Math Camp

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2) What is the pmf? How would we derive it?
3) What does iid mean?
4) Define $E[X]$, $\text{var}(X)$
5) What does it mean for a random variable, $Y \sim \text{Poisson}(\lambda)$?

Where We've Been, Where We're Going

Continuous Random Variables:

- Random variables that are not discrete
- Widely used:
  - Approval ratings
  - Vote Share
  - GDP
  - ...
- Many analogues to distributions used on Friday

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- Stock price $X(p) = p$ for all p
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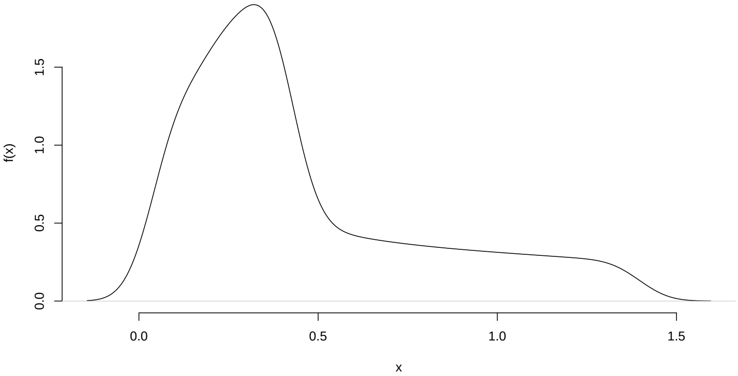
We'll need calculus to answer questions about probability.

Integration

Suppose we have some function $f(x)$



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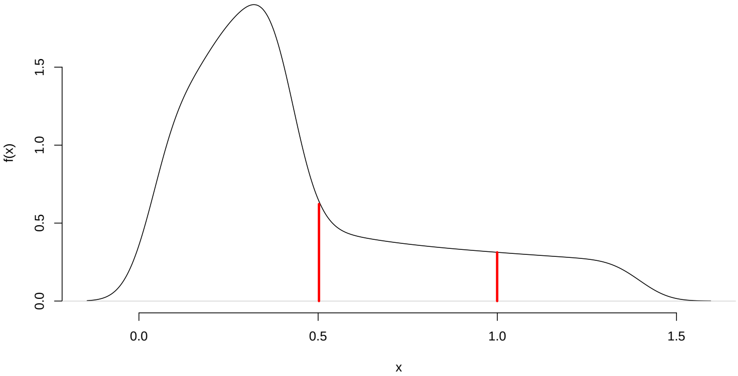

What is the area under $f(x)$ between $\frac{1}{2}$ and 1?

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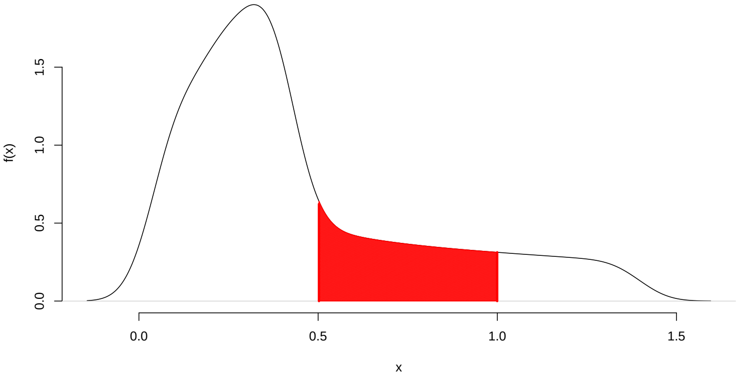


What is the area under $f(x)$ between $\frac{1}{2}$ and 1?

Area under curve $= \int_{1/2}^{1} f(x)dx = F(1) - F(1/2)$

Continuous Random Variable

Definition

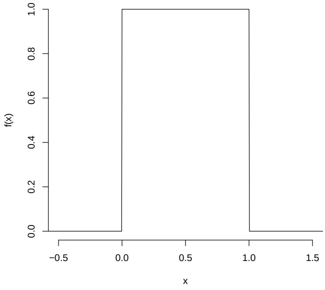
X is a continuous random variable if there exists a nonnegative function defined for all $x \in \Re$ having the property for any (measurable) set of real numbers B,

$$P(X \in B) = \int_B f(x) dx$$

We'll call $f(\cdot)$ the probability density function for X.

Example: Uniform Random Variable

$X \sim \text{Uniform}(0, 1)$ if



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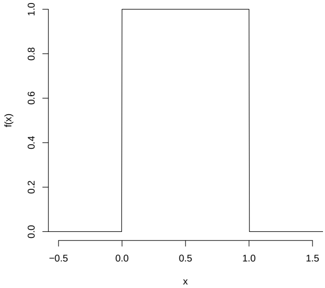
$X \sim \text{Uniform}(0, 1)$ if

$$f(x) \;=\; 1 \text{ if } x \in [0, 1]$$



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$X \sim \text{Uniform}(0,1)$ if

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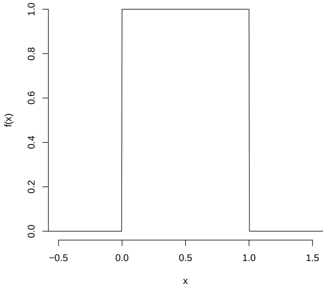

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$$\begin{aligned} P(X \in [0.2, 0.5]) &= \int_{0.2}^{0.5} 1 dx \\ &= X|_{0.2}^{0.5} \\ &= 0.5 - 0.2 \\ &= 0.3 \end{aligned}$$

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\begin{aligned}
f(x) &= 1 \text{ if } x \in [0,1] \\
f(x) &= 0 \text{ otherwise}
\end{aligned}
$$

$$
\begin{aligned}
P(X \in \{[0,0.2] \cup [0.5,1]\}) &= \int_0^{0.2} 1 dx + \int_{0.5}^{1} 1 dx \\
&= X_0^{0.2} + X_{0.5}^{1} \\
&= 0.2 - 0 + 1 - 0.5 \\
&= 0.7
\end{aligned}
$$

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To summarize

- $P(X = a) = 0$
- $P(X \in (-\infty, \infty)) = 1$
- If F is antiderivative of f, then $P(X \in [c, d]) = F(d) - F(c)$ (Fundamental theorem of calculus)

Cumulative Mass Function

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\text{pdf} \arrow[r, bend left, "\text{Integrate}"] & \text{cdf} \arrow[l, bend left, "\text{Differentiate}"]
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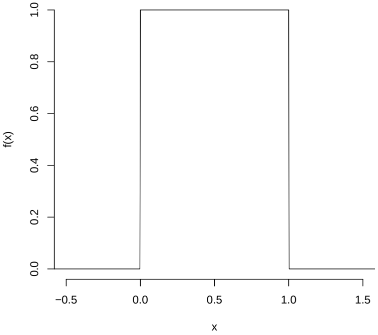

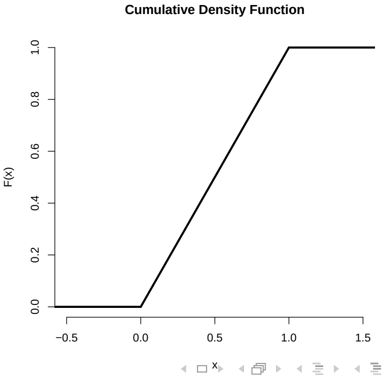

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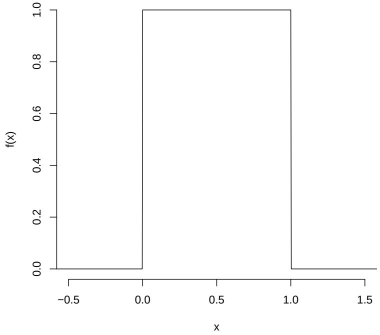

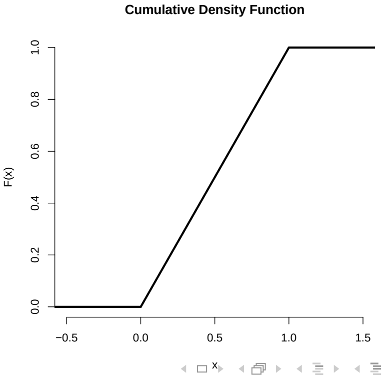

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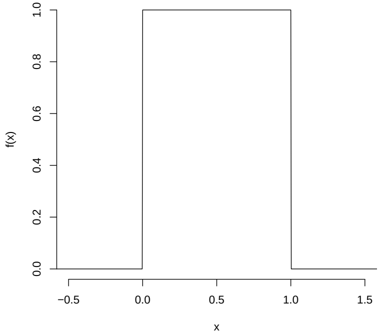

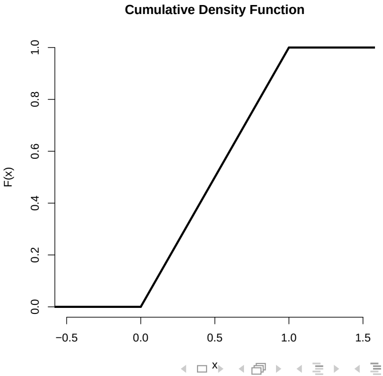

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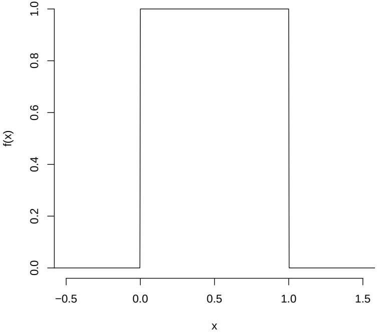

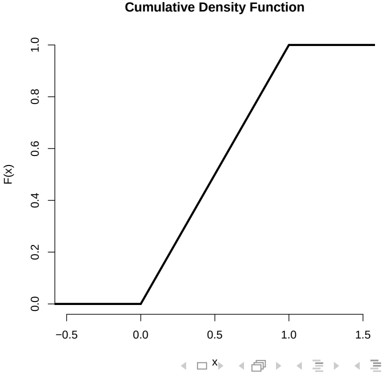

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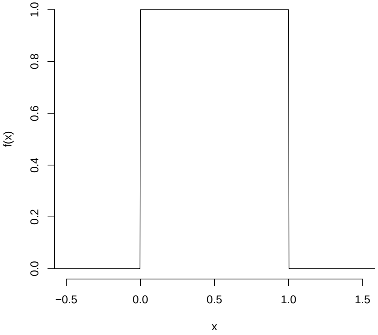

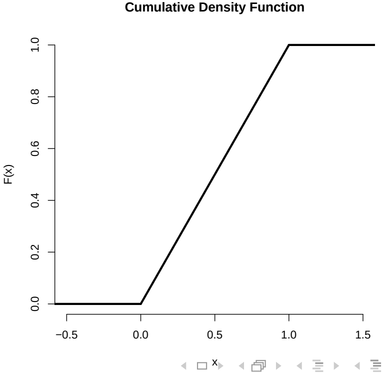

Expectation With Continuous Random Variables

Definition

If X is a continuous random variable then,

$$E[X] \;=\; \int_{-\infty}^{\infty} x f(x) dx$$

Suppose $X \sim \textit{Uniform}(0, 1)$. What is $E[X]$?

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\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} x f(x) dx \\
&= \int_{-\infty}^{0} x0 dx + \int_{0}^{1} x1 dx + \int_{1}^{\infty} x0 dx
\end{aligned}
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&= 0 + \frac{1}{2} + 0 \\
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\end{aligned}
$$

Expectations of Functions

Proposition

Suppose X is a continuous random variable and $g : \Re \rightarrow \Re$ (that isn't crazy). Then,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx$$

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Suppose $g(X) = X^2$ and $X \sim \text{Uniform}(0, 1)$. What is E[g(X)]?

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$$= \int_0^1 x^2 dx$$

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\begin{aligned}
E[g(X)] &= \int_{-\infty}^{\infty} g(x) f(x) dx \\
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Corollary

Suppose X is a continuous random variable. Then,

$$E[aX + b] = aE[X] + b$$

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$$\begin{aligned} E[aX + b] &= \int_{-\infty}^{\infty} (ax + b) f(x) dx \\ &= a \int_{-\infty}^{\infty} x f(x) dx + b \int_{-\infty}^{\infty} f(x) dx \end{aligned}$$

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□

Definition

Variance. If X is a continuous random variable, define its variance, $Var(X)$,

$$\begin{aligned} Var(X) &= E[(X - E[X])^2] \\ &= \int_{-\infty}^{\infty} (x - E[X])^2 f(x) dx \\ &= E[X^2] - E[X]^2 \end{aligned}$$

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$$Var(X) = E[X^2] - E[X]^2$$

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$$\begin{aligned} Var(X) &= E[X^2] - E[X]^2 \\ &= \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \end{aligned}$$

Famous Continuous Distributions

- Normal Distribution
- Gamma distribution
- χ^2 Distribution
- t Distribution
- Beta, Dirichlet distributions (not today!)
- F-distribution (not today!)

Definition

Suppose X is a random variable with $X \in \Re$ and *density*

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Then X is a *normally* *distributed random variable with parameters μ and σ^2.*
Equivalently, we'll write

$$X \sim \mathit{Normal}(\mu, \sigma^2)$$

Support for President Obama

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$$
\begin{aligned}
Y & \sim \text{Normal}(\mu, \sigma^2) \\
f(y) & = \frac{\exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)}{\sqrt{2\pi\sigma^2}}
\end{aligned}
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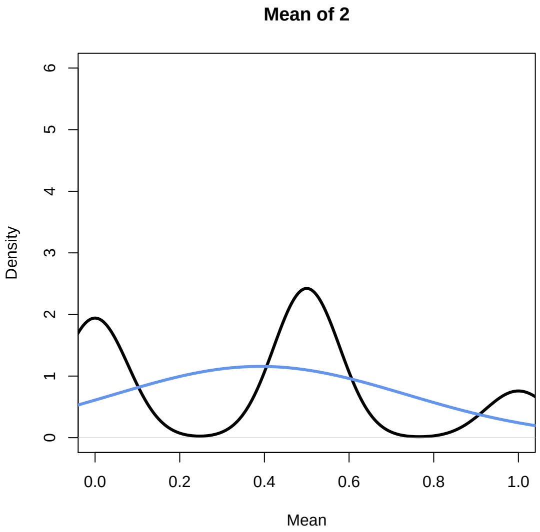
Central Limit Theorem

We'll prove it on Friday.



