RM 294 - Optimization I
FA Track Project 1: Linear Programming
Dedicated / Cash Flow Matched Portfolio

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Goal:

Create a dedicated portfolio of bonds and forwards to hedge against interest rate risk.

Process:

Our group formulated the optimal portfolio using a linear programming problem. We set our objective to minimize the amount of money spent on bonds, with the constraint that assets will equal expected liabilities each year.

CSV files

We have 8 years of liabilities where we want cash outflows to equal cash inflows. We have 13 bonds and forwards we are looking to purchase, where their annual payments are equal to annual liabilities.

Details of our assets:

Bond	Price	Coupon	StartTime	Maturity
1	102	5.0	0	1
2	100	3.0	1	2
3	99	3.5	0	2
4	101	4.0	0	2
5	98	2.5	0	3
6	98	4.0	0	4
7	98	2.0	2	4
8	104	9.0	0	5
9	100	6.0	0	5
10	101	8.0	0	6
11	102	9.0	0	7
12	94	7.0	0	8
13	91	3.0	3	8

Details of our liabilities:

Year	Liability
1	1200000
2	1800000
3	2000000
4	2000000
5	1600000
6	1500000
7	1200000
8	1000000

Project Details

Objective Function

The objective function is the initial investment in bonds in Year 0, which we try to minimize. Note that there is no cash outflow for forward bonds (bonds starting after Year 0). Therefore, for N bonds:

$$minimize \sum_{i=1}^{N}(P_i * x_i)$$

where
i = ith bond
P_i = Price of the ith bond
x_i = Quantity of the ith bond (to be optimized)

The objective matrix then becomes:

```
[102.  -0.  99. 101.  98.  98.  -0. 104. 100. 101. 102.  94.  -0.]
```

Constraint Function

Our constraint entails that the cash outflow should equal the cash inflow. The cash outflow is given by L(t) which is the liability at year t. The cash inflow is a combination of the coupon prices received from bonds, the maturity prices of bonds at the expiration date and the outflow from paying the forward prices for forward bonds. Therefore, the constraint at time 't' becomes:

$$L_t = \sum_{i=1}^{N}(C_i * x_i) \mid (M_i > t-1, S_i > t-1) - \sum_{i=1}^{N}(P_i - x_i) \mid (S_i = t) + \sum_{i=1}^{N}(100 * x_i) \mid (M_i = t)$$

where
t = tth year
C_i = Coupon price of the ith bond
M_i = Maturity price of the ith bond

The constraint Matrix is thus:

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	105.0	-100.0	3.5	4.0	2.5	4.0	0.0	9.0	6.0	8.0	9.0	7.0	0.0
1	0.0	103.0	103.5	104.0	2.5	4.0	-98.0	9.0	6.0	8.0	9.0	7.0	0.0
2	0.0	0.0	0.0	0.0	102.5	4.0	2.0	9.0	6.0	8.0	9.0	7.0	-91.0
3	0.0	0.0	0.0	0.0	0.0	104.0	102.0	9.0	6.0	8.0	9.0	7.0	3.0
4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	109.0	106.0	8.0	9.0	7.0	3.0
5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	108.0	9.0	7.0	3.0
6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	109.0	7.0	3.0
7	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	107.0	103.0

After solving the problem using Gurobi, we got the following results:

Optimal cost of the portfolio: **$9,447,500.76**

We will buy the following bonds at year 0 to satisfy liabilities for the following 8 years. Although we can buy 13 bonds, we only buy 8 of the 13 bonds. It is important to note that we do not invest in any of the forward contracts. We would have done so had the price of the forwards been lower. The below table and graph shows the amount bought of each bond.

