ENRICHED GROTHENDIECK TOPOLOGIES UNDER CHANGE OF BASE

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ABSTRACT. In the presence of a monoidal adjunction $F \dashv G : \mathcal{U} \leftrightarrows \mathcal{V}$ between locally finitely presentable Bénabou cosmoi, we examine the behavior of $\mathcal{V}$-Grothendieck topologies on a $\mathcal{V}$-category $\mathcal{C}$, and that of their constituent covering sieves, under the change of enriching category $G_* : \mathcal{V}\text{-Cat} \to \mathcal{U}\text{-Cat}$ induced by G. We prove in particular that when G is faithful and conservative, any $\mathcal{V}$-Grothendieck topology on $\mathcal{C}$ corresponds uniquely to a $\mathcal{U}$-Grothendieck topology on $G_*\mathcal{C}$, and that when G is fully faithful, base change commutes with enriched sheafification in the sense of Borceux-Quinteiro.

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1. INTRODUCTION

As outgrowths of the move to formalize algebraic geometry in terms of abelian categories, Grothendieck topologies and their accompanying categories of sheaves arose in the early 1960s as a framework for defining cohomology theories on schemes. Roughly speaking, a Grothendieck topology on a category $\mathcal{C}$ can be regarded as a way to specify, for all objects U of $\mathcal{C}$, which objects of $\mathcal{C}$ cover U. This is in exactly the same sense as, given a topological space X and an open set $U \subset X$, we might ask when $\bigcup_{i \in I} U_i = U$ for some family $\{U_i : i \in I\}$ of opens of X. Enriched categories, where the hom-sets of ordinary category theory are replaced, more generally, by objects of a closed monoidal category $\mathcal{V}$, were first introduced in the mid-1960s in the work of Maranda [21] and Bénabou [2], among others. Around the same time, Gabriel introduced in [12, V.2, p. 411] the notion of a (right) linear topology (*topologie linéaire à droite*) on a ring - an early example of an enriched Grothendieck topology, in the particular case of a category with one object enriched over $\mathcal{V} = \mathsf{Ab}$.

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The definition of a Grothendieck topology admits a number of different formulations, but the definition in terms of sieves on objects $U \in \mathcal{C}$ - that is, subfunctors of $\mathcal{C}(-, U)$ - is perhaps the most straightforwardly generalizable to the enriched setting. For a nice enough base category $\mathcal{V}$, enriched Grothendieck topologies on a $\mathcal{V}$-category $\mathcal{C}$ (now taken to be families of subfunctors of enriched hom-functors), their accompanying sheaves, and their correspondence with localizations of and universal closure operations on $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$, were introduced by Borceux and Quinteiro in 1996 with the publication of [4]. Their paper greatly inspires the current work. More recently, details of the theory of enriched sheaves in the case $\mathcal{V} = \mathsf{Ab}$ were established in the 2000s by Lowen in [17] and [18]; and in 2020 by Coulembier [7].

Given a category $\mathcal{C}$ enriched over $(\mathcal{V}, \otimes, I)$ and a lax monoidal functor $G : \mathcal{V} \to \mathcal{U}$, G canonically induces a 2-functor

$$G_* : \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{U}\text{-}\mathsf{Cat}$$

which acts via an operation called 'base change' or 'change of base,' changing $\mathcal{V}$-categories into $\mathcal{U}$-categories, $\mathcal{V}$-functors into $\mathcal{U}$-functors, and $\mathcal{V}$-natural transformations into $\mathcal{U}$-natural transformations. Base change first appeared in the literature around the same time as enriched categories themselves, with Eilenberg and Kelly's publication of [9], and is fundamental to the theory of enriched categories, in part because it allows one to view a $\mathcal{V}$-category $\mathcal{C}$ as an ordinary category by applying the functor

$$\mathrm{Hom}_{\mathcal{V}}(I, -) : \mathcal{V} \to \mathsf{Set}$$

to the hom-objects of $\mathcal{C}$. Many of the technical results in Section 3 of the current work rely heavily on the results and style of argument developed in Cruttwell's 2008 doctoral thesis [8], which, toward understanding normed spaces, addressed in detail the question of how base change interacts with the monoidal structures on $\mathcal{V}$ and $\mathcal{U}$.

A central theme of this work is the following: Changing base via a particular G may result in more or less loss of information about the hom-objects of $\mathcal{C}$. To illustrate, two examples of G considered early in the process of the current work included

$$\mathrm{Hom}_{\mathsf{Ab}}(\mathbb{Z}, -) : \mathsf{Ab} \to \mathsf{Set} \quad \text{and} \quad \mathrm{Hom}_{\mathsf{grMod}_k}(k, -) : \mathsf{grMod}_k \to \mathsf{Set},$$

where k is a field, and gradings are taken over $\mathbb{Z}$. Letting $\mathcal{V}$ be either of Ab or grMod_k, we define the hom-objects of the Set-category $G_*\mathcal{C}$ to be

$$G_*\mathcal{C}(x, y) := G(\mathcal{C}(x, y)).$$

In the former case, the hom-sets resulting from base change are (in bijection with) the underlying sets of the original hom-objects, and the $\mathcal{U}$-topology resulting from changing the base of a $\mathcal{V}$-topology is no coarser than the one we started with. In the latter case, however, for a graded k-module $M := \mathcal{C}(x, y)$, we only recover the set

$$\mathrm{Hom}_{\mathcal{V}}(k, M) \cong \mathrm{Hom}_k(k, M_0) \cong M_0$$

of degree-preserving k-linear maps $k \to M_0$ after changing base - in this case, the $\mathcal{U}$-topology resulting from a given $\mathcal{V}$-topology is much coarser. The key difference

between these two examples lies in whether or not $\mathrm{Hom}_{\mathcal{V}}(I, -)$ is faithful; or equivalently, whether $\{I\}$ is a separating family for $\mathcal{V}$.

Below, we examine situations where this 'loss' is minimal, as in our Theorems 3.5 and 4.10, and situations where changing base results in topologies which are radically coarser than the ones we started with, as in 5.9.

1.1. **Summary of non-technical results.**

§**3**. Working in the presence of a monoidal right adjoint $G : \mathcal{V} \to \mathcal{U}$, we define the $\mathcal{U}$-sieve canonically induced by G from a $\mathcal{V}$-sieve (3.1), and prove that when G is faithful, there is an injective assignment from $\mathcal{V}$-sieves on $U \in \mathcal{C}$ to $\mathcal{U}$-sieves on $U \in G_*\mathcal{C}$ (3.5).

§**4**. We prove that the $\mathcal{V}$-Grothendieck topologies on $\mathcal{C}$ form a complete lattice (4.9), and that when G is faithful and conservative, there is an injective assignment from $\mathcal{V}$-Grothendieck topologies on $\mathcal{C}$ to $\mathcal{U}$-Grothendieck topologies on $G_*\mathcal{C}$ (4.10). As a corollary to [4, 1.5], we derive similar injectivity results for localizations and universal closure operations on $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ (4.11). Finally, we show that when G is fully faithful, change of base via G commutes with enriched sheafification in the sense of Borceux-Quinteiro (4.15).

§**5**. We examine the special case of $\mathcal{V}$-sieves and $\mathcal{V}$-topologies on a monoid object in $\mathcal{V}$. Via an example, we show that when $\mathcal{V} = \mathsf{grMod}_k$, $\mathcal{C}$ is a graded k-algebra, and $G = \mathrm{Hom}_{\mathcal{V}}(k, -)$, the injectivity results of §3 and §4 do not hold (5.9). Generalizing the notions for $\mathcal{V} = \mathsf{Ab}$ and $\mathcal{V} = \mathsf{grMod}_k$, we propose a definition for a $\mathcal{V}$-Gabriel topology (5.3), and prove that $\mathcal{V}$-Gabriel topologies on monoid objects in $\mathcal{V}$ are exactly $\mathcal{V}$-Grothendieck topologies on one-object $\mathcal{V}$-categories (5.4).

2. Preliminaries

We begin by addressing some questions of size in the categories at hand. By a **small** $\mathcal{V}$-category, we mean one which is equivalent to a $\mathcal{V}$-category with a small set of objects. $\mathcal{C}$ will always denote a small $\mathcal{V}$-category unless otherwise indicated.

For this work, we care only about those enriching categories $\mathcal{V}$ whose objects are built from 'finite' objects - for example, in the same sense that any object of Set is the union of its finite subsets, or that any object of $R\text{-}\mathsf{Mod}$, for a commutative ring R, is the filtered colimit of its finitely presented submodules. More precisely, we recall:

Definition 2.1. [1, 1.A, 1.1 and 1.9] An object x of a category $\mathcal{V}$ is called **finitely presentable** if the functor

$$\mathrm{Hom}_{\mathcal{V}}(x, -) : \mathcal{V} \to \mathsf{Set}$$

preserves filtered colimits. $\mathcal{V}$ is **locally finitely presentable** if it is cocomplete and has a set $\mathcal{V}_{fp}$ of finitely presentable objects such that every object of $\mathcal{V}$ is a directed colimit of objects from $\mathcal{V}_{fp}$.

Borceux-Quinteiro [4] and Kelly [14] require $\mathcal{V}$ to be locally finitely presentable to ensure that their results are sensibly analogous to more classical results. To ensure continuity of this work with theirs, we make the same assumptions on $\mathcal{V}$:

Remark 2.2. Unless otherwise indicated,

(i) $\mathcal{V}$ is locally finitely presentable;

(ii) $(\mathcal{V}, \otimes, I)$ is closed symmetric monoidal;

(iii) $\mathrm{Hom}_{\mathcal{V}}(I, A)$ is a small set for all objects $A \in \mathcal{V}$ (in other words, $\mathcal{V}_0$ is locally small);

(iv) $\mathcal{V}$ admits all small conical limits and colimits, or equivalently, $\mathcal{V}_0$ is bi-complete (hence $\mathcal{V}$ as an enriched category is tensored and cotensored over itself);

(v) a finite tensor product of finitely presentable objects of $\mathcal{V}$ is again finitely presentable.

Examples of categories which satisfy these conditions include

- Set, Ab, Mod_k for k a commutative ring, and the category grMod_k of $\mathbb{Z}$-graded k-modules;
- the category dgMod_k of differential graded k-modules, and by isomorphism, the category $\mathsf{Ch}_\bullet(\mathsf{Mod}_k)$ of chain complexes of k-modules;
- the category sSet of simplicial sets.

For a locally small category $\mathcal{C}$, the collection of set functions $\{\bullet\} \to \mathcal{C}(x, y)$ encodes all available information about the structure of the hom-object $\mathcal{C}(x, y)$ as a set, in the sense that anytime we have $fg = hg$ for all $g : \{\bullet\} \to \mathcal{C}(x, y)$, we know that $f = h$. In a $\mathcal{V}$-category $\mathcal{C}$, it is no longer necessarily true that having $fg = hg$ for all $g : I \to \mathcal{C}(x, y)$ implies $f = h$ (for example, in the case where $\mathcal{V} = \mathsf{grMod}_k$ for k a field), so to capture all the information we want about hom-objects in our categories, we need a more general notion:

Definition 2.3. By a **separating family** for $\mathcal{V}$, we mean a family $\mathcal{G}$ of objects of $\mathcal{V}$ such that if $fg = hg$ for any g with domain in $\mathcal{G}$, then $f = h$; or equivalently, that the family $\{\mathrm{Hom}_{\mathcal{V}}(G, -) : G \in \mathcal{G}\}$ is jointly faithful. We say that $\mathcal{G}$ is an **extremal** separating family if for each object K of $\mathcal{V}$ and each proper subobject L of K there exists a morphism $G \to K$ with $G \in \mathcal{G}$ which does not factor through L.

