MATH-3580 Test Two Review

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Test Two covers Chapter 2 - Parts 2 and 3 and Chapter 3 - Part 1 (up to Theorem 5)

Contents

1 Definitions

1.1 Metric Space

In mathematics, space = set + structure(s).

Definition 1.1. Let X be a set. A function $d : X \times X \to [0, \infty)$ is called a **metric** (or distance) on X if

1. $\forall x, y \in X, d(x, y) = 0 \iff x = y$;
2. $\forall x, y \in X, d(x, y) = d(y, x)$;
3. $\forall x, y, z \in X, d(x, z) \leq d(x, y) + d(y, z)$. (Triangle inequality)

In this case, (X, d) is called a **metric space**.

1.2 Open Ball and Bounded Set

Definition 1.2. Let (X, d) be a metric space, $x \in X$, and $r > 0$.

1. Define $B(x, r) = \{y \in X : d(x, y) < r\}$, called the **open ball** centered at x with radius r, or the **r-neighborhood of x**.
2. A general **neighborhood of x** is a subset U of X such that $B(x, r) \subseteq U$ for some $r > 0$.
3. A subset $E \subseteq X$ of a metric space (X, d) is **bounded** if there exists some $x_0 \in X$ and $M > 0$ such that $d(x, x_0) \leq M$ for all $x \in E$.

1.3 Interior Points and Interior

Definition 1.3. Let $E \subseteq X$. The **closure** of E is the set $\overline{E} = E \cup E'$.

1. An element x of X is called an **interior point** of E if $\exists r > 0, B(x, r) \subseteq E$.
2. The **interior** of E is the set E° of all interior points of E.

By definition, we have $E^\circ \subseteq E \subseteq \overline{E}$.

1.4 Open Set

Definition 1.4. A subset $G \subseteq X$ is called **open** if $\forall x \in G, \exists r > 0, B(x,r) \subseteq G$. Note, $\emptyset$ and X are open in X. That is, a set is open if every point in it has an open ball around it that's still entirely inside the set.

1.5 Limit Points and Derived Set

Definition 1.5. Let $E \subseteq X$.

1. x is called a **limit point** of E (or cluster point, or accumulation point) if

   $$\forall r > 0, B(x,r) \cap E \text{ contains some } y \neq x$$

   Equivalently,

   $$(B(x,r) - \{x\}) \cap E \neq \emptyset$$

2. We let $E' =$ the set of all limit points of E, called the **derived set** of E.

1.6 Closed Set

Definition 1.6. A subset $E \subseteq X$ is called **closed** if $E' \subseteq E$, where E' is the derived set of E (the set of all limit points of E). Note, $\emptyset$ and X are closed in X. That is, a subset $E \subseteq X$ is closed if every limit point of E belongs to E.

1. E is open $\iff X - E$ is closed;
2. E is closed $\iff X - E$ is open.

1.7 Closure

Definition 1.7. Let $E \subseteq X$. The **closure** of E is the set $\overline{E} = E \cup E'$.

1.8 Boundary Points and Boundary

Definition 1.8. Let X be a metric space, $E \subseteq X$ and $x \in X$.

1. x is called a **boundary point** of E if $\forall r > 0, B(x,r) \cap E \neq \emptyset$ and $B(x,r) \cap (X - E) \neq \emptyset$.
2. We use ∂E to denote the set of all boundary points of E, called the **boundary** of E.

1.9 Open Cover

Definition 1.9. Let X be a metric space and let $K \subseteq X$. A family $\{G_\alpha\}$ of open sets in X is called an **open cover** of K if $K \subseteq \bigcup_\alpha G_\alpha$. That is, every point of K lies in at least one of the open sets G_α.

1.10 Compact Set

Definition 1.10. The set K is called **compact** if every open cover of K has a finite **subcover** of K. That is, if $K \subseteq \bigcup_\alpha G_\alpha$, then $\exists \alpha_1, \ldots, \alpha_n$ such that $K \subseteq \bigcup_{i=1}^n G_\alpha$.

1.11 Convergent Sequence

Definition 1.11. Let (X, d) be a metric space and let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is **convergent** if

$$\exists x \in X \text{ such that } \underline{\forall \epsilon > 0, \exists N = N(\epsilon) \in \mathbb{N}, \forall n \geq N, d(x_n, x) < \epsilon}.$$

In this case, we say that $\{x_n\}$ converges to x, and write $x_n \to x$.

