

MATH-3580 Test Two Review

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Test Two covers Chapter 2 - Parts 2 and 3 and Chapter 3 - Part 1 (up to Theorem 5)

Contents

1	Definitions	2
1.1	Metric Space	2
1.2	Open Ball and Bounded Set	2
1.3	Interior Points and Interior	2
1.4	Open Set	3
1.5	Limit Points and Derived Set	3
1.6	Closed Set	3
1.7	Closure	3
1.8	Boundary Points and Boundary	3
1.9	Open Cover	3
1.10	Compact Set	4
1.11	Convergent Sequence	4
1.12	Divergent Sequence	4
1.13	Bounded Sequence	4
1.14	The ϵ - N description of $x_n \rightarrow x$	4
1.15	The ϵ - N description of $x_n \not\rightarrow x$	4
2	Results	5
2.1	Chapter 2: Theorem 6	5
2.2	Characterization (I) of $x \in \overline{E}$	5
2.3	Chapter 2: Theorem 8	5
2.4	Corollary 8	5
2.5	Chapter 3: Theorem 1	5
2.6	Chapter 3: Theorem 2	5
2.7	Chapter 3: Theorem 3	5
3	Questions/Proofs	5
3.1	Assignment 3: Question #6	5
3.2	Assignment 4: Question #2	7
3.3	Assignment 4: Question #3	8

3.4	Assignment 4: Question #5	8
3.5	Assignment 4: Question #8	9
3.6	Assignment 5: Question #1(i)(ii)	9
3.7	Chapter 2: the proof of Theorem 9	9
3.8	Chapter 3: the proof of Theorem 2	9
3.9	Chapter 3: the proof of Theorem 3	10

1 Definitions

1.1 Metric Space

In mathematics, space = set + structure(s).

Definition 1.1. Let X be a set. A function $d : X \times X \rightarrow [0, \infty)$ is called a **metric** (or distance) on X if

1. $\forall x, y \in X, d(x, y) = 0 \iff x = y;$
2. $\forall x, y \in X, d(x, y) = d(y, x);$
3. $\forall x, y, z \in X, d(x, z) \leq d(x, y) + d(y, z).$ (Triangle inequality)

In this case, (X, d) is called a **metric space**.

1.2 Open Ball and Bounded Set

Definition 1.2. Let (X, d) be a metric space, $x \in X$, and $r > 0$.

1. Define $B(x, r) = \{y \in X : d(x, y) < r\}$, called the **open ball** centered at x with radius r , or the **r-neighborhood of x**.
2. A general **neighborhood of x** is a subset U of X such that $B(x, r) \subseteq U$ for some $r > 0$.
3. A subset $E \subseteq X$ of a metric space (X, d) is **bounded** if there exists some $x_0 \in X$ and $M > 0$ such that $d(x, x_0) \leq M$ for all $x \in E$.

1.3 Interior Points and Interior

Definition 1.3. Let $E \subseteq X$. The **closure** of E is the set $\overline{E} = E \cup E'$.

1. An element x of X is called an **interior point** of E if $\exists r > 0, B(x, r) \subseteq E$.
2. The **interior** of E is the set E° of all interior points of E .

By definition, we have $E^\circ \subseteq E \subseteq \overline{E}$.

1.4 Open Set

Definition 1.4. A subset $G \subseteq X$ is called **open** if $\forall x \in G, \exists r > 0, B(x, r) \subseteq G$. Note, \emptyset and X are open in X . That is, A set is open if every point in it has an open ball around it that's still entirely inside the set.

1.5 Limit Points and Derived Set

Definition 1.5. Let $E \subseteq X$.

1. x is called a **limit point** of E (or cluster point, or accumulation point) if

$$\forall r > 0, B(x, r) \cap E \text{ contains some } y \neq x$$

Equivalently,

$$(B(x, r) - \{x\}) \cap E \neq \emptyset$$

2. We let $E' =$ the set of all limit points of E , called the **derived set** of E .

1.6 Closed Set

Definition 1.6. A subset $E \subseteq X$ is called **closed** if $E' \subseteq E$, where E' is the derived set of E (the set of all limit points of E). Note, \emptyset and X are closed in X . That is, a subset $E \subseteq X$ is closed if every limit point of E belongs to E .

