Differentiable Manifolds Notes

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1 Definitions

Hausdorff space: Suppose M is a topological space. For every pair of distinct points $p, q \in M \; \exists \; U, V \subset M$ open, such that $U \cap V = \varnothing$ and $p \in U, \; q \in V$

Second countable: Suppose M is a topological space. Then M is second countable if $\exists$ a countable basis for the topology of M.

Locally euclidean: Suppose M is a topological space. Then M is locally euclidean of dimension n if $\forall \; p \in M$ we have:

1.) $p \in U \subseteq M$ where U is open.

2.) An open subset $\widehat{U} \subseteq \mathbb{R}^n$.

3.) A homeomorphism $\phi : U \to \widehat{U}$.

Topological n-Manifold: A topological space M that has the following properties:

1.) A Hausdorff space.

2.) Second countable.

3.) Locally euclidean of dimension n.

Coordinate chart: Let M be a topological n-Manifold, then a coordinate chart on M is a pair (U, ϕ) with $U \subset M$ open, $\phi(U) \subset \mathbb{R}^n$ open and $\phi : U \to \phi(U)$ a homeomorphism.

- if $\phi(p) = 0$ the chart is centered at $p \in M$
- if $p \in (U, \phi)$ we are always capable of centering the chart at p by defining $\phi : U \to \phi(U)$ by $\phi(q) = \phi(q) - \phi(p), \; \forall \; q \in (U, \phi)$

Coordinate Ball: for the chart (U, ϕ) if $\phi(U) = B \subset \mathbb{R}^n$, where B is an n-dimensional ball, then U is a coordinate ball (in M).

Local coordinates: for a topological n-Manifold with chart (U, ϕ) the local coordinate map ϕ is actually comprised of n functions $(x^1, \ldots, x^n)$ so that

$$\phi(U) = \big(x^1(U), \ldots, x^n(U)\big) \subset \mathbb{R}^n$$

and for each $p \in U$, we have $\phi(p) = (x^1(p), \ldots, x^n(p))$, which are the local coordinates on U.

Disconnected: A topological space X is disconnected if it has 2 disjoint nonempty open subsets whose union is X; i.e. $U, V \subset X$ open, such that

$$U \neq \varnothing, \quad V \neq \varnothing, \text{ where } U \cap V = \varnothing, \quad \text{and } U \cup V = X$$

Connected: A topological space X is connect if it is not disconnected. Equivalently it is connected iff its' only subsets which are both open and closed are: $\varnothing$, and X itself.

A connected subset of X is a subset that is a connected space when endowed with the subspace topology.

Component: For topological space X a component of X is a maximal connected subset of X; i.e. a connected subset that is not properly contained in any larger connected subset.

Path: If X is a topological space and $p, q \in X$, a path in X from p to q is a continuous map $f : [0, 1] \to X$ such that $f(0) = p$ and $f(1) = q$.

If for every pair of points $p, q \in X$, $\exists$ a path in X from p to q then X is path-connected.

The path components of X are its maximal path-connected subsets.

Locally Path-Connected: A topological space X is said to be locally path-connected if it admits a basis $\mathcal{B}$ of path-connected open subsets.

Locally Compact: let (X, τ) be a topological space. Then X is locally compact if $\forall\, x \in X$, $\exists\, O \in \tau$ with $x \in O$ such that $\overline{O}$ is compact.

Locally finite: Let M be a topological space, and let $\mathcal{S}$ be a collection of subsets of M. $\mathcal{S}$ is locally finite if $\forall\, p \in M$ $\exists$ a neighborhood U_p such that $U_p \cap S \neq \varnothing$ for finitely many $S \in \mathcal{S}$.

Refinement: Let M be a topological space and $\mathcal{U}$ a cover for M. If there is another cover $\mathcal{V}$ of M such that $\forall\, V \in \mathcal{V}$ $\exists\, U \in \mathcal{U}$ with $V \subseteq U$ then $\mathcal{V}$ is a refinement of $\mathcal{U}$.

Paracompact: Let M be a topological space. M is paracompact if every open cover of M admits an open, locally finite refinement.

Paracompactness is a consequence of local compactness and second countability.

Exhaustion: Let X be a topological space, a sequence $\{K_i\}_{i=1}^{\infty}$ of compact subsets of X, is an exhaustion of X by compact sets if,

$$\bigcup_{i=1}^{\infty} K_i = X$$

and

$$K_i \subseteq \operatorname{Int}(K_{i+1}) \ \forall\, i$$

Transition Map: Let M be a n-dimensional topological manifold, and let $(U, \phi), (V, \psi)$ be two charts on M such that

$$U \cap V \neq \varnothing$$

then the transition map from ϕ to ψ is the composite function

$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V) \subseteq \psi(V) \subseteq \mathbb{R}^n$$

it is the composition of homeomorphisms, and is therefore itself a homeomorphism.

Smooth Compatibility: Let M be a n-dimensional topological manifold, and let $(U, \phi), (V, \psi)$ be two charts on M. Then (U, ϕ) and (V, ψ) are smoothly compatible if either

$$U \cap V = \varnothing$$

or, $\psi \circ \phi^{-1}$ is a diffeomorphism; i.e. a bi-smooth bijection. And since $\psi(U \cap V), \phi(U \cap V) \subseteq \mathbb{R}^n$ smooth means $\psi \circ \phi^{-1} \in C^\infty(\mathbb{R}^n)$ and so has continuous partial derivatives of all orders.

Smooth Atlas: Let M be a n-dimensional topological manifold. A smooth atlas $\mathcal{A}_M$ for M is a collection of charts covering M such that any two charts in $\mathcal{A}_M$ are smoothly compatible; i.e. either their domains have empty intersection, or their composition is $C^\infty(\mathbb{R}^n)$.

Smooth Structure: Let M be a topological manifold. A smooth structure on M is a maximal smooth atlas.

Smooth Manifold: A smooth manifold is a pair $(M, \mathcal{A})$, where M is a topological manifold and $\mathcal{A}$ is a smooth structure on M.

Regular coordinate balls: Let M be a smooth manifold. Then $B \subset M$ is a regular coordinate ball if $\exists\ (B', \phi) \in \mathcal{A}_M$ such that $B' \supseteq \overline{B}$ and

$$\phi : B' \to \mathbb{R}^n$$

such that for $r, r' \in \mathbb{R}^+$ with $r < r'$ we have:

$$\phi(B) = B_r(\mathbf{0})$$
$$\phi(\overline{B}) = \overline{B_r(\mathbf{0})}$$
$$\phi(B') = B_{r'}(\mathbf{0})$$

Because $\overline{B}$ is homeomorphic to $\overline{B_r(\mathbf{0})}$, it is compact, and so every regular coordinate ball is precompact in M.

For a manifold with boundary, we make the following adjustments for regular coordinate half-balls

$$\phi : B' \to \mathbb{H}^n$$

such that for $0 < r < r'$ we have:

$$\phi(B) = B_r(\mathbf{0}) \cap \mathbb{H}^n$$
$$\phi(\overline{B}) = \overline{B_r(\mathbf{0})} \cap \mathbb{H}^n$$
$$\phi(B') = B_{r'}(\mathbf{0}) \cap \mathbb{H}^n$$

Smooth Function: Let M be a smooth n-manifold, $k > 0$ and

$$f : M \to \mathbb{R}^k$$

any function. Then f is a smooth function if $\forall\ p \in M,\ \exists\ (U, \phi) \in \mathcal{A}_M$ such that

$$f \circ \phi^{-1} : \phi(U) \to f(U)$$

is smooth; i.e. the coordinate representation of f is smooth.

Note: smooth functions have smooth coordinate representations in every smooth chart.

Smooth Map between Manifolds: Let M, N be smooth manifolds and

$$F : M \to N$$

be any map. Then F is a smooth map if $\forall\ p \in M,\ \exists\ (U, \phi) \in \mathcal{A}_M$ containing p, and $(V, \psi) \in \mathcal{A}_N$ containing $F(p)$, such that $F(U) \subseteq V$ and

$$\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$$

is smooth.

Let $A \subseteq M$ be an arbitrary subset, then

$$F : A \to N$$

is smooth on A if $\forall\ p \in A\ \exists\ U_p \subseteq M$ open, and a smooth map

$$\widetilde{F} : U_p \to N$$

such that $\widetilde{F}|_{U_p \cap A} = F$.

Coordinate Representation: Let M, N be smooth manifolds. For a smooth map $F : M \to N$ and $(U, \phi) \in \mathcal{A}_M$, $(V, \psi) \in \mathcal{A}_N$

$$\widehat{F} := \psi \circ F \circ \phi^{-1} : \phi\left(U \cap F^{-1}(V)\right) \to \psi(V)$$

is the coordinate representation of F.

Cutoff function: for $r_1, r_2 \in \mathbb{R}$ such that $r_1 < r_2$, a cutoff function is a smooth function $h : \mathbb{R} \to \mathbb{R}$ having the properties

$$h(t) = \begin{cases} 1, & t \leq r_1 \\ 0 < h(t) < 1, & r_1 < t < r_2 \\ 0, & t \geq r_2 \end{cases}$$

Support: Let M be a topological space. For any real, or vector-valued function

$$f : M \to \mathbb{R}$$

or

$$f : M \to \mathbb{R}^n$$

the support of f is the closure of the set of points in M where f in nonzero:

$$\operatorname{supp}(f) = \overline{\{p \ \in M : f(p) \neq \mathbf{0}\}}$$

Bump function: Let M be a topological space with closed subset $A \subset M$ and open subset $U \subset M$ such that $U \supset A$. A bump function for A supported in U is a continuous function $\psi : M \to \mathbb{R}$ such that:

$$\begin{aligned}
&0 \leq \psi(M) \leq 1 \\
&\psi(a) \equiv 1 \quad \forall\ a \in A \\
&\operatorname{supp}(\psi) \subseteq U
\end{aligned}$$

Partition of Unity: Suppose M is a topological space, and let $\{U_\alpha\}_{\alpha \in \Lambda} := \mathcal{U}$ be an arbitrary open cover of M indexed by a set Λ. A partition of unity subordinate to $\mathcal{U}$ is an indexed family of continuous functions $\{\psi_\alpha\}_{\alpha \in \Lambda}$ with $\psi_\alpha : M \to \mathbb{R}$ such that:

(i) $0 \leq \psi_\alpha(p) \leq 1 \quad \forall\ \alpha \in \Lambda; \forall\ p \in M$

(ii) $\operatorname{supp}(\psi_\alpha) \subseteq U_\alpha$ for each $\alpha \in \Lambda$

(iii) The family of supports $\{\operatorname{supp}(\psi_\alpha)\}_{\alpha \in \Lambda}$ is locally finite; i.e. $\forall\ p \in M\ \exists$ a neighborhood U_p such that

$$U_p \cap \operatorname{supp}(\psi_\alpha) \neq \varnothing$$

for finitely many $\alpha \in \Lambda$

(iv) $\sum_{\alpha \in \Lambda} \psi_\alpha(p) = 1 \quad \forall\ p \in M$.

Note: local finiteness gives only finitely many of the terms in the sum are non-zero, and so there is no issue of convergence.

A **Smooth Partition of Unity** has the family $\{\psi_\alpha\}_{\alpha \in \Lambda}$ being smooth, instead of just continuous. Used to blend Local Smooth objects into Global Smooth objects, without the assumption of agreement on overlaps.

Level set: for manifolds M, N and any map $\Phi : M \to N$ for each $c \in N$, $\Phi^{-1}(c)$ is a level set of Φ.

- special case $N = \mathbb{R}^k$ then for $c = \mathbf{0} \in \mathbb{R}^k$, $\Phi^{-1}(\mathbf{0})$ is the zero set of Φ

Sublevel set: Let M be a topological space. The sublevel sets for the function

$$f : M \to \mathbb{R}$$

are $f^{-1}\big((-\infty, c]\big)$ for each $c \in \mathbb{R}$.

Exhaustion function: If M is a topological space, an exhaustion function for M is a continuous function

$$f : M \to \mathbb{R}$$

such that $f^{-1}((-\infty, c]) \subseteq M$ is compact for each $c \in \mathbb{R}$; i.e. all of f's sublevel sets are compact.

Note: If M is compact, any continuous $\mathbb{R}$-valued function on M is an exhaustion function.

Examples:

- $f : \mathbb{R}^n \to \mathbb{R}$ by $f(\mathbf{x}) = |\mathbf{x}|^2$
- $f : \mathbb{B}^n \to \mathbb{R}$ by $f(\mathbf{x}) = \frac{1}{1-|\mathbf{x}|^2}$

Geometric Tangent Space: Let $\mathbf{a} \in \mathbb{R}^n$ then the geometric tangent space of $\mathbb{R}^n$ at $\mathbf{a}$ is

$$\{\mathbf{a}\} \times \mathbb{R}^n = \{(\mathbf{a}, \mathbf{v}) : \mathbf{v} \in \mathbb{R}^n\} := \mathbb{R}^n_{\mathbf{a}}$$

Which is to say a translation of $\mathbb{R}^n$ by $\mathbf{a}$ so that the origin coincides with $\mathbf{a}$; i.e.

$$\mathbb{R}^n_{\mathbf{a}} = \mathbb{R}^n - \mathbf{a}$$

Elements $\mathbf{v}_{\mathbf{a}} \in \mathbb{R}^n_{\mathbf{a}}$ are referred to as geometric tangent vectors

Directional Derivative: Let $\mathbf{v}_{\mathbf{a}} \in \mathbb{R}^n_{\mathbf{a}}$ then the directional derivative defined by

$$D_{\mathbf{v}_{\mathbf{a}}} = D_{\mathbf{v}}\big|_{\mathbf{a}} : C^\infty(\mathbb{R}^n) \to \mathbb{R}; \text{ by } D_{\mathbf{v}_{\mathbf{a}}}(f) = D_{\mathbf{v}}\big(f(\mathbf{a})\big) = \left.\frac{d}{dt}\right|_{t=0} f(\mathbf{a} + t\mathbf{v})$$

Which is the derivative at $\mathbf{a}$ in the direction of $\mathbf{v}$. If $\mathbf{v}_{\mathbf{a}} = v^i e_i|_{\mathbf{a}}$ we have

$$D_{\mathbf{v}}(f)|_{\mathbf{a}} = v^i \frac{\partial f}{\partial x^i}(\mathbf{a}) = \sum_{i=1}^n v^i \frac{\partial f}{\partial x^i}(\mathbf{a})$$

Derivation: Let $\mathbf{a} \in \mathbb{R}^n$, then the map $w : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is a derivation at $\mathbf{a}$ if it is linear over $\mathbb{R}$ and satisfies

$$w(fg)|_{\mathbf{a}} = f(\mathbf{a})w(g)|_{\mathbf{a}} + g(\mathbf{a})w(f)|_{\mathbf{a}}$$

Denote the set of all derivations $C^\infty(\mathbb{R}^n)|_{\mathbf{a}}$ as $T_{\mathbf{a}}\mathbb{R}^n$

Tangent Space: Let M be a manifold with or without boundary and let $p \in M$, then the set of all derivations $C^\infty(M)|_p := T_pM$ is the tangent space of M at p. And any element of T_pM is referred to as a tangent vector at p.

T_pM has a basis given by $\left\{\frac{\partial}{\partial x^1}\big|_p, \ldots, \frac{\partial}{\partial x^n}\big|_p\right\}$, and so each $v_p \in T_pM$ can be written uniquely as

$$v_p = \sum_{i=1}^n v^i \left.\frac{\partial}{\partial x^i}\right|_p$$

with $x^j \in C^\infty(U)$ we get the components of v_p by

$$v_p(x^j) = \sum_{i=1}^n v^i \left.\frac{\partial}{\partial x^i}\right|_p (x^j) = \sum_{i=1}^n v^i \frac{\partial x^j}{\partial x^i}(p) = \sum_{i=1}^n v^i \delta^j_i = v^j$$

Coordinate Vectors: Let $p \in M$, the coordinate vectors at p are a basis for T_pM defined as follows. Let $(U, \phi) \in \mathcal{A}_M$ be a chart containing p. Since ϕ is a diffeomorphism, and both $d\iota|_p : T_pU \to T_pM$ and $d\iota|_{\phi(p)} : T_{\phi(p)}\phi(U) \to T_{\phi(p)}\mathbb{R}^n$ are isomorphisms we get the induced isomorphism

$$d\phi|_p : T_pM \to T_{\phi(p)}\mathbb{R}^n$$

with $\left\{\frac{\partial}{\partial x^1}\big|_{\phi(p)}, \dots, \frac{\partial}{\partial x^n}\big|_{\phi(p)}\right\}$ forming a basis for $T_{\phi(p)}\mathbb{R}^n$ and since isomorphisms map basis vectors to basis vectors we have

$$\left.\frac{\partial}{\partial x^i}\right|_p := (d\phi|_p)^{-1}\left(\left.\frac{\partial}{\partial x^i}\right|_{\phi(p)}\right) = d\phi^{-1}|_{\phi(p)}\left(\left.\frac{\partial}{\partial x^i}\right|_{\phi(p)}\right)$$

for $i \in \{1, \dots, n\}$ form a basis for T_pM. So for $f \in C^\infty(U)$ with coordinate representation given by

$$\widehat{f} := f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$$

we get that the derivation $\frac{\partial}{\partial x^i}\big|_p \in T_pM$ acts on f by

$$\left.\frac{\partial}{\partial x^i}\right|_p (f) = d\phi^{-1}|_{\phi(p)}\left(\left.\frac{\partial}{\partial x^i}\right|_{\phi(p)}\right)(f) = \left.\frac{\partial}{\partial x^i}\right|_{\phi(p)}(f \circ \phi^{-1})|_{\phi(p)} = \frac{\partial \widehat{f}}{\partial x^i}\big(\phi(p)\big)$$

and so $\frac{\partial}{\partial x^i}\big|_p$ takes the i^{th} partial derivative of $\widehat{f}$ at $\phi(p)$.

Differential: Let M, N be smooth manifold with or without boundary, and $F : M \to N$ a smooth map. Then for each $p \in M$ the differential of F at p is the map

$$dF|_p : T_pM \to T_{F(p)}N \text{ by } dF|_p(v_p) = v|_{F(p)} := v_{F(p)}$$

so for $v_p \in T_pM$, $dF|_p(v_p)$ is the derivation at $F(p) \in N$ that acts on functions $f \in C^\infty(N)$. And so

$$dF|_p(v_p) : C^\infty(N)|_{F(p)} \to \mathbb{R}, \text{ by } dF|_p(v_p)(f) = v(f \circ F)|_p$$

and is a derivation since for any $f, g \in C^\infty(N)|_{F(p)}$ we have

$$\begin{aligned} dF|_p(v_p)(fg) &= v\big((fg) \circ F\big)|_p \\ &= v\big((f \circ F) \cdot (g \circ F)\big)|_p \\ &= f\big(F(p)\big)v(g \circ F)|_p + g\big(F(p)\big)v(f \circ F)|_p \\ &= f\big(F(p)\big)dF|_p(v_p)(g) + g\big(F(p)\big)dF|_p(v_p)(f) \end{aligned}$$

Next, noting that for $(U, \phi) \in \mathcal{A}_M$ containing p, and $(V, \psi) \in \mathcal{A}_N$ containing $F(p)$, and for the coordinate representation of F we have

$$\widehat{F} = \psi \circ F \circ \phi^{-1} \implies \psi^{-1} \circ \widehat{F} = F \circ \phi^{-1}$$

and from the definition of basis vectors for $T_{F(p)}N$ we have

$$\left.\frac{\partial}{\partial y^j}\right|_{F(p)} = d\psi^{-1}|_{\widehat{F}(\phi(p))}\left(\left.\frac{\partial}{\partial y^j}\right|_{\widehat{F}(\phi(p))}\right)$$

and for the special case where $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$ with $\big(U, (x^1, \dots, x^m)\big) \in \mathcal{A}_{\mathbb{R}^m}$ containing p, and $\big(V, (y^1, \dots, y^n)\big) \in \mathcal{A}_{\mathbb{R}^n}$ containing $F(p)$, and $f \in C^\infty(\mathbb{R}^n)$ we get

$$dF|_p \left(\frac{\partial}{\partial x^i}\bigg|_p \right)(f) = \frac{\partial}{\partial x^i}\bigg|_p (f \circ F)|_p = \sum_{j=1}^n \frac{\partial f}{\partial y^j}\big(F(p)\big)\frac{\partial(y^j \circ F)}{\partial x^i}(p) = \left(\sum_{j=1}^n \frac{\partial F^j}{\partial x^i}(p) \frac{\partial}{\partial y^j}\bigg|_{F(p)} \right)(f)$$

so the action of $dF|_p$ on the basis vector $\frac{\partial}{\partial x^i}\big|_p \in T_pM$ is

$$\begin{aligned}
dF|_p \left(\frac{\partial}{\partial x^i}\bigg|_p \right) &= dF|_p \left(d\phi^{-1}|_{\phi(p)} \left(\frac{\partial}{\partial x^i}\bigg|_{\phi(p)} \right) \right) \\
&= d(F \circ \phi^{-1})|_{\phi(p)} \left(\frac{\partial}{\partial x^i}\bigg|_{\phi(p)} \right) \\
&= d(\psi^{-1} \circ \widehat{F})|_{\phi(p)} \left(\frac{\partial}{\partial x^i}\bigg|_{\phi(p)} \right) \\
&= d\psi^{-1}|_{\widehat{F}(\phi(p))} \left(d\widehat{F}|_{\phi(p)} \left(\frac{\partial}{\partial x^i}\bigg|_{\phi(p)} \right) \right) \\
&= d\psi^{-1}|_{\widehat{F}(\phi(p))} \left(\sum_{j=1}^n \frac{\partial \widehat{F}^j}{\partial x^i}\big(\phi(p)\big) \frac{\partial}{\partial y^j}\bigg|_{\widehat{F}(\phi(p))} \right) && \widehat{F} \text{ map between Euclidean spaces} \\
&= \sum_{j=1}^n \frac{\partial \widehat{F}^j}{\partial x^i}\big(\phi(p)\big) \cdot d\psi^{-1}|_{\widehat{F}(\phi(p))} \left(\frac{\partial}{\partial y^j}\bigg|_{\widehat{F}(\phi(p))} \right) && \text{linearity} \\
&= \sum_{j=1}^n \frac{\partial \widehat{F}^j}{\partial x^i}\big(\phi(p)\big) \frac{\partial}{\partial y^j}\bigg|_{F(p)}
\end{aligned}$$

and therefore $dF|_p$ is represented by the Jacobian matrix of $\widehat{F}$ at $\phi(p)$. Or,

$$dF|_p = \begin{bmatrix} \frac{\partial \widehat{F}^1}{\partial x^1}\big(\phi(p)\big) & \cdots & \frac{\partial \widehat{F}^1}{\partial x^m}\big(\phi(p)\big) \\ \vdots & \ddots & \vdots \\ \frac{\partial \widehat{F}^n}{\partial x^1}\big(\phi(p)\big) & \cdots & \frac{\partial \widehat{F}^n}{\partial x^m}\big(\phi(p)\big) \end{bmatrix}$$

Equivalently, for $f \in C^\infty(M)$, with $f : M \to \mathbb{R}$ we can define the differential

$$df|_p : T_pM \to \mathbb{R} \cong T_{f(p)}\mathbb{R}, \text{ by } df|_p(v_p) = v_p(f)|_p$$

Change of Coordinates Let M be a manifold with or without boundary, let $p \in M$, and let $(U, \phi), (V, \psi) \in \mathcal{A}_M$ be two charts containing p. Then considering the smooth map $Id_M : M \to M$ with coordinate representation

$$\widehat{Id}_M = \psi \circ Id_M \circ \phi^{-1} : \phi\big(U \cap Id_M^{-1}(V)\big) = \phi(U \cap V) \to \psi(U \cap V)$$

so for $\phi = (x^1, \ldots, x^n)$, and $\psi = (y^1, \ldots, y^n)$, and recalling that $d(Id_M)|_p = Id_{T_pM}$, we get

$$\begin{aligned}
\left.\frac{\partial}{\partial x^i}\right|_p &= d(Id_M)|_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) \\
&= \sum_{j=1}^n \frac{\partial(y^j \circ \widehat{Id}_M)}{\partial x^i}\big(\phi(p)\big)\left.\frac{\partial}{\partial y^j}\right|_{Id_M(p)} \\
&= \sum_{j=1}^n \frac{\partial y^j}{\partial x^i}\big(\phi(p)\big)\left.\frac{\partial}{\partial y^j}\right|_p
\end{aligned}$$

so for $v_p \in T_pM$ where

$$v_p = \sum_{i=1}^n v_x^i \left.\frac{\partial}{\partial x^i}\right|_p = \sum_{j=1}^n v_y^j \left.\frac{\partial}{\partial y^j}\right|_p$$

we get that components of v_p are related by

$$v_y^j = \sum_{i=1}^n \frac{\partial y^j}{\partial x^i}\big(\phi(p)\big)v_x^i$$

Tangent Bundle: Given a smooth manifold M with or without boundary the tangent bundle, denoted TM, is the disjoint union of the tangent spaces $\forall\ p \in M$. That is

$$TM = \bigsqcup_{p\in M} T_pM$$

TM comes equipped with a natural projection

$$\pi : TM \to M \text{ by } \pi(v_p) = p$$

so for each vector $v_p \in T_pM$, π sends v_p to the point $p \in M$ at which it is tangent.

