approach a unique number I as the subintervals of partitions shrink) even if f is not continuous.

a. Let $f(x) = 1$ for all x in [0, 1] except $x = \frac{1}{2}$, and $f(\frac{1}{2}) = 0$. Show that I exists, and find its value.

b. Let $g(x) = 1$ for all x in [0, 1] except for a finite collection of numbers $m_1, m_2, \ldots, m_k$ in (0, 1), and $g(m_j) = 0$ for $j = 1, 2, \ldots, k$. Show that I exists, and find its value.

c. Let $h(x) = 1$ for all rational numbers x in [0, 1], and $h(x) = 0$ for all irrational numbers x in [0, 1]. Show that there is *no* number I that all Riemann sums approach. Thus the integral $\int_0^1 h(x)\,dx$ cannot be defined by Riemann sums. (*Hint:* Every interval $[c, d]$ with $c < d$ contains both rational and irrational numbers.)

5.3 SPECIAL PROPERTIES OF THE DEFINITE INTEGRAL

In this section we return to the three basic properties of area from which our definition of integral was derived, and we present them in terms of integrals. The theorems of this section will be used repeatedly throughout the remainder of the book.

Before reformulating the three basic properties in terms of integrals, we recall that in Definition 5.4 we defined $\int_a^b f(x)\,dx$ under the stipulation that $a < b$. For theoretical purposes and for later applications it will be convenient to give meaning to $\int_a^a f(x)\,dx$ and $\int_b^a f(x)\,dx$ when $a < b$.

DEFINITION 5.6

Let f be continuous on $[a, b]$. Then

$$\int_a^a f(x)\,dx = 0 \quad \text{and} \quad \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

The definition $\int_a^a f(x)\,dx = 0$ is consistent with our expectation that a line segment has area 0 (Figure 5.21).

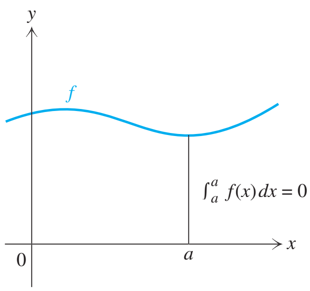


FIGURE 5.21 The area of a line segment is 0.

EXAMPLE 1 Evaluate $\displaystyle\int_4^1 x^2\,dx$.

Solution By Definition 5.6 we have

$$\int_4^1 x^2\,dx = -\int_1^4 x^2\,dx$$

and by (4) in Section 5.2 we know that

$$\int_1^4 x^2\,dx = \frac{1}{3}(4^3 - 1^3) = 21$$

Therefore we conclude that

$$\int_4^1 x^2\,dx = -21 \qquad \square$$

Integral Forms of the Three Basic Properties

The first property to appear is an integral form of the Rectangle Property, which is related to Example 1 of Section 5.2.

THEOREM 5.7
Rectangle Property

For any numbers a, b, and c,

$$\int_a^b c\,dx = c(b-a)$$

Proof If $a < b$, then the result follows directly from Example 1 of Section 5.2. If $a = b$, then by Definition 5.6,

$$\int_a^b c\,dx = \int_a^a c\,dx = 0 = c(b-a)$$

Finally, if $a > b$, then by combining Definition 5.6 and Example 1 of Section 5.2, we obtain

$$\int_a^b c\,dx = -\int_b^a c\,dx = -c(a-b) = c(b-a) \quad ■$$

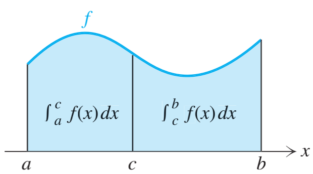


FIGURE 5.22

For the Addition Property we give a version that in special cases can be interpreted geometrically as follows: The area of a region composed of two smaller regions overlapping only in a line segment is the sum of the areas of the two regions (Figure 5.22).

Since the proof is technical, we have placed it in the Appendix.

THEOREM 5.8
Addition Property

Let f be continuous on an interval containing a, b, and c. Then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

EXAMPLE 2 Evaluate $\int_1^3 f(x)\,dx$, where

$$f(x) = \begin{cases} x & \text{for } 1 \le x \le 2 \\ 2 & \text{for } 2 \le x \le 3 \end{cases}$$

(See Figure 5.23.)

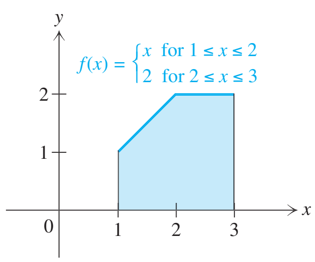


FIGURE 5.23

Solution Notice that f is continuous on [1, 3]. By the Addition Property,

$$\int_1^3 f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx$$

Now by Examples 1 and 2 of Section 5.2,

$$\int_1^2 f(x)\,dx = \int_1^2 x\,dx = \frac{1}{2}(2^2 - 1^2) = \frac{3}{2}$$

and

$$\int_2^3 f(x)\,dx = \int_2^3 2\,dx = 2(3-2) = 2$$

We conclude that

$$\int_1^3 f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx = \frac{3}{2} + 2 = \frac{7}{2} \qquad \square$$

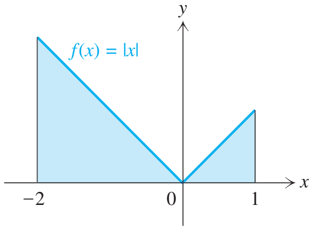


FIGURE 5.24

EXAMPLE 3 Evaluate $\int_1^{-2} |x|\,dx$ (see Figure 5.24).

Solution With the help of the Addition Property and Definition 5.6, we find that

$$\int_1^{-2} |x|\,dx = \int_1^0 |x|\,dx + \int_0^{-2} |x|\,dx = \int_1^0 x\,dx + \int_0^{-2} -x\,dx$$

$$= -\int_0^1 x\,dx + \int_{-2}^0 x\,dx = -\frac{1}{2}(1^2 - 0^2) + \frac{1}{2}[0^2 - (-2)^2]$$

$$= -\frac{1}{2} - 2 = -\frac{5}{2} \qquad \square$$

It turns out that the Addition Property is valid not only for functions that are continuous on $[a, b]$ but also functions that are continuous on $[a, b]$ except at a finite number of points in $[a, b]$, provided that at such points of discontinuity the function has both left and right limits. Such functions are said to be **piecewise continuous** because they are composed of continuous "pieces." For example, consider

$$f(x) = \begin{cases} 2 & \text{if } -1 \le x \le 0 \\ x & \text{if } 0 < x \le 1 \end{cases}$$

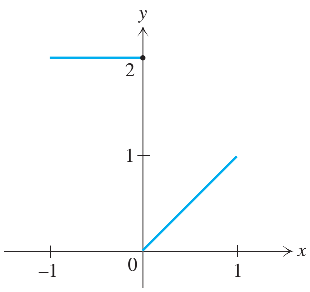


FIGURE 5.25

(See Figure 5.25.) Then f consists of two separate parts, one part defined on [–1, 0] and the other part defined on (0, 1]. Thus f is piecewise continuous on [–1, 1].

It is possible to prove that if f is piecewise continuous on $[a, b]$ and is continuous except at $\{c_1, c_2, \ldots, c_n\}$ in (a, b), then $\int_a^b f(x)\,dx$ exists, and that

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \cdots + \int_{c_n}^b f(x)\,dx$$

where each of the integrals on the right is evaluated by considering the continuous extension of f to the interval of integration.

For example, if

$$f(x) = \begin{cases} 2 & \text{if } -1 \le x \le 0 \\ x & \text{if } 0 < x \le 1 \end{cases}$$

then

$$\int_{-1}^1 f(x)\,dx = \int_{-1}^0 f(x)\,dx + \int_0^1 f(x)\,dx = \int_{-1}^0 2\,dx + \int_0^1 x\,dx = 2 + \frac{1}{2} = \frac{5}{2}$$

The third property of area is the Comparison Property. We will need only a special case of the Comparison Property. In geometric terms it implies that the area of any region is at least as large as that of any inscribed rectangle and no larger than that of any circumscribed rectangle (Figure 5.26).

THEOREM 5.9
Comparison Property

Let f be continuous on $[a, b]$, and suppose $m \le f(x) \le M$ for all x in $[a, b]$. Then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

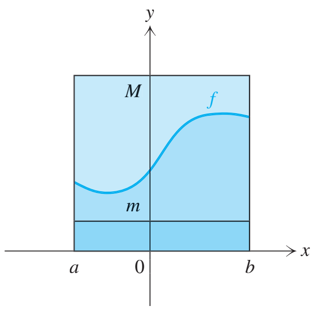


FIGURE 5.26

Proof Let $P = \{x_0, x_1, \ldots, x_n\}$ be any partition of $[a, b]$, and let t_k be in $[x_{k-1}, x_k]$ for each k. Since $m \le f(x) \le M$ for all x in $[a, b]$, it follows that necessarily $m \le f(t_k) \le M$ for all $k = 1, 2, \ldots, n$. Therefore since

$$\sum_{k=1}^{n} \Delta x_k = b - a$$

we have

$$m(b-a) = \sum_{k=1}^{n} m\Delta\, x_k \le \sum_{k=1}^{n} f(t_k)\Delta\, x_k \le \sum_{k=1}^{n} M\Delta\, x_k = M(b-a)$$

The inequality above is valid for *any* Riemann sum, and hence is valid for the limit of the Riemann sums, which is the integral $\int_a^b f(x)\,dx$. Consequently

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a) \quad \blacksquare$$

The Comparison Property is frequently used to estimate the value of an integral that cannot be computed exactly or easily. A number less than or equal to a given integral is called a **lower bound** for the integral, and a number greater than or equal to the integral is an **upper bound** for the integral.

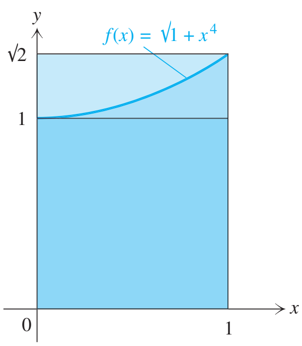


FIGURE 5.27

EXAMPLE 4 Using the Comparison Property, find lower and upper bounds for

$$\int_0^1 \sqrt{1 + x^4}\,dx$$

Solution If $0 \le x \le 1$, then $1 \le 1 + x^4 \le 2$, so that

$$1 \le \sqrt{1 + x^4} \le \sqrt{2} \quad \text{for } 0 \le x \le 1$$

(See Figure 5.27.) By the Comparison Property it follows that

$$1(1-0) \le \int_0^1 \sqrt{1 + x^4}\,dx \le \sqrt{2}(1-0)$$

so that

$$1 \le \int_0^1 \sqrt{1 + x^4}\,dx \le \sqrt{2}$$

Thus 1 is a lower bound and $\sqrt{2}$ is an upper bound for the integral. ❑

In Example 4 of Section 5.5 we will obtain a better, that is, smaller upper bound for $\int_0^1 \sqrt{1 + x^4}\, dx$.

Consequences of the Comparison Property

Our first consequence of the Comparison Property tells us that the integral of a nonnegative function is nonnegative.

COROLLARY 5.10

Let f be nonnegative and continuous on $[a, b]$. Then

$$\int_a^b f(x)\, dx \geq 0$$

Proof By hypothesis, $f(x) \geq 0$ for $a \leq x \leq b$. Therefore it is permissible to let $m = 0$ in Theorem 5.9. This implies that

$$\int_a^b f(x)\, dx \geq 0(b - a) = 0 \quad \blacksquare$$

From Corollary 5.10 it follows that the area of any region covered by Definition 5.5 is a nonnegative number. A second consequence of the Comparison Property is an integral form of the Mean Value Theorem.

THEOREM 5.11
Mean Value Theorem for Integrals

Let f be continuous on $[a, b]$. Then there is a number c in $[a, b]$ such that

$$\int_a^b f(x)\, dx = f(c)(b - a)$$

Proof If $a = b$, then the result is obvious. Thus assume that $a < b$, and let m and M be the minimum and the maximum values of f on $[a, b]$. By the Comparison Property we know that

$$m(b - a) \leq \int_a^b f(x)\, dx \leq M(b - a)$$

Since $a < b$, and hence $b - a > 0$, this means that

$$m \leq \frac{\int_a^b f(x)\, dx}{b - a} \leq M$$

Since f is continuous on $[a, b]$, the Intermediate Value Theorem asserts that there is a number c in $[a, b]$ such that

$$\frac{\int_a^b f(x)\, dx}{b - a} = f(c)$$

that is,

$$\int_a^b f(x)\, dx = f(c)(b - a) \quad \blacksquare$$

It follows from the Mean Value Theorem for Integrals that if f is nonnegative and continuous on $[a, b]$, then the area of the region bounded above by the graph of f is the same as the area of a rectangle whose height is $f(c)$ for some properly chosen c in $[a, b]$ (Figure 5.28).

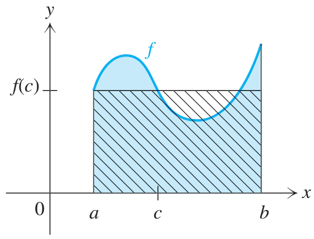


FIGURE 5.28

EXAMPLE 5 Let $f(x) = x^2$, and let A be the area of the region R between the graph of f and the x axis on $[0, 2]$. Find the value of c such that A is the same as the area of the rectangle with base $[0, 2]$ and height $f(c)$ (Figure 5.29).

Solution By (4) of Section 5.2,

$$A = \int_0^2 x^2\,dx = \frac{1}{3}(2^3 - 0^3) = \frac{8}{3}$$

Thus the rectangle must have area $\frac{8}{3}$ and the base must have length 2. Therefore the height $f(c)$ of the rectangle must be $\frac{4}{3}$. Consequently $c^2 = f(c) = \frac{4}{3}$, so that $c = \sqrt{4/3}$. ❑

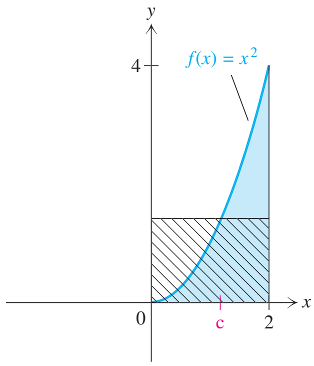


FIGURE 5.29

The value

$$\frac{1}{b-a}\int_a^b f(x)\,dx \qquad (1)$$

that arises from the Mean Value Theorem for Integrals is called the **mean** (or **average**) **value** of f on the interval $[a, b]$. When the interval is obvious, the mean or average value of f is denoted f_{av}.

EXAMPLE 6 Suppose that a ball drops into a hole 4 feet deep. Find the mean height during its fall.

Solution Measured in terms of feet, the height $h(t)$ of the ball after t seconds is given by $h(t) = -16t^2$ until the ball hits the bottom of the hole, which occurs after $\frac{1}{2}$ second. Why? It follows that the mean value h_{av} of the height is given by

$$h_{av} = \frac{1}{1/2 - 0}\int_0^{1/2} (-16t^2)\,dt = 2\int_0^{1/2} (-16t^2)\,dt$$

By (4) in Section 5.2,

$$\int_0^{1/2} (-16t^2)\,dt = -\frac{16}{3}\left[\left(\frac{1}{2}\right)^3 - 0^3\right] = -\frac{2}{3}$$

Therefore

$$h_{av} = 2\left(-\frac{2}{3}\right) = -\frac{4}{3}$$

We conclude that the mean height of the ball is $-\frac{4}{3}$ feet. ❑

The reason that the mean height of the ball in Example 6 is not –2 feet is that the ball drops more slowly at the beginning of its journey and more swiftly at the end.

Later we will see that if f represents velocity, then the mean value of velocity on $[a, b]$ is the same as the average velocity defined in Section 2.1. Corresponding statements apply to mean cost and mean revenue.

EXERCISES 5.3

In Exercises 1–4 use the Rectangle Property to evaluate the integral.

1. $\int_3^5 7\,dx$ **2.** $\int_{-1}^2 -3\,dx$

3. $\int_2^{-1} -10\,du$ **4.** $\int_1^{-1} 5\,dx$

In Exercises 5–8 evaluate the integrals to corroborate the Addition Property for integrals. Use the results of Section 5.2 in your calculations.

5. $\int_0^1 x\,dx + \int_1^2 x\,dx = \int_0^2 x\,dx$

6. $\int_3^4 x^2\,dx + \int_4^3 x^2\,dx = \int_3^3 x^2\,dx$

7. $\int_1^0 y^2\,dy + \int_0^2 y^2\,dy = \int_1^2 y^2\,dy$

8. $\int_{-2}^{-3} -y\,dy + \int_{-3}^{-6} -y\,dy = \int_{-2}^{-6} -y\,dy$

In Exercises 9–12 let f be a continuous function on $(-\infty, \infty)$. Use the Addition Property to find the values of a and b that make the equation true.

9. $\int_0^2 f(x)\,dx + \int_3^0 f(x)\,dx = \int_a^b f(x)\,dx$

10. $\int_{1/2}^{-1/2} f(x)\,dx + \int_{-1}^{1/2} f(x)\,dx = \int_a^b f(x)\,dx$

11. $\int_a^b f(t)\,dt - \int_5^3 f(t)\,dt = \int_3^1 f(t)\,dt$

12. $\int_\pi^{2\pi} f(t)\,dt - \int_a^b f(t)\,dt = \int_{3\pi}^{2\pi} f(t)\,dt$

In Exercises 13–16 find the maximum and minimum values of the given function on the given interval. Then use the Comparison Property to find upper and lower bounds for the area of the region between the graph of the function and the x axis on the given interval.

13. $f(x) = 1/x$; $[2, 3]$ **14.** $g(x) = \sin x$; $[\pi/4, \pi/2]$

15. $g(x) = \cos x$; $[\pi/4, \pi/3]$ **16.** $h(t) = \tan t$; $[0, \pi/3]$

In Exercises 17–20 find the mean value of f on the given interval.

17. $f(x) = x$; $[0, 1]$ **18.** $f(x) = x$; $[-2, 2]$

19. $f(x) = x^2$; $[-1, 1]$ **20.** $f(x) = |x|$; $[-2, 3]$

21. Let $f(x) = x$ for $a \le x \le b$. Show that the mean value of f on $[a, b]$ is $(a + b)/2$.

***22.** Let $0 \le a < b$ and let $f(x) = x^2$.

a. Show that the mean value of f on $[a, b]$ is
$\frac{1}{3}(a^2 + ab + b^2)$

b. Show that there is a number c in $[a, b]$ such that

$$c^2 = \frac{1}{3}(a^2 + ab + b^2)$$

23. Let

$$f(x) = \begin{cases} -x & \text{for } x < 0 \\ x^2 & \text{for } x \ge 0 \end{cases}$$

Find the area A of the region between the graph of f and the x axis on $[-1, 1]$.

24. Let

$$f(x) = \begin{cases} 1 & \text{for } 0 \le x \le 1 \\ x^2 & \text{for } x > 1 \end{cases}$$

Find the area A of the region between the graph of f and the x axis on $[0, 4]$.

25. Consider the graph of f in Figure 5.30.

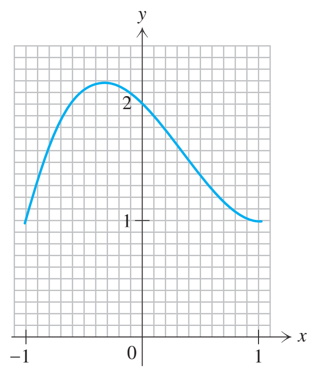


FIGURE 5.30 Graph for Exercise 25.

a. By the graph alone, predict the mean value of f on $[-1, 1]$.

b. Use the left sum with 10 subintervals to approximate $\int_{-1}^{1} f(x)\,dx$ and then the mean value of f. (The actual mean value of f on $[-1, 1]$ is $\frac{5}{3}$.)

26. Let $f(x) = \sin x^2$ for $0 \le x \le \sqrt{\pi}$.

a. Plot the graph of f.

b. Predict the mean value f_{av} of f on $[0, \sqrt{\pi}]$. Call your prediction A.

c. Use the midpoint sum with $n = 100$ in order to find an approximate value B of f_{av}.

d. Compare the values A and B.

27. Prove the Addition Property for the cases $c < a < b$ and $b < a < c$.

28. Show that the Comparison Property is a consequence of the Mean Value Theorem for Integrals.

Applications

29. If T represents the temperature at time t, then the mean temperature during an interval of length b is $\frac{1}{b}\int_0^b T(t)\,dt$. In degrees Fahrenheit the average temperature for each of the months of the year in New York City is as follows:

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
32	33	41	53	62	71	77	75	68	58	47	36

First prove, using the Addition Property, that the mean temperature during the year is the average of the mean temperatures during the twelve months. Then use the values in the table to find the mean temperature in New York during the year.

30. Suppose that the temperature T is taken regularly during a 24-hour period. By definition the average of n successive readings is given by

$$\frac{1}{n}\sum_{k=1}^{n} T\left(\frac{24k}{n}\right)$$

By using an appropriate Riemann sum, show that if n is large then the average temperature over n readings approximates the mean temperature over the same period.

Project

1. This project focuses on the integral of a function f that is continuous on $[a, b]$ and whose values are positive except possibly at the endpoints a and b.

a. Show that $\int_{\pi/4}^{\pi/2} \sin x\,dx > 0$.

b. Show that $\int_0^{\pi/2} \sin x\,dx > 0$ (*Hint:* Use the Addition Property with $a = 0$, $b = \pi/2$, and $c = \pi/4$, and then apply the Comparison Property to each of the resulting integrals.)

c. Suppose f is continuous on $[a, b]$. Assume that $f(x) \ge 0$ for all x in $[a, b]$ and $f(x) > 0$ for at least one value of x in $[a, b]$. Show that $\int_a^b f(x)\,dx > 0$.

d. Let f be a continuous and nonnegative on $[a, b]$, and assume that $\int_a^b f(x)\,dx = 0$. Show that $f(x) = 0$ for *all* x in $[a, b]$.

