1 Lecture 5: February 3, 2026

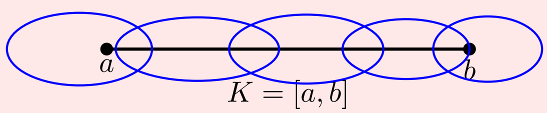
1.1 Compactness

Definition 1.1. Let $E \subseteq X$ be a topological space. If $\{G_\alpha\}$ is a collection of open sets of X such that $E \subseteq \bigcup G_\alpha$, then $\{G_\alpha\}$ is an **open cover** of E.

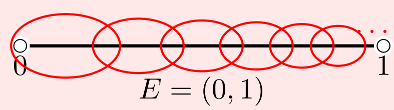
Definition 1.2. A subset $K \subseteq X$ is **compact** if every open cover contains a finite subcover. That is, given $\{G_\alpha\}$, there exist $G_{\alpha_1}, \ldots, G_{\alpha_n} \subseteq \{G_\alpha\}$ such that $K \subseteq G_{\alpha_1} \cup G_{\alpha_2} \cup \cdots \cup G_{\alpha_n}$.

Understanding covers and compactness. An open cover is a collection of open sets that together contain every point of E. Think of it as "blanketing" the set with open sets. The cover may have infinitely many sets — in fact, that's the interesting case.
Compactness says: no matter how you cover K with open sets, you can always throw away all but finitely many and still cover K. This is a strong condition — it fails for many sets.

Open cover: finitely many suffice



Not compact: needs infinitely many



The closed interval $[a, b]$ is compact: any open cover has a finite subcover. The open interval $(0, 1)$ is not compact: the cover $\{(\frac{1}{n}, 1) : n \geq 2\}$ has no finite subcover, since points near 0 escape any finite subcollection.

1.2 Subspace Topology

$Y \subseteq X$. U is open in Y if and only if there exists V open in X such that $U = V \cap Y$.

Theorem 1.3. *If $K \subseteq Y \subseteq X$, then K compact relative to X $\Leftrightarrow$ K compact relative to Y.*

Proof. ($\Rightarrow$) Suppose K is compact relative to X. Let $\{U_\alpha\}$ be an open cover of K in Y. By the subspace topology, for each U_α there exists V_α open in X such that $U_\alpha = V_\alpha \cap Y$. Then $\{V_\alpha\}$ is an open cover of K in X. Since K is compact relative to X, there exist $V_{\alpha_1}, \ldots, V_{\alpha_n}$ such that $K \subseteq V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$. Then

$$K \subseteq (V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}) \cap Y = U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}.$$

So K is compact relative to Y.

($\Leftarrow$) Suppose K is compact relative to Y. Let $\{V_\alpha\}$ be an open cover of K in X. Then $\{V_\alpha \cap Y\}$ is an open cover of K in Y (each $V_\alpha \cap Y$ is open in Y by the subspace topology). Since K is compact relative to Y, there exist $V_{\alpha_1} \cap Y, \ldots, V_{\alpha_n} \cap Y$ such that $K \subseteq (V_{\alpha_1} \cap Y) \cup \cdots \cup (V_{\alpha_n} \cap Y)$. Since $K \subseteq Y$, we have $K \subseteq V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$. So K is compact relative to X. $\square$

Theorem 1.4. *Compact subsets of metric spaces are closed.*

Proof. It suffices to show that K^c is open. Since we are in a metric space, we can use the open ball definition of open sets. We will show for every point $p \in K^c$, there exists $\varepsilon > 0$ such that $B(p, \varepsilon) \subseteq K^c$.