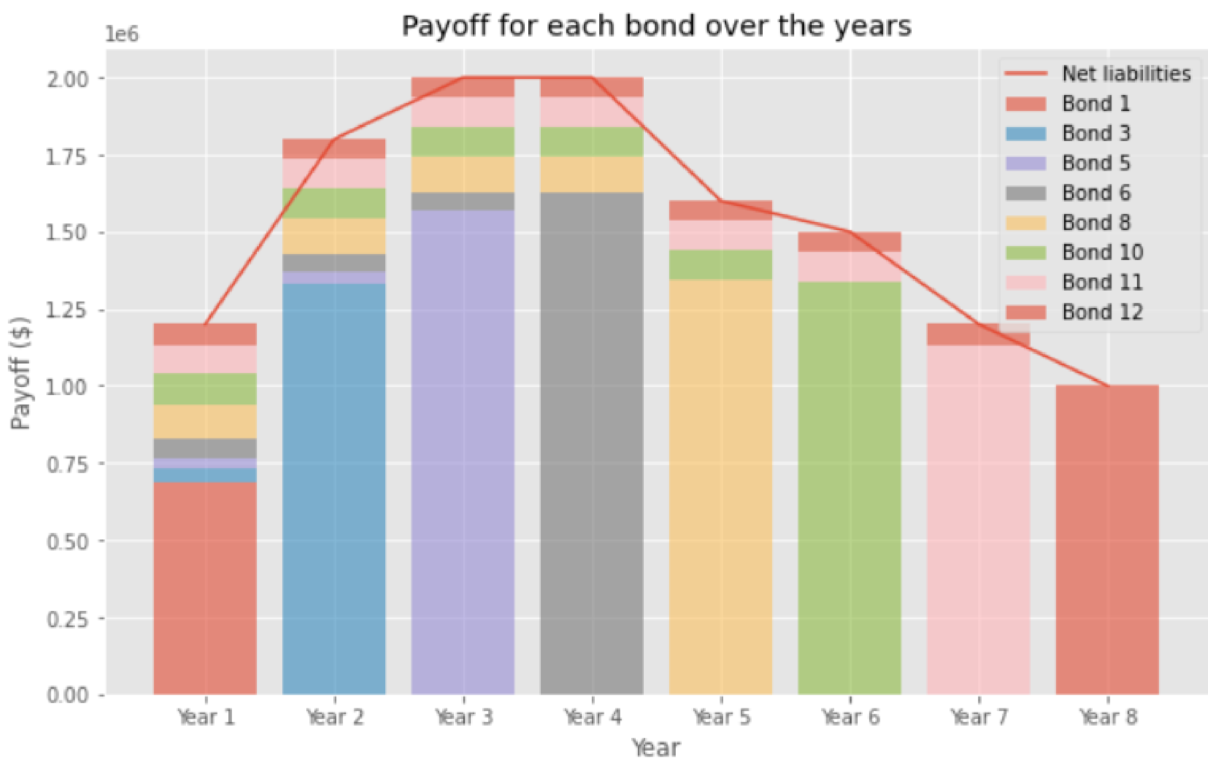
```
Cost of portfolio: $ 9447500.76

We buy the following bond quantities at Year 0:

Bond 1 (@ $102):  6522.49
Bond 3 (@ $99):  12848.62
Bond 5 (@ $98):  15298.32
Bond 6 (@ $98):  15680.78
Bond 8 (@ $104):  12308.01
Bond 10 (@ $101):  12415.73
Bond 11 (@ $102):  10408.99
Bond 12 (@ $94):  9345.79
```

As a final check, we coded an equality checking if cash flows equal liabilities. This ensures our optimal solution listed above meets the client's required goal.

```
:hed?:", all(list(map(int, A_bonds@(modX.x.T))) == b_liabilities.T[0]))
```

```
All Cash Flows matched?: True
```

We also created the following graph to show the payouts of each bond over the 8 years. Since the bonds have different maturity years, we see that each year's liabilities are mainly matched by the bond maturing at that year and once that bond pays out, you do not see it again. The line across the top is net liabilities, showing how each year the portfolio matches cash outflow needs. The bars are stacked with the respective bonds.



Flexibility of the Portfolio

To understand the flexibility of the Portfolio in case of changes in liability, we first look at the following outcomes.

Outcome 1
If some of our employees were to quit without notice and don't need a replacement, this will reduce the liabilities. In the case of our annual liabilities decreasing, the current portfolio would have a surplus and our liabilities set at time zero could still be met.

Outcome 2
If some employees were to quit without notice and the new replacement requests a higher salary than the previous employee, then our annual liabilities will increase. In the case of increasing annual liabilities, our portfolio would have a deficit and we would need to adjust the portfolio at that year in order to meet future liability payments.

The magnitude at which the Portfolio cost will change according to a change in Liability depends on the year and can be understood from the Shadow Prices of the respective year. The Shadow Prices are calculated below.

	Year	Shadow Price	Lower Bound	Upper Bound
0	1	0.971429	5.151384e+05	inf
1	2	0.923671	4.701682e+05	2.205234e+07
2	3	0.909876	4.319224e+05	3.106210e+07
3	4	0.834424	3.691993e+05	2.089037e+07
4	5	0.653628	2.584273e+05	1.075133e+07
5	6	0.617183	1.591014e+05	1.261887e+07
6	7	0.530350	6.542056e+04	1.197295e+07
7	8	0.522580	-1.164153e-10	1.582050e+07

WSJ Treasury Bonds

Lastly, we conducted the same analysis as asked in our original goal, this time with real data on Treasury Bonds from the WSJ. We began by creating a CSV from the webpage https://www.wsj.com/market-data/bonds/treasuries and extracting 18 bonds, listed below. The Treasury Bond data from WSJ is current as of 6th October 2021.

