Section 6.1: Volume

1. A region R is bounded by $y = e^x$ and $y = 0$ on the interval $[-1, 1]$. Set up the integral to find the volume V of the solid formed by rotating R around the x-axis and then find the volume.

Solution:

$$V = \pi \int_{-1}^{1} (e^x)^2 \, dx = \pi \int_{-1}^{1} e^{2x} dx = \pi \left[\frac{1}{2} e^{2x} \right]_{-1}^{1} = \frac{\pi}{2} \left(e^2 - e^{-2} \right)$$

2. Let R be the region between the graphs of $f(x) = x + 1$ and $g(x) = x - 1$ on the interval $[1, 4]$. Find the volume V of the solid obtained by revolving R about the x axis.

Solution:

$$V = \int_{1}^{4} \pi \left[(x+1)^2 - (x-1)^2 \right] dx = \pi \int_{1}^{4} 4x dx = 2\pi x^2 \Big|_1^4 = 30\pi$$

3. A region R is bounded by $y = \sqrt{x}$ and $y = x^4$. Set up the integral to find the volume V of the solid formed by rotating R around the x-axis and then find the volume.

Solution:

$$V = \pi \int_{0}^{1} \left((\sqrt{x})^2 - \left(x^4\right)^2 \right) dx = \pi \int_{0}^{1} \left(x - x^8 \right) dx = \pi \left[\frac{x^2}{2} - \frac{x^9}{9} \right]_0^1 = \pi \left(\frac{1}{2} - \frac{1}{9} \right) = \frac{7\pi}{18}$$

4. Let R be the region between the graph of $f(x) = \sqrt{x+2}$ and $g(x) = \sqrt{x-2}$ on the interval $[2, 6]$. Set up an appropriate integral to find the volume V of the solid obtained by revolving R about the x axis and find the volume (exact value).

Solution:

$$V = \pi \int_{2}^{6} \left[(\sqrt{x+2})^2 - (\sqrt{x-2})^2 \right] dx = \pi \int_{2}^{6} 4 dx = \pi \ (4x)|_2^6 = \pi(24 - 8) = 16\pi$$

5. Let $f(x) = x^2$ and $g(x) = 3x$, and let R be the bounded region in the xy plane between the graphs of f and g. Set up the integral for the volume V of the solid generated by revolving R around the line $y = -1$. You do **not** have the evaluate the integral.

Solution:

$$V = \int_1^2 \pi\left(\sqrt{1+y^3}\right)^2 dy = \pi \int_1^2 \left(1+y^3\right) dy = \pi\left(y + \frac{1}{4}y^4\right)\Bigg|_1^2 = \frac{19}{4}\pi$$

6. Let R be the region between the graph of $y^2 = x$ and $x = 2y$. Find the volume V of the solid obtained by revolving R about the y axis.

Solution:

$$V = \pi \int_0^2 \left[(2y)^2 - \left(y^2\right)^2\right] dy = \pi \int_0^2 \left(4y^2 - y^4\right) dy = \pi\left[\frac{4}{3}y^3 - \frac{1}{5}y^5\right]_0^2 = \pi\left(\frac{32}{3} - \frac{32}{5}\right)$$
$$= \frac{64}{15}\pi$$

7. The region bounded by the curves $y = x^2 - 2x$ and $y = 3x$ is revolved about the line $y = -1$. Find the volume of the resulting solid.

Solution: The curves intersect when $x^2 - 2x = 3x$, which is at $x = 0$ and $x = 5$. At a test point $x = 1$ in the interval $[0, 5]$, we see that $(1)^2 - 2(1) = -1$ and $3(1) = 3$, so $3x \geq x^2 - 2x$ on this interval. Then

$$V = \int_0^5 \pi\left[(3x+1)^2 - (x^2 - 2x + 1)^2\right] dx = \pi \int_0^5 (3x+1)^2 - \left((x-1)^2\right)^2 dx$$
$$= \pi \int_0^5 (3x+1)^2 - (x-1)^4 dx = \pi\left(\frac{(3x+1)^3}{9} - \frac{(x-1)^5}{5}\right)\Bigg|_0^5 = 250\pi.$$

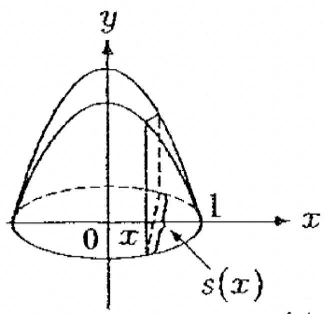
8. The region bounded by the curves $y = x^3, y = 1$, and $x = 2$ is revolved about the line $y = -3$. Find the volume of the resulting solid.

