Span, linear independence, dimension

Professor K. A. Ribet

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Office Hours

885 Evans Hall

Mondays, 1:30–3 PM
Thursdays, 10:30–noon

Office full $\Longrightarrow$ possible move to a nearby classroom

Lunch

I plan to come to the DCs at least once per week. There will be official Residential Life "lunches with Professor Ribet" at noon at Foothill DC on September 18, September 26, October 3 and October 9. There will also be additional lunch gatherings at DCs and the Faculty Club.

Gatherings are optional and not part of Math 110, but I'll continue to list them on slides for those who are interested. Also, you can send me email to subscribe to email announcements.

- Faculty Club lunch Monday, September 15 at noon
- First official Lunch with Prof. Ribet on Thursday, September 18 after office hour ends.

Wednesday

We made a list of items concerning lists and talked through all of the items. Here's a quick recap:

- Lists
- Span of a List
- Linear maps (reminder)
- Linear map defined by a list
- Linear independence
- Lists that span
- Bases (certain lists)
- Finite-dimensional spaces

The only thing wrong with this list is that it wasn't numbered. My bad! Lists be like

$$v_1, \ldots, v_\ell, \quad \ell \geq 0, \quad \text{all } v_j \in V.$$

Now it's Friday, so let's go!

Lemma (2.19)

Let $v_1, \ldots, v_\ell$ be a linearly dependent list of vectors of V. Then there is some index k such that v_k lies in the span of $v_1, \ldots, v_{k-1}$. For this k, the span of the list with v_k deleted is the same as the span of the list $v_1, \ldots, v_\ell$.

A nerdy remark is that if $\ell = 0$, then the list is linearly independent, so there is nothing to prove. Also, if $v_1 = 0$, then we can take $k = 1$ and note that 0 is in the span of the empty list (of length $k - 1 = 0$). Thus we can prove the lemma under the assumption that ℓ is at least 1 and v_1 is nonzero. If $\ell = 1$ and v_1 is nonzero, then the list v_1 is linearly independent and there is nothing to prove. Thus we can and will suppose $\ell \geq 2$ and that v_1 is nonzero.

Proof of the lemma

Lemma (2.19)

Let $v_1, \ldots, v_\ell$ be a linearly dependent list of vectors of V. Then there is some index k such that v_k lies in the span of $v_1, \ldots, v_{k-1}$. For this k, the span of the list with v_k deleted is the same as the span of the list $v_1, \ldots, v_\ell$.

The start of the proof is to note that linear dependence means that there are scalars λ_j, not all equal to 0, so that

$$0 = \lambda_1 v_1 + \cdots + \lambda_\ell v_\ell.$$

Let k be the largest index so that λ_k is nonzero. Then

$$0 = \lambda_1 v_1 + \cdots + \lambda_k v_k, \quad \lambda_k \neq 0.$$

The equation

$$v_k = -1/\lambda_k \left(\lambda_1 v_1 + \cdots + \lambda_{k-1} v_{k-1} \right)$$

shows that $v_k \in \text{span}(v_1, \ldots, v_{k-1})$.

Proof of the lemma

Lemma (2.19)

Let $v_1, \ldots, v_\ell$ be a linearly dependent list of vectors of V. Then there is some index k such that v_k lies in the span of $v_1, \ldots, v_{k-1}$. For this k, the span of the list with v_k deleted is the same as the span of the list $v_1, \ldots, v_\ell$.

It remains to show that the span of $v_1, \ldots, v_\ell$ does not shrink if we remove v_k from the list. Each element of the span is a linear combination

$$a_1 v_1 + \cdots + a_{k-1} v_{k-1} + a_k v_k + a_{k+1} v_{k+1} + \cdots + a_\ell v_\ell.$$

In this expression, replace v_k by $-1/\lambda_k \left(\lambda_1 v_1 + \cdots + \lambda_{k-1} v_{k-1} \right)$. After collecting terms, we see that the linear combination $a_1 v_1 + \cdots + a_\ell v_\ell$ has an alternative expression as a linear combination of the vectors in the list other than v_k.

Linearly ind. list is no longer than spanning list

Theorem (2.22, p. 35)

In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

I've copied Axler's 2.22 and will now copy his notation for your convenience. Let $w_1, \ldots, w_n$ be a list of vectors of V that spans V. Let $u_1, \ldots, u_m$ be a linearly independent list. The theorem asserts

$$m \overset{?}{\leq} n,$$

and this is what we must prove.

Lin. independent list is no longer than spanning list

Theorem (2.22, p. 35)

If $w_1, \ldots, w_n$ is a spanning list and $u_1, \ldots, u_m$ is a linearly independent list, then $m \leq n$.

The proof involves *replacement*. We can do m replacements of a w vector by a u vector. There are only n w vectors to replace; thus if we are able to make more replacements than there are objects to replace, we're in big trouble.

This sounds nuts, but really works.