1. E is open $\iff X - E$ is closed;
2. E is closed $\iff X - E$ is open.

1.7 Closure

Definition 1.7. Let $E \subseteq X$. The **closure** of E is the set $\overline{E} = E \cup E'$.

1.8 Boundary Points and Boundary

Definition 1.8. Let X be a metric space, $E \subseteq X$ and $x \in X$.

1. x is called a **boundary point** of E if $\forall r > 0, B(x, r) \cap E \neq \emptyset$ and $B(x, r) \cap (X - E) \neq \emptyset$.
2. We use ∂E to denote the set of all boundary points of E , called the **boundary** of E .

1.9 Open Cover

Definition 1.9. Let X be a metric space and let $K \subseteq X$. A family $\{G_\alpha\}$ of open sets in X is called an **open cover** of K if $K \subseteq \bigcup_\alpha G_\alpha$. That is, every point of K lies in at least one of the open sets G_α .

1.10 Compact Set

Definition 1.10. The set K is called **compact** if every open cover of K has a finite **sub-cover** of K . That is, if $K \subseteq \bigcup_{\alpha} G_{\alpha}$, then $\exists \alpha_1, \dots, \alpha_n$ such that $K \subseteq \bigcup_{i=1}^n G_{\alpha_i}$.

1.11 Convergent Sequence

Definition 1.11. Let (X, d) be a metric space and let $\{x_n\}$ be a sequence in X . We say that $\{x_n\}$ is **convergent** if

$$\exists x \in X \text{ such that } \underline{\forall \epsilon > 0, \exists N = N(\epsilon) \in \mathbb{N}, \forall n \geq N, d(x_n, x) < \epsilon}.$$

In this case, we say that $\{x_n\}$ converges to x , and write $x_n \rightarrow x$.

1.12 Divergent Sequence

Definition 1.12. We say that $\{x_n\}$ is **divergent** if $\{x_n\}$ is not convergent. That is, $\forall x \in X$, $\{x_n\}$ does not converge to x . The ϵ - N description of divergence of $\{x_n\}$ is

$$\underline{\forall x \in X, \exists \epsilon_0 > 0, \forall N, \exists n \geq N, d(x_n, x) \geq \epsilon_0}.$$

1.13 Bounded Sequence

Definition 1.13. A sequence $\{x_n\}$ in a metric space (X, d) is **bounded** if there exists a point $x_0 \in X$ and a number $M > 0$ such that

$$d(x_n, x_0) \leq M \text{ for all } n \in \mathbb{N}$$

In other words, all terms of the sequence lie within some fixed distance M of a single point x_0 . In simplest terms, a sequence is bounded if all its terms lie inside some ball of finite radius.

1.14 The ϵ - N description of $x_n \rightarrow x$

Definition 1.14. The ϵ - N description of $x_n \rightarrow x$ is

$$\underline{\forall \epsilon > 0, \exists N = N(\epsilon) \in \mathbb{N}, \forall n \geq N, d(x_n, x) < \epsilon}.$$

1.15 The ϵ - N description of $x_n \not\rightarrow x$

Definition 1.15. The ϵ - N description of $x_n \not\rightarrow x$ is

$$\underline{\exists \epsilon_0 > 0, \forall N, \exists n \in \mathbb{N}, d(x_n, x) \geq \epsilon_0}.$$

2 Results

2.1 Chapter 2: Theorem 6

Theorem. Let (X, d) be a metric space.

1. If $\{G_\alpha\}$ is a family of open sets in X , then $\bigcup_\alpha G_\alpha$ is open in X .
2. If G_1, \dots, G_n are open sets in X , then $\bigcap_{i=1}^n G_i$ is open in X .

2.2 Characterization (I) of $x \in \overline{E}$

Comparing with $x \in \overline{E'} \iff \forall r > 0, (B(x, r) - \{x\}) \cap E' \neq \emptyset$ (definition), we have

$$\underline{x \in E' \iff \forall r > 0, B(x, r) \cap E \neq \emptyset. \text{ Therefore, } E_1 \subseteq E_2 \implies \overline{E_1} \subseteq \overline{E_2}.}$$

2.3 Chapter 2: Theorem 8

Theorem. Every compact set in a metric space is bounded and closed. Therefore, if E is either unbounded or non-closed, then E is not compact.

2.4 Corollary 8

Corollary. Let $E \subseteq \mathbb{R}$. Then E is compact $\iff E$ is bounded and closed.

