \arrow[ur, dashed, "\tau"'] & \end{tikzcd}$$

commutes. If G is full, 5.4 says that $\widetilde{G}(i\ell P)$ is a sheaf for $\widetilde{G}J$, so the same argument yields a unique factorization of $\widetilde{\eta_P}$ through $\eta_{\widetilde{P}}$, say $\sigma\eta_{\widetilde{P}} = \widetilde{\eta_P}$. We then have, for example,

$$\tau\sigma\eta_{\widetilde{P}} = \tau\widetilde{\eta_P} = \eta_{\widetilde{P}},$$

so since $\eta_{\widetilde{P}}$ is an isomorphism, $\tau\sigma$ is an identity. The same argument shows that $\sigma\tau$ is an identity, so we have $\widetilde{G}(i\ell P) \cong i_G\ell_G(\widetilde{G}P)$. □

6. Gabriel topologies

Our goal in this section is to illustrate via an example (namely 6.9) that the conclusion of Theorem 4.6 may fail if the functor $G : \mathcal{V} \to \mathcal{U}$ is not faithful. Toward that end, we generalize Definition 2.20 of a Gabriel topology on a ring - that is, on a monoid object in Ab - to monoid objects in an arbitrary $\mathcal{V}$ satisfying 2.2.

Perhaps among the easiest $\mathcal{V}$-categories to understand are one-object $\mathcal{V}$-categories, which are easily seen to coincide with the monoid objects in $\mathcal{V}$—that is to say, those objects A of $\mathcal{V}$ equipped with suitably coherent morphisms $m : A \otimes A \to A$ and $\nu : *_{\mathcal{V}} \to A$. Denoting the opposite monoid of A by A^{op}, we can use any such A to define a **right A-module** in $\mathcal{V}$ - an object M of $\mathcal{V}$ equipped with a morphism

$$\psi : A^{\mathrm{op}} \otimes M \to M,$$

called a **right A-action** on M, satisfying coherence conditions encoding associativity and unitality of the action. (For brevity, we do not discuss the coherence of these morphisms in detail; the uninitiated reader may consult [21, VII.3-4].) In particular, a monoid object (A, m, ν) of $\mathcal{V}$ is always a right module over itself. To emphasize that we are viewing A as a right A-module, we will sometimes use the notation A_A. By an **A-submodule** of M, we mean an A-module N admitting a monomorphism $\iota : N \rightarrowtail M$ in $\mathcal{V}$, and whose A-action is compatible with that of M in a sense that we will make precise below.

When $\mathcal{V}$ is closed monoidal, as in the present setting, we can 'transpose' a right action and its requisite coherence diagrams, obtaining a morphism

$$\varphi : A^{\mathrm{op}} \to \mathcal{V}(M, M)$$

in $\mathcal{V}_0$ which satisfies conditions encoding compatibility of the monoidal structure on A^{op} with the composition and identities in $\mathcal{V}$. If we shift our perspective and view A^{op} as a $\mathcal{V}$-category with a single object $\bullet$, the coherence of φ expresses $\mathcal{V}$-functoriality of the assignment $\bullet \mapsto M$. From this perspective, $\mathcal{V}$-sieves have straightforward descriptions in terms of subobjects of A.

Proposition 6.1. *If $\mathcal{V}$ is closed monoidal and $\mathcal{A}$ is a one-object $\mathcal{V}$-category with $\mathcal{A}(\bullet, \bullet) = A \in Mon(\mathcal{V})$, a $\mathcal{V}$-sieve on $\bullet$—that is, a subfunctor of $\mathcal{A}(-, \bullet) : \mathcal{A}^{op} \to \mathcal{V}$—is equivalently an A-submodule of A_A.*

