The Laplace Transform

Definition and properties of Laplace Transform, piecewise continuous functions, the Laplace Transform method of solving initial value problems

The method of Laplace transforms is a system that relies on algebra (rather than calculus-based methods) to solve linear differential equations. While it might seem to be a somewhat cumbersome method at times, it is a very powerful tool that enables us to readily deal with linear differential equations with discontinuous forcing functions.

Definition: Let $f(t)$ be defined for $t \geq 0$. The Laplace transform of $f(t)$, denoted by $F(s)$ or $\mathcal{L}\{f(t)\}$, is an *integral transform* given by the *Laplace integral*:

$$\boxed{\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt\,.}$$

Provided that this (improper) integral exists, i.e. that the integral is convergent.

The Laplace transform is an operation that transforms a function of t (i.e., a function of *time domain*), defined on $[0, \infty)$, to a function of s (i.e., of *frequency domain*)[*]. $F(s)$ is the *Laplace transform*, or simply *transform*, of $f(t)$. Together the two functions $f(t)$ and $F(s)$ are called a *Laplace transform pair*.

For functions of t continuous on $[0, \infty)$, the above transformation to the frequency domain is *one-to-one*. That is, different continuous functions will have different transforms.

[*] The *kernel* of the Laplace transform, e^{-st} in the integrand, is unit-less. Therefore, the unit of s is the reciprocal of that of t. Hence s is a variable denoting (complex) frequency.

Example: Let $f(t) = 1$, then $F(s) = \dfrac{1}{s}$, $\qquad s > 0.$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-st}\,dt = \frac{-1}{s} e^{-st}\Big|_0^\infty$$

The integral is divergent whenever $s \le 0$. However, when $s > 0$, it converges to

$$\frac{-1}{s}\left(0 - e^0\right) = \frac{-1}{s}(-1) = \frac{1}{s} = F(s).$$

Example: Let $f(t) = t$, then $F(s) = \dfrac{1}{s^2}$, $\qquad s > 0.$

[This is left to you as an exercise.]

Example: Let $f(t) = e^{at}$, then $F(s) = \dfrac{1}{s-a}$, $\qquad s > a.$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{(a-s)t}\,dt = \frac{1}{a-s} e^{(a-s)t}\Big|_0^\infty$$

The integral is divergent whenever $s \le a$. However, when $s > a$, it converges to

$$\frac{1}{a-s}\left(0 - e^0\right) = \frac{1}{a-s}(-1) = \frac{1}{s-a} = F(s).$$

Definition: A function $f(t)$ is called *piecewise continuous* if it only has finitely many (or none whatsoever – a continuous function is considered to be "piecewise continuous"!) discontinuities on any interval $[a, b]$, and that both one-sided limits exist as t approaches each of those discontinuity from <u>within</u> the interval. The last part of the definition means that f could have removable and/or jump discontinuities only; it cannot have any infinity discontinuity.

Theorem: Suppose that

1. f is piecewise continuous on the interval $0 \le t \le A$ for any $A > 0$.

2. $|f(t)| \le K e^{at}$ when $t \ge M$, for any real constant a, and some positive constants K and M. (This means that f is "of *exponential order*", i.e. its rate of growth is no faster than that of exponential functions.)

Then the Laplace transform, $F(s) = \mathcal{L}\{f(t)\}$, exists for $s > a$.

Note: The above theorem gives a <u>sufficient</u> condition for the existence of Laplace transforms. It is <u>not</u> a <u>necessary</u> condition. A function does not need to satisfy the two conditions in order to have a Laplace transform. Examples of such functions that nevertheless have Laplace transforms are logarithmic functions and the unit impulse function.

Some properties of the Laplace Transform

1. $\mathcal{L}\{0\} = 0$

2. $\mathcal{L}\{f(t) \pm g(t)\} = \mathcal{L}\{f(t)\} \pm \mathcal{L}\{g(t)\}$

3. $\mathcal{L}\{cf(t)\} = c\,\mathcal{L}\{f(t)\}$, for any constant c.

Properties 2 and 3 together means that the Laplace transform is *linear*.