Note that the terminology **strong** is sometimes used in the literature where we use the word extremal (for example, in [1, 0.6]); and that in any locally finitely presentable category, the finitely presentable objects form an extremal separating family.

2.1. **Change of base.** A very detailed treatment of this topic can be found in [8, 4], but for convenience, we recount the bare rudiments here. Let

$$(\mathcal{U}, \otimes, \mathbf{1}) \quad \text{and} \quad (\mathcal{V}, \times, *)$$

be closed symmetric monoidal categories, and let $\mathcal{C}$ be a $\mathcal{V}$-category. We denote an identity morphism in an enriched category $\mathcal{X}$ by $\mathrm{id}^{\mathcal{X}}$, and a composition morphism

in $\mathcal{X}$ by $\circ^{\mathcal{X}}$. For visual simplicity, we will often omit subscripts which would ordinarily indicate the domain objects of the morphisms id and $\circ$.

We frequently refer to a special case of base change, namely the underlying category construction, in which the lax monoidal functor $\mathrm{Hom}_{\mathcal{V}}(*, -) : \mathcal{V} \to \mathsf{Set}$ is used to turn a $\mathcal{V}$-category into an ordinary one.

Definition 2.4. Given a $\mathcal{V}$-category $\mathcal{C}$, define an ordinary category $\mathcal{C}_0$ by setting $\mathrm{Ob}(\mathcal{C}_0) = \mathrm{Ob}(\mathcal{C})$ and $\mathcal{C}_0(x, y) = \mathrm{Hom}_{\mathcal{V}}(*, \mathcal{C}(x, y))$. Given morphisms $g : x \to y$ and $f : y \to z$ in $\mathcal{C}_0$, we define the composite $f \cdot g$ by

$$* \xrightarrow{\ \sim\ } * \times * \xrightarrow{\ f \times g\ } \mathcal{C}(y, z) \times \mathcal{C}(x, y) \xrightarrow{\ \circ^{\mathcal{C}}\ } \mathcal{C}(x, z) \ .$$

In light of the above, we note that having a morphism $* \to \mathcal{C}(x, y)$ in $\mathcal{V}$ no longer necessarily specifies an element of $\mathcal{C}(x, y)$ in the set-theoretic sense, and so referring to an 'arrow' in $\mathcal{C}$ is mildly nonsensical. Any diagrams in the work below should therefore be interpreted as living in the underlying category of the relevant $\mathcal{V}$-category.

In general, given a lax monoidal functor $G : \mathcal{V} \to \mathcal{U}$, we can form $\mathcal{U}$-categories, $\mathcal{U}$-functors, and $\mathcal{U}$-natural transformations in a canonical way.

Definition 2.5. Let $G : \mathcal{V} \to \mathcal{U}$ be a lax monoidal functor with coherence morphisms

$$u : \mathbf{1} \to G(*), \quad m_{xy} : G(x) \otimes G(y) \to G(x \times y).$$

(i) Form a $\mathcal{U}$-category $G_*\mathcal{C}$ by setting

$$\mathrm{Ob}(G_*\mathcal{C}) := \mathrm{Ob}(\mathcal{C}),$$
$$G_*\mathcal{C}(x, y) := G(\mathcal{C}(x, y)),$$
$$\mathrm{id}^{G_*\mathcal{C}} := G(\mathrm{id}^{\mathcal{C}}) \cdot u$$
$$\circ^{G_*\mathcal{C}} := G(\circ^{\mathcal{C}}) \cdot m.$$

(ii) For a $\mathcal{V}$-functor $A : \mathcal{C} \to \mathcal{D}$, let

$$G_*A : G_*\mathcal{C} \to G_*\mathcal{D}$$

denote the $\mathcal{U}$-functor defined by

$$G_*Ax := Ax \ \text{ and } \ (G_*A)_{xy} := GA_{xy} : G(\mathcal{C}(x, y)) \to G(\mathcal{D}(Ax, Ay)).$$

(iii) For a $\mathcal{V}$-natural transformation

$$\{\alpha_x : * \to \mathcal{D}(Ax, Bx)\},$$

let $G_*\alpha$ denote the $\mathcal{U}$-natural transformation

$$\{G(\alpha_x) \cdot u : \mathbf{1} \to G(\mathcal{D}(Ax, Bx))\}.$$

We will often be concerned with the case where the functor $G : \mathcal{U} \to \mathcal{V}$ is half of a monoidal adjunction, rather than merely lax monoidal. To give a fully rigorous definition of a monoidal adjunction, we require a few elementary notions from the theory of 2-categories, which we recall in abbreviated form below.

Definition 2.6. [25, B.1.1] A (strict) **2-category** is a Cat-category. More explicitly, a 2-category $\mathbb{C}$ consists of

- a class of objects;

- for each pair a, b of objects, a category $\mathbb{C}(a, b)$, whose objects are called **1-cells**;
- for each pair $f, g : a \to b$ of 1-cells, a collection of arrows $f \Rightarrow g$ in $\mathbb{C}(a, b)$, called **2-cells**;

such that

- the objects and 1-cells form a 1-category;
- the objects and 2-cells form a 1-category;
- the composition laws in each of these 1-categories are compatible with one another, and with the category structure on $\mathbb{C}(a, b)$ for each pair of objects a, b.

Important examples include the 2-category MonCat_ℓ of monoidal categories with lax monoidal functors, as well as $\mathcal{V}\text{-}\mathsf{Cat}$.

We omit the associated notions of 2-functors and 2-natural transformations, as knowledge of the definitions in full detail is not necessary for our discussion - the reader may consult [25, B.2.1, B.2.2], or simply think of them as Cat-enriched functors and natural transformations. The important fact is that, given a monoidal functor G as above, change of base as outlined in 2.5 defines a 2-functor

$$\mathcal{V}\text{-}\mathsf{Cat} \xrightarrow{G_*} \mathcal{U}\text{-}\mathsf{Cat} \ .$$

Moreover, we have an assignment

$$\mathsf{MonCat}_\ell \xrightarrow{(-)_*} 2\text{-}\mathsf{Cat}$$

which takes a monoidal category $\mathcal{V}$ to the 2-category $\mathcal{V}\text{-}\mathsf{Cat}$, a monoidal functor G to the 2-functor G_*, et cetera. Proof that this assignment defines a 2-functor is [8, 4.3.2].

We note here, if only for the sake of the resulting nice algebraic expression, that given $\mathcal{V}$-categories $\mathcal{X}, \mathcal{Y}$,

$$G_* : [\mathcal{X}, \mathcal{Y}] \to [G_*\mathcal{X}, G_*\mathcal{Y}]$$

itself being a 1-functor means that for composable morphisms α, β in $[\mathcal{X}, \mathcal{Y}]$, we have

$$G_*(\alpha \cdot \beta) = G_*\alpha \cdot G_*\beta,$$

where the components of the natural transformations on both the left-hand and right-hand sides of the equality are simply composites in the underlying category.

Definition 2.7. [25, B.3] An **adjunction** internal to a 2-category $\mathbb{C}$ is

- a pair of objects a, b;
- a pair of 1-cells $u : a \to b$ and $f : b \to a$, called the right and left adjoint, respectively;
- a pair of 2-cells $\eta : 1_b \Rightarrow uf$, $\varepsilon : fu \Rightarrow 1_a$, called the unit and counit of the adjunction, respectively;

satisfying the triangle identities

$$(\varepsilon \cdot f)(f \cdot \eta) = \mathrm{id}_f, \ (u \cdot \varepsilon)(\eta \cdot u) = \mathrm{id}_u$$

in the hom-categories $\mathbb{C}(b, a)$ and $\mathbb{C}(a, b)$, respectively.

With the notions above in hand, we define a **monoidal adjunction** to be an adjunction internal to the 2-category MonCat_ℓ. For the remainder of this section, we suppose given a monoidal adjunction

$$\mathcal{U} \underset{G}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathcal{V} \ . \tag{2.8}$$

The last two results we will need regarding monoidal adjunctions are the following:

Theorem 2.9.   *(i) [13, 1.4] The left adjoint of a monoidal adjunction is necessarily strong monoidal.*

    (ii) The monoidal adjunction 2.8 induces an adjunction

$$\mathcal{U}\text{-}\mathsf{Cat} \underset{G_*}{\overset{F_*}{\underset{\perp}{\rightleftarrows}}} \mathcal{V}\text{-}\mathsf{Cat}$$

    in 2-Cat *via the 2-functor* $(-)_*$ *mentioned above.*

Proof. (ii). Any 2-functor preserves adjunctions - this is [25, 2.1.3]. $\square$

2.2. $\mathcal{V}$-**limits.** We will often need to deal with enriched limits. The cases we encounter in this work are as simple as possible, in that they behave for the most part like limits in an ordinary category.

Definition 2.10. Let $* : \mathcal{D} \to \mathcal{V}_0$ be an ordinary functor constant at the monoidal unit $*$ of $\mathcal{V}$, and let $F : \mathcal{D} \to \mathcal{C}$ be a $\mathcal{V}$-functor. The **conical limit** of F, if it exists, is an object $\lim^* F$ of $\mathcal{C}$ defined by the universal property

$$\mathcal{C}(m, \lim{}^* F) \cong [\mathcal{D}, \mathcal{V}](*, \mathcal{C}(m, F(-))).$$

Though we will not need the definition of the latter object in full detail, we note for the curious that it is in fact a $\mathcal{V}$-enriched end, as defined in [24, 7.3], so the isomorphisms above are truly isomorphisms as objects of $\mathcal{V}$.

In the setting of Remark 2.2, conical limits in $\mathcal{V}$ coincide with ordinary limits in $\mathcal{V}_0$, as noted in [15, p. 50]. We note here that conical limits are a special case of the more general notion of $\mathcal{V}$-limit, defined in [15, 3] and [24, 7.4], and that they do not encompass the full theory of limits in a $\mathcal{V}$-category.

In the presence of a monoidal adjunction 2.8 and a cotensored $\mathcal{V}$-category $\mathcal{C}$, change of base makes $G_*\mathcal{C}$ cotensored over $\mathcal{U}$ as follows:

Definition 2.11. Given a cotensored $\mathcal{V}$-category $\mathcal{C}$, $G_*\mathcal{C}$ is cotensored over $\mathcal{U}$ via

$$\{u, x\} := \{Fu, x\}$$

for $u \in \mathcal{U}$ and $x \in \mathcal{C}$.

That the above object satisfies the appropriate universal property is a consequence of 2.9, (i).

2.3. **Sieves.** A Grothendieck topology on an ordinary category is made up of so-called 'sieves,' which should be thought of as admissible coverings for each object in the category. In the $\mathcal{V}$-enriched case, we start with a definition:

Definition 2.12. Let $\mathcal{C}$ be a $\mathcal{V}$-category, and let $U \in \mathcal{C}$ be an object. A **sieve** on $U \in \mathcal{C}$ is $\mathcal{V}$-subfunctor of $\mathcal{C}(-, U)$ - in other words, a $\mathcal{V}$-functor $R : \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$ admitting a $\mathcal{V}$-natural transformation

$$r : R \to \mathcal{C}(-, U)$$

whose components are monomorphisms in $\mathcal{V}_0$.

To justify this definition, we make an expository detour into the theory of sieves in an ordinary category. Initiated readers may skip the rest of this section with no detriment to understanding.

Recall that a sieve on an object U in a locally small category $\mathcal{X}$ is a family R of morphisms with codomain U such that

($\blacktriangle$) $f \in R$ implies $fg \in R$ whenever the composite fg is defined.

