STAT UN1201 – Chapter 2

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Probability

In 1654, writer Antoine Gombaud "Chevalier de Méré" wanted to know if the following bets are profitable:

- getting at least one six on 4 dice rolls
- getting at least one double-six on 24 dice rolls

Vocabulary (2.1)

- **experiment** – process whose outcome is subject to uncertainty
  (ex. rolling a die)
- **sample space** – set of all possible outcomes of an experiment
  S = {1, 2, 3, 4, 5, 6}
- **event** – collection of outcomes contained in the sample space

Experiment with an infinite sample space

- ex. flip a coin until you get tails
- **sample space**
  S = {T,   HT,   HHT,   HHHT, …}
- **event**
  you get tails in fewer than 8 flips
  A = {T,   HT,   HHT,   HHHT,   HHHHT,   HHHHHT,   HHHHHHT}

Set theory 1

- **complement** of an event – all outcomes in the sample space that are not in the event
  S = {1, 2, 3, 4, 5, 6}
  A = {1, 3}
  A' ("not A") = {2, 4, 5, 6}

Set theory 2

- **union** of two events: all outcomes in either event or in both $A \cup B$ ("A or B")
  $A = \{1, 3\}$
  $B = \{3, 5\}$
  $A \cup B = \{1, 3, 5\}$

Set theory 3

- **intersection** of two events: all outcomes in both events
  $A \cap B$ ("A and B")
  $A = \{1, 3\}$
  $B = \{3, 5\}$
  $A \cap B = \{3\}$

Set theory 4

- **null event**: no outcomes $\varnothing$ or {}
  C = {1, 2}
  D = {3, 4}
  $C \cap D = \varnothing$
- **mutually exclusive** – events that cannot occur at the same time
  if $A \cap B = \varnothing$, then A and B are mutually exclusive or disjoint

Set theory: more than two events 1

- $A \cup B \cup C$: all outcomes in at least one of A, B, & C

$A = \{1, 2, 3\}$
$B = \{5\}$
$C = \{1, 5, 10\}$

$A \cup B \cup C = \{1, 2, 3, 5, 10\}$

- $A \cap B \cap C$: all outcomes in A, B, and C
  $A \cap B \cap C = \{\}$

Set theory: more than two events 2

- **mutually exclusive** or **pairwise disjoint** – no two events have any outcomes in common
  A = {1, 3}
  B = {2, 4}
  C = {5, 6}
  A, B, & C are mutually exclusive
- Are the following events mutually exclusive?
  D = {H, T}
  E = {HH, TT, TH, HT}
  F = {T, TT}

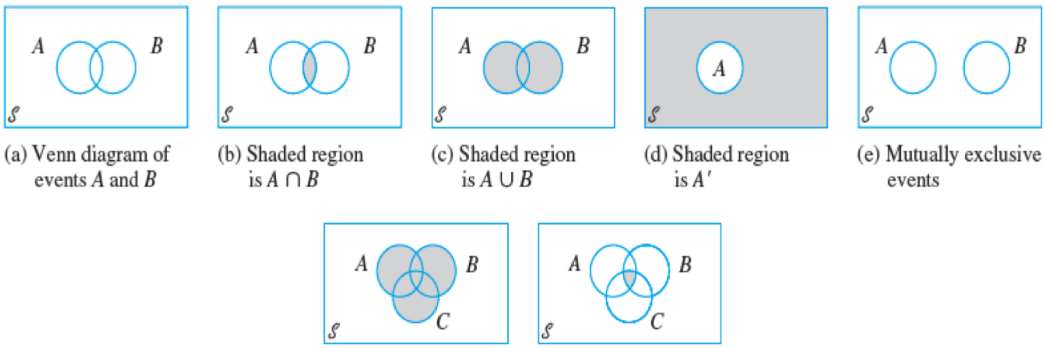
Venn diagrams

Denzel Washington Venn diagram

Denzel Washington Venn diagram

- **Sample Space**: all Denzel Washington movies
- **Events**: Hat, Glasses, Facial Hair
- Hat ∩ Glasses ∩ Facial Hair = {“Malcolm X”}

Venn diagrams of **events**



(a) Venn diagram of events A and B

(b) Shaded region is $A \cap B$

(c) Shaded region is $A \cup B$

(d) Shaded region is A'

(e) Mutually exclusive events

(f) Shaded region is $A \cup B \cup C$

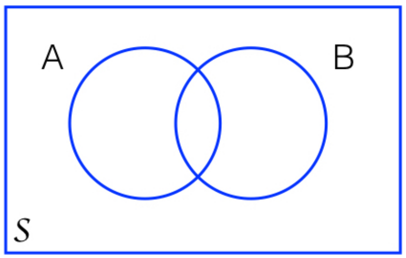
(g) Shaded region is $A \cap B \cap C$

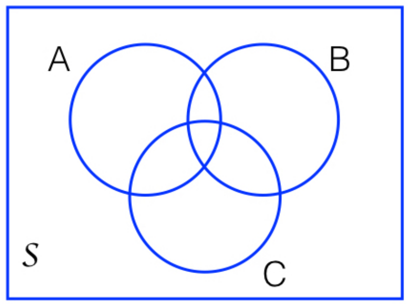
Figure 2.1 Venn diagrams

EXERCISE

Shade:

