Logic

Aria Ruler(Forouzandeh)

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Chapter 1

Atomes

1.1 Basic tools

we start the book by building logic step by step.block by block in first it might seem a little annoying but I advice you to be patient.book will show its power soon. like euclidian geometry this book has undefined concepts. I tried to explain them for you to understand.but they don't have any definition.

Undefined 1.1.1. Urelements are the most fundamental entities in a logical. They cannot be constructed, defined, or described in terms of other things. Their existence is primitive—accepted without internal structure or relational properties. Urelements simply are.

In this book, we will denote individual urelements using lowercase letters such as a, b, c,...

Undefined 1.1.2. Type parallel lines used to show that an object exists (you will undrestnd what is object soon.) (we will show you how to use them).

1.	a		
2.		b	
3.			c

Rule 1.1.1. fusion it's the Rule that explain how sets made. (this rule is schema.it means it works everywhere for every object.)

$$\begin{array}{r|l|l} m. & a_0 & \\ & \vdots & \\ m+n. & a_n & \\ o. & & \{a_0, a_1 ... a_n\} \end{array}$$

Undefined 1.1.3. Set we can't say what exactly sets are but we know sets can be made by fusion (from now we denote sets with uppercase letters).

Definition 1.1.1. Object urelements and sets.(we write obj instead of object).

Rule 1.1.2. fission it show us how to break an obj. (this rule is scheme it means it works everywhere for every obj.)

$$\begin{array}{r|l|l} o. & & \{a_0, a_1 ... a_n\} \\ m. & a_0 & \\ & \vdots & \\ m+n. & a_n & \end{array}$$

Example 1.1.1. $\{a, b, c\}/\therefore\{a, b\}$ (prove if $\{a, b, c\}$ exists then $\{a, c\}$ exists)

1.		$\{a, b, c\}$	
2.	a		fi1
3.	b		fi1
4.	c		fi1
5.		$\{a, c\}$	fu2,3,4

Example 1.1.2. $\{a, b\}/\therefore\{b, a\}$

1.		$\{a, b\}$	
2.	a		fi1
3.	b		fi1
4.		$\{b, a\}$	fu2,3

Rule 1.1.3. repetition you can repeat an obj (this rule is scheme it means it works everywhere for every obj.)

$n.$	a	
$m.$	a	rp

Undefined 1.1.4. Gutter is a tool to make an object isolated. (nothing can get in or out and the focused obj is the first obj in gutter) with Gutter we can undrestnd object More precisely we denote Gutter like this.

Rule 1.1.4. repetition in Gutter you can only repeat an obj in Gutter once(Gutter isolated obj).

$$
\begin{array}{r|l l}
n. & a & \\
\hline
m. & a & \text{rpg.}n \\
\hline
\end{array}
$$

1.2 Identity

Undefined 1.2.1. identity (=) one of the basice concepts that talk about two things are the same.

Rule 1.2.1. $I =$ (this rule is scheme it means it works everywhere for every obj.)

$$
\begin{array}{r|l}
\hline
n. & a \\
 & \vdots \\
m. & b \\
\hline
\hline
m+1. & b \\
 & \vdots \\
o. & a \\
\hline
o+1. & a = b
\end{array}
$$

Rule 1.2.2. $E =$ (this rule is scheme it means it works everywhere for every obj.)

$$
\begin{array}{r|l l}
n. & a & \\
m. & a = b & \\
o. & b & E = n, m
\end{array}
$$

Example 1.2.1. $a / \therefore a = a$

1.	a	
2.	a	rp.1
3.	a	
4.	a	rp.3
5.	$a = a$	

Example 1.2.2. $a, b / \therefore \{a, b\} = \{b, a\}$

1.	a		
2.	b		
3.		$\{a, b\}$	fu1,2
4.		$\{b, a\}$	fu1,2
5.		$\{a, b\}$	rpg3
6.	a		fi1
7.	b		fi1
8.		$\{b, a\}$	fu2,3
9.		$\{b, a\}$	rpg4
10.	b		fi5
11.	a		fi5
12.		$\{a, b\}$	fu6,7
13.		$\{a, b\} = \{b, a\}$	$I = 4, 8$

Example 1.2.3. $a, b, c, b = c / \therefore \{a, b\} = \{a, c\}$

1.	a		ass
2.	b		ass
3.	c		ass
4.		$\{a, b\}$	fu1,2
5.		$\{a, c\}$	fu2,3
6.		$\{a, b\}$	rp4
7.	a		fi6
8.	b		fi6
9.		$b = c$	ass
10.	c		$E = 8, 9$
11.		$\{a, c\}$	fi7,10
12.		$\{a, c\}$	rp5
13.	a		fi12
14.	c		fi12
15.		$b = c$	ass
16.	b		$E = 15, 16$
17.		$\{a, b\}$	fu13,15
18.		$\{a, b\} = \{a, c\}$	$I = 11, 17$

1.3 Membership

Undefined 1.3.1. membership ($\in$) a binary relation show that an obj is a member of another obj.

