AT LEAST ONE ~~TWO~~ OPEN PROBLEM~~S~~ IN CATEGORICAL QUANTUM THEORY

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WHAT IS THIS?

ORIGINAL PLAN: TALK ABOUT TWO PROJECTS I STARTED, BUT DIDN'T FINISH, WHICH I THOUGHT COULD BE ACCESSIBLE TO PEOPLE WHO DON'T KNOW TONS OF CATEGORY THEORY.

ONE PROBLEM IS NO LONGER OPEN :(

IT'S A VERY PRETTY FAMILY OF RESULTS, SO I'LL TELL YOU ABOUT IT ANYWAY.

LED TO A NEW OPEN PROBLEM, BUT IT MIGHT BE HARD.

WHAT IS THIS?

MAIN GOAL:

IF ANY OF THIS SOUNDS FUN AND/OR YOU'RE TRYING TO LEARN SOME CATEGORY THEORY AND MAYBE PUBLISH A MATH PAPER, LET'S TALK AFTER!

WHY CQT?

- CT:  

"STUDY THE TRANSFORMATIONS A CLASS OF OBJECTS CAN UNDERGO, AS LONG AS YOU CAN COMPOSE THEM."

- CQT: "REFORMULATE QUANTUM THEORY IN A GENERAL DAGGER COMPACT CATEGORY." Hilb IS ONE OF THESE.

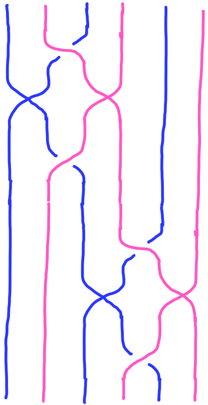
WHY CQT?

- FOR COMPUTING: WANT TO VIEW QUANTUM INFORMATION THEORY AS ONE AMONGST MANY COMPUTATIONAL THEORIES, SEE WHAT DISTINGUISHES IT.

- PRACTICAL APPLICATIONS: ??? :)

① MODELING CONCURRENCY IN SYMMETRIC MONOIDAL CATEGORIES

EXAMPLE OF MORPHISM IN $\mathcal{X} \amalg \mathcal{Y}$



(j/w
CARMEN CONSTANTIN
CHRIS HEUNEN
NESTA v.d. SCHAAF)

BACKGROUND

TRADITIONALLY, QUANTUM COMPUTATION HAPPENS IN THE CATEGORY

Hilb {

OBJECTS: FINITE DIMENSIONAL HILBERT SPACES

(QUDIT REGISTERS)

MORPHISMS: BOUNDED $\mathbb{C}$-LINEAR MAPS

(QUANTUM CIRCUITS)

BACKGROUND

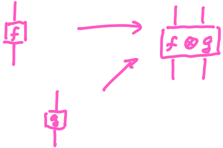
Hilb HAPPENS TO BE A SYMMETRIC MONOIDAL CATEGORY:

MONOIDAL:

CAN PUT TWO REGISTERS SIDE-BY-SIDE TO FORM A NEW ONE:

$$\mathbb{C}^2, \ \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \otimes \mathbb{C}^2$$

SAME WITH CIRCUITS:



AND THERE'S A "TRIVIAL REGISTER" W/R/T $\otimes$

SYMMETRIC:

SWITCHING LEFT WITH RIGHT IS UNPROBLEMATIC. (i.e., $\otimes$ IS COMMUTATIVE.)

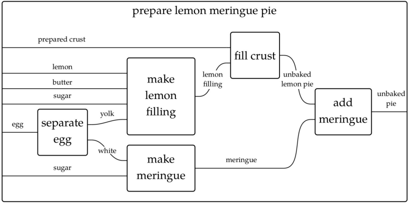
BACKGROUND

SYMMETRIC MONOIDAL CATEGORY	RESOURCE THEORY
OBJECT	TYPE OF RESOURCE
MORPHISM	PROCESS

BACKGROUND

ANY SYMMETRIC MONOIDAL CATEGORY $(\mathcal{X}, \otimes, I)$ COMES WITH:

(a) A STRING-DIAGRAMMATIC CALCULUS FOR ITS MORPHISMS.



MORPHISM
= STRING DIAGRAM
= CIRCUIT DIAGRAM

BUT NOW WE IMPOSE RULES FOR SENSIBLY TRANSFORMING ONE TO ANOTHER.

BACKGROUND

ANY SYMMETRIC MONOIDAL CATEGORY $(\mathcal{X}, \otimes, I)$ COMES WITH:

LATTICE OF OPENS OF A

(b) AN INTERNAL TOPOLOGICAL SPACE CALLED $\mathrm{ISub}(\mathcal{X})$.

IT'S A LATTICE OF DISTINGUISHED MONOMORPHISMS INTO THE TENSOR UNIT I, CALLED SUBUNITS.