Natural Coordinates on the Tangent Bundle: Let M be a smooth manifold. Given any smooth chart $(U, \phi) \in \mathcal{A}_M$ and letting $\phi = (x^1, \ldots, x^n)$. Then

$$\pi^{-1}(U) = \bigsqcup_{p\in U} T_pM$$

Define the map

$$\widetilde{\phi} : \pi^{-1}(U) \to \mathbb{R}^{2n}, \text{ by } \widetilde{\phi}\left(\sum_{i=1}^n v^i \left.\frac{\partial}{\partial x^i}\right|_p\right) = \big(x^1(p), \ldots, x^n(p), v^1, \ldots, v^n\big)$$

With natural coordinates on TM being given by $\widetilde{\phi} = (x^1, \ldots, x^n, v^1, \ldots, v^n)$.

Global Differential: Let M and N be smooth manifolds, and $F : M \to N$ a smooth map. the global differential

$$dF : TM \to TN$$

is defined by

$$dF(v_p) = dF|_p(v_p) = v|_{F(p)}$$

so that $\forall\ v_p \in T_pM \subset TM$ we have $dF(v_p) \in T_{F(p)}N \subset TN$.

Curve: Let M be a manifold with or without boundary, and $J \subseteq \mathbb{R}$ an interval. A curve is a continuous map

$$\gamma : J \to M$$

Velocity of a Curve: Let M be a manifold with or without boundary, $J \subseteq \mathbb{R}$, and $t_0 \in J$ the velocity of the curve $\gamma : J \to M$ at t_0 is the derivation

$$\gamma'(t_0) = d\gamma|_{t_0}\left(\frac{d}{dt}\Big|_{t_0}\right) \in T_{\gamma(t_0)}M$$

so for $\big(U,(x^1,\ldots,x^n)\big) \in \mathcal{A}_M$, $\gamma(t_0) \in U \subseteq M$ and $f \in C^\infty(M)$

$$\begin{aligned}
\gamma'(t_0)(f) &= d\gamma|_{t_0}\left(\frac{d}{dt}\Big|_{t_0}\right)(f) \\
&= \frac{d}{dt}\Big|_{t_0}(f\circ\gamma)|_{t_0} \\
&= \sum_{i=1}^{n}\frac{\partial f}{\partial x^i}\big(\gamma(t_0)\big)\frac{d(x^i\circ\gamma)}{dt}(t_0) \\
&= \sum_{i=1}^{n}\left(\frac{d\gamma^i}{dt}(t_0)\frac{\partial}{\partial x^i}\Big|_{\gamma(t_0)}\right)(f)
\end{aligned}$$

so that

$$\gamma'(t_0) = \sum_{i=1}^{n}\frac{d\gamma^i}{dt}(t_0)\frac{\partial}{\partial x^i}\Big|_{\gamma(t_0)} \text{ in Local Coordintates}$$

Equivalently for $f \in C^\infty(M)$ we have

$$df|_{\gamma(t_0)} : T_{\gamma(t_0)}M \to \mathbb{R} \cong T_{f(\gamma(t_0))}\mathbb{R},\ \text{by } df|_{\gamma(t_0)}(\gamma'(t_0)) = (f\circ\gamma)'(t_0)$$

Rank of Smooth map: let M, N be smooth manifolds with or without boundary. Given a smooth map $F : M \to N$ and a point $p \in M$ the rank of F at p is the rank of the linear map

$$dF|_p : T_pM \to T_{F(p)}N, \quad \text{recall } dF|_p = \begin{bmatrix} \frac{\partial \widehat{F}^1}{\partial x^1}\big(\phi(p)\big) & \cdots & \frac{\partial \widehat{F}^1}{\partial x^m}\big(\phi(p)\big) \\ \vdots & \ddots & \vdots \\ \frac{\partial \widehat{F}^n}{\partial x^1}\big(\phi(p)\big) & \cdots & \frac{\partial \widehat{F}^n}{\partial x^m}\big(\phi(p)\big) \end{bmatrix}$$

Which is the rank of the Jacobian of $\widehat{F}$ at $\phi(p)$ in any smooth chart. If F has the same rank at every point then we say F has **Constant Rank**.

Note: $\text{rank}(F) \leq \min\{\dim(M), \dim(N)\}$; if $\text{rank}(dF|_p) = \min\{\dim(M), \dim(N)\}$ then F has **Full Rank** at p

Smooth Submersion: A smooth map $F : M \to N$ is a smooth submersion if its differential

$$dF|_p : T_pM \to T_{F(p)}N$$

is surjective at each point; i.e. $\text{rank}(F) = \dim(N)$.

Smooth Immersion: A smooth map $F : M \to N$ is a smooth immersion if its differential

$$dF|_p : T_pM \to T_{F(p)}N$$

is injective at each point; i.e. $\text{rank}(F) = \dim(M)$.

Local Diffeomorphism: Let M and N be smooth manifolds with or without boundary, a map

$$F : M \to N$$

is a local diffeomorphism if $\forall\ p \in M, \exists\ U_p \subseteq M$ open such that $F(U_p) \subseteq N$ is open and

$$F|_{U_p} : U_p \to F(U_p)$$

is a diffeomorphism; i.e. a map whose differential is invertible at each point.

2 Notes

Lemma 1. Every topological manifold has a countable basis of precompact coordinate balls.

Proof. Let M be a topological n-manifold. First, considering the special case where M is covered by a single chart (M, ϕ), so that

$$\phi : M \to \mathbb{R}^n$$

is a global coordinate map, and let

$$\mathcal{B} = \{B_r(\mathbf{x}) \subseteq \mathbb{R}^n : r \in \mathbb{Q}; \mathbf{x} \in \mathbb{Q}^n\}$$

then for

$$B_{r'}(\mathbf{x}) \subseteq \phi(M) \quad \text{for some } r' > r$$

we have that $\overline{B_{r'}(\mathbf{x})}$ is compact in $\phi(M)$, and hence is precompact. Furthermore, $\mathcal{B}$ is a countable basis for the topology of $\mathbb{R}^n$, and hence is a countable basis for the relative topology of $\phi(M)$. Then since ϕ is a homeomorphism and

$$\mathcal{B} \supseteq \mathbb{R}^n \implies \phi^{-1}(\mathcal{B}) = \{\phi^{-1}(B) : B \in \mathcal{B}\} \supseteq \phi(\mathbb{R}^n) = M$$

and so $\{\phi^{-1}(B) : B \in \mathcal{B}\}$ is a countable base for the topology on M. Moreover, ϕ^{-1} is continuous, and since continuous functions map compact sets into compact sets we have that $\phi^{-1}(\overline{B}) \subseteq M$ is compact, and

$$\overline{\phi^{-1}(B)} \subseteq \phi^{-1}(\overline{B})$$

and so $\overline{\phi^{-1}(B)}$ is compact as the closed subset of a compact set, and thus we see that $\{\phi^{-1}(B) : B \in \mathcal{B}\}$ is a countable base for M of precompact coordinate balls.

Next, letting M be an arbitrary topological n-manifold, by definition each point of M belongs to the domain of some coordinate chart, and we also have that M is second countable, and since every open cover of a second countable space has a countable subcover we have

$$\bigcup_{i\geq 0}\{(U_i, \phi_i)\} \supseteq M$$

Now, from above each U_i has a countable base $\phi_i^{-1}(\mathcal{B}_i)$ of precompact coordinate balls for U_i, where we then get

$$\bigcup_{i\geq 0}\phi_i^{-1}(\mathcal{B}_i)$$

is a countable base for the topology on M. Then for

$$V \subseteq U_i \subseteq M$$

we have that $\overline{V} \subseteq U_i$ is compact in U_i, and hence in M, and since M is Hausdorff and $\overline{V} \subseteq M$ is compact we have that $\overline{V}$ is closed in M. Thus, the closure of V in M is the same as its closer in U_i, and therefore, V is precompact in M. $\square$

Proposition 2. Let M be a topological manifold. Then

(a) M is locally path-connected.

(b) M is connected if and only if it is path-connected.

(c) The components of M are the same as its path components.

(d) M has countably many components, each of which is an open subset of M and a connected topological manifold.

Proof.

(a) Since M has a countable basis of coordinate balls, and each ball is path-connected, M is locally path-connected.

(b) First since path-connected implies connected in any topological space, it suffices to show that for a manifold M, connected implies path-connected.

So let M be a connected manifold of dimension n, let $p \in M$ be arbitrary, and let $V \subseteq M$ be the set of points $q \in M$ such that p can be joined to q by a path in M.

Take any $q \in V$, since $V \subseteq M$ and since M is a manifold it is locally euclidean, and so $\exists$ (U_q, ϕ) such that

$$\phi : U_q \to \phi(U_q) \subseteq \mathbb{R}^n$$

where $\phi(U_q) \subseteq \mathbb{R}^n$ is open, and since $\phi(q) \in \phi(U_q)$, $\exists\ B_\epsilon\big(\phi(q)\big) \subset \phi(U_q)$ for $\epsilon > 0$, and since $\mathbb{R}^n$ is path connected, $\forall\ \mathbf{x} \in B_\epsilon\big(\phi(q)\big)\ \exists$ a path

$$\gamma_{\mathbf{x},\phi(q)} : [0,1] \to B_\epsilon\big(\phi(q)\big) \text{ such that } \gamma_{\mathbf{x},\phi(q)}(0) = \mathbf{x},\ \gamma_{\mathbf{x},\phi(q)}(1) = \phi(q)$$

and since ϕ is a homeomorphism, it is continuous as is it's inverse, and as $B_\epsilon\big(\phi(q)\big) \subset \phi(U_q)$ is open, we also have $\phi^{-1}\big(B_\epsilon\big(\phi(q)\big)\big) \subset M$ is open. Then since $\phi^{-1} : \phi(U_q) \to M$ is continuous and $B_\epsilon\big(\phi(q)\big)$ is path connected, and since the image of a path-connected set under a continuous map is path-connected, we have $\phi^{-1}\big(B_\epsilon\big(\phi(q)\big)\big) \ni q$ is also path connected, that is $\forall\ \phi^{-1}(\mathbf{x}) \in \phi^{-1}\big(B_\epsilon\big(\phi(q)\big)\big)\ \exists$

$$\gamma_{p,\phi^{-1}(\mathbf{x})} : [0,2] \to M,\ \text{by } \gamma_{p,\phi^{-1}(\mathbf{x})}(t) = \begin{cases} \gamma_{p,q}(t), & 0 \le t \le 1 \\ \gamma_{q,\phi^{-1}(\mathbf{x})}(t), & 1 \le t \le 2 \end{cases}$$

which is a path connecting p to q and then q to $\phi^{-1}(\mathbf{x})$ and therefore $\phi^{-1}\big(B_\epsilon\big(\phi(q)\big)\big) \subseteq V$ and V is open.

Similarly for any $q \in V^c$, there exists a chart (W_q, ψ) with $q \in W_q$ which is locally euclidean, and for which $\psi^{-1}\big(B_\epsilon\big(\psi(q)\big)\big) \subset M$ is open. Now,

$$V \cap \psi^{-1}\big(B_\epsilon\big(\psi(q)\big)\big)$$

must be empty, if not, since $B_\epsilon\big(\psi(q)\big)$ is path connected, and ψ^{-1} is continuous, we know that the image of a path connected set under a continuous map is also path connected, and so if there was some $r \in V \cap \psi^{-1}\big(B_\epsilon\big(\psi(q)\big)\big)$ then we could find a path $\gamma_{p,r} \subset V$ joining p to r, and a path $\gamma_{r,q} \subset \psi^{-1}\big(B_\epsilon\big(\psi(q)\big)\big)$ joining r to q and concatenation would give us a path

$$\gamma_{p,r} \circ \gamma_{r,q}$$

joining p to q and hence $q \in V \Rightarrow\Leftarrow$. And so $\psi^{-1}\big(B_\epsilon\big(\psi(q)\big)\big) \subseteq V^c$.

That is, for each $q \in V^c$ we can find an open neighborhood of q entirely contained V^c, and so V^c is also open.

And thus we have $M = V \sqcup V^c$ where both V, V^c are open, and hence also both closed as the compliments of open sets. Yet $V \cap V^c = \varnothing$. Now, $p \in V \implies V \neq \varnothing$, and thus $V^c = \varnothing$ and therefore

$$M = V \sqcup V^c = V \sqcup \varnothing = V$$

and so M is path-connected through p.

Since $p \in M$ was arbitrary we conclude that for each $p \in M$, M is path-connected through p, and therefore M is path-connected.

(c) First, we show the result for a Locally path-connected topological space, then since a topological manifold is a special type of topological space the result follows.

Let X be a locally path-connected topological space, let $x \in X$ be arbitrary, and let $x \in P_x$ where P_x is the path component in X containing x. Since X is locally path-connected it has a basis $\mathcal{B}$ of path-connected open sets. So for any $y \in P_x$, $\exists\ B_y \in \mathcal{B}$ such that $y \in B_y \subset P_x$, since B_y is path-connected. Then as P_x contains an open neighborhood around each of its points it must me open. Since $x \in X$ was arbitrary we conclude that all path components of X are open.

Let C be an arbitrary component of X, let $x \in C$, and let P_x be the path component of X such that $x \in P_x$. Since path-connected implies connected we have P_x is connected and thus $P_x \subseteq C$.

Next suppose that $P_x \neq C$ that is P_x is strictly contained in C, then as the path components partition X there must be other path components contained in C, as path-connected implies connected they must lie entirely in C, so define

$$V := \bigcup_{\substack{y \in C \\ y \neq x}} P_y$$

Then we have

$$C = P_x \sqcup V$$

Since X is locally path-connected, each path component of X is open in X from above. So that P_x is open in X and thus $P_x \cap C$ is open in the relative topology of C, and V which is the union of open sets is open in $X \implies V \cap C$ is open in C. Additionally both P_x, V were defined so that $P_x \cap C \neq \varnothing$, and $V \cap C \neq \varnothing$ and again by their definition $(P_x \cap C) \cap (V \cap C) = \varnothing$. And as noted above

$$(P_x \cap C) \cup (V \cap C) = (P_x \cup V) \cap C = C \cap C = C$$

So that $P_x, V \subseteq C$ form a separation of C which contradicts the fact that C is a connected component.

And therefore we must have that $P_x = C$.

(d) First, we show that for any topological space X, the components of X are open. Then, since this holds for arbitrary topological spaces, it also holds for topological manifolds.

Let X be locally path-connected, and let C be an arbitrary component of X. Since X is locally path-connected $\exists$ a basis $\mathcal{B}$ of path-connected subsets, each which are open as they belong to a basis. So for any $x \in C$, $\exists\ B_x \in \mathcal{B}$ such that $x \in B_x \subset C$, where B_x is path-connected, and hence, connected. Then as each point of C has an open neighborhood contained in $C \implies C$ is open. Then as the component C of X was arbitrary we conclude that all of the components of X must be open.

Now, since the components of M are open, we have that the components of M form an open cover of M, and since M is second countable it admits a countable subcover. Yet, since the components which partition M are disjoint, this tells us that the open cover must have been countable to begin with, and so M has only countably many components. Then, since the components are open, they are connected topological manifolds in the relative topology of M.

$\square$

Proposition 3 (**Manifolds are Locally Compact**)**.** Every topological manifold is locally compact.

Proof. From Lemma 1 we know that every topological manifold has a countable basis of precompact coordinate balls. And since the closure of these balls is compact in M, and each $p \in M$ belongs to at least one of these balls, we have that M is locally compact. □

Lemma 4. Suppose $\mathcal{S}$ is a locally finite collection of subsets of a topological space M. Then

(a) The collection
$$\left\{\overline{S} : S \in \mathcal{S}\right\}$$
is also locally finite.

(b)
$$\overline{\bigcup_{S\in\mathcal{S}} S} = \bigcup_{S\in\mathcal{S}} \overline{S}$$

Proof.

(a) Let $p \in M$ be arbitrary, then by the local finiteness of $\mathcal{S}$ $\exists$ U_p open such that
$$U_p \cap S = \varnothing$$
for all but finitely many $S \in \mathcal{S}$, say $\{S_1, \ldots, S_n\}$. Next note that for each $j \neq \{1, \ldots, n\}$ since
$$U_p \cap S_j = \varnothing \implies S_j \subseteq U_p^c = M \setminus U_p$$
Then since U_p is open, U_p^c must be closed, and thus
$$\overline{S_j} \subseteq \overline{U_p^c} = U_p^c = M \setminus U_p$$
this implies that for each $j \neq \{1, \ldots, n\}$ we have
$$\overline{S}_j \cap U_p = \varnothing$$
while for $i = \{1, \ldots, n\}$ we have $S_i \cap U_p \neq \varnothing$ where $S_i \subseteq \overline{S}_i$ and so
$$\overline{S}_i \cap U_p \neq \varnothing$$
and therefore $\left\{\overline{S} : S \in \mathcal{S}\right\}$ is locally finite.

(b) First since for each $S \in \mathcal{S}$ we have
$$S \subseteq \bigcup_{S\in\mathcal{S}} S \implies \overline{S} \subseteq \overline{\bigcup_{S\in\mathcal{S}} S} \implies \bigcup_{S\in\mathcal{S}} \overline{S} \subseteq \overline{\bigcup_{S\in\mathcal{S}} S}$$
Next, let $p \in M$ be arbitrary, then by the local finiteness of $\mathcal{S}$ $\exists$ U_p open such that
$$U_p \cap S = \varnothing$$

for all but finitely many $S \in \mathcal{S}$, say $\{S_1, \ldots, S_n\}$. Suppose further that

$$p \notin \bigcup_{S \in \mathcal{S}} \overline{S} \implies p \in \left(\bigcup_{S \in \mathcal{S}} \overline{S} \right)^c$$

then since $p \notin \bigcup_{S \in \mathcal{S}} \overline{S}$ this implies that $p \notin \{\overline{S}_1, \ldots, \overline{S}_n\}$ and this implies that

$$p \in U_p \setminus \bigcup_{i=1}^{n} \overline{S}_i = U_p \cap \left(\bigcup_{i=1} \overline{S}_i \right)^c = U_p \cap \bigcap_{i=1}^{n} \overline{S}_i^c$$

which is open as the finite intersection of open sets. That is $U_p \cap \bigcap_{i=1}^n \overline{S}_i^c$ is an open neighborhood of p in $\left(\bigcup_{S \in \mathcal{S}} \overline{S}\right)^c$.

Since $p \in \left(\bigcup_{S \in \mathcal{S}} \overline{S}\right)^c$ was arbitrary, we conclude that each point in the compliment has an open neighborhood, and therefore $\left(\bigcup_{S \in \mathcal{S}} \overline{S}\right)^c$ is open and hence

$$\left(\left(\bigcup_{S \in \mathcal{S}} \overline{S} \right)^c \right)^c = \bigcup_{S \in \mathcal{S}} \overline{S}$$

must be closed and since for each $S \in \mathcal{S}$ we have $\overline{S} \supseteq S$, this gives

$$\bigcup_{S \in \mathcal{S}} S \subseteq \bigcup_{S \in \mathcal{S}} \overline{S} \implies \overline{\bigcup_{S \in \mathcal{S}} S} \subseteq \overline{\bigcup_{S \in \mathcal{S}} \overline{S}} = \bigcup_{S \in \mathcal{S}} \overline{S}$$

and thus we can conclude

$$\overline{\bigcup_{S \in \mathcal{S}} S} = \bigcup_{S \in \mathcal{S}} \overline{S}$$

$\square$

Theorem 5 (**Manifolds are Paracompact**)**.** Every topological manifold is paracompact. Furthermore, given a topological manifold M; an open cover $\mathcal{U}$ of M; and any basis $\mathcal{B}$ for the topology of M; there exists a countable, locally finite open refinement of $\mathcal{U}$ consisting of elements of $\mathcal{B}$.

Proof. Let M,$\mathcal{U}$ and $\mathcal{B}$ be given, since M is a manifold it is second countable, Hausdorff, and locally compact from Proposition 3, and so it admits an exhaustion by compact sets. So let $\{K_i\}_{i=1}^\infty$ we the exhaustion of M by compact sets and recall that since $\{K_i\}_{i=1}^\infty$ is an exhaustion of M this implies

$$M = \bigcup_{i=1}^{\infty} K_i$$

as well as

$$\cdots K_{i-1} \subseteq \mathrm{Int}(K_i) \subseteq K_i \subseteq \mathrm{Int}(K_{i+1}) \subseteq K_{i+1} \subseteq \mathrm{Int}(K_{i+2}) \subseteq \cdots$$

so for each $i \geq 0$ define

$$\begin{aligned} V_i &= K_{i+1} \setminus \mathrm{Int}(K_i) \\ W_i &= \mathrm{Int}(K_{i+2}) \setminus K_{i-1} \end{aligned}$$

where $K_i = \varnothing$ for $i < 1$. Then since $\text{Int}(K_i)$ is open, we have that V_i is a closed subset of compact K_{i+1} and is therefore compact. And since since K_{i-1} is a compact set in a Hausdorff space, it is closed, and so W_i is open, and furthermore, by construction

$$V_i \subseteq W_i$$

Now, for each $p \in V_i$, $\exists\ U_p \in \mathcal{U}$ containing p, as $\mathcal{U}$ is a cover. Then since $\mathcal{B}$ is a basis $\exists\ B_p \in \mathcal{B}$ such that

$$p \in B_p \subseteq U_p \cap W_i$$

Then the collection of all such B_p as p ranges over V_i is an open cover of V_i; that is

$$\bigcup_{p \in V_i} B_p \supseteq V_i$$

and by the compactness of V_i, there must be a finite subcover

$$\bigcup_{j=1}^{n} B_{p_j} = V_i$$

and the union of all such finite subcovers as i ranges over $\mathbb{N}$, is a countable open cover for M that refines $\mathcal{U}$; i.e.

$$\bigcup_{i \in \mathbb{N}} \bigcup_{j=1}^{n_i} B_{p_j} \supseteq M$$

Then, because

$$\bigcup_{j=1}^{n} B_{p_j} = V_i \subseteq W_i$$

where

$$W_i \cap W_k = \varnothing \quad \text{unless } k \in \{i-2, i-1, i, i+1, i+2\}$$

we have that for each $p \in M$, $\exists\ U_p$ open such that

$$U_p \cap B_p = \varnothing$$

for all but finitely many B_p, and thus, the cover

$$\mathcal{V} = \bigcup_{i \in \mathbb{N}} \bigcup_{j=1}^{n_i} B_{p_j}$$

is locally finite. □

Proposition 6. The Fundamental group of a topological manifold is countable.

