Topology and Analysis Class Notes

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1 Definitions

Topology: Let X be a set, then a topology τ on X is a collection of open subsets such that:

1. $\varnothing$ and X are open. Or, $\varnothing, X \in \tau$.
2. A finite intersection of open sets is open; i.e. for $U_1, \ldots, U_n \in \tau$
$$\bigcap_{i=1}^{n} U_i \in \tau$$
3. An arbitrary union of open sets is open; i.e. $\forall\ U \in \tau$
$$\bigcup_{U \in \tau} U \in \tau$$

in any topological space, the closed sets satisfy the following.

1. $\varnothing$ and X are closed. Or, $\varnothing, X \in \tau^c$.
2. A finite union of closed sets is closed; i.e. for $A_1, \ldots, A_n \in \tau^c$
$$\bigcup_{i=1}^{n} A_i \in \tau^c$$
3. An arbitrary intersection of closed sets is closed; i.e. $\forall\ A \in \tau^c$
$$\bigcap_{A \in \tau^c} A \in \tau^c$$

Discrete Space: A space with the discrete topology; that is, the topology on a set X where each $U \subseteq X$ is declared open, in particular each $\{x\} \in X$ is open.

Ordinary Topology: Let $X = \mathbb{R}$ then a subset $U \subseteq \mathbb{R}$ is open if $\forall\ x \in U\ \exists\ J = (a, b)$ such that $x \in J \subseteq U$.

Normed Vector Space: A normed vector space V over $\mathbb{R}$ is a vector space with a mapping

$$V \to \mathbb{R}$$
$$v \mapsto ||v||$$

such that

1. $||v|| \geq 0$ and $||v|| = 0 \iff v = 0$.
2. If $c \in \mathbb{R}$ and $v \in V$, then $||cv|| = |c| \cdot ||v||$.
3. If $v, u \in V$, then
$$||v + u|| \leq ||v|| + ||u||$$

denoted $(V, || \cdot ||)$.

Cauchy Sequence: let $\{x_n\}_{n\in\mathbb{N}}$ be any sequence in a normed vector space $(V, || \cdot ||)$. The sequence is cauchy if $\forall\ \epsilon > 0\ \exists\ N$ such that $\forall\ n, m \geq N$ we have

$$||x_n - x_m|| < \epsilon$$

Converge: let $\{x_n\}_{n\in\mathbb{N}}$ be any sequence in a normed vector space $(V, || \cdot ||)$. The sequence converges to $v \in V$ if $\forall\ \epsilon > 0\ \exists\ N$ such that $\forall\ n \geq N$ we have

$$||v - x_n|| < \epsilon$$

Sup Norm: Let S be a set. A map

$$f : S \to (V, || \cdot ||)$$

into a normed vector space V is bounded if $\exists\ c \in \mathbb{R}$ with $c > 0$ such that $||f(x)|| \leq c\ \forall\ x \in S$. If f is bounded, define

$$||f||_S := \sup_{x\in S} ||f(x)||$$

called the sup norm.

***L*^{1}-Norm**: Let $C([0, 1])$ be the space of continuous functions on $[0, 1]$. For $f \in C([0, 1])$ define

$$||f||_1 = \int_0^1 |f(x)|dx$$

then $|| \cdot ||_1$ is a norm on $C([0, 1])$ called the L^1-norm.

Uniformly Cauchy Map: A sequence of maps $\{f_n\}_{n\in\mathbb{N}}$ with $f_n : S \to (V, ||\cdot||)$ is uniformly cauchy on a set S if given $\epsilon > 0\ \exists\ N$ such that $\forall\ n, m \geq N$ we have

$$||f_n - f_m||_S < \epsilon$$

Uniformly Convergent Map: A sequence of maps $\{f_n\}_{n\in\mathbb{N}}$ with

$$f_n : S \to (V, ||\cdot||)$$

is uniformly convergent to a map f, if given $\epsilon > 0\ \exists\ N$ such that $\forall\ n \geq N$ we have

$$||f_n - f||_S < \epsilon$$

Uniformly Continuous:

$$f : (X, d_X) \to (Y, d_Y)$$

is uniformly continuous if $\forall\ \epsilon > 0,\ \exists\ \delta > 0$ such that

$$d_X(x_1, x_2) < \delta \implies d_Y\big(f(x_1), f(x_2)\big) < \epsilon$$

Continuous:

$$f : (X, d_X) \to (Y, d_Y)$$

is continuous at $x_0 \in X$ if $\forall\ \epsilon > 0,\ \exists\ \delta > 0$ such that

$$d_X(x, x_0) < \delta \implies d_Y\big(f(x), f(x_0)\big) < \epsilon$$

f is continuous on X if it is continuous at x_0 for all $x_0 \in X$.

Metric Space: Let X be a set, a metric on X is map d with

$$\begin{aligned} d : X \times X &\to \mathbb{R} \\ (x, y) &\mapsto d(x, y) \end{aligned}$$

such that

1. $d(x, y) \geq 0\ \forall\ x, y \in X$ and $d(x, y) = 0 \iff x = y$.
2. $\forall\ x, y \in X$ we have $d(x, y) = d(y, x)$.
3. $\forall\ x, y, z \in X$ we have

$$d(x, z) \leq d(x, y) + d(y, z)$$

a set with a metric is a metric space (X, d).

If $U \subseteq X$ such that $U \neq \varnothing$ then we can define $(U, d|_{U\times U})$ as a metric subspace.

For a normed vector space $(V, ||\cdot||)$, the norm $||\cdot||$ induces a metric

$$d(v, u) := ||v - u||$$

If $A, B \subseteq V$ then

$$d(A, B) = \inf ||a - b||,\ \text{such that } a \in A, b \in B$$

Semi-Metric space: Let X be a set, a semi-metric on X is map d with

$$d : X \times X \to \mathbb{R}$$
$$(x, y) \mapsto d(x, y)$$

such that

1. $d(x, y) \geq 0 \ \forall \ x, y \in X$ and $d(x, x) = 0$. The distinction here being $d(x, y) = 0 \nRightarrow x = y$
2. $\forall \ x, y \in X$ we have $d(x, y) = d(y, x)$.
3. $\forall \ x, y, z \in X$ we have

$$d(x, z) \leq d(x, y) + d(y, z)$$

a set with a semi-metric is a semi-metric space (X, d).

Isometric: For metric spaces (X, d_X) and (Y, d_Y) a map

$$f : X \to Y$$

is isometric if

$$d_X(v, w) = d_Y\big(f(v), f(w)\big) \quad \forall \ v, w \in X$$

if in addition f is surjective, then f is an **Isometric Isomorphism**.

Lipschitz: A function

$$f : (X, d_X) \to (Y, d_Y)$$

is Lipschitz if $\exists \ C \geq 0$ with $C \in \mathbb{R}$, such that

$$d_Y\big(f(x), f(y)\big) \leq C d_X(x, y) \quad \forall \ x, y \in X$$

the smallest such

$$C := L(f)$$

is the Lipschitz constant.

Complete: A metric space X is complete if every Cauchy sequence converges to a point in X; i.e. $\forall \ \{x_i\}_{i=1}^{\infty} \in X, \ x_i \to x \in X$.

Completion: For (X, d) a metric space, the completion of (X, d) is a complete metric space $(X_\sim, d_\sim)$ together with an isometric function

$$f : X \to X_\sim$$

where $f(X) \subseteq X_\sim$ is dense in $X_\sim$.

Profinite Topology: Let G be a group, then $U \subseteq G$ is open if $\forall\ x \in U\ \exists$ a subgroup H of G, of finite index, such that $xH \subseteq U$.

Ideal Topology: Let R be a commutative ring with unity, then $U \subseteq R$ is open if $\forall\ x \in U\ \exists$ an ideal I of R such that $x + I \subseteq U$.

Zariski Topology: An algebraic topology. For instance let $X = \mathbb{R}^n$ and

$$f : \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \to \mathbb{R}$$

be a polynomial in n variables, $\mathbf{a} \in \mathbb{R}^n$ is a zero of f if $f(\mathbf{a}) = \mathbf{0}$, then a subset $S \subseteq \mathbb{R}^n$ is closed if $\exists$ a family $\{f_i\}_{i \in I}$ of polynomials in n variables such that S is the zero set of $\{f_i\}_{i \in I}$. That is

$$S = \{\mathbf{a} \in \mathbb{R}^n : f_i(\mathbf{a}) = \mathbf{0}\ \forall\ i \in I\}$$

Boundary Point: Let (X, τ) be a topological space and $S \subseteq X$ a subset of X, then $x \in X$ is a boundary point of S if $\forall\ U \in \tau$ such that $x \in U$ we have $x \neq s \in S$ and $y \notin S$ such that $s, y \in U$. That is, U contains both a point in S, and a point not in S.

Dense: Let (X, τ) be a topological space and $S \subseteq X$, then S is dense in X if $\overline{S} = X$.

equivalently, S is dense iff for each open $U \subseteq X$ such that $U \neq \varnothing$ there is some $s \in S$ such that $s \in U$.

In terns of metrics, this is $\forall\ x \in X$ and $\epsilon > 0,\ \exists\ s \in S$ such that $d(x, s) < \epsilon$

Base: A collection $\mathcal{B} = \{B_\alpha : \alpha \in I\} \subseteq X$ of open subsets is a base for the topology on X if for every $U \subseteq X$ open, we have $U = \cup_{B_\alpha \in \mathcal{B}} B_\alpha$ for some $\alpha \in I$.
If X is a set and $\mathcal{B}$ a collection of subsets of X satisfying

1.) $X = \displaystyle\bigcup_{B \in \mathcal{B}} B$

2.) if $B_1, B_2 \in \mathcal{B}, x \in B_1 \cap B_2 \implies \exists\ B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$

Then the collection of all unions of elements in $\mathcal{B}$ is a unique topology on X generated by base $\mathcal{B}$

Sub-Base: If $\mathcal{S}$ is a collection of subsets of X such that

$$\bigcup_{V \in \mathcal{S}} V = X$$

and the finite intersection of elements of $\mathcal{S}$ is a base for X, then $\mathcal{S}$ is a sub-base for τ.

Refinement: let X be a set and τ, σ topologies on X then σ is a refinement of τ if for each $U \in \tau$ we also have $U \in \sigma$.

This can also be stated as τ is coarser than σ.

Coarse: Let X be a topological space and let τ_1, τ_2 be two topologies for X. If $\tau_1 \subseteq \tau_2$ then τ_1 is coarser than τ_2.

Fine: Let X be a topological space and let τ_1, τ_2 be two topologies for X. If $\tau_1 \subseteq \tau_2$ then τ_2 is finer than τ_1.

Quotient Topology: If X is a topological space, Y is a set, and $\pi : X \to Y$ is a surjective map, the Quotient Topology on Y determined by π is defined by declaring a subset $U \subseteq Y$ to be open iff $\pi^{-1}(U) \subseteq X$ is open in X. or

$$\tau_Y = \{U \subseteq Y : \pi^{-1}(U) \in \tau_X\}$$

we need the surjectiveness here otherwise if $y \notin \pi(X)$, then $\pi^{-1}(\{y\}) = \varnothing \implies \{y\}$ is open.

equivalently if we define $x_1 \sim x_2$ iff $\pi(x_1) = \pi(x_2)$ then for $Y = X/\sim$ we have

$$\pi : X \to X/\sim$$

is the quotient topology determined by π.

Final Topology: Given $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in \Lambda}$ and a set Y the final topology is the finest topology on Y such that the family

$$\mathcal{F} = \{f_\alpha : X_\alpha \to Y : \alpha \in \Lambda\}$$

is continuous $\forall\ \alpha$; i.e. $U \in \tau_Y$ iff $f_\alpha^{-1}(U) \in \tau_\alpha\ \forall\ \alpha$.

Weak Topology: Let Y be a topological space and let $\mathcal{F}$ be a family of mappings

$$f : X \to Y$$

let

$$\tau_X = \{f^{-1}(W) \subseteq X : W \subseteq Y \text{ is open } ; f \in \mathcal{F}\}$$

then τ_X is the weak topology on X determined by $\mathcal{F}$ and is the coarsest topology on X such that each $f \in \mathcal{F}$ is continuous.

equivalently, let X be a set and $\{Y_\alpha\}$ a family of topological spaces. For each α, let

$$f_\alpha : X \to Y_\alpha$$

be a map. The weak topology on X is the corsest topology making each f_α continuous.

Note: the sub-base for the weak topology has all sets of the form $f_\alpha^{-1}(U)$ where $U \subseteq Y_\alpha$ is open.

Relative Topology: If (X, τ) is a topological space and $S \subseteq X$ is arbitrary, the relative topology is defined by declaring $U \subseteq S$ to be open iff $\exists\ V \in \tau$ such that $U = V \cap S$.

Hausdorff: Suppose X is a topological space. If for every pair of distinct points $x, y \in X$ $\exists\ U, V \subset X$ open, such that $U \cap V = \varnothing$ and $x \in U,\ y \in V$, then X is hausdorff.

Separable: A topological (X, τ) space is separable if it has a countable base.

If (X, d) is a metric space, and has a countable dense subset, then X is separable; i.e. if $A \subset X$ is a countable dense subset then X is separable.

Continuous Map: Let X, Y be topological spaces, a map $f : X \to Y$ is continuous if $\forall$ open $V \subseteq Y$ we have $f^{-1}(V) \subseteq X$ is open.

Note, that if $U \subseteq X$ is open, then $f(U) \subseteq Y$ may not be open.

Product Topology: Let $\{X_i\}_{i \in I}$ be a family of topological spaces and let

$$X = \prod_{i \in I} X_i$$

a topology on X is determined by declaring $U \subseteq X$ to be open if $\forall\ x \in U,\ \exists$ a finite number of indices $i_1, \ldots i_n$ and open subsets $U_{i_j} \subseteq X_{i_j}$ for $i \leq j \leq n$ such that

$$x \in U_{i_1} \times \cdots \times U_{i_n} \times \prod_{i \neq i_1, \ldots i_n} X_i \subseteq U$$

that is the product topology has as base all sets of the form

$$U_{i_1} \times \cdots \times U_{i_n} \times \prod_{i \neq i_1, \ldots i_n} X_i$$

which is to say, arbitrary open sets at a finite number of components and the full space in all other components.

The product topology is the coarsest topology on X such that each projection map

$$\pi_j : X \to X_j$$

is continuous.

Regular: Suppose that one-point sets are closed in (X, τ). Then X is said to be regular if for each pair consisting of a point x and a closed set $A \subset X$ such that $A \cap x = \varnothing$, there exist $U, V \in \tau$ where $U \cap V = \varnothing$, such that

$$x \in U, \text{ and } A \subset V$$

i.e. for closed $A \subseteq X$ with $x \notin A$, $\exists$ disjoint $U, V \in \tau$ with $x \in U$ and $A \subseteq V$.

Normal: Suppose that one-point sets are closed in (X, τ). Then X is normal if for $A, B \subset X$ closed such that $A \cap B = \varnothing$, $\exists\, U, V \in \tau$ with $U \cap V = \varnothing$, such that

$$A \subset U, \text{ and } B \subset V$$

Banach Space: A complete normed vector space.

Topological Convergence: A sequence $\{x_n\}$ in a topological space X is said to converge to $x \in X$, denoted $x_n \to x$, iff for each neighborhood U_x of x, there is some positive integer $N \in \mathbb{N}$ such that $n > N \implies x_n \in U_x$. In this case, we say $\{x_n\}$ is eventually in U_x.

Directed Set: A set Λ is a directed set iff there is a relation $\leq$ on Λ satisfying:

1. $\lambda \leq \lambda$, for each $\lambda \in \Lambda$.
2. If $\lambda_1 \leq \lambda_2$ and $\lambda_2 \leq \lambda_3$ then $\lambda_1 \leq \lambda_3$.
3. If $\lambda_1, \lambda_2 \in \Lambda$ then there is some $\lambda_3 \in \Lambda$ with $\lambda_1 \leq \lambda_3$ and $\lambda_2 \leq \lambda_3$.

Net: A net in a set X is a function

$$\begin{aligned} \Lambda &\to X \\ \lambda &\to x_\lambda \end{aligned}$$

where Λ is some directed set.

If $\{x_\lambda\}_{\lambda \in \Lambda}$ is a net in X, then $x_\lambda \to x$ if for each neighborhood U_x there is some $\lambda_0 \in \Lambda$ such that

$$\lambda \geq \lambda_0 \implies x_\lambda \in U_x$$

so $x_\lambda \to x$ if for every neighborhood U of x we have x_λ is eventually in U.

Cover: Let X be a topological space. A cover of X is a collection $\mathcal{U}$ of subsets of X whose union is X; i.e.

$$\bigcup_{U \in \mathcal{U}} U = X$$

a subcover is a subcollection of $\mathcal{U}$ that is still a cover, i.e. $\mathcal{U}' \subset \mathcal{U}$ where

$$\bigcup_{U \in \mathcal{U}'} U = X$$

$\mathcal{U}$ is an open cover if each $U \in \mathcal{U}$ is open.

Compact: A topological space X is compact if every open cover; i.e. $\bigcup_{U \in \mathcal{U}} U = X$, has a finite subcover.

A compact subset $S \subseteq X$ of a topological space X, is one that is a compact space in the relative topology.

Finite Intersection Property: Let X be a topological space, and $\{A_\alpha\}_{\alpha \in I}$ a family of nonempty subsets of X. Then $\{A_\alpha\}_{\alpha \in I}$ has the finite intersection property if every finite subcollection of $\{A_\alpha\}_{\alpha \in I}$ has nonempty intersection; i.e. $\{A_{i_1}, \ldots, A_{i_n}\} \subset \{A_\alpha\}_{\alpha \in I}$ gives

$$\bigcap_{j=1}^{i_n} A_{i_j} \neq \varnothing$$

for all subsets such that $|\{A_{i_1}, \ldots, A_{i_n}\}| < \infty$.

Disconnected: A topological space X is disconnected if it has 2 disjoint nonempty open subsets whose union is X; i.e. $U, V \subset X$ open, such that

$$U \neq \varnothing, \quad V \neq \varnothing, \text{ where } U \cap V = \varnothing, \quad \text{and } U \cup V = X$$

Connected: A topological space X is connect if it is not disconnected. Equivalently it is connected iff its' only subsets which are both open and closed are: $\varnothing$, and X itself.

A connected subset of X is a subset that is a connected space when endowed with the subspace topology.

Axiom of Choice: For any collection $\mathcal{C}$ of non-empty sets, there's is a set that contains exactly one element for each $A \in \mathcal{C}$.

Partially Ordered Set: A pair $(P, \leq)$ such that.

1. $x \leq x \; \forall \; x \in P$.
2. $x \leq y$ and $y \leq z \implies x \leq z$.
3. If $x \leq y$ and $y \leq x$, then $x = y$

a totally ordered set also satisfies: $\forall \; x, y \in P$

$$x \leq y \text{ or } y \leq x$$

.

Chain: A chain in P is a subset $\mathcal{C}$ of P that is totally ordered in the partial order of P.

Inductively Ordered: Say that P is inductively ordered if for any chain $\mathcal{C}$ in P there is an $a \in P$, possibly in $\mathcal{C}$, such that $c \leq a \;\forall\; c \in \mathcal{C}$ so a is an upper bound for $\mathcal{C}$.

i.e. a partially ordered set P is inductively ordered if every chain has an upper bound.

Maximal: $m \in P$ is a maximal element if $a \geq m \implies a = m$. Not unique, can have many maximal elements.

Zorn's Lemma: if a partially ordered set P is inductively ordered then P has at least one maximal element.

Bounded: let (X, d) be a metric space. A subset $A \subseteq X$ is bounded if $\exists\; C \in \mathbb{R}^+$ such that

$$d(x, y) \leq C \quad \forall\; x, y \in A$$

if X is a set and (Y, d) a metric space, then

$$f : X \to Y$$

is bounded if $f(X) \subseteq Y$ is bounded.

Equicontinuous: let (X, τ) be a topological space and (Y, d) a metric space, and let $\mathcal{F} \subseteq C(X, Y)$. Then $\mathcal{F}$ is equicontinuous at x if $\forall\; \epsilon > 0 \;\exists\; O_x \in \tau$ such that $\forall\; f \in \mathcal{F}$ and any $y \in O_x$ we have

$$d\big(f(x), f(y)\big) < \epsilon$$

$\mathcal{F}$ is equicontinuous if it is equicontinuous at x, $\forall\; x \in X$.

Totally Bounded: let (X, d) be a metric space a subset A is totally bounded if $\forall\; \epsilon > 0$, A can be covered by a finite number of open ϵ-balls; i.e.

$$A \subseteq \bigcup_{i=1}^{n} B_\epsilon^i$$

Any subset of a totally bounded set is totally bounded.

Pointwise Totally Bounded: let (X, τ) be a topological space and (Y, d) a metric space. Given $\epsilon > 0$ and $x \in X$ if $\exists\; g_j \in C_B(X, Y)$ such that

$$d\big(f(x), g_j(x)\big) < \epsilon$$

Then $\big\{B_\epsilon(g_j(x))\big\}_{i=1}^{n}$ covers $\{f(x) \,:\, f \in \mathcal{F}\}$ and so $\mathcal{F}$ is pointwise totally bounded.

Locally Compact: let (X, τ) be a topological space. then X is locally compact if $\forall\ x \in X,\ \exists\ O \in \tau$ with $x \in O$ such that $\overline{O}$ is compact.

Ring: Let X be a set, a nonempty collection of subsets $\mathcal{R} \subseteq \mathcal{P}(X)$ is a ring if

1. $E, F \in \mathcal{R} \implies E \cup F \in \mathcal{R}$. Closure under set union.
2. $E, F \in \mathcal{R} \implies E \setminus F \in \mathcal{R}$. Closure under set difference.
   This also implies that $\mathcal{R}$ is closed under intersection as

$$\begin{aligned} E \setminus (E \setminus F) &= E \setminus (E \cap F^c) \\ &= E \cap \big(E \cap F^c\big)^c \\ &= E \cap (E^c \cup F) \\ &= (E \cap E^c) \cup (E \cap F) \\ &= \varnothing \cup (E \cap F) \\ &= (E \cap F) \end{aligned}$$

This also implies, by induction, that a ring $\mathcal{R}$ is closed under finite unions and intersections; i.e. if $E_1, \ldots, E_n \in \mathcal{R}$ then

$$\bigcup_{i=1}^{n} E_i \in \mathcal{R}$$

and

$$\bigcap_{i=1}^{n} E_i \in \mathcal{R}$$

as well as $\varnothing \in \mathcal{R}$. Since if $E \in \mathcal{R}$ then

$$E \setminus E = \varnothing \in \mathcal{R}$$

If, in addition, $X \in \mathcal{R}$, then $\mathcal{R}$ is a **Field** or **Algebra**.

$\boldsymbol{\sigma}$-Ring: Let X be a set, a nonempty collection of subsets $\mathcal{S} \subseteq \mathcal{P}(X)$ is a σ-ring if it is a ring and, in addition, is closed under countable unions; i.e. if $E_1, E_2, \cdots \in \mathcal{S}$ then

$$\bigcup_{i=1}^{\infty} E_i \in \mathcal{S}$$

where this also implies closure under countable intersection since if $F = \bigcup_{i=1}^{\infty} E_i$ then

$$\bigcap_{i=1}^{\infty} E_i = F \setminus \left(\bigcup_{i=1}^{\infty} (F \setminus E_i) \right)$$

If, in addition, $X \in \mathcal{S}$, then $\mathcal{S}$ is a **$\boldsymbol{\sigma}$-Field** or **$\boldsymbol{\sigma}$-Algebra**.

$\boldsymbol{\sigma}$-Algebra: Let X be a set, a collection of subsets $\mathcal{A} \subseteq \mathcal{P}(X)$ is a σ-algebra in X if it satisfies

1. Nonemptiness: $\mathcal{A} \neq \varnothing$.
2. Closure under Compliments: If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$.
3. Closure under Countable Unions: If $A_1, A_2 \cdots \in \mathcal{A}$, then

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$$

   this also implies closure under countable intersection as

$$\bigcap_{i=1}^{\infty} A_i = \left(\bigcup_{i=1}^{\infty} A_i^c \right)^c \in \mathcal{A}$$

Generated σ-Algebra: Let X be a set and $\mathcal{S}$ a collection of subsets of X, then the σ-algebra generated by $\mathcal{S}$ is the intersection of all σ-algebras containing $\mathcal{S}$ denoted $\sigma(\mathcal{S})$; that is

$$\sigma(\mathcal{S}) = \bigcap_{\mathcal{S} \subseteq \mathcal{A}} \mathcal{A}$$

Borel Sets: Let (X, τ) be a topological space, then $\sigma(\tau)$ is the σ-ring of Borel sets of X.

Measure: Let X be a set with σ-ring $\mathcal{R}$. A measure is a function

$$\mu : \mathcal{R} \to [0, \infty]$$

satisfying

1. $\mu(\varnothing) = 0$.
2. **Countable Additivity**: If $E_1, E_2, \cdots \in \mathcal{R}$ are mutually disjoint; i.e. $E_i \cap E_j = \varnothing$ whenever $i \neq j$. Then

$$\mu\left(\bigsqcup_{i=1}^{\infty} E_i \right) = \sum_{i=1}^{\infty} \mu(E_i)$$

   this also holds for finite additivity; i.e. for $E_1, \dots E_n \in \mathcal{R}$ mutually disjoint we have $\mu\left(\bigsqcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu(E_i)$ by simply setting $E_k = \varnothing \ \forall \ k > n$.

Semiring: Let X be a set, a collection of subsets $\mathcal{S} \subseteq \mathcal{P}(X)$ is a semiring if

1. $\varnothing \in \mathcal{S}$.
2. If $E, F \in \mathcal{S} \implies E \cap F \in \mathcal{S}$.
3. If $E, F \in \mathcal{S}$ then $\exists \ E_1, \dots, E_n \in \mathcal{S}$ such that

$$E \setminus F = \bigsqcup_{i=1}^{n} E_i$$

Premeasure: Let $\mathcal{S}$ be a semiring, then the function

$$\mu_0 : \mathcal{S} \to [0, \infty]$$

is a premeasure if it is countably additive.

