Math Camp

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September 12th, 2017

Where we're at

- Conditional Probability/Bayes' Rule

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- Today: Random Variables

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- Expectation, Variance
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- A Brief Introduction to Markov Chains

Random Variable: Intuition

Recall the three parts of our probability model

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Random variables: functions defined on the sample space

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$$X : \textit{Sample Space} \rightarrow \mathcal{R}$$

- X's domain are all outcomes (Sample Space)
- X's range is the Real line (or some subset of it)
- Because X is defined on outcomes, makes sense to write $p(X)$ (we'll talk about this soon)

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Treatment assignment:

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Big Question: How do we compute P(X=1), P(X=0), etc?

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$p(X = a) = 0$, for all $a \notin (0, 1, 2, 3)$

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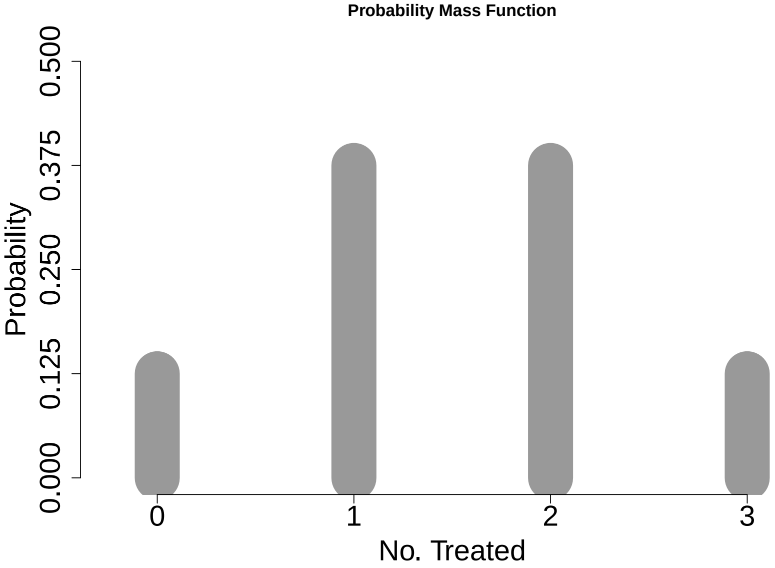

Probability Mass Function: Intuition

Consider outcome of election:

- $X(v) = 1$ if $v > 0.5$ otherwise $X(v) = 0$
- $P(X = 1)$ then is equal to $P(v > 0.5)$

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(Brief aside) Countable: A set is countable if there is a function that can map all its elements to the natural numbers $\{1, 2, 3, 4, \ldots\}$ (one-to-one, injective). If it is onto (from S to all natural numbers, surjective), then we say the set is countably infinite

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Definition

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$$p(x) \quad = \quad P(X = x)$$

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P(afghanistan) = 0.3; P(fire) = 0.0001; P(department) = 0.0001; P(soldier) = 0.2; P(troop) = 0.2; P(war)=0.2997; P(grant)=0.0001

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Topic Models: take a set of documents and estimate topics.

Definition

Cumulative Mass (distribution) Function: For a random variable X, define the cumulative mass function $F(x)$ as,

$$F(x) = P(X \leq x)$$

- Characterizes how probability cumulates as X gets larger
- $F(x) \in [0, 1]$
- $F(x)$ is non-decreasing

Cumulative Mass Function: Example

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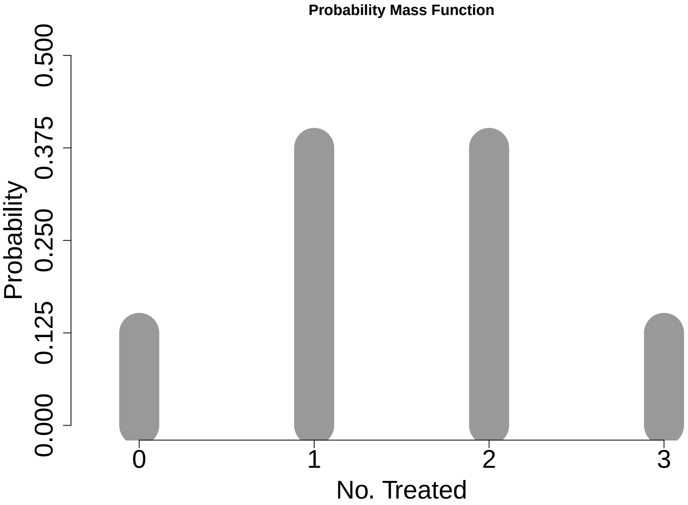
Cumulative Mass Function

There is a close relationship between pmf's and cmf's.