Proof. Let M be a topological manifold, by Lemma 1 M has a countable basis $\mathcal{B}$ of precompact coordinate balls. Hence, for any $B, B' \in \mathcal{B}$

$$B \cap B'$$

has at most countably many components each of which are path-connected by Proposition 2 (c).

let $\mathcal{U}$ be a set containing a point from each component of $B \cap B'$ for each $B, B' \in \mathcal{B}$; i.e. if $p \in B \cap B'$ and C_p is the component containing p, then since there are only countably many components

$$\bigcup_{i=1}^{\infty} C_{p_i} = B \cap B'$$

and so

$$\mathcal{U} = \bigcup_{B, B' \in \mathcal{B}} \left(\bigcup_{i=1}^{\infty} C_{p_i} \right)$$

including when $B' = B$. For each $B \in \mathcal{B}$ with $x, x' \in \mathcal{U}$ such that $x, x' \in B$ define $h^B_{x,x'}$ to be a path connecting x to x' in B; i.e.

$$h^B_{x,x'} : [0,1] \to B, \text{ such that } h^B_{x,x'}(0) = x, \ h^B_{x,x'}(1) = x'$$

since the fundamental groups based at any two points of a connected component are isomorphic, and $\mathcal{U}$ contains at least one point from each component of M, we may consider our base point p as belonging to $\mathcal{U}$.

Next we define a special loop to be a loop based at p, or $\gamma \in \pi_1(M, p)$, such that

$$\gamma = \prod_{i=1}^{n} \left(h^{B_i}_{x_i, x_i'} \right)_i$$

that is γ is equal to a finite product of $h^B_{x,x'}$'s. Then since each special loop determines an element in $\pi_1(M, p)$, and there will be countably many special loops, it suffices to show that each element of $\pi_1(M, p)$ is represented by a special loop.

So let $f \in \pi_1(M, p)$ be an arbitrary loop based at p; that is

$$f : [0,1] \to M, \text{ such that } f(0) = p = f(1)$$

then $\{f^{-1}(B) : B \in \mathcal{B}\}$ is an open cover of compact $[0,1]$, and thus admits a finite subcover, and by the Lebesgue number Lemma we may partition $[0,1]$ into

$$0 = a_0 < a_1 < \cdots < a_k = 1$$

such that $[a_i, a_{i+1}] \subseteq f^{-1}(B)$ for some $B \in \mathcal{B}$. So for each i define

$$f_i := f|_{[a_i, a_{i+1}]} : [0,1] \to B_i$$

where he have reparameterized $[a_i, a_{i+1}]$ to be $[0,1]$ and where B_i denotes the element of $\mathcal{B}$ that f_i maps into.

Now for each i we have

$$f(a_i) \in C_p \subseteq B_i \cap B_{i+1}$$

that is $f(a_i)$ belongs to some component of the intersection of two basis elements, and there is some $x_i \in \mathcal{U}$ such that $x_i \in C_p$. So we define a map g_i from x_i to $f(a_i)$; that is

$$g_i : [0,1] \to C_p, \text{ such that } g_i(0) = x_i, \; g_i(1) = f(a_i)$$

where $x_0 = p = x_k$ and $g_0 = c_p = g_k$, with c_p denoting the constant path based at p. Recall that the reverse path is defined by $g_i^{-1}(t) = g_i(1-t)$. And by the path-connectedness of each C_p we have

$$g_i^{-1} \cdot g_i \sim c_{f(a_i)}$$

then

$$\begin{aligned} f &\sim f_1 \cdot f_2 \cdot \cdots \cdot f_k \\ &\sim c_p \cdot f_1 \cdot c_{f(a_1)} \cdot f_2 \cdot \cdots \cdot c_{f(a_{k-1})} \cdot f_k \cdot c_p \\ &\sim g_0 \cdot f_1 \cdot (g_1^{-1} \cdot g_1) \cdot f_2 \cdot \cdots \cdot (g_{k-1}^{-1} \cdot g_{k-1}) \cdot f_k \cdot g_k^{-1} \\ &\sim (g_0 \cdot f_1 \cdot g_1^{-1}) \cdot (g_1 \cdot f_2 \cdot g_2^{-1}) \cdot \cdots \cdot (g_{k-1} \cdot f_k \cdot g_k^{-1}) \\ &\sim h_1 \cdot h_2 \cdot \cdots \cdot h_k \end{aligned}$$

where each h_i is entirely contained in B_i, which is a coordinate ball, and thus, simply connected. So that $h_i \sim h^{B_i}_{x_{i-1},x_i}$, and therefore we have

$$\begin{aligned} f &\sim h_1 \cdot h_2 \cdot \cdots \cdot h_k \\ &\sim h^{B_1}_{p,x_1} \cdot h^{B_2}_{x_1,x_2} \cdot \cdots \cdot h^{B_k}_{x_{k-1},p} \end{aligned}$$

and thus $\pi_1(M,p)$ is countable. □

Proposition 7. Let M be a topological manifold. Then

(a) Every smooth atlas $\mathcal{A}$ for M is contained in a unique maximal smooth atlas, called the smooth structure determined by $\mathcal{A}$.

(b) Two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas.

Proof.

(a) Let $\mathcal{A}$ be a smooth atlas for M, and let $\mathcal{A}_M$ be the set of all charts that are smoothly compatible with every chart of $\mathcal{A}$.

Let $(U,\phi),(V,\psi) \in \mathcal{A}_M$ be arbitrary charts such that $U \cap V \neq \varnothing$, we wish to show that

$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

is smooth. So let $\phi(p) \in \phi(U \cap V)$ be arbitrary. Then since $\mathcal{A}$ is an atlas for M, its charts cover M, and hence there exists some chart $(W,\xi) \in \mathcal{A}$ such that $p \in W$. Then, since every chart in $\mathcal{A}_M$ is smoothly compatible with (W,ξ) we have both

$$\begin{aligned} \xi \circ \phi^{-1} &: \phi(U \cap W) \to \xi(U \cap W) \\ \psi \circ \xi^{-1} &: \xi(W \cap V) \to \psi(W \cap V) \end{aligned}$$

are smooth. And by construction, since $p \in U \cap V \cap W$ we have

$$\psi \circ \phi^{-1} = (\psi \circ \xi^{-1}) \circ (\xi \circ \phi^{-1}) : \phi(U \cap V) \to \psi(U \cap V)$$

is a smooth neighborhood of $\phi(p)$. Since $\phi(p)$ is arbitrary we conclude that $\psi \circ \phi^{-1}$ is a smooth neighborhood for each point in $\phi(U \cap V)$. And by the arbitrariness of $(U, \phi), (V, \psi) \in \mathcal{A}_M$ we conclude that $\mathcal{A}_M$ is a smooth atlas.

Next we note that any chart that is smoothly compatible with every chart of $\mathcal{A}_M$ is, in particular, smoothly compatible with every chart in $\mathcal{A}$, and so must already be contained in $\mathcal{A}_M$. And so a maximal atlas containing $\mathcal{A}$ exists.

Suppose that $\mathcal{B}$ is another maximal smooth atlas containing $\mathcal{A}$, then each chart in $\mathcal{B}$ is smoothly compatible with each chart in $\mathcal{A}$ and hence

$$\mathcal{B} \subseteq \mathcal{A}_M$$

then by the maximality of $\mathcal{B}$ we must have

$$\mathcal{B} = \mathcal{A}_M$$

and so $\mathcal{A}_M$ is the unique maximal atlas containing $\mathcal{A}$.

(b) First suppose $\mathcal{A}_1, \mathcal{A}_2$ are two atlas's for M determining the same smooth structure $\mathcal{A}_M$. By part (a) $\mathcal{A}_M$ is a unique smooth structure and hence all charts in $\mathcal{A}_M$ are smoothly compatible. Yet, $\mathcal{A}_1 \subseteq \mathcal{A}_M$ and $\mathcal{A}_2 \subseteq \mathcal{A}_M$, and so in particular, for any $(U, \phi) \in \mathcal{A}_1$ and any $(V, \phi) \in \mathcal{A}_2$ we must have that

$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

is smoothly compatible, and hence $\mathcal{A}_1 \cup \mathcal{A}_2 \subseteq \mathcal{A}_M$ and thus, is smoothly compatible.

Next suppose that $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_1 \cup \mathcal{A}_2$ are all smooth atlas's. Then considering the smooth atlas $\mathcal{A}_M$ determined by $\mathcal{A}_1 \cup \mathcal{A}_2$, we know from (a) that $\mathcal{A}_M$ is unique, and that in particular

$$\mathcal{A}_1 \subseteq \mathcal{A}_1 \cup \mathcal{A}_2$$
$$\mathcal{A}_2 \subseteq \mathcal{A}_1 \cup \mathcal{A}_2$$

and so $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{A}_M$. That is, both $\mathcal{A}_1$ and $\mathcal{A}_2$ determine the same maximal atlas $\mathcal{A}_M$, and therefore the same smooth structure.

□

Proposition 8. Every smooth manifold has a countable basis of regular coordinate balls.

Lemma 9 (Smooth Manifold Chart Lemma). Let M be a set, and suppose we are given a collection $\{U_\alpha\}$ of subsets of M together with maps

$$\phi_\alpha : U_\alpha \to \mathbb{R}^n$$

such that the following properties are satisfied:

(i) For each α, ϕ_α is a bijection between U_α and an open subset $\phi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$.

(ii) For each α and β, the sets $\phi_\alpha(U_\alpha \cap U_\beta), \phi_\beta(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^n$ are open.

(iii) Whenever $U_\alpha \cap U_\beta \neq \varnothing$, the map

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is smooth.

(iv) Countably many of the sets U_α cover M.

(v) Whenever $p, q \in M$ are distinct; either there exists some U_α such that

$$p, q \in U_\alpha$$

or there exist sets U_α, U_β such that

$$U_\alpha \cap U_\beta = \varnothing, \qquad p \in U_\alpha, \qquad q \in U_\beta$$

Then M has a unique smooth manifold structure such that each (U_α, ϕ_α) is a smooth chart.

Proof. We begin by giving M the initial topology; i.e.

$$\tau_M = \{\phi_\alpha^{-1}(V) : V \subseteq \mathbb{R}^n \text{ is open } \forall\ \alpha\}$$

so let $\phi_\alpha^{-1}(V), \phi_\beta^{-1}(W) \in \tau_M$ be arbitrary and let

$$p \in \phi_\alpha^{-1}(V) \cap \phi_\beta^{-1}(W) \implies \phi_\alpha^{-1}(V) \cap \phi_\beta^{-1}(W) \neq \varnothing$$

where (iii) gives the smoothness of $\phi_\beta \circ \phi_\alpha^{-1}$ which implies continuous and so

$$(\phi_\beta \circ \phi_\alpha^{-1})^{-1}(W) \subseteq \phi_\alpha(U_\alpha \cap U_\beta)$$

must be open, and (ii) implies

$$\phi_\alpha(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^n$$

is open, and since the finite intersection of open sets is open; that is, $V \cap (\phi_\beta \circ \phi_\alpha^{-1})^{-1}(W) \subseteq \mathbb{R}^n$ is open, we have

$$\begin{aligned} \phi_\alpha^{-1}(V) \cap \phi_\beta^{-1}(W) &= \phi_\alpha^{-1}(V) \cap (\phi_\alpha^{-1} \circ \phi_\alpha \circ \phi_\beta^{-1})(W) \\ &= \phi_\alpha^{-1}\big(V \cap (\phi_\beta \circ \phi_\alpha^{-1})^{-1}(W)\big) \in \tau_M \end{aligned}$$

and so $\phi_\alpha^{-1}(V) \cap \phi_\beta^{-1}(W) \in \tau_M$, and so $\{\phi_\alpha^{-1}(V) : V \subseteq \mathbb{R}^n \text{ is open } \forall\ \alpha\}$ is a base for τ_M.

Next, since

$$\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$$

is a homeomorphism, M is locally euclidean of dimension n. Where (v) tells us that M is Hausdorff. Then since each $\phi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$ which is second countable, and since ϕ_α is a homeomorphism, this tells us that each

$$U_\alpha \subseteq \phi_\alpha^{-1}(\mathbb{R}^n) = M$$

is second countable in M. Then (iv) tells us that finitely many U_α cover M, and so M itself is second countable.

Therefore M is a topological manifold. Where (iii) says that $\{(U_\alpha, \phi_\alpha)\} = \mathcal{A}$ is a smooth atlas. And since each chart is smoothly compatible with every chart in $\mathcal{A}$, this must be the unique smooth structure determined by $\{(U_\alpha, \phi_\alpha)\}$. $\square$

Theorem 10 (**Topological Invariance of Boundary**)**.** If M is a topological manifold with boundary, then each point of M is either a boundary point or an interior point, but not both. Thus, ∂M and $\mathrm{Int}(M)$ form a separation of M; that is

$$M = \partial M \sqcup \mathrm{Int}(M)$$

Proposition 11. Let M be a topological n-manifold with boundary. Then

(a) $\mathrm{Int}(M)$ is an open subset of M and a topological n-manifold without boundary.

(b) ∂M is a closed subset of M and a topological $(n-1)$-manifold without boundary.

(c) M is a topological manifold iff $\partial M = \varnothing$.

(d) If $n = 0$, then $\partial M = \varnothing$ and M is a 0-manifold.

Proof.

(a) Let $p \in \mathrm{Int}(M)$ be arbitrary, and let (U, ϕ) be an interior chart for M containing p. We wish to show that $U \subseteq \mathrm{Int}(M)$ to demonstrate that arbitrary points have neighborhoods contained in $\mathrm{Int}(M)$. To that end, let $q \in U$ be a point distinct from p. Then

$$\phi(q) \in \phi(U) \subseteq \mathbb{R}^n$$

and since $\phi(U)$ is open, there exists an open neighborhood $V_{\phi(q)}$ containing $\phi(q)$, and since ϕ is a homeomorphism $\phi^{-1}\big(\phi(U)\big) = U$, which implies

$$q \in \phi^{-1}\big(V_{\phi(q)}\big) \subseteq U$$

where the continuity of ϕ^{-1} implies that $\phi^{-1}\big(V_{\phi(q)}\big)$ is an open neighborhood of q in U, giving the interior chart

$$\Big(\phi^{-1}(V_{\phi(q)}), \phi|_{\phi^{-1}(V_{\phi(q)})}\Big)$$

thus $q \in \mathrm{Int}(M)$, and since this can be done for each point of U we can conclude that $U \subseteq \mathrm{Int}(M)$, and hence $\mathrm{Int}(M) \subseteq M$ is open.

Next we note that $\mathrm{Int}(M)$ inherits second countability and Hausdorffness as a subspace of M, and by definition, each point of $\mathrm{Int}(M)$ has an interior chart homeomorphic to an open subset of $\mathbb{R}^n$ and so is locally euclidean of dimension n. Thus, we can conclude that $\mathrm{Int}(M)$ is an n-manifold.

To see that $\mathrm{Int}(M)$ has no boundary we simply note that since each point of $\mathrm{Int}(M)$ is contained in an interior chart, the Topological Invariance of Boundary then tells us that no point of $\mathrm{Int}(M)$ can be contained in a boundary chart, and thus $\mathrm{Int}(M)$ is an n-manifold without boundary.

(b) First we observe that by the Topological Invariance of Boundary we have

$$M = \partial M \sqcup \operatorname{Int}(M) \implies \partial M = M \setminus \operatorname{Int}(M) = \operatorname{Int}(M)^c$$

and by (a) we know that $\operatorname{Int}(M)$ is open in M, and hence $\partial M \subseteq M$ must be closed.

Next, let $p \in \partial M$ and (U, ϕ) be a boundary chart for M containing p, then $\phi(p) \in \partial\mathbb{H}^n$. Let

$$V = \phi(U) \cap \partial\mathbb{H}^n$$

and note that $\phi(p) \in V$, and since $\phi(U) \subseteq \mathbb{H}^n$ is open by definition, then V is open in the relative topology of $\partial\mathbb{H}^n$ where

$$\partial\mathbb{H}^n = \{(x_1, \ldots, x_n) : x_n = 0\} \cong \mathbb{R}^{n-1} \times \{0\} \cong \mathbb{R}^{n-1}$$

then since ϕ is a homeomorphism we have that ϕ^{-1} is continuous and $\phi^{-1}(\phi(U)) = U$, and so

$$\phi^{-1}(V) = U \cap \phi^{-1}(\partial\mathbb{H}^n) = U \cap \partial M$$

and since $U \subseteq M$ is open, we have $U \cap \partial M$ is open in the relative topology of ∂M, and thus $\phi^{-1}(V)$ is open in ∂M and contains p, giving the chart

$$\left(\phi^{-1}(V), \phi|_{\phi^{-1}(V)}\right)$$

and since $p \in \partial M$ was arbitrary, we can find a chart for each point of ∂M, and thus ∂M is locally euclidean of dimension $(n-1)$. Then, since ∂M inherits second countability and Hausdorffness as a subspace of M we can conclude that ∂M is an $(n-1)$-manifold.

To see that ∂M has no boundary we simply note that since each point of ∂M is contained in a chart locally euclidean of dimension $(n-1)$, this implies that each chart for ∂M is contained in $\operatorname{Int}(\partial M)$; i.e. each chart for ∂M is an interior chart. Where the Topological Invariance of Boundary then tells us that no point of ∂M can be contained in a boundary chart, and thus ∂M is an $(n-1)$-manifold without boundary.

(c) First, let M be a topological n-manifold, then by definition M is locally euclidean of dimension n, which implies that each point of M is contained in an interior chart, since from (b) ∂M is locally euclidean of dimension $(n-1)$ and so contains no points locally euclidean of dimension n. Where the Topological Invariance of Boundary then tells us that no point of M can be contained in a boundary chart, and so $\partial M = \varnothing$.

Next, suppose that $\partial M = \varnothing$. Then, by the Topological Invariance of Boundary

$$M = \partial M \sqcup \operatorname{Int}(M) = \varnothing \sqcup \operatorname{Int}(M) = \operatorname{Int}(M)$$

where (a) then tells us that $M = \operatorname{Int}(M)$ is a topological n-manifold.

(d) Let M be a 0-manifold with boundary. Then each point of $p \in M$ is contained in a chart $\big(\{p\}, \phi\big)$ with

$$\phi : \{p\} \to \mathbb{R}^0 = \{0\}$$

and since

$$\mathrm{Int}(\mathbb{H}^0) = \mathbb{R}^0 \implies \partial\mathbb{H}^0 = \varnothing$$

where the homeomorphism ϕ then tells us

$$\partial M = \phi^{-1}(\partial\mathbb{H}^0) = \phi^{-1}(\varnothing) = \varnothing$$

then, since M has empty boundary, (c) tells us that $M = \mathrm{Int}(M)$ is a topological 0-manifold.

□

Proposition 12. Let M be a topological manifold with boundary. Then

(a) M has a countable basis of precompact coordinate balls and half-balls.

(b) M is locally compact.

(c) M is paracompact.

(d) M is locally path-connected.

(e) M has countably many components, each of which is an open subset of M and a connected topological manifold with boundary.

(f) The fundamental group of M is countable.

Proposition 13. Suppose $M_1, \ldots, M_k$ are smooth manifolds and N is a smooth manifold with boundary. Then

$$M_1 \times \cdots \times M_k \times N$$

is a smooth manifold with boundary, and

$$\partial(M_1 \times \cdots \times M_k \times N) = M_1 \times \cdots \times M_k \times \partial N$$

Proposition 14. Every smooth map between manifolds is continuous.

Proof. Let M, N be smooth manifolds with or without boundary and

$$F : M \to N$$

a smooth map. Given $p \in M$, the smoothness of F implies $\exists\ (U, \phi) \in \mathcal{A}_M$ containing p, and $(V, \psi) \in \mathcal{A}_N$ containing $F(p)$, such that $F(U) \subseteq V$ and

$$\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$$

is smooth, in the usual sense of calculus, and so is continuous.

Next, since ψ and ϕ are both homeomorphism we have

$$F|_U = \psi^{-1} \circ (\psi \circ F \circ \phi^{-1}) \circ \phi : U \to V$$

is continuous as the composition of continuous maps. That is F is continuous on a neighborhood of p.

Since $p \in M$ was arbitrary we conclude that F is continuous on a neighborhood of every point of M, and thus, F is continuous on M. □

Proposition 15 (**Smoothness is Local**)**.** Let M and N be smooth manifolds with or without boundary, and let $F : M \to N$ be a map. Then

(a) If every point $p \in M$ has a neighborhood U_p such that the restriction $F|_{U_p}$ is smooth, then F is smooth.

(b) Conversely, if F is smooth, then its restriction to every open subset is smooth.

Proof.

(a) Let $p \subseteq M$ be arbitrary and U_p a neighborhood containing p such that

$$F|_{U_p} : U_P \to F(U_p)$$

is smooth. Since U_p is open, $\exists\ B_p \subseteq U_p$ which is an open subset in U_p containing p. And by the smoothness of F on U_p we have the charts $(B_p, \phi) \in \mathcal{A}_{U_p} \subseteq \mathcal{A}_M$ containing p, and $(V, \psi) \in \mathcal{A}_N$ containing $F(p)$ where $F(B_p) \subseteq V$ and

$$\psi \circ F \circ \phi^{-1} : \phi(B_p) \to \psi(V)$$

is smooth. And we note that since $B_p \subseteq U_p$ is open, where $U_p \subseteq M$ is open, gives $B_p \subseteq M$ is open, so the chart $(B_p, \phi) \in \mathcal{A}_M$. Thus, we can conclude that F is smooth.

(b) let

$$F : M \to N$$

be smooth, and let $U \subseteq M$ be an arbitrary open, nonempty, subset. Then for any $p \in U$ there is some chart $(W, \phi) \in \mathcal{A}_M$ with $p \in W$, and by the smoothness of F there exists $(V, \psi) \in \mathcal{A}_N$ such that $F(p) \in V$ and $F(W) \subseteq V$ where

$$\psi \circ F \circ \phi^{-1} : \phi(W) \to \psi(V)$$

is smooth. And by the smoothness of both ϕ and ψ, where the smooth structures on M and N tell us that $\psi^{-1} \circ \psi$ and $\phi^{-1} \circ \phi$ are both smoothly compatible, we have

$$F|_W = \psi^{-1} \circ (\psi \circ F \circ \phi^{-1}) \circ \phi : W \to V$$

is smooth as the composition of smooth maps. Moreover we have $p \in U \cap W$, and $U \cap W$ is open in the relative topology of U, and so

$$F|_{U \cap W} = \psi^{-1} \circ (\psi \circ F \circ \phi^{-1}) \circ \phi|_{U \cap W} : U \cap W \to V$$

is a smooth neighborhood of p in U and so $\big(U \cap W, \phi|_{U \cap W}\big) \in \mathcal{A}_U \subseteq \mathcal{A}_M$ and where

$$U \cap W \subseteq W \implies F(U \cap W) \subseteq F(W) \subseteq V$$

and thus, $F|_{U \cap W}$ is smooth. By the arbitrariness of $p \in U$ we conclude that for each point in U we can construct a neighborhood where F is smooth and hence F is smooth on U, or $F|_U$ is smooth.