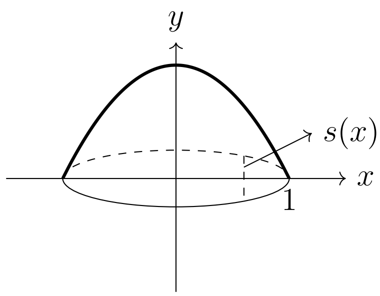
Solution: A cross-section is a washer with inner radius $1 - (-3) = 4$ and outer radius $x^3 - (-3) = x^3 + 3$, so its area is $A(x) = \pi\left(x^3 + 3\right)^2 - \pi(4)^2 = \pi\left(x^6 + 6x^3 - 7\right)$

$$V = \int_1^2 A(x)dx = \int_1^2 \pi\left(x^6 + 6x^3 - 7\right) dx = \pi\left[\frac{1}{7}x^7 + \frac{3}{2}x^4 - 7x\right]_1^2$$
$$= \pi\left[\left(\frac{128}{7} + 24 - 14\right) - \left(\frac{1}{7} + \frac{3}{2} - 7\right)\right] = \frac{471\pi}{14}$$

9. Find the volume V of the solid that has a circular base with radius 1, and the cross sections perpendicular to a fixed diameter of the base are squares.

Solution: Let a square cross-section x units from the center have a side $s(x)$ units long. Note that $s(x) = 2y$ and the circle is $x^2 + y^2 = 1$. Solving for y we have $y = \sqrt{1 - x^2}$. Together we have $s(x) = 2\sqrt{1 - x^2}$.





Thus the cross-sectional area is given by $A(x) = (s(x))^2 = 4\left(1 - x^2\right)$, so

$$V = \int_{-1}^{1} 4\left(1 - x^2\right) dx = \left(4x - \frac{4}{3}x^3\right)\Big|_{-1}^{1} = \frac{16}{3}$$

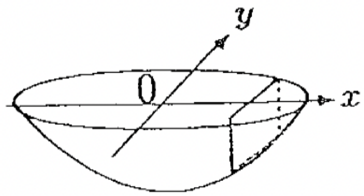
.

10. As viewed from above, a swimming pool has the shape of the ellipse

$$\frac{x^2}{400} + \frac{y^2}{100} = 1$$

The cross sections of the pool perpendicular to the ground and parallel to the y axis are squares. If the units are feet, determine the volume V of the pool.

Solution: Let a square cross-section x feet from the center have a side $s(x)$ feet long. Note that $s(x) = 2y$ and the ellipse is $\dfrac{x^2}{400} + \dfrac{y^2}{100} = 1$. Solving for y we have $y = \sqrt{100 - x^2/4}$. Together we have $s(x) = 2\sqrt{100 - x^2/4}$.



Thus the cross-sectional area is $A(x) = (s(x))^2 = 4\left(100 - x^2/4\right) = 400 - x^2$, so that

$$V = \int_{-20}^{20} \left(400 - x^2\right) dx = \left(400x - \frac{1}{3}x^3\right)\Big|_{-20}^{20} = \frac{32,000}{3}\text{(cubic feet).}$$

11. The base of a solid is the semicircle $y = \sqrt{9 - x^2}$ centered at the origin with a radius of 3 so that the diameter of the semicircle runs along the x-axis. The cross sections are isosceles right triangles with one leg perpendicular to the diameter and other leg extending vertically upward from the semicircle. Find the volume V of the solid.

Solution: Note that $A(x) = \frac{1}{2} \times \text{leg} \times \text{leg} = \frac{1}{2} \times \sqrt{9 - x^2} \times \sqrt{9 - x^2} = \frac{1}{2}\left(9 - x^2\right)$

$$V = \frac{1}{2}\int_{-3}^{3} \left(9 - x^2\right) dx = \frac{1}{2}\left[9x - \frac{x^3}{3}\right]_{-3}^{3} = \frac{1}{2}(54 - 18) = 18$$

12. Find the volume of the described solid S where the base of S is the triangular region with vertices $(0,0)$, $(1,0)$, and $(0,1)$ and the cross-sections perpendicular to the x-axis are squares.

Solution: This triangle lies in the first quadrant, with its hypotenuse being the line from $(0,1)$ to $(1,0)$. Note that the line connecting those two points has a slope of -1, so its equation is $y = 1 - x$.

$$V = \int_0^1 (1 - x)^2 dx = \left[x - x^2 + \frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

Section 6.4: Work

1. A 10-pound bag of groceries is to be carried up a flight of stairs 8 feet tall. Find the work W done on the bag.

Solution: $f(x) = 10$ so

$$W = \int_0^8 10\,dx = 10x|_0^8 = 80 \text{ (foot-pounds)}$$

2. When a certain spring is expanded 10 centimeters from its natural position and held fixed, the force necessary to hold it is 4×10^6 dynes. Find the work W required to stretch the spring an additional 10 centimeters.

