

# Test 1 Review

MATH-2251  
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This document contains all relevant definitions, theorems, and methods for Test 1 from chapters 8-10.

## 1 Chapter 8: Diagonalization of Matrices

### Eigenvalues and Eigenvectors

We begin by defining the notion of an eigenvalue and an eigenvector for both linear maps and  $n \times n$  matrices.

**Def 8.1.1.** Let  $V$  be an  $n$  dimensional vector space over  $\mathbb{F}$  and let  $T : V \rightarrow V$  be a linear map.

1. A scalar  $\lambda \in \mathbb{F}$  is said to be an **eigenvalue** of  $T$  if and only if there exists a nonzero vector  $\vec{v} \in V$ , such that

$$T(\vec{v}) = \lambda\vec{v}$$

2. Every nonzero vector  $\vec{v} \in V$  such that  $T(\vec{v}) = \lambda\vec{v}$  is said to be an **eigenvector** of  $T$  corresponding  $\lambda$  and the  $\lambda$ -eigenspace of  $T$  is defined to be  $\{\vec{v} \in V \mid T(\vec{v}) = \lambda\vec{v}\}$  which is simply the nullspace of  $T - \lambda I_V$ .

$$\mathcal{N}(T - \lambda I_V) = \{\vec{v} \in V \mid (T - \lambda I_V)\vec{v} = \vec{0}\}$$

3. The generalized  $\lambda$ -eigenspace of  $T$  is defined to be

$$V_\lambda = \{\vec{v} \in V \mid \text{there exists a positive integer } n \text{ with } (T - \lambda I_V)^n \vec{v} = \vec{0}\}$$

**Def 8.1.2.** Let  $A$  be an  $n \times n$  matrix over  $\mathbb{F}$ .

1. A scalar  $\lambda \in \mathbb{F}$  is said to be an **eigenvalue** of  $A$  if and only if there exists a nonzero vector  $\vec{v} \in \mathbb{F}^n$ , such that

$$A\vec{v} = \lambda\vec{v}$$

2. Every nonzero vector  $\vec{v} \in \mathbb{F}^n$  such that  $A\vec{v} = \lambda\vec{v}$  is said to be an **eigenvector** of  $A$  corresponding  $\lambda$  and the  $\lambda$ -eigenspace of  $A$  is defined to be  $\{\vec{v} \in V \mid A\vec{v} = \lambda\vec{v}\}$  which is simply the nullspace of the matrix  $\lambda I_n - A$ .

$$\mathcal{N}(\lambda I_n - A) = \{\vec{v} \in V \mid (\lambda I_n - A)\vec{v} = \vec{0}\}$$

3. The generalized  $\lambda$ -eigenspace of  $A$  is defined to be

$$V_\lambda = \{\vec{v} \in V \mid \text{there exists a positive integer } n \text{ with } (\lambda I_n - A)^n \vec{v} = \vec{0}\}$$

**Def 8.1.3.**

1. If  $V$  is a finite dimensional vector space over  $\mathbb{F}$  and  $T : V \rightarrow V$  is a linear map then  $T$  is said to be **diagonalizable** provided there exists a basis  $\mathcal{B}$  of  $V$  such that  $[T]_{\mathcal{B}}$  is diagonal.
2. An  $n \times n$  matrix  $A$  over  $\mathbb{F}$  is said to be **diagonalizable over  $\mathbb{F}$**  provided it is similar to a diagonal matrix - i.e. there is an invertible matrix  $P \in \mathcal{M}_{n \times n}(\mathbb{F})$  such that

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$$

where  $\lambda_i \in \mathbb{F}$ .

### Eigenvalues Related to the Characteristic Polynomial

**Def 8.3.1.** If  $A \in \mathcal{M}_{n \times n}(\mathbb{F})$  the **characteristic polynomial** of  $A$  is defined to be

$$\chi_A(x) = \det(xI_n - A)$$

### Theorem 8.3.1. Fundamental Theorem of Algebra

Every polynomial  $f(x) \in \mathbb{C}[x]$  has at least one root in  $\mathbb{C}$ . In other words, every complex polynomial can be factored into a product of linear polynomials in  $\mathbb{C}[x]$ . Alternately, the only irreducible polynomials in  $\mathbb{C}[x]$  have degree one.

