Differentiable Manifolds Notes

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1 Definitions

Hausdorff space: Suppose M is a topological space. For every pair of distinct points $p, q \in M$ $\exists$ $U, V \subset M$ open, such that $U \cap V = \varnothing$ and $p \in U,\ q \in V$

Second countable: Suppose M is a topological space. Then M is second countable if $\exists$ a countable basis for the topology of M.

Locally euclidean: Suppose M is a topological space. Then M is locally euclidean of dimension n if $\forall$ $p \in M$ we have:

1.) $p \in U \subseteq M$ where U is open.

2.) An open subset $\widehat{U} \subseteq \mathbb{R}^n$.

3.) A homeomorphism $\phi : U \to \widehat{U}$.

Topological n-Manifold: A topological space M that has the following properties:

1.) A Hausdorff space.

2.) Second countable.

3.) Locally euclidean of dimension n.

Coordinate chart: Let M be a topological n-Manifold, then a coordinate chart on M is a pair (U, ϕ) with $U \subset M$ open, $\phi(U) \subset \mathbb{R}^n$ open and $\phi : U \to \phi(U)$ a homeomorphism.

- if $\phi(p) = 0$ the chart is centered at $p \in M$
- if $p \in (U, \phi)$ we are always capable of centering the chart at p by defining $\phi : U \to \phi(U)$ by $\phi(q) = \phi(q) - \phi(p),\ \forall\ q \in (U, \phi)$

Coordinate Ball: for the chart (U, ϕ) if $\phi(U) = B \subset \mathbb{R}^n$, where B is an n-dimensional ball, then U is a coordinate ball (in M).

Local coordinates: for a topological n-Manifold with chart (U, ϕ) the local coordinate map ϕ is actually comprised of n functions $(x^1, \ldots, x^n)$ so that

$$\phi(U) = \big(x^1(U), \ldots, x^n(U)\big) \subset \mathbb{R}^n$$

and for each $p \in U$, we have $\phi(p) = (x^1(p), \dots, x^n(p))$, which are the local coordinates on U.

Disconnected: A topological space X is disconnected if it has 2 disjoint nonempty open subsets whose union is X; i.e. $U, V \subset X$ open, such that

$$U \neq \varnothing, \quad V \neq \varnothing, \text{ where } U \cap V = \varnothing, \quad \text{and } U \cup V = X$$

Connected: A topological space X is connect if it is not disconnected. Equivalently it is connected iff its' only subsets which are both open and closed are: $\varnothing$, and X itself.

A connected subset of X is a subset that is a connected space when endowed with the relative topology.

Component: For topological space X a component of X is a maximal connected subset of X; i.e. a connected subset that is not properly contained in any larger connected subset.

Path: If X is a topological space and $p, q \in X$, a path in X from p to q is a continuous map $f : [0, 1] \to X$ such that $f(0) = p$ and $f(1) = q$.

If for every pair of points $p, q \in X$, $\exists$ a path in X from p to q then X is path-connected.

The path components of X are its maximal path-connected subsets.

Locally Path-Connected: A topological space X is said to be locally path-connected if it admits a basis $\mathcal{B}$ of path-connected open subsets.

Locally Compact: let (X, τ) be a topological space. Then X is locally compact if $\forall\, x \in X$, $\exists\, O \in \tau$ with $x \in O$ such that $\overline{O}$ is compact.

Locally finite: Let M be a topological space, and let $\mathcal{S}$ be a collection of subsets of M. $\mathcal{S}$ is locally finite if $\forall\, p \in M$ $\exists$ a neighborhood U_p such that $U_p \cap S \neq \varnothing$ for finitely many $S \in \mathcal{S}$.

Refinement: Let M be a topological space and $\mathcal{U}$ a cover for M. If there is another cover $\mathcal{V}$ of M such that $\forall\, V \in \mathcal{V}\ \exists\, U \in \mathcal{U}$ with $V \subseteq U$ then $\mathcal{V}$ is a refinement of $\mathcal{U}$.

Paracompact: Let M be a topological space. M is paracompact if every open cover of M admits an open, locally finite refinement.

Paracompactness is a consequence of local compactness and second countability.

Exhaustion: Let X be a topological space, a sequence $\{K_i\}_{i=1}^{\infty}$ of compact subsets of X, is an exhaustion of X by compact sets if,

$$\bigcup_{i=1}^{\infty} K_i = X$$

and

$$K_i \subseteq \mathrm{Int}(K_{i+1})\ \forall\, i$$

Transition Map: Let M be a n-dimensional topological manifold, and let $(U, \phi), (V, \psi)$ be two charts on M such that

$$U \cap V \neq \varnothing$$

then the transition map from ϕ to ψ is the composite function

$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V) \subseteq \psi(V) \subseteq \mathbb{R}^n$$

it is the composition of homeomorphisms, and is therefore itself a homeomorphism.

Smooth Compatibility: Let M be a n-dimensional topological manifold, and let $(U, \phi), (V, \psi)$ be two charts on M. Then (U, ϕ) and (V, ψ) are smoothly compatible if either

$$U \cap V = \varnothing$$

or, $\psi \circ \phi^{-1}$ is a diffeomorphism; i.e. a bi-smooth bijection. And since $\psi(U \cap V), \phi(U \cap V) \subseteq \mathbb{R}^n$ smooth means $\psi \circ \phi^{-1} \in C^\infty(\mathbb{R}^n)$ and so has continuous partial derivatives of all orders.

Smooth Atlas: Let M be a n-dimensional topological manifold. A smooth atlas $\mathcal{A}_M$ for M is a collection of charts covering M such that any two charts in $\mathcal{A}_M$ are smoothly compatible; i.e. either their domains have empty intersection, or their composition is $C^\infty(\mathbb{R}^n)$.

Smooth Structure: Let M be a topological manifold. A smooth structure on M is a maximal smooth atlas.

Smooth Manifold: A smooth manifold is a pair $(M, \mathcal{A})$, where M is a topological manifold and $\mathcal{A}$ is a smooth structure on M.

Regular coordinate balls: Let M be a smooth manifold. Then $B \subset M$ is a regular coordinate ball if $\exists\ (B', \phi) \in \mathcal{A}_M$ such that $B' \supseteq \overline{B}$ and

$$\phi : B' \to \mathbb{R}^n$$

such that for $r, r' \in \mathbb{R}_+$ with $r < r'$ we have:

$$\phi(B) = B_r(\mathbf{0})$$
$$\phi(\overline{B}) = \overline{B_r(\mathbf{0})}$$
$$\phi(B') = B_{r'}(\mathbf{0})$$

Because $\overline{B}$ is homeomorphic to $\overline{B_r(\mathbf{0})}$, it is compact, and so every regular coordinate ball is precompact in M.

For a manifold with boundary, we make the following adjustments for regular coordinate half-balls

$$\phi : B' \to \mathbb{H}^n$$

such that for $0 < r < r'$ we have:

$$\phi(B) = B_r(\mathbf{0}) \cap \mathbb{H}^n$$
$$\phi(\overline{B}) = \overline{B_r(\mathbf{0})} \cap \mathbb{H}^n$$
$$\phi(B') = B_{r'}(\mathbf{0}) \cap \mathbb{H}^n$$

Smooth Function: Let M be a smooth n-manifold, $k > 0$ and

$$f : M \to \mathbb{R}^k$$

any function. Then f is a smooth function if $\forall\, p \in M,\ \exists\, (U, \phi) \in \mathcal{A}_M$ such that

$$f \circ \phi^{-1} : \phi(U) \to f(U)$$

is smooth; i.e. the coordinate representation of f is smooth.

Note: smooth functions have smooth coordinate representations in every smooth chart.

Smooth Map between Manifolds: Let M, N be smooth manifolds and

$$F : M \to N$$

be any map. Then F is a smooth map if $\forall\, p \in M,\ \exists\, (U, \phi) \in \mathcal{A}_M$ containing p, and $(V, \psi) \in \mathcal{A}_N$ containing $F(p)$, such that $F(U) \subseteq V$ and

$$\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$$

is smooth.

Let $A \subseteq M$ be an arbitrary subset, then

$$F : A \to N$$

is smooth on A if $\forall\, p \in A,\ \exists\, U_p \subseteq M$ open, and a smooth map

$$\widetilde{F} : U_p \to N$$

such that $\widetilde{F}|_{U_p \cap A} = F$.

Coordinate Representation: Let M, N be smooth manifolds. For a smooth map $F : M \to N$ and $(U, \phi) \in \mathcal{A}_M$, $(V, \psi) \in \mathcal{A}_N$

$$\widehat{F} := \psi \circ F \circ \phi^{-1} : \phi\left(U \cap F^{-1}(V)\right) \to \psi(V)$$

is the coordinate representation of F.

Cutoff function: for $r_1, r_2 \in \mathbb{R}$ such that $r_1 < r_2$, a cutoff function is a smooth function $h : \mathbb{R} \to \mathbb{R}$ having the properties

$$h(t) = \begin{cases} 1, & t \leq r_1 \\ 0 < h(t) < 1, & r_1 < t < r_2 \\ 0, & t \geq r_2 \end{cases}$$

Support: Let M be a topological space. For any real, or vector-valued function

$$f : M \to \mathbb{R}$$

or

$$f : M \to \mathbb{R}^n$$

the support of f is the closure of the set of points in M where f in nonzero:

$$\operatorname{supp}(f) = \overline{\{p \ \in M : f(p) \neq \mathbf{0}\}}$$

Bump function: Let M be a topological space with closed subset $A \subset M$ and open subset $U \subset M$ such that $U \supset A$. A bump function for A supported in U is a continuous function $\psi : M \to \mathbb{R}$ such that:

$$\begin{aligned}
&0 \leq \psi(M) \leq 1 \\
&\psi(a) \equiv 1 \quad \forall\ a \in A \\
&\operatorname{supp}(\psi) \subseteq U
\end{aligned}$$

Partition of Unity: Suppose M is a topological space, and let $\{U_\alpha\}_{\alpha\in\Lambda} := \mathcal{U}$ be an arbitrary open cover of M indexed by a set Λ. A partition of unity subordinate to $\mathcal{U}$ is an indexed family of continuous functions $\{\psi_\alpha\}_{\alpha\in\Lambda}$ with $\psi_\alpha : M \to \mathbb{R}$ such that:

(i) $0 \leq \psi_\alpha(p) \leq 1 \quad \forall\ \alpha \in \Lambda; \forall\ p \in M$

(ii) $\operatorname{supp}(\psi_\alpha) \subseteq U_\alpha$ for each $\alpha \in \Lambda$

(iii) The family of supports $\{\operatorname{supp}(\psi_\alpha)\}_{\alpha\in\Lambda}$ is locally finite; i.e. $\forall\ p \in M\ \exists$ a neighborhood U_p such that

$$U_p \cap \operatorname{supp}(\psi_\alpha) \neq \varnothing$$

for finitely many $\alpha \in \Lambda$

(iv) $\sum_{\alpha\in\Lambda} \psi_\alpha(p) = 1 \quad \forall\ p \in M$.

Note: local finiteness gives only finitely many of the terms in the sum are non-zero, and so there is no issue of convergence.

A **Smooth Partition of Unity** has the family $\{\psi_\alpha\}_{\alpha\in\Lambda}$ being smooth, instead of just continuous. Used to blend Local Smooth objects into Global Smooth objects, without the assumption of agreement on overlaps.

Level set: For manifolds M, and N and any map

$$\Phi : M \to N$$

for each $c \in N$, $\Phi^{-1}(c)$ is a level set of Φ.

- special case $N = \mathbb{R}^k$ then for $c = \mathbf{0} \in \mathbb{R}^k$, $\Phi^{-1}(\mathbf{0})$ is the zero set of Φ

Sublevel set: Let M be a topological space. The sublevel sets for the function

$$f : M \to \mathbb{R}$$

are $f^{-1}\big((-\infty, c]\big)$ for each $c \in \mathbb{R}$.

Exhaustion function: If M is a topological space, an exhaustion function for M is a continuous function

$$f : M \to \mathbb{R}$$

such that $f^{-1}((-\infty, c]) \subseteq M$ is compact for each $c \in \mathbb{R}$; i.e. all of f's sublevel sets are compact.

Note: If M is compact, any continuous $\mathbb{R}$-valued function on M is an exhaustion function.

Examples:

- $f : \mathbb{R}^n \to \mathbb{R}$ by $f(\mathbf{x}) = |||\mathbf{x}|||^2$
- $f : \mathbb{B}^n \to \mathbb{R}$ by $f(\mathbf{x}) = \frac{1}{1-|||\mathbf{x}|||^2}$

Geometric Tangent Space: Let $\mathbf{a} \in \mathbb{R}^n$ then the geometric tangent space of $\mathbb{R}^n$ at $\mathbf{a}$ is

$$\{\mathbf{a}\} \times \mathbb{R}^n = \{(\mathbf{a}, \mathbf{v}) : \mathbf{v} \in \mathbb{R}^n\} := \mathbb{R}^n_{\mathbf{a}}$$

Which is to say a translation of $\mathbb{R}^n$ by $\mathbf{a}$ so that the origin coincides with $\mathbf{a}$; i.e.

$$\mathbb{R}^n_{\mathbf{a}} = \mathbb{R}^n - \mathbf{a}$$

Elements $\mathbf{v}_{\mathbf{a}} \in \mathbb{R}^n_{\mathbf{a}}$ are referred to as geometric tangent vectors

Directional Derivative: Let $\mathbf{v}_{\mathbf{a}} \in \mathbb{R}^n_{\mathbf{a}}$ then the directional derivative defined by

$$D_{\mathbf{v}_{\mathbf{a}}} = D_{\mathbf{v}}\big|_{\mathbf{a}} : C^\infty(\mathbb{R}^n) \to \mathbb{R}; \text{ by } D_{\mathbf{v}_{\mathbf{a}}}(f) = D_{\mathbf{v}}\big(f(\mathbf{a})\big) = \left.\frac{d}{dt}\right|_{t=0} f(\mathbf{a} + t\mathbf{v})$$

Which is the derivative at $\mathbf{a}$ in the direction of $\mathbf{v}$. If $\mathbf{v}_{\mathbf{a}} = v^i e_i|_{\mathbf{a}}$ we have

$$D_{\mathbf{v}}(f)|_{\mathbf{a}} = v^i \frac{\partial f}{\partial x^i}(\mathbf{a}) = \sum_{i=1}^{n} v^i \frac{\partial f}{\partial x^i}(\mathbf{a})$$

Derivation: Let $\mathbf{a} \in \mathbb{R}^n$, then the map $w : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is a derivation at $\mathbf{a}$ if it is linear over $\mathbb{R}$ and satisfies

$$w(fg)|_{\mathbf{a}} = f(\mathbf{a})w(g)|_{\mathbf{a}} + g(\mathbf{a})w(f)|_{\mathbf{a}}$$

Denote the set of all derivations $C^\infty(\mathbb{R}^n)|_{\mathbf{a}}$ as $T_{\mathbf{a}}\mathbb{R}^n$.

Tangent Space: Let M be a manifold with or without boundary and let $p \in M$, then the set of all derivations $C^\infty(M)|_p := T_pM$ is the tangent space of M at p. And any element of T_pM is referred to as a tangent vector at p.

T_pM has a basis given by $\left\{\frac{\partial}{\partial x^1}\big|_p, \ldots, \frac{\partial}{\partial x^n}\big|_p\right\}$, and so each $v_p \in T_pM$ can be written uniquely as

$$v_p = \sum_{i=1}^{n} v^i \left.\frac{\partial}{\partial x^i}\right|_p$$

with $x^j \in C^\infty(U)$ we get the components of v_p by

$$v_p(x^j) = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}\bigg|_p (x^j) = \sum_{i=1}^n v^i \frac{\partial x^j}{\partial x^i}(p) = \sum_{i=1}^n v^i \delta_i^j = v^j$$

Coordinate Vectors: Let $p \in M$, the coordinate vectors at p are a basis for T_pM defined as follows. Let $(U, \phi) \in \mathcal{A}_M$ be a chart containing p. Since ϕ is a diffeomorphism, and both $d\iota|_p : T_pU \to T_pM$ and $d\iota|_{\phi(p)} : T_{\phi(p)}\phi(U) \to T_{\phi(p)}\mathbb{R}^n$ are isomorphisms we get the induced isomorphism

$$d\phi|_p : T_pM \to T_{\phi(p)}\mathbb{R}^n$$

with $\left\{ \frac{\partial}{\partial x^1}\big|_{\phi(p)}, \dots, \frac{\partial}{\partial x^n}\big|_{\phi(p)} \right\}$ forming a basis for $T_{\phi(p)}\mathbb{R}^n$ and since isomorphisms map basis vectors to basis vectors we have

$$\frac{\partial}{\partial x^i}\bigg|_p := (d\phi|_p)^{-1}\left(\frac{\partial}{\partial x^i}\bigg|_{\phi(p)} \right) = d\phi^{-1}|_{\phi(p)}\left(\frac{\partial}{\partial x^i}\bigg|_{\phi(p)} \right)$$

for $i \in \{1, \dots, n\}$ form a basis for T_pM. So for $f \in C^\infty(U)$ with coordinate representation given by

$$\widehat{f} := f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$$

we get that the derivation $\frac{\partial}{\partial x^i}\big|_p \in T_pM$ acts on f by

$$\frac{\partial}{\partial x^i}\bigg|_p (f) = d\phi^{-1}|_{\phi(p)}\left(\frac{\partial}{\partial x^i}\bigg|_{\phi(p)} \right)(f) = \frac{\partial}{\partial x^i}\bigg|_{\phi(p)} (f \circ \phi^{-1})|_{\phi(p)} = \frac{\partial \widehat{f}}{\partial x^i}\big(\phi(p)\big)$$

and so $\frac{\partial}{\partial x^i}\big|_p$ takes the i^{th} partial derivative of $\widehat{f}$ at $\phi(p)$.

Differential: Let M, N be smooth manifold with or without boundary, and $F : M \to N$ a smooth map. Then for each $p \in M$ the differential of F at p is the map

$$dF|_p : T_pM \to T_{F(p)}N \text{ by } dF|_p(v_p) = v|_{F(p)} := v_{F(p)}$$

so for $v_p \in T_pM$, $dF|_p(v_p)$ is the derivation at $F(p) \in N$ that acts on functions $f \in C^\infty(N)$. And so

$$dF|_p(v_p) : C^\infty(N)|_{F(p)} \to \mathbb{R}, \text{ by } dF|_p(v_p)(f) = v(f \circ F)|_p$$

and is a derivation since for any $f, g \in C^\infty(N)|_{F(p)}$ we have

$$\begin{aligned} dF|_p(v_p)(fg) &= v\big((fg) \circ F\big)|_p \\ &= v\big((f \circ F) \cdot (g \circ F)\big)|_p \\ &= f\big(F(p)\big)v(g \circ F)|_p + g\big(F(p)\big)v(f \circ F)|_p \\ &= f\big(F(p)\big)dF|_p(v_p)(g) + g\big(F(p)\big)dF|_p(v_p)(f) \end{aligned}$$

Next, noting that for $(U, \phi) \in \mathcal{A}_M$ containing p, and $(V, \psi) \in \mathcal{A}_N$ containing $F(p)$, and for the coordinate representation of F we have

$$\widehat{F} = \psi \circ F \circ \phi^{-1} \implies \psi^{-1} \circ \widehat{F} = F \circ \phi^{-1}$$

and from the definition of basis vectors for $T_{F(p)}N$ we have

$$\left.\frac{\partial}{\partial y^j}\right|_{F(p)} = d\psi^{-1}|_{\widehat{F}(\phi(p))}\left(\left.\frac{\partial}{\partial y^j}\right|_{\widehat{F}(\phi(p))}\right)$$

and for the special case where $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$ with $\big(U, (x^1, \dots, x^m)\big) \in \mathcal{A}_{\mathbb{R}^m}$ containing p, and $\big(V, (y^1, \dots, y^n)\big) \in \mathcal{A}_{\mathbb{R}^n}$ containing $F(p)$, and $f \in C^\infty(\mathbb{R}^n)$ we get

$$dF|_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right)(f) = \left.\frac{\partial}{\partial x^i}\right|_p (f\circ F)|_p = \sum_{j=1}^n \frac{\partial f}{\partial y^j}\big(F(p)\big)\frac{\partial(y^j\circ F)}{\partial x^i}(p) = \left(\sum_{j=1}^n \frac{\partial F^j}{\partial x^i}(p)\left.\frac{\partial}{\partial y^j}\right|_{F(p)}\right)(f)$$

so the action of $dF|_p$ on the basis vector $\left.\frac{\partial}{\partial x^i}\right|_p \in T_pM$ is

$$\begin{aligned}
dF|_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) &= dF|_p\left(d\phi^{-1}|_{\phi(p)}\left(\left.\frac{\partial}{\partial x^i}\right|_{\phi(p)}\right)\right)\\
&= d(F\circ\phi^{-1})|_{\phi(p)}\left(\left.\frac{\partial}{\partial x^i}\right|_{\phi(p)}\right)\\
&= d(\psi^{-1}\circ\widehat{F})|_{\phi(p)}\left(\left.\frac{\partial}{\partial x^i}\right|_{\phi(p)}\right)\\
&= d\psi^{-1}|_{\widehat{F}(\phi(p))}\left(d\widehat{F}|_{\phi(p)}\left(\left.\frac{\partial}{\partial x^i}\right|_{\phi(p)}\right)\right)\\
&= d\psi^{-1}|_{\widehat{F}(\phi(p))}\left(\sum_{j=1}^n\frac{\partial\widehat{F}^j}{\partial x^i}\big(\phi(p)\big)\left.\frac{\partial}{\partial y^j}\right|_{\widehat{F}(\phi(p))}\right) && \widehat{F}\text{ map between Euclidean spaces}\\
&= \sum_{j=1}^n\frac{\partial\widehat{F}^j}{\partial x^i}\big(\phi(p)\big)\cdot d\psi^{-1}|_{\widehat{F}(\phi(p))}\left(\left.\frac{\partial}{\partial y^j}\right|_{\widehat{F}(\phi(p))}\right) && \text{linearity}\\
&= \sum_{j=1}^n\frac{\partial\widehat{F}^j}{\partial x^i}\big(\phi(p)\big)\left.\frac{\partial}{\partial y^j}\right|_{F(p)}
\end{aligned}$$

and therefore $dF|_p$ is represented by the Jacobian matrix of $\widehat{F}$ at $\phi(p)$. Or,

$$dF|_p = \begin{bmatrix} \frac{\partial\widehat{F}^1}{\partial x^1}\big(\phi(p)\big) & \cdots & \frac{\partial\widehat{F}^1}{\partial x^m}\big(\phi(p)\big)\\ \vdots & \ddots & \vdots \\ \frac{\partial\widehat{F}^n}{\partial x^1}\big(\phi(p)\big) & \cdots & \frac{\partial\widehat{F}^n}{\partial x^m}\big(\phi(p)\big)\end{bmatrix}$$

Equivalently, for $f \in C^\infty(M)$, with $f : M \to \mathbb{R}$ we can define the differential

$$df|_p : T_pM \to \mathbb{R} \cong T_{f(p)}\mathbb{R},\ \text{by } df|_p(v_p) = v_p(f)|_p$$

Change of Coordinates Let M be a manifold with or without boundary, let $p \in M$, and let $(U, \phi), (V, \psi) \in \mathcal{A}_M$ be two charts containing p. Then we have the transition map

$$\psi\circ\phi^{-1} : \phi(U\cap V) \to \psi(U\cap V)$$

so for $\phi = (x^1, \dots, x^n)$, and $\psi = (y^1, \dots, y^n)$, then

$$
\begin{aligned}
d(\psi \circ \phi^{-1})|_{\phi(p)} \left(\left.\frac{\partial}{\partial x^i}\right|_{\phi(p)} \right) &= \left.\frac{\partial}{\partial x^i}\right|_{\phi(p)} (\psi \circ \phi^{-1})|_{\phi(p)} \\
&= \sum_{j=1}^{n} \frac{\partial (y^j \circ \phi^{-1})}{\partial x^i} (\phi(p)) \left.\frac{\partial}{\partial y^j}\right|_{\psi \circ \phi(\phi(p))} \\
&= \sum_{j=1}^{n} \frac{\partial (y^j \circ \phi^{-1})}{\partial x^i} (\phi(p)) \left.\frac{\partial}{\partial y^j}\right|_{\psi(p)}
\end{aligned}
$$

then we have

$$
\begin{aligned}
\left.\frac{\partial}{\partial x^i}\right|_{p} &= d\phi^{-1}|_{\phi(p)} \left(\left.\frac{\partial}{\partial x^i}\right|_{\phi(p)} \right) \\
&= d(\psi^{-1} \circ \psi \circ \phi^{-1})|_{\phi(p)} \left(\left.\frac{\partial}{\partial x^i}\right|_{\phi(p)} \right) \\
&= d\psi^{-1}|_{\psi(p)} \circ d(\psi \circ \phi^{-1})|_{\phi(p)} \left(\left.\frac{\partial}{\partial x^i}\right|_{\phi(p)} \right) \\
&= d\psi^{-1}|_{\psi(p)} \left(\sum_{j=1}^{n} \frac{\partial (y^j \circ \phi^{-1})}{\partial x^i} (\phi(p)) \left.\frac{\partial}{\partial y^j}\right|_{\psi(p)} \right) \\
&= \sum_{j=1}^{n} \frac{\partial (y^j \circ \phi^{-1})}{\partial x^i} (\phi(p)) \cdot d\psi^{-1}|_{\psi(p)} \left(\left.\frac{\partial}{\partial y^j}\right|_{\psi(p)} \right) \\
&= \sum_{j=1}^{n} \frac{\partial (y^j \circ \phi^{-1})}{\partial x^i} (\phi(p)) \left.\frac{\partial}{\partial y^j}\right|_{p}
\end{aligned}
$$

so for $v_p \in T_pM$ where

$$
v_p = \sum_{i=1}^{n} v_x^i \left.\frac{\partial}{\partial x^i}\right|_{p} = \sum_{j=1}^{n} v_y^j \left.\frac{\partial}{\partial y^j}\right|_{p}
$$

we get that components of v_p are related by

$$
v_y^j = \sum_{i=1}^{n} \frac{\partial y^j}{\partial x^i} \big(\phi(p)\big) v_x^i
$$

Tangent Bundle: Given a smooth manifold M with or without boundary the tangent bundle, denoted TM, is the disjoint union of the tangent spaces $\forall\, p \in M$. That is

$$
TM = \bigsqcup_{p \in M} T_pM
$$

TM comes equipped with a natural projection

$$\pi : TM \to M \text{ by } \pi(v_p) = p$$

so for each vector $v_p \in T_pM$, π sends v_p to the point $p \in M$ at which it is tangent.

Natural Coordinates on the Tangent Bundle: Let M be a smooth manifold. Given any smooth chart $(U, \phi) \in \mathcal{A}_M$ and letting $\phi = (x^1, \ldots, x^n)$. Then

$$\pi^{-1}(U) = \bigsqcup_{p \in U} T_pM$$

Define the map

$$\widetilde{\phi} : \pi^{-1}(U) \to \mathbb{R}^{2n}, \text{ by } \widetilde{\phi}\left(\sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\bigg|_p\right) = \big(x^1(p), \ldots, x^n(p), v^1, \ldots, v^n\big)$$

With natural coordinates on TM being given by $\widetilde{\phi} = (x^1, \ldots, x^n, v^1, \ldots, v^n)$. And the atlas on TM is given by $\mathcal{A}_{TM} = \big\{\big(\pi^{-1}(U), \widetilde{\phi}\big) : (U, \phi) \in \mathcal{A}_M\big\}$

Global Differential: Let M and N be smooth manifolds, and $F : M \to N$ a smooth map. the global differential

$$dF : TM \to TN$$

is defined by

$$dF(v_p) = dF|_p(v_p) = v|_{F(p)}$$

so that $\forall\ v_p \in T_pM \subset TM$ we have $dF(v_p) \in T_{F(p)}N \subset TN$.

Curve: Let M be a manifold with or without boundary, and $J \subseteq \mathbb{R}$ an interval. A curve is a continuous map

$$\gamma : J \to M$$

Velocity of a Curve: Let M be a manifold with or without boundary, $J \subseteq \mathbb{R}$, and $t_0 \in J$ the velocity of the curve $\gamma : J \to M$ at t_0 is the derivation

$$\gamma'(t_0) = d\gamma|_{t_0}\left(\frac{d}{dt}\bigg|_{t_0}\right) \in T_{\gamma(t_0)}M$$

so for $\big(U, (x^1, \ldots, x^n)\big) \in \mathcal{A}_M$, $\gamma(t_0) \in U \subseteq M$ and $f \in C^\infty(M)$

$$\begin{aligned}
\gamma'(t_0)(f) &= d\gamma|_{t_0}\left(\frac{d}{dt}\bigg|_{t_0}\right)(f) \\
&= \frac{d}{dt}\bigg|_{t_0}(f \circ \gamma)|_{t_0} \\
&= \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\big(\gamma(t_0)\big)\frac{d(x^i \circ \gamma)}{dt}(t_0) \\
&= \sum_{i=1}^{n}\left(\frac{d\gamma^i}{dt}(t_0)\frac{\partial}{\partial x^i}\bigg|_{\gamma(t_0)}\right)(f)
\end{aligned}$$

so that

$$\gamma'(t_0) = \sum_{i=1}^{n} \frac{d\gamma^i}{dt}(t_0)\frac{\partial}{\partial x^i}\Big|_{\gamma(t_0)} \text{ in Local Coordintates}$$

Equivalently for $f \in C^\infty(M)$ we have

$$df|_{\gamma(t_0)} : T_{\gamma(t_0)}M \to \mathbb{R} \cong T_{f(\gamma(t_0))}\mathbb{R}, \text{ by } df|_{\gamma(t_0)}(\gamma'(t_0)) = (f \circ \gamma)'(t_0)$$

Rank of Smooth map: let M, N be smooth manifolds with or without boundary. Given a smooth map $F : M \to N$ and a point $p \in M$ the rank of F at p is the rank of the linear map

$$dF|_p : T_pM \to T_{F(p)}N, \quad \text{recall} \quad dF|_p = \begin{bmatrix} \frac{\partial \widehat{F}^1}{\partial x^1}\big(\phi(p)\big) & \cdots & \frac{\partial \widehat{F}^1}{\partial x^m}\big(\phi(p)\big) \\ \vdots & \ddots & \vdots \\ \frac{\partial \widehat{F}^n}{\partial x^1}\big(\phi(p)\big) & \cdots & \frac{\partial \widehat{F}^n}{\partial x^m}\big(\phi(p)\big) \end{bmatrix}$$

Which is the rank of the Jacobian of $\widehat{F}$ at $\phi(p)$ in any smooth chart. If F has the same rank at every point then we say F has **Constant Rank**.

Note: $\text{rank}(F) \leq \min\{\dim(M), \dim(N)\}$; if $\text{rank}(dF|_p) = \min\{\dim(M), \dim(N)\}$ then F has **Full Rank** at p

Smooth Submersion: A smooth map $F : M \to N$ is a smooth submersion if its differential

$$dF|_p : T_pM \to T_{F(p)}N$$

is surjective at each point; i.e. $\text{rank}(F) = \dim(N)$.

Smooth Immersion: A smooth map $F : M \to N$ is a smooth immersion if its differential

$$dF|_p : T_pM \to T_{F(p)}N$$

is injective at each point; i.e. $\text{rank}(F) = \dim(M)$.

Local Diffeomorphism: Let M and N be smooth manifolds with or without boundary, a map

$$F : M \to N$$

is a local diffeomorphism if $\forall\, p \in M, \exists\, U_p \subseteq M$ open such that $F(U_p) \subseteq N$ is open and

$$F|_{U_p} : U_p \to F(U_p)$$

is a diffeomorphism; i.e. a map whose differential is invertible at each point.

Smooth Embedding: If M, N are smooth manifolds with or without boundary, and $F : M \to N$ is a smooth immersion; $dF|_p$ is injective for each $p \in M$, with the additional property that $F : M \to F(M)$ is a homeomorphism then F is a smooth embedding.

Note, the additional property can be viewed as a topological embedding where $F : M \to F(M)$ is a homeomorphism onto its image $F(M) \subseteq N$ in the relative topology.

every smooth embedding is a topological embedding, but a topological embedding need not be a smooth embedding.

Proper Map: Suppose X, Y are topological spaces and

$$f : X \to Y$$

is a map. Then f is proper if $\forall\, K \subseteq Y$ compact, we have $f^{-1}(K) \subseteq X$ is compact.

Section: For any continuous map $\pi : M \to N$ a section for π is a continuous right inverse; i.e. a map $\sigma : N \to M$ such that

$$\pi \circ \sigma = Id_N$$

Local Section: For any continuous map $\pi : M \to N$, and for any open subset $V \subseteq N$, a local section of π is a continuous map $\sigma : V \to M$ such that

$$\pi \circ \sigma = Id_V$$

so that σ is a right inverse for π on V.

Smooth Covering Map: If E, M are connected smooth manifolds with or without boundary, a map $\pi : E \to M$ is a smooth covering map if:

- π is smooth and surjective.
- $\forall\, p \in M$, $\exists\, U_p \subseteq M$ open and containing p, such that each connected component of $\pi^{-1}(U_p) \subseteq E$ is mapped diffeomorphically onto U_p by π; i.e.

  $$\pi|_{V_i} : V_i \to U_p$$

  is a diffeomorphism for each component of $\pi^{-1}(U_p) = \bigsqcup_{i \in I} V_i$.

M is called the base of the covering.
E is called a covering manifold of M. If in addition E is simply connected, E is called the universal covering manifold of M.

Note, every smooth covering map is a (topological) covering map, but a covering map, that is smooth, need not be a smooth covering map.

Embedded Submanifold: Suppose M is a smooth manifold with or without boundary. An embedded submanifold of M is a subset $S \subseteq M$, which is a manifold without boundary in the relative topology, endowed with a smooth structure to which

$$\iota : S \hookrightarrow M$$

is a smooth embedding

Note, every embedded submanifold is also an immersed submanifold.

Note: by Proposition 77 embedded submanifolds are exactly the images of smooth embeddings.

Codimension: If M is a smooth manifold and $S \subseteq M$ is an immersed submanifold for which an embedded smooth submanifold is a special case, then $\dim(M) - \dim(S)$ is the codimension of S in M.

M is referred to as the **Ambient Manifold** for S.

An immersed hypersurface (or smooth hypersurface), which has, as a special case, an embedded hypersurface, is an embedded submanifold of codimension 1; i.e. $\dim(M) - \dim(S) = 1$.

Properly Embedded: An embedded submanifold $S \subseteq M$ is properly embedded if

$$\iota : S \hookrightarrow M$$

is a proper map; i.e. the preimage of compact sets is compact.

Slice: For $U \subseteq \mathbb{R}^n$ open, and $k \in \{0, 1, \ldots, n\}$ a k-slice of U is any subset $S \subseteq U$ of the form:

$$S = \{(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) \in U : x_{k+1} = c_{k+1}, \ldots, x_n = c_n\}$$

for some constants $c_{k+1}, c_{k+2}, \ldots, c_n \in \mathbb{R}$. If $k = n$ then $S = U$.

For a smooth n-manifold M with smooth chart (U, ϕ), if $S \subseteq U$ is such that $\phi(S)$ is a k-slice of $\phi(U)$, then S is a k-slice of U.

Note: the constants $c_{k+1}, \ldots, c_n \in \mathbb{R}$ can all be made zero by taking
$\phi = (x^1, \ldots, x^k, x^{k+1} - c_{k+1}, \ldots, x^n - c_n)$

Slice Chart: Let M be a smooth manifold and $S \subseteq M$ a subset of M. For $(U, \phi = (x^1, \ldots, x^n)) \in \mathcal{A}_M$, if $S \cap U$ is a single k-slice in U, then (U, ϕ) is a slice chart for S in M; that is

$$S \cap U = \{p \in U : x^{k+1}(p) = c_{k+1}, \ldots, x^n(p) = c_n\}$$

$\phi = (x^1, \ldots, x^n)$ are the slice coordinates.

local k-slice condition: Given $S \subseteq M$ if $\forall\, p \in S$ we have p is in some smooth chart (U, ϕ) such that $S \cap U$ is a single k-slice in U.

Regular Point: If M, N are smooth manifolds, and $F : M \to N$ a smooth map, a point $p \in M$ is a regular point of F if

$$dF|_p : T_pM \to T_{F(p)}N$$

is surjective.

In particular, every point is regular iff F is a submersion.
By Proposition 50, the set of regular points of F is always an open set.

Regular Value: If M, N are smooth manifolds, and $F : M \to N$ a smooth map, then a point $c \in N$ is a regular value for F, if each $p \in F^{-1}(c) \subseteq M$ is a regular point; i.e.

$$dF|_p : T_pM \to T_{F(p)}N$$

is surjective.

Critical Point: If M, N are smooth manifolds, and $F : M \to N$ a smooth map, a point $p \in M$ is critical if it is not regular; i.e.

$$dF|_p : T_pM \to T_{F(p)}N$$

is not surjective.

In particular, every point of M is critical if $\dim(M) < \dim(N)$, (smooth immersions which are not local diffeomorphism).

Critical Value: If M, N are smooth manifolds, and $F : M \to N$ a smooth map, then a point $c \in N$ is a critical value for F, if it is not a regular value; i.e. if not every $p \in F^{-1}(c) \subseteq M$ has the property that

$$dF|_p : T_pM \to T_{F(p)}N$$

is surjective.

Note, there are many more critical points than critical values. For instance consider the range where a function F is constant, say $[a, b]$, then the differential $dF|_{[a,b]} = 0$ by lemma 33, then all of $[a, b]$ are critical points of F, yet its image is a single point; i.e $F([a, b]) = c$.

Regular Level Set: If M, N are smooth manifolds, and $F : M \to N$ a smooth map, then $F^{-1}(c) \subset M$ is a regular level set if $c \in N$ is a regular value of F; i.e. a level set for which $\forall\ p \in F^{-1}(c)$ we have

$$dF|_p : T_pM \to T_{F(p)}N$$

is surjective.

Defining Map: If M, N are smooth manifolds, $S \subseteq M$ is an embedded submanifold, and

$$F : M \to N$$

is a smooth function such that S is a regular level set of F; that is, for some $q \in F(M) \subseteq N$ we have

$$F^{-1}(q) = S \subseteq M$$

and for each $p \in S$ we have

$$dF|_p : T_pM \to T_{F(p)}N$$

is surjective. Then F is a defining map for S.

Local Defining Map: If M, N are smooth manifolds, $S \subseteq M$ is an embedded submanifold, $U \subseteq M$ is an open subset, and

$$F : U \to N$$

is a smooth map such that $S \cap U$ is a regular level set of F, that is, for some $q \in F(U) \subseteq N$ we have

$$F^{-1}(q) = S \cap U \subseteq M$$

and for each $p \in S \cap U$ we have

$$dF|_p : T_pM \to T_{F(p)}N$$

is surjective. Then F is a local defining map for S.

Immersed Submanifold: Let M be a smooth manifold with or without boundary. An immersed submanifold of M is a subset $S \subseteq M$ endowed with a topology (not necessarily the subspace topology as with embedded submanifolds), with respect to which it is a topological manifold without boundary, and with a smooth structure to which

$$\iota : S \hookrightarrow M$$

is an (injective) smooth immersion.

Local Parameterization: Let M be a smooth manifold, and suppose $S \subseteq M$ is an immersed k-dimensional submanifold. A local parameterization of S is a continuous map

$$F : U \to M$$

where $U \subseteq \mathbb{R}^k$ is open, $F(U) \subseteq S$ is open, and where, considered as a mapping into S; i.e.

$$F : U \to F(U) \subseteq S$$

is a homeomorphism. If $F(U) = S$, then it is a **global parameterization**.

Smooth Local Parameterization: Let M be a smooth manifold, and suppose $S \subseteq M$ is an immersed k-dimensional submanifold. A smooth local parameterization of S is a continuous map

$$F : U \to M$$

where $U \subseteq \mathbb{R}^k$ is open, $F(U) \subseteq S$ is open, and where, considered as a mapping into S; i.e.

$$F : U \to F(U) \subseteq S$$

is a diffeomorphism with respect to the smooth manifold structure on S.

Weakly Embedded: If N is a smooth manifold, and $S \subseteq N$ is an immersed submanifold, then S is weakly embedded in N if every smooth map

$$F : M \to N$$

such that $F(M) \subseteq S$, is smooth as a map from M to S; i.e.

$$F : M \to S$$

is smooth.

By Corollary 95, every embedded submanifold is weakly embedded.

Tangent Space to a Smooth Submanifold: Let M be a smooth manifold with or without boundary, and $S \subseteq M$ an embedded or immersed submanifold. In either case

$$\iota : S \hookrightarrow M$$

is a smooth immersion, and thus, for each $p \in S$ we have

$$d\iota|_p : T_pS \to T_{\iota(p)}M$$

is an injective linear map. So for $v_p \in T_pS$ we have $d\iota|_p(v_p) \in T_pM$, and so for any $f \in C^\infty(M)$ we have the action

$$d\iota|_p(v_p)(f) = v(f \circ \iota)|_p = v(f|_S)|_p$$

Inward Pointing: If M is smooth manifold with boundary, and $p \in \partial M$. A vector $v_p \in T_pM \setminus T_p\partial M$ is inward pointing if for some $\epsilon > 0$, $\exists$ a smooth curve

$$\gamma : [0, \epsilon) \to M, \quad \text{such that} \quad \gamma(0) = p, \text{ and } \gamma'(0) = v_p$$

Outward Pointing: If M is smooth manifold with boundary, and $p \in \partial M$. A vector $v_p \in T_pM \setminus T_p\partial M$ is inward pointing if for some $\epsilon > 0$, $\exists$ a smooth curve

$$\gamma : (-\epsilon, 0] \to M, \quad \text{such that} \quad \gamma(0) = p, \text{ and } \gamma'(0) = v_p$$

Boundary Defining Function: If M is a smooth manifold with boundary, a boundary defining function for M is a smooth function $f : M \to [0, \infty)$ such that

$$f^{-1}(0) = \partial M, \quad \text{and} \quad df|_p \neq 0 \ \forall \ p \in \partial M$$

Example: $f(\mathbf{x}) = 1 - ||\mathbf{x}||^2$ is a boundary defining function for $\overline{\mathbb{B}}^n$.

Regular Domain: If M is a smooth manifold with or without boundary, a regular domain in M is a properly embedded codimension-0 submanifold with boundary.

Examples: $\overline{\mathbb{B}}^n \subset \mathbb{R}^n$, $\overline{\mathbb{H}}^n \subset \mathbb{R}^n$

Regular Sublevel Set: Let M be a smooth manifold, and

$$f : M \to \mathbb{R}$$

a function with regular value $b \in \mathbb{R}$; that is, $\forall \ p \in f^{-1}(b) \subseteq M$ we have

$$df|_p : T_pM \to T_{f(p)}\mathbb{R}$$

is surjective. Then, a set of the form $f^{-1}\big((-\infty, b]\big) \subseteq M$, is a regular sublevel set of f.

every regular sublevel set of a smooth real-valued function is a regular domain

If $D \subseteq M$ is a regular domain, and $f \in C^\infty(M)$ is a smooth function such that D is a regular sublevel set for f; i.e $D = f^{-1}\big((-\infty, b]\big)$ for some regular value $b \in \mathbb{R}$, then f is a defining function for D.

Half-Slice: Let M be a smooth manifold without boundary, and $(U, \phi = (x^1, \ldots, x^n)) \in \mathcal{A}_M$. A k-dimensional half-slice of U is a subset $S \subseteq U$ of the form

$$S = \{p \in U : x^{k+1}(p) = c_{k+1}, \ldots, x^n(p) = c_n;\ x^k(p) \geq 0\}$$

for some constants $c_{k+1}, c_{k+2}, \ldots, c_n \in \mathbb{R}$.

the local k-slice condition for submanifolds with boundary: Given $S \subseteq M$ if $\forall\ p \in S$ we have p is in some smooth chart (U, ϕ) such that $S \cap U$ is either:

1. An ordinary single k-dimensional slice in U, then U is an interior slice chart for S in M.
2. A single k-dimensional half-slice in U, then U is a boundary slice chart for S in M.

Measure Zero: A subset $X \subseteq \mathbb{R}^n$ is of measure zero if $\forall\ \epsilon > 0\ \exists$ a countable collection $U_1, U_2, \ldots$. where each U_i is an open cube; i.e. a product of open intervals

$$U_i = (a_1, b_1)_i \times (a_2, b_2)_i \times \cdots \times (a_n, b_n)_i$$

that has a volume

$$\text{vol}(U_i) = (b_1 - a_1)_i (b_2 - a_2)_i \ldots (b_n - a_n)_i$$

such that

$$X \subseteq \bigcup_{i=1}^{\infty} U_i, \text{ and } \sum_{i=1}^{\infty} \text{vol}(U_i) < \epsilon$$

Manifold Measure Zero: If M is a smooth n-manifold with or without boundary, and $A \subseteq M$. Then A has measure zero in M if $\forall\ (U, \phi) \in \mathcal{A}_M$ we have $\phi(A \cap U) \subseteq \mathbb{R}^n$ has n-dimensional measure zero.

$\boldsymbol{\delta}$-Close: Let M be a smooth manifold, and

$$\delta : M \to \mathbb{R}$$

be a continuous function such that $\delta(p) \geq 0\ \forall\ p \in M$. Two continuous functions $F, G : M \to \mathbb{R}^k$ are δ-close if

$$|F(p) - G(p)| < \delta(p) \quad \forall\ p \in M$$

Normal Space: Let $M \subseteq \mathbb{R}^n$ be an embedded m-dimensional submanifold. For each $\mathbf{x} \in M$ the normal space to M at $\mathbf{x}$ is the $(n-m)$-dimensional subspace $N_{\mathbf{x}}M \subseteq T_{\mathbf{x}}\mathbb{R}^n$ of all vectors orthogonal to $T_{\mathbf{x}}M$ with respect to the euclidean dot product; i.e.

$$N_{\mathbf{x}}M = (T_{\mathbf{x}}M)^\perp$$

Note: By proposition 3.2 $\forall\ \mathbf{x} \in \mathbb{R}^n, T_{\mathbf{x}}\mathbb{R}^n$ is isomorphic to $\mathbb{R}^n$ leading to $T\mathbb{R}^n$ being diffeomorphic to $\mathbb{R}^n \times \mathbb{R}^n$. So that each $T_{\mathbf{x}}\mathbb{R}^n$ inherits a euclidean dot product.

Normal Bundle: Let $M \subseteq \mathbb{R}^n$ be an embedded submanifold. The normal bundle of M is the subset $NM \subseteq T\mathbb{R}^n \approx \mathbb{R}^n \times \mathbb{R}^n$ consisting of all vectors orthogonal to TM. That is

$$NM = \{(\mathbf{x}, v_{\mathbf{x}}) \in \mathbb{R}^n \times \mathbb{R}^n : \mathbf{x} \in M; v_{\mathbf{x}} \in N_{\mathbf{x}}M\} = \bigsqcup_{\mathbf{x} \in M} N_{\mathbf{x}}M$$

from the natural projection

$$\pi : T\mathbb{R}^n \to \mathbb{R}^n$$

NM inherits the natural projection

$$\pi|_{NM} : NM \to M$$

Tubular Neighborhood: Let $M \subseteq \mathbb{R}^n$ be an embedded submanifold, and considering $NM \subseteq \mathbb{R}^n \times \mathbb{R}^n$ as an embedded submanifold. The tubular neighborhood of M, $U_M \subseteq \mathbb{R}^n$ is the diffeomorphic image of the smooth map

$$E : NM \to \mathbb{R}^n, \text{ by } E(\mathbf{x}, v_{\mathbf{x}}) = \mathbf{x} + v_{\mathbf{x}}$$

for some open subset $V \subseteq NM$ of the form $V = \{(\mathbf{x}, v_{\mathbf{x}}) : ||v_{\mathbf{x}}|| < \delta(\mathbf{x})\}$ for some positive continuous function $\delta : M \to \mathbb{R}$. Thus, the Tubular neighborhood of M satisfies

$$E|_V : V \to U_M$$

is a diffeomorphism.

Retraction: Let X be a topological space and $U \subseteq X$ a subspace of X. The retraction of X onto U is a continuous map

$$r : X \to U, \text{ such that } r|_U = Id_U$$

Smooth Homotopy: If N and M are two smooth manifolds with or without boundary, a homotopy

$$H : M \times [0, 1] \to N$$

is a smooth homotopy if it extends to a smooth map on some neighborhood of $M \times [0, 1] \subseteq M \times \mathbb{R}$.

If there exists $F_0, F_1 \in C^\infty(M, N)$ with $H \in C^\infty(M \times [0, 1], N)$ such that

$$H(\cdot, 0) = F_0, \quad H(\cdot, 1) = F_1$$

then F_0, and F_1 are smoothly homotopic.

Transversality: Suppose M is a smooth manifold and $S, S' \subseteq M$ are embedded submanifolds, then S, S' intersect transversely if $\forall\, p \in S \cap S'$ we have

$$T_pS + T_pS' = T_pM$$

where $T_pS, T_pS' \subseteq T_pM$ are considered as subspaces.

Additionally, if N is a smooth manifold,

$$F : M \to N$$

a smooth map, and $S \subseteq N$ an embedded submanifold, then F is transverse to S if $\forall\, q \in F^{-1}(S)$ we have

$$dF|_q(T_q M) + T_{F(q)}S = T_{F(q)}N$$

Note:

- If F is a smooth submersion, then it is automatically transverse to every embedded submanifold of N.
- $S, S' \subseteq M$ intersect transversely iff
$$\iota : S \hookrightarrow M$$
is transverse to S'; that is, $\forall\, p \in \iota^{-1}(S') \subseteq S$ we have
$$d\iota|_p(T_p S) + T_{\iota(p)}S = T_{\iota(p)}M$$

2 Notes

Lemma 1. Every topological manifold has a countable basis of precompact coordinate balls.

Proof. Let M be a topological n-manifold. First, considering the special case where M is covered by a single chart (M, ϕ), so that

$$\phi : M \to \mathbb{R}^n$$

is a global coordinate map, and let

$$\mathcal{B} = \{B_r(\mathbf{x}) \subseteq \mathbb{R}^n : r \in \mathbb{Q}; \mathbf{x} \in \mathbb{Q}^n\}$$

then for

$$B_{r'}(\mathbf{x}) \subseteq \phi(M) \quad \text{for some } r' > r$$

we have that $\overline{B_{r'}(\mathbf{x})}$ is compact in $\phi(M)$, and hence is precompact. Furthermore, $\mathcal{B}$ is a countable basis for the topology of $\mathbb{R}^n$, and hence is a countable basis for the relative topology of $\phi(M)$. Then since ϕ is a homeomorphism and

$$\mathcal{B} \supseteq \mathbb{R}^n \implies \phi^{-1}(\mathcal{B}) = \{\phi^{-1}(B) : B \in \mathcal{B}\} \supseteq \phi^{-1}(\mathbb{R}^n) = M$$

and so $\{\phi^{-1}(B) : B \in \mathcal{B}\}$ is a countable base for the topology on M. Moreover, ϕ^{-1} is continuous, and since continuous functions map compact sets into compact sets we have that $\phi^{-1}(\overline{B}) \subseteq M$ is compact, and

$$\overline{\phi^{-1}(B)} \subseteq \phi^{-1}(\overline{B})$$

and so $\overline{\phi^{-1}(B)}$ is compact as the closed subset of a compact set, and thus we see that $\{\phi^{-1}(B) : B \in \mathcal{B}\}$ is a countable base for M of precompact coordinate balls.

Next, letting M be an arbitrary topological n-manifold, by definition each point of M belongs to the domain of some coordinate chart, and we also have that M is second countable, and since every open cover of a second countable space has a countable subcover we have

$$\bigcup_{i \geq 0} \{(U_i, \phi_i)\} \supseteq M$$

Now, from above each U_i has a countable base $\phi_i^{-1}(\mathcal{B}_i)$ of precompact coordinate balls for U_i, where we then get

$$\bigcup_{i\geq 0} \phi_i^{-1}(\mathcal{B}_i)$$

is a countable base for the topology on M. Then for

$$V \subseteq U_i \subseteq M$$

we have that $\overline{V} \subseteq U_i$ is compact in U_i, and hence in M, and since M is Hausdorff and $\overline{V} \subseteq M$ is compact we have that $\overline{V}$ is closed in M. Thus, the closure of V in M is the same as its closer in U_i, and therefore, V is precompact in M. □

Proposition 2. Let M be a topological manifold. Then

(a) M is locally path-connected.

(b) M is connected if and only if it is path-connected.

(c) The components of M are the same as its path components.

(d) M has countably many components, each of which is an open subset of M and a connected topological manifold.

Proof.

(a) Since M has a countable basis of coordinate balls, and each ball is path-connected, M is locally path-connected.

(b) First since path-connected implies connected in any topological space, it suffices to show that for a manifold M, connected implies path-connected.

So let M be a connected manifold of dimension n, let $p \in M$ be arbitrary, and let $V \subseteq M$ be the set of points $q \in M$ such that p can be joined to q by a path in M.

Take any $q \in V$, since $V \subseteq M$ and since M is a manifold it is locally euclidean, and so $\exists\ (U_q, \phi)$ such that

$$\phi : U_q \to \phi(U_q) \subseteq \mathbb{R}^n$$

where $\phi(U_q) \subseteq \mathbb{R}^n$ is open, and since $\phi(q) \in \phi(U_q)$, $\exists\ B_\epsilon\big(\phi(q)\big) \subset \phi(U_q)$ for $\epsilon > 0$, and since $\mathbb{R}^n$ is path connected, $\forall\ \mathbf{x} \in B_\epsilon\big(\phi(q)\big)\ \exists$ a path

$$\gamma_{\mathbf{x},\phi(q)} : [0,1] \to B_\epsilon\big(\phi(q)\big) \text{ such that } \gamma_{\mathbf{x},\phi(q)}(0) = \mathbf{x},\ \gamma_{\mathbf{x},\phi(q)}(1) = \phi(q)$$

and since ϕ is a homeomorphism, it is continuous as is it's inverse, and as $B_\epsilon\big(\phi(q)\big) \subset \phi(U_q)$ is open, we also have $\phi^{-1}\big(B_\epsilon\big(\phi(q)\big)\big) \subset M$ is open. Then since $\phi^{-1} : \phi(U_q) \to M$ is continuous and $B_\epsilon\big(\phi(q)\big)$ is path connected, and since the image of a path-connected set under a continuous map is path-connected, we have $\phi^{-1}\big(B_\epsilon\big(\phi(q)\big)\big) \ni q$ is also path connected, that is $\forall\ \phi^{-1}(\mathbf{x}) \in \phi^{-1}\big(B_\epsilon\big(\phi(q)\big)\big)\ \exists$

$$\gamma_{p,\phi^{-1}(\mathbf{x})} : [0,2] \to M,\ \text{by } \gamma_{p,\phi^{-1}(\mathbf{x})}(t) = \begin{cases} \gamma_{p,q}(t), & 0 \leq t \leq 1 \\ \gamma_{q,\phi^{-1}(\mathbf{x})}(t), & 1 \leq t \leq 2 \end{cases}$$

which is a path connecting p to q and then q to $\phi^{-1}(\mathbf{x})$ and therefore $\phi^{-1}\big(B_\epsilon\big(\phi(q)\big)\big) \subseteq V$ and V is open.

Similarly for any $q \in V^c$, there exists a chart (W_q, ψ) with $q \in W_q$ which is locally euclidean, and for which $\psi^{-1}\big(B_\epsilon\big(\psi(q)\big)\big) \subset M$ is open. Now,

$$V \cap \psi^{-1}\big(B_\epsilon\big(\psi(q)\big)\big)$$

must be empty, if not, since $B_\epsilon\big(\psi(q)\big)$ is path connected, and ψ^{-1} is continuous, we know that the image of a path connected set under a continuous map is also path connected, and so if there was some $r \in V \cap \psi^{-1}\big(B_\epsilon\big(\psi(q)\big)\big)$ then we could find a path $\gamma_{p,r} \subset V$ joining p to r, and a path $\gamma_{r,q} \subset \psi^{-1}\big(B_\epsilon\big(\psi(q)\big)\big)$ joining r to q and concatenation would give us a path

$$\gamma_{p,r} \circ \gamma_{r,q}$$

joining p to q and hence $q \in V \Rightarrow\Leftarrow$. And so $\psi^{-1}\big(B_\epsilon\big(\psi(q)\big)\big) \subseteq V^c$.

That is, for each $q \in V^c$ we can find an open neighborhood of q entirely contained V^c, and so V^c is also open.

And thus we have $M = V \sqcup V^c$ where both V, V^c are open, and hence also both closed as the compliments of open sets. Yet $V \cap V^c = \varnothing$. Now, $p \in V \implies V \neq \varnothing$, and thus $V^c = \varnothing$ and therefore

$$M = V \sqcup V^c = V \sqcup \varnothing = V$$

and so M is path-connected through p.

Since $p \in M$ was arbitrary we conclude that for each $p \in M$, M is path-connected through p, and therefore M is path-connected.

(c) First, we show the result for any locally path-connected topological space, then since a topological manifold is a special type of topological space the result follows.

Let X be a locally path-connected topological space, let $x \in X$ be arbitrary, and let $x \in P_x$ where P_x is the path component in X containing x. Since X is locally path-connected it has a basis $\mathcal{B}$ of path-connected open sets. So for any $y \in P_x$, $\exists\ B_y \in \mathcal{B}$ such that $y \in B_y \subset P_x$, since B_y is path-connected. Then as P_x contains an open neighborhood around each of its points it must me open. Since $x \in X$ was arbitrary we conclude that all path components of X are open.

Let C be an arbitrary component of X, let $x \in C$, and let P_x be the path component of X such that $x \in P_x$. Since path-connected implies connected we have P_x is connected and thus $P_x \subseteq C$.

Next suppose that $P_x \neq C$ that is P_x is strictly contained in C, then as the path components partition X there must be other path components contained in C, as path-connected implies connected they must lie entirely in C, so define

$$V := \bigcup_{\substack{y \in C \\ y \neq x}} P_y$$

Then we have

$$C = P_x \sqcup V$$

Since X is locally path-connected, each path component of X is open in X from above. So that P_x is open in X and thus $P_x \cap C$ is open in the relative topology of C, and V which is the union of open sets is open in $X \implies V \cap C$ is open in C. Additionally both P_x, V were defined so that $P_x \cap C \neq \varnothing$, and $V \cap C \neq \varnothing$ and again by their definition $(P_x \cap C) \cap (V \cap C) = \varnothing$. And as noted above

$$(P_x \cap C) \cup (V \cap C) = (P_x \cup V) \cap C = C \cap C = C$$

So that $P_x, V \subseteq C$ form a separation of C which contradicts the fact that C is a connected component.

And therefore we must have that $P_x = C$.

(d) First, we show that for any locally path-connected topological space X, the components of X are open. Then, since this holds for arbitrary topological spaces, it also holds for topological manifolds.

Let X be locally path-connected, and let C be an arbitrary component of X. Since X is locally path-connected $\exists$ a basis $\mathcal{B}$ of path-connected subsets, each of which are open as they belong to a basis. So for any $x \in C$, $\exists\ B_x \in \mathcal{B}$ such that $x \in B_x \subset C$, where B_x is path-connected, and hence, connected. Then as each point of C has an open neighborhood contained in $C \implies C$ is open. Then as the component C of X was arbitrary we conclude that all of the components of X must be open.

Now, since the components of M are open, we have that the components of M form an open cover of M, and since M is second countable it admits a countable subcover. Yet, since the components which partition M are disjoint, this tells us that the open cover must have been countable to begin with, and so M has only countably many components. Then, since the components are open, they are connected topological manifolds in the relative topology of M.

□

Proposition 3 (Manifolds are Locally Compact). Every topological manifold is locally compact.

Proof. From Lemma 1 we know that every topological manifold has a countable basis of precompact coordinate balls. And since the closure of these balls is compact in M, and each $p \in M$ belongs to at least one of these balls, we have that M is locally compact. □

Lemma 4. Suppose $\mathcal{S}$ is a locally finite collection of subsets of a topological space M. Then

(a) The collection

$$\{\overline{S} : S \in \mathcal{S}\}$$

is also locally finite.

(b)

$$\overline{\bigcup_{S \in \mathcal{S}} S} = \bigcup_{S \in \mathcal{S}} \overline{S}$$

Proof.

(a) Let $p \in M$ be arbitrary, then by the local finiteness of $\mathcal{S}$ $\exists$ U_p open such that

$$U_p \cap S = \varnothing$$

for all but finitely many $S \in \mathcal{S}$, say $\{S_1, \dots, S_n\}$. Next note that for each $j \neq \{1, \dots, n\}$ since

$$U_p \cap S_j = \varnothing \implies S_j \subseteq U_p^c = M \setminus U_p$$

Then since U_p is open, U_p^c must be closed, and thus

$$\overline{S_j} \subseteq \overline{U_p^c} = U_p^c = M \setminus U_p$$

this implies that for each $j \neq \{1, \dots, n\}$ we have

$$\overline{S}_j \cap U_p = \varnothing$$

while for $i = \{1, \dots, n\}$ we have $S_i \cap U_p \neq \varnothing$ where $S_i \subseteq \overline{S}_i$ and so

$$\overline{S}_i \cap U_p \neq \varnothing$$

and therefore $\left\{\overline{S} : S \in \mathcal{S}\right\}$ is locally finite.

(b) First since for each $S \in \mathcal{S}$ we have

$$S \subseteq \bigcup_{S \in \mathcal{S}} S \implies \overline{S} \subseteq \overline{\bigcup_{S \in \mathcal{S}} S} \implies \bigcup_{S \in \mathcal{S}} \overline{S} \subseteq \overline{\bigcup_{S \in \mathcal{S}} S}$$

Next, let $p \in M$ be arbitrary, then by the local finiteness of $\mathcal{S}$ $\exists$ U_p open such that

$$U_p \cap S = \varnothing$$

for all but finitely many $S \in \mathcal{S}$, say $\{S_1, \dots, S_n\}$. Suppose further that

$$p \notin \bigcup_{S \in \mathcal{S}} \overline{S} \implies p \in \left(\bigcup_{S \in \mathcal{S}} \overline{S}\right)^c$$

then since $p \notin \bigcup_{S \in \mathcal{S}} \overline{S}$ this implies that $p \notin \{\overline{S}_1, \dots, \overline{S}_n\}$ and this implies that

$$p \in U_p \setminus \bigcup_{i=1}^{n} \overline{S}_i = U_p \cap \left(\bigcup_{i=1} \overline{S}_i\right)^c = U_p \cap \bigcap_{i=1}^{n} \overline{S}_i^c$$

which is open as the finite intersection of open sets. That is $U_p \cap \bigcap_{i=1}^n \overline{S}_i^c$ is an open neighborhood of p in $\left(\bigcup_{S \in \mathcal{S}} \overline{S}\right)^c$.

Since $p \in \left(\bigcup_{S \in \mathcal{S}} \overline{S}\right)^c$ was arbitrary, we conclude that each point in the compliment has an open neighborhood, and therefore $\left(\bigcup_{S \in \mathcal{S}} \overline{S}\right)^c$ is open and hence

$$\left(\left(\bigcup_{S \in \mathcal{S}} \overline{S}\right)^c\right)^c = \bigcup_{S \in \mathcal{S}} \overline{S}$$

must be closed and since for each $S \in \mathcal{S}$ we have $\overline{S} \supseteq S$, this gives

$$\bigcup_{S\in\mathcal{S}} S \subseteq \bigcup_{S\in\mathcal{S}} \overline{S} \implies \overline{\bigcup_{S\in\mathcal{S}} S} \subseteq \overline{\bigcup_{S\in\mathcal{S}} \overline{S}} = \bigcup_{S\in\mathcal{S}} \overline{S}$$

and thus we can conclude

$$\overline{\bigcup_{S\in\mathcal{S}} S} = \bigcup_{S\in\mathcal{S}} \overline{S}$$

□

Theorem 5 (Manifolds are Paracompact). Every topological manifold is paracompact. Furthermore, given a topological manifold M; an open cover $\mathcal{U}$ of M; and any basis $\mathcal{B}$ for the topology of M; there exists a countable, locally finite open refinement of $\mathcal{U}$ consisting of elements of $\mathcal{B}$.

Proof. Let M,$\mathcal{U}$ and $\mathcal{B}$ be given, since M is a manifold it is second countable, Hausdorff, and locally compact from Proposition 3, and so it admits an exhaustion by compact sets. So let $\{K_i\}_{i=1}^{\infty}$ we the exhaustion of M by compact sets and recall that since $\{K_i\}_{i=1}^{\infty}$ is an exhaustion of M this implies

$$M = \bigcup_{i=1}^{\infty} K_i$$

as well as

$$\cdots K_{i-1} \subseteq \text{Int}(K_i) \subseteq K_i \subseteq \text{Int}(K_{i+1}) \subseteq K_{i+1} \subseteq \text{Int}(K_{i+2}) \subseteq \cdots$$

so for each $i \geq 0$ define

$$V_i = K_{i+1} \setminus \text{Int}(K_i)$$
$$W_i = \text{Int}(K_{i+2}) \setminus K_{i-1}$$

where $K_i = \varnothing$ for $i < 1$. Then since $\text{Int}(K_i)$ is open, we have that V_i is a closed subset of compact K_{i+1} and is therefore compact. And since since K_{i-1} is a compact set in a Hausdorff space, it is closed, and so W_i is open, and furthermore, by construction

$$V_i \subseteq W_i$$

Now, for each $p \in V_i$, $\exists\ U_p \in \mathcal{U}$ containing p, as $\mathcal{U}$ is a cover. Then since $\mathcal{B}$ is a basis $\exists\ B_p \in \mathcal{B}$ such that

$$p \in B_p \subseteq U_p \cap W_i$$

Then the collection of all such B_p as p ranges over V_i is an open cover of V_i; that is

$$\bigcup_{p\in V_i} B_p \supseteq V_i$$

and by the compactness of V_i, there must be a finite subcover

$$\bigcup_{j=1}^{n} B_{p_j} = V_i$$

and the union of all such finite subcovers as i ranges over $\mathbb{N}$, is a countable open cover for M that refines $\mathcal{U}$; i.e.

$$\bigcup_{i\in\mathbb{N}}\bigcup_{j=1}^{n_i} B_{p_j} \supseteq M$$

Then, because

$$\bigcup_{j=1}^{n} B_{p_j} = V_i \subseteq W_i$$

where

$$W_i \cap W_k = \varnothing \quad \text{unless } k \in \{i-2, i-1, i, i+1, i+2\}$$

we have that for each $p \in M$, $\exists\ U_p$ open such that

$$U_p \cap B_p = \varnothing$$

for all but finitely many B_p, and thus, the cover

$$\mathcal{V} = \bigcup_{i\in\mathbb{N}}\bigcup_{j=1}^{n_i} B_{p_j}$$

is locally finite. □

Proposition 6. The Fundamental group of a topological manifold is countable.

Proof. Let M be a topological manifold, by Lemma 1 M has a countable basis $\mathcal{B}$ of precompact coordinate balls. Hence, for any $B, B' \in \mathcal{B}$

$$B \cap B'$$

has at most countably many components each of which are path-connected by Proposition 2 (c).

let $\mathcal{U}$ be a set containing a point from each component of $B \cap B'$ for each $B, B' \in \mathcal{B}$; i.e. if $p \in B \cap B'$ and C_p is the component containing p, then since there are only countably many components

$$\bigcup_{i=1}^{\infty} C_{p_i} = B \cap B'$$

and so

$$\mathcal{U} = \bigcup_{B,B'\in\mathcal{B}} \left(\bigcup_{i=1}^{\infty} C_{p_i}\right)$$

including when $B' = B$. For each $B \in \mathcal{B}$ with $x, x' \in \mathcal{U}$ such that $x, x' \in B$ define $h^B_{x,x'}$ to be a path connecting x to x' in B; i.e.

$$h^B_{x,x'} : [0,1] \to B, \text{ such that } h^B_{x,x'}(0) = x,\ h^B_{x,x'}(1) = x'$$

since the fundamental groups based at any two points of a connected component are isomorphic, and $\mathcal{U}$ contains at least one point from each component of M, we may consider our base point p as belonging to $\mathcal{U}$.

Next we define a special loop to be a loop based at p, or $\gamma \in \pi_1(M,p)$, such that

$$\gamma = \prod_{i=1}^{n} \left(h^{B_i}_{x_i,x_i'}\right)_i$$

that is γ is equal to a finite product of $h^{B}_{x,x'}$'s. Then since each special loop determines an element in $\pi_1(M,p)$, and there will be countably many special loops, it suffices to show that each element of $\pi_1(M,p)$ is represented by a special loop.

So let $f \in \pi_1(M,p)$ be an arbitrary loop based at p; that is

$$f : [0,1] \to M, \text{ such that } f(0) = p = f(1)$$

then $\{f^{-1}(B) : B \in \mathcal{B}\}$ is an open cover of compact $[0,1]$, and thus admits a finite subcover, and by the Lebesgue number Lemma we may partition $[0,1]$ into

$$0 = a_0 < a_1 < \cdots < a_k = 1$$

such that $[a_i, a_{i+1}] \subseteq f^{-1}(B)$ for some $B \in \mathcal{B}$. So for each i define

$$f_i := f|_{[a_i,a_{i+1}]} : [0,1] \to B_i$$

where he have reparameterized $[a_i, a_{i+1}]$ to be $[0,1]$ and where B_i denotes the element of $\mathcal{B}$ that f_i maps into.

