

# MATH-3580 Final Exam Review

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The Final Exam contains eight questions covering topics in Chapter 3 (Parts 1 - 3, 5, and 6) and Chapter 4 (Parts 1 - 3, up to Example 5) in the lecture outlines.

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# 1 Definitions

## 1.1 Metric Space

In mathematics, space = set + structure(s).

**Definition 1.1.** Let  $X$  be a set. A function  $d : X \times X \rightarrow [0, \infty)$  is called a **metric** (or distance) on  $X$  if

1.  $\forall x, y \in X, d(x, y) = 0 \iff x = y;$
2.  $\forall x, y \in X, d(x, y) = d(y, x);$
3.  $\forall x, y, z \in X, d(x, z) \leq d(x, y) + d(y, z).$  (Triangle inequality)

In this case,  $(X, d)$  is called a **metric space**.

## 1.2 Open Ball and Bounded Set

**Definition 1.2.** Let  $(X, d)$  be a metric space,  $x \in X$ , and  $r > 0$ .

1. Define  $B(x, r) = \{y \in X : d(x, y) < r\}$ , called the **open ball** centered at  $x$  with radius  $r$ , or the **r-neighborhood of  $x$** .

2. A general **neighborhood of  $x$**  is a subset  $U$  of  $X$  such that  $B(x, r) \subseteq U$  for some  $r > 0$ .
3. A subset  $E \subseteq X$  of a metric space  $(X, d)$  is **bounded** if there exists some  $x_0 \in X$  and  $M > 0$  such that  $d(x, x_0) \leq M$  for all  $x \in E$ .

### 1.3 Interior Points and Interior

**Definition 1.3.** Let  $E \subseteq X$ . The **closure** of  $E$  is the set  $\overline{E} = E \cup E'$ .

1. An element  $x$  of  $X$  is called an **interior point** of  $E$  if  $\exists r > 0, B(x, r) \subseteq E$ .
2. The **interior** of  $E$  is the set  $E^\circ$  of all interior points of  $E$ .

By definition, we have  $E^\circ \subseteq E \subseteq \overline{E}$ .

### 1.4 Open Set

**Definition 1.4.** A subset  $G \subseteq X$  is called **open** if  $\forall x \in G, \exists r > 0, B(x, r) \subseteq G$ . Note,  $\emptyset$  and  $X$  are open in  $X$ . That is, a set is open if every point in it has an open ball around it that's still entirely inside the set.

### 1.5 Limit Points and Derived Set

**Definition 1.5.** Let  $E \subseteq X$ .

1.  $x$  is called a **limit point** of  $E$  (or cluster point, or accumulation point) if

$$\forall r > 0, B(x, r) \cap E \text{ contains some } y \neq x$$

Equivalently,

$$(B(x, r) - \{x\}) \cap E \neq \emptyset$$

2. We let  $E' =$  the set of all limit points of  $E$ , called the **derived set** of  $E$ .

### 1.6 Closed Set

**Definition 1.6.** A subset  $E \subseteq X$  is called **closed** if  $E' \subseteq E$ , where  $E'$  is the derived set of  $E$  (the set of all limit points of  $E$ ). Note,  $\emptyset$  and  $X$  are closed in  $X$ . That is, a subset  $E \subseteq X$  is closed if every limit point of  $E$  belongs to  $E$ .

1.  $E$  is open  $\iff X - E$  is closed;
2.  $E$  is closed  $\iff X - E$  is open.

### 1.7 Closure

**Definition 1.7.** Let  $E \subseteq X$ . The **closure** of  $E$  is the set  $\overline{E} = E \cup E'$ .

## 1.8 Boundary Points and Boundary

**Definition 1.8.** Let  $X$  be a metric space,  $E \subseteq X$  and  $x \in X$ .

1.  $x$  is called a **boundary point** of  $E$  if  $\forall r > 0, B(x, r) \cap E \neq \emptyset$  and  $B(x, r) \cap (X - E) \neq \emptyset$ .
2. We use  $\partial E$  to denote the set of all boundary points of  $E$ , called the **boundary** of  $E$ .