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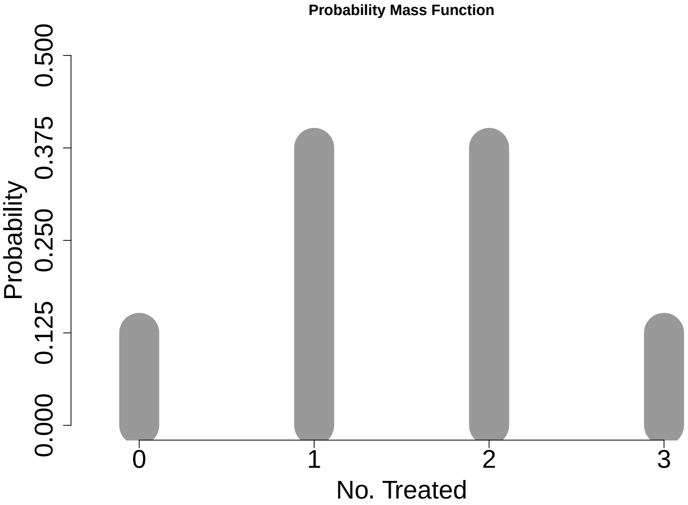

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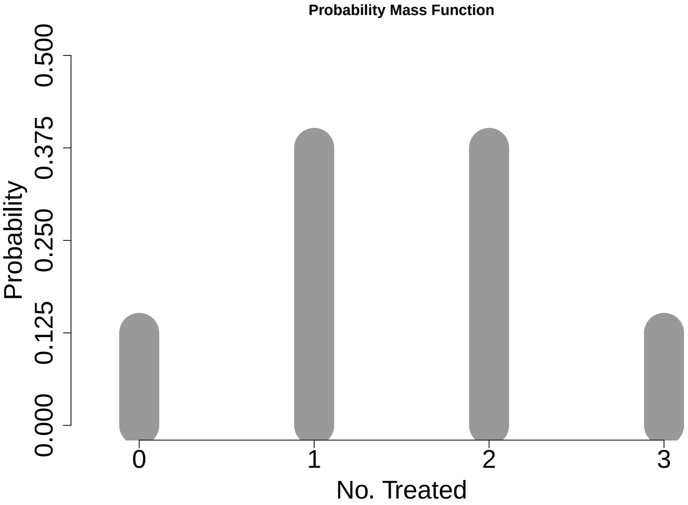

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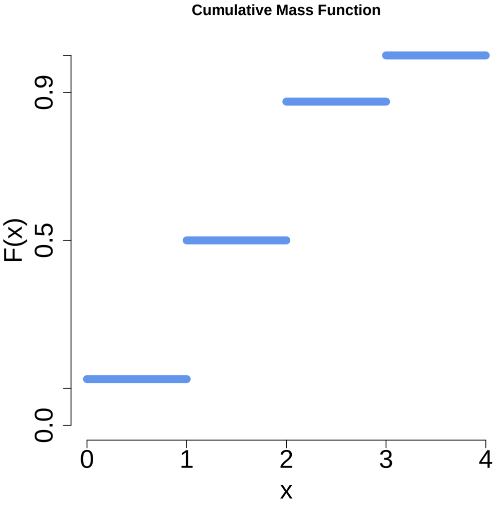

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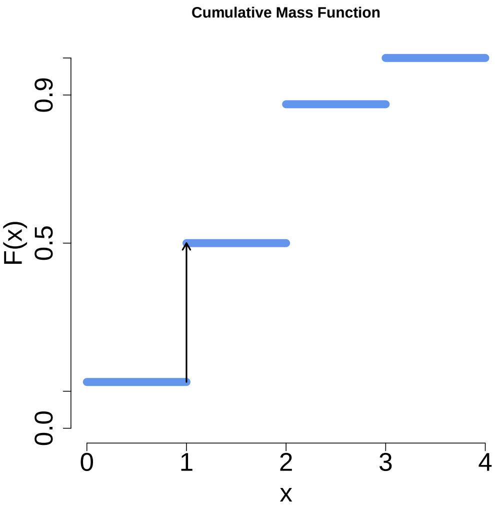