Now for each i we have

$$f(a_i) \in C_p \subseteq B_i \cap B_{i+1}$$

that is $f(a_i)$ belongs to some component of the intersection of two basis elements, and there is some $x_i \in \mathcal{U}$ such that $x_i \in C_p$. So we define a map g_i from x_i to $f(a_i)$; that is

$$g_i : [0,1] \to C_p, \text{ such that } g_i(0) = x_i,\ g_i(1) = f(a_i)$$

where $x_0 = p = x_k$ and $g_0 = c_p = g_k$, with c_p denoting the constant path based at p. Recall that the reverse path is defined by $g_i^{-1}(t) = g_i(1-t)$. And by the path-connectedness of each C_p we have

$$g_i^{-1} \cdot g_i \sim c_{f(a_i)}$$

then

$$\begin{aligned}
f &\sim f_1 \cdot f_2 \cdot \cdots \cdot f_k \\
&\sim c_p \cdot f_1 \cdot c_{f(a_1)} \cdot f_2 \cdot \cdots \cdot c_{f(a_{k-1})} \cdot f_k \cdot c_p \\
&\sim g_0 \cdot f_1 \cdot (g_1^{-1} \cdot g_1) \cdot f_2 \cdot \cdots \cdot (g_{k-1}^{-1} \cdot g_{k-1}) \cdot f_k \cdot g_k^{-1} \\
&\sim (g_0 \cdot f_1 \cdot g_1^{-1}) \cdot (g_1 \cdot f_2 \cdot g_2^{-1}) \cdot \cdots \cdot (g_{k-1} \cdot f_k \cdot g_k^{-1}) \\
&\sim h_1 \cdot h_2 \cdot \cdots \cdot h_k
\end{aligned}$$

where each h_i is entirely contained in B_i, which is a coordinate ball, and thus, simply connected. So that $h_i \sim h^{B_i}_{x_{i-1},x_i}$, and therefore we have

$$\begin{aligned} f &\sim h_1 \cdot h_2 \cdot \cdots \cdot h_k \\ &\sim h^{B_1}_{p,x_1} \cdot h^{B_2}_{x_1,x_2} \cdot \cdots \cdot h^{B_k}_{x_{k-1},p} \end{aligned}$$

and thus $\pi_1(M, p)$ is countable. □

Proposition 7. Let M be a topological manifold. Then

(a) Every smooth atlas $\mathcal{A}$ for M is contained in a unique maximal smooth atlas, called the smooth structure determined by $\mathcal{A}$.

(b) Two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas.

Proof.

(a) Let $\mathcal{A}$ be a smooth atlas for M, and let $\mathcal{A}_M$ be the set of all charts that are smoothly compatible with every chart of $\mathcal{A}$.

Let $(U, \phi), (V, \psi) \in \mathcal{A}_M$ be arbitrary charts such that $U \cap V \neq \varnothing$, we wish to show that

$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

is smooth. So let $\phi(p) \in \phi(U \cap V)$ be arbitrary. Then since $\mathcal{A}$ is an atlas for M, its charts cover M, and hence there exists some chart $(W, \xi) \in \mathcal{A}$ such that $p \in W$. Then, since every chart in $\mathcal{A}_M$ is smoothly compatible with (W, ξ) we have both

$$\begin{aligned} \xi \circ \phi^{-1} &: \phi(U \cap W) \to \xi(U \cap W) \\ \psi \circ \xi^{-1} &: \xi(W \cap V) \to \psi(W \cap V) \end{aligned}$$

are smooth. And by construction, since $p \in U \cap V \cap W$ we have

$$\psi \circ \phi^{-1} = (\psi \circ \xi^{-1}) \circ (\xi \circ \phi^{-1}) : \phi(U \cap V) \to \psi(U \cap V)$$

is a smooth neighborhood of $\phi(p)$. Since $\phi(p)$ is arbitrary we conclude that $\psi \circ \phi^{-1}$ is a smooth neighborhood for each point in $\phi(U \cap V)$. And by the arbitrariness of $(U, \phi), (V, \psi) \in \mathcal{A}_M$ we conclude that $\mathcal{A}_M$ is a smooth atlas.

Next we note that any chart that is smoothly compatible with every chart of $\mathcal{A}_M$ is, in particular, smoothly compatible with every chart in $\mathcal{A}$, and so must already be contained in $\mathcal{A}_M$. And so a maximal atlas containing $\mathcal{A}$ exists.

Suppose that $\mathcal{B}$ is another maximal smooth atlas containing $\mathcal{A}$, then each chart in $\mathcal{B}$ is smoothly compatible with each chart in $\mathcal{A}$ and hence

$$\mathcal{B} \subseteq \mathcal{A}_M$$

then by the maximality of $\mathcal{B}$ we must have

$$\mathcal{B} = \mathcal{A}_M$$

and so $\mathcal{A}_M$ is the unique maximal atlas containing $\mathcal{A}$.

(b) First suppose $\mathcal{A}_1, \mathcal{A}_2$ are two atlas's for M determining the same smooth structure $\mathcal{A}_M$. By part (a) $\mathcal{A}_M$ is a unique smooth structure and hence all charts in $\mathcal{A}_M$ are smoothly compatible. Yet, $\mathcal{A}_1 \subseteq \mathcal{A}_M$ and $\mathcal{A}_2 \subseteq \mathcal{A}_M$, and so in particular, for any $(U, \phi) \in \mathcal{A}_1$ and any $(V, \phi) \in \mathcal{A}_2$ we must have that

$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

is smoothly compatible, and hence $\mathcal{A}_1 \cup \mathcal{A}_2 \subseteq \mathcal{A}_M$ and thus, is smoothly compatible.

Next suppose that $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_1 \cup \mathcal{A}_2$ are all smooth atlas's. Then considering the smooth atlas $\mathcal{A}_M$ determined by $\mathcal{A}_1 \cup \mathcal{A}_2$, we know from (a) that $\mathcal{A}_M$ is unique, and that in particular

$$\mathcal{A}_1 \subseteq \mathcal{A}_1 \cup \mathcal{A}_2$$
$$\mathcal{A}_2 \subseteq \mathcal{A}_1 \cup \mathcal{A}_2$$

and so $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{A}_M$. That is, both $\mathcal{A}_1$ and $\mathcal{A}_2$ determine the same maximal atlas $\mathcal{A}_M$, and therefore the same smooth structure.

$\square$

Proposition 8. Every smooth manifold has a countable basis of regular coordinate balls.

Lemma 9 (**Smooth Manifold Chart Lemma**)**.** Let M be a set, and suppose we are given a collection $\{U_\alpha\}$ of subsets of M together with maps

$$\phi_\alpha : U_\alpha \to \mathbb{R}^n$$

such that the following properties are satisfied:

(i) For each α, ϕ_α is a bijection between U_α and an open subset $\phi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$.

(ii) For each α and β, the sets $\phi_\alpha(U_\alpha \cap U_\beta), \phi_\beta(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^n$ are open.

(iii) Whenever $U_\alpha \cap U_\beta \neq \varnothing$, the map

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is smooth.

(iv) Countably many of the sets U_α cover M.

(v) Whenever $p, q \in M$ are distinct; either there exists some U_α such that

$$p, q \in U_\alpha$$

or there exist sets U_α, U_β such that

$$U_\alpha \cap U_\beta = \varnothing, \qquad p \in U_\alpha, \qquad q \in U_\beta$$

Then M has a unique smooth manifold structure such that each (U_α, ϕ_α) is a smooth chart.

Proof. We begin by giving M the initial topology; i.e.

$$\tau_M = \{\phi_\alpha^{-1}(V) : V \subseteq \mathbb{R}^n \text{ is open } \forall\ \alpha\}$$

so let $\phi_\alpha^{-1}(V), \phi_\beta^{-1}(W) \in \tau_M$ be arbitrary and let

$$p \in \phi_\alpha^{-1}(V) \cap \phi_\beta^{-1}(W) \implies \phi_\alpha^{-1}(V) \cap \phi_\beta^{-1}(W) \neq \varnothing$$

where (iii) gives the smoothness of $\phi_\beta \circ \phi_\alpha^{-1}$ which implies continuous and so

$$(\phi_\beta \circ \phi_\alpha^{-1})^{-1}(W) \subseteq \phi_\alpha(U_\alpha \cap U_\beta)$$

must be open, and (ii) implies

$$\phi_\alpha(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^n$$

is open, and since the finite intersection of open sets is open; that is, $V \cap (\phi_\beta \circ \phi_\alpha^{-1})^{-1}(W) \subseteq \mathbb{R}^n$ is open, we have

$$\begin{aligned} \phi_\alpha^{-1}(V) \cap \phi_\beta^{-1}(W) &= \phi_\alpha^{-1}(V) \cap (\phi_\alpha^{-1} \circ \phi_\alpha \circ \phi_\beta^{-1})(W) \\ &= \phi_\alpha^{-1}\big(V \cap (\phi_\beta \circ \phi_\alpha^{-1})^{-1}(W)\big) \in \tau_M \end{aligned}$$

and so $\phi_\alpha^{-1}(V) \cap \phi_\beta^{-1}(W) \in \tau_M$, and so $\{\phi_\alpha^{-1}(V) : V \subseteq \mathbb{R}^n \text{ is open } \forall\ \alpha\}$ is a base for τ_M.

Next, since

$$\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$$

is a homeomorphism, M is locally euclidean of dimension n. Where (v) tells us that M is Hausdorff. Then since each $\phi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$ which is second countable, and since ϕ_α is a homeomorphism, this tells us that each

$$U_\alpha \subseteq \phi_\alpha^{-1}(\mathbb{R}^n) = M$$

is second countable in M. Then (iv) tells us that finitely many U_α cover M, and so M itself is second countable.

Therefore M is a topological manifold. Where (iii) says that $\{(U_\alpha, \phi_\alpha)\} = \mathcal{A}$ is a smooth atlas. And since each chart is smoothly compatible with every chart in $\mathcal{A}$, this must be the unique smooth structure determined by $\{(U_\alpha, \phi_\alpha)\}$. $\square$

Theorem 10 (**Topological Invariance of Boundary**). If M is a topological manifold with boundary, then each point of M is either a boundary point or an interior point, but not both. Thus, ∂M and $\mathrm{Int}(M)$ form a separation of M; that is

$$M = \partial M \sqcup \mathrm{Int}(M)$$

Proposition 11. Let M be a topological n-manifold with boundary. Then

(a) $\mathrm{Int}(M)$ is an open subset of M and a topological n-manifold without boundary.

(b) ∂M is a closed subset of M and a topological $(n-1)$-manifold without boundary.

(c) M is a topological manifold iff $\partial M = \varnothing$.

(d) If $n = 0$, then $\partial M = \varnothing$ and M is a 0-manifold.

Proof.

(a) Let $p \in \text{Int}(M)$ be arbitrary, and let (U, ϕ) be an interior chart for M containing p. We wish to show that $U \subseteq \text{Int}(M)$ to demonstrate that arbitrary points have neighborhoods contained in $\text{Int}(M)$. To that end, let $q \in U$ be a point distinct from p. Then

$$\phi(q) \in \phi(U) \subseteq \mathbb{R}^n$$

and since $\phi(U)$ is open, there exists an open neighborhood $V_{\phi(q)}$ containing $\phi(q)$, and since ϕ is a homeomorphism $\phi^{-1}\big(\phi(U)\big) = U$, which implies

$$q \in \phi^{-1}\big(V_{\phi(q)}\big) \subseteq U$$

where the continuity of ϕ^{-1} implies that $\phi^{-1}\big(V_{\phi(q)}\big)$ is an open neighborhood of q in U, giving the interior chart

$$\Big(\phi^{-1}(V_{\phi(q)}), \phi|_{\phi^{-1}(V_{\phi(q)})}\Big)$$

thus $q \in \text{Int}(M)$, and since this can be done for each point of U we can conclude that $U \subseteq \text{Int}(M)$, and hence $\text{Int}(M) \subseteq M$ is open.

Next we note that $\text{Int}(M)$ inherits second countability and Hausdorffness as a subspace of M, and by definition, each point of $\text{Int}(M)$ has an interior chart homeomorphic to an open subset of $\mathbb{R}^n$ and so is locally euclidean of dimension n. Thus, we can conclude that $\text{Int}(M)$ is a topological n-manifold.

To see that $\text{Int}(M)$ has no boundary we simply note that since each point of $\text{Int}(M)$ is contained in an interior chart, where Topological Invariance of Boundary then tells us that no point of $\text{Int}(M)$ can be contained in a boundary chart, and thus $\text{Int}(M)$ is an n-manifold without boundary.

(b) First we observe that by the Topological Invariance of Boundary we have

$$M = \partial M \sqcup \text{Int}(M) \implies \partial M = M \setminus \text{Int}(M) = \text{Int}(M)^c$$

and by (a) we know that $\text{Int}(M)$ is open in M, and hence $\partial M \subseteq M$ must be closed.

Next, let $p \in \partial M$ and (U, ϕ) be a boundary chart for M containing p, then $\phi(p) \in \partial\mathbb{H}^n$. Let

$$V = \phi(U) \cap \partial\mathbb{H}^n$$

and note that $\phi(p) \in V$, and since $\phi(U) \subseteq \mathbb{H}^n$ is open by definition, then V is open in the relative topology of $\partial\mathbb{H}^n$ where

$$\partial\mathbb{H}^n = \{(x_1, \dots, x_n) : x_n = 0\} \cong \mathbb{R}^{n-1} \times \{0\} \cong \mathbb{R}^{n-1}$$

then since ϕ is a homeomorphism we have that ϕ^{-1} is continuous and $\phi^{-1}(\phi(U)) = U$, and so

$$\phi^{-1}(V) = U \cap \phi^{-1}(\partial\mathbb{H}^n) = U \cap \partial M$$

and since $U \subseteq M$ is open, we have $U \cap \partial M$ is open in the relative topology of ∂M, and thus $\phi^{-1}(V)$ is open in ∂M and contains p, giving the boundary chart

$$\left(\phi^{-1}(V), \phi|_{\phi^{-1}(V)}\right)$$

and since $p \in \partial M$ was arbitrary, we can find a chart for each point of ∂M, and thus ∂M is locally euclidean of dimension $(n-1)$. Then, since ∂M inherits second countability and Hausdorffness as a subspace of M we can conclude that ∂M is a topological $(n-1)$-manifold.

To see that ∂M has no boundary we simply note that since each point of ∂M is contained in a chart locally euclidean of dimension $(n-1)$, this implies that each chart for ∂M is contained in $\text{Int}(\partial M)$; i.e. each chart for ∂M is an interior chart. Where the Topological Invariance of Boundary then tells us that no point of ∂M can be contained in a boundary chart, and thus ∂M is an $(n-1)$-manifold without boundary.

(c) First, let M be a topological n-manifold, then by definition M is locally euclidean of dimension n, which implies that each point of M is contained in an interior chart, since from (b) ∂M is locally euclidean of dimension $(n-1)$ and so contains no points locally euclidean of dimension n. Where the Topological Invariance of Boundary then tells us that no point of M can be contained in a boundary chart, and so $\partial M = \varnothing$.

Next, suppose that $\partial M = \varnothing$. Then, by the Topological Invariance of Boundary

$$M = \partial M \sqcup \text{Int}(M) = \varnothing \sqcup \text{Int}(M) = \text{Int}(M)$$

where (a) then tells us that $M = \text{Int}(M)$ is a topological n-manifold.

(d) Let M be a 0-manifold with boundary. Then each point $p \in M$ is contained in a chart $\left(\{p\}, \phi\right)$ with

$$\phi : \{p\} \to \mathbb{R}^0 = \{0\}$$

and since

$$\text{Int}(\mathbb{H}^0) = \mathbb{R}^0 \implies \partial\mathbb{H}^0 = \varnothing$$

where the homeomorphism ϕ then tells us

$$\partial M = \phi^{-1}(\partial\mathbb{H}^0) = \phi^{-1}(\varnothing) = \varnothing$$

then, since M has empty boundary, (c) tells us that $M = \text{Int}(M)$ is a topological 0-manifold.

$\square$

Proposition 12. Let M be a topological manifold with boundary. Then

(a) M has a countable basis of precompact coordinate balls and half-balls.

(b) M is locally compact.

(c) M is paracompact.

(d) M is locally path-connected.

(e) M has countably many components, each of which is an open subset of M and a connected topological manifold with boundary.

(f) The fundamental group of M is countable.

Proposition 13. Suppose $M_1, \dots, M_k$ are smooth manifolds and N is a smooth manifold with boundary. Then

$$M_1 \times \cdots \times M_k \times N$$

is a smooth manifold with boundary, and

$$\partial(M_1 \times \cdots \times M_k \times N) = M_1 \times \cdots \times M_k \times \partial N$$

Proposition 14. Every smooth map between manifolds is continuous.

Proof. Let M, N be smooth manifolds with or without boundary and

$$F : M \to N$$

a smooth map. Given $p \in M$, the smoothness of F implies $\exists\ (U, \phi) \in \mathcal{A}_M$ containing p, and $(V, \psi) \in \mathcal{A}_N$ containing $F(p)$, such that $F(U) \subseteq V$ and

$$\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$$

is smooth, in the usual sense of calculus, and so is continuous.

Next, since ψ and ϕ are both homeomorphism we have

$$F|_U = \psi^{-1} \circ (\psi \circ F \circ \phi^{-1}) \circ \phi : U \to V$$

is continuous as the composition of continuous maps. That is F is continuous on a neighborhood of p.

Since $p \in M$ was arbitrary we conclude that F is continuous on a neighborhood of every point of M, and thus, F is continuous on M. $\square$

Proposition 15 (Smoothness is Local). Let M and N be smooth manifolds with or without boundary, and let $F : M \to N$ be a map. Then

(a) If every point $p \in M$ has a neighborhood U_p such that the restriction $F|_{U_p}$ is smooth, then F is smooth.

(b) Conversely, if F is smooth, then its restriction to every open subset is smooth.

Proof.

(a) Let $p \subseteq M$ be arbitrary and U_p a neighborhood containing p such that

$$F|_{U_p} : U_p \to F(U_p)$$

is smooth. Since U_p is open, $\exists\ B_p \subseteq U_p$ which is an open subset in U_p containing p. And by the smoothness of F on U_p we have the charts $(B_p, \phi) \in \mathcal{A}_{U_p} \subseteq \mathcal{A}_M$ containing p, and $(V, \psi) \in \mathcal{A}_N$ containing $F(p)$ where $F(B_p) \subseteq V$ and

$$\psi \circ F \circ \phi^{-1} : \phi(B_p) \to \psi(V)$$

is smooth. And we note that since $B_p \subseteq U_p$ is open, where $U_p \subseteq M$ is open, gives $B_p \subseteq M$ is open, and so is a chart for the smooth structure on M; that is $(B_p, \phi) \in \mathcal{A}_M$. Thus, we can conclude that F is smooth.

(b) let

$$F : M \to N$$

be smooth, and let $U \subseteq M$ be an arbitrary open, nonempty, subset. Then for any $p \in U$ there is some chart $(W, \phi) \in \mathcal{A}_M$ with $p \in W$, and by the smoothness of F there exists $(V, \psi) \in \mathcal{A}_N$ such that $F(p) \in V$ and $F(W) \subseteq V$ where

$$\psi \circ F \circ \phi^{-1} : \phi(W) \to \psi(V)$$

is smooth. And by the smoothness of both ϕ and ψ, where the smooth structures on M and N tell us that $\psi^{-1} \circ \psi$ and $\phi^{-1} \circ \phi$ are both smoothly compatible, we have

$$F|_W = \psi^{-1} \circ (\psi \circ F \circ \phi^{-1}) \circ \phi : W \to V$$

is smooth as the composition of smooth maps. Moreover we have $p \in U \cap W$, and $U \cap W$ is open in the relative topology of U, and so

$$F|_{U \cap W} = \psi^{-1} \circ (\psi \circ F \circ \phi^{-1}) \circ \phi|_{U \cap W} : U \cap W \to V$$

is a smooth neighborhood of p in U and so $\big(U \cap W, \phi|_{U \cap W}\big) \in \mathcal{A}_U \subseteq \mathcal{A}_M$ and where

$$U \cap W \subseteq W \implies F(U \cap W) \subseteq F(W) \subseteq V$$

and thus, $F|_{U \cap W}$ is smooth. By the arbitrariness of $p \in U$ we conclude that for each point in U we can construct a neighborhood where F is smooth and hence F is smooth on U, or $F|_U$ is smooth.

□

Corollary 16. Let M and N be smooth manifolds with or without boundary, and let $\{U_\alpha\}_{\alpha \in \Lambda}$ be an open cover of M. Suppose that for each $\alpha \in \Lambda$, we are given a smooth map

$$F_\alpha : U_\alpha \to N$$

such that the maps agree on overlaps:

$$F_\alpha|_{U_\alpha \cap U_\beta} = F_\beta|_{U_\alpha \cap U_\beta} \quad \forall\ \alpha, \beta$$

Then there exists a unique smooth map

$$F : M \to N$$

such that $F|_{U_\alpha} = F_\alpha,\ \forall\ \alpha \in \Lambda$.

Proposition 17. Let M and N be smooth manifolds with or without boundary, and $F : M \to N$ a smooth map. Then the coordinate representation of F with respect to every pair of smooth charts for M and N is smooth.

Proof. Let $(U, \phi) \in \mathcal{A}_M$ and $(V, \psi) \in \mathcal{A}_N$ be given. First, if

$$U \cap F^{-1}(V) = \varnothing$$

then by the definition of smooth compatibility the charts $(U \cap F^{-1}(V), \phi), (V, \psi)$ will be smoothly compatible.

So suppose $U \cap F^{-1}(V) \neq \varnothing$ and let $p \in U \cap F^{-1}(V)$. Then, by the smoothness of F, $\exists\, (W, \xi) \in \mathcal{A}_M$ containing p and $(Q, \eta) \in \mathcal{A}_N$ containing $F(p)$ with $F(W) \subseteq Q$ and

$$\eta \circ F \circ \xi^{-1} : \xi(W) \to \eta(Q)$$

smooth. Then, since all charts in $\mathcal{A}_M$ and $\mathcal{A}_N$ are smoothly compatible we have

$$\xi \circ \phi^{-1} : \phi(U \cap W) \to \xi(U \cap W)$$
$$\psi \circ \eta^{-1} : \eta(V \cap Q) \to \psi(V \cap Q)$$

are both smooth. So we then get

$$\psi \circ F \circ \phi^{-1} = (\psi \circ \eta^{-1}) \circ (\eta \circ F \circ \xi^{-1}) \circ (\xi \circ \phi^{-1}) : \phi(U \cap F^{-1}(V) \cap W) \to \psi(V \cap Q)$$

is smooth as the composition of smooth maps. Where, by the arbitrariness of $p \in U \cap F^{-1}(V)$, we conclude that $\psi \circ F \circ \phi^{-1}$ is smooth on $\phi(U \cap F^{-1}(V))$. And the arbitrariness of the charts (U, ϕ) and (V, ψ) allow us to conclude that the coordinate representation of F is smooth for every pair of charts in $\mathcal{A}_M$ and $\mathcal{A}_N$. ☐

Proposition 18. Let M, N and P be smooth manifolds with or without boundary. Then

(a) Every constant map $f_c : M \to N$ is smooth.

(b) The identity map of M is smooth.

(c) If $U \subseteq M$ is an open submanifold with or without boundary, then the inclusion map

$$\iota : U \hookrightarrow M$$

is smooth.

(d) If $F : M \to N$ and $G : M \to P$ are smooth, then so is

$$G \circ F : M \to P$$

Proof.

(a) Let $p \in M$ be given, and $(U, \phi) \in \mathcal{A}_M$ be an arbitrary chart containing p. Then for any chart $(V, \psi) \in \mathcal{A}_N$ containing $f_c(p) = c$ we have

$$U \cap f_c^{-1}(V) = U \cap M = U$$

where we then get the coordinate representation of f_c

$$\psi \circ f_c \circ \phi^{-1} : \phi(U) \to \psi(c)$$

is smooth as a constant map between euclidean spaces. And thus, we have that f_c must also be smooth.

(b) Let $p \in M$ be given, and $(U, \phi) \in \mathcal{A}_M$ be an arbitrary chart containing p, then (U, ϕ) also contains $Id(p) = p$, and
$$U \cap Id^{-1}(U) = U \cap U = U$$
where the coordinate representation of Id then gives
$$\phi \circ Id \circ \phi^{-1} : \phi(U) \to \phi(U)$$
is smooth as the identity map in a euclidean space. And thus, we have that Id must also be smooth.

(c) Let $p \in U$ be given, and $(V, \phi) \in \mathcal{A}_U$ be an arbitrary chart containing p, then (V, ϕ) is also a chart in $\mathcal{A}_M$ containing $\iota(p) = p$. Where the coordinate representation of ι then gives
$$\phi \circ \iota \circ \phi^{-1} : \phi(V) \to \phi(V)$$
is smooth as the inclusion map in a euclidean space. And thus, we have that ι must also be smooth.

(d) Let $p \in M$ be given. By the smoothness of G, $\exists$ $(V, \psi) \in \mathcal{A}_N$ containing $F(p)$, and $(W, \xi) \in \mathcal{A}_P$ containing $G(F(p))$, such that $G(V) \subseteq W$ and
$$\xi \circ G \circ \psi^{-1} : \psi(V) \to \xi(W)$$
is smooth.

Next, since F is smooth, and hence continuous, $F^{-1}(V) \subseteq M$ is an open neighborhood containing p. So, $\exists$ $(U, \phi) \in \mathcal{A}_M$ such that $p \in U \subseteq F^{-1}(V)$, and so $U \cap F^{-1}(V) = U$. Then by Proposition 17, and the smoothness of F we have
$$\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$$
is smooth.

Then as
$$U \subseteq F^{-1}(V) \implies F(U) \subseteq V$$
we get
$$G(F(U)) \subseteq G(V) \subseteq W$$
where the coordinate representation of $G \circ F$ gives
$$\xi \circ (G \circ F) \circ \phi^{-1} = (\xi \circ G \circ \psi^{-1}) \circ (\psi \circ F \circ \phi^{-1}) : \phi(U) \to \xi(W)$$
is smooth as the compositions of smooth maps between euclidean spaces. And thus, we have that $G \circ F$ must also be smooth.

$\square$

Proposition 19. Suppose $M_1, \ldots, M_k$ and N are smooth manifolds with or without boundary, such that at most one of $M_1, \ldots, M_k$ has nonempty boundary. For each i, let

$$\pi_i : M_1 \times \cdots \times M_k \to M_i$$

denote the projection onto the M_i factor. A map

$$F : N \to M_1 \times \cdots \times M_k$$

is smooth iff each of the component maps

$$F_i := \pi_i \circ F : N \to M_i$$

is smooth.

Proof. First suppose that F is smooth, by Proposition 18 the composition of smooth maps are smooth, so to show $\pi_i \circ F$ is smooth it suffices to show that π_i is smooth. So given $(p_1, \ldots, p_k) \in M_1 \times \cdots \times M_k$, since $M_1 \times \cdots \times M_k$ is a smooth manifold there exists

$$\big(U_1 \times \cdots \times U_k, \phi_1 \times \cdots \times \phi_k\big) \in \mathcal{A}_{M_1 \times \cdots \times M_k}$$

containing $(p_1, \ldots, p_k)$. Then by definition of the projection, we will have that chart $(U_i, \phi_i) \in \mathcal{A}_{M_i}$ containing $\pi_i(p_1, \ldots, p_k) = p_i$, where

$$\pi_i\big(U_1 \times \cdots \times U_k\big) = U_i \implies \pi_i\big(U_1 \times \cdots \times U_k\big) \subseteq U_i$$

and the coordinate representation of π_i, since $(U_1 \times \cdots \times U_k) \cap \pi_i^{-1}(U_i) = U_i$, gives

$$\phi_i \circ \pi_i \circ (\phi_1 \times \cdots \times \phi_k)^{-1} : \phi_i(U_i) \to \phi_i(U_i)$$

which is smooth as the identity map between euclidean spaces. And thus, we have that π_i must also be smooth. Since i and $(p_1, \ldots, p_k)$ were arbitrary we conclude that π_i and hence $\pi_i \circ F$ is smooth for each i.

Next, suppose that $\pi_i \circ F$ is smooth for each i. Given $q \in N$, since N is a smooth manifold $\exists\ (V, \psi) \in \mathcal{A}_N$ containing q, and by the smoothness of $\pi_i \circ F$ there is a chart $(U_i, \phi_i) \in \mathcal{A}_{M_i}$ containing $(\pi_i \circ F)(q)$ where $\big(\pi_i \circ F\big)(V) \subseteq U_i$ and

$$\phi_i \circ (\pi_i \circ F) \circ \psi^{-1} : \psi(V) \to \phi_i(U_i)$$

is smooth for each i. Then since $\big(U_1 \times \cdots \times U_k, \phi_1 \times \cdots \times \phi_k\big) \in \mathcal{A}_{M_1 \times \cdots \times M_k}$ is a chart containing U_i, and

$$\big(\pi_i \circ F\big)(V) \subseteq U_i \implies F(V) \subseteq \pi_i^{-1}(U_i) = U_i$$

we have $F(V) \subseteq U_1 \times \cdots \times U_k$, where the coordinate representation of F gives

$$(\phi_1 \times \cdots \times \phi_k) \circ F \circ \psi^{-1} : \psi\big(V \cap F^{-1}(U_1 \times \cdots \times U_k)\big) \to \phi_1(U_1) \times \cdots \times \phi_k(U_k)$$

now since we know the composition with the projection is smooth between euclidean spaces; that is

$$\pi_i \circ \Big((\phi_1 \times \cdots \times \phi_k) \circ F \circ \psi^{-1}\Big) : \psi\big(V \cap F^{-1}(U_1 \times \cdots \times U_k)\big) \to \phi_i(U_i)$$

is smooth for each i. And since the composition of smooth maps is smooth we also have that $\pi_i^{-1} \circ \phi_i$ is smooth for each i, where we note that we know π_i^{-1} is smooth since it is just the inclusion into $M_1 \times \cdots \times M_k$. And so

$$(\phi_1 \times \cdots \times \phi_k) \circ F \circ \psi^{-1} = \pi_i^{-1} \circ \phi_i$$

must be smooth as well, as a map between euclidean spaces. And thus, we have that F must also be smooth.

<u>Or</u>

Then noting that since π_i is smooth for each i, and by definition $\phi_1 \times \cdots \times \phi_k$ is smooth, where Proposition 18 tells us the composition of smooth maps is smooth we have $(\phi_1 \times \cdots \times \phi_k) \circ \pi_i^{-1}$ is smooth as well as $(\pi_i \circ F) \circ \psi^{-1}$. Where the coordinate representation of F then gives

$$\begin{aligned}&(\phi_1 \times \cdots \times \phi_k) \circ F \circ \psi^{-1} \\ &\quad = \Big((\phi_1 \times \cdots \times \phi_k) \circ \pi_i^{-1}\Big) \circ \Big(\pi_i \circ F \circ \psi^{-1}\Big) : \psi\big(V \cap F^{-1}(U_1 \times \cdots \times U_k)\big) \to \phi_1(U_1) \times \cdots \times \phi_k(U_k)\end{aligned}$$

is smooth as the composition of smooth maps between euclidean spaces. And thus, we have that F must also be smooth. ☐

Proposition 20 (**Properties of Diffeomorphisms**)**.**

(a) Every composition of diffeomorphisms is a diffeomorphism.

(b) Every finite product of diffeomorphisms between smooth manifolds is a diffeomorphism.

(c) Every diffeomorphism is a homeomorphism and an open map.

(d) The restriction of a diffeomorphism to an open submanifold with or without boundary is a diffeomorphism onto its image.

(e) "Diffeomorphic" is an equivalence relation on the class of all smooth manifolds with or without boundary.

Proof.

(a) Let

$$\begin{aligned}F &: M \to N \\ G &: N \to P\end{aligned}$$

be diffeomorphisms, then by definition they are bijective and smooth. As are F^{-1} and G^{-1}, where Proposition 18 tells us the composition of smooth maps are smooth and so we have both

$$\begin{aligned}&G \circ F : M \to P \\ &F^{-1} \circ G^{-1} = (G \circ F)^{-1} : P \to M\end{aligned}$$

are smooth, and bijective. And thus, the composition of diffeomorphisms is a diffeomorphism.

(b) Let

$$\begin{aligned} F_1 &: M_1 \to N_1 \\ F_2 &: M_2 \to N_2 \\ &\vdots \\ F_k &: M_k \to N_k \end{aligned}$$

be diffeomorphisms. By Proposition 19

$$F := F_1 \times \cdots \times F_k : M_1 \times \cdots \times M_k \to N_1 \times \cdots \times N_k$$

is smooth iff $\pi_i \circ F$ is smooth for each i. Let π_i^N denote projection from $N_1 \times \cdots \times N_k$, and similarly for π_i^M. Given $(p_1, \ldots, p_k) \in M_1 \times \cdots \times M_k$ we have

$$\big(\pi_i^N \circ F\big)(p_1, \ldots, p_k) = \pi_i^N\big(F_1(p_1), \ldots, F_k(p_k)\big) = F_i(p_i)$$

yet we also have

$$F_i(p_i) = F_i\big(\pi_i^M(p_1, \ldots, p_k)\big) = \big(F_i \circ \pi_i^M\big)(p_1, \ldots, p_k)$$

where the projection maps are all smooth, and by Proposition 18 the composition of smooth maps are smooth and so $F_i \circ \pi_i^M$ is smooth for each i, where we then have

$$\pi_i^N \circ F = F_i \circ \pi_i^M \quad \forall\, i$$

and thus $\pi_i^N \circ F$ is smooth for each i, and thus F must also be smooth. Bijectivity comes from each component function F_i being bijective.

For

$$F^{-1} := F_1^{-1} \times \cdots \times F_k^{-1} : N_1 \times \cdots \times N_k \to M_1 \times \cdots \times M_k$$

a similar argument holds except we now have

$$\pi_i^M \circ F^{-1} = F_i^{-1} \circ \pi_i^N \quad \forall\, i$$

giving the smoothness for F^{-1}. And so we conclude that

$$F = F_1 \times \cdots \times F_k$$

is a diffeomorphism.

(c) Since smooth implies continuous, any diffeomorphism is trivially a homeomorphism. And since homeomorphisms are open maps, we have diffeomorphisms must be as well.

(d) Let $F : M \to N$ be a diffeomorphism, and let $U \subseteq M$ be an open submanifold. From Proposition 18 we know that the inclusion map is smooth and since it is the restriction of the identity map it is bijective onto its image. Then

$$F|_U = F \circ \iota : U \to F(U)$$

is smooth as the composition of smooth maps, and is also bijective.

Then by (c), F is an open mapping and so $F(U) \subseteq N$ is an open submanifold where a similar argument gives

$$(F|_U)^{-1} = \iota \circ F^{-1}|_{F(U)} : F(U) \to U$$

is the composition of smooth bijective maps. And so the restriction of F to an open submanifold with or without boundary is a diffeomorphism onto its image.

(e) Reflexive: Since $Id : M \to M$ is a smooth bijective map which is its own inverse M is diffeomorphic to M.

Symmetric: If M is diffeomorphic to N there exists a diffeomorphism

$$F : M \to N$$

then $F^{-1} : N \to M$ is also a smooth bijection, and so N is diffeomorphic to M.

Transitive: If M is diffeomorphic to N, and N is diffeomorphic to P, then there exists diffeomorphisms

$$F : M \to N$$
$$G : N \to P$$

Yet, (a) then tells us that $G \circ F : M \to P$ is also a diffeomorphism and so M is diffeomorphic to P.

□

Theorem 21 (**Diffeomorphism Invariance of Dimension**). A nonempty smooth manifold of dimension m cannot be diffeomorphic to an n-dimensional smooth manifold unless $m = n$.

Proof. Suppose M is a nonempty smooth m-dimensional manifold, and N is a nonempty smooth n-dimensional manifold where

$$F : M \to N$$

is a diffeomorphism. Given $p \in M$ choose smooth charts $(U, \phi) \in \mathcal{A}_M$ containing p, and $(V, \psi) \in \mathcal{A}_N$ containing $F(p)$, then the coordinate representation of F gives

$$\psi \circ F \circ \phi^{-1} : \phi\big(U \cap F^{-1}(V)\big) \to \psi(V)$$

is a diffeomorphism between open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, and thus, we must have $m = n$. □

Theorem 22 (**Diffeomorphism Invariance of Boundary**). Suppose M and N are smooth manifolds with boundary and

$$F : M \to N$$

is a diffeomorphism. Then

$$F(\partial M) = \partial N$$

and F restricts to a diffeomorphism from $\mathrm{Int}(M)$ to $\mathrm{Int}(N)$.

Proof. Let $F(p) \in F(\partial M) \subseteq N$ be arbitrary, then $p \in \partial M$, and by the smoothness of M there exists a boundary chart $(U, \phi) \in \mathcal{A}_M$ containing p, and by the smoothness of F there is a chart $(V, \psi) \in \mathcal{A}_N$ containing $F(p)$. Then by the Topological Invariance of Boundary, homeomorphisms send boundary points of the manifold to $\partial \mathbb{H}^n$. Then the coordinate representation of F gives

$$\psi \circ F \circ \phi^{-1} : \partial \mathbb{H}^n \to \partial \mathbb{H}^n$$

and since the composition of diffeomorphisms is a diffeomorphism and hence a homeomorphism. We have that (V, ψ) must be a boundary chart mapping $\phi(p) \in \partial \mathbb{H}^n$ to $\psi(F(p)) \in \partial \mathbb{H}^n$, that is, we must have that $F(p) \in \partial N$. And so

$$F(\partial M) \subseteq \partial N$$

Then applying similar reasoning to the diffeomorphism F^{-1}, with $q \in F^{-1}(\partial N) \subseteq M$, gives

$$F^{-1}(\partial N) \subseteq \partial M \implies \partial N \subseteq F(\partial M)$$

and thus, $F(\partial M) = \partial N$.

Since $\mathrm{Int}(M) \subseteq M$ is an open submanifold by Proposition 11 (a), and by Proposition 20 (d) F restricts to a diffeomorphism onto its image i.e.

$$F|_{\mathrm{Int}(M)} : \mathrm{Int}(M) \to F(\mathrm{Int}(M))$$

is a diffeomorphism, yet from above we know that $F(\partial M) = \partial N$, and by the Topological Invariance of Boundary each point is either a boundary point, or an interior point, and so we conclude that $F(\mathrm{Int}(M)) = \mathrm{Int}(N)$. And thus

$$F|_{\mathrm{Int}(M)} : \mathrm{Int}(M) \to \mathrm{Int}(N)$$

is a diffeomorphism. □

Lemma 23 (**Existence of Cutoff Functions**)**.** Given any real numbers $r_1, r_2 \in \mathbb{R}$ such that $r_1 < r_2$, there exists a smooth function

$$h : \mathbb{R} \to \mathbb{R}, \text{ such that } h(t) = \begin{cases} 1, & t \leq r_1 \\ 0 < h(t) < 1, & r_1 < t < r_2 \\ 0, & t \geq r_2 \end{cases}$$

Proof. Consider the function

$$f : \mathbb{R} \to \mathbb{R}, \text{ by } f(t) = \begin{cases} e^{-1/t}, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

Which is smooth, then define h by

$$h(t) = \frac{f(r_2 - t)}{f(r_2 - t) + f(t - r_1)}$$

note that the denominator is always defined since either $r_2 - t > 0$ or $t - r_1 > 0$. Where the smoothness is inherited from f, and for

$t \geq r_2$

$$f(r_2 - t) = 0 \implies h(t) = 0$$

$t \leq r_1$

$$f(t - r_1) = 0 \implies h(t) = \frac{f(r_2 - t)}{f(r_2 - t)} = 1$$

$r_1 < t < r_2$

$$f(r_2 - t) > 0 \text{ and } f(t - r_1) > 0 \implies 0 < h(t) < 1$$

$\square$

Lemma 24 (**Existence of Smooth Bump Functions**)**.** Given any positive real numbers $r_1, r_2 \in \mathbb{R}^+$ such that $r_1 < r_2$, there is a smooth function

$$H : \mathbb{R}^n \to \mathbb{R}, \text{ such that } H(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in \overline{B_{r_1}(\mathbf{0})} \\ 0 < H(\mathbf{x}) < 1, & \mathbf{x} \in B_{r_2}(\mathbf{0}) \setminus \overline{B_{r_1}(\mathbf{0})} \\ 0, & \mathbf{x} \in \mathbb{R}^n \setminus B_{r_2}(\mathbf{0}) \end{cases}$$

Proof. Utilizing the smooth cutoff function

$$h : \mathbb{R} \to \mathbb{R}, \text{ such that } h(t) = \begin{cases} 1, & t \leq r_1 \\ 0 < h(t) < 1, & r_1 < t < r_2 \\ 0, & t \geq r_2 \end{cases}$$

we can define H by

$$\begin{aligned} H := h \circ ||\cdot|| : \mathbb{R}^n &\to \mathbb{R} \\ \mathbf{x} &\mapsto h(||\mathbf{x}||) \end{aligned}$$

Then H is smooth on $\mathbb{R}^n \setminus \{\mathbf{0}\}$ as the composition of smooth functions.

Now for any $\mathbf{x} \in \overline{B_{r_1}(\mathbf{0})}$

$$\mathbf{x} \mapsto t \leq r_1 \implies h(||\mathbf{x}||) = 1$$

and so we must also have for $\mathbf{0} \mapsto 0 < r_1$ that $h(||\mathbf{0}||) = 1$.

And so H is smooth on all of $\mathbb{R}^n$. $\square$

Theorem 25 (**Existence of Partitions of Unity**)**.** Suppose M is a smooth manifold with or without boundary, and $\mathcal{U} := \{U_\alpha\}_{\alpha \in \Lambda}$ is any indexed open cover of M. Then there exists a smooth partition of unity subordinate to $\mathcal{U}$.

Proof. Suppose first that M is a smooth manifold without boundary. Each $U_\alpha \subseteq M$ is itself a smooth manifold, and thus, by Proposition 8 has a countable base of regular coordinate balls $\mathcal{B}_\alpha$. Taking

$$\mathcal{B} = \bigcup_{\alpha \in \Lambda} \mathcal{B}_\alpha$$

we have that $\mathcal{B}$ is a base for M. From Theorem 5 manifolds are paracompact, and since we have both a cover $\mathcal{U}$, and a base $\mathcal{B}$, there exists a countable locally finite refinement $\{B_i\}$ of $\mathcal{U}$ consisting of elements from $\mathcal{B}$. From Lemma 4, the closure of sets in a locally finite collection is locally finite and so $\{\overline{B}_i\}$ is also locally finite.

Now, since each B_i is a regular coordinate ball in some U_α, $\exists$ $(B_i', \phi_i) \in \mathcal{A}_{U_\alpha}$ with $B_i' \supseteq \overline{B}_i$ such that for $r_i, r_i' \in \mathbb{R}^+$ with $r_i < r_i'$ we have:

$$\begin{aligned} \phi_i(B_i) &= B_{r_i}(\mathbf{0}) \\ \phi_i(\overline{B}_i) &= \overline{B_{r_i}(\mathbf{0})} \\ \phi_i(B_i') &= B_{r_i'}(\mathbf{0}) \end{aligned}$$

For each i define the smooth bump function

$$H_i : \mathbb{R}^n \to \mathbb{R}, \text{ such that } H_i(\mathbf{x}) = \begin{cases} 0 < H_i(\mathbf{x}) < 1, & \mathbf{x} \in B_{r_i}(\mathbf{0}) \\ 0, & \mathbf{x} \in \mathbb{R}^n \setminus B_{r_i}(\mathbf{0}) \end{cases}$$

then defining, for each i, the function

$$f_i : M \to \mathbb{R}, \text{ by } f_i(p) = \begin{cases} H_i \circ \phi_i, & p \in B_i' \\ 0, & p \in M \setminus \overline{B}_i \end{cases}$$

where for $p \in B_i' \setminus \overline{B}_i$, continuity of ϕ tells us that

$$\phi(p) \in B_{r_i'}(\mathbf{0}) \setminus \overline{B_{r_i}(\mathbf{0})} \subset \mathbb{R}^n \setminus B_{r_i}(\mathbf{0})$$

and so we have $f_i(p) = 0 = H_i(\phi_i(p))$. Thus, f_i and the smooth function $H_i \circ \phi_i$ agree on the overlap, and so f_i is well defined, smooth, and $\text{supp}(f) \subseteq \overline{B}_i$.

Next, define

$$f : M \to \mathbb{R}, \text{ by } f(p) = \sum_i f_i(p)$$

where, by the local finiteness of $\{\overline{B}_i\}$, for each $p \in M$, $\exists$ U_p such that $U_p \cap \overline{B}_i = \varnothing$ for all but finitely many i and so $f_i(p) = 0$ for all but finitely many i, and so f is smooth.

Then as $f_i \geq 0$ on M, and $f_i > 0$ on B_i, where each $p \in M$ belongs to some B_i we have $f > 0$ on M, and so the functions

$$g_i : M \to \mathbb{R}, \text{ by } g_i(p) = \frac{f_i(p)}{f(p)}$$

are all well defined and smooth. Moreover

$$0 \leq g_i(p) \leq 1 \qquad \text{and} \qquad \sum_i g_i(p) = 1$$

Finally, reindexing so that our functions are indexed by the same Λ as our open cover $\{B_i\}$. Since $\{B_i'\}$ is a refinement of $\mathcal{U}$ we may choose $\sigma(i) \in \Lambda$ for each i such that $B_i' \subseteq U_{\sigma(i)}$. Then for each $\alpha \in \Lambda$ define

$$\psi_\alpha : M \to \mathbb{R}, \text{ by } \psi_\alpha(p) = \sum_{\{i:\sigma(i)=\alpha\}} g_i(p)$$

if $\sigma(i) \neq \alpha \; \forall \; i$ then $\psi_\alpha := 0$. Then by the local finiteness of $\{B_i\}$ Lemma 4 tells us that

$$\operatorname{supp}(\psi_\alpha) = \overline{\bigcup_{\{i:\sigma(i)=\alpha\}} B_i} = \bigcup_{\{i:\sigma(i)=\alpha\}} \overline{B}_i \subseteq B_i' \subseteq U_\alpha$$

Then each ψ_α is a smooth function such that $0 \leq \psi_\alpha(p) \leq 1$, where the family of supports $\{\operatorname{supp}(\psi_\alpha)\}_{\alpha\in\Lambda}$ is locally finite and

$$\sum_\alpha \psi_\alpha(p) = \sum_i g_i(p) = 1 \quad \forall \; p \in M$$

and so $\{\psi_\alpha\}_{\alpha\in\Lambda}$ is a smooth partition of unity subordinate to $\mathcal{U}$. $\square$

Proposition 26 (**Existence of Smooth Bump Functions on Manifolds**)**.** Let M be a smooth manifold with or without boundary. For any closed subset $A \subseteq M$ and any open subset U such that $U \supseteq A$, there exists a smooth bump function ψ for A supported in U.

Proof. Let

$$\begin{aligned} &U_0 = U \\ &U_1 = M \setminus A = A^c \end{aligned}$$

then $\mathcal{U} = \{U_0, U_1\}$ is an indexed open cover of M, and so there exists a smooth partition of unity $\{\psi_0, \psi_1\}$ subordinate to $\mathcal{U}$. Where

$$\operatorname{supp}(\psi_1) \subseteq U_1 \implies \psi_1(p) = 0 \quad \forall \; p \in A$$

and since $\{\psi_0, \psi_1\}$ is a smooth partition of unity this implies

$$1 = \sum_{i=0}^{1} \psi_i(p) = \psi_0(p) + \psi_1(p) = \psi_0(p) + 0 = \psi_0(p) \quad \forall \; p \in A$$

and so

$$\begin{aligned} &0 \leq \psi_0(M) \leq 1 \\ &\psi_0(p) = 1, \qquad \forall \; p \in A \\ &\operatorname{supp}(\psi_0) \subseteq U_0 = U \end{aligned}$$

and so ψ_0 is therefore a bump function for A supported in U. $\square$

Lemma 27 (**Extension Lemma for Smooth Functions**)**.** Suppose M is a smooth manifold with or without boundary, $A \subseteq M$ is a closed subset, and

$$f : A \to \mathbb{R}^k$$

is a smooth function. For any open subset $U \subseteq M$ such that $U \supseteq A$, there exists a smooth function

$$\widetilde{f} : M \to \mathbb{R}^k$$

such that $\widetilde{f}|_A = f$ and $\operatorname{supp}(\widetilde{f}) \subseteq U$.

Proof. For each $p \in A$ choose $U_p \subseteq M$ containing p and a smooth function

$$\widetilde{f}_p : U_p \to \mathbb{R}^k$$

such that $\widetilde{f}_p|_{U_p \cap A} = f|_{U_p \cap A}$. Then replacing U_p by

$$U_p = U_p \cap U \implies U_p \subseteq U$$

then by the closedness of A we have $M \setminus A = A^c$ must be open, and thus

$$\mathcal{U} := \{U_p : p \in A\} \cup \{A^c\}$$

is an indexed open cover for M, here considering the p's in A to be our index, and so there exists a smooth partition of unity $\{\psi_p : p \in A\} \cup \{\psi_0\}$ subordinate to $\mathcal{U}$, such that

$$\operatorname{supp}(\psi_p) \subseteq U_p, \quad \text{and} \quad \operatorname{supp}(\psi_0) \subseteq A^c$$

then for each $p \in A$ we have

$$\psi_p \widetilde{f}_p : U_p \to \mathbb{R}^k$$

is smooth and has a smooth extention to M by the gluing lemma for smooth maps since

$$\psi_p \widetilde{f}_p\big(U_p \setminus \operatorname{supp}(\psi_p)\big) = 0 = \psi_p \widetilde{f}_p\big(M \setminus \operatorname{supp}(\psi_p)\big)$$

i.e. the functions agree on their overlap. And so we define

$$\widetilde{f} : M \to \mathbb{R}^k, \text{ by } \widetilde{f}(q) = \sum_{p \in A} \psi_p(q) \widetilde{f}_p(q)$$

now since $\{\psi_p : p \in A\} \cup \{\psi_0\}$ is a partition of unity, we have $\{\operatorname{supp}(\psi_p)\}$ is locally finite, and so only finitely many of the terms in the sum are non-zero in the neighborhood of any point of M, and therefore $\widetilde{f}$ is smooth.

To see that $\widetilde{f}$ is an extension of f, note that for each $q \in A$ we have $\psi_0(q) = 0$, and $\widetilde{f}_p(q) = f(q)$ for each $q \in U_p \cap A$. Then since $\{\psi_p : p \in A\} \cup \{\psi_0\}$ is a partition of unity, we have

$$\psi_0(q) + \sum_{p \in A} \psi_p(q) = 1 \qquad \forall\ q \in M$$

putting it all together we get for each $q \in A$

$$\begin{aligned}
\widetilde{f}(q) &= \sum_{p \in A} \psi_p(q) \widetilde{f}_p(q) \\
&= 0 + \sum_{p \in A} \psi_p(q) \widetilde{f}_p(q) \\
&= \left(\psi_0(q) + \sum_{p \in A} \psi_p(q)\right) f(q) \\
&= f(q)
\end{aligned}$$

and therefore $\widetilde{f}|_A = f$.

Finally, since $\{\text{supp}(\psi_p)\}$ is a locally finite collection of subsets Lemma 4 (b) tells us that

$$\text{supp}(\widetilde{f}) \subseteq \overline{\bigcup_{p \in A} \text{supp}(\psi_p)} = \bigcup_{p \in A} \overline{\text{supp}(\psi_p)} = \bigcup_{p \in A} \text{supp}(\psi_p) \subseteq \bigcup_{p \in A} U_p \subseteq U$$

□

Proposition 28 (**Existence of Smooth Exhaustion Functions**)**.** Every smooth manifold with or without boundary admits a smooth positive exhaustion function.

Proof. Let M be a smooth manifold with or without boundary, and let $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$ be an arbitrary countable open cover of M by precompact subsets, as it is indexed, let $\{\psi_i\}$ be the smooth partition of unity subordinate to $\mathcal{U}$. Next, define

$$f : M \to \mathbb{R}, \text{ by } f(p) = \sum_{i=1}^{\infty} i\psi_i(p)$$

then f is smooth since $\{\text{supp}(\psi_i)\}$ is a locally finite collection of subsets, and so only finitely many of the terms in the sum are non-zero in the neighborhood of any point of M.

and since $\{\psi_i\}$ is a partition of unity $\sum_{i=1}^{\infty} \psi_i(p) = 1$ for all $p \in M$ and so

$$f(p) = \sum_{i=1}^{\infty} i\psi_i(p) \geq \sum_{i=1}^{\infty} \psi_i(p) = 1$$

hence f is positive.

To see that f is an exhaustion function, note that since the U_i's are precompact each $\overline{U}_i$ is compact in M. Then, given $c \in \mathbb{R}$ we can choose $N \in \mathbb{N}$ such that $N > c$. Then if

$$p \notin \bigcup_{i=1}^{N} \overline{U}_i \implies \psi_i(p) = 0 \text{ for } 1 \leq i \leq N$$

since $\text{supp}(\psi_i) \subseteq U_i$. And so

$$f(p) = \sum_{i=N+1}^{\infty} i\psi_i(p) \geq \sum_{i=N+1}^{\infty} N\psi_i(p) = N \sum_{i=1}^{\infty} \psi_i(p) = N > c$$

and so $p \notin f^{-1}\big((-\infty, c]\big)$. That is,

$$\text{if } p \notin \bigcup_{i=1}^{N} \overline{U}_i \text{ then } p \notin f^{-1}\big((-\infty, c]\big)$$

taking the contrapositive gives

$$\text{if } p \in f^{-1}\big((-\infty, c]\big) \text{ then } p \in \bigcup_{i=1}^{N} \overline{U}_i$$

or $f^{-1}\big((-\infty, c]\big) \subseteq \bigcup_{i=1}^{N} \overline{U}_i$, which is compact as a finite union of compact sets, were the continuity of f tells us that $f^{-1}\big((-\infty, c]\big) \subseteq M$ is closed, and therefore must be compact as a closed subset of a compact set. ☐

Theorem 29 (**Level Sets of Smooth Functions**)**.** Let M be a smooth manifold. If $A \subseteq M$ is closed, there is a smooth non-negative function

$$f : M \to \mathbb{R}$$

such that $f^{-1}(0) = A$.

Lemma 30 (**Properties of Derivations**)**.** Suppose $\mathbf{a} \in \mathbb{R}^n, w \in T_{\mathbf{a}}\mathbb{R}^n$ and $f, g \in C^\infty(\mathbb{R}^n)$, then

(a) If $f = cnst$ then
$$w(f)|_{\mathbf{a}} = 0$$

(b) If $f(\mathbf{a}) = g(\mathbf{a}) = 0$ then
$$w(fg)|_{\mathbf{a}} = 0$$

Proof.

(a) First note that for $f_1(\mathbf{x}) = 1$ i.e. the constant function equal to 1, and that for any other constant function $f(\mathbf{x}) = c$ we have

$$\begin{aligned} w(f)|_{\mathbf{a}} &= w(cf_1)|_{\mathbf{a}} \\ &= cw(f_1)|_{\mathbf{a}} && \text{linearity} \end{aligned}$$

so it suffices to show the property for the the case $f = f_1$. Then since

$$f_1(\mathbf{x})^2 = 1^2 = 1 = f_1(\mathbf{x})$$

the product rule gives

$$\begin{aligned} w(f_1)|_{\mathbf{a}} &= w(f_1 \cdot f_1)|_{\mathbf{a}} \\ &= f_1(\mathbf{a})w(f_1)|_{\mathbf{a}} + f_1(\mathbf{a})w(f_1)|_{\mathbf{a}} \\ &= 1 \cdot w(f_1)|_{\mathbf{a}} + 1 \cdot w(f_1)|_{\mathbf{a}} \\ &= 2w(f_1)|_{\mathbf{a}} \end{aligned}$$

which is only possible if $w(f_1)|_{\mathbf{a}} = 0$.

(b) From the product rule we have

$$w(fg)|_{\mathbf{a}} = f(\mathbf{a})w(g)|_{\mathbf{a}} + g(\mathbf{a})w(f)|_{\mathbf{a}} = 0 + 0 = 0$$

☐

Proposition 31. let $\mathbf{a} \in \mathbb{R}^n$, then

(a) For each geometric tangent vector $\mathbf{v_a} \in \mathbb{R}^n_{\mathbf{a}}$ the map

$$D_{\mathbf{v}}\Big|_{\mathbf{a}} : C^\infty(\mathbb{R}^n) \to \mathbb{R}, \text{ by } D_{\mathbf{v}}(f)|_{\mathbf{a}} = \frac{d}{dt}\bigg|_{t=0} f(\mathbf{a} + t\mathbf{v})$$

is a derivation at $\mathbf{a}$.

(b) The mapping

$$\begin{aligned} \eta : \mathbb{R}^n_{\mathbf{a}} &\to T_{\mathbf{a}}\mathbb{R}^n \\ \mathbf{v_a} &\mapsto D_{\mathbf{v}}\Big|_{\mathbf{a}} \end{aligned}$$

is an isomorphism.

Proof.

(a) Since $D_{\mathbf{v}}\Big|_{\mathbf{a}}$ is defined as the directional derivative at $\mathbf{a}$ in the direction of $\mathbf{v}$ which is defined to be linear over $\mathbb{R}$ and satisfy the product rule we get that $D_{\mathbf{v}}\Big|_{\mathbf{a}}$ is a derivation.

(b) First note the mapping $\mathbf{v_a} \mapsto D_{\mathbf{v}}\Big|_{\mathbf{a}}$ is linear since $D_{\mathbf{v}}\Big|_{\mathbf{a}}$ is.

Next, suppose that $\mathbf{v_a} \in \ker(\eta)$; that is

$$D_{\mathbf{v}}(f)|_{\mathbf{a}} = 0 \quad \forall\ f \in C^\infty(\mathbb{R}^n)$$

so letting $f = x^j \in C^\infty(\mathbb{R}^n)$, and letting

$$\mathbf{v_a} = \sum_{i=1}^n v^i e_i|_{\mathbf{a}}$$

in terms of the standard basis, we then have

$$\begin{aligned} 0 &= D_{\mathbf{v}}(x^j)|_{\mathbf{a}} \\ &= \sum_{i=1}^n v^i \frac{\partial x^j}{\partial x^i}\bigg|_{\mathbf{a}} \\ &= \sum_{i=1}^n v^i \delta^j{}_i \\ &= v^j \end{aligned}$$

and since this can be done for each component of $\mathbf{v_a}$ for $j \in \{1, \ldots, , n\}$, we conclude that $\mathbf{v_a} = \mathbf{0}$, and therefore η is injective.

Now, let $w_{\mathbf{a}} \in T_{\mathbf{a}}\mathbb{R}^n$ be arbitrary, and let the output of $w_{\mathbf{a}}$ acting on a "basis element" $x^i \in C^\infty(\mathbb{R}^n)$ be given by

$$v^i = w(x^i)$$

and let $f \in C^\infty(\mathbb{R}^n)$. Then by Taylor's Theorem

$$f(\mathbf{x}) = f(\mathbf{a}) + \sum_{i=1}^{n}(x_i - a_i)\frac{\partial f}{\partial x^i}(\mathbf{a}) + \frac{1}{2}\sum_{i,j=1}^{n}(x_i - a_i)(x_j - a_j)\int_0^1 (1-t)\frac{\partial f}{\partial x^i \partial x^j}\big(\mathbf{a} + t(\mathbf{x} - \mathbf{a})\big)dt$$

Then as

$$(x_i - a_i)|_{\mathbf{a}} = 0 = (x_j - a_j)|_{\mathbf{a}}$$

where Lemma 30 then says

$$w\Big((x_i - a_i)\cdot(x_j - a_j)\Big)\Big|_{\mathbf{a}} = 0$$

then since both $f(\mathbf{a})$ and the components a_i are constants Lemma 30 also tells us that

$$w\big(f(\mathbf{a})\big) = 0, \quad \text{and} \quad w(a_i) = 0\ \forall\ i$$

and so we have

$$\begin{aligned}
w(f)|_{\mathbf{a}} &= w\big(f(\mathbf{a})\big) + \sum_{i=1}^{n} w\left((x_i - a_i)\frac{\partial f}{\partial x^i}(\mathbf{a})\right) + 0 \\
&= 0 + \sum_{i=1}^{n}\frac{\partial f}{\partial x^i}(\mathbf{a})\big(w(x_i) - w(a_i)\big) && \text{linearity} \\
&= \sum_{i=1}^{n}\frac{\partial f}{\partial x^i}(\mathbf{a})\big(v^i - 0\big) && x_i = x^i \in C^\infty(\mathbb{R}^n) \\
&= \sum_{i=1}^{n}\frac{\partial f}{\partial x^i}(\mathbf{a})\cdot v^i \\
&= \sum_{i=1}^{n} v^i\frac{\partial f}{\partial x^i}(\mathbf{a}) \\
&= D_{\mathbf{v}}(f)|_{\mathbf{a}}
\end{aligned}$$

that is

$$\eta(\mathbf{v}_{\mathbf{a}}) = D_{\mathbf{v}}\big|_{\mathbf{a}} = w$$

and so is surjective.

So we have a linear map between vector spaces that is both injective and surjective. Thus, η is an isomorphism between $\mathbb{R}^n_{\mathbf{a}}$ and $T_{\mathbf{a}}\mathbb{R}^n$.

$\square$

Corollary 32. For any $\mathbf{a} \in \mathbb{R}^n$, the n derivations

$$\frac{\partial}{\partial x^1}\bigg|_{\mathbf{a}}, \dots, \frac{\partial}{\partial x^n}\bigg|_{\mathbf{a}} : C^\infty(\mathbb{R}^n)|_{\mathbf{a}} \to \mathbb{R},\ \text{by}\ \frac{\partial}{\partial x^i}\bigg|_{\mathbf{a}}(f) = \frac{\partial f}{\partial x^i}(\mathbf{a})$$

form a basis for $T_{\mathbf{a}}\mathbb{R}^n$, and therefore $\dim\big(T_{\mathbf{a}}\mathbb{R}^n\big) = n$.

Proof. From Proposition 31, $T_{\mathbf{a}}\mathbb{R}^n \cong \mathbb{R}^n_{\mathbf{a}}$ and $\mathbb{R}^n_{\mathbf{a}} \cong \mathbb{R}^n$ and so

$$n = \dim(\mathbb{R}^n) = \dim(\mathbb{R}^n_{\mathbf{a}}) = \dim(T_{\mathbf{a}}\mathbb{R}^n)$$

and any isomorphism must map basis vectors to basis vectors

$$e_i \mapsto e_i - a_i \mapsto D_{e_i}|_{\mathbf{a}} = e_i \cdot \left.\frac{\partial}{\partial x^i}\right|_{\mathbf{a}} = \left.\frac{\partial}{\partial x^i}\right|_{\mathbf{a}}$$

□

Lemma 33 (**Properties of Tangent Vectors on Manifolds**). Suppose M is a smooth manifold with or without boundary, $p \in M$; $v_p \in T_pM$; and $f, g \in C^\infty(M)$. Then

(a) If $f = cnst$, then $v(f)|_p = 0$

(b) If $f(p) = g(p) = 0$, then $v(fg)|_p = 0$

Proof.

(a) It is sufficient to consider the constant function $f_1(p) = 1$ then we have

$$v(f_1)|_p = v(f_1 f_1)|_p = f_1(p)v(f_1)|_p + f_1(p)v(f_1)|_p = 2v(f_1)|_p$$

and so we must have $v(f_1)|_p = 0$. The reason $f_1(p)$ is sufficient is because for any other constant function $g(p) = c$ we may simply define $g(p) = (cf_1(p))$ where by linearity we get

$$v(g)|_p = v(cf_1)|_p = cv(f_1)_p = 0$$

(b) From the product rule we have

$$v(fg)|_p = f(p)v(g)|_p + g(p)v(f)|_p = 0 + 0 = 0$$

□

Proposition 34 (**Properties of Differentials**). Let M, N, and P be smooth manifolds with or without boundary, let $F : M \to N$ and $G : N \to P$ be smooth maps, and let $p \in M$. Then

(a) $dF|_p : T_pM \to T_{F(p)}N$ is linear

(b) $d(G \circ F)|_p = dG|_{F(p)} \circ dF|_p : T_pM \to T_{G\circ F(p)}P$

(c) $d(Id_M)|_p = Id_{T_pM} : T_pM \to T_pM$

(d) If F is a diffeomorphism, then

$$dF|_p : T_pM \to T_{F(p)}N$$

is an isomorphism, and

$$(dF|_p)^{-1} = d(F^{-1})|_{F(p)}$$

Proof.

(a) Let $v_1, v_2 \in T_pM$ and $f \in C^\infty(N)$ then by the linearity of T_pM we have $(v_1 + v_2)(F) = v_1(F) + v_2(F)$ so that we get

$$\begin{aligned} dF|_p(v_1+v_2)(f) &= [v_1+v_2](f\circ F)|_p \\ &= v_1(f\circ F)|_p + v_2(f\circ F)|_p \\ &= dF|_p(v_1)(f) + dF|_p(v_2)(f) \end{aligned}$$

and similarly since $(cv)(F) = cv(F)$ we get

$$dF|_p(cv)(f) = cv(f\circ F)|_p = cdF|_p(v)(f)$$

(b) Let $v \in T_pM$ then $d(G\circ F)|_p(v) : C^\infty(P) \to \mathbb{R}$ is linear because v is, so let $f, g \in C^\infty(P)|_{G(F(p))}$ Then

$$\begin{aligned} d(G\circ F)|_p(v)(fg) &= v(fg\circ G\circ F)|_p \\ &= v\big((f\circ G\circ F)\cdot(g\circ G\circ F)\big)|_p \\ &= f\circ G\circ F(p)v(g\circ G\circ F)|_p + g\circ G\circ F(p)v(f\circ G\circ F)|_p \\ &= f\circ G(F(p))v(g\circ G\circ F)|_p + g\circ G(F(p))v(f\circ G\circ F)|_p \\ &= f\Big(G\big(F(p)\big)\Big)v(g\circ G\circ F)|_p + g\Big(G\big(F(p)\big)\Big)v(f\circ G\circ F)|_p \\ &= f\Big(G\big(F(p)\big)\Big)d(G\circ F)|_p(v)(g) + g\Big(G\big(F(p)\big)\Big)d(G\circ F)|_p(v)(f) \end{aligned}$$

and so $d(G\circ F)|_p(v)$ is a derivation at $G(F(p))$, and therefore $d(G\circ F)|_p(v) \in T_{G(F(p))}P$. Since $v \in T_pM$ was arbitrary we conclude

$$d(G\circ F)|_p : T_pM \to T_{G\circ F(p)}P$$

Also,

$$d(G\circ F)|_p(v)(f) = v(f\circ G\circ F)|_p = dF|_p(v)(f\circ G)|_{F(p)} = dG|_{F(p)}(dF|_p(v))(f)$$

And hence $d(G\circ F)|_p = dG|_{F(p)} \circ dF|_p$

(c) Let $v \in T_pM$ and $f \in C^\infty(M)$ and since $Id_M : M \to M$ we have

$$d(Id_M)|_p(v)(f) = v(f\circ Id_M)|_p = v(f)|_p$$

and $d(Id_M)|_p : T_pM \to T_pM$. Also

$$d(Id_M)|_p(v) = v \qquad \forall\ v \in T_pM$$

and so $d(Id_M)|_p = Id_{T_pM}$

(d) First $v|_p \mapsto v|_{F(p)}$ is linear by (a). Next since F is a diffeomorphism it is bijective and it's inverse exists. And from (b) and (c) we have

$$d(F^{-1})|_{F(p)} \circ dF|_p = d(F^{-1}\circ F)|_p = d(Id_M)|_p = Id_{T_pM}$$

and

$$dF|_p \circ d(F^{-1})|_{F(p)} = d(F \circ F^{-1})|_{F(p)} = d(Id_N)|_{F(p)} = Id_{T_{F(p)}N}$$

and so $dF|_p : T_pM \to T_{F(p)}N$ is a bijective linear map between vector spaces, and hence, is an isomorphism, with

$$(dF|_p)^{-1} = d(F^{-1})|_{F(p)}$$

□

Proposition 35. Let M be a smooth manifold with or without boundary, $p \in M$; and $v_p \in T_pM$. If $f, g \in C^\infty(M)$ agree on some neighborhood U_p of p; i.e.

$$f|_{U_p} = g|_{U_p}$$

then

$$v(f)|_p = v(g)|_p$$

Proof. Define

$$h = f - g \implies h|_{U_p} = 0$$

and is smooth where $\text{supp}(h) \subseteq M \setminus U_p$ is closed, and $V = M \setminus \{p\}$ is an open subset such that $V \supseteq \text{supp}(h)$. And so we may define a bump function ψ for $\text{supp}(h)$ supported in V, such that

$$\begin{aligned}
&0 \leq \psi(M) \leq 1 \\
&\psi(q) = 1 \qquad \forall\ q \in \text{supp}(h) \\
&\text{supp}(\psi) \subseteq V
\end{aligned}$$

so if $h(q) \neq 0$, then

$$\big(\psi h\big)(q) = 1 \cdot h(q) = h(q)$$

and if $h(q) = 0$ then

$$\big(\psi h\big)(q) = 0$$

and therefore $\psi h = h$ identically. Then since

$$h(p) = 0 = \psi(p)$$

Lemma 33 says

$$v(\psi h)|_p = 0$$

where we then get

$$\begin{aligned}
0 &= v(\psi h)|_p \\
&= v(h)|_p && \psi h = h \text{ identically} \\
&= v(f - g)|_p \\
&= v(f)|_p - v(g)|_p && \text{linearity}
\end{aligned}$$

which then implies

$$v(f)|_p = v(g)|_p$$

□

Proposition 36 (**The Tangent Space to an Open Submanifold**)**.** Let M be a smooth manifold with or without boundary, let $U \subseteq M$ be an open subset, and let

$$\iota : U \hookrightarrow M$$

be the inclusion map. For every $p \in U$, the differential

$$d\iota|_p : T_pU \to T_pM$$

is an isomorphism.

