Dual Spaces

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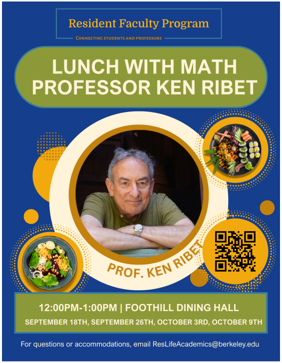
Announcements

My office hours are Mondays 1:30–3 PM and Thursday 10:30 AM–noon

Last Lunch with Math Professor Ken Ribet is *Thursday* at noon.

One or two *unofficial* lunches each week. Feel free to request a day, time, venue.



Lunch at Crossroads at 11:40 AM *today*

Dual space

It's Monday morning — time to recall what we did at the end of last week.

If V is a vector space over $\mathbf{F}$, then $V' = \mathcal{L}(V, \mathbf{F})$ is the space dual to V. For some reason, the elements of V' are called linear functionals, and they're typically written with Greek letters.

Example: if $V = \mathcal{P}(\mathbf{R})$, the map

$$f \longmapsto \int_{110}^{155} f(x)\,dx$$

is an element of V'.

Another element is

$$f \longmapsto f^{(155)}(110).$$

Dual space

If V has dimension n, V' has dimension n. If $v_1, \ldots, v_n$ is a basis of V, a basis of V' is $\varphi_1, \ldots, \varphi_n$, where

$$\varphi_k : a_1 v_1 + \cdots + a_n v_n \longmapsto a_k.$$

The list $\varphi_1, \ldots, \varphi_n$ is the "dual basis," the basis of V' dual to $v_1, \ldots, v_n$.

Note that

$$\varphi_k(v_j) = \delta_{kj}$$

for $j = 1, \ldots, n$, $k = 1, \ldots, n$.

Special case where $V = \mathbf{F}^n$

If $V = \mathbf{F}^n$ with standard basis $e_1, \ldots, e_n$, then

$$\varphi_k(a_1, \ldots, a_n) = a_k.$$

Dot product, ish

If V is a vector space and V' is its dual, there is a natural map

$$V \times V' \longrightarrow \mathbf{F}, \qquad (v, \varphi) \mapsto \varphi(v).$$

If we choose a basis $v_1, \ldots, v_n$ for V and let $\varphi_1, \ldots, \varphi_n$ be the dual basis, then we can write

$$v = \sum_i a_i v_i, \quad \varphi = \sum_j b_j \varphi_j.$$

The number $\varphi(v)$ is nothing other than the dot product $a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$.

Dual map

If $T : V \to W$ is a linear map, there is an induced linear map

$$\mathcal{L}(W, \mathbf{F}) \to \mathcal{L}(V, \mathbf{F}), \qquad \psi \in \mathcal{L}(W, \mathbf{F}) \longmapsto \psi \circ T \in \mathcal{L}(V, \mathbf{F}).$$

This map is called T' and is said to be the map *dual to* T. Thus T' is a linear map $W' \to V'$.

Properties of dual maps

Here are some basic properties of dual maps (3.120 in LADR):

- $(S+T)' = S' + T'$ for $S, T \in \mathcal{L}(V, W)$;
- $(\lambda T)' = \lambda T'$ for $T \in \mathcal{L}(V, W)$;
- $(ST)' = T'S'$ for $T \in \mathcal{L}(V, W)$ and $S \in \mathcal{L}(W, X)$.

These are all things that you should check. For example, $(ST)'$ is the map from X' to V' taking $\varphi \in \mathcal{L}(X, \mathbf{F})$ to $\varphi \circ (ST)$. But

$$\varphi \circ (ST) = (\varphi \circ S) \circ T = T'(\varphi \circ S) = T'(S'\varphi).$$

Matrix of dual map

Let $T : V \to W$ be a linear map with V and W finite-dimensional. Choose bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ of V and W. The matrix of T relative to these bases is

$$\mathcal{M}(T) = (a_{ij}), \quad Tv_j = \sum_{i=1}^{m} a_{ij} w_i \text{ for } j = 1, \ldots n.$$

The dual of T is the map $T' : W' \to V'$, $\psi \mapsto \psi \circ T$.