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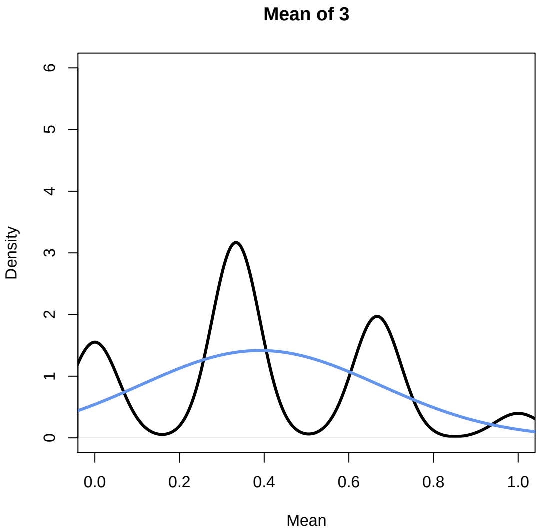

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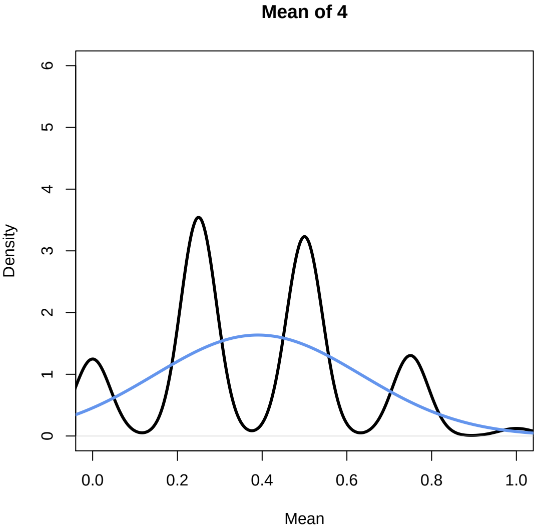

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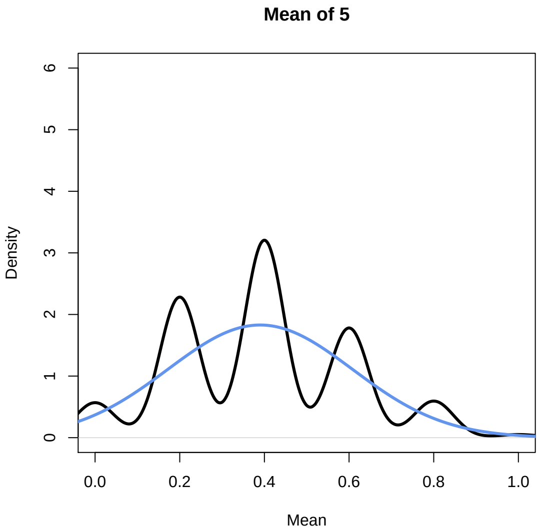

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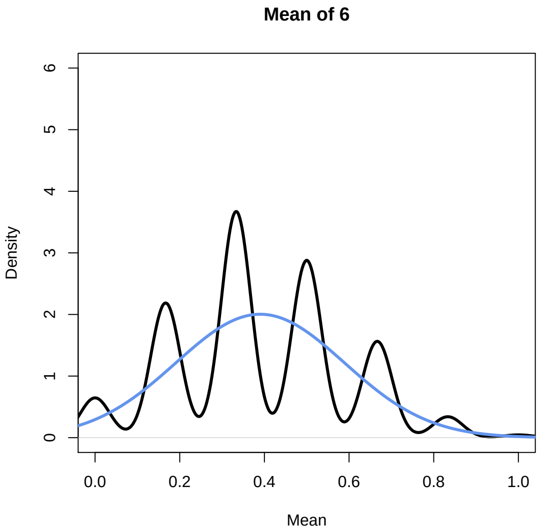

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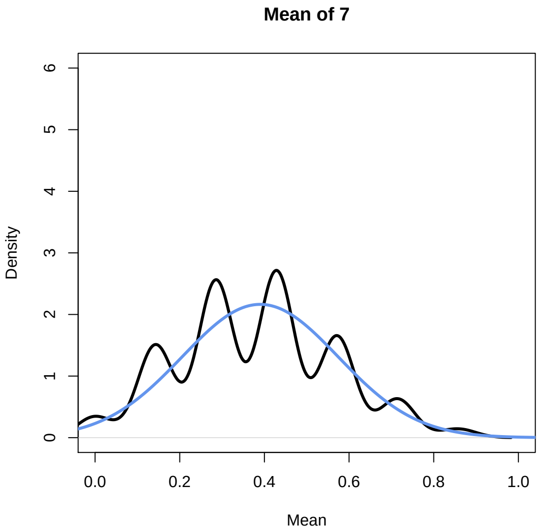

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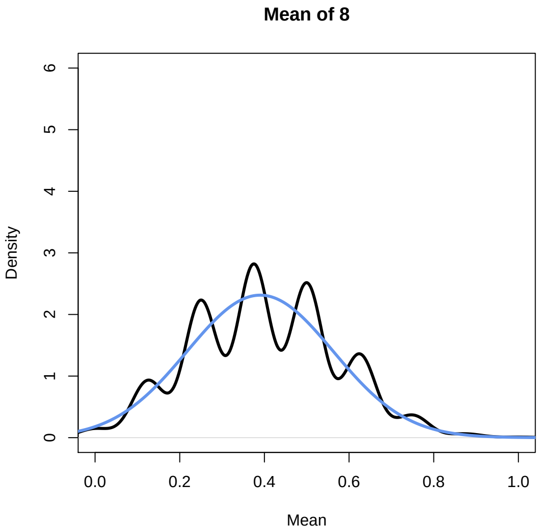

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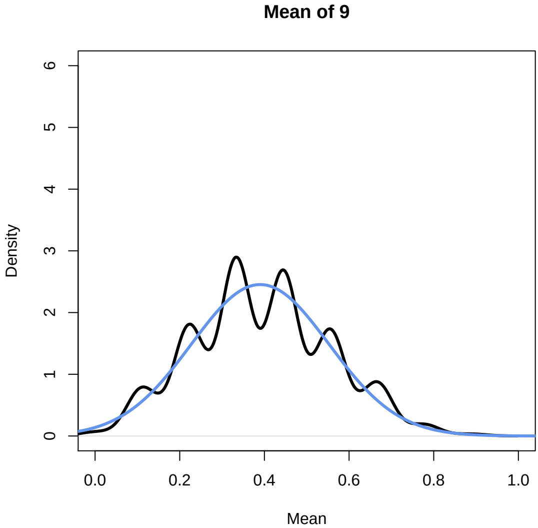

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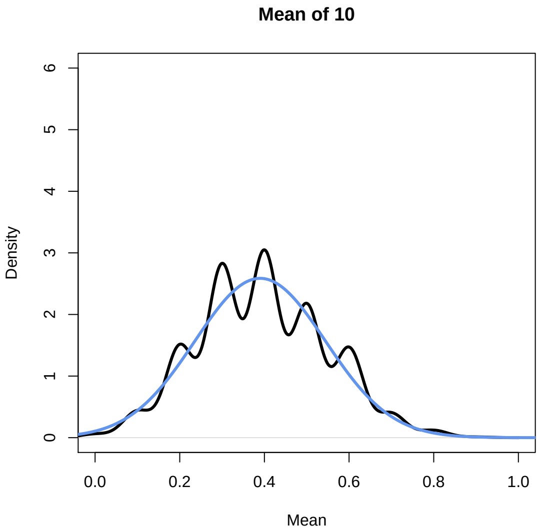

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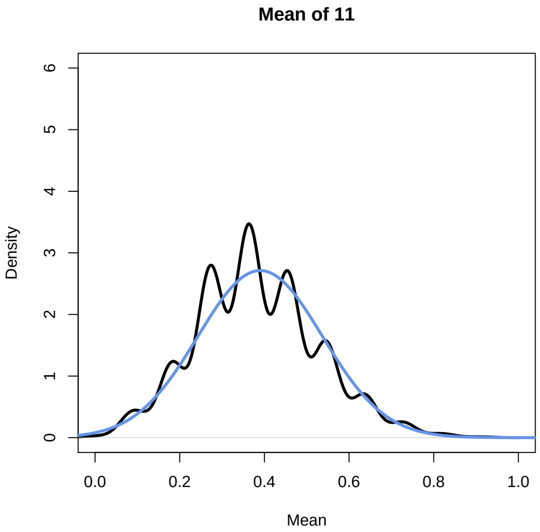

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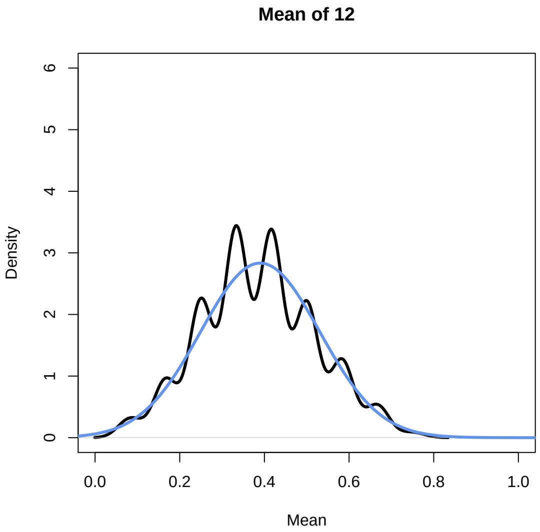

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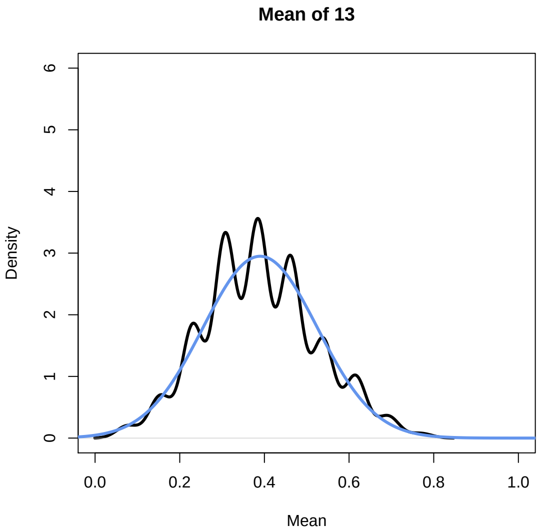

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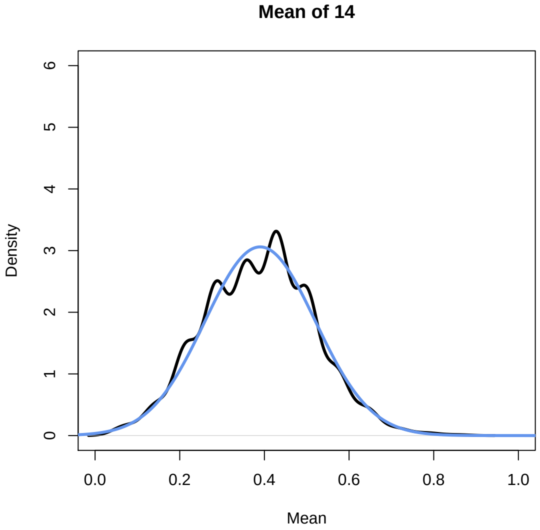

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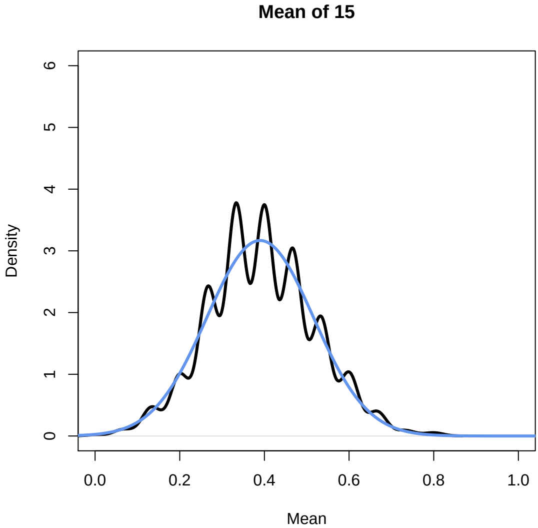

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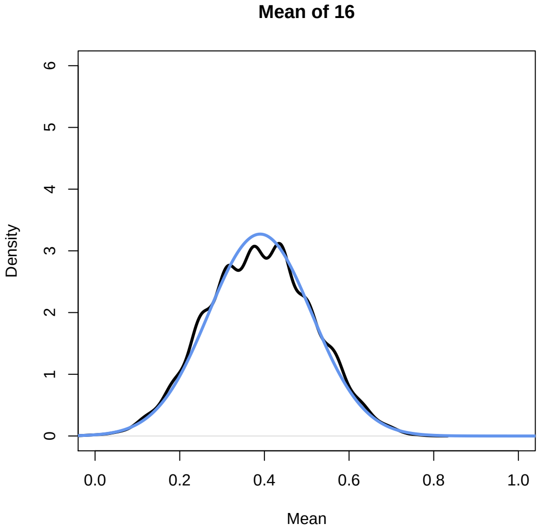

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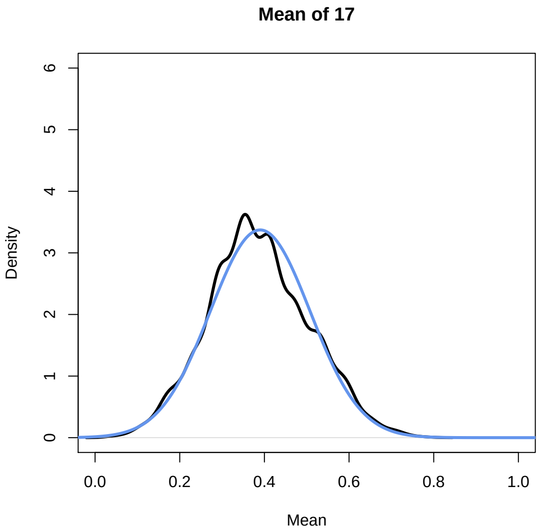

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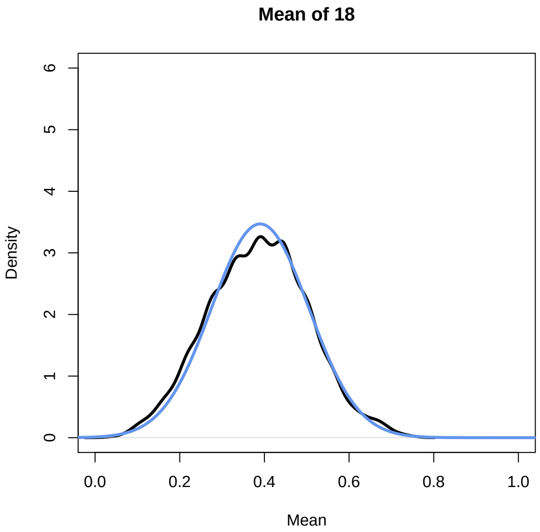

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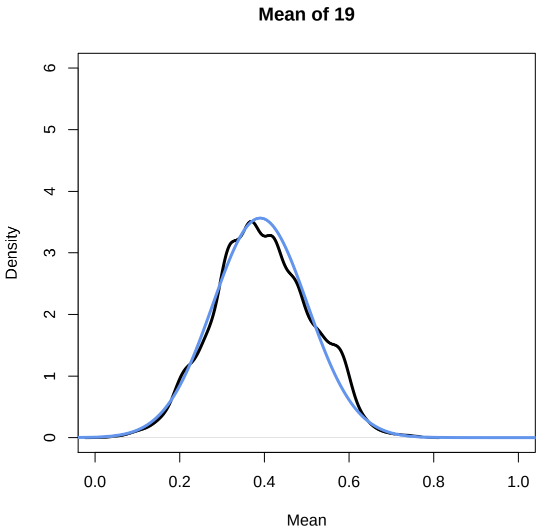

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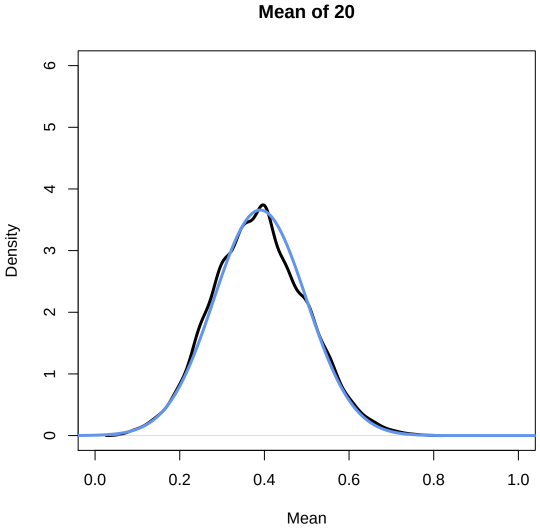

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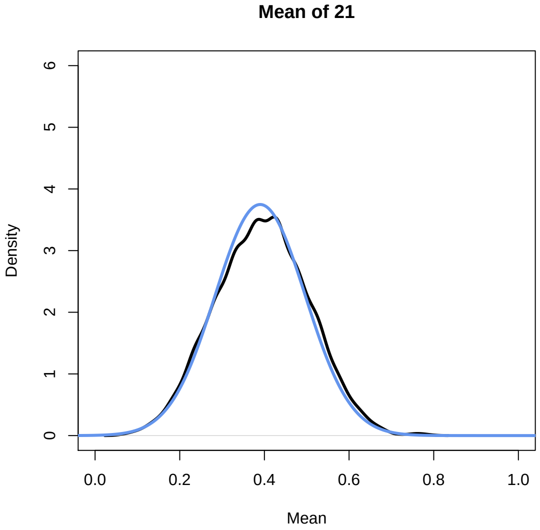

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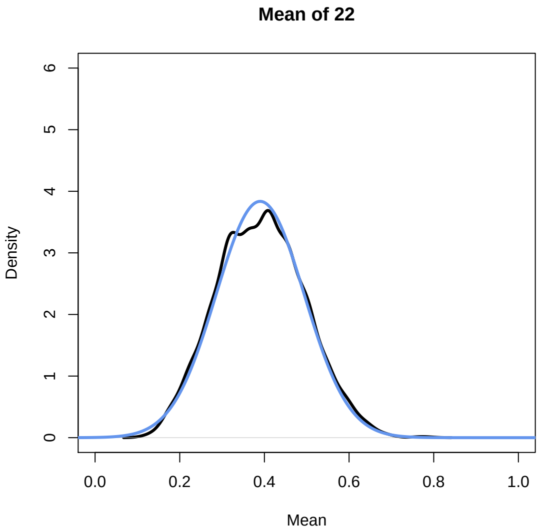

