Topology and Measure Theory Notes

Alexander Richardson

1 Metric Spaces

In topology, a major goal is to find those traits which are invariant. So we begin with a special kind of topology, a Metric Space, which uses the idea of distance to define our topology. In a metric space, and in particular a mapping between two metric spaces, this naturally leads to an interest in distance preserving maps. So first we need a few definitions.

Definition 1. Let X be a set, a **Metric** on X is map d with

$$\begin{aligned} d : X \times X &\to \mathbb{R} \\ (x, y) &\mapsto d(x, y) \end{aligned}$$

such that

1. $d(x, y) \geq 0 \ \forall \ x, y \in X$ and $d(x, y) = 0 \iff x = y$.
2. $\forall \ x, y \in X$ we have $d(x, y) = d(y, x)$.
3. $\forall \ x, y, z \in X$ we have

$$d(x, z) \leq d(x, y) + d(y, z)$$

a set with a metric is a metric space, denoted (X, d). Additionally If $U \subseteq X$ such that $U \neq \varnothing$ then we can define $(U, d|_{U \times U})$ as a metric subspace. We also recall a definition familiar from linear algebra

Definition 2. A **Normed Vector Space** V over $\mathbb{R}$ is a vector space with a mapping

$$\begin{aligned} V &\to \mathbb{R} \\ v &\mapsto ||v|| \end{aligned}$$

such that

1. $||v|| \geq 0$ and $||v|| = 0 \iff v = 0$.
2. If $c \in \mathbb{R}$ and $v \in V$, then $||cv|| = |c| \cdot ||v||$.
3. If $v, u \in V$, then

$$||v + u|| \leq ||v|| + ||u||$$

A normed vector space is similarly denoted $(V, ||\cdot||)$. The interesting fact here is that for a normed vector space $(V, ||\cdot||)$, the norm $||\cdot||$ induces a metric

$$d(v,u) := ||v-u||$$

If we are interested in comparing the distance between subsets, as opposed to individual elements, we have the following description: for $A, B \subseteq V$ then

$$d(A,B) = \inf ||a-b||, \text{ such that } a \in A, b \in B$$

We also have a slight variation on our notion of a metric space

Definition 3. Let X be a set, a **Semi-Metric** on X is map d with

$$\begin{aligned} d : X \times X &\to \mathbb{R} \\ (x,y) &\mapsto d(x,y) \end{aligned}$$

such that

1. $d(x,y) \geq 0 \; \forall \; x,y \in X$ and $d(x,x) = 0$.
2. $\forall \; x,y \in X$ we have $d(x,y) = d(y,x)$.
3. $\forall \; x,y,z \in X$ we have

$$d(x,z) \leq d(x,y) + d(y,z)$$

The distinction here being $d(x,y) = 0 \not\Rightarrow x = y$. And with similar notation a set with a semi-metric is a semi-metric space (X,d). Now we can define those maps between metric spaces which preserve distance.

Definition 4. For metric spaces (X, d_X) and (Y, d_Y) a map

$$f : X \to Y$$

is **Isometric** if

$$d_X(v,w) = d_Y\big(f(v), f(w)\big) \quad \forall \; v,w \in X$$

if in addition f is surjective, then f is an **Isometric Isomorphism**.

These isometric mapping have a few very important properties that are essentially inherited just from the definition.

Proposition 5. Isometries are injective and uniformly continuous.

Proof. Let

$$f : (X, d_X) \to (Y, d_Y)$$

be an isometric map between metric spaces and let $\epsilon > 0$ be given. Select $\delta = \epsilon > 0$, then for any $x, y \in X$ such that $d_X(x,y) < \delta$ gives

$$d_Y\big(f(x), f(y)\big) = d_X(x,y) < \delta = \epsilon$$

and therefore f is uniformly continuous.

Next, take $a, b \in X$ such that $f(a) = f(b)$, then

$$d_X(a,b) = d_Y\big(f(a), f(b)\big) = 0 \implies a = b$$

and so f is injective. $\square$

Proposition 6. If

$$f : (X, d_X) \to (Y, d_Y)$$

is an isometry, then

$$f^{-1} : \big(f(X), d_Y\big) \to (X, d_X)$$

is an isometry.

Proof. Let f be an isometry and let $x, y \in f(X)$, then $\exists\ a, b \in X$ such that

$$f(a) = x \text{ and } f(b) = y \implies a = f^{-1}(x) \text{ and } b = f^{-1}(y)$$

then

$$\begin{aligned} d_Y(x,y) &= d_Y\big(f(a), f(b)\big) \\ &= d_X(a,b) && f \text{ is an isometry} \\ &= d_X\big(f^{-1}(x), f^{-1}(y)\big) \end{aligned}$$

and hence, f^{-1} is an isometry. $\square$

Not surprisingly, metric spaces come with several different notions of continuity, many inherited from analysis. Which we recall here.

Definition 7.

$$f : (X, d_X) \to (Y, d_Y)$$

is **Continuous** at $x_0 \in X$ if $\forall\ \epsilon > 0,\ \exists\ \delta(x_0, \epsilon) > 0$, depending on both x_0 and ϵ, such that

$$d_X(x, x_0) < \delta(x_0, \epsilon) \implies d_Y\big(f(x), f(x_0)\big) < \epsilon$$

And f is continuous on X if it is continuous at x_0 for all $x_0 \in X$.

Definition 8.

$$f : (X, d_X) \to (Y, d_Y)$$

is **Uniformly Continuous** if $\forall\ \epsilon > 0,\ \exists\ \delta(\epsilon) > 0$, depending only on ϵ, such that

$$d_X(x_1, x_2) < \delta(\epsilon) \implies d_Y\big(f(x_1), f(x_2)\big) < \epsilon$$

The next, possibly familiar notion, gives us an idea of continuity when our mapping is bounded.

Definition 9. let (X, d) be a metric space. A subset $A \subseteq X$ is bounded if $\exists\ C \in \mathbb{R}^+$ such that

$$d(x, y) \leq C \quad \forall\ x, y \in A$$

if X is a set and (Y, d) a metric space, then

$$f : X \to Y$$

is bounded if $f(X) \subseteq Y$ is bounded.

Definition 10. A function

$$f : (X, d_X) \to (Y, d_Y)$$

is **Lipschitz** if $\exists\ C \geq 0$ with $C \in \mathbb{R}$, such that

$$d_Y\big(f(x), f(y)\big) \leq C d_X(x, y) \quad \forall\ x, y \in X$$

the smallest such

$$C := L(f)$$

is the Lipschitz constant.

Proposition 11. If (M_1, d_1) and (M_2, d_2) are metric spaces, then Lipschitz continuous implies uniformly continuous.

Proof. Let (M_1, d_1) and (M_2, d_2) be metric spaces and $f : M_1 \to M_2$ a lipschitz continuous map. Since f is lipschitz $\exists\ L(f) \in \mathbb{R}^+$ such that for any $x, y \in M_1$ we have

$$d_2\big(f(x), f(y)\big) \leq L(f) \cdot d_1(x, y)$$

if $y = x$ then $d_2\big(f(x), f(x)\big) = 0$ as well as $d_1(x, x) = 0$ so that for any $\epsilon > 0, \exists\ \delta > 0$ where we have

$$d_1(x, x) = 0 < \delta \implies L(f) d_2\big(f(x), f(x)\big) = 0 < \epsilon$$

so let $y \neq x$, then $d_1(x, y) \neq 0$, so for $\delta(\epsilon) > 0$ such that $d_1(x, y) < \delta(\epsilon)$, selecting $\delta(\epsilon) = \frac{\epsilon}{L(f)} > 0$ we have

$$d_2\big(f(x), f(y)\big) \leq L(f) \cdot d_1(x, y) < L(f) \cdot \delta(\epsilon) = L(f) \cdot \frac{\epsilon}{L(f)} = \epsilon$$

and so

$$d_1(x, y) < \delta(\epsilon) \implies d_2\big(f(x), f(y)\big) < \epsilon$$

and so f is uniformly continuous. □

Next we recall a few definitions familiar from analysis, and there corresponding notions of continuity, which or course gives us continuity in a metric space.

Definition 12. let $\{x_n\}_{n \in \mathbb{N}}$ be any sequence in a normed vector space $(V, ||\cdot||)$. The sequence **Converges** to $v \in V$ if $\forall\ \epsilon > 0,\ \exists\ N$ such that $\forall\ n \geq N$ we have

$$||v - x_n|| < \epsilon$$

Definition 13. let $\{x_n\}_{n\in\mathbb{N}}$ be any sequence in a normed vector space $(V, ||\cdot||)$. The sequence is **Cauchy** if $\forall\ \epsilon > 0,\ \exists\ N$ such that $\forall\ n, m \geq N$ we have

$$||x_n - x_m|| < \epsilon$$

Definition 14. A sequence of maps $\{f_n\}_{n\in\mathbb{N}}$, from a set S, with

$$f_n : S \to (V, ||\cdot||)$$

is **Uniformly Convergent** to a map f, if given $\epsilon > 0,\ \exists\ N$ such that $\forall\ n \geq N$ we have

$$||f_n - f||_S < \epsilon$$

Definition 15. A sequence of maps $\{f_n\}_{n\in\mathbb{N}}$ with $f_n : S \to (V, ||\cdot||)$ is **Uniformly Cauchy** on a set S if given $\epsilon > 0,\ \exists\ N$ such that $\forall\ n, m \geq N$ we have

$$||f_n - f_m||_S < \epsilon$$

Proposition 16.

$$f : (X, d_X) \to (Y, d_Y)$$

is continuous iff

$$x_n \to x \implies f(x_n) \to f(x)$$

Proof. First suppose f is continuous and that $x_n \to x \in X$. Let $\epsilon > 0$ be given and $B_\epsilon\big(f(x)\big) \subseteq Y$ be open such that $f(x) \in B_\epsilon\big(f(x)\big)$. Then since f is continuous $f^{-1}\Big(B_\epsilon\big(f(x)\big)\Big) \subseteq X$ is open and contains x. Then, since $x_n \to x, \forall\ \delta > 0,\ \exists\ N \in \mathbb{N}$ such that $n \geq N \implies d_X(x_n, x) < \delta$ which implies

$$\begin{aligned} B_\delta(x) \subseteq f^{-1}\Big(B_\epsilon\big(f(x)\big)\Big) &\implies x_n \in f^{-1}\Big(B_\epsilon\big(f(x)\big)\Big) \\ &\implies f(x_n) \in B_\epsilon\big(f(x)\big) \\ &\implies d_Y\big(f(x_n), f(x)\big) < \epsilon \end{aligned}$$

and so $f(x_n) \to f(x)$.

Next suppose $x_n \to x \implies f(x_n) \to f(x)$. And assume, for contradiction, that f is not continuous. Then $\forall\ \epsilon > 0$ with $\delta = \frac{1}{n}$ we have

$$d_X(x_n, x) < \frac{1}{n}$$

yet,

$$d_Y\big(f(x_n), f(x)\big) \geq \epsilon$$

and doing this for each n we have $d(x_n, x) \to 0$ while $d_Y\big(f(x_n), f(x)\big) \geq \epsilon\ \forall\ n \Rightarrow\Leftarrow$. And so f must be continuous. $\square$

Proposition 17. If $f : X \to Y$ is uniformly continuous, and $\{x_n\} \in X$ is a cauchy sequence, then $\big\{f(x_n)\big\}$ is a cauchy sequence in Y.

Proof. Since $f : X \to Y$ is uniformly continuous, $\forall\ \epsilon > 0,\ \exists\ \delta > 0$ such that

$$d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \epsilon$$

so for any cauchy sequence $\{x_n\} \in X$, $\exists\ N$ such that $n, m > N \implies d_X(x_n, x_m) < \delta$, yet this then gives

$$d_Y(f(x_n), f(x_m)) < \epsilon$$

by uniform continuity, and so $\{f(x_n)\}$ is cauchy in Y. $\square$

Our first major aim, is to develop the completion of a metric space, and for this we need the following definitions

Definition 18. A metric space X is **Complete** if every Cauchy sequence converges to a point in X; i.e. $\forall\ \{x_i\}_{i=1}^{\infty} \in X,\ x_i \to x \in X$.

Definition 19. For (X, d) a metric space, the **Completion** of (X, d) is a complete metric space $(X_\sim, d_\sim)$ together with an isometric function

$$f : X \to X_\sim$$

where $f(X) \subseteq X_\sim$ is dense in $X_\sim$.

And since the isometric function is required to have a dense image, we must have a definition for this as well.

Definition 20. Let (X, τ) be a topological space and $S \subseteq X$, then S is **Dense** in X if $\overline{S} = X$. Where the over-bar denotes closure.

Equivalently, this can be reformulated as follows: S is dense in X if and only if for each open $U \subseteq X$ such that $U \neq \varnothing$ there is some $s \in S$ such that $s \in U$. In terns of metrics, this is $\forall\ x \in X$ and $\epsilon > 0,\ \exists\ s \in S$ such that $d(x, s) < \epsilon$.

Proposition 21. If S is dense in X, and

$$f, g : X \to Y$$

are continuous maps such that $f(s) = g(s)\ \forall\ s \in S$, then $f = g$ on X.

Proof. Let $x \in X \setminus S = S^c$ and let $\epsilon > 0$ be given. Then by continuity of f and g, $\exists\ \delta > 0$ and by density of S, $\exists\ s \in S$ such that

$$d_X(x, s) < \delta \implies d_Y\big(f(x), f(s)\big) < \frac{\epsilon}{2} \text{ and } d_Y\big(g(x), g(s)\big) < \frac{\epsilon}{2}$$

then

$$\begin{aligned} d_Y\big(f(x), g(x)\big) &\leq d_Y\big(f(x), f(s)\big) + d_Y\big(f(s), g(s)\big) + d_Y\big(g(s), g(x)\big) \\ &< \frac{\epsilon}{2} + 0 + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

and thus $f(x) = g(x)$. Since $x \in S^c$ was arbitrary we conclude $f = g$ on S^c, and we are given that $f = g$ on S, and since $X = S \cup S^c$ we conclude that $f = g$ on X. $\square$

The next few lemmas and theorem provide us with the tools necessary to prove that every metric space has a completion.

Lemma 22. If $\{s_n\}, \{t_n\} \in X$ are cauchy sequences, then $\{d(s_n, t_n)\}$ converges in $\mathbb{R}$.

Proof. Let $\{s_n\}, \{t_n\}$ be cauchy sequences in X, then $\forall\ \epsilon > 0,\ \exists\ N_s, N_t$ such that

$$\begin{aligned} n_s, m_s \geq N_s &\implies d(s_{n_s}, s_{m_s}) < \frac{\epsilon}{2} \\ n_t, m_t \geq N_t &\implies d(t_{n_t}, t_{m_t}) < \frac{\epsilon}{2} \end{aligned}$$

so let $N = \max\{N_s, N_t\}$ then

$$\begin{aligned} n, m \geq N &\implies d(s_n, t_n) \leq d(s_n, s_m) + d(s_m, t_m) + d(t_m, t_n) \\ &\implies \big|d(s_n, t_n) - d(s_m, t_m)\big| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

with a symmetric argument giving

$$\big|d(s_m, t_m) - d(s_n, t_n)\big| < \epsilon$$

and so $\{d(s_n, t_n)\} \in \mathbb{R}$ is cauchy, and since $\mathbb{R}$ is complete we can conclude that $\{d(s_n, t_n)\}$ converges in $\mathbb{R}$. $\square$

Lemma 23. Cauch(X), denoting the cauchy sequences in X, has the equivalence relation $\{s_n\} \sim \{t_n\}$ iff $d(s_n, t_n) \to 0$.

Proof.
Reflexive: Trivially, $d(s_n, s_n) \to 0$, so $\{s_n\} \sim \{s_n\}$

Symmetric: If $d(s_n, t_n) \to 0$, then $d(s_n, t_n) = d(t_n, s_n) \to 0$. Giving $\{s_n\} \sim \{t_n\}$.

Transitive: Suppose $d(s_n, r_n) \to 0$ and $d(r_n, t_n) \to 0$, then $\forall\ n$

$$d(s_n, t_n) \leq d(s_n, r_n) + d(r_n, t_n) \to 0$$

and so $\{s_n\} \sim \{t_n\}$. $\square$

Lemma 24. If $X_\sim = \text{Cauch}(X)/\sim$ then

$$d_\sim : X_\sim \times X_\sim \to [0, \infty), \text{ by } d_\sim\big(\{s_n\}, \{t_n\}\big) = \lim_{n \to \infty} d(s_n, t_n)$$

is a metric on $X_\sim$.

Proof. First, since $\{d(s_n, t_n)\}$ converges in $\mathbb{R}$, we have that $d_\sim$ is always defined. To see that $d_\sim$ is well defined, let $\alpha, \beta \in X_\sim$ with $\{x_n\}, \{s_n\} \in \alpha$ and $\{y_n\}, \{t_n\} \in \beta$. Then

$$\lim_{n\to\infty} d(x_n, s_n) = \lim_{n\to\infty} d(y_n, t_n) = 0$$

and so $\forall\ \epsilon > 0,\ \exists\ N \in \mathbb{N}$ such that

$$n \geq N \implies d(x_n, s_n) < \frac{\epsilon}{2} \text{ and } d(y_n, t_n) < \frac{\epsilon}{2}$$

then for $n > N$ we have

$$\begin{aligned} d(s_n, t_n) &\leq d(s_n, x_n) + d(x_n, y_n) + d(y_n, t_n) \\ \implies \big|d(s_n, t_n) - d(x_n, y_n)\big| &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

$\therefore \lim_{n\to\infty} d(s_n, t_n) = \lim_{n\to\infty} d(x_n, y_n)$, or

$$d_\sim(\alpha, \beta) = \lim_{n\to\infty} d(s_n, t_n) = \lim_{n\to\infty} d(x_n, y_n)$$

and so $d_\sim$ is well-defined.

To see that $d_\sim$ it is a metric, for symmetry we have

$$d_\sim(\alpha, \beta) = \lim_{n\to\infty} d(s_n, t_n) = \lim_{n\to\infty} d(t_n, s_n) = d_\sim(\beta, \alpha)$$

now for $\alpha, \beta, \gamma \in X_\sim$ with $\{x_n\} \in \alpha,\ \{y_n\} \in \beta,\ \{z_n\} \in \gamma$, then $\forall\ n$

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, z_n) + d(z_n, y_n) \\ \implies \lim_{n\to\infty} d(x_n, y_n) &\leq \lim_{n\to\infty} d(x_n, z_n) + \lim_{n\to\infty} d(z_n, y_n) \\ \implies d_\sim(\alpha, \beta) &\leq d_\sim(\alpha, \gamma) + d_\sim(\gamma, \beta) \end{aligned}$$

and so satisfies the triangle inequality.

Next, if $d_\sim(\alpha, \beta) = 0$, then $\forall\ \{x_n\} \in \alpha,\ \{y_n\} \in \beta$ we have

$$\implies \lim_{n\to\infty} d(x_n, y_n) = 0$$

and so $\{x_n\} \sim \{y_n\} \implies \{y_n\} \in \alpha$ and thus $\alpha = \beta$. ☐

Theorem 25. Let (X, d_X) and (Y, d_Y) be metric spaces, with (Y, d_Y) being complete. If $S \subseteq X$ is dense and

$$f : X \to Y$$

is uniformly continuous, then there exists a unique continuous extension

$$F : X \to Y$$

such that $F|_S = f$.

Proof. Given $x \in X$, by the denseness of S in X, let $\{s_n\} \in S$ be a cauchy sequence such that $s_n \to x$. That is $\forall\ \epsilon > 0,\ \exists\ N \in \mathbb{N}$ such that

$$n \geq N \implies d(s_n, x) < \epsilon$$

Next, since f is uniformly continuous we have $\{f(s_n)\}$ is cauchy in Y, and by the completeness of Y this implies $f(s_n) \to p \in Y$. So set

$$F(x) := p$$

This is well-defined, since if $\{t_n\} \in \text{Cauch}(S)$ is another cauchy sequence in S such that $t_n \to x$, then

$$\begin{aligned}
&\lim_{n\to\infty} d_X(s_n, t_n) = 0 \\
\implies &\lim_{n\to\infty} d_Y\big(f(s_n), f(t_n)\big) = 0 \\
\implies &\lim_{n\to\infty} d_Y\big(p, f(t_n)\big) = 0 \\
\implies &f(t_n) \to p
\end{aligned}$$

and so is well defined and

$$F|_S = f : S \to Y$$

Now, since f is uniformly continuous, this implies that $\forall\ \epsilon > 0,\ \exists\ \delta > 0$ such that $\forall\ x, y \in X$ with

$$d_X(x, y) < \frac{\delta}{3} \implies d_Y\big(f(x), f(y)\big) < \frac{\epsilon}{3}$$

and $\forall\ p, q \in Y$ there exists cauchy sequences $\{f(x_n)\}, \{f(y_n)\} \in Y$ such that

$$f(x_n) \to p, \qquad f(y_n) \to q$$

so for $N = \max\{N_p, N_q, N_{x_n}, N_{y_n}\}$, we have $n \geq N$ implies

$$\begin{aligned}
d_X(x, x_n) < \frac{\delta}{3} &\implies d_Y\big(p, f(x_n)\big) < \frac{\epsilon}{3} \\
d_X(y, y_n) < \frac{\delta}{3} &\implies d_Y\big(q, f(y_n)\big) < \frac{\epsilon}{3}
\end{aligned}$$

and so $n \geq N$ implies

$$\begin{aligned}
&d_X(x, y) \leq d_X(x, x_n) + d_X(x_n, y_n) + d_X(y_n, y) < \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta \\
&\implies d_Y(p, q) \leq d_Y\big(p, f(x_n)\big) + d_Y\big(f(x_n), f(y_n)\big) + d_Y\big(f(y_n), q\big) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon
\end{aligned}$$

and therefore F is uniformly continuous. $\square$

Now, with all the tools in place, we can establish our goal of proving that every metric space has a completion.

Theorem 26. Every metric space (X, d) has a completion.

Proof. We begin by defining the completion; namely the completion of (X, d) is $(X_\sim, d_\sim)$, where

$$X_\sim = \text{Cauch}(X)/\sim$$

and

$$d_\sim : X_\sim \times X_\sim \to \mathbb{R}$$

is a well defined metric on $X_\sim$ from Lemma 24. For our isometry we choose the inclusion map

$$\iota : X \to X_\sim, \text{ by } \iota(x) = \{x, x, x, \dots\}$$

i.e. ι maps $x \in X$ to the constant cauchy sequence $\{x\} \in X_\sim$. Where we note that for any $x, y \in X$ we have

$$d_\sim\big(\iota(x), \iota(y)\big) = d_\sim\big(\{x, x, \dots\}, \{y, y, \dots\}\big) = \lim_{n\to\infty} d(x, y) = d(x, y)$$

and so $d_\sim|_X = d$, and hence is isometric.

To see that $\iota(X) \subseteq X_\sim$ is dense, let $\alpha \in X_\sim$, $\epsilon > 0$ and $\{x_n\} \in \alpha$ be given. Then $\exists\ N \in \mathbb{N}$ such that

$$n, m \geq N \implies d(x_n, x_m) < \epsilon$$

Fixing $n > N$ we have,

$$d_\sim\big(\iota(x_n), \alpha\big) = \lim_{m\to\infty} d(x_n, x_m) < \epsilon$$

and so for n sufficiently large each neighborhood of α will contain $\iota(x_n)$, and since $\alpha \in X_\sim$ was arbitrary we conclude that for each $\alpha \in X_\sim$ we can find n sufficiently large so that $\iota(x_n) \in U_\alpha$ and thus $\iota(X)$ is dense in $X_\sim$.

To see that $(X_\sim, d_\sim)$ is complete, let $\{\alpha_n\}$ be a cauchy sequence in $X_\sim$, since $\iota(X)$ is dense in $X_\sim$, we may select n sufficiently large so that

$$d_\sim\big(\iota(x_n), \alpha_n\big) < \frac{1}{n}$$

Let $\epsilon > 0$ be given. Since $\{\alpha_n\}$ is cauchy $\exists\ N \in \mathbb{N}$ such that

$$n, m \geq N \implies d(\alpha_n, \alpha_m) < \frac{\epsilon}{3}$$

letting N be larger if necessary so that $\frac{1}{N} < \frac{\epsilon}{3}$, we then have

$$\begin{aligned} n, m \geq N \implies d(x_n, x_m) &= d_\sim\big(\iota(x_n), \iota(x_m)\big) \\ &\leq d_\sim\big(\iota(x_n), \alpha_n\big) + d_\sim\big(\alpha_n, \alpha_m\big) + d_\sim\big(\alpha_m, \iota(x_m)\big) \\ &< \frac{1}{n} + \frac{\epsilon}{3} + \frac{1}{m} \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

and therefore $\{x_n\} \in X$ is cauchy. So let $\{x_n\} \in \beta \in X_\sim$, and $\epsilon' > 0$ be given. Then selecting $N' > N$ such that

$$n, m \geq N' \implies d(x_n, x_m) < \frac{\epsilon'}{2}$$

letting N' be larger if necessary so that $\frac{1}{N'} < \frac{\epsilon'}{2}$, we then have

$$\begin{aligned}
n, m \geq N' \implies d_\sim\big(\alpha_n, \beta\big) &\leq d_\sim\big(\alpha_n, \iota(x_n)\big) + d_\sim\big(\iota(x_n), \beta\big) \\
&= d_\sim\big(\alpha_n, \iota(x_n)\big) + \lim_{m\to\infty} d\big(\iota(x_n), x_m\big) \\
&< \frac{1}{n} + \frac{\epsilon'}{2} \\
&< \frac{\epsilon'}{2} + \frac{\epsilon'}{2} \\
&= \epsilon'
\end{aligned}$$

and so $\alpha_n \to \beta$ and since $\{\alpha_n\}$ was arbitrary, we conclude that $(X_\sim, d_\sim)$ is complete. □

Proposition 27. The uniform limit of continuous functions is continuous.

