Groups and Geometries Notes

Alexander Richardson

1 Definitions

Inner Product: Euclidean space is endowed with and inner product $\langle \cdot, \cdot \rangle$ with the following properties

(i) $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$.

(ii) $\langle \lambda \mathbf{x}, \mathbf{y} \rangle = \lambda \langle \mathbf{x}, \mathbf{y} \rangle$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.

(iii) $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

(iv) $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ iff $\mathbf{x} = 0$.

The inner product induces a norm $|| \cdot ||$, which in turn induces a metric d. By

$$||\mathbf{x}||^2 := \langle \mathbf{x}, \mathbf{x} \rangle, \qquad d(\mathbf{x}, \mathbf{y}) := ||\mathbf{x} - \mathbf{y}|| = \sqrt{\langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle}$$

if $\mathbf{x} = \sum_{i=1}^n x_i \boldsymbol{e}_i$ and $\mathbf{y} = \sum_{i=1}^n y_i \boldsymbol{e}_i$, then

$$\langle \mathbf{x}, \mathbf{y} \rangle = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \mathbf{x}^T \boldsymbol{\cdot} \mathbf{y}$$

Orthogonal Linear Transformation: A linear transformation

$$T : \mathbb{R}^n \to \mathbb{R}^n$$

is orthogonal if

$$\langle T(\mathbf{x}), T(\mathbf{y}) \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. For the matrix representation A of T we have

$$\mathbf{x}^T A^T \boldsymbol{\cdot} A\mathbf{y} = (A\mathbf{x})^T \boldsymbol{\cdot} (A\mathbf{y}) = \langle A\mathbf{x}, A\mathbf{y} \rangle = \langle T(\mathbf{x}), T(\mathbf{y}) \rangle = \langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \boldsymbol{\cdot} \mathbf{y}$$

Orthogonal Basis: A basis $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ for $\mathbb{R}^n$ is orthogonal if

$$\langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

Isometry: For metric spaces (X, d_X) and (Y, d_Y) a map

$$f : X \to Y$$

is an isometry if

$$d_X(x, y) = d_Y\big(f(x), f(y)\big) \quad \forall\ x, y \in X$$

if in addition f is surjective, then f is an **Isometric Isomorphism**.

Isometries are affine maps.

Note: The isometric isometries of a metric space X, denoted $I(X)$ form a group, with

1. $e = Id \in I(X)$.
2. Since f is surjective, and isometries are always injective we have f^{-1} exists and is also an isometry.
3. If $f, g \in I(X)$, then $f \circ g \in I(X)$.

Orthogonal Group: The set of all orthogonal $n \times n$ matrices

$$\mathrm{O}(n) := \{A \in \mathrm{GL}(n, \mathbb{R}) : A^T A = I_n\}$$

if $A \in \mathrm{O}(n)$, and since $\det(A^T) = \det(A)$ we have

$$1 = \det(I_n) = \det(A^T A) = \det(A^T)\det(A) = \det(A)^2$$

and thus, $\det(A) = \pm 1$

Special Orthogonal Group: $\mathrm{SO}(n) \subset \mathrm{O}(n)$ is the subset of orthogonal matrices such that

$$A \in \mathrm{SO}(n) \implies \det(A) = 1$$

that is

$$\mathrm{SO}(n) = \mathrm{O}(n) \cap \mathrm{SL}(n, \mathbb{R})$$

and $[\mathrm{O}(n) : \mathrm{SO}(n)] = 2$. As an example

$$\mathrm{O}(2) = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \right\}$$

and

$$\mathrm{SO}(2) = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \right\}$$

with

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

being the reflection through the line making an angle of $\frac{\theta}{2}$ with the x-axis.

Upper Triangular: $\mathrm{UT}_+(n)$ is the set of upper triangular matrices with positive diagonal entries.

Affine Subspace: $A \subset \mathbb{R}^n$ is an affine subspace if

$$\lambda\mathbf{a} + \mu\mathbf{b} \in A, \quad \forall\ \mathbf{a}, \mathbf{b} \in A,\ \lambda, \mu \in \mathbb{R} \text{ such that } \lambda + \mu = 1$$

similarly

$$\sum_{i=1}^{k} \lambda_i \mathbf{a}_i \in A, \quad \forall\ \mathbf{a}_i \in A,\ \lambda_i \in \mathbb{R} \text{ such that } \sum_{i=1}^{k} \lambda_i = 1$$

If $V \subset \mathbb{R}^n$ is a linear subspace; i.e. $\mathbf{0} \in V$, then for any fixed $\mathbf{x} \in \mathbb{R}^n$

$$V + \mathbf{x} = \{\mathbf{v} + \mathbf{x} : \mathbf{v} \in V\}$$

is an affine subspace of $\mathbb{R}^n$. Furthermore, every affine subspace A is of this form.

1. If $\mathbf{a} \in A$, then
$$V = A - \mathbf{a} = \{\mathbf{b} - \mathbf{a} : \mathbf{b} \in A\}$$
is an affine subspace and $\mathbf{a} - \mathbf{a} = \mathbf{0} \in V$, and so V is a linear subspace of $\mathbb{R}^n$.

2. Let $\lambda \in \mathbb{R}$ and $\mathbf{x} - \mathbf{a} \in V$, then note that
$$\lambda(\mathbf{x} - \mathbf{a}) + \mathbf{a} = \lambda\mathbf{x} + (1 - \lambda)\mathbf{a}, \quad \text{where } \mathbf{x}, \mathbf{a} \in A, \text{ and } \lambda + (1 - \lambda) = 1$$
that is $\lambda(\mathbf{x} - \mathbf{a}) + \mathbf{a} \in A$ and so
$$\big(\lambda(\mathbf{x} - \mathbf{a}) + \mathbf{a}\big) - \mathbf{a} = \lambda(\mathbf{x} - \mathbf{a}) \in V$$

3. Now let $\mathbf{x} - \mathbf{a}, \mathbf{y} - \mathbf{a} \in V$, then
$$(\mathbf{x} - \mathbf{a}) + (\mathbf{y} - \mathbf{a}) + \mathbf{a} = \mathbf{x} + \mathbf{y} - \mathbf{a} \quad \text{where } \mathbf{x}, \mathbf{y}, \mathbf{a} \in A, \text{ and } 1 + 1 - 1 = 1$$
and so $(\mathbf{x} - \mathbf{a}) + (\mathbf{y} - \mathbf{a}) + \mathbf{a} \in A$ which tells us that
$$(\mathbf{x} - \mathbf{a}) + (\mathbf{y} - \mathbf{a}) \in V$$

and

$$\dim(A) = \dim(A - \mathbf{a}) = \dim(V) \subset \mathbb{R}^n$$

Affine Span: For any subset $X \subset \mathbb{R}^n$ its affine span is defined to be

$$\mathrm{Aff}(X) = \left\{ \sum_{i=1}^{k} \lambda_i \mathbf{x}_i : \mathbf{x}_i \in X,\ \sum_{i=1}^{k} \lambda_i = 1 \right\}$$

and $\mathrm{Aff}(X)$ is the smallest affine subspace containing X.

Affine Independence: A set $X = \{\mathbf{x}_0, \ldots, \mathbf{x}_k\}$ is affinely independent if

$$\sum_{i=0}^{k} \lambda_i \mathbf{x}_i = \mathbf{0}, \text{ and } \sum_{i=0}^{k} \lambda_i = 0 \implies \lambda_0 = \lambda_1 = \cdots = \lambda_k = 0$$

$\{\mathbf{x}_0, \ldots, \mathbf{x}_k\}$ is affinely independent iff

$$\{\mathbf{x}_1 - \mathbf{x}_0, \ldots, \mathbf{x}_k - \mathbf{x}_0\}$$

is linearly independent.

Affine Basis: If $X = \{\mathbf{x}_0, \ldots, \mathbf{x}_k\}$ is affinely independent, then

$$\dim\big(\mathrm{Aff}(X)\big) = k$$

and X is a basis for $\mathrm{Aff}(X)$. Note that an affine basis for a k-dimensional affine space has $k+1$ elements.

If $V \subset \mathbb{R}^n$ is a linear subspace with basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$, then an affine basis for V is $\{\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_k\}$.

Hyperplane: An affine subspace $H \subset \mathbb{R}^n$ with $\dim(H) = n-1$ is a hyperplane. If H is a linear hyperplane for $\mathbb{R}^n$, then $\exists\ \mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x} \neq \mathbf{0}$, such that

$$H = \{\mathbf{x}\}^{\perp} = \{n-1 \text{ vectors perpendicular to } \mathbf{x}\}$$

Affine Map: If $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ are both affine subspaces, a map

$$f : A \to B$$

is an affine map if

$$f\big(\lambda\mathbf{a} + \mu\mathbf{b}\big) = \lambda f(\mathbf{a}) + \mu f(\mathbf{b}), \quad \forall\ \mathbf{a}, \mathbf{b} \in A,\ \lambda, \mu \in \mathbb{R} \text{ such that } \lambda + \mu = 1$$

an affine map takes straight lines to straight lines. If $\mathbf{a} \in A$, and $\mathbf{b} \in B$ then

$$A - \mathbf{a}, \qquad B - \mathbf{b}$$

are linear subspaces, and if

$$L : A - \mathbf{a} \to B - \mathbf{b}$$

is a linear map, then the map

$$f = T_{\mathbf{b}} \circ L \circ T_{-\mathbf{a}} : A \to B,\ \text{by } f(\mathbf{x}) = L(\mathbf{x} - \mathbf{a}) + \mathbf{b}$$

is an affine map.

Reflection: Let $H \subset \mathbb{R}^n$ be a hyperplane, so for any $\mathbf{x} \in \mathbb{R}^n$ we have

$$\mathbf{x} = \mathbf{y} + \mathbf{z} \quad \text{such that } \mathbf{y} \in H,\ \mathbf{z} \perp (H - \mathbf{y})$$

then the reflection through H is the isometry

$$R_H(\mathbf{x}) = \mathbf{y} - \mathbf{z}$$

Direct and Opposite: With the mapping

$$\begin{aligned} I(\mathbb{R}^n) &\to \mathrm{O}(n) \\ f &\mapsto \widetilde{f} = T_{-f(\mathbf{0})} \circ f \end{aligned}$$

Since $\widetilde{f}$ fixes $\mathbf{0}$, we know it is orthogonal and so $\det(\widetilde{f}) = \pm 1$. If $\det(\widetilde{f}) = 1$ then f is direct, and if $\det(\widetilde{f}) = -1$ then f is opposite.

Inversion: For a rotatory inversion

$$R_H \circ R(\vec{l}, \alpha) = R(\vec{l}, \alpha) \circ R_H$$

if $\alpha = \pi$, and $\mathbf{a} = H \cap \vec{l}$ then

$$R_H \circ R(\vec{l}, \pi) = I_{\mathbf{a}}$$

and

$$I_{\mathbf{a}} = R_{H_1} \circ R_{H_2} \circ R_{H_3}$$

where H_1, H_2, H_3 are all mutually perpendicular and $\mathbf{a} = H_1 \cap H_2 \cap H_3$.

If $\mathbf{a} = \mathbf{0}$, then $I_{\mathbf{0}}$ is linear and

$$I_{\mathbf{0}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Symmetry Group: If $X \subset \mathbb{R}^n$ contains $n+1$ affinely independent points its symmetry group $S(X) \leq I(\mathbb{R}^n)$ is defined by

$$S(X) = \{f \in I(\mathbb{R}^n) : f(X) = X\}$$

Orbit: For a finite group G, and $\mathbf{x} \in \mathbb{R}^n$, the orbit of $\mathbf{x}$ under the action of G is

$$\mathrm{Orb}(\mathbf{x}) = \{g\mathbf{x} : g \in G\}$$

Convex: A subset $X \subset \mathbb{R}^n$ is convex if, for every pair of points $\mathbf{x}, \mathbf{y} \in X$ the line segment joining them is contained in X; i.e.

$$t \cdot \mathbf{x} + (1-t)\mathbf{y} \subset X \quad \forall\, t \in [0,1]$$

so X is closed under taking non-negative affine combinations.

Convex Polyhedron: A subset $X \subset \mathbb{R}^n$ defined by a set of linear inequalities

$$\sum_{i=1}^{n} a_i \mathbf{x}_i \geq c$$

a finite union of convex polyhedron is a **Polyhedron**.

Regular: A polyhedron is regular if, all faces, vertices, and edges are identical. With an identical vertex meaning: there are the same number of edges at each vertex and that the angles between them are all congruent.

Platonic Solid: A Regular convex polyhedron in $\mathbb{R}^3$.

Lattice: A lattice $L \subseteq \mathbb{R}^n$, is a discrete subgroup of $\mathbb{R}^n$ containing n linearly independent vectors. Where by discrete, we mean no accumulation point; that is, $\forall\, \mathbf{x} \in L,\ \exists\, \epsilon > 0$ such that

$$d(\mathbf{x}, \mathbf{y}) \geq \epsilon, \quad \forall\, \mathbf{y} \in L \text{ such that } \mathbf{y} \neq \mathbf{x}$$

alternatively for a basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ of $\mathbb{R}^n$ we have

$$L = \left\{ \sum_{i=1}^{n} n_i \mathbf{e}_i : n_i \in \mathbb{Z} \right\}$$

Point Group: A crystal group $G \subseteq I(\mathbb{R}^n)$ has the normal subgroup

$$G_T = G \cap T \cong G \cap \mathbb{R}^n \lhd G$$

which forms a lattice in $\mathbb{R}^n$. The point group is the quotient

$$G/G_T \leq \mathrm{O}(n)$$

and is finite.

Unitary Group: The set of $n \times n$ complex matrices

$$\mathrm{U}(n) := \{A \in \mathrm{M}(n, \mathbb{C}) : A\overline{A}^T = I_n\}$$

the rows are orthonormal with respect to the hermitian inner product

$$\langle \mathbf{z}, \mathbf{w} \rangle_{\mathbb{C}^n} = \sum_{i=1}^{n} z_i \overline{w}_i, \quad \text{on } \mathbb{C}^n$$

therefore, for $A \in \mathrm{U}(2)$ we have

$$A = \begin{bmatrix} \alpha & \beta \\ -\lambda\overline{\beta} & \lambda\overline{\alpha} \end{bmatrix} \quad \text{such that } \alpha\overline{\alpha} + \beta\overline{\beta} = 1, \ |\lambda| = 1$$

Special Unitary Group: $\mathrm{SU}(n) \subset \mathrm{U}(n)$ is the subset of the unitary matrices such that

$$A \in \mathrm{SU}(n) \implies \det(A) = 1$$

for $A \in \mathrm{SU}(2)$ we have

$$A = \begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix}$$

i.e. $\lambda = 1$, and has inverse

$$A^{-1} = \begin{bmatrix} \overline{\alpha} & -\beta \\ \overline{\beta} & \alpha \end{bmatrix}$$

Homogeneous Coordinates: Representing the points of $\mathbb{RP}^1$ as ratios of elements of $\mathbb{R}$, that is for $x, y \in \mathbb{R}$ if $y \neq 0$, then $\frac{x}{y} \in \mathbb{RP}^1$, and if $y = 0$ then $\frac{x}{y} = \{\infty\} \in \mathbb{RP}^1$. And so

$$\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{\mathbf{0}\})/\sim \quad \text{where} \quad \mathbf{x} \sim \mathbf{y} \iff \mathbf{y} = \lambda\mathbf{x}, \ \text{for } \lambda \in \mathbb{R} \setminus \{0\}$$

similarly for any vector space V, over any field $\mathbb{F}$ we have

$$\mathbb{P}(V) = (V \setminus \{\mathbf{0}\})/\sim \text{ where} \quad \mathbf{v} \sim \mathbf{w} \iff \mathbf{w} = \lambda\mathbf{v}, \ \text{for } \lambda \in \mathbb{F} \setminus \{0\}$$

Space of Linear Transformations: For two vector spaces V and W, over a common field $\mathbb{F}$, the space of all linear transformations from V to W is denoted

$$\mathrm{Hom}(V, W)$$

Dual: For a vector space V, over a field $\mathbb{F}$, the dual space V^* is the space of all linear functionals

$$\mathrm{Hom}(V, \mathbb{F})$$

a basis $\{v_1, \ldots, v_n\}$ of V determines a basis $\{f_1, \ldots, f_n\}$ of V^* where the dual basis acts on $\{v_1, \ldots, v_n\}$ by

$$f_i(v_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Projective Group: The quotient of the group of linear isomorphism from a Vector Space, by scalar multiplication

$$\mathrm{PGL}(n, \mathbb{F}) = \mathrm{GL}(n, \mathbb{F})/\{\lambda \cdot Id_V\}$$

2 Notes

$\mathrm{GL}(n, \mathbb{R}) \subset \mathrm{M}(n, \mathbb{R})$ is the set of units, elements with multiplicative inverse.

$$\det : \mathrm{M}(n, \mathbb{R}) \to \mathbb{R}$$

is a continuous map. since it is a polynomial in its entries. Then, since $\{0\} \in \mathbb{R}$ is a singleton and hence closed, we have that $\mathbb{R} \setminus \{0\}$ must therefore be open, and hence

$$\det{}^{-1}\Big(\mathbb{R} \setminus \{0\}\Big) = \mathrm{GL}(n, \mathbb{R}) \subset \mathrm{M}(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$$

must be open. Since the determinant is multiplicative

$$\det|_{\mathrm{GL}(n,\mathbb{R})} : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R} \setminus \{0\}$$

is a homomorphism with

$$\ker\Big(\det|_{\mathrm{GL}(n,\mathbb{R})}\Big) = \mathrm{SL}(n, \mathbb{R}) = \{A \in \mathrm{GL}(n, \mathbb{R}) : \det(A) = 1\}$$

which then tells us that

$$\det{}^{-1}(\{1\}) = \mathrm{SL}(n, \mathbb{R}) \subseteq \mathrm{GL}(n, \mathbb{R})$$

must be closed.

Lemma 1.

(i) If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear isometry, then T is orthogonal.

(ii) If $T : \mathbb{R}^n \to \mathbb{R}^n$ is linear and norm-preserving, then T is orthogonal.

Proof. First we note that any linear isometry is norm-preserving since it satisfies

$$\begin{aligned}
\langle T(\mathbf{x}-\mathbf{y}), T(\mathbf{x}-\mathbf{y})\rangle &= \langle T(\mathbf{x})-T(\mathbf{y}), T(\mathbf{x})-T(\mathbf{y})\rangle && \text{linearity} \\
&= ||T(\mathbf{x})-T(\mathbf{y})||^2 && \langle\cdot,\cdot\rangle \text{ induces } ||\cdot|| \\
&= d\big(T(\mathbf{x}), T(\mathbf{y})\big)^2 && ||\cdot|| \text{ induces } d \\
&= d(\mathbf{x},\mathbf{y})^2 && T \text{ is an isometry} \\
&= ||\mathbf{x}-\mathbf{y}||^2 \\
&= \langle \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y}\rangle
\end{aligned}$$

and so, letting $\mathbf{y} = \mathbf{0} \in \mathbb{R}^n$ then gives

$$||T(\mathbf{x})||^2 = ||\mathbf{x}||^2$$

That is (i) $\implies$ (ii), and so it suffices to prove (ii). Next we note that

$$\begin{aligned}
\langle T(\mathbf{x})-T(\mathbf{y}), T(\mathbf{x})-T(\mathbf{y})\rangle &= \langle T(\mathbf{x}), T(\mathbf{x})-T(\mathbf{y})\rangle - \langle T(\mathbf{y}), T(\mathbf{x})-T(\mathbf{y})\rangle \\
&= \langle T(\mathbf{x}), T(\mathbf{x})\rangle - \langle T(\mathbf{x}), T(\mathbf{y})\rangle - \langle T(\mathbf{y}), T(\mathbf{x})\rangle + \langle T(\mathbf{y}), T(\mathbf{y})\rangle \\
&= \langle T(\mathbf{x}), T(\mathbf{x})\rangle - 2\langle T(\mathbf{x}), T(\mathbf{y})\rangle + \langle T(\mathbf{y}), T(\mathbf{y})\rangle \\
&= ||T(\mathbf{x})||^2 - 2\langle T(\mathbf{x}), T(\mathbf{y})\rangle + ||T(\mathbf{y})||^2
\end{aligned}$$

Now, from above we have

$$\begin{aligned}
&\langle T(\mathbf{x}-\mathbf{y}), T(\mathbf{x}-\mathbf{y})\rangle = \langle \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y}\rangle \\
&\implies ||T(\mathbf{x})||^2 - 2\langle T(\mathbf{x}), T(\mathbf{y})\rangle + ||T(\mathbf{y})||^2 = ||\mathbf{x}||^2 - 2\langle \mathbf{x}, \mathbf{y}\rangle + ||\mathbf{y}||^2 \\
&\implies -2\langle T(\mathbf{x}), T(\mathbf{y})\rangle = -2\langle \mathbf{x}, \mathbf{y}\rangle && T \text{ is norm-preserving} \\
&\implies \langle T(\mathbf{x}), T(\mathbf{y})\rangle = \langle \mathbf{x}, \mathbf{y}\rangle
\end{aligned}$$

and thus, T is orthogonal. □

Theorem 2. The isometries of $\mathbb{R}^n$ are given by

$$I(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R}^n \text{ such that } f(\mathbf{x}) = A\mathbf{x} + \mathbf{a}, A \in \mathrm{O}(n), \mathbf{a} \in \mathbb{R}^n\}$$

Proof. First let

$$f : \mathbb{R}^n \to \mathbb{R}^n, \text{ by } f(\mathbf{x}) = A\mathbf{x} + \mathbf{a}$$

with $A \in \mathrm{O}(n)$ and $\mathbf{a} \in \mathbb{R}^n$, then

$$\begin{aligned}
d\big(f(\mathbf{x}), f(\mathbf{y})\big) &= ||f(\mathbf{x}) - f(\mathbf{y})|| \\
&= ||A\mathbf{x} + \mathbf{a} - (A\mathbf{y} + \mathbf{a})|| \\
&= ||A\mathbf{x} - A\mathbf{y}|| \\
&= ||A(\mathbf{x}-\mathbf{y})|| \\
&= \sqrt{\langle A(\mathbf{x}-\mathbf{y}), A(\mathbf{x}-\mathbf{y})\rangle} \\
&= \sqrt{\langle \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y}\rangle} && A \text{ orthogonal} \\
&= ||\mathbf{x}-\mathbf{y}|| \\
&= d(\mathbf{x},\mathbf{y})
\end{aligned}$$

and so $f \in I(\mathbb{R}^n)$.

