Notes from MWG's *Microeconomic Theory*

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Brief Introduction

The following are select notes from *Micoreconomic Theory* by Mas-Colell, Whinston, and Green. I have tried to synthesize these into the key axioms, theorems, propositions, and proofs introduced in the text. Overall we mirror the Definition-Theorem-Proof style employed by MWG. Examples and notes are provided where needed. Some proofs are my own work and thus prone to error. Notice this is an individual effort and very much a work in progress. It should not be viewed as a replacement for the original text.

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Ch. 1 Preference and Choice

1.1 Preference Relations

Axioms of Preference:

i. $x \succ y$. x is "strictly" preferred to y *if and only if (iff)* $x \succeq y$ but $y \not\succeq x$. *Note:* $x \succeq y$ is an at least as good as relation. That is, x is at least as good as y.

ii. $x \sim y$. Then, x is indifferent to y. Thus, $x \succeq y$ and $y \succeq x$.

Definition: We call a relation *rational* if it is both **complete** and **transitive**. Consider the set of goods, X.

i. Completeness: $\forall\ x, y \in X$ either $x \succeq y$ or $y \succeq x$ or both.

ii. Transitivity: $\forall\ x, y, z \in X$ if $x \succeq y$ and $y \succeq z$ then $x \succeq z$.

Proposition 1.B.1: It follows that if $\succeq$ is rational then:

i. $\succ$ is irreflexive and transitive

ii. $\sim$ is reflexive and transitive

iii. If $x \succ y \succeq z$ then $x \succ z$

Proof. **Proposition 1.B.1**
Consider the set of goods X. Then take $x, y, z \in X$.
Let $\succeq$ be rational so it is both complete and transitive.

i. Suppose by contradiction that $x \succ x$. Then, since $x \succeq x$ and $x \not\succeq x$. Both these facts cannot be true by the def of a strict pref relation. relation, so $\succ$ is irreflexive. Furthermore, consider $x \succ y$ and $y \succ x$. Then, $x \succeq y$ and $y \not\succeq x$. Also, $y \succeq z$ and $z \not\succeq y$. So, since $\succeq$ is rational, by transitivity we have $x \succ z$.

ii. Consider $x \sim x$. Then, $x \succeq x$ and $x \succeq x$. If condition one holds than condition two must hold. So, $\sim$ is reflexive. Also, if $x \sim y$ and $y \sim z$ then, $x \succeq y$ and $y \succeq x$, and $z \succeq y$ and $y \succeq z$. So, by transitivity $x \sim z$.

iii. If $x \succ y$ then $x \succeq y$ and $y \not\succeq x$. Also, $y \succeq z$. Then, by transitivity $x \succeq z$. But, since $y \not\succeq x$ then by transitivity $z \not\succeq x$. So, $x \succ z$.

□

If a relation is rational we can also represent it with a **utility function**:

Definition $u : X \to \mathbb{R}$ is a utility function if $\forall\, x, y \in X,\ x \succeq y \quad \text{iff} \quad u(x) \geq u(y)$. Notice, that this map is not unique.

Proposition: A preference relation must be rational to represent it with a utility function.

Proof Sketch. Since, $u(\cdot)$ is real-valued $\forall\, x, y \in X$ $u(x) \geq u(y)$ or $u(y) \geq u(x)$. This implies completeness. Then, $\forall\, x, y, z \in X$ if $u(x) \geq u(y)$ and $u(y) \geq u(z)$ then, $u(x) \geq u(z)$. This implies transitivity.

1.2 Choice

Definitions A choice structure is defined by $(\mathcal{B}, C(\cdot))$.

1. $\mathcal{B}$ is a family on nonempty subsets of X. That is, $B \in \mathcal{B}, B \subset X$. Intuitively, $\mathcal{B}$ is a family of comprehensive budget sets that that are possible in a society. However, it is not necessarily all possible subsets in X.

2. $C(\cdot)$ is a choice rule. It assigns a nonempty set of chosen elements, $C(B) \subset B$. These are the decision makers chosen alternatives given a budget set.

Example: $X = \{x, y, z\}, \mathcal{B} = \{\{x, y\}, \{x, z\}\}$
Then one possible $C(\{x, y\}) = \{x\}$ and $C(\{x, z\}) = \{x, z\}$.
We can apply the weak axiom of revealed preference (Samuelson 1947) so that we may expect some consistency. We can formally write this as:
Definition of Weak Axiom of Revealed Preference: Given $(\mathcal{B}, C(\cdot))$. The choice structure satisfies the weak axiom of revealed preference if $x, y \in B$ and $x \in C(B)$ then $\forall\ B' \in \mathcal{B}\ s.t.\ x, y \in B'$. If $y \in C(B')$ then, $x \in C(B')$.
Perhaps, a simpler yet equivalent statement is that x is revealed at least as good as y if and

only if there is some budget set, B, where x is in C(B).
More formally, the weak axiom states $x \succeq \star\, y$ iff $B \in \mathcal{B}\ s.t.\ x, y \in B$ and $x \in C(B)$.