5.4 THE FUNDAMENTAL THEOREM OF CALCULUS

The purpose of this section is to develop a general method for evaluating $\int_a^b f(x)\,dx$ that does not necessitate computing various sums. The method will allow us to evaluate many (but not all) of the integrals that arise in applications.

We begin by letting I be any interval and c any number in I. Suppose f is continuous on I. Then for each x in I, f is necessarily continuous on the closed interval whose endpoints are c and x. Consequently we can associate with any such x the number $\int_c^x f(t)\,dt$ in order to obtain a function G that is defined by

$$G(x) = \int_c^x f(t)\,dt \quad \text{for } x \text{ in } I \tag{1}$$

Here we have used t inside the integral, rather than x, because x appears as a limit of integration.

For example, if $f(x) = x$, $I = (-\infty, \infty)$, and $c = 1$, then by (3) in Section 5.2,

$$G(0) = \int_1^0 t\,dt = -\int_0^1 t\,dt = -\frac{1}{2}(1^2 - 0^2) = -\frac{1}{2}$$

$$G(1) = \int_1^1 t\,dt = 0$$

$$G(2) = \int_1^2 t\,dt = \frac{1}{2}(2^2 - 1^2) = \frac{3}{2}$$

In fact, for any x,

$$G(x) = \int_1^x t\,dt = \frac{1}{2}(x^2 - 1)$$

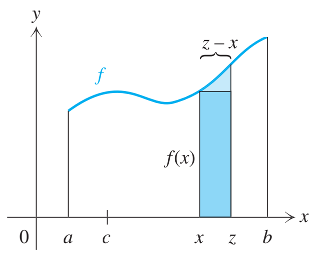


FIGURE 5.31

Notice that in this example, $G'(x) = x = f(x)$ for all x. We will prove more generally that if f is *any* function that is continuous on an interval I, and if G is defined as in (1), then $G'(x) = f(x)$ for all x in I, so that G is an antiderivative of f on I.

To see geometrically why we can expect to have $G' = f$, let us suppose that f is continuous and nonnegative on I, and let x and z be any numbers in I with $x < z$, as in Figure 5.31. Since $f \geq 0$, it follows from the definition of G in (1) that

$G(z)$ = area of the region between the graph of f and the x axis on $[c, z]$

$G(x)$ = area of the region between the graph of f and the x axis on $[c, x]$

Therefore $G(z) - G(x)$ is the area of the entire shaded region in Figure 5.31, and if z is close to x, this area appears to be close to the area $f(x)(z - x)$ of the darkly shaded rectangle in the figure. We deduce that if $z > x$ and z is close to x, then $G(z) - G(x)$ should be close to $f(x)(z - x)$, and hence that

$$\frac{G(z) - G(x)}{z - x} \quad \text{should be close to } f(x)$$

A similar analysis would show that this approximation also holds if $z < x$ and z is close to x. We conclude intuitively that

$$\lim_{z \to x} \frac{G(z) - G(x)}{z - x} = f(x)$$

that is,

$$G'(x) = f(x)$$

We now state the result in the preceding paragraph formally and prove it.

THEOREM 5.12

Let f be continuous on an interval I (containing more than one point) and let c be any point in I. Define G by the equation

$$G(x) = \int_c^x f(t)\,dt \quad \text{for all } x \text{ in } I$$

Then G is differentiable on I, and

$$G'(x) = f(x) \quad \text{for all } x \text{ in } I \tag{2}$$

In particular, f has an antiderivative on I.

Proof Let x be any number interior to I. We must show that

$$\lim_{z\to x} \frac{G(z) - G(x)}{z - x} = f(x)$$

To that end, let z be any number in I with $z \neq x$. The Addition Property tells us that

$$G(z) - G(x) = \int_c^z f(t)\,dt - \int_c^x f(t)\,dt = \int_x^z f(t)\,dt$$

If ε is any positive number, then because f is assumed to be continuous at x, there is a $\delta > 0$ such that if $|t - x| < \delta$, then $|f(t) - f(x)| < \varepsilon$, that is, $f(x) - \varepsilon < f(t) < f(x) + \varepsilon$. Now suppose that $0 < |z - x| < \delta$. By the Comparison Property, we find that if $x < z$, then

$$(f(x) - \varepsilon)(z - x) \le \int_x^z f(t)\,dt = G(z) - G(x) \le (f(x) + \varepsilon)(z - x)$$

and if $x > z$, then

$$(f(x) - \varepsilon)(x - z) \le \int_z^x f(t)\,dt = G(x) - G(z) \le (f(x) + \varepsilon)(x - z)$$

If we divide through the first set of inequalities by $z - x$, and the second set by $x - z$, then each set of inequalities reduces to

$$f(x) - \varepsilon \le \frac{G(z) - G(x)}{z - x} \le f(x) + \varepsilon, \text{ that is, } \left| \frac{G(z) - G(x)}{z - x} - f(x) \right| \le \varepsilon$$

Therefore

$$G'(x) = \lim_{z\to x} \frac{G(z) - G(x)}{z - x} = f(x)$$

If x is an endpoint of I, then $G'(x)$ is just the appropriate one-sided limit, and the foregoing argument can be altered to apply. ■

EXAMPLE 1 Let $G(x) = \int_1^x (1/t^2)\,dt$ for $x > 0$. Find $G'(x)$.

Solution By Theorem 5.12, with $I = (0, \infty)$ and $c = 1$, we have

$$G'(x) = \frac{1}{x^2} \quad \text{for } x > 0$$ □

EXAMPLE 2 Let $G(x) = \int_0^x t \sin t^3\,dt$ for all x. Find $G'(x)$.

Solution Again by Theorem 5.12,

$$G'(x) = x \sin x^3$$ □

At this point let us recall from Theorem 4.6 that any two antiderivatives of a function f differ by a constant. For example, the antiderivatives of $2x$ all have the form $x^2 + C$, where C is a constant. For easy reference we list in Table 5.1 a few of the basic functions, along with their antiderivatives. To verify any entry in the column of antiderivatives, simply differentiate it and observe that its derivative is the same as the corresponding entry in the left column.

TABLE 5.1
Table of Antiderivatives

Function	Antiderivative
c (c a constant)	$cx + C$ (C any constant)
x	$\frac{1}{2}x^2 + C$
x^2	$\frac{1}{3}x^3 + C$
$px + q$ (p and q constants)	$\frac{1}{2}px^2 + qx + C$
x^r ($r \neq -1$)	$\frac{1}{r+1}x^{r+1} + C$
$\frac{1}{x}$ ($x > 0$)	$\ln x + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
e^x	$e^x + C$

Now we are ready for the most important theorem in calculus. It shows that the notions of derivative and integral are intimately related, and also provides an effective method of evaluating numerous integrals.

THEOREM 5.13
Fundamental Theorem of Calculus

Let f be continuous on $[a, b]$.

a. The function G defined by

$$G(x) = \int_a^x f(t)\,dt, \quad \text{for} \quad x \quad in \quad [a, b]$$

is an antiderivative of f on $[a, b]$.

b. If F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Proof Part (a) follows immediately from Theorem 5.12. To prove (b), we let G be the function defined in (a). Then

$$G(a) = \int_a^a f(t)\,dt = 0 \quad \text{and} \quad G(b) = \int_a^b f(t)\,dt$$

Now if F is any antiderivative of f, then by Theorem 4.6 we know that $F = G + C$

for some constant C. Consequently

$$\int_a^b f(t)\,dt = G(b) = G(b) - G(a) = [F(b) - C] - [F(a) - C]$$

$$= F(b) - F(a) \quad ■$$

EXAMPLE 3 Evaluate $\int_0^2 x^2\,dx$.

Solution From Table 5.1 we know that if $F(x) = \frac{1}{3}x^3$ then F is an antiderivative of x^2, so that by the Fundamental Theorem,

$$\int_0^2 x^2\,dx = F(2) - F(0) = \frac{1}{3}\cdot 2^3 - \frac{1}{3}\cdot 0^3 = \frac{8}{3} \quad □$$

It is usually simplest to dispense with the symbol F when evaluating integrals. Instead we write, for example,

$$\int_0^2 x^2\,dx = \left.\frac{x^3}{3}\right|_0^2 = \frac{1}{3}\cdot 2^3 - \frac{1}{3}\cdot 0^3 = \frac{8}{3}$$

where the expression $|_0^2$ indicates the numbers at which the antiderivative is to be evaluated, in this case 0 and 2.

EXAMPLE 4 Evaluate $\int_1^4 x^{1/2}\,dx$.

Solution Since $\frac{2}{3}x^{3/2}$ is an antiderivative of $x^{1/2}$, the Fundamental Theorem asserts that

$$\int_1^4 x^{1/2}\,dx = \left.\frac{2}{3}x^{3/2}\right|_1^4 = \frac{2}{3}(4^{3/2}) - \frac{2}{3}(1^{3/2}) = \frac{16}{3} - \frac{2}{3} = \frac{14}{3} \quad □$$

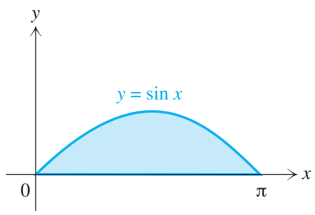


FIGURE 5.32 The area under one arch of the sine curve is 2.

EXAMPLE 5 Evaluate $\int_0^\pi \sin x\,dx$.

Solution Because $-\cos x$ is an antiderivative of $\sin x$, we know from the Fundamental Theorem that

$$\int_0^\pi \sin x\,dx = -\cos x\,\Big|_0^\pi = -\cos\pi - (-\cos 0) = -(-1) - (-1) = 2 \quad □$$

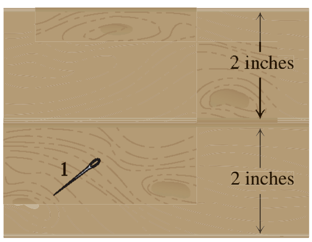


FIGURE 5.33 Buffon's needle problem.

From Example 5 it follows that the area of the region in Figure 5.32 is 2. Without the Fundamental Theorem this result would have been very hard to obtain. The integral in Example 5 arises in the famous eighteenth-century problem known as **Buffon's needle problem.** In this problem a 1-inch-long needle is dropped onto a hardwood floor whose boards are 2 inches wide. The question is: What is the probability that the needle will come to rest lying across two boards (Figure 5.33)?

See the project for this section for the details.

EXAMPLE 6 Evaluate $\int_{-1}^{1} (1 + 2x)\, dx$.

Solution Using Table 5.1 or some trial and error, we find that $x + x^2$ is an antiderivative of $1 + 2x$. Thus

$$\int_{-1}^{1} (1 + 2x)\, dx = (x + x^2)\Big|_{-1}^{1} = (1 + 1^2) - [-1 + (-1)^2] = 2 \quad \square$$

EXAMPLE 7 Evaluate $\int_{-1}^{1} e^x\, dx$.

Solution As indicated in Table 5.1, e^x is an antiderivative of e^x. Therefore

$$\int_{-1}^{1} e^x\, dx = e^x\Big|_{-1}^{1} = e^1 - e^{-1} = e - 1/e \quad \square$$

We can extend the Fundamental Theorem of Calculus to the case in which the lower limit of integration is greater than the upper limit.

COROLLARY 5.14 Let f be continuous on $[a, b]$. Then for any antiderivative F of f,

$$\int_{b}^{a} f(t)\, dt = F(a) - F(b)$$

Proof By the Fundamental Theorem,

$$\int_{a}^{b} f(t)\, dt = F(b) - F(a)$$

Therefore by Definition 5.6,

$$\int_{b}^{a} f(t)\, dt = -\int_{a}^{b} f(t)\, dt = -[F(b) - F(a)] = F(a) - F(b) \quad \blacksquare$$

As a consequence of the Fundamental Theorem and Corollary 5.14, once we know an antiderivative F of f, we can compute $\int_a^b f(t)\, dt$ by the formula

$$\int_{a}^{b} f(t)\, dt = F(b) - F(a) \tag{3}$$

whether $a < b$, $a > b$, or $a = b$.

EXAMPLE 8 Find $\int_{2}^{1} 2t^3\, dt$.

Solution Since $\dfrac{d}{dt}\left(\dfrac{1}{2}t^4\right) = 2t^3$, we deduce from (3) that

$$\int_{2}^{1} 2t^3\,dt = \frac{1}{2}t^4\Big|_2^1 = \frac{1}{2}\cdot 1^4 - \frac{1}{2}\cdot 2^4 = \frac{1}{2} - 8 = -\frac{15}{2} \quad \square$$

Formula (2) can be restated as

$$\boxed{\frac{d}{dx}\int_c^x f(t)\,dt = f(x)} \tag{4}$$

Using the Fundamental Theorem and the Chain Rule we can evaluate

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt$$

where g and h are differentiable functions of x, and where f is continuous between $g(x)$ and $h(x)$ for all appropriate values of x. Indeed, let F be any antiderivative of f. By (3) with $g(x)$ replacing a and $h(x)$ replacing b, we have

$$\int_{g(x)}^{h(x)} f(t)\,dt = F(h(x)) - F(g(x))$$

Therefore using the Chain Rule and the fact that $F' = f$, we find that

$$\begin{aligned}\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt &= \frac{d}{dx}[(F(h(x)) - F(g(x))] = F'(h(x))\,h'(x) - F'(g(x))\,g'(x)\\ &= f(h(x))\,h'(x) - f(g(x))\,g'(x)\end{aligned}$$

We conclude that

$$\boxed{\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))\,h'(x) - f(g(x))\,g'(x)} \tag{5}$$

EXAMPLE 9 Let $K(x) = \int_{x^2}^{x^3} \sin t^7\,dt$. Find $K'(x)$.

Solution From (5) with $f(x) = \sin x^7$, $g(x) = x^2$, and $h(x) = x^3$, we deduce that

$$K'(x) = \frac{d}{dx}\int_{x^2}^{x^3} \sin t^7\,dt = [\sin (x^3)^7](3x^2) - [\sin (x^2)^7](2x) = 3x^2 \sin (x^{21}) - 2x \sin (x^{14}) \quad \square$$

Differentiation and Integration as Inverse Processes

According to (4), if we start with a continuous function f, integrate it to obtain $\int_c^x f(t)\,dt$, and then differentiate, the result is the original function f. Thus the differentiation has nullified the integration. On the other hand, if we start with a function F having a continuous derivative, first differentiate, and then integrate, we obtain $\int_c^x F'(t)\,dt$. But by the Fundamental Theorem of Calculus,

$$\boxed{\int_c^x F'(t)\,dt = F(x) - F(c)} \tag{6}$$

so we obtain the original function F altered by at most a constant. This time the integration has essentially nullified the differentiation. Thus the two basic processes of calculus, differentiation and integration, are inverses of each other.

Furthermore, whenever we know the derivative F' of a function F, (6) gives us an integration formula. For example, we know already that

$$\frac{d}{dx}\tan x = \sec^2 x$$

Therefore (6) tells us that

$$\int_{\pi/4}^{x} \sec^2 t\, dt = \tan x - \tan\frac{\pi}{4} = \tan x - 1$$

provided that x is in the interval $(-\pi/2, \pi/2)$.

Formula (6) has numerous applications. For example, in economics the marginal revenue function m_R is by definition the derivative of the total revenue function R. Thus by (6),

$$R(x) - R(c) = \int_c^x R'(t)\, dt = \int_c^x m_R(t)\, dt \tag{7}$$

Likewise, the marginal cost function m_C is by definition the derivative of the total cost function C. Consequently by (6),

$$C(x) - C(c) = \int_c^x C'(t)\, dt = \int_c^x m_C(t)\, dt \tag{8}$$

In physics, the velocity of a particle moving along a straight line is the derivative of the position function. If we use t for the independent variable representing time, f for the position, v for velocity, and u for the variable of integration, we obtain, by suitable substitution in (6),

$$f(t) - f(t_0) = \int_{t_0}^{t} v(u)\, du \tag{9}$$

In (9) the number t_0 is arbitrary, and it plays the same role as c does in (6). In applications t_0 is usually a special instant of time. When t_0 is the moment at which motion begins, it is called the **initial time.**

The acceleration a of a particle is the derivative of the velocity v. Hence we obtain

$$v(t) - v(t_0) = \int_{t_0}^{t} a(u)\, du \tag{10}$$

Near the surface of the earth, the acceleration due to gravity is essentially constant, approximately –9.8 meters per second per second. Assuming that an object is under the sole influence of gravity, we can derive the formula

$$h(t) = -4.9t^2 + v_0 t + h_0 \tag{11}$$

for the height of the object at time t. Formula (11) was first presented (without proof) in Section 1.3.

Isaac Barrow (1630–1677)
Though unruly in his school days, Barrow achieved recognition in mathematics, physics, astronomy, theology, and Greek. In fact, before he accepted the newly created Lucasian mathematics chair at Cambridge University, he was nominated for a Greek professorship. Barrow's most important work, *Lectiones Geometricae,* was based on his lectures during the Great Plague years. During that time Newton was laying the foundation for future monumental discoveries in mathematics, optics, and astronomy. When Barrow resigned his professorship in 1669 in order to serve as chaplain to Charles II, he proposed Newton to be his successor.

EXAMPLE 10 Suppose an object experiences a constant acceleration of –9.8 meters per second per second. Assume that at time $t = 0$ its initial height is h_0 and initial velocity is v_0. Show that the height $h(t)$ of the object at any time $t > 0$ is given by (11).

Solution Using (10) with $t_0 = 0$ and $a(u) = -9.8$, we find that

$$v(t) - v_0 = v(t) - v(0) = \int_0^t -9.8\, du = -9.8u \Big|_0^t = -9.8t$$

Thus

$$v(t) = v_0 - 9.8t$$

From this equation, and from (9) with $t_0 = 0$ and f replaced by h, we find that

$$h(t) - h_0 = h(t) - h(0) = \int_0^t (v_0 - 9.8u)\, du = (v_0 u - 4.9u^2) \Big|_0^t = v_0 t - 4.9t^2$$

Therefore $h(t) = -4.9t^2 + v_0 t + h_0$, which is (11). ❑

Because differentiation and integration arose from apparently unrelated problems (such as tangents and areas), it was only after mathematicians had worked for centuries with concepts related to derivatives and integrals separately that Isaac Barrow, who was Newton's teacher, discovered and proved the Fundamental Theorem. His proof was completely geometric, and his terminology far different from ours. Beginning with the work of Newton and Leibniz, the theorem grew in importance, eventually becoming the cornerstone for the study of integration.

EXERCISES 5.4

In Exercises 1–10 find the derivative of each function.

1. $F(x) = \displaystyle\int_0^x t(1 + t^3)^{29}\, dt$
2. $F(x) = \displaystyle\int_3^x \frac{1}{(t + t^3)^{16}}\, dt$
3. $F(y) = \displaystyle\int_y^2 \frac{1}{t^3}\, dt$
4. $F(t) = \displaystyle\int_t^0 x \sin x\, dx$
5. $F(x) = \displaystyle\int_0^{x^2} t \sin t\, dt$
6. $F(x) = \displaystyle\int_0^{-x} e^{(t^2)}\, dt$
7. $G(y) = \displaystyle\int_y^{y^2} (1 + t^2)^{1/2}\, dt$
8. $F(x) = \displaystyle\int_{x^2}^{x^3} (1 + t^2)^{1/2}\, dt$
9. $F(x) = \displaystyle\frac{d}{dx}\int_0^{4x} (1 + t^2)^{4/5}\, dt$
10. $G(y) = \displaystyle\frac{d}{dy}\int_{\sin y}^{2y} \cos t\, dt$

In Exercises 11–40 use (3) to evaluate the integral.

11. $\displaystyle\int_0^1 4\, dx$
12. $\displaystyle\int_1^{12} 0\, dx$
13. $\displaystyle\int_1^3 -y\, dy$
14. $\displaystyle\int_5^2 -4t\, dt$
15. $\displaystyle\int_1^{-3} 3u\, du$
16. $\displaystyle\int_{-b}^b x^5\, dx$, b a constant

17. $\int_0^1 x^{100}\,dx$ **18.** $\int_0^2 u^{1/2}\,du$

19. $\int_{-1}^1 u^{1/3}\,du$ **20.** $\int_{16}^2 x^{5/4}\,dx$

21. $\int_1^4 x^{-7/9}\,dx$ **22.** $\int_0^1 x^{12/5}\,dx$

23. $\int_{-1.5}^{2\pi} (5-x)\,dx$ **24.** $\int_0^3 \left(\frac{1}{2}x - 4\right)dx$

25. $\int_{-4}^{-1} (5x + 14)\,dx$ **26.** $\int_0^{\pi/6} \cos x\,dx$

27. $\int_{-\pi}^{\pi/3} \cos x\,dx$ **28.** $\int_{\pi/3}^{\pi/4} \sin t\,dt$

29. $\int_{\pi/3}^{-\pi/4} \sin t\,dt$ **30.** $\int_2^3 \frac{1}{x^3}\,dx$

31. $\int_1^2 \frac{1}{y^4}\,dy$ **32.** $\int_{-1}^{-2} \left(x - \frac{5}{x^3}\right)dx$

33. $\int_2^4 \frac{1}{x}\,dx$ **34.** $\int_1^e \frac{2}{x}\,dx$

35. $\int_0^2 e^x\,dx$ **36.** $\int_1^{\ln 3} e^x\,dx$

37. $\int_{\pi/6}^{\pi/2} \csc^2 t\,dt$ **38.** $\int_0^{\pi/4} \sec x \tan x\,dx$

39. $\int_0^{\pi/2} \left(\frac{d}{dx}\sin^5 x\right)dx$ **40.** $\int_{-1}^1 \left(\frac{d}{dx}\sqrt{1 + x^4}\right)dx$

In Exercises 41–52 compute the area A of the region between the graph of f and the x axis on the given interval.

