Math 104: Introduction to Real Analysis

Lecture Notes

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Contents

1 Syllabus

Course Overview: Math 104 is an introduction to real analysis, covering the rigorous foundations of calculus. We study the real number system, sequences and series, continuity, differentiation, and Riemann integration, with emphasis on proofs. The primary text is Walter Rudin's *Principles of Mathematical Analysis* (3rd edition).

1.1 Course Information

At a Glance:

- **Course**: Math 104 — Introduction to Real Analysis
- **Term**: Spring 2026
- **Schedule**: MWF 10:00–10:50 AM, Evans Hall 3
- **Textbook**: Rudin, *Principles of Mathematical Analysis*, 3rd ed.
- **Prerequisites**: Math 53 and Math 54 (or equivalents)

1.2 Course Requirements

- **Homework (40%)**: Weekly problem sets from Rudin. Due Fridays. Lowest score dropped.
- **Midterm (25%)**: In-class, Week 8. Covers Chapters 1–4.
- **Final (35%)**: Comprehensive. Covers Chapters 1–7.

1.3 Learning Objectives

By the end of this course, students will be able to:

1. Construct rigorous proofs involving properties of the real numbers.
2. State and apply the completeness axiom (Least Upper Bound Property).
3. Prove convergence or divergence of sequences and series.
4. Define and prove properties of continuous functions on metric spaces.
5. Apply compactness and connectedness in proofs.
6. Develop and present mathematical arguments with clarity and precision.

1.4 Assignments

Homework Schedule:

- **HW 1** — Rudin Ch. 1: Problems 1, 5, 9, 18 (Bonus: 7, 8, 20) *Due: Jan 31*
- **HW 2** — Rudin Ch. 2: Problems 6, 22, 27, 29 (Bonus: Kuratowski) *Due: Feb 7*
- **HW 3** — Rudin Ch. 2–3: TBD *Due: Feb 14*
- **HW 4** — Rudin Ch. 3: TBD *Due: Feb 21*
- **HW 5** — Rudin Ch. 4: TBD *Due: Feb 28*

1.5 Course Calendar

Week	Dates	Topics	Reading/Due
1	Jan 20–24	Ordered sets, LUB property, fields	Rudin 1.1–1.4
2	Jan 27–31	Construction of $\mathbb{R}$, Archimedean property	Rudin 1.5–1.8; **HW 1 due**
3	Feb 3–7	Countability, metric spaces	Rudin 2.1–2.3; **HW 2 due**
4	Feb 10–14	Topology, open/closed sets	Rudin 2.4–2.6; **HW 3 due**
5	Feb 17–21	Compactness, Heine-Borel	Rudin 2.7–2.8; **HW 4 due**
6	Feb 24–28	Sequences, subsequences	Rudin 3.1–3.3; **HW 5 due**
7	Mar 3–7	Series, convergence tests	Rudin 3.4–3.7
8	Mar 10–14	**Midterm (Wed)**	Covers Ch. 1–4

2 Lectures

2.1 Lecture 1: January 20, 2026

Lecture Overview: We begin by proving $\sqrt{2}$ is irrational, motivating the need for a number system without "gaps." This leads us to define **ordered sets** and the crucial **Least Upper Bound Property (LUBP)**—the defining feature of $\mathbb{R}$ that $\mathbb{Q}$ lacks. We then introduce **fields** as algebraic structures with addition and multiplication, and combine these ideas into **ordered fields**. The real numbers are the unique complete ordered field.

2.1.1 Ordered sets and the least-upper-bound property

Section Overview: This section motivates the need for the real numbers by showing that $\mathbb{Q}$ has "gaps"—$\sqrt{2}$ is irrational, yet we can get arbitrarily close to it with rationals. We develop the machinery of **ordered sets**: partial orders, total orders, upper/lower bounds, and the supremum/infimum. The central concept is the **Least Upper Bound Property (LUBP)**: every non-empty bounded-above subset has a supremum. This property distinguishes $\mathbb{R}$ from $\mathbb{Q}$ and is the foundation for all of real analysis. We prove that LUBP implies GLBP.

Consider the ancient problem from Greek times: can we write $\sqrt{2}$ as a quotient of two natural numbers?

Theorem 2.1. *$\sqrt{2}$ is irrational; that is, there do not exist $p, q \in \mathbb{N}$ such that $\sqrt{2} = \frac{p}{q}$.*

Proof. Suppose, for contradiction, that $\sqrt{2} = \frac{p}{q}$ for some $p, q \in \mathbb{N}$ with $\gcd(p, q) = 1$ (i.e., the fraction is in lowest terms).

Then $2 = \frac{p^2}{q^2}$, so $p^2 = 2q^2$.

This means p^2 is even, so p is even. Write $p = 2k$ for some $k \in \mathbb{N}$.

Then $(2k)^2 = 2q^2$, so $4k^2 = 2q^2$, hence $q^2 = 2k^2$.

This means q^2 is even, so q is even.

But then both p and q are even, contradicting $\gcd(p, q) = 1$. $\square$

Now consider two sets:

$$A = \{p \in \mathbb{Q} : p > 0 \text{ and } p^2 < 2\}, \quad B = \{p \in \mathbb{Q} : p > 0 \text{ and } p^2 > 2\}.$$

Proposition 2.2. *A contains no largest element and B contains no smallest element.*

Proof. Let $p_0 \in A$. Define

$$q = p_0 + \frac{2 - p_0^2}{p_0^2 + 2}.$$

Since $p_0 \in A$, we have $p_0^2 < 2$, so $2 - p_0^2 > 0$. Thus $q > p_0$.

We claim $q \in A$, i.e., $q^2 < 2$. One can verify that

$$q^2 - 2 = \frac{(p_0^2 - 2)^2 \cdot (\text{positive})}{(p_0^2 + 2)^2}$$

which shows $q^2 < 2$ when $p_0^2 < 2$.

Hence A has no largest element.

A similar argument shows B has no smallest element. $\square$

Definition 2.3 (1.3)**.** If A is any set, we write $x \in A$ to say that x is a **member** of A. Otherwise, $x \notin A$. The set that contains no elements is called the **empty set**, denoted $\emptyset$. If $A \neq \emptyset$, we say that A is **non-empty**.

If A, B are sets and $\forall x \in A$ we have $x \in B$, we say that $A \subset B$, or A is a **subset** of B. If there exists an element $x \in B$ with $x \notin A$, then A is a **proper subset** of B, denoted $A \subsetneq B$.

Example. $3 \in \mathbb{N}$, but $-1 \notin \mathbb{N}$. We have $\mathbb{N} \subset \mathbb{Z}$ and $\mathbb{N} \subsetneq \mathbb{Z}$ (since $-1 \in \mathbb{Z}$ but $-1 \notin \mathbb{N}$).

Definition 2.4. A **binary relation** on a set S is a set of ordered pairs $\langle x, y\rangle$ with $x, y \in S$.

Example. On $\mathbb{Z}$, the relation $\leq$ is the set $\{\langle x, y\rangle : x, y \in \mathbb{Z}, x \leq y\}$, e.g., $\langle 2, 5\rangle$ is in the relation.

Definition 2.5. A **partial order** is a binary relation $\leq$ on S such that:

1. **Reflexive:** $\forall x \in S$, $x \leq x$.
2. **Anti-symmetric:** $\forall x, y \in S$, if $x \leq y$ and $y \leq x$, then $x = y$.
3. **Transitive:** $\forall x, y, z \in S$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

Example. On the power set $\mathcal{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, the subset relation $\subseteq$ is a partial order (but not a total order, since $\{1\} \nsubseteq \{2\}$ and $\{2\} \nsubseteq \{1\}$).

Definition 2.6. A **total order** is a partial order with the additional axiom that any two elements are comparable. That is, for any $x, y \in S$, either $x \leq y$ or $y \leq x$ (non-exclusive).

Example. The usual $\leq$ on $\mathbb{R}$ is a total order: for any $x, y \in \mathbb{R}$, either $x \leq y$ or $y \leq x$.

Definition 2.7. An **ordered set** is a set equipped with a total order.

Example. $(\mathbb{Q}, \leq)$ and $(\mathbb{R}, \leq)$ are ordered sets.

Definition 2.8. Suppose S is an ordered set and $E \subset S$. If there exists $\beta \in S$ such that $x \leq \beta$ for all $x \in E$, we say β is an **upper bound** of E. Similarly, if there exists $\alpha \in S$ such that $\alpha \leq x$ for all $x \in E$, we say α is a **lower bound** of E.

Example. Let $E = (0, 1) \subset \mathbb{R}$. Then $1, 2, 100$ are all upper bounds of E, and $0, -5$ are lower bounds of E.

Definition 2.9. Suppose S is an ordered set and $E \subset S$ is bounded above. If there exists $\alpha \in S$ such that:

1. α is an upper bound of E, and
2. if $\gamma < \alpha$, then γ is not an upper bound of E,

then α is called the **least upper bound** of E (or **supremum**), denoted $\sup E$.

Example. $\sup(0, 1) = 1$ and $\sup[0, 1] = 1$ in $\mathbb{R}$.

Definition 2.10. Suppose S is an ordered set and $E \subset S$ is bounded below. If there exists $\alpha \in S$ such that:

1. α is a lower bound of E, and
2. if $\gamma > \alpha$, then γ is not a lower bound of E,

then α is called the **greatest lower bound** of E (or **infimum**), denoted $\inf E$.

Example. $\inf(0,1) = 0$ and $\inf[0,1] = 0$ in $\mathbb{R}$.

Remark 2.11. If $\sup E$ or $\inf E$ exists, it need not be an element of E. For example, the set $A = \{p \in \mathbb{Q} : p > 0, p^2 < 2\}$ has $\sup A = \sqrt{2}$ (in $\mathbb{R}$), but $\sqrt{2} \notin A$ since $\sqrt{2} \notin \mathbb{Q}$.

Definition 2.12. Let S be an ordered set.

1. S has the **least upper bound property** if for any non-empty $E \subset S$ that is bounded above, $\sup E$ exists in S.

2. S has the **greatest lower bound property** if for any non-empty $E \subset S$ that is bounded below, $\inf E$ exists in S.

Example. $\mathbb{R}$ has the LUBP (and hence GLBP). However, $\mathbb{Q}$ does not: the set $A = \{p \in \mathbb{Q} : p > 0, p^2 < 2\}$ is bounded above in $\mathbb{Q}$, but $\sup A = \sqrt{2} \notin \mathbb{Q}$.

Theorem 2.13 (LUBP implies GLBP)**.** *Suppose S is an ordered set with the least upper bound property. Let $B \subset S$, $B \neq \emptyset$, and suppose B is bounded below. Let L be the set of all lower bounds of B. Then $\alpha = \sup L$ exists in S, and $\alpha = \inf B$.*

Proof. First, $L \neq \emptyset$ since B is bounded below.

Second, L is bounded above: every $b \in B$ is an upper bound for L (since if $\ell \in L$, then $\ell \leq b$ by definition of lower bound).

By the LUBP, $\alpha = \sup L$ exists in S.

We claim $\alpha = \inf B$:

1. α is a lower bound of B: For any $b \in B$, b is an upper bound of L, so $\alpha \leq b$ (since α is the *least* upper bound of L).

2. α is the greatest lower bound: If $\gamma > \alpha$ and γ were a lower bound of B, then $\gamma \in L$, so $\gamma \leq \sup L = \alpha$, contradicting $\gamma > \alpha$. Thus γ is not a lower bound of B.

Thus $\alpha = \inf B$. ☐

2.1.2 Fields

Section Overview: This section introduces the algebraic structure underlying $\mathbb{R}$. We define **groups** (sets with an operation having identity, inverses, and associativity) and **fields** (sets with addition and multiplication that behave like we expect from $\mathbb{Q}$ or $\mathbb{R}$). We sketch how to construct $\mathbb{N} \to \mathbb{Z} \to \mathbb{Q}$ using equivalence relations. The key definition is an **ordered field**: a field that is also an ordered set, allowing us to combine algebraic operations with comparison. $\mathbb{R}$ is the unique complete ordered field.

Definition 2.14. A **binary operation** on S is a map $S \times S \to S$.

Definition 2.15. A **group** is a set G with a binary operation $+$ satisfying the following axioms:

1. **Identity:** There exists $0 \in G$ such that $a + 0 = 0 + a = a$ for all $a \in G$.

2. **Existence of inverse:** For every $a \in G$, there exists $-a \in G$ such that $a + (-a) = 0$.

3. **Associativity:** For all $a, b, c \in G$, $(a + b) + c = a + (b + c)$.

If we add a fourth axiom:

4. **Commutativity:** For all $a, b \in G$, $a + b = b + a$,

then G is called an **abelian group**.

Definition 2.16. A **field** is a set F with two binary operations, addition $(+)$ and multiplication $(\cdot)$, such that:

1. $(F, +)$ is an abelian group with identity 0.
2. $(F \setminus \{0\}, \cdot)$ is an abelian group with identity 1.
3. **Distributivity:** For all $a, b, c \in F$, $a \cdot (b + c) = a \cdot b + a \cdot c$.

Example. $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are fields. $\mathbb{Z}$ is not a field (e.g., 2 has no multiplicative inverse in $\mathbb{Z}$).

Zooming out, we can construct the number systems as follows:
The **natural numbers** $\mathbb{N}$ can be defined by the cardinality of iterated power sets of $\emptyset$:

$$0 = |\emptyset|, \quad 1 = |\mathcal{P}(\emptyset)|, \quad 2 = |\mathcal{P}(\mathcal{P}(\emptyset))|, \quad \ldots$$

The **integers** are defined as:

$$\mathbb{Z} = \{a - b : a, b \in \mathbb{N}\}.$$

Definition 2.17. An **equivalence relation** $\sim$ on a set S has the following properties:

1. **Reflexive:** $x \sim x$ for all $x \in S$.
2. **Symmetric:** If $x \sim y$, then $y \sim x$.
3. **Transitive:** If $x \sim y$ and $y \sim z$, then $x \sim z$.