4. **[The derivative of Laplace transforms]**

$$\mathcal{L}\{(-t)f(t)\} = F'(s) \qquad \text{or, equivalently} \qquad \mathcal{L}\{t\,f(t)\} = -F'(s)$$

Example: $\mathcal{L}\{t^2\} = -\left(\mathcal{L}\{t\}\right)' = -\dfrac{d}{ds}\dfrac{1}{s^2} = -\dfrac{-2}{s^3} = \dfrac{2}{s^3}$

In general, the derivatives of Laplace transforms satisfy

$$\mathcal{L}\{(-t)^n f(t)\} = F^{(n)}(s) \quad \text{or, equivalently} \quad \mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$$

Warning: The Laplace transform, while a linear operation, is **not** *multiplicative*. That is, in general

$$\mathcal{L}\{f(t)\,g(t)\} \neq \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\}.$$

Exercise: (a) Use property 4 above, and the fact that $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$, to deduce that $\mathcal{L}\{te^{at}\} = \dfrac{1}{(s-a)^2}$. (b) What will $\mathcal{L}\{t^2 e^{at}\}$ be?

Exercises C-1.1:

1 – 5 Use the (integral transformation) definition of the Laplace transform to find the Laplace transform of each function below.

1. t^2

2. te^{6t}

3. $\cos 3t$

4. $e^{-t}\sin 2t$

5.* $e^{i\alpha t}$, where i and α are constants, $i = \sqrt{-1}$.

6 – 8 Each function $F(s)$ below is defined by a definite integral. Without integrating, find an explicit expression for each $F(s)$.
[Hint: each expression is the Laplace transform of a certain function. Use your knowledge of Laplace Transformation, or with the help of a table of common Laplace transforms to find the answer.]

6. $\int_0^\infty e^{-(s+7)t}\,dt$

7. $\int_0^\infty t^2 e^{-(s-3)t}\,dt$

8. $\int_0^\infty 4e^{-st}\sin 6t\,dt$

Answers C-1.1:

1. $\dfrac{2}{s^3}$

2. $\dfrac{1}{(s-6)^2}$

3. $\dfrac{s}{s^2+9}$

4. $\dfrac{2}{s^2+2s+5}$

5. $\dfrac{s}{s^2+\alpha^2} + i\dfrac{\alpha}{s^2+\alpha^2}$

Note: Since the Euler's formula says that $e^{i\alpha t} = \cos \alpha t + i\sin \alpha t$, therefore, $\mathcal{L}\{e^{i\alpha t}\} = \mathcal{L}\{\cos \alpha t + i\sin \alpha t\}$. That is, the real part of its Laplace transform corresponds to that of $\cos \alpha t$, the imaginary part corresponds to that of $\sin \alpha t$. (Check it for yourself!)

6. $\dfrac{1}{s+7}$

7. $\dfrac{2}{(s-3)^3}$

8. $\dfrac{24}{s^2+36}$

Solution of Initial Value Problems

We now shall meet "the new System": how the Laplace transforms can be used to solve linear differential equations algebraically.

Theorem: [**Laplace transform of derivatives**] Suppose f is of exponential order, and that f is continuous and f' is piecewise continuous on any interval $0 \le t \le A$. Then

$$\boxed{\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)}$$

Applying the theorem multiple times yields:

$$\mathcal{L}\{f''(t)\} = s^2\,\mathcal{L}\{f(t)\} - sf(0) - f'(0),$$

$$\mathcal{L}\{f'''(t)\} = s^3\,\mathcal{L}\{f(t)\} - s^2 f(0) - sf'(0) - f''(0),$$

$$\vdots$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\,\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots$$
$$- s^2 f^{(n-3)}(0) - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

This is an extremely useful aspect of the Laplace transform: that it changes differentiation with respect to t into multiplication by s (and, as seen a little earlier, differentiation with respect to s into multiplication by $-t$, on the other hand). Equally importantly, it says that the Laplace transform, when applied to a differential equation, would change derivatives into algebraic expressions in terms of s and (the transform of) the dependent variable itself. Thus, <u>it can transform a differential equation into an algebraic equation</u>.

We are now ready to see how the Laplace transform can be used to solve differentiation equations.