### Theorem 8.5.1.

If  $A \in \mathcal{M}_{n \times n}(\mathbb{F})$  and  $\{\vec{v}_1, \dots, \vec{v}_k\}$  are eigenvectors of  $A$  corresponding to distinct eigenvalues  $\lambda_1, \dots, \lambda_k \in \mathbb{F}$ , then  $\{\vec{v}_1, \dots, \vec{v}_k\}$  is linearly independent.

### Theorem 8.5.2.

If  $A \in \mathcal{M}_{n \times n}(\mathbb{F})$  and  $\lambda_1, \dots, \lambda_r \in \mathbb{F}$  are distinct eigenvalues of  $A$  then the sum of the eigenspaces

$$\mathcal{N}(\lambda_1 I_n - A) + \cdots + \mathcal{N}(\lambda_r I_n - A)$$

is a direct sum

$$\mathcal{N}(\lambda_1 I_n - A) \oplus \cdots \oplus \mathcal{N}(\lambda_r I_n - A)$$

### Diagonalizability of a Matrix

Let  $A \in \mathcal{M}_{n \times n}(\mathbb{F})$  then  $A$  is diagonalizable over  $\mathbb{F}$  provided the following algorithm is successful.

1. Compute the characteristic polynomial:  $\chi_A(\lambda) = \det(\lambda I_n - A)$ .
2. Factor  $\chi_A(\lambda)$  over  $\mathbb{F}$ . In order for  $A$  to be diagonalizable over  $\mathbb{F}$ ,  $\chi_A(\lambda)$  must factor linearly over  $\mathbb{F}$ .
3. For each distinct eigenvalue  $\lambda_i$  determine a basis  $\mathcal{B}_i = \{\vec{v}_{i1}, \dots, \vec{v}_{il_i}\}$  of the  $\lambda_i$  eigenspace  $\mathcal{N}(\lambda_i I_n - A)$  of  $A$ .
4. Set

$$\mathcal{B} = \bigcup_{i=1}^r \mathcal{B}_i = \{\vec{v}_{11}, \dots, \vec{v}_{1l_1}, \vec{v}_{21}, \dots, \vec{v}_{2l_2}, \dots, \vec{v}_{r1}, \dots, \vec{v}_{rl_r}\}.$$

If  $\mathcal{B}$  is a basis of  $\mathbb{F}^n$ , then  $A$  is diagonalizable and

$$P^{-1}AP = \begin{bmatrix} \lambda_1 I_{l_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_r I_{l_r} \end{bmatrix}$$

where

$$P = [\vec{v}_{11} \quad \dots \quad \vec{v}_{1l_1} \quad \dots \quad \vec{v}_{rl_r}]$$

## 2 Chapter 9: Inner Product Spaces

### Def 9.1.1. Complex and Real Inner Product Spaces

A **complex** ( $\mathbb{C}$ ) inner product space is a complex vector space  $V$  together with a map  $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{C}$  which satisfies the following properties:

1.  $\langle \vec{v}, \vec{v} \rangle \geq 0$  for all  $\vec{v} \in V$ , and equal to 0 exactly when  $\vec{v} = \vec{0}$
2.  $\langle \vec{v}, \vec{w} \rangle = \overline{\langle \vec{w}, \vec{v} \rangle}$  for all  $\vec{v}, \vec{w} \in V$
3.  $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$  for all  $\vec{u}, \vec{v}, \vec{w} \in V$
4.  $\langle \alpha \vec{v}, \vec{w} \rangle = \bar{\alpha} \langle \vec{v}, \vec{w} \rangle = \langle \vec{v}, \bar{\alpha} \vec{w} \rangle$  for all  $\vec{v}, \vec{w} \in V$  and  $\alpha \in \mathbb{C}$

A **real** ( $\mathbb{R}$ ) inner product space is a real vector space  $V$  together with a map  $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{R}$  which satisfies the following properties:

1.  $\langle \vec{v}, \vec{v} \rangle \geq 0$  for all  $\vec{v} \in V$ , and equal to 0 exactly when  $\vec{v} = \vec{0}$
2.  $\langle \vec{v}, \vec{w} \rangle = \overline{\langle \vec{w}, \vec{v} \rangle}$  for all  $\vec{v}, \vec{w} \in V$
3.  $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$  for all  $\vec{u}, \vec{v}, \vec{w} \in V$
4.  $\langle r\vec{v}, \vec{w} \rangle = r \langle \vec{v}, \vec{w} \rangle = \langle \vec{v}, r\vec{w} \rangle$  for all  $\vec{v}, \vec{w} \in V$  and  $r \in \mathbb{R}$

### Def 9.1.2. Hermitian Conjugation

Let  $A = (a_{ij}) \in \mathcal{M}_{m \times n}(\mathbb{C})$ . Then the **hermitian conjugation** of  $A$ , denoted  $A^H$  or  $A^*$ , is the  $n \times m$  matrix obtained from  $A$  by conjugating each entry and then taking the transpose.  $A^H = A^* = (\bar{a}_{ij})^T = (\bar{a}_{ji})$ .