2.5 Chapter 3: Theorem 1

Theorem. (Uniqueness of Limit). Let (X, d) be a metric space and let $\{x_n\}$ be a sequence in X . If $x_n \rightarrow x$ and $x_n \rightarrow y$, then $x = y$.

2.6 Chapter 3: Theorem 2

Theorem. Any convergent sequence in a metric space is bounded.

2.7 Chapter 3: Theorem 3

Theorem. (Squeeze Theorem in \mathbb{R}). Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be sequences in \mathbb{R} such that $a_n \rightarrow x, c_n \rightarrow x$, and $\forall n, a_n \leq b_n \leq c_n$. Then $b_n \rightarrow x$.

3 Questions/Proofs

3.1 Assignment 3: Question #6

Question: Find $E^\circ, E', \overline{E}$, and ∂E for the following subsets $E \subseteq \mathbb{R}$:

1. $E = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

2. $E = (0, \pi) \cap \mathbb{Q}$
3. $E = [0, 2) \cup (4, 7) \cup \{8\}$
4. $E = \mathbb{Z}$

Recall:

- $E^\circ =$ set of all interior points of E
- $E' =$ set of all limit points of E
- $\overline{E} =$ the closure set of $E = E \cup E'$
- $\partial E =$ the set of all boundary points of $E = \overline{E} \cap \overline{(X - E)}$

Solution.

1. $E = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\} = \{\frac{1}{n}\}_{n=1}^\infty$
So we get,

$$\begin{cases} E^\circ = \emptyset \\ E' = \{0\} \\ \overline{E} = E \cup E' = \{0\} \cup \{\frac{1}{n}\}_{n=1}^\infty \\ \partial E = \overline{E} \cap \overline{(\mathbb{R} - E)} = \overline{E} = \{0\} \cup \{\frac{1}{n}\}_{n=1}^\infty \end{cases}$$

2. $E = (0, \pi) \cap \mathbb{Q}$
So we get,

$$\begin{cases} E^\circ = \emptyset \\ E' = [0, \pi] \\ \overline{E} = E \cup E' = [0, \pi] \\ \partial E = \overline{E} \cap \overline{(\mathbb{R} - E)} = [0, \pi] \end{cases}$$

3. $E = [0, 2) \cup (4, 7) \cup \{8\}$
So we get,

$$\begin{cases} E^\circ = (0, 2) \cup (4, 7) \\ E' = [0, 2] \cup [4, 7] \\ \overline{E} = E \cup E' = [0, 2] \cup [4, 7] \cup \{8\} \\ \partial E = \overline{E} \cap \overline{(\mathbb{R} - E)} = \{0, 2, 4, 7, 8\} \end{cases}$$

Further analysis on $\mathbb{R} - E$:

$$\begin{aligned} \mathbb{R} - E &= (-\infty, 0) \cup [2, 4] \cup [7, 8) \cup (8, \infty) \\ \overline{\mathbb{R} - E} &= (-\infty, 0] \cup [2, 4] \cup [7, \infty) \end{aligned}$$

4. $E = \mathbb{Z}$
So we get,

$$\begin{cases} E^\circ = \emptyset \\ E' = \emptyset \\ \overline{E} = E \cup E' = \mathbb{Z} \\ \partial E = \overline{E} \cap \overline{(\mathbb{R} - E)} = \overline{E} \cap \mathbb{R} = \mathbb{Z} \end{cases}$$

□

3.2 Assignment 4: Question #2

Question: Find E° , E' , \overline{E} , and ∂E for the following subsets $E \subseteq \mathbb{R}$:

1. $E = (-\infty, 0) \cup (1, 2) \cup \{5 + \frac{1}{n} : n = 1, 2, 3, \dots\}$
2. $E = \{\pi\} \cup \mathbb{N}$
3. $E = \{\pi\} \cup \mathbb{Q}$

Solution.