The **rational numbers** are defined as:

$$\mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z},\, q \in \mathbb{N},\, q \neq 0 \right\} / \sim$$

We can verify this is an equivalence relation: $\frac{p}{q} \sim \frac{r}{s}$ if and only if $ps = rq$.

Definition 2.18. An **ordered field** is a field F which is also an ordered set such that:

1. If $x, y, z \in F$ and $y < z$, then $x + y < x + z$.
2. If $x, y \in F$, $x > 0$, and $y > 0$, then $xy > 0$.

Proposition 2.19. *If $x > 0$ and $y < z$, then $xy < xz$.*

Proof. Since $y < z$, we have $z - y > 0$. Since $x > 0$ and $z - y > 0$, we have $x(z - y) > 0$. Thus $xz - xy > 0$, so $xy < xz$. $\square$

2.2 Lecture 2: January 22, 2026

Lecture Overview: We construct the real numbers $\mathbb{R}$ as an ordered field with the Least Upper Bound Property (LUBP) containing $\mathbb{Q}$ as a subfield, using Dedekind cuts. We prove key properties of $\mathbb{R}$: the Archimedean property and density of $\mathbb{Q}$ in $\mathbb{R}$. Using the LUBP, we establish existence of nth roots of positive reals via a supremum argument. We discuss decimal/ternary representations and the Cantor set. We introduce the complex numbers $\mathbb{C}$ and prove $\mathbb{C}$ is not an ordered field. Finally, we define Euclidean spaces $\mathbb{R}^n$ with inner products and norms, and prove the Cauchy-Schwarz inequality.

2.2.1 Dedekind Cuts

Section Overview: We define Dedekind cuts as a way to construct the real numbers from the rationals.

Definition 2.20. A **cut** $\alpha \subset \mathbb{Q}$ is a nonempty, proper subset such that:

1. **Downward closed:** If $p \in \alpha$ and $q < p$, then $q \in \alpha$.
2. If $\sup \alpha$ exists, then $\sup \alpha \notin \alpha$.

The set of all cuts is ordered by inclusion: $\alpha \leq \beta$ if and only if $\alpha \subseteq \beta$.

2.2.2 Field Operations on Cuts

Section Overview: We define addition and multiplication on cuts to make them into an ordered field.

Addition:

$$\alpha + \beta = \{r + s \mid r \in \alpha, s \in \beta\}$$

Additive identity:

$$0^* = \{p \in \mathbb{Q} \mid p < 0\}$$

Multiplication: For $\alpha > 0^*$ and $\beta > 0^*$:

$$\alpha\beta = \{p \in \mathbb{Q} \mid p \leq rs \text{ for some } r \in \alpha, r > 0 \text{ and } s \in \beta, s > 0\}$$

2.2.3 Least Upper Bound Property

Section Overview: We show that the set of cuts has the LUBP.

For a nonempty set E of cuts that is bounded above:

$$\sup E = \bigcup_{\alpha \in E} \alpha$$

2.2.4 Embedding $\mathbb{Q}$ into $\mathbb{R}$

Section Overview: We embed the rationals into the reals as a subfield.

For $p \in \mathbb{Q}$, define the cut:

$$p^* := \{q \in \mathbb{Q} \mid q < p\}$$

This embedding $p \mapsto p^*$ identifies $\mathbb{Q}$ as a subfield of $\mathbb{R}$.

2.2.5 Properties of $\mathbb{R}$

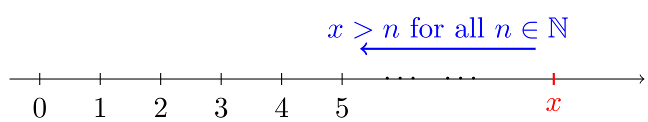
Section Overview: Having constructed $\mathbb{R}$, we now explore its key properties.

Theorem 2.21 (Archimedean Property)**.** *For any $x, y \in \mathbb{R}$ with $x > 0$, there exists $n \in \mathbb{N}$ such that $nx > y$.*

Proof. By contradiction. Suppose no such n exists, i.e., $nx \leq y$ for all $n \in \mathbb{N}$. Then the set $A = \{nx : n \in \mathbb{N}\}$ is bounded above by y. By the LUBP, $\sup A$ exists. Let $\alpha = \sup A$. Since $x > 0$, we have $\alpha - x < \alpha$, so $\alpha - x$ is not an upper bound for A. Thus there exists $m \in \mathbb{N}$ with $mx > \alpha - x$, which gives $(m+1)x > \alpha$. But $(m+1)x \in A$, contradicting that $\alpha = \sup A$. $\square$

Remark 2.22. The Archimedean property ensures there are no **infinitely large** elements (every element is bounded by some natural number) and no **infinitesimals** (positive elements smaller than $1/n$ for all n). This property is essential for proving that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Example of a non-Archimedean field: Consider the field of rational functions $\mathbb{R}(x)$ with the ordering where x is declared to be larger than every real number (i.e., $x > r$ for all $r \in \mathbb{R}$). Then $x > n$ for all $n \in \mathbb{N}$, so the Archimedean property fails. In this field, $1/x$ is an infinitesimal: it is positive but smaller than $1/n$ for all $n \in \mathbb{N}$.



Theorem 2.23 (Density of $\mathbb{Q}$ in $\mathbb{R}$)**.** *For any $a, b \in \mathbb{R}$ with $a < b$, there exists $q \in \mathbb{Q}$ such that $a < q < b$.*

Proof. Since $b - a > 0$, by the Archimedean property there exists $n \in \mathbb{N}$ such that $n(b-a) > 1$, i.e., $nb - na > 1$. Thus there exists an integer m with $na < m < nb$. Then $a < \frac{m}{n} < b$, and $q = \frac{m}{n} \in \mathbb{Q}$. $\square$

2.2.6 The Roots of Reals

Section Overview: Having constructed $\mathbb{R}$ with the LUBP, we can now prove that nth roots of positive reals exist, resolving the gap in $\mathbb{Q}$ where $\sqrt{2}$ was missing.

Previously, we showed that $\sqrt{2} \notin \mathbb{Q}$. Now that we have constructed $\mathbb{R}$ with the LUBP, we can prove that nth roots exist.

Theorem 2.24 (Existence of nth Roots)**.** *For all $x \in \mathbb{R}_{>0}$ and for all $n \in \mathbb{Z}_{>0}$, there exists a unique $y \in \mathbb{R}_{>0}$ such that $y^n = x$.*

Proof. Let $E = \{t \in \mathbb{R}_{>0} : t^n < x\}$.

E **is non-empty:** We have

$$\left(\frac{x}{x+1}\right)^n < \frac{x}{x+1} < x,$$

so $\frac{x}{x+1} \in E$.

E **is bounded above:** (to be shown)

By the LUBP, $y = \sup E$ exists.

Claim: $y^n = x$.

Aside (Trichotomy): Since $\mathbb{R}$ is a totally ordered set, for any $a, b \in \mathbb{R}$, exactly one of the following holds: $a < b$, $a = b$, or $a > b$. Thus for y^n and x, exactly one of $y^n < x$, $y^n = x$, or $y^n > x$ holds. We show the first and third cases lead to contradictions.

Case 1: Suppose $y^n < x$. Then there exists $h > 0$ small enough such that $(y+h)^n < x$. But then $y + h \in E$, contradicting that $y = \sup E$.

Case 2: Suppose $y^n > x$. Then there exists $h > 0$ small enough such that $(y-h)^n > x$. But then $y - h$ is still an upper bound for E, contradicting that $y = \sup E$ (the *least* upper bound).

Therefore $y^n = x$. □

Note to the reader: This proof employs a fundamental technique in real analysis called a *supremum argument*. The strategy is:

1. **Define a set:** Construct a set E of elements that are "too small" (i.e., $t^n < x$).
2. **Apply LUBP:** Since E is nonempty and bounded above, $\sup E$ exists—this is where we crucially use that $\mathbb{R}$ has the Least Upper Bound Property.
3. **Use trichotomy:** By the trichotomy of total orders, the supremum y satisfies exactly one of $y^n < x$, $y^n = x$, or $y^n > x$.
4. **Eliminate by contradiction:** Show that $y^n < x$ contradicts y being an *upper* bound (we can go higher), and $y^n > x$ contradicts y being the *least* upper bound (we can find a smaller upper bound).

This technique appears repeatedly throughout analysis whenever we need to prove existence of a value with a specific property. A similar technique is employed in Exercise 7 (showing the existence of the logarithm).

2.2.7 Decimals, Binaries, Ternaries

Section Overview: We discuss representations of real numbers in different bases.

Observe that decimal representations come in the form

$$n_0 + \frac{n_1}{10} + \frac{n_2}{100} + \cdots = n_0 + \sum_{k=1}^{\infty} \frac{n_k}{10^k}$$

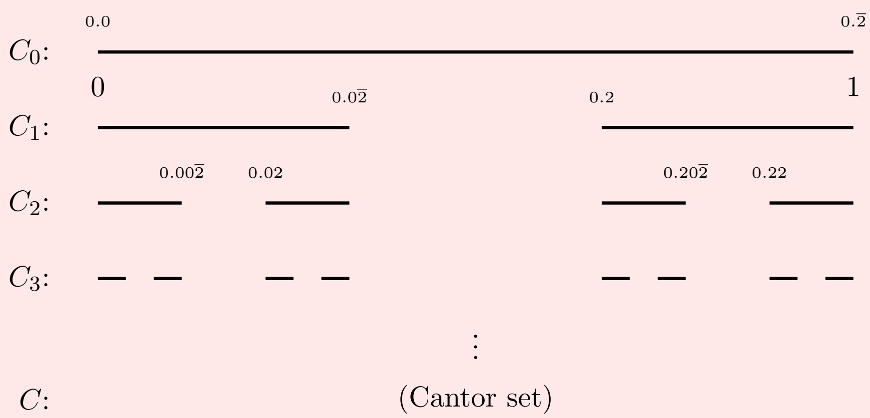
where $n_0 \in \mathbb{Z}$ and $n_k \in \{0, 1, 2, \ldots, 9\}$ for $k \geq 1$.

If we consider the set of partial sums

$$E = \left\{ n_0,\ n_0 + \frac{n_1}{10},\ n_0 + \frac{n_1}{10} + \frac{n_2}{100},\ \ldots \right\}$$

then $x = \sup E$.

Note: This construction is used to build the **Cantor set**. Starting with the interval $[0, 1]$, we iteratively remove the open middle third of each remaining interval:



Here the labels are *ternary* (base-3) expansions: e.g., $0.2_3 = \frac{2}{3}$, $0.02_3 = \frac{2}{9}$, $0.22_3 = \frac{8}{9}$, and $0.\overline{2}_3 = 0.222\ldots_3 = 1$.

The Cantor set $C = \bigcap_{n=0}^{\infty} C_n$ consists of all points in $[0, 1]$ whose ternary (base-3) expansion contains only the digits 0 and 2.

2.2.8 The Complex Field

Section Overview: We introduce the complex numbers $\mathbb{C}$ as an extension of $\mathbb{R}$.

Definition 2.25. The **complex numbers** are defined as

$$\mathbb{C} = \mathbb{R}[x]/(x^2 + 1)$$

where each element is of the form $z = a + bi$ with $a, b \in \mathbb{R}$ and $i^2 = -1$.

Theorem 2.26. $\mathbb{C}$ *is not an ordered field.*

Proof. By contradiction. Suppose $\mathbb{C}$ is an ordered field. By trichotomy, either $i > 0$ or $i < 0$ (since $i \neq 0$).

Case 1: If $i > 0$, then $i^2 > 0$ (since squares of nonzero elements are positive in an ordered field). But $i^2 = -1 < 0$, a contradiction.

Case 2: If $i < 0$, then $-i > 0$, so $(-i)^2 > 0$. But $(-i)^2 = i^2 = -1 < 0$, a contradiction.

Therefore $\mathbb{C}$ cannot be an ordered field. □

2.2.9 The Euclidean Spaces

Section Overview: We introduce Euclidean spaces $\mathbb{R}^n$ as spaces of ordered n-tuples.

Definition 2.27. The **Euclidean space** $\mathbb{R}^n$ is the set of all ordered n-tuples

$$\vec{x} = (x_1, x_2, x_3, \dots, x_n)$$

where $x_i \in \mathbb{R}$ for each $i = 1, \dots, n$.

For $\vec{x} = (x_1, \dots, x_n)$, $\vec{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ and $c \in \mathbb{R}$:

Addition:

$$\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$

Scalar multiplication:

$$c\vec{x} = (cx_1, cx_2, \dots, cx_n)$$

Inner product:

$$\langle \vec{x}, \vec{y} \rangle = \vec{x} \cdot \vec{y} = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

Definition 2.28. An **inner product** over $\mathbb{R}$ is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ that is:

- Symmetric bilinear and positive definite (for real vector spaces), or
- Hermitian sesquilinear and positive definite (for complex vector spaces).