The hermitian transpose satisfies the following properties:

1.  $(A + B)^H = A^H + B^H$
2.  $(A^H)^H = A$
3.  $(AB)^H = B^H A^H$
4.  $(\alpha A)^H = \bar{\alpha} A^H$
5.  $A^H = A^T$  when  $A \in \mathcal{M}_{n \times n}(\mathbb{R})$

**Def 9.1.4. Norm, Orthogonal, and Orthonormal Inner Product Spaces**

Let  $V$  be an inner product space.

1. The **norm** of a vector  $\vec{v} \in V$ , denoted  $\|\vec{v}\|$ , is defined to be  $\sqrt{\langle \vec{v}, \vec{v} \rangle}$ .
2. Two vectors  $\vec{v}, \vec{u} \in V$  are said to be **orthogonal** provided  $\langle \vec{v}, \vec{u} \rangle = 0$ .
3. If  $\{\vec{v}_1, \dots, \vec{v}_k\} \subseteq V$ , then  $\{\vec{v}_1, \dots, \vec{v}_k\}$  is said to be an **orthogonal** set of vectors provided  $\vec{v}_1, \dots, \vec{v}_k$  are mutually orthogonal vectors - i.e.  $\langle \vec{v}_i, \vec{v}_j \rangle = 0$  for  $i \neq j$  and  $\{\vec{v}_1, \dots, \vec{v}_k\}$  is orthonormal provided it is orthogonal and each  $\vec{v}_i$  is a unit vector - i.e.  $\|\vec{v}_i\| = 1$ .

**Def 9.1.6. Unitary and Orthogonal Matrices**

An  $n \times n$  complex matrix  $U$  is said to be **unitary** provided  $U^H = U^{-1}$ . An  $n \times n$  real matrix  $Q$  is said to be **orthogonal** provided  $Q^T = Q^{-1}$ .

**Theorem 9.3.1. Gram-Schmidt Orthogonalization Process**

Given any linearly independent set of vectors  $\{\vec{v}_1, \dots, \vec{v}_m\}$  in an inner product space  $(V, \langle \cdot, \cdot \rangle)$  there exists an orthonormal set of vectors  $\{\vec{u}_1, \dots, \vec{u}_m\}$  such that  $\text{span}\{\vec{v}_1, \dots, \vec{v}_k\} = \text{span}\{\vec{u}_1, \dots, \vec{u}_k\}$  for all  $k = 1, \dots, m$ .

**GSOP Algorithm**

For practical implementations of the GSOP we often break the process into two steps.

1. Adjust the angles - construct an orthogonal set  $\{\vec{w}_1, \dots, \vec{w}_m\}$  from  $\{\vec{v}_1, \dots, \vec{v}_m\}$  as follows.

$$\begin{aligned} \vec{w}_1 &= \vec{v}_1 \\ \vec{w}_2 &= \vec{v}_2 - \frac{\langle \vec{w}_1, \vec{v}_2 \rangle}{\langle \vec{w}_1, \vec{w}_1 \rangle} \vec{w}_1 \\ &\vdots \\ \vec{w}_m &= \vec{v}_m - \frac{\langle \vec{w}_{m-1}, \vec{v}_m \rangle}{\langle \vec{w}_{m-1}, \vec{w}_{m-1} \rangle} \vec{w}_{m-1} - \cdots - \frac{\langle \vec{w}_1, \vec{v}_m \rangle}{\langle \vec{w}_1, \vec{w}_1 \rangle} \vec{w}_1 \end{aligned}$$

2. Adjust the lengths - normalize the vectors  $\vec{w}_1, \dots, \vec{w}_m$  produced in step 1 to produce an orthonormal set,  $\{\vec{u}_1, \dots, \vec{u}_m\}$  with

$$\vec{u}_i = \frac{\vec{w}_i}{\|\vec{w}_i\|}$$

**Def 9.5.1. U-perp**

If  $U$  is a subspace of an inner product space  $(V, \langle \cdot, \cdot \rangle)$ , then the **orthogonal complement** of  $U$  in  $V$ , denoted  $U^\perp$ , is defined by

$$U^\perp = \{\vec{v} \in \mathbb{R}^n \mid \langle \vec{u}, \vec{v} \rangle = 0 \text{ for all } \vec{u} \in U\}$$

We often read  $U^\perp$  as U-perp.

