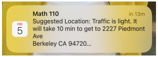
Sums of subspaces, some linear maps. . .

Professor K. A. Ribet



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Office Hours

885 Evans Hall

Mondays, 1:30–3 PM
Thursdays, 10:30–noon.

Office full $\Longrightarrow$ possible move to a nearby classroom

No more wrinkles involving Labor Day, Res Life symposium

Lunch schedule

I plan to come to the DCs at least once per week. There will be official Residential Life “lunches with Professor Ribet” at noon at Foothil DC on September 18, September 26, October 3 and October 9. There will also be occasional totally optional lunch gatherings at the Faculty Club. Send me email to subscribe to announcements.

Faculty Club lunch today at noon for the curious. Don’t all come at once. Please read description on `bCourses`,

`Files -> Slides -> previews.pdf.`

Sums of subspaces

On Friday, we discussed the sum of subspaces X and Y of V. More generally (p. 19 of LADR), consider subspaces $U_1, \dots, U_m$ of V ($m \geq 1$). The sum of these subspaces is the image of the summation map

$$S : U_1 \times \cdots \times U_m \to V, \quad (u_1, \dots, u_m) \mapsto u_1 + u_2 + \cdots + u_m.$$

In general, this map is neither onto (surjective) nor 1-1 (injective).

If $m = 1$, it's just the inclusion of a subspace into V.

Sum of subspaces

Observation

The sum of subspaces is the smallest subspace that contains their union.

How come: A subspace of V that contains all the U_j needs to contain all sums $u_1 + u_2 + \cdots + u_m$. The set of these sums is a subspace.

Direct sums

The sum of U_1, U_2,..., U_m is denoted $U_1 + \cdots + U_m$.

There is a special word and a special notation for the case where the summation map is 1-1: we say that the sum of the U_j is a *direct sum* and write $U_1 \oplus \cdots \oplus U_m$ for the sum.

As a first example, let $V = \mathbf{F}^m$ and let U_j be the set of m-tuples whose entries are 0 except for the jth place. Thus $U_j = \mathbf{F} \cdot e_j$, where e_j has a 1 in the jth place and 0s elsewhere. Each m-tuple $(a_1, \ldots, a_m)$ is *uniquely* a linear combination of the standard basis vectors e_j: $(a_1, \ldots, a_m) = a_1 e_1 + \cdots + a_m e_m$. Equivalently, each m-tuple is uniquely a sum of elements of the various subspaces U_j. Hence the summation map is 1-1 (and onto). The sum of the subspaces is direct, and in fact

$$V = U_1 \oplus \cdots \oplus U_m.$$

A two-dimensional example

Let $V = \mathbf{F}^2$, and let $X = \mathbf{F} \cdot (a, b)$, $Y = \mathbf{F} \cdot (c, d)$, where the two vectors (a, b) and (c, d) are both nonzero. Then X and Y are lines in the plane.

If $X = Y$, the sum $X + Y$ is X (or Y), and this sum is not direct. Indeed, we can write $(0, 0) \in V$ as the sum of $(0, 0) \in X$ and $(0, 0) \in Y$, but also as $x - x$, where x is a nonzero vector in $X = Y$.

If $X \neq Y$, then there is no nonzero vector that's a multiple of both (a, b) and (c, d). (This requires a sentence or two of explanation.) It follows that the sum is direct, as we'll see on the next slide(s).

Injectivity of the summation map

For this slide, check out 1.45 of page 23 of LADR.

The summation map

$$S : U_1 \times \cdots \times U_m \longrightarrow V$$

is a *linear map*. This means that S of a sum is the sum of the S's and that S of λ times something is λ times the value of S on that something. More generally, if W is an $\mathbf{F}$-vector space, a function

$$T : W \to V$$

is *linear* if $T(w + w') = Tw + Tw'$ for all $w, w' \in W$ and $T(\lambda w) = \lambda Tw$ for all $w \in W$ and $\lambda \in \mathbf{F}$.

