LESSON 1

MARKOV'S PROPERTY AND GBM

Derivative Pricing - Module 4

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Outline

- Stochastic Processes and the Markov Property
- (Geometric) Brownian Motion (GBM)
- Continuous-Time Stochastic Process for Stock Prices

Stochastic Processes and Markov

Now that we have mastered option pricing in discrete time, let's move to a continuous-time framework.

$\Rightarrow$ To that end, we'll use a special type of stochastic process, a **Markov Process:**

- In a continuous-time stochastic process, the underlying variable can take any value within a range.
- In a Markov process, only the current value of the variable matters for predicting future.
  $\Rightarrow$ i.e., Past path is irrelevant and/or current price contains all info $\rightarrow$ Weak-form efficiency
- For stock price modeling, we will use a special type of Markov: a **Wiener process.**
  - This is a process to model changes in stock price, also called Brownian Motion
  - Brownian Motion processes have a mean change of 0 and a variance of 1 $\rightarrow N(0,1)$
  - We can fit this distribution to, for example, 1-year stock returns (change in prices)
  - To assess stock returns in 2 years, we can sum up two (independent) Normal distributions.

    $\rightarrow$ Mean of the sum distr. equals sum of the means: $\mu = 0 + 0 = 0$

    $\rightarrow$ σ^2 of the sum is sum of σ_t^2, so: $\sigma = \sqrt{1+1} = \sqrt{2}$

Wiener Processes / Brownian Motion

Formally, Wiener processes have 2 properties:

- **Property 1:** If z follows a Wiener process, then the change, Δz, during a small period, Δt is:

$$\Delta z = \epsilon \sqrt{\Delta t}$$

  where ϵ follows a standard normal distribution $N(0, 1)$

- **Property 2:** The values of Δz for any 2 short intervals of time, Δt, are independent.

From the previous 2 properties, it follows that Δz *follows* $N(0, \sqrt{\Delta t})$

Thus, Δz, or dz as we will call it later, serves us well to account for uncertainty (change in prices)

- But we also need to account for the level and return of stock prices.

  Picture a stock with constant annual return of 10% (μ). Then, if no uncertainty, change in price ΔS:

$$\Delta S = \mu S \Delta t, \quad \text{which as } \Delta t \to 0, \;\; dS = \mu S dt$$

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(Geometric) Brownian Motion (GBM)

From the previous expression, we now know how to account for:

(i) Regular/expected changes in stock price (μ) $\rightarrow$ Drift

(ii) Uncertainty/unexpected change in prices ($\sigma, \Delta z$) $\rightarrow$ Volatility

- Combining both, we can model changes in stock prices, dS, as:

$$dS = S\left(\mu dt + \sigma dW_t\right)$$

  where μ and σ are expected return and volatility, and $dW_t = \epsilon\sqrt{dt}$, with $\epsilon \sim \mathcal{N}(0, 1)$

$\Rightarrow$ We refer to this process as **(Geometric) Brownian Motion** $\rightarrow$ We'll see the geometric part next lesson.

- Interestingly, now that we know how to model prices, we can aggregate changes to construct a stock price path:

$$S_1 = S_0 + dS_1; \quad S_2 = S_1 + dS_2; \quad \ldots$$

Summary of Lesson 1

In Lesson 1, we have:

- Introduced a continuous-time framework for option pricing.
- Covered the basics of Markov and Wiener processes.
- Learned a way to simulate future stock prices according to GBM process.

$\Rightarrow$ **TO-DO NEXT**: Now, please go to the associated Jupyter Notebook for this lesson to learn how to implement the previous GBM process in Python. This will be the basis for all the remaining pricing in the course (and future courses), so make sure you understand it perfectly.

$\Rightarrow$ In the next lesson, we will continue developing the GBM intuition and introduce Itö calculus to solve the differential equation on dS. This is the first step towards understanding the famous Black-Scholes option pricing model.