Proof. Suppose $v_p \in T_pU$ is such that $v_p \in \ker(d\iota|_p)$; that is

$$d\iota|_p(v_p)(f) = v(f \circ \iota)|_p = 0 \quad \forall\ f \in C^\infty(M)$$

Next, let B_p be a neighborhood of p such that $\overline{B}_p \subseteq U$, and let $f \in C^\infty(U)$ be arbitrary. Then since $\overline{B}_p \subseteq M$ is closed and contained in open U, the Extension Lemma for Smooth Functions then says $\exists\ \widetilde{f} \in C^\infty(M)$ such that $\widetilde{f}|_{\overline{B}_p} = f|_{\overline{B}_p}$ and $\operatorname{supp}(\widetilde{f}) \subseteq U$.

Next we note that $f, \widetilde{f}|_U \in C^\infty(U)$, and agree on the open neighborhood B_p; that is

$$\widetilde{f}|_U\big|_{B_p} = f|_{B_p}$$

where Proposition 35 then says that

$$v(f)|_p = v\big(\widetilde{f}|_U\big)\big|_p$$

yet

$$\widetilde{f}|_U = \widetilde{f} \circ \iota$$

and so we have

$$v(f)|_p = v\big(\widetilde{f}|_U\big)\big|_p = v(\widetilde{f} \circ \iota)|_p = d\iota|_p(v_p)(\widetilde{f}) = 0 \qquad \widetilde{f} \in C^\infty(M)$$

and since this holds for any $f \in C^\infty(U)$ we conclude that $v_p = 0$ and so $d\iota|_p$ is injective.

Now, let $w_p \in T_pM$ be given, and choose $v_p \in T_pU$ such that for $f \in C^\infty(U)$ and $\widetilde{f} \in C^\infty(M)$ with $\overline{B}_p \subseteq U$ where

$$\widetilde{f}|_{\overline{B}_p} = f|_{\overline{B}_p}$$

we have

$$v(f)|_p = w(\widetilde{f})|_p$$

by Proposition 35 $v(f)$ is independent of the choice $\widetilde{f}$, and so is well defined, and is a derivation since for any $f, g \in C^\infty(U)$ we have

$$\begin{aligned} v(fg)|_p &= w(\widetilde{f} \cdot \widetilde{g})|_p \\ &= \widetilde{f}(p)w(\widetilde{g})|_p + \widetilde{g}(p)w(\widetilde{f})|_p \\ &= f(p)v(g)|_p + g(p)v(f)|_p \qquad\qquad \widetilde{f}|_{\overline{B}_p} = f|_{\overline{B}_p},\ \text{and}\ \widetilde{g}|_{\overline{B}_p} = g|_{\overline{B}_p} \end{aligned}$$

So let $g \in C^\infty(M)$ be given, then

$$d\iota|_p(v_p)(g) = v(g \circ \iota)|_p = w(\widetilde{g \circ \iota})|_p = w(g)|_p \qquad \text{since } (g \circ \iota)|_{B_p} = (\widetilde{g \circ \iota})|_{B_p} = g|_{B_p}$$

and by the arbitrariness of g we see that

$$d\iota|_p(v_p) = w_p$$

and so $d\iota|_p$ is surjective.

Hence, we have a linear map between vector spaces that is both injective and surjective. Thus, $d\iota|_p$ is an isomorphism between T_pU and T_pM. $\square$

Proposition 37 (Dimension of the Tangent Space). If M is an n-dimensional smooth manifold, then for each $p \in M$; the tangent space T_pM is an n-dimensional vector space.

Proof. Let $p \in M$ be given, and let $(U, \phi) \in \mathcal{A}_M$ be a smooth chart containing p. Then since

$$\phi : U \to \phi(U)$$

is a diffeomorphism Proposition 34 then tells us that

$$d\phi|_p : T_pU \to T_{\phi(p)}\phi(U)$$

is an isomorphism. Since derivations are defined locally Proposition 36 tells us that both

$$T_pM \cong T_pU$$
$$T_{\phi(p)}\phi(U) \cong T_{\phi(p)}\mathbb{R}^n$$

and from Proposition 31 the set of derivations at a point in euclidean space is isomorphic to the euclidean space with shifted origin; i.e. the space of geometric tangent vectors is isomorphic to the space of tangent vectors in euclidean space, we have

$$T_{\phi(p)}\mathbb{R}^n \cong \mathbb{R}^n_{\phi(p)}$$

and canonically $\mathbb{R}^n_{\phi(p)} \cong \mathbb{R}^n$. Putting it all together we get

$$T_pM \cong T_pU \cong T_{\phi(p)}\phi(U) \cong T_{\phi(p)}\mathbb{R}^n \cong \mathbb{R}^n_{\phi(p)} \cong \mathbb{R}^n$$

and since isomorphisms are structure preserving we must have

$$\dim(\mathbb{R}^n) = n = \dim(T_pM)$$

$\square$

Lemma 38. Let

$$\iota : \mathbb{H}^n \hookrightarrow \mathbb{R}^n$$

denote the inclusion map. For any $\mathbf{a} \in \partial\mathbb{H}^n$, the differential

$$d\iota|_{\mathbf{a}} : T_{\mathbf{a}}\mathbb{H}^n \to T_{\mathbf{a}}\mathbb{R}^n$$

is an isomorphism.

Proof. Let $\mathbf{a} \in \partial\mathbb{H}^n$ be given, and suppose that $v_{\mathbf{a}} \in T_{\mathbf{a}}\mathbb{H}^n$ is such that $v_{\mathbf{a}} \in \ker(d\iota|_{\mathbf{a}})$; that is

$$d\iota|_{\mathbf{a}}(v_{\mathbf{a}})(f) = v(f \circ \iota)|_{\mathbf{a}} = 0 \quad \forall\ f \in C^\infty(\mathbb{R}^n)$$

and let

$$f : \mathbb{H}^n \to \mathbb{R}^n$$

be smooth, and since $\mathbb{H}^n \subseteq \mathbb{R}^n$ is closed, and we may consider $\mathbb{R}^n$ as an open subset containing $\mathbb{H}^n$, where the Extension Lemma for Smooth Functions then tells us that $\exists\ \widetilde{f} \in C^\infty(\mathbb{R}^n)$ such that $\widetilde{f}|_{\mathbb{H}^n} = f$ and $\text{supp}(\widetilde{f}) \subseteq \mathbb{R}^n$. Yet we also have

$$f = \widetilde{f}|_{\mathbb{H}^n} = \widetilde{f} \circ \iota$$

and since $\widetilde{f}$ and f agree on $\mathbb{H}^n$, which is some neighborhood of $\mathbf{a}$, and since agreement on some neighborhood of $\mathbf{a}$ implies that the tangent vectors are equal by Proposition 35 we get

$$v(f)|_{\mathbf{a}} = v(\widetilde{f} \circ \iota)|_{\mathbf{a}} = d\iota|_{\mathbf{a}}(v_{\mathbf{a}})(\widetilde{f}) = 0$$

and since this holds for any $f \in C^\infty(\mathbb{H}^n)$ we conclude that $v_{\mathbf{a}} = 0$, and so $d\iota|_{\mathbf{a}}$ is injective.

Next, let $w_{\mathbf{a}} \in T_{\mathbf{a}}\mathbb{R}^n$ be given, and choose $v_{\mathbf{a}} \in T_{\mathbf{a}}\mathbb{H}^n$ such that for $f \in C^\infty(\mathbb{H}^n)$ with extension $\widetilde{f} \in C^\infty(\mathbb{R}^n)$, we have

$$v(f)|_{\mathbf{a}} = w(\widetilde{f})|_{\mathbf{a}}$$

where Proposition 35 assures us that $v(f)$ is independent of the choice $\widetilde{f}$, and so is well defined, and is a derivation since for an any $f, g \in C^\infty(\mathbb{H}^n)$ we have

$$\begin{aligned} v(fg)|_{\mathbf{a}} &= w(\widetilde{f}\widetilde{g})|_{\mathbf{a}} \\ &= \widetilde{f}(\mathbf{a})w(\widetilde{g})|_{\mathbf{a}} + \widetilde{g}(\mathbf{a})w(\widetilde{f})|_{\mathbf{a}} \\ &= f(\mathbf{a})v(g)|_{\mathbf{a}} + g(\mathbf{a})v(f)|_{\mathbf{a}} && \widetilde{f}|_{\mathbb{H}^n} = f, \text{ and } \widetilde{g}|_{\mathbb{H}^n} = g \end{aligned}$$

so for any $h \in C^\infty(\mathbb{R}^n)$ we see that

$$d\iota|_{\mathbf{a}}(v_{\mathbf{a}})(h) = v(h \circ \iota)|_{\mathbf{a}} = w(\widetilde{h \circ \iota})|_{\mathbf{a}} = w(h)|_{\mathbf{a}} \qquad \text{since } (h \circ \iota)|_{\mathbb{H}^n} = (\widetilde{h \circ \iota})|_{\mathbb{H}^n} = (h)|_{\mathbb{H}^n}$$

and by the arbitrariness of h we have that

$$d\iota|_{\mathbf{a}}(v_{\mathbf{a}}) = w_{\mathbf{a}}$$

and so $d\iota|_{\mathbf{a}}$ is surjective.

Hence we have a linear map between vector spaces that is both injective and surjective. Thus, $d\iota|_{\mathbf{a}}$ is an isomorphism between $T_{\mathbf{a}}\mathbb{H}^n$ and $T_{\mathbf{a}}\mathbb{R}^n$. □

Proposition 39 (**Dimension of Tangent Spaces on a Manifold with Boundary**)**.** Suppose M is an n-dimensional smooth manifold with boundary. For each $p \in M$; T_pM is an n-dimensional vector space.

Proof. Let $p \in M$ be arbitrary.

Case 1: If $p \in \text{Int}(M)$, then since $\text{Int}(M) \subseteq M$ is an open submanifold, and derivations are defined locally, Proposition 36 tells us

$$T_p\text{Int}(M) \cong T_pM$$

and since $\text{Int}(M)$ is an n-dimensional smooth manifold without boundary, each of its tangent spaces has dimension n by Proposition 37. And so we have

$$n = \dim(T_p\text{Int}(M)) = \dim(T_pM)$$

Case 2: If $p \in \partial M$ let $(U, \phi) \in \mathcal{A}_M$ be a boundary chart containing p. Since

$$\phi : U \to \phi(U) \subseteq \mathbb{H}^n$$

is a diffeomorphism, Proposition 34 tells us that

$$d\phi|_p : T_pU \to T_{\phi(p)}\phi(U)$$

is an isomorphism. Since derivations are defined locally, Proposition 36 then tells us that both

$$T_pM \cong T_pU$$
$$T_{\phi(p)}\phi(U) \cong T_{\phi(p)}\mathbb{H}^n$$

Then since derivations of the boundary are also defined locally, Lemma 38 then gives

$$T_{\phi(p)}\mathbb{H}^n \cong T_{\phi(p)}\mathbb{R}^n$$

and from Proposition 31 the set of derivations at a point in euclidean space is isomorphic to the euclidean space with shifted origin; i.e. the space of geometric tangent vectors is isomorphic to the space of tangent vectors in euclidean space, we have

$$T_{\phi(p)}\mathbb{R}^n \cong \mathbb{R}^n_{\phi(p)}$$

and canonically $\mathbb{R}^n_{\phi(p)} \cong \mathbb{R}^n$. Putting it all together we get

$$T_pM \cong T_pU \cong T_{\phi(p)}\phi(U) \cong T_{\phi(p)}\mathbb{H}^n \cong T_{\phi(p)}\mathbb{R}^n \cong \mathbb{R}^n_{\phi(p)} \cong \mathbb{R}^n$$

and since isomorphisms are structure preserving we must have

$$\dim(\mathbb{R}^n) = n = \dim(T_pM)$$

□

Proposition 40 (The Tangent Space to a Vector Space). Suppose V is a finite dimensional vector space with its standard smooth manifold structure. For each point $a \in V$, the mapping

$$\begin{aligned} \eta : V|_a &\to T_aV \\ v_a &\mapsto D_v\big|_a \end{aligned}$$

with

$$D_v\big|_a : C^\infty(V) \to \mathbb{R}, \text{ by } D_v\big|_a(f) = \frac{d}{dt}\bigg|_{t=0} f(a+tv)$$

is a canonical isomorphism from V to T_aV, such that for any linear map

$$L : V \to W$$

we have the following relation

$$dL|_a \circ \eta = \eta \circ L|_a : V|_a \to T_{L(a)}W$$

Proof. Once a basis is chosen for V, since by Proposition 31 the set of derivations at a point in euclidean space is isomorphic to the euclidean space with shifted origin, where both $T_{\mathbf{a}}\mathbb{R}^n$ and $\mathbb{R}^n_{\mathbf{a}}$ are vector spaces, we have η is an isomorphism between V and T_aV.

Next suppose

$$L : V \to W$$

is a linear map. Since its' components with respect to any choice of bases for V and W are linear functions of the coordinates, L is smooth, and so for any $f \in C^\infty(W)$

$$\begin{aligned}
dL|_a\big(D_v\big|_a\big)(f) &= D_v(f \circ L)|_a && D_v\big|_a \in T_aV \\
&= \frac{d}{dt}\bigg|_{t=0} f\big(L(a+tv)\big) \\
&= \frac{d}{dt}\bigg|_{t=0} f\big(L(a)+tL(v)\big) && L \text{ is linear} \\
&= D_{L(v)}(f)|_{L(a)} \\
&= D_{L(v)}\big|_{L(a)}(f)
\end{aligned}$$

and thus, since $\eta(v_a) = D_v\big|_a$, and $\eta(L(v_a)) = D_{L(v)}\big|_{L(a)}$ we see that

$$dL|_a \circ \eta = \eta \circ L|_a$$

□

Proposition 41 (The Tangent Space to a Product Manifold). Let $M_1, \ldots, M_k$ be smooth manifolds, and for each j, let

$$\pi_j : M_1 \times \cdots \times M_k \to M_j$$

be the projection onto the M_j factor. For any point $p = (p_1, \ldots, p_k) \in M_1 \times \cdots \times M_k$, the map

$$\alpha : T_p(M_1 \times \cdots \times M_k) \to T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k$$

defined by

$$\alpha(v) = \big(d\pi_1|_p(v), \ldots, d\pi_k|_p(v)\big)$$

is an isomorphism. The same is true if one of the spaces M_i is a smooth manifold with boundary.

Proof. The idea here is to come up with something that resembles an inverse so that we can use dimensionality and surjection, or injection to show an isomorphism. First we note that α is linear since each $d\pi_i|_p$ is. So given $p = (p_1, \dots, p_k) \in M_1 \times \cdots \times M_k$ define

$$\iota_j : M_j \to M_1 \times \cdots \times M_k \text{ by } \iota_j(x) = (p_1, \dots, p_{j-1}, x, p_{j+1}, \dots, p_k)$$

for $1 \leq j \leq k$ so that ι_j induces the linear map

$$d\iota_j|_{p_j} : T_{p_j} M_j \to T_p(M_1 \times \cdots \times M_k), \quad \iota_j(p_j) = (p_1, \dots, p_j, \dots, p_k) = p$$

where we next define the map

$$\beta : T_{p_1} M_1 \oplus \cdots \oplus T_{p_k} M_k \to T_p(M_1 \times \cdots \times M_k)$$

defined by

$$\beta\big(v_{1p_1} \oplus v_{2p_2} \oplus \cdots \oplus v_{kp_k}\big) = d\iota_1|_{p_1}\big(v_{1p_1}\big) \times d\iota_2|_{p_2}\big(v_{2p_2}\big) \times \cdots \times d\iota_k|_{p_k}\big(v_{kp_k}\big)$$

so that for any $v_{1p_1} \oplus \cdots \oplus v_{kp_k} \in T_{p_1} M_1 \oplus \cdots \oplus T_{p_k} M_k$ and $f \in C^\infty(M_1 \oplus \cdots \oplus M_k)$ we get

$$\begin{aligned}
(\alpha \circ \beta)\big(v_{1p_1} \oplus \cdots \oplus v_{kp_k}\big)(f) &= \alpha\Big(d\iota_1|_{p_1}\big(v_{1p_1}\big), \dots, d\iota_k|_{p_k}\big(v_{kp_k}\big)\Big)(f) \\
&= \Big[d\pi_1|_p\Big(d\iota_1|_{p_1}\big(v_{1p_1}\big), \dots, d\iota_k|_{p_k}\big(v_{kp_k}\big)\Big), \dots, \\
&\qquad d\pi_k|_p\Big(d\iota_1|_{p_1}\big(v_{1p_1}\big), \dots, d\iota_k|_{p_k}\big(v_{kp_k}\big)\Big)\Big](f) \\
&= \Big[d\pi_1|_{\iota_1(p_1)}\Big(d\iota_1|_{p_1}\big(v_{1p_1}\big), \dots, d\iota_k|_{p_k}\big(v_{kp_k}\big)\Big), \dots, \\
&\qquad d\pi_k|_{\iota_k(p_k)}\Big(d\iota_1|_{p_1}\big(v_{1p_1}\big), \dots, d\iota_k|_{p_k}\big(v_{kp_k}\big)\Big)\Big](f) \\
&= \Big(d(\pi_1 \circ \iota_1)|_{p_1}\big(v_{1p_1}\big)(f), \dots, d(\pi_k \circ \iota_k)|_{p_k}\big(v_{kp_k}\big)(f)\Big) && \text{Proposition 34 (b)} \\
&= \Big(v_1(f \circ \pi_1 \circ \iota_1)|_{p_1}, \dots, v_k(f \circ \pi_k \circ \iota_k)|_{p_k}\Big) \\
&= \Big(v_1(f \circ Id_{M_1})|_{p_1}, \dots, v_k(f \circ Id_{M_k})|_{p_k}\Big) && \pi_i \circ \iota_i = Id_{M_i} \\
&= \Big(v_1(f)|_{p_1}, \dots, v_k(f)|_{p_k}\Big) && \text{Proposition 34 (c)} \\
&= \big(v_{1p_1}, \dots, v_{kp_k}\big)(f) \\
&= \big(v_{1p_1} \oplus \cdots \oplus v_{kp_k}\big)(f)
\end{aligned}$$

so that

$$\alpha \circ \beta = Id_{T_{p_1} M_1 \oplus \cdots \oplus T_{p_k} M_k}$$

And so α is surjective.

Next we note by Proposition 37, the tangent space at any point of the manifold has the same dimension as the manifold and so

$$\begin{aligned}
\dim\big(T_p(M_1 \times \cdots \times M_k)\big) &= n_1 + n_2 + \cdots + n_k \\
&= \dim\big(T_{p_1} M_1 \oplus \cdots \oplus T_{p_k} M_k\big)
\end{aligned}$$

Thus, we have a surjective linear map α between vector spaces of equal dimension. Therefore α must be an isomorphism between $T_p(M_1 \times \cdots \times M_k)$ and $T_{p_1} M_1 \oplus \cdots \oplus T_{p_k} M_k$. $\square$

Proposition 42. Let M be a smooth n-manifold with or without boundary, and let $p \in M$. Then T_pM is an n-dimensional vector space, and for any smooth chart $\big(U, (x^1, \dots x^n)\big) \in \mathcal{A}_M$ containing p, the coordinate vectors $\big\{ \frac{\partial}{\partial x^1}\big|_p, \dots, \frac{\partial}{\partial x^n}\big|_p \big\}$ form a basis for T_pM.

Proposition 43. For any smooth n-manifold M; the tangent bundle TM has a natural topology and smooth structure that make it into a $2n$-dimensional smooth manifold. With respect to this structure, the projection

$$\pi : TM \to M$$

is smooth.

Proof. First, let $\{U_i\}_{i=1}^{\infty}$ be a cover for M with corresponding smooth structure $\{(U_i, \phi_i)\} = \mathcal{A}_M$. Now, for any $(U, \phi) \in \mathcal{A}_M$ we have

$$\pi^{-1}(U) = \bigsqcup_{p \in U} T_pM \subseteq TM$$

letting $\phi = (x^1, \dots, x^n)$ so that any element $v_p \in \pi^{-1}(U)$ may be written in the form

$$v_p = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\bigg|_p$$

and so we can define

$$\widetilde{\phi} : \pi^{-1}(U) \to \mathbb{R}^{2n}, \text{ by } \widetilde{\phi}\left(\sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\bigg|_p \right) = \big(x^1(p), \dots, x^n(p), v^1, \dots, v^n\big)$$

where

$$\widetilde{\phi}\big(\pi^{-1}(U)\big) = \phi(U) \times \mathbb{R}^n \subseteq \mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{R}^{2n}$$

is an open subset. So we propose the smooth charts $\big(\pi^{-1}(U), \widetilde{\phi}\big)$ for TM. With explicit inverse given by

$$\widetilde{\phi}^{-1} : \phi(U) \times \mathbb{R}^n \to TM, \text{ by } \widetilde{\phi}^{-1}(\mathbf{x}, \mathbf{v}) = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\bigg|_{\phi^{-1}(\mathbf{x})}$$

and so is bijective onto its image, which is a subset of a euclidean space.

Next, for any charts $(U, \phi), (V, \psi) \in \mathcal{A}_M$, such that $U \cap V \neq \varnothing$, with corresponding smooth charts $\big(\pi^{-1}(U), \widetilde{\phi}\big), \big(\pi^{-1}(V), \widetilde{\psi}\big)$ for TM, and we note that

$$\widetilde{\phi}\big(\pi^{-1}(U) \cap \pi^{-1}(V)\big) = \phi(U \cap V) \times \mathbb{R}^n \subseteq \mathbb{R}^n \times \mathbb{R}^n$$
$$\widetilde{\psi}\big(\pi^{-1}(U) \cap \pi^{-1}(V)\big) = \psi(U \cap V) \times \mathbb{R}^n \subseteq \mathbb{R}^n \times \mathbb{R}^n$$

are both open, and letting $\phi = (x^1, \dots, x^n)$ and $\psi = (y^1, \dots, y^n)$, with the change of coordinates giving

$$\widetilde{\phi}^{-1}(\mathbf{x}, \mathbf{v}) = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\bigg|_{\phi^{-1}(\mathbf{x})} = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} v^i \frac{\partial (y^j \circ \phi^{-1})}{\partial x^i}(\phi(\mathbf{x})) \right) \frac{\partial}{\partial y^j}\bigg|_{\phi^{-1}(\mathbf{x})}$$

we have the transition map

$$\widetilde{\psi} \circ \widetilde{\phi}^{-1} : \phi(U \cap V) \times \mathbb{R}^n \to \psi(U \cap V) \times \mathbb{R}^n$$

defined by

$$\big(\widetilde{\psi} \circ \widetilde{\phi}^{-1}\big)(\mathbf{x}, \mathbf{v}) = \left(y^1\big(\phi^{-1}(\mathbf{x})\big), \dots, y^n\big(\phi^{-1}(\mathbf{x})\big), \sum_{i=1}^{n} v^i \frac{\partial(y^1 \circ \phi^{-1})}{\partial x^i}(\phi(\mathbf{x})), \dots, \sum_{i=1}^{n} v^i \frac{\partial(y^n \circ \phi^{-1})}{\partial x^i}(\phi(\mathbf{x})) \right)$$

which is smooth as a function of $(\mathbf{x}, \mathbf{v})$.

Now, since for each $p \in M$ we have $p \in U_i$ for some i we get that

$$TM = \bigsqcup_{p \in M} T_pM = \bigcup_{i=1}^{\infty} \bigsqcup_{p \in U_i} T_pM$$

which is to say that $\{\pi^{-1}(U_i)\}_{i=1}^{\infty}$ is a countable cover for TM.

Now if $v_p, w_q \in TM$, then

Case 1: If $p = q$, then $p \in U$ gives $v_p, w_p \in \pi^{-1}(U)$.

Case 2: If $p \neq q$, then since M is Hausdorff, there are disjoint open sets $U, V \subseteq M$ such that $p \in U$ and $q \in V$. Where we then have

$$v_p \in \pi^{-1}(U), \text{ and } w_q \in \pi^{-1}(V) \quad \text{ where } \pi^{-1}(U) \cap \pi^{-1}(V) = \varnothing$$

Thus, we have a collection $\{\pi^{-1}(U_i)\}$ of subsets of TM, together with maps

$$\widetilde{\phi}_i : \pi^{-1}(U_i) \to \mathbb{R}^{2n}$$

such that

1. For each i, $\widetilde{\phi}_i$ is a bijection between $\pi^{-1}(U_i)$ and $\widetilde{\phi}_i\big(\pi^{-1}(U_i)\big) \subseteq \mathbb{R}^{2n}$.
2. For each i and j, the sets $\widetilde{\phi}_i\big(\pi^{-1}(U_i) \cap \pi^{-1}(U_j)\big), \widetilde{\phi}_j\big(\pi^{-1}(U_i) \cap \pi^{-1}(U_j)\big) \subseteq \mathbb{R}^{2n}$ are open.
3. Whenever $U_i \cap U_j \neq \varnothing$ the map

$$\begin{aligned} \widetilde{\phi}_j \circ \widetilde{\phi}_i^{-1} &: \widetilde{\phi}_i\big(\pi^{-1}(U_i) \cap \pi^{-1}(U_j)\big) \to \widetilde{\phi}_j\big(\pi^{-1}(U_i) \cap \pi^{-1}(U_j)\big) \\ &= \phi_i(U_i \cap U_j) \times \mathbb{R}^n \to \phi_j(U_i \cap U_j) \times \mathbb{R}^n \end{aligned}$$

   is smooth.
4. Countably many of the sets $\pi^{-1}(U_i)$ cover TM.
5. Whenever $v_p, w_q \in TM$ are distinct; then either there exists some $\pi^{-1}(U_i)$ such that

$$v_p, w_q \in \pi^{-1}(U_i)$$

   or there exists open sets $\pi^{-1}(U_i), \pi^{-1}(U_j)$ such that

$$\pi^{-1}(U_i) \cap \pi^{-1}(U_j) = \varnothing, \quad v_p \in \pi^{-1}(U_i), \quad w_q \in \pi^{-1}(U_j)$$

and so, by the Smooth Manifold Chart Lemma, $\{(\pi^{-1}(U_i), \widetilde{\phi}_i)\} = \mathcal{A}_{TM}$ is a smooth structure for TM.

Moreover, for any $v_p \in TM$ with the chart $(\pi^{-1}(U), \widetilde{\phi}) \in \mathcal{A}_{TM}$ containing v_p, and $(U, \phi) \in \mathcal{A}_M$ containing $\pi(v_p) = p$, we have

$$\pi\big(\pi^{-1}(U)\big) = U \implies \pi\big(\pi^{-1}(U)\big) \subseteq U$$

where the coordinate representation of π then gives

$$\phi \circ \pi \circ \widetilde{\phi}^{-1} : \phi(U) \times \mathbb{R}^n \to \phi(U)$$

is smooth as the projection map between euclidean spaces. And thus, we have that π must also be smooth. $\square$

Proposition 44. If M is a smooth n-manifold with or without boundary, and M can be covered by a single smooth chart, then TM is diffeomorphic to $M \times \mathbb{R}^n$.

Proof. Let $(M, \phi) = \mathcal{A}_M$, then by definition

$$\phi : M \to \phi(M)$$

is a diffeomorphism with $\phi(M) \subseteq \mathbb{R}^n$ or $\mathbb{H}^n$. And from Proposition 43,

$$\widetilde{\phi} : \pi^{-1}(M) \to \phi(M) \times \mathbb{R}^n$$

is a diffeomorphism , where $\pi^{-1}(M) = TM$. Thus we have

$$TM \cong \phi(M) \times \mathbb{R}^n \cong M \times \mathbb{R}^n$$

$\square$

Proposition 45. let M be a smooth m-Manifold, and N a smooth n-Manifold. If

$$F : M \to N$$

is a smooth map, then its global differential

$$dF : TM \to TN$$

is a smooth map.

Proof. Since

$$dF|_p = \begin{bmatrix} \frac{\partial \widehat{F}^1}{\partial x^1}\big(\phi(p)\big) & \cdots & \frac{\partial \widehat{F}^1}{\partial x^m}\big(\phi(p)\big) \\ \vdots & \ddots & \vdots \\ \frac{\partial \widehat{F}^n}{\partial x^1}\big(\phi(p)\big) & \cdots & \frac{\partial \widehat{F}^n}{\partial x^m}\big(\phi(p)\big) \end{bmatrix}$$

given $p \in M$, let $(U, \phi) \in \mathcal{A}_M$ be a chart containing p, and $(V, \psi) \in \mathcal{A}_N$ be a chart containing $F(p)$, such that $F(U) \subseteq V$. Then we have $(\pi_M^{-1}(U), \widetilde{\phi}) \in \mathcal{A}_{TM}$ is a chart containing v_p, and $(\pi_N^{-1}(V), \widetilde{\psi}) \in$

$\mathcal{A}_{TN}$ is a chart containing $dF(v_p) = v_{F(p)}$. Letting $\phi = (x^1, \dots, x^m)$ and $\psi = (y^1, \dots, y^n)$, we get that dF has coordinate representation

$$\widetilde{\psi} \circ dF \circ \widetilde{\phi}^{-1} : \phi(U) \times \mathbb{R}^m \to \psi(V) \times \mathbb{R}^n$$

defined by

$$\begin{aligned}
\widehat{dF}(\mathbf{x}, \mathbf{v}) &= \big(\widetilde{\psi} \circ dF\big) \left(\sum_{i=1}^{m} v^i \frac{\partial}{\partial x^i}\bigg|_{\phi^{-1}(\mathbf{x})} \right) \\
&= \widetilde{\psi} \left(\sum_{j=1}^{n} \left(\sum_{i=1}^{m} v^i \frac{\partial (y^j \circ \widehat{F})}{\partial x^i} \big(\phi^{-1}(\mathbf{x})\big) \right) \frac{\partial}{\partial y^j}\bigg|_{F(\phi^{-1}(\mathbf{x}))} \right) \\
&= \left(y^1\big(F(\phi^{-1}(\mathbf{x}))\big), \dots, y^n\big(F(\phi^{-1}(\mathbf{x}))\big), \sum_{i=1}^{m} v^i \frac{\partial \widehat{F}^1}{\partial x^i}\big(\phi^{-1}(\mathbf{x})\big), \dots, \sum_{i=1}^{m} v^i \frac{\partial \widehat{F}^n}{\partial x^i}\big(\phi^{-1}(\mathbf{x})\big) \right)
\end{aligned}$$

or letting $p \in U$ be such that $\phi(p) = \mathbf{x} \in \phi(U)$ we get

$$\widehat{dF}(\mathbf{x}, \mathbf{v}) = \left(y^1\big(F(p)\big), \dots, y^n\big(F(p)\big), \sum_{i=1}^{m} v^i \frac{\partial \widehat{F}^1}{\partial x^i}(p), \dots, \sum_{i=1}^{m} v^i \frac{\partial \widehat{F}^n}{\partial x^i}(p) \right)$$

and since F is smooth, its coordinate representation $\widehat{F}$ is therefore smooth. And thus, $\widehat{dF}$ is smooth in each of it's components as a map between euclidean spaces, and hence is smooth, and so dF must also be smooth. $\square$

Proposition 46 (**Properties of the Global Differential**)**.** Suppose both

$$\begin{aligned} F &: M \to N \\ G &: N \to P \end{aligned}$$

are smooth maps. Then

(a) $d(G \circ F) = dG \circ dF$

(b) $d(Id_M) = Id_{TM} : TM \to TM$

(c) If F is a diffeomorphism, then

$$dF : TM \to TN$$

is also a diffeomorphism, and

$$(dF)^{-1} = d(F^{-1})$$

Proposition 47. Suppose M is a smooth manifold with or without boundary and $p \in M$. Every derivation $v_p \in T_pM$ is the velocity of some smooth curve

$$\gamma : (-\epsilon, \epsilon) \to M$$

in M; i.e.

$$\gamma(0) = p, \quad \text{and} \quad \gamma'(0) = v_p$$

Proof. Let $(U, \phi) \in \mathcal{A}_M$ be a smooth chart centered at p, and in terms of the basis vectors for T_pM we have

$$v_p = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}\bigg|_p$$

since $\phi(U) \subseteq \mathbb{R}^n$ is open and $\phi(p) \in \phi(U)$, for some $\epsilon > 0$ we have

$$\phi \circ \gamma : (-\epsilon, \epsilon) \to \phi(U), \text{ by } (\phi \circ \gamma)(t) = t(v^1, \ldots, v^n)$$

and so we may define our smooth curve

$$\gamma : (-\epsilon, \epsilon) \to M, \text{ by } \gamma(t) = \phi^{-1}\big(tv^1, \ldots, tv^n\big)$$

since (U, ϕ) was chosen to be centered at p this implies $\phi(p) = \mathbf{0}$, which then implies

$$p = \phi^{-1}(\mathbf{0}) = \phi^{-1}(0 \cdot \mathbf{x}) = \gamma(0)$$

and

$$\gamma'(0) = \sum_{i=1}^n \frac{d(x^i \circ \gamma)}{dt}(0) \frac{\partial}{\partial x^i}\bigg|_{\gamma(0)} = \sum_{i=1}^n \frac{d(x^i \circ \gamma)}{dt}(0) \frac{\partial}{\partial x^i}\bigg|_p = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}\bigg|_p = v_p$$

□

Proposition 48 (**Velocity of a Composite Curve**)**.** Let M, N be smooth manifolds,

$$F : M \to N$$

a smooth map, and let

$$\gamma : J \to M$$

be a smooth curve. For any $t_0 \in J$, the velocity at t_0 of the composite curve

$$F \circ \gamma : J \to N$$

is given by

$$(F \circ \gamma)'(t_0) = dF|_{\gamma(t_0)}\big(\gamma'(t_0)\big)$$

Proof. Since the composition $F \circ \gamma$ is itself a smooth curve into the manifold N we have, from the definition of velocity curve

$$(F \circ \gamma)'(t_0) = d(F \circ \gamma)|_{t_0} \left(\frac{d}{dt}\bigg|_{t_0} \right) = dF|_{\gamma(t_0)} \left(d\gamma|_{t_0} \left(\frac{d}{dt}\bigg|_{t_0} \right) \right) = dF|_{\gamma(t_0)}\big(\gamma'(t_0)\big)$$

□

Corollary 49. Let M, N be smooth manifolds, and

$$F : M \to N$$

a smooth map. Fix $p \in M$, and $v_p \in T_pM$. Then

$$dF|_p(v_p) = (F \circ \gamma)'(0)$$

for any smooth curve

$$\gamma : (-\epsilon, \epsilon) \to M, \quad \text{such that} \quad \gamma(0) = p, \text{ and } \gamma'(0) = v_p$$

Proposition 50. Suppose $F : M \to N$ is a smooth map and $p \in M$. If

$$dF|_p : T_pM \to T_{F(p)}N$$

is surjective, then p has a neighborhood U_p such that $dF|_{U_p}$ is a submersion. If

$$dF|_p : T_pM \to T_{F(p)}N$$

is injective, then p has a neighborhood U_p such that $dF|_{U_p}$ is an immersion.

Proof. Let M be of dimension m, and N be of dimension n, and let $(U, \phi) \in \mathcal{A}_M$ be a chart containing p, and $(V, \psi) \in \mathcal{A}_N$ be a chart containing $F(p)$. Either of the hypothesis imply that the coordinate representation $\widehat{F}$ of F has full rank at $\phi(p)$, since

$$dF|_p = \begin{bmatrix} \frac{\partial \widehat{F}^1}{\partial x^1}\big(\phi(p)\big) & \cdots & \frac{\partial \widehat{F}^1}{\partial x^m}\big(\phi(p)\big) \\ \vdots & \ddots & \vdots \\ \frac{\partial \widehat{F}^n}{\partial x^1}\big(\phi(p)\big) & \cdots & \frac{\partial \widehat{F}^n}{\partial x^m}\big(\phi(p)\big) \end{bmatrix}$$

has either $\operatorname{rank}(dF|_p) = n$ if $dF|_p$ is surjective, or $\operatorname{rank}(dF|_p) = m$ if $dF|_p$ is injective. And we know that the matrices of full rank $\mathrm{M}_m(n\times m, \mathbb{R}) \subseteq \mathrm{M}(n\times m, \mathbb{R})$ of rank m, or $\mathrm{M}_n(n\times m, \mathbb{R}) \subseteq \mathrm{M}(n\times m, \mathbb{R})$ of rank n, are open subsets. Which tells us that each element in either $\mathrm{M}_m(n\times m, \mathbb{R})$ or $\mathrm{M}_n(n\times m, \mathbb{R})$ has a neighborhood which must also be full rank by the continuity of the determinant function (on the $n \times n$ or $m \times m$ sub-matrices which have non-zero determinant). And so there is a neighborhood U_p of p, such that $dF|_{U_p}$ which is represented by the Jacobian matrix of $\widehat{F}$ on $\phi(U_p)$ has full rank. □

Proposition 51. A composition of smooth submersions is a smooth submersion, and a composition of smooth immersions is a smooth immersion. Yet, a composition of maps of constant rank need not have constant rank.

Proof. Let

$$F : M \to N$$
$$G : N \to P$$

be smooth submersions, and let $p \in M$ be given. By Proposition 34 (b) we have

$$d(G \circ F)|_p = dG|_{F(p)} \circ dF|_p$$

and since F is a smooth submersion

$$dF|_p : T_pM \to T_{F(p)}N$$

is surjective. Similarly, since G is a smooth submersion

$$dG|_{F(p)} : T_{F(p)}N \to T_{G(F(p))}P$$

is surjective. And thus,

$$dG|_{F(p)} \circ dF|_p : T_pM \to T_{G(F(p))}P$$

is surjective at p.

Since $p \in M$ was arbitrary we conclude that

$$d(G \circ F)|_p : T_pM \to T_{G(F(p))}P$$

is surjective at each point of M and therefore

$$G \circ F : M \to P$$

is a smooth submersion.

The same argument holds with "submersion" replaced by "immersion" so that the composition of smooth immersions is a smooth immersion.

Next, we produce a counter-example to show that the composition of maps of constant rank need not have constant rank. Define

$$F : \mathbb{R} \to \mathbb{R}^2, \text{ by } F(x) = (x, x^2)$$

so that

$$DF = \begin{bmatrix} 1 \\ 2x \end{bmatrix}$$

and F has constant rank with $\operatorname{rank}(F) = 1$. And define

$$G : \mathbb{R}^2 \to \mathbb{R}, \text{ by } G(x, y) = y$$

then

$$DG = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

and so G has constant rank with $\operatorname{rank}(G) = 1$. Then we have

$$G \circ F : \mathbb{R} \to \mathbb{R}, \text{ by } \big(G \circ F\big)(x) = x^2$$

which gives

$$D(G \circ F) = 2x$$

which has

$$\operatorname{rank}(G \circ F) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

and therefore, does not have constant rank. □

Theorem 52 (**Inverse Function Theorem for Manifolds**)**.** Suppose M and N are smooth manifolds, and $F : M \to N$ is a smooth map. If $p \in M$ is a point such that

$$dF|_p : T_pM \to T_{F(p)}N$$

is invertible, then there are connected neighborhoods U_p of p, and $V_{F(p)}$ of $F(p)$, such that

$$F|_{U_p} : U_p \to V_{F(p)}$$

is a diffeomorphism.

Proof. First, since $dF|_p$ is represented by the Jacobian matrix of $\widehat{F}$ at $\phi(p)$, for any charts $(U, \phi) \in \mathcal{A}_M$ containing p, and $(V, \psi) \in \mathcal{A}_N$ containing $F(p)$, we have

$$dF|_p = \begin{bmatrix} \frac{\partial \widehat{F}^1}{\partial x^1}\big(\phi(p)\big) & \cdots & \frac{\partial \widehat{F}^1}{\partial x^m}\big(\phi(p)\big) \\ \vdots & \ddots & \vdots \\ \frac{\partial \widehat{F}^n}{\partial x^1}\big(\phi(p)\big) & \cdots & \frac{\partial \widehat{F}^n}{\partial x^m}\big(\phi(p)\big) \end{bmatrix}$$

so $dF|_p$ being invertible implies that the Jacobian matrix of $\widehat{F}$ at $\phi(p)$ is invertible, and only square matrices are invertible, thus we have

$$\dim(M) = \dim(N) = n$$

Next, by the smoothness of F, we may choose charts $(U, \phi) \in \mathcal{A}_M$ centered at p, so $\phi(p) = \mathbf{0}$, and $(V, \psi) \in \mathcal{A}_N$ centered at $F(p)$, so $\psi(F(p)) = \mathbf{0}$, such that $F(U) \subseteq V$. Where

$$F(U) \subseteq V \implies U \subseteq F^{-1}(V) \implies U \cap F^{-1}(V) = U$$

then we have the coordinate representation of F given by

$$\widehat{F} : \phi(U) \to \psi(V)$$

which is smooth, and where

$$\widehat{F}(\mathbf{0}) = \psi \circ F \circ \phi^{-1}(\mathbf{0}) = \psi(F(p)) = \mathbf{0}$$

Now, since both ϕ and ψ are diffeomorphisms, and hence invertible, the differential

$$d\widehat{F}|_{\mathbf{0}} = d\psi|_{F(p)} \circ dF|_p \circ d\phi^{-1}|_{\mathbf{0}} : T_{\mathbf{0}}\mathbb{R}^n \to T_{\mathbf{0}}\mathbb{R}^n$$

is nonsingular, as the composition of 3 non-singular matrices between euclidean spaces. Where the ordinary Inverse Function Theorem then says that there are connected open subsets

$$U_{\mathbf{0}} \subseteq \phi(U), \quad V_{\mathbf{0}} \subseteq \psi(V)$$

containing $\mathbf{0}$, such that

$$\widehat{F}|_{U_{\mathbf{0}}} : U_{\mathbf{0}} \to V_{\mathbf{0}}$$

is a diffeomorphism. Then $U_p = \phi^{-1}(U_{\mathbf{0}})$ is a connected neighborhood containing p, and $V_{F(p)} = \psi^{-1}(V_{\mathbf{0}})$ is connected neighborhood containing $F(p)$. And so

$$F|_{U_p} = \psi^{-1} \circ \widehat{F} \circ \phi|_{U_p} : U_p \to V_{F(p)}$$

is a diffeomorphism, since the composition of diffeomorphisms is a diffeomorphism. □

Proposition 53 (Elementary Properties of Local Diffeomorphisms).

(a) Every composition of local diffeomorphisms is a local diffeomorphism.

(b) Every finite product of local diffeomorphisms between smooth manifolds is a local diffeomorphism.

(c) Every local diffeomorphism is a local homeomorphism and an open map.

(d) The restriction of a local diffeomorphism to an open submanifold with or without boundary is a local diffeomorphism.

(e) Every diffeomorphism is a local diffeomorphism.

(f) Every bijective local diffeomorphism is a diffeomorphism.

(g) A map between smooth manifolds with or without boundary is a local diffeomorphism if and only if in a neighborhood of each point of its domain, it has a coordinate representation that is a local diffeomorphism.

Proof.

(a) Let M, N, and P be smooth manifolds and let

$$\begin{aligned} F &: M \to N \\ G &: N \to P \end{aligned}$$

be local diffeomorphisms. Let $p \in M$, then since F is a local diffeomorphism, $\exists\ U_p \subseteq M$ open and containing p, such that $F(U_p) \subseteq N$ is open, and

$$F|_{U_p} : U_p \to F(U_p)$$

is a diffeomorphism. Similarly, since G is a local diffeomorphism, $\exists\ V_{F(p)} \subseteq N$ open and containing $F(p)$, such that $G\big(V_{F(p)}\big) \subseteq P$ is open, and

$$G|_{V_{F(p)}} : V_{F(p)} \to G\big(V_{F(p)}\big)$$

is a diffeomorphism. And since $F|_{U_p}$ is a diffeomorphism we have

$$F^{-1}\big(F(U_p) \cap V_{F(p)}\big) = U_p \cap F^{-1}\big(V_{F(p)}\big) \subseteq U_p$$

So that the map

$$(G \circ F)|_{U_p \cap F^{-1}(V_{F(p)})} : U_p \cap F^{-1}\big(V_{F(p)}\big) \to G\big(\overbrace{F(U_p) \cap V_{F(p)}}^{\subseteq V_{F(p)}}\big)$$

is a local diffeomorphism.

(b) Let $M_1, \ldots, M_k$, and $N_1, \ldots, N_k$ be smooth manifolds such that for each i we have

$$F_i : M_i \to N_i$$

is a local diffeomorphism then for each $p_i \in M_i, \exists\ U_{p_i} \subseteq M_i$ open and containing p_i, such that $F_i(U_{p_i}) \subseteq N_i$ is open and

$$F_i|_{U_{p_i}} : U_{p_i} \to F_i(U_{p_i})$$

is a diffeomorphism, so the finite product

$$F_1 \times \cdots \times F_k : M_1 \times \cdots \times M_k \to N_1 \times \cdots \times N_k$$

has a local diffeomorphism given by

$$F_1|_{U_{p_1}} \times \cdots \times F_k|_{U_{p_k}} : U_{p_1} \times \cdots \times U_{p_k} \to F_1(U_{p_1}) \times \cdots \times F_k(U_{p_k})$$

since it is a local diffeomorphism in each of its components.

(c) Let

$$F : M \to N$$

be a local diffeomorphism, then for every $p \in M$, $\exists\, U_p \subseteq M$ open and containing p, such that $F(U_p) \subseteq N$ is open and

$$F|_{U_p} : U_p \to F(U_p)$$

is a diffeomorphism. Since differentiability implies continuity we have

$$F|_{U_p} : U_p \to F(U_p)$$

is a homeomorphism, and since this is true for each $p \in M$ we have that F is a local homeomorphism.

By definition we have that $F(U_p) \subseteq N$ is open, and so a local diffeomorphism is defined to be an open map.

(d) Let

$$F : M \to N$$

be a local diffeomorphism, let $U \subseteq M$ be an open submanifold, and let $p \in U$. Then $p \in M$, and there exists $V_p \subseteq M$ open and containing p, such that $F(V_p) \subseteq N$ is open and

$$F|_{V_p} : V_p \to F(V_p)$$

is a diffeomorphism. Since an open subset of a smooth manifold is again a smooth manifold, and $U \cap V_p \subset M$ is open as the finite intersection of open sets, it is also a smooth submanifold, where Proposition 20 (d) then tells us that the restriction of a diffeomorphism to an open submanifold is a diffeomorphism; that is

$$F|_{V_p}\big|_{U \cap V_p} : U \cap V_p \to F(U \cap V_p)$$

is a diffeomorphism. And by Proposition 20 (c) every diffeomorphism is an open map, and so $F(U \cap V_p) \subseteq N$ is open.

Thus, we have found a neighborhood $U \cap V_p \subseteq U$ of p in U, such that $F(U \cap V_p) \subseteq N$ is open and

$$F|_{V_p}\big|_{U \cap V_p} : U \cap V_p \to F(U \cap V_p)$$

is a diffeomorphism. Since $p \in U$ was arbitrary we conclude that this can be done for each point of the open submanifold U, and thus we conclude that the restriction of a local diffeomorphism is a local diffeomorphism.

(e) Let
$$F : M \to N$$
be a diffeomorphism then for each $p \in M, \exists\ U_p \subseteq M$ open and containing p, and since diffeomorphisms are open mappings we must have $F(U_p) \subseteq N$ is open. Then since open subsets of smooth manifolds are again smooth manifolds, Proposition 20 (d) then tells us that the restriction of a diffeomorphism to an open submanifold is a diffeomorphism and so
$$F|_{U_p} : U_p \to F(U_p)$$
is a diffeomorphism. Thus, we see that the restriction of F to an open subset around any $p \in M$ induces a local diffeomorphism.

(f) Let
$$F : M \to N$$
be a bijective local diffeomorphism, since F is bijective
$$F^{-1} : N \to M$$
exists. Since F is a local diffeomorphism for every $p \in M$, $\exists\ U_p \subseteq M$ open and containing p, such that $F(U_p) \subseteq N$ is open and
$$F|_{U_p} : U_p \to F(U_p)$$
is a diffeomorphism. Yet, by the openness of $F(U_p)$ for each $q \in F(U_p)$ there is an open neighborhood $V_q \subseteq F(U_p)$ containing q, where (d) tells us that the restriction of a diffeomorphism to an open submanifold is a local diffeomorphism and so
$$F^{-1}|_{F(U_p)}\big|_{F(U_p)\cap V_p} : F(U_p) \cap V_p \to U_p \cap F^{-1}(V_p)$$
is a diffeomorphism and hence smooth. And since $q \in F(U_p)$ arbitrary we conclude that F^{-1} is smooth in a neighborhood of each point of $F(U_p)$, and since F is bijective we then conclude that F^{-1} will be smooth in a neighborhood of each point of N, and thus F^{-1} is smooth, where F is already smooth in a neighborhood of each point of M by definition, and thus F is a diffeomorphism.

(g) First let
$$F : M \to N$$
be a local diffeomorphism and let $p \in M$ with smooth chart $(U, \phi) \in \mathcal{A}_M$ containing p, and smooth chart $(V, \psi) \in \mathcal{A}_N$ containing $F(p)$, then the coordinate representation
$$\widehat{F}|_{\phi(U\cap F^{-1}(V))} = \psi \circ F \circ \phi^{-1}|_{\phi(U\cap F^{-1}(V))} : \phi\big(U \cap F^{-1}(V)\big) \to \psi\big(F(U) \cap V\big)$$
is a local diffeomorphism.

Next suppose F has coordinate representation $\widehat{F}$ which is locally a diffeomorphism, then since the smooth maps ϕ, ψ are diffeomorphisms and the the composition of diffeomorphisms is a diffeomorphism we have

$$F|_{\phi^{-1}(U\cap\widehat{F}^{-1}(\psi(V)))}$$
$$= \psi^{-1} \circ \widehat{F} \circ \phi|_{\phi^{-1}(U\cap\widehat{F}^{-1}(\psi(V)))} : \phi^{-1}\big(U \cap \widehat{F}^{-1}(\psi(V))\big) \to F(\phi^{-1}(U)) \cap V$$

is a local diffeomorphism.

□

Proposition 54. Suppose M and N are smooth manifolds (without boundary), and $F : M \to N$ is a map.

(a) F is a local diffeomorphism if and only if it is both a smooth immersion and a smooth submersion.

(b) If $\dim(M) = \dim(N)$ and F is either a smooth immersion or a smooth submersion, then it is a local diffeomorphism.

Proof.

(a) First suppose that F is a local diffeomorphism. Given any $p \in M$, $\exists\ U_p \subseteq M$ open and containing p, such that $F(U_p) \subseteq N$ is open and

$$F|_{U_p} : U_p \to F(U_p)$$

is a diffeomorphism, and since it is a diffeomorphism, Proposition 34 (d) tell us that

$$dF|_p : T_pU_p \cong T_pM \to T_{F(p)}F(U_p) \cong T_{F(p)}N$$

is an isomorphism, which tells us that $\dim(T_pM) = \dim\big(T_{F(p)}N\big)$. Yet, $\dim(T_pM) = \dim(M)$ and $\dim\big(T_{F(p)}N\big) = \dim(N)$, and so $\dim(M) = \dim(N) = \operatorname{rank}(dF|_p)$, and since this can be done for each $p \in M$, we conclude that dF has full rank at each point of M, and hence is both injective and surjective at each point of M. And therefore dF is both a smooth immersion and a smooth submersion.

Next, suppose that F, or dF, is both a smooth immersion and a smooth submersion, then $\dim(M) = \dim(N)$, and for each $p \in M$

$$dF|_p : T_pM \to T_{F(p)}N$$

is an isomorphism, and so is invertible, where the Inverse Function Theorem for Manifolds says that there exists an open neighborhood U_p of p, and $V_{F(p)}$ of $F(p)$, where we may assume that $F(U_p) = V_{F(p)}$, such that

$$F|_{U_p} : U_p \to F(U_p)$$

is a diffeomorphism. Thus, F is a local diffeomorphism.

(b) Note that if $\dim(M) = \dim(N)$ then either injectivity, or surjectivity of

$$dF|_p : T_pM \to T_{F(p)}N$$

implies bijectivity, and so F, or dF, is a smooth immersion/submersion iff it is a smooth submersion/immersion, then (a) tells us that F must also be a local diffeomorphism.

$\square$

Proposition 55. Suppose M, N, and P are smooth manifolds with or without boundary, and $F : M \to N$ is a local diffeomorphism. Prove the following:

(a) If $G : P \to M$ is continuous, then G is smooth if and only if $F \circ G$ is smooth.

(b) If in addition F is surjective and $G : N \to P$ is any map, then G is smooth if and only if $G \circ F$ is smooth.

Proof.

(a) First suppose

$$G : P \to M$$

is smooth. Then, since local diffeomorphisms are smooth, and the composition of smooth functions is smooth

$$F \circ G : P \to N$$

is smooth.

Next, suppose

$$F \circ G : P \to N$$

is smooth. Let $q \in P$, then $G(q) \in M$ and since F is a local diffeomorphism $\exists\, U_{G(q)} \subseteq M$ open and containing $G(q)$, such that

$$F|_{U_{G(q)}} : U_{G(q)} \to F(U_{G(q)})$$

is a diffeomorphism. Yet, this also implies

$$(F|_{U_{G(q)}})^{-1} = F^{-1}|_{F(U_{G(q)})} : F(U_{G(q)}) \to U_{G(q)}$$

is a diffeomorphism.

Next, since G is continuous and $U_{G(q)} \subseteq M$ is open we have $G^{-1}(U_{G(q)}) \subseteq P$ is open and contains q, and since

$$(F \circ G)|_{G^{-1}(U_{G(q)})} : G^{-1}(U_{G(q)}) \to F(U_{G(q)})$$

is smooth and the composition of smooth functions is smooth, we get

$$F^{-1}|_{F(U_{G(q)})} \circ (F \circ G)|_{G^{-1}(U_{G(q)})} = G|_{G^{-1}(U_{G(q)})} : G^{-1}(U_{G(q)}) \to U_{G(q)}$$

is smooth. And since $q \in P$ was arbitrary we can conclude that G is smooth.

(b) First suppose
$$G : N \to P$$
is smooth. Then since since local diffeomorphisms are smooth and, the composition of smooth functions is smooth
$$G \circ F : M \to P$$
is smooth.

Next, suppose
$$G \circ F : M \to P$$
is smooth, and let $q \in N$ be arbitrary. Since F is surjective, WLOG let $q = F(p)$. Since F is a local diffeomorphism $\exists\, U_p \subseteq M$ open and containing p, with $F(U_p) \subseteq N$ open and containing q, such that
$$F|_{U_p} : U_p \to F(U_p)$$
is a diffeomorphism. Yet, this implies
$$(F|_{U_p})^{-1} = F^{-1}|_{F(U_p)} : F(U_p) \to U_p$$
is a diffeomorphism. Then since
$$(G \circ F)|_{U_p} : U_p \to G(F(U_p))$$
is smooth by assumption, and the composition of smooth functions is smooth, we get
$$(G \circ F)|_{U_p} \circ F^{-1}|_{F(U_p)} = G|_{F(U_p)} : F(U_p) \to G(F(U_p))$$
is smooth. And since $q \in N$ was arbitrary we can conclude that G is smooth.

$\square$

Theorem 56 (**Rank Theorem**)**.** Suppose M and N are smooth manifolds with $\dim(M) = m$, $\dim(N) = n$ and
$$F : M \to N$$
is a smooth map with constant rank r. For each $p \in M$ there exist smooth charts $(U, \phi) \in \mathcal{A}_M$ centered at p and $(V, \psi) \in \mathcal{A}_N$ centered at $F(p)$, such that $F(U) \subseteq V$, in which F has a coordinate representation of the form

$$\begin{aligned}
&\widehat{F}(x_1, \ldots, x_r, x_{r+1}, \ldots, x_m) = (x_1, \ldots, x_r, 0, \ldots, 0) && \\
&\widehat{F}(x_1, \ldots, x_n, x_{n+1}, \ldots, x_m) = (x_1, \ldots, x_n) && F \text{ a smooth submersion} \\
&\widehat{F}(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0) && F \text{ a smooth immersion}
\end{aligned}$$

Proof. By the smoothness of F, we may choose charts $(U, \phi) \in \mathcal{A}_M$ centered at p, so $\phi(p) = \mathbf{0}$, and $(V, \psi) \in \mathcal{A}_N$ centered at $F(p)$, so $\psi(F(p)) = \mathbf{0}$, such that $F(U) \subseteq V$. Where
$$F(U) \subseteq V \implies U \subseteq F^{-1}(V) \implies U \cap F^{-1}(V) = U$$

then we have the coordinate representation of F given by

$$\widehat{F}:\phi(U)\to\psi(V)$$

which is smooth, and where

$$\widehat{F}(\mathbf{0})=\psi\circ F\circ\phi^{-1}(\mathbf{0})=\psi(F(p))=\mathbf{0}$$

Since the theorem is local we may associate M with $\phi(U)\subseteq\mathbb{R}^m$, and N with $\psi(V)\subseteq\mathbb{R}^n$. Since F, or dF, has constant rank r, this means that the Jacobian matrix of $\widehat{F}$ at $\phi(p)=\mathbf{0}$ has an $r\times r$ submatrix with non-zero determinant, re-ordering the coordinates if necessary we may assume this is the upper left submatrix; that is

$$D\widehat{F}(\mathbf{0})=\begin{bmatrix}\left(\frac{\partial\widehat{F}^j}{\partial x^i}(\mathbf{0})\right)_{\substack{i=1,\dots,r\\ j=1,\dots,r}} & * \\ * & *\end{bmatrix}$$

relabeling the standard coordinates

$$\begin{aligned}(\mathbf{x},\mathbf{y})&=(x_1,\dots x_r,y_1\dots,y_{m-r})\text{ in }\mathbb{R}^m\\ (\mathbf{u},\mathbf{v})&=(u_1,\dots u_r,v_1\dots,v_{n-r})\text{ in }\mathbb{R}^n\end{aligned}$$

then writing

$$\widehat{F}(\mathbf{x},\mathbf{y})=\big(Q(\mathbf{x},\mathbf{y}),R(\mathbf{x},\mathbf{y})\big),\text{ for }Q:\phi(U)\to\mathbb{R}^r,\ R:\phi(U)\to\mathbb{R}^{n-r}$$

our hypothesis then becomes

$$\left(\frac{\partial Q^j}{\partial x^i}(\mathbf{0},\mathbf{0})\right)_{\substack{i=1,\dots,r\\ j=1,\dots,r}},\text{ is non-singular}$$

Defining

$$\eta:\phi(U)\to\mathbb{R}^m,\text{ by }\eta(\mathbf{x},\mathbf{y})=\big(Q(\mathbf{x},\mathbf{y}),\mathbf{y}\big)$$

we have

$$D\eta(\mathbf{0},\mathbf{0})=\begin{bmatrix}\left(\frac{\partial Q^j}{\partial x^i}(\mathbf{0},\mathbf{0})\right)_{\substack{i=1,\dots,r\\ j=1,\dots,r}} & \left(\frac{\partial Q^i}{\partial y^j}(\mathbf{0},\mathbf{0})\right)_{\substack{i=1,\dots,r\\ j=1,\dots,m-r}} \\ O & I_{m-r}\end{bmatrix}$$

and so $D\eta(\mathbf{0},\mathbf{0})$ is non-singular, since its upper left and lower right sub-matrices are both non-singular, where the Inverse Function Theorem then says there are connected neighborhoods $U_{(\mathbf{0},\mathbf{0})}\subseteq\phi(U)$ containing $(\mathbf{0},\mathbf{0})$, and $\widetilde{U}_{(\mathbf{0},\mathbf{0})}\subseteq\eta(\phi(U))$ containing $\eta(\mathbf{0},\mathbf{0})=(\mathbf{0},\mathbf{0})$, such that

$$\eta|_{U_{(\mathbf{0},\mathbf{0})}}:U_{(\mathbf{0},\mathbf{0})}\to\widetilde{U}_{(\mathbf{0},\mathbf{0})}$$

is a diffeomorphism. Shrinking both $U_{(\mathbf{0},\mathbf{0})}$ and $\widetilde{U}_{(\mathbf{0},\mathbf{0})}$ if necessary we may assume that $\widetilde{U}_{(\mathbf{0},\mathbf{0})}\subseteq\mathbb{R}^m$ is a cube. Then, writing the inverse map of $\eta|_{U_{(\mathbf{0},\mathbf{0})}}$ as

$$\eta|^{-1}_{\widetilde{U}_{(\mathbf{0},\mathbf{0})}}(\mathbf{x},\mathbf{y})=\big(A(\mathbf{x},\mathbf{y}),B(\mathbf{x},\mathbf{y})\big),\text{ for }A:\widetilde{U}_{(\mathbf{0},\mathbf{0})}\to\mathbb{R}^r,\ B:\widetilde{U}_{(\mathbf{0},\mathbf{0})}\to\mathbb{R}^{m-r}$$

this implies that

$$
\begin{aligned}
(\mathbf{x},\mathbf{y}) &= \eta|_{U_{(\mathbf{0},\mathbf{0})}}\Big(\big(A(\mathbf{x},\mathbf{y}),B(\mathbf{x},\mathbf{y})\big)\Big)\\
&= \Big(Q\big(A(\mathbf{x},\mathbf{y}),B(\mathbf{x},\mathbf{y})\big),B(\mathbf{x},\mathbf{y})\Big)
\end{aligned}
$$

where we see that $\mathbf{y}=B(\mathbf{x},\mathbf{y})$ and so

$$\eta|^{-1}_{\widetilde{U}_{(\mathbf{0},\mathbf{0})}}(\mathbf{x},\mathbf{y}) = \big(A(\mathbf{x},\mathbf{y}),\mathbf{y}\big)$$

yet, we also have

$$\eta|_{U_{(\mathbf{0},\mathbf{0})}}\circ\eta|^{-1}_{\widetilde{U}_{(\mathbf{0},\mathbf{0})}} = Id_{\widetilde{U}_{(\mathbf{0},\mathbf{0})}} \implies \mathbf{x} = Q\big(A(\mathbf{x},\mathbf{y}),\mathbf{y}\big)$$

and therefore

$$
\begin{aligned}
\widehat{F}\circ\eta|^{-1}_{\widetilde{U}_{(\mathbf{0},\mathbf{0})}}(\mathbf{x},\mathbf{y}) &= \widehat{F}\circ\big(A(\mathbf{x},\mathbf{y}),\mathbf{y}\big)\\
&= \Big(Q\big(A(\mathbf{x},\mathbf{y}),\mathbf{y}\big),R\big(A(\mathbf{x},\mathbf{y}),\mathbf{y}\big)\Big)\\
&= \Big(\mathbf{x},R\big(A(\mathbf{x},\mathbf{y}),\mathbf{y}\big)\Big)
\end{aligned}
$$

Letting

$$\widetilde{R} = R\big(A(\mathbf{x},\mathbf{y}),\mathbf{y}\big)|_{\widetilde{U}_{(\mathbf{0},\mathbf{0})}} : \widetilde{U}_{(\mathbf{0},\mathbf{0})} \to \mathbb{R}^{n-r}$$

Then for any $(\mathbf{x},\mathbf{y})\in\widetilde{U}_{(\mathbf{0},\mathbf{0})}$ we have

$$D\Big(\widehat{F}\circ\eta|^{-1}_{\widetilde{U}_{(\mathbf{0},\mathbf{0})}}\Big)(\mathbf{x},\mathbf{y}) = \begin{bmatrix} I_r & O \\ \Big(\frac{\partial\widetilde{R}^i}{\partial x^j}(\mathbf{x},\mathbf{y})\Big)_{\substack{i=1,\dots,n-r\\ j=1,\dots,r}} & \Big(\frac{\partial\widetilde{R}^i}{\partial y^j}(\mathbf{x},\mathbf{y})\Big)_{\substack{i=1,\dots,n-r\\ j=1,\dots,n-r}} \end{bmatrix}$$

and since composition with a diffeomorphism does not change the rank of a map, this means Jacobian matrix of $\widehat{F}\circ\eta|^{-1}_{\widetilde{U}_{(\mathbf{0},\mathbf{0})}}$ has rank r throughout $\widetilde{U}_{(\mathbf{0},\mathbf{0})}$. Since I_r is linearly independent, this can only be the case if

$$\sum_{i,j=1}^{n-r}\frac{\partial\widetilde{R}^i}{\partial y^j}(\mathbf{x},\mathbf{y}) = O_{n-r\times n-r}$$

identically, that is $\widetilde{R}$, and hence $\widehat{F}|_{U_{(\mathbf{0},\mathbf{0})}}$ is independent of $\mathbf{y}=(y_1,\dots,y_{m-r})$, so setting

$$S(\mathbf{x}) = \widetilde{R}(\mathbf{x},\mathbf{0}) = R\big(A(\mathbf{x},\mathbf{0}),\mathbf{0}\big)|_{\widetilde{U}_{(\mathbf{0},\mathbf{0})}}$$

we get

$$\widehat{F}\circ\eta|^{-1}_{\widetilde{U}_{(\mathbf{0},\mathbf{0})}}(\mathbf{x},\mathbf{y}) = \big(\mathbf{x},S(\mathbf{x})\big)$$

Next, let $V_{(\mathbf{0},\mathbf{0})}\subseteq\psi(V)$ be defined by

$$V_{(\mathbf{0},\mathbf{0})} = \{(\mathbf{u},\mathbf{v})\in\psi(V) : (\mathbf{u},\mathbf{0})\in\widetilde{U}_{(\mathbf{0},\mathbf{0})}\}$$

and so $(\mathbf{0},\mathbf{0}) \in V_{(\mathbf{0},\mathbf{0})}$. Furthermore, since $\widehat{F} \circ \eta|^{-1}_{\widetilde{U}_{(\mathbf{0},\mathbf{0})}}$ is independent of $\mathbf{y}$ we have

$$\widehat{F} \circ \eta|^{-1}_{\widetilde{U}_{(\mathbf{0},\mathbf{0})}}(\widetilde{U}_{(\mathbf{0},\mathbf{0})}) = \widehat{F}(U_{(\mathbf{0},\mathbf{0})}) \subseteq V_{(\mathbf{0},\mathbf{0})}$$

so define

$$\xi : V_{(\mathbf{0},\mathbf{0})} \to \mathbb{R}^n, \text{ by } \xi(\mathbf{u},\mathbf{v}) = \big(\mathbf{u}, \mathbf{v} - S(\mathbf{u})\big)$$

which has an inverse given by

$$\xi|^{-1}_{\xi(V_{(\mathbf{0},\mathbf{0})})} : \xi(V_{(\mathbf{0},\mathbf{0})}) \to V_{(\mathbf{0},\mathbf{0})}, \text{ by } \xi|^{-1}_{\xi(V_{(\mathbf{0},\mathbf{0})})}(\mathbf{u},\mathbf{v}) = \big(\mathbf{u}, \mathbf{v} + S(\mathbf{u})\big)$$

and so is diffeomorphic onto its image, and hence $(V_{(\mathbf{0},\mathbf{0})}, \xi)$ is a smooth chart. And so we get

$$\begin{aligned}\xi \circ \widehat{F} \circ \eta|^{-1}_{\widetilde{U}_{(\mathbf{0},\mathbf{0})}}(\mathbf{x},\mathbf{y}) &= \xi\big(\mathbf{x}, S(\mathbf{x})\big) \\ &= \big(\mathbf{x}, S(\mathbf{x}) - S(\mathbf{x})\big) \\ &= (\mathbf{x}, \mathbf{0}) \\ &= (x_1, \ldots, x_r, 0, \ldots, 0)\end{aligned}$$

or

$$\xi \circ \widehat{F} \circ \eta|^{-1}_{\widetilde{U}_{(\mathbf{0},\mathbf{0})}} = \xi \circ (\psi \circ F \circ \phi^{-1}) \circ \eta|^{-1}_{\widetilde{U}_{(\mathbf{0},\mathbf{0})}} = (\xi \circ \psi) \circ F \circ (\eta \circ \phi)|^{-1}_{\widetilde{U}_{(\mathbf{0},\mathbf{0})}}$$

so letting $U_p = \phi^{-1}(U_{(\mathbf{0},\mathbf{0})})$, and $V_{F(p)} = \psi^{-1}(V_{(\mathbf{0},\mathbf{0})})$, we have the charts $(U_p, \eta \circ \phi|_{U_p}) \in \mathcal{A}_M$ containing p, and $(V_{F(p)}, \xi \circ \psi|_{V_{F(p)}}) \in \mathcal{A}_N$ containing $F(p)$ such that

$$\begin{aligned}&\widehat{F}(x_1, \ldots, x_r, x_{r+1}, \ldots, x_m) \\ &\quad = (\xi \circ \psi) \circ F \circ (\eta \circ \phi)|^{-1}_{\widetilde{U}_{(\mathbf{0},\mathbf{0})}}(x_1, \ldots, x_r, x_{r+1}, \ldots, x_m) \\ &\quad = (x_1, \ldots, x_r, 0, \ldots, 0)\end{aligned}$$

Since $p \in M$ was arbitrary, we conclude that for each point of M there exists charts such that

$$\widehat{F}(x_1, \ldots, x_r, x_{r+1}, \ldots, x_m) = (x_1, \ldots, x_r, 0, \ldots, 0)$$

$\square$

Corollary 57. Let M and N be smooth manifolds, let $F : M \to N$ be a smooth map, and suppose M is connected. Then the following are equivalent:

(a) For each $p \in M$ there exist smooth charts $(U, \phi) \in \mathcal{A}_M$ containing p, and $(V, \psi) \in \mathcal{A}_N$ containing $F(p)$ in which the coordinate representation of F, namely $\widehat{F}$, is linear.

(b) F has constant rank.

