MATH-3580 Final Exam Review

Aidan Richer

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The Final Exam contains eight questions covering topics in Chapter 3 (Parts 1 - 3, 5, and 6) and Chapter 4 (Parts 1 - 3, up to Example 5) in the lecture outlines.

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1 Definitions

1.1 Metric Space

In mathematics, space = set + structure(s).

Definition 1.1. Let X be a set. A function $d : X \times X \to [0, \infty)$ is called a **metric** (or distance) on X if

1. $\forall x, y \in X, d(x, y) = 0 \iff x = y$;
2. $\forall x, y \in X, d(x, y) = d(y, x)$;
3. $\forall x, y, z \in X, d(x, z) \le d(x, y) + d(y, z)$. (Triangle inequality)

In this case, (X, d) is called a **metric space**.

1.2 Open Ball and Bounded Set

Definition 1.2. Let (X, d) be a metric space, $x \in X$, and $r > 0$.

1. Define $B(x, r) = \{y \in X : d(x, y) < r\}$, called the **open ball** centered at x with radius r, or the **r-neighborhood of x**.

2. A general **neighborhood of x** is a subset U of X such that $B(x,r) \subseteq U$ for some $r > 0$.

3. A subset $E \subseteq X$ of a metric space (X,d) is **bounded** if there exists some $x_0 \in X$ and $M > 0$ such that $d(x, x_0) \leq M$ for all $x \in E$.

1.3 Interior Points and Interior

Definition 1.3. Let $E \subseteq X$. The **closure** of E is the set $\overline{E} = E \cup E'$.

1. An element x of X is called an **interior point** of E if $\exists r > 0, B(x,r) \subseteq E$.

2. The **interior** of E is the set E° of all interior points of E.

By definition, we have $E^\circ \subseteq E \subseteq \overline{E}$.

1.4 Open Set

Definition 1.4. A subset $G \subseteq X$ is called **open** if $\forall x \in G, \exists r > 0, B(x,r) \subseteq G$. Note, $\emptyset$ and X are open in X. That is, a set is open if every point in it has an open ball around it that's still entirely inside the set.

1.5 Limit Points and Derived Set

Definition 1.5. Let $E \subseteq X$.

1. x is called a **limit point** of E (or cluster point, or accumulation point) if

$$\forall r > 0, B(x,r) \cap E \text{ contains some } y \neq x$$

Equivalently,

$$(B(x,r) - \{x\}) \cap E \neq \emptyset$$

2. We let $E' =$ the set of all limit points of E, called the **derived set** of E.

1.6 Closed Set

Definition 1.6. A subset $E \subseteq X$ is called **closed** if $E' \subseteq E$, where E' is the derived set of E (the set of all limit points of E). Note, $\emptyset$ and X are closed in X. That is, a subset $E \subseteq X$ is closed if every limit point of E belongs to E.

1. E is open $\iff X - E$ is closed;

2. E is closed $\iff X - E$ is open.

1.7 Closure

Definition 1.7. Let $E \subseteq X$. The **closure** of E is the set $\overline{E} = E \cup E'$.

1.8 Boundary Points and Boundary

Definition 1.8. Let X be a metric space, $E \subseteq X$ and $x \in X$.

1. x is called a **boundary point** of E if $\forall r > 0, B(x,r) \cap E \neq \emptyset$ and $B(x,r) \cap (X - E) \neq \emptyset$.
2. We use ∂E to denote the set of all boundary points of E, called the **boundary** of E.

1.9 Open Cover

Definition 1.9. Let X be a metric space and let $K \subseteq X$. A family $\{G_\alpha\}$ of open sets in X is called an **open cover** of K if $K \subseteq \bigcup_\alpha G_\alpha$. That is, every point of K lies in at least one of the open sets G_α.

1.10 Compact Set

Definition 1.10. The set K is called **compact** if every open cover of K has a finite **subcover** of K. That is, if $K \subseteq \bigcup_\alpha G_\alpha$, then $\exists \alpha_1, \dots, \alpha_n$ such that $K \subseteq \bigcup_{i=1}^n G_\alpha$.

1.11 Convergent Sequence

Definition 1.11. Let (X, d) be a metric space and let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is **convergent** if

$$\exists x \in X \text{ such that } \underline{\forall \epsilon > 0, \exists N = N(\epsilon) \in \mathbb{N}, \forall n \geq N, d(x_n, x) < \epsilon}.$$

In this case, we say that $\{x_n\}$ converges to x, and write $x_n \to x$.

