

# MATH-3580 Test Two Review

Aidan Richer

October 2025

Test Two covers Chapter 2 - Parts 2 and 3 and Chapter 3 - Part 1 (up to Theorem 5)

## Contents

<b>1</b>	<b>Definitions</b>	<b>1</b>
1.1	Metric Space . . . . .	1
1.2	Open Ball and Bounded Set . . . . .	1
1.3	Interior Points and Interior . . . . .	2
1.4	Open Set . . . . .	2
1.5	Limit Points and Derived Set . . . . .	2
1.6	Closed Set . . . . .	2
1.7	Closure . . . . .	2
1.8	Boundary Points and Boundary . . . . .	2
1.9	Open Cover . . . . .	3
1.10	Compact Set . . . . .	3
1.11	Convergent Sequence . . . . .	3
1.12	Divergent Sequence . . . . .	3
1.13	Bounded Sequence . . . . .	3
1.14	The $\epsilon$ - $N$ description of $x_n \rightarrow x$ . . . . .	3
1.15	The $\epsilon$ - $N$ description of $x_n \not\rightarrow x$ . . . . .	3
<b>2</b>	<b>Results</b>	<b>4</b>
2.1	Chapter 2: Theorem 6 . . . . .	4
2.2	Characterization (I) of $x \in \overline{E}$ . . . . .	4
2.3	Chapter 2: Theorem 8 . . . . .	4
2.4	Corollary 8 . . . . .	4
2.5	Chapter 3: Theorem 1 . . . . .	4
2.6	Chapter 3: Theorem 2 . . . . .	4
2.7	Chapter 3: Theorem 3 . . . . .	4
<b>3</b>	<b>Questions/Proofs</b>	<b>4</b>
3.1	Assignment 3: Question #6 . . . . .	4
3.2	Assignment 4: Question #2 . . . . .	6
3.3	Assignment 4: Question #3 . . . . .	7

3.4	Assignment 4: Question #5 . . . . .	7
3.5	Assignment 4: Question #8 . . . . .	9
3.6	Assignment 5: Question #1(i)(ii) . . . . .	10
3.7	Chapter 2: the proof of Theorem 9 . . . . .	10
3.8	Chapter 3: the proof of Theorem 2 . . . . .	10
3.9	Chapter 3: the proof of Theorem 3 . . . . .	11

# 1 Definitions

## 1.1 Metric Space

In mathematics, space = set + structure(s).

**Definition 1.1.** Let  $X$  be a set. A function  $d : X \times X \rightarrow [0, \infty)$  is called a **metric** (or distance) on  $X$  if

1.  $\forall x, y \in X, d(x, y) = 0 \iff x = y$ ;
2.  $\forall x, y \in X, d(x, y) = d(y, x)$ ;
3.  $\forall x, y, z \in X, d(x, z) \leq d(x, y) + d(y, z)$ . (Triangle inequality)

In this case,  $(X, d)$  is called a **metric space**.

## 1.2 Open Ball and Bounded Set

**Definition 1.2.** Let  $(X, d)$  be a metric space,  $x \in X$ , and  $r > 0$ .

1. Define  $B(x, r) = \{y \in X : d(x, y) < r\}$ , called the **open ball** centered at  $x$  with radius  $r$ , or the **r-neighborhood of x**.
2. A general **neighborhood of x** is a subset  $U$  of  $X$  such that  $B(x, r) \subseteq U$  for some  $r > 0$ .
3. A subset  $E \subseteq X$  of a metric space  $(X, d)$  is **bounded** if there exists some  $x_0 \in X$  and  $M > 0$  such that  $d(x, x_0) \leq M$  for all  $x \in E$ .

## 1.3 Interior Points and Interior

**Definition 1.3.** Let  $E \subseteq X$ . The **closure** of  $E$  is the set  $\overline{E} = E \cup E'$ .

1. An element  $x$  of  $X$  is called an **interior point** of  $E$  if  $\exists r > 0, B(x, r) \subseteq E$ .
2. The **interior** of  $E$  is the set  $E^\circ$  of all interior points of  $E$ .