## 1.9 Open Cover

**Definition 1.9.** Let  $X$  be a metric space and let  $K \subseteq X$ . A family  $\{G_\alpha\}$  of open sets in  $X$  is called an **open cover** of  $K$  if  $K \subseteq \bigcup_\alpha G_\alpha$ . That is, every point of  $K$  lies in at least one of the open sets  $G_\alpha$ .

## 1.10 Compact Set

**Definition 1.10.** The set  $K$  is called **compact** if every open cover of  $K$  has a finite **sub-cover** of  $K$ . That is, if  $K \subseteq \bigcup_\alpha G_\alpha$ , then  $\exists \alpha_1, \dots, \alpha_n$  such that  $K \subseteq \bigcup_{i=1}^n G_\alpha$ .

## 1.11 Convergent Sequence

**Definition 1.11.** Let  $(X, d)$  be a metric space and let  $\{x_n\}$  be a sequence in  $X$ . We say that  $\{x_n\}$  is **convergent** if

$$\exists x \in X \text{ such that } \underline{\forall \epsilon > 0, \exists N = N(\epsilon) \in \mathbb{N}, \forall n \geq N, d(x_n, x) < \epsilon}.$$

In this case, we say that  $\{x_n\}$  converges to  $x$ , and write  $x_n \rightarrow x$ .

## 1.12 Divergent Sequence

**Definition 1.12.** We say that  $\{x_n\}$  is **divergent** if  $\{x_n\}$  is not convergent. That is,  $\forall x \in X, \{x_n\}$  does not converge to  $x$ . The  $\epsilon$ - $N$  description of divergence of  $\{x_n\}$  is

$$\underline{\forall x \in X, \exists \epsilon_0 > 0, \forall N, \exists n \geq N, d(x_n, x) \geq \epsilon_0.}$$

## 1.13 Bounded Sequence

**Definition 1.13.** A sequence  $\{x_n\}$  in a metric space  $(X, d)$  is **bounded** if there exists a point  $x_0 \in X$  and a number  $M > 0$  such that

$$d(x_n, x_0) \leq M \text{ for all } n \in \mathbb{N}$$

In other words, all terms of the sequence lie within some fixed distance  $M$  of a single point  $x_0$ . In simplest terms, a sequence is bounded if all its terms lie inside some ball of finite radius.

### 1.14 The $\epsilon$ - $N$ description of $x_n \rightarrow x$

**Definition 1.14.** The  $\epsilon$ - $N$  description of  $x_n \rightarrow x$  is

$$\underline{\forall \epsilon > 0, \exists N = N(\epsilon) \in \mathbb{N}, \forall n \geq N, d(x_n, x) < \epsilon.}$$

### 1.15 The $\epsilon$ - $N$ description of $x_n \not\rightarrow x$

**Definition 1.15.** The  $\epsilon$ - $N$  description of  $x_n \not\rightarrow x$  is

$$\underline{\exists \epsilon_0 > 0, \forall N, \exists n \in N, d(x_n, x) \geq \epsilon_0.}$$

### 1.16 The $\epsilon$ - $N$ definition of a Cauchy sequence and its negation

**Definition 1.16.** Let  $\{x_n\}$  be a sequence in a metric space  $(X, d)$ .  $\{x_n\}$  is called Cauchy if

$$\underline{\forall \epsilon > 0, \exists N, \forall m, n \geq N, d(x_m, x_n) < \epsilon}$$

So we can say it's negation:  $\{x_n\}$  is not Cauchy if and only if

$$\underline{\exists \epsilon_0 > 0, \forall N, \exists m, n \geq N, d(x_m, x_n) \geq \epsilon_0}$$

### 1.17 Subsequence

**Definition 1.17.** Let  $\{x_n\}$  be a sequence. If

$\{n_k\}_{k \in \mathbb{N}}$  is a sequence in  $\mathbb{N}$  such that  $n_1 < n_2 < \dots$

then  $\{x_{n_k}\}_{k=1}^{\infty}$  is called a subsequence of  $\{x_n\}$ .