1.12 Divergent Sequence

Definition 1.12. We say that $\{x_n\}$ is **divergent** if $\{x_n\}$ is not convergent. That is, $\forall x \in X, \{x_n\}$ does not converge to x. The ϵ-N description of divergence of $\{x_n\}$ is

$$\underline{\forall x \in X, \exists \epsilon_0 > 0, \forall N, \exists n \geq N, d(x_n, x) \geq \epsilon_0}.$$

1.13 Bounded Sequence

Definition 1.13. A sequence $\{x_n\}$ in a metric space (X, d) is **bounded** if there exists a point $x_0 \in X$ and a number $M > 0$ such that

$$d(x_n, x_0) \leq M \text{ for all } n \in \mathbb{N}$$

In other words, all terms of the sequence lie within some fixed distance M of a single point x_0. In simplest terms, a sequence is bounded if all its terms lie inside some ball of finite radius.

1.14 The ϵ-N description of $x_n \to x$

Definition 1.14. The ϵ-N description of $x_n \to x$ is

$$\underline{\forall \epsilon > 0, \exists N = N(\epsilon) \in \mathbb{N}, \forall n \geq N, d(x_n, x) < \epsilon}.$$

1.15 The ϵ-N description of $x_n \not\to x$

Definition 1.15. The ϵ-N description of $x_n \not\to x$ is

$$\underline{\exists \epsilon_0 > 0, \forall N, \exists n \in N, d(x_n, x) \geq \epsilon_0}.$$

1.16 The ϵ-N definition of a Cauchy sequence and its negation

Definition 1.16. Let $\{x_n\}$ be a sequence in a metric space (X, d). $\{x_n\}$ is called <u>Cauchy</u> if

$$\underline{\forall \epsilon > 0, \exists N, \forall m, n \geq N, d(x_m, x_n) < \epsilon}$$

So we can say it's negation: $\{x_n\}$ is not Cauchy if and only if

$$\underline{\exists \epsilon_0 > 0, \forall N, \exists m, n \geq N, d(x_m, x_n) \geq \epsilon_0}$$

1.17 Subsequence

Definition 1.17. Let $\{x_n\}$ be a sequence. If

$$\{n_k\}_{k\in\mathbb{N}} \text{ is a sequence in } \mathbb{N} \text{ such that } n_1 < n_2 < \dots$$

then $\{x_{n_k}\}_{k=1}^{\infty}$ is called a <u>subsequence</u> of $\{x_n\}$.

1.18 Convergence, Divergence, Absolute Convergence of a Series

Definition 1.18. Let $\{a_n\}$ be a sequence in $\mathbb{R}$. For each $n \in \mathbb{N}$, let

$$s_n = a_1 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

called the <u>n^{th} partial sum</u> of the series $\displaystyle\sum_{k=1}^{\infty} a_k$.

Convergence: If $s_n \to s \in \mathbb{R}$, then we say that the series $\displaystyle\sum_{k=1}^{\infty} a_k$ is <u>convergent</u>, and we write

$$\sum_{k=1}^{\infty} a_k = s \text{ (called the sum of } \sum_{k=1}^{\infty} a_k)$$

Divergence: If the sequence of partial sums $\{s_n\}$ is divergent (does not approach a finite limit), then we say that the series $\displaystyle\sum_{k=1}^{\infty} a_k$ is <u>divergent</u>.

Absolute Convergence: A series $\displaystyle\sum_{n=1}^{\infty} a_n$ is <u>absolutely convergent</u> if $\displaystyle\sum_{n=1}^{\infty} |a_n|$ is convergent.

1.19 Geometric series, P-series, Alternating series, and their Convergence

Definition 1.19. We observe the following,

Geometric series: The geometric series $\sum_{n=0}^{\infty} x_n$ is convergent $\iff |x| < 1$. When $|x| < 1$, we have $\sum_{n=0}^{\infty} x_n = \dfrac{1}{1-x}$.

P-series: Let $p \in \mathbb{R}$. For a series of the form $\sum_{n=1}^{\infty} \dfrac{1}{n^p}$ called the p-series, we have

(i) If $p \leq 1$, then $\sum_{n=1}^{\infty} \dfrac{1}{n^p}$ is divergent

(ii) If $p > 1$, then $\sum_{n=1}^{\infty} \dfrac{1}{n^p}$ is convergent.

Alternating series: Suppose $\{b_n\}$ is a sequence in $\mathbb{R}$ such that $b_1 \geq b_2 \geq \cdots \geq 0$ and $b_n \to 0$. Then the alternating series of the form $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ and $\sum_{n=1}^{\infty} (-1)^n b_n$ are convergent.

