Lexicographic Order on $\mathbb{C}$

A visual guide to Rudin 1.9

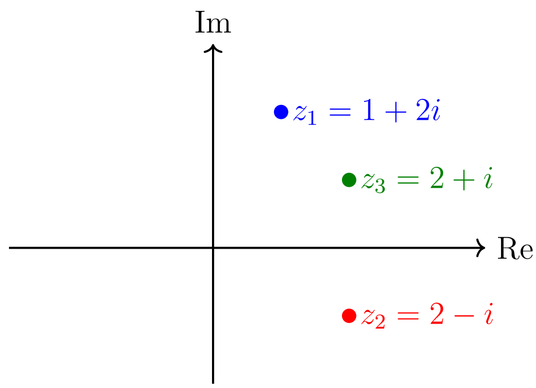
1 What is Lexicographic Order?

"Lexicographic" means "dictionary order." Compare the first component; if equal, compare the second.

For $z = a + bi$ and $w = c + di$:

$$z < w \iff (a < c) \text{ or } (a = c \text{ and } b < d)$$

2 Visual: How the Order Works



Order: $z_1 < z_2 < z_3$
$z_1 < z_2$ because $1 < 2$ (real parts)
$z_2 < z_3$ because real parts
equal, $-1 < 1$ (imag parts)

3 The "Dictionary" Analogy

Words:

"apple" < "banana"
(compare first letter)

"cat" < "cow"
(first letter same, compare second)

Complex numbers:

$1 + 2i < 3 + 0i$
(compare real parts)

$2 + 1i < 2 + 5i$
(real parts same, compare imaginary)

4 Is This a Total Order?

We need to verify four properties:

4.1 Reflexive: $z \leq z$

Trivially true since $z = z$.

4.2 Anti-symmetric: $z \leq w$ and $w \leq z$ implies $z = w$

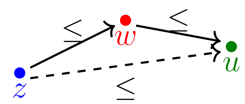
$$\text{If } z \leq w \text{ and } w \leq z\text{:}$$

$$\text{Then } a \leq c \text{ and } c \leq a, \text{ so } a = c$$

$$\text{With } a = c\text{: } b \leq d \text{ and } d \leq b, \text{ so } b = d$$

$$\text{Therefore } z = w \checkmark$$

4.3 Transitive: $z \leq w$ and $w \leq u$ implies $z \leq u$



4.4 Comparable: For any z, w, either $z \leq w$ or $w \leq z$

Since $\mathbb{R}$ is totally ordered, we can always compare a with c:

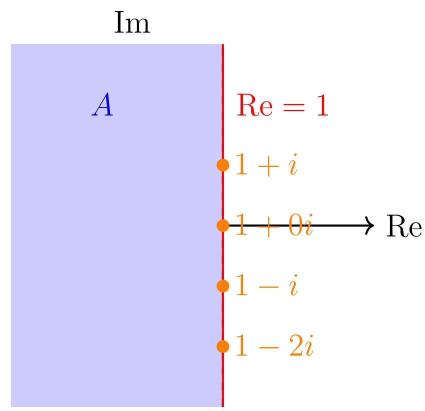
- If $a < c$: then $z < w$
- If $a > c$: then $w < z$
- If $a = c$: compare b with d (same logic)

5 Does It Have the Least Upper Bound Property?

NO!

5.1 The Counterexample

Consider $A = \{z = a + bi : a < 1\}$ (everything to the left of the line $\mathrm{Re}(z) = 1$).

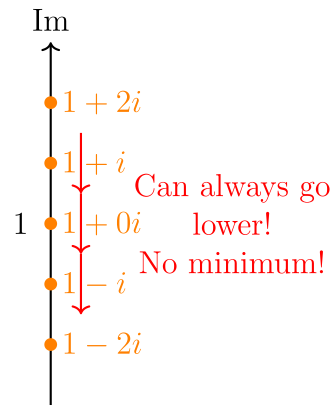


All points on the red line
are upper bounds!

5.2 Why No Least Upper Bound?

Upper bounds of A are points with real part ≥ 1.

Among those with real part exactly 1: $\{1 + bi : b \in \mathbb{R}\}$



For any candidate $1 + ci$, we have $1 + (c-1)i < 1 + ci$, and $1 + (c-1)i$ is still an upper bound.

So there's no **least** upper bound!

6 Summary

Is $\mathbb{C}$ with lex order a total order?

YES
(reflexive, anti-symmetric, transitive, comparable)

Does it have the LUB property?

NO
(bounded sets can lack a least upper bound)