**Homework (fair game for quiz).**

§6.1: Exercises 1-22

**Challenge problems (won't be on the quiz)!**

1. 🌶️ *(Ball with a bite taken out)* Let $S_1$ and $S_2$ be solid balls of radius 1. Say $S_1$ is centered at $(-1/2, 0, 0)$ and $S_2$ is centered at $(1/2, 0, 0)$. Find the volume of the solid that results from removing all points belonging to both $S_2$ and $S_1$ from $S_1$.

2. 🌶️🌶️ *(Hollow ball with thick wall)* Let $S_1$ be a ball of radius 1 and $S_2$ be a ball of radius 2, both centered at the origin. Find the volume of the solid that results from removing $S_1$ from $S_2$.

3. 🌶️🌶️🌶️ In the following exercise, we compute the surface area of the unit sphere as the limit of a family of surfaces with regular polygonal cross-sections.

   (a) Use integration to compute the surface area $A$ of the unit sphere centered at the origin.

   (b) Use integration to compute the surface area $T_n$ of a solid inscribed in the unit sphere whose cross-sections perpendicular to the $x$-axis are regular $n$-sided polygons. *(Hint: Use the law of sines or law of cosines to find the perimeter of a given cross-section. Treat $n$ as a constant while integrating.)*

   (c) Show that $\lim_{n\to\infty} T_n = A$. *(Hint: To compute the limit, you'll need to read ahead and use L'Hôpital's rule.)*

## EXERCISES 6.1

In Exercises 1–8 let $R$ be the region between the graph of the given function and the $x$ axis on the given interval. Find the volume $V$ of the solid obtained by revolving $R$ about the $x$ axis.

**1.** $f(x) = x^{3/2}$; $[0, 1]$ **2.** $f(x) = 1 + x^2$; $[-1, 2]$

**3.** $f(x) = \sqrt{\cos x}$; $[0, \pi/6]$

**4.** $f(x) = \sqrt{1 + \sin^2 x}$; $[-\pi/2, \pi/2]$

**5.** $f(x) = \sec x$; $[-\pi/4, 0]$ **6.** $f(x) = (\ln x)/\sqrt{x}$; $[1, e]$

**7.** $f(x) = x(x^3 + 1)^{1/4}$; $[1, 2]$ **8.** $f(x) = \sqrt{x}(1 - x)^{1/4}$; $[0, 1]$

In Exercises 9–10 let $R$ be the region between the graph of $f$ and the $y$ axis on the given interval. Find the volume $V$ of the solid obtained by revolving $R$ about the $y$ axis.

**9.** $f(y) = \sqrt{1 + y^3}$; $[1, 2]$

**10.** $f(y) = \sqrt{\sin y \cos y}\,(1 + \cos^2 y)^{1/3}$; $[0, \pi/2]$

In Exercises 11–14 let $R$ be the region between the graphs of $f$ and $g$ on the given interval. Find the volume $V$ of the solid obtained by revolving $R$ about the $x$ axis.

**11.** $f(x) = \sqrt{x + 1}$, $g(x) = \sqrt{x - 1}$; $[1, 3]$

**12.** $f(x) = x + 1$, $g(x) = x - 1$; $[1, 4]$

**13.** $f(x) = \cos x + \sin x$, $g(x) = \cos x - \sin x$; $[0, \pi/4]$

***14.** $f(x) = 2x - x^2$, $g(x) = x^2 - 2x$; $[0, 1]$ (*Hint:* The washer method will not work. Why?)

In Exercises 15–18 find the volume $V$ of the solid generated by revolving about the $x$ axis the region between the graphs of the given equations.

**15.** $y = \frac{1}{2}x^2 + 3$ and $y = 12 - \frac{1}{2}x^2$

**16.** $y = x^{1/2}$ and $y = 2x^{1/4}$

**17.** $y = 5x$ and $y = x^2 + 2x + 2$

***18.** $y = x^3 + 2$ and $y = x^2 + 2x + 2$

In Exercises 19–22 let $R$ be the region between the graph of the function and the $x$ axis on the given interval. Use the shell method to find the volume $V$ of the solid generated by revolving $R$ about the $y$ axis.