The algebraically-minded reader might like to think of this condition as saying that R is a 'right ideal' in $\mathrm{Mor}(\mathcal{X})$; we will see this perspective further justified below in Examples 2.17 and 5.8.

We can express a family $\{f : x_i \to U\}_{i \in I}$ of morphisms with common codomain U alternatively as a union

$$\bigcup_{i \in I} \{g \in \{f\}_{i \in I} : \mathrm{dom}(g) = x_i\},$$

where $\{g \in \{f\}_{i \in I} : \mathrm{dom}(g) = x_i\} \subset \mathcal{X}(x_i, U)$ for each i. For an object y of $\mathcal{X}$, denote

$$Ry := \{g \in \{f\}_{i \in I} : \mathrm{dom}(g) = y\},$$

where Ry may be empty for some particular y. When $\{f : x_i \to U\}_{i \in I}$ is a sieve on $U \in \mathcal{X}$, we have, for any morphism $h : x \to y$ in $\mathcal{X}$, a function $Ry \to Rx$ in Set, namely h^* (the fact that this is a function with codomain Rx and not merely $\mathcal{C}(x, U)$ follows from the condition ($\blacktriangle$)). Since $(gh)^* = h^* g^*$, the assignment $x \mapsto Rx$ is functorial. Moreover, observe that for any object z of $\mathcal{X}$, ($\blacktriangle$) is equivalent to the set-theoretic image $\{g \circ h : g \in Ry, \ h \in \mathcal{X}(z, y)\}$ of the function

$$\mathcal{X}(z, y) \times Ry \xrightarrow{(-)^* \times \mathrm{id}} \mathsf{Set}(\mathcal{X}(y, U), \mathcal{X}(z, U)) \times Ry \xrightarrow{(g^*, f) \mapsto fg} \mathcal{X}(z, U)$$

being contained in Rz, which in turn is true if and only if we have a commuting square

$$
\begin{array}{ccc}
\mathcal{X}(z, y) \times Ry & \xrightarrow{\ \ \circ\ \ } & Rz \\
{\scriptstyle (-)^* \times Ry} \downarrow & & \downarrow {\scriptstyle \mathrm{inc}} \\
\mathsf{Set}(\mathcal{X}(y, U), \mathcal{X}(z, U)) \times Ry & \xrightarrow{\ \mathrm{ev}\ } & \mathcal{X}(z, U)
\end{array}
$$

Since Set is closed monoidal, the above square commutes exactly when

$$
\begin{array}{ccc}
\mathcal{X}(z, y) & \xrightarrow{\ \ R\ \ } & \mathsf{Set}(Ry, Rz) \\
{\scriptstyle (-)^*} \downarrow & & \downarrow {\scriptstyle \mathrm{inc}_*} \\
\mathsf{Set}(\mathcal{X}(y, U), \mathcal{X}(z, U)) & \xrightarrow{\ \mathrm{inc}^*\ } & \mathsf{Set}(Ry, \mathcal{X}(z, U))
\end{array}
$$

does. Note that this latter diagram expresses naturality of the family of monomorphisms $\{\text{inc} : Rx \hookrightarrow \mathcal{C}(x, U)\}$, so we see that a sieve on $U \in \mathcal{X}$ is exactly a subfunctor of $\mathcal{C}(-, U)$.

If we systematically replace Set above by a category $\mathcal{V}$ as in 2.2, take $\mathcal{X}$ to be a small $\mathcal{V}$-category, and R a $\mathcal{V}$-functor $\mathcal{X}^{\mathrm{op}} \to \mathcal{V}$, the discussion above still makes sense: Since $\mathcal{V}_0$ is cocomplete, it has images of morphisms, and since $\mathcal{V}$ is closed monoidal, we can transpose the former commuting square above into the latter.

2.4. **Enriched Grothendieck topologies.** Lastly, we outline [4, 1.2] and a few of the notions surrounding it.

The enriched functor category $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ has all small conical limits and colimits if $\mathcal{V}$ does, as explained in [15, 3.3]. Thus $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ is cotensored over $\mathcal{V}$:

Definition 2.13. The **cotensor** $\{v, A\}$ of $A \in [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ by $v \in \mathcal{V}$ is the $\mathcal{V}$-functor whose value at $x \in \mathcal{C}$ is $\{v, Ax\} \in \mathcal{V}$, together with $\mathcal{V}$-natural isomorphisms

$$[\mathcal{C}^{\mathrm{op}}, \mathcal{V}](B, \{v, A\}) \cong \mathcal{V}(v, [\mathcal{C}^{\mathrm{op}}, \mathcal{V}](A, B)).$$

Note that for any $v \in \mathcal{V}$, a monomorphism $R \rightarrowtail \mathcal{C}(-, U)$ of $\mathcal{V}$-functors - that is, a $\mathcal{V}$-natural transformation each of whose components $Rx \to \mathcal{C}(x, U)$ is a monomorphism in $\mathcal{V}_0$ - induces, by naturality of cotensoring, a monomorphism $\{v, R\} \rightarrowtail \{v, \mathcal{C}(-, U)\}$, which we denote by ι. Moreover, the enriched Yoneda lemma [24, 7.3.5] tells us that any $f : v \to \mathcal{C}(V, U)$ induces a map $v \to \mathsf{Nat}_{\mathcal{V}}(\mathcal{C}(-, V), \mathcal{C}(-, U))$, which in turn induces a $\mathcal{V}$-natural transformation $f : \mathcal{C}(-, V) \to \{v, \mathcal{C}(-, U)\}$.

The morphisms f and ι above, along with the fact that $\mathcal{V}$ is complete, allow us to define the pullback $f^* R$ of a sieve R as follows:

Definition 2.14. The limit $f^* R$ of the diagram

$$\mathcal{C}(-, V) \xrightarrow{\ \ f\ \ } \{v, \mathcal{C}(-, U)\} \xleftarrow{\ \ \iota\ \ } \{v, R\}$$

in $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]_0$ is defined pointwise as the functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{V}$ whose value $f^* Rx$ at $x \in \mathcal{C}$ is the pullback of the diagram

$$\mathcal{C}(x, V) \xrightarrow{\ \ f_x\ \ } \{v, \mathcal{C}(x, U)\} \xleftarrow{\ \ \iota_x\ \ } \{v, R\}$$

in $\mathcal{V}_0$.

Recall that one condition for an ordinary Grothendieck topology is that covering sieves are 'pullback-stable,' meaning that the pullback of a covering sieve is itself a covering sieve [20, III.2.1]. Definition 2.14 allows us to state an enriched analogue of this condition, namely (T2) below. We denote by $\mathcal{G}$ the collection of finitely presentable objects of $\mathcal{V}$.

Definition 2.15. [4, 1.2] A $\mathcal{V}$-**Grothendieck topology** on a small $\mathcal{V}$-category $\mathcal{C}$ is, to each object $U \in \mathcal{C}$, the assignment $J(U)$ of a collection of subobjects $R \rightarrowtail \mathcal{C}(-, U)$ satisfying:

(T1) $\mathcal{C}(-, U) \in J(U)$ for each object U;

(T2) Fixing an object U, for any $R \in J(U)$ and $f : G \to \mathcal{C}(V, U)$, where $G \in \mathcal{G}$, we have $f^* R \in J(V)$.

(T3) For $S \in J(U)$ and a subobject R of $\mathcal{C}(-, U)$ such that $f^* R \in J(V)$ for any $f : G \to S(V)$, we have $R \in J(U)$.

We will sometimes say $\mathcal{V}$-**topology** to mean $\mathcal{V}$-Grothendieck topology.

A very simple example of 2.15 occurs in the case where $\mathcal{V}$ is the monoidal preorder $([0,\infty], \geq, +, 0)$.

Example 2.16. Denote the monoidal preorder $([0,\infty], \geq, +, 0)$ by Cost. As described in [11, 2.51], we can view the real numbers $\mathbb{R}$ as a Cost-category whose hom-objects are defined by

$$\mathbb{R}(x, y) := |x - y|.$$

Cost-functors are exactly (1-)Lipschitz functions, and there is a unique Cost-sieve on each $U \in \mathbb{R}$, namely the maximal sieve $\mathbb{R}(-, U)$, which sends

$$x \longmapsto |x - U|.$$

There is thus a unique Cost-Grothendieck topology on $\mathbb{R}$, namely that with

$$J(U) = \{\mathbb{R}(-, U)\}$$

for each $U \in \mathbb{R}$. (In this case, since there is a unique subobject of $\mathbb{R}(-, U)$, the 'discrete' and 'indiscrete' topologies on $\mathbb{R}$, which we describe in more detail in §4.1, coincide.)

Toward an algebraic example of 2.15, take an associative, unital, not-necessarily commutative ring A, and think of it as a one-object Ab-category.

Example 2.17. Let A be a ring and let $\mathfrak{R}$ be a non-empty set of right ideals of A. $\mathfrak{R}$ is a **(right) Gabriel topology** on A if

(R1) $I \in \mathfrak{R}$ and $I \subset J$ implies $J \in \mathfrak{R}$;
(R2) if $I \in \mathfrak{R}$ and $x \in A$, then

$$(I : x) := \{r \in A : xr \in I\} \in \mathfrak{R};$$

(R3) if I is a right ideal and there exists $J \in \mathfrak{R}$ such that $(I : x) \in \mathfrak{R}$ for every $x \in J$, then $I \in \mathfrak{R}$.

Denoting the lone object of A by $\bullet$, an Ab-sieve on $\bullet$ is a right A-submodule of A, or in other words, a right ideal of A. The pullback f^*I of 2.15, (T2) is the right ideal $(I : f)$, where the group homomorphism $f : \mathbb{Z} \to A$ is identified with the element $f(1) \in A$, so (R2) is equivalent to (T2). Moreover (R1) and (R3) are respectively equivalent to (T1) and (T3). As remarked by Lowen in [17, 2.4], we see that a Gabriel topology on A is the same thing as an Ab-Grothendieck topology on A.

In light of 2.17, we see that Definition 2.15 is a generalization of what is alternately called a Gabriel topology [26, VI.5] or topologizing filter [10, p. 520] on A, to a setting where the category A might have many objects and be enriched over some general $\mathcal{V}$. In §5, we will address $\mathcal{V}$-Grothendieck topologies on one-object $\mathcal{V}$-categories in greater detail.

3. Sieves under change of base

Below, we consider categories $\mathcal{U}$ and $\mathcal{V}$ satisfying the hypotheses in 2.2. We denote the unit objects in $\mathcal{U}, \mathcal{V}$ by $*_{\mathcal{U}}, *_{\mathcal{V}}$, the monoidal operation on both categories by $\times$, and refer to a fixed lax monoidal functor $G : \mathcal{V} \to \mathcal{U}$, whose coherence

morphisms we denote by

$$u : *_{\mathcal{U}} \to G(*_{\mathcal{V}}), \quad m_{ab} : G(a) \times G(b) \to G(a \times b).$$

For an enriched category $\mathcal{X}$ (over either $\mathcal{U}$ or $\mathcal{V}$), we will continue to denote composition in $\mathcal{X}_0$, as defined in 2.4, by $\cdot$. Toward answering the question of how base change affects $\mathcal{V}$-Grothendieck topologies, we first address the behavior of enriched sieves on objects of a $\mathcal{V}$-category $\mathcal{C}$, defined in 2.12, under the change of base induced by G.

