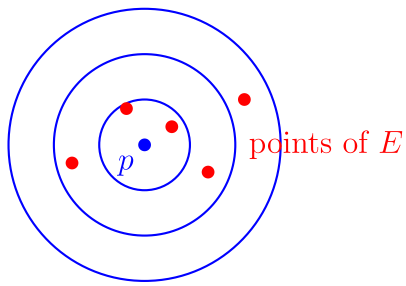
Understanding Limit Points and Closure

A visual guide to Rudin 2.6

1 What is a Limit Point?

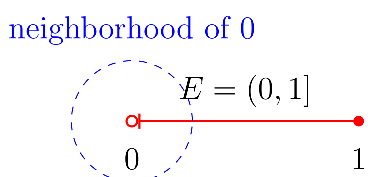
A point p is a **limit point** of a set E if every neighborhood of p contains at least one point of E *different from* p.



Every circle around p contains
a red point $\neq p$
$\Rightarrow p$ is a limit point of E

1.1 Limit Point vs Element of the Set

Important: A limit point doesn't have to be *in* the set!

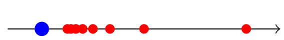


$0 \notin E$, but 0 is a limit point of E
(every neighborhood of 0 hits $(0, 1]$)

2 The Set of Limit Points: E'

We write E' for the set of *all* limit points of E.

Example 1



$E = \{1/n : n \in \mathbb{N}\}$
$E' = \{0\}$

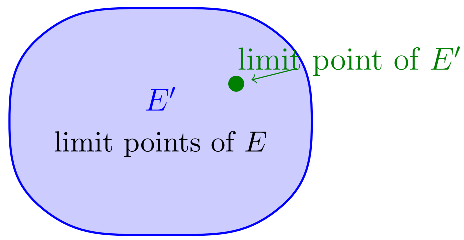
Example 2

$E = [0, 1]$
$E' = [0, 1]$

3 What Does "E' is Closed" Mean?

A set is **closed** if it contains all its limit points.

So "E' is closed" means: $(E')' \subseteq E'$

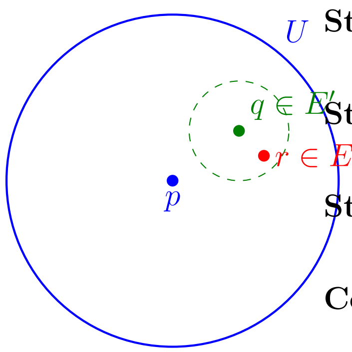


Claim: Any limit point of E'
is already inside E'

4 The Proof Strategy

To show E' is closed, we show $(E')' \subseteq E'$.

Setup: Let $p \in (E')'$ (a limit point of E'). We must show $p \in E'$.



Step 1: $p \in (E')'$, so U contains
some $q \in E'$ with $q \neq p$

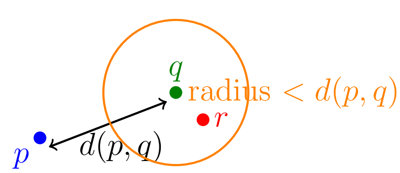
Step 2: $q \in E'$, so near q there's
some $r \in E$ with $r \neq q$

Step 3: Choose ball around q small
enough that $r \neq p$ too

Conclusion: $r \in U \cap E$, $r \neq p$
$\Rightarrow p$ is a limit point of E

5 The Key Trick: Excluding p

Why do we need $r \neq p$? Because the definition of limit point requires a point *different from* p.

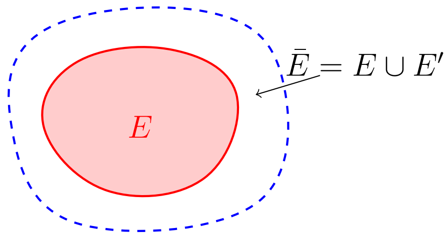


Pick a ball around q
with radius $< d(p,q)$
Then any r in this
ball satisfies $r \neq p$

6 Part 2: E and $\bar{E}$ Have the Same Limit Points

Recall: $\bar{E} = E \cup E'$ (closure = set $\cup$ limit points)

Claim: $E' = (\bar{E})'$



Adding E' to E doesn't
create new limit points!

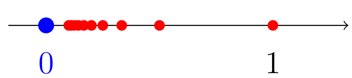
Why?

- $E \subseteq \bar{E}$, so limit points of E are limit points of $\bar{E}$: $E' \subseteq (\bar{E})'$
- If p is a limit point of $\bar{E}$, nearby points are in E or E'. Either way, we can find points of E nearby (using the Part 1 trick), so $p \in E'$: $(\bar{E})' \subseteq E'$

7 Part 3: Do E and E' Have the Same Limit Points?

No! Counterexample: $E = \{1/n : n \in \mathbb{N}\}$

$$E = \{1, 1/2, 1/3, \ldots\}$$



$E' = \{0\}$

$(E')' = \varnothing$

A single point has
no limit points!
So $E' \neq (E')'$