Fix $p \in K^c$. For each $q \in K$, let $r_q = \frac{1}{2}d(p,q) > 0$. The balls $B(p, r_q)$ and $B(q, r_q)$ are disjoint. The collection $\{B(q, r_q) : q \in K\}$ is an open cover of K. By compactness, there exist $q_1, \dots, q_n \in K$ such that

$$K \subseteq B(q_1, r_{q_1}) \cup \cdots \cup B(q_n, r_{q_n}).$$

Let $\varepsilon = \min(r_{q_1}, \dots, r_{q_n}) > 0$. Then $B(p, \varepsilon) \subseteq K^c$, since $B(p, \varepsilon) \subseteq B(p, r_{q_i})$ is disjoint from $B(q_i, r_{q_i})$ for each i, and K is covered by these balls. □

Theorem 1.5. *Closed subsets of compact sets are compact.*

Proof. Let F be a closed subset of a compact set K in X. Let $\{G_\alpha\}$ be an open cover of F. Then $\{G_\alpha\} \cup \{F^c\}$ is an open cover of K (F is closed so F^c is open, and $\{G_\alpha\}$ covers F). Since K is compact, there exists a finite subcover: $G_{\alpha_1}, \dots, G_{\alpha_n}$, and possibly F^c. Removing F^c (if present), we have $F \subseteq G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}$. Thus F is compact. □

Corollary 1.6. *The intersection of a closed set and a compact set is compact (in a metric space).*

Theorem 1.7. *Suppose $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of any finite subcollection is nonempty. Then $\bigcap_\alpha K_\alpha \neq \emptyset$.*

Proof. Suppose $\bigcap_\alpha K_\alpha = \emptyset$. Each K_α^c is open. By De Morgan's law,

$$\left(\bigcap_\alpha K_\alpha\right)^c = \bigcup_\alpha K_\alpha^c = X.$$

Fix $K_1 \in \{K_\alpha\}$. Then $\{K_\alpha^c\}$ is an open cover of K_1. By compactness, there exist $K_{\alpha_1}, \dots, K_{\alpha_n}$ such that $K_1 \subseteq K_{\alpha_1}^c \cup \cdots \cup K_{\alpha_n}^c$. By De Morgan's law,

$$K_1 \subseteq (K_{\alpha_1} \cap \cdots \cap K_{\alpha_n})^c.$$

Thus $K_1 \cap K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} = \emptyset$, contradicting the finite intersection property. □

Corollary 1.8. *If $\{K_n\}$ is a sequence of compact subsets of a metric space X such that $K_n \supseteq K_{n+1}$, then $\bigcap_n K_n \neq \emptyset$.*

Theorem 1.9. *If E is an infinite subset of a compact set K, then E has a limit point in K.*

Proof. (By contradiction) Assume that E has no limit points. Then for all $p \in K$, there exists a neighborhood U_p of p where either $U_p \cap E = \emptyset$ or $U_p \cap E = \{p\}$. Therefore, $\{U_p\}$ forms an open cover of K. By compactness of K, there is a finite subcover $U_{p_1} \cup \cdots \cup U_{p_n} \supseteq K$. Each U_{p_i} contains at most one point of E, so E has at most n points. This contradicts the fact that E is infinite. □

The Cantor set is nonempty. Recall the Cantor set is defined as $C = \bigcap_{n=0}^\infty C_n$, where each C_n is a finite union of closed intervals. Each C_n is compact. The sets are nested: $C_0 \supseteq C_1 \supseteq C_2 \supseteq \cdots$, so any finite intersection equals the smallest set in the subcollection, which is nonempty. By the theorem above, $C = \bigcap_{n=0}^\infty C_n \neq \emptyset$.

Theorem 1.10. *Suppose $\{I_n\}$ is a sequence of closed intervals in $\mathbb{R}$ such that $I_n \supseteq I_{n+1}$. Then $\bigcap_k I_k \neq \emptyset$.*

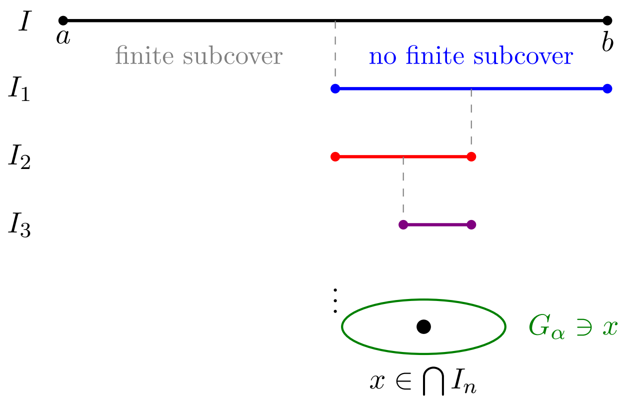
Proof. Let $I_n = [a_n, b_n]$. Let $E = \{a_n\}$. Since the intervals are nested, $a_n \leq b_m$ for all n, m, so E is bounded above. There exists $x = \sup E$.