Solving initial value problems using the method of Laplace transforms

To solve a linear differential equation using Laplace transforms, there are only 3 basic steps:

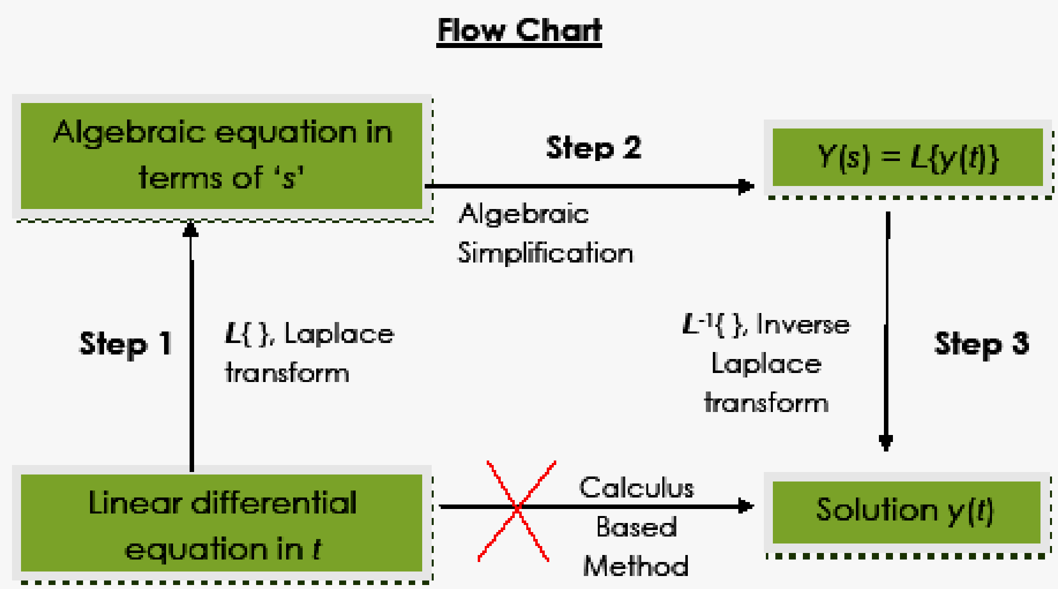
1. Take the Laplace transforms of both sides of an equation.

2. Simplify algebraically the result to solve for $\mathcal{L}\{y\} = Y(s)$ in terms of s.

3. Find the inverse transform of $Y(s)$. (Or, rather, find a function $y(t)$ whose Laplace transform matches the expression of $Y(s)$.) This inverse transform, $y(t)$, is the solution of the given differential equation.

The nice thing is that the same 3-step procedure works whether or not the differential equation is homogeneous or nonhomogeneous. The first two steps in the procedure are rather mechanical. The last step is the heart of the process, and it will take some practice. Let's get started.

The Laplace Transform Method of Solving an Initial Value Problem



Example: $y'' - 6y' + 5y = 0, \qquad y(0) = 1, \;\; y'(0) = -3$

[Step 1] Transform both sides

$$\mathcal{L}\{y'' - 6y' + 5y\} = \mathcal{L}\{0\}$$

$$(s^2\mathcal{L}\{y\} - s\,y(0) - y'(0)) - 6(s\mathcal{L}\{y\} - y(0)) + 5\mathcal{L}\{y\} = 0$$

[Step 2] Simplify to find $Y(s) = \mathcal{L}\{y\}$

$$(s^2\mathcal{L}\{y\} - s - (-3)) - 6(s\,\mathcal{L}\{y\} - 1) + 5\mathcal{L}\{y\} = 0$$

$$(s^2 - 6\,s\; + 5)\,\mathcal{L}\{y\} - s + 9 = 0$$

$$(s^2 - 6\,s\; + 5)\,\mathcal{L}\{y\} = s - 9$$

$$\mathcal{L}\{y\} = \frac{s-9}{s^2 - 6s + 5}$$

[Step 3] Find the inverse transform $y(t)$

Use partial fractions to simplify,

$$\mathcal{L}\{y\} = \frac{s-9}{s^2 - 6s + 5} = \frac{a}{s-1} + \frac{b}{s-5}$$

$$\frac{s-9}{s^2 - 6s + 5} = \frac{a(s-5)}{(s-1)(s-5)} + \frac{b(s-1)}{(s-5)(s-1)}$$

$$s - 9 = a(s-5) + b(s-1) = (a+b)s + (-5a - b)$$

Equating the corresponding coefficients:

$$\begin{aligned} 1 &= a + b \\ -9 &= -5a - b \end{aligned} \qquad\qquad \begin{aligned} a &= 2 \\ b &= -1 \end{aligned}$$

Hence,

$$\mathcal{L}\{y\} = \frac{s-9}{s^2 - 6s + 5} = \frac{2}{s-1} - \frac{1}{s-5}.$$

The last expression corresponds to the Laplace transform of $2e^t - e^{5t}$. Therefore, it must be that

$$y(t) = 2e^t - e^{5t}.$$

Many of the observant students no doubt have noticed an interesting aspect (out of many) of the method of Laplace transform: that it finds the particular solution of an initial value problem directly, without solving for the general solution first. Indeed, it usually takes more effort to find the general solution of an equation than it takes to find a particular solution!

