# 1 Lecture 2: January 22, 2026

**Lecture Overview:** We construct the real numbers $\mathbb{R}$ as an ordered field with the Least Upper Bound Property (LUBP) containing $\mathbb{Q}$ as a subfield, using Dedekind cuts. We prove key properties of $\mathbb{R}$: the Archimedean property and density of $\mathbb{Q}$ in $\mathbb{R}$. Using the LUBP, we establish existence of $n$th roots of positive reals via a supremum argument. We discuss decimal/ternary representations and the Cantor set. We introduce the complex numbers $\mathbb{C}$ and prove $\mathbb{C}$ is not an ordered field. Finally, we define Euclidean spaces $\mathbb{R}^n$ with inner products and norms, and prove the Cauchy-Schwarz inequality.

## 1.1 Dedekind Cuts

**Section Overview:** We define Dedekind cuts as a way to construct the real numbers from the rationals.

**Definition 1.1.** A **cut** $\alpha \subset \mathbb{Q}$ is a nonempty, proper subset such that:

1. **Downward closed:** If $p \in \alpha$ and $q < p$, then $q \in \alpha$.
2. If $\sup \alpha$ exists, then $\sup \alpha \notin \alpha$.

The set of all cuts is ordered by inclusion: $\alpha \leq \beta$ if and only if $\alpha \subseteq \beta$.

## 1.2 Field Operations on Cuts

**Section Overview:** We define addition and multiplication on cuts to make them into an ordered field.

**Addition:**

$$\alpha + \beta = \{r + s \mid r \in \alpha, s \in \beta\}$$

**Additive identity:**

$$0^* = \{p \in \mathbb{Q} \mid p < 0\}$$

**Multiplication:** For $\alpha > 0^*$ and $\beta > 0^*$:

$$\alpha\beta = \{p \in \mathbb{Q} \mid p \leq rs \text{ for some } r \in \alpha, r > 0 \text{ and } s \in \beta, s > 0\}$$

## 1.3 Least Upper Bound Property

**Section Overview:** We show that the set of cuts has the LUBP.

For a nonempty set $E$ of cuts that is bounded above:

$$\sup E = \bigcup_{\alpha \in E} \alpha$$

## 1.4 Embedding $\mathbb{Q}$ into $\mathbb{R}$

**Section Overview:** We embed the rationals into the reals as a subfield.

For $p \in \mathbb{Q}$, define the cut:

$$p^* := \{q \in \mathbb{Q} \mid q < p\}$$

This embedding $p \mapsto p^*$ identifies $\mathbb{Q}$ as a subfield of $\mathbb{R}$.

## 1.5 Properties of $\mathbb{R}$

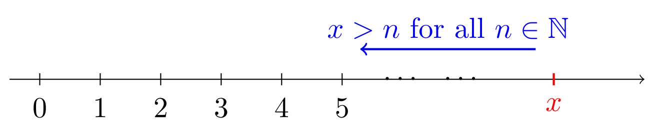
**Section Overview:** Having constructed $\mathbb{R}$, we now explore its key properties.

**Theorem 1.2** (Archimedean Property)**.** *For any $x, y \in \mathbb{R}$ with $x > 0$, there exists $n \in \mathbb{N}$ such that $nx > y$.*

*Proof.* By contradiction. Suppose no such $n$ exists, i.e., $nx \leq y$ for all $n \in \mathbb{N}$. Then the set $A = \{nx : n \in \mathbb{N}\}$ is bounded above by $y$. By the LUBP, $\sup A$ exists. Let $\alpha = \sup A$. Since $x > 0$, we have $\alpha - x < \alpha$, so $\alpha - x$ is not an upper bound for $A$. Thus there exists $m \in \mathbb{N}$ with $mx > \alpha - x$, which gives $(m+1)x > \alpha$. But $(m+1)x \in A$, contradicting that $\alpha = \sup A$. $\square$

*Remark* 1.3. The Archimedean property ensures there are no **infinitely large** elements (every element is bounded by some natural number) and no **infinitesimals** (positive elements smaller than $1/n$ for all $n$). This property is essential for proving that $\mathbb{Q}$ is dense in $\mathbb{R}$.

**Example of a non-Archimedean field:** Consider the field of rational functions $\mathbb{R}(x)$ with the ordering where $x$ is declared to be larger than every real number (i.e., $x > r$ for all $r \in \mathbb{R}$). Then $x > n$ for all $n \in \mathbb{N}$, so the Archimedean property fails. In this field, $1/x$ is an infinitesimal: it is positive but smaller than $1/n$ for all $n \in \mathbb{N}$.




**Theorem 1.4** (Density of $\mathbb{Q}$ in $\mathbb{R}$)**.** *For any $a, b \in \mathbb{R}$ with $a < b$, there exists $q \in \mathbb{Q}$ such that $a < q < b$.*

*Proof.* Since $b - a > 0$, by the Archimedean property there exists $n \in \mathbb{N}$ such that $n(b-a) > 1$, i.e., $nb - na > 1$. Thus there exists an integer $m$ with $na < m < nb$. Then $a < \frac{m}{n} < b$, and $q = \frac{m}{n} \in \mathbb{Q}$. $\square$

## 1.6 The Roots of Reals

**Section Overview:** Having constructed $\mathbb{R}$ with the LUBP, we can now prove that $n$th roots of positive reals exist, resolving the gap in $\mathbb{Q}$ where $\sqrt{2}$ was missing.