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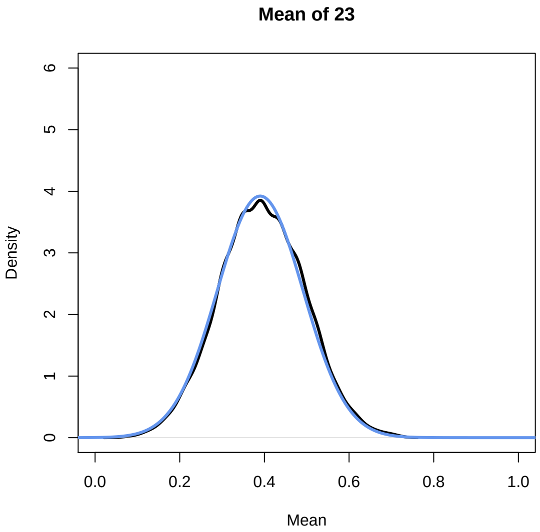

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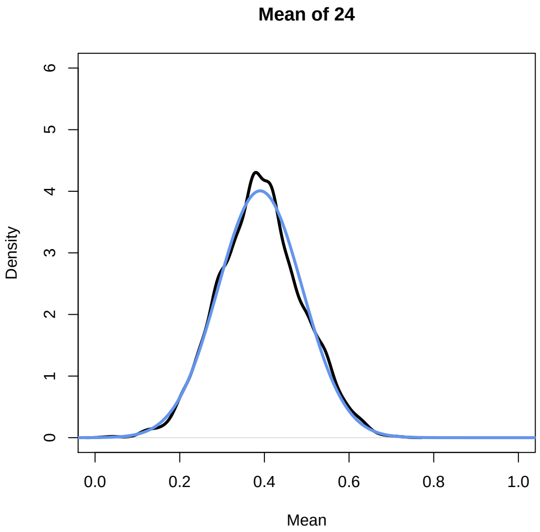

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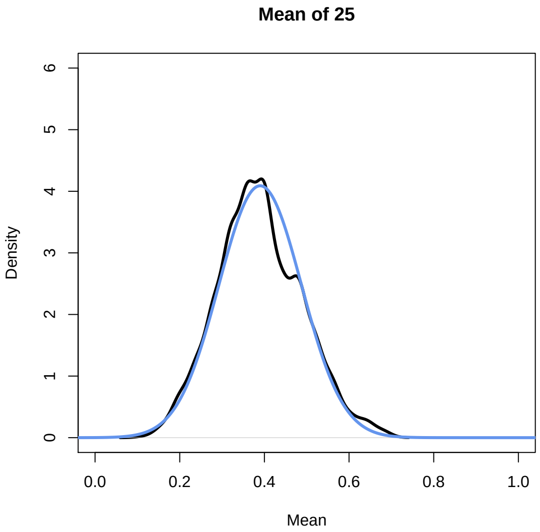

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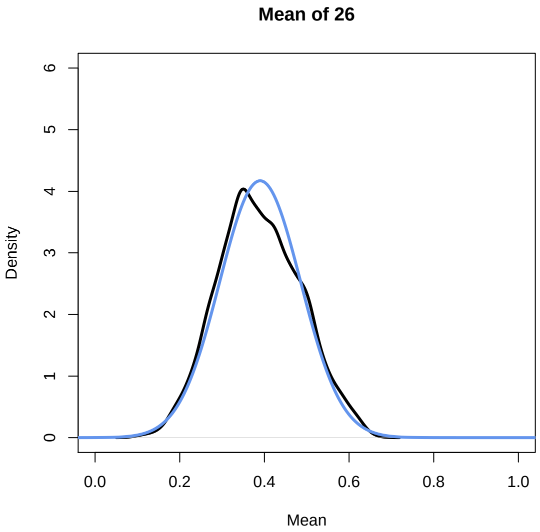

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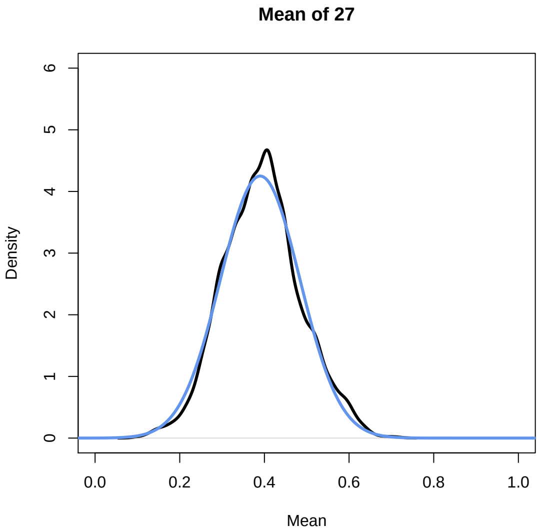

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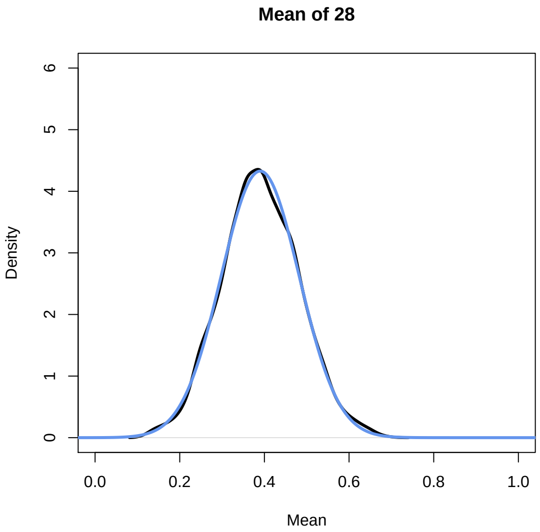

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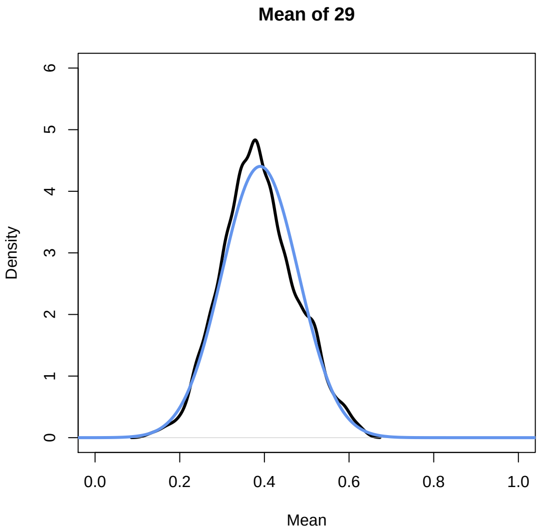

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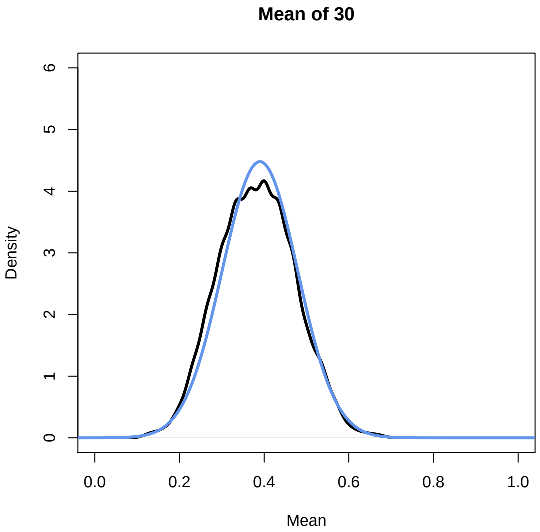

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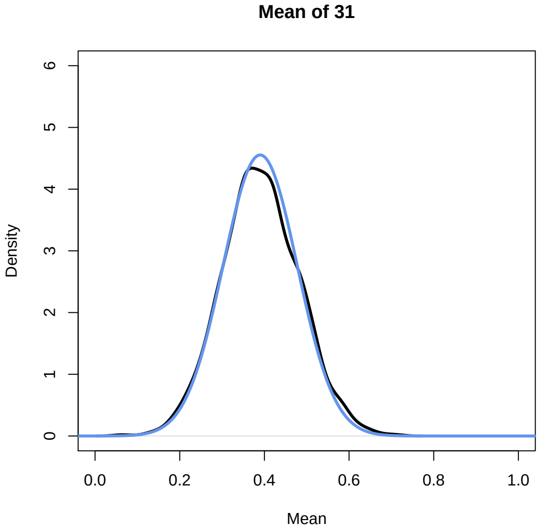

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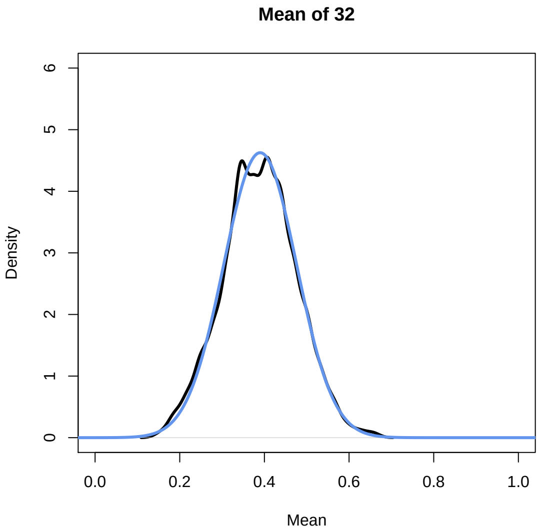

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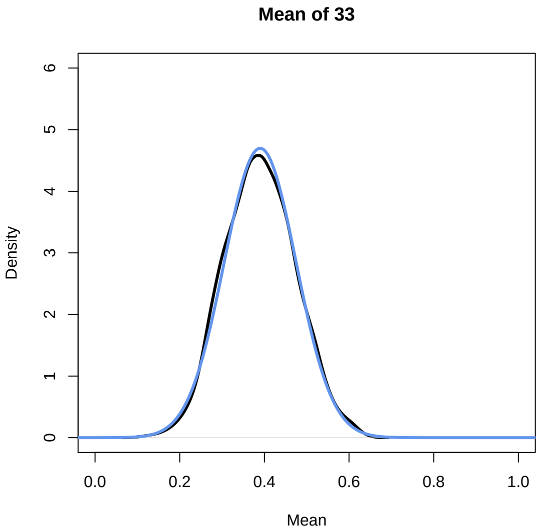

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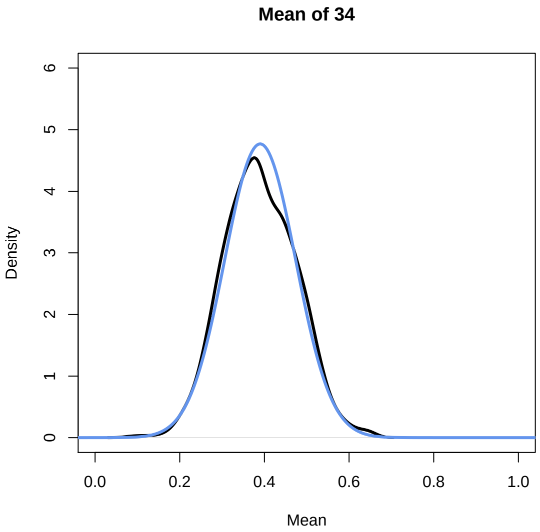

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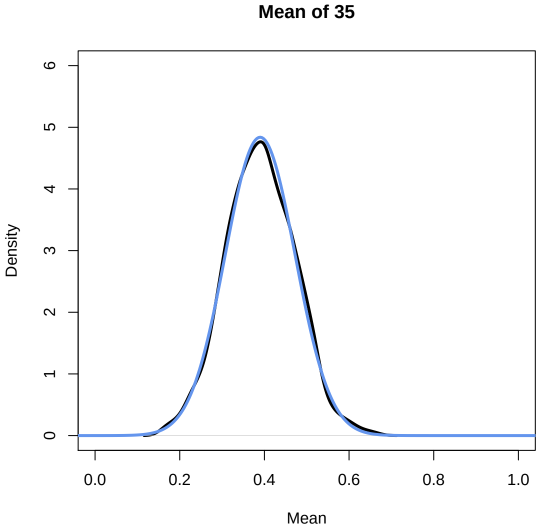

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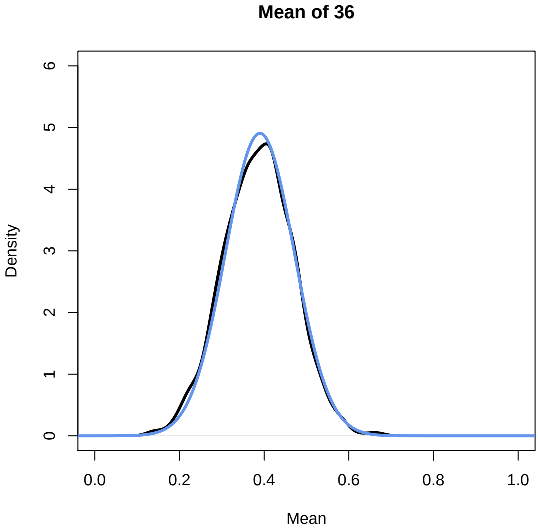

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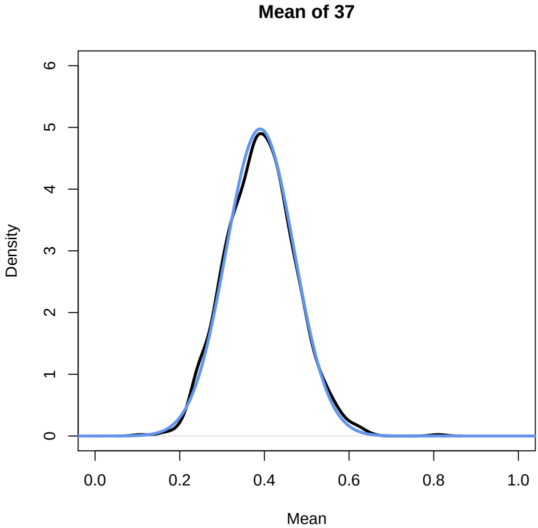

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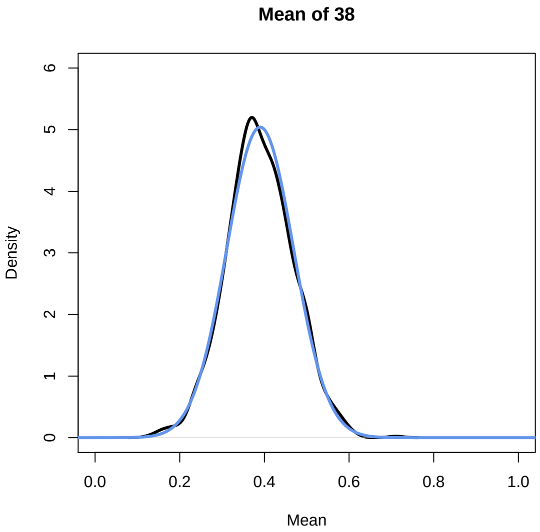

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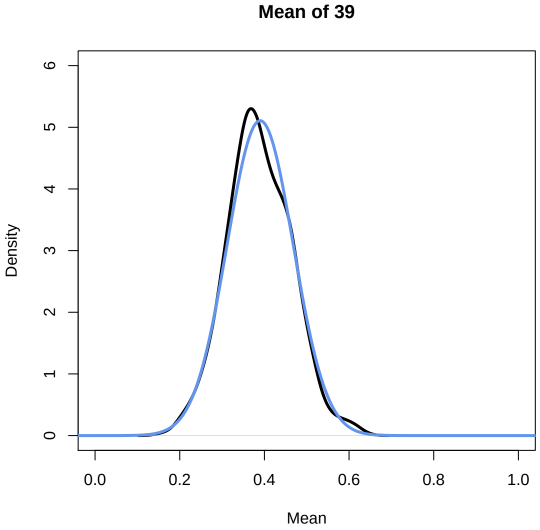

