Open Sets in $\mathbb{R}$ are Countable Unions of Disjoint Segments

A visual guide to Rudin 2.29

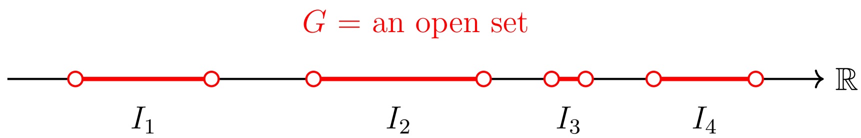
1 The Claim

Every open set $G \subseteq \mathbb{R}$ can be written as:

$$G = \bigcup_{n=1}^{\infty} I_n$$

where the I_n are **disjoint open intervals** (segments), and there are **at most countably many** of them.

2 What Do Open Sets in $\mathbb{R}$ Look Like?

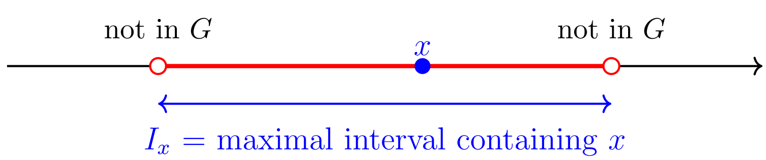


Open sets in $\mathbb{R}$ can have "gaps" — points not in the set. The set naturally breaks into disjoint intervals.

3 The Key Idea: Maximal Intervals

For any point $x \in G$, define the **maximal interval** containing x:

$$I_x = \text{the largest open interval containing } x \text{ that is still contained in } G$$



How to construct I_x:

$$a_x = \inf\{a : (a, x) \subseteq G\}$$
$$b_x = \sup\{b : (x, b) \subseteq G\}$$

Then $I_x = (a_x, b_x)$.

4 Key Properties of Maximal Intervals

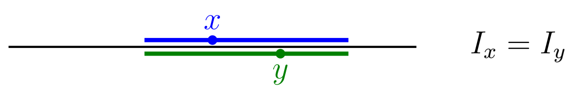
4.1 Property 1: They Cover G

Every point $x \in G$ is in its maximal interval I_x, so:

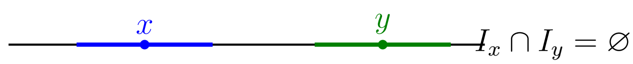
$$G = \bigcup_{x \in G} I_x$$

4.2 Property 2: They're Either Equal or Disjoint

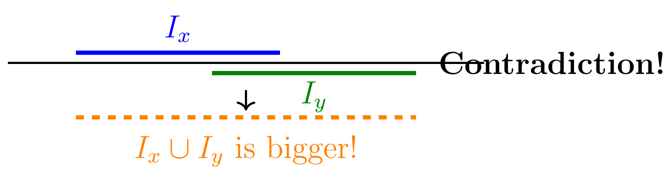
Case 1:



Case 2:

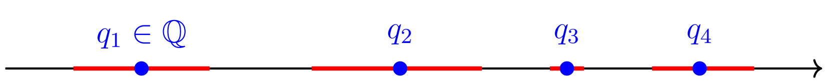


Why? If I_x and I_y overlap but aren't equal, their union $I_x \cup I_y$ would be a larger interval contained in G — contradicting maximality!



5 So We Have Disjoint Intervals — Why Countably Many?

This is where the hint comes in: use the fact that $\mathbb{R}$ is **separable** (from Problem 2.22).



Each interval contains a rational (density of $\mathbb{Q}$)

Different intervals $\Rightarrow$ different rationals (disjoint!)

$\therefore$ at most countably many intervals (since $\mathbb{Q}$ is countable)

The Argument:

1. Each maximal interval I is nonempty and open, so it contains a rational $q_I \in \mathbb{Q}$ (by density).

2. If $I \neq J$ are two distinct maximal intervals, they're disjoint, so $q_I \neq q_J$.
3. This gives an injection from {maximal intervals} to $\mathbb{Q}$.
4. Since $\mathbb{Q}$ is countable, there are at most countably many maximal intervals.

6 Summary of the Proof

1. For each $x \in G$, define $I_x = (a_x, b_x)$ where:

$$a_x = \inf\{a : (a, x) \subseteq G\}$$
$$b_x = \sup\{b : (x, b) \subseteq G\}$$

2. Show $I_x \subseteq G$ (need to verify this!)
3. Show: if $I_x \cap I_y \neq \varnothing$, then $I_x = I_y$
4. Conclude: the distinct maximal intervals partition G
5. Each interval contains a rational; different intervals contain different rationals
6. Therefore: at most countably many intervals

7 A Subtlety: Why is $I_x \subseteq G$?

We need to check that $(a_x, b_x) \subseteq G$.

Let $y \in (a_x, b_x)$. Then $a_x < y < b_x$.

- Since $y > a_x = \inf\{a : (a, x) \subseteq G\}$, there exists $a < y$ with $(a, x) \subseteq G$.
- Since $y < b_x = \sup\{b : (x, b) \subseteq G\}$, there exists $b > y$ with $(x, b) \subseteq G$.
- So $(a, x) \cup (x, b) = (a, b) \setminus \{x\} \subseteq G$, and $x \in G$.
- Therefore $(a, b) \subseteq G$, and since $a < y < b$, we have $y \in G$.

8 Visual Summary