**19.** $f(x) = (x - 1)^2$; $[0, 2]$

**20.** $f(x) = \sin x^2$; $[\sqrt{\pi/2}, \sqrt{\pi}]$

**21.** $g(x) = \dfrac{1}{\sqrt{1 - x^2}}$; $[0, \sqrt{3}/2]$

***22.** $g(x) = \sqrt{1 + \sqrt{x}}$; $[0, 4]$

In Exercises 23–24 let $R$ be the region between the graph of $f$ and the $y$ axis on the given interval. By interchanging the roles of $x$ and $y$ in (7), find the volume $V$ of the solid generated by revolving $R$ about the $x$ axis.

**23.** $f(y) = \dfrac{\ln y}{y^2}$; $[1, 2]$ **24.** $f(y) = y^2\sqrt{1 + y^4}$; $[0, 1]$

In Exercises 25–26 let $R$ be the region between the graphs of $f$ and $g$ on the given interval. Use the shell method to find the volume $V$ of the solid obtained by revolving $R$ about the $y$ axis.

**25.** $f(x) = 1$, $g(x) = x - 2$; $[1, 3]$

**26.** $f(x) = \sqrt{1 - x^2}$, $g(x) = -2 + \sqrt{1 + x^2}$; $[0, 1]$

In Exercises 27–28 let $R$ be the region between the graphs of $f$ and $g$ on the given interval. Use the shell method to find the volume $V$ of the solid obtained by revolving $R$ about the $x$ axis.

**27.** $f(y) = y^2 + 1$, $g(y) = y\sqrt{1 + y^3}$; $[0, 1]$

**28.** $f(y) = \dfrac{1}{(y + 2)^2}$, $g(y) = \dfrac{1}{y + 2}$; $[0, 2]$ (*Hint:* To evaluate the integral, make the substitution $u = y + 2$.)

In Exercises 29–30 find the volume $V$ of the solid generated by revolving the region between the graphs of the equations about the $y$ axis.

**29.** $y = 2x$ and $y = x^2$ **30.** $y^2 = x$ and $y = 3x$

In Exercises 31–36 find the volume $V$ of the solid with the given information about its cross-sections.

**31.** The base of the solid is an isosceles right triangle whose legs $L_1$ and $L_2$ are each 4 units long. Any cross section perpendicular to $L_1$ is semicircular (see Figure 6.15).

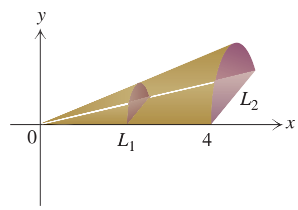



**FIGURE 6.15** The solid for Exercise 31.

**32.** The base of the solid is a square centered at the origin, with sides of length 6 and parallel to the axes. The area of each cross section perpendicular to the edge of the base equals the distance from the cross section to the origin.

**33.** The solid has a circular base with radius 1, and the cross sections perpendicular to a fixed diameter of the base are squares. (*Hint:* Center the base at the origin.)

**34.** The base of a solid is a circle with radius 2, and the cross sections perpendicular to a fixed diameter of the base are equilateral triangles.

**35.** The base is an equilateral triangle each side of which has length 10. The cross sections perpendicular to a given altitude of the triangle are squares.

**36.** The base is an equilateral triangle with altitude 10. The cross sections perpendicular to a given altitude of the triangle are semicircles.

**37.** Suppose $f$ is continuous on $[a, b]$ and the graph of $f$ lies above the line $y = c$. Write down a formula for the volume $V$ of the solid obtained by revolving about the line $y = c$ the region between the graph of $f$ and the line $y = c$ on $[a, b]$.

**38.** Use the result of Exercise 37 to find the volume $V$ of the solid obtained by revolving about the line $y = -1$ the region between the graph of the equation $y = \sqrt{x + 1}$ and the line $y = -1$ on the interval $[0, 1]$.