Let $\varphi_1, \ldots, \varphi_n$ be the basis of V' dual to $v_1, \ldots, v_n$. Analogously, let $\psi_1, \ldots, \psi_m$ be the basis of W' dual to $w_1, \ldots, w_m$.

Then T and T' are represented by matrices $\mathcal{M}(T)$ and $\mathcal{M}(T')$ of dimensions $m \times n$ and $n \times m$, respectively.

Formula (3.123)

The matrices $\mathcal{M}(T)$ and $\mathcal{M}(T')$ are transposes of each other.

Matrix of dual map

Formula

The matrices $\mathcal{M}(T)$ and $\mathcal{M}(T')$ are transposes of each other.

If $\mathcal{M}(T') = (b_{ij})$, then $T'(\psi_j) = \psi_j T = \sum_i b_{ij}\varphi_i$ for each $j = 1, \dots, m$. The formula to be proved is $b_{ij} = a_{ji}$ for each i and j. To verify this equality is to check the following equation for each j:

$$\psi_j T \stackrel{?}{=} \sum_i a_{ji}\varphi_i.$$

The two sides of this equation are linear maps $V \to W$. We use the fact that two linear maps $V \to W$ are equal if they agree on the basis vectors $v_1, \dots, v_n$.

Thus the formula is equivalent to the equality

$$\psi_j(Tv_k) \stackrel{?}{=} \sum_i a_{ji}\varphi_i(v_k), \qquad j = 1, \dots, m, \quad k = 1, \dots, n.$$

Matrix of dual map

Formula

The matrices $\mathcal{M}(T)$ and $\mathcal{M}(T')$ are transposes of each other.

The left-hand side of the equality to be verified is

$$\psi_j(\sum_i a_{ik} w_i) = \sum_i a_{ik} \psi_j(w_i) = a_{jk},$$

the point being that $\psi_j(w_i)$ is 0 except when $i = j$, when it's 1.

The right-hand side $\sum_i a_{ji} \varphi_i(v_k)$ also collapses to a single term, and for the same reason: $\varphi_i(v_k)$ is 0 except when $i = k$ (when it's 1). The single term is a_{jk}, as for the left-hand side.

Column rank = row rank

Let A be an $m \times n$ matrix. The matrix A defines a linear map $T : \mathbf{F}^n \to \mathbf{F}^m$ whose dual T' has matrix A^{t}. The statement that the column ranks of A and A^{t} are the same is the statement that range T and range T' have the same dimension.

Proposition

If T is a linear map between finite-dimensional vector spaces, then the ranges of T and T' have equal dimensions.

After we prove this proposition, we will have a conceptual proof of the coincidence between the row and column ranks of a matrix.

Annihilator

The *annihilator* of a subspace $U \subseteq V$ of V is the subspace

$$U^0 = \{\, \varphi \in V' \mid \varphi_{|U} = 0 \,\}.$$

This is the space of vectors of V' that are "perpendicular to U" in the dot product language that relates to $V \times V' \longrightarrow \mathbf{F}$.

Proposition (3.125)

If U has dimension d and V has dimension n, then U^0 has dimension $n - d$.

Proof: Choose a basis $v_1, \ldots, v_d$ of U and extend it to a basis $v_1 \ldots, v_n$ of V. Let $\varphi_1, \ldots, \varphi_n$ be the basis of V' dual to the chosen basis of V. If $\varphi = b_1\varphi_1 + \cdots + b_n\varphi_n$ is an element of V', its restriction to U is 0 if and only if $\varphi(v_j) = 0$ for $j = 1, \ldots, d$. Since $\varphi(v_j) = b_j$, φ is in the annihilator of U if and only if $b_1 = \cdots = b_d = 0$, i.e., if and only if $\varphi \in \operatorname{span} \varphi_{d+1}, \ldots, \varphi_n$. This span has dimension $n - d$; thus $\dim U^0 = n - d$.

Annihilator

Proposition

If V has finite dimension, then $\dim U^0 = \dim V - \dim U$.