1. $(A \cap B) \cup (A' \cap B')$ 2. $(A \cup B) \cap C \cap (A \cap B)'$





Basic properties of probability (axioms) (2.2)

P(A) = measure of the chance that A will occur
(multiple interpretations)

1. For any event A, P(A) $\geq$ 0.
2. P(S) = 1
3. If $A_1, A_2, A_3, \ldots$ is an infinite collection of disjoint events, then
   $P(A_1 \cup A_2 \cup A_3 \cup \ldots)$ = $\sum_{i=1}^{\infty} P(A_i)$

Propositions

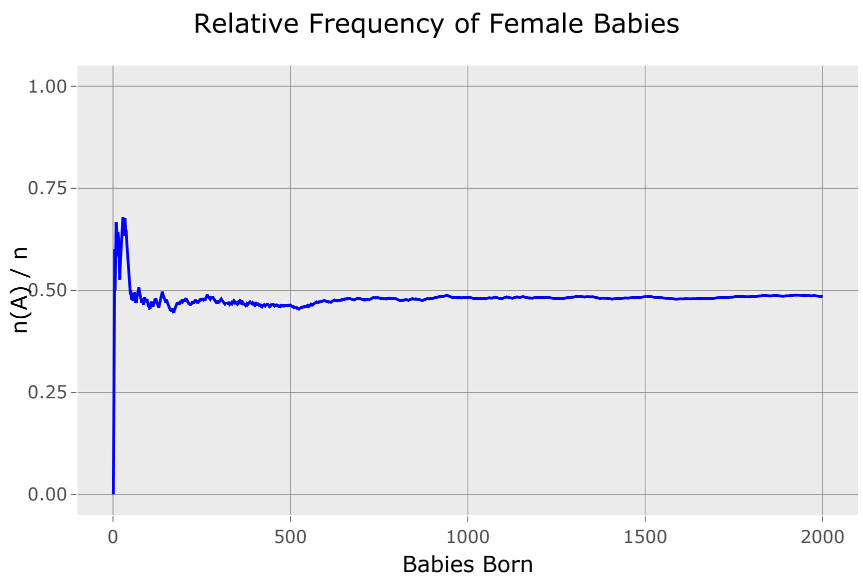
- $P(\emptyset) = 0$
- Axiom 3 is valid for **finite** disjoint events
  $P(A \cup B) = P(A) + P(B)$

Relative frequency

- probability = relative frequency = $\frac{n(A)}{n}$ = number of times A occurs in n replications
- Experiments that can be repeated
- Example: What is the probability that a baby born in the U.S. is female?

Observation

BabiesBorn	Females	RelFreq
1	0	0.000
2	1	0.500
3	1	0.333
4	2	0.500
5	3	0.600
10	6	0.600
50	25	0.500
100	46	0.460
500	232	0.464
1000	482	0.482
2000	969	0.484



Objective vs. subjective interpretations of probability

- objective interpretation of probability is based on long-run relative frequency
- subjective interpretation – assignment of probability to nonrepeatable events

More probability properties

- For any event A, P(A) + P(A’) = 1, from which P(A) = 1 - P(A’).
- For any event A, P(A) $\leq$ 1.
- For any two events A and B, P($A \cup B$) = P(A) + P(B) - P($A \cap B$)

Counting techniques (2.3)

- Are items being replaced?
- Does order matter?
- Add (“or”, alternatives) or multiply (“and”)?

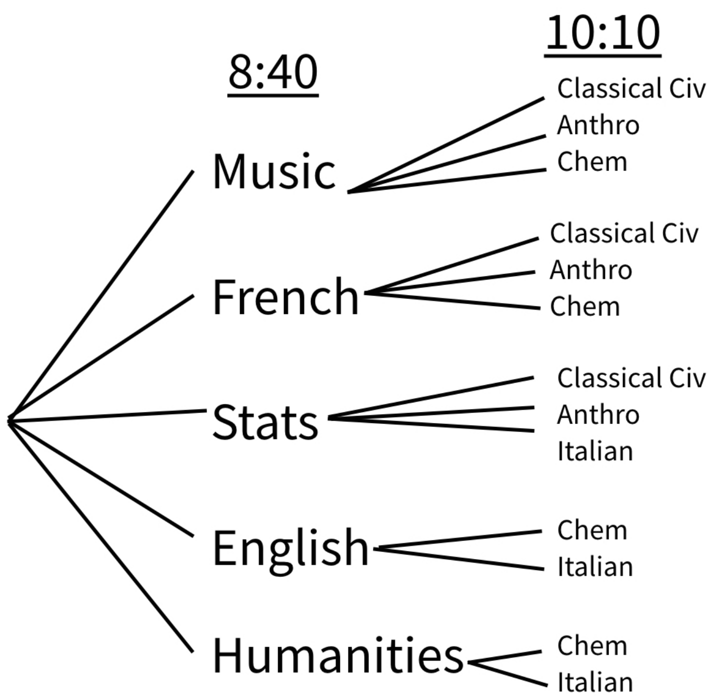
The Product Rule

If one element of a pair can be selected in n_1 ways, and for each of these n_1 ways the other element of the pair can be selected n_2 ways, then the number of pairs is $n_1 n_2$.