By definition, we have  $E^\circ \subseteq E \subseteq \overline{E}$ .

## 1.4 Open Set

**Definition 1.4.** A subset  $G \subseteq X$  is called **open** if  $\forall x \in G, \exists r > 0, B(x, r) \subseteq G$ . Note,  $\emptyset$  and  $X$  are open in  $X$ . That is, a set is open if every point in it has an open ball around it that's still entirely inside the set.

## 1.5 Limit Points and Derived Set

**Definition 1.5.** Let  $E \subseteq X$ .

1.  $x$  is called a **limit point** of  $E$  (or cluster point, or accumulation point) if

$$\forall r > 0, B(x, r) \cap E \text{ contains some } y \neq x$$

Equivalently,

$$(B(x, r) - \{x\}) \cap E \neq \emptyset$$

2. We let  $E' =$  the set of all limit points of  $E$ , called the **derived set** of  $E$ .

## 1.6 Closed Set

**Definition 1.6.** A subset  $E \subseteq X$  is called **closed** if  $E' \subseteq E$ , where  $E'$  is the derived set of  $E$  (the set of all limit points of  $E$ ). Note,  $\emptyset$  and  $X$  are closed in  $X$ . That is, a subset  $E \subseteq X$  is closed if every limit point of  $E$  belongs to  $E$ .

1.  $E$  is open  $\iff X - E$  is closed;
2.  $E$  is closed  $\iff X - E$  is open.

## 1.7 Closure

**Definition 1.7.** Let  $E \subseteq X$ . The **closure** of  $E$  is the set  $\overline{E} = E \cup E'$ .

## 1.8 Boundary Points and Boundary

**Definition 1.8.** Let  $X$  be a metric space,  $E \subseteq X$  and  $x \in X$ .

1.  $x$  is called a **boundary point** of  $E$  if  $\forall r > 0, B(x, r) \cap E \neq \emptyset$  and  $B(x, r) \cap (X - E) \neq \emptyset$ .
2. We use  $\partial E$  to denote the set of all boundary points of  $E$ , called the **boundary** of  $E$ .

## 1.9 Open Cover

**Definition 1.9.** Let  $X$  be a metric space and let  $K \subseteq X$ . A family  $\{G_\alpha\}$  of open sets in  $X$  is called an **open cover** of  $K$  if  $K \subseteq \bigcup_\alpha G_\alpha$ . That is, every point of  $K$  lies in at least one of the open sets  $G_\alpha$ .

## 1.10 Compact Set

**Definition 1.10.** The set  $K$  is called **compact** if every open cover of  $K$  has a finite **sub-cover** of  $K$ . That is, if  $K \subseteq \bigcup_{\alpha} G_{\alpha}$ , then  $\exists \alpha_1, \dots, \alpha_n$  such that  $K \subseteq \bigcup_{i=1}^n G_{\alpha_i}$ .

## 1.11 Convergent Sequence

**Definition 1.11.** Let  $(X, d)$  be a metric space and let  $\{x_n\}$  be a sequence in  $X$ . We say that  $\{x_n\}$  is **convergent** if

$$\exists x \in X \text{ such that } \underline{\forall \epsilon > 0, \exists N = N(\epsilon) \in \mathbb{N}, \forall n \geq N, d(x_n, x) < \epsilon}.$$

In this case, we say that  $\{x_n\}$  converges to  $x$ , and write  $x_n \rightarrow x$ .

## 1.12 Divergent Sequence

**Definition 1.12.** We say that  $\{x_n\}$  is **divergent** if  $\{x_n\}$  is not convergent. That is,  $\forall x \in X$ ,  $\{x_n\}$  does not converge to  $x$ . The  $\epsilon$ - $N$  description of divergence of  $\{x_n\}$  is

$$\underline{\forall x \in X, \exists \epsilon_0 > 0, \forall N, \exists n \geq N, d(x_n, x) \geq \epsilon_0}.$$

## 1.13 Bounded Sequence

**Definition 1.13.** A sequence  $\{x_n\}$  in a metric space  $(X, d)$  is **bounded** if there exists a point  $x_0 \in X$  and a number  $M > 0$  such that

$$d(x_n, x_0) \leq M \text{ for all } n \in \mathbb{N}$$

In other words, all terms of the sequence lie within some fixed distance  $M$  of a single point  $x_0$ . In simplest terms, a sequence is bounded if all its terms lie inside some ball of finite radius.