1.12 Divergent Sequence

Definition 1.12. We say that $\{x_n\}$ is **divergent** if $\{x_n\}$ is not convergent. That is, $\forall x \in X, \{x_n\}$ does not converge to x. The ϵ-N description of divergence of $\{x_n\}$ is

$$\underline{\forall x \in X, \exists \epsilon_0 > 0, \forall N, \exists n \geq N, d(x_n, x) \geq \epsilon_0}.$$

1.13 Bounded Sequence

Definition 1.13. A sequence $\{x_n\}$ in a metric space (X, d) is **bounded** if there exists a point $x_0 \in X$ and a number $M > 0$ such that

$$d(x_n, x_0) \leq M \text{ for all } n \in \mathbb{N}$$

In other words, all terms of the sequence lie within some fixed distance M of a single point x_0. In simplest terms, a sequence is bounded if all its terms lie inside some ball of finite radius.

1.14 The ϵ-N description of $x_n \to x$

Definition 1.14. The ϵ-N description of $x_n \to x$ is

$$\underline{\forall \epsilon > 0, \exists N = N(\epsilon) \in \mathbb{N}, \forall n \geq N, d(x_n, x) < \epsilon}.$$

1.15 The ϵ-N description of $x_n \not\to x$

Definition 1.15. The ϵ-N description of $x_n \not\to x$ is

$$\underline{\exists \epsilon_0 > 0, \forall N, \exists n \in N, d(x_n, x) \geq \epsilon_0}.$$

2 Results

2.1 Chapter 2: Theorem 6

Theorem. Let (X, d) be a metric space.

1. If $\{G_\alpha\}$ is a family of open sets in X, then $\bigcup_\alpha G_\alpha$ is open in X.
2. If $G_1, \ldots, G_n$ are open sets in X, then $\bigcap_{i=1}^n G_i$ is open in X.

2.2 Characterization (I) of $x \in \overline{E}$

Comparing with $x \in \overline{E'} \iff \forall r > 0, (B(x,r) - \{x\}) \cap E \neq \emptyset$ (definition), we have

$\underline{x \in E' \iff \forall r > 0, B(x,r) \cap E \neq \emptyset}$. Therefore, $E_1 \subseteq E_2 \implies \overline{E_1} \subseteq \overline{E_2}$.

2.3 Chapter 2: Theorem 8

Theorem. Every compact set in a metric space is bounded and closed. Therefore, if E is either unbounded or non-closed, them E is not compact.

2.4 Corollary 8

Corollary. Let $E \subseteq \mathbb{R}$. Then E is compact $\iff$ E is bounded and closed.

2.5 Chapter 3: Theorem 1

Theorem. (Uniqueness of Limit). Let (X, d) be a metric space and let $\{x_n\}$ be a sequence in X. If $x_n \to x$ and $x_n \to y$, then $x = y$.

2.6 Chapter 3: Theorem 2

Theorem. Any convergent sequence in a metric space is bounded.

2.7 Chapter 3: Theorem 3

Theorem. (Squeeze Theorem in $\mathbb{R}$). Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be sequences in $\mathbb{R}$ such that $a_n \to x, c_n \to x$, and $\forall n, a_n \leq b_n \leq c_n$. Then $b_n \to x$.

3 Questions/Proofs

3.1 Assignment 3: Question #6

Question: Find $E^\circ, E', \overline{E},$ and $, \partial E$ for the following subsets $E \subseteq \mathbb{R}$: **Recall:**

- $E^\circ =$ set of all interior points of E
- $E' =$ set of all limit points of E
- $\overline{E} =$ the closure set of $E = E \cup E'$
- $\partial E =$ the set of all boundary points of $E = \overline{E} \cap \overline{(X - E)}$

1. $E = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\} \ = \{\frac{1}{n}\}_{n=1}^{\infty}$
   So we get,
$$\begin{cases} E^\circ = \emptyset \\ E' = \{0\} \\ \overline{E} = E \cup E' = \{0\} \cup \{\frac{1}{n}\}_{n=1}^{\infty} \\ \partial E = \overline{E} \cap (\overline{\mathbb{R} - E}) = \overline{E} = \{0\} \cup \{\frac{1}{n}\}_{n=1}^{\infty} \end{cases}$$

2. $E = (0, \pi) \cap \mathbb{Q}$
   So we get,
$$\begin{cases} E^\circ = \emptyset \\ E' = [0, \pi] \\ \overline{E} = E \cup E' = [0, \pi] \\ \partial E = \overline{E} \cap (\overline{\mathbb{R} - E}) = [0, \pi] \end{cases}$$