Monotone: If $\mathcal{C}$ is any collection of subsets of a set X, and if $\mu : \mathcal{C} \to \mathbb{R}^+$ is any function, we say that μ is monotone if whenever $E, F \in \mathcal{C}$ and $E \subseteq F$, then $\mu(E) \leq \mu(F)$

Countable Sub-Additive: Let $\mathcal{C}$ be a family of subsets of X and $\mu : \mathcal{C} \to \mathbb{R}^+$ a mapping. We say that μ is countably sub-additive if whenever $E \subseteq \bigcup_{j=1}^{\infty} F_j$ not necessarily disjoint with $E, \{F_j\}_{j=1}^{\infty} \in \mathcal{C}$, then

$$\mu(E) \leq \sum_{j=1}^{\infty} \mu(F_j)$$

Countably Covered: Let $\mathcal{S}$ be a collection of subsets of the set X. Then $A \subset X$ is countably covered by $\mathcal{S}$ if $\exists \{E_i\}_{i=1}^{\infty} \in \mathcal{S}$ such that

$$A \subseteq \bigcup_{i=1}^{\infty} E_i$$

Let $\mathcal{H}(\mathcal{S})$ be the collection of all sets countably covered by $\mathcal{S}$, then $\mathcal{H}(\mathcal{S})$ is a σ-ring and is **Hereditary** meaning if $E \in \mathcal{H}(\mathcal{S})$ and $F \subseteq E$ then $F \in \mathcal{H}(\mathcal{S})$.

Outer Measure: Let $\mathcal{H}$ be a hereditary σ-ring of subsets of X, then

$$\mu^* : \mathcal{H} \to [0, \infty]$$

is an outer measure if

1. $\mu^*(\varnothing) = 0$
2. μ^* is monotone; i.e. if $F \subseteq E$ and $E \in \mathcal{H}$, then

$$\mu^*(F) \leq \mu^*(E)$$

3. μ^* is countably subadditive; i.e. if $F \subseteq \bigcup_{i=1}^{\infty} E_i$ where $E_i \in \mathcal{H}$, then

$$\mu^*(F) \leq \sum_{i=1}^{\infty} \mu^*(E_i)$$

If $\mathcal{S}$ is a semiring and μ_0 a premeasure on $\mathcal{S}$, and μ^* the outer measure on $\mathcal{H}(\mathcal{S})$ determined by μ_0 then

1. $\mu^*|_{\sigma(\mathcal{S})}$ is a measure on the σ-ring generated by $\mathcal{S}$ which extends μ_0.

2. $\mu^*|_{M(\mu^*)}$ is a complete measure on the σ-ring $M(\mu^*)$ which extends $\mu^*|_{\sigma(\mathcal{S})}$ and hence μ_0.

Measurable: Given a hereditary σ-ring $\mathcal{H}$ and an outer measure μ^* on $\mathcal{H}$, $E \in \mathcal{H}$ is μ^*-measurable if for every $A \in \mathcal{H}$ we have

$$\mu^*(A) = \mu^*\big(A \cap E\big) + \mu^*\big(A \setminus E\big)$$

the collection of all μ^*-measurable sets is denoted $M(\mu^*)$.

Note: by the subadditivity of μ^* we always have $\mu^*(A) \leq \mu^*\big(A \cap E\big) + \mu^*\big(A \cap E^c\big)$

Complete Measure: Let $\mathcal{R}$ be a σ-ring and μ a measure on $\mathcal{R}$. Then μ is complete if whenever $E \in \mathcal{R}$ and $\mu(E) = 0$, then for all $F \subseteq E$ we have $F \in \mathcal{R}$ and $\mu(F) = 0$

$\boldsymbol{\sigma}$-Finite: Let $\mathcal{S}$ be a collection of subsets of X, and let $\mu : \mathcal{S} \to [0, \infty]$ be a set function. Then $E \subseteq X$ is σ-finite if $\exists\ \{F_i\} \in \mathcal{S}$ such that $E \subseteq \bigcup_{i=1}^{\infty} F_i$ and $\mu(F_i) < \infty\ \forall\ i$.

If each $E \in \mathcal{S}$ is σ-finite, then μ is σ-finite.

If X is σ-finite, then μ is **Totally $\boldsymbol{\sigma}$-Finite**.

Simple $\boldsymbol{\mathcal{S}}$-Measurable Function: Let X be a set and $\mathcal{S}$ a σ-ring of subsets of X, and B a Banach Space. Then a function

$$f : X \to B$$

is a simple $\mathcal{S}$-measurable function if

1. $\mathrm{Im}(f) = \{b_1, \ldots, b_n\} \in B$ is finite.
2. For each $b_i \in B$ such that $b_i \neq 0$ we have $f^{-1}(b_i) = E_i \in \mathcal{S}$.

the family $\mathcal{F}$ of B-valued simple $\mathcal{S}$-measurable functions are functions of the form

$$f = \sum_{i=1}^{n} b_i \chi_{E_i}, \text{ with } \chi_{E_i}(x) = \begin{cases} 1, & x \in E_i \\ 0, & \text{otherwise} \end{cases}$$

where the b_i's are distinct and the E_i's $\in \mathcal{S}$ are disjoint.

Note: simple $\mathcal{S}$-measurable $\implies$ simple μ-measurable.

$\boldsymbol{\mathcal{S}}$-Measurable Function: Let X be a set and $\mathcal{S}$ a σ-ring of subsets of X, and B a Banach Space. Then a function

$$f : X \to B$$

is a $\mathcal{S}$-measurable function if $\exists\ \{f_n\}_{n\in\mathbb{N}}$ of simple $\mathcal{S}$-measurable functions such that $f_n \to f$ pointwise; i.e. $\forall\ x \in X$ we have $f_n(x) \to f(x)$.

Note: $\mathcal{S}$-measurable $\implies$ μ-measurable.

Null-Set: Let X be a set, $\mathcal{S}$ a σ-ring of subsets of X, and μ a measure on $\mathcal{S}$. A subset $E \subset X$ is a null-set with respect to μ if $\exists\ F \in \mathcal{S}$ such that $E \subseteq F$ and $\mu(F) = 0$. The null-sets form a hereditary σ-ring denoted $N(\mu)$.

that is E is contained in some set of $\mathcal{S}$ of measure zero.

Almost Everywhere: Let X be a set, $\mathcal{S}$ a σ-ring of subsets of X, and μ a measure on $\mathcal{S}$. A property P on X is said to hold almost everywhere if $\exists\ N(\mu)$ such that P is true $\forall\ x \in X \setminus N(\mu)$.

Simple $\boldsymbol{\mu}$-Measurable: Let X be a set, $\mathcal{S}$ a σ-ring of subsets of X, μ a measure on $\mathcal{S}$, and let B be a Banach space. Then a function

$$f : X \to B$$

is a simple μ-measurable function if f is a simple $\big(\mathcal{S} \sqcup N(\mu)\big)$-measurable function. where

$$\mathcal{S} \sqcup N(\mu) = \big\{E \sqcup F : E \in \mathcal{S},\ F \in N(\mu)\big\}$$

$\boldsymbol{\mu}$-Measurable: Let X be a set, $\mathcal{S}$ a σ-ring of subsets of X, μ a measure on $\mathcal{S}$, and let B be a Banach space. Then a function defined almost everywhere on X

$$f : X \setminus N(\mu) \to B$$

is a μ-measurable function if $\exists\ \{f_n\}_{n\in\mathbb{N}}$ of simple μ-measurable functions such that $f_n \to f$ pointwise; i.e. $\forall\ x \in X \setminus N(\mu)$ we have $f_n(x) \to f(x)$.

Carrier: Let X be a set and let B be a Banach space. For any function

$$f : X \to B$$

the carrier of f denoted

$$\operatorname{car}(f) = \{x \in X : f(x) \neq 0 \in B\}$$

similar to the support.

Almost Uniformly: Let $(X, \mathcal{S}, \mu)$ be a measure space, let $\{f_n\}$ be a sequence of μ-measurable functions, and let $E \in \mathcal{S}$. Then $f_n \to f$ almost uniformly on E, if $\forall\ \epsilon > 0\ \exists\ F \in \mathcal{S}$ such that $F \subseteq E$ where

$$\mu(E \setminus F) < \epsilon$$

and $f_n \to f$ uniformly on F.

By Egoroff's Theorem, if we have a sequence $\{f_n\}$ of μ-measurable functions such that $f_n \to f$ pointwise on a set of finite measure, then $f_n \to f$ almost uniformly; i.e. if $\forall\ x \in E \setminus N(\mu)$ we have $f_n(x) \to f(x)$, then $f_n \to f$ almost uniformly on E.

Almost Uniformly Cauchy: Let $(X, \mathcal{S}, \mu)$ be a measure space, let B a Banach space, let $\{f_n\}$ be a sequence of μ-measurable functions, and let $E \in \mathcal{S}$. Then $f_n \to f$ almost uniformly cauchy on E, if $\forall\ \epsilon > 0\ \exists\ F \in \mathcal{S}$ such that $F \subseteq E$ where

$$\mu(E \setminus F) < \epsilon$$

such that $\{f_n\}$ is uniformly cauchy on F; i.e. $\forall\ \delta > 0\ \exists\ N$ such that

$$m, n \geq N \implies ||f_m(x) - f_n(x)||_B < \delta \quad \forall\ x \in F$$

Converges in Measure: Let $(X, \mathcal{S}, \mu)$ be a measure space with $E \in \mathcal{S}$, let B a Banach space, and let $\{f_n\}$ be a sequence of $\mathcal{S}$-measurable B-valued functions, then $\{f_n\}$ converges in measure on E to $f \in \mathcal{S}$-measurable if $\forall\ \epsilon > 0$

$$\mu(\{x \in E : ||f(x) - f_n(x)|| \geq \epsilon\}) \to 0 \text{ as } n \to \infty$$

Note: when dealing with these sets we must have

$$\begin{aligned}\{x \in E :& ||f(x) - g(x)||_B > \epsilon\} \\ &\subseteq \left\{x \in E : ||f(x)||_B > \frac{\epsilon}{2}\right\} \cup \left\{x \in E : ||g(x)||_B > \frac{\epsilon}{2}\right\}\end{aligned}$$

and NOT

$$\begin{aligned}\{x \in E :& ||f(x) - g(x)||_B > \epsilon\} \\ &\subseteq \{x \in E : ||f(x)||_B > \epsilon\} \cup \{x \in E : ||g(x)||_B > \epsilon\}\end{aligned}$$

consider

$$|a| < \frac{\epsilon}{2} \text{ and } |b| < \frac{\epsilon}{2} \implies |a + b| \leq |a| + |b| < \epsilon$$

then taking the negation we have

$$|a + b| \geq \epsilon \implies |a| \geq \frac{\epsilon}{2} \text{ or } |b| \geq \frac{\epsilon}{2}$$

for a concrete example in our case note that if $f(x) = \frac{\epsilon}{2}$ and $g(x) = -\frac{\epsilon}{2}$, then

$$f(x) - g(x) = \epsilon \implies x \in \{x \in E : ||f(x) - g(x)||_B > \epsilon\}$$

yet

$$x \notin \{x \in E : ||f(x)||_B > \epsilon\} \text{ and } x \notin \{x \in E : ||g(x)||_B > \epsilon\}$$

and so

$$\{x \in E : ||f(x) - g(x)||_B > \epsilon\} \supset \{x \in E : ||f(x)||_B > \epsilon\} \cup \{x \in E : ||g(x)||_B > \epsilon\}$$

Cauchy in Measure: Let $(X, \mathcal{S}, \mu)$ be a measure space with $E \in \mathcal{S}$, let B a Banach space, and let $\{f_n\}$ be a sequence of $\mathcal{S}$-measurable B-valued functions, then $\{f_n\}$ is cauchy in measure on E if $\forall\ \epsilon > 0$

$$\mu(\{x \in E : ||f_m(x) - f_n(x)|| \geq \epsilon\}) \to 0 \text{ as } n, m \to \infty$$

Simple Integrable Function: Let X be a set and $\mathcal{S}$ a σ-ring of subsets of X, and B a Banach Space. Then a function

$$f : X \to B$$

if it is a simple $\mathcal{S}$-measurable function and the preimage of each $b \in \text{Im}(f)$ has finite measure; i.e. for each $f^{-1}(b) = E \in \mathcal{S}$ we have $\mu(E) < \infty$. Then the integral of $f = \sum_{i=1}^{n} b_i \chi_{E_i}$ is

$$\int f d\mu = \sum_{i=1}^{n} b_i \mu(E_i)$$

L^1 Semi-norm: Let $(X, \mathcal{S}, \mu)$ be a measure space and let B a Banach Space. Then a function

$$f : X \to B$$

that is a simple integrable function, has semi-norm $|| \cdot ||_1$ defined by

$$||f||_1 = \int ||f(x)||_B d\mu(x)$$

Mean Cauchy: Let $(X, \mathcal{S}, \mu)$ be a measure space and let B a Banach Space. Then a sequence $\{f_n\}$ of simple integrable functions is mean cauchy if it is a cauchy sequence with respect to $|| \cdot ||_1$; i.e.

$$\lim_{n,m} ||f_n - f_m||_1 = 0$$

μ-integrable: Let f be a $\mathcal{S}$-measurable B-valued function, then f is μ-integrable if it satisfies one, and hence all, of the conditions.

1. There is a mean cauchy sequence $\{f_n\}$ of ISFs that converge in measure to f.
2. There is a mean cauchy sequence $\{f_n\}$ of ISFs such that $f_n \to f$ almost uniformly.
3. There is a mean cauchy sequence $\{f_n\}$ of ISFs such that $f_n \to f$ pointwise almost everywhere.

with the μ-integral of f defined by

$$\int f d\mu = \lim_{n \to \infty} \int f_n d\mu$$

$\boldsymbol{\mathcal{L}^1(X, \mathcal{S}, \mu, B)}$: The vector space of μ-integrable B-valued functions; i.e. if $f \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$, then $\exists$ a mean cauchy sequence $\{f_n\}$ of simple integrable functions such that $f_n \to f$ in measure, almost uniformly, and pointwise almost everywhere.

If $f \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$ then $x \mapsto ||f(x)||_B \in \mathcal{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$.

Convergence in Mean: Let $(X, \mathcal{S}, \mu)$ be a measure space and let B a Banach Space. Then a sequence $\{f_n\}$ of simple integrable functions converges in mean to a μ-integrable function f if

$$\lim_n ||f - f_n||_1 = 0$$

$\boldsymbol{L^1(X, \mathcal{S}, \mu, B)}$: The complete normed vector space defined by

$$L^1(X, \mathcal{S}, \mu, B) = \mathcal{L}^1(X, \mathcal{S}, \mu, B)/\sim$$

where $\sim$ is the equivalence class of simple integrable functions which are mean cauchy.

Indefinite Integral: Let $(X, \mathcal{S}, \mu)$ be a measure space and let B a Banach Space. for $f \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$ and $E \in \mathcal{S}$ the indefinite integral of f is

$$\mu_f(E) = \int_E f(x) d\mu(x) = \int f \chi_E d\mu$$

$\boldsymbol{L^p}$**-Norm**: Let $(X, \mathcal{S}, \mu)$ be a measure space and let B a Banach Space. For $0 < p < \infty$ the space of μ-measurable, B-valued functions f such that $||f(\cdot)||^p$ is μ-integrable is denoted $\mathcal{L}^p(X, \mathcal{S}, \mu, B)$, then the function

$$|| \cdot ||_p : \mathcal{L}^p(X, \mathcal{S}, \mu, B) \to \mathbb{R}$$

defined by

$$||f||_p = \left(\int ||f(x)||^p d\mu(x) \right)^{1/p}$$

is the L^p-norm.

Note: if $f \in \mathcal{L}^p(X, \mathcal{S}, \mu, B)$, then $x \mapsto ||f(x)||^p \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$.

2 Notes

Equality of topologies. If τ, τ' are two topologies on X, then they are equal iff

- $\forall\, x \in X;\ U \in \tau$ with $x \in U,\ \exists\, U' \in \tau'$ such that $x \in U' \subseteq U$.

- $\forall\ x \in X;\ U' \in \tau'$ with $x \in U',\ \exists\ U \in \tau$ such that $x \in U \subseteq U'$.

Two norms $|\cdot|_1$ and $|\cdot|_2$ are equivalent on a vector space V iff $\exists\ c_1, c_2 > 0$, such $\forall\ v \in V$ we have

$$c_1|v|_1 \leq |v|_2 \leq c_2|v|_1$$

Proposition 1.1: Let X, Y be normed vector spaces and let $f : X \to Y$ be a map. Then f is continuous iff the usual (ϵ, δ) definition is satisfied at every point of X.

Proposition 1.2: Let X be a metric space (or a subset of a normed vector space) and let $f : X \to Y$ be a map into a normed vector space Y. Then f is continuous iff the following condition is satisfied. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X converging to a point $x \in X$. Then $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x) \in Y$.

Proposition 1. Proposition 2.12. Let (X, d) be a pseudometric space. If members x and y of X are called equivalent whenever $d(x, y) = 0$, then the result is an equivalence relation. Denote by $[x]$ the equivalence class of x and by $X\backslash \sim$ the set of all equivalence classes. The definition $d_\sim([x], [y]) = d(x, y)$ consistently defines a function

$$d_\sim : \Big(X\backslash \sim\Big) \times \Big(X\backslash \sim\Big) \to \mathbb{R}$$

and $(X\backslash \sim, d_\sim)$ is a metric space. A subset $U \subseteq X$ is open if and only if two conditions are satisfied: U is a union of equivalence classes; i.e $U = \cup_{i\in I}[x_i]$, and the set $U_\sim = \{[x_{i_1}], \ldots, \}$ of such classes is an open subset of $X\backslash \sim$.

Proof. The reflexive, symmetric, and transitive properties of $\sim$ are immediate from the defining properties of a metric.
Next let $x, x' \in [x]$ and $y, y' \in [y]$ then

$$\begin{aligned} d(x, y) &\leq d(x, x') + d(x', y') + d(y', y) \\ &= 0 + d(x', y') + 0 \\ &= d(x', y') \end{aligned}$$

similarly $d(x', y') \leq d(x, y)$ and thus, $d(x', y') = d(x, y)$ and so $d_\sim$ is well defined, where $d_\sim$ inherits the properties of a metric from d.
Next $x \in X$ be arbitrary, suppose $U \subseteq X$ is open such that $x \in U$ and let $x' \sim x$. Since U is open $\exists\ B_r(x) \subseteq U$, since $x' \sim x$ we also have $d(x, x') = 0$ and hence $x' \in B_r(x) \implies x' \in U$ that is $[x] \in U$, since x was arbitrary we conclude that $U = \cup_{i\in I}[x_i]$.
Now let $U = \cup_{i\in I}[x_i]$ and $U_\sim = \{[x_{i_1}], \ldots, \}$, if $x \in U$ then for all $y \in [x]$ we have $U_\sim \supseteq B_r([x]) = B_r(x) \subseteq U$, and so $U \subseteq X$ is open iff $U_\sim \subseteq X\backslash \sim$ is open. $\square$

Proposition 2. Proposition 2.22. If (X, d) is a metric space, then

(a) for any subset U of X and limit point x of U, there exists a sequence in $U \setminus \{x\}$ converging to x.

(b) Any convergent sequence in X with limit $x \in X$ either has infinite image, with x as a limit point of the image, or else is eventually constantly equal to x.

Proof.

(a) Let $U \subseteq X$ and let x be a limit point of U, then for each $n \geq 1$

$$B_{\frac{1}{n}}(x)$$

is an open neighborhood of x, and since x is a limit point of U, $\exists\ x_n \in U$ such that $x_n \in B_{\frac{1}{n}}(x)$. Then

$$d(x_n, x) = \frac{1}{n}$$

thus, we have $(x_n)_{n\in\mathbb{N}} \in U \setminus \{x\}$ where $x_n \to x$.

(b) Suppose $x_n \to x$ and has infinite image. So

$$\{x_n\}_{n\in\mathbb{N}} \setminus \{x_i : x_i = x\} := \{x_{n_k}\}$$

is a subsequence such that $x_{n_k} \to x$. If U_x is an open neighborhood of x, then $\{x_{n_k}\}$ is eventually in U_x, by the assumption of convergence. Since by construction $\nexists\ x_{n_i} \in \{x_{n_k}\}$ such that $x_{n_i} = x$ and so $x_{n_i} \in \{x_{n_k}\} \subseteq \{x_n\}$ where $x \neq x_{n_i} \in U_x$, since U_x was arbitrary we conclude x is a limit point of $\{x_n\}$, or x is a limit point of the image of $\{x_n\}$.
Next suppose that $x_n \to x$, yet has finite image, say $\{p_1, \ldots, p_i\}$, meaning the sequence repeats values as n ranges over $\mathbb{N}$. If for some particular index j we have $x_n = p_j$ for infinitely many n, then $\exists$ a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p_j$. Yet, since $x_n \to x$, every convergent subsequence must also converge to x and thus $p_j = x$. So, for $m \neq j$ there are only finitely many $x_n = p_m$, since the image of $\{x_n\}$ is finite, and so $\{x_n\}$ must eventually be $p_j = x$ constantly.

□

Corollary 3. Corollary 2.23. If (X, d) is a metric space, then a subset A of X is closed if and only if every convergent sequence in A has its limit in A.

Proof. Suppose that A is closed and that $(x_n)_{n\in\mathbb{N}} \in A$ such that x is a limit point of (x_n). Then by Proposition 2 (b) either $x \in (x_n) \in A$ or $x_n \to x \in A$ as a closed subset contains all of its limit points. Therefore, the limit of any convergent sequence $(x_n)_{n\in\mathbb{N}} \in A$ also belongs to A.
Next suppose that every convergent sequence $(x_n)_{n\in\mathbb{N}} \in A$ also has its limit in A. If x is a limit point of A, then by Proposition 2 (a) $\exists\ (x_n) \in A \setminus \{x\}$ such that $x_n \to x$, and by assumption $x \in A$ and therefore A contains all its limit points and thus, is closed. □

Proposition 4. Isometries are injective and uniformly continuous.

Proof. Let
$$f : (X, d_X) \to (Y, d_Y)$$
be an isometric map between metric spaces and let $\epsilon > 0$ be given. Select $\delta = \epsilon > 0$, then for any $x, y \in X$ such that $d_X(x, y) < \delta$ gives

$$d_Y\big(f(x), f(y)\big) = d_X(x, y) < \delta = \epsilon$$

and therefore f is uniformly continuous.

Next, take $a, b \in X$ such that $f(a) = f(b)$, then

$$d_X(a, b) = d_Y\big(f(a), f(b)\big) = 0 \implies a = b$$

and so f is injective. □

Proposition 5. If
$$f : (X, d_X) \to (Y, d_Y)$$
is an isometry, then
$$f^{-1} : \big(f(X), d_Y\big) \to (X, d_X)$$
is an isometry.

Proof. Let f be an isometry and let $x, y \in f(X)$, then $\exists\ a, b \in X$ such that

$$f(a) = x \text{ and } f(b) = y \implies a = f^{-1}(x) \text{ and } b = f^{-1}(y)$$

then

$$\begin{aligned} d_Y(x, y) &= d_Y\big(f(a), f(b)\big) \\ &= d_X(a, b) && f \text{ is an isometry} \\ &= d_X\big(f^{-1}(x), f^{-1}(y)\big) \end{aligned}$$

and hence, f^{-1} is an isometry. □

Proposition 6. If (M_1, d_1) and (M_2, d_2) are metric spaces, then Lipschitz continuous implies uniformly continuous.

Proof. Let (M_1, d_1) and (M_2, d_2) be metric spaces and $f : M_1 \to M_2$ a lipschitz continuous map. Since f is lipschitz $\exists\ L(f) \in \mathbb{R}^+$ such that for any $x, y \in M_1$ we have
$$d_2\big(f(x), f(y)\big) \leq L(f) \cdot d_1(x, y)$$
if $y = x$ then $d_2\big(f(x), f(x)\big) = 0$ as well as $d_1(x, x) = 0$ so that for any $\epsilon > 0, \exists\ \delta > 0$ where we have

$$d_1(x, x) = 0 < \delta \implies L(f) d_2\big(f(x), f(x)\big) = 0 < \epsilon$$

so let $y \neq x$, then $d_1(x,y) \neq 0$, so for $\delta(\epsilon) > 0$ such that $d_1(x,y) < \delta(\epsilon)$, selecting $\delta(\epsilon) = \frac{\epsilon}{L(f)} > 0$ we have

$$d_2\big(f(x), f(y)\big) \leq L(f) \cdot d_1(x,y) < L(f) \cdot \delta(\epsilon) = L(f) \cdot \frac{\epsilon}{L(f)} = \epsilon$$

and so

$$d_1(x,y) < \delta(\epsilon) \implies d_2\big(f(x), f(y)\big) < \epsilon$$

and so f is uniformly continuous. $\square$

Proposition 7.

$$f : (X, d_X) \to (Y, d_Y)$$

is continuous iff

$$x_n \to x \implies f(x_n) \to f(x)$$

Proof. First suppose f is continuous and that $x_n \to x \in X$. Let $\epsilon > 0$ be given and $B_\epsilon\big(f(x)\big) \subseteq Y$ be open such that $f(x) \in B_\epsilon\big(f(x)\big)$. Then since f is continuous $f^{-1}\Big(B_\epsilon\big(f(x)\big)\Big) \subseteq X$ is open and contains x. Then, since $x_n \to x, \forall\ \delta > 0\ \exists\ N \in \mathbb{N}$ such that $n \geq N \implies d_X(x_n, x) < \delta$ which implies

$$\begin{aligned} B_\delta(x) \subseteq f^{-1}\Big(B_\epsilon\big(f(x)\big)\Big) &\implies x_n \in f^{-1}\Big(B_\epsilon\big(f(x)\big)\Big) \\ &\implies f(x_n) \in B_\epsilon\big(f(x)\big) \\ &\implies d_Y\big(f(x_n), f(x)\big) < \epsilon \end{aligned}$$

and so $f(x_n) \to f(x)$.