41. $f(x) = x^4$; $[-1, 1]$ **42.** $f(x) = 1/x^2$; $[-2, -1]$

43. $f(x) = \sin x$; $[0, 2\pi/3]$

44. $f(x) = \cos x$; $[-\pi/2, \pi/3]$

45. $f(x) = x^{1/2}$; $[1, 4]$ **46.** $f(x) = x^{1/3}$; $[1, 8]$

47. $f(x) = \sec^2 x$; $[0, \pi/4]$ **48.** $f(x) = \csc x \cot x$; $[\pi/4, \pi/2]$

49. $f(x) = 1/x$; $[1/e, 1]$ **50.** $f(x) = 2 + e^x$; $[0, 4]$

***51.** $f(x) = \cos^2 x$; $[0, \pi/2]$
(*Hint:* What is the derivative of $x/2 + (\sin 2x)/4$?)

52. $f(x) = \sec x \tan^3 x$; $[0, \pi/3]$
(*Hint:* What is the derivative of $(\sec^3 x)/3 - \sec x$?)

53. **a.** Show that if n is an odd positive integer, then

$$\int_{-1}^1 x^n\,dx = 0.$$

b. Show that if n is an even nonnegative integer, then

$$\int_{-1}^1 x^n\,dx = \frac{2}{n+1}.$$

***54.** Suppose $f(1) = 10$ and the graph of the derivative f' is as in Figure 5.34. Find $f(3)$.

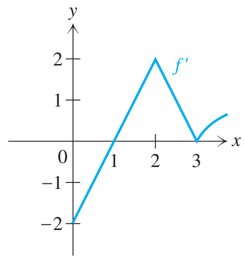


FIGURE 5.34 Graph for Exercise 54

55. Evaluate $\int_0^x f(t)\,dt$ for each of the following functions. By differentiating the resulting function, verify formula (4) of this section.

a. $f(x) = x$ **b.** $f(x) = -2x^2$
c. $f(x) = -\sin x$ **d.** $f(x) = 10x^4$

56. In each of the following, verify formula (6) by first differentiating F, then integrating F' from a to x, and finally comparing that result with $f(x)$.

a. $F(x) = x + 2$; $a = 1$ **b.** $F(x) = x^3$; $a = 1$
c. $F(x) = x^4$; $a = -1$

57. Find the number I satisfying

$$(x_0^2 + 4x_0)\,\Delta x_1 + \cdots + (x_{n-1}^2 + 4x_{n-1})\,\Delta x_n \le I \le (x_1^2 + 4x_1)\,\Delta x_1 + \cdots + (x_n^2 + 4x_n)\,\Delta x_n$$

for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[1, 2]$.

58. Find the number I satisfying

$$(\cos x_1 - \sin x_1)\,\Delta x_1 + \cdots + (\cos x_n - \sin x_n)\,\Delta x_n \le I \le (\cos x_0 - \sin x_0)\,\Delta x_1 + \cdots + (\cos x_{n-1} - \sin x_{n-1})\,\Delta x_n$$

for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[0, \pi/2]$.

Applications

59. Suppose the velocity of a car, which starts from the origin at $t = 0$ and moves along the x axis, is given by

$$v(t) = 10t - t^2 \quad \text{for } 0 \le t \le 10$$

Find the position of the car
a. at any time t, with $0 \le t \le 10$.
b. when its acceleration is 0.

60. The velocity of a bob moving along the x axis on a spring varies with time according to the equation

$$v(t) = 2\sin t + 3\cos t$$

At $t = 0$ the position of the bob is 1. Express the position of the bob as a function of time.

61. The flow of water through a dam is controlled so that the rate $F'(t)$ of flow in tons per hour is given by the equation

$$F'(t) = 14{,}000 \sin \frac{\pi t}{24} \quad \text{for } 0 \leq t \leq 24$$

How many tons of water flow through the dam per day? (*Hint:* Use formula (6) and the fact that

$$\frac{d}{dt}\left(-\frac{336{,}000}{\pi} \cos \frac{\pi t}{24}\right) = F'(t)$$

for $0 \leq t \leq 24$.)

62. Suppose the velocity of a person walking up and down Main Street is as shown in Figure 5.35.

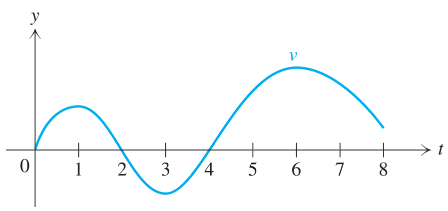


FIGURE 5.35 Graph for Exercise 62.

a. At what time(s) did the person change directions?

b. At what time(s) was the person farthest from the starting position?

c. At what time(s) was the acceleration equal to zero?

63. Suppose the revenue from the first pound of soap sold is \$4 and the marginal revenue is given by

$$m_R(x) = 4 - 0.02x \quad \text{for } 1 \leq x \leq 400$$

a. What is the revenue accruing from a sale of 30 pounds of soap? (*Hint:* Use formula (7).)

b. What is the revenue at the point at which the marginal revenue is 0?

64. Assume that the cost of producing the first two pounds of soap is \$10.98 and that the marginal cost is given by

$$m_C(x) = 3 - 0.1x \quad \text{for } 0 \leq x \leq 30$$

Find the total cost involved in producing 30 pounds of soap. (*Hint:* Use formula (8).)

65. To make a balloon rise faster, the balloonist drops a sandbag from the bottom of the balloon.

a. If the sandbag is ejected at an elevation of 528 feet and takes 6 seconds to hit the ground, what was the speed of the balloon at the time of ejection?

b. If the balloon rises at 4 feet per second and is 992 feet above the ground when the sandbag is ejected, how long will it take the sandbag to hit the ground?

A hot-air balloon. (*Larry Brownstein/Rainbow*)

66. A train is cruising at 60 miles per hour when suddenly the engineer notices a cow on the track ahead of the train. The engineer applies the brakes, causing a constant deceleration in the train. Two minutes later the train grinds to a halt, barely touching the cow, which is too frightened to move. How far back was the train when the brakes were applied?

67. Radium disintegrates at a variable rate. Let $R(t)$ be the rate at which radium disintegrates at time t. Express the total amount lost between times t_1 and t_2 as an integral.

68. a. For $a \leq t \leq b$ let $v(t)$ be the velocity of an object at time t. Using the Fundamental Theorem, show that the mean value

$$\frac{\int_a^b v(t)\, dt}{b - a}$$

of the velocity is the average velocity of the object as defined in Section 2.1.

b. Suppose a ball thrown from the top of a cliff has the velocity

$$v(t) = -20 - 32t \quad \text{for } 0 \leq t \leq 3$$

Find the ball's average velocity during its flight.

69. a. Let $C(x)$ be the cost of producing x units of a product for $a \leq x \leq b$. Using the Fundamental Theorem, show that the mean value

$$\frac{\int_a^b m_C(x)\, dx}{b - a}$$

of the cost is the average cost between the ath and bth units produced, as defined in Section 3.1.

b. Suppose an umbrella manufacturing company has a marginal cost function given by

$$m_C(x) = \frac{1}{x^{1/2}} \quad \text{for } 1 \le x \le 4$$

where x represents thousands of umbrellas produced and $m_C(x)$ is measured in thousands of dollars. Find the mean cost between the one-thousandth umbrella produced and the four-thousandth umbrella produced.

70. Suppose the voltage in an electrical circuit varies with time according to the formula

$$V(t) = 110 \sin t \quad 0 \le t \le \pi$$

Find the mean voltage in the circuit during the given time interval.

71. A cylindrical tank 100 feet high and 100 feet in diameter is full of water. The work W (in foot-pounds) required to pump all the water out of the tank is given by

$$W = (2500\pi)(62.5)\int_0^{100} (100 - y)\, dy$$

Compute the work required.

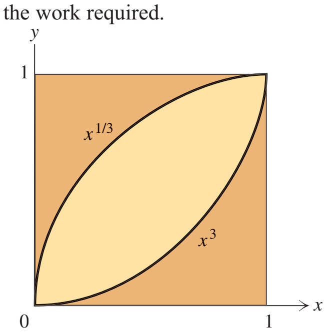


FIGURE 5.36 Graph for Exercise 72.

72. A tile manufacturing company plans to produce enamel tiles with the design and color scheme shown in Figure 5.36. Will more peach enamel be required than cream?

73. A landing strip for an airport is to be built so as to accommodate airplanes landing at 150 miles per hour and decelerating at the rate of 10 miles per hour per second. How long should the landing strip be if it is required to be 60% longer than the stopping distance of the planes?

74. Suppose a carton of chocolate bars slides down an inclined plane with an initial velocity of v_0. The force F acting on the carton is given by

$$F = mg \sin \theta - \mu mg \cos \theta$$

where m is the mass of the carton, g is the acceleration due to gravity, θ is the angle between the plane and the horizontal, and μ is the coefficient of friction between the carton and the inclined plane. Show that the distance s the carton travels in t seconds is

$$s = \frac{g}{2}(\sin \theta - \mu \cos \theta)\, t^2 + v_0 t$$

(*Hint:* Use Newton's Second Law of Motion, $F = ma$, and the ideas of Example 10.)

75. The **linear momentum** of an object is the product of its mass and velocity. Newton's Second Law of Motion is sometimes expressed in the form

$$F = \frac{dp}{dt} \tag{12}$$

where F is the force and p is the linear momentum, both expressed as functions of time t. When a force acts on an object during a time interval $[t_1, t_2]$, as when a baseball is hit by a bat, the change $p(t_2) - p(t_1)$ in the linear momentum of the object is called the **impulse** of the force.

a. Use (12) to express the impulse between t_1 and t_2 as an integral.

b. A ball with mass 0.1 kilogram falls vertically and hits the floor with a speed of 5 meters per second. It remains in contact with the floor 10^{-3} seconds, and rebounds with a speed of 4.5 meters per second. First find the impulse of the force exerted on the ball by the floor, and then use it and part (a) to determine the average force exerted on the ball by the floor during the time of contact. (*Note:* A force of 1 Newton equals 1 kilogram meter per second per second.)

Project

1. In a famous eighteenth-century problem known as **Buffon's needle problem,** a needle 1 inch long is dropped onto a hardwood floor whose boards are 2 inches wide. We are asked to determine the probability that the needle will come to rest lying across two boards. As in Figure 5.37(a), we let P be the southernmost point of the needle (or the left-hand endpoint of the needle if the needle lies horizontally). Let y denote the distance from P to the next crack northward, so that $0 \le y < 2$. Let

θ denote the positive angle the needle makes with the horizontal to the right of P, with $0 \le \theta < \pi$.

a. Convince yourself that the needle crosses the crack *only* when the pair (θ, y) satisfies $y \le \sin\theta$, that is, when (θ, y) lies in the darker shaded region in Figure 5.37(b).

b. Convince yourself that the total set of possibilities for the needle can be identified with the rectangular region with $0 \le y < 2$ and $0 \le \theta < \pi$.

c. The proportion of times that the needle crosses the crack is the fraction

$$\frac{\text{area of the darker shaded region}}{\text{area of rectangle}}$$

It is reasonable to take this proportion to be the probability sought. Calculate this probability, that is, proportion.

d. Suppose that the needle is 2 inches long (instead of 1 inch long). Draw the counterpart to Figure 5.37(b), and find the probability that a dropped needle would come to rest lying across two boards.

e. Suppose that the needle is 2 inches long and the boards 5 inches wide. Then find the corresponding probability.

f. What happens to the picture and the probability if the needle is longer than 2 inches, and the boards are 2 inches wide?

Buffon's needle problem

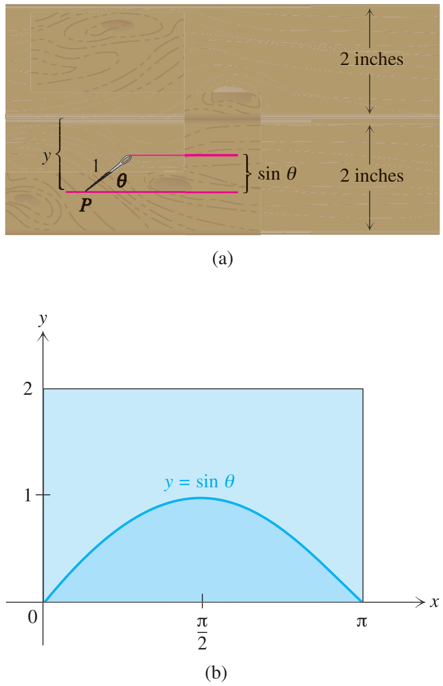


FIGURE 5.37 Figures for the project.

5.5 INDEFINITE INTEGRALS AND INTEGRATION RULES

When we evaluate a definite integral $\int_a^b f(x)\,dx$ using the Fundamental Theorem of Calculus, the basic problem is to find an antiderivative of f. In this section we present some elementary rules that will help us find antiderivatives of combinations of functions. Recall that if F is an antiderivative of f, then for any constant C the function $F + C$ is also an antiderivative of f, since

$$(F + C)' = F' + C' = f$$

Consequently every function f that has an antiderivative has infinitely many of them—one for each choice of C. By Theorem 4.6 the functions of the form $F + C$, where C is an arbitrary constant, are the *only* antiderivatives of f. As a result, mathematicians group the antiderivatives of f together and call them the indefinite integral of f.

DEFINITION 5.15 Let f have an antiderivative on an interval I. The collection of antiderivatives of f is called the **indefinite integral** of f on I and is denoted by $\int f(x)\,dx$.

Mathematicians use the notation

$$\int f(x)\,dx = F(x) + C$$

when finding indefinite integrals. For example,

$$\int x^2\,dx = \frac{1}{3}x^3 + C$$

This equation means that $\frac{1}{3}x^3$ is a function whose derivative is x^2 and that the only other functions having derivative x^2 are of the form $\frac{1}{3}x^3 + C$, where C is a constant.

In the notation of indefinite integrals, Table 5.1 in Section 5.4 yields Table 5.2. In the table c, C, p, q, and r are constants.

TABLE 5.2

Some Common Indefinite Integrals

$\int c\,dx = cx + C$	$\int \frac{1}{x}\,dx = \ln x + C$
$\int x\,dx = \frac{1}{2}x^2 + C$	$\int \sin x\,dx = -\cos x + C$
$\int x^2\,dx = \frac{1}{3}x^3 + C$	$\int \cos x\,dx = \sin x + C$
$\int (px + q)\,dx = \frac{1}{2}px^2 + qx + C$	$\int e^x\,dx = e^x + C$
$\int x^r\,dx = \frac{1}{r+1}x^{r+1} + C\ (r \neq -1)$	

We now show how to obtain the indefinite integrals of certain combinations of functions from the indefinite integrals of the individual functions.

THEOREM 5.16 If f and g have antiderivatives on an interval I, then

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

Proof This formula means that the sum of an antiderivative of f and an antiderivative of g is an antiderivative of $f + g$. To prove this, let F and G be antiderivatives of f and g, respectively. Then

$$(F + G)' = F' + G' = f + g$$

Hence $F + G$ is an antiderivative of $f + g$. ■

If f and g are continuous on $[a, b]$, then by the Fundamental Theorem we have the corresponding equation for definite integrals:

$$\int_a^b [f(x) + g(x)]\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx \tag{1}$$

Moreover, Theorem 5.16 can be extended to the sum of more than two functions (see Exercise 63).

THEOREM 5.17 If f has an antiderivative on an interval I, and c is a real number, then

$$\int cf(x)\, dx = c \int f(x)\, dx$$

Proof This formula means that c times an antiderivative of f is an antiderivative of cf. But if F is an antiderivative of f, then

$$(cF)' = c(F') = cf$$

so that cF is an antiderivative of cf. ■

If f is continuous on $[a, b]$, then the Fundamental Theorem of Calculus implies that the corresponding formula for definite integrals holds:

$$\int_a^b cf(x)\, dx = c \int_a^b f(x)\, dx \tag{2}$$

From Theorems 5.16 and 5.17 we immediately have the following corollary, whose proof is left as an exercise (see Exercise 62).

COROLLARY 5.18 If f and g have antiderivatives on an interval I, then

$$\int [f(x) - g(x)]\, dx = \int f(x)\, dx - \int g(x)\, dx$$

If f and g are continuous on $[a, b]$, then the Fundamental Theorem yields

$$\int_a^b [f(x) - g(x)]\, dx = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx \tag{3}$$

EXAMPLE 1 Evaluate the indefinite integral $\int (5x - 3\cos x)\, dx$.

Solution Using first Corollary 5.18 and then Theorem 5.17, we obtain

$$\begin{aligned}\int (5x - 3\cos x)\, dx &= \int 5x\, dx - \int 3\cos x\, dx \\ &= 5\int x\, dx - 3\int \cos x\, dx = 5\left(\frac{1}{2}x^2\right) - 3\sin x + C \\ &= \frac{5}{2}x^2 - 3\sin x + C\end{aligned}$$

□

EXAMPLE 2 Evaluate the definite integral $\int_0^1 (6x^2 + 5e^x)\,dx$.

Solution First method: We break the integral into its components and then integrate. Using (1) and (2), we obtain

$$\begin{aligned}\int_0^1 (6x^2 + 5e^x)\,dx &\overset{(1)}{=} \int_0^1 6x^2\,dx + \int_0^1 5e^x\,dx \\ &\overset{(2)}{=} 6\int_0^1 x^2\,dx + 5\int_0^1 e^x\,dx \\ &= 6\left(\frac{1}{3}x^3\right)\Bigg|_0^1 + 5(e^x)\Bigg|_0^1 \\ &= 6\left(\frac{1}{3} - 0\right) + 5(e - 1) = 5e - 3\end{aligned}$$

Second method: Here we first find an antiderivative of $6x^2 + 5e^x$ and then evaluate it at the limits of integration, which are 0 and 1:

$$\int_0^1 (6x^2 + 5e^x)\,dx = (2x^3 + 5e^x)\Big|_0^1 = (2 + 5e) - 5 = 5e - 3$$

Repeated applications of Theorems 5.16 and 5.17 yield the indefinite integral of a general polynomial:

$$\begin{aligned}&\int [c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0]\,dx \\ &= \frac{c_n}{n+1}x^{n+1} + \frac{c_{n-1}}{n}x^n + \cdots + \frac{c_1}{2}x^2 + c_0 x + C\end{aligned}$$

Thus to find the integral of a polynomial

$$c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$$

just integrate the terms $c_n x^n$, $c_{n-1}x^{n-1}$, . . . , $c_1 x$ and c_0 one by one and add the results.

EXAMPLE 3 Evaluate $\int (x^4 - 3x^2 + 4x - 2)\,dx$.

Solution Integrating the terms one by one, we find that

$$\int (x^4 - 3x^2 + 4x - 2)\,dx = \frac{1}{5}x^5 - 3\left(\frac{1}{3}x^3\right) + 4\left(\frac{1}{2}x^2\right) - 2x + C$$

$$= \frac{1}{5} x^5 - x^3 + 2x^2 - 2x + C \qquad \square$$

Formula (3) yields a simple proof of the following result, which we could also have proved, with more difficulty, in Section 5.3.

COROLLARY 5.19
General Comparison Property

Let f and g be continuous on $[a, b]$, with $g(x) \leq f(x)$ for $a \leq x \leq b$. Then

$$\int_a^b g(x)\,dx \leq \int_a^b f(x)\,dx$$

Proof From the hypotheses it follows that $f(x) - g(x) \geq 0$ for $a \leq x \leq b$. Then by (3) and Corollary 5.10 with $f - g$ replacing f,

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx \overset{(3)}{=} \int_a^b [f(x) - g(x)]\,dx \overset{\substack{\text{Corollary}\\ 5.10}}{\geq} 0$$

Consequently

$$\int_a^b g(x)\,dx \leq \int_a^b f(x)\,dx \qquad \blacksquare$$

In effect, Corollary 5.19 says that the larger the function, the larger the definite integral. Of course if $b < a$, then $\int_a^b g(x)\,dx \geq \int_a^b f(x)\,dx$ (see Exercise 61).

Another consequence of Corollary 5.19 is that if $h(x) \leq g(x) \leq f(x)$ for $a \leq x \leq b$, then

$$\int_a^b h(x)\,dx \leq \int_a^b g(x)\,dx \leq \int_a^b f(x)\,dx \tag{4}$$

These inequalities can lead to good lower and upper bounds for integrals that are difficult or impossible to find directly.

EXAMPLE 4 Show that $1 \leq \int_0^1 \sqrt{1 + x^4}\,dx \leq \frac{4}{3}$.

Solution We have

$$1 \leq 1 + x^4 \leq 1 + 2x^2 + x^4 = (1 + x^2)^2$$

which implies that

$$1 \leq \sqrt{1 + x^4} \leq \sqrt{(1 + x^2)^2} = 1 + x^2$$

From (4) we conclude that

$$1 = \int_0^1 1\,dx \leq \int_0^1 \sqrt{1 + x^4}\,dx \leq \int_0^1 (1 + x^2)\,dx = \left(x + \frac{x^3}{3}\right)\Bigg|_0^1 = \frac{4}{3} \qquad \square$$

Since $\frac{4}{3} < \sqrt{2}$, Example 4 yields a smaller upper bound for the integral $\int_0^1 \sqrt{1 + x^4}\,dx$ than the one found in Example 4 of Section 5.3. An even smaller upper bound will be obtained in Section 8.6.

Since $-|f(x)| \le f(x) \le |f(x)|$ for $a \le x \le b$, we obtain as a special case of (4) the inequalities

$$-\int_a^b |f(x)|\,dx = \int_a^b -|f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx$$

Therefore, by the definition of the absolute value,

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx \tag{5}$$

EXERCISES 5.5

In Exercises 1–14 evaluate the indefinite integral.