(CATEGORIFICATION OF CENTRAL IDEMPOTENTS OF A RING.)

BACKGROUND

FACT: EACH OF THESE CORRESPONDS TO A SYMMETRIC MONOIDAL SUBCATEGORY OF $\mathcal{X}$, AND VICE VERSA. ("SUBUNIT" BECAUSE EACH IS THE TENSOR UNIT FOR A SYMM- MONOIDAL SUBCATEGORY.)

FOR NARRATIVE PURPOSES, THINK OF THEM AS ADMISSIBLE LOCATIONS FOR PROCESSES TO OCCUR AT.

BACKGROUND

ANY SYMMETRIC MONOIDAL CATEGORY $(\mathcal{X}, \otimes, I)$ COMES WITH:

C NOTIONS OF RESTRICTION OF A MORPHISM TO A SUBUNIT AND SUPPORT OF A MORPHISM ON A SUBUNIT.

"THERE'S A PLACE IN $\mathcal{X}$ WHERE A GIVEN PROCESS OCCURS, AND IT ONLY OCCURS THERE."

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ELSEVIER

Tensor topology

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BACKGROUND

SAY THAT TWO PROCESSES CAN EXCHANGE RESOURCES IF THEY HAVE COMMON SUPPORT.

THE PROBLEM

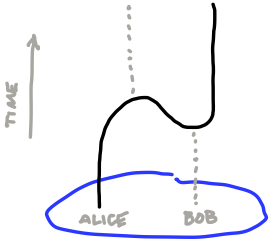
WE'D EXPECT SYMMETRIC MONOIDAL CAT's TO BE GOOD FOR MODELING COMPUTATIONAL CONCURRENCY:

- SEQUENTIAL PROCESSES ✓
  (COMPOSING MORPHISMS)
- PARALLEL PROCESSES ✓
  (TENSORING MORPHISMS)

THE PROBLEM

WE'D LIKE PROCESSES IN A GIVEN $\mathcal{X}$ TO BE ABLE TO EXCHANGE RESOURCES.

e.g., TELEPORTATION:



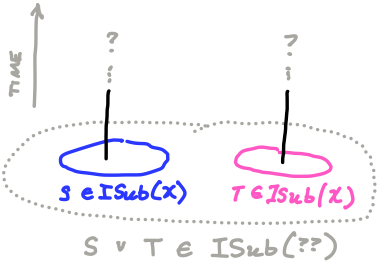
$S \in \mathrm{ISub}(\mathcal{X})$

(PRESENCE OF DUAL OBJECTS)

NICE PROPERTIES OF Hilb_{fd} LET US ASSUME ALL PROCESSES INVOLVED HAVE COMMON SUPPORT.

"LIVE IN THE SAME UNIVERSE."

THE PROBLEM



THOSE NICE PROPERTIES DON'T HOLD FOR A GENERAL $(\mathcal{X}, \otimes, I)$.

MIGHT NOT BE ABLE TO FORM JOINS IN $\mathrm{ISub}(\mathcal{X})$.

CAN REMEDY THIS BY FINDING A NEW CATEGORY ?? WITH A NICER SUBUNIT LATTICE.

THE PROBLEM

WHY WOULD WE WANT TO FIX THIS PROBLEM?

BECAUSE WE WANT TO DO SHEAF THEORY.

Advances in Mathematics 416 (2023) 108900

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ELSEVIER

Sheaf representation of monoidal categories ☆

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ARTICLE INFO

ABSTRACT

BUT THAT WOULD BE A MUCH LONGER TALK.

SEE FOR MORE ON THIS

SO FAR

CONSTRUCTED A CANDIDATE FOR ?? :

GIVEN $\mathcal{X}, \mathcal{Y} \in \mathsf{SymMonCat}$, FORM

$$\mathcal{X} \amalg \mathcal{Y} \in \mathsf{SymMonCat}$$ WHOSE

MORPHISMS ARE 2-COLORED STRING DIAGRAMS.

(HARD PART: ALGEBRAIC FORMALISM. MORPHISMS ARE ACTUALLY MATRICES WITH ENTRIES IN A JUDICIOUSLY CHOSEN SEMIRING.)

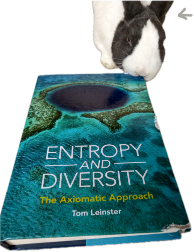
TO - DO

- VERIFY THAT $\mathcal{X} \amalg \mathcal{Y}$ IS A PUSHOUT IN SymMonCat. (REQUIRES SOME LIGHT HIGHER CT.)

- FIND A NORMAL FORM FOR $\mathrm{Mor}(\mathcal{X} \amalg \mathcal{Y})$.

- GENERALIZE TO $n > 2$. (i.e., WRITE DOWN WHAT AN n-COLORED STRING-DIAGRAM CALCULUS SHOULD BE.)

② →



← TOM L'S RABBIT?

