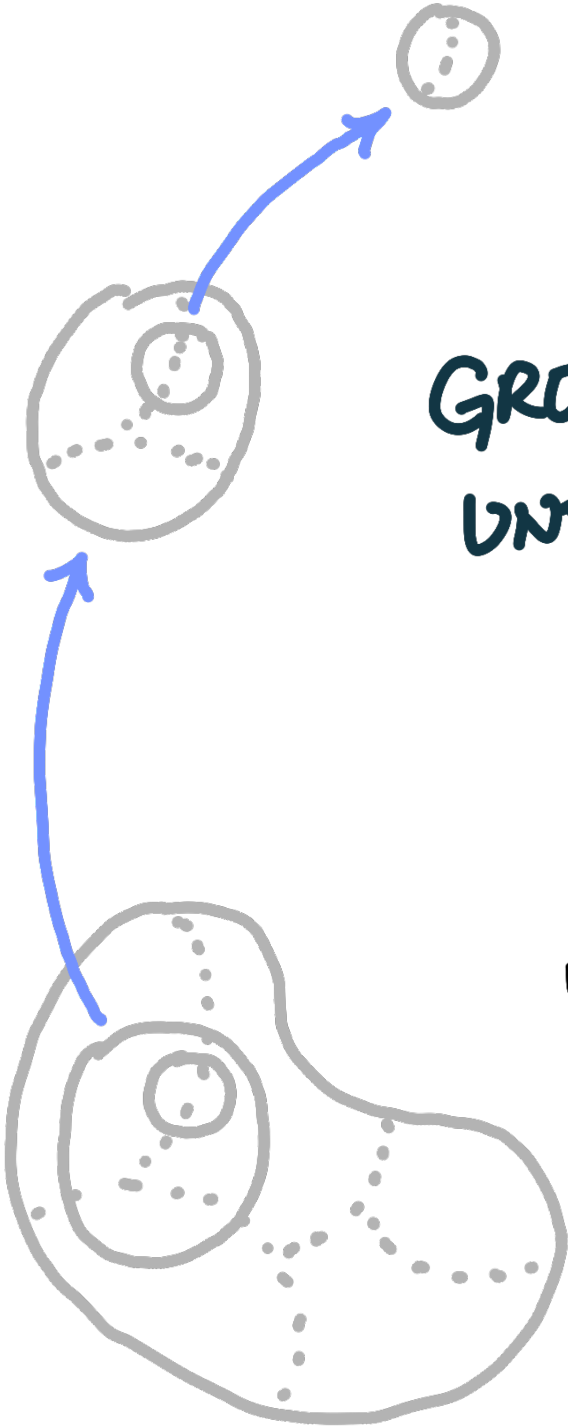

ENRICHED GROTHENDIECK TOPOLOGIES UNDER CHANGE OF BASE

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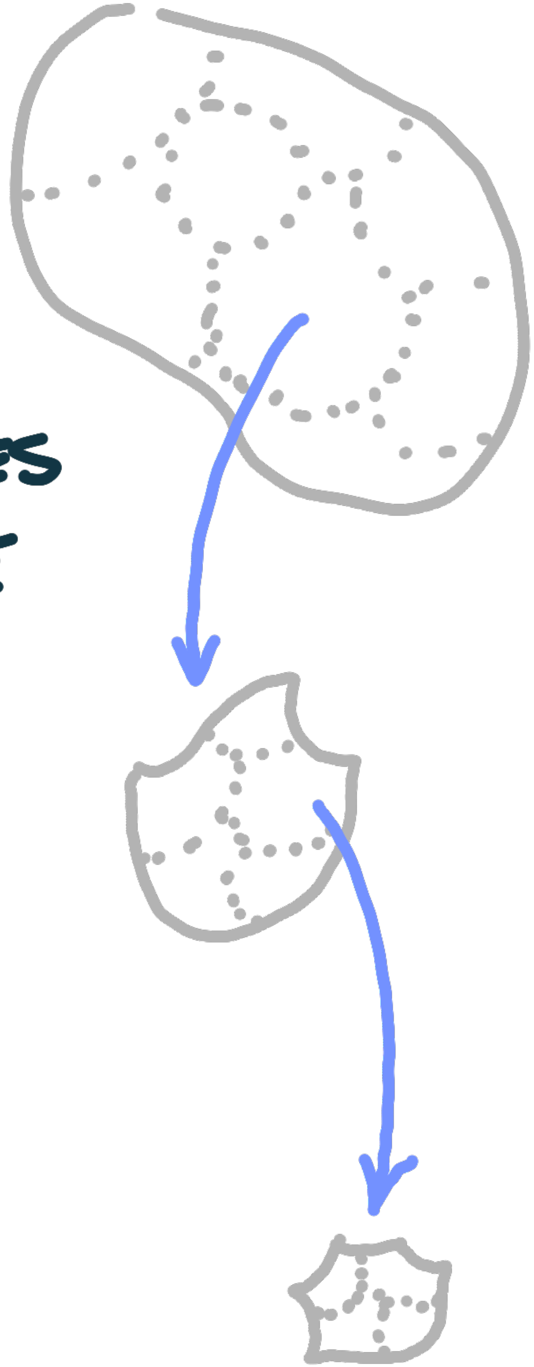