Rule 1.3.1. $I\in$ (this rule is scheme it means it works everywhere for every obj.)

$$
\begin{array}{l|l|ll}
n. & & A & \\
m. & a & & fi, n \\
o. & & a \in A & I\in, n, m
\end{array}
$$

$$
\begin{array}{l|l|ll}
n. & a & & \\
m. & & A & fu, n \\
o. & & a \in A & I\in, n, m
\end{array}
$$

Rule 1.3.2. $E\in$ (this rule is scheme it means it works everywhere for every obj.)

$$
\begin{array}{l|l|ll}
n. & & a \in A & \\
m. & & A & \\
o. & a & & \text{fiss}, n, m
\end{array}
$$

$$
\begin{array}{l|l|ll}
n. & & a \in A & \\
m. & a & & \\
o. & & A & \text{fus}, n, m
\end{array}
$$

Rule 1.3.3. $E=$ (this rule is scheme it means it works everywhere for every obj.)

$$\begin{array}{r|ll} n. & a \in A & \\ m. & a = b & \\ o. & b \in A & E= \end{array}$$

$$\begin{array}{r|ll} n. & a \in A & \\ m. & A = B & \\ o. & a \in B & E= \end{array}$$

1.4 Bot

Undefined 1.4.1. bot ($\perp$) also called "bottom".means impossible, or a logical symbol denoting a impossible situation.

also called "bottom".means impossible, or a logical symbol denoting a impossible situation.

Rule 1.4.1. $I\perp$ (this rule is scheme it means it works everywhere for every obj.) assume that $a_0, a_1, a_2 ... a_n$ are distinct obj.if someone says $a_0 \in \{a_1, a_2\}$ or if someone says $a_0 = a_1$. we know it's not right but how do we must show that? (see attention 1.1).

for (rp):

$$\begin{array}{r|ll} n. & a_0 & \\ m. & a_1 & \text{rp1} \\ o. & \perp & I\perp, n, m \end{array}$$

for (fu):

$$\begin{array}{l|l|ll}
n. & a_0 & & \\
m. & & \{a_1, a_2...a_n\} & \text{fu1} \\
o. & & \perp & I\perp, n, m
\end{array}$$

for (fi):

$$\begin{array}{l|l|ll}
n. & & \{a_1, a_2...a_n\} & \\
m. & a_0 & & \text{fi1} \\
o. & & \perp & I\perp, n, m
\end{array}$$

Rule 1.4.2. $E\perp$ (this rule is scheme it means it works everywhere for every obj.) (see attention 1.1). for (rp):

$$\begin{array}{l|ll}
n. & \perp & \\
m. & a_0 & \\
o. & a_1 & \text{rp}, n, m
\end{array}$$

for (fu):

$$\begin{array}{l|l|ll}
n. & & \perp & \\
m. & a_0 & & \\
o. & & \{a_1, a_2...a_n\} & \text{fu}, n, m
\end{array}$$

for (fi):

$n.$		$\perp$	
$m.$		$\{a_1, a_2...a_n\}$	
$o.$	a_0		fiss,n, m

1.5 constants and variables

obj are two types.we know them and we know what exactly are we talking about and unknowns are mystery.all the obj we have used until now were known ones we call them **"constants"**.

Undefined 1.5.1. variables A variable is a symbol (usually a letter like x, y, orz) that represents an unknown obj.

Definition 1.5.1. variables sets A variable that is a set (usually a letter like X, Y, orZ) that represents an unknown set.

Definition 1.5.2. predicate identity and membership symbols are predicates. we will introduce them all to you.

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Chapter 2

propositional logic(zeroth-order logic)

2.1 What is proposition?

before answering to this question let me show you what is well formed formula(WWF).

Definition 2.1.1. Atomic formula every two obj(conatants and variable) and a binary predicate $(=, \in)$ between them like $x = b, y \in C, ...$
we denote atomic formulas with A_i.

Undefined 2.1.1. logical operators $(\neg, \rightarrow, \wedge, \vee)$ tool that used between atoms to make more complex formulas (all of the opratores are binary except "$\neg$" that is unary).

Definition 2.1.2. well formed formula(WWF) these are formulas.

1. atomic formulas (A_i)
2. if A is a formula then $\neg A$ is a formula too.
3. if A, B are two formulas then $A ** B$ is a formula too.(** can be $\rightarrow, \wedge, \vee$)

for example $(A_0 \rightarrow A_1, \neg A_3)$
we call WWF easily formula and also we denote WWF with $A, B, C, ...$

you can easily see that formula are made up by atomic formulas.in general we say formula is an atomic formula or combination of them.