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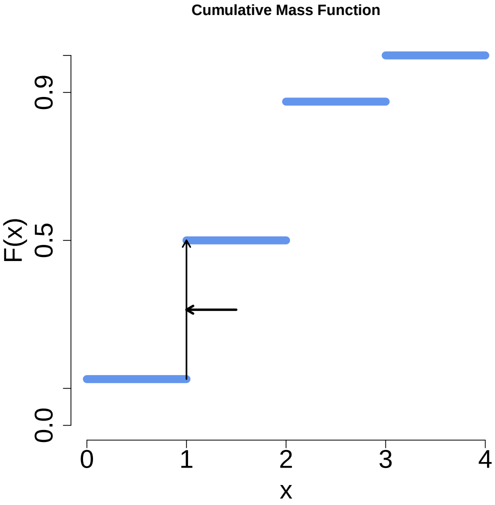

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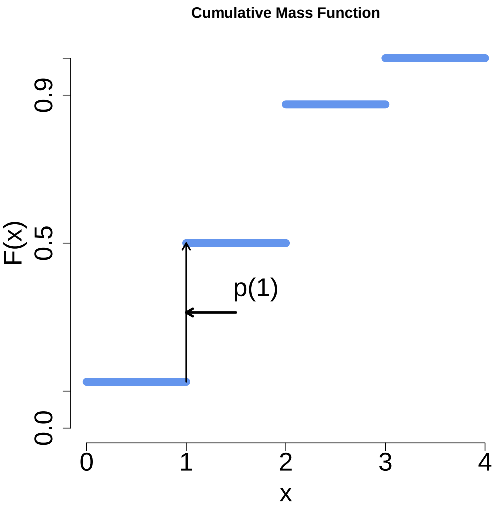

Expectation

What can we expect from a trial?

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Expected Value: define the expected value of a function X as,

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In words: for all values of x with $p(x)$ greater than zero, take the weighted average of the values

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Suppose that there is a group of N people.

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Indicator Variables and Probabilities

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Functions of Random Variables

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$$E[g(X)] = \sum_i g(x_i)p(x_i)$$

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Let's suppose that X is the number of observations assigned to treatment (from our previous example).

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&= \frac{24}{8} = 3
\end{aligned}
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Functions of Random Variables: Corollary

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Suppose X is a random variable and a and b are constants (not random variables). Then,

$$E[aX + b] \quad = \quad aE[X] + b$$

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&= aE[X] + b(1)
\end{aligned}$$

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&= E[X^2] - 2E[X]^2 + E[X]^2 \\
&= E[X^2] - E[X]^2 \\
&= \text{Var}(X)
\end{aligned}
$$

Variance

Definition

The variance of a random variable X, $var(X)$, is

$$\begin{aligned} var(X) &= E[(X - E[X])^2] \\ &= E[X^2] - E[X]^2 \end{aligned}$$

- We will define the standard deviation of X, $\text{sd}(X) = \sqrt{\text{var}(X)}$
- $\text{var}(X) \geq 0$.

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Continue the three person experiment, with $P(T) = P(C) = 1/2$.

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E[X^2] &= 3 \\
E[X]^2 &= 1.5^2 = 2.25 \\
\text{Var}(X) &= E[X^2] - E[X]^2 \\
&= 3 - 2.25 = 0.75
\end{aligned}$$

Variance Corollary

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$Var(aX + b) = a^2 Var(X)$

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$\square$

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Var(aX + b) $= a^2$ *Var(X)*

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Var(Y) &= E[(aX + b - aE[X] - b)^2] \\
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&= a^2E[X^2] - 2a^2E[X]^2 + a^2E[X]^2 \\
&= a^2(E[X^2] - E[X]^2) \\
&= a^2 Var(X)
\end{aligned}
$$

□

Famous Distributions

- Bernoulli
- Binomial
- Multinomial
- Poisson

Models of how world works.