Now, given arbitrary $f \in I(\mathbb{R}^n)$ let us choose

$$f(\mathbf{0}) = \mathbf{a}$$

and so for $T_{-\mathbf{a}} \circ f$ we have

$$\big(T_{-\mathbf{a}} \circ f\big)(\mathbf{0}) = T_{-\mathbf{a}}(\mathbf{a}) = \mathbf{a} - \mathbf{a} = \mathbf{0}$$

and note that

$$\begin{aligned}
d\Big(\big(T_{-\mathbf{a}} \circ f\big)(\mathbf{x}), \big(T_{-\mathbf{a}} \circ f\big)(\mathbf{y})\Big) &= ||f(\mathbf{x}) - \mathbf{a} - (f(\mathbf{y}) - \mathbf{a})|| \\
&= ||f(\mathbf{x}) - f(\mathbf{y})|| \\
&= d\big(f(\mathbf{x}), f(\mathbf{y})\big) \\
&= d(\mathbf{x}, \mathbf{y})
\end{aligned}$$

and so $T_{-\mathbf{a}} \circ f \in I(\mathbb{R}^n)$. Let

$$g = T_{-\mathbf{a}} \circ f \implies f = T_{\mathbf{a}} \circ g$$

then $\forall\ \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

$$\begin{aligned}
&d\big(g(\mathbf{x}), g(\mathbf{y})\big)^2 = d(\mathbf{x}, \mathbf{y})^2 \\
\implies &||g(\mathbf{x}) - g(\mathbf{y})||^2 = ||\mathbf{x} - \mathbf{y}||^2
\end{aligned}$$

and setting $\mathbf{y} = \mathbf{0} \in \mathbb{R}^n$ gives

$$||g(\mathbf{x})||^2 = ||\mathbf{x}||^2 \quad \forall\ \mathbf{x} \in \mathbb{R}^n$$

now

$$\begin{aligned}
&||g(\mathbf{x}) - g(\mathbf{y})||^2 = ||\mathbf{x} - \mathbf{y}||^2 \\
&\implies \langle g(\mathbf{x}) - g(\mathbf{y}), g(\mathbf{x}) - g(\mathbf{y})\rangle = \langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y}\rangle \\
&\implies \langle g(\mathbf{x}), g(\mathbf{x})\rangle + \langle g(\mathbf{y}), g(\mathbf{y})\rangle - 2\langle g(\mathbf{x}), g(\mathbf{y})\rangle = \langle \mathbf{x}, \mathbf{x}\rangle + \langle \mathbf{y}, \mathbf{y}\rangle - 2\langle \mathbf{x}, \mathbf{y}\rangle \\
&\implies ||g(\mathbf{x})||^2 + ||g(\mathbf{y})||^2 - 2\langle g(\mathbf{x}), g(\mathbf{y})\rangle = ||\mathbf{x}||^2 + ||\mathbf{y}||^2 - 2\langle \mathbf{x}, \mathbf{y}\rangle \\
&\implies \langle g(\mathbf{x}), g(\mathbf{y})\rangle = \langle \mathbf{x}, \mathbf{y}\rangle
\end{aligned}$$

and so if g is linear, then g is orthogonal. Now to see that g is linear we note that

$$\begin{aligned}
&||g(\mathbf{x} + \mathbf{y}) - g(\mathbf{x}) - g(\mathbf{y})||^2 \\
&\quad = \langle g(\mathbf{x} + \mathbf{y}) - g(\mathbf{x}) - g(\mathbf{y}), g(\mathbf{x} + \mathbf{y}) - g(\mathbf{x}) - g(\mathbf{y})\rangle \\
&\quad = \langle g(\mathbf{x} + \mathbf{y}), g(\mathbf{x} + \mathbf{y})\rangle + \langle g(\mathbf{x}), g(\mathbf{x})\rangle + \langle g(\mathbf{y}), g(\mathbf{y})\rangle \\
&\qquad\quad - 2\langle g(\mathbf{x} + \mathbf{y}), g(\mathbf{x})\rangle - 2\langle g(\mathbf{x} + \mathbf{y}), g(\mathbf{y})\rangle + 2\langle g(\mathbf{x}), g(\mathbf{y})\rangle \\
&\quad = \langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y}\rangle + \langle \mathbf{x}, \mathbf{x}\rangle + \langle \mathbf{y}, \mathbf{y}\rangle - 2\langle \mathbf{x} + \mathbf{y}, \mathbf{x}\rangle - 2\langle \mathbf{x} + \mathbf{y}, \mathbf{y}\rangle + 2\langle \mathbf{x}, \mathbf{y}\rangle \\
&\quad = ||\mathbf{x} + \mathbf{y} - \mathbf{x} - \mathbf{y}||^2 \\
&\quad = \mathbf{0}
\end{aligned}$$

and therefore we must have

$$g(\mathbf{x}+\mathbf{y}) = g(\mathbf{x}) + g(\mathbf{y}) \quad \forall\ \mathbf{x},\mathbf{y} \in \mathbb{R}^n$$

and similarly for any $c \in \mathbb{R}$ we have

$$\begin{aligned}
||g(c\mathbf{x}) - cg(\mathbf{x})||^2 &= \langle g(c\mathbf{x}) - cg(\mathbf{x}), g(c\mathbf{x}) - cg(\mathbf{x})\rangle \\
&= \langle g(c\mathbf{x}), g(c\mathbf{x})\rangle + c^2 \langle g(\mathbf{x}), g(\mathbf{x})\rangle - 2c \langle g(c\mathbf{x}), g(\mathbf{x})\rangle \\
&= \langle c\mathbf{x}, c\mathbf{x}\rangle + c^2 \langle \mathbf{x}, \mathbf{x}\rangle - 2c \langle c\mathbf{x}, \mathbf{x}\rangle \\
&= \mathbf{0}
\end{aligned}$$

and hence

$$g(c\mathbf{x}) = cg(\mathbf{x})$$

and thus we see that g is a linear transformation, which tells us that $g \in \mathrm{M}(n,\mathbb{R})$, and since

$$\langle g(\mathbf{x}), g(\mathbf{y})\rangle = \langle \mathbf{x}, \mathbf{y}\rangle$$

we see further, that g is orthogonal, and hence $g \in \mathrm{O}(n)$. So let $g = A \in \mathrm{O}(n)$ where we then get

$$f(\mathbf{x}) = \big(T_{\mathbf{a}} \circ g\big)(\mathbf{x}) = A\mathbf{x} + \mathbf{a}, \quad \text{with } A \in \mathrm{O}(n),\ \mathbf{a} \in \mathbb{R}^n$$

$\square$

Corollary 3.

$$\begin{aligned}
T : \mathbb{R}^n &\to I(\mathbb{R}^n) \\
\mathbf{a} &\mapsto T_{\mathbf{a}}
\end{aligned}$$

realizes $\mathbb{R}^n$ as a subgroup of $I(\mathbb{R}^n)$, which is also a normal subgroup. and so

$$I(\mathbb{R}^n)/\mathbb{R}^n \cong \mathrm{O}(n)$$

Proof. Let $g \in I(\mathbb{R}^n)$ we wish to show

$$g^{-1} \circ T_{\mathbf{a}} \circ g \in T \cong \mathbb{R}^n$$

by Theorem 2, we know that

$$g(\mathbf{x}) = A\mathbf{x} + \mathbf{b}, \quad \text{with } A \in \mathrm{O}(n),\ \mathbf{b} \in \mathbb{R}^n$$

and

$$T_{\mathbf{a}}(g(\mathbf{x})) = A\mathbf{x} + \mathbf{b} + \mathbf{a}$$

now,

$$\begin{aligned}
&\mathbf{y} = g(\mathbf{x}) = A\mathbf{x} + \mathbf{b} \\
&\implies \mathbf{y} - \mathbf{b} = A\mathbf{x} \\
&\implies \mathbf{x} = A^{-1}\mathbf{y} - A^{-1}\mathbf{b} \\
&\implies g^{-1}(\mathbf{y}) = A^{-1}\mathbf{y} - A^{-1}\mathbf{b}
\end{aligned}$$

then

$$\begin{aligned}
\big(g^{-1} \circ T_{\mathbf{a}} \circ g\big)(\mathbf{x}) &= A^{-1}(A\mathbf{x} + \mathbf{b} + \mathbf{a}) - A^{-1}\mathbf{b} \\
&= \mathbf{x} + A^{-1}\mathbf{b} + A^{-1}\mathbf{a} - A^{-1}\mathbf{b} \\
&= \mathbf{x} + A^{-1}\mathbf{a} \\
\implies g^{-1} \circ T_{\mathbf{a}} \circ g &= T_{A^{-1}\mathbf{a}} \in T \cong \mathbb{R}^n
\end{aligned}$$

and so $T \cong \mathbb{R}^n$ is a normal subgroup; that is, $\mathbb{R}^n \lhd I(\mathbb{R}^n)$.

Next, to see that the quotient is $\mathrm{O}(n)$ define

$$\phi : I(\mathbb{R}^n) \to \mathrm{O}(n), \text{ by } \phi\big(f(\mathbf{x})\big) = \phi\big(A\mathbf{x} + \mathbf{a}\big) = A$$

then for $f(\mathbf{x}) = A\mathbf{x} + \mathbf{a}$, $g(\mathbf{x}) = B\mathbf{x} + \mathbf{b} \in I(\mathbb{R}^n)$ we have

$$f(g(\mathbf{x})) = A(B\mathbf{x} + \mathbf{b}) + \mathbf{a} = AB\mathbf{x} + A\mathbf{b} + \mathbf{a}$$

which implies

$$\phi(f \circ g) = AB = \phi(f)\phi(g)$$

and so ϕ is a homomorphism, with

$$\ker(\phi) = \{f(\mathbf{x}) = A\mathbf{x} + \mathbf{a} : A = I_n\} = \{T_{\mathbf{a}} : \mathbf{a} \in \mathbb{R}^n\} \cong \mathbb{R}^n$$

and so, by the Fundamental Homomorphism Theorem we have

$$I(\mathbb{R}^n)/\ker(\phi) = I(\mathbb{R}^n)/\mathbb{R}^n \cong \mathrm{O}(n)$$

□

For the relationship between $\mathrm{O}(n)$ and $\mathrm{GL}(n, \mathbb{R})$, we note that $A \in \mathrm{GL}(n, \mathbb{R})$ has independent columns, while $B \in \mathrm{O}(n)$ has orthonormal columns. Where we know that the Gram-Schmidt process transforms a set of independent vectors into a set of orthonormal vectors, and so should define a mapping

$$\begin{aligned}
\mathrm{GL}(n, \mathbb{R}) &\to \mathrm{O}(n) \\
A &\mapsto B
\end{aligned}$$

Proposition 4. $\mathrm{UT}_+(n)$ is a subgroup of $\mathrm{GL}(n, \mathbb{R})$, or $\mathrm{UT}_+(n) \leq \mathrm{GL}(n, \mathbb{R})$.

Proof. If $A \in \mathrm{UT}_+(n)$, then

$$\det(A) = a_{11} \cdot a_{22} \cdots a_{nn} > 0$$

and so $\mathrm{UT}_+(n) \subset \mathrm{GL}(n, \mathbb{R})$. Now

$$A \in \mathrm{UT}_+(n) \iff a_{ij} = \begin{cases} 0, & i > j \\ > 0, & i = j \end{cases}$$

and so for $A, B \in \mathrm{UT}_+(n)$ we have

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

now if $i > j$ then we have

Case 1: $i > k$ in which case $a_{ik} = 0$.

Case 2: $k \geq i > j$ in which case $b_{kj} = 0$

and if $i = j$ then

$$(AB)_{ii} = a_{ii} b_{ii} > 0$$

and so $AB \in UT(n)$, and so we have closure. For the identity element

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \implies I_n \in \mathrm{UT}_+(n)$$

Now for $A \in \mathrm{UT}_+(n)$ we have $\det(A) \neq 0$, and so A^{-1} exists. Where

$$I_n = AA^{-1} = 1_{ii} \implies a_{ii}^{-1} > 0 \ \forall \ i$$

and

$$\sum_{k=1}^{n} a_{ik} b_{kj} = 0 \qquad \text{if } i \neq k \neq j$$

and so fixing $i = 2$ and $j = 1$ we get

$$a_{21}^{-1} a_{11} + a_{22}^{-1} a_{21} + \cdots + a_{2n}^{-1} a_{n1} = a_{21}^{-1} a_{11} + 0 \cdots + 0 = 0$$

yet

$$a_{11} \neq 0 \implies a_{21}^{-1} = 0$$

which then gives

$$a_{21}^{-1} = a_{31}^{-1} = \cdots = a_{n1}^{-1} = 0$$

For the 2^{nd} column we get

$$\cancelto{0}{a_{31}^{-1} a_{12}} + a_{32}^{-1} a_{22} + 0 \cdots + 0 = 0$$

yet

$$a_{22} \neq 0 \implies a_{32}^{-1} = 0$$

which then gives

$$a_{32}^{-1} = a_{42}^{-1} = \cdots = a_{n2}^{-1} = 0$$

so assume the result holds for $j = l$, then for the $(l+1)^{th}$ column we have

$$a_{(l+2)1}^{-1} a_{1(l+1)} + a_{(l+2)2}^{-1} a_{2(l+1)} + \cdots + a_{(l+2)(l+1)}^{-1} a_{(l+1)(l+1)} + 0 \cdots + 0 = 0$$

and by hypothesis we have

$$a_{(l+2)1}^{-1} = \cdots = a_{(l+2)l}^{-1} = 0$$

and so we have

$$a_{(l+2)(l+1)}^{-1} a_{(l+1)(l+1)} = 0$$

yet

$$a_{(l+1)(l+1)} \neq 0 \implies a_{(l+2)(l+1)}^{-1} = 0$$

and so by induction we get that $A^{-1} \in \mathrm{UT}_+(n)$.

So $\mathrm{UT}_+(n)$ is a subset, closed under the binary operation of $\mathrm{GL}(n,\mathbb{R})$ which contains the identity element, and all of its inverses, and thus, we can conclude that $\mathrm{UT}_+(n) \leq \mathrm{GL}(n,\mathbb{R})$. $\square$

Theorem 5. For a given $A \in \mathrm{GL}(n,\mathbb{R})$, there are unique matrices $B \in \mathrm{O}(n)$, and $C \in \mathrm{UT}_+(n)$ such that

$$A = BC$$

Proof. Let $A \in \mathrm{GL}(n,\mathbb{R})$ be given and let $\mathbf{a}_1, \dots, \mathbf{a}_n$ be the columns of A, so that

$$A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix}$$

since each column is independent, using the Gram-Schmidt process we can construct vectors $\mathbf{f}_1, \dots, \mathbf{f}_n$ from $\mathbf{a}_1, \dots, \mathbf{a}_n$ such that the $\mathbf{f}_i$'s are orthogonal, by

$$\begin{aligned} \mathbf{f}_1 &= \mathbf{a}_1 \\ \mathbf{f}_k &= \mathbf{a}_k - \sum_{i=1}^{k-1} \frac{\langle \mathbf{a}_k, \mathbf{f}_i \rangle}{\langle \mathbf{f}_i, \mathbf{f}_i \rangle} \mathbf{f}_i \end{aligned}$$

or

$$\begin{aligned} \mathbf{a}_1 &= 1 \cdot \mathbf{f}_1 \\ \mathbf{a}_k &= 1 \cdot \mathbf{f}_k + \sum_{i=1}^{k-1} t^1_{ki} \mathbf{f}_i \end{aligned}$$

where we have designated the constant $\frac{\langle \mathbf{a}_k, \mathbf{f}_i \rangle}{\langle \mathbf{f}_i, \mathbf{f}_i \rangle} = t^1_{ki}$ and since the vectors depend only on the previous ones, we get an upper triangular matrix with the t^1_{ij} entries. That is, if F is the matrix with the $\mathbf{f}_i$'s as columns then

$$A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix} \begin{bmatrix} 1 & t^1_{12} & \cdots & t^1_{1n} \\ 0 & 1 & \cdots & t^1_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = FT_1 \quad \text{with } T_1 \in \mathrm{UT}_+(n)$$

which then gives

$$F = AT_1^{-1}$$

Next we normalize by setting

$$\mathbf{b}_i = \frac{\mathbf{f}_1}{||\mathbf{f}_i||}$$

then if B is the matrix with the $\mathbf{b}_i$'s as columns we have

$$B = \begin{bmatrix} \mathbf{b}_1 & \dots & \mathbf{b}_n \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix} \begin{bmatrix} \frac{1}{||\mathbf{f}_1||} & t^2_{12} & \cdots & t^2_{1n} \\ 0 & \frac{1}{||\mathbf{f}_2||} & \cdots & t^2_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{||\mathbf{f}_n||} \end{bmatrix} = FT_2 \quad \text{with } T_2 \in \mathrm{UT}_+(n)$$

then since $\mathrm{UT}_+(n)$ is a subgroup of $\mathrm{GL}(n, \mathbb{R})$ we know that the product of elements in $\mathrm{UT}_+(n)$ is also in $\mathrm{UT}_+(n)$ so letting

$$C = (T_1^{-1}T_2)^{-1} \in \mathrm{UT}_+(n)$$

then we have

$$B = FT_2 = (AT_1^{-1})T_2 \implies A = B(T_1^{-1}T_2)^{-1} = BC$$

where $B \in \mathrm{O}(n)$ and $C \in \mathrm{UT}_+(n)$.

For uniqueness, suppose that A has two such decomposition's, say

$$A = B_1C_1 = B_2C_2 \implies B_2^{-1}B_1 = C_1C_2^{-1} \in \mathrm{O}(n) \cap \mathrm{UT}_+(n)$$

so let $D = B_2^{-1}B_1$, and so we also have $D = C_1C_2^{-1}$ where $D \in \mathrm{O}(n) \cap \mathrm{UT}_+(n)$. Now since $D \in \mathrm{O}(n)$ we have in particular that $D^{-1} \in \mathrm{O}(n)$ and that

$$D^{-1} = D^T$$

and since $\mathrm{UT}_+(n)$ is a subgroup and $D \in \mathrm{UT}_+(n)$ we have that $D^{-1} = D^T \in \mathrm{UT}_+(n)$. Now if D is upper triangular, then D^T must be lower triangular, and since $D^T \in \mathrm{UT}_+(n)$ we must have that D is diagonal. Then as

$$DD^{-1} = DD^T = D^2 = I_n$$

each diagonal entry $d_{ii} = \pm 1$. Yet, since $D \in \mathrm{UT}_+(n)$ its diagonal entries must be positive, and so

$$D = I_n$$

which then tells us that

$$B_1 = B_2, \qquad \text{and} \qquad C_1 = C_2$$

and thus, the decomposition is unique. $\square$

Corollary 6. $\mathrm{GL}(n, \mathbb{R})$ is homeomorphic to $\mathrm{O}(n) \times \mathrm{UT}_+(n)$.

Proof. we construct the homeomorphism with the following sequence of maps

$$\begin{gathered} \mathrm{GL}(n, \mathbb{R}) \to \mathrm{O}(n) \times \mathrm{UT}_+(n) \to \mathrm{GL}(n, \mathbb{R}) \\ A \mapsto (B, C) \mapsto BC \end{gathered}$$

With continuity being given by the factoring, and then the product of polynomials (in the entries of the matrices).

Next we note that $\mathrm{UT}_+(n)$ has $\frac{n(n-1)}{2}$ off-diagonal components which are nonzero, and n diagonal components which are strictly non-negative. And so

$$\mathrm{UT}_+(n) \cong \mathbb{R}^{n(n-1)/2} \times \mathbb{R}^n_{>0}$$

yet we also have the homeomorphism

$$\log : \mathbb{R}_{>0} \to \mathbb{R}$$

with inverse

$$\exp : \mathbb{R} \to \mathbb{R}_{>0}$$

which gives

$$\mathrm{UT}_+(n) \cong \mathbb{R}^{n(n-1)/2} \times \mathbb{R}^n_{>0} \cong \mathbb{R}^{n(n-1)/2} \times \mathbb{R}^n \cong \mathbb{R}^{n(n+1)/2}$$

and therefore $\mathrm{GL}(n, \mathbb{R})$ is homeomorphic to $\mathrm{O}(n) \times \mathbb{R}^{n(n+1)/2}$. $\square$

Lemma 7. If $A \subset \mathbb{R}^n$ is an affine subspace and $\mathbf{a}, \mathbf{b} \in A$, then

$$A - \mathbf{a} = A - \mathbf{b}$$

Proof. First, let $\mathbf{x} - \mathbf{a} \in A - \mathbf{a}$, then we note that

$$\mathbf{x} - \mathbf{a} + \mathbf{b}, \quad \text{where } \mathbf{x}, \mathbf{a}, \mathbf{b} \in A, \text{ and } 1 - 1 + 1 = 1$$

and so $\mathbf{x} - \mathbf{a} + \mathbf{b} \in A$ which then gives

$$\big(\mathbf{x} - \mathbf{a} + \mathbf{b}\big) - \mathbf{b} = \mathbf{x} - \mathbf{a} \in A - \mathbf{b}$$

and so

$$A - \mathbf{a} \subseteq A - \mathbf{b}$$

Next, let $\mathbf{y} - \mathbf{b} \in A - \mathbf{b}$, then we note that

$$\mathbf{y} - \mathbf{b} + \mathbf{a}, \quad \text{where } \mathbf{y}, \mathbf{b}, \mathbf{a} \in A, \text{ and } 1 - 1 + 1 = 1$$

and so $\mathbf{y} - \mathbf{b} + \mathbf{a} \in A$ which then gives

$$\big(\mathbf{y} - \mathbf{b} + \mathbf{a}\big) - \mathbf{a} = \mathbf{y} - \mathbf{b} \in A - \mathbf{a}$$

and so

$$A - \mathbf{b} \subseteq A - \mathbf{a}$$

which gives

$$A - \mathbf{a} = A - \mathbf{b}$$

□

Lemma 8. If $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ with $\mathbf{a} \neq \mathbf{b}$, then

$$B = \{\mathbf{x} \in \mathbb{R}^n : d(\mathbf{x}, \mathbf{a}) = d(\mathbf{x}, \mathbf{b})\}$$

is a hyperplane in $\mathbb{R}^n$.

