Orthogonal Vectors in $\mathbb{R}^k$

A visual guide to Rudin 1.18

1 The Claim

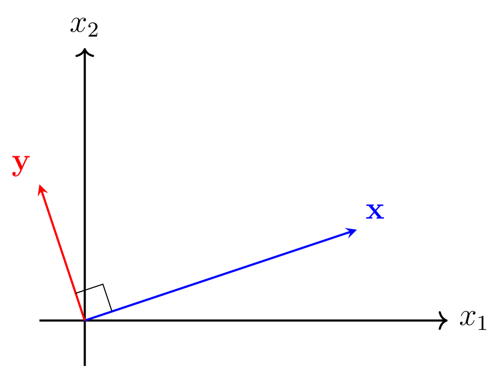
If $k \geq 2$ and $\mathbf{x} \in \mathbb{R}^k$, there exists $\mathbf{y} \neq \mathbf{0}$ such that $\mathbf{x} \cdot \mathbf{y} = 0$.

This says: in 2 or more dimensions, every vector has a nonzero **perpendicular** vector.

2 What Does $\mathbf{x} \cdot \mathbf{y} = 0$ Mean?

The dot product $\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ky_k$.

When $\mathbf{x} \cdot \mathbf{y} = 0$, we say $\mathbf{x}$ and $\mathbf{y}$ are **orthogonal** (perpendicular).



$\mathbf{x} \cdot \mathbf{y} = 0$ means $\mathbf{x} \perp \mathbf{y}$

3 The Construction in $\mathbb{R}^2$

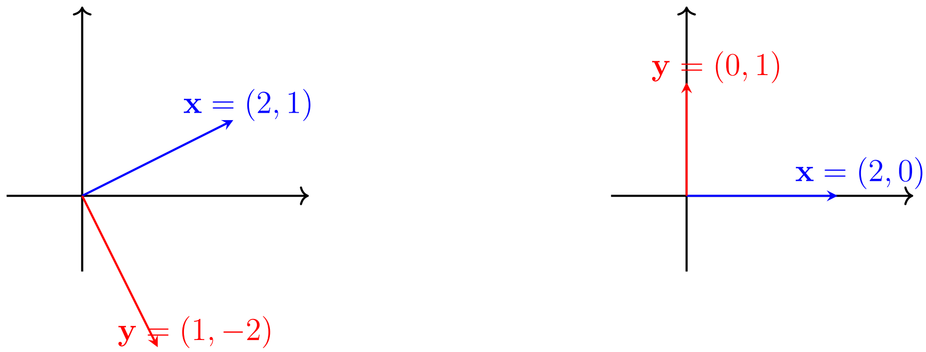
Given $\mathbf{x} = (x_1, x_2)$, we want $\mathbf{y} = (y_1, y_2)$ with $\mathbf{y} \neq \mathbf{0}$ and $x_1y_1 + x_2y_2 = 0$.

Case 1: If $x_2 \neq 0$, let $\mathbf{y} = (1, -x_1/x_2)$

Check: $\mathbf{x} \cdot \mathbf{y} = x_1 \cdot 1 + x_2 \cdot (-x_1/x_2) = x_1 - x_1 = 0$ ✓

Case 2: If $x_2 = 0$, let $\mathbf{y} = (0, 1)$

Check: $\mathbf{x} \cdot \mathbf{y} = x_1 \cdot 0 + 0 \cdot 1 = 0$ ✓



Case 1: $x_2 \neq 0$ Case 2: $x_2 = 0$

4 Extending to $\mathbb{R}^k$ ($k \geq 2$)

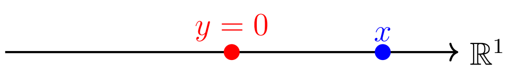
The same idea works! We only use the first two coordinates.

Given $\mathbf{x} = (x_1, x_2, x_3, \ldots, x_k)$:
Case 1: If $x_2 \neq 0$, let $\mathbf{y} = (1, -x_1/x_2, 0, 0, \ldots, 0)$
Case 2: If $x_2 = 0$, let $\mathbf{y} = (0, 1, 0, 0, \ldots, 0)$
In both cases, $\mathbf{x} \cdot \mathbf{y} = 0$ and $\mathbf{y} \neq \mathbf{0}$.

5 Why Does $k \geq 2$ Matter?

In $\mathbb{R}^1$, vectors are just numbers. If $x \neq 0$:

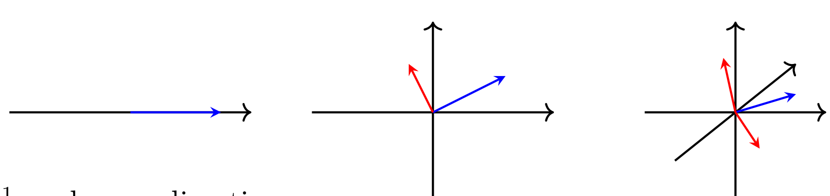
$$xy = 0 \quad \Rightarrow \quad y = 0$$



The only "perpendicular" to a nonzero number is zero itself!

In higher dimensions, there's "room" to be perpendicular.

6 Geometric Intuition



$\mathbb{R}^1$: only one direction
No room for perpendicular $\mathbb{R}^2$: perpendicular exists $\mathbb{R}^3$: infinitely many!

In $\mathbb{R}^k$ with $k \geq 2$, the set of vectors perpendicular to $\mathbf{x}$ forms a $(k-1)$-dimensional subspace — plenty of nonzero options!

7 Summary

For $k \geq 2$: Given any $\mathbf{x} \in \mathbb{R}^k$, we can always find $\mathbf{y} \neq \mathbf{0}$ with $\mathbf{x} \cdot \mathbf{y} = 0$.

For $k = 1$: If $x \neq 0$, the only solution to $xy = 0$ is $y = 0$.