Extra Practice 1

3/3/2019

Question

1. In how many ways can n people sit around a round table? Count A seated to the left of B as different from B seated to the left of A. *Hint: if everyone shifts the same number of seats in the same direction, it's the same arrangement.*

Solution

$(n-1)!$ ways. This occurs because in a table, the first person seated has to have a fixed position while the rest move therefore removing 1 from the total number of people.

Another way to think about this is that if the n people were sitting in a row, we would have $n!$ ways in which they could be seated. If we tried that method for a round table we would be overcounting by a factor of n. For example, for 4 people siting in a row ABCD, BCDA, CDAB, and DABC are four different arrangements. But for a circular table they are all the same:

$$A$$

$$D \bigcirc B$$

$$C$$

Question

2. The pmf of random variable x is $p(x) = 0.05x$ for all integers $[1, 4], 0.05x - 0.2$ for all integers $[5, 8]$, and 0 otherwise. Determine the cdf, $F(x)$, including all values of x for which $F(x)$ is defined. Then, find the following:

a. $P(3 \leq x \leq 7)$

b. $P(3 \leq x < 7)$

c. $P(3 < x \leq 7)$

d. $P(3 < x < 7)$

Solution

pmf:

x	p(x)
1	0.05
2	0.10
3	0.15
4	0.20
5	0.05
6	0.10
7	0.15
8	0.20

cdf:

x	F(x)
x<1	0
$1 \leq$ x<2	.05
$2 \leq$ x<3	.15
$3 \leq$ x<4	.3
$4 \leq$ x<5	.5
$5 \leq$ x<6	.55
$6 \leq$ x<7	.65
$7 \leq$ x<8	.8
$8 \leq$ x	1

a. P($3 \leq$ x ≤ 7) = P(x ≤ 7) - P(x ≤ 2) = 0.80 - 0.15 = **0.65**
b. P($3 \leq$ x<7) = P(x ≤ 6) - P(x ≤ 2) = 0.65 - 0.15 = **0.50**
c. P(3<x ≤ 7) = P(x ≤ 7) - P(x ≤ 3) = 0.80 - 0.30 = **0.50**
d. P(3<x<7) = P(x ≤ 6) - P(x ≤ 3) = 0.65 - 0.30= **0.35**

Question

3. Suppose that the temperature of the ocean in a particular area is known to be normally distributed with mean 60 degrees and standard deviation of 7 degrees. If X = the temperature of one reading taken from this area, find:

a. P(X $\leq$ 53)
b. P(X $\leq$ 44)
c. P(X $\geq$ 70)

Solution

1. P(X $\leq$ 53) = P(Z $\leq \frac{53-60}{7}$) = P(Z $\leq$ -1) = .1587
2. P(X $\leq$ 44) = P(Z $\leq \frac{44-60}{7}$) = P(Z $\leq$ -2.29) = .011
3. P(X $\geq$ 70) = 1 - P(Z $\leq \frac{70-60}{7}$) = 1 - P(Z $\leq$ 1.43) = 1 - .9236 = .0764

Question

4. If I am a safety inspector for a paint production plant, and the probability that a paint color does not meet expectations is p = .05, what is the probability that I will inspect 10 good paint colors before 3 poor ones?

Solution

This can be solved with the negative binomial. Remember that X represents the number of failures – good paint colors.

P(bad paint color) = .05 r = 3

$$P(X=x) = \left(\begin{array}{c} x+r-1 \\ r-1 \end{array} \right) p^r(1-p)^x = \left(\begin{array}{c} 12 \\ 2 \end{array} \right) .05^3 .95^1 0 = .005$$

Or in R:

```
dnbinom(10, 3, .05)
```

```
## [1] 0.00493958
```

Question

5. Robbins Bakery sells croissants. The pmf for daily sales is:

X = daily croissant sales (in dozens)

x	p(x)
0	.1
1	.1
2	.3
3	.4
4	.1

The retail price of the croissants is $30/dozen. The cost to produce each item is half of that. What is the expected daily profit for Robbins Bakery?

Solution

E(X) = $\Sigma x \cdot p(x)$ = 0 + .1 + .6 + 1.2 + .4 = 2.3

profit function = 30 * .5X = 15X

E(15X) = 15*E(X) = $34.50

Question

6. An American roulette wheel has 38 numbers upon which you can bet. 2 are green, 18 are black, and 18 are red. Each number is equally likely to be chosen.

a) You pay $1 to bet on one number. If you win you get $35 plus the original dollar that you bet. What are your expected winnings on each game?

b) You pay $1 to bet on red or black. If you win you get $1 plus the original dollar that you bet. What are your expected winnings on each game?

c) You pay $1 to bet on green. If you win you get $17 plus the original dollar that you bet. What are your expected winnings on each game?