Previously, we showed that $\sqrt{2} \notin \mathbb{Q}$. Now that we have constructed $\mathbb{R}$ with the LUBP, we can prove that $n$th roots exist.

**Theorem 1.5** (Existence of $n$th Roots)**.** *For all $x \in \mathbb{R}_{>0}$ and for all $n \in \mathbb{Z}_{>0}$, there exists a unique $y \in \mathbb{R}_{>0}$ such that $y^n = x$.*

*Proof.* Let $E = \{t \in \mathbb{R}_{>0} : t^n < x\}$.

$E$ **is non-empty:** We have

$$\left(\frac{x}{x+1}\right)^n < \frac{x}{x+1} < x,$$

so $\frac{x}{x+1} \in E$.

$E$ **is bounded above:** (to be shown)

By the LUBP, $y = \sup E$ exists.

**Claim:** $y^n = x$.

*Aside (Trichotomy):* Since $\mathbb{R}$ is a totally ordered set, for any $a, b \in \mathbb{R}$, exactly one of the following holds: $a < b$, $a = b$, or $a > b$. Thus for $y^n$ and $x$, exactly one of $y^n < x$, $y^n = x$, or $y^n > x$ holds. We show the first and third cases lead to contradictions.

**Case 1:** Suppose $y^n < x$. Then there exists $h > 0$ small enough such that $(y + h)^n < x$. But then $y + h \in E$, contradicting that $y = \sup E$.

**Case 2:** Suppose $y^n > x$. Then there exists $h > 0$ small enough such that $(y - h)^n > x$. But then $y - h$ is still an upper bound for $E$, contradicting that $y = \sup E$ (the *least* upper bound).

Therefore $y^n = x$. $\square$

**Note to the reader:** This proof employs a fundamental technique in real analysis called a *supremum argument*. The strategy is:

1. **Define a set:** Construct a set $E$ of elements that are "too small" (i.e., $t^n < x$).

2. **Apply LUBP:** Since $E$ is nonempty and bounded above, $\sup E$ exists—this is where we crucially use that $\mathbb{R}$ has the Least Upper Bound Property.

3. **Use trichotomy:** By the trichotomy of total orders, the supremum $y$ satisfies exactly one of $y^n < x$, $y^n = x$, or $y^n > x$.

4. **Eliminate by contradiction:** Show that $y^n < x$ contradicts $y$ being an *upper* bound (we can go higher), and $y^n > x$ contradicts $y$ being the *least* upper bound (we can find a smaller upper bound).

This technique appears repeatedly throughout analysis whenever we need to prove existence of a value with a specific property. A similar technique is employed in Exercise 7 (showing the existence of the logarithm).

## 1.7 Decimals, Binaries, Ternaries

**Section Overview:** We discuss representations of real numbers in different bases.

Observe that decimal representations come in the form

$$n_0 + \frac{n_1}{10} + \frac{n_2}{100} + \cdots = n_0 + \sum_{k=1}^{\infty} \frac{n_k}{10^k}$$

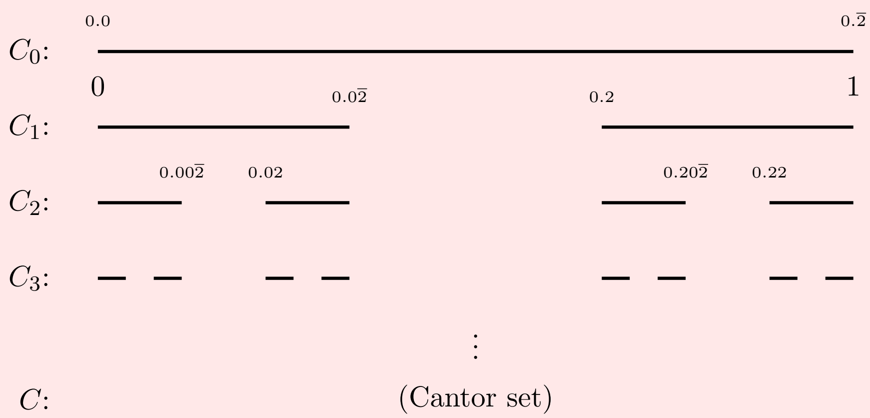
where $n_0 \in \mathbb{Z}$ and $n_k \in \{0, 1, 2, \ldots, 9\}$ for $k \geq 1$.

If we consider the set of partial sums

$$E = \left\{ n_0,\ n_0 + \frac{n_1}{10},\ n_0 + \frac{n_1}{10} + \frac{n_2}{100},\ \ldots \right\}$$

then $x = \sup E$.

