5 Waves and free surface flows

Free surface flows are in some ways very different from what we have done before. In all problems considered so far, the domain $\mathcal{D}$ in which to solve the problem is given (for example some box or the exterior of an aeroplane wing). A free surface, on the other hand, moves, so the domain varies in time. The key feature, however, is that the domain does not vary according to some predetermined program. Instead, it moves in response to the flow itself, that is in response to the flow solution (which of course itself depends on the domain).

This leads to solutions which are quite different in character to the fixed-boundary solutions we were considering so far (and generally speaking much more interesting solutions)! Another nice feature of free surface flows is that they are easy to observe experimentally, one just needs to track the motion of the free surface.

5.1 Kinematic boundary condition

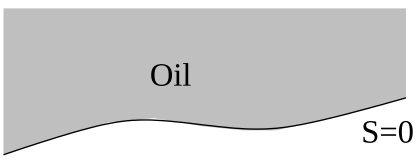
If a fluid particle is adjacent to a boundary then we must impose a condition which links the velocity of the boundary to that of the particle. This is known as the kinematic boundary condition.

Let $S(\mathbf{x}, t) = 0$ describe the equation of a surface in (or on the boundary of) the fluid. As the flow evolves, particles remain on the surface S if

$$\frac{DS}{Dt} = \frac{\partial S}{\partial t} + \mathbf{u} \cdot \nabla S = 0. \tag{52}$$

The proof is analogous to the arguments of (1.7). Namely, according to Taylor's theorem:

$$0 = S(\mathbf{x}, t) - S(\mathbf{x} + \mathbf{u}\delta t, t + \delta t) = \delta t \left(\mathbf{u} \cdot \nabla S + \frac{\partial S}{\partial t} \right).$$



Example: Consider two fluids bounded by an interface $S(\mathbf{x}, t) = 0$ (e.g. water/oil). Then

$$\left.\begin{aligned} \frac{DS}{Dt} &= \frac{\partial S}{\partial t} + \mathbf{u}^{(oil)} \cdot \nabla S = 0, \qquad \text{on } S = 0 \text{ from above} \\ \frac{DS}{Dt} &= \frac{\partial S}{\partial t} + \mathbf{u}^{(water)} \cdot \nabla S = 0, \qquad \text{on } S = 0 \text{ from below} \end{aligned}\right\}$$

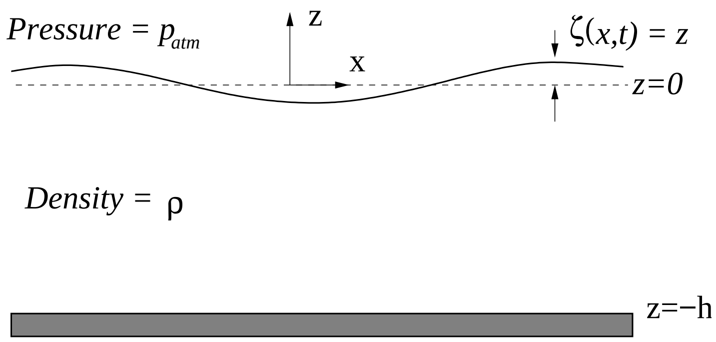
Now ∇S is normal to the surface $S = const$, so $\mathbf{n} = \nabla S / |\nabla S|$ is the unit normal to the surface $S = 0$. By subtracting one of the two equations from the other, it follows that

$$\mathbf{u}^{(oil)} \cdot \mathbf{n} = \mathbf{u}^{(water)} \cdot \mathbf{n}, \qquad \text{on } S = 0. \tag{53}$$

This means the normal component of the velocity on either side of the interface must be equal, which is very intuitive.

5.2 Nonlinear free-surface motion

We begin with the general equations for free-surface flow, assuming that the flow inside the fluid is potential, and the fluid is incompressible. We assume for now that the flow is 2D.



Hence the fluid flow is described by

$$\mathbf{u} = \nabla\phi, \quad \nabla^2\phi = 0,$$

where $\mathbf{u} = (u, 0, w)$ and $\phi = \phi(x, z, t)$.
We choose $z = 0$ to coincide with the undisturbed free surface, and the bottom of the fluid is at $z = -h$.
Let the surface of the water in motion be given by $z = \zeta(x, t)$.