**Theorem 9.5.1**

Let  $U$  be a subspace of an inner product space  $(V, \langle \cdot, \cdot \rangle)$ . Then

- $U^\perp$  is a subspace of  $V$
- $\dim U^\perp = \dim V - \dim U$
- $(U^\perp)^\perp = U$
- $V = U \oplus U^\perp$

**Def 9.7.1 Orthogonal Projections**

We define the **orthogonal projection** of  $V$  onto  $U$  to be the map

$$\text{proj}_U : V \rightarrow U$$

such that for any  $\vec{v} \in V$ ,  $\text{proj}_U(\vec{v})$  is the unique vector in  $U$  such that  $\vec{v} - \text{proj}_U(\vec{v}) \in U^\perp$ . The **perpendicular component** of  $V$  with respect to  $U$  is the map

$$\text{perp}_U : V \rightarrow U^\perp$$

such that  $\text{perp}_U(\vec{v}) = \vec{v} - \text{proj}_U(\vec{v})$ .

**Important Remark:**

To better visualize this, if  $\{\vec{u}_1, \dots, \vec{u}_n\}$  is any orthonormal basis of  $V$  such that  $\{\vec{u}_1, \dots, \vec{u}_k\}$  is a basis of  $U$  then we have for any vector  $\vec{v} \in V$

$$\text{proj}_U(\vec{v}) = \langle \vec{u}_1, \vec{v} \rangle \vec{u}_1 + \dots + \langle \vec{u}_k, \vec{v} \rangle \vec{u}_k$$

and

$$\text{perp}_U(\vec{v}) = \langle \vec{u}_{k+1}, \vec{v} \rangle \vec{u}_{k+1} + \dots + \langle \vec{u}_n, \vec{v} \rangle \vec{u}_n$$

Adding them we get,

$$\vec{v} = \text{proj}_U(\vec{v}) + \text{perp}_U(\vec{v})$$

**Fundamental Theorem of Linear Algebra Part II**(swap  $H$  for  $\perp$  when moving between  $\mathbb{C}$  and  $\mathbb{R}$ )1. If  $A \in \mathcal{M}_{m \times n}(\mathbb{C})$  then

- $\mathcal{N}(A)^\perp = \mathcal{R}(A^H)$
- $\mathcal{R}(A^H)^\perp = \mathcal{N}(A)$
- $\mathcal{N}(A^H)^\perp = \mathcal{R}(A)$
- $\mathcal{R}(A)^\perp = \mathcal{N}(A^H)$
- $\mathbb{C}^n = \mathcal{R}(A^H) \oplus \mathcal{N}(A)$
- $\mathbb{C}^m = \mathcal{R}(A) \oplus \mathcal{N}(A^H)$

2. If  $A \in \mathcal{M}_{m \times n}(\mathbb{R})$  then

- $\mathcal{N}(A)^\perp = \mathcal{R}(A^T)$
- $\mathcal{R}(A^T)^\perp = \mathcal{N}(A)$
- $\mathcal{N}(A^T)^\perp = \mathcal{R}(A)$
- $\mathcal{R}(A)^\perp = \mathcal{N}(A^T)$
- $\mathbb{C}^n = \mathcal{R}(A^T) \oplus \mathcal{N}(A)$
- $\mathbb{C}^m = \mathcal{R}(A) \oplus \mathcal{N}(A^T)$

**3 Chapter 10: The Spectral Theorem****Def 10.1.1 Unitary and Orthogonal Similarity**

1.  $\mathbb{C}$ : If  $A$  and  $B$  are  $n \times n$  complex matrices then we say that  $A$  is **unitarily similar** to  $B$  if and only if there exists a unitary matrix  $U$  over  $\mathbb{C}$  such that  $B = U^{-1}AU = U^HAU$ . Further  $A$  is **unitarily diagonalizable** if and only if it is unitarily similar to a diagonal matrix.
2.  $\mathbb{R}$ : If  $A$  and  $B$  are  $n \times n$  real matrices then we say that  $A$  is **orthogonally similar** to  $B$  if and only if there exists an orthogonal matrix  $Q$  over  $\mathbb{R}$  such that  $B = Q^{-1}AQ = Q^TAQ$ . Further  $A$  is **orthogonally diagonalizable** if and only if it is orthogonally similar to a diagonal matrix.

