We need to talk about lists, linear independence. . .

Professor K. A. Ribet

September 10, 2025

Office Hours

885 Evans Hall

Mondays, 1:30–3 PM
Thursdays, 10:30–noon.

Office full $\Longrightarrow$ possible move to a nearby classroom

Lots of people at the last two office hours. See you tomorrow at 10:30 AM

Lunch

I plan to come to the DCs at least once per week. There will be official Residential Life “lunches with Professor Ribet” at noon at Foothill DC on September 18, September 26, October 3 and October 9. There will also be additional lunch gatherings at DCs and the Faculty Club.

Gatherings are optional and not part of Math 110, but I’ll continue to list them on slides for those who are interested. Also, you can send me email to subscribe to email announcements.

- Lunch *today* at Crossroads at 11:45 AM
- Lunch after office hour on Thursday at Foothill DC (starting around 12:15
- Faculty Club lunch Monday, September 15 at noon PM)

Monday

We discussed sums of subspaces, direct sums, criterion for the sum of two subspaces to be a direct sum.

Two subspaces X and Y of V are complements of each other if $V = X \oplus Y$. I stated but did not prove that every subspace has at least one complement.

The subspace $\{0\}$ has V as its complement, while V has $\{0\}$ as its complement. If $\{0\} \subset X \subset V$, with $X \neq \{0\}, V$, then X likely has a whole bunch of complements.

Monday's last slide

Theorem (only implicit in LADR)

Let V be an $\mathbf{F}$-vector space, and let X be a subspace of V. Then X has a complement in V. In other words, there is a subspace Y of V so that $V = X \oplus Y$.

We will prove this theorem pretty soon.

Wednesday

Yes, that is today. I hope to discuss:

- Lists
- Span of a List
- Linear maps (reminder)
- Linear map defined by a list
- Linear independence
- Lists that span
- Bases (certain lists)
- Finite-dimensional spaces

Lists and sets

On page 5 of LADR, Axler defines *lists* of vectors in V: a list is a sequence

$$v_1, v_2, \ldots, v_\ell, \quad v_j \in V \text{ for all } j.$$

The *length* of the list is ℓ. The empty sequence

(nothing here)

is the list of length 0.

Lists are ordered, and repetition is allowed. For example,

$$0, 0, 0, 0, \ldots, 0$$

is a list of length 155 if there are 155 0's in a row.

Every list $v_1, v_2, \ldots, v_\ell$ gives rise to a set $\{ v_1, v_2, \ldots, v_\ell \}$, which will have fewer than ℓ elements if there is repetition. For example, $\{ 0, 0, 0, 0, \ldots, 0 \} = \{0\}$.

Lists

A length-ℓ list may be viewed as an element of

$$V^\ell = V \times V \times \cdots \times V \text{ (ℓ factors).}$$

Lists

A list of length ℓ is also the same thing as a linear map $\mathbf{F}^\ell \to V$:
A list $v_1, v_2, \ldots, v_\ell$ defines the linear map

$$\mathbf{F}^\ell \longrightarrow V, \quad (\lambda_1, \ldots, \lambda_\ell) \mapsto \lambda_1 v_1 + \cdots + \lambda_\ell v_\ell.$$

In the other direction, a linear map $T : \mathbf{F}^\ell \to V$ yields the list

$$Te_1, Te_2, \ldots, Te_\ell; \qquad e_j = (0, 0, \ldots, 0, \underset{\substack{\uparrow \\ \text{1 in } j\text{th place}}}{1}, 0, \ldots 0).$$

The dictionary between lists and linear maps appears as 3.4 on p. 54 of LADR.

Smallest subspace containing a list

If $v_1, v_2, \ldots, v_\ell$ is a list, then the set of linear combinations

$$\lambda_1 v_1 + \cdots + \lambda_\ell v_\ell$$

is a subspace that contains each vector v_j that appears in the list. This subspace is the smallest subspace of V containing each of the vectors. Indeed, if $U \subseteq V$ is a subspace that contains each v_j, then it contains each product $\lambda_j v_j$ and thus each linear combination $\lambda_1 v_1 + \cdots + \lambda_\ell v_\ell$.

As we've seen, the list defines a linear map

$$\mathbf{F}^\ell \longrightarrow V, \quad (\lambda_1, \ldots, \lambda_\ell) \mapsto \lambda_1 v_1 + \cdots + \lambda_\ell v_\ell.$$

The span of the list (i.e., the span of the set defined by the list) is the image of this map.