Properties:

- **Symmetric:** $\langle x, y \rangle = \langle y, x \rangle$
- **Hermitian:** $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (conjugate symmetry)
- **Bilinear:** Linear in both arguments:

$$\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$$
$$\langle x, ay + bz \rangle = a\langle x, y \rangle + b\langle x, z \rangle$$

- **Sesquilinear:** Linear in one argument, conjugate-linear in the other:

$$\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$$
$$\langle x, ay + bz \rangle = \bar{a}\langle x, y \rangle + \bar{b}\langle x, z \rangle$$

- **Positive definite:** $\langle x, x \rangle \geq 0$, with equality if and only if $x = 0$

Definition 2.29. The **norm** of a vector $\vec{x}$ is defined as

$$\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

Properties of a norm:

- **Triangle inequality:** $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$
- **Absolute homogeneity:** $\|c\vec{x}\| = |c| \cdot \|\vec{x}\|$ for all $c \in \mathbb{R}$
- **Positive definite:** $\|\vec{x}\| \geq 0$, with equality if and only if $\vec{x} = \vec{0}$

Theorem 2.30 (Schwarz Inequality)**.** *If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are complex numbers, then*

$$\left| \sum_{j=1}^{n} a_j \bar{b}_j \right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2.$$

Proof. Put $A = \sum |a_j|^2$, $B = \sum |b_j|^2$, $C = \sum a_j \bar{b}_j$ (all sums run over $j = 1, \ldots, n$). If $B = 0$, then $b_1 = \cdots = b_n = 0$ and the conclusion is trivial. Assume therefore that $B > 0$. We have

$$\begin{aligned}
\sum |Ba_j - Cb_j|^2 &= \sum (Ba_j - Cb_j)\overline{(Ba_j - Cb_j)} \\
&= B^2 \sum |a_j|^2 - B\bar{C} \sum a_j \bar{b}_j - BC \sum \bar{a}_j b_j + |C|^2 \sum |b_j|^2 \\
&= B^2 A - B|C|^2 \\
&= B(AB - |C|^2).
\end{aligned}$$

Since the left side is a sum of non-negative terms, $B(AB - |C|^2) \geq 0$. Since $B > 0$, we conclude $AB - |C|^2 \geq 0$, i.e.,

$$\left| \sum a_j \bar{b}_j \right|^2 \leq \sum |a_j|^2 \sum |b_j|^2. \qquad \square$$

2.3 Lecture 3: January 27, 2026

Lecture Overview: We develop the set-theoretic foundations needed for analysis. We define **functions** as special relations and introduce key properties: surjectivity, injectivity, and bijectivity. Using bijections, we define when two sets have the same **cardinality**, leading to the notions of **finite**, **infinite**, and **countable** sets.

2.3.1 Functions

Section Overview: We define functions as relations with a uniqueness property, introduce images and preimages, and classify functions as injective (one-to-one), surjective (onto), or bijective (both).

Definition 2.31. A **function** $f : A \to B$ is a relation $R \subseteq A \times B$ such that for all $x \in A$, there exists a unique $y \in B$ with $(x, y) \in R$. This y is denoted $f(x)$. The set A is called the **domain** of f, and B is the **codomain** (or **range**).

Remark 2.32. Not all relations are functions. A relation fails to be a function if some element of A maps to multiple elements of B, or to no element at all.

Definition 2.33. Let $f : A \to B$ be a function.

- If $E \subseteq A$, the **image** of E under f is $f(E) = \{f(x) : x \in E\}$.
- If $F \subseteq B$, the **preimage** of F under f is $f^{-1}(F) = \{x \in A : f(x) \in F\}$.

Definition 2.34. Let $f : A \to B$ be a function.

- f is **surjective** (or **onto**) if $f(A) = B$, i.e., for every $y \in B$, there exists $x \in A$ with $f(x) = y$.
- f is **injective** (or **one-to-one**) if $f(x_1) = f(x_2)$ implies $x_1 = x_2$, i.e., distinct inputs map to distinct outputs.
- f is **bijective** if it is both injective and surjective.

2.3.2 Equivalence Relations

Section Overview: We review equivalence relations and observe how function properties relate to composition and inverses.

Remark 2.35. Recall from Lecture 1 that an equivalence relation satisfies:

- **Reflexivity:** $x \sim x$ (like the identity function)
- **Symmetry:** $x \sim y \Rightarrow y \sim x$ (like invertible functions: if $f : A \to B$, then $f^{-1} : B \to A$)
- **Transitivity:** $x \sim y$ and $y \sim z \Rightarrow x \sim z$ (like composition: $f : A \to B$ and $g : B \to C$ give $g \circ f : A \to C$)

2.3.3 Cardinality

Section Overview: We define cardinality using bijections, then classify sets as finite, infinite, or countable.

Definition 2.36. Two sets A and B have the same **cardinality**, written $A \sim B$, if there exists a bijection $f : A \to B$.

Definition 2.37. Let $[n] = \{1, 2, 3, \ldots, n\}$. We say that a set A is:

1. **finite** if there exists $n \in \mathbb{N}$ such that $A \sim [n]$,
2. **infinite** if A is not finite,
3. **countable** if $A \sim \mathbb{N}$.

Example. $\mathbb{Z}$ is countable. We can list the integers as:

$$0, 1, -1, 2, -2, 3, -3, \ldots$$

This defines a bijection $f : \mathbb{N} \to \mathbb{Z}$ given by

$$f(n) = \begin{cases} -\frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

(where we start $\mathbb{N}$ at 0). Thus $\mathbb{Z} \sim \mathbb{N}$, so $\mathbb{Z}$ is countable.

Definition 2.38. A **sequence** is a function $f : \mathbb{N} \to X$ where X is some set. If we denote $f(n)$ by x_n, then we write the sequence as $(x_n)_{n=1}^{\infty}$.

Remark 2.39. If A is a countable set, then there exists a surjection $f : \mathbb{N} \to A$. We say that A can be **arranged in a sequence**: the elements of A can be listed as $f(1), f(2), f(3), \ldots$

Note: The **set difference** $A \setminus B$ (read "A minus B") is defined as

$$A \setminus B = \{x \in A : x \notin B\}.$$

Theorem 2.40. *Every infinite subset of a countable set is countable.*

Proof. Let A be a countable set and let $B \subseteq A$ be infinite. Since A is countable, we can arrange its elements as a sequence $(x_n)_{n=1}^{\infty}$.

We construct a bijection $f : \mathbb{N} \to B$ inductively. Let $B_0 = \emptyset$. For each $k \geq 1$:

- Let $n_k = \min\{n \in \mathbb{N} : x_n \in B \setminus B_{k-1}\}$
- Define $f(k) = x_{n_k}$
- Set $B_k = B_{k-1} \cup \{x_{n_k}\}$

At each step, x_{n_k} is assigned position $k = |B_{k-1}| + 1$ in our enumeration of B. Since B is infinite, we never exhaust it, so each n_k exists.

This f is a bijection: it is injective since each element is added exactly once, and surjective since every element of B appears at some position n in the sequence (x_n) and will eventually be enumerated.

Thus $B \sim \mathbb{N}$, so B is countable. □

2.3.4 Unions and Intersections of Sets

Section Overview: We define unions and intersections of collections of sets, including arbitrary (possibly infinite) unions and intersections.

$A \cup B$ (union) $A \cap B$ (intersection)

Definition 2.41. Let A and B be sets, and let $f : A \to \mathcal{P}(B)$ be a function assigning to each $\alpha \in A$ a subset $f(\alpha) \subseteq B$. We define:

- The **union** of the family is

$$\bigcup_{\alpha \in A} f(\alpha) = \{x \in B : x \in f(\alpha) \text{ for some } \alpha \in A\}.$$

- The **intersection** of the family is

$$\bigcap_{\alpha \in A} f(\alpha) = \{x \in B : x \in f(\alpha) \text{ for all } \alpha \in A\}.$$

Theorem 2.42 (De Morgan's Laws)**.** *Let A and B be subsets of a universal set U. Then:*

1. $(A \cup B)^c = A^c \cap B^c$
2. $(A \cap B)^c = A^c \cup B^c$

Proof. We prove (1); the proof of (2) is similar.

($\subseteq$) Let $x \in (A \cup B)^c$. Then $x \notin A \cup B$, so $x \notin A$ and $x \notin B$. Thus $x \in A^c$ and $x \in B^c$, so $x \in A^c \cap B^c$.

($\supseteq$) Let $x \in A^c \cap B^c$. Then $x \in A^c$ and $x \in B^c$, so $x \notin A$ and $x \notin B$. Thus $x \notin A \cup B$, so $x \in (A \cup B)^c$. □

Note: To prove two sets are equal, $X = Y$, a standard technique is the **mutual subset argument**: show $X \subseteq Y$ and $Y \subseteq X$. For each direction, take an arbitrary element of one set and show it belongs to the other.

Theorem 2.43 (Distributive Law)**.** *Let A, B, and C be sets. Then*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Proof. ($\subseteq$) Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. Since $x \in B \cup C$, either $x \in B$ or $x \in C$.

- If $x \in B$, then $x \in A \cap B$, so $x \in (A \cap B) \cup (A \cap C)$.

- If $x \in C$, then $x \in A \cap C$, so $x \in (A \cap B) \cup (A \cap C)$.

($\supseteq$) Let $x \in (A \cap B) \cup (A \cap C)$. Then $x \in A \cap B$ or $x \in A \cap C$.

- If $x \in A \cap B$, then $x \in A$ and $x \in B \subseteq B \cup C$, so $x \in A \cap (B \cup C)$.
- If $x \in A \cap C$, then $x \in A$ and $x \in C \subseteq B \cup C$, so $x \in A \cap (B \cup C)$.

$\square$

2.3.5 The (Un)countability of Number Systems

Section Overview: We apply our results to the classical number systems. We show $\mathbb{Q}$ is countable (as a countable union of countable sets), but $\mathbb{R}$ is **uncountable** using Cantor's diagonal argument on binary sequences. This reveals a hierarchy of infinities: $|\mathbb{N}| = |\mathbb{Q}| < |\mathbb{R}|$.

Theorem 2.44. *Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of countable sets. Then*

$$\bigcup_{n=1}^{\infty} E_n$$

is countable.

Proof. Since each E_n is countable, we can enumerate its elements as

$$E_n = \{x_{n,1}, x_{n,2}, x_{n,3}, \ldots\}.$$

Arrange all elements in an infinite grid:

$$\begin{array}{cccccc}
E_1: & \overset{1}{x_{1,1}} & \overset{2}{x_{1,2}} & \overset{4}{x_{1,3}} & \overset{7}{x_{1,4}} & \cdots \\
E_2: & \overset{3}{x_{2,1}} & \overset{5}{x_{2,2}} & \overset{8}{x_{2,3}} & x_{2,4} & \cdots \\
E_3: & \overset{6}{x_{3,1}} & \overset{9}{x_{3,2}} & x_{3,3} & x_{3,4} & \cdots \\
E_4: & \overset{10}{x_{4,1}} & x_{4,2} & x_{4,3} & x_{4,4} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

We enumerate the union by traversing diagonals: first $x_{1,1}$, then $x_{1,2}, x_{2,1}$, then $x_{1,3}, x_{2,2}, x_{3,1}$, and so on. The k-th diagonal contains all $x_{n,m}$ with $n + m = k + 1$.

This gives a surjection from $\mathbb{N}$ onto $\bigcup_{n=1}^{\infty} E_n$ (skipping repeats if sets overlap). Thus the union is countable. $\square$

Note: The **diagonal argument** is a powerful technique for enumerating countable unions. By arranging elements in a grid and traversing along diagonals, we reduce a "two-dimensional" infinite collection to a "one-dimensional" sequence.

Theorem 2.45. *If A is countable, then A^n is countable for all $n \in \mathbb{N}$.*

Proof. By induction on n.

Base case ($n = 1$): $A^1 = A$ is countable by assumption.

Inductive hypothesis: Assume A^n is countable for some $n \geq 1$.

Inductive step: We show A^{n+1} is countable. Observe that

$$A^{n+1} = A^n \times A.$$

By the inductive hypothesis, A^n is countable, so we can enumerate it as $A^n = \{b_1, b_2, b_3, \ldots\}$. Since A is countable by assumption, we can write $A = \{a_1, a_2, a_3, \ldots\}$. The Cartesian product $A^n \times A$ can then be arranged in a grid:

$$\begin{array}{cccc}
(b_1, a_1) & (b_1, a_2) & (b_1, a_3) & \cdots \\
(b_2, a_1) & (b_2, a_2) & (b_2, a_3) & \cdots \\
(b_3, a_1) & (b_3, a_2) & (b_3, a_3) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}$$

By the diagonal argument, $A^n \times A$ is countable.

Conclusion: By the principle of mathematical induction, A^n is countable for all $n \in \mathbb{N}$. $\square$

Corollary 2.46. $\mathbb{Q}$ *is countable.*

Proof. For each $n \in \mathbb{N}$, let $E_n = \left\{\frac{m}{n} : m \in \mathbb{Z}\right\}$ be the set of rationals with denominator n. Each E_n is countable (since $E_n \sim \mathbb{Z}$ and $\mathbb{Z}$ is countable). Then

$$\mathbb{Q} = \bigcup_{n=1}^{\infty} E_n$$

is a countable union of countable sets, hence countable. $\square$

Theorem 2.47 ($\mathbb{R}$ is uncountable)**.** *The set of binary sequences $\{0,1\}^{\mathbb{N}}$ is uncountable.*

Proof. By contradiction. Suppose $\{0,1\}^{\mathbb{N}}$ is countable. Then we can list all binary sequences as $s_1, s_2, s_3, \ldots$ where each $s_n = (s_{n,1}, s_{n,2}, s_{n,3}, \ldots)$:

$$\begin{array}{lcccccc}
 & \text{pos 1} & \text{pos 2} & \text{pos 3} & \text{pos 4} & \text{pos 5} & \\
s_1\colon & 0 & 1 & 0 & 1 & 1 & \cdots \\
s_2\colon & 1 & 1 & 0 & 0 & 1 & \cdots \\
s_3\colon & 0 & 0 & 1 & 1 & 0 & \cdots \\
s_4\colon & 1 & 0 & 1 & 0 & 0 & \cdots \\
s_5\colon & 0 & 1 & 1 & 1 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
d\colon & 1 & 0 & 0 & 1 & 0 & \cdots
\end{array}$$

flip diagonal

Construct a new sequence $d = (d_1, d_2, d_3, \ldots)$ by flipping the diagonal entries:

$$d_n = \begin{cases} 1 & \text{if } s_{n,n} = 0 \\ 0 & \text{if } s_{n,n} = 1 \end{cases}$$

Then d differs from s_n in the n-th position for every $n \in \mathbb{N}$. Thus $d \neq s_n$ for all n, so d is not in our list. But $d \in \{0,1\}^{\mathbb{N}}$, contradicting that our list contains all binary sequences.

