Problem Set #2

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Exercise 1

Proof. $1/4(||x+y||^2 - ||x-y||^2) = 1/4(\langle x+y, x+y\rangle - \langle x-y, x-y\rangle) = 1/4(2\langle x,y\rangle + 2\langle x,y\rangle) = \langle x,y\rangle$ □

Proof. $1/2(||x+y||^2 + ||x-y||^2) = 1/2(2\langle x,x\rangle + 2\langle y,y\rangle) = \langle x,x\rangle + \langle y,y\rangle$ □

Exercise 2

$1/4(||x+y||^2 - ||x-y||^2 + i||x-iy||^2 - i||x+iy||^2) = 1/4(\langle x+y, x+y\rangle - \langle x-y, x-y\rangle + i\langle x-iy, x-iy\rangle - i\langle x+iy, x+iy\rangle)$
Consider,

$$i(\langle x+iy, x+iy\rangle - \langle x-iy, x-iy\rangle) = i(\langle x,x\rangle + \langle x,iy\rangle + \langle iy,x\rangle + \langle iy,iy\rangle - \langle x,x\rangle + \langle x,iy\rangle + \langle iy,x\rangle - \langle iy,iy\rangle)$$

$$= -2 - \langle x,y\rangle + \overline{\langle x,y\rangle}$$

$$\implies 4\langle x,y\rangle$$

Exercise 3

1. $\cos\theta = \frac{\langle x,x^5\rangle}{||x^5||||x||} = \frac{1/7}{\sqrt{1/33}} \implies 35^o$

2. $\cos\theta = \frac{\langle x^2,x^4\rangle}{||x^2||||||x^4||} = \frac{1/7}{\sqrt{1/45}} \implies 16^o$

Exercise 8

1. $\langle S_i, S_j, \ \rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

2. $||t|| = \sqrt{\langle t,t\rangle} = \sqrt{\frac{1}{\pi}\int t^2} = 0$

3. $proj_x cos(3t) = \langle x, cos(3t)\rangle = 0$

4. $proj_x cos(3t) = 2sin(t) - sin(2t)$

Exercise 9

$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ Then, $\langle a_i, a_j\rangle = \begin{bmatrix} \frac{1}{\pi}(\frac{x}{2} + \frac{1}{4}sin(2x))|_{-\pi}^{\pi} & -\frac{1}{\pi}(\frac{1}{2}cos^2(x))|_{-\pi}^{\pi} \\ \frac{1}{\pi}(\frac{1}{2}cos^2(x))|_{-\pi}^{\pi} & \frac{1}{\pi}(\frac{x}{2} + \frac{1}{4}sin(2x))|_{-\pi}^{\pi} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Exercise 10

1. $\rightarrow$ If a matrix is orthonormal then it represents an orthonormal transformation, then $Q : \mathbb{F}^n \rightarrow \mathbb{F}^n$ preserves the inner product implies $|Q| = 1$ or -1 implies $Q^H = Q^{-1}$.
   $\leftarrow QQ^H = I \implies Q^H = Q^{-1} \implies |Q^H| = 1$ or -1 which implies that the lin transformation defined by Q preserves the inner product.

2. Notice from the above proof we have $|Q| = 1$ or $-1 \implies ||Qx|| = ||x||$.

3. Again from (1) we see that $QQ^H \implies Q^H = Q^{-1} \implies Q^{-1H} = Q \implies Q^{-1H}Q = I$.

4. Since $Q^{-1} = Q^H$ we have that the columns are lin independent (from existence), then since $\langle Q, Q \rangle = \delta_{ij} = \sum_j q_{ij}q_{kj}$ thus we have an orthonormal basis.

5. The first part follows directly from (1). But this is not iff, $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$.

6. $Q_1^H Q_1 Q_2^H Q_2 = I \implies Q_1 Q_2 (Q_1 Q_2)^H = I$

Exercise 11

You get 0.

Exercise 16

1. Simply consider D to be a diagonal matrix with negative entries, then the decomposition is not unique.

2. Suppose by

Exercise 17

Proof.

$$A^H Ax = A^H b$$
$$(QR)^H ARx = (QR)^H b$$
$$R^H Q^H QRx = R^H Q^H$$
$$R^H Rx = R^H Q^H b$$
$$Rx = Q^H b$$

$\square$

Exercise 23

$$|x| + |y - x| \geq |x + y - x| = |y| \implies |y - x| \geq |y| - |x|$$

$$|y| + |x - y| \geq |y + x - y| = |x| \implies |y - x| \geq |x| - |y|$$

But then since $|y - x| = |x - y|$ we get $|x - y| \geq ||x| - |y|| \implies ||x - y|| \geq |||x|| - ||y|||$

Exercise 24

1. $|f(t)| \geq 0$ implies positivity. $\int |\alpha f(t)| = |\alpha| \int |f(t)|$. $||f + g|| = \int |f(t) + g(t)| \leq \int |f(t)| + \int |g(t)|$
2. Positivity follows from the previous part. $||\alpha f|| = (\int_a^b |\alpha f(t)|^2 dt)^{1/2} = (|\alpha|^2 \int |f(t)|^2)^{1/2} = |\alpha|(\int |f(t)|^2)^{1/2}$. $||f+g|| = \int |f(t)+g(t)|^2 = \int |f(t)|^2 + 2|f(t)g(t)| + |g(t)|^2 \leq ||f|| + 2||f||||g|| + ||g|| = (||f|| + ||g||)^2$
3. Again, positivity follows from the previous parts. $||\alpha f|| = |\alpha| sup|f|$. And, $sup|f + g| = sup|f(x) + g(x) \leq sup(|f| + |g|) = ||f|| + ||g||$.

