PHYS-E0412 Computational Physics :: Homework 7

Due date 5.3.2019 at 10.00 am

Finite-difference discretizations for elliptic PDEs

In this exercise we study finite-difference discretization of the operator $-\frac{d^2}{dx^2}$ or $-\Delta$ with varying boundary conditions and spatial dimension. Such a discretization plays a central role when solving, e.g., the Poisson, heat, wave, or Schrödinger equation.

a) Dimension 1. Implement a solver for the one-dimensional problem

$$\begin{cases} -u''(x) = (x-0.5)^3 - 2(x-0.5), \quad 0 < x < 1 \\ u(0) = u(1) = 0 \end{cases}$$

using the three-point stencil

$$-u''(x_i) \approx \frac{1}{h^2}\left(-u_{i-1} + 2u_i - u_{i+1}\right)$$

Check that when varying the grid spacing h you converge to the exact solution at the rate h^2.

b) Change the boundary conditions to $u(0) = 0$, $u'(1) = 0$. Does the speed of convergence change? What happens when you use Neumann conditions at both endpoints $u'(0) = 0$, $u'(1) = 0$? (2 p for a+b)

c) Dimension 2. Implement a solver for the two-dimensional problem

$$\begin{cases} -\Delta u(x,y) = \exp\left(-\frac{(x-0.5)^2+(y-0.5)^2}{18}\right), \quad (x,y) \in [0,1] \times [0,1] \\ u(0,y) = u(1,y) = u(x,0) = u(x,1) = 0 \end{cases}$$

using the five-point stencil (see, e.g.,
`https://en.wikipedia.org/wiki/Discrete_Poisson_equation`)

$$-\Delta u(x_i, y_j) \approx \frac{1}{h^2}\left(4u_{i,j} - u_{i-1,j} - u_{i+1,j} - u_{i,j-1} - u_{i,j+1}\right)$$

What can you say about the convergence with respect to the grid specing h? (3 p)

d) How many hours did you spend working on this exercise?