The interesting part of the problem are its boundary conditions:

(i) At the lower boundary $z = -h$, the vertical velocity must vanish (kineamtic b.c.):

$$0 = \mathbf{n} \cdot \nabla\phi = \frac{\partial\phi}{\partial z} = w.$$

In other words,

$$\frac{\partial\phi}{\partial z} = 0 \qquad \text{on } z = -h. \tag{54}$$

(ii) On $z = \zeta(x, t)$, the kinematic boundary condition on a moving surface is $\dfrac{DS}{Dt} = 0$, where the free surface is the zero set of $S(\mathbf{x}, t)$. If the surface can be represented by a height function $\zeta(x, t)$ (i.e. we are not allowed overhangs) we can define a function

$$S(x, z, t) = z - \zeta(x, t)$$

which is indeed zero at the free surface. Thus

$$0 = \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z}\right)(z - \zeta(x, t)) = -\frac{\partial\zeta}{\partial t} - \frac{\partial\phi}{\partial x}\frac{\partial\zeta}{\partial x} + \frac{\partial\phi}{\partial z} \qquad \text{on } z = \zeta,$$

so the kinematic boundary condition becomes

$$\frac{\partial\zeta}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial\zeta}{\partial x} = \frac{\partial\phi}{\partial z}, \qquad \text{on } z = \zeta. \tag{55}$$

(iii) The dynamic boundary condition at the free surface is that the pressure equals the exterior athmospheric pressure: $p = p_{atm}$ (const). on $z = \zeta$. The unsteady Bernoulli equation (28) for a constant force of gravity $\Phi = gz$ is

$$p/\rho + \tfrac{1}{2}|\nabla\phi|^2 + \frac{\partial \phi}{\partial t} + gz = C(t),$$

and so the third boundary condition becomes

$$p_{atm}/\rho + \tfrac{1}{2}|\nabla\phi|^2 + \frac{\partial \phi}{\partial t} + g\zeta = C(t), \qquad \text{on } z = \zeta. \tag{56}$$

This concludes our description of the fully nonlinear problem.

5.3 Linear gravity waves

By the small amplitude assumption, $|\zeta| \ll h$ and so $\left|\frac{\partial \zeta}{\partial x}\right| \ll 1$. If we assume that the flow is driven by the motion of the free surface (no additional driving from below the surface), it must be that the flow is also week: $|\phi| \ll 1$. Thus we will throw away all terms *quadratic* in the two variables ζ and ϕ, i.e. containing ζ^2, ϕ^2, or products $\zeta\phi$. Note that (56) contains the velocity at quadratic (nonlinear) order, which is a remnant of the nonlinear character of the Euler equation, as it remains in Bernoulli's equation. The kinematic boundary condition (55) contains $\frac{\partial \phi}{\partial x}\frac{\partial \zeta}{\partial x}$, which is quadratic as well. Both terms will be neglected in our linearised treatment.

But represents only one type of nonlinearity. The most significant problem is that in the formulation of the boundary conditions, the position of the free surface is part of the solution, so don't know where to apply the boundary conditions ! We can deal with this difficulty by linearising about $z = 0$ using Taylor's expansion about $z = 0$ for all z-dependent terms, for example

$$\left.\frac{\partial \phi}{\partial t}\right|_{z=\zeta} = \left.\frac{\partial \phi}{\partial t}\right|_{z=0} + \zeta \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial t}\right)_{z=0} + \ldots.$$

The second term on the right is quadratic, and will be neglected. Thus we observe that linearizing the boundary conditions implies that instead of at $z = \zeta(x,t)$, the boundary condition is imposed at the equilibrium position $z = 0$. Thus we obtain

(i) On $z = -h$, $\dfrac{\partial \phi}{\partial z} = 0$

(ii) On $z = 0$, $\dfrac{\partial \zeta}{\partial t} = \dfrac{\partial \phi}{\partial z}$

(iii) On $z = 0$, $\dfrac{\partial \phi}{\partial t} + g\zeta = C(t) - p_{atm}/\rho$.