Proof. First note that

$$d\left(\frac{\mathbf{a}+\mathbf{b}}{2}, \mathbf{a}\right) = \left|\left|\frac{\mathbf{a}+\mathbf{b}}{2} - \mathbf{a}\right|\right| = \left|\left|\frac{\mathbf{b}-\mathbf{a}}{2}\right|\right| = \left|\left|\frac{\mathbf{a}-\mathbf{b}}{2}\right|\right| = d\left(\frac{\mathbf{a}+\mathbf{b}}{2}, \mathbf{b}\right)$$

and so $\frac{\mathbf{a}+\mathbf{b}}{2} \in B$, so we must show that

$$H = B - \frac{\mathbf{a}+\mathbf{b}}{2}$$

is an $(n-1)$-dimensional linear subspace. Now, for $\mathbf{c} = \frac{\mathbf{a}-\mathbf{b}}{2}$ and any $\mathbf{y} \in H$ we have

$$d(\mathbf{y}, \mathbf{c}) = d\left(\mathbf{x} - \frac{\mathbf{a}+\mathbf{b}}{2}, \frac{\mathbf{a}-\mathbf{b}}{2}\right) = \left|\left|\mathbf{x} - \frac{\mathbf{a}+\mathbf{b}}{2} - \frac{\mathbf{a}-\mathbf{b}}{2}\right|\right| = ||\mathbf{x} - \mathbf{a}|| = d(\mathbf{x}, \mathbf{a})$$

and similarly

$$d(\mathbf{y}, -\mathbf{c}) = d(\mathbf{x}, \mathbf{b})$$

and so

$$H = \{\mathbf{y} \in \mathbb{R}^n : d(\mathbf{y}, \mathbf{c}) = d(\mathbf{y}, -\mathbf{c})\}$$

so if $\{\mathbf{c}, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is an orthogonal basis for $\mathbb{R}^n$, then $\{\mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis for H. □

If $H \subset \mathbb{R}^n$ is any hyperplane, then $\forall\ \mathbf{x} \in \mathbb{R}^n$ we have

$$\mathbf{x} = \mathbf{y} + \mathbf{z}, \quad \text{with } \mathbf{y} \in H,\ \mathbf{z} \in H^\perp$$

to see this, let $\mathbf{a} \in H$, then

$$H - \mathbf{a} = \{\mathbf{h} - \mathbf{a} : \mathbf{h} \in H\}$$

is a linear hyperplane and so

$$H - \mathbf{a} = \{\mathbf{b}\}^\perp, \quad \text{for some } \mathbf{b} \in \mathbb{R}^n$$

and so, for any $\mathbf{x} \in \mathbb{R}^n$ we have

$$\mathbf{x} - \mathbf{a} = \lambda\mathbf{b} + \mathbf{c}, \quad \text{where } \mathbf{c} \in H - \mathbf{a}$$

so let $\mathbf{y} = \mathbf{c} + \mathbf{a}$ and $\mathbf{z} = \lambda\mathbf{b}$, then we have

$$\mathbf{x} = \lambda\mathbf{b} + \mathbf{c} + \mathbf{a} = \mathbf{z} + \mathbf{y}, \quad \text{with } \mathbf{y} \in H,\ \mathbf{z} \in H^\perp$$

For uniqueness, suppose that

$$\mathbf{y}_1 + \mathbf{z}_1 = \mathbf{x} = \mathbf{y}_2 + \mathbf{z}_2 \quad \text{with } \mathbf{y}_1, \mathbf{y}_2 \in H,\ \mathbf{z}_1, \mathbf{z}_2 \in H^\perp$$

which then implies that

$$\mathbf{z}_2 - \mathbf{z}_1 = \mathbf{y}_1 - \mathbf{y}_2$$

and that

$$H - \mathbf{y}_1 = H - \mathbf{y}_2$$

and so

$$\mathbf{z}_2, \mathbf{z}_1 \in \perp (H - \mathbf{y}_2) \implies \mathbf{z}_2 - \mathbf{z}_1 \in \perp (H - \mathbf{y}_2)$$

and $\mathbf{y}_1 - \mathbf{y}_2 \in (H - \mathbf{y}_2)$, which then implies

$$\langle \mathbf{z}_2 - \mathbf{z}_1, \mathbf{y}_1 - \mathbf{y}_2 \rangle = \mathbf{0} \implies \mathbf{z}_2 - \mathbf{z}_1 = \mathbf{y}_1 - \mathbf{y}_2 = \mathbf{0}$$

and so we must have $\mathbf{z}_2 = \mathbf{z}_1$ and $\mathbf{y}_1 = \mathbf{y}_2$

Lemma 9. If $f : A \to B$ is an affine map, then the map

$$L_f : A - \mathbf{a} \to B - f(\mathbf{a}), \text{ by } L_f(\mathbf{x}) = f(\mathbf{x} + \mathbf{a}) - f(\mathbf{a})$$

is a linear map. And f is determined by L_f as

$$f(\mathbf{x}) = L_f(\mathbf{x} - \mathbf{a}) + \mathbf{b}$$

Proof. First, let $(\mathbf{x} + \mathbf{a}), (\mathbf{y} + \mathbf{a}), \mathbf{a} \in A$, then

$$\mathbf{x} + \mathbf{y} + \mathbf{a} = (\mathbf{x} + \mathbf{a}) + (\mathbf{y} + \mathbf{a}) - \mathbf{a}, \quad \text{where } 1 + 1 - 1 = 1$$

and so $\mathbf{x}+\mathbf{y}+\mathbf{a} \in A$, then

$$\begin{aligned}
L_f(\mathbf{x}+\mathbf{y}) &= f(\mathbf{x}+\mathbf{y}+\mathbf{a}) - f(\mathbf{a}) \\
&= f\big((\mathbf{x}+\mathbf{a}) + (\mathbf{y}+\mathbf{a}) - \mathbf{a}\big) - f(\mathbf{a}) \\
&= f(\mathbf{x}+\mathbf{a}) + f(\mathbf{y}+\mathbf{a}) - f(\mathbf{a}) - f(\mathbf{a}) && f \text{ is affine} \\
&= f(\mathbf{x}+\mathbf{a}) - f(\mathbf{a}) + f(\mathbf{y}+\mathbf{a}) - f(\mathbf{a}) \\
&= L_f(\mathbf{x}) + L_f(\mathbf{y})
\end{aligned}$$

and

$$\lambda\mathbf{x}+\mathbf{a} = \lambda(\mathbf{x}+\mathbf{a}) + (1-\lambda)\mathbf{a}, \quad \text{where } \lambda + (1-\lambda) = 1$$

and so $\lambda\mathbf{x}+\mathbf{a} \in A$, and

$$\begin{aligned}
L_f(\lambda\mathbf{x}) &= f(\lambda\mathbf{x}+\mathbf{a}) - f(\mathbf{a}) \\
&= f\big(\lambda(\mathbf{x}+\mathbf{a}) + (1-\lambda)\mathbf{a}\big) - f(\mathbf{a}) \\
&= \lambda f(\mathbf{x}+\mathbf{a}) + (1-\lambda)f(\mathbf{a}) - f(\mathbf{a}) \\
&= \lambda f(\mathbf{x}+\mathbf{a}) - \lambda f(\mathbf{a}) \\
&= \lambda\Big(f(\mathbf{x}+\mathbf{a}) - f(\mathbf{a})\Big) \\
&= \lambda L_f(\mathbf{x})
\end{aligned}$$

and so L_f is linear.

Moreover, letting $f(\mathbf{a}) = \mathbf{b}$ we get

$$\begin{aligned}
L_f(\mathbf{x}-\mathbf{a}) &= f(\mathbf{x}-\mathbf{a}+\mathbf{a}) - f(\mathbf{a}) \\
&= f(\mathbf{x}) - \mathbf{b} \\
\implies f(\mathbf{x}) &= L_f(\mathbf{x}-\mathbf{a}) + \mathbf{b}
\end{aligned}$$

□

Corollary 10. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is an affine map, then $\exists\, \mathbf{a} \in \mathbb{R}^n$ such that

$$L : \mathbb{R}^n \to \mathbb{R}^n, \text{ by } L(\mathbf{x}) = f(\mathbf{x}) - \mathbf{a}$$

is linear, and so

$$f(\mathbf{x}) = L(\mathbf{x}) + \mathbf{a}$$

Theorem 11. An isometry

$$f : \mathbb{R}^n \to \mathbb{R}^n$$

is uniquely determined by the images $f(\mathbf{a}_0), \ldots, f(\mathbf{a}_n)$ of a set $\{\mathbf{a}_0, \ldots, \mathbf{a}_n\}$ of $n+1$ affinely independent points.

Proof. Let $\{\mathbf{a}_0, \ldots, \mathbf{a}_n\}$ be affinely independent and let $f, g \in I(\mathbb{R}^n)$ be such that

$$f(\mathbf{a}_i) = g(\mathbf{a}_i) \quad \forall\ 0 \le i \le n$$

Then since the compositions of isometries is an isometry we have $g^{-1} \circ f$ is an isometry such that

$$g^{-1}(f(\mathbf{a}_i)) = \mathbf{a}_i \quad \forall\ 0 \le i \le n$$

Defining the translation $T_{-\mathbf{a}_0}$ we have

$$T_{-\mathbf{a}_0}(\{\mathbf{a}_0, \dots, \mathbf{a}_n\}) = \{\mathbf{a}_0 - \mathbf{a}_0, \dots, \mathbf{a}_n - \mathbf{a}_0\} = \{\mathbf{0}, \mathbf{a}_1 - \mathbf{a}_0 \dots, \mathbf{a}_n - \mathbf{a}_0\} := \{\mathbf{0}, \mathbf{b}_1, \dots, \mathbf{b}_n\}$$

then the set $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ forms a basis for $\mathbb{R}^n$. Next, we define the map

$$h = T_{-\mathbf{a}_0} \circ g^{-1} \circ f \circ T_{-\mathbf{a}_0}^{-1}$$

where

$$\begin{aligned} h(\mathbf{b}_i) &= T_{-\mathbf{a}_0} \circ g^{-1} \circ f \circ T_{\mathbf{a}_0}(\mathbf{a}_i - \mathbf{a}_0) \\ &= T_{-\mathbf{a}_0} \circ g^{-1} \circ f(\mathbf{a}_i) \\ &= T_{-\mathbf{a}_0}(\mathbf{a}_i) \\ &= \mathbf{a}_i - \mathbf{a}_0 \\ &= \mathbf{b}_i & \forall\ 1 \le i \le n \end{aligned}$$

and

$$h(\mathbf{0}) = T_{-\mathbf{a}_0} \circ g^{-1} \circ f(\mathbf{a}_0) = T_{-\mathbf{a}_0}(\mathbf{a}_0) = \mathbf{0}$$

so if $\mathbf{y} = h(\mathbf{x})$, since h is the composition of isometries, and hence an isometry, we have

$$d\big(h(\mathbf{x}), h(\mathbf{0})\big) = d(\mathbf{y}, \mathbf{0}) = d(\mathbf{x}, \mathbf{0})$$

and

$$d(\mathbf{x}, \mathbf{b}_i) = d\big(h(\mathbf{x}), h(\mathbf{b}_i)\big) = d(\mathbf{y}, \mathbf{b}_i) \quad \forall\ 1 \le i \le n$$

yet

$$\begin{aligned} & d(\mathbf{y}, \mathbf{0}) = d(\mathbf{x}, \mathbf{0}) \\ & \implies ||\mathbf{y} - \mathbf{0}|| = ||\mathbf{x} - \mathbf{0}|| \\ & \implies ||\mathbf{y}|| = ||\mathbf{x}|| \\ & \implies \sqrt{\langle \mathbf{y}, \mathbf{y} \rangle} = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} \\ & \implies \langle \mathbf{y}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle \end{aligned}$$

and similarly

$$\begin{aligned} & d(\mathbf{y}, \mathbf{b}_i) = d(\mathbf{x}, \mathbf{b}_i) \\ & \implies ||\mathbf{y} - \mathbf{b}_i|| = ||\mathbf{x} - \mathbf{b}_i|| \\ & \implies \sqrt{\langle \mathbf{y} - \mathbf{b}_i, \mathbf{y} - \mathbf{b}_i \rangle} = \sqrt{\langle \mathbf{x} - \mathbf{b}_i, \mathbf{x} - \mathbf{b}_i \rangle} \\ & \implies \langle \mathbf{y} - \mathbf{b}_i, \mathbf{y} - \mathbf{b}_i \rangle = \langle \mathbf{x} - \mathbf{b}_i, \mathbf{x} - \mathbf{b}_i \rangle \\ & \implies \langle \mathbf{y}, \mathbf{y} \rangle - 2 \langle \mathbf{y}, \mathbf{b}_i \rangle + \langle \mathbf{b}_i, \mathbf{b}_i \rangle = \langle \mathbf{x}, \mathbf{x} \rangle - 2 \langle \mathbf{x}, \mathbf{b}_i \rangle + \langle \mathbf{b}_i, \mathbf{b}_i \rangle \\ & \implies \langle \mathbf{x}, \mathbf{x} \rangle - 2 \langle h(\mathbf{x}), \mathbf{b}_i \rangle = \langle \mathbf{x}, \mathbf{x} \rangle - 2 \langle \mathbf{x}, \mathbf{b}_i \rangle \\ & \implies \langle h(\mathbf{x}), \mathbf{b}_i \rangle = \langle \mathbf{x}, \mathbf{b}_i \rangle & \forall\ 1 \le i \le n \end{aligned}$$

and since $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis, we then have

$$\langle h(\mathbf{x}), \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle \quad \forall\, \mathbf{z} \in \mathbb{R}^n$$

which then implies that $h(\mathbf{x}) = \mathbf{x}$ and so $h = Id$; that is

$$Id = T_{-\mathbf{a}_0} \circ g^{-1} \circ f \circ T_{-\mathbf{a}_0}^{-1} \implies Id = g^{-1} \circ f \implies g = f$$

and so f is uniquely determined by its image of $n+1$ affinely independent points. $\square$

This also demonstrates that a point $\mathbf{x} \in \mathbb{R}^n$ is uniquely determined by its distance from $n+1$ affinely independent points.

Theorem 12. If $\{\mathbf{a}_0, \ldots, \mathbf{a}_n\}$ and $\{\mathbf{b}_0, \ldots, \mathbf{b}_n\}$ are two sets of $n+1$ affinely independent points in $\mathbb{R}^n$ with

$$d(\mathbf{a}_i, \mathbf{a}_j) = d(\mathbf{b}_i, \mathbf{b}_j) \quad \forall\, 0 \le i, j \le n$$

then $\exists\, f \in I(\mathbb{R}^n)$ such that

$$f(\mathbf{a}_i) = \mathbf{b}_i \quad \forall\, 0 \le i \le n$$

Proof. Using a translation if necessary let us assume that $\mathbf{a}_0 = \mathbf{0} = \mathbf{b}_0$. which then implies that

$$d(\mathbf{a}_i, \mathbf{a}_0) = d(\mathbf{a}_i, \mathbf{0}) = ||\mathbf{a}_i - \mathbf{0}|| = ||\mathbf{a}_i|| = ||\mathbf{b}_i|| = ||\mathbf{b}_i - \mathbf{0}|| = d(\mathbf{b}_i, \mathbf{0}) = d(\mathbf{b}_i, \mathbf{b}_0)$$

Then both $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ and $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ are bases for $\mathbb{R}^n$ and by assumption we have

$$\begin{aligned}
&d(\mathbf{a}_i, \mathbf{a}_j) = d(\mathbf{b}_i, \mathbf{b}_j) \\
&\implies ||\mathbf{a}_1 - \mathbf{a}_j|| = ||\mathbf{b}_i - \mathbf{b}_j|| \\
&\implies \sqrt{\langle \mathbf{a}_i - \mathbf{a}_j, \mathbf{a}_i - \mathbf{a}_j \rangle} = \sqrt{\langle \mathbf{b}_i - \mathbf{b}_j, \mathbf{b}_i - \mathbf{b}_j \rangle} \\
&\implies \langle \mathbf{a}_i - \mathbf{a}_j, \mathbf{a}_i - \mathbf{a}_j \rangle = \langle \mathbf{b}_i - \mathbf{b}_j, \mathbf{b}_i - \mathbf{b}_j \rangle \\
&\implies \langle \mathbf{a}_i, \mathbf{a}_i \rangle - 2\langle \mathbf{a}_i, \mathbf{a}_j \rangle + \langle \mathbf{a}_j, \mathbf{a}_j \rangle = \langle \mathbf{b}_i, \mathbf{b}_i \rangle - 2\langle \mathbf{b}_i, \mathbf{b}_j \rangle + \langle \mathbf{b}_j, \mathbf{b}_j \rangle \\
&\implies ||\mathbf{a}_i|| - 2\langle \mathbf{a}_i, \mathbf{a}_j \rangle + ||\mathbf{a}_j|| = ||\mathbf{b}_i|| - 2\langle \mathbf{b}_i, \mathbf{b}_j \rangle + ||\mathbf{b}_j|| \\
&\implies \langle \mathbf{a}_i, \mathbf{a}_j \rangle = \langle \mathbf{b}_i, \mathbf{b}_j \rangle \qquad \forall\, 1 \le i \le n
\end{aligned}$$

so let g be the unique linear transformation such that

$$g(\mathbf{a}_i) = \mathbf{b}_i \quad \forall\, 1 \le i \le n$$

next, let

$$\mathbf{x} - \mathbf{y} = \sum_{i=1}^{n} \lambda_i \mathbf{a}_i \quad \text{since } \{\mathbf{a}_i\}_{i=1}^{n} \text{ a basis for } \mathbb{R}^n$$

then by the linearity of g we have

$$g(\mathbf{x}) - g(\mathbf{y}) = g(\mathbf{x} - \mathbf{y}) = g\left(\sum_{i=1}^{n} \lambda_i \mathbf{a}_i\right) = \sum_{i=1}^{n} \lambda_i g(\mathbf{a}_i) = \sum_{i=1}^{n} \lambda_i \mathbf{b}_i$$

which gives

$$\begin{aligned}
d\big(g(\mathbf{x}), g(\mathbf{y})\big)^2 &= ||g(\mathbf{x}) - g(\mathbf{y})||^2 \\
&= ||g(\mathbf{x} - \mathbf{y})||^2 \\
&= \left|\left|\sum_{i=1}^{n} \lambda_i \mathbf{b}_i\right|\right|^2 \\
&= \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{n} \lambda_j \langle \mathbf{b}_i, \mathbf{b}_j \rangle \\
&= \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{n} \lambda_j \langle \mathbf{a}_i, \mathbf{a}_j \rangle \\
&= \left|\left|\sum_{i=1}^{n} \lambda_i \mathbf{a}_i\right|\right|^2 \\
&= ||\mathbf{x} - \mathbf{y}||^2 \\
&= d(\mathbf{x}, \mathbf{y})^2
\end{aligned}$$

and so g is a linear isometry. And our f will be

$$f = T_{\mathbf{b}_0} \circ g \circ T_{-\mathbf{a}_0}$$

which is affine. $\square$

Theorem 13. let $A \subseteq \mathbb{R}^n$ be an affine subspace of dimension $n - r$. If $f \in I(\mathbb{R}^n)$, such that $f|_A = Id$, then f is a product of at most r reflections.

Proof. Proof by induction on r. Base case: $r = 1$, then $\dim(A) = n - 1$ and A is a hyperplane. If $f = Id_{\mathbb{R}^n}$ we are done. If $f \neq Id_{\mathbb{R}^n}$, then pick $\mathbf{x} \notin A$ such that $f(\mathbf{x}) \neq \mathbf{x}$, and note that since A is a hyperplane we have

$$\mathbf{x} = \mathbf{a} + \mathbf{b}, \quad \mathbf{a} \in A,\ \mathbf{b} \perp (A - \mathbf{a})$$

then, since A is a hyperplane, and f and isometry, it can be defined by

$$A = \{\mathbf{a} \in A : d(f(\mathbf{x}), \mathbf{a}) = d(\mathbf{x}, \mathbf{a})\}, \qquad \text{since } f(\mathbf{a}) = \mathbf{a}$$

and so $f(\mathbf{x}) = \mathbf{a} - \mathbf{b}$, that is

$$R_A = f$$

where R_A fixes $\{\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}, \mathbf{x}\}$, or $(n-1)+1$ affinely independent points. And so f is the composition of 1 reflection.

Now suppose the result holds for $r = k > 1$, then we must check for $r = k + 1$ where $\dim(A) = n - (k+1)$, and $f|_A = Id_A$. If $f = Id_{\mathbb{R}^n}$ we are done. If $f \neq Id_{\mathbb{R}^n}$, then pick $\mathbf{x} \notin A$ such that $f(\mathbf{x}) \neq \mathbf{x}$, and consider the hyperplane defined by

$$H = \{\mathbf{y} \in \mathbb{R}^n : d(f(\mathbf{x}), \mathbf{y}) = d(\mathbf{x}, \mathbf{y})\}$$

since $f|_A = Id_A$ this implies

$$d(f(\mathbf{a}), \mathbf{y}) = d(\mathbf{a}, \mathbf{y}) \implies A \subset H$$

so let $f' = R_H \circ f$, then

$$f'(\mathbf{x}) = R_H \circ f(\mathbf{x}) = \mathbf{x}$$

then we have that f' fixes $\{\mathbf{a}_0, \ldots, \mathbf{a}_{n-(k+1)}, \mathbf{x}\}$, or $n - (k+1) + 1 = n - k$ affinely independent points, so that f' is the identity map on an affine subspace of dimension $n-k$, and so, by hypothesis we have that $f' = R_1 \circ \cdots \circ R_k$ is a composition of at most k reflections giving

$$f = R_H \circ f' = R_H \circ R_1 \circ \cdots \circ R_k$$

is the composition of at most $k+1$ reflections.

So by the PMI we conclude the result holds for all r. $\square$

Corollary 14. If $f \in I(\mathbb{R}^n)$ such that $f(\mathbf{0}) = \mathbf{0}$, then f is orthogonal.