**Theorem 10.1.1. Unitary and Orthogonal Diagonalizability**

1.  $\mathbb{C}$ : An  $n \times n$  complex matrix  $A$  is unitarily diagonalizable if and only if there exists an orthonormal basis  $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$  of  $\mathbb{C}^n$  consisting of eigenvectors of  $A$  - i.e.  $A\vec{v}_i = \lambda_i \vec{v}_i$  for  $i = 1, \dots, n$  with  $\lambda_i \in \mathbb{C}$ . In this case the matrix  $U = [\vec{v}_1, \dots, \vec{v}_n]$  is a unitary matrix which diagonalizes  $A$ .
2.  $\mathbb{R}$ : An  $n \times n$  real matrix  $A$  is orthogonally diagonalizable if and only if there exists an orthonormal basis  $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$  of  $\mathbb{R}^n$  consisting of eigenvectors of  $A$  - i.e.  $A\vec{v}_i = \lambda_i \vec{v}_i$  for  $i = 1, \dots, n$  with  $\lambda_i \in \mathbb{R}$ . In this case the matrix  $U = [\vec{v}_1, \dots, \vec{v}_n]$  is an orthogonal matrix which diagonalizes  $A$ .

**Theorem 10.1.2. Schur's Lemma**

Every  $n \times n$  matrix  $A$  over the complex numbers ( $\mathbb{C}$ ) is unitarily similar to an upper triangular matrix with the eigenvalues of  $A$  along the diagonal. In the case of an  $n \times n$  matrix  $A$  over the real numbers ( $\mathbb{R}$ ),  $A$  is orthogonally similar to an upper triangular matrix with the eigenvalues of  $A$  along the diagonal provided the  $\chi_A(\lambda)$  factors linearly over  $\mathbb{R}$ .

**Def 10.1.2. Normal Complex Matrices**

An  $(n \times n)$  complex matrix ( $\mathbb{C}$ )  $A$  is said to be **normal** if and only if

$$A^H A = A A^H$$

**Theorem 10.1.3. The Spectral Theorem**

An  $n \times n$  complex ( $\mathbb{C}$ ) matrix  $A$  is unitarily diagonalizable if and only if  $A$  is normal. Alternately,  $A$  is normal if and only if there exists an orthonormal basis  $\mathcal{B} = \{\vec{u}_1, \dots, \vec{u}_n\}$  of  $\mathbb{C}^n$  consisting of eigenvectors of  $A$  (- i.e.  $A\vec{u}_i = \lambda_i \vec{u}_i$ ) and hence a unitary matrix  $U = [\vec{u}_1, \dots, \vec{u}_n]$  such that

$$A = UDU^H = \lambda_1 \vec{u}_1 \vec{u}_1^H + \cdots + \lambda_n \vec{u}_n \vec{u}_n^H$$

Notice that  $\vec{u}_i \vec{u}_i^H$  is the orthogonal projection of  $\mathbb{C}^n$  onto  $\mathbb{C}\vec{u}_i$ .

**Def 10.1.3. Special Matrices**

There are three special subclasses of normal complex matrices:

1. **hermitian matrices** ( $A^H = A$ )  $\implies$  all of its eigenvalues are real.
2. **skew hermitian matrices** ( $A^H = -A$ )  $\implies$  all of its eigenvalues are purely imaginary.
3. **unitary matrices** ( $A^H = A^{-1}$ )  $\implies$  all of its eigenvalues have modulus one.

**Theorem 10.2.1. Principal Axis Theorem**

An  $n \times n$  real ( $\mathbb{R}$ ) matrix  $A$  is orthogonally diagonalizable over the real numbers if and only if  $A$  is symmetric. Moreover,  $A$  is symmetric if and only if there exists an orthonormal basis  $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$  of  $\mathbb{R}^n$  such that

$$A = \lambda_1 \vec{v}_1 \vec{v}_1^T + \cdots + \lambda_n \vec{v}_n \vec{v}_n^T$$

where  $A\vec{v}_i = \lambda_i \vec{v}_i$ .