Homework (fair game for quiz).

§6.2: Exercises 1-17, §6.3: Exercises 1-7

Challenge problems (won't be on the quiz)!

1. 🌶️ 6.2, Exercise 18
2. 🌶️ 🌶️ 6.2, Project 1
3. 🌶️ 🌶️ 🌶️ (for length!) 6.2, Project 2

EXERCISES 6.2

In Exercises 1–11 find the length L of the graph of the given function.

1. $f(x) = 2x + 3$ for $1 \le x \le 5$
2. $f(x) = 2/3x^{3/2}$ for $1 \le x \le 4$
3. $f(x) = x^2 - \frac{1}{8}\ln x$ for $2 \le x \le 3$
4. $g(x) = x^3 + \dfrac{1}{12x}$ for $1 \le x \le 3$
5. $k(x) = x^4 + \dfrac{1}{32x^2}$ for $1 \le x \le 2$
6. $g(x) = \ln(x^2 - 1)$ for $2 \le x \le 5$
7. $f(x) = \ln(1 + x^2) - \dfrac{1}{8}\left(\dfrac{x^2}{2} + \ln x\right)$ for $1 \le x \le 2$
8. $f(x) = \ln(1 + x^3) + \dfrac{1}{12}\left(\dfrac{1}{x} - \dfrac{x^2}{2}\right)$ for $1 \le x \le 2$
9. $f(x) = -\frac{1}{4}\sin x + \ln(\sec x + \tan x)$ for $\pi/4 \le x \le \pi/3$
10. $f(x) = \frac{1}{2}(e^x + e^{-x})$ for $0 \le x \le \ln 2$

*11. $f(x) = \tan x - \frac{1}{8}(x + \frac{1}{2}\sin 2x)$ for $0 \le x \le \pi/4$

In Exercises 12–17 let f be a function defined on the given interval and having the indicated derivative f'. (Such a function f exists by the Fundamental Theorem of Calculus.) Find the length L of the graph of f.

12. $f'(x) = \sqrt{x^2 - 1}$; $[2, 3]$
13. $f'(x) = \tan x$; $[0, \pi/4]$
14. $f'(x) = \sqrt{\tan^2 x - 1}$; $[\frac{2}{3}\pi, \frac{3}{4}\pi]$
15. $f'(x) = \sqrt{x^n - 1}$; $[2, 4]$; n a positive integer
16. $f'(x) = \sqrt{\sqrt{x} - 1}$; $[25, 100]$
17. $f'(x) = \sqrt{x^{1/n} - 1}$; $[1, 2^n]$; n a positive integer
18. **a.** Let $f(x) = x^{1+1/(2n)}$, for $a \le x \le b$. Show that the length L of the graph of f is given by

$$L = \int_a^b \sqrt{1 + (2n+1)^2 x^{1/n}/(2n)^2}\, dx$$

   ***b.** Show that if $u = \sqrt{1 + (2n+1)^2 x^{1/n}/(2n)^2}$, then the integral in part (a) becomes an integral of a polynomial, and hence can (at least in theory) be evaluated.

   c. Using the substitution in part (b), evaluate the integral in part (a) when $n = 1$, $a = 0$, and $b = 1$.

19. Let $r > 0$. The graph of the equation

$$x^{2/3} + y^{2/3} = r^{2/3}$$

   is called an **astroid** (Figure 6.24).

   a. For the portion of the astroid in the first quadrant, express y as a function of x.

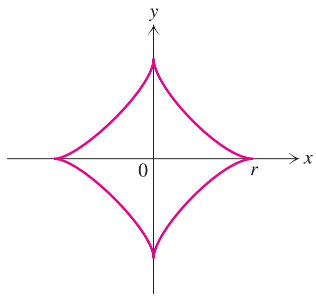


FIGURE 6.24 The astroid $x^{2/3} + y^{2/3} = r^{2/3}$ for Exercise 19.

   b. Let $\epsilon > 0$. Find the length L_ϵ of the portion of the astroid in the first quadrant for $\epsilon \le x \le r$. Then find the limit $L = \lim_{\epsilon \to 0^+} L_\epsilon$, and thereby determine the length of one-fourth of the astroid and hence the length of the whole astroid. (Later we will have an easier way to find the length of the astroid. See Exercise 10 in Section 6.8.)

Application

*20. An electric wire connecting a telephone pole to a house hangs in the shape of a catenary $y = \frac{c}{2}(e^{x/c} + e^{-x/c})$, where the units are feet (Figure 6.25). Find the length of the wire. (*Hint:* First find the value of c.)