## 1.14 The $\epsilon$ - $N$ description of $x_n \rightarrow x$

**Definition 1.14.** The  $\epsilon$ - $N$  description of  $x_n \rightarrow x$  is

$$\underline{\forall \epsilon > 0, \exists N = N(\epsilon) \in \mathbb{N}, \forall n \geq N, d(x_n, x) < \epsilon}.$$

## 1.15 The $\epsilon$ - $N$ description of $x_n \not\rightarrow x$

**Definition 1.15.** The  $\epsilon$ - $N$  description of  $x_n \not\rightarrow x$  is

$$\underline{\exists \epsilon_0 > 0, \forall N, \exists n \in \mathbb{N}, d(x_n, x) \geq \epsilon_0}.$$

## 2 Results

### 2.1 Chapter 2: Theorem 6

**Theorem.** Let  $(X, d)$  be a metric space.

1. If  $\{G_\alpha\}$  is a family of open sets in  $X$ , then  $\bigcup_\alpha G_\alpha$  is open in  $X$ .
2. If  $G_1, \dots, G_n$  are open sets in  $X$ , then  $\bigcap_{i=1}^n G_i$  is open in  $X$ .

### 2.2 Characterization (I) of $x \in \overline{E}$

Comparing with  $x \in \overline{E'} \iff \forall r > 0, (B(x, r) - \{x\}) \cap E \neq \emptyset$  (definition), we have

$$\underline{x \in E' \iff \forall r > 0, B(x, r) \cap E \neq \emptyset. \text{ Therefore, } E_1 \subseteq E_2 \implies \overline{E_1} \subseteq \overline{E_2}.}$$

### 2.3 Chapter 2: Theorem 8

**Theorem.** Every compact set in a metric space is bounded and closed. Therefore, if  $E$  is either unbounded or non-closed, then  $E$  is not compact.

### 2.4 Corollary 8

**Corollary.** Let  $E \subseteq \mathbb{R}$ . Then  $E$  is compact  $\iff E$  is bounded and closed.

### 2.5 Chapter 3: Theorem 1

**Theorem.** (Uniqueness of Limit). Let  $(X, d)$  be a metric space and let  $\{x_n\}$  be a sequence in  $X$ . If  $x_n \rightarrow x$  and  $x_n \rightarrow y$ , then  $x = y$ .

### 2.6 Chapter 3: Theorem 2

**Theorem.** Any convergent sequence in a metric space is bounded.

### 2.7 Chapter 3: Theorem 3

**Theorem.** (Squeeze Theorem in  $\mathbb{R}$ ). Let  $\{a_n\}, \{b_n\}$  and  $\{c_n\}$  be sequences in  $\mathbb{R}$  such that  $a_n \rightarrow x, c_n \rightarrow x$ , and  $\forall n, a_n \leq b_n \leq c_n$ . Then  $b_n \rightarrow x$ .

## 3 Questions/Proofs

### 3.1 Assignment 3: Question #6

**Question:** Find  $E^\circ, E', \overline{E}$ , and  $\partial E$  for the following subsets  $E \subseteq \mathbb{R}$ :

1.  $E = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

2.  $E = (0, \pi) \cap \mathbb{Q}$
3.  $E = [0, 2) \cup (4, 7) \cup \{8\}$
4.  $E = \mathbb{Z}$

**Recall:**

- $E^\circ =$  set of all interior points of  $E$
- $E' =$  set of all limit points of  $E$
- $\overline{E} =$  the closure set of  $E = E \cup E'$
- $\partial E =$  the set of all boundary points of  $E = \overline{E} \cap \overline{(X - E)}$