### 1.18 Convergence, Divergence, Absolute Convergence of a Series

**Definition 1.18.** Let  $\{a_n\}$  be a sequence in  $\mathbb{R}$ . For each  $n \in \mathbb{N}$ , let

$$s_n = a_1 + \cdots + a_n = \sum_{k=1}^n a_k$$

called the  $n^{th}$  partial sum of the series  $\sum_{k=1}^{\infty} a_k$ .

**Convergence:** If  $s_n \rightarrow s \in \mathbb{R}$ , then we say that the series  $\sum_{k=1}^{\infty} a_k$  is convergent, and we write

$$\sum_{k=1}^{\infty} a_k = s \text{ (called the sum of } \sum_{k=1}^{\infty} a_k)$$

**Divergence:** If the sequence of partial sums  $\{s_n\}$  is divergent (does not approach a finite limit), then we say that the series  $\sum_{k=1}^{\infty} a_k$  is divergent.

**Absolute Convergence:** A series  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent if  $\sum_{n=1}^{\infty} |a_n|$  is convergent.

## 1.19 Geometric series, P-series, Alternating series, and their Convergence

**Definition 1.19.** We observe the following,

**Geometric series:** The geometric series  $\sum_{n=0}^{\infty} x_n$  is convergent  $\iff |x| < 1$ . When  $|x| < 1$ , we have  $\sum_{n=0}^{\infty} x_n = \frac{1}{1-x}$ .

**P-series:** Let  $p \in \mathbb{R}$ . For a series of the form  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  called the p-series, we have

(i) If  $p \leq 1$ , then  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  is divergent

(ii) If  $p > 1$ , then  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  is convergent.

**Alternating series:** Suppose  $\{b_n\}$  is a sequence in  $\mathbb{R}$  such that  $b_1 \geq b_2 \geq \dots \geq 0$  and  $b_n \rightarrow 0$ . Then the alternating series of the form  $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$  and  $\sum_{n=1}^{\infty} (-1)^n b_n$  are convergent.

## 1.20 The $\epsilon$ - $\delta$ definition of $\lim_{x \rightarrow c} f(x)$ and its negation

**Definition 1.20.** Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces,  $E \subseteq X$ ,  $c \in E'$ ,  $f : E \rightarrow Y$ , and  $q \in Y$ . We write the  $\epsilon$ - $\delta$  definition of  $\lim_{x \rightarrow c} f(x) = q$  as

$$\underline{\forall \epsilon > 0, \exists \delta > 0, \forall x \in E \text{ with } 0 < d_X(x, c) < \delta, d_Y(f(x), q) < \epsilon}$$

That is,

$$x \in (B_X(c, \delta) - \{c\}) \cap E \implies f(x) \in B_Y(q, \epsilon), \text{ or equivalently,}$$

$$f((B_X(c, \delta) - \{c\}) \cap E) \subseteq B_Y(q, \epsilon)$$

**Negation:** The  $\epsilon$ - $\delta$  description of the negation of  $\lim_{x \rightarrow c} f(x) = q$  is

$$\underline{\exists \epsilon_0 > 0, \forall \delta > 0, \exists x \in E \text{ with } 0 < d_X(x, c) < \delta, d_Y(f(x), q) \geq \epsilon_0}$$

## 1.21 The $\epsilon$ - $\delta$ definition of continuity of $f$ at $c$ and its negation

**Definition 1.21.** Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces,  $c \in X$ , and  $f : X \rightarrow Y$ . We say that  $f$  is continuous at  $c$  and write the  $\epsilon$ - $\delta$  definition if

$$\underline{\forall \epsilon > 0, \exists \delta = \delta(\epsilon, c) > 0, \forall x \text{ with } d_X(x, c) < \delta, d_Y(f(x), f(c)) < \epsilon}$$

That is,  $f(B_X(c, \delta)) \subseteq B_Y(f(c), \epsilon)$ . If  $f$  is continuous at every point in  $X$ , then we say that  $f$  is continuous on  $X$ .