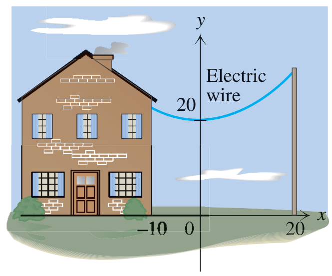


FIGURE 6.25 Figure for Exercise 20.

Projects

1. This project is parallel to Project 2 in Section 5.1, where we found approximations to the area of the unit circle. We all know that the circumference of the unit circle centered at the origin is 2π. In order to approximate the circumference, Archimedes inscribed polygons, as in

Figures 6.26 (a)–(c), and found the lengths of those polygons.

a. Find the length L_4 of the inscribed square.

b. Find the length L_6 of the inscribed hexagon. (*Hint:* What are the coordinates of a vertex that is neighbor to (1, 0)?)

c. Find the length L_8 of the inscribed octagon. (*Hint:* What are the coordinates of a vertex that is neighbor to (1, 0)?)

d. Find a formula for the length L_n of the inscribed polygon of n sides, and then determine the minimum number n such that the length L_n is greater than 6.

e. Find the minimum number n such that the length L_n of the inscribed polygon of n sides is greater than 6.28.

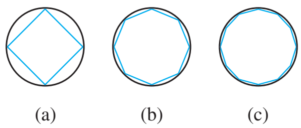


FIGURE 6.26 Figure for Project 1

2. In this project we look at the length of curves that have an infinite number of wobbles.

a. Let $f(x) = \sin(\pi/x)$ for $0 < x \le 1$ (Figure 6.27 (a))

i. Show that $f(\frac{1}{n}) = 0$ and $|f(\frac{2}{2n+1})| = 1$ for all $n \ge 1$. (Thus $|f(2/3)| = |f(2/5)| = |f(2/7)| = \cdots = 1$).

ii. Let L_2 = the length of the graph of f for $1/2 \le x \le 1$. Using the fact that $1/2 < 2/3 < 1$ and the fact that a straight line is the shortest distance between two points in the plane, show that $L_2 > 2$.

iii. Let L_3 = the length of the graph of f for $1/3 \le x \le 1$. Using the fact that $1/3 < 2/5 < 1/2 < 2/3 < 1$, show that $L_3 > 4$.

iv. Let L_n = the length of the graph of f for $1/n \le x \le 1$. Use the ideas of (ii) and (iii) to show that $L_n > 2n - 2$.

Note: It follows from (iv) that the length of the graph of f for $0 < x \le 1$ could not be finite, even if it were defined.

b. Let

$$g(x) = \begin{cases} x \sin(\pi/x) & \text{for } 0 < x \le 1 \\ 0 & \text{for } x = 0 \end{cases}$$

(See Figure 6.27 (b)).

i. Show that $g(\frac{1}{n}) = 0$ and $|g(\frac{2}{2n+1})| = \frac{2}{2n+1}$, for all $n \ge 1$.

ii. Let M_2 = the length of the graph of g for $1/2 \le x \le 1$. Show that $M_2 > 2(2/3) = 4/3$.

iii. Let M_3 = the length of the graph of g for $1/3 \le x \le 1$. Show that $M_3 > 2(2/3) + 2(2/5) = 4/3 + 4/5$.

iv. Let M_n = the length of the graph of g for $1/n \le x \le 1$. Show that

$$M_n > 2\left(\frac{2}{3}\right) + 2\left(\frac{2}{5}\right) + 2\left(\frac{2}{7}\right) + \cdots$$
$$+ 2\left(\frac{2}{2n-1}\right) > 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n-1}$$

v. Show that $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n-1}$ is a left sum (and upper sum also) for $\int_1^n 1/x\,dx$.

vi. Using the fact that $\int_1^n 1/x\,dx = \ln n$, as well as your knowledge about $\lim_{x\to\infty} \ln x$, show that the graph of g would have infinite length (if it were defined), despite the fact that g is continuous on [0, 1]. This shows that we cannot define the length of the graph of every continuous function.

c. Let

$$h(x) = \begin{cases} x^3 \sin(\pi/x) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

(See Figure 6.27(c)).

i. Prove that h is not only continuous but also differentiable at 0 (as well as $x \neq 0$).

ii. From Definition 6.2 we know that the length L of the portion of the graph of h for $0 \le x \le 1$ is defined and finite. First estimate the value of L by using Riemann sums with partitions having a large number of subintervals. Then use a technique similar to that in part (b) and see if the estimates you get by this technique remains finite as n increases without bound.

