Gearing and Motor Sizing

The mechanical power produced by a DC motor is a product of its torque and angular velocity at its output shaft. Even if a DC motor provides enough power for a given application, it may rotate at too high a speed (up to thousands of rpm), and too low a torque, to be useful. In this case, we can add a gearhead to the output shaft to decrease the speed by a factor of $G > 1$ and to increase the torque by a similar factor. In rare cases, we can choose $G < 1$ to actually increase the output speed.

In this chapter we discuss options for gearing the output of a motor, and how to choose a DC motor and gearing combination that works for your application.

26.1 Gearing

Gearing takes many forms, including different kinds of rotating meshing gears, belts and pulleys, chain drives, cable drives, and even methods for converting rotational motion to linear motion, such as racks and pinions, lead screws, and ball screws. All transform torques/forces and angular/linear velocities while ideally preserving mechanical power. For specificity, we refer to a gearhead with an input shaft (attached to the motor shaft) and an output shaft.

Figure 26.1 shows the basic idea. The input shaft is attached to an input gear A with N teeth, and the output shaft is attached to an output gear B with GN teeth, where typically $G > 1$. The meshing of these teeth enforces the relationship

$$\omega_{\text{out}} = \frac{1}{G}\omega_{\text{in}}.$$

Ideally the meshing gears preserve mechanical power, so $P_{\text{in}} = P_{\text{out}}$, which implies

$$\tau_{\text{in}}\omega_{\text{in}} = P_{\text{in}} = P_{\text{out}} = \frac{1}{G}\omega_{\text{in}}\tau_{\text{out}} \quad \rightarrow \quad \tau_{\text{out}} = G\tau_{\text{in}}.$$

It is common to have multiple stages of gearing (Figure 26.2(a)), so the output shaft described above has a second, smaller gear which becomes the input of the next stage. If the gear ratios of the two stages are G_1 and G_2, the total gear ratio is $G = G_1G_2$.

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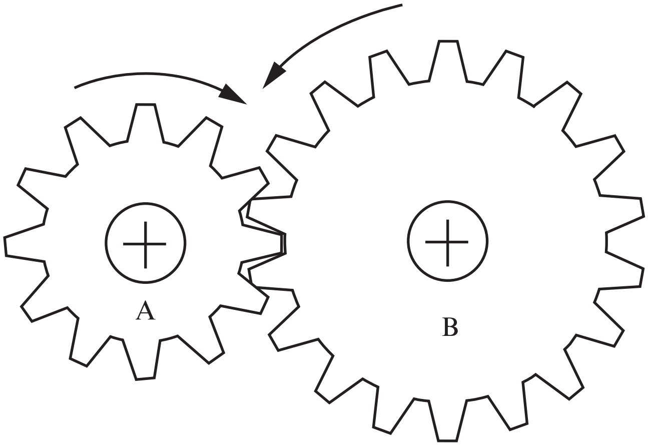

Figure 26.1
The input gear A has 12 teeth and the output gear B has 18, making the gear ratio $G = 1.5$.

Multi-stage gearheads can make huge reductions in speed and increases in torque, up to ratios of hundreds or more.

26.1.1 Practical Issues

Efficiency

In practice, some power is lost due to friction and impacts between the teeth. This power loss is often modeled by an efficiency coefficient $\eta < 1$, such that $P_{out} = \eta P_{in}$. Since the teeth enforce the ratio G between input and output velocities, the power loss must appear as a decrease in the available output torque, i.e.,

$$\omega_{out} = \frac{1}{G}\omega_{in} \quad \tau_{out} = \eta G \tau_{in}.$$

The total efficiency of a multi-stage gearhead is the product of the efficiencies of each stage individually, i.e., $\eta = \eta_1 \eta_2$ for a two-stage gearhead. As a result, high ratio gearheads may have relatively low efficiency.

