Infimum and Supremum Duality

A visual guide to Rudin 1.5

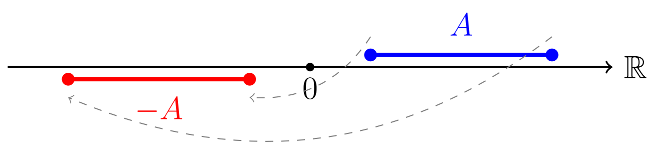
1 The Claim

If $A \subseteq \mathbb{R}$ is nonempty and bounded below, and $-A = \{-x : x \in A\}$, then:

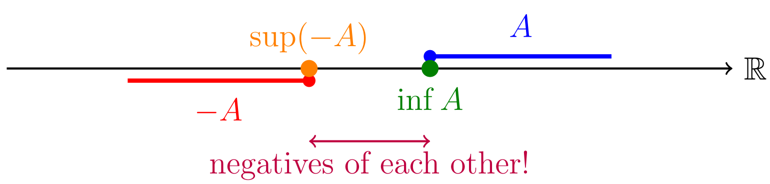
$$\inf A = -\sup(-A)$$

2 What is $-A$?

$-A$ is the set A "reflected" across zero.



3 The Key Relationship



4 Why Does This Work?

Negation **reverses inequalities**:

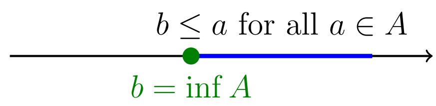
$$a \leq b \iff -a \geq -b$$

So:

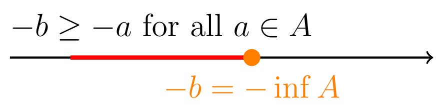
- Lower bounds of A become upper bounds of $-A$
- The **greatest** lower bound of A becomes the **least** upper bound of $-A$

5 The Proof in Pictures

Step 1: $\inf A$ is a lower bound of A



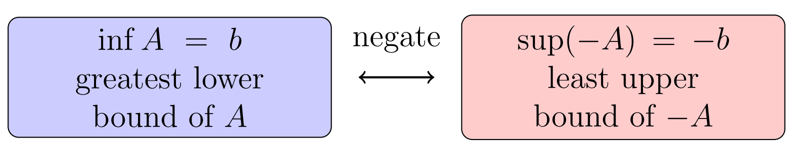
Step 2: So $-\inf A$ is an upper bound of $-A$



Step 3: $-\inf A$ is the **least** upper bound of $-A$

Why? If c is any upper bound of $-A$, then $-c$ is a lower bound of A. Since $\inf A$ is the greatest lower bound: $-c \le \inf A$, so $c \ge -\inf A$.

6 Summary



$$\therefore \inf A = -\sup(-A)$$

7 Intuition

Think of it as a mirror:

- The infimum of A is on the "left edge" of A
- When you reflect across zero, left becomes right
- So the infimum of A becomes the supremum of $-A$ (negated)