1. $E = (-\infty, 0) \cup (1, 2) \cup \{5 + \frac{1}{n} : n = 1, 2, 3, \dots\}$

So we get,

$$\begin{cases} E^\circ = (-\infty, 0) \cup (1, 2) \\ E' = (-\infty, 0] \cup [1, 2] \cup \{5\} \\ \overline{E} = E \cup E' = (-\infty, 0] \cup [1, 2] \cup \{5 + \frac{1}{n}\}_{n=1}^\infty \cup \{5\} \\ \partial E = \overline{E} \cap (\mathbb{R} - \overline{E}) = \{0, 1, 2, 5\} \cup \{5 + \frac{1}{n}\}_{n=1}^\infty \end{cases}$$

Further analysis on ∂E :

$$\begin{aligned} &= \overline{E} \cap ([0, 1] \cup ([2, \infty] - \{5 + \frac{1}{n}\}_{n=1}^\infty)) \\ &= \overline{E} \cap ([0, 1] \cup [2, \infty)) \\ &= \{0, 1, 2, 5\} \cup \{5 + \frac{1}{n}\}_{n=1}^\infty \end{aligned}$$

2. $E = \{\pi\} \cup \mathbb{N}$

So we get,

$$\begin{cases} E^\circ = \emptyset \\ E' = \emptyset \\ \overline{E} = E \cup E' = \{\pi\} \cup \mathbb{N} \text{ (so just } E) \\ \partial E = \overline{E} \cap (\mathbb{R} - \overline{E}) = E \cap \mathbb{R} = \{\pi\} \cup \mathbb{N} \text{ (so just } E) \end{cases}$$

3. $E = \{\pi\} \cup \mathbb{Q}$

So we get,

$$\begin{cases} E^\circ = \emptyset \\ E' = \mathbb{R} \text{ (since } \mathbb{Q} \text{ is dense in } \mathbb{R}) \\ \overline{E} = E \cup E' = \mathbb{R} \\ \partial E = \overline{E} \cap (\mathbb{R} - \overline{E}) = \mathbb{R} \end{cases}$$

Further analysis on $\mathbb{R} - E$:

$$\begin{aligned} &= \mathbb{R} - E = \{x \in \mathbb{R} \mid x \text{ is irrational and } x \neq \pi\} \\ &= \overline{\mathbb{R} - E} = \mathbb{R}, \text{ (since } (\mathbb{R} - \mathbb{Q}) - \{\pi\} \text{ is dense in } \mathbb{R}) \end{aligned}$$

□

3.3 Assignment 4: Question #3

Question: Give an open cover of the set $[0, 1] \cap \mathbb{Q}$ in \mathbb{R} that has no finite subcover.

Proof.

$$\implies \text{Note: } \frac{\pi}{4} \in [0, 1] \text{ and } \frac{\pi}{4} \notin \mathbb{Q}$$

$$\implies \text{For } n \in \mathbb{N}, \text{ let } G_n = (-1, \frac{\pi}{4} - \frac{1}{2^n}) \cup (\frac{\pi}{4}, 2)$$

$$\implies \text{Each } G_n \text{ is open, and } G_1 \subseteq G_2 \subseteq G_3 \subseteq \dots \text{ and } [0, 1] \cap \mathbb{Q} \subseteq \bigcup_{n=1}^{\infty} G_n \text{ is an open cover}$$

$$\implies \{G_n\}_{n=1}^{\infty} \text{ no finite subcover of } [0, 1] \cap \mathbb{Q}$$

$$\implies \forall n_0 \in \mathbb{N}, \bigcup_{n=1}^{n_0} G_n = G_{n_0} \text{ does not contain the points } (\frac{\pi}{4} - \frac{1}{2^{n_0}}, \frac{\pi}{4}) \cap \mathbb{Q}$$

Therefore, there does not exist a finite subcover. □

3.4 Assignment 4: Question #5

Question: Use the ϵ - N definition to prove the following:

1.

$$\lim_{n \rightarrow \infty} (1 + \frac{1}{n^2}) = 1$$

2.

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$$

3. The sequence $\{1 + (-1)^n\}$ is divergent in \mathbb{R} .

Solution.

1.

$$\lim_{n \rightarrow \infty} (1 + \frac{1}{n^2}) = 1$$

$$\implies \text{Given } \epsilon > 0, \text{ choose } N = \left\lceil \frac{1}{\sqrt{\epsilon}} \right\rceil + 1 \text{ (integer part)}$$

$$\implies \text{Then } N > \frac{1}{\sqrt{\epsilon}} \implies \frac{1}{N^2} < \epsilon$$

$$\implies \forall n \in \mathbb{N}, \text{ we have } \left| (1 + \frac{1}{n^2}) - 1 \right| = \frac{1}{n^2} \leq \frac{1}{N^2} < \epsilon$$