Proof. $(a) \implies (b)$

Suppose that $\widehat{F}$ is linear in a neighborhood of each point of M. Since all linear maps have constant rank, and since

$$dF|_p = \begin{bmatrix} \frac{\partial \widehat{F}^1}{\partial x^1}\big(\phi(p)\big) & \cdots & \frac{\partial \widehat{F}^1}{\partial x^m}\big(\phi(p)\big) \\ \vdots & \ddots & \vdots \\ \frac{\partial \widehat{F}^n}{\partial x^1}\big(\phi(p)\big) & \cdots & \frac{\partial \widehat{F}^n}{\partial x^m}\big(\phi(p)\big) \end{bmatrix}$$

we have $\operatorname{rank}(D\widehat{F}|_{\phi(p)}) = \operatorname{rank}(dF|_p)$, and so F, or dF, has constant rank in a neighborhood of each point of M. Thus, by the connectedness of M, dF has constant rank on M.

$(b) \implies (a)$

Now suppose that F, or dF, has constant rank. Then the Rank Theorem says that F has the coordinate representation

$$\widehat{F}(x_1, \ldots, x_r, x_{r+1}, \ldots, x_m) = (x_1, \ldots, x_r, 0, \ldots, 0)$$

in a neighborhood of each point, and this map is linear. □

Theorem 58 (Global Rank Theorem). Let M and N be smooth manifolds, and suppose $F : M \to N$ is a smooth map of constant rank. Then

(a) If F is surjective, then it is a smooth submersion.

(b) If F is injective, then it is a smooth immersion.

(c) If F is bijective, then it is a diffeomorphism.

Proof. Let $\dim(M) = m, \dim(N) = n$ and $\operatorname{rank}(dF) = r$.

(a) Suppose that

$$F : M \to N$$

is surjective, but that F is not a smooth submersion; i.e. $dF|_p$ is not surjective for each $p \in M$. Then $r < n$. Since F, or dF, has constant rank, the Rank Theorem says that for each $p \in M$, $\exists\ (U, \phi) \in \mathcal{A}_M$ centered at p, and $(V, \psi) \in \mathcal{A}_N$ centered at $F(p)$ such that $F(U) \subseteq V$, where the coordinate representation of F is given by

$$\widehat{F}(x_1, \ldots, x_r, x_{r+1}, \ldots, x_m) = (x_1, \ldots, x_r, 0, \ldots, 0)$$

by shrinking U if necessary we may assume that it is a coordinate ball and hence $\overline{U}$ is compact, and since the continuous image of a compact set is compact, $F(\overline{U}) \subseteq V$ is compact. Moreover it is the compact subset of

$$\{q \in V : y^{r+1}(q) = \cdots = y^n(q) = 0\}$$

that is the set of points in V, such that for the local coordinates of $\psi = (y^1, \ldots, y^n)$, the last $n - r$ coordinate functions each of which is the zero-mapping. That is

$$\psi\big(\{q \in V : y^{r+1}(q) = \cdots = y^n(q) = 0\}\big) \subseteq \mathbb{R}^r \times \{\mathbf{0}\}$$

which has empty interior as a subset of $\mathbb{R}^n$, and so is nowhere dense. And since ψ is a diffeomorphism, $\{q \in V : y^{r+1}(q) = \cdots = y^n(q) = 0\} \supseteq F(\overline{U})$ is an r-dimensional subset in N and hence contains no non-trivial; i.e. no non-empty open subsets. So $F(\overline{U})$ is closed in N and contains no open subset of N, and thus, is nowhere dense.

Then, since every open cover of a manifold has a countable subcover, we may choose countably many charts $\{(U_i, \phi_i)\}$ covering M, with corresponding charts $\{(V_i, \psi_i)\}$ covering $F(M)$ such that $F(\overline{U}_i) \subseteq V_i$ is compact, and nowhere dense in N for each i, then since

$$F(M) \subseteq \bigcup_i F(\overline{U}_i)$$

that is, $F(M)$ is contained in the countable union of nowhere dense sets, where the Baire Category Theorem then tells us that it too is nowhere dense, and so has empty interior in N, and thus $F(M) \neq N$ and so F is not surjective $\Rightarrow\Leftarrow$. Therefore F must be a smooth submersion.

(b) Suppose that

$$F : M \to N$$

is injective, but that F is not a smooth immersion; i.e. $dF|_p$ is not injective for each $p \in M$. Then $r < m$. Since F, or dF, has constant rank, the Rank Theorem says that for each $p \in M$, $\exists\ (U, \phi) \in \mathcal{A}_M$ centered at p, and $(V, \psi) \in \mathcal{A}_N$ centered at $F(p)$ such that $F(U) \subseteq V$, where the coordinate representation of F is given by

$$\widehat{F}(x_1, \ldots, x_r, x_{r+1}, \ldots, x_m) = (x_1, \ldots, x_r, 0, \ldots, 0)$$

in particular we have

$$\widehat{F}(0, \ldots, 0, x_{r+1}, \ldots, x_m) = (0, \ldots, 0)$$

so for ϵ sufficiently small, so that the point is still in our local chart, we have

$$\widehat{F}(0, \ldots, 0, \epsilon, \ldots, \epsilon) = (0, \ldots, 0) = \widehat{F}(\mathbf{0})$$

and so $\widehat{F}$ is not injective as a map between euclidean spaces, and hence F must also not be injective $\Rightarrow\Leftarrow$. And so F must be a smooth immersion.

(c) Suppose that

$$F : M \to N$$

is bijective. Then F is a surjective map with constant rank, and so (a) says F is a smooth submersion. Similarly F is an injective map with constant rank, and so (b) says F is a smooth immersion. And so we have a map which is both a smooth submersion and a smooth immersion where Proposition 54 F is a local diffeomorphism. Then we have a bijective local diffeomorphism and so by Proposition 53 (f) F is a diffeomorphism.

□

Theorem 59 (Local Immersion Theorem for Manifolds with Boundary). Suppose M is a smooth m-manifold with boundary, N is a smooth n-manifold, and

$$F : M \to N$$

is a smooth immersion. For any $p \in \partial M$; there exist a smooth boundary chart $(U, \phi) \in \mathcal{A}_M$ centered at p, and a smooth coordinate chart $(V, \psi) \in \mathcal{A}_N$ centered at $F(p)$ with $F(U) \subseteq V$, and where F has the coordinate representation

$$\widehat{F}(x_1, \dots, x_m) = (x_1, \dots, x_m, 0, \dots, 0)$$

Proposition 60. Every composition of smooth embeddings is a smooth embedding.

Proof. Let

$$\begin{aligned} F &: M \to N \\ G &: N \to P \end{aligned}$$

be smooth embeddings. Then both F and G are smooth immersions, and by Proposition 51, the composition of smooth immersions is a smooth immersion, and so

$$d(G \circ F)|_p = dG|_{F(p)} \circ dF|_p$$

is injective for each $p \in M$ and so is a smooth immersion.

Next since both F, G are smooth embeddings we have

$$\begin{aligned} F &: M \to F(M) \subseteq N \\ G &: N \to G(N) \subseteq P \end{aligned}$$

are both homeomorphisms. As $F(M) \subseteq N$ we also have

$$G|_{F(M)} : F(M) \to G(F(M))$$

is a homeomorphism, as the restriction of a homeomorphisms, which in turn implies

$$G \circ F : M \to G(F(M))$$

is a homeomorphism, as a composition of homeomorphisms, and thus

$$G \circ F : M \to P$$

is a homeomorphism onto its image, whose differential at each point in its domain is injective, and therefore is a smooth embedding. $\square$

Lemma 61 (**Dirichlet's Approximation Theorem**)**.** Given $r \in \mathbb{R}$ and any positive integer N, there exist integers n, m with

$$1 \leq n \leq N, \quad \text{such that} \quad |nr - m| < \frac{1}{N}$$

Proof. For any $x \in \mathbb{R}$, let

$$f(x) = x - \lfloor x \rfloor$$

then as the $N + 1$ numbers

$$\{f(0), f(r), f(2r), \dots, f(Nr)\} \in [0, 1)$$

by the pigeonhole principle $\exists\ i, j$ with $0 \leq i < j \leq N$ such that both $f(ir), f(jr)$ belong to the same sub-interval

$$\left[0, \frac{1}{N}\right), \left[\frac{1}{N}, \frac{2}{n}\right), \ldots, \left[\frac{N-1}{n}, 1\right)$$

which implies

$$|f(jr) - f(ir)| < \frac{1}{N}$$

so taking $n = j - i$, and $m = \lfloor jr \rfloor - \lfloor ir \rfloor$ we get

$$\big|(j-i)r - (\lfloor jr \rfloor - \lfloor ir \rfloor)\big| = \big|rj - \lfloor jr \rfloor - (ri - \lfloor ir \rfloor)\big| = |f(jr) - f(ir)| < \frac{1}{N}$$

□

Proposition 62. Suppose M and N are smooth manifolds with or without boundary, and

$$F : M \to N$$

is an injective smooth immersion. If any of the following holds, then F is a smooth embedding.

(a) F is an open or closed map.

(b) F is a proper map; i.e. the preimage of compact sets is compact.

(c) M is compact.

(d) M has empty boundary and $\dim(M) = \dim(N)$.

Proof.

(a) Suppose F is open, or closed, and injective. Since F is smooth, it is continuous, and so we have an open, or closed, injective continuous map and therefore $F : M \to F(M)$ is a homeomorphism onto its image, and thus a topological embedding.

We are given that $dF|_p$ is injective at each $p \in M$, and so we have a smooth immersion which is also a topological embedding, and thus F is a smooth embedding.

(b) Suppose F is proper, then since N is a manifold it is locally compact by Proposition 3, and is Hausdorff; i.e. is locally compact Hausdorff. Since F is smooth, and hence continuous, we have that F is a proper continuous map into a locally compact Hausdorff space, and therefore F is a closed map, and by (a), we have that F is a smooth embedding.

(c) Suppose that M is compact, then since M is Hausdorff, each closed subset $K \subseteq M$ is also compact. And since F is continuous, and the continuous image of a compact set is compact we have that $F(K) \subseteq N$ is compact, then since N is Hausdorff, we have that $F(K)$ must also be closed, and so F is a closed mapping. Where (a) tells us that F is a smooth embedding.

(d) Suppose that $\partial M = \varnothing$ and that $\dim(M) = \dim(N)$. Then, since dF is a smooth immersion

$$dF|_p : T_pM \to T_{F(p)}N$$

is injective for each $p \in M$, and also has full rank, and so $F(M) \subseteq \mathrm{Int}(N)$, and so

$$F : M \to \mathrm{Int}(N)$$

is a smooth immersion between manifolds without boundary, where Proposition 54 tells us that F is therefore a local diffeomorphism, and since every local diffeomorphism is an open map by Proposition 53, part (a) tells us that F is a smooth embedding. Therefore

$$\iota \circ F : M \to N$$

is the composition of smooth embeddings, and hence, is a smooth embedding.

$\square$

Theorem 63 (**Local Embedding Theorem**)**.** Suppose M and N are smooth manifolds with or without boundary, and

$$F : M \to N$$

is a smooth map. Then F is a smooth immersion iff every point in M has a neighborhood $U \subseteq M$ such that

$$F|_U : U \to N$$

is a smooth embedding.

Proof. First suppose that for every $p \in M$, $\exists\ U_p \subseteq M$ such that

$$F|_{U_p} : U_p \to N$$

is a smooth embedding, then $dF|_p$ has full rank for each $p \in M$, and hence is a smooth immersion.

Next, suppose that dF is a sooth immersion, and let $p \in M$ be given. Then by

The Rank Theorem,	$p \in \mathrm{Int}(M)$
Local Immersion Theorem for Manifolds with Boundary,	$p \in \partial M$

$\exists\ (U, \phi) \in \mathcal{A}_M$ centered at p, and $(V, \psi) \in \mathcal{A}_N$ centered at $F(p)$, such that $F(U) \subseteq V$, and F has coordinate representation

$$\widehat{F}(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0)$$

and so $\widehat{F}|_{\phi(U)}$ is injective as the inclusion map between euclidean spaces, and therefore $F|_U$ must be injective as well.

If however, $F(p) \in \partial N$, let $(W, \psi) \in \mathcal{A}_N$ be any smooth boundary chart centered at $F(p)$. Then defining $U_p = F^{-1}(W)$ which is a neighborhood containing p, Then by

The Rank Theorem,	$p \in \mathrm{Int}(M)$
Local Immersion Theorem for Manifolds with Boundary,	$p \in \partial M$

$\exists\ (U, \phi) \in \mathcal{A}_M$ centered at p where $U \subseteq U_p$, and with

$$\iota : \mathbb{H}^n \hookrightarrow \mathbb{R}^n$$

we have the chart $(W, \iota \circ \psi) \in \mathcal{A}_N$ centered at $F(p)$, and $U \subseteq U_p \implies F(U) \subseteq W$. Such that F has coordinate representation

$$\iota \circ \psi \circ F \circ \phi^{-1}(x_1, \ldots, x_m) = \widehat{F}(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0)$$

and so $\widehat{F}|_{\phi(U)}$ is injective as the inclusion in euclidean space, and therefore $F|_U$ must be as well. Thus, there exists a neighborhood U such that F is injective.

Now let $p \in M$ be given, and let U_p be a neighborhood of p for which F is injective. Since M admits a basis of precompact coordinate balls, choose B_p open and containing p, such that $\overline{B}_p \subseteq U_p$. Since B_p is precompact we have that $\overline{B}_p$ will be compact, and since F is smooth, and hence continuous and the continuous image of a compact set is compact, we will have that $F(\overline{B}_p) \subseteq N$ is compact, then since N is a topological manifold it is Hausdorff, and so $F(\overline{B}_p)$ must also be closed, and therefore

$$F|_{\overline{B}_p} : \overline{B}_p \to N$$

is a closed injective map and where $dF|_p$ is given to be injective at each $p \in M$, so by Proposition 62, is a smooth embedding. Then since $B_p \subseteq \overline{B}_p$ and the restriction of a smooth embedding is a smooth embedding we have

$$F|_{\overline{B}_p}\Big|_{B_p} : B_p \to N$$

is also a smooth embedding. $\square$

Theorem 64 (**Local Section Theorem**)**.** Suppose M and N are smooth manifolds, with $\dim(M) = m$ and $\dim(N) = n$, and that

$$\pi : M \to N$$

is a smooth map. Then π is a smooth submersion iff every $p \in M$ is in the image of a smooth local section of π.

Proof. First suppose that π is a smooth submersion. Given $p \in M$ so that $\pi(p) \in N$, and since π is a smooth submersion, we have $d\pi|_p$ is surjective $\forall\ p \in M$ and so π has full rank, so by the Rank Theorem there are charts $\big(U, \phi = (x^1, \ldots, x^m)\big) \in \mathcal{A}_M$ centered at p, and $\big(V, \psi = (y^1, \ldots, y^n)\big) \in \mathcal{A}_N$ centered at $\pi(p)$ such that $\pi(U) \subseteq V$ with coordinate representation

$$\psi \circ \pi \circ \phi^{-1} := \widehat{\pi} : \phi(U) \to \psi(V) \text{ by } \widehat{\pi}(x_1, \ldots, x_n, x_{n+1}, \ldots, x_m) = (x_1, \ldots, x_n)$$

If $\epsilon > 0$ is sufficiently small, the coordinate cube

$$C_\epsilon = \{\mathbf{x} : |x_i| < \epsilon \text{ for } 1 \le i \le m\}$$

is a neighborhood of $\phi(p)$ and so by making ϵ smaller if necessary we may assume that $C_\epsilon \subseteq \phi(U)$ so that we have

$$\widehat{\pi}(C_\epsilon) = C'_\epsilon = \{\mathbf{y} : |y_i| < \epsilon \text{ for } 1 \le i \le n\}$$

so defining the map σ, with coordinate representation

$$\widehat{\sigma} : C'_\epsilon \to C_\epsilon \text{ by } \sigma(x_1, \dots, x_n) = (x_1, \dots, x_n, 0, \dots, 0)$$

then for any $\mathbf{y} \in C'_\epsilon$ we have

$$(\widehat{\pi} \circ \widehat{\sigma})(\mathbf{y}) = \widehat{\pi}(y_1, \dots, y_n, 0, \dots, 0) = (y_1, \dots, y_n) = \mathbf{y}$$

so that in the coordinate representation $(\widehat{\pi} \circ \widehat{\sigma}) = Id_{C'_\epsilon}$ and hence $\pi \circ \sigma = Id_W$ for some $W \subseteq V$ such that $\psi(W) = C'_\epsilon$ and so

$$\sigma : \overbrace{\psi^{-1}(C'_\epsilon)}^{W} \to M$$

is a smooth local section of π.

Conversely, suppose that each $p \in M$ is in the image of a smooth local section. Given $p \in M$ let $U \subseteq N$ and

$$\sigma : U \to M$$

be a smooth local section such that $\sigma(\pi(p)) = p$. Then

$$\pi \circ \sigma = Id_U \implies d\pi|_{\sigma(\pi(p))} \circ d\sigma|_{\pi(p)} = Id_{T_{\pi(p)}N}$$

by Proposition 34 (c), so that

$$d\pi|_{\sigma(\pi(p))} = d\pi|_p$$

is surjective and since $p \in M$ was arbitrary we conclude π is a smooth submersion. $\square$

Proposition 65 (**Properties of Smooth Submersions**)**.** Let M and N be smooth manifolds, and suppose

$$\pi : M \to N$$

is a smooth submersion. Then

(a) π is an open map.

(b) If π is surjective, then it is a quotient map.

Proof.

(a) Given $U \subseteq M$ is an open subset, and let $q \in \pi(U) \subseteq N$. Since π is a smooth submersion, the Local Section Theorem tells us that each $p \in M$ belongs to the image of a smooth local section of π. So, in particular, for $p \in U$ such that $\pi(p) = q$, $\exists\ V_q \subseteq N$, open and containing q, such that

$$\sigma : V_q \to M$$

is a local section of π. That is

$$\pi \circ \sigma = Id_{V_q} \implies \pi(\sigma(q)) = q \implies \sigma(q) = p, \text{ for some } p \in U \text{ such that } \pi(p) = q$$

That is

$$p \in U \implies \sigma^{-1}(p) = q \in \sigma^{-1}(U)$$

Now, since σ is continuous, and $U \subseteq M$ is open, we know that $\sigma^{-1}(U) \subseteq N$ must also be open. Next, for each $r \in \sigma^{-1}(U) \subseteq N$ we have

$$\sigma(r) \in U \subseteq M \implies \pi(\sigma(r)) = r \in \pi(U)$$

thus $q \in \sigma^{-1}(U) \subseteq \pi(U) \subseteq N$. And so we have found an open neighborhood of q contained in $\pi(U)$.

And since $q \in \pi(U)$ was arbitrary, we conclude that each point in $\pi(U)$ has an open neighborhood contained in $\pi(U)$, therefore we can conclude that $\pi(U)$ is open.

Since $U \subseteq M$ was arbitrary, we may thus conclude that π maps open sets to opens sets, and hence is an open mapping.

(b) Let π be a surjective smooth submersion. Since π is smooth it is therefore continuous, and from (a) we know that π is an open mapping. And so we have that π is a continuous surjective open mapping, and thus, is a quotient map.

$\square$

Theorem 66 (**Characteristic Property of Surjective Smooth Submersions**)**.** Suppose M and N are smooth manifolds, and

$$\pi : M \to N$$

is a surjective smooth submersion. For any smooth manifold P with or without boundary, a map $F : N \to P$ is smooth if and only if $F \circ \pi$ is smooth

Proof. First suppose that F is smooth, then, since π is smooth, and the composition of smooth maps is smooth we have that

$$F \circ \pi : M \to P$$

is smooth.

Next suppose that

$$F \circ \pi : M \to P$$

is smooth, and let $q \in N$ be arbitrary. Since π is surjective, WLOG we may assume that $q = \pi(p)$.

Now since π is a smooth submersion, the Local Section Theorem says $\exists\, U_q \subseteq N$ such that

$$\sigma : U_q \to M$$

is a smooth local section of π, where $\pi \circ \sigma = Id_{U_q}$. Then, since the composition of smooth maps is smooth

$$F \circ \pi \circ \sigma = F \circ Id_{U_q} = F|_{U_q} : U_q \to F(U_q)$$

is smooth, so that F is smooth in a neighborhood of each point of N and thus, is smooth. $\square$

Theorem 67 (**Passing Smoothly to the Quotient**)**.** Suppose M and N are smooth manifolds and

$$\pi : M \to N$$

is a surjective smooth submersion. If P is a smooth manifold with or without boundary and

$$F : M \to P$$

is a smooth map that is constant on the fibers of π; that is, $\pi(q) = \pi(p) \implies F(q) = F(p)$, then there exists a unique smooth map

$$\widetilde{F} : N \to P$$

such that $F = \widetilde{F} \circ \pi$.

Proof. Since a surjective smooth submersion is a quotient map by Proposition 65, we have

$$\pi : M \to N$$

is a surjective quotient map. So, for each $q \in N$ simply select $p \in M$ such that $\pi(p) = q$, and define

$$\widetilde{F} : N \to P \text{ by } \widetilde{F}(q) = F(p)$$

Since

$$F : M \to P$$

is constant on the fibers of π; i.e.

$$\pi(p) = \pi(r) \in N \implies F(p) = F(r) \in P$$

this is independent of the choice of $p \in M$, and $\widetilde{F}$ is well defined. And we see

$$F = \widetilde{F} \circ \pi : M \to P$$

since F is smooth this implies $F = \widetilde{F} \circ \pi$ is. And by the Characteristic Property of Surjective Smooth Submersions,

$$\widetilde{F} \circ \pi : M \to P$$

being smooth implies $\widetilde{F} : N \to P$ is smooth.

To check uniqueness, suppose that there exists another smooth map

$$\widetilde{G} : N \to P \quad \text{such that} \quad F = \widetilde{G} \circ \pi$$

then for any $r \in N$, $\exists\, p \in M$ such that $\pi(p) = r$, and so

$$\widetilde{G}(r) = \widetilde{G}(\pi(p)) = F(p) = (\widetilde{F} \circ \pi)(p) = \widetilde{F}(\pi(p)) = \widetilde{F}(r)$$

and thus,

$$\widetilde{F} : N \to P$$

is the unique smooth map such that the smooth map

$$F : M \to P$$

has the form $F = \widetilde{F} \circ \pi$. $\square$

Theorem 68 (Uniqueness of Smooth Quotients). Suppose that M, N_1, and N_2 are smooth manifolds, and that

$$\pi_1 : M \to N_1 \quad \text{and} \quad \pi_2 : M \to N_2$$

are surjective smooth submersions that are constant on each other's fibers; i.e.

$$\pi_1(p) = \pi_1(q) \implies \pi_2(p) = \pi_2(q)$$

Then there exists a unique diffeomorphism

$$F : N_1 \to N_2$$

such that $F \circ \pi_1 = \pi_2$.

Proof. As π_1 is a surjective smooth submersion and π_2 is a smooth submersion and hence a smooth map constant on the fibers of π_1, the Passing Smoothly to the Quotient Theorem says $\exists$ a unique smooth map

$$\widetilde{\pi}_2 : N_1 \to N_2 \quad \text{such that} \quad \pi_2 = \widetilde{\pi}_2 \circ \pi_1$$

Similarly, since π_2 is a surjective smooth submersion and π_1 is a smooth submersion and hence a smooth map constant on the fibers of π_2 the Passing Smoothly to the Quotient Theorem says $\exists$ a unique smooth map

$$\widetilde{\pi}_1 : N_2 \to N_1 \quad \text{such that} \quad \pi_1 = \widetilde{\pi}_1 \circ \pi_2$$

Now, if we now consider π_2 as a surjective smooth submersion and a smooth map constant on its own fibers, the Passing Smoothly to the Quotient Theorem says $\exists$ a unique smooth map

$$\widetilde{\pi}_2' : N_2 \to N_2 \quad \text{such that} \quad \pi_2 = \widetilde{\pi}_2' \circ \pi_2$$

Yet,

$$Id_{N_2} : N_2 \to N_2$$

is a smooth map that has the form $\pi_2 = Id_{N_2} \circ \pi_2$. While we also have,

$$(\widetilde{\pi}_2 \circ \widetilde{\pi}_1) \circ \pi_2 = \widetilde{\pi}_2 \circ (\widetilde{\pi}_1 \circ \pi_2) = \widetilde{\pi}_2 \circ \pi_1 = \pi_2$$

where

$$\widetilde{\pi}_2 \circ \widetilde{\pi}_1 : N_2 \to N_2$$

is smooth, as the composition of smooth maps. So by the uniqueness of $\widetilde{\pi}_2'$ we must have

$$\widetilde{\pi}_2' = Id_{N_2} = \widetilde{\pi}_2 \circ \widetilde{\pi}_1$$

Likewise, if we consider π_1 as a surjective smooth submersion and a smooth map constant on its own fibers, the Passing Smoothly to the Quotient Theorem says $\exists$ a unique smooth map

$$\widetilde{\pi}_1' : N_1 \to N_1 \quad \text{such that} \quad \pi_1 = \widetilde{\pi}_1' \circ \pi_1$$

Yet,

$$Id_{N_1} : N_1 \to N_1$$

is a smooth map that has the form $\pi_1 = Id_{N_1} \circ \pi_1$. While we also have,

$$(\widetilde{\pi}_1 \circ \widetilde{\pi}_2) \circ \pi_1 = \widetilde{\pi}_1 \circ (\widetilde{\pi}_2 \circ \pi_1) = \widetilde{\pi}_1 \circ \pi_2 = \pi_1$$

where

$$\widetilde{\pi}_1 \circ \widetilde{\pi}_2 : N_1 \to N_1$$

is smooth, as the composition of smooth maps. So by the uniqueness of $\widetilde{\pi}_1'$ we must have

$$\widetilde{\pi}_1' = Id_{N_1} = \widetilde{\pi}_1 \circ \widetilde{\pi}_2$$

So if we set $F = \widetilde{\pi}_2$ we have that $\pi_2 = F \circ \pi_1$ which is unique by the Passing Smoothly to the Quotient Theorem, further F has a smooth inverse given by $\widetilde{\pi}_1$, since

$$F \circ \widetilde{\pi}_1 = Id_{N_2}, \quad \text{and} \quad \widetilde{\pi}_1 \circ F = Id_{N_1}$$

which allows us to conclude that

$$F : N_1 \to N_2$$

is a bi-smooth bijection and hence a diffeomorphism. □

Proposition 69 (Properties of Smooth Coverings).

(a) Every smooth covering map is a local diffeomorphism, a smooth submersion, an open map, and a quotient map.

(b) An injective smooth covering map is a diffeomorphism.

(c) A topological covering map is a smooth covering map if and only if it is a local diffeomorphism.

Proof.

(a) Let

$$\pi : E \to M$$

be a smooth covering map, and let $p \in M$ be given. Since π is surjective we may assume that $\pi(q) = p$ for some $q \in E$. Since π is a smooth covering $\exists\, U_p \subseteq M$ open and containing p, such that $q \in V_q \subseteq \pi^{-1}(U_p) \subseteq E$ where $V_q \subseteq E$ is the component of $\pi^{-1}(U_p)$ containing q, and since π is a smooth covering map this tells us that

$$\pi|_{V_q} : V_q \to U_p$$

is a diffeomorphism. As, $p \in M$, and $q \in E$ such that $\pi(q) = p$, were arbitrary we conclude that each component of E containing $\pi^{-1}(p)$, has a neighborhood diffeomorphic to U_p. Thus, π is a local diffeomorphism.

Since π is a local diffeomorphism, Proposition 54 (a) says that π is also a smooth submersion.

Next as π is a local diffeomorphism, Proposition 53 (c) says that π is an open map.

Finally, since π is a surjective smooth submersion, it is a quotient map by 65.

(b) Let

$$\pi : E \to M$$

be an injective smooth covering map, since all covering maps are surjective, π is bijective. Then, from (a) we also have that π is a local diffeomorphism, so by Proposition 53 (f) π is a diffeomorphism.

(c) Let

$$\pi : E \to M$$

be a smooth covering map, and hence a topological covering map. Then by (a) π is a local diffeomorphism.

Next, let

$$\pi : E \to M$$

be a topological covering map that is also a local diffeomorphism. Let $p \in M$ be arbitrary and since π is a topological covering $\exists\ U_p \subseteq M$ open and containing p, that is evenly covered. Let $\pi^{-1}(U_p) = \bigsqcup_{i \in I} V_i$ where $V_i \subseteq E$ is a component of $\pi^{-1}(U_p)$ for each $i \in I$. Then

$$\pi|_{V_i} : V_i \to U_p$$

is a homeomorphism for each $i \in I$ since π is a topological cover. And hence each

$$\pi|_{V_i} : V_i \to U_p$$

is bijective for each $i \in I$, and since π is also a local diffeomorphism, Proposition 53 (f) tells us that

$$\pi|_{V_i} : V_i \to U_p$$

is a diffeomorphism for each $i \in I$.

Thus, we have each component of $\pi^{-1}(U_p)$ is mapped diffeomorphically to U_p.

Since $p \in M$ was arbitrary we conclude for each point of M there exists a neighborhood U, such that each component of $\pi^{-1}(U) \subseteq E$ is mapped diffeomorphically onto U, and therefore π is a smooth covering map.

$\square$

Theorem 70 (**Local Section Theorem for Smooth Covering Maps**)**.** Suppose E and M are smooth manifolds with or without boundary, and

$$\pi : E \to M$$

is a smooth covering map. Given any evenly covered open subset $U \subseteq M$, any $q \in U$, and any p in the fiber of π over q; i.e. $p \in \pi^{-1}(q)$, there exists a unique smooth local section

$$\sigma : U \to E$$

such that $\sigma(q) = p$.

Proof. Suppose $q \in U_q \subseteq M$ is evenly covered, and $p \in \pi^{-1}(q)$. Let $V_p \subseteq E$ be the component of $\pi^{-1}(U_q)$ containing p. Since π is a smooth covering

$$\pi|_{V_p} : V_p \to U_q$$

is a diffeomorphism. Thus $\pi|_{V_p}^{-1}$ is both a smooth left and right inverse. So let

$$\sigma = \pi|_{V_p}^{-1} : U_q \to V_p$$

and since σ is injective, we get

$$\sigma(q) = \pi|_{V_p}^{-1}(q) = p$$

Next suppose

$$\sigma' : U_q \to E$$

is another smooth local section satisfying $\sigma'(q) = p$. Since U_q is connected, and σ' is continuous, and since the continuous image of a connected set is connected we have $\sigma'(U_q) \subset E$ is connect. Yet, since $p \in V_p$, and any every connected subset of the connected space E, must be contained in a single connected component we must have $\sigma'(U_q) \subseteq V_p$.

Then, since σ' is a local section $\pi|_{V_p} \circ \sigma' = Id_{U_q}$, and as $\pi|_{V_p}$ is bijective, inverses are unique, giving

$$\pi|_{V_p} \circ \sigma' = Id_{U_q} = \pi|_{V_p} \circ \sigma$$

and so $\sigma' = \sigma$ and is the unique smooth local section

$$\sigma : U \to E$$

such that $\sigma(q) = p$. □

Proposition 71. Let

$$\pi : E \to M$$

be a smooth covering map, then every local section of π is smooth.

Proof. Let open $U \subseteq M$ be given, and let

$$\sigma : U \to E$$

be a local section section of π. Since π is a smooth covering map, $\forall\ p \in U$, $\exists\ U_p \subseteq U$ open and containing p, which is evenly covered by π. Then, by the Local Section Theorem for Smooth Covering Maps $\exists$ a unique smooth local section

$$\tau : U_p \to E$$

of π and by the uniqueness of τ this implies $\tau(p) = \sigma(p)$. Which is to say

$$\pi \circ \sigma|_{U_p} = Id_{U_p} = \pi \circ \tau$$

Yet, since π is bijective when restricted to each of its components in $\pi^{-1}(U_p)$, we have that inverses are unique and hence

$$\sigma|_{U_p} = \tau$$

and so $\sigma|_{U_p}$ must be smooth.

Since $p \in U$ was arbitrary, we conclude that each point of U has a neighborhood such that the restriction of σ to that neighborhood is smooth. Where the Smoothness is Local Lemma then tells us that since each point of U has a neighborhood V such that $\sigma|_V$ is smooth, then σ is smooth. $\square$

Proposition 72 (**Covering Spaces of Smooth Manifolds**)**.** Suppose M is a connected smooth n-manifold, and

$$\pi : E \to M$$

is a topological covering map. Then E is a topological n-manifold, and has a unique smooth structure such that π is a smooth covering map.

Proposition 73 (**Covering Spaces of Smooth Manifolds with Boundary**)**.** Suppose M is a connected smooth n-manifold with boundary, and

$$\pi : E \to M$$

is a topological covering map. Then E is a topological n-manifold with boundary such that

$$\partial E = \pi^{-1}(\partial M)$$

and it has a unique smooth structure such that π is a smooth covering map.

Corollary 74 (**Existence of a Universal Covering Manifold**)**.** If M is a connected smooth manifold, there exists a simply connected smooth manifold $\widetilde{M}$, called the universal covering manifold of M, and a smooth covering map

$$\pi : \widetilde{M} \to M$$

The universal covering manifold is unique in the following sense: if $\widetilde{M}'$ is any other simply connected smooth manifold that admits a smooth covering map

$$\pi' : \widetilde{M}' \to M$$

then there exists a diffeomorphism

$$\Phi : \widetilde{M} \to \widetilde{M}'$$

such that $\pi' \circ \Phi = \pi$.

Proposition 75. Suppose E and M are nonempty connected smooth manifolds with or without boundary. If

$$\pi : E \to M$$

is a proper local diffeomorphism, then π is a smooth covering map.

Proposition 76 (**Open Submanifolds**)**.** Suppose M is a smooth manifold. The embedded submanifolds of codimension-0 in M are exactly the open submanifolds.

Proof. Suppose that $U \subseteq M$ is an open submanifold. Then U is a smooth manifold with $\dim(U) = \dim(M)$, and thus

$$\dim(M) - \dim(U) = 0$$

and U has codimension 0. For the smooth map

$$\iota : U \hookrightarrow M$$

and any chart $(U \cap V, \phi|_{U \cap V}) \in \mathcal{A}_U$, which is determined by the relative topology of U, has the corresponding chart $(U, \phi) \in \mathcal{A}_M$. So, have the coordinate representation of ι

$$\phi \circ \iota \circ \phi^{-1}|_{\phi(U \cap V)} : \phi(U \cap V) \to \phi(U)$$

is given by

$$\phi \circ \iota(\phi^{-1}(\mathbf{x})) = \phi(\phi^{-1}(\mathbf{x})) = \mathbf{x}$$

and is locally the identity map between euclidean spaces, and so has full rank, and since $d\iota|_p$ is represented by the Jacobian matrix of $\widehat{\iota}$ at $\phi(p)$, and $\dim(U) = \dim(M)$ we have that

$$d\iota|_p : T_pU \to T_pM$$

is injective at each $p \in U$, and so ι, or $d\iota$, is a smooth immersion. Furthermore

$$\iota : U \to \iota(U) = U$$

is a homeomorphism, and so ι is a smooth embedding. Thus, U is a subset of M that is a manifold in the relative topology endowed with a smooth structure to which ι is a smooth embedding, and therefore, U is an embedded submanifold.

Next suppose that U is any embedded submanifold of codimension 0. Then, since U is an embedded submanifold

$$\iota : U \hookrightarrow M$$

is a smooth embedding, and hence a smooth immersion. And since U has codimension 0 this implies that $\dim(U) = \dim(M)$ where Proposition 54 then says that ι is a local diffeomorphism, and by Proposition 53, local diffeomorphisms are open maps. Then since $U \subseteq U$ is open in U, we thus have $\iota(U) = U \subseteq M$ must be open. $\square$

Proposition 77 (**Images of Embeddings as Submanifolds**). Suppose N is a smooth manifold with or without boundary, M is a smooth manifold, and

$$F : M \to N$$

is a smooth embedding. Let $S = F(M)$ with the subspace topology, S is a topological manifold, and it has a unique smooth structure making it into an embedded submanifold of N with the property that F is a diffeomorphism onto its image.

Proof. Giving $F(M)$ the relative topology inherited from N, and since F is a smooth embedding, it is a topological embedding, and thus

$$F : M \to F(M)$$

is a homeomorphism, and therefore $F(M)$ is a topological manifold.

Next we give $F(M)$ a smooth structure be defining its charts to be $\big\{\big(F(U), \phi \circ F^{-1}\big) : (U, \phi) \in \mathcal{A}_M\big\}$.

Then for any $(U,\phi),(V,\psi)\in\mathcal{A}_M$ we have that charts $\big(F(U),\phi\circ F^{-1}\big),\big(F(V),\psi\circ F^{-1}\big)$ where we get

$$\phi\circ F^{-1}\circ\big(\psi\circ F^{-1}\big)^{-1}=\phi\circ F^{-1}\circ F\circ\psi^{-1}=\phi\circ\psi^{-1}$$

and so the charts for $F(M)$ are smoothly compatible since the charts of $\mathcal{A}_M$ are, and so determine a smooth structure $\mathcal{A}_{F(M)}$. Further, since F is an embedding it is smooth, and bijective onto its image, and so F^{-1} exists, so we have

$$F^{-1}|_{F(M)}:F(M)\to M$$

and for any $(\xi,W)\in\mathcal{A}_N$ we have the coordinate representation of F^{-1}

$$\phi\circ F^{-1}\circ\xi^{-1}=(\phi\circ F^{-1})\circ\xi^{-1}$$

which is the composition of smooth maps between euclidean spaces, and so is smooth, and thus, F^{-1} must also be smooth, and therefore we can conclude that

$$F:M\to F(M)$$

is a diffeomorphism. And since any other smooth structure that makes F a diffeomorphism onto its image will determine the same unique maximal smooth atlas and hence smooth structure. And for the inclusion we have

$$\iota=F\circ F^{-1}|_{F(M)}:F(M)\hookrightarrow N$$

which is the composition of a diffeomorphism and a smooth embedding and is hence a smooth embedding.

Thus we have $F(M)\subseteq N$ is a subset, that is a manifold in the relative topology inherited from N, and endowed with a smooth structure to which ι is a smooth embedding. Therefore, we have that $F(M)$ is an embedded submanifold. □

Note: this shows that embedded submanifolds are exactly the images of smooth embeddings

Proposition 78 (**Slices of Product Manifolds**)**.** Suppose M and N are smooth manifolds. For each $q\in N$, the subset (slice)

$$M\times\{q\}\subseteq M\times N$$

is an embedded submanifold of $M\times N$ diffeomorphic to M.

Proof. Define the map

$$\iota_M:M\to M\times N,\ \text{ by }\iota_M(p)=(p,q)$$

which is a smooth embedding, and so

$$\iota_M(M)=M\times\{q\}$$

is the image of a smooth embedding and is hence, an embedded submanifold. □

Proposition 79 (**Graphs as Submanifolds**)**.** Suppose M is a smooth m-manifold without boundary, N is a smooth n-manifold with or without boundary, $U\subseteq M$ is open, and

$$f:U\to N$$

is a smooth map. Let $\Gamma(f) \subseteq M \times N$ denote the graph of f :

$$\Gamma(f) = \{(p, f(p)) \in M \times N : p \in U\}$$

Then $\Gamma(f)$ is an embedded m-dimensional submanifold of $M \times N$.

Proof. Define the map

$$\gamma_f : U \to M \times N, \text{ by } \gamma_f(p) = (p, f(p))$$

which is smooth, since it is smooth in each of its components; i.e.

$$\gamma_f = Id_M \times f$$

both of which are smooth. Then for the smooth projection map given by

$$\pi_M : M \times N \to M$$

we note that for each $p \in U$ we have

$$\pi_M \circ \gamma_f(p) = \pi_M(p, f(p)) = p \implies \pi_M \circ \gamma_f = Id_U$$

and so be Proposition 34 we have

$$d\pi_M|_{(p,f(p))} \circ d\gamma_f|_p = d(\pi|_M \circ \gamma_f)|_p = d(Id_U)|_p = Id_{T_pU} \cong Id_{T_pM}$$

and thus, $d\gamma_f|_p$ is injective for each $p \in U$, so $d\gamma_f$ is a smooth immersion.

Next we note that for each $(q, f(q)) \in \Gamma(f)$ we have

$$\gamma_f \circ \pi_M\big|_{\Gamma(f)}(q, f(q)) = \gamma_f(q) = (q, f(q)) \implies \gamma_f \circ \pi_M\big|_{\Gamma(f)} = Id_{\Gamma(f)}$$

and so γ_f has an explicit inverse, and thus is homeomorphic onto it's image that is

$$\gamma_f : U \to \gamma_f(U)$$

is a homeomorphism, which is also a smooth immersion, and therefore is a smooth embedding. Hence,

$$\gamma_f(U) = \Gamma(f)$$

is the image of a smooth embedding and is therefore an embedded submanifold diffeomorphic to its domain, namely U. $\square$

Proposition 80. Suppose M is a smooth manifold with or without boundary and $S \subseteq M$ is an embedded submanifold. Then S is properly embedded if and only if it is a closed subset of M.

Proof. First suppose that $S \subseteq M$ is properly embedded. Then, since M is a manifold, it is a locally compact Hausdorff space. And since every proper continuous map into a locally compact Hausdorff space is a closed map, and the inclusion is a continuous map assumed to be proper, and $S \subseteq S$ is closed in S, we must have $\iota(S) = S \subseteq M$ is closed.

Next suppose that $S \subseteq M$ is closed. Since S is an embedded submanifold we have that

$$\iota : S \hookrightarrow M$$

is a smooth embedding, and hence a topological embedding. Then since $\iota(S) = S$ we have that ι is a topological embedding with closed image, and therefore ι is a proper map. And thus, $S \subseteq M$ is properly embedded. $\square$

Corollary 81. If $S \subseteq M$ is an embedded submanifold that is compact, then it is properly embedded.

Proof. Since $S \subseteq M$ is compact, and M is a manifold and therefore Hausdorff, we have that S is a compact set in a Hausdorff space, and is therefore closed. And since S is an embedded submanifold which is closed in M, it is properly embedded. □

Proposition 82 (Global Graphs Are Properly Embedded). Suppose M is a smooth manifold, N is a smooth manifold with or without boundary, and

$$f : M \to N$$

is a smooth map. Let $\Gamma(f) \subseteq M \times N$ denote the graph of f :

$$\Gamma(f) = \{(p, f(p)) \in M \times N : p \in M\}$$

Then $\Gamma(f)$ is properly embedded in $M \times N$.

Proof. Define the map

$$\gamma_f : M \to M \times N, \text{ by } \gamma_f(p) = (p, f(p))$$

which is smooth, since it is smooth in each of its components; i.e.

$$\gamma_f = Id_M \times f$$

both of which are smooth. Then for the smooth projection map given by

$$\pi_M : M \times N \to M$$

we note that for each $p \in M$ we have

$$\pi_M \circ \gamma_f(p) = \pi_M(p, f(p)) = p \implies \pi_M \circ \gamma_f = Id_M$$

and so be Proposition 34 we have

$$d\pi_M|_{(p,f(p))} \circ d\gamma_f|_p = d(\pi|_M \circ \gamma_f)|_p = d(Id_M)|_p = Id_{T_pM}$$

and thus, $d\gamma_f|_p$ is injective for each $p \in M$, so $d\gamma_f$ is a smooth immersion.

Next we note that for each $(q, f(q)) \in \Gamma(f)$ we have

$$\gamma_f \circ \pi_M\big|_{\Gamma(f)}(q, f(q)) = \gamma_f(q) = (q, f(q)) \implies \gamma_f \circ \pi_M\big|_{\Gamma(f)} = Id_{\Gamma(f)}$$

and so γ_f has an explicit inverse, and thus is homeomorphic onto it's image that is

$$\gamma_f : M \to \gamma_f(M)$$

is a homeomorphism, which is also a smooth immersion, and therefore is a smooth embedding.

Next, we note that since γ_f is continuous, and $M \times N$ is a manifold, and is therefore Hausdorff, and π_M is a continuous left inverse. And since continuous maps into Hausdorff spaces with continuous left inverses are proper, we have that γ_f is a proper map.

Hence,

$$\gamma_f(M) = \Gamma(f)$$

is the image of a smooth proper embedding and is therefore a properly embedded submanifold diffeomorphic to its domain, namely M. $\square$

Theorem 83 (**Local Slice Criterion for Embedded Submanifolds**). Let M be a smooth n-manifold. If $S \subseteq M$ is an embedded k-dimensional submanifold, then S satisfies the local k-slice condition. Conversely, if $S \subseteq M$ is a subset that satisfies the local k-slice condition, then with the subspace topology, S is a topological k-manifold, and it has a smooth structure making it into a k-dimensional embedded submanifold of M.

Proof. First suppose that $S \subseteq M$ is an embedded k-dimensional submanifold. Then the inclusion

$$\iota : S \hookrightarrow M$$

is a smooth embedding, and hence a smooth immersion, and thus, has full rank. Where the Rank Theorem then says that for each $p \in S$ we map find charts $(U, \phi) \in \mathcal{A}_S$ centered at p, and $(V, \psi) \in \mathcal{A}_M$ centered at $\iota(p) = p$, such that $\iota(U) \subseteq V$ and ι has coordinate representation

$$\widehat{\iota}(x_1, \ldots, x_k) = (x_1, \ldots, x_k, 0, \ldots, 0)$$

select $\epsilon, \epsilon' > 0$ so that $B_{\epsilon'}(\phi(p)) \subseteq \phi(U)$, $B_\epsilon(\psi(p)) \subseteq \psi(V)$ and $B_{\epsilon'}(\phi(p)) \subseteq B_\epsilon(\psi(p))$ then

$$\begin{aligned} U_p &:= \phi^{-1}\big(B_{\epsilon'}(\phi(p))\big) \subseteq U \\ V_p &:= \psi^{-1}\big(B_{\epsilon}(\psi(p))\big) \subseteq V \end{aligned}$$

are coordinate balls in U and V respectively such that $U_p \subseteq V_p$. Then, as S has the relative topology inherited from M, $U_p \subseteq S$ open, implies $\exists\ W \subseteq M$ open and containing p, such that

$$U_p = S \cap W$$

so setting $V_p' = V_p \cap W$ which is open as the finite intersection of open sets, we then have

$$\begin{aligned} V_p' \cap S &= V_p \cap W \cap S \\ &= V_p \cap (W \cap S) \\ &= V_p \cap U_p \\ &= U_p \end{aligned}$$

and so $(V_p', \psi|_{V_p'}) \in \mathcal{A}_M$ is a smooth chart containing p, such that $V_p' \cap S = U_p$ is a single k-slice in V_p'. And since this is possible for each $p \in S$, S satisfies the local k-slice condition.

Next suppose that S is a subset satisfying the local k-slice condition, and has the relative topology inherited from M. Then S also inherits Hausdorffness and second countablility as these are inherited by subspaces.

next let
$$\pi^k : \mathbb{R}^n \to \mathbb{R}^k, \text{ by } \pi^k(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) = (x_1, \ldots, x_k)$$
be the projection onto the first k coordinates, and let $(U, \phi) \in \mathcal{A}_M$ be any slice chart for S in M. And set $V = U \cap S$, and since (U, ϕ) is a slice chart, we have that

$$\phi(V) = \phi(U) \cap A, \quad \text{with } A \subset \mathbb{R}^n \text{ such that } \{x_{k+1} = c_{k+1}, \ldots, x_n = c_n\}$$

and therefore $\phi(V) \subseteq A$ is open, in the relative topology of A as $\phi(U) \subseteq \mathbb{R}^n$ is open, and

$$\pi^k|_A : A \to \mathbb{R}^k$$

is a diffeomorphism, and since diffeomorphism are open mappings by Proposition 20 (c), we have $\pi^k(\phi(V)) \subseteq \mathbb{R}^k$ is open. Moreover the mapping

$$\pi^k \circ \phi|_V : V \to \pi^k(\phi(V))$$

has a continuous inverse given by

$$\phi^{-1} \circ \iota^k|_{\pi^k(\phi(V))} : \pi^k(\phi(V)) \to V$$

where
$$\iota^k : \mathbb{R}^k \to \mathbb{R}^n, \text{ by } \iota^k(x_1, \ldots, x_k) = (x_1, \ldots, x_k, c_{k+1}, \ldots, c_n)$$
is in fact smooth. So, for any $\mathbf{x} \in \pi^k(\phi(V))$ we have

$$\begin{aligned}
(\pi^k \circ \phi) \circ (\phi^{-1} \circ \iota^k)(x_1, \ldots, x_k) &= \pi^k \circ \iota^k(x_1, \ldots, x_k) \\
&= \pi^k(x_1, \ldots, x_k, c_{k+1}, \ldots, c_n) \\
&= (x_1, \ldots, x_k) \\
&= \mathbf{x}
\end{aligned}$$

similarly for any $p \in V$ we have

$$\begin{aligned}
(\phi^{-1} \circ \iota^k) \circ (\pi^k \circ \phi)(p) &= \phi^{-1} \circ \iota^k \circ \pi^k(x^1(p), \ldots, x^k(p), c_{k+1}, \ldots c_n) \\
&= \phi^{-1} \circ \iota^k(x^1(p), \ldots, x^k(p)) \\
&= \phi^{-1}(x^1(p), \ldots, x^k(p), c_{k+1}, \ldots c_n) \\
&= p
\end{aligned}$$

and so S is locally euclidean of dimension k.

That is, S is second countable, Hausdorff, and locally euclidean of dimension k, and is therefore a topological k-manifold. And the inclusion map

$$\iota : S \hookrightarrow M$$

is a homeomorphism onto its image, and so is a topological embedding.

To see that S has a smooth structure, let $(U, \phi), (V, \psi) \in \mathcal{A}_M$ we any two slice charts for S in

M with nonempty intersection, and let $(U\cap S, \pi^k\circ\phi|_{U\cap S}), (V\cap S, \pi^k\circ\psi|_{V\cap S})$ be the corresponding charts in S. Then the transition map

$$(\pi^k\circ\phi)\circ(\pi^k\circ\psi)^{-1}|_{\pi^k(\psi(V\cap U\cap S))}:\pi^k(\psi(V\cap U\cap S))\to\pi^k(\phi(V\cap U\cap S))$$

is given by

$$(\pi^k\circ\phi)\circ(\pi^k\circ\psi)^{-1}=\pi^k\circ\phi\circ\psi^{-1}\circ\iota^k$$

which is the composition of smooth maps, and therefore is smooth. And thus, the charts $\{(U\cap S, \pi^k\circ\phi|_{U\cap S})\}$ are all smoothly compatible, and so define a smooth structure $\mathcal{A}_S$ on S.

Finally, for the slice chart $(U,\phi)\in\mathcal{A}_M$ for S in M, with corresponding chart $\{(U\cap S,\pi^k\circ\phi|_{U\cap S})\}\in\mathcal{A}_S$, ι has coordinate representation

$$\psi\circ\iota\circ\phi^{-1}\circ\iota^k|_{\pi^k(\phi(U\cap S))}:\pi^k(\phi(U\cap S))\to\phi(U)$$

given by

$$\begin{aligned}\psi\circ\iota\circ\phi^{-1}\circ\iota^k(x_1,\dots,x_k)&=\psi\circ\phi^{-1}\circ\iota^k(x_1,\dots,x_k)\\&=\iota^k(x_1,\dots,x_k)\\&=(x_1,\dots,x_k,c_{k+1},\dots,c_n)\end{aligned}$$

which is smooth as a map between euclidean spaces, and thus, so is ι. Further, we see that the coordinate representation for ι has full rank, and since $\dim(S)\leq\dim(M)$ full rank implies that ι, or $d\iota$, is a smooth immersion.

Then as ι is a smooth immersion, which is also a topological embedding, we thus conclude that ι is a smooth embedding.

And so we have that $S\subseteq M$ is a subset of M that is a manifold in the relative topology, endowed with a smooth structure to which ι is a smooth embedding, and therefore, S is an embedded submanifold. □

Theorem 84. If M is a smooth n-manifold with boundary, then with the subspace topology, ∂M is a topological $(n-1)$-dimensional manifold (without boundary), and has a smooth structure such that it is a properly embedded submanifold of M.

Theorem 85 (Constant-Rank Level Set Theorem). Let M and N be smooth manifolds, and let

$$F:M\to N$$

be a smooth map with constant rank r. Each level set of F is a properly embedded submanifold of codimension r in M, or equivalently each level set of F is a properly embedded $(m-r)$-dimensional submanifold.

Proof. Let $\dim(M)=m$ and $\dim(N)=n$, and let $q\in F(M)\subseteq N$ be arbitrary, we wish to consider $F^{-1}(q)\subseteq M$. As F has constant rank, the Rank Theorem says $\forall\ p\in F^{-1}(q)$, $\exists\ (U,\phi)\in\mathcal{A}_M$ centered at p, and $(V,\psi)\in\mathcal{A}_N$ centered at $F(p)=q$, such that $F(U)\subseteq V$ and where F has coordinate representation

$$\widehat{F}(x_1,\dots,x_r,x_{r+1},\dots,x_m)=(x_1,\dots,x_r,0,\dots,0)$$

and so for the restriction to $F^{-1}(q) \cap U \subseteq U$ we have

$$\psi \circ F \circ \phi^{-1}|_{\phi(F^{-1}(q)\cap U)} : \phi(F^{-1}(q) \cap U) \to \psi(V)$$

is given by

$$\begin{aligned} \psi \circ F \circ \phi^{-1}(\mathbf{x}) &= \psi \circ F(\phi^{-1}(\mathbf{x})) \\ &= \psi(q) \\ &= (c_1, \ldots, c_r, 0 \ldots, 0) \end{aligned}$$

for some $c_1, \ldots, c_r \in \mathbb{R}$. And this holds for all $\mathbf{x} \in \phi(F^{-1}(q) \cap U)$ and therefore

$$F^{-1}(q) \cap U = \{p \in U : x^1(p) = c_1, \ldots, x^r(p) = c_r\}$$

is an $(m-r)$-slice in U.

Then as $p \in F^{-1}(q)$ was arbitrary, we conclude that each point in the preimage of q under F has a chart $(U, \phi) \in \mathcal{A}_M$ such that $F^{-1}(q) \cap U$ is a single $(m-r)$-slice in U, and therefore $F^{-1}(q) \subseteq M$ satisfies the local $(m-r)$-slice condition. Thus, by Theorem 83, $F^{-1}(q) \subseteq M$, with the relative topology, is an embedded submanifold of dimension $m - r$. And hence has codimension

$$\dim(M) - \dim(F^{-1}(q)) = m - (m - r) = r$$

Then as the singleton $\{q\} \subset N$ is closed, since N is Hausdorff, where the continuity of F then implies that $F^{-1}(q) \subseteq M$ is closed, and therefore $F^{-1}(q)$ is a closed embedded submanifold, and hence is properly embedded by Proposition 80. □

Corollary 86 (**Submersion Level Set Theorem**)**.** If M and N are smooth manifolds and

$$F : M \to N$$

is a smooth submersion, then each level set of F is a properly embedded submanifold whose codimension is equal to the dimension of N.

Proof. Since F is a smooth submersion, $dF|_p$ is surjective for each $p \in M$, and so dF has full rank with $\operatorname{rank}(dF) = \dim(N) = n$, and so the Constant-Rank Level Set Theorem then says that preimage of each point of N under F is a properly embedded $(m-n)$-dimensional submanifold of M, and so has codimension n. □

Corollary 87 (**Regular Level Set Theorem**)**.** Every regular level set of a smooth map between smooth manifolds is a properly embedded submanifold whose codimension is equal to the dimension of the codomain.

Proof. Let

$$F : M \to N$$

be a smooth map, and let $q \in F(M) \subseteq N$ be a regular value of F, then for each $p \in F^{-1}(q) \subseteq M$ we have

$$dF|_p : T_pM \to T_{F(p)}N$$

is surjective. Next, if $U \subseteq M$ is the set of all points $p \in M$ such that

$$dF|_p : T_pM \to T_{F(p)}N$$

is surjective, then U is open since a smooth submersion in an open condition by Proposition 50. Further, we have that $U \supseteq F^{-1}(q)$, and by construction

$$F|_U : U \to N$$

is a smooth submersion, where the Submersion Level Set Theorem then implies that $F^{-1}(q) \subseteq U$ is a properly embedded submanifold of codimension

$$\dim(U) - \dim(F^{-1}(q)) = m = (m-n) = n$$

then as ι is a smooth embedding, and the composition of smooth embeddings is a smooth embedding we have

$$\iota_U \circ \iota_{F^{-1}(q)} : F^{-1}(q) \hookrightarrow M$$

is a smooth embedding, and thus $F^{-1}(q) \subseteq M$ is the image of a smooth embedding and so is an embedded submanifold. Furthermore, since $F^{-1}(q) \subseteq U$ is proper, $F^{-1}(q)$ closed in $U \subseteq M$, and hence, is also closed in M. Therefore $F^{-1}(q)$ is a closed embedded submanifold, and hence is properly embedded by Proposition 80. $\square$

Proposition 88. Let S be a subset of a smooth m-manifold M. Then S is an embedded k-submanifold of M if and only if every point of $p \in S$ has a neighborhood $U_p \subseteq M$ such that $S \cap U_p$ is a level set of a smooth submersion

$$f : U_p \to \mathbb{R}^{m-k}$$

Proof. First suppose that S is an embedded submanifold of dimension k, and since S is an embedded submanifold, it satisfies the local k-slice condition. let $p \in S$ be given, and choose a slice chart $\big(U, (x^1, \ldots, x^m)\big) \in \mathcal{A}_M$ containing p. So that

$$S \cap U = \{q \in U : x^{k+1}(q) = c_{k+1}, \ldots, x^m(q) = c_m\}$$

for constants $c_{k+1}, \ldots c_m \in \mathbb{R}$. Then, defining the map

$$f : U \to \mathbb{R}^{m-k}, \text{ by } f(p) = (x^{k+1}(p), \ldots x^m(p))$$

then by definition coordinate functions are diffeomorphisms onto their image, and so for each $p \in U$, we have

$$df|_p : T_pU \to T_{f(p)}\big(x^{k+1}(U) \times \cdots \times x^m(U)\big) \cong \mathbb{R}^{m-k}$$

is a smooth submersion with the level set

$$f^{-1}(c_{k+1}, \ldots c_m) = S \cap U$$

Next, suppose that $S \subseteq M$ is a subset such that for each $p \in S$, $\exists\, U_p \subseteq M$ such that $S \cap U_p$ is the level set of a smooth submersion

$$f : U \to \mathbb{R}^{m-k}$$

then since U_p and $\mathbb{R}^{m-k}$ are both smooth manifolds, and f is a smooth submersion, the Submersion Level Set Theorem then says, that each level set of f is a properly embedded submanifold of codimension $m-k$. So in particular for some $\mathbf{c} \in \mathbb{R}^{m-k}$ we have

$$f^{-1}(\mathbf{c}) = S \cap U_p = \{q \in U : x^{k+1}(q) = c_{k+1}, \dots, x^m(q) = c_m\}$$

which is a properly embedded submanifold submanifold of U_p, which can be smoothly embedding into M by the inclusion. With dimension

$$m - k = \dim(U_p) - \dim(f^{-1}(\mathbf{c})) = m - \dim(f^{-1}(\mathbf{c})) \implies \dim(f^{-1}(\mathbf{c})) = m - (m-k) = k$$

Then as $S \cap U_p$ is a properly embedded submanifold of dimension k, it satisfies the local k-slice condition, and so for each $q \in S \cap U_p$, $\exists\ (V, \psi) \in \mathcal{A}_M$ such that $(S \cap U_p) \cap V$ is a single k-slice.

Then as $p \in S$ was arbitrary, we conclude that for each point of S the intersection $S \cap U_p$ satisfies the local k-slice condition, then as

$$S = \bigcup_{p \in S} (S \cap U_p)$$

is the union of sets all satisfying the local k-slice condition, we conclude that S itself satisfies the local k-slice condition, and is therefore an embedded submanifold of dimension k. $\square$

Proposition 89 (**Images of Immersions as Submanifolds**)**.** Suppose N is a smooth manifold with or without boundary, M is a smooth manifold, and

$$F : M \to N$$

is an injective smooth immersion. Let $S = F(M)$. Then S has a unique topology and smooth structure such that it is a smooth submanifold of N and such that

$$F : M \to F(M)$$

is a diffeomorphism onto its image.

Proposition 90. Suppose M is a smooth manifold with or without boundary, and $S \subseteq M$ is an immersed submanifold. If any of the following holds, then S is embedded.

(a) S has codimension 0 in M.

(b) The inclusion map $\iota : S \hookrightarrow M$ is proper.

(c) S is compact.

Proposition 91 (**Immersed Submanifolds Are Locally Embedded**)**.** If M is a smooth manifold with or without boundary, and $S \subseteq M$ is an immersed submanifold, then for each $p \in S$ there exists a neighborhood U_p of p in S that is an embedded submanifold of M.

Note this says that given an immersed submanifold $S \subseteq M$ and $p \in S$ it is possible to find $U_p \subseteq S$ that is embedded in M, but it may not be possible to find $V_p \subseteq M$ such that $V_p \cap S$ is embedded.

Proposition 92. Suppose M is a smooth manifold with or without boundary, $S \subseteq M$ is an immersed k-submanifold,

$$\iota : S \hookrightarrow M$$

is the inclusion map, and U is an open subset of $\mathbb{R}^k$. A map

$$F : U \to M$$

is a smooth local parameterization of S if and only if there is a smooth coordinate chart $(V, \phi) \in \mathcal{A}_S$ for S such that

$$F = \iota \circ \phi^{-1}$$

Therefore, every point of S is in the image of some local parameterization.

Theorem 93 (**Restricting the Domain of a Smooth Map**)**.** If M and N are smooth manifolds with or without boundary,

$$F : M \to N$$

is a smooth map, and $S \subseteq M$ is an immersed or embedded submanifold, then

$$F|_S : S \to N$$

is smooth.

Proof. If $S \subseteq M$ is either an immersed or embedded submanifold, then by definition

$$\iota : S \hookrightarrow M$$

is a smooth immersion, and therefore smooth. Then as F is given to be smooth, and the composition of smooth maps is smooth we have that

$$F|_S = F \circ \iota : S \to N$$

is also smooth. □

Theorem 94 (**Restricting the Codomain of a Smooth Map**)**.** Suppose N is a smooth manifold without boundary, $S \subseteq N$ is an immersed submanifold, and

$$F : M \to N$$

is a smooth map whose image is contained in S; i.e. $F(M) \subseteq S$. If F is continuous as a map from M to S, then

$$F : M \to S$$

is smooth.

Proof. Let $\dim(M) = m,\ \dim(N) = n$ and $\dim(S) = k$, and let $p \in M$ be given, then $F(p) \in S$, and since immersed submanifolds are locally embedded by Proposition 91, $\exists\ V_{F(p)} \subseteq S$ such that

$$\iota : V_{F(p)} \hookrightarrow N$$

is a smooth embedding, and so $V_{F(p)}$ is the image of a smooth embedding and thus an embedded submanifold. As an embedded submanifold $V_{F(p)}$ satisfies the local slice condition, and so there exists a slice chart $(W, \psi) \in \mathcal{A}_N$ for $V_{F(p)}$ in N centered at $F(p)$. Then

$$V_{F(p)} \cap W = \{q \in W : y^{k+1}(q) = c_{k+1}, \dots y^n = c_n\}$$

and since the composition of smooth maps is smooth we have

$$\pi^k \circ \psi : V_{F(p)} \to \mathbb{R}^k$$

is smooth. And so

$$\left(V_{F(p)} \cap W, \pi^k \circ \psi|_{V_{F(p)} \cap W}\right) \in \mathcal{A}_{V_{F(p)}}$$

then by the continuity of ι, since $W \subseteq N$ is open, we have

$$\iota|_{V_{F(p)}}^{-1}(W) = V_{F(p)} \cap W \subseteq V_{F(p)}$$

is open in $V_{F(p)}$, and as $V_{F(p)} \subseteq S$ we have that $V_{F(p)} \cap W$ is also open in S in its given topology, and hence

$$\left(V_{F(p)} \cap W, \pi^k \circ \psi|_{V_{F(p)} \cap W}\right) \in \mathcal{A}_S$$

Now, by the continuity of F we also have that $F^{-1}(V_{F(p)} \cap W) \subseteq M$ is open and contains p. Since M is a smooth manifold $\exists$ $(U, \phi) \in \mathcal{A}_M$ centered at p, such that $U \subseteq F^{-1}(V_{F(p)} \cap W)$, where we have the coordinate representation of F, given by

$$\pi^k \circ \psi \circ F \circ \phi^{-1}|_{\phi(U)} : \phi(U) \to \pi^k(\psi(V_{F(p)} \cap W))$$

is smooth as the composition of smooth maps between euclidean spaces, and therefore

$$F : M \to S$$

must also be smooth. $\square$

Corollary 95 (Embedded Case). Let N be a smooth manifold and $S \subseteq N$ be an embedded submanifold. Then every smooth map

$$F : M \to N$$

whose image is contained in S; i.e. $F(M) \subseteq S$, is also smooth as a map from M to S.

Proof. Since $S \subseteq M$ is an embedded submanifold, the topology on S is the relative topology, then by the characteristic property of the relative topology

$$\iota \circ F : M \to N$$

being continuous, implies

$$F : M \to S$$

is continuous, then by Theorem 94

$$F : M \to S$$

is smooth. $\square$

Theorem 96. Suppose M is a smooth manifold and $S \subseteq M$ is an immersed submanifold. For the given topology on S, there is only one smooth structure making S into an immersed submanifold.

Here, the smooth structure is unique, once the topology is known.

Theorem 97. If M is a smooth manifold and $S \subseteq M$ is a weakly embedded submanifold, then S has only one topology and smooth structure with respect to which it is an immersed submanifold.

Theorem 98 (**Extension Lemma for Functions on Submanifolds**)**.** Suppose M is a smooth manifold, $S \subseteq M$ is a smooth submanifold, and $f \in C^\infty(S)$. Then

(a) If S is embedded, then there exist a neighborhood $U_S \subseteq M$ open and containing S and a smooth function $\widetilde{f} \in C^\infty(U_S)$, such that $\widetilde{f}|_S = f$.

(b) If S is properly embedded, then the neighborhood $U_S = M$.

Proposition 99. Suppose M is a smooth manifold with or without boundary, $S \subseteq M$ is an immersed or embedded submanifold, and $p \in S$. A vector $v_p \in T_pM$ is in T_pS if and only if there is a smooth curve

$$\gamma : J \to M$$

whose image is contained in S; i.e $\gamma(J) \subseteq S$, and which is also smooth as a map into S; i.e.

$$\gamma : J \to S, \quad \text{such that } 0 \in J,\ \gamma(0) = p,\ \gamma'(0) = v_p$$

Proposition 100. Suppose M is a smooth manifold, $S \subseteq M$ is an embedded submanifold, and $p \in S$. As a subspace of T_pM; the tangent space T_pS is characterized by

$$T_pS = \{v_p \in T_pM : v_p(f) = 0 \text{ whenever } f \in C^\infty(M) \text{ and } f|_S = 0\}$$

Proof. First suppose $v_p \in T_pS \subseteq T_pM$, then $v_p = d\iota|_p(w_p)$ for some $w_p \in T_pS$. And suppose we have any $f \in C^\infty(M)$ such that $f|_S = 0$. This then implies

$$f|_S = f \circ \iota = 0 \text{ identically}$$

and therefore

$$v_p(f) = d\iota|_p(w_p)(f) = w(f \circ \iota)|_p = 0$$

Next, let $n = \dim(M),\ k = \dim(S)$, and suppose that $v_p \in T_pM$ satisfies

$$v_p(f) = 0, \quad \text{whenever } f \in C^\infty(M) \text{ such that } f|_S = 0$$

Since S is an embedded submanifold we may choose a slice chart $\big(U, (x^1, \dots, x^n)\big) \in \mathcal{A}_M$ centered at p, such that

$$S \cap U = \{q \in U : x^{k+1}(q) = \cdots = x^n(q) = 0\}$$

then $\big(S \cap U, (x^1, \dots, x^k)\big) \in \mathcal{A}_S$ and

$$\iota : S \cap U \hookrightarrow M$$

has the coordinate representation

$$(x^1, \dots, x^n) \circ \iota \circ (x^1, \dots, x^k)^{-1}|_{S\cap U} : (x^1, \dots, x^k)^{-1}(S \cap U) \to (x^1, \dots, x^n)(U)$$

given by

$$\widehat{\iota}(x_1,\ldots,x_k)=(x_1,\ldots,x_k,0,\ldots,0)$$

and therefore $d\iota|_p(T_pS)\subseteq T_pM$ is the subspace spanned by

$$\left\{\frac{\partial}{\partial x^1}\bigg|_p,\ldots,\frac{\partial}{\partial x^k}\bigg|_p\right\}\subseteq\left\{\frac{\partial}{\partial x^1}\bigg|_p,\ldots,\frac{\partial}{\partial x^n}\bigg|_p\right\}$$

and so, taking the coordinate representation for $v_p\in T_pM$ we have

$$v_p=\sum_{i=1}^n v^i\frac{\partial}{\partial x^i}\bigg|_p$$

and so

$$v_p\in T_pS\iff v^i=0,\quad \text{for } k+1\leq i\leq n$$

Now choose a bump function ψ, supported in U such that $\psi(\overline{B})=1$ for some closed neighborhood containing p such that $\overline{B}\subseteq U$. So that we have

$$\begin{aligned}&0\leq\psi(M)\leq 1\\&\psi(q)\equiv 1\quad\forall\ q\in\overline{B}\\&\operatorname{supp}(\psi)\subseteq U\end{aligned}$$

then for each $j>k$ consider the function $f=\psi x^j\in C^\infty(U)$ which extends to a smooth function

$$f=\psi x^j:M\to\mathbb{R},\ \text{by}\ f(q)=\begin{cases}\psi(q)x^j(q), & q\in U\\ 0, & q\in M\setminus\operatorname{supp}(\psi)\end{cases}$$

then $\forall\ q\in S\cap U$ we have

$$x^j=0\implies f(q)=0$$

and further since,

$$S\setminus U\subseteq M\setminus U\subseteq M\setminus\operatorname{supp}(\psi)$$

we have $q\in S\setminus U$ that $f(q)=0$ by definition, and thus

$$f|_S=0\text{ identically}$$

and thus we have

$$\begin{aligned}
0 &= v(f)|_p \\
&= \sum_{i=1}^n v^i \frac{\partial f}{\partial x^i}(p) \\
&= \sum_{i=1}^n v^i \frac{\partial (\psi x^j)}{\partial x^i}(p) \\
&= \sum_{i=1}^n v^i \left(\psi(p) \frac{\partial x^j}{\partial x^i}(p) + x^j(p) \frac{\partial \psi}{\partial x^i}(p) \right) \\
&= \sum_{i=1}^n v^i \left(1 \cdot \frac{\partial x^j}{\partial x^i}(p) + 0 \cdot \frac{\partial \psi}{\partial x^i}(p) \right) \\
&= \sum_{i=1}^n v^i \delta^j{}_i \\
&= v^j
\end{aligned}$$

and thus we have that for each $j \in \{k+1, \ldots, n\}$ that $v^j = 0$, and therefore $v_p \in T_pS$. $\square$

Proposition 101. Suppose M, N are smooth manifolds and $S \subseteq M$ is an embedded submanifold. If $F : U \to N$ is any local defining map for S, then

$$T_pS = \ker(dF|_p) : T_pM \to T_{F(p)}N, \quad \forall\, p \in S \cap U$$

Proof. Since F is a local defining map for S, there exists $U \subseteq M$ open, such that for some $q \in N$ we have $F^{-1}(q) = S \cap U$, that is

$$F|_{S \cap U} = F \circ \iota = q$$

and since $F \circ \iota$ is constant we have

$$d(F \circ \iota)|_p = dF|_{\iota(p)} \circ d\iota|_p = 0 \quad \forall\, p \in S \cap U$$

and so, since $S \subseteq M$ is embedded we have the identification $d\iota|_p(T_pS) \subseteq T_pM$, for each $v_p \in T_pS \subseteq T_pM$, meaning $v_p = d\iota|_p(w_p)$ for some $w_p \in T_pS$ we have

$$dF|_p(v_p) = d(F \circ \iota)|_p(w_p) = 0$$

and therefore $\mathrm{Im}(d\iota|_p) \subseteq \ker(dF|_p)$.