Definition 2.1.3. open formula formulas that variables used in one of its atomic formulas.

the formulas that are not open call **"closed formulas"** or **"sentece"**. Now question is "are very closed formulas proposition?" (propositions in classic logic are sentences that are true or false?) the answer is unfortunately **No**

Definition 2.1.4. Atomic proposition every atomic formula that can be proven by tools introduced in chapter 1($I =, E =, I \in, E \in, I\perp, E\perp$).
we denote atomic formulas with p_i.

Definition 2.1.5. proposition every WWF formula that its atomes are all atomic propositions. we denote propositions with $p, q, r, \ldots$.

Definition 2.1.6. Substitution we define Substitution for a formula like this.

1. for atomic formulas like (A_i) if obj was not used before just replace it. (example $(a \in A)[b/a]$ is $b \in A$ and if the alternative was in formula. it will not change.)

2. if formula A is complex do it for all atomes.

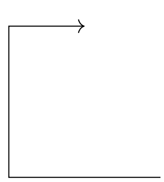
simply means in a formula put an obj like b instead of another like a.we denote it by $A[b/a]$ or $A[\frac{b}{a}]$

Definition 2.1.7. open proposition assume formula of a proposition but Substitute some obj and replace them with variables.we call it open proposition and denote it with capital letter (P, Q, R) showes other types of **predicates** and variables like$(Px, Qxy, ..)$

2.2 Rules of propositions.

Definition 2.2.1. proposition schema(variable proposition) a proposition that is unknown for us and it works as variable.(we show them with $\varphi, \psi, \theta, \ldots$)

Undefined 2.2.1. Assumption for conditional proof(acp) is a tool for assuming a proposition. (any proposition can't get out but it can get in.two lines of assumption must not intersect and every hypothesis that opens must be closed.)



Rule 2.2.1. rules of logical operations $(I \rightarrow, E \rightarrow, I\wedge, E\wedge, I\vee, E\vee, I\neg, E\neg, I\perp, E\perp)$ this rule is scheme it means it works everywhere for every propositions. (thats why we use φ, ψ instead of p, q)

φ $\vdots$ ψ $\therefore \varphi \to \psi \quad I \to$	$\varphi \to \psi$ φ $\therefore \psi \quad E \to$
φ ψ $\therefore \varphi \wedge \psi \quad I\wedge$	$\varphi \wedge \psi$ $\therefore \varphi \quad \therefore \psi \quad E\wedge$
φ $\therefore \varphi \vee \psi \quad \therefore \psi \vee \varphi \quad I\vee$	$\varphi \vee \psi$ φ $\vdots$ θ ψ $\vdots$ θ $\therefore \theta \quad E\vee$
φ $\vdots$ $\bot$ $\therefore \neg\varphi \quad I\neg$	$\neg\varphi$ $\vdots$ $\bot$ $\therefore \varphi \quad E\neg$
φ $\neg\varphi$ $\therefore \bot \quad I\bot$	$\bot$ $\therefore \varphi \quad E\bot$

Definition 2.2.2. $\varphi \leftrightarrow \psi : (\varphi \to \psi) \wedge (\psi \to \varphi)$

Definition 2.2.3. **schema of prove** you've got knowed by now what is schema of a rule. if you want to prove a theorem that it's schema form.you can't prove it directly. here we use a technique calls **"schema of prove"**. it let us prove for all situations at the same time.

Theorem 2.2.1. *(schema)* $(I \leftrightarrow, E \leftrightarrow)$

φ $\vdots$ ψ ψ $\vdots$ φ $\therefore \varphi \leftrightarrow \psi \quad I \leftrightarrow$	$\varphi \leftrightarrow \psi$ $\varphi \qquad \psi$ $\therefore \psi \quad \therefore \varphi \qquad E \leftrightarrow$

/	$I \leftrightarrow$	
1.	φ	
	$\vdots$	
n.	ψ	
n+1.	ψ	
	$\vdots$	
m.	φ	
m+1.	$\varphi \rightarrow \psi$	$I \rightarrow, 1, n$
m+2.	$\psi \rightarrow \varphi$	$I \rightarrow, n+1, m$
m+3.	$(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$	$I\wedge, m+1, m+2$
m+4.	$\varphi \leftrightarrow \psi$	$Def., m+3$

/	$E \leftrightarrow$	
1.	$\varphi \leftrightarrow \psi$	
2.	φ	
3.	$(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$	$Def.1$
4.	$\varphi \rightarrow \psi$	$E \wedge 3$
5.	ψ	$E \rightarrow 1, 4$

Example 2.2.1. $a_0 \notin \{a_1, a_2 ... a_n\}$

1.		$a_0 \in \{a_1, a_2...a_n\}$	acp
2.		$\{a_1, a_2...a_n\}$	ass
3.	a		fiss $1, 2$
4.		$\perp$	$I\perp 2, 3$
5.		$a_0 \notin \{a_1, a_2...a_n\}$	$I \in$
6.			

