Dual spaces and polynomials

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October 8, 2025

SLC

I am coming to the SLC today at noon to see how upper-division tutoring goes. If you were thinking of going down there some time, why not join me today?

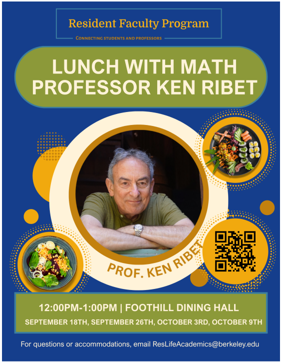
I'll be leaving around 12:35 to head over to a Faculty Club lunch (which you can join as well).

More announcements

My office hours are Mondays 1:30–3 PM and Thursday 10:30 AM–noon in 732 Evans.

Last Lunch with Math Professor Ken Ribet is *Tomorrow* at noon.

One or two *unofficial* lunches each week. Feel free to request a day, time, venue.



Faculty Club lunch *today* at 12:45 PM

Dual map

If $T : V \to W$ is a linear map, there is an induced linear map

$$\mathcal{L}(W, \mathbf{F}) \to \mathcal{L}(V, \mathbf{F}), \qquad \psi \in \mathcal{L}(W, \mathbf{F}) \longmapsto \psi \circ T \in \mathcal{L}(V, \mathbf{F}).$$

This map is called T' and is said to be the map *dual to* T. Thus T' is a linear map $W' \to V'$.

Column rank = row rank

Let A be an $m \times n$ matrix. The matrix A defines a linear map $T : \mathbf{F}^n \to \mathbf{F}^m$ whose dual T' has matrix A^{t}. The statement that the column ranks of A and A^{t} are the same is the statement that range T and range T' have the same dimension.

Proposition (A coming attraction)

If T is a linear map between finite-dimensional vector spaces, then the ranges of T and T' have equal dimensions.

Annihilator

The *annihilator* of a subspace $U \subseteq V$ of V is the subspace

$$U^0 = \{\, \varphi \in V' \mid \varphi_{|U} = 0 \,\}.$$

Proposition (3.125)

If U has dimension d and V has dimension n, then U^0 has dimension $n - d$.

Annihilator

Proposition

If V has finite dimension, then $\dim U^0 = \dim V/U$.

This is the same proposition as the previous one.

There is a natural linear map

$$U^0 \to \mathcal{L}(V/U, \mathbf{F}) = (V/U)' :$$

If φ is a linear functional in U^0, then φ is a linear map $V \to \mathbf{F}$ that is 0 on U and therefore is of the form $\psi \circ \pi$, where $\pi : V \to V/U$ is the quotient map and ψ is a linear map $V/U \to \mathbf{F}$, i.e., an element of $(V/U)'$. The map ψ is unique, given φ. In other words, $\varphi \in U^0 \to \psi \in (V/U)'$ is a well defined linear map.

Exercise: Check that this association is an isomorphism of vector spaces, i.e., an invertible linear map.

T and T': null spaces and ranges

Proposition

If $T : V \to W$ is a linear map, then the null space of T' is the annihilator of the range of T.

This proposition is the last result that we obtained on Monday. The proof is elementary — it just amounts to unraveling some definitions.

T and T': null spaces and ranges

In the context of the proposition on the previous slide, suppose now that V and W have finite dimension.

Corollary

The dimension of null T' is $\dim \operatorname{null} T + \dim W - \dim V$.

Proof: Since the nullspace of T' is the annihilator of the range of T, $\dim \operatorname{null} T' = \dim W - \operatorname{rank} T$. By the rank–nullity formula, $\operatorname{rank} T = \dim V - \dim \operatorname{null} T$. The desired formula follows.

The formula

$$\dim \operatorname{null} T' = \dim \operatorname{null} T + \dim W - \dim V$$

is equivalent to the relation

$$\dim W - \dim \operatorname{null} T' = \dim V - \dim \operatorname{null} T,$$

which becomes

$$\operatorname{rank} T' = \operatorname{rank} T$$

if we use rank–nullity for T and T'.

T and T': null spaces and ranges

Corollary

The dimension of null T' is $\dim \operatorname{null} T + \dim W - \dim V$.

Corollary (3.129)

The linear map T is onto if and only if its dual T' is 1-1.

Proof: The map T is onto $\iff$ range $T = W$, which is true if and only if the annihilator of range T is $\{0\}$. The annihilator of range T is the null space of T'. Thus T is onto if and only if T' has $\{0\}$ as its null space, which is true if and only if T' is injective.

Row rank = column rank

Corollary

If $T : V \to W$ is a linear map between finite-dimensional vector spaces, then T' and T have equal ranks.

We already proved this on a previous slide. This was the “coming attraction” at the start of the slide deck.

range T', annihilator of null T

Corollary (3.130)

If $T : V \to W$ is a linear map between finite-dimensional vector spaces, the range of T' is the annihilator of the null space of T.

The two spaces being compared are subspaces of V'. They have equal dimension: Indeed, the dimension of the annihilator of the null space of T is $\dim V - \dim \operatorname{null} T = \dim \operatorname{range} T$. Since $\dim \operatorname{range} T' = \dim \operatorname{range} T$, the dimensions agree.

It follows that the equality range $T' = (\operatorname{null} T)^0$ is equivalent to the *inclusion* range $T' \subseteq (\operatorname{null} T)^0$. This inclusion is the statement that the range of T' annihilates the null space of T. Now an element of the range of T' is a linear functional $T'\psi = \psi \circ T$, where ψ is a linear functional on W. The annihilation is the statement $(\psi \circ T)v = 0$ if $Tv = 0$. This is now clear because $(\psi \circ T)v = \psi(Tv) = \psi(0) = 0$.