3. $E = [0, 2) \cup (4, 7) \cup \{8\}$
   So we get,
$$\begin{cases} E^\circ = (0, 2) \cup (4, 7) \\ E' = [0, 2] \cup [4, 7] \\ \overline{E} = E \cup E' = [0, 2] \cup [4, 7] \cup \{8\} \\ \partial E = \overline{E} \cap (\overline{\mathbb{R} - E}) = \{0, 2, 4, 7, 8\} \end{cases}$$
   Further analysis on $\mathbb{R} - E$:
$$\mathbb{R} - E = (-\infty, 0) \cup [2, 4] \cup [7, 8) \cup (8, \infty)$$
$$\overline{\mathbb{R} - E} = (-\infty, 0] \cup [2, 4] \cup [7, \infty)$$

4. $E = \mathbb{Z}$
   So we get,
$$\begin{cases} E^\circ = \emptyset \\ E' = \emptyset \\ \overline{E} = E \cup E' = \mathbb{Z} \\ \partial E = \overline{E} \cap (\overline{\mathbb{R} - E}) = \overline{E} \cap \mathbb{R} = \mathbb{Z}\} \end{cases}$$

3.2 Assignment 4: Question #2

Question: Find $E^\circ, E', \overline{E},$ and $, \partial E$ for the following subsets $E \subseteq \mathbb{R}$:

1. $E = (-\infty, 0) \cup (1, 2) \cup \{5 + \frac{1}{n} : n = 1, 2, 3, \ldots\}$
   So we get,

$$\begin{cases} E^\circ = (-\infty, 0) \cup (1, 2) \\ E' = (-\infty, 0] \cup [1, 2] \cup \{5\} \\ \overline{E} = E \cup E' = (-\infty, 0] \cup [1, 2] \cup \{5 + \frac{1}{n}\}_{n=1}^{\infty} \cup \{5\} \\ \partial E = \overline{E} \cap (\overline{\mathbb{R} - E}) = \{0, 1, 2, 5\} \cup \{5 + \frac{1}{n}\}_{n=1}^{\infty} \end{cases}$$

   Further analysis on ∂E:

$$= \overline{E} \cap \overline{([0, 1] \cup ([2, \infty] - \{5 + \frac{1}{n}\}_{n=1}^{\infty})}$$

$$= \overline{E} \cap ([0, 1] \cup [2, \infty))$$

$$= \{0, 1, 2, 5\} \cup \{5 + \frac{1}{n}\}_{n=1}^{\infty}$$

2. $E = \{\pi\} \cup \mathbb{N}$
   So we get,

$$\begin{cases} E^\circ = \emptyset \\ E' = \emptyset \\ \overline{E} = E \cup E' = \{\pi\} \cup \mathbb{N} \text{ (so just E)} \\ \partial E = \overline{E} \cap (\overline{\mathbb{R} - E}) = E \cap \mathbb{R} = \{\pi\} \cup \mathbb{N} \text{ (so just E)} \end{cases}$$

3. $E = \{\pi\} \cup \mathbb{Q}$
   So we get,

$$\begin{cases} E^\circ = \emptyset \\ E' = \mathbb{R} \text{ (since } \mathbb{Q} \text{ is dense in } \mathbb{R}) \\ \overline{E} = E \cup E' = \mathbb{R} \\ \partial E = \overline{E} \cap (\overline{\mathbb{R} - E}) = \mathbb{R} \end{cases}$$

   Further analysis on $\mathbb{R} - E$:

$$= \mathbb{R} - E = \{x \in \mathbb{R} \mid x \text{ is irrational and } x \neq \pi\}$$

$$= \overline{\mathbb{R} - E} = \mathbb{R}, \text{ (since } (\mathbb{R} - \mathbb{Q}) - \{\pi\} \text{ is dense in } \mathbb{R})$$

3.3 Assignment 4: Question #3

Question: Give an open cover of the set $[0,1] \cap \mathbb{Q}$ in $\mathbb{R}$ that has no finite subcover.

Proof.