Proof. Let $\epsilon > 0$, and $x, y \in X$, then $\forall\ n$ we have

$$\big|f(x) - f(y)\big| \leq \big|f(x) - f_n(x)\big| + \big|f_n(x) - f_n(y)\big| + \big|f_n(y) - f(y)\big|$$

then. by uniform continuity $\exists\ N \in \mathbb{N}$ such that

$$n \geq N \implies \big|f(x) - f_n(x)\big| < \frac{\epsilon}{3} \quad \forall\ x \in X$$

and by continuity $\forall\ \epsilon > 0, \exists\ \delta > 0$ such that

$$|x - y| < \delta \implies \big|f_n(x) - f_n(y)\big| < \frac{\epsilon}{3}$$

and thus $\forall\ x, y \in X$ such that $|x - y| < \delta$ and $n \geq N$ we have

$$\big|f(x) - f(y)\big| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

and so f is continuous. □

Theorem 28. $C\big([0,1]\big)$ is complete for $||\cdot||_\infty$.

Proof. Let $\{f_n\} \in C\big([0,1]\big)$ be a cauchy sequence, then $\forall\ \epsilon > 0,\ \exists\ N \in \mathbb{N}$ such that

$$n, m \geq N \implies ||f_n - f_m||_\infty < \epsilon$$

Now, for each fixed $x \in [0,1]$ we have

$$\big|f_n(x) - f_m(x)\big| < \epsilon \quad \forall\ n, m \geq N$$

and this implies $\{f_n(x)\}$ is cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete $\{f_n(x)\}$ converges, so set

$$f(x) = \lim_{n\to\infty} f_n(x)$$

now, since $\{f_n\} \in C\big([0,1]\big)$ is cauchy $\exists\ N$ such that

$$\begin{aligned} &\big|f_n(x) - f_m(x)\big| < \epsilon \quad \forall\ n,m \geq N \\ \implies &\big|f(x) - f_m(x)\big| < \epsilon \quad \forall\ m \geq N;\ x \in [0,1] \end{aligned}$$

and this in turn implies that $f_m \to f$ uniformly. Since f is the uniform limit of continuous functions, f is continuous; that is $f_n \to f \in C\big([0,1]\big)$, and so $C\big([0,1]\big)$ is complete. □

2 Topology

we begin by relating continuity from our knowledge of metric spaces to that of the more abstract concept of topology.

Proposition 29. Let $(X, d_X), (Y, d_Y)$ be metric spaces. Then a map

$$f : X \to Y$$

is continuous at $x_0 \in X$ if $\forall\ \epsilon > 0,\ \exists\ \delta > 0$ such that

$$f\big(B_\delta(x_0)\big) \subseteq B_\epsilon\big(f(x_0)\big)$$

Proof. Let $f(x) \in B_\epsilon\big(f(x_0)\big)$ for some $x \in X$, and let

$$\epsilon' = \epsilon - d\big(f(x), f(x_0)\big) > 0$$

then $B_{\epsilon'}\big(f(x)\big) \subseteq B_\epsilon\big(f(x_0)\big) \implies \exists\ \delta' > 0$ such that

$$f\big(B_{\delta'}(x)\big) \subseteq B_{\epsilon'}\big(f(x)\big) \subseteq B_\epsilon\big(f(x_0)\big)$$

if $x_1 \in f^{-1}\Big(B_{\epsilon'}\big(f(x)\big)\Big)$ then $\exists\ B_{\delta'}(x_1)$ such that

$$B_{\delta'}(x_1) \subseteq f^{-1}\Big(B_{\epsilon'}\big(f(x)\big)\Big) \subseteq X$$

and so is open. □

now we define a topology, and introduce what the notions of base and subbase, and what is the actual definition of continuity on a topological space.

Definition 30. Let X be a set, then a topology τ on X is a collection of open subsets such that:

1. $\varnothing$ and X are open. Or, $\varnothing, X \in \tau$.

2. A finite intersection of open sets is open; i.e. for $U_1, \ldots, U_n \in \tau$

$$\bigcap_{i=1}^{n} U_i \in \tau$$

3. An arbitrary union of open sets is open; i.e. $\forall\ U \in \tau$

$$\bigcup_{U \in \tau} U \in \tau$$

it should be noted that in any topological space, the closed sets satisfy the following.

1. $\varnothing$ and X are closed. Or, $\varnothing, X \in \tau^c$.
2. A finite union of closed sets is closed; i.e. for $A_1, \ldots, A_n \in \tau^c$

$$\bigcup_{i=1}^{n} A_i \in \tau^c$$

3. An arbitrary intersection of closed sets is closed; i.e. $\forall\ A \in \tau^c$

$$\bigcap_{A \in \tau^c} A \in \tau^c$$

so that a topology can just as easily be defied in terms of the closed sets.

Definition 31. Let X, Y be topological spaces, a map $f : X \to Y$ is continuous if $\forall$ open $V \subseteq Y$ we have $f^{-1}(V) \subseteq X$ is open.

be alert that the definition makes no reference to the image of a map, and that if $U \subseteq X$ is open, then $f(U) \subseteq Y$ may not be open.

Definition 32. Let $X = \mathbb{R}$ then a subset $U \subseteq \mathbb{R}$ is open if $\forall\ x \in U\ \exists\ J = (a, b)$ such that $x \in J \subseteq U$ is the Ordinary Topology.

Definition 33. A collection $\mathcal{B} = \{B_\alpha : \alpha \in I\} \subseteq X$ of open subsets is a base for the topology on X if for every $U \subseteq X$ open, we have $U = \cup_{B_\alpha \in \mathcal{B}} B_\alpha$ for some $\alpha \in I$.
If X is a set and $\mathcal{B}$ a collection of subsets of X satisfying

1.) $X = \bigcup_{B \in \mathcal{B}} B$

2.) if $B_1, B_2 \in \mathcal{B}, x \in B_1 \cap B_2 \implies \exists\ B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$

Then the collection of all unions of elements in $\mathcal{B}$ is a unique topology on X generated by base $\mathcal{B}$

Definition 34. If $\mathcal{S}$ is a collection of subsets of X such that

$$\bigcup_{V \in \mathcal{S}} V = X$$

and the finite intersection of elements of $\mathcal{S}$ is a base for X, then $\mathcal{S}$ is a sub-base for τ.

Proposition 35. Let (X, τ_X) and (Y, τ_Y) be topological spaces, then a map

$$f : X \to Y$$

is continuous iff for a base, or sub-base $\mathcal{B}_Y \subseteq \tau_Y$ we have

$$f^{-1}(B) \subseteq \tau_X \quad \forall\ B \in \mathcal{B}_Y$$

Proof. First suppose f is continuous. Then $\forall\ B \in \mathcal{B}_Y$ since $\mathcal{B}_Y$ is a base we have $B \in \tau_Y$ and so is open, then $f^{-1}(B) \in \tau_X$ by continuity.

Next suppose that $f^{-1}(B) \subseteq \tau_X\ \forall\ B \in \mathcal{B}_Y$, and let $V \in \tau_Y$. Since $\mathcal{B}_Y = \{B_i : i \in I\}$ is a base we have

$$V = \bigcup_{B_i \in \mathcal{B}_Y} B_i \quad \text{for some } i \in I$$

then

$$f^{-1}(V) = f^{-1}\left(\bigcup_{B_i \in \mathcal{B}_Y} B_i\right) = \bigcup_{B_i \in \mathcal{B}_Y} f^{-1}(B_i) \in \tau_X$$

and so f is continuous. $\square$

next we introduce a few different kinds of topologies

Definition 36. The Profinite Topology is defined on a group as follows: Let G be a group, then $U \subseteq G$ is open if $\forall\ x \in U\ \exists$ a subgroup H of G, of finite index, such that $xH \subseteq U$.

Definition 37. We also have the Ideal Topology, defined on rings. Let R be a commutative ring with unity, then $U \subseteq R$ is open if $\forall\ x \in U\ \exists$ an ideal I of R such that $x + I \subseteq U$.

Definition 38. Zariski Topology. An algebraic topology. For instance let $X = \mathbb{R}^n$ and

$$f : \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \to \mathbb{R}$$

be a polynomial in n variables, $\mathbf{a} \in \mathbb{R}^n$ is a zero of f if $f(\mathbf{a}) = \mathbf{0}$, then a subset $S \subseteq \mathbb{R}^n$ is closed if $\exists$ a family $\{f_i\}_{i \in I}$ of polynomials in n variables such that S is the zero set of $\{f_i\}_{i \in I}$. That is

$$S = \{\mathbf{a} \in \mathbb{R}^n : f_i(\mathbf{a}) = \mathbf{0}\ \forall\ i \in I\}$$

Definition 39. If X is a topological space, Y is a set, and $\pi : X \to Y$ is a surjective map, the Quotient Topology on Y determined by π is defined by declaring a subset $U \subseteq Y$ to be open iff $\pi^{-1}(U) \subseteq X$ is open in X. or

$$\tau_Y = \{U \subseteq Y : \pi^{-1}(U) \in \tau_X\}$$

Notice that we need the surjectiveness here otherwise if $y \notin \pi(X)$, then $\pi^{-1}(\{y\}) = \varnothing \implies \{y\}$ is open. Equivalently, if we define $x_1 \sim x_2$ iff $\pi(x_1) = \pi(x_2)$ then for $Y = X/\sim$ we have

$$\pi : X \to X/\sim$$

is the quotient topology determined by π.

Definition 40. If (X, τ) is a topological space and $S \subseteq X$ is arbitrary, the relative topology is defined by declaring $U \subseteq S$ to be open iff $\exists\, V \in \tau$ such that $U = V \cap S$.

Proposition 41. Let X be a topological space. If $A \subseteq X$ is closed and $C \subseteq A$ is closed in the relative topology of A, then C is closed in X.

Proof. Since $A \setminus C = A \cap C^c$ is open in the relative topology of A, then $\exists\, U \in \tau$ such that

$$A \cap C^c = A \cap U \implies C = A \cap U^c$$

is closed in X. □

we also might be curious how two topologies compare to one another on a given set.

Definition 42. let X be a set and τ, σ topologies on X then σ is a refinement of τ if for each $U \in \tau$ we also have $U \in \sigma$.

This can also be stated as τ is coarser than σ. Which leads way to the next two definitions

Definition 43. Let X be a topological space and let τ_1, τ_2 be two topologies for X. If $\tau_1 \subseteq \tau_2$ then τ_1 is coarser than τ_2.

Definition 44. Let X be a topological space and let τ_1, τ_2 be two topologies for X. If $\tau_1 \subseteq \tau_2$ then τ_2 is finer than τ_1.

Definition 45. Given $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in \Lambda}$ and a set Y the final topology is the finest topology on Y such that the family

$$\mathcal{F} = \{f_\alpha : X_\alpha \to Y : \alpha \in \Lambda\}$$

is continuous $\forall\, \alpha$; i.e. $U \in \tau_Y$ iff $f_\alpha^{-1}(U) \in \tau_\alpha\ \forall\, \alpha$.

Definition 46. Let Y be a topological space and let $\mathcal{F}$ be a family of mappings

$$f : X \to Y$$

let

$$\tau_X = \{f^{-1}(W) \subseteq X : W \subseteq Y \text{ is open } ; f \in \mathcal{F}\}$$

then τ_X is the weak/initial topology on X determined by $\mathcal{F}$ and is the coarsest topology on X such that each $f \in \mathcal{F}$ is continuous.

equivalently, let X be a set and $\{Y_\alpha\}$ a family of topological spaces. For each α, let

$$f_\alpha : X \to Y_\alpha$$

be a map. The weak topology on X is the coarsest topology making each f_α continuous. It should be noted that the sub-base for the weak topology has all sets of the form $f_\alpha^{-1}(U)$ where $U \subseteq Y_\alpha$ is open.

Proposition 47. Consider

$$f_i : X \to Y_i \quad \text{for } i \in I$$

let τ_X be the initial/weak topology on X, let (Z, τ_Z) be a topological space and

$$g : Z \to X$$

then g is continuous iff

$$f_i \circ g$$

is continuous $\forall\ i$.

Proof. First suppose $f_i \circ g$ is continuous $\forall\ i$. It suffices to check on a sub-base, so let $U \in \tau_i$ for some i, then

$$(f_i \circ g)^{-1}(U)$$

is open by the continuity if $f_i \circ g$, yet

$$(f_i \circ g)^{-1}(U) = g^{-1}(f_i^{-1}(U))$$

and so $g^{-1}(f_i^{-1}(U)) \subseteq Z$ is open, and since the topology on X implies that $f_i^{-1}(U)$ is open in X, we then have that the preimage under g of an open set is open, and so g must be continuous.

Next suppose that g is continuous. Then by the continuity of g and the f_i's we have for any $i \in I$ and $U \in \tau_i$ that

$$g^{-1}(f_i^{-1}(U)) = (f_i \circ g)^{-1}(U)$$

is open and thus $f_i \circ g$ is continuous for each i. $\square$

the next definition, and the propositions and lemmas associated with it lead us to the very useful Urysohn's Lemma, and Tietze Extension Theorem.

Definition 48. Suppose that one-point sets are closed in (X, τ). Then X is normal if for $A, B \subset X$ closed such that $A \cap B = \varnothing$, $\exists\, U, V \in \tau$ with $U \cap V = \varnothing$, such that

$$A \subset U, \text{ and } B \subset V$$

Proposition 49. Every metrizable topological space is normal.

Proof. It suffices to consider a metric space (M, d). Let $C_1, C_2 \subseteq M$ be closed and disjoint. For each $x \in C_1$ choose $\epsilon_x > 0$ such that

$$B_{\epsilon_x}(x) \subseteq C_2^c$$

and for each $y \in C_2$ choose $\epsilon_y > 0$ such that

$$B_{\epsilon_y}(y) \subseteq C_1^c$$

let

$$O_1 = \bigcup_{x \in C_1} B_{\frac{\epsilon_x}{3}}(x) \text{ and } O_2 = \bigcup_{y \in C_2} B_{\frac{\epsilon_y}{3}}(y)$$

then O_1, O_2 are open as arbitrary unions of open sets, and since $C_1 \cap C_2 = \varnothing \implies C_1 \subseteq C_2^c$ and $C_2 \subseteq C_1^c$ so that

$$C_1 \subseteq O_1 \text{ and } C_2 \subseteq O_2$$

so suppose, for contradiction, that $O_1 \cap O_2 \neq \varnothing \implies \exists\, z \in O_1 \cap O_2$. Then $\exists\, x' \in C_1$ and $y' \in C_2$ such that $z \in B_{\frac{\epsilon_{x'}}{3}}(x')$ and $z \in B_{\frac{\epsilon_{y'}}{3}}(y')$, then

$$\begin{aligned} d(x', y') &\leq d(x', z) + d(z, y') \\ &< \frac{\epsilon_{x'}}{3} + \frac{\epsilon_{y'}}{3} \\ &\leq \frac{2}{3} \max\{\epsilon_{x'}, \epsilon_{y'}\} \quad \Rightarrow\Leftarrow \end{aligned}$$

as this implies $z \in C_1 \cap C_2 = \varnothing$. Thus $O_1 \cap O_2 = \varnothing$, and so M is normal. $\square$

Lemma 50. If (X, τ) is normal, $C \subset X$ is closed and $O \subseteq X$ is open and $C \subseteq O$, then $\exists\, U$ open with

$$C \subseteq U \subseteq \overline{U} \subseteq O$$

Proof. Since O is open, then O^c is closed and $C \subset O$ gives $O^c \cap C = \varnothing$. So, by normality, $\exists$ open U, V where $U \cap V = \varnothing$ such that $C \subseteq U$, and $O^c \subseteq V$.
Then $O^c \subseteq V \implies V^c \subseteq O$, and since $U \cap V = \varnothing$ we must have $U \subseteq V^c$ where V^c is closed. So $\overline{U} \subseteq \overline{V^c} = V^c$. Then

$$C \subseteq U \subseteq \overline{U} \subseteq O$$

$\square$

Lemma 51 (**Urysohn's Lemma**)**.** Let (X, τ) be normal, and let C_0, C_1 be disjoint closed subsets. Then $\exists\ f : X \to [0,1]$ continuous such that $f(C_0) = \{0\},\ f(C_1) = \{1\}$

Proof. Set $O_1 = X \setminus C_1 = C_1^c$ which is open as C_1 is closed in X. And since $C_0 \cap C_1 = \varnothing$ we have $C_0 \subseteq O_1$. Then, by Lemma 50 $\exists$ open O_0 such that

$$C_0 \subseteq O_0 \subseteq \overline{O}_0 \subseteq O_1$$

Then, by Lemma 50 $\exists$ open $O_{1/2}$ with

$$C_0 \subseteq O_0 \subseteq \overline{O}_0 \subseteq O_{1/2} \subseteq \overline{O}_{1/2} \subseteq O_1$$

so by Lemma 50 $\exists$ open $O_{1/4}, O_{3/4}$ so that

$$C_0 \subseteq O_0 \subseteq \overline{O}_0 \subseteq O_{1/4} \subseteq \overline{O}_{1/4} \subseteq O_{1/2} \subseteq \overline{O}_{1/2} \subseteq O_{3/4} \subseteq \overline{O}_{3/4} \subseteq O_1$$

So by Lemma 50 $\exists$ open $O_{1/8}, O_{3/8}, O_{5/8}, O_{7/8}$ such that

$$C_0 \subseteq O_0 \subseteq \overline{O}_0 \subseteq O_{1/8} \subseteq \overline{O}_{1/8} \subseteq O_{1/4} \subseteq \overline{O}_{1/4} \subseteq O_{3/8} \subseteq \overline{O}_{3/8} \subseteq \cdots$$

so by induction, for each dyadic rational

$$\left\{ \frac{m}{2^n} : 1 \leq m \leq 2^n - 1; n, m \in \mathbb{N} \right\} =: \Delta$$

we get open $O_{\frac{m}{2^n}}$ such that if $r, s \in \Delta$, with $r < s$ then $\overline{O}_r \subseteq O_s$ and $C_0 \subseteq O_r\ \forall\ r$. Define $f : X \to [0,1]$ by

$$\begin{aligned} &f(x) = \inf\{r \in \Delta : x \in O_r\} \text{ for } x \in O_1 \\ &f(x) = 1 \text{ for } x \in C_1 \end{aligned}$$

Then if $x \in C_0$, then $x \in O_r\ \forall\ r \in \Delta$ including $r = 0$, so we have $f(x) = 0$. To check continuity, use as a sub-base

$$\{(-\infty, a) : a \in \mathbb{R}\} \cup \{(b, \infty) : b \in \mathbb{R}\}$$

If $a \in \mathbb{R}$, then

$$f^{-1}\big((-\infty, a)\big) = \begin{cases} \varnothing, & a \leq 0 \\ X, & a > 1 \end{cases}$$

Suppose $0 < a \leq 1$. If $x \in X$ and $f(x) < a\ \exists\ r \in \Delta$ such that $f(x) < r < a$ and so $x \in O_r$ and thus $f^{-1}\big((-\infty, a)\big) = \bigcup_{r<a} O_r$ which is the union of open sets and hence is open.

If $f(x) > b$ then

$$f^{-1}\big((b, \infty)\big) = \begin{cases} X, & b < 0 \\ \varnothing, & b \geq 1 \end{cases}$$

for $0 \leq b < 1$ we claim $f^{-1}\big((b,\infty)\big) = \bigcup_{r>b} \overline{O}_r^c$.
If $f(x) > b$, then $\exists\ s \in \Delta$ with $f(x) > s > b \implies x \notin O_s$. Then $\exists\ r \in \Delta$ such that $s > r > b$ where $\overline{O}_r \subseteq O_s$ with $x \notin \overline{O}_r \implies x \in \overline{O}_r^c$ which is open, and so $f^{-1}\big((b,\infty)\big) = \bigcup_{r>b} \overline{O}_r^c$ which is open as the union of open sets. And so in all cases we see that f is continuous. □

Definition 52. A Banach Space is a complete normed vector space.

Proposition 53. If $(V, ||\cdot||)$ is a banach space, then $\big(B(X,V), ||\cdot||_\infty\big)$ is a banach space. Where $B(X,V)$ is the set of all bounded functions from X to V.

Proof. Let $\{f_n\} \in B(X,V)$ be a cauchy sequence. For each $x \in X$, $\{f_n(x)\}$ is cauchy in V, and by the completeness of V converges in V, say $f_n(x) \to f(x)$. Let $\epsilon > 0$ be given, since $\{f_n\}$ is cauchy $\exists\ N_1 \in \mathbb{N}$ such that

$$n, m \geq N_1 \implies ||f_n - f_m||_\infty < \frac{\epsilon}{2}$$

so for $x \in X$ and $n, m \geq N$ we have $||f_n(x) - f_m(x)|| < \frac{\epsilon}{2}$, so for fixed $m > N$ we have

$$||f_m(x) - f(x)|| = \lim_{n\to\infty} ||f_m(x) - f_n(x)|| < \frac{\epsilon}{2}$$

and so f is bounded. Next, fix $x \in X$ then since $f_n(x) \to f(x)$ $\exists\ N_2 \in \mathbb{N}$ such that

$$n \geq N_2 \implies ||f_n(x) - f(x)||_\infty < \frac{\epsilon}{2}$$

so for $n > \max\{N_1, N_2\}$ we have

$$\begin{aligned} ||f_n - f||_\infty &\leq ||f_n - f_{n+1}||_\infty + ||f_{n+1} - f||_\infty \\ &\leq ||f_n - f_{n+1}||_\infty + ||f_{n+1}(x) - f(x)|| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

and so $f_n \to f \in B(X,V)$, and hence is complete. □

Proposition 54. Let (X,τ) be a topological space and Y a metric space. Then the set of continuous bounded functions $C_B(X,Y)$ is a closed subset of $\big(B(X,Y), ||\cdot||_\infty\big)$.

Proof. Let $\{f_n\} \in C_B(X,Y)$ be a cauchy sequence such that $f_n \to f \in B(X,Y)$ under $||\cdot||_\infty$. We wish to show that $f \in C_B(X,Y)$. So let $\epsilon > 0$ and be given and $x_0 \in X$ be arbitrary. Then $\exists\ N \in \mathbb{N}$ such that

$$n \geq N \implies ||f - f_n||_\infty < \frac{\epsilon}{3}$$

Then since $f_n \in C_B(X,Y)$ is continuous $\exists$ open $B_\delta(x_0) \ni x_0$ such that if $y \in B_\delta(x_0)$ then $||f_n(y) - f_n(x_0)|| < \frac{\epsilon}{3}$ and so

$$\begin{aligned}||f(y) - f(x_0)|| &\le ||f(y) - f_n(y)|| + ||f_n(y) - f_n(x_0)|| + ||f_n(x_0) - f(x_0)|| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon\end{aligned}$$

and thus we have $f \in C_B(X,Y)$; that is $C_B(X,Y) \subset \big(B(X,Y), ||\cdot||_\infty\big)$ is closed. $\square$

Theorem 55 (**Tietze Extension Theorem**)**.** Let (X,τ) be a normal topological space and let $A \subset X$ be closed, and $f : A \to \mathbb{R}$ be continuous. Then $\exists\ F : X \to \mathbb{R}$ continuous, where $F|_A = f$. If $f(A) \subseteq [a,b]$ then we can arrange $F(X) \subseteq [a,b]$.

Proof. First, suppose that

$$f : A \to [-1,1]$$

and let

$$\begin{aligned}A_1 &= \left\{x \in A : f(x) \ge \tfrac{1}{3}\right\} = f^{-1}\left(\left[\tfrac{1}{3}, 1\right]\right) \\ B_1 &= \left\{x \in A : f(x) \le -\tfrac{1}{3}\right\} = f^{-1}\left(\left[-1, -\tfrac{1}{3}\right]\right)\end{aligned}$$

where by the continuity of f we have B_1, A_1 are closed in A where $B_1 \cap A_1 = \varnothing$, and thus are also closed and disjoint in X. So by Urysohn's lemma we have that there exists continuous

$$f_1 : X \to \left[-\tfrac{1}{3}, \tfrac{1}{3}\right]$$

such that

$$f_1(A_1) = \tfrac{1}{3},\ \text{and } f_1(B_1) = -\tfrac{1}{3}$$

Thus, for any $x \in A$ we have $|f(x) - f_1(x)| \le \frac{2}{3}$ so that

$$g_1 := f - f_1 : A \to \left[-\tfrac{2}{3}, \tfrac{2}{3}\right]$$

and let

$$\begin{aligned}A_2 &= \left\{x \in A : g_1(x) \ge \tfrac{1}{3}\left(\tfrac{2}{3}\right)\right\} = g_1^{-1}\left(\left[\tfrac{2}{9}, \tfrac{2}{3}\right]\right) \\ B_2 &= \left\{x \in A : g_1(x) \le -\tfrac{1}{3}\left(\tfrac{2}{3}\right)\right\} = g_1^{-1}\left(\left[-\tfrac{2}{3}, -\tfrac{2}{9}\right]\right)\end{aligned}$$

where by the continuity of g_1 we have B_2, A_2 are closed in A where $B_2 \cap A_2 = \varnothing$, and thus are also closed and disjoint in X. So by Urysohn's lemma we have that there exists continuous

$$f_2 : X \to \left[-\tfrac{2}{9}, \tfrac{2}{9}\right]$$

such that

$$f_2(A_2) = \tfrac{2}{9},\ \text{and } f_2(B_2) = -\tfrac{2}{9}$$

Thus, for any $x \in A$ we have $|f(x) - f_1(x) - f_2(x)| \leq \left(\frac{2}{3}\right)^2$ so that

$$g_2 := f - f_1 - f_2 : A \to \left[-\tfrac{4}{9}, \tfrac{4}{9}\right]$$

continuing inductively we can construct a sequence of continuous functions $f_1, f_2, \ldots$ such that

$$\left| f(x) - \sum_{i=1}^{n} f_i(x) \right| \leq \left(\tfrac{2}{3}\right)^n \to 0, \text{ as } n \to \infty$$

on A, so defining $F := \sum_{i=1}^{\infty} f_i$, then by construction we have $F|_A = f$.
For continuity let $\epsilon > 0$ and $x \in X$ be given, then pick $N \in \mathbb{N}$ such that $\sum_{i=N+1}^{\infty} \left(\frac{2}{3}\right)^i < \frac{\epsilon}{2}$. Then, since each Urysohn function f_i is continuous on X for $1 \leq i \leq N$ select $U_i \in \tau$ such that $x \in U_i$ where

$$y \in U_i \implies |f_i(x) - f_i(y)| < \frac{\epsilon}{2N}$$

then

$$U := \bigcap_{j=1}^{N} U_j$$

is open as the finite intersection of open sets and $y \in U$ implies

$$\begin{aligned}
|F(x) - F(y)| &\leq \sum_{i=1}^{N} |f_i(x) - f_i(y)| + \sum_{i=N+1}^{\infty} \left(\tfrac{2}{3}\right)^i \\
&< \frac{\epsilon}{2N} \sum_{i=1}^{N} 1 + \frac{\epsilon}{2} \\
&= \frac{\epsilon}{2N} \cdot N + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

and so F is continuous as x, since $x \in X$ was arbitrary we conclude that F is continuous on X.