1.3 Interplay of Preference and Choice

Consider $C^*(B, \succeq) = \{x \in B | x \succeq y\ \forall\ y \in B\}$. This represents the most preferred good in the budget set (could be null in theory). We assume it is nonempty.
Thus, we generate the choice structure $(\mathcal{B}, C^*(B, \succeq))$.

Proposition: $(\mathcal{B}, C^*(B, \succeq))$ satisfies the weak axiom of revealed preference if $\succeq$ is a rational preference relation.

Proof. Take $x \in C^*(B, \succeq)$ for some $x, y \in B \in \mathcal{B}$. Then, $x \succeq y$. Now consider by contradiction $x, y \in B'$ and $y \in C^*(B', \succeq)$. Then, $y \succeq z\ \forall\ z \in B'$. Then, this implies that $y \succeq x$ which contradicts our initial construction. Thus, if $y \in C^*(B', \succeq)$ then also $x \in C^*(B', \succeq)$. Thus, this relation satisfies the weak axiom. □

Definition: Given a choice structure we say a rational preference relation rationalizes $C(\cdot)$ relative to $\mathcal{B}$ if

$$C(B) = C^*(B, \succeq)\ \forall\ B \in \mathcal{B}$$

.
That is, the rational preference relation $\succeq$ rationalizes the choice rule on $\mathcal{B}$ if it generates the same output as the preference maximizing choice rule $(C^*(\cdot, \succeq))$. We then call the choice maker a preference maximizer.

Example: Notice that the following choice structure satisfies the weak axiom but is not rationalizing preferences: $\mathcal{B} = \{\{x, y\}, \{y, z\}, \{z, x\}\},\ C(\{x, y\}) = x,\ C(\{y, z\}) = y,\ C(\{z, x\}) = z$.

Proposition: If $(\mathcal{B}, C(\cdot))$ is a choice structure such that:

1. the weak axiom is satisfied,
2. $\mathcal{B}$ includes all subsets of X up to three elements,

then there exists a unique rational preference relation $\succeq$ that rationalizes $C(\cdot)$ such that

$$C(B) = C^*(B, \succeq)\ \forall\ B \in \mathcal{B}$$

Proof Sketch: To prove the above result we must (1) show $\succeq *$ is rational, (2) $\succeq *$ rationalizes $C(\cdot)$, and (3) the solution is unique (this falls from every binary pairing being included in $\mathcal{B}$).

Ch. 2 Consumer Choice

The consumer is the most basic decision unit in the economy. We assume a market economy where commodities (good and services) are traded and available for purchase.

2.1 Commodities

We assume a finite number of commodities, L.
The commodity vector lists amounts of each commodity:

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_L \end{bmatrix}$$

2.2 Consumption Set

The consumption set is an element of the commodity space, $X \subset \mathbb{R}^L$.
The consumption set represents an individuals conceivable consumption given particular constraints.
The consumption set is defined as $X = \mathbb{R}^L_+ = \{x \in \mathbb{R}^L | x_l \geq 0 \ \forall \ 1, \ldots, L\}$
It is a set of all nonnegative bundles of commodities.
Notice that $\mathbb{R}^L$ is convex. Take $x, x' \in \mathbb{R}^L$ then $x'' = \alpha x + (1-\alpha)x' \in \mathbb{R}^L \ \forall \ \alpha \in [0,1]$.
This design allows us to reflect physical constraints on the consumption bundle.

2.3 Budgets

We also add a monetary constraint to the consumption set.
We assume completeness or universality of markets and that consumers are price takers.
Then we define the price vector as an element in $\mathbb{R}^L$

$$p = \begin{bmatrix} p_1 \\ \vdots \\ p_L \end{bmatrix} \in \mathbb{R}^L$$

For simplicity we assume non-negativity of prices.
Finally, we introduce a consumers total wealth, w. That is,

$$\mathbf{p} \cdot \mathbf{x} = p_1 x_1 + \cdots + p_L x_L \leq w^1$$

The consumer problem is simply to choose a consumption bundle $x \in B_{\mathbf{p},w}$.

Definition: Since there are multiple consumption bundles that exhibit this property for

[1] Notice we use the dot product here.

a given price vector and wealth we define the *Walrasian* budget set or *competitive budget set* as:

$$B_{\mathbf{p},w} = \{\mathbf{x} \in \mathbb{R}^L_+ \mid \mathbf{p} \cdot \mathbf{x} \leq w\}$$

Definition: We define the budget hyperplane (or line in $L = 2$) to be the set of consumption bundles defined by $\{\mathbf{x} \in \mathbb{R}^L_+ \mid \mathbf{p} \cdot \mathbf{x} = w\}$.

There are multiple interesting phenomena associated with the budget hyperplane. First, in $\mathbb{R}^2$ the slope of the line is $m = -\frac{(w/p_2)}{(w/p_1)} = -(p_1/p_2)$. Notice that the negativity of the slope reflects the budget constraint.