1.20 The ϵ-δ definition of $\lim_{x \to c} f(x)$ and its negation

Definition 1.20. Let (X, d_X) and (Y, d_Y) be metric spaces, $E \subseteq X$, $c \in E'$, $f : E \to Y$, and $q \in Y$. We write the ϵ-δ definition of $\lim_{x \to c} f(x) = q$ as

$$\underline{\forall \epsilon > 0, \exists \delta > 0, \forall x \in E \text{ with } 0 < d_X(x, c) < \delta, d_Y(f(x), q) < \epsilon}$$

That is,

$$x \in \Big(B_X(c, \delta) - \{c\}\Big) \cap E \implies f(x) \in B_Y(q, \epsilon), \text{ or equivalently,}$$

$$f\Big((B_X(c, \delta) - \{c\}) \cap E\Big) \subseteq B_Y(q, \epsilon)$$

Negation: The ϵ-δ description of the negation of $\lim_{x \to c} f(x) = q$ is

$$\underline{\exists \epsilon_0 > 0, \forall \delta > 0, \exists x \in E \text{ with } 0 < d_X(x, c) < \delta,\ d_Y(f(x), q) \geq \epsilon_0}$$

1.21 The ϵ-δ definition of continuity of f at c and its negation

Definition 1.21. Let (X, d_X) and (Y, d_Y) be metric spaces, $c \in X$, and $f : X \to Y$. We say that f is continuous at c and write the ϵ-δ definition if

$$\underline{\forall \epsilon > 0, \exists \delta = \delta(\epsilon, c) > 0, \forall x \text{ with } d_X(x, c) < \delta,\ d_Y(f(x), f(c)) < \epsilon}$$

That is, $f\Big(B_X(c,\delta)\Big) \subseteq B_Y\Big(f(c),\epsilon\Big)$. If f is continuous at every point in X, then we say that f is <u>continuous on X</u>.
Negation: Therefore, we can write the negation and say that f is not continuous at c if and only if

$$\underline{\exists \epsilon_0 > 0, \forall \delta > 0, \exists x \text{ with } d_X(x,c) < \delta,\ d_Y(f(x), f(c)) \geq \epsilon_0}$$

1.22 The ϵ-δ definition of uniform continuity of f

Definition 1.22. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$. We say that f is <u>uniformly continuous on X</u> if

$$\underline{\forall \epsilon > 0, \exists \delta = \delta(\epsilon) > 0, \forall p, q \in X \text{ with } d_X(p,q) < \delta,\ d_Y(f(p), f(q)) < \epsilon}$$

Negation: We say that the negation, $f : X \to Y$ is <u>not uniformly continuous on X</u> if and only if

$$\underline{\exists \epsilon_0 > 0, \forall \delta > 0, \exists p, q \in X \text{ with } d_X(p,q) < \delta,\ d_Y(f(p), f(q)) \geq \epsilon_0}$$

2 Results

2.1 Chapter 3: Theorem 11

Let (X, d) be a metric space and let $\{x_n\}$ be a sequence in X.

(i) If $\{x_n\}$ is convergent, then $\{x_n\}$ is Cauchy.

(ii) If $\{x_n\}$ is Cauchy, then $\{x_n\}$ is bounded.

2.2 Chapter 3: Theorem 14

(Cauchy Criterion for Series). Let $\sum_{n=1}^{\infty} a_n$ be a series in $\mathbb{R}$. Then

$$\sum_{n=1}^{\infty} a_n \text{ is convergent} \iff \forall \epsilon > 0, \exists N, \forall m \geq n \geq N, \left|\sum_{n=1}^{\infty} a_k\right| < \epsilon$$

2.3 Chapter 3: Corollary 7

Let $\sum_{n=1}^{\infty} a_n$ be a series in $\mathbb{R}$. Then

$$\sum_{n=1}^{\infty} a_n \text{ is divergent} \iff \exists \epsilon_0 > 0, \forall N, \exists m \geq n \geq N, \left|\sum_{k=n}^{m} a_k\right| \geq \epsilon_0$$

Note that if $\sum_{n=1}^{\infty} a_n$ is convergent, then $a_n \to 0$ since

$$a_n = s_n - s_{n-1} \to s - s = 0$$

Therefore, we have the following test for divergence.

2.4 Chapter 3: Theorem 15

(Divergence Test). If $a_n \not\to 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

2.5 Chapter 3: Theorem 16

If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

2.6 Chapter 3: Corollary 6

(Cauchy Criterion). Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Then

$$\{x_n\} \text{ is convergent } \iff \{x_n\} \text{ is Cauchy.}$$

2.7 Chapter 3: Corollary 8

(Comparison Theorem). Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be nonnegative series. Suppose that $\exists n_0 \in \mathbb{N}$ such that $0 \leq a_n \leq b_n$ for all $n \geq n_0$.