Proof. Since $f \in I(\mathbb{R}^n)$ has the form $f = T_{\mathbf{a}} \circ A$ for $A \in \mathrm{O}(n)$ and $\mathbf{a} \in \mathbb{R}^n$, then

$$f(\mathbf{0}) = \mathbf{0} \implies A\mathbf{0} + \mathbf{a} = \mathbf{0} \implies \mathbf{a} = \mathbf{0}$$

and so $f = A$ is an orthogonal linear transformation. $\square$

A metric can be defined on $I(\mathbb{R}^n)$, by choosing a set $\{\mathbf{x}_0, \ldots, \mathbf{x}_n\} \in \mathbb{R}^n$ of $n+1$ independent points and defining

$$d(f, g) = \max_{0 \le i \le n} d\big(f(\mathbf{x}_i), g(\mathbf{x}_i)\big)$$

this metric is also left-invariant; i.e. $\forall\ f, g, h \in I(\mathbb{R}^n)$

$$d(hf, hg) = d(f, g)$$

if $\{\mathbf{a}_0, \ldots, \mathbf{a}_n\}$ and $\{\mathbf{b}_0, \ldots, \mathbf{b}_n\}$ are two sets $n+1$ affinely independent points in $\mathbb{R}^n$, then first writing the $\mathbf{a}_i$'s in terms of the $\mathbf{b}_i$'s we get

$$\mathbf{a}_i = \sum_{j=1}^{n} \lambda_{ij} \mathbf{b}_j, \quad \text{with } \sum_{j=1}^{n} \lambda_{ij} = 1$$

Given $\epsilon > 0$ choose M such that

$$|\lambda_{ij}| \le M \quad \forall\ i, j$$

and choose $\delta = \frac{\epsilon}{(n+1)M}$. Then

$$d_{\mathbf{b}}(f, g) < \delta \implies ||f(\mathbf{b}_i) - g(\mathbf{b}_i)|| < \delta \quad \forall\ i$$

now as both f, g are affine maps we have

$$\begin{aligned} f(\mathbf{a}_i) - g(\mathbf{a}_i) &= f\left(\sum_{j=1}^{n} \lambda_{ij} \mathbf{b}_j\right) - g\left(\sum_{j=1}^{n} \lambda_{ij} \mathbf{b}_j\right) \\ &= \sum_{j=1}^{n} \lambda_{ij} f(\mathbf{b}_j) - \sum_{j=1}^{n} \lambda_{ij} g(\mathbf{b}_j) \\ &= \sum_{j=1}^{n} \lambda_{ij} \big(f(\mathbf{b}_j) - g(\mathbf{b}_j)\big) \end{aligned}$$

and so

$$\begin{aligned}
||f(\mathbf{a}_i) - g(\mathbf{a}_i)|| &= \left|\left|\sum_{j=1}^{n} \lambda_{ij}\big(f(\mathbf{b}_j) - g(\mathbf{b}_j)\big)\right|\right| \\
&\leq \sum_{j=1}^{n} |\lambda_{ij}| \cdot ||f(\mathbf{b}_i) - g(\mathbf{b}_i)|| \\
&\leq (n+1)M\delta \\
&= (n+1)M\frac{\epsilon}{(n+1)M} \\
&= \epsilon
\end{aligned}$$

and thus $\forall\ \epsilon > 0,\ \exists\ \delta > 0$ such that

$$d_{\mathbf{b}}(f,g) < \delta \implies d_{\mathbf{a}}(f,g) < \epsilon$$

and so both of the metrics induce the same topology on $I(\mathbb{R}^n)$.

Theorem 15. $I(\mathbb{R}^n)$ is homeomorphic to $\mathrm{O}(n) \times \mathbb{R}^n$.

Proof. Given $f \in I(\mathbb{R}^n)$ define

$$\widetilde{f} : \mathbb{R}^n \to \mathbb{R}^n,\ \text{by}\ \widetilde{f}(\mathbf{x}) = f(\mathbf{x}) - f(\mathbf{0})$$

so that $\widetilde{f} = T_{-f(\mathbf{0})} \circ f$ and

$$\widetilde{f}(\mathbf{0}) = T_{-f(\mathbf{0})} \circ f(\mathbf{0}) = f(\mathbf{0}) - f(\mathbf{0}) = \mathbf{0}$$

that is $\widetilde{f} \in I(\mathbb{R}^n)$ as the composition of isometries and fixes $\mathbf{0}$, and hence $\widetilde{f}$ is orthogonal by Corollary 14. Then we have the mapping

$$\begin{aligned}
I(\mathbb{R}^n) &\to \mathrm{O}(n) \times \mathbb{R}^n \\
f &\mapsto (\widetilde{f}, f(\mathbf{0}))
\end{aligned}$$

and its inverse

$$\begin{aligned}
\mathrm{O}(n) \times \mathbb{R}^n &\to I(\mathbb{R}^n) \\
(A, \mathbf{a}) &\mapsto f = T_{\mathbf{a}} \circ A
\end{aligned}$$

□

Lemma 16. The mapping

$$\begin{aligned}
I(\mathbb{R}^n) &\to \{\pm 1\} \\
f &\mapsto \det(\widetilde{f})
\end{aligned}$$

is a group homomorphism.

Proof. Since the determinant is multiplicative we will show that $f \mapsto \widetilde{f} = T_{-f(\mathbf{0})} \circ f$ is a homomorphism. so, for any $f, g \in I(\mathbb{R}^n)$ we have

$$\begin{aligned}
(\widetilde{f} \circ \widetilde{g})(\mathbf{x}) &= T_{-f(\mathbf{0})} \circ f \circ T_{-g(\mathbf{0})} \circ g(\mathbf{x}) \\
&= T_{-f(\mathbf{0})} \circ f\big(g(\mathbf{x}) - g(\mathbf{0})\big) \\
&= T_{-f(\mathbf{0})} \circ f(g(\mathbf{x})) - T_{-f(\mathbf{0})} \circ f(g(\mathbf{0})) && f \text{ is linear} \\
&= f(g(\mathbf{x})) - f(\mathbf{0}) - f(g(\mathbf{0})) + f(\mathbf{0}) \\
&= f(g(\mathbf{x})) - f(g(\mathbf{0})) \\
&= T_{-f(g(\mathbf{0}))} \circ (f \circ g)(\mathbf{x}) \\
&= \widetilde{f \circ g}(\mathbf{x})
\end{aligned}$$

and so the mapping is a homomorphism. □

Isometries of $\mathbb{R}^2$

1. Translations: Direct, group of translations is a normal subgroup of $I(\mathbb{R}^2)$ isomorphic to $\mathbb{R}^2$

$$T_{\mathbf{a}} = R_l \circ R_m, \quad \text{where } l||m,\ d(l,m) = \frac{||\mathbf{a}||}{2}$$

   the elements $T_{\mathbf{a}}$ have infinite order.

2. Rotations: Direct, denoted $R(\mathbf{a}, \alpha)$ for a rotation through the angle α at the point $\mathbf{a}$.

$$\begin{aligned}
f \circ R(\mathbf{a},\alpha) \circ f^{-1} &= R(f(\mathbf{a}),\alpha) && \text{if } f \text{ is a direct isometry} \\
g \circ R(\mathbf{a},\alpha) \circ g^{-1} &= R(g(\mathbf{a}),-\alpha) && \text{if } g \text{ is an opposite isometry}
\end{aligned}$$

   For fixed $\mathbf{a}$ the set of rotations $R(\mathbf{a},\alpha)$ about $\mathbf{a}$ forms a subgroup $\mathrm{SO}(2)|_{\mathbf{a}} \leq I(\mathbb{R}^2)$ where

$$\mathrm{SO}(2)|_{\mathbf{a}} \cong \mathrm{SO}(2) = \mathrm{SO}(2)|_{\mathbf{0}}$$

$$R(\mathbf{a},\alpha) = R_l \circ R_m, \quad \text{where } l, m \text{ are concurrent and the angle between them is } \frac{\alpha}{2}$$

$$|R(\mathbf{a},\alpha)| = \begin{cases} \infty, & \frac{2\pi}{\alpha} \in \mathbb{I} \\ n, & \frac{2\pi}{\alpha} \in \mathbb{Q} \text{ where } \frac{2\pi}{\alpha} = \frac{n}{m} \implies \alpha = \frac{2\pi m}{n} \end{cases}$$

3. Reflections: Opposite, for any pair of lines $l, m \in \mathbb{R}^n$, $\exists\ f \in I(\mathbb{R}^n)$ such that

$$f(l) = m$$

   and

$$f \circ R_l \circ f^{-1} = R_{f(l)} \qquad \forall\ f \in I(\mathbb{R}^2)$$

4. Glide: Opposite,

$$G(l,\mathbf{a}) = R_l \circ T_{\mathbf{a}} = T_{\mathbf{a}} \circ R_l, \quad \text{where } \mathbf{a}||l$$

   and so

$$G(l,\mathbf{a})^2 = R_l \circ T_{\mathbf{a}} \circ R_l \circ T_{\mathbf{a}} = T_{\mathbf{a}} \circ R_l \circ R_l \circ T_{\mathbf{a}} = T_{\mathbf{a}} \circ T_{\mathbf{a}} = T_{2\mathbf{a}}$$

   $|G(l,\mathbf{a})| = \infty$.

Lemma 17. $T_{\mathbf{a}} \circ R_l$ is a glide if $l \not\perp \mathbf{a}$, and a reflection if $l \perp \mathbf{a}$.

Proof. Choose a point $\mathbf{p}$ on l and let

$$\mathbf{q} = \mathbf{p} + \frac{\mathbf{a}}{2}$$

and let $\mathbf{r}$ be the point on l perpendicular to $\mathbf{p} + \mathbf{a}$. Let m be the line parallel to l passing through $\mathbf{p} + \frac{\mathbf{a}}{2} = \mathbf{q}$. Now since any isometry of $\mathbb{R}^2$ is uniquely determined by its image of 3 independent points by Theorem 11, namely $\mathbf{p}, \mathbf{q}, \mathbf{r}$, so we wish to show that

$$T_{\mathbf{a}} \circ R_l = R_m \circ T_{\mathbf{r}-\mathbf{p}}$$

since $\mathbf{r} - \mathbf{p}||l$ and so by definition is a glide.

first we recognise that we can break $\mathbf{a}$ into is components parallel to l and perpendicular to l, so that

$$\mathbf{a} = \mathbf{c} + \mathbf{d}, \quad \text{where } \mathbf{c}||l,\ \mathbf{d} \perp l$$

And recall that $\mathbf{r}$ is the point on l perpendicular to $\mathbf{p} + \mathbf{a}$, so it is the translation of the component of $\mathbf{a}$ parallel to the line l; i.e. $\mathbf{r} = \mathbf{p} + \mathbf{c}$,

1: Since $\mathbf{p} \in l$ it is a fixed point,

$$\big(T_{\mathbf{a}} \circ R_l\big)(\mathbf{p}) = T_{\mathbf{a}}(\mathbf{p}) = \mathbf{p} + \mathbf{a}$$

and since $m||l$ and at a distance of $\frac{\mathbf{d}}{2}$ above $\mathbf{r} = \mathbf{p} + \mathbf{c}$ we get

$$\begin{aligned}
\big(R_m \circ T_{\mathbf{r}-\mathbf{p}}\big)(\mathbf{p}) &= R_m(\mathbf{p} + \mathbf{r} - \mathbf{p}) \\
&= R_m(\mathbf{r}) \\
&= R_m(\mathbf{p} + \mathbf{c}) \\
&= \mathbf{p} + \mathbf{c} + 2\frac{\mathbf{d}}{2} \\
&= \mathbf{p} + \mathbf{c} + \mathbf{d} \\
&= \mathbf{p} + \mathbf{a}
\end{aligned}$$

2: Now we check $\mathbf{q}$, where we get

$$\begin{aligned}
\big(T_{\mathbf{a}} \circ R_l\big)(\mathbf{q}) &= \big(T_{\mathbf{a}} \circ R_l\big)\left(\mathbf{p} + \frac{\mathbf{c} + \mathbf{d}}{2}\right) \\
&= T_{\mathbf{a}}\left(\mathbf{p} + \frac{\mathbf{c}}{2} - \frac{\mathbf{d}}{2}\right) \\
&= \mathbf{p} + \frac{\mathbf{c}}{2} - \frac{\mathbf{d}}{2} + \mathbf{a} \\
&= \mathbf{p} + \frac{\mathbf{c}}{2} - \frac{\mathbf{d}}{2} + \mathbf{c} + \mathbf{d} \\
&= \mathbf{p} + \frac{3}{2}\mathbf{c} + \frac{1}{2}\mathbf{d}
\end{aligned}$$

And recall that $m||l$ and so $\mathbf{q}+\mathbf{c} \in m$ and so is a fixed point.

$$\begin{aligned}
\big(R_m \circ T_{\mathbf{r}-\mathbf{p}}\big)(\mathbf{q}) &= R_m\big(\mathbf{q}+(\mathbf{p}+\mathbf{c})-\mathbf{p}\big) \\
&= R_m(\mathbf{q}+\mathbf{c}) \\
&= \mathbf{q}+\mathbf{c} \\
&= \mathbf{p}+\frac{\mathbf{c}+\mathbf{d}}{2}+\mathbf{c} \\
&= \mathbf{p}+\frac{3}{2}\mathbf{c}+\frac{1}{2}\mathbf{d}
\end{aligned}$$

3: $\mathbf{r}=\mathbf{p}+\mathbf{c} \in l$ and so is a fixed point, giving

$$\big(T_{\mathbf{a}} \circ R_l\big)(\mathbf{r}) = T_{\mathbf{a}}(\mathbf{r}) = \mathbf{r}+\mathbf{a}$$

and

$$\begin{aligned}
\big(R_m \circ T_{\mathbf{r}-\mathbf{p}}\big)(\mathbf{r}) &= R_m(2\mathbf{r}-\mathbf{p}) \\
&= R_m\left(2\mathbf{p}+2\mathbf{c}-\mathbf{p}\right) \\
&= \mathbf{p}+2\mathbf{c}+2\frac{\mathbf{d}}{2} \\
&= (\mathbf{p}+\mathbf{c})+(\mathbf{c}+\mathbf{d}) \\
&= \mathbf{r}+\mathbf{a}
\end{aligned}$$

and so if $l \not\angle \mathbf{a}$ then $T_{\mathbf{a}} \circ R_l$ is a glide. □

Theorem 18. Any isometry $f \in I(\mathbb{R}^2)$ is the identity, a translation, a rotation, a reflection or a glide.

Proof. First suppose that f has a fixed point, translating the fixed point to the origin if necessary we have by Corollary 14, that $f \in \mathrm{O}(2)$ and so f is of the form

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

in which case f is a rotation by an angle of α. Or

$$\begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}$$

in which case f is a reflection through the line

$$y = \tan\left(\tfrac{\alpha}{2}\right)x$$

Next, suppose that f has no fixed point, so let $\mathbf{a} \in \mathbb{R}^2$ be such that

$$f(\mathbf{a}) = \mathbf{b}$$

and consider the hyperplane defined by

$$H = \{\mathbf{x} \in \mathbb{R}^2 : d(\mathbf{a},\mathbf{x}) = d(\mathbf{b},\mathbf{x})\}$$

then

$$R_H \circ f(\mathbf{a}) = R_H(\mathbf{b}) = \mathbf{a}$$

and so $\mathbf{a}$ is a fixed point of $R_H \circ f$, where, from above, we know that $R_H \circ f$ is either a reflection across a line through $\mathbf{a}$, or a rotation about $\mathbf{a}$.

Case 1: If $R_H \circ f$ is a reflection across a line say m, then

$$f = R_H \circ R_m$$

where $H||m$, otherwise, if $H \not|| m$, then they meet at a point say $\mathbf{c}$, which could b a fixed point for $R_H \circ R_m$ and hence f, contradicting the assumption that f has no fixed point. Then, since the reflections must be parallel we know that

$$f = T_{2d(H,m)}$$

and so is a translation.

Case 2: If $R_H \circ f$ is a rotation, then

$$R_H \circ f = R(\mathbf{a}, \alpha)$$

where $R(\mathbf{a}, \alpha) = R_m \circ R_n$ with m, n concurrent with angle $\frac{\alpha}{2}$ between them, and we may choose m such that $\mathbf{a} \in m$ and $m||H$, then

$$f = R_H \circ R_m \circ R_n = T_{2d(H,m)} \circ R_n$$

then from Lemma 17, f is either a reflection, or a glide. Yet, since f has no fixed point, it must be a glide.

$\square$

so we get the following summarization of the isometries of $\mathbb{R}^2$.

$I(\mathbb{R}^2)$	Fixed point	No fixed point
Direct	Rotation	Translation
Opposite	Reflection	Glide

Isometries of $\mathbb{R}^3$

(i) **Direct**:

1: Translation.

2: Rotation: Let $\vec{l}$ be a directed line in $\mathbb{R}^3$, then $R(\vec{l}, \alpha)$ is the rotation about $\vec{l}$, through an angle of α.

3: Screw: The composition of a translation and a rotation. let $\mathbf{a}||\vec{l}$, then

$$T_{\mathbf{a}} \circ R(\vec{l}, \alpha) = R(\vec{l}, \alpha) \circ T_{\mathbf{a}}$$

is a screw.

(ii) **Opposite**:

4: Reflection: If H is a plane in $\mathbb{R}^3$, then R_H is a reflection through H.

5: Glides: If H is a plane in $\mathbb{R}^3$, and $\mathbf{a} \in \mathbb{R}^3$ such that $\mathbf{a}||H$, then

$$T_{\mathbf{a}} \circ R_H = R_H \circ T_{\mathbf{a}}$$

is a glide.

6: Rotatory Reflection: If H is a plane in $\mathbb{R}^3$, and $\vec{l}$ a line such that $\vec{l} \perp H$, then

$$R_H \circ R(\vec{l}, \alpha) = R(\vec{l}, \alpha) \circ R_H$$

is a rotatory reflection.

7: Rotatory Inversion: Is the composition of $R(\vec{l}, \alpha)$, a rotation about a line $\vec{l}$, and an inversion $I_{\mathbf{a}}$ where $\mathbf{a} \in \vec{l}$; that is

$$R(\vec{l}, \alpha) \circ I_{\mathbf{a}}$$

is a rotatory inversion.

Lemma 19. For any $\mathbf{a}$ and $\vec{l}$, if $\mathbf{a} \not\perp \vec{l}$, then

$$T_{\mathbf{a}} \circ R(\vec{l}, \alpha)$$

is a screw.

Proof. Decomposing $\mathbf{a}$ into

$$\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2, \quad \text{where} \quad \mathbf{a}_1||\vec{l},\ \mathbf{a}_2 \perp \vec{l}$$

we get

$$T_{\mathbf{a}} \circ R(\vec{l}, \alpha) = T_{\mathbf{a}_1} \circ T_{\mathbf{a}_2} \circ R(\vec{l}, \alpha)$$

where

$$T_{\mathbf{a}_2} \circ R(\vec{l}, \alpha) = R(\vec{m}, \beta), \quad \text{with } \vec{m}||\vec{l}$$

and hence

$$T_{\mathbf{a}_1} \circ T_{\mathbf{a}_2} \circ R(\vec{l}, \alpha) = T_{\mathbf{a}_1} \circ R(\vec{m}, \beta)$$

is the composition of a translation and a rotation where $\mathbf{a}_1||\vec{m}$ and hence is a screw. Unless $\mathbf{a}_1 = \mathbf{0}$. □

Lemma 20. Let $H \subset \mathbb{R}^3$ be any plane and $\mathbf{a} \in \mathbb{R}^3$ any vector. Then $R_H \circ T_{\mathbf{a}}$ is a glide if $\mathbf{a} \not\perp H$, and a reflection if $\mathbf{a} \perp H$.

Proof. Decomposing $\mathbf{a}$ into

$$\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2, \quad \text{where} \quad \mathbf{a}_1 \perp H,\ \mathbf{a}_2||H$$

we get

$$R_H \circ T_{\mathbf{a}} = R_H \circ T_{\mathbf{a}_1} \circ T_{\mathbf{a}_2}$$

where

$$R_H \circ T_{\mathbf{a}_1} = R_{H'} \quad \text{with } H'||H$$

and hence

$$R_H \circ T_{\mathbf{a}_1} \circ T_{\mathbf{a}_2} = R_{H'} \circ T_{\mathbf{a}_2}$$

is the composition of a reflection and a translation where $\mathbf{a}_2||H'$, and therefore is a glide, unless $\mathbf{a}_2 = \mathbf{0}$, in which case it is a reflection. □

Proposition 21. Every rotatory inversion can be written as a rotatory reflection and every rotatory reflection can be written as a rotatory inversion.

Proof. First, let

$$R(\vec{l}, \alpha) \circ I_{\mathbf{a}}$$

be a rotary inversion. Then write

$$R(\vec{l}, \alpha) = R_{H_1} \circ R_{H_2}, \quad \text{where } \vec{l} = H_1 \cap H_2$$

and the angle between H_1 and H_2 is $\frac{\alpha}{2}$. Since $R(\vec{l}, \alpha) \circ I_{\mathbf{a}}$ is a rotary inversion, $\mathbf{a} \in \vec{l}$, and we may decompose $I_{\mathbf{a}}$ as

$$I_{\mathbf{a}} = R_{H_2} \circ R_{H_3} \circ R_{H_4}$$

where H_2, H_3, H_4 are mutually perpendicular, and $\mathbf{a} = H_2 \cap H_3 \cap H_4$. Then

$$R(\vec{l}, \alpha) \circ I_{\mathbf{a}} = R_{H_1} \circ R_{H_2} \circ R_{H_2} \circ R_{H_3} \circ R_{H_4} = R_{H_1} \circ R_{H_3} \circ R_{H_4}$$

Now, since H_1, H_2 were not perpendicular, we will also have $H_1 \not\perp H_3$, and so $\vec{m} = H_1 \cap H_3$ where $\vec{m} \perp H_4$ since $H_1, H_3 \perp H_4$ and so

$$R_{H_1} \circ R_{H_3} \circ R_{H_4} = R(\vec{m}, \beta) \circ R_{H_4}$$

is the composition of a rotation about $\vec{m}$, and a reflections through a plane perpendicular to $\vec{m}$, and thus, is a rotatory reflection.