Therefore $\{0,1\}^{\mathbb{N}}$ is uncountable. □

Note: This is **Cantor's diagonal argument**. Since there is a bijection between $\{0,1\}^{\mathbb{N}}$ and $\mathbb{R}$ (via binary expansions), this proves $\mathbb{R}$ is uncountable. The key insight is that any proposed enumeration can be "diagonalized" to produce a missing element.

Exercise: An **algebraic number** is a solution to a polynomial equation with coefficients in $\mathbb{Q}$:

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0, \quad a_i \in \mathbb{Q}.$$

Is the set of algebraic numbers countable?

Proof: Yes. We proceed by induction on the degree of the polynomial.

Base case ($n = 1$): A degree-1 polynomial $mx + b = 0$ has solution $x = -b/m \in \mathbb{Q}$. Since $\mathbb{Q}$ is countable, the set of algebraic numbers of degree 1 is countable.

Inductive step: Let A_n denote the set of algebraic numbers that are roots of some polynomial of degree at most n. Assume A_n is countable. Consider numbers of the form

$$\{a + b \cdot \sqrt[n+1]{z} : a, b, z \in A_n\}.$$

This set is countable since A_n^3 is countable (as a finite Cartesian product of a countable set). More generally, the set of polynomials of degree $n+1$ with coefficients in $\mathbb{Q}$ is $\mathbb{Q}^{n+2}$, which is countable. Each such polynomial has at most $n+1$ roots, so the roots form a countable union of finite sets, hence A_{n+1} is countable.

Conclusion: The set of all algebraic numbers is $\bigcup_{n=1}^{\infty} A_n$, a countable union of countable sets, hence countable.

Note: A real number that is *not* algebraic is called **transcendental**. Since the algebraic numbers are countable but $\mathbb{R}$ is uncountable, transcendental numbers must exist—in fact, "most" real numbers are transcendental! Examples include π, e, and $\tau = 2\pi$. Proving that a specific number is transcendental is typically very difficult: Lindemann proved π is transcendental in 1882, and Hermite proved e is transcendental in 1873.

Remark 2.48. Recall the **Cantor set** from Lecture 2: the set of all points in $[0, 1]$ whose ternary expansion uses only the digits 0 and 2. The Cantor set is in bijection with $\{0,2\}^{\mathbb{N}} \sim \{0,1\}^{\mathbb{N}}$ (just map $0 \mapsto 0$ and $2 \mapsto 1$). By the same diagonal argument, the Cantor set is uncountable—despite being "sparse" (it contains no intervals and has measure zero).

2.4 Lecture 4: January 29, 2026

Lecture Overview: We introduce the abstract notion of a topological space and its key building blocks: open sets, bases, and the topology they generate. We then specialize to metric spaces, where open balls form a natural basis, and develop the metric topology with epsilon-delta techniques. Finally, we study closed sets, closures, and limit points — the dual perspective to open sets — along with boundedness, convexity, convergence, and the equivalence of the product and metric topologies on $\mathbb{R}^n$.

Symbol	Meaning
$\mathcal{T}$	A topology (collection of open sets) on X
$\mathcal{B}$	A basis for a topology
(X, d)	A metric space (set X with metric d)
$B(x, \varepsilon)$	Open ball of radius ε centered at x
E	A subset of X
$\overline{E}$	Closure of E (smallest closed set containing E)
E'	Derived set (set of all limit points of E)
$E \setminus E'$	Isolated points of E
Open set	A set $U \in \mathcal{T}$; every point has a neighborhood inside U
Closed set	Complement of an open set; contains all its limit points ($E' \subseteq E$)

2.4.1 Topological Spaces

Section Overview: We define a topology on a set X via three axioms governing open sets, then explore examples (discrete and cofinite topologies). We introduce the notion of a basis for a topology and show that every open set is a union of basis elements.

Definition 2.49. A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X such that:

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.
2. For all $\mathcal{S} \subseteq \mathcal{T}$, $\displaystyle\bigcup_{S \in \mathcal{S}} S \in \mathcal{T}$ (arbitrary unions).
3. If $\mathcal{S} \subseteq \mathcal{T}$ is finite, then $\displaystyle\bigcap_{S \in \mathcal{S}} S \in \mathcal{T}$ (finite intersections).

Remark 2.50. The elements of $\mathcal{T}$ are called **open sets**.

Example. The **discrete topology** on a set X is $\mathcal{T} = \mathcal{P}(X)$, the power set of X. In this topology, every subset of X is open. For instance, $\mathbb{Z} \subseteq \mathbb{R}$ inherits the discrete topology from the standard topology on $\mathbb{R}$: every subset of $\mathbb{Z}$ is open.

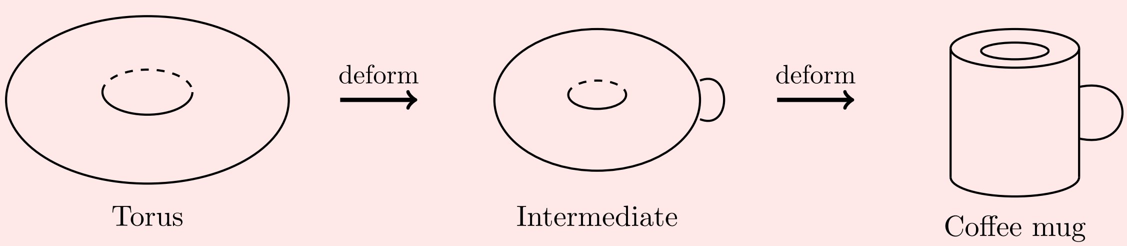
Why is $\mathbb{Z}$ discrete in $\mathbb{R}$? For any $n \in \mathbb{Z}$, the interval $(n - \frac{1}{2}, n + \frac{1}{2}) \cap \mathbb{Z} = \{n\}$, so every singleton $\{n\}$ is open in the subspace topology on $\mathbb{Z}$. Since arbitrary unions of open sets are open, every subset of $\mathbb{Z}$ is open. Intuitively, the integers are "isolated" — there is space around each one with no other integers nearby. The discrete topology is the topology where every set is open, which is the finest (most open sets) possible topology on a set.

Example. The finite-complement topology (or cofinite topology) on a set X: a subset $U \subseteq X$ is open if and only if $X \setminus U$ is finite (or $U = \emptyset$).

We verify the three topology axioms:

1. $\emptyset$ is open by convention. X is open since $X \setminus X = \emptyset$ is finite.
2. Let $\{U_\alpha\}_{\alpha \in A}$ be a collection of open sets. Then
$$X \setminus \bigcup_{\alpha \in A} U_\alpha = \bigcap_{\alpha \in A} (X \setminus U_\alpha).$$
Each $X \setminus U_\alpha$ is finite, so the intersection is a subset of any one of them, hence finite. Thus $\bigcup U_\alpha$ is open.
3. Let $U_1, \dots, U_n$ be finitely many open sets. Then
$$X \setminus \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X \setminus U_i).$$
Each $X \setminus U_i$ is finite, and a finite union of finite sets is finite. Thus $\bigcap_{i=1}^{n} U_i$ is open.

How does this relate to shapes like a torus? The axioms above define topology in full generality — they tell us what it means for sets to be "open" without any reference to distance or geometry. A shape like a torus is a *topological space*: a set of points (the surface) equipped with a topology (which subsets count as open). Two shapes are "the same" topologically if there is a continuous bijection with a continuous inverse (a homeomorphism) between them. The famous "coffee mug = donut" equivalence means there is such a map between them. The abstract axioms here are the foundation: they capture exactly the structure needed to define continuity, connectedness, and compactness, which are the properties that let us distinguish a torus from a sphere without ever measuring distances or angles.



Both have exactly one hole — they are homeomorphic (genus 1 surfaces).

Definition 2.51. A basis for a topology on X is a collection $\mathcal{B}$ of subsets of X such that:

1. For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.
2. For all $B_1, B_2 \in \mathcal{B}$ and for all $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ with $B_3 \subseteq B_1 \cap B_2$ and $x \in B_3$.

The topology generated by $\mathcal{B}$ is defined as follows: $U \in \mathcal{T}$ if and only if for all $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq U$.

Example. The order topology on $\mathbb{R}$ is generated by the basis of open intervals (a, b) for $a < b$.

Lemma 2.52. *Let $\mathcal{B}$ be a basis for the topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.*

Proof. Let $\mathcal{U}$ denote the collection of all unions of elements of $\mathcal{B}$. We show $\mathcal{T} = \mathcal{U}$.

($\supseteq$) Each $B \in \mathcal{B}$ is open (since for any $x \in B$, the element B itself witnesses $x \in B \subseteq B$). Since $\mathcal{T}$ is closed under arbitrary unions, every union of elements of $\mathcal{B}$ is in $\mathcal{T}$. Thus $\mathcal{U} \subseteq \mathcal{T}$.

($\subseteq$) Let $U \in \mathcal{T}$. For each $x \in U$, by definition of the topology generated by $\mathcal{B}$, there exists $B_x \in \mathcal{B}$ with $x \in B_x \subseteq U$. Then

$$U = \bigcup_{x \in U} B_x,$$

which is a union of elements of $\mathcal{B}$. Thus $U \in \mathcal{U}$, so $\mathcal{T} \subseteq \mathcal{U}$. □

2.4.2 Metric Topology

Section Overview: We define metrics, open balls, and the metric topology they generate. Key results include the characterization of open sets via epsilon-balls, the product topology equaling the metric topology on $\mathbb{R}^n$, and the epsilon-delta proof technique that recurs throughout analysis.

Definition 2.53. Let X and Y be topological spaces. The **product topology** on $X \times Y$ is the topology generated by the basis

$$\mathcal{B} = \{U \times V \mid U \text{ open in } X,\ V \text{ open in } Y\}.$$

Definition 2.54. A **metric** on a set X is a function $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$:

1. $d(x, y) \geq 0$, with equality if and only if $x = y$ (positive definiteness).
2. $d(x, y) = d(y, x)$ (symmetry).
3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

Example. The **Euclidean metric** on $\mathbb{R}$ (or $\mathbb{R}^n$) is defined by $d(x, y) = |x - y|$. The topology induced by this metric is called the **Euclidean topology**.

Definition 2.55. Let d be a metric on X. The **ε-open ball** centered at $x \in X$ is

$$B(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}.$$

Theorem 2.56. *The collection $\{B(x, \varepsilon) \mid x \in X,\ \varepsilon > 0\}$ forms a basis for a topology on X. The resulting topology is called the* ***metric topology*** *induced by d.*

Proof. We verify the two basis axioms for $\mathcal{B} = \{B(x, \varepsilon) \mid x \in X,\ \varepsilon > 0\}$.

1. For any $x \in X$, we have $x \in B(x, 1)$, so every point of X is contained in some basis element.
2. Let $B(x_1, \varepsilon_1), B(x_2, \varepsilon_2) \in \mathcal{B}$ and let $y \in B(x_1, \varepsilon_1) \cap B(x_2, \varepsilon_2)$. By the lemma, there exist $\delta_1, \delta_2 > 0$ such that $B(y, \delta_1) \subseteq B(x_1, \varepsilon_1)$ and $B(y, \delta_2) \subseteq B(x_2, \varepsilon_2)$. Setting $\delta = \min(\delta_1, \delta_2)$, we have
$$y \in B(y, \delta) \subseteq B(x_1, \varepsilon_1) \cap B(x_2, \varepsilon_2).$$
Since $B(y, \delta) \in \mathcal{B}$, the second basis axiom is satisfied.

$\square$

Epsilon-delta proof technique. This is a recurring pattern in analysis. The goal is to show some property holds "locally" — that is, within a small neighborhood of a point. The approach:

1. **Identify what you need:** You want to find a $\delta > 0$ such that something (a ball, a set, a bound) holds within distance δ of your point.
2. **Use what you're given:** You typically start with some $\varepsilon > 0$ that gives you room to work with (e.g., your point lies inside a ball of radius ε).
3. **Compute the gap:** Figure out how much room you have — often $\delta = \varepsilon - d(\text{point}, \text{center})$ or $\delta = \min(\delta_1, \delta_2)$ when intersecting constraints.
4. **Verify with the triangle inequality:** Chain distances together to show your choice of δ works.

When you see "for all ... there exists $\delta > 0$," think: *How much room do I have, and how do I stay within it?*

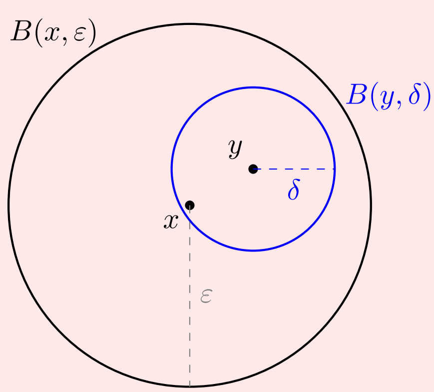
Lemma 2.57. *For every $y \in B(x, \varepsilon)$, there exists $\delta > 0$ such that $B(y, \delta) \subseteq B(x, \varepsilon)$.*

Proof. Let $y \in B(x, \varepsilon)$, so $d(x, y) < \varepsilon$. Set $\delta = \varepsilon - d(x, y) > 0$. For any $z \in B(y, \delta)$, we have $d(y, z) < \delta$, and by the triangle inequality:

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + \delta = d(x, y) + (\varepsilon - d(x, y)) = \varepsilon.$$

Thus $z \in B(x, \varepsilon)$, so $B(y, \delta) \subseteq B(x, \varepsilon)$. $\square$

Intuition: If y is inside the open ball $B(x, \varepsilon)$, then y is strictly closer than ε to x, so there is some "room left over." We can fit a smaller ball around y that still stays inside the original ball. Concretely, $\delta = \varepsilon - d(x, y) > 0$ works.



Theorem 2.58. *A set U is open in the metric topology on X induced by d if and only if for all $y \in U$, there exists $\delta > 0$ such that $B_d(y, \delta) \subseteq U$.*

Proof. ($\Rightarrow$) This follows directly from the definition of the topology generated by a basis.