Backlash

Backlash refers to the angle that the output shaft of a gearhead can rotate without the input shaft moving. Backlash arises due to tolerance in manufacturing; the gear teeth need some play to avoid jamming when they mesh. An inexpensive gearhead may have backlash of a degree or more, while more expensive precision gearheads have nearly zero backlash. Backlash typically increases with the number of gear stages. Some gear types, notably harmonic drive gears (see Section 26.1.2), are specifically designed for near-zero backlash, usually by using flexible elements.

Backlash can be a serious issue in controlling endpoint motions, due to the limited resolution of sensing the gearhead output shaft angle using an encoder attached to the motor shaft (the input of the gearhead).

Backdrivability

Backdrivability refers to the ability to drive the output shaft of a gearhead with an external device (or by hand), i.e., to backdrive the gearing. Typically the motor and gearhead combination is less backdrivable for higher gear ratios, due to the higher friction in the gearhead and the higher apparent inertia of the motor (see Section 26.2.2). Backdrivability also depends on the type of gearing. In some applications we do not want the motor and gearhead to be backdrivable (e.g., if we want the gearhead to act as a kind of brake that prevents motion when the motor is turned off), and in others backdrivability is highly desirable (e.g., in virtual environment haptics applications, where the motor is used to create forces on a user's hand).

Input and output limits

The input and output shafts and gears, and the bearings that support them, are subject to practical limits on how fast they can spin and how much torque they can support. Gearheads will often have maximum input velocity and maximum output torque specifications reflecting these limits. For example, you cannot assume that you get a 10 Nm actuator by adding a $G = 10$ gearhead to a 1 Nm motor; you must make sure that the gearhead is rated for 10 Nm output torque.

26.1.2 Examples

Figure 26.2 shows several different gear types. Not shown are cable, belt, and chain drives, which can also be used to implement a gear ratio while transmitting torques over distances.

Spur gearhead

Figure 26.2(a) shows a multi-stage *spur* gearhead. To keep the spur gearhead package compact, typically each stage has a gear ratio of only 2 or 3; larger gear ratios would require large gears.

Planetary gearhead

A planetary gearhead has an input rotating a *sun* gear and an output attached to a *planet carrier* (Figure 26.2(b)). The sun gear meshes with the planets, which also mesh with an internal gear. An advantage of the planetary gearhead is that more teeth mesh, allowing higher torques.

Bevel gears

Bevel gears (Figure 26.2(c)) can be used to implement a gear ratio as well as to change the axis of rotation.

Worm gears

The screw-like input *worm* interfaces with the output *worm gear* in Figure 26.2(d), making for a large gear ratio in a compact package.

Harmonic drive

The *harmonic drive* gearhead (Figure 26.2(e)) has an elliptical *wave generator* attached to the input shaft and a flexible *flexspline* attached to the output shaft. Ball bearings between the wave generator and the flexibility of the flexspline allow them to move smoothly relative to each other. The flexspline teeth engage with a rigid external *circular spline*. As the wave generator completes a full revolution, the teeth of the flexspline may have moved by as little as one tooth relative to the circular spline. Thus the harmonic drive can implement a high gear ratio (for example, $G = 50$ or 100) in a single stage with essentially zero backlash. Harmonic drives can be quite expensive.

Ball screw and lead screw

A ball screw or lead screw (Figure 26.2(f)) is aligned with the axis of, and coupled to, the motor's shaft. As the screw rotates, a nut on the screw translates along the screw. The nut is prevented from rotating (and therefore must translate) by linear guide rods passing through the nut. The holes in the nuts in Figure 26.2(f) are clearly visible. A lead screw and a ball screw are basically the same, but a ball screw has ball bearings in the nut to reduce friction with the screw.

Ball and lead screws convert rotational motion to linear motion. The ratio of the linear motion to the rotational motion is specified by the *lead* of the screw.

Rack and pinion

The rack and pinion (Figure 26.2(g)) is another way to convert angular to linear motion. The rack is typically mounted to a part on a linear slide.