**39.** Find the volume $V$ of the solid obtained by revolving about the line $y = 1$ the region between the graph of the equation $y = e^{-2x}$ and the $x$ axis on the interval $[0, 1]$. (*Hint:* Pattern your solution after Exercise 37.)

**40.** Suppose $f$ and $g$ are continuous on $[a, b]$, and let $c$ be such that $c \le g(x) \le f(x)$ for $a \le x \le b$. Write down a formula for the volume $V$ of the solid obtained by revolving about the line $y = c$ the region between the graphs of $f$ and $g$ on $[a, b]$.

**41.** Use the result of Exercise 40 to find the volume $V$ of the solid obtained by revolving about the line $y = 1$ the region between the graphs of $y = x^2 - x + 1$ and $y = 2x^2 - 4x + 3$.

**42.** Use the result of Exercise 40 to find the volume $V$ of the solid obtained by revolving about the line $y = 2$ the region between the graphs of $y = x + 1$ and $y = x^2 - 2x + 2$.

**43.** Suppose the Great Pyramid of Cheops had been built with equilateral triangular cross sections instead of square cross sections but had the same height of 482 feet and base 754 feet on a side. What percentage of the original volume would have resulted?

**44.** Let a sphere of radius $r$ be sliced at a distance $h$ from its center. Show that the volume $V$ of the smaller piece cut off is given by

$$V = \frac{\pi(r - h)^2}{3}(2r + h)$$

**45.** Let $f$ be continuous and nonnegative on $[a, \infty)$. If

$$\lim_{b\to\infty} \int_a^b \pi[f(x)]^2\, dx$$

converges, then we say that the volume of the solid obtained by revolving the graph of $f$ about the $x$ axis is finite. Otherwise, we say that the volume is infinite. Determine whether the graph of each of the following functions generates a solid with finite volume when the graph is revolved about the $x$ axis. For any that is finite, find the volume $V$.

**a.** $f(x) = 1/x$ for $x \ge 1$
**b.** $f(x) = 1/x^2$ for $x \ge 1$
**c.** $f(x) = 1/x^{1/2}$ for $x \ge 1$

**46.** Cavalieri's Principle for volume states that if two solids have the same cross-sectional area at each $x$ between $a$ and $b$, then the two solids have the same volume. Prove Cavalieri's Principle for volume by using (1).

**47.** Let $f$ be continuous and nonnegative on $[a, b]$, and assume that $c \le a$. Let $R$ be the region between the graph of $f$ and the $x$ axis on $[a, b]$. Find a formula for the volume $V$ of the solid obtained by revolving $R$ about the line $x = c$.

**48.** Use the result of Exercise 47 to find the volume $V$ of the solid obtained by revolving about the line $x = -1$ the region between the graph of $y = x^4$ and the $x$ axis on $[0, 1]$.

**49.** Find the volume $V$ of the solid obtained by revolving about the line $x = 2$ the region between the graph of $y = x^2$ and the $x$ axis on $[2, 3]$. (*Hint:* Pattern your solution after Exercise 47.)

**50.** Let $f$ and $g$ be continuous on $[a, b]$, with $g(x) \le f(x)$ for $a \le x \le b$. Let $R$ be the region between the graphs of $f$ and $g$ on $[a, b]$, and assume that $c \le a$. Find a formula for the volume $V$ of the solid obtained by revolving $R$ about the line $x = c$.

**51.** Use the result of Exercise 50 to find the volume $V$ of the solid obtained by revolving about the line $x = -1$ the region between the graphs of $y = 2x$ and $y = -2x^2 + 4x$.

**52.** Use the result of Exercise 50 to find the volume $V$ of the solid obtained by revolving about the line $x = -5$ the region between the graphs of $y = x^2 + 4$ and $y = 2x^2 + x + 2$.

## Applications

**53.** As viewed from above, a swimming pool has the shape of the ellipse

$$\frac{x^2}{400} + \frac{y^2}{100} = 1$$

The cross sections of the pool perpendicular to the ground and parallel to the $y$ axis are squares. If the units are feet, determine the volume $V$ of the pool.