Proof. We unpack the definition of a subfunctor $\mathcal{I}(-)$ of $\mathcal{A}(-, \bullet) : \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$. Say $\mathcal{I}(-) : \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ sends $\bullet \mapsto I$, and let $\varphi : \mathcal{A}^{\mathrm{op}}(\bullet, \bullet) = A^{\mathrm{op}} \to \mathcal{V}(I, I)$ be the hom-component of $\mathcal{I}(-)$. Functoriality of $\mathcal{I}(-)$ says that the diagrams

$$\begin{tikzcd}
A^{\mathrm{op}} \otimes A^{\mathrm{op}} \arrow[r, "m"] \arrow[d, "\varphi\otimes\varphi"'] & A^{\mathrm{op}} \arrow[d, "\varphi"] \\
\mathcal{V}(I,I) \otimes \mathcal{V}(I,I) \arrow[r, "\circ"'] & \mathcal{V}(I,I)
\end{tikzcd}
\qquad
\begin{tikzcd}
*_{\mathcal{V}} \arrow[r, "\nu"] \arrow[dr, "\mathrm{id}"'] & A^{\mathrm{op}} \arrow[d, "\varphi"] \\
 & \mathcal{V}(I,I)
\end{tikzcd}$$

commute. Denoting the transpose of φ by $\psi : A^{\mathrm{op}} \otimes I \to I$, commutativity of the diagrams above is equivalent to commutativity of

$$\begin{tikzcd}
A^{\mathrm{op}} \otimes I \arrow[r, "\psi"] & I \\
(A^{\mathrm{op}} \otimes A^{\mathrm{op}}) \otimes I \arrow[u, "m\otimes\mathrm{id}"] \arrow[r, "h"] \arrow[d, "(\psi\otimes\psi)\otimes\mathrm{id}"'] & I \arrow[u, equal] \arrow[d, equal] \\
(\mathcal{V}(I,I) \otimes \mathcal{V}(I,I)) \otimes I \arrow[r, "\circ^\flat"'] & I
\end{tikzcd}
\qquad
\begin{tikzcd}
*_{\mathcal{V}} \otimes I \arrow[r, "\nu\otimes\mathrm{id}"] \arrow[dr, "\lambda^{-1}"'] & A^{\mathrm{op}} \otimes I \arrow[d, "\psi"] \\
& I
\end{tikzcd},$$

where $h = \psi(1 \otimes \psi)\alpha$, and with α and λ respectively denoting the associator and left-unitor in $\mathcal{V}$. Commutativity of the top square in the left-hand diagram above is equivalent to associativity of ψ as a right action of A on I, and the triangle is equivalent to unitality. We see that I is a right A-module.

Having a $\mathcal{V}$-natural transformation $\iota : \mathcal{I}(-) \Rightarrow \mathcal{A}(-, \bullet)$ with monic components says that we have a monomorphism $I \rightarrowtail A$ in $\mathcal{V}_0$ which satisfies

$$\begin{tikzcd}
A^{\mathrm{op}} \arrow[r, "\varphi"] \arrow[d, "m^\flat"'] & \mathcal{V}(I,I) \arrow[d, "\iota_*"] \\
\mathcal{V}(A,A) \arrow[r, "\iota^*"'] & \mathcal{V}(I,A)
\end{tikzcd},$$

expressing compatibility of the right A-action on I with the right A-action of A on itself.

In the converse direction, say given a right A-submodule I of A_A, it is easy to check (by showing that commutativity is satisfied in the diagrams above) that $\bullet \mapsto I$ determines a $\mathcal{V}$-subfunctor of $\mathcal{A}(-, \bullet)$. $\square$