Therefore,

$$\lim_{n \rightarrow \infty} (1 + \frac{1}{n^2}) = 1$$

Further analysis on choosing N :

$$\left| \left(1 + \frac{1}{n^2}\right) - 1 \right| < \epsilon \implies \left| \frac{1}{n^2} \right| < \epsilon \implies \frac{1}{n^2} < \epsilon$$

when $n \geq N$, $\frac{1}{n^2} \leq \frac{1}{N^2} < \epsilon$

$$\frac{1}{\epsilon} < N^2 \implies \frac{1}{\sqrt{\epsilon}} < N$$

□

3.5 Assignment 4: Question #8

Question: Let $\{x_n\}$ and $\{y_n\}$ be sequences in \mathbb{R} such that $\{x_n\}$ is bounded and $y_n \rightarrow 0$. Prove that $x_n y_n \rightarrow 0$.

Solution.

□

3.6 Assignment 5: Question #1(i)(ii)

Question: Use the ϵ - N definition to prove the following:

1.

$$\lim_{n \rightarrow \infty} (\sqrt{n^2 + 2n} - n) = 1$$

2. The sequence $\{n^{(-1)^n - 1}\}$ is divergent in \mathbb{R} .

Solution.

□

3.7 Chapter 2: the proof of Theorem 9

Theorem. Let K be a compact set in metric space X and let F be a closed subset of K . Then F is compact.

Proof. Let $\{G_\alpha\}$ be an open cover of F . Then $\{X - F\} \cup \{G_\alpha\}$ is an open cover of K . Since K is compact, $\exists \alpha_1, \dots, \alpha_n$ such that

$$K \subseteq (X - F) \cup (G_{\alpha_1} \cup \dots \cup G_{\alpha_n}).$$

Now $F = F \cap K \subseteq (F \cap (X - F)) \cup (G_{\alpha_1} \cup \dots \cup G_{\alpha_n}) = G_{\alpha_1} \cup \dots \cup G_{\alpha_n}$. Hence, $\{G_\alpha\}$ has a finite subcover $\{G_{\alpha_1}, \dots, G_{\alpha_n}\}$ of F . Therefore, F is compact. □

3.8 Chapter 3: the proof of Theorem 2

Theorem. Any convergent sequence in a metric space is bounded.

Proof. Suppose $x_n \rightarrow x$ in a metric space (X, d) . We prove that

$$\exists M > 0, \forall n, x_n \in B(x, M)$$

For $\epsilon = 1$, $\exists N, \forall n \geq N, d(x_n, x) < 1$. Let $M = 1 + \max\{d(x_1, x), \dots, d(x_{N-1}, x)\}$. Then $\forall n$, we have $d(x_n, x) < M$. □

3.9 Chapter 3: the proof of Theorem 3

Theorem. (Squeeze Theorem in \mathbb{R}). Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences in \mathbb{R} such that $a_n \rightarrow x$, $c_n \rightarrow x$, and $\forall n, a_n \leq b_n \leq c_n$. Then, $b_n \rightarrow x$.

Note: Here either $x \in \mathbb{R}$ or $x = \pm\infty$. We only prove the case when $x \in \mathbb{R}$. Also, in the theorem, we can only require that $a_n \leq b_n \leq c_n$ holds for all $n \geq n_0$ for some n_0 .

Proof. Given $\epsilon > 0$. Since $a_n \rightarrow x$ and $c_n \rightarrow x$, $\exists N_1, N_2 \in \mathbb{N}$ such that

$$\forall n \geq N_1, |a_n - x| < \epsilon \text{ and } \forall n \geq N_2, |c_n - x| < \epsilon.$$

Let $N = \max\{N_1, N_2\}$. Then $\forall n \geq N$, we have $|a_n - x| < \epsilon$ and $|c_n - x| < \epsilon$. That is,

$$x - \epsilon < a_n < x + \epsilon \text{ and } x - \epsilon < c_n < x + \epsilon$$

In this case, $\forall n \geq N$, we have $x - \epsilon < a_n \leq b_n \leq c_n < x + \epsilon$. That is,

$$x - \epsilon < b_n < x + \epsilon, \text{ or } |b_n - x| < \epsilon$$

So, we prove that $\forall \epsilon > 0, \exists N, \forall n \geq N, |b_n - x| < \epsilon$. Therefore, $b_n \rightarrow x$. □