Quotients

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Homework #5

The problems in §3D have been moved to HW #6. Less to do over the next three days.

Office hour

Tomorrow at 10:30 (as usual)

Optional lunch meetings

Today at 11:45 at Café 3

Friday at noon at Foothill Dining (official Residential Life event)

Quotient spaces

Let U be a subspace of a vector space V. Last Friday, we described a set V/U, along with an addition law on this set. We will continue by completing the description of the vector space structure associated with V/U.

As mentioned on Friday, V/U comes equipped with a surjective linear map

$$\pi : V \to V/U$$

whose null space is U. If V has finite dimension, then rank–nullity implies that

$$\dim V/U = \dim V - \dim U.$$

Quotient spaces

If V is an $\mathbf{F}$-vector space and U is a subspace of V, V/U is the set of *translates* of U by elements of V:

$$v + U := \{\, v + u \mid u \in U\}.$$

These are subsets of V but typically not subspaces; for example, $v + U$ contains 0 if and only if v is an element of U

Quotient spaces

We saw the proof of the following result on Friday of last week.

Proposition

For v and v' in V and U a subspace of V,

$$v + U = v' + U \iff v - v' \in U.$$

Discrete math courses like Math 55 study *equivalence relations*. A natural equivalence relation on the set V has $v \sim v'$ if and only if $v - v' \in U$.

Proposition

The set $v + U$ is the equivalence class of v.

A vector v' is in $v + U$ if and only if it is $v + u$ for some $u \in U$, which is true if and only if $v' - v$ is in U.

Quotient spaces

As is true for equivalence relations in general, two translates (i.e., equivalence classes) are either identical or disjoint.

The equivalence classes fill up V (because v is in the translate $v + U$).

Thus every vector in V belongs to exactly one equivalence class.

The map $\pi : V \to V/U$ sends v to the translate of U that contains v:

$$\pi(v) = v + U \in V/U.$$

It's a surjective function because V/U is the set of all $v + U$.

Addition of two translates

Far, we have defined V/U as a set. We wish to turn it into a vector space.

Addition (defined last Friday):

$$(v_1 + U) + (v_2 + U) := (v_1 + v_2) + U.$$

A key point is that this addition is well defined. Indeed, imagine that $v_1 + U$ is also $v_1' + U$. Is it true that $(v_1 + v_2) + U$ is also $(v_1' + v_2) + U$? Yes because $v_1' - v_1$ is in U, and therefore so is $(v_1' + v_2) - (v_1 + v_2)$.

Scalar multiplication

Define $\lambda \cdot (v + U) := \lambda v + U$. This is again well defined because if $v + U = v' + U$, then $v' - v$ is in U, so that $\lambda(v' - v) = \lambda v' - \lambda v$ is in U.

Are the axioms verified?

Yes because they're verified for addition of vectors together with scalar multiplication of vectors. The operations for V/U are derived from the operations for V by simple non-threatening formulas.

More about π

The function

$$\pi : V \to V/U, \quad v \mapsto v + U$$

is a *linear* map because of the vector space operations that we defined for V/U. Its null space is the set of vectors $v \in V$ such that $v + U = 0 + U$ (which is the 0 element of V/U, by the way). That set is U.

Dimensions

If V has finite dimension, then

$$\dim V = \dim \operatorname{null} \pi + \dim \operatorname{range} \pi = \dim U + \dim V/U.$$

Thus

$$\dim V/U = \dim V - \dim U.$$

Another perspective: if $v_1 + U, \ldots, v_t + U$ is a basis of V/U and if $u_1, \ldots, u_d$ is a basis of U, then

$$u_1, \ldots, u_d; v_1, \ldots, v_t$$

is a basis of V. (The semicolon is my way of emphasizing the separation between vectors that came from two different bins.)

Relation to complements

Suppose that $U \subseteq V$ is a subspace and that $X \subseteq V$ is a vector space complement to U in V in the sense that $V = U \oplus X$. Then the restriction of π to X is an isomorphism

$$X \xrightarrow{\sim} V/U.$$

It's 1-1 because its null space is $U \cap X$, which is $\{0\}$. It's onto because each $v \in V$ is a sum $u + x$ with $u \in U, x \in X$. With v written this way, $\pi v = \pi u + \pi x = \pi x$.