Now for the case when f is not bounded, since $\mathbb{R}$ is homeomorphic to $(-1, 1)$ via the mapping

$$\frac{2}{\pi} \tan^{-1} : \mathbb{R} \to (-1, 1)$$

so let us consider

$$f : A \to (-1, 1) \subset [-1, 1]$$

Then from above there exists continuous $\widetilde{f} : X \to [1, -1]$ such that $\widetilde{f}|_A = f$. So, let

$$B = \widetilde{f}^{-1}(\{1\}) \cup \widetilde{f}^{-1}(\{-1\})$$

where by the continuity of $\widetilde{f}$ we have that $B \subset X$ is closed as the union of singletons which are closed, and since

$$\widetilde{f}(A) = f(A) \subseteq (-1, 1)$$

we have that $A \cap B = \varnothing$. So by Urysohn's lemma there exists continuous

$$g : X \to [0, 1]$$

such that

$$g(A) = 1, \text{ and } g(B) = 0$$

so define

$$F := g \cdot \widetilde{f} : X \to (-1, 1)$$

Then F is continuous as the product of two continuous functions, and for any $x \in A$ we have

$$F(x) = g(x) \cdot \widetilde{f}(x) = 1 \cdot \widetilde{f}(x) = f(x)$$

so $F|_A = f$. For $y \in B$ we have

$$F(y) = g(y) \cdot \widetilde{f}(y) = 0 \cdot \widetilde{f}(y) = 0$$

and for $z \notin A \cup B$, then since $|\widetilde{f}(z)| < 1$ we have

$$|F(z)| \leq 1 \cdot |\widetilde{f}(z)| < 1$$

and so $\mathrm{Im}(F) = (-1, 1)$, and F is an extension of f. $\square$

Definition 56. Let X be a topological space. A cover of X is a collection $\mathcal{U}$ of subsets of X whose union is X; i.e.

$$\bigcup_{U \in \mathcal{U}} U = X$$

a subcover is a subcollection of $\mathcal{U}$ that is still a cover, i.e. $\mathcal{U}' \subset \mathcal{U}$ where

$$\bigcup_{U \in \mathcal{U}'} U = X$$

$\mathcal{U}$ is an open cover if each $U \in \mathcal{U}$ is open.

Definition 57. A topological space X is compact if every open cover; i.e. $\bigcup_{U \in \mathcal{U}} U = X$, has a finite subcover.

A compact subset $S \subseteq X$ of a topological space X, is one that is a compact space in the relative topology.

Definition 58. Finite Intersection Property: Let X be a topological space, and $\{A_\alpha\}_{\alpha\in I}$ a family of nonempty subsets of X. Then $\{A_\alpha\}_{\alpha\in I}$ has the finite intersection property if every finite subcollection of $\{A_\alpha\}_{\alpha\in I}$ has nonempty intersection; i.e. $\{A_{i_1},\dots,A_{i_n}\}\subset\{A_\alpha\}_{\alpha\in I}$ gives

$$\bigcap_{j=1}^{i_n} A_{i_j}\neq\varnothing$$

for all subsets such that $|\{A_{i_1},\dots,A_{i_n}\}|<\infty$.

Now we prove an interesting relationship between the last two definitions

Proposition 59 (Equivalent Definition of Compact).

(a) X is compact if every open cover of X has a finite subcover.

(b) Every collection $\{K_\alpha\}_{\alpha\in I}$ of closed sets with the finite intersection property, has nonempty intersection; i.e. $\bigcap_{\alpha\in I}K_\alpha\neq\varnothing$.

Proof. $(a) \implies (b)$
Let X be compact, and let $\{K_\alpha\}_{\alpha\in I}$ be a collection of closed sets with the finite intersection property, and assume for contradiction, that $\bigcap_{\alpha\in I}K_\alpha=\varnothing$. Then for each K_α we have $X\setminus K_\alpha=K_\alpha^c$ is open. So,

$$\begin{aligned}\bigcap_{\alpha\in I}K_\alpha&=\varnothing\\ \implies\left(\bigcap_{\alpha\in I}K_\alpha\right)^c&=\varnothing^c\\ \implies\bigcup_{\alpha\in I}K_\alpha^c&=X\end{aligned}$$

That is $\bigcup_{\alpha\in I}K_\alpha^c$ is an open cover for X, and since X is compact, it admits a finite subcover, giving

$$\begin{aligned}\bigcup_{i=1}^{n}K_\alpha^c&=X\\ \implies\left(\bigcup_{i=1}^{n}K_\alpha^c\right)^c&=X^c\\ \bigcap_{i=1}^{n}K_\alpha&=\varnothing\quad\Rightarrow\Leftarrow\end{aligned}$$

A contradiction to our assumption that for finite K_α we have $\bigcap_{i=1}^n K_\alpha\neq\varnothing$. And therefore me must have $\bigcap_{\alpha\in I}K_\alpha\neq\varnothing$.

$(b) \implies (a)$

Let X be a topological space, and suppose that for every collection $\{K_\alpha\}_{\alpha\in I}$ of closed sets with the finite intersection property, we have $\bigcap_{\alpha\in I} K_\alpha \neq \varnothing$.
Next let $\mathcal{U}$ be an open cover of X and assume, for contradiction, that $\mathcal{U}$ has no finite subcover of X. That is

$$\bigcup_{j=1}^{i_n} U_{i_j} \neq X$$

so we must have at least one $p \in X$ such that

$$\begin{aligned} p &\notin \bigcup_{j=1}^{i_n} U_{i_j} \\ \implies p &\in \left(\bigcup_{j=1}^{i_n} U_{i_j}\right)^c \\ &= \bigcap_{j=1}^{i_n} U_{i_j}^c \\ \implies \varnothing &\neq \bigcap_{j=1}^{i_n} U_{i_j}^c \end{aligned}$$

where each $U_{i_k}^c$ is closed in X, and since this is true for each finite subcollection of $\mathcal{U}$, we have the family $\{X \setminus U\}_{U\in\mathcal{U}} = \{U^c\}_{U\in\mathcal{U}}$ satisfies the finite intersection property. Where by our assumption we have

$$\begin{aligned} \bigcap_{U\in\mathcal{U}} U^c &\neq \varnothing \\ \implies \left(\bigcap_{U\in\mathcal{U}} U^c\right)^c &\neq \varnothing^c \\ \implies \bigcup_{U\in\mathcal{U}} U &\neq X \quad \Rightarrow\Leftarrow \end{aligned}$$

A contradiction to the assumption that $\mathcal{U}$ was an open cover for X. Thus, we conclude that $\mathcal{U}$ must admit a finite subcover of X.

Since $\mathcal{U}$ was an arbitrary open cover for X, we conclude that every open cover of X admits a finite subcover, and therefore X is compact. $\square$

Definition 60. A space with the discrete topology; that is, the topology on a set X where each $U \subseteq X$ is declared open, in particular each $\{x\} \in X$ is open.

Definition 61. A topological space X is disconnected if it has 2 disjoint nonempty open subsets whose union is X; i.e. $U, V \subset X$ open, such that

$$U \neq \varnothing, \quad V \neq \varnothing, \text{ where } U \cap V = \varnothing, \quad \text{and } U \cup V = X$$

Definition 62. A topological space X is connect if it is not disconnected. Equivalently it is connected iff its' only subsets which are both open and closed are: $\varnothing$, and X itself.

A connected subset of X is a subset that is a connected space when endowed with the relative topology.

Proposition 63. A topological space X is connected if and only if every continuous map of X into a discrete space having at least two elements is constant.

Proof. First assume that X is connected, and that $f : X \to Y$ is a continuous map, where Y is a discrete space with at least 2 elements. WLOG suppose $Y = \{y, y'\}$.
If $f(X) \neq$ constant, then $f(x) = y$ and $f(x') = y'$ where $y, y' \in Y$ are disjoint and open by the discrete topology, yet this implies that for $U_x, U_{x'} \in X$ we have

$$f(U_x) \cap f(U_{x'}) = \varnothing$$

where $f(U_x), f(U_{x'}) \neq \varnothing$ and so form a separation of Y, which contradicts the continuity of f, since the image of a connected set under a continuous map must be connected.

Next suppose that X is not connected; i.e. $X = U \cup V$ where $V, U \neq \varnothing$ are open and $U \cap V = \varnothing$. Then let $p \neq q$ and endow $\{p, q\}$ with the discrete topology. If we define

$$f : X \to \{p, q\}, \text{ by } \begin{cases} f(U) = \{p\} \\ f(V) = \{q\} \end{cases}$$

then f is continuous and non-constant. $\square$

Proposition 64. If a topological space (X, τ) is compact, and $A \subseteq X$ is closed, then A is compact.

Proof. Let $\mathcal{U} \subseteq \tau$ be an open cover of A, then since $A \subseteq X$ is closed, we have $A^c \subseteq X$ is open, and so

$$\mathcal{U} \cup A^c$$

is an open cover for X. Since X is compact, it admits a finite subcover which must contain A. $\square$

The next few definitions lead us into Tychonoff's Theorem which is a fairly big statement about the product topology

Definition 65. Let $\{X_i\}_{i \in I}$ be a family of topological spaces and let

$$X = \prod_{i \in I} X_i$$

a topology on X is determined by declaring $U \subseteq X$ to be open if $\forall\, x \in U$, $\exists$ a finite number of indices $i_1, \ldots i_n$ and open subsets $U_{i_j} \subseteq X_{i_j}$ for $i \leq j \leq n$ such that

$$x \in U_{i_1} \times \cdots \times U_{i_n} \times \prod_{i \neq i_1, \ldots i_n} X_i \subseteq U$$

that is the product topology has as base all sets of the form

$$U_{i_1} \times \cdots \times U_{i_n} \times \prod_{i \neq i_1, \ldots i_n} X_i$$

which is to say, arbitrary open sets at a finite number of components and the full space in all other components.

it is also worth noting that The product topology is the coarsest topology on X such that each projection map

$$\pi_j : X \to X_j$$

is continuous.

we also need the notion of **Axiom of Choice**. Which states that: For any collection $\mathcal{C}$ of non-empty sets, there's is a set that contains exactly one element for each $A \in \mathcal{C}$.

Definition 66. A **Partially Ordered Set** is a pair $(P, \leq)$ such that.

1. $x \leq x \ \forall\, x \in P$.
2. $x \leq y$ and $y \leq z \implies x \leq z$.
3. If $x \leq y$ and $y \leq x$, then $x = y$

a totally ordered set also satisfies: $\forall\, x, y \in P$

$$x \leq y \text{ or } y \leq x$$

Definition 67. A chain in P is a subset $\mathcal{C}$ of P that is totally ordered in the partial order of P.

Definition 68. We say that P is inductively ordered if for any chain $\mathcal{C}$ in P there is an $a \in P$, possibly in $\mathcal{C}$, such that $c \leq a \ \forall\, c \in \mathcal{C}$ so a is an upper bound for $\mathcal{C}$.

i.e. a partially ordered set P is inductively ordered if every chain has an upper bound.

Definition 69. $m \in P$ is a maximal element if $a \geq m \implies a = m$. Not unique, can have many maximal elements.

The last key element we need before tackling Tychonoff's Theorem is **Zorn's Lemma**. Which says: if a partially ordered set P is inductively ordered then P has at least one maximal element.

Proposition 70. Properties of maximal FIP family $\mathcal{F}^*$

(a) $\mathcal{F}^*$ is closed/stable under finite intersections.

(b) If $B \subseteq X$ and $B \cap A \neq \varnothing$, $\forall\ A \in \mathcal{F}^*$ then $B \in \mathcal{F}^*$.

Proof.

(a) Given $B, C \in \mathcal{F}^*$, then taking finite $A_1, \dots, A_k \in \mathcal{F}^*$ we have by FIP,

$$(B \cap C) \bigcap (A_1 \cap \cdots \cap A_k) \neq \varnothing$$

and so $\mathcal{F}^* \cup \{B \cap C\}$ is an FIP family, yet by the maximality of $\mathcal{F}^*$ we must have

$$\mathcal{F}^* = \mathcal{F}^* \cup \{B \cap C\}$$

and so $B \cap C \in \mathcal{F}^*$, and $\mathcal{F}^*$ is stable under finite intersections.

(b) Consider $\mathcal{F}^{'} = \mathcal{F}^* \cup \{B\}$. Then, $\mathcal{F}^{'}$ has FIP, as any finite subcollection of $\mathcal{F}^{'}$ is either of the form

$$A_1, \dots, A_n$$

which has nonempty intersection, or

$$B, A_1, \dots, A_n$$

where

$$B \bigcap \big(\overbrace{\cap_{j=1}^n A_j}^{\in \mathcal{F}^*}\big) \neq \varnothing$$

and thus by maximality $\mathcal{F}^* = \mathcal{F}^{'}$, otherwise $\mathcal{F}^{'}$ would be a larger set with the FIP property and $\mathcal{F}^*$ would not be maximal. Thus, $B \in \mathcal{F}^*$.

$\square$

Theorem 71 (Tychonoff's Theorem). Let I be some index set. For each $i \in I$ let (X_i, τ_i) be a topological space. If all the (X_i, τ_i)'s are compact then

$$X = \prod_{i \in I} X_i$$

with the product topology is compact. (Need the axiom of choice)

Proof. First, given a set $X \neq \varnothing$ and some FIP family of closed subsets $\mathcal{S}$ on X, consider as a partially ordered set

$$\mathcal{W} := \Big\{\mathcal{F} \subseteq \mathcal{P}(X) : \mathcal{S} \subseteq \mathcal{F}; \mathcal{F} \text{ is an FIP family on } X\Big\}$$

with the partial ordering on $\mathcal{W}$ given by set inclusion, and note that $\mathcal{S} \in \mathcal{W} \implies \mathcal{W} \neq \varnothing$. Now let $\mathcal{C}$ be a non-empty chain in $\mathcal{W}$, so that $\mathcal{C}$ is a collection of FIP families in $\mathcal{W}$ and is totally ordered by inclusion. Let us set

$$\mathcal{F}_0 = \bigcup_{\mathcal{F} \in \mathcal{C}} \mathcal{F}$$

so let $n \in \mathbb{N}$ and $A_1, \ldots, A_n$ be subsets of X such that $A_1, \ldots, A_n \in \mathcal{F}_0$. Since $\mathcal{F}_0$ is the union of elements in $\mathcal{C}$, for $A_i \in \mathcal{F}_0$ we must have $A_i \in \mathcal{F}$ for some $\mathcal{F} \in \mathcal{C}$, and so, for each $i \in \{1, \ldots, n\}$ $\exists$ $\mathcal{F}_i \in \mathcal{C}$ such that $A_i \in \mathcal{F}_i$ for each i. Then, in particular,

$$\{\mathcal{F}_1, \ldots, \mathcal{F}_n\} \in \mathcal{C}$$

and hence is totally ordered by set inclusion, and so one of $\{\mathcal{F}_1, \ldots, \mathcal{F}_n\}$ must be maximal, let this be $\mathcal{F}_j$ so that

$$\mathcal{F}_j \supseteq \mathcal{F}_i, \text{ for } 1 \leq i \leq n$$

and thus $A_1, \ldots, A_n \in \mathcal{F}_j$, and since $\mathcal{F}_j$ is an FIP family we have

$$\bigcap_{i=1}^{n} A_i \neq \varnothing$$

and since each $\mathcal{F} \supseteq \mathcal{S}$ we trivially have that $\mathcal{F}_0 \supseteq \mathcal{S}$ and so $\mathcal{F}_0 \in \mathcal{W}$ and $\mathcal{F}_0 = \bigcup_{\mathcal{F} \in \mathcal{C}} \mathcal{F}$ is an upper bound for the chain $\mathcal{C}$.
Since the chain $\mathcal{C} \in \mathcal{W}$ was arbitrary we conclude that every chain in $\mathcal{W}$ has an upper bound in $\mathcal{W}$, and hence $\mathcal{W}$ is inductively ordered.

Thus, by Zorn's Lemma $\mathcal{W}$ has a maximal element $\mathcal{F}^*$ which contains $\mathcal{S}$.

Now, for each $i \in I$ consider

$$\mathcal{F}_i = \{\pi_i(A) : A \in \mathcal{F}^*\}$$

then $\mathcal{F}_i \subseteq \mathcal{P}(X_i)$, now for $A_1, \ldots, A_n \in \mathcal{F}^*$ we have

$$\bigcap_{j=1}^{n} A_j \neq \varnothing$$

which implies that there exists at least one $x \in \bigcap_{j=1}^{n} A_i$, and so

$$\pi_i(x) \in \pi_i\left(\bigcap_{j=1}^{n} A_j\right) \subseteq \bigcap_{j=1}^{n} \pi_i(A_j)$$

and so $\mathcal{F}_i$ is an FIP family on X_i, and since each $\pi_i(A_j) \subseteq \overline{\pi_i(A_j)}$ we also have that

$$\left\{\overline{\pi_i(A)} : A \in \mathcal{F}^*\right\}$$

is an FIP family on X_i of closed subsets, and since X_i is compact we have that

$$\bigcap_{A \in \mathcal{F}^*} \overline{\pi_i(A)} \neq \varnothing$$

and so by the axiom of choice we may select $x_i \in \bigcap_{A\in\mathcal{F}^*} \overline{\pi_i(A)} \subseteq X_i$ and set

$$x = (x_i) \in \prod_{i\in I} X_i$$

and let O_x be an open neighbourhood of x in X. It suffices to consider O_x as a basis element of X so that

$$x \in O_x = \prod_{i_j \neq \{i_1,\ldots,i_k\}} X_{i_j} \times \prod_{j=1}^{k} U_{i_j}$$

or. equivalently

$$x \in O_x = \bigcap_{j=1}^{k} \pi_{i_j}^{-1}(U_{i_j})$$

and note for each $j \in \{1,\ldots,k\}$ we have $x_{i_j} \in U_{i_j}$ and by construction $x_{i_j} \in \bigcap_{A\in\mathcal{F}^*} \overline{\pi_{i_j}(A)}$ and since $U_{ij} \subseteq X_{i_j}$ is open and contains x_{i_j} by the definition of a limit point we must have that $U_{i_j} \cap \pi_{i_j}(A) \neq \varnothing$ for each $A \in \mathcal{F}^*$ and hence

$$\pi_{i_j}^{-1}(U_{i_j}) \cap A \neq \varnothing, \quad \forall\ A \in \mathcal{F}^*$$

and hence $\pi_{i_j}^{-1}(U_{i_j}) \in \mathcal{F}^*$ by maximality for each $j \in \{1,\ldots,k\}$. Where maximality then gives

$$\bigcap_{j=1}^{k} \pi_{i_j}^{-1}(U_{i_j}) = O_x \in \mathcal{F}^*$$

and therefore $O_x \cap A \neq \varnothing$, $\forall\ A \in \mathcal{F}^*$, and in particular since $\mathcal{S} \subseteq \mathcal{F}^*$ we have that $O_x \cap A \neq \varnothing$, $\forall\ A \in \mathcal{S}$ and hence

$$\bigcap_{A\in\mathcal{S}} A \neq \varnothing$$

and thus, X is compact. □

not only does Tychonoff's theorem tell us about the infinite product of compact spaces which requires the axiom of choice, but given Tychonoff's theorem we can then get the axiom of choice!

Theorem 72. Tychonoff's Theorem implies the Axiom of Choice.

Proof. Let $\{X_i\}_{i\in I}$ be a non-empty family and let

$$X = \prod_{i\in I} X_i$$

let ω be some set not in X.

Next, for each i set $Y_i = X_i \cup \{\omega\}$ and define

$$\tau_{Y_i} = \{Y_i, X_i, \{\omega\}, \varnothing\}$$

then (Y_i, τ_{Y_i}) is finite and hence compact. So let

$$Y = \prod_{i \in I} Y_i$$

which is then compact by Tychonoff's Theorem.

Since $\omega \in Y_i$ is open, this implies $\omega^c = X_i$ is closed in Y_i, and hence is clopen. So by the continuity of the projection maps π_i we have

$$\pi_i^{-1}(X_i) \subseteq Y$$

is closed for each i. To see that $\{\pi_i^{-1}(X_i)\}$ has FIP, let $\pi_{i_1}^{-1}(X_{i_1}), \ldots, \pi_{i_n}^{-1}(X_{i_n}) \subset \{\pi_i^{-1}(X_i)\}$ be given and note that $\exists\ x_{i_j} \in X_{i_j}\ \forall\ i_j$, so define $y \in Y$ by

$$y_i = \begin{cases} x_{i_j}, & i = i_j \\ \omega, & i \neq i_j\ \forall\ j \end{cases}$$

then

$$y \in \bigcap_{j=i}^{n} \pi_{i_j}^{-1}(X_{i_j}) \implies \{\pi_i^{-1}(X_i)\} \text{ is FIP}$$

then since $\{\pi_i^{-1}(X_i)\}$ is an FIP family and Y is compact this gives

$$\bigcap_{i \in I} \pi_i^{-1}(X_i) \neq \varnothing$$

so let $z \in \bigcap_{i \in I} \pi_i^{-1}(X_i)$, then $z \in X_i$ for each i and therefore

$$z \in \prod_{i \in I} X_i$$

□

Definition 73. Suppose X is a topological space. If for every pair of distinct points $x, y \in X\ \exists\ U, V \subset X$ open, such that $U \cap V = \varnothing$ and $x \in U,\ y \in V$, then X is hausdorff.

Definition 74. Suppose that one-point sets are closed in (X, τ). Then X is said to be regular if for each pair consisting of a point x and a closed set $A \subset X$ such that $A \cap x = \varnothing$, there exist $U, V \in \tau$ where $U \cap V = \varnothing$, such that

$$x \in U, \text{ and } A \subset V$$

I.e. a space X is regular if, for closed $A \subseteq X$ with $x \notin A$, $\exists$ disjoint $U, V \in \tau$ with $x \in U$ and $A \subseteq V$. Now clearly Hausdorff spaces are Regular.

Proposition 75. If (X, τ) is compact and Hausdorff, then it is normal.

Proof. Let $A, B \subseteq X$ be closed and disjoint. Since X is compact and A, B are closed subsets of a compact space we have that A, B are also compact. Since X is Hausdorff, it is regular. Thus, for $x \in A \ \exists \ U_x, V_x \in \tau$ disjoint with

$$x \in U_x \quad \text{and} \quad B \subseteq V_x$$

then $\{U_x\}_{x \in A}$ is an open cover for A, and by compactness of A admits a finite subcover giving

$$A \subseteq \bigcup_{i=1}^{n} U_{x_i} =: U$$

and

$$V := \bigcap_{i=1}^{n} V_{x_i} \supseteq B$$

which are both open as the union and finite intersection of open sets, where $U \cap V = \varnothing$. Hence, X is normal. $\square$

Theorem 76. If (X, τ_X) is compact and (Y, τ_Y) is Hausdorff, and if

$$f : X \to Y$$

is continuous, injective and surjective. Then f is a homeomorphism.

Proof. Since f is continuous, injective and surjective, we have

$$f^{-1} : Y \to X$$

exists, so let $A \subseteq X$ be closed, then A is compact as the closed subset of a compact space, and by the continuity of f we also have that $F(A) \subseteq Y$ is compact. Since Y is Hausdorff $f(A)$ is closed as a compact set in a Hausdorff space. Since f is injective and surjective we also have

$$f(A)^c = Y \setminus f(A) = f(X) \setminus f(A) = f(X \setminus A) = f(A^c)$$

where $f(a)^c = f(A^c)$ is open in Y and so

$$f^{-1}\big(f(A^c)\big) = A^c \subseteq X$$

is open and thus f^{-1} is continuous. Therefore, f is a homeomorphism. $\square$

On the way to Arzela-Ascoli's Theorem we will need a few more definitions.

Definition 77. let (X, τ) be a topological space and (Y, d) a metric space, and let $\mathcal{F} \subseteq C(X, Y)$. Then $\mathcal{F}$ is equicontinuous at x if $\forall \ \epsilon > 0 \ \exists \ O_x \in \tau$ such that $\forall \ f \in \mathcal{F}$ and any $y \in O_x$ we have

$$d\big(f(x), f(y)\big) < \epsilon$$

$\mathcal{F}$ is equicontinuous if it is equicontinuous at x, $\forall \ x \in X$.

Definition 78. let (X, d) be a metric space a subset A is totally bounded if $\forall\ \epsilon > 0$, A can be covered by a finite number of open ϵ-balls; i.e.

$$A \subseteq \bigcup_{i=1}^{n} B_\epsilon^i$$

And it should be noted that any subset of a totally bounded set is totally bounded.

Definition 79. let (X, τ) be a topological space and (Y, d) a metric space. Given $\epsilon > 0$ and $x \in X$ if $\exists\ g_j \in C_B(X, Y)$ such that

$$d\big(f(x), g_j(x)\big) < \epsilon$$

Then $\big\{B_\epsilon\big(g_j(x)\big)\big\}_{j=1}^n$ covers $\{f(x) : f \in \mathcal{F}\}$ and so $\mathcal{F}$ is pointwise totally bounded.

Proposition 80. let (X, d) be a metric space and $A \subseteq X$ be totally bounded, then $\overline{A}$ is totally bounded.

Proof. Let $\epsilon > 0$ be given, since A is totally bounded $\exists\ x_i, \ldots, x_n \in A$ such that $\{B_{\frac{\epsilon}{2}}(x_i)\}_{i=1}^n$ cover A. For each $z \in \overline{A}\ \exists\ y \in A$ such that $z \in B_{\frac{\epsilon}{2}}(y)$, by the definition of a limit point, and there is some j such that $y \in B_{\frac{\epsilon}{2}}(x_j)$ since the $B_{\frac{\epsilon}{2}}(x_j)$'s cover A and so

$$d(z, x_j) \leq d(z, y) + d(y, x_j) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and so $z \in B_\epsilon(x_j)$ and hence $\{B_\epsilon(x_i)\}_{i=1}^n$ cover $\overline{A}$. □

Proposition 81. let (X, d) be a metric space. If X is compact, then it is complete.