For all n, we have $a_n \leq x$ (since x is an upper bound of E). Also $x \leq b_n$ for all n (since each b_n is an upper bound for E, and x is the least upper bound). Thus $a_n \leq x \leq b_n$, so $x \in I_n$ for all n. Therefore $x \in \bigcap_k I_k$. □

Theorem 1.11. *Closed intervals (and therefore closed boxes) are compact.*

Proof. Let $I = [a, b]$ and let $\{G_\alpha\}$ be an open cover of I. Suppose this open cover does not reduce to a finite subcover. Cut the interval in half: at least one half cannot be covered by finitely many G_α (if both halves could, we could combine them to cover I). Call this half I_1. Repeat: bisect I_1 and choose a half I_2 with no finite subcover. Continuing, we obtain nested closed intervals $I \supseteq I_1 \supseteq I_2 \supseteq \cdots$ with $|I_n| = (b - a)/2^n$, each having no finite subcover.

By the nested intervals theorem, there exists $x \in \bigcap_n I_n$. Since $\{G_\alpha\}$ covers I, we have $x \in G_\alpha$ for some α. Since G_α is open, there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq G_\alpha$. For large n, $|I_n| < \varepsilon$ and $x \in I_n$, so $I_n \subseteq (x - \varepsilon, x + \varepsilon) \subseteq G_\alpha$. But then I_n is covered by a single open set, contradicting that I_n has no finite subcover.



□

Key ideas in this proof:

1. **Proof by contradiction:** Assume no finite subcover exists and derive a contradiction.
2. **Bisection argument:** If a set has no finite subcover, at least one half doesn't either. This lets us build nested intervals.
3. **Nested intervals theorem:** The intersection $\bigcap I_n \neq \emptyset$, giving us a point x.
4. **Open set definition:** Since $x \in G_\alpha$ and G_α is open, there exists $\varepsilon > 0$ with $(x - \varepsilon, x + \varepsilon) \subseteq G_\alpha$.
5. **Intervals shrink to zero:** $|I_n| = (b - a)/2^n \to 0$, so eventually I_n fits inside the ε-neighborhood, giving a single-set cover — contradiction.

Theorem 1.12 (**Heine-Borel**)**.** *Let $E \subseteq \mathbb{R}^n$. The following are equivalent:*

1. *E is closed and bounded.*
2. *E is compact.*
3. *Every infinite subset of E has a limit point in E.*

Proof. $(1 \Rightarrow 2)$: Since E is bounded, $E \subseteq [-M, M]^n$ for some $M > 0$. The closed box $[-M, M]^n$ is compact. Since E is a closed subset of a compact set, E is compact.

$(2 \Rightarrow 3)$: This follows from the theorem: if E is an infinite subset of a compact set K, then E has a limit point in K. Taking $K = E$, every infinite subset of E has a limit point in E.

$(3 \Rightarrow 1)$: *Closed:* Let p be a limit point of E. Every neighborhood of p contains a point of E distinct from p. We can construct a sequence (x_n) in E with $x_n \to p$. The set $\{x_n\}$ is infinite, so by (3) it has a limit point in E. This limit point must be p, so $p \in E$. Thus E contains all its limit points, so E is closed.

Bounded: Suppose E is unbounded. Then for each $n \in \mathbb{N}$, there exists $x_n \in E$ with $|x_n| > n$. The set $\{x_1, x_2, \dots\}$ is infinite. By (3), it has a limit point $p \in E$. But for any $\varepsilon > 0$, only finitely many x_n lie in $B(p, \varepsilon)$ (since $|x_n| \to \infty$), contradicting that p is a limit point. Thus E is bounded. □

What Heine-Borel means and how to use it.
In $\mathbb{R}^n$, compactness has a simple characterization: *closed and bounded.* This is easy to check! You don't need to verify that every open cover has a finite subcover — just check two conditions.