Example 2.2.2. for conatants p,q,r we can prove these.Proof Is Left As An Exercise To The Reader.😃

1. $p \to q, q \to r / \therefore p \to r$
2. $q / \therefore p \to q$
3. $(p \wedge q) \wedge r / \therefore p \wedge (q \wedge r)$
4. $p \wedge q / \therefore q \wedge p$
5. $p \to q / \therefore (p \wedge r) \to q$
6. $(p \wedge q) \to r / \therefore p \to (q \to r)$
7. $p \to (q \to r) / \therefore (p \wedge r) \to q$
8. $p \vee q / \therefore q \vee p$
9. $(p \vee q) \vee r / \therefore p \vee (q \vee r)$
10. $p \wedge (q \vee r) / \therefore (p \wedge q) \vee (p \wedge r)$
11. $(p \to q) \wedge (r \to s) / \therefore (p \vee r) \to (q \vee s)$
12. $p \to q, \neg q / \therefore \neg p$
13. $p \to q / \therefore \neg p \vee q$

Exercise 2.2.1. for conatants p,q,r prove.

1. $p / \therefore p$
2. $/ \therefore p \vee \neg p$
3. $\neg p, p \vee q / \therefore q$
4. $\neg p / \therefore p \to q$
5. $p \to q \therefore / \therefore \neg q \to \neg p$

6. $(p \vee q) \wedge (p \vee r) \therefore / \therefore (p \vee (q \wedge r))$
7. $\neg p \wedge \neg q \therefore / \therefore \neg(p \vee q)$
8. $\neg p \vee \neg q \therefore / \therefore \neg(p \wedge q)$
9. $(\neg p \to q) / \therefore ((p \to q) \to q)$
10. $/ \therefore (p \wedge q \to r) \wedge (p \wedge s \to r) \to (p \wedge (q \vee s) \to r)$
11. $/ \therefore (p \wedge q \to r) \vee (p \wedge s \to r) \to (p \wedge (q \wedge s) \to r)$
12. $/ \therefore (p \to (q \to r)) \to ((p \to q) \to (p \to r))$
13. $/ \therefore (p \to q) \to ((q \to r) \to (p \to r))$

Attention 2.2.1. we can rewrite **Example** and **Exercise** propositions and proofs with proposition schemas and schema of proves.so they are all rules.

Chapter 3

first-order logic

3.1 what is predicate?

Definition 3.1.1. Big AND and OR assume a set like $\{a_0, a_1, a_2, ..., a_n\}$ we define **big AND** like this.

$$\bigwedge_{i=0}^{n} Pa_i : Pa_0 \wedge Pa_1 \wedge Pa_2 \wedge ... \wedge Pa_n$$

and **big OR**

$$\bigvee_{i=0}^{n} Pa_i : Pa_0 \vee Pa_1 \vee Pa_2 \vee ... \vee Pa_n$$

Definition 3.1.2. Universe of discourse we specify a set like $\{a_0, a_1, a_2, ..., a_n\}$ as universe of a Logic world.

Rule 3.1.1. use of open propositions assume universe of discourse is $\{a_0, a_1, a_2, ..., a_n\}$

$c.\quad Px$

means :

$$\begin{array}{ll} c. & Pa_0 \\ c+1. & Pa_1 \\ c+2. & Pa_2 \\ & \vdots \\ c+n. & Pa_n \end{array}$$

if Px was been assemed then like this

$\rightarrow c.\quad Px$

$\vdots$

means :

$\rightarrow c_0.\quad Pa_0$

$\vdots$

$\rightarrow c_1.\quad Pa_1$

$\vdots$

$\rightarrow c_2.\quad Pa_2$

$\vdots$

$\vdots$

$\rightarrow c_n.\quad Pa_n$

$\vdots$

Definition 3.1.3. Universal quantification assume universe of discourse is $\{a_0, a_1, a_2, ..., a_n\}$ we define $\forall x Px$

$$\forall x Px : \bigwedge_{i=0}^{n} Pa_i$$

Definition 3.1.4. Existential quantification assume universe of discourse is $\{a_0, a_1, a_2, ..., a_n\}$ we define $\exists x Px$

$$\exists x Px : \bigvee_{i=0}^{n} Pa_i$$

Definition 3.1.5. predicate schema(variable predicate) a predicate that is unknown for us and it works as variable.(we show them with $\Phi, \Psi, \Theta, ...$)

as you can see $\forall x \Phi x$ and $\exists x \Phi x$ are sentences.