**Negation:** Therefore, we can write the negation and say that  $f$  is not continuous at  $c$  if and only if

$$\exists \epsilon_0 > 0, \forall \delta > 0, \exists x \text{ with } d_X(x, c) < \delta, d_Y(f(x), f(c)) \geq \epsilon_0$$

## 1.22 The $\epsilon$ - $\delta$ definition of uniform continuity of $f$

**Definition 1.22.** Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces, and let  $f : X \rightarrow Y$ . We say that  $f$  is uniformly continuous on  $X$  if

$$\forall \epsilon > 0, \exists \delta = \delta(\epsilon) > 0, \forall p, q \in X \text{ with } d_X(p, q) < \delta, d_Y(f(p), f(q)) < \epsilon$$

**Negation:** We say that the negation,  $f : X \rightarrow Y$  is not uniformly continuous on  $X$  if and only if

$$\exists \epsilon_0 > 0, \forall \delta > 0, \exists p, q \in X \text{ with } d_X(p, q) < \delta, d_Y(f(p), f(q)) \geq \epsilon_0$$

## 2 Results

### 2.1 Chapter 3: Theorem 11

Let  $(X, d)$  be a metric space and let  $\{x_n\}$  be a sequence in  $X$ .

- (i) If  $\{x_n\}$  is convergent, then  $\{x_n\}$  is Cauchy.
- (ii) If  $\{x_n\}$  is Cauchy, then  $\{x_n\}$  is bounded.

### 2.2 Chapter 3: Theorem 14

**(Cauchy Criterion for Series).** Let  $\sum_{n=1}^{\infty} a_n$  be a series in  $\mathbb{R}$ . Then

$$\sum_{n=1}^{\infty} a_n \text{ is convergent} \iff \forall \epsilon > 0, \exists N, \forall m \geq n \geq N, \left| \sum_{k=n}^{\infty} a_k \right| < \epsilon$$

### 2.3 Chapter 3: Corollary 7

Let  $\sum_{n=1}^{\infty} a_n$  be a series in  $\mathbb{R}$ . Then

$$\sum_{n=1}^{\infty} a_n \text{ is divergent} \iff \exists \epsilon_0 > 0, \forall N, \exists m \geq n \geq N, \left| \sum_{k=n}^m a_k \right| \geq \epsilon_0$$

Note that if  $\sum_{n=1}^{\infty} a_n$  is convergent, then  $a_n \rightarrow 0$  since

$$a_n = s_n - s_{n-1} \rightarrow s - s = 0$$

Therefore, we have the following test for divergence.

## 2.4 Chapter 3: Theorem 15

**(Divergence Test).** If  $a_n \not\rightarrow 0$ , then  $\sum_{n=1}^{\infty} a_n$  is divergent.

## 2.5 Chapter 3: Theorem 16

If  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent, then  $\sum_{n=1}^{\infty} |a_n|$  is convergent.

## 2.6 Chapter 3: Corollary 6

**(Cauchy Criterion).** Let  $\{x_n\}$  be a sequence in  $\mathbb{R}$ . Then

$$\{x_n\} \text{ is convergent} \iff \{x_n\} \text{ is Cauchy.}$$

## 2.7 Chapter 3: Corollary 8

**(Comparison Theorem).** Let  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  be nonnegative series. Suppose that  $\exists n_0 \in \mathbb{N}$  such that  $0 \leq a_n \leq b_n$  for all  $n \geq n_0$ .

(i) If  $\sum_{n=1}^{\infty} a_n$  is divergent, then  $\sum_{n=1}^{\infty} b_n$  is divergent.

(ii) If  $\sum_{n=1}^{\infty} b_n$  is convergent, then  $\sum_{n=1}^{\infty} a_n$  is convergent.

