Problem Set #1

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Exercise 1.3

1. $\mathcal{G}_1$ Not Algebra - not closed under complements
2. $\mathcal{G}_2$ Algebra - not closed under countable unions.
3. $\mathcal{G}_3$ $\sigma-$algebra

Exercise 1.6

Proof. First, notice that the empty set is in $P(X)$. Next, notice that if $a \in X$ then $\{a\} \in P(X)$. Also, since the power set has every set it also has the set of all elements in X not a $\implies \{a\}^c \in P(X)$. Finally, take $a, b \in P(X)$. Then, since both sets contain elements of X this implies that $x \in a \cup b \implies a \cup b \in P(X)$. Thus, it is also closed under unions. □

Exercise 1.7

$\{\varnothing, X\}$ is the smallest $\sigma-$alg since it contains only two sets, it is still a sigma algebra, clearly the complement of the empty set is the entire set, X and vice-versa. Thus, we need both. Clealry its also closed under finite unions and contains the empty set. Since we need the empty set we also need X. So we can't have a smaller sigma alg. The power set is clearly the largest as it contaisn every possible set of elements of X.

Exercise 1.10

Proof. Let $F = \cap_\alpha S_\alpha$. Take $x \in F \implies x \in \text{every} S_1, \ldots, S_\alpha$. Then, since each S_α is a $\sigma-$alg $\implies x^c \in F$. Furthermore, $\varnothing \in S_\alpha \; \forall \; \alpha \implies \varnothing \in F$. Finally, notice that each S_α closed under finite unions, thus if $x = a_1 \cup a_2 \cup \cdots \cup a_n$ and $a_1, \ldots, a_n \in F \implies a_1, \ldots, a_n \in S_\alpha \; \forall \; \alpha$ then $x \in S_\alpha \; \forall \; \alpha \implies x \in F$. □

Exercise 1.17

- Since $A \subset B$, $\forall \; x \in A \implies x \in B$. Consider the cover of B of the form $\{x_1\} \ldots \{x_n\}$ where $x_1, \ldots, x_n \in B, x_i \neq x_j \; \forall \; i \neq j, n = |B|$. Then, there exists some subset such that covers A of the form $\{x_1\} \ldots \{x_k\}$. Then, $u(\cup^k \{x_k\}) = \sum^k u(x_k)$ and $u(\cup^n \{x_n\}) = \sum^n u(x_n)$. Then since $n > k$ and every element in A is also in B and $\mu : \mathcal{S} \to \mathbb{R}_+ \implies \sum^n u(x_n) < \sum^n u(x_k)$.
- Notice, that if each $\{A_i\}^\infty$ is disjoint then from the definition of a measure we have $u(\cup\{A_i\}) = \sum^\infty u(A_i)$. Now, we consider when $\exists \; i, j \; s.t. \; A_j \cap A_i = a \neq \varnothing$. Then, since $a \in A_j$ and A_i then $u(A_j \cup A_i) = u(A_j) + u(A_i) - u(a)$ since a is included in both sets.

Exercise 1.18

First notice, if B or $A = \varnothing \implies A \cap B = \varnothing \implies u(A \cap B) = 0 \implies \lambda(A) = 0$. Similarly if $A \cap B = \varnothing \implies \lambda(A) = \mu(\varnothing) = 0$. Next, suppose $\{A_i\}^\infty$ is a collection of disjoint sets.

$$\lambda(\cup A_i) = u((\cup A_i) \cap B) = u(\cup(A_i \cap B)) = \sum u(A_i \cap B) = \cup\lambda(A_i)$$

Exercise 1.20

First, notice that $u(\cap A_n) \leq \infty$. Now consider,

$$u(A_1 \ /(\cap A_n)) = u(A_1) - u(\cap A_n)$$

Also,

$$\begin{aligned} A_1 \ /(\cap A_n)) &= A_1 \cap (\cap A_n)^c \\ &= A_1 \cap (\cup A_n^c) \\ &= \cup(A_n^c \cap A_1) \\ &= \cup(A_1 \ /A_n) \end{aligned}$$