$\square$

Corollary 16. Let M and N be smooth manifolds with or without boundary, and let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover of M. Suppose that for each $\alpha \in \Lambda$, we are given a smooth map

$$F_\alpha : U_\alpha \to N$$

such that the maps agree on overlaps:

$$F_\alpha|_{U_\alpha\cap U_\beta} = F_\beta|_{U_\alpha\cap U_\beta} \quad \forall\ \alpha, \beta$$

Then there exists a unique smooth map

$$F : M \to N$$

such that $F|_{U_\alpha} = F_\alpha,\ \forall\ \alpha \in \Lambda$.

Proposition 17. Let M and N be smooth manifolds with or without boundary, and $F : M \to N$ a smooth map. Then the coordinate representation of F with respect to every pair of smooth charts for M and N is smooth.

Proof. Let $(U, \phi) \in \mathcal{A}_M$ and $(V, \psi) \in \mathcal{A}_N$ be given. First, if

$$U \cap F^{-1}(V) = \varnothing$$

then by the definition of smooth compatibility the charts $(U \cap F^{-1}(V), \phi), (V, \psi)$ will be smoothly compatible.

So suppose $U \cap F^{-1}(V) \neq \varnothing$ and let $p \in U \cap F^{-1}(V)$. Then, by the smoothness of F, $\exists\ (W, \xi) \in \mathcal{A}_M$ containing p and $(Q, \eta) \in \mathcal{A}_N$ containing $F(p)$ with $F(W) \subseteq Q$ and

$$\eta \circ F \circ \xi^{-1} : \xi(W) \to \eta(Q)$$

smooth. Then since all charts in $\mathcal{A}_M$ and $\mathcal{A}_N$ are smoothly compatible we have

$$\xi \circ \phi^{-1} : \phi(U \cap W) \to \xi(U \cap W)$$
$$\psi \circ \eta^{-1} : \eta(V \cap Q) \to \psi(V \cap Q)$$

are both smooth. So we then get

$$\psi \circ F \circ \phi^{-1} = (\psi \circ \eta^{-1}) \circ (\eta \circ F \circ \xi^{-1}) \circ (\xi \circ \phi^{-1}) : \phi(U \cap F^{-1}(V) \cap W) \to \psi(V \cap Q)$$

is smooth as the composition of smooth maps. Where, by the arbitrariness of $p \in U \cap F^{-1}(V)$, we conclude that $\psi \circ F \circ \phi^{-1}$ is smooth on $\phi(U \cap F^{-1}(V))$. And the arbitrariness of the charts (U, ϕ) and (V, ψ) allow us to conclude that the coordinate representation of F is smooth for every pair of charts in $\mathcal{A}_M$ and $\mathcal{A}_N$. $\square$

Proposition 18. Let M, N and P be smooth manifolds with or without boundary. Then

(a) Every constant map $f_c : M \to N$ is smooth.

(b) The identity map of M is smooth.

(c) If $U \subseteq M$ is an open submanifold with or without boundary, then the inclusion map

$$\iota : U \hookrightarrow M$$

is smooth.

(d) If $F : M \to N$ and $G : M \to P$ are smooth, then so is

$$G \circ F : M \to P$$

Proof.

(a) Let $p \in M$ be given, and $(U, \phi) \in \mathcal{A}_M$ be an arbitrary chart containing p. Then for any chart $(V, \psi) \in \mathcal{A}_N$ containing $f_c(p) = c$ we have

$$U \cap f_c^{-1}(V) = U \cap M = U$$

where we then get the coordinate representation of f_c

$$\psi \circ f_c \circ \phi^{-1} : \phi(U) \to \psi(c)$$

is smooth as a constant map between euclidean spaces. And thus, we have that f_c must also be smooth.

(b) Let $p \in M$ be given, and $(U, \phi) \in \mathcal{A}_M$ be an arbitrary chart containing p, then (U, ϕ) also contains $Id(p) = p$, and

$$U \cap Id^{-1}(U) = U \cap U = U$$

where the coordinate representation of Id then gives

$$\phi \circ Id \circ \phi^{-1} : \phi(U) \to \phi(U)$$

is smooth as the identity map in a euclidean space. And thus, we have that Id must also be smooth.

(c) Let $p \in U$ be given, and $(V, \phi) \in \mathcal{A}_U$ be an arbitrary chart containing p, then (V, ϕ) is also a chart in $\mathcal{A}_M$ containing $\iota(p) = p$. Where the coordinate representation of ι then gives

$$\phi \circ \iota \circ \phi^{-1} : \phi(V) \to \phi(V)$$

is smooth as the inclusion map in a euclidean space. And thus, we have that ι must also be smooth.

(d) Let $p \in M$ be given. By the smoothness of G, $\exists\ (V, \psi) \in \mathcal{A}_N$ containing $F(p)$, and $(W, \xi) \in \mathcal{A}_P$ containing $G(F(p))$, such that $G(V) \subseteq W$ and

$$\xi \circ G \circ \psi^{-1} : \psi(V) \to \xi(W)$$

is smooth.

Next, since F is smooth, and hence continuous, $F^{-1}(V) \subseteq M$ is an open neighborhood containing p. So, $\exists\ (U, \phi) \in \mathcal{A}_M$ such that $p \in U \subseteq F^{-1}(V)$, and so $U \cap F^{-1}(V) = U$. Then by Proposition 17, and the smoothness of F we have

$$\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$$

is smooth.

Then as

$$U \subseteq F^{-1}(V) \implies F(U) \subseteq V$$

we get

$$G(F(U)) \subseteq G(V) \subseteq W$$

where the coordinate representation of $G \circ F$ gives

$$\xi \circ (G \circ F) \circ \phi^{-1} = (\xi \circ G \circ \psi^{-1}) \circ (\psi \circ F \circ \phi^{-1}) : \phi(U) \to \xi(W)$$

is smooth as the compositions of smooth maps between euclidean spaces. And thus, we have that $G \circ F$ must also be smooth.

$\square$

Proposition 19. Suppose $M_1, \ldots, M_k$ and N are smooth manifolds with or without boundary, such that at most one of $M_1, \ldots, M_k$ has nonempty boundary. For each i, let

$$\pi_i : M_1 \times \cdots \times M_k \to M_i$$

denote the projection onto the M_i factor. A map

$$F : N \to M_1 \times \cdots \times M_k$$

is smooth iff each of the component maps

$$F_i := \pi_i \circ F : N \to M_i$$

is smooth.

Proof. First suppose that F is smooth, by Proposition 18 the composition of smooth maps are smooth, so to show $\pi_i \circ F$ is smooth it suffices to show that π_i is smooth. So given $(p_1, \ldots, p_k) \in M_1 \times \cdots \times M_k$, since $M_1 \times \cdots \times M_k$ is a smooth manifold there exists

$$\big(U_1 \times \cdots \times U_k, \phi_1 \times \cdots \times \phi_k\big) \in \mathcal{A}_{M_1 \times \cdots \times M_k}$$

containing $(p_1, \ldots, p_k)$. Then by definition of the projection, we will have that chart $(U_i, p_i) \in \mathcal{A}_{M_i}$ containing $\pi_i(p_1, \ldots, p_k) = p_i$, where

$$\pi_i\big(U_1 \times \cdots \times U_k\big) = U_i \implies \pi_i\big(U_1 \times \cdots \times U_k\big) \subseteq U_i$$

and the coordinate representation of π_i, since $(U_1 \times \cdots \times U_k) \cap \pi_i^{-1}(U_i) = U_i$, gives

$$\phi_i \circ \pi_i \circ (\phi_1 \times \cdots \times \phi_k)^{-1} : \phi_i(U_i) \to \phi_i(U_i)$$

which is smooth as the identity map between euclidean spaces. And thus, we have that π_i must also be smooth. Since i and $(p_1, \ldots, p_k)$ were arbitrary we conclude that π_i and hence $\pi_i \circ F$ is smooth for each i.

Next, suppose that $\pi_i \circ F$ is smooth for each i. Given $q \in N$, since N is a smooth manifold $\exists\ (V, \psi) \in \mathcal{A}_N$ containing q, and by the smoothness of $\pi_i \circ F$ there is a chart $(U_i, \phi_i) \in \mathcal{A}_{M_i}$ containing $(\pi_i \circ F)(q)$ where $\big(\pi_i \circ F\big)(V) \subseteq U_i$ and

$$\phi_i \circ (\pi_i \circ F) \circ \psi^{-1} : \psi(V) \to \phi_i(U_i)$$

are smooth for each i. Then since $\big(U_1 \times \cdots \times U_k, \phi_1 \times \cdots \times \phi_k\big) \in \mathcal{A}_{M_1 \times \cdots \times M_k}$ is a chart containing U_i, and

$$\big(\pi_i \circ F\big)(V) \subseteq U_i \implies F(V) \subseteq \pi_i^{-1}(U_i) = U_i$$

we have $F(V) \subseteq U_1 \times \cdots \times U_k$, where the coordinate representation of F gives

$$(\phi_1 \times \cdots \times \phi_k) \circ F \circ \psi^{-1} : \psi\big(V \cap F^{-1}(U_1 \times \cdots \times U_k)\big) \to \phi_1(U_1) \times \cdots \times \phi_k(U_k)$$

now since we know the composition with the projection is smooth between euclidean spaces; that is

$$\pi_i \circ \Big((\phi_1 \times \cdots \times \phi_k) \circ F \circ \psi^{-1}\Big) : \psi\big(V \cap F^{-1}(U_1 \times \cdots \times U_k)\big) \to \phi_i(U_i)$$

is smooth for each i. And since the composition of smooth maps is smooth we also have that $\pi_i^{-1} \circ \phi_i$ is smooth for each i, where we note that we know π_i^{-1} is smooth since it is just the inclusion into $M_1 \times \cdots \times M_k$. And so

$$(\phi_1 \times \cdots \times \phi_k) \circ F \circ \psi^{-1} = \pi_i^{-1} \circ \phi_i$$

must be smooth as well, as a map between euclidean spaces. And thus, we have that F must also be smooth.

<u>Or</u>

Then noting that since π_i is smooth for each i, and by definition $\phi_1 \times \cdots \times \phi_k$ is smooth, where Proposition 18 tells us the composition of smooth maps is smooth we have $(\phi_1 \times \cdots \times \phi_k) \circ \pi_i^{-1}$ is smooth as well as $(\pi_i \circ F) \circ \psi^{-1}$. Where the coordinate representation of F then gives

$$\begin{aligned}&(\phi_1 \times \cdots \times \phi_k) \circ F \circ \psi^{-1} \\ &\ = \Big((\phi_1 \times \cdots \times \phi_k) \circ \pi_i^{-1}\Big) \circ \Big(\pi_i \circ F \circ \psi^{-1}\Big) : \psi\big(V \cap F^{-1}(U_1 \times \cdots \times U_k)\big) \to \phi_1(U_1) \times \cdots \times \phi_k(U_k)\end{aligned}$$

is smooth as the compositions of smooth maps between euclidean spaces. And thus, we have that F must also be smooth. $\square$

Proposition 20 (**Properties of Diffeomorphisms**).

(a) Every composition of diffeomorphisms is a diffeomorphism.

(b) Every finite product of diffeomorphisms between smooth manifolds is a diffeomorphism.

(c) Every diffeomorphism is a homeomorphism and an open map.

(d) The restriction of a diffeomorphism to an open submanifold with or without boundary is a diffeomorphism onto its image.

(e) "Diffeomorphic" is an equivalence relation on the class of all smooth manifolds with or without boundary.

Proof.

(a) Let

$$\begin{aligned} F &: M \to N \\ G &: N \to P \end{aligned}$$

be diffeomorphisms, then by definition they are bijective and smooth. As are F^{-1} and G^{-1}, where Proposition 18 tells us the composition of smooth maps are smooth and so we have both

$$\begin{aligned} &G \circ F : M \to P \\ &F^{-1} \circ G^{-1} = (G \circ F)^{-1} : P \to M \end{aligned}$$

are smooth, and bijective. And thus, the composition of diffeomorphisms is a diffeomorphism.

(b) Let

$$\begin{aligned} F_1 &: M_1 \to N_1 \\ F_2 &: M_2 \to N_2 \\ &\vdots \\ F_k &: M_k \to N_k \end{aligned}$$

be diffeomorphisms. By Proposition 19

$$F := F_1 \times \cdots \times F_k : M_1 \times \cdots \times M_k \to N_1 \times \cdots \times N_k$$

is smooth iff $\pi_i \circ F$ is smooth for each i. Let π_i^N denote projection on $N_1 \times \cdots \times N_k$, and similarly for π_i^M. Given $(p_1, \ldots, p_k) \in M_1 \times \cdots \times M_k$ we have

$$\big(\pi_i^N \circ F\big)(p_1, \ldots, p_k) = \pi_i^N \big(F_1(p_1), \ldots, F_k(p_k)\big) = F_i(p_i)$$

yet we also have

$$F_i(p_i) = F_i\big(\pi_i^M(p_1, \ldots, p_k)\big) = \big(F_i \circ \pi_i^M\big)(p_1, \ldots, p_k)$$

where the projection maps are all smooth, and by Proposition 18 the composition of smooth maps are smooth and so $F_i \circ \pi_i^M$ are smooth for each i, where we then have

$$\pi_i^N \circ F = F_i \circ \pi_i^M \quad \forall\ i$$

and thus $\pi_i^N \circ F$ is smooth for each i, and thus F must also be smooth. Bijectivity comes from each component function F_i being bijective.

For

$$F^{-1} := F_1^{-1} \times \cdots \times F_k^{-1} : N_1 \times \cdots \times N_k \to M_1 \times \cdots \times M_k$$

a similar argument holds except we now have

$$\pi_i^M \circ F^{-1} = F_i^{-1} \circ \pi_i^N \quad \forall\ i$$

giving the smoothness for F^{-1}. And so we conclude that

$$F = F_1 \times \cdots \times F_k$$

is a diffeomorphism.

(c) Since smooth implies continuous, any diffeomorphism is trivially a homeomorphism. And since homeomorphisms are open maps, we have diffeomorphisms must be as well.

(d) Let $F : M \to N$ be a diffeomorphism, and let $U \subseteq M$ be an open submanifold. From Proposition 18 we know that the inclusion map is smooth and since it the restriction of the identity map it is bijective onto its image. Then

$$F|_U = F \circ \iota : U \to F(U)$$

is smooth as the composition of smooth maps, and is also bijective.

Then by (c), F is an open mapping and so $F(U) \subseteq N$ is an open submanifold where a similar argument gives

$$F|_U^{-1} = F^{-1} \circ \iota : F(U) \to U$$

is the composition of smooth bijective maps. And so the restriction of F to an open submanifold with or without boundary is a diffeomorphism onto its image.

(e) Reflexive: Since $Id : M \to M$ is a smooth bijective map which is its own inverse M is diffeomorphic to M.

Symmetric: If M is diffeomorphic to N there exists a diffeomorphism

$$F : M \to N$$

then $F^{-1} : N \to M$ is also a smooth bijection, and so N is diffeomorphic to M.

Transitive: If M is diffeomorphic to N, and N is diffeomorphic to P, then there exists diffeomorphisms

$$F : M \to N$$
$$G : N \to P$$

Yet, (a) then tells us that $G \circ F : M \to P$ is also a diffeomorphism and so M is diffeomorphic to P.

□

Theorem 21 (**Diffeomorphism Invariance of Dimension**)**.** A nonempty smooth manifold of dimension m cannot be diffeomorphic to an n-dimensional smooth manifold unless $m = n$.

Proof. Suppose M is a nonempty smooth m-dimensional manifold, and N is a nonempty smooth n-dimensional manifold where

$$F : M \to N$$

is a diffeomorphism. Given $p \in M$ choose smooth charts $(U, \phi) \in \mathcal{A}_M$ containing p, and $(V, \psi) \in \mathcal{A}_N$ containing $F(p)$, then the coordinate representation of F gives

$$\psi \circ F \circ \phi^{-1} : \phi\big(U \cap F^{-1}(V)\big) \to \psi(V)$$

is a diffeomorphism between open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, and thus, we must have $m = n$. □

Theorem 22 (**Diffeomorphism Invariance of Boundary**)**.** Suppose M and N are smooth manifolds with boundary and

$$F : M \to N$$

is a diffeomorphism. Then

$$F(\partial M) = \partial N$$

and F restricts to a diffeomorphism from $\mathrm{Int}(M)$ to $\mathrm{Int}(N)$.

Proof. Let $F(p) \in F(\partial M) \subseteq N$ be arbitrary, then $p \in \partial M$, and by the smoothness of M there exists a boundary chart $(U, \phi) \in \mathcal{A}_M$ containing p, and by the smoothness of F there is a chart $(V, \psi) \in \mathcal{A}_N$ containing $F(p)$. Then by the Topological Invariance of Boundary, homeomorphisms send boundary points of the manifold to $\partial\mathbb{H}^n$. Then the coordinate representation of F gives

$$\psi \circ F \circ \phi^{-1} : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$$

since the composition of diffeomorphisms is a diffeomorphism and hence a homeomorphism. And so (V, ψ) must be a boundary chart mapping $\phi(p) \in \partial\mathbb{H}^n$ to $\psi(F(p)) \in \partial\mathbb{H}^n$, that is, we must have that $F(p) \in \partial N$. And so

$$F(\partial M) \subseteq \partial N$$

Then applying similar reasoning to the diffeomorphism F^{-1}, with $q \in F^{-1}(\partial N) \subseteq M$, gives

$$F^{-1}(\partial N) \subseteq \partial M \implies \partial N \subseteq F(\partial M)$$

and thus, $F(\partial M) = \partial N$.

Since $\mathrm{Int}(M) \subseteq M$ is an open submanifold by Proposition 11 (a), and by Proposition 20 (d) F restricts to a diffeomorphism onto its image i.e.

$$F|_{\mathrm{Int}(M)} : \mathrm{Int}(M) \to F(\mathrm{Int}(M))$$

is a diffeomorphism, yet from above we know that $F(\partial M) = \partial N$, and by the Topological Invariance of Boundary each point is either a boundary point, or an interior point, and so we conclude that $F(\mathrm{Int}(M)) = \mathrm{Int}(N)$. And thus

$$F|_{\mathrm{Int}(M)} : \mathrm{Int}(M) \to \mathrm{Int}(N)$$

is a diffeomorphism. □

Lemma 23 (**Existence of Cutoff Functions**)**.** Given any real numbers $r_1, r_2 \in \mathbb{R}$ such that $r_1 < r_2$, there exists a smooth function

$$h : \mathbb{R} \to \mathbb{R}, \text{ such that } h(t) = \begin{cases} 1, & t \leq r_1 \\ 0 < h(t) < 1, & r_1 < t < r_2 \\ 0, & t \geq r_2 \end{cases}$$

Proof. Consider the function

$$f : \mathbb{R} \to \mathbb{R}, \text{ by } f(t) = \begin{cases} e^{-1/t}, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

Which is smooth, then define h by

$$h(t) = \frac{f(r_2 - t)}{f(r_2 - t) + f(t - r_2)}$$

note that the denominator is always defined since either $r_2 - t > 0$ or $t - r_1 > 0$. Where the smoothness is inherited from f, and for

$t \geq r_2$

$$f(r_2 - t) = 0 \implies h(t) = 0$$

$t \leq r_1$

$$f(t - r_1) = 0 \implies h(t) = \frac{f(r_2 - t)}{f(r_2 - t)} = 1$$

$r_1 < t < r_2$

$$f(r_2 - t) > 0 \text{ and } f(t - r_1) > 0 \implies 0 < h(t) < 1$$

$\square$

Lemma 24 (**Existence of Smooth Bump Functions**)**.** Given any positive real numbers $r_1, r_2 \in \mathbb{R}^+$ such that $r_1 < r_2$, there is a smooth function

$$H : \mathbb{R}^n \to \mathbb{R}, \text{ such that } H(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in \overline{B_{r_1}(\mathbf{0})} \\ 0 < H(\mathbf{x}) < 1, & \mathbf{x} \in B_{r_2}(\mathbf{0}) \setminus \overline{B_{r_1}(\mathbf{0})} \\ 0, & \mathbf{x} \in \mathbb{R}^n \setminus B_{r_2}(\mathbf{0}) \end{cases}$$

Proof. Utilizing the smooth cutoff function

$$h : \mathbb{R} \to \mathbb{R}, \text{ such that } h(t) = \begin{cases} 1, & t \leq r_1 \\ 0 < h(t) < 1, & r_1 < t < r_2 \\ 0, & t \geq r_2 \end{cases}$$

we can define H by

$$\begin{aligned} H := h \circ ||\cdot|| &: \mathbb{R}^n \to \mathbb{R} \\ \mathbf{x} &\mapsto h(||\mathbf{x}||) \end{aligned}$$

Then H is smooth on $\mathbb{R}^n \setminus \{\mathbf{0}\}$ as the composition of smooth functions.

Now for any $\mathbf{x} \in \overline{B_{r_1}(\mathbf{0})}$
$$\mathbf{x} \mapsto t \leq r_1 \implies h(||\mathbf{x}||) = 1$$
and so we must also have for $\mathbf{0} \mapsto 0 < r_1$ that $h(||\mathbf{0}||) = 1$.

And so H is smooth on all of $\mathbb{R}^n$. $\square$

Theorem 25 (**Existence of Partitions of Unity**). Suppose M is a smooth manifold with or without boundary, and $\mathcal{U} := \{U_\alpha\}_{\alpha \in \Lambda}$ is any indexed open cover of M. Then there exists a smooth partition of unity subordinate to $\mathcal{U}$.

Proof. Suppose first that M is a smooth manifold without boundary. Each $U_\alpha \subseteq M$ is itself a smooth manifold, and thus, by Proposition 8 has a countable base of regular coordinate balls $\mathcal{B}_\alpha$. Taking
$$\mathcal{B} = \bigcup_{\alpha \in \Lambda} \mathcal{B}_\alpha$$
we have that $\mathcal{B}$ is a base for M. From Theorem 5 manifolds are paracompact, and since we have both a cover $\mathcal{U}$, and a base $\mathcal{B}$, there exists a countable locally finite refinement $\{B_i\}$ of $\mathcal{U}$ consisting of elements from $\mathcal{B}$. From Lemma 4, the closure of sets in a locally finite collection is locally finite and so $\left\{\overline{B}_i\right\}$ is also locally finite.

Now, since each B_i is a regular coordinate ball in some U_α, $\exists\ (B_i', \phi_i) \in \mathcal{A}_{U_\alpha}$ with $B_i' \supseteq \overline{B}_i$ such that for $r_i, r_i' \in \mathbb{R}^+$ with $r_i < r_i'$ we have:

$$\phi_i(B_i) = B_{r_i}(\mathbf{0})$$
$$\phi_i(\overline{B}_i) = \overline{B_{r_i}(\mathbf{0})}$$
$$\phi_i(B_i') = B_{r_i'}(\mathbf{0})$$

For each i define the smooth bump function

$$H_i : \mathbb{R}^n \to \mathbb{R}, \text{ such that } H_i(\mathbf{x}) = \begin{cases} 0 < H_i(\mathbf{x}) < 1, & \mathbf{x} \in B_{r_i}(\mathbf{0}) \\ 0, & \mathbf{x} \in \mathbb{R}^n \setminus B_{r_i}(\mathbf{0}) \end{cases}$$

then defining, for each i, the function

$$f_i : M \to \mathbb{R}, \text{ by } f_i(p) = \begin{cases} H_i \circ \phi_i, & p \in B_i' \\ 0, & p \in M \setminus \overline{B}_i \end{cases}$$

where for $p \in B_i' \setminus \overline{B}_i$, continuity of ϕ tells us that

$$\phi(p) \in B_{r_i'}(\mathbf{0}) \setminus \overline{B_{r_i}(\mathbf{0})} \subset \mathbb{R}^n \setminus B_{r_i}(\mathbf{0})$$

and so we have $f_i(p) = 0 = H_i(\phi_i(p))$. Thus, f_i and the smooth function $H_i \circ \phi_i$ agree on the overlap, and so f_i is well defined, smooth, and $\operatorname{supp}(f) \subseteq \overline{B}_i$.

