MATH-3580 Final Exam Review

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The Final Exam contains eight questions covering topics in Chapter 3 (Parts 1 - 3, 5, and 6) and Chapter 4 (Parts 1 - 3, up to Example 5) in the lecture outlines.

Contents

1 Definitions

1.1 Metric Space

In mathematics, space = set + structure(s).

Definition 1.1. Let X be a set. A function $d : X \times X \to [0, \infty)$ is called a **metric** (or distance) on X if

1. $\forall x, y \in X, d(x, y) = 0 \iff x = y$;
2. $\forall x, y \in X, d(x, y) = d(y, x)$;
3. $\forall x, y, z \in X, d(x, z) \leq d(x, y) + d(y, z)$. (Triangle inequality)

In this case, (X, d) is called a **metric space**.

1.2 Open Ball and Bounded Set

Definition 1.2. Let (X, d) be a metric space, $x \in X$, and $r > 0$.

1. Define $B(x, r) = \{y \in X : d(x, y) < r\}$, called the **open ball** centered at x with radius r, or the **r-neighborhood of x**.
2. A general **neighborhood of x** is a subset U of X such that $B(x, r) \subseteq U$ for some $r > 0$.
3. A subset $E \subseteq X$ of a metric space (X, d) is **bounded** if there exists some $x_0 \in X$ and $M > 0$ such that $d(x, x_0) \leq M$ for all $x \in E$.

1.3 Interior Points and Interior

Definition 1.3. Let $E \subseteq X$. The **closure** of E is the set $\overline{E} = E \cup E'$.

1. An element x of X is called an **interior point** of E if $\exists r > 0, B(x, r) \subseteq E$.
2. The **interior** of E is the set E° of all interior points of E.

By definition, we have $E^\circ \subseteq E \subseteq \overline{E}$.

1.4 Open Set

Definition 1.4. A subset $G \subseteq X$ is called **open** if $\forall x \in G, \exists r > 0, B(x, r) \subseteq G$. Note, $\emptyset$ and X are open in X. That is, a set is open if every point in it has an open ball around it that's still entirely inside the set.

1.5 Limit Points and Derived Set

Definition 1.5. Let $E \subseteq X$.

1. x is called a **limit point** of E (or cluster point, or accumulation point) if

$$\forall r > 0, B(x, r) \cap E \text{ contains some } y \neq x$$

   Equivalently,

$$(B(x, r) - \{x\}) \cap E \neq \emptyset$$

2. We let E' = the set of all limit points of E, called the **derived set** of E.

1.6 Closed Set

Definition 1.6. A subset $E \subseteq X$ is called **closed** if $E' \subseteq E$, where E' is the derived set of E (the set of all limit points of E). Note, $\emptyset$ and X are closed in X. That is, a subset $E \subseteq X$ is closed if every limit point of E belongs to E.

1. E is open $\iff$ $X - E$ is closed;
2. E is closed $\iff$ $X - E$ is open.

1.7 Closure

Definition 1.7. Let $E \subseteq X$. The **closure** of E is the set $\overline{E} = E \cup E'$.

1.8 Boundary Points and Boundary

Definition 1.8. Let X be a metric space, $E \subseteq X$ and $x \in X$.

1. x is called a **boundary point** of E if $\forall r > 0, B(x,r) \cap E \neq \emptyset$ and $B(x,r) \cap (X - E) \neq \emptyset$.
2. We use ∂E to denote the set of all boundary points of E, called the **boundary** of E.

1.9 Open Cover

Definition 1.9. Let X be a metric space and let $K \subseteq X$. A family $\{G_\alpha\}$ of open sets in X is called an **open cover** of K if $K \subseteq \bigcup_\alpha G_\alpha$. That is, every point of K lies in at least one of the open sets G_α.

1.10 Compact Set

Definition 1.10. The set K is called **compact** if every open cover of K has a finite **subcover** of K. That is, if $K \subseteq \bigcup_\alpha G_\alpha$, then $\exists \alpha_1, \ldots, \alpha_n$ such that $K \subseteq \bigcup_{i=1}^n G_\alpha$.