Next, let

$$R_H \circ R(\vec{l}, \alpha)$$

be a rotatory reflection. Then write

$$R(\vec{l}, \alpha) = R_{H_1} \circ R_{H_2}, \quad \text{where } \vec{l} = H_1 \cap H_2$$

and the angle between H_1 and H_2 is $\frac{\alpha}{2}$, and $\vec{l} \perp H$ implies $H_1, H_2 \perp H$. Then they all intersect at a point, say $\mathbf{a} = H \cap H_1 \cap H_2$, then we have

$$\begin{aligned} R_H \circ R(\vec{l}, \alpha) &= R_H \circ R_{H_1} \circ R_{H_2} \\ &= R_H \circ R_{H_1} \circ (Id) \circ R_{H_2} \\ &= R_H \circ R_{H_1} \circ R_{H_3} \circ R_{H_3} \circ R_{H_2} \end{aligned}$$

and we may choose H_3 to be perpendicular to both H, H_1 at $\mathbf{a}$ and so

$$R_H \circ R_{H_1} \circ R_{H_3} = I_{\mathbf{a}}$$

and since $H_1 \not\perp H_2$ we will have $H_3 \not\perp H_2$, since we have chosen $H_3 \perp H_1$. Yet, since we have chosen $H_3 \perp H_1$ such that $H_3 \ni \mathbf{a}$, and $\vec{l} \perp H$ to begin with, we will have $\vec{l} = H_3 \cap H_2$, and clearly $\mathbf{a}$ still belongs to $\vec{l}$. And so $R_{H_3} \circ R_{H_2} = R(\vec{l}, \beta)$, and so we have

$$R_H \circ R_{H_1} \circ R_{H_3} \circ R_{H_3} \circ R_{H_2} = I_{\mathbf{a}} \circ R(\vec{l}, \beta)$$

which is a rotation about a line $\vec{l}$, and an inversion about a point $\mathbf{a} \in \vec{l}$, and thus is a rotatory inversion. ☐

Lemma 22. Let $H \subset \mathbb{R}^3$ be a plane, and if $\vec{l} \in \mathbb{R}^3$ is a line such that $\vec{l} \not\subset H$, but $\vec{l} \cap H \neq \varnothing$, then $R_H \circ R(\vec{l}, \alpha)$ is a rotatory reflection.

Note: from Lemma 20, if $\vec{l} || H$, then it is a glide.

Proof. First decomposing

$$R(\vec{l}, \alpha) = R_{H_1} \circ R_{H_2}, \quad \text{where } \vec{l} = H_1 \cap H_2$$

and $\mathbf{a} = H \cap H_1 \cap H_2$, where we may choose $H_1 \perp H$. Then, there is a plane $H_3 \ni \mathbf{a}$, perpendicular to both H, H_1, such that all three are mutually perpendicular, and so

$$\begin{aligned} R_H \circ R(\vec{l}, \alpha) &= R_H \circ R_{H_1} \circ R_{H_2} \\ &= R_H \circ R_{H_1} \circ (Id) \circ R_{H_2} \\ &= R_H \circ R_{H_1} \circ R_{H_3} \circ R_{H_3} \circ R_{H_2} \\ &= I_{\mathbf{a}} \circ R_{H_3} \circ R_{H_2} \end{aligned}$$

and since $H_1 \not\perp H_2$ we have $H_3 \not\perp H_2$, since H_3 was chosen to be perpendicular to H_1, and so $\vec{m} = H_3 \cap H_2$, (not necessarily $\vec{l}$, since $\vec{l}$ may not have been perpendicular H) and thus we get

$$I_{\mathbf{a}} \circ R_{H_3} \circ R_{H_2} = I_{\mathbf{a}} \circ R(\vec{m}, \beta)$$

which is a rotation about a line $\vec{m}$, and an inversion about a point $\mathbf{a} \in \vec{m}$, and thus is a rotatory inversion, and from Proposition 21 must also be a rotatory reflection. ☐

Theorem 23. Any isometry $f \in I(\mathbb{R}^3)$ is the identity, a translation, a rotation, a screw, a reflection, a glide, or a rotatory inversion/reflection.

Proof. Since we know that any isometry $f \in I(\mathbb{R}^3)$ is the composition of at most 4 reflections, we may simply check the cases.

Case 1: 0 Reflections: Then $f = Id$.

Case 2: 1 Reflection: Then $f = R_H$, and is a reflection.

Case 3: 2 Reflections: Then

$$f = R_{H_1} \circ R_{H_2}$$

Sub-case 1: $H_1 || H_2$, then

$$f = R_{H_1} \circ R_{H_2} = T_{2d(H_1, H_2)}$$

and is a translation.

Sub-case 2: $H_1 \not\parallel H_2$, then they meet in a line $\overrightarrow{l} = H_1 \cap H_2$, and have angle α between them. Then

$$f = R_{H_1} \circ R_{H_2} = R(\overrightarrow{l}, 2\alpha)$$

and is a rotation about $\overrightarrow{l}$ through an angle of 2α.

Case 4: 3 Reflections: Then

$$f = R_{H_1} \circ R_{H_2} \circ R_{H_3}$$

where from Case 2 we get

$$R_{H_1} \circ R_{H_2} \circ R_{H_3} = \begin{cases} R_{H_1} \circ T_{2d(H_2,H_3)} \\ R_{H_1} \circ R(\overrightarrow{l}, 2\alpha) \end{cases}$$

Sub-case 1: $f = R_{H_1} \circ T_{2d(H_2,H_3)}$, then Lemma 20, tells us that f is a glide, unless $2d(H_2, H_3) \perp H_1$, in which case it is a reflection.

Sub-case 2: $f = R_{H_1} \circ R(\overrightarrow{l}, 2\alpha)$, then by Lemma 22, f must be a rotatory reflection, unless $\overrightarrow{l} || H_1$, in which case it is a glide.

Case 5: 4 Reflections: Then

$$f = R_{H_1} \circ R_{H_2} \circ R_{H_3} \circ R_{H_4}$$

where

$$\det(f) = \det(R_{H_1})\det(R_{H_2})\det(R_{H_3})\det(R_{H_4}) = (-1)(-1)(-1)(-1) = 1$$

and so f is direct.

Sub-case 1: f has a fixed point. Then $f|_A = Id$ for an affine subspace of dimension $3 - k$ with $1 \leq k \leq 3$, and so by Theorem 13, f is the product of at most 3 reflections. Yet, since f is direct, it cannot be the product of 1 or 3 reflections, and hence must be the product of either 0 reflections, and so $f = Id$; or 2 reflections where Case 2 tells us f is a translation or a rotation. Yet, since f has a fixed point it cannot be a translation, and so must be a rotation.

Sub-case 2: f has no fixed points, then

$$T_{-f(\mathbf{x})} \circ f$$

will have at least one fixed point, and so from Sub-case 1

$$T_{-f(\mathbf{x})} \circ f = \begin{cases} Id & \Longrightarrow\ f = T_{f(\mathbf{x})} \\ \text{Translation} & \Longrightarrow\ f = T_{f(\mathbf{x})} \circ T_{\mathbf{a}} \\ \text{Rotation} & \Longrightarrow\ f = T_{f(\mathbf{x})} \circ R(\overrightarrow{l}, \alpha) \end{cases}$$

in the first 2 instances we have a translation and in the last, from Lemma 19, we have that f is a screw, unless $f(\mathbf{x}) \perp \overrightarrow{l}$, in which case it is a rotation.

$\square$

Corollary 24. If $f \in I(\mathbb{R}^3)$ is direct, and has a fixed point, then f has a fixed line.

For any regular n-gon, P_n centered for convenience at the origin in $\mathbb{R}^2$, we have

$$S(P_n) = D_n$$

and and $f \in S(P_n)$ has the property that

$$f(\mathbf{0}) = \mathbf{0} \implies S(P_n) \subset \mathrm{O}(2)$$

and

$$S(P_n) = \left\langle R\left(0, \tfrac{2\pi}{n}\right), R_l \middle| R\left(0, \tfrac{2\pi}{n}\right)^n, R_l^2, R_l R\left(0, \tfrac{2\pi}{n}\right) R_l = R\left(0, \tfrac{2\pi}{n}\right)^{-1} \right\rangle$$

where l is a line through $\mathbf{0}$, and one of the vertices of P_n. With

$$\begin{aligned} R\left(0, \tfrac{2\pi}{n}\right) &= \text{ direct} \\ R_l &= \text{ opposite} \end{aligned}$$

and

$$\left\langle R\left(0, \tfrac{2\pi}{n}\right)\right\rangle = S(P_n) \cap \mathrm{SO}(2) \quad \text{and} \quad \frac{|S(P_n)|}{\left|\left\langle R\left(0, \frac{2\pi}{n}\right)\right\rangle\right|} = 2$$

and so $S(P_n) = \left\{\left\langle R\left(0, \frac{2\pi}{n}\right)\right\rangle, R_l\left\langle R\left(0, \frac{2\pi}{n}\right)\right\rangle\right\}$, where $R_l\left\langle R\left(0, \frac{2\pi}{n}\right)\right\rangle$ is the coset consisting of reflections through $\mathbf{0}$ and a vertex of P_n, or the midpoint of an edge of P_n.

Lemma 25. If $G < I(\mathbb{R}^n)$ is a finite subgroup, then there exists a fixed point $\mathbf{a} \in \mathbb{R}^n$ such that $g\mathbf{a} = \mathbf{a},\ \forall\ g \in G$.

Proof. The key observation here being that isometries respect the center of mass of a finite set of points. We will first use induction, on the number of points in a given set $X \subset \mathbb{R}^n$, to demonstrate that this is the case. Let $C(X)$ be the center of mass, or centroid of X, so that

$$C(\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i$$

Base case: If $X = \{\mathbf{x}_1, \mathbf{x}_2\}$ then

$$C(\{\mathbf{x}_1, \mathbf{x}_2\}) = \frac{\mathbf{x}_1 + \mathbf{x}_2}{2}$$

which is the midpoint of the line segment $[\mathbf{x}_1, \mathbf{x}_2]$. Which is the unique point such that

$$d(\mathbf{x}_1, C(X)) = d(\mathbf{x}_2, C(X)) = \frac{1}{2}d(\mathbf{x}_1, \mathbf{x}_2)$$

if $f \in I(\mathbb{R}^n)$, then

$$d\big(f(\mathbf{x}_1), f(C(X))\big) = d\big(f(\mathbf{x}_2), f(C(X))\big) = \frac{1}{2}d\big(f(\mathbf{x}_1), f(\mathbf{x}_2)\big)$$

and so $f(C(X))$ is the midpoint of the line segment $[f(\mathbf{x}_1), f(\mathbf{x}_2)]$, that is

$$f(C(X)) = C(f(X))$$

So assume the result holds for $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}\}$, and we must check the case when $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$. First, let

$$\mathbf{y} = C(\{\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}\})$$

then $C(\{\mathbf{x}_1, \ldots, \mathbf{x}_n\})$ is the unique point on the line segment $[\mathbf{y}, \mathbf{x}_n]$ such that

$$\begin{aligned}
d(\mathbf{y}, C(X)) &= \frac{1}{n-1}\sum_{i=1}^{n-1}\mathbf{x}_i - \frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i \\
&= \frac{1}{n(n-1)}\sum_{i=1}^{n-1}\mathbf{x}_i - \frac{\mathbf{x}_n}{n} \\
&= \frac{1}{n}\left(\frac{1}{n-1}\sum_{i=1}^{n-1}\mathbf{x}_i - \mathbf{x}_n\right) \\
&= \frac{1}{n}d(\mathbf{y}, \mathbf{x}_n)
\end{aligned}$$

and

$$\begin{aligned}
d(C(X), \mathbf{x}_n) &= \frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i - \mathbf{x}_n \\
&= \frac{1}{n}\sum_{i=1}^{n-1}\mathbf{x}_i - \frac{n-1}{n}\mathbf{x}_n \\
&= \frac{n-1}{n}\left(\frac{1}{n-1}\sum_{i=1}^{n-1}\mathbf{x}_i - \mathbf{x}_n\right) \\
&= \frac{n-1}{n}d(\mathbf{y}, \mathbf{x}_n)
\end{aligned}$$

and so for any $f \in I(\mathbb{R}^n)$ we then have

$$d\big(f(\mathbf{y}), f(C(X))\big) = \frac{1}{n}d\big(f(\mathbf{y}), f(\mathbf{x}_n)\big), \quad d\big(f(C(X)), f(\mathbf{x}_n)\big) = \frac{n-1}{n}d\big(f(\mathbf{y}), f(\mathbf{x}_n)\big)$$

and by hypothesis we have

$$f(C(\{\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}\})) = C(\{f(\mathbf{x}_1), \ldots, f(\mathbf{x}_{n-1})\})$$

and so we then get

$$f(C(X)) = C(f(X))$$

and so by the principal of mathematical induction we have that the result holds $\forall\ n$.

Next, we observe that $G < I(\mathbb{R}^n)$ is finite iff $\mathrm{Orb}(\mathbf{x})$ is finite for every $\mathbf{x} \in \mathbb{R}^n$. So let $\mathbf{x} \in \mathbb{R}^n$ be given, and note that, for any $h \in G$

$$h(\mathrm{Orb}(\mathbf{x})) = \{h(g\mathbf{x}) : g \in G\} = \{(hg)\mathbf{x} : hg \in G\} = \mathrm{Orb}(\mathbf{x})$$

that is h merely permutes the points of the orbit. Then from above, since h is also an isometry, we then have

$$h(C(\mathrm{Orb}(\mathbf{x})) = C(h(\mathrm{Orb}(\mathbf{x}))) = C(\mathrm{Orb}(\mathbf{x}))$$

and since $h \in G$ was arbitrary we see that the centroid is fixed by every element of G, so set

$$\mathbf{a} = C(\mathrm{Orb}(\mathbf{x}))$$

$\square$

Theorem 26. Every finite subgroup of $I(\mathbb{R}^2)$ is either cyclic or dihedral.

Proof. Let $G < I(\mathbb{R}^2)$ be a subgroup of finite order, say $|G| = n$. Let $\mathbf{a} \in \mathbb{R}^2$ be given, its orbit under the action of G is

$$\mathrm{Orb}(\mathbf{a}) = \{g\mathbf{a} : g \in G\}$$

and is a finite subset of $\mathbb{R}^2$, its centroid is given by

$$C(\mathrm{Orb}(\mathbf{a})) = \frac{1}{n} \sum_{g \in G} g\mathbf{a}$$

from Lemma 25, we know for any $f \in I(\mathbb{R}^2)$ we have $f(C(\mathrm{Orb}(\mathbf{a})) = C(f(\mathrm{Orb}(\mathbf{a}))$, and that $C(\mathrm{Orb}(\mathbf{a}))$ is a fixed point of G. And therefore

$$G < \mathrm{O}(2)|_{C(\mathrm{Orb}(\mathbf{a}))}$$

that is, G is a subgroup of the orthogonal transformations centered at $C(\mathrm{Orb}(\mathbf{a}))$.

Consider first, the direct subgroup of G

$$G_d = G \cap \mathrm{SO}(2)|_{C(\mathrm{Orb}(\mathbf{a}))} = \{R(C(\mathrm{Orb}(\mathbf{a})), \alpha) : \alpha \in [0, \pi)\}$$

and let

$$\alpha_0 = \min\{\alpha : \alpha > 0,\ g = R(C(\mathrm{Orb}(\mathbf{a})), \alpha) \in G_d\}$$

if $\alpha_0 \neq \frac{2\pi}{m}$ for some $m \in \mathbb{N}$, then $\exists\ k$ such that

$$k\alpha_0 \in (2\pi, 2\pi + \alpha_0)$$

and consequently

$$R(C(\mathrm{Orb}(\mathbf{a})), \alpha_0)^k = R(C(\mathrm{Orb}(\mathbf{a})), k\alpha_0) = R(C(\mathrm{Orb}(\mathbf{a})), k\alpha_0 - 2\pi) \in G_d$$

but then

$$0 < k\alpha_0 - 2\pi < \alpha_0 \quad \Rightarrow\Leftarrow$$

contradicting the minimality of α_0. And therefore $\alpha_0 = \frac{2\pi}{m}$. Furthermore, $\forall\ g \in G_d$ we have $g = R\left(C(\mathrm{Orb}(\mathbf{a})), k\frac{2\pi}{m}\right)$, for some $0 \leq k < n$. To see this, fix

$$\beta \in \{R(C(\mathrm{Orb}(\mathbf{a})), \alpha) : \alpha \in [0, \pi)\}$$

and observe, that if $\beta = k\alpha_0 + \beta'$ for some $0 \leq \beta' < \alpha_0$, then $\beta' = \beta - k\alpha_0$ and

$$R(C(\mathrm{Orb}(\mathbf{a})), \beta') = R(C(\mathrm{Orb}(\mathbf{a})), \beta) \circ R(C(\mathrm{Orb}(\mathbf{a})), \alpha_0)^{-k} \in G_d$$

as both $\beta, k\alpha_0 \in \{R(C(\mathrm{Orb}(\mathbf{a})), \alpha) : \alpha \in [0, \pi)\}$ and hence $\beta' \in \{R(C(\mathrm{Orb}(\mathbf{a})), \alpha) : \alpha \in [0, \pi)\}$ and by the minimality of α_0 we must therefore have $\beta' = 0$, and so $\beta = k\alpha_0$. Therefore we get

$$G_d = \left\langle R\left(C(\mathrm{Orb}(\mathbf{a})), \tfrac{2\pi}{m}\right)\right\rangle$$

and is cyclic. If $G = G_d$ then $m = n$ and $|G| = |G_d| = n$, and we are done. If not, then if $h, h' \in G$ are opposite, we note that $h'h^{-1} \in G$ is direct, that is $h'h^{-1} \in G_d$. So let $k = h'h^{-1}$ then

$$h' = kh \in h'G_d = h'\left\langle R\left(C(\mathrm{Orb}(\mathbf{a})), \tfrac{2\pi}{m}\right)\right\rangle$$

and so

$$G = G_d \cup h'G_d = \left\langle R\left(C(\mathrm{Orb}(\mathbf{a})), \tfrac{2\pi}{m}\right)\right\rangle \cup h'\left\langle R\left(C(\mathrm{Orb}(\mathbf{a})), \tfrac{2\pi}{m}\right)\right\rangle$$

and so G_d is a subgroup of index 2 and $m = \frac{n}{2}$. And since h' has the fixed point $C(\mathrm{Orb}(\mathbf{a}))$, it cannot be a translation or glide, and as h' is opposite, we therefore must have that it is a reflection in a line l containing $C(\mathrm{Orb}(\mathbf{a}))$.

Thus G is generated by R_l and $R\left(C(\mathrm{Orb}(\mathbf{a})), \frac{2\pi}{m}\right)$. And we then observe that if $R\left(C(\mathrm{Orb}(\mathbf{a})), k\frac{2\pi}{m}\right) \in G_d$ is any rotation we have

$$\begin{aligned}
\left(R_l \circ R\left(C(\mathrm{Orb}(\mathbf{a})), k\tfrac{2\pi}{m}\right)\right)^2 &= R_l \circ R\left(C(\mathrm{Orb}(\mathbf{a})), k\tfrac{2\pi}{m}\right) \circ R_l \circ R\left(C(\mathrm{Orb}(\mathbf{a})), k\tfrac{2\pi}{m}\right) \\
&= R_l \circ R\left(C(\mathrm{Orb}(\mathbf{a})), k\tfrac{2\pi}{m}\right) \circ R\left(C(\mathrm{Orb}(\mathbf{a})), -k\tfrac{2\pi}{m}\right) \circ R_l \\
&= R_l \circ R_l \\
&= e
\end{aligned}$$

yet this implies that

$$R_l \circ R\left(C(\mathrm{Orb}(\mathbf{a})), k\tfrac{2\pi}{m}\right) \circ R_l = R\left(C(\mathrm{Orb}(\mathbf{a})), k\tfrac{2\pi}{m}\right)^{-1}$$

and so we have

$$\begin{aligned}
G &= \left\langle R_l, R\left(C(\mathrm{Orb}(\mathbf{a})), \tfrac{2\pi}{m}\right) \middle| R_l^2, R\left(C(\mathrm{Orb}(\mathbf{a})), \tfrac{2\pi}{m}\right)^{\frac{n}{2}}, R_l R\left(C(\mathrm{Orb}(\mathbf{a})), \tfrac{2\pi}{m}\right) R_l = R\left(C(\mathrm{Orb}(\mathbf{a})), \tfrac{2\pi}{m}\right)^{-1}\right\rangle \\
&\cong D_{\frac{n}{2}}
\end{aligned}$$

and thus we see, that if $G < I(\mathbb{R}^2)$ is a subgroup of finite order, then G is cyclic or dihedral. $\square$

A polyhedron in $\mathbb{R}^3$ is bounded by planes $P_1 \ldots, P_k$, with a 2-dimensional subset contained in one of the P_i being a face, an edge being the intersection of 2 faces or $P_i \cap P_j$, and a vertex the intersection of 2 edges.

Face homeomorphic to a closed 2-ball: $\overline{B}$
edge homeomorphic to a closed interval: $[\mathbf{a}, \mathbf{b}]$
vertex homeomorphic to a point: $\mathbf{x}$

Platonic Solids:

1. Tetrahedron:
   - i: $|V| = 4$ each with 3 edges and angle $\frac{\pi}{3}$
   - ii: $|E| = 6$
   - iii: $|F| = 4$ each an equilateral triangle
2. Cube:
   - i: $|V| = 8$ each with 3 edges and angle $\frac{\pi}{2}$
   - ii: $|E| = 12$
   - iii: $|F| = 6$ each a square
3. Octahedron:
   - i: $|V| = 6$ each with 4 edges and angle $\frac{\pi}{3}$
   - ii: $|E| = 12$
   - iii: $|F| = 8$ each an equilateral triangle
4. Dodecahedron:
   - i: $|V| = 20$ each with 3 edges and angle $\frac{3\pi}{5}$
   - ii: $|E| = 30$
   - iii: $|F| = 12$ each a regular pentagon
5. Icosahedron:
   - i: $|V| = 12$ each with 5 edges and angle $\frac{\pi}{3}$
   - ii: $|E| = 30$
   - iii: $|F| = 20$ each an equilateral triangle

Theorem 27. There are precisely 5 Platonic Solids.