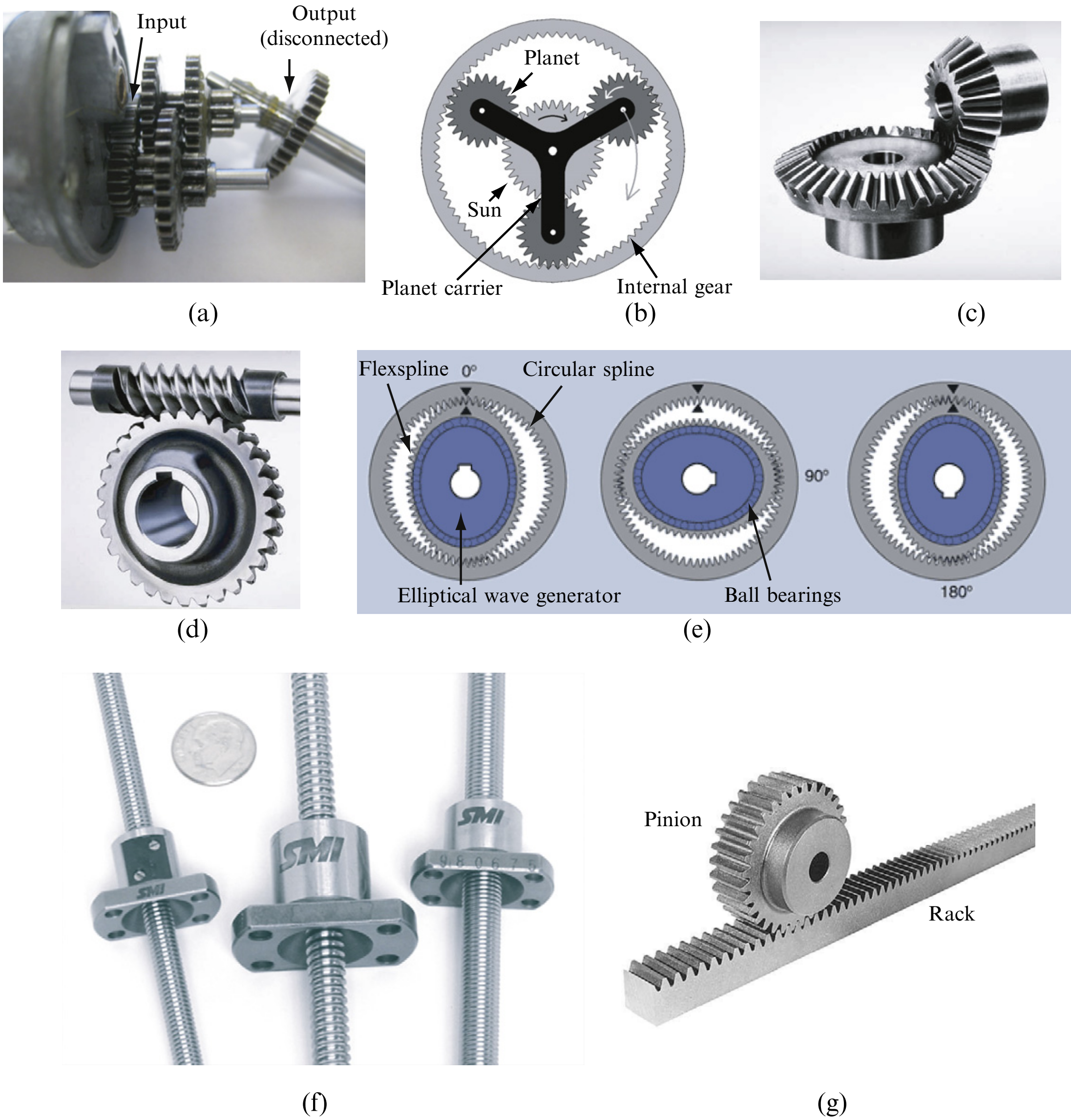

Figure 26.2

(a) Multi-stage spur gearhead. (b) A planetary gearhead. (c) Bevel gears. (d) Worm gears. (e) Harmonic drive gearhead. (f) Ball screws. (g) Rack and pinion.