1.11 Convergent Sequence

Definition 1.11. Let (X, d) be a metric space and let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is **convergent** if

$$\exists x \in X \text{ such that } \underline{\forall \epsilon > 0, \exists N = N(\epsilon) \in \mathbb{N}, \forall n \geq N, d(x_n, x) < \epsilon}.$$

In this case, we say that $\{x_n\}$ converges to x, and write $x_n \to x$.

1.12 Divergent Sequence

Definition 1.12. We say that $\{x_n\}$ is **divergent** if $\{x_n\}$ is not convergent. That is, $\forall x \in X, \{x_n\}$ does not converge to x. The ϵ-N description of divergence of $\{x_n\}$ is

$$\underline{\forall x \in X, \exists \epsilon_0 > 0, \forall N, \exists n \geq N, d(x_n, x) \geq \epsilon_0}.$$

1.13 Bounded Sequence

Definition 1.13. A sequence $\{x_n\}$ in a metric space (X, d) is **bounded** if there exists a point $x_0 \in X$ and a number $M > 0$ such that

$$d(x_n, x_0) \leq M \text{ for all } n \in \mathbb{N}$$

In other words, all terms of the sequence lie within some fixed distance M of a single point x_0. In simplest terms, a sequence is bounded if all its terms lie inside some ball of finite radius.

1.14 The ϵ-N description of $x_n \to x$

Definition 1.14. The ϵ-N description of $x_n \to x$ is

$$\underline{\forall \epsilon > 0, \exists N = N(\epsilon) \in \mathbb{N}, \forall n \geq N, d(x_n, x) < \epsilon}.$$

1.15 The ϵ-N description of $x_n \not\to x$

Definition 1.15. The ϵ-N description of $x_n \not\to x$ is

$$\underline{\exists \epsilon_0 > 0, \forall N, \exists n \in N, d(x_n, x) \geq \epsilon_0}.$$

1.16 The ϵ-N definition of a Cauchy sequence and its negation

Definition 1.16. Let $\{x_n\}$ be a sequence in a metric space (X, d). $\{x_n\}$ is called Cauchy if

$$\underline{\forall \epsilon > 0, \exists N, \forall m, n \geq N, d(x_m, x_n) < \epsilon}$$

So we can say it's negation: $\{x_n\}$ is not Cauchy if and only if

$$\underline{\exists \epsilon_0 > 0, \forall N, \exists m, n \geq N, d(x_m, x_n) \geq \epsilon_0}$$

1.17 Subsequence

Definition 1.17. Let $\{x_n\}$ be a sequence. If

$$\{n_k\}_{k\in\mathbb{N}} \text{ is a sequence in } \mathbb{N} \text{ such that } n_1 < n_2 < \dots$$

then $\{x_{n_k}\}_{k=1}^{\infty}$ is called a subsequence of $\{x_n\}$.

1.18 Convergence, Divergence, Absolute Convergence of a Series

Definition 1.18. Let $\{a_n\}$ be a sequence in $\mathbb{R}$. For each $n \in \mathbb{N}$, let

$$s_n = a_1 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

called the n^{th} partial sum of the series $\sum_{k=1}^{\infty} a_k$.

Convergence: If $s_n \to s \in \mathbb{R}$, then we say that the series $\sum_{k=1}^{\infty} a_k$ is convergent, and we write

$$\sum_{k=1}^{\infty} a_k = s \text{ (called the sum of } \sum_{k=1}^{\infty} a_k)$$

Divergence: If the sequence of partial sums $\{s_n\}$ is divergent (does not approach a finite limit), then we say that the series $\sum_{k=1}^{\infty} a_k$ is divergent.

Absolute Convergence: A series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ is convergent.

1.19 Geometric series, P-series, Alternating series, and their Convergence

Definition 1.19. We observe the following,

Geometric series: The geometric series $\sum_{n=0}^{\infty} x_n$ is convergent $\iff |x| < 1$. When $|x| < 1$, we have $\sum_{n=0}^{\infty} x_n = \dfrac{1}{1-x}$.