Furthermore, the price vector starting at any $x \in \{\mathbf{x} \in \mathbb{R}^L_+ \mid \mathbf{p} \cdot \mathbf{x} = w\}$ is orthogonal to any vector on the budget line. This is because $p \cdot x = p \cdot x' = w$. That is,

$$p \cdot x - p \cdot x' = 0$$

$$p \cdot (x - x') = 0$$

$$p \cdot \Delta x = 0$$

Furthermore, $B_{\mathbf{p},w}$ is a convex set. Consider $\mathbf{p} \cdot \mathbf{x}'' = \mathbf{p} \cdot \alpha\mathbf{x} + \mathbf{p} \cdot (1-\alpha)\mathbf{x}' \leq w$.[2]

2.4 Comparative Statics

$x(\mathbf{p}, w)$ defines a demand correspondence.
We assume the demand correspondence is *homogeneous of degree zero* and satisfies *Walrus' Law.*
Definition: $x(\mathbf{p}, w)$ is homogeneous of degree zero if $x(\alpha\mathbf{p}, \alpha w) = x(\mathbf{p}, w)\ \forall\ \mathbf{p}, w, \alpha > 0$. That is, if price and wealth change in the same proportion there is no change since demand is based on feasibility. This entails that:

$$B_{\alpha\mathbf{p},\alpha w} = B_{\mathbf{p},w}$$

Definition: $x(\mathbf{p}, w)$ satisfies Walrus' Law if $\forall\ p >> 0$ and $w > 0$:

$$\mathbf{p} \cdot \mathbf{x} = w\ \forall\ \mathbf{x} \in x(\mathbf{p}, w)$$

Simply, this says that the consumer fully expends their wealth over a lifetime.
We can write the demand function in matrix form with an entry for each good as such:

$$x(\mathbf{p}, w) = \begin{bmatrix} x_1(\mathbf{p}, w) \\ \vdots \\ x_L(\mathbf{p}, w) \end{bmatrix}$$

Notice, that this defines a choice structure $(\mathcal{B}^w, x(\cdot))$ given $\mathcal{B}^w = \{B_{\mathbf{p},w} | \mathbf{p} >> 0, w > 0\}$.

[2] This is easily proven if we recall $\forall\ \mathbf{x} \in B_{\mathbf{p},w},\ \mathbf{p} \cdot \mathbf{x} \leq w$

Comparative statics study choices under changes in various economic parameters such as wealth and price.

Wealth Effects
First, some definitions:
Definition: If we fix $\mathbf{p}^*$ then $x(\mathbf{p}^*, w)$ is called the *Engel function.*

Definition: The image of the Engel function, $E_{\mathbf{p}^*} = \{x(\mathbf{p}^*, w) \mid w > 0\}$, is called the *wealth expansion function.*

Definition: $\frac{\partial x_l(\mathbf{p},w)}{\partial w}$ is the *wealth effect* for the l'th good.

Definition: If $\frac{\partial x_l(\mathbf{p},w)}{\partial w} > 0$ then the good is *normal* otherwise if < 0 it is *inferior* at $(\mathbf{p}, w)$. Furthermore, if every good is normal we say demand is normal.
We can express the wealth effects at any $(\mathbf{p}, w)$ for goods using matrix notation:[3]

$$D_w x(\mathbf{p}, w) = \begin{bmatrix} \frac{\partial x_1(\mathbf{p},w)}{\partial w} \\ \vdots \\ \frac{\partial x_L(\mathbf{p},w)}{\partial w} \end{bmatrix} \in R^L$$

Price Effects
Definition: $\partial x_l(\mathbf{p}, w)/\partial p_k$ is the price effect of the price of good k on demand for good l. As the price of good k changes how does the price of good l?

Definition: If $\partial x_l(\mathbf{p}, w)/\partial p_l > 0$ then we call it a *giffen good.* That is as price rises, so too does consumption.
Notice, that we can conveniently represent price effects in matrix form:

$$D_p x(\mathbf{p}, w) = \begin{bmatrix} \frac{\partial x_1(\mathbf{p},w)}{\partial p_1} & \cdots & \frac{\partial x_1(\mathbf{p},w)}{\partial p_L} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_L(\mathbf{p},w)}{\partial p_1} & \cdots & \frac{\partial x_L(\mathbf{p},w)}{\partial p_L} \end{bmatrix}$$

Proposition: If the Walrasian demand function is homogeneous of degree 0 then for all p and w:

$$\sum_{j=1}^{L} p_j \frac{\partial x(\mathbf{p}, w)}{\partial p_j} + w \frac{\partial x(\mathbf{p}, w)}{\partial w} = 0$$

$$D_p x(\mathbf{p}, w) p + D_w x(\mathbf{p}, w) w = 0$$

Proof. If the demand function is homogeneous of degree 0 then,

$$x(\alpha \mathbf{p}, \alpha w) = x(\mathbf{p}, w)$$

$$\frac{\alpha \partial x(\mathbf{p}, w)}{\partial \alpha} = 0$$

[3]We use the $D_x f(x, y)$ notation to denote the matrix of partial derivaties of f with respect to x.

Set $\alpha = \mathbf{p}$

$$\sum_{j=1}^{L} p_j \frac{\partial x(\mathbf{p}, w)}{\partial p_j} = 0$$

The same follows for $\alpha = w$. $\square$