Our main examples of interest occur when G is part of a monoidal adjunction 2.8, whose unit and counit we denote respectively by $\varepsilon : FG \to \mathbb{1}_{\mathcal{V}}$ and $\eta : \mathbb{1}_{\mathcal{U}} \to GF$, since we will require the existence of a natural family ε to make sense of a '$\mathcal{U}$-sieve induced by a $\mathcal{V}$-sieve.' In this setting, we denote the induced 2-adjunction by

$$\mathcal{U}\text{-Cat} \quad \overset{F_*}{\underset{G_*}{\rightleftarrows}} \quad \bot \quad \mathcal{V}\text{-Cat} .$$

The unenriched adjunction 2.8 induces a $\mathcal{U}$-adjunction: For $x, y \in \mathcal{U}$, we have (unenriched) natural isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{U}}(-, \mathcal{U}(x, Gy)) &\cong \mathrm{Hom}_{\mathcal{U}}(- \otimes x, Gy) \\
&\cong \mathrm{Hom}_{\mathcal{V}}(F(- \otimes x), y) \\
&\cong \mathrm{Hom}_{\mathcal{V}}(F(-) \otimes Fx, y) \\
&\cong \mathrm{Hom}_{\mathcal{V}}(F(-), \mathcal{V}(Fx, y)) \\
&\cong \mathrm{Hom}_{\mathcal{U}}(-, G(\mathcal{V}(Fx, y))),
\end{aligned}$$

whence $\mathcal{U}(x, Gy) \cong G(\mathcal{V}(Fx, y))$ as objects of $\mathcal{U}$ by Yoneda's lemma, and naturally in x and y. Denote the components (in $\mathcal{U}_0$) of this natural isomorphism by

$$\Phi_{xy} : G(\mathcal{V}(Fx, y)) \to \mathcal{U}(x, Gy).$$

Following the discussion in [15, 1.11], the family Φ corresponds uniquely to an adjunction in $\mathcal{U}$-Cat in the sense of 2.7. Moreover, the right adjoint necessarily has the following form:

Definition 3.1.   (1) The right adjoint $\mathbf{t}$ of the $\mathcal{U}$-adjunction induced by

$$(F \dashv G, \varepsilon, \eta)$$

is the $\mathcal{U}$-functor

$$\mathbf{t} : G_* \mathcal{V} \to \mathcal{U}$$

defined on objects by $\mathbf{t}(x) = Gx$ and with hom-components

$$\mathbf{t}_{xy} : G(\mathcal{V}(x, y)) \to \mathcal{U}(Gx, Gy) := \Phi_{(Gx)y} \cdot G(\varepsilon_x^*).$$

  (2) Given a $\mathcal{V}$-sieve $R \rightarrowtail \mathcal{C}(-, U)$, define

$$\widetilde{G}R := \mathbf{t} G_* R : G_* \mathcal{C}^{\mathrm{op}} \to \mathcal{U},$$

where juxtaposition denotes composition of $\mathcal{U}$-functors.

Our goal for this section is to prove that the assignment $R \mapsto \widetilde{G}R$ is injective (Theorem 3.5), for which we need a handful of technical results. The first of these, which will ensure that we can sensibly pass between $\mathcal{U}$-sieves and $\mathcal{V}$-sieves, is a

generalization of the observation made in [15, 1.3] that if $\mathrm{Hom}_{\mathcal{V}}(I, -)$ is faithful, any $\mathcal{V}$-natural transformation corresponds uniquely to an ordinary natural transformation.

Proposition 3.2. (i) Suppose $G : \mathcal{V} \to \mathcal{U}$ is faithful. For $\mathcal{V}$-functors $A, B : \mathcal{C} \to \mathcal{D}$, the family

$$\{\alpha_x : *_{\mathcal{V}} \to \mathcal{D}(Ax, Bx)\}$$

is $\mathcal{V}$-natural if and only if the family

$$\{(G_*\alpha)_x : *_{\mathcal{U}} \to G_*\mathcal{D}(Ax, Bx)\}$$

is $\mathcal{U}$-natural.

(ii) Suppose G is as in 2.8. For $\mathcal{V}$-presheaves $A, B : \mathcal{C}^{op} \to \mathcal{V}$, the family

$$\{\iota_x : *_{\mathcal{U}} \to G(\mathcal{V}(Ax, Bx))\}$$

is $\mathcal{U}$-natural if and only if the family

$$\{\Phi_{(GAx)(Bx)} \cdot G_*(\varepsilon^*)_{Ax} \cdot \iota_x : *_{\mathcal{U}} \to \mathcal{U}(GAx, GBx)\}$$

is $\mathcal{U}$-natural.

Proof. (i) Denote $\beta_x := (G_*\alpha)_x$ for brevity, and denote the left and right unitors in a monoidal category $\mathcal{X}$ by $\lambda_{\mathcal{X}}, \rho_{\mathcal{X}}$. If $\{\alpha_x\}$ is $\mathcal{V}$-natural, $\mathcal{U}$-naturality of $\{\beta_x\}$ follows from [8, 4.1.1]. Conversely, suppose $\{\beta_x\}$ is $\mathcal{U}$-natural, so that

$$
\begin{array}{ccc}
G\mathcal{C}(x,y) & \xrightarrow{\quad GA_{xy} \quad} & G\mathcal{D}(Ax, Ay) \\
{\scriptstyle GB_{xy}}\downarrow & & \downarrow{\scriptstyle (\varepsilon \times \mathrm{id})\cdot\lambda_{\mathcal{U}}^{-1}} \\
G\mathcal{D}(Bx, By) & & G(*_{\mathcal{V}}) \times G\mathcal{D}(Ax, Ay) \\
{\scriptstyle (\mathrm{id}\times\varepsilon)\cdot\rho_{\mathcal{U}}^{-1}}\downarrow & & \downarrow{\scriptstyle \eta\cdot(G\alpha_y \times \mathrm{id})} \\
G\mathcal{D}(Bx, By) \times G(*_{\mathcal{V}}) & & G(\mathcal{D}(Ay, By) \times \mathcal{D}(Ax, Ay)) \\
{\scriptstyle \eta\cdot(\mathrm{id}\times G\alpha_x)}\downarrow & & \downarrow{\scriptstyle G\circ} \\
G(\mathcal{D}(Bx, By) \times \mathcal{D}(Ax, Bx)) & \xrightarrow{\quad G\circ \quad} & G\mathcal{D}(Ax, By)
\end{array}
$$

commutes. Suppressing subscripts, naturality of η implies that

$$\eta \cdot (G\alpha \times \mathrm{id}) = G(\alpha \times \mathrm{id}) \cdot \eta,$$

so the above diagram becomes

$$
\begin{array}{ccc}
G\mathcal{C}(x,y) & \xrightarrow{\quad GA_{xy} \quad} & G\mathcal{D}(Ax, Ay) \\
{\scriptstyle GB_{xy}}\downarrow & & \downarrow{\scriptstyle \eta\cdot(\varepsilon \times \mathrm{id})\cdot\lambda_{\mathcal{U}}^{-1}} \\
G\mathcal{D}(Bx, By) & & G(*_{\mathcal{V}} \times \mathcal{D}(Ax, Ay)) \quad \cdot \\
{\scriptstyle \eta\cdot(\mathrm{id}\times\varepsilon)\cdot\rho_{\mathcal{U}}^{-1}}\downarrow & & \downarrow{\scriptstyle G(\circ\cdot(\alpha_y \times \mathrm{id}))} \\
G(\mathcal{D}(Bx, By) \times *_{\mathcal{V}}) & \xrightarrow{\ G(\circ\cdot(\mathrm{id}\times\alpha_x))\ } & G\mathcal{D}(Ax, By)
\end{array}
$$

Finally, coherence of the monoidal functor G means that we have

$$\eta \cdot (\varepsilon \times \mathrm{id}) \cdot \lambda_{\mathcal{U}}^{-1} = G\lambda_{\mathcal{V}}^{-1} \quad \text{and} \quad \eta \cdot (\mathrm{id} \times \varepsilon) \cdot \rho_{\mathcal{U}}^{-1} = G\rho_{\mathcal{V}}^{-1}.$$

Since G is faithful,

$$
\begin{array}{ccc}
\mathcal{C}(x,y) & \xrightarrow{\ \ A_{xy}\ \ } & \mathcal{D}(Ax, Ay) \\
{\scriptstyle B_{xy}}\downarrow & & \downarrow{\scriptstyle \lambda_{\mathcal{V}}^{-1}} \\
\mathcal{D}(Bx, By) & & *_{\mathcal{V}} \times \mathcal{D}(Ax, Ay) \\
{\scriptstyle \rho_{\mathcal{V}}^{-1}}\downarrow & & \downarrow{\scriptstyle \circ\cdot(\alpha_y \times \mathrm{id})} \\
\mathcal{D}(Bx, By) \times *_{\mathcal{V}} & \xrightarrow[{\scriptstyle \circ\cdot(\mathrm{id}\times\alpha_x)}]{} & \mathcal{D}(Ax, By)
\end{array}
$$

commutes, which is exactly $\mathcal{V}$-naturality of $\{\alpha_x\}$.

(ii) For visual simplicity, we omit alphanumeric subscripts. Naturality of the counit ε for $F \dashv G$ implies that the top-right square in the diagram

$$
\begin{array}{ccc}
G(\mathcal{C}(x,y)) \xrightarrow{\ G_*B\ } G(\mathcal{V}(Bx, By)) \xrightarrow{\ G_*(\varepsilon^*)\ } G(\mathcal{V}(FGBx, By)) \\
{\scriptstyle G_*A}\downarrow \qquad\qquad {\scriptstyle \iota}\downarrow \qquad\qquad\qquad {\scriptstyle \iota}\downarrow \\
G(\mathcal{V}(Ax, Ay)) \xrightarrow{\ \iota\ } G(\mathcal{V}(Ax, By)) \xrightarrow{\ G_*(\varepsilon^*)\ } G(\mathcal{V}(FGAx, By)) \\
{\scriptstyle G_*(\varepsilon^*)}\downarrow \qquad\qquad {\scriptstyle G_*(\varepsilon^*)}\downarrow \qquad\qquad \| \| \\
G(\mathcal{V}(FGAx, Ay)) \xrightarrow{\ \iota\ } G(\mathcal{V}(FGAx, By)) =\!=\!= G(\mathcal{V}(FGAx, By))
\end{array}
$$

commutes for any x, y, while commutativity of the bottom-left square follows from associativity of composition in $\mathcal{V}$. Thus commutativity of the outer square, expressing $\mathcal{U}$-naturality of $G_*(\varepsilon^*) \cdot \iota$, is equivalent to commutativity of the upper-left square, expressing $\mathcal{U}$-naturality of $G_*\iota$. Postcomposing each instance of $G_*(\varepsilon^*)$ above with the appropriate component of Φ yields squares which trivially commute (they are of the form $(\Phi \cdot \iota \cdot \Phi^{-1}) \cdot \Phi = \Phi \cdot \iota$), so commutativity of the diagram above is sufficient. $\qquad\square$

Proposition 3.2 shows that $\mathcal{V}$-naturality of $\alpha : A \to B$ is equivalent to $\mathcal{U}$-naturality of

$$
\Phi \cdot (G_*\varepsilon^*) \cdot (G_*\alpha)
$$

as long as G is faithful and a right adjoint, so we define the following:

Definition 3.3. Suppose G is faithful and satisfies 2.8. If $\alpha : A \to B$ is a $\mathcal{V}$-natural transformation between sieves $A, B \rightarrowtail \mathcal{C}(-, U)$, denote the induced $\mathcal{U}$-natural transformation $\widetilde{G}A \to \widetilde{G}B$, as in 3.2, by $\widetilde{G}\alpha$, with components

$$
(\widetilde{G}\alpha)_x := \Phi_{(GAx)(Bx)} \cdot (G_*\varepsilon^*)_{Ax} \cdot (G_*\alpha)_x : *_{\mathcal{U}} \to \mathcal{U}(GAx, GBx).
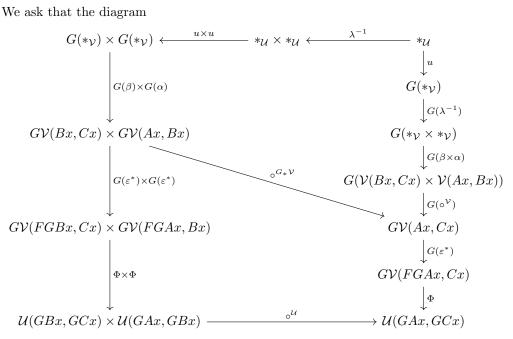$$

Referring to Definition 2.5 (iii), note that when $\alpha : A \to \mathcal{C}(-, U)$ is $\mathcal{V}$-natural and monic, the $\mathcal{U}$-natural transformation $G_*\alpha$ is not necessarily monic unless the coherence morphism $u : *_{\mathcal{V}} \to G(*_{\mathcal{U}})$ is. Thus, to ensure that $\widetilde{G}\alpha$ is monic (that is, to ensure that $\widetilde{G}A$ is a sieve on $U \in G_*\mathcal{C}$), we will often add the assumption that u is a monomorphism. Before proving the main result of this section, we check one last technicality.