Proof. Let $\{x_n\} \in X$ be a cauchy sequence and suppose, for contradiction, that X is not complete. Then $\{x_n\}$ does not converge in X. So $\forall x \in X\ \exists\ \epsilon_x > 0$ such that $\forall\ N \in \mathbb{N}\ \exists\ n \geq N$ where $d(x, x_n) \geq \epsilon_x$.

Then since $\{x_n\}$ is cauchy $\exists\ M \in \mathbb{N}$ such that

$$n, m \geq M \implies d(x_n, x_m) < \epsilon_x$$

pick $M_x > M$ such that $n_x \geq M_x$ gives $d(x, x_{n_x}) \geq \epsilon_x$. So for $n > M_x$ we have $d(x, x_n) \geq \frac{\epsilon_x}{2}$. Thus, $\forall\ x \in X$, $B_{\epsilon_x}(x)$ contains at most finite $x_i \in \{x_n\}$. Now $\{B_{\epsilon_x}(x)\}_{x \in X}$ cover X, yet it does not admit a finite subcover, contradicting the compactness of X. □

Theorem 82. let (X, d) be a complete metric space. If X is totally bounded, then it is compact.

Proof. Let $\mathcal{U}$ be an open cover of X, and since X is totally bounded let $\overline{B}_1^1, \ldots, \overline{B}_n^1$ be a finite cover of X by closed balls of radius 1. Suppose, for contradiction, that X is not compact. So at least one ball say A^1 has no finite subcover and let $\overline{B}_1^2, \ldots, \overline{B}_{n_2}^2$ be closed balls of radius $\frac{1}{2}$ covering A^1, then at least one, say B_*^2 has no finite subcover so let

$$A^2 = A^1 \cap B_*^2$$

let $\overline{B}_1^3, \ldots, \overline{B}_{n_3}^3$ be closed balls of radius $\frac{1}{4}$ covering A^2, then at least one has no finite subcover, say B_*^3 so let

$$A^3 = A^2 \cap B_*^3$$

continuing inductively we get a sequence $\{A^n\}$ such that

$$A^{n+1} \subseteq A^n \quad \forall\, n$$

and

$$\operatorname{diam}(A^n) \to 0$$

and each A^n is not finitely covered.

For each n select $x_n \in A^n$, then $\{x_n\}$ is cauchy, and by the completeness of X, $\exists\, x \in X$ such that $x_n \to x$. Since $\mathcal{U}$ covers X there exists $U \in \mathcal{U}$ such that $x \in U$, then given $\epsilon > 0$ $\exists\, B_\epsilon(x) \subseteq U$. So choose n such that $\operatorname{diam}(A^n) < \epsilon$ then

$$A^n \subset B_\epsilon(x) \subseteq U \qquad \Rightarrow\Leftarrow$$

contradicting the assumption that A^n was not finitely covered. □

Theorem 83 (**Arzela-Ascoli**)**.** Let (X, τ) be a compact topological space, (Y, d) be a metric space, and let $\mathcal{F} \subseteq C_B(X, Y)$ be pointwise totally bounded and equicontinuous, then $\mathcal{F}$ is totally bounded for d_∞.

Proof. let $\epsilon > 0$ be given. Since $\mathcal{F}$ is equicontinuous $\forall\, x \in X$, $\exists\, O_x \ni x$ such that

$$y \in O_x \implies d\big(f(x), f(y)\big) < \epsilon \quad \forall\, f \in \mathcal{F}$$

since X is compact $\exists\, x_1, \ldots, x_n \in X$ such that

$$X \subseteq \bigcup_{i=1}^{n} O_{x_i}$$

for each j, since $\mathcal{F}$ is pointwise totally bounded, we have $\{f(x_j) : f \in \mathcal{F}\}$ is totally bounded. Let

$$S_j \subseteq \{f(x_j) : f \in \mathcal{F}\} \subseteq Y$$

be a finite subset such that

$$\bigcup_{y \in S_j} B_\epsilon(y) \supseteq \{f(x_j) : f \in \mathcal{F}\}$$

and let

$$S = \bigcup_{j=1}^{n} S_j$$

also let

$$\Psi = \{\psi : \{1, \ldots, n\} \to S\}$$

which is finite and set

$$B_\psi = \big\{ f \in \mathcal{F} : d\big(f(x_j), \psi(j)\big) < \epsilon \ \forall\ j \big\}$$

then

$$\mathcal{F} = \bigcup_{\psi \in \Psi} B_\psi$$

So let $\psi \in \Psi$ be given and let $f, g \in B_\psi$, and $x \in X$ be such that $x \in O_{x_j}$ for some j, then

$$\begin{aligned}
d\big(f(x), g(x)\big) &\leq d\big(f(x), f(x_j)\big) + d\big(f(x_j), g(x_j)\big) + d\big(g(x_j), g(x)\big) \\
&\leq d\big(f(x), f(x_j)\big) + d\big(f(x_j), \psi(j)\big) + d\big(\psi(j), g(x_j)\big) + d\big(g(x_j), g(x)\big) \\
&= \epsilon + \epsilon + \epsilon + \epsilon \\
&= 4\epsilon
\end{aligned}$$

and therefore

$$B_\psi \subseteq \bigcup_{y \in B_\psi} B_{4\epsilon}(y)$$

and since Ψ is finite, $\mathcal{F}$ is totally bounded. $\square$

Corollary 84. Let (X, τ) be a compact topological space, (Y, d) be a complete metric space, and let $\mathcal{F} \subseteq C_B(X, Y)$. Then $\mathcal{F}$ is compact iff it is pointwise totally bounded, equicontinuous, and closed in $C_B(X, Y)$.

Definition 85. let (X, τ) be a topological space. then X is locally compact if $\forall\ x \in X,\ \exists\ O \in \tau$ with $x \in O$ such that $\overline{O}$ is compact.

Proposition 86. Let (X, τ) be a locally compact topological space, and let $C \subseteq X$ be compact. Then $\exists\ O \in \tau$ such that $C \subseteq O$ where $\overline{O}$ is compact.

Proof. $\forall\ x \in C$, by local compactness $\exists\ O_x \in \tau$ with $x \in O_x$ such that $\overline{O}_x$ is compact. Then $\{O_x\}_{x \in C}$ is an open cover for C, and since C is compact it admits a finite subcover and so

$$C \subseteq \bigcup_{i=1}^{n} O_{x_i} \subseteq \bigcup_{i=1}^{n} \overline{O}_{x_i} \subseteq \overline{\bigcup_{i=1}^{n} O_{x_i}}$$

which is compact as the finite union of compact sets. $\square$

Proposition 87. Let (X, τ) be a locally compact Hausdorff space. Then every $x \in X$ has a neighborhood base consisting of compact neighborhoods; i.e. $\forall\ x \in O_x\ \exists\ U \in \tau$, with $x \in U$ such that $\overline{U} \subseteq O_x$ where $\overline{U}$ is compact.

Proof. Given $x \in O_x$, let $V \in \tau$ with $x \in V$ where $\overline{V}$ is compact by local compactness. Then we can replace O_x with

$$O = O_x \cap V \subseteq V$$

so that $\overline{O}$ is compact as a closed subset of a compact set. Let

$$\partial O := \overline{O} \setminus O$$

which is closed in the relative topology of $\overline{O}$, since $O \notin \partial O \implies x \notin \partial O$. Since $\overline{O}$ is compact Hausdorff, it is normal, and hence regular. So $\exists\ U, W$ relatively open in $\overline{O}$ such that $U \cap W = \varnothing$ with

$$x \in U \text{ and } \partial O \subseteq W$$

then

$$U \cap W = \varnothing \implies W^c = \overline{O} \setminus W \supseteq U$$

and since $W \supseteq \partial O \implies W^c \subseteq \partial O^c$, which then implies that $W^c \subseteq O$
Now $\overline{O} \setminus W$ is relatively closed in $\overline{O}$, which gives

$$\overline{U} \subseteq \overline{O} \setminus W = W^c \subseteq O$$

so $\overline{U} \subseteq O$ and hence is compact as a closed subset of a compact set. $\square$

Proposition 88. Let (X, τ) be a locally compact Hausdorff space, and let $C \subseteq X$ be compact, and $O \in \tau$ with $C \subseteq O$. Then $\exists$ open U such that

$$C \subseteq U \subseteq \overline{U} \subseteq O$$

with $\overline{U}$ compact.

Proof. Since X is a locally compact hausdorff space and $C \subseteq X$ is compact we can find $V \in \tau$ such that $C \subseteq V$ with $\overline{V}$ compact. Then we have both $C \subseteq V$ and $C \subseteq O$ so let

$$W = V \cap O$$

then $C \subseteq W$ and since

$$V \cap O \subseteq V \implies W \subseteq V$$

and so $\overline{W} \subseteq \overline{V}$ which tells us that $\overline{W}$ is compact as the closed subset of a compact set. Then ∂W is closed in the relative topology of $\overline{W}$ and since $\partial W = \overline{W} \setminus W$ we have that $C \nsubseteq \partial W$, and since X is Hausdorff, $\overline{W}$ is compact Hausdorff, and so it is normal. Then as $C, \partial W$ are closed and disjoint, by normality $\exists$ disjoint $U, Q \in \tau$ such that

$$C \subseteq U, \text{ and } \partial W \subseteq Q$$

then since $U \cap Q = \varnothing$ we have $Q^c \supseteq U$ and also

$$U \subseteq Q^c \cap \overline{W}$$

which implies

$$\overline{U} \subseteq \overline{Q^c \cap \overline{W}} = Q^c \cap \overline{W}$$

since both $Q^c, \overline{W}$ are closed, and the intersection of closed sets is closed. Next we note that $Q^c \cap \overline{W} \subseteq Q^c$ and $\partial W^c \supseteq Q^c$, and in the relative topology of $\overline{W}$ we have

$$\partial W^c = \left(\overline{W} \cap W^c\right)^c \cap \overline{W} = \left(\overline{W}^c \cup W\right) \cap \overline{W} = W$$

and so we have

$$\overline{U} \subseteq Q^c \subseteq \partial W^c = W$$

and so $\overline{U}$ will be compact as the closed subset of compact $\overline{W}$. And so we have

$$C \subseteq U \subseteq \overline{U} \subseteq W \subseteq O$$

and hence

$$C \subseteq U \subseteq \overline{U} \subseteq O$$

□

Proposition 89 (**Urysohn for Locally Compact Hausdorff**)**.** Let (X, τ) be a locally compact Hausdorff space, and let $C \subseteq X$ be compact, and $O \in \tau$ with $C \subseteq O$. Then $\exists$ continuous $f : X \to [0,1]$ such that $f(C) = \{1\}$, and $\mathrm{supp}(f) = \overline{\{x : f(x) \neq 0\}} \subseteq O$ is compact.

Proof. Since X is locally compact Hausdorff and $C \subseteq X$ is compact, we may choose $U \in \tau$ such that

$$C \subseteq U \subseteq \overset{\text{compact}}{\overline{U}} \subseteq O$$

where $C, \partial U$ are closed and disjoint in compact $\overline{U}$, so by Urysohn's Lemma $\exists$ continuous $g : \overline{U} \to [0,1]$ with $g(C) = \{1\}$ and $g(\partial U) = \{0\}$. So set

$$f : X \to [0,1], \text{ by } f(x) = \begin{cases} g(x), & x \in \overline{U} \\ 0, & x \notin \overline{U} \end{cases}$$

then $\mathrm{supp}(f) \subseteq \overline{U}$ and is compact as the closed subset of a compact set. So we need to check that f is continuous on X. f is continuous on $\overline{U}$ and continuous on $\overline{U}^c$, if $x \in \partial U$, then $f(x) = g(x) = 0$. Now $[0, \epsilon)$ is open in $[0,1]$, where the continuity of g tells us that $g^{-1}\big([0,\epsilon)\big)$ is open in $\overline{U}$. And so

$$f^{-1}\big([0,\epsilon)\big) = g^{-1}\big([0,\epsilon)\big) \cup \overline{U}^c$$

is open as the union of open sets, and so f is continuous. □

3 Measure Theory

We begin with a few definitions, and remind ourselves that these definitions should not be confused with definitions of those with identical names coming from algebra.

Definition 90. Let X be a set, a nonempty collection of subsets $\mathcal{R} \subseteq \mathcal{P}(X)$ is a Ring if

1. $E, F \in \mathcal{R} \implies E \cup F \in \mathcal{R}$. Closure under set union.
2. $E, F \in \mathcal{R} \implies E \setminus F \in \mathcal{R}$. Closure under set difference.

The second condition also implies that $\mathcal{R}$ is closed under intersection as

$$\begin{aligned} E \setminus (E \setminus F) &= E \setminus (E \cap F^c) \\ &= E \cap \big(E \cap F^c\big)^c \\ &= E \cap (E^c \cup F) \\ &= (E \cap E^c) \cup (E \cap F) \\ &= \varnothing \cup (E \cap F) \\ &= (E \cap F) \end{aligned}$$

This also implies, by induction, that a ring $\mathcal{R}$ is closed under finite unions and intersections; i.e. if $E_1, \ldots, E_n \in \mathcal{R}$ then

$$\bigcup_{i=1}^{n} E_i \in \mathcal{R}$$

and

$$\bigcap_{i=1}^{n} E_i \in \mathcal{R}$$

as well as $\varnothing \in \mathcal{R}$. Since if $E \in \mathcal{R}$ then

$$E \setminus E = \varnothing \in \mathcal{R}$$

If, in addition, $X \in \mathcal{R}$, then $\mathcal{R}$ is a **Field** or **Algebra**.

Definition 91. Let X be a set, a nonempty collection of subsets $\mathcal{S} \subseteq \mathcal{P}(X)$ is a σ-ring if it is a ring and, in addition, is closed under countable unions; i.e. if $E_1, E_2, \dots \in \mathcal{S}$ then

$$\bigcup_{i=1}^{\infty} E_i \in \mathcal{S}$$

where this also implies closure under countable intersection since if $F = \bigcup_{i=1}^{\infty} E_i$ then

$$\bigcap_{i=1}^{\infty} E_i = F \setminus \left(\bigcup_{i=1}^{\infty} (F \setminus E_i) \right)$$

If, in addition, $X \in \mathcal{S}$, then $\mathcal{S}$ is a **σ-Field** or **σ-Algebra**, which also has an equivalent definition.

Definition 92. Let X be a set, a collection of subsets $\mathcal{A} \subseteq \mathcal{P}(X)$ is a σ-algebra in X if it satisfies

1. Nonemptiness: $\mathcal{A} \neq \varnothing$.
2. Closure under Compliments: If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$.
3. Closure under Countable Unions: If $A_1, A_2 \cdots \in \mathcal{A}$, then
$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$$

Where this also implies closure under countable intersection as
$$\bigcap_{i=1}^{\infty} A_i = \left(\bigcup_{i=1}^{\infty} A_i^c \right)^c \in \mathcal{A}$$

Definition 93. Let X be a set and $\mathcal{S}$ a collection of subsets of X, then the σ-algebra generated by $\mathcal{S}$ is the intersection of all σ-algebras containing $\mathcal{S}$ denoted $\sigma(\mathcal{S})$; that is
$$\sigma(\mathcal{S}) = \bigcap_{\mathcal{S} \subseteq \mathcal{A}} \mathcal{A}$$

Definition 94. Let (X, τ) be a topological space, then $\sigma(\tau)$ is the σ-ring of Borel sets of X.

Proposition 95. The intersection of any collection of rings/fields/σ-algebras/σ-rings on a set X is a ring/field/σ-algebra/σ-ring on X.

Proof. We give a proof for rings with the proofs for the others being similar.

Let $\{\mathcal{R}_i\}_{i \in I}$ be a collection of rings on X where I is an indexing set and let
$$\mathcal{R} = \bigcap_{i \in I} \mathcal{R}_i$$
so if $E, F \in \mathcal{R}$, then $E, F \in \mathcal{R}_i, \ \forall\ i \in I$ and since each $\mathcal{R}_i$ is a ring we have
$$E \cup F \in \mathcal{R}_i, \quad \forall\ i \in I$$
and
$$E \setminus F \in \mathcal{R}_i, \quad \forall\ i \in I$$
and thus $E \cup F, E \setminus F \in \mathcal{R}$, and so $\mathcal{R}$ is a ring. □

Theorem 96. Let $\mathcal{P} = \{[a, b) : a < b; a, b \in \mathbb{R}\}$ and let $\alpha : \mathbb{R} \to \mathbb{R}$ be a non-decreasing left continuous function and define
$$\mu_\alpha : \mathcal{R} \to \mathbb{R}, \text{ by } \mu_\alpha\big([a, b)\big) = \alpha(b) - \alpha(a)$$
then μ_α is countably additive.

Proof. Given $[a_0, b_0) \in \mathcal{P}$ such that

$$[a_0, b_0) = \bigsqcup_{i=1}^{\infty} [a_i, b_i)$$

we note that for the ($\geq$) direction it suffices to show that for each $n \in \mathbb{N}$ we have

$$\mu_\alpha\big([a_0, b_0)\big) \geq \sum_{i=1}^{n} \mu_\alpha\big([a_i, b_i)\big)$$

Given any n, re-index the intervals so that $a_i < a_{i+1} \ \forall\ 1 \leq i \leq n-1$. Since the intervals are disjoint, we have that $b_i < a_{i+1}$. Now since

$$a_0 \leq a_i,\ \ b_i \leq b_0 \quad \forall\ i$$

we have

$$\alpha(b_0) - \alpha(a_0) \geq \alpha(b_n) - \alpha(a_1)$$

then

$$\begin{aligned}
\sum_{i=1}^{n} \mu_\alpha\big([a_i, b_i)\big) = \sum_{i=1}^{n} \big(\alpha(b_i) - \alpha(a_i)\big) \\
&= \alpha(b_1) - \alpha(a_1) + \alpha(b_2) - \alpha(a_2) + \cdots + \alpha(b_n) - \alpha(a_n) \\
&= \alpha(b_n) - \alpha(a_1) + \alpha(b_1) - \alpha(a_2) + \cdots + \alpha(b_{n-1}) - \alpha(a_n) \\
&= \alpha(b_n) - \alpha(a_1) + \sum_{i=1}^{n-1} \big(\alpha(b_i) - \alpha(a_{i+1})\big)
\end{aligned}$$

and since each $b_i < a_{i+1}$ and α is non-decreasing we have that $\sum_{i=1}^{n-1} \big(\alpha(b_i) - \alpha(a_{i+1})\big) \leq 0$ and therefore

$$\mu_\alpha\big([a_0, b_0)\big) = \alpha(b_0) - \alpha(a_0) \geq \alpha(b_n) - \alpha(a_1) \geq \sum_{i=1}^{n} \mu_\alpha\big([a_i, b_i)\big)$$

Next, let $\epsilon > 0$ be given and choose $b_0' < b_0$ such that

$$\alpha(b_0') \geq \alpha(b_0) - \frac{\epsilon}{2}$$

and by the left continuity of α for each i choose $a_i' < a_i$ such that

$$\alpha(a_i') \geq \alpha(a_i) - \epsilon_i$$

where each $\epsilon_i > 0$ such that $\sum_{i=1}^{\infty} \epsilon_i = \frac{\epsilon}{2}$. Then we have

$$[a_0, b_0'] \subseteq [a_0, b_0) = \bigsqcup_{i=1}^{\infty} [a_i, b_i) \subseteq \bigcup_{i=1}^{\infty} (a_i', b_i)$$

then, since $\bigcup_{i=1}^{\infty}(a_i', b_i)$ is an open cover of $[a_0, b_0']$ which is compact, we know that $[a_0, b_0']$ admits a finite subcover, so that

$$[a_0, b_0'] \subseteq \bigcup_{i=1}^{m}(a_i', b_i)$$

then re-indexing the intervals so that

$$a_0 \in (a_1', b_1) \text{ and } b_1 \in (a_2', b_2), \cdots, b_0' \in (a_m', b_m)$$

then

$$\begin{aligned}
\alpha(b_0) - \alpha(a_0) &\leq \alpha(b_0') + \frac{\epsilon}{2} - \alpha(a_0) && b_0' < b_0 \\
&\leq \alpha(b_m) - \alpha(a_1') + \frac{\epsilon}{2} && b_0' \leq b_m,\ a_1' \leq a_0 \\
&\leq \alpha(b_m) - \alpha(a_1') + \frac{\epsilon}{2} + \sum_{i=1}^{m-1}\big(\alpha(b_i) - \alpha(a_{i+1}')\big) && b_i \geq a_{i+1}' \\
&= \sum_{i=1}^{m}\big(\alpha(b_i) - \alpha(a_i')\big) + \frac{\epsilon}{2} \\
&\leq \sum_{i=1}^{m}\big(\alpha(b_i) - (\alpha(a_i) - \epsilon_i)\big) + \frac{\epsilon}{2} \\
&= \sum_{i=1}^{m}\big(\alpha(b_i) - \alpha(a_i) + \epsilon_i\big) + \frac{\epsilon}{2} \\
&\leq \sum_{i=1}^{\infty}\big(\alpha(b_i) - \alpha(a_i)\big) + \sum_{i=1}^{\infty}\epsilon_i + \frac{\epsilon}{2} \\
&= \sum_{i=1}^{\infty}\big(\alpha(b_i) - \alpha(a_i)\big) + \epsilon
\end{aligned}$$

and since ϵ was arbitrary we conclude

$$\mu_\alpha\big([a_0, b_0)\big) = \alpha(b_0) - \alpha(a_0) \leq \sum_{i=1}^{\infty}\big(\alpha(b_i) - \alpha(a_i)\big) = \sum_{i=1}^{\infty}\mu_\alpha\big([a_i, b_i)\big)$$

and thus we conclude that $\mu_\alpha\big([a_0, b_0)\big) = \sum_{i=1}^{\infty}\mu_\alpha\big([a_i, b_i)\big)$. And so μ_α is countably additive. $\square$

Definition 97. Let X be a set, a collection of subsets $\mathcal{S} \subseteq \mathcal{P}(X)$ is a semiring if

1. $\varnothing \in \mathcal{S}$.
2. If $E, F \in \mathcal{S} \implies E \cap F \in \mathcal{S}$.

3. If $E, F \in \mathcal{S}$ then $\exists\ E_1, \ldots, E_n \in \mathcal{S}$ such that

$$E \setminus F = \bigsqcup_{i=1}^{n} E_i$$

Lemma 98. Let $\mathcal{S}$ be a semiring. If $E, E_1, \ldots, E_n \in \mathcal{S}$, then $\exists\ F_1, \ldots, F_k \in \mathcal{S}$ such that

$$((\ldots(E \setminus E_1) \setminus E_2) \setminus \ldots) \setminus E_n) = \bigsqcup_{i=1}^{k} F_i$$

Proof. By induction. Base case: if $n = 1$ then $E \setminus E_1 = \bigsqcup_{i=1}^{k} F_i$ with $F_1, \ldots, F_k \in \mathcal{S}$ by the definition of semiring.

So suppose the result holds for $n-1$ with $n > 1$. Then $\exists\ G_1, \ldots G_m$ such that

$$\begin{aligned}
((\ldots(E \setminus E_1) \setminus E_2) \setminus \ldots) \setminus E_{n-1}) \setminus E_n) &= E \setminus \bigsqcup_{i=1}^{n} E_i \\
&= \left(E \setminus \bigsqcup_{i=1}^{n-1} E_i \right) \setminus E_n \\
&= \left(\bigsqcup_{i=1}^{m} G_i \right) \setminus E_n \\
&= \bigsqcup_{i=1}^{m} (G_i \setminus E_n) \\
&= \bigsqcup_{i=1}^{m} \bigsqcup_{j=1}^{l} G_{ij}
\end{aligned}$$

where by the definition of a semiring we have that each $G_{ij} \in \mathcal{S}$. □

Definition 99. Let X be a set with σ-ring $\mathcal{R}$. A measure is a function

$$\mu : \mathcal{R} \to [0, \infty]$$

satisfying

1. $\mu(\varnothing) = 0$.
2. **Countable Additivity**: If $E_1, E_2, \dots \in \mathcal{R}$ are mutually disjoint; i.e. $E_i \cap E_j = \varnothing$ whenever $i \neq j$. Then

$$\mu\left(\bigsqcup_{i=1}^{\infty} E_i \right) = \sum_{i=1}^{\infty} \mu(E_i)$$

This also holds for finite additivity; i.e. for $E_1, \dots E_n \in \mathcal{R}$ mutually disjoint we have $\mu\left(\bigsqcup_{i=1}^n E_i\right) = \sum_{i=1}^n \mu(E_i)$ by simply setting $E_k = \varnothing \ \forall\ k > n$.

Definition 100. Let $\mathcal{C}$ be a family of subsets of X and $\mu : \mathcal{C} \to \mathbb{R}^+$ a mapping. We say that μ is countably sub-additive if whenever $E \subseteq \bigcup_{j=1}^\infty F_j$ not necessarily disjoint with $E, \{F_j\}_{j=1}^\infty \in \mathcal{C}$, then

$$\mu(E) \leq \sum_{j=1}^\infty \mu(F_j)$$

Definition 101. Let $\mathcal{S}$ be a semiring, then the function

$$\mu_0 : \mathcal{S} \to [0, \infty]$$

is a premeasure if it is countably additive.