Solution

a) bet on a single number

p(win) = 1/38 = .0263

pmf of h(X):

h(x)	p(x)
-1	.9737
35	.0263

E(h(x)) = (-1)(.9737) + (35)(.0263) = -$0.0532

b) bet on red or black

p(win) = 18/38 = .4737

pmf of h(X):

h(x)	p(x)
-1	.5263
1	.4737

E(h(x)) = (-1)(.5263) + (1)(.4737) = -\$0.0526

c) bet on green

p(win) = 2/38 = .0526

pmf of h(X):

h(x)	p(x)
-1	.9474
17	.0526

E(h(x)) = (-1)(.9474) + (17)(.0526) = -\$0.0532

Challenge problems (beyond the level of this class)

Question

7. A fair coin is tossed 12 times. What is the probability that no two consecutive tosses are heads?

Solution

Let s_i be the number of total possible cases where no two consecutive tosses are heads when the coin is tossed i times. (i should be a natural number.)

1 toss: cases with no two consecutive heads: {H, T}

2 tosses: {HT, TT, TH}

3 tosses: {HTT, TTT, THT, TTH, HTH}

So far we have: $s_1 = 2; s_2 = 3; s_3 = 5$.

For 4 tosses, we can add T to the end of the sequence of 3 cases, and TH to the end of the sequence of 2 cases:

{HTTT, TTTT, THTT, TTHT, HTHT} + {HTTH, TTTH, THTH}

Thus $s_4 = s_2 + s_3 = 8$.

In general, $s_i = s_{i-2} + s_{i-1}$.

Therefore, s_i are the Fibonacci numbers, so $s_{12} = 377$. Total possible cases of 12 coin tosses are 2^{12}. Hence, the probability is: $\frac{377}{4096}$.

Question

8. Suppose we toss a fair coin 20 times. Annie wins if there are odd number of heads and Ben wins if there are even number of heads. What is the probability that Ben will win? (This is more a reasoning than a calculating question, though you can use R to figure it out.)

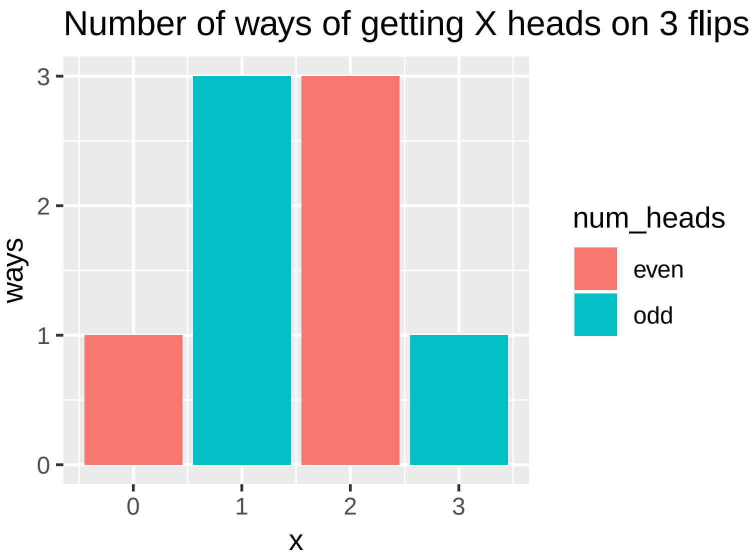
Solution

With R:

```r
sum(dbinom(seq(0, 20, 2), 20, .5))
```

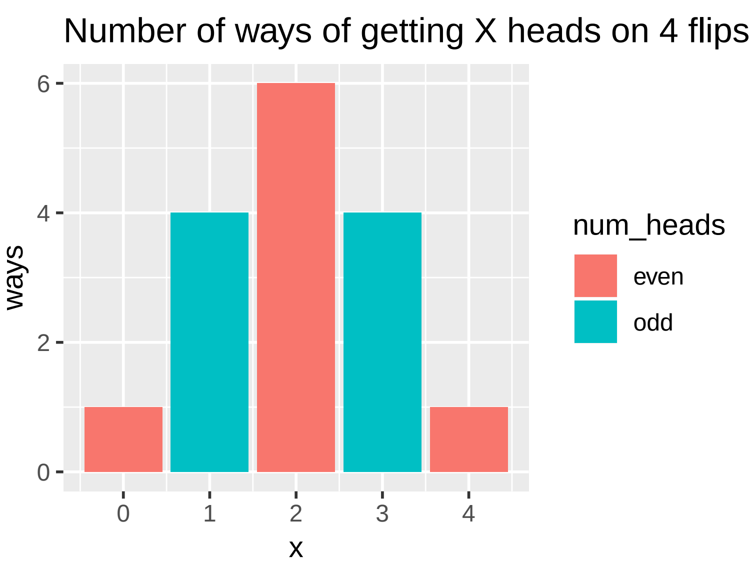
```
## [1] 0.5
```

Another approach: It is easier to see that the solution must be .5 for an odd number of flips, based on the symmetry of the binomial distribution. Let's start simple and graph the number of ways of getting 0, 1, 2, or 3 heads on 3 flips:



Clearly, odd and even are identical, and therefore the probability of either is 1/2.