Lemma 3.4. *Suppose G is faithful and satisfies 2.8. The assignment $\widetilde{G}(-)$ defined as the composite*

$$[\mathcal{C}^{op}, \mathcal{V}]_0 \xrightarrow{\ G_* \ } [G_*\mathcal{C}^{op}, G_*\mathcal{V}]_0 \xrightarrow{\ \mathbf{t}\circ - \ } [G_*\mathcal{C}^{op}, \mathcal{U}]_0$$

is (unenriched) functorial. In particular, for $A, B, C : \mathcal{C}^{op} \to \mathcal{V}$, $\alpha : A \to B$ and $\beta : B \to C$, we have $\widetilde{G}(\beta \cdot \alpha) = \widetilde{G}(\beta) \cdot \widetilde{G}(\alpha)$.

Proof. We ask that the diagram

commutes in $\mathcal{U}_0$. That the upper octagon commutes is proven in [8, 4.2.4]; that the lower hexagon commutes is $\mathcal{U}$-functoriality of $\mathbf{t}$. Similarly, unitality of $\widetilde{G}(-)$ follows from [8, 4.1.1] together with unitality of $\mathbf{t}$. □

Finally, we have the machinery to prove our main result on sieves.

Theorem 3.5. *If $G : \mathcal{V} \to \mathcal{U}$ is faithful, satisfies 2.8, and $u : *_\mathcal{U} \to G(*_\mathcal{V})$ is both a monomorphism and an epimorphism, then*

$$\mathsf{Sub}_\mathcal{V}(\mathcal{C}(-, U)) \xrightarrow{\ \widetilde{G} \ } \mathsf{Sub}_\mathcal{U}(\widetilde{G}\mathcal{C}(-, U))$$

is injective on objects.

Proof. Let $a : A \rightarrowtail \mathcal{C}(-, U)$ and $b : B \rightarrowtail \mathcal{C}(-, U)$ be such that

$$\widetilde{G}A, \widetilde{G}B : G_*\mathcal{C}^{\mathrm{op}} \to \mathcal{U}$$

represent the same subobject of $\widetilde{G}\mathcal{C}(-, U)$, and let

$$\alpha : \widetilde{G}A \to \widetilde{G}B, \ \beta : \widetilde{G}B \to \widetilde{G}A$$

be $\mathcal{U}$-natural transformations instantiating the isomorphism $\widetilde{G}A \cong \widetilde{G}B$, so that $\alpha\beta = 1_{\widetilde{G}B}$, $\beta\alpha = 1_{\widetilde{G}A}$, and $\widetilde{G}a = \widetilde{G}b \cdot \alpha$.

Since $\widetilde{G}A$ and $\widetilde{G}B$ are in the essential image of $\widetilde{G}(-)$, we know that both of α, β are in the essential image of $\widetilde{G}(-)$, whence $\alpha = \widetilde{G}\mathfrak{a}$ and $\beta = \widetilde{G}\mathfrak{b}$ for some $\mathcal{V}$-natural transformations $\mathfrak{a} : A \to B$ and $\mathfrak{b} : B \to A$. Then

$$\widetilde{G}(\mathfrak{a} \cdot \mathfrak{b}) = \widetilde{G}\mathfrak{a} \cdot \widetilde{G}\mathfrak{b} = \widetilde{G}(1_B).$$

Dropping subscripts, we expand the latter expression using Definition 3.3 to obtain

$$\Phi \cdot G_*(\varepsilon^*) \cdot G_*(\mathfrak{a} \cdot \mathfrak{b}) = \Phi \cdot G_*(\varepsilon^*) \cdot G_*(1_B)$$
$$\implies \quad G_*(\mathfrak{a} \cdot \mathfrak{b}) = G_*(1_B)$$
$$\implies \quad \mathfrak{a} \cdot \mathfrak{b} = 1_B.$$

The first implication is justified by the fact that since G is faithful, ε is an epimorphism, so that ε^* is monic; and that since G is monomorphism-preserving, $G(\varepsilon^*)$ is monic. An identical argument proves that $\mathfrak{b} \cdot \mathfrak{a} = 1_A$, so $A \cong B$ as subfunctors of $\mathcal{C}(-, U)$. $\qquad\square$

4. Enriched Grothendieck topologies under change of base

Here we prove the main theorem of this work, namely that 3.5, where we showed that change of base is injective on sieves for 'nice enough' G, extends to injectivity on Grothendieck topologies (Theorem 4.10). Below, we refer to the monoidal adjunction 2.8 of the previous sections. Since $\mathcal{U}$ and $\mathcal{V}$ are locally finitely presentable, the collections of finitely presentable objects in each category, denoted respectively by $\mathcal{G}_\mathcal{U}$ and $\mathcal{G}_\mathcal{V}$, are extremally separating. In this situation, we want to be able to say that the left adjoint F preserves extremally separating families, a property we can ensure if we impose some additional requirements on G.

Lemma 4.1. *[5, 2.2.1] The following are equivalent:*

(a) G is faithful and conservative;
(b) the family

$$\{Fx : x \in \mathcal{H}\}$$

is (extremally) separating in $\mathcal{V}$ whenever $\mathcal{H}$ is (extremally) separating in $\mathcal{U}$.

Since F is a left adjoint functor between locally finitely presentable (hence $\aleph_0$-accessible) categories, F is $\aleph_0$-accessible. By [1, 2.19], F preserves finitely presentable objects, and thus

$$\{Fx : x \in \mathcal{G}_\mathcal{U}\}$$

is an extremally separating family of finitely presentable objects in $\mathcal{V}$. For the rest of this section, we assume that G is faithful and conservative.

Before proving our main result, we require two technical lemmas, the first of which allows us to pass between conical $\mathcal{U}$-limits and conical $\mathcal{V}$-limits.

Lemma 4.2. *Let $\mathcal{C}$ be a $\mathcal{V}$-category.*

(i) G_* preserves pointwise limits in $[\mathcal{C}^{op}, \mathcal{V}]_0$
(ii) If G is conservative, then G_* reflects pointwise limits in the category $[\mathcal{C}^{op}, \mathcal{V}]_0$.

Thus, if G is conservative, $\widetilde{G} = \mathfrak{t}_* G_*$ preserves and reflects conical limits in $[\mathcal{C}^{op}, \mathcal{V}]$.

Proof. (i). Let $\mathcal{L}$ be a locally small category and $T : \mathcal{L} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]_0$ be an ordinary functor such that

$$\lim T(-)(x) := \lim_{\ell \in \mathcal{L}}(T(\ell)(x))$$

exists in $\mathcal{V}$ for every $x \in \mathcal{C}$. We have $\mathcal{V}$-natural isomorphisms

$$\mathcal{V}(v, \lim(T(-)(x))) \cong \lim \mathcal{V}(v, T(-)(x))$$

in $\mathcal{V}_0$. Since G is a right adjoint, we then have $\mathcal{U}$-natural isomorphisms

$$G\left(\mathcal{V}(v, \lim(T(-)(x)))\right) \cong G\left(\lim \mathcal{V}(v, T(-)(x))\right) \tag{4.3}$$
$$\cong \lim G\left(\mathcal{V}(v, T(-)(x))\right) \tag{4.4}$$

in $\mathcal{U}_0$. Thus $\lim T$ exists pointwise in $[G_*\mathcal{C}, G_*\mathcal{V}]_0$.

(ii). With $\mathcal{L}$ and T as above, suppose that 4.4 holds for each $x \in \mathcal{C}$, so that 4.3 holds. Since G is conservative, we have $\mathcal{V}$-natural isomorphisms

$$\mathcal{V}(v, \lim(T(-)(x))) \cong \lim \mathcal{V}(v, T(-)(x)),$$

whence $\lim T$ exists pointwise in $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]_0$.

Since conical limits in the $\mathcal{V}$-category $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ coincide with ordinary limits in the category $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]_0$ as long as $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ is tensored over $\mathcal{V}$ (as noted in [15, §3.8]), (i) implies that G_* preserves pointwise conical limits in $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$. Since $\mathbf{t}$ is a right $\mathcal{U}$-adjoint, it preserves $\mathcal{U}$-limits, and thus the composite $\widetilde{G}(-) = \mathbf{t}_*G_*$ preserves conical limits.

To see that

$$\mathbf{t}_* : [G_*\mathcal{C}^{\mathrm{op}}, G_*\mathcal{V}] \to [G_*\mathcal{C}^{\mathrm{op}}, \mathcal{U}]$$

reflects pointwise conical limits, observe that if $\mathbf{t}_*k$ is the limit of $\mathbf{t}_*T : \mathcal{L} \to [G_*\mathcal{C}^{\mathrm{op}}, G_*\mathcal{V}]_0 \to [G_*\mathcal{C}^{\mathrm{op}}, \mathcal{U}]_0$, so that

$$\mathbf{t}_*k(x) \cong \lim_{\ell \in \mathcal{L}} \mathbf{t}_*T(\ell)(x),$$

then we have

$$G(k(x)) \cong \lim_{\ell \in \mathcal{L}} G(T(\ell)(x))$$

by definition of $\mathbf{t}$. Since G preserves limits, we have

$$\lim_{\ell \in \mathcal{L}} G(T(\ell)(x)) \cong G\left(\lim_{\ell \in \mathcal{L}} T(\ell)(x)\right).$$

Since G is conservative, $k(x) \cong \lim_{\ell \in \mathcal{L}} T(\ell)(x)$. $\qquad\square$

Corollary 4.5. *Suppose G satisfies 2.8. For $y \in \mathcal{G}_\mathcal{U}$ and $R \in [\mathcal{C}^{op}, \mathcal{V}]$,*

$$\widetilde{G}\{Fy, R\} := \mathbf{t}_*G_*\{Fy, R\} = \{y, \widetilde{G}R\}.$$

Proof. Cotensors in enriched functor categories can be realized as pointwise conical limits - see [24, 7.4.3]. $\qquad\square$

To shorten the statements of the results below, we collect all of the conditions we might require G to satisfy.

Remark 4.6.   (i) G is faithful;
   (ii) G is conservative;
   (iii) G is the right adjoint of the pair 2.8;
   (iv) The coherence morphism $u : *_\mathcal{V} \to G(*_\mathcal{U})$ is a monomorphism;

(v) The coherence morphism $u : *_\mathcal{V} \to G(*_\mathcal{U})$ is an epimorphism.