Definition 3.1.6. Free variables variables that are not bound by a quantifier. (you can easily see that every formula with no free variables are sentences) (previously we sayed that every sentece is not necessary a proposition)

3.2 Rules of predicates.

Theorem 3.2.1. (*schema*) $(I\forall, E\forall, I\exists, E\exists)$

Rules	Terms and Conditions
$\dfrac{\Phi y}{\therefore \forall x \Phi x \quad I\forall}$	1. y is a variable. 2. Φy must not be assumed.
$\dfrac{\forall x \Phi x}{\therefore \Phi a \quad E\forall}$	1. a is an obj in universe of discourse.(it can be constant or variable.)
$\dfrac{\Phi a}{\therefore \exists x \Phi x \quad I\exists}$	1. a is an obj in universe of discourse.(it can be conatant or variable.)
$\exists x \Phi x$ Φy $\vdots$ θ $\therefore \theta \quad E\exists$	1. y is a variable. 2. Φy must not be assemed in previous lines.(except ones that are closed) 3. y is not free in θ

$$
\begin{array}{rll}
/ & I\forall & \\
1. & \Phi y & ass \\
2. & \Phi a_0 & Def.1 \\
 & \vdots & \\
n+2. & \Phi a_n & Def.1 \\
n+3. & \bigwedge\limits_{i=0}^{n} \Phi a_i & I\wedge, 2, 3, ..., n+2 \\
n+4. & \forall x \Phi x & Def.n+3
\end{array}
$$

$$
\begin{array}{rll}
/ & E\forall & \\
1. & \forall x \Phi x & ass \\
2. & \bigwedge\limits_{i=0}^{n} \Phi a_i & Def.1 \\
3. & \Phi a_i & E\wedge 2
\end{array}
$$

$$
\begin{array}{rll}
/ & I\exists & \\
1. & \exists x \Phi x & ass \\
2. & \bigvee\limits_{i=0}^{n} \Phi a_i & Def2 \\
3. & \Phi a_0 & \\
 & \vdots & \\
m. & \theta & \\
m+1. & \Phi a_1 & \\
 & \vdots & \\
l. & \theta & \\
 & \vdots & \\
l+1. & \Phi a_n & \\
 & \vdots & \\
r. & \theta & \\
r+1. & \theta & E\vee 1
\end{array}
$$

$$
\begin{array}{rll}
/ & I\exists & \\
1. & \Phi a_i & \\
2. & \bigvee\limits_{i=0}^{n} \Phi a_i & I\vee 1 \\
3. & \exists x \Phi x & Def2
\end{array}
$$

Example 3.2.1. for conatants P,Q,R we can prove these.Proof Is Left As An Exercise To The Reader.😃

1. $\forall x(Px \wedge Qx) \therefore / \therefore \forall x Px \wedge \forall x Qx$

2. $\exists x(Px \vee Qx) \therefore / \therefore \exists xPx \vee \exists xQx$
3. $\forall xPx \vee \forall xQx / \therefore \forall x(Px \vee Qx)$
4. $\exists x(Px \wedge Qx) / \therefore \exists xPx \wedge \exists xQx$
5. $(\exists xPx \to \exists xQx) / \therefore \exists x(Px \to Qx)$
6. $\forall xPx \wedge Q \therefore / \therefore \forall x(Px \wedge Q)$
7. $\forall xPx \vee Q \therefore / \therefore \forall x(Px \vee Q)$
8. $\exists xPx \wedge Q \therefore / \therefore \exists x(Px \wedge Q)$
9. $\exists xPx \vee Q \therefore / \therefore \exists x(Px \vee Q)$
10. $(Q \to \forall xPx) \therefore / \therefore \forall x(Q \to Px)$
11. $(Q \to \exists xPx) \therefore / \therefore \exists x(Q \to Px)$
12. $(\forall xPx \to Q) \therefore / \therefore \exists x(Px \to Q)$
13. $(\exists xPx \to Q) \therefore / \therefore \forall x(Px \to Q)$
14. $\exists x \neg Px \therefore / \therefore \neg \forall xPx$
15. $\forall x \neg Px \therefore / \therefore \neg \exists xPx$

Exercise 3.2.1. for conatants P,Q,R prove.

1. $\exists x(Px \wedge \exists yQy \to \forall zRz) / \therefore \forall y \forall z(\forall xPx \wedge Qy \to Rz)$
2. $\forall x \exists y(Px \wedge Qy \to \forall zRz) / \therefore \forall z(\exists xPx \wedge \forall yQy \to Rz)$
3. $(\forall xPx \to \exists yQy) / \therefore \exists x \exists y(Px \to Qy)$
4. $\forall x \forall z \exists y \exists w(Pyz \vee Qz \to Rxw) / \therefore (\exists z \forall yPyz \vee \exists zQz) \to \forall x \exists wRxw$
5. $\exists xPx \vee \exists x(Qx \wedge Rx) / \therefore \exists x(Px \vee Qx) \wedge \exists x(Px \vee Rx)$
6. $\exists x \exists z \forall y(Pxy \to Qz \vee Rxz) / \therefore \forall x \exists yPxy \to (\exists zQz \vee \exists x \exists zRxz)$
7. $\forall x \exists y(\exists zPxyz \wedge Qxy) \vee \forall x \exists y \exists z(Pxyz \wedge Rxy) / \therefore \forall x \exists y \exists z(Pxyz \wedge (Qxy \vee Rxy))$

Attention 3.2.1. we can rewrite **Example** and **Exercise** propositions and proofs with proposition schemas and schema of proves.so they are all rules.