The *null space* of T is the set of $w \in W$ such that $Tw = 0$. It's a subspace of W.

Lemma

The map T is 1-1 if and only if the null space of T is $\{0\}$.

Proof of lemma

Lemma

Let $T : W \to V$ be a linear map. Then T is 1-1 if and only if the null space of T is $\{0\}$.

Suppose that T is 1-1. If w is in the null space of T, then

$$0 = T0 = Tw \implies 0 = w.$$

Hence the null space consists only of 0.

Suppose that the null space of T is 0. To check that T is 1-1, we must show

$$Tw = Tw' \implies w = w'$$

for $w, w' \in W$. Assuming that $Tw = Tw'$, we use the linearity of T to write

$$0 = Tw - Tw' = T(w - w')$$

and conclude that $0 = w - w'$ because $w - w'$ is in the null space of T. This conclusion is the statement that $w = w'$. Hence T is indeed 1-1.

Consequence for sums

Because the summation map is a linear map, the lemma implies the following statement:

Proposition

If $U_1, \ldots U_m$ are subspaces of V, then the sum $U_1 + \cdots + U_m$ inside V is a direct sum if and only if the null space of the summation map is $\{0\}$.

Concretely, the null space statement means this:

> *If* $0 = u_1 + \cdots + u_m$ *with* $u_j \in U_j$ *for all* j*, then each summand* u_j *is 0.*

Direct sum of two subspaces

Proposition (1.46, p. 23)

If X and Y are subspaces of V, then the sum $X + Y$ is direct if and only if $X \cap Y = \{0\}$.

We have seen that $X + Y = X \oplus Y$ if and only if $x + y = 0$ (for $x \in X$, $y \in Y$) implies that $x = y = 0$. We need to translate that equivalence into the statement of the proposition.

If $t \in X \cap Y$, then $(t, -t) \in X \times Y$ is in the null space of the summation map. Hence if the sum is direct, so that the null space is $\{0\}$, then $t = 0$ for each $t \in X \cap Y$. Thus $X \cap Y = \{0\}$.

Conversely, suppose that $X \cap Y = \{0\}$ and that (x, y) is in the null space of the sum map. Then $x + y = 0$. Since $y = -x$, $y \in X$. Similarly, $x \in Y$. Thus x and y are in $X \cap Y$, which is $\{0\}$. Hence $(x, y) = (0, 0)$.

V as direct sum of two subspaces

If X and Y are subspaces of V, we might have

$$X + Y \stackrel{?}{=} V, \qquad X + Y \stackrel{?}{=} X \oplus Y.$$

The first equality states that V is the sum of X and Y. The second says that the sum $X + Y$ inside V is *direct*.

If both statements are true, we say that V is the direct sum of its subspaces X and Y.

We say that X and Y are *complementary* subspaces of V and that Y is a complement of X (and vice versa).

Example, maybe for later

Let $V = \mathbf{F}^2$ and let $X = \{\, (x, 0) \mid x \in \mathbf{F} \,\}$. If y is an element of V that is not in X, the line $\mathbf{F} \cdot y$ is a complement of X in V.

A peek into the future

Once we know about dimensions, we'll be able to have the following discussion:

If X and Y are subspaces of V, we know that their sum is direct if $X \cap Y = \{0\}$. Suppose that this is the case.

Then the dimension of $X \oplus Y$ is $\dim X + \dim Y$. Further, $V = X \oplus Y$ if and only if

$$\dim V = \dim X + \dim Y.$$

In other words, complements inside V are subspaces with trivial intersection whose dimensions are complementary relative to $\dim V$.

Big caution: we have no idea yet what dimension is, and we haven't yet imposed the hypothesis that all vector spaces in the book are going to be finite-dimensional.

A main theorem of linear algebra

Theorem

Let V be an $\mathbf{F}$-vector space, and let X be a subspace of V. Then X has a complement in V. In other words, there is a subspace Y of V so that $V = X \oplus Y$.