Proof. Since we already have 5, it suffices to show that there are no more. So suppose that r faces meet at each vertex, with each face a regular n-gon. Then both $r, n \geq 3$, and the sum of the angles α at each each vertex is $\sum \alpha < 2\pi$. Since it is a regular n-gon each angle is identical and given by

$$\alpha = \frac{(n-2)\pi}{n}$$

and so we have

$$
\begin{aligned}
&r \cdot \frac{(n-2)\pi}{n} < 2\pi \\
&\implies r(n-2) < 2n \\
&\implies r(n-2) - 2n < 0 \\
&\implies r(n-2) - 2n + 4 < 4 \\
&\implies r(n-2) - 2(n-2) < 4 \\
&\implies (r-2)(n-2) < 4
\end{aligned}
$$

and the only integer solutions to this equation with $r, n \geq 3$ require at least one of r or n to be 3, and so we have the solutions

$$(r, n) = (3,3), (3,4), (3,5), (4,3), (5,3)$$

and these are the only solutions, and thus, there are precisely 5 platonic solids. $\square$

If $X \subset \mathbb{R}^3$ is a platonic solid, with centroid $C(X) = \mathbf{0}$, then its symmetry group is given by

$$S(X) = \{f \in \mathrm{O}(3) : f(X) = X\}$$

and its direct symmetry group, or rotation symmetry group, is given by

$$S_d(X) = \{f \in \mathrm{SO}(3) : f(X) = X\}$$

where $S_d(X) \lhd S(X)$ and has index 2. All platonic solids, except the regular tetrahedron have central symmetry, or central inversion

$$\eta : \mathbb{R}^3 \to \mathbb{R}^3, \text{ by } \eta(\mathbf{x}) = -\mathbf{x}$$

and η commutes with every linear transformation of $\mathbb{R}^3$.

Proposition 28.

$$\mathrm{O}(3) \cong \mathrm{SO}(3) \times \{\pm 1\}$$

Proof. This is realized by the mapping

$$\xi : \mathrm{O}(3) \to \mathrm{SO}(3) \times \{\pm 1\}, \text{ by } \xi(A) = \begin{cases} (A, 1), & \det(A) = 1 \\ (\eta(A), -1), & \det(A) = -1 \end{cases}$$

with explicit inverse given by

$$\xi^{-1} : \mathrm{SO}(3) \times \{\pm 1\} \to \mathrm{O}(3), \text{ by } \begin{cases} \xi^{-1}(A, 1) = A \\ \xi^{-1}(A, -1) = \eta(A) \end{cases}$$

now if $A, B \in \mathrm{O}(3)$ are such that $\det(A) = 1$ and $\det(B) = -1$ we have

$$
\begin{aligned}
\xi(AB) &= \big(\det(AB) \cdot AB, \det(AB)\big) \\
&= \big(\det(A)\det(B) \cdot AB, \det(A)\det(B)\big) \\
&= \big(\det(A)A \cdot \det(B)B, \det(A)\det(B)\big) \\
&= \big(\det(A)A, \det(A)\big) \cdot \big(\det(B)B, \det(B)\big) \\
&= (A, 1) \cdot (\eta(B), -1) \\
&= \xi(A) \cdot \xi(B)
\end{aligned}
$$

with all other cases being similar, and so ξ is a bijective homomorphism, and thus an isomorphism between O(3) and SO(3) $\times$ $\{\pm 1\}$ $\square$

So if $X \subseteq \mathbb{R}^3$ is a centrally symmetric solid we then have

$$S(X) \cong S_d(X) \times \{\pm 1\}$$

Proposition 29. Let T be the regular tetrahedron, then

$$S(T) \cong S_4 \quad \text{and} \quad S_d(T) \cong A_4$$

Proof. Let $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ be the set of vertices of T. For $i \neq j \in \{1, 2, 3, 4\}$ let

$$H_{i,j} = \text{Hyperplane} \perp \frac{\mathbf{x}_i + \mathbf{x}_j}{2}$$

that is, the hyperplane perpendicular to the bisector of the segment $[\mathbf{x}_i, \mathbf{x}_j]$. Then $H_{i,j}$ will contain the other two vertices, and hence these will be fixed points of the reflection $R_{H_{i,j}}$, and so

$$R_{H_{i,j}} = (i, j)$$

is equivalent to the transposition of the the two vertices $\mathbf{x}_i, \mathbf{x}_j$. By taking products of all such reflections we may generate every permutation in S_4 as an element of $S(T)$. And since an element $f \in I(\mathbb{R}^3)$ is determined by its action on 4 affinely independent points, we have that the action on the vertices determines the isometry, and thus

$$S(T) \cong S_4$$

furthermore, direct isometries correspond to even permutations of vertices, and thus

$$S_d(T) \cong A_4$$

$\square$

Proposition 30. Let C be the cube then

$$S_d(C) \cong A_4$$

Proof. First, WLOG we may suppose that C is centered at $\mathbf{0}$, that is $C(C) = \mathbf{0}$. Fix a face of the cube and label its vertices $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$. For each $i \in \{1, 2, 3, 4\}$ let l_i be the line passing through $\mathbf{0}$ and joining $\mathbf{x}_i$ to $-\mathbf{x}_i$, or l_i is the line such that $-\mathbf{x}_i, \mathbf{x}_i, \mathbf{0} \in l_i$.

Next, if $i \neq j$ then

Case 1: $\mathbf{x}_i$ and $\mathbf{x}_j$ are adjacent. In this case let

$$\mathbf{y} = \frac{\mathbf{x}_i + \mathbf{x}_j}{2}$$

that is $\mathbf{y}$ is the midpoint of the edge defined by $[\mathbf{x}_i, \mathbf{x}_j]$.

Case 2: $\mathbf{x}_i$ and $\mathbf{x}_j$ are not adjacent. In this case $\mathbf{x}_i$ and $-\mathbf{x}_j$ are adjacent, so let

$$\mathbf{y} = \frac{\mathbf{x}_i - \mathbf{x}_j}{2}$$

then $\mathbf{y}$ is the midpoint of the edge defined by $[\mathbf{x}_i, -\mathbf{x}_j]$.

then we let $l_{i,j}$ be the line such that

$$-\mathbf{y}, \mathbf{y}, \mathbf{0} \in l_{i,j}$$

then we note that if $k \neq i, j$ then

$$l_k \perp l_{i,j}$$

and therefore l_k is a fixed line of the rotation $R(\overrightarrow{l}_{i,j}, \pi)$, and so

$$R(\overrightarrow{l}_{i,j}, \pi) = (i, j)$$

acts as the transposition of vertices $\mathbf{x}_i$ and $\mathbf{x}_j$. Thus, every permutation of the diagonals can be realized by a rotation in $S(C)$; that is every permutation of $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ can be realized by a rotation, and since rotation are direct we have

$$S_d(C) \cong S_4$$

we note that these can be the only direct isometries of $S(C)$, since $\mathbf{0}$ is a fixed point, and the other direct isometries of $\mathbb{R}^3$, namely translations and screws, have no fixed points. □

Lemma 31. If $A \in \mathrm{SO}(n)$ and n is odd, then 1 is an eigenvalue of A.

Proof. First we observe that $\det(A) = \det(A^T) = 1$ and so

$$\begin{aligned}
\det(A - I_n) &= \det(A - I_n) \cdot \det(A^T) \\
&= \det(AA^T - A^T) \\
&= \det(I_n - A^T) \\
&= \det(I_n - A)^T \\
&= \det(I_n - A) \\
&= (-1)^n \det(A - I_n) \\
&= -\det(A - I_n) && n \text{ is odd}
\end{aligned}$$

and therefore $\det(A - I_n) = 0$, and thus, 1 is an eigenvalue of A. □

Corollary 32. If $A \in \mathrm{SO}(3)$, then there is an orthogonal matrix B such that

$$B^{-1}AB = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Proof. From Lemma 31, we have 1 is an eigenvalue of A, so let $\mathbf{v}_3$ be the eigenvector corresponding to the eigenvalue 1. We may assume it is a unit vector and can find orthonormal vectors $\mathbf{v}_1, \mathbf{v}_2$ that are a basis for $\mathbf{v}_3^{\perp}$, and forming a matrix $B = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$ which is by construction an orthogonal matrix, and by switching $\mathbf{v}_1$ and $\mathbf{v}_2$ if necessary we may assume that $B \in \mathrm{SO}(3)$, then

$$AB = [A\mathbf{v}_1\ A\mathbf{v}_2\ A\mathbf{v}_3] = [A\mathbf{v}_1\ A\mathbf{v}_2\ \mathbf{v}_3] \in \mathrm{SO}(3)$$

where $A\mathbf{v}_1, A\mathbf{v}_2 \in \mathbf{v}_3^{\perp}$ and so we can write

$$\begin{aligned} A\mathbf{v}_1 &= a_1\mathbf{v}_1 + a_2\mathbf{v}_2 \\ A\mathbf{v}_1 &= b_1\mathbf{v}_1 + b_2\mathbf{v}_2 \end{aligned}$$

which then gives

$$B^{-1}AB = \begin{bmatrix} a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{SO}(3)$$

since the determinant of the product is the product of the determinants, and so the upper 2×2 matrix $\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$ must belong to SO(2) which we know to be the plane rotations, an so A must be a rotation. $\square$

This Corollary simply states that every rotation in $\mathbb{R}^3$ has an axis.

Theorem 33. Let $G < \mathrm{SO}(3)$ be a finite subgroup, then G is either cyclic, dihedral or the direct symmetry group of a Platonic solid. That is G is isomorphic to one of the following

$$C_n, D_n, A_4, S_4, A_5$$

Proof. First, we note that the group SO(3) preserves distances and $\mathbf{0}$, and so acts on $\mathbb{S}^2$, WLOG we will assume $\mathbf{0}$ to be our centroid, since by Lemma 25 our finite group will have some fixed point. So let $S \subset \mathbb{S}^2$ be the set of points $\mathbf{x} \in \mathbb{S}^2$ such that $l_{\mathbf{x}}$, the line containing $\mathbf{x}, \mathbf{0}$, is an axis of rotation for some $g \in G$. Thus,

$$S = \bigcup_{R(\overrightarrow{l_{\mathbf{x}}}, \alpha) \in G} l_{\mathbf{x}} \cap \mathbb{S}^2 = \left\{ \mathbf{x} \in \mathbb{S}^2 : R(\overrightarrow{l_{\mathbf{x}}}, \alpha)\mathbf{x} = \mathbf{x},\ \text{for } R(\overrightarrow{l_{\mathbf{x}}}, \alpha) \in G \text{ such that } R(\overrightarrow{l_{\mathbf{x}}}, \alpha) \neq e \right\}$$

Since G is finite, $|S| < \infty$, and for every $\mathbf{x} \in S$ we also have

$$\left| \left\langle R(\overrightarrow{l_{\mathbf{x}}}, \alpha) \right\rangle \right| < \infty$$

in fact, there exists $n_{\mathbf{x}} \in \mathbb{N}$ such that every rotation $R(\overrightarrow{l_{\mathbf{x}}}, \alpha) \in G$ with $l_{\mathbf{x}}$ as its axis of rotation is of the form

$$R\left(\overrightarrow{l_{\mathbf{x}}}, k\tfrac{2\pi}{n_{\mathbf{x}}}\right), \quad \text{with } 0 \leq k < n_{\mathbf{x}}$$

then defining the equivalence relation on S as follows

$$\mathbf{y} \sim \mathbf{x} \iff \exists\, g \in G \text{ such that } g\mathbf{y} = \mathbf{x}$$

now choose $\mathbf{z} \in \mathbb{S}^2$ such that $\mathbf{z} \notin S$, then

$$\mathrm{Orb}(\mathbf{z}) = \{g\mathbf{z} : g \in G\}$$

will be a set $\{\mathbf{z}_1, \ldots, \mathbf{z}_n\}$ of n distinct points, where $n = |G|$, otherwise, if

$$g\mathbf{z} = h\mathbf{z} \implies h^{-1}g\mathbf{z} = \mathbf{z} \implies \mathbf{z} \in S$$

thus for each $\mathbf{z}_i \in \mathrm{Orb}(\mathbf{z})$, $\exists$ unique $g \in G$ such that $g\mathbf{z} = \mathbf{z}_i$. Define the mapping

$$\sigma : \mathrm{Orb}(\mathbf{z}) \to \{\mathbf{y} \in \mathbb{S}^2 : \mathbf{y} \sim \mathbf{x}\}, \text{ by } \sigma(\mathbf{z}_i) = \sigma(g\mathbf{z}) = g\mathbf{x}$$

then we have

$$\sigma(\mathbf{z}_i) = g\mathbf{x} \implies g^{-1}\sigma(\mathbf{z}_i) = \mathbf{x} \implies \sigma(\mathbf{z}_i) \sim \mathbf{x} \quad \forall\ i$$

now, if $g\mathbf{x} = \mathbf{y} \implies \mathbf{x} = g^{-1}\mathbf{y}$ then we have

$$\sigma(\mathbf{z}_i) = \mathbf{y} \iff \sigma(g^{-1}\mathbf{z}_i) = \mathbf{x}$$

which then implies that $g^{-1}\mathbf{z}_i = R\left(\overrightarrow{l_{\mathbf{x}}}, k\frac{2\pi}{n_{\mathbf{x}}}\right)$ for some $0 \leq k < n_{\mathbf{x}}$.

Therefore $\mathrm{Orb}(\mathbf{z}) = \{\mathbf{z}_1, \ldots, \mathbf{z}_n\}$ and for each $\mathbf{y} \in S$ there are $\{\mathbf{z}_1, \ldots, \mathbf{z}_{n_{\mathbf{x}}}\} \subset \{\mathbf{z}_1, \ldots, \mathbf{z}_n\}$ such that

$$\sigma : \{\mathbf{z}_1, \ldots, \mathbf{z}_{n_{\mathbf{x}}}\} \to \mathbf{y}$$

and therefore the number of points $\mathbf{y}$ equivalent to $\mathbf{x}$ must be $\frac{n}{n_{\mathbf{x}}}$.

Now $G\backslash\{e\}$ contains $n-1$ nontrivial rotations $R(\overrightarrow{l_{\mathbf{x}}}, \alpha)$ where each $l_{\mathbf{x}}$ contains the points $-\mathbf{x}, \mathbf{x} \in S$, and for each $l_{\mathbf{x}}$ there are $n_{\mathbf{x}} - 1$ non-zero rotations about $l_{\mathbf{x}}$. Thus, we have that the total number of non-trivial rotations must be

$$n - 1 = \frac{1}{2}\sum_{\mathbf{x} \in S}(n_{\mathbf{x}} - 1), \quad l_{\mathbf{x}} = l_{-\mathbf{x}} \text{ and so each } n_{\mathbf{x}} - 1 \text{ is counted twice}$$

Now, as S is finite let $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\} = S/\sim$, then

$$2(n-1) = \sum_{i=1}^{k} \Big|\{\mathbf{y} \in S : \mathbf{y} \sim \mathbf{x}_i\}\Big| \cdot (n_{\mathbf{x}_i} - 1) = \sum_{i=1}^{k} \frac{n}{n_{\mathbf{x}_i}}(n_{\mathbf{x}_i} - 1)$$

that is

$$2\left(1 - \frac{1}{n}\right) = \sum_{i=1}^{k}\left(1 - \frac{1}{n_{\mathbf{x}_i}}\right)$$

and we note that $2(1 - \frac{1}{n}) < 2$ for all n, while $1 - \frac{1}{p} \geq \frac{1}{2}$ for $p \geq 2$ and so

$$\left(1 - \frac{1}{n_{\mathbf{x}_1}}\right) + \left(1 - \frac{1}{n_{\mathbf{x}_2}}\right) + \left(1 - \frac{1}{n_{\mathbf{x}_3}}\right) + \left(1 - \frac{1}{n_{\mathbf{x}_4}}\right) \geq 2 \implies k \leq 3$$

If $n = 1$ then $G = \{e\}$, so we may assume $n \geq 2$, and thus

$$2\left(1 - \frac{1}{n}\right) \geq 1 \implies k \geq 2$$

Case 1: $k = 2$, then

$$2 - \frac{2}{n} = \left(1 - \frac{1}{n_{\mathbf{x}_1}}\right) + \left(1 - \frac{1}{n_{\mathbf{x}_2}}\right) = 2 - \frac{1}{n_{\mathbf{x}_1}} - \frac{1}{n_{\mathbf{x}_2}}$$

which implies

$$\frac{n}{n_{\mathbf{x}_1}} + \frac{n}{n_{\mathbf{x}_2}} = 2$$

yet since $\frac{n}{n_{\mathbf{x}}} \in \mathbb{N},\ \forall\ \mathbf{x} \in S$ we must have $n_{\mathbf{x}_1} = n_{\mathbf{x}_2} = n$, and thus

$$S = \{\mathbf{x}_1, \mathbf{x}_2\} = \{\mathbf{x}, -\mathbf{x}\}$$

since any line through $\mathbf{x} \in \mathbb{S}^2$ must also go through $-\mathbf{x} \in \mathbb{S}^2$, and so

$$G = C_n = \left\langle R\left(\vec{l}_{\mathbf{x}}, \tfrac{2\pi}{n}\right)\right\rangle$$

Case 2: $k = 3$, then

$$2 - \frac{2}{n} = \left(1 - \frac{1}{n_{\mathbf{x}_1}}\right) + \left(1 - \frac{1}{n_{\mathbf{x}_2}}\right) + \left(1 - \frac{1}{n_{\mathbf{x}_3}}\right) = 3 - \frac{1}{n_{\mathbf{x}_1}} - \frac{1}{n_{\mathbf{x}_2}} - \frac{1}{n_{\mathbf{x}_3}}$$

which implies

$$\frac{1}{n_{\mathbf{x}_1}} + \frac{1}{n_{\mathbf{x}_2}} + \frac{1}{n_{\mathbf{x}_3}} = 1 + \frac{2}{n} > 1$$

Now if $n_{\mathbf{x}_i} \geq 3, \forall\ i$, then

$$\frac{1}{n_{\mathbf{x}_1}} + \frac{1}{n_{\mathbf{x}_2}} + \frac{1}{n_{\mathbf{x}_3}} \leq 3$$

so WLOG let us assume that $n_{\mathbf{x}_3} = 2$, then

$$\frac{1}{n_{\mathbf{x}_1}} + \frac{1}{n_{\mathbf{x}_2}} + \frac{1}{n_{\mathbf{x}_3}} = 1 + \frac{2}{n} \longrightarrow \frac{1}{n_{\mathbf{x}_1}} + \frac{1}{n_{\mathbf{x}_2}} = \frac{1}{2} + \frac{2}{n}$$

then

$$\begin{aligned} 2n_{\mathbf{x}_1}n_{\mathbf{x}_2}\left(\frac{1}{n_{\mathbf{x}_1}} + \frac{1}{n_{\mathbf{x}_2}}\right) &= 2n_{\mathbf{x}_1} + 2n_{\mathbf{x}_2} \\ &= 2n_{\mathbf{x}_1}n_{\mathbf{x}_2}\left(\frac{1}{2} + \frac{2}{n}\right) \\ &= n_{\mathbf{x}_1}n_{\mathbf{x}_2} + 4\frac{n_{\mathbf{x}_1}n_{\mathbf{x}_2}}{n} \end{aligned}$$

and so

$$\begin{aligned} 4 - 4\frac{n_{\mathbf{x}_1}n_{\mathbf{x}_2}}{n} &= 4 + n_{\mathbf{x}_1}n_{\mathbf{x}_2} - 2n_{\mathbf{x}_1} - 2n_{\mathbf{x}_2} \\ &= n_{\mathbf{x}_1}(n_{\mathbf{x}_2} - 2) - 2(n_{\mathbf{x}_2} - 2) \\ &= (n_{\mathbf{x}_1} - 2)(n_{\mathbf{x}_2} - 2) \end{aligned}$$

and if $n_{\mathbf{x}_2} = 2$, then

$$4 - 8\frac{n_{\mathbf{x}_1}}{n} = 0 \implies n_{\mathbf{x}_1} = \frac{n}{2}$$

if $n_{\mathbf{x}_1}, n_{\mathbf{x}_2} \geq 4$, then

$$(n_{\mathbf{x}_1} - 2)(n_{\mathbf{x}_2} - 2) \geq 4$$

While LHS < 4.

so if $n_{\mathbf{x}_2} = 3$ then

$$\begin{aligned} 4 - 12\frac{n_{\mathbf{x}_1}}{n} &= n_{\mathbf{x}_1} - 2 \\ \implies 4n - 12n_{\mathbf{x}_1} &= nn_{\mathbf{x}_1} - 2n \\ \implies n(6 - n_{\mathbf{x}_1}) &= 12n_{\mathbf{x}_1} \\ \implies n &= \frac{12n_{\mathbf{x}_1}}{6 - n_{\mathbf{x}_1}} \end{aligned}$$

and so $3 \leq n_{\mathbf{x}_1} \leq 5$, where we exclude the case $n_{\mathbf{x}_1} = 2$, since from above this would mean $n_{\mathbf{x}_2} = \frac{n}{2}$, which would include the case $n = 6$. Thus, we have

$n_{\mathbf{x}_3}$	$n_{\mathbf{x}_2}$	$n_{\mathbf{x}_1}$	n
2	2	$n/2$	arbitrary even
2	3	3	12
2	3	4	24
2	3	5	60

Next we partition S into its equivalence classes

$$S = S_1 \sqcup S_2 \sqcup S_3$$

and note that for every rotation $R(\overrightarrow{l}_{\mathbf{x}}, \alpha) \in G$ we have

$$R(\overrightarrow{l}_{\mathbf{x}}, \alpha)(S_i) = S_i$$

and also we have

$$G = R\left(\overrightarrow{l_{\mathbf{x}_1}}, k_1\tfrac{2\pi}{n_{\mathbf{x}_1}}\right) \sqcup R\left(\overrightarrow{l_{\mathbf{x}_2}}, k_2\tfrac{2\pi}{n_{\mathbf{x}_2}}\right) \sqcup R\left(\overrightarrow{l_{\mathbf{x}_3}}, k_3\tfrac{2\pi}{n_{\mathbf{x}_3}}\right)$$

with $\mathbf{x}_1 \in S_1,\ \mathbf{x}_2 \in S_2,\ \mathbf{x}_3 \in S_3$, and also we have $|S_i| = \frac{n}{n_{\mathbf{x}_i}}$.