② FUNCTORIAL ENTROPY

(j/w MAICOL OCHOA)

I LIED. THIS IS MERELY CQT-ADJACENT. NEEDED CATCHY TITLE

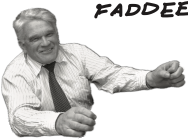
BACKGROUND

OVERARCHING PROJECT: FIND SETS OF AXIOMS WHICH CHARACTERIZE ENTROPY (RELATIVE ENTROPY, ENTROPY DIFFERENCE) AS A FUNCTION FROM SOME CLASS OF (GENERALIZED) PROBABILITY SPACES INTO

$[0,\infty)$ OR $[0,\infty]$.

FROM 1950S:

FADDEEV,



KHINCHIN.

BACKGROUND

REMINDER: SAY $(X,\mu) \xrightarrow{f} (Y,\nu)$ IS A MEASURE-PRESERVING MAP OF FINITE PROBABILITY SPACES.

THE SHANNON ENTROPY DIFFERENCE OF f IS

$$H(\mu) - H(\nu) = \sum_{Y} \nu_i \ln \nu_i - \sum_{X} \mu_j \ln \mu_j$$

(ALSO RECALL: ALWAYS NON NEGATIVE.)

BACKGROUND

CT PERSPECTIVE: WE LIKE WHEN "MAPS" BETWEEN CATEGORIES ARE FUNCTORIAL, RESPECTING COMPOSITIONAL STRUCTURE.

ANALOGY:

- FOR GROUP THEORY, WE LIKE OUR MAPS TO BE HOMOMORPHISMS
- FOR MEASURE THEORY, WE LIKE OUR MAPS TO BE MEASURE-PRESERVING

... ETC.

BACKGROUND

FROM THAT PERSPECTIVE, NATURAL TO ASK WHAT HAPPENS IF WE REQUIRE ENTROPY TO BEHAVE FUNCTORIALLY.

TURNS OUT, WE NECESSARILY GET SHANNON ENTROPY DIFFERENCE:

BACKGROUND

(BUILDING ON FADDEEV'S CHARACTERIZATION)

THEOREM. (BAEZ - FRITZ - LEINSTER, 2011)

IF $F : \mathrm{FinProb} \longrightarrow [0,\infty)$ IS ANY CONTINUOUS FUNCTION WHICH IS

- CONVEX LINEAR

  $$F(\lambda f \oplus (1-\lambda) g) = \lambda Ff + (1-\lambda) Fg$$

- FUNCTORIAL

  $$F(fg) = Ff + Fg$$

THEN F IS A REAL MULTIPLE OF THE SHANNON ENTROPY DIFFERENCE.

BACKGROUND

" The charm of this result is that the first two hypotheses look like linear conditions, and none of the hypotheses hint at any special role for the function $-p \ln p$, but it emerges in the conclusion. "

THEN, ANALOGUES FOR:

- FINITE MEASURE SPACES
- TSALLIS ENTROPY (MATH. ECOLOGY?)

PARZYGNAT, 2022:

- VON NEUMANN ENTROPY OF FINITE-DIMENSIONAL NONCOMMUTATIVE PROBABILITY SPACES

"PROBLEM"

CAN WE EXTEND ANY OF THESE RESULTS TO THE COUNTABLY INFINITE /INFINITE-DIMENSIONAL CASE? YES:

"SOLUTION"

THEOREM. (GAGNÉ–PANANGADEN, 2023)

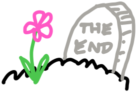
IF A FUNCTOR $F : \mathsf{StdBorel} \longrightarrow [0, \infty]$ IS LOWER-SEMICONTINUOUS, CONVEX LINEAR, + 1 OTHER THING, THEN ON MORPHISMS, F IS A REAL MULTIPLE OF KULLBACK-LEIBLER DIVERGENCE.

GENERALIZES B-F-L; ANSWERS OUR QUESTION SINCE A COUNTABLE SET /w DISCRETE σ-ALGEBRA GIVES A StdBorel.

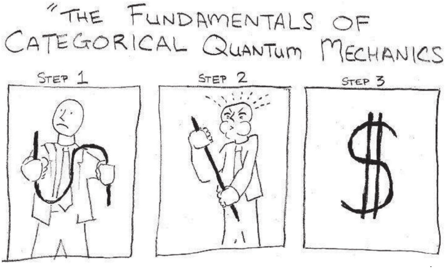
OBVIOUS NEW QUESTION: WHAT'S THE ANALOGUE FOR VON NEUMANN ENTROPY?

AT THE MOMENT, I DON'T HAVE A CONJECTURE (DON'T UNDERSTAND THE PAPER WELL ENOUGH YET!)

LIKELY OPEN, MAY BE HARD,
CERTAINLY WORTHWHILE.



THANKS FOR LISTENING!



SLIDES:

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