Lemma 102. Let $\mathcal{S}$ be a semiring, μ_0 a premeasure on $\mathcal{S}$, and let $E, F_i \in \mathcal{S}$ such that $E \subseteq \bigsqcup_{i=1}^\infty F_i$ then

$$\mu_0(E) \leq \sum_{i=1}^\infty \mu_0(F_i)$$

Proof. First we note that it is sufficient to show that

$$\mu_0(E) \leq \sum_{i=1}^n \mu_0(F_i)$$

for each finite n, that is for each $n \in \mathbb{N}$. Then

$$\bigsqcup_{i=1}^n F_i = E \sqcup \left(\bigsqcup_{i=1}^n F_i \setminus E\right) = E \sqcup \left(\bigsqcup_{i=1}^{n_1} \bigsqcup_{j=1}^{n_2} G_{ij}\right)$$

where each $G_{ij} \in \mathcal{S}$ and are disjoint by the previous Lemma, and by construction E and $\bigsqcup_{i=1}^{n_1} \bigsqcup_{j=1}^{n_2} G_{ij}$ are disjoint, so we have

$$\sum_{i=1}^n \mu_0(F_i) = \mu_0(E) + \sum_{i}^{n_1} \sum_{j=1}^{n_2} \mu_0(G_{ij}) \geq \mu_0(E)$$

□

Lemma 103. Let $\mathcal{S}$ be a semiring, μ_0 a premeasure on $\mathcal{S}$, then μ_0 is countably subadditive; i.e. if $E, F_i \in \mathcal{S}$ such that $E \subseteq \bigcup_{i=1}^\infty F_i$ then

$$\mu_0(E) \leq \sum_{i=1}^\infty \mu_0(F_i)$$

Proof. First note that

$$E = \bigcup_{i=1}^{\infty}(E \cap F_i)$$

letting $H_i = E \cap F_i$ where by definition we have that each $H_i \in \mathcal{S}$, so that by $E \setminus \bigsqcup_{i=1}^{n} E_i = \bigsqcup_{i=1}^{k} F_i$ we have

$$\begin{aligned}
E &= \bigcup_{i=1}^{\infty} H_i \\
&= H_1 \sqcup (H_2 \setminus H_1) \sqcup \cdots \sqcup \left(H_m \setminus \bigcup_{j=1}^{m-1} H_j\right) \sqcup \ldots \\
&= H_1 \sqcup \left(\bigsqcup_{i=1}^{n_1} G_{2_i}\right) \sqcup \cdots \sqcup \left(\bigsqcup_{i=1}^{n_m} G_{m_i}\right) \sqcup \ldots
\end{aligned}$$

then

$$\mu_0(E) = \mu_0(H_1) + \sum_{i=1}^{n_1} \mu_0(G_{2_i}) + \sum_{i=1}^{n_m} \mu_0(G_{m_i}) + \ldots$$

yet,

$$\bigsqcup_{i=1}^{n_m} G_{m_i} \subseteq \overbrace{E \cap F_m}^{=H_m} \subseteq F_m$$

so that $\sum_{i=1}^{n_m} \mu_0(G_{m_i}) \leq \mu_0(F_m)$, and therefore,

$$\mu_0(E) \leq \sum_{i=1}^{\infty} \mu_0(F_i)$$

□

Definition 104. Let $\mathcal{S}$ be a collection of subsets of the set X. Then $A \subset X$ is countably covered by $\mathcal{S}$ if $\exists\ \{E_i\}_{i=1}^{\infty} \in \mathcal{S}$ such that

$$A \subseteq \bigcup_{i=1}^{\infty} E_i$$

Let $\mathcal{H}(\mathcal{S})$ be the collection of all sets countably covered by $\mathcal{S}$, then $\mathcal{H}(\mathcal{S})$ is a σ-ring and is **Hereditary** meaning if $E \in \mathcal{H}(\mathcal{S})$ and $F \subseteq E$ then $F \in \mathcal{H}(\mathcal{S})$.

Definition 105. If $\mathcal{C}$ is any collection of subsets of a set X, and if $\mu : \mathcal{C} \to \mathbb{R}^+$ is any function, we say that μ is monotone if whenever $E, F \in \mathcal{C}$ and $E \subseteq F$, then $\mu(E) \leq \mu(F)$

Definition 106. Let $\mathcal{H}$ be a hereditary σ-ring of subsets of X, then

$$\mu^* : \mathcal{H} \to [0, \infty]$$

is an outer measure if

1. $\mu^*(\varnothing) = 0$
2. μ^* is monotone; i.e. if $F \subseteq E$ and $E \in \mathcal{H}$, then
$$\mu^*(F) \leq \mu^*(E)$$
3. μ^* is countably subadditive; i.e. if $F \subseteq \bigcup_{i=1}^{\infty} E_i$ where $E_i \in \mathcal{H}$, then
$$\mu^*(F) \leq \sum_{i=1}^{\infty} \mu^*(E_i)$$

Theorem 107. Let $\mathcal{S}$ be a semiring and μ_0 a premeasure on $\mathcal{S}$, then defining

$$\mu^* : \mathcal{H}(\mathcal{S}) \to [0, \infty]$$

by

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} \mu_0(E_i) : A \subseteq \bigcup_{i=1}^{\infty} E_i; E_i \in \mathcal{S} \right\}$$

then μ^* is an outer measure which extends μ_0; i.e. $\mu^*|_{\mathcal{S}} = \mu_0$.

Proof. First, since $\varnothing \in \mathcal{S}$, so setting $E_i = \varnothing \ \forall i$ gives

$$\mu^*(\varnothing) \leq \sum_{i=1}^{\infty} \mu_0(\varnothing) = 0$$

and so $\mu^*(\varnothing) = 0$.

Now, if $A \subseteq B$ then $B \subseteq \bigcup_{i=1}^{\infty} E_i \implies A \subseteq \bigcup_{i=1}^{\infty} E_i$. So

$$\begin{aligned}
\mu^*(A) &= \inf \left\{ \sum_{i=1}^{\infty} \mu_0(E_i) : A \subseteq \bigcup_{i=1}^{\infty} E_i; E_i \in \mathcal{S} \right\} \\
&\leq \inf \left\{ \sum_{i=1}^{\infty} \mu_0(E_i) : B \subseteq \bigcup_{i=1}^{\infty} E_i; E_i \in \mathcal{S} \right\} \\
&= \mu^*(B)
\end{aligned}$$

and so μ^* is monotone.

Next, given $\epsilon > 0$, and $A \subseteq \bigcup_{i=1}^{\infty} E_i$ for each E_i choose $E_{ij} \in \mathcal{S}$ for each $j \in \mathbb{N}$ such that $E_i \subseteq \bigcup_{j=1}^{\infty} E_{ij}$ and

$$\sum_{i=1}^{\infty} \mu_0(E_{ij}) \leq \mu^*(E_i) + \frac{\epsilon}{2^i}$$

then

$$A \subseteq \bigcup_{i=1}^{\infty} E_i \subseteq \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} E_{ij}$$

and

$$\begin{aligned}\mu^*(A) &\leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\mu_0(E_{ij})\\ &\leq \sum_{i=1}^{\infty}\left[\mu^*(E_i)+\frac{\epsilon}{2^i}\right]\\ &\leq \sum_{i=1}^{\infty}\mu^*(E_i)+\epsilon\end{aligned}$$

Since ϵ was arbitrary we conclude that $\mu^*(A) \leq \sum_{i=1}^{\infty}\mu^*(E_i)$ and μ^* is countably subadditive.

Now let $E \in \mathcal{S}$, by the definition of μ^* we have that

$$\mu^*(E) \leq \mu_0(E)$$

now if $E \subseteq \bigcup_{i=1}^{\infty} F_i$ for $F_i \in \mathcal{S}$, then by countable subadditivity we have

$$\mu_0(E) \leq \sum_{i=1}^{\infty}\mu_0(F_i)$$

and in particular this holds for the infimum and so

$$\mu_0(E) \leq \inf\left\{\sum_{i=1}^{\infty}\mu_0(F_i) : E \subseteq \bigcup_{i=1}^{\infty} F_i; F_i \in \mathcal{S}\right\} = \mu^*(E)$$

and thus $\mu^*|_{\mathcal{S}} = \mu_0$ $\square$

Definition 108. Given a hereditary σ-ring $\mathcal{H}$ and an outer measure μ^* on $\mathcal{H}$, $E \in \mathcal{H}$ is μ^*-measurable if for every $A \in \mathcal{H}$ we have

$$\mu^*(A) = \mu^*\big(A \cap E\big) + \mu^*\big(A \setminus E\big)$$

the collection of all μ^*-measurable sets is denoted $M(\mu^*)$.

Observe, by the subadditivity of μ^* we always have

$$\mu^*(A) \leq \mu^*\big(A \cap E\big) + \mu^*\big(A \cap E^c\big)$$

so only one direction needs to be shown.

Definition 109. Let $\mathcal{R}$ be a σ-ring and μ a measure on $\mathcal{R}$. Then μ is complete if whenever $E \in \mathcal{R}$ and $\mu(E) = 0$, then for all $F \subseteq E$ we have $F \in \mathcal{R}$ and $\mu(F) = 0$

Definition 110. Let $\mathcal{S}$ be a collection of subsets of X, and let $\mu : \mathcal{S} \to [0, \infty]$ be a set function. Then $E \subseteq X$ is σ-finite if $\exists \; \{F_i\} \in \mathcal{S}$ such that $E \subseteq \bigcup_{i=1}^{\infty} F_i$ and $\mu(F_i) < \infty \; \forall \; i$.

If each $E \in \mathcal{S}$ is σ-finite, then μ is σ-finite. If X is σ-finite, then μ is **Totally σ-Finite**.

Theorem 111 (Caratheodory's Theorem). Let μ^* be an outer measure on a hereditary σ-ring $\mathcal{H}$. Let $M(\mu^*)$ be the set of μ^*-measurable sets in $\mathcal{H}$. Then $M(\mu^*)$ is a σ-ring and $\mu^*|_{M(\mu^*)}$ is a measure.

Proof. First we show that $M(\mu^*)$ is a ring, so let $E, F \in M(\mu^*)$, and $A \in \mathcal{H}$ be arbitrary. Then

$$\begin{aligned}
&\mu^*\big(A \cap (E \cup F)\big) + \mu^*\big(A \cap (E \cup F)^c\big) \\
&\qquad = \mu^*\big((A \cap E) \cup (A \cap F)\big) + \mu^*\big(A \cap E^c \cap F^c\big) \\
&\qquad = \mu^*\big((A \cap E) \sqcup ((A \setminus E) \cap F)\big) + \mu^*\big((A \setminus E) \cap F^c\big) \\
&\qquad \leq \mu^*\big(A \cap E\big) + \mu^*\big((A \setminus E) \cap F\big) + \mu^*\big((A \setminus E) \cap F^c\big) \\
&\qquad = \mu^*\big(A \cap E\big) + \mu^*\big(A \setminus E\big) && F\ \mu^*\text{-measurable} \\
&\qquad = \mu^*(A) && E\ \mu^*\text{-measurable}
\end{aligned}$$

that is $\mu^*\big(A \cap (E \cup F)\big) + \mu^*\big(A \cap (E \cup F)^c\big) \leq \mu^*(A)$ and since we always have $\mu^*\big(A \cap (E \cup F)\big) + \mu^*\big(A \cap (E \cup F)^c\big) \geq \mu^*(A)$ by the subadditivity of μ^*, we have

$$\mu^*(A) = \mu^*\big(A \cap (E \cup F)\big) + \mu^*\big(A \cap (E \cup F)^c\big)$$

and so $E \cup F \in M(\mu^*)$.

Next we check set difference where we have

$$\begin{aligned}
&\mu^*\big(A \cap (E \setminus F)\big) + \mu^*\big(A \cap (E \setminus F)^c\big) \\
&\qquad = \mu^*\big(A \cap (E \cap F^c)\big) + \mu^*\big(A \cap (E \cap F^c)^c\big) \\
&\qquad = \mu^*\big(A \cap E \cap F^c\big) + \mu^*\big(A \cap (E^c \cup F)\big) \\
&\qquad = \mu^*\big((A \cap E) \setminus F\big) + \mu^*\big((A \cap E^c) \cup (A \cap F)\big) \\
&\qquad = \mu^*\big((A \cap E) \setminus F\big) + \mu^*\big((A \cap E^c) \sqcup ((A \setminus E^c) \cap F)\big) \\
&\qquad \leq \mu^*\big((A \cap E) \setminus F\big) + \mu^*\big(A \cap E^c\big) + \mu^*\big(A \cap E \cap F\big) \\
&\qquad = \mu^*\big(A \cap E\big) + \mu^*\big(A \cap E^c\big) && F\ \mu^*\text{-measurable} \\
&\qquad = \mu^*(A) && E\ \mu^*\text{-measurable}
\end{aligned}$$

that is $\mu^*\big(A \cap (E \setminus F)\big) + \mu^*\big(A \cap (E \setminus F)^c\big) \leq \mu^*(A)$ and thus

$$\mu^*(A) = \mu^*\big(A \cap (E \setminus F)\big) + \mu^*\big(A \cap (E \setminus F)^c\big)$$

and so $E \setminus F \in M(\mu^*)$.

And so $M(\mu^*)$ is a ring.

Now we note that if $E, F \in M(\mu^*)$ are disjoint that

$$\mu^*\big(A \cap (E \sqcup F)\big) = \mu^*\big((A \cap E) \sqcup (A \cap F)\big) = \mu^*\big(A \cap E\big) + \mu^*\big(A \cap F\big)$$

since $F \sqcup E$ is μ^*-measurable and $A \cap (E \sqcup F) \in \mathcal{H}$ so that

$$
\begin{aligned}
&\mu^*\big(A \cap (E \sqcup F)\big) \\
&\qquad = \mu^*\Big(\big(A \cap (E \sqcup F)\big) \cap E\Big) + \mu^*\Big(\big(A \cap (E \sqcup F)\big) \cap E^c\Big) && \text{measurability} \\
&\qquad = \mu^*\Big(A \cap \big((E \cap E) \sqcup (F \cap E)\big)\Big) + \mu^*\Big(A \cap \big((E \cap E^c) \sqcup (F \cap E^c)\big)\Big) \\
&\qquad = \mu^*\Big(A \cap \big(E \sqcup \varnothing\big)\Big) + \mu^*\Big(A \cap \big(\varnothing \sqcup F\big)\Big) && E \cap F = \varnothing \\
&\qquad = \mu^*\big(A \cap E\big) + \mu^*\big(A \cap F\big)
\end{aligned}
$$

Next suppose $E = \bigcup_{i=1}^\infty E_i$ where each $E_i \in M(\mu^*)$ defining $F_1 = E_1$ and $F_k = E_k \setminus \bigcup_{i=1}^{k-1} E_i$ for each $k > 1$ we see that

$$
E = \bigcup_{i=1}^\infty E_i = \bigsqcup_{i=1}^\infty F_i
$$

where each $F_i \in M(\mu^*)$ since $M(\mu^*)$ is a ring, and we note

$$
E \supseteq \bigsqcup_{i=1}^n F_i \implies E^c \subseteq \left(\bigsqcup_{i=1}^n F_i\right)^c
$$

Then for any $A \in \mathcal{H}$

$$
\begin{aligned}
\mu^*(A) &= \mu^*\left(A \cap \bigsqcup_{i=1}^n F_i\right) + \mu^*\left(A \cap \left(\bigsqcup_{i=1}^n F_i\right)^c\right) \\
&\geq \mu^*\left(A \cap \bigsqcup_{i=1}^n F_i\right) + \mu^*\big(A \cap E^c\big) && \text{subadditivity} \\
&= \sum_{i=1}^n \mu^*(A \cap F_i) + \mu^*\big(A \cap E^c\big)
\end{aligned}
$$

where only the RHS depends on n to taking the limit to infinity gives

$$
\begin{aligned}
\mu^*(A) &\geq \sum_{i=1}^\infty \mu^*(A \cap F_i) + \mu^*\big(A \cap E^c\big) \\
&\geq \mu^*\left(\bigsqcup_{i=1}^\infty (A \cap F_i)\right) + \mu^*\big(A \cap E^c\big) && \text{subadditivity} \\
&= \mu^*\big(A \cap E\big) + \mu^*\big(A \cap E^c\big)
\end{aligned}
$$

and therefore we have that $E \in M(\mu^*)$ and so $M(\mu^*)$ is closed under countable unions, and thus $M(\mu^*)$ is a σ-ring.

Now we note from

$$
\mu^*(A) \geq \sum_{i=1}^\infty \mu^*(A \cap F_i) + \mu^*\big(A \cap E^c\big) \geq \mu^*\big(A \cap E\big) + \mu^*\big(A \cap E^c\big)
$$

yet we always have $\mu^*(A) \leq \mu^*\big(A \cap E\big) + \mu^*\big(A \cap E^c\big)$, so that we actually have

$$\mu^*(A) = \sum_{i=1}^{\infty} \mu^*(A \cap F_i) + \mu^*\big(A \cap E^c\big)$$

Then since this holds for any $A \in \mathcal{H}$ letting $A = E = \bigsqcup_{i=1}^{\infty} F_i$ where each $F_i \in M(\mu^*)$ gives

$$\begin{aligned}\mu^*|_{M(\mu^*)}\left(\bigsqcup_{i=1}^{\infty} F_i\right) &= \sum_{j=1}^{\infty} \mu^*\left(\bigsqcup_{i=1}^{\infty} (F_i \cap F_j)\right) + \mu^*(\varnothing) \\ &= \sum_{j=1}^{\infty} \mu^*(F_i)\end{aligned}$$

and thus $\mu^*|_{M(\mu^*)}$ is a measure on the σ-ring $M(\mu^*)$. □

Proposition 112. Let μ^* be an outer measure on a hereditary σ-ring $\mathcal{H}$. Then $\mu^*|_{M(\mu^*)}$ is a complete measure, if $M(\mu^*) \neq \varnothing$.

Proof. It suffices to show that if $\mu^*(E) = 0$ then $E \in M(\mu^*)$. So let $A \in \mathcal{H}$, then since $A \cap E \subseteq E$ monotonicity gives $\mu^*(A \cap E) = 0$ and $A \cap E^c \subseteq A$ so again by monotonicity we get

$$\mu^*(A \cap E) + \mu^*(A \cap E^c) = 0 + \mu^*(A \cap E^c) \leq \mu^*(A)$$

and thus

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

and so E is μ^*-measurable and hence $E \in M(\mu^*)$. And therefore $\mu^*|_{M(\mu^*)}$ is complete. □

The last few results have thus given us the following: If $\mathcal{S}$ is a semiring and μ_0 a premeasure on $\mathcal{S}$, and μ^* the outer measure on $\mathcal{H}(\mathcal{S})$ determined by μ_0 then

1. $\mu^*|_{\sigma(\mathcal{S})}$ is a measure on the σ-ring generated by $\mathcal{S}$ which extends μ_0.
2. $\mu^*|_{M(\mu^*)}$ is a complete measure on the σ-ring $M(\mu^*)$ which extends $\mu^*|_{\sigma(\mathcal{S})}$ and hence μ_0.

Theorem 113. If μ_0 is a premeasure on a semiring $\mathcal{S}$, and if μ^* is the outer measure on $\mathcal{H}(\mathcal{S})$ determined by μ_0, then $\mathcal{S} \subseteq M(\mu^*)$.

Proof. We must show this if $E \in \mathcal{S}$, then $E \in M(\mu^*)$; that is, $\forall\ A \in \mathcal{H}(\mathcal{S})$

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

If $\mu^*(A) = \infty$ then $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$ and we are done.

So let us assume that $\mu^*(A) < \infty$. Given $\epsilon > 0$, since $A \in \mathcal{H}(\mathcal{S})$, we may select $F_i \in \mathcal{S}$ such that $A \subseteq \bigcup_{i=1}^{\infty} F_i$ and

$$\sum_{i=1}^{\infty} \mu_0(F_i) \leq \mu^*(A) + \epsilon$$

now $F_i = (F_i \cap E) \sqcup (F_i \setminus E)$, and since $\mathcal{S}$ is a semiring $\exists\, G_{ij} \in \mathcal{S}$ such that $F_i \setminus E = \bigsqcup_{j=1}^{n_j} G_{ij}$ so that

$$\begin{aligned}
\sum_{i=1}^{\infty} \mu_0(F_i) &= \sum_{i=1}^{\infty} \mu_0 \left((F_i \cap E) \sqcup \bigsqcup_{j=1}^{n_j} G_{ij} \right) \\
&= \sum_{i=1}^{\infty} \left[\mu_0(F_i \cap E) + \sum_{j=1}^{n_j} \mu_0(G_{ij}) \right] \\
&= \sum_{i=1}^{\infty} \mu_0(F_i \cap E) + \sum_{i=1}^{\infty} \sum_{j=1}^{n_j} \mu_0(G_{ij})
\end{aligned}$$

and $A \subseteq \bigcup_{i=1}^{\infty} F_i$ implies

$$A \cap E \subseteq \bigcup_{i=1}^{\infty} (F_i \cap E) \quad \text{and} \quad A \setminus E \subseteq \bigcup_{i=1}^{\infty} (F_i \setminus E) = \bigcup_{i=1}^{\infty} \bigsqcup_{j=1}^{n_j} G_{ij}$$

and thus we have

$$\begin{aligned}
\mu^*(A) + \epsilon &\geq \sum_{i=1}^{\infty} \mu_0(F_i) \\
&= \sum_{i=1}^{\infty} \mu_0(F_i \cap E) + \sum_{i=1}^{\infty} \sum_{j=1}^{n_j} \mu_0(G_{ij}) \\
&\geq \inf \left\{ \sum_{i=1}^{\infty} \mu_0(F_i \cap E) : A \cap E \subseteq \bigcup_{i=1}^{\infty} (F_i \cap E); F_i \cap E \in \mathcal{S} \right\} \\
&\quad + \inf \left\{ \sum_{i=1}^{\infty} \sum_{j=1}^{n_j} \mu_0(G_{ij}) : A \setminus E \subseteq \bigcup_{i=1}^{\infty} \bigsqcup_{j=1}^{n_j} G_{ij}; G_{ij} \in \mathcal{S} \right\} \\
&= \mu^*(A \cap E) + \mu^*(A \setminus E)
\end{aligned}$$

and since ϵ is arbitrary we conclude that $\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \setminus E)$ giving

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E)$$

and so E is μ^*-measurable and thus $E \in M(\mu^*)$. And therefore $\mathcal{S} \subseteq M(\mu^*)$. $\square$

Proposition 114. Let μ_0 be a premeasure on a semiring $\mathcal{S}$, and μ^* the outer measure on $\mathcal{H}(\mathcal{S})$ determined by μ_0. Then $\mathcal{H}(\mathcal{S}) = \mathcal{H}\big(M(\mu^*)\big)$ and if $E \in \mathcal{H}(\mathcal{S})$ then

$$\mu^*(E) = \inf\left\{\mu^*|_{\sigma(\mathcal{S})}(F) : E \subseteq F; F \in \sigma(\mathcal{S})\right\} = \inf\left\{\mu^*|_{M(\mu^*)}(F) : E \subseteq F; F \in M(\mu^*)\right\}$$

which is to say that $\mu^*|_{\sigma(\mathcal{S})} = \mu^* = \mu^*|_{M(\mu^*)}$

Proof. First since

$$\mathcal{S} \subseteq M(\mu^*) \subseteq \mathcal{H}(\mathcal{S})$$

we have $\mathcal{H}(\mathcal{S}) = \mathcal{H}\big(M(\mu^*)\big)$.

Next, let $E \in \mathcal{H}(\mathcal{S})$ then

$$\begin{aligned}
\mu^*(E) &= \inf\left\{\sum_{i=1}^{\infty} \mu_0(F_i) : E \subseteq \bigcup_{i=1}^{\infty} F_i; F_i \in \mathcal{S}\right\} && \text{def of } \mu^* \\
&\geq \inf\left\{\mu^*|_{\sigma(\mathcal{S})}(F) : E \subseteq F; F \in \sigma(\mathcal{S})\right\} && \text{countable subadditivity of } \mu^* \\
&\geq \inf\left\{\mu^*|_{M(\mu^*)}(F) : E \subseteq F; F \in M(\mu^*)\right\} && M(\mu^*) \supseteq \sigma(\mathcal{S}) \\
&\geq \mu^*(E) && \text{monotonicity of } \mu^*
\end{aligned}$$

and thus the inner inequalities must be equalities. □

Theorem 115 (**Uniqueness of Extensions**)**.** If μ_0 is a σ-finite premeasure on a semiring $\mathcal{S}$, and if $\mathcal{R}$ is a σ-ring such that $\mathcal{S} \subseteq \mathcal{R} \subseteq M(\mu^*)$, and if ν is a non-negative extension of μ_0 to a measure on $\mathcal{R}$, then $\nu = \mu^*|_{\mathcal{R}}$.

Proof. If $E \in \mathcal{R}$, and $E \subseteq \bigcup_{i=1}^{\infty} F_i$ where each $F_i \in \mathcal{S}$, then

$$\begin{aligned}
\nu(E) &\leq \sum_{i=1}^{\infty} \nu(F_i) && \text{non-negative measures are countably subadditive} \\
&= \sum_{i=1}^{\infty} \mu_0(F_i) && \nu \text{ an extension of } \mu_0 \text{ and } F_i \in \mathcal{S}
\end{aligned}$$

and thus $\nu(E) \leq \mu^*(E) \; \forall\, E \in \mathcal{R}$, so it remains to show that $\nu(E) \geq \mu^*(E) \; \forall\, E \in \mathcal{R}$

Case 1: Suppose $E \in \mathcal{R}$, and that $\exists\, F \in \mathcal{S}$ such that $E \subseteq F$, and $\mu_0(F) < \infty$. Then, since

$$F = (F \cap E) \sqcup (F \setminus E) = E \sqcup (F \setminus E)$$

we have, by the measurability of E

$$\begin{aligned}
\nu(F) &= \nu(E) + \nu(F \setminus E) \\
&\leq \mu^*(E) + \mu^*(F \setminus E) \\
&= \mu^*(F) \\
&= \mu_0(F) \\
&= \nu(F)
\end{aligned}$$

and thus

$$\nu(E) + \nu(F \setminus E) = \mu^*(E) + \mu^*(F \setminus E)$$

yet,

$$\nu(E) \leq \mu^*(E) < \infty \quad \text{and} \quad \nu(F \setminus E) \leq \mu^*(F \setminus E) < \infty$$

and thus we must have $\mu^*(E) = \nu(E)$

Case 2: Let $E \in \mathcal{R}$ be arbitrary. Then, since μ_0 is assumed to be σ-finite. $\exists \{F_i\}_{i=1}^{\infty} \in \mathcal{S}$ such that $\mu_0(F_i) < \infty$ for each i and $E \subseteq \bigcup_{i=1}^{\infty} F_i$, since $E \in \mathcal{R} \subseteq M(\mu^*) \subseteq \mathcal{H}(\mathcal{S})$ and $\mathcal{H}(\mathcal{S})$ is defined to be the collection of all sets countably covered by elements of $\mathcal{S}$. Then disjointizing we get $\{G_{ij}\} \in \mathcal{S}$ such that $\mu_0(G_{ij}) < \infty \ \forall \ i, j$, with $E \subseteq \bigsqcup_{i,j \geq 1} G_{ij}$ and $E = \bigsqcup_{i,j \geq 1}(E \cap G_{ij})$. Then since $E \cap G_{ij} \subseteq G_{ij}$ so Case 1 gives

$$\begin{aligned}
\nu(E) &= \sum_{i,j \geq 1} \nu(E \cap G_{ij}) \\
&= \sum_{i,j \geq 1} \mu^*(E \cap G_{ij}) \\
&= \mu^*(E)
\end{aligned}$$

and hence, $\mu^*(E) = \nu(E)$.

and therefore we conclude that $\nu = \mu^*|_{\mathcal{R}}$. □

Now we begin to relate these ideas to functions, as one of the major goals of measure theory is to generalize the idea of integration.