Chapter 4

The beginning of set theory

4.1 Axioms of set theory

we emphasis before that set has no definition.but we can describe what set is with rules and axioms.

Axiom 4.1.1. Extensionality Two sets are equal (are the same set) if they have the same elements.

$$\forall X, Y(\forall x(x \in X \leftrightarrow x \in Y) \leftrightarrow X = Y)$$

Definition 4.1.1. Subset a subset is a set where every element is also an element of another, larger set called the superset

$$A \subseteq B : \forall x(x \in A \to x \in B)$$

so we can rewrite axiom of Extensionality like this.

$$\forall X, Y((X \subseteq Y \wedge Y \subseteq X) \leftrightarrow X = Y)$$

Theorem 4.1.1. (*schema*) $(I \subseteq, E \subseteq)$

$\quad x \in A$ $\quad \vdots$ $\quad x \in B$ $\therefore A \subseteq B \quad I \subseteq$	$A \subseteq B$ $a \in A$ $\therefore a \in B \quad E \subseteq$

/	$I \subseteq$	
1.	$x \in A$	acp
	$\vdots$	
n.	$x \in B$	
$n+1$.	$x \in A \to x \in B$	$I \to, 1, n$
$n+2$.	$\forall x(x \in A \to x \in B)$	$I\forall, n+1$
$n+3$.	$A \subseteq B$	Def.$n+2$

/	$E \subseteq$	
2.	$A \subseteq B$	ass
3.	$a \in A$	ass
4.	$\forall x(x \in A \to x \in A)$	def.1
5.	$a \in A \to a \in B$	$E\forall, 4$
6.	$a \in B$	$E \to 3, 5$

Theorem 4.1.2. (*schema*) $(I =, E =)$

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<tr>
<td>
$x \in A$
$\vdots$
$x \in B$
<hr>
$x \in B$
$\vdots$
$x \in A$
<hr>
$\therefore A = B \quad I =$
</td>
<td>
$A = B$
$a \in A$
<hr>
$\therefore a \in B \quad E =$
</td>
</tr>
</table>

/	$I\subseteq$	
1.	$x \in A$	acp
	$\vdots$	
n.	$x \in B$	
$n+1$.	$A \subseteq B$	$I\subseteq, 1, n$
$n+2$.	$x \in B$	acp
	$\vdots$	
m.	$x \in A$	
$m+1$.	$B \subseteq A$	$I\subseteq, n+2, m$
$m+2$.	$\forall X, Y((X \subseteq Y \land Y \subseteq X) \leftrightarrow X = Y)$	Extensionality
$m+3$.	$(A \subseteq B \land B \subseteq A) \leftrightarrow A = B$	$E\forall, m+2$
$m+4$.	$A \subseteq B \land B \subseteq A$	$I\land, n+1, m+1$
$m+5$.	$A = B$	$E\leftrightarrow, m+3, m+4$

/	$E\subseteq$	
1.	$A = B$	
2.	$a \in A$	
3.	$\forall X, Y((X \subseteq Y \land Y \subseteq X) \leftrightarrow X = Y)$	Extensionality
4.	$(A \subseteq B \land B \subseteq A) \leftrightarrow A = B$	$E\forall, 3$
5.	$A \subseteq B \land B \subseteq A$	$E\leftrightarrow, 1, 4$
6.	$A \subseteq B$	$E\land 5$
7.	$a \in B$	$E\subseteq, 2, 6$

Axiom 4.1.2. Weak comprehension this axioms says for every open proposition with just one variable like ϕx there is a set such that every x is a member of that set if and only if it holds ϕx.

$$\exists X \forall x(\Phi x \leftrightarrow x \in X)$$

by axiom of extentionality we can show that y is uniq. so we denote it as $\{x|\Phi x\}$.

this axiom show that why Russell's paradox doesn't happened here because if ou assume $R = \{x|x \notin x\}$ is a set.you can't prove or disprove $R \in R$. (we don't need to prove that now.we will prove it in **'MetaLogic'**.)

4.2 Definitions of set theory

Let's talk about some definitions in set theoty.These definitions will come in handy later.

