Math 104 - Homework 2

Problem 1 (Rudin 2.6). *Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\bar{E}$ have the same limit points. (Recall that $\bar{E} = E \cup E'$.) Do E and E' always have the same limit points?*

Proof. **Part 1:** We show E' is closed by showing $(E')' \subseteq E'$.

Let $p \in (E')'$ and let U be any neighborhood of p. Since p is a limit point of E', there exists $q \in U \cap E'$ with $q \neq p$. Since $q \neq p$, we have $d(p,q) > 0$. Let $V = U \cap B(q, d(p,q))$. Then V is a neighborhood of q contained in U, and $p \notin V$.

Since $q \in E'$, the neighborhood V contains some $r \in E$ with $r \neq q$. Since $r \in V$ and $p \notin V$, we have $r \neq p$. Thus $r \in U \cap E$ with $r \neq p$.

Since U was an arbitrary neighborhood of p, every neighborhood of p contains a point of E different from p. Therefore $p \in E'$, and so $(E')' \subseteq E'$. Hence E' is closed.

Part 2: We show $E' = (\bar{E})'$ by proving both inclusions.

($\subseteq$) Let $p \in E'$. Then every neighborhood U of p contains some $q \in E$ with $q \neq p$. Since $E \subseteq \bar{E}$, we have $q \in \bar{E}$. Thus every neighborhood of p contains a point of $\bar{E}$ different from p, so $p \in (\bar{E})'$.

($\supseteq$) Let $p \in (\bar{E})'$ and let U be any neighborhood of p. Then U contains some $q \in \bar{E} = E \cup E'$ with $q \neq p$. We consider two cases.

Case 1: $q \in E$. Then U contains a point of E different from p.

Case 2: $q \in E' \setminus E$. Since q is a limit point of E and $q \neq p$, we have $d(p,q) > 0$. Let $V = U \cap B(q, d(p,q))$. Then V is a neighborhood of q, so V contains some $x \in E$ with $x \neq q$. Since $x \in V$ and $p \notin V$, we have $x \neq p$. Thus $x \in U \cap E$ with $x \neq p$.

In either case, U contains a point of E different from p. Since U was arbitrary, $p \in E'$.

Part 3: No, E and E' do not always have the same limit points.

Counterexample: Let $E = \{1/n : n \in \mathbb{N}\}$. Then $E' = \{0\}$, since 0 is the only limit point of E. But $(E')' = \varnothing$, since a single point has no limit points. Thus $E' \neq (E')'$. □

Problem 2 (Rudin 2.22). *A metric space is called* separable *if it contains a countable dense subset. Show that $\mathbb{R}^k$ is separable. Hint: Consider the set of points which have only rational coordinates.*

Proof. Let $\mathbb{Q}^k = \{(x_1, \ldots, x_k) : x_i \in \mathbb{Q} \text{ for all } i\}$ be the set of points in $\mathbb{R}^k$ with rational coordinates. We show $\mathbb{Q}^k$ is countable and dense in $\mathbb{R}^k$.

Countable: $\mathbb{Q}$ is countable, and the finite Cartesian product of countable sets is countable. Thus $\mathbb{Q}^k$ is countable.

Dense: Let U be a nonempty open set in $\mathbb{R}^k$. Then U contains an open ball $B(\mathbf{x}, \varepsilon)$ for some $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$ and $\varepsilon > 0$. By the density of $\mathbb{Q}$ in $\mathbb{R}$ (Theorem 1.20), for each i there exists $q_i \in \mathbb{Q}$ with $|q_i - x_i| < \varepsilon/\sqrt{k}$. Then $\mathbf{q} = (q_1, \ldots, q_k) \in \mathbb{Q}^k$ and

$$|\mathbf{q} - \mathbf{x}| = \sqrt{\sum_{i=1}^{k} (q_i - x_i)^2} < \sqrt{k \cdot \frac{\varepsilon^2}{k}} = \varepsilon.$$

Thus $\mathbf{q} \in B(\mathbf{x}, \varepsilon) \subseteq U$, so $U \cap \mathbb{Q}^k \neq \varnothing$. Since every nonempty open set intersects $\mathbb{Q}^k$, we have $\overline{\mathbb{Q}^k} = \mathbb{R}^k$, so $\mathbb{Q}^k$ is dense.

Therefore $\mathbb{R}^k$ is separable. □

Problem 3 (Rudin 2.27)**.** *Define a point p in a metric space X to be a* condensation point *of a set $E \subset X$ if every neighborhood of p contains uncountably many points of E.*

Suppose $E \subset \mathbb{R}^k$, E is uncountable, and let P be the set of all condensation points of E. Prove that P is perfect and that at most countably many points of E are not in P. In other words, show that $P^c \cap E$ is at most countable. Hint: Let $\{V_n\}$ be a countable base of $\mathbb{R}^k$, let W be the union of those V_n for which $E \cap V_n$ is at most countable, and show that $P = W^c$.