The last condition can be prettyfied by redefining the potential according to

$$\phi = \phi' - p_{atm}t/\rho + \int^t C(t')dt'.$$

The extra terms are time dependent only, so they do not affect the flow velocities, Laplace's equation or the boundary conditions. So now (dropping primes) (iii) is replaced with

(iii) On $z = 0$, $\dfrac{\partial \phi}{\partial t} + g\zeta = 0$.

The linearised pressure *in* the fluid is

$$\frac{p(x,z,t)}{\rho} + \frac{\partial \phi}{\partial t} + gz = C(t)$$

and because of $\phi \to \phi'$ this transforms to

$$\frac{p(x,z,t) - p_{atm}}{\rho} = -\frac{\partial \phi}{\partial t} - gz. \tag{57}$$

Now we look for particular solutions to $\nabla^2 \phi = 0$ with (i), (ii), (iii) motivated by observations, which describe a propagating wave.

Such a propagating wave of amplitude H is described by a solution of the form

$$\zeta(x,t) = H \sin(kx - \omega t + \delta). \tag{58}$$

The phase factor δ is inconsequential and will not be considered in the future. The solution (58) is periodic both in time (with period $\tau = 2\pi/\omega$) and space (with wavelength $\lambda = 2\pi/k$). Most importantly, it moves to the *right* with speed $c = \omega/k$, where c is called the phase speed. To see that, let $\xi = x - ct$, so that

$$\zeta = H \sin(k\xi + \delta)$$

is unchanged for ξ constant. But this means that

$$0 = \frac{d\xi}{dt} = \frac{dx}{dt} - c,$$

so c is the speed at which the point x is moving.

Now we need the potential ϕ, which describes the velocity field. Since $\dfrac{\partial \phi}{\partial t} = -g\zeta$ reasonable to write

$$\phi(x,z,t) = \cos(kx - \omega t) Z(z)$$

for some $Z(z)$ – this is a separation solution $\phi = X(x)Z(z)$. Then from $\nabla^2 \phi = 0$,

$$-k^2 \cos(kx - \omega t + \delta) Z(z) + \cos(kx - \omega t + \delta) Z''(z) = 0$$

and so $Z''(z) - k^2 Z(z) = 0$. Then

$$Z(z) = A \cosh k(z+h) + B \sinh k(z+h),$$

for constants A, B. Because of (i), need $B = 0$, so

$$\phi(x,z,t) = A \cos(kx - \omega t) \cosh k(z+h).$$

We still have (ii) and (iii) to apply, but only A to find. The second condition will lead to an equation for ω. From (iii) first, $g\zeta = -\partial\phi/\partial t|_{z=0}$

$$gH \sin(kx - \omega t + \delta) = -A\omega \sin(kx - \omega t + \delta) \cosh kh$$

and then $A = -gH/(\omega \cosh kh)$. Then from (ii), $\partial\phi/\partial z|_{z=0} = \partial\zeta/\partial t$

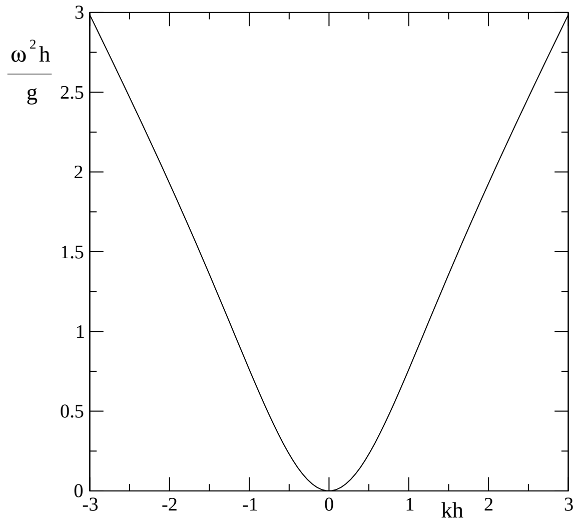
$$-\omega H \cos(kx - \omega t + \delta) = kA\cos(kx - \omega t + \delta)\sinh kh$$

implies

$$-\omega H = k\left(\frac{-gH}{\omega \cosh kh}\right)\sinh kh$$

or

$$\frac{\omega^2 h}{g} = kh \tanh kh \tag{59}$$



Notes:

- Changing $k \to -k$, (59) still holds. So then

$$\zeta(x,t) = H\sin(kx + \omega t - \delta)$$

  is a wave travelling to the left, with speed $c = -\omega/k$.