Solution: $k(10) = 4 \times 10^6$, so that $k = 4 \times 10^5$. Thus

$$W = \int_{10}^{20} 4\left(10^5\right) x\,dx = 2\left(10^5\right) x^2\Big|_{10}^{20} = 6 \times 10^7 \text{(ergs)}.$$

3. If 6×10^7 ergs of work are required to compress a spring from its natural length of 10 centimeters to a length of 5 centimeters, find the work W necessary to stretch the spring from its natural length to a length of 12 centimeters.

Solution: $6 \times 10^7 = W = \int_0^{-5} kx\,dx$. Since $\int_0^{-5} kx\,dx = \frac{1}{2}kx^2\Big|_0^{-5} = \frac{25}{2}k$, it follows that $k = \frac{12}{25} \times 10^7$. Thus the work necessary to stretch the spring 2 centimeters is given by

$$W = \int_0^2 \left(\frac{12}{25} \times 10^7\right) x\,dx = \left(\frac{12}{25} \times 10^7\right)\left(\frac{1}{2}x^2\right)\Big|_0^2 = \frac{24}{25} \times 10^7 = 9.6 \times 10^6 \text{ (ergs)}$$

4. A bottle of wine has a cork 5 centimeters long. Uncorking the bottle exerts a force to overcome the friction force between the cork and the bottle. The applied force in dynes is given by $F(x) = 2 \times 10^6(5 - x)$ for $0 \le x \le 5$ where x represents the length in centimeters of the cork extending from the bottle. Determine the work W done in removing the cork.

Solution:

$$W = \int_0^5 2\left(10^6\right)(5-x)dx = 2\left(10^6\right)\left(5x - \frac{1}{2}x^2\right)\Big|_0^5 = 2\left(10^6\right)(25-12.5) = 2.5 \times 10^7 \text{(ergs)}$$

5. A swimming pool has the shape of a right circular cylinder with radius 28 feet and height 10 feet. Suppose that the pool is full of water weighing 62.5 pounds per cubic foot. Find the work W required to pump all the water to a platform 2 feet above the top of the pool.

Solution: The integral is taken from $y = 0$ (the bottom of the pool) to $y = 10$ (the top of the pool):

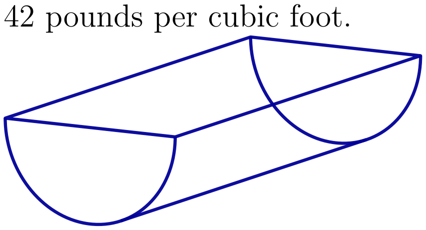
$$W = \int_0^{10} 62.5(12 - y)28^2\pi dy = 62.5\pi(28^2)\int_0^{10}(12 - y)dy = 62.5\pi(28^2)\left[12y - \frac{y^2}{2}\right]_0^{10}$$
$$= 62.5\pi(28^2)70 = 3,430,000\pi\text{ft-pounds}$$

6. A tank in the shape of an inverted cone 12 feet tall and 3 feet in radius is full of water. Calculate the work W required to pump all the water over the edge of the tank.

Solution: We position the origin at the vertex of the cone. Then $l = 12$, and a particle of water x feet from the vertex is to be raised $12 - x$ feet. Moreover, the water to be pumped extends from 0 to 12 on the x axis. By similar triangles, we have $\frac{r}{x} = \frac{3}{12} = \frac{1}{4}$, so that $r = x/4$ and hence $A(x) = \pi x^2/16$.

$$W = \int_0^{12} 62.5(12 - x)\frac{\pi}{16}x^2 dx = \frac{62.5\pi}{16}\int_0^{12}\left(12x^2 - x^3\right)dx$$
$$= \frac{62.5\pi}{16}\left(4x^3 - \frac{1}{4}x^4\right)\Bigg|_0^{12} = 6750\pi \text{ (foot-pounds)}$$

7. Suppose a large gasoline tank has the shape of a half cylinder 8 feet in diameter and 10 feet long. If the tank is full, set-up the integral to find the work W necessary to pump all the gasoline to the top of the tank. Do NOT evaluate the integral. Assume the gasoline weighs 42 pounds per cubic foot.



Solution: If the tank is positioned with respect to the x-axis with the origin at the top of the tank, then $l = 0$ and the gasoline to be pumped extends from -4 to 0 on the x-axis. Moreover, the width $w(x)$ of the cross-section at x is given by $w(x) = 2\sqrt{16 - x^2}$, so $A(x) = 10w(x) = 20\sqrt{16 - x^2}$.