1. $\int (2x - 7)\,dx$

2. $\int (2x^2 - 7x^3 + 4x^4)\,dx$

3. $\int (2x^{1/3} - 3x^{3/4} + x^{2/5})\,dx$

4. $\int (x^{3/2} + 4x^{1/2} - \pi)\,dx$

5. $\int \left(t^5 - \frac{1}{t^4}\right)dt$

6. $\int \left(\sqrt{y} + \frac{1}{\sqrt{y}}\right)dy$

7. $\int (2\cos x - 5x)\,dx$

8. $\int (\theta^2 + \sec^2\theta)\,d\theta$

9. $\int (3\csc^2 x - x)\,dx$

10. $\int \left(\frac{1}{y^3} - \frac{1}{y} + 2y\right)dy$

11. $\int (2t + 1)^2\,dt$ (*Hint:* Expand the binomial.)

12. $\int \left(t + \frac{1}{t}\right)^2 dt$

13. $\int \left(1 + \frac{1}{x}\right)^2 dx$

14. $\int (\sqrt{x} - 3e^x)\,dx$

In Exercises 15–34 evaluate the definite integral.

15. $\int_{-1}^{2} (3x - 4)\,dx$

16. $\int_1^4 \left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right)^2 dx$

17. $\int_{\pi/4}^{\pi/2} (-7\sin x + 3\cos x)\,dx$

18. $\int_0^{\pi} (\sin x - 8x^2)\,dx$

19. $\int_{-\pi/4}^{-\pi/2} \left(3x - \frac{1}{x^2} + \sin x\right)dx$

20. $\int_{-1/4}^{1/4} (4t - 3)^2\,dt$

21. $\int_{\pi/3}^{\pi/4} (3\sec^2\theta + 4\csc^2\theta)\,d\theta$

22. $\int_0^1 (3 - 2e^t)\,dt$

23. $\int_1^{1/3} (3t + 2)^3\,dt$

24. $\int_{-\pi/4}^{0} (\sec\theta)(\tan\theta + \sec\theta)\,d\theta$

(*Hint:* First multiply out the product.)

25. $\int_{\pi/2}^{\pi} \left(\pi\sin x - 2x + \frac{5}{x^2} + 2\pi\right)dx$

26. $\int_1^0 x(2x + 5)\,dx$

27. $\int_{-1}^{1} (2x + 5)(2x - 5)\,dx$

28. $\int_2^1 (x + 3)^2(x + 1)\,dx$

29. $\int_4^7 |x - 5|\,dx$

30. $\int_2^1 \left(4x - \frac{1}{x}\right)dx$

31. $\int_0^{\pi} (\sin x - 2e^x)\,dx$

32. $\int_{1/2}^{2} \frac{x - 1}{x}\,dx$

33. $\int_4^6 f(x)\,dx$, where

$$f(x) = \begin{cases} 2x & \text{for } 4 \le x \le 5 \\ 20 - 2x & \text{for } 5 < x \le 6 \end{cases}$$

34. $\int_0^{\pi/2} f(x)\,dx$, where

$$f(x) = \begin{cases} \sec^2 x & \text{for } 0 \le x \le \pi/4 \\ \csc^2 x & \text{for } \pi/4 < x \le \pi/2 \end{cases}$$

In Exercises 35–42 differentiate the function F. Then give F in terms of an indefinite integral. For example, if $F(x) = \cos x^2$, then since $d(\cos x^2)/dx = -2x \sin x^2$, we obtain

$$\int -2x \sin x^2\,dx = \cos x^2 + C$$

35. $F(x) = (1 + x^2)^{10}$

36. $F(x) = \frac{1}{2}(1 + x)^2$

37. $F(x) = x \sin x - \cos x$

38. $F(x) = \tan(2x + 1)$

39. $F(x) = 3 \sin^7 x$

40. $F(x) = x \sin x \cos x$

41. $F(x) = e^{(x^2)} - e^{-x}$

42. $F(x) = x \ln x - x$

In Exercises 43–50 find the area A of the region between the graph of f and the x axis on the given interval.

43. $f(x) = 3x^2 + 4$; $[-1, 1]$

44. $f(x) = \frac{1}{2}x^3 + 3x$; $[1, 2]$

45. $f(x) = 3\sqrt{x} - 1/\sqrt{x}$; $[1, 4]$

46. $f(x) = 8x^{1/3} - x^{-1/3}$; $[1, 8]$

47. $f(x) = 2 \sin x + 3 \cos x$; $[\pi/4, \pi/2]$

48. $f(x) = |x + 1|$; $[-2, 0]$

49. $f(x) = 2x - 4/x$; $[2, 4]$

50. $f(x) = \frac{1}{2}e^x - \frac{1}{3}x$; $[0, 3]$

51. Use the inequalities $0 \le \sin x \le x$ for $0 \le x \le 1$ to show that

a. $0 \le \int_0^1 \sin x^2\,dx \le \frac{1}{3}$

b. $0 \le \int_0^{\pi/6} \sin^{3/2} x\,dx \le \frac{2}{5}(\pi/6)^{5/2}$

52. Use the inequalities $0 \le \sin x \le x$ for $0 \le x \le \frac{1}{2}$ to show that

$$0 \le \int_0^{1/2} x \sin x\,dx \le \frac{1}{24}$$

53. Let $f(x) = 1 - x$. Show that $\left|\int_0^2 f(x)\,dx\right| < \int_0^2 |f(x)|\,dx$. (This inequality shows that the inequality sign in (5) cannot in general be replaced by an equals sign.)

54. Let f be continuous on $[a, b]$, and let $|f(x)| \le M$ for $a \le x \le b$. Prove that

$$\left|\int_a^b f(x)\,dx\right| \le M(b - a)$$

55. Use Exercise 54 to find an upper bound for

$$\left|\int_{-\pi/3}^{-\pi/4} \tan x\,dx\right|$$

56. Show that

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_0^{\varepsilon} x \sin x\,dx = 0$$

(*Hint:* $0 \le \sin x \le x$ for $x \ge 0$.)

***57.** Suppose f has a bounded derivative on $[a, b]$, so that there is a number M such that $|f'(x)| \le M$ for $a \le x \le b$. Assume that $f(a) = 0$.

a. Using the Mean Value Theorem, show that

$$|f(x)| \le M(x - a) \quad \text{for } a \le x \le b$$

b. Using the result of (a) and formula (5), show that

$$\left|\int_a^b f(x)\,dx\right| \le \frac{M}{2}(b - a)^2$$

58. Using Exercise 57(b), find upper bounds for the absolute values of the following integrals.

a. $\int_1^{\sqrt{5}} (x^2 - 1)^{3/2}\,dx$

***b.** $\int_0^{\pi/6} \sin^{3/2} x\,dx$ (*Hint:* Show first that $d(\sin^{3/2} x)/dx \le \frac{3}{4}\sqrt{2}$ for $0 \le x \le \pi/6$.)

***59.** Show that for every polynomial f of degree 3 or less,

$$\int_{-1}^{1} f(x)\,dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \tag{6}$$

Formula (6) is known as the two-point **Gauss quadrature formula** for $[-1, 1]$. For functions other than polynomials of degree 3 or less, the right side of (6) yields an approximation of the left side.

60. Use the two-point Gauss quadrature formula in Exercise 59 to approximate $\int_{-1}^{1} e^{-x^2/2}\,dx$.

61. Suppose that $b < a$ and that f and g are continuous, with $g(x) \le f(x)$ for $b \le x \le a$. Use Corollary 5.19 to prove that

$$\int_a^b g(x)\,dx \ge \int_a^b f(x)\,dx$$

62. Prove Corollary 5.18.

63. Prove that if $f_1, f_2, \ldots, f_n$ have antiderivatives on an interval I, then

$$\int [f_1(x) + f_2(x) + \cdots + f_n(x)]\,dx$$
$$= \int f_1(x)\,dx + \int f_2(x)\,dx + \cdots + \int f_n(x)\,dx$$

***64.** Find an upper bound for the area $\pi/4$ of the quarter of the unit circle in the first quadrant by calculating the area of the region formed by the tangent to the circle $x^2 + y^2 = 1$ at $(\sqrt{3}/2, \frac{1}{2})$, the x and y axes, and the line $x = 1$.

***65** **a.** Use the method of Exercise 64 to find an upper bound for the area $\pi/2$ of the quarter of the ellipse

$$\frac{x^2}{4} + y^2 = 1$$

in the first quadrant. (*Hint:* Consider the tangent line at $(\sqrt{2}, 1/\sqrt{2})$.)

b. Find a lower bound for the area $\pi/2$ by constructing a triangle inside the ellipse.

66. Let a, b, c, and d be positive numbers with $a < b$. The region R bounded by the graph of a function of the form $f(x) = cx + d$, the x axis, and the lines $x = a$ and $x = b$ is a right trapezoid. Show that the area A of R is given by

$$\left(\frac{f(a) + f(b)}{2}\right)(b - a)$$

***67.** Let f and g be continuous on $[a, b]$. Show that

$$\left(\int_a^b f(x)g(x)\,dx\right)^2 \le \int_a^b [f(x)]^2\,dx \int_a^b [g(x)]^2\,dx$$

(*Hint:* Let $p(r) = \int_a^b [f(x) + rg(x)]^2\,dx$ for all real numbers r. Show that $p(r)$ has the form $Ar^2 + Br + C$, where A, B, and C are constants. Then show that $p(r) \ge 0$ for all r and deduce that $B^2 - 4AC \le 0$, which yields the desired inequality.)

Application

68. Let r denote the radius of a cylindrical artery of length l and let x denote the distance of a given blood cell from the center of a cross section of the artery. The volume V per unit time of the flow of blood through the artery is given by the formula

$$V = \int_0^r \frac{k}{l}x(r^2 - x^2)\,dx$$

where k is a constant depending on the difference in pressure at the two ends of the artery and on the viscosity of the blood. Calculate V.

Project

1. In each part of this project, assume that f is differentiable on $[0, \infty)$, with $f(0) = 0$ but $f(x) \ne 0$ for $x \ne 0$.

a. Assume that $\int_0^x f(t)\,dt = (f(x))^2$ for all x. First show that $f(x) = 2f(x)f'(x)$. Then show that f is unique, and find a formula for $f(x)$.

b. Assume that $\int_0^x f(t)\,dt = (f(x))^3$ for all x. First show that $f(x) = 3[f(x)]^2 f'(x)$. Then show that f is unique, and find a formula for $f(x)$. (*Hint:* What is the derivative of $\frac{1}{2}f^2$?)

c. Assume that $\int_0^x f(t)\,dt = (f(x))^4$ for all x. First show that $f(x) = 4[f(x)]^3 f'(x)$. Then show that f is unique, and find a formula for $f(x)$.

5.6 INTEGRATION BY SUBSTITUTION

In the preceding section we transformed addition and constant multiple theorems for derivatives into corresponding theorems for integrals (see Theorems 5.16 and 5.17). Presently we will transform the Chain Rule in the form of

$$\frac{d}{dx}G(f(x)) = G'(f(x))f'(x)$$

into a theorem for integrals. The result we will obtain will be as useful in integration as the Chain Rule is in differentiation, and it will allow us to express many integrals, such as

$$\int x\sqrt{2x+1}\,dx \quad \text{and} \quad \int \sin 3x \cos 3x\,dx$$

in terms of functions familiar to us. The latter integral appears in the study of electric power. (See Exercise 66.)

THEOREM 5.20 Let f and g be functions, with both $g \circ f$ and f' continuous on an interval I. If G is an antiderivative of g, then

$$\int g(f(x))f'(x)\,dx = G(f(x)) + C \tag{1}$$

Proof Since G is an antiderivative of g, we have $G'(x) = g(x)$. Therefore the Chain Rule implies that

$$\frac{d}{dx}G(f(x)) = G'(f(x))f'(x) = g(f(x))f'(x)$$

In terms of indefinite integrals this becomes

$$\int g(f(x))f'(x)\,dx = G(f(x)) + C$$

which is (1). ■

In the process of applying (1) it is usually convenient to substitute u for $f(x)$ and du for $f'(x)\,dx$. Thus we obtain

$$\int g(\overbrace{f(x)}^{u})\,\overbrace{f'(x)\,dx}^{du} = \int g(u)\,du = G(u) + C = G(f(x)) + C$$

which shows clearly the integration of g to obtain G. For this reason, evaluating an integral by means of (1) is called **integration by substitution.** We illustrate the method in the examples that follow.

EXAMPLE 1 Find $\int 3x^2(x^3+5)^9\,dx$.

Solution We let

$$u = x^3 + 5, \quad \text{so that} \quad du = 3x^2\,dx$$

Then

$$\int 3x^2(x^3+5)^9\,dx = \int \overbrace{(x^3+5)^9}^{u^9}\,\overbrace{(3x^2)\,dx}^{du} = \int u^9\,du = \frac{1}{10}u^{10} + C$$

$$= \frac{1}{10}(x^3+5)^{10} + C \quad \square$$

Notice that after the integration was performed we resubstituted $x^3 + 5$ for u, so the answer would be expressed in terms of the original variable x. The variable u is only a temporary convenience.

We observe that it would also be possible to find $\int 3x^2(x^3+5)^9\,dx$ by expanding the polynomial $3x^2(x^3+5)^9$ and integrating term by term. However, integrating by substitution is much more efficient.

EXAMPLE 2 Find $\int \sin^4 x \cos x \, dx$.

Solution We let

$$u = \sin x, \quad \text{so that} \quad du = \cos x \, dx$$

Then

$$\int \sin^4 x \cos x \, dx = \int \overbrace{(\sin x)^4}^{u^4} \overbrace{\cos x \, dx}^{du} = \int u^4 \, du = \frac{1}{5} u^5 + C = \frac{1}{5} \sin^5 x + C \quad \square$$

The substitution of u for $x^3 + 5$ worked well in Example 1 because $du = 3x^2 \, dx$ and because $3x^2 \, dx$ appeared in the original integral. Similarly, the substitution of u for $\sin x$ worked well in Example 2 because $du = \cos x \, dx$ and because $\cos x \, dx$ appeared in the original integral. The method of substitution can still be applied if merely a constant multiple of du appears in the original integral.

EXAMPLE 3 Find $\int \frac{1}{2} \cos 2x \, dx$.

Solution We let

$$u = 2x, \quad \text{so that} \quad du = 2 \, dx, \quad \text{and thus} \quad dx = \tfrac{1}{2} \, du$$

Then

$$\int \frac{1}{2} \cos 2x \, dx = \int \frac{1}{2} (\cos \overbrace{2x}^{u}) \overbrace{dx}^{\frac{1}{2} du} = \int \frac{1}{2} (\cos u) \frac{1}{2} \, du$$

$$= \frac{1}{4} \int \cos u \, du = \frac{1}{4} \sin u + C = \frac{1}{4} \sin 2x + C \quad \square$$

We can use the result of Example 3 to evaluate $\int \cos^2 x \, dx$. Indeed, by using the trigonometric identity

$$\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x$$

along with Example 3 and the integration rules of Section 5.5, we find that

$$\int \cos^2 x \, dx = \int \left(\frac{1}{2} + \frac{1}{2} \cos 2x \right) dx = \int \frac{1}{2} \, dx + \int \frac{1}{2} \cos 2x \, dx$$

$$= \frac{1}{2} x + \frac{1}{4} \sin 2x + C$$

Therefore

$$\boxed{\int \cos^2 x \, dx = \frac{1}{2} x + \frac{1}{4} \sin 2x + C} \qquad (2)$$

By a similar argument,

$$\int \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C \tag{3}$$

(See Exercise 59.) The integrals $\int \sin^2 x\,dx$ and $\int \cos^2 x\,dx$ will occur from time to time throughout the remainder of this book.

EXAMPLE 4 Find $\displaystyle\int e^x\sqrt{1-e^x}\,dx$.

Solution We let

$$u = 1 - e^x, \quad \text{so that} \quad du = -e^x\,dx$$

Then

$$\int e^x\sqrt{1-e^x}\,dx = \int \overbrace{\sqrt{1-e^x}}^{\sqrt{u}}\;\overbrace{e^x\,dx}^{(-1)\,du} = \int \sqrt{u}\,(-1)\,du = -\int u^{1/2}\,du$$

$$= -\frac{2}{3}u^{3/2} + C = -\frac{2}{3}(1-e^x)^{3/2} + C \qquad \square$$

Occasionally it is convenient to solve for x (or some expression involving x) in terms of u in order to complete the substitution in the original integral.

EXAMPLE 5 Find $\displaystyle\int x\sqrt{2x+1}\,dx$.

Solution To simplify the expression $\sqrt{2x+1}$, we let

$$u = 2x + 1, \quad \text{so that} \quad du = 2\,dx$$

Then

$$\int x\sqrt{2x+1}\,dx = \int x\overbrace{\sqrt{2x+1}}^{u^{1/2}}\;\overbrace{dx}^{\frac{1}{2}\,du}$$

Thus we still need to find x in terms of u. From the equation $u = 2x + 1$ we deduce that

$$x = \frac{1}{2}(u-1)$$

Therefore

$$\int x\sqrt{2x+1}\,dx = \int \overbrace{x}^{\frac{1}{2}(u-1)}\;\overbrace{\sqrt{2x+1}}^{u^{1/2}}\;\overbrace{dx}^{\frac{1}{2}\,du}$$

$$= \int \frac{1}{2}(u-1)u^{1/2} \cdot \frac{1}{2}\,du = \frac{1}{4}\int (u^{3/2} - u^{1/2})\,du$$

$$= \frac{1}{4}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C$$

$$= \frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C \qquad \square$$

Frequently there is more than one substitution that will work. For instance, in Example 5 we could have let $u = \sqrt{2x+1}$. Then

$$du = \frac{1}{\sqrt{2x+1}}\,dx, \quad \text{so that} \quad u\,du = dx$$

Solving for x in the equation $u = \sqrt{2x+1}$, we obtain $x = \frac{1}{2}(u^2 - 1)$. Thus

$$\int x\sqrt{2x+1}\,dx = \int \overbrace{x}^{\frac{1}{2}(u^2-1)}\ \overbrace{\sqrt{2x+1}}^{u}\ \overbrace{dx}^{u\,du}$$

$$= \int \frac{1}{2}(u^2 - 1)u^2\,du$$

$$= \int \left(\frac{1}{2}u^4 - \frac{1}{2}u^2\right)du = \frac{1}{10}u^5 - \frac{1}{6}u^3 + C$$

$$= \frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C$$

Even though we used a different substitution, the final answer remains the same.

EXAMPLE 6 Find $\displaystyle\int x^5\sqrt{x^2-1}\,dx$.

Solution In order to simplify the integrand, we let

$$u = \sqrt{x^2-1}, \quad \text{so that} \quad du = \frac{x}{\sqrt{x^2-1}}\,dx, \quad \text{and thus} \quad u\,du = x\,dx$$

Then we factor out an x from x^5 so that $x\,dx$, which equals $u\,du$, appears in the integral:

$$\int x^5\sqrt{x^2-1}\,dx = \int x^4\ \overbrace{\sqrt{x^2-1}}^{u}\ \overbrace{x\,dx}^{u\,du}$$

Now we need to write x^4 in terms of u:

$$u = \sqrt{x^2-1}, \quad \text{so} \quad u^2 = x^2 - 1$$

Thus

$$x^2 = u^2 + 1, \quad \text{so} \quad x^4 = (u^2+1)^2$$

Therefore

$$\int x^5\sqrt{x^2-1}\,dx = \int \overbrace{x^4}^{(u^2+1)^2}\ \overbrace{\sqrt{x^2-1}}^{u}\ \overbrace{x\,dx}^{u\,du} = \int (u^2+1)^2\,u\cdot u\,du$$
$$= \int (u^6+2u^4+u^2)\,du = \frac{1}{7}u^7 + \frac{2}{5}u^5 + \frac{1}{3}u^3 + C$$
$$= \frac{1}{7}\left(\sqrt{x^2-1}\right)^7 + \frac{2}{5}\left(\sqrt{x^2-1}\right)^5 + \frac{1}{3}\left(\sqrt{x^2-1}\right)^3 + C \quad \square$$

Substitution with Definite Integrals

Suppose we wish to evaluate a definite integral of the form $\int_a^b g(f(x))f'(x)\,dx$. Using (1) and the Fundamental Theorem of Calculus, we find that

$$\int_a^b g(f(x))f'(x)\,dx = G(f(x))\Big|_a^b = G(f(b)) - G(f(a)) \qquad (4)$$

However, since G is an antiderivative of g, we also have

$$\int_{f(a)}^{f(b)} g(u)\,du = G(u)\Big|_{f(a)}^{f(b)} = G(f(b)) - G(f(a)) \qquad (5)$$

From (4) and (5) it follows that

$$\boxed{\int_a^b g(f(x))f'(x)\,dx = \int_{f(a)}^{f(b)} g(u)\,du} \qquad (6)$$

Thus we now have two methods of evaluating a definite integral of the form $\int_a^b g(f(x))f'(x)\,dx$. One way is to find the indefinite integral $\int g(f(x))f'(x)\,dx$ by substitution, and then evaluate it between the limits a and b, as in (4). The other way is to use (6), which involves using substitution, but with the limits of integration changed before we integrate. Formula (6) is called the **change of variable formula** in integration.

We will illustrate both methods of evaluating a definite integral by substitution in our final example.

EXAMPLE 7 Evaluate $\displaystyle\int_0^1 \frac{x^5}{(x^6+1)^3}\,dx$.