Common uses:

- **Proving a set is compact:** Show it's closed (contains its limit points) and bounded (fits in some ball). Examples: $[0, 1]$, closed balls $\overline{B}(x, r)$, the Cantor set.
- **Proving a set is NOT compact:** Show it's either not closed or not bounded. Examples: $(0, 1)$ is not closed; $\mathbb{R}$ is not bounded.
- **Extracting convergent subsequences:** Condition (3) says infinite subsets have limit points. This is the key to proving the Bolzano-Weierstrass theorem: every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Warning: Heine-Borel is specific to $\mathbb{R}^n$. In general metric spaces, compact implies closed and bounded, but the converse can fail.

Theorem 1.13 (**Weierstrass**)**.** *Every bounded infinite subset has a limit point in $\mathbb{R}^n$.*

1.3 Perfect Sets

Recall: E is **perfect** if E is closed and has no isolated points. If E is perfect, then $E = \overline{E} = E'$.

Theorem 1.14. *Every nonempty perfect subset of $\mathbb{R}^n$ is uncountable.*

Proof. Let $P \subseteq \mathbb{R}^n$ be nonempty and perfect. Suppose for contradiction that P is countable, say $P = \{x_1, x_2, x_3, \dots\}$.

We construct nested closed sets $V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots$ such that:

1. $V_n \cap P \neq \emptyset$ for all n,
2. $x_n \notin V_n$ for all n.

Base case: Since P has no isolated points, x_1 is a limit point of P. Choose $y_1 \in P$ with $y_1 \neq x_1$. Let $V_1 = \overline{B}(y_1, r_1)$ where $r_1 = \frac{1}{2}d(x_1, y_1)$. Then $y_1 \in V_1 \cap P$ and $x_1 \notin V_1$.

Inductive step: Suppose V_n is constructed with $V_n \cap P \neq \emptyset$ and $x_n \notin V_n$. Pick any $y \in V_n \cap P$. Since P is perfect, y is a limit point of P, so there exists $y_{n+1} \in P \cap V_n$ with $y_{n+1} \neq x_{n+1}$ (if $x_{n+1} \notin V_n$, any point works; if $x_{n+1} \in V_n$, choose a different point). Let $V_{n+1} = \overline{B}(y_{n+1}, r_{n+1}) \cap V_n$ where r_{n+1} is small enough that $x_{n+1} \notin V_{n+1}$ and $V_{n+1} \subseteq V_n$.

Each V_n is closed and bounded, hence compact. The V_n are nested and nonempty, so by the finite intersection property, $\bigcap_n V_n \neq \emptyset$. Let $x \in \bigcap_n V_n$. Since each $V_n \cap P$ is closed (intersection of closed sets) and the V_n are nested, we have $x \in P$. But $x \neq x_n$ for all n (since $x_n \notin V_n$). This contradicts $P = \{x_1, x_2, \dots\}$. □

Theorem 1.15. *The Cantor set is perfect.*

Proof. The Cantor set C is closed. Additionally, using the ternary expansion: for any $x \in C$, we can truncate x at the n-th digit and define a sequence (x_n) in C with $|x - x_n| \leq 3^{-n}$. Thus $x_n \to x$, so x is a limit point of C. Hence C has no isolated points, and C is perfect.

$$x = 0.\underbrace{0202\overset{\text{truncate here}}{0}}_{n\text{ digits}}\,2002\cdots$$

$$x_n = 0.02020\,\underbrace{0000\cdots}_{\text{zeros}}$$

$$|x - x_n| = 0.\;\underbrace{00000}_{\text{first } n \text{ digits}}\;2002\cdots$$

$$\leq 0.00000\overline{22} = \tfrac{2}{3^{n+1}} \cdot \tfrac{1}{1-1/3} = \tfrac{1}{3^n}$$

□