The Short Version:

- THINK OF AN ENRICHED CATEGORY AS A CATEGORY WHOSE HOM SETS HAVE EXTRA STRUCTURE.

  ENRICHED FUNCTORS, TRANSFORMATIONS RESPECT THIS EXTRA STRUCTURE.

  e.g., A RING IS A ONE-OBJECT Ab-CATEGORY. A RING HOMOMORPHISM IS AN Ab-FUNCTOR.

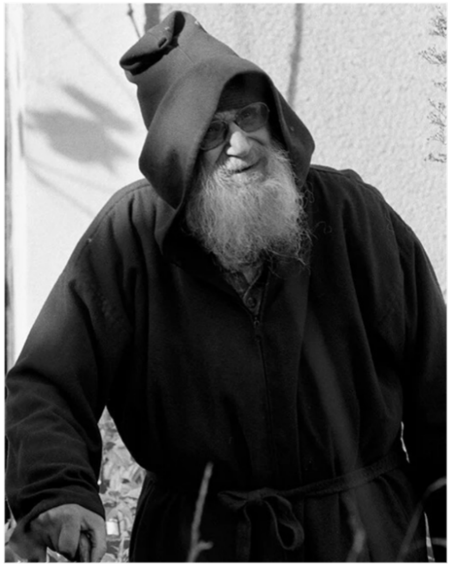
- GIVEN ENRICHING CAT'S $\mathcal{U}$, $\mathcal{V}$, A $\mathcal{V}$-CAT. $\mathcal{C}$, AND A FUNCTOR

$$G : \mathcal{V} \longrightarrow \mathcal{U},$$

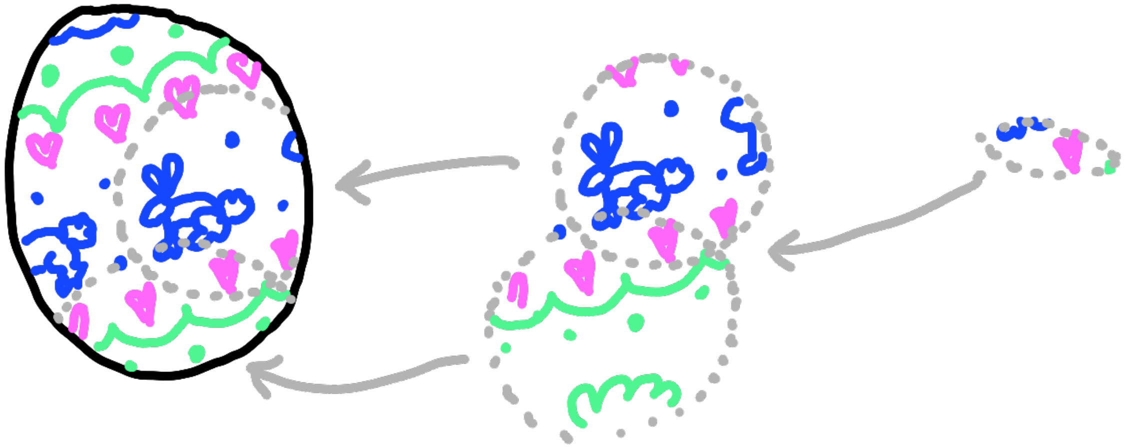
WE CAN CHANGE THE BASE OF $\mathcal{C}$ VIA G, OBTAINING A $\mathcal{U}$-CAT.

e.g., USING $\mathrm{Hom}_{Ab}(\mathbb{Z}, -) : Ab \longrightarrow Set$, WE CAN "FORGET" THE ADDITIVE STRUCTURE ON A RING TO OBTAIN A MERE MULTIPLICATIVE MONOID.

- A GROTHENDIECK TOPOLOGY IS A WAY TO ENDOW ANY CATEGORY (NOT JUST $\mathcal{O}(X)$) WITH A NOTION OF "CLOSE-TOGETHER-NESS" OF ITS OBJECTS.