Solution For the first method we find the indefinite integral $\int x^5/(x^6+1)^3\,dx$ and then evaluate it between 0 and 1. To achieve this, we let

$$u = x^6 + 1, \quad \text{so that} \quad du = 6x^5\,dx$$

Then

$$\int \frac{x^5}{(x^6+1)^3}\,dx = \int \overbrace{\frac{1}{(x^6+1)^3}}^{1/u^3}\ \overbrace{x^5\,dx}^{\frac{1}{6}du} = \int \frac{1}{u^3}\cdot\frac{1}{6}\,du$$
$$= \frac{1}{6}\left(-\frac{1}{2}\cdot\frac{1}{u^2}\right) + C = -\frac{1}{12}\cdot\frac{1}{(x^6+1)^2} + C$$

Therefore

$$\int_0^1 \frac{x^5}{(x^6+1)^3}\,dx = -\frac{1}{12}\cdot\frac{1}{(x^6+1)^2}\bigg|_0^1 = -\frac{1}{12}\cdot\frac{1}{4} - \left(-\frac{1}{12}\right) = \frac{1}{16}$$

For the second method we make the same substitution as before but accompany it with a change in limits of integration. Since $u = x^6 + 1$, it follows that

$$\text{if } x = 0 \quad \text{then } u = 1, \quad \text{and} \quad \text{if } x = 1 \quad \text{then } u = 2$$

Consequently

$$\int_0^1 \frac{x^5}{(x^6+1)^3}\,dx = \int_1^2 \frac{1}{u^3}\cdot\frac{1}{6}\,du = \frac{1}{6}\left(-\frac{1}{2}\cdot\frac{1}{u^2}\right)\bigg|_1^2$$

$$= \frac{1}{6}\left(-\frac{1}{8}\right) - \frac{1}{6}\left(-\frac{1}{2}\right) = \frac{1}{16} \qquad \square$$

EXERCISES 5.6

In Exercises 1–18 evaluate the integral by making the indicated substitution.

1. $\int \sqrt{4x-5}\,dx;\ u = 4x-5$
2. $\int (1-5x^2)^{2/3}(10x)\,dx;\ u = 1-5x^2$
3. $\int \cos \pi x\,dx;\ u = \pi x$
4. $\int 3\sin(-2x)\,dx;\ u = -2x$
5. $\int x\cos x^2\,dx;\ u = x^2$
6. $\int \sin^3 t\cos t\,dt;\ u = \sin t$
7. $\int \cos^{-4} t\sin t\,dt;\ u = \cos t$
8. $\int \frac{2t-3}{(t^2-3t+1)^2}\,dt;\ u = t^2-3t+1$
9. $\int \frac{2t-3}{(t^2-3t+1)^{7/2}}\,dt;\ u = t^2-3t+1$
10. $\int x\sqrt{x-1}\,dx;\ v = x-1$
11. $\int (x-1)\sqrt{x+1}\,dx;\ v = x+1$
12. $\int x^2\sqrt{x+3}\,dx;\ v = x+3$
13. $\int \sec x\tan x\sqrt{3+\sec x}\,dx;\ u = 3+\sec x$
14. $\int \frac{\sqrt[3]{x}}{(\sqrt[3]{x}+1)^5}\,dx;\ u = \sqrt[3]{x}+1$
15. $\int x\,e^{(x^2)}\,dx;\ u = x^2$
16. $\int \frac{e^x}{1+e^x}\,dx;\ u = 1+e^x$
17. $\int \frac{x}{x^2+1}\,dx;\ u = x^2+1$

*18. $\int \frac{1}{x+\sqrt{x}}\,dx;\ u = \sqrt{x}+1$

In Exercises 19–44 evaluate the integral.

19. $\int 3x^2(x^3+1)^{12}\,dx$
20. $\int x^3(2-5x^4)^7\,dx$
21. $\int \sqrt{x}\,(4+x^{3/2})\,dx$
22. $\int \left(1-\frac{1}{x^2}\right)\left(x+\frac{1}{x}\right)^{-3}dx$
23. $\int \sqrt{3x+7}\,dx$
24. $\int \sqrt{4-2x}\,dx$
25. $\int (1+4x)\sqrt{1+2x+4x^2}\,dx$

26. $\int \cos 7x\, dx$ **27.** $\int_{-1}^{3} \sin \pi x\, dx$

28. $\int_0^1 t^9 \sin t^{10}\, dt$ **29.** $\int \sin^6 t \cos t\, dt$

30. $\int \cos^{-3} t \sin t\, dt$ **31.** $\int \sqrt{\sin 2z} \cos 2z\, dz$

32. $\int \sin 3z \sqrt{1 - \cos 3z}\, dz$

33. $\int_0^{\pi/4} \frac{\sin z}{\cos^2 z}\, dz$ **34.** $\int_{\pi/2}^{\pi/6} \frac{\cos z}{\sin^3 z}\, dz$

35. $\int \frac{1}{\sqrt{z}} \sec^2 \sqrt{z}\, dz$ **36.** $\int \frac{1}{z^2} \csc^2 \frac{1}{z}\, dz$

37. $\int w\left(\sqrt{w^2+1} + \frac{1}{\sqrt{w^2+1}}\right) dw$

38. $\int_4^1 \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\, dx$ **39.** $\int_1^8 x^{-2/3} \sqrt{1 + 4x^{1/3}}\, dx$

40. $\int_{-1}^{0} w(\sqrt{1-w^2} + \sin \pi w^2)\, dw$

41. $\int e^{2x} \sin (1 + e^{2x})\, dx$ **42.** $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$

43. $\int \frac{x^3}{1+x^4}\, dx$ **44.** $\int \frac{(\ln x)^2}{x}\, dx$

In Exercises 45–52 evaluate the integral.

45. $\int x\sqrt{x+2}\, dx$ **46.** $\int \frac{x}{\sqrt{x+3}}\, dx$

47. $\int_1^3 4x\sqrt{6-2x}\, dx$ **48.** $\int x^2\sqrt{x+4}\, dx$

49. $\int t^2 \sqrt{1-8t}\, dt$ **50.** $\int e^{3t}\sqrt{1+e^t}\, dt$

51. $\int_{-1}^{2} \frac{t^2}{\sqrt{t+2}}\, dt$ **52.** $\int_0^1 \frac{\sqrt{x}}{\sqrt{1+\sqrt{x}}}\, dx$

In Exercises 53–58 find the area A of the region between the graph of f and the x axis on the given interval.

53. $f(x) = \sqrt{x+1}$; $[0, 3]$ **54.** $f(x) = \sin \pi x$; $[0, 1]$

55. $f(x) = \frac{x}{(x^2+1)^2}$; $[1, 2]$ **56.** $f(x) = x\sqrt{x^2-9}$; $[3, 5]$

57. $f(x) = \frac{1}{x^2}\left(1 + \frac{1}{x}\right)^{1/2}$; $\left[\frac{1}{8}, \frac{1}{3}\right]$

58. $f(x) = -x^{1/3}(1 + x^{4/3})^{1/3}$; $[-1, 0]$

59. By using a trigonometric identity, show that

$$\int \sin^2 x\, dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

60. Let a be any number and k any integer.

a. Verify that $\int_a^{a+k\pi} \sin^2 x\, dx = k\pi/2$.

b. Verify that $\int_a^{a+k\pi} \cos^2 x\, dx = k\pi/2$.

61. a. By making the substitution $u = 1 - x$, show that

$$\int_0^1 x^n (1-x)^m\, dx = \int_0^1 x^m (1-x)^n\, dx$$

for any nonnegative integers m and n.

b. Use part (a) to evaluate

$$\int_0^1 x^2 (1-x)^{10}\, dx$$

62. Let $a > 0$, and let f be continuous on $[0, a]$.

a. Making the substitution $u = a - x$, show that

$$\int_0^a \frac{f(x)}{f(x) + f(a-x)}\, dx = \int_0^a \frac{f(a-u)}{f(u) + f(a-u)}\, du$$

b. Use part (a) to show that

$$\int_0^a \frac{f(x)}{f(x) + f(a-x)}\, dx = \frac{a}{2}$$

(The answer is independent of f!)

c. Use part (b) to evaluate

$$\int_0^1 \frac{x^4}{x^4 + (1-x)^4}\, dx$$

63. Let f be continuous, and suppose $\int f(x)\, dx = F(x) + C$.

a. Prove that $\int f(ax+b)\, dx = [F(ax+b)]/a + C$ for any $a \neq 0$ and any b.

b. Use (a) to evaluate $\int \sin (ax+b)\, dx$.

c. Use (a) to evaluate $\int (ax+b)^n\, dx$, for any integer $n \neq 0, -1$.

64. Show that $\int_{ca}^{cb} (1/x)\, dx = \int_a^b (1/x)\, dx$ for any positive numbers a, b, and c. (*Hint:* Let $u = x/c$.)

65. Find the fallacy in the following argument: Since $(1+x^2)^{-1} > 0$ we have $\int_{-1}^{1} (1+x^2)^{-1}\, dx > 0$. However, by substituting $u = 1/x$ we obtain

$$\begin{aligned}\int_{-1}^{1} (1+x^2)^{-1}\, dx &= \int_{-1}^{1} \left(1 + \frac{1}{u^2}\right)^{-1}\left(-\frac{1}{u^2}\right) du \\ &= -\int_{-1}^{1} (1+u^2)^{-1}\, du \\ &= -\int_{-1}^{1} (1+x^2)^{-1}\, dx\end{aligned}$$

which implies that $\int_{-1}^{1} (1+x^2)^{-1}\, dx = 0$.

Applications

66. In an electric circuit, the **electric power** P produced by the voltage V and current I is given by $P = VI$. Frequently both voltage and current are sinusoidal as functions of time. Assume that V_0 and I_0 are constants. Find the mean power

$$\frac{1}{\pi}\int_0^{\pi} V(t)\, I(t)\, dt$$

during the time interval $[0, \pi]$, assuming that $V(t) = V_0 \sin 3t$ and

a. $I(t) = I_0 \sin 3t$ **b.** $I(t) = I_0 \cos 3t$

67. According to the theory of quantum mechanics, an electron moving along the x axis does not have a definite position at any particular instant, but instead is represented by a wave function ψ. The probability of finding the electron in an interval $[a, b]$ is

$$\int_a^b [\psi(x)]^2\, dx$$

For an electron confined to the interval $[0, 1]$ with impenetrable barriers at 0 and 1, the wave function is given by $\psi(x) = \sqrt{2} \sin(2\pi x)$.

a. Find the value of $\int_0^{1/4} 2 \sin^2(2\pi x)\, dx$, which is the probability of finding the electron in the interval $[0, 1/4]$.

b. Show that

$$\int_0^1 |\psi(x)|^2\, dx = 1$$

and explain the physical significance of this result.

68. The probability of "breakdowns" of certain systems, such as automobile accidents at a busy intersection, cars arriving at a toll booth, lifetime of a battery, or earthquakes, can be modeled by the **exponential density function** f, given by

$$f(t) = \frac{1}{\lambda} e^{-t/\lambda} \quad \text{for } t \geq 0$$

where λ is a positive constant associated with the system. The probability $P(t)$ that such a breakdown will occur during the time interval $[0, t]$ is given by

$$P(t) = \int_0^t f(s)\, ds \quad \text{for } t \geq 0$$

a. Show that $\lim_{t\to\infty} P(t) = 1$.

b. Assume that the formula for P holds for the lifetime of car batteries, with $\lambda = 2$. Find the probability that a randomly selected car battery will last at most 1 year.

c. Using $\lambda = 2$ in part (b), find the value of t^* such that $P(t^*) = 0.5$.

69. Suppose an object with mass m_1 is located at a on the x axis, and another object with mass m_2 is to the left of the first object on the x axis. Newton's Law of Gravitation implies that the work W required to move the second object from $x_2 < a$ to $x_1 < a$ is given by

$$W = \int_{x_1}^{x_2} \frac{Gm_1m_2}{(x-a)^2}\, dx$$

where G is a constant. Show that

$$W = \frac{Gm_1m_2(x_2 - x_1)}{(x_1 - a)(x_2 - a)}$$

Project

1. Let f be continuous on the interval $[-a, a]$. This project concerns functions that are even (that is, $f(-x) = f(x)$ for $-a \leq x \leq a$) and those that are odd (that is, $f(-x) = -f(x)$ for $-a \leq x \leq a$). We start with two supporting results, in (a) and (b).

a. Prove that

$$\int_{-a}^0 f(x)\, dx = \int_0^a f(-x)\, dx$$

b. Prove that

$$\int_{-a}^a f(x)\, dx = \int_0^a [f(x) + f(-x)]\, dx$$

c. Show that if f is odd, then $\int_{-a}^a f(x)dx = 0$

d. Show that if f is even, then $\int_{-a}^a f(x)\, dx = 2\int_0^a f(x)\, dx$.

e. Suppose that $\int_{-c}^c f(x)\, dx = 0$ for all c such that $0 \leq c \leq a$. Does this imply that f is an odd function on $[-a, a]$? Explain your answer.

5.7 THE LOGARITHM

Until now our discussion of the natural logarithm has been based on an informal definition of the logarithm. Having introduced integrals, we are now in a position to give a formal definition of the natural logarithm and derive several of its properties, including three that we have exploited in earlier chapters:

$$\text{its value at 1:} \quad \ln 1 = 0 \tag{1}$$

$$\text{its derivative:} \quad \frac{d}{dx} \ln x = \frac{1}{x} \tag{2}$$

$$\text{the law of logarithms:} \quad \ln bc = \ln b + \ln c \tag{3}$$

To prepare for the definition of the natural logarithm, let $f(x) = 1/x$. Then f is continuous on $(0, \infty)$, so we may define a function G by the formula

$$G(x) = \int_1^x \frac{1}{t}\, dt \quad \text{for all } x > 0 \tag{4}$$

Then

$$G(1) = \int_1^1 \frac{1}{t}\, dt = 0 \tag{5}$$

By Theorem 5.12, G is differentiable on $(0, \infty)$, and

$$\frac{dG}{dx} = \frac{1}{x} \tag{6}$$

Comparing (1) with (5) and (2) with (6), we see that G and $\ln x$ have the same derivative and also the same value at 1. However, Theorem 4.6 implies that there is exactly one function defined on $(0, \infty)$ with a given derivative and a given value at 1. Thus it is reasonable to formally define the **natural logarithm function** by

$$\boxed{\ln x = \int_1^x \frac{1}{t}\, dt} \tag{7}$$

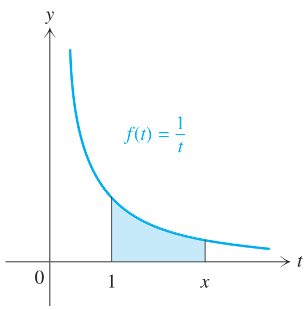


FIGURE 5.38

For $x > 1$, $\ln x$ may be considered as the area of the region in Figure 5.38.

It is immediate from (7) that

$$\ln 1 = 0 \quad \text{and} \quad \frac{d}{dx} \ln x = \frac{1}{x} \tag{8}$$

That the well-known Law of Logarithms is valid for $\ln x$ will be proved in Theorem 5.21.

The indefinite integral form of (7) is

$$\int \frac{1}{x}\, dx = \ln x + C \tag{9}$$

As a result, we have a formula for $\int t^r\, dt$ when $r = -1$. This means that finally we have found a formula for the indefinite integral $\int t^r\, dt$ for every real number r.

From (8) we know that $\ln x$ is an increasing function (because $x > 0$). Thus

$$x < z \quad \text{if and only if} \quad \ln x < \ln z \tag{10}$$

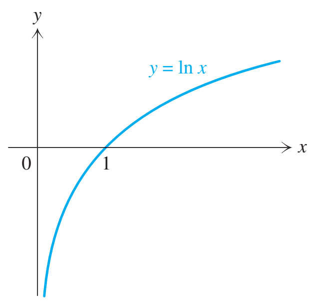


FIGURE 5.39

Moreover,

$$\frac{d^2}{dx^2} \ln x = \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

so that the graph of $\ln x$ is concave downward. One can also prove that

$$\lim_{x\to 0^+} \ln x = -\infty \quad \text{and} \quad \lim_{x\to\infty} \ln x = \infty$$

(See Exercise 50.) Using these facts, along with the equation $\ln 1 = 0$, we deduce that the graph of $\ln x$ has the shape in Figure 5.39.

Since $\ln 1 = 0$ and $\lim_{x\to\infty} \ln x = \infty$, the Intermediate Value Theorem implies that there is a (unique) number e such that $\ln e = 1$. It can be shown that e is irrational and that

$$e = 2.71828182845904523536\ldots$$

To prove that the expansion is nonrepeating is not easy. However, using the Addition and Comparison Properties of the integral, we can show that $e < 3$.

EXAMPLE 1 Show that $e < 3$.

Solution By (10), the inequality $e < 3$ is equivalent to $\ln e < \ln 3$, that is, $1 < \ln 3$. Thus we need only show that $\ln 3 > 1$. First we notice that $1/t$ is a decreasing function on $[1, \infty)$, so that the minimum value of $1/t$ on any subinterval $[r, s]$ of $[1, \infty)$ is $1/s$. Thus from the Addition and Comparison Properties of the integral,

$$\begin{aligned}
\ln 3 &= \int_1^{5/4} \frac{1}{t}\,dt + \int_{5/4}^{6/4} \frac{1}{t}\,dt + \int_{6/4}^{7/4} \frac{1}{t}\,dt + \cdots + \int_{11/4}^{3} \frac{1}{t}\,dt \\
&\geq \int_1^{5/4} \frac{1}{5/4}\,dt + \int_{5/4}^{6/4} \frac{1}{6/4}\,dt + \int_{6/4}^{7/4} \frac{1}{7/4} + \cdots + \int_{11/4}^{3} \frac{1}{3}\,dt \\
&= \frac{1}{4}\left(\frac{4}{5} + \frac{4}{6} + \frac{4}{7} + \cdots + \frac{4}{12}\right) \\
&\approx 1.019877345
\end{aligned}$$

Therefore $\ln 3 > 1$, which is what we wished to prove. ❑

Since $\ln x$ is an antiderivative of $1/x$, we can easily evaluate any integral of the form $\int_a^b 1/x\,dx$ in terms of the natural logarithm, whenever a and b are positive.

EXAMPLE 2 Evaluate $\displaystyle\int_2^6 \frac{1}{x}\,dx$ in terms of logarithms.

Solution By the preceding comment,

$$\int_2^6 \frac{1}{x}\,dx = (\ln x)\Big|_2^6 = \ln 6 - \ln 2 \qquad \square$$

Next we will state and prove the Law of Logarithms.

THEOREM 5.21
Law of Logarithms

For all $b > 0$ and $c > 0$,

$$\ln bc = \ln b + \ln c$$

Proof Fix $b > 0$. For any $x > 0$ let

$$g(x) = \ln bx$$

The Chain Rule yields

$$g'(x) = \left(\frac{1}{bx}\right)b = \frac{1}{x}$$

Therefore g and $\ln x$ have the same derivative. By Theorem 4.6 they differ by a constant C, that is,

$$\ln bx = \ln x + C$$

Substituting $x = 1$ in this equation and noting that $\ln 1 = 0$, we obtain

$$\ln b = \ln 1 + C = C$$

As a result,

$$\ln bx = \ln x + \ln b$$

When $x = c$, this is equivalent to the equation that was to be verified. ■

The following properties of logarithms can be proved from the Law of Logarithms:

$$\ln b^r = r \ln b \qquad \text{for } b > 0 \tag{11}$$

$$\ln \frac{1}{b} = -\ln b \qquad \text{for } b > 0 \tag{12}$$

$$\ln \frac{b}{c} = \ln b - \ln c \qquad \text{for } b, c > 0 \tag{13}$$

The formulas can also be proved by using derivatives (see Exercises 47–49).

Using (13), we can simplify the answer obtained in Example 2. Indeed (13) implies that

$$\ln 6 - \ln 2 = \ln \frac{6}{2} = \ln 3$$

Thus

$$\int_2^6 \frac{1}{x}\,dx = \ln 3$$

If f is any positive differentiable function, then we can apply the Chain Rule to conclude that

$$\boxed{\frac{d}{dx}\ln f(x) = \frac{1}{f(x)}f'(x) = \frac{f'(x)}{f(x)}} \tag{14}$$

EXAMPLE 3 Find

$$\frac{d}{dx}\ln(x^2+x)^{1/3}$$

Solution By using first (11) and then (14) we find that

$$\begin{aligned}\frac{d}{dx}\ln(x^2+x)^{1/3} &\overset{(11)}{=} \frac{d}{dx}\left[\frac{1}{3}\ln(x^2+x)\right] \overset{(14)}{=} \frac{1}{3}\cdot\frac{1}{x^2+x}\cdot\frac{d}{dx}(x^2+x)\\ &= \frac{1}{3}\cdot\frac{1}{x^2+x}(2x+1) = \frac{2x+1}{3(x^2+x)} \quad \square\end{aligned}$$

Since the domain of the function $\ln x$ is $(0, \infty)$, the domain of $\ln(-x)$ is $(-\infty, 0)$. Hence by the Chain Rule we have

$$\frac{d}{dx}\ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}$$

Thus the function $\ln|x|$, which is equal to $\ln(-x)$ on $(-\infty, 0)$, and equal to $\ln x$ on $(0, \infty)$, is an antiderivative of $1/x$ on $(-\infty, 0)$ and on $(0, \infty)$. This gives us the formula

$$\boxed{\int \frac{1}{x}\,dx = \ln|x| + C} \tag{15}$$

EXAMPLE 4 Express $\int_{-8}^{-7} \frac{1}{x}\,dx$ in terms of logarithms.