WSJ Treasury Bonds

	Maturity	StartTime	Expiration	Coupon	Price
0	1	0	2022-08-15	1.500	101.066
1	1	0	2022-08-15	1.625	101.104
2	1	0	2022-08-15	7.250	106.032
3	2	0	2023-08-15	0.125	99.242
4	2	0	2023-08-15	2.500	104.050
5	2	0	2023-08-15	6.250	111.036
6	3	0	2024-08-15	0.375	99.204
7	3	0	2024-08-15	2.375	105.106
8	4	0	2025-08-15	2.000	104.240
9	4	0	2025-08-15	6.875	123.110
10	5	0	2026-08-15	1.500	102.176
11	5	0	2026-08-15	6.750	127.162
12	6	0	2027-08-15	2.250	106.110
13	6	0	2027-08-15	6.375	129.284
14	7	0	2028-08-15	2.875	110.164
15	7	0	2028-08-15	5.500	127.262
16	8	0	2029-08-15	1.625	101.280
17	8	0	2029-08-15	6.125	135.262

We computed the Objective and Constraint matrices in the same manner as above. The only difference is the absence of forward contracts in this case.

Objective Matrix

```
[101.066 101.104 106.032  99.242 104.05   111.036  99.204 105.106 104.24
 123.11   102.176 127.162 106.11   129.284 110.164 127.262 101.28   135.262]
```

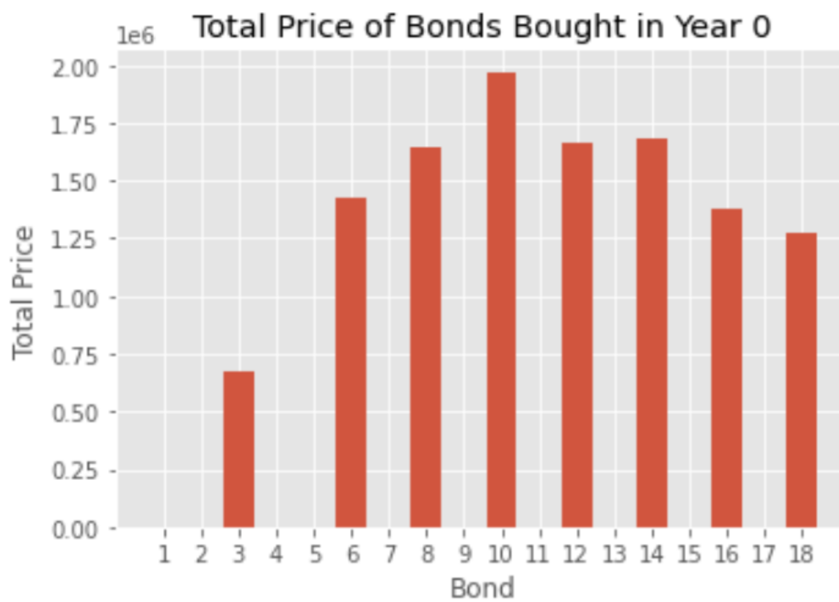
Constraint Matrix

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	101.5	101.625	107.25	0.125	2.5	6.25	0.375	2.375	2.0	6.875	1.5	6.75	2.25	6.375	2.875	5.5	1.625	6.125
1	0.0	0.000	0.00	100.125	102.5	106.25	0.375	2.375	2.0	6.875	1.5	6.75	2.25	6.375	2.875	5.5	1.625	6.125
2	0.0	0.000	0.00	0.000	0.0	0.00	100.375	102.375	2.0	6.875	1.5	6.75	2.25	6.375	2.875	5.5	1.625	6.125
3	0.0	0.000	0.00	0.000	0.0	0.00	0.000	0.000	102.0	106.875	1.5	6.75	2.25	6.375	2.875	5.5	1.625	6.125
4	0.0	0.000	0.00	0.000	0.0	0.00	0.000	0.000	0.0	0.000	101.5	106.75	2.25	6.375	2.875	5.5	1.625	6.125
5	0.0	0.000	0.00	0.000	0.0	0.00	0.000	0.000	0.0	0.000	0.0	0.00	102.25	106.375	2.875	5.5	1.625	6.125
6	0.0	0.000	0.00	0.000	0.0	0.00	0.000	0.000	0.0	0.000	0.0	0.00	0.00	0.000	102.875	105.5	1.625	6.125
7	0.0	0.000	0.00	0.000	0.0	0.00	0.000	0.000	0.0	0.000	0.0	0.00	0.00	0.000	0.000	0.0	101.625	106.125