$\implies$ Note: $\frac{\pi}{4} \in [0,1]$ and $\frac{\pi}{4} \notin \mathbb{Q}$

$\implies$ For $n \in \mathbb{N},$ let $G_n = (-1, \frac{\pi}{4} - \frac{1}{2^n}) \cup (\frac{\pi}{4}, 2)$

$\implies$ Each G_n is open, and $G_1 \subseteq G_2 \subseteq G_3 \subseteq$ and $[0,1] \cap \mathbb{Q} \subseteq \bigcup_{n=1}^{\infty} G_n$ is an open cover

$\implies$ $\{G_n\}_{n=1}^{\infty}$ no finite subcover of $[0,1] \cap \mathbb{Q}$

$\implies$ $\forall n_0 \in \mathbb{N}, \bigcup_{n=1}^{n_0} G_n = G_{n_0}$ does not contain the points $(\frac{\pi}{4} - \frac{1}{2^{n_0}}, \frac{\pi}{4}) \cap \mathbb{Q}$

Therefore, there does not exist a finite subcover. □

3.4 Assignment 4: Question #5

Question: Use the ϵ-N definition to prove the following:

1.
$$\lim_{n \to \infty} (1 + \frac{1}{n^2}) = 1$$

Proof. For convergence, $|x_n - x| < \epsilon$, so we need to show

$$\left| \left(1 + \frac{1}{n^2}\right) - 1 \right| = \frac{1}{n^2} < \epsilon$$

This is equivalent to

$$n > \frac{1}{\sqrt{\epsilon}} \text{ for } \epsilon > 0$$

Choose

$$N = \left[\frac{1}{\sqrt{\epsilon}} \right] + 1 \text{ (integer part)}$$

Then for all $n \geq N$

$$\frac{1}{n^2} \leq \frac{1}{N^2} < \epsilon$$

Thus we have

$$\left| \left(1 + \frac{1}{n^2}\right) - 1 \right| = \frac{1}{n^2} \leq \frac{1}{N^2} < \epsilon$$

Therefore,

$$\lim_{n\to\infty}(1+\frac{1}{n^2})=1$$

□

Further analysis on choosing N for $n \geq N$:

$$\left|(1+\frac{1}{n^2})-1\right|<\epsilon \implies \left|\frac{1}{n^2}\right|<\epsilon \implies \frac{1}{n^2}<\epsilon$$

when $n \geq N$, $\frac{1}{n^2} \leq \frac{1}{N^2} < \epsilon$

$$\frac{1}{\epsilon}<N^2 \implies \frac{1}{\sqrt{\epsilon}}<N$$

2.
$$\lim_{n\to\infty}\frac{n}{n+1}=1$$

Proof. For every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\left|\frac{n}{n+1}-1\right|<\epsilon \text{ whenever } n \geq N$$

Take

$$N=\left[\frac{1}{\epsilon}\right]+1 \text{ (integer part)}$$

Then $N \in \mathbb{N}$ and $\forall n \geq N$, we get

$$\left|\frac{n}{n+1}-1\right|=\frac{1}{n+1}<\frac{1}{n}\leq\frac{1}{N}<\epsilon$$

Thus,

$$\left|\frac{n}{n+1}-1\right|<\epsilon \text{ for all } n \geq N$$

Therefore,

$$\lim_{n\to\infty}\frac{n}{n+1}=1$$

□

Further analysis on choosing N for $n \geq N$:

$$\left|\frac{n}{n+1} - 1\right| = \left|\frac{-1}{n+1}\right| = \frac{1}{n+1} < \frac{1}{n} \leq \frac{1}{N} < \epsilon \iff N > \frac{1}{\epsilon}$$

Note:

$$\frac{1}{\epsilon} < \left[\frac{1}{\epsilon}\right] + 1$$

where $\left[...\right]$ represents the integer part.

3.

$$\text{The sequence } \{1 + (-1)^n\} \text{ is divergent in } \mathbb{R}$$

Proof. For even $n, x_n = 1 + 1 = 2$ and for odd $n, x_n = 1 - 1 = 0$. So the sequence alternates between 2 and 0. To test convergence, assume $x_n \to x$ for some $x \in \mathbb{R}$.
Case 1: $x = 0$
For even $n, x_n = 2$, then

$$|x_n - x| = |2 - 0| = 2$$

Choose $\epsilon_0 = 1$. For all $N \in \mathbb{N}$, if we take $n = 2N$, we have

$$n \geq N \text{ but } |x_n - x| = 2 \geq \epsilon_0$$

Hence, the limit condition fails.
Case 2: $x \neq 0$
For odd $n, x_n = 0$, then

$$|x_n - x| = |0 - x| = |x|$$

Choose $\epsilon_0 = \frac{|x|}{2}$. For all $N \in \mathbb{N}$, if we take $n = 2N + 1$, we have