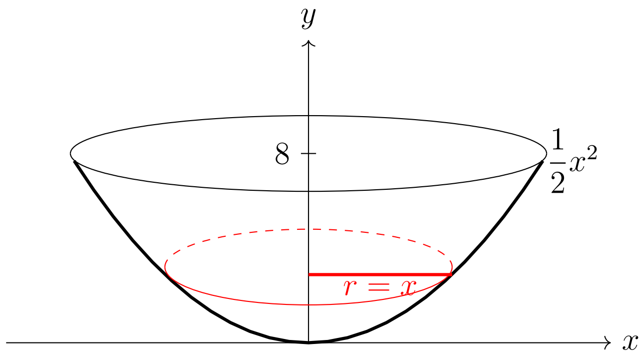
$$W = \int_{-4}^{0} 42(0 - x)20\sqrt{16 - x^2}\,dx$$

8. A tank has the shape of the surface generated by revolving the parabolic segment $y = \frac{1}{2}x^2$ for $0 \le x \le 4$ about the y axis. If the tank is full of a fluid weighing 80 pounds per cubic

foot, set-up the integral to find the work W required to pump the contents of the tank to a level 4 feet above the top of the tank. Do NOT evaluate the integral.

Solution: We place the bottom of the tank at the origin, as in the figure. Then $l = 12$. Moreover, the fluid to be pumped extends from 0 to 8 on the y axis, and the cross-sectional area $A(y)$ at y is given by $A(y) = \pi x^2 = 2\pi y$. With x replaced by y and 62.5 by 80, we have

$$W = \int_0^8 80(12 - y)2\pi y \, dy$$



9. The ends of a "parabolic" water tank are the shape of the region inside the graph of $y = x^2$ for $0 \le y \le 4$; the cross sections parallel to the top of the tank (and the ground) are rectangles. The tank is 6 feet long. Rain has filled the tank and water is removed by pumping it up to a spout that is 4 feet above the top of the tank. Set up a definite integral to find the work W that is done to lower the water to a depth of 3 feet.

Solution: Note that we position the origin at the bottom of the tank and the cross-sections are rectangles so $A(x) = length \times width = 6 \times 2x = 12\sqrt{y}$.

$$W = \int_3^4 62.5(8 - y)12\sqrt{y} dy$$

Section 6.5: Moments

1. Calculate the center of gravity of the region R between the graphs of $f(x) = 2x - 1$ and $g(x) = x - 2$ on the interval $[2, 5]$.

Solution:

$$M_x = \int_2^5 \frac{1}{2}\left[(2x-1)^2 - (x-2)^2\right]\,dx = \int_2^5 \frac{1}{2}\left(3x^2-3\right)\,dx = \frac{1}{2}\left(x^3-3x\right)\Big|_2^5 = 54$$
$$M_y = \int_2^5 x[(2x-1)-(x-2)]\,dx = \int_2^5 \left(x^2+x\right)\,dx = \left(\frac{1}{3}x^3+\frac{1}{2}x^2\right)\Big|_2^5 = \frac{99}{2}$$
$$A = \int_2^5 [(2x-1)-(x-2)]\,dx = \int_2^5 (x+1)\,dx = \left(\frac{1}{2}x^2+x\right)\Big|_2^5 = \frac{27}{2}$$
$$\bar{x} = \frac{M_y}{A} = \frac{99/2}{27/2} = \frac{11}{3}; \bar{y} = \frac{M_x}{A} = \frac{54}{27/2} = 4; (\bar{x},\bar{y}) = \left(\frac{11}{3},4\right)$$

2. Find the center of gravity of the region bounded by the graphs of f and g, where $f(x) = x$ and $g(x) = x^2$.

Solution: The graphs of f and g intersect for (x, y) such that $x = y = x^2$, which means that $x = 0$ or $x = 1$.

$$M_x = \int_0^1 \frac{1}{2}\left[x^2 - \left(x^2\right)^2\right]\,dx = \frac{1}{2}\left(\frac{1}{3}x^3 - \frac{1}{5}x^5\right)\Big|_0^1 = \frac{1}{15}$$
$$M_y = \int_0^1 x\left(x-x^2\right)\,dx = \left(\frac{1}{3}x^3-\frac{1}{4}x^4\right)\Big|_0^1 = \frac{1}{12}$$
$$A = \int_0^1 \left(x-x^2\right)\,dx = \left(\frac{1}{2}x^2-\frac{1}{3}x^3\right)\Big|_0^1 = \frac{1}{6}$$
$$\bar{x} = \frac{1/12}{1/6} = \frac{1}{2}; \quad \bar{y} = \frac{1/15}{1/6} = \frac{2}{5}; \quad (\bar{x},\bar{y}) = \left(\frac{1}{2},\frac{2}{5}\right)$$

3. Let $f(x) = 23 - x^2$ and $g(x) = x^2 + 5$. Find the center of gravity, $(\bar{x}, \bar{y})$, of the bounded region enclosed by the graphs of f and g.