(⇐) For each $y \in U$, choose $\delta_y > 0$ such that $B_d(y, \delta_y) \subseteq U$. Then

$$U = \bigcup_{y \in U} B_d(y, \delta_y).$$

Each $B_d(y, \delta_y)$ is open (it is a basis element), and an arbitrary union of open sets is open by the second topology axiom. Thus U is open. □

2.4.3 Closed Sets

Section Overview: We define closed sets as complements of open sets and establish their dual properties (arbitrary intersections, finite unions). We then introduce closures, limit points, and isolated points, proving that $\overline{E} = E \cup E'$. Examples include the Cantor set and sets with isolated points. We also define boundedness (a metric-dependent property), convexity, and convergence of sequences in topological spaces.

Definition 2.59. A set $C \subseteq X$ is **closed** if its complement $X \setminus C$ is open.

Properties of closed sets (dual to the open set axioms):

1. $\emptyset$ and X are closed.
2. Arbitrary intersections of closed sets are closed.
3. Finite unions of closed sets are closed.

Definition 2.60. The **closure** of a set $E \subseteq X$ is

$$\overline{E} = \bigcap \{C \subseteq X \mid C \text{ is closed and } E \subseteq C\}.$$

Theorem 2.61. *$x \in \overline{E}$ if and only if for every open set U containing x, $U \cap E \neq \emptyset$. (We say U is a* ***neighborhood*** *of x.)*

Proof. We prove the contrapositive: $x \notin \overline{E}$ if and only if there exists an open set U containing x with $U \cap E = \emptyset$.

(⇒) If $x \notin \overline{E}$, then there exists a closed set $C \supseteq E$ with $x \notin C$. Let $U = X \setminus C$. Then U is open, $x \in U$, and $U \cap E \subseteq U \cap C = \emptyset$.

(⇐) If there exists an open set U with $x \in U$ and $U \cap E = \emptyset$, then $C = X \setminus U$ is closed, $E \subseteq C$, and $x \notin C$. Thus $x \notin \overline{E}$. □

Proof by contrapositive. To prove $P \Rightarrow Q$, it is equivalent to prove $\neg Q \Rightarrow \neg P$. This is often easier when the negation is more concrete to work with. In the proof above, showing "$x \notin \overline{E}$ implies there exists an open set missing E" is more direct than working with the original statement, because $x \notin \overline{E}$ hands us a specific closed set to work with. As a rule of thumb: if the statement you want to prove starts with "for all," its contrapositive starts with "there exists" — and existential statements are often easier to prove since you only need to produce one witness.

P	Q	$P \Rightarrow Q$	$\neg Q \Rightarrow \neg P$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

The columns for $P \Rightarrow Q$ and $\neg Q \Rightarrow \neg P$ are identical — the two statements are logically equivalent.

Definition 2.62. A point x is a **limit point** of E if every neighborhood U of x contains some $y \neq x$ with $y \in U \cap E$.

Theorem 2.63. *$\overline{E} = E \cup E'$, where E' denotes the set of all **limit points** of E.*

Proof. ($\supseteq$) If $x \in E$, then for every open U containing x, we have $x \in U \cap E \neq \emptyset$, so $x \in \overline{E}$. If $x \in E'$, then every neighborhood U of x contains some $y \neq x$ with $y \in E$, so $U \cap E \neq \emptyset$, and thus $x \in \overline{E}$.

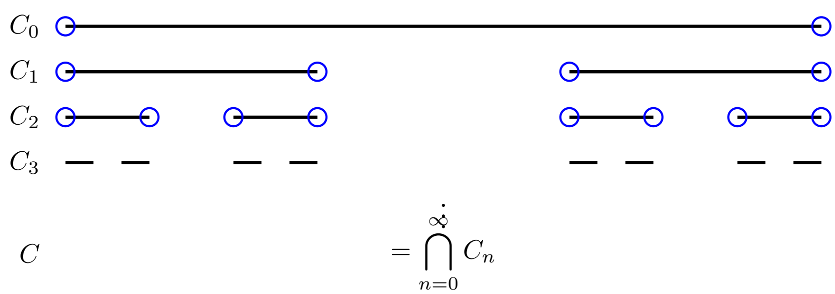
($\subseteq$) Suppose $x \in \overline{E}$ and $x \notin E$. Then for every neighborhood U of x, $U \cap E \neq \emptyset$. Since $x \notin E$, the point witnessing $U \cap E \neq \emptyset$ must be some $y \neq x$ with $y \in E$. Thus $x \in E'$. □

Keeping E, $\overline{E}$, and E' straight.

- E — the original set.
- E' — the *limit points* (or *derived set*) of E: points x such that every neighborhood of x contains some *other* point of E. Note x need not belong to E itself.
- $\overline{E} = E \cup E'$ — the *closure* of E: the smallest closed set containing E. It includes the points of E together with any "boundary" points that E accumulates toward.

For example, if $E = (0, 1) \subset \mathbb{R}$, then $E' = [0, 1]$ (every neighborhood of 0 or 1 hits $(0, 1)$), and $\overline{E} = E \cup E' = [0, 1]$.

Example. The Cantor set. Let C be the Cantor set, constructed by repeatedly removing the open middle third of each interval. Despite the intervals shrinking at each stage, the endpoints of every removed interval remain in C and are limit points of C.



The circled endpoints at each stage are never removed — they persist through every C_n and thus belong to $C = \bigcap_{n=0}^{\infty} C_n$. Moreover, every neighborhood of such an endpoint contains points from the remaining intervals, making them limit points of C. In fact, C is closed ($C = \overline{C}$) and every point of C is a limit point: $C' = C$.

Corollary 2.64. *E is closed if and only if $E' \subseteq E$.*

Points of $E \setminus E'$ are called **isolated points**.

Example. Let $E = (0, \frac{1}{2}) \cup \{1\}$. The set of limit points is $E' = [0, \frac{1}{2}]$. Note that $1 \in E$ but $1 \notin E'$: for instance, the neighborhood $(\frac{3}{4}, \frac{5}{4})$ contains no point of E other than 1 itself. Thus 1 is an isolated point of E.

Conversely, $0 \notin E$ and $\frac{1}{2} \notin E$, yet both are limit points: for any $\varepsilon > 0$, the neighborhoods $(0 - \varepsilon, 0 + \varepsilon)$ and $(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon)$ each contain points of $(0, \frac{1}{2}) \subseteq E$. The key takeaway: limit points can lie *outside* the set, and points *inside* the set need not be limit points.

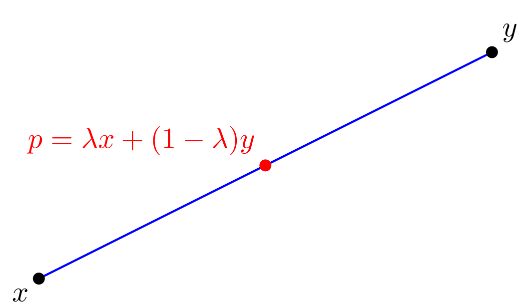
Example. Consider the set $E = (0,1) \setminus \{\frac{1}{2}\}$: the open interval $(0,1)$ with a hole at $\frac{1}{2}$. Although $\frac{1}{2} \notin E$, it is a limit point of E: for any $\varepsilon > 0$, the neighborhood $(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon)$ contains points of E other than $\frac{1}{2}$. Being in the set and being a limit point are independent properties.

Definition 2.65. A subset E of a metric space (X, d) is **bounded** if there exist $x \in X$ and $M > 0$ such that $E \subseteq B(x, M)$, i.e., $d(x, y) < M$ for all $y \in E$.

Remark 2.66. Boundedness is a metric-dependent property, not a topological one. The same set can be bounded under one metric and unbounded under another. For example, $\mathbb{R}$ with the usual metric $d(x,y) = |x - y|$ is unbounded. However, define the *bounded metric* $\bar{d}(x,y) = \min(|x-y|, 1)$. This induces the same topology on $\mathbb{R}$ (the same sets are open), but now $\mathbb{R}$ is bounded: $\bar{d}(x,y) \leq 1$ for all x, y, so $\mathbb{R} \subseteq B(0, 2)$. Since two metrics can generate the same topology yet disagree on boundedness, boundedness is not a topological invariant — it depends on the choice of metric.

Definition 2.67. A set $E \subseteq \mathbb{R}$ is **convex** if for all $x, y \in E$ and for all $\lambda \in (0, 1)$,

$$\lambda x + (1 - \lambda) y \in E.$$



Definition 2.68. A sequence (x_n) in a topological space X **converges** to $y \in X$ if for every neighborhood U of y, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in U$.

Theorem 2.69. *The product topology on $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$ equals the metric topology induced by the Euclidean metric.*

Proof. We show both topologies have the same open sets by showing each basis element of one is open in the other.

Product basis elements are open in the metric topology. A basic open set in the product topology is $(a_1, b_1) \times \cdots \times (a_n, b_n)$. Let $x = (x_1, \ldots, x_n)$ be a point in this set. For each i, choose $\varepsilon_i = \min(x_i - a_i,\, b_i - x_i) > 0$. Set $\varepsilon = \min(\varepsilon_1, \ldots, \varepsilon_n) > 0$. Then $B(x, \varepsilon) \subseteq (a_1, b_1) \times \cdots \times (a_n, b_n)$, since if $d(x, y) < \varepsilon$ then $|x_i - y_i| \leq d(x, y) < \varepsilon \leq \varepsilon_i$ for each i, so $y_i \in (a_i, b_i)$.

Open balls are open in the product topology. Let $x \in B(y, \varepsilon)$ and set $\delta = \varepsilon - d(x, y) > 0$. Consider the product of intervals

$$U = \left(x_1 - \tfrac{\delta}{\sqrt{n}},\, x_1 + \tfrac{\delta}{\sqrt{n}}\right) \times \cdots \times \left(x_n - \tfrac{\delta}{\sqrt{n}},\, x_n + \tfrac{\delta}{\sqrt{n}}\right).$$

This is open in the product topology, $x \in U$, and for any $z \in U$ we have

$$d(x, z) = \sqrt{\sum_{i=1}^{n} (x_i - z_i)^2} < \sqrt{n \cdot \tfrac{\delta^2}{n}} = \delta,$$

so $z \in B(x, \delta) \subseteq B(y, \varepsilon)$. Thus $U \subseteq B(y, \varepsilon)$, and $B(y, \varepsilon)$ is open in the product topology. $\square$

Theorem 2.70. *If $p \in E'$, then every neighborhood U of p contains infinitely many points of E.*

Proof. By contradiction. Suppose U is a neighborhood of p containing only finitely many points $q_1, \ldots, q_n \in U \cap E$ with $q_i \neq p$. Let $r = \min_{1 \leq i \leq n} d(p, q_i) > 0$. Then $B(p, r)$ contains no point of E other than p itself. But $p \in E'$, so every neighborhood of p must contain some $y \neq p$ with $y \in E$ — a contradiction. $\square$

2.5 Lecture 5: February 3, 2026

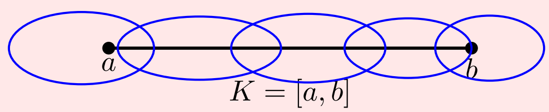
2.5.1 Compactness

Definition 2.71. Let $E \subseteq X$ be a topological space. If $\{G_\alpha\}$ is a collection of open sets of X such that $E \subseteq \bigcup G_\alpha$, then $\{G_\alpha\}$ is an **open cover** of E.

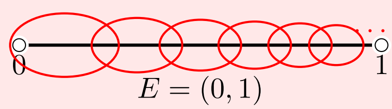
Definition 2.72. A subset $K \subseteq X$ is **compact** if every open cover contains a finite subcover. That is, given $\{G_\alpha\}$, there exist $G_{\alpha_1}, \ldots, G_{\alpha_n} \subseteq \{G_\alpha\}$ such that $K \subseteq G_{\alpha_1} \cup G_{\alpha_2} \cup \cdots \cup G_{\alpha_n}$.

Understanding covers and compactness. An open cover is a collection of open sets that together contain every point of E. Think of it as "blanketing" the set with open sets. The cover may have infinitely many sets — in fact, that's the interesting case.
Compactness says: no matter how you cover K with open sets, you can always throw away all but finitely many and still cover K. This is a strong condition — it fails for many sets.

Open cover: finitely many suffice



Not compact: needs infinitely many



The closed interval $[a, b]$ is compact: any open cover has a finite subcover. The open interval $(0, 1)$ is not compact: the cover $\{(\frac{1}{n}, 1) : n \geq 2\}$ has no finite subcover, since points near 0 escape any finite subcollection.

2.5.2 Subspace Topology

$Y \subseteq X$. U is open in Y if and only if there exists V open in X such that $U = V \cap Y$.

Theorem 2.73. *If $K \subseteq Y \subseteq X$, then K compact relative to $X \Leftrightarrow K$ compact relative to Y.*

Proof. ($\Rightarrow$) Suppose K is compact relative to X. Let $\{U_\alpha\}$ be an open cover of K in Y. By the subspace topology, for each U_α there exists V_α open in X such that $U_\alpha = V_\alpha \cap Y$. Then $\{V_\alpha\}$ is an open cover of K in X. Since K is compact relative to X, there exist $V_{\alpha_1}, \ldots, V_{\alpha_n}$ such that $K \subseteq V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$. Then

$$K \subseteq (V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}) \cap Y = U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}.$$

So K is compact relative to Y.

($\Leftarrow$) Suppose K is compact relative to Y. Let $\{V_\alpha\}$ be an open cover of K in X. Then $\{V_\alpha \cap Y\}$ is an open cover of K in Y (each $V_\alpha \cap Y$ is open in Y by the subspace topology). Since K is compact relative to Y, there exist $V_{\alpha_1} \cap Y, \ldots, V_{\alpha_n} \cap Y$ such that $K \subseteq (V_{\alpha_1} \cap Y) \cup \cdots \cup (V_{\alpha_n} \cap Y)$. Since $K \subseteq Y$, we have $K \subseteq V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$. So K is compact relative to X. $\square$

Theorem 2.74. *Compact subsets of metric spaces are closed.*

Proof. It suffices to show that K^c is open. Since we are in a metric space, we can use the open ball definition of open sets. We will show for every point $p \in K^c$, there exists $\varepsilon > 0$ such that $B(p, \varepsilon) \subseteq K^c$.