## 2.8 Chapter 4: Theorem 2

**(Sequential Criterion for Convergence).** Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces,  $E \subseteq X$ ,  $c \in E'$ ,  $f : E \rightarrow Y$ , and  $q \in Y$ . Then

$$\lim_{x \rightarrow c} f(x) = q \iff \forall \text{ sequence } \{c_n\} \text{ in } E - \{c\}, \ c_n \rightarrow c \implies f(c_n) \rightarrow q.$$

## 2.9 Chapter 4: Theorem 8

Let  $X$  and  $Y$  be metric spaces with  $X$  compact and let  $f : X \rightarrow Y$  be continuous. Then  $f(X)$  is a compact set in  $Y$ . Therefore,  $f(X)$  is bounded and closed in  $Y$ .

## 2.10 Chapter 4: Corollary 1

We have

$$f(x) \not\rightarrow q \text{ as } x \rightarrow c \iff \exists \text{ sequence } \{c_n\} \text{ in } E - \{c\} \text{ such that } c_n \rightarrow c \text{ but } f(c_n) \not\rightarrow q.$$

## 2.11 Chapter 4: Corollary 2

If there exist sequences  $\{x_n\}$  and  $\{y_n\}$  in  $E - \{c\}$  such that

$$x_n \rightarrow c, y_n \rightarrow c, f(x_n) \rightarrow p, f(y_n) \rightarrow q \text{ and } p \neq q, \text{ then } \lim_{x \rightarrow c} f(x) \text{ DNE.}$$

## 2.12 Chapter 4: Corollary 3

(Sequential Criterion for Continuity). Let  $c \in X$ . Then

$$f : X \rightarrow Y \text{ is continuous at } c \iff [c_n \rightarrow c \text{ in } X \implies f(c_n) \rightarrow f(c) \text{ in } Y].$$

## 2.13 Chapter 4: Corollary 7

If  $f : X \rightarrow Y$  is continuous, then  $\forall$  compact  $E \subseteq X$ ,  $f(E)$  is compact.

# 3 Questions/Proofs

## 3.1 Assignment 6: Question #1

**Question:** Suppose that  $a_n > 0$  ( $n = 1, 2, \dots$ ).

- (i) Prove that if  $\sum a_n$  is convergent, then  $\sum \frac{\sqrt{a_n}}{n}$  is convergent.

**Solution.** Analysis on the property we will be using:

$$(a - b)^2 = a^2 - 2ab + b^2 \geq 0 \implies ab \leq \frac{1}{2}(a^2 + b^2)$$

*Proof.* We have

$$\frac{\sqrt{a_n}}{n} = \left(\sqrt{a_n}\right)\left(\frac{1}{n}\right) \leq \frac{1}{2}\left(a_n + \frac{1}{n^2}\right)$$

Since

$$\sum a_n \text{ and } \sum \frac{1}{n^2} \text{ converge} \implies \sum \frac{1}{2}\left(a_n + \frac{1}{n^2}\right) \text{ converges.}$$

Therefore, since  $\sum \frac{\sqrt{a_n}}{n} \leq \sum \frac{1}{2}\left(a_n + \frac{1}{n^2}\right)$ , then  $\sum \frac{\sqrt{a_n}}{n}$  converges. □

□

(ii) Prove that if  $\sum a_n$  is divergent, then  $\sum \frac{a_n}{1+a_n}$  is divergent.

**Solution.** We will use the contrapositive form:

$$\sum \frac{a_n}{1+a_n} \text{ converges} \implies \sum a_n \text{ converges}$$

*Proof.* Assume  $\sum \frac{a_n}{1+a_n}$  converges.

$$\begin{aligned} &\implies \lim_{n \rightarrow \infty} \frac{a_n}{1+a_n} = 0, \text{ so } \exists N, \forall n \geq N, \frac{a_N}{1+a_n} < \frac{1}{2} \\ &\implies 2a_n < 1 + a_n \text{ or } a_n < 1 \text{ for all } n \geq N \\ &\implies 1 + a_n < 2 \\ &\implies \frac{1}{2} < \frac{1}{1+a_n} \\ &\implies \frac{1}{2}a_n < \frac{a_n}{1+a_n} \end{aligned}$$

So, we have that  $\sum \frac{1}{2}a_n$  converges since  $\sum \frac{a_n}{1+a_n}$  converges.