Next suppose $x_n \to x \implies f(x_n) \to f(x)$. And assume, for contradiction, that f is not continuous. Then $\forall\ \epsilon > 0$ with $\delta = \frac{1}{n}$ we have

$$d_X(x_n, x) < \frac{1}{n}$$

yet,

$$d_Y\big(f(x_n), f(x)\big) \geq \epsilon$$

and doing this for each n we have $d(x_n, x) \to 0$ while $d_Y\big(f(x_n), f(x)\big) \geq \epsilon\ \forall\ n \Rightarrow\Leftarrow$. And so f must be continuous. $\square$

Proposition 8. If S is dense in X, and

$$f, g : X \to Y$$

are continuous maps such that $f(s) = g(s)\ \forall\ s \in S$, then $f = g$ on X.

Proof. Let $x \in X \setminus S = S^c$ and let $\epsilon > 0$ be given. Then by continuity of f and g, $\exists\ \delta > 0$ and by density of S, $\exists\ s \in S$ such that

$$d_X(x, s) < \delta \implies d_Y\big(f(x), f(s)\big) < \frac{\epsilon}{2} \text{ and } d_Y\big(g(x), g(s)\big) < \frac{\epsilon}{2}$$

then

$$\begin{aligned} d_Y\big(f(x), g(x)\big) &\leq d_Y\big(f(x), f(s)\big) + d_Y\big(f(s), g(s)\big) + d_Y\big(g(s), g(x)\big) \\ &< \frac{\epsilon}{2} + 0 + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

and thus $f(x) = g(x)$. Since $x \in S^c$ was arbitrary we conclude $f = g$ on S^c, and we are given that $f = g$ on S, and since $X = S \cup S^c$ we conclude that $f = g$ on X. □

Proposition 9. If $f : X \to Y$ is uniformly continuous, and $\{x_n\} \in X$ is a cauchy sequence, then $\big\{f(x_n)\big\}$ is a cauchy sequence in Y.

Proof. Since $f : X \to Y$ is uniformly continuous, $\forall\ \epsilon > 0,\ \exists\ \delta > 0$ such that

$$d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \epsilon$$

so for any cauchy sequence $\{x_n\} \in X$, $\exists\ N$ such that $n, m > N \implies d_X(x_n, x_m) < \delta$, yet this then gives

$$d_Y(f(x_n), f(x_m)) < \epsilon$$

by uniform continuity, and so $\{f(x_n)\}$ is cauchy in Y. □

Lemma 10. If $\{s_n\}, \{t_n\} \in X$ are cauchy sequences, then $\{d(s_n, t_n)\}$ converges in $\mathbb{R}$.

Proof. Let $\{s_n\}, \{t_n\}$ be cauchy sequences in X, then $\forall\ \epsilon > 0,\ \exists\ N_s, N_t$ such that

$$\begin{aligned} n_s, m_s \geq N_s &\implies d(s_{n_s}, s_{m_s}) < \frac{\epsilon}{2} \\ n_t, m_t \geq N_t &\implies d(t_{n_t}, t_{m_t}) < \frac{\epsilon}{2} \end{aligned}$$

so let $N = \max\{N_s, N_t\}$ then

$$\begin{aligned} n, m \geq N &\implies d(s_n, t_n) \leq d(s_n, s_m) + d(s_m, t_m) + d(t_m, t_n) \\ &\implies \big|d(s_n, t_n) - d(s_m, t_m)\big| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

with a symmetric argument giving

$$\big|d(s_m, t_m) - d(s_n, t_n)\big| < \epsilon$$

and so $\{d(s_n, t_n)\} \in \mathbb{R}$ is cauchy, and since $\mathbb{R}$ is complete we can conclude that $\{d(s_n, t_n)\}$ converges in $\mathbb{R}$. □

Lemma 11. Cauch(X) has $\{s_n\} \sim \{t_n\}$ iff $d(s_n, t_n) \to 0$.

Proof.
Reflexive: Trivially, $d(s_n, s_n) \to 0$, so $\{s_n\} \sim \{s_n\}$

Symmetric: If $d(s_n, t_n) \to 0$, then $d(s_n, t_n) = d(t_n, s_n) \to 0$. Giving $\{s_n\} \sim \{t_n\}$.

Transitive: Suppose $d(s_n, r_n) \to 0$ and $d(r_n, t_n) \to 0$, then $\forall\ n$

$$d(s_n, t_n) \leq d(s_n, r_n) + d(r_n, t_n) \to 0$$

and so $\{s_n\} \sim \{t_n\}$. $\square$

Lemma 12. If $X_\sim = \text{Cauch}(X)/\sim$ then

$$d_\sim : X_\sim \to [0, \infty),\ \text{by}\ d_\sim\big(\{s_n\}, \{t_n\}\big) = \lim_{n\to\infty} d(s_n, t_n)$$

is a metric on $X_\sim$.

Proof. First, since $\{d(s_n, t_n)\}$ converges in $\mathbb{R}$, we have that $d_\sim$ is always defined. To see that $d_\sim$ is well defined, let $\alpha, \beta \in X_\sim$ with $\{x_n\}, \{s_n\} \in \alpha$ and $\{y_n\}, \{t_n\} \in \beta$. Then

$$\lim_{n\to\infty} d(x_n, s_n) = \lim_{n\to\infty} d(y_n, t_n) = 0$$

and so $\forall\ \epsilon > 0,\ \exists N \in \mathbb{N}$ such that

$$n \geq N \implies d(x_n, s_n) < \frac{\epsilon}{2} \text{ and } d(y_n, t_n) < \frac{\epsilon}{2}$$

then for $n > N$ we have

$$\begin{aligned} d(s_n, t_n) &\leq d(s_n, x_n) + d(x_n, y_n) + d(y_n, t_n) \\ \implies \big|d(s_n, t_n) - d(x_n, y_n)\big| &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

$\therefore \lim_{n\to\infty} d(s_n, t_n) = \lim_{n\to\infty} d(x_n, y_n)$, or

$$d_\sim(\alpha, \beta) = \lim_{n\to\infty} d(s_n, t_n) = \lim_{n\to\infty} d(x_n, y_n)$$

and so $d_\sim$ is well-defined.

To see that $d_\sim$ it is a metric, for symmetry we have

$$d_\sim(\alpha, \beta) = \lim_{n\to\infty} d(s_n, t_n) = \lim_{n\to\infty} d(t_n, s_n) = d_\sim(\beta, \alpha)$$

now for $\alpha, \beta, \gamma \in X_\sim$ with $\{x_n\} \in \alpha,\ \{y_n\} \in \beta,\ \{z_n\} \in \gamma$, then $\forall\ n$

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, z_n) + d(z_n, y_n) \\ \implies \lim_{n\to\infty} d(x_n, y_n) &\leq \lim_{n\to\infty} d(x_n, z_n) + \lim_{n\to\infty} d(z_n, y_n) \\ \implies d_\sim(\alpha, \beta) &\leq d_\sim(\alpha, \gamma) + d_\sim(\gamma, \beta) \end{aligned}$$

and so satisfies the triangle inequality.

Next, if $d_{\sim}(\alpha,\beta)=0$, then $\forall\ \{x_n\}\in\alpha,\ \{y_n\}\in\beta$ we have

$$\implies \lim_{n\to\infty} d(x_n,y_n)=0$$

and so $\{x_n\}\sim\{y_n\} \implies \{y_n\}\in\alpha$ and thus $\alpha=\beta$. □

Proposition 13. The uniform limit of continuous functions is continuous.

Proof. Let $\epsilon>0$, and $x,y\in X$, then $\forall\ n$ we have

$$\big|f(x)-f(y)\big| \le \big|f(x)-f_n(x)\big| + \big|f_n(x)-f_n(y)\big| + \big|f_n(y)-f(y)\big|$$

then. by uniform continuity $\exists\ N\in\mathbb{N}$ such that

$$n\ge N \implies \big|f(x)-f_n(x)\big| < \frac{\epsilon}{3} \quad \forall\ x\in X$$

and by continuity $\forall\ \epsilon>0, \exists\ \delta>0$ such that

$$|x-y|<\delta \implies \big|f_n(x)-f_n(y)\big| < \frac{\epsilon}{3}$$

and thus $\forall\ x,y\in X$ such that $|x-y|<\delta$ and $n\ge N$ we have

$$\big|f(x)-f(y)\big| \le \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}=\epsilon$$

and so f is continuous. □

Theorem 14. $C\big([0,1]\big)$ is complete for $||\cdot||_{\infty}$.

Proof. Let $\{f_n\}\in C\big([0,1]\big)$ be a cauchy sequence, then $\forall\ \epsilon>0,\ \exists\ N\in\mathbb{N}$ such that

$$n,m\ge N \implies ||f_n-f_m||_{\infty}<\epsilon$$

Now, for each fixed $x\in[0,1]$ we have

$$\big|f_n(x)-f_m(x)\big|<\epsilon \quad \forall\ n,m\ge N$$

and this implies $\{f_n(x)\}$ is cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete $\{f_n(x)\}$ converges, so set

$$f(x)=\lim_{n\to\infty} f_n(x)$$

now, since $\{f_n\}\in C\big([0,1]\big)$ is cauchy $\exists\ N$ such that

$$\begin{aligned} &\big|f_n(x)-f_m(x)\big|<\epsilon \quad \forall\ n,m\ge N \\ \implies &\big|f(x)-f_m(x)\big|<\epsilon \quad \forall\ m\ge N;\ x\in[0,1] \end{aligned}$$

and this in turn implies that $f_m\to f$ uniformly. Since f is the uniform limit of continuous functions, f is continuous; that is $f_n\to f\in C\big([0,1]\big)$, and so $C\big([0,1]\big)$ is complete. □

Proposition 15. Let $(X, d_X), (Y, d_Y)$ be metric spaces. Then a map

$$f : X \to Y$$

is continuous at $x_0 \in X$ if $\forall\ \epsilon > 0,\ \exists\ \delta > 0$ such that

$$f\big(B_\delta(x_0)\big) \subseteq B_\epsilon\big(f(x_0)\big)$$

Proof. Let $f(x) \in B_\epsilon\big(f(x_0)\big)$ for some $x \in X$, and let

$$\epsilon' = \epsilon - d\big(f(x), f(x_0)\big) > 0$$

then $B_{\epsilon'}\big(f(x)\big) \subseteq B_\epsilon\big(f(x_0)\big) \implies \exists\ \delta' > 0$ such that

$$f\big(B_{\delta'}(x)\big) \subseteq B_{\epsilon'}\big(f(x)\big) \subseteq B_\epsilon\big(f(x_0)\big)$$

if $x_1 \in f^{-1}\Big(B_{\epsilon'}\big(f(x)\big)\Big)$ then $\exists\ B_{\delta'}(x_1)$ such that

$$B_{\delta'}(x_1) \subseteq f^{-1}\Big(B_{\epsilon'}\big(f(x)\big)\Big) \subseteq X$$

and so is open. $\square$

Proposition 16. Let (X, τ_X) and (Y, τ_Y) be topological spaces, then a map

$$f : X \to Y$$

is continuous iff for a base, or sub-base $\mathcal{B}_Y \subseteq \tau_Y$ we have

$$f^{-1}(B) \subseteq \tau_X \quad \forall\ B \in \mathcal{B}_Y$$

Proof. First suppose f is continuous. Then $\forall\ B \in \mathcal{B}_Y$ since $\mathcal{B}_Y$ is a base we have $B \in \tau_Y$ and so is open, then $f^{-1}(B) \in \tau_X$ by continuity.

Next suppose that $f^{-1}(B) \subseteq \tau_X\ \forall\ B \in \mathcal{B}_Y$, and let $V \in \tau_Y$. Since $\mathcal{B}_Y = \{B_i : i \in I\}$ is a base we have

$$V = \bigcup_{B_i \in \mathcal{B}_Y} B_i \quad \text{for some } i \in I$$

then

$$f^{-1}(V) = f^{-1}\left(\bigcup_{B_i \in \mathcal{B}_Y} B_i\right) = \bigcup_{B_i \in \mathcal{B}_Y} f^{-1}(B_i) \in \tau_X$$

and so f is continuous. $\square$

Proposition 17. Let X be a topological space. If $A \subseteq X$ is closed and $C \subseteq A$ is closed in the relative topology of A, then C is closed in X.

Proof. Since $A \setminus C = A \cap C^c$ is open in the relative topology of A, then $\exists\, U \in \tau$ such that

$$A \cap C^c = A \cap U \implies C = A \cap U^c$$

is closed in X. $\square$

Proposition 18. Consider

$$f_i : X \to Y_i \quad \text{for } i \in I$$

let τ_X be the initial/weak topology on X, let (Z, τ_Z) be a toplological space and

$$g : Z \to X$$

then g is continuous iff

$$f_i \circ g$$

is continuous $\forall\, i$.

Proof. First suppose $f_i \circ g$ is continuous $\forall\, i$. It suffices to check on a sub-base, so let $U \in \tau_i$ for some i, then

$$(f_i \circ g)^{-1}(U)$$

is open by the continuity if $f_i \circ g$, yet

$$(f_i \circ g)^{-1}(U) = g^{-1}(f_i^{-1}(U))$$

and so $g^{-1}(f_i^{-1}(U)) \subseteq Z$ is open, and since the the topology on X implies that $f_i^{-1}(U)$ is open in X, we then have that the preimage under g of an open set is open, and so g must be continuous.

Next suppose that g is continuous. Then by the continuity of g and the f_i's we have for any $i \in I$ and $U \in \tau_i$ that

$$g^{-1}(f_i^{-1}(U)) = (f_i \circ g)^{-1}(U)$$

is open and thus $f_i \circ g$ is continuous for each i. $\square$

Proposition 19. Every metrizable topological space is normal.

Proof. It suffices to consider a metric space (M, d). Let $C_1, C_2 \subseteq M$ be closed and disjoint. For each $x \in C_1$ choose $\epsilon_x > 0$ such that

$$B_{\epsilon_x}(x) \subseteq C_2^c$$

and for each $y \in C_2$ choose $\epsilon_y > 0$ such that

$$B_{\epsilon_y}(y) \subseteq C_1^c$$

let

$$O_1 = \bigcup_{x \in C_1} B_{\frac{\epsilon_x}{3}}(x) \text{ and } O_2 = \bigcup_{y \in C_2} B_{\frac{\epsilon_y}{3}}(y)$$

then O_1, O_2 are open as arbitrary unions of open sets, and since $C_1 \cap C_2 = \varnothing \implies C_1 \subseteq C_2^c$ and $C_2 \subseteq C_1^c$ so that

$$C_1 \subseteq O_1 \text{ and } C_2 \subseteq O_2$$

so suppose, for contradiction, that $O_1 \cap O_2 \neq \varnothing \implies \exists\, z \in O_1 \cap O_2$. Then $\exists\, x' \in C_1$ and $y' \in C_2$ such that $z \in B_{\frac{\epsilon_{x'}}{3}}(x')$ and $z \in B_{\frac{\epsilon_{y'}}{3}}(y')$, then

$$\begin{aligned} d(x', y') &\leq d(x', z) + d(z, y') \\ &< \frac{\epsilon_{x'}}{3} + \frac{\epsilon_{y'}}{3} \\ &\leq \frac{2}{3} \max\{\epsilon_{x'}, \epsilon_{y'}\} \quad \Rightarrow\Leftarrow \end{aligned}$$

as this implies $z \in C_1 \cap C_2 = \varnothing$. Thus $O_1 \cap O_2 = \varnothing$, and so M is normal. $\square$

Lemma 20. If (X, τ) is normal, $C \subset X$ is closed and $O \subseteq X$ is open and $C \subseteq O$, then $\exists\, U$ open with

$$C \subseteq U \subseteq \overline{U} \subseteq O$$

Proof. Since O is open, then O^c is closed and $C \subset O$ gives $O^c \cap C = \varnothing$. So, by normality, $\exists$ open U, V where $U \cap V = \varnothing$ such that $C \subseteq U$, and $O^c \subseteq V$.
Then $O^c \subseteq V \implies V^c \subseteq O$, and since $U \cap V = \varnothing$ we must have $U \subseteq V^c$ where V^c is closed. So $\overline{U} \subseteq \overline{V^c} = V^c$. Then

$$C \subseteq U \subseteq \overline{U} \subseteq O$$

$\square$

Lemma 21 (Urysohn's Lemma). Let (X, τ) be normal, and let C_0, C_1 be disjoint closed subsets. Then $\exists\, f : X \to [0, 1]$ continuous such that $f(C_0) = \{0\},\ f(C_1) = \{1\}$

Proof. Set $O_1 = X \setminus C_1 = C_1^c$ which is open as C_1 is closed in X. And since $C_0 \cap C_1 = \varnothing$ we have $C_0 \subseteq O_1$. Then, by Lemma 20 $\exists$ open O_0 such that

$$C_0 \subseteq O_0 \subseteq \overline{O}_0 \subseteq O_1$$

Then, by Lemma 20 $\exists$ open $O_{1/2}$ with

$$C_0 \subseteq O_0 \subseteq \overline{O}_0 \subseteq O_{1/2} \subseteq \overline{O}_{1/2} \subseteq O_1$$

so by Lemma 20 $\exists$ open $O_{1/4}, O_{3/4}$ so that

$$C_0 \subseteq O_0 \subseteq \overline{O}_0 \subseteq O_{1/4} \subseteq \overline{O}_{1/4} \subseteq O_{1/2} \subseteq \overline{O}_{1/2} \subseteq O_{3/4} \subseteq \overline{O}_{3/4} \subseteq O_1$$

So by Lemma 20 $\exists$ open $O_{1/8}, O_{3/8}, O_{5/8}, O_{7/8}$ such that

$$C_0 \subseteq O_0 \subseteq \overline{O}_0 \subseteq O_{1/8} \subseteq \overline{O}_{1/8} \subseteq O_{1/4} \subseteq \overline{O}_{1/4} \subseteq O_{3/8} \subseteq \overline{O}_{3/8} \subseteq \cdots$$

so by induction, for each dyadic rational

$$\left\{\frac{m}{2^n} : 1 \leq m \leq 2^n - 1; n, m \in \mathbb{N}\right\} =: \Delta$$

we get open $O_{\frac{m}{2^n}}$ such that if $r, s \in \Delta$, with $r < s$ then $\overline{O}_r \subseteq O_s$ and $C_0 \subseteq O_r\ \forall\, r$.
Define $f : X \to [0,1]$ by

$$f(x) = \inf\{r \in \Delta : x \in O_r\} \text{ for } x \in O_1$$
$$f(x) = 1 \text{ for } x \in C_1$$

Then if $x \in C_0$, then $x \in O_r\ \forall\ r \in \Delta$ including $r = 0$, so we have $f(x) = 0$.
To check continuity, use as a sub-base

$$\{(-\infty, a) : a \in \mathbb{R}\} \cup \{(b, \infty) : b \in \mathbb{R}\}$$

If $a \in \mathbb{R}$, then

$$f^{-1}\big((-\infty, a)\big) = \begin{cases} \varnothing, & a \leq 0 \\ X, & a > 1 \end{cases}$$

Suppose $0 < a \leq 1$. If $x \in X$ and $f(x) < a$ $\exists$ $r \in \Delta$ such that $f(x) < r < a$ and so $x \in O_r$ and thus $f^{-1}\big((-\infty, a)\big) = \bigcup_{r<a} O_r$ which is the union of open sets and hence is open.

If $f(x) > b$ then

$$f^{-1}\big((b, \infty)\big) = \begin{cases} X, & b < 0 \\ \varnothing, & b \geq 1 \end{cases}$$

for $0 \leq b < 1$ we claim $f^{-1}\big((b,\infty)\big) = \bigcup_{r>b} \overline{O}_r^c$.
If $f(x) > b$, then $\exists$ $s \in \Delta$ with $f(x) > s > b \implies x \notin O_s$. Then $\exists$ $r \in \Delta$ such that $s > r > b$ where $\overline{O}_r \subseteq O_s$ with $x \notin \overline{O}_r \implies x \in \overline{O}_r^c$ which is open, and so $f^{-1}\big((b,\infty)\big) = \bigcup_{r>b} \overline{O}_r^c$ which is open as the union of open sets. And so in all cases we see that f is continuous. $\square$

Proposition 22. If $(V, ||\cdot||)$ is a banach space, then $\big(B(X,V), ||\cdot||_\infty\big)$ is a banach space. Where $B(X,V)$ is the set of all bounded functions from X to V.

Proof. Let $\{f_n\} \in B(X,V)$ be a cauchy sequence. For each $x \in X$, $\{f_n(x)\}$ is cauchy in V, and by the completeness of V converges in V, say $f_n(x) \to f(x)$. Let $\epsilon > 0$ be given, since $\{f_n\}$ is cauchy $\exists$ $N_1 \in \mathbb{N}$ such that

$$n, m \geq N_1 \implies ||f_n - f_m||_\infty < \frac{\epsilon}{2}$$

so for $x \in X$ and $n, m \geq N$ we have $||f_n(x) - f_m(x)|| < \frac{\epsilon}{2}$, so for fixed $m > N$ we have

$$||f_m(x) - f(x)|| = \lim_{n\to\infty} ||f_m(x) - f_n(x)|| < \frac{\epsilon}{2}$$

and so f is bounded. Next, fix $x \in X$ then since $f_n(x) \to f(x)$ $\exists N_2 \in \mathbb{N}$ such that

$$n \geq N_2 \implies ||f_n(x) - f(x)||_\infty < \frac{\epsilon}{2}$$

so for $n > \max\{N_1, N_2\}$ we have

$$\begin{aligned}||f_n - f||_\infty &\leq ||f_n - f_{n+1}||_\infty + ||f_{n+1} - f||_\infty \\ &\leq ||f_n - f_{n+1}||_\infty + ||f_{n+1}(x) - f(x)|| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon\end{aligned}$$

and so $f_n \to f \in B(X, V)$, and hence is complete. $\square$

Proposition 23. Let (X, τ) be a topological space and Y a metric space. Then $C_B(X, Y)$ is a closed subset of $\big(B(X, Y), ||\cdot||_\infty\big)$.

Proof. Let $\{f_n\} \in C_B(X, Y)$ be a cauchy sequence such that $f_n \to f \in B(X, Y)$ under $||\cdot||_\infty$. We wish to show that $f \in C_B(X, Y)$. So let $\epsilon > 0$ and be given and $x_0 \in X$ be arbitrary. Then $\exists N \in \mathbb{N}$ such that

$$n \geq N \implies ||f - f_n||_\infty < \frac{\epsilon}{3}$$

Then since $f_n \in C_B(X, Y)$ is continuous $\exists$ open $B_\delta(x_0) \ni x_0$ such that if $y \in B_\delta(x_0)$ then $||f_n(y) - f_n(x_0)|| < \frac{\epsilon}{3}$ and so

$$\begin{aligned}||f(y) - f(x_0)|| &\leq ||f(y) - f_n(y)|| + ||f_n(y) - f_n(x_0)|| + ||f_n(x_0) - f(x_0)|| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon\end{aligned}$$

and thus we have $f \in C_B(X, Y)$; that is $C_B(X, Y) \subset \big(B(X, Y), ||\cdot||_\infty\big)$ is closed. $\square$

Theorem 24 (**Tietze Extension Theorem**)**.** Let (X, τ) be a normal topological space and let $A \subset X$ be closed, and $f : A \to \mathbb{R}$ be continuous. Then $\exists$ $F : X \to \mathbb{R}$ continuous, where $F|_A = f$. If $f(A) \subseteq [a, b]$ then we can arrange $F(X) \subseteq [a, b]$.

Proof. First, suppose that

$$f : A \to [-1, 1]$$

and let

$$\begin{aligned}A_1 &= \left\{x \in A : f(x) \geq \tfrac{1}{3}\right\} = f^{-1}\left(\left[\tfrac{1}{3}, 1\right]\right) \\ B_1 &= \left\{x \in A : f(x) \leq -\tfrac{1}{3}\right\} = f^{-1}\left(\left[-1, -\tfrac{1}{3}\right]\right)\end{aligned}$$

where by the continuity of f we have B_1, A_1 are closed in A where $B_1 \cap A_1 = \varnothing$, and thus are also closed and disjoint in X. So by Urysohn's lemma we have that there exists continuous

$$f_1 : X \to \left[-\tfrac{1}{3}, \tfrac{1}{3}\right]$$

such that

$$f_1(A_1) = \tfrac{1}{3}, \text{ and } f_1(B_1) = -\tfrac{1}{3}$$

Thus, for any $x \in A$ we have $|f(x) - f_1(x)| \leq \frac{2}{3}$ so that

$$g_1 := f - f_1 : A \to \left[-\tfrac{2}{3}, \tfrac{2}{3}\right]$$

and let

$$\begin{aligned} A_2 &= \left\{x \in A : g_1(x) \geq \tfrac{1}{3}\left(\tfrac{2}{3}\right)\right\} = g_1^{-1}\left(\left[\tfrac{2}{9}, \tfrac{2}{3}\right]\right) \\ B_2 &= \left\{x \in A : g_1(x) \leq -\tfrac{1}{3}\left(\tfrac{2}{3}\right)\right\} = g_1^{-1}\left(\left[-\tfrac{2}{3}, -\tfrac{2}{9}\right]\right) \end{aligned}$$

where by the continuity of g_1 we have B_2, A_2 are closed in A where $B_2 \cap A_2 = \varnothing$, and thus are also closed and disjoint in X. So by Urysohn's lemma we have that there exists continuous

$$f_2 : X \to \left[-\tfrac{2}{9}, \tfrac{2}{9}\right]$$

such that

$$f_2(A_2) = \tfrac{2}{9}, \text{ and } f_2(B_2) = -\tfrac{2}{9}$$

Thus, for any $x \in A$ we have $|f(x) - f_1(x) - f_2(x)| \leq \left(\frac{2}{3}\right)^2$ so that

$$g_2 := f - f_1 - f_2 : A \to \left[-\tfrac{4}{9}, \tfrac{4}{9}\right]$$

continuing inductively we can construct a sequence of continuous functions $f_1, f_2, \ldots$ such that

$$\left| f(x) - \sum_{i=1}^{n} f_i(x) \right| \leq \left(\tfrac{2}{3}\right)^n \to 0, \text{ as } n \to \infty$$

on A, so defining $F := \sum_{i=1}^{\infty} f_i$, then by construction we have $F|_A = f$.
For continuity let $\epsilon > 0$ and $x \in X$ be given, then pick $N \in \mathbb{N}$ such that $\sum_{i=N+1}^{\infty} \left(\frac{2}{3}\right)^i < \frac{\epsilon}{2}$. Then, since each Urysohn function f_i is continuous on X for $1 \leq i \leq N$ select $U_i \in \tau$ such that $x \in U_i$ where

$$y \in U_i \implies |f_i(x) - f_i(y)| < \frac{\epsilon}{2N}$$

then

$$U := \bigcap_{j=1}^{N} U_j$$

is open as the finite intersection of open sets and $y \in U$ implies

$$\begin{aligned}
|F(x) - F(y)| &\leq \sum_{i=1}^{N} |f_i(x) - f_i(y)| + \sum_{i=N+1}^{\infty} \left(\tfrac{2}{3}\right)^i \\
&< \frac{\epsilon}{2N} \sum_{i=1}^{N} 1 + \frac{\epsilon}{2} \\
&= \frac{\epsilon}{2N} \cdot N + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

and so F is continuous as x, since $x \in X$ was arbitrary we conclude that F is continuous on X.