Are we ready for Chapter 4?

We consider polynomials over **F**: real polynomials or complex polynomials.

Polynomials have a division algorithm just like positive integers. (This is 4.9 in LADR, by the way.) All of us can divide a polynomial $p = p(z)$ by a nonzero polynomial d, getting a quotient and a remainder:

$$p = qd + r,$$

where r is a polynomial of degree less than the degree of d.

The degree of the polynomial 0 is $-\infty$, which is deemed to be less than any integer ≥ 0.

Factorization of integers

An *irreducible polynomial* is a nonconstant polynomial that does not factor into a product of two nonconstant polynomials. Irreducible polynomials are like the prime numbers, except that usually we insist that prime numbers be *positive*. In analogy, we insist that irreducible polynomials be *monic*. The monic irreducible polynomials over **F** are then like the prime numbers.

Theorem (Fundamental theorem of arithmetic)

Every integer $\neq 0, 1, -1$ is $\pm$ the product of prime numbers. The factorization of an integer into such a product in unique, up to permutation of the prime factors.

This theorem is proved in Math 55. The same proof gives the unique factorization theorem for polynomials. The units among the polynomials are the polynomials with polynomial inverses; these are the nonzero constant polynomials.

Factorization of polynomials

Theorem

Each polynomial $\neq 0$ *or a unit is the product of irreducible polynomials times a unit (i.e., a nonzero constant). Factorizations of such polynomials are unique up to the order of the factors.*

Roots and divisors

Divide polynomials by $d = z - \lambda$, a polynomial of degree 1. If p is a polynomial, then $p = q(z - \lambda) + r$, where r is a "polynomial" of degree ≤ 0, i.e., a number. Set $z = \lambda$ to get $p(\lambda) = r$.

Proposition (4.6)

The linear factor $z - \lambda$ divides a polynomial if and only if λ is a root of the polynomial.

Polynomials

Proposition (4.6)

The linear factor $z - \lambda$ divides a polynomial if and only if λ is a root of the polynomial.

Corollary (4.8)

A polynomial of degree m has at most m roots.

We have discussed the fact that a polynomial that's identically 0 as a function is the 0 polynomial. The point is that a nonzero polynomial some degree m and then cannot have more than m roots. A polynomial that's identically zero has infinitely many roots!

Fundamental theorem of algebra

Theorem

A nonconstant polynomial over $\mathbf{C}$ has at least one root.

A telegraphic proof of this theorem is on page 125 of LADR. Last semester, I wrote a more detailed version of the proof. You'll find it in `Files` on `bCourses`.

Theorem

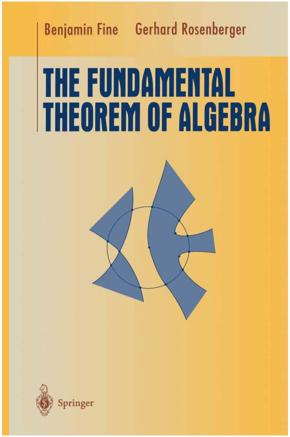
The irreducible polynomials over $\mathbf{C}$ are the polynomials of degree 1.

In other words, the irreducible polynomials over $\mathbf{C}$ are the various $z - \lambda$, multiplied by nonzero constants.

Fundamental theorem of algebra

You can grab this book for free if you're in `berkeley.edu`. Just click on the image to get to the right web page.

Proofs of the Fundamental Theorem are usually given in Math 185. Axler's proof uses "only" Math 104.



Summary of Axler's proof

Lemma

All complex numbers have complex kth roots for all $k \geq 1$.

Proof: Since 0 has the kth root 0, we can consider only nonzero complex numbers. If $z \neq 0$, $z \in \mathbf{C}$, then $z = |z| \cdot \dfrac{z}{|z|}$.

The first factor is a positive real number and thus has a kth root. The second factor has absolute value 1. Thus it suffices to find kth roots of complex numbers of absolute value 1. These numbers lie on the unit circle in the complex plane and are of the form $e^{i\theta}$. A kth root of $e^{i\theta}$ is $e^{i\theta/k}$.

Summary of Axler's proof

Axler's proof starts with a nonconstant polynomial f. We want to prove that f has a root. We can divide f by its top coefficient without changing anything of significance. Then $f = z^n +$ lower-degree terms for some $n \geq 1$. It should be clear to all that $|f(z)| \to \infty$ as $|z| \to \infty$.

Choose a (random) complex number a and let $M = |f(a)|$. If $M = 0$, a is a root of f and we're done.

Suppose that M is positive. Then we can find some radius $R > 0$ so that $|z| \geq R$ implies $|f(z)| \geq 2M$. A consequence is that all values of $|f|$ less than or equal to M occur on the disc $D = \{\, z \in \mathbf{Z} \mid |z| \leq R \,\}$. The aim is to show that 0 is one of those values.

Summary of Axler's proof

The big gun here is Math 104, which will imply that the set of values $|f(D)|$ is a closed interval $[A, B]$ with $A \leq B$ and A, B nonnegative real numbers. Once again, the aim is to show that $A = 0$.

The proof is by contradiction: Assuming that A is positive, we make estimates to show that there is some value of $|f|$ that's less than A. In this final step, we use the lemma about kth roots.