Case 1: $n_{\mathbf{x}_1} = n/2$ and $n_{\mathbf{x}_2} = 2 = n_{\mathbf{x}_3}$, then

$$|S_1| = \frac{n}{\frac{n}{2}} = 2$$

and so has only 2 points, while

$$\left|R\left(\overrightarrow{l_{\mathbf{x}_1}}, k_1\tfrac{2\pi}{n_{\mathbf{x}_1}}\right)\right| = \frac{n}{2}$$

around each of the corresponding axes. And

$$|S_2| = \frac{n}{2} = |S_3| \quad \text{with} \quad \left|R\left(\overrightarrow{l_{\mathbf{x}_2}}, k_2\tfrac{2\pi}{n_{\mathbf{x}_2}}\right)\right| = 2 = \left|R\left(\overrightarrow{l_{\mathbf{x}_3}}, k_3\tfrac{2\pi}{n_{\mathbf{x}_3}}\right)\right|$$

fix $\mathbf{x} \in S_1$ and we note that as long as $n > 4$, $n_{\mathbf{x}_1} \neq n_{\mathbf{x}_2}, n_{\mathbf{x}_3}$. Further since $n_{\mathbf{x}} = n_{-\mathbf{x}}$ we have

$$S_1 = \{\mathbf{x}, -\mathbf{x}\}$$

since all rotations in G preserve S_1, and all rotation in $R\left(\overrightarrow{l_{\mathbf{x}_2}}, k_2 \frac{2\pi}{n_{\mathbf{x}_2}}\right)$ and $R\left(\overrightarrow{l_{\mathbf{x}_3}}, k_3 \frac{2\pi}{n_{\mathbf{x}_3}}\right)$ have order 2, and preserve $\{\mathbf{x}, -\mathbf{x}\}$, the points in S_2 and S_3 must lie on the plane through $\mathbf{0}$ perpendicular to $l_{\mathbf{x}}$, or on the equator of $\mathbb{S}^2$, if we orient $\mathbf{x}$ to be the north and $-\mathbf{x}$ to be the south poles.

Since this holds for any even n, let $\mathbf{y} \in S_2$ be given and since any rotation in G preserves S_2, in particular any rotation in $R\left(\overrightarrow{l_{\mathbf{x}}}, k_1 \frac{2\pi}{n_{\mathbf{x}}}\right)$ will preserve S_2, and thus

$$S_2 = \left\{R\left(\overrightarrow{l_{\mathbf{x}}}, k_1 \tfrac{2\pi}{n_{\mathbf{x}}}\right)(\mathbf{y}) : 0 \leq k_1 < n/2\right\}$$

further, for any $\mathbf{z} \in S_3$ by similar reasoning we have

$$S_3 = \left\{R\left(\overrightarrow{l_{\mathbf{x}}}, k_1 \tfrac{2\pi}{n_{\mathbf{x}}}\right)(\mathbf{z}) : 0 \leq k_1 < n/2\right\}$$

and since $R\left(\overrightarrow{l_{\mathbf{y}}}, \pi\right)$ preserves S_3 we have $R\left(\overrightarrow{l_{\mathbf{y}}}, \pi\right)(\mathbf{z}) \in S_3$. Furthermore,

$$R\left(\overrightarrow{l_{\mathbf{y}}}, \pi\right)^2 = e = R\left(\overrightarrow{l_{\mathbf{z}}}, \pi\right)^2$$

so one of S_2, S_3 say S_2 may be considered the vertices, and $l_{\mathbf{z}}$ must then be a line through the midpoint of 2 vertices. And therefore we see that

$$G = D_{\frac{n}{2}}$$

Case 2: $n_{\mathbf{x}_1} = 3,\ n_{\mathbf{x}_2} = 3,\ n_{\mathbf{x}_3} = 2$ and $n = 12$. Then

$$|S_1| = \frac{12}{3} = 4 = |S_2| \quad \text{with} \quad \left|R\left(\overrightarrow{l_{\mathbf{x}_2}}, k_2 \tfrac{2\pi}{n_{\mathbf{x}_2}}\right)\right| = 3 = \left|R\left(\overrightarrow{l_{\mathbf{x}_1}}, k_1 \tfrac{2\pi}{n_{\mathbf{x}_1}}\right)\right|$$

and

$$|S_3| = \frac{12}{2} = 6 \quad \text{with} \quad \left|R\left(\overrightarrow{l_{\mathbf{x}_3}}, k_3 \tfrac{2\pi}{n_{\mathbf{x}_3}}\right)\right| = 2$$

fix $\mathbf{x} \in S_1$, and since $|S_1| = 4,\ \exists\ \mathbf{x}' \in S_1$ such that $\mathbf{x}' \neq \mathbf{x}, -\mathbf{x}$, that is $\mathbf{x}' \notin l_{\mathbf{x}}$, and since rotations fix the elements in S_1 we also have that $R\left(\overrightarrow{l_{\mathbf{x}}}, \frac{2\pi}{3}\right)(\mathbf{x}'), R\left(\overrightarrow{l_{\mathbf{x}}}, \frac{2\pi}{3}\right)^2(\mathbf{x}') \in S_1$ and thus

$$S_1 = \left\{\mathbf{x}, \mathbf{x}', R\left(\overrightarrow{l_{\mathbf{x}}}, \tfrac{2\pi}{3}\right)(\mathbf{x}'), R\left(\overrightarrow{l_{\mathbf{x}}}, \tfrac{4\pi}{3}\right)(\mathbf{x}')\right\}$$

next we note that

$$R\left(\overrightarrow{l_{\mathbf{x}'}}, \tfrac{2\pi}{3}\right)\left(R\left(\overrightarrow{l_{\mathbf{x}}}, \tfrac{2\pi}{3}\right)(\mathbf{x}')\right) \in S_1$$

as it is preserved under rotation and that is, and that and that

$$R\left(\overrightarrow{l_{\mathbf{x}'}}, \tfrac{2\pi}{3}\right)\left(R\left(\overrightarrow{l_{\mathbf{x}}}, \tfrac{2\pi}{3}\right)(\mathbf{x}')\right) \neq \mathbf{x}', R\left(\overrightarrow{l_{\mathbf{x}}}, \tfrac{2\pi}{3}\right)(\mathbf{x}'), R\left(\overrightarrow{l_{\mathbf{x}}}, \tfrac{4\pi}{3}\right)(\mathbf{x}')$$

and so we must have
$$R\left(\overrightarrow{l_{\mathbf{x}'}}, \tfrac{2\pi}{3}\right)\left(R\left(\overrightarrow{l_{\mathbf{x}}}, \tfrac{2\pi}{3}\right)(\mathbf{x}')\right) = \mathbf{x}$$
and so $\mathbf{x}, \mathbf{x}', R\left(\overrightarrow{l_{\mathbf{x}}}, \tfrac{2\pi}{3}\right)(\mathbf{x}')$ form the vertices of an equilateral triangle. Using similar reasoning for each set of 3 points in S_1, we see that each set of 3 points forms an equilateral triangle, and so S_1 forms the vertices of a regular tetrahedron.

Next, let $\mathbf{y} \in S_3$, and since rotations in G fix S_1 we must have
$$R\left(\overrightarrow{l_{\mathbf{y}}}, \pi\right)(\mathbf{x}) \in \left\{\mathbf{x}', R\left(\overrightarrow{l_{\mathbf{x}}}, \tfrac{2\pi}{3}\right)(\mathbf{x}'), R\left(\overrightarrow{l_{\mathbf{x}}}, \tfrac{4\pi}{3}\right)(\mathbf{x}')\right\}$$
and so $l_{\mathbf{y}}$ must contain the midpoint of the segment $\left[\mathbf{x}, R\left(\overrightarrow{l_{\mathbf{y}}}, \pi\right)(\mathbf{x})\right]$; i.e.
$$\frac{\mathbf{x} + R\left(\overrightarrow{l_{\mathbf{y}}}, \pi\right)(\mathbf{x})}{2} := \mathbf{w} \in l_{\mathbf{y}}$$
that is $l_{\mathbf{y}}$ must pass through the midpoint of one of the edges defined by the vertices in S_1. Thus, S_3 is comprised of the points on $\mathbb{S}^2$, obtained by taking the radial projection of the midpoints of the edges defined by S_1. Hence
$$S_3 = \left\{\frac{\mathbf{w}_i}{||\mathbf{w}_i||}\right\}_{i=1}^{6}$$
now since S_1 does not contain any antipodal pairs, it must be the case that
$$S_2 = \left\{-\mathbf{x}, -\mathbf{x}', -R\left(\overrightarrow{l_{\mathbf{x}}}, \tfrac{2\pi}{3}\right)(\mathbf{x}'), -R\left(\overrightarrow{l_{\mathbf{x}}}, \tfrac{4\pi}{3}\right)(\mathbf{x}')\right\}$$
and therefore
$$G = A_4 = S_d(T)$$
is the group of direct isometries of the regular tetrahedron.

Case 3: $n_{\mathbf{x}_1} = 4,\ n_{\mathbf{x}_2} = 3,\ n_{\mathbf{x}_3} = 2$ and $n = 24$. Then
$$|S_1| = \frac{24}{4} = 6 \qquad \left|R\left(\overrightarrow{l_{\mathbf{x}_1}}, k_1\tfrac{2\pi}{n_{\mathbf{x}_1}}\right)\right| = 4$$
$$|S_2| = \frac{24}{3} = 8 \qquad \left|R\left(\overrightarrow{l_{\mathbf{x}_2}}, k_2\tfrac{2\pi}{n_{\mathbf{x}_2}}\right)\right| = 3$$
$$|S_3| = \frac{24}{2} = 12 \qquad \left|R\left(\overrightarrow{l_{\mathbf{x}_3}}, k_3\tfrac{2\pi}{n_{\mathbf{x}_3}}\right)\right| = 2$$
choose $\mathbf{x}, \mathbf{y} \in S_1$ such that $\mathbf{y} \notin l_{\mathbf{x}}$, then $R(\overrightarrow{l}_{\mathbf{x}})(\mathbf{y}), R^2(\overrightarrow{l}_{\mathbf{x}})(\mathbf{y}), R^3(\overrightarrow{l}_{\mathbf{x}})(\mathbf{y}) \in S_1$, none of which are equal to either $\mathbf{x}$, or $-\mathbf{x}$, and since $n_{\mathbf{x}} = 4 = n_{-\mathbf{x}}$ and so $-\mathbf{x} \in S_1$, and so
$$S_1 = \left\{\mathbf{x}, -\mathbf{x}, \mathbf{y}, R\left(\overrightarrow{l}_{\mathbf{x}}, \tfrac{\pi}{2}\right)(\mathbf{y}), R\left(\overrightarrow{l}_{\mathbf{x}}, \pi\right)(\mathbf{y}), R\left(\overrightarrow{l}_{\mathbf{x}}, \tfrac{3\pi}{2}\right)(\mathbf{y})\right\}$$
and next we not that $R^k(\overrightarrow{l}_{\mathbf{y}})(\mathbf{x}) \in S_1$ for $1 \le k \le 4$, as S_1 is fixed under rotations, we may identify S_1 with the vertices of an octahedron.

Now, given $\mathbf{z} \in S_3$, since S_1 is fixed under rations we have

$$R(\overrightarrow{l}_{\mathbf{z}}, \pi)(\mathbf{x}) \in \left\{-\mathbf{x}, \mathbf{y}, R(\overrightarrow{l}_{\mathbf{x}})(\mathbf{y}), R^2(\overrightarrow{l}_{\mathbf{x}})(\mathbf{y}), R^3(\overrightarrow{l}_{\mathbf{x}})(\mathbf{y})\right\}$$

and so $l_{\mathbf{z}}$ is a line which passes through the midpoint of one of the edges defined by the vertices of S_1, and so is comprised of the points of $\mathbb{S}^2$ determined by the radial projection of the midpoints of the edges defined by S_1, giving

$$S_3 = \left\{\frac{\mathbf{w}_i}{||\mathbf{w}_i||}\right\}_{i=1}^{12}$$

Next, labeling the vertices of S_1

$$\left\{\mathbf{x}, -\mathbf{x}, \mathbf{y}, R(\overrightarrow{l}_{\mathbf{x}})(\mathbf{y}), R^2(\overrightarrow{l}_{\mathbf{x}})(\mathbf{y}), R^3(\overrightarrow{l}_{\mathbf{x}})(\mathbf{y})\right\} \mapsto \{1, 6, 2, 3, 4, 5\}$$

then

$$\begin{aligned} R\left(\overrightarrow{R(\overrightarrow{l}_{\mathbf{x}})(\mathbf{y})}, \tfrac{\pi}{2}\right) &\mapsto (1, 2, 6, 4) \\ R\left(\overrightarrow{l_{\mathbf{y}}}, \tfrac{\pi}{2}\right) &\mapsto (1, 5, 6, 3) \end{aligned}$$

where

$$(1, 5, 6, 3)(1, 2, 6, 4) = (1, 2, 3)(4, 5, 6)$$

and so has order 3, and hence belongs to S_2. Further, the axis of rotation is precisely a the center of a face determined by the vertices of S_1. And so S_2 is comprised of the points of $\mathbb{S}^2$ determined by the radial projection of the centers of faces defined by S_1, giving 8 such points, and therefore

$$G = S_4 = S_d(O)$$

Case 4: $n_{\mathbf{x}_1} = 5,\ n_{\mathbf{x}_2} = 3,\ n_{\mathbf{x}_3} = 2$ and $n = 60$. Then

$$\begin{aligned} |S_1| &= \frac{60}{5} = 12 & \left|R\left(\overrightarrow{l_{\mathbf{x}_1}}, k_1 \tfrac{2\pi}{n_{\mathbf{x}_1}}\right)\right| &= 5 \\ |S_2| &= \frac{60}{3} = 20 & \left|R\left(\overrightarrow{l_{\mathbf{x}_2}}, k_2 \tfrac{2\pi}{n_{\mathbf{x}_2}}\right)\right| &= 3 \\ |S_3| &= \frac{60}{2} = 30 & \left|R\left(\overrightarrow{l_{\mathbf{x}_3}}, k_3 \tfrac{2\pi}{n_{\mathbf{x}_3}}\right)\right| &= 2 \end{aligned}$$

Fix $\mathbf{x} \in S_1$ and since $|S_1| = 12$, $\exists\ \mathbf{z}, \mathbf{z} \in \S_1$ such that $\mathbf{yz}$ and $\mathbf{z}, \mathbf{yx}, -\mathbf{x}$

□

Given a discrete subgroup $L \subset \mathbb{R}^n$ we obtain a basis for L as follows. Select $\mathbf{e}_1 \in L$ such that

$$||\mathbf{e}_1|| = \min_{\mathbf{e} \in L} ||\mathbf{e}||, \quad \text{and } \mathbf{e}_1 \neq \mathbf{0}$$

then choose $\mathbf{e}_2 \in L$, such that

$$||\mathbf{e}_2|| = \min_{\substack{\mathbf{e}_2 \notin \text{span}\{\mathbf{e}_1\} \\ \mathbf{e} \in L}} d(\mathbf{e}, \text{span}\{\mathbf{e}_1\})$$

then for instance if $n = 2$ we have that

$$L = \mathbf{e}_1 \mathbb{Z} \oplus \mathbf{e}_2 \mathbb{Z}$$

otherwise, considering the diamon shaped region determined by the segments $[\mathbf{e}_2, \mathbf{e}_1], [\mathbf{e}_2, -\mathbf{e}_1], [-\mathbf{e}_2, \mathbf{e}_1], [-\mathbf{e}_2, -\mathbf{e}_1] := D$, then if $\mathbf{x} \in L$ such that $\mathbf{x} \notin \mathbf{e}_1 \mathbb{Z} \oplus \mathbf{e}_2 \mathbb{Z}$, then there is a translation of $\mathbf{x}$ so that $\mathbf{x} \in D$, then if $\mathbf{x} \neq \{\pm\mathbf{e}_1, \pm\mathbf{e}_2\}$ then

$$||\mathbf{x}|| < ||\mathbf{e}_2||$$

further, if $\mathbf{x} \notin l_{\mathbf{e}_1}$, where $l_{\mathbf{e}_1}$ is the line containing $\mathbf{0}, \mathbf{e}_1$, then we also have

$$||\mathbf{x}|| < ||\mathbf{e}_1|| \quad \Rightarrow\Leftarrow$$

contradicting our choice for $\mathbf{e}_1$.

continuing inductively, once $\mathbf{e}_k$ has been defined select $\mathbf{e}_{k+1} \in L$ such that

$$||\mathbf{e}_{k+1}|| = \min_{\substack{\mathbf{e}_{k+1} \notin \text{span}\{\mathbf{e}_1, \ldots, \mathbf{e}_k\} \\ \mathbf{e} \in L}} d(\mathbf{e}, \text{span}\{\mathbf{e}_1, \ldots, \mathbf{e}_k\})$$

we obtain

$$L = \mathbf{e}_1 \mathbb{Z} \oplus \mathbf{e}_2 \mathbb{Z} \oplus \cdots \oplus \mathbf{e}_n \mathbb{Z} = \left\{ \sum_{i=1}^{n} n_i \mathbf{e}_i : n_i \in \mathbb{Z} \right\}$$

Theorem 34 (**Crystallographic Restriction**)**.** If $L \subset \mathbb{R}^n$ with $n \in \{2, 3\}$, and $R(\mathbf{a}, \alpha) \in S(L)$, then

$$\left|R(\mathbf{a}, \alpha)\right| \in \{2, 3, 4, 6\}$$

Proof. Let

$$L = \mathbf{e}_1 \mathbb{Z} \oplus \mathbf{e}_2 \mathbb{Z} \oplus \cdots \oplus \mathbf{e}_n \mathbb{Z}$$

then for $f \in S(L) \subseteq I(\mathbb{R}^n)$ we have

$$f(\mathbf{x}) = A\mathbf{x} + \mathbf{a}, \quad \text{with } \mathbf{a} \in L, \ A \in \mathrm{O}(n)$$

and so for the basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of L then

$$f(\mathbf{0}) = \mathbf{a} \in L \implies \mathbf{a} = \sum_{i=1}^{n} n_i \mathbf{e}_i$$

$$A \in \mathrm{O}(n) \text{ is such that } A\mathbf{e}_i = \sum_{k=1}^{n} a_{ik} e_{ki} = \begin{bmatrix} n_1 \\ \vdots \\ n_n \end{bmatrix}$$

and so we have

$$f(\mathbf{e}_i) = A\mathbf{e}_i + \mathbf{a} \implies f(\mathbf{e}_i) - \mathbf{a} = \sum_{i=1}^{n} b_{ij} e_j$$

where $B = (B_{ij}) \in \mathrm{M}(n, \mathbb{Z})$ and so $\mathrm{Tr}(A) = \mathrm{Tr}(B) \in \mathbb{Z}$ since the trace is preserved under conjugation.

n=2

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

or $A = R_l$ is which case $|A| = |R_l| = 2$. And so $\mathrm{Tr}(A) = 2\cos\alpha \in \mathbb{Z}$ and thus we must have

$$\cos\alpha = 0, \pm 1, \pm\tfrac{1}{2}$$

which implies that

$$\alpha \in \{0, \tfrac{\pi}{3}.\tfrac{\pi}{2}, \tfrac{2\pi}{3}, \pi, \tfrac{4\pi}{3}, \tfrac{3\pi}{2}, \tfrac{5\pi}{3}, 2\pi\}$$

hence

$$\big|R(\mathbf{a}, \alpha)\big| \in \{2, 3, 4, 6\}$$

n=3

similarly under conjugation we get

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $\mathrm{Tr}(A) = 2\cos\alpha + 1$, where the same argument above shows that

$$\big|R(\mathbf{a}, \alpha)\big| \in \{2, 3, 4, 6\}$$

$\square$

recalling that $T \lhd I(R^n)$ where T is the group of translations we get that $I(\mathbb{R}^n)$ is the semi-direct product of T and $\mathrm{O}(n)$

$$I(\mathbb{R}^n) = \mathrm{O}(n) \ltimes T \cong \mathrm{O}(n) \ltimes \mathbb{R}^n$$

with elements

$$(A, \mathbf{x}) \in I(\mathbb{R}^n), \quad \text{for } A \in \mathrm{O}(n),\ \mathbf{x} \in \mathbb{R}^n$$

with group operations

$$(A, \mathbf{x})(B, \mathbf{y}) = (AB, \mathbf{x} + A\mathbf{y})$$
$$(A, \mathbf{x})^{-1} = (A^{-1}, -A^{-1}\mathbf{x})$$

where the action on $\mathbb{R}^n$ is given by

$$(A, \mathbf{x})(\mathbf{v}) = \mathbf{x} + A\mathbf{v}$$

Lemma 35. With $G_T = G \cap T$ being the translation group of G, and

$$G/G_T \leq \mathrm{O}(n)$$

then G/G_T acts on G_T

Proof. let $f \in G_T$, then

$$f(\mathbf{x}) = \mathbf{x} + \mathbf{a}$$

then for $g \in G$ we have

$$g(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$$

and so

$$\begin{aligned} g^{-1}fg(\mathbf{x}) &= A(A^{-1}\mathbf{x} - A^{-1}\mathbf{b} + \mathbf{a}) - \mathbf{b} \\ &= \mathbf{x} + A\mathbf{a} \in G_T \end{aligned}$$

□

and so G/G_T acts on G_T by orthogonal transformation. If $G/G_T \leq \mathrm{O}(3)$ is a subgroup with each element $g \in G/G_T$ having order $|g| \in \{2, 3, 4, 6\}$, if $G/G_T \subseteq \mathrm{SO}(3)$, then by Theorem 33 we have G/G_T is one of the following

$$1, \overbrace{\mathbb{Z}/2\mathbb{Z}}^{C_2}, \overbrace{\mathbb{Z}/3\mathbb{Z}}^{C_3}, \overbrace{\mathbb{Z}/4\mathbb{Z}}^{C_4}, \overbrace{\mathbb{Z}/6\mathbb{Z}}^{C_6}, D_2, D_4, D_6, \overbrace{S_d(T)}^{A_4}, \overbrace{S_d(O)}^{S_4}$$