The Laplace Transform method can be used to solve linear differential equations of any order, rather than just second order equations as in the previous example. The method will also solve a nonhomogeneous linear differential equation directly, using the exact same three basic steps, without having to separately solve for the complementary and particular solutions. These points are illustrated in the next two examples.

Example: $y' + 2y = 4t\,e^{-2t}, \qquad y(0) = -3.$

[Step 1] Transform both sides

$$\mathcal{L}\{y' + 2y\} = \mathcal{L}\{4t\,e^{-2t}\}$$

$$(s\mathcal{L}\{y\} - y(0)) + 2\mathcal{L}\{y\} = \mathcal{L}\{4t\,e^{-2t}\} = \frac{4}{(s+2)^2}$$

[Step 2] Simplify to find $Y(s) = \mathcal{L}\{y\}$

$$(s\mathcal{L}\{y\} - (-3)) + 2\mathcal{L}\{y\} = \frac{4}{(s+2)^2}$$

$$(s+2)\,\mathcal{L}\{y\} + 3 = \frac{4}{(s+2)^2}$$

$$(s+2)\,\mathcal{L}\{y\} = \frac{4}{(s+2)^2} - 3$$

$$\mathcal{L}\{y\} = \frac{4}{(s+2)^3} - \frac{3}{s+2} = \frac{4-3(s+2)^2}{(s+2)^3} = \frac{-3s^2 - 12s - 8}{(s+2)^3}$$

[Step 3] Find the inverse transform $y(t)$

By partial fractions,

$$\mathcal{L}\{y\} = \frac{-3s^2 - 12s - 8}{(s+2)^3} = \frac{a}{(s+2)^3} + \frac{b}{(s+2)^2} + \frac{c}{s+2}.$$

$$\frac{-3s^2-12s-8}{(s+2)^3} = \frac{a}{(s+2)^3} + \frac{b(s+2)}{(s+2)^3} + \frac{c(s+2)^2}{(s+2)^3}$$

$$= \frac{a+bs+2b+cs^2+4cs+4c}{(s+2)^3} = \frac{cs^2+(b+4c)s+(a+2b+4c)}{(s+2)^3}$$

$$\begin{aligned} -3 &= c \\ -12 &= b + 4c \\ -8 &= a + 2b + 4c \end{aligned} \qquad\qquad \begin{aligned} a &= 4 \\ b &= 0 \\ c &= -3 \end{aligned}$$

$$\mathcal{L}\{y\} = \frac{-3s^2-12s-8}{(s+2)^3} = \frac{4}{(s+2)^3} - \frac{3}{s+2}.$$

This expression corresponds to the Laplace transform of $2t^2 e^{-2t} - 3e^{-2t}$. Therefore,

$$y(t) = 2t^2 e^{-2t} - 3e^{-2t}.$$

Note: $\mathcal{L}\{t^n e^{at}\} = \dfrac{n!}{(s-a)^{n+1}}$

Example: $y'' - 3y' + 2y = e^{3t}$, $y(0) = 1$, $y'(0) = 0$

[Step 1] Transform both sides

$$(s^2 \mathcal{L}\{y\} - s\,y(0) - y'(0)) - 3(s\mathcal{L}\{y\} - y(0)) + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{3t}\}$$

[Step 2] Simplify to find $Y(s) = \mathcal{L}\{y\}$

$$(s^2 \mathcal{L}\{y\} - s - 0) - 3(s\mathcal{L}\{y\} - 1) + 2\mathcal{L}\{y\} = 1/(s-3)$$

$$(s^2 - 3s + 2)\mathcal{L}\{y\} - s + 3 = 1/(s-3)$$

$$(s^2 - 3s + 2)\mathcal{L}\{y\} = s - 3 + \frac{1}{s-3} = \frac{(s-3)^2 + 1}{s-3}$$

$$\mathcal{L}\{y\} = \frac{s^2 - 6s + 10}{(s^2 - 3s + 2)(s-3)} = \frac{s^2 - 6s + 10}{(s-1)(s-2)(s-3)}$$

[Step 3] Find the inverse transform $y(t)$

By partial fractions,

$$\mathcal{L}\{y\} = \frac{s^2 - 6s + 10}{(s-1)(s-2)(s-3)} = \frac{5}{2}\frac{1}{s-1} - 2\frac{1}{s-2} + \frac{1}{2}\frac{1}{s-3}.$$

Therefore, $y(t) = \frac{5}{2}e^{t} - 2e^{2t} + \frac{1}{2}e^{3t}$.