Fix $p \in K^c$. For each $q \in K$, let $r_q = \frac{1}{2} d(p, q) > 0$. The balls $B(p, r_q)$ and $B(q, r_q)$ are disjoint. The collection $\{B(q, r_q) : q \in K\}$ is an open cover of K. By compactness, there exist $q_1, \ldots, q_n \in K$ such that

$$K \subseteq B(q_1, r_{q_1}) \cup \cdots \cup B(q_n, r_{q_n}).$$

Let $\varepsilon = \min(r_{q_1}, \ldots, r_{q_n}) > 0$. Then $B(p, \varepsilon) \subseteq K^c$, since $B(p, \varepsilon) \subseteq B(p, r_{q_i})$ is disjoint from $B(q_i, r_{q_i})$ for each i, and K is covered by these balls. □

Theorem 2.75. *Closed subsets of compact sets are compact.*

Proof. Let F be a closed subset of a compact set K in X. Let $\{G_\alpha\}$ be an open cover of F. Then $\{G_\alpha\} \cup \{F^c\}$ is an open cover of K (F is closed so F^c is open, and $\{G_\alpha\}$ covers F). Since K is compact, there exists a finite subcover: $G_{\alpha_1}, \ldots, G_{\alpha_n}$, and possibly F^c. Removing F^c (if present), we have $F \subseteq G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}$. Thus F is compact. □

Corollary 2.76. *The intersection of a closed set and a compact set is compact (in a metric space).*

Theorem 2.77. *Suppose $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of any finite subcollection is nonempty. Then $\bigcap_\alpha K_\alpha \neq \emptyset$.*

Proof. Suppose $\bigcap_\alpha K_\alpha = \emptyset$. Each K_α^c is open. By De Morgan's law,

$$\left(\bigcap_\alpha K_\alpha \right)^c = \bigcup_\alpha K_\alpha^c = X.$$

Fix $K_1 \in \{K_\alpha\}$. Then $\{K_\alpha^c\}$ is an open cover of K_1. By compactness, there exist $K_{\alpha_1}, \ldots, K_{\alpha_n}$ such that $K_1 \subseteq K_{\alpha_1}^c \cup \cdots \cup K_{\alpha_n}^c$. By De Morgan's law,

$$K_1 \subseteq (K_{\alpha_1} \cap \cdots \cap K_{\alpha_n})^c.$$

Thus $K_1 \cap K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} = \emptyset$, contradicting the finite intersection property. □

Corollary 2.78. *If $\{K_n\}$ is a sequence of compact subsets of a metric space X such that $K_n \supseteq K_{n+1}$, then $\bigcap_n K_n \neq \emptyset$.*

Theorem 2.79. *If E is an infinite subset of a compact set K, then E has a limit point in K.*

Proof. (By contradiction) Assume that E has no limit points. Then for all $p \in K$, there exists a neighborhood U_p of p where either $U_p \cap E = \emptyset$ or $U_p \cap E = \{p\}$. Therefore, $\{U_p\}$ forms an open cover of K. By compactness of K, there is a finite subcover $U_{p_1} \cup \cdots \cup U_{p_n} \supseteq K$. Each U_{p_i} contains at most one point of E, so E has at most n points. This contradicts the fact that E is infinite. □

The Cantor set is nonempty. Recall the Cantor set is defined as $C = \bigcap_{n=0}^\infty C_n$, where each C_n is a finite union of closed intervals. Each C_n is compact. The sets are nested: $C_0 \supseteq C_1 \supseteq C_2 \supseteq \cdots$, so any finite intersection equals the smallest set in the subcollection, which is nonempty. By the theorem above, $C = \bigcap_{n=0}^\infty C_n \neq \emptyset$.

Theorem 2.80. *Suppose $\{I_n\}$ is a sequence of closed intervals in $\mathbb{R}$ such that $I_n \supseteq I_{n+1}$. Then $\bigcap_k I_k \neq \emptyset$.*

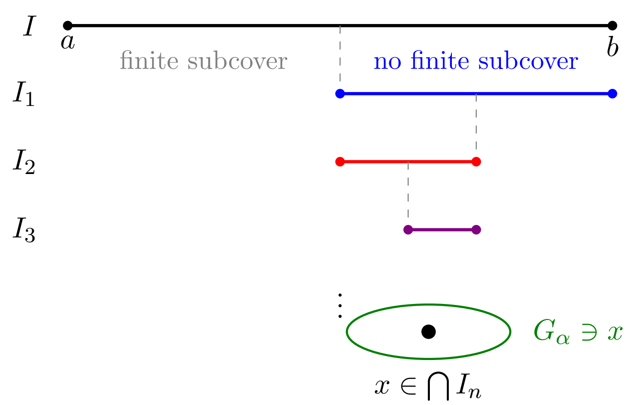
Proof. Let $I_n = [a_n, b_n]$. Let $E = \{a_n\}$. Since the intervals are nested, $a_n \leq b_m$ for all n, m, so E is bounded above. There exists $x = \sup E$.

For all n, we have $a_n \leq x$ (since x is an upper bound of E). Also $x \leq b_n$ for all n (since each b_n is an upper bound for E, and x is the least upper bound). Thus $a_n \leq x \leq b_n$, so $x \in I_n$ for all n. Therefore $x \in \bigcap_k I_k$. □

Theorem 2.81. *Closed intervals (and therefore closed boxes) are compact.*

Proof. Let $I = [a, b]$ and let $\{G_\alpha\}$ be an open cover of I. Suppose this open cover does not reduce to a finite subcover. Cut the interval in half: at least one half cannot be covered by finitely many G_α (if both halves could, we could combine them to cover I). Call this half I_1. Repeat: bisect I_1 and choose a half I_2 with no finite subcover. Continuing, we obtain nested closed intervals $I \supseteq I_1 \supseteq I_2 \supseteq \cdots$ with $|I_n| = (b-a)/2^n$, each having no finite subcover.

By the nested intervals theorem, there exists $x \in \bigcap_n I_n$. Since $\{G_\alpha\}$ covers I, we have $x \in G_\alpha$ for some α. Since G_α is open, there exists $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subseteq G_\alpha$. For large n, $|I_n| < \varepsilon$ and $x \in I_n$, so $I_n \subseteq (x-\varepsilon, x+\varepsilon) \subseteq G_\alpha$. But then I_n is covered by a single open set, contradicting that I_n has no finite subcover.



□

Key ideas in this proof:

1. **Proof by contradiction:** Assume no finite subcover exists and derive a contradiction.
2. **Bisection argument:** If a set has no finite subcover, at least one half doesn't either. This lets us build nested intervals.
3. **Nested intervals theorem:** The intersection $\bigcap I_n \neq \emptyset$, giving us a point x.
4. **Open set definition:** Since $x \in G_\alpha$ and G_α is open, there exists $\varepsilon > 0$ with $(x-\varepsilon, x+\varepsilon) \subseteq G_\alpha$.
5. **Intervals shrink to zero:** $|I_n| = (b-a)/2^n \to 0$, so eventually I_n fits inside the ε-neighborhood, giving a single-set cover — contradiction.

Theorem 2.82 (**Heine-Borel**)**.** *Let $E \subseteq \mathbb{R}^n$. The following are equivalent:*

1. *E is closed and bounded.*

2. *E is compact.*

3. *Every infinite subset of E has a limit point in E.*

Proof. ($1 \Rightarrow 2$): Since E is bounded, $E \subseteq [-M, M]^n$ for some $M > 0$. The closed box $[-M, M]^n$ is compact. Since E is a closed subset of a compact set, E is compact.

($2 \Rightarrow 3$): This follows from the theorem: if E is an infinite subset of a compact set K, then E has a limit point in K. Taking $K = E$, every infinite subset of E has a limit point in E.

($3 \Rightarrow 1$): *Closed:* Let p be a limit point of E. Every neighborhood of p contains a point of E distinct from p. We can construct a sequence (x_n) in E with $x_n \to p$. The set $\{x_n\}$ is infinite, so by (3) it has a limit point in E. This limit point must be p, so $p \in E$. Thus E contains all its limit points, so E is closed.

Bounded: Suppose E is unbounded. Then for each $n \in \mathbb{N}$, there exists $x_n \in E$ with $|x_n| > n$. The set $\{x_1, x_2, \dots\}$ is infinite. By (3), it has a limit point $p \in E$. But for any $\varepsilon > 0$, only finitely many x_n lie in $B(p, \varepsilon)$ (since $|x_n| \to \infty$), contradicting that p is a limit point. Thus E is bounded. ∎

What Heine-Borel means and how to use it.
In $\mathbb{R}^n$, compactness has a simple characterization: *closed and bounded*. This is easy to check! You don't need to verify that every open cover has a finite subcover — just check two conditions.
Common uses:

- **Proving a set is compact:** Show it's closed (contains its limit points) and bounded (fits in some ball). Examples: $[0, 1]$, closed balls $\overline{B}(x, r)$, the Cantor set.

- **Proving a set is NOT compact:** Show it's either not closed or not bounded. Examples: $(0, 1)$ is not closed; $\mathbb{R}$ is not bounded.

- **Extracting convergent subsequences:** Condition (3) says infinite subsets have limit points. This is the key to proving the Bolzano-Weierstrass theorem: every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Warning: Heine-Borel is specific to $\mathbb{R}^n$. In general metric spaces, compact implies closed and bounded, but the converse can fail.

Theorem 2.83 (**Weierstrass**)**.** *Every bounded infinite subset has a limit point in $\mathbb{R}^n$.*

2.5.3 Perfect Sets

Recall: E is **perfect** if E is closed and has no isolated points. If E is perfect, then $E = \overline{E} = E'$.

Theorem 2.84. *Every nonempty perfect subset of $\mathbb{R}^n$ is uncountable.*

Proof. Let $P \subseteq \mathbb{R}^n$ be nonempty and perfect. Suppose for contradiction that P is countable, say $P = \{x_1, x_2, x_3, \dots\}$.

We construct nested closed sets $V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots$ such that:

1. $V_n \cap P \neq \emptyset$ for all n,
2. $x_n \notin V_n$ for all n.

Base case: Since P has no isolated points, x_1 is a limit point of P. Choose $y_1 \in P$ with $y_1 \neq x_1$. Let $V_1 = \overline{B}(y_1, r_1)$ where $r_1 = \frac{1}{2}d(x_1, y_1)$. Then $y_1 \in V_1 \cap P$ and $x_1 \notin V_1$.

Inductive step: Suppose V_n is constructed with $V_n \cap P \neq \emptyset$ and $x_n \notin V_n$. Pick any $y \in V_n \cap P$. Since P is perfect, y is a limit point of P, so there exists $y_{n+1} \in P \cap V_n$ with $y_{n+1} \neq x_{n+1}$ (if $x_{n+1} \notin V_n$, any point works; if $x_{n+1} \in V_n$, choose a different point). Let $V_{n+1} = \overline{B}(y_{n+1}, r_{n+1}) \cap V_n$ where r_{n+1} is small enough that $x_{n+1} \notin V_{n+1}$ and $V_{n+1} \subseteq V_n$.

Each V_n is closed and bounded, hence compact. The V_n are nested and nonempty, so by the finite intersection property, $\bigcap_n V_n \neq \emptyset$. Let $x \in \bigcap_n V_n$. Since each $V_n \cap P$ is closed (intersection of closed sets) and the V_n are nested, we have $x \in P$. But $x \neq x_n$ for all n (since $x_n \notin V_n$). This contradicts $P = \{x_1, x_2, \dots\}$. $\square$

Theorem 2.85. *The Cantor set is perfect.*

Proof. The Cantor set C is closed. Additionally, using the ternary expansion: for any $x \in C$, we can truncate x at the n-th digit and define a sequence (x_n) in C with $|x - x_n| \leq 3^{-n}$. Thus $x_n \to x$, so x is a limit point of C. Hence C has no isolated points, and C is perfect.

truncate here

$$x = 0.\underbrace{02020}_{n\text{ digits}}\,2002\cdots$$

$$x_n = 0.02020\,\underbrace{0000\cdots}_{\text{zeros}}$$

$$|x - x_n| = 0.\ \underbrace{00000}_{\text{first } n \text{ digits}}\ 2002\cdots$$

$$\leq 0.00000\overline{22} = \tfrac{2}{3^{n+1}} \cdot \tfrac{1}{1-1/3} = \tfrac{1}{3^n}$$

$\square$

3 Assignments

Assignments Overview: This section tracks homework assignments for the course. Each assignment includes the problem set, relevant lecture material, and submission status. Full solutions and explainer documents are maintained in the `hw/` directory.

3.1 Homework 1 — Rudin Chapter 1

Status: Submitted
Due: January 31, 2026
Problems: 1, 5, 9, 18 *Bonus:* 7, 8, 20
Related Lectures: Lecture 1 (Ordered sets, LUBP, fields), Lecture 2 (Construction of $\mathbb{R}$)

Key concepts tested:

- Irrationality proofs and closure properties of $\mathbb{Q}$ (Problem 1)
- Ordered field axioms (Problem 5)
- Supremum and infimum computations (Problem 9)
- Properties of the complex field (Problem 18)

Math 104 - Homework 1

Problem 1 (Rudin 1.1)**.** *If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.*

Proof. **Part 1:** We show $r + x$ is irrational. Suppose for contradiction that $r + x = s$ for some $s \in \mathbb{Q}$. Then $x = s - r = s + (-r)$. Since $r \in \mathbb{Q}$, we have $-r \in \mathbb{Q}$, and since $\mathbb{Q}$ is closed under addition, $x = s + (-r) \in \mathbb{Q}$. This contradicts the assumption that x is irrational.