Solution From (15) we deduce that

$$\int_{-8}^{-7}\frac{1}{x}\,dx = \ln|x|\,\Big|_{-8}^{-7} = \ln|-7| - \ln|-8| = \ln 7 - \ln 8 = \ln\frac{7}{8} \quad \square$$

Next we will evaluate

$$\int \frac{f'(x)}{f(x)}\,dx$$

by substitution. To accomplish that, we let

$$u = f(x), \quad \text{so that} \quad du = f'(x)\,dx$$

Then by (15),

$$\int \frac{f'(x)}{f(x)}\,dx = \int \overbrace{\frac{1}{f(x)}}^{1/u}\,\overbrace{f'(x)\,dx}^{du} = \int \frac{1}{u}\,du \overset{(15)}{=} \ln|u| + C = \ln|f(x)| + C$$

which yields the formula

$$\boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C} \tag{16}$$

It is not necessary to memorize (16) if you remember that when you see an integral of the form in (16), you should think about substituting $u = f(x)$.

EXAMPLE 5 Find $\displaystyle\int \frac{x^4}{x^5+1}\,dx$.

Solution Let

$$u = x^5 + 1, \quad \text{so that} \quad du = 5x^4\,dx$$

Then

$$\int \frac{x^4}{x^5+1}\,dx = \int \overbrace{\frac{1}{x^5+1}}^{1/u}\,\overbrace{x^4\,dx}^{\frac{1}{5}du} = \int \frac{1}{u}\cdot\frac{1}{5}\,du$$

$$= \frac{1}{5}\ln|u| + C = \frac{1}{5}\ln|x^5+1| + C \qquad \square$$

Formula (16) can be used to find the integrals of the four remaining basic trigonometric functions that we have not yet found: tan x, sec x, cot x, and csc x.

To find $\int \tan x\,dx$, we first recall that

$$\tan x = \frac{\sin x}{\cos x}$$

Since the numerator is essentially the derivative of the denominator, we can apply (16) by letting

$$u = \cos x, \quad \text{so that} \quad du = -\sin x\,dx$$

We find that

$$\int \tan x\,dx = \int \overbrace{\frac{1}{\cos x}}^{1/u}\,\overbrace{\sin x\,dx}^{(-1)\,du} = \int \frac{1}{u}(-1)\,du = -\ln|u| + C = -\ln|\cos x| + C$$

Therefore

$$\boxed{\int \tan x\,dx = -\ln|\cos x| + C}$$

The evaluation of $\int \cot x\,dx$ is analogous. However, the evaluations of $\int \sec x\,dx$ and $\int \csc x\,dx$ are not so transparent. For $\int \sec x\,dx$ recall that

$$\frac{d}{dx}\tan x = \sec^2 x \quad \text{and} \quad \frac{d}{dx}\sec x = \sec x \tan x$$

Thus

$$\frac{d}{dx}(\sec x + \tan x) = \sec x \tan x + \sec^2 x = (\sec x)(\sec x + \tan x)$$

Since

$$\int \sec x\, dx = \int \frac{(\sec x)(\sec x + \tan x)}{\sec x + \tan x}\, dx$$

the calculations above tell us that the numerator of the latter integral is the derivative of the denominator, so we can once again use (16). We let

$$u = \sec x + \tan x, \quad \text{so that} \quad du = (\sec x)(\sec x + \tan x)\, dx$$

Then

$$\begin{aligned} \int \sec x\, dx &= \int \overbrace{\frac{1}{\sec x + \tan x}}^{1/u} \overbrace{[\sec x(\tan x + \sec x)]\, dx}^{du} \\ &= \int \frac{1}{u}\, du = \ln|u| + C = \ln|\sec x + \tan x| + C \end{aligned}$$

Consequently

$$\boxed{\int \sec x\, dx = \ln|\sec x + \tan x| + C} \tag{17}$$

The evaluation of $\int \csc x\, dx$ is similar (see Exercise 36).

Logarithmic Differentiation

The formula in (16) can be useful in the differentiation of complicated functions involving products, quotients, and powers. Indeed, from (16),

$$\frac{d}{dx} \ln|f(x)| = \frac{f'(x)}{f(x)}$$

so that

$$f'(x) = f(x) \frac{d}{dx} \ln|f(x)| \tag{18}$$

or in the Leibniz notation,

$$\frac{dy}{dx} = y \frac{d}{dx} \ln|y| \tag{19}$$

The idea is to first use the properties of logarithms to rewrite $\ln|f(x)|$ as a combination of logarithms of the component functions of f. For example, let

$$f(x) = \frac{(x^3 - 1)^8 (2x + \sin x)^{1/3}}{(x^2 + 1)^{1/2}}$$

Then

$$\ln|f(x)| = \ln|x^3 - 1|^8 + \ln|2x + \sin x|^{1/3} - \ln(x^2 + 1)^{1/2}$$

$$= 8 \ln |x^3 - 1| + \frac{1}{3} \ln |2x + \sin x| - \frac{1}{2} \ln (x^2 + 1)$$

Then we differentiate the right–hand side and use (18) to obtain $f'(x)$. This method of finding $f'(x)$ is called **logarithmic differentiation,** and is illustrated in the next example.

EXAMPLE 6 Let

$$f(x) = \frac{(x^3 - 1)^8 (2x + \sin x)^{1/3}}{(x^2 + 1)^{1/2}}$$

Use logarithmic differentiation to find $f'(x)$.

Solution By the comments preceding the example,

$$\ln |f(x)| = 8 \ln |x^3 - 1| + \frac{1}{3} \ln |2x + \sin x| - \frac{1}{2} \ln (x^2 + 1)$$

Next we differentiate $\ln |f(x)|$ to obtain

$$\frac{d}{dx} \ln |f(x)| = \frac{24x^2}{x^3 - 1} + \frac{1}{3} \frac{2 + \cos x}{2x + \sin x} - \frac{x}{x^2 + 1}$$

Now (18) yields

$$f'(x) = \frac{(x^3 - 1)^8 (2x + \sin x)^{1/3}}{(x^2 + 1)^{1/2}} \left(\frac{24x^2}{x^3 - 1} + \frac{2 + \cos x}{6x + 3 \sin x} - \frac{x}{x^2 + 1} \right)$$ ❑

We will return to the study of logarithms in Section 7.3, where we will establish relationships between the natural logarithm and other logarithms.

EXERCISES 5.7

In Exercises 1–4 evaluate the integral. Express your answer in terms of logarithms.

1. $\displaystyle\int_2^8 \frac{1}{x}\, dx$ **2.** $\displaystyle\int_{1/9}^{1/4} \frac{-1}{3x}\, dx$

3. $\displaystyle\int_{-4}^{-12} \frac{2}{t}\, dt$ **4.** $\displaystyle\int_{-1/16}^{-1/8} \frac{1}{t}\, dt$

In Exercises 5–12 find the domain and the derivative of the function.

5. $f(x) = \ln (x + 1)$ **6.** $k(t) = \ln (t^2 + 4)^3$

7. $f(x) = \ln \sqrt{\dfrac{x - 3}{x - 2}}$ **8.** $f(x) = \dfrac{\ln x}{x - 1}$

9. $f(t) = \sin (\ln t)$ **10.** $g(u) = \ln (\sin u)$

11. $f(x) = \ln (\ln x)$ **12.** $f(x) = \ln (x + \sqrt{x^2 - 1})$

In Exercises 13–14 find dy/dx by implicit differentiation.

13. $x \ln (y^2 + x) = 1 + 5y$ **14.** $y \ln \dfrac{y}{x} = \sin y^2$

In Exercises 15–18 find the domain, intercepts, relative extreme values, inflection points, concavity, and asymptotes for the given function. Then draw its graph.

15. $f(x) = \ln |x|$ **16.** $f(x) = \ln (x - 2)$

17. $f(x) = \ln (1 + x^2)$ **18.** $f(x) = (\ln x)^2$

19. Let $f(x) = x^{-2} - \ln x$. Use the Newton-Raphson method to approximate a zero of f. Continue until successive iterations obtained by calculator are identical.

20. Let $f(x) = x^2 + x \ln x$. Use the Newton-Raphson method to approximate a relative extreme value of f. Continue until successive iterations obtained by calculator are identical.

In Exercises 21–35 evaluate the integral.

21. $\displaystyle\int \frac{1}{x-1}\,dx$ **22.** $\displaystyle\int \frac{2}{1-4x}\,dx$

23. $\displaystyle\int \frac{x}{x^2+4}\,dx$ **24.** $\displaystyle\int \frac{x^2}{1-x^3}\,dx$

25. $\displaystyle\int_0^{\pi/3} \frac{\sin x}{1-3\cos x}\,dx$ **26.** $\displaystyle\int \frac{1}{x(1+\ln x)}\,dx$

27. $\displaystyle\int_{-1}^{0} \frac{x+2}{x^2+4x-1}\,dx$ **28.** $\displaystyle\int_1^4 \frac{1}{\sqrt{x}\,(1+\sqrt{x})}\,dx$

29. $\displaystyle\int \frac{\ln z}{z}\,dz$ **30.** $\displaystyle\int \frac{(\ln z)^5}{z}\,dz$

31. $\displaystyle\int \frac{\ln(\ln t)}{t\ln t}\,dt$ **32.** $\displaystyle\int \frac{\tan\sqrt{t}}{\sqrt{t}}\,dt$

33. $\displaystyle\int \cot t\,dt$

34. $\displaystyle\int \frac{1}{1+x^{1/3}}\,dx$ (*Hint:* Substitute $u = 1 + x^{1/3}$.)

***35.** $\displaystyle\int \frac{x}{1+x\tan x}\,dx$ (*Hint:* Substitute $u = x\sin x + \cos x$.)

***36.** Evaluate $\int \csc x\,dx$. (*Hint:* Pattern the solution after the evaluation of $\int \sec x\,dx$ in the text.)

In Exercises 37–39 find the area A of the region between the graph of f and the x axis on the given interval.

37. $f(x) = \dfrac{1}{x}$; $[e, e^2]$ **38.** $f(x) = \dfrac{x}{2-x^2}$; $[-2, -\sqrt{3}]$

***39.** $f(x) = \dfrac{\sin^3 x}{\cos x}$; $\left[\dfrac{\pi}{4}, \dfrac{\pi}{3}\right]$

40. **a.** Adapt the ideas used in the solution of Example 1 in order to show that $\int_1^{2.9} 1/t\,dt > 1$. (*Hint:* Find a suitable partition P of $[1, 2.9]$, and use the Comparison Property.)

b. Use part (a) to prove that $e < 2.9$.

In Exercises 41–46 use logarithmic differentiation to find the derivative of the given function.

41. $f(x) = (x+1)^{1/5}(2x+3)^2(7-4x)^{-1/2}$

42. $f(x) = (1+\cos x)^{2/3}(x^2+x+1)^{4/5}x^{1.1}$

43. $y = \sqrt[3]{\dfrac{(x+3)^2(2x-1)}{(4x+5)^4}}$ **44.** $y = \dfrac{\sqrt{x^2+1}\,\sin^3 x}{x^2\sqrt{2x^2+1}}$

45. $y = \dfrac{x^{3/2}e^{-x^2}}{1-e^x}$ **46.** $y = \dfrac{x^2\ln x}{(2x+1)^{3/2}\cos x}$

47. Prove that $\ln b^r = r\ln b$ for $b > 0$ and r rational by showing that the functions $\ln x^r$ and $r\ln x$ have the same derivative and the same value at 1.

48. Prove that $\ln(1/b) = -\ln b$ for $b > 0$

a. by using (11).

b. by showing that the functions $\ln(1/x)$ and $-\ln x$ have the same derivative and the same value at 1.

49. Prove that $\ln(b/c) = \ln b - \ln c$ for $b, c > 0$

a. by using (12) and the Law of Logarithms.

b. by showing that the functions $\ln(x/c)$ and $\ln x - \ln c$ have the same derivative and the same value at c.

***50.** Use Exercises 47–48 to show that

$$\lim_{x\to\infty} \ln x = \infty \quad\text{and}\quad \lim_{x\to 0^+} \ln x = -\infty$$

51. Using lower and upper sums, show that for each integer $n \ge 2$,

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln n < 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}$$

52. Exercise 50 and the inequality in Exercise 51 together show that if n is large enough, then the sum

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

is as large as we please. Use this information along with Exercise 51 to find a positive integer n such that the sum is larger than

a. 20 **b.** 100

53. **a.** Use the inequality $1/t \le 1/\sqrt{t}$ for $t \ge 1$ to show that

$$\ln x = \int_1^x \frac{1}{t}\,dt \le 2(\sqrt{x} - 1) \quad \text{for } x \ge 1$$

b. Use part (a) to show that

$$\lim_{x\to\infty} \frac{\ln x}{x} = 0$$

c. Use part (b) and (12) to show that $\lim_{x\to 0^+} x\ln x = 0$.

54. Sketch the graph of $x\ln x$. Use Exercise 53(c) and the fact that $\ln x = -1$ for $x \approx 0.37$.

55. Recall that $\ln e = 1$, and let

$$f(x) = \frac{\ln x}{x}$$

a. Show that f is increasing on $(0, e]$ and decreasing on $[e, \infty)$.

b. Find the maximum value of f on $(0, \infty)$.

c. Sketch the graph of f. Use the second derivative of f, Exercise 53, and the fact that $\ln x = 3/2$ for $x \approx 4.5$.

56. Use Exercise 57(b) of Section 5.5 to find an upper bound for the integral $\int_1^2 \ln x\,dx$. (The exact value is $(2\ln 2) - 1$.)

57. When a quantity of gas expands from an initial volume V_1 to a final volume V_2, the amount of work W done by the gas during the expansion is given by

$$W = \int_{V_1}^{V_2} P\, dV \tag{20}$$

where P is the pressure expressed as a function of the volume V. During an expansion in which the temperature remains constant, P is related to the volume by means of Boyle's Law:

$$P = \frac{c}{V} \tag{21}$$

where c is a constant. Using (20) and (21), obtain a formula for W that involves logarithms.

Applications

58. A beanbag factory has a marginal revenue function

$$m_R(x) = \frac{2}{x+1}$$

where x denotes thousands of beanbags sold and $m_R(x)$ denotes dollars received per beanbag.

a. Determine the total revenue function R. (*Hint:* Use (7) of Section 5.4 and the fact that $R(0) = 0$. What is the derivative of $\ln(x+1)$?)

b. Demonstrate that R is a reasonable total revenue function by showing that R is increasing and concave downward on the interval $(0, \infty)$.

59. The equation

$$\ln w = 4.4974 + 3.135 \ln s$$

has been used to relate the weight w (in kilograms) to the sitting height s (in meters) of people. Using this equation, find the weight of a person whose sitting height is

a. 1 meter **b.** $\frac{1}{2}$ meter

60. Let y represent the weight in ounces of a baby and x the age in months. It has been conjectured that y and x are related by the equation

$$\ln y - \ln(341.5 - y) = c(x - 1.66) \quad \text{for } 0 \le x \le 9$$

where c is a positive constant.

a. Show that dy/dx, the rate of weight increase with respect to age of the baby, is a positive function.

b. Find the age x_0 at which dy/dx is maximum, and find the corresponding weight. According to the equation, it is at this age that the baby gains weight fastest.

Projects

1. This project relates approximate values of the natural logarithm to parabolic approximation as discussed in Section 3.8.

a. Find an equation of the line tangent to the curve $y = 1/t$ at $(1, 1)$.

b. Show that for small $h > 0$, the area below the tangent line on $[1, 1 + h]$ is $h - h^2/2$ (which is approximately $\int_1^{1+h} (1/t)\, dt$).

c. Show that for $h < 0$ with h near 0, the area below the tangent line on $[1 + h, 1]$ is $-(h - h^2/2)$. (which is approximately $\int_{1+h}^{1} (1/t)\, dt$).

d. Together (b) and (c) yield

$$\ln(1 + h) \approx h - \frac{1}{2}h^2$$

when $|h|$ is small. Show that this formula is precisely the parabolic approximation formula for $\ln x$ with h near 0 (see Section 3.8).

2. In this project we will show that under mild conditions, a "law of logarithms" kind of condition yields the natural logarithm function. To that end, suppose f is defined on $(0, \infty)$, is differentiable at 1, and satisfies

$$f(xy) = f(x) + f(y) \tag{22}$$

for all x and y in $(0, \infty)$.

a. Prove that $f(1) = 0$.

b. Show that

$$f\left(\frac{y}{x}\right) = f(y) - f(x)$$

for all x and y in $(0, \infty)$. (Hint: $y = x(y/x)$.)

c. Deduce from (a) and (b) that if $0 < |h| < x$, then

$$\frac{f(x+h) - f(x)}{h} = \frac{f\left(\frac{x+h}{x}\right)}{h} = \frac{1}{x}\,\frac{f(1 + h/x) - f(1)}{h/x}$$

d. From (c) and the hypothesis that f is differentiable at 1, conclude that f' exists on $(0, \infty)$ and that

$$f'(x) = \frac{1}{x} f'(1) \quad \text{for } x > 0$$

It follows from (a) and (d) and the definition of $\ln x$ that if f is a differentiable function satisfying (22), then

$$f(x) = f'(1) \ln x \quad \text{for } x > 0$$

Finally, if we assume that $f'(1) = 1$, then $f(x) = \ln x$ for $x > 0$. Thus with mild extra conditions, the "law of logarithms" type of formula in (22) yields the natural logarithm function.

5.8 ANOTHER LOOK AT AREA

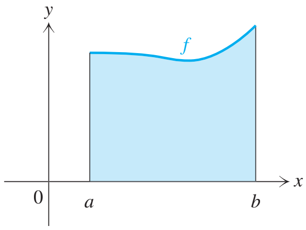
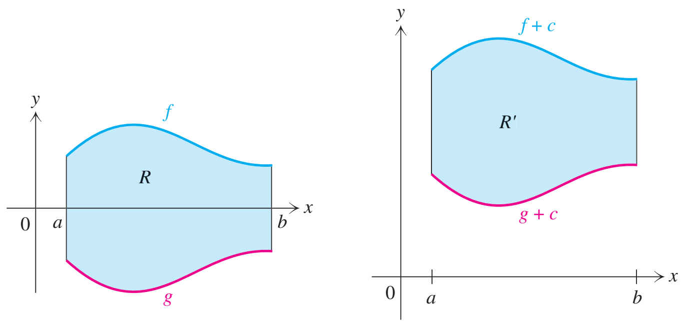


FIGURE 5.40

In Section 5.2 we defined the area of a region of the type shown in Figure 5.40 to be $\int_a^b f(x)\,dx$. However, this definition does not apply to regions like that shown in Figure 5.41(a) (which might, for instance, represent the surface of a lake). Using further properties that we naturally expect area to have, we will now extend the earlier definition of area to include regions whose lower boundaries are not necessarily horizontal.



(a) (b)

FIGURE 5.41

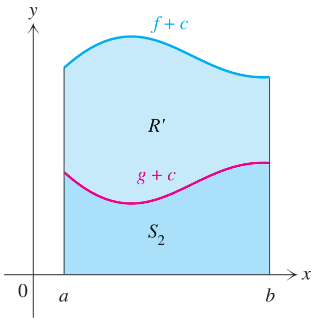


FIGURE 5.42

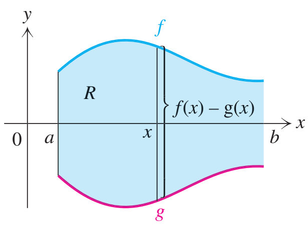


FIGURE 5.43

Let f and g be continuous on an interval $[a, b]$, and assume that $f(x) \ge g(x)$ for $a \le x \le b$. We will define the area of the region R that is bounded above by the graph of f, below by the graph of g, and on the sides by the lines $x = a$ and $x = b$ (Figure 5.41(a)). We call R the **region between the graphs of f and g on $[a, b]$.**

Since the area of a region should not be affected by shifting the region vertically, the area A of the region R in Figure 5.41(a) should be the same as the area of the region R' in Figure 5.41(b), which lies above the x axis and is bounded by the graphs of $f + c$ and $g + c$ for an appropriate constant c. However, in Figure 5.42 the area of the region S_1 (which is bounded above by the graph of $f + c$ and below by the x axis) should be the sum of the areas of R' and S_2. Now by Definition 5.5, we have the formulas for the areas of S_1 and S_2. Therefore we would expect that

$$A = \overbrace{\int_a^b [f(x) + c]\,dx}^{\text{area of } S_1} - \overbrace{\int_a^b [g(x) + c]\,dx}^{\text{area of } S_2} = \int_a^b [f(x) - g(x)]\,dx$$

Thus we are led to the following formal definition of the area A:

DEFINITION 5.22

Let f and g be continuous on $[a, b]$, with $f(x) \ge g(x)$ for $a \le x \le b$. The area A of the region R between the graphs of f and g on $[a, b]$ is given by

$$A = \int_a^b [f(x) - g(x)]\,dx \tag{1}$$

Notice that for $a \le x \le b$ the integrand $f(x) - g(x)$ represents the height of R at x (Figure 5.43).

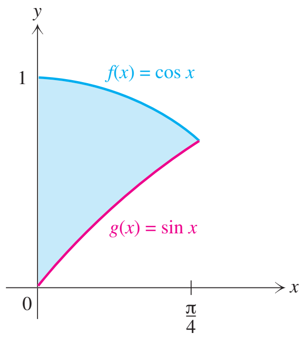


FIGURE 5.44

EXAMPLE 1 Let $f(x) = \cos x$ and $g(x) = \sin x$. Find the area A of the region between the graphs of f and g on $[0, \pi/4]$ (Figure 5.44).

Solution Since $\cos x \geq \sin x$ on $[0, \pi/4]$, it follows from (1) that

$$A = \int_0^{\pi/4} [f(x) - g(x)]\, dx = \int_0^{\pi/4} (\cos x - \sin x)\, dx$$

$$= (\sin x + \cos x)\Big|_0^{\pi/4} = \left(\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}\right) - (0 + 1) = \sqrt{2} - 1 \quad \square$$

If we seek the area of the region between two graphs that cross exactly twice but we are not given the interval over which to integrate, we first determine where the graphs cross and which graph lies above the other, and then integrate over the corresponding (bounded) interval.