Now for the case when f is not bounded, since $\mathbb{R}$ is homeomorphic to $(-1, 1)$ via the mapping

$$\frac{2}{\pi} \tan^{-1} : \mathbb{R} \to (-1, 1)$$

so let us consider

$$f : A \to (-1, 1) \subset [-1, 1]$$

Then from above there exists continuous $\widetilde{f} : X \to [1, -1]$ such that $\widetilde{f}|_A = f$. So, let

$$B = \widetilde{f}^{-1}(\{1\}) \cup \widetilde{f}^{-1}(\{-1\})$$

where by the continuity of $\widetilde{f}$ we have that $B \subset X$ is closed as the union of singletons which are closed, and since

$$\widetilde{f}(A) = f(A) \subseteq (-1, 1)$$

we have that $A \cap B = \varnothing$. So by Urysohn's lemma there exists continuous

$$g : X \to [0, 1]$$

such that

$$g(A) = 1, \text{ and } g(B) = 0$$

so define

$$F := g \cdot \widetilde{f} : X \to (-1, 1)$$

Then F is continuous as the product of two continuous functions, and for any $x \in A$ we have

$$F(x) = g(x) \cdot \widetilde{f}(x) = 1 \cdot \widetilde{f}(x) = f(x)$$

so $F|_A = f$. For $y \in B$ we have

$$F(y) = g(y) \cdot \widetilde{f}(y) = 0 \cdot \widetilde{f}(y) = 0$$

and for $z \notin A \cup B$, then since $|\widetilde{f}(z)| < 1$ we have

$$|F(z)| \leq 1 \cdot |\widetilde{f}(z)| < 1$$

and so $\mathrm{Im}(F) = (-1, 1)$, and F is an extension of f. $\square$

Proposition 25 (**Equivalent Definition of Compact**).

(a) X is compact if every open cover of X has a finite subcover.

(b) Every collection $\{K_\alpha\}_{\alpha\in I}$ of closed sets with the finite intersection property, has nonempty intersection; i.e. $\bigcap_{\alpha\in I} K_\alpha \neq \varnothing$.

Proof. $(a) \implies (b)$
Let X be compact, and let $\{K_\alpha\}_{\alpha\in I}$ be a collection of closed sets with the finite intersection property, and assume for contradiction, that $\bigcap_{\alpha\in I} K_\alpha = \varnothing$. Then for each K_α we have $X \setminus K_\alpha = K_\alpha^c$ is open. So,

$$\begin{aligned} \bigcap_{\alpha\in I} K_\alpha &= \varnothing \\ \implies \left(\bigcap_{\alpha\in I} K_\alpha\right)^c &= \varnothing^c \\ \implies \bigcup_{\alpha\in I} K_\alpha^c &= X \end{aligned}$$

That is $\bigcup_{\alpha\in I} K_\alpha^c$ is an open cover for X, and since X is compact, it admits a finite subcover, giving

$$\begin{aligned} \bigcup_{i=1}^{n} K_\alpha^c &= X \\ \implies \left(\bigcup_{i=1}^{n} K_\alpha^c\right)^c &= X^c \\ \bigcap_{i=1}^{n} K_\alpha &= \varnothing \quad\quad \Rightarrow\Leftarrow \end{aligned}$$

A contradiction to our assumption that for finite K_α we have $\bigcap_{i=1}^{n} K_\alpha \neq \varnothing$.
And therefore me must have $\bigcap_{\alpha\in I} K_\alpha \neq \varnothing$.

$(b) \implies (a)$

Let X be a topological space, and suppose that for every collection $\{K_\alpha\}_{\alpha\in I}$ of closed sets with the finite intersection property, we have $\bigcap_{\alpha\in I} K_\alpha \neq \varnothing$.
Next let $\mathcal{U}$ be an open cover of X and assume, for contradiction, that $\mathcal{U}$ has no finite subcover of X. That is

$$\bigcup_{j=1}^{i_n} U_{i_j} \neq X$$

so we must have at least one $p \in X$ such that

$$\begin{aligned} p &\notin \bigcup_{j=1}^{i_n} U_{i_j} \\ \implies p &\in \left(\bigcup_{j=1}^{i_n} U_{i_j} \right)^c \\ &= \bigcap_{j=1}^{i_n} U_{i_j}^c \\ \implies \varnothing &\neq \bigcap_{j=1}^{i_n} U_{i_j}^c \end{aligned}$$

where each $U_{i_k}^c$ is closed in X, and since this is true for each finite subcollection of $\mathcal{U}$, we have the family $\{X \setminus U\}_{U \in \mathcal{U}} = \{U^c\}_{U \in \mathcal{U}}$ satisfies the finite intersection property. Where by our assumption we have

$$\begin{aligned} \bigcap_{U \in \mathcal{U}} U^c &\neq \varnothing \\ \implies \left(\bigcap_{U \in \mathcal{U}} U^c \right)^c &\neq \varnothing^c \\ \implies \bigcup_{U \in \mathcal{U}} U &\neq X \quad \Rightarrow\Leftarrow \end{aligned}$$

A contradiction to the assumption that $\mathcal{U}$ was an open cover for X. Thus, we conclude that $\mathcal{U}$ must admit a finite subcover of X.

Since $\mathcal{U}$ was an arbitrary open cover for X, we conclude that every open cover of X admits a finite subcover, and therefore X is compact. $\square$

Proposition 26. A topological space X is connected if and only if every continuous map of X into a discrete space having at least two elements is constant.

Proof. First assume that X is connected, and that $f : X \to Y$ is a continuous map, where Y is a discrete space with at least 2 elements. WLOG suppose $Y = \{y, y'\}$.
If $f(X) \neq$ constant, then $f(x) = y$ and $f(x') = y'$ where $y, y' \in Y$ are disjoint and open by the discrete topology, yet this implies that for $U_x, U_{x'} \in X$ we have

$$f(U_x) \cap f(U_{x'}) = \varnothing$$

where $f(U_x), f(U_{x'}) \neq \varnothing$ and so form a separation of Y, which contradicts the continuity of f, since the image of a connected set under a continuous map must be connected.

Next suppose that X is not connected; i.e. $X = U \cup V$ where $V, U \neq \varnothing$ are open and $U \cap V = \varnothing$. Then let $p \neq q$ and endow $\{p, q\}$ with the discrete topology. If we define

$$f : X \to \{p, q\}, \text{ by } \begin{cases} f(U) = \{p\} \\ f(V) = \{q\} \end{cases}$$

then f is continuous and non-constant. □

Proposition 27. If a topological space (X, τ) is compact, and $A \subseteq X$ is closed, then A is compact.

Proof. Let $\mathcal{U} \subseteq \tau$ be an open cover of A, then since $A \subseteq X$ is closed, we have $A^c \subseteq X$ is open, and so

$$\mathcal{U} \cup A^c$$

is an open cover for X. Since X is compact, it admits a finite subcover which must contain A. □

Proposition 28. Properties of maximal FIP family $\mathcal{F}^*$

(a) $\mathcal{F}^*$ is closed/stable under finite intersections.

(b) If $B \subseteq X$ and $B \cap A \neq \varnothing$, $\forall\ A \in \mathcal{F}^*$ then $B \in \mathcal{F}^*$.

Proof.

(a) Given $B, C \in \mathcal{F}^*$, then taking finite $A_1, \dots, A_k \in \mathcal{F}^*$ we have by FIP,

$$(B \cap C) \bigcap (A_1 \cap \cdots \cap A_k) \neq \varnothing$$

and so $\mathcal{F}^* \cup \{B \cap C\}$ is an FIP family, yet by the maximality of $\mathcal{F}^*$ we must have

$$\mathcal{F}^* = \mathcal{F}^* \cup \{B \cap C\}$$

and so $B \cap C \in \mathcal{F}^*$, and $\mathcal{F}^*$ is stable under finite intersections.

(b) Consider $\mathcal{F}^{'} = \mathcal{F}^* \cup \{B\}$. Then, $\mathcal{F}^{'}$ has FIP, as any finite subcollection of $\mathcal{F}^{'}$ is either of the form

$$A_1, \dots, A_n$$

which has nonempty intersection, or

$$B, A_1, \dots, A_n$$

where

$$B \bigcap \big(\overset{\in \mathcal{F}^*}{\cap_{j=1}^{n} A_j}\big) \neq \varnothing$$

and thus by maximality $\mathcal{F}^* = \mathcal{F}^{'}$, otherwise $\mathcal{F}^{'}$ would be a larger set with the FIP property and $\mathcal{F}^*$ would not be maximal. Thus, $B \in \mathcal{F}^*$.

□

Theorem 29 (**Tychonoff's Theorem**)**.** Let I be some index set. For each $i \in I$ let (X_i, τ_i) be a topological space. If all the (X_i, τ_i)'s are compact then

$$X = \prod_{i \in I} X_i$$

with the product topology is compact. (Need the axiom of choice)

Proof. First, given a set $X \neq \varnothing$ and some FIP family of closed subsets $\mathcal{S}$ on X, consider as a partially ordered set

$$\mathcal{W} := \left\{\mathcal{F} \subseteq \mathcal{P}(X) : \mathcal{S} \subseteq \mathcal{F}; \mathcal{F} \text{ is an FIP family on } X\right\}$$

with the partial ordering on $\mathcal{W}$ given by set inclusion, and note that $\mathcal{S} \in \mathcal{W} \implies \mathcal{W} \neq \varnothing$. Now let $\mathcal{C}$ be a non-empty chain in $\mathcal{W}$, so that $\mathcal{C}$ is a collection of FIP families in $\mathcal{W}$ and is totally ordered by inclusion. Let us set

$$\mathcal{F}_0 = \bigcup_{\mathcal{F} \in \mathcal{C}} \mathcal{F}$$

so let $n \in \mathbb{N}$ and $A_1, \dots, A_n$ be subsets of X such that $A_1, \dots, A_n \in \mathcal{F}_0$. Since $\mathcal{F}_0$ is the union of elements in $\mathcal{C}$, for $A_i \in \mathcal{F}_0$ we must have $A_i \in \mathcal{F}$ for some $\mathcal{F} \in \mathcal{C}$, and so, for each $i \in \{1, \dots, n\}$ $\exists$ $\mathcal{F}_i \in \mathcal{C}$ such that $A_i \in \mathcal{F}_i$ for each i. Then, in particular,

$$\{\mathcal{F}_1, \dots, \mathcal{F}_n\} \in \mathcal{C}$$

and hence is totally ordered by set inclusion, and so one of $\{\mathcal{F}_1, \dots, \mathcal{F}_n\}$ must be maximal, let this be $\mathcal{F}_j$ so that

$$\mathcal{F}_j \supseteq \mathcal{F}_i, \text{ for } 1 \leq i \leq n$$

and thus $A_1, \dots, A_n \in \mathcal{F}_j$, and since $\mathcal{F}_j$ is an FIP family we have

$$\bigcap_{i=1}^{n} A_i \neq \varnothing$$

and since each $\mathcal{F} \supseteq \mathcal{S}$ we trivially have that $\mathcal{F}_0 \supseteq \mathcal{S}$ and so $\mathcal{F}_0 \in \mathcal{W}$ and $\mathcal{F}_0 = \bigcup_{\mathcal{F} \in \mathcal{C}} \mathcal{F}$ is an upper bound for the chain $\mathcal{C}$.
Since the chain $\mathcal{C} \in \mathcal{W}$ was arbitrary we conclude that every chain in $\mathcal{W}$ has an upper bound in $\mathcal{W}$, and hence $\mathcal{W}$ is inductively ordered.

Thus, by Zorn's Lemma $\mathcal{W}$ has a maximal element $\mathcal{F}^*$ which contains $\mathcal{S}$.

Now, for each $i \in I$ consider

$$\mathcal{F}_i = \{\pi_i(A) : A \in \mathcal{F}^*\}$$

then $\mathcal{F}_i \subseteq \mathcal{P}(X_i)$, now for $A_1, \dots, A_n \in \mathcal{F}^*$ we have

$$\bigcap_{j=1}^{n} A_j \neq \varnothing$$

which implies that there exists at least one $x \in \bigcap_{j=1}^{n} A_i$, and so

$$\pi_i(x) \in \pi_i\left(\bigcap_{j=1}^{n} A_j\right) \subseteq \bigcap_{j=1}^{n} \pi_i(A_j)$$

and so $\mathcal{F}_i$ is an FIP family on X_i, and since each $\pi_i(A_j) \subseteq \overline{\pi_i(A_j)}$ we also have that

$$\left\{\overline{\pi_i(A)} : A \in \mathcal{F}^*\right\}$$

is an FIP family on X_i of closed subsets, and since X_i is compact we have that

$$\bigcap_{A \in \mathcal{F}^*} \overline{\pi_i(A)} \neq \varnothing$$

and so by the axiom of choice we may select $x_i \in \bigcap_{A \in \mathcal{F}^*} \overline{\pi_i(A)} \subseteq X_i$ and set

$$x = (x_i) \in \prod_{i \in I} X_i$$

and let O_x be an open neighbourhood of x in X. It suffices to consider O_x as a basis element of X so that

$$x \in O_x = \prod_{i_j \neq \{i_1, \dots, i_k\}} X_{i_j} \times \prod_{j=1}^{k} U_{i_j}$$

or. equivalently

$$x \in O_x = \bigcap_{j=1}^{k} \pi_{i_j}^{-1}(U_{i_j})$$

and note for each $j \in \{1, \dots, k\}$ we have $x_{i_j} \in U_{i_j}$ and by construction $x_{i_j} \in \bigcap_{A \in \mathcal{F}^*} \overline{\pi_{i_j}(A)}$ and since $U_{ij} \subseteq X_{i_j}$ is open and contains x_{i_j} by the definition of a limit point we must have that $U_{i_j} \cap \pi_{i_j}(A) \neq \varnothing$ for each $A \in \mathcal{F}^*$ and hence

$$\pi_{i_j}^{-1}(U_{i_j}) \cap A \neq \varnothing, \quad \forall\ A \in \mathcal{F}^*$$

and hence $\pi_{i_j}^{-1}(U_{i_j}) \in \mathcal{F}^*$ by maximality for each $j \in \{1, \dots, k\}$. Where maximality then gives

$$\bigcap_{j=1}^{k} \pi_{i_j}^{-1}(U_{i_j}) = O_x \in \mathcal{F}^*$$

and therefore $O_x \cap A \neq \varnothing$, $\forall\ A \in \mathcal{F}^*$, and in particular since $\mathcal{S} \subseteq \mathcal{F}^*$ we have that $O_x \cap A \neq \varnothing$, $\forall\ A \in \mathcal{S}$ and hence

$$\bigcap_{A \in \mathcal{S}} A \neq \varnothing$$

and thus, X is compact. □

Theorem 30. Tychonoff's Theorem implies the Axiom of Choice.

Proof. Let $\{X_i\}_{i \in I}$ be a non-empty family and let

$$X = \prod_{i \in I} X_i$$

let ω be some set not in X.

Next, for each i set $Y_i = X_i \cup \{\omega\}$ and define

$$\tau_{Y_i} = \{Y_i, X_i, \{\omega\}, \varnothing\}$$

then (Y_i, τ_{Y_i}) is finite and hence compact. So let

$$Y = \prod_{i \in I} Y_i$$

which is then compact by Tychonoff's Theorem.

Since $\omega \in Y_i$ is open, this implies $\omega^c = X_i$ is closed in Y_i, and hence is clopen. So by the continuity of the projection maps π_i we have

$$\pi_i^{-1}(X_i) \subseteq Y$$

is closed for each i. To see that $\{\pi_i^{-1}(X_i)\}$ has FIP, let $\pi_{i_1}^{-1}(X_{i_1}), \ldots, \pi_{i_n}^{-1}(X_{i_n}) \subset \{\pi_i^{-1}(X_i)\}$ be given and note that $\exists\ x_{i_j} \in X_{i_j}\ \forall\ i_j$, so define $y \in Y$ by

$$y_i = \begin{cases} x_{i_j}, & i = i_j \\ \omega, & i \neq i_j\ \forall\ j \end{cases}$$

then

$$y \in \bigcap_{j=i}^{n} \pi_{i_j}^{-1}(X_{i_j}) \implies \{\pi_i^{-1}(X_i)\} \text{ is FIP}$$

then since $\{\pi_i^{-1}(X_i)\}$ is an FIP family and Y is compact this gives

$$\bigcap_{i \in I} \pi_i^{-1}(X_i) \neq \varnothing$$

so let $z \in \bigcap_{i \in I} \pi_i^{-1}(X_i)$, then $z \in X_i$ for each i and therefore

$$z \in \prod_{i \in I} X_i$$

□

Proposition 31. If (X, τ) is compact and Hausdorff, then it is normal.

Proof. Let $A, B \subseteq X$ be closed and disjoint. Since X is compact and A, B are closed subsets of a compact space we have that A, B are also compact. Since X is Hausdorff, it is regular. Thus, for $x \in A\ \exists\ U_x, V_x \in \tau$ disjoint with

$$x \in U_x \quad \text{and} \quad B \subseteq V_x$$

then $\{U_x\}_{x\in A}$ is an open cover for A, and by compactness of A admits a finite subcover giving

$$A \subseteq \bigcup_{i=1}^{n} U_{x_i} =: U$$

and

$$V := \bigcap_{i=1}^{n} V_{x_i} \supseteq B$$

which are both open as the union and finite intersection of open sets, where $U \cap V = \varnothing$. Hence, X is normal. $\square$

Theorem 32. If (X, τ_X) is compact and (Y, τ_Y) is Hausdorff, and if

$$f : X \to Y$$

is continuous, injective and surjective. Then f is a homeomorphism.

Proof. Since f is continuous, injective and surjective, we have

$$f^{-1} : Y \to X$$

exists, so let $A \subseteq X$ be closed, then A is compact as the closed subset of a compact space, and by the continuity of f we also have that $F(A) \subseteq Y$ is compact. Since Y is Hausdorff $f(A)$ is closed as a compact set in a Hausdorff space. Since f is injective and surjective we also have

$$f(A)^c = Y \setminus f(A) = f(X) \setminus f(A) = f(X \setminus A) = f(A^c)$$

where $f(a)^c = f(A^c)$ is open in Y and so

$$f^{-1}\big(f(A^c)\big) = A^c \subseteq X$$

is open and thus f^{-1} is continuous. Therefore, f is a homeomorphism. $\square$

Proposition 33. let (X, d) be a metric space and $A \subseteq X$ be totally bounded, then $\overline{A}$ is totally bounded.

Proof. Let $\epsilon > 0$ be given, since A is totally bounded $\exists\ x_i, \ldots, x_n \in A$ such that $\{B_{\frac{\epsilon}{2}}(x_i)\}_{i=1}^n$ cover A. For each $z \in \overline{A}\ \exists\ y \in A$ such that $z \in B_{\frac{\epsilon}{2}}(y)$, by the definition of a limit point, and there is some j such that $y \in B_{\frac{\epsilon}{2}}(x_j)$ since the $B_{\frac{\epsilon}{2}}(x_j)$'s cover A and so

$$d(z, x_j) \leq d(z, y) + d(y, x_j) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and so $z \in B_\epsilon(x_j)$ and hence $\{B_\epsilon(x_i)\}_{i=1}^n$ cover $\overline{A}$. $\square$

Proposition 34. let (X, d) be a metric space. If X is compact, then it is complete.

Proof. Let $\{x_n\} \in X$ be a cauchy sequence and suppose, for contradiction, that X is not complete. Then $\{x_n\}$ does not converge in X. So $\forall x \in X \ \exists \ \epsilon_x > 0$ such that $\forall \ N \in \mathbb{N} \ \exists \ n \geq N$ where $d(x, x_n) \geq \epsilon_x$.

Then since $\{x_n\}$ is cauchy $\exists \ M \in \mathbb{N}$ such that

$$n, m \geq M \implies d(x_n, x_m) < \epsilon_x$$

pick $M_x > M$ such that $n_x \geq M_x$ gives $d(x, x_{n_x}) \geq \epsilon_x$. So for $n > M_x$ we have $d(x, x_n) \geq \frac{\epsilon_x}{2}$. Thus, $\forall \ x \in X$, $B_{\epsilon_x}(x)$ contains at most finite $x_i \in \{x_n\}$. Now $\{B_{\epsilon_x}(x)\}_{x \in X}$ cover X, yet it does not admit a finite subcover, contradicting the compactness of X. $\square$

Theorem 35. let (X, d) be a complete metric space. If X is totally bounded, then it is compact.

Proof. Let $\mathcal{U}$ be an open cover of X, and since X is totally bounded let $\overline{B}_1^1, \ldots, \overline{B}_n^1$ be a finite cover of X by closed balls of radius 1. Suppose, for contradiction, that X is not compact. So at least one ball say A^1 has no finite subcover and let $\overline{B}_1^2, \ldots, \overline{B}_{n_2}^2$ be closed balls of radius $\frac{1}{2}$ covering A^1, then at least one, say B_*^2 has no finite subcover so let

$$A^2 = A^1 \cap B_*^2$$

let $\overline{B}_1^3, \ldots, \overline{B}_{n_3}^3$ be closed balls of radius $\frac{1}{4}$ covering A^2, then at least one has no finite subcover, say B_*^3 so let

$$A^3 = A^2 \cap B_*^3$$

continuing inductively we get a sequence $\{A^n\}$ such that

$$A^{n+1} \subseteq A^n \quad \forall \ n$$

and

$$\operatorname{diam}(A^n) \to 0$$

and each A^n is not finitely covered.

For each n select $x_n \in A^n$, then $\{x_n\}$ is cauchy, and by the completeness of X, $\exists \ x \in X$ such that $x_n \to x$. Since $\mathcal{U}$ covers X there exists $U \in \mathcal{U}$ such that $x \in U$, then given $\epsilon > 0 \ \exists \ B_\epsilon(x) \subseteq U$. So choose n such that $\operatorname{diam}(A^n) < \epsilon$ then

$$A^n \subset B_\epsilon(x) \subseteq U \quad \Rightarrow\Leftarrow$$

contradicting the assumption that A^n was not finitely covered. $\square$

Theorem 36 (**Arzela-Ascoli**)**.** Let (X, τ) be a compact topological space, (Y, d) be a metric space, and let $\mathcal{F} \subseteq C_B(X, Y)$ be pointwise totally bounded and equicontinuous, then $\mathcal{F}$ is totally bounded for d_∞.

Proof. let $\epsilon > 0$ be given. Since $\mathcal{F}$ is equicontinuous $\forall\ x \in X,\ \exists\ O_x \ni x$ such that

$$y \in O_x \implies d\big(f(x), f(y)\big) < \epsilon \quad \forall\ f \in \mathcal{F}$$

since X is compact $\exists\ x_1, \dots, x_n \in X$ such that

$$X \subseteq \bigcup_{i=1}^{n} O_{x_i}$$

for each j, since $\mathcal{F}$ is pointwise totally bounded, we have $\{f(x_j) : f \in \mathcal{F}\}$ is totally bounded. Let

$$S_j \subseteq \{f(x_j) : f \in \mathcal{F}\} \subseteq Y$$

be a finite subset such that

$$\bigcup_{y \in S_j} B_\epsilon(y) \supseteq \{f(x_j) : f \in \mathcal{F}\}$$

and let

$$S = \bigcup_{j=1}^{n} S_j$$

also let

$$\Psi = \{\psi : \{1, \dots, n\} \to S\}$$

which is finite and set

$$B_\psi = \big\{f \in \mathcal{F} : d\big(f(x_j), \psi(j)\big) < \epsilon\ \forall\ j\big\}$$

then

$$\mathcal{F} = \bigcup_{\psi \in \Psi} B_\psi$$

So let $\psi \in \Psi$ be given and let $f, g \in B_\psi$, and $x \in X$ be such that $x \in O_{x_j}$ for some j, then

$$\begin{aligned} d\big(f(x), g(x)\big) &\leq d\big(f(x), f(x_j)\big) + d\big(f(x_j), g(x_j)\big) + d\big(g(x_j), g(x)\big) \\ &\leq d\big(f(x), f(x_j)\big) + d\big(f(x_j), \psi(j)\big) + d\big(\psi(j), g(x_j)\big) + d\big(g(x_j), g(x)\big) \\ &= \epsilon + \epsilon + \epsilon + \epsilon \\ &= 4\epsilon \end{aligned}$$

and therefore

$$B_\psi \subseteq \bigcup_{y \in B_\psi} B_{4\epsilon}(y)$$

and since Ψ is finite, $\mathcal{F}$ is totally bounded. $\square$

Corollary 37. Let (X, τ) be a compact topological space, (Y, d) be a complete metric space, and let $\mathcal{F} \subseteq C_B(X, Y)$. Then $\mathcal{F}$ is compact iff it is pointwise totally bounded, equicontinuous, and closed in $C_B(X, Y)$.

Proposition 38. Let (X, τ) be a locally compact topological space, and let $C \subseteq X$ be compact. Then $\exists\ O \in \tau$ such that $C \subseteq O$ where $\overline{O}$ is compact.