Definition 116. Let X be a set and $\mathcal{S}$ a σ-ring of subsets of X, and B a Banach Space. Then a function

$$f : X \to B$$

is a simple $\mathcal{S}$-measurable function if

1. $\operatorname{Im}(f) = \{b_1, \ldots, b_n\} \in B$ is finite.
2. For each $b_i \in B$ such that $b_i \neq 0$ we have $f^{-1}(b_i) = E_i \in \mathcal{S}$.

the family $\mathcal{F}$ of B-valued simple $\mathcal{S}$-measurable functions are functions of the form

$$f = \sum_{i=1}^{n} b_i \chi_{E_i}, \text{ with } \chi_{E_i}(x) = \begin{cases} 1, & x \in E_i \\ 0, & \text{otherwise} \end{cases}$$

where the b_i's are distinct and the E_i's $\in \mathcal{S}$ are disjoint.

Definition 117. Let X be a set and $\mathcal{S}$ a σ-ring of subsets of X, and B a Banach Space. Then a function

$$f : X \to B$$

is a $\mathcal{S}$-measurable function if $\exists \{f_n\}_{n \in \mathbb{N}}$ of simple $\mathcal{S}$-measurable functions such that $f_n \to f$ pointwise; i.e. $\forall \ x \in X$ we have $f_n(x) \to f(x)$.

Definition 118. Let X be a set, $\mathcal{S}$ a σ-ring of subsets of X, and μ a measure on $\mathcal{S}$. A subset $E \subset X$ is a null-set with respect to μ if $\exists\ F \in \mathcal{S}$ such that $E \subseteq F$ and $\mu(F) = 0$. The null-sets form a hereditary σ-ring denoted $N(\mu)$.

That is, E is contained in some set of $\mathcal{S}$ of measure zero.

Definition 119. Let X be a set, $\mathcal{S}$ a σ-ring of subsets of X, and μ a measure on $\mathcal{S}$. A property P on X is said to hold almost everywhere if $\exists\ N(\mu)$ such that P is true $\forall\ x \in X \setminus N(\mu)$.

Definition 120. Let X be a set, $\mathcal{S}$ a σ-ring of subsets of X, μ a measure on $\mathcal{S}$, and let B be a Banach space. Then a function

$$f : X \to B$$

is a simple μ-measurable function if f is a simple $\big(\mathcal{S} \sqcup N(\mu)\big)$-measurable function. where

$$\mathcal{S} \sqcup N(\mu) = \big\{E \sqcup F : E \in \mathcal{S},\ F \in N(\mu)\big\}$$

Note, that simple $\mathcal{S}$-measurable $\implies$ simple μ-measurable.

Definition 121. Let X be a set, $\mathcal{S}$ a σ-ring of subsets of X, μ a measure on $\mathcal{S}$, and let B be a Banach space. Then a function defined almost everywhere on X

$$f : X \setminus N(\mu) \to B$$

is a μ-measurable function if $\exists\ \{f_n\}_{n\in\mathbb{N}}$ of simple μ-measurable functions such that $f_n \to f$ pointwise; i.e. $\forall\ x \in X \setminus N(\mu)$ we have $f_n(x) \to f(x)$.

Note, that $\mathcal{S}$-measurable $\implies$ μ-measurable.

Proposition 122. Let $(X, \mathcal{S}, \mu)$ be a measure space and B a Banach space. A function f defined almost everywhere, i.e. defined on $X \setminus N(\mu)$, is μ-measurable iff $\exists\ \{f_n\}$ of simple $\mathcal{S}$-measurable such that $f_n \to f$ pointwise almost everywhere; i.e. $\forall\ x \in X \setminus N(\mu)$ we have $f_n(x) \to f(x)$.

Proof. Suppose that f is μ-measurable, then $\exists\ \{f_n\}$ of simple μ-measurable functions and a null-set $N_0(\mu)$, such that $\forall\ x \in X \setminus N_0(\mu)$ we have $f_n(x) \to f(x)$. Since each f_n is simple μ-measurable we have for each n that

$$f_n = \sum_{i=1}^{k_n} b_i^n \chi_{F_i^n}$$

where each $b_i^n \in B$ and each $F_i^n \in \mathcal{S} \sqcup N(\mu)$, that is

$$F_i^n = E_i^n \sqcup N_i^n, \text{ where } E_i^n \in \mathcal{S},\ N_i^n \in N(\mu)$$

so let

$$N = N_0(\mu) \cup \left(\bigcup_{n,i} N_i^n\right)$$

then N is a null-set, and letting

$$\varepsilon_n = \sum_{i=1}^{k_n} b_i^n \chi_{E_i^n}$$

then each ε_n is a simple $\mathcal{S}$-measurable function.

Then since $\varepsilon_n|_{X \setminus N} = f_n$, then $\forall\, x \in X \setminus N$ we have $\varepsilon_n(x) \to f(x)$.

Conversely, if $\exists\, \{f_n\}$ of simple $\mathcal{S}$-measurable functions such that $\forall\, x \in X \setminus N(\mu)$ we have $f_n(x) \to f(x)$, then f is $\mathcal{S}$-measurable on $X \setminus N(\mu)$. Then since $\mathcal{S}$-measurable implies μ-measurable we have that f is μ-measurable. $\square$

Proposition 123. Let $(X, \mathcal{S}, \mu)$ be a measure space and B a Banach space. If f, g are simple $\mathcal{S}$-measurable functions, then $f + g$ is a simple $\mathcal{S}$-measurable function.

Proof. First suppose $f = \sum_{i=1}^{n} b_i \chi_{E_i}$, and $g = c\chi_F$, to get F contained in the E_i's let us set $E_{n+1} = F \setminus \bigsqcup_{i=1}^{n} E_i$ and $b_{n+1} = 0$, then

$$F \subseteq \bigsqcup_{i=1}^{n+1} E_i \implies F = \bigsqcup_{i=1}^{n+1} (F \cap E_i)$$

and

$$f = \sum_{i=1}^{n+1} b_i \chi_{E_i} = \sum_{i=1}^{n+1} b_i \Big(\chi_{E_i \cap F} + \chi_{E_i \setminus F} \Big)$$
$$g = \sum_{i=1}^{n+1} c\chi_{E_i \cap F}$$

and so

$$f + g = \sum_{i=1}^{n+1} (b_i + c)\chi_{E_i \cap F} + \sum_{i=1}^{n+1} b_i \chi_{E_i \setminus F}$$

and so $f + g$ is a simple $\mathcal{S}$-measurable function. The general case follows inductively. $\square$

Proposition 124. Let $(X, \mathcal{S}, \mu)$ be a measure space and B a Banach space. Let

$$f, g : X \to B$$

be $\mathcal{S}$-measurable/μ-measurable functions, and let c be a scalar. Then $f + g, cf, ||f(\cdot)||$ are $\mathcal{S}$-measurable/μ-measurable functions. If f is scalar valued, then fg is $\mathcal{S}$-measurable/μ-measurable. If f and g are $\mathbb{R}$ valued functions, then $\max(f, g)$ and $\min(f, g)$ are $\mathcal{S}$-measurable/μ-measurable functions.

Proof. If $\{f_n\}, \{g_n\}$ are sequences of simple $\mathcal{S}$-measurable such that $\forall\ x \in X$

$$\begin{aligned} f_n(x) &\to f(x) \\ g_n(x) &\to g(x) \end{aligned}$$

then $\forall\ x \in X$ we have

$$(f_n + g_n)(x) = f_n(x) + g_n(x) \to f(x) + g(x) = (f+g)(x)$$

the next follows as $\{cf_n\} = c\{f_n\}$, and if $f_n \to f\ \forall\ x \in X$, then $||f_n(x)|| = \sum_{i=1}^n ||b_i||\chi_{E_i}(x) = ||b_i|| = ||f(x)||$. Then fg follows from cf

the last two follow from the first 4 and the fact that

$$\begin{aligned} \max(f,g) &= \frac{f+g+|f-g|}{2} \\ \min(f,g) &= \frac{f+g-|f-g|}{2} \end{aligned}$$

□

Lemma 125. If $\{f_n\}$ is a sequence of functions from a set X to a Banach space B which converge to f pointwise, and if for any open set $U \subseteq B$ we define

$$U_n = \left\{y \in U : d\big(y, U^c\big) > \tfrac{1}{n}\right\}$$

then

$$f^{-1}(U) = \bigcup_{n=1}^{\infty} \bigcup_{K=1}^{\infty} \bigcap_{k=K}^{\infty} f_k^{-1}(U_n)$$

for all open $U \subseteq B$.

Proof.

$$\begin{aligned} x \in f^{-1}(U) &\iff f(x) \in U \\ &\iff \exists\ n, K \in \mathbb{N} \text{ such that} \\ &f_k(x) \in U_n\ \forall\ k \geq K \\ &\iff \exists\ n, K \in \mathbb{N} \text{ such that} \\ &x \in f_k^{-1}(U_n)\ \forall\ k \geq K \\ &\iff \exists\ n, K \in \mathbb{N} \text{ such that} \\ &x \in \bigcap_{k=K}^{\infty} f_k^{-1}(U_n) \qquad\qquad \overline{U}_n \subseteq U_{n+1} \\ &\iff x \in \bigcup_{n=1}^{\infty} \bigcup_{K=1}^{\infty} \bigcap_{k=K}^{\infty} f_k^{-1}(U_n) \end{aligned}$$

□

Definition 126. Let X be a set and let B be a Banach space. For any function

$$f : X \to B$$

the carrier of f denoted

$$\operatorname{car}(f) = \{x \in X : f(x) \neq 0 \in B\}$$

This is very similar to the support, except we do not take the closure, however they are related by

$$\operatorname{supp}(f) = \overline{\operatorname{car}(f)}$$

Theorem 127. Let X be a set, $\mathcal{S}$ a σ-ring of subsets of X, B be a Banach space, and let

$$f : X \to B$$

be a function, then f is $\mathcal{S}$-measurable if

1. $f(X) \subseteq B$ is separable.
2. $f^{-1}(U) \cap \operatorname{car}(f) \in \mathcal{S}$ for all open $U \subseteq B$.

Proof. Suppose that f is $\mathcal{S}$-measurable, then $\exists\ \{f_n\}$ of simple $\mathcal{S}$-measurable functions such that $\forall\, x \in X$ we have $f_n(x) \to f(x)$. Since each f_n is simple $\mathcal{S}$-measurable its range is finite so for each n let

$$\operatorname{Im}(f_n) = \{b_1^n, \ldots, b_{k_n}^n\}$$

and let

$$R = \overline{\bigcup_{n=1}^{\infty} \operatorname{Im}(f_n)}$$

so given $\epsilon > 0$, then

$$\begin{aligned} b \in \operatorname{Im}(f) &\iff \exists\ x \in X \text{ such that } f(x) = b \\ &\iff f_n(x) \to f(x) = b \\ &\iff \ \exists\ n \in \mathbb{N} \text{ such that } ||f_n(x) - b|| < \epsilon \end{aligned}$$

and therefore $B_\epsilon(b) \cap R \neq \varnothing$. Since $b \in \operatorname{Im}(f)$ was arbitrary we conclude that $\forall\, b \in \operatorname{Im}(f)$ there is a ball containing b which has nonempty intersection with R, and so $f(X) \subseteq R$. And for each n there is some $A_n \subseteq B$ such that $A_n \subseteq \operatorname{Im}(f_n)$ is countably dense in the range of f_n, then

$$\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} \operatorname{Im}(f_n)$$

is countably dense, and so $\bigcup_{n=1}^{\infty} \operatorname{Im}(f_n)$ is separable, and hence so is R, thereby making $\operatorname{Im}(f) = f(X)$ separable as the subset of a separable set.

Now let $U \subseteq B$ be any open set, then since

$$f^{-1}(U) \cap \operatorname{car}(f) = f^{-1}\big(U \setminus \{0\}\big)$$

it suffices to show that if U is any open set such that $U \not\ni 0$, then $f^{-1}(U) \in \mathcal{S}$, then with

$$U_n = \big\{y \in U : d\big(y, (U \setminus \{0\})^c\big) > \tfrac{1}{n}\big\}$$

we will have each $U_n \not\ni 0$ and

$$f^{-1}(U) = \bigcup_{n=1}^{\infty} \bigcup_{K=1}^{\infty} \bigcap_{k=K}^{\infty} f_k^{-1}(U_n)$$

by the previous lemma, and since the f_k's are simple $\mathcal{S}$-measurable there preimages $f_k^{-1}(U_n) \in \mathcal{S}$, and as $\mathcal{S}$ is a σ-ring, it is closed under countable unions and intersections, and so $f^{-1}(U) \in \mathcal{S}$.

Conversely, suppose that f is such that $f(X) \subseteq B$ is separable and $f^{-1}(U) \in \mathcal{S}$. So we may choose a sequence $\{b_i\} \in B$ which is dense in $f(X)$ since $f(X)$ is separable. So let

$$C_{ij} = \Big\{x \in X : x \in \operatorname{car}(f);\ ||f(x) - b_i|| < \tfrac{1}{j}\Big\} = f^{-1}\Big(B_{\frac{1}{j}}(b_i) \setminus \{0\}\Big)$$

for all $i, j \in \mathbb{N}$, and since each $B_{\frac{1}{j}}(b_i) \setminus \{0\} \in B$ is open, by hypothesis we have that $f^{-1}\Big(B_{\frac{1}{j}}(b_i) \setminus \{0\}\Big) \in \mathcal{S}$. Then, ordering the pairs (i, j) lexicographically; that is

$$(i, j) \leq (k, n) \text{ if } \begin{cases} i < k \\ i = k, \text{ and } j < n \end{cases}$$

so for each fixed n defining

$$E_{ij}^n = C_{ij} \setminus \bigcup \big\{C_{kl} : (i, j) < (k, l) \leq (n, n)\big\}$$

then the sets E_{ij}^n are disjoint and $E_{ij}^n \subseteq C_{ij}\ \forall\ i, j$. So let

$$f_n = \sum_{i,j=1}^{n} b_i \chi_{E_{ij}^n}$$

and suppose we are given $\epsilon > 0$ and $x \in X$. If $x \notin \operatorname{car}(f)$, then $f(x) = 0$ and so $f_n(x) = 0\ \forall\ n$ and we are done. So suppose that $x \in \operatorname{car}(f)$. Choose j_0 such that $\frac{1}{j_0} < \epsilon$, and choose i_0 so that

$$||f(x) - b_{i_0}|| < \tfrac{1}{j_0}$$

next we note that

$$x \in C_{i_0 j_0} = f^{-1}\Big(B_{\frac{1}{j_0}}(b_{i_0}) \setminus \{0\}\Big)$$

by the definition of j_0 and i_0. So setting $N = \max\{i_0, j_0\}$, then if $n > N$ we have $x \in E^n_{kl}$ where

$$(k,l) = \max\big\{(i,j) : x \in C_{ij};\ (i_0, j_0) \leq (i,j) \leq (n,n)\big\}$$

then

$$||f(x) - b_k|| < \tfrac{1}{l} \leq \tfrac{1}{j_0} < \epsilon$$

and by construction we have

$$f_n(x) = \sum_{i,j=1}^{n} b_i \chi_{E^n_{ij}}(x) = b_k$$

so that

$$||f(x) - b_k|| = ||f(x) - f_n(x)|| < \epsilon$$

and so $f_n \to f$ pointwise, and thus f is $\mathcal{S}$-measurable. □

Proposition 128. If $\{f_n\}$ is a sequence of $\mathcal{S}$-measurable/μ-measurable functions which converge to a function f pointwise/almost everywhere pointwise; i.e. $\forall\, x \in X / \forall\, x \in X \backslash N(\mu)$ we have $f_n(x) \to f(x)$. Then f is $\mathcal{S}$-measurable/μ-measurable.

Proof. Since $\mathcal{S}$-measurable $\implies$ μ-measurable we will prove the case with $\mathcal{S}$-measurable functions.

Since $\{f_n\}$ are $\mathcal{S}$-measurable, for each n we have that $f_n(X) \subset B$ is separable. Since the closure of a separable set is separable we also have that $\overline{\bigcup_{n=1}^{\infty} f_n(X)} \subseteq B$ is separable, and

$$f(X) \subseteq \overline{\bigcup_{n=1}^{\infty} f_n(X)}$$

and so $f(X)$ is separable as the subset of a separable set.

Then since for any open $U \subseteq B$ we have

$$f^{-1}(U) = \bigcup_{n=1}^{\infty} \bigcup_{K=1}^{\infty} \bigcap_{k=K}^{\infty} f_k^{-1}(U_n)$$

where

$$U_n = \big\{y \in U : d\big(y, (U \setminus \{0\})^c\big) > \tfrac{1}{n}\big\}$$

and since the f_k's are $\mathcal{S}$-measurable there preimages $f_k^{-1}(U_n) \in \mathcal{S}$, and as $\mathcal{S}$ is a σ-ring, it is closed under countable unions and intersections, and so $f^{-1}(U) \in \mathcal{S}$.

Then since $f(X)$ is separable, and for each open set $U \subset B$ we have $f^{-1}(U) \in \mathcal{S}$, we can conclude that f is $\mathcal{S}$-measurable. □

Theorem 129 (Egoroff). Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space, if $E \in \mathcal{S}$ such that $\mu(E) < \infty$ and if $\{f_n\}$ is a sequence of μ-measurable functions such that $\forall\ x \in E \setminus N(\mu)$ we have $f_n(x) \to f(x)$. Then for every $\epsilon > 0$ $\exists$ measurable $F \subseteq E$, and so $F \in \mathcal{S}$, such that

$$\mu(E \setminus F) < \epsilon$$

and $f_n \to f$ uniformly on F; i.e. given $\delta > 0$, $\exists\ N$ such that

$$n \geq N \implies ||f(x) - f_n(x)|| < \delta \quad \forall\ x \in F$$

Proof. For any k and m, let

$$\begin{aligned} G_m^k &= \left\{x \in E : ||f(x) - f_k(x)|| > \tfrac{1}{m}\right\} \in \mathcal{S} \\ F_m^n &= \bigcup_{k \geq n} G_m^k = \left\{x \in E : \exists\ k \geq n; ||f(x) - f_k(x)|| > \tfrac{1}{m}\right\} \in \mathcal{S} \end{aligned}$$

for fixed m, as $n \to \infty$, since $f_n \to f$, we have $F_m^n \to \varnothing$ and therefore

$$\mu(F_m^n) \to \mu(\varnothing) = 0$$

Let $\epsilon > 0$ be given and for each m choose n_m such that

$$\mu(F_m^{n_m}) < \frac{\epsilon}{2^m}$$

let $H = \bigcup_m F_m^{n_m}$, then

$$\mu(H) = \mu\left(\bigcup_m F_m^{n_m}\right) \leq \sum_m \mu(F_m^{n_m}) < \sum_m \frac{\epsilon}{2^m} = \epsilon$$

let $F = E \setminus H$, then

$$\begin{aligned} \mu(E \setminus F) &= \mu(E \cap F^c) \\ &= \mu\big(E \cap (E \cap H^c)^c\big) \\ &= \mu\big(E \cap (E^c \cup H)\big) \\ &= \mu\big(\varnothing \cup (E \cap H)\big) \\ &= \mu(H) \\ &< \epsilon \end{aligned}$$

so let $\delta > 0$ be given, and choose m_0 such that $\frac{1}{m_0} < \delta$. Then $\forall\ x \in F$ by the definition of F we must have $x \notin H$, and in particular we have $x \notin F_{m_0}^{n_{m_0}}$. Thus, for all $k \geq n_{m_0}$ we also have that $x \notin G_{m_0}^k$ which is to say

$$||f(x) - f_k(x)|| \leq \frac{1}{m_0} < \delta$$

and since this is independent of $x \in F$ we have that $f_n \to f$ uniformly on F. $\square$

Egoroff's Theorem gives inspiration to the next definition.

Definition 130. Let $(X, \mathcal{S}, \mu)$ be a measure space, let $\{f_n\}$ be a sequence of μ-measurable functions, and let $E \in \mathcal{S}$. Then $f_n \to f$ almost uniformly on E, if $\forall\ \epsilon > 0\ \exists\ F \in \mathcal{S}$ such that $F \subseteq E$ where

$$\mu(E \setminus F) < \epsilon$$

and $f_n \to f$ uniformly on F.

By Egoroff's Theorem, if we have a sequence $\{f_n\}$ of μ-measurable functions such that $f_n \to f$ pointwise on a set of finite measure, then $f_n \to f$ almost uniformly; i.e. if $\forall\ x \in E \setminus N(\mu)$ we have $f_n(x) \to f(x)$, then $f_n \to f$ almost uniformly on E.

Proposition 131. Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space. If $\{f_n\}$ is a sequence of μ-measurable functions such that $f_n \to f$ almost uniformly on $E \in \mathcal{S}$, then $f_n \to f$ pointwise on $E \setminus N(\mu)$.

Proof. For each m choose $F_m \subseteq E$ such that

$$\mu(E \setminus F_m) < \frac{1}{m}$$

and $f_m \to f$ uniformly on F_m. Let $G = \bigcup_{m=1}^{\infty} F_m$, then

$$E \setminus G \subseteq E \setminus F_m \quad \forall\ m$$

which implies

$$\mu(E \setminus G) = 0$$

yet $f_m \to f$ uniformly on each $F_m \implies f_m \to f$ pointwise on each F_m and so $f_m \to f$ pointwise on $\bigcup_{m=1}^{\infty} F_m = G$ and hence on E almost everywhere. $\square$

Definition 132. Let $(X, \mathcal{S}, \mu)$ be a measure space, let B a Banach space, let $\{f_n\}$ be a sequence of μ-measurable functions, and let $E \in \mathcal{S}$. Then $f_n \to f$ almost uniformly cauchy on E, if $\forall\ \epsilon > 0\ \exists\ F \in \mathcal{S}$ such that $F \subseteq E$ where

$$\mu(E \setminus F) < \epsilon$$

such that $\{f_n\}$ is uniformly cauchy on F; i.e. $\forall\ \delta > 0\ \exists\ N$ such that

$$m, n \geq N \implies ||f_m(x) - f_n(x)||_B < \delta \quad \forall\ x \in F$$

Proposition 133. Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space. If $\{f_n\}$ is a sequence of μ-measurable functions which are almost uniformly Cauchy on $E \in \mathcal{S}$, then $\exists\ f$ such that $f_n \to f$ almost uniformly on E.

Proof. Given $\epsilon > 0$, then since $\{f_n\}$ is almost uniformly Cauchy on E, $\exists\ F \in \mathcal{S}$ such that $F \subseteq E$, $\mu(E \setminus F) < \epsilon$, and $\{f_n\}$ is uniformly Cauchy on F.

Since $\{f_n\}$ is uniformly Cauchy on F, $\forall\ x \in F$ we have $\{f_n(x)\}$ is cauchy in B. Since B is a Banach space it is complete, and so $\{f_n(x)\}$ converges in B, so define

$$f : E \to B, \text{ by } f(x) = \begin{cases} \lim f_n(x), & x \in F \\ 0, & x \in E \setminus F \end{cases}$$

to show that $f_n \to f$ uniformly on F, we note that since $\{f_n\}$ is uniformly Cauchy on F, for any $\delta > 0$, $\exists\ N_1$ such that

$$n, m \geq N_1 \implies ||f_m(x) - f_n(x)||_B < \frac{\delta}{2} \quad \forall\ x \in F$$

in addition, for each $x \in F$ since $f_n(x) \to f(x)$, $\exists\ N_2$ such that

$$n \geq N_2 \implies ||f_n(x) - f(x)||_B < \frac{\delta}{2}$$

so letting $N = \max\{N_1, N_2\}$, and fixing $m > N$, we have for any $n \geq N$ that

$$\begin{aligned} ||f(x) - f_n(x)||_B &\leq ||f(x) - f_m(x)||_B + ||f_m(x) - f_n(x)||_B \\ &\leq \frac{\delta}{2} + \frac{\delta}{2} \\ &= \delta \end{aligned}$$

and so $f_n \to f$ uniformly on F, and thus almost uniformly on E. $\square$

Definition 134. Let $(X, \mathcal{S}, \mu)$ be a measure space with $E \in \mathcal{S}$, let B a Banach space, and let $\{f_n\}$ be a sequence of $\mathcal{S}$-measurable B-valued functions, then $\{f_n\}$ converges in measure on E to $f \in \mathcal{S}$-measurable if $\forall\ \epsilon > 0$

$$\mu(\{x \in E : ||f(x) - f_n(x)|| \geq \epsilon\}) \to 0 \text{ as } n \to \infty$$

be careful to note that when dealing with these sets we must have

$$\begin{aligned} &\{x \in E : ||f(x) - g(x)||_B > \epsilon\} \\ &\qquad \subseteq \left\{x \in E : ||f(x)||_B > \frac{\epsilon}{2}\right\} \cup \left\{x \in E : ||g(x)||_B > \frac{\epsilon}{2}\right\} \end{aligned}$$

and <u>NOT</u>

$$\begin{aligned} &\{x \in E : ||f(x) - g(x)||_B > \epsilon\} \\ &\qquad \subseteq \{x \in E : ||f(x)||_B > \epsilon\} \cup \{x \in E : ||g(x)||_B > \epsilon\} \end{aligned}$$

consider

$$|a| < \frac{\epsilon}{2} \text{ and } |b| < \frac{\epsilon}{2} \implies |a + b| \leq |a| + |b| < \epsilon$$

then taking the negation we have

$$|a + b| \geq \epsilon \implies |a| \geq \frac{\epsilon}{2} \text{ or } |b| \geq \frac{\epsilon}{2}$$

for a concrete example in our case note that if $f(x) = \frac{\epsilon}{2}$ and $g(x) = -\frac{\epsilon}{2}$, then

$$f(x) - g(x) = \epsilon \implies x \in \{x \in E : ||f(x) - g(x)||_B > \epsilon\}$$

yet

$$x \notin \{x \in E : ||f(x)||_B > \epsilon\} \text{ and } x \notin \{x \in E : ||g(x)||_B > \epsilon\}$$

and so

$$\{x \in E : ||f(x) - g(x)||_B > \epsilon\} \supset \{x \in E : ||f(x)||_B > \epsilon\} \cup \{x \in E : ||g(x)||_B > \epsilon\}$$

Proposition 135. Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space. If $\{f_n\}$ is a sequence of μ-measurable functions such that $f_n \to f$ almost uniformly on $E \in \mathcal{S}$, then $\{f_n\}$ converges to f in measure.