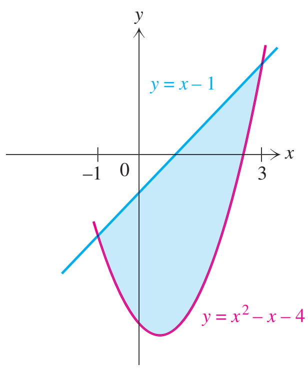


FIGURE 5.45

EXAMPLE 2 Find the area A of the region between the graphs of $y = x^2 - x - 4$ and $y = x - 1$ (Figure 5.45).

Solution First we determine the x coordinates of the points at which the two curves intersect:

$$x^2 - x - 4 = x - 1$$

$$x^2 - 2x - 3 = 0$$

$$(x + 1)(x - 3) = 0$$

$$x = -1 \qquad \text{or} \qquad x = 3$$

Thus the region whose area we seek lies between the graphs of $y = x^2 - x - 4$ and $y = x - 1$ on $[-1, 3]$. Since

$$x - 1 \geq x^2 - x - 4 \quad \text{for } -1 \leq x \leq 3$$

(see Figure 5.45), it follows that the height of the region at x is $[(x - 1) - (x^2 - x - 4)]$, so by (1),

$$A = \int_{-1}^{3} [(x - 1) - (x^2 - x - 4)]\, dx = \int_{-1}^{3} (-x^2 + 2x + 3)\, dx$$

$$= \left(-\frac{1}{3}x^3 + x^2 + 3x\right)\Big|_{-1}^{3} = (-9 + 9 + 9) - \left(\frac{1}{3} + 1 - 3\right)$$

$$= \frac{32}{3} \quad \square$$

Now suppose the graphs of f and g cross at one or more points in (a, b). Then the region R between the graphs of f and g on $[a, b]$ is composed of several regions R_1, R_2, . . . , each of the type whose area we have already defined (Figure 5.46). We naturally define the total area A of R to be the sum of the areas of those regions. In order to calculate A, that is, the area of the region between the graphs of f and g on $[a, b]$, we first determine those subintervals on which $f - g \geq 0$ and those on which $f - g \leq 0$. Then we integrate over those subintervals separately. Example 3 illustrates this technique.

EXAMPLE 3 Let $f(x) = \sin x$ and $g(x) = \cos x$. Find the area A of the region between the graphs of f and g on $[0, 2\pi]$ (Figure 5.47).

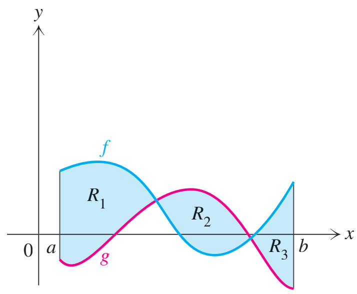


FIGURE 5.46

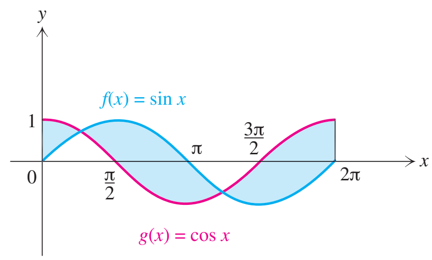


FIGURE 5.47

Solution In order to determine where $\sin x \geq \cos x$ and where $\sin x \leq \cos x$, we first find the values of x in $[0, 2\pi]$ for which $\sin x = \cos x$, which is equivalent to solving $\tan x = 1$ for x. But $\tan x = 1$ for $x = \pi/4$ and $x = 5\pi/4$. From this you can verify that $\sin x \geq \cos x$ on $[\pi/4, 5\pi/4]$ and $\cos x \geq \sin x$ on $[0, \pi/4]$ and $[5\pi/4, 2\pi]$. Therefore

$$\begin{aligned} A &= \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\, dx + \int_{5\pi/4}^{2\pi} (\cos x - \sin x)\, dx \\ &= (\sin x + \cos x)\Big|_0^{\pi/4} + (-\cos x - \sin x)\Big|_{\pi/4}^{5\pi/4} + (\sin x + \cos x)\Big|_{5\pi/4}^{2\pi} \\ &= (\sqrt{2} - 1) + (2\sqrt{2}) + (1 + \sqrt{2}) = 4\sqrt{2} \qquad \square \end{aligned}$$

The region in the next example may at first appear not to be of the form covered by our discussion, but in fact it is.

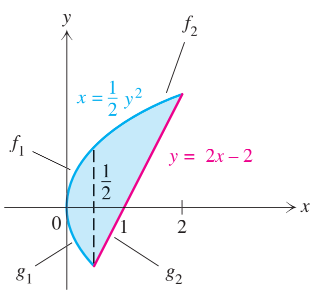


FIGURE 5.48

EXAMPLE 4 Find the area A of the region R between the parabola $x = \frac{1}{2}y^2$ and the line $y = 2x - 2$ (Figure 5.48).

Solution First we determine the x coordinates of the points at which the parabola and the line intersect:

$$x = \frac{1}{2}y^2 = \frac{1}{2}(2x-2)^2 = 2x^2 - 4x + 2$$

which means that

$$2x^2 - 5x + 2 = 0$$

or

$$(2x - 1)(x - 2) = 0$$

Consequently $x = \frac{1}{2}$ or $x = 2$. Next, observe from Figure 5.48 that R may be broken up into two parts: the part between $x = 0$ and $x = \frac{1}{2}$ on the x axis and the part between

$x = \frac{1}{2}$ and $x = 2$ on the x axis. The part of R between $x = 0$ and $x = \frac{1}{2}$ lies between the graphs of f_1 and g_1, where

$$f_1(x) = \sqrt{2x} \quad \text{and} \quad g_1(x) = -\sqrt{2x}$$

The part of R between $x = \frac{1}{2}$ and $x = 2$ lies between the graphs of f_2 and g_2, where

$$f_2(x) = \sqrt{2x} \quad \text{and} \quad g_2(x) = 2x - 2$$

Therefore

$$\begin{aligned} A &= \int_0^{1/2} (f_1(x) - g_1(x))\, dx + \int_{1/2}^{2} (f_2(x) - g_2(x))\, dx \\ &= \int_0^{1/2} [\sqrt{2x} - (-\sqrt{2x})]\, dx + \int_{1/2}^{2} [\sqrt{2x} - (2x - 2)]\, dx \\ &= \int_0^{1/2} 2\sqrt{2x}\, dx + \int_{1/2}^{2} (\sqrt{2x} - 2x + 2)\, dx \\ &= \frac{4\sqrt{2}}{3} x^{3/2} \bigg|_0^{1/2} + \left(\frac{2\sqrt{2}}{3} x^{3/2} - x^2 + 2x\right)\bigg|_{1/2}^{2} \\ &= \left(\frac{2}{3} - 0\right) + \left[\left(\frac{8}{3} - 4 + 4\right) - \left(\frac{1}{3} - \frac{1}{4} + 1\right)\right] = \frac{9}{4} \quad \square \end{aligned}$$

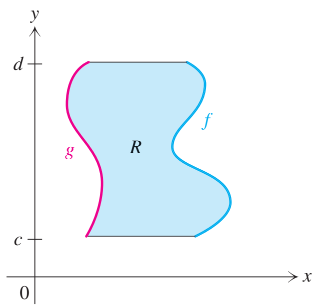


FIGURE 5.49

Reversing the Roles of x and y

Instead of considering a region R as the region between the graphs of two functions of x, it is sometimes convenient to consider R as the region between the graphs of two functions of y (Figure 5.49). Then the area is computed by integrating along the y axis, instead of along the x axis.

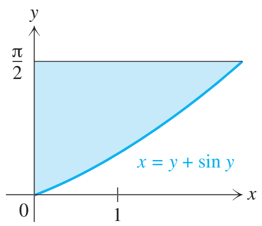


FIGURE 5.50

EXAMPLE 5 Let R be the region between the y axis and the graph of the equation $x = y + \sin y$ on $[0, \pi/2]$ (Figure 5.50). Find the area A of R.

Solution Since $y + \sin y \geq 0$ for $0 \leq y \leq \pi/2$, it follows that

$$\begin{aligned} A &= \int_0^{\pi/2} (y + \sin y)\, dy = \left(\frac{1}{2} y^2 - \cos y\right)\bigg|_0^{\pi/2} \\ &= \frac{1}{2}\left(\frac{\pi}{2}\right)^2 - (-1) = \frac{\pi^2}{8} + 1 \quad \square \end{aligned}$$

In Example 5 we have no way of describing y in terms of x; therefore it would be impossible to use (1) to determine the area of the region.

If a region can be described both in terms of x and in terms of y, the area is the same, whether we integrate with respect to x or with respect to y. To support this claim, we now return to the region described in Example 4 and find its area by integrating along the y axis.

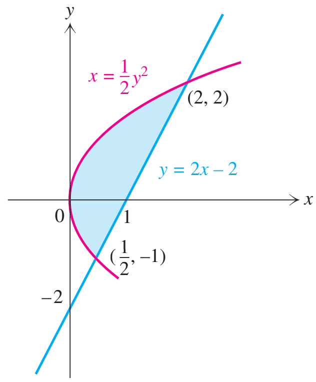


FIGURE 5.51

EXAMPLE 6 Find the area A of the region R between the parabola $x = \frac{1}{2} y^2$ and the line $y = 2x - 2$ (Figure 5.51).

Solution First we determine the y coordinates of points at which the parabola and the line intersect:

$$\frac{1}{2}y^2 = x = \frac{1}{2}(y+2)$$

$$y^2 - y - 2 = 0$$

$$(y+1)(y-2) = 0$$

$$y = -1 \quad \text{or} \quad y = 2$$

Since $\frac{1}{2}(y+2) \geq \frac{1}{2}y^2$ for $-1 \leq y \leq 2$, it follows that

$$\begin{aligned} A &= \int_{-1}^{2} \left[\frac{1}{2}(y+2) - \frac{1}{2}y^2\right] dy = \frac{1}{2}\int_{-1}^{2} (y + 2 - y^2)\, dy \\ &= \frac{1}{2}\left(\frac{1}{2}y^2 + 2y - \frac{1}{3}y^3\right)\Big|_{-1}^{2} \\ &= \frac{1}{2}\left[\left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)\right] = \frac{9}{4} \end{aligned}$$

(the same value for the area we found in Example 4). ❑

Cavalieri's Principle

We continue this section with a discussion of Cavalieri's Principle. Let f and g be continuous on an interval $[a, b]$, and let A be the area of the region between the graphs of f and g. Using the fact that

$$|f(x) - g(x)| = \begin{cases} f(x) - g(x) & \text{if } f(x) \geq g(x) \\ g(x) - f(x) & \text{if } g(x) \geq f(x) \end{cases}$$

we obtain the formula

$$A = \int_a^b |f(x) - g(x)|\, dx \tag{2}$$

which holds regardless of the relationship between f and g.

Now suppose f is any continuous function defined on $[a, b]$, and let $g = f + c$, where c is a fixed positive constant. Then the area of the region between the graphs of f and g on $[a, b]$ is

$$\int_a^b |f(x) - g(x)|\, dx = \int_a^b c\, dx = c(b-a)$$

In other words, if the distance between the lower and upper boundaries of a region is constant, then the region's area is the product of its length and height, just as in the case of rectangles (Figure 5.52(a)).

Bonaventura Cavalieri (1598–1647)
Cavalieri was born in Milan, studied under Galileo, and was professor of mathematics at the University of Bologna. In his treatise *Geometricia indivisibilibus,* he introduced the "method of indivisibles," which led to Cavalieri's Principle and strongly influenced Leibniz. Although the notion of "indivisible" was vague and was roundly criticized by contemporary mathematicians, the method enabled Cavalieri and his successors to determine the areas and volumes of many figures encountered in geometry. Moreover, Cavalieri in effect integrated x^n for $n = 1, 2, \ldots, 9$, and suggested that a pattern existed that would be valid for any positive integer n.

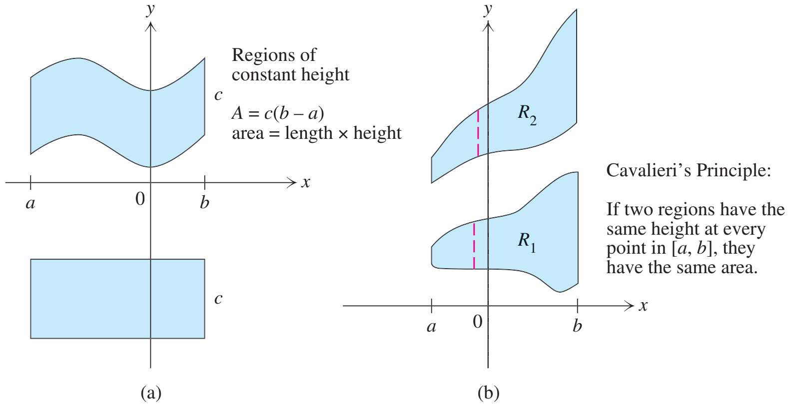


FIGURE 5.52

The result just obtained is a special case of Cavalieri's Principle. This principle was stated in 1635 by the Milanese mathematician Bonaventura Cavalieri, who used it to help derive formulas for areas of plane regions. In effect Cavalieri's Principle says that if R_1 and R_2 are two regions like those in Figure 5.52(b), with side boundaries $x = a$ and $x = b$, and if the height of R_1 (measured perpendicular to the x axis) is the same as the height of R_2 at every point in $[a, b]$ (Figure 5.52(b)), then R_1 and R_2 have the same area. Cavalieri came to this conclusion several decades before Newton and Leibniz introduced the general concept of integral. However, by using (2), we can easily prove Cavalieri's Principle for the case in which the upper boundaries of R_1 and R_2 are the graphs of continuous functions f_1 and f_2 and the lower boundaries are the graphs of continuous functions g_1 and g_2. In that case the fact that R_1 and R_2 have identical height at every point in $[a, b]$ means that

$$|f_1(x) - g_1(x)| = |f_2(x) - g_2(x)| \qquad \text{for } a \le x \le b$$

Consequently

$$\int_a^b |f_1(x) - g_1(x)|\, dx = \int_a^b |f_2(x) - g_2(x)|\, dx$$

which means, by (2), that the area of R_1 and the area of R_2 are the same.

Other Interpretations of Area

The area of the region between the graphs of f and g on an interval $[a, b]$ has many interpretations, depending on the particular meaning attached to f and g.

For example, if v represents the velocity of an object during the time period $[a, b]$ and if $v(t) \ge 0$ for $a \le t \le b$, then the net distance the object has traveled during the period is given by $\int_a^b v(t)\, dt$ (see Figure 5.53(a)). More generally, suppose v_1 and v_2 denote the velocities of two objects that start from the same position at the same time and suppose $v_2(t) \ge v_1(t) \ge 0$ for $a \le t \le b$, as in Figure 5.53(b). Then the area of the shaded region represents the distance between the objects at the end of that time interval.

Areas of regions such as the one shown in Figure 5.53(c) play a role in the economic study of the distribution of national income. In the figure, R is bounded above by the line $y = x$ and below by the so-called **Lorenz curve** $y = L(x)$. For any x

between 0 and 100, $L(x)$ is the percentage of the national income owned by the lowest x percentage of families in the country. Values for income shares in various years have been surprisingly stable, which means that over the years L has changed very little. If the income were distributed equally, then the Lorenz curve would be the line $y = x$. Thus the shaded region in Figure 5.53(c) represents the inequity in income distribution. The larger the area is, the larger the inequality in income distribution.

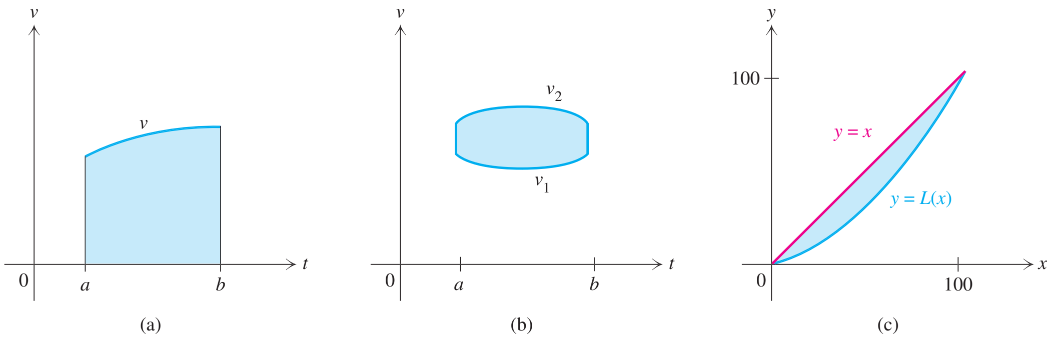
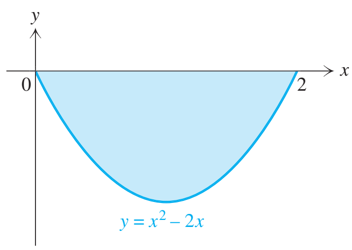


FIGURE 5.53 (a) The area under the velocity curve represents distance. (b) The area between the velocity curves represents distance. (c) A Lorenz diagram.

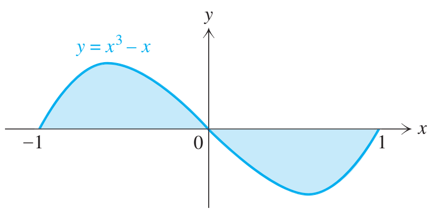
EXERCISES 5.8

In Exercises 1–6 find the area A of the shaded region.

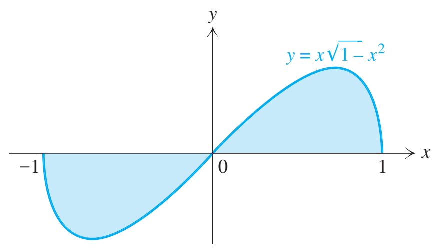
1.



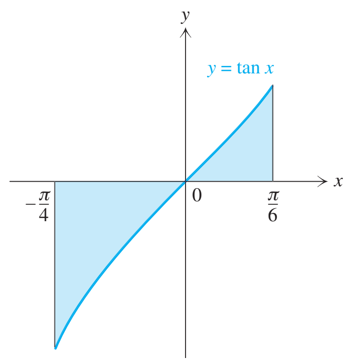
2.



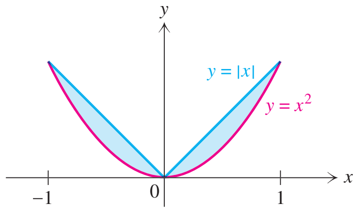
3.



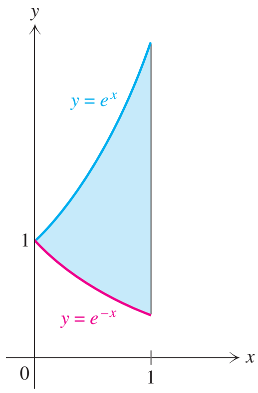
4.



5.



6.



In Exercises 7–12 find the area A of the region between the graph of f and the x axis on the given interval.

7. $f(x) = x^2 + 2x$; $[-1, 3]$

8. $f(x) = \cos x - \sin x$; $[0, \pi/3]$

9. $f(x) = \dfrac{x}{\sqrt{1+x^2}}$; $[-1, \sqrt{7}]$

10. $f(x) = \dfrac{x}{x^2-1}$; $[-\frac{1}{2}, \frac{1}{3}]$

11. $f(x) = \dfrac{\ln x}{x}$; $[\frac{1}{2}, 2]$

12. $f(x) = e^{-x} - 1$; $[0, 3]$

In Exercises 13–24 find the area A of the region between the graphs of the functions on the given interval.

13. $f(x) = x^2$, $g(x) = x^3$; $[-2, 1]$

14. $f(x) = 1/x$, $g(x) = 1/x^2$; $[\frac{1}{2}, 2]$

15. $g(x) = x^2 + 4x$, $k(x) = x - 2$; $[-3, 0]$

16. $g(x) = 2x^3 + 2x^2$, $k(x) = 2x^3 - 2x$; $[-4, 2]$

17. $f(x) = \sec^2 x$, $g(x) = \sec x \tan x$; $[-\pi/3, \pi/6]$

18. $f(x) = \sin 2x$, $g(x) = 2 \cot x$; $[\pi/3, 2\pi/3]$

19. $g(x) = \sin^2 x$, $k(x) = \tan x$; $[-\pi/4, \pi/4]$

20. $g(x) = x^3 + 3x + \cos^2 x$, $k(x) = 4x^2 - \sin^2 x + 1$; $[-1, 2]$

21. $f(x) = x\sqrt{2x+3}$, $g(x) = x^2$; $[-1, 3]$

22. $f(x) = x(x^2+1)^5$, $g(x) = x^2(x^3+1)^5$; $[-1, 1]$

23. $f(x) = e^{2x}$, $g(x) = e^x$; $[-1, 1]$

24. $f(x) = e^x$, $g(x) = 1/e^x$; $[-1, 2]$

In Exercises 25–30 the graphs of f and g enclose a region. Determine the area A of that region.

25. $f(x) = x^3$, $g(x) = x^{1/3}$

26. $f(x) = x^2 + 3$, $g(x) = 12 - x^2$

27. $f(x) = x^2 + 1$, $g(x) = 2x + 9$

28. $f(x) = x^3 + x$, $g(x) = 3x^2 - x$

29. $f(x) = x^3 + 1$, $g(x) = (x+1)^2$

30. $f(x) = 2 - \sqrt{x}$, $g(x) = \dfrac{\sqrt{x}+1}{2\sqrt{x}}$

In Exercises 31–34 find the area A of the region between the graphs of the given equations.