Solution: Draw it out and use symmetry, otherwise it gets a bit messy with the numbers. $(\bar{x}, \bar{y}) = (0, 14)$

4. Determine the center of gravity of R where R is bounded by the hexagon with vertices at $(0, 0), (0, 6), (1, 1), (1, 5), (-1, 1)$, and $(-1, 5)$.

Solution: Draw the hexagonal shape to get $(\bar{x}, \bar{y}) = (0, 3)$

5. Find the center of gravity of the region bounded by $y = 2 - x^2$ and $y = x$.

Solution: The points of intersection: $2 - x^2 = x$, which simplifies to $x^2 + x - 2 = 0$, giving $x = -2$ and $x = 1$. Then:

$$A = \int_{-2}^{1} \left(2 - x^2 - x\right) dx = \left[2x - \frac{x^3}{3} - \frac{x^2}{2}\right]_{-2}^{1} = \frac{9}{2}$$

$$M_x = \frac{1}{2}\int_{-2}^{1} \left(4 - 5x^2 + x^4\right) dx = \frac{1}{2}\left[4x - \frac{5x^3}{3} + \frac{x^5}{5}\right]_{-2}^{1} = \frac{9}{5}$$

$$M_y = \int_{-2}^{1} \left(2x - x^3 - x^2\right) dx = \left[x^2 - \frac{x^4}{4} - \frac{x^3}{3}\right]_{-2}^{1} = \frac{-9}{4}$$

$$\bar{x} = \frac{\frac{-9}{4}}{\frac{9}{2}} = -\frac{1}{2}, \qquad \bar{y} = \frac{\frac{9}{5}}{\frac{9}{2}} = \frac{2}{5}$$

6. Let R be the region between the graphs of $f(x) = (x-1)^2$ and $g(x) = (x+1)^2$ on $[0, 3]$. Find the moments, M_x and M_y and the area A of the region R. Then find the center of gravity, $(\bar{x}, \bar{y})$ of R.

Solution:

$$A = \int_0^3 \left[(x+1)^2 - (x-1)^2\right] dx = \int_0^3 4x dx = 4\left[\frac{x^2}{2}\right]_0^3 = 4 \times \frac{9}{2} = 18$$

$$M_x = \frac{1}{2}\int_0^3 \left[(x+1)^4 - (x-1)^4\right] dx = \frac{1}{2}\int_0^3 \left(8x^3 + 8x\right) dx = 4\left[\frac{x^4}{4} + \frac{x^2}{2}\right]_0^3 = 81 + 18 = 99$$

$$M_y = \int_0^3 x\left[(x+1)^2 - (x-1)^2\right] dx = \int_0^3 x(4x) dx = 4\int_0^3 x^2 dx = 4\left[\frac{x^3}{3}\right]_0^3 = 36$$

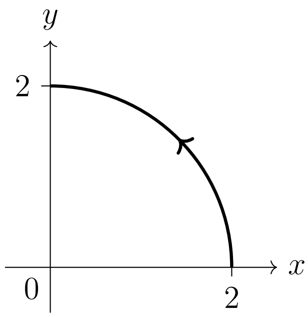
$$\bar{x} = \frac{M_y}{A} = \frac{36}{18} = 2, \quad \bar{y} = \frac{M_x}{A} = \frac{99}{18} = \frac{11}{2}$$

Section 6.7: Parametric Equations

1. First find an equation relating x and y, when possible. Then sketch the curve C whose parametric equations are given, and indicate the direction $P(t)$ moves as t increases.