For the next example, we will need the following Laplace transforms:

$$\mathcal{L}\{\cos bt\} = \frac{s}{s^2 + b^2} \qquad , \qquad s > 0$$

$$\mathcal{L}\{\sin bt\} = \frac{b}{s^2 + b^2} \qquad , \qquad s > 0$$

$$\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2 + b^2} \qquad , \qquad s > a$$

$$\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2 + b^2} \qquad , \qquad s > a$$

Note: The values of a and b in the last two expressions' denominators can be determined without using the method of completing the squares. Any irreducible quadratic polynomial $s^2 + Bs + C$ can always be written in the required from of $(s - a)^2 + b^2$ by using the quadratic formula to find (necessarily complex-valued roots) s. The value a is the real part of s, and the value b is just the absolute value of the imaginary part of s. That is, if $s = \lambda \pm \mu i$, then $a = \lambda$ and $b = \mu$.

Example: $y'' - 2y' + 2y = \cos(t)$, $y(0) = 1$, $y'(0) = 0$

[Step 1] Transform both sides

$$(s^2 \mathcal{L}\{y\} - s\,y(0) - y'(0)) - 2(s\mathcal{L}\{y\} - y(0)) + 2\mathcal{L}\{y\} = \mathcal{L}\{\cos(t)\}$$

[Step 2] Simplify to find $Y(s) = \mathcal{L}\{y\}$

$$(s^2 \mathcal{L}\{y\} - s - 0) - 2(s\mathcal{L}\{y\} - 1) + 2\mathcal{L}\{y\} = s\,/\,(s^2 + 1)$$

$$(s^2 - 2s + 2)\,\mathcal{L}\{y\} - s + 2 = s\,/\,(s^2 + 1)$$

$$(s^2 - 2s + 2)\,\mathcal{L}\{y\} = s - 2 + \frac{s}{s^2+1} = \frac{(s-2)(s^2+1)+s}{s^2+1}$$

$$\mathcal{L}\{y\} = \frac{s^3 - 2s^2 + s - 2 + s}{(s^2+1)(s^2-2s+2)} = \frac{s^3 - 2s^2 + 2s - 2}{(s^2+1)(s^2-2s+2)}$$

[Step 3] Find the inverse transform $y(t)$

By partial fractions,

$$\mathcal{L}\{y\} = \frac{s^3 - 2s^2 + 2s - 2}{(s^2+1)(s^2-2s+2)} = \frac{1}{5}\left[\frac{s-2}{s^2+1} + \frac{4s-6}{s^2-2s+2}\right]$$

$$= \frac{1}{5}\left[\frac{s}{s^2+1} - \frac{2}{s^2+1} + \frac{4(s-1)}{s^2-2s+2} - \frac{2}{s^2-2s+2}\right]$$

which corresponds to

$$y(t) = \frac{1}{5}\left[\cos(t) - 2\sin(t) + 4e^t\cos(t) - 2e^t\sin(t)\right]$$

Examples: Find the inverse Laplace transform of each

(i) $F(s) = \dfrac{2s-5}{s^2+4s+8}$

Rewrite $F(s)$ as:

$$F(s) = \frac{2s-5}{(s+2)^2+2^2} = \frac{2(s+2)}{(s+2)^2+2^2} - \frac{9}{2}\frac{2}{(s+2)^2+2^2}$$

Answer: $f(t) = 2e^{-2t}\cos(2t) - \dfrac{9}{2}e^{-2t}\sin(2t)$

(ii) $F(s) = \dfrac{s+4}{(s-2)^3}$

Use partial fractions to rewrite $F(s)$ as:

$$F(s) = \frac{1}{(s-2)^2} + \frac{6}{(s-2)^3}$$

Answer: $f(t) = te^{2t} + 3t^2e^{2t}$

Appendix A

Some Additional Properties of Laplace Transforms

I. Suppose $f(t)$ is discontinuous at $t = 0$, then the Laplace transform of its derivative becomes

$$\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - \lim_{t \to 0^-} f(t).$$

II. Suppose $f(t)$ is a periodic function of period T, that is, $f(t + T) = f(t)$, for all t in its domain, then

$$\mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\,dt.$$

Comment: This property is convenient when finding the Laplace transform of a discontinuous periodic function, whose discontinuities (necessarily there are infinitely many, due to the periodic nature of the function) would make the usual approach of integrating from 0 to ∞ unwieldy.