We will prove this theorem *very soon* if V is a finite-dimensional vector space.

What does that mean? We'll say (on page 30) that V is finite-dimensional if there is a finite subset of V whose span is all of V.

Lists and sets

On page 5 of LADR, Axler defines *lists* of vectors in V: a list is a sequence

$$v_1, v_2, \ldots, v_\ell, \quad v_j \in V \text{ for all } j.$$

The *length* of the list is ℓ. The empty sequence

(nothing here)

is the list of length 0.

Lists are ordered, and repetition is allowed. For example,

$$0, 0, 0, 0, \ldots, 0$$

is a list of length 155 if there are 155 0's in a row.

Every list $v_1, v_2, \ldots, v_\ell$ gives rise to a set $\{ v_1, v_2, \ldots, v_\ell \}$, which will have fewer than ℓ elements if there is repetition. For example, $\{ 0, 0, 0, 0, \ldots, 0 \} = \{0\}$.

Smallest subspace containing a list

If $v_1, v_2, \ldots, v_\ell$ is a list, then the set of linear combinations

$$\lambda_1 v_1 + \cdots + \lambda_\ell v_\ell$$

is a subspace that contains each vector v_j that appears in the list. This subspace is the smallest subspace of V containing each of the vectors. Indeed, if $U \subseteq V$ is a subspace that contains each v_j, then it contains each product $\lambda_j v_j$ and thus each linear combination $\lambda_1 v_1 + \cdots + \lambda_\ell v_\ell$.

A structural way to think about this is that the list defines a map

$$T : \mathbf{F}^\ell \longrightarrow V, \quad (\lambda_1, \ldots, \lambda_\ell) \mapsto \lambda_1 v_1 + \cdots + \lambda_\ell v_\ell.$$

The span of the list (i.e., the span of the set defined by the list) is the image of T.

A linear map

Scrolling all the way down to page 52, you'll see that T is a *linear map* from $\mathbf{F}^\ell$ to V. This means that the value of T on the sum of two elements of $\mathbf{F}^\ell$ is the sum of the values of T on the two elements and that the value of T on a scalar multiple of a tuple $(\lambda_1, \ldots, \lambda_\ell)$ is the same scalar multiple of the value of T on $(\lambda_1, \ldots, \lambda_\ell)$.

Mantra: lists define linear maps.

I wanted to write more about linear maps, and I did. But most of what I wrote is at the end of this slide deck and is not intended to be discussed this week.

A linear map (said another way)

Let $v_1, v_2, \dots, v_\ell$ be a list of vectors in V. Then

$$T : \mathbf{F}^\ell \longrightarrow V, \quad (\lambda_1, \dots, \lambda_\ell) \mapsto \lambda_1 v_1 + \cdots + \lambda_\ell v_\ell$$

is a linear map that takes the standard basis vectors $e_1, e_2, \dots, e_\ell$ of $\mathbf{F}^\ell$ to the list vectors $v_1, v_2, \dots, v_\ell$ in V.

The function T is the *unique linear map* taking e_j to v_j for each $j = 1, \dots, \ell$.

For each j,

$$e_j = (0, \dots, 0, 1, 0 \dots, 0), \qquad 1 \text{ in the } j\text{th place.}$$

Even though we say that the e_j are standard basis vectors, we haven't yet said what a basis is.

Coming attractions

If $v_1, \ldots, v_\ell$ is a list of vectors in V, the *span* of the list is the image of the map T on the previous slide or two.

The list *spans* V if the span of the list is all of V. This means that T is onto (i.e., surjective).

A vector space is *finite-dimensional* if there is a (finite) list that spans the space. We will see later on that finite-dimensional vector spaces have a well-defined *dimension*.

Is there interest in lists for which T is injective (1-1). You bet! We say that a list is *linearly independent* if T is 1-1.