Therefore,  $\sum a_n$  is convergent. □

□

### 3.2 Assignment 6: Question #2

**Question:** Discuss the (absolute) convergence/divergence of the series  $\sum_{n=1}^{\infty} a_n$ , where  $a_n$  is given by

$$(i) \quad a_n = (-1)^n \left(1 + \frac{1}{n}\right) \quad (v) \quad a_n = \frac{(-1)^n}{1+x^n}, (x \geq 0)$$

$$(ii) \quad a_n = \sqrt{n+1} - \sqrt{n} \quad (vi) \quad a_n = \frac{n!}{1000^n}$$

$$(iii) \quad a_n = (-1)^n \frac{1}{\sqrt{n}} \quad (vii) \quad a_n = \frac{n!}{n^n}$$

$$(iv) \quad a_n = \frac{\sqrt{n+1} - \sqrt{n}}{2n} \quad (viii) \quad a_n = \frac{\cos n}{n^{5/4}}$$

**Solution.**

□

### 3.3 Assignment 7: Question #2

**Question:** Use the  $\epsilon$ - $\delta$  definition to prove that

$$(i) \lim_{x \rightarrow 2} (x^2 - x + 1) = 3 \quad (ii) \lim_{x \rightarrow 0} \frac{2x}{1+x^2} = 0 \quad (iii) \lim_{x \rightarrow 1} \sqrt{x^2 + 2} = \sqrt{3}$$

**Solution.**

□

### 3.4 Assignment 7: Question #5

**Question:** Let  $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$  Use the  $\epsilon$ - $\delta$  definition to prove that  $f : \mathbb{R} \times \mathbb{R}$

is continuous at 0.

**Solution.**

□

### 3.5 Assignment 7: Question #6

**Question:** Use the  $\epsilon$ - $\delta$  definition to prove that  $f(x) = \sqrt{x}$  is continuous at every  $c \geq 0$ .

**Solution.** To prove  $f$  is continuous at every  $c \geq 0$ , let's do some analysis on  $|f(x) - f(c)|$  and relate it with  $M|x - c|$  where  $M > 0$ . By doing so, we can find the proper delta to use in our formal proof.

*Analysis:*

$$|f(x) - f(c)| = |\sqrt{x} - \sqrt{c}| = \begin{cases} \sqrt{x} & c = 0 \\ \frac{1}{\sqrt{x} + \sqrt{c}} |x - c| \leq \frac{1}{\sqrt{c}} |x - c| & c > 0 \end{cases}$$

When  $c = 0$ , we need to choose a delta such that

$$\sqrt{x} = \epsilon \implies \delta = \epsilon^2$$

When  $c > 0$ , we need to choose a delta such that

$$\frac{1}{\sqrt{x} + \sqrt{c}} |x - c| \leq \frac{1}{\sqrt{c}} |x - c| \implies |x - c| < \epsilon \sqrt{c} \implies \delta = \epsilon \sqrt{c}$$

Further expanding

$$|\sqrt{x} - \sqrt{c}| = \frac{|x - c|}{\sqrt{x} + \sqrt{c}}$$

*Proof.* Given  $\epsilon > 0$ .

**Case 1:**  $c = 0$

Take  $\delta = \epsilon^2$ . Then  $\delta > 0$ . Now,

$$\forall x \in [0, \infty) \text{ with } |x - 0| = x < \delta$$

we have

$$|f(x) - f(0)| = \sqrt{x} < \sqrt{\delta} = \epsilon$$

Therefore,  $f$  is continuous at 0.