Next, since F is a local defining map for S we have

$$dF|_p : T_pM \to T_{F(p)}N$$

is surjective for each $p \in S \cap U$, and thus has full rank, and so $\mathrm{rank}(dF)$ is constant, and so by the Constant-Rank Level Set Theorem

$$\dim(F^{-1}(q)) = \dim(S) = \dim(M) - \dim(N)$$

and by the Rank-Nullity Theorem we also have

$$\begin{aligned}
\dim\big(\ker(F|_p)\big) &= \dim(T_pM) - \dim\big(T_{F(p)}N\big) \\
&= \dim(M) - \dim(N) \\
&= \dim(S) \\
&= \dim(T_pS) \\
&= \dim\big(\operatorname{Im}(d\iota|_p)\big) && d\iota|_p(T_pS) \subseteq T_pM
\end{aligned}$$

and therefore $\operatorname{Im}(d\iota|_p)$ is a vector subspace of $\ker(F|_p)$ of equal dimension, and therefore $\operatorname{Im}(d\iota|_p) = \ker(dF|_p)$. Thus

$$d\iota|_p(T_pS) = \ker(dF|_p) : T_pM \to T_{F(p)}N$$

$\square$

Corollary 102. Suppose $S \subseteq M$ is a level set of a smooth submersion

$$F = (F^1, \dots, F^k) : M \to \mathbb{R}^k$$

A vector $v_p \in T_pM$ is tangent to S if and only if

$$v_p(F^1) = \cdots = v_p(F^k) = 0$$

Proposition 103. Suppose M is a smooth n-dimensional manifold with boundary, $p \in \partial M$ and $(x^1, \dots, x^n)$ are any smooth boundary coordinates defined on a neighborhood of p. The inward-pointing vectors in T_pM are precisely those with positive x^n-component ($x^n > 0$), the outward-pointing ones are those with negative x^n-component ($x^n < 0$), and the ones tangent to ∂M are those with zero x^n-component ($x^n = 0$). Thus, T_pM is the disjoint union of $T_p\partial M$; the set of inward-pointing vectors, and the set of outward-pointing vectors; i.e. for $p \in \partial M$ we have

$$T_pM = T_p\partial M \sqcup \{T_pM : x^n > 0\} \sqcup \{T_pM : x^n < 0\}$$

and $v_p \in T_pM$ is inward-pointing if and only if $-v_p$ is outward-pointing.

Proposition 104. Every smooth manifold with boundary admits a boundary defining function.

Proof. Let $\big\{\big(U_i, (x^1, \dots, x^n)_i\big)\big\}_{i\in I}$ be a collection of smooth charts such that

$$\bigcup_{i\in I} U_i \supseteq M$$

and for each i define the smooth functions

$$f_i : U_i \to [0,\infty), \text{ by } f_i(p) = \begin{cases} 1, & U_i \text{ an interior chart} \\ x_i^n(p) = 0, & U_i \text{ a boundary chart} \end{cases}$$

and therefore for $p \in \operatorname{Int}(M)$ we have $f_i(p) > 0$, and for $p \in \partial M$ we have $f_i(p) = x_i^n(p) = 0$. Since $\{U_i\}_{i\in I}$ is an indexed cover of M, let $\{\psi_i\}_{i\in I}$ be a smooth partition of unity subordinate to this cover, and so

(i) $0 \leq \psi_i(p) \leq 1 \quad \forall\, i \in I; \forall\, p \in M$

(ii) $\operatorname{supp}(\psi_i) \subseteq U_i$ for each $i \in I$

(iii) The family of supports $\{\operatorname{supp}(\psi_i)\}_{i \in I}$ is locally finite; i.e. $\forall\, p \in M\ \exists$ a neighborhood U_p such that

$$U_p \cap \operatorname{supp}(\psi_i) \neq \varnothing$$

for finitely many $i \in I$

(iv) $\sum_{i \in I} \psi_i(p) = 1 \quad \forall\, p \in M$.

then for each i

$$\psi_i f_i : U_i \to \mathbb{R}$$

is smooth and has a smooth extention to M by the gluing lemma for smooth maps since

$$\psi_i f_i\big(U_i \setminus \operatorname{supp}(\psi_i)\big) = 0 = \psi_i f_i\big(M \setminus \operatorname{supp}(\psi_i)\big)$$

i.e. the functions agree on their overlap. And so we may define

$$f : M \to [0, \infty), \text{ by } f = \sum_{i \in I} \psi_i f_i$$

then f is smooth, and for $p \in \partial M$

$$f(p) = \sum_{i \in I} \psi_i(p) f_i(p) = 0, \text{ identically}$$

and for $p \in \operatorname{Int}(M)$, $f(p) > 0$.

Now let $p \in \partial M$, and let $v_p \in \{T_pM : x^n > 0\}$ so that v_p is inward pointing. Then for each i, we have

$$df_i|_p(v_p) = dx_i^n|_p(v_p) > 0 \quad \text{Proposition 103}$$

and therefore for $Id_{\mathbb{R}} \in C^{\infty}(\mathbb{R})$ we have

$$\begin{aligned}
df|_p(v_p)(Id) &= v(Id \circ f)|_p \\
&= v(f)|_p \\
&= v\left(\sum_{i \in I} \psi_i f_i\right)\Bigg|_p \\
&= \sum_{i \in I} v(\psi_i f_i)|_p && \text{linearity} \\
&= \sum_{i \in I} (f_i(p) v(\psi_i)|_p + \psi(p) v(f_i)|_p) \\
&= \sum_{i \in I} (0 \cdot v(\psi_i)|_p + \psi(p) v(f_i)|_p) \\
&= \sum_{i \in I} \psi(p) v(f_i)|_p \\
&= \sum_{i \in I} \psi(p) v(Id \circ f_i)|_p \\
&= \sum_{i \in I} \psi(p) df_i|_p(v_p)(Id) \\
&= \left(\sum_{i \in I} \psi(p) df_i|_p(v_p)\right)(Id)
\end{aligned}$$

and so

$$df|_p(v_p) = \sum_{i \in I} \psi(p) df_i|_p(v_p) = \sum_{i \in I} \psi(p) dx_i^n|_p(v_p) > 0$$

thus we have $df|_p \neq 0$, while $f(p) = 0 \implies f^{-1}(0) = \partial M$, since $f(p) \neq 0, \forall\, p \in \operatorname{Int}(M)$. Hence, f is a boundary defining map. $\square$

Proposition 105. Suppose M is a smooth manifold without boundary and $D \subseteq M$ is a regular domain. The topological interior and boundary of D are equal to its manifold interior and boundary, respectively.

Proof. Let $p \in D$ be given. If p is in the manifold boundary of D, then as D is an embedded submanifold of M of codimension-0, there exists a boundary slice-chart $\big(U, (x^1, \ldots, x^n)\big) \in \mathcal{A}_M$ centered at p, such that

$$D \cap U = \{q \in U : x^n(q) \geq 0\}$$

Now, since M is a manifold without boundary we have

$$\phi(U) \subseteq \mathbb{R}^n$$

and so there must exists $r \in U$ such that $x^n(r) < 0$, yet this then implies that $r \notin D$ as $r \in D \cap U \implies x^n(r) \geq 0$. And therefore U is a neighborhood of p in M such that

$$\begin{aligned}
U \cap D &\neq \varnothing \\
U \cap (M \setminus D) &\neq \varnothing
\end{aligned}$$

Since U was an arbitrary neighborhood of p in M we conclude that each neighborhood of p in U contains points both in D and D^c, and therefore p is in the topological boundary of D.

Next, if p is in the manifold interior of D, then as the manifold interior of D is an embedded submanifold of codimension-0 without boundary in M, and by Proposition 76, the codimension-0 submanifolds are precisely the open subsets of M, and so p belongs to the topological interior.

Conversely, if p is in the topological interior of D, then by the Topological Invariance of Boundary, p does not belong to the topological boundary of D, and so the topological interior of D is an open subset of M containing p, and thus by Proposition 76, is an embedded submanifold of codimension-0, and so p belongs to the manifold interior of D.

If p is in the topological boundary of D, then each neighborhood of p intersects both D and D^c, and so we may find a boundary-slice chart in M for D about p, and so p belongs to the manifold boundary of D.

□

Proposition 106. Suppose M is a smooth manifold and $f \in C^\infty(M)$. Then

(a) For each regular value $b \in \mathbb{R}$ of f , the sublevel set $f^{-1}\big((-\infty, b]\big)$ is a regular domain in M.

(b) If a and b are two regular values of f with $a < b$, then $f^{-1}\big([a, b]\big)$ is a regular domain in M.

Theorem 107. If M is a smooth manifold and $D \subseteq M$ is a regular domain, then there exists a defining function f, for D. If D is compact, then f can be taken to be a smooth exhaustion function for M.

Proof. Let $n = \dim(M)$ and suppose that $D \subseteq M$ is a regular domain in M. So that D is a properly embedded codimension-0 submanifold with boundary. Let $\big\{\big(U_i, (x^1, \dots, x^n)_i\big)\big\}_{i\in I}$ be a countable collection of smooth charts such that

$$\bigcup_{i\in I} U_i \supseteq M$$

and for each i define the smooth functions

$$f_i : U_i \to \mathbb{R}, \text{ by } f_i(p) = \begin{cases} -1, & U_i \text{ Interior slice chart for } D \text{ in } M \\ 1, & U_i \subseteq M \setminus D \\ x_i^n(p), & U_i \text{ Boundary slice chart for } D \text{ in } M \end{cases}$$

then since D has codimension-0, an interior chart is just a subset of the coordinate chart; i.e. $(D \cap U_i, \phi_i|_{D\cap U_i})$, with no further restrictions, while a boundary slice chart is the 0-dimensional half-slice

$$D \cap U_i = \{q \in U_i : x_i^n(q) \geq 0\}$$

and therefore for $p \in \mathrm{Int}(D)$ we have $f_i(p) < 0$, and for $p \in \partial D$ we have $f_i(p) = x_i^n(p) = 0$. Since $\{U_i\}_{i\in I}$ is an indexed cover of M, let $\{\psi_i\}_{i\in I}$ be a smooth partition of unity subordinate to this cover, and so

(i) $0 \leq \psi_i(p) \leq 1 \quad \forall\, i \in I; \forall\, p \in M$

(ii) $\text{supp}(\psi_i) \subseteq U_i$ for each $i \in I$

(iii) The family of supports $\{\text{supp}(\psi_i)\}_{i\in I}$ is locally finite; i.e. $\forall\, p \in M\ \exists$ a neighborhood U_p such that

$$U_p \cap \text{supp}(\psi_i) \neq \varnothing$$

for finitely many $i \in I$

(iv) $\sum_{i\in I} \psi_i(p) = 1 \quad \forall\, p \in M$.

then for each i

$$\psi_i f_i : U_i \to \mathbb{R}$$

is smooth and has a smooth extention to M by the gluing lemma for smooth maps since

$$\psi_i f_i\big(U_i \setminus \text{supp}(\psi_i)\big) = 0 = \psi_i f_i\big(M \setminus \text{supp}(\psi_i)\big)$$

i.e. the functions agree on their overlap. And so we may define

$$f : M \to \mathbb{R},\ \text{by } f = \sum_{i\in I} \psi_i f_i$$

then f is smooth, and for $p \in \partial M$

$$f(p) = \sum_{i\in I} \psi_i(p) f_i(p) = \sum_{i\in I} \psi_i(p) x_i^n(p) = 0,\ \text{identically}$$

and for $p \in \text{Int}(M)$,

$$f(p) = \sum_{i\in I} \psi_i(p) f_i(p) = \sum_{i\in I} \psi_i(p) \cdot (-1) = -\sum_{i\in I} \psi_i(p) < 0$$

while for $p \in M \setminus D = D^c$ we get

$$f(p) = \sum_{i\in I} \psi_i(p) f_i(p) = \sum_{i\in I} \psi_i(p) \cdot (1) = \sum_{i\in I} \psi_i(p) > 0$$

Now let $p \in \partial D$, and let $v_p \in \{T_p D : x^n > 0\}$ so that v_p is inward pointing. Then for each i, we have

$$df_i|_p(v_p) = dx_i^n|_p(v_p) > 0 \quad \text{Proposition 103}$$

and therefore for $Id_{\mathbb{R}} \in C^\infty(\mathbb{R})$ we have

$$\begin{aligned}
df|_p(v_p)(Id) &= v(Id \circ f)|_p \\
&= v(f)|_p \\
&= v\left(\sum_{i\in I} \psi_i f_i\right)\Bigg|_p \\
&= \sum_{i\in I} v(\psi_i f_i)|_p && \text{linearity} \\
&= \sum_{i\in I} (f_i(p)v(\psi_i)|_p + \psi(p)v(f_i)|_p) \\
&= \sum_{i\in I} (0 \cdot v(\psi_i)|_p + \psi(p)v(f_i)|_p) \\
&= \sum_{i\in I} \psi(p)v(f_i)|_p \\
&= \sum_{i\in I} \psi(p)v(Id \circ f_i)|_p \\
&= \sum_{i\in I} \psi(p)df_i|_p(v_p)(Id) \\
&= \left(\sum_{i\in I} \psi(p)df_i|_p(v_p)\right)(Id)
\end{aligned}$$

and so

$$df|_p(v_p) = \sum_{i\in I} \psi(p)df_i|_p(v_p) = \sum_{i\in I} \psi(p)dx_i^n|_p(v_p) > 0$$

thus we have $df|_p \neq 0$, that is

$$df|_p = \begin{bmatrix} \frac{\partial \widehat{f}}{\partial x^1}(\phi(p)) & \dots & \frac{\partial \widehat{f}}{\partial x^n}(\phi(p)) \end{bmatrix} \neq \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix}$$

and so has full rank, and since $\dim(M) \geq \dim(\mathbb{R})$ we have that f is surjective, and therefore p is a regular point of f.

Since $p \in \partial D$ was arbitrary we conclude that each $p \in \partial D$ is a regular point, and thus $f(p) = 0$, is a regular value of f, and by construction the only positive values of f are for $p \in D^c$ and therefore we have

$$f^{-1}\big((-\infty, 0]\big) = D$$

and so f is a defining function for D.

If $D \subseteq M$ is compact, then for the cover $\big\{\big(U_i, (x^1, \dots, x^n)_i\big)\big\}_{i\in I}$; Here, since M is a manifold it admits a basis of precompact coordinate balls, so let us suppose this cover is precompact. And since it is a cover for M it will also cover D, and by the compactness of D we have a finite subcover, say

$$\bigcup_{i=1}^{k} U_i \supseteq D$$

while

$$\bigcup_{i=k+1}^{\infty} U_i \supseteq D^c$$

Then, following the proof of Proposition 2.28, we take a partition of unity $\{\psi_i\}_{i=1}^{\infty}$ subordinate to the cover $\{U_i\}_{i=1}^{k} \cup \{U_i\}_{i=k+1}^{\infty}$ and define

$$f : M \to \mathbb{R}, \text{ by } f(p) = \sum_{i=1}^{k} f_i(p)\psi_i(p) + \sum_{i=k+1}^{\infty} i f_i(p)\psi_i(p)$$

then f is smooth since $\{\text{supp}(\psi_i)\}$ is a locally finite collection of subsets, and so only finitely many of the terms in the sum are non-zero in the neighborhood of any point of M. Then since $\sum_{i\in I} \psi_i(p) = 1 \quad \forall\ p \in M$, then for any $p \in \text{Int}(D)$ we have

$$f(p) = \sum_{i=1}^{k} f_i(p)\psi_i(p) + \sum_{i=k+1}^{\infty} i f_i(p)\psi_i(p) = -\sum_{i=1}^{k} \psi_i(p) < 0$$

and again for $p \in \partial D$ we have

$$f(p) = \sum_{i=1}^{k} f_i(p)\psi_i(p) + \sum_{i=k+1}^{\infty} i f_i(p)\psi_i(p) = \sum_{i=1}^{k} x_i^n(p)\psi_i(p) = 0$$

and since $\sum_{i=k+1}^{\infty} i f_i(p)\psi_i(p)$ is strictly positive, therefore we have

$$f^{-1}\big((-\infty, 0]\big) = D$$

and so f is a defining function for D, where from above, 0 is a regular value of f.

Then as D is compact in a Hausdorff space, it is closed and note that since the U_i's are precompact each $\overline{U}_i$ is compact in M, so we have

$$D = \overline{D} \subseteq \overline{\bigcup_{i=1}^{k} U_i} = \bigcup_{i=1}^{k} \overline{U}_i$$

and $\bigcup_{i=1}^{k} \overline{U}_i$ is compact as the finite union of compact sets. Then Since f is smooth, it is continuous, and the continuous image of a compact set is compact and so

$$f\left(\bigcup_{i=1}^{k} \overline{U}_i\right) = \bigcup_{i=1}^{k} f(\overline{U}_i) \subseteq \mathbb{R}$$

is also compact, and so by Heine Borel is bounded. Thus, there exists some $c \in \mathbb{R}$ such that

$$\bigcup_{i=1}^{k} f(\overline{U}_i) \subseteq (-\infty, c] \implies \bigcup_{i=1}^{k} \overline{U}_i \subseteq f^{-1}\big((-\infty, c]\big)$$

To see that f is an exhaustion function, let us now focus on the U_i's covering D^c. Then, given $a \in \mathbb{R}$ we can choose $N \in \mathbb{N}$ such that $N > a$. Then if

$$p \notin \bigcup_{i=k+1}^{N} \overline{U}_i \implies \psi_i(p) = 0 \text{ for } k+1 \leq i \leq N, \quad \text{since } \operatorname{supp}(\psi_i) \subseteq U_i$$

And so

$$f(p) = \sum_{i=N+1}^{\infty} i\psi_i(p) \geq \sum_{i=N+1}^{\infty} N\psi_i(p) = N\sum_{i=1}^{\infty} \psi_i(p) = N > c$$

and so $p \notin f^{-1}\big((-\infty, a]\big)$. That is,

$$\text{if } p \notin \bigcup_{i=k+1}^{N} \overline{U}_i \text{ then } p \notin f^{-1}\big((-\infty, a]\big)$$

taking the contrapositive gives

$$\text{if } p \in f^{-1}\big((-\infty, a]\big) \text{ then } p \in \bigcup_{i=k+1}^{N} \overline{U}_i$$

or $f^{-1}\big((-\infty, a]\big) \subseteq \bigcup_{i=k+1}^{N} \overline{U}_i$, which is compact as a finite union of compact sets, were the continuity of f tells us that $f^{-1}\big((-\infty, c]\big) \subseteq M$ is closed, and therefore must be compact as a closed subset of a compact set.

Finally we note, that since f is continuous, and strictly positive on D^c, we get $f^{-1}\big((-\infty, c]\big) \subseteq f^{-1}\big((-\infty, a]\big)$, and so is compact. $\square$

Proposition 108 (Properties of Submanifolds with Boundary). Suppose M is a smooth manifold with or without boundary. Then

(a) Every open subset of M is an embedded codimension-0 submanifold with (possibly empty) boundary.

(b) If M is a smooth manifold with boundary and

$$F : M \to N$$

is a smooth embedding, then with the subspace topology $F(M) \subseteq N$ is a topological manifold with boundary, and it has a smooth structure making it into an embedded submanifold with boundary in N.

(c) An embedded submanifold with boundary in M is properly embedded if and only if it is closed.

(d) If $S \subseteq M$ is an immersed submanifold with boundary, then for each $p \in S$, $\exists\, U_p \subseteq S$ such that U_p is embedded in M.

Theorem 109. Let M be a smooth n-manifold without boundary. If $S \subseteq M$ is an embedded k-dimensional submanifold with boundary, then S satisfies the local k-slice condition for submanifolds with boundary. Conversely, if $S \subseteq M$ is a subset that satisfies the local k-slice condition for submanifolds with boundary, then with the subspace topology, S is a topological k-manifold with boundary, and it has a smooth structure making it into an embedded submanifold with boundary in M.

Theorem 110 (Restricting Maps to Submanifolds with Boundary). Suppose M and N are smooth manifolds with boundary and $S \subseteq M$ is an embedded submanifold with boundary. Then

(a) RESTRICTING THE DOMAIN: If
$$F : M \to N$$
is a smooth map, then
$$F|_S : S \to N$$
is smooth.

(b) RESTRICTING THE CODOMAIN: If $\partial M = \varnothing$ and
$$F : N \to M$$
is a smooth map whose image is contained in S; i.e $F(N) \subseteq S$, then
$$F : N \to S$$
is smooth.

Lemma 111. Suppose $X \subseteq \mathbb{R}^n$ is a compact subset whose intersection with $\{c\} \times \mathbb{R}^{n-1}$ has $(n-1)$-dimensional measure zero, that is,
$$X \cap (\{c\} \times \mathbb{R}^{n-1})$$
has $(n-1)$-dimensional measure zero, for every $c \in \mathbb{R}$. Then X has n-dimensional measure zero.

Proposition 112. Suppose X is an open or closed subset of $\mathbb{R}^{n-1}$ or $\mathbb{H}^{n-1}$, and
$$f : X \to \mathbb{R}$$
is a continuous function. Then the graph of f has measure zero in $\mathbb{R}^n$.

Corollary 113. Every proper affine subspace of $\mathbb{R}^n$ has measure zero in $\mathbb{R}^n$.

Proposition 114. Suppose $X \subseteq \mathbb{R}^n$ has measure zero and
$$F : X \to \mathbb{R}^n$$
is a smooth map. Then $F(X)$ has measure zero.

Lemma 115. Let M be a smooth n-manifold with or without boundary and $A \subseteq M$. Suppose that for some collection $\{(U_\alpha, \phi_\alpha\}_{\alpha \in I}$ of smooth charts whose domains cover A; i.e such that
$$\bigcup_{\alpha \in I} U_\alpha = A$$
we have $\phi_\alpha(A \cap U_\alpha) \subseteq \mathbb{R}^n$ has measure zero in $\mathbb{R}^n$ for each $\alpha \in I$. Then A has measure zero in M.

Proposition 116. Suppose M is a smooth manifold with or without boundary and $A \subseteq M$ has measure zero in M. Then $M \setminus A$ is dense in M.

Theorem 117. Suppose M and N are smooth n-manifolds with or without boundary,

$$F : M \to N$$

is a smooth map, and $A \subseteq M$ is a subset of measure zero. Then $F(A) \subseteq N$ has measure zero in N.

Theorem 118 (Sard's Theorem). Suppose M and N are smooth manifolds with or without boundary and

$$F : M \to N$$

is a smooth map. Then the set of critical values of F has measure zero in N.

Proof. What we are looking to show is that for

$$\{\text{critical points of } F\} := C = \{p \in M : dF|_p \text{ not surjective}\} \subseteq M$$

we have

$$F(C) =: \{\text{critical values of } F\} \subseteq N$$

has measure zero. Let $\dim(N) = n$ and $\dim(M) = m$, we will go by induction on m.

For the base case $m = 0$, if $n = 0$, then

$$dF|_p : T_pM \cong \{\mathbf{0}_p\} \to T_{F(p)}N \cong \{\mathbf{0}_{F(p)}\}$$

is always invertible, and so $C = \varnothing$ and so

$$F(C) = F(\varnothing) = \varnothing$$

and $\mu(\varnothing) = 0$. If $n > 0$, then since M is 0-dimensional manifold it is a set of countably many points which means

$$F(M) = F(\{p_1, p_2, \dots\}) = \{F(p_1), F(p_2), \dots\}$$

and the measure of countably many singletons has measure zero, and so trivially

$$\mu(F(C)) = \sum_i \mu(F(p_i)) = \sum_i 0 = 0$$

finally if $n = 0$ and $m \geq 1$, then

$$dF|_p : T_pM \to T_{F(p)}N \cong \{\mathbf{0}_{F(p)}\}$$

is always surjective, and so again $C = \varnothing$.

So suppose $m \geq 1$ and the result holds for maps F with domains of dimension less than m. Since M, N are manifolds, and hence second countable, we may cover them by countably many smooth charts $(U, \phi) \in \mathcal{A}_M$, and $(V, \psi) \in \mathcal{A}_N$ such that

$$M \subseteq \bigcup_{i \in I} U_i, \quad \text{and} \quad N \subseteq \bigcup_{j \in J} V_j$$

and by the smoothness of F for each $U \in \{U_i\}$ we will have $F(U) \subseteq V$ for some $V \in \{V_j\}$. Where we may then consider the coordinate representation

$$\widehat{F} : \phi(U) \subseteq \mathbb{R}^m \to \psi(V) \subseteq \mathbb{R}^n$$

and since $dF|_p$ is equal to the Jacobian matrix of $\widehat{F}$ at $\phi(p)$, it suffices to consider, $M = U = \phi(U) \subseteq \mathbb{R}^m$ which is open, and $N = \mathbb{R}^n$ and the smooth map

$$F : U \to \mathbb{R}^n$$

With

$$\{\text{critical points of } F\} := C = \{p \in U : dF|_p \text{ not surjective}\} \subseteq U$$

next define

$$C_k := \left\{ \mathbf{x} \in C : \frac{\partial^r F^j}{\partial x^{i_1} \partial x^{i_2} \ldots \partial x^{i_r}}\bigg|_{\mathbf{x}} = 0,\ \forall\ r \leq k,\ i_1, \ldots, i_r \in \{1, \ldots, m\} \right\}$$

which produces the nested sequence

$$C \supseteq C_1 \supseteq C_2 \supseteq \cdots$$

As an explicit example, note that C_1 is the set of values for which the Jacobian matrix is equal to $0_{n \times m}$, that is

$$\begin{bmatrix} \frac{\partial(y^1 \circ F)}{\partial x^1}(\mathbf{x}) & \cdots & \frac{\partial(y^1 \circ F)}{\partial x^m}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial(y^n \circ F)}{\partial x^1}(\mathbf{x}) & \cdots & \frac{\partial(y^n \circ F)}{\partial x^m}(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

and C_2, is the set of values for which the matrix of all 2^{nd} partials is equal to $0_{n \times m}$ etc. And we must show that

$$F(C) = \overbrace{F(C \setminus C_1)}^{(1)} \cup \Big(\overbrace{F(C_1 \setminus C_2) \cup \cdots \cup F(C_k \setminus C_{k+1})}^{(2)} \Big) \cup \overbrace{F(C_{k+1})}^{(3)}$$

has measure zero. Where we note that by Proposition 50, the set of regular points of F is always an open set, and so

$$\{\text{regular points of F}\} := R \subseteq U$$

is open, then R^c is the set of $p \in U$ where F is not regular; i.e. the critical points of F must be closed, that is $R^c = C$ is closed in U as is each C_i.

Case 1: First note $F(C \setminus C_1)$ is the set of points where the Jacobian of F is not surjective yet, $\left(\frac{\partial F^j}{\partial x^i} \right) \neq 0_{n \times m}$, and that since $C_1 \subset U$ is closed, we have $U \setminus C_1$ is open, so we can make the adjustment

$$U \to U \setminus C_1$$

and consider $C_1 = \varnothing$ so that

$$U \setminus C_1 = U$$
$$C \setminus C_1 = C$$

so let $\mathbf{x} \in C$ be arbitrary, and since $\left(\frac{\partial F^j}{\partial x^i}\right) \neq 0_{n\times m}$, by a permutation of the coordinates if necessary we may assume

$$\left.\frac{\partial(y^1 \circ F)}{\partial x^1}\right|_{\mathbf{x}} \neq 0$$

So for $V_{\mathbf{x}} \subseteq U$ we define smooth coordinate functions by

$$\begin{aligned}
&u^1 = y^1 \circ F : V_{\mathbf{x}} \to \mathbb{R}\\
&u^2 = x^2 : V_{\mathbf{x}} \to \mathbb{R}\\
&\quad\vdots\\
&u^m = x^m : V_{\mathbf{x}} \to \mathbb{R}
\end{aligned}$$

and since the Jacobian of a change of coordinates is nonsingular and

$$\left.\frac{\partial F^1}{\partial x^1}\right|_{\mathbf{x}} \neq 0 \implies \begin{bmatrix} \frac{\partial u^1}{\partial x^1}(\mathbf{x}) & \cdots & \frac{\partial u^1}{\partial x^m}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial u^m}{\partial x^1}(\mathbf{x}) & \cdots & \frac{\partial u^m}{\partial x^m}(\mathbf{x}) \end{bmatrix} \neq \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

that is $\left(\frac{\partial u^j}{\partial x^i}\right)$ nonsingular near $\mathbf{x}$, and so an inverse exists. So define

$$\Phi := (u^1, \ldots, u^m) : V_{\mathbf{x}} \to \mathbb{R}^m$$

and since the Jacobian matrix is invertible near $\mathbf{x}$, the Inverse Function Theorem then tells us that Φ is a local diffeomorphism at $\mathbf{x}$. Shrinking $V_{\mathbf{x}}$ if necessary so that $\overline{V}_{\mathbf{x}} \subset U$ is compact where the coordinates extend smoothly to $\overline{V}_{\mathbf{x}}$ we can consider

$$\Phi : \overline{V}_{\mathbf{x}} \to \Phi(\overline{V}_{\mathbf{x}})$$

to be a diffeomorphism with inverse

$$\Phi^{-1}|_{\Phi(\overline{V}_{\mathbf{x}})} : \Phi(\overline{V}_{\mathbf{x}}) \to \overline{V}_{\mathbf{x}}$$

Setting

$$\widetilde{F} = F \circ \Phi^{-1}|_{\Phi(\overline{V}_{\mathbf{x}})} : \Phi(\overline{V}_{\mathbf{x}}) \to \mathbb{R}^n$$

Then

$$F|_{\overline{V}_{\mathbf{x}}} = \widetilde{F} \circ \Phi : \overline{V}_{\mathbf{x}} \to \mathbb{R}^n$$

and so for $\mathbf{p} \in \overline{V}_{\mathbf{x}}$ we have $\widetilde{F}(\Phi(\mathbf{p})) = F(\mathbf{p})$, so if $\mathbf{z} \in \overline{V}_{\mathbf{x}}$ is a critical point of F, then $\Phi(\mathbf{z}) \in \Phi(\overline{V}_{\mathbf{x}}) \subseteq \mathbb{R}^m$ is a critical point of $\widetilde{F}$. And note

$$\begin{aligned}
F(p_1, \ldots p_m) &= \widetilde{F}(\Phi(p_1, \ldots p_m))\\
&= \widetilde{F}\big(u^1(p_1, \ldots p_m), \ldots, u^m(p_1, \ldots p_m)\big)\\
&= \widetilde{F}\big(y^1 \circ F(p_1, \ldots p_m), \ldots, x^m(p_1, \ldots p_m)\big)\\
&= \Big(y^1 \circ F(p_1, \ldots p_m), y^2 \circ \widetilde{F}(p_1, \ldots p_m), \ldots, y^n \circ \widetilde{F}(p_1, \ldots p_m)\Big)
\end{aligned}$$

and therefore

$$y^1 \circ F(p_1, \dots p_m) = x^1(p_1, \dots p_m) \implies y^1 \circ F = x^1$$

And so writing

$$\widetilde{F} = \begin{bmatrix} x^1 \\ y^2 \circ \widetilde{F} \\ \vdots \\ y^2 \circ \widetilde{F} \end{bmatrix}$$

we see that

$$d\widetilde{F}|_{\mathbf{p}} = \begin{bmatrix} \frac{\partial x^1}{\partial x^1}(\mathbf{p}) & \cdots & \frac{\partial x^1}{\partial x^m}(\mathbf{p}) \\ \vdots & \ddots & \vdots \\ \frac{\partial (y^n \circ \widetilde{F})}{\partial x^1}(\mathbf{p}) & \cdots & \frac{\partial (y^n \circ \widetilde{F})}{\partial x^m}(\mathbf{p}) \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \frac{\partial (y^2 \circ \widetilde{F})}{\partial x^1}(\mathbf{p}) & \frac{\partial (y^2 \circ \widetilde{F})}{\partial x^2}(\mathbf{p}) & \cdots & \frac{\partial (y^2 \circ \widetilde{F})}{\partial x^m}(\mathbf{p}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial (y^n \circ \widetilde{F})}{\partial x^1}(\mathbf{p}) & \frac{\partial (y^n \circ \widetilde{F})}{\partial x^2}(\mathbf{p}) & \cdots & \frac{\partial (y^n \circ \widetilde{F})}{\partial x^m}(\mathbf{p}) \end{bmatrix}$$

and so the critical points of $\widetilde{F}$, and hence those of F in $C \cap \overline{V}_{\mathbf{x}}$ belong to the inner $(n-1) \times (m-1)$ matrix. Then for any $c \in \mathbb{R}$ we have

$$\begin{aligned} (C \cap \overline{V}_{\mathbf{x}}) \cap \big(\{c\} \times \mathbb{R}^{m-1}\big) &= \Big\{(c, p_2, \dots, p_m) \in \overline{V}_{\mathbf{x}} : (c, \mathbf{p}) \text{ a critical point of } F\Big\} \\ &= \left\{(c, p_2, \dots, p_m) \in \Phi(\overline{V}_{\mathbf{x}}) : \left(\frac{\partial \widetilde{F}^j}{\partial x^i}\right)_{\substack{2 \le j \le n \\ 2 \le i \le m}} \text{ not surjective}\right\} \\ &= \Big\{\text{critical points of } \widetilde{F}_c\Big\} \end{aligned}$$

where $\widetilde{F}_c$ is the mapping

$$\widetilde{F}_c : \Phi(\overline{V}_{\mathbf{x}}) \cap \big(\{c\} \times \mathbb{R}^{m-1}\big) \to \mathbb{R}^n$$

And by the induction hypothesis, since $(m-1) < m$ each $\widetilde{F}_c$ has $(n-1)$-dimensional measure zero $\forall\ c \in \mathbb{R}$.

Then since $\overline{V}_{\mathbf{x}}$ is compact in $\mathbb{R}^n$ it is closed, and $C \cap \overline{V}_{\mathbf{x}} \subseteq \overline{V}_{\mathbf{x}}$ is closed as the finite intersection of closed sets, and is compact as the closed subset of a compact set. Then, since continuous image of a compact set is compact we have

$$\widetilde{F}_c\Big((C \cap \overline{V}_{\mathbf{x}}) \cap \big(\{c\} \times \mathbb{R}^{m-1}\big)\Big) \subseteq \mathbb{R}^n$$

is compact. Therefore, by Lemma 111, letting

$$\widetilde{C}_c = (C \cap \overline{V}_{\mathbf{x}}) \cap \big(\{c\} \times \mathbb{R}^{m-1}\big)$$

we have

$$\begin{aligned} F(C \cap \overline{V}_{\mathbf{x}}) &= \bigcup_{c \in \mathbb{R}} \widetilde{F}_c\Big((C \cap \overline{V}_{\mathbf{x}}) \cap \big(\{c\} \times \mathbb{R}^{m-1}\big)\Big) \cap \big(\{c\} \times \mathbb{R}^{n-1}\big) \\ &= \bigcup_{c \in \mathbb{R}} \widetilde{F}_c(\widetilde{C}_c) \cap \big(\{c\} \times \mathbb{R}^{n-1}\big) \end{aligned}$$

has n-dimensional measure zero, and so

$$\mu\big(F(C \cap \overline{V}_{\mathbf{x}})\big) = 0$$

Then as U is open, $\forall\ \mathbf{x} \in C \subseteq U, \exists\ V_{\mathbf{x}} \subseteq U$ which is open, and since $C \subseteq U$ is closed we have

$$\{U \setminus C\} \cup \{V_{\mathbf{x}} : \mathbf{x} \in C\}$$

is an open cover for U. And since $\mathbb{R}^m$ is second countable, we have $U \subseteq \mathbb{R}^m$ is second countable, and since every open cover of a second countable topological space has a countable subcover, we have

$$(U \setminus C) \cup \bigcup_{\substack{\mathbf{x}_i \in C \\ i \in I}} V_{\mathbf{x}_i} \supseteq U$$

then

$$\begin{aligned}
C \cap U &= C \cap \left(\bigcup_{\substack{\mathbf{x}_i \in C \\ i \in I}} V_{\mathbf{x}_i} \cup (U \setminus C) \right) \\
&= \left(C \cap \bigcup_{\substack{\mathbf{x}_i \in C \\ i \in I}} V_{\mathbf{x}_i} \right) \cup \big(C \cap (U \setminus C)\big) \\
&= \bigcup_{\substack{\mathbf{x}_i \in C \\ i \in I}} \big(C \cap V_{\mathbf{x}_i}\big)
\end{aligned}$$

and thus

$$\begin{aligned}
F(C \cap U) &= F\left(\bigcup_{\substack{\mathbf{x}_i \in C \\ i \in I}} \big(C \cap V_{\mathbf{x}_i}\big) \right) \\
&= \bigcup_{\substack{\mathbf{x}_i \in C \\ i \in I}} F\big(C \cap V_{\mathbf{x}_i}\big) \\
&= \bigcup_{\substack{\mathbf{x}_i \in C \\ i \in I}} \left(\bigcup_{c \in \mathbb{R}} \widetilde{F}_c\Big((C \cap \overline{V}_{\mathbf{x}_i}) \cap \big(\{c\} \times \mathbb{R}^{m-1}\big)\Big) \cap \big(\{c\} \times \mathbb{R}^{n-1}\big) \right) \\
&= \bigcup_{\substack{\mathbf{x}_i \in C \\ i \in I}} \left(\bigcup_{c \in \mathbb{R}} \widetilde{F}_c(\widetilde{C}_c) \cap \big(\{c\} \times \mathbb{R}^{n-1}\big) \right)
\end{aligned}$$

is a countable union of sets of measure zero and therefore has measure zero; i.e.

$$\mu\big(F(C \cap U)\big) = \mu\left(\bigcup_{\substack{\mathbf{x}_i \in C \\ i \in I}} F\big(C \cap V_{\mathbf{x}_i}\big) \right) \leq \sum_{i \in I} \mu\big(F(C \cap \overline{V}_{\mathbf{x}_i})\big) = 0$$

Case 2: Note that $F(C_k \setminus C_{k+1})$ is the set of points where the first k^{th} order partial derivatives vanish, yet

$$\Big(\frac{\partial^{k+1} F}{\partial x^{i_1} \dots \partial x^{i_{k+1}}}\Big) \neq 0_{n\times m}$$

(to see how this may be possible consider x^{k+1} on $\mathbb{R}$ which has the property that the first k partials vanish at $x = 0$, yet the $(k+1)^{st}$ partial does not).

Again since C_{k+1} is closed in U we have $U \setminus C_{k+1}$ is open, so we can make the adjustment

$$U \to U \setminus C_{k+1}$$

and consider $C_{k+1} = \varnothing$, so that we have

$$U \setminus C_{k+1} = U$$
$$C_k \setminus C_{k+1} = C_k$$

where

$$C_k = \left\{ \mathbf{x} \in C : \frac{\partial^r F}{\partial x^{i_1} \partial x^{i_2} \dots \partial x^{i_r}}\Big|_{\mathbf{x}} = 0,\ \forall\ r \leq k;\ \text{some}\ \frac{\partial^{k+1} F^j}{\partial x^{i_1} \dots \partial x^{i_{k+1}}}\Big|_{\mathbf{x}} \neq 0 \right\}$$

so let $\mathbf{x} \in C_k$ be arbitrary, which implies

$$\frac{\partial^{k+1} F}{\partial x^{i_1} \dots \partial x^{i_{k+1}}}\Bigg|_{\mathbf{x}} \neq 0_{n\times m}$$

So define

$$H := \frac{\partial^k F^j}{\partial x^{i_1} \partial x^{i_2} \dots \partial x^{i_k}} : U \to \mathbb{R}$$

then

$$H(\mathbf{x}) = \frac{\partial^r F^j}{\partial x^{i_1} \partial x^{i_2} \dots \partial x^{i_r}}\Bigg|_{\mathbf{x}} = 0, \quad \text{and} \quad \frac{\partial H}{\partial x^{i_{k+1}}}\Bigg|_{\mathbf{x}} \neq 0$$

so the Jacobian of H has full rank at $\mathbf{x}$, and thus, $\mathbf{x}$ is a regular point for H. Since $\mathbf{x}$ is a regular point for H this implies

$$dH|_{\mathbf{x}} : T_{\mathbf{x}} U \to T_{H(\mathbf{x})}\mathbb{R} = T_0\mathbb{R}$$

is surjective, this is an open condition by Proposition 50, and so $\exists\ V_{\mathbf{x}} \subseteq U$ such that

$$H|_{V_{\mathbf{x}}} : V_{\mathbf{x}} \to \mathbb{R}$$

is a smooth submersion, and hence each $\mathbf{z} \in V_{\mathbf{x}}$ is also a regular point of H. So by the Submersion Level Set Theorem, $H^{-1}(0) \cap V_{\mathbf{x}} \subseteq U$ is a properly embedded submanifold of codimension $= \dim(\mathbb{R}) = 1$, or equivalently

$$\dim(H^{-1}(0) \cap V_{\mathbf{x}}) = \dim(U) - \dim(\mathbb{R}) = m - 1$$

And recall that $\forall\ \mathbf{p} \in C_k$ we have $H(\mathbf{p}) = 0$, which is to say

$$C_k \cap V_{\mathbf{x}} \subseteq H^{-1}(0) \cap V_{\mathbf{x}}$$

Then as

$$H^{-1}(0) = \left\{ \mathbf{x} \in U : \frac{\partial^r F^j}{\partial x^{i_1} \partial x^{i_2} \ldots \partial x^{i_r}} \bigg|_{\mathbf{x}} = 0 \right\} \subset C_1 = \left\{ \mathbf{x} \in U : \left(\frac{\partial F^j}{\partial x^i} \right) = 0_{n \times m} \right\}$$

we have

$$C_k \cap V_{\mathbf{x}} \subseteq H^{-1}(0) \cap V_{\mathbf{x}} \subseteq C_1 \cap V_{\mathbf{x}} \subseteq C_1$$

so for any $\mathbf{p} \in C_k \cap V_{\mathbf{x}}$ we have

$$dF|_{\mathbf{p}} = 0_{n \times m}$$

and so is not surjective, thus its restriction

$$\big(dF|_{H^{-1}(0) \cap V_{\mathbf{x}}} \big) \big|_{\mathbf{p}} = (dF|_{\mathbf{p}}) \big|_{T_{\mathbf{p}}(H^{-1}(0) \cap V_{\mathbf{x}})} \subseteq dF|_{\mathbf{p}}$$

must not be surjective. And so we have

$$F(C_k \cap V_{\mathbf{x}}) \subseteq F|_{H^{-1}(0) \cap V_{\mathbf{x}}} : H^{-1}(0) \cap V_{\mathbf{x}} \to \mathbb{R}^n$$

then since

$$\dim(H^{-1}(0) \cap V_{\mathbf{x}}) = \dim(U) - \dim(\mathbb{R}) = m - 1 < m$$

our inductive hypothesis says

$$F|_{H^{-1}(0) \cap V_{\mathbf{x}}} \Big(\overbrace{H^{-1}(0) \cap V_{\mathbf{x}}}^{\text{critical points of } F|_{H^{-1}(0) \cap V_{\mathbf{x}}}} \Big) = \{\text{critical values of } F|_{H^{-1}(0) \cap V_{\mathbf{x}}}\}$$

has measure zero. Or, letting

$$S = H^{-1}(0) \cap V_{\mathbf{x}}$$

we have

$$F|_S(S) = \{\text{critical values of } F|_S\}$$

has measure zero, and so

$$\mu\big(F|_S(S)\big) = 0$$

Then as U is open, $\forall\ \mathbf{x} \in C_k \subseteq U, \exists\ V_{\mathbf{x}} \subseteq U$ which is open, and since $C_k \subseteq U$ is closed we have

$$\{U \setminus C_k\} \cup \{V_{\mathbf{x}} : \mathbf{x} \in C_k\}$$

is an open cover for U. And since $\mathbb{R}^m$ is second countable, we have $U \subseteq \mathbb{R}^m$ is second countable, and since every open cover of a second countable topological space has a countable subcover, we have

$$(U \setminus C) \cup \bigcup_{\substack{\mathbf{x}_i \in C \\ i \in I}} V_{\mathbf{x}_i} \supseteq U$$

then

$$\begin{aligned}
C \cap U &= C \cap \left(\bigcup_{\substack{\mathbf{x}_i \in C \\ i \in I}} V_{\mathbf{x}_i} \cup (U \setminus C) \right) \\
&= \left(C \cap \bigcup_{\substack{\mathbf{x}_i \in C \\ i \in I}} V_{\mathbf{x}_i} \right) \cup \big(C \cap (U \setminus C) \big) \\
&= \bigcup_{\substack{\mathbf{x}_i \in C \\ i \in I}} \big(C \cap V_{\mathbf{x}_i} \big)
\end{aligned}$$

and thus

$$\begin{aligned}
F(C_k \setminus C_{k+1}) &\subseteq F(C_k \cap U) \\
&= F \left(\bigcup_{\substack{\mathbf{x}_i \in C \\ i \in I}} \big(C_k \cap V_{\mathbf{x}_i} \big) \right) \\
&= \bigcup_{\substack{\mathbf{x}_i \in C \\ i \in I}} F\big(C_k \cap V_{\mathbf{x}_i} \big) \\
&\subseteq \bigcup_{\substack{\mathbf{x}_i \in C \\ i \in I}} F|_{H^{-1}(0) \cap V_{\mathbf{x}_i}} (H^{-1}(0) \cap V_{\mathbf{x}_i}) \\
&= \bigcup_{\substack{\mathbf{x}_i \in C \\ i \in I}} F|_{S_i}(S_i)
\end{aligned}$$

is a countable union of sets of measure zero and therefore has measure zero, which gives

$$\begin{aligned}
\mu\big(F(C_k \setminus C_{k+1})\big) &\leq \mu\big(F(C_k \cap U)\big) \\
&= \mu \left(\bigcup_{\substack{\mathbf{x}_i \in C \\ i \in I}} F|_{S_i}(S_i) \right) \\
&\leq \sum_{i \in I} \mu\big(F(C \cap \overline{V}_{\mathbf{x}_i})\big) \\
&= 0
\end{aligned}$$

and therefore $F(C_k \setminus C_{k+1})$ has measure zero.

Case 3: Taking $k+1$ to be sufficiently large, so that $k+1 > \frac{m}{n}$, and decomposing U, into the union of countably many closed cubes of the form

$$Q = [a_1, b_1] \times \cdots \times [a_m, b_m] \text{ where each } a_i, b_i \in \mathbb{Q}$$

so that

$$U = \bigcup_{\substack{Q_i \subset U \\ i \in I}} Q_i$$

then for each $\mathbf{x} \in U$ we have that $\mathbf{x} \in Q_i \subseteq U$ for some i. So let $\mathbf{x} \in C_{k+1}$ be arbitrary, and since $C_{k+1} \subset U$ we also have $\mathbf{x} \in U$ which then implies that $\mathbf{x} \in Q_i$ for some i.

Next, since F is smooth and Q_i is closed, for and other $\mathbf{y} \in Q_i$ we have for each $1 \leq j \leq n$, $\exists\ M > 0$ such that

$$|F^j(\mathbf{x}) - F^j(\mathbf{y})| \leq M \cdot |\mathbf{x} - \mathbf{y}|^{k+2}$$

to see this let $f(t) = F^j\big(t\mathbf{y} + (1-t)\mathbf{x}\big) - F^j(\mathbf{x})$ then

$$\begin{aligned}
f(1) &= F^j(\mathbf{y}) - F^j(\mathbf{x}) \\
f(0) &= F^j(\mathbf{x}) - F^j(\mathbf{x}) = 0
\end{aligned}$$

and

$$\begin{aligned}
f'(0) &= \left.\frac{\partial F^j}{\partial x^{i_1}}\right|_{\mathbf{x}} \cdot (\mathbf{y} - \mathbf{x}) = 0 && \text{as } \mathbf{x} \in C_{k+1} \\
f''(0) &= \left.\frac{\partial^2 F^j}{\partial x^{i_1} \partial x^{i_2}}\right|_{\mathbf{x}} \cdot (\mathbf{y} - \mathbf{x})^2 = 0 \\
&\vdots \\
f^{(k+1)}(0) &= \left.\frac{\partial^{k+1} F^j}{\partial x^{i_1} \dots \partial x^{i_{k+1}}}\right|_{\mathbf{x}} \cdot (\mathbf{y} - \mathbf{x})^{k+1} = 0 \\
|f^{(k+2)}(t)| &= \left| \sum_{i_1, \dots, i_{k+1} = 1}^{m} \left.\frac{\partial^{k+2} F^j}{\partial x^{i_1} \dots \partial x^{i_{k+2}}}\right|_{t\mathbf{y} + (1-t)\mathbf{x}} \cdot (\mathbf{y} - \mathbf{x})^{k+2} \right| \\
&\leq (cnst) \cdot |\mathbf{y} - \mathbf{x}|^{k+2}
\end{aligned}$$

with this bound, we then get

$$\begin{aligned}
|f^{(k+1)}(t)| &\leq (cnst) \cdot |\mathbf{y}-\mathbf{x}|^{k+2} \int_0^t ds \\
&= (cnst) \cdot |\mathbf{y}-\mathbf{x}|^{k+2} \cdot t \\
\implies |f^{(k)}(t)| &\leq (cnst) \cdot |\mathbf{y}-\mathbf{x}|^{k+2} \int_0^t s ds \\
&= \frac{(cnst)}{2} |\mathbf{y}-\mathbf{x}|^{k+2} \cdot t^2 \\
&\vdots \\
\implies |f(t)| &\leq \frac{(cnst)}{(k+2)!} |\mathbf{y}-\mathbf{x}|^{k+2} \cdot t^{k+2} \\
\implies |f(1)| = |F^j(\mathbf{y}) - F^j(\mathbf{x})| &\leq M \cdot |\mathbf{y}-\mathbf{x}|^{k+2}
\end{aligned}$$

then allowing j to run over all values $1 \leq j \leq n$ we get

$$|F(\mathbf{x}) - F(\mathbf{y})| \leq \left(M \cdot |\mathbf{x}-\mathbf{y}|^{k+2}\right)^n$$

Now if the side length of $Q_i = L$, subdividing Q_i into cubes of side length $\frac{L}{N}$ so that

$$Q_i = \left(\left[\frac{a_{1_1}}{N}, \frac{a_{1_2}}{N}\right] \times \cdots \times \left[\frac{a_{1_{N-1}}}{N}, \frac{b_1}{N}\right]\right) \times \cdots \times \left(\left[\frac{a_{m_1}}{N}, \frac{a_{m_2}}{N}\right] \times \cdots \times \left[\frac{a_{m_{N-1}}}{N}, \frac{b_m}{N}\right]\right)$$

Then labeling the subcubes of Q_i as $(Q_{i,1}, \ldots, Q_{i,N^m})$ so if $\mathbf{a} \in C_{k+1} \cap Q_{i,j}$ and for any other $\mathbf{y} \in Q_{i,j}$ we have

$$|F(\mathbf{y}) - F(\mathbf{a})| \leq \left(2M\sqrt{m}\left(\frac{L}{N}\right)^{k+2}\right)^n$$

and so

$$
\begin{aligned}
F(C_{k+1} \cap U) &= F\left(C_{k+1} \cap \bigcup_{\substack{Q_i \subset U \\ i \in I}} Q_i\right) \\
&= \bigcup_{\substack{Q_i \subset U \\ i \in I}} F(C_{k+1} \cap Q_i) \\
&= \bigcup_{\substack{Q_i \subset U \\ i \in I}} F\left(C_{k+1} \cap \bigcup_{j=1}^{N^m} Q_{i,j}\right) \\
&= \bigcup_{\substack{Q_i \subset U \\ i \in I}} \bigcup_{j=1}^{N^m} F(C_{k+1} \cap Q_{i,j}) \\
&\leq \bigcup_{\substack{Q_i \subset U \\ i \in I}} \bigcup_{j=1}^{N^m} \left(\left(2M\sqrt{m}\right)^n \left(\frac{L}{N}\right)^{n(k+2)}\right) \\
&\leq \bigcup_{\substack{Q_i \subset U \\ i \in I}} \left(\left(2M\sqrt{m}\right)^n \left(\frac{L}{N}\right)^{n(k+2)}\right) \cdot N^m \\
&= \bigcup_{\substack{Q_i \subset U \\ i \in I}} \left(2M\sqrt{m}L^{k+2}\right)^n \cdot N^{-n(k+2)+m} \\
&= \bigcup_{\substack{Q_i \subset U \\ i \in I}} M' \cdot N^{-n(k+2)+m}
\end{aligned}
$$

where we recall

$$k+1 > \frac{m}{n} \implies -(k+1) < -\frac{m}{n}$$

then we note

$$-n(k+2)+m = -n(k+1) - n + m < -n\left(\frac{m}{n}\right) - n + m = -n$$

so our constant

$$M' \cdot N^{-n(k+2)+m}$$

can be made as small as desired by taking N sufficiently large and therefore

$$\mu\big(F(C_{k+1} \cap Q_{i,j})\big) = 0$$

furthermore, $F(C_{k+1})$ is the countable union of sets of measure zero and thus has measure zero, as

$$\begin{aligned}
\mu\big(F(C_{k+1}\cap U)\big) &= \mu\left(\bigcup_{\substack{Q_i\subset U\\ i\in I}}\bigcup_{j=1}^{N^m} F(C_{k+1}\cap Q_{i,j})\right)\\
&\leq \sum_{i\in I}\sum_{j=1}^{N^m}\mu\big(F(C_{k+1}\cap Q_{i,j})\big)\\
&= 0
\end{aligned}$$

□

Corollary 119. Suppose M and N are smooth manifolds with or without boundary, and

$$F : M \to N$$

is a smooth map. If $\dim(M) < \dim(N)$, then $F(M) \subseteq N$ has measure zero in N.

Proof. If $\dim(M) < \dim(N)$ then since $\dim(T_pM) = \dim(M)$ and $\dim(T_{F(p)}N) = \dim(N)$, each point of M must be critical. And so by Sard's theorem $F(M) \subseteq N$ has measure zero. □

Corollary 120. Suppose M is a smooth manifold with or without boundary, and $S \subseteq M$ is an immersed submanifold with or without boundary. If $\dim(S) < \dim(M)$; then S has measure zero in M.

Proof. Since $S \subseteq M$ is immersed we have

$$\iota : S \hookrightarrow M$$

is smooth, then by Corollary 119, since $\dim(S) < \dim(M)$, we have that $\iota(S)$ will have measure zero in M. □

Lemma 121. If $M \subseteq \mathbb{R}^N$ is a smooth submanifold with or without boundary of dimension n and $N > 2n + 1$, then M admits an injective immersion into $\mathbb{R}^{N-1}$.

Proof. First, since $M \subseteq \mathbb{R}^N$ is a smooth submanifold, it is at least an immersed submanifold, and so

$$\iota : M \to \mathbb{R}^N$$

is an injective smooth immersion. Next we note that every $(N-1)$-dimensional hyperplane in $\mathbb{R}^N$, is uniquely determined by its 1-dimensional orthogonal compliment, which can be taken to be a 1-dimensional line through the origin. The space of all such 1-dimensional line through the origin in $\mathbb{R}^N$, is $\mathbb{RP}^{N-1}$.

Next, for any $[\mathbf{v}] \in \mathbb{RP}^{N-1}$ let

$$O_{[\mathbf{v}]} = \big\{\mathbf{u} \in \mathbb{R}^N : \langle \mathbf{u}, \mathbf{v}\rangle = 0\big\} \cong \mathbb{R}^{N-1}$$

that is $O_{[\mathbf{v}]}$ is the orthogonal compliment of $\mathbf{v}$ in $\mathbb{R}^N$. And define

$$\pi_{[\mathbf{v}]} : \mathbb{R}^N \to O_{[\mathbf{v}]}, \text{ by } \pi_{[\mathbf{v}]}(\mathbf{x}) = \mathbf{x} - \frac{\langle \mathbf{x}, \mathbf{v} \rangle}{||\mathbf{v}||}\mathbf{v}$$

to be the orthogonal projection to the hyperplane $O_{[\mathbf{v}]}$.

Next we consider the map

$$\pi_{[\mathbf{v}]} \circ \iota : M \to O_{[\mathbf{v}]}$$

which is smooth as the composition of smooth maps. Now, if $\pi_{[\mathbf{v}]} \circ \iota$ is not injective, then there exists $p, q \in M$ with $p \neq q$ such that

$$\pi_{[\mathbf{v}]} \circ \iota(p) = \pi_{[\mathbf{v}]}(p) = \pi_{[\mathbf{v}]}(q) \implies \pi_{[\mathbf{v}]}(p) - \pi_{[\mathbf{v}]}(q) = \pi_{[\mathbf{v}]}(p - q) = \mathbf{0} \quad \text{by linearity}$$

that is $p - q \neq 0$, yet $t(p - q) = \mathbf{v}$, and so $[p - q] = [\mathbf{v}]$. Thus, for the smooth map

$$\alpha : (M \times M) \setminus \Delta_M \to \mathbb{RP}^{N-1}, \text{ by } \alpha(p, q) = [p - q]$$

we have $[\mathbf{v}] \in \alpha\big((M \times M) \setminus \Delta_M\big) \subseteq \mathbb{RP}^{N-1}$. Yet, by Corollary 119, since

$$\dim\big((M \times M) \setminus \Delta_M\big) = 2m < N - 1 = \dim(\mathbb{RP}^{N-1})$$

we have $\alpha\big((M \times M) \setminus \Delta_M\big) \subseteq \mathbb{RP}^{N-1}$ has measure zero in $\mathbb{RP}^{N-1}$. Thus we have the set of $[\mathbf{v}]$'s such that $\pi_{[\mathbf{v}]} \circ \iota$ is not injective has measure zero.

Next, if $\pi_{[\mathbf{v}]} \circ \iota$ is not a smooth immersion, then $\exists\ p \in M$ such that $v_p \in T_pM$, where $v_p \neq 0$ gives

$$d(\pi_{[\mathbf{v}]} \circ \iota)|_p(v_p) = \mathbf{0}$$

now since the differential of a smooth map is a linear approximation of the smooth map and $\pi_{[\mathbf{v}]}$ is linear, we have $d\pi_{[\mathbf{v}]} = \pi_{[\mathbf{v}]}$. And so we have

$$d\pi_{[\mathbf{v}]}|_{\iota(p)} \circ d\iota|_p(v_p) = \pi_{[\mathbf{v}]} \circ d\iota|_p(v_p) = \mathbf{0}$$

so we have $d\iota|_p(v_p) \neq 0$, yet $d\iota|_p(v_p) = t\mathbf{v}$, which gives $[d\iota|_p(v_p)] = [\mathbf{v}]$, so for the set defined by

$$M_0 = \{(p, 0) \in TM : p \in M\}$$

we have

$$TM \setminus M_0 = \{(p, v_p) \in TM : v_p \neq 0\}$$

thus, for the smooth map

$$\beta : TM \setminus M_0 \to \mathbb{RP}^{N-1}, \text{ by } \beta(p, v_p) = [d\iota|_p(v_p)]$$

we have $[\mathbf{v}] \in \beta\big(TM \setminus M_0\big) \subseteq \mathbb{RP}^{N-1}$, Yet again, by Sard's Theorem since

$$\dim\big(\beta\big(TM \setminus M_0\big)\big) = 2m < N - 1 = \dim(\mathbb{RP}^{N-1})$$

we have $\beta\big(TM \setminus M_0\big) \subseteq \mathbb{RP}^{N-1}$ has measure zero in $\mathbb{RP}^{N-1}$. Thus we have the set of $[\mathbf{v}]$'s such that $\pi_{[\mathbf{v}]} \circ \iota$ is not a smooth submersion has measure zero. And sine the union of sets of measure zero has measure zero we have

$$\mu\Big(\alpha\big((M \times M) \setminus \Delta_M\big) \cup \beta\big(TM \setminus M_0\big)\Big) = 0 \quad \text{in } \mathbb{RP}^{N-1}$$

and therefore $\mathbb{RP}^{N-1} \setminus \Big(\alpha\big((M \times M) \setminus \Delta_M\big) \cup \beta\big(TM \setminus M_0\big)\Big)$ is dense in $\mathbb{RP}^{N-1}$.

Therefore, the set of $[\mathbf{v}]$'s such that $\pi_{[\mathbf{v}]} \circ \iota$ is an injective immersion are dense. $\square$

Iterating this argument repeatedly we see that if M is a smooth n-manifold that admits an injective smooth immersion into some euclidean space, it admits an injective smooth immersion into $\mathbb{R}^{2n+1}$

Lemma 122. Let M be a smooth n-manifold with or without boundary. If M admits a smooth embedding into $\mathbb{R}^N$ for some $N \in \mathbb{N}$, then it admits a proper smooth embedding into $\mathbb{R}^{2n+1}$.

Proof. Let

$$F : M \to \mathbb{R}^N$$

be an arbitrary smooth embedding, and consider the diffeomorphism

$$\Phi : \mathbb{R}^N \to \mathbb{B}^N, \text{ by } \Phi(\mathbf{x}) = \frac{\mathbf{x}}{1 + ||\mathbf{x}||^2}$$

Since M is a smooth manifold, by Proposition 28, it admits a positive exhaustion function $f \in C^\infty(M)$, such that

$$f : M \to \mathbb{R}_+$$

and define

$$\Psi : M \to \mathbb{R}^N \times \mathbb{R}, \text{ by } \Psi(p) = (\Phi \circ F(p), f(p))$$

since $\Phi \circ F$ is the composition of a smooth embedding and a diffeomorphism, and so is a smooth embedding. Thus Ψ is injective and

$$d\Psi|_p : T_pM \to T_{\Psi(p)}(\mathbb{R}^N \times \mathbb{R})$$

is injective for each $p \in M$ and so $d\Psi$ is an injective smooth immersion. Furthermore, for any $K \subseteq \mathbb{R}^{N+1}$ compact, we will have $\Psi^{-1}(K) \subseteq f^{-1}\big((-\infty, c]\big)$, since f is an exhaustion of M, and since Ψ is continuous the preimage of a compact, hence closed, set K in $\mathbb{R}^{N+1}$ will be closed in M, so $\Psi^{-1}(K) \subseteq M$ is a closed subset contained in compact $f^{-1}\big((-\infty, c]\big)$, and thus is compact. Therefore, the preimage of compact sets under Ψ are compact, and so Ψ is a proper map. Thus, by Proposition 62, Ψ is a smooth embedding. And further we note

$$\Psi(M) \subseteq \mathbb{B}^N \times \mathbb{R}$$

Since Ψ is an injective smooth immersion, by Lemma 121, if $N+1 > 2n+1$, M admits the injective smooth immersion

$$\pi_{[\mathbf{v}]} \circ \Psi : M \to O_{[\mathbf{v}]}$$

where

$$O_{[\mathbf{v}]} = \big\{\mathbf{u} \in \mathbb{R}^{N+1} : \langle \mathbf{u}, \mathbf{v} \rangle = 0\big\} \cong \mathbb{R}^N$$

and since the set of $[\mathbf{v}] \in \mathbb{RP}^N$ such that $\pi_{[\mathbf{v}]} \circ \Psi$ is an injective smooth immersion is dense in $\mathbb{RP}^N$, we may choose $[\mathbf{v}]$ such that

$$[\mathbf{v}] \neq [0 : 0 : \cdots : 0 : 1]$$

so WLOG let $\mathbf{v} = (v_1, \ldots, v_{N+1}) \in \mathbb{S}^N$, then

$$[\mathbf{v}] \neq [0 : 0 : \cdots : 0 : 1] \implies |v_{N+1}| < 1$$

and as $||\mathbf{v}|| = 1$ we then get

$$\pi_{[\mathbf{v}]}(\mathbf{x}) = \mathbf{x} - \langle \mathbf{x}, \mathbf{v} \rangle \mathbf{v}$$

and so

$$\begin{aligned}
\pi_{[\mathbf{v}]} \circ \Psi(p) &= \pi_{[\mathbf{v}]}(\Phi \circ F(p), f(p)) \\
&= (\Phi \circ F(p), f(p)) - \langle (\Phi \circ F(p), f(p)), (\mathbf{v}', v_{N+1}) \rangle (\mathbf{v}', v_{N+1}) \\
&= (*, f(p)(1 - v_{N+1}^2) - (\Phi \circ F(p) \cdot \mathbf{v}')v_{N+1})
\end{aligned}$$

Now, for any compact $K \subseteq \mathbb{R}^{N+1}$ since it is closed and bounded $\exists\, C \in \mathbb{R}$ such that

$$K \subseteq \left\{ \mathbf{x} \in \mathbb{R}^{N+1} : |x_{N+1}| < C \right\}$$

and since

$$\Psi(M) \subseteq \mathbb{B}^N \times \mathbb{R} \implies ||\Phi \circ F(p)|| < 1 \quad \forall\, p \in M$$

and so we have

$$\begin{aligned}
&||\Phi \circ F(p)|| < 1 \\
&||\mathbf{v}'|| \leq 1 \\
&|v_{N+1}| < 1
\end{aligned}$$

then

$$\begin{aligned}
&\left| f(p)(1 - v_{N+1}^2) - (\Phi \circ F(p) \cdot \mathbf{v}')v_{N+1} \right| < C \\
&\implies \left| f(p)(1 - v_{N+1}^2) \right| < C + \left| (\Phi \circ F(p) \cdot \mathbf{v}')v_{N+1} \right| \\
&\implies \left| f(p)(1 - v_{N+1}^2) \right| < C + 1 \\
&\implies |f(p)| < \frac{C+1}{1 - v_{N+1}^2}
\end{aligned}$$

and thus we have

$$\pi_{[\mathbf{v}]} \circ \Psi(K) \subseteq f^{-1}\left(\Big(-\infty, \tfrac{C+1}{1 - v_{N+1}^2} \Big] \right)$$

then as $\pi_{[\mathbf{v}]} \circ \Psi$ is an injective smooth immersion it is continuous, and hence $\pi_{[\mathbf{v}]} \circ \Psi(K) \subseteq M$ is a closed subset contained in compact $f^{-1}\left(\Big(-\infty, \frac{C+1}{1 - v_{N+1}^2} \Big] \right)$, and is therefore compact. Thus, the preimage of compact sets under $\pi_{[\mathbf{v}]} \circ \Psi$ is compact, and therefore $4\pi_{[\mathbf{v}]} \circ \Psi$ is a proper map.

Thus we have, $\pi_{[\mathbf{v}]} \circ \Psi$ is a proper injective smooth immersion, and thus by Proposition 62, $\pi_{[\mathbf{v}]} \circ \Psi$

is a smooth embedding. And since every proper continuous map into a locally compact Hausdorff space is a closed map, and $M \subseteq M$ is closed, we have that

$$\pi_{[\mathbf{v}]} \circ \Psi(M) \subseteq \mathbb{R}^N$$

is a properly embedded submanifold.

We can now iterate this process until we have a proper smooth embedding of M into $\mathbb{R}^{2n+1}$. $\square$

Theorem 123 (**Whitney Embedding Theorem**). Every smooth n-manifold with or without boundary admits a proper smooth embedding into $\mathbb{R}^{2n+1}$.

Proof. Let M be a smooth n-manifold with or without boundary. By Lemma 122, it suffices to show that M admits a smooth embedding into some euclidean space.