Pullbacks of sieves on $\bullet \in \mathcal{A}$, as in 2.17 (T2), are somewhat simpler to describe than in the general case. Given $f : G \to \mathcal{A}(\bullet, \bullet) = A$, f induces a morphism

$$G \to [\mathcal{A}^{\mathrm{op}}, \mathcal{V}](\mathcal{A}(-, \bullet), \mathcal{A}(-, \bullet))$$

by the enriched Yoneda lemma, and thus a morphism

$$\mathcal{A}(-, \bullet) \to \{G, \mathcal{A}(-, \bullet)\}.$$

Let $\iota : \mathcal{I}(-) \rightarrowtail \mathcal{A}(-, \bullet)$. Since $\mathcal{A}$ has only one object, the pullback of the diagram

$$\begin{tikzcd}
\mathcal{A}(-, \bullet) \arrow[r, "f"] & \{G, \mathcal{A}(-, \bullet)\} & \{G, \mathcal{I}(-)\} \arrow[l, tail, "\iota"']
\end{tikzcd}$$

in $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]_0$ is uniquely determined by the pullback

$$\begin{tikzcd}
A \arrow[r, "f"] & \mathcal{V}(G, A) & \mathcal{V}(G, I) \arrow[l, tail, "\iota"']
\end{tikzcd} \tag{6.2}$$

in $\mathcal{V}_0$. In the case where $\mathcal{A}$ has only one object, we identify the pullback f^*I in the functor category with the pullback of the diagram 6.2 in $\mathcal{V}_0$.

In light of the discussion above, we see that 2.20 is the case $\mathcal{V} = \mathsf{Ab}$ of the following:

Definition 6.3. Given a monoid object A of $\mathcal{V}$, a **(right) $\mathcal{V}$-Gabriel topology** on A is a non-empty family $\mathfrak{R}$ of right A-submodules of A_A such that

(V1) if $I \in \mathfrak{R}$ and J is a right A-submodule of A_A such that I is a right A-submodule of J, then $J \in \mathfrak{R}$;
(V2) for any $(\iota : I \rightarrowtail A) \in \mathfrak{R}$, $G \in \mathcal{V}_{fp}$, and $f : G \to A$ in $\mathcal{V}_0$, the pullback f^*I of the diagram 6.2 is in $\mathfrak{R}$;
(V3) if $I \in \mathfrak{R}$ and J is a right A-submodule of A_A such that $f^*J \in \mathfrak{R}$ for all $f : G \to I$, then $J \in \mathfrak{R}$.

Squinting at 6.3, the reader might guess that the following is true, although it may not be obviously apparent that (V1) is a perfect analogue of (T1) in 2.17. We provide a bit more detail:

Proposition 6.4. *Let $A \in Mon(\mathcal{V})$, and let $\mathfrak{R}$ be a set of right A-submodules of A_A. Denote by $\mathcal{A}$ the one-object $\mathcal{V}$-category with $\mathcal{A}(\bullet, \bullet) = A$. Given a right A-submodule $I \rightarrowtail A$, denote the $\mathcal{V}$-subfunctor $\bullet \mapsto I$ of $\mathcal{A}(-, \bullet)$ by $\mathcal{I}(-)$. The following are equivalent:*

(i) $\mathfrak{R}$ is a $\mathcal{V}$-Gabriel topology on A;
(ii) $\mathcal{T} := \{\mathcal{I}(-) : I \in \mathfrak{R}\}$ is a $\mathcal{V}$-topology on $\mathcal{A}$.

Proof. That (T2) and (T3) are respectively equivalent to (V2) and (V3) follows directly from the definitions 6.1 and 6.2. Moreover if (V1) holds for $\mathfrak{R}$, the fact that $\mathfrak{R}$ is nonempty immediately implies (T1).

The only subtlety is in proving that (V1) holds for $\mathfrak{R}$, given (ii). Following [4, V2, 3.2.5], suppose that $I \in \mathfrak{R}$ is such that $\iota : I \to A$ factors as

$$I \xrightarrow{i} J \xrightarrow{j} A$$

for some A-submodule J of A_A. If $f : G \to I$ has $G \in \mathcal{V}_{fp}$, then $\iota f = jif : A \to \mathcal{V}(G, A)$, so that the pullback $f^*\mathcal{J}$ of

$$\mathcal{A}(-, \bullet) \xrightarrow{\iota f = jif} \{G, \mathcal{A}(-, \bullet)\} \xleftarrow{j_*} \{G, \mathcal{J}(-)\}$$

is $\mathcal{A}(-, \bullet) \in \mathcal{T}$. Since $\mathcal{T}$ is a $\mathcal{V}$-topology, we have $\mathcal{J}(-) \in \mathcal{T}$, so that $J \in \mathfrak{R}$. $\square$