Next, define

$$f : M \to \mathbb{R}, \text{ by } f(p) = \sum_i f_i(p)$$

where, by the local finiteness of $\{\overline{B}_i\}$, for each $p \in M$, $\exists\, U_p$ such that $U_p \cap \overline{B}_i = \varnothing$ for all but finitely many i and so $f_i(p) = 0$ for all but finitely many i, and so f is smooth.

Then as $f_i \geq 0$ on M, and $f_i > 0$ on B_i, where each $p \in M$ belongs to some B_i we have $f > 0$ on M, and so the functions

$$g_i : M \to \mathbb{R}, \text{ by } g_i(p) = \frac{f_i(p)}{f(p)}$$

are all well defined and smooth. Moreover

$$0 \leq g_i(p) \leq 1 \qquad \text{and} \qquad \sum_i g_i(p) = 1$$

Finally, reindexing so that our functions are indexed by the same Λ as our open cover $\{B_i\}$. Since $\{B_i'\}$ is a refinement of $\mathcal{U}$ we may choose $\sigma(i) \in \Lambda$ for each i such that $B_i' \subseteq U_{\sigma(i)}$. Then for each $\alpha \in \Lambda$ define

$$\psi_\alpha : M \to \mathbb{R}, \text{ by } \psi_\alpha(p) = \sum_{\{i:\sigma(i)=\alpha\}} g_i(p)$$

if $\sigma(i) \neq \alpha\ \forall\ i$ then $\psi_\alpha := 0$. Then by the local finiteness of $\{B_i\}$ Lemma 4 tells us that

$$\operatorname{supp}(\psi_\alpha) = \overline{\bigcup_{\{i:\sigma(i)=\alpha\}} B_i} = \bigcup_{\{i:\sigma(i)=\alpha\}} \overline{B}_i \subseteq B_i' \subseteq U_\alpha$$

Then each ψ_α is a smooth function such that $0 \leq \psi_\alpha(p) \leq 1$, where the family of supports $\{\operatorname{supp}(\psi_\alpha)\}_{\alpha\in\Lambda}$ is locally finite and

$$\sum_\alpha \psi_\alpha(p) = \sum_i g_i(p) = 1 \quad \forall\ p \in M$$

and so $\{\psi_\alpha\}_{\alpha\in\Lambda}$ is a smooth partition of unity subordinate to $\mathcal{U}$. □

Proposition 26 (**Existence of Smooth Bump Functions on Manifolds**)**.** Let M be a smooth manifold with or without boundary. For any closed subset $A \subseteq M$ and any open subset U such that $U \supseteq A$, there exists a smooth bump function ψ for A supported in U.

Proof. Let

$$U_0 = U$$
$$U_1 = M \setminus A = A^c$$

then $\mathcal{U} = \{U_0, U_1\}$ is an indexed open cover of M, and so there exists a smooth partition of unity $\{\psi_0, \psi_1\}$ subordinate to $\mathcal{U}$. Where

$$\operatorname{supp}(\psi_1) \subseteq U_1 \implies \psi_1(p) = 0 \quad \forall\ p \in A$$

and since $\{\psi_0, \psi_1\}$ is a smooth partition of unity this implies

$$1 = \sum_{i=0}^{1} \psi_i(p) = \psi_0(p) + \psi_1(p) = \psi_0(p) + 0 = \psi_0(p) \quad \forall\ p \in A$$

and so

$$\begin{aligned} &0 \le \psi_0(M) \le 1 \\ &\psi_0(p) = 1, \qquad \forall\ p \in A \\ &\operatorname{supp}(\psi_0) \subseteq U_0 = U \end{aligned}$$

and so ψ_0 is therefore a bump function for A supported in U. $\square$

Lemma 27 (**Extension Lemma for Smooth Functions**)**.** Suppose M is a smooth manifold with or without boundary, $A \subseteq M$ is a closed subset, and

$$f : A \to \mathbb{R}^k$$

is a smooth function. For any open subset $U \subseteq M$ such that $U \supseteq A$, there exists a smooth function

$$\widetilde{f} : M \to \mathbb{R}^k$$

such that $\widetilde{f}|_A = f$ and $\operatorname{supp}(\widetilde{f}) \subseteq U$.

Proof. For each $p \in A$ choose $U_p \subseteq M$ containing p and a smooth function

$$\widetilde{f}_p : U_p \to \mathbb{R}^k$$

such that $\widetilde{f}_p|_{U_p \cap A} = f|_{U_p \cap A}$. Then replacing U_p by

$$U_p = U_p \cap U \implies U_p \subseteq U$$

then by the closedness of A we have $M \setminus A = A^c$ must be open, and thus

$$\mathcal{U} := \{U_p : p \in A\} \cup \{A^c\}$$

is an indexed open cover for M, here considering the p's in A to be our index, and so there exists a smooth partition of unity $\{\psi_p : p \in A\} \cup \{\psi_0\}$ subordinate to $\mathcal{U}$, such that

$$\operatorname{supp}(\psi_p) \subseteq U_p, \quad \text{and} \quad \operatorname{supp}(\psi_0) \subseteq A^c$$

then for each $p \in A$ we have

$$\psi_p \widetilde{f}_p : U_p \to \mathbb{R}^k$$

is smooth and has a smooth extention to M by the gluing lemma for smooth maps since

$$\psi_p \widetilde{f}_p\big(U_p \setminus \operatorname{supp}(\psi_p)\big) = 0 = \psi_p \widetilde{f}_p\big(M \setminus \operatorname{supp}(\psi_p)\big)$$

i.e. functions agree on their overlap. And so we define

$$\widetilde{f} : M \to \mathbb{R}^k, \text{ by } \widetilde{f}(q) = \sum_{p \in A} \psi_p(q) \widetilde{f}_p(q)$$

now since $\{\psi_p : p \in A\} \cup \{\psi_0\}$ is a partition of unity, we have $\{\text{supp}(\psi_p)\}$ is locally finite, and so only finitely many of the terms in the sum are non-zero in the neighborhood of any point of M, and therefore $\widetilde{f}$ is smooth.

To see that $\widetilde{f}$ is an extension of f, note that for each $q \in A$ we have $\psi_0(q) = 0$, and $\widetilde{f}_p(q) = f(q)$ for each $q \in U_p \cap A$. Then since $\{\psi_p : p \in A\} \cup \{\psi_0\}$ is a partition of unity, we have

$$\psi_0(q) + \sum_{p \in A} \psi_p(q) = 1 \qquad \forall\ q \in M$$

putting it all together we get for each $q \in A$

$$\begin{aligned}
\widetilde{f}(q) &= \sum_{p \in A} \psi_p(q) \widetilde{f}_p(q) \\
&= 0 + \sum_{p \in A} \psi_p(q) \widetilde{f}_p(q) \\
&= \left(\psi_0(q) + \sum_{p \in A} \psi_p(q) \right) f(q) \\
&= f(q)
\end{aligned}$$

and therefore $\widetilde{f}|_A = f$.

Finally, since $\{\text{supp}(\psi_p)\}$ is a locally finite collection of subsets Lemma 4 (b) tells us that

$$\text{supp}(\widetilde{f}) \subseteq \overline{\bigcup_{p \in A} \text{supp}(\psi_p)} = \bigcup_{p \in A} \overline{\text{supp}(\psi_p)} = \bigcup_{p \in A} \text{supp}(\psi_p) \subseteq \bigcup_{p \in A} U_p \subseteq U$$

$\square$

Proposition 28 (**Existence of Smooth Exhaustion Functions**)**.** Every smooth manifold with or without boundary admits a smooth positive exhaustion function.

Proof. Let M be a smooth manifold with or without boundary, and let $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$ be an arbitrary countable open cover of M be precompact subsets, as it is indexed, let $\{\psi_i\}$ be the smooth partition of unity subordinate to $\mathcal{U}$. Next, define

$$f : M \to \mathbb{R}, \text{ by } f(p) = \sum_{i=1}^{\infty} i\psi_i(p)$$

then f is smooth since $\{\text{supp}(\psi_i)\}$ is a locally finite collection of subsets, and so only finitely many of the terms in the sum are non-zero in the neighborhood of any point of M.

and since $\{\psi_i\}$ is a partition of unity $\sum_{i=1}^{\infty} \psi_i(p) = 1$ for all $p \in M$ and so

$$f(p) \sum_{i=1}^{\infty} i\psi_i(p) \geq \sum_{i=1}^{\infty} \psi_i(p) = 1$$

hence f is positive.

To see that f is an exhaustion function, note that since the U_i's are precompact each $\overline{U}_i$ is compact in M. Then, given $c \in \mathbb{R}$ we can choose $N \in \mathbb{N}$ such that $N > c$. Then if

$$p \notin \bigcup_{i=1}^{N} \overline{U}_i \implies \psi_i(p) = 0 \text{ for } 1 \leq i \leq N$$

since $\operatorname{supp}(\psi_i) \subseteq U_i$. And so

$$f(p) = \sum_{i=N+1}^{\infty} i\psi_i(p) \geq \sum_{i=N+1}^{\infty} N\psi_i(p) = N \sum_{i=1}^{\infty} \psi_i(p) = N > c$$

and so $p \notin f^{-1}\big((-\infty, c]\big)$. That is,

$$\text{if } p \notin \bigcup_{i=1}^{N} \overline{U}_i \text{ then } p \notin f^{-1}\big((-\infty, c]\big)$$

taking the contrapositive gives

$$\text{if } p \in f^{-1}\big((-\infty, c]\big) \text{ then } p \in \bigcup_{i=1}^{N} \overline{U}_i$$

or $f^{-1}\big((-\infty, c]\big) \subseteq \bigcup_{i=1}^{N} \overline{U}_i$, which is compact as a finite union of compact sets, were the continuity of f tells us that $f^{-1}\big((-\infty, c]\big) \subseteq M$ is closed, and therefore must be compact as a closed subset of a compact set. □

Theorem 29 (**Level Sets of Smooth Functions**)**.** Let M be a smooth manifold. If $A \subseteq M$ is closed, there is a smooth non-negative function

$$f : M \to \mathbb{R}$$

such that $f^{-1}(0) = A$.

Lemma 30 (**Properties of Derivations**)**.** Suppose $\mathbf{a} \in \mathbb{R}^n, w \in T_{\mathbf{a}}\mathbb{R}^n$ and $f, g \in C^\infty(\mathbb{R}^n)$, then

(a) If $f = cnst$ then

$$w(f)|_{\mathbf{a}} = 0$$

(b) If $f(\mathbf{a}) = g(\mathbf{a}) = 0$ then

$$w(fg)|_{\mathbf{a}} = 0$$

Proof.

(a) First note that for $f_1(\mathbf{x}) = 1$ i.e. the constant function equal to 1, and that for any other constant function $f(\mathbf{x}) = c$ we have

$$\begin{aligned} w(f)|_{\mathbf{a}} &= w(cf_1)|_{\mathbf{a}} \\ &= cw(f_1)|_{\mathbf{a}} && \text{linearity} \end{aligned}$$

so it suffices to show the property for the the case $f = f_1$. Then since

$$f_1(\mathbf{x})^2 = 1^2 = 1 = f_1(\mathbf{x})$$

the product rule gives

$$\begin{aligned} w(f_1)|_{\mathbf{a}} &= w(f_1 \cdot f_1)|_{\mathbf{a}} \\ &= f_1(\mathbf{a})w(f_1)|_{\mathbf{a}} + f_1(\mathbf{a})w(f_1)|_{\mathbf{a}} \\ &= 1 \cdot w(f_1)|_{\mathbf{a}} + 1 \cdot w(f_1)|_{\mathbf{a}} \\ &= 2w(f_1)|_{\mathbf{a}} \end{aligned}$$

which is only possible if $w(f_1)|_{\mathbf{a}} = 0$.

(b) From the product rule we have

$$w(fg)|_{\mathbf{a}} = f(\mathbf{a})w(g)|_{\mathbf{a}} + g(\mathbf{a})w(f)|_{\mathbf{a}} = 0 + 0 = 0$$

□

Proposition 31. let $\mathbf{a} \in \mathbb{R}^n$, then

(a) For each geometric tangent vector $\mathbf{v_a} \in \mathbb{R}^n_{\mathbf{a}}$ the map

$$D_{\mathbf{v}}\big|_{\mathbf{a}} : C^\infty(\mathbb{R}^n) \to \mathbb{R}, \text{ by } D_{\mathbf{v}}(f)|_{\mathbf{a}} = \left.\frac{d}{dt}\right|_{t=0} f(\mathbf{a} + t\mathbf{v})$$

is a derivation at $\mathbf{a}$.

(b) The mapping

$$\begin{aligned} \eta : \mathbb{R}^n_{\mathbf{a}} &\to T_{\mathbf{a}}\mathbb{R}^n \\ \mathbf{v_a} &\mapsto D_{\mathbf{v}}\big|_{\mathbf{a}} \end{aligned}$$

is an isomorphism.

Proof.

(a) Since $D_{\mathbf{v}}\big|_{\mathbf{a}}$ is defined as the directional derivative at $\mathbf{a}$ in the direction of $\mathbf{v}$ which is defined to be linear over $\mathbb{R}$ and satisfy the product rule we get that $D_{\mathbf{v}}\big|_{\mathbf{a}}$ is a derivation.

(b) First note the mapping $\mathbf{v_a} \mapsto D_{\mathbf{v}}\big|_{\mathbf{a}}$ is linear since $D_{\mathbf{v}}\big|_{\mathbf{a}}$ is.

Next, suppose that $\mathbf{v_a} \in \ker(\eta)$; that is

$$D_{\mathbf{v}}(f)|_{\mathbf{a}} = 0 \quad \forall\ f \in C^\infty(\mathbb{R}^n)$$

so letting $f = x^j \in C^\infty(\mathbb{R}^n)$, and letting

$$\mathbf{v_a} = \sum_{i=1}^n v^i e_i|_{\mathbf{a}}$$

in terms of the standard basis, we then have

$$\begin{aligned} 0 &= D_{\mathbf{v}}(x^j)|_{\mathbf{a}} \\ &= \sum_{i=1}^n v^i \frac{\partial x^j}{\partial x^i}\bigg|_{\mathbf{a}} \\ &= \sum_{i=1}^n v^i \delta^j{}_i \\ &= v^j \end{aligned}$$

and since this can be done for each component of $\mathbf{v_a}$ for $j \in \{1, \ldots, , n\}$, we conclude that $\mathbf{v_a} = \mathbf{0}$, and therefore η is injective.

Now, let $w_{\mathbf{a}} \in T_{\mathbf{a}}\mathbb{R}^n$ be arbitrary, $\mathbf{v_a} = \sum_{i=1}^n v^i e_i|_{\mathbf{a}}$ so that

$v^i = w(x^i)$doesn't defining how w evaluates x^i no longer make it arbitrary?

and let $f \in C^\infty(\mathbb{R}^n)$. Then by Taylor's Theorem

$$f(\mathbf{x}) = f(\mathbf{a}) + \sum_{i=1}^n (x_i - a_i)\frac{\partial f}{\partial x^i}(\mathbf{a}) + \frac{1}{2}\sum_{i,j=1}^n (x_i - a_i)(x_j - a_j)\int_0^1 (1-t)\frac{\partial f}{\partial x^i \partial x^j}\big(\mathbf{a} + t(\mathbf{x} - \mathbf{a})\big)dt$$

Then as

$$(x_i - a_i)|_{\mathbf{a}} = 0 = (x_j - a_j)|_{\mathbf{a}}$$

where Lemma 30 then says

$$w\Big((x_i - a_i) \cdot (x_j - a_j)\Big)\Big|_{\mathbf{a}} = 0$$

then since both $f(\mathbf{a})$ and the components a_i are both constants Lemma 30 also tells us that

$$w\big(f(\mathbf{a})\big) = 0, \quad \text{and} \quad w(a_i) = 0\ \forall\ i$$

and so we have

$$\begin{aligned}
w(f)|_{\mathbf{a}} &= w\big(f(\mathbf{a})\big) + \sum_{i=1}^{n} w\left((x_i - a_i)\frac{\partial f}{\partial x^i}(\mathbf{a})\right) + 0 \\
&= 0 + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(\mathbf{a})\big(w(x_i) - w(a_i)\big) && \text{linearity} \\
&= \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(\mathbf{a})\big(v^i - 0\big) && x_i = x^i \in C^\infty(\mathbb{R}^n) \\
&= \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(\mathbf{a}) \cdot v^i \\
&= \sum_{i=1}^{n} v^i \frac{\partial f}{\partial x^i}(\mathbf{a}) \\
&= D_{\mathbf{v}}(f)|_{\mathbf{a}}
\end{aligned}$$

that is

$$\eta(\mathbf{v_a}) = D_{\mathbf{v}}\big|_{\mathbf{a}} = w$$

and so is surjective.

So we have a linear map between vector spaces that is both injective and surjective. Thus, η is an isomorphism between $\mathbb{R}^n_{\mathbf{a}}$ and $T_{\mathbf{a}}\mathbb{R}^n$.

$\square$

Corollary 32. For any $\mathbf{a} \in \mathbb{R}^n$, the n derivations

$$\frac{\partial}{\partial x^1}\bigg|_{\mathbf{a}}, \dots, \frac{\partial}{\partial x^n}\bigg|_{\mathbf{a}} : C^\infty(\mathbb{R}^n)|_{\mathbf{a}} \to \mathbb{R}, \text{ by } \frac{\partial}{\partial x^i}\bigg|_{\mathbf{a}}(f) = \frac{\partial f}{\partial x^i}(\mathbf{a})$$

form a basis for $T_{\mathbf{a}}\mathbb{R}^n$, and therefore $\dim\big(T_{\mathbf{a}}\mathbb{R}^n\big) = n$.

Proof. From Proposition 31 $T_{\mathbf{a}}\mathbb{R}^n \cong \mathbb{R}^n_{\mathbf{a}}$ and $\mathbb{R}^n_{\mathbf{a}} \cong \mathbb{R}^n$ and so

$$n = \dim(\mathbb{R}^n) = \dim(\mathbb{R}^n_{\mathbf{a}}) = \dim(T_{\mathbf{a}}\mathbb{R}^n)$$

and any isomorphism must map basis vectors to basis vectors

$$e_i \mapsto e_i - a_i \mapsto D_{e_i}|_{\mathbf{a}} = e_i \cdot \frac{\partial}{\partial x^i}\bigg|_{\mathbf{a}} = \frac{\partial}{\partial x^i}\bigg|_{\mathbf{a}}$$

$\square$

Lemma 33 (Properties of Tangent Vectors on Manifolds). Suppose M is a smooth manifold with or without boundary, $p \in M$; $v_p \in T_pM$; and $f, g \in C^\infty(M)$. Then

(a) If $f = cnst$, then $v(f)|_p = 0$

(b) If $f(p) = g(p) = 0$, then $v(fg)|_p = 0$

Proof.

(a) It is sufficient to consider the constant function $f_1(p) = 1$ then we have

$$v(f_1)|_p = v(f_1 f_1)|_p = f_1(p)v(f_1)|_p + f_1(p)v(f_1)|_p = 2v(f_1)|_p$$

and so we must have $v(f_1)|_p = 0$. The reason $f_1(p)$ is sufficient is because for any other constant function $g(p) = c$ we may simply define $g(p) = (cf_1(p))$ where by linearity we get

$$v(g)|_p = v(cf_1)|_p = cv(f_1)_p = 0$$

(b) From the product rule we have

$$v(fg)|_p = f(p)v(g)|_p + g(p)v(f)|_p = 0 + 0 = 0$$

□

Proposition 34 (**Properties of Differentials**)**.** Let M, N, and P be smooth manifolds with or without boundary, let $F : M \to N$ and $G : N \to P$ be smooth maps, and let $p \in M$. Then

(a) $dF|_p : T_pM \to T_{F(p)}N$ is linear

(b) $d(G \circ F)|_p = dG|_{F(p)} \circ dF|_p : T_pM \to T_{G\circ F(p)}P$

(c) $d(Id_M)|_p = Id_{T_pM} : T_pM \to T_pM$

(d) If F is a diffeomorphism, then

$$dF|_p : T_pM \to T_{F(p)}N$$

is an isomorphism, and

$$(dF|_p)^{-1} = d(F^{-1})|_{F(p)}$$

Proof.

(a) Let $v_1, v_2 \in T_pM$ and $f \in C^\infty(N)$ then by the linearity of T_pM we have $(v_1 + v_2)f = v_1 f + v_2 f$ so that we get

$$\begin{aligned} dF|_p(v_1 + v_2)f &= [v_1 + v_2](f \circ F)|_p \\ &= v_1(f \circ F)|_p + v_2(f \circ F)|_p \\ &= dF|_p(v_1)f + dF|_p(v_2)f \end{aligned}$$

and similary since $(cv)f = cv(f)$ we get

$$dF|_p(cv)f = cv(f \circ F)|_p = cdF|_p(v)f$$

(b) Let $v \in T_pM$ then $d(G\circ F)|_p(v) : C^\infty(P) \to \mathbb{R}$ is linear because v is, so let $f, g \in C^\infty(P)|_{G(F(p))}$ Then

$$\begin{aligned} d(G\circ F)|_p(v)fg &= v(fg\circ G\circ F)|_p \\ &= v\big((f\circ G\circ F)\cdot(g\circ G\circ F)\big)|_p \\ &= f\circ G\circ F(p)v(g\circ G\circ F)|_p + g\circ G\circ F(p)v(f\circ G\circ F)|_p \\ &= f\circ G(F(p))v(g\circ G\circ F)|_p + g\circ G(F(p))v(f\circ G\circ F)|_p \\ &= f\Big(G\big(F(p)\big)\Big)v(g\circ G\circ F)|_p + g\Big(G\big(F(p)\big)\Big)v(f\circ G\circ F)|_p \\ &= f\Big(G\big(F(p)\big)\Big)d(G\circ F)|_p(v)(g) + g\Big(G\big(F(p)\big)\Big)d(G\circ F)|_p(v)(f) \end{aligned}$$

and so $d(G\circ F)|_p(v)$ is a derivation at $G(F(p))$, and therefore $d(G\circ F)|_p(v) \in T_{G(F(p))}P$. Since $v \in T_pM$ was arbitrary we conclude

$$d(G\circ F)|_p : T_pM \to T_{G\circ F(p)}P$$

Also,

$$d(G\circ F)|_p(v)f = v(f\circ G\circ F)|_p = dF|_p(v)(f\circ G)|_{F(p)} = dG|_{F(p)}(dF|_p(v))f$$

And hence $d(G\circ F)|_p = dG|_{F(p)} \circ dF|_p$

(c) Let $v \in T_pM$ and $f \in C^\infty(M)$ and since $Id_M : M \to M$ we have

$$d(Id_M)|_p(v)f = v(f\circ Id_M)|_p = v(f)|_p$$

and $d(Id_M)|_p : T_pM \to T_pM$. Also

$$d(Id_M)|_p(v) = v \qquad \forall\ v \in T_pM$$

and so $d(Id_M)|_p = Id_{T_pM}$

(d) First $v|_p \mapsto v|_{F(p)}$ is linear by (a). Next since F is a diffeomorphism it is bijective and it's inverse exists. And from (b) and (c) we have

$$d(F^{-1})|_{F(p)} \circ dF|_p = d(F^{-1}\circ F)|_p = d(Id_M)|_p = Id_{T_pM}$$

and

$$dF|_p \circ d(F^{-1})|_{F(p)} = d(F\circ F^{-1})|_{F(p)} = d(Id_N)|_{F(p)} = Id_{T_{F(p)}N}$$

and so $dF|_p : T_pM \to T_{F(p)}N$ is an isomorphism, with

$$(dF|_p)^{-1} = d(F^{-1})|_{F(p)}$$

□

Proposition 35. Let M be a smooth manifold with or without boundary, $p \in M$; and $v_p \in T_pM$. If $f, g \in C^\infty(M)$ agree on some neighborhood U_p of p; i.e.

$$f|_{U_p} = g|_{U_p}$$

then

$$v(f)|_p = v(g)|_p$$

Proof. Define

$$h = f - g \implies h|_{U_p} = 0$$

and is smooth where $\operatorname{supp}(h) \subseteq M$ is closed, and $V = M \setminus \{p\}$ is an open subset such that $V \supseteq \operatorname{supp}(h)$. And so we may define a bump function ψ for $\operatorname{supp}(h)$ supported in V, such that

$$\begin{aligned}
&0 \leq \psi(M) \leq 1 \\
&\psi(q) = 1 \qquad \forall\, q \in \operatorname{supp}(h) \\
&\operatorname{supp}(\psi) \subseteq V
\end{aligned}$$

so if $h(q) \neq 0$, then

$$\big(\psi h\big)(q) = 1 \cdot h(q) = h(q)$$

and if $h(q) = 0$ then

$$\big(\psi h\big)(q) = 0$$

and therefore $\psi h = h$ identically. Then since

$$h(p) = 0 = \psi(p)$$

Lemma 33 says

$$v(\psi h)|_p = 0$$

where we then get

$$\begin{aligned}
0 &= v(\psi h)|_p \\
&= v(h)|_p && \psi h = h \text{ identically} \\
&= v(f - g)|_p \\
&= v(f)|_p - v(g)|_p && \text{linearity}
\end{aligned}$$

which then implies

$$v(f)|_p = v(g)|_p$$

□

Proposition 36 (The Tangent Space to an Open Submanifold). Let M be a smooth manifold with or without boundary, let $U \subseteq M$ be an open subset, and let

$$\iota : U \hookrightarrow M$$

be the inclusion map. For every $p \in U$, the differential

$$d\iota|_p : T_pU \to T_pM$$

is an isomorphism.

