LESSON 2

PRICING IN THE TRINOMIAL MODEL

Derivative Pricing - Module 3

WORLDQUANT UNIVERSITY

Outline

- Overview of Option Pricing in a Trinomial Model
- The Martingale Concept.
- Using Risk-Neutral Probabilities in a Trinomial Setting to Force Completeness!

Option Pricing in a Trinomial Model

We have already seen the problems related to extracting risk-neutral probabilities in order to perform option pricing in a Trinomial tree.

Apart from these, pricing is analogous to the Binomial model case! $\rightarrow$ **Backwards induction**

Let's start with an overview of the pricing problem, which is extracting risk-neutral probabilities.

We know the **2 conditions** that must meet in order to get an arbitrage-free market according to FTAP I:

(1) $p_u + p_m + p_d = 1$

(2) $S_0 = e^{-rdt}[p_u \times S_u + p_m \times S_m + p_d \times S_d]$

Condition (2) is generally called the **Martingale condition** $\rightarrow$ What is a Martingale?

- A **Martingale** is a stochastic process for which the conditional expectation of its future value, given the information known up today, is the current value today. Formally:

$$E(X_{t+1}|F_t) = X_t, \text{ where } F_t = \{X_0, X_1, ..., X_t\}$$

The concept of Martingale

A **Martingale** is a stochastic process such that:

$$E(X_{t+1}|F_t) = X_t, \text{ where } F_t = \{X_0, X_1, ..., X_t\}$$

- One implication of this is the following:

$$E(X_{t+1} - X_t|F_t) = E(X_{t+1}|F_t) - X_t = 0 \rightarrow \text{Expected Return} = 0$$

- **Example:** Think of a 'fair casino' (unlike in real life). You bet on a color (red/black) and there is a 50% probability of each outcome. You bet \$1 and win another \$1 if you guessed right and lose the bet if wrong.

  $\rightarrow$ What is the expected return of such a bet?

- In our setting:

  With risk-neutral probabilities, the only return made by an investor must be the risk-free rate, r

Obtaining Risk-Neutral Probabilities: Complete Market?

We already know the requirements for absence of arbitrage in a market, the first step towards our pricing.

- But with only the previous 2 equations, we do not have a unique set of probabilities, $\{p_u, p_d, p_m\}$.
- A set of risk-neutral probs $\{p_u, p_d, p_m\}$ is formally referred to as **Equivalent Martingale Measure (EMM).**
- When we do **not have a unique EMM**, we say that the **market is not complete.**
  - $\rightarrow$ Because there is not perfect (unique) replicability of payoffs given multiple EMMs.

Fundamental Theorem of Asset Pricing (FTAP) II

Market completeness (perfect replicability) is only achieved when there is a unique EMM.

$\Rightarrow$ Thus, we need to **"force" completeness** somehow in order to have a unique EMM for our pricing

$\rightarrow$ In short, we need another equation for the previous system

Risk-Neutral Probabilities: Forcing Completeness

There are different ways to achieve completeness in a Trinomial tree (e.g., adding an extra risky asset)

Here, we will implement the most commonly used in practice: matching the second moment of the distribution.

- This is, essentially, a matching volatility approach $\rightarrow$ (3) $E(Var(S_{t+1}|S_t)) = Var(S_t)$
- This and forcing a recombining tree ($d = \frac{1}{u}$) will produce the following values:

$$u = e^{\sigma\sqrt{2dt}},\ d = e^{-\sigma\sqrt{2dt}},\ m = 1$$

- Which, in turn, will produce the following expressions for risk-neutral probabilities:

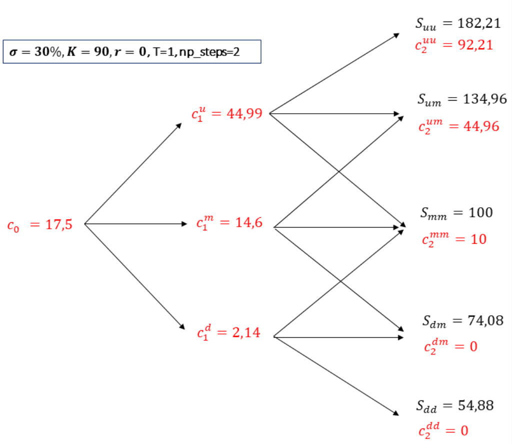
$$p_u = \left(\frac{e^{\frac{rdt}{2}} - e^{-\sigma\sqrt{\frac{dt}{2}}}}{e^{\sigma\sqrt{\frac{dt}{2}}} - e^{-\sigma\sqrt{\frac{dt}{2}}}}\right)^2$$

$$p_d = \left(\frac{e^{\sigma\sqrt{\frac{dt}{2}}} - e^{\frac{rdt}{2}}}{e^{\sigma\sqrt{\frac{dt}{2}}} - e^{-\sigma\sqrt{\frac{dt}{2}}}}\right)^2$$

$$p_m = 1 - p_u - p_d$$

Pricing in the Trinomial Tree

Here, you have a pricing example of a call option with underlying asset $S_0 = 100$ in a trinomial tree:



Summary of Lesson 2

In lesson 2, we have:

- Understood pricing in the Trinomial tree framework.
- Provided potential solutions to the completeness/replicability issue in Trinomial models.
- Performed pricing of options in the Trinomial tree under assumptions of completeness.

⇒ **TO-DO NEXT**: Go to the next Lesson and see its associated **Jupyter Notebook** to go over a complete pricing example (like the one you have seen conceptually in the slides).

⇒ In the next lessons, we will dig deeper into all these concepts. Also, you will see more realistic coding examples with object-oriented programming that comes in very handy for more complex code down the road.

⇒ Also, please check the extra material provided if you want to see the step-by-step development of the Trinomial tree formulas.