III. Let $c > 0$ be a constant, the *time-scaling* property of Laplace transform states that

$$\mathcal{L}\{f(ct)\} = \frac{1}{c} F\left(\frac{s}{c}\right).$$

IV. We have known that $\mathcal{L}\{-tf(t)\} = F'(s)$. Taking the inverse transforms on both sides yields $-tf(t) = \mathcal{L}^{-1}\{F'(s)\}$. Therefore,

$$f(t) = \frac{-1}{t}\mathcal{L}^{-1}\{F'(s)\}.$$

That is, if we know how to invert the function $F'(s)$, then we also know how to find the inverse of its anti-derivative $F(s)$. Formally,

$$\mathcal{L}\{\frac{f(t)}{t}\} = \int_s^\infty F(u)\,du\,.$$

V. Similarly, dividing $F(s)$ by s corresponding an integral with respect to t.

$$\mathcal{L}\{\int_0^t f(u)\,du\} = \frac{F(s)}{s}.$$

Comment: Therefore, dividing a function by its independent variable has the effect of anti-differentiation with respect to the other independent variable for Laplace transforms. These two properties (IV and V) are the counter parts of the multiplication-corresponds-to-differentiation properties seen earlier.

VI. After learning the fact that $\mathcal{L}\{f(t)\,g(t)\} \neq \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\}$, one might have wondered whether there is an operation of two functions f and g whose result has a Laplace transform equal to the product of the individual transforms of f and g. Such operation does exist:

$$\mathcal{L}\{\int_0^t f(\tau)g(t-\tau)\,d\tau\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}.$$

The integral on the left is called *convolution*, usually denoted by $f * g$ (the asterisk is the *convolution operator*, not a multiplication sign!). Hence,

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\}.$$

Furthermore,

$$\mathcal{L}\{f * g\} = \mathcal{L}\{g * f\}.$$

Example: Let $f(t) = t$ and $g(t) = e^{4t}$, find $f * g$ and $g * f$.

$$f * g = \int_0^t \tau\, e^{4(t-\tau)} d\tau = \left(\frac{-1}{4} \tau e^{4(t-\tau)} \Big|_0^t + \frac{1}{4} \int_0^t e^{4(t-\tau)} d\tau \right)$$

$$= \frac{-\tau}{4} e^{4(t-\tau)} - \frac{1}{16} e^{4(t-\tau)} \Bigg|_0^t = \frac{-1}{4} t - \frac{1}{16} + \frac{1}{16} e^{4t}$$

$$g * f = \int_0^t (t-\tau) e^{4\tau} d\tau = \left(\frac{t-\tau}{4} e^{4\tau} \Big|_0^t + \frac{1}{4} \int_0^t e^{4\tau} d\tau \right)$$

$$= \frac{t-\tau}{4} e^{4\tau} + \frac{1}{16} e^{4\tau} \Bigg|_0^t = \frac{-1}{4} t + \frac{1}{16} e^{4t} - \frac{1}{16}$$

In addition, observe that, by taking the inverse Laplace transform of both sides the property VI, we have

$$\int_0^t f(\tau) g(t-\tau)\, d\tau = \mathcal{L}^{-1}\{\mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}\}.$$

In other words, we can obtain the inverse Laplace transform of a (simple) function of s that is itself a product of Laplace transforms of two known functions of t by taking the convolution of the two functions of t.

Example: Find the inverse Laplace transform of

$$F(s) = \frac{1}{s(s^2+4)}.$$

First, note that

$$F(s) = \frac{1}{s} \times \frac{1}{s^2+4} = \frac{1}{2} \mathcal{L}\{1\}\mathcal{L}\{\sin 2t\}$$

Therefore, its inverse can be found by the convolution integral (letting $f(t) = 1$ and $g(t) = \sin 2t$):

$$\frac{1}{2}\int_0^t \sin 2(t-\tau)\,d\tau = \frac{1}{4}\cos 2(t-\tau)\Big|_0^t = \frac{1}{4}\left(\cos(0)-\cos(2t)\right)$$
$$= \frac{1}{4} - \frac{1}{4}\cos 2t\,.$$

The answer can be easily verified using the usual inverse technique.