Proof. Suppose $v_p \in T_pU$ is such that $v_p \in \ker(d\iota|_p)$; that is

$$d\iota|_p(v_p)(f) = v(f \circ \iota)|_p = 0 \quad \forall\, f \in C^\infty(M)$$

Next, let B_p be a neighborhood of p such that $\overline{B}_p \subseteq U$, and let $f \in C^\infty(U)$ be arbitrary. Then since $\overline{B}_p \subseteq M$ is closed and contained in U, the Extension Lemma for Smooth Functions then says

$\exists\ \widetilde{f} \in C^\infty(M)$ such that $\widetilde{f}|_{\overline{B}_p} = f|_{\overline{B}_p}$ and $\text{supp}(\widetilde{f}) \subseteq U$.

Next we note that $f, \widetilde{f}|_U \in C^\infty(U)$, and agree on the open neighborhood B_p; that is

$$\widetilde{f}|_U\big|_{B_p} = f|_{B_p}$$

where Proposition 35 then says that

$$v(f)|_p = v\big(\widetilde{f}|_U\big)\big|_p$$

yet

$$\widetilde{f}|_U = \widetilde{f} \circ \iota$$

and so we have

$$v(f)|_p = v\big(\widetilde{f}|_U\big)\big|_p = v(\widetilde{f} \circ \iota)|_p = d\iota|_p(v_p)(\widetilde{f}) = 0 \qquad \widetilde{f} \in C^\infty(M)$$

and since this holds for any $f \in C^\infty(U)$ we conclude that $v_p = 0$ and so $d\iota|_p$ is injective.

Now, let $w_p \in T_pM$ be given, and choose $v_p \in T_pU$ such that for $f \in C^\infty(U)$ and $\widetilde{f} \in C^\infty(M)$ with $\overline{B}_p \subseteq U$ where

$$\widetilde{f}|_{\overline{B}_p} = f|_{\overline{B}_p}$$

we have

$$v(f)|_p = w(\widetilde{f})|_p$$

by Proposition 35 $v(f)$ is independent of the choice $\widetilde{f}$, and so is well defined, and is a derivation since for any $f, g \in C^\infty(U)$ we have

$$\begin{aligned} v(fg)|_p &= w(\widetilde{f} \cdot \widetilde{g})|_p \\ &= \widetilde{f}(p)w(\widetilde{g})|_p + \widetilde{g}(p)w(\widetilde{f})|_p \\ &= f(p)v(g)|_p + g(p)v(f)|_p \qquad\qquad \widetilde{f}|_{\overline{B}_p} = f|_{\overline{B}_p},\ \text{and}\ \widetilde{g}|_{\overline{B}_p} = g|_{\overline{B}_p} \end{aligned}$$

So let $g \in C^\infty(M)$ be given, then

$$d\iota|_p(v_p)(g) = v(g \circ \iota)|_p = w(\widetilde{g \circ \iota})|_p = w(g)|_p \qquad \text{since } (g \circ \iota)|_{B_p} = (\widetilde{g \circ \iota})|_{B_p} = g|_{B_p}$$

and by the arbitrariness of g we see that

$$d\iota|_p(v_p) = w_p$$

and so $d\iota|_p$ is surjective.

Hence, we have a linear map between vector spaces that is both injective and surjective. Thus, $d\iota|_p$ is an isomorphism between T_pU and T_pM. □

Proposition 37 (**Dimension of the Tangent Space**)**.** If M is an n-dimensional smooth manifold, then for each $p \in M$; the tangent space T_pM is an n-dimensional vector space.

Proof. Let $p \in M$ be given, and let $(U, \phi) \in \mathcal{A}_M$ be a smooth chart containing p. Then since

$$\phi : U \to \phi(U)$$

is a diffeomorphism Proposition 34 then tells us that

$$d\phi|_p : T_pU \to T_{\phi(p)}\phi(U)$$

is an isomorphism. Since derivations are defined locally Proposition 36 tells us that both

$$T_pM \cong T_pU$$
$$T_{\phi(p)}\phi(U) \cong T_{\phi(p)}\mathbb{R}^n$$

and from Proposition 31 the set of derivations at a point in euclidean space is isomorphic to the euclidean space with shifted origin; i.e. the space of geometric tangent vectors is isomorphic to the space of tangent vectors in euclidean space, we have

$$T_{\phi(p)}\mathbb{R}^n \cong \mathbb{R}^n_{\phi(p)}$$

and canonically $\mathbb{R}^n_{\phi(p)} \cong \mathbb{R}^n$. Putting it all together we get

$$T_pM \cong T_pU \cong T_{\phi(p)}\phi(U) \cong T_{\phi(p)}\mathbb{R}^n \cong \mathbb{R}^n_{\phi(p)} \cong \mathbb{R}^n$$

and since isomorphism are structure preserving we must have

$$\dim(\mathbb{R}^n) = n = \dim(T_pM)$$

□

Lemma 38. Let

$$\iota : \mathbb{H}^n \hookrightarrow \mathbb{R}^n$$

denote the inclusion map. For any $\mathbf{a} \in \partial\mathbb{H}^n$, the differential

$$d\iota|_{\mathbf{a}} : T_{\mathbf{a}}\mathbb{H}^n \to T_{\mathbf{a}}\mathbb{R}^n$$

is an isomorphism.

Proof. Let $\mathbf{a} \in \partial\mathbb{H}^n$ be given, and suppose that $v_{\mathbf{a}} \in T_{\mathbf{a}}\mathbb{H}^n$ is such that $v_{\mathbf{a}} \in \ker(d\iota|_{\mathbf{a}})$; that is

$$d\iota|_{\mathbf{a}}(v_{\mathbf{a}})(f) = v(f \circ \iota)|_{\mathbf{a}} = 0 \quad \forall\ f \in C^\infty(\mathbb{R}^n)$$

and let

$$f : \mathbb{H}^n \to \mathbb{R}^n$$

be smooth, and since $\mathbb{H}^n \subseteq \mathbb{R}^n$ is closed, and we may consider $\mathbb{R}^n$ as an open subset containing $\mathbb{H}^n$, where the Extension Lemma for Smooth Functions then tells us $\exists\ \widetilde{f} \in C^\infty(\mathbb{R}^n)$ such that $\widetilde{f}|_{\mathbb{H}^n} = f$ and $\mathrm{supp}(\widetilde{f}) \subseteq \mathbb{R}^n$. Yet we also have

$$f = \widetilde{f}|_{\mathbb{H}^n} = \widetilde{f} \circ \iota$$

and since $\widetilde{f}$ and f agree on $\mathbb{H}^n$, which is some neighborhood of $\mathbf{a}$, and since agreement on some neighborhood of the $\mathbf{a}$ implies the tangent vectors are equal by Proposition 35 we get

$$v(f)|_{\mathbf{a}} = v(\widetilde{f} \circ \iota)|_{\mathbf{a}} = d\iota|_{\mathbf{a}}(v_{\mathbf{a}})(\widetilde{f}) = 0$$

and since this holds for any $f \in C^\infty(\mathbb{H}^n)$ we conclude that $v_{\mathbf{a}} = 0$, and so $d\iota|_{\mathbf{a}}$ is injective.

Next, let $w_{\mathbf{a}} \in T_{\mathbf{a}}\mathbb{R}^n$ be given, and choose $v_{\mathbf{a}} \in T_{\mathbf{a}}\mathbb{H}^n$ such that for $f \in C^\infty(\mathbb{H}^n)$ with extension $\widetilde{f} \in C^\infty(\mathbb{R}^n)$, so that

$$v(f)|_{\mathbf{a}} = w(\widetilde{f})|_{\mathbf{a}}$$

where Proposition 35 assures us that $v(f)$ is independent of the choice $\widetilde{f}$, and so is well defined, and is a derivation since for an any $f, g \in C^\infty(\mathbb{H}^n)$ we have

$$\begin{aligned} v(fg)|_{\mathbf{a}} &= w(\widetilde{fg})|_{\mathbf{a}} \\ &= \widetilde{f}(\mathbf{a})w(\widetilde{g})|_{\mathbf{a}} + \widetilde{g}(\mathbf{a})w(\widetilde{f})|_{\mathbf{a}} \\ &= f(\mathbf{a})v(g)|_{\mathbf{a}} + g(\mathbf{a})v(f)|_{\mathbf{a}} \qquad\qquad \widetilde{f}|_{\mathbb{H}^n} = f, \text{ and } \widetilde{g}|_{\mathbb{H}^n} = g \end{aligned}$$

so for any $h \in C^\infty(\mathbb{R}^n)$ we see that

$$d\iota|_{\mathbf{a}}(v_{\mathbf{a}})(h) = v(h \circ \iota)|_{\mathbf{a}} = w(\widetilde{h \circ \iota})|_{\mathbf{a}} = w(h)|_{\mathbf{a}} \qquad \text{since } (h \circ \iota)|_{\mathbb{H}^n} = (\widetilde{h \circ \iota})|_{\mathbb{H}^n} = (h)|_{\mathbb{H}^n}$$

and by the arbitrariness of h we have that

$$d\iota|_{\mathbf{a}}(v_{\mathbf{a}}) = w_{\mathbf{a}}$$

and so $d\iota|_{\mathbf{a}}$ is surjective.

Hence we have a linear map between vector spaces that is both injective and surjective. Thus, $d\iota|_{\mathbf{a}}$ is an isomorphism between $T_{\mathbf{a}}\mathbb{H}^n$ and $T_{\mathbf{a}}\mathbb{R}^n$. □

Proposition 39 (**Dimension of Tangent Spaces on a Manifold with Boundary**)**.** Suppose M is an n-dimensional smooth manifold with boundary. For each $p \in M$; T_pM is an n-dimensional vector space.

Proof. Let $p \in M$ be arbitrary.

Case 1: If $p \in \mathrm{Int}(M)$, then since $\mathrm{Int}(M) \subseteq M$ is an open, and derivations are defined locally, Proposition 36 tells us

$$T_p\mathrm{Int}(M) \cong T_pM$$

and since $\mathrm{Int}(M)$ is an n-dimensional smooth manifold without boundary, each of its tangent spaces has dimension n by Proposition 37. And so we have

$$n = \dim(T_p\mathrm{Int}(M)) = \dim(T_pM)$$

Case 2: If $p \in \partial M$ let $(U, \phi) \in \mathcal{A}_M$ be a boundary chart containing p. Since

$$\phi : U \to \phi(U) \subseteq \mathbb{H}^n$$

is a diffeomorphism, Proposition 34 tells us that

$$d\phi|_p : T_pU \to T_{\phi(p)}\phi(U)$$

is an isomorphism. Since derivation are defined locally Proposition 36 then tells us that both

$$T_pM \cong T_pU$$
$$T_{\phi(p)}\phi(U) \cong T_{\phi(p)}\mathbb{H}^n$$

Then since derivations of the boundary are also defined locally Lemma 38 then gives

$$T_{\phi(p)}\mathbb{H}^n \cong T_{\phi(p)}\mathbb{R}^n$$

and from Proposition 31 the set of derivations at a point in euclidean space is isomorphic to the euclidean space with shifted origin; i.e. the space of geometric tangent vectors is isomorphic to the space of tangent vectors in euclidean space, we have

$$T_{\phi(p)}\mathbb{R}^n \cong \mathbb{R}^n_{\phi(p)}$$

and canonically $\mathbb{R}^n_{\phi(p)} \cong \mathbb{R}^n$. Putting it all together we get

$$T_pM \cong T_pU \cong T_{\phi(p)}\phi(U) \cong T_{\phi(p)}\mathbb{H}^n \cong T_{\phi(p)}\mathbb{R}^n \cong \mathbb{R}^n_{\phi(p)} \cong \mathbb{R}^n$$

and since isomorphism are structure preserving we must have

$$\dim(\mathbb{R}^n) = n = \dim(T_pM)$$

□

Proposition 40 (**The Tangent Space to a Vector Space**)**.** Suppose V is a finite dimensional vector space with its standard smooth manifold structure. For each point $a \in V$, the mapping

$$\begin{aligned} \eta : V|_a &\to T_aV \\ v_a &\mapsto D_v\big|_a \end{aligned}$$

with

$$D_v\big|_a : C^\infty(V) \to \mathbb{R}, \text{ by } D_v\big|_a(f) = \frac{d}{dt}\bigg|_{t=0} f(a+tv)$$

is a canonical isomorphism from V to T_aV, such that for any linear map

$$L : V \to W$$

we have the following relation

$$dL|_a \circ \eta = \eta \circ L|_a : V|_a \to T_{L(a)}W$$

Proof. Once a basis is chosen for V, since by Proposition 31 the set of derivations at a point in euclidean space is isomorphic to the euclidean space with shifted origin, where both $T_{\mathbf{a}}\mathbb{R}^n$ and $\mathbb{R}^n_{\mathbf{a}}$ are vector spaces, we have η is an isomorphism between V and T_aV.

Next suppose

$$L : V \to W$$

is a linear map. Since its' components with respect to any choice of bases for V and W are linear functions of the coordinates, L is smooth, and so for any $f \in C^\infty(W)$

$$\begin{aligned}
dL|_a\big(D_v\big|_a\big)(f) &= D_v(f \circ L)|_a & D_v\big|_a \in T_aV \\
&= \frac{d}{dt}\bigg|_{t=0} f\big(L(a+tv)\big) \\
&= \frac{d}{dt}\bigg|_{t=0} f\big(L(a)+tL(v)\big) & L \text{ is linear} \\
&= D_{L(v)}(f)|_{L(a)} \\
&= D_{L(v)}\big|_{L(a)}(f)
\end{aligned}$$

and thus, since $\eta(v_a) = D_v\big|_a$, and $\eta(L(v_a)) = D_{L(v)}\big|_{L(a)}$ we see that

$$dL|_a \circ \eta = \eta \circ L|_a$$

□

Proposition 41 (The Tangent Space to a Product Manifold). Let $M_1, \dots, M_k$ be smooth manifolds, and for each j, let

$$\pi_j : M_1 \times \cdots \times M_k \to M_j$$

be the projection onto the M_j factor. For any point $p = (p_1, \dots, p_k) \in M_1 \times \cdots \times M_k$, the map

$$\alpha : T_p(M_1 \times \cdots \times M_k) \to T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k$$

defined by

$$\alpha(v) = \big(d\pi_1|_p(v), \dots, d\pi_k|_p(v)\big)$$

is an isomorphism. The same is true if one of the spaces M_i is a smooth manifold with boundary.

Proof. The idea here is to come up with something that resembles an inverse so that we can use dimensionality and surjection, or injection to show an isomorphism. First we note that α is linear since each $d\pi_i|_p$ is. So given $p = (p_1, \dots, p_k) \in M_1 \times \cdots \times M_k$ define

$$\iota_j : M_j \to M_1 \times \cdots \times M_k \text{ by } \iota_j(x) = (p_1, \dots, p_{j-1}, x, p_{j+1}, \dots, p_k)$$

for $1 \leq j \leq k$ so that ι_j induces the linear map

$$d\iota_j|_{p_j} : T_{p_j}M_j \to T_p(M_1 \times \cdots \times M_k) \quad \iota_j(p_j) = (p_1, \dots, p_j, \dots, p_k) = p$$

where we next define the map

$$\beta : T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k \to T_p(M_1 \times \cdots \times M_k)$$

defined by

$$\beta\big(v_{1p_1} \oplus v_{2p_2} \oplus \cdots \oplus v_{kp_k}\big) = d\iota_1|_{p_1}\big(v_{1p_1}\big) \times d\iota_2|_{p_2}\big(v_{2p_2}\big) \times \cdots \times d\iota_k|_{p_k}\big(v_{kp_k}\big)$$

so that for any $v_{1p_1} \oplus \cdots \oplus v_{kp_k} \in T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k$ and $f \in C^\infty(M_1 \oplus \cdots \oplus M_k)$ we get

$$\begin{aligned}
(\alpha \circ \beta)\big(v_{1p_1} \oplus \cdots \oplus v_{kp_k}\big)(f) &= \alpha\Big(d\iota_1|_{p_1}\big(v_{1p_1}\big), \ldots, d\iota_k|_{p_k}\big(v_{kp_k}\big)\Big)(f) \\
&= \Big[d\pi_1|_p\Big(d\iota_1|_{p_1}\big(v_{1p_1}\big), \ldots, d\iota_k|_{p_k}\big(v_{kp_k}\big)\Big), \ldots, \\
&\qquad d\pi_k|_p\Big(d\iota_1|_{p_1}\big(v_{1p_1}\big), \ldots, d\iota_k|_{p_k}\big(v_{kp_k}\big)\Big)\Big](f) \\
&= \Big[d\pi_1|_{\iota_1(p_1)}\Big(d\iota_1|_{p_1}\big(v_{1p_1}\big), \ldots, d\iota_k|_{p_k}\big(v_{kp_k}\big)\Big), \ldots, \\
&\qquad d\pi_k|_{\iota_k(p_k)}\Big(d\iota_1|_{p_1}\big(v_{1p_1}\big), \ldots, d\iota_k|_{p_k}\big(v_{kp_k}\big)\Big)\Big](f) \\
&= \Big(d(\pi_1 \circ \iota_1)|_{p_1}\big(v_{1p_1}\big)(f), \ldots, d(\pi_k \circ \iota_k)|_{p_k}\big(v_{kp_k}\big)(f)\Big) && \text{Proposition 34 (b)} \\
&= \Big(v_1(f \circ \pi_1 \circ \iota_1)|_{p_1}, \ldots, v_k(f \circ \pi_k \circ \iota_k)|_{p_k}\Big) \\
&= \Big(v_1(f \circ Id_{M_1})|_{p_1}, \ldots, v_k(f \circ Id_{M_k})|_{p_k}\Big) && \pi_i \circ \iota_i = Id_{M_i} \\
&= \Big(v_1(f)|_{p_1}, \ldots, v_k(f)|_{p_k}\Big) && \text{Proposition 34 (c)} \\
&= \big(v_{1p_1}, \ldots, v_{kp_k}\big)(f) \\
&= \big(v_{1p_1} \oplus \cdots \oplus v_{kp_k}\big)(f)
\end{aligned}$$

so that

$$\alpha \circ \beta = Id_{T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k}$$

And so α is surjective.

Next we note by Proposition 37, the tangent space at any point of the manifold has the same dimension as the manifold and so

$$\begin{aligned}
\dim\big(T_p(M_1 \times \cdots \times M_k)\big) &= n_1 + n_2 + \cdots + n_k \\
&= \dim\big(T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k\big)
\end{aligned}$$

Thus, we have a surjective linear map α between vector spaces of equal dimension. Therefore α must be an isomorphism between $T_p(M_1 \times \cdots \times M_k)$ and $T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k$. □

Proposition 42. Let M be a smooth n-manifold with or without boundary, and let $p \in M$. Then T_pM is an n-dimensional vector space, and for any smooth chart $\big(U, (x^1, \ldots x^n)\big) \in \mathcal{A}_M$ containing p, the coordinate vectors $\big\{\frac{\partial}{\partial x^1}\big|_p, \ldots, \frac{\partial}{\partial x^n}\big|_p\big\}$ form a basis for T_pM.

Proposition 43. For any smooth n-manifold M; the tangent bundle TM has a natural topology and smooth structure that make it into a $2n$-dimensional smooth manifold. With respect to this structure, the projection

$$\pi : TM \to M$$

is smooth.

Proof. First, let $\{U_i\}_{i=1}^{\infty}$ be a cover for M with corresponding smooth structure $\{(U_i, \phi_i)\} = \mathcal{A}_M$. Now, for any $(U, \phi) \in \mathcal{A}_M$ we have

$$\pi^{-1}(U) = \bigsqcup_{p \in U} T_pM \subseteq TM$$

letting $\phi = (x^1, \dots, x^n)$ so that any element $v_p \in \pi^{-1}(U)$ may be written in the form

$$v_p = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\bigg|_p$$

and so we can define

$$\widetilde{\phi} : \pi^{-1}(U) \to \mathbb{R}^{2n}, \text{ by } \widetilde{\phi}\left(\sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\bigg|_p\right) = \big(x^1(p), \dots, x^n(p), v^1, \dots, v^n\big)$$

where

$$\widetilde{\phi}\big(\pi^{-1}(U)\big) = \phi(U) \times \mathbb{R}^n \subseteq \mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{R}^{2n}$$

is an open subset. So we propose the chart smooth chart $\big(\pi^{-1}(U), \widetilde{\phi}\big)$ for TM. With inverse

$$\widetilde{\phi}^{-1} : \phi(U) \times \mathbb{R}^n \to TM, \text{ by } \widetilde{\phi}^{-1}(\mathbf{x}, \mathbf{v}) = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\bigg|_{\phi^{-1}(\mathbf{x})}$$

and so is bijective onto its image, which is a subset of euclidean space.