Bernoulli Random Variable

Definition

Suppose X *is a random variable, with* $X \in \{0,1\}$ *and* $P(X=1)=\pi$*.*
Then we will say that X *is* *Bernoulli* *random variable,*

$$p(k) = \pi^k(1-\pi)^{1-k}$$

for $k \in \{0,1\}$ *and* $p(k)=0$ *otherwise.*
We will (equivalently) say that

$$Y \sim \textit{Bernoulli}(\pi)$$

Bernoulli Random Variable

Suppose we flip a fair coin and $Y = 1$ if the outcome is Heads .

$$\begin{aligned}
Y &\sim \text{Bernoulli}(1/2) \\
p(1) &= (1/2)^{1}(1-1/2)^{1-1} = 1/2 \\
p(0) &= (1/2)^{0}(1-1/2)^{1-0} = (1-1/2)
\end{aligned}$$

Bernoulli Random Variable Moments

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$E[Y] = \pi$
$\text{var}(Y) = \pi(1-\pi)$ What is the maximum variance?

Example: Winning a War

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&= B \times \pi - c
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Definition

Suppose Y is a random variable that counts the number of successes in N independent and identically distributed Bernoulli trials. Then Y is a Binomial random variable,

$$p(k) \quad = \quad \binom{N}{k}\pi^k(1-\pi)^{1-k}$$

for $k \in \{0, 1, 2, \ldots, N\}$ and $p(k) = 0$ otherwise.
Equivalently,

$$Y \quad \sim \quad Binomial(N, \pi)$$

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\begin{aligned}
Z &\sim \text{Binomial}(1/2) \\
p(0) &= \binom{3}{0}(1/2)^0(1-1/2)^{3-0} = 1 \times \frac{1}{8}
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Suppose we have a set N voters, with iid turnout decisions
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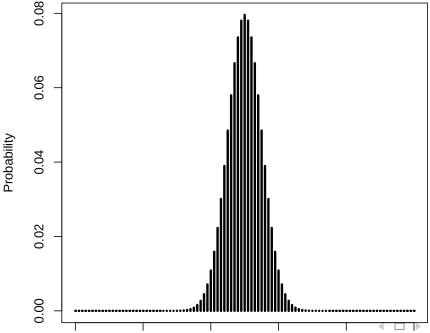

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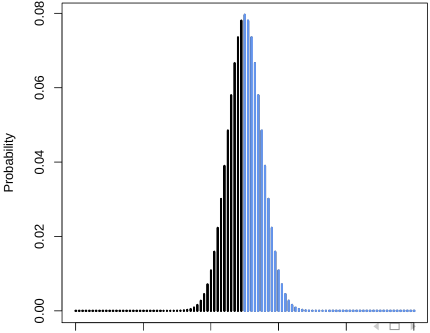

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`R Code!`

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$$Y_i \sim \text{Bernoulli}(0.9) \text{ if } j \text{ votes}$$

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Suppose follower i depends on only one leader j (and each follower has their own leader)

$$\begin{aligned}
Y_i &\sim \text{Bernoulli}(0.9) \text{ if } j \text{ votes} \\
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\end{aligned}$$

Voter Turnout, with Spillovers

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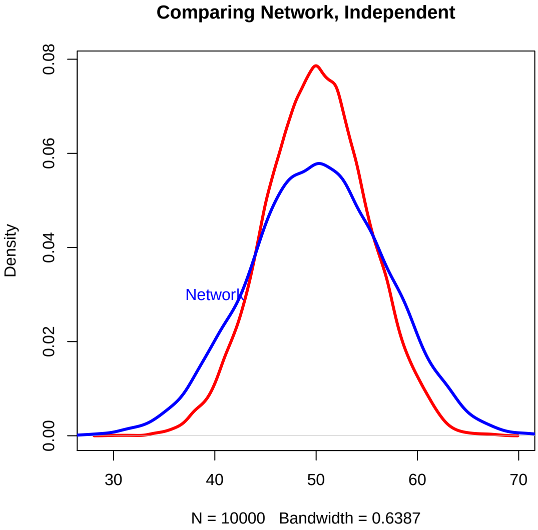

Trials with More than Two Outcomes

Definition

Suppose we observe a trial, which might result in J outcomes.
And that $P(outcome \ = i) = \pi_i$
$\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_J)$ *where* $Y_j = 1$ *if outcome j occurred and 0 otherwise.*
Then $\boldsymbol{Y}$ *follows a* *multinomial* *distribution, with*

$$p(\boldsymbol{y}) \quad = \quad \pi_1^{y_1}\pi_2^{y_2}\ldots\pi_k^{y_k}$$

if $\sum_{i=1}^{k} y_i = 1$ *and the pmf is* 0 *otherwise.*
Equivalently, we'll write

$$\begin{aligned} \boldsymbol{Y} &\sim Multnomial(1, \boldsymbol{\pi}) \\ \boldsymbol{Y} &\sim Categorial(\boldsymbol{\pi}) \end{aligned}$$

Multinomial Properties + Notes

Computer scientists: commonly call Multinomial$(1, \boldsymbol{\pi})$ Discrete$(\boldsymbol{\pi})$.

$$
\begin{aligned}
E[X_i] &= N\pi_i \\
\text{var}(X_i) &= N\pi_i(1 - \pi_i)
\end{aligned}
$$

Investigate Further in Homework!

