Since $x \in A_z$ and $y \in B_z$, $A$ and $B$ are nonempty. Since $A_z \subset (-\infty, z)$ and $B_z \subset (z, \infty)$, they are separated. Hence $E$ is not connected.

To prove the converse, suppose $E$ is not connected. Then there are nonempty separated sets $A$ and $B$ such that $A \cup B = E$. Pick $x \in A, y \in B$, and assume (without loss of generality) that $x < y$. Define

$$z = \sup(A \cap [x, y]).$$

By Theorem 2.28, $z \in \bar{A}$; hence $z \notin B$. In particular, $x \leq z < y$.
If $z \notin A$, it follows that $x < z < y$ and $z \notin E$.
If $z \in A$, then $z \notin \bar{B}$, hence there exists $z_1$ such that $z < z_1 < y$ and $z_1 \notin B$. Then $x < z_1 < y$ and $z_1 \notin E$.

# EXERCISES

1. Prove that the empty set is a subset of every set.

2. A complex number $z$ is said to be algebraic if there are integers $a_0, \ldots, a_n$, not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$

Prove that the set of all algebraic numbers is countable. Hint: For every positive integer $N$ there are only finitely many equations with

$$n + |a_0| + |a_1| + \cdots + |a_n| = N.$$

3. Prove that there exist real numbers which are not algebraic.

4. Is the set of all irrational real numbers countable?

5. Construct a bounded set of real numbers with exactly three limit points.

6. Let $E'$ be the set of all limit points of a set $E$. Prove that $E'$ is closed. Prove that $E$ and $\bar{E}$ have the same limit points. (Recall that $\bar{E} = E \cup E'$.) Do $E$ and $E'$ always have the same limit points?

7. Let $A_1, A_2, A_3, \ldots$ be subsets of a metric space.
(a) If $B_n = \bigcup_{i=1}^{n} A_i$, prove that $\bar{B}_n = \bigcup_{i=1}^{n} \bar{A}_i$, for $n = 1, 2, 3, \ldots$.
(b) If $B = \bigcup_{i=1}^{\infty} A_i$, prove that $\bar{B} \supset \bigcup_{i=1}^{\infty} \bar{A}_i$.

Show, by an example, that this inclusion can be proper.
8. Is every point of every open set $E \subset R^2$ a limit point of $E$ ? Answer the same question for closed sets in $R^2$.
9. Let $E^\circ$ denote the set of all interior points of a set $E$. [See Definition 2.18(e); $E^\circ$ is called the interior of $E$.]
(a) Prove that $E^\circ$ is always open.
(b) Prove that $E$ is open if and only if $E^\circ = E$.

(c) If $G \subset E$ and $G$ is open, prove that $G \subset E^\circ$.
(d) Prove that the complement of $E^\circ$ is the closure of the complement of $E$.
(e) Do $E$ and $\bar{E}$ always have the same interiors?
(f) Do $E$ and $E^\circ$ always have the same closures?
10. Let $X$ be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p,q) = \begin{cases} 1 & (\text{ if } p \neq q) \\ 0 & (\text{ if } p = q) \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?
11. For $x \in R^1$ and $y \in R^1$, define

$$\begin{aligned} d_1(x,y) &= (x-y)^2 \\ d_2(x,y) &= \sqrt{|x-y|} \\ d_3(x,y) &= \left|x^2 - y^2\right| \\ d_4(x,y) &= |x - 2y| \\ d_5(x,y) &= \frac{|x-y|}{1+|x-y|} \end{aligned}$$

Determine, for each of these, whether it is a metric or not.
12. Let $K \subset R^1$ consist of 0 and the numbers $1/n$, for $n = 1, 2, 3, \ldots$. Prove that $K$ is compact directly from the definition (without using the Heine-Borel theorem).
13. Construct a compact set of real numbers whose limit points form a countable set.
14. Give an example of an open cover of the segment $(0,1)$ which has no finite subcover.
15. Show that Theorem 2.36 and its Corollary become false (in $R^1$, for example) if the word "compact" is replaced by "closed" or by "bounded."
16. Regard $Q$, the set of all rational numbers, as a metric space, with $d(p,q) = |p-q|$. Let $E$ be the set of all $p \in Q$ such that $2 < p^2 < 3$. Show that $E$ is closed and bounded in $Q$, but that $E$ is not compact. Is $E$ open in $Q$ ?
17. Let $E$ be the set of all $x \in [0,1]$ whose decimal expansion contains only the digits 4 and 7 . Is $E$ countable? Is $E$ dense in $[0,1]$ ? Is $E$ compact? Is $E$ perfect?
18. Is there a nonempty perfect set in $R^1$ which contains no rational number?
19. (a) If $A$ and $B$ are disjoint closed sets in some metric space $X$, prove that they are separated.
(b) Prove the same for disjoint open sets.
(c) Fix $p \in X, \delta > 0$, define $A$ to be the set of all $q \in X$ for which $d(p,q) < \delta$, define $B$ similarly, with $>$ in place of $<$. Prove that $A$ and $B$ are separated.
(d) Prove that every connected metric space with at least two points in uncountable. Hint: Use (c).
20. Are closures and interiors of connected sets always connected? (Look at

subsets of $R^2$.)