(i) If $\sum_{n=1}^{\infty} a_n$ is divergent, then $\sum_{n=1}^{\infty} b_n$ is divergent.

(ii) If $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

2.8 Chapter 4: Theorem 2

(Sequential Criterion for Convergence). Let (X, d_X) and (Y, d_Y) be metric spaces, $E \subseteq X$, $c \in E'$, $f : E \to Y$, and $q \in Y$. Then

$$\lim_{x \to c} f(x) = q \iff \forall \text{ sequence } \{c_n\} \text{ in } E - \{c\},\ c_n \to c \implies f(c_n) \to q.$$

2.9 Chapter 4: Theorem 8

Let X and Y be metric spaces with X compact and let $f : X \to Y$ be continuous. Then $f(X)$ is a compact set in Y. Therefore, $f(X)$ is bounded and closed in Y.

2.10 Chapter 4: Corollary 1

We have

$$f(x) \not\to q \text{ as } x \to c \iff \exists \text{ sequence } \{c_n\} \text{ in } E - \{c\} \text{ such that } c_n \to c \text{ but } f(c_n) \not\to q.$$

2.11 Chapter 4: Corollary 2

If there exist sequences $\{x_n\}$ and $\{y_n\}$ in $E - \{c\}$ such that

$$x_n \to c,\ y_n \to c,\ f(x_n) \to p,\ f(y_n) \to q \text{ and } p \neq q,\ \text{ then } \lim_{x\to c} f(x) \text{ DNE.}$$

2.12 Chapter 4: Corollary 3

(Sequential Criterion for Continuity). Let $c \in X$. Then

$$f : X \to Y \text{ is continuous at } c \iff \Big[c_n \to c \text{ in } X \implies f(c_n) \to f(c) \text{ in } Y\Big].$$

2.13 Chapter 4: Corollary 7

If $f : X \to Y$ is continuous, then $\forall$ compact $E \subseteq X$, $f(E)$ is compact.

3 Questions/Proofs

3.1 Assignment 6: Question #1

Question: Suppose that $a_n > 0 \ \ (n = 1, 2, \dots)$.

(i) Prove that if $\sum a_n$ is convergent, then $\sum \frac{\sqrt{a_n}}{n}$ is convergent.

Solution. *Analysis on the property we will be using:*

$$(a-b)^2 = a^2 - 2ab + b^2 \geq 0 \implies ab \leq \frac{1}{2}(a^2 + b^2)$$

Proof. We have

$$\frac{\sqrt{a_n}}{n} = \left(\sqrt{a_n}\right)\left(\frac{1}{n}\right) \leq \frac{1}{2}\left(a_n + \frac{1}{n^2}\right)$$

Since

$$\sum a_n \text{ and } \sum \frac{1}{n^2} \text{ converge } \implies \sum \frac{1}{2}\left(a_n + \frac{1}{n^2}\right) \text{ converges.}$$

Therefore, since $\sum \frac{\sqrt{a_n}}{n} \leq \sum \frac{1}{2}\left(a_n + \frac{1}{n^2}\right)$, then $\sum \frac{\sqrt{a_n}}{n}$ converges. □

□

(ii) Prove that if $\sum a_n$ is divergent, then $\sum \dfrac{a_n}{1+a_n}$ is divergent.

Solution. We will use the contrapositive form:

$$\sum \frac{a_n}{1+a_n} \text{ converges} \implies \sum a_n \text{ converges}$$

Proof. Assume $\sum \dfrac{a_n}{1+a_n}$ converges.

$$\begin{aligned}
&\implies \lim_{n\to\infty} \frac{a_n}{1+a_n} = 0, \text{ so } \exists N, \forall n \geq N, \frac{a_N}{1+a_n} < \frac{1}{2} \\
&\implies 2a_n < 1 + a_n \text{ or } a_n < 1 \text{ for all } n \geq N \\
&\implies 1 + a_n < 2 \\
&\implies \frac{1}{2} < \frac{1}{1+a_n} \\
&\implies \frac{1}{2}a_n < \frac{a_n}{1+a_n}
\end{aligned}$$

So, we have that $\sum \frac{1}{2}a_n$ converges since $\sum \dfrac{a_n}{1+a_n}$ converges.