Proof. $\forall\ x \in C$, by local compactness $\exists\ O_x \in \tau$ with $x \in O_x$ such that $\overline{O}_x$ is compact. Then $\{O_x\}_{x \in C}$ is an open cover for C, and since C is compact it admits a finite subcover and so

$$C \subseteq \bigcup_{i=1}^{n} O_{x_i} \subseteq \bigcup_{i=1}^{n} \overline{O}_{x_i} \subseteq \overline{\bigcup_{i=1}^{n} O_{x_i}}$$

which is compact as the finite union of compact sets. □

Proposition 39. Let (X, τ) be a locally compact Hausdorff space. Then every $x \in X$ has a neighborhood base consisting of compact neighborhoods; i.e. $\forall\ x \in O_x\ \exists\ U \in \tau$, with $x \in U$ such that $\overline{U} \subseteq O_x$ where $\overline{U}$ is compact.

Proof. Given $x \in O_x$, let $V \in \tau$ with $x \in V$ where $\overline{V}$ is compact by local compactness. Then we can replace O_x with

$$O = O_x \cap V \subseteq V$$

so that $\overline{O}$ is compact as a closed subset of a compact set. Let

$$\partial O := \overline{O} \setminus O$$

which is closed in the relative topology of $\overline{O}$, since $O \notin \partial O \implies x \notin \partial O$. Since $\overline{O}$ is compact Hausdorff, it is normal, and hence regular. So $\exists\ U, W$ relatively open in $\overline{O}$ such that $U \cap W = \varnothing$ with

$$x \in U \text{ and } \partial O \subseteq W$$

then

$$U \cap W = \varnothing \implies W^c = \overline{O} \setminus W \supseteq U$$

and since $W \supseteq \partial O \implies W^c \subseteq \partial O^c$, which then implies that $W^c \subseteq O$
Now $\overline{O} \setminus W$ is relatively closed in $\overline{O}$, which gives

$$\overline{U} \subseteq \overline{O} \setminus W = W^c \subseteq O$$

so $\overline{U} \subseteq O$ and hence is compact as a closed subset of a compact set. □

Proposition 40. Let (X, τ) be a locally compact Hausdorff space, and let $C \subseteq X$ be compact, and $O \in \tau$ with $C \subseteq O$. Then $\exists$ open U such that

$$C \subseteq U \subseteq \overline{U} \subseteq O$$

with $\overline{U}$ compact.

Proof. Since X is a locally compact hausdorff space and $C \subseteq X$ is compact we can find $V \in \tau$ such that $C \subseteq V$ with $\overline{V}$ compact. Then we have both $C \subseteq V$ and $C \subseteq O$ so let

$$W = V \cap O$$

then $C \subseteq W$ and since

$$V \cap O \subseteq V \implies W \subseteq V$$

and so $\overline{W} \subseteq \overline{V}$ which tells us that $\overline{W}$ is compact as the closed subset of a compact set. Then ∂W is closed in the relative topology of $\overline{W}$ and since $\partial W = \overline{W} \setminus W$ we have that $C \not\subseteq \partial W$, and since X is Hausdorff, $\overline{W}$ is compact Hausdorff, and so it is normal. Then as $C, \partial W$ are closed and disjoint, by normality $\exists$ disjoint $U, Q \in \tau$ such that

$$C \subseteq U, \text{ and } \partial W \subseteq Q$$

then since $U \cap Q = \varnothing$ we have $Q^c \supseteq U$ and also

$$U \subseteq Q^c \cap \overline{W}$$

which implies

$$\overline{U} \subseteq \overline{Q^c \cap \overline{W}} = Q^c \cap \overline{W}$$

since both $Q^c, \overline{W}$ are closed, and the intersection of closed sets is closed. Next we note that $Q^c \cap \overline{W} \subseteq Q^c$ and $\partial W^c \supseteq Q^c$, and in the relative topology of $\overline{W}$ we have

$$\partial W^c = \left(\overline{W} \cap W^c\right)^c \cap \overline{W} = \left(\overline{W}^c \cup W\right) \cap \overline{W} = W$$

and so we have

$$\overline{U} \subseteq Q^c \subseteq \partial W^c = W$$

and so $\overline{U}$ will be compact as the closed subset of compact $\overline{W}$. And so we have

$$C \subseteq U \subseteq \overline{U} \subseteq W \subseteq O$$

and hence

$$C \subseteq U \subseteq \overline{U} \subseteq O$$

□

Proposition 41 (Urysohn for Locally Compact Hausdorff). Let (X, τ) be a locally compact Hausdorff space, and let $C \subseteq X$ be compact, and $O \in \tau$ with $C \subseteq O$. Then $\exists$ continuous $f : X \to [0,1]$ such that $f(C) = \{1\}$, and $\text{supp}(f) = \overline{\{x : f(x) \neq 0\}} \subseteq O$ is compact.

Proof. Since X is locally compact Hausdorff and $C \subseteq X$ is compact, we may choose $U \in \tau$ such that

$$C \subseteq U \subseteq \overset{\text{compact}}{\overline{U}} \subseteq O$$

where $C, \partial U$ are closed and disjoint in compact $\overline{U}$, so by Urysohn's Lemma $\exists$ continuous $g : \overline{U} \to [0,1]$ with $g(C) = \{1\}$ and $g(\partial U) = \{0\}$. So set

$$f : X \to [0,1], \text{ by } f(x) = \begin{cases} g(x), & x \in \overline{U} \\ 0, & x \notin \overline{U} \end{cases}$$

then $\operatorname{supp}(f) \subseteq \overline{U}$ and is compact as the closed subset of a compact set. So we need to check that f is continuous on X. f is continuous on $\overline{U}$ and continuous on $\overline{U}^c$, if $x \in \partial U$, then $f(x) = g(x) = 0$. Now $[0, \epsilon)$ is open in $[0,1]$, where the continuity of g tells us that $g^{-1}\big([0,\epsilon)\big)$ is open in $\overline{U}$. And so

$$f^{-1}\big([0,\epsilon)\big) = g^{-1}\big([0,\epsilon)\big) \cup \overline{U}^c$$

is open as the union of open sets, and so f is continuous. □

Proposition 42. The intersection of any collection of rings/fields/σ-algebras/σ-rings on a set X is a ring/field/σ-algebra/σ-ring on X.

Proof. We give a proof for rings with the proofs for the others being similar.

Let $\{\mathcal{R}_i\}_{i \in I}$ be a collection of rings on X where I is an indexing set and let

$$\mathcal{R} = \bigcap_{i \in I} \mathcal{R}_i$$

so if $E, F \in \mathcal{R}$, then $E, F \in \mathcal{R}_i,\ \forall\, i \in I$ and since each $\mathcal{R}_i$ is a ring we have

$$E \cup F \in \mathcal{R}_i, \quad \forall\, i \in I$$

and

$$E \setminus F \in \mathcal{R}_i, \quad \forall\, i \in I$$

and thus $E \cup F, E \setminus F \in \mathcal{R}$, and so $\mathcal{R}$ is a ring. □

Theorem 43. Let $\mathcal{P} = \{[a,b) : a < b; a, b \in \mathbb{R}\}$ and let $\alpha : \mathbb{R} \to \mathbb{R}$ be a non-decreasing left continuous function and define

$$\mu_\alpha : \mathcal{R} \to \mathbb{R}, \text{ by } \mu_\alpha\big([a,b)\big) = \alpha(b) - \alpha(a)$$

then μ_α is countably additive.

Proof. Given $[a_0, b_0) \in \mathcal{P}$ such that

$$[a_0, b_0) = \bigsqcup_{i=1}^{\infty} [a_i, b_i)$$

we note that for the ($\geq$) direction it suffices to show that for each $n \in \mathbb{N}$ we have

$$\mu_\alpha\big([a_0, b_0)\big) \geq \sum_{i=1}^{n} \mu_\alpha\big([a_i, b_i)\big)$$

Given any n, re-index the intervals so that $a_i < a_{i+1}$ $\forall\ 1 \le i \le n-1$. Since the intervals are disjoint, we have that $b_i < a_{i+1}$. Now since

$$a_0 \le a_i,\ b_i \le b_0 \quad \forall\ i$$

we have

$$\alpha(b_0) - \alpha(a_0) \ge \alpha(b_n) - \alpha(a_1)$$

then

$$\begin{aligned}
\sum_{i=1}^{n} \mu_\alpha\big([a_i, b_i)\big) &= \sum_{i=1}^{n} \big(\alpha(b_i) - \alpha(a_i)\big) \\
&= \alpha(b_1) - \alpha(a_1) + \alpha(b_2) - \alpha(a_2) + \cdots + \alpha(b_n) - \alpha(a_n) \\
&= \alpha(b_n) - \alpha(a_1) + \alpha(b_1) - \alpha(a_2) + \cdots + \alpha(b_{n-1}) - \alpha(a_n) \\
&= \alpha(b_n) - \alpha(a_1) + \sum_{i=1}^{n-1} \big(\alpha(b_i) - \alpha(a_{i+1})\big)
\end{aligned}$$

and since each $b_i < a_{i+1}$ and α is non-decreasing we have that $\sum_{i=1}^{n-1} \big(\alpha(b_i) - \alpha(a_{i+1})\big) \le 0$ and therefore

$$\mu_\alpha\big([a_0, b_0)\big) = \alpha(b_0) - \alpha(a_0) \ge \alpha(b_n) - \alpha(a_1) \ge \sum_{i=1}^{n} \mu_\alpha\big([a_i, b_i)\big)$$

Next, let $\epsilon > 0$ be given and choose $b_0' < b_0$ such that

$$\alpha(b_0') \ge \alpha(b_0) - \frac{\epsilon}{2}$$

and by the left continuity of α for each i choose $a_i' < a_i$ such that

$$\alpha(a_i') \ge \alpha(a_i) - \epsilon_i$$

where each $\epsilon_i > 0$ such that $\sum_{i=1}^{\infty} \epsilon_i = \frac{\epsilon}{2}$. Then we have

$$[a_0, b_0'] \subseteq [a_0, b_0) = \bigsqcup_{i=1}^{\infty} [a_i, b_i) \subseteq \bigcup_{i=1}^{\infty} (a_i', b_i)$$

then, since $\bigcup_{i=1}^{\infty} (a_i', b_i)$ is an open cover of $[a_0, b_0']$ which is compact, we know that $[a_0, b_0']$ admits a finite subcover, so that

$$[a_0, b_0'] \subseteq \bigcup_{i=1}^{m} (a_i', b_i)$$

then re-indexing the intervals so that

$$a_0 \in (a_1', b_1) \text{ and } b_1 \in (a_2', b_2), \cdots, b_0' \in (a_m', b_m)$$

then

$$
\begin{aligned}
\alpha(b_0) - \alpha(a_0) &\leq \alpha(b_0') + \frac{\epsilon}{2} - \alpha(a_0) && b_0' < b_0 \\
&\leq \alpha(b_m) - \alpha(a_1') + \frac{\epsilon}{2} && b_0' \leq b_m,\ a_1' \leq a_0 \\
&\leq \alpha(b_m) - \alpha(a_1') + \frac{\epsilon}{2} + \sum_{i=1}^{m-1} \big(\alpha(b_i) - \alpha(a_{i+1}')\big) && b_i \geq a_{i+1}' \\
&= \sum_{i=1}^{m} \big(\alpha(b_i) - \alpha(a_i')\big) + \frac{\epsilon}{2} \\
&\leq \sum_{i=1}^{m} \big(\alpha(b_i) - (\alpha(a_i) - \epsilon_i)\big) + \frac{\epsilon}{2} \\
&= \sum_{i=1}^{m} \big(\alpha(b_i) - \alpha(a_i) + \epsilon_i\big) + \frac{\epsilon}{2} \\
&\leq \sum_{i=1}^{\infty} \big(\alpha(b_i) - \alpha(a_i)\big) + \sum_{i=1}^{\infty} \epsilon_i + \frac{\epsilon}{2} \\
&= \sum_{i=1}^{\infty} \big(\alpha(b_i) - \alpha(a_i)\big) + \epsilon
\end{aligned}
$$

and since ϵ was arbitrary we conclude

$$
\mu_\alpha\big([a_0, b_0)\big) = \alpha(b_0) - \alpha(a_0) \leq \sum_{i=1}^{\infty} \big(\alpha(b_i) - \alpha(a_i)\big) = \sum_{i=1}^{\infty} \mu_\alpha\big([a_i, b_i)\big)
$$

and thus we conclude that $\mu_\alpha\big([a_0, b_0)\big) = \sum_{i=1}^{\infty} \mu_\alpha\big([a_i, b_i)\big)$. And so μ_α is countably additive. $\square$

Lemma 44. Let $\mathcal{S}$ be a semiring. If $E, E_1, \ldots, E_n \in \mathcal{S}$, then $\exists\ F_1, \ldots, F_k \in \mathcal{S}$ such that

$$
((\ldots(E \setminus E_1) \setminus E_2) \setminus \ldots) \setminus E_n) = \bigsqcup_{i=1}^{k} F_i
$$

Proof. By induction. Base case: if $n = 1$ then $E \setminus E_1 = \bigsqcup_{i=1}^{k} F_i$ with $F_1, \ldots, F_k \in \mathcal{S}$ by the definition of semiring.

So suppose the result holds for $n - 1$ with $n > 1$. Then $\exists\ G_1, \ldots G_m$ such that

$$
\begin{aligned}
((\dots(E \setminus E_1) \setminus E_2) \setminus \dots) \setminus E_{n-1}) \setminus E_n) &= E \setminus \bigsqcup_{i=1}^{n} E_i \\
&= \left(E \setminus \bigsqcup_{i=1}^{n-1} E_i \right) \setminus E_n \\
&= \left(\bigsqcup_{i=1}^{m} G_i \right) \setminus E_n \\
&= \bigsqcup_{i=1}^{m} (G_i \setminus E_n) \\
&= \bigsqcup_{i=1}^{m} \bigsqcup_{j=1}^{l} G_{ij}
\end{aligned}
$$

where by the definition of a semiring we have that each $G_{ij} \in \mathcal{S}$. □

Lemma 45. Let $\mathcal{S}$ be a semiring, μ_0 a premeasure on $\mathcal{S}$, and let $E, F_i \in \mathcal{S}$ such that $E \subseteq \bigsqcup_{i=1}^{\infty} F_i$ then

$$\mu_0(E) \leq \sum_{i=1}^{\infty} \mu_0(F_i)$$

Proof. First we note that it is sufficient to show that

$$\mu_0(E) \leq \sum_{i=1}^{n} \mu_0(F_i)$$

for each finite n, that is for each $n \in \mathbb{N}$. Then

$$\bigsqcup_{i=1}^{n} F_i = E \sqcup \left(\bigsqcup_{i=1}^{n} F_i \setminus E \right) = E \sqcup \left(\bigsqcup_{i=1}^{n_1} \bigsqcup_{j=1}^{n_2} G_{ij} \right)$$

where each $G_{ij} \in \mathcal{S}$ and are disjoint by the previous Lemma, and by construction E and $\bigsqcup_{i=1}^{n_1} \bigsqcup_{j=1}^{n_2} G_{ij}$ are disjoint, so we have

$$\sum_{i=1}^{n} \mu_0(F_i) = \mu_0(E) + \sum_{i}^{n_1} \sum_{j=1}^{n_2} \mu_0(G_{ij}) \geq \mu_0(E)$$

□

Lemma 46. Let $\mathcal{S}$ be a semiring, μ_0 a premeasure on $\mathcal{S}$, then μ_0 is countably subadditive; i.e. if $E, F_i \in \mathcal{S}$ such that $E \subseteq \bigcup_{i=1}^{\infty} F_i$ then

$$\mu_0(E) \leq \sum_{i=1}^{\infty} \mu_0(F_i)$$

Proof. First note that

$$E = \bigcup_{i=1}^{\infty} (E \cap F_i)$$

letting $H_i = E \cap F_i$ where by definition we have that each $H_i \in \mathcal{S}$, so that by $E \setminus \bigsqcup_{i=1}^{n} E_i = \bigsqcup_{i=1}^{k} F_i$ we have

$$\begin{aligned} E &= \bigcup_{i=1}^{\infty} H_i \\ &= H_1 \sqcup (H_2 \setminus H_1) \sqcup \cdots \sqcup \left(H_m \setminus \bigcup_{j=1}^{m-1} H_j \right) \sqcup \ldots \\ &= H_1 \sqcup \left(\bigsqcup_{i=1}^{n_1} G_{2_i} \right) \sqcup \cdots \sqcup \left(\bigsqcup_{i=1}^{n_m} G_{m_i} \right) \sqcup \ldots \end{aligned}$$

then

$$\mu_0(E) = \mu_0(H_1) + \sum_{i=1}^{n_1} \mu_0(G_{2_i}) + \sum_{i=1}^{n_m} \mu_0(G_{m_i}) + \ldots$$

yet,

$$\bigsqcup_{i=1}^{n_m} G_{m_i} \subseteq \overbrace{E \cap F_m}^{=H_m} \subseteq F_m$$

so that $\sum_{i=1}^{n_m} \mu_0(G_{m_i}) \leq \mu_0(F_m)$, and therefore,

$$\mu_0(E) \leq \sum_{i=1}^{\infty} \mu_0(F_i)$$

□

Theorem 47. Let $\mathcal{S}$ be a semiring and μ_0 a premeasure on $\mathcal{S}$, then defining

$$\mu^* : \mathcal{H}(\mathcal{S}) \to [0, \infty]$$

by

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} \mu_0(E_i) : A \subseteq \bigcup_{i=1}^{\infty} E_i ; E_i \in \mathcal{S} \right\}$$

then μ^* is an outer measure which extends μ_0; i.e. $\mu^*|_{\mathcal{S}} = \mu_0$.

Proof. First, since $\varnothing \in \mathcal{S}$, so setting $E_i = \varnothing \ \forall i$ gives

$$\mu^*(\varnothing) \leq \sum_{i=1}^{\infty} \mu_0(\varnothing) = 0$$

and so $\mu^*(\varnothing) = 0$.

Now, if $A \subseteq B$ then $B \subseteq \bigcup_{i=1}^{\infty} E_i \implies A \subseteq \bigcup_{i=1}^{\infty} E_i$. So

$$\begin{aligned}
\mu^*(A) &= \inf\left\{\sum_{i=1}^{\infty} \mu_0(E_i) : A \subseteq \bigcup_{i=1}^{\infty} E_i; E_i \in \mathcal{S}\right\} \\
&\leq \inf\left\{\sum_{i=1}^{\infty} \mu_0(E_i) : B \subseteq \bigcup_{i=1}^{\infty} E_i; E_i \in \mathcal{S}\right\} \\
&= \mu^*(B)
\end{aligned}$$

and so μ^* is monotone.

Next, given $\epsilon > 0$, and $A \subseteq \bigcup_{i=1}^{\infty} E_i$ for each E_i choose $E_{ij} \in \mathcal{S}$ for each $j \in \mathbb{N}$ such that $E_i \subseteq \bigcup_{j=1}^{\infty} E_{ij}$ and

$$\sum_{i=1}^{\infty} \mu_0(E_{ij}) \leq \mu^*(E_i) + \frac{\epsilon}{2^i}$$

then

$$A \subseteq \bigcup_{i=1}^{\infty} E_i \subseteq \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} E_{ij}$$

and

$$\begin{aligned}
\mu^*(A) &\leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mu_0(E_{ij}) \\
&\leq \sum_{i=1}^{\infty} \left[\mu^*(E_i) + \frac{\epsilon}{2^i}\right] \\
&\leq \sum_{i=1}^{\infty} \mu^*(E_i) + \epsilon
\end{aligned}$$

Since ϵ was arbitrary we conclude that $\mu^*(A) \leq \sum_{i=1}^{\infty} \mu^*(E_i)$ and μ^* is countably subadditive.

Now let $E \in \mathcal{S}$, by the definition of μ^* we have that

$$\mu^*(E) \leq \mu_0(E)$$

now if $E \subseteq \bigcup_{i=1}^{\infty} F_i$ for $F_i \in \mathcal{S}$, then by countable subadditivity we have

$$\mu_0(E) \leq \sum_{i=1}^{\infty} \mu_0(F_i)$$

and in particular this holds for the infimum and so

$$\mu_0(E) \leq \inf\left\{\sum_{i=1}^{\infty}\mu_0(F_i) : E \subseteq \bigcup_{i=1}^{\infty} F_i; F_i \in \mathcal{S}\right\} = \mu^*(E)$$

and thus $\mu^*|_{\mathcal{S}} = \mu_0$ $\square$

Theorem 48 (**Caratheodory's Theorem**)**.** Let μ^* be an outer measure on a hereditary σ-ring $\mathcal{H}$. Let $M(\mu^*)$ be the set of μ^*-measurable sets in $\mathcal{H}$. Then $M(\mu^*)$ is a σ-ring and $\mu^*|_{M(\mu^*)}$ is a measure.

Proof. First we show that $M(\mu^*)$ is a ring, so let $E, F \in M(\mu^*)$, and $A \in \mathcal{H}$ be arbitrary. Then

$$\begin{aligned}
\mu^*\big(A \cap (E \cup F)\big) &+ \mu^*\big(A \cap (E \cup F)^c\big) \\
&= \mu^*\big((A \cap E) \cup (A \cap F)\big) + \mu^*\big(A \cap E^c \cap F^c\big) \\
&= \mu^*\big((A \cap E) \sqcup ((A \setminus E) \cap F)\big) + \mu^*\big((A \setminus E) \cap F^c\big) \\
&\leq \mu^*\big(A \cap E\big) + \mu^*\big((A \setminus E) \cap F\big) + \mu^*\big((A \setminus E) \cap F^c\big) \\
&= \mu^*\big(A \cap E\big) + \mu^*\big(A \setminus E\big) && F\ \mu^*\text{-measurable} \\
&= \mu^*(A) && E\ \mu^*\text{-measurable}
\end{aligned}$$

that is $\mu^*\big(A \cap (E \cup F)\big) + \mu^*\big(A \cap (E \cup F)^c\big) \leq \mu^*(A)$ and since we always have $\mu^*\big(A \cap (E \cup F)\big) + \mu^*\big(A \cap (E \cup F)^c\big) \geq \mu^*(A)$ by the subadditivity of μ^*, we have

$$\mu^*(A) = \mu^*\big(A \cap (E \cup F)\big) + \mu^*\big(A \cap (E \cup F)^c\big)$$

and so $E \cup F \in M(\mu^*)$.

Next we check set difference where we have

$$\begin{aligned}
\mu^*\big(A \cap (E \setminus F)\big) &+ \mu^*\big(A \cap (E \setminus F)^c\big) \\
&= \mu^*\big(A \cap (E \cap F^c)\big) + \mu^*\big(A \cap (E \cap F^c)^c\big) \\
&= \mu^*\big(A \cap E \cap F^c\big) + \mu^*\big(A \cap (E^c \cup F)\big) \\
&= \mu^*\big((A \cap E) \setminus F\big) + \mu^*\big((A \cap E^c) \cup (A \cap F)\big) \\
&= \mu^*\big((A \cap E) \setminus F\big) + \mu^*\big((A \cap E^c) \sqcup ((A \setminus E^c) \cap F)\big) \\
&\leq \mu^*\big((A \cap E) \setminus F\big) + \mu^*\big(A \cap E^c\big) + \mu^*\big(A \cap E \cap F\big) \\
&= \mu^*\big(A \cap E\big) + \mu^*\big(A \cap E^c\big) && F\ \mu^*\text{-measurable} \\
&= \mu^*(A) && E\ \mu^*\text{-measurable}
\end{aligned}$$

that is $\mu^*\big(A \cap (E \setminus F)\big) + \mu^*\big(A \cap (E \setminus F)^c\big) \leq \mu^*(A)$ and thus

$$\mu^*(A) = \mu^*\big(A \cap (E \setminus F)\big) + \mu^*\big(A \cap (E \setminus F)^c\big)$$

and so $E \setminus F \in M(\mu^*)$.

And so $M(\mu^*)$ is a ring.

Now we note that if $E, F \in M(\mu^*)$ are disjoint that

$$\mu^*\big(A \cap (E \sqcup F)\big) = \mu^*\big((A \cap E) \sqcup (A \cap F)\big) = \mu^*\big(A \cap E\big) + \mu^*\big(A \cap F\big)$$

since $F \sqcup E$ is μ^*-measurable and $A \cap (E \sqcup F) \in \mathcal{H}$ so that

$$\begin{aligned}
\mu^*\big(A \cap & (E \sqcup F)\big) \\
&= \mu^*\Big(\big(A \cap (E \sqcup F)\big) \cap E\Big) + \mu^*\Big(\big(A \cap (E \sqcup F)\big) \cap E^c\Big) && \text{measurability} \\
&= \mu^*\Big(A \cap \big((E \cap E) \sqcup (F \cap E)\big)\Big) + \mu^*\Big(A \cap \big((E \cap E^c) \sqcup (F \cap E^c)\big)\Big) \\
&= \mu^*\Big(A \cap \big(E \sqcup \varnothing\big)\Big) + \mu^*\Big(A \cap \big(\varnothing \sqcup F\big)\Big) && E \cap F = \varnothing \\
&= \mu^*\big(A \cap E\big) + \mu^*\big(A \cap F\big)
\end{aligned}$$

Next suppose $E = \bigcup_{i=1}^{\infty} E_i$ where each $E_i \in M(\mu^*)$ defining $F_1 = E_1$ and $F_k = E_k \setminus \bigcup_{i=1}^{k-1} E_i$ for each $k > 1$ we see that

$$E = \bigcup_{i=1}^{\infty} E_i = \bigsqcup_{i=1}^{\infty} F_i$$

where each $F_i \in M(\mu^*)$ since $M(\mu^*)$ is a ring, and we note

$$E \supseteq \bigsqcup_{i=1}^{n} F_i \implies E^c \subseteq \left(\bigsqcup_{i=1}^{n} F_i\right)^c$$

Then for any $A \in \mathcal{H}$

$$\begin{aligned}
\mu^*(A) &= \mu^*\left(A \cap \bigsqcup_{i=1}^{n} F_i\right) + \mu^*\left(A \cap \left(\bigsqcup_{i=1}^{n} F_i\right)^c\right) \\
&\geq \mu^*\left(A \cap \bigsqcup_{i=1}^{n} F_i\right) + \mu^*\big(A \cap E^c\big) && \text{subadditivity} \\
&= \sum_{i=1}^{n} \mu^*(A \cap F_i) + \mu^*\big(A \cap E^c\big)
\end{aligned}$$

where only the RHS depends on n to taking the limit to infinity gives

$$\begin{aligned}
\mu^*(A) &\geq \sum_{i=1}^{\infty} \mu^*(A \cap F_i) + \mu^*\big(A \cap E^c\big) \\
&\geq \mu^*\left(\bigsqcup_{i=1}^{\infty} (A \cap F_i)\right) + \mu^*\big(A \cap E^c\big) && \text{subadditivity} \\
&= \mu^*\big(A \cap E\big) + \mu^*\big(A \cap E^c\big)
\end{aligned}$$

and therefore we have that $E \in M(\mu^*)$ and so $M(\mu^*)$ is closed under countable unions, and thus $M(\mu^*)$ is a σ-ring.