Next, for any charts $(U, \phi), (V, \psi) \in \mathcal{A}_M$, such that $U \cap V \neq \varnothing$, with corresponding smooth charts $\big(\pi^{-1}(U), \widetilde{\phi}\big), \big(\pi^{-1}(V), \widetilde{\psi}\big)$ for TM, and we note that

$$\widetilde{\phi}\big(\pi^{-1}(U) \cap \pi^{-1}(V)\big) = \phi(U \cap V) \times \mathbb{R}^n \subseteq \mathbb{R}^n \times \mathbb{R}^n$$
$$\widetilde{\psi}\big(\pi^{-1}(U) \cap \pi^{-1}(V)\big) = \psi(U \cap V) \times \mathbb{R}^n \subseteq \mathbb{R}^n \times \mathbb{R}^n$$

are both open, and letting $\phi = (x^1, \dots, x^n)$ and $\psi = (y^1, \dots, y^n)$, with the change of coordinates giving

$$\widetilde{\phi}^{-1}(\mathbf{x}, \mathbf{v}) = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\bigg|_{\phi^{-1}(\mathbf{x})} = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} v^i \frac{\partial y^j}{\partial x^i}(\mathbf{x})\right) \frac{\partial}{\partial y^j}\bigg|_{\phi^{-1}(\mathbf{x})}$$

we have the transition map

$$\widetilde{\psi} \circ \widetilde{\phi}^{-1} : \phi(U \cap V) \times \mathbb{R}^n \to \psi(U \cap V) \times \mathbb{R}^n$$

defined by

$$\big(\widetilde{\psi} \circ \widetilde{\phi}^{-1}\big)(\mathbf{x}, \mathbf{v}) = \left(y^1\big(\phi^{-1}(\mathbf{x})\big), \dots, y^n\big(\phi^{-1}(\mathbf{x})\big), \sum_{i=1}^{n} v^i \frac{\partial y^1}{\partial x^i}(\mathbf{x}), \dots, \sum_{i=1}^{n} v^i \frac{\partial y^n}{\partial x^i}(\mathbf{x})\right)$$

which is smooth as a function of $(\mathbf{x}, \mathbf{v})$.

Now, since for each $p \in M$ we have $p \in U_i$ for some i we get that

$$TM = \bigsqcup_{p \in M} T_pM = \bigcup_{i=1}^{\infty} \bigsqcup_{p \in U_i} T_pM$$

which is to say that $\{\pi^{-1}(U_i)\}_{i=1}^{\infty}$ is a countable cover for TM.

Now if $v_p, w_q \in TM$, then

Case 1: If $p = q$, then $p \in U$ gives $v_p, w_p \in \pi^{-1}(U)$.

Case 2: If $p \neq q$, then since M is Hausdorff, there are disjoint open sets $U, V \subseteq M$ such that $p \in U$ and $q \in V$. Where we then have

$$v_p \in \phi^{-1}(U), \text{ and } w_q \in \pi^{-1}(V) \quad \text{ where } \phi^{-1}(U) \cap \pi^{-1}(V) = \varnothing$$

Thus, we have a collection $\{\pi^{-1}(U_i)\}$ of subsets of TM, together with maps

$$\widetilde{\phi}_i : \pi^{-1}(U_i) \to \mathbb{R}^{2n}$$

such that

1. For each i, $\widetilde{\phi}_i$ is a bijection between $\pi^{-1}(U_i)$ and $\widetilde{\phi}_i\big(\pi^{-1}(U_i)\big) \subseteq \mathbb{R}^{2n}$.
2. For each i and j, the sets $\widetilde{\phi}_i\big(\pi^{-1}(U_i) \cap \pi^{-1}(U_j)\big), \widetilde{\phi}_j\big(\pi^{-1}(U_i) \cap \pi^{-1}(U_j)\big) \subseteq \mathbb{R}^{2n}$ are open.
3. Whenever $U_i \cap U_j \neq \varnothing$ the map

$$\begin{aligned} \widetilde{\phi}_j \circ \widetilde{\phi}_i^{-1} : \widetilde{\phi}_i\big(\pi^{-1}(U_i) \cap \pi^{-1}(U_j)\big) &\to \widetilde{\phi}_j\big(\pi^{-1}(U_i) \cap \pi^{-1}(U_j)\big) \\ = \phi_i(U_i \cap U_j) \times \mathbb{R}^n &\to \phi_j(U_i \cap U_j) \times \mathbb{R}^n \end{aligned}$$

   is smooth.
4. Countably many of the sets $\pi^{-1}(U_i)$ cover TM.
5. Whenever $v_p, w_q \in TM$ are distinct; then either there exists some $\pi^{-1}(U_i)$ such that

$$v_p, w_q \in \pi^{-1}(U_i)$$

   or there exists open sets $\pi^{-1}(U_i), \pi^{-1}(U_j)$ such that

$$\pi^{-1}(U_i) \cap \pi^{-1}(U_j) = \varnothing, \quad v_p \in \pi^{-1}(U_i), \quad w_q \in \pi^{-1}(U_j)$$

and so by the Smooth Manifold Chart Lemma $\left\{\big(\pi^{-1}(U_i), \widetilde{\phi}_i\big)\right\} = \mathcal{A}_{TM}$ is a smooth structure for TM.

Moreover, for any $v_p \in TM$ with the chart $\big(\pi^{-1}(U), \widetilde{\phi}\big) \in \mathcal{A}_{TM}$ containing v_p, and $(U, \phi) \in \mathcal{A}_M$ containing $\pi(v_p) = p$, we have

$$\pi\big(\pi^{-1}(U)\big) = U \implies \pi\big(\pi^{-1}(U)\big) \subseteq U$$

where the coordinate representation of π gives

$$\phi \circ \pi \circ \widetilde{\phi}^{-1} : \phi(U) \times \mathbb{R}^n \to \phi(U)$$

is smooth as the projection map between euclidean spaces. And thus, we have that π must also be smooth. □

Proposition 44. If M is a smooth n-manifold with or without boundary, and M can be covered by a single smooth chart, then TM is diffeomorphic to $M \times \mathbb{R}^n$.

Proof. Let $(M, \phi) = \mathcal{A}_M$, then by definition

$$\phi : M \to \phi(M)$$

is a diffeomorphism with $\phi(M) \subseteq \mathbb{R}^n$ or $\mathbb{H}^n$. And from Proposition 43,

$$\widetilde{\phi} : \pi^{-1}(M) \to \phi(M) \times \mathbb{R}^n$$

is a diffeomorphism , where $\pi^{-1}(M) = TM$. Thus we have

$$TM \cong \phi(M) \times \mathbb{R}^n \cong M \times \mathbb{R}^n$$

□

Proposition 45. let M be a smooth m-Manifold, and N a smooth n-Manifold. If

$$F : M \to N$$

is a smooth map, then its global differential

$$dF : TM \to TN$$

is a smooth map.

Proof. Since

$$dF|_p = \begin{bmatrix} \frac{\partial \widehat{F}^1}{\partial x^1}\big(\phi(p)\big) & \cdots & \frac{\partial \widehat{F}^1}{\partial x^m}\big(\phi(p)\big) \\ \vdots & \ddots & \vdots \\ \frac{\partial \widehat{F}^n}{\partial x^1}\big(\phi(p)\big) & \cdots & \frac{\partial \widehat{F}^n}{\partial x^m}\big(\phi(p)\big) \end{bmatrix}$$

given $p \in M$ let $(U, \phi) \in \mathcal{A}_T$ we a chart containing p, and $(V, \psi) \in \mathcal{A}_N$ be a chart containing $F(p)$, such that $F(U) \subseteq V$. Then we have $\big(\pi_M^{-1}(U), \widetilde{\phi}\big) \in \mathcal{A}_{TM}$ is a chart containing v_p, and $\big(\pi_N^{-1}(V), \widetilde{\psi}\big) \in \mathcal{A}_{TN}$ is a chart containing $dF(v_p) = v_{F(p)}$. Letting $\phi = (x^1, \ldots, x^m)$ and $\psi = (y^1, \ldots, y^n)$, we get that dF has coordinate representation

$$\widetilde{\psi} \circ dF \circ \widetilde{\phi}^{-1} : \phi(U) \times \mathbb{R}^m \to \psi(V) \times \mathbb{R}^n$$

defined by

$$\begin{aligned}
\widehat{dF}(\mathbf{x}, \mathbf{v}) &= \big(\widetilde{\psi} \circ dF\big) \left(\sum_{i=1}^{m} v^i \frac{\partial}{\partial x^i}\bigg|_{\phi^{-1}(\mathbf{x})} \right) \\
&= \widetilde{\psi} \left(\sum_{j=1}^{n} \left(\sum_{i=1}^{m} v^i \frac{\partial (y^j \circ \widehat{F})}{\partial x^i}\big(\phi^{-1}(\mathbf{x})\big) \right) \frac{\partial}{\partial y^j}\bigg|_{F(\phi^{-1}(\mathbf{x}))} \right) \\
&= \left(y^1\big(F(\phi^{-1}(\mathbf{x}))\big), \dots, y^n\big(F(\phi^{-1}(\mathbf{x}))\big), \sum_{i=1}^{m} v^i \frac{\partial \widehat{F}^1}{\partial x^i}\big(\phi^{-1}(\mathbf{x})\big), \dots, \sum_{i=1}^{m} v^i \frac{\partial \widehat{F}^n}{\partial x^i}\big(\phi^{-1}(\mathbf{x})\big) \right)
\end{aligned}$$

or letting $p \in U$ be such that $\phi(p) = \mathbf{x} \in \phi(U)$ we get

$$\widehat{dF}(\mathbf{x}, \mathbf{v}) = \left(y^1\big(F(p)\big), \dots, y^n\big(F(p)\big), \sum_{i=1}^{m} v^i \frac{\partial \widehat{F}^1}{\partial x^i}(p), \dots, \sum_{i=1}^{m} v^i \frac{\partial \widehat{F}^n}{\partial x^i}(p) \right)$$

which is smooth because F is. And thus, so must be dF. □

Proposition 46 (**Properties of the Global Differential**)**.** Suppose both

$$\begin{aligned} F &: M \to N \\ G &: N \to P \end{aligned}$$

are smooth maps. Then

(a) $d(G \circ F) = dG \circ dF$

(b) $d(Id_M) = Id_{TM} : TM \to TM$

(c) If F is a diffeomorphism, then

$$dF : TM \to TN$$

is also a diffeomorphism, and

$$(dF)^{-1} = d(F^{-1})$$

Proposition 47. Suppose M is a smooth manifold with or without boundary and $p \in M$. Every derivation $v_p \in T_pM$ is the velocity of some smooth

$$\gamma : (-\epsilon, \epsilon) \to M$$

curve in M; i.e.

$$\gamma(0) = p, \quad \text{and} \quad \gamma'(0) = v_p$$

Proof. Let $(U, \phi) \in \mathcal{A}_M$ be a smooth chart centered at p, and in terms of the basis vector for T_pM we have

$$v_p = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\bigg|_p$$

since $\phi(U) \subseteq \mathbb{R}^n$ is open and $\phi(p) \in \phi(U)$, for some $\epsilon > 0$ we have

$$\phi \circ \gamma : (-\epsilon, \epsilon) \to \phi(U), \text{ by } (\phi \circ \gamma)(t) = t(v^1, \dots, v^n)$$

and so we may define our smooth curve

$$\gamma : (-\epsilon, \epsilon) \to M, \text{ by } \gamma(t) = \phi^{-1}\big(tv^1, \ldots, tv^n\big)$$

since (U, ϕ) was chosen to be centered at p this implies $\phi(p) = \mathbf{0}$, which then implies

$$p = \phi^{-1}(\mathbf{0}) = \phi^{-1}(0 \cdot \mathbf{x}) = \gamma(0)$$

and

$$\gamma'(0) = \sum_{i=1}^{n} \frac{d(x^i \circ \gamma)}{dt}(0) \frac{\partial}{\partial x^i}\bigg|_{\gamma(0)} = \sum_{i=1}^{n} \frac{d(x^i \circ \gamma)}{dt}(0) \frac{\partial}{\partial x^i}\bigg|_{p} = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\bigg|_{p} = v_p$$

□

Proposition 48 (**Velocity of a Composite Curve**)**.** Let M, N be smooth manifolds,

$$F : M \to N$$

a smooth map, and let

$$\gamma : J \to M$$

be a smooth curve. For any $t_0 \in J$, the velocity at t_0 of the composite curve

$$F \circ \gamma : J \to N$$

is given by

$$(F \circ \gamma)'(t_0) = dF|_{\gamma(t_0)}\big(\gamma'(t_0)\big)$$

Proof. Since the composition $F \circ \gamma$ is itself a smooth curve into the manifold N we have, from the definition of velocity curve

$$(F \circ \gamma)'(t_0) = d(F \circ \gamma)|_{t_0}\left(\frac{d}{dt}\bigg|_{t_0}\right) = dF|_{\gamma(t_0)}\left(d\gamma|_{t_0}\left(\frac{d}{dt}\bigg|_{t_0}\right)\right) = dF|_{\gamma(t_0)}\big(\gamma'(t_0)\big)$$

□

Corollary 49. Let M, N be smooth manifolds, and

$$F : M \to N$$

a smooth map. Fix $p \in M$, and $v_p \in T_pM$. Then

$$dF|_p(v_p) = (F \circ \gamma)'(0)$$

for any smooth curve

$$\gamma : (-\epsilon, \epsilon) \to M, \quad \text{such that} \quad \gamma(0) = p, \text{ and } \gamma'(0) = v_p$$

Proposition 50. Suppose $F : M \to N$ is a smooth map and $p \in M$. If

$$dF|_p : T_pM \to T_{F(p)}N$$

is surjective, then p has a neighborhood U_p such that $dF|_{U_p}$ is a submersion. If

$$dF|_p : T_pM \to T_{F(p)}N$$

is injective, then p has a neighborhood U_p such that $dF|_{U_p}$ is an immersion.

Proof. Let M be of dimension m, and N be of dimension n, and let $(U, \phi) \in \mathcal{A}_M$ be a chart containing p, and $(V, \psi) \in \mathcal{A}_N$ be a chart containing $F(p)$. Either of the hypothesis imply that the coordinate representation $\widehat{F}$ of F has full rank at $\phi(p)$, since

$$dF|_p = \begin{bmatrix} \frac{\partial \widehat{F}^1}{\partial x^1}\big(\phi(p)\big) & \cdots & \frac{\partial \widehat{F}^1}{\partial x^m}\big(\phi(p)\big) \\ \vdots & \ddots & \vdots \\ \frac{\partial \widehat{F}^n}{\partial x^1}\big(\phi(p)\big) & \cdots & \frac{\partial \widehat{F}^n}{\partial x^m}\big(\phi(p)\big) \end{bmatrix}$$

has either $\mathrm{rank}(dF|_p) = n$ if $dF|_p$ is surjective, or $\mathrm{rank}(dF|_p) = m$ if $dF|_p$ is injective. And we know that the matrices of full rank $\mathrm{M}_m(n \times m, \mathbb{R}) \subseteq \mathrm{M}(n \times m, \mathbb{R})$ of rank m, or $\mathrm{M}_n(n \times m, \mathbb{R}) \subseteq \mathrm{M}(n \times m, \mathbb{R})$ of rank n, are open subsets. Which tells us that each element in either $\mathrm{M}_m(n \times m, \mathbb{R})$ or $\mathrm{M}_m(n \times m, \mathbb{R})$ has a neighborhood which must also be full rank by continuity of the determinant function (on the $n \times n$ or $m \times m$ sub-matrices which have non-zero determinant). And so there is a neighborhood U_p such that $dF|_{U_p}$ which is represented by the Jacobian matrix of $\widehat{F}$ on $\phi(U_p)$ has full rank. $\square$

Proposition 51. A composition of smooth submersions is a smooth submersion, and a composition of smooth immersions is a smooth immersion. Yet, a composition of maps of constant rank need not have constant rank.

Proof. Let

$$F : M \to N$$
$$G : N \to P$$

be smooth submersions, and let $p \in M$ be given. By Proposition 34 (b) we have

$$d(G \circ F)|_p = dG|_{F(p)} \circ dF|_p$$

and since F is a smooth submersion

$$dF|_p : T_pM \to T_{F(p)}N$$

is surjective. Similarly, since G is a smooth submersion

$$dG|_{F(p)} : T_{F(p)}N \to T_{G(F(p))}P$$

is surjective. And thus,

$$dG|_{F(p)} \circ dF|_p : T_pM \to T_{G(F(p))}P$$

is surjective at p.

Since $p \in M$ was arbitrary we conclude that

$$d(G \circ F)|_p : T_pM \to T_{G(F(p))}P$$

is surjective at each point of M and therefore

$$G \circ F : M \to P$$

is a smooth submersion.

The same argument holds with "submersion" replaced by "immersion" so that the composition of smooth immersions is a smooth immersion.

Next, we product a counter-example to show that the composition of maps of constant rank need not have constant rank. Define

$$F : \mathbb{R} \to \mathbb{R}^2, \text{ by } F(x) = (x, x^2)$$

so that

$$DF = \begin{bmatrix} 1 \\ 2x \end{bmatrix}$$

and F has constant rank with $\operatorname{rank}(F) = 1$. And let

$$G : \mathbb{R}^2 \to \mathbb{R}, \text{ by } G(x, y) = y$$

where

$$DG = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

and so G has constant rank with $\operatorname{rank}(G) = 1$. Then we have

$$G \circ F : \mathbb{R} \to \mathbb{R}, \text{ by } \big(G \circ F\big)(x) = x^2$$

which gives

$$D(G \circ F) = 2x$$

which has

$$\operatorname{rank}(G \circ F) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

and therefore, does not have constant rank. $\square$

Theorem 52 (**Inverse Function Theorem for Manifolds**)**.** Suppose M and N are smooth manifolds, and $F : M \to N$ is a smooth map. If $p \in M$ is a point such that

$$dF|_p : T_pM \to T_{F(p)}N$$

is invertible, then there are connected neighborhoods U_p of p and $V_{F(p)}$ of $F(p)$ such that

$$F|_{U_p} : U_p \to V_{F(p)}$$

is a diffeomorphism.

Proof. First, since $dF|_p$ is represented by the Jacobian matrix of $\widehat{F}$ at $\phi(p)$, for any charts $(U, \phi) \in \mathcal{A}_M$ containing p, and $(V, \psi) \in \mathcal{A}_N$ containing $F(p)$; that is

$$dF|_p = \begin{bmatrix} \frac{\partial \widehat{F}^1}{\partial x^1}\big(\phi(p)\big) & \cdots & \frac{\partial \widehat{F}^1}{\partial x^m}\big(\phi(p)\big) \\ \vdots & \ddots & \vdots \\ \frac{\partial \widehat{F}^n}{\partial x^1}\big(\phi(p)\big) & \cdots & \frac{\partial \widehat{F}^n}{\partial x^m}\big(\phi(p)\big) \end{bmatrix}$$

so $dF|_p$ being invertible implies that the Jacobian matrix of $\widehat{F}$ at $\phi(p)$ is invertible, and only square matrices are invertible, thus we have

$$\dim(M) = \dim(N) = n$$

Next, by the smoothness of F, we may choose charts $(U, \phi) \in \mathcal{A}_M$ centered at p, so $\phi(p) = \mathbf{0}$, and $(V, \psi) \in \mathcal{A}_N$ centered at $F(p)$, so $\psi(F(p)) = \mathbf{0}$, such that $F(U) \subseteq V$. Where

$$F(U) \subseteq V \implies U \subseteq F^{-1}(V) \implies U \cap F^{-1}(V) = U$$

then we have the coordinate representation of F given by

$$\widehat{F} : \phi(U) \to \psi(V)$$

which is smooth, and where

$$\widehat{F}(\mathbf{0}) = \psi \circ F \circ \phi^{-1}(\mathbf{0}) = \psi(F(p)) = \mathbf{0}$$

Now, since both ϕ and ψ are diffeomorphisms, and hence invertible, the differential

$$d\widehat{F}|_{\mathbf{0}} = d\psi|_{F(p)} \circ dF|_p \circ d\phi^{-1}|_{\mathbf{0}} : T_{\mathbf{0}}\mathbb{R}^n \to T_{\mathbf{0}}\mathbb{R}^n$$

is nonsingular, as the composition of 3 non-singular matrices between euclidean spaces. Where the ordinary Inverse Function Theorem then says that there are connected open subsets

$$U_{\mathbf{0}} \subseteq \phi(U), \quad V_{\mathbf{0}} \subseteq \psi(V)$$

containing $\mathbf{0}$, such that

$$\widehat{F}|_{U_{\mathbf{0}}} : U_{\mathbf{0}} \to V_{\mathbf{0}}$$

is a diffeomorphism. Then $U_p = \phi^{-1}(U_{\mathbf{0}})$ is a connected neighborhood containing p, and $V_{F(p)} = \psi^{-1}(V_{\mathbf{0}})$ is connected neighborhood containing $F(p)$. And so

$$F|_{U_p} = \psi^{-1} \circ \widehat{F} \circ \phi|_{U_p} : U_p \to V_{F(p)}$$

is a diffeomorphism, since the composition of diffeomorphisms is a diffeomorphism. □

Proposition 53 (Elementary Properties of Local Diffeomorphisms).

(a) Every composition of local diffeomorphisms is a local diffeomorphism.

(b) Every finite product of local diffeomorphisms between smooth manifolds is a local diffeomorphism.

(c) Every local diffeomorphism is a local homeomorphism and an open map.

(d) The restriction of a local diffeomorphism to an open submanifold with or without boundary is a local diffeomorphism.

(e) Every diffeomorphism is a local diffeomorphism.

(f) Every bijective local diffeomorphism is a diffeomorphism.

(g) A map between smooth manifolds with or without boundary is a local diffeomorphism if and only if in a neighborhood of each point of its domain, it has a coordinate representation that is a local diffeomorphism.

Proof.

(a) Let M, N, and P be smooth manifolds and let

$$F : M \to N$$
$$G : N \to P$$

be local diffeomorphisms. Let $p \in M$, then since F is a local diffeomorphism, $\exists\ U_p \subseteq M$ open and containing p such that $F(U_p) \subseteq N$ is open, and

$$F|_{U_p} : U_p \to F(U_p)$$

is a diffeomorphism. Similarly, since G is a local diffeomorphism, $\exists\ V_{F(p)} \subseteq N$ open and containing $F(p)$ such that $G\big(V_{F(p)}\big) \subseteq P$ is open, and

$$G|_{V_{F(p)}} : V_{F(p)} \to G\big(V_{F(p)}\big)$$

is a diffeomorphism. And since F is a diffeomorphism we have

$$F^{-1}\big(F(U_p) \cap V_{F(p)}\big) = U_p \cap F^{-1}\big(V_{F(p)}\big)$$

So that the map

$$(G \circ F)|_{U_p \cap F^{-1}(V_{F(p)})} : U_p \cap F^{-1}\big(V_{F(p)}\big) \to G\big(F(U_p) \cap V_{F(p)}\big)$$

is a local diffeomorphism.

(b) Let $M_1, \dots, M_k$, and $N_1, \dots, N_k$ be smooth manifolds such that for each i we have

$$F_i : M_i \to N_i$$

is a local diffeomorphism then for each $p_i \in M_i, \exists\ U_{p_i} \subseteq M_i$ open and containing p_i such that $F_i(U_{p_i}) \subseteq N_i$ is open and

$$F_i|_{U_{p_i}} : U_{p_i} \to F_i(U_{p_i})$$

is a diffeomorphism, so the finite product

$$F_1 \times \cdots \times F_k : M_1 \times \cdots \times M_k \to N_1 \times \cdots \times N_k$$

has a local diffeomorphism given by

$$F_1|_{U_{p_1}} \times \cdots \times F_k|_{U_{p_k}} : U_{p_1} \times \cdots \times U_{p_k} \to F_1(U_{p_1}) \times \cdots \times F_k(U_{p_k})$$

(c) Let

$$F : M \to N$$

be a local diffeomorphism, then for every $p \in M$, $\exists\ U_p \subseteq M$ open and containing p such that $F(U_p) \subseteq N$ is open and

$$F|_{U_p} : U_p \to F(U_p)$$

is a diffeomorphism. Since differentiability implies continuity we have

$$F|_{U_p} : U_p \to F(U_p)$$

is a homeomorphism, and since this is true for each $p \in M$ we have that F is a local homeomorphism.