   (a) $x = 2\cos t$ and $y = 2\sin t$ for $0 \leq t \leq \pi/2$

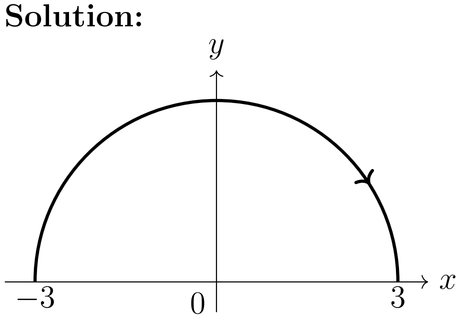
Solution:



Counter clockwise

(b) $x = 3\sin t$ and $y = 3\cos t$ for $-\pi/2 \le t \le \pi/2$

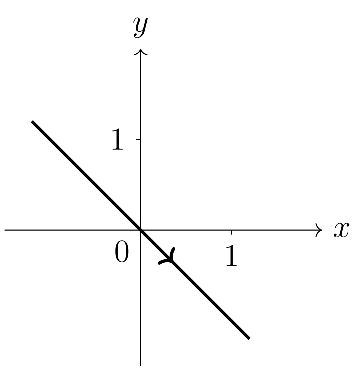
Solution:



Clockwise

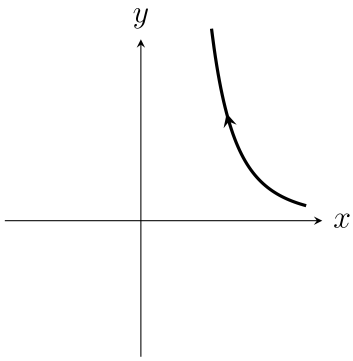
(c) $x = -2 + 3t$ and $y = 2 - 3t$ for all t

Solution:



(d) $x = e^{-t}$ and $y = e^{3t}$ for all t

Solution:



2. **Fill in the blank:** Consider the curve C parametrized by $x = -1 + 9\sin t$ and $y = \dfrac{1}{2} - 9\cos t$ for $-\pi \le t \le 3\pi$. Then C is a circle of radius BLANK with center $(x, y) =$ (BLANK , BLANK), and is traversed BLANK times in the BLANK direction.

Solution: $9, -1, \dfrac{1}{2}, 2$, counterclockwise

Sections 6.2 and 6.8: Arclength

1. Find the length L of the graph of $f(x) = 2x + 3$ for $1 \le x \le 5$.

Solution:
$$L = \int_1^5 \sqrt{1 + (2)^2}\, dx = \int_1^5 \sqrt{5}\, dx = 4\sqrt{5}$$

2. Find the length L of the graph of $f(x) = 2/3x^{3/2}$ for $1 < x < 4$.

Solution:
$$L = \int_1^4 \sqrt{1 + \left(x^{1/2}\right)^2}\, dx = \int_1^4 \sqrt{1+x}\, dx = \frac{2}{3}(1+x)^{3/2}\Big|_1^4 = \frac{2}{3}(5\sqrt{5} - 2\sqrt{2})$$

3. Find the length L of the graph of $y = \dfrac{\left(x^2+2\right)^{3/2}}{3}$ from $x = 0$ to $x = 3$.

Solution:

$$
\begin{aligned}
&\frac{dy}{dx} = \frac{1}{3} \cdot \frac{3}{2} \left(x^2+2\right)^{1/2} \cdot 2x = \sqrt{(x^2+2)} \cdot x \\
&\Rightarrow L = \int_0^3 \sqrt{1+(x^2+2)\,x^2}dx = \int_0^3 \sqrt{1+2x^2+x^4}dx \\
&= \int_0^3 \sqrt{(1+x^2)^2}dx = \int_0^3 \left(1+x^2\right) dx = \left[x+\frac{x^3}{3}\right]_0^3 = 3+\frac{27}{3} = 12
\end{aligned}
$$

4. Find the length L of the graph of $f(x) = \dfrac{1}{3}\sqrt{x}(x-3)$ for $0 \leq x \leq 3$.

Solution:

$$
L = \int_0^3 \sqrt{1+\left(\frac{1}{2}\sqrt{x}-\frac{1}{2\sqrt{x}}\right)^2}dx = \int_0^3 \left(\frac{1}{2}\sqrt{x}+\frac{1}{2\sqrt{x}}\right) dx = \left[\frac{1}{3}x^{3/2}+x^{1/2}\right]_0^3 = 2\sqrt{3}
$$

5. Find the length L of the graph of $f(x) = \dfrac{1}{8}x^2 - \ln x$ for $1 \leq x \leq 3$.

Solution:

$$
\begin{aligned}
L &= \int_1^3 \sqrt{1+[f'(x)]^2}dx = \int_1^3 \sqrt{1+\left(\frac{x}{4}-\frac{1}{x}\right)^2}dx = \int_1^3 \sqrt{1+\left(\frac{x^2}{16}-\frac{1}{2}+\frac{1}{x^2}\right)}dx \\
&= \int_1^3 \sqrt{\frac{x^2}{16}+\frac{1}{2}+\frac{1}{x^2}}dx = \int_1^3 \sqrt{\left(\frac{x}{4}+\frac{1}{x}\right)^2}dx = \int_1^3 \frac{x}{4}+\frac{1}{x}dx = \left[\frac{x^2}{8}+\ln x\right]_1^3 \\
&= 1+\ln 3
\end{aligned}
$$

6. **Spring 2016 Final Exam** Let $g(x) = x^3 + \dfrac{1}{12x}$, for $1 \leq x \leq 3$. Find the length L of the graph of g.