**Case 2:**  $c > 0$

Take  $\delta = \epsilon\sqrt{c}$ . Then  $\delta > 0$ . Now,

$$\forall x \in [0, \infty) \text{ with } |x - c| < \delta$$

we have

$$|f(x) - f(c)| = |\sqrt{x} - \sqrt{c}| = \frac{1}{\sqrt{x} + \sqrt{c}}|x - c| \leq \frac{1}{\sqrt{c}}|x - c| < \frac{1}{\sqrt{c}}\delta = \epsilon$$

Therefore,  $f$  is continuous at  $c$ . □

□

□

### 3.6 Assignment 7: Question #7

**Question:** Use the  $\epsilon$ - $\delta$  definition to prove the uniform continuity of the following functions on the given intervals.

$$(i) \quad f(x) = \frac{1}{x^2} \text{ on } [2, \infty)$$

**Solution.** To prove uniform continuity of  $f$ , let's do some analysis on  $|f(p) - f(q)|$  and relate it with  $M|p - q|$  where  $M > 0$ . By doing so, we can find the proper delta to use in our formal proof.

*Analysis:*

$$\begin{aligned} |f(p) - f(q)| &= \left| \frac{1}{p^2} - \frac{1}{q^2} \right| = \left| \frac{q^2 - p^2}{p^2 q^2} \right| = \frac{(q + p)|p - q|}{p^2 q^2} \\ &= \left( \frac{1}{p^2 q^2} + \frac{1}{pq^2} \right) |p - q| \leq \left( \frac{1}{8} + \frac{1}{8} \right) |p - q| \text{ (since } p, q \geq 2) \\ &= \frac{1}{4} |p - q| < \frac{\delta}{4} = \epsilon \implies \delta = 4\epsilon \end{aligned}$$

Now we have found the correct delta to use in our formal proof.

*Proof.* Given  $\epsilon > 0$ . Take  $\delta = 4\epsilon$ . Then  $\delta > 0$ . Now,

$$\forall p, q \in [2, \infty) \text{ with } |p - q| < \delta$$

we have

$$|f(p) - f(q)| = \left( \frac{1}{p^2 q^2} + \frac{1}{pq^2} \right) |p - q| \leq \frac{1}{4} |p - q| < \frac{\delta}{4} = \epsilon$$

Therefore,  $f$  is uniform continuous. □

□

□

$$(ii) \ f(x) = \frac{x-1}{x+1} \text{ on } [0, \infty)$$

**Solution.** To prove uniform continuity of  $f$ , let's do some analysis on  $|f(p) - f(q)|$  and relate it with  $M|p - q|$  where  $M > 0$ . By doing so, we can find the proper delta to use in our formal proof.

*Analysis:*

$$\begin{aligned} |f(p) - f(q)| &= \left| \frac{p-1}{p+1} - \frac{q-1}{q+1} \right| \\ &= \frac{2|p-q|}{(p+1)(q+1)} \leq 2|p-q| < 2\delta = \epsilon \implies \delta = \frac{\epsilon}{2} \end{aligned}$$

Now we have found the correct delta to use in our formal proof.

*Proof.* Given  $\epsilon > 0$ . Take  $\delta = \frac{\epsilon}{2}$ . Then  $\delta > 0$ . Now,

$$\forall p, q \in [0, \infty) \text{ with } |p - q| < \delta$$

we have

$$\frac{2|p-q|}{(p+1)(q+1)} \leq 2|p-q| < 2\delta = \epsilon$$

Therefore,  $f$  is uniform continuous. □

□

$$(iii) \ f(x) = \frac{x}{1+x^2} \text{ on } (-\infty, \infty)$$

**Solution.** To prove uniform continuity of  $f$ , let's do some analysis on  $|f(p) - f(q)|$  and relate it with  $M|p - q|$  where  $M > 0$ . By doing so, we can find the proper delta to use in our formal proof.

