Quotients, dual spaces

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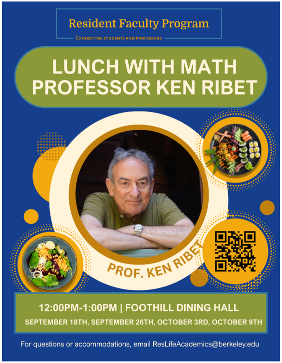
Announcements

The problems in §3D have been moved to HW #6. The bonus problem has not budged.

My office hours are Mondays 1:30–3 PM and Thursday 10:30 AM–noon

Next to last Foothill DC lunch with Prof. Ribet is *today* at noon



U.S. Cardiologist Warns Aging Seniors About Blueberries for Breakfast

Quotient spaces

If U is a subspace of V, we have defined a vector space V/U along with a surjective map ("quotient map")

$$\pi : V \to VU.$$

If W is a vector space, there is a natural linear map $\mathcal{L}(V/U, W) \to \mathcal{L}(V, W)$, $f : S \longmapsto S \circ \pi$. A linear map $V \to W$ of the form $S \circ \pi$ is said to *factor through* π.

Proposition

The function f is an injective linear map $\mathcal{L}(V/U, W) \to \mathcal{L}(V, W)$ whose image is the set of linear maps $V \to W$ whose restriction to U is 0.

A paraphrase: a linear map $V \to W$ factors through π if and only if its null space contains U.

Quotients

Proposition

The function f is an injective linear map $\mathcal{L}(V/U, W) \to \mathcal{L}(V, W)$ whose image is the set of linear maps $V \to W$ whose restriction to U is 0.

"A linear map $V \to W$ factors through π if and only if its null space contains U."

On Wednesday, we proved this. If $T : V \to W$ is identically 0 on U, then $T = S \circ \pi$, where $S : V/U \to W$ is defined by the formula $S(v + U) = Tv$. That $Tu = 0$ for all $u \in U$ ensures that S is well defined. So we're in good shape, and the class on Wednesday ended at a good breakpoint.

The range and null space of S

Let $T : V \to W$ be a linear map and let $U \subseteq V$ be a subspace that is *contained in* null T. Let S be the unique linear map $V/U \to W$ such that $T = S \circ \pi$.

Proposition

The range of S is the range of T.

The range of S is the set of all $S(v + U)$, but $S(v + U) = Tv$. Thus the range of S consists of all vectors $Tv \in W$ and is therefore the range of T.

Proposition

The null space of S is the quotient (null $T)/U$.

The null space of S is the set of all $v + U \in V/U$ such that $S(v + U) = 0$. This is the set of all $v + U$ for which $Tv = 0$, i.e., the set of all $v + U$ with $v \in$ null T. This is the quotient (null $T)/U$ of the proposition.

LADR's map $\tilde{T}$

A slightly different perspective. Start with a linear map $T : V \to W$, and let $U = \text{null}\, T$. Then $T = S \circ \pi$ for some $S : V/(\text{null}\, T) \to W$.

In LADR, the map S in this situation is called $\tilde{T}$.

Proposition (3.107)

If $U = \text{null}\, T$, the map $\tilde{T} : V/U \to W$ is injective. Its range is the range of T.

This follows from our more general discussion, since the null space of $\tilde{T}$ is $(\text{null}\, T)/U = (\text{null}\, T)/(\text{null}\, T) = 0$.

Functoriality

Starting with $\pi : V \to V/U$, we produced and studied a linear map $\mathcal{L}(V/U, W) \to \mathcal{L}(V, W)$.

More generally, if $\pi : V \to X$ is some linear map (not necessarily surjective), there's an induced linear map

$$\mathcal{L}(X, W) \to \mathcal{L}(V, W), \qquad S \mapsto S \circ \pi.$$

Note that the displayed map goes in the opposite direction from π in the sense that π goes from $V \to X$ and the induced map goes from an object related to X to an object related to V.

Denizens of Evans Hall would write π^* for the induced map and say that $\mathcal{L}(\bullet, W)$ depends *contravariantly* on the first argument $\bullet$. Remember that "contra" means "against or contrary to"; mathematicians interpret this as "opposite of."

