Col rank = row rank and more

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Office Hours

☛ 732 Evans Hall

Monday, 1:30–3 PM

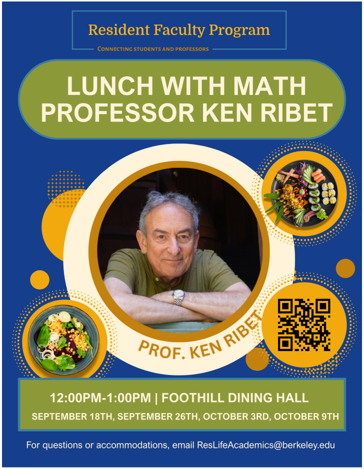
Thursday, 10:30–noon

Foothill lunches

Noon at Foothill this Friday, and also on October 3, October 9

Meet *today* at 12:45 at Crossroads

Thursday at 12:20 at the Faculty Club



Math Monday talk on Fermat’s Last Theorem

By request: I arranged to give a Math Mondays talk on Fermat’s Last Theorem this semester.

Talk on October 20 in 1015 Evans from 5 to 6 PM.

Later that evening: Res Life Academic Empowerment Series event on *office hours*, 8–10 PM in Anchor House

Column rank, row rank, transpose

The *column rank* of an $m \times n$ matrix A is the dimension of the span of the n columns of A. It's the rank of the linear map $T_A : \mathbf{F}^n \to \mathbf{F}^m$ defined by A.

The *row rank* of A is the dimension of the span of the m different rows of A, each row being in $\mathbf{F}^n$.

The transpose of an $m \times n$ matrix $A = (a_{jk})$ is the $n \times m$ matrix (a_{kj}). We exchange columns and rows to pass from A to its transpose A^{t}. Thus the column rank of A is the row rank of A^{t}, and vice versa.

Theorem

The row and column ranks of a matrix are equal. Equivalently, a matrix and its transpose have equal column ranks.

Three proofs of the theorem in LADR

Theorem

Row rank = column rank.

- The theorem appears as 3.57 on page 78.
- It's 3.133 on page 114.
- It's proved in Exercises 7 and 8 on page 239.

Column–row factorization

Proposition

Let A be an $m \times n$ matrix, and let c be the column rank of A. If $c \geq 1$, then $A = CR$, where C is an $m \times c$ matrix and R is a $c \times n$ matrix.

Proof: The matrix A is the matrix of the map $T_A =$ multiplication by A from $\mathbf{F}^n$ to $\mathbf{F}^m$ with respect to the standard bases of those spaces. Let $U \subseteq \mathbf{F}^m$ be the range of T_A, so that $c = \dim U$ is the column rank of A.

Let $\pi : \mathbf{F}^n \to U$ be T, thought of as taking values in U. Let $\iota : U \to \mathbf{F}^m$ be the inclusion map. By construction, $T = \iota \circ \pi$. Choose a basis of U (needed to represent π and ι by matrices). Then

$$A = \mathcal{M}(T) = \mathcal{M}(\iota)\mathcal{M}(\pi).$$

The right-hand matrices are respectively of sizes $m \times c$ and $c \times n$ (with $c = \dim U = \operatorname{rank} T$).

A takeaway

We have written $T_A : \mathbf{F}^n \to \mathbf{F}^m$ as a surjection $\pi : \mathbf{F}^n \to U$ followed by an injection $\iota : U \to \mathbf{F}^m$. You can do this for every linear map $V \to W$: just let U be the range of the map and follow the same procedure.

You can do it for every function from a set to a set. In other words, this is a basic construction.

Canonicity

The space $U \subseteq \mathbf{F}^m$ is the span of the n columns of A. A choice-free way to find a basis of U is to prune down the list of the n columns by the method of 2.30: “Every spanning list in a vector space can be reduced to a basis of the vector space.” Then the decomposition $A = CR$ becomes entirely canonical (choice free).

With this method, the $m \times c$ matrix is gotten from the $m \times n$ matrix A by chucking away some of A's columns.

Example

Take $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 5 & 5 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$, which encodes our room and course numbers. The first three columns of A form a linearly independent list (in $\mathbf{F}^3$), but the fourth column is 4 times the third column minus 4 times the second column. Thus the column rank is 3 and the first three columns of A form a basis of U. The matrix of ι is the first chunk of A: $C = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 5 & 5 \\ 0 & 1 & 1 \end{pmatrix}$.

To calculate R, we express each of the four columns of A as linear combinations of the three basis vectors of U (i.e., columns of C). The first three columns of A are the columns of C. As already mentioned, the fourth column of A is 4 times the third column of C, minus 4 times the second column of C.

Thus $R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 4 \end{pmatrix}$, and $A = CR$ (check?).

A somewhat silly example?

Let A be an $m \times n$ matrix whose columns are linearly independent. This means that $T_A : \mathbf{F}^n \to \mathbf{F}^m$ is injective. Let $X = \text{range } T_A$, as usual. The map $\pi : \mathbf{F}^n \to X$ is an isomorphism. The basis that we are using for X is the list $Ae_1, \ldots, Ae_n$. Hence the matrix for π is the $n \times n$ identity matrix. Meanwhile, C, which holds that pared down list of columns of A in general, is the entire matrix A.

Hence the decomposition $A = CR$ becomes the equation $A = AI$, where I is the $n \times n$ identity matrix.

Check that you really understand this and can reproduce it.

Column rank = row rank

Theorem

The row and column ranks of a matrix are equal.