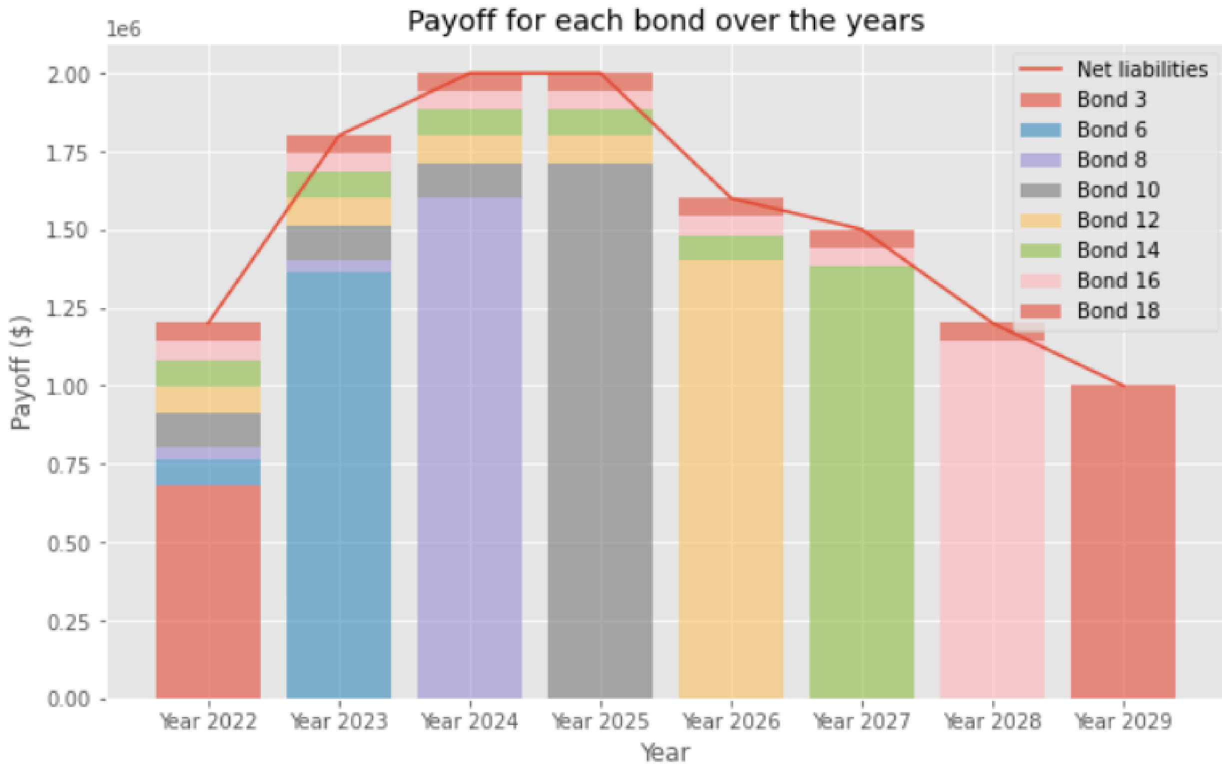
We then used gurobi to determine which bonds and how much of each bond to invest in. The cost and optimal portfolio bonds are listed below.

Optimal cost of the portfolio: **$11,717,495.64**

```
Cost of portfolio: $ 11717495.64

We buy the following bond quantities at Year 0:

Bond 3 (Year 2022) (@ $106.032): 6376.46
Bond 6 (Year 2023) (@ $111.036): 12838.75
Bond 8 (Year 2024) (@ $105.106): 15641.17
Bond 10 (Year 2025) (@ $123.11): 16012.65
Bond 12 (Year 2026) (@ $127.162): 13113.52
Bond 14 (Year 2027) (@ $129.284): 12998.68
Bond 16 (Year 2028) (@ $127.262): 10827.35
Bond 18 (Year 2029) (@ $135.262): 9422.85
```



We see the same usage of bonds in the below stacked bar graph as in our first analysis. 1 bond covers the majority of the liability in it's payout year, and afterwards we do not see the bond again.



We again did our equality check to ensure cash flows = liabilities.

```
1 :hed?:", all(list(map(int, A_bonds@(modX.x.T))) == b_liabilities.T[0]))
```

```
All Cash Flows matched?: True
```

Conclusion

After doing the initial analysis and then repeating it with real world data from the WSJ we notice that for a dedicated portfolio, we only buy one bond for each maturity year, specifically the bond with the highest price. The model also recognizes there are certain bonds that are not optimal, and only invests in the bonds that will most cost-efficiently cover expected liabilities. It is crucial to have accurately forecasted liabilities in order for the dedicated portfolio method to work. If liabilities change, and we have already purchased our bonds, the annual cash outflows will not match our cash inflows. Annual cash outflows & inflows could possibly differ if we 1) do not have a liquidity issue and 2) are able to use leftover investment money and allocate it to next year's liabilities. Overall for this 'matching' method to be effective, it is best to use it to hedge against predictable and known future liabilities.