- $k = 2\pi/\lambda$ is called the wavenumber (the number of wavelengths that can be fit into 2π).

The principal result of this calculation is the so called *dispersion relation* (59) of the wave problem. It establishes how the frequency of a monochromatic wave is related to its wavelength. Very roughly, large $\lambda \Rightarrow$ small $k \Rightarrow$ small $\omega \Rightarrow$ large $\tau = 2\pi/\omega$. So long wavelengths imply long periods, and vice versa.

If the dispersion relation between ω and k is non-linear, one speaks of a dispersive problem: parts of a wave containing different frequencies travel at different speeds, and thus *disperse*. If the relation is linear (as for exlectromagnetic waves), the phase speed is a constant.

Two important special cases of (59) exist:

5.3.1 Shallow depth

This is the case $kh \ll 1$ (or $\lambda/h \gg 1$), which means that the wave length is long compared to the depth. Then $\tanh kh \approx kh$ or

$$\omega = \sqrt{gh}k,$$

so the dispersion relation is a linear function. The constant of proportionality is the (constant) wave speed $c = \omega/k = \sqrt{gh}$. Evidently, the shallow water problem has *no dispersion.*

5.3.2 Infinite depth

If on the other hand $kh \gg 1$ (or $\lambda/h \ll 1$), the water is deep relative to the length of the wave. Then $\tanh kh \approx 1$ and the dispersion relation is

$$\omega = \sqrt{gk}.$$

Thus the wave speed $c = \omega/k = \sqrt{g/k} = \sqrt{g\lambda/2\pi}$ does depend on the wavelength, and there is dispersion. The velocity potential corresponding to the wave form (58) is

$$\phi = -\frac{\omega H}{k} e^{kz} \cos(kx - \omega t + \delta). \tag{60}$$

Neglecting the irrelevant phase factor δ, the velocity components are

$$u = \omega H e^{kz} \sin(kx - \omega t), \quad w = -\omega H e^{kz} \cos(kx - \omega t).$$

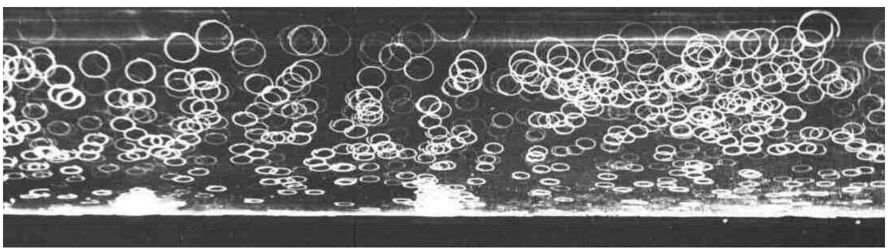
Assuming that a particle with position (x', z') in the flow only departs by a small amount from its mean position $(\overline{x}, \overline{z})$, one can write

$$\frac{dx'}{dt} = \omega H e^{k\overline{z}} \sin(k\overline{x} - \omega t), \quad \frac{dz'}{dt} = -\omega H e^{k\overline{z}} \cos(k\overline{x} - \omega t).$$

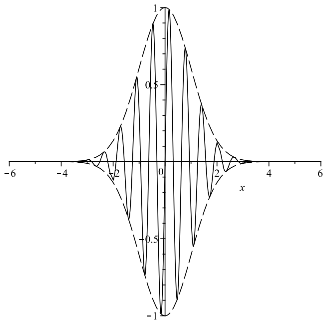
This is integrated easily to yield the particle trajectory

$$x' = H e^{k\overline{z}} \cos(k\overline{x} - \omega t), \quad z' = H e^{k\overline{z}} \sin(k\overline{x} - \omega t).$$

Thus a particle moves along a circular path, whose radius decreases exponentially with depth. Most of the motion is concentrated near the surface, and the wave energy decreases exponentially as one descends into the water, as seen in the Figure. Near the bottom, the effect of the wall is fealt, and the trajectories deform into ellipses.