Therefore, $\sum a_n$ is convergent. □

□

3.2 Assignment 6: Question #2

Question: Discuss the (absolute) convergence/divergence of the series $\sum_{n=1}^{\infty} a_n$, where a_n is given by

(i) $a_n = (-1)^n \left(1 + \frac{1}{n}\right)$

(ii) $a_n = \sqrt{n+1} - \sqrt{n}$

(iii) $a_n = (-1)^n \frac{1}{\sqrt{n}}$

(iv) $a_n = \frac{\sqrt{n+1}-\sqrt{n}}{2n}$

(v) $a_n = \frac{(-1)^n}{1+x^n}, (x \geq 0)$

(vi) $a_n = \frac{n!}{1000^n}$

(vii) $a_n = \frac{n!}{n^n}$

(viii) $a_n = \frac{\cos n}{n^{5/4}}$

Solution.

(i) We have that

$$|a_n| = 1 + \frac{1}{n} \to 1 \neq 0$$

So the terms do not go to 0, thus the series diverges immediately.

(ii) We can rewrite this as

$$\sqrt{n+1}-\sqrt{n}=\frac{1}{\sqrt{n+1}+\sqrt{n}}\sim\frac{1}{2\sqrt{n}}<\frac{1}{\sqrt{n}}=\frac{1}{n^{1/2}}\text{ (divergent p-series)}$$

Since the corresponding p-series has $p=\frac{1}{2}<1$, the series is divergent.

(iii) We have

$$|a_n|=\frac{1}{\sqrt{n}}=\frac{1}{n^{1/2}}$$

Which is a divergent p-series since $p=\frac{1}{2}<1$. But for the alternating series test, the terms $\to 0$. So we can say that the series is conditionally convergent, not absolutely convergent.

(iv) We can simplify this as

$$\frac{\sqrt{n+1}-\sqrt{n}}{2n}\cdot\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}=\frac{1}{2n}\cdot\frac{1}{\sqrt{n+1}+\sqrt{n}}$$

Since

$$\frac{1}{\sqrt{n+1}+\sqrt{n}}\sim\frac{1}{2\sqrt{n}}$$

We can rewrite this as

$$a_n\sim\frac{1}{2n}\cdot\frac{1}{2\sqrt{n}}=\frac{1}{4n^{3/2}}$$

Since this can be compared to the p-series with $p=\frac{3}{2}>1$, the series is absolutely convergent.

(v) We have

$$|a_n|=\frac{1}{1+2^n}\sim\frac{1}{2^n}$$

Since $\dfrac{1}{2^n}$ is a convergent geometric series with ratio $\frac{1}{2}$, the Comparison Test shows that $|a_n|$ also converges. So, we can say that the series is absolutely convergent.

(vi) Using the Ratio Test, we get

$$\left|\frac{a_{n+1}}{a_n}\right|=\frac{(n+1)!}{1000^{n+1}}\cdot\frac{1000^n}{n!}=\frac{(n+1)!}{n!}\cdot 1000^{n-n-1}=(n+1)\cdot 1000^{-1}=\frac{n+1}{1000}\to\infty>1$$

So, by the Ratio Test, the series is divergent since the limit is >1.

(vii) Using the Ratio Test, we get

$$\left|\frac{a_{n+1}}{a_n}\right|=\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}=\frac{n+1}{\left(1+\frac{1}{n}\right)^{n+1}}\to\frac{1}{e}<1$$

So, by the Ratio Test, the series is absolutely convergent since the limit is <1.

(viii) We can compare this to

$$|a_n| = \frac{1}{n^{5/4}}$$

Which is a convergent p-series since $p = \frac{5}{4} > 1$. So, we can say that the series is absolutely convergent.

$\square$

3.3 Assignment 7: Question #2

Question: Use the ϵ-δ definition to prove that

(i) $\lim_{x\to 2}(x^2 - x + 1) = 3$ (ii) $\lim_{x\to 0} \frac{2x}{1+x^2} = 0$ (iii) $\lim_{x\to 1} \sqrt{x^2+2} = \sqrt{3}$

Solution.

(i) *Analysis:* We start with

$$|(x^2 - x + 1) - 3| = |x^2 - x - 2| = |(x-2)(x+1)| = |x-2| \cdot |x+1|$$

To control $|x+1|$, we restrict x near 2
Pick $\delta \leq 1$ Then

$$|x-2| < 1 \implies 1 < x < 3 \implies |x+1| < 4$$

So

$$|(x^2 - x + 1) - 3| < 4|x-2| < \epsilon \implies |x-2| < \frac{\epsilon}{4}$$

So, we have found the correct delta to use in our proof $\delta = \min\left(1, \frac{\epsilon}{4}\right)$.

