Problem Set #2

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Exercise 2

$$\lambda I - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 2 \\ 0 & 0 & \lambda \end{bmatrix} \implies \lambda = 0$$

Exercise 4

a. Since $A = A^H \implies diag(A) \in \mathbb{R}^n \implies p(\lambda) = \lambda^2 - \mathrm{tr}(A)\lambda + det(A) \implies \lambda \in \mathbb{C} \iff (\mathrm{tr}(A)\lambda)^2 - 4\lambda \det A < 0$ but, $A_{12}A_{21} \geq 0 \implies$ real eigenvalues.
b. In the same vain, if A is skew hermitian then $A_{12} = -\overline{A_{21}} \implies A_{12}A_{21} \leq 0$ thus we have imaginary eigenvalues.

Exercise 6

Simply, consider $\det(\lambda I - A) = \prod_{i=1}^{n}(\lambda - A_{ii}) = 0 \iff \lambda = a_{ii} \implies a_{ii}$ are eigenvalues. Notice we implicitly use the fact that A is full rank.

Exercise 8

1. Consider that $\sin(x), \cos(x) \in C^\infty \ \forall\ x$. Furthermore notice that every continuously differentiable function maybe expressed in terms of sin and cos thus we have the result.
2. $D = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$
3. $\{sin(x), cos(x)\}$ and $\{sin(2x), cos(2x)\}$

Exercise 13

$P = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$

Exercise 15

$f(A) = \{a_0 I + a_1 b_i + \cdots + a_n b_i^n\}_{i=1}^n = \{a_0 + a_1\lambda_i b_i + \cdots + \lambda + i^n b_i\}_i = f(\lambda_i)b_i \implies f(A)f(b_i) = f(\lambda)f(b_i) \implies f(A) = f(\lambda)$ where b_i is an eigenbasis.

Exercise 16

1. $Diag(A) = D = P^{-1}AP \implies A^k = PD^kP^{-1} = \begin{bmatrix} 2/3 & 2/3 \\ 1/3 & 1/3 \end{bmatrix}$
2. No change - topologically equivalent norms.
3. We can just plug in the eigenvalues of A. Thus, we get $\lambda_i = 8$ and 5.96

Exercise 18

Suppose λ is an eigenvalue of A. Then, $Ax = \lambda x \implies A^T x = \lambda x \implies x^T A = \lambda x^T$.

Exercise 20

$B^H = (QAQ^H)^H = QAQ^H = B$ where Q, Q^H orthonormal.

Exercise 24

If Hermitian $\langle Ax, x \rangle = \overline{\langle x, Ax \rangle}$ thus it is equivalent under complex conjugates thus it is real. If skew-hermitian then $\langle -Ax, x \rangle = -\overline{\langle x, Ax \rangle}$ which implies that it is imaginary.

Exercise 25

1. $(x_1 x_1^T + \cdots + x_n x_n^T)x_j = x_j$
2. $(\lambda_1 x_1 x_1^T + \cdots + \lambda_n x_n x_n^T)x_j = \lambda_j x_j = Ax_j$

Exercise 27

Consider that a matrix that is positive definite has all positive eigenvalues, hermitian and $\langle x, Ax \rangle > 0$. Then, this implies that all roots of p(A) are positive. Then, take e_i to be the standard basis. Then since $\langle e_i, Ae_i \rangle > 0 \implies a_{ii} > 0$.

Exercise 28

If, A, B positive semidefinite then AB is also positive semidefinite. Then it is clear that $0 < tr(AB)$. Furthermore, $tr(A)tr(B) = \sum_i \sum_j \lambda_{1,i}\lambda_{2,j}$ while, $tr(AB) = \sum_k \lambda_{1,k}\lambda_{2,k} \implies 0 \leq tr(AB \leq tr(A)tr(B))$

Exercise 31

1. $||A|| = max\frac{||Ax||}{||x||}$ Letting, $||x|| = 1$ we get $||A|| \leq max\sqrt{|\lambda_j|}$.
2. $A^{-1} = (U\Sigma V^T)^{-1} = (V^T)\Sigma^{-1}U^{-1} = VS^{-1}U^T \implies ||A^{-1}|| = \sigma^{-1}$
3. $\Sigma^H = \Sigma^T = \Sigma$ combined with $A^HA = U^H\Sigma^2V$implies the result.
4. Letting $UAV = (UU)\Sigma(V^HV^H) = A$ and taking norms implies the result.

Exercise 32

1. This follows directly from the last part of the last problem.
2. $||A||_F = \text{tr}(()A^HA)^{1/2} = \text{tr}(()\Sigma^H\Sigma)^{1/2} = (||\sigma_i||)$

Exercise 33

$||A||_2 = \frac{||y^HAx||}{|||x|} = ||y^HAx||$

Exercise 36

$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

Exercise 38

1. $AA^\dagger A = (U\Sigma V^H)V\Sigma^{-1}U^H(U\Sigma V^H) = A$
2. $A^\dagger AA^\dagger = V\Sigma^{-1}U^H = A^\dagger$
3. $(AA^\dagger)^H = (V\Sigma U^H)(V\Sigma^{-1}U^H) = UU^H = AA^\dagger$
4. $(A^\dagger A)^H = VV^H = A^\dagger A$
5. Notice that $\mathbb{R}(AA^\dagger) \subset \mathbb{R}(A)$ and vice versa. Then combining this with (i) gives the result.
6. Using the result from the last part and (ii) yields the result.