Solution:

$$
\begin{aligned}
L &= \int_1^3 \sqrt{1+\left(3x^2-\frac{1}{12x^2}\right)^2}dx = \int_1^3 \sqrt{\left(3x^2+\frac{1}{12x^2}\right)^2}dx = \int_1^3 \left(3x^2+\frac{1}{12x^2}\right) dx \\
&= \left[x^3-\frac{1}{12x}\right]_1^3 = 27-\frac{1}{36}-\left(1-\frac{1}{12}\right) = 26+\frac{2}{36} = 26+\frac{1}{18} = \frac{469}{18}
\end{aligned}
$$

7. **Spring 2008 Final Exam** Find the length L of the curve described parametrically by $x(t) = 1 - t^2$ and $y(t) = 1 + t^3$ for $0 \leq t \leq 1$.

Solution:

$$L = \int_0^1 \sqrt{(-2t)^2 + (3t^2)^2}dt = \int_0^1 t\sqrt{4+9t^2}dt \overset{u=4+9t^2}{=} \int_4^{13} \sqrt{u} \cdot \frac{1}{18}du$$
$$= \frac{1}{18}\left(\frac{2}{3}u^{3/2}\right)\Big|_4^{13} = \frac{1}{27}(13\sqrt{13} - 8) \text{ or } \frac{1}{27}(13^{3/2} - 8)$$

8. Find the length L of the curve described parametrically by $x(t) = \dfrac{3t^2}{2} + 4$, and $y(t) = 7 + t^3$ for $0 \leqslant t \leqslant \sqrt{3}$.

Solution: Note that $x'(t) = 3t \quad y'(t) = 3t^2$

$$L = \int_0^{\sqrt{3}} \sqrt{9t^2 + 9t^4}dt = \int_0^{\sqrt{3}} \sqrt{9t^2\left(1+t^2\right)}dt = \int_0^{\sqrt{3}} 3t\sqrt{1+t^2}dt \overset{u=1+t^2}{=} \int_1^4 3tu^{1/2}\frac{du}{2t}$$
$$= \frac{3}{2}\int_1^4 u^{1/2}du = \frac{3}{2} \cdot \frac{2}{3}u^{3/2}\Big|_1^4 = u^{3/2}\big|_1^4 = (\sqrt{u})^3\big|_1^4 = 2^3 - 1 = 7$$

9. Find the length L of the curve described parametrically by $x(t) = \dfrac{1}{2}t^2$ and $y(t) = \dfrac{1}{9}(6t + 9)^{3/2}$ from $t = 0$ to $t = 4$.

Solution: Note $x'(t) = t$ and $y'(t) = (6t + 9)^{1/2}$

$$L = \int_0^4 \sqrt{t^2 + 6t + 9}dt = \int_0^4 \sqrt{(t+3)^2}dt = \int_0^4 (t+3)dt = \left[\frac{1}{2}t^2 + 3t\right]_0^4 = 20$$

10. Find the length L of the curve described parametrically by $x(t) = \sin t - t\cos t$ and $y(t) = t\sin t + \cos t$ for $0 \leq t \leq \pi/2$.

Solution:

$$L = \int_0^{\pi/2} \sqrt{(\cos t - \cos t + t\sin t)^2 + (\sin t + t\cos t - \sin t)^2}dt$$
$$= \int_0^{\pi/2} \sqrt{t^2\left(\sin^2 t + \cos^2 t\right)}dt = \int_0^{\pi/2} tdt = \frac{1}{2}t^2\Big|_0^{\pi/2} = \frac{1}{8}\pi^2$$

Answers and Hints

Section 6.1: Volume

1. $\frac{\pi}{2}\left(e^2 - e^{-2}\right)$

2. 30π

3. $\frac{7\pi}{18}$

4. 16π

5. $\frac{19\pi}{4}$

6. $\frac{64}{15}\pi$

7. *Hint: Curves intersect at $x = 0$ and $x = 5$.*
   250π

8. $\pi\left[\left(\frac{128}{7} + 24 - 14\right) - \left(\frac{1}{7} + \frac{3}{2} - 7\right)\right] = \frac{471\pi}{14}$

9. $\frac{16}{3}$

10. $\frac{32,000}{3}$ (cubic feet)

11. 18

12. *Hint: Draw the region and give the region some structure (a formula)*
    $\frac{1}{3}$