6.1. **Graded Gabriel topologies on a graded algebra.** We turn to an example of categories $\mathcal{U}, \mathcal{V}$, a $\mathcal{V}$-category $\mathcal{C}$, and a functor $G : \mathcal{V} \to \mathcal{U}$ where the injectivity results of sections 3 and 4 do not hold. For the rest of this section, we consider a field k, and set $\mathcal{V} = \mathsf{grMod}_k$, the category of $\mathbb{Z}$-graded k-modules. Recall that the monoidal unit in $\mathcal{V}$ is k, viewed as a $\mathbb{Z}$-graded algebra concentrated in degree 0, and the internal hom in $\mathcal{V}$ is defined as

$$\mathcal{V}(M, N) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_i(M, N),$$

where $\mathrm{Hom}_i(M, N)$ denotes the collection of k-module homomorphisms f for which $f(M_j) \subset N_{j+i}$, which we call **morphisms of degree** i. Uninitiated readers can find a detailed treatment of graded algebras in [24] or [23].

The functor

$$\mathrm{Hom}_{\mathcal{V}}(k, -) : \mathcal{V} \to \mathsf{Set}$$

has a left adjoint $k[-]$ in Cat which takes a set X to the free graded k-module $k[X]$ generated in degree 0 by the elements of X. Since the functor $\mathrm{Hom}_{\mathcal{V}}(k, -)$ is lax and the functor $k[-]$ is strong monoidal, they comprise an adjunction in $\mathbf{MonCat}_\ell$

by [14, 1.5]. We will see that $k[-] \dashv \mathrm{Hom}_{\mathcal{V}}(k,-)$ yields an example where the assignment $\widetilde{G}(-)$ of Theorem 4.6 is not injective.

Example 6.5. $\mathrm{Hom}_{\mathcal{V}}(k,-) : \mathcal{V} \to \mathsf{Set}$ is not faithful - to see this, take any two distinct graded k-modules, say M and N, with $M_0 = N_0 = 0$, and recall that

$$\mathrm{Hom}_{\mathcal{V}}(k, M) \cong \mathrm{Hom}_k(k, M_0) \cong \{0\}$$

(and similarly for N). As long as there exists a non-trivial graded module homomorphism $M \to N$, for example, in the case of M and N with homogeneous components defined by

$$M_i = \begin{cases} 0 & i < 2 \\ k & i \geq 2 \end{cases}, \quad N_i = \begin{cases} 0 & i < 1 \\ k & i \geq 1 \end{cases},$$

the map

$$\mathcal{V}(M, N) \to \mathsf{Set}(\mathrm{Hom}_{\mathcal{V}}(k, M), \mathrm{Hom}_{\mathcal{V}}(k, N)) \cong \{0\}$$

is not injective.

Below, we construct an example of two $\mathcal{V}$-coverages which correspond to the same Set-coverage under change of base, toward which our first task is to describe $\mathcal{V}$-sieves and their pullbacks. Recall that a left or right ideal I of a graded ring $A = \bigoplus_{i \in \mathbb{Z}} A_i$ is called **homogeneous** if whenever $\sum a_i \in I$, where $a_i \in A_i$, each homogeneous element a_i in the sum is itself an element of I; or equivalently, if it is a graded A-submodule of A. As a corollary to 6.1, we have the following:

Corollary 6.6. *Given $A \in \mathsf{grAlg}_k$, viewed as a grMod_k-category with one object $\bullet$, the $\mathcal{V}$-sieves on $\bullet$ are exactly the homogeneous right ideals of A.*

As described in [24, p. 21], $\mathcal{V}$ admits a separating family: For $i \in \mathbb{Z}$, define the homogeneous components of a graded k-module $k(-i)$ by

$$k(-i)_j := k_{j-i},$$

so that

$$k(-i)_i = k_0 = k,$$

and $k(-i)_j = 0$ otherwise.