First let us suppose that M is compact, and for each $p \in M$ and select $(U_p, \phi_p) \in \mathcal{A}_M$ containing p, since M is manifold it admits a basis of regular coordinate balls, or half-balls, for each p select $B_p \subseteq U_p$ containing p such that $\overline{B_p} \subseteq U_p$ and $\overline{B_p}$ is compact. Then $\{B_p\}_{p \in M}$ form a cover of M, and since M is compact it admits a finite subcover

$$\bigcup_{i=1}^{m} B_{p_i} \supseteq M$$

where

$$\phi_{p_i} : U_{p_i} \to \phi p_i(U_{p_i}) \subseteq \mathbb{R}^n$$

is a diffeomorphism, and hence $\phi_{p_i}(\overline{B_{p_i}}) \subseteq \mathbb{R}^n$ is compact. Then, for each i choose a smooth bump function ψ_i for $\overline{B_{pi}}$ supported in U_{p_i}, then we have, for each i

$$\begin{aligned} &0 \leq \psi_i(M) \leq 1 \\ &\psi_i(q) \equiv 1 \quad \forall\, q \in \overline{B_{p_i}} \\ &\operatorname{supp}(\psi_i) \subseteq U_{p_i} \end{aligned}$$

Then for each i we have $\psi_i \phi_{p_i} \in C^\infty(U_{p_i})$ which extends to a smooth function

$$\psi_i \phi_{p_i} = M \to \mathbb{R}, \text{ by } (\psi_i \phi_{p_i})(q) = \begin{cases} \psi_i(q)\phi_{p_i}(q), & q \in U_{p_i} \\ 0, & q \in M \setminus \operatorname{supp}(\psi_i) \end{cases}$$

and so we may define the smooth map

$$F : M \to \mathbb{R}^{nm+m}, \text{ by } F(p) = \Big(\psi_1(p)\phi_{p_1}(p), \ldots, \psi_m(p)\phi_{p_m}(p), \psi_1(p), \ldots, \psi_m(p)\Big)$$

now suppose that $p, q \in M$ are such that $F(p) = F(q)$, then there is some i, since the B_{p_i}'s cover M, such that $q \in B_{p_i}$, then since $\psi_i \equiv 1$ on $\overline{B_{p_i}}$ we have $\psi_i(q) = 1$ then

$$F(p) = F(q) \implies \psi_i(q) = \psi_i(p) = 1$$

and so $q \in \operatorname{supp}(\psi_i) \subseteq U_{p_i}$. Thus we have

$$\phi_{p_i}(q) = \psi_i(q)\phi_{p_i}(q) = \psi_i(p)\phi_{p_i}(p) = \phi_{p_i}(p)$$

and since ϕ_{p_i} is a diffeomorphism onto its image, it is injective and therefore we have that $p = q$, and so F is injective.

Next, let $p \in M$ be given and fix j such that $p \in B_{p_j}$, then since $\psi_j = 1 = cnst$ on B_{p_j} we have
$$d(\psi_j \phi_{p_j})|_p = \phi_{p_j}(p) d\psi_i|_p + \psi_i(p) d\phi_{p_j}|_p = 0 + 1 \cdot d\phi_{p_j}|_p = d\phi_{p_j}|_p$$
which is injective since ϕ_{p_j} is a local diffeomorphism, by Proposition 54. Then, since if any component of a linear map is injective the map is injective, we have that $dF|_p$ is injective, as it will have full rank in row j of its Jacobian matrix at $\phi_{p_j}(p)$.

Thus, F is an injective smooth immersion, and since M is compact, by Proposition 62, F is a smooth embedding.

The argument above still works when M is an arbitrary compact subset of a larger manifold $\widetilde{M}$ with or without boundary, by covering M with finitely many coordinate balls or half-balls for $\widetilde{M}$. The result is a smooth injective map

$$F : M \to \mathbb{R}^{nm+m}$$

whose differential is injective at each point.

Now suppose that M is non-compact, and let

$$f : M \to \mathbb{R}$$

be a smooth exhaustion function for M, since f is smooth, by Sard's theorem its set of critical values has measure zero in $\mathbb{R}$, so for each $i \in \mathbb{N}$, there are regular values $a_i, b_i \in \mathbb{R}$, for f, such that

$$i < a_i < b_i < i + 1$$

and so we define subsets $A_i, B_i \subseteq M$ by

$$\begin{aligned} A_0 &= f^{-1}\big((-\infty, 1]\big) \\ B_0 &= f^{-1}\big((-\infty, a_1]\big) \\ A_i &= f^{-1}\big([i, i+1]\big) \\ B_i &= f^{-1}\big([b_{i-1}, a_{i+1}]\big) \end{aligned}$$

therefore, for each i we have
$$A_i \subseteq \mathrm{Int}(B_i)$$
and $M = \bigcup_i A_i$, with
$$B_i \cap B_j = \varnothing, \quad \text{unless } j \in \{i-1, i, i+1\}$$
Then from above, since each B_i is compact, for each i, there is a smooth embedding of B_i into some euclidean space and so by Lemma 122, there is an embedding

$$\phi_i : B_i \to \mathbb{R}^{2n+1}$$

for each i choose a smooth bump function ψ_i for A_i supported in $\text{Int}(B_i)$, then we have, for each i

$$\begin{aligned}
&0 \le \psi_i(M) \le 1 \\
&\psi_i(p) \equiv 1 \quad \forall\, p \in A_i \\
&\text{supp}(\psi_i) \subseteq \text{Int}(B_i)
\end{aligned}$$

Then for each i we have $\psi_i\phi_i \in C^\infty(\text{Int}(B_i))$ which extends to a smooth function

$$\psi_i\phi_i = M \to \mathbb{R}, \text{ by } (\psi_i\phi_i)(q) = \begin{cases} \psi_i(q)\phi_i(q), & q \in \text{Int}(B_i) \\ 0, & q \in M \setminus \text{supp}(\psi_i) \end{cases}$$

and so we may define the smooth map

$$F : M \to \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1}\mathbb{R}, \text{ by } F(p) = \left(\sum_{i \text{ even}} \psi_i(p)\phi_i(p), \sum_{i \text{ odd}} \psi_i(p)\phi_i(p), f(p) \right)$$

noting that for each $p \in M$ at most one term in each sum is non-zero in a neighborhood of p, we have that F is smooth, and is proper because f is.

Suppose that $F(p) = F(q)$ for some $p, q \in M$, then $\exists\, i \in \mathbb{N}$ such that

$$f(p) = f(q) \in [i, i+1] \implies p, q \in A_i \subseteq \text{Int}(B_i)$$

and therefore $\phi_i(p) = \phi_i(q)$, and since ϕ_i is a diffeomorphism onto its image, it is injective and therefore we have that $p = q$, and so F is injective.

Next, let $p \in M$ be given and fix j such that $p \in A_j$, WLOG let j be even, then since $\psi_j = 1 = cnst$ on A_j we have

$$d(\psi_j\phi_j|_p = \phi_j(p)d\psi_i|_p + \psi_i(p)d\phi_j|_p = 0 + 1 \cdot d\phi_j|_p = d\phi_j|_p$$

which is injective since ϕ_j is a local diffeomorphism, by Proposition 54 and so $d\phi_j|_p \neq 0$, and therefore

$$dF|_p = (d\phi_j|_p, *, *) \neq 0$$

and so $dF|_p$ is injective.

Thus we have, F is a proper injective smooth immersion, and thus by Proposition 62, F is a smooth embedding. $\square$

Corollary 124. Every smooth n-dimensional manifold with or without boundary is diffeomorphic to a properly embedded submanifold (with or without boundary) of $\mathbb{R}^{2n+1}$. How do we get diffeomorphism out of a proper smooth embedding?

Corollary 125. Suppose M is a compact smooth n-manifold with or without boundary. If $N \geq 2n+1$, then every smooth map from M to $\mathbb{R}^N$ can be uniformly approximated by embeddings.

Proof. Let $N \geq 2n+1$ and let

$$f : M \to \mathbb{R}^N$$

be a smooth map. By Whitney's Embedding Theorem, M admits a proper smooth embedding

$$F : M \to \mathbb{R}^{2n+1}$$

then the map

$$G := f \times F : M \to \mathbb{R}^N \times \mathbb{R}^{2n+1}$$

is also a smooth embedding. Where

$$f = \pi^{\mathbb{R}^N} \circ G : M \to \mathbb{R}^N$$

then with $G(M) \subseteq \mathbb{R}^N \times \mathbb{R}^{2n+1}$ as an embedded submanifold, and hence an immersed submanifold, Lemma 121, then says that the $\exists\ \mathbf{v}_{N+2n+1} = (v_1, \dots v_N, v_{N+1}, \dots, v_{N+2n+1}) \in \mathbb{R}^N \times \mathbb{R}^{2n+1}$ such that $\mathbf{v}_{N+2n+1}$ is arbitrarily close to the standard basis element $\mathbf{e}_{N+2n+1}$; that is, given $\epsilon > 0$, $\mathbf{v}_{N+2n+1}$ can be chosen such that

$$\sup_{\mathbf{x} \in G(M)} \left| \pi_{[\mathbf{v}_{N+2n+1}]}(\mathbf{x}) - \pi_{[\mathbf{e}_{N+2n+1}]}(\mathbf{x}) \right| < \epsilon$$

so we have the smooth embedding

$$\pi_{[\mathbf{v}_{N+2n+1}]} : G(M) \to \mathbb{R}^N \times \mathbb{R}^{2n}$$

which then gives $\pi_{[\mathbf{v}_{N+2n+1}]}(G(M)) \subseteq \mathbb{R}^N \times \mathbb{R}^{2n}$ is an embedded submanifold, and hence an immersed submanifold, so by Lemma 121, $\exists\ \mathbf{v}_{N+2n} = (v_1, \dots v_N, v_{N+1}, \dots, v_{N+2n}) \in \mathbb{R}^N \times \mathbb{R}^{2n}$ such that $\mathbf{v}_{N+2n}$ is arbitrarily close to the standard basis element $\mathbf{e}_{N+2n}$; that is, given $\epsilon > 0$, $\mathbf{v}_{N+2n}$ can be chosen such that

$$\sup_{\mathbf{x} \in \pi_{[\mathbf{v}_{N+2n+1}]}(G(M))} \left| \pi_{[\mathbf{v}_{N+2n}]}(\mathbf{x}) - \pi_{[\mathbf{e}_{N+2n}]}(\mathbf{x}) \right| < \epsilon$$

so we have the smooth embedding

$$\pi_{[\mathbf{v}_{N+2n}]} : \pi_{[\mathbf{v}_{N+2n+1}]}(G(M)) \to \mathbb{R}^N \times \mathbb{R}^{2n-1}$$

which then gives $\pi_{[\mathbf{v}_{N+2n}]}\big(\pi_{[\mathbf{v}_{N+2n+1}]}(G(M))\big) \subseteq \mathbb{R}^N \times \mathbb{R}^{2n-1}$ is an embedded submanifold.

Iterating this process, we get a set of vectors $\{\mathbf{v}_i\}_{i=N+1}^{N+2n+1}$ arbitrarily close to the set of basis vectors $\{\mathbf{e}_i\}_{i=N+1}^{N+2n+1}$, such that given $\epsilon > 0$

$$\sup_{\mathbf{x} \in G(M)} \left| \pi_{[\mathbf{v}_{N+1}]} \circ \dots \circ \pi_{[\mathbf{v}_{N+2n+1}]}(\mathbf{x}) - \pi_{[\mathbf{e}_{N+1}]} \circ \dots \circ \pi_{[\mathbf{e}_{N+2n+1}]}(\mathbf{x}) \right| < \epsilon$$

yet $\pi_{[\mathbf{e}_{N+1}]} \circ \dots \circ \pi_{[\mathbf{e}_{N+2n+1}]} = \pi^{\mathbb{R}^N}$, and so we have the smooth embedding

$$\pi_{[\mathbf{v}_{N+1}]} \circ \dots \circ \pi_{[\mathbf{v}_{N+2n+1}]} \circ G : M \to \mathbb{R}^N$$

arbitrarily close to f; that is given any $\epsilon > 0$ we have

$$\sup_{p \in M} \left| \pi_{[\mathbf{v}_{N+1}]} \circ \dots \circ \pi_{[\mathbf{v}_{N+2n+1}]} \circ G(p) - f(p) \right| < \epsilon$$

$\square$

Theorem 126 (**Whitney Immersion Theorem**)**.** Every smooth n-manifold with or without boundary admits a smooth immersion into $\mathbb{R}^{2n}$.

Theorem 127 (**Strong Whitney Embedding Theorem**)**.** If $n > 0$, every smooth n-manifold admits a smooth embedding into $\mathbb{R}^{2n}$.

Theorem 128 (**Strong Whitney Immersion Theorem**)**.** If $n > 1$, every smooth n-manifold admits a smooth immersion into $\mathbb{R}^{2n-1}$.

Theorem 129 (**Whitney Approximation Theorem for Functions**)**.** Suppose M is a smooth manifold with or without boundary, and

$$F : M \to \mathbb{R}^k$$

is a continuous function. Given any positive continuous function

$$\delta : M \to \mathbb{R}_+$$

there exists a smooth function

$$\widetilde{F} : M \to \mathbb{R}^k$$

that is δ-close to F. If F is smooth on a closed subset $A \subseteq M$; then $\widetilde{F}$ can be chosen to be equal to F on A.

Proof. Let $A \subseteq M$ be closed, such that

$$F : A \to \mathbb{R}^k$$

is smooth. Then, by the Extension Lemma for Smooth Functions, $\exists\ F'$ such that

$$F' : M \to \mathbb{R}^k$$

is smooth and $F'|_A = F$. So define

$$U_0 = \{p \in M : |F'(p) - F(p)| < \delta(p)\}$$

then $U_0 \subseteq M$ is open and $U_0 \supseteq A$. If $\nexists\ A$, then $U_0 = \varnothing$, and $F' \equiv 0$.

Next, for any $p \in M \setminus A$, by the continuity of F and δ there exists an open neighborhood

$$U_p = \{q \in M \setminus A : |F(q) - F(p)| < \delta(q)\}$$

then the collection $\{U_p\}_{p \in M \setminus A}$ is an open cover of $M \setminus A$, and since M is second countable, we may take countably many such that

$$\bigcup_{i=1}^{\infty} U_{p_i} \supseteq M \setminus A$$

and therefore $\{U_0, U_i\}$ is an indexed open cover for M, so we may select a smooth partition of unity $\{\psi_0, \psi_i\}$ subordinate to $\{U_0, U_i\}$, and we can define the map

$$\widetilde{F} : M \to \mathbb{R}^k, \text{ by } \widetilde{F}(q) = \psi_0(q)F'(q) + \sum_{i=1}^{\infty} \psi_i(q)F(p_i)$$

which is smooth since the summation is locally finite, and for each $q \in A$ we have

$$\widetilde{F}(q) = 1 \cdot F'(q) + \sum_{i=1}^{\infty} 0 \cdot F(p_i) = F'(q) = F(q)$$

Further, for any $q \in M$ we will have

$$\begin{aligned}
|\widetilde{F}(q) - F(q)| &= \left|\psi_0(q)F'(q) + \sum_{i=1}^{\infty}\psi_i(q)F(p_i) - 1 \cdot F(q)\right| \\
&= \left|\psi_0(q)F'(q) + \sum_{i=1}^{\infty}\psi_i(q)F(p_i) - \left(\psi_0(q) + \sum_{i=1}^{\infty}\psi_i(q)\right)(q)\right| \qquad \sum_{i=0}^{\infty}\psi_i(q) = 1 \ \forall \ q \in M \\
&\le \psi_0(q)\big|F'(q) - F(q)\big| + \sum_{i=1}^{\infty}\psi_i(q)\big|F(p_i) - F(q)\big| \\
&< \psi_0(q)\delta(q) + \sum_{i=1}^{\infty}\psi_i(q)\delta(q) \\
&= \delta(q)
\end{aligned}$$

and so $\widetilde{F}$ is δ-close to F. □

Corollary 130. If M is a smooth manifold with or without boundary and

$$\delta : M \to \mathbb{R}_+$$

is a positive continuous function, there is a smooth function

$$\mathfrak{e} : M \to \mathbb{R}_+$$

such that $0 < \mathfrak{e}(p) < \delta(p), \ \forall \ p \in M$.

Proof. Since δ is continuous, by the Whitney Approximation Theorem $\exists \ \mathfrak{e}$ such that

$$\mathfrak{e} : M \to R$$

is smooth, and is δ-close to δ. In particular we may take

$$\left|\mathfrak{e}(p) - \frac{\delta(p)}{2}\right| < \frac{\delta(p)}{2}, \quad \forall \ p \in M$$

□

Theorem 131. If $M \subseteq \mathbb{R}^n$ is an embedded m-dimensional submanifold, then NM is an embedded n-dimensional submanifold of $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$.

Proof. Let $\mathbf{x}_0 \in M$ and let $(U, \phi) \in \mathcal{A}_M$ be a slice chart for M in $\mathbb{R}^n$, centered at $\mathbf{x}_0$, then

$$M \cap U = \{\mathbf{z} \in U : y^{m+1}(\mathbf{z}) = \cdots y^n(\mathbf{z}) = 0\}$$

then

$$\left.\frac{\partial}{\partial y^i}\right|_{\mathbf{z}} = d\phi^{-1}|_{\phi(\mathbf{z})}\left(\left.\frac{\partial}{\partial y^i}\right|_{\phi(\mathbf{z})}\right)$$

yet with the chart $(\mathbb{R}^n, Id_{\mathbb{R}^n}) \in \mathcal{A}_{\mathbb{R}^n}$ letting $Id_{\mathbb{R}^n} = (x^1, \ldots, x^n)$ we have another basis given by

$$\left.\frac{\partial}{\partial x^i}\right|_{\mathbf{z}} = d(Id_{\mathbb{R}^n})^{-1}|_{Id_{\mathbb{R}^n}(\mathbf{z})}\left(\left.\frac{\partial}{\partial y^i}\right|_{Id_{\mathbb{R}^n}(\mathbf{z})}\right)$$

then the two bases are related by

$$\begin{aligned}
\left.\frac{\partial}{\partial y^i}\right|_{\mathbf{z}} &= d\phi^{-1}|_{\phi(\mathbf{z})}\left(\left.\frac{\partial}{\partial y^i}\right|_{\phi(\mathbf{z})}\right)\\
&= d\Big(Id_{\mathbb{R}^n}^{-1} \circ Id_{\mathbb{R}^n} \circ \phi^{-1}\Big)\Big|_{\phi(\mathbf{z})}\left(\left.\frac{\partial}{\partial y^i}\right|_{\phi(\mathbf{z})}\right)\\
&= d(Id_{\mathbb{R}^n}^{-1})|_{Id_{\mathbb{R}^n}(\mathbf{z})} \circ d(Id_{\mathbb{R}^n} \circ \phi^{-1})|_{\phi(\mathbf{z})}\left(\left.\frac{\partial}{\partial y^i}\right|_{\phi(\mathbf{z})}\right)\\
&= d(Id_{\mathbb{R}^n}^{-1})|_{Id_{\mathbb{R}^n}(\mathbf{z})}\left(\sum_{j=1}^{n}\frac{\partial(x^j \circ \phi^{-1})}{\partial y^i}(\phi(\mathbf{z}))\left.\frac{\partial}{\partial x^j}\right|_{Id_{\mathbb{R}^n}\circ\phi^{-1}(\phi(\mathbf{z}))}\right)\\
&= \sum_{j=1}^{n}\frac{\partial(x^j \circ \phi^{-1})}{\partial y^i}(\phi(\mathbf{z})) \cdot d(Id_{\mathbb{R}^n}^{-1})|_{Id_{\mathbb{R}^n}(\mathbf{z})}\left(\left.\frac{\partial}{\partial x^j}\right|_{Id_{\mathbb{R}^n}(\mathbf{z})}\right)\\
&= \sum_{j=1}^{n}\frac{\partial(x^j \circ \phi^{-1})}{\partial y^i}(\phi(\mathbf{z}))\left.\frac{\partial}{\partial x^j}\right|_{\mathbf{z}}
\end{aligned}$$

and hence are smooth. Then defining the map

$$\Phi : U \times \mathbb{R}^n \to \phi(U) \times \mathbb{R}^n$$

given by

$$\Phi(\mathbf{x}, \mathbf{v}) = \left(y^1(\mathbf{x}), \cdots y^n(\mathbf{x}), \cdot \sum_{i=1}^{n} v^i \frac{\partial(x^1 \circ \phi^{-1})}{\partial y^i}(\phi(\mathbf{x})), \cdots, \sum_{i=1}^{n} v^i \frac{\partial(x^n \circ \phi^{-1})}{\partial y^i}(\phi(\mathbf{x}))\right)$$

then the total derivative of Φ at $(\mathbf{x}, \mathbf{v})$ is

$$D\Phi(\mathbf{x}, \mathbf{v}) = \begin{bmatrix} \left(\frac{\partial y^j}{\partial x^i}(\mathbf{x})\right)_{\substack{i=1,\ldots,m\\ j=1,\ldots,m}} & O \\ * & \left(v^i \frac{\partial(x^n \circ \phi^{-1})}{\partial y^i}(\phi(\mathbf{x}))\right)_{\substack{i=1,\ldots,n-m\\ j=1,\ldots,n-m}} \end{bmatrix}$$

which is invertible, and so Φ is a local diffeomorphism. If $\Phi(\mathbf{x}, \mathbf{v}) = \Phi(\mathbf{x}', \mathbf{v}')$, then since ϕ is injective we have $\mathbf{x} = \mathbf{x}'$, and

$$\sum_{i=1}^{n} v^i \frac{\partial(x^j \circ \phi^{-1})}{\partial y^i}(\phi(\mathbf{x})) = \sum_{i=1}^{n} (v')^i \frac{\partial(x^j \circ \phi^{-1})}{\partial y^i}(\phi(\mathbf{x})) \quad \forall\ j$$

implies

$$\sum_{i=1}^{n}(v-v')^i \frac{\partial(x^j \circ \phi^{-1})}{\partial y^i}(\phi(\mathbf{x})) = 0 \quad \forall\ j$$

which then implies that $(\mathbf{v}-\mathbf{v}') \perp \text{span}\left\{\sum_{i=1}^{n} v^i \frac{\partial(x^1\circ\phi^{-1})}{\partial y^i}(\phi(\mathbf{x})), \cdots, \sum_{i=1}^{n} v^i \frac{\partial(x^n\circ\phi^{-1})}{\partial y^i}(\phi(\mathbf{x}))\right\}$ and so $\mathbf{v}-\mathbf{v}'=\mathbf{0}$. Thus we have that Φ is injective, and defines a smooth chart on $U\times\mathbb{R}^n$. Then for $NM \subseteq \mathbb{R}^n \times \mathbb{R}^n$, we have

$$\Phi\big((U\times\mathbb{R}^n)\cap NM\big) = \{(\mathbf{x},\mathbf{z}) \in \mathbb{R}^n\times\mathbb{R}^n : x_{m+1} = \cdots = x_n = 0,\ z_1 = \cdots = z_m = 0\}$$

or

$$U\times\mathbb{R}^n\cap NM =$$
$$\left\{(\mathbf{x},\mathbf{z}) \in U\times\mathbb{R}^n : y^{m+1}(\mathbf{x}) = \cdots = y^n(\mathbf{x}) = 0,\ \sum_{i=1}^{n} v^i \frac{\partial(x^1\circ\phi^{-1})}{\partial y^i}(\phi(\mathbf{x})) = \cdots = \sum_{i=1}^{n} v^i \frac{\partial(x^m\circ\phi^{-1})}{\partial y^i}(\phi(\mathbf{x})) = 0\right\}$$

and so $(U\times\mathbb{R}^n,\Phi) \in \mathcal{A}_{\mathbb{R}^n\times\mathbb{R}^n}$ is an n-slice chart for NM, and so satisfies the local slice condition, and so NM is an embedded n-submanifold of $\mathbb{R}^n\times\mathbb{R}^n$. □

Theorem 132 (Tubular Neighborhood Theorem). Every embedded submanifold of $\mathbb{R}^n$ has a tubular neighborhood.

Proof. Let $M_{\mathbf{0}} \subseteq NM \subseteq \mathbb{R}^n\times\mathbb{R}^n$ be defined by $\{(\mathbf{x},\mathbf{0}) : \mathbf{x}\in M\}$ and with the smooth mapping

$$E : NM \to \mathbb{R}^n,\ \text{by}\ E(\mathbf{x},\mathbf{v}) = \mathbf{x}+\mathbf{v}$$

we have

$$E|_{M_{\mathbf{0}}} : M_{\mathbf{0}} \to M$$

acts as the identity map, and so is a diffeomorphism, and so

$$dE|_{(\mathbf{x},\mathbf{0})} : T_{(\mathbf{x},\mathbf{0})}M_{\mathbf{0}} \subseteq T_{(\mathbf{x},\mathbf{0})}NM \to T_{E(\mathbf{x},\mathbf{0})}M = T_{\mathbf{x}}M$$

is an isomorphism by Proposition 34. We also have

$$E|_{N_{\mathbf{x}}M} : N_{\mathbf{x}}M \to N_{\mathbf{x}}M + \mathbf{v}$$

is a diffeomorphism, and so

$$dE|_{(\mathbf{x},\mathbf{0})} : T_{(\mathbf{x},\mathbf{0})}N_{\mathbf{x}}M \subseteq T_{(\mathbf{x},\mathbf{0})}NM \to T_{E(\mathbf{x},\mathbf{0})}N_{\mathbf{x}}M = N_{\mathbf{x}}M$$

is also an isomorphism. Then as

$$T_{\mathbf{x}}\mathbb{R}^n = T_{\mathbf{x}}M \oplus N_{\mathbf{x}}M$$

and so $dE|_{(\mathbf{x},\mathbf{0})}$ is surjective onto $T_{\mathbf{x}}\mathbb{R}^n$, and since

$$\dim\left(T_{(\mathbf{x},\mathbf{0})}NM\right) = n = \dim(T_{\mathbf{x}}\mathbb{R}^n)$$

we have the E is bijective, and so is invertible. So by the Inverse Function Theorem, there exists a neighborhood $V_{(\mathbf{x},\mathbf{0})} \subseteq NM$ containing $(\mathbf{x},\mathbf{0})$, such that

$$E|_{V_{(\mathbf{x},\mathbf{0})}} : V_{(\mathbf{x},\mathbf{0})} \to E(V_{(\mathbf{x},\mathbf{0})})$$

is a diffeomorphism. And we may take this neighborhood to be

$$V_{(\mathbf{x},\mathbf{0})} = V_{\delta_{\mathbf{x}}} = \{(\mathbf{x}',\mathbf{v}') \in NM : ||\mathbf{x}-\mathbf{x}'|| < \delta_{\mathbf{x}},\ ||\mathbf{v}'|| < \delta_{\mathbf{x}}\}$$

for some $\delta_{\mathbf{x}>0}$. And since $(\mathbf{x},\mathbf{0}) \in M_{\mathbf{0}}$ was arbitrary we conclude that such a neighborhood may be found for each point in $M_{\mathbf{0}}$.

Next, for each $\mathbf{x} \in M$, let $\rho(\mathbf{x}) = \sup_{\delta\leq 1}\{V_{\delta_{\mathbf{x}}}\}$ such that

$$E|_{V_{\delta_{\mathbf{x}}}} : V_{\delta_{\mathbf{x}}} \to E(V_{\delta_{\mathbf{x}}})$$

is a diffeomorphism. Then

$$\rho : M \to \mathbb{R}_+$$

is positive. Now let $\mathbf{x},\mathbf{y} \in M$ be given, and suppose that $||\mathbf{x}-\mathbf{y}|| < \rho(\mathbf{x})$, then by the triangle inequality we have $V_{\delta_{\mathbf{y}}} \subseteq V_{\rho(\mathbf{x})}$ for $\delta_{\mathbf{y}} = \rho(\mathbf{x}) - ||\mathbf{x}-\mathbf{y}||$ which then implies that

$$\rho(\mathbf{y}) \geq \rho(\mathbf{x}) - ||\mathbf{x}-\mathbf{y}|| \implies \rho(\mathbf{x}) - \rho(\mathbf{y}) \leq ||\mathbf{x}-\mathbf{y}||$$

and reversing the roles of $\mathbf{x}$ and $\mathbf{y}$ we get

$$||\rho(\mathbf{x}) - \rho(\mathbf{y})|| \leq ||\mathbf{x}-\mathbf{y}||$$

and so ρ is Lipschitz and is hence continuous. Further, for any $(\mathbf{x},\mathbf{v}),(\mathbf{y},\mathbf{v}') \in V_{\rho(\mathbf{x})}$, there is some $\delta < \rho(\mathbf{x})$ such that $(\mathbf{x},\mathbf{v}),(\mathbf{y},\mathbf{v}') \in V_{\delta}$, and so E is injective on $V_{\rho(\mathbf{x})}$.

Now define

$$V = \left\{(\mathbf{x},\mathbf{v}) \in NM : ||\mathbf{v}|| < \frac{\rho(\mathbf{x})}{2}\right\}$$

then for any $(\mathbf{x},\mathbf{v}),(\mathbf{y},\mathbf{v}') \in V$ if

$$\begin{aligned} &E(\mathbf{x},\mathbf{v}) = E(\mathbf{y},\mathbf{v}') \\ &\implies \mathbf{x}+\mathbf{v} = \mathbf{y}+\mathbf{v}' \\ &\implies ||\mathbf{x}-\mathbf{y}|| = ||\mathbf{v}'-\mathbf{v}|| \end{aligned}$$

where

$$||\mathbf{v}'|| + ||\mathbf{v}|| \leq \frac{\rho(\mathbf{y})}{2} + \frac{\rho(\mathbf{x})}{2} \leq \max\{\rho(\mathbf{y}),\rho(\mathbf{x})\}$$

WLOG suppose that $\rho(\mathbf{y}) \leq \rho(\mathbf{x})$, then $(\mathbf{x},\mathbf{v}),(\mathbf{y},\mathbf{v}') \in V_{\rho(\mathbf{x})}$, and E is injective on $V_{\rho(\mathbf{x})}$, and therefore $(\mathbf{x},\mathbf{v}) = (\mathbf{y},\mathbf{v}')$. Then since $E|_V$ is a local diffeomorphism, it is an open map by Proposition 53 (c), and so we have $E(V) \subseteq \mathbb{R}^n$ is open.

Finally we have that E is a smooth, bijective local diffeomorphism, and so is therefore a diffeomorphism by Proposition 53 (f), and so $E(V)$ is a tubular neighborhood for M. □

Proposition 133. Let $M \subseteq \mathbb{R}^n$ be an embedded submanifold. If $U_M \subseteq \mathbb{R}^n$ is any tubular neighborhood of M; there exists a smooth map

$$r : U_M \to M$$

that is both a retraction and a smooth submersion.

Proof. Let $NM \subseteq T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$, be the normal bundle of M, and let $M_{\mathbf{0}} \subseteq NM$ be defined by $\{(\mathbf{x}, \mathbf{0}) : \mathbf{x} \in M\}$, then as U_M is a tubular neighborhood, by definition, there exists $V \subseteq NM$ such that

$$E : V \to U_M$$

is a diffeomorphism. So define

$$r = \pi_{NM} \circ E^{-1}|_{U_M} : U_M \to M$$

then since both E and π_{NM} are smooth, r is smooth as the composition of smooth functions. And for any $\mathbf{x} \in M$ we have

$$r(\mathbf{x}) = \pi_{NM} \circ E^{-1}(\mathbf{x}) = \pi_{NM}(\mathbf{x}, \mathbf{0}) = \mathbf{x}$$

and so $r|_M = Id_M$, and so r is a retraction. Then as $\pi|_{NM}$ is a smooth submersion and E is a diffeomorphism, we have that r is a smooth submersion as the composition of a smooth submersion and a diffeomorphism. ☐

Theorem 134 (**Whitney Approximation Theorem**)**.** Suppose M is a smooth manifold with or without boundary, N is a smooth manifold (without boundary), and

$$F : M \to N$$

is a continuous map. Then F is homotopic to a smooth map. If F is already smooth on a closed subset $A \subseteq M$, then the homotopy can be taken to be relative to A.

Proof. By the Whitney Embedding Theorem, we may suppose that $N \subseteq \mathbb{R}^n$, then as an embedded submanifold of euclidean space the Tubular Neighborhood Theorem says that there exists $V \subseteq NN \subseteq \mathbb{R}^n \times \mathbb{R}^n$ of the form $\{(\mathbf{x}, \mathbf{v}) \in NN : ||\mathbf{v}|| < \delta(\mathbf{x})\}$ such that the map

$$E|_V : V \to U_N, \text{ by } E(\mathbf{x}, \mathbf{v}) = \mathbf{x} + \mathbf{v}$$

is a diffeomorphism and, where $N \subseteq U_N$, that is U_N is a tubular neighborhood of N. Then as U_N is a tubular neighborhood of an embedded submanifold in euclidean space, by Proposition 133 there exists a smooth map

$$r : U_N \to N$$

that is both a retraction and a smooth submersion. Then, for any $\mathbf{x} \in N$ let

$$\delta(\mathbf{x}) = \sup\{\epsilon \leq 1 : B_\epsilon(\mathbf{x}) \subseteq U_N\}$$

and for $\mathbf{x}, \mathbf{y} \in N$ such that $||\mathbf{x} - \mathbf{y}|| < \delta(\mathbf{x})$ we will have $B_\epsilon(\mathbf{y}) \subseteq B_{\delta(\mathbf{x})}(\mathbf{x})$, for $\epsilon = \delta(\mathbf{x}) - ||\mathbf{x} - \mathbf{y}||$ which implies

$$\delta(\mathbf{y}) \geq \delta(\mathbf{x}) - ||\mathbf{x} - \mathbf{y}|| \implies \delta(\mathbf{x}) - \delta(\mathbf{y}) \leq ||\mathbf{x} - \mathbf{y}||$$

and reversing the roles of $\mathbf{x}$ and $\mathbf{y}$ gives

$$||\delta(\mathbf{x}) - \delta(\mathbf{y})|| \leq ||\mathbf{x} - \mathbf{y}||$$

and so

$$\delta : N \to \mathbb{R}_+$$

is continuous. Then as the composition of continuous maps, we have the positive continuous map

$$\widetilde{\delta} := \delta \circ F : M \to \mathbb{R}_+$$

and so, by the Whitney Approximation Theorem for Functions, $\exists\ \widetilde{F} \in C^\infty(M)$ such that $\widetilde{F}$ is δ-close to F; i.e.

$$\big|\big|\widetilde{F}(p) - F(p)\big|\big| < \delta(F(p)) \quad \forall\ p \in M$$

and so therefore $\widetilde{F}(p) \in B_{\delta(F(p))}(F(p)) \subseteq U_N$, and as the open ball $B_{\delta(F(p))}(F(p))$ in euclidean space is path-connected we have the line segment

$$(1-t)F(p) + t \cdot \widetilde{F}(p) \in B_{\delta(F(p))}(F(p)) \quad \forall\ t \in [0,1]$$

and so we define

$$H : M \times [0,1] \to N, \text{ by } H(p,t) = r\Big((1-t)F(p) + t\widetilde{F}(p)\Big)$$

then $\forall\ p \in M$

$$\begin{aligned} H(p,0) &= r(F(p)) = F(p) && r|_N = Id_N \\ H(p,1) &= r(\widetilde{F}(p)) \end{aligned}$$

where $r \circ \widetilde{F}$ is smooth as the composition of smooth maps. Thus, H is a homotopy connecting the continuous function F to the smooth function $r \circ \widetilde{F}$.

If $A \subseteq M$ is closed and F is smooth on A, then the Whitney Approximation Theorem for Functions says that $\widetilde{F}|_A = F$, and so for any $q \in A$ we have

$$H(q,t) = F(q) \quad \forall\ t \in [0,1]$$

and thus, the homotopy is relative to A. $\square$

Corollary 135 (Extension Lemma for Smooth Maps). Suppose M is a smooth manifold with or without boundary, N is a smooth manifold, $A \subseteq M$ is a closed subset, and

$$f : A \to N$$

is a smooth map. Then f has a smooth extension to M if and only if it has a continuous extension to M.

Proof. Suppose that f has the continuous extention

$$F : M \to N$$

then, the Whitney Approximation Theorem says that $\exists\ \widetilde{F} \in C^{\text{inf}}(M)$ which is homotopic to F such that $\widetilde{F}|_A = F|_A = f$, and thus, $\widetilde{F}$ is a smooth extension of f.

If f admits a smooth extension, then since smooth implies continuity the extension is also continuous. $\square$

Lemma 136. If M and N are smooth manifolds with or without boundary, smooth homotopy is an equivalence relation on the set of all smooth maps from N to M.

Theorem 137. Suppose M is a smooth manifold with or without boundary, N is a smooth manifold, and

$$F, G : M \to N$$

are smooth maps. If F and G are homotopic, then they are smoothly homotopic. If F and G are homotopic relative to some closed subset $A \subseteq M$, then they are smoothly homotopic relative to A.

Proof. Let

$$F, G : M \to N$$

be smooth and let

$$H : M \times [0,1] \to N$$

be a homotopy joining F to G, relative to A with may be empty. Then define the extension

$$\widetilde{H} : M \times \mathbb{R} \to N, \text{ by } \widetilde{H}(p,t) = \begin{cases} H(p,0), & t \le 0 \\ H(p,t), & 0 \le t \le 1 \\ H(p,1), & t \ge 1 \end{cases}$$

then $\widetilde{H}$ is continuous by the gluing lemma for continuous maps. And

$$\widetilde{H}|_{M\times\{0\}} = F \circ \pi_1$$
$$\widetilde{H}|_{M\times\{1\}} = G \circ \pi_1$$

and so $\widetilde{H}$ is smooth on the closed subsets $M \times \{0\}, M \times \{1\} \subseteq M \times \mathbb{R}$, if H is a homotopy relative to A, then $F|_A = G|_A$ and so

$$\widetilde{H}|_{A\times[0,1]} = F \circ \pi_1$$

is smooth. Since $M \times \mathbb{R}$ and N are both smooth manifolds, and $\widetilde{H}$ a continuous map, the Whitney Approximation Theorem says $\exists\ \widehat{H} \in C^\infty(M \times \mathbb{R}, N)$ which is homotopic to $\widetilde{H}$, such that

$$\widehat{H}|_{M\times\{0\}\cup M\times\{1\}\cup A\times[0,1]} = \widetilde{H}|_{M\times\{0\}\cup M\times\{1\}\cup A\times[0,1]} = H$$

and therefore

$$\widehat{H}|_{M\times[0,1]} : M \times [0,1] \to N$$

has a smooth extention on a neighborhood of $M \times [0,1]$ and is therefore a smooth homotopy, relative to A, between F and G. $\square$

3 Problems

1-5 *Suppose M is a locally Euclidean Hausdorff space. Show that M is second countable if and only if it is paracompact and has countably many connected components. [Hint: assuming M is paracompact, show that each component of M has a locally finite cover by precompact coordinate domains, and extract from this a countable subcover.]*

Proof. First suppose that M is second countable. Then by Proposition 1.11 M has countably many components, and by Theorem 1.15 M is also paracompact.

Next, suppose that M has countably many connected components and is paracompact. And we note that since the connected components $\{C_i\}_{i=1}^{\infty}$ of M partition M, they are all disjoint; that is

$$C_i \cap C_j = \varnothing, \quad \text{if } i \neq j$$

And we also note that connected components are closed. So if C_i is a connected component of M, and if $\mathcal{U}_{C_i}$ is any open cover of C_i, then, since connected components are closed $M \setminus C_i$ must be open and so

$$\mathcal{U}_{C_i} \cup M \setminus C_i$$

is an open cover of M, and by paracompactness has a locally finite open refinement, say $\mathcal{U}_M$. Now since C_i and $M \setminus C_i = C_i^c$ are disjoint, any refinement $\mathcal{V}_{C_i^c}$ of the open set C_i^c will remain disjoint from C_i, and so

$$\mathcal{U}_M \setminus \mathcal{V}_{C_i^c}$$

is a refinement of $\mathcal{U}_{C_i}$, and so we have that each connected component of M, is individually paracompact.

Next, for any $p \in C_i$ since $p \in M$, and M is locally euclidean, say of dimension n, so for each $p \in M$ there exists an open subset $U \subseteq M$ such that $p \in U$, and an open subset $\widehat{U} \subseteq \mathbb{R}^n$ such that

$$\phi : U \to \widehat{U} = \phi(U)$$

is a homeomorphism. Then since

$$\mathcal{B} = \{B_r(\mathbf{x}) \subseteq \mathbb{R}^n : r \in \mathbb{Q}; \mathbf{x} \in \mathbb{Q}^n\}$$

is a countable basis of precompact open sets for the topology of $\mathbb{R}^n$, it will also be a countable base of precompact open sets for the relative topology of $\phi(U)$. Then since ϕ is a homeomorphism and

$$\mathcal{B} \supseteq \phi(U) \implies \phi^{-1}(\mathcal{B}) = \{\phi^{-1}(B) : B \in \mathcal{B}\} \supseteq \phi^{-1}(\phi(U)) = U$$

and so $\{\phi^{-1}(B) : B \in \mathcal{B}\}$ is a countable base for the topology on U. Moreover, ϕ^{-1} is continuous, and since continuous functions map compact sets into compact sets we have that $\phi^{-1}(\overline{B}) \subseteq U$ is compact, and

$$\overline{\phi^{-1}(B)} \subseteq \phi^{-1}(\overline{B})$$

and so $\overline{\phi^{-1}(B)}$ is compact as the closed subset of a compact set, and thus we see that $\{\phi^{-1}(B) : B \in \mathcal{B}\}$ is a countable base for U of precompact coordinate balls. So the union of all such sets for each $p \in C_i$ will be an open cover of C_i by precompact coordinate balls; that is

$$\bigcup_{p \in C_i} U_p = \bigcup_{p \in C_i} \{\phi_p^{-1}(B) : B \in \mathcal{B}\} \supseteq C_i$$

and by the paracompactness of C_i, this open cover admits a locally finite open refinement $\mathcal{U}_{C_i}$. And since $\mathcal{U}_{C_i}$ refines $\bigcup_{p \in C_i} \{\phi_p^{-1}(B) : B \in \mathcal{B}\}$, this means that $\forall\, V \in \mathcal{U}_{C_i},\ \exists\, \phi_p^{-1}(B) \in \bigcup_{p \in C_i} \{\phi_p^{-1}(B) : B \in \mathcal{B}\}$ such that

$$V \subseteq \phi_p^{-1}(B) \implies \overline{V} \subseteq \overline{\phi_p^{-1}(B)}$$

which will be compact as the closed subset of a compact set, and therefore $\mathcal{U}_{C_i}$, consists of precompact sets.

Now fix $U_1 \in \mathcal{U}_{C_i}$, we wish to consider the closer $\overline{U}_1$, since any cover of $\overline{U}_1$ will also be a cover of U_1. For any $p \in \overline{U}_1$, by the local finiteness of $\mathcal{U}_{C_i}$, $\exists\ B_p$ open such that

$$B_p \cap V = \varnothing$$

for all but finitely many $V \in \mathcal{U}_{C_i}$, say $\{V_1, \dots, V_n\}$. doing this for each $p \in \overline{U}_1$ we get an open cover of $\overline{U}_1$, namely

$$\bigcup_{p \in \overline{U}_1} \bigcup_{i=1}^{n} V_{p_i} \supseteq \overline{U}_1$$

Now, since we know, from above, that $\overline{U}_1$ is compact, it admits a finite subcover; i.e.

$$\bigcup_{j=1}^{m_1} \bigcup_{i=1}^{n} V_{p_{i_j}} \supseteq \overline{U}_1$$

and since this is a cover of $\overline{U}_1$ it must also be a cover of U_1. So set

$$U_2 = \bigcup_{j=1}^{m_1} \bigcup_{i=1}^{n} V_{p_{i_j}}$$

and since this was a cover for U_1 we know that $U_1 \subseteq U_2$, and so our goal is to find a cover of the additional points contained in the $V_{p_{i_j}}$'s that were not contained in U_1.

To that end, we similarly considering the closer $\overline{U}_2$, which will also be compact as the union of compact sets, where we can find an open cover where only finitely many elements in $\mathcal{U}_{C_i}$ contain each point of $\overline{U}_2$. Where the compactness of $\overline{U}_2$ then allows us to find a finite subcover, and we can set

$$U_3 = \bigcup_{j=1}^{m_2} \bigcup_{i=1}^{n} V_{p_{i_j}}$$

continuing inductively, once U_k is defined, using the local finiteness of our open cover and the compactness of the closure of U_k we can define

$$U_{k+1} = \bigcup_{j=1}^{m_k} \bigcup_{i=1}^{n} V_{p_{i_j}}$$

Then setting

$$\mathcal{V}_i = \bigcup_{k=1}^{\infty} U_k$$

Which will be open as the the union of open sets, and $\mathcal{V}_i \subseteq C_i$ since it was built from elements in C_i. Now consider arbitrary $q \in C_i \setminus \mathcal{V}_i$, since $\mathcal{U}_{C_i}$ is a cover of C_i, $\exists\ V_q \in \mathcal{U}_{C_i}$ open and containing q. And suppose, for contradiction, that

$$V_q \cap \mathcal{V}_i \neq \varnothing$$

yet, this then implies that there is some U_k with $p \in U_k$ such that

$$B_p \cap V_q \neq \varnothing$$

and by our construction this then implies that $V_q \in U_{k+1} \Rightarrow\Leftarrow$. And so we must have that

$$V_q \cap \mathcal{V}_i = \varnothing$$

That is, for each $q \in \mathcal{V}_i^c$ in the relative topology of C_i, we can find an open neighborhood entirely contained in $\mathcal{V}_i^c$, and therefore

$$\mathcal{V}_i^c = C_i \setminus \mathcal{V}_i$$

must be open, and therefore $\mathcal{V}_i \subseteq C_i$ is closed. Hence, we have that $\mathcal{V}_i$ is both open and closed in connected C_i and hence

$$\mathcal{V}_i = C_i$$

and since $\mathcal{V}_i$ is countable, by construction, we have a countable base for C_i, and therefore C_i is second countable.

Finally, since the union of countable sets is countable, and since countably many connected components partition M, we have

$$\bigsqcup_{i=1}^{\infty} C_i = M \implies \bigsqcup_{i=1}^{\infty} \mathcal{V}_i = M$$

and so M has a countable base, and is therefore second countable. $\square$

1-7 *Let N denote the north pole $(0,\dots,0,1) \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$, and let S denote the south pole $(0,\dots,0,-1)$ Define the stereographic projection*

$$\phi_N : \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n, \text{ by } \phi_N(x_1,\dots,x_n) = \frac{(x_1,\dots,x_n)}{1-x_{n+1}}$$

and let $\phi_S(\mathbf{x}) = -\phi_N(-\mathbf{x})$ for $\mathbf{x} \in \mathbb{S}^n \setminus \{S\}$

(a) *For any $\mathbf{x} \in \mathbb{S}^n \setminus \{N\}$, show that $\phi_N(\mathbf{x}) = \mathbf{u}$, where $(\mathbf{u},0)$ is the point where the line through N and $\mathbf{x}$ intersects the linear subspace where $x_{n+1}=0$. Similarly, show that $\phi_S(\mathbf{x})$ is the point where the line through S and $\mathbf{x}$ intersects the same subspace. (For this reason, ϕ_S is called stereographic projection from the south pole.)*

(b) *Show that ϕ_N is bijective, and*

$$\phi_N^{-1}(u_1,\dots,u_n) = \frac{(2u_1,\dots 2u_n, ||\mathbf{u}||^2-1)}{||\mathbf{u}||^2+1}$$

(c) *Compute the transition map $\phi_S \circ \phi_N^{-1}$ and verify that the atlas consisting of the two charts $(\mathbb{S}^n \setminus \{N\}, \phi_N)$ and $(\mathbb{S}^n \setminus \{S\}, \phi_S)$ defines a smooth structure on $\mathbb{S}^n$. (The coordinates defined by ϕ_N or ϕ_S are called stereographic coordinates.)*

(d) *Show that this smooth structure is the same as the one defined in Example 1.31.*

Proof.

(a) Let $\mathbf{x} \in \mathbb{S}^n \setminus \{N\}$ be arbitrary, to find the line connecting N to $\mathbf{x}$ we parameterize the line by

$$\mathbf{n}(t) = (\mathbf{x}-N)t + N, \quad \text{then} \quad \mathbf{n}(0) = N,\ \mathbf{n}(1) = \mathbf{x}$$

yet we also have

$$\mathbf{n}(t) = \big((x_1,\dots,x_{n+1}) - (0,\dots,1)\big)t + (0,\dots,1) = \big(x_1t, x_2t, \dots, (x_{n+1}-1)t+1\big)$$

then the point where this line intersects the hyperplane with $x_{n+1}=0$ is given by

$$(x_{n+1}-1)t_0 + 1 = 0 \implies t_0 = \frac{-1}{x_{n+1}-1} = \frac{1}{1-x_{n+1}}$$

then we have

$$\begin{aligned}
\mathbf{n}\left(\tfrac{1}{1-x_{n+1}}\right) &= \left(x_1\left(\tfrac{1}{1-x_{n+1}}\right), \dots, x_n\left(\tfrac{1}{1-x_{n+1}}\right), (x_{n+1}-1)\left(\tfrac{1}{1-x_{n+1}}\right)+1\right) \\
&= \left(\tfrac{x_1}{1-x_{n+1}}, \dots, \tfrac{x_n}{1-x_{n+1}}, 0\right) \\
&= \frac{(x_1,\dots,x_n,0)}{1-x_{n+1}} \\
&= \big(\phi_N(\mathbf{x}), 0\big) \\
&= (\mathbf{u},0)
\end{aligned}$$

Similarly, take arbitrary $\mathbf{y} \in \mathbb{S}^n \setminus \{S\}$, where to find the line connecting S to $\mathbf{y}$ we parameterize by

$$\mathbf{s}(t) = (\mathbf{y} - S)t + S, \quad \text{then} \quad \mathbf{s}(0) = S,\ \mathbf{s}(1) = \mathbf{y}$$

and

$$\mathbf{s}(t) = \big((y_1, \ldots, y_{n+1}) - (0, \ldots, -1)\big)t + (0, \ldots, -1) = \big(y_1 t, \ldots, y_n t, (y_{n+1} + 1)t - 1\big)$$

and this line will intersect the hyperplane where $y_{n+1} = 0$ when

$$(y_{n+1} + 1)t_0 - 1 = 0 \implies t_0 = \frac{1}{1 + y_{n+1}}$$

and so we get

$$\begin{aligned}
\mathbf{s}\left(\tfrac{1}{1+y_{n+1}}\right) &= \left(y_1\left(\tfrac{1}{1+y_{n+1}}\right), \ldots, y_n\left(\tfrac{1}{1+y_{n+1}}\right), (1 + y_{n+1})\left(\tfrac{1}{1+y_{n+1}}\right) - 1\right) \\
&= \left(\tfrac{y_1}{1+y_{n+1}}, \ldots, \tfrac{y_n}{1+y_{n+1}}, 0\right) \\
&= \frac{(y_1, \ldots, y_n, 0)}{1 + y_{n+1}} \\
&= -\frac{(-y_1, \ldots, -y_n, 0)}{1 - (-y_{n+1})} \\
&= \big(-\phi_N(-\mathbf{y}), 0\big) \\
&= \big(\phi_S(\mathbf{y}), 0\big)
\end{aligned}$$

(b) First we note, that we can show that ϕ_N is a homeomorphism, and hence bijective, by computing its explicit inverse. So for

$$\phi_N^{-1} : \mathbb{R}^n \to \mathbb{S}^n \setminus \{(0, \ldots, 1)\}$$

defined by

$$\phi_N^{-1}(x_1, \ldots, x_n) = \left(\frac{2x_1}{x_1^2 + \cdots + x_n^2 + 1}, \ldots, \frac{2x_n}{x_1^2 + \cdots + x_n^2 + 1}, \frac{x_1^2 + \cdots + x_n^2 - 1}{x_1^2 + \cdots + x_n^2 + 1}\right)$$

so consider any $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, then we get

$$\begin{aligned}
\phi_N \circ \phi_N^{-1}(x_1, \dots, x_n) &= \phi_N \left(\frac{2x_1}{x_1^2 + \cdots + x_n^2 + 1}, \dots, \frac{x_1^2 + \cdots + x_n^2 - 1}{x_1^2 + \cdots + x_n^2 + 1} \right) \\
&= \left(\frac{\frac{2x_1}{x_1^2+\cdots+x_n^2+1}}{1 - \frac{x_1^2+\cdots+x_n^2-1}{x_1^2+\cdots+x_n^2+1}}, \dots, \frac{\frac{2x_n}{x_1^2+\cdots+x_n^2+1}}{1 - \frac{x_1^2+\cdots+x_n^2-1}{x_1^2+\cdots+x_n^2+1}} \right) \\
&= \left(\frac{\frac{2x_1}{x_1^2+\cdots+x_n^2+1}}{\frac{x_1^2+\cdots+x_n^2+1-x_1^2-\cdots-x_n^2+1}{x_1^2+\cdots+x_n^2+1}}, \dots, \frac{\frac{2x_n}{x_1^2+\cdots+x_n^2+1}}{\frac{x_1^2+\cdots+x_n^2+1-x_1^2-\cdots-x_n^2+1}{x_1^2+\cdots+x_n^2+1}} \right) \\
&= \left(\frac{\frac{2x_1}{x_1^2+\cdots+x_n^2+1}}{\frac{2}{x_1^2+\cdots+x_n^2+1}}, \dots, \frac{\frac{2x_n}{x_1^2+\cdots+x_n^2+1}}{\frac{2}{x_1^2+\cdots+x_n^2+1}} \right) \\
&= (x_1, \dots, x_n) \\
&= \mathbf{x}
\end{aligned}$$

next for any $\mathbf{x} = (x_1, \dots, x_{n+1}) \in \mathbb{S}^n \setminus \{N\}$ we have

$$\begin{aligned}
\phi_N^{-1} \circ \phi_N(x_1, \dots, x_{n+1}) &= \phi_N^{-1} \left(\frac{x_1}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}} \right) \\
&= \left(\frac{2\left(\frac{x_1}{1-x_{n+1}}\right)}{\left(\frac{x_1}{1-x_{n+1}}\right)^2+\cdots+\left(\frac{x_n}{1-x_{n+1}}\right)^2+1}, \dots, \frac{\left(\frac{x_1}{1-x_{n+1}}\right)^2+\cdots+\left(\frac{x_n}{1-x_{n+1}}\right)^2-1}{\left(\frac{x_1}{1-x_{n+1}}\right)^2+\cdots+\left(\frac{x_n}{1-x_{n+1}}\right)^2+1} \right) \\
&= \left(\frac{2\left(\frac{x_1}{1-x_{n+1}}\right)}{\frac{x_1^2+\cdots+x_n^2+(1-x_{n+1})^2}{(1-x_{n+1})^2}}, \dots, \frac{\frac{x_1^2+\cdots+x_n^2-(1-x_{n+1})^2}{(1-x_{n+1})^2}}{\frac{x_1^2+\cdots+x_n^2+(1-x_{n+1})^2}{(1-x_{n+1})^2}} \right) \\
&= \left(\frac{2x_1(1-x_{n+1})}{x_1^2+\cdots+x_n^2+(1-x_{n+1})^2}, \dots, \frac{x_1^2+\cdots+x_n^2-(1-x_{n+1})^2}{x_1^2+\cdots+x_n^2+(1-x_{n+1})^2} \right) \\
&= \left(\frac{2x_1(1-x_{n+1})}{x_1^2+\cdots+x_n^2+x_{n+1}^2+1-2x_{n+1}}, \dots, \frac{(1-x_{n+1}^2)-(1-x_{n+1})^2}{x_1^2+\cdots+x_n^2+x_{n+1}^2+1-2x_{n+1}} \right) \\
&= \left(\frac{2x_1(1-x_{n+1})}{2-2x_{n+1}}, \dots, \frac{1-x_{n+1}^2-1-x_{n+1}^2+2x_{n+1}}{2-2x_{n+1}} \right) \\
&= \left(\frac{2x_1(1-x_{n+1})}{2(1-x_{n+1})}, \dots, \frac{2x_{n+1}(1-x_{n+1})}{2(1-x_{n+1})} \right) \\
&= (x_1, \dots, x_{n+1}) \\
&= \mathbf{x}
\end{aligned}$$

and so

$$\phi_N \circ \phi_N^{-1} = Id_{\mathbb{R}^n}$$

and

$$\phi_N^{-1} \circ \phi_N = Id_{\mathbb{S}^n \setminus \{N\}}$$

and therefore ϕ_N is a homeomorphism between $\mathbb{S}^n \setminus \{N\}$ and $\mathbb{R}^n$, and hence ϕ_N is bijective.

(c) For the transition maps between ϕ_N and ϕ_S, where we are careful to note, that though $\mathbf{0} \in \operatorname{Im}\big(\phi_N(\mathbb{S}^n \setminus \{N\})\big)$

$$\phi_N^{-1}(\mathbf{0}) = (0, \dots, -1) = S$$

and $\phi_S(S)$ is not defined, so we must take care not to include $\mathbf{0}$ in our domain. So we have

$$\phi_S \circ \phi_N^{-1} : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}^n \setminus \{\mathbf{0}\}$$

given by

$$\begin{aligned}
(\phi_S \circ \phi_N^{-1})(x_1, \dots, x_n) &= \phi_S\left(\frac{2x_1}{x_1^2 + \cdots + x_n^2 + 1}, \dots, \frac{x_1^2 + \cdots + x_n^2 - 1}{x_1^2 + \cdots + x_n^2 + 1}\right) \\
&= \left(\frac{\frac{2x_1}{x_1^2+\cdots+x_n^2+1}}{1 + \frac{x_1^2+\cdots+x_n^2-1}{x_1^2+\cdots+x_n^2+1}}, \dots, \frac{\frac{2x_n}{x_1^2+\cdots+x_n^2+1}}{1 + \frac{x_1^2+\cdots+x_n^2-1}{x_1^2+\cdots+x_n^2+1}}\right) \\
&= \left(\frac{\frac{2x_1}{x_1^2+\cdots+x_n^2+1}}{\frac{2(x_1^2+\cdots+x_n^2)}{x_1^2+\cdots+x_n^2+1}}, \dots, \frac{\frac{2x_n}{x_1^2+\cdots+x_n^2+1}}{\frac{2(x_1^2+\cdots+x_n^2)}{x_1^2+\cdots+x_n^2+1}}\right) \\
&= \left(\frac{x_1}{x_1^2 + \cdots + x_n^2}, \dots, \frac{x_n}{x_1^2 + \cdots + x_n^2}\right)
\end{aligned}$$

which is smooth on $\mathbb{R}^n \setminus \{\mathbf{0}\}$. And we note that

$$\mathbb{S}^n \setminus \{N\} \cup \mathbb{S}^n \setminus \{S\} = \mathbb{S}$$

thus. since the charts $(\mathbb{S}^n \setminus \{N\}, \phi_N)$ and $(\mathbb{S}^n \setminus \{S\}, \phi_S)$ have coordinate domains which cover $\mathbb{S}$ and are smoothly compatible, they define a smooth structure on $\mathbb{S}$; i.e. $\big\{(\mathbb{S}^n \setminus \{N\}, \phi_N), (\mathbb{S}^n \setminus \{S\}, \phi_S)\big\} = \mathcal{A}_{\mathbb{S}}$.

(d) To show that this generates the same smooth structure, we need to show that the charts $\{(U_i^\pm, \phi_i^\pm)\}$ are smoothly compatible with $\big\{(\mathbb{S}^n \setminus \{N\}, \phi_N), (\mathbb{S}^n \setminus \{S\}, \phi_S)\big\}$ recall the definitions

$$\begin{aligned}
U_i^+ &= \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_i > 0\} \\
U_i^- &= \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_i < 0\}
\end{aligned}$$

and the mappings

$$\phi_i^\pm : U_i^\pm \cap \mathbb{S}^n \to \mathbb{B}^n, \text{ by } \phi_i^\pm(\mathbf{x}) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots x_n)$$

First we note that to ensure our coordinate domains are well defined we have $(U_i^\pm \cap \mathbb{S}^n) \bigcap ((\mathbb{S}^n \setminus \{N\}) = U_i^\pm \setminus \{N\}$ and so we will have the transition maps

$$\phi_N \circ (\phi_i^\pm)^{-1} : \phi_i^\pm(U_i^\pm \setminus \{N\}) \to \phi_N(U_i^\pm \setminus \{N\})$$

given by

$$\begin{aligned}
\phi_N \circ (\phi_i^\pm)^{-1}(x_1, \dots, x_n) &= \phi_N(x_1, \dots, x_{i-1}, \pm\sqrt{1 - ||\mathbf{x}||^2}, x_i, \dots, x_n) \\
&= \left(\frac{x_1}{1 - x_n}, \dots, \frac{x_{i-1}}{1 - x_n}, \pm\frac{\sqrt{1 - ||\mathbf{x}||^2}}{1 - x_n}, \frac{x_i}{1 - x_n}, \dots, \frac{x_{n-1}}{1 - x_n}\right)
\end{aligned}$$

for $i \neq n+1$, and when $i = n+1$ by

$$\begin{aligned}\phi_N \circ (\phi_{n+1}^{\pm})^{-1}(x_1, \ldots, x_n) &= \phi_N(x_1, \ldots, x_n, \pm\sqrt{1 - ||\mathbf{x}||^2}) \\ &= \left(\frac{x_1}{1 \mp \sqrt{1 - ||\mathbf{x}||^2}}, \ldots, \frac{x_n}{1 \mp \sqrt{1 - ||\mathbf{x}||^2}}\right)\end{aligned}$$

and so is smooth and for

$$\phi_i^{\pm} \circ \phi_N^{-1} : \phi_N(U_i^{\pm} \setminus \{N\}) \to \phi_i^{\pm}(U_i^{\pm} \setminus \{N\})$$

we get

$$\begin{aligned}\phi_i^{\pm} \circ \phi_N^{-1}(x_1, \ldots, x_n) &= \phi_i^{\pm}\left(\frac{2x_1}{x_1^2 + \cdots + x_n^2 + 1}, \ldots, \frac{x_1^2 + \cdots + x_n^2 - 1}{x_1^2 + \cdots + x_n^2 + 1}\right) \\ &= \left(\frac{2x_1}{||\mathbf{x}||^2 + 1}, \ldots, \frac{2x_{i-1}}{||\mathbf{x}||^2 + 1}, \frac{2x_{i+1}}{||\mathbf{x}||^2 + 1}, \ldots, \frac{||\mathbf{x}||^2 - 1}{||\mathbf{x}||^2 + 1}\right)\end{aligned}$$

where we need no special case when $i = n+1$ here to see that the mapping is smooth.

Similarly, for the transition maps with ϕ_S the domain will be $U_i^{\pm} \setminus \{S\}$. So for the transition

$$\phi_S \circ (\phi_i^{\pm})^{-1} : \phi_i^{\pm}(U_i^{\pm} \setminus \{S\}) \to \phi_S(U_i^{\pm} \setminus \{S\})$$

we get

$$\begin{aligned}\phi_S \circ (\phi_i^{\pm})^{-1}(x_1, \ldots, x_n) &= \phi_S(x_1, \ldots, x_{i-1}, \pm\sqrt{1 - ||\mathbf{x}||^2}, x_i, \ldots, x_n) \\ &= \left(\frac{x_1}{1 + x_n}, \ldots, \frac{x_{i-1}}{1 + x_n}, \pm\frac{\sqrt{1 - ||\mathbf{x}||^2}}{1 + x_n}, \frac{x_i}{1 - x_n}, \ldots, \frac{x_{n-1}}{1 + x_n}\right)\end{aligned}$$

for $i \neq n+1$, and when $i = n+1$ by

$$\begin{aligned}\phi_S \circ (\phi_{n+1}^{\pm})^{-1}(x_1, \ldots, x_n) &= \phi_S(x_1, \ldots, x_n, \pm\sqrt{1 - ||\mathbf{x}||^2}) \\ &= \left(\frac{x_1}{1 \pm \sqrt{1 - ||\mathbf{x}||^2}}, \ldots, \frac{x_n}{1 \pm \sqrt{1 - ||\mathbf{x}||^2}}\right)\end{aligned}$$

and so is smooth, while for

$$\phi_i^{\pm} \circ \phi_S^{-1} : \phi_S(U_i^{\pm} \setminus \{S\}) \to \phi_i^{\pm}(U_i^{\pm} \setminus \{S\})$$

we get

$$\begin{aligned}\phi_i^{\pm} \circ \phi_S^{-1}(x_1, \ldots, x_n) &= \phi_i^{\pm}\left(\frac{2x_1}{x_1^2 + \cdots + x_n^2 + 1}, \ldots, \frac{2x_n}{x_1^2 + \cdots + x_n^2 + 1}, \frac{-x_1^2 - \cdots - x_n^2 + 1}{x_1^2 + \cdots + x_n^2 + 1}\right) \\ &= \left(\frac{2x_1}{||\mathbf{x}||^2 + 1}, \ldots, \frac{2x_{i-1}}{||\mathbf{x}||^2 + 1}, \frac{2x_{i+1}}{||\mathbf{x}||^2 + 1}, \ldots, \frac{1 - ||\mathbf{x}||^2}{||\mathbf{x}||^2 + 1}\right)\end{aligned}$$

and again need no special case for $i = n+1$ to see that the map is smooth.

Thus we see that that charts $\{(U_i^{\pm}, \phi_i^{\pm})\}$ are all smoothly compatible with the charts $\big\{(\mathbb{S}^n \setminus \{N\}, \phi_N), (\mathbb{S}^n \setminus \{S\}, \phi_S)\big\}$, and therefore define the same smooth structure for $\mathbb{S}^n$.

$\square$

1-9 *Complex projective n-space, denoted by* $\mathbb{CP}^n$*, is the set of all 1-dimensional complex-linear subspaces of* $\mathbb{C}^{n+1}$*, with the quotient topology inherited from the natural projection*

$$\pi : \mathbb{C}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{CP}^n$$

Show that $\mathbb{CP}^n$ *is a compact* $2n$*-dimensional topological manifold, and show how to give it a smooth structure analogous to the one we constructed for* $\mathbb{RP}^n$*. (We use the correspondence*

$$(x_1 + iy_1, \dots, x_{n+1} + iy_{n+1}) \leftrightarrow (x_1, y_1, \dots, x_{n+1}, y_{n+1})$$

to identify $\mathbb{C}^{n+1}$ *with* $\mathbb{R}^{2n+2}$*.)*

Proof. First, since

$$\pi : \mathbb{C}^{n+1} \setminus \{\mathbf{0}\} \to \left(\mathbb{C}^{n+1} \setminus \{\mathbf{0}\}\right) / \sim, \quad \mathbf{z} \sim \mathbf{w} \iff \mathbf{w} = \lambda \mathbf{z},\ \lambda \in \mathbb{C} \setminus \{0\}$$

where $\left(\mathbb{C}^{n+1} \setminus \{\mathbf{0}\}\right) / \sim$ is identified with $\mathbb{CP}^n$. Now for $1 \le i \le n+1$ we define $U_i \subseteq \mathbb{C}^{n+1} \setminus \{\mathbf{0}\}$ by

$$U_i = \{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} : z_i \neq 0\}$$

Then we have

$$\pi(U_i) = \{[z_1 : \cdots : z_{n+1}] \in \mathbb{CP}^n : z_i \neq 0\}$$

then

$$\pi^{-1}\big(\pi(U_i)\big) = \{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} : z_i \neq 0\} = U_i$$

and so U_i is saturated, and hence $\pi(U_i) \subseteq \mathbb{CP}^n$ is open. Then, for each i, we define

$$\phi_i : \pi(U_i) \to \mathbb{C}^n, \text{ by } \phi_i([\mathbf{z}]) = \left(\frac{z_1}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_{n+1}}{z_i}\right)$$

and since $z_i \neq 0$, this is well defined, since by definition of the equivalence, $[\mathbf{z}] = [\lambda \mathbf{z}]$, $\forall\, \lambda \in \mathbb{C} \setminus \{0\}$. Further we have

$$\phi_i \circ \pi|_{U_i} : U_i \to \phi_i(\pi(U_i))$$

is continuous, and so each ϕ_i is continuous with explicit inverse given by

$$\phi_i^{-1} : \phi_i(\pi(U_i)) \to \pi(U_i), \text{ by } \phi_i^{-1}(w_1, \dots, w_n) = [w_1 : \cdots : w_{i-1} : 1 : w_i : \cdots : w_n]$$

to see this, simply note that

$$\begin{aligned}
\phi_i^{-1} \circ \phi_i([\mathbf{z}]) &= \phi_i^{-1}\left(\frac{z_1}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_{n+1}}{z_i}\right) \\
&= \left[\frac{z_1}{z_i} : \cdots : \frac{z_{i-1}}{z_i} : 1 : \frac{z_{i+1}}{z_i} : \cdots : \frac{z_{n+1}}{z_i}\right] \\
&= z_i \cdot \left[\frac{z_1}{z_i} : \cdots : \frac{z_{i-1}}{z_i} : 1 : \frac{z_{i+1}}{z_i} : \cdots : \frac{z_{n+1}}{z_i}\right] && \text{multiplication by a constant} \\
&= [z_1 : \cdots : z_{i-1} : z_i : z_{i+1} : \cdots : z_{n+1}] \\
&= [\mathbf{z}]
\end{aligned}$$

and

$$\begin{aligned}\phi_i \circ \phi_i^{-1}(\mathbf{w}) &= \phi([w_1 : \cdots : w_{i-1} : 1 : w_i : \cdots : w_n]) \\ &= \Big(\frac{w_1}{1}, \ldots, \frac{w_{i-1}}{1}, \frac{w_i}{1}, \ldots, \frac{w_n}{1}\Big) \\ &= (w_1, \ldots, w_n) \\ &= \mathbf{w}\end{aligned}$$

Then as the U_i's cover $\mathbb{C}^{n+1} \setminus \{\mathbf{0}\}$ that is

$$\bigcup_{i=1}^{n+1} U_i \supseteq \mathbb{C}^{n+1} \setminus \{\mathbf{0}\}$$

we in turn have that the $\pi(U_i)$'s cover $\mathbb{CP}^n$;

$$\bigcup_{i=1}^{n+1} \pi(U_i) \supseteq \mathbb{CP}^n$$

then we have to recall the identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$, where our homeomorphisms then become

$$\begin{aligned}&\phi_i([\mathbf{z}]) \mapsto \left(\frac{x_1 + iy_1}{x_i + iy_i}, \ldots, \frac{x_{i-1} + iy_{i-1}}{x_i + iy_i}, \frac{x_{i+1} + iy_{i+1}}{x_i + iy_i}, \ldots, \frac{x_{n+1} + iy_{n+1}}{x_i + iy_i}\right) \\ &= \left(\frac{x_1x_i + y_1y_i}{x_i^2 + y_i^2} + i\frac{y_1x_i - x_1y_i}{x_i^2 + y_i^2}, \ldots, \frac{x_nx_i + y_ny_i}{x_i^2 + y_i^2} + i\frac{y_nx_i - x_ny_i}{x_i^2 + y_i^2}\right) \\ &= \Big(\tfrac{x_1x_i+y_1y_i}{x_i^2+y_i^2}, \tfrac{y_1x_i-x_1y_i}{x_i^2+y_i^2}, \ldots, \tfrac{x_{i-1}x_i+y_{i-1}y_i}{x_i^2+y_i^2}, \tfrac{y_{i-1}x_i-x_{i-1}y_i}{x_i^2+y_i^2}, \tfrac{x_{i+1}x_i+y_{i+1}y_i}{x_i^2+y_i^2}, \tfrac{y_{i+1}x_i-x_{i+1}y_i}{x_i^2+y_i^2} \ldots, \tfrac{x_nx_i+y_ny_i}{x_i^2+y_i^2}, \tfrac{y_nx_i-x_ny_i}{x_i^2+y_i^2}\Big)\end{aligned}$$

and

$$\phi_i^{-1}(x_1, y_1, \ldots, x_n, y_n) \mapsto [x_1 + iy_1 : \cdots : x_{i-1} + iy_{i-1} : 1 : x_i + iy_i : \cdots : x_n + iy_n]$$

and so we have that $\mathbb{CP}^n$ is locally euclidean of dimension $2n$.