Proof. Let $\epsilon > 0$. Take $\delta = \min\left(1, \frac{\epsilon}{4}\right)$. Then $\delta > 0$. So we have

$$|f(p) - f(q)| = |(x^2 - x + 1) - 3| = |x-2||x+1| < 4|x-2| < 4\left(\frac{\epsilon}{4}\right) = \epsilon$$

Therefore, $\lim_{x\to 2}(x^2 - x + 1) = 3$. $\square$

(ii) *Analysis:* We start with

$$\left|\frac{2x}{1+x^2}\right| = \frac{2|x|}{1+x^2} \leq 2|x| < \epsilon \implies |x| < \frac{\epsilon}{2}$$

So, we have found the correct delta to use in our proof $\delta = \frac{\epsilon}{2}$.

Proof. Let $\epsilon > 0$. Take $\delta = \frac{\epsilon}{2}$. Then $\delta > 0$. So we have

$$|f(p) - f(q)| = \left|\frac{2x}{1+x^2} - 0\right| = \frac{2|x|}{1+x^2} \leq 2|x| < 2\left(\frac{\epsilon}{2}\right) = \epsilon$$

Therefore, $\lim_{x\to 0} \frac{2x}{1+x^2} = 0$. $\square$

(iii) *Analysis:* We start with

$$|\sqrt{x^2+2} - \sqrt{3}| = |\sqrt{x^2+2} - \sqrt{3}| \cdot \frac{\sqrt{x^2+2}+\sqrt{3}}{\sqrt{x^2+2}+\sqrt{3}} = \frac{|x^2+2-3|}{\sqrt{x^2+2}+\sqrt{3}} = \frac{x^2-1}{\sqrt{x^2+2}+\sqrt{3}}$$
$$= \frac{|x-1||x+1|}{\sqrt{x^2+2}+\sqrt{3}}$$

Choose $\delta \leq 1$ so

$$|x-1| < 1 \implies 0 < x < 2$$

Then

$$|\sqrt{x^2+2} - \sqrt{3}| = \frac{|x-1||x+1|}{\sqrt{x^2+2}+\sqrt{3}} < \frac{|x-1| \cdot 3}{3} = |x-1| < \epsilon \implies \delta = \min(1, \epsilon)$$

So, we have found the correct delta to use in our proof $\delta = \min(1, \epsilon)$.

Proof. Let $\epsilon > 0$. Take $\delta = \min(1, \epsilon)$. Then $\delta > 0$. So if

$$0 < |x-1| < \delta, \text{ then } |x+1| < 3 \text{ and } \sqrt{x^2+2} + \sqrt{3} > 3$$

So we have

$$|f(p) - f(q)| = |\sqrt{x^2+2} - \sqrt{3}| = \frac{|x-1||x+1|}{\sqrt{x^2+2}+\sqrt{3}} < \frac{|x-1| \cdot 3}{3} < \epsilon$$

Therefore, $\lim_{x\to 1} \sqrt{x^2+2} = \sqrt{3}$ $\square$

$\square$

3.4 Assignment 7: Question #5

Question: Let $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ Use the ϵ-δ definition to prove that $f : \mathbb{R} \times \mathbb{R}$ is continuous at 0.

Solution. *Analysis:* We want to show continuity at $x = 0$. Since $f(0) = 0$, we need to control

$$|f(x) - f(0)| = |x\sin\left(\frac{1}{x}\right)|$$

Using the standard bound

$$|\sin\left(\frac{1}{x}\right)| \leq 1$$

So

$$|x\sin\left(\frac{1}{x}\right)| \leq |x| < \epsilon \implies \delta = \epsilon$$

So, we have found the correct delta to use in our proof $\delta = \epsilon$.

Proof. Let $\epsilon > 0$. Take $\delta = \epsilon$. Then $\delta > 0$. So if

$$0 < |x| < \delta$$

Then we have

$$|f(x) - f(0)| = |x\sin\left(\frac{1}{x}\right)| \leq |x| < \epsilon$$

Therefore, f is continuous at 0. □

□

3.5 Assignment 7: Question #6

Question: Use the ϵ-δ definition to prove that $f(x) = \sqrt{x}$ is continuous at every $c \geq 0$.
Solution. To prove f is continuous at every $c \geq 0$, let's do some analysis on $|f(x) - f(c)|$ and relate it with $M|x - c|$ where $M > 0$. By doing so, we can find the proper delta to use in our formal proof.
Analysis:

$$|f(x) - f(c)| = |\sqrt{x} - \sqrt{c}| = \begin{cases} \sqrt{x} & c = 0 \\ \frac{1}{\sqrt{x}+\sqrt{c}}|x - c| \leq \frac{1}{\sqrt{c}}|x - c| & c > 0 \end{cases}$$