By definition we have that $F(U_p) \subseteq N$ is open, and so a local diffeomorphism is defined to be an open map.

(d) Let

$$F : M \to N$$

be a local diffeomorphism, let $U \subseteq M$ be an open submanifold, and let $p \in U$. Then $p \in M$, and there exists $V_p \subseteq M$ open and containing p such that $F(V_p) \subseteq N$ is open and

$$F|_{V_p} : V_p \to F(V_p)$$

is a diffeomorphism. Since an open subset of a smooth manifold is again a smooth manifold, and $U \cap V_p \subset M$ is open as the finite intersection of open sets, it is also a submanifold, where Proposition 20 (d) then tells us that the restriction of a diffeomorphism to an open submanifold is a diffeomorphism; that is

$$F|_{V_p}\big|_{U\cap V_p} : U \cap V_p F(U \cap V_p)$$

is a diffeomorphism. And by Proposition 20 (c) every diffeomorphism is an open map, and so $F(U \cap V_p) \subseteq N$ is open.

Thus, we have found a neighborhood of $U \cap V_p \subseteq U$ of p in U, such that $F(U \cap V_p) \subseteq N$ is open and

$$F|_{V_p}\big|_{U\cap V_p} : U \cap V_p F(U \cap V_p)$$

is a diffeomorphism. Since $p \in U$ was arbitrary we conclude that this can be done for each point of the open submanifold U, and thus we conclude that the restriction of a local diffeomorphism is a local diffeomorphism.

(e) Let

$$F : M \to N$$

be a diffeomorphism then for each $p \in M, \exists\ U_p \subseteq M$ open and containing p and since diffeomorphisms are open mappings we must have $F(U_p) \subseteq N$ is open. Then since open subsets of smooth manifolds are again smooth manifolds, Proposition 20 (d) then tells us that the restriction of a diffeomorphism to an open submanifold is a diffeomorphism and so

$$F|_{U_p} : U_p \to F(U_p)$$

is a diffeomorphism. Thus, we see that the restriction of F to an open subset around any $p \in M$ induces a local diffeomorphism.

(f) Let

$$F : M \to N$$

be a bijective local diffeomorphism, since F is bijective

$$F^{-1} : N \to M$$

exists. Since F is a local diffeomorphism for every $p \in M$, $\exists\ U_p \subseteq M$ open and containing p such that $F(U_p) \subseteq N$ is open and

$$F|_{U_p} : U_p \to F(U_p)$$

is a diffeomorphism. Yet, by the openness of $F(U_p)$ for each $q \in F(U_p)$ there is an open neighborhood $V_q \subseteq F(U_p)$ containing q, where (d) tells us that the restriction of a diffeomorphism to a submanifold is a local diffeomorphism and so

$$F^{-1}|_{F(U_p)}\big|_{F(U_p)\cap V_p} : F(U_p) \cap V_p \to U_p \cap F^{-1}(V_p)$$

is a diffeomorphism and hence smooth. And since $q \in F(U_p)$ arbitrary we conclude that F^{-1} is smooth in a neighborhood of each point of $F(U_p)$, and since F is bijective we then conclude that F^{-1} will be smooth in a neighborhood of each point of N, and thus F^{-1} is smooth, where F is already smooth in a neighborhood of each point of M by definition, and thus F is a diffeomorphism.

(g) First let

$$F : M \to N$$

be a local diffeomorphism and let $p \in M$ with smooth chart $(U, \phi) \in \mathcal{A}_M$ containing p, and smooth chart $(V, \psi) \in \mathcal{A}_N$ containing $F(p)$, then the coordinate representation

$$\widehat{F}|_{\phi(U\cap F^{-1}(V))} = \psi \circ F \circ \phi^{-1}|_{\phi(U\cap F^{-1}(V))} : \phi\big(U \cap F^{-1}(V)\big) \to \psi\big(F(U) \cap V\big)$$

is a local diffeomorphism.

Next suppose F has coordinate representation $\widehat{F}$ which is locally diffeomorphic, then since the smooth maps ϕ, ψ are diffeomorphic and the the composition of diffeomorphic functions is diffeomorphic we have

$$\begin{aligned} &F|_{\phi^{-1}(U\cap\widehat{F}^{-1}(\psi(V)))} \\ &\quad = \psi^{-1} \circ \widehat{F} \circ \phi|_{\phi^{-1}(U\cap\widehat{F}^{-1}(\psi(V)))} : \phi^{-1}\big(U \cap \widehat{F}^{-1}(\psi(V))\big) \to F(\phi^{-1}(U)) \cap V \end{aligned}$$

is a local diffeomorphism.

□

Proposition 54. Suppose M and N are smooth manifolds (without boundary), and $F : M \to N$ is a map.

(a) F is a local diffeomorphism if and only if it is both a smooth immersion and a smooth submersion.

(b) If $\dim(M) = \dim(N)$ and F is either a smooth immersion or a smooth submersion, then it is a local diffeomorphism.

Proof.

(a) First suppose that F is a local diffeomorphism. Given any $p \in M$, $\exists\ U_p \subseteq M$ open and containing p such that $F(U_p) \subseteq N$ is open and

$$F|_{U_p} : U_p \to F(U_p)$$

is a diffeomorphism, and since it is a diffeomorphism, Proposition 34 (d) tell us that

$$dF|_p : T_pU_p \cong T_pM \to T_{F(p)}F(U_p) \cong T_{F(p)}N$$

is an isomorphism, which tells us that $\dim(T_pM) = \dim\big(T_{F(p)}N\big)$. Yet, $\dim(T_pM) = \dim(M)$ and $\dim\big(T_{F(p)}N\big) = \dim(N)$, and so $\dim(M) = dim(N) = \text{rank}(dF|_p)$, and since this can be done for each $p \in M$, we conclude that dF has full rank at each point of M, and hence is both injective and surjective at each point of M. And therefore dF is both a smooth immersion and a smooth submersion.

Next, suppose that F, or dF, is both a smooth immersion and a smooth submersion, then $\dim(M) = \dim(N)$, and for each $p \in M$

$$dF|_p : T_pM \to T_{F(p)}N$$

is an isomorphism, and so invertible, where the Inverse Function Theorem for Manifolds says that there exists an open neighborhood U_p of p, and $V_{F(p)}$ of $F(p)$, where we may assume that $F(U_p) = V_{F(p)}$, such that

$$F|_{U_p} : U_p \to F(U_p)$$

is a diffeomorphism. Thus, F is a local diffeomorphism.

(b) Note that if $\dim(M) = \dim(N)$ then either injectivity, or surjectivity of

$$dF|_p : T_pM \to T_{F(p)}N$$

implies bijectivity, and so F, or dF, is a smooth immersion/submersion iff it is a smooth submersion/immersion, then (a) tells us that F must also be a local diffeomorphism.

$\square$

Proposition 55. Suppose M, N, and P are smooth manifolds with or without boundary, and $F : M \to N$ is a local diffeomorphism. Prove the following:

(a) If $G : P \to M$ is continuous, then G is smooth if and only if $F \circ G$ is smooth.

(b) If in addition F is surjective and $G : N \to P$ is any map, then G is smooth if and only if $G \circ F$ is smooth.

Proof.

(a) First suppose

$$G : P \to M$$

is smooth. Then, since local diffeomorphisms are smooth, and the composition of smooth functions is smooth

$$F \circ G : P \to N$$

is smooth.

Next, suppose

$$F \circ G : P \to N$$

is smooth. Let $q \in P$, then $G(q) \in M$ and since F is a local diffeomorphism $\exists\, U_{G(q)} \subseteq M$ open and containing $G(q)$, such that

$$F|_{U_{G(q)}} : U_{G(q)} \to F(U_{G(q)})$$

is a diffeomorphism. Yet, this also implies

$$(F|_{U_{G(q)}})^{-1} = F^{-1}|_{F(U_{G(q)})} : F(U_{G(q)}) \to U_{G(q)}$$

is a diffeomorphism.

Next, since G is continuous and $U_{G(q)} \subseteq M$ is open we have $G^{-1}(U_{G(q)}) \subseteq P$ is open and contains q, and since

$$(F \circ G)|_{G^{-1}(U_{G(q)})} : G^{-1}(U_{G(q)}) \to F(U_{G(q)})$$

is smooth and the composition of smooth functions is smooth, we get

$$F^{-1}|_{F(U_{G(q)})} \circ (F \circ G)|_{G^{-1}(U_{G(q)})} = G|_{G^{-1}(U_{G(q)})} : G^{-1}(U_{G(q)}) \to U_{G(q)}$$

is smooth. And since $q \in P$ was arbitrary we can conclude that G is smooth.

(b) First suppose

$$G : N \to P$$

is smooth. Then since since local diffeomorphisms are smooth and, the composition of smooth functions is smooth

$$G \circ F : M \to P$$

is smooth.

Next, suppose

$$G \circ F : M \to P$$

is smooth, and let $q \in N$ be arbitrary. Since F is surjective, WLOG let $q = F(p)$. Since F is a local diffeomorphism $\exists\, U_p \subseteq M$ open and containing p, with $F(U_p) \subseteq N$ open and containing q, such that

$$F|_{U_p} : U_p \to F(U_p)$$

is a diffeomorphism. Yet, this implies

$$(F|_{U_p})^{-1} = F^{-1}|_{F(U_p)} : F(U_p) \to U_p$$

is a diffeomorphism. Then since

$$(G \circ F)|_{U_p} : U_p \to G(F(U_p))$$

is smooth by assumption, and the composition of smooth functions is smooth, we get

$$(G \circ F)|_{U_p} \circ F^{-1}|_{F(U_p)} = G|_{F(U_p)} : F(U_p) \to G(F(U_p))$$

is smooth. And since $q \in N$ was arbitrary we can conclude that G is smooth.

$\square$

Theorem 56 (**Rank Theorem**)**.** Suppose M and N are smooth manifolds with $\dim(M) = m$, $\dim(N) = n$ and

$$F : M \to N$$

is a smooth map with constant rank r. For each $p \in M$ there exist smooth charts $(U, \phi) \in \mathcal{A}_M$ centered at p and $(V, \psi) \in \mathcal{A}_N$ centered at $F(p)$, such that $F(U) \subseteq V$, in which F has a coordinate representation of the form

$$\begin{aligned}
&\widehat{F}(x_1, \ldots, x_r, x_{r+1}, \ldots, x_m) = (x_1, \ldots, x_r, 0, \ldots, 0) && \\
&\widehat{F}(x_1, \ldots, x_n, x_{n+1}, \ldots, x_m) = (x_1, \ldots, x_n) && F \text{ a smooth submersion} \\
&\widehat{F}(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0) && F \text{ a smooth immersion}
\end{aligned}$$

Proof. By the smoothness of F, we may choose charts $(U, \phi) \in \mathcal{A}_M$ centered at p, so $\phi(p) = \mathbf{0}$, and $(V, \psi) \in \mathcal{A}_N$ centered at $F(p)$, so $\psi(F(p)) = \mathbf{0}$, such that $F(U) \subseteq V$. Where

$$F(U) \subseteq V \implies U \subseteq F^{-1}(V) \implies U \cap F^{-1}(V) = U$$

then we have the coordinate representation of F given by

$$\widehat{F} : \phi(U) \to \psi(V)$$

which is smooth, and where

$$\widehat{F}(\mathbf{0}) = \psi \circ F \circ \phi^{-1}(\mathbf{0}) = \psi(F(p)) = \mathbf{0}$$

Since the theorem is local we may associate M with $\phi(U) \subseteq \mathbb{R}^m$, and N with $\psi(V) \subseteq \mathbb{R}^n$. Since F, or dF, has constant rank r, this means that the Jacobian matrix of $\widehat{F}$ at $\phi(p) = \mathbf{0}$ has an $r \times r$ submatrix with non-zero determinant, re-ordering the coordinates if necessary we may assume this is the upper left submatrix; that is

$$D\widehat{F}(\mathbf{0}) = \begin{bmatrix} \sum_{i,j=1}^{r} \frac{\partial \widehat{F}^j}{\partial x^i}(\mathbf{0}) & * \\ * & * \end{bmatrix}$$

relabeling the standard coordinates

$$(\mathbf{x},\mathbf{y}) = (x_1,\dots x_r, y_1\dots,y_{m-r}) \text{ in } \mathbb{R}^m$$
$$(\mathbf{u},\mathbf{v}) = (u_1,\dots u_r, v_1\dots,v_{n-r}) \text{ in } \mathbb{R}^n$$

so writing

$$\widehat{F}(\mathbf{x},\mathbf{y}) = \big(Q(\mathbf{x},\mathbf{y}), R(\mathbf{x},\mathbf{y})\big),\ \text{for } Q:\phi(U)\to\mathbb{R}^r,\ R:\phi(U)\to\mathbb{R}^{n-r}$$

our hypothesis then becomes

$$\sum_{i,j=1}^{r}\frac{\partial Q^j}{\partial x^i}(\mathbf{0},\mathbf{0}),\ \text{ is non-singular}$$

Defining

$$\eta:\phi(U)\to\mathbb{R}^m,\ \text{by } \eta(\mathbf{x},\mathbf{y}) = \big(Q(\mathbf{x},\mathbf{y}),\mathbf{y}\big)$$

we have

$$D\eta(\mathbf{0},\mathbf{0}) = \begin{bmatrix} \sum_{i,j=1}^{r}\frac{\partial Q^j}{\partial x^i}(\mathbf{0},\mathbf{0}) & \sum_{j=1}^{m-r}\sum_{i=1}^{r}\frac{\partial Q^i}{\partial y^j}(\mathbf{0},\mathbf{0}) \\ O & I_{m-r}\end{bmatrix}$$

and so $D\eta(\mathbf{0},\mathbf{0})$ is non-singular, since its upper left and lower right sub-matrices are both non-singular, where the Inverse Function Theorem then says there are connected neighborhoods $U_{(\mathbf{0},\mathbf{0})}\subseteq\phi(U)$ containing $(\mathbf{0},\mathbf{0})$, and $\widetilde{U}_{(\mathbf{0},\mathbf{0})}\subseteq\eta(\phi(U))$ containing $\eta(\mathbf{0},\mathbf{0}) = (\mathbf{0},\mathbf{0})$, such that

$$\eta|_{U_{(\mathbf{0},\mathbf{0})}}: U_{(\mathbf{0},\mathbf{0})}\to\widetilde{U}_{(\mathbf{0},\mathbf{0})}$$

is a diffeomorphism. Shirking both $U_{(\mathbf{0},\mathbf{0})}$ and $\widetilde{U}_{(\mathbf{0},\mathbf{0})}$ if necessary we may assume that $\widetilde{U}_{(\mathbf{0},\mathbf{0})}\subseteq\mathbb{R}^m$ is a cube. Then, writing the inverse map of $\eta|_{U_{(\mathbf{0},\mathbf{0})}}$ as

$$\eta|^{-1}_{\widetilde{U}_{(\mathbf{0},\mathbf{0})}}(\mathbf{x},\mathbf{y}) = \big(A(\mathbf{x},\mathbf{y}), B(\mathbf{x},\mathbf{y})\big),\ \text{for } A:\widetilde{U}_{(\mathbf{0},\mathbf{0})}\to\mathbb{R}^r,\ B:\widetilde{U}_{(\mathbf{0},\mathbf{0})}\to\mathbb{R}^{m-r}$$

this implies that

$$\begin{aligned}(\mathbf{x},\mathbf{y}) &= \eta|_{U_{(\mathbf{0},\mathbf{0})}}\Big(\big(A(\mathbf{x},\mathbf{y}), B(\mathbf{x},\mathbf{y})\big)\Big)\\ &= \Big(Q\big(A(\mathbf{x},\mathbf{y}), B(\mathbf{x},\mathbf{y})\big), B(\mathbf{x},\mathbf{y})\Big)\end{aligned}$$

where we see that $\mathbf{y} = B(\mathbf{x},\mathbf{y})$ and so

$$\eta|^{-1}_{\widetilde{U}_{(\mathbf{0},\mathbf{0})}}(\mathbf{x},\mathbf{y}) = \big(A(\mathbf{x},\mathbf{y}),\mathbf{y}\big)$$

yet, we also have

$$\eta|_{U_{(\mathbf{0},\mathbf{0})}}\circ\eta|^{-1}_{\widetilde{U}_{(\mathbf{0},\mathbf{0})}} = Id_{\widetilde{U}_{(\mathbf{0},\mathbf{0})}} \implies \mathbf{x} = Q\big(A(\mathbf{x},\mathbf{y}),\mathbf{y}\big)$$

and therefore

$$\begin{aligned}\widehat{F}\circ\eta|^{-1}_{\widetilde{U}_{(\mathbf{0},\mathbf{0})}}(\mathbf{x},\mathbf{y}) &= \widehat{F}\circ\big(A(\mathbf{x},\mathbf{y}),\mathbf{y}\big)\\ &= \Big(Q\big(A(\mathbf{x},\mathbf{y}),\mathbf{y}\big), R\big(A(\mathbf{x},\mathbf{y}),\mathbf{y}\big)\Big)\\ &= \Big(\mathbf{x}, R\big(A(\mathbf{x},\mathbf{y}),\mathbf{y}\big)\Big)\end{aligned}$$

Letting

$$\widetilde{R} = R\big(A(\mathbf{x},\mathbf{y}),\mathbf{y}\big)|_{\widetilde{U}_{(\mathbf{0},\mathbf{0})}} : \widetilde{U}_{(\mathbf{0},\mathbf{0})} \to \mathbb{R}^{n-r}$$

Then for any $(\mathbf{x},\mathbf{y}) \in \widetilde{U}_{(\mathbf{0},\mathbf{0})}$ we have

$$D\Big(\widehat{F}\circ\eta|^{-1}_{\widetilde{U}_{(\mathbf{0},\mathbf{0})}}\Big)(\mathbf{x},\mathbf{y}) = \begin{bmatrix} I_r & O \\ \sum_{j=1}^{r}\sum_{i=1}^{n-r}\frac{\partial\widetilde{R}^i}{\partial x^j}\big(\mathbf{x},\mathbf{y}\big) & \sum_{i,j=1}^{n-r}\frac{\partial\widetilde{R}^i}{\partial y^j}\big(\mathbf{x},\mathbf{y}\big)\end{bmatrix}$$

and since composition with a diffeomorphism does not change the rank of a map, this means Jacobian matrix of $\widehat{F}\circ\eta|^{-1}_{\widetilde{U}_{(\mathbf{0},\mathbf{0})}}$ has rank r throughout $\widetilde{U}_{(\mathbf{0},\mathbf{0})}$. Since I_r is linearly independent, this can only be the case if

$$\sum_{i,j=1}^{n-r}\frac{\partial\widetilde{R}^i}{\partial y^j}\big(\mathbf{x},\mathbf{y}\big) = O_{n-r\times n-r}$$

identically, that is $\widetilde{R}$, and hence $\widehat{F}|_{U_{(\mathbf{0},\mathbf{0})}}$ is independent of $\mathbf{y} = (y_1,\ldots,y_{m-r})$, so setting

$$S(\mathbf{x}) = \widetilde{R}(\mathbf{x},\mathbf{0}) = R\big(A(\mathbf{x},\mathbf{0}),\mathbf{0}\big)|_{\widetilde{U}_{(\mathbf{0},\mathbf{0})}}$$

we get

$$\widehat{F}\circ\eta|^{-1}_{\widetilde{U}_{(\mathbf{0},\mathbf{0})}}(\mathbf{x},\mathbf{y}) = \big(\mathbf{x},S(\mathbf{x})\big)$$

Next, let $V_{(\mathbf{0},\mathbf{0})} \subseteq \psi(V)$ be defined by

$$V_{(\mathbf{0},\mathbf{0})} = \{(\mathbf{u},\mathbf{v}) \in \psi(V) : (\mathbf{u},\mathbf{0}) \in \widetilde{U}_{(\mathbf{0},\mathbf{0})}\}$$

and so $(\mathbf{0},\mathbf{0}) \in V_{(\mathbf{0},\mathbf{0})}$. Furthermore, since $\widehat{F}\circ\eta|^{-1}_{\widetilde{U}_{(\mathbf{0},\mathbf{0})}}$ is independent of $\mathbf{y}$ we have

$$\widehat{F}\circ\eta|^{-1}_{\widetilde{U}_{(\mathbf{0},\mathbf{0})}}(\widetilde{U}_{(\mathbf{0},\mathbf{0})}) = \widehat{F}(U_{(\mathbf{0},\mathbf{0})}) \subseteq V_{(\mathbf{0},\mathbf{0})}$$

so define

$$\xi : V_{(\mathbf{0},\mathbf{0})} \to \mathbb{R}^n, \text{ by } \xi(\mathbf{u},\mathbf{v}) = \big(\mathbf{u},\mathbf{v}-S(\mathbf{u})\big)$$

which has an inverse given by

$$\xi|^{-1}_{\xi(V_{(\mathbf{0},\mathbf{0})})} : \xi(V_{(\mathbf{0},\mathbf{0})}) \to V_{(\mathbf{0},\mathbf{0})}, \text{ by } \xi|^{-1}_{\xi(V_{(\mathbf{0},\mathbf{0})})}(\mathbf{u},\mathbf{v}) = \big(\mathbf{u},\mathbf{v}+S(\mathbf{u})\big)$$

and so is diffeomorphic onto its image, and hence $(V_{(\mathbf{0},\mathbf{0})},\xi)$ is a smooth chart. And so we get

$$\begin{aligned}\xi\circ\widehat{F}\circ\eta|^{-1}_{\widetilde{U}_{(\mathbf{0},\mathbf{0})}}(\mathbf{x},\mathbf{y}) &= \xi\big(\mathbf{x},S(\mathbf{x})\big)\\ &= \big(\mathbf{x},S(\mathbf{x})-S(\mathbf{x})\big)\\ &= (\mathbf{x},\mathbf{0})\\ &= (x_1,\ldots,x_r,0,\ldots,0)\end{aligned}$$

or

$$\xi\circ\widehat{F}\circ\eta|^{-1}_{\widetilde{U}_{(\mathbf{0},\mathbf{0})}} = \xi\circ(\psi\circ F\circ\phi^{-1})\circ\eta|^{-1}_{\widetilde{U}_{(\mathbf{0},\mathbf{0})}} = (\xi\circ\psi)\circ F\circ(\eta\circ\phi)|^{-1}_{\widetilde{U}_{(\mathbf{0},\mathbf{0})}}$$

so letting $U_p = \phi^{-1}(U_{(\mathbf{0},\mathbf{0})})$, and $V_{F(p)} = \psi^{-1}(V_{(\mathbf{0},\mathbf{0})})$, we have the charts $(U_p, \eta \circ \phi|_{U_p}) \in \mathcal{A}_M$ containing p, and $(V_{F(p)}, \xi \circ \psi|_{V_{F(p)}}) \in \mathcal{A}_N$ containing $F(p)$ such that

$$\begin{aligned}
&\widehat{F}(x_1, \ldots, x_r, x_{r+1}, \ldots, x_m) \\
&\quad = (\xi \circ \psi) \circ F \circ (\eta \circ \phi)|_{\widetilde{U}_{(\mathbf{0},\mathbf{0})}}^{-1}(x_1, \ldots, x_r, x_{r+1}, \ldots, x_m) \\
&\quad = (x_1, \ldots, x_r, 0, \ldots, 0)
\end{aligned}$$

Since $p \in M$ was arbitrary, we conclude that for each point of M there exists charts such that

$$\widehat{F}(x_1, \ldots, x_r, x_{r+1}, \ldots, x_m) = (x_1, \ldots, x_r, 0, \ldots, 0)$$

□