Understanding Separable Metric Spaces

A visual guide to Rudin 2.22

1 What Does "Separable" Mean?

A metric space is **separable** if it contains a **countable dense subset**.

Let's break this down:

Countable

Can be listed: $a_1, a_2, a_3, \dots$

+

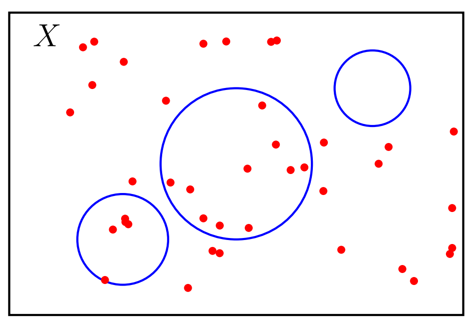
Dense

Gets arbitrarily close to ev-
ery point

2 What Does "Dense" Mean?

A set S is **dense** in X if $\bar{S} = X$.

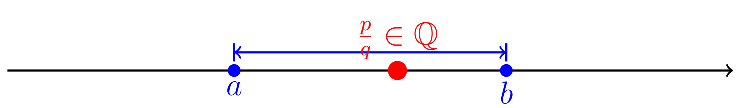
Equivalently: Every nonempty open set contains a point of S.



Red points are **dense**: every blue circle contains at least one

3 The Classic Example: $\mathbb{Q}$ is Dense in $\mathbb{R}$

Between any two real numbers, there's a rational number.



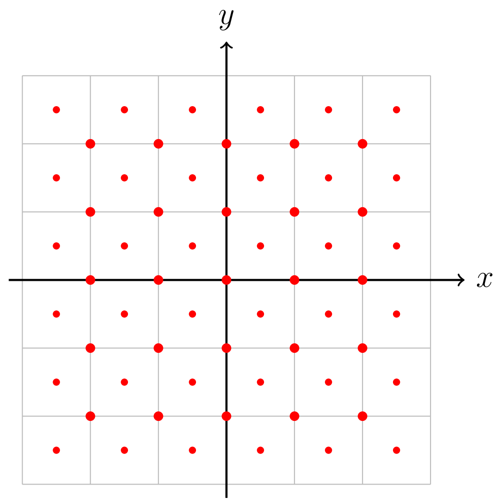
No matter how small the interval (a, b),
there's always a rational inside!

Why? The Archimedean property: for any $\varepsilon > 0$, there exists $n \in \mathbb{N}$ with $1/n < \varepsilon$. So rationals can get arbitrarily close to any real.

4 The Problem: Show $\mathbb{R}^k$ is Separable

We need a countable dense subset of $\mathbb{R}^k$.

Natural choice: $\mathbb{Q}^k$ = points with all rational coordinates.

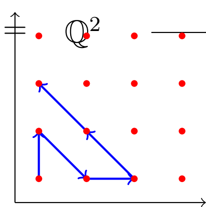


$\mathbb{Q}^2$: points with rational coordinates
(shown: some of infinitely many)

5 Step 1: $\mathbb{Q}^k$ is Countable

Key fact: Finite products of countable sets are countable.

$\mathbb{Q} \times \mathbb{Q} = \mathbb{Q}^2 \longrightarrow$ countable

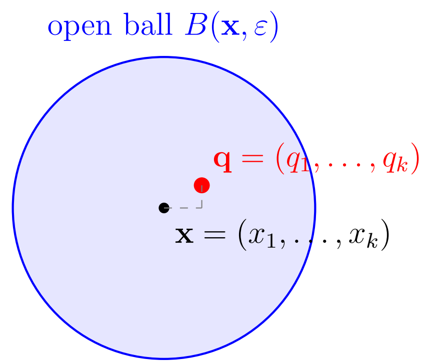


diagonal counting argument

Similarly, $\mathbb{Q}^k = \mathbb{Q} \times \mathbb{Q} \times \cdots \times \mathbb{Q}$ (k times) is countable.

6 Step 2: $\mathbb{Q}^k$ is Dense in $\mathbb{R}^k$

We need: every open set in $\mathbb{R}^k$ contains a point of $\mathbb{Q}^k$.



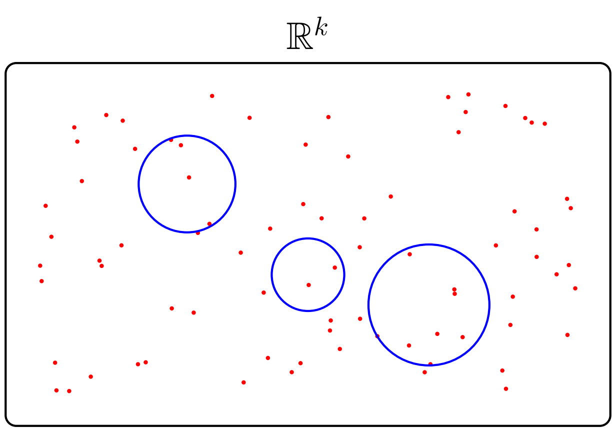
For each coordinate i:
find $q_i \in \mathbb{Q}$ with $|q_i - x_i| < \varepsilon/\sqrt{k}$
(possible by density of $\mathbb{Q}$ in $\mathbb{R}$)

Why does this work?

$$\begin{aligned} |\mathbf{q} - \mathbf{x}| &= \sqrt{\sum_{i=1}^{k} (q_i - x_i)^2} \\ &< \sqrt{\sum_{i=1}^{k} \left(\frac{\varepsilon}{\sqrt{k}}\right)^2} \\ &= \sqrt{k \cdot \frac{\varepsilon^2}{k}} = \varepsilon \end{aligned}$$

So $\mathbf{q} \in B(\mathbf{x}, \varepsilon)$. Done!

7 Visual Summary



$\mathbb{Q}^k$ (red dots) is **countable** and **dense**
$\Rightarrow \mathbb{R}^k$ is **separable**

8 Why Does This Matter?

Separability is important because:

- It means the space isn't "too big" — it can be approximated by a countable set
- Separable spaces have countable bases (see Rudin 2.23)
- Many theorems in analysis require separability