Linear maps

We saw the linear map

$$S : U_1 \times \cdots \times U_m \to V$$

on Monday and a just now linear map

$$\mathbf{F}^{\ell} \longrightarrow V.$$

It might be helpful to have the definition of "linear map" on the screen: if W and V are **F**-vector spaces, a function $T : W \to V$ is a *linear map* if it respects addition and scalar multiplication:

$$T(w_1 + w_2) = Tw_1 + Tw_2 \text{ for all } w_1, w_2 \in W;$$
$$T(\lambda W) = \lambda TW \text{ for all } W \in W, \lambda \in \mathbf{F}.$$

Linear independence

A list $v_1, v_2, \ldots, v_\ell$ is *linearly independent* (page 31) if the map

$$T : \mathbf{F}^\ell \longrightarrow V, \quad (\lambda_1, \ldots, \lambda_\ell) \mapsto \lambda_1 v_1 + \cdots + \lambda_\ell v_\ell$$

is 1-1. In words: if a vector in v can be expressed as a linear combination $\lambda_1 v_1 + \cdots + \lambda_\ell v_\ell$, then the scalars $\lambda_1, \ldots, \lambda_\ell$ are unique.

Note that T is 1-1 if and only if its null space is $\{0\}$. This means

$$\lambda_1 v_1 + \cdots + \lambda_\ell v_\ell = 0 \implies \lambda_1 = \lambda_2 = \cdots = \lambda_\ell = 0.$$

That's the usual definition of linear independence.

A slide for nerds about the empty list

If $\ell = 0$ (empty list), T is a map $\mathbf{F}^0 \to V$. Now $\mathbf{F}^0$ is the zero vector space (and has exactly one element). Hence the null space of T is $\{0\}$ and the list is linearly independent.

Lists of length 1

A list v is linearly independent if and only if v is nonzero.

Lists of length 2

A list of length 2 is linearly independent if and only if neither vector is a scalar multiple of the other.

Spanning lists

A list $v_1, v_2, \ldots, v_\ell$ *spans* V if its span is all of V (and not some smaller subspace).

The list spans if

$$T : \mathbf{F}^\ell \longrightarrow V, \quad (\lambda_1, \ldots, \lambda_\ell) \mapsto \lambda_1 v_1 + \cdots + \lambda_\ell v_\ell$$

is surjective (onto). This means that every vector in V is some linear combination $\lambda_1 v_1 + \cdots + \lambda_\ell v_\ell$ of the vectors in the list.

Basis

A list $v_1, v_2, \ldots, v_\ell$ is a *basis* of V if it is both linear independent and a spanning list (page 39). The two conditions together mean that every vector in V may be written *uniquely* as a linear combination $\lambda_1 v_1 + \cdots + \lambda_\ell v_\ell$ or (more formally) that T is both onto and 1-1.

Standard basis

If $V = \mathbf{F}^n$, the list $e_1, \ldots, e_n$ (defined before) is a basis of $\mathbf{F}^n$. People call it the *standard basis*.

Most vector spaces don't have specially defined bases. Nothing is standard in the abstract setting! Typical random vector spaces on the ground have infinitely many bases, all with their own special claim to fame.

Finite-dimensionality

A vector space V is *finite-dimensional* if there a list $v_1, v_2, \ldots, v_\ell$ (of finite length) that spans it.

An example of an $\mathbf{F}$-vector space that is *not* finite-dimensional is $\mathcal{P}(\mathbf{F})$, the space of polynomials with coefficients in $\mathbf{F}$.

How come $\mathcal{P}(\mathbf{F})$ isn't finite-dimensional?

Let $p_1, \ldots, p_\ell$ be a list of polynomials. If m is the largest degree of gthe polynomials p_j, then all p_j are contained in $\mathcal{P}_m(\mathbf{F})$, so that the span of the list is contained in $\mathcal{P}_m(\mathbf{F})$. That's a proper subspace of $\mathcal{P}(\mathbf{F})$.

An amazing theorem

Theorem

All subspaces of a finite-dimensional $\mathbf{F}$*-vector space are finite-dimensional.*

This is for the future, but in fact for the near future.

The slides for each lecture are gotten by combining undiscussed slides from the previous two lectures. This is the Fibonacci method of class preparation.