This is the same proposition as on the previous slide. Another view of the proof:

By definition, U^0 is the null space of the restriction map $\varphi \in V' \mapsto \varphi_{|U} \in U'$. This map is surjective; in other words, every linear funtional $\alpha : U \to \mathbf{F}$ can be extended to a linear functional $V \to \mathbf{F}$. Indeed, if X is a vector space complement to U in V, then an extension of α to V is given by $v = u + x \mapsto \alpha(u)$. Hence rank–nullity implies that $\dim U^0 = \dim V' - \dim U' = \dim V - \dim U$.

Given that $\dim U^0 = \dim V/U$, one might suspect that there is a relation between these two spaces. Can you guess the relation?

Annihilators

If V is finite-dimensional, then

$$U = V \iff \dim U^0 = 0 \iff U^0 = 0$$

and

$$U = \{0\} \iff \dim U^0 = \dim V' \iff U^0 = V'.$$

If you're keeping score at home, this is 3.127 on page 111 of LADR.

T and T': null spaces and ranges

Proposition

If $T : V \to W$ is a linear map, then the null space of T' is the annihilator of the range of T.

Since range $T \subseteq W$, $(\text{range } T)^0$ is a subspace of W'. So is the null space of T'. Thus the two space being compared are subspaces of the same vector space.

The null space of T' consists of all $\varphi \in W'$ satisfying $0 = \varphi \circ T$. This condition means that $(\varphi T)v = 0$ for all $v \in V$. Rewrite this condition as $\varphi(Tv) = 0$ for all $v \in V$ and then $\varphi w = 0$ for all $w \in$ range T. Finally, this is now the condition that $\varphi_{|\text{ range } T} = 0$, i.e., that φ belongs to the annihilator of range T.

T and T': null spaces and ranges

In the context of the proposition on the previous slide, suppose now that V and W have finite dimension.

Corollary

The dimension of null T' is $\dim \operatorname{null} T + \dim W - \dim V$.

Proof: Since the nullspace of T' is the annihilator of the range of T, $\dim \operatorname{null} T' = \dim W - \operatorname{rank} T$. By the rank–nullity formula, $\operatorname{rank} T = \dim V - \dim \operatorname{null} T$. The desired formula follows.

T and T': null spaces and ranges

Corollary

The dimension of null T' is $\dim \operatorname{null} T + \dim W - \dim V$.

Corollary

The linear map T is onto if and only if its dual T' is 1-1.

Proof: The map T' is 1-1 if and only if its nullspace has dimension 0. This is true if and only if

$$\dim V \stackrel{?}{=} \dim \operatorname{null} T + \dim W.$$

The rank–nullity formula gives

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T.$$

Hence the formula $\dim V \stackrel{?}{=} \dim \operatorname{null} T + \dim W$ is equivalent to the equality $\dim W = \dim \operatorname{range} T$, i.e., to the surjectivity of T.

Row rank = column rank

Corollary

If $T : V \to W$ is a linear map between finite-dimensional vector spaces, then T' and T have equal ranks.

Proof: We proved $\dim \operatorname{null} T' = \dim \operatorname{null} T + \dim W - \dim V$, which is eqivalent to

$$\dim V - \dim \operatorname{null} T = \dim W - \dim \operatorname{null} T'.$$

Note that $\dim W = \dim W'$. Hence the right-hand side is rank T', while the left-hand side is rank T. (We used rank–nullity on each side.)

range T', annihilator of null T

Corollary

In the context above, the range of T' is the annihilator of the null space of T.

The two spaces being compared are subspaces of V'. They have equal dimension: Indeed, the dimension of the annihilator of the null space of T is $\dim V - \dim \operatorname{null} T = \dim \operatorname{range} T$. Since $\dim \operatorname{range} T' = \dim \operatorname{range} T$, the dimensions agree.

It follows that the equality $\operatorname{range} T' = (\operatorname{null} T)^0$ is equivalent to the *inclusion* $\operatorname{range} T' \subseteq (\operatorname{null} T)^0$. This inclusion is the statement that the range of T' annihilates the null space of T. Now an element of the range of T' is a linear functional $T'\psi = \psi \circ T$, where ψ is a linear functional on W. The annihilation is the statement $(\psi \circ T)v = 0$ if $Tv = 0$. This is now clear because $(\psi \circ T)v = \psi(Tv) = \psi(0) = 0$.