**54.** Find the volume $V$ of the solid generated when the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is revolved about the $x$ axis. With $a = 100$ and $b = 25$, your answer could be the volume of gas needed to fill a dirigible having these dimensions.

**55.** The ivory stones used in the Oriental game of *go* have approximately the shape of the solid generated by revolving a certain ellipse about the $x$ axis. If the length of a *go* stone is 2 centimeters and its height is 1 centimeter, what is its volume? (*Hint:* Let $a = \frac{1}{2}$ and $b = 1$ in Exercise 54.)

**56.** Suppose a ring is obtained by revolving about the $x$ axis the region between the curve $y = 4 - x^2$ and the line $y = 1$ for $-\sqrt{3} \le x \le \sqrt{3}$. Determine the volume $V$ of the ring.

**57.** A soda glass has the shape of the surface generated by revolving the graph of $y = 6x^2$ for $0 \le x \le 1$ about the $y$ axis. Soda is extracted from the glass through a straw at the rate of $\frac{1}{2}$ cubic inch per second. How fast is the depth of soda decreasing when the depth is $\frac{3}{2}$ inches?

**58.** The Washington Monument has the shape of an obelisk. Its sides slant gradually inward as they rise to the base of

The Washington Monument. (*Rob Crandall/Stock Boston*)

the small pyramid at the top. The base of the monument is square, 16.80 meters on a side. At the base of the small pyramid, 152.49 meters above ground, the walls are 10.50 meters on a side. Finally, the small pyramid is 16.79 meters tall. Determine the total volume $V$ of the monument.

**59.** A wooden golf tee has approximately the dimensions of the solid obtained by revolving about the $x$ axis the region bounded by the graphs of $f$ and $g$, where

$$f(x) = \begin{cases} \frac{1}{2}x & \text{for } 0 \le x \le \frac{1}{2} \\ \frac{1}{4} & \text{for } \frac{1}{2} \le x \le \frac{7}{2} \\ \frac{1}{4}\left[1 + \left(x - \frac{7}{2}\right)^2\right] & \text{for } \frac{7}{2} \le x \le \frac{9}{2} \\ \frac{1}{2} & \text{for } \frac{9}{2} \le x \le 5 \end{cases}$$

and

$$g(x) = \begin{cases} 0 & \text{for } 0 \le x \le \frac{9}{2} \\ x - \frac{9}{2} & \text{for } \frac{9}{2} \le x \le 5 \end{cases}$$

(Figure 6.16). Here $x$ and $g(x)$ are measured in centimeters. Determine how much wood goes into a golf tee.

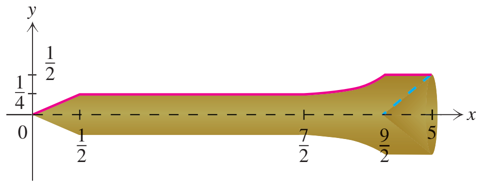



**FIGURE 6.16** Figure for Exercise 59.

**60.** In order to create a ring, a hole with radius 1 centimeter is drilled through the center of a sphere of radius 2 centimeters. Find the volume $V$ removed.

**61.** A wok is in the shape of a solid obtained by revolving about the $y$ axis the curve shown in Figure 6.17. Assuming that the units are centimeters, find the capacity of the wok.

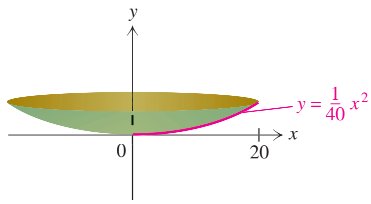



**FIGURE 6.17** The wok for Exercise 61.

## Project

**1.** This project concerns solids $D$ in the shape of a "pyramid" such as that in Figure 6.18, whose cross section at $x$ perpendicular to the $x$ axis is a regular polygon (or is a circle), for $0 \le x \le h$, where $h$ is a positive number.

**a.** Let $A(x)$ = the area of the cross section perpendicular to the $x$ axis at $x$, and let $A$ be the area of the largest cross section of $D$. Find a formula for $A(x)$ in terms of $x$, $h$, and $A$.