**Solution.**

1.  $E = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\} = \{\frac{1}{n}\}_{n=1}^\infty$   
So we get,

$$\begin{cases} E^\circ = \emptyset \\ E' = \{0\} \\ \overline{E} = E \cup E' = \{0\} \cup \{\frac{1}{n}\}_{n=1}^\infty \\ \partial E = \overline{E} \cap \overline{(\mathbb{R} - E)} = \overline{E} = \{0\} \cup \{\frac{1}{n}\}_{n=1}^\infty \end{cases}$$

2.  $E = (0, \pi) \cap \mathbb{Q}$   
So we get,

$$\begin{cases} E^\circ = \emptyset \\ E' = [0, \pi] \\ \overline{E} = E \cup E' = [0, \pi] \\ \partial E = \overline{E} \cap \overline{(\mathbb{R} - E)} = [0, \pi] \end{cases}$$

3.  $E = [0, 2) \cup (4, 7) \cup \{8\}$   
So we get,

$$\begin{cases} E^\circ = (0, 2) \cup (4, 7) \\ E' = [0, 2] \cup [4, 7] \\ \overline{E} = E \cup E' = [0, 2] \cup [4, 7] \cup \{8\} \\ \partial E = \overline{E} \cap \overline{(\mathbb{R} - E)} = \{0, 2, 4, 7, 8\} \end{cases}$$

Further analysis on  $\mathbb{R} - E$ :

$$\begin{aligned} \mathbb{R} - E &= (-\infty, 0) \cup [2, 4] \cup [7, 8) \cup (8, \infty) \\ \overline{\mathbb{R} - E} &= (-\infty, 0] \cup [2, 4] \cup [7, \infty) \end{aligned}$$

4.  $E = \mathbb{Z}$   
So we get,

$$\begin{cases} E^\circ = \emptyset \\ E' = \emptyset \\ \overline{E} = E \cup E' = \mathbb{Z} \\ \partial E = \overline{E} \cap \overline{(\mathbb{R} - E)} = \overline{E} \cap \mathbb{R} = \mathbb{Z} \end{cases}$$

□

### 3.2 Assignment 4: Question #2

**Question:** Find  $E^\circ, E', \overline{E}$ , and  $\partial E$  for the following subsets  $E \subseteq \mathbb{R}$ :

1.  $E = (-\infty, 0) \cup (1, 2) \cup \{5 + \frac{1}{n} : n = 1, 2, 3, \dots\}$
2.  $E = \{\pi\} \cup \mathbb{N}$
3.  $E = \{\pi\} \cup \mathbb{Q}$

**Solution.**

1.  $E = (-\infty, 0) \cup (1, 2) \cup \{5 + \frac{1}{n} : n = 1, 2, 3, \dots\}$

So we get,

$$\begin{cases} E^\circ = (-\infty, 0) \cup (1, 2) \\ E' = (-\infty, 0] \cup [1, 2] \cup \{5\} \\ \overline{E} = E \cup E' = (-\infty, 0] \cup [1, 2] \cup \{5 + \frac{1}{n}\}_{n=1}^\infty \cup \{5\} \\ \partial E = \overline{E} \cap (\mathbb{R} - \overline{E}) = \{0, 1, 2, 5\} \cup \{5 + \frac{1}{n}\}_{n=1}^\infty \end{cases}$$

Further analysis on  $\partial E$ :

$$\begin{aligned} &= \overline{E} \cap ([0, 1] \cup ([2, \infty] - \{5 + \frac{1}{n}\}_{n=1}^\infty)) \\ &= \overline{E} \cap ([0, 1] \cup [2, \infty)) \\ &= \{0, 1, 2, 5\} \cup \{5 + \frac{1}{n}\}_{n=1}^\infty \end{aligned}$$

2.  $E = \{\pi\} \cup \mathbb{N}$

So we get,

$$\begin{cases} E^\circ = \emptyset \\ E' = \emptyset \\ \overline{E} = E \cup E' = \{\pi\} \cup \mathbb{N} \text{ (so just } E) \\ \partial E = \overline{E} \cap (\mathbb{R} - \overline{E}) = E \cap \mathbb{R} = \{\pi\} \cup \mathbb{N} \text{ (so just } E) \end{cases}$$