Functoriality

Suppose that α is a linear map $W \to Y$. Then composition with α gives rise to a linear map $\mathcal{L}(V, W) \to \mathcal{L}(V, Y)$:

$$\alpha_* : \mathcal{L}(V, W) \to \mathcal{L}(V, Y), \qquad T \mapsto \alpha \circ T.$$

The Evans folk like to say that $\mathcal{L}(V, \bullet)$ is covariant in the second variable.

Dual space

If V and W are vector spaces, we live happily with $\mathcal{L}(V, W)$. It's a space of linear maps. It's a space of $m \times n$ matrices. We're totally comfortable.

Make the choice $W = \mathbf{F}$. It's a special case, so we're still comfortable. It's a space of $1 \times n$ matrices. Chill.

Now say that $\mathcal{L}(V, \mathbf{F})$ is the *dual space* V'. Suddenly we're uneasy.

Refer to the linear maps $V \to \mathbf{F}$ in $\mathcal{L}(V, \mathbf{F})$ as *linear functionals*. Our palms are sweaty.

Use Greek letters like φ for the linear maps $W \to \mathbf{F}$ and we're quaking.

Welcome to §3F.

Dual space

The vector space dual to a space V is $V' = \mathcal{L}(V, \mathbf{F})$. If V has dimension n, V' has dimension $1 \cdot n = n$.

Suppose that V has a basis $v_1, \ldots, v_n$. Is there a natural basis of V' that results from $v_1, \ldots, v_n$?

This question becomes *easier* if we replace $\mathbf{F}$ by W and say that $w_1, \ldots, w_m$ is a basis for W. Then $\mathcal{L}(V, W)$ is the space of $m \times n$ matrices; it has a basis consisting of the matrices with a 1 in the kth row and ℓth column and 0s elsewhere (for $k = 1, \ldots, m$ and $\ell = 1, \ldots, n$). The matrix with a 1 in place ℓ, k and 0s elsewhere represents the linear map sending v_k to w_ℓ and all other basis vectors of V to 0.

Dual space

Again: a basis of $\mathcal{L}(V, W)$ is given by the linear maps sending v_k to w_ℓ and all other basis vectors of V to 0. There is one for each k between 1 and n and each ℓ between 1 and m.

Now take $W = \mathbf{F}$ and use the list of length one "1" as a basis of $\mathbf{F}$. Then $V' = \mathcal{L}(V, \mathbf{F})$ has a basis consisting of the n different linear maps gotten by sending some v_k to 1 and all other v_j to 0. If k is between 1 and n, the linear map taking v_k to 1 and the other basis vectors to 0 is called φ_k.

The linear maps φ_1, φ_2,. . . , φ_n form a basis of $\mathcal{L}(V, \mathbf{F})$ that is called the "dual basis"; it's the basis dual to $v_1, \ldots, v_n$.

Dual basis

Again, if V has dimension n, $V' = \mathcal{L}(V, \mathbf{F})$ also has dimension n.

If $v_1, \ldots, v_n$ is a basis of V, $\varphi_1, \ldots, \varphi_n$ is the basis of V' that is dual to $v_1, \ldots, v_n$. Here's a nifty formula:

$$\varphi_k : v_j \mapsto \begin{cases} 1 & \text{if } j = k; \\ 0 & \text{if } j \neq k. \end{cases}$$

A more compact version:

$$\varphi_k(v_j) = \delta_{kj}.$$

Dual basis gives coordinates

Now suppose that v is a vector in V. Then there are unique scalars $\lambda_1, \ldots, \lambda_n$ such that $v = \lambda_1 v_1 + \cdots + \lambda_n v_n$.

Formula (3.114)

For $k = 1, \ldots, n$, $\lambda_k = \varphi_k(v)$.

Compute to see this:

$$\varphi_k(v) = \varphi_k(\sum_j \lambda_j v_j) = \sum_j \lambda_j \varphi_k(v_j) = \sum_j \lambda_j \delta_{kj} = \lambda_k.$$

Special case where $V = \mathbf{F}^n$

If $V = \mathbf{F}^n$ and the basis is the standard basis $e_1, \ldots, e_n$, then the λ_j for a vector $v = (x_1, \ldots, x_n)$ are the coordinates x_1, x_2, etc. Thus

$$\varphi_k((x_1, \ldots, x_n)) = x_k$$

for all $k = 1, \ldots, n$.

