LESSON 4

MATCHING VOLATILITY AND RISK MEASURES

Derivative Pricing - Module 1

Outline

- Risk-neutral Valuation: Irrelevance of expected returns
- Real World vs. Risk-Neutral Measures: Girsanov's Theorem
- Intro to Calibration: Adjusting with Volatility when Increasing N in Binomial Tree

Risk-Neutral Valuation and Expected Stock Return

Throughout this module, we have worked with absence of arbitrage conditions and portfolio replication to obtain a fair price for an option (call or put).

One important feature that allows us to do that is risk-neutral valuation:

$\rightarrow$ Derive risk-neutral probabilities and discount at risk-free rate.

$$S_0 = e^{-rT}[\mathbf{p}S_1^u + (1-\mathbf{p})S_1^d] \rightarrow \text{with } \mathbf{p} \text{ risk-neutral prob}$$

$$\text{Lesson 1 example: } 100 = e^{-0.1\times 1}[120\mathbf{p} + 80(1-\mathbf{p})] \rightarrow \text{with } \mathbf{p} = 0.7629.$$

Notice that this implies expected stock return is completely irrelevant for valuation (which is exactly what we want, given how difficult it is to know this in practice).

However, the price should be exactly the same. For example, for $E(R) = \mu = 15\%$:

$$S_0 = e^{-\mu T}[\mathbf{p^*}S_1^u + (1-\mathbf{p^*})S_1^d] \rightarrow \text{with } \mathbf{p^*} \text{ real-world problem}$$

$$\text{Lesson 1 example: } 100 = e^{-0.15\times 1}[120\mathbf{p^*} + 80(1-\mathbf{p^*})] \rightarrow \text{with } \mathbf{p^*} = 0.9046.$$

Real-World vs. Risk-Neutral Measures

If all information is available, we should be **indifferent between risk measures**:

- Real-world probabilities (p^*) are typically referred to as <u>P-measure</u>.
- Risk-neutral probabilities (p) are typically referred to as <u>Q-measure</u>.
- Be careful about matching risk-measure & discount rate.

How do we move between risk-neutral & real-world measures, and what implications does this have?

$\rightarrow$ **Girsanov's Theorem!**

- To change the measure (move between risk-neutral & real worlds)
- When we change from one world to another, only two things change: discount rate & probability.
- **Volatility** of the underlying (e.g., stock) **does not change**.

$\Rightarrow$ We can avoid taking a stand on E(R) of underlying, μ, with risk-neutral valuation.
$\Rightarrow$ What about volatility? $\rightarrow$**Volatility will not change** when changing "world"

- *Match underlying stock volatility with u and d (and time-step, dt)*

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Matching Volatility with u and d

In Lesson 3, we observed the absurd behavior of option prices when N increased... **Why?**

- We need to choose u and d to match the volatility of the underlying asset, σ!

So, u and d must be such that they match the variance of the underlying:

Variance $\rightarrow E(X^2) - [E(X)]^2 = \sigma^2 dt$, for dt equal time-step:

$$p(u-1)^2 + (1-p)(d-1)^2 - [p(u-1) + (1-p)(d-1)]^2 = \sigma^2 dt$$

Substituting for the value of p:

$$e^{rdt}(u+d) - ud - e^{2rdt} = \sigma^2 dt$$

Using series expansion of e^x and ignoring power terms dt^2 and higher ($\rightarrow 0$):

$$u = e^{\sigma\sqrt{dt}} \text{ and } d = e^{-\sigma\sqrt{dt}}$$

$\Rightarrow$ Remember, these expressions are **the same** regardless of risk-neutral or real worlds!

$$p = \frac{e^{rdt} - d}{u-d} \text{ and } p^* = \frac{e^{\mu dt} - d}{u-d}$$

Summary of Lesson 4

In Lesson 4, we covered the following topics:

- Differences between risk-neutral vs. real-world probability measures
- Importance of matching discount rate & probability measure
- Volatility does not change with world $\rightarrow$ Need to adjust u and d

You can find more information on the derivation of binomial model formulas for u and d (and more) in the original paper:

- Cox, John C., Stephen A. Ross, and Mark Rubinstein. "Option Pricing: A Simplified Approach." *Journal of Financial Economics*, vol. 7, 1979, 229–263.

$\Rightarrow$ **TO-DO NEXT**: In the Jupyter Notebook accompanying this lesson, you will find code developing the adjustments of u and d to underlying stock volatility. This will serve as a first introduction to the concept of calibration.

$\Rightarrow$ In the next module, we will revisit and reinforce these concepts for other types of derivatives, such as American options.