

Rationals + Irrationals = Irrationals

A visual guide to Rudin 1.1

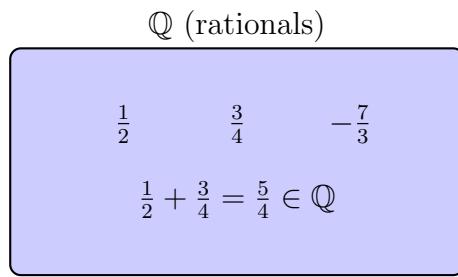
1 The Claim

If $r \in \mathbb{Q}$ with $r \neq 0$ and $x \notin \mathbb{Q}$ (irrational), then:

1. $r + x$ is irrational
2. rx is irrational

2 Key Insight: Closure of \mathbb{Q}

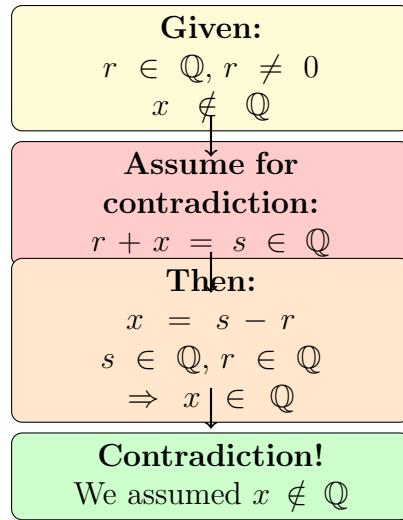
The rationals \mathbb{Q} are **closed** under addition, subtraction, multiplication, and division (except by zero).



Adding, subtracting, multiplying, or dividing two rationals always gives a rational

3 The Proof Strategy: Contradiction

We use **proof by contradiction**. Assume the conclusion is false, then derive something impossible.



4 Part 1: $r + x$ is Irrational

Proof:

1. Suppose $r + x = s$ for some $s \in \mathbb{Q}$
2. Then $x = s - r = s + (-r)$
3. Since $r \in \mathbb{Q}$, we have $-r \in \mathbb{Q}$
4. Since $s \in \mathbb{Q}$ and $-r \in \mathbb{Q}$, and \mathbb{Q} is closed under addition: $x = s + (-r) \in \mathbb{Q}$
5. But we assumed x is irrational — contradiction!

5 Part 2: rx is Irrational

Proof:

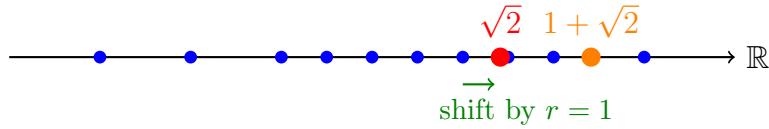
1. Suppose $rx = s$ for some $s \in \mathbb{Q}$
2. Since $r \neq 0$, the inverse $\frac{1}{r}$ exists
3. Since $r \in \mathbb{Q}$ and $r \neq 0$, we have $\frac{1}{r} \in \mathbb{Q}$
4. Then $x = s \cdot \frac{1}{r}$
5. Since $s \in \mathbb{Q}$ and $\frac{1}{r} \in \mathbb{Q}$, and \mathbb{Q} is closed under multiplication: $x \in \mathbb{Q}$
6. But we assumed x is irrational — contradiction!

6 Why $r \neq 0$ Matters for Part 2

If $r = 0$:
 $rx = 0 \cdot x = 0$
 But $0 \in \mathbb{Q}!$

So $r = 0$ breaks the claim for multiplication.

7 Visual Summary: The Number Line



Shifting an irrational by a rational
gives another irrational

8 The Contrapositive View

The proof by contradiction is equivalent to proving the **contrapositive**:

| Original | Contrapositive |
|---|---|
| $x \text{ irrational} \Rightarrow r + x \text{ irrational}$ | $r + x \text{ rational} \Rightarrow x \text{ rational}$ |

The contrapositive is what we actually showed: if $r + x \in \mathbb{Q}$, then $x = (r + x) - r \in \mathbb{Q}$.