Our second technical result ensures that change of base gives a well-defined assignment from $\mathcal{V}$-topologies to $\mathcal{U}$-topologies.

Proposition 4.7. *Suppose G satisfies (i)-(iv) in 4.6. For a $\mathcal{V}$-Grothendieck topology J on $\mathcal{C}$, the assignment to each object $U \in \mathcal{C}$ of the family*

$$\widetilde{G}J(U) := \{\widetilde{G}R : R \in J(U)\}$$

is a $\mathcal{U}$-Grothendieck topology, which we denote by $\widetilde{G}J$.

Proof. We verify that axioms (T1)-(T3) of Definition 2.15 hold. In light of Definition 2.14, we make heavy use of 4.2.

(T1) Immediate from the definition of $\widetilde{G}J$.

(T2) Take any $\mathcal{V}$-sieve $r : R \rightarrowtail \mathcal{C}(-, U)$, any $y \in \mathcal{G}_\mathcal{U}$, and any $a : y \to G(\mathcal{C}(V, U))$. We first show that the pullback $a^*(\widetilde{G}R)$ defined by

$$
\begin{array}{ccc}
a^*(\widetilde{G}R) & \longrightarrow & \{y, \widetilde{G}R\} \\
\downarrow & & \downarrow r \\
\widetilde{G}\mathcal{C}(-, V) & \xrightarrow{\ a\ } & \{y, \widetilde{G}\mathcal{C}(-, U)\}
\end{array}
$$

is in $\widetilde{G}J(U)$. Take the transpose $a^\flat : Fy \to \mathcal{C}(V, U)$ of a - since the adjunction $F \dashv G$ satisfies the conditions in 4.1, we have $Fy \in \mathcal{G}_\mathcal{V}$. Forming the pullback

$$
\begin{array}{ccc}
(a^\flat)^*R & \longrightarrow & \{Fy, R\} \\
\downarrow & & \downarrow r \\
\mathcal{C}(-, V) & \xrightarrow{\ b\ } & \{Fy, \mathcal{C}(-, U)\}
\end{array}
$$

in $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$, we have $(a^\flat)^*R \in J(V)$, since J is a $\mathcal{V}$-topology. Applying $\widetilde{G}$ to the diagram above, Prop. 4.2 and Corollary 4.5 imply that the resulting square

$$
\begin{array}{ccc}
\widetilde{G}((a^\flat)^*R) & \longrightarrow & \{y, \widetilde{G}R\} \\
\downarrow & & \downarrow \\
\widetilde{G}\mathcal{C}(-, V) & \longrightarrow & \{y, \widetilde{G}\mathcal{C}(-, U)\}
\end{array}
$$

is a pullback, whence $\widetilde{G}((a^\flat)^*R) \cong a^*(\widetilde{G}R)$, since they are pullbacks of the same diagram.

(T3) Suppose that $\widetilde{G}S \in \widetilde{G}J(U)$, and that $Q \rightarrowtail \widetilde{G}\mathcal{C}(-, U)$ is naturally isomorphic to a functor of the form $\widetilde{G}R$ for some $R \rightarrowtail \mathcal{C}(-, U)$ (if Q is not of this form, then manifestly $Q \notin \widetilde{G}J(U)$). We want to show that if $\widetilde{G}S$ and Q are such that

$$f^*Q \in \widetilde{G}J(V) \text{ for all } f : y \to G(S(V))$$

for any $y \in \mathcal{G}_\mathcal{U}$, then $Q \in \widetilde{G}J(U)$; for which it suffices to show that $R \in J(U)$. Given $f : y \to G(S(V))$, we have by assumption that for some

$T_f \in J(V)$, the square

$$\widetilde{G}T_f \cong f^*Q \longrightarrow \{y, \widetilde{G}R\}$$
$$\downarrow \qquad\qquad\qquad \downarrow$$
$$\widetilde{G}\mathcal{C}(-, V) \longrightarrow \{y, \widetilde{G}\mathcal{C}(-, U)\}$$

is a pullback, where the bottom arrow is induced by

$$y \xrightarrow{f} G(S(V)) \rightarrowtail G(\mathcal{C}(V, U)).$$

Since G is faithful and conservative, 4.2 implies that

$$T_f \longrightarrow \{Fy, R\}$$
$$\downarrow \qquad\qquad \downarrow$$
$$\mathcal{C}(-, V) \longrightarrow \{Fy, \mathcal{C}(-, U)\}$$

is a pullback in $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$, where the bottom arrow is induced by

$$Fy \xrightarrow{f^\flat} S(V) \rightarrowtail \mathcal{C}(V, U).$$

Thus $S \in J(U)$ and $R \rightarrowtail \mathcal{C}(-, U)$ are such that for all $f^\flat : Fy \to S(V)$, we have $(f^\flat)^*R \cong T_f \in J(V)$. Since J is a $\mathcal{V}$-Grothendieck topology on $\mathcal{C}$, we have $R \in J(U)$, whence $\widetilde{G}R \cong Q \in \widetilde{G}J(U)$.

$\square$

4.1. **Lattices of $\mathcal{V}$-Grothendieck topologies on $\mathcal{C}$.** If $\mathcal{C}$ is small, and $\mathcal{V}$ is complete and well-powered, as is true in the case where $\mathcal{V}$ satisfies 2.2, then $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ is well-powered, as proven in [6, 4.15]. It follows that the collection of $\mathcal{V}$-Grothendieck topologies on $\mathcal{C}$ is a small set, which we will denote by $\Sigma(\mathcal{C}, \mathcal{V})$.

Exactly as for ordinary topologies on a set of points, as in [16], and Grothendieck topologies on an ordinary category, as in [3, 3.2.13], $\mathcal{V}$-Grothendieck topologies form a complete lattice:

Definition 4.8. Let J, K be two $\mathcal{V}$-Grothendieck topologies on $\mathcal{C}$. K is a **refinement** of J (and J is **coarser** than K) if

$$J(U) \subseteq K(U)$$

for all objects U. Say $J = K$ if $J(U) = K(U)$ for all U.

It is routine to check that $\Sigma(\mathcal{C}, \mathcal{V})$ is partially ordered under refinement, with top element the discrete topology

$$D(U) := \mathrm{Sub}(\mathcal{C}(-, U))$$

and bottom element the indiscrete topology

$$I(U) := \{\mathcal{C}(-, U)\}.$$

Moreover, given a family $\{J_\alpha\}_{\alpha \in A} \subset \Sigma(\mathcal{C}, \mathcal{V})$, the assignment

$$S(U) := \bigcap_\alpha J_\alpha(U)$$

defines a $\mathcal{V}$-Grothendieck topology, which is easily seen to be the finest one which is coarser than any of the J_α. Using the fact that the greatest lower bound property implies the least upper bound property on a small set proves the following:

Proposition 4.9. *For $\mathcal{C}$ small and $\mathcal{V}$ satisfying 2.2, the set $\Sigma(\mathcal{C}, \mathcal{V})$ of $\mathcal{V}$-Grothendieck topologies is a complete lattice.*

Finally, we are in a position to prove our main result on Grothendieck topologies.

Theorem 4.10. *Suppose G satisfies all conditions in 4.6. The assignment*

$$\widetilde{G} : \Sigma(\mathcal{C}, \mathcal{V}) \to \Sigma(\widetilde{G}\mathcal{C}, \mathcal{U})$$

$$J \longmapsto \widetilde{G}J$$

is an injective morphism of lattices.

Proof. Monotonicity and preservation of meets follow immediately from the definition of $\widetilde{G}J$.

To prove injectivity, suppose J, K are $\mathcal{V}$-Grothendieck topologies such that $\widetilde{G}J = \widetilde{G}K$. For all U, we thus have that (i) for each $\widetilde{G}R \in \widetilde{G}J(U)$, there exists an $S \in K(U)$ such that $\widetilde{G}R = \widetilde{G}S$; (ii) for each $\widetilde{G}S \in \widetilde{G}K(U)$, there exists an $R \in J(U)$ such that $\widetilde{G}S = \widetilde{G}R$. By 3.5, (i) implies that $J(U) \subset K(U)$, and (ii) implies that $K(U) \subset J(U)$. Thus $J(U) = K(U)$ for all U, whence $J = K$. $\qquad\square$

As a consequence of the main theorem in [4], we immediately obtain the following corollary.

Corollary 4.11. *If G satisfies all conditions in 4.6, then:*

(1) *There is an injective map from the localizations of $[\mathcal{C}^{op}, \mathcal{V}]$ to the localizations of $[G_*\mathcal{C}^{op}, \mathcal{U}]$;*

(2) *There is an injective map from the universal closure operations on $[\mathcal{C}^{op}, \mathcal{V}]$ to the universal closure operations on $[G_*\mathcal{C}^{op}, \mathcal{U}]$.*

4.2. **$\mathcal{V}$-sheaves under change of base.** We make a few observations on how change of base interacts with enriched sheaves in the sense of [4]. Throughout this section, we assume $\mathcal{C}$ is a small $\mathcal{V}$-category equipped with a $\mathcal{V}$-topology J, and that $G : \mathcal{V} \to \mathcal{U}$ is faithful and lax monoidal.

Definition 4.12. [4, 1.3] A presheaf $P \in [\mathcal{C}^{op}, \mathcal{V}]$ is a **sheaf** for J when, given R and α as in

$$
\begin{array}{ccc}
R & \xrightarrow{\ \ r\ \ } & \mathcal{C}(-, U) \\
{\scriptstyle \alpha}\downarrow & \overset{\exists! \beta}{\cdots\cdots\nearrow} & \\
\{g, P\} & &
\end{array}
\quad ,
$$

with $g \in \mathcal{G}_\mathcal{V}$ and $R \in J(U)$, there exists a unique β for which the diagram commutes.

Definition 4.13. [4, 4.1, 4.4] Given a presheaf $P \in [\mathcal{C}^{op}, \mathcal{V}]$, define a new presheaf ΣP on objects by

$$\Sigma P(x) = \mathrm{colim}_{R \in J(x)}[R, P],$$

where square brackets denote the internal $\mathcal{V}$-hom in $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$. The **sheafification** or **associated sheaf** of P with respect to J is $\Sigma\Sigma P$. We will refer to the right adjoint

$$\ell : [\mathcal{C}^{\mathrm{op}}, \mathcal{V}] \longrightarrow \mathrm{Sh}_{\mathcal{V}}(\mathcal{C}, J)$$

to the inclusion functor $i : \mathrm{Sh}_{\mathcal{V}}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$, where $\ell(P) = \Sigma\Sigma P$.

A classical example is the case where $\mathcal{V} = \mathsf{Ab}$ and J is a $\mathcal{V}$-topology as in 2.17.

Example 4.14. [26, IX.1] Given a commutative ring A equipped with an Ab-topology (that is, Gabriel topology) $\mathfrak{R}$, and viewing A as a right A-module, the module

$$A_{\mathfrak{R}} := \mathrm{colim}_{I \in \mathfrak{R}} \mathrm{Hom}_A(I, A/t(A)),$$

where

$$t(A) := \{a \in A : aJ = 0 \text{ for some } J \in \mathfrak{R}\},$$

is the sheafification of A with respect to $\mathfrak{R}$. In particular, if S is a multiplicatively closed subset of A containing no zero divisors and such that for $s \in S$ and $a \in A$, there exist $t \in S$ and $b \in A$ such that $sb = at$, the family

$$\mathfrak{R} := \{I \triangleleft A : I \cap S \neq \varnothing\}$$

(where $I \triangleleft A$ means that I is an ideal of A) defines a Gabriel topology on A, and $A_{\mathfrak{R}}$ is isomorphic to the ring of fractions $A[S^{-1}]$.