- WE WANT TO KNOW WHEN OBJECTS ARE CLOSE TOGETHER BECAUSE THEN WE CAN DEFINE SHEAVES: "CLOSENESS-RESPECTING" ASSIGNMENTS OF DATA TO THE "LOCATIONS" IN (i.e., OBJECTS OF) OUR CAT'S.



"A SHEAF IS A CONTINUOUS PRESHEAF."

WE HAVE ENRICHED GROTHENDIECK TOPOLOGIES AND ENRICHED SHEAVES, TOO (SAY, OVER $\mathcal{V}$).

e.g., A GABRIEL LOCALIZING SYSTEM ON A RING IS AN Ab-TOPOLOGY. AN Ab-SHEAF ON A RING W/R/T SUCH A LOCALIZING SYSTEM IS ITS GABRIEL LOCALIZATION.

WHAT I WANTED TO KNOW:

1. HOW CAN I USE BASE CHANGE TO TURN A $\mathcal{V}$-TOPOLOGY INTO A $\mathcal{U}$-TOPOLOGY?

2. UNDER WHAT CONDITIONS IS SUCH AN ASSIGNMENT INJECTIVE?

$xy \neq yx$???

ASK ME LATER!

"ARI, WHAT IS THIS TALK DOING IN THE SPECIAL SESSION ON NONCOMMUTATIVE ALGEBRA?"

SOMETHING SOMETHING FUNCTORIAL SPECTRA OF NONCOMMUTATIVE RINGS . . .

The Long Version:

WHAT IS A GROTHENDIECK TOPOLOGY?

BORCEUX - QUINTEIRO, 1996:

Definition 1.2 *Let $\mathcal{C}$ be a small $\mathcal{V}$-category. A $\mathcal{V}$-Grothendieck topology on $\mathcal{C}$ is the choice, for every object $C \in \mathcal{C}$, of a family $\mathcal{T}(C)$ of subobjects of the representable $\mathcal{V}$-presheaf $\mathcal{C}(-,C)$. Those data must satisfy the following axioms:*

(T1) $\mathcal{C}(-,C) \in \mathcal{T}(C)$ *for every object* $C \in \mathcal{C}$;

(T2) given $R \in \mathcal{T}(C)$ *and* $f \in_G \mathcal{C}(D,C)$, *one has* $f^{-1}(R) \in \mathcal{T}(D)$, *where* $f^{-1}(R)$ *is defined by the following pullback:*

$$\begin{tikzcd}
f^{-1}(R) \arrow[r] \arrow[d, tail] & \{G,R\} \arrow[d, tail] \\
\mathcal{C}(-,D) \arrow[r, "f"'] & \{G,\mathcal{C}(-,C)\};
\end{tikzcd}$$

(T3) given $S \in \mathcal{T}(C)$ *and a subobject* $R \rightarrowtail \mathcal{C}(-,C)$ *such that* $f^{-1}(R) \in \mathcal{T}(D)$ *for all* $f \in_G S(D)$, *one has* $R \in \mathcal{T}(C)$.

WHAT IS A GROTHENDIECK TOPOLOGY?

$\mathcal{O}(X)$

Definition 1.2 *Let $\mathcal{C}$ be a small category. A Grothendieck topology on $\mathcal{C}$ is the choice, for every object $C \in \mathcal{C}$, of a family $\mathcal{T}(C)$ of subobjects of the representable presheaf $\mathcal{C}(-,C)$. Those data must satisfy the following axioms:*

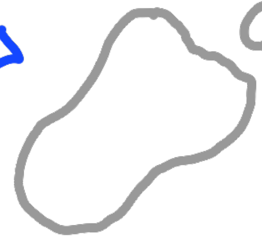
FAMILIES $\{\mathcal{U}_i\} \subseteq \mathcal{O}(X)$

S.T. $C = \bigcup_i \mathcal{U}_i$

(T1) $\mathcal{C}(-,C) \in \mathcal{T}(C)$ for every object $C \in \mathcal{C}$;

(T2) given $R \in \mathcal{T}(C)$ and $f \in_G \mathcal{C}(D,C)$, one has $f^{-1}(R) \in \mathcal{T}(D)$, where $f^{-1}(R)$ is defined by the following pullback:

$$\begin{tikzcd} f^{-1}(R) \arrow[r] \arrow[d, tail] & \{G,R\} \arrow[d, tail] \\ \mathcal{C}(-,D) \arrow[r, "f"'] & \{G,\mathcal{C}(-,C)\}; \end{tikzcd}$$