Next, let $[\mathbf{z}], [\mathbf{w}] \in \mathbb{CP}^n$ be arbitrary distinct points.

Case 1: $[\mathbf{z}], [\mathbf{w}] \in \pi(U_i)$, that is they both belong to the same chart. Then since the points are distinct, their image will be distinct under a homeomorphic mapping and so we will have $\phi_i([\mathbf{z}]), \phi_i([\mathbf{w}]) \in \phi_i(\pi(U_i)) \subseteq \mathbb{R}^{2n}$ such that $\phi_i([\mathbf{z}]) \neq \phi_i([\mathbf{w}])$, an since $\mathbb{R}^{2n}$ is Hausdorff, $\exists\ B_{\epsilon_\mathbf{z}}\big(\phi_i([\mathbf{z}])\big), B_{\epsilon_\mathbf{w}}\big(\phi_i([\mathbf{w}])\big) \subset \phi_i(\pi(U_i))$ open, such that

$$\phi_i([\mathbf{z}]) \in B_{\epsilon_\mathbf{z}}\big(\phi_i([\mathbf{z}])\big), \quad \phi_i([\mathbf{w}]) \in B_{\epsilon_\mathbf{w}}\big(\phi_i([\mathbf{w}])\big), \quad \text{and } B_{\epsilon_\mathbf{z}}\big(\phi_i([\mathbf{z}])\big) \cap B_{\epsilon_\mathbf{w}}\big(\phi_i([\mathbf{w}])\big) = \varnothing$$

and since ϕ_i is a homeomorphism, we know that $\phi_i^{-1}\big(B_{\epsilon_\mathbf{z}}\big(\phi_i([\mathbf{z}])\big)\big), \phi_i^{-1}\big(B_{\epsilon_\mathbf{w}}\big(\phi_i([\mathbf{w}])\big)\big) \subseteq \pi(U_i)$ are both open and

$$\begin{aligned}&B_{\epsilon_\mathbf{z}}\big(\phi_i([\mathbf{z}])\big) \cap B_{\epsilon_\mathbf{w}}\big(\phi_i([\mathbf{w}])\big) = \varnothing \\ &\quad\implies \phi_i^{-1}\Big(B_{\epsilon_\mathbf{z}}\big(\phi_i([\mathbf{z}])\big) \cap B_{\epsilon_\mathbf{w}}\big(\phi_i([\mathbf{w}])\big)\Big) \\ &\quad\quad = \phi_i^{-1}\big(B_{\epsilon_\mathbf{z}}\big(\phi_i([\mathbf{z}])\big)\big) \cap \phi_i^{-1}\big(B_{\epsilon_\mathbf{w}}\big(\phi_i([\mathbf{w}])\big)\big) = \phi_i^{-1}(\varnothing) = \varnothing\end{aligned}$$

and so we have disjoint open neighborhoods

$$\phi_i^{-1}\big(B_{\epsilon_{\mathbf{z}}}\big(\phi_i([\mathbf{z}])\big)\big) \ni [\mathbf{z}]$$
$$\phi_i^{-1}\big(B_{\epsilon_{\mathbf{w}}}\big(\phi_i([\mathbf{w}])\big)\big) \ni [\mathbf{w}]$$

Case 2: $[\mathbf{z}] \in \pi(U_i)$, and $[\mathbf{w}] \in \pi(U_j)$ for $j \neq i$. And we may assume the points don't belong to the intersection, as then that would fall into Case 1. So now we consider the restriction of a continuous map which is also continuous, that is we consider

$$\mathbf{x} \in \pi|_{\mathbb{S}^{2n+1}}^{-1}([\mathbf{z}]) = \pi^{-1}([\mathbf{z}]) \cap \mathbb{S}^{2n+1}, \quad \text{and} \quad \mathbf{y} \in \pi|_{\mathbb{S}^{2n+1}}^{-1}([\mathbf{w}]) = \pi^{-1}([\mathbf{w}]) \cap \mathbb{S}^{2n+1}$$

both of which are subsets of $\mathbb{R}^{2n+2} \setminus \{\mathbf{0}\}$. Where we choose

$$\epsilon = \frac{1}{10}\min\{||\mathbf{x}-\mathbf{y}||, ||\mathbf{x}+\mathbf{y}||\}$$

then we consider the opens subsets in the relative topology of $\mathbb{S}^{2n+1}$

$$B_\epsilon(\mathbf{x}) \cap \mathbb{S}^{2n+1}$$
$$B_\epsilon(\mathbf{y}) \cap \mathbb{S}^{2n+1}$$

neither of which will contain $\mathbf{0}$, by our choice of ϵ, and so are well-defined. Then since multiplication by a constant is a homeomorphism we can define the open sets of $\mathbb{R}^{2n+2}\setminus\{\mathbf{0}\}$

$$U = \{t\cdot\mathbf{p} : \mathbf{p} \in B_\epsilon(\mathbf{x}) \cap \mathbb{S}^{2n+1};\ t \in \mathbb{R}\setminus\{0\}\}$$
$$V = \{t\cdot\mathbf{q} : \mathbf{q} \in B_\epsilon(\mathbf{x}) \cap \mathbb{S}^{2n+1};\ t \in \mathbb{R}\setminus\{0\}\}$$

then $V \cap U = \varnothing$ since $(B_\epsilon(\mathbf{x}) \cap \mathbb{S}^{2n+1}) \cap (B_\epsilon(\mathbf{y}) \cap \mathbb{S}^{2n+1}) = \varnothing$; i.e. $\mathbf{q} \in (B_\epsilon(\mathbf{x}) \cap \mathbb{S}^{2n+1}) \cap (B_\epsilon(\mathbf{y}) \cap \mathbb{S}^{2n+1})$ implies

$$d(\mathbf{x},\mathbf{y}) \leq d(\mathbf{x},\mathbf{q}) + d(\mathbf{q},\mathbf{y}) \leq \frac{1}{5}\min\{||\mathbf{x}-\mathbf{y}||, ||\mathbf{x}+\mathbf{y}||\} > \epsilon \quad \Rightarrow\Leftarrow$$

and again since $V, U \subset \mathbb{R}^{2n+2}\setminus\{\mathbf{0}\} \cong \mathbb{C}^{n+1}\setminus\{\mathbf{0}\}$ are open, and saturated, so that

$$\pi^{-1}(\pi(U)) = U, \quad \text{and} \quad \pi^{-1}(\pi(V)) = V$$

we have that $\pi(U), \pi(V) \subset \mathbb{CP}^n$ must also be open, since their preimage is open. Furthermore, we have that they are disjoint, otherwise

$$\begin{aligned} \pi(U) \cap \pi(V) \neq \varnothing &\implies \pi^{-1}\big(\pi(U) \cap \pi(V)\big) \neq \pi^{-1}(\varnothing) \\ &\implies \pi^{-1}(\pi(U)) \cap \pi^{-1}(\pi(V)) \neq \varnothing \\ &\implies U \cap V \neq \varnothing \quad \Rightarrow\Leftarrow \end{aligned}$$

and so we have found open disjoint sets $\pi(U), \pi(V)$ which separate $[\mathbf{z}]$ and $[\mathbf{w}]$.

And therefore we see that $\mathbb{CP}^n$ is Hausdorff.

Next, since each coordinate domain is locally euclidean, and so second countable in the relative topology of $\mathbb{R}^{2n}$, say the countable base for $\phi_i(\pi(U_i))$ is $\mathcal{B}_i$, gives the countable base $\phi_i^{-1}(\mathcal{B}_i)$ for $\pi(U_i) \subset \mathbb{CP}^n$, then as the union of finitely many countable sets is countable we have

$$\bigcup_{i=1}^{n+1} \phi_i^{-1}(\mathcal{B}_i)$$

is a countable base for $\mathbb{CP}^n$, and so $\mathbb{CP}^n$ is second countable.

Thus we have shown that $\mathbb{CP}^n$ is locally euclidean, Hausdorff, and second countable. Therefore $\mathbb{CP}^n$ is a topological $2n$-manifold.

To see that $\mathbb{CP}^n$ is compact, we note that $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1} \setminus \{\mathbf{0}\} \cong \mathbb{R}^{2n+2} \setminus \{\mathbf{0}\}$ is a compact subset since it is closed and bounded by Heine-Borel, and since the natural projection π is continuous, and the restriction of a continuous map is continuous, and further we have the restriction of π surjects onto $\mathbb{CP}^n$; that is

$$\pi|_{\mathbb{S}^{2n+1}} : \mathbb{S}^{2n+1} \to \pi(\mathbb{S}^{2n+1}) = \mathbb{CP}^n$$

and finally we note, that the continuous image of a compact set is compact, and therefore, $\mathbb{CP}^n$ is compact.

To see that the charts are all smoothly compatible we simply compute the transition maps

$$\phi_i \circ \phi_j^{-1}|_{\phi_j(U_i \cap U_j)} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

WLOG let $i > j$

$$\begin{aligned} \phi_i \circ \phi_j^{-1}(w_i, \dots, w_n) &= \phi_i([w_1 : \dots : w_{j-1} : 1 : w_j : \dots : w_n]) \\ &= \left(\frac{w_1}{w_i}, \dots, \frac{w_{j-1}}{w_i}, \frac{1}{w_i}, \frac{w_j}{w_i}, \dots, \frac{w_{i-1}}{w_i}, \frac{w_{i+1}}{w_i}, \dots \frac{w_n}{w_i}\right) \end{aligned}$$

which is smooth, when considered in $\mathbb{R}^{2n}$, and as i, j arbitrary we conclude the charts are all smoothly compatible and hence determine a smooth structure $\mathcal{A}_{\mathbb{CP}^n}$ on $\mathbb{CP}^n$. □

1-11 *Let $M = \overline{\mathbb{B}}^n$, the closed unit ball in $\mathbb{R}^n$. Show that M is a topological manifold with boundary in which each point in $\mathbb{S}^{n-1}$ is a boundary point and each point in $\mathbb{B}^n$ is an interior point. Show how to give it a smooth structure such that every smooth interior chart is a smooth chart for the standard smooth structure on $\mathbb{B}^n$. [Hint: consider the map*

$$\pi \circ \phi_N : \mathbb{R}^n \to \mathbb{R}^n$$

where $\phi_N : \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n$ is the stereographic projection and $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is a projection that omits some coordinate other than the last.]

Proof. First we note that in the relative topology of $\overline{\mathbb{B}}^n$ as a subset of $\mathbb{R}^n$, $\overline{\mathbb{B}}^n$ inherits second countablility and Hausdorffness, also we note that for any

$$\mathbf{x} \in \operatorname{Int}(\overline{\mathbb{B}}^n) = \mathbb{B}^n$$

we have the smooth chart $(\mathbb{B}^n, Id_{\mathbb{B}^n})$ which will also be a smooth chart for the standard smooth structure on $\mathbb{B}^n$. So we must find charts for points belonging to

$$\overline{\mathbb{B}}^n \setminus \mathbb{B}^n = \mathbb{S}^{n-1}$$

from the hint of using $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$, we will have to consider $\overline{\mathbb{B}}^n \subseteq \mathbb{R}^{n+1}$, but instead we use the stereographic projection from the south

$$\phi_S : \mathbb{S}^n \setminus \{(0, \ldots, -1)\} \to \mathbb{R}^n, \text{ by } \phi_S(x_1, \ldots, x_{n+1}) = \left(\frac{x_1}{1 + x_{n+1}}, \ldots, \frac{x_n}{1 + x_{n+1}} \right)$$

and its' inverse

$$\phi_S^{-1} : \mathbb{R}^n \to \mathbb{S}^n \setminus \{(0, \ldots, -1)\}$$

defined by

$$\phi_S^{-1}(x_1, \ldots, x_n) = \left(\frac{2x_1}{x_1^2 + \cdots + x_n^2 + 1}, \ldots, \frac{2x_n}{x_1^2 + \cdots + x_n^2 + 1}, \frac{1 - x_1^2 - \cdots - x_n^2}{x_1^2 + \cdots + x_n^2 + 1} \right)$$

then for $1 \leq i \leq n$ we define

$$\begin{aligned}
{}^nU_i^+ &= \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i > 0\} \\
{}^nU_i^- &= \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i < 0\} \\
{}^{n+1}U_i^+ &= \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_i > 0\} \\
{}^{n+1}U_i^- &= \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_i < 0\}
\end{aligned}$$

then

$$\phi_S^{-1}({}^nU_i^+) = \mathbb{S}^n \cap^{n+1} U_i^+, \quad \text{and} \quad \phi_S^{-1}({}^nU_i^-) = \mathbb{S}^n \cap^{n+1} U_i^-$$

and for any $\mathbf{x} \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$ we have

$$\begin{aligned}
\phi_S^{-1}(x_1, \ldots, x_n) &= \left(\frac{2x_1}{x_1^2 + \cdots + x_n^2 + 1}, \ldots, \frac{2x_n}{x_1^2 + \cdots + x_n^2 + 1}, \frac{1 - x_1^2 - \cdots - x_n^2}{x_1^2 + \cdots + x_n^2 + 1} \right) \\
&\left(\frac{2x_1}{1 + 1}, \ldots, \frac{2x_n}{1 + 1}, \frac{1 - 1}{1 + 1} \right) \\
&= (x_1, \ldots, x_n, 0) \\
&= (\mathbf{x}, 0)
\end{aligned}$$

and so

$$\phi_S^{-1}|_{\mathbb{S}^{n-1}} = Id_{\mathbb{S}^{n-1}} \times \{0\}$$

then we note, for $\mathbf{x} \in \mathbb{B}^n \cap^n U_i^{\pm}$, we will have

$$\phi_S^{-1}(x_1, \dots, x_n) = \left(\frac{2x_1}{x_1^2 + \cdots + x_n^2 + 1}, \dots, \frac{2x_n}{x_1^2 + \cdots + x_n^2 + 1}, \overbrace{\frac{1 - x_1^2 - \cdots - x_n^2}{x_1^2 + \cdots + x_n^2 + 1}}^{>0} \right)$$

while for $\mathbf{x} \in^n U_i^{\pm} \cap (\overline{\mathbb{B}}^n)^c$, we will have

$$\phi_S^{-1}(x_1, \dots, x_n) = \left(\frac{2x_1}{x_1^2 + \cdots + x_n^2 + 1}, \dots, \frac{2x_n}{x_1^2 + \cdots + x_n^2 + 1}, \overbrace{\frac{1 - x_1^2 - \cdots - x_n^2}{x_1^2 + \cdots + x_n^2 + 1}}^{<0} \right)$$

then since the composition of continuous maps is continuous we have

$$\pi_i \circ \phi_S^{-1} : \mathbb{R}^n \to \mathbb{R}^n$$

will be continuous for each i, then as the restriction of a continuous map is continuous we have

$$\pi_i \circ \phi_S^{-1}|_{^nU_i^{\pm} \cap \overline{\mathbb{B}}^n} :^n U_i^{\pm} \cap \overline{\mathbb{B}}^n \to \mathbb{H}^n$$

is continuous for each i. Thus, we have constructed a collection of $2n$ charts that cover $\mathbb{S}^{n-1}$; that is

$$\bigcup_{i=1}^{n} {}^nU_i^{\pm} \cap \overline{\mathbb{B}}^n \supseteq \mathbb{S}^{n-1}$$

such that for any $\mathbf{x} \in \mathbb{S}^{n-1}$ we have

$$\pi_i \circ \phi_S^{-1}(\mathbf{x}) = \pi_i(x_1, \dots, x_n, 0) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n, 0) \in \partial\mathbb{H}^n$$

with $\pi_i \circ \phi_S^{-1}\left({}^nU_i^{\pm} \cap \mathbb{B}^n\right) \subset \mathrm{Int}(\mathbb{H}^n)$ and $\pi_i \circ \phi_S^{-1}\left({}^nU_i^{\pm} \cap \overline{\mathbb{B}}^n\right)$ an open half-ball. And so the collection $\{({}^nU_i^{\pm} \cap \overline{\mathbb{B}}^n, \pi_i \circ \phi_S^{-1}|_{^nU_i^{\pm} \cap \overline{\mathbb{B}}^n})\}_{i=1}^n$ form our boundary charts for $\partial\overline{\mathbb{B}}^n = \mathbb{S}^{n-1}$, which are smoothly compatible since for any two charts we will have

$$\pi_i \circ \phi_S^{-1} \circ (\pi_j \circ \phi_S^{-1})^{-1} = \pi_i \circ \phi_S^{-1} \circ \phi_S \circ \pi_j^{-1} = \pi_i \circ \pi_j^{-1}$$

which will map any point $\mathbf{x}$ in the intersection, letting $i < j$, as

$$\pi_i \circ \phi_S^{-1} \circ (\pi_j \circ \phi_S^{-1})^{-1}(x_1, \dots, x_n) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, 1, x_j, \dots, x_n)$$

which is smooth.

The identity map is clearly smooth, and so we have a smooth atlas $\{(\mathbb{B}^n, Id_{\mathbb{B}^n})\} \cup \{({}^nU_i^{\pm} \cap \overline{\mathbb{B}}^n, \pi_i \circ \phi_S^{-1}|_{^nU_i^{\pm} \cap \overline{\mathbb{B}}^n})\}_{i=1}^n$ for our manifold with boundary $\overline{\mathbb{B}}^n$, giving $\overline{\mathbb{B}}^n$ a smooth structure. $\square$

2-1 *Define*

$$f : \mathbb{R} \to \mathbb{R}, \ \textit{by} \ f(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Show that for every $x \in \mathbb{R}$, there are smooth coordinate charts (U, ϕ) containing x, and (V, ψ) containing $f(x)$ such that

$$\psi \circ f \circ \phi^{-1} : \phi\big(U \cap f^{-1}(V)\big) \to \psi(V)$$

is smooth, but f is not smooth in the sense we have defined in this chapter.

Proof. Let $x \in \mathbb{R}$ be given, and let

$$V = \left(f(x) - \frac{1}{2}, f(x) + \frac{1}{2}\right)$$
$$U = \mathbb{R}$$

then considering the charts $(U, Id_{\mathbb{R}}), (V, Id_V) \in \mathcal{A}_{\mathbb{R}}$, we have

Case 1: if $x < 0$, then

$$f(x) = 0 \implies V = \left(-\frac{1}{2}, \frac{1}{2}\right)$$

and so $f^{-1}(V) = (-\infty, 0)$ which gives

$$U \cap f^{-1}(V) = \mathbb{R} \cap (-\infty, 0) = (-\infty, 0)$$

and so $(U, Id_{\mathbb{R}})$ is a chart containing x, and (V, Id_V) is a chart containing $f(x)$. Where for any $y \in (-\infty, 0)$ we then have

$$Id_V \circ f \circ Id_{\mathbb{R}}^{-1}|_{(-\infty,0)}(y) = Id_V \circ f(y) = Id_V(0) = 0$$

which is a constant mapping, and therefore is smooth.

Case 2: $x \geq 0$, then

$$f(x) = 1 \implies V = \left(\frac{1}{2}, \frac{3}{2}\right)$$

and so $f^{-1}(V) = [0, \infty)$ which gives

$$U \cap f^{-1}(V) = \mathbb{R} \cap [0, \infty) = [0, \infty)$$

and so $(U, Id_{\mathbb{R}})$ is a chart containing x, and (V, Id_V) is a chart containing $f(x)$. Where for any $z \in [0, \infty)$ we then have

$$Id_V \circ f \circ Id_{\mathbb{R}}^{-1}|_{[0,\infty)}(z) = Id_V \circ f(z) = Id_V(1) = 1$$

which is a constant mapping, and therefore is smooth.

Since $x \in \mathbb{R}$ was arbitrary we conclude that for each $x \in \mathbb{R}$ we may find charts (U, ϕ) containing x, and (V, ψ) containing $f(x)$ such that

$$\psi \circ f \circ \phi^{-1} : \phi\big(U \cap f^{-1}(V)\big) \to \psi(V)$$

is smooth.

Yet, f is not smooth as we have defined it, since from Proposition 2.4, smooth implies continuous, and f is not continuous as the the preimage of the open set $\left(\frac{1}{2}, \frac{3}{2}\right)$ gave $f^{-1}\left(\frac{1}{2}, \frac{3}{2}\right) = [0, \infty)$ which is not open. It should be noted that this stems from the fact that we cannot find charts which cover $\mathbb{R}$ such that $f(U) \subseteq V$, the best we can do is $U_1 = (-\infty, 0), U_2 = (0, \infty)$. $\square$

2-3 *For each of the following maps between spheres, compute sufficiently many coordinate representations to prove that it is smooth.*

(a) *n^{th} power map*

$$p_n : \mathbb{S}^1 \to \mathbb{S}^1, \quad \textit{for } n \in \mathbb{Z}$$

given in complex notation by $p_n(z) = z^n$.

(b) *antipodal map*

$$\alpha : \mathbb{S}^n \to \mathbb{S}^n, \ \textit{by } \alpha(\mathbf{x}) = -\mathbf{x}$$

(c) *The mapping*

$$f : \mathbb{S}^3 \to \mathbb{S}^2, \ \textit{by } f(\mathbf{w}, \mathbf{z}) = (\mathbf{z}\overline{\mathbf{w}} + \mathbf{w}\overline{\mathbf{z}}, i\mathbf{w}\overline{\mathbf{z}} - i\mathbf{z}\overline{\mathbf{w}}, \mathbf{z}\overline{\mathbf{z}} - \mathbf{w}\overline{\mathbf{w}})$$

where $\mathbb{S}^3 = \{(\mathbf{w}, \mathbf{z}) \in \mathbb{C}^2 : |\mathbf{w}|^2 + |\mathbf{z}|^2 = 1\} \subset \mathbb{C}^2$.

Proof.

(a) Considering $\mathbb{S}^1 \subset \mathbb{C}$, where in $\mathbb{C}$ we have that $\mathbb{S}^1$ can be described by the set of points $\{z = e^{it} \in \mathbb{C} : 0 \leq t \leq 2\pi\}$, so if we take as our coordinate domains

$$U_1 = \{z = e^{it} \in \mathbb{C} : 0 < t < 2\pi\}$$
$$U_2 = \{z = e^{it} \in \mathbb{C} : -\pi < t < \pi\}$$

these will certainly cover $\mathbb{S}^1$, and for our coordinate functions if we let

$$\phi_1 = \phi_2 = \frac{1}{i} \ln(z)$$

which are continuous on their branch cuts, and have explicit inverses given by e^{iz}, and thus, are homeomorphisms, where we also have

$$\phi_1(U_1) = (0, 2\pi)$$
$$\phi_2(U_2) = (-\pi, \pi)$$

and so the charts $(U_1, \phi_1), (U_2, \phi_2)$ are locally euclidean, and we note that the intersection gives

$$U_1 \cap U_2 = \{z = e^{it} \in \mathbb{C} : 0 < t < \pi\} \cup \{z = e^{it} \in \mathbb{C} : -\pi < t < 0\}$$

and for the transition maps

$$\phi_1 \circ \phi_2^{-1}|_{\phi_2(U_1 \cap U_2)} : \phi_2(U_1 \cap U_2) \to \phi_1(U_1 \cap U_2)$$

we get

$$\phi_1 \circ \phi_2^{-1}(t) = \phi_1(e^{it}) = \begin{cases} t, & t \in (0, \pi) \\ t + 2\pi, & t \in (-\pi, 0) \end{cases}$$

and similarly for

$$\phi_2 \circ \phi_1^{-1}|_{\phi_1(U_1 \cap U_2)} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$$

we get

$$\phi_2 \circ \phi_1^{-1}(t) = \phi_2(e^{it}) = \begin{cases} t, & t \in (0, \pi) \\ t - 2\pi, & t \in (\pi, 2\pi) \end{cases}$$

and so the charts $(U_1, \phi_1), (U_2, \phi_2)$ are smoothly compatible, and thus, they determine a smooth structure for $\mathbb{S}^1$.

Next we note, that for any $n \in \mathbb{Z}$ we have $p_n(z) = z^n = e^{int}$, which is continuous, and so by Proposition 2.5, to show that p_n is smooth, it suffices to show that its coordinate representation with charts in $\mathcal{A}_{\mathbb{S}^1}$ are smooth.

Case 1: Considering the local coordinates with respect to the single chart (U_1, ϕ_1)

$$\phi_1 \circ p_n \circ \phi_1^{-1}|_{\phi_1(U_1 \cap p_n^{-1}(U_1))} : \phi_1(U_1 \cap p_n^{-1}(U_1)) \to \phi_1(U_1)$$

is given by

$$\begin{aligned} \phi_1 \circ p_n \circ \phi_1^{-1}(t) &= \phi_1 \circ p_n(e^{it}) \\ &= \phi_1(e^{int}) \\ &= nt \mod 2\pi \end{aligned}$$

Case 2: Similarly for the single chart (U_2, ϕ_2)

$$\phi_2 \circ p_n \circ \phi_2^{-1}|_{\phi_2(U_2 \cap p_n^{-1}(U_2))} : \phi_2(U_2 \cap p_n^{-1}(U_2)) \to \phi_2(U_2)$$

is given by

$$\begin{aligned} \phi_2 \circ p_n \circ \phi_2^{-1}(t) &= \phi_1 \circ p_n(e^{it}) \\ &= \phi_2(e^{int}) \\ &= nt \mod 2\pi \end{aligned}$$

Case 3: Now considering both charts, we have

$$\phi_2 \circ p_n \circ \phi_1^{-1}|_{\phi_1(U_1 \cap p_n^{-1}(U_2))} : \phi_1(U_1 \cap p_n^{-1}(U_2)) \to \phi_2(U_2)$$

is given by

$$\begin{aligned} \phi_2 \circ p_n \circ \phi_1^{-1}(t) &= \phi_2 \circ p_n(e^{it}) \\ &= \phi_2(e^{int}) \\ &= \begin{cases} nt \mod 2\pi, & t \in (0, \pi) \\ nt - 2\pi \mod 2\pi, & t \in (\pi, 2\pi) \end{cases} \end{aligned}$$

Case 4: And lastly we get

$$\phi_1 \circ p_n \circ \phi_2^{-1}|_{\phi_2(U_2 \cap p_n^{-1}(U_1))} : \phi_2(U_2 \cap p_n^{-1}(U_1)) \to \phi_1(U_1)$$

is given by

$$\begin{aligned}\phi_1 \circ p_n \circ \phi_2^{-1}(t) &= \phi_1 \circ p_n(e^{it}) \\ &= \phi_1(e^{int}) \\ &= \begin{cases} nt \mod 2\pi, & t \in (0,\pi) \\ nt + 2\pi \mod 2\pi, & t \in (-\pi, 0) \end{cases}\end{aligned}$$

and then we note that

$$U_1 \cap p_n^{-1}(U_1) = \bigsqcup_{i=1}^{n} \left(\frac{i-1}{|n|}2\pi, \frac{i}{|n|}2\pi \right)$$

is a union of open sets and is therefore open, and for each t in any such interval $\left(\frac{k-1}{|n|}2\pi, \frac{k}{|n|}2\pi\right)$ we get that $nt - 2\pi k$ is smooth. With the remaining cases giving a similar disjoint union of open intervals, and in each case, we just have a linear function which is smooth. Therefore the coordinate representation of p_n is smooth, and thus, p_n must be as well.

Since $n \in \mathbb{Z}$ was arbitrary, we see that p_n is smooth for each n.

(b) With the standard smooth structure $\{(U_i^{\pm}, \phi_i^{\pm})\}$ on $\mathbb{S}^n$ given by

$$\begin{aligned} U_i^+ &= \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_i > 0\} \\ U_i^- &= \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_i < 0\} \end{aligned}$$

and the mappings

$$\phi_i^{\pm} : U_i^{\pm} \cap \mathbb{S}^n \to \mathbb{B}^n, \text{ by } \phi_i^{\pm}(\mathbf{x}) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_n)$$

and inverse's

$$(\phi_i^{\pm})^{-1} : \mathbb{B}^n \to \mathbb{S}^n \cap U_i^{\pm}$$

given by

$$(\phi_i^{\pm})^{-1}(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, \pm\sqrt{1 - ||\mathbf{x}||^2}, x_i, \ldots, x_n)$$

So, WLOG let

$$\mathbf{x} \in U_i^+ \implies \alpha(\mathbf{x}) = -\mathbf{x} \in U_i^-$$

and further we get $\alpha(U_i^+) = U_i^-$ and hence $\alpha(U_i^+) \subseteq U_i^-$; that is, we have a chart (U_i^+, ϕ_i^+) containing $\mathbf{x}$, and a chart (U_i^-, ϕ_i^-) containing $\alpha(\mathbf{x})$, such that $\alpha(U_i^+) \subseteq U_i^-$. And noting that

$$\alpha(U_i^+) \subseteq U_i^- \implies U_i^+ \subseteq \alpha^{-1}(U_i^-) \implies U_i^+ \cap \alpha^{-1}(U_i^-) = U_i^+$$

So for the coordinate representation of α we get

$$\phi_i^- \circ \alpha \circ (\phi_i^+)^{-1}|_{\phi_i^+(U_i^+)} : \phi_i^+(U_i^+) \to \phi_i^-(U_i^-)$$

is given by

$$\begin{aligned}\phi_i^- \circ \alpha \circ (\phi_i^+)^{-1}(x_1, \ldots, x_n) &= \phi_i^- \circ \alpha(x_1, \ldots, x_{i-1}, -\sqrt{1 - ||\mathbf{x}||^2}, x_i, \ldots, x_n) \\ &= \phi_i^-(-x_1, \ldots, -x_{i-1}, \sqrt{1 - ||\mathbf{x}||^2}, -x_i, \ldots, -x_n) \\ &= (-x_1, \ldots, -x_{i-1}, -x_i, \ldots, -x_n) \\ &= -\mathbf{x}\end{aligned}$$

which is smooth, and hence α must also be smooth on the charts $(U_i^+, \phi_i^+), (U_i^-, \phi_i^-)$.

And since the same process can be done for each point of $\mathbb{S}^n$, we can conclude that α is smooth on $\mathbb{S}^n$.

(c) First realizing $\mathbb{C}^2 \cong \mathbb{R}^4$, so that for $\mathbf{w} = a + ib$ and $\mathbf{z} = c + id$ where we then get

$$\begin{aligned}
&\Big((c+id)(a-ib) + (a+ib)(c-id), i(a+ib)(c-id) - i(c+id)(a-ib), (c+id)(c-id) - (a+ib)(a-ib)\Big) \\
&= \Big(ac - ibc + iab + bd + ac - iad + ibc + bd, i(ac - iad + ibc + bd - ac + ibc - iad - bd), c^2 + d^2 - a^2 - b^2\Big) \\
&= \Big(2ac + 2bd, 2ad - 2bc, c^2 + d^2 - a^2 - b^2) \\
&= f(a, b, c, d)
\end{aligned}$$

where we see that f is continuous as it is a polynomial in each of its components. So by proposition 2.5 to show that f is smooth, it suffices to show that the coordinate representation of f is smooth, with charts in $\mathcal{A}_{\mathbb{S}^3}$ and $\mathcal{A}_{\mathbb{S}^2}$.

Choosing the standard smooth structure $\{({}^3U_i^\pm, {}^3\phi_i^\pm)\}_{i=1}^4$ for $\mathbb{S}^3$, and also the standard smooth structure $\{({}^2U_i^\pm, {}^2\phi_i^\pm)\}_{i=1}^3$ for $\mathbb{S}^2$ defined by

$$\begin{aligned}
&{}^3U_i^+ = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_i > 0\} \quad {}^3U_i^- = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_i < 0\} \quad 1 \le i \le 4 \\
&{}^2U_j^+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_j > 0\} \qquad\quad {}^2U_j^+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_j < 0\} \quad 1 \le j \le 3
\end{aligned}$$

and the mappings

$$ {}^3\phi_i^\pm : U_i^\pm \cap \mathbb{S}^3 \to \mathbb{B}^3, \qquad {}^3(\phi_i^\pm)^{-1} : \mathbb{B}^3 \to \mathbb{S}^3 \cap U_i^\pm $$

given by

$$\begin{aligned}
{}^3\phi_1^\pm(x_1, x_2, x_3, x_4) &= (x_2, x_3, x_4) \qquad & {}^3(\phi_1^\pm)^{-1}(u_1, u_2, u_3) &= (\pm\sqrt{1 - ||\mathbf{u}||^2}, u_2, u_3, u_4) \\
{}^3\phi_2^\pm(x_1, x_2, x_3, x_4) &= (x_1, x_3, x_4) \qquad & {}^3(\phi_2^\pm)^{-1}(u_1, u_2, u_3) &= (u_1, \pm\sqrt{1 - ||\mathbf{u}||^2}, u_3, u_4) \\
{}^3\phi_3^\pm(x_1, x_2, x_3, x_4) &= (x_1, x_2, x_4) \qquad & {}^3(\phi_3^\pm)^{-1}(u_1, u_2, u_3) &= (u_1, u_2, \pm\sqrt{1 - ||\mathbf{u}||^2}, u_4) \\
{}^3\phi_4^\pm(x_1, x_2, x_3, x_4) &= (x_1, x_2, x_3) \qquad & {}^3(\phi_4^\pm)^{-1}(u_1, u_2, u_3) &= (u_1, u_2, u_3, \pm\sqrt{1 - ||\mathbf{u}||^2})
\end{aligned}$$

and similarly we have

$$ {}^2\phi_i^\pm : U_i^\pm \cap \mathbb{S}^2 \to \mathbb{B}^2, \qquad {}^2(\phi_i^\pm)^{-1} : \mathbb{B}^2 \to \mathbb{S}^2 \cap U_i^\pm $$

given by

$$\begin{aligned}
{}^2\phi_1^\pm(x_1, x_2, x_3) &= (x_2, x_3) \qquad & {}^2(\phi_1^\pm)^{-1}(u_1, u_2) &= (\pm\sqrt{1 - ||\mathbf{u}||^2}, u_2, u_3) \\
{}^2\phi_2^\pm(x_1, x_2, x_3, x_4) &= (x_1, x_3) \qquad & {}^2(\phi_2^\pm)^{-1}(u_1, u_2) &= (u_1, \pm\sqrt{1 - ||\mathbf{u}||^2}, u_3) \\
{}^2\phi_3^\pm(x_1, x_2, x_3) &= (x_1, x_2) \qquad & {}^2(\phi_3^\pm)^{-1}(u_1, u_2) &= (u_1, u_2, \pm\sqrt{1 - ||\mathbf{u}||^2})
\end{aligned}$$

and so we have

$$ {}^2\phi_1^{\pm} \circ f \circ ({}^3\phi_1^{\pm})^{-1}|_{{}^3\phi_1^{\pm}(U_1^{\pm}\cap f^{-1}(U_1^{\pm}))} :^3 \phi_1^{\pm}\big(U_1^{\pm} \cap f^{-1}(U_1^{\pm})\big) \to^2 \phi_1^{\pm}(U_1^{\pm}) $$

is given by

$$ \begin{aligned} {}^2\phi_1^{\pm} \circ f \circ ({}^3\phi_1^{\pm})^{-1}(u_1, u_2, u_3) &=^2 \phi_1^{\pm} \circ f(\pm\sqrt{1-||\mathbf{u}||^2}, u_2, u_3, u_4) \\ &=^2 \phi_1^{\pm}(\pm 2u_3\sqrt{1-||\mathbf{u}||^2} + 2u_2u_4, \pm 2u_4\sqrt{1-||\mathbf{u}||^2} - 2u_2u_3, u_3^2 + u_4^2 - (1-||\mathbf{u}||^2) - u_2^2) \\ &= (\pm 2u_4\sqrt{1-||\mathbf{u}||^2} - 2u_2u_3, u_3^2 + u_4^2 + ||\mathbf{u}||^2 - u_2^2 - 1) \end{aligned} $$

which is smooth since the domain will have $1 - ||\mathbf{u}||^2 \geq 0$, and similarly, when $i \neq j$ we get

$$ {}^2\phi_2^{\pm} \circ f \circ ({}^3\phi_4^{\pm})^{-1}|_{{}^3\phi_4^{\pm}(U_4^{\pm}\cap f^{-1}(U_2^{\pm}))} :^3 \phi_4^{\pm}\big(U_4^{\pm} \cap f^{-1}(U_2^{\pm})\big) \to^2 \phi_2^{\pm}(U_2^{\pm}) $$

is given by

$$ \begin{aligned} {}^2\phi_2^{\pm} \circ f \circ ({}^3\phi_4^{\pm})^{-1}(u_1, u_2, u_3) &=^2 \phi_2^{\pm} \circ f(u_1, u_2, u_3, \pm\sqrt{1-||\mathbf{u}||^2}) \\ &=^2 \phi_2^{\pm}(2u_1u_3 + 2u_2u_4, \pm 2u_1\sqrt{1-||\mathbf{u}||^2} - 2u_2u_3, u_3^2 + (1-||\mathbf{u}||^2) - u^1 - u_2^2) \\ &= (2u_1u_3 + 2u_2u_4, u_3^2 + 1 - ||\mathbf{u}||^2 - u^1 - u_2^2) \end{aligned} $$

which again is smooth, with all other combinations coming out similarly.

In each case we either have no square root, in which case our function is smooth, as a polynomial in each component, or if there is a square root, our domain ensures that $1 - ||\mathbf{u}||^2 \geq 0$, and so our function remains smooth.

Thus we can conclude that the coordinate representation of f is smooth, and hence, so is f.

$\square$

2-4 *Show that the inclusion map*

$$\iota : \overline{B}^n \hookrightarrow \mathbb{R}^n$$

is smooth when $\overline{B}^n$ is regarded as a smooth manifold with boundary.

Proof. First, Since the inclusion is continuous, by Proposition 2.5, to show that ι is smooth, is suffices to show that the coordinate representation of ι is smooth, with charts in $\mathcal{A}_{\overline{B}^n}$ and $\mathcal{A}_{\mathbb{R}^n}$.

Taking the smooth atlas we constructed for $\overline{\mathbb{B}}^n$ from last time, given by $\{(\mathbb{B}^n, Id|_{\mathbb{B}^n})\} \cup \{({}^nU_i^\pm \cap \overline{\mathbb{B}}^n, \pi_i \circ \phi_S^{-1}|_{{}^nU_i^\pm \cap \overline{\mathbb{B}}^n})\}_{i=1}^n$ where we recall that for $1 \le i \le n$ we defined

$$\begin{aligned} {}^nU_i^+ &= \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i > 0\} \\ {}^nU_i^- &= \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i < 0\} \end{aligned}$$

and had our mappings defined by

$$\begin{aligned} \phi_S(x_1, \dots, x_{n+1}) &= \left(\frac{x_1}{1 + x_{n+1}}, \dots, \frac{x_n}{1 + x_{n+1}} \right) \\ \phi_S^{-1}(x_1, \dots, x_n) &= \left(\frac{2x_1}{x_1^2 + \cdots + x_n^2 + 1}, \dots, \frac{2x_n}{x_1^2 + \cdots + x_n^2 + 1}, \frac{1 - x_1^2 - \cdots - x_n^2}{x_1^2 + \cdots + x_n^2 + 1} \right) \end{aligned}$$

and where π_i was deletion of the i^{th} component.

For our chart in $\mathbb{R}^n$ we choose $(\mathbb{R}^n, Id_{\mathbb{R}^n})$. Where for our interior chart we get

$$Id_{\mathbb{R}^n} \circ \iota \circ Id|_{\mathbb{B}^n}^{-1} : \mathbb{B}^n \to \mathbb{R}^n$$

is given by

$$Id_{\mathbb{R}^n} \circ \iota \circ Id|_{\mathbb{B}^n}^{-1}(\mathbf{x}) = Id_{\mathbb{R}^n} \circ \iota(\mathbf{x}) = Id_{\mathbb{R}^n}(\mathbf{x}) = \mathbf{x}$$

and so is smooth.

For our boundary charts we get

$$Id_{\mathbb{R}^n} \circ \iota \circ (\pi_i \circ \phi_S^{-1}|_{{}^nU_i^\pm \cap \overline{\mathbb{B}}^n}))^{-1}|_{\pi_i \circ \phi_S^{-1}({}^nU_i^\pm \cap \overline{\mathbb{B}}^n \cap \iota^{-1}(\mathbb{R}^n))} : \pi_i \circ \phi_S^{-1}\left({}^nU_i^\pm \cap \overline{\mathbb{B}}^n \cap \iota^{-1}(\mathbb{R}^n)\right) \to \mathbb{R}^n$$

is given by

$$\begin{aligned} Id_{\mathbb{R}^n} \circ \iota \circ \phi_S \circ \pi_i^{-1}(x_1, \dots, x_n) &= Id_{\mathbb{R}^n} \circ \iota \circ \phi_S(x_1, \dots, x_{i-1}, 1, x_i, \dots, x_n) \\ &= Id_{\mathbb{R}^n} \circ \iota \left(\frac{x_1}{1 + x_n}, \dots, \frac{x_{i-1}}{1 + x_n}, \frac{1}{1 + x_n}, \frac{x_i}{1 + x_n}, \dots, \frac{x_{n-1}}{1 + x_n} \right) \\ &= Id_{\mathbb{R}^n} \left(\frac{x_1}{1 + x_n}, \dots, \frac{x_{i-1}}{1 + x_n}, \frac{1}{1 + x_n}, \frac{x_i}{1 + x_n}, \dots, \frac{x_{n-1}}{1 + x_n} \right) \\ &= \left(\frac{x_1}{1 + x_n}, \dots, \frac{x_{i-1}}{1 + x_n}, \frac{1}{1 + x_n}, \frac{x_i}{1 + x_n}, \dots, \frac{x_{n-1}}{1 + x_n} \right) \end{aligned}$$

which is again smooth for each i, since the domain of ϕ_S does not contain the point $(0, \dots, -1)$. And thus we can conclude that the inclusion map

$$\iota : \overline{B}^n \hookrightarrow \mathbb{R}^n$$

is smooth. $\square$

2-6 *Let*
$$P : \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{R}^{k+1} \setminus \{\mathbf{0}\}$$
be a smooth function, and suppose that for some $d \in \mathbb{Z}$,
$$P(\lambda \mathbf{x}) = \lambda^d P(\mathbf{x}), \quad \forall\ \lambda \in \mathbb{R} \setminus \{\mathbf{0}\},\ \mathbf{x} \in \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$$
(Such a function is said to be homogeneous of degree d.) Show that the map
$$\widetilde{P} : \mathbb{RP}^n \to \mathbb{RP}^k,\ \textit{by}\ \widetilde{P}([\mathbf{x}]) = [P(\mathbf{x})]$$
is well defined and smooth.

Proof. First recall
$$\pi : \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \to \left(\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}\right) / \sim, \quad \mathbf{x} \sim \mathbf{y} \iff \mathbf{y} = \lambda \mathbf{x},\ \lambda \in \mathbb{R} \setminus \{0\}$$
where $\left(\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}\right) / \sim$ is identified with $\mathbb{RP}^n$. Therefore, if $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ are such that
$$[\mathbf{x}] = [\mathbf{y}] \implies \mathbf{x} = \lambda \mathbf{y}$$
where by the definition of P we have
$$P(\mathbf{x}) = P(\lambda \mathbf{y}) = \lambda^d P(\mathbf{y})$$
which then tells us that, under the mapping
$$\pi : \mathbb{R}^{k+1} \setminus \{\mathbf{0}\} \to \mathbb{RP}^k$$
we have
$$P(\mathbf{x}) \sim P(\mathbf{y})$$
which will give
$$\pi(P(\mathbf{x})) = [P(\mathbf{x})] = [P(\mathbf{y})] = \pi(\lambda^d P(\mathbf{y}))$$
and so
$$\widetilde{P} : \mathbb{RP}^n \to \mathbb{RP}^k$$
is well-defined. Introducing the notation π^n for the projection from $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$, and similarly for π^k.

Now for $1 \le i \le n+1$ we define, as before, $U_i \subseteq \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ by
$$U_i = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_i \neq 0\}$$
Then we have
$$\pi^n(U_i) = \{[x_1 : \cdots : x_{n+1}] \in \mathbb{RP}^n : x_i \neq 0\}$$
and for $1 \le i \le k+1$, define $U_i \subseteq \mathbb{R}^{k+1} \setminus \{\mathbf{0}\}$ by
$$U_i = \{(x_1, \ldots, x_{k+1}) \in \mathbb{R}^{k+1} : x_i \neq 0\}$$
Then we have
$$\pi^k(U_i) = \{[x_1 : \cdots : x_{k+1}] \in \mathbb{RP}^k : x_i \neq 0\}$$

and these sets are saturated and hence open as we saw before, and we also have

$$\begin{aligned}(\pi^n)^{-1}\big(\widetilde{P}^{-1}(\pi^k(U_i))\big) &= (\widetilde{P}\circ\pi^n)^{-1}(\pi^k(U_i))\\ &= (\pi^k\circ P)^{-1}(\pi^k(U_i))\\ &= P^{-1}\circ(\pi^k)^{-1}(\pi^k(U_i))\\ &= P^{-1}(U_i)\end{aligned}$$

which is open, since P is smooth and hence continuous, and π^n is a smooth map and thus continuous, which means that $\widetilde{P}^{-1}(\pi^k(U_i))$ must also be open, which tells us that, the preimage of an open set under $\widetilde{P}$ is open, and so $\widetilde{P}$ is continuous. So by Proposition 2.5, to show that $\widetilde{P}$ is smooth, it suffices to show that its coordinate representation with charts in $\mathcal{A}_{\mathbb{RP}^n}$ and $\mathcal{A}_{\mathbb{RP}^k}$ are smooth.

As before, for each $1 \le i \le n$, we define

$$\phi_i : \pi^n(U_i) \to \mathbb{R}^n, \text{ by } \phi_i([\mathbf{x}]) = \left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_{n+1}}{x_i}\right)$$

with inverse given by

$$\phi_i^{-1} : \phi_i(\pi^n(U_i)) \to \pi^n(U_i), \text{ by } \phi_i^{-1}(y_1, \dots, y_n) = [y_1 : \cdots : y_{i-1} : 1 : y_i : \cdots : y_n]$$

and for each $1 \le i \le k$, we define

$$\psi_i : \pi^k(U_i) \to \mathbb{R}^k, \text{ by } \psi_i([\mathbf{x}]) = \left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_{k+1}}{x_i}\right)$$

with inverse given by

$$\psi_i^{-1} : \psi_i(\pi^k(U_i)) \to \pi^k(U_i), \text{ by } \psi_i^{-1}(y_1, \dots, y_k) = [y_1 : \cdots : y_{i-1} : 1 : y_i : \cdots : y_k]$$

Next, writing

$$P(\mathbf{x}) = (P_1(\mathbf{x}), \dots, P_{k+1}(\mathbf{x}))$$

then for any $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, k\}$ we have

$$\psi_j \circ \widetilde{P} \circ \phi_i^{-1}|_{\phi_i(\pi^n(U_i)\cap\widetilde{P}^{-1}(\pi^k(U_j)))} : \phi_i\big(\pi^n(U_i)\cap\widetilde{P}^{-1}(\pi^k(U_j))\big) \to \psi_j(\pi^k(U_j))$$

is given by

$$\begin{aligned}\psi_j\circ\widetilde{P}\circ\phi_i^{-1}(y_1,\dots,y_n) &= \psi_j\circ\widetilde{P}([y_1:\cdots:y_{i-1}:1:y_i:\cdots:y_n])\\ &= \psi_j\big([P_1(y_1,\dots,y_{i-1},1,y_i,\dots,y_n):\cdots:P_k(y_1,\dots,y_{i-1},1,y_i,\dots,y_n)]\big)\\ &= \left(\frac{P_1(y_1,\dots,y_{i-1},1,y_i,\dots,y_n)}{P_j(y_1,\dots,y_{i-1},1,y_i,\dots,y_n)},\dots,\frac{P_{j-1}(y_1,\dots,y_{i-1},1,y_i,\dots,y_n)}{P_j(y_1,\dots,y_{i-1},1,y_i,\dots,y_n)},\right.\\ &\qquad \left.,\frac{P_{j+1}(y_1,\dots,y_{i-1},1,y_i,\dots,y_n)}{P_j(y_1,\dots,y_{i-1},1,y_i,\dots,y_n)},\dots,\frac{P_{k+1}(y_1,\dots,y_{i-1},1,y_i,\dots,y_n)}{P_j(y_1,\dots,y_{i-1},1,y_i,\dots,y_n)}\right)\end{aligned}$$

and by the definition of $\pi^k(U_j)$ we have that $P_j(y_1, \dots, y_{i-1}, 1, y_i, \dots, y_n) \ne 0$, and so is well-defined, and smooth, since P is smooth.

Since $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, k\}$ were arbitrary, we conclude that the coordinate representation of $\widetilde{P}$ is smooth for each pair of charts as a map between euclidean spaces, and thus $\widetilde{P}$ must also be smooth. $\square$

2-10 *For any topological space M, let $C(M)$ denote the algebra of continuous functions*

$$f : M \to \mathbb{R}$$

Given a continuous map $F : M \to N$, define

$$F^* : C(N) \to C(M), \text{ by } F^*(f) = f \circ F$$

(a) Show that F^* is a linear map.

(b) Suppose M and N are smooth manifolds. Show that $F : M \to N$ is smooth if and only if $F^*\big(C^\infty(N)\big) \subseteq C^\infty(M)$.

(c) Suppose $F : M \to N$ is a homeomorphism between smooth manifolds. Show that it is a diffeomorphism if and only if F^* restricts to an isomorphism from $C^\infty(N)$ to $C^\infty(M)$.

Proof.

(a) Let $f, g \in C(N)$ and $c \in \mathbb{R}$ be given. then we have

$$F^*(f + cg) = (f + cg) \circ F : M \to \mathbb{R}$$

so for any $p \in M$ we have

$$\begin{aligned}
\big((f + cg) \circ F\big)(p) &= (f + cg)(F(p)) \\
&= f(F(p)) + cg(F(p)) \\
&= (f \circ F)(p) + c(g \circ F)(p) \\
&= F^*(f)(p) + cF^*(g)(p) \\
&= \big(F^*(f) + cF^*(g)\big)(p)
\end{aligned}$$

and therefore we get

$$F^*(f + cg) = F^*(f) + cF^*(g)$$

and thus, is linear.

(b) First suppose that

$$F : M \to N$$

is smooth. Then for any $f \in C^\infty(N)$ we will have

$$F^*(f) = f \circ F : M \to \mathbb{R}$$

which is smooth, as the composition of smooth functions, and so $f \circ F = F^*(f) \in C^\infty(M)$ and hence $F^*\big(C^\infty(N)\big) \subseteq C^\infty(M)$

Next, suppose that $F^*\big(C^\infty(N)\big) \subseteq C^\infty(M)$, that is for every $f \in C^\infty(N)$ we have

$$F^*(f) = f \circ F \in C^\infty(M)$$

Let $p \in M$ be given. Since M is a smooth manifold, $\exists$ $(U, p) \in \mathcal{A}_M$ containing p, and since N is a smooth manifold $\exists$ $(V, \psi) \in \mathcal{A}_N$ containing $F(p)$. Shrinking U if necessary we may assume that $F(U) \subseteq V$. Now since every topological manifold has a countable basis of precompact coordinate balls by lemma 1.10, so we may find $B_{F(p)}$ containing $F(p)$ such that $\overline{B_{F(p)}} \subseteq V$, and since

$$\psi : V \to \psi(V) \subseteq \mathbb{R}^n$$

is smooth as it is a diffeomorphism, by the definition of smooth chart, and the restriction of a diffeomorphism is smooth, we will have

$$\psi|_{\overline{B_{F(p)}}} : \overline{B_{F(p)}} \to \psi(\overline{B_{F(p)}}) \subseteq \mathbb{R}^n$$

is also smooth. Thus we have a closed set $\overline{B_{F(p)}} \subseteq N$ and a smooth map $\psi|_{\overline{B_{F(p)}}}$ on $\overline{B_{F(p)}}$, and an open set $V \subseteq N$ such that $V \supseteq \overline{B_{F(p)}}$. So, by the Extension Lemma for Smooth maps there exists a smooth extension

$$\widetilde{\psi} : M \to \mathbb{R}^n$$

such that $\widetilde{\psi}|_{\overline{B_{F(p)}}} = \psi|_{\overline{B_{F(p)}}}$. Next, since the projection maps

$$\pi_i : \mathbb{R}^n = \mathbb{R}_1 \times \mathbb{R}_2 \times \cdots \times \mathbb{R}_n \to \mathbb{R}_i$$

are all smooth and the composition of smooth maps is smooth we will have

$$\pi_i \circ \widetilde{\psi} : M \to \mathbb{R}$$

is smooth $\forall\ 1 \leq i \leq n$; that is $\pi_i \circ \widetilde{\psi} \in C^\infty(M)$, and so by our assumption $(\pi_i \circ \widetilde{\psi}) \circ F$ is smooth $\forall\ 1 \leq i \leq n$. Yet,

$$(\pi_i \circ \widetilde{\psi}) \circ F = \pi_i \circ (\widetilde{\psi} \circ F)$$

and by Proposition 2.12 since $\pi_i \circ (\widetilde{\psi} \circ F)$ is smooth for each i, we have that $\widetilde{\psi} \circ F$ is also smooth.

Now, since F is continuous, we will have $U \cap F^{-1}(B_{F(p)}) \subseteq U$ is open in M, and contains p, and since the restriction of a diffeomorphism is a diffeomorphism we will have the chart

$$\Big(U \cap F^{-1}(B_{F(p)}), \phi|_{U \cap F^{-1}(B_{F(p)})}\Big) \in \mathcal{A}_M$$

then since

$$F(U) \cap B_{F(p)} \subseteq B_{F(p)} \subseteq \overline{B_{F(p)}} \quad \text{and} \quad \widetilde{\psi}|_{\overline{B_{F(p)}}} = \psi|_{\overline{B_{F(p)}}}$$

we have the coordinate representation of F

$$\psi \circ F \circ \phi^{-1}|_{\phi(U \cap F^{-1}(B_{F(p)}))} : \phi\big(U \cap F^{-1}(B_{F(p)})\big) \to \psi\big(F(U) \cap B_{F(p)}\big)$$

is smooth, as the composition of smooth maps, on a neighborhood of $\phi(p)$ as a map between euclidean spaces, and thus, F must be smooth on a neighborhood of p.

Since $p \in M$ was arbitrary, we conclude that for each $p \in M$ we may find a neighborhood of p where F is smooth, that is, F is smooth in a neighborhood of each point of M, and thus we conclude that

$$F : M \to N$$

is smooth

(c) Let $F : M \to N$ be a homeomorphism, and we note from (a), we know that F^* is linear, and since F is a homeomorphism we know that

$$F^{-1} : N \to M$$

exists, and so the mapping

$$(F^{-1})^* : C(M) \to C(N)$$

is defined. So let $f \in C(M)$ be arbitrary then $(F^{-1})^*(f) \in C(N)$, and

$$F^*\big((F^{-1})^*(f)\big) = (F^{-1})^*(f) \circ F = f \circ F^{-1} \circ F = f$$

similarly for any $g \in C(N)$ we will have $F^*(g) \in C(M)$ and

$$(F^{-1})^*\big(F^*(g)\big) = F^*(g) \circ F^{-1} = g \circ F \circ F^{-1} = g$$

giving

$$(F^{-1})^* \circ F^* = Id_{C(N)}, \quad \text{and} \quad F^* \circ (F^{-1})^* = Id_{C(M)}$$

and so both $(F^{-1})^*, F^*$ are bijective and inverses of each other. Finally, let $f, g \in C(N)$ be given, and $p \in M$ be arbitrary, then

$$\begin{aligned}
\big(F^*(fg)\big)(p) &= fg \circ F(p) \\
&= f(F(p)) \cdot g(F(p)) \\
&= (f \circ F)(p) \cdot (g \circ F)(p) \\
&= F^*(f)(p) \cdot F^*(g)(p) \\
&= \big(F^*(f) \cdot F^*(g)\big)(p)
\end{aligned}$$

and therefore $F^*(fg) = F^*(f)F^*(g)$. And so be have a bijective homomorphism between rings, and therefore F^* is an isomorphism.

Now suppose that $F : M \to N$ is a diffeomorphism, then F and F^{-1} are both smooth and from (b), we have

$$\begin{aligned}
&F^*\big(C^\infty(N)\big) \subseteq C^\infty(M) \\
&(F^{-1})^*\big(C^\infty(M)\big) = (F^*)^{-1}\big(C^\infty(M)\big) \subseteq C^\infty(N)
\end{aligned}$$

then as

$$F^* : C(N) \to C(M)$$

is an isomorphism, we know that

$$F^*|_{C^\infty(N)} : C^\infty(N) \to C^\infty(M)$$

is an injective homomorphism. Yet, for any $g \in C^\infty(M)$ we have that $(F^*)^{-1}(g) \in C^\infty(N)$, where we then get

$$F^*\big((F^*)^{-1}(g)\big) = g$$

and thus, $F^*|_{C^\infty(N)}$ is surjective, and therefore an isomorphism.

Next, suppose that

$$F^*|_{C^\infty(N)} : C^\infty(N) \to C^\infty(M)$$

is an isomorphism. Then trivially we have

$$\begin{aligned} &F^*\big(C^\infty(N)\big) \subseteq C^\infty(M) \\ &(F^{-1})^*\big(C^\infty(M)\big) = (F^*)^{-1}\big(C^\infty(M)\big) \subseteq C^\infty(N) \end{aligned}$$

and so by part (b) F is smooth, and so F is a smooth isomorphism, and hence, is a diffeomorphism.

$\square$

2-14 *Suppose A and B are disjoint closed subsets of a smooth manifold M. Show that there exists $f \in C^\infty(M)$ such that*

$$\begin{aligned} &0 \le f(p) \le 1, \quad \forall\, p \in M \\ &f^{-1}(0) = A \\ &f^{-1}(1) = B \end{aligned}$$

Proof. Since both A and B are closed, the Level Sets of Smooth Functions Theorem says that $\exists\, f_A, f_B \in C^\infty(M)$ such that both f_A and f_B are non-negative; that is,

$$f_A, f_B : M \to \mathbb{R}_+$$

and

$$\begin{aligned} &f_A^{-1}(0) = A \\ &f_B^{-1}(0) = B \end{aligned}$$

and so we may define

$$f : M \to \mathbb{R}_+, \text{ by } f(p) = \frac{f_A(p)}{f_A(p) + f_B(p)}$$

which is well defined, since $A \cap B = \varnothing$, and so the denominator is never zero. And is smooth since both f_A, and f_B are. Then for

$p \in A$

$$f_A(0) = 0 \implies f(p) = \frac{0}{0 + f_B(p)} = 0$$

$p \in B$

$$f_B(0) = 0 \implies f(p) = \frac{f_A(p)}{f_A(p) + 0} = 1$$

$p \in M \setminus (A \sqcup B)$

$$f_A(0) > 0, \text{ and } f_B(0) > 0 \implies 0 < f(p) < 1$$

and thus we have $f \in C^\infty(M)$ such that

$$\begin{aligned} &0 \le f(p) \le 1, \quad \forall\, p \in M \\ &f^{-1}(0) = A \\ &f^{-1}(1) = B \end{aligned}$$

□

3-1 *Suppose M and N are smooth manifolds with or without boundary, and*

$$F : M \to N$$

is a smooth map. Show that

$$dF|_p : T_pM \to T_{F(p)}N$$

is the zero map for each $p \in M$ if and only if F is constant on each component of M.

Proof. WLOG suppose that M is connected, since if M is not connected, then by Proposition 1.11, M has countably many connected components, each of which are open and topological manifolds so we get

$$M = \bigsqcup_{i=1}^{\infty} C_i$$

giving each the smooth structure induced by the relative topology, then each is also a smooth manifold, and we can define

$$F = \sum_{i=1}^{\infty} F|_{C_i}$$

So, first suppose that

$$F : M \to N$$

is constant on M, and let $p \in M$ be given. Then for any $v_p \in T_pM$, and any $f \in C^\infty(N)$ we have

$$dF|_p(v_p)(f) = v_p(f \circ F)$$

and since $F = cnst$ we will have $f \circ F \in C^\infty(M)$ is constant, and so

$$v_p(f \circ F) = 0$$

By Lemma 3.4. Then, as $f \in C^\infty(N)$ arbitrary we have

$$dF|_p(v_p) = 0$$

and as $v_p \in T_pM$ was abritrary, we have that

$$dF|_p = 0$$

and finally since $p \in M$ was arbitrary we have $dF|_p = 0$ for each $p \in M$

Next suppose that

$$dF|_p : T_pM \to T_{F(p)}N$$

is the zero map for each $p \in M$. Let $p \in M$ be given, and by the smoothness of F select charts $(U, \phi) \in \mathcal{A}_M$ containing p, and $(V, \psi) \in \mathcal{A}_N$ containing $F(p)$, such that $F(U) \subseteq V$. Since F is smooth we have the coordinate representation of F, given by

$$\psi \circ F \circ \phi^{-1}|_{\phi(U)} : \phi(U) \to \psi(V)$$

is smooth. Now since, by assumption M is connected, and as it in a manifold, it is locally path-connect, so we may arrange for $U \subseteq M$, such that U is connected, and since continuous image of a connected set is connected, we have $\phi(U)$ is connected as is $\psi(V)$ as it is the image of a composition of smooth, and hence continuous, maps.

Next, we note that $dF|_p$ is the Jacobian matrix of the coordinate representation $\widehat{F}$ at $\phi(p)$; i.e.

$$dF|_p = \begin{bmatrix} \frac{\partial \widehat{F}^1}{\partial x^1}\big(\phi(p)\big) & \cdots & \frac{\partial \widehat{F}^1}{\partial x^m}\big(\phi(p)\big) \\ \vdots & \ddots & \vdots \\ \frac{\partial \widehat{F}^n}{\partial x^1}\big(\phi(p)\big) & \cdots & \frac{\partial \widehat{F}^n}{\partial x^m}\big(\phi(p)\big) \end{bmatrix}$$

and so

$$dF|_p = 0 \implies \begin{bmatrix} \frac{\partial \widehat{F}^1}{\partial x^1}\big(\phi(p)\big) & \cdots & \frac{\partial \widehat{F}^1}{\partial x^m}\big(\phi(p)\big) \\ \vdots & \ddots & \vdots \\ \frac{\partial \widehat{F}^n}{\partial x^1}\big(\phi(p)\big) & \cdots & \frac{\partial \widehat{F}^n}{\partial x^m}\big(\phi(p)\big) \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

Thus, all of $\widehat{F}$'s directional derivatives at $\phi(p)$ are zero. Yet, for any $\mathbf{x} \in \phi(U)$ we also have by proposition 3.6

$$d\widehat{F}|_{\mathbf{x}} = d\psi_{F(\phi^{-1}(\mathbf{x}))} \circ \overbrace{dF_{\phi^{-1}(\mathbf{x})}}^{0} \circ d\phi^{-1}|_{\mathbf{x}} = 0$$

and thus, we have that all directional derivatives of $\widehat{F}$ for all points $\mathbf{x}$ contained in the connected set $\phi(U)$ are zero, and thus $\widehat{F}$ is constant on $\phi(U)$, say $\widehat{F}(\mathbf{x}) = \mathbf{c},\ \forall\ \mathbf{x} \in \phi(U)$. Then as

$$\widehat{F} = \psi \circ F \circ \phi^{-1} \implies \psi^{-1} \circ \widehat{F} \circ \phi = F$$

we have for each $p \in U$ that

$$F(p) = \psi^{-1} \circ \widehat{F}(\phi(p)) = \psi^{-1}(\mathbf{c}) = cnst$$

and so F is constant on U.

To see that F is constant on the connected space M assume, for contradiction, that F assumes two values in N, say q_1, q_2, then as the singleton $\{q_1\} \subseteq N$ is closed we have $N \setminus \{q_1\}$ is open, where the continuity of F then says that

$$F^{-1}(N \setminus \{q_1\}) \subseteq M$$

must be open, and nonempty as there is at least one $p \in M$ such that $F(p) = q_2 \in N \setminus \{q_1\}$. Also, we have that $F^{-1}(\{q_1\}) \subseteq M$ and $F^{-1}(\{q_1\}) \neq \varnothing$. Then, from above, for any $r \in F^{-1}(\{q_1\})$ we may find a chart $(U, \phi) \in \mathcal{A}_M$ containing r, such that $F(U) = cnst$. Yet, since $r \in F^{-1}(\{q_1\})$ we have

$$F(r) = q_1 \implies F(s) = q_1\ \forall\ s \in U \implies U \subseteq F^{-1}(\{q_1\})$$

and so, around each point of $F^{-1}(\{q_1\})$, we can find a neighborhood entirely contained in $F^{-1}(\{q_1\})$, and therefore $F^{-1}(\{q_1\})$ must be open. Thus we have found $F^{-1}(\{q_1\}), F^{-1}(N \setminus \{q_1\}) \subseteq M$ open such that

$$\begin{aligned} &F^{-1}(\{q_1\}) \neq \varnothing \\ &F^{-1}(N \setminus \{q_1\}) \neq \varnothing \\ &F^{-1}(\{q_1\}) \cap F^{-1}(N \setminus \{q_1\}) = \varnothing \\ &F^{-1}(\{q_1\}) \cup F^{-1}(N \setminus \{q_1\}) = M \end{aligned}$$

and so $\{F^{-1}(\{q_1\}), F^{-1}(N \setminus \{q_1\})\}$ form a separation of M, contradicting the assumption that M is connected.

Therefore we can conclude that F is constant on M. $\square$

3-2 *Let $M_1, \dots, M_k$ be smooth manifolds, and for each j, let $\pi_j : M_1 \times \cdots \times M_k \to M_j$ be the projection onto the M_j factor. For any point $p = (p_1, \dots, p_k) \in M_1 \times \cdots \times M_k$, the map*

$$\alpha : T_p(M_1 \times \cdots \times M_k) \to T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k$$

defined by

$$\alpha(v) = \big(d(\pi_1)|_p(v), \dots, d(\pi_k)|_p(v)\big)$$

is an isomorphism. The same is true if one of the spaces M_i is a smooth manifold with boundary.

Proof. The idea here is to come up with something that resembles an inverse so that we can use dimensionality and surjectivity, or injectivity to show an isomorphism. First we note that α is linear since each $d\pi_i|_p$ is. So given $p = (p_1, \dots, p_k) \in M_1 \times \cdots \times M_k$ define

$$\iota_j : M_j \to M_1 \times \cdots \times M_k \text{ by } \iota_j(x) = (p_1, \dots, p_{j-1}, x, p_{j+1}, \dots, p_k)$$

for $1 \le j \le k$ so that ι_j induces the linear map

$$d\iota_j|_{p_j} : T_{p_j}M_j \to T_p(M_1 \times \cdots \times M_k), \quad \iota_j(p_j) = (p_1, \dots, p_j, \dots, p_k) = p$$

where we next define the map

$$\beta : T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k \to T_p(M_1 \times \cdots \times M_k)$$

defined by

$$\beta\big(v_{1p_1} \oplus v_{2p_2} \oplus \cdots \oplus v_{kp_k}\big) = d\iota_1|_{p_1}\big(v_{1p_1}\big) \times d\iota_2|_{p_2}\big(v_{2p_2}\big) \times \cdots \times d\iota_k|_{p_k}\big(v_{kp_k}\big)$$

so that for any $v_{1p_1} \oplus \cdots \oplus v_{kp_k} \in T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k$ and $f \in C^\infty(M_1 \oplus \cdots \oplus M_k)$ we get

$$\begin{aligned}
(\alpha \circ \beta)\big(v_{1p_1} \oplus \cdots \oplus v_{kp_k}\big)(f) &= \alpha\Big(d\iota_1|_{p_1}\big(v_{1p_1}\big), \dots, d\iota_k|_{p_k}\big(v_{kp_k}\big)\Big)(f) \\
&= \Big[d\pi_1|_p\Big(d\iota_1|_{p_1}\big(v_{1p_1}\big), \dots, d\iota_k|_{p_k}\big(v_{kp_k}\big)\Big), \dots, \\
&\qquad d\pi_k|_p\Big(d\iota_1|_{p_1}\big(v_{1p_1}\big), \dots, d\iota_k|_{p_k}\big(v_{kp_k}\big)\Big)\Big](f) \\
&= \Big[d\pi_1|_{\iota_1(p_1)}\Big(d\iota_1|_{p_1}\big(v_{1p_1}\big), \dots, d\iota_k|_{p_k}\big(v_{kp_k}\big)\Big), \dots, \\
&\qquad d\pi_k|_{\iota_k(p_k)}\Big(d\iota_1|_{p_1}\big(v_{1p_1}\big), \dots, d\iota_k|_{p_k}\big(v_{kp_k}\big)\Big)\Big](f) \\
&= \Big(d(\pi_1 \circ \iota_1)|_{p_1}\big(v_{1p_1}\big)(f), \dots, d(\pi_k \circ \iota_k)|_{p_k}\big(v_{kp_k}\big)(f)\Big) && \text{Proposition 3.6 (b)} \\
&= \Big(v_1(f \circ \pi_1 \circ \iota_1)|_{p_1}, \dots, v_k(f \circ \pi_k \circ \iota_k)|_{p_k}\Big) \\
&= \Big(v_1(f \circ Id_{M_1})|_{p_1}, \dots, v_k(f \circ Id_{M_k})|_{p_k}\Big) && \pi_i \circ \iota_i = Id_{M_i} \\
&= \Big(v_1(f)|_{p_1}, \dots, v_k(f)|_{p_k}\Big) && \text{Proposition 3.6 (c)} \\
&= \big(v_{1p_1}, \dots, v_{kp_k}\big)(f) \\
&= \big(v_{1p_1} \oplus \cdots \oplus v_{kp_k}\big)(f)
\end{aligned}$$

so that

$$\alpha \circ \beta = Id_{T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k}$$

And so α is surjective.