When $c = 0$, we need to choose a delta such that

$$\sqrt{x} = \epsilon \implies \delta = \epsilon^2$$

When $c > 0$, we need to choose a delta such that

$$\frac{1}{\sqrt{x} + \sqrt{c}}|x - c| \leq \frac{1}{\sqrt{c}}|x - c| \implies |x - c| < \epsilon\sqrt{c} \implies \delta = \epsilon\sqrt{c}$$

Further expanding

$$|\sqrt{x} - \sqrt{c}| = \frac{|x - c|}{\sqrt{x} + \sqrt{c}}$$

Proof. Given $\epsilon > 0$.
Case 1: $c = 0$
Take $\delta = \epsilon^2$. Then $\delta > 0$. Now,

$$\forall x \in [0, \infty) \text{ with } |x - 0| = x < \delta$$

we have

$$|f(x) - f(c)| = \sqrt{x} < \sqrt{\delta} = \epsilon$$

Therefore, f is continuous at 0.
Case 2: $c > 0$
Take $\delta = \epsilon\sqrt{c}$. Then $\delta > 0$. Now,

$$\forall x \in [0, \infty) \text{ with } |x - c| < \delta$$

we have

$$|f(x) - f(c)| = |\sqrt{x} - \sqrt{c}| = \frac{1}{\sqrt{x} + \sqrt{c}}|x - c| \leq \frac{1}{\sqrt{c}}|x - c| < \frac{1}{\sqrt{c}}\delta = \epsilon$$

Therefore, f is continuous at c. □

□

3.6 Assignment 7: Question #7

Question: Use the ϵ-δ definition to prove the uniform continuity of the following functions on the given intervals.

(i) $f(x) = \dfrac{1}{x^2}$ on $[2, \infty)$

Solution. To prove uniform continuity of f, let's do some analysis on $|f(p) - f(q)|$ and relate it with $M|p - q|$ where $M > 0$. By doing so, we can find the proper delta to use in our formal proof.
Analysis:

$$\begin{aligned} |f(p) - f(q)| &= \left|\frac{1}{p^2} - \frac{1}{q^2}\right| = \left|\frac{q^2 - p^2}{p^2q^2}\right| = \frac{(q+p)|p-q|}{p^2q^2} \\ &= \left(\frac{1}{p^2q^2} + \frac{1}{pq^2}\right)|p - q| \leq \left(\frac{1}{8} + \frac{1}{8}\right)|p - q| \text{ (since } p, q \geq 2) \\ &= \frac{1}{4}|p - q| < \frac{\delta}{4} = \epsilon \implies \delta = 4\epsilon \end{aligned}$$

Now we have found the correct delta to use in our formal proof.

Proof. Given $\epsilon > 0$. Take $\delta = 4\epsilon$. Then $\delta > 0$. Now,

$$\forall p, q \in [2, \infty) \text{ with } |p - q| < \delta$$

we have

$$|f(p) - f(q)| = \left(\frac{1}{p^2q} + \frac{1}{pq^2}\right)|p - q| \leq \frac{1}{4}|p - q| < \frac{\delta}{4} = \epsilon$$

Therefore, f is uniform continuous. □

□

(ii) $f(x) = \dfrac{x-1}{x+1}$ on $[0, \infty)$

Solution. To prove uniform continuity of f, let's do some analysis on $|f(p) - f(q)|$ and relate it with $M|p-q|$ where $M > 0$. By doing so, we can find the proper delta to use in our formal proof.

Analysis:

$$\begin{aligned} |f(p) - f(q)| &= \left|\frac{p-1}{p+1} - \frac{q-1}{q+1}\right| \\ &= \frac{2|p-q|}{(p+1)(q+1)} \leq 2|p-q| < 2\delta = \epsilon \implies \delta = \frac{\epsilon}{2} \end{aligned}$$

Now we have found the correct delta to use in our formal proof.

Proof. Given $\epsilon > 0$. Take $\delta = \frac{\epsilon}{2}$. Then $\delta > 0$. Now,

$$\forall p, q \in [0, \infty) \text{ with } |p - q| < \delta$$

we have

$$\frac{2|p-q|}{(p+1)(q+1)} \leq 2|p-q| < 2\delta = \epsilon$$

Therefore, f is uniform continuous. □

□

(iii) $f(x) = \dfrac{x}{1+x^2}$ on $(-\infty, \infty)$

Solution. To prove uniform continuity of f, let's do some analysis on $|f(p) - f(q)|$ and relate it with $M|p-q|$ where $M > 0$. By doing so, we can find the proper delta to use in our formal proof.