$$n \geq N \text{ but } |x_n - x| = |x| > \epsilon_0$$

Hence, the limit condition fails again.
Therefore,

$$\text{The series } \{1 + (-1)^n\} \text{ is divergent in } \mathbb{R}$$

□

Some analysis on the result:

$$\{1 + (-1)^n\} = 0, 2, 0, 2, 0, \ldots \text{ for infinity, hence is divergent.}$$

$$\text{Recall: } \{x_n\} \text{ is divergent in } \mathbb{R} : \forall x \in \mathbb{R}, x_n \not\to x$$

$$\text{i.e., } \forall x \in \mathbb{R}, \exists \epsilon_0 > 0, \forall N \in \mathbb{N}, \exists n \geq N, |x_n - x| \geq \epsilon_0$$

3.5 Assignment 4: Question #8

Question: Let $\{x_n\}$ and $\{y_n\}$ be sequences in $\mathbb{R}$ such that $\{x_n\}$ is bounded and $y_n \to 0$. Prove that $x_n y_n \to 0$.

Proof.

$$\begin{aligned}
&\implies \text{Since } \{x_n\} \text{ is bounded, there exists } M > 0 \text{ such that } |x_n| \leq M \text{ for all } n \in \mathbb{N} \\
&\implies \text{Given } \epsilon > 0. \text{ Since } y_n \to 0, \\
&\implies \exists N \in \mathbb{N}, \forall n \geq N, |y_n - 0| < \frac{\epsilon}{M} \\
&\implies \text{Then } \forall n \geq N, \text{ we have} \\
&\implies |x_n y_n - 0| = |x_n||y_n| \leq M|y_n| < M \times \frac{\epsilon}{M} = \epsilon
\end{aligned}$$

Therefore,

$$\lim_{n \to \infty} x_n y_n = 0$$

□

Some analysis on $|x_n y_n - 0| < \epsilon$ for $n \geq N$:

$$\text{(Boundedness): } \exists M > 0 \text{ such that } \forall n \in \mathbb{N},\ |x_n| \leq M.$$

$$|x_n y_n - 0| = |x_n||y_n| \leq M|y_n|.$$

Since $y_n \to 0$, for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|y_n| < \frac{\epsilon}{M}.$$

Therefore,

$$|x_n y_n - 0| \leq M|y_n| < M \cdot \frac{\epsilon}{M} = \epsilon.$$

This satisfies the definition of the limit:

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } n \geq N \implies |x_n y_n - 0| < \epsilon.$$

3.6 Assignment 5: Question #1(i)(ii)

1.
$$\lim_{n\to\infty}(\sqrt{n^2+2n}-n)=1$$

Proof.

$$\implies \text{Given } \epsilon>0,\ \text{choose } N=\left[\frac{1}{\epsilon}\right]+1 \text{ (integer part)}$$
$$\implies \text{Then, } N>\frac{1}{\epsilon} \implies \frac{1}{N}<\epsilon$$
$$\implies \text{For } n\in\mathbb{N},\ \text{we have } x_n=\sqrt{n^2+2n}-n$$

Rationalizing,

$$x_n=\frac{(\sqrt{n^2+2n}-n)(\sqrt{n^2+2n}+n)}{\sqrt{n^2+2n}+n}=\frac{2n}{\sqrt{n^2+2n}+n}$$

Dividing the numerator and denominator by n,

$$x_n=\frac{2}{\sqrt{1+\frac{2}{n}}+1}$$

Then,

$$|x_n-1|=\left|\frac{2}{\sqrt{1+\frac{2}{n}}+1}-1\right|=\frac{\left|1-\sqrt{1+\frac{2}{n}}\right|}{\sqrt{1+\frac{2}{n}}+1}$$

For the denominator, since $n>0$,

$$\sqrt{1+\frac{2}{n}}+1>2$$

For the numerator,

$$\left|1-\sqrt{1+\frac{2}{n}}\right|=\frac{\frac{2}{n}}{1+\sqrt{1+\frac{2}{n}}}<\frac{\frac{2}{n}}{n}=\frac{1}{n}$$

So substituting these in, we get

$$|x_n-1|=\frac{\left|1-\sqrt{1+\frac{2}{n}}\right|}{\sqrt{1+\frac{2}{n}}+1}<\frac{\frac{1}{n}}{2}<\frac{1}{n}$$

For $n\geq N$,

$$|x_n-1|<\frac{1}{n}\leq\frac{1}{N}<\epsilon$$

Therefore,

$$\lim_{n\to\infty}(\sqrt{n^2+2n}-n)=1$$

□

2.