3.  $E = \{\pi\} \cup \mathbb{Q}$

So we get,

$$\begin{cases} E^\circ = \emptyset \\ E' = \mathbb{R} \text{ (since } \mathbb{Q} \text{ is dense in } \mathbb{R}) \\ \overline{E} = E \cup E' = \mathbb{R} \\ \partial E = \overline{E} \cap (\mathbb{R} - \overline{E}) = \mathbb{R} \end{cases}$$

Further analysis on  $\mathbb{R} - E$ :

$$\begin{aligned} &= \mathbb{R} - E = \{x \in \mathbb{R} \mid x \text{ is irrational and } x \neq \pi\} \\ &= \overline{\mathbb{R} - E} = \mathbb{R}, \text{ (since } (\mathbb{R} - \mathbb{Q}) - \{\pi\} \text{ is dense in } \mathbb{R}) \end{aligned}$$

□

### 3.3 Assignment 4: Question #3

**Question:** Give an open cover of the set  $[0, 1] \cap \mathbb{Q}$  in  $\mathbb{R}$  that has no finite subcover.

*Proof.*

$$\implies \text{Note: } \frac{\pi}{4} \in [0, 1] \text{ and } \frac{\pi}{4} \notin \mathbb{Q}$$

$$\implies \text{For } n \in \mathbb{N}, \text{ let } G_n = (-1, \frac{\pi}{4} - \frac{1}{2^n}) \cup (\frac{\pi}{4}, 2)$$

$$\implies \text{Each } G_n \text{ is open, and } G_1 \subseteq G_2 \subseteq G_3 \subseteq \dots \text{ and } [0, 1] \cap \mathbb{Q} \subseteq \bigcup_{n=1}^{\infty} G_n \text{ is an open cover}$$

$$\implies \{G_n\}_{n=1}^{\infty} \text{ no finite subcover of } [0, 1] \cap \mathbb{Q}$$

$$\implies \forall n_0 \in \mathbb{N}, \bigcup_{n=1}^{n_0} G_n = G_{n_0} \text{ does not contain the points } (\frac{\pi}{4} - \frac{1}{2^{n_0}}, \frac{\pi}{4}) \cap \mathbb{Q}$$

Therefore, there does not exist a finite subcover. □

### 3.4 Assignment 4: Question #5

**Question:** Use the  $\epsilon$ - $N$  definition to prove the following:

1.

$$\lim_{n \rightarrow \infty} (1 + \frac{1}{n^2}) = 1$$

2.

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$$

3. The sequence  $\{1 + (-1)^n\}$  is divergent in  $\mathbb{R}$ .

**Solution.**

1.

$$\lim_{n \rightarrow \infty} (1 + \frac{1}{n^2}) = 1$$

*Proof.*

$$\implies \text{Given } \epsilon > 0, \text{ choose } N = \left\lceil \frac{1}{\sqrt{\epsilon}} \right\rceil + 1 \text{ (integer part)}$$

$$\implies \text{Then } N > \frac{1}{\sqrt{\epsilon}} \implies \frac{1}{N^2} < \epsilon$$

$$\implies \forall n \in \mathbb{N}, \text{ we have } \left| \left(1 + \frac{1}{n^2}\right) - 1 \right| = \frac{1}{n^2} \leq \frac{1}{N^2} < \epsilon$$



Therefore,

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n^2}\right) = 1$$

□

Further analysis on choosing  $N$  for  $n \geq N$ :

$$\left| \left(1 + \frac{1}{n^2}\right) - 1 \right| < \epsilon \implies \left| \frac{1}{n^2} \right| < \epsilon \implies \frac{1}{n^2} < \epsilon$$

when  $n \geq N$ ,  $\frac{1}{n^2} \leq \frac{1}{N^2} < \epsilon$

$$\frac{1}{\epsilon} < N^2 \implies \frac{1}{\sqrt{\epsilon}} < N$$

2.