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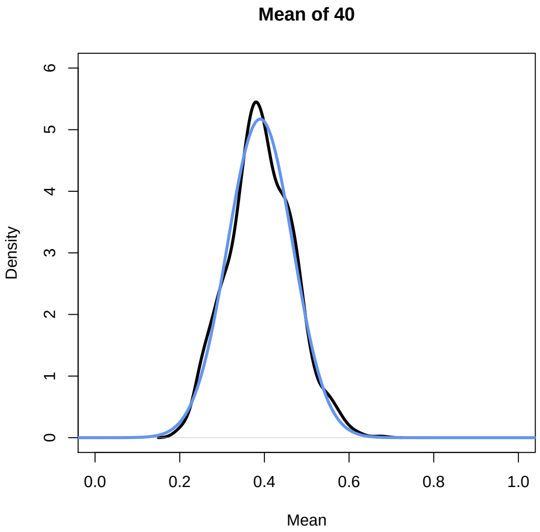

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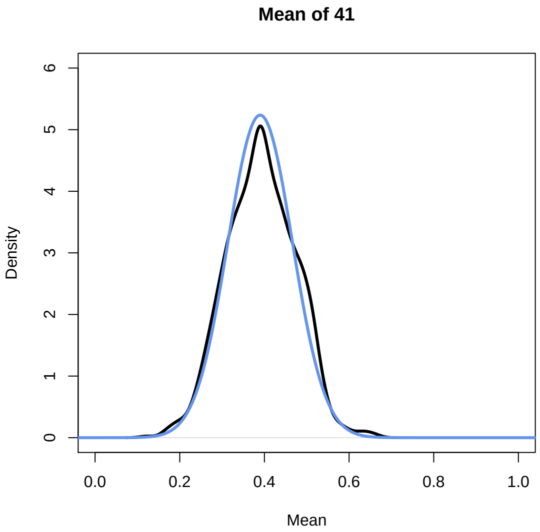

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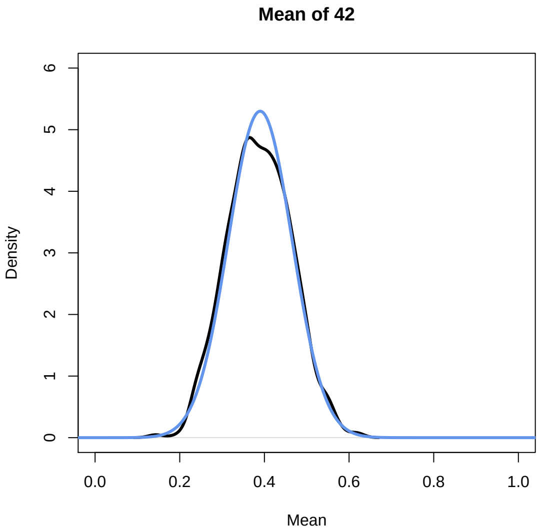

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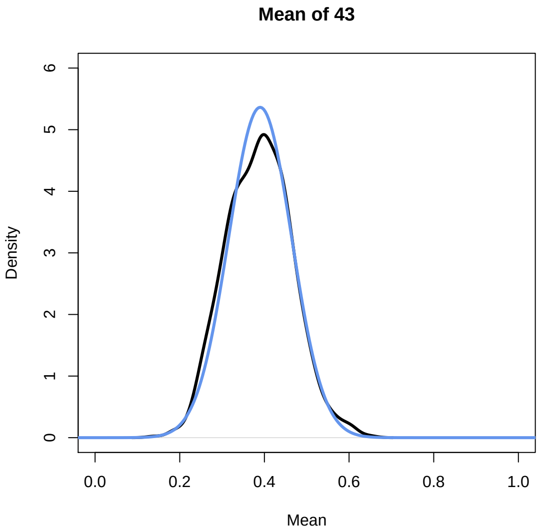

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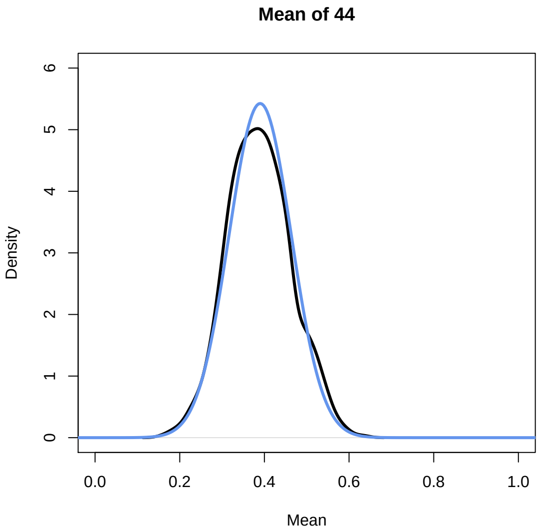

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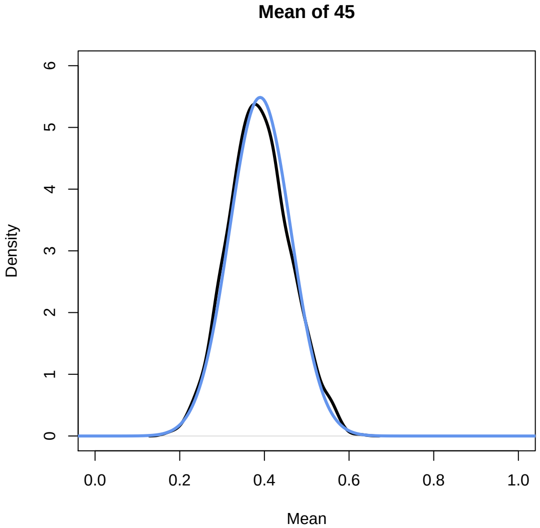

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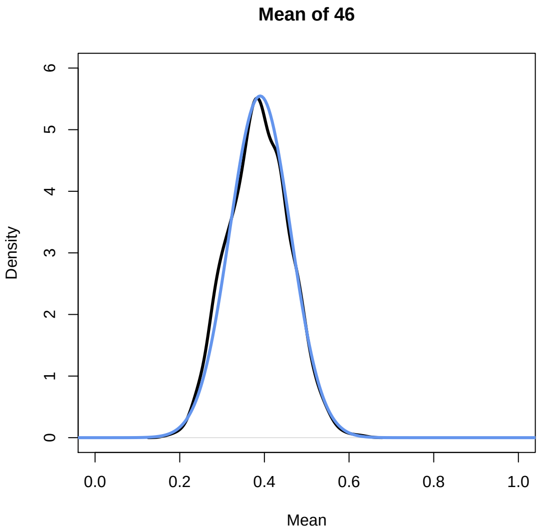

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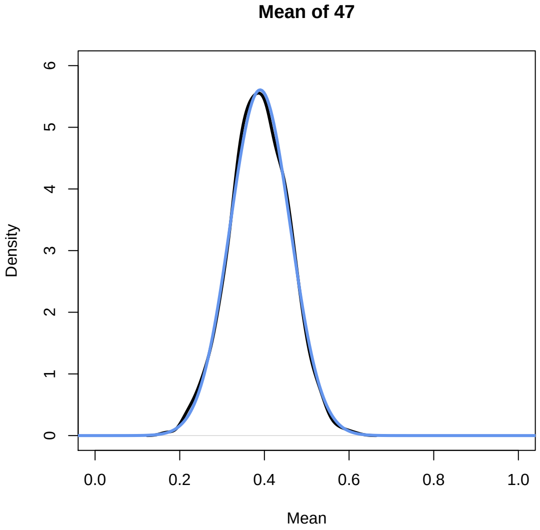

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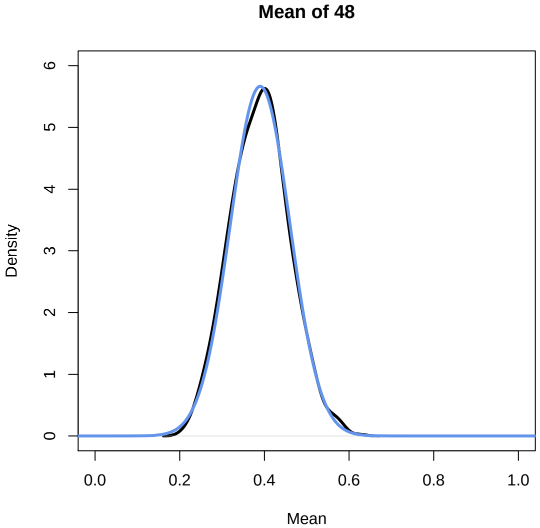

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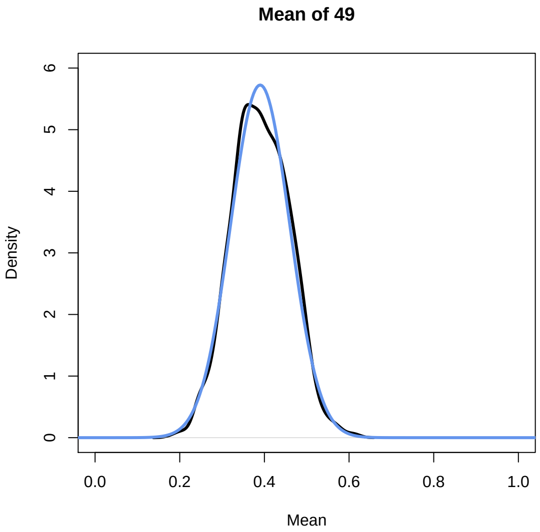

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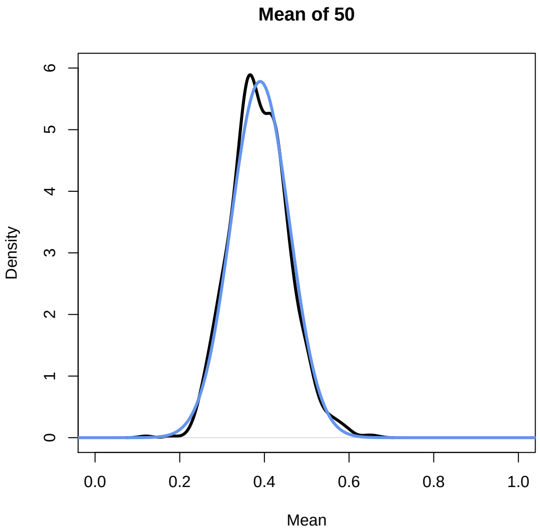

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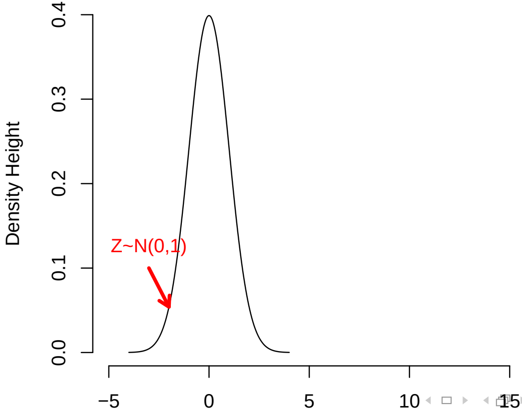
Proposition

Scale/Location. If $Z \sim N(0,1)$, then $X = aZ + b$ is,

$$X \sim Normal(b, a^2)$$

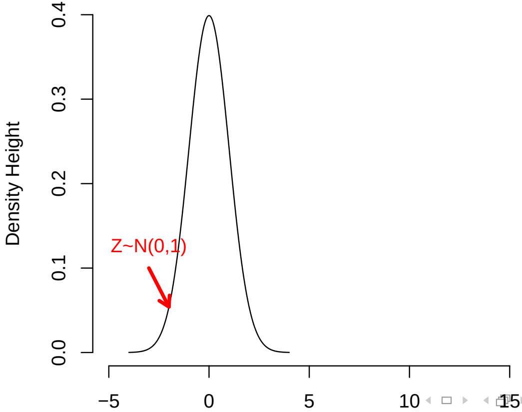
Intuition

Suppose $Z \sim \text{Normal}(0, 1)$.



Intuition

Suppose $Z \sim \text{Normal}(0, 1)$.
$Y = 2Z + 6$



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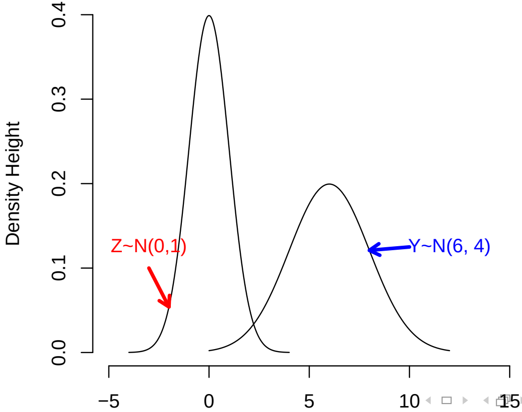

Proof: $Z \sim N(0,1)$ and $Y = aZ + b$, then $Y \sim N(b, a^2)$

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Expectation and Variance

Assume we know:

$$E[Z] = 0$$
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Assume we know:

$$\begin{aligned} E[Z] &= 0 \\ Var(Z) &= 1 \end{aligned}$$

This implies that, for $Y \sim \text{Normal}(\mu, \sigma^2)$

Expectation and Variance

Assume we know:

$$\begin{aligned} E[Z] &= 0 \\ Var(Z) &= 1 \end{aligned}$$

This implies that, for $Y \sim \text{Normal}(\mu, \sigma^2)$

$$E[Y] = E[\sigma Z + \mu]$$

Expectation and Variance

Assume we know:

$$
\begin{aligned}
E[Z] &= 0 \\
Var(Z) &= 1
\end{aligned}
$$

This implies that, for $Y \sim \text{Normal}(\mu, \sigma^2)$

$$
\begin{aligned}
E[Y] &= E[\sigma Z + \mu] \\
&= \sigma E[Z] + \mu
\end{aligned}
$$

Expectation and Variance

Assume we know:

$$\begin{aligned} E[Z] &= 0 \\ Var(Z) &= 1 \end{aligned}$$

This implies that, for $Y \sim \text{Normal}(\mu, \sigma^2)$

$$\begin{aligned} E[Y] &= E[\sigma Z + \mu] \\ &= \sigma E[Z] + \mu \\ &= \mu \end{aligned}$$

Expectation and Variance

Assume we know:

$$\begin{aligned} E[Z] &= 0 \\ Var(Z) &= 1 \end{aligned}$$

This implies that, for $Y \sim \text{Normal}(\mu, \sigma^2)$

$$\begin{aligned} E[Y] &= E[\sigma Z + \mu] \\ &= \sigma E[Z] + \mu \\ &= \mu \\ Var(Y) &= Var(\sigma Z + \mu) \end{aligned}$$

Expectation and Variance

Assume we know:

$$\begin{aligned} E[Z] &= 0 \\ Var(Z) &= 1 \end{aligned}$$

This implies that, for $Y \sim \text{Normal}(\mu, \sigma^2)$

$$\begin{aligned} E[Y] &= E[\sigma Z + \mu] \\ &= \sigma E[Z] + \mu \\ &= \mu \\ Var(Y) &= Var(\sigma Z + \mu) \\ &= \sigma^2 Var(Z) + Var(\mu) \end{aligned}$$

Expectation and Variance

Assume we know:

$$\begin{aligned} E[Z] &= 0 \\ Var(Z) &= 1 \end{aligned}$$

This implies that, for $Y \sim \text{Normal}(\mu, \sigma^2)$

$$\begin{aligned} E[Y] &= E[\sigma Z + \mu] \\ &= \sigma E[Z] + \mu \\ &= \mu \\ Var(Y) &= Var(\sigma Z + \mu) \\ &= \sigma^2 Var(Z) + Var(\mu) \\ &= \sigma^2 + 0 \end{aligned}$$

Expectation and Variance

Assume we know:

$$\begin{aligned} E[Z] &= 0 \\ Var(Z) &= 1 \end{aligned}$$

This implies that, for $Y \sim \text{Normal}(\mu, \sigma^2)$

$$\begin{aligned} E[Y] &= E[\sigma Z + \mu] \\ &= \sigma E[Z] + \mu \\ &= \mu \\ Var(Y) &= Var(\sigma Z + \mu) \\ &= \sigma^2 Var(Z) + Var(\mu) \\ &= \sigma^2 + 0 \\ &= \sigma^2 \end{aligned}$$

Back To Obama

Suppose $\mu = 0.39$ and $\sigma^2 = 0.0025$

Back To Obama

Suppose $\mu = 0.39$ and $\sigma^2 = 0.0025$
$P(Y \geq 0.45)$ (What is the probability it isn't that bad?) ?

