The Kuratowski Closure-Complement Problem

Understanding the two operations

The Problem

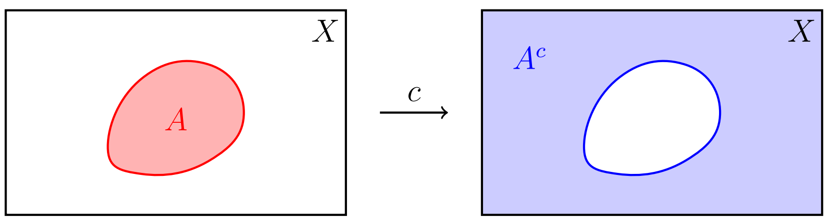
> *Consider the collection of all subsets of a topological space. The operations of taking closure and complement produce at most 14 sets. Show this and give an example of a subset of the reals that produces exactly 14 sets.*

1 The Two Operations

We have two operations we can apply to any subset A of a topological space X:

1.1 Complement: A^c

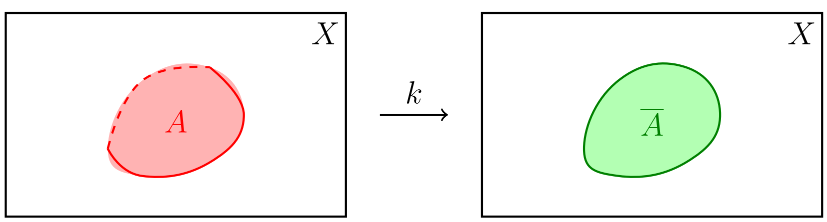
The **complement** $A^c = X \setminus A$ consists of all points *not* in A.



Key property: $(A^c)^c = A$ (complementing twice gives back the original)

1.2 Closure: $\overline{A}$

The **closure** $\overline{A}$ is the smallest closed set containing A. Equivalently, $\overline{A} = A \cup A'$ where A' is the set of limit points.



(dashed = not in A) (boundary filled in)

Key property: $\overline{\overline{A}} = \overline{A}$ (closure is idempotent—closing twice is the same as closing once)

2 The Game: Apply Operations Repeatedly

Starting with any set A, we can keep applying closure (k) and complement (c):

$$\begin{tikzcd}
 & \overline{A} \arrow[r, "c"] & (\overline{A})^c \arrow[r] & \cdots \\
A \arrow[ur, "k"] \arrow[dr, "c"'] & & & \\
 & A^c \arrow[r, "k"'] & \overline{A^c} \arrow[r] & \cdots
\end{tikzcd}$$

The Question: How many *distinct* sets can we produce this way?

3 Why Not Infinitely Many?

At first, it seems like we could get infinitely many sets by alternating operations forever. But the key properties limit us:

- $c \circ c = \text{id}$ (complement twice = do nothing)
- $k \circ k = k$ (closure twice = closure once)

So we never need two c's in a row, and we never need two k's in a row.
This means every sequence of operations can be written as an alternating sequence:

$$k, \quad c, \quad kc, \quad ck, \quad kck, \quad ckc, \quad kckc, \quad ckck, \quad \ldots$$

But even this doesn't obviously stop! The claim is that this process eventually stabilizes after at most 14 distinct sets.

4 The Two Parts of the Problem

Part 1: Prove that starting from *any* set in *any* topological space, applying closure and complement repeatedly produces **at most 14** distinct sets.

Part 2: Find a specific subset of $\mathbb{R}$ (with usual topology) that achieves **exactly 14** distinct sets.

5 What Makes a Set "Interesting"?

For a set to produce many distinct sets under these operations, it needs to have interesting structure. Consider:

Boring: $A = [0, 1]$ **Better:** $A = (0, 1)$

$\overline{A} = A$ (already closed)

$\overline{A} = [0, 1] \neq A$

To get many sets, you want A where:

- $A \neq \overline{A}$ (not closed)
- $A^c \neq \overline{A^c}$ (complement not closed)
- The pattern keeps producing new sets for as long as possible

6 Notation Shorthand

It's convenient to write k for closure and c for complement:

$$\begin{aligned} kA &= \overline{A} \\ cA &= A^c \\ kcA &= \overline{A^c} \\ ckA &= (\overline{A})^c \end{aligned}$$

The 14 sets (when all distinct) come from applying these operators in all possible alternating patterns.