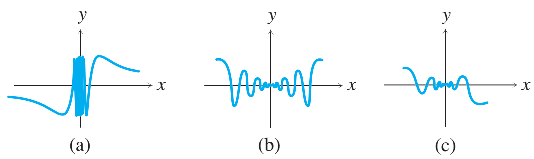


FIGURE 6.27 Figure for Project 2

EXERCISES 6.3

In Exercises 1–7 find the area S of the surface generated by revolving about the x axis the graph of f on the given interval.

1. $f(x) = \sqrt{4 - x^2}$; $[-\frac{1}{2}, \frac{3}{2}]$
2. $f(x) = \frac{1}{3}x^3$; $[0, \sqrt{2}]$
3. $f(x) = \sqrt{x}$; $[2, 6]$
4. $f(x) = 2\sqrt{1 - x}$; $[-1, 0]$
5. $f(x) = \frac{1}{2}(e^x + x^{-x})$; $[0, 1]$
6. $f(x) = \frac{1}{2}x^3 + \frac{1}{6x}$; $[1/\sqrt{2}, 1]$
7. $f(x) = \frac{1}{4}x^4 + \frac{1}{8x^2}$; $[1, \sqrt{2}]$
8. Derive (1) from the formula $S = \pi rl$ for the surface area of a cone with slant height l and radius r. (*Hint:* The surface area of the frustum is the difference of the surface areas of two cones, one with radius r_2 and one with radius r_1.)
9. Let $0 < a < r$. The portion of the sphere of radius r obtained by revolving the graph of

$$y = \sqrt{r^2 - x^2} \quad \text{for } -a \le x \le a$$

about the x axis is called a **zone** of the sphere.

a. Determine the surface area S of the zone.

b. Check your result in part (a) by letting $a = r$ and seeing whether you obtain the surface area of the sphere.

10. Assume that $[a, b]$ is a closed interval contained in the open interval $(-1, 1)$. Let $f(x) = \sqrt{1 - x^2}$, and let S be the area of the surface obtained by revolving the graph of f on $[a, b]$ about the x axis (refer to Figure 6.32). Show that

$$S = 2\pi(b - a)$$

(Thus the surface area depends only on the width of the interval $[a, b]$ and not on its location within $(-1, 1)$.)

11. The graph of the equation $x^{2/3} + y^{2/3} = 1$ is an astroid (Figure 6.34). Let $0 < \epsilon < 1$. Find the area S_ϵ of the surface obtained by revolving the portion of the top half of the astroid for $\epsilon \le x \le 1$ about the x axis. Then find the limit $S = \lim_{\epsilon \to 0^+} S_\epsilon$. (Later we will have an easier way of finding the area S. See Exercise 10 in Section 6.8.)

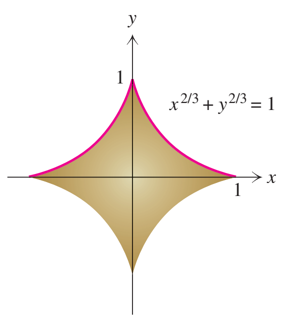


FIGURE 6.34 The surface for Exercise 11.

Applications

12. A yo-yo is made from a sphere by removing a slice from its center, as in Figure 6.35. Suppose the yo-yo is cut from a sphere of radius 3 centimeters in such a way that its width is

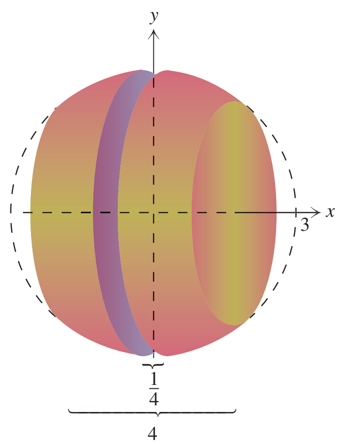


FIGURE 6.35 The yo-yo for Exercise 12.

4 centimeters, with a slit $\frac{1}{4}$ centimeter wide in its middle (Figure 6.35). Find the surface area S of the outer surfaces of the yo-yo.

13. A wok is in the shape of a solid obtained by revolving about the y axis the curve shown in Figure 6.36. Assume that the units are centimeters. Find the interior surface area of the wok. (*Hint:* Use (4) with the roles of x and y reversed.)