**b.** Find a formula for the volume $V$ of $D$ in terms of $h$ and $A$.

**c.** Suppose that the cross sections are square (as for the Great Pyramid of Cheops), and that $A = a^2$. Use (b) to show that $V = \frac{1}{3}a^2h$ (which conforms to the result of Example 2).

**d.** Suppose that $D$ is a circular cone, with base (largest) radius equal to $a$. Show that $V = \frac{\pi}{3}\, a^2 h$, a formula known to the ancient Greeks.

**e.** Suppose that $D$ is an equilateral triangular pyramid, with largest triangle having side length $a$. Find a simple formula for the volume $V$ of $D$, and show that the volume is smaller than the volume in (c).

**f.** Finally, suppose that $D$ is a hexagonal pyramid, with largest hexagon having side length $a$. Find a simple formula for the volume $V$ of $D$.

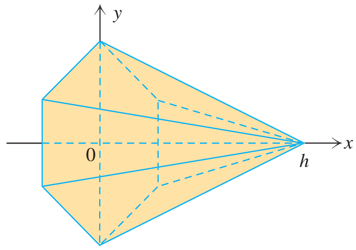



**FIGURE 6.18** Figure for the project.

## 6.2 LENGTH OF A CURVE

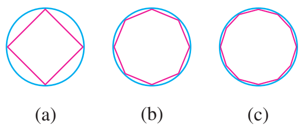



**FIGURE 6.19** The circumference of a circle is approximated by the perimeters of inscribed polygons.

The ancient Greeks estimated the circumference of a circle by inscribing a polygon of $n$ sides and then computing the perimeter of the polygon. They surmised that the larger $n$ was, the better the perimeter of the polygon approximated the actual circumference of the circle (Figure 6.19). In this section we will use this basic idea to define and compute the lengths of many curves.

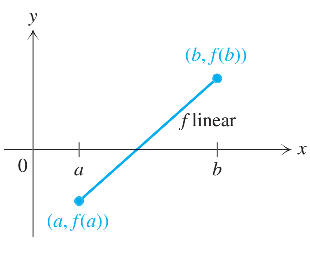
Each curve that we consider will be the graph of a function $f$ with a continuous derivative on a closed interval $[a, b]$. If $f$ is linear, that is, if the graph of $f$ is a line segment, then the length $L$ of the graph is the distance between $(a, f(a))$ and $(b, f(b))$, so that

$$L = \sqrt{(b - a)^2 + [f(b) - f(a)]^2}$$

(Figure 6.20(a)).

When $f$ is not necessarily linear, we let $P = \{x_0, x_1, \ldots, x_n\}$ be any partition of $[a, b]$, and approximate the graph of $f$ by a polygonal line $l$ whose vertices are $(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n))$ (Figure 6.20(b)). Let $\Delta L_k$ be the length of the portion of the graph of $f$ joining $(x_{k-1}, f(x_{k-1}))$ and $(x_k, f(x_k))$. If $\Delta x_k$ is small, $\Delta L_k$ is approximately equal to the length of the line segment joining $(x_{k-1}, f(x_{k-1}))$ and $(x_k, f(x_k))$. In other words,

$$\Delta L_k \approx \sqrt{(x_k - x_{k-1})^2 + [f(x_k) - f(x_{k-1})]^2} \tag{1}$$



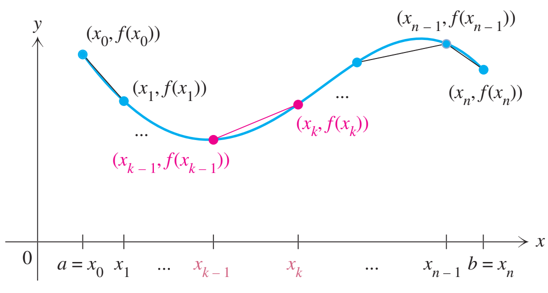




**FIGURE 6.20** (a) The length of a line segment is the distance between its endpoints. (b) The length of a curve is approximated by the length of a polygonal line.