Now we note from

$$\mu^*(A) \geq \sum_{i=1}^{\infty} \mu^*(A \cap F_i) + \mu^*\big(A \cap E^c\big) \geq \mu^*\big(A \cap E\big) + \mu^*\big(A \cap E^c\big)$$

yet we always have $\mu^*(A) \leq \mu^*\big(A \cap E\big) + \mu^*\big(A \cap E^c\big)$, so that we actually have

$$\mu^*(A) = \sum_{i=1}^{\infty} \mu^*(A \cap F_i) + \mu^*\big(A \cap E^c\big)$$

Then since this holds for any $A \in \mathcal{H}$ letting $A = E = \bigsqcup_{i=1}^{\infty} F_i$ where each $F_i \in M(\mu^*)$ gives

$$\begin{aligned}\mu^*|_{M(\mu^*)}\left(\bigsqcup_{i=1}^{\infty} F_i\right) &= \sum_{j=1}^{\infty} \mu^*\left(\bigsqcup_{i=1}^{\infty}(F_i \cap F_j)\right) + \mu^*(\varnothing)\\ &= \sum_{j=1}^{\infty} \mu^*(F_i)\end{aligned}$$

and thus $\mu^*|_{M(\mu^*)}$ is a measure on the σ-ring $M(\mu^*)$. □

Proposition 49. Let μ^* be an outer measure on a hereditary σ-ring $\mathcal{H}$. Then $\mu^*|_{M(\mu^*)}$ is a complete measure, if $M(\mu^*) \neq \varnothing$.

Proof. It suffices to show that if $\mu^*(E) = 0$ then $E \in M(\mu^*)$. So let $A \in \mathcal{H}$, then since $A \cap E \subsetneq E$ monotonicity gives $\mu^*(A \cap E) = 0$ and $A \cap E^c \subsetneq A$ so again by monotonicity we get

$$\mu^*(A \cap E) + \mu^*(A \cap E^c) = 0 + \mu^*(A \cap E^c) \leq \mu^*(A)$$

and thus

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

and so E is μ^*-measurable and hence $E \in M(\mu^*)$. And therefore $\mu^*|_{M(\mu^*)}$ is complete. □

Theorem 50. If μ_0 is a premeasure on a semiring $\mathcal{S}$, and if μ^* is the outer measure on $\mathcal{H}(\mathcal{S})$ determined by μ_0, then $\mathcal{S} \subsetneq M(\mu^*)$.

Proof. We must show this if $E \in \mathcal{S}$, then $E \in M(\mu^*)$; that is, $\forall\ A \in \mathcal{H}(\mathcal{S})$

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

If $\mu^*(A) = \infty$ then $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$ and we are done.

So let us assume that $\mu^*(A) < \infty$. Given $\epsilon > 0$, since $A \in \mathcal{H}(\mathcal{S})$, we may select $F_i \in \mathcal{S}$ such that $A \subseteq \bigcup_{i=1}^{\infty} F_i$ and

$$\sum_{i=1}^{\infty} \mu_0(F_i) \leq \mu^*(A) + \epsilon$$

now $F_i = (F_i \cap E) \sqcup (F_i \setminus E)$, and since $\mathcal{S}$ is a semiring $\exists\ G_{ij} \in \mathcal{S}$ such that $F_i \setminus E = \bigsqcup_{j=1}^{n_j} G_{ij}$ so that

$$\begin{aligned}
\sum_{i=1}^{\infty} \mu_0(F_i) &= \sum_{i=1}^{\infty} \mu_0 \left((F_i \cap E) \sqcup \bigsqcup_{j=1}^{n_j} G_{ij} \right) \\
&= \sum_{i=1}^{\infty} \left[\mu_0(F_i \cap E) + \sum_{j=1}^{n_j} \mu_0(G_{ij}) \right] \\
&= \sum_{i=1}^{\infty} \mu_0(F_i \cap E) + \sum_{i=1}^{\infty} \sum_{j=1}^{n_j} \mu_0(G_{ij})
\end{aligned}$$

and $A \subseteq \bigcup_{i=1}^{\infty} F_i$ implies

$$A \cap E \subseteq \bigcup_{i=1}^{\infty} (F_i \cap E) \quad \text{and} \quad A \setminus E \subseteq \bigcup_{i=1}^{\infty} (F_i \setminus E) = \bigcup_{i=1}^{\infty} \bigsqcup_{j=1}^{n_j} G_{ij}$$

and thus we have

$$\begin{aligned}
\mu^*(A) + \epsilon &\geq \sum_{i=1}^{\infty} \mu_0(F_i) \\
&= \sum_{i=1}^{\infty} \mu_0(F_i \cap E) + \sum_{i=1}^{\infty} \sum_{j=1}^{n_j} \mu_0(G_{ij}) \\
&\geq \inf \left\{ \sum_{i=1}^{\infty} \mu_0(F_i \cap E) : A \cap E \subseteq \bigcup_{i=1}^{\infty} (F_i \cap E); F_i \cap E \in \mathcal{S} \right\} \\
&\quad + \inf \left\{ \sum_{i=1}^{\infty} \sum_{j=1}^{n_j} \mu_0(G_{ij}) : A \setminus E \subseteq \bigcup_{i=1}^{\infty} \bigsqcup_{j=1}^{n_j} G_{ij}; G_{ij} \in \mathcal{S} \right\} \\
&= \mu^*(A \cap E) + \mu^*(A \setminus E)
\end{aligned}$$

and since ϵ is arbitrary we conclude that $\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \setminus E)$ giving

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E)$$

and so E is μ^*-measurable and thus $E \in M(\mu^*)$. And therefore $\mathcal{S} \subseteq M(\mu^*)$. $\square$

Proposition 51. Let μ_0 be a premeasure on a semiring $\mathcal{S}$, and μ^* the outer measure on $\mathcal{H}(\mathcal{S})$ determined by μ_0. Then $\mathcal{H}(\mathcal{S}) = \mathcal{H}\big(M(\mu^*)\big)$ and if $E \in \mathcal{H}(\mathcal{S})$ then

$$\mu^*(E) = \inf\big\{\mu^*|_{\sigma(\mathcal{S})}(F) : E \subseteq F; F \in \sigma(\mathcal{S})\big\} = \inf\big\{\mu^*|_{M(\mu^*)}(F) : E \subseteq F; F \in M(\mu^*)\big\}$$

which is to say that $\mu^*|_{\sigma(\mathcal{S})} = \mu^* = \mu^*|_{M(\mu^*)}$

Proof. First since

$$\mathcal{S} \subseteq M(\mu^*) \subseteq \mathcal{H}(\mathcal{S})$$

we have $\mathcal{H}(\mathcal{S}) = \mathcal{H}\big(M(\mu^*)\big)$.

Next, let $E \in \mathcal{H}(\mathcal{S})$ then

$$\begin{aligned}
\mu^*(E) &= \inf\left\{\sum_{i=1}^{\infty} \mu_0(F_i) : E \subseteq \bigcup_{i=1}^{\infty} F_i; F_i \in \mathcal{S}\right\} && \text{def of } \mu^* \\
&\geq \inf\big\{\mu^*|_{\sigma(\mathcal{S})}(F) : E \subseteq F; F \in \sigma(\mathcal{S})\big\} && \text{countable subadditivity of } \mu^* \\
&\geq \inf\big\{\mu^*|_{M(\mu^*)}(F) : E \subseteq F; F \in M(\mu^*)\big\} && M(\mu^*) \supseteq \sigma(\mathcal{S}) \\
&\geq \mu^*(E) && \text{monotonicity of } \mu^*
\end{aligned}$$

and thus the inner inequalities must be equalities. □

Theorem 52 (Uniqueness of Extensions). If μ_0 is a σ-finite premeasure on a semiring $\mathcal{S}$, and if $\mathcal{R}$ is a σ-ring such that $\mathcal{S} \subseteq \mathcal{R} \subseteq M(\mu^*)$, and if ν is a non-negative extension of μ_0 to a measure on $\mathcal{R}$, then $\nu = \mu^*|_{\mathcal{R}}$.

Proof. If $E \in \mathcal{R}$, and $E \subseteq \bigcup_{i=1}^{\infty} F_i$ where each $F_i \in \mathcal{S}$, then

$$\begin{aligned}
\nu(E) &\leq \sum_{i=1}^{\infty} \nu(F_i) && \text{non-negative measures are countably subadditive} \\
&= \sum_{i=1}^{\infty} \mu_0(F_i) && \nu \text{ an extension of } \mu_0 \text{ and } F_i \in \mathcal{S}
\end{aligned}$$

and thus $\nu(E) \leq \mu^*(E)\ \forall\ E \in \mathcal{R}$, so it remains to show that $\nu(E) \geq \mu^*(E)\ \forall\ E \in \mathcal{R}$

Case 1: Suppose $E \in \mathcal{R}$, and that $\exists\ F \in \mathcal{S}$ such that $E \subseteq F$, and $\mu_0(F) < \infty$. Then, since

$$F = (F \cap E) \sqcup (F \setminus E) = E \sqcup (F \setminus E)$$

we have, by the measurability of E

$$\begin{aligned}
\nu(F) &= \nu(E) + \nu(F \setminus E) \\
&\leq \mu^*(E) + \mu^*(F \setminus E) \\
&= \mu^*(F) \\
&= \mu_0(F) \\
&= \nu(F)
\end{aligned}$$

and thus

$$\nu(E) + \nu(F \setminus E) = \mu^*(E) + \mu^*(F \setminus E)$$

yet,

$$\nu(E) \leq \mu^*(E) < \infty \quad \text{and} \quad \nu(F \setminus E) \leq \mu^*(F \setminus E) < \infty$$

and thus we must have $\mu^*(E) = \nu(E)$

Case 2: Let $E \in \mathcal{R}$ be arbitrary. Then, since μ_0 is assumed to be σ-finite. $\exists\ \{F_i\}_{i=1}^{\infty} \in \mathcal{S}$ such that $\mu_0(F_i) < \infty$ for each i and $E \subseteq \bigcup_{i=1}^{\infty} F_i$, since $E \in \mathcal{R} \subseteq M(\mu^*) \subseteq \mathcal{H}(\mathcal{S})$ and $\mathcal{H}(\mathcal{S})$ is defined to be the collection of all sets countably covered by elements of $\mathcal{S}$. Then disjointizing we get $\{G_{ij}\} \in \mathcal{S}$ such that $\mu_0(G_{ij}) < \infty\ \forall\ i, j$, with $E \subseteq \bigsqcup_{i,j\geq 1} G_{ij}$ and $E = \bigsqcup_{i,j\geq 1}(E \cap G_{ij})$. Then since $E \cap G_{ij} \subseteq G_{ij}$ so Case 1 gives

$$\begin{aligned} \nu(E) &= \sum_{i,j\geq 1} \nu(E \cap G_{ij}) \\ &= \sum_{i,j\geq 1} \mu^*(E \cap G_{ij}) \\ &= \mu^*(E) \end{aligned}$$

and hence, $\mu^*(E) = \nu(E)$.

and therefore we conclude that $\nu = \mu^*|_{\mathcal{R}}$. □

Proposition 53. Let $(X, \mathcal{S}, \mu)$ be a measure space and B a Banach space. A function f defined almost everywhere, i.e. defined on $X \setminus N(\mu)$, is μ-measurable iff $\exists\ \{f_n\}$ of simple $\mathcal{S}$-measurable such that $f_n \to f$ pointwise almost everywhere; i.e. $\forall\ x \in X \setminus N(\mu)$ we have $f_n(x) \to f(x)$.

Proof. Suppose that f is μ-measurable, then $\exists\ \{f_n\}$ of simple μ-measurable functions and a null-set $N_0(\mu)$, such that $\forall\ x \in X \setminus N_0(\mu)$ we have $f_n(x) \to f(x)$. Since each f_n is simple μ-measurable we have for each n that

$$f_n = \sum_{i=1}^{k_n} b_i^n \chi_{F_i^n}$$

where each $b_i^n \in B$ and each $F_i^n \in \mathcal{S} \sqcup N(\mu)$, that is

$$F_i^n = E_i^n \sqcup N_i^n, \text{ where } E_i^n \in \mathcal{S},\ N_i^n \in N(\mu)$$

so let

$$N = N_0(\mu) \cup \left(\bigcup_{n,i} N_i^n \right)$$

then N is a null-set, and letting

$$\varepsilon_n = \sum_{i=1}^{k_n} b_i^n \chi_{E_i^n}$$

then each ε_n is a simple $\mathcal{S}$-measurable function.

Then since $\varepsilon_n|_{X\setminus N} = f_n$, then $\forall\ x \in X \setminus N$ we have $\varepsilon_n(x) \to f(x)$.

Conversely, if $\exists\ \{f_n\}$ of simple $\mathcal{S}$-measurable functions such that $\forall\ x \in X \setminus N(\mu)$ we have $f_n(x) \to f(x)$, then f is $\mathcal{S}$-measurable on $X \setminus N(\mu)$. Then since $\mathcal{S}$-measurable implies μ-measurable we have that f is μ-measurable. $\square$

Proposition 54. Let $(X, \mathcal{S}, \mu)$ be a measure space and B a Banach space. If f, g are simple $\mathcal{S}$-measurable functions, then $f + g$ is a simple $\mathcal{S}$-measurable function.

Proof. First suppose $f = \sum_{i=1}^n b_i \chi_{E_i}$, and $g = c\chi_F$, to get F contained in the E_i's let us set $E_{n+1} = F \setminus \bigsqcup_{i=1}^n E_i$ and $b_{n+1} = 0$, then

$$F \subseteq \bigsqcup_{i=1}^{n+1} E_i \implies F = \bigsqcup_{i=1}^{n+1} (F \cap E_i)$$

and

$$f = \sum_{i=1}^{n+1} b_i \chi_{E_i} = \sum_{i=1}^{n+1} b_i \Big(\chi_{E_i \cap F} + \chi_{E_i \setminus F} \Big)$$
$$g = \sum_{i=1}^{n+1} c \chi_{E_i \cap F}$$

and so

$$f + g = \sum_{i=1}^{n+1} (b_i + c) \chi_{E_i \cap F} + \sum_{i=1}^{n+1} b_i \chi_{E_i \setminus F}$$

and so $f + g$ is a simple $\mathcal{S}$-measurable function. The general case follows inductively. $\square$

Proposition 55. Let $(X, \mathcal{S}, \mu)$ be a measure space and B a Banach space. Let

$$f, g : X \to B$$

be $\mathcal{S}$-measurable/μ-measurable functions, and let c be a scalar. Then $f + g, cf, ||f(\cdot)||$ are $\mathcal{S}$-measurable/μ-measurable functions. If f is scalar valued, then fg is $\mathcal{S}$-measurable/μ-measurable. If f and g are $\mathbb{R}$ valued functions, then $\max(f, g)$ and $\min(f, g)$ are $\mathcal{S}$-measurable/μ-measurable functions.

Proof. If $\{f_n\}, \{g_n\}$ are sequences of simple $\mathcal{S}$-measurable such that $\forall\ x \in X$

$$f_n(x) \to f(x)$$
$$g_n(x) \to g(x)$$

then $\forall\ x \in X$ we have

$$(f_n + g_n)(x) = f_n(x) + g_n(x) \to f(x) + g(x) = (f+g)(x)$$

the next follows as $\{cf_n\} = c\{f_n\}$, and if $f_n \to f\ \forall\ x \in X$, then $||f_n(x)|| = \sum_{i=1}^{n} ||b_i|| \chi_{E_i}(x) = ||b_i|| = ||f(x)||$. Then fg follows from cf

the last two follow from the first 4 and the fact that

$$\max(f,g) = \frac{f+g+|f-g|}{2}$$
$$\min(f,g) = \frac{f+g-|f-g|}{2}$$

□

Lemma 56. If $\{f_n\}$ is a sequence of functions from a set X to a Banach space B which converge to f pointwise, and if for any open set $U \subseteq B$ we define

$$U_n = \left\{y \in U : d\big(y, U^c\big) > \tfrac{1}{n}\right\}$$

then

$$f^{-1}(U) = \bigcup_{n=1}^{\infty} \bigcup_{K=1}^{\infty} \bigcap_{k=K}^{\infty} f_k^{-1}(U_n)$$

for all open $U \subseteq B$.

Proof.

$$\begin{aligned}
x \in f^{-1}(U) &\iff f(x) \in U \\
&\iff \exists\ n, K \in \mathbb{N} \text{ such that} \\
&f_k(x) \in U_n\ \forall\ k \geq K \\
&\iff \exists\ n, K \in \mathbb{N} \text{ such that} \\
&x \in f_k^{-1}(U_n)\ \forall\ k \geq K \\
&\iff \exists\ n, K \in \mathbb{N} \text{ such that} \\
&x \in \bigcap_{k=K}^{\infty} f_k^{-1}(U_n) \qquad\qquad \overline{U}_n \subseteq U_{n+1} \\
&\iff x \in \bigcup_{n=1}^{\infty} \bigcup_{K=1}^{\infty} \bigcap_{k=K}^{\infty} f_k^{-1}(U_n)
\end{aligned}$$

□

Theorem 57. Let X be a set, $\mathcal{S}$ a σ-ring of subsets of X, B be a Banach space, and let

$$f : X \to B$$

be a function, then f is $\mathcal{S}$-measurable if

1. $f(X) \subseteq B$ is separable.
2. $f^{-1}(U) \cap \operatorname{car}(f) \in \mathcal{S}$ for all open $U \subseteq B$.

Proof. Suppose that f is $\mathcal{S}$-measurable, then $\exists\ \{f_n\}$ of simple $\mathcal{S}$-measurable functions such that $\forall\, x \in X$ we have $f_n(x) \to f(x)$. Since each f_n is simple $\mathcal{S}$-measurable its range is finite so for each n let

$$\operatorname{Im}(f_n) = \{b_1^n, \dots, b_{k_n}^n\}$$

and let

$$R = \overline{\bigcup_{n=1}^{\infty} \operatorname{Im}(f_n)}$$

so given $\epsilon > 0$, then

$$\begin{aligned} b \in \operatorname{Im}(f) &\iff \exists\ x \in X \text{ such that } f(x) = b \\ &\iff f_n(x) \to f(x) = b \\ &\iff \ \exists\ n \in \mathbb{N} \text{ such that } ||f_n(x) - b|| < \epsilon \end{aligned}$$

and therefore $B_\epsilon(b) \cap R \neq \varnothing$. Since $b \in \operatorname{Im}(f)$ was arbitrary we conclude that $\forall\, b \in \operatorname{Im}(f)$ there is a ball containing b which has nonempty intersection with R, and so $f(X) \subseteq R$. And for each n there is some $A_n \subseteq B$ such that $A_n \subseteq \operatorname{Im}(f_n)$ is countably dense in the range of f_n, then

$$\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} \operatorname{Im}(f_n)$$

is countably dense, and so $\bigcup_{n=1}^{\infty} \operatorname{Im}(f_n)$ is separable, and hence so is R, thereby making $\operatorname{Im}(f) = f(X)$ separable as the subset of a separable set.

Now let $U \subseteq B$ be any open set, then since

$$f^{-1}(U) \cap \operatorname{car}(f) = f^{-1}\big(U \setminus \{0\}\big)$$

it suffices to show that if U is any open set such that $U \not\ni 0$, then $f^{-1}(U) \in \mathcal{S}$, then with

$$U_n = \big\{y \in U : d\big(y, (U \setminus \{0\})^c\big) > \tfrac{1}{n}\big\}$$

we will have each $U_n \not\ni 0$ and

$$f^{-1}(U) = \bigcup_{n=1}^{\infty} \bigcup_{K=1}^{\infty} \bigcap_{k=K}^{\infty} f_k^{-1}(U_n)$$

by the previous lemma, and since the f_k's are simple $\mathcal{S}$-measurable there preimages $f_k^{-1}(U_n) \in \mathcal{S}$, and as $\mathcal{S}$ is a σ-ring, it is closed under countable unions and intersections, and so $f^{-1}(U) \in \mathcal{S}$.

Conversely, suppose that f is such that $f(X) \subseteq B$ is separable and $f^{-1}(U) \in \mathcal{S}$. So we may choose a sequence $\{b_i\} \in B$ which is dense in $f(X)$ since $f(X)$ is separable. So let

$$C_{ij} = \left\{x \in X : x \in \mathrm{car}(f);\ ||f(x) - b_i|| < \tfrac{1}{j}\right\} = f^{-1}\Big(B_{\frac{1}{j}}(b_i) \setminus \{0\}\Big)$$

for all $i, j \in \mathbb{N}$, and since each $B_{\frac{1}{j}}(b_i) \setminus \{0\} \in B$ is open, by hypothesis we have that $f^{-1}\Big(B_{\frac{1}{j}}(b_i) \setminus \{0\}\Big) \in \mathcal{S}$. Then, ordering the pairs (i, j) lexicographically; that is

$$(i,j) \le (k,n) \text{ if } \begin{cases} i < k \\ i = k, \text{ and } j < n \end{cases}$$

so for each fixed n defining

$$E_{ij}^n = C_{ij} \setminus \bigcup \{C_{kl} : (i,j) < (k,l) \le (n,n)\}$$

then the sets E_{ij}^n are disjoint and $E_{ij}^n \subseteq C_{ij}\ \forall\ i, j$. So let

$$f_n = \sum_{i,j=1}^{n} b_i \chi_{E_{ij}^n}$$

and suppose we are given $\epsilon > 0$ and $x \in X$. If $x \notin \mathrm{car}(f)$, then $f(x) = 0$ and so $f_n(x) = 0\ \forall\ n$ and we are done. So suppose that $x \in \mathrm{car}(f)$. Choose j_0 such that $\frac{1}{j_0} < \epsilon$, and choose i_0 so that

$$||f(x) - b_{i_0}|| < \tfrac{1}{j_0}$$

next we note that

$$x \in C_{i_0 j_0} = f^{-1}\Big(B_{\frac{1}{j_0}}(b_{i_0}) \setminus \{0\}\Big)$$

by the definition of j_0 and i_0. So setting $N = \max\{i_0, j_0\}$, then if $n > N$ we have $x \in E_{kl}^n$ where

$$(k,l) = \max\left\{(i,j) : x \in C_{ij};\ (i_0, j_0) \le (i,j) \le (n,n)\right\}$$

then

$$||f(x) - b_k|| < \tfrac{1}{l} \le \tfrac{1}{j_0} < \epsilon$$

and by construction we have

$$f_n(x) = \sum_{i,j=1}^{n} b_i \chi_{E_{ij}^n}(x) = b_k$$

so that

$$||f(x) - b_k|| = ||f(x) - f_n(x)|| < \epsilon$$

and so $f_n \to f$ pointwise, and thus f is $\mathcal{S}$-measurable. $\square$

Proposition 58. If $\{f_n\}$ is a sequence of $\mathcal{S}$-measurable/μ-measurable functions which converge to a function f pointwise/almost everywhere pointwise; i.e. $\forall\, x \in X / \forall\, x \in X \backslash N(\mu)$ we have $f_n(x) \to f(x)$. Then f is $\mathcal{S}$-measurable/μ-measurable.

Proof. Since $\mathcal{S}$-measurable $\implies$ μ-measurable we will prove the case with $\mathcal{S}$-measurable functions.

Since $\{f_n\}$ are $\mathcal{S}$-measurable, for each n we have that $f_n(X) \subset B$ is separable. Since the closure of a separable set is separable we also have that $\overline{\bigcup_{n=1}^{\infty} f_n(X)} \subseteq B$ is separable, and

$$f(X) \subseteq \overline{\bigcup_{n=1}^{\infty} f_n(X)}$$

and so $f(X)$ is separable as the subset of a separable set.

Then since for any open $U \subseteq B$ we have

$$f^{-1}(U) = \bigcup_{n=1}^{\infty} \bigcup_{K=1}^{\infty} \bigcap_{k=K}^{\infty} f_k^{-1}(U_n)$$

where

$$U_n = \left\{y \in U : d\big(y, (U \setminus \{0\})^c\big) > \tfrac{1}{n}\right\}$$

and since the f_k's are $\mathcal{S}$-measurable there preimages $f_k^{-1}(U_n) \in \mathcal{S}$, and as $\mathcal{S}$ is a σ-ring, it is closed under countable unions and intersections, and so $f^{-1}(U) \in \mathcal{S}$.