Proof: Let A be an $m \times n$ matrix. To prove that the row rank of A is the column rank of A is to prove that the column ranks of A and A^{t} are equal. As explained on Monday, it suffices to prove

$$\text{col-rank}\, A^{\mathrm{t}} \overset{?}{\leq} \text{col-rank}\, A,$$

where "col-rank" stands for "column rank." Indeed, if we know this, we can replace A by its transpose to get

$$\text{col-rank}\, A = \text{col-rank}(A^{\mathrm{t}})^{\mathrm{t}} \leq \text{col-rank}\, A^{\mathrm{t}}$$

which gives the equality of the two column ranks.

Column rank = row rank

Proposition

Let A be a matrix. Then col-rank $A^t \leq$ col-rank A.

The proposition is true if the column rank of A is 0, since col-rank A^t is nonnegative.

Anyway, if the column rank of A is 0, A is then the 0 matrix, and its row rank is clearly 0 as well. On the next slide, we can and will assume that the column rank c of A is *positive*.

Column rank = row rank

Proposition

Let A be an $m \times n$ matrix. Then col-rank $A^{\mathrm{t}} \leq$ col-rank A.

Because the column rank of A is positive, we may write $A = CR$, where C is an $m \times c$ matrix and R is a $c \times n$ matrix.

As you agreed on Monday, the transpose of a product is the product of transposes *in the opposite order*:

$$A^{\mathrm{t}} = R^{\mathrm{t}} C^{\mathrm{t}} \implies T_{A^{\mathrm{t}}} = T_{R^{\mathrm{t}}} T_{C^{\mathrm{t}}}.$$

One again, "T" refers to the linear map arising from multiplication by a matrix. The column rank of A^{t} is the dimension of the range of $T_{A^{\mathrm{t}}}$, which is contained in the range of $T_{R^{\mathrm{t}}}$. Since $T_{R^{\mathrm{t}}}$ is a linear map $\mathbf{F}^c \to \mathbf{F}^n$, its rank is at most c.

We have thus shown

$$\text{col-rank}\, A^{\mathrm{t}} = \text{rank}\, T_{A^{\mathrm{t}}} \leq \text{rank}\, T_{R^{\mathrm{t}}} \leq c = \text{col-rank}\, A.$$

Rank of a matrix

If A is a matrix, its column and row ranks are equal. The common value is called the *rank* of A.

Stuff in Chapter 3

We have already discussed invertibility of linear maps. If $T : V \to W$ is a linear map between finite-dimensional vector spaces, we showed:

- T injective $\Rightarrow$ $\dim V \leq \dim W$.
- T surjective $\Rightarrow$ $\dim V \geq \dim W$.
- If $\dim V = \dim W$, then T is surjective if and only if T is injective.

If $f : A \to B$ is a function between finite sets:

- f injective $\Rightarrow$ $|A| \leq |B|$.
- f surjective $\Rightarrow$ $|A|V \geq |B|$.
- If |A|=|B|, then f is surjective if and only if f is injective.

Change of basis

If $T : V \to W$ is a linear map between finite-dimensional vector spaces, we get a matrix $\mathcal{M}(T)$ by choosing bases for V and W.

What happens if we change one of the bases? The three natural questions are:

- How does $\mathcal{M}(T)$ change if we change the basis of V?
- How does $\mathcal{M}(T)$ change if we change the basis of W?
- How does $\mathcal{M}(T)$ change if we change both bases?

If we know the answer to the first two questions, we get the answer to the third (combining the answers). If we know the answer to the first question, we're pretty close to knowing the answer to the second.

Two bases of V

Suppose that $v_1, \ldots, v_n$ and $v'_1, \ldots, v'_n$ are bases for V. As an example, V might be $\mathbf{F}^n$, the v_j might be the standard basis vectors e_j, and the v'_j might be weird af.

It is natural to imagine writing the primed vectors v'_j in terms of the unprimed vectors v_j. Say

$$v'_j = c_{1j} v_1 + \cdots + c_{nj} v_n \text{ for j=1,} \ldots \text{, n.}$$

The scalars c_{ij} form an $n \times n$ matrix, whose jth column pertains to v'_j..

Insight of the moment: The matrix $C = (c_{ij})$ is the matrix of the identity map $I : V \to V$ if we use $v'_1, \ldots, v'_n$ as the basis of the left-hand copy of V and $v_1, \ldots, v_n$ for the right-hand copy of V. This just follows from the definition of the matrix of a linear map between spaces, each furnished with a basis.

This will knock you out

Again, imagine $T : V \to W$ with $\mathcal{M}(T)$ made from $v_1, \ldots, v_n$ and some basis $w_1, \ldots, w_m$ of W. Let $\mathcal{M}'(T)$ be the matrix of T using the bases $v'_1, \ldots, v'_n$ and $w_1, \ldots, w_m$. View T as the composite

$$V \xrightarrow{I} V \xrightarrow{T} W,$$

and use the bases

$$v'_1, \ldots, v'_n; \qquad v_1, \ldots, v_n; \qquad w_1, \ldots, w_m$$

on V, V and W (reading from left to right). Then with the right mindset

$$\mathcal{M}'(T) = \mathcal{M}(T \circ I) = \mathcal{M}(T)\mathcal{M}(I) = \mathcal{M}(T)C.$$

The answer to the first question:

> *Moving from the first basis of V to the second basis of V multiplies $\mathcal{M}(T)$ on the right by the change of basis matrix C.*