Back To Obama

Suppose $\mu = 0.39$ and $\sigma^2 = 0.0025$
$P(Y \geq 0.45)$ (What is the probability it isn't that bad?) ?

$$P(Y \geq 0.45) \quad = \quad 1 - P(Y \leq 0.45)$$

Back To Obama

Suppose $\mu = 0.39$ and $\sigma^2 = 0.0025$
$P(Y \geq 0.45)$ (What is the probability it isn't that bad?) ?

$$\begin{aligned} P(Y \geq 0.45) &= 1 - P(Y \leq 0.45) \\ &= 1 - P(0.05Z + 0.39 \leq 0.45) \end{aligned}$$

Back To Obama

Suppose $\mu = 0.39$ and $\sigma^2 = 0.0025$
$P(Y \geq 0.45)$ (What is the probability it isn't that bad?) ?

$$
\begin{aligned}
P(Y \geq 0.45) &= 1 - P(Y \leq 0.45) \\
&= 1 - P(0.05Z + 0.39 \leq 0.45) \\
&= 1 - P(Z \leq \frac{0.45 - 0.39}{0.05})
\end{aligned}
$$

Back To Obama

Suppose $\mu = 0.39$ and $\sigma^2 = 0.0025$
$P(Y \geq 0.45)$ (What is the probability it isn't that bad?) ?

$$
\begin{aligned}
P(Y \geq 0.45) &= 1 - P(Y \leq 0.45) \\
&= 1 - P(0.05Z + 0.39 \leq 0.45) \\
&= 1 - P(Z \leq \frac{0.45 - 0.39}{0.05}) \\
&= 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{6/5} \exp(-z^2/2) dz
\end{aligned}
$$

Back To Obama

Suppose $\mu = 0.39$ and $\sigma^2 = 0.0025$
$P(Y \geq 0.45)$ (What is the probability it isn't that bad?) ?

$$\begin{aligned}
P(Y \geq 0.45) &= 1 - P(Y \leq 0.45) \\
&= 1 - P(0.05Z + 0.39 \leq 0.45) \\
&= 1 - P(Z \leq \frac{0.45 - 0.39}{0.05}) \\
&= 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{6/5} \exp(-z^2/2) dz \\
&= 1 - F_Z(\frac{6}{5})
\end{aligned}$$

Back To Obama

Suppose $\mu = 0.39$ and $\sigma^2 = 0.0025$
$P(Y \geq 0.45)$ (What is the probability it isn't that bad?) ?

$$
\begin{aligned}
P(Y \geq 0.45) &= 1 - P(Y \leq 0.45) \\
&= 1 - P(0.05Z + 0.39 \leq 0.45) \\
&= 1 - P(Z \leq \frac{0.45 - 0.39}{0.05}) \\
&= 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{6/5} \exp(-z^2/2) dz \\
&= 1 - F_Z(\frac{6}{5}) \\
&= 0.1150697
\end{aligned}
$$

Back To Obama

Via simulation:

< code >

```
draws<- rnorm(1e7, mean = 0.39, sd = sqrt(0.0025) )
greater<- which(draws>0.45)
p.45 <- length(greater)/1e7
print(p.45)
[1] 0.1149824
```

< / code >

The Gamma Function

Definition

Suppose $\alpha > 0$*. Then define* $\Gamma(\alpha)$ *as*

$$\Gamma(\alpha) \quad = \quad \int_0^{\infty} y^{\alpha-1} e^{-y} dy$$

- For $\alpha \in \{1, 2, 3, \ldots\}$
  $\Gamma(\alpha) = (\alpha - 1)!$
- $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Gamma Distribution

Suppose we have $\Gamma(\alpha)$,

Gamma Distribution

Suppose we have $\Gamma(\alpha)$,

$$\begin{aligned}
\frac{\Gamma(\alpha)}{\Gamma(\alpha)} &= \frac{\int_0^\infty y^{\alpha-1}e^{-y}dy}{\Gamma(\alpha)} \\
1 &= \int_0^\infty \frac{1}{\Gamma(\alpha)} y^{\alpha-1}e^{-y}dy
\end{aligned}$$

Gamma Distribution

Suppose we have $\Gamma(\alpha)$,

$$\begin{aligned}
\frac{\Gamma(\alpha)}{\Gamma(\alpha)} &= \frac{\int_0^\infty y^{\alpha-1}e^{-y}dy}{\Gamma(\alpha)} \\
1 &= \int_0^\infty \frac{1}{\Gamma(\alpha)} y^{\alpha-1}e^{-y}dy
\end{aligned}$$

Set $X = Y/\beta$

Gamma Distribution

Suppose we have $\Gamma(\alpha)$,

$$\begin{aligned}\frac{\Gamma(\alpha)}{\Gamma(\alpha)} &= \frac{\int_0^{\infty} y^{\alpha-1} e^{-y} dy}{\Gamma(\alpha)} \\ 1 &= \int_0^{\infty} \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y} dy\end{aligned}$$

Set $X = Y/\beta$

$$F(x) = P(X \leq x) \;=\; P(Y/\beta \leq x)$$

Gamma Distribution

Suppose we have $\Gamma(\alpha)$,

$$\begin{aligned} \frac{\Gamma(\alpha)}{\Gamma(\alpha)} &= \frac{\int_0^{\infty} y^{\alpha-1} e^{-y} dy}{\Gamma(\alpha)} \\ 1 &= \int_0^{\infty} \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y} dy \end{aligned}$$

Set $X = Y/\beta$

$$\begin{aligned} F(x) = P(X \leq x) &= P(Y/\beta \leq x) \\ &= P(Y \leq x\beta) \end{aligned}$$

Gamma Distribution

Suppose we have $\Gamma(\alpha)$,

$$\begin{aligned}
\frac{\Gamma(\alpha)}{\Gamma(\alpha)} &= \frac{\int_0^\infty y^{\alpha-1}e^{-y}dy}{\Gamma(\alpha)} \\
1 &= \int_0^\infty \frac{1}{\Gamma(\alpha)} y^{\alpha-1}e^{-y}dy
\end{aligned}$$

Set $X = Y/\beta$

$$\begin{aligned}
F(x) = P(X \leq x) &= P(Y/\beta \leq x) \\
&= P(Y \leq x\beta) \\
&= F_Y(x\beta)
\end{aligned}$$

Gamma Distribution

Suppose we have $\Gamma(\alpha)$,

$$\begin{aligned}
\frac{\Gamma(\alpha)}{\Gamma(\alpha)} &= \frac{\int_0^{\infty} y^{\alpha-1} e^{-y} dy}{\Gamma(\alpha)} \\
1 &= \int_0^{\infty} \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y} dy
\end{aligned}$$

Set $X = Y/\beta$

$$\begin{aligned}
F(x) = P(X \leq x) &= P(Y/\beta \leq x) \\
&= P(Y \leq x\beta) \\
&= F_Y(x\beta) \\
\frac{\partial F_Y(x\beta)}{\partial x} &= f_Y(x\beta)\beta
\end{aligned}$$

Gamma Distribution

Suppose we have $\Gamma(\alpha)$,

$$\begin{aligned}
\frac{\Gamma(\alpha)}{\Gamma(\alpha)} &= \frac{\int_0^\infty y^{\alpha-1} e^{-y} dy}{\Gamma(\alpha)} \\
1 &= \int_0^\infty \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y} dy
\end{aligned}$$

Set $X = Y/\beta$

$$\begin{aligned}
F(x) = P(X \leq x) &= P(Y/\beta \leq x) \\
&= P(Y \leq x\beta) \\
&= F_Y(x\beta) \\
\frac{\partial F_Y(x\beta)}{\partial x} &= f_Y(x\beta)\beta
\end{aligned}$$

The result is:

Gamma Distribution

Suppose we have $\Gamma(\alpha)$,

$$\begin{aligned}
\frac{\Gamma(\alpha)}{\Gamma(\alpha)} &= \frac{\int_0^{\infty} y^{\alpha-1} e^{-y} dy}{\Gamma(\alpha)} \\
1 &= \int_0^{\infty} \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y} dy
\end{aligned}$$

Set $X = Y/\beta$

$$\begin{aligned}
F(x) = P(X \leq x) &= P(Y/\beta \leq x) \\
&= P(Y \leq x\beta) \\
&= F_Y(x\beta) \\
\frac{\partial F_Y(x\beta)}{\partial x} &= f_Y(x\beta)\beta
\end{aligned}$$

The result is:

$$f(x|\alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-x\beta}$$

Definition

Suppose X is a continuous random variable, with $X \geq 0$. Then if the pdf of X is

$$f(x|\alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-x\beta}$$

if $x \geq 0$ and 0 otherwise, we will say X is a Gamma distribution.

$$X \sim Gamma(\alpha, \beta)$$

Gamma Distribution

Suppose $X \sim \text{Gamma}(\alpha, \beta)$

Gamma Distribution

Suppose $X \sim \text{Gamma}(\alpha, \beta)$

$$E[X] = \frac{\alpha}{\beta}$$

Gamma Distribution

Suppose $X \sim \text{Gamma}(\alpha, \beta)$

$$
\begin{aligned}
E[X] &= \frac{\alpha}{\beta} \\
\text{var}(X) &= \frac{\alpha}{\beta^2}
\end{aligned}
$$

Gamma Distribution

Suppose $X \sim \text{Gamma}(\alpha, \beta)$

$$\begin{aligned} E[X] &= \frac{\alpha}{\beta} \\ \text{var}(X) &= \frac{\alpha}{\beta^2} \end{aligned}$$

Suppose $\alpha = 1$ and $\beta = \lambda$. If

Gamma Distribution

Suppose $X \sim \text{Gamma}(\alpha, \beta)$

$$\begin{aligned} E[X] &= \frac{\alpha}{\beta} \\ \text{var}(X) &= \frac{\alpha}{\beta^2} \end{aligned}$$

Suppose $\alpha = 1$ and $\beta = \lambda$. If

$$X \sim \text{Gamma}(1, \lambda)$$

Gamma Distribution

Suppose $X \sim \text{Gamma}(\alpha, \beta)$

$$\begin{aligned} E[X] &= \frac{\alpha}{\beta} \\ \text{var}(X) &= \frac{\alpha}{\beta^2} \end{aligned}$$

Suppose $\alpha = 1$ and $\beta = \lambda$. If

$$\begin{aligned} X &\sim \text{Gamma}(1, \lambda) \\ f(x|1, \lambda) &= \lambda e^{-x\lambda} \end{aligned}$$

Gamma Distribution

Suppose $X \sim \text{Gamma}(\alpha, \beta)$

$$\begin{aligned} E[X] &= \frac{\alpha}{\beta} \\ \text{var}(X) &= \frac{\alpha}{\beta^2} \end{aligned}$$

Suppose $\alpha = 1$ and $\beta = \lambda$. If

$$\begin{aligned} X &\sim \text{Gamma}(1, \lambda) \\ f(x|1, \lambda) &= \lambda e^{-x\lambda} \end{aligned}$$

We will say

Gamma Distribution

Suppose $X \sim \text{Gamma}(\alpha, \beta)$

$$\begin{aligned} E[X] &= \frac{\alpha}{\beta} \\ \text{var}(X) &= \frac{\alpha}{\beta^2} \end{aligned}$$

Suppose $\alpha = 1$ and $\beta = \lambda$. If

$$\begin{aligned} X &\sim \text{Gamma}(1, \lambda) \\ f(x|1, \lambda) &= \lambda e^{-x\lambda} \end{aligned}$$

We will say

$$X \sim \text{Exponential}(\lambda)$$

Properties of Gamma Distributions

Proposition

Suppose we have a sequence of independent random variables, with

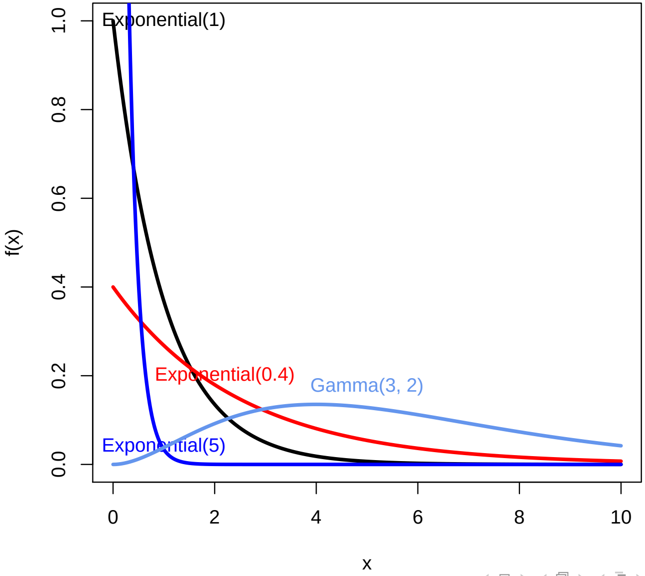
$$X_i \sim Gamma(\alpha_i, \beta)$$

Then

$$Y = \sum_{i=1}^{N} X_i$$

$Y \sim Gamma(\sum_{i=1}^{N} \alpha_i, \beta)$

We can evaluate in R with `dgamma` and simulate with `rgamma`
$X \sim \text{Gamma}(3, 5)$ and we evaluate at 3,
```
dgamma(3, shape= 3, rate = 5)
```
and we can simulate with
```
rgamma(1000, shape = 3, rate = 5)
```


Exponential(1)
Exponential(0.4)
Gamma(3, 2)
Exponential(5)
f(x)
x
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10

χ^2 Distribution

Suppose $Z \sim \text{Normal}(0, 1)$.

χ^2 Distribution

Suppose $Z \sim \text{Normal}(0, 1)$.
Consider $X = Z^2$

χ^2 Distribution

Suppose $Z \sim \text{Normal}(0, 1)$.
Consider $X = Z^2$

$$F_X(x) = P(X \leq x)$$

χ^2 Distribution

Suppose $Z \sim \text{Normal}(0, 1)$.
Consider $X = Z^2$

$$\begin{aligned} F_X(x) &= P(X \leq x) \\ &= P(Z^2 \leq x) \end{aligned}$$

χ^2 Distribution

Suppose $Z \sim \text{Normal}(0, 1)$.
Consider $X = Z^2$

$$\begin{aligned} F_X(x) &= P(X \leq x) \\ &= P(Z^2 \leq x) \\ &= P(-\sqrt{x} \leq Z \leq x) \end{aligned}$$

χ^2 Distribution

Suppose $Z \sim \text{Normal}(0, 1)$.
Consider $X = Z^2$

$$
\begin{aligned}
F_X(x) &= P(X \leq x) \\
&= P(Z^2 \leq x) \\
&= P(-\sqrt{x} \leq Z \leq x) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{x}}^{\sqrt{x}} e^{-\frac{z^2}{2}} dz
\end{aligned}
$$

χ^2 Distribution

Suppose $Z \sim \text{Normal}(0, 1)$.
Consider $X = Z^2$

$$
\begin{aligned}
F_X(x) &= P(X \leq x) \\
&= P(Z^2 \leq x) \\
&= P(-\sqrt{x} \leq Z \leq x) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{x}}^{\sqrt{x}} e^{-\frac{z^2}{2}} dz \\
&= F_Z(\sqrt{x}) - F_Z(-\sqrt{x})
\end{aligned}
$$

χ^2 Distribution

Suppose $Z \sim \text{Normal}(0, 1)$.
Consider $X = Z^2$

$$
\begin{aligned}
F_X(x) &= P(X \leq x) \\
&= P(Z^2 \leq x) \\
&= P(-\sqrt{x} \leq Z \leq x) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{x}}^{\sqrt{x}} e^{-\frac{z^2}{2}} dz \\
&= F_Z(\sqrt{x}) - F_Z(-\sqrt{x})
\end{aligned}
$$

The pdf then is

χ^2 Distribution

Suppose $Z \sim \text{Normal}(0, 1)$.
Consider $X = Z^2$

$$
\begin{aligned}
F_X(x) &= P(X \leq x) \\
&= P(Z^2 \leq x) \\
&= P(-\sqrt{x} \leq Z \leq x) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{x}}^{\sqrt{x}} e^{-\frac{z^2}{2}} dz \\
&= F_Z(\sqrt{x}) - F_Z(-\sqrt{x})
\end{aligned}
$$

The pdf then is

$$
\frac{\partial F_X(x)}{\partial x} = f_Z(\sqrt{x})\frac{1}{2\sqrt{x}} + f_Z(-\sqrt{x})\frac{1}{2\sqrt{x}}
$$

χ^2 Distribution

$$\frac{\partial F_X(x)}{\partial x} = f_Z(\sqrt{x})\frac{1}{2\sqrt{x}} + f_Z(-\sqrt{x})\frac{1}{2\sqrt{x}}$$