Definition 4.2.1. Empty set also known as the null set, is the unique mathematical set containing no elements at all. (uniqueness can be proven.)

$$\varnothing := \{x \mid \perp\}$$

therefore.

$$\forall x(x \in \varnothing \leftrightarrow \perp)$$

therefore.

$$\forall x(x \notin \varnothing)$$

Lemma 4.2.1. $A = \varnothing \therefore / \therefore \forall y(y \notin A)$

$\forall y(y \notin A) / \therefore A = \varnothing$

1.	$\forall y(y \notin A)$	ass
2.	$y \in A$	acp
3.	$y \notin A$	$E\forall, 1$
4.	$\perp$	$I\perp, 2, 3$
5.	$y \in \varnothing$	$E\perp, 4$
6.	$y \in \varnothing$	acp
7.	$\forall y(y \notin \varnothing)$	Def.
8.	$y \notin \varnothing$	$E\forall, 7$
9.	$\perp$	$E \leftrightarrow, 6, 8$
10.	$y \in A$	$I\perp, 9$
11.	$y \in A \leftrightarrow y \in \varnothing$	$I \leftrightarrow, 5, 10$
12.	$\forall y(y \in A \leftrightarrow y \in \varnothing)$	$I\forall, 11$
13.	$\forall X, Y(\forall y(y \in X \leftrightarrow y \in Y) \leftrightarrow X = Y)$	extentionality
14.	$(\forall y(y \in A \leftrightarrow y \in \varnothing) \leftrightarrow A = \varnothing)$	$E\forall, 13$
15.	$A = \varnothing$	$E \leftrightarrow, 12, 14$

$A = \varnothing / \therefore \forall y (y \notin A)$

1.	$A = \varnothing$	ass
2.	$\forall X, Y (\forall y (y \in X \leftrightarrow y \in Y) \leftrightarrow X = Y)$	extentionality
3.	$(\forall y (y \in A \leftrightarrow y \in \varnothing) \leftrightarrow A = \varnothing)$	$E\forall, 2$
4.	$\forall y (y \in A \leftrightarrow y \in \varnothing)$	$E \leftrightarrow, 1, 3$
5.	$y \in A$	acp
6.	$y \in A \leftrightarrow y \in \varnothing$	$E\forall, 4$
7.	$y \in \varnothing$	$E \leftrightarrow, 5, 6$
8.	$\forall y (y \notin \varnothing)$	Def.
9.	$y \notin \varnothing$	$E\forall, 8$
10.	$\perp$	$E \leftrightarrow, 7, 9$
11.	$y \notin A$	$I\neg, 5, 10$
12.	$\forall y (y \notin A)$	$I\forall, 11$

Theorem 4.2.1. *(schema)* $(I\varnothing \neq, E\varnothing \neq, I\varnothing =, E\varnothing =)$

$\dfrac{a \in A}{A \neq \varnothing} \quad I\varnothing \neq$	$\begin{array}{ll} A \neq \varnothing & \\ \quad x \in A & \\ \quad \vdots & \\ \quad \theta & \\ \hline \theta & E\varnothing \neq \end{array}$
$\begin{array}{ll} \quad x \in A & \\ \quad \vdots & \\ \quad \perp & \\ \hline A = \varnothing & I\varnothing = \end{array}$	$\dfrac{A = \varnothing}{a \notin A} \quad E\varnothing =$

1.	$a \in A$	
2.	$\exists x(x \in A)$	$I\exists, 1$
3.	$\neg(\forall x(x \notin A))$	De Morgan 2
4.	$A \neq \varnothing$	lemma 4.4.1, 3

1.	$A \neq \varnothing$	
2.	$\neg(\forall x(x \notin A))$	lemma 4.4.1, 2
3.	$\exists x(x \in A)$	De Morgan 2
4.	$x \in A$	acp
	$\vdots$	
n.	θ	
$n+1$.	θ	$E\exists, 4, n$

1.	$x \in A$	acp
	$\vdots$	
n.	$\perp$	
$n+1$.	$x \notin A$	$I\neg, n$
$n+2$.	$\forall x(x \notin A)$	$I\forall, n+1$
$n+3$.	$A = \varnothing$	lemma 4.4.1, 1

1.	$A = \varnothing$	
2.	$\forall x(x \notin A)$	lemma 4.4.1, 1
1.	$a \notin A$	$E\forall 2$

Example 4.2.1. $/\therefore \forall X(\varnothing \subseteq X)$

1.	$x \in \varnothing$	acp
2.	$\forall x(x \notin \varnothing)$	lemma 4.4.1
3.	$x \notin \varnothing$	$E\forall, 2$
4.	$\perp$	$I\perp, 1, 3$
5.	$x \in X$	$E\perp, 4$
6.	$\varnothing \subseteq X$	$I\subseteq, 1, 5$
7.	$\forall X(\varnothing \subseteq X)$	lemma 4.4.1, 1

Example 4.2.2. $A \subseteq B, B \subseteq C / \therefore A \subseteq C$

1.	$A \subseteq B$	ass
2.	$B \subseteq C$	ass
3.	$x \in A$	acp
4.	$x \in B$	$E\subseteq, 1, 3$
5.	$x \in C$	$E\subseteq, 2, 4$
6.	$A \subseteq C$	$I\subseteq, 3, 5$

Definition 4.2.2. pairing set

$$\{a_0, a_1, a_2, ..., a_n\} := \{x \mid x = a_0 \lor x = a_1 \lor x = a_2 \lor ... \lor x = a_n\}$$

therefore.