Given $G : \mathcal{V} \to U$ satisfying (i)-(iv) of 4.6, we can also sheafify objects of $[G_*\mathcal{C}^{\mathrm{op}}, \mathcal{U}]$ with respect to $\widetilde{G}J$. We will use the notation

$$\ell_G \dashv i_G : \mathrm{Sh}_{\mathcal{U}}(G_*\mathcal{C}, \widetilde{G}J) \leftrightarrows [G_*\mathcal{C}, \mathcal{U}]$$

for the resulting localization, and denote the units of both adjunctions $i \dashv \ell$ and $i_G \dashv \ell_G$ by η.

It seems natural to ask whether sheafification 'commutes' with change of base, in the sense that $\widetilde{G}(i\ell P) \cong i_G\ell_G(\widetilde{G}P)$ as sheaves. We will see that in the case where G is only faithful, we at least obtain a distinguished morphism $\widetilde{G}(i\ell P) \to i_G\ell_G(\widetilde{G}P)$; but when G is also full, the isomorphism is guaranteed.

Lemma 4.15. *Let J be a $\mathcal{V}$-topology on $\mathcal{C}$ and $P \in [\mathcal{C}^{op}, \mathcal{V}]$ be a sheaf for J. If G satisfies (i)-(iv) of 4.6, and is additionally full, then $\widetilde{G}P$ is a sheaf for $\widetilde{G}J$.*

Proof. Say $P \in \mathrm{Sh}_{\mathcal{V}}(\mathcal{C}, J)$, and suppose that $\gamma : \widetilde{G}\mathcal{C}(-, U) \to \{y, \widetilde{G}P\}$ is such that $\widetilde{G}\alpha = \gamma \circ \widetilde{G}r$, so that

$$\gamma_x Gr_x = G\beta_x \circ Gr_x = G\alpha_x$$

in $\mathcal{U}_0$ for each object $x \in \mathcal{C}$. Since G is full, γ_x has the form $G\delta_x$ for some $\delta_x : \mathcal{C}(x, U) \to \{Fy, Px\}$. Since G is faithful, uniqueness of β implies that $\delta_x = \beta_x$, whence $\gamma = \widetilde{G}\beta$. $\square$

Given $S \in [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ and $r : R \rightarrowtail S$, define $\overline{R}$ to be the pullback

$$
\begin{array}{ccc}
\overline{R} & \longrightarrow & i\ell(R) \\
\downarrow & & \downarrow{\scriptstyle i\ell(r)} \quad\cdot \\
S & \xrightarrow{\;\eta_S\;} & i\ell(S)
\end{array}
$$

The operation $R \mapsto \overline{R}$ is a universal closure operation on $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ in the sense of [4, 1.4]. R is called **dense** if $\overline{R} = S$.

For visual simplicity, we define

$$\widetilde{G}(\eta_Q) := \widetilde{\eta_Q} \quad \text{and} \quad \eta_{\widetilde{Q}} := \eta_{\widetilde{G}Q}.$$

Theorem 4.16. *Suppose G satisfies (i)-(iv) of 4.6. For $P \in [\mathcal{C}^{op}, \mathcal{V}]$, the unit*

$$\eta_{\widetilde{P}} : \widetilde{G}P \to i_G \ell_G(\widetilde{G}P)$$

factors uniquely through $\widetilde{G}(i\ell P)$; and if G is full, $\widetilde{G}(i\ell P) \cong i_G \ell_G(\widetilde{G}P)$.

Proof. Since i is fully faithful, we have for any $Q \in [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ that the unit $\eta_Q : Q \to i\ell Q$ is an isomorphism. Then $i\ell(\eta_Q)$ is an isomorphism, and since isomorphisms are pullback stable, we have $\overline{Q} \cong i\ell Q$; in other words, η_Q is dense. Since $\widetilde{G}$ preserves conical limits, we have

$$\widetilde{G}\overline{Q} \cong \overline{\widetilde{G}Q} \cong \widetilde{G}(i\ell Q),$$

so that $\widetilde{\eta_Q}$ is dense.

The result [4, 2.2] says that P is (isomorphic to) a sheaf for J exactly when, for every dense monomorphism $r : R \rightarrowtail Q$ and morphism $s : R \to P$, there is a unique $t : Q \to P$ for which $r = ts$. In particular, since $i_G \ell_G(\widetilde{G}P)$ is a sheaf for $\widetilde{G}J$ and $\widetilde{\eta_P} : \widetilde{G}P \to \widetilde{G}(i\ell P)$ is dense, there is a unique morphism τ for which

$$
\begin{array}{ccc}
\widetilde{G}P & \xrightarrow{\ \eta_{\widetilde{P}}\ } & i_G \ell_G(\widetilde{G}P) \\
{\scriptstyle \widetilde{\eta_P}}\Big\downarrow & \nearrow{\scriptstyle \tau} & \\
\widetilde{G}(i\ell P) & &
\end{array}
$$

commutes. If G is full, 4.15 says that $\widetilde{G}(i\ell P)$ is a sheaf for $\widetilde{G}J$, so the same argument yields a unique factorization of $\eta_{\widetilde{P}}$ through $\widetilde{\eta_P}$, say $\sigma\widetilde{\eta_P} = \eta_{\widetilde{P}}$. Since both of the unit morphisms η are isomorphisms, $\sigma\tau$ and $\tau\sigma$ are identities, and we have $\widetilde{G}(i\ell P) \cong i_G \ell_G(\widetilde{G}P)$. $\qquad\square$

5. Gabriel topologies

Our goal in this section is to illustrate via an example (namely 5.9) that 4.10 does not always hold. Toward that end, we generalize Definition 2.17 of a Gabriel topology on a ring - that is, on a monoid object in Ab - to monoid objects in an arbitrary $\mathcal{V}$ satisfying 2.2.

Perhaps among the easiest $\mathcal{V}$-categories to understand are one-object $\mathcal{V}$-categories, which are easily seen to coincide with the monoid objects in $\mathcal{V}$ - that is to say, those objects A of $\mathcal{V}$ equipped with suitably coherent morphisms $m : A \times A \to A$ and $u : *_\mathcal{V} \to A$. We can use any such A to define a **right A-module** in $\mathcal{V}$ - an object M of $\mathcal{V}$ equipped with a morphism

$$\psi : A^{\mathrm{op}} \times M \to M,$$

called a **right A-action** on M, satisfying coherence conditions encoding associativity and unitality of the action. (For brevity, we do not discuss the coherence of these morphisms in detail; the uninitiated reader may consult [19, VII.3-4].) In particular, a monoid object (A, m, u) of $\mathcal{V}$ is always a right module over itself. To

emphasize that we are viewing A as a right A-module, we will sometimes use the notation A_A. By an A-**submodule** of M, we mean an A-module N admitting a monomorphism $\iota : N \rightarrowtail M$, and whose A-action is compatible with that of M in a sense that we will make precise below.

When $\mathcal{V}$ is closed monoidal, as in the present setting, we can 'transpose' a right action and its requisite coherence diagrams, obtaining a morphism

$$\varphi : A^{\mathrm{op}} \to \mathcal{V}(M, M)$$

in $\mathcal{V}_0$ which satisfies conditions encoding compatibility of the monoidal structure on A^{op} with the composition and identities in $\mathcal{V}$. If we shift our perspective and view A^{op} as a $\mathcal{V}$-category with a single object $\bullet$, the coherence of φ expresses $\mathcal{V}$-functoriality of the assignment $\bullet \to M$. From this perspective, $\mathcal{V}$-sieves have straightforward descriptions in terms of subobjects of A.

Proposition 5.1. *If $\mathcal{V}$ is closed monoidal and $\mathcal{A}$ is a one-object $\mathcal{V}$-category with $\mathcal{A}(\bullet, \bullet) = A \in Mon(\mathcal{V})$, a $\mathcal{V}$-sieve on $\bullet$ - that is, a subfunctor of $\mathcal{A}(-, \bullet) : \mathcal{A}^{op} \to \mathcal{V}$ - is the same thing as an A-submodule of A_A.*

Proof. We unpack the definition of a subfunctor $\mathcal{I}(-)$ of $\mathcal{A}(-, \bullet) : \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$. Say $\mathcal{I}(-) : \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ sends $\bullet \mapsto I$, and let $\varphi : \mathcal{A}^{\mathrm{op}}(\bullet, \bullet) = A^{\mathrm{op}} \to \mathcal{V}(I, I)$ be the hom-component of $\mathcal{I}(-)$. Functoriality of $\mathcal{I}(-)$ says that the diagrams

$$
\begin{array}{ccc}
A^{\mathrm{op}} \times A^{\mathrm{op}} & \xrightarrow{\;\;m\;\;} & A^{\mathrm{op}} \\
{\scriptstyle \varphi \times \varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\
\mathcal{V}(I, I) \times \mathcal{V}(I, I) & \xrightarrow[\;\;\circ\;\;]{} & \mathcal{V}(I, I)
\end{array}
\qquad\qquad
\begin{array}{ccc}
*_{\mathcal{V}} & \xrightarrow{\;\;u\;\;} & A^{\mathrm{op}} \\
& {\scriptstyle \mathrm{id}}\searrow & \downarrow{\scriptstyle \varphi} \\
& & \mathcal{V}(I, I)
\end{array}
$$

commute. Denoting the transpose of φ by $\psi : A^{\mathrm{op}} \times I \to I$, commutativity of the diagrams above is equivalent to commutativity of

$$
\begin{array}{ccc}
A^{\mathrm{op}} \times I & \xrightarrow{\;\;\;\psi\;\;\;} & I \\
{\scriptstyle m \times \mathrm{id}}\uparrow & & \| \\
(A^{\mathrm{op}} \times A^{\mathrm{op}}) \times I & \xrightarrow{\;\;h\;\;} & I \\
{\scriptstyle (\psi \times \psi) \times \mathrm{id}}\downarrow & & \| \\
(\mathcal{V}(I,I) \times \mathcal{V}(I,I)) \times I & \xrightarrow[\;\;\circ^{\flat}\;\;]{} & I
\end{array}
\qquad
\begin{array}{ccc}
*_{\mathcal{V}} \times I & \xrightarrow{u \times \mathrm{id}} & A^{\mathrm{op}} \times I \\
& {\scriptstyle \lambda^{-1}}\searrow & \downarrow{\scriptstyle \psi} \\
& & I
\end{array}\;,
$$

where $h = \psi(1 \times \psi)\alpha$, and with α and λ respectively denoting the associator and left-unitor in $\mathcal{V}$. Commutativity of the top square in the left-hand diagram above is equivalent to associativity of ψ as a right action of A on I, and the triangle is equivalent to unitality. We see that I is a right A-module.

Having a $\mathcal{V}$-natural transformation $\iota : \mathcal{I}(-) \Rightarrow \mathcal{A}(-, \bullet)$ with monic components says that we have a monomorphism $I \rightarrowtail A$ in $\mathcal{V}_0$ which satisfies

$$
\begin{array}{ccc}
A^{\mathrm{op}} & \xrightarrow{\;\;\varphi\;\;} & \mathcal{V}(I, I) \\
{\scriptstyle m^{\flat}}\downarrow & & \downarrow{\scriptstyle \iota_*} \\
\mathcal{V}(A, A) & \xrightarrow[\;\;\iota^*\;\;]{} & \mathcal{V}(I, A)
\end{array}\;,
$$

expressing compatibility of the right A-action on I with the right A-action of A on itself.