Proof. Let $\{V_n\}_{n=1}^{\infty}$ be the collection of all open balls in $\mathbb{R}^k$ with rational centers and rational radii. This collection is countable since it is indexed by $\mathbb{Q}^k \times \mathbb{Q}^+$, a finite product of countable sets. It forms a base for $\mathbb{R}^k$: every neighborhood of a point contains some V_n.

Define

$$W = \bigcup \{V_n : E \cap V_n \text{ is at most countable}\}.$$

We show that $P = W^c$.

($\subseteq$) Let $p \in P$. Then every neighborhood of p contains uncountably many points of E. In particular, for any V_n containing p, the set $E \cap V_n$ is uncountable, so V_n does not contribute to W. Thus $p \notin W$, i.e., $p \in W^c$.

($\supseteq$) Let $p \in W^c$. Then p is not in any V_n with $E \cap V_n$ countable, so for every V_n containing p, the set $E \cap V_n$ is uncountable. Now let U be any neighborhood of p. There exists V_n with $p \in V_n \subseteq U$, and $E \cap V_n$ is uncountable. Since $V_n \subseteq U$, we have $E \cap U$ is uncountable. Thus $p \in P$.

Therefore $P = W^c$.

Now we show P is perfect.

P is closed: W is a union of open sets, so W is open. Thus $P = W^c$ is closed.

P has no isolated points: Let $p \in P$ and let U be a neighborhood of p. Since p is a condensation point, $E \cap U$ is uncountable. We can write

$$E \cap U = (E \cap U \cap W) \cup (E \cap U \cap P).$$

Now $E \cap U \cap W \subseteq E \cap W$, and $E \cap W = \bigcup \{E \cap V_n : E \cap V_n \text{ is countable}\}$ is a countable union of countable sets, hence countable. So $E \cap U \cap W$ is countable.

Since $E \cap U$ is uncountable and $E \cap U \cap W$ is countable, the set $E \cap U \cap P$ must be uncountable. In particular, it contains a point different from p. This point is in P and in U, so p is a limit point of P.

Since every point of P is a limit point of P, the set P has no isolated points. Combined with P being closed, P is perfect.

Finally, $E \setminus P = E \cap W$ is countable (as shown above), so at most countably many points of E are not in P. $\square$

Problem 4 (Rudin 2.29)**.** *Prove that every open set in $\mathbb{R}^1$ is the union of an at most countable collection of disjoint segments. Hint: Use Exercise 22.*

Proof. Let $G \subseteq \mathbb{R}$ be open. For each $x \in G$, define the maximal interval containing x as $I_x = (a_x, b_x)$, where

$$a_x = \inf\{a : (a, x) \subseteq G\} \quad \text{and} \quad b_x = \sup\{b : (x, b) \subseteq G\}.$$

The inf and sup exist: Since G is open and $x \in G$, there exists $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subseteq G$. Thus $(x-\varepsilon, x) \subseteq G$ and $(x, x+\varepsilon) \subseteq G$, so both sets above are non-empty. By the least upper bound property, a_x and b_x exist.

$I_x \subseteq G$**:** Let $y \in (a_x, b_x)$. Since $y > a_x$, there exists $a < y$ with $(a, x) \subseteq G$. Since $y < b_x$, there exists $b > y$ with $(x, b) \subseteq G$. Then $(a, x) \cup \{x\} \cup (x, b) = (a, b) \subseteq G$, and since $a < y < b$, we have $y \in G$. Thus $I_x \subseteq G$.

Maximal intervals are equal or disjoint: Suppose $I_x \cap I_y \neq \varnothing$. Then $I_x \cup I_y$ is an interval (the union of overlapping intervals is an interval) contained in G. Since I_x is maximal and $I_x \cup I_y$ contains x, we have $I_x \cup I_y \subseteq I_x$, so $I_y \subseteq I_x$. By symmetry, $I_x \subseteq I_y$. Thus $I_x = I_y$.

At most countably many: Each maximal interval is non-empty and open, so by density of $\mathbb{Q}$ in $\mathbb{R}$ (Exercise 22 shows $\mathbb{R}$ is separable), each contains a rational. Distinct maximal intervals are disjoint, so they contain distinct rationals. This defines an injection from the set of maximal intervals into $\mathbb{Q}$. Since $\mathbb{Q}$ is countable, there are at most countably many maximal intervals.

Conclusion: The distinct maximal intervals $\{I_\alpha\}$ are disjoint, and $G = \bigcup_\alpha I_\alpha$ since every $x \in G$ is in its maximal interval I_x. Thus G is a union of at most countably many disjoint segments. $\square$

Bonus Problem

Problem 5 (Bonus: Kuratowski's Closure-Complement Theorem)**.** *Consider the collection of all subsets of a topological space. The operations of taking closure and complement produce at most 14 sets. Show this and give an example of a subset of the reals that produces exactly 14 sets.*

Proof. □

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