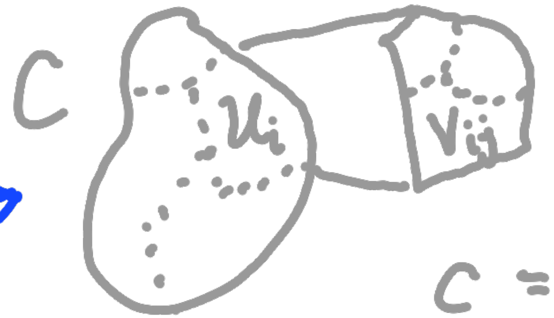


$C = \bigcup C$

(T3) given $S \in \mathcal{T}(C)$ and a subobject $R \rightarrowtail \mathcal{C}(-,C)$ such that $f^{-1}(R) \in \mathcal{T}(D)$ for all $f \in_G S(D)$, one has $R \in \mathcal{T}(C)$.

$C = \bigcup_i \mathcal{U}_i\,,\ D \subseteq C \Rightarrow$

$D = \bigcup_i (D \cap \mathcal{U}_i)$



$C = \bigcup_i \mathcal{U}_i\,,\ \mathcal{U}_i = \bigcup_j V_{ij} \Rightarrow C = \bigcup_{i,j} V_{ij}$

WHAT IS A GROTHENDIECK TOPOLOGY?

A RING R

Definition 1.2 *Let $\mathcal{C}$ be a small category. A Grothendieck topology on $\mathcal{C}$ is the choice, for every object $C \in \mathcal{C}$, of a family $\mathcal{T}(C)$ of subobjects of the representable presheaf $\mathcal{C}(-,C)$. Those data must satisfy the following axioms:*

RIGHT IDEALS OF R

(T1) $\mathcal{C}(-,C) \in \mathcal{T}(C)$ *for every object* $C \in \mathcal{C}$;

$R \in \mathcal{T}$

(T2) given $R \in \mathcal{T}(C)$ *and* $f \in_G \mathcal{C}(D,C)$, *one has* $f^{-1}(R) \in \mathcal{T}(D)$, *where* $f^{-1}(R)$ *is defined by the following pullback:*

$$\begin{array}{ccc} f^{-1}(R) & \longrightarrow & \{G,R\} \\ \downarrow & & \downarrow \\ \mathcal{C}(-,D) & \xrightarrow[f]{} & \{G,\mathcal{C}(-,C)\}; \end{array}$$

$I \in \mathcal{T} \Rightarrow$
$\forall x \in R,\ (I:x) \in \mathcal{T}$

(T3) given $S \in \mathcal{T}(C)$ *and a subobject* $R \rightarrowtail \mathcal{C}(-,C)$ *such that* $f^{-1}(R) \in \mathcal{T}(D)$ *for all* $f \in_G S(D)$, *one has* $R \in \mathcal{T}(C)$.

$I \in \mathcal{T}$, J s.t. $(J:x) \in \mathcal{T}$ $\forall x \in I$

$\Rightarrow J \in \mathcal{T}$

WHAT IS A GROTHENDIECK TOPOLOGY?

COVERINGS
IDEALS
FUNCTORS

" A FAMILY OF ________

SATISFYING SOME CLOSURE

CONDITIONS."

WHAT IS A GROTHENDIECK TOPOLOGY?

COVERAGE: ASSIGNMENT SATISFYING ONLY (T1) AND (T2).

DENOTE

$$\Sigma(\mathcal{C}, \mathcal{V}) = \{ \text{ALL } \mathcal{V}\text{-COVERAGES ON } \mathcal{C} \}$$

$$\tau(\mathcal{C}, \mathcal{V}) = \{ \text{ALL } \mathcal{V}\text{-TOPOLOGIES ON } \mathcal{C} \}$$

BASE CHANGE

- FOR OUR ENRICHING CAT'S $\mathcal{U}, \mathcal{V}$, WE CONSIDER CLOSED SYMMETRIC MONOIDAL CAT'S (+ SEVERAL OTHER PROPERTIES).

- WE CONSIDER LAX MONOIDAL FUNCTORS $G : \mathcal{V} \longrightarrow \mathcal{U}$ WHICH MAY BE

  i. FAITHFUL
  ii. CONSERVATIVE
  iii. RIGHT ADJOINTS

  ($\blacktriangle$)

BASE CHANGE

FIX $\mathcal{U}, \mathcal{V}$, AND $G : \mathcal{V} \to \mathcal{U}$.