Next we note by Proposition 3.10, the tangent space at any point of the manifold has the same dimension as the manifold and so

$$\begin{aligned} \dim\big(T_p(M_1 \times \cdots \times M_k)\big) &= n_1 + n_2 + \cdots + n_k \\ &= \dim\big(T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k\big) \end{aligned}$$

Thus, we have a surjective linear map α between vector spaces of equal dimension. Therefore α must be an isomorphism between $T_p(M_1 \times \cdots \times M_k)$ and $T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k$. $\square$

3-4 *Show that $T\mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$.*

Proof. First we note that

$$T\mathbb{S}^1 = \{(x, y, -ty, tx) \in \mathbb{S}^1 \times \mathbb{R}^2 : t \in \mathbb{R}\}$$

so defining the map

$$f : T\mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{R}, \text{ by } f(x, y, -ty, tx) = (x, y, t)$$

with explicit inverse given by

$$f^{-1} : \mathbb{S}^1 \times \mathbb{R} \to T\mathbb{S}^1, \text{ by } f^{-1}(x, y, t) = (x, y, -ty, tx)$$

as

$$\begin{aligned} f^{-1} \circ f(x, y, -ty, tx) &= f^{-1}(x, y, t) \\ &= (x, y, -ty, tx) \end{aligned}$$

and similarly

$$\begin{aligned} f \circ f^{-1}(x, y, t) &= f(x, y, -ty, tx) \\ &= (x, y, t) \end{aligned}$$

and both f and f^{-1} are smooth as they are polynomials in each component. And thus, we have a bi-smooth bijection and so $T\mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$. $\square$

3-6 *Consider $\mathbb{S}^3$ as the unit sphere in $\mathbb{C}^2$ under the usual identification $\mathbb{C}^2 \leftrightarrow \mathbb{R}^4$. For each $z = (z_1, z_2) \in \mathbb{S}^3$, define a curve*

$$\gamma_z : \mathbb{R} \to \mathbb{S}^3, \text{ by } \gamma_z(t) = (e^{it}z_1, e^{it}z_2)$$

Show that γ_z is a smooth curve whose velocity is never zero.

Proof. First, making the identification $\mathbb{C}^2 \cong \mathbb{R}^4$ and letting $\mathbf{z} = (z_1, z_2) = (a, b, c, d)$ with $z_1 = a + ib$ and $z_2 = c + id$. Then noting that

$$\begin{aligned}(\cos t + i \sin t)(a + bi) &= a\cos t + ai \sin t + ib\cos t - b \sin t \\ &= a \cos t - b \sin t + i\big(a \sin t + b \cos t\big)\end{aligned}$$

we get

$$\begin{aligned}\gamma_{\mathbf{z}}(t) &= (e^{it}z_1, e^{it}z_2) \\ &\mapsto \big(a\cos t - b\sin t, a\sin t + b\cos t, c\cos t - d\sin t, c\sin t + d\cos t\big)\end{aligned}$$

which is continuous in each component as both $\sin t, \cos t$ are, and multiplication by a constant is continuous. So by proposition 2.5 to show that f is smooth, it suffices to show that the coordinate representation of $\gamma_{\mathbf{z}}$ is smooth, with charts in $\mathcal{A}_{\mathbb{S}^3}$, and $\mathcal{A}_{\mathbb{R}}$.

Choosing the standard smooth structure $\{(U_i^{\pm}, \phi_i^{\pm})\}_{i=1}^4$ for $\mathbb{S}^3$, defined by

$$U_i^+ = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_i > 0\} \qquad U_i^- = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_i < 0\} \quad 1 \le i \le 4$$

and the mappings

$$\phi_i^{\pm} : U_i^{\pm} \cap \mathbb{S}^3 \to \mathbb{B}^3, \qquad (\phi_i^{\pm})^{-1} : \mathbb{B}^3 \to \mathbb{S}^3 \cap U_i^{\pm}$$

given by

$$\begin{aligned}
\phi_1^{\pm}(x_1, x_2, x_3, x_4) &= (x_2, x_3, x_4) & (\phi_1^{\pm})^{-1}(u_1, u_2, u_3) &= (\pm\sqrt{1 - ||\mathbf{u}||^2}, u_2, u_3, u_4) \\
\phi_2^{\pm}(x_1, x_2, x_3, x_4) &= (x_1, x_3, x_4) & (\phi_2^{\pm})^{-1}(u_1, u_2, u_3) &= (u_1, \pm\sqrt{1 - ||\mathbf{u}||^2}, u_3, u_4) \\
\phi_3^{\pm}(x_1, x_2, x_3, x_4) &= (x_1, x_2, x_4) & (\phi_3^{\pm})^{-1}(u_1, u_2, u_3) &= (u_1, u_2, \pm\sqrt{1 - ||\mathbf{u}||^2}, u_4) \\
\phi_4^{\pm}(x_1, x_2, x_3, x_4) &= (x_1, x_2, x_3) & (\phi_4^{\pm})^{-1}(u_1, u_2, u_3) &= (u_1, u_2, u_3, \pm\sqrt{1 - ||\mathbf{u}||^2})
\end{aligned}$$

and $(\mathbb{R}, Id_{\mathbb{R}}) \in \mathcal{A}_{\mathbb{R}}$ we have the coordinate representation

$$\phi_i^{\pm} \circ \gamma_{\mathbf{z}}|_{\gamma_{\mathbf{z}}^{-1}(U_i^{\pm})} : \gamma_{\mathbf{z}}^{-1}(U_i^{\pm}) \to \phi_i^{\pm}(U_i^{\pm})$$

is given by, for example let $i = 1$, with all other cases being similar

$$\begin{aligned}\phi_1^{\pm} \circ \gamma_{\mathbf{z}}(t) &= \phi_1^{\pm}\big(a\cos t - b\sin t, a\sin t + b\cos t, c\cos t - d\sin t, c\sin t + d\cos t\big) \\ &= \big(a\sin t + b\cos t, c\cos t - d\sin t, c\sin t + d\cos t\big)\end{aligned}$$

which is smooth, as a map between euclidean spaces, and thus, $\gamma_{\mathbf{z}}$ must also be smooth.

Next, since we know that that the inclusion

$$\iota : \mathbb{S}^3 \to \mathbb{R}^4$$

is smooth, and the composition of smooth functions is smooth, so we can consider the smooth map

$$\iota \circ \gamma_{\mathbf{z}} : \mathbb{R} \to \mathbb{R}^4$$

Let $t_0 \in \mathbb{R}$ be given, and note that

$$(\iota \circ \gamma_{\mathbf{z}})'(t_0) = d(\iota \circ \gamma_{\mathbf{z}})|_{t_0} : T_{t_0}\mathbb{R} \to T_{\iota(\gamma_{\mathbf{z}}(t_0))}\mathbb{R}^4$$

and since the deferential at $d(\iota \circ \gamma_{\mathbf{z}})|_{t_0}$ is the Jacobean matrix of the coordinate representation $\widehat{\iota \circ \gamma_{\mathbf{z}}} = \widehat{\gamma}_{\mathbf{z}}$ at $\gamma_{\mathbf{z}}(t_0)$ we get

$$d(\iota \circ \gamma_{\mathbf{z}})|_{t_0} = \begin{bmatrix} \frac{\partial \gamma^1_{\mathbf{z}}}{dt}\Big|_{\gamma_{\mathbf{z}}(t_0)} \\ \frac{\partial \gamma^2_{\mathbf{z}}}{dt}\Big|_{\gamma_{\mathbf{z}}(t_0)} \\ \frac{\partial \gamma^3_{\mathbf{z}}}{dt}\Big|_{\gamma_{\mathbf{z}}(t_0)} \\ \frac{\partial \gamma^4_{\mathbf{z}}}{dt}\Big|_{\gamma_{\mathbf{z}}(t_0)} \end{bmatrix} = \begin{bmatrix} -a\sin t_0 - b\cos t_0 \\ a\cos t_0 - b\sin t_0 \\ -c\sin t_0 - d\cos t_0 \\ c\cos t_0 - d\sin t_0 \end{bmatrix}$$

where we note that $(a, b), (c, d) \neq (0, 0)$ since $z_1, z_2 \in \mathbb{S}^3$, and there is no value of $t_0 \in \mathbb{R}$ which will make all 4 rows equal to zero. Since

$$\begin{aligned} -a\sin t_0 - b\cos t_0 = 0 &\iff a = -b, \text{ and } \sin t_0 = \cos t_0 \implies a\cos t_0 - b\sin t_0 \neq 0 \\ c\cos t_0 - d\sin t_0 = 0 &\iff c = d, \text{ and } \sin t_0 = \cos t_0 \implies -c\sin t_0 - d\cos t_0 \neq 0 \end{aligned}$$

thus we can conclude that $(\iota \circ \gamma_{\mathbf{z}})'(t_0) \neq 0 \ \forall \ t_0 \in \mathbb{R}$. This, then implies that for any $f \in C^\infty(\mathbb{R}^4)$ we have

$$(\iota \circ \gamma_{\mathbf{z}})'(t_0)(f) = d(\iota \circ \gamma_{\mathbf{z}})|_{t_0}(f) = (f \circ \iota \circ \gamma_{\mathbf{z}})|_{t_0} \neq 0$$

yet, by the smoothness of ι, we may also consider $f \circ \iota \in C^\infty(\mathbb{S}^3)$ which then gives

$$\gamma'_{\mathbf{z}}(t_0)(f \circ \iota) = d\gamma_{\mathbf{z}}|_{t_0}(f \circ \iota) = (f \circ \iota \circ \gamma_{\mathbf{z}})|_{t_0} \neq 0$$

and therefore we can conclude that $\gamma'_{\mathbf{z}}(t_0) \neq 0 \ \forall \ t_0 \in \mathbb{R}$.

Therefore $\gamma_{\mathbf{z}}$ is a smooth curve whose velocity is never zero. $\square$

3-8 *Let M be a smooth manifold with or without boundary and $p \in M$. Let $\mathcal{V}_pM$ denote the set of equivalence classes of smooth curves starting at p under the relation*

$$\gamma_1 \sim \gamma_2 \iff (f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$$

for every smooth real-valued function f defined in a neighborhood of p. Show that the map

$$\Psi : \mathcal{V}_pM \to T_pM, \ \textit{by} \ \Psi[\gamma] = \gamma'(0)$$

is well defined and bijective.

Proof. First, suppose that γ_1 and γ_2 are two smooth curves where

$$\gamma_1(0) = p = \gamma_2(0)$$

such that $\gamma_1 \sim \gamma_2$, then

$$(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$$

for every smooth real-valued function f defined in a neighborhood of p, so let $f \in C^\infty(M)$, then f is smooth on the neighborhood M of p, and therefore

$$\begin{aligned}
\Psi[\gamma_1](f) &= (\gamma_1)'(0)(f) \\
&= d\gamma_1|_0(f) \\
&= (f \circ \gamma_1)|_0 \\
&= (f \circ \gamma_1)'(0) \\
&= (f \circ \gamma_2)'(0) \\
&= (f \circ \gamma_2)|_0 \\
&= d\gamma_2|_0(f) \\
&= (\gamma_2)'(0)(f) \\
&= \Psi[\gamma_2](f)
\end{aligned}$$

and so

$$\Psi : \mathcal{V}_pM \to T_pM, \ \text{by} \ \Psi[\gamma] = \gamma'(0)$$

is well defined.

Next suppose that

$$\begin{aligned}
&\Psi[\gamma_1] = \Psi[\gamma_2] \\
&\implies (\gamma_1)'(0) = (\gamma_2)'(0)
\end{aligned}$$

and let $f \in C^\infty(U)$ where U is a neighborhood of p, we have

$$\begin{aligned}
(\gamma_1)'(0)(f) &= d\gamma_1|_0(f) \\
&= (f \circ \gamma_1)|_0 \\
&= (f \circ \gamma_1)'(0) \\
&= (f \circ \gamma_2)'(0) \\
&= (f \circ \gamma_2)|_0 \\
&= d\gamma_2|_0(f) \\
&= (\gamma_2)'(0)(f)
\end{aligned}$$

and so the velocities are equal on the neighborhood U. Since M is a manifold, it has a basis of precompact coordinate balls, let B be a neighborhood of p in U, then $\overline{B}$ is a closed neighborhood of p, where f is smooth, such that $\overline{B} \subseteq U$, and so the Extension Lemma for smooth maps says there exists $\widetilde{f} \in C^\infty(M)$ such that $\widetilde{f}|_{\overline{B}} = f$ then

$$(\gamma_1)'(0)(\widetilde{f}) = (\widetilde{f} \circ \gamma_1)'(0) = (f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0) = (\widetilde{f} \circ \gamma_2)'(0) = (\gamma_2)'(0)(\widetilde{f})$$

and so, the velocities are equal on M. Since f arbitrary, we have $[\gamma_1] = [\gamma_2]$, and so Ψ is injective.

Next, let $v_p \in T_pM$ be given, then by Proposition 3.23, there exists

$$\gamma : J \to M$$

such that

$$\gamma(0) = p, \quad \text{and} \quad \gamma'(0) = v_p$$

and therefore $\Psi[\gamma] = v_p$, and so Ψ is surjective.

Thus, Ψ is well defined and bijective. $\square$

4-5 *Let* $\mathbb{CP}^n$ *denote the n-dimensional complex projective space*

(a) *Show that the quotient map* $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ *is a surjective smooth submersion.*

(b) *Show that* $\mathbb{CP}^1$ *is diffeomorphic to* $\mathbb{S}^2$.

Proof.

(a) First, since
$$\pi : \mathbb{C}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{CP}^n$$
is a quotient map, thus, it is by definition continuous and surjective.

Then, with the standard charts $\{(\pi(U_i), \phi_i)\}_{i=1}^n \in \mathcal{A}_{\mathbb{CP}^n}$ with coordinate domains defined by
$$\pi(U_i) = \{[z_1 : \cdots : z_{n+1}] \in \mathbb{CP}^n : z_i \neq 0\}$$
and the coordinate functions given by
$$\phi_i : \pi(U_i) \to \mathbb{C}^n, \text{ by } \phi([\mathbf{z}]) = \left(\frac{z_1}{z_i}, \dots \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_{n+1}}{z_i}\right)$$
and again me make the observation that $U_i \subseteq \mathbb{C}^{n+1} \setminus \{\mathbf{0}\}$ is given by
$$U_i = \{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} : z_i \neq 0\}, \quad \text{for } 1 \leq i \leq n+1$$
which gives
$$\pi^{-1}\big(\pi(U_i)\big) = \{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} : z_i \neq 0\} = U_i$$
and so, for each i we have the coordinate representation of π, with $(\mathbb{C}^{n+1} \setminus \{\mathbf{0}\}, Id_{\mathbb{C}^{n+1}\setminus\{\mathbf{0}\}}) \in \mathcal{A}_{\mathbb{C}^{n+1}\setminus\{\mathbf{0}\}}$, is
$$\widehat{\pi}_i = \phi_i \circ \pi|_{\pi^{-1}(\pi(U_i))} : U_i \to \phi(\pi(U_i))$$
defined by
$$\begin{aligned}\phi_i \circ \pi(z_1, \dots, z_{n+1}) &= \phi_i([z_1 : \cdots : z_{n+1}]) \\ &= \left(\frac{z_1}{z_i}, \dots \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_{n+1}}{z_i}\right)\end{aligned}$$
which is well defined and smooth for each i, since $z_i \neq 0$, as a map between euclidean spaces, and so π must also be smooth.

Next, we note that
$$d\pi|_{\mathbf{z}} : T_{\mathbf{z}}\mathbb{C}^{n+1} \setminus \{\mathbf{0}\} \to T_{\pi(\mathbf{z})}\mathbb{CP}^n$$
is given by the Jacobian matrix of the coordinate representation $\widehat{\pi}$ at $Id_{\mathbb{C}^{n+1}\setminus\{\mathbf{0}\}}(\mathbf{z}) = \mathbf{z}$, so let

$(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \setminus \{\mathbf{0}\}$ be arbitrary, WLOG say $(z_1, \ldots, z_{n+1}) \in U_i$, then we have

$$
\begin{aligned}
d\pi|_{\mathbf{z}} &= \begin{bmatrix} \frac{\partial \widehat{\pi}_i^1}{\partial z^1}(\mathbf{z}) & \cdots & \frac{\partial \widehat{\pi}_i^1}{\partial z^{n+1}}(\mathbf{z}) \\ \vdots & \ddots & \vdots \\ \frac{\partial \widehat{\pi}_i^n}{\partial z^1}(\mathbf{z}) & \cdots & \frac{\partial \widehat{\pi}_i^n}{\partial z^{n+1}}(\mathbf{z}) \end{bmatrix} \\
&= \begin{bmatrix}
\frac{1}{z_i} & 0 & \ldots & 0 & -\frac{z_1}{z_i^2} & 0 & \ldots & 0 & 0 \\
0 & \frac{1}{z_i} & \ldots & 0 & -\frac{z_2}{z_i^2} & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & \frac{1}{z_i} & -\frac{z_{i-1}}{z_i^2} & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & -\frac{z_{i+1}}{z_i^2} & \frac{1}{z_i} & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & -\frac{z_n}{z_i^2} & 0 & \ldots & \frac{1}{z_i} & 0 \\
0 & 0 & \ldots & 0 & -\frac{z_{n+1}}{z_i^2} & 0 & \ldots & 0 & \frac{1}{z_i}
\end{bmatrix}
\end{aligned}
$$

which has full rank, and since $\mathbf{z} \in \mathbb{C}^{n+1} \setminus \{\mathbf{0}\}$ was arbitrary, we conclude that $d\pi|_{\mathbf{z}}$ has full rank for each $\mathbf{z} \in \mathbb{C}^{n+1} \setminus \{\mathbf{0}\}$.

Therefore, we have that π is a surjective map, whose differential has full rank at each point in its domain, and finally we note that

$$\dim\left(\mathbb{C}^{n+1} \setminus \{\mathbf{0}\}\right) > \dim(\mathbb{CP}^n)$$

and so having full rank implies that π, or $d\pi$, is a smooth submersion. And thus, π is a surjective smooth submersion.

(b) Recalling the stereographic projection charts $\{(\mathbb{S}^2 \setminus \{(0,0,1)\}, \phi_N), (\mathbb{S}^2 \setminus \{(0,0,-1)\}, \phi_S)\}$ with the coordinate functions and their inverses given by

$$
\begin{aligned}
\phi_N(x,y,z) &= \left(\frac{x}{1-z}, \frac{y}{1-z}\right), \qquad \phi^{-1}(x,y) = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right) \\
\phi_S(x,y,z) &= \left(\frac{x}{1+z}, \frac{y}{1+z}\right), \qquad \phi_S^{-1}(x,y) = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{1-x^2-y^2}{x^2+y^2+1}\right)
\end{aligned}
$$

and using the standard charts from (a) for $\mathbb{CP}^1$, namely $\{(U_1, \psi_1), (U_2, \overline{\psi}_2)\}$, where

$$
\begin{aligned}
\pi(U_1) &:= \{[z:w] \in \mathbb{CP}^1 : z \neq 0\} \\
\pi(U_2) &:= \{[z:w] \in \mathbb{CP}^1 : w \neq 0\}
\end{aligned}
$$

and where we have made a slight modification to the second coordinate map, so that we have

$$
\begin{aligned}
\psi_1([z:w]) &= \frac{w}{z}, & \psi^{-1}(z) &= [1:z] \\
\overline{\psi}_2([z:w]) &= \frac{\overline{z}}{\overline{w}}, & \overline{\psi}_2^{-1}(z) &= [\overline{z}:1]
\end{aligned}
$$

and so the change being complex conjugation. Then making the usual realization $\mathbb{C} \cong \mathbb{R}^2$, we have the coordinate maps

$$\psi_1([x_1 + iy_1 : x_2 + iy_2]) = \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}, \frac{y_1x_2 - y_2x_1}{x_2^2 + y_2^2} \right), \qquad \psi^{-1}(z) = [1 : x + iy]$$
$$\overline{\psi}_2([x_1 + iy_1 : x_2 + iy_2]) = \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}, \frac{y_2x_1 - y_1x_2}{x_2^2 + y_2^2} \right), \qquad \overline{\psi}_2^{-1}(z) = [x - iy : 1]$$

which are both still smooth. Let us just double check to see that adjusting the maps to conjugation still gives us a legitimate inverse. We have,

$$\begin{aligned}
(\overline{\psi}_2 \circ \overline{\psi}_2^{-1})(x,y) &= \overline{\psi}_2([x - iy : 1 - i0]) \\
&= \left(\frac{x(1) + (-y)(0)}{1^2 + 0^2}, \frac{(0)x - (-y)(1)}{1^2 + 0^2} \right) \\
&= (x,y)
\end{aligned}$$

and

$$\begin{aligned}
&(\overline{\psi}_2^{-1} \circ \overline{\psi}_2)([x_1 + iy_1 : x_2 + iy_2]) = \overline{\psi}_2^{-1} \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}, \frac{y_1x_2 - y_2x_1}{x_2^2 + y_2^2} \right) \\
&\quad = \left[\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} - i\frac{y_2x_1 - y_1x_2}{x_2^2 + y_2^2} : 1 \right] \\
&\quad = \left[\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} - i\frac{y_2x_1 - y_1x_2}{x_2^2 + y_2^2} : 1 \right] (x_2 + iy_2) && \text{mulitplying by a constant} \\
&\quad = \left[\frac{x_1x_2 + y_1y_2 - i(y_2x_1 - y_1x_2)}{x_2^2 + y_2^2} (x_2 + iy_2) : x_2 + iy_2 \right] \\
&\quad = \left[\frac{x_1(x_2 - iy_2) + y_1(y_2 + ix_2)}{(x_2 + iy_2)(x_2 - iy_2)} (x_2 + iy_2) : x_2 + iy_2 \right] \\
&\quad = \left[\frac{x_1(x_2 - iy_2) + iy_1(x_2 - iy_2)}{x_2 - iy_2} : x_2 + iy_2 \right] \\
&\quad = \left[\frac{(x_1 + iy_1)(x_2 - iy_2)}{x_2 - iy_2} : x_2 + iy_2 \right] \\
&\quad = [x_1 + iy_1 : x_2 + iy_2]
\end{aligned}$$

and so we have

$$\overline{\psi}_2 \circ \overline{\psi}_2^{-1} = Id_{\mathbb{R}^2}, \quad \text{and} \quad \overline{\psi}_2^{-1} \circ \overline{\psi}_2 = Id_{\mathbb{CP}^1}$$

and so are indeed inverses. We must also check that the transition maps are still smoothly compatible, first we have

$$\psi_1 \circ \overline{\psi}_2^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$$

given by

$$\begin{aligned}
\psi_1 \circ \overline{\psi}^{-1}(x,y) &= \psi_1([x - iy : 1]) \\
&= \left(\frac{x(1) + (-1)0}{1^2 + 0^2}, \frac{-y(1) - x(0)}{1^2} \right) \\
&= (x, -y)
\end{aligned}$$

and is smooth, and we also have
$$\overline{\psi}_2 \circ \psi_1^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$$
given by
$$\begin{aligned}(\overline{\psi}_2 \circ \psi_1^{-1})(x, y) &= \overline{\psi}_2([1, x + iy]) \\ &= \left(\frac{(1)x + (0)y}{x^2 + y^2}, \frac{y(1) - (0)x}{x^2 + y^2} \right) \\ &= \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right)\end{aligned}$$
which is smooth, and so the transition maps are still smoothly compatible. But here we can see the reason for the adjustment in the coordinate map, we observe that
$$(\overline{\psi}_2 \circ \psi_1^{-1})(x, y) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right) = (\phi_S \circ \phi_N^{-1})(x, y)$$
suggesting the correspondence we are looking for, so we define
$$F : \mathbb{CP}^1 \to \mathbb{S}^2, \text{ by } F([z : w]) = \begin{cases} \phi_N^{-1} \circ \psi_1([x_1 + iy_1 : x_2 + iy_2]), & x_2 + iy_2 \neq 0 \\ (0, 0, 1) & x_2 + iy_2 = 0 \end{cases}$$

so for any $[z:w] \in \mathbb{CP}^1$ such that $x_2 + iy_2 \neq 0$; i.e. $[x_1 + iy_1 : x_2 + iy_2] \in \pi(U_2)$ we have

$$
\begin{aligned}
F([x_1+iy_1 : x_2+iy_2]) &= \phi_N^{-1}\left(\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}, \frac{y_1x_2-y_2x_1}{x_2^2+y_2^2}\right) \\
&= \left(\frac{2\left(\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}\right)}{\left(\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}\right)^2+\left(\frac{y_1x_2-y_2x_1}{x_2^2+y_2^2}\right)^2+1}, \frac{2\left(\frac{y_1x_2-y_2x_1}{x_2^2+y_2^2}\right)}{\left(\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}\right)^2+\left(\frac{y_1x_2-y_2x_1}{x_2^2+y_2^2}\right)^2+1}, \frac{\left(\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}\right)^2+\left(\frac{y_1x_2-y_2x_1}{x_2^2+y_2^2}\right)^2-1}{\left(\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}\right)^2+\left(\frac{y_1x_2-y_2x_1}{x_2^2+y_2^2}\right)^2+1}\right) \\
&= \left(\frac{2\left(\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}\right)}{\frac{x_1^2x_2^2+2x_1x_2y_1y_2+y_1^2y_2^2}{x_2^4+2x_2^2y_2^2+y_2^4}+\frac{y_2^2x_1^2-2y_1y_2x_1x_2+y_1^2x_2^2}{x_2^4+2x_2^2y_2^2+y_2^4}+1}, \frac{2\left(\frac{y_1x_2-y_2x_1}{x_2^2+y_2^2}\right)}{\frac{x_1^2x_2^2+2x_1x_2y_1y_2+y_1^2y_2^2}{x_2^4+2x_2^2y_2^2+y_2^4}+\frac{y_1^2x_2^2-2y_1y_2x_1x_2+y_2^2x_1^2}{x_2^4+2x_2^2y_2^2+y_2^4}+1},\right. \\
&\qquad \left.\frac{\frac{x_1^2x_2^2+2x_1x_2y_1y_2+y_1^2y_2^2}{x_2^4+2x_2^2y_2^2+y_2^4}+\frac{y_1^2x_2^2-2y_1y_2x_1x_2+y_2^2x_1^2}{x_2^4+2x_2^2y_2^2+y_2^4}-1}{\frac{x_1^2x_2^2+2x_1x_2y_1y_2+y_1^2y_2^2}{x_2^4+2x_2^2y_2^2+y_2^4}+\frac{y_1^2x_2^2-2y_1y_2x_1x_2+y_2^2x_1^2}{x_2^4+2x_2^2y_2^2+y_2^4}+1}\right) \\
&= \left(\frac{2\left(\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}\right)}{\frac{x_1^2x_2^2+y_1^2y_2^2+y_1^2x_2^2+y_2^2x_1^2}{x_2^4+2x_2^2y_2^2+y_2^4}+1}, \frac{2\left(\frac{y_1x_2-y_2x_1}{x_2^2+y_2^2}\right)}{\frac{x_1^2x_2^2+y_1^2y_2^2+y_1^2x_2^2+y_2^2x_1^2}{x_2^4+2x_2^2y_2^2+y_2^4}+1}, \frac{\frac{x_1^2x_2^2+y_1^2y_2^2+y_1^2x_2^2+y_2^2x_1^2}{x_2^4+2x_2^2y_2^2+y_2^4}-1}{\frac{x_1^2x_2^2+y_1^2y_2^2+y_1^2x_2^2+y_2^2x_1^2}{x_2^4+2x_2^2y_2^2+y_2^4}+1}\right) \\
&= \left(\frac{2\left(\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}\right)}{\frac{x_1^2(x_2^2+y_2^2)+y_1^2(y_2^2+x_2^2)}{(x_2^2+y_2^2)^2}+1}, \frac{2\left(\frac{y_1x_2-y_2x_1}{x_2^2+y_2^2}\right)}{\frac{x_1^2(x_2^2+y_2^2)+y_1^2(y_2^2+x_2^2)}{(x_2^2+y_2^2)^2}+1}, \frac{\frac{x_1^2(x_2^2+y_2^2)+y_1^2(y_2^2+x_2^2)}{(x_2^2+y_2^2)^2}-1}{\frac{x_1^2(x_2^2+y_2^2)+y_1^2(y_2^2+x_2^2)}{(x_2^2+y_2^2)^2}+1}\right) \\
&= \left(\frac{2\left(\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}\right)}{\frac{x_1^2+y_1^2}{x_2^2+y_2^2}+1}, \frac{2\left(\frac{y_1x_2-y_2x_1}{x_2^2+y_2^2}\right)}{\frac{x_1^2+y_1^2}{x_2^2+y_2^2}+1}, \frac{\frac{x_1^2+y_1^2}{x_2^2+y_2^2}-1}{\frac{x_1^2+y_1^2}{x_2^2+y_2^2}+1}\right) \\
&= \left(\frac{2\left(\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}\right)\cdot(x_2^2+y_2^2)}{x_1^2+y_1^2+x_2^2+y_2^2}, \frac{2\left(\frac{y_1x_2-y_2x_1}{x_2^2+y_2^2}\right)\cdot(x_2^2+y_2^2)}{x_1^2+y_1^2+x_2^2+y_2^2}, \frac{\frac{x_1^2+y_1^2-x_2^2-y_2^2}{x_2^2+y_2^2}}{\frac{x_1^2+y_1^2+x_2^2+y_2^2}{x_2^2+y_2^2}}\right) \\
&= \left(\frac{2(x_1x_2+y_1y_2)}{x_1^2+y_1^2+x_2^2+y_2^2}, \frac{2(y_1x_2-y_2x_1)}{x_1^2+y_1^2+x_2^2+y_2^2}, \frac{x_1^2+y_1^2-x_2^2-y_2^2}{x_1^2+y_1^2+x_2^2+y_2^2}\right)
\end{aligned}
$$

which is smooth since $x_2, y_2 \neq 0$, and in particular if $x_1 = y_1 = 0$ we have

$$F([0, x_2 + iy_2]) = \phi_N^{-1}(0,0) = (0,0,-1)$$

and thus we have

$$F|_{\pi(U_2)} : \pi(U_2) \to \mathbb{S}^2 \setminus \{(0,0,1)\}$$

and since, by our definition of F we have $F([x_1 + iy_1 : 0]) = (0,0,1)$, we similarly see that

$$F|_{\pi(U_1)} : \pi(U_1) \to \mathbb{S}^2 \setminus \{(0,0,-1)\}$$

and therefore, F is surjective.

Next suppose that

$$
\begin{aligned}
&F([x_1+iy_1 : x_2+iy_2]) = F([u_1+iv_1 : u_2+iv_2]) \\
&\implies \phi_N^{-1} \circ \psi_1([x_1+iy_1 : x_2+iy_2]) = \phi_N^{-1} \circ \psi_1([u_1+iv_1 : u_2+iv_2]) \\
&\implies \psi_1([x_1+iy_1 : x_2+iy_2]) = \psi_1([u_1+iv_1 : u_2+iv_2]) \\
&\implies [x_1+iy_1 : x_2+iy_2] = [u_1+iv_1 : u_2+iv_2]
\end{aligned}
$$

and so F is injective.

Next, let us consider the explicit inverse for F

$$F^{-1} : \mathbb{S}^2 \to \mathbb{CP}^1, \text{ by } F^{-1}(x,y,z) = \begin{cases} \psi_1^{-1} \circ \phi_N(x,y,z), & (x,y,z) \neq (0,0,1) \\ [x_1 + iy_1 : 0], & (x,y,z) = (0,0,1) \end{cases}$$

since

$$F \circ F^{-1}(0,0,1) = F([x_1 + iy_1 : 0]) = (0,0,1)$$

and for any $(x,y,z) \neq (0,0,1)$ we have

$$F \circ F^{-1}(x,y,z) = (\phi_N^{-1} \circ \psi_1) \circ (\psi_1^{-1} \circ \phi_N)(x,y,z) = (x,y,z)$$

and similarly

$$F^{-1} \circ F([x_1 + iy_1 : 0]) = F^{-1}(0,0,1) = [x_1 + iy_1 : 0]$$

while for any $[x_1 + iy_1 : x_2 + iy_2] \in \mathbb{CP}^1$ such that $x_2 + iy_2 \neq 0$ we get

$$F^{-1} \circ F([z : w]) = (\psi_1^{-1} \circ \phi_N) \circ (\phi_N^{-1} \circ \psi_1)([x : w]) = [x_1 + iy_1 : x_2 + iy_2]$$

and so F^{-1} is bijective. To see if F^{-1} is smooth we check the coordinate representation of F^{-1} with the charts $\{(\mathbb{S}^2 \setminus \{(0,0,1)\}, \phi_N), (\mathbb{S}^2 \setminus \{(0,0,-1)\}, \phi_S)\} \in \mathcal{A}_{\mathbb{S}^2}$, and $\{(U_1, \psi_1), (U_2, \overline{\psi}_2)\} \in \mathcal{A}_{\mathbb{CP}^1}$, where we get

Case 1:

$$\psi_1 \circ F^{-1} \circ \phi_N^{-1}|_{\phi_N(\mathbb{S}^2 \setminus \{N\})} : \phi_N\big(\mathbb{S}^2 \setminus \{N\}\big) \to \psi_1(U_1)$$

given by

$$(\psi_1 \circ F^{-1} \circ \phi_N^{-1})(x,y) = (\psi_1 \circ \psi_1^{-1} \circ \phi_N \circ \phi_N^{-1})(x,y) = (x,y)$$

Case 2:

$$\psi_1 \circ F^{-1} \circ \phi_S^{-1}|_{\phi_S(\mathbb{S}^2 \setminus \{S\})} : \phi_S\big(\mathbb{S}^2 \setminus \{S\}\big) \to \psi_1(U_1)$$

given by

$$\begin{aligned}(\psi_1 \circ F^{-1} \circ \phi_S^{-1})(x,y) &= (\psi_1 \circ \psi_1^{-1} \circ \phi_N \circ \phi_S^{-1})(x,y) \\ &= (\psi_1 \circ \psi_1^{-1}) \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right) \\ &= \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right)\end{aligned}$$

Case 3:

$$\overline{\psi}_2 \circ F^{-1} \circ \phi_N^{-1}|_{\phi_N(\mathbb{S}^2 \setminus \{N\})} : \phi_N\big(\mathbb{S}^2 \setminus \{N\}\big) \to \overline{\psi}_2(U_2)$$

given by

$$\begin{aligned}(\overline{\psi}_2 \circ F^{-1} \circ \phi_N^{-1})(x,y) &= (\overline{\psi}_2 \circ \psi_1^{-1} \circ \phi_N \circ \phi_N^{-1})(x,y) \\ &= (\overline{\psi}_2 \circ \psi_1^{-1})(x,y) \\ &= \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right)\end{aligned}$$

Case 4:

$$\overline{\psi}_2 \circ F^{-1} \circ \phi_S^{-1}|_{\phi_S(\mathbb{S}^2 \setminus \{S\})} : \phi_S\big(\mathbb{S}^2 \setminus \{S\}\big) \to \overline{\psi}_2(U_2)$$

given by

$$\begin{aligned}
(\overline{\psi}_2 \circ F^{-1} \circ \phi_S^{-1})(x,y) &= (\overline{\psi}_2 \circ \psi_1^{-1} \circ \phi_N \circ \phi_S^{-1})(x,y) \\
&= (\overline{\psi}_2 \circ \psi_1^{-1}) \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right) \\
&= \overline{\psi} \left(\left[1, \frac{x+iy}{x^2+y^2} \right] \right) \\
&= \left(\frac{x}{\frac{x^2+y^2}{(x^2+y^2)^2}}, \frac{y}{\frac{x^2+y^2}{(x^2+y^2)^2}} \right) \\
&= \big(x(x^2+y^2), y(x^2+y^2) \big)
\end{aligned}$$

all of which are smooth if $(x,y) \neq (0,0) \in \mathbb{R}^2$, yet we also have

$$\begin{aligned}
(\psi_1 \circ F^{-1} \circ \phi_N^{-1})(0,0) &= (\psi_1 \circ \psi_1^{-1} \circ \phi_N \circ \phi_N^{-1})(0,0) \\
&= (0,0) \\
(\psi_1 \circ F^{-1} \circ \phi_S^{-1})(0,0) &= (\psi_1 \circ F^{-1})(0,0,1) \\
&= \psi_1([x+iy:0]) \\
&= (0,0) \\
(\overline{\psi}_2 \circ F^{-1} \circ \phi_N^{-1})(0,0) &= (\overline{\psi}_2 \circ \psi_1^{-1} \circ \phi_N \circ \phi_N^{-1})(0,0) \\
&= (\overline{\psi}_2 \circ \psi_1^{-1})(0,0) \\
&= \overline{\psi}_2([1:0]) \\
&= (0,0) \\
(\overline{\psi}_2 \circ F^{-1} \circ \phi_S^{-1})(0,0) &= (\overline{\psi}_2 \circ F^{-1})(0,0,1) \\
&= \overline{\psi}_2([x+iy:0]) \\
&= (0,0)
\end{aligned}$$

which are all smooth, and so for all $(x,y) \in \mathbb{R}^2$, the coordinate representation of F^{-1} is smooth as a map between euclidean spaces, and therefore F^{-1} must also be smooth.

And thus, we have a bi-smooth bijection, and so F is a diffeomorphism between $\mathbb{S}^2$ and $\mathbb{CP}^1$, and therefore they are diffeomorphic.

$\square$

4-6 *Let M be a nonempty smooth compact manifold. Show that there is no smooth submersion*

$$F : M \to \mathbb{R}^k$$

for any $k > 0$.

Proof. Suppose, for contradiction, that $M \neq \varnothing$, and is compact, and that

$$F : M \to \mathbb{R}^k$$

is a smooth submersion. Then, by the Properties of Smooth Submersions, we have that F is an open map, and since $M \subseteq M$ is open in M tautologically, we have that

$$F(M) \subseteq \mathbb{R}^k$$

must be open.

Next, since M is a manifold, it is Hausdorff, and since M is compact, we have a compact set in a Hausdorff space, and so M is closed. Further, since F is smooth, it is continuous, and the continuous image of a compact set is compact and so

$$F(M) \subseteq \mathbb{R}^k$$

is compact. Since $\mathbb{R}^k$ is Hausdorff, again we have a compact set in a Hausdorff space, and so $F(M)$, must also be closed.

Now, since $\mathbb{R}^k$ is connected the only sets in $\mathbb{R}^k$ which are both open and closed are $\varnothing$ and $\mathbb{R}^k$. And

$$M \neq \varnothing \implies F(M) \neq F(\varnothing) = \varnothing$$

and so we must have

$$F(M) = \mathbb{R}^k$$

Yet, this then implies that $F(M) = \mathbb{R}^k$ is compact $\Rightarrow\Leftarrow$.

And therefore we can conclude that there does not exists a function

$$F : M \to \mathbb{R}^k$$

where M is smooth, compact, and non-empty, which is a smooth submersion for any $k > 0$. $\square$

4-7 *Suppose M and N are smooth manifolds, and*

$$\pi : M \to N$$

is a surjective smooth submersion. Show that there is no other smooth manifold structure on N that satisfies the conclusion of Theorem 4.29; in other words, assuming that N' represents the same set as N with a possibly different topology and smooth structure, and that for every smooth manifold P with or without boundary, a map

$$F : N' \to P$$

is smooth if and only if $F \circ \pi$ is smooth, show that Id_N is a diffeomorphism between N and N'.

Proof. Suppose that N' is the same set as N, but with a different topology and smooth structure, and that N' also satisfies the Characteristic Property of Surjective Smooth Submersions, and that

$$\pi : M \to N$$

is a surjective smooth submersions, and so satisfies the condition that for every smooth manifold P, the maps

$$F : N \to P$$
$$F' : N' \to P$$

are smooth iff

$$F \circ \pi : M \to P$$
$$F' \circ \pi : M \to P$$

are smooth respectively. Next we note that

$$Id_{N'} : N' \to N'$$

is always smooth, so by the Characteristic Property of Surjective Smooth Submersions, we have

$$\pi' = Id_{N'} \circ \pi : M \to N'$$

is smooth. Then considering the map

$$Id_N : N \to N'$$

since

$$Id_{N'} \circ \pi = \pi' : M \to N'$$

is smooth, from above, where the Characteristic Property then tells us that Id_N must also be smooth. Then, considering

$$Id_N^{-1} : N' \to N$$

we have that

$$Id_N^{-1} \circ \pi' = \pi : M \to N$$

is smooth, and so by the Characteristic Property Id_N^{-1}, must also be smooth.

Then as Id_N^{-1}, Id_N, are both smooth, where

$$Id_N^{-1} \circ Id_N = Id_N, \quad \text{and} \quad Id_N \circ Id_N^{-1} = Id_{N'}$$

we have a bi-smooth bijection, and so Id_N is a diffeomorphism between N and N', and so the spaces are diffeomorphic. $\square$

4-8 *This problem shows that the converse of Theorem 4.29 is false. Let*

$$\pi : \mathbb{R}^2 \to \mathbb{R}, \text{ by } \pi(x, y) = xy$$

Show that π is surjective and smooth, and for each smooth manifold P, a map

$$F : \mathbb{R} \to P$$

is smooth if and only if $F \circ \pi$ is smooth; but π is not a smooth submersion.

Proof. First, let $x \in \mathbb{R}$ be arbitrary, then

$$\pi(x, 1) = x \cdot 1 = x$$

and so π is surjective. Furthermore, since π is a polynomial, it is smooth.

Next suppose that P is any smooth manifold, and that

$$F : \mathbb{R} \to P$$

is smooth. Since the composition of smooth maps is smooth, we then have that

$$F \circ \pi : \mathbb{R}^2 \to P$$

must also be smooth.

Next suppose that

$$F \circ \pi : \mathbb{R}^2 \to P$$

is smooth, and consider the inclusion

$$\iota : \mathbb{R} \to \mathbb{R}^2, \text{ by } \iota(x) = (x, 1)$$

which is the restriction of the identity, and therefore diffeomorphic onto its image, and hence, smooth. Then since the composition of smooth maps is smooth we have

$$(F \circ \pi) \circ \iota : \mathbb{R} \to P$$

must be smooth. Yet, for each $x \in \mathbb{R}$ we have

$$(F \circ \pi) \circ \iota(x) = F \circ \pi(x, 1) = F(x)$$

and therefore F must also be smooth.

Thus, we have that π is surjective, smooth, and that for any smooth manifold P, a map

$$F : \mathbb{R} \to P$$

is smooth iff $F \circ \pi$ is smooth.

Yet, the differential

$$d\pi|_{(x,y)} : T_{(x,y)}\mathbb{R}^2 \to T_{xy}\mathbb{R}$$

is not surjective as

$$d\pi|_{(x,y)} = \frac{\partial \pi(x, y)}{\partial(x, y)} = \begin{bmatrix} y & x \end{bmatrix}$$

does not have full rank at $(0, 0) \in \mathbb{R}$, and thus π cannot be a smooth submersion. $\square$

5-4 *Show that the image of the curve*

$$\beta : (-\pi, \pi) \to \mathbb{R}^2, \textit{ by } \beta(t) = (\sin 2t, \sin t)$$

is not an embedded submanifold of $\mathbb{R}^2$. *[Be careful: this is not the same as showing that* β *is not an embedding.]*

Proof. Suppose, for contradiction, that $\beta(-\pi, \pi) \subseteq \mathbb{R}^2$ is an embedded submanifold. Then we note that $\beta(-\pi, \pi)$ must satisfies the local slice condition, then as

$$\beta(0) = (\sin 20, \sin 0) = (0, 0)$$

in particular we have that there exists a slice chart $(U, \phi) \in \mathcal{A}_{\mathbb{R}^2}$ containing $(0, 0)$ such that $\beta(-\pi, \pi) \cap U$ is open in the relative topology of $\beta(-\pi, \pi)$, yet $\beta(-\pi, \pi) \cap U \setminus \{(0, 0)\}$ has four connected components whereas

$$B_r(0) \subseteq \mathbb{R} \implies B_r(0) \setminus \{0\} \text{ has 2 connected components}$$
$$B_r(0,0) \subseteq \mathbb{R}^2 \implies B_r(0,0) \setminus \{(0,0)\} \text{ has 1 connected components}$$

and so $\beta(-\pi, \pi) \cap U$ is not locally euclidean of dimension 1 or 2, and therefore is not a topological manifold in the relative topology, and therefore cannot be an embedded submanifold. □

5-6 *Suppose $M \subseteq \mathbb{R}^n$ is an embedded m-dimensional submanifold, and let $UM \subseteq T\mathbb{R}^n$ be the set of all unit tangent vectors to M:*

$$UM = \{(\mathbf{x}, \mathbf{v}) \in T\mathbb{R}^n : \mathbf{x} \in M,\ \mathbf{v} \in T_{\mathbf{x}}M,\ ||\mathbf{v}|| = 1\}$$

It is called the unit tangent bundle of M. Prove that UM is an embedded $(2m-1)$-dimensional submanifold of $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$.

Proof. Let $M \subseteq \mathbb{R}^n$ be an embedded submanifold of dimension m. Then we have

$$TM = \bigsqcup_{\mathbf{x} \in M} T_{\mathbf{x}}M \subseteq T\mathbb{R}^n$$

is an embedded submanifold of dimension $2m$. To see this, we note that since M is an embedded submanifold it satisfies the local m-slice condition, so for each $\mathbf{x} \in M$, $\exists\ (U, \phi) \in \mathcal{A}_{\mathbb{R}^n}$ centered at $\mathbf{x}$, such that

$$M \cap U = \{\mathbf{y} \in U : x^{m+1}(\mathbf{y}) = 0, \ldots, x^n(\mathbf{y}) = 0\}$$

and so with the associated chart $(\pi^{-1}(U), \widetilde{\phi}) \in \mathcal{A}_{T\mathbb{R}^n}$ we have

$$TM \cap \pi^{-1}(U) = \big\{(\mathbf{y}, \mathbf{v}) \in \pi^{-1}(U) : x^{m+1}(\mathbf{y}) = 0, \ldots, x^n(\mathbf{y}) = 0, v^{m+1} = 0, \cdots, v^n = 0\big\}$$

and so TM satisfies the local $2m$-slice condition. And we may also make the identification

$$TM = \{(\mathbf{x}, \mathbf{v}) \in T\mathbb{R}^n : \mathbf{x} \in M,\ \mathbf{v} \in T_{\mathbf{x}}M\} = \{(\mathbf{x}, \mathbf{v}) \in \mathbb{R}^n_1 \times \mathbb{R}^n_2 : \mathbf{x} \in M,\ \mathbf{v} \in T_{\mathbf{x}}M\}$$

then with projection mapping

$$\pi_2 : \mathbb{R}^n_1 \times \mathbb{R}^n_2 \to \mathbb{R}^n_2, \text{ by } \pi_2(\mathbf{x}, \mathbf{v}) = \mathbf{v}$$

which is continuous, and the restriction of a continuous map is continuous and so $\pi_2|_{TM}$ will also be continuous. Further, we also have the norm mapping

$$||\cdot|| : \mathbb{R}^n \to \mathbb{R}, \text{ by } ||\mathbf{x}|| = \sqrt{x_1^2 + \cdots + x_n^2}$$

is continuous, since $\forall\ \epsilon > 0$ we may choose $\delta = \epsilon$, and then for all $\mathbf{x}, \mathbf{y}$ such that $||\mathbf{x} - \mathbf{y}|| < \delta$ we have by the Reverse Triangle Inequality

$$\big|||\mathbf{x}|| - ||\mathbf{y}||\big| \le ||\mathbf{x} - \mathbf{y}|| < \delta = \epsilon$$

then as the composition of continuous maps is continuous we have

$$||\cdot|| \circ \pi_2|_{TM} : TM \to \mathbb{R}$$

is continuous. Further, since the singleton $\{\mathbf{0}\} \in \mathbb{R}^n$ is closed we have $\mathbb{R}^n \setminus \{\mathbf{0}\}$ is open, and since the projection mapping is smooth and the norm mapping is smooth on $\mathbb{R}^n \setminus \{\mathbf{0}\}$ we can define the subset

$$S = \{(\mathbf{x}, \mathbf{v}) \in \mathbb{R}^n_1 \times \mathbb{R}^n_2 : \mathbf{x} \in M,\ \mathbf{v} \in T_{\mathbf{x}}M,\ ||\mathbf{v}|| \neq 0\} \subseteq TM$$

and since this is an open subset of TM is is also a submanifold of dimension $2m$. Then since the composition of smooth maps is smooth we will have

$$||\cdot|| \circ \pi_2|_S : S \to \mathbb{R}$$

is smooth.

Next, let us denote $F = ||\cdot|| \circ \pi_2|_S$, and consider

$$F^{-1}(1) = \{(\mathbf{x}, \mathbf{v}) \in T\mathbb{R}^n : \mathbf{x} \in M,\ \mathbf{v} \in T_{\mathbf{x}}M,\ ||\mathbf{v}|| = 1\} = UM \subseteq TM$$

then for any $(\mathbf{x}, \mathbf{v}) \in F^{-1}(1)$ we have

$$\begin{aligned} dF|_{(\mathbf{x},\mathbf{v})} &= \frac{\partial F(x_1, \ldots, x_m, v_1, \ldots, v_m)}{\partial(x_1, \ldots, x_m, v_1, \ldots, v_m)} \\ &= \begin{bmatrix} 0 & \cdots & 0 & \frac{v_1}{||\mathbf{v}||} & \cdots & \frac{v_m}{||\mathbf{v}||} \end{bmatrix} \\ &= \begin{bmatrix} 0 & \cdots & 0 & v_1 & \cdots & v_m \end{bmatrix} && ||\mathbf{v}|| = 1 \end{aligned}$$

and since $||\mathbf{v}|| = 1$ not all the components $v_i = 0$, and thus, $dF|_{(\mathbf{x},\mathbf{v})}$ has full rank.

Since $(\mathbf{x}, \mathbf{v}) \in F^{-1}(1)$ was arbitrary we conclude that each point in the preimage of 1 under F has full rank, and since S and $\mathbb{R}$ are smooth manifolds and F a smooth map we can conclude that 1 is a regular value of F. And hence $F^{-1}(1)$ is a regular level set of S and hence of TM. The Regular Level Set Theorem then says that $F^{-1}(1) = UM \subseteq TM$ is a properly embedded submanifold of codimension equal to the dimension of its codomain; that is of codimension $\dim(\mathbb{R}) = 1$. And therefore

$$1 = \dim(TM) - \dim(UM) \implies \dim(UM) = \dim(TM) - 1 = 2m - 1$$

and so UM is an embedded submanifold of dimension $(2m - 1)$. $\square$

5-7 *Let*

$$F : \mathbb{R}^2 \to \mathbb{R}, \text{ by } F(x,y) = x^3 + xy + y^3$$

Which level sets of F are embedded submanifolds of $\mathbb{R}^2$? For each level set, prove either that it is or that it is not an embedded submanifold.

Proof. First, noting that both $\mathbb{R}^2$ and $\mathbb{R}$ are smooth manifolds, and F is smooth since it is a polynomial. And so the differential

$$dF|_{(x,y)} : T_{(x,y)}\mathbb{R}^2 \to T_{F(x,y)}\mathbb{R}$$

is given by the Jacobian matrix of F at (x,y), that is

$$dF|_{(x,y)} = \frac{\partial F(x,y)}{\partial (x,y)} = \begin{bmatrix} 3x^2 + y & x + 3y^2 \end{bmatrix}$$

which has full rank unless $(x,y) = (0,0)$ in which case

$$dF|_{(0,0)} = \begin{bmatrix} 3(0)^2 + 0 & 0 + 3(0)^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

to see if there are any other possibilities we compute

$$3x^2 + y = 0 \implies y = -3x^2 \tag{1}$$

$$x + 3y^2 = 0 \tag{2}$$

plugging (1) into (2) gives

$$x + 3(-3x^2)^2 = x + 27x^4 = x(1 + 27x^3) = 0$$

the $x = 0$ case was handled above so we are left with

$$x^3 = -\frac{1}{27} \implies x = -\frac{1}{3}$$

plugging back into (1), then gives

$$y = -3\left(-\frac{1}{3}\right)^2 = -\frac{1}{3}$$

therefore dF has full rank except at the points $\{(0,0), (-\frac{1}{3}, -\frac{1}{3})\}$, and since $\dim(\mathbb{R}^2) > \dim(\mathbb{R})$, F having full rank means that F is surjective. Thus, we have, for any $(x,y) \in \mathbb{R}^2 \setminus \{(0,0), (-\frac{1}{3}, -\frac{1}{3})\}$ we have that $dF|_{(x,y)}$ is surjective and therefore (x,y) is a regular point, and since

$$F(0,0) = 0$$

$$F(-\tfrac{1}{3}, -\tfrac{1}{3}) = \left(-\frac{1}{3}\right)^3 + \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right) + \left(-\frac{1}{3}\right)^3 = -\frac{2}{27} + \frac{1}{9} = \frac{1}{27}$$

are the only critical points of dF. We may conclude that for each $x \in \mathbb{R} \setminus \{0, \frac{1}{27}\}$, that x is a regular value of F, and therefore

$$F^{-1}(x) \subseteq \mathbb{R}^2$$

is a regular level set, and so, by the Regular Level Set Theorem $F^{-1}(x) \subseteq \mathbb{R}^2$ is a properly embedded submanifold of codimension 1. And so is a properly embedded submanifold of dimension

$$1 = \dim(\mathbb{R}^2) - \dim(F^{-1}(x)) \implies \dim(F^{-1}(x)) = \dim(\mathbb{R}^2) - 1 = 2 - 1 = 1$$

Case 1: $F^{-1}\left(\frac{1}{27}\right)$, Noting that since $F(-\frac{1}{3},-\frac{1}{3})=\frac{1}{27}$ we have

$$\begin{aligned}
0 &= x^3+xy+y^3-\frac{1}{27}\\
&= x^3+xy+y^3-\frac{1}{27}-x^2+x^2-\frac{x}{3}+\frac{x}{3}\\
&= \left(x^3-x^2+\frac{x}{3}-\frac{1}{27}\right)+xy+y^3-\frac{x}{3}+x^2\\
&= \left(x-\tfrac{1}{3}\right)^3+y^3+xy-\frac{x}{3}+x^2\\
&= \left(x-\tfrac{1}{3}+y\right)\left(\left(x-\tfrac{1}{3}\right)^2-y\left(x-\tfrac{1}{3}\right)+y^2\right)+x\left(y-\tfrac{1}{3}+x\right)\\
&= \left(x+y-\tfrac{1}{3}\right)\left(\left(x-\tfrac{1}{3}\right)^2-y\left(x-\tfrac{1}{3}\right)+y^2+x\right)\\
&= \left(x+y-\tfrac{1}{3}\right)\left(x^2-\tfrac{2}{3}x+\tfrac{1}{9}-yx+\tfrac{y}{3}+y^2+x\right)\\
&= \left(x+y-\tfrac{1}{3}\right)\left(x^2+\tfrac{x}{3}+\tfrac{1}{9}-yx+\tfrac{y}{3}+y^2\right)\\
&= \left(x+y-\tfrac{1}{3}\right)\left(x^2+x\left(\tfrac{1}{3}-y\right)+\tfrac{1}{9}+\tfrac{y}{3}+y^2\right)
\end{aligned}$$

where in the second factor, using the quadratic formula we get

$$\begin{aligned}
x &= \frac{y-\frac{1}{3}\pm\sqrt{y^2-\frac{2}{3}y+\frac{1}{9}-4\cdot 1\cdot\left(\frac{1}{9}+\frac{y}{3}+y^2\right)}}{2}\\
&= \frac{y-\frac{1}{3}\pm\sqrt{-3y^2-2y-\frac{1}{3}}}{2}\\
&= \frac{y-\frac{1}{3}\pm\sqrt{-3\left(y^2+\frac{2}{3}y+\frac{1}{9}\right)}}{2}\\
&= \frac{y-\frac{1}{3}\pm\sqrt{-3\left(y+\frac{1}{3}\right)^2}}{2}
\end{aligned}$$

which only have a real solution when

$$y=-\frac{1}{3}\implies x=\frac{-\frac{1}{3}-\frac{1}{3}\pm\sqrt{-3\left(0\right)^2}}{2}=-\tfrac{1}{3}$$

and so

$$\begin{gathered}
y+x=\frac{1}{3}\\
\left(-\frac{1}{3},-\frac{1}{3}\right)
\end{gathered}$$

is the solution set to $F(x,y)=\frac{1}{27}$, so that

$$F^{-1}\left(\tfrac{1}{27}\right)=\left\{x+y=\tfrac{1}{3}\right\}\sqcup\left\{\left(-\tfrac{1}{3},-\tfrac{1}{3}\right)\right\}$$

which is the disjoint union of a line and a point, and so has connected components locally euclidean to spaces of differing dimension, and so therefore cannot be a manifold, and so, is not an embedded submanifold.

Case 2: $F^{-1}(0)$. Graphing the solution, we see that it makes and loop forming a cross at the origin. So, supposing for contradiction, that $F^{-1}(0) \subseteq \mathbb{R}^2$ is an embedded submanifold, then it satisfies the local slice condition, and so there exists a slice chart $(U, \phi) \in \mathcal{A}_{\mathbb{R}^2}$ containing $(0,0)$ such that $F^{-1}(0) \cap U$ is open in the relative topology of $F^{-1}(0)$. Yet $F^{-1}(0) \cap U \setminus \{(0,0)\}$ has three connected components whereas

$$B_r(0) \subseteq \mathbb{R} \implies B_r(0) \setminus \{0\} \text{ has 2 connected components}$$
$$B_r(0,0) \subseteq \mathbb{R}^2 \implies B_r(0,0) \setminus \{(0,0)\} \text{ has 1 connected components}$$

and so $F^{-1}(0) \cap U$ is not locally euclidean of dimension 1 or 2, and therefore is not topological manifold in the relative topology, and therefore cannot be an embedded submanifold.

□

5-22 *If M is a smooth manifold and $D \subseteq M$ is a regular domain, then there exists a defining function f for D. If D is compact, then f can be taken to be a smooth exhaustion function for M.*

Proof. Let $n = \dim(M)$ and suppose that $D \subseteq M$ is a regular domain in M. So that D is a properly embedded codimension-0 submanifold with boundary. Let $\big\{\big(U_i, (x^1, \ldots, x^n)_i\big)\big\}_{i \in I}$ be a countable collection of smooth charts such that

$$\bigcup_{i \in I} U_i \supseteq M$$

and for each i define the smooth functions

$$f_i : U_i \to \mathbb{R}, \text{ by } f_i(p) = \begin{cases} -1, & U_i \text{ Interior slice chart for } D \text{ in } M \\ 1, & U_i \subseteq M \setminus D \\ x_i^n(p), & U_i \text{ Boundary slice chart for } D \text{ in } M \end{cases}$$

then since D has codimension-0, an interior chart is just a subset of the coordinate chart; i.e. $(D \cap U_i, \phi_i|_{D \cap U_i})$, with no further restrictions, while a boundary slice chart is the 0-dimensional half-slice

$$D \cap U_i = \{q \in U_i : x_i^n(q) \geq 0\}$$

and therefore for $p \in \text{Int}(D)$ we have $f_i(p) < 0$, and for $p \in \partial D$ we have $f_i(p) = x_i^n(p) = 0$. Since $\{U_i\}_{i \in I}$ is an indexed cover of M, let $\{\psi_i\}_{i \in I}$ be a smooth partition of unity subordinate to this cover, and so

(i) $0 \leq \psi_i(p) \leq 1 \quad \forall\, i \in I; \forall\, p \in M$

(ii) $\text{supp}(\psi_i) \subseteq U_i$ for each $i \in I$

(iii) The family of supports $\{\text{supp}(\psi_i)\}_{i \in I}$ is locally finite; i.e. $\forall\, p \in M$ $\exists$ a neighborhood U_p such that
$$U_p \cap \text{supp}(\psi_i) \neq \varnothing$$
for finitely many $i \in I$

(iv) $\sum_{i \in I} \psi_i(p) = 1 \quad \forall\, p \in M$.

then for each i

$$\psi_i f_i : U_i \to \mathbb{R}$$

is smooth and has a smooth extention to M by the gluing lemma for smooth maps since

$$\psi_i f_i\big(U_i \setminus \text{supp}(\psi_i)\big) = 0 = \psi_i f_i\big(M \setminus \text{supp}(\psi_i)\big)$$

i.e. the functions agree on their overlap. And so we may define

$$f : M \to \mathbb{R}, \text{ by } f = \sum_{i \in I} \psi_i f_i$$

then f is smooth, and for $p \in \partial M$

$$f(p) = \sum_{i \in I} \psi_i(p) f_i(p) = \sum_{i \in I} \psi_i(p) x_i^n(p) = 0, \text{ identically}$$

and for $p \in \mathrm{Int}(M)$,

$$f(p) = \sum_{i\in I} \psi_i(p) f_i(p) = \sum_{i\in I} \psi_i(p) \cdot (-1) = -\sum_{i\in I} \psi_i(p) < 0$$

while for $p \in M \setminus D = D^c$ we get

$$f(p) = \sum_{i\in I} \psi_i(p) f_i(p) = \sum_{i\in I} \psi_i(p) \cdot (1) = \sum_{i\in I} \psi_i(p) > 0$$

Now let $p \in \partial D$, and let $v_p \in \{T_p D : x^n > 0\}$ so that v_p is inward pointing. Then for each i, we have

$$df_i|_p(v_p) = dx_i^n|_p(v_p) > 0 \quad \text{Proposition 5.41}$$

and therefore for $Id_{\mathbb{R}} \in C^\infty(\mathbb{R})$ we have

$$\begin{aligned}
df|_p(v_p)(Id) &= v(Id \circ f)|_p \\
&= v(f)|_p \\
&= v\left(\sum_{i\in I} \psi_i f_i\right)\Bigg|_p \\
&= \sum_{i\in I} v(\psi_i f_i)|_p && \text{linearity} \\
&= \sum_{i\in I} (f_i(p) v(\psi_i)|_p + \psi(p) v(f_i)|_p) \\
&= \sum_{i\in I} (0 \cdot v(\psi_i)|_p + \psi(p) v(f_i)|_p) \\
&= \sum_{i\in I} \psi(p) v(f_i)|_p \\
&= \sum_{i\in I} \psi(p) v(Id \circ f_i)|_p \\
&= \sum_{i\in I} \psi(p) df_i|_p(v_p)(Id) \\
&= \left(\sum_{i\in I} \psi(p) df_i|_p(v_p)\right)(Id)
\end{aligned}$$

and so

$$df|_p(v_p) = \sum_{i\in I} \psi(p) df_i|_p(v_p) = \sum_{i\in I} \psi(p) dx_i^n|_p(v_p) > 0$$

thus we have $df|_p \neq 0$, that is

$$df|_p = \begin{bmatrix} \frac{\partial \widehat{f}}{\partial x^1}(\phi(p)) & \dots & \frac{\partial \widehat{f}}{\partial x^n}(\phi(p)) \end{bmatrix} \neq \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix}$$

and so has full rank, and since $\dim(M) \geq \dim(\mathbb{R})$ we have that f is surjective, and therefore p is a regular point of f.

Since $p \in \partial D$ was arbitrary we conclude that each $p \in \partial D$ is a regular point, and thus $f(p) = 0$, is a regular value of f, and by construction the only positive values of f are for $p \in D^c$ and therefore we have

$$f^{-1}\big((-\infty, 0]\big) = D$$

and so f is a defining function for D.

If $D \subseteq M$ is compact, then for the cover $\big\{\big(U_i, (x^1, \dots, x^n)_i\big)\big\}_{i \in I}$; Here, since M is a manifold it admits a basis of precompact coordinate balls, so let us suppose this cover is precompact. And since it is a cover for M it will also cover D, and by the compactness of D we have a finite subcover, say

$$\bigcup_{i=1}^{k} U_i \supseteq D$$

while

$$\bigcup_{i=k+1}^{\infty} U_i \supseteq D^c$$

Then, following the proof of Proposition 2.28, we take a partition of unity $\{\psi_i\}_{i=1}^{\infty}$ subordinate to the cover $\{U_i\}_{i=1}^{k} \cup \{U_i\}_{i=k+1}^{\infty}$ and define

$$f : M \to \mathbb{R}, \text{ by } f(p) = \sum_{i=1}^{k} f_i(p)\psi_i(p) + \sum_{i=k+1}^{\infty} i f_i(p)\psi_i(p)$$

then f is smooth since $\{\operatorname{supp}(\psi_i)\}$ is a locally finite collection of subsets, and so only finitely many of the terms in the sum are non-zero in the neighborhood of any point of M. Then since $\sum_{i \in I} \psi_i(p) = 1 \quad \forall\, p \in M$, then for any $p \in \operatorname{Int}(D)$ we have

$$f(p) = \sum_{i=1}^{k} f_i(p)\psi_i(p) + \sum_{i=k+1}^{\infty} i f_i(p)\psi_i(p) = -\sum_{i=1}^{k} \psi_i(p) < 0$$

and again for $p \in \partial D$ we have

$$f(p) = \sum_{i=1}^{k} f_i(p)\psi_i(p) + \sum_{i=k+1}^{\infty} i f_i(p)\psi_i(p) = \sum_{i=1}^{k} x_i^n(p)\psi_i(p) = 0$$

and since $\sum_{i=k+1}^{\infty} i f_i(p)\psi_i(p)$ is strictly positive, therefore we have

$$f^{-1}\big((-\infty, 0]\big) = D$$

and so f is a defining function for D, where from above, 0 is a regular value of f.

Then as D is compact in a Hausdorff space, it is closed and note that since the U_i's are precompact each $\overline{U}_i$ is compact in M, so we have

$$D = \overline{D} \subseteq \overline{\bigcup_{i=1}^{k} U_i} = \bigcup_{i=1}^{k} \overline{U}_i$$

and $\bigcup_{i=1}^{k} \overline{U}_i$ is compact as the finite union of compact sets. Then Since f is smooth, it is continuous, and the continuous image of a compact set is compact and so

$$f\left(\bigcup_{i=1}^{k} \overline{U}_i\right) = \bigcup_{i=1}^{k} f(\overline{U}_i) \subseteq \mathbb{R}$$

is also compact, and so by Heine Borel is bounded. Thus, there exists some $c \in \mathbb{R}$ such that

$$\bigcup_{i=1}^{k} f(\overline{U}_i) \subseteq (-\infty, c] \implies \bigcup_{i=1}^{k} \overline{U}_i \subseteq f^{-1}\big((-\infty, c]\big)$$

To see that f is an exhaustion function, let us now focus on the U_i's covering D^c. Then, given $a \in \mathbb{R}$ we can choose $N \in \mathbb{N}$ such that $N > a$. Then if

$$p \notin \bigcup_{i=k+1}^{N} \overline{U}_i \implies \psi_i(p) = 0 \text{ for } k+1 \leq i \leq N, \quad \text{since } \operatorname{supp}(\psi_i) \subseteq U_i$$

And so

$$f(p) = \sum_{i=N+1}^{\infty} i\psi_i(p) \geq \sum_{i=N+1}^{\infty} N\psi_i(p) = N\sum_{i=1}^{\infty} \psi_i(p) = N > c$$

and so $p \notin f^{-1}\big((-\infty, a]\big)$. That is,

$$\text{if } p \notin \bigcup_{i=k+1}^{N} \overline{U}_i \text{ then } p \notin f^{-1}\big((-\infty, a]\big)$$

taking the contrapositive gives

$$\text{if } p \in f^{-1}\big((-\infty, a]\big) \text{ then } p \in \bigcup_{i=k+1}^{N} \overline{U}_i$$

or $f^{-1}\big((-\infty, a]\big) \subseteq \bigcup_{i=k+1}^{N} \overline{U}_i$, which is compact as a finite union of compact sets, were the continuity of f tells us that $f^{-1}\big((-\infty, c]\big) \subseteq M$ is closed, and therefore must be compact as a closed subset of a compact set.

Finally we note, that since f is continuous, and strictly positive on D^c, we get $f^{-1}\big((-\infty, c]\big) \subseteq f^{-1}\big((-\infty, a]\big)$, and so is compact. $\square$