LESSON 2

ITÔ'S LEMMA AND BLACK-SCHOLES MODEL

Derivative Pricing - Module 4

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Outline

- Log-normal property of stock prices
- Itô's Lemma and the solution to GBM SDE
- Black-Scholes model

Log-Normal Property of Stock Prices

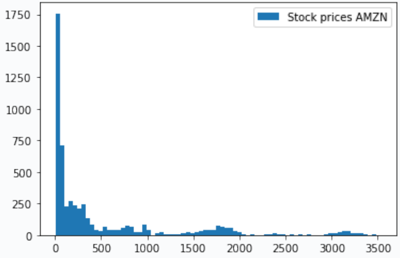
In order to analytically solve the SDE in the previous lesson, we are going to make use of two things:

(i) Log-normal property of stock prices

(ii) Itö's Lemma to solve the SDE

Let's explore the first of these $\Rightarrow$ Is it realistic to assume stock prices follow a log-normal distribution?

Go to the first part of the Jupyter Notebook accompanying this lesson to see this:



Itô calculus for GBM

Conveniently for us, a log-normal transform of a log-normally distributed variable will follow a normal distribution.

- By taking $S_t = Ln(S_t)$ and applying **Itô's Lemma**, we can get a closed-form solution for S_t from our SDE:

$$Ln\frac{S_t}{S_0} = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t,$$

$$LnS_t \sim \mathcal{N}\left(LnS_0 + \left(\mu - \frac{\sigma^2}{2}\right)t\ ,\ \sigma^2 t\right)$$

which yields:

$$S_t = S_0\ e^{\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)}$$

where,

$$E(S_t) = S_0 e^{\mu t}$$

$$Var(S_t) = S_0^2 e^{2\mu t}\left(e^{\sigma^2 t} - 1\right)$$

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Black-Scholes-Merton Model

The last SDE gave rise to one of the most important option pricing models ever: **Black-Scholes**.

The model departs from some important assumptions (some will be relaxed in the future):

(i) Prices follow a GBM with μ and σ constant.

(ii) Perfect capital markets: No taxes, transaction costs, no dividends, absence of arbitrage, etc.

(iii) Risk-free rate is constant and equal for all maturities.

- Next, apply the same intuition as in binomial setting: No-arbitrage and riskless portfolio replica!
- This yields the following **Black-Scholes-Merton SDE** (see additional references for proof):

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

  for f being the claim associated with any derivative instrument.

- Importantly, note how the previous equation only depends on $r \rightarrow$ Risk-neutral valuation

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Black-Scholes Formulas for Option Pricing

Using the Black-Scholes-Merton SDE, we can get closed-form solutions for call and put option prices.

- The formulas are the following:

$$Call\ option \rightarrow c = S_0\mathcal{N}(d_1) - Ke^{-rT}\mathcal{N}(d_2)$$

$$Put\ option \rightarrow p = Ke^{-rT}\mathcal{N}(-d_2) - S_0\mathcal{N}(-d_1)$$

where $\mathcal{N}()$ is the cumulative density function (CDF) of a standard normal and:

$$d_1 = \frac{Ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{Ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

- Again, only discounting on $r \rightarrow$ Risk-neutral valuation

Greeks in Black-Scholes

In the final part of the lesson, we are going to deal with the **Greeks in a Black-Scholes model.**

We can interpret Greeks as **the sensitivity of option price to different variables** $\rightarrow$ remember Delta?

- Sensitivity to underlying stock price (**Delta**):

$$Call\ \ Option\ \Delta = \frac{\partial C}{\partial S} = \mathcal{N}(d_1); \quad Put\ \ Option\ \Delta = \frac{\partial P}{\partial S} = \mathcal{N}(d_1) - 1$$

- Sensitivity to changes in underlying stock price (**Gamma**):

$$\Gamma = \frac{\partial^2 V}{\partial S^2} = \frac{\mathcal{N}^{'}(d_1)}{S\sigma\sqrt{T-t}}$$

- Sensitivity to volatility (**Vega**):

$$\nu = \frac{\partial^2 V}{\partial \sigma} = S\mathcal{N}^{'}(d_1)\sqrt{T-t}$$

Greeks in Black-Scholes (con.)

- Sensitivity to time (**Theta**):

$$Call\ \ Option\ \Theta = \frac{\partial C}{\partial t} = -\frac{S\mathcal{N}'(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}\mathcal{N}(d_2)$$

$$Put\ \ Option\ \Theta = \frac{\partial P}{\partial t} = -\frac{S\mathcal{N}'(d_1)\sigma}{2\sqrt{T-t}} + rKe^{-r(T-t)}\mathcal{N}(-d_2)$$

- Sensitivity to risk-free rate (**Rho**):

$$Call\ \ Option\ \rho = \frac{\partial C}{\partial r} = K(T-t)e^{-r(T-t)}\mathcal{N}(d_2)$$

$$Put\ \ Option\ \rho = \frac{\partial P}{\partial r} = -K(T-t)e^{-r(T-t)}\mathcal{N}(-d_2)$$

$\Rightarrow$ You can consult additional references to see more on the derivation of each of these.

Summary of Lesson 2

In Lesson 2, we have looked at:

- Log-normality of stock prices
- Derivation of GBM process for stock prices using Itô's Lemma
- Black-Scholes-Merton SDE and option pricing model
- Greeks (sensitivities) in Black-Scholes model

$\Rightarrow$ **TO-DO NEXT**: Now, please go to the associated Jupyter Notebook for this lesson to learn how to implement the Black-Scholes option pricing model in Python. Also, you have a working example of the different Greeks there. Lastly, please make sure that you **check the additional material in this lesson** to ensure that you have a comprehensive understanding of how to derive the previous formulas (Itô's Lemma, GBM, Black-Scholes, etc.)

$\Rightarrow$ In the next lesson, we will see the application of a previously introduced method, **Monte Carlo**, for pricing options under the Black-Scholes framework.