31. $y^2 = 6x$ and $x^2 = 6y$

32. $y^2 = x$ and $y = x - 2$

33. $y^2 = 2x - 5$ and $y = x - 4$

34. $y^2 = 3x$ and $y = x^2 - 2x$ (*Hint:* The curves intersect at the points $(0, 0)$ and $(3, 3)$.)

In Exercises 35–36 the graphs of the three equations enclose a region. Determine the area A of that region.

35. $y = x + 2$, $y = -3x + 6$, $y = (2 - x)/3$

36. $y = \frac{3}{2}x$ for $x \geq 0$; $y = -\frac{3}{2}x$ for $x \leq 0$; $y = -x^2 - \frac{3}{2}x + 4$

In Exercises 37–39 find the area A of the region between the graphs of the given equations.

37. $x = y^2 - y$ and $x = y - y^2$

38. $x = 0$ and $x = \cos y$ for $-\pi/2 \leq y \leq 3\pi/2$

39. $x = y^2$ and $x = 6 - y - y^2$

40. Calculate the area A_b of the region shaded in Figure 5.54. Then find the limit of the area as b tends to ∞.

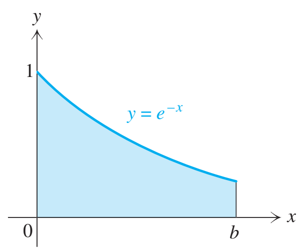


FIGURE 5.54 Graph for Exercise 40.

41. For any positive number a, let R_a and S_a denote the regions colored light blue and dark blue, respectively, in Figure 5.55, and let r_a be the ratio of the area of R_a to the area of S_a. Show that r_a is independent of a.

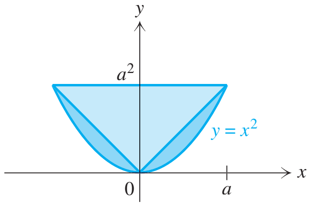


FIGURE 5.55 Graph for Exercise 41.

42. Let f, g, and h be continuous on $[a, b]$. Using (2), show that the area A of the region between the graphs of f and g on $[a, b]$ is the same as the area of the region between the graphs of $f + h$ and $g + h$ on $[a, b]$.

Application

43. Suppose the revenue per unit time and the cost per unit time of an athletic equipment company are given, respectively, by

$$f(t) = \sqrt{t} + 3 \quad \text{and} \quad g(t) = t^{1/3} + 2 \quad \text{for } 0 \le t \le 4$$

where t is in months and $f(t)$ and $g(t)$ are in thousands of dollars per month. Determine if the company is able to earn a profit of \$5000 during the four-month period.

Project

1. Let f be differentiable on $[0, \infty)$ and $f(x) > 0$ for $x > 0$. In this project we will compare the area $A(x)$ under the graph of f on $[0, x]$ with the area $B(x)$ of the rectangle with vertices $(0, 0)$, $(x, 0)$, $(x, f(x))$, and $(0, f(x))$ (Figure 5.56).

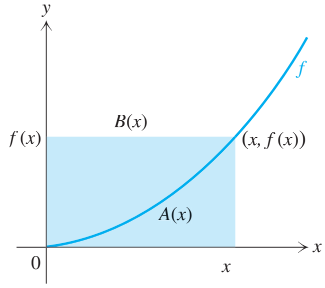


FIGURE 5.56 Figure for the project.

a. Let $c > 0$ and $r > 1$, and let $f(x) = cx^{r-1}$ for $x \ge 0$. Show that for each $x > 0$, $B(x)/A(x)$ depends only on r (and not on either x or c).

b. This part demonstrates that the functions f described in (a) are the only ones for which $B(x)/A(x)$ does not depend on x. To that end, suppose that g is differentiable on $[0, \infty)$ and $g(x) > 0$ for $x > 0$. Then $A(x) = \int_0^x g(t)\,dt$ and $B(x) = xg(x)$. Suppose $B(x)/A(x) = r$, or equivalently, $B(x) = rA(x)$, for all $x > 0$.

i. By taking derivatives of both sides of the equation $B(x) = rA(x)$, show that $g(x) + xg'(x) = rg(x)$, or equivalently, $g'(x)/g(x) = (r - 1)/x$ for all $x > 0$.

ii. Integrate the two sides of the last equation in (i), and deduce that there is a constant c such that $\ln g(x) = \ln cx^{r-1}$. (*Hint:* Let $\ln c = C$, where C is the constant appearing from indefinite integration, and use the Law of Logarithms.)

iii. Conclude that $g(x) = cx^{r-1}$ for all $x \ge 0$.

5.9 WHO INVENTED CALCULUS?

It is often said that Newton and Leibniz invented calculus. From the vantage point of our knowledge of the derivative and the integral—as well as of the Fundamental Theorem of Calculus, which unites them—we may ask why these two great mathematicians are credited with founding calculus.

Long before Newton and Leibniz, mathematicians knew how to calculate tangents to various curves. Using horizontal tangents, they produced a general procedure for determining maxima and minima of curves. As regards areas, even Archimedes was able to find areas of regions bounded by a few of the more common curves. By the early seventeenth century, the foremost mathematicians had developed very refined ways of evaluating areas and computing what we now call integrals.

41. m even, $n = 3m$ **43.** e^{-1}

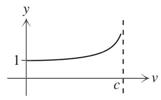
45. **a.** approximately 4.5 billion years
b. approximately 99.8%

47. 48 miles per hour

49. side length of base: 10 feet; height: 8 feet

51. $125,000

53.



55. 160 feet

59. **a.** $T = \dfrac{a \sec \theta}{v} + \dfrac{k - a \tan \theta}{w}$ **c.** Jo

61. radius: $\frac{1}{6}\sqrt{6l}$; length: $\frac{2}{3}\sqrt{3l}$

63. 60 feet from the house closest to the street

65. no

Cumulative Review Exercises (Chapters 1–3)

1. union of $(-1/\sqrt[3]{2}, 0)$ and $(1/\sqrt[3]{2}, \infty)$

2. $[-7, 7]$

3. union of $(-\sqrt[4]{15}, -1]$ and $[1, \sqrt[4]{15})$

4. $3\pi/2 + 2n\pi$ and $[\pi/6 + 2n\pi, 5\pi/6 + 2n\pi]$ for any integer n

5. **a.** union of $(-\infty, \frac{1}{3})$ and $(\frac{1}{2}, \infty)$ **b.** $\sqrt{\dfrac{2x-1}{6x-2}}$

6. $f(x) = 1 + x$ and $g(x) = \sin(3x^2 - 2)$

7. $-\infty$ **8.** ∞ **9.** 2 **10.** ∞

11. $(\frac{15}{4}, \frac{17}{16})$

12. **a.** It is. **b.** It is not.

13. $-\frac{1}{8}$

14. $-(3x^2 \sin x^3)\cos(\cos x^3)$

15. $\dfrac{(\sec^2 x)\, e^{\tan x} - 1}{(x - e^{\tan x})^2}$

16. **a.** $(0, \infty)$ **b.** $\dfrac{e^x}{1 - e^x}$ **c.** ∞

18. **a.** $t = 1$ **b.** $-\frac{1}{2}$

19. $\dfrac{4 - y^2}{2xy + 3}$

20. $2\sqrt{3}$ units per second

21. $\dfrac{\pi}{16}$ square feet

22. 1.308571201

23. approximately 4.5×10^8 kilometers

CHAPTER 5

Section 5.1

1. $L_f(P) = 3/2$; $U_f(P) = 9/2$

3. $L_f(P) = -\pi/2$; $U_f(P) = \pi/2$

5. left sum: 11/6; right sum: 13/12

7. $L_f(P) = 27/4$; $U_f(P) = 33/4$

9. $L_f(P) = 13/12$; $U_f(P) = 11/6$

11. $L_f(P) = \pi/2$; $U_f(P) = \sqrt{2}\pi/2$

13. $L_f(P) \approx 0.6039259454$; $U_f(P) \approx 0.7150370565$

15. $L_f(P) \approx 0.0518487986$; $U_f(P) \approx 0.3984223889$

17. left sum: -15; right sum: -9; midpoint sum: -12

19. left sum: approximately 0.927222222; right sum: approximately 0.482777778; midpoint sum: approximately 0.6481464407

21. left sum: approximately 4.163897657; right sum: approximately 4.163897657; midpoint sum: approximately 4.15942373

23. left sum: approximately 2.12401742; right sum: approximately 2.193332139; midpoint sum: approximately 2.158987223

25. left sum: approximately 0.927222222

27. upper sum: approximately 0.7113948156

29. We cannot find $U_f(P)$ since f has no maximum value on $[0, x_1]$. No problem for $L_f(P)$.

33. **a.** Approximately 0.9352941176×10^4, that is, $\$9,352.94 \times 10^4$
b. Approximately 1.202905554×10^4, that is, $12,029.06. Adding partition points increases the lower sum.

Section 5.2

1. $L_f(P) = 4$; $U_f(P) = 12$

3. $L_f(P) = -3\sqrt{2}\pi/8$; $U_f(P) = 3\sqrt{2}\pi/8$

5. left sum: 2; right sum: 8; midpoint sum: 9/2

7. left sum: $-\pi/3$; right sum: $\pi/3$; midpoint sum: 0

9. approximately 2.008248408

11. approximately 0.1427055747

13. approximately 2.762521203

15. 0 **17.** -125 **19.** approximately 2/3

21. 25/2 **23.** 15/2

33. **a.** 16 **b.** -8 **c.** 0 **d.** $-1/(2\pi)$

37. $\displaystyle\int_a^b x^n dx = \frac{1}{n+1}(b^{n+1} - a^{n+1})$

39. **a.** $1/(12n^2)$ **b.** $9/(8n^2)$ **c.** $1/(12n^2)$

43. **a.** $A \approx 18.85539331$ **b.** $A = \pi ab$

45. 69,800 square feet

Section 5.3

1. 14 **3.** 30

5. $\frac{1}{2} + \frac{3}{2} = 2$ **7.** $-\frac{1}{3} + \frac{8}{3} = \frac{7}{3}$

9. $a = 3, b = 2$ **11.** $a = 5, b = 1$

13. $m = \frac{1}{3}, M = \frac{1}{2}; \frac{1}{3} \le \int_2^3 \frac{1}{x}\,dx \le \frac{1}{2}$

15. $m = \frac{1}{2}, M = \frac{\sqrt{2}}{2}; \frac{\pi}{24} \le \int_{\pi/4}^{\pi/3} \cos x\,dx \le \frac{\sqrt{2}\pi}{24}$

17. $\frac{1}{2}$ **19.** $\frac{1}{3}$ **23.** $\frac{5}{6}$

25. b. approximately 1.65

29. approximately 54.4 degrees Fahrenheit

Section 5.4

1. $x(1 + x^3)^{29}$ **3.** $-1/y^3$ **5.** $2x^3 \sin x^2$

7. $-(1 + y^2)^{1/2} + 2y(1 + y^4)^{1/2}$

9. $\frac{512}{5}x(1 + 16x^2)^{-1/5}$ **11.** 4

13. -4 **15.** 12

17. $\frac{1}{101}$ **19.** 0

21. $\frac{9}{2}(4^{2/9} - 1)$ **23.** $10\pi - 2\pi^2 + 8.625$

25. $\frac{9}{2}$ **27.** $\frac{1}{2}\sqrt{3}$ **29.** $\frac{1}{2} - \frac{1}{2}\sqrt{2}$

31. $\frac{7}{24}$ **33.** $\ln 2$ **35.** $e^2 - 1$

37. $\sqrt{3}$ **39.** 1 **41.** $\frac{2}{5}$

43. $\frac{3}{2}$ **45.** $\frac{14}{3}$ **47.** 1

49. 1 **51.** $\frac{1}{4}\pi$ **57.** $\frac{25}{3}$

59. a. $f(t) = 5t^2 - \frac{1}{3}t^3$ **b.** $f(5) = \frac{250}{3}$

61. $672{,}000/\pi$ tons

63. a. \$111.01 dollars **b.** \$400.01 dollars

65. a. 8 feet per second **b.** 8 seconds

67. $\int_{t_1}^{t_2} R(t)\,dt$

71. $781{,}250{,}000\pi$ foot-pounds

73. 1800 feet

75. a. $p(t_2) - p(t_1) = \int_{t_1}^{t_2} F\,dt$

b. The impulse is 0.95 kilogram meter per second. The average force is 950 Newtons.

Section 5.5

1. $x^2 - 7x + C$

3. $\frac{3}{2}x^{4/3} - \frac{12}{7}x^{7/4} + \frac{5}{7}x^{7/5} + C$

5. $\frac{t^6}{6} + \frac{1}{3t^3} + C$ **7.** $2 \sin x - \frac{5}{2}x^2 + C$

9. $-3 \cot x - \frac{1}{2}x^2 + C$ **11.** $\frac{4}{3}t^3 + 2t^2 + t + C$

13. $x + 2 \ln x - 1/x + C$ **15.** $-\frac{15}{2}$

17. $3 - 5\sqrt{2}$ **19.** $\frac{9\pi^2}{32} + \frac{2}{\pi} + \frac{\sqrt{2}}{2}$

21. $-1 - \frac{5}{3}\sqrt{3}$ **23.** $-\frac{136}{3}$

25. $\pi + \frac{5}{\pi} + \frac{\pi^2}{4}$ **27.** $-\frac{142}{3}$

29. $\frac{5}{2}$ **31.** $4 - 2e^{\pi}$ **33.** 18

35. $\int 20x(1 + x^2)^9\,dx = (1 + x^2)^{10} + C$

37. $\int (x \cos x + 2 \sin x)\,dx = x \sin x - \cos x + C$

39. $\int 21 \sin^6 x \cos x\,dx = 3 \sin^7 x + C$

41. $e^{(x^2)} - e^{-x} + C$ **43.** 10

45. 12 **47.** $3 - \sqrt{2}/2$

49. $12 - 4 \ln 2$ **55.** $\frac{1}{12}\sqrt{3}\pi$

65. a. $2\sqrt{2} - 1$ **b.** 1

Section 5.6

1. $\frac{1}{6}(4x - 5)^{3/2} + C$ **3.** $(\sin \pi x)/\pi + C$

5. $\frac{1}{2}\sin x^2 + C$ **7.** $\frac{1}{3}\cos^{-3} t + C$

9. $-\frac{2}{5} \cdot \frac{1}{(t^2 - 3t + 1)^{5/2}} + C$

11. $\frac{2}{5}(x + 1)^{5/2} - \frac{4}{3}(x + 1)^{3/2} + C$

13. $\frac{2}{3}(3 + \sec x)^{3/2} + C$ **15.** $\frac{1}{2}e^{(x^2)} + C$

17. $\frac{1}{2}\ln(x^2 + 1) + C$ **19.** $\frac{1}{13}(x^3 + 1)^{13} + C$

21. $\frac{1}{3}(4 + x^{3/2})^2 + C$ **23.** $\frac{2}{9}(3x + 7)^{3/2} + C$

25. $\frac{1}{3}(1 + 2x + 4x^2)^{3/2} + C$

27. 0 **29.** $\frac{1}{7}\sin^7 t + C$

31. $\frac{1}{3}(\sin 2z)^{3/2} + C$ **33.** $\sqrt{2} - 1$

35. $2 \tan \sqrt{z} + C$

37. $\frac{1}{3}(w^2 + 1)^{3/2} + (w^2 + 1)^{1/2} + C$

39. $\frac{1}{2}(27 - 5\sqrt{5})$

41. $-\frac{1}{2}\cos(1 + e^{2x}) + C$

43. $\frac{1}{4}\ln(1 + x^4) + C$

45. $\frac{2}{5}(x + 2)^{5/2} - \frac{4}{3}(x + 2)^{3/2} + C$

47. $\frac{96}{5}$

49. $-\frac{1}{256}[\frac{1}{3}(1 - 8t)^{3/2} - \frac{2}{5}(1 - 8t)^{5/2} + \frac{1}{7}(1 - 8t)^{7/2}] + C$

51. $\frac{26}{15}$ **53.** $\frac{14}{3}$ **55.** $\frac{3}{20}$ **57.** $\frac{38}{3}$

61. b. $1/858 \approx 0.0011655012$

65. $1/x$ is not continuous (or even defined) on $[-1, 1]$.

67. a. 1/4

b. The electron is (essentially) certain to be in the interval $[0, 1]$.

Section 5.7

1. $\ln 4$ **3.** $2 \ln 3$

5. domain: $(-1, \infty)$; $f'(x) = \frac{1}{x + 1}$

7. domain: union of $(-\infty, 2)$ and $(3, \infty)$;

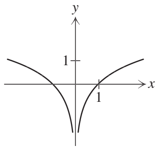
$f'(x) = \frac{1}{2(x - 3)(x - 2)}$

9. domain: $(0, \infty)$; $f'(t) = [\cos(\ln t)]\dfrac{1}{t}$

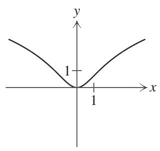
11. domain: $(1, \infty)$; $f'(x) = \dfrac{1}{x \ln x}$

13. $\dfrac{dy}{dx} = \dfrac{(y^2 + x)\ln(y^2 + x) + x}{5(y^2 + x) - 2xy}$

15.



17. relative minimum value: $f(0) = 0$; inflection points: $(-1, \ln 2)$ and $(1, \ln 2)$



19. 1.531584394

21. $\ln|x - 1| + C$

23. $\frac{1}{2}\ln(x^2 + 4) + C$

25. $-\frac{2}{3}\ln 2$

27. $-\ln 2$

29. $\frac{1}{2}(\ln z)^2 + C$

31. $\frac{1}{2}(\ln(\ln t))^2 + C$

33. $\ln|\sin t| + C$

35. $\ln|x \sin x + \cos x| + C$

37. 1

39. $\frac{1}{2}\ln 2 - \frac{1}{8}$

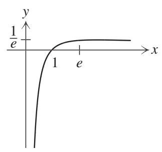
41. $[(x + 1)^{1/5}(2x + 3)^2(7 - 4x)^{-1/2}] \cdot \left[\dfrac{1}{5(x + 1)} + \dfrac{4}{2x + 3} + \dfrac{2}{7 - 4x}\right]$

43. $\sqrt[3]{(x + 3)^2(2x - 1)/(4x + 5)^4} \cdot \left(\dfrac{2}{3x + 9} + \dfrac{2}{6x - 3} - \dfrac{16}{12x + 15}\right)$

45. $\dfrac{x^{3/2}e^{-x^2}}{1 - e^x}\left(\dfrac{3}{2x} - 2x + \dfrac{e^x}{1 - e^x}\right)$

53. $y = -x + 2$

55. b. $1/e$ **c.**



57. $W = c \ln(V_2/V_1)$

59. a. approximately 89.7834 kilograms
b. approximately 10.2204 kilograms

Section 5.8

1. $\frac{4}{3}$ **3.** $\frac{2}{3}$ **5.** $\frac{1}{3}$

7. $\frac{56}{3}$ **9.** $3\sqrt{2} - 2$ **11.** $(\ln 2)^2$

13. $\frac{27}{4}$ **15.** $\frac{11}{6}$ **17.** $\frac{2}{3}\sqrt{3} + 2$

19. $\ln 2$ **21.** $\frac{6}{5}\sqrt{3} + \frac{26}{15}$

23. $1 - e^{-1} + \frac{1}{2}e^{-2} + \frac{1}{2}e^2 - e$

25. 1 **27.** 36 **29.** $\frac{37}{12}$

31. 12 **33.** $\frac{16}{3}$ **35.** 4

37. $\frac{1}{3}$ **39.** $\frac{343}{24}$ **43.** It is not.

Chapter 5 Review Exercises

1. left sum: $\frac{1}{6}\pi^2\sqrt{3}$; right sum: $\frac{1}{6}\pi^2\sqrt{3} - \frac{3}{4}\pi^2$; midpoint sum: $\frac{1}{3}\pi^2 - \frac{5}{16}\sqrt{2}\pi^2$

3. $\frac{5}{8}x^{8/5} - 3x^{8/3} + C$

5. $\frac{1}{4}x^4 - \frac{3}{2}x^2 + 2x - 2\ln|x| + C$

7. $(1 + \sqrt{x + 1})^2 + C$

9. $-\frac{1}{12}\cos^4 3t + C$

11. $\frac{4}{5}(1 + \sqrt{x})^{5/2} - \frac{4}{3}(1 + \sqrt{x})^{3/2} + C$

13. $-\frac{1}{2} - 2^{5/3}$ **15.** 20

17. $\frac{2}{5}\ln 2$ **19.** $\ln(1 + e)$

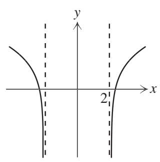
21. $2\ln 2 - \ln(2 - \sqrt{2})$ **23.** $\frac{195}{4}$

25. $68 - 6\sqrt{2}$

27. $3 - e + \ln\dfrac{(1 + e)^2}{16} \approx 0.1356528243$

29. $\frac{128}{3}$ **31.** $\frac{1}{2}$

33.



35. $x\sqrt{1 + x^5}$

37. $1/(x \ln x)$ **39.** 0

41. $\dfrac{(4 - \cos x)^{1/3}\sqrt{2x - 5}}{\sqrt[3]{x + 5}} \cdot \left(\dfrac{\sin x}{12 - 3\cos x} + \dfrac{1}{2x - 5} - \dfrac{1}{3x + 15}\right)$

43. b. $\frac{1}{3}(x^2 + 6)^{3/2} + C$

45. c. $\frac{1}{2}[\ln(x + 1)]^2 + C$

49. c. lower bound: 0; upper bound: $\frac{5}{6}$

53. a. $f(x) > 0$ for x in I
b. b such that $f(b) = 0$
c. f increasing on I

55. 242 feet **57.** 6050 R **59. b.** $e^{-(10^4)}$

61. approximately 28 joules per mole