Dual map

If $T : V \to W$ is a linear map, there is an induced linear map

$$T^* : \mathcal{L}(W, \mathbf{F}) \to \mathcal{L}(V, \mathbf{F}), \qquad \psi \in \mathcal{L}(W, \mathbf{F}) \longmapsto \psi \circ T \in \mathcal{L}(V, \mathbf{F}).$$

This map is called T' and is said to be the map *dual to* T. We write

$$T' : W' \to V'.$$

Dual maps

Axler lists a number of basic properties of dual maps (3.120 in LADR):

- $(S+T)' = S' + T'$ for $S, T \in \mathcal{L}(V, W)$;
- $(\lambda T)' = \lambda T'$ for $T \in \mathcal{L}(V, W)$;
- $(ST)' = T'S'$ for $T \in \mathcal{L}(V, W)$ and $S \in \mathcal{L}(W, X)$.

These are all things that you should check. For example, $(ST)'$ is the map from X' to V' taking $\varphi \in \mathcal{L}(X, \mathbf{F})$ to $\varphi \circ (ST)$. But

$$\varphi \circ (ST) = (\varphi \circ S) \circ T = T'(\varphi \circ S) = T'(S'\varphi).$$

Matrix of dual map

Let $T : V \to W$ be a linear map with V and W finite-dimensional. Choose bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ of V and W. The matrix of T relative to these bases is

$$\mathcal{M}(T) = (a_{ij}), \quad Tv_j = \sum_{i=1}^{m} a_{ij} w_i \text{ for } j = 1, \ldots n.$$

The dual of T is the map $T' : W' \to V'$, $\psi \mapsto \psi \circ T$.

Let $\varphi_1, \ldots, \varphi_n$ be the basis of V' dual to $v_1, \ldots, v_n$. Analogously, let $\psi_1, \ldots, \psi_m$ be the basis of W' dual to $w_1, \ldots, w_m$.

Then T and T' are represented by matrices $\mathcal{M}(T)$ and $\mathcal{M}(T')$ of dimensions $m \times n$ and $n \times m$, respectively.

Formula

The matrices $\mathcal{M}(T)$ and $\mathcal{M}(T')$ are transposes of each other.

Matrix of dual map

Formula

The matrices $\mathcal{M}(T)$ and $\mathcal{M}(T')$ are transposes of each other.

If $\mathcal{M}(T') = (b_{ij})$, then $T'(\psi_j) = \psi_j T = \sum_i b_{ij}\varphi_i$ for each $j = 1, \ldots, m$. The formula to be proved is $b_{ij} = a_{ji}$ for each i and j. I.e., the formula states (for each j)

$$\psi_j T \stackrel{?}{=} \sum_i a_{ji}\varphi_i.$$

The two sides of the desired equality are linear maps $V \to W$. A key fact is that two linear maps $V \to W$ are equal if they agree on the basis vectors $v_1, \ldots, v_n$.

Thus the formula is equivalent to the equality

$$\psi_j(Tv_k) \stackrel{?}{=} \sum_i a_{ji}\varphi_i(v_k)$$

for all $j = 1, \ldots, m$ and $k = 1, \ldots, n$.

Matrix of dual map

Formula

The matrices $\mathcal{M}(T)$ and $\mathcal{M}(T')$ are transposes of each other.

The left-hand side of the equality to be verified is

$$\psi_j(\sum_i a_{ik} w_i) = \sum_i a_{ik} \psi_j(w_i) = a_{jk},$$

the point being that $\psi_j(w_i)$ is 0 except when $i = j$, when it's 1.

The right-hand side $\sum_i a_{ji} \varphi_i(v_k)$ also collapses to a single term, and for the same reason: $\varphi_i(v_k)$ is 0 except when $i = k$ (when it's 1). The single term is a_{jk}, as for the left-hand side.