Part 2: We show rx is irrational. Suppose for contradiction that $rx = s$ for some $s \in \mathbb{Q}$. Since $r \neq 0$ and $r \in \mathbb{Q}$, the multiplicative inverse $1/r$ exists and $1/r \in \mathbb{Q}$. Then $x = s \cdot (1/r)$, and since $\mathbb{Q}$ is closed under multiplication, $x \in \mathbb{Q}$. This contradicts the assumption that x is irrational. □

Problem 2 (Rudin 1.5)**.** *Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that*

$$\inf A = -\sup(-A).$$

Proof. Let $A \subseteq \mathbb{R}$ be nonempty and bounded below, and define $-A = \{-x : x \in A\}$. Since A is nonempty and bounded below, and $\mathbb{R}$ has the GLBP, we know $b := \inf A$ exists.

Claim: $-b = \sup(-A)$.

First, we show $-b$ is an upper bound of $-A$. Since $b = \inf A$, we have $b \leq a$ for all $a \in A$. By properties of ordered fields, $-b \geq -a$ for all $a \in A$. Thus $-b \geq x$ for all $x \in -A$, so $-b$ is an upper bound of $-A$.

Next, we show $-b$ is the least upper bound. Let f be any lower bound of A. Then $-f$ is an upper bound of $-A$. Since $b = \inf A$ is the greatest lower bound, $f \leq b$, which implies $-f \geq -b$. Thus $-b$ is less than or equal to every upper bound of $-A$, so $-b = \sup(-A)$.

Therefore, $\inf A = b = -(-b) = -\sup(-A)$. □

Problem 3 (Rudin 1.9)**.** *Suppose $z = a + bi$, $w = c + di$. Define $z < w$ if $a < c$, and also if $a = c$ but $b < d$. Prove that this turns the set of all complex numbers into an ordered set. (This type of order relation is called a dictionary order, or lexicographic order, for obvious reasons.) Does this ordered set have the least-upper-bound property?*

Proof. Let $z = a + bi$, $w = c + di$, and $u = f + gi$. Define $z \leq w$ as "$z < w$ or $z = w$." We show this is a total order.

Reflexive: $z \leq z$ since $z = z$.

Anti-symmetric: Suppose $z \leq w$ and $w \leq z$. Then $a \leq c$ and $c \leq a$, so $a = c$. Since $a = c$, if $b < d$ then $z < w$, contradicting $w \leq z$. If $b > d$ then $w < z$, contradicting $z \leq w$. Therefore $b = d$, so $z = w$.

Transitive: Suppose $z \leq w$ and $w \leq u$. Then $a \leq c$ and $c \leq f$, so $a \leq f$. If $a < f$, then $z < u$, so $z \leq u$. If $a = f$, then $a = c = f$. Since $a = c$ and $z \leq w$, we have $b \leq d$. Since $c = f$ and $w \leq u$, we have $d \leq g$. Thus $b \leq g$, so $z \leq u$.

Comparable: Let z, w be any two complex numbers. Since $\mathbb{R}$ is ordered, either $a \leq c$ or $a \geq c$. If $a < c$, then $z < w$, so $z \leq w$. If $a > c$, then $w < z$, so $w \leq z$. If $a = c$, then since $\mathbb{R}$ is ordered, either $b \leq d$ or $b > d$. If $b \leq d$, then $z \leq w$. If $b > d$, then $w < z$, so $w \leq z$.

Thus $\mathbb{C}$ with the lexicographic order is an ordered set.

Least-upper-bound property: No. Consider $A = \{a + bi : a \in [0, 1)\}$. Then $1 + 0i$ is an upper bound of A, but so is $1 - i$, $1 - 2i$, and so on. Any upper bound must have real part ≥ 1, but among those with real part exactly 1, there is no least element (since $1 + ci > 1 + (c - 1)i$ for all c). Thus A has no least upper bound. $\square$

Problem 4 (Rudin 1.18)**.** *If $k \geq 2$ and $\mathbf{x} \in \mathbb{R}^k$, prove that there exists $\mathbf{y} \in \mathbb{R}^k$ such that $\mathbf{y} \neq \mathbf{0}$ but $\mathbf{x} \cdot \mathbf{y} = 0$. Is this also true if $k = 1$?*

Proof. Let $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ with $k \geq 2$. We construct $\mathbf{y} \neq \mathbf{0}$ such that $\mathbf{x} \cdot \mathbf{y} = 0$.

Case 1: If $x_2 \neq 0$, let $\mathbf{y} = (1, -x_1/x_2, 0, \ldots, 0)$. Then $\mathbf{y} \neq \mathbf{0}$ and

$$\mathbf{x} \cdot \mathbf{y} = x_1 \cdot 1 + x_2 \cdot (-x_1/x_2) + 0 + \cdots + 0 = x_1 - x_1 = 0.$$

Case 2: If $x_2 = 0$, let $\mathbf{y} = (0, 1, 0, \ldots, 0)$. Then $\mathbf{y} \neq \mathbf{0}$ and

$$\mathbf{x} \cdot \mathbf{y} = x_1 \cdot 0 + x_2 \cdot 1 + 0 + \cdots + 0 = 0 + 0 = 0.$$

For $k = 1$: No. If $x \neq 0$ and $xy = 0$, we can divide both sides by x to get $y = 0$. Thus no nonzero y exists. $\square$

Bonus Problems

Problem 5 (Bonus, Rudin 1.7)**.** *Fix $b > 1$, $y > 0$, and prove that there is a unique real x such that $b^x = y$, by completing the following outline. (This x is called the logarithm of y to the base b.)*

(a) For any positive integer n, $b^n - 1 \geq n(b-1)$.

(b) Hence $b - 1 \geq n(b^{1/n} - 1)$.

(c) If $t > 1$ and $n > (b-1)/(t-1)$, then $b^{1/n} < t$.

(d) If w is such that $b^w < y$, then $b^{w+(1/n)} < y$ for sufficiently large n; to see this, apply part (c) with $t = y \cdot b^{-w}$.

(e) If $b^w > y$, then $b^{w-(1/n)} > y$ for sufficiently large n.

(f) Let A be the set of all w such that $b^w < y$, and show that $x = \sup A$ satisfies $b^x = y$.

(g) Prove that this x is unique.

Proof. □

Problem 6 (Bonus, Rudin 1.8)**.** *Prove that no order can be defined in the complex field that turns it into an ordered field. Hint: -1 is a square.*

Proof. □

Problem 7 (Bonus, Rudin 1.20)**.** *With reference to the Appendix, suppose that property (III) were omitted from the definition of a cut. Keep the same definitions of order and addition. Show that the resulting ordered set has the least-upper-bound property, that addition satisfies axioms (A1) to (A4) (with a slightly different zero-element!) but that (A5) fails.*

Proof. □

AI Use Disclaimer: Claude (Anthropic) was used in the preparation of this assignment. Claude served solely as a transcription and formatting tool, taking verbal dictation of my solutions and converting them into LaTeX. Claude did not provide answers, solve problems, or generate proofs. It was used only as a guide to help structure my own reasoning, never as a solver.

3.2 Homework 2 — Rudin Chapter 2

Status: Submitted
Due: February 7, 2026
Problems: 6, 22, 27, 29 *Bonus:* Kuratowski closure complement
Related Lectures: Lecture 3 (Countability), Lecture 4 (Topology), Lecture 5 (Compactness)

Key concepts tested:

- Countability of specific sets (Problem 6)
- Closure and interior in metric spaces (Problem 22)
- Compactness arguments (Problem 27)
- Perfect sets and the Cantor set (Problem 29)

Math 104 - Homework 2

Problem 1 (Rudin 2.6)**.** *Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\bar{E}$ have the same limit points. (Recall that $\bar{E} = E \cup E'$.) Do E and E' always have the same limit points?*

Proof. **Part 1:** We show E' is closed by showing $(E')' \subseteq E'$.

Let $p \in (E')'$ and let U be any neighborhood of p. Since p is a limit point of E', there exists $q \in U \cap E'$ with $q \neq p$. Since $q \neq p$, we have $d(p, q) > 0$. Let $V = U \cap B(q, d(p, q))$. Then V is a neighborhood of q contained in U, and $p \notin V$.

Since $q \in E'$, the neighborhood V contains some $r \in E$ with $r \neq q$. Since $r \in V$ and $p \notin V$, we have $r \neq p$. Thus $r \in U \cap E$ with $r \neq p$.

Since U was an arbitrary neighborhood of p, every neighborhood of p contains a point of E different from p. Therefore $p \in E'$, and so $(E')' \subseteq E'$. Hence E' is closed.

Part 2: We show $E' = (\bar{E})'$ by proving both inclusions.

($\subseteq$) Let $p \in E'$. Then every neighborhood U of p contains some $q \in E$ with $q \neq p$. Since $E \subseteq \bar{E}$, we have $q \in \bar{E}$. Thus every neighborhood of p contains a point of $\bar{E}$ different from p, so $p \in (\bar{E})'$.

($\supseteq$) Let $p \in (\bar{E})'$ and let U be any neighborhood of p. Then U contains some $q \in \bar{E} = E \cup E'$ with $q \neq p$. We consider two cases.

Case 1: $q \in E$. Then U contains a point of E different from p.

Case 2: $q \in E' \setminus E$. Since q is a limit point of E and $q \neq p$, we have $d(p, q) > 0$. Let $V = U \cap B(q, d(p, q))$. Then V is a neighborhood of q, so V contains some $x \in E$ with $x \neq q$. Since $x \in V$ and $p \notin V$, we have $x \neq p$. Thus $x \in U \cap E$ with $x \neq p$.

In either case, U contains a point of E different from p. Since U was arbitrary, $p \in E'$.

Part 3: No, E and E' do not always have the same limit points.

Counterexample: Let $E = \{1/n : n \in \mathbb{N}\}$. Then $E' = \{0\}$, since 0 is the only limit point of E. But $(E')' = \varnothing$, since a single point has no limit points. Thus $E' \neq (E')'$. □

Problem 2 (Rudin 2.22)**.** *A metric space is called* separable *if it contains a countable dense subset. Show that $\mathbb{R}^k$ is separable. Hint: Consider the set of points which have only rational coordinates.*

Proof. Let $\mathbb{Q}^k = \{(x_1, \ldots, x_k) : x_i \in \mathbb{Q} \text{ for all } i\}$ be the set of points in $\mathbb{R}^k$ with rational coordinates. We show $\mathbb{Q}^k$ is countable and dense in $\mathbb{R}^k$.

Countable: $\mathbb{Q}$ is countable, and the finite Cartesian product of countable sets is countable. Thus $\mathbb{Q}^k$ is countable.

Dense: Let U be a nonempty open set in $\mathbb{R}^k$. Then U contains an open ball $B(\mathbf{x}, \varepsilon)$ for some $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$ and $\varepsilon > 0$. By the density of $\mathbb{Q}$ in $\mathbb{R}$ (Theorem 1.20), for each i there exists $q_i \in \mathbb{Q}$ with $|q_i - x_i| < \varepsilon/\sqrt{k}$. Then $\mathbf{q} = (q_1, \ldots, q_k) \in \mathbb{Q}^k$ and

$$|\mathbf{q} - \mathbf{x}| = \sqrt{\sum_{i=1}^{k} (q_i - x_i)^2} < \sqrt{k \cdot \frac{\varepsilon^2}{k}} = \varepsilon.$$

Thus $\mathbf{q} \in B(\mathbf{x}, \varepsilon) \subseteq U$, so $U \cap \mathbb{Q}^k \neq \varnothing$. Since every nonempty open set intersects $\mathbb{Q}^k$, we have $\overline{\mathbb{Q}^k} = \mathbb{R}^k$, so $\mathbb{Q}^k$ is dense.

Therefore $\mathbb{R}^k$ is separable. □

Problem 3 (Rudin 2.27)**.** *Define a point p in a metric space X to be a* condensation point *of a set $E \subset X$ if every neighborhood of p contains uncountably many points of E.*

Suppose $E \subset \mathbb{R}^k$, E is uncountable, and let P be the set of all condensation points of E. Prove that P is perfect and that at most countably many points of E are not in P. In other words, show that $P^c \cap E$ is at most countable. Hint: Let $\{V_n\}$ be a countable base of $\mathbb{R}^k$, let W be the union of those V_n for which $E \cap V_n$ is at most countable, and show that $P = W^c$.

Proof. Let $\{V_n\}_{n=1}^{\infty}$ be the collection of all open balls in $\mathbb{R}^k$ with rational centers and rational radii. This collection is countable since it is indexed by $\mathbb{Q}^k \times \mathbb{Q}^+$, a finite product of countable sets. It forms a base for $\mathbb{R}^k$: every neighborhood of a point contains some V_n.

Define

$$W = \bigcup \{V_n : E \cap V_n \text{ is at most countable}\}.$$

We show that $P = W^c$.

($\subseteq$) Let $p \in P$. Then every neighborhood of p contains uncountably many points of E. In particular, for any V_n containing p, the set $E \cap V_n$ is uncountable, so V_n does not contribute to W. Thus $p \notin W$, i.e., $p \in W^c$.

($\supseteq$) Let $p \in W^c$. Then p is not in any V_n with $E \cap V_n$ countable, so for every V_n containing p, the set $E \cap V_n$ is uncountable. Now let U be any neighborhood of p. There exists V_n with $p \in V_n \subseteq U$, and $E \cap V_n$ is uncountable. Since $V_n \subseteq U$, we have $E \cap U$ is uncountable. Thus $p \in P$.

Therefore $P = W^c$.

Now we show P is perfect.

P is closed: W is a union of open sets, so W is open. Thus $P = W^c$ is closed.

P has no isolated points: Let $p \in P$ and let U be a neighborhood of p. Since p is a condensation point, $E \cap U$ is uncountable. We can write

$$E \cap U = (E \cap U \cap W) \cup (E \cap U \cap P).$$

Now $E \cap U \cap W \subseteq E \cap W$, and $E \cap W = \bigcup\{E \cap V_n : E \cap V_n \text{ is countable}\}$ is a countable union of countable sets, hence countable. So $E \cap U \cap W$ is countable.