χ^2 Distribution

$$\begin{aligned}
\frac{\partial F_X(x)}{\partial x} &= f_Z(\sqrt{x})\frac{1}{2\sqrt{x}} + f_Z(-\sqrt{x})\frac{1}{2\sqrt{x}} \\
&= \frac{1}{\sqrt{x}}\frac{1}{2\sqrt{2\pi}}(2e^{-\frac{x}{2}})
\end{aligned}$$

χ^2 Distribution

$$
\begin{aligned}
\frac{\partial F_X(x)}{\partial x} &= f_Z(\sqrt{x})\frac{1}{2\sqrt{x}} + f_Z(-\sqrt{x})\frac{1}{2\sqrt{x}} \\
&= \frac{1}{\sqrt{x}}\frac{1}{2\sqrt{2\pi}}(2e^{-\frac{x}{2}}) \\
&= \frac{1}{\sqrt{x}}\frac{1}{\sqrt{2\pi}}(e^{-\frac{x}{2}})
\end{aligned}
$$

χ^2 Distribution

$$
\begin{aligned}
\frac{\partial F_X(x)}{\partial x} &= f_Z(\sqrt{x})\frac{1}{2\sqrt{x}} + f_Z(-\sqrt{x})\frac{1}{2\sqrt{x}} \\
&= \frac{1}{\sqrt{x}}\frac{1}{2\sqrt{2\pi}}(2e^{-\frac{x}{2}}) \\
&= \frac{1}{\sqrt{x}}\frac{1}{\sqrt{2\pi}}(e^{-\frac{x}{2}}) \\
&= \frac{(\frac{1}{2})^{1/2}}{\Gamma(\frac{1}{2})}\left(x^{1/2-1}e^{-\frac{x}{2}}\right)
\end{aligned}
$$

χ^2 Distribution

$$
\begin{aligned}
\frac{\partial F_X(x)}{\partial x} &= f_Z(\sqrt{x})\frac{1}{2\sqrt{x}} + f_Z(-\sqrt{x})\frac{1}{2\sqrt{x}} \\
&= \frac{1}{\sqrt{x}}\frac{1}{2\sqrt{2\pi}}(2e^{-\frac{x}{2}}) \\
&= \frac{1}{\sqrt{x}}\frac{1}{\sqrt{2\pi}}(e^{-\frac{x}{2}}) \\
&= \frac{(\frac{1}{2})^{1/2}}{\Gamma(\frac{1}{2})}\left(x^{1/2-1}e^{-\frac{x}{2}}\right)
\end{aligned}
$$

$X \sim \text{Gamma}(1/2, 1/2)$

χ^2 Distribution

$$
\begin{aligned}
\frac{\partial F_X(x)}{\partial x} &= f_Z(\sqrt{x})\frac{1}{2\sqrt{x}} + f_Z(-\sqrt{x})\frac{1}{2\sqrt{x}} \\
&= \frac{1}{\sqrt{x}}\frac{1}{2\sqrt{2\pi}}(2e^{-\frac{x}{2}}) \\
&= \frac{1}{\sqrt{x}}\frac{1}{\sqrt{2\pi}}(e^{-\frac{x}{2}}) \\
&= \frac{(\frac{1}{2})^{1/2}}{\Gamma(\frac{1}{2})}\left(x^{1/2-1}e^{-\frac{x}{2}}\right)
\end{aligned}
$$

$X \sim \text{Gamma}(1/2, 1/2)$
Then if $X = \sum_{i=1}^{N} Z^2$

χ^2 Distribution

$$
\begin{aligned}
\frac{\partial F_X(x)}{\partial x} &= f_Z(\sqrt{x})\frac{1}{2\sqrt{x}} + f_Z(-\sqrt{x})\frac{1}{2\sqrt{x}} \\
&= \frac{1}{\sqrt{x}}\frac{1}{2\sqrt{2\pi}}(2e^{-\frac{x}{2}}) \\
&= \frac{1}{\sqrt{x}}\frac{1}{\sqrt{2\pi}}(e^{-\frac{x}{2}}) \\
&= \frac{(\frac{1}{2})^{1/2}}{\Gamma(\frac{1}{2})}\left(x^{1/2-1}e^{-\frac{x}{2}}\right)
\end{aligned}
$$

$X \sim \text{Gamma}(1/2, 1/2)$
Then if $X = \sum_{i=1}^{N} Z^2$
$X \sim \text{Gamma}(n/2, 1/2)$

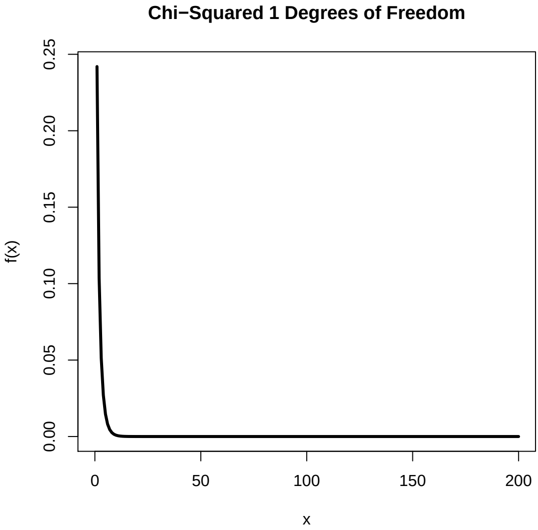
Definition

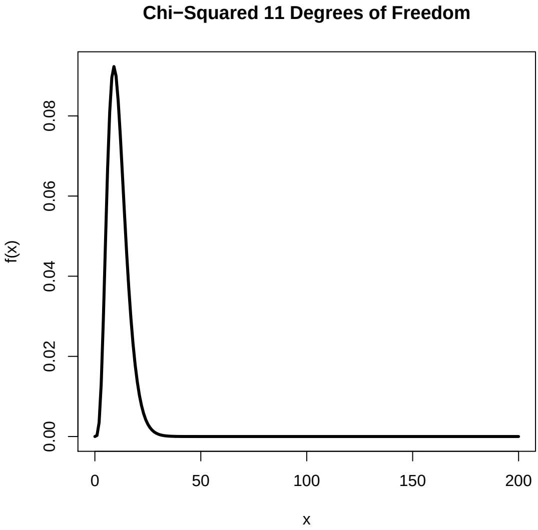
Suppose X *is a continuous random variable with* $X \geq 0$, *with pdf*

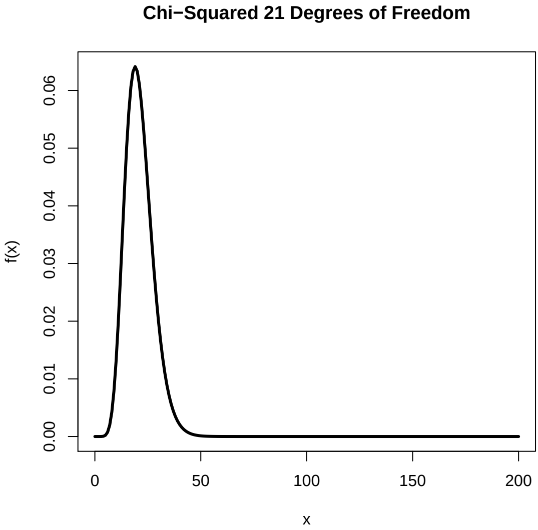
$$f(x) \;=\; \frac{1}{2^{n/2}\Gamma(n/2)} x^{n/2-1} e^{-x/2}$$

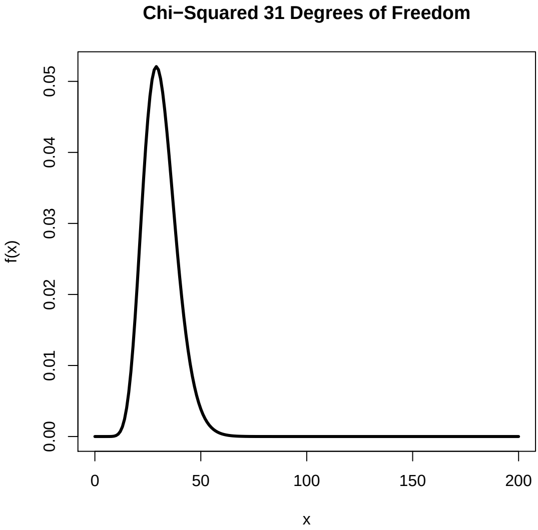
Then we will say X *is a* χ^2 *distribution with n degrees of freedom. Equivalently,*

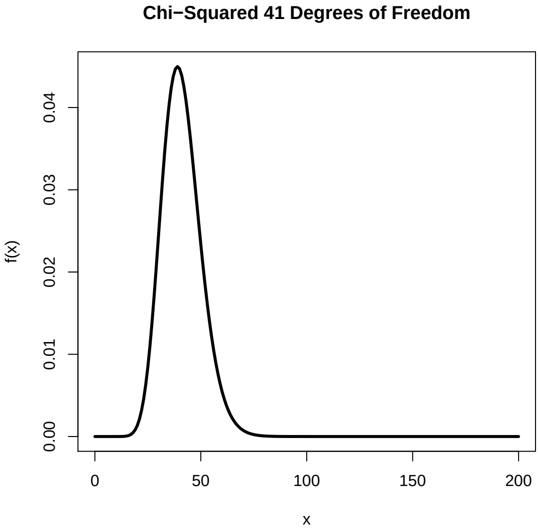
$$X \;\sim\; \chi^2(n)$$

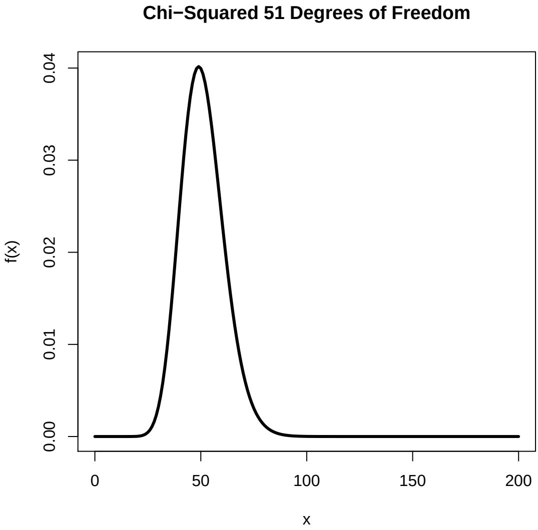

Chi−Squared 1 Degrees of Freedom
f(x)
0.00
0.05
0.10
0.15
0.20
0.25
0
50
100
150
200
x

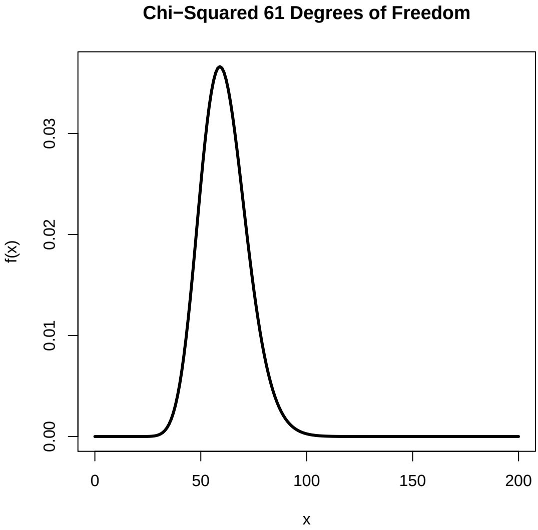

Chi−Squared 11 Degrees of Freedom
f(x)
0.00
0.02
0.04
0.06
0.08
0
50
100
150
200
x

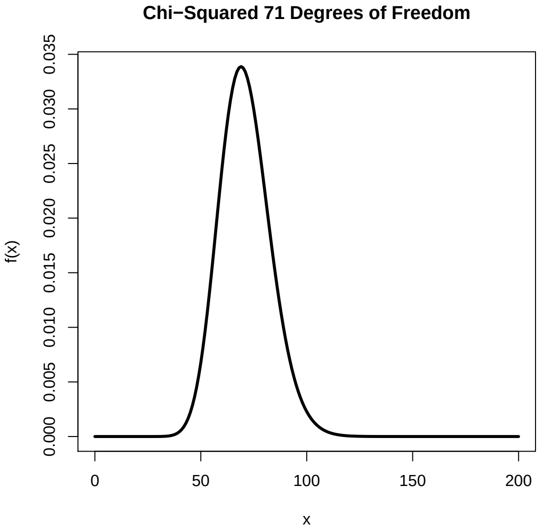

Chi−Squared 21 Degrees of Freedom
f(x)
0.00
0.01
0.02
0.03
0.04
0.05
0.06
0
50
100
150
200
x

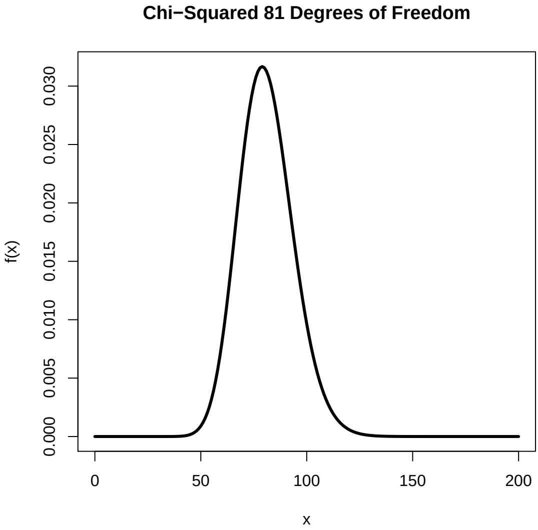

Chi−Squared 31 Degrees of Freedom
f(x)
0.00
0.01
0.02
0.03
0.04
0.05
0
50
100
150
200
x

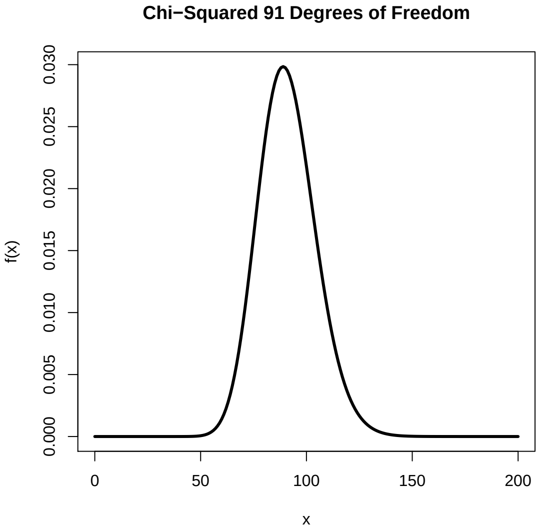

Chi−Squared 41 Degrees of Freedom
f(x)
0.00
0.01
0.02
0.03
0.04
0
50
100
150
200
x


Chi−Squared 51 Degrees of Freedom
f(x)
0.00
0.01
0.02
0.03
0.04
0
50
100
150
200
x


Chi−Squared 61 Degrees of Freedom
f(x)
0.00
0.01
0.02
0.03
0
50
100
150
200
x


Chi−Squared 71 Degrees of Freedom
f(x)
0.000
0.005
0.010
0.015
0.020
0.025
0.030
0.035
0
50
100
150
200
x


Chi−Squared 81 Degrees of Freedom
f(x)
0.000
0.005
0.010
0.015
0.020
0.025
0.030
0
50
100
150
200
x


Chi−Squared 91 Degrees of Freedom
f(x)
0.000
0.005
0.010
0.015
0.020
0.025
0.030
0
50
100
150
200
x

χ^2 Properties

Suppose $X \sim \chi^2(n)$

$$
\begin{aligned}
E[X] &= E\left[\sum_{i=1}^{N} Z_i^2\right] \\
&= \sum_{i=1}^{N} E[Z_i^2] \\
\text{var}(Z_i) &= E[Z_i^2] - E[Z_i]^2 \\
1 &= E[Z_i^2] - 0 \\
E[X] &= n
\end{aligned}
$$

χ^2 Properties

$$
\begin{aligned}
\text{var}(X) &= \sum_{i=1}^{N} \text{var}(Z_i^2) \\
&= \sum_{i=1}^{N} \left(E[Z_i^4] - E[Z_i]^2\right) \\
&= \sum_{i=1}^{N} (3-1) = 2n
\end{aligned}
$$

We will use the χ^2 across statistics.