Proof. Let $\epsilon > 0$ and $\delta > 0$ be given. Since $f_n \to f$ almost uniformly on E, choose $F \subseteq E$ such that

$$\mu(E \setminus F) < \delta$$

and $f_n \to f$ uniformly on F. Since B is a Banach space it is complete, and so $\{f_n(x)\}$ converges in B, say to $f(x) = \lim f_n(x)$, for each $x \in F$. So $\exists$ N such that

$$n \geq N \implies ||f_n(x) - f(x)||_B < \epsilon$$

then for $n \geq N$ we have

$$\{x \in E : ||f(x) - f_n(x)||_B > \epsilon\} \subseteq E \setminus F$$
$$\mu\Big(\{x \in E : ||f(x) - f_n(x)||_B > \epsilon\} \Big) \leq \mu(E \setminus F) < \delta \to 0$$

and so $\{f_n\}$ converges in measure to f. □

Proposition 136. Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space and let $E \in \mathcal{S}$. If $\{f_n\}$ is a sequence of $\mathcal{S}$-measurable functions such that $\{f_n\}$ converges to f in measure on E, and $\{f_n\}$ converges to g in measure in E, then $f = g$ almost everywhere on E.

Proof. By the triangle inequality we have

$$||f(x) - g(x)||_B \leq ||f(x) - f_n(x)||_B + ||f_n(x) - g(x)||_B$$

and so for any $\epsilon > 0$ we have

$$\begin{aligned}
\{x \in E :& ||f(x) - g(x)||_B > \epsilon\} \\
&\subseteq \Big\{x \in E : ||f(x) - f_n(x)||_B > \frac{\epsilon}{2}\Big\} \cup \Big\{x \in E : ||f_n(x) - g(x)||_B > \frac{\epsilon}{2}\Big\} \\
\implies \mu&\Big(\{x \in E : ||f(x) - g(x)||_B > \epsilon\}\Big) \\
&\leq \mu\Big(\{x \in E : ||f(x) - f_n(x)||_B > \tfrac{\epsilon}{2}\} \Big) + \mu\Big(\{x \in E : ||f_n(x) - g(x)||_B > \tfrac{\epsilon}{2}\} \Big)
\end{aligned}$$

then since $\{f_n\}$ converges to f in measure on E, and $\{f_n\}$ converges to g in measure in E we have

$$\mu\Big(\ \{x \in E : ||f(x) - f_n(x)||_B > \tfrac{\epsilon}{2}\}\ \Big) \to 0 \text{ as } n \to \infty$$
$$\mu\Big(\ \{x \in E : ||f_n(x) - g(x)||_B > \tfrac{\epsilon}{2}\}\ \Big) \to 0 \text{ as } n \to \infty$$

and hence $\mu\Big(\{x \in E : ||f(x) - g(x)||_B > \epsilon\}\Big) \to 0$; i.e.

$$\mu\Big(\{x \in E : f(x) \neq g(x)\}\Big) \to 0$$

so that $f = g$ almost everywhere on E. □

Definition 137. Let $(X, \mathcal{S}, \mu)$ be a measure space with $E \in \mathcal{S}$, let B a Banach space, and let $\{f_n\}$ be a sequence of $\mathcal{S}$-measurable B-valued functions, then $\{f_n\}$ is cauchy in measure on E if $\forall\ \epsilon > 0$

$$\mu(\{x \in E : ||f_m(x) - f_n(x)|| \geq \epsilon\}) \to 0 \text{ as } n, m \to \infty$$

Theorem 138 (Riesz-Weyl). Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space and let $E \in \mathcal{S}$. If $\{f_n\}$ is a sequence of $\mathcal{S}$-measurable B-valued functions which are cauchy in measure on E, then there is a subsequence $\{f_{n_k}\}$ that is almost uniformly cauchy.

Proof. Defining the integers n_k inductively, which we may do since $\{f_n\}$ is cauchy in measure, by $n_1 = 1$ and for $k > 1$ choosing n_k such that $n_k > n_{k-1}$, and so that

$$m, n \geq n_k \implies \mu\left(\left\{x \in E : ||f_m(x) - f_n(x)||_B \geq \frac{1}{2^k}\right\}\right) \leq \frac{1}{2^k}$$

given $\epsilon > 0$ select K such that

$$\sum_{k=K}^{\infty} \frac{1}{2^k} < \epsilon$$

and let

$$F = E \setminus \bigcup_{k=K}^{\infty} \left\{x \in E : ||f_{n_k}(x) - f_{n_{k+1}}(x)||_B \geq \frac{1}{2^k}\right\}$$

so by constructions we have

$$\begin{aligned}
\mu(E \setminus F) &= \mu\left(E \cap \left(E \cap \left(\bigcup_{k=K}^{\infty}\left\{x \in E : ||f_{n_k}(x) - f_{n_{k+1}}(x)||_B \geq \frac{1}{2^k}\right\}\right)^c\right)^c\right) \\
&= \mu\left(E \cap \left(E^c \cup \left(\bigcup_{k=K}^{\infty}\left\{x \in E : ||f_{n_k}(x) - f_{n_{k+1}}(x)||_B \geq \frac{1}{2^k}\right\}\right)\right)\right) \\
&= \mu\left(\varnothing \cup \left(\bigcup_{k=K}^{\infty}\left\{x \in E : ||f_{n_k}(x) - f_{n_{k+1}}(x)||_B \geq \frac{1}{2^k}\right\}\right)\right) \\
&\leq \sum_{k=K}^{\infty} \mu\left(\left\{x \in E : ||f_{n_k}(x) - f_{n_{k+1}}(x)||_B \geq \frac{1}{2^k}\right\}\right) \\
&\leq \sum_{k=K}^{\infty} \frac{1}{2^k} \\
&< \epsilon
\end{aligned}$$

to see that $\{f_{n_k}\}$ is uniformly cauchy on F, let $\delta > 0$ be given, and choose $N > K$ such that

$$\sum_{n=N}^{\infty} \frac{1}{2^n} < \delta$$

then for any $x \in F$ and $k > l > N$ we have

$$\begin{aligned}
||f_{n_k}(x) - f_{n_l}(x)||_B &\leq ||f_{n_k}(x) - f_{n_{k-1}}(x)||_B + ||f_{n_{k-1}}(x) - f_{n_{k-2}}(x)||_B \\
&\quad + \cdots + ||f_{n_{l+1}}(x) - f_{n_l}(x)||_B \\
&= \sum_{m=l}^{k-1} ||f_{n_{m+1}}(x) - f_{n_m}(x)||_B \\
&\leq \sum_{m=l}^{k-1} \frac{1}{2^m} \\
&\leq \sum_{m=N}^{\infty} \frac{1}{2^m} \\
&< \delta
\end{aligned}$$

and therefore $\{f_{n_k}\}$ is almost uniformly cauchy on E. □

Proposition 139. Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space and let $E \in \mathcal{S}$. If $\{f_n\}$ is a sequence of function which are cauchy in measure on E such that some subsequence $\{f_{n_k}\}$ converges almost uniformly to f on E, then $\{f_n\}$ converges in measure to f.

Proof. Given $\epsilon > 0$, note that

$$\begin{aligned}
&\{x \in E : ||f(x) - f_n(x)||_B > \epsilon\} \\
&\qquad \subseteq \left\{x \in E : ||f(x) - f_{n_k}(x)||_B > \frac{\epsilon}{2}\right\} \cup \left\{x \in E : ||f_{n_k}(x) - f_n(x)||_B > \frac{\epsilon}{2}\right\}
\end{aligned}$$

and since $f_{n_k} \to f$ almost uniformly on E, given $\delta > 0$, $\exists\ N_1$ such that

$$n_k \geq N_1 \implies \mu\Big(\{x \in E : ||f(x) - f_{n_k}(x)||_B > \tfrac{\epsilon}{2}\} \Big) < \frac{\delta}{2}$$

then as $\{f_n\}$ are cauchy in measure on E, $\exists\ N_2$ such that

$$n_k, n \geq N_2 \implies \mu\Big(\{x \in E : ||f_n(x) - f_{n_k}(x)||_B > \tfrac{\epsilon}{2}\} \Big) < \frac{\delta}{2}$$

so letting $N = \max\{N_1, N_2\}$, and fix $n_k > N$, then for any $n \geq N$ we have

$$\begin{aligned}
&\mu\Big(\{x \in E : ||f(x) - f_n(x)||_B > \epsilon\}\Big) \\
&\quad \leq \mu\Big(\{x \in E : ||f(x) - f_{n_k}(x)||_B > \tfrac{\epsilon}{2}\} \Big) + \mu\Big(\{x \in E : ||f_{n_k}(x) - f_n(x)||_B > \tfrac{\epsilon}{2}\} \Big) \\
&\quad < \frac{\delta}{2} + \frac{\delta}{2} \\
&\quad = \delta \to 0
\end{aligned}$$

and so $\{f_n\}$ converges in measure on E to f. □

now we finally have enough in place to get to our main objective, integration.

Definition 140. Let X be a set and $\mathcal{S}$ a σ-ring of subsets of X, and B a Banach Space. Then a function

$$f : X \to B$$

is a Simple Integrable Function, if it is a simple $\mathcal{S}$-measurable function and the preimage of each $b \in \mathrm{Im}(f)$ has finite measure; i.e. for each $f^{-1}(b) = E \in \mathcal{S}$ we have $\mu(E) < \infty$. Then the integral of $f = \sum_{i=1}^{n} b_i \chi_{E_i}$ is

$$\int f d\mu = \sum_{i=1}^{n} b_i \mu(E_i)$$

Proposition 141. If f, g are simple integrable functions then $f + g$ is a simple integrable function and

$$\int (f+g) d\mu = \int f d\mu + \int g d\mu$$

Proof. First suppose $f = \sum_{i=1}^{n} b_i \chi_{E_i}$, and $g = c\chi_F$, to get F contained in the E_i's let us set $E_{n+1} = F \setminus \bigsqcup_{i=1}^{n} E_i$ and $b_{n+1} = 0$, then

$$F \subseteq \bigsqcup_{i=1}^{n+1} E_i \implies F = \bigsqcup_{i=1}^{n+1} (F \cap E_i)$$

and

$$f = \sum_{i=1}^{n+1} b_i \chi_{E_i} = \sum_{i=1}^{n+1} b_i \Big(\chi_{E_i \cap F} + \chi_{E_i \setminus F} \Big)$$

$$g = \sum_{i=1}^{n+1} c \chi_{E_i \cap F}$$

and so

$$f + g = \sum_{i=1}^{n+1} (b_i + c) \chi_{E_i \cap F} + \sum_{i=1}^{n+1} b_i \chi_{E_i \setminus F}$$

and so $f + g$ is a simple $\mathcal{S}$-measurable function. The general case follows inductively. Where we then have

$$\begin{aligned}
\int (f+g) d\mu &= \sum_{i=1}^{n+1} (b_i + c) \mu(E_i \cap F) + \sum_{i=1}^{n+1} b_i \mu(E_i \setminus F) \\
&= \sum_{i=1}^{n+1} b_i \Big[\mu(E_i \cap F) + \mu(E_i \setminus F) \Big] + \sum_{i=1}^{n+1} c \mu(E_i \cap F) \\
&= \int f d\mu + \int g d\mu
\end{aligned}$$

□

Definition 142. Let $(X, \mathcal{S}, \mu)$ be a measure space and let B a Banach Space. Then a function

$$f : X \to B$$

that is a simple integrable function, has the L^1 semi-norm $|| \cdot ||_1$ defined by

$$||f||_1 = \int ||f(x)||_B d\mu(x)$$

Proposition 143. Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space. If f, g are simple integrable functions then

$$||f + g||_1 \leq ||f||_1 + ||g||_1$$

Proof. First note that for all $x \in X$ we have

$$||f(x) + g(x)||_B \leq ||f(x)||_B + ||g(x)||_B$$

and therefore

$$\begin{aligned}
||f + g||_1 &= \int ||f(x) + g(x)||_B d\mu(x) \\
&\leq \int \Big(||f(x)||_B + ||g(x)||_B \Big) d\mu(x) \\
&= \int ||f(x)||_B d\mu(x) + \int ||g(x)||_B d\mu(x) \\
&= ||f||_1 + ||g||_1
\end{aligned}$$

$\square$

Proposition 144. Let $(X, \mathcal{S}, \mu)$ be measure space and let $\{f_n\}$ be a sequence of simple integrable functions that is cauchy for $||\cdot||_1$. Then $\{f_n\}$ is cauchy in measure.

Proof. Since $\{f_n\}$ is cauchy for $||\cdot||_1$ we have

$$||f_n - f_m||_1 = \int ||f_n(x) - f_m(x)||_B d\mu(x) \to 0 \text{ as } n, m \to \infty$$

let $\epsilon > 0$ be given and let

$$E^\epsilon_{mn} = \{x \in E : ||f_m(x) - f_n(x)|| \geq \epsilon\}$$

then

$$\chi_{E^\epsilon_{mn}} \leq \frac{||f_m(x) - f_n(x)||_B}{\epsilon}$$

so

$$\mu(E^\epsilon_{mn}) = \int \chi_{E^\epsilon_{mn}} d\mu(x) \leq \int \frac{||f_m(x) - f_n(x)||}{\epsilon} d\mu(x) \to 0 \text{ as } m, n \to \infty$$

and so $\{f_n\}$ is cauchy in measure on E. $\square$

Proposition 145. Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space. If $\{f_n\}, \{g_n\}$ are sequences of simple integrable functions which are equivalent under $||\cdot||_1$; i.e.

$$||f_n - g_n||_1 \to 0 \text{ as } n \to \infty$$

and if $\{f_n\}$ converges to f is measure, then $\{g_n\}$ also converges to f in measure.

Proof. Given $\epsilon > 0$, note that

$$\begin{aligned}\{x \in X :&||f(x) - g_n(x)||_B > \epsilon\} \\ &\subseteq \left\{x \in X : ||f(x) - f_n(x)||_B > \frac{\epsilon}{2}\right\} \cup \left\{x \in X : ||f_n(x) - g_n(x)||_B > \frac{\epsilon}{2}\right\}\end{aligned}$$

and since $\{f_n\}$ converges to f in measure we have

$$\mu\Big(\left\{x \in X : ||f(x) - f_n(x)||_B > \tfrac{\epsilon}{2}\right\}\Big) \to 0 \text{ as } n \to \infty$$

additionally since $\{f_n\}, \{g_n\}$ are equivalent under $||\cdot||_1$, we have

$$\begin{aligned}\mu\Big(\left\{x \in X : ||f_n(x) - g_n(x)||_B > \tfrac{\epsilon}{2}\right\}\Big) &= \int \chi_{\left\{x \in X : ||f_n(x) - g_n(x)||_B > \frac{\epsilon}{2}\right\}} d\mu(x) \\ &\leq 2 \int \frac{||f_n(x) - g_n(x)||_B}{\epsilon} d\mu(x) \to 0 \text{ as } n \to \infty\end{aligned}$$

and so

$$\begin{aligned}
&\mu\Big(\{x \in X : ||f(x) - g_n(x)||_B > \epsilon\}\Big) \\
&\qquad \leq \mu\Big(\ \{x \in X : ||f(x) - f_n(x)||_B > \tfrac{\epsilon}{2}\}\ \Big) + \mu\Big(\ \{x \in X : ||f_n(x) - g_n(x)||_B > \tfrac{\epsilon}{2}\}\ \Big) \\
&\qquad \to 0 \text{ as } n \to \infty
\end{aligned}$$

and so $\{g_n\}$ also converges to f in measure. □

This gives rise to the next definition

Definition 146. Let $(X, \mathcal{S}, \mu)$ be a measure space and let B a Banach Space. Then a sequence $\{f_n\}$ of simple integrable functions is mean cauchy if it is a cauchy sequence with respect to $|| \cdot ||_1$; i.e.

$$\lim_{n,m} ||f_n - f_m||_1 = 0$$

Lemma 147. Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space. If $\{f_n\}$ is a mean cauchy sequence of simple integrable functions such that $f_n \to 0$ almost uniformly, then

$$||f_n||_1 \to 0$$

Proof. Let $\epsilon > 0$ be given. Then since $\{f_n\}$ is mean cauchy, choose $N \in \mathbb{N}$ such that

$$n, m \geq N \implies ||f_n - f_m||_1 < \epsilon$$

and let

$$E = \{x \in X : f_N(x) \neq 0\} = \operatorname{car}(f_N)$$

and since f_N is simple integrable we have $\mu(E) < \infty$. Now for $n \geq N$ we have

$$\begin{aligned}
\int_{E^c} ||f_n(x)||_B d\mu(x) &= \int_{E^c} ||f_n(x) - 0||_B d\mu(x) \\
&= \int_{E^c} ||f_n(x) - f_N(x)||_B d\mu(x) \quad f_N(x) = 0 \text{ for } x \in E^c \\
&\leq \int_X ||f_n(x) - f_N(x)||_B d\mu(x) \\
&= ||f_n - f_N||_1 \\
&< \epsilon
\end{aligned}$$

Now since $f_n \to 0$ almost uniformly, $\exists\ F \in \mathcal{S}$ such that $F \subseteq E$ where

$$\mu(E \setminus F) < \frac{\epsilon}{1 + ||f_N||_\infty}$$

and $f_n \to 0$ uniformly on F. And so we may choose $M > N$ such that for $n > M$ and $x \in F$ we have

$$||f_n(x)||_B < \frac{\epsilon}{1 + \mu(F)}$$

and so

$$\begin{aligned}
\int_F ||f_n(x)||_B d\mu(x) &\leq \int_F \frac{\epsilon}{1+\mu(F)} d\mu(x) \\
&= \frac{\epsilon}{1+\mu(F)} \cdot \mu(F) \\
&< \epsilon
\end{aligned}$$

and lastly, using the triangle inequality

$$\begin{aligned}
\int_{E\setminus F} ||f_n(x)||_B d\mu(x) &\leq \int_{E\setminus F} ||f_n(x) - f_N(x)||_B d\mu(x) + \int_{E\setminus F} ||f_N(x)||_B d\mu(x) \\
&\leq \int_X ||f_n(x) - f_N(x)||_B d\mu(x) + \int_{E\setminus F} ||f_N(x)||_B d\mu(x) \\
&\leq ||f_n - f_N||_1 + ||f_N||_\infty \int_{E\setminus F} d\mu(x) \qquad ||f_N(x)||_B \leq ||f_N||_\infty \\
&= ||f_n - f_N||_1 + ||f_N||_\infty \mu(E\setminus F) \\
&< \epsilon + ||f_N||_\infty \frac{\epsilon}{1+||f_N||_\infty} \\
&< 2\epsilon
\end{aligned}$$

then putting all the piece together we get for $n > M$

$$\begin{aligned}
||f_n||_1 &= \int_X ||f_n(x)||_B d\mu(x) \\
&= \int_{E^c} ||f_n(x)||_B d\mu(x) + \int_{E\setminus F} ||f_n(x)||_B d\mu(x) + \int_F ||f_n(x)||_B d\mu(x) \\
&< \epsilon + 2\epsilon + \epsilon \\
&= 4\epsilon
\end{aligned}$$

and so $||f_n||_1 \to 0$. $\square$

Proposition 148. Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space. If $\{f_n\}$ and $\{g_n\}$ are mean cauchy sequences of simple integrable functions, such that $f_n, g_n \to h$ is measure, then $\{f_n\}$ and $\{g_n\}$ are equivalent cauchy sequences; i.e.

$$\lim_{n,m\to\infty} ||f_n - g_m||_1 = 0$$

Proof. Since $\{f_n\}, \{g_m\}$ converge in measure to h and are mean cauchy, Riesz-Weyl says that $\exists$ subsequences $\{f_{n_k}\}, \{g_{m_k}\}$ that converge to h are almost uniformly. So it suffices to show that

$$\lim_{k\to\infty} ||f_{n_k} - g_{m_k}||_1 = 0$$

So define

$$h_k = f_{n_k} - g_{m_k}$$

then $\{h_n\}$ is a mean cauchy sequence of simple integrable functions such that $h_n \to 0$ almost uniformly, and from the previous Lemma we then have

$$||h_k||_1 \to 0$$

and therefore

$$\lim_{k\to\infty} ||f_{n_k} - g_{m_k}||_1 = 0$$

and so $\{f_n\}$ and $\{g_m\}$ are equivalent cauchy sequences. $\square$

Theorem 149. Let f be a $\mathcal{S}$-measurable B-valued function, then the following are equivalent

1. There is a mean cauchy sequence $\{f_n\}$ of ISFs that converge in measure to f.
2. There is a mean cauchy sequence $\{f_n\}$ of ISFs such that $f_n \to f$ almost uniformly.
3. There is a mean cauchy sequence $\{f_n\}$ of ISFs such that $f_n \to f$ pointwise almost everywhere.

f is μ-integrable if it satisfies one, and hence all, of these conditions.

Proof. (1) $\implies$ (2).

Riezs-Weyl gives a subsequence that converges almost uniformly.

(2) $\implies$ (3).

Riezs-Weyl gives a subsequence that converges almost uniformly, and hence pointwise.

(3) $\implies$ (1).

Since $\{f_n\}$ is mean cauchy we know that $\forall\ \epsilon > 0,\ \exists\ N \in \mathbb{N}$ such that

$$n, m \geq N \implies ||f_n - f_m||_1 = \int_X ||f_n - f_m||_B d\mu(x) < \epsilon$$

and hence, for any $\delta > 0$

$$n, m \geq N \implies ||f_n - f_m||_1 = \int_X ||f_n - f_m||_B d\mu(x) < \epsilon\delta$$

so suppose, for contradiction, that $\{f_n\}$ is not cauchy in measure, this implies that $\exists\ \epsilon, \delta$ such that $\forall\ N \in \mathbb{N}$ there exists $m, n \geq N$ where

$$\mu\Big(x \in X : ||f_n(x) - f_m(x)||_B \geq \epsilon\Big) \geq \delta$$

let $A \subset X$ be the set of points which satisfy $||f_n(x) - f_m(x)||_B \geq \epsilon$. Then

$$\begin{aligned}\int_X ||f_n - f_m||_B d\mu(x) &\geq \int_A ||f_n - f_m||_B d\mu(x)\\ &\geq \int_A \epsilon d\mu(x)\\ &= \epsilon\mu(A)\\ &\geq \epsilon\delta \quad \Rightarrow\Leftarrow\end{aligned}$$

and so we can conclude that $\{f_n\}$ is cauchy in measure. Then Riesz-Weyl says $\exists$ $\{f_{n_k}\}$ which converges almost uniformly, and hence almost everywhere and in measure, to an $\mathcal{S}$-measurable function g. Yet, $f_n \to f$ pointwise almost everywhere and thus $f_{n_k} \to f$ pointwise almost everywhere, and so $f = g$ almost everywhere. That is $\{f_{n_k}\}$ converges in measure to f. $\square$

from this we get the next definition.

Definition 150. Let f be a $\mathcal{S}$-measurable B-valued function, then f is μ-integrable if it satisfies one, and hence all, of the conditions.

1. There is a mean cauchy sequence $\{f_n\}$ of ISFs that converge in measure to f.
2. There is a mean cauchy sequence $\{f_n\}$ of ISFs such that $f_n \to f$ almost uniformly.
3. There is a mean cauchy sequence $\{f_n\}$ of ISFs such that $f_n \to f$ pointwise almost everywhere.

with the μ-integral of f defined by

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$$

Definition 151. $\mathcal{L}^1(X, \mathcal{S}, \mu, B)$ is the vector space of μ-integrable B-valued functions; i.e. if $f \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$, then $\exists$ a mean cauchy sequence $\{f_n\}$ of simple integrable functions such that $f_n \to f$ in measure, almost uniformly, and pointwise almost everywhere.

Theorem 152. $\mathcal{L}^1(X, \mathcal{S}, \mu, B)$ is a vector space.

Proof. Let $f, g \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$, then $\exists$ sequences $\{f_n\}, \{g_n\}$ of simple integrable functions which are mean cauchy such that $f_n \to f$ and $g_n \to g$ pointwise almost everywhere. Then $\{f_n + g_n\}$ is a sequence of simple integrable functions which is mean cauchy and $f_n + g_n \to f + g$ pointwise almost everywhere and so

$$\begin{aligned}\int (f+g)d\mu &= \lim_{n\to\infty} \int (f_n + g_n)d\mu\\ &= \lim_{n\to\infty} \int f_n d\mu + \lim_{n\to\infty} \int g_n d\mu\\ &= \int f d\mu + \int g d\mu\end{aligned}$$

and so $f+g \in \mathcal{L}^1(X,\mathcal{S},\mu,B)$.

Next if $c \in \mathbb{R}$, then $\{cf_n\}$ is a sequence of simple integrable functions which are mean cauchy such that $cf_n \to cf$ pointwise almost everywhere, then

$$\int cf d\mu = c\int f = c\lim_{n\to\infty}\int f_n d\mu = \lim_{n\to\infty}\int cf_n d\mu$$

thus $cf \in \mathcal{L}^1(X,\mathcal{S},\mu,B)$.

Finally if $\{O_n\}$ is a sequence of simple integrable functions which are mean cauchy such that $O_n \to 0$ pointwise almost everywhere, then

$$0 = \int 0 d\mu = \lim_{n\to\infty}\int O_n d\mu$$

and so $0 \in \mathcal{L}^1(X,\mathcal{S},\mu,B)$.

$\therefore \mathcal{L}^1(X,\mathcal{S},\mu,B)$ is a vector space. $\square$

We should also note that if $f \in \mathcal{L}^1(X,\mathcal{S},\mu,B)$ then $x \mapsto ||f(x)||_B \in \mathcal{L}^1(X,\mathcal{S},\mu,\mathbb{R})$. And as we proved in the first section this space has a completion.