Example: Find the inverse Laplace transform of

$$F(s) = \frac{1}{(s+2)(s-1)}\,.$$

Since $F(s)$ is a product of the Laplace transforms of e^{-2t} and e^t, it follows that

$$f(t) = e^{-2t} * e^t = \int_0^t e^{-2\tau}\, e^{t-\tau}\,d\tau = \int_0^t e^{t-3\tau}\,d\tau$$
$$= \frac{-1}{3}e^{t-3\tau}\Big|_0^t = \frac{-1}{3}\left(e^{-2t} - e^t\right) = \frac{1}{3}e^t - \frac{1}{3}e^{-2t}\,.$$

Exercises C-1.2:

1 – 9 Find the Laplace transform of each function below.

1. $f(t) = t^3 - t^2 + 5t + 2$

2. $f(t) = -2\cos 6t + 5\sin 6t$

3. $f(t) = 3e^t - 4e^{2t} + 2e^{-4t}$

4. $f(t) = 7t + 6e^t - 2e^{-t} - 10$

5. (a) $f(t) = \sin 2t \sin 3t$ (b) $f(t) = \sin 2t \cos 2t$

6. (a) $f(t) = \cos^2 5t$ (b) $f(t) = t\cos^2 5t$

7. (a) $f(t) = t\,e^{at}\cos bt$, (b) $f(t) = t\,e^{at}\sin bt$

8. $f(t) = t^3 \cos 2t$

9. (a) $f(t) = \cos(\alpha t + \beta)$ (b) $f(t) = \sin(\alpha t + \beta)$

10 – 18 Find the inverse Laplace transform of each function below.

10. $F(s) = \dfrac{4s+2}{s^2+6s+34}$ 11. $F(s) = \dfrac{1}{(s-2)(s-4)(s-8)}$

12. $F(s) = \dfrac{3s-7}{4s^2+1}$ 13. $F(s) = \dfrac{s^2+s-6}{s^3+2s^2+s}$

14. $F(s) = \dfrac{1}{s^4-81}$ 15. $F(s) = \dfrac{s^3}{s^4-16}$

16. $F(s) = \dfrac{10}{s^4-s^3}$ 17. $F(s) = \dfrac{12}{s^3-8}$

18. $F(s) = \dfrac{1}{(s-\alpha)(s-\beta)}$

19 – 32 Use the method of Laplace transforms to solve each IVP.

19. $y' + 10y = t^2$, $\quad y(0) = 0$

20. $y' + 2y = t\,e^{-t}$, $\quad y(0) = 2$

21. $y' - 6y = 2\sin 3t$, $\quad y(0) = -1$

22. $y'' + 2y' = t\,e^{-t}$, $\quad y(0) = 6, \quad y'(0) = -1$

23. $y'' + 4y' - 5y = 0$, $\quad y(0) = 5, \quad y'(0) = -1$

24. $y'' - 2y' + y = 2t - 3$, $\quad y(0) = 5, \quad y'(0) = 11$

25. $y'' + 8y' + 25y = 13e^{-2t}$, $\quad y(0) = -1, \quad y'(0) = 18$

26. $y'' + 6y' + 34y = 0$, $\quad y(0) = -1, \quad y'(0) = 13$

27. $y'' + 4y = 8\cos 2t - 8e^{-2t}$, $\quad y(0) = -2, \quad y'(0) = 0$

28. $y'' - 4y' + 4y = 8t^2 - 16t + 4$, $\quad y(0) = 0, \quad y'(0) = 1$

29. $y'' - 4y' - 5y = 3t^3$, $\quad y(0) = 3, \quad y'(0) = 3$

30. $y''' + 3y'' + 3y' + y = 0$, $\quad y(0) = 7, \quad y'(0) = -7, \quad y''(0) = 11$

31. $y''' + 4y'' - 5y' = 0$, $\quad y(0) = 4, \quad y'(0) = -7, \quad y''(0) = 23$

32. $y''' - y'' + 4y' - 4y = 26e^{3t}$, $\quad y(0) = -2, \quad y'(0) = 3, \quad y''(0) = 1$

33. Prove the time-scaling property (property III, Appendix A).

Hint: Let $u = ct$, and $v = s/c$, then show $\mathcal{L}\{f(ct)\} = \frac{1}{c}\int_0^\infty e^{-vu} f(u)\,du$.