$$\forall x(x \in \{a_0, a_1, a_2, ..., a_n\} \leftrightarrow (x = a_0 \lor x = a_1 \lor x = a_2 \lor ... \lor x = a_n))$$

Theorem 4.2.2. $(schema)$ $(I\{\}, E\{\})$

$a = a_i$ $\therefore a \in \{a_0, a_1, a_2, ..., a_n\}$ $I\{\}$	$a \in \{a_0, a_1, a_2, ..., a_n\}$ $a = a_0$ $\vdots$ θ $\vdots$ $a = a_n$ $\vdots$ θ $\therefore \theta$ $E\{\}$

Proof Is Left As An Exercise To The Reader.😀

Definition 4.2.3. Union and Intersection

Intersection:

$$A \cap B := \{x \mid x \in A \land x \in B\}$$

therefore.

$$\forall x(x \in A \cap B \leftrightarrow (x \in A \land x \in B))$$

Union:

$$A \cup B := \{x \mid x \in A \vee x \in B\}$$

therefore.

$$\forall x(x \in A \cup B \leftrightarrow (x \in A \vee x \in B))$$

Theorem 4.2.3. (*schema*) $(I\cap, E\cap, I\cup, E\cup)$

$\begin{array}{l} a \in A \\ a \in B \\ \hline \therefore a \in A \cap B \quad I\cap \end{array}$	$\begin{array}{l} a \in A \cap B \\ \hline \therefore a \in A \quad \therefore a \in B \quad E\cap \end{array}$
$\begin{array}{l} a \in A \\ \hline \therefore a \in A \cup B \quad \therefore a \in B \cup A \quad I\cup \end{array}$	$\begin{array}{l} a \in A \cup B \\ \rightarrow a \in A \\ \vdots \\ \theta \\ \hline \rightarrow a \in B \\ \vdots \\ \theta \\ \hline \therefore \theta \qquad E\cup \end{array}$

Proof Is Left As An Exercise To The Reader.😀

Definition 4.2.4. Big Union and Intersection
Intersection:

$$\bigcap A := \{x \mid \forall X(X \in A \rightarrow x \in X)\}$$

therefore.

$$\forall x(x \in \bigcap A \leftrightarrow (\forall X(X \in A \rightarrow x \in X)))$$

Union:

$$\bigcup A := \{x \mid \exists X(X \in A \wedge x \in X)\}$$

therefore.

$$\forall x(x \in \bigcup A \leftrightarrow (\exists X(X \in A \wedge x \in X)))$$

Theorem 4.2.4. (*schema*) $(I\bigcap, E\bigcap, I\bigcup, E\bigcup)$

$\begin{array}{l} \rightarrow X \in A \\ \vdots \\ a \in X \\ \hline \therefore a \in \bigcap A \quad I\bigcap \end{array}$	$\begin{array}{l} a \in \bigcap A \\ B \in A \\ \hline \therefore a \in B \quad E\bigcap \end{array}$
$\begin{array}{l} a \in B \\ B \in A \\ \hline \therefore a \in \bigcup A \quad I\bigcup \end{array}$	$\begin{array}{l} a \in \bigcup A \\ \rightarrow a \in X \wedge X \in A \\ \vdots \\ \theta \\ \hline \therefore \theta \qquad\qquad E\bigcup \end{array}$

Proof Is Left As An Exercise To The Reader.😀

Definition 4.2.5. Complement

$$A^c := \{x \mid x \notin A\}$$

therefore.

$$\forall x(x \in A^c \leftrightarrow x \notin A)$$

Definition 4.2.6. Universal set(we denote it as V.it simply means the set of all sets)

$$V := \{x \mid \top\}$$

therefore.

$$\forall x(x \in V \leftrightarrow \top)$$

therefore.

$$\forall x(x \in V)$$

Definition 4.2.7. Subtraction

$$A - B := \{x \mid x \in A \wedge x \notin B\}$$

therefore.

$$\forall x(x \in A - B \leftrightarrow (x \in A \wedge x \notin B))$$

Theorem 4.2.5. $(schema)$ $(I-, E-)$

$$\frac{a \in A \quad a \notin B}{\therefore a \in A - B} \quad I-$$	$$\frac{a \in A - B}{\therefore a \in A \quad \therefore a \notin B} \quad E-$$

Proof Is Left As An Exercise To The Reader.😀

Lemma 4.2.2. $/\therefore A - B = A \cap B^c$

Proof Is Left As An Exercise To The Reader.😀

Definition 4.2.8. Powerset

$$P(A) := \{X \mid X \subseteq A\}$$

therefore.

$$\forall X(X \in P(A) \leftrightarrow (X \subseteq A))$$

Theorem 4.2.6. $(schema)$ (IP, EP)

$$\frac{A \subseteq B}{\therefore A \in P(B)} \quad IP$$	$$\frac{A \in P(B)}{\therefore A \subseteq B} \quad EP$$

Proof Is Left As An Exercise To The Reader.😀