P-series: Let $p \in \mathbb{R}$. For a series of the form $\sum_{n=1}^{\infty} \dfrac{1}{n^p}$ called the p-series, we have

(i) If $p \leq 1$, then $\sum_{n=1}^{\infty} \dfrac{1}{n^p}$ is divergent

(ii) If $p > 1$, then $\sum_{n=1}^{\infty} \dfrac{1}{n^p}$ is convergent.

Alternating series: Suppose $\{b_n\}$ is a sequence in $\mathbb{R}$ such that $b_1 \geq b_2 \geq \cdots \geq 0$ and $b_n \to 0$. Then the alternating series of the form $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ and $\sum_{n=1}^{\infty} (-1)^n b_n$ are convergent.

1.20 The ϵ-δ definition of $\lim_{x \to c} f(x)$ and its negation

Definition 1.20. Let (X, d_X) and (Y, d_Y) be metric spaces, $E \subseteq X$, $c \in E'$, $f : E \to Y$, and $q \in Y$. We write the ϵ-δ definition of $\lim_{x \to c} f(x) = q$ as

$$\underline{\forall \epsilon > 0, \exists \delta > 0, \forall x \in E \text{ with } 0 < d_X(x, c) < \delta, d_Y(f(x), q) < \epsilon}$$

That is,

$$x \in \Big(B_X(c, \delta) - \{c\}\Big) \cap E \implies f(x) \in B_Y(q, \epsilon), \text{ or equivalently,}$$

$$f\Big((B_X(c, \delta) - \{c\}) \cap E\Big) \subseteq B_Y(q, \epsilon)$$

Negation: The ϵ-δ description of the negation of $\lim_{x \to c} f(x) = q$ is

$$\underline{\exists \epsilon_0 > 0, \forall \delta > 0, \exists x \in E \text{ with } 0 < d_X(x, c) < \delta,\ d_Y(f(x), q) \geq \epsilon_0}$$

1.21 The ϵ-δ definition of continuity of f at c and its negation

Definition 1.21. Let (X, d_X) and (Y, d_Y) be metric spaces, $c \in X$, and $f : X \to Y$. We say that f is continuous at c and write the ϵ-δ definition if

$$\underline{\forall \epsilon > 0, \exists \delta = \delta(\epsilon, c) > 0, \forall x \text{ with } d_X(x, c) < \delta,\ d_Y(f(x), f(c)) < \epsilon}$$

That is, $f\Big(B_X(c,\delta)\Big) \subseteq B_Y\Big(f(c),\epsilon\Big)$. If f is continuous at every point in X, then we say that f is continuous on X.
Negation: Therefore, we can write the negation and say that f is not continuous at c if and only if

$$\underline{\exists\epsilon_0 > 0, \forall\delta > 0, \exists x \text{ with } d_X(x,c) < \delta,\ d_Y(f(x), f(c)) \geq \epsilon_0}$$

1.22 The ϵ-δ definition of uniform continuity of f

Definition 1.22. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$. We say that f is uniformly continuous on X if

$$\underline{\forall\epsilon > 0, \exists\delta = \delta(\epsilon) > 0, \forall p, q \in X \text{ with } d_X(p,q) < \delta,\ d_Y(f(p), f(q)) < \epsilon}$$

Negation: We say that the negation, $f : X \to Y$ is not uniformly continuous on X if and only if

$$\underline{\exists\epsilon_0 > 0, \forall\delta > 0, \exists p, q \in X \text{ with } d_X(p,q) < \delta,\ d_Y(f(p), f(q)) \geq \epsilon_0}$$

2 Results

2.1 Chapter 3: Theorem 11

Let (X, d) be a metric space and let $\{x_n\}$ be a sequence in X.

(i) If $\{x_n\}$ is convergent, then $\{x_n\}$ is Cauchy.

(ii) If $\{x_n\}$ is Cauchy, then $\{x_n\}$ is bounded.