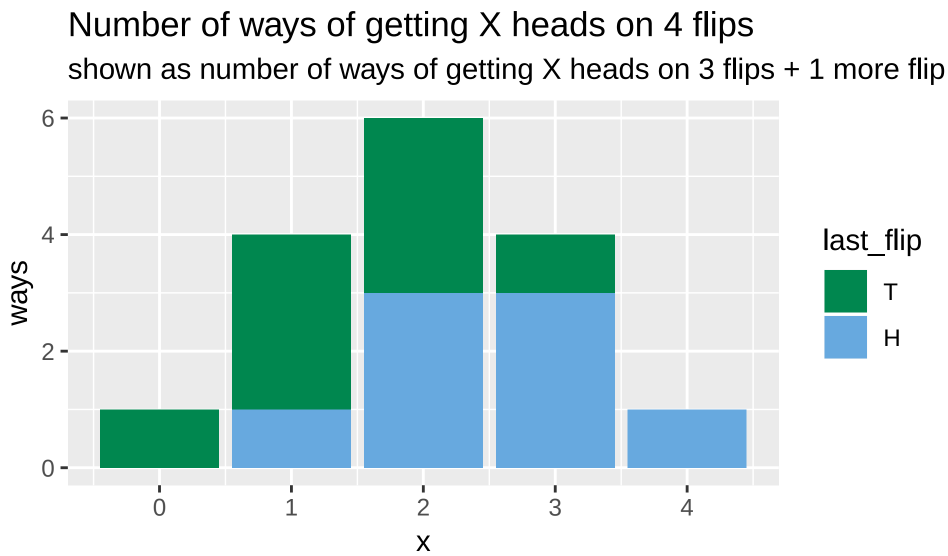
With 4 flips, it's not quite so obvious that the number of ways to get an even number of heads {0, 2, or 4} is the same as the number of ways of getting an odd number of heads {1 or 3}:



But let's look at it another way. The number of different ways of flipping 4 coins is double the number of ways of flipping 3 coins. The outcomes can be summarized as:

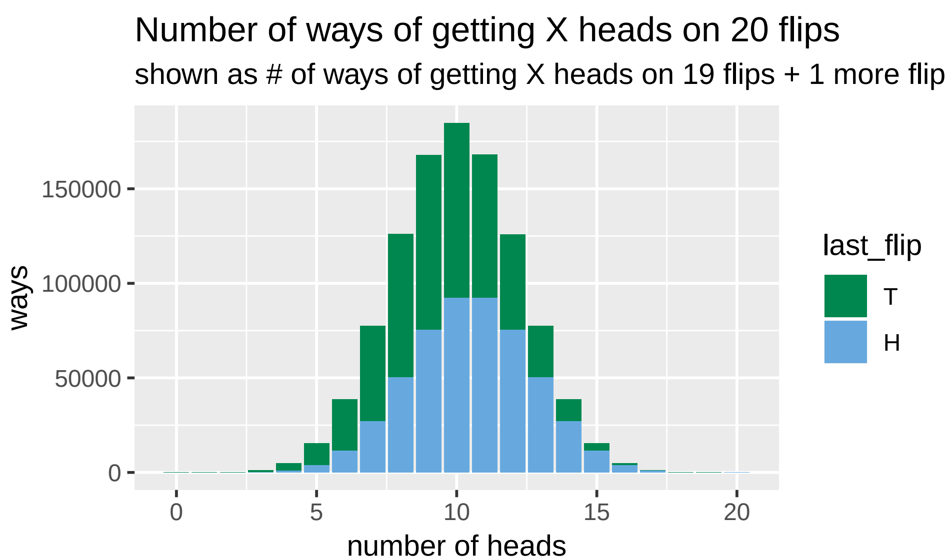
outcomes for 3 flips + "T" –> same number of heads as with 3 flips

outcomes for 3 flips + "H" –> one more heads than with 3 flips Graphically we have:



The green represents the number of ways on getting X heads on 4 flips if the last flip is tails. It is the same as the distribution of getting X heads on 3 flips, since none of the counts increase if the last flip is tails. The blue represents the number of ways of getting X heads on 4 flips if the last flip is heads. It is the same as the distribution we had before shifted to the right. If you look carefully at the rectangles you'll see that both the even number of heads {0, 2, 4} are made up of 2 short and 2 long rectangles as are the odd number of heads {1, 3}.

The same pattern occurs for n = 20: we can think of it as two n = 19 distributions combined:



If you look carefully you'll see that the rectangles for the even number of heads (Ben wins) match with the rectangles for the odd number of heads (Annie wins) and therefore the probability of Ben winning is 1/2.