GIVEN A $\mathcal{V}$-CATEGORY $\mathcal{C}$, DENOTE THE CORRESPONDING $\mathcal{U}$-CATEGORY BY $G_*\mathcal{C}$.

IF $R : \mathcal{C}^{op} \longrightarrow \mathcal{V}$ IS A $\mathcal{V}$-PRESHEAF, WE OBTAIN A $\mathcal{U}$-FUNCTOR $G_*R : G_*\mathcal{C}^{op} \longrightarrow G_*\mathcal{V}$ – NOT QUITE A $\mathcal{U}$-PRESHEAF!

DENOTE THE INDUCED $\mathcal{U}$-PRESHEAF BY

$$\tilde{G}R : G_*\mathcal{C}^{op} \longrightarrow \mathcal{U}.$$

BASE CHANGE

PROPOSITION. SUPPOSE G IS FAITHFUL AND A RIGHT ADJOINT. GIVEN A $\mathcal{V}$-SUBFUNCTOR $R \rightarrowtail \mathcal{C}(-,x)$, G INDUCES A $\mathcal{U}$-SUBFUNCTOR

$$\tilde{G}R \rightarrowtail \tilde{G}\mathcal{C}(-,x).$$

SUPPOSE $\mathcal{C}$ IS SMALL.

PROPOSITION. GIVEN A $\mathcal{V}$-COVERAGE J ON $\mathcal{C}$, G INDUCES A $\mathcal{U}$-COVERAGE $\tilde{G}J$ ON $G_*\mathcal{C}$.

PROPOSITION. GIVEN A $\mathcal{V}$-TOPOLOGY J ON $\mathcal{C}$, G INDUCES A $\mathcal{U}$-TOPOLOGY $\overline{G}J$ ON $G_*\mathcal{C}$.

THEOREM. $\Sigma(\mathcal{C},\mathcal{V})$ AND $\mathcal{T}(\mathcal{C},\mathcal{V})$ ARE COMPLETE LATTICES.

THEOREM. SUPPOSE G SATISFIES ALL CONDITIONS IN ($\blacktriangle$). THE ASSIGNMENT

$$\Sigma(\mathcal{C}, \mathcal{V}) \xrightarrow{\tilde{G}(-)} \Sigma(G_*\mathcal{C}, \mathcal{U})$$

IS AN INJECTIVE LATTICE MORPHISM.

"DISTINCT $\mathcal{V}$-COVERAGES GIVE RISE TO DISTINCT $\mathcal{U}$-COVERAGES."

CONJECTURE. SUPPOSE G SATISFIES ALL CONDITIONS IN ($\blacktriangle$), PLUS ONE MORE (SECRET) CONDITION.

THE ASSIGNMENT

$$\tau(\mathscr{C}, \mathcal{V}) \xrightarrow{\overline{G(-)}} \tau(G_*\mathscr{C}, \mathcal{U})$$

IS AN INJECTIVE LATTICE MORPHISM.

IT APPEARS THAT FAITHFULNESS IS NECESSARY!

THEOREM. FOR A FIELD k, $\mathcal{V} = \mathrm{grMod}_k$, $\mathcal{U} = \mathrm{Set}$, AND

NOT FAITHFUL!

$$G = \mathrm{Hom}_{\mathcal{V}}(k, -) : \mathcal{V} \to \mathcal{U},$$

THERE EXIST DISTINCT $\mathcal{V}$-COVERAGES ON $k[x,y]$ WHICH GIVE RISE TO THE SAME $\mathcal{U}$-COVERAGE.

ONE MORE THING

THEOREM. IN CASE $G : \mathcal{V} \to \mathcal{U}$ IS FULLY FAITHFUL, BASE CHANGE "COMMUTES" WITH ENRICHED SHEAFIFICATION.

"CANONICAL WAY TO TURN A PRESHEAF INTO A SHEAF."

UP NEXT

- THE $\mathcal{V}$-SHEAVES ON A $\mathcal{V}$-CATEGORY THEMSELVES ARE A $\mathcal{V}$-CATEGORY, WHICH IS ONE WAY TO SAY WHAT A $\mathcal{V}$-TOPOS IS. CAN I CHARACTERIZE $\mathcal{V}$-TOPOI AXIOMATICALLY?

- ORE SETS SHOULD GIVE RISE TO SPECIAL Ab-TOPOLOGIES. WHAT NICE PROPERTIES DO THEY HAVE? CAN I FIND ANALOGUES FOR $\mathcal{V} \neq Ab$?

THANKS FOR LISTENING!

SLIDES: ari-rosenfield.github.io