Then since $f(X)$ is separable, and for each open set $U \subset B$ we have $f^{-1}(U) \in \mathcal{S}$, we can conclude that f is $\mathcal{S}$-measurable. $\square$

Theorem 59 (Egoroff). Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space, if $E \in \mathcal{S}$ such that $\mu(E) < \infty$ and if $\{f_n\}$ is a sequence of μ-measurable functions such that $\forall\, x \in E \setminus N(\mu)$ we have $f_n(x) \to f(x)$. Then for every $\epsilon > 0$ $\exists$ measurable $F \subseteq E$, and so $F \in \mathcal{S}$, such that

$$\mu(E \setminus F) < \epsilon$$

and $f_n \to f$ uniformly on F; i.e. given $\delta > 0$, $\exists\, N$ such that

$$n \geq N \implies ||f(x) - f_n(x)|| < \delta \quad \forall\, x \in F$$

Proof. For any k and m, let

$$\begin{aligned} G_m^k &= \left\{x \in E : ||f(x) - f_k(x)|| > \tfrac{1}{m}\right\} \in \mathcal{S} \\ F_m^n &= \bigcup_{k \geq n} G_m^k = \left\{x \in E : \exists\ k \geq n; ||f(x) - f_k(x)|| > \tfrac{1}{m}\right\} \in \mathcal{S} \end{aligned}$$

for fixed m, as $n \to \infty$, since $f_n \to f$, we have $F_m^n \to \varnothing$ and therefore

$$\mu(F_m^n) \to \mu(\varnothing) = 0$$

Let $\epsilon > 0$ be given and for each m choose n_m such that

$$\mu(F_m^{n_m}) < \frac{\epsilon}{2^m}$$

let $H = \bigcup_m F_m^{n_m}$, then

$$\mu(H) = \mu\left(\bigcup_m F_m^{n_m}\right) \leq \sum_m \mu(F_m^{n_m}) < \sum_m \frac{\epsilon}{2^m} = \epsilon$$

let $F = E \setminus H$, then

$$\begin{aligned} \mu(E \setminus F) &= \mu(E \cap F^c) \\ &= \mu\big(E \cap (E \cap H^c)^c\big) \\ &= \mu\big(E \cap (E^c \cup H)\big) \\ &= \mu\big(\varnothing \cup (E \cap H)\big) \\ &= \mu(H) \\ &< \epsilon \end{aligned}$$

so let $\delta > 0$ be given, and choose m_0 such that $\frac{1}{m_0} < \delta$. Then $\forall\ x \in F$ by the definition of F we must have $x \notin H$, and in particular we have $x \notin F_{m_0}^{n_{m_0}}$. Thus, for all $k \geq n_{m_0}$ we also have that $x \notin G_{m_0}^k$ which is to say

$$||f(x) - f_k(x)|| \leq \frac{1}{m_0} < \delta$$

and since this is independent of $x \in F$ we have that $f_n \to f$ uniformly on F. $\square$

Proposition 60. Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space. If $\{f_n\}$ is a sequence of μ-measurable functions such that $f_n \to f$ almost uniformly on $E \in \mathcal{S}$, then $f_n \to f$ pointwise on $E \setminus N(\mu)$.

Proof. For each m choose $F_m \subseteq E$ such that

$$\mu(E \setminus F_m) < \frac{1}{m}$$

and $f_m \to f$ uniformly on F_m. Let $G = \bigcup_{m=1}^{\infty} F_m$, then

$$E \setminus G \subseteq E \setminus F_m \quad \forall\, m$$

which implies

$$\mu(E \setminus G) = 0$$

yet $f_m \to f$ uniformly on each $F_m \implies f_m \to f$ pointwise on each F_m and so $f_m \to f$ pointwise on $\bigcup_{m=1}^{\infty} F_m = G$ and hence on E almost everywhere. ☐

Proposition 61. Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space. If $\{f_n\}$ is a sequence of μ-measurable functions which are almost uniformly Cauchy on $E \in \mathcal{S}$, then $\exists\, f$ such that $f_n \to f$ almost uniformly on E.

Proof. Given $\epsilon > 0$, then since $\{f_n\}$ is almost uniformly Cauchy on E, $\exists\, F \in \mathcal{S}$ such that $F \subseteq E$, $\mu(E \setminus F) < \epsilon$, and $\{f_n\}$ is uniformly Cauchy on F.

Since $\{f_n\}$ is uniformly Cauchy on F, $\forall\, x \in F$ we have $\{f_n(x)\}$ is cauchy in B. Since B is a Banach space it is complete, and so $\{f_n(x)\}$ converges in B, so define

$$f : E \to B,\ \text{by}\ f(x) = \begin{cases} \lim f_n(x), & x \in F \\ 0, & x \in E \setminus F \end{cases}$$

to show that $f_n \to f$ uniformly on F, we note that since $\{f_n\}$ is uniformly Cauchy on F, for any $\delta > 0$, $\exists\, N_1$ such that

$$n, m \geq N_1 \implies ||f_m(x) - f_n(x)||_B < \frac{\delta}{2} \quad \forall\, x \in F$$

in addition, for each $x \in F$ since $f_n(x) \to f(x)$, $\exists\, N_2$ such that

$$n \geq N_2 \implies ||f_n(x) - f(x)||_B < \frac{\delta}{2}$$

so letting $N = \max\{N_1, N_2\}$, and fixing $m > N$, we have for any $n \geq N$ that

$$\begin{aligned} ||f(x) - f_n(x)||_B &\leq ||f(x) - f_m(x)||_B + ||f_m(x) - f_n(x)||_B \\ &\leq \frac{\delta}{2} + \frac{\delta}{2} \\ &= \delta \end{aligned}$$

and so $f_n \to f$ uniformly on F, and thus almost uniformly on E. ☐

Proposition 62. Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space. If $\{f_n\}$ is a sequence of μ-measurable functions such that $f_n \to f$ almost uniformly on $E \in \mathcal{S}$, then $\{f_n\}$ converges to f in measure.

Proof. Let $\epsilon > 0$ and $\delta > 0$ be given. Since $f_n \to f$ almost uniformly on E, choose $F \subseteq E$ such that

$$\mu(E \setminus F) < \delta$$

and $f_n \to f$ uniformly on F. Since B is a Banach space it is complete, and so $\{f_n(x)\}$ converges in B, say to $f(x) = \lim f_n(x)$, for each $x \in F$. So $\exists\, N$ such that

$$n \geq N \implies ||f_n(x) - f(x)||_B < \epsilon$$

then for $n \geq N$ we have

$$\{x \in E : ||f(x) - f_n(x)||_B > \epsilon\} \subseteq E \setminus F$$
$$\mu\Big(\{x \in E : ||f(x) - f_n(x)||_B > \epsilon\} \Big) \leq \mu(E \setminus F) < \delta \to 0$$

and so $\{f_n\}$ converges in measure to f. $\square$

Proposition 63. Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space and let $E \in \mathcal{S}$. If $\{f_n\}$ is a sequence of $\mathcal{S}$-measurable functions such that $\{f_n\}$ converges to f in measure on E, and $\{f_n\}$ converges to g in measure in E, then $f = g$ almost everywhere on E.

Proof. By the triangle inequality we have

$$||f(x) - g(x)||_B \leq ||f(x) - f_n(x)||_B + ||f_n(x) - g(x)||_B$$

and so for any $\epsilon > 0$ we have

$$\begin{aligned}
&\{x \in E : ||f(x) - g(x)||_B > \epsilon\} \\
&\qquad \subseteq \left\{x \in E : ||f(x) - f_n(x)||_B > \frac{\epsilon}{2}\right\} \cup \left\{x \in E : ||f_n(x) - g(x)||_B > \frac{\epsilon}{2}\right\} \\
&\implies \mu\Big(\{x \in E : ||f(x) - g(x)||_B > \epsilon\}\Big) \\
&\qquad \leq \mu\Big(\{x \in E : ||f(x) - f_n(x)||_B > \tfrac{\epsilon}{2}\} \Big) + \mu\Big(\{x \in E : ||f_n(x) - g(x)||_B > \tfrac{\epsilon}{2}\} \Big)
\end{aligned}$$

then since $\{f_n\}$ converges to f in measure on E, and $\{f_n\}$ converges to g in measure in E we have

$$\mu\Big(\{x \in E : ||f(x) - f_n(x)||_B > \tfrac{\epsilon}{2}\} \Big) \to 0 \text{ as } n \to \infty$$
$$\mu\Big(\{x \in E : ||f_n(x) - g(x)||_B > \tfrac{\epsilon}{2}\} \Big) \to 0 \text{ as } n \to \infty$$

and hence $\mu\Big(\{x \in E : ||f(x) - g(x)||_B > \epsilon\}\Big) \to 0$; i.e.

$$\mu\Big(\{x \in E : f(x) \neq g(x)\}\Big) \to 0$$

so that $f = g$ almost everywhere on E. $\square$

Theorem 64 (**Riesz-Weyl**)**.** Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space and let $E \in \mathcal{S}$. If $\{f_n\}$ is a sequence of $\mathcal{S}$-measurable B-valued functions which are cauchy in measure on E, then there is a subsequence $\{f_{n_k}\}$ that is almost uniformly cauchy.

Proof. Defining the integers n_k inductively, which we may do since $\{f_n\}$ is cauchy in measure, by $n_1 = 1$ and for $k > 1$ choosing n_k such that $n_k > n_{k-1}$, and so that

$$m, n \geq n_k \implies \mu\Bigg(\left\{x \in E : ||f_m(x) - f_n(x)||_B \geq \frac{1}{2^k}\right\}\Bigg) \leq \frac{1}{2^k}$$

given $\epsilon > 0$ select K such that

$$\sum_{k=K}^{\infty} \frac{1}{2^k} < \epsilon$$

and let

$$F = E \setminus \bigcup_{k=K}^{\infty} \left\{x \in E : ||f_{n_k}(x) - f_{n_{k+1}}(x)||_B \geq \frac{1}{2^k}\right\}$$

so by constructions we have

$$\begin{aligned}
\mu(E \setminus F) &= \mu\left(E \cap \left(E \cap \left(\bigcup_{k=K}^{\infty} \left\{x \in E : ||f_{n_k}(x) - f_{n_{k+1}}(x)||_B \geq \frac{1}{2^k}\right\}\right)^c\right)^c\right) \\
&= \mu\left(E \cap \left(E^c \cup \left(\bigcup_{k=K}^{\infty} \left\{x \in E : ||f_{n_k}(x) - f_{n_{k+1}}(x)||_B \geq \frac{1}{2^k}\right\}\right)\right)\right) \\
&= \mu\left(\varnothing \cup \left(\bigcup_{k=K}^{\infty} \left\{x \in E : ||f_{n_k}(x) - f_{n_{k+1}}(x)||_B \geq \frac{1}{2^k}\right\}\right)\right) \\
&\leq \sum_{k=K}^{\infty} \mu\left(\left\{x \in E : ||f_{n_k}(x) - f_{n_{k+1}}(x)||_B \geq \frac{1}{2^k}\right\}\right) \\
&\leq \sum_{k=K}^{\infty} \frac{1}{2^k} \\
&< \epsilon
\end{aligned}$$

to see that $\{f_{n_k}\}$ is uniformly cauchy on F, let $\delta > 0$ be given, and choose $N > K$ such that

$$\sum_{n=N}^{\infty} \frac{1}{2^n} < \delta$$

then for any $x \in F$ and $k > l > N$ we have

$$\begin{aligned}
||f_{n_k}(x) - f_{n_l}(x)||_B &\leq ||f_{n_k}(x) - f_{n_{k-1}}(x)||_B + ||f_{n_{k-1}}(x) - f_{n_{k-2}}(x)||_B \\
&\quad + \cdots + ||f_{n_{l+1}}(x) - f_{n_l}(x)||_B \\
&= \sum_{m=l}^{k-1} ||f_{n_{m+1}}(x) - f_{n_m}(x)||_B \\
&\leq \sum_{m=l}^{k-1} \frac{1}{2^m} \\
&\leq \sum_{m=N}^{\infty} \frac{1}{2^m} \\
&< \delta
\end{aligned}$$

and therefore $\{f_{n_k}\}$ is almost uniformly cauchy on E. $\square$

Proposition 65. Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space and let $E \in \mathcal{S}$. If $\{f_n\}$ is a sequence of function which are cauchy in measure on E such that some subsequence $\{f_{n_k}\}$ converges almost uniformly to f on E, then $\{f_n\}$ converges in measure to f.

Proof. Given $\epsilon > 0$, note that

$$\begin{aligned}
\{x \in E :&||f(x) - f_n(x)||_B > \epsilon\} \\
&\subseteq \left\{x \in E : ||f(x) - f_{n_k}(x)||_B > \frac{\epsilon}{2}\right\} \cup \left\{x \in E : ||f_{n_k}(x) - f_n(x)||_B > \frac{\epsilon}{2}\right\}
\end{aligned}$$

and since $f_{n_k} \to f$ almost uniformly on E, given $\delta > 0$, $\exists\ N_1$ such that

$$n_k \geq N_1 \implies \mu\Big(\{x \in E : ||f(x) - f_{n_k}(x)||_B > \tfrac{\epsilon}{2}\} \Big) < \frac{\delta}{2}$$

then as $\{f_n\}$ are cauchy in measure on E, $\exists\ N_2$ such that

$$n_k, n \geq N_2 \implies \mu\Big(\{x \in E : ||f_n(x) - f_{n_k}(x)||_B > \tfrac{\epsilon}{2}\} \Big) < \frac{\delta}{2}$$

so letting $N = \max\{N_1, N_2\}$, and fix $n_k > N$, then for any $n \geq N$ we have

$$\begin{aligned}
\mu\Big(&\{x \in E : ||f(x) - f_n(x)||_B > \epsilon\}\Big) \\
&\leq \mu\Big(\{x \in E : ||f(x) - f_{n_k}(x)||_B > \tfrac{\epsilon}{2}\} \Big) + \mu\Big(\{x \in E : ||f_{n_k}(x) - f_n(x)||_B > \tfrac{\epsilon}{2}\} \Big) \\
&< \frac{\delta}{2} + \frac{\delta}{2} \\
&= \delta \to 0
\end{aligned}$$

and so $\{f_n\}$ converges in measure on E to f. $\square$

Proposition 66. If f, g are simple integrable functions then $f+g$ is simple integrable function and

$$\int (f+g)d\mu = \int f d\mu + \int g d\mu$$

Proof. First suppose $f = \sum_{i=1}^{n} b_i \chi_{E_i}$, and $g = c\chi_F$, to get F contained in the E_i's let us set $E_{n+1} = F \setminus \bigsqcup_{i=1}^{n} E_i$ and $b_{n+1} = 0$, then

$$F \subseteq \bigsqcup_{i=1}^{n+1} E_i \implies F = \bigsqcup_{i=1}^{n+1} (F \cap E_i)$$

and

$$\begin{aligned} f &= \sum_{i=1}^{n+1} b_i \chi_{E_i} = \sum_{i=1}^{n+1} b_i \Big(\chi_{E_i \cap F} + \chi_{E_i \setminus F} \Big) \\ g &= \sum_{i=1}^{n+1} c \chi_{E_i \cap F} \end{aligned}$$

and so

$$f + g = \sum_{i=1}^{n+1} (b_i + c)\chi_{E_i \cap F} + \sum_{i=1}^{n+1} b_i \chi_{E_i \setminus F}$$

and so $f+g$ is a simple $\mathcal{S}$-measurable function. The general case follows inductively. Where we then have

$$\begin{aligned} \int (f+g)d\mu &= \sum_{i=1}^{n+1} (b_i + c)\mu(E_i \cap F) + \sum_{i=1}^{n+1} b_i \mu(E_i \setminus F) \\ &= \sum_{i=1}^{n+1} b_i \Big[\mu(E_i \cap F) + \mu(E_i \setminus F) \Big] + \sum_{i=1}^{n+1} c\mu(E_i \cap F) \\ &= \int f d\mu + \int g d\mu \end{aligned}$$

$\square$

Proposition 67. Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space. If f, g are simple integrable functions then

$$||f + g||_1 \leq ||f||_1 + ||g||_1$$

Proof. First note that for all $x \in X$ we have

$$||f(x) + g(x)||_B \leq ||f(x)||_B + ||g(x)||_B$$

and therefore

$$
\begin{aligned}
||f+g||_1 &= \int ||f(x)+g(x)||_B d\mu(x) \\
&\leq \int \Big(||f(x)||_B + ||g(x)||_B \Big) d\mu(x) \\
&= \int ||f(x)||_B d\mu(x) + \int ||g(x)||_B d\mu(x) \\
&= ||f||_1 + ||g||_1
\end{aligned}
$$

□

Proposition 68. Let $(X, \mathcal{S}, \mu)$ be measure space and let $\{f_n\}$ be a sequence of simple integrable functions that is cauchy for $||\cdot||_1$. Then $\{f_n\}$ is cauchy in measure.

Proof. Since $\{f_n\}$ is cauchy for $||\cdot||_1$ we have

$$||f_n - f_m||_1 = \int ||f_n(x) - f_m(x)||_B d\mu(x) \to 0 \text{ as } n, m \to \infty$$

let $\epsilon > 0$ be given and let

$$E_{mn}^{\epsilon} = \{x \in E : ||f_m(x) - f_n(x)|| \geq \epsilon\}$$

then

$$\chi_{E_{mn}^{\epsilon}} \leq \frac{||f_m(x) - f_n(x)||_B}{\epsilon}$$

so

$$\mu(E_{mn}^{\epsilon}) = \int \chi_{E_{mn}^{\epsilon}} d\mu(x) \leq \int \frac{||f_m(x) - f_n(x)||}{\epsilon} d\mu(x) \to 0 \text{ as } m, n \to \infty$$

and so $\{f_n\}$ is cauchy in measure on E. □

Proposition 69. Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space. If $\{f_n\}, \{g_n\}$ are sequences of simple integrable functions which are equivalent under $||\cdot||_1$; i.e.

$$||f_n - g_n||_1 \to 0 \text{ as } n \to \infty$$

and if $\{f_n\}$ converges to f is measure, then $\{g_n\}$ also converges to f in measure.

Proof. Given $\epsilon > 0$, note that

$$
\begin{aligned}
&\{x \in X : ||f(x) - g_n(x)||_B > \epsilon\} \\
&\qquad \subseteq \left\{x \in X : ||f(x) - f_n(x)||_B > \frac{\epsilon}{2}\right\} \cup \left\{x \in X : ||f_n(x) - g_n(x)||_B > \frac{\epsilon}{2}\right\}
\end{aligned}
$$

and since $\{f_n\}$ converges to f in measure we have

$$\mu\Big(\{x \in X : ||f(x) - f_n(x)||_B > \tfrac{\epsilon}{2}\} \Big) \to 0 \text{ as } n \to \infty$$

additionally since $\{f_n\}, \{g_n\}$ are equivalent under $|| \cdot ||_1$, we have

$$\begin{aligned}\mu\Big(\{x \in X : ||f_n(x) - g_n(x)||_B > \tfrac{\epsilon}{2}\} \Big) &= \int \chi_{\left\{x \in X : ||f_n(x) - g_n(x)||_B > \frac{\epsilon}{2}\right\}} d\mu(x) \\ &\leq 2 \int \frac{||f_n(x) - g_n(x)||_B}{\epsilon} d\mu(x) \to 0 \text{ as } n \to \infty\end{aligned}$$

and so

$$\begin{aligned}&\mu\Big(\{x \in X : ||f(x) - g_n(x)||_B > \epsilon\}\Big) \\ &\quad \leq \mu\Big(\{x \in X : ||f(x) - f_n(x)||_B > \tfrac{\epsilon}{2}\} \Big) + \mu\Big(\{x \in X : ||f_n(x) - g_n(x)||_B > \tfrac{\epsilon}{2}\} \Big) \\ &\quad \to 0 \text{ as } n \to \infty\end{aligned}$$

and so $\{g_n\}$ also converges to f in measure. □

Lemma 70. Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space. If $\{f_n\}$ is a mean cauchy sequence of simple integrable functions such that $f_n \to 0$ almost uniformly, then

$$||f_n||_1 \to 0$$

Proof. Let $\epsilon > 0$ be given. Then since $\{f_n\}$ is mean cauchy, choose $N \in \mathbb{N}$ such that

$$n, m \geq N \implies ||f_n - f_m||_1 < \epsilon$$

and let

$$E = \{x \in X : f_N(x) \neq 0\} = \text{car}(f_N)$$

and since f_N is simple integrable we have $\mu(E) < \infty$. Now for $n \geq N$ we have

$$\begin{aligned}\int_{E^c} ||f_n(x)||_B d\mu(x) &= \int_{E^c} ||f_n(x) - 0||_B d\mu(x) \\ &= \int_{E^c} ||f_n(x) - f_N(x)||_B d\mu(x) \quad f_N(x) = 0 \text{ for } x \in E^c \\ &\leq \int_X ||f_n(x) - f_N(x)||_B d\mu(x) \\ &= ||f_n - f_N||_1 \\ &< \epsilon\end{aligned}$$

Now since $f_n \to 0$ almost uniformly, $\exists\ F \in \mathcal{S}$ such that $F \subseteq E$ where

$$\mu(E \setminus F) < \frac{\epsilon}{1 + ||f_N||_\infty}$$

and $f_n \to 0$ uniformly on F. And so we may choose $M > N$ such that for $n > M$ and $x \in F$ we have

$$||f_n(x)||_B < \frac{\epsilon}{1+\mu(F)}$$

and so

$$\begin{aligned}\int_F ||f_n(x)||_B d\mu(x) &\leq \int_F \frac{\epsilon}{1+\mu(F)} d\mu(x) \\ &= \frac{\epsilon}{1+\mu(F)} \cdot \mu(F) \\ &< \epsilon\end{aligned}$$

and lastly, using the triangle inequality

$$\begin{aligned}\int_{E\setminus F} ||f_n(x)||_B d\mu(x) &\leq \int_{E\setminus F} ||f_n(x) - f_N(x)||_B d\mu(x) + \int_{E\setminus F} ||f_N(x)||_B d\mu(x) \\ &\leq \int_X ||f_n(x) - f_N(x)||_B d\mu(x) + \int_{E\setminus F} ||f_N(x)||_B d\mu(x) \\ &\leq ||f_n - f_N||_1 + ||f_N||_\infty \int_{E\setminus F} d\mu(x) \qquad ||f_N(x)||_B \leq ||f_N||_\infty \\ &= ||f_n - f_N||_1 + ||f_N||_\infty \mu(E \setminus F) \\ &< \epsilon + ||f_N||_\infty \frac{\epsilon}{1+||f_N||_\infty} \\ &< 2\epsilon\end{aligned}$$

then putting all the piece together we get for $n > M$

$$\begin{aligned}||f_n||_1 &= \int_X ||f_n(x)||_B d\mu(x) \\ &= \int_{E^c} ||f_n(x)||_B d\mu(x) + \int_{E\setminus F} ||f_n(x)||_B d\mu(x) + \int_F ||f_n(x)||_B d\mu(x) \\ &< \epsilon + 2\epsilon + \epsilon \\ &= 4\epsilon\end{aligned}$$

and so $||f_n||_1 \to 0$. □

Proposition 71. Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space. If $\{f_n\}$ and $\{g_n\}$ are mean cauchy sequences of simple integrable functions, such that $f_n, g_n \to h$ is measure, then $\{f_n\}$ and $\{g_n\}$ are equivalent cauchy sequences; i.e.

$$\lim_{n,m\to\infty} ||f_n - g_m||_1 = 0$$

Proof. Since $\{f_n\}, \{g_m\}$ converge in measure to h and are mean cauchy, Riesz-Weyl says that $\exists$ subsequences $\{f_{n_k}\}, \{g_{m_k}\}$ that converge to h are almost uniformly. So it suffices to show that

$$\lim_{k\to\infty} ||f_{n_k} - g_{m_k}||_1 = 0$$

So define

$$h_k = f_{n_k} - g_{m_k}$$

then $\{h_n\}$ is a mean cauchy sequence of simple integrable functions such that $h_n \to 0$ almost uniformly, and from the previous Lemma we then have

$$||h_k||_1 \to 0$$

and therefore

$$\lim_{k\to\infty} ||f_{n_k} - g_{m_k}||_1 = 0$$

and so $\{f_n\}$ and $\{g_m\}$ are equivalent cauchy sequences. □

Theorem 72. Let f be a $\mathcal{S}$-measurable B-valued function, then the following are equivalent

1. There is a mean cauchy sequence $\{f_n\}$ of ISFs that converge in measure to f.
2. There is a mean cauchy sequence $\{f_n\}$ of ISFs such that $f_n \to f$ almost uniformly.
3. There is a mean cauchy sequence $\{f_n\}$ of ISFs such that $f_n \to f$ pointwise almost everywhere.

f is μ-integrable if it satisfies one, and hence all, of these conditions.

Proof. (1) $\implies$ (2).

Riezs-Weyl gives a subsequence that converges almost uniformly.

(2) $\implies$ (3).

Riezs-Weyl gives a subsequence that converges almost uniformly, and hence pointwise.

(3) $\implies$ (1).

Since $\{f_n\}$ is mean cauchy we know that $\forall\ \epsilon > 0,\ \exists\ N \in \mathbb{N}$ such that

$$n, m \geq N \implies ||f_n - f_m||_1 = \int_X ||f_n - f_m||_B d\mu(x) < \epsilon$$

and hence, for any $\delta > 0$

$$n, m \geq N \implies ||f_n - f_m||_1 = \int_X ||f_n - f_m||_B d\mu(x) < \epsilon\delta$$

so suppose, for contradiction, that $\{f_n\}$ is not cauchy in measure, this implies that $\exists\ \epsilon, \delta$ such that $\forall\ N \in \mathbb{N}$ there exists $m, n \geq N$ where

$$\mu\Big(x \in X : ||f_n(x) - f_m(x)||_B \geq \epsilon\Big) \geq \delta$$

let $A \subset X$ be the set of points which satisfy $||f_n(x) - f_m(x)||_B \geq \epsilon$. Then

$$\begin{aligned}
\int_X ||f_n - f_m||_B d\mu(x) &\geq \int_A ||f_n - f_m||_B d\mu(x) \\
&\geq \int_A \epsilon d\mu(x) \\
&= \epsilon\mu(A) \\
&\geq \epsilon\delta \quad \Rightarrow\Leftarrow
\end{aligned}$$

and so we can conclude that $\{f_n\}$ is cauchy in measure. Then Riesz-Weyl says $\exists\ \{f_{n_k}\}$ which converges almost uniformly, and hence almost everywhere and in measure, to an $\mathcal{S}$-measurable function g. Yet, $f_n \to f$ pointwise almost everywhere and thus $f_{n_k} \to f$ pointwise almost everywhere, and so $f = g$ almost everywhere. That is $\{f_{n_k}\}$ converges in measure to f. $\square$

Theorem 73. $\mathcal{L}^1(X, \mathcal{S}, \mu, B)$ is a vector space.