**Note:** This construction is used to build the **Cantor set**. Starting with the interval $[0, 1]$, we iteratively remove the open middle third of each remaining interval:




Here the labels are *ternary* (base-3) expansions: e.g., $0.2_3 = \frac{2}{3}$, $0.02_3 = \frac{2}{9}$, $0.22_3 = \frac{8}{9}$, and $0.\overline{2}_3 = 0.222\ldots_3 = 1$.

The Cantor set $C = \bigcap_{n=0}^{\infty} C_n$ consists of all points in $[0, 1]$ whose ternary (base-3) expansion contains only the digits 0 and 2.

## 1.8 The Complex Field

**Section Overview:** We introduce the complex numbers $\mathbb{C}$ as an extension of $\mathbb{R}$.

**Definition 1.6.** The **complex numbers** are defined as

$$\mathbb{C} = \mathbb{R}[x]/(x^2 + 1)$$

where each element is of the form $z = a + bi$ with $a, b \in \mathbb{R}$ and $i^2 = -1$.

**Theorem 1.7.** $\mathbb{C}$ *is not an ordered field.*

*Proof.* By contradiction. Suppose $\mathbb{C}$ is an ordered field. By trichotomy, either $i > 0$ or $i < 0$ (since $i \neq 0$).

**Case 1:** If $i > 0$, then $i^2 > 0$ (since squares of nonzero elements are positive in an ordered field). But $i^2 = -1 < 0$, a contradiction.

**Case 2:** If $i < 0$, then $-i > 0$, so $(-i)^2 > 0$. But $(-i)^2 = i^2 = -1 < 0$, a contradiction.

Therefore $\mathbb{C}$ cannot be an ordered field. □

## 1.9 The Euclidean Spaces

**Section Overview:** We introduce Euclidean spaces $\mathbb{R}^n$ as spaces of ordered $n$-tuples.

**Definition 1.8.** The **Euclidean space** $\mathbb{R}^n$ is the set of all ordered $n$-tuples

$$\vec{x} = (x_1, x_2, x_3, \dots, x_n)$$

where $x_i \in \mathbb{R}$ for each $i = 1, \dots, n$.

For $\vec{x} = (x_1, \dots, x_n)$, $\vec{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ and $c \in \mathbb{R}$:

**Addition:**

$$\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$

**Scalar multiplication:**

$$c\vec{x} = (cx_1, cx_2, \dots, cx_n)$$

**Inner product:**

$$\langle \vec{x}, \vec{y} \rangle = \vec{x} \cdot \vec{y} = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

**Definition 1.9.** An **inner product** over $\mathbb{R}$ is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ that is:

- Symmetric bilinear and positive definite (for real vector spaces), or
- Hermitian sesquilinear and positive definite (for complex vector spaces).

**Properties:**

- **Symmetric:** $\langle x, y \rangle = \langle y, x \rangle$
- **Hermitian:** $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (conjugate symmetry)
- **Bilinear:** Linear in both arguments:

$$\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$$
$$\langle x, ay + bz \rangle = a\langle x, y \rangle + b\langle x, z \rangle$$

- **Sesquilinear:** Linear in one argument, conjugate-linear in the other:

$$\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$$
$$\langle x, ay + bz \rangle = \bar{a}\langle x, y \rangle + \bar{b}\langle x, z \rangle$$

- **Positive definite:** $\langle x, x \rangle \geq 0$, with equality if and only if $x = 0$

**Definition 1.10.** The **norm** of a vector $\vec{x}$ is defined as

$$\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

**Properties of a norm:**

- **Triangle inequality:** $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$
- **Absolute homogeneity:** $\|c\vec{x}\| = |c| \cdot \|\vec{x}\|$ for all $c \in \mathbb{R}$
- **Positive definite:** $\|\vec{x}\| \geq 0$, with equality if and only if $\vec{x} = \vec{0}$

**Theorem 1.11** (Cauchy-Schwarz Inequality)**.** *For all $\vec{x}, \vec{y} \in \mathbb{R}^n$:*

$$|\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\| \|\vec{y}\|$$

*Proof.* Consider the function $f(t) = \|\vec{x} + t\vec{y}\|^2$ for $t \in \mathbb{R}$. By positive definiteness, $f(t) \geq 0$ for all $t$.

Expanding:

$$f(t) = \langle \vec{x} + t\vec{y}, \vec{x} + t\vec{y} \rangle = \|\vec{x}\|^2 + 2t\langle \vec{x}, \vec{y} \rangle + t^2\|\vec{y}\|^2$$

This is a quadratic in $t$ that is always non-negative. For a quadratic $at^2 + bt + c \geq 0$ for all $t$, the discriminant must satisfy $b^2 - 4ac \leq 0$.

Here $a = \|\vec{y}\|^2$, $b = 2\langle \vec{x}, \vec{y} \rangle$, $c = \|\vec{x}\|^2$, so:

$$4\langle \vec{x}, \vec{y} \rangle^2 - 4\|\vec{x}\|^2\|\vec{y}\|^2 \leq 0$$

Therefore $\langle \vec{x}, \vec{y} \rangle^2 \leq \|\vec{x}\|^2\|\vec{y}\|^2$, and taking square roots gives the result. □