

Existence of Walrasian Equilibrium with Production*

Price Theory II

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1 Walrasian Equilibrium with Production

Theorem 1.1 (Existence of WE with Production).

Let an economy, $\mathcal{E} = (u^i, e^i, \theta^{i,j}, Y^j)_{i \in \mathcal{I}, j \in \mathcal{J}}$. If each individual utility is continuous, strictly quasi-concave, and strongly increasing, and if the production possibilities set for each firm is closed and bounded, strongly convex, and $0 \in Y^j \subset \mathbb{R}^n$, and $y + \sum_{i \in \mathcal{I}} e^i \gg 0$ for some $y \in \sum_{j \in \mathcal{J}} Y^j$ then there exists (atleast one and possibly many) a price vector $p^* \gg 0$ s.t. $z(p^*) = 0$.

Proof. We proceed by showing that under the conditions listed in the theorem, our excess demand function, $z(\cdot)$ satisfies the conditions for equilibrium. Recall, these conditions will allow us to utilize Brouwer's fixed point theorem. The first of these conditions is continuity,

Claim 1.1. $z(\cdot)$ is continuous.

Consider that $z(p) = \sum_{i \in \mathcal{I}} x^i(p, m^i(p)) - \sum_{i \in \mathcal{J}} y^j(p) - \sum_{i \in \mathcal{I}} e^i$. First notice that the summation of endowments is constant. Thus, we are concerned with our individual demand functions, and firm production plans.

Now, consider the production plan $y^j(p)$. Recall, that we assume that Y^j is closed and bounded, and the firm chooses to maximize the convex set at $y^j(p)$ then, from the theory of the maximum we have that $y^j(p)$ is continuous.

Now, consider $x^i(p, m^i(p))$. Consider that $m^i(p) = \sum \theta^{ij} \Pi^j(p)$. Again because we assume compactness and convexity it is clear the profit function is continuous – this is again by theory of the maximum. Then, since all the components of $x^i(\cdot)$ are continuous, and the demand function itself is continuous on $R_{++}^n \times R_+^n$. We can establish the continuity of $z(\cdot)$ via each of its components.

Claim 1.2 (Walras' Law). $p \cdot z(p) = 0, \forall p \gg 0$

To see this consider,

$$pz(p) = p \sum_{\forall k} z_k(p) = p \sum_{\forall k} \left(\sum_{i \in \mathcal{I}} x^i(p, m^i(p)) - \sum_{i \in \mathcal{J}} y^j(p) - \sum_{i \in \mathcal{I}} e^i \right)$$

*This proof follows the outline given by Jehle and Reny (2011). It relies heavily on insight derived from Prof. Reny's Price Theory II course at the University of Chicago. For educational purposes only.

$$= p \sum_{\forall k} \sum_{i \in \mathcal{I}} x^i(\mathbf{p}, m^i(\mathbf{p})) - p \sum_{\forall k} \sum_{i \in \mathcal{J}} y^j(\mathbf{p}) - p \sum_{\forall k} \sum_{i \in \mathcal{I}} e^i$$

Notice, we have just defined our profit function $\Pi^j(p) = \max y^j p$. And, $\sum_{i \in \mathcal{I}} \Pi^j(p) = \sum_{j \in \mathcal{J}} \sum_{i \in \mathcal{I}} \theta^{ij} \Pi^j$
Thus,

$$= p \sum_{\forall k} \sum_{i \in \mathcal{I}} x^i(\mathbf{p}, m^i(\mathbf{p})) - p \sum_{\forall k} \sum_{i \in \mathcal{J}} y^j(\mathbf{p}) - p \sum_{\forall k} \sum_{i \in \mathcal{I}} e^i$$

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