Proof. Let $f, g \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$, then $\exists$ sequences $\{f_n\}, \{g_n\}$ of simple integrable functions which are mean cauchy such that $f_n \to f$ and $g_n \to g$ pointwise almost everywhere. Then $\{f_n + g_n\}$ is a sequence of simple integrable functions which is mean cauchy and $f_n + g_n \to f + g$ pointwise almost everywhere and so

$$\begin{aligned}
\int (f+g) d\mu &= \lim_{n\to\infty} \int (f_n + g_n) d\mu \\
&= \lim_{n\to\infty} \int f_n d\mu + \lim_{n\to\infty} \int g_n d\mu \\
&= \int f d\mu + \int g d\mu
\end{aligned}$$

and so $f + g \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$.

Next if $c \in \mathbb{R}$, then $\{cf_n\}$ is a sequence of simple integrable functions which are mean cauchy such that $cf_n \to cf$ pointwise almost everywhere, then

$$\int cf d\mu = c \int f = c \lim_{n\to\infty} \int f_n d\mu = \lim_{n\to\infty} \int cf_n d\mu$$

thus $cf \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$.

Finally if $\{O_n\}$ is a sequence of simple integrable functions which are mean cauchy such that $O_n \to 0$ pointwise almost everywhere, then

$$0 = \int 0 d\mu = \lim_{n\to\infty} \int O_n d\mu$$

and so $0 \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$.

$\therefore \mathcal{L}^1(X, \mathcal{S}, \mu, B)$ is a vector space. □

Lemma 74. Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space. If $\{f_n\}$ is a mean cauchy sequence of simple integrable functions such that $f_n \to f$ in measure, or almost uniformly, or pointwise almost everywhere, then $f_n \to f$ in mean.

Proof. For each fixed n $\{f_m - f_n\}$ is a mean cauchy sequence of simple integrable functions such that

$$f_m - f_n \to f - f_n$$

in measure, or almost uniformly, or pointwise almost everywhere, so that

$$\begin{aligned}
||f - f_n||_1 &= \int ||f(x) - f_n(x)||_B d\mu(x) \\
&= \lim_{m\to\infty} \int ||f_m(x) - f_n(x)||_B d\mu(x) \\
&= \lim_{m\to\infty} ||f_m - f_n||_1
\end{aligned}$$

Given $\epsilon > 0$ choose $N \in \mathbb{N}$ such that

$$n, m > N \implies ||f_m - f_n||_1 < \epsilon$$

that is for $n > N$ we have

$$||f - f_n||_1 < \epsilon$$

and so $f_n \to f$ in mean. □

Proposition 75. If $f \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$, then $\text{car}(f)$ is σ-finite.

Proof. Since $f \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$, $\exists$ a mean cauchy sequence $\{f_n\}$ of simple integrable functions such that $f_n \to f$ pointwise almost everywhere. Let

$$E_n = \text{car}(f_n)$$

then since the f_n's are simple integrable functions we have

$$\mu(E_n) < \infty$$

then

$$\operatorname{car}(f) \subseteq \bigcup_{n=1}^{\infty} \operatorname{car}(f_n) < \infty$$

and so $\operatorname{car}(f)$ is σ-finite. □

Proposition 76. Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space. If $f \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$, then $\forall\ \epsilon > 0,\ \exists\ E \in \mathcal{S}$ such that

$$\mu(E) < \infty$$

and

$$\left|\left| \int_{X \setminus E} f(x) d\mu(x) \right|\right|_B < \epsilon$$

Proof. Since f is μ-integrable, and from Lemma 74 this implies convergence in mean so from our mean cauchy sequence $\{f_n\}$ of simple integrable functions choose f_n such that

$$||f - f_n||_1 = \int ||f(x) - f_n(x)||_B d\mu(x) < \epsilon$$

since f_n is a simple integrable function we have

$$\mu\big(\operatorname{car}(f_n)\big) < \infty$$

so let $E = \operatorname{car}(f_n)$, then since $f_n(x) = 0\ \forall\ x \in X \setminus E = E^c$ we have

$$\begin{aligned}
\left|\left| \int_{X \setminus E} f(x) d\mu(x) \right|\right|_B &= \left|\left| \int_{X \setminus E} f(x) d\mu(x) - 0 \right|\right|_B \\
&= \left|\left| \int_{X \setminus E} \big(f(x) - f_n(x)\big) d\mu(x) \right|\right|_B \\
&\leq \int_{X \setminus E} \big|\big|f(x) - f_n(x)\big|\big|_B d\mu(x) \\
&\leq \int_X \big|\big|f(x) - f_n(x)\big|\big|_B d\mu(x) \\
&= ||f - f_n||_1 \\
&< \epsilon
\end{aligned}$$

□

Proposition 77 (Absolute Continuity). Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space. If $f \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$, then $\forall\ \epsilon > 0,\ \exists\ \delta > 0$ such that if

$$\mu(E) < \delta$$

then

$$\big|\big|\mu_f(E)\big|\big|_B < \epsilon$$

Proof. Let $\epsilon > 0$ be given and choose a simple integrable function g such that

$$||f - g|| < \frac{\epsilon}{2}$$

and select $\delta = \frac{\epsilon}{2||g||_\infty}$ that is

$$\mu(E) < \frac{\epsilon}{2||g||_\infty}$$

then

$$\begin{aligned}
\big|\big|\mu_f(E)\big|\big|_B &= \big|\big|\mu_f(E) - \mu_g(E) + \mu_g(E)\big|\big|_B \\
&\leq \big|\big|\mu_f(E) - \mu_g(E)\big|\big|_B + \big|\big|\mu_g(E)\big|\big|_B \\
&= \left|\left|\int_E f(x)d\mu(x) - \int_E g(x)d\mu(x)\right|\right|_B + \left|\left|\int_E g(x)d\mu(x)\right|\right|_B \\
&\leq \int_E \big|\big|f(x) - g(x)\big|\big|_B d\mu(x) + \int_E \big|\big|g(x)\big|\big|_B d\mu(x) \\
&\leq \int_X \big|\big|f(x) - g(x)\big|\big|_B d\mu(x) + \int_E ||g||_\infty d\mu(x) \\
&= ||f - g||_1 + ||g||_\infty \mu(E) \\
&< \frac{\epsilon}{2} + ||g||_\infty \frac{\epsilon}{2||g||_\infty} \\
&= \epsilon
\end{aligned}$$

□

Proposition 78. Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space. If $f \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$, then μ_f is a B-valued measure on $\mathcal{S}$.

Proof. To do this we must show that μ_f is countably additive. First we note that for any $g, f \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$ and any $E \in \mathcal{S}$ we have

$$\begin{aligned}
\big|\big|\mu_f(E) - \mu_g(E)\big|\big|_B &= \left|\left|\int_E f(x)d\mu(x) - \int_E g(x)d\mu(x)\right|\right|_B \\
&\leq \int_E \big|\big|f(x) - g(x)\big|\big|_B d\mu(x) \\
&\leq \int_X \big|\big|f(x) - g(x)\big|\big|_B d\mu(x) \\
&= ||f - g||_1
\end{aligned}$$

let $E \in \mathcal{S}$ such that $\mu(E) < \infty$ and $\epsilon > 0$ be given, and let

$$E = \bigsqcup_{i=1}^{\infty} E_i$$

since f is μ-integrable, by Lemma 74 this implies convergence in mean so from our mean cauchy sequence $\{f_n\}$ of simple integrable functions choose f_n such that

$$||f - f_n||_1 = \int ||f(x) - f_n(x)||_B d\mu(x) < \frac{\epsilon}{3}$$

since f_n is a simple integrable function μ_{f_n} is countably additive, so choose $N \in \mathbb{N}$ such that

$$m > N \implies \left|\left|\mu_{f_n}(E) - \mu_{f_n}\left(\bigsqcup_{i=1}^{m} E_i\right)\right|\right|_B < \frac{\epsilon}{3}$$

and so for $m > N$ we have

$$\begin{aligned}
&\left|\left|\mu_f(E) - \mu_f\left(\bigsqcup_{i=1}^{m} E_i\right)\right|\right|_B \\
&\leq \big|\big|\mu_f(E) - \mu_{f_n}(E)\big|\big|_B + \left|\left|\mu_{f_n}(E) - \mu_{f_n}\left(\bigsqcup_{i=1}^{m} E_i\right)\right|\right|_B + \left|\left|\mu_{f_n}\left(\bigsqcup_{i=1}^{m} E_i\right) - \mu_f\left(\bigsqcup_{i=1}^{m} E_i\right)\right|\right|_B \\
&= \left|\left|\int_E f(x)d\mu(x) - \int_E f_n(x)d\mu(x)\right|\right|_B + \left|\left|\mu_{f_n}(E) - \mu_{f_n}\left(\bigsqcup_{i=1}^{m} E_i\right)\right|\right|_B \\
&\qquad + \left|\left|\int_{\bigsqcup_{i=1}^{m} E_i} f_n(x)d\mu(x) - \int_{\bigsqcup_{i=1}^{m} E_i} f(x)d\mu(x)\right|\right|_B \\
&< \int_E \big|\big|f(x) - f_n(x)\big|\big|_B d\mu(x) + \frac{\epsilon}{3} + \int_{\bigsqcup_{i=1}^{m} E_i} \big|\big|f_n(x) - f(x)\big|\big|_B d\mu(x) \\
&\leq \int_X \big|\big|f(x) - f_n(x)\big|\big|_B d\mu(x) + \frac{\epsilon}{3} + \int_X \big|\big|f_n(x) - f(x)\big|\big|_B d\mu(x) \\
&< ||f - f_n||_1 + \frac{\epsilon}{3} + ||f_n - f||_1 \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon
\end{aligned}$$

□

Theorem 79 (**Lebesgue Dominated Convergence**)**.** Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space, and let $\{f_n\} \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$; i.e. a sequence of μ-integrable functions, that converge pointwise almost everywhere to a function f. Suppose there $\exists$ $g \in \mathcal{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$ such that

$$||f_n(x)||_B \leq g(x)$$

for all n and for all x, or almost everywhere for each n. Then $\{f_n\}$ is a mean cauchy sequence. And so $\{f_n\}$ converges to f in mean, $f \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$, and

$$\int f d\mu = \lim \int f_n d\mu$$

Proof. Let $\epsilon > 0$ be given, then since $g \in \mathcal{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$ Proposition 76 says $\exists\ E \in \mathcal{S}$ such that

$$\mu(E) < \infty \quad \text{and} \quad \left| \int_{X \setminus E} g(x) d\mu(x) \right| < \frac{\epsilon}{6}$$

then $\forall\ n, m$ we have

$$\begin{aligned}
\int_{X \setminus E} \big|\big| f_m(x) - f_n(x) \big|\big|_B d\mu(x) &\leq \int_{X \setminus E} \Big(\big|\big| f_m(x) \big|\big|_B + \big|\big| f_n(x) \big|\big|_B \Big) d\mu(x) \\
&= \int_{X \setminus E} \big|\big| f_m(x) \big|\big|_B d\mu(x) + \int_{X \setminus E} \big|\big| f_n(x) \big|\big|_B d\mu(x) \\
&\leq \int_{X \setminus E} g(x) d\mu(x) + \int_{X \setminus E} g(x) d\mu(x) \qquad \text{since } ||f_n(x)||_B \leq g(x)\ \forall\ n \\
&= 2 \int_{X \setminus E} g(x) d\mu(x) \\
&< 2 \frac{\epsilon}{6} \\
&= \frac{\epsilon}{3}
\end{aligned}$$

Next, since $g \in \mathcal{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$ we also have that the indefinite integral μ_g is absolutely continuous, so we may choose $\delta > 0$ such that for any $G \in \mathcal{S}$

$$\mu(G) < \delta \implies \big|\mu_g(G)\big| = \left| \int_G g(x) d\mu(x) \right| < \frac{\epsilon}{6}$$

Now, since $f_n \to f$ pointwise almost everywhere and $\mu(E) < \infty$, Egoroff's Theorem then says that $f_n \to f$ almost uniformly on E. Therefore we may choose $F \in \mathcal{S}$ with $F \subseteq E$ such that

$$\mu(E \setminus F) < \delta$$

and $f_n \to f$ uniformly on F. Then $\forall\ n, m$ we have

$$\begin{aligned}
\int_{E \setminus F} \big|\big| f_m(x) - f_n(x) \big|\big|_B d\mu(x) &\leq \int_{E \setminus F} \Big(\big|\big| f_m(x) \big|\big|_B + \big|\big| f_n(x) \big|\big|_B \Big) d\mu(x) \\
&= \int_{E \setminus F} \big|\big| f_m(x) \big|\big|_B d\mu(x) + \int_{E \setminus F} \big|\big| f_n(x) \big|\big|_B d\mu(x) \\
&\leq \int_{E \setminus F} g(x) d\mu(x) + \int_{E \setminus F} g(x) d\mu(x) \qquad \text{since } ||f_n(x)||_B \leq g(x)\ \forall\ n \\
&= 2 \int_{E \setminus F} g(x) d\mu(x) \\
&= 2\mu_g(E \setminus F) \\
&< 2 \frac{\epsilon}{6} \\
&= \frac{\epsilon}{3}
\end{aligned}$$

Finally, since $f_n \to f$ uniformly on F we may choose $N \in \mathbb{N}$ such that

$$n, m \geq N \implies \big|\big|f_m(x) - f_n(x)\big|\big|_B < \frac{\epsilon}{3\mu(F)}$$

then $\forall\ x \in F$ and $\forall\ n, m > N$ we have

$$\int_F \big|\big|f_m(x) - f_n(x)\big|\big|_B d\mu(x) < \int_F \frac{\epsilon}{3\mu(F)} d\mu(x) = \frac{\epsilon}{3\mu(F)}\mu(F) = \frac{\epsilon}{3}$$

and so, for all $n, m > N$ we get

$$\begin{aligned}
||f_n - f_m||_1 &= \int_X \big|\big|f_m(x) - f_n(x)\big|\big|_B d\mu(x) \\
&= \int_{X \setminus E} \big|\big|f_m(x) - f_n(x)\big|\big|_B d\mu(x) + \int_{E \setminus F} \big|\big|f_m(x) - f_n(x)\big|\big|_B d\mu(x) \\
&\quad + \int_F \big|\big|f_m(x) - f_n(x)\big|\big|_B d\mu(x) \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon
\end{aligned}$$

And thus, $\{f_n\}$ is a mean cauchy sequence.

Now, since $\{f_n\}$ is a mean cauchy sequence of simple integrable functions such that $f_n \to f$ pointwise almost everywhere then f is μ-integrable, or $f \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$ and

$$\int f d\mu = \lim_{n \to \infty} \int f_n d\mu$$

where Lemma 74 then says that $\{f_n\}$ converges to f in mean. □

Proposition 80. Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space, and let f be a μ-measurable B-valued function. If $\exists\ g \in \mathcal{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$ such that

$$||f(x)||_B \leq g(x)$$

almost everywhere, then $f \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$; i.e. f is μ-integrable.

Proof. Since f is μ-measurable, $\exists\ \{f_n\}$ of simple $\mathcal{S}$-measurable such that $f_n \to f$ almost everywhere. For each n choose

$$E_n = \Big\{x \in X : 2g(x) - \big|\big|f_n(x)\big|\big|_B \geq 0\Big\}$$

and define

$$h_n(x) = \begin{cases} f_n(x), & \big|\big|f_n(x)\big|\big|_B \leq 2g(x) \\ 0, & \text{otherwise} \end{cases}$$

then

$$h_n = f_n \chi_{E_n}$$

since $g \in \mathcal{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$ we have car(g) is σ-finite, and so, by construction, for each E_n we have

$$\mu(E_n) < \infty$$

and so each h_n is a simple integrable function, and $\{h_n\} \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$. And note, since $f_n \to f$ almost everywhere, and the h_n's are defined in terms of the f_n's this implies that $h_n \to f$ almost everywhere, or pointwise almost everywhere. Then since

$$||h_n(x)||_B \leq 2g(x)$$

for all n and for all x, Lebesgue Dominated Convergence says that $\{h_n\}$ is a mean cauchy sequence of simple integrable functions and therefore the limit function $f \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$. $\square$

Theorem 81 (**Monotone Convergence Theorem**). Let $(X, \mathcal{S}, \mu)$ be measure space, and let $\{f_n\} \in \mathcal{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$ such that $f_n \geq 0$ and is non-decreasing; i.e.

$$f_{n+1} \geq f_n \quad \forall\ n$$

if $\exists\ C \in \mathbb{R}$ such that

$$||f_n||_1 = \int f_n(x) d\mu(x) < C \quad \forall\ n$$

then $\{f_n\}$ is a mean cauchy sequence and $\exists\ f \in \mathcal{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$ such that $f_n \to f$ pointwise almost everywhere. That is

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$$

Proof. Since $f_n \leq f_{n+1}\ \forall\ n$ we have

$$\int f_n(x) d\mu \leq \int f_{n+1}(x) d\mu \quad \forall\ n$$

and since

$$\int f_n(x) d\mu(x) < C \quad \forall\ n$$

we have $\left\{ \int f_n d\mu \right\}$ is a sequence which converges and so is cauchy.

Let $\epsilon > 0$ be given, then $\exists\ N \in \mathbb{N}$ such that

$$n, m > N \implies \left| \int f_n(x) d\mu - \int f_m(x) d\mu \right| < \epsilon$$

so let $n > m$, then since $f_k > 0 \ \forall \ k$ we have

$$\begin{aligned}
\left|\int f_n(x)d\mu - \int f_m(x)d\mu\right| &= \left|\int \big(f_n(x) - f_m(x)\big)d\mu\right| \\
&= \int |f_n(x) - f_m(x)|d\mu \\
&= ||f_n - f_m||_1 \\
&< \epsilon
\end{aligned}$$

and so $\{f_n\}$ is mean cauchy. Then since $\mathcal{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$ is complete, $\exists \ f \in \mathcal{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$ such that

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu$$

□

Theorem 82 (More general Monotone Convergence Theorem). Let $(X, \mathcal{S}, \mu)$ be measure space with Banach space $\mathbb{R}$, and let $\{f_n\} \in \mathcal{S}$ satisfying

$$0 \leq f_1(x) \leq f_2(x) \leq \cdots f_n(x) \leq \cdots \quad \forall \ x \in X$$

let

$$f(x) = \lim_{n\to\infty} f_n(x)$$

then $\lim_{n\to\infty} \int f_n d\mu$ and $\int f d\mu$ both exist and

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$$

Proof. First, since f is the pointwise limit of measurable functions and

$$f \geq 0$$

f is measurable and

$$\int f d\mu$$

exists in $\mathbb{R} \setminus \{0\}$.

Since $\{f_n(x)\}$ is a monotone increasing sequence and each $f_n \geq 0$, the same is true for $\{\int f_n d\mu\}$, and so

$$\lim_{n\to\infty} \int f_n d\mu$$

exists in $\mathbb{R} \setminus \{0\}$. Moreover we have

$$\int f_n d\mu \leq \int f_{n+1} d\mu \leq \int f d\mu \quad \forall \ n$$

and so

$$\lim_{n\to\infty}\int f_n d\mu \leq \int f d\mu$$

For the reverse inequality let

$$g : X \to [0,\infty)$$

be a simple measurable function such that

$$0 \leq g \leq f$$

and fix $0 < t < 1$. Then defining

$$E_n = \{x \in X : f_n(x) \geq tg(x)\}$$

we have an increasing sequence of measurable sets such that

$$E_1 \subseteq E_2 \subseteq \cdots \subseteq X$$

Then, for any $x \in X$ if

$$f(x) = 0 \implies f_n(x) = 0 \; \forall \; n$$

and since $g \leq f$ we also have

$$tg(x) = 0 \implies x \in E_n \; \forall \; n$$

if $f(x) > 0$, then

$$f(x) \geq g(x) \implies f(x) > tg(x) \qquad \text{since } 0 < t < 1$$

and since $f_n \to f$ monotonically $f_n(x) > tg(x)$ eventually, thus $x \in E_n$ for some n. And so, for any $x \in X$ we have that

$$x \in \bigcup_{n=1}^{\infty} E_n \implies \bigcup_{n=1}^{\infty} E_n = X$$

then for every n we have

$$\int_X f_n d\mu \geq \int_{E_n} f_n d\mu \geq t\int_{E_n} g d\mu$$

and since $\int_{E_n} g d\mu = \mu_g(E_n)$ where μ_g is a measure and hence countably additive, so disjointizing the E_n's if necessary, and by the simplicity of $g = \sum_{i=1}^{N} c_i \chi_{A_i}$ we have

$$\begin{aligned}
\lim_{n\to\infty}\mu_g(E_n) = \lim_{n\to\infty}\sum_{i=1}^{N} c_i\mu(A_i \cap E_n) \to \sum_{i=1}^{N} c_i\mu(A_i \cap X) \\
= \sum_{i=1}^{N} c_i\mu(A_i) \\
= \int_X g d\mu
\end{aligned}$$

giving

$$\lim_{n\to\infty}\int_X f_n d\mu \geq \lim_{n\to\infty} t\int_{E_n} g d\mu = t\int_X g d\mu$$

then since $t \in (0,1)$ is arbitrary we conclude that

$$\lim_{n\to\infty}\int_X f_n d\mu \geq \int_X g d\mu$$

and since $g \leq f$ is an arbitrary simple function, taking

$$\sup_g\{g \in \mathcal{S} : 0 \leq g \leq f\}$$

we get

$$\lim_{n\to\infty}\int_X f_n d\mu \geq \int_X f d\mu$$

and thus we can conclude

$$\int f d\mu = \lim_{n\to\infty}\int f_n d\mu$$

$\square$

Lemma 83 (**Fatou's Lemma**)**.** Let $(X, \mathcal{S}, \mu)$ be measure space, and let $\{f_n\} \in \mathcal{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$ such that $f_n \geq 0\ \forall\ n$. Then

$$\int \liminf\{f_n\} d\mu \leq \liminf \int f_n d\mu$$

Proof. Set

$$g_n(x) = \inf\{f_i(x) : n \leq i < \infty\}$$

then

$$\lim_{n\to\infty} g_n(x) = \liminf f_n(x)$$

and since $g_1(x) \leq g_2(x) \leq \cdots$ we have $\{g_n\}$ is non-decreasing, or monotonic, and so the general version of the Monotone Convergence Theorems says

$$\int \liminf_{n\to\infty} f_n(x) d\mu = \int \lim_{n\to\infty} g_n(x) d\mu = \lim_{n\to\infty}\int g_n(x) d\mu$$

yet, since $g_n(x) \leq f_n(x)$ pointwise $\forall\ n$ we then have

$$\int g_n(x) d\mu \leq \int f_n(x) d\mu \quad \forall\ n$$

and thus,

$$\liminf \int f_n(x) d\mu \geq \lim_{n\to\infty}\int g_n(x) d\mu = \int \liminf_{n\to\infty} f_n(x) d\mu$$

and so we have

$$\int \liminf\{f_n\} d\mu \leq \liminf \int f_n d\mu$$

$\square$

Theorem 84. Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space. For $1 \leq p \leq \infty$, if $f, g \in \mathcal{L}^p(X, \mathcal{S}, \mu, B)$ then $f+g \in \mathcal{L}^p(X, \mathcal{S}, \mu, B)$, and so $\mathcal{L}^p(X, \mathcal{S}, \mu, B)$ is a vector space of functions.

Proof.

$$\begin{aligned} ||f(x) + g(x)||^p &\leq (||f(x)|| + ||g(x)||)^p \\ &\leq (2 \max\{f(x), g(x)\})^p \\ &\leq 2^p(||\overset{\in \mathcal{L}^1}{f(x)}||^p + ||\overset{\in \mathcal{L}^1}{g(x)}||^p) \in \mathcal{L}^1 \end{aligned}$$

and so $||f(x) + g(x)||^p$ is dominated by an integrable function and so must also be integrable by Lebesgue Dominated Convergence Theorem. □

Proposition 85. Let $(X, \mathcal{S}, \mu)$ be measure space with Banach space $\mathbb{R}$, and let $f \in \mathcal{L}^2(X, \mathcal{S}, \mu, \mathbb{R} \text{ or } \mathbb{C})$. Then

$$x \mapsto |f(x)|^2 \in \mathcal{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$$

and $\mathcal{L}^2(X, \mathcal{S}, \mu, \mathbb{R} \text{ or } \mathbb{C})$ satisfies Cauchy-Schwartz; i.e. for $f, g \in \mathcal{L}^2(X, \mathcal{S}, \mu, \mathbb{R} \text{ or } \mathbb{C})$

$$|\langle f, g \rangle| \leq ||f||_2 ||g||_2$$

Proof. For $r, s \in \mathbb{R} \setminus \{0\}$

$$\begin{aligned} 0 &\leq (r-s)^2 = r^2 - 2rs + s^2 \\ \implies 2rs &\leq r^2 + s^2 \end{aligned}$$

which implies

$$2\left|f(x)\overline{g(x)}\right| \leq |f(x)|^2 + |g(x)|^2 \in \mathcal{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$$

and so by Lebesgue Dominated Converge $x \mapsto \left|f(x)\overline{g(x)}\right| \in \mathcal{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$. So set

$$\langle f, g \rangle := \int f(x)\overline{g(x)} d\mu(x)$$

then

$$2|\langle f, g \rangle| \leq \int |f(x)|^2 d\mu(x) + \int |g(x)|^2 d\mu(x) = ||f||_2^2 + ||g||_2^2$$

if, in addition, $||f||_2 = 1$ and $||g||_2 = 1$, then

$$|\langle f, g \rangle| \leq 1$$

so for any $f, g \in \mathcal{L}^2(X, \mathcal{S}, \mu, \mathbb{R} \text{ or } \mathbb{C})$ scale by setting $f = \frac{f}{||f||_2}$ and $g = \frac{g}{||g||_2}$, then

$$\begin{aligned} \frac{|\langle f, g \rangle|}{||f||_2 ||g||_2} &\leq 1 \\ \implies |\langle f, g \rangle| &\leq ||f||_2 ||g||_2 \end{aligned}$$

□