2.2 Chapter 3: Theorem 14

(Cauchy Criterion for Series). Let $\displaystyle\sum_{n=1}^{\infty} a_n$ be a series in $\mathbb{R}$. Then

$$\sum_{n=1}^{\infty} a_n \text{ is convergent} \iff \forall\epsilon > 0, \exists N, \forall m \geq n \geq N, \left|\sum_{n=1}^{\infty} a_k\right| < \epsilon$$

2.3 Chapter 3: Corollary 7

Let $\displaystyle\sum_{n=1}^{\infty} a_n$ be a series in $\mathbb{R}$. Then

$$\sum_{n=1}^{\infty} a_n \text{ is divergent} \iff \exists\epsilon_0 > 0, \forall N, \exists m \geq n \geq N, \left|\sum_{k=n}^{m} a_k\right| \geq \epsilon_0$$

Note that if $\sum_{n=1}^{\infty} a_n$ is convergent, then $a_n \to 0$ since

$$a_n = s_n - s_{n-1} \to s - s = 0$$

Therefore, we have the following test for divergence.

2.4 Chapter 3: Theorem 15

(Divergence Test). If $a_n \not\to 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

2.5 Chapter 3: Theorem 16

If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

2.6 Chapter 3: Theorem 23

(The Root Test). Given $\sum_{n=1}^{\infty} a_n$ in $\mathbb{R}$, let $\alpha = \limsup_n |a_n|^{\frac{1}{n}}$. Then $\alpha \in [0, \infty]$.

(i) If $\alpha < 1$, then $\sum_{n=1}^{\infty} |a_n|$ is convergent (and hence $\sum_{n=1}^{\infty} a_n$ is convergent).

(ii) If $\alpha > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\alpha = 1$, then there is no conclusion.

2.7 Chapter 3: Theorem 24

(The Ratio Test). Given $\sum_{n=1}^{\infty} a_n$ in $\mathbb{R}$.

(i) If $\limsup_n \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum_{n=1}^{\infty} |a_n|$ is convergent (and hence $\sum_{n=1}^{\infty} a_n$ is convergent).

(ii) If $\exists n_0, \forall n \geq n_0, \left|\frac{a_{n+1}}{a_n}\right| \geq 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

2.8 Chapter 3: Corollary 6

(Cauchy Criterion). Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Then

$$\{x_n\} \text{ is convergent} \iff \{x_n\} \text{ is Cauchy.}$$

2.9 Chapter 3: Corollary 8

(Comparison Theorem). Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be nonnegative series. Suppose that $\exists n_0 \in \mathbb{N}$ such that $0 \leq a_n \leq b_n$ for all $n \geq n_0$.

(i) If $\sum_{n=1}^{\infty} a_n$ is divergent, then $\sum_{n=1}^{\infty} b_n$ is divergent.

(ii) If $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

2.10 Chapter 4: Theorem 2

(Sequential Criterion for Convergence). Let (X, d_X) and (Y, d_Y) be metric spaces, $E \subseteq X$, $c \in E'$, $f : E \to Y$, and $q \in Y$. Then

$$\lim_{x \to c} f(x) = q \iff \forall \text{ sequence } \{c_n\} \text{ in } E - \{c\},\ c_n \to c \implies f(c_n) \to q.$$

2.11 Chapter 4: Theorem 8

Let X and Y be metric spaces with X compact and let $f : X \to Y$ be continuous. Then $f(X)$ is a compact set in Y. Therefore, $f(X)$ is bounded and closed in Y.

2.12 Chapter 4: Corollary 1

We have

$$f(x) \not\to q \text{ as } x \to c \iff \exists \text{ sequence } \{c_n\} \text{ in } E - \{c\} \text{ such that } c_n \to c \text{ but } f(c_n) \not\to q.$$

2.13 Chapter 4: Corollary 2

If there exist sequences $\{x_n\}$ and $\{y_n\}$ in $E - \{c\}$ such that

$$x_n \to c,\ y_n \to c,\ f(x_n) \to p,\ f(y_n) \to q \text{ and } p \neq q,\ \text{then } \lim_{x \to c} f(x) \text{ DNE}.$$

2.14 Chapter 4: Corollary 3

(Sequential Criterion for Continuity). Let $c \in X$. Then

$$f : X \to Y \text{ is continuous at } c \iff \Big[c_n \to c \text{ in } X \implies f(c_n) \to f(c) \text{ in } Y\Big].$$

2.15 Chapter 4: Corollary 7

If $f : X \to Y$ is continuous, then $\forall$ compact $E \subseteq X$, $f(E)$ is compact.