34. Use property IV of Appendix A to verify that

$$\mathcal{L}^{-1}\{\arctan(a/s)\} = \frac{1}{t}\sin at .$$

35. Apply property V of Appendix A to
(a) $f(t) = \cos 5t$ $\qquad$ (b) $f(t) = t^n$, $\quad n$ = a positive integer

Answers C-1.2:

1. $F(s)=\dfrac{2s^3+5s^2-2s+6}{s^4}$

2. $F(s)=\dfrac{-2s+30}{s^2+36}$

3. $F(s)=\dfrac{s^2-12s-4}{(s-1)(s-2)(s+4)}$

4. $F(s)=\dfrac{-6s^3+15s^2+10s-7}{s^4-s^2}$

5. (a) $F(s)=\dfrac{s}{2(s^2+1)}-\dfrac{s}{2(s^2+25)}$ (b) $F(s)=\dfrac{2}{s^2+16}$

6. (a) $F(s)=\dfrac{1}{2}\left[\dfrac{1}{s}+\dfrac{s}{s^2+100}\right]$, (b) $F(s)=\dfrac{1}{2}\left[\dfrac{1}{s^2}+\dfrac{s^2-100}{(s^2+100)^2}\right]$

7. (a) $F(s)=\dfrac{(s-a)^2-b^2}{((s-a)^2+b^2)^2}$, (b) $F(s)=\dfrac{2b(s-a)}{((s-a)^2+b^2)^2}$

8. $F(s)=\dfrac{6s^4-144s^2+96}{(s^2+4)^4}$

9. (a) $F(s)=\dfrac{s\cos(\beta)-\alpha\sin(\beta)}{s^2+\alpha^2}$, (b) $F(s)=\dfrac{s\sin(\beta)+\alpha\cos(\beta)}{s^2+\alpha^2}$

10. $f(t)=4e^{-3t}\cos 5t-2e^{-3t}\sin 5t$

11. $f(t)=\dfrac{1}{12}e^{2t}-\dfrac{1}{8}e^{4t}+\dfrac{1}{24}e^{8t}$

12. $f(t)=\dfrac{3}{4}\cos\dfrac{t}{2}-\dfrac{7}{2}\sin\dfrac{t}{2}$

13. $f(t)=7e^{-t}+6te^{-t}-6$

14. $f(t)=\dfrac{1}{108}e^{3t}-\dfrac{1}{108}e^{-3t}-\dfrac{1}{54}\sin 3t$

15. $f(t)=\dfrac{1}{4}e^{2t}+\dfrac{1}{4}e^{-2t}-\dfrac{1}{2}\cos 2t$

16. $f(t)=10e^{t}-5t^2-10t-10$

17. $f(t)=e^{2t}-e^{-t}\cos(\sqrt{3}\,t)-\sqrt{3}\,e^{-t}\sin(\sqrt{3}\,t)$

18. $f(t)=\dfrac{1}{\alpha-\beta}(e^{\alpha t}-e^{\beta t})$

19. $y=\dfrac{t^2}{10}-\dfrac{t}{50}+\dfrac{1}{500}-\dfrac{1}{500}e^{-10t}$

20. $y=te^{-t}-e^{-t}+3e^{-2t}$

21. $y = \frac{-13}{15}e^{6t} - \frac{2}{15}\cos 3t - \frac{4}{15}\sin 3t$
22. $y = -t\,e^{-t} + 6$
23. $y = 4e^{t} + e^{-5t}$
24. $y = 4e^{t} + 5t\,e^{t} + 2t + 1$
25. $y = -2\,e^{-4t}\cos 3t + 4\,e^{-4t}\sin 3t + e^{-2t}$
26. $y = -e^{-3t}\cos 5t + 2e^{-3t}\sin 5t$
27. $y = -\cos 2t - \sin 2t + 2t\sin 2t - e^{-2t}$
28. $y = t\,e^{2t} + 2t^{2}$
29. $y = e^{5t} + 2e^{-t} + 3t^{3}$
30. $y = 7e^{-t} + 2t^{2}e^{-t}$
31. $y = 5 - 2e^{t} + e^{-5t}$
32. $y = -4e^{t} + e^{3t} + \cos 2t + 2\sin 2t$