5.4 Group velocity



Now we investigate the propagation of so-called wave-packet, that is a burst of finite spatial extend, as it might be produced by throwing a stone into the water. We write this situation as a superposition of infinitely extended plane waves:

$$\zeta = \int_{-\infty}^{\infty} a(k) \mathrm{e}^{i(kx-\omega t)} dk.$$

The amplitude of each part of the wave is given by $a(k)$. It is easy to show (a version of the uncertainty relation), that the more peaked the distribution $a(k)$ is around some mean value k_0, the more oscillations the wave packet contains, and thus the broader it becomes.

Let us assume that $a(k)$ is sufficiently narrow relative to k_0. In other words, it is safe to write

$$\omega(k) = \omega(k_0) + \left.\frac{d\omega}{dk}\right|_{k=k_0} (k - k_0).$$

But this means the wave packet can be written

$$\zeta(x,t) = \mathrm{e}^{i(k_0 x - \omega(k_0)t)} \int_{-\infty}^{\infty} a(k) \mathrm{e}^{i(k-k_0)(x-c_g t)} dk,$$

where

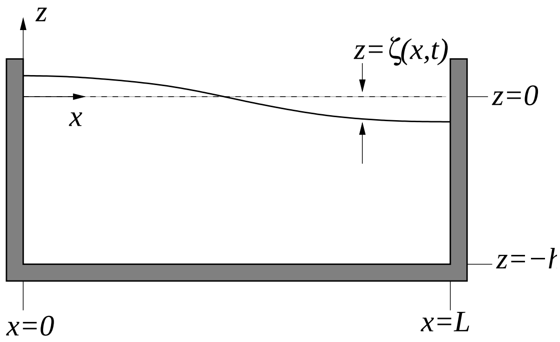
$$c_g \equiv \left.\frac{d\omega}{dk}\right|_{k=k_0} \tag{61}$$

is called the group velocity. The first factor in the representation of the wave packet represents a harmonic modulation with a high-frequency wave. The second factor is envelope, giving the shape of the wave packet. The key observation is that x only appears in the combination $x - c_g t$, and so the entire wave packet moves with the group velocity c_g. This velocity is usually more significant than the phase velocity $c = \omega/k$. For example, the energy of a wave is concentrated where the wave packet is localised. Therefore, energy transported by a wave moves at speed c_g.

Taking waves of infinite depth as an example, the phase velocity is $c = \sqrt{g/k}$, while $c_g = \sqrt{g/k}/2$, or half the value!

5.5 Oscillations in a container

Liquids readily slosh back and forth in a closed container (e.g. tea in a tea-cup). There are many associated important practical problems: sloshing in road tankers, water on decks of ships, resonance in harbours, etc...



Example: consider a two-dimensional rectangular box with rigid walls at $x = 0, L$ and a bottom on $z = -h$, filled with fluid to $z = 0$.

Use small-amplitude theory from before, so that linearised equations are $\nabla^2 \phi = 0$ for (x, y, z) in box,

(i) $\dfrac{\partial \phi}{\partial z} = 0$ on $z = -h$.

(ii) Kinematic: $\dfrac{\partial \zeta}{\partial t} = \dfrac{\partial \phi}{\partial z}$ on $z = 0$.

(iii) Dynamic: $\dfrac{\partial \phi}{\partial t} = -g\zeta$ on $z = 0$.