26.2 Choosing a Motor and Gearhead

26.2.1 Speed-Torque Curve

Figure 26.3 illustrates the effect of a gearhead with $G = 2$ and efficiency $\eta = 0.75$ on the speed-torque curve. The continuous operating torque also increases by a factor ηG, or 1.5 in

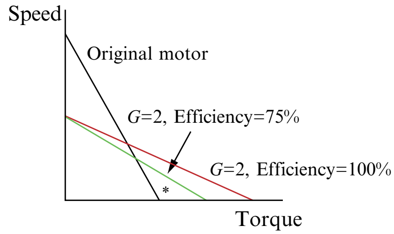

Figure 26.3
The effect of gearing on the speed-torque curve. The operating point * is possible with the gearhead, but not without.

this example. When choosing a motor and gearing combination, the expected operating points should lie under the geared speed-torque curve, and continuous operating points should have torques less than $\eta G \tau_c$, where τ_c is the continuous torque of the motor alone.

26.2.2 Inertia and Reflected Inertia

If you spin the shaft of a motor by hand, you can feel its rotor inertia directly. If you spin the output shaft of a gearhead attached to the motor, however, you feel the *reflected inertia* of the rotor through the gearbox. Say J_m is the inertia of the motor, ω_m is the angular velocity of the motor, and $\omega_{\text{out}} = \omega_m / G$ is the output velocity of the gearhead. Then we can write the kinetic energy of the motor as

$$KE = \frac{1}{2} J_m \omega_m^2 = \frac{1}{2} J_m G^2 \omega_{\text{out}}^2 = \frac{1}{2} J_{\text{ref}} \omega_{\text{out}}^2,$$

and $J_{\text{ref}} = G^2 J_m$ is called the reflected (or apparent) inertia of the motor. (We ignore the inertia of the gears.)

Commonly the gearbox output shaft is attached to a rigid-body load. For a rigid body consisting of point masses, the inertia J_{load} about the axis of rotation is calculated as

$$J_{\text{load}} = \sum_{i=1}^{N} m_i r_i^2,$$

where m_i is the mass of point i and r_i is its distance from the axis of rotation. In the case of a continuous body, the discrete sum becomes the integral

$$J_{\text{load}} = \int_V \rho(\mathbf{r}) r^2 \, \mathrm{d}V(\mathbf{r}),$$

where $\mathbf{r}$ refers to the location of a point on the body, r is the distance of that point to the rotation axis, $\rho(\mathbf{r})$ is the mass density at that point, V is the volume of the body, and $\mathrm{d}V$ is a differential volume element. Solutions to this equation are given in Figure 26.4 for some simple bodies of mass m and uniform density.

If the inertia of a body about its center of mass is J_{cm}, then the inertia J' about a parallel axis a distance r from the center of mass is

$$J' = J_{\mathrm{cm}} + mr^2. \tag{26.1}$$

Equation (26.1) is called the *parallel-axis theorem*. With the parallel-axis theorem and the formulas in Figure 26.4, we can approximately calculate the inertia of a load consisting of multiple bodies (Figure 26.5). Typically J_{load} is significantly larger than J_m, but with the gearing, the reflected inertia of the motor $G^2 J_m$ may be as large or larger than J_{load}.

Given a load of mass m and inertia J_{load} (about the gearhead axis) in gravity as shown in Figure 26.6, and a desired acceleration $\alpha > 0$ (counterclockwise), we can calculate the torque needed to achieve the acceleration (using Newton's second law):

$$\tau = (G^2 J_m + J_{\mathrm{load}})\alpha + mgr \sin\theta.$$

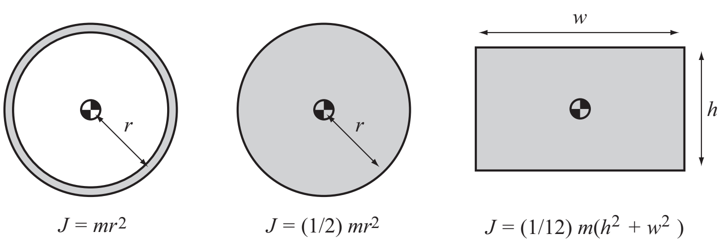

Figure 26.4
Inertia for an annulus, a solid disk, and a rectangle, each of mass m, about an axis out of the page and through the center of mass.