The sequence $\{n(-1)^{n-1}\}$ is divergent in $\mathbb{R}$

Proof. Assume that the sequence $\{x_n\}$ converges to M. By the ϵ-N definition of convergence, we have

$$\forall\epsilon>0, \exists N\in\mathbb{N} \text{ such that for all } n\geq N, |x_n-M|<\epsilon$$

The series terms are as follows,

$$x_1=1, x_2=-2, x_3=3, x_4=-4, \dots \text{ (alternating in sign and magnitude infinitely)}$$

Take $\epsilon=1$. For any $N\in\mathbb{N}$, $x_{2N}=-2N$ and $x_{2N+1}=2N+1$. Then,

$$|x_{2N+1}-x_{2N}|=|(2N+1)-(-2N)|=4N+1>1=\epsilon$$

Thus, for all N, there exists some $n\geq N$ such that $|x_n-M|\geq\epsilon$, which is a contradiction.

Therefore, the sequence $\{n(-1)^{n-1}\}$ is divergent in $\mathbb{R}$

□

3.7 Chapter 2: the proof of Theorem 9

Theorem. Let K be a compact set in metric space X and let F be a closed subset of K. Then F is compact.

Proof. Let $\{G_\alpha\}$ be an open cover of F. Then $\{X-F\}\cup\{G_\alpha\}$ is an open cover of K. Since K is compact, $\exists\alpha_1,\dots,\alpha_n$ such that

$$K\subseteq(X-F)\cup(G_\alpha\cup\cdots\cup G_{\alpha_n}).$$

Now $F=F\cap K\subseteq(F\cap(X-F))\cup(G_{\alpha_1}\cup\cdots\cup G_{\alpha_n})=G_{\alpha_1}\cup\cdots\cup G_{\alpha_n}$. Hence, $\{G_\alpha\}$ has a finite subcover $\{G_{\alpha_1},\dots,G_{\alpha_n}\}$ of F. Therefore, F is compact. □

3.8 Chapter 3: the proof of Theorem 2

Theorem. Any convergent sequence in a metric space is bounded.

Proof. Suppose $x_n\to x$ in a metric space (X,d). We prove that

$$\exists M>0, \forall n, x_n\in B(x,M)$$

For $\epsilon=1, \exists N, \forall n\geq N, d(x_n,x)<1$. Let $M=1+\max\{d(x_1,x),\dots,d(x_{N-1},x)\}$. Then $\forall n$, we have $d(x_n,x)<M$. □

3.9 Chapter 3: the proof of Theorem 3

Theorem. (Squeeze Theorem in $\mathbb{R}$). Let $\{a_n\}, \{b_n\},$ and $, \{c_n\}$ be sequences in $\mathbb{R}$ such that $a_n \to x, c_n \to x,$ and $, \forall n, a_n \leq b_n \leq c_n$. Then, $b_n \to x$.

Note: Here either $x \in \mathbb{R}$ or $x = \pm\infty$. We only prove the case when $x \in \mathbb{R}$. Also, in the theorem, we can only require that $a_n \leq b_n \leq c_n$ holds for all $n \geq n_0$ for some n_0.

Proof. Given $\epsilon > 0$. Since $a_n \to x$ and $c_n \to x, \exists N_1, N_2 \in \mathbb{N}$ such that

$$\forall n \geq N_1, |a_n - x| < \epsilon \text{ and } \forall n \geq N_2, |c_n - x| < \epsilon.$$

Let $N = \max\{N_1, N_2\}$. Then $\forall n \geq N$, we have $|a_n - x| < \epsilon$ and $|c_n - x| < \epsilon$. That is,

$$x - \epsilon < a_n < x + \epsilon \text{ and } x - \epsilon < c_n < x + \epsilon$$

In this case, $\forall n \geq N$, we have $x - \epsilon < a_n \leq b_n \leq c_n < x + \epsilon$. That is,

$$x - \epsilon < b_n < x + \epsilon, \text{ or } |b_n - x| < \epsilon$$

So, we prove that $\forall \epsilon > 0, \exists N, \forall n \geq N, |b_n - x| < \epsilon$. Therefore, $b_n \to x$. □