Counting the Number of Events

Often interested in counting number of events that occur:

1) Number of wars started
2) Number of speeches made
3) Number of bribes offered
4) Number of people waiting for license

Generally referred to as event counts

Stochastic processes: a course provide introduction to many processes (Queing Theory)

Poisson Distribution

Definition

Suppose X is a random variable that takes on values $X \in \{0, 1, 2, \ldots, \}$ and that $P(X = k) = p(k)$ is,

$$p(k) \quad = \quad e^{-\lambda}\frac{\lambda^k}{k!}$$

for $k \in \{0, 1, \ldots, \}$ and 0 otherwise. Then we will say that X follows a Poisson distribution with rate parameter λ.

$$X \quad \sim \quad Poisson(\lambda)$$

Example: Poisson Distribution

Suppose the number of threats a president makes in a term is given by $X \sim \text{Poisson}(5)$.

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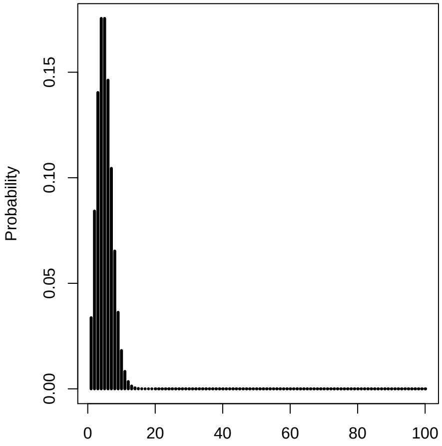

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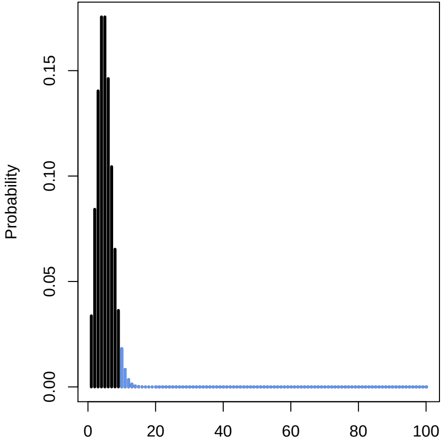

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`R code!`

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Very useful distribution, with strong assumptions. We'll explore in homework!

Often interested in how processes evolve over time

- Given voting history, probability of voting in the future
- Given history of candidate support, probability of future support
- Given prior conflicts, probability of future war
- Given previous words in a sentence, probability of next word

Potentially complex history

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Stochastic Process

Definition

Suppose we have a sequence of random variables $\{X\}_{i=0}^{M} = X_0, X_1, X_2, \ldots, X_M$ *that take on the countable values of* S. *We will call* $\{X\}_{i=0}^{M}$ *a stochastic process with state space* S.

If index gives time, then we might condition on history to obtain probability

$$\text{PMF } X_t \text{, given history} \quad = \quad P(X_t | X_{t-1}, X_{t-2}, \ldots, X_1, X_0)$$

Still Complex

Markov Chain

Definition

Suppose we have a stochastic process $\{X\}_{i=0}^{M}$ with countable state space S. Then $\{X\}_{i=0}^{M}$ is a markov chain if:

$$P(X_t|X_{t-1}, X_{t-2}, \ldots, X_1, X_0) \quad = \quad P(X_t|X_{t-1})$$

A Markov chain's future depends only on its current state

Transition Matrix

Habitual turnout?

$$\boldsymbol{T} \;=\; \begin{pmatrix} & \text{Vote}_t & \text{Not Vote}_t \\ \text{Vote}_{t-1} & 0.8 & 0.2 \\ \text{Not Vote}_{t-1} & 0.3 & 0.7 \end{pmatrix}$$

- Suppose someone starts as a voter—what is their behavior after
- 1 iteration?
- 2 interations?
- The long run?

`R` Code!

Tomorrow: Continuous Random Variables!