Analysis:

$$\begin{aligned} |f(p) - f(q)| &= \left|\frac{p}{1+p^2} - \frac{q}{1+q^2}\right| = \frac{|p + pq^2 - q - p^2q|}{(1+p^2)(1+q^2)} = \frac{|(p-q) + pq(q-p)|}{(1+p^2)(1+q^2)} \\ &= \frac{|1-pq||p-q|}{1+p^2+q^2+p^2q^2} \leq \frac{1+|p||q|}{1+p^2+q^2+p^2q^2}|p-q| \\ &\leq \frac{1+\frac{1}{2}(p^2+q^2)}{1+p^2+q^2}|p-q| \leq |p-q| < \delta = \epsilon \implies \delta = \epsilon \end{aligned}$$

Now we have found the correct delta to use in our formal proof.

Proof. Given $\epsilon > 0$. Take $\delta = \epsilon$. Then $\delta > 0$. Now,

$$\forall p, q \in (-\infty, \infty) \text{ with } |p - q| < \delta$$

we have

$$|f(p) - f(q)| \leq \frac{1+|p||q|}{1+p^2+q^2+p^2q^2}|p-q| \leq |p-q| < \delta = \epsilon$$

Therefore, f is uniform continuous. □

□

3.7 Chapter 2: the proof of Theorem 9

Theorem: Let K be a compact set in metric space X and let F be a closed subset of K. Then F is compact.

Proof. Let $\{G_\alpha\}$ be an open cover of F. Then $\{X - F\} \cup \{G_\alpha\}$ is an open cover of K. Since K is compact, $\exists \alpha_1, \dots, \alpha_n$ such that

$$K \subseteq (X - F) \cup \Big(G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}\Big).$$

Now $F = F \cap K \subseteq \Big(F \cap (X - F)\Big) \cup \Big(G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}\Big) = G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}$. Hence, $\{G_\alpha\}$ has a finite subcover $\{G_{\alpha_1}, \dots, G_{\alpha_n}\}$ of F. Therefore, F is compact. □

3.8 Chapter 3: the proof of Theorem 11

Theorem: Let (X, d) be a metric space and let $\{x_n\}$ be a sequence in X.

(i) If $\{x_n\}$ is convergent, then $\{x_n\}$ is Cauchy.

(ii) If $\{x_n\}$ is Cauchy, then $\{x_n\}$ is bounded.

Proof. (i) Suppose $x_n \to x$. Then $\forall \epsilon > 0, \exists N, \forall n \geq N, d(x_n, x) < \frac{\epsilon}{2}$, and thus $\forall m, n \geq N$, we have

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) = d(x_m, x) + d(x_n, x) < \epsilon.$$

Therefore, $\{x_n\}$ is Cauchy.

(ii) Suppose $\{x_n\}$ is Cauchy. For $\epsilon = 1, \exists N,\ \forall m, n \geq N, d(x_m, x_n) < \epsilon$. Let $x = x_N$ and $r = 1 + \max\limits_{1 \leq i \leq N} d(x_i, x)$. Then $d(x_n, x) < r$ for all n; that is, $\{x_n\}$ is in the ball $B(x, r)$. Therefore, $\{x_n\}$ is bounded.

□

3.9 Chapter 4: the proof of Theorem 8

Theorem: Let X and Y be metric spaces with X compact and let $f : X \to Y$ be continuous. Then $f(X)$ is a compact set in Y. Therefore, $f(X)$ is bounded and closed in Y.

Proof. Let $\mathcal{U}$ be an open cover of $f(X)$ in Y; that is, $\mathcal{U}$ is a family of open sets on Y such that $f(X) \subseteq \bigcup\limits_{U \in \mathcal{U}} U$. Then

$$X \subseteq f^{-1}\Big(\bigcup_{U \in \mathcal{U}} U\Big) = \bigcup_{U \in \mathcal{U}} f^{-1}(U)$$

with each $f^{-1}(U)$ open in X (since f is continuous). Thus $\left\{f^{-1}(U) : U \in \mathcal{U}\right\}$ is an open cover of X. By the compactness of $X, \exists U_1, \ldots, U_n \in \mathcal{U}$ such that $X = \bigcup_{k=1}^{n} f^{-1}(U_k)$. Hence,

$$f(X) = f\Big(\bigcup_{k=1}^{n} f^{-1}(U_k)\Big) = \bigcup_{k=1}^{n} f(f^{-1}(U_k)) \subseteq \bigcup_{k=1}^{n} U_k.$$

So, $\mathcal{U}$ has a finite subcover of $f(X)$. Therefore, $f(X)$ is compact. $\square$