We say that a list of vectors of V is a *basis* of V if T is both 1-1 and onto.

The slides for each lecture are gotten by combining undiscussed slides from the previous two lectures. This is the Fibonacci method of class preparation.

Going rogue with linear maps

I can't refrain from telling you about the functions that we study in linear algebra. The reference for this is the beginning of Chapter 3 of LADR (pp. 51–).

If V and W are $\mathbf{F}$-vector spaces, a function $T : V \to W$ is a linear map if it respects addition and scalar multiplication:

$$T(v_1 + v_2) = Tv_1 + Tv_2 \text{ for all } v_1, v_2 \in V;$$
$$T(\lambda v) = \lambda Tv \text{ for all } v \in V, \lambda \in \mathbf{F}.$$

Familiar examples

Let V be the space of differentiable functions on $\mathbf{R}$, and let W be the space of all functions $\mathbf{R} \to \mathbf{R}$. The differentiation map $f \mapsto f'$ is a linear map $V \to W$. (On this slide, $\mathbf{F} = \mathbf{R}$.)

Let V be the space of integrable functions on $[0, 1]$, and let $W = \mathbf{R}$. The association

$$f \mapsto \int_0^1 f(x)\,dx$$

is a linear map $V \to W$.

Matrix multiplication

Let $V = \mathbf{F}^n$ and $W = \mathbf{F}^m$, with the vectors in both spaces thought of as vertical tuples. Let A be an $m \times n$ matrix of elements of $\mathbf{F}$. The map

$$x \in \mathbf{F}^n \longmapsto Ax \in \mathbf{F}^m$$

is linear.

More about the case $V = \mathbf{F}^n$

Let $T : \mathbf{F}^n \to W$ be a linear map. Recall the familiar "standard basis vectors" $e_1, e_2, \ldots, e_n$ from Math 54:

$$e_j = (0, 0, \ldots, 0, 1, 0, \ldots 0), \text{ the 1 in the } j\text{th place.}$$

Then the vectors

$$Te_1, Te_2, \ldots, Te_n \in W$$

form a *list* of vectors of length n in W.

Lists are introduced on page 5 of LADR:

> *A list of length n is an ordered collection of n elements. . . .*

A list of vectors in W of length n may be viewed as an element of $W^n = W \times \cdots \times W$ (n copies).

The case $V = \mathbf{F}^n$

Thus we have an association

$$\{\text{ linear maps } \mathbf{F}^n \to W\,\} \longrightarrow W^n, \quad T \longmapsto (Te_1, \dots, Te_n).$$

Theorem (3.4 on page 54)

This association is a 1-1 correspondence between the set of linear maps $\mathbf{F}^n \to W$ and the set W^n.

The “association” is a function

$$F : \{\text{ linear maps } \mathbf{F}^n \to W\,\} \to W^n.$$

One way to show that F is a bijection is to exhibit a function

$$G : W^n \to \{\text{ linear maps } \mathbf{F}^n \to W\,\}$$

such that $G \circ F$ is the identity map on the set of linear maps $\mathbf{F}^n \to W$ and $F \circ G$ is the identity map on W^n.

The case $V = \mathbf{F}^n$

The function G from W^n to the space of linear maps $\mathbf{F}^n \to W$ is defined by

$$(w_1, \ldots, w_n) \mapsto T : \mathbf{F}^n \to W,$$

$$T(a_1, \ldots, a_n) := a_1 w_1 + a_2 w_2 + \cdots + a_n w_n.$$

The linearity of T amounts to two compatibilities:

$$T\left((a_1, \ldots, a_n) + (b_1, \ldots, b_n)\right) \stackrel{?}{=} T(a_1, \ldots, a_n) + T(b_1, \ldots, b_n),$$
$$T\left(\lambda(a_1, \ldots, a_n)\right) = \lambda T(a_1, \ldots, a_n).$$

We'll check them live and in person in 155 Dwinelle.