Exercise 26

Clearly, $m||x||_a \leq M||x||_a$, $1/M||x||_a \leq ||x||_b \leq 1/m||x_b||$ and $m||x||_b \leq ||x||_c \leq M||x||b \implies ||x||_a \sim ||x||_b$. Thus it is an equivalence relation.

Exercise 28

1.
$$||Ax||_2 \leq ||Ax||_1 \leq \sqrt{n}||Ax||_2$$
$$||A||_1 \geq \frac{||Ax||}{||x||_1} \geq \frac{||Ax||_2}{\sqrt{n}||x||_2} \implies \frac{1}{\sqrt{n}}||A||_2 \leq ||A||_1 \leq ||A||_2$$
2. Note we have $||A||_2 \geq \frac{||Ax||_2}{||x||} \geq \frac{||Ax||_\infty}{\sqrt{n}||x||_\infty} \implies ||A||_2 \geq \frac{1}{\sqrt{n}}||A||_\infty \implies \frac{1}{\sqrt{n}}||A||_\infty \leq ||A||_2 \leq ||A||_\infty$

Exercise 29

Take Q to be orthonormal. Then, $||Q|| = 1$, then if A is orthonormal we have equality since $||R_x|| = ||x|| \implies ||Ax|| = ||A||||x|| = ||x_2||$.

Exercise 30

Positivity, since $||\cdot||$ is a norm with positivity the change of basis in the inside will not affect this. The same statement holds for scale preservation as $||\alpha A|| = ||\alpha S^{-1}AS|| = \alpha||A||_S$. Finally, triangle ineq:

$$||A+B||_s = ||S^{-1}(A+B)S|| \leq ||S^{-1}AS|| + ||S^{-1}BS||$$

. Similarly we have by Cauchy-Schwartz that $||S^{-1}ABS|| \leq ||A_S||||B_S|| \implies ||\cdot||$ is a matrix norm.

Exercise 37

Take p such that $p = ax^2 + bx + c$ then $L(p) = p'(1) = 2a + b \implies q = (2, 1, 0)$.

Exercise 38

$D = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $D^H = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Exercise 39

1. $\langle S + T(v), w\rangle = \langle Sv, w\rangle + \langle Tv, w\rangle \implies (S+T)^{=}S^* + T^*$ and $\langle \alpha Tv, w\rangle = \alpha\langle v, Tw\rangle \implies (aT^*) = \alpha T^*$
2. $\langle S * (w), v\rangle = \overline{\langle S(v), w\rangle} = \langle w, Sv\rangle$
3. $\langle STv, w\rangle = \langle Tv, S^*w\rangle = \langle v, T^*S^*w\rangle = \langle v, (T^*S^*)w\rangle$
4. $T(T^*)^{-1} = I \implies (T^*)^{-1} = (T^{-1})^*$

Exercise 40

Given Frobenius inner product:

1. $\langle A, B\rangle = \langle tr(A^HB), \rangle = \langle A^HB, \rangle$
2. $\langle A_2, A_3A_1\rangle = tr(A_2A_1^HA_3) \implies \langle A_2, A_3A_1\rangle = \langle A_2A_1^*, A_3\rangle$
3. If we combine both parts and note we have a linear operator we get $(T_A)^* = T_A$.

Exercise 44

Consider $Ax = b$ has a solution $\hat{x}$. Then, $y \in \mathcal{N}(A^H) \implies \langle y, b\rangle = 0$ since $\langle A^Hy, x\rangle = \langle 0, x\rangle = 0 \implies A^Hy = 0$. Then by contradiction assume we do not have a solution. Then, $\langle b, b\rangle = 0$ but then $x = 0$ so we must have a solution.

Exercise 45

Take A from the skew and B from the sym. And consider them with the forebenous inner product such that $\langle A,B\rangle = \langle -A,B\rangle = tr(AB)$. Now notice that $\langle A+A^T,B\rangle = \langle A,B\rangle = tr(A^TB) = 0. Also tr(AB) + tr(AB) = 0$ so, $A^T = -A \implies A \in skew_n(\mathbb{R}) \implies skew(\mathbb{R})^{\perp} = skew(\mathbb{R})$

Exercise 46

1. $A^H Ax = 0 = A^H(Ax) \implies Ax \in$ null space.
2. $\langle Ax, Ax\rangle = x^H A^H Ax = 0 \implies ||Ax|| = 0 \implies Ax = 0$
3. $\mathcal{A}^{\mathcal{H}}\mathcal{A} = \mathcal{A} \implies A, A^H A$ have same rank. From previous part.
4. Since $A^H A$ is square we only need tot have linearly independent cols to be nonsingular, since this is assumed we have the result.

Exercise 47

$P = A(A^H A)^{-1}A^H \implies rk(P) = rk(A(A^H A)^{-1}A^H) \implies rk(P) = n$ since $tr(P) = tr(I)$.

Exercise 48

1. $P(xA+yB) = \frac{xA+A^t}{2} + \frac{yB+B^T}{2} = xP(A) + yP(B)$.
2. $P^2(A) = \frac{A+A^T}{2} = P(A)$
3. $\langle A, P(A)\rangle = \langle P^*A, A\rangle = tr(A^T P^*(A)) = \langle A, P^*(A)\rangle$
4. $P(A)A = \frac{A+A^T}{2}A = \frac{AA-A^TA}{2} = 0$
5. $P(A) = \frac{A+A^T}{2} = \frac{2A}{A} = A \implies A^T = A$.
6. $||A - P(A||^2 = \frac{tr(A^TA)-tr(A^2)}{2} \implies \sqrt{\frac{tr(A^TA)-tr(A^2)}{2}}$

Exercise 50

$A = [1...1, X_1^2, ...X_n^2]^T, x = [1/s, -r/s]^T, b = [y_1^2, .., y_n^2]$