Example

TuTh 8:40 classes	**TuTh 10:10 classes**
Music BC1002	Classical Civilization UN3230
French UN2102	Anthropology UN2003
Statistics UN1201	Chemistry S1404
English BC1211	Italian UN1102
Humanities UN1123	

Tree diagram



Permutations

order matters

$$P_{k,n} = \frac{n!}{(n-k)!}$$

- 10 people, 1st, 2nd, 3rd place
- $P_{3,10} = \frac{10!}{(10-3)!} = \frac{10!}{7!} = 10(9)(8) = 720$

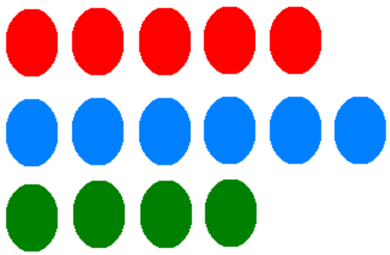
Combinations

order doesnft matter

Handshake problem: 5 people, everyone must shake everyone else's hand, how many handshakes?

- $C_{k,n} = \binom{n}{k} = \frac{P_{k,n}}{k!} = \frac{n!}{k!(n-k)!}$
- 10 people, how many distinct groups of 3 can be formed?
- $C_{3,10} = \binom{10}{3} = \frac{10!}{3!(10-3)!} = \frac{10(9)(8)}{3(2)(1)} = 120$
- Permutations: ABC, ACB, BAC, BCA, CAB, CBA are all different, for combinations, all the same

EXERCISE

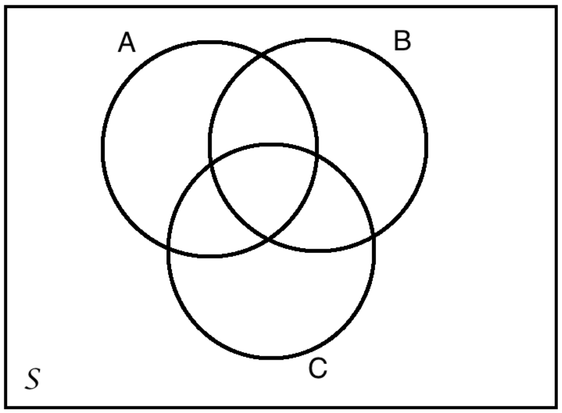
(based on #39) 

A box has 5 red, 6 blue, and 4 green lightbulbs. **Three** are randomly selected.

1. What is the probability that exactly two are green?
2. What is the probability that all three are the same color?
3. What is the probability that one of each color is selected?

Union of three events

- What is P($A \cup B \cup$ C) expressed in terms of intersection rather than union? (Extending $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ to three events)



Independence (2.5)

- Formal definition: A & B are independent if $P(A|B) = P(A)$
- Informal: the outcome of one has no impact on the outcome of the other (knowing the odds for one doesn't change the odds for the other)
- often assumed in statistical problems
- not the same as disjoint / mutually exclusive

Multiplication Rule for P($A \cap B$)

A and B are independent iű P($A \cap B$) = P(A) · P(B)

- **More than two events**: If events $A_1, A_2, ... A_n$ are **mutually independent**, then the probability of the intersection of any subset of the events is equal to the product of the individual probabilities.
- ex. if $A_1, A_2, ... A_{50}$ are **mutually independent**,
  $P(A_{23} \cap A_{27} \cap A_{45}) = P(A_{23}) \cdot P(A_{27}) \cdot P(A_{45})$

EXERCISE (#87)

Consider randomly selecting a single individual to test drive three different vehicles. Define events A_1, A_2, and A_3 by

A_1 = likes vehicle #1 A_2 = likes vehicle #2 A_3 = likes vehicle #3

Suppose P(A_1) = .55; P(A_2) = .65; P(A_3) = .70

P($A_1 \cup A_2$) = .80; P($A_2 \cap A_3$) = .40, and

P($A_1 \cup A_2 \cup A_3$) = .88

1. Are A_2 and A_3 independent events?

2. What is the probability that the individual likes both vehicle #1 and vehicle #2?

The Dice Problems

1. What is the probability of getting at least one six on 4 dice rolls?

2. What is the probability of getting at least one double six on 24 dice rolls?