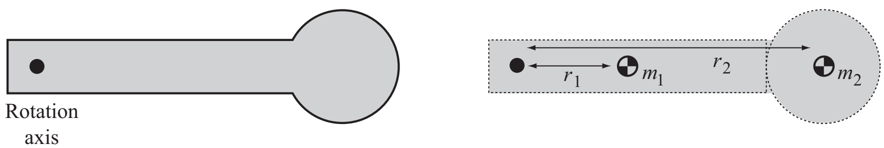

Figure 26.5
The body on the left can be approximated by the rectangle and disk on the right. If the inertias of the two bodies (about their centers of mass) are J_1 and J_2, then the approximate inertia of the compound body about the rotation axis is $J = J_1 + m_1 r_1^2 + J_2 + m_2 r_2^2$ by the parallel-axis theorem.

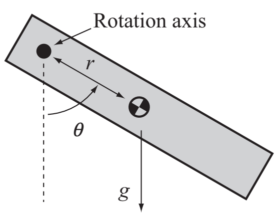

Figure 26.6
A load in gravity.

Given angular velocities at which we would like this acceleration to be possible, we have a set of speed-torque points that must lie under the speed-torque curve (transformed by the gearhead).

26.2.3 Choosing a Motor and Gearhead

To choose a motor and gearing combination, consider the following factors:

- The motor can be chosen based on the peak mechanical power required for the task. If the motor's power rating is sufficient, then theoretically we can follow by choosing a gearhead to give the necessary speed and torque. Our choice of motor might also be constrained by the voltage supply available for the application.
- The maximum velocity needed for the task should be less than ω_0/G, where ω_0 is the no-load speed of the motor.
- The maximum torque needed for the task should be less than $G\tau_{\text{stall}}$, where τ_{stall} is the motor's stall torque.
- Any required operating point (τ, ω) must lie below the gearing-transformed speed-torque curve.
- If the motor will be used continuously, then the torques during this continuous operation should be less than $G\tau_c$, where τ_c is the maximum continuous torque of the motor.

To account for the efficiency η of the gearhead and other uncertain factors, it is a good idea to oversize the motor by a safety factor of 1.5 or 2.

Subject to the hard constraints specified above, we might wish to find an "optimal" design, e.g., to minimize the cost of the motor and gearing, its weight, or the electrical power consumed by the motor. One type of optimization is called *inertia matching*.

Inertia matching

Given the motor inertia J_m and the load inertia J_{load}, the system is inertia matched if the gearing G is chosen so that the load acceleration α is maximized for any given motor torque τ_m. We can express the load acceleration as

$$\alpha = \frac{G\tau_m}{J_{\text{load}} + G^2 J_m}.$$

The derivative with respect to G is

$$\frac{\mathrm{d}\alpha}{\mathrm{d}G} = \frac{(J_{\text{load}} - G^2 J_m)\tau_m}{(J_{\text{load}} + G^2 J_m)^2}$$

and solving $\mathrm{d}\alpha/\mathrm{d}G = 0$ yields

$$G = \sqrt{\frac{J_{\text{load}}}{J_m}},$$

or $G^2 J_m = J_{\text{load}}$, hence the term “inertia matched.” With this choice of gearing, half of the torque goes to accelerating the motor’s inertia and half goes to accelerating the load inertia.

26.3 Chapter Summary

- For gearing with a gear ratio G, the output angular velocity is $\omega_{\text{out}} = \omega_{\text{in}}/G$ and the ideal output torque is $\tau_{\text{out}} = G\tau_{\text{in}}$, where ω_{in} and τ_{in} are the input angular velocity and torque, respectively. If the gear efficiency $\eta < 1$ is taken into account, the output torque is $\tau_{\text{out}} = \eta G\tau_{\text{in}}$.
- For a two-stage gearhead with gear ratios G_1 and G_2 and efficiencies η_1 and η_2 for the individual stages, the total gear ratio is $G_1 G_2$ and total efficiency is $\eta_1\eta_2$.
- Backlash refers to the amount the output of the gearing can move without motion of the input.
- The reflected inertia of the motor (the apparent inertia of the motor from the output of the gearhead) is $G^2 J_m$.
- A motor and gearing system is inertia matched with its load if