3 Questions/Proofs

3.1 Assignment 6: Question #1

3.2 Assignment 6: Question #2

3.3 Assignment 7: Question #2

3.4 Assignment 7: Question #5

3.5 Assignment 7: Question #6

3.6 Assignment 7: Question #7

3.7 Chapter 2: the proof of Theorem 9

Theorem: Let K be a compact set in metric space X and let F be a closed subset of K. Then F is compact.

Proof. Let $\{G_\alpha\}$ be an open cover of F. Then $\{X - F\} \cup \{G_\alpha\}$ is an open cover of K. Since K is compact, $\exists \alpha_1, \dots, \alpha_n$ such that

$$K \subseteq (X - F) \cup \Big(G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}\Big).$$

Now $F = F \cap K \subseteq \Big(F \cap (X - F)\Big) \cup \Big(G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}\Big) = G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}$. Hence, $\{G_\alpha\}$ has a finite subcover $\{G_{\alpha_1}, \dots, G_{\alpha_n}\}$ of F. Therefore, F is compact. $\square$

3.8 Chapter 3: the proof of Theorem 11

Theorem: Let (X, d) be a metric space and let $\{x_n\}$ be a sequence in X.

(i) If $\{x_n\}$ is convergent, then $\{x_n\}$ is Cauchy.

(ii) If $\{x_n\}$ is Cauchy, then $\{x_n\}$ is bounded.

Proof. (i) Suppose $x_n \to x$. Then $\forall \epsilon > 0, \exists N, \forall n \geq N, d(x_n, x) < \frac{\epsilon}{2}$, and thus $\forall m, n \geq N$, we have

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) = d(x_m, x) + d(x_n, x) < \epsilon.$$

Therefore, $\{x_n\}$ is Cauchy.

(ii) Suppose $\{x_n\}$ is Cauchy. For $\epsilon = 1, \exists N,\ \forall m, n \geq N, d(x_m, x_n) < \epsilon$. Let $x = x_N$ and $r = 1 + \max\limits_{1 \leq i \leq N} d(x_i, x)$. Then $d(x_n, x) < r$ for all n; that is, $\{x_n\}$ is in the ball $B(x, r)$. Therefore, $\{x_n\}$ is bounded.

$\square$

3.9 Chapter 4: the proof of Theorem 8

Theorem: Let X and Y be metric spaces with X compact and let $f : X \to Y$ be continuous. Then $f(X)$ is a compact set in Y. Therefore, $f(X)$ is bounded and closed in Y.

Proof. Let $\mathcal{U}$ be an open cover of $f(X)$ in Y; that is, $\mathcal{U}$ is a family of open sets on Y such that $f(X) \subseteq \bigcup_{U \in \mathcal{U}} U$. Then

$$X \subseteq f^{-1}\Big(\bigcup_{U \in \mathcal{U}} U\Big) = \bigcup_{U \in \mathcal{U}} f^{-1}(U)$$

with each $f^{-1}(U)$ open in X (since f is continuous). Thus $\Big\{f^{-1}(U) : U \in \mathcal{U}\Big\}$ is an open cover of X. By the compactness of $X, \exists U_1, \ldots, U_n \in \mathcal{U}$ such that $X = \bigcup_{k=1}^{n} f^{-1}(U_k)$. Hence,

$$f(X) = f\Big(\bigcup_{k=1}^{n} f^{-1}(U_k)\Big) = \bigcup_{k=1}^{n} f(f^{-1}(U_k)) \subseteq \bigcup_{k=1}^{n} U_k.$$

So, $\mathcal{U}$ has a finite subcover of $f(X)$. Therefore, $f(X)$ is compact. □