21. Let $A$ and $B$ be separated subsets of some $R^k$, suppose $\mathbf{a} \in A, \mathbf{b} \in B$, and define

$$\mathbf{p}(t) = (1-t)\mathbf{a} + t\mathbf{b}$$

for $t \in R^1$. Put $A_0 = \mathbf{p}^{-1}(A), B_0 = \mathbf{p}^{-1}(B)$. [Thus $t \in A_0$ if and only if $\mathbf{p}(t) \in A$.]

(a) Prove that $A_0$ and $B_0$ are separated subsets of $R^1$.

(b) Prove that there exists $t_0 \in (0,1)$ such that $\mathbf{p}(t_0) \notin A \cup B$.

(c) Prove that every convex subset of $R^k$ is connected.

22. A metric space is called separable if it contains a countable dense subset. Show that $R^k$ is separable. Hint: Consider the set of points which have only rational coordinates.

23. A collection $\{V_\alpha\}$ of open subsets of $X$ is said to be a base for $X$ if the following is true: For every $x \in X$ and every open set $G \subset X$ such that $x \in G$, we have $x \in V_\alpha \subset G$ for some $\alpha$. In other words, every open set in $X$ is the union of a subcollection of $\{V_\alpha\}$.

Prove that every separable metric space has a countable base. Hint: Take all neighborhoods with rational radius and center in some countable dense subset of $X$.

24. Let $X$ be a metric space in which every infinite subset has a limit point. Prove that $X$ is separable. Hint: Fix $\delta > 0$, and pick $x_1 \in X$. Having chosen $x_1, \ldots, x_j \in X$, choose $x_{j+1} \in X$, if possible, so that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, \ldots, j$. Show that this process must stop after a finite number of steps, and that $X$ can therefore be covered by finitely many neighborhoods of radius $\delta$. Take $\delta = 1/n (n = 1, 2, 3, \ldots)$, and consider the centers of the corresponding neighborhoods.

25. Prove that every compact metric space $K$ has a countable base, and that $K$ is therefore separable. Hint: For every positive integer $n$, there are finitely many neighborhoods of radius $1/n$ whose union covers $K$.

26. Let $X$ be a metric space in which every infinite subset has a limit point. Prove that $X$ is compact. Hint: By Exercises 23 and 24, $X$ has a countable base. It follows that every open cover of $X$ has a countable subcover $\{G_n\}, n = 1, 2, 3, \ldots$. If no finite subcollection of $\{G_n\}$ covers $X$, then the complement $F_n$ of $G_1 \cup \cdots \cup G_n$ is nonempty for each $n$, but $\bigcap F_n$ is empty. If $E$ is a set which contains a point from each $F_n$, consider a limit point of $E$, and obtain a contradiction.

27. Define a point $p$ in a metric space $X$ to be a condensation point of a set $E \subset X$ if every neighborhood of $p$ contains uncountably many points of $E$.

Suppose $E \subset R^k, E$ is uncountable, and let $P$ be the set of all condensation points of $E$. Prove that $P$ is perfect and that at most countably many points of $E$ are not in $P$. In other words, show that $P^c \cap E$ is at most countable. Hint: Let $\{V_n\}$ be a countable base of $R^k$, let $W$ be the union of those $V_n$ for which $E \cap V_n$ is at most countable, and show that $P = W^c$.

28. Prove that every closed set in a separable metric space is the union of a

(possibly empty) perfect set and a set which is at most countable. (Corollary: Every countable closed set in $R^k$ has isolated points.) Hint: Use Exercise 27.

29. Prove that every open set in $R^1$ is the union of an at most countable collection of disjoint segments. Hint: Use Exercise 22.

30. Imitate the proof of Theorem 2.43 to obtain the following result:

If $R^k = \bigcup_1^\infty F_n$, where each $F_n$ is a closed subset of $R^k$, then at least one $F_n$ has a nonempty interior.

Equivalent statement: If $G_n$ is a dense open subset of $R^k$, for $n = 1, 2, 3, \ldots$, then $\bigcap_1^\infty G_n$ is not empty (in fact, it is dense in $R^k$ ).

(This is a special case of Baire's theorem; see Exercise 22, Chap. 3, for the general case.)

# 3

# NUMERICAL SEQUENCES AND SERIES

As the title indicates, this chapter will deal primarily with sequences and series of complex numbers. The basic facts about convergence, however, are just as easily explained in a more general setting. The first three sections will therefore be concerned with sequences in euclidean spaces, or even in metric spaces.

## CONVERGENT SEQUENCES

3.1 Definition A sequence $\{p_n\}$ in a metric space $X$ is said to converge if there is a point $p \in X$ with the following property: For every $\varepsilon > 0$ there is an integer $N$ such that $n \geq N$ implies that $d(p_n, p) < \varepsilon$. (Here $d$ denotes the distance in $X$.)

In this case we also say that $\{p_n\}$ converges to $p$, or that $p$ is the limit of $\{p_n\}$ [see Theorem 3.2(b)], and we write $p_n \to p$, or

$$\lim_{n\to\infty} p_n = p$$

If $\{p_n\}$ does not converge, it is said to diverge.