Enriched Grothendieck topologies under change of base

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Abstract

In the presence of a monoidal adjunction $F \dashv G : \mathcal{V}_1 \leftrightarrows \mathcal{V}_2$ between locally finitely presentable Bénabou cosmoi, we examine the behavior of $\mathcal{V}_2$-Grothendieck topologies on a $\mathcal{V}_2$-category $\mathcal{C}$, and that of their constituent covering sieves, under the change of enriching category induced by G. We prove in particular that when G is faithful and U is an object of $\mathcal{C}$, G induces an injection from the poset of $\mathcal{V}_2$-sieves on U to the poset of $\mathcal{V}_1$-sieves on U. A similar result for $\mathcal{V}_2$-Grothendieck topologies on $\mathcal{C}$ is still in progress, and examples where these induced maps are not injective are still to be written down.

Introduction

This work was inspired by the hunt for good notions of functorial spectra for noncommutative structures, and particularly by a family of results of the following flavor:

Theorem [Rey12]. Suppose we have a functor $F : \mathsf{Ring}^{\mathrm{op}} \to \mathsf{Set}$ which is an extension of the Zariski spectrum on commutative rings, in the sense that the diagram

$$\begin{tikzcd} \mathsf{cRing}^{\mathrm{op}} \arrow[r, "\mathsf{Spec}"] \arrow[d, hook, "\mathsf{inc}"'] & \mathsf{Set} \\ \mathsf{Ring}^{\mathrm{op}} \arrow[ur, "F"'] & \end{tikzcd}$$

commutes. Then $F(\mathsf{Mat}_{n\times n}(\mathbb{C})) = \varnothing$ when $n \geq 3$.

Similar obstructions arise for the Zariski spectrum viewed as a functor into spaces, locales, or toposes, as well as for other spectra. All of these results are corollaries of the main theorem from [vdBH14], which roughly says that obstructions in one category must persist in another under certain conditions on the limit behavior of a pair of 'transporting' functors.

In [Rey24], the maximal spectrum

$$\mathsf{cAff}_k^{\mathrm{op}} \xrightarrow{\mathsf{Max}} \mathsf{Set}$$

was extended, in a weak sense, to a certain nice class of noncommutative k-algebras via the finite dual coalgebra construction

$$\mathsf{Alg}_k^{\mathrm{op}} \xrightarrow{(-)^\circ} \mathsf{Coalg}_k \ ,$$

viewed as a Coalg_k-enriched functor. This result, together with earlier results of [KS09] and [Bru08], suggest that for an algebra A, we can imagine A° as a "quantized" version of the maximal spectrum, and Coalg_k as a category of "quantized sets."

This success inspires hope that a similar approach could yield an extension of the Zariski spectrum

$$\mathsf{cRing}^{\mathrm{op}} \xrightarrow{\mathsf{Spec}} \mathsf{Topos} \ ,$$

or some restriction of it, to a sufficiently nice class of noncommutative rings. The current work is part of an endeavor to locate a suitable category of "quantized Grothendieck topoi."

Enriched Grothendieck topologies

For convenience, we recall some useful definitions:

Change of base for enrichment

Suppose given two closed symmetric monoidal categories $\mathcal{V}_1, \mathcal{V}_2$, and a lax monoidal functor

$$G : \mathcal{V}_2 \to \mathcal{V}_1,$$

which we will often refer to as a **change of base.**

- Given a $\mathcal{V}_2$-category $\mathcal{C}$, G induces a $\mathcal{V}_1$-category $G_*\mathcal{C}$ whose objects are the same as those of $\mathcal{C}$ and whose hom-objects are $G_*\mathcal{C}(x,y) := G(\mathcal{C}(x,y))$.
- Given a $\mathcal{V}_2$-functor $A : \mathcal{C} \to \mathcal{D}$, G induces a $\mathcal{V}_1$-functor G_*A whose action on objects is $x \mapsto Ax$ and with
$$(G_*A)_{xy} := G(A_{xy}) : G_*\mathcal{C}(x,y) \to G_*\mathcal{D}(Ax, Ay).$$

Locally presentable categories

An object X in a category $\mathcal{V}$ is called **finitely presentable** if $\mathcal{V}(X,-)$ preserves filtered colimits. $\mathcal{V}$ is called **locally finitely presentable** if

- $\mathcal{V}$ has small colimits;
- the subcategory $\mathcal{V}_{fp}$ of finitely presentable objects is essentially small; and
- every object of $\mathcal{V}$ is a filtered colimit of finitely presentable objects.

A **separating family** for $\mathcal{V}$ is a family $\{X_\alpha\}_{\alpha\in A}$ of objects of $\mathcal{V}$ such that the family $\{\mathcal{V}(X_\alpha, -) : \mathcal{V} \to \mathsf{Set}\}_{\alpha \in A}$ of hom-functors is jointly faithful.

$\mathcal{V}$-Grothendieck topologies

Let $\mathcal{V}$ be a locally finitely presentable, closed symmetric monoidal category, and let $\mathcal{C}$ be a $\mathcal{V}$-category.