Also need no-flow conditions (kinematic) on walls:

(iv) $\dfrac{\partial \phi}{\partial x} = 0$ on $x = 0, L$;

However now we no longer have travelling waves, but are interested in describing a phenomenon that is periodic in time, so we write:

$$\zeta(x, y, t) = \zeta(x, y) \sin \omega t$$

and assume we can separate variables in ϕ:

$$\phi(x, y, z, t) = X(x)Z(z) \cos \omega t.$$

Then Laplace's equation gives

$$X''(x)Z(z) \cos \omega t + Z''(z)X(x) \cos \omega t = 0$$

which implies

$$X''(x)Z(z) + Z''(z)X(x) = 0$$

This separates:

$$\frac{X''(x)}{X(x)} = -\frac{Z''(z)}{Z(z)} = -k^2$$

where $-k^2$ is the separation constant.

Solving for $Z(z)$, with (i) gives $Z(z) = A\cosh k(z+h)$ as before. Combining (ii) and (iii) to eliminate ζ (that's g(ii) $+\partial/\partial t$(iii) gives

$$\frac{\partial^2\phi}{\partial t^2} = -g\frac{\partial \phi}{\partial z}, \qquad \text{on } z = 0$$

which implies

$$-\omega^2 X(x)Z(0)\cos\omega t = -gX(x)Z'(0)\cos\omega t$$

and it follows that $\omega^2/g = k\tanh kh$ as before. (We expect this, as the vertical structure of the fluid is independent of the vertical lateral walls)

Now solving for $X(x)$ gives

$$X(x) = B\cos(kx) + C\sin(kx)$$

subject to (from (iv)), $X'(0) = X'(L) = 0$. Easy to show that must have $C = 0$ and $k = n\pi/L$ and so

$$X(x) = B\cos(n\pi x/L)$$

So pulling everything together we have

$$\phi(x,z,t) = A'\cos(n\pi x/L)\cosh k(z+h)\cos\omega t$$

for some constant $A' = AB$ while from (iii) the free surface is given by

$$\zeta(x,t) = \frac{\omega A'\cosh kh}{g}\cos(n\pi x/L)\sin\omega t$$

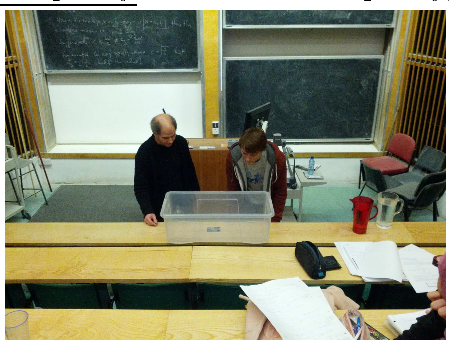
and here, $k = n\pi/L$, so that (59) reads

$$\omega^2/g = (n\pi/L)\tanh(n\pi h/L) \equiv \omega_n^2/g, \qquad n = 0, 1, 2, \ldots$$

and therefore defines a set of discrete wave frequencies at which these oscillations may occur. Here, n is a mode number and tells you how many oscillations are occurring across the box. Crucially, by considering waves in a finite box, we have selceted a *discrete spectrum* of allowed frequencies.

We cannot have $n = 0$ as then $\omega = 0$ and $\phi = 1$, and $\zeta = 0$. So there is no motion in the fluid. (In fact, you can discount a sloshing mode with no x-dependence – a flat surface oscillating up and down – as it would violate mass conservation in the tank).

The Fundamental frequency is the lowest frequency, given here by $n = 1$.



T_1 [s]	h [cm]	$\frac{\omega_1^2 L}{g}$	$\pi \tanh \frac{\pi h}{L}$
4.15	0.7	0.17	0.093
2.7	1.4	0.41	0.19
2.1	2.3	0.68	0.31

The Table gives experimental data for measurements in a rectangular tank (see picture), whose length was $L = 74$ cm. One end of the tank was lifted, to excite the fundamental sloshing mode. The third and fourth columns compare experiment and theory, as

$$\frac{\omega_1^2 L}{g} = \pi \tanh \frac{\pi h}{L}.$$

The trend is reproduced correctly, but there is a consistent discrepancy by a factor of two, perhaps due to the fact that the tank is too long, so we didn't really excite the findamental mode...

The fundamental and higher order modes are illustrated in the Figure below. The higher n, the higher the frequency. As $n \to \infty$, $\tanh(n\pi h/L) \to 1$ and so $\omega_n \to \sqrt{gn\pi/L}$.