Linearly independent list is no longer than span list

I'll explain this with small numbers: $n = 2$, $m = 3$. We assume that V is spanned by w_1, w_2 and consider a linearly independent list $u_1, \ldots, u_m$. The aim is to prove $m \geq 2$. We want to see what is wrong with the contrary case $m > 2$. It's enough to show that there cannot be a linearly independent list with three vectors. If that's true, there can't be a linearly independent list with four or more vectors.

Linearly independent list is short

Again: assume that V is spanned by w_1, w_2 and that u_1, u_2, u_3 is linearly independent.

Because w_1, w_2 spans, u_3 is a linear combination of w_1, w_2:

$$u_3 = aw_1 + bw_2.$$

By linear independence of the 1-element list u_3, a and b are not both 0. We can and will assume b is nonzero (explanation on site). Multiplying by $1/b$ and moving terms around, we find that w_2 is in the span of w_1 and u_3. So is w_1. Thus V is spanned by w_1, u_3.

Next step: u_2 is a linear combination of w_1, u_3:

$$u_2 = \lambda w_1 + \mu u_3.$$

By linear independence of u_2, u_3, λ is nonzero. Repeat previous manipulation to show that V is spanned by u_2, u_3. In particular, u_1 is a linear combination of u_2, u_3. This is not possible because u_1, u_2, u_3 is linearly independent.

Subspaces of finite-dimensional spaces

Theorem (2.25)

Every subspace of a finite-dimensional vector space is finite-dimensional.

Assume that U is a subspace of V and that V is finite-dimensional. This means that V is spanned by a list of vectors of some length n. If $U = \{0\}$, it's finite-dimensional.

Assume $U \neq \{0\}$. Then U has a nonzero vector u_1. Is U spanned by u_1? If so, done! If not, there is $u_2 \in U$ not in the span of u_1. This means that u_1, u_2 is linearly independent. Does this list span? If so, excellent! If not, there is a u_3 not in the span; then u_1, u_2, u_3 is linearly independent. Does it span? If so, fabulous!! If not, there's a linearly independent list in U of length 4.

We can't keep going like this because a linearly independent list can have no more than n elements.

Bases

Recall that a *basis* of *V* is a linearly independent list that spans *V*. To say that $v_1, \ldots, v_n$ is a basis is to say that the linear map

$$\mathbf{F}^n \to V, \quad (a_1, \ldots, a_n) \mapsto \sum_j a_j v_j$$

is 1-1 and onto. In words, this means that every vector in *V* can be written uniquely as a linear combination of $v_1, \ldots, v_n$.

Proposition

If a vector space has two bases, they are of the same length.

In other words, if *V* has a basis of length *n* and also a basis of length *m*, then $m = n$.

A basis is both a spanning list and a linearly independent list. View the basis of length *m* as a spanning list and the basis of length *n* as a linearly independent list. We get $n \leq m$. Reversing roles gives $m \leq n$. Thus $m = n$.

Spanning lists shrink to bases

Theorem (2.30)

Every spanning list in a vector space can be reduced to a basis of the vector space.

This means that if $v_1, \ldots, v_m$ spans, we can get a basis of V by ejecting some of the v_j from the list.

Refining a spanning list to get a basis

Theorem (2.30)

Every spanning list in a vector space can be reduced to a basis of the vector space.

We can view the proof as an induction on the length of the spanning list $v_1, \ldots, v_m$. If $m = 0$, $V = \{0\}$ and the spanning list (which is empty) is also a basis. If $m = 1$, the list is a single vector, say v. Because it spans, $V = \mathbf{F} \cdot v$. If v is nonzero, the list v is linearly independent and represents a basis. If $v = 0$, remove it to get the empty list, which is a basis of $V = \{0\}$.

Refining a spanning list to get a basis

In the induction step, take a spanning list $v_1, \ldots, v_m$ of length > 1. If it's linearly independent, it's a basis and we're done.

If it's linearly dependent, some vector in the list is a linear combination of the previous vectors. We can chuck it without disturbing the span of the list — which is all of V. The pruned list is still a spanning list, but now it has length $m - 1$. Assuming the desired result for lists of that length, we deduce that the pruned list can be pruned further, if necessary, to yield a basis of V.

Finite-dimensional spaces have bases

Corollary (2.31)

If *V* is finite-dimensional, it has a basis.

To prove it, take a spanning list and prune it if necessary to get a basis.

Extending linearly independent lists

Proposition (2.32)

Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.

Start with a linearly independent list $v_1, \ldots, v_n$ in a finite-dimensional vector space V. The idea is to extend the list incrementally until it spans (and thus is a basis).

Ask first whether $v_1, \ldots, v_n$ is already a basis. If not, there is $v_{n+1} \notin \text{span}(v_1, \ldots, v_n)$. The key is that $v_1, \ldots, v_n, v_{n+1}$ is then again linearly independent (to be explained in class). Does it span? If not, it can be grown a second time to a longer linearly independent list $v_1, \ldots, v_n, v_{n+1}, v_{n+2}$.

list doesn't span? $\implies$ it can grow.

However, a linearly independent list can't be longer than $\dim V$. Growth has to stop. When it does, we have a linearly independent spanning list (= a basis).

California

PLEASE HANG UP COATS

They do not need to be washed after every use.

Thank You & Go Bears!