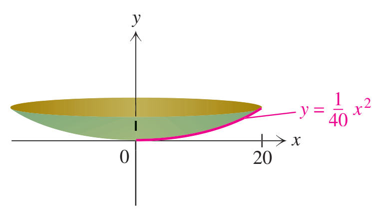


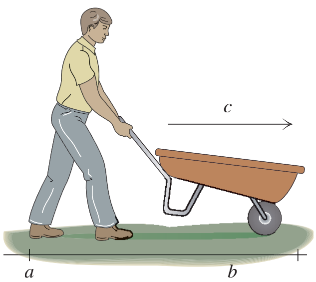
FIGURE 6.36 The wok for Exercise 13.

Project

1. With the results on Gabriel's horn under our belt, we will consider another hornlike object. Let $g(x) = 1/x^2$ for $x \geq 1$, and for $b > 1$ let D_b denote the solid obtained by revolving the graph of g on $[1, b]$ about the x axis.
   a. Find the volume V_b of D_b, and find $\lim_{b\to\infty} V_b$.
   b. Prove that the surface area S_b of D_b satisfies $S_b \leq 15$ for all $b > 1$. (*Hint:* First find a reasonable positive number M such that $\sqrt{1 + (g'(x))^2} \leq M$ for all $x \geq 1$.)
   c. Can you find a number $S < 14$ such that $S_b \leq S$ for all $b > 1$?
   d. Suppose that $g_r(x) = 1/x^r$ for $x \geq 1$, where $r \leq 1$. Let D_{rb} denote the solid obtained by revolving the graph of g_r on $[1, b]$ about the x axis, and S_{rb} the surface area of D_{rb}. On the one hand, we know that the surface area S_{1b} of Gabriel's horn approaches ∞ as b approaches ∞. On the other hand, we know from part (b) that the surface area S_{2b} of D_{2b} is no larger than 15 as b increases without bound. Is the surface area S_{rb} of D_{rb} bounded as a function of b, for each r with $r > 1$? Explain your answer.

6.4 WORK

Next we consider the physical concept of work. To introduce this concept, let us imagine a person pushing a wheelbarrow across a yard, from point a to point b. If a constant force c is exerted all the way across the yard, then the amount of work done is defined to be $c(b - a)$, that is, the force times the distance traveled by the wheelbarrow (Figure 6.37). (The product $c(b - a)$ should remind you of the area of a rectangle.) This is a reasonable definition, since we would expect the work done to increase if the person pushes the wheelbarrow farther or with a greater force.



work = force × distance = $c(b - a)$

FIGURE 6.37

How should the work be defined when the force is variable? The integral will enable us to do this. In order to relate integrals to work, we make certain assumptions.

1. The object on which the force acts moves in a straight line, so that we may think of the object as moving along the x axis from a point a to a point b.
2. At each point x between a and b a certain force $F(x)$ is exerted on the object.
3. The function F is continuous on $[a, b]$.

Furthermore, we adopt the convention that force is positive if exerted in the direction of the positive x axis and negative if exerted in the direction of the negative axis.

As we have seen, if the force F exerted on an object moving from a to b is constant, say $F(x) = c$, then the work W done is $W = c(b - a)$. When F is not necessarily constant, we let $P = \{x_0, x_1, \ldots, x_n\}$ be any partition of $[a, b]$, and for each k between 1 and n we let t_k be an arbitrary point in the subinterval $[x_{k-1}, x_k]$. If Δx_k is small, then as the object moves from x_{k-1} to x_k, the amount of work ΔW_k done by the force on the object is approximately $F(t_k)\,\Delta x_k$. It is also reasonable to expect the work W done by the force when the object moves from a to b to be the sum of $\Delta W_1, \Delta W_2, \ldots, \Delta W_n$, each one of which represents the work done on the object as it travels over one of the successive subintervals of $[a, b]$. Therefore W should be approximately equal to

$$\sum_{k=1}^{n} \overbrace{F(t_k)}^{\text{force}}\ \overbrace{\Delta x_k}^{\text{distance}}$$

Cumulative Review Exercises (Chapters 1–4)

1. union of $(-\infty, 2)$ and $(4, \infty)$

2. a. domain of $f \circ g$: all numbers except $\frac{2}{3}$; domain of $g \circ f$: all numbers except $-\frac{2}{3}$
c. no

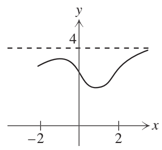
3. $-\infty$ **4.** $-\frac{2}{5}$ **5.** 2

6. a. It is. **b.** It is not.