- A **sieve** on an object $U \in \mathcal{C}$ is a $\mathcal{V}$-subfunctor of $\mathcal{C}(-,U)$; i.e., a $\mathcal{V}$-functor R with a $\mathcal{V}$-natural transformation $R \Rightarrow \mathcal{C}(-,U)$ in which every component is monic.
- [BQ96] A **$\mathcal{V}$-Grothendieck topology** on $\mathcal{C}$ is, to each object $U \in \mathcal{C}$, the assignment of a family $J(U)$ of sieves on U such that
  a. $\mathcal{C}(-,U) \in J(U)$;
  b. For any G in a dense generating family for $\mathcal{V}$, any map $f : G \to \mathcal{C}(V,U)$, and any $R \in J(U)$, the pullback $f^*(R)$ defined by
  $$\begin{tikzcd} f^*(R) \arrow[r] \arrow[d, hook] & \{G, R\} \arrow[d, hook] \\ \mathcal{C}(-,V) \arrow[r] & \{G, \mathcal{C}(-,U)\} \end{tikzcd}$$
  is an element of $J(V)$;
  c. For $S \in J(U)$ and a subobject R of $\mathcal{C}(-,U)$ such that $f^*(R) \in J(V)$ for any $f : G \to S(V)$, we have $R \in J(U)$.

Work in Progress

To avoid the obstructions following from [vdBH14], we'd like to be able to tell whether or not a $\mathcal{V}$-Grothendieck topology somehow reduces to an ordinary (i.e., Set-enriched) Grothendieck topology. The current work develops methods for comparing Grothendieck topologies over different enriching categories, not just for a change of base $\mathcal{V} \to \mathsf{Set}$, but for a general change of base $\mathcal{V}_2 \to \mathcal{V}_1$, where $\mathcal{V}_1, \mathcal{V}_2$ are locally finitely presentable, closed symmetric monoidal categories

Below, let $\mathcal{C}$ be a $\mathcal{V}_2$-category, and suppose given a lax monoidal functor $G : \mathcal{V}_2 \to \mathcal{V}_1$.

Proposition A. *(Sieves are preserved under change of base)* If G is faithful, $\mathcal{V}_2$-naturality of

$$\{\alpha_x : * \to \mathcal{D}(Ax, Bx)\}$$

is equivalent to $\mathcal{V}_1$-naturality of

$$\{G\alpha_x : * \to G_*\mathcal{D}(Ax, Bx)\}.$$

Proposition B. *(A $\mathcal{V}_2$-sieve is uniquely a $\mathcal{V}_1$-sieve)* If G is faithful and preserves monomorphisms, there is an injective morphism of posets

$$\beta : \mathsf{Sub}(\mathcal{C}(-,U)) \to \mathsf{Sub}(G_*\mathcal{C}(-,U)).$$

With $\mathcal{V}_1, \mathcal{V}_2$, and $\mathcal{C}$ as above, we now consider a monoidal adjunction

$$\begin{tikzcd} \mathcal{V}_1 \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{V}_2 \arrow[l, bend left, "G"] \end{tikzcd} .$$

This allows us to make use of a correspondence outlined in [BT91] between the respective separating families for $\mathcal{V}_1$ and $\mathcal{V}_2$.

Theorems C and D, below, are the analogues of Props. A and B for Grothendieck topologies, which allow us to make the comparison we want, and whose proofs are in progress. Theorem E requires an example yet to be found.

Theorem C. For a $\mathcal{V}_2$-Grothendieck topology J on $\mathcal{C}$, the assignment to each object $U \in G_*(\mathcal{C})$ of the family

$$\{G_*R : R \in J(U)\}$$

of $\mathcal{V}_1$-sieves on U is a $\mathcal{V}_1$-Grothendieck topology.

Status. Proof is straightforward but technical; the author is currently mired in notation issues.

Theorem D. There is an injection $\mathcal{B}$ from the (possibly large) set of $\mathcal{V}_2$-Grothendieck topologies on $\mathcal{C}$ to the (possibly large) set of $\mathcal{V}_1$-Grothendieck topologies on $G_*\mathcal{C}$.

Idea for proof. Leverage a bijection outlined in [BQ96] between the collection of $\mathcal{V}$-Grothendieck topologies on $\mathcal{C}$ and $\mathcal{V}$-localizations of $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$.

Theorem E. There exist $\mathcal{V}_1, \mathcal{V}_2$, and $\mathcal{C}$ as above such that the maps β in Prop. A and $\mathcal{B}$ in Theorem D are not injective.

Idea for proof. Look at cases where any separating family for $\mathcal{V}_2$ contains at least two objects; for example, $\mathcal{V}_2 = \mathsf{Ch}_\bullet(\mathcal{A})$ for $\mathcal{A}$ an abelian category.

Future questions

- What should a $\mathcal{V}$-Grothendieck pretopology on $\mathcal{C}$ be?
  - For this, I'd need a sufficiently general notion of a "$\mathcal{V}$-enriched pullback." Can I find such a thing?
- What are the images of β and $\mathcal{B}$ under different conditions on G? When these maps are injective, what extra conditions on a sieve or Grothendieck topology guarantee that it is in the image?
- What are some specific use cases for these results (specific $\mathcal{V}_1, \mathcal{V}_2$, and $\mathcal{C}$)?
  - I don't know enough geometry yet to know where to look!

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