Thus V/U behaves like a choice-free complement to U that lives externally to U and V.

Relation to linear maps that are 0 on U

Let W be a vector space. If $S : V//U \to W$ is a linear map, $S \circ \pi$ is a linear map $V \to W$ whose resriction to U is 0. View $S \mapsto S \circ \pi$ as a function

$$\mathcal{L}(V/U, W) \xrightarrow{f} \mathcal{L}(V, W).$$

Proposition

The function f is an injective linear map $\mathcal{L}(V/U, W) \to \mathcal{L}(V, W)$ whose image is the set of linear maps $V \to W$ whose resriction to U is 0.

A linear map $V \to W$ of the form $S \circ \pi$ is said to *factor through* π. The proposition states that a linear map $V \to W$ factors through π if and only if its null space contains U.

Relation to linear maps that are 0 on U

Proposition

The function f is an injective linear map $\mathcal{L}(V/U, W) \to \mathcal{L}(V, W)$ whose image is the set of linear maps $V \to W$ whose null spaces contain U.

The linearity of f just results from definitions. One of the two conditions for the linearity of f is this: if S_1 and S_2 are two linear maps $V/U \to W$, then $(S_1 + S_2) \circ \pi = S_1 \circ \pi + S_2 \circ \pi$.

The map $S \mapsto S \circ \pi$ is injective: if $S \circ \pi = 0$, then $S(v + U) = 0$ for all $v \in V$. But this equation just means that S is 0 on all elements of V/U, so S is the 0 map $V/U \to W$.

If $S : V/U \to W$ is a linear map, then $S \circ \pi$ is 0 on the subspace U of V because π is 0 on U. Thus the null space of $S \circ \pi$ contains U. Said otherwise: the image of f is contained in the set of linear maps $V \to W$ whose null spaces contain U.

Relation to linear maps that are 0 on U

Proposition

The function f is an injective linear map $\mathcal{L}(V/U, W) \to \mathcal{L}(V, W)$ whose image is the set of linear maps $V \to W$ whose resrictions to U are 0.

We have seen that f is a linear map whose image is contained in the set of linear maps $V \to W$ whose null spaces contain U. It remains to show that if $T : V \to W$ is a linear map whose restriction $T_{|U}$ to U is 0, then T is in the image of f. This means that $T = S \circ \pi$ for some linear $S : V/U \to W$.

If T is given with $T_{|U} = 0$, we define $S : V/U \to W$ by $S(v + U) = Tv$. This is a *well defined* linear map $V/U \to W$: if $v + U = v' + U$, then $Tv' = T(v' - v) + Tv = Tv$ (since $v' - v \in U$ is in the null space of T).

The range and null space of S

Let $T : V \to W$ be a linear map and let $U \subseteq V$ be a subspace that is *contained in* null T. Let S be the unique linear map $V/U \to W$ such that $T = S \circ \pi$.

Proposition

The range of S is the range of T.

The range of S is the set of all $S(v + U)$, but $S(v + U) = Tv$. Thus the range of S consists of all vectors $Tv \in W$ and is therefore the range of T.

Proposition

The null space of S is the quotient $(\text{null } T)/U$.

The null space of S is the set of all $v + U \in V/U$ such that $S(v + U) = 0$. This is the set of all $v + U$ for which $Tv = 0$, i.e., the set of all $v + U$ with $v \in \text{null } T$. This is just the quotient $(\text{null } T)/U$ of the proposition.

LADR's map $\tilde{T}$

A slightly different perspective. Start with a linear map $T : V \to W$, and let $U = \text{null}\, T$. Then $T = S \circ \pi$ for some $S : V/(\text{null}\, T) \to W$.

In LADR, the map S in this situation is called $\tilde{T}$.

Proposition (3.107)

If $U = \text{null}\, T$, the map $\tilde{T} : V/U \to W$ is injective. Its range is the range of T.

This follows from our more general discussion, since the null space of $\tilde{T}$ is $(\text{null}\, T)/U = (\text{null}\, T)/(\text{null}\, T) = 0$.