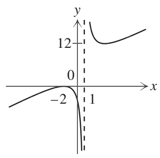
7. $e^{1/(x^2+1)}\left[\dfrac{-2x}{(x^2+1)^2}\right]$

8. $-\frac{1}{2}$ **11.** $-\frac{3}{16}$ **12.** $f''(x) = 4f(x)$

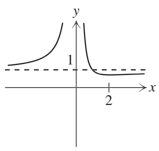
13.



14. relative maximum value: $f(-2) = 0$; relative minimum value: $f(4) = 12$



15. relative minimum value: $f(2) = \frac{3}{4}$; inflection point: $(3, \frac{7}{9})$



16. $-1/(24\pi)$ inch per second

17. a. -64 feet per second **b.** 2 seconds

18. $\frac{5}{2}$ **19.** \$3.50

20. 1 and 3 inches

21. approximately 0.2766633735 radian (or 15.9°) per second

CHAPTER 6

Section 6.1

1. $\pi/4$ **3.** $\pi/2$ **5.** π

7. $\frac{2}{9}\pi(27 - 2\sqrt{2})$ **9.** $\frac{19}{4}\pi$ **11.** 4π

13. π **15.** 540π **17.** $\frac{37}{15}\pi$

19. $\frac{4}{3}\pi$ **21.** π **23.** $\pi(\ln 2)^2$

25. $\frac{20}{3}\pi$ **27.** $\frac{1}{18}\pi(35 - 16\sqrt{2})$

29. $\frac{8}{3}\pi$ **31.** $\frac{8}{3}\pi$ **33.** $\frac{16}{3}\pi$

35. $\frac{500}{3}\sqrt{3}$ **37.** $V = \int_a^b \pi[f(x) - c]^2\,dx$

39. $\frac{\pi}{4}(3 - 4e^{-2} + e^{-4})$ **41.** $\frac{7}{30}\pi$

43. approximately 43.3%

45. a. π **b.** $\pi/3$ **c.** infinite volume

47. $V = \int_a^b 2\pi(x - c)f(x)\,dx$

49. $43\pi/6$ **51.** π

53. $\frac{32{,}000}{3}$ cubic feet **55.** $\frac{2}{3}\pi$ cubic centimeters

57. $2/\pi$ inches per second **59.** $\frac{191}{480}\pi$ cubic centimeters

61. 2000π cubic centimeters

Section 6.2

1. $4\sqrt{5}$ **3.** $5 + \frac{1}{8}\ln\frac{3}{2}$

5. $\frac{1923}{128}$ **7.** $\ln 5 - \frac{7}{8}\ln 2 + \frac{3}{16}$

9. $\frac{1}{8}(\sqrt{3} - \sqrt{2}) + \ln\dfrac{2 + \sqrt{3}}{\sqrt{2} + 1}$

11. $\frac{17}{16} + \frac{1}{32}\pi$ **13.** $\ln(\sqrt{2} + 1)$

15. $\dfrac{2}{n+2}(4^{(n+2)/2} - 2^{(n+2)/2})$

17. $\dfrac{2n}{2n+1}(\sqrt{2}\,2^n - 1)$

19. a. $y = (r^{2/3} - x^{2/3})^{2/3}$ for $0 \le x \le r$
b. $L_\varepsilon = \frac{3}{2}r - \frac{3}{2}\varepsilon^{2/3}r^{1/3}$; $L = \frac{3}{2}r$

Section 6.3

1. 8π **3.** $\frac{49}{3}\pi$

5. $\frac{\pi}{4}(e^2 - 2 + e^{-2}) = \frac{\pi}{4}(e - e^{-1})^2$ **7.** $\frac{291}{256}\pi$

9. a. $S = 4\pi ra$ **b.** yes

11. $S_\varepsilon = \frac{12}{5}\pi(1 - \varepsilon^{2/3})^{5/2}$; $S = \frac{12}{5}\pi$

13. $\frac{800}{3}\pi(2\sqrt{2} - 1) \approx 1531.78$ square centimeters

Section 6.4

1. 3384 joules **3.** 80 foot-pounds

5. 2×10^7 joules **7.** 2.5×10^7 ergs

9. 10^8 ergs **11.** 9.6×10^6 ergs

13. 1,850,000 foot-pounds **15.** 6750π foot-pounds

17. 17,920 foot-pounds **19.** 0

21. 60.28 ergs

23. a. 1.4 foot-pounds **b.** 0.2 foot-pounds

25. a. approximately 16,000 foot-pounds
b. 19,200 foot-pounds

27. 880 foot-pounds

29. approximately 1.651×10^{12} foot-pounds

31. 16 foot-pounds