Since $E \cap U$ is uncountable and $E \cap U \cap W$ is countable, the set $E \cap U \cap P$ must be uncountable. In particular, it contains a point different from p. This point is in P and in U, so p is a limit point of P.

Since every point of P is a limit point of P, the set P has no isolated points. Combined with P being closed, P is perfect.

Finally, $E \setminus P = E \cap W$ is countable (as shown above), so at most countably many points of E are not in P. □

Problem 4 (Rudin 2.29)**.** *Prove that every open set in $\mathbb{R}^1$ is the union of an at most countable collection of disjoint segments. Hint: Use Exercise 22.*

Proof. Let $G \subseteq \mathbb{R}$ be open. For each $x \in G$, define the maximal interval containing x as $I_x = (a_x, b_x)$, where

$$a_x = \inf\{a : (a, x) \subseteq G\} \quad \text{and} \quad b_x = \sup\{b : (x, b) \subseteq G\}.$$

The inf and sup exist: Since G is open and $x \in G$, there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq G$. Thus $(x - \varepsilon, x) \subseteq G$ and $(x, x + \varepsilon) \subseteq G$, so both sets above are non-empty. By the least upper bound property, a_x and b_x exist.

$I_x \subseteq G$**:** Let $y \in (a_x, b_x)$. Since $y > a_x$, there exists $a < y$ with $(a, x) \subseteq G$. Since $y < b_x$, there exists $b > y$ with $(x, b) \subseteq G$. Then $(a, x) \cup \{x\} \cup (x, b) = (a, b) \subseteq G$, and since $a < y < b$, we have $y \in G$. Thus $I_x \subseteq G$.

Maximal intervals are equal or disjoint: Suppose $I_x \cap I_y \neq \varnothing$. Then $I_x \cup I_y$ is an interval (the union of overlapping intervals is an interval) contained in G. Since I_x is maximal and $I_x \cup I_y$ contains x, we have $I_x \cup I_y \subseteq I_x$, so $I_y \subseteq I_x$. By symmetry, $I_x \subseteq I_y$. Thus $I_x = I_y$.

At most countably many: Each maximal interval is non-empty and open, so by density of $\mathbb{Q}$ in $\mathbb{R}$ (Exercise 22 shows $\mathbb{R}$ is separable), each contains a rational. Distinct maximal intervals are disjoint, so they contain distinct rationals. This defines an injection from the set of maximal intervals into $\mathbb{Q}$. Since $\mathbb{Q}$ is countable, there are at most countably many maximal intervals.

Conclusion: The distinct maximal intervals $\{I_\alpha\}$ are disjoint, and $G = \bigcup_\alpha I_\alpha$ since every $x \in G$ is in its maximal interval I_x. Thus G is a union of at most countably many disjoint segments. $\square$

Bonus Problem

Problem 5 (Bonus: Kuratowski's Closure-Complement Theorem)**.** *Consider the collection of all subsets of a topological space. The operations of taking closure and complement produce at most 14 sets. Show this and give an example of a subset of the reals that produces exactly 14 sets.*

Proof. □

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4 Student Progress

Learning Journey: This section is a living document maintained by the YouLearn AI companion. It synthesizes observations from study sessions, lecture notes, and homework into a narrative portrait of the student's evolving understanding of Real Analysis. Rather than a checklist of topics covered, this is a reflective assessment — where intuition runs deep, where mechanical procedures still substitute for genuine understanding, and where the next breakthrough might come. Updated automatically after each study session.

4.1 Where We Are

The student has built a solid foundational vocabulary for Real Analysis through five lectures spanning the construction of the real numbers, metric space topology, and the beginnings of compactness theory. What stands out most is a developing geometric intuition: when we worked through why $[0, 1]$ is compact but $(0, 1)$ is not, the student's instinct was to "picture the open cover leaking out the endpoints" — a visual metaphor that, while informal, reveals genuine conceptual engagement rather than rote absorption.

Current Conceptual Landscape:

- **Strong foundations:** The **least upper bound property**, **ordered fields**, and the construction of $\mathbb{R}$ from $\mathbb{Q}$ via Dedekind cuts (Lectures 1–2). The student can articulate *why* completeness matters, not just state it.
- **Growing comfort:** **Metric spaces**, **open sets**, **closed sets**, and the topology of $\mathbb{R}^n$ (Lectures 3–4). Definitions are solid; the student is beginning to develop intuition for how these concepts interact.
- **Active frontier:** **Compactness**, the **Heine-Borel theorem**, and **perfect sets** (Lecture 5). The student can apply Heine-Borel mechanically to identify compact subsets of $\mathbb{R}^n$, but the deeper "why" — the connection between finite subcovers and the geometry of closed-and-bounded sets — is still forming.

The ε-δ formalism remains an area where procedure outpaces intuition. The student can execute ε-δ proofs by following templates, but when asked to *construct* a proof from scratch — particularly when the choice of δ requires ingenuity — there is visible hesitation. This is entirely normal at this stage; the gap between "I can follow this proof" and "I can write this proof" is where the real learning happens.

4.2 The Journey So Far

Our work began with the algebraic foundations: why $\sqrt{2}$ is irrational, what it means for a field to be ordered, and why the rationals alone are insufficient for analysis. These early lectures (1–2) established a pattern that has served us well — the student engages most deeply when a definition is motivated by a *problem*. The irrationality of $\sqrt{2}$ wasn't just a proof exercise; it was the reason we needed to build $\mathbb{R}$ in the first place.

The transition to topology (Lectures 3–4) marked a shift in the kind of thinking required. Where the algebraic material was concrete and computational, metric spaces demand a new level of abstraction — reasoning about arbitrary open sets, neighborhoods, and limit points rather than specific numbers. The student navigated this transition with characteristic curiosity, asking pointed questions about the relationship between open balls in $\mathbb{R}^n$ and the more abstract metric space framework. The question "Is every metric space secretly just $\mathbb{R}^n$ in disguise?" was particularly revealing — it showed both engagement with the material and a productive misconception worth exploring further.

Turning Point: The review session on February 6 marked an important consolidation. Rather than pushing into new material, we spent time making connections: compactness relates to closedness and boundedness (Heine-Borel), the Cantor set is simultaneously nowhere dense and uncountable, and sequential compactness offers an alternative lens on the same phenomenon. The student's comfort with these ideas is growing, though the formal equivalence between open-cover and sequential compactness remains an edge that needs sharpening.

4.3 Edges of Understanding

Several concepts sit at the productive boundary between "understood" and "still forming":

1. **Sequential vs. open-cover compactness.** The student understands both definitions individually but has not yet internalized why they are equivalent in metric spaces. The Bolzano-Weierstrass theorem provides the bridge, but we haven't walked across it together yet. This is a natural next step.
2. **The role of Hausdorff separation.** When we discussed compact subsets of Hausdorff spaces being closed (Theorem 2.34 in Rudin), the student accepted the result but didn't fully engage with *why* the Hausdorff property matters. What goes wrong without it? This is worth revisiting with a counterexample from a non-Hausdorff space.
3. **Proof construction vs. proof comprehension.** The student can follow sophisticated proofs and identify the key steps, but constructing proofs independently — especially those requiring a clever choice of auxiliary object (a particular open cover, a specific sequence, a well-chosen δ) — remains challenging. This is the central skill to develop over the coming weeks.
4. **The Cantor set as archetype.** We touched on the Cantor set as a perfect set, but there is rich territory here: its uncountability, its measure-zero property, and its role as a universal compact metric space. The student showed curiosity but we didn't have time to explore deeply.

4.4 Looking Forward

The student's growing comfort with compactness suggests readiness to move toward **connectedness** and the beginnings of **sequences and series** — the analytical core of the course. However, before pushing ahead, two consolidation steps would pay dividends:

- **Immediate priority:** Work through HW 2, Problem 27, which requires a compactness argument. This will test whether the Heine-Borel intuition translates into proof-writing ability. Struggling here is expected and productive — it's exactly the kind of exercise that builds the "proof construction" muscle.
- **Short-term goal:** Solidify the connection between sequential compactness and the Bolzano-Weierstrass theorem before Lecture 6 introduces new material. A focused 20-minute review session walking through the proof of equivalence would be ideal.
- **Medium-term arc:** As we move into sequences, series, and continuity, the student's geometric intuition will be a tremendous asset. The challenge will be channeling that intuition

into rigorous ε-δ arguments — turning "I can see why this is true" into "I can prove this is true."

In a sentence: A student with strong conceptual instincts and genuine curiosity, building the technical machinery to match their mathematical intuition. The foundations are solid; the next phase is about developing fluency in proof-writing and deepening the connections between the big ideas of real analysis.

5 Sessions

Study Sessions: This section logs study sessions with the YouLearn AI companion. Each entry records the date, mode, topics covered, and what was accomplished. Sessions are automatically logged when ending a study session with `/Done`.

5.1 February 6, 2026 — Review Session

Session Summary
Date: February 6, 2026
Mode: Review (`/Rev`)
Duration: 45 minutes
Topics: Compactness, Heine-Borel theorem, perfect sets

What we covered:

- Reviewed the definition of **compactness** and open covers from Lecture 5
- Worked through why $[0, 1]$ is compact but $(0, 1)$ is not
- Practiced applying the Heine-Borel theorem to identify compact subsets of $\mathbb{R}^n$
- Discussed the relationship between compactness, closedness, and boundedness
- Reviewed the Cantor set as an example of a perfect set

Areas for further review:

- Sequential compactness vs. open-cover compactness (Lecture 5, §5.1)
- Proof that compact subsets of Hausdorff spaces are closed (Theorem 2.34)

Next session: Work through HW 2, Problem 27 (compactness argument).

6 Resources

Course Resources: Textbooks, supplementary materials, and references for Math 104. This section is updated throughout the course as new resources are discovered.

6.1 Primary Textbook

Walter Rudin, *Principles of Mathematical Analysis*, 3rd edition. McGraw-Hill, 1976. ISBN: 978-0-07-054235-8.
The standard reference for undergraduate real analysis. Chapters 1–7 are covered in this course. Known for its concise, rigorous style. Expect to read proofs multiple times.

6.2 Supplementary Materials

- **Abbott**, *Understanding Analysis*, 2nd ed. — More accessible introduction. Good for building intuition before tackling Rudin.
- **Tao**, *Analysis I & II* — Builds analysis from the ground up. Excellent for students who want to see every detail.
- **Pugh**, *Real Mathematical Analysis* — Beautiful exposition with great exercises and pictures.

6.3 Online Resources

- Francis Su's *Real Analysis* lecture series (Harvey Mudd, YouTube) — Exceptional lectures covering Rudin chapter by chapter.
- MIT OCW 18.100A — Problem sets and lecture notes for a similar course.

7 Glossary

Key Definitions: A glossary of important terms and definitions from the course, organized by topic. Terms marked in **red bold** in the lecture notes appear here with their formal definitions and the lecture where they were introduced.

7.1 Ordered Sets & Real Numbers (Lectures 1–2)

Member
$x \in A$ means x is an element of the set A. (Lecture 1)

Empty set
The set $\emptyset$ containing no elements. (Lecture 1)

Subset
$A \subseteq B$ if every element of A is also in B. (Lecture 1)

Proper subset
$A \subset B$ if $A \subseteq B$ and $A \neq B$. (Lecture 1)

Partial order
A relation $\leq$ on S that is reflexive, antisymmetric, and transitive. (Lecture 1)

Total order
A partial order where every two elements are comparable. (Lecture 1)

Upper bound
b is an upper bound of $E \subseteq S$ if $x \leq b$ for all $x \in E$. (Lecture 1)

Lower bound
b is a lower bound of E if $b \leq x$ for all $x \in E$. (Lecture 1)

Supremum
The least upper bound of a set E, written $\sup E$. (Lecture 1)

Infimum
The greatest lower bound of a set E, written $\inf E$. (Lecture 1)

Least Upper Bound Property
Every non-empty subset bounded above has a supremum. (Lecture 1)

Field
A set F with addition and multiplication satisfying the field axioms. (Lecture 1)

Ordered field
A field with a total order compatible with the field operations. (Lecture 1)

Dedekind cut
A partition of $\mathbb{Q}$ into two non-empty sets $A|B$ where every element of A is less than every element of B, and A has no maximum. (Lecture 2)

Archimedean property
For any $x, y \in \mathbb{R}$ with $x > 0$, there exists $n \in \mathbb{N}$ such that $nx > y$. (Lecture 2)

7.2 Set Theory & Countability (Lecture 3)

Countable

A set A is countable if there exists a bijection $f : A \to \mathbb{N}$ (or A is finite). (Lecture 3)

Uncountable

A set that is not countable. (Lecture 3)

Cardinality

Two sets have the same cardinality if there is a bijection between them, written $A \sim B$. (Lecture 3)

Equivalence relation

A relation that is reflexive, symmetric, and transitive. (Lecture 3)

7.3 Topology & Metric Spaces (Lectures 4–5)

Metric space

A set X with a distance function $d : X \times X \to \mathbb{R}$ satisfying positivity, symmetry, and the triangle inequality. (Lecture 4)

Open ball

$B_r(x) = \{y \in X : d(x, y) < r\}$ for $r > 0$. Also called a neighborhood. (Lecture 4)

Open set

A set G is open if every point of G is an interior point. (Lecture 4)

Closed set

A set F is closed if its complement F^c is open. Equivalently, F contains all its limit points. (Lecture 4)

Limit point

p is a limit point of E if every neighborhood of p contains a point $q \in E$ with $q \neq p$. (Lecture 4)

Interior point

p is an interior point of E if there exists $r > 0$ such that $B_r(p) \subseteq E$. (Lecture 4)

Closure

$\overline{E} = E \cup E'$ where E' is the set of limit points of E. (Lecture 4)

Dense

E is dense in X if $\overline{E} = X$. (Lecture 4)

Open cover

A collection of open sets $\{G_\alpha\}$ such that $E \subseteq \bigcup G_\alpha$. (Lecture 5)

Compact

A set K is compact if every open cover has a finite subcover. (Lecture 5)

Perfect set

A closed set in which every point is a limit point. (Lecture 5)

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