$$G = \sqrt{\frac{J_{\text{load}}}{J_m}}.$$

26.4 Exercises

1. You are designing gearheads using gears with 10, 15, and 20 teeth. When the 10- and 15-teeth gears mesh, you have $\eta = 85\%$. When the 15-and 20-teeth gear mesh, you have $\eta = 90\%$. When the 10- and 20-teeth gear mesh, you have $\eta = 80\%$.
   a. For a one-stage gearhead, what gear ratios $G > 1$ can you achieve, and what are their efficiencies?
   b. For a two-stage gearhead, what gear ratios $G > 1$ can you achieve, and what are their efficiencies? Consider all possible combinations of one-stage gearheads.
2. The inertia of the motor’s rotor is J_m, and its load is a uniform solid disk, which will be centered on the gearhead output shaft. The disk has a mass m and a radius R. For what gear ratio G is the system inertia matched?
3. The inertia of the motor’s rotor is J_m, and its load is a propeller with three blades. You model the propeller as a simple planar body consisting of a uniform-density solid disk of

radius R and mass M, with each blade a uniform-density solid rectangle extending from the disk. Each blade has mass m, length ℓ, and (small) width w.

a. What is the inertia of the propeller? (Since a propeller must push air to be effective, ideally our model of the propeller inertia would include the *added mass* of the air being pushed, but we leave that out here.)

b. What gear ratio G provides inertia matching?

4. You are working for a startup robotics company designing a small differential-drive mobile robot, and your job is to choose the motors and gearing. A diff-drive robot has two wheels, each driven directly by its own motor, as well as a caster wheel or two for balance. Your design specifications say that the robot should be capable of continuously climbing a 20° slope at 20 cm/s. To simplify the problem, assume that the mass of the whole robot, including motor amplifiers, motors, and gearing, will be 2 kg, regardless of the motors and gearing you choose. Further assume that the robot must overcome a viscous damping force of (10 Ns/m)$\times v$ when it moves forward at a constant velocity v, regardless of the slope. The radius of the wheels has already been chosen to be 4 cm, and you can assume they never slip. If you need to make other assumptions to complete the problem, clearly state them.

You will choose among the 15 V motors in Table 25.1, as well as gearheads with $G = 1$, 10, 20, 50, or 100. Assume the gearing efficiency η for $G = 1$ is 100%, and for the others, 75%. (Do not combine gearheads! You get to use only one.)

a. Provide a list of all combinations of motor and gearhead that satisfy the specifications, and explain your reasoning. (There are 20 possible combinations: four motors and five gearheads.) "Satisfy the specifications" means that the motor and gearhead can provide at least what is required by the specifications. Remember that each motor only needs to provide half of the total force needed, since there are two wheels.

b. To optimize your design, you decide to use the motor with the lowest power rating, since it is the least expensive. You also decide to use the lowest gear ratio that works with this motor. (Even though we are not modeling it, a lower gear ratio likely means higher efficiency, less backlash, less mass in a smaller package, a higher top-end speed (though lower top-end torque), and lower cost.) Which motor and gearing do you choose?

c. Instead of optimizing the cost, you decide to optimize the power efficiency—the motor and gearing combination that uses the least electrical power when climbing up the 20° slope at a constant 20 cm/s. This is in recognition that battery life is very important to your customers. Which motor and gearhead do you choose?

d. Forget about your previous answers, satisfying the specifications, or the limited set of gear ratios. If the motor you choose has rotor inertia J_m, half of the mass of the robot (including the motors and gearheads) is M, and the mass of the wheels is negligible, what gear ratio would you choose to achieve inertia matching? If you need to make other assumptions to complete the problem, clearly state them.

Further Reading

Budynas, R., & Nisbett, K. (2014). *Shigley's mechanical engineering design.* New York: McGraw-Hill.