Student's t-Distribution

Definition

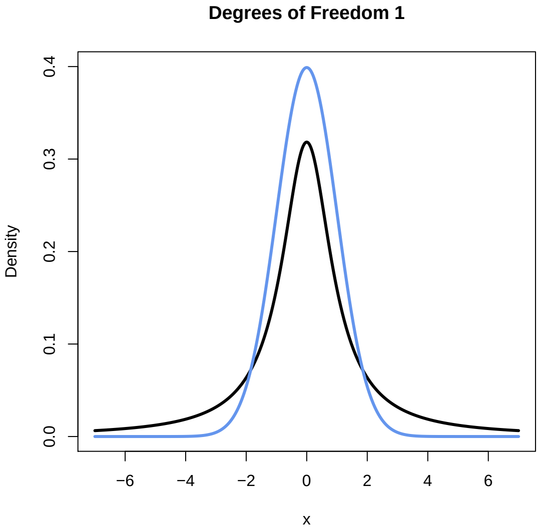
Suppose $Z \sim Normal(0,1)$ and $U \sim \chi^2(n)$. Define the random variable Y as,

$$Y = \frac{Z}{\sqrt{\frac{U}{n}}}$$

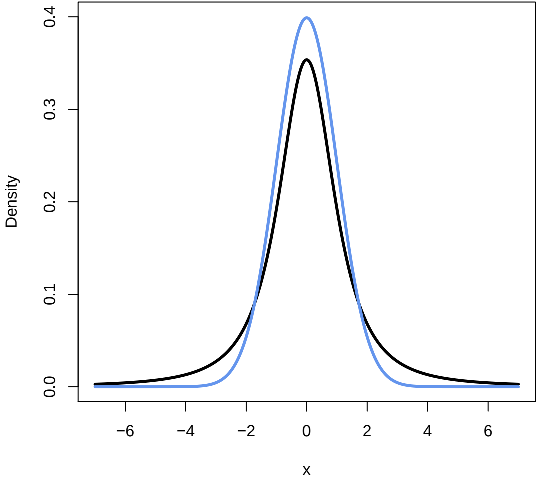
If Z and U are independent then $Y \sim t(n)$, with pdf

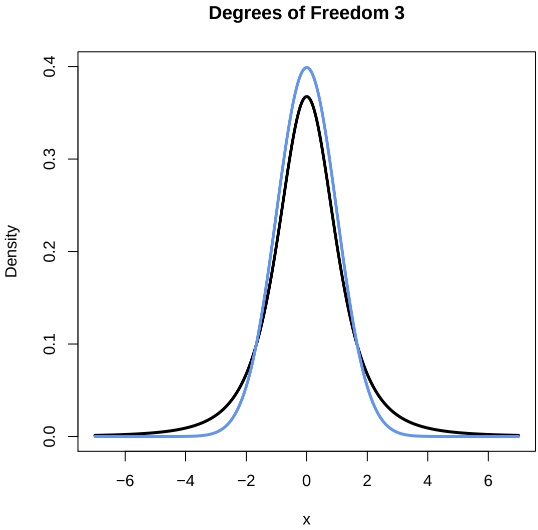
$$f(x) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n}\Gamma(\frac{n}{2})} \left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}}$$

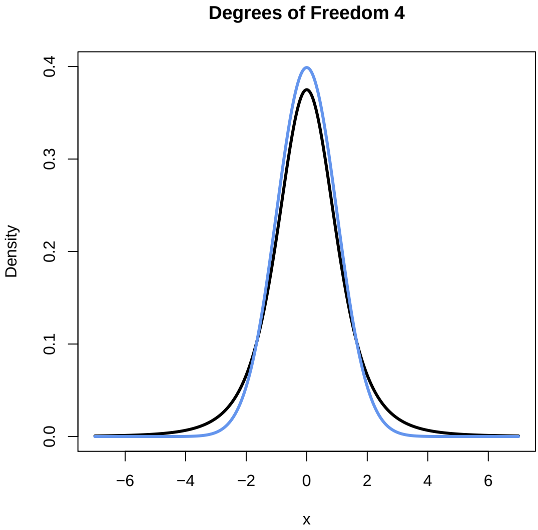
We will use the t-distribution extensively for test-statistics

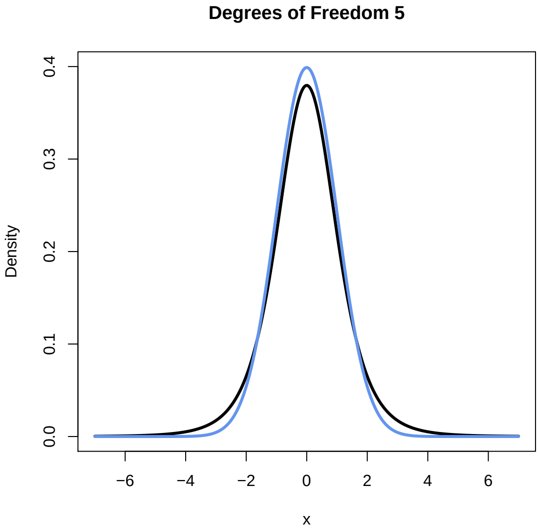

Degrees of Freedom 1
Density
0.0
0.1
0.2
0.3
0.4
−6
−4
−2
0
2
4
6
x

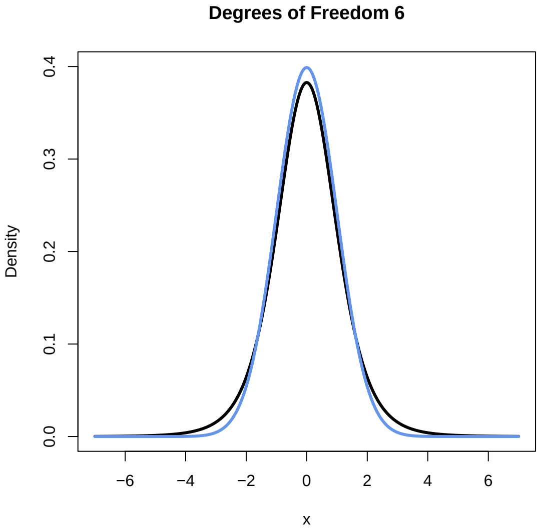
Degrees of Freedom 2

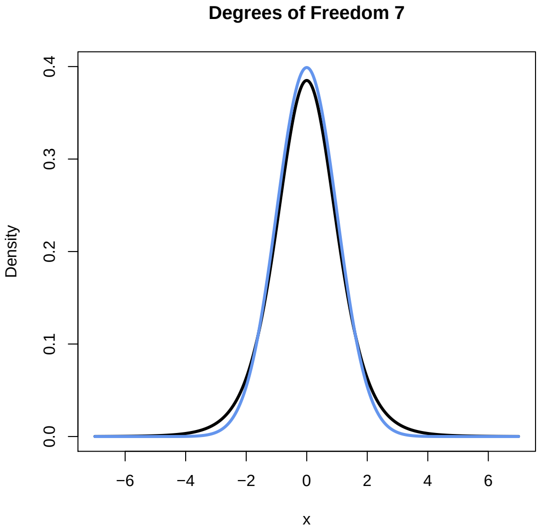


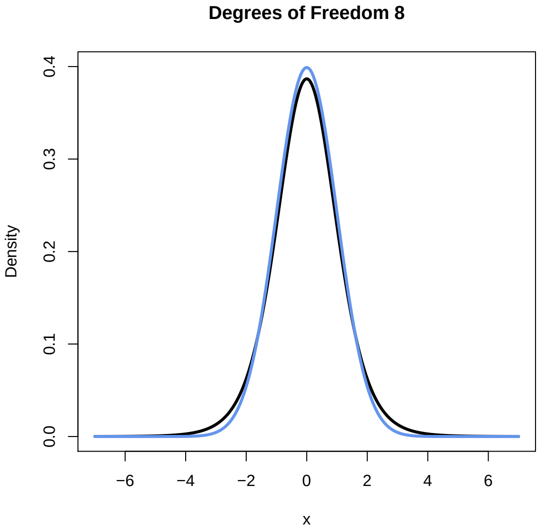

Degrees of Freedom 3
Density
0.0
0.1
0.2
0.3
0.4
−6
−4
−2
0
2
4
6
x

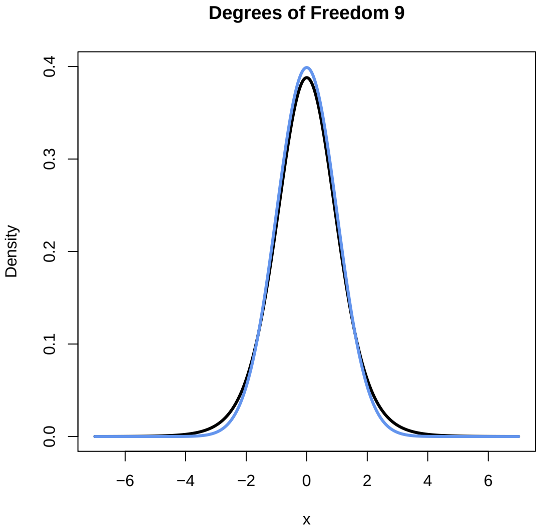

Degrees of Freedom 4
Density
0.0
0.1
0.2
0.3
0.4
−6
−4
−2
0
2
4
6
x


Degrees of Freedom 5
Density
0.0
0.1
0.2
0.3
0.4
−6
−4
−2
0
2
4
6
x


Degrees of Freedom 6
Density
0.0
0.1
0.2
0.3
0.4
−6
−4
−2
0
2
4
6
x


Degrees of Freedom 7
Density
0.0
0.1
0.2
0.3
0.4
−6
−4
−2
0
2
4
6
x


Degrees of Freedom 8
Density
0.0
0.1
0.2
0.3
0.4
−6
−4
−2
0
2
4
6
x


Degrees of Freedom 9
Density
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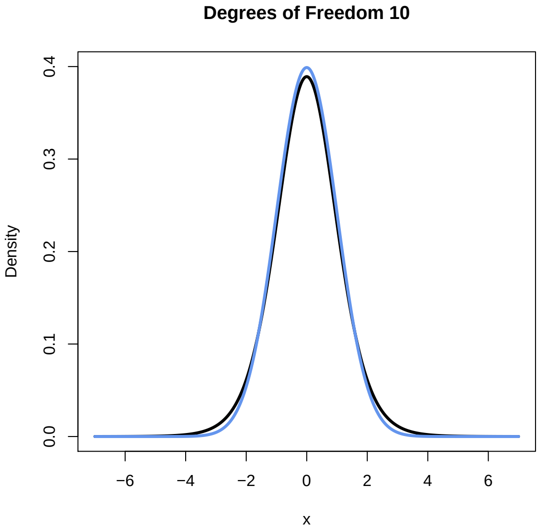
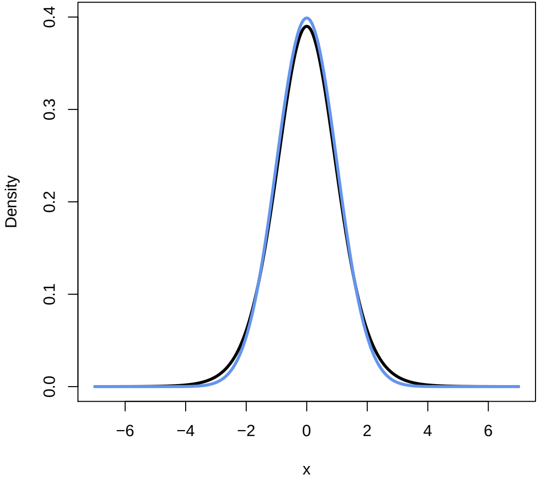
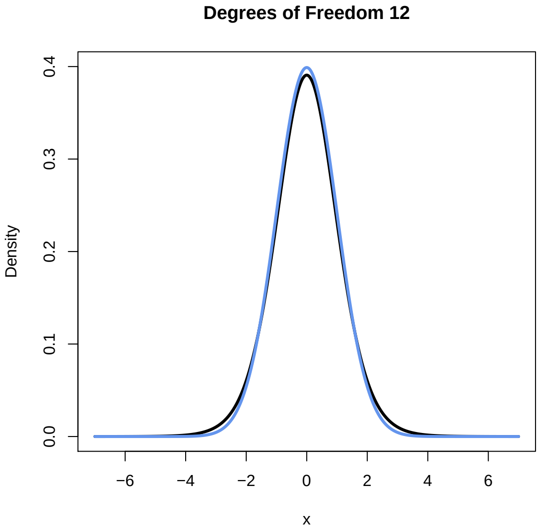
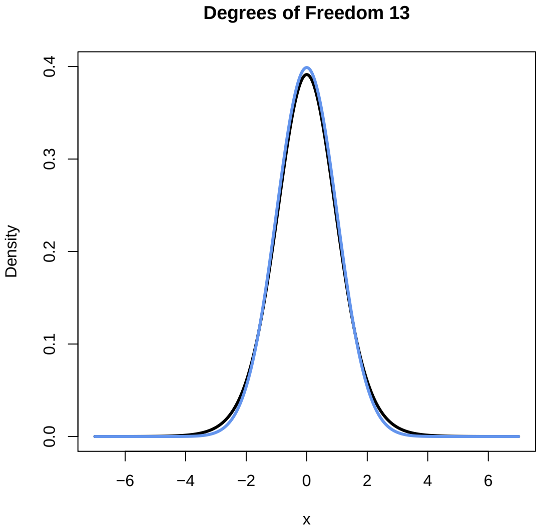
0.1
0.2
0.3
0.4
−6
−4
−2
0
2
4
6
x

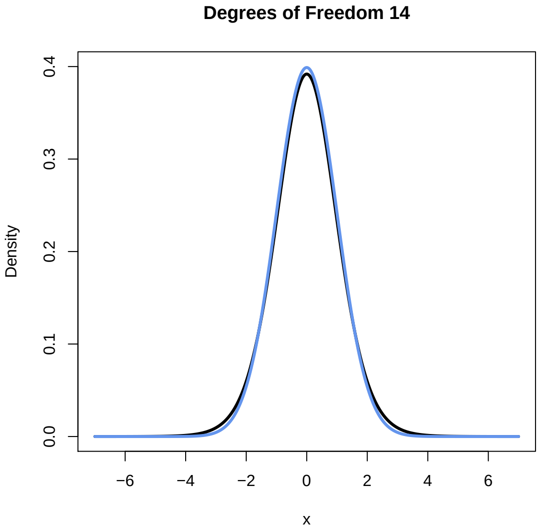

Degrees of Freedom 10
Density
0.0
0.1
0.2
0.3
0.4
−6
−4
−2
0
2
4
6
x

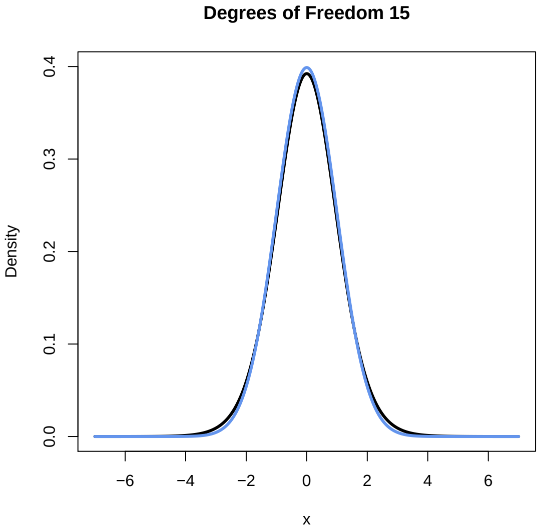
Degrees of Freedom 11

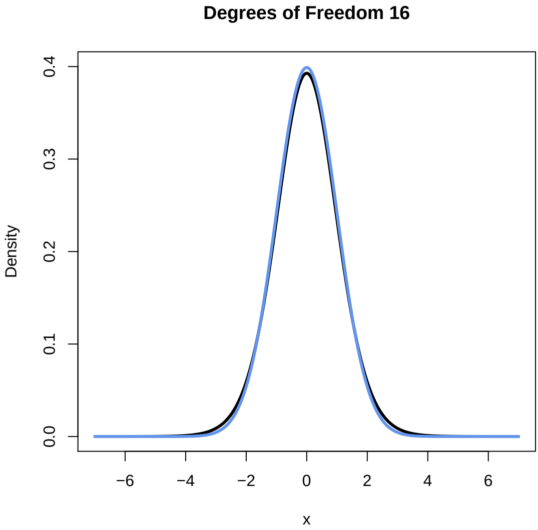


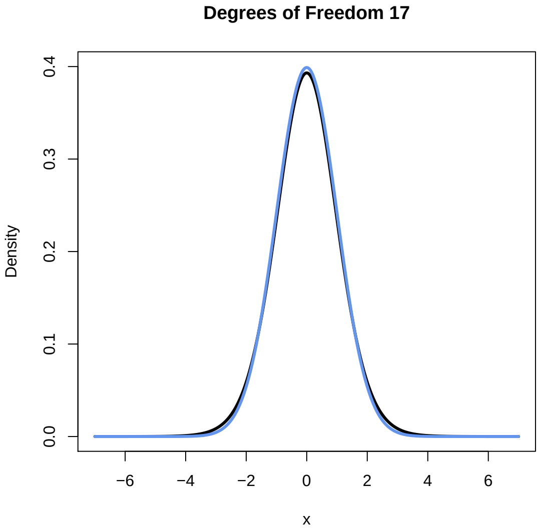

Degrees of Freedom 12
Density
0.0
0.1
0.2
0.3
0.4
−6
−4
−2
0
2
4
6
x


Degrees of Freedom 13
Density
0.0
0.1
0.2
0.3
0.4
−6
−4
−2
0
2
4
6
x


Degrees of Freedom 14
Density
0.0
0.1
0.2
0.3
0.4
−6
−4
−2
0
2
4
6
x


Degrees of Freedom 15
Density
0.0
0.1
0.2
0.3
0.4
−6
−4
−2
0
2
4
6
x


Degrees of Freedom 16
Density
0.0
0.1
0.2
0.3
0.4
−6
−4
−2
0
2
4
6
x


Degrees of Freedom 17
Density
0.0
0.1
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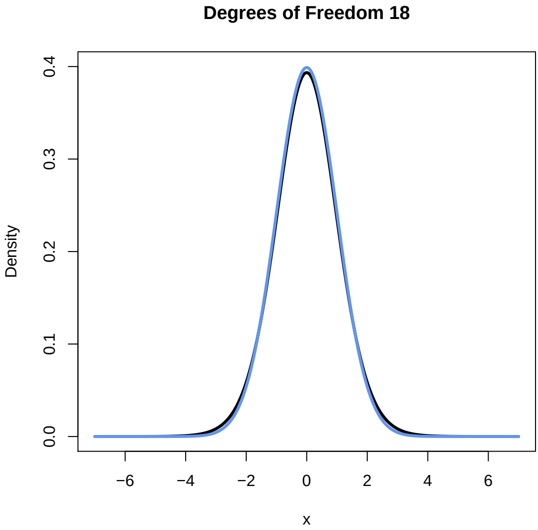
0.3
0.4
−6
−4
−2
0
2
4
6
x


