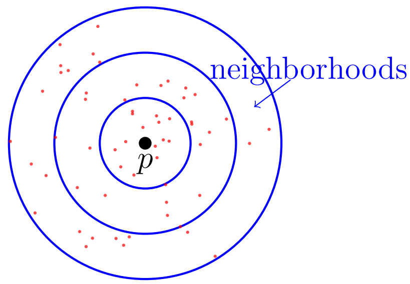
Understanding Condensation Points

A visual guide to Rudin 2.27

1 What is a Condensation Point?

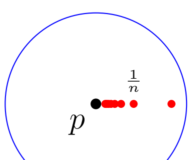
Intuition: A condensation point is where points of your set are "extremely crowded" — not just infinitely many, but *uncountably* many.



Condensation point: Every circle around p contains *uncountably many* red points

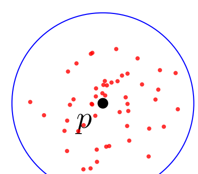
1.1 Comparison: Limit Point vs Condensation Point

Limit Point



Only needs *one* point of E (other than p) in each neighborhood

Condensation Point



Needs *uncountably many* points of E in *every* neighborhood

2 Quick Review: Countable vs Uncountable

Countable

Can list them: $a_1, a_2, a_3, \ldots$
Examples: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, finite sets

Uncountable

"Too big" to list

Examples: $\mathbb{R}$, $[0, 1]$, Cantor set

Key fact: A countable union of countable sets is countable.

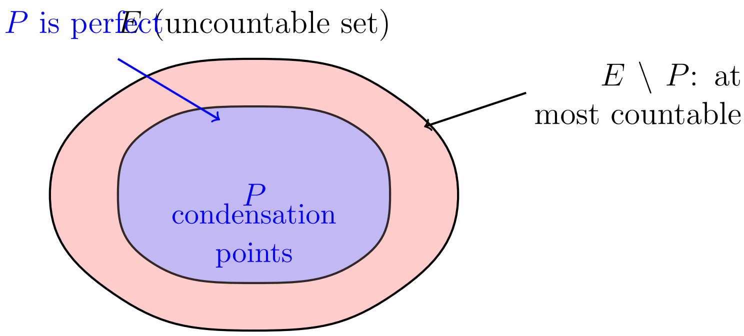
3 What the Problem Asks

Given: $E \subset \mathbb{R}^k$ is uncountable.

Define: $P = \{\text{all condensation points of } E\}$.

Prove two things:

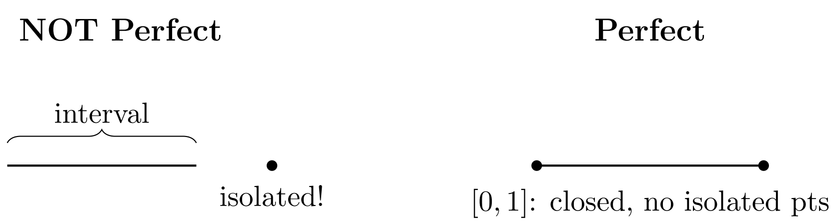
1. P is **perfect** (closed + every point is a limit point)
2. At most countably many points of E are *not* in P



4 What is a Perfect Set?

A set P is **perfect** if:

1. P is **closed** (contains all its limit points), AND
2. Every point of P is a **limit point** of P (no isolated points)



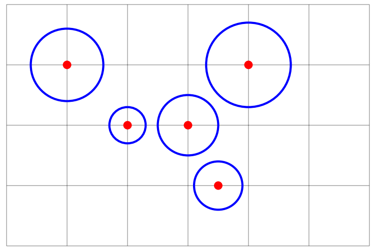
Classic example: The Cantor set is perfect (and uncountable!).

5 The Hint Explained

The hint says: Let $\{V_n\}$ be a countable base of $\mathbb{R}^k$...

5.1 What is a Countable Base?

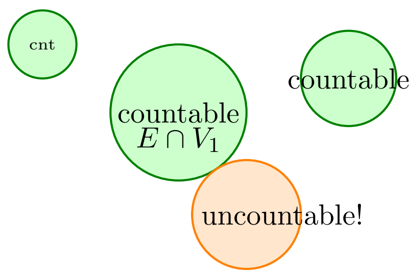
A **base** is a collection of open sets such that every open set is a union of sets from the base.
For $\mathbb{R}^k$: Use open balls with **rational** centers and **rational** radii.



Balls centered at rational points with rational radii
There are **countably many** such balls (from 2.22!)

5.2 The Strategy

Define W = union of all V_n where $E \cap V_n$ is countable.



W = green regions (where E is "thin")
W^c = everything else = where E is "thick" = P!

Key insight:

- W is a countable union of sets, each containing only countably many points of E
- So $E \cap W$ is countable (countable union of countable sets)
- The "interesting" points of E (uncountably many) must be in $W^c = P$

6 Summary of the Proof Strategy

1. Show $P = W^c$ (condensation points are exactly the complement of W)
2. W is open (union of open sets), so $P = W^c$ is **closed**
3. Show every point of P is a limit point of P (so P is **perfect**)
4. $E \cap W$ is countable, so $E \setminus P = E \cap W$ is countable