If G/G_T is not entirely contained in SO(3), then $(G/G_T)_d \lhd G/G_T$ has index 2, and

$$G/G_T = (G/G_T)_d \cap g(G/G_T)_d \quad \text{for any } g \in G \setminus (G/G_T)_d$$

if

$$\eta = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \in G/G_T$$

then

$$G/G_T = (G/G_T)_d \times \{1, \eta\}$$

Theorem 36. The space SO(3) is homeomorphic to the quotient space $\overline{B}^3_\pi(\mathbf{0})/\sim$ where $\sim$ is the equivalence relation given by

$$\mathbf{x} \sim \mathbf{y} \iff \begin{cases} \mathbf{x} = \mathbf{y} \\ \mathbf{x} = -\mathbf{y}, \text{ and } ||\mathbf{x}|| = \pi \end{cases}$$

Proof. Form Corollary 32, each $A \in \mathrm{SO}(3)$ is determined by its angel and axis of rotation. So define

$$f : \overline{B}^3_\pi(\mathbf{0}) \to \mathrm{SO}(3), \text{ by } f(\mathbf{x}) = R(\overrightarrow{\mathbf{0x}}, ||\mathbf{x}||)$$

where $f(\mathbf{0}) = Id$, and so f is well-defined. Since each $A \in \mathrm{SO}(3)$ is determined by an angle and axis of rotation f is surjective. And injective since for $\mathbf{x}$ such that $||\mathbf{x}|| \neq \pi$ we have

$$\begin{aligned}
&f(\mathbf{x}) = f(\mathbf{y}) \\
&\implies R(\overrightarrow{\mathbf{0x}}, ||\mathbf{x}||) = R(\overrightarrow{\mathbf{0y}}, ||\mathbf{y}||) \\
&\implies \overrightarrow{\mathbf{0x}} = \overrightarrow{\mathbf{0y}} \\
&\implies ||\mathbf{x}|| = ||\mathbf{y}|| \\
&\implies \mathbf{x} = \mathbf{y}
\end{aligned}$$

Now if $||\mathbf{x}|| = \pi$ we have

$$f(\mathbf{x}) = R(\overrightarrow{\mathbf{0x}}, \pi) = R(\overrightarrow{-\mathbf{0x}}, \pi) = f(-\mathbf{x})$$

and so

$$f : \overline{B}^3_\pi(\mathbf{0})/\sim \to \mathrm{SO}(3)$$

is a continuous bijection, and as each space is compact, this is a homeomorphism. To see that SO(3) is compact first note that the mapping

$$\phi : \mathrm{M}(n, \mathbb{R}) \to \mathrm{M}(n, \mathbb{R}), \text{ by } \phi(A) = A^T A$$

is continuous as the multiplication of matrices. Where $\{I_n\} \in \mathrm{M}(n, \mathbb{R})$ is closed as a singleton, and the continuity of ϕ tells us that $\phi^{-1}(I_n) = \mathrm{O}(n)$ must also be closed. Then for each $\mathbf{x} \in \mathbb{R}$, and for any $Q \in \mathrm{O}(n)$ we have

$$||Q\mathbf{x}|| = \sqrt{\langle Q\mathbf{x}, Q\mathbf{x}\rangle} = \sqrt{\langle \mathbf{x}, \mathbf{x}\rangle} = ||\mathbf{x}|| \implies \mathrm{O}(n) \subseteq B^n_1(\mathbf{x})$$

and so $\mathrm{O}(n)$ is bounded, where by Heine-Borel, since $\mathrm{O}(n)$ is closed and bounded in $\mathbb{R}^n$ it is compact. Then we simply note that under the continuous determinant mapping

$$\det : \mathrm{O}(n) \to \mathbb{R}$$

that the closed singleton $\{1\} \in \mathbb{R}$ has preimage $\det^{-1}(\{1\}) = \mathrm{SO}(n)$, and so $\mathrm{SO}(n) \subset \mathrm{O}(n)$ is closed, and therefore is a closed subset of a compact space, and is thus compact. □

The Quaternions $\mathbb{H}$ are a four dimensional vector space over $\mathbb{R}$ with basis $\{1, i, j, k\}$, with the following multiplication rules

$$\begin{aligned}
&i^2 = j^2 = k^2 = -1 \\
&ij = -ji = k \\
&jk = -kj = i \\
&ki = -ik = j
\end{aligned}$$

for $x + iy + jz + ku \in \mathbb{H}$ we can also consider the subset of $\mathrm{M}(2, \mathbb{C})$ given by

$$\begin{bmatrix} x + yi & z + ui \\ -z + ui & x - yi \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix} \subset \mathrm{M}(2, \mathbb{C})$$

we also have the norm $(\mathbb{H}, ||\cdot||)$ given by

$$||\mathbf{q}||^2 = \mathbf{q} \cdot \overline{\mathbf{q}} = (x + iy + jz + ku) \cdot (x - iy - jz - ku) = x^2 + y^2 + z^2 + u^2$$

when $\mathbf{q} \neq \mathbf{0}$, then $\mathbf{q} \cdot \frac{\overline{\mathbf{q}}}{||\mathbf{q}||^2} = 1$ which implies that $\mathbf{q}$ is invertible with inverse

$$\mathbf{q}^{-1} = \frac{\overline{\mathbf{q}}}{||\mathbf{q}||^2}$$

and so $\mathbb{H}$ is a division ring, or skew field. When regarded as an element of $\mathrm{M}(2, \mathbb{C})$ with $\mathbf{q} = A$ we have

$$||\mathbf{q}||^2 = \det(A)$$

where we then see

$$||\mathbf{qp}||^2 = \det(AB) = \det(A)\det(B) = ||\mathbf{q}||^2 \cdot ||\mathbf{p}||^2$$

Proposition 37. $\mathbb{S}^3$ acts orthogonally on $\mathrm{Im}(\mathbb{H}) \cong \mathbb{R}^3$ by conjugation.

Proof. Let $\mathbf{q} \in \mathrm{Im}(\mathbb{H})$, then its representation in $\mathrm{M}(2, \mathbb{C})$ is

$$\begin{bmatrix} yi & ui \\ ui & -yi \end{bmatrix}$$

and so $\mathrm{Tr}(A) = 0$, and since traces are preserved by conjugation; i.e.

$$\mathrm{Tr}\big(BAB^{-1}\big) = \mathrm{Tr}(A)$$

so if $\mathbf{p} \in \mathbb{S}^3$ and $\mathbf{q} \in \mathrm{Im}(\mathbb{H})$ then $\mathbf{pqp}^{-1} \in \mathrm{Im}(\mathbb{H})$, so for each $\mathbf{p} \in \mathbb{S}^3$ we may define the map

$$\phi_{\mathbf{p}} : \mathbb{R}^3 \to \mathbb{R}^3, \text{ by } \phi_{\mathbf{p}}(\mathbf{q}) = \mathbf{pqp}^{-1}$$

then $\phi_{\mathbf{p}}$ is linear, as

$$\phi_{\mathbf{p}}(\mathbf{q}_1 + \mathbf{q_2}) = (\mathbf{q}_1 + \mathbf{q_2})\mathbf{p}^{-1} = \mathbf{pq}_1\mathbf{p}^{-1} + \mathbf{pq}_2\mathbf{p}^{-1}$$

and since

$$||\mathbf{pqp}^{-1}|| = ||\mathbf{p}|| \cdot ||\mathbf{q}|| \cdot ||\mathbf{p}^{-1}|| = ||\mathbf{q}||, \quad \mathbf{p}, \mathbf{p}^{-1} \in \mathbb{S}^3$$

and therefore $\phi_{\mathbf{p}} \in \mathrm{O}(3)$. Also

$$\phi_{\mathbf{p}}(\mathbf{q}_1 \cdot \mathbf{q}_2) = \mathbf{p}(\mathbf{q}_1 \cdot \mathbf{q_2})\mathbf{p}^{-1} = (\mathbf{pq}_1\mathbf{p}^{-1})(\mathbf{pq}_2\mathbf{p}^{-1}) = \phi_{\mathbf{p}}(\mathbf{q}_1)\phi_{\mathbf{p}}(\mathbf{q}_2)$$

and so $\phi_{\mathbf{p}}$ is a homomorphism. $\square$

Theorem 38. Conjugation of $\mathbb{H}$ induces an isomorphism

$$\phi : \mathbb{S}^3/\{\pm 1\} \to \mathrm{SO}(3)$$

Proof. From Proposition 37 we know that

$$\phi : \mathbb{S}^3 \to \mathrm{O}(3)$$

is a homomorphism. Then, as $\mathbb{S}^3$ is connected, ϕ is continuous, and the continuous image of a connected set is connected, we have $\phi(\mathbb{S}^3) \subseteq \mathrm{O}(3)$ is connected. Now,

$$\mathrm{O}(3) = \mathrm{SO}(3) \sqcup \mathrm{O}(3) \setminus \mathrm{SO}(3)$$

and since $I_3 \in \mathrm{SO}(3)$ we have $\phi(\mathbb{S}^3) \subseteq \mathrm{SO}(3)$. Furthermore,

$$\ker(\phi) = \{\mathbf{p} \in \mathbb{S}^3 : \mathbf{p}\mathbf{q}\mathbf{p}^{-1} = \mathbf{q} \;\forall\; \mathbf{q} \in \mathbb{R}^3\} = \{p = x + iy + jz + ku \in \mathbb{S}^3 : p = x\} = \{\pm 1\}$$

since $||x|| = 1$.

Next consider $\mathbf{q} \in \mathbb{S}^3$ such that $\mathbf{q} = \lambda + \mathbf{a}\mu$ with $\mathbf{a} \in \mathbb{R}^3$ such that $||\mathbf{a}|| = 1$, and $\mu, \lambda \in \mathbb{R}$ such that $\lambda^2 + \mu^2 = 1$. Then

$$\mathbf{q}\overline{\mathbf{q}} = 1 \implies (\lambda + \mathbf{a}\mu)(\lambda - \mathbf{a}\mu) = \lambda^2 - \mu\lambda\mathbf{a} + \mu\lambda\mathbf{a} - (\mu\mathbf{a})^2 = \lambda^2 - (\mu\mathbf{a})^2 = 1$$

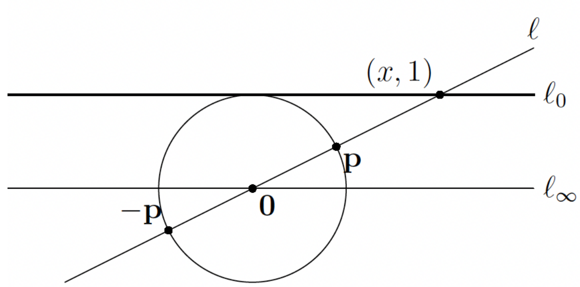
and so

$$\begin{aligned}
\mathbf{q}\mathbf{a}\mathbf{q}^{-1} &= (\lambda + \mathbf{a}\mu)\mathbf{a}\,\overbrace{(\lambda - \mathbf{a}\mu)}^{\mathbf{q}^{-1} = \frac{\overline{\mathbf{q}}}{||\mathbf{q}||^2}} \\
&= (\lambda\mathbf{a} + \mu\mathbf{a}^2)(\lambda - \mu\mathbf{a}) \\
&= \lambda^2\mathbf{a} + \mu\lambda\mathbf{a}^2 - \lambda\mu\mathbf{a}^2 - \mu^2\mathbf{a}^3 \\
&= \lambda^2\mathbf{a} - \mu^2\mathbf{a}^3 \\
&= (\lambda^2 - \mu^2\mathbf{a}^2)\mathbf{a} \\
&= \mathbf{a}
\end{aligned}$$

so $\mathbf{a}$ is a fixed point, that is $\phi_{\mathbf{q}}(\mathbf{a}) = \mathbf{q}\mathbf{a}\mathbf{q}^{-1} = R(\mathbf{a}, \alpha)$ □

To consider the points of $\mathbb{RP}^1$, fix a line $l_0 \subset \mathbb{R}^2$, such that $\mathbf{0} \notin l_0$, this being the affine subspace of $\mathbb{RP}^1$, then for each line $l \in \mathbb{RP}^1$ we have $l \cap l_0$ at a unique point, except the line $l_\infty \in \mathbb{RP}^1$, which is the line in $\mathbb{RP}^1$ such that $l_\infty || l_0$ which we associate to $\{\infty\}$ then we obtain the following bijection

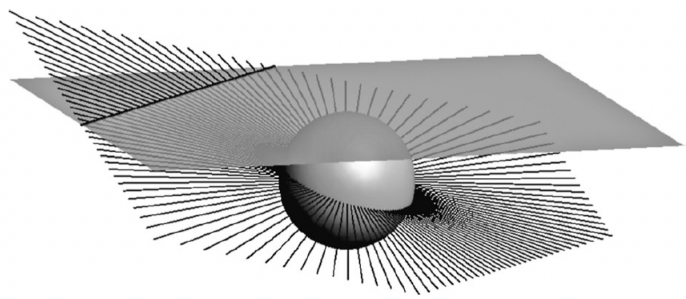
$$\begin{aligned}
\mathbb{RP}^1 &\to l_0 \\
l &\mapsto \begin{cases} l \cap l_0, & l \nparallel l_0 \\ \infty, & l \parallel l_0 \end{cases}
\end{aligned}$$



similarly let $P \subset \mathbb{R}^3$ be any plane in $\mathbb{R}^3$, such that $\mathbf{0} \notin P$, then every element $l \in \mathbb{RP}^2$ such that $l \nparallel P$ intersects P in exactly one point. Yet, the set of $l \in \mathbb{RP}^2$ which are parallel to P, is the set of lines in $\mathbb{R}^2$ through $\mathbf{0}$ parallel to P, but this is precisely $\mathbb{RP}^1$, and so we have

$$\mathbb{RP}^2 = \mathbb{R}^2 \cup \mathbb{RP}^1 = \mathbb{R}^2 \cup \mathbb{R}^1 \cup \{\infty\}$$
$$[x_1 : x_2 : x_3] \mapsto \left(\frac{x_1}{x_3}, \frac{x_2}{x_3}\right) \cup \left(\frac{x_1}{x_2}\right) \cup \{\infty\}$$

For the lines in $\mathbb{RP}^2$ consider $P \subset \mathbb{R}^3$ to be any plane in $\mathbb{R}^3$, such that $\mathbf{0} \notin P$, and let $l \in P$ be a line in P, and let Q_l be the plane in $\mathbb{R}^3$ which contains both l and $\mathbf{0}$, therefore, Q_l is the union of all lines in $\mathbb{R}^3$ passing through $\mathbf{0}$ and a point of l. Thus, lines in $\mathbb{RP}^2$ correspond to planes in $\mathbb{R}^3$.



Proposition 39. Any pair of distinct lines in $\mathbb{RP}^2$ meet at a point.

Proof. First, we note that a line $l \in \mathbb{RP}^2$ is given by the equation

$$a_1x_1 + a_2x_2 + a_3x_3 = 0$$

then if the point $[\mathbf{x}] \in \mathbb{RP}^2$ satisfies this equation, then so does $[\lambda\mathbf{x}] \in \mathbb{RP}^2$. Then we note that two lines $l_1, l_2 \in \mathbb{RP}^2$

$$a_1x_1 + a_2x_2 + a_3x_3 = 0$$
$$b_1x_1 + b_2x_2 + b_3x_3 = 0$$

are distinct if

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

has rank 2. Now $[\mathbf{x}] \in \mathbb{RP}^2$ belongs to the intersection $[\mathbf{x}] \in l_1 \cap l_2$ if

$$A[\mathbf{x}] = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$$

has a solution, yet by the Rank-Nullity Theorem, since $\operatorname{rank}(A) = 2$ we have

$$3 = \operatorname{rank}(A) + \dim(\ker(A)) \implies \dim(\ker(A)) = 1$$

and so has a 1-dimensional solution $(\lambda x_1, \lambda x_2, \lambda x_3) \sim [x_1 : x_2 : x_3] \in \mathbb{RP}^2$. $\square$

Recall that $\mathbb{R}$ is homeomorphic to an open interval via the map

$$\mathbb{R} \to \mathbb{B}$$
$$x \mapsto \frac{1}{1+|x|}$$

and as $\mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$ we have $\mathbb{RP}^1$ is homeomorphic to an open interval with a point at infinity, or $\mathbb{RP}^1$ is homeomorphic to a closed interval with the endpoints identified. That is, $\mathbb{RP}^1$ is homeomorphic to $\mathbb{S}^1$. We can see this with stereographic projection mapping

$$\mathbb{S}^1 \to \mathbb{R} \cup \{\infty\} = \mathbb{RP}^1 \quad \text{where } (0,1) \mapsto \infty$$

we may also consider the projective space, corresponding to a vector space, over any field $\mathbb{F}$ is

$$\mathbb{P}(V) = (V \setminus \{0\})/\sim \quad \text{where} \quad \mathbf{v} \sim \mathbf{w} \iff \mathbf{w} = \lambda \mathbf{v}, \text{ for } \lambda \in \mathbb{F} \setminus \{\mathbf{0}\}$$

For a vector space V, and its dual V^*, we have that the projective spaces $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$ are also dual. For instance $[f] \in \mathbb{P}(V^*)$ determines a codimension 1 subspace of V. So if $v_1 \in V$ is an a basis element and $f_1 \in V^*$ is its corresponding dual basis element then

$$\ker(f_1) = \{v_2, \dots, v_n\}$$

is an $(n-1)$-dimensional subspace of V. And so the point $[f] \in \mathbb{P}(V^*)$ determines a codimension 1 projective subspace $\mathbb{P}(\ker(f)) \subset \mathbb{P}(V)$.

Theorem 40. If $\Delta A_1A_2A_3$ and $\Delta B_1B_2B_3$ are two triangles in $\mathbb{RP}^2$, such that the three lines $l_{A_1B_1}, l_{A_2B_2}, l_{A_3B_3} \in \mathbb{RP}^2$ are concurrent, as a point say $[\mathbf{S}] \in \mathbb{RP}^2$, then the points

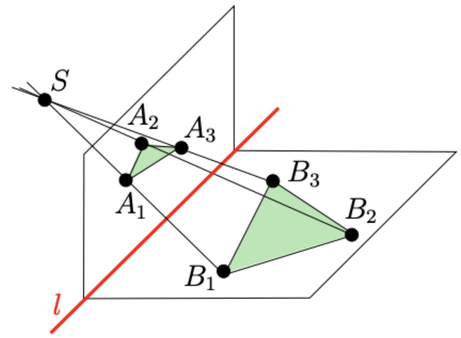
$$[\mathbf{p}_{12}] = l_{A_1A_2} \cap l_{B_1B_2}$$
$$[\mathbf{p}_{23}] = l_{A_2A_3} \cap l_{B_2B_3}$$
$$[\mathbf{p}_{13}] = l_{A_1A_3} \cap l_{B_1B_3}$$

are collinear.

Proof. First considering $\mathbb{RP}^2 \subset \mathbb{RP}^3$, and let $P_A \subset \mathbb{R}^3$ be a plane containing the points $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ such that

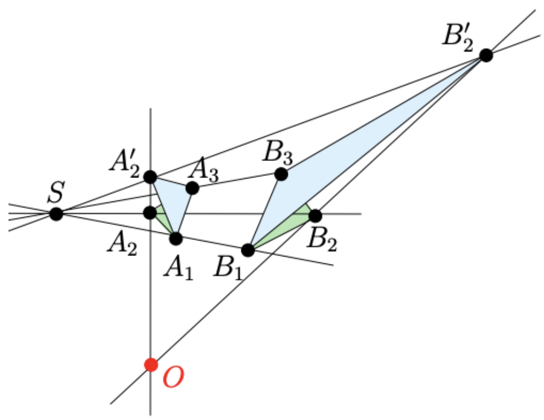
$$\mathbf{A}_1 \mapsto [\mathbf{A}_1]$$
$$\mathbf{A}_2 \mapsto [\mathbf{A}_2]$$
$$\mathbf{A}_3 \mapsto [\mathbf{A}_3]$$

giving the triangle $\Delta A_1A_2A_3 \in \mathbb{RP}^2$, and similarly for $P_B \subseteq \mathbb{R}^3$. These planes will be nonparallel as their projections meet outside of ∞. So let $l = P_A \cap P_B$, and if each line $l_{A_1B_1}, l_{A_2B_2}, l_{A_3B_3} \in \mathbb{R}^3$ meets at $\mathbf{S}$, then we will have $\mathbf{p}_{12}, \mathbf{p}_{23}, \mathbf{p}_{13} \in l$



and so $[\mathbf{p}_{12}], [\mathbf{p}_{23}], [\mathbf{p}_{23}] \in l \subset \mathbb{RP}^2$

Now if $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3$ belong to the same plane $P \subset \mathbb{R}^3$, then we choose a point $\mathbf{O} \notin P$, such that the plane containing $\mathbf{O}, \mathbf{A}_2, \mathbf{B}_2$ is perpendicular to P.



then we can choose a point $\mathbf{A}_2' \in \overrightarrow{\mathbf{OA}_2}$, and find a corresponding point $\mathbf{B}_2'$, such that

$$\mathbf{B}_2' \in \overrightarrow{\mathbf{OB}_2} \cap \overrightarrow{\mathbf{SA}_2'}$$

then we have the triangles $\Delta\mathbf{A}_1\mathbf{A}_2'\mathbf{A}_3 \subset P_{A'} \subseteq \mathbb{R}^3$, and $\Delta\mathbf{B}_1\mathbf{B}_2'\mathbf{B}_3 \subset P_{B'} \subset \mathbb{R}^3$, with the lines $l_{A_1B_1}, l_{A_2'B_2'}, l_{A_3B_3} \in \mathbb{R}^3$ meeting at $\mathbf{S}$ where from above, we know that $\mathbf{p}_{12}', \mathbf{p}_{23}', \mathbf{p}_{13} \in l = P_{A'} \cap P_{B'}$. Then taking the limit as $\mathbf{A}_2' \to \mathbf{A}_2$ we get $P_{A'}, P_{B'} \to P$ and $l \to L \in P$ where $\mathbf{p}_{12}, \mathbf{p}_{23}, \mathbf{p}_{13} \in L \subset P$. Thus, $[\mathbf{p}_{12}], [\mathbf{p}_{23}], [\mathbf{p}_{23}] \in L \subset \mathbb{RP}^2$. $\square$

If

$$T : V \to V$$

is a linear isomorphism, it induces an isomorphism

$$\begin{aligned} \mathbb{P}(V) &\to \mathbb{P}(V) \\ [\mathbf{v}] &\mapsto [T(\mathbf{v})] \end{aligned}$$