In the converse direction, say given a right A-submodule I of A_A, it is easy to check (by showing that commutativity is satisfied in the diagrams above) that $\bullet \mapsto I$ determines a $\mathcal{V}$-subfunctor of $\mathcal{A}(-, \bullet)$. $\hfill\square$

Pullbacks of sieves on $\bullet \in \mathcal{A}$, as in 2.15 (T2), are somewhat simpler to describe than in the general case. Given $f : G \to \mathcal{A}(\bullet, \bullet) = A$, f induces a morphism

$$G \to [\mathcal{A}^{\mathrm{op}}, \mathcal{V}](\mathcal{A}(-, \bullet), \mathcal{A}(-, \bullet))$$

by the enriched Yoneda lemma, and thus a morphism

$$\mathcal{A}(-, \bullet) \to \{G, \mathcal{A}(-, \bullet)\}.$$

Let $\iota : \mathcal{I}(-) \rightarrowtail \mathcal{A}(-, \bullet)$. Since $\mathcal{A}$ has only one object, the pullback of the diagram

$$\mathcal{A}(-, \bullet) \xrightarrow{\quad f \quad} \{G, \mathcal{A}(-, \bullet)\} \xleftarrow{\quad \iota \quad} \{G, \mathcal{I}(-)\}$$

in $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]_0$ is uniquely determined by the pullback

$$A \xrightarrow{\quad f \quad} \mathcal{V}(G, A) \xleftarrow{\quad \iota \quad} \mathcal{V}(G, I) \qquad (5.2)$$

in $\mathcal{V}_0$. In the case where $\mathcal{A}$ has only one object, we identify the pullback f^*I in the functor category with the pullback of the diagram 5.2 in $\mathcal{V}_0$.

In light of the discussion above, we see that 2.17 is the case $\mathcal{V} = \mathsf{Ab}$ of the following:

Definition 5.3. Given a monoid object A of $\mathcal{V}$, a **(right) $\mathcal{V}$-Gabriel topology** on A is a non-empty family $\mathfrak{R}$ of right A-submodules of A_A such that

(V1) if $I \in \mathfrak{R}$ and J is a right A-submodule of A_A such that I is a right A-submodule of J, then $J \in \mathfrak{R}$;

(V2) for any $(\iota : I \rightarrowtail A) \in \mathfrak{R}$, $G \in \mathcal{G}_{\mathcal{V}}$, and $f : G \to A$ in $\mathcal{V}_0$, the pullback f^*I of the diagram 5.2 is in $\mathfrak{R}$;

(V3) if $I \in \mathfrak{R}$ and J is a right A-submodule of A_A such that $f^*J \in \mathfrak{R}$ for all $f : G \to I$, then $J \in \mathfrak{R}$.

Squinting at 5.3, the reader might guess that the following is true, although it may not be obviously apparent that (V1) is a perfect analogue of (T1) in 2.15. We provide a bit more detail:

Proposition 5.4. *Let $A \in Mon(\mathcal{V})$, and let $\mathfrak{R}$ be a set of right A-submodules of A_A. Denote by $\mathcal{A}$ the one-object $\mathcal{V}$-category with $\mathcal{A}(\bullet, \bullet) = A$. Given a right A-submodule $I \rightarrowtail A$, denote the $\mathcal{V}$-subfunctor $\bullet \mapsto I$ of $\mathcal{A}(-, \bullet)$ by $\mathcal{I}(-)$. The following are equivalent:*

(i) $\mathfrak{R}$ is a $\mathcal{V}$-Gabriel topology on A;

(ii) $\mathcal{T} := \{\mathcal{I}(-) : I \in \mathfrak{R}\}$ is a $\mathcal{V}$-topology on $\mathcal{A}$.

Proof. That (T2) and (T3) are respectively equivalent to (V2) and (V3) follows directly from the definitions 5.1 and 5.2. Moreover if (V1) holds for $\mathfrak{R}$, the fact that $\mathfrak{R}$ is nonempty immediately implies (T1).

The only subtlety is in proving that (V1) holds for $\mathfrak{R}$, given (ii). Following [3, 3.2.5], suppose that $I \in \mathfrak{R}$ is such that $\iota : I \to A$ factors as

$$I \xrightarrowtail{i} J \xrightarrowtail{j} A$$

for some A-submodule J of A_A. If $f : G \to I$ has $G \in \mathcal{G}_{\mathcal{V}}$, then ιf and jif induce the same morphism $A \to \mathcal{V}(G, A)$, so that the pullback $f^*\mathcal{J}$ of

$$\mathcal{A}(-, \bullet) \xrightarrow{\iota f = jif} \{G, \mathcal{A}(-, \bullet)\} \xleftarrow{j_*} \{G, \mathcal{J}(-)\}$$

is $\mathcal{A}(-, \bullet) \in \mathcal{T}$. Since $\mathcal{T}$ is a $\mathcal{V}$-topology, we have $\mathcal{J}(-) \in \mathcal{T}$, so that $J \in \mathfrak{R}$. $\square$

5.1. **Graded Gabriel topologies on a graded algebra.** We turn to an example of categories $\mathcal{U}, \mathcal{V}$, a $\mathcal{V}$-category $\mathcal{C}$, and a functor $G : \mathcal{V} \to \mathcal{U}$ where the injectivity results of sections 3 and 4 do not hold. For the rest of this section, we consider a field k, and set $\mathcal{V} = \mathsf{grMod}_k$, the category of $\mathbb{Z}$-graded k-modules. Recall that the monoidal unit in $\mathcal{V}$ is k, viewed as a $\mathbb{Z}$-graded algebra concentrated in degree 0, and the internal hom in $\mathcal{V}$ is defined as

$$\mathcal{V}(M, N) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_i(M, N),$$

where $\mathrm{Hom}_i(M, N)$ denotes the collection of k-module homomorphisms f for which $f(M_j) \subset N_{j+i}$, which we call **morphisms of degree** i. Uninitiated readers can find a detailed treatment of graded algebras in [23] or [22].

The functor

$$\mathrm{Hom}_{\mathcal{V}}(k, -) : \mathcal{V} \to \mathsf{Set}$$

has a left adjoint $k[-]$ in Cat which takes a set X to the free graded k-module $k[X]$ generated in degree 0 by the elements of X. The functor $\mathrm{Hom}_{\mathcal{V}}(k, -)$ is lax and the functor $k[-]$ is strong monoidal, so by [13, 1.5], they comprise an adjunction in $\mathbf{MonCat}_\ell$. We will see that $k[-] \dashv \mathrm{Hom}_{\mathcal{V}}(k, -)$ yields an example where the assignment $\widetilde{G}$ of 4.10 is not injective.

Example 5.5. $\mathrm{Hom}_{\mathcal{V}}(k, -) : \mathcal{V} \to \mathsf{Set}$ is not faithful - to see this, take any two distinct graded k-modules, say M and N, with $M_0 = N_0 = 0$, and recall that

$$\mathrm{Hom}_{\mathcal{V}}(k, M) \cong \mathrm{Hom}_k(k, M_0) \cong \{0\}$$

(and similarly for N). As long as there exists a non-trivial graded module homomorphism $M \to N$, for example, in the case of M and N with homogeneous components defined by

$$M_i = \begin{cases} 0 & i < 2 \\ k & i \geq 2 \end{cases}, \quad N_i = \begin{cases} 0 & i < 1 \\ k & i \geq 1 \end{cases},$$

the map

$$\mathcal{V}(M, N) \to \mathsf{Set}(\mathrm{Hom}_{\mathcal{V}}(k, M), \mathrm{Hom}_{\mathcal{V}}(k, N)) \cong \{0\}$$

is not injective.

Below, we construct an example of two $\mathcal{V}$-topologies which correspond to the same Set-topology under change of base, toward which our first task is to describe $\mathcal{V}$-sieves and their pullbacks. As a corollary to 5.1, we have the following:

Corollary 5.6. *Given $A \in \mathsf{grAlg}_k$, viewed as a grMod_k-category with one object $\bullet$, the $\mathcal{V}$-sieves on $\bullet$ are exactly the homogeneous right ideals of A.*

As described in [23, p. 21], $\mathcal{V}$ admits a separating family: For $i \in \mathbb{Z}$, define the homogeneous components of a graded k-module $k(-i)$ by

$$k(-i)_j := k_{j-i},$$

so that

$$k(-i)_i = k_0 = k,$$

and $k(-i)_j = 0$ otherwise.

Note that any graded k-module is the filtered colimit of its finite-dimensional graded subspaces, and any finite-dimensional graded k-module is the direct sum of the objects in $\{k(-j)\}_{j \in J}$ for some $J \subset \mathbb{Z}$. Thus, to construct the pullback as in 5.3 (V2), we need only consider pullbacks along graded module morphisms $f : k(-i) \to A$, where we identify f with $f(1_k) \in A_i$. Denote the set of homogeneous elements of A by

$$h(A) := \bigcup_{i \in \mathbb{Z}} A_i.$$

Definition 5.7. Given a morphism $f : k(-i) \to A$ of graded k-modules and a homogeneous right ideal $I \subset A$, the pullback of the diagram

$$A \xrightarrow{\;\;f\;\;} \mathcal{V}(k(-i), A) \xleftarrow{\;\;\text{inc}\;\;} \mathcal{V}(k(-i), I) \ ,$$

where f is identified with the map $1_A \mapsto f(1_k)$, is the homogeneous right ideal $(I : f(1_k))$.

With 5.6 and 5.7 in hand, we can define an analogue of 2.17 for a graded k-algebra A, as in [22].

Definition 5.8. A **graded (right) Gabriel topology** on A is a non-empty set $\mathfrak{R}$ of homogeneous right ideals of A satisfying

(G1) if $I \in \mathfrak{R}$ and J is a homogeneous right ideal of A for which $I \subset J$, then $J \in \mathfrak{R}$;

(G2) if $I \in \mathfrak{R}$, then $(I : x) \in \mathfrak{R}$ for all $x \in h(A)$;

(G3) if $I \in \mathfrak{R}$ and J is a homogeneous right ideal of A such that $(J : x) \in \mathfrak{R}$ for all $x \in h(I)$, then $J \in \mathfrak{R}$.

Given a graded algebra A, any multiplicatively closed set S of homogeneous elements of A gives rise to a graded Gabriel topology by letting H_S be the collection of homogeneous right ideals defined by

$$H_S := \{I \mid (I : a) \cap S \neq \varnothing \text{ for all homogeneous elements } a \in A\},$$

as in [22, II.9.11].

Example 5.9. For a field k, take A to be the commutative ring $k[x, y]$, graded by polynomial degree. Set

$$S := \{1, x, x^2, ...\} \text{ and } T := \{1, y, y^2, ...\},$$

and consider the change of base given by

$$G = \mathrm{Hom}_{\mathcal{V}}(k, -).$$

The families S and T generate distinct $\mathcal{V}$-Grothendieck topologies on A, namely

$$H_S = \{I \lhd A : x^n \in I \text{ for some } n\} \text{ and } H_T = \{I \lhd A : y^n \in I \text{ for some } n\},$$

where the notation $I \lhd A$ means I is an ideal of A. Given any $M \in \mathcal{V}$, we have

$$\mathrm{Hom}_\mathcal{V}(k, M) \cong \mathrm{Hom}_k(k, M_0) \cong M_0,$$

so in particular, we have $\widetilde{G}I \cong \mathrm{Hom}_k(k, I_0)$ for any I in H_S or H_T. Recall that the degree-0 elements of A are exactly the scalars k; thus, if $I \neq A$, we have $I_0 = \{0\}$ (otherwise I contains a unit of A), and if $I = A$, we have $I_0 = k$. Then

$$\widetilde{G}H_S \cong \widetilde{G}H_T \cong \{k, \{0\}\}.$$

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