Degrees of Freedom 18
Density
x
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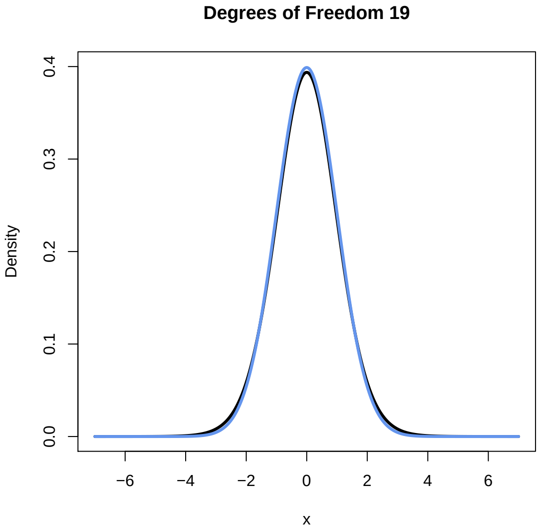
0.1
0.2
0.3
0.4
−6
−4
−2
0
2
4
6


Degrees of Freedom 19
Density
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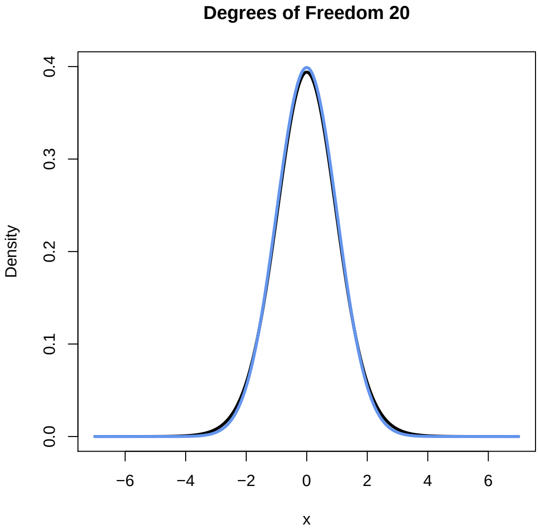
0.3
0.4
−6
−4
−2
0
2
4
6
x


Degrees of Freedom 20
Density
0.0
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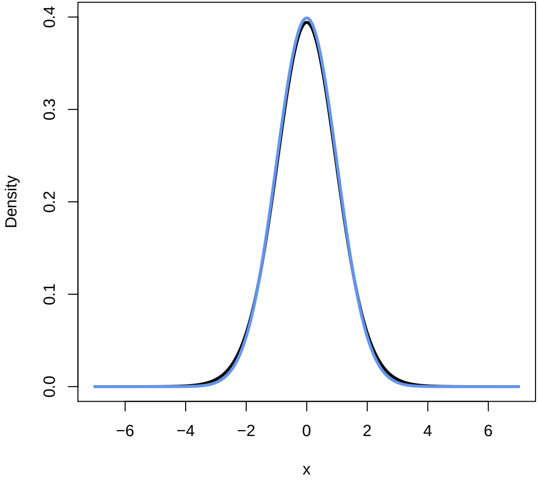
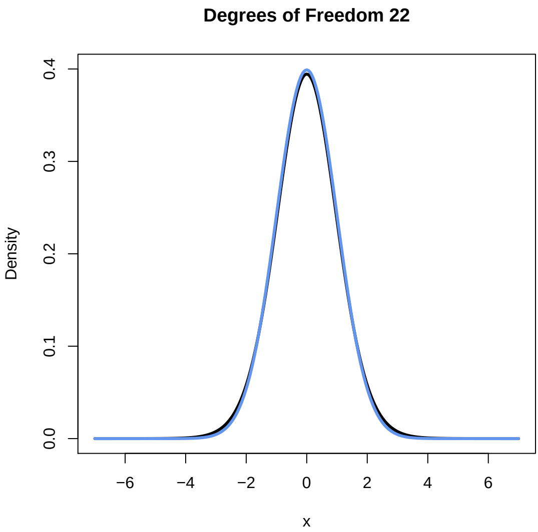
0.2
0.3
0.4
−6
−4
−2
0
2
4
6
x

Degrees of Freedom 21




Degrees of Freedom 22
Density
0.0
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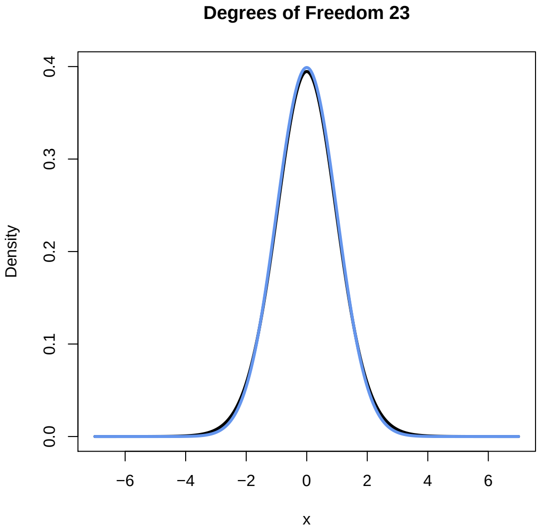
0.2
0.3
0.4
−6
−4
−2
0
2
4
6
x


Degrees of Freedom 23
Density
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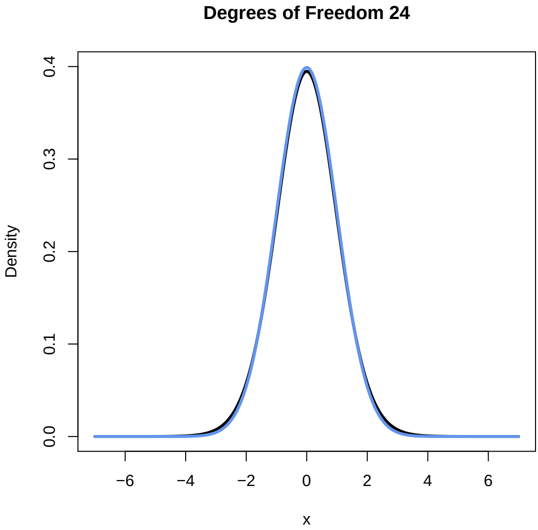
0.1
0.2
0.3
0.4
−6
−4
−2
0
2
4
6
x


Degrees of Freedom 24
Density
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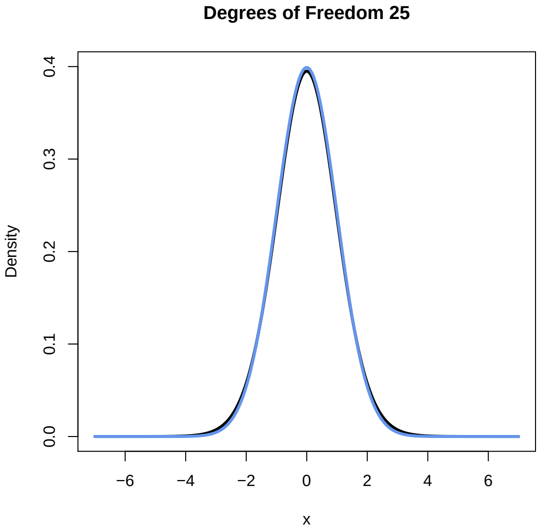
0.1
0.2
0.3
0.4
−6
−4
−2
0
2
4
6
x


Degrees of Freedom 25
Density
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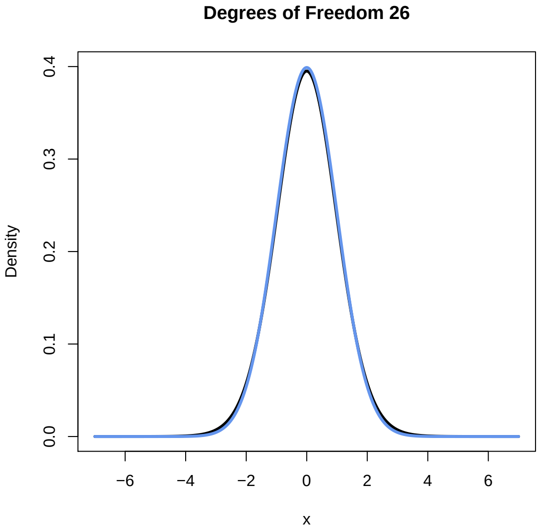
0.2
0.3
0.4
−6
−4
−2
0
2
4
6
x


Degrees of Freedom 26
Density
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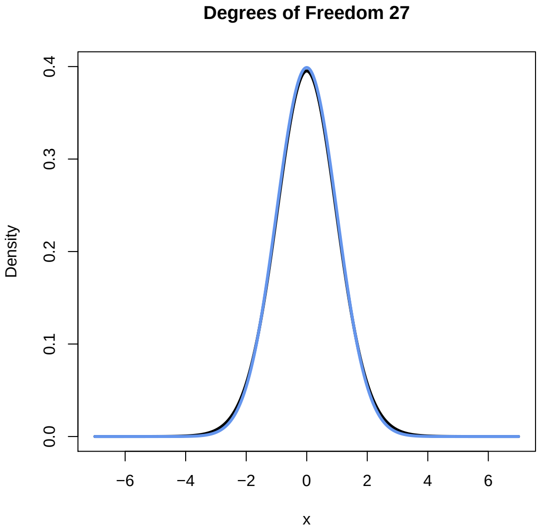
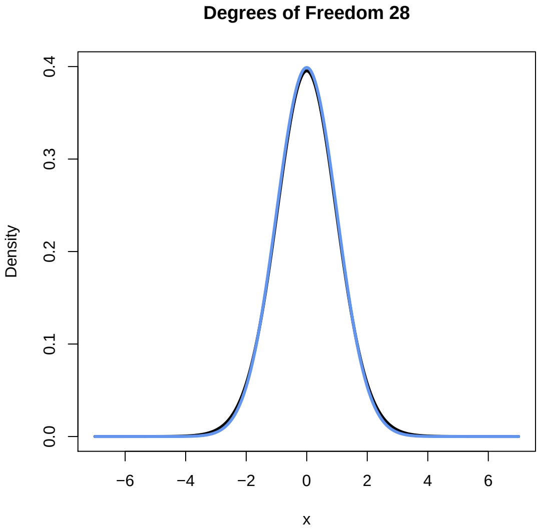
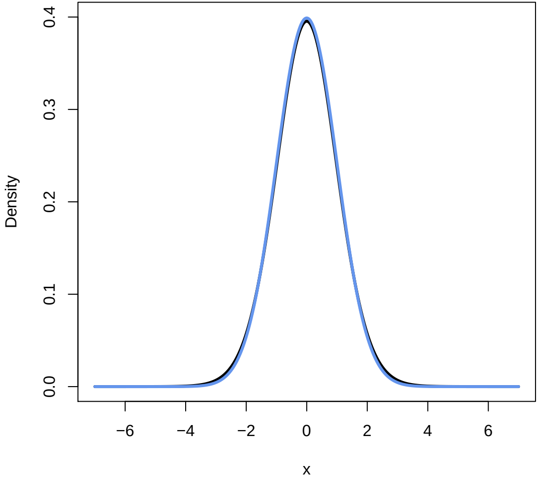
0.3
0.4
−6
−4
−2
0
2
4
6
x

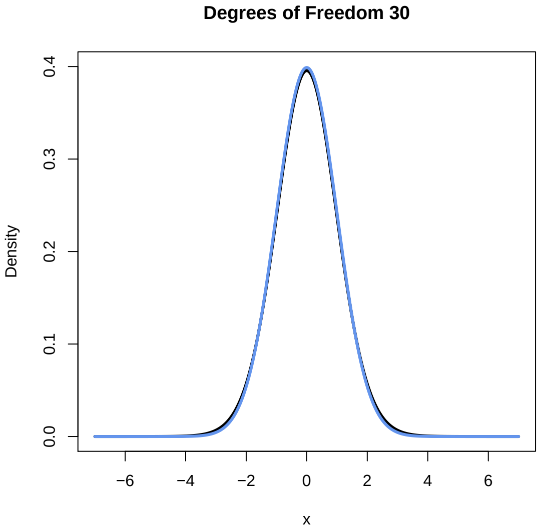

Degrees of Freedom 27
Density
0.0
0.1
0.2
0.3
0.4
−6
−4
−2
0
2
4
6
x



Degrees of Freedom 28
Density
0.0
0.1
0.2
0.3
0.4
−6
−4
−2
0
2
4
6
x

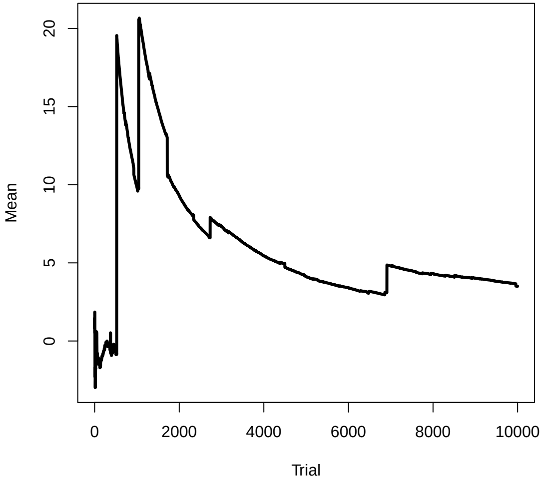
Degrees of Freedom 29




Degrees of Freedom 30
Density
0.0
0.1
0.2
0.3
0.4
−6
−4
−2
0
2
4
6
x

Student's t-Distribution, Properties

Suppose $n = 1$, Cauchy distribution



Student's t-Distribution, Properties

Suppose $n = 1$, Cauchy distribution
If $X \sim \text{Cauchy}(1)$, then:
$E[X] = \text{undefined}$
$\text{var}(X) = \text{undefined}$
If $X \sim t(2)$
$\text{E[X]} = 0$
$\text{var}(X) = \text{undefined}$

Student's t-Distribution, Properties

Suppose $n > 2$, then
$var(X) = \frac{n}{n-2}$
As $n \to \infty$ var$(X) \to 1$.

Tomorrow: Joint Distributions and Multivariate Normal Distribution