Note that any graded k-module is the filtered colimit of its finite-dimensional graded subspaces, and any finite-dimensional graded k-module is isomorphic to the direct sum of the objects in $\{k(-j)\}_{j \in J}$ for some $J \subset \mathbb{Z}$. Thus, to construct the pullback as in 6.3 (V2), we need only consider pullbacks along graded module morphisms $f : k(-i) \to A$, where we identify f with $f(1_k) \in A_i$. Denote the set of homogeneous elements of A by

$$h(A) := \bigcup_{i \in \mathbb{Z}} A_i.$$

Definition 6.7. Given a morphism $f : k(-i) \to A$ of graded k-modules and a homogeneous right ideal $I \subset A$, the pullback of the diagram

$$A \xrightarrow{\ f\ } \mathcal{V}(k(-i), A) \xleftarrow{\ \mathrm{inc}\ } \mathcal{V}(k(-i), I) \ ,$$

where f is identified with the map $1_A \mapsto f(1_k)$, is the homogeneous right ideal $(I : f(1_k))$.

With 6.6 and 6.7 in hand, we can define an analogue of 2.20 for a graded k-algebra A, as in [23].

Definition 6.8. A **graded (right) Gabriel topology** on A is a non-empty set $\mathfrak{R}$ of homogeneous right ideals of A satisfying

(G1) if $I \in \mathfrak{R}$ and J is a homogeneous right ideal of A for which $I \subset J$, then $J \in \mathfrak{R}$;
(G2) if $I \in \mathfrak{R}$, then $(I : x) \in \mathfrak{R}$ for all $x \in h(A)$;
(G3) if $I \in \mathfrak{R}$ and J is a homogeneous right ideal of A such that $(J : x) \in \mathfrak{R}$ for all $x \in h(I)$, then $J \in \mathfrak{R}$.

Given a graded algebra A, any multiplicatively closed set S of homogeneous elements of A gives rise to a graded Gabriel topology by letting H_S be the collection of homogeneous right ideals defined by

$$H_S := \{I \mid (I : a) \cap S \neq \varnothing \text{ for all homogeneous elements } a \in A\},$$

as in [23, II.9.11].

Example 6.9. For a field k, take A to be the commutative ring $k[x, y]$, graded by polynomial degree. Set

$$S := \{1, x, x^2, ...\} \text{ and } T := \{1, y, y^2, ...\},$$

and consider the change of base given by

$$G = \mathrm{Hom}_{\mathcal{V}}(k, -) : \mathcal{V} \to \mathsf{Set}.$$

The families S and T generate distinct $\mathcal{V}$-coverages on A, namely

$$H_S = \{I \triangleleft A : I \text{ is homogeneous and } x^n \in I \text{ for some } n\}$$

and

$$H_T = \{I \triangleleft A : I \text{ is homogeneous and } y^n \in I \text{ for some } n\},$$

where the notation $I \triangleleft A$ means I is an ideal of A.

On the other hand, given any $M \in \mathcal{V}$, we have

$$\mathrm{Hom}_{\mathcal{V}}(k, M) \cong \mathrm{Hom}_k(k, M_0) \cong M_0,$$

so in particular, we have $\widetilde{G}I \cong \mathrm{Hom}_k(k, I_0)$ for any I in H_S or H_T. Recall that the degree-0 elements of A are exactly the scalars k; thus, if $I \neq A$, we have $I_0 = \{0\}$ (otherwise I contains a unit of A), and if $I = A$, we have $I_0 = k$. Then

$$\widetilde{G}H_S = \widetilde{G}H_T = \{k, (0)\}.$$

We see that the conclusion of Theorem 4.6 fails in this case.

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