Section 6.4: Work

1. 80 (foot-pounds)

2. 6×10^7 (ergs)

3. $\frac{24}{25} \times 10^7 = 9.6 \times 10^6$ (ergs)

4. 2.5×10^7(ergs)

5. $3,430,000\pi$ (ft-pounds)

6. $\frac{62.5\pi}{16}\left(4x^3 - \frac{1}{4}x^4\right)\Big|_0^{12} = 6750\pi$ (ft-pounds)

7. $W = \int_{-4}^{0} 42(0 - x)20\sqrt{16 - x^2}\,dx$

8. *Hint: Integrate along the y axis.* $W = \int_0^8 80(12-y)2\pi y\,dy$

9. *Hint: Integrate along the y axis.* $W = \int_3^4 62.5(8-y)12\sqrt{y}dy$

Section 6.5: Moments

1. $A = \frac{27}{2}, \quad M_x = 54, \quad M_y = \frac{99}{2}, \quad (\bar{x}, \bar{y}) = \left(\frac{11}{3}, 4\right)$

2. $A = \frac{1}{6}, \quad M_x = \frac{1}{15}, \quad M_y = \frac{1}{12}, \quad (\bar{x}, \bar{y}) = \left(\frac{1}{2}, \frac{2}{5}\right)$

3. *Hint: draw it out and use symmetry.* $(\bar{x}, \bar{y}) = (0, 14)$

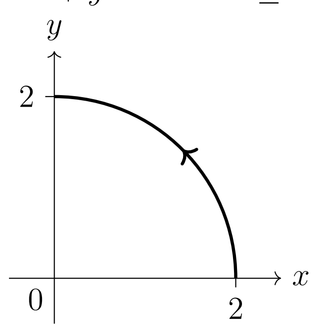
4. *Hint: draw it out and use symmetry.* $(\bar{x}, \bar{y}) = (0, 3)$

5. $(\bar{x}, \bar{y}) = \left(-\frac{1}{2}, \frac{2}{5}\right)$

6. $A = 18, \quad M_x = 99, \quad M_y = 36, \quad (\bar{x}, \bar{y}) = \left(2, \frac{11}{2}\right)$

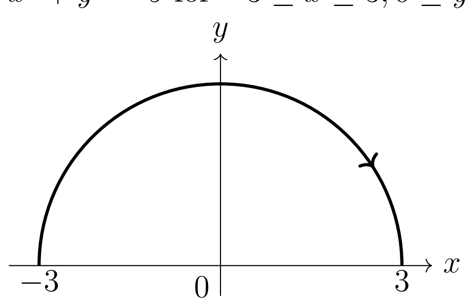
Section 6.7: Parametric Equations

1. (a) $x^2 + y^2 = 4$ for $0 \le x \le 2, 0 \le y \le 2$



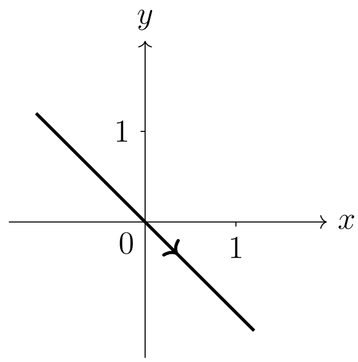
Counter clockwise

(b) $x^2 + y^2 = 9$ for $-3 \le x \le 3, 0 \le y \le 3$

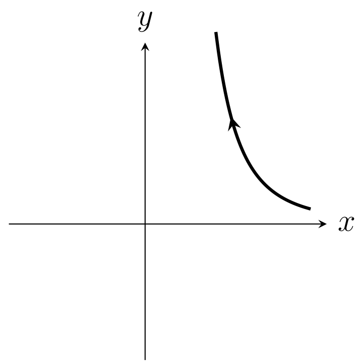


Clockwise

(c) $x = -y$



(d) $y = 1/x^3$ for $x > 0$



2. $9, -1, \dfrac{1}{2}, 2$, counterclockwise

Sections 6.2 and 6.8: Arclength

1. $4\sqrt{5}$
2. $\dfrac{2}{3}(5\sqrt{5} - 2\sqrt{2})$
3. 12
4. *Hint: Get a perfect square under the square root*
   $2\sqrt{3}$
5. *Hint: Get a perfect square under the square root*
   $1 + \ln 3$
6. *Hint: Get a perfect square under the square root*
   $26 + \dfrac{1}{18} = \dfrac{469}{18}$
7. $\dfrac{1}{27}(13\sqrt{13} - 8)$ or $\dfrac{1}{27}(13^{3/2} - 8)$
8. 7
9. 20
10. $\dfrac{\pi^2}{8}$