Then plugging in our result into our first equation yields,

$$u(A_1 \ /(\cap A_n))) = u(\cup A_1 \ /A_n) = \lim_{n\to\infty} u(A_1 \ /A_n) = u(A_1) - \lim_{n\to\infty} u(A_n)$$

Then we have,

$$u(A_1) - \lim_{n\to\infty} u(A_n) = u(A_1) - u(\cap A_n) \implies \lim_{n\to\infty} u(A_n) = u(\cap A_n)$$

Exercise 2.10

Simply, because the outer measure is a measure that is restricted such that $u = u^*|_{\mathcal{S}}$ and of course regardless of where E is, $B = (B \cap E) \cup (B \cap E^c)$. Furthermore, since E and E^c are disjoint we have that $u(B) = u(B \cap E) + u(B \cap E^c)$. However, because we use the outer measure our relation condition for sub additivity becomes $\leq$.

Exercise 2.14

The Borel-algebra is the sigma algebra generated by all open sets in $\mathbb{R}$. Then, it is only left to show that $\sigma(\mathcal{A}) = \sigma(\mathcal{O})$. Since the Borel-algebra is generated from all the open sets it is equivalent to say that it can also be generated by closed sets. Simmilary it can be generated by half open sets then its clear it is equivalent to $\sigma(\mathcal{A})$.

Exercise 3.1

Consider the Lebesgue measure of a single point, that is consider $u^*(\{a\}) \leq u*(\{a-e, a+e\})\ \forall\ e > 0$ then $u*(\{a-e, a+e\}) = 2e$ but since e is arbitrarily small and $e > 0 \implies u^*\{a\} = 0$. But then since every countable subset of the real line is a collection of finite points and the Lebesgue measure is countably additive we have that the every countable set has Lebesgue measure 0.

Exercise 3.4

Because, we can simply need that the preimage is in the sigma algebra, which we can check using either inequality to get a subset of the preimage.

Exercise 3.7

Simply define $F(f(x), g(x)) = f + g$ or $f \cdot g$ proving the first two as this transformation is cts. Then since $supf_n$ and $inff_n$ are measurable this clearly implies the max and min of f, as the sup implies the maximum (min) value of $f(x)$, $\forall\ x, n$. Then, $F(f(x)) = |f(x)|$ since this is also cts.

Exercise 3.14

Since, f is measurable and bounded we can construct two simple function such that $\forall\ \epsilon > 0\ \exists\ g_\epsilon, q_\epsilon$ s.t. $g_\epsilon < f < q_\epsilon$ such that $|g_\epsilon - q_\epsilon| < \epsilon \implies f_n \to g_\epsilon$.

Exercise 4.13

Consider that $f \in \mathcal{L}^1(E, \mu) \iff \int_E ||f|| d\mu < \infty \implies \int_E f^+ + \int_E f^-$. Furthermore, we have that $\mu(E) < \infty \implies \int_E f d\mu < \infty \implies \int_E ||f|| d\mu < \infty$. Thus we have the result.

Exercise 4.14

Since $f \in \mathcal{L}^1(E, \mu)$ we have that it is measurable. Then we can simply define $E_n = \{x \in E, f(x) \geq n\}$ such that we then have $\int_E f d\mu < \int_{E_n} n d\mu = n\mu(E_n) < \infty$

Exercise 4.15

Notice that if $f, g \in \mathcal{L}^1$ then by definition it is measurable. Then simply applying Proposition 4.7 yields the result.

Exercise 4.16

Since $A \subset E$ and more so $A \in \mathcal{M}$ since we already know that f is measurable we get $\int_A f = \int_A f^+ - \int_A f^-$. Furthermore, since we had absolute Lebesgue integrability on E we know that the integral of the subset is also finite, that is $||f|| = f^+ + f^- \implies f \in \mathcal{L}^1(\mu, A)$. More simply we can just consider E the union of disjoint sets A_i such that $A_i \subset E$.

Exercise 4.21

$A = (A - B) \cup B \implies \mu(A) = \mu(A - B) + \mu(B) \implies \mu(A) - \mu(B) = 0 \implies \int_A f = \mu(A) - \mu(B) = \int_B f \implies \int_A f - \mu(A) = \int_B f + \mu(B) \implies \int_A f \leq \int_B f$. By additive separability.