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$$

*Proof.*

$$\implies \text{Given } \epsilon > 0. \text{ Take } N = \left\lceil \frac{1}{\epsilon} \right\rceil + 1 \text{ (integer part)}$$

$$\implies \text{Then } N \in \mathbb{N} \text{ and } \forall n \geq N, \text{ we get}$$

$$\implies \left| \frac{n}{n+1} - 1 \right| = \frac{1}{n+1} < \frac{1}{n} \leq \frac{1}{N} < \epsilon$$

Therefore,

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$$

□

Further analysis on choosing  $N$  for  $n \geq N$ :

$$\left| \frac{n}{n+1} - 1 \right| = \left| \frac{-1}{n+1} \right| = \frac{1}{n+1} < \frac{1}{n} \leq \frac{1}{N} < \epsilon \iff N > \frac{1}{\epsilon}$$

Note:

$$\frac{1}{\epsilon} < \left\lceil \frac{1}{\epsilon} \right\rceil + 1$$

where  $\left\lceil \dots \right\rceil$  represents the integer part.

3.

The sequence  $\{1 + (-1)^n\}$  is divergent in  $\mathbb{R}$

*Proof.* Case 1:  $x = 0$

$$\begin{aligned} \implies |x_n - x| &= |x|, \text{ if } n \text{ is even, then this implies } 2 \geq \epsilon_0 \\ \implies \text{Take } \epsilon_0 &= 2. \text{ Then } \forall N \in \mathbb{N}, \text{ taking } n = 2N, \text{ then } n \geq N \\ \implies |x_n - x| &= |x_n| = 2 \geq \epsilon_0 \end{aligned}$$

Case 2:  $x \neq 0$

$$\begin{aligned} \implies |x_n - x|, & \text{ if } n \text{ is odd, then this implies } |x| \geq \epsilon_0 \\ \implies \text{Take } \epsilon_0 &= 2. \text{ Then } \forall N \in \mathbb{N}, \text{ taking } n = 2N + 1, \text{ then } n \geq N \\ \implies |x_n - x| &= |0 - x| = |x| \geq \epsilon_0 \end{aligned}$$

Therefore,

The series  $\{1 + (-1)^n\}$  is divergent in  $\mathbb{R}$

□

Some analysis on the result:

$$\{1 + (-1)^n\} = 0, 2, 0, 2, 0, \dots \text{ for infinity, hence is divergent.}$$

Recall:  $\{x_n\}$  is divergent in  $\mathbb{R} : \forall x \in \mathbb{R}, x_n \not\rightarrow x$

i.e.,  $\forall x \in \mathbb{R}, \exists \epsilon_0 > 0, \forall N \in \mathbb{N}, \exists n \geq N, |x_n - x| \geq \epsilon_0$

□

### 3.5 Assignment 4: Question #8

**Question:** Let  $\{x_n\}$  and  $\{y_n\}$  be sequences in  $\mathbb{R}$  such that  $\{x_n\}$  is bounded and  $y_n \rightarrow 0$ . Prove that  $x_n y_n \rightarrow 0$ .

**Solution.**

*Proof.*

$$\begin{aligned} \implies \text{Since } \{x_n\} & \text{ is bounded, there exists } M > 0 \text{ such that } |x_n| \leq M \text{ for all } n \in \mathbb{N} \\ \implies \text{Given } \epsilon > 0. & \text{ Since } y_n \rightarrow 0, \\ \implies \exists N \in \mathbb{N}, \forall n \geq N, & |y_n - 0| < \frac{\epsilon}{M} \\ \implies \text{Then } \forall n \geq N, & \text{ we have} \\ \implies |x_n y_n - 0| = |x_n| |y_n| & \leq M |y_n| < M \times \frac{\epsilon}{M} = \epsilon \end{aligned}$$

Therefore,

$$\lim_{n \rightarrow \infty} x_n y_n = 0$$

□

Some analysis on  $|x_n y_n - 0| < \epsilon$  for  $n \geq N$ :

(Boundedness):  $\exists M > 0$  such that  $\forall n \in \mathbb{N}, |x_n| \leq M$ .

$$|x_n y_n - 0| = |x_n| |y_n| \leq M |y_n|.$$

Since  $y_n \rightarrow 0$ , for any  $\epsilon > 0$  there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,

$$|y_n| < \frac{\epsilon}{M}.$$

Therefore,

$$|x_n y_n - 0| \leq M |y_n| < M \cdot \frac{\epsilon}{M} = \epsilon.$$

This satisfies the definition of the limit:

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } n \geq N \implies |x_n y_n - 0| < \epsilon.$$

□

### 3.6 Assignment 5: Question #1(i)(ii)

**Question:** Use the  $\epsilon$ - $N$  definition to prove the following:

1.

$$\lim_{n \rightarrow \infty} (\sqrt{n^2 + 2n} - n) = 1$$

2. The sequence  $\{n^{(-1)^n - 1}\}$  is divergent in  $\mathbb{R}$ .

**Solution.**

□

### 3.7 Chapter 2: the proof of Theorem 9

**Theorem.** Let  $K$  be a compact set in metric space  $X$  and let  $F$  be a closed subset of  $K$ . Then  $F$  is compact.

*Proof.* Let  $\{G_\alpha\}$  be an open cover of  $F$ . Then  $\{X - F\} \cup \{G_\alpha\}$  is an open cover of  $K$ . Since  $K$  is compact,  $\exists \alpha_1, \dots, \alpha_n$  such that

$$K \subseteq (X - F) \cup (G_{\alpha_1} \cup \dots \cup G_{\alpha_n}).$$

Now  $F = F \cap K \subseteq (F \cap (X - F)) \cup (G_{\alpha_1} \cup \dots \cup G_{\alpha_n}) = G_{\alpha_1} \cup \dots \cup G_{\alpha_n}$ . Hence,  $\{G_\alpha\}$  has a finite subcover  $\{G_{\alpha_1}, \dots, G_{\alpha_n}\}$  of  $F$ . Therefore,  $F$  is compact. □

### 3.8 Chapter 3: the proof of Theorem 2

**Theorem.** Any convergent sequence in a metric space is bounded.

*Proof.* Suppose  $x_n \rightarrow x$  in a metric space  $(X, d)$ . We prove that

$$\exists M > 0, \forall n, x_n \in B(x, M)$$

For  $\epsilon = 1, \exists N, \forall n \geq N, d(x_n, x) < 1$ . Let  $M = 1 + \max\{d(x_1, x), \dots, d(x_{N-1}, x)\}$ . Then  $\forall n$ , we have  $d(x_n, x) < M$ . □

### 3.9 Chapter 3: the proof of Theorem 3

**Theorem.** (Squeeze Theorem in  $\mathbb{R}$ ). Let  $\{a_n\}$ ,  $\{b_n\}$ , and  $\{c_n\}$  be sequences in  $\mathbb{R}$  such that  $a_n \rightarrow x$ ,  $c_n \rightarrow x$ , and  $\forall n, a_n \leq b_n \leq c_n$ . Then,  $b_n \rightarrow x$ .

**Note:** Here either  $x \in \mathbb{R}$  or  $x = \pm\infty$ . We only prove the case when  $x \in \mathbb{R}$ . Also, in the theorem, we can only require that  $a_n \leq b_n \leq c_n$  holds for all  $n \geq n_0$  for some  $n_0$ .

*Proof.* Given  $\epsilon > 0$ . Since  $a_n \rightarrow x$  and  $c_n \rightarrow x$ ,  $\exists N_1, N_2 \in \mathbb{N}$  such that

$$\forall n \geq N_1, |a_n - x| < \epsilon \text{ and } \forall n \geq N_2, |c_n - x| < \epsilon.$$

Let  $N = \max\{N_1, N_2\}$ . Then  $\forall n \geq N$ , we have  $|a_n - x| < \epsilon$  and  $|c_n - x| < \epsilon$ . That is,

$$x - \epsilon < a_n < x + \epsilon \text{ and } x - \epsilon < c_n < x + \epsilon$$

In this case,  $\forall n \geq N$ , we have  $x - \epsilon < a_n \leq b_n \leq c_n < x + \epsilon$ . That is,

$$x - \epsilon < b_n < x + \epsilon, \text{ or } |b_n - x| < \epsilon$$

So, we prove that  $\forall \epsilon > 0, \exists N, \forall n \geq N, |b_n - x| < \epsilon$ . Therefore,  $b_n \rightarrow x$ . □