Definition 153. $L^1(X,\mathcal{S},\mu,B)$ is the complete normed vector space defined by

$$L^1(X,\mathcal{S},\mu,B) = \mathcal{L}^1(X,\mathcal{S},\mu,B)/\sim$$

where $\sim$ is the equivalence class of simple integrable functions which are mean cauchy.

Definition 154. Let $(X,\mathcal{S},\mu)$ be a measure space and let B a Banach Space. Then a sequence $\{f_n\}$ of simple integrable functions converges in mean to a μ-integrable function f if

$$\lim_n ||f-f_n||_1 = 0$$

Lemma 155. Let $(X,\mathcal{S},\mu)$ be measure space and B a Banach space. If $\{f_n\}$ is a mean cauchy sequence of simple integrable functions such that $f_n \to f$ in measure, or almost uniformly, or pointwise almost everywhere, then $f_n \to f$ in mean.

Proof. For each fixed n, $\{f_m - f_n\}$ is a mean cauchy sequence of simple integrable functions such that

$$f_m - f_n \to f - f_n$$

in measure, or almost uniformly, or pointwise almost everywhere, so that

$$\begin{aligned} ||f-f_n||_1 &= \int ||f(x)-f_n(x)||_B d\mu(x) \\ &= \lim_{m\to\infty}\int ||f_m(x)-f_n(x)||_B d\mu(x) \\ &= \lim_{m\to\infty} ||f_m - f_n||_1 \end{aligned}$$

Given $\epsilon > 0$ choose $N \in \mathbb{N}$ such that

$$n, m > N \implies ||f_m - f_n||_1 < \epsilon$$

that is for $n > N$ we have

$$||f - f_n||_1 < \epsilon$$

and so $f_n \to f$ in mean. $\square$

Proposition 156. If $f \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$, then $\text{car}(f)$ is σ-finite.

Proof. Since $f \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$, $\exists$ a mean cauchy sequence $\{f_n\}$ of simple integrable functions such that $f_n \to f$ pointwise almost everywhere. Let

$$E_n = \text{car}(f_n)$$

then since the f_n's are simple integrable functions we have

$$\mu(E_n) < \infty$$

then

$$\text{car}(f) \subseteq \bigcup_{n=1}^{\infty} \text{car}(f_n) < \infty$$

and so $\text{car}(f)$ is σ-finite. $\square$

Proposition 157. Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space. If $f \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$, then $\forall\ \epsilon > 0,\ \exists\ E \in \mathcal{S}$ such that

$$\mu(E) < \infty$$

and

$$\left|\left| \int_{X \setminus E} f(x) d\mu(x) \right|\right|_B < \epsilon$$

Proof. Since f is μ-integrable, and from Lemma 155 this implies convergence in mean so from our mean cauchy sequence $\{f_n\}$ of simple integrable functions choose f_n such that

$$||f - f_n||_1 = \int ||f(x) - f_n(x)||_B d\mu(x) < \epsilon$$

since f_n is a simple integrable function we have

$$\mu\big(\text{car}(f_n)\big) < \infty$$

so let $E = \text{car}(f_n)$, then since $f_n(x) = 0 \;\forall\; x \in X \setminus E = E^c$ we have

$$\begin{aligned}
\left|\left| \int_{X\setminus E} f(x)d\mu(x) \right|\right|_B &= \left|\left| \int_{X\setminus E} f(x)d\mu(x) - 0 \right|\right|_B \\
&= \left|\left| \int_{X\setminus E} \big(f(x) - f_n(x)\big)d\mu(x) \right|\right|_B \\
&\le \int_{X\setminus E} \big|\big|f(x) - f_n(x)\big|\big|_B d\mu(x) \\
&\le \int_X \big|\big|f(x) - f_n(x)\big|\big|_B d\mu(x) \\
&= ||f - f_n||_1 \\
&< \epsilon
\end{aligned}$$

□

Definition 158. Let $(X, \mathcal{S}, \mu)$ be a measure space and let B a Banach Space. for $f \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$ and $E \in \mathcal{S}$ the indefinite integral of f is defined to be

$$\mu_f(E) = \int_E f(x)d\mu(x) = \int f\chi_E d\mu$$

Proposition 159 (**Absolute Continuity**)**.** Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space. If $f \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$, then $\forall\; \epsilon > 0,\; \exists\; \delta > 0$ such that if

$$\mu(E) < \delta$$

then

$$\big|\big|\mu_f(E)\big|\big|_B < \epsilon$$

Proof. Let $\epsilon > 0$ be given and choose a simple integrable function g such that

$$||f - g|| < \frac{\epsilon}{2}$$

and select $\delta = \frac{\epsilon}{2||g||_\infty}$ that is

$$\mu(E) < \frac{\epsilon}{2||g||_\infty}$$

then

$$\begin{aligned}
\big|\big|\mu_f(E)\big|\big|_B &= \big|\big|\mu_f(E) - \mu_g(E) + \mu_g(E)\big|\big|_B \\
&\leq \big|\big|\mu_f(E) - \mu_g(E)\big|\big|_B + \big|\big|\mu_g(E)\big|\big|_B \\
&= \left|\left|\int_E f(x)d\mu(x) - \int_E g(x)d\mu(x)\right|\right|_B + \left|\left|\int_E g(x)d\mu(x)\right|\right|_B \\
&\leq \int_E \big|\big|f(x) - g(x)\big|\big|_B d\mu(x) + \int_E \big|\big|g(x)\big|\big|_B d\mu(x) \\
&\leq \int_X \big|\big|f(x) - g(x)\big|\big|_B d\mu(x) + \int_E ||g||_\infty d\mu(x) \\
&= ||f-g||_1 + ||g||_\infty \mu(E) \\
&< \frac{\epsilon}{2} + ||g||_\infty \frac{\epsilon}{2||g||_\infty} \\
&= \epsilon
\end{aligned}$$

□

Proposition 160. Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space. If $f \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$, then μ_f is a B-valued measure on $\mathcal{S}$.

Proof. To do this we must show that μ_f is countably additive. First we note that for any $g, f \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$ and any $E \in \mathcal{S}$ we have

$$\begin{aligned}
\big|\big|\mu_f(E) - \mu_g(E)\big|\big|_B &= \left|\left|\int_E f(x)d\mu(x) - \int_E g(x)d\mu(x)\right|\right|_B \\
&\leq \int_E \big|\big|f(x) - g(x)\big|\big|_B d\mu(x) \\
&\leq \int_X \big|\big|f(x) - g(x)\big|\big|_B d\mu(x) \\
&= ||f-g||_1
\end{aligned}$$

let $E \in \mathcal{S}$ such that $\mu(E) < \infty$ and $\epsilon > 0$ be given, and let

$$E = \bigsqcup_{i=1}^{\infty} E_i$$

since f is μ-integrable, by Lemma 155 this implies convergence in mean so from our mean cauchy sequence $\{f_n\}$ of simple integrable functions choose f_n such that

$$||f - f_n||_1 = \int ||f(x) - f_n(x)||_B d\mu(x) < \frac{\epsilon}{3}$$

since f_n is a simple integrable function μ_{f_n} is countably additive, so choose $N \in \mathbb{N}$ such that

$$m > N \implies \left|\left|\mu_{f_n}(E) - \mu_{f_n}\left(\bigsqcup_{i=1}^{m} E_i\right)\right|\right|_B < \frac{\epsilon}{3}$$

and so for $m > N$ we have

$$
\begin{aligned}
&\left|\left|\mu_f(E) - \mu_f\left(\bigsqcup_{i=1}^{m} E_i\right)\right|\right|_B \\
&\quad \leq \big|\big|\mu_f(E) - \mu_{f_n}(E)\big|\big|_B + \left|\left|\mu_{f_n}(E) - \mu_{f_n}\left(\bigsqcup_{i=1}^{m} E_i\right)\right|\right|_B + \left|\left|\mu_{f_n}\left(\bigsqcup_{i=1}^{m} E_i\right) - \mu_f\left(\bigsqcup_{i=1}^{m} E_i\right)\right|\right|_B \\
&\quad = \left|\left|\int_E f(x)d\mu(x) - \int_E f_n(x)d\mu(x)\right|\right|_B + \left|\left|\mu_{f_n}(E) - \mu_{f_n}\left(\bigsqcup_{i=1}^{m} E_i\right)\right|\right|_B \\
&\qquad + \left|\left|\int_{\bigsqcup_{i=1}^{m} E_i} f_n(x)d\mu(x) - \int_{\bigsqcup_{i=1}^{m} E_i} f(x)d\mu(x)\right|\right|_B \\
&\quad < \int_E \big|\big|f(x) - f_n(x)\big|\big|_B d\mu(x) + \frac{\epsilon}{3} + \int_{\bigsqcup_{i=1}^{m} E_i} \big|\big|f_n(x) - f(x)\big|\big|_B d\mu(x) \\
&\quad \leq \int_X \big|\big|f(x) - f_n(x)\big|\big|_B d\mu(x) + \frac{\epsilon}{3} + \int_X \big|\big|f_n(x) - f(x)\big|\big|_B d\mu(x) \\
&\quad < ||f - f_n||_1 + \frac{\epsilon}{3} + ||f_n - f||_1 \\
&\quad < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&\quad = \epsilon
\end{aligned}
$$

□

Theorem 161 (**Lebesgue Dominated Convergence**)**.** Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space, and let $\{f_n\} \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$; i.e. a sequence of μ-integrable functions, that converge pointwise almost everywhere to a function f. Suppose there $\exists\ g \in \mathcal{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$ such that

$$||f_n(x)||_B \leq g(x)$$

for all n and for all x, or almost everywhere for each n. Then $\{f_n\}$ is a mean cauchy sequence. And so $\{f_n\}$ converges to f in mean, $f \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$, and

$$\int f d\mu = \lim \int f_n d\mu$$

Proof. Let $\epsilon > 0$ be given, then since $g \in \mathcal{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$ Proposition 157 says $\exists\ E \in \mathcal{S}$ such that

$$\mu(E) < \infty \quad \text{and} \quad \left|\int_{X \setminus E} g(x)d\mu(x)\right| < \frac{\epsilon}{6}$$

then $\forall\ n, m$ we have

$$\begin{aligned}
\int_{X\setminus E} \big|\big|f_m(x) - f_n(x)\big|\big|_B d\mu(x) &\leq \int_{X\setminus E} \Big(\big|\big|f_m(x)\big|\big|_B + \big|\big|f_n(x)\big|\big|_B\Big) d\mu(x) \\
&= \int_{X\setminus E} \big|\big|f_m(x)\big|\big|_B d\mu(x) + \int_{X\setminus E} \big|\big|f_n(x)\big|\big|_B d\mu(x) \\
&\leq \int_{X\setminus E} g(x)d\mu(x) + \int_{X\setminus E} g(x)d\mu(x) \qquad \text{since } ||f_n(x)||_B \leq g(x)\ \forall\ n \\
&= 2\int_{X\setminus E} g(x)d\mu(x) \\
&< 2\frac{\epsilon}{6} \\
&= \frac{\epsilon}{3}
\end{aligned}$$

Next, since $g \in \mathcal{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$ we also have that the indefinite integral μ_g is absolutely continuous, so we may choose $\delta > 0$ such that for any $G \in \mathcal{S}$

$$\mu(G) < \delta \implies \big|\mu_g(G)\big| = \left|\int_G g(x)d\mu(x)\right| < \frac{\epsilon}{6}$$

Now, since $f_n \to f$ pointwise almost everywhere and $\mu(E) < \infty$, Egoroff's Theorem then says that $f_n \to f$ almost uniformly on E. Therefore we may choose $F \in \mathcal{S}$ with $F \subseteq E$ such that

$$\mu(E \setminus F) < \delta$$

and $f_n \to f$ uniformly on F. Then $\forall\ n, m$ we have

$$\begin{aligned}
\int_{E\setminus F} \big|\big|f_m(x) - f_n(x)\big|\big|_B d\mu(x) &\leq \int_{E\setminus F} \Big(\big|\big|f_m(x)\big|\big|_B + \big|\big|f_n(x)\big|\big|_B\Big) d\mu(x) \\
&= \int_{E\setminus F} \big|\big|f_m(x)\big|\big|_B d\mu(x) + \int_{E\setminus F} \big|\big|f_n(x)\big|\big|_B d\mu(x) \\
&\leq \int_{E\setminus F} g(x)d\mu(x) + \int_{E\setminus F} g(x)d\mu(x) \qquad \text{since } ||f_n(x)||_B \leq g(x)\ \forall\ n \\
&= 2\int_{E\setminus F} g(x)d\mu(x) \\
&= 2\mu_g(E \setminus F) \\
&< 2\frac{\epsilon}{6} \\
&= \frac{\epsilon}{3}
\end{aligned}$$

Finally, since $f_n \to f$ uniformly on F we may choose $N \in \mathbb{N}$ such that

$$n, m \geq N \implies \big|\big|f_m(x) - f_n(x)\big|\big|_B < \frac{\epsilon}{3\mu(F)}$$

then $\forall\ x \in F$ and $\forall\ n, m > N$ we have

$$\int_F \left|\left|f_m(x) - f_n(x)\right|\right|_B d\mu(x) < \int_F \frac{\epsilon}{3\mu(F)} d\mu(x) = \frac{\epsilon}{3\mu(F)}\mu(F) = \frac{\epsilon}{3}$$

and so, for all $n, m > N$ we get

$$\begin{aligned}
||f_n - f_m||_1 &= \int_X \left|\left|f_m(x) - f_n(x)\right|\right|_B d\mu(x) \\
&= \int_{X\setminus E} \left|\left|f_m(x) - f_n(x)\right|\right|_B d\mu(x) + \int_{E\setminus F} \left|\left|f_m(x) - f_n(x)\right|\right|_B d\mu(x) \\
&\quad + \int_F \left|\left|f_m(x) - f_n(x)\right|\right|_B d\mu(x) \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon
\end{aligned}$$

And thus, $\{f_n\}$ is a mean cauchy sequence.

Now, since $\{f_n\}$ is a mean cauchy sequence of simple integrable functions such that $f_n \to f$ pointwise almost everywhere then f is μ-integrable, or $f \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$ and

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$$

where Lemma 155 then says that $\{f_n\}$ converges to f in mean. □

Proposition 162. Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space, and let f be a μ-measurable B-valued function. If $\exists\ g \in \mathcal{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$ such that

$$||f(x)||_B \leq g(x)$$

almost everywhere, then $f \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$; i.e. f is μ-integrable.

Proof. Since f is μ-measurable, $\exists\ \{f_n\}$ of simple $\mathcal{S}$-measurable such that $f_n \to f$ almost everywhere. For each n choose

$$E_n = \left\{x \in X : 2g(x) - \left|\left|f_n(x)\right|\right|_B \geq 0\right\}$$

and define

$$h_n(x) = \begin{cases} f_n(x), & \left|\left|f_n(x)\right|\right|_B \leq 2g(x) \\ 0, & \text{otherwise} \end{cases}$$

then

$$h_n = f_n \chi_{E_n}$$

since $g \in \mathcal{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$ we have $\text{car}(g)$ is σ-finite, and so, by construction, for each E_n we have

$$\mu(E_n) < \infty$$

and so each h_n is a simple integrable function, and $\{h_n\} \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$. And note, since $f_n \to f$ almost everywhere, and the h_n's are defined in terms of the f_n's this implies that $h_n \to f$ almost everywhere, or pointwise almost everywhere. Then since

$$||h_n(x)||_B \leq 2g(x)$$

for all n and for all x, Lebesgue Dominated Convergence says that $\{h_n\}$ is a mean cauchy sequence of simple integrable functions and therefore the limit function $f \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$. □

Theorem 163 (**Monotone Convergence Theorem**)**.** Let $(X, \mathcal{S}, \mu)$ be measure space, and let $\{f_n\} \in \mathcal{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$ such that $f_n \geq 0$ and is non-decreasing; i.e.

$$f_{n+1} \geq f_n \quad \forall\ n$$

if $\exists\ C \in \mathbb{R}$ such that

$$||f_n||_1 = \int f_n(x) d\mu(x) < C \quad \forall\ n$$

then $\{f_n\}$ is a mean cauchy sequence and $\exists\ f \in \mathcal{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$ such that $f_n \to f$ pointwise almost everywhere. That is

$$\int f d\mu = \lim_{n \to \infty} \int f_n d\mu$$

Proof. Since $f_n \leq f_{n+1}\ \forall\ n$ we have

$$\int f_n(x) d\mu \leq \int f_{n+1}(x) d\mu \quad \forall\ n$$

and since

$$\int f_n(x) d\mu(x) < C \quad \forall\ n$$

we have $\left\{ \int f_n d\mu \right\}$ is a sequence which converges and so is cauchy.

Let $\epsilon > 0$ be given, then $\exists\ N \in \mathbb{N}$ such that

$$n, m > N \implies \left| \int f_n(x) d\mu - \int f_m(x) d\mu \right| < \epsilon$$

so let $n > m$, then since $f_k > 0\ \forall\ k$ we have

$$\begin{aligned} \left| \int f_n(x) d\mu - \int f_m(x) d\mu \right| &= \left| \int \big(f_n(x) - f_m(x)\big) d\mu \right| \\ &= \int |f_n(x) - f_m(x)| d\mu \\ &= ||f_n - f_m||_1 \\ &< \epsilon \end{aligned}$$

and so $\{f_n\}$ is mean cauchy. Then since $\mathcal{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$ is complete, $\exists\ f \in \mathcal{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$ such that

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu$$

$\square$

But there is a more general version of this theorem which we now give.

Theorem 164 (**More general Monotone Convergence Theorem**). Let $(X, \mathcal{S}, \mu)$ be measure space with Banach space $\mathbb{R}$, and let $\{f_n\} \in \mathcal{S}$ satisfying

$$0 \le f_1(x) \le f_2(x) \le \cdots f_n(x) \le \cdots \quad \forall\ x \in X$$

let

$$f(x) = \lim_{n\to\infty} f_n(x)$$

then $\lim_{n\to\infty} \int f_n d\mu$ and $\int f d\mu$ both exist and

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$$

Proof. First, since f is the pointwise limit of measurable functions and

$$f \ge 0$$

f is measurable and

$$\int f d\mu$$

exists in $\mathbb{R} \setminus \{0\}$.

Since $\{f_n(x)\}$ is a monotone increasing sequence and each $f_n \ge 0$, the same is true for $\left\{ \int f_n d\mu \right\}$, and so

$$\lim_{n\to\infty} \int f_n d\mu$$

exists in $\mathbb{R} \setminus \{0\}$. Moreover we have

$$\int f_n d\mu \le \int f_{n+1} d\mu \le \int f d\mu \quad \forall\ n$$

and so

$$\lim_{n\to\infty} \int f_n d\mu \le \int f d\mu$$

For the reverse inequality let

$$g : X \to [0, \infty)$$

be a simple measurable function such that

$$0 \le g \le f$$

and fix $0 < t < 1$. Then defining

$$E_n = \{x \in X : f_n(x) \ge tg(x)\}$$

we have an increasing sequence of measurable sets such that

$$E_1 \subseteq E_2 \subseteq \cdots \subseteq X$$

Then, for any $x \in X$ if

$$f(x) = 0 \implies f_n(x) = 0 \ \forall \ n$$

and since $g \le f$ we also have

$$tg(x) = 0 \implies x \in E_n \ \forall \ n$$

if $f(x) > 0$, then

$$f(x) \ge g(x) \implies f(x) > tg(x) \qquad \text{since } 0 < t < 1$$

and since $f_n \to f$ monotonically $f_n(x) > tg(x)$ eventually, thus $x \in E_n$ for some n. And so, for any $x \in X$ we have that

$$x \in \bigcup_{n=1}^{\infty} E_n \implies \bigcup_{n=1}^{\infty} E_n = X$$

then for every n we have

$$\int_X f_n d\mu \ge \int_{E_n} f_n d\mu \ge t \int_{E_n} g d\mu$$

and since $\int_{E_n} g d\mu = \mu_g(E_n)$ where μ_g is a measure and hence countably additive, so disjointizing the E_n's if necessary, and by the simplicity of $g = \sum_{i=1}^{N} c_i \chi_{A_i}$ we have

$$\begin{aligned}
\lim_{n\to\infty} \mu_g(E_n) = \lim_{n\to\infty} \sum_{i=1}^{N} c_i \mu(A_i \cap E_n) \to \sum_{i=1}^{N} c_i \mu(A_i \cap X) \\
= \sum_{i=1}^{N} c_i \mu(A_i) \\
= \int_X g d\mu
\end{aligned}$$

giving

$$\lim_{n\to\infty} \int_X f_n d\mu \ge \lim_{n\to\infty} t \int_{E_n} g d\mu = t \int_X g d\mu$$

then since $t \in (0, 1)$ is arbitrary we conclude that

$$\lim_{n\to\infty} \int_X f_n d\mu \geq \int_X g d\mu$$

and since $g \leq f$ is an arbitrary simple function, taking

$$\sup_g \{g \in \mathcal{S} : 0 \leq g \leq f\}$$

we get

$$\lim_{n\to\infty} \int_X f_n d\mu \geq \int_X f d\mu$$

and thus we can conclude

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$$

$\square$

Lemma 165 (**Fatou's Lemma**). Let $(X, \mathcal{S}, \mu)$ be measure space, and let $\{f_n\} \in \mathcal{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$ such that $f_n \geq 0 \; \forall \; n$. Then

$$\int \liminf\{f_n\} d\mu \leq \liminf \int f_n d\mu$$

Proof. Set

$$g_n(x) = \inf\{f_i(x) : n \leq i < \infty\}$$

then

$$\lim_{n\to\infty} g_n(x) = \liminf f_n(x)$$

and since $g_1(x) \leq g_2(x) \leq \cdots$ we have $\{g_n\}$ is non-decreasing, or monotonic, and so the general version of the Monotone Convergence Theorems says

$$\int \liminf_{n\to\infty} f_n(x) d\mu = \int \lim_{n\to\infty} g_n(x) d\mu = \lim_{n\to\infty} \int g_n(x) d\mu$$

yet, since $g_n(x) \leq f_n(x)$ pointwise $\forall \; n$ we then have

$$\int g_n(x) d\mu \leq \int f_n(x) d\mu \quad \forall \; n$$

and thus,

$$\liminf \int f_n(x) d\mu \geq \lim_{n\to\infty} \int g_n(x) d\mu = \int \liminf_{n\to\infty} f_n(x) d\mu$$

and so we have

$$\int \liminf\{f_n\} d\mu \leq \liminf \int f_n d\mu$$

$\square$

Definition 166. Let $(X, \mathcal{S}, \mu)$ be a measure space and let B a Banach Space. For $0 < p < \infty$ the space of μ-measurable, B-valued functions f such that $||f(\cdot)||^p$ is μ-integrable is denoted $\mathcal{L}^p(X, \mathcal{S}, \mu, B)$, then the function

$$|| \cdot ||_p : \mathcal{L}^p(X, \mathcal{S}, \mu, B) \to \mathbb{R}$$

defined by

$$||f||_p = \left(\int ||f(x)||^p d\mu(x) \right)^{1/p}$$

is the L^p-norm.

Theorem 167. Let $(X, \mathcal{S}, \mu)$ be measure space and B a Banach space. For $1 \leq p \leq \infty$, if $f, g \in \mathcal{L}^p(X, \mathcal{S}, \mu, B)$ then $f+g \in \mathcal{L}^p(X, \mathcal{S}, \mu, B)$, and so $\mathcal{L}^p(X, \mathcal{S}, \mu, B)$ is a vector space of functions.

Proof.

$$\begin{aligned} ||f(x) + g(x)||^p &\leq (||f(x)|| + ||g(x)||)^p \\ &\leq (2 \max\{f(x), g(x)\})^p \\ &\leq 2^p(||\overset{\in \mathcal{L}^1}{f(x)}||^p + ||\overset{\in \mathcal{L}^1}{g(x)}||^p) \in \mathcal{L}^1 \end{aligned}$$

and so $||f(x) + g(x)||^p$ is dominated by an integrable function and so must also be integrable by Lebesgue Dominated Convergence Theorem. $\square$

And we take note that, if $f \in \mathcal{L}^p(X, \mathcal{S}, \mu, B)$, then $x \mapsto ||f(x)||^p \in \mathcal{L}^1(X, \mathcal{S}, \mu, B)$.

Proposition 168. Let $(X, \mathcal{S}, \mu)$ be measure space with Banach space $\mathbb{R}$, and let $f \in \mathcal{L}^2(X, \mathcal{S}, \mu, \mathbb{R} \text{ or } \mathbb{C})$. Then

$$x \mapsto |f(x)|^2 \in \mathcal{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$$

and $\mathcal{L}^2(X, \mathcal{S}, \mu, \mathbb{R} \text{ or } \mathbb{C})$ satisfies Cauchy-Schwartz; i.e. for $f, g \in \mathcal{L}^2(X, \mathcal{S}, \mu, \mathbb{R} \text{ or } \mathbb{C})$

$$|\langle f, g \rangle| \leq ||f||_2 ||g||_2$$

Proof. For $r, s \in \mathbb{R} \setminus \{0\}$

$$\begin{aligned} 0 \leq (r-s)^2 &= r^2 - 2rs + s^2 \\ \implies 2rs &\leq r^2 + s^2 \end{aligned}$$

which implies

$$2\left|f(x)\overline{g(x)}\right| \leq |f(x)|^2 + |g(x)|^2 \in \mathcal{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$$

and so by Lebesgue Dominated Converge $x \mapsto \left|f(x)\overline{g(x)}\right| \in \mathcal{L}^1(X, \mathcal{S}, \mu, \mathbb{R})$. So set

$$\langle f, g \rangle := \int f(x)\overline{g(x)} d\mu(x)$$

then

$$2|\langle f,g\rangle| \leq \int |f(x)|^2 d\mu(x) + \int |g(x)|^2 d\mu(x) = ||f||_2^2 + ||g||_2^2$$

if, in addition, $||f||_2 = 1$ and $||g||_2 = 1$, then

$$|\langle f,g\rangle| \leq 1$$

so for any $f, g \in \mathcal{L}^2(X, \mathcal{S}, \mu, \mathbb{R} \text{ or } \mathbb{C})$ scale by setting $f = \frac{f}{||f||_2}$ and $g = \frac{g}{||g||_2}$, then

$$\frac{|\langle f,g\rangle|}{||f||_2||g||_2} \leq 1$$
$$\implies |\langle f,g\rangle| \leq ||f||_2||g||_2$$

$\square$