*Analysis:*

$$\begin{aligned} |f(p) - f(q)| &= \left| \frac{p}{1+p^2} - \frac{q}{1+q^2} \right| = \frac{|p + pq^2 - q - p^2q|}{(1+p^2)(1+q^2)} = \frac{|(p-q) + pq(q-p)|}{(1+p^2)(1+q^2)} \\ &= \frac{|1-pq||p-q|}{1+p^2+q^2+p^2q^2} \leq \frac{1+|p||q|}{1+p^2+q^2+p^2q^2}|p-q| \\ &\leq \frac{1+\frac{1}{2}(p^2+q^2)}{1+p^2+q^2}|p-q| \leq |p-q| < \delta = \epsilon \implies \delta = \epsilon \end{aligned}$$

Now we have found the correct delta to use in our formal proof.

*Proof.* Given  $\epsilon > 0$ . Take  $\delta = \epsilon$ . Then  $\delta > 0$ . Now,

$$\forall p, q \in (-\infty, \infty) \text{ with } |p - q| < \delta$$

we have

$$|f(p) - f(q)| \leq \frac{1+|p||q|}{1+p^2+q^2+p^2q^2}|p-q| \leq |p-q| < \delta = \epsilon$$

Therefore,  $f$  is uniform continuous. □

□

### 3.7 Chapter 2: the proof of Theorem 9

**Theorem:** Let  $K$  be a compact set in metric space  $X$  and let  $F$  be a closed subset of  $K$ . Then  $F$  is compact.

*Proof.* Let  $\{G_\alpha\}$  be an open cover of  $F$ . Then  $\{X - F\} \cup \{G_\alpha\}$  is an open cover of  $K$ . Since  $K$  is compact,  $\exists \alpha_1, \dots, \alpha_n$  such that

$$K \subseteq (X - F) \cup (G_{\alpha_1} \cup \dots \cup G_{\alpha_n}).$$

Now  $F = F \cap K \subseteq (F \cap (X - F)) \cup (G_{\alpha_1} \cup \dots \cup G_{\alpha_n}) = G_{\alpha_1} \cup \dots \cup G_{\alpha_n}$ . Hence,  $\{G_\alpha\}$  has a finite subcover  $\{G_{\alpha_1}, \dots, G_{\alpha_n}\}$  of  $F$ . Therefore,  $F$  is compact.  $\square$

### 3.8 Chapter 3: the proof of Theorem 11

**Theorem:** Let  $(X, d)$  be a metric space and let  $\{x_n\}$  be a sequence in  $X$ .

- (i) If  $\{x_n\}$  is convergent, then  $\{x_n\}$  is Cauchy.
- (ii) If  $\{x_n\}$  is Cauchy, then  $\{x_n\}$  is bounded.

*Proof.* (i) Suppose  $x_n \rightarrow x$ . Then  $\forall \epsilon > 0, \exists N, \forall n \geq N, d(x_n, x) < \frac{\epsilon}{2}$ , and thus  $\forall m, n \geq N$ , we have

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) = d(x_m, x) + d(x_n, x) < \epsilon.$$

Therefore,  $\{x_n\}$  is Cauchy.

- (ii) Suppose  $\{x_n\}$  is Cauchy. For  $\epsilon = 1, \exists N, \forall m, n \geq N, d(x_m, x_n) < \epsilon$ . Let  $x = x_N$  and  $r = 1 + \max_{1 \leq i \leq N} d(x_i, x)$ . Then  $d(x_n, x) < r$  for all  $n$ ; that is,  $\{x_n\}$  is in the ball  $B(x, r)$ . Therefore,  $\{x_n\}$  is bounded.

$\square$

### 3.9 Chapter 4: the proof of Theorem 8

**Theorem:** Let  $X$  and  $Y$  be metric spaces with  $X$  compact and let  $f : X \rightarrow Y$  be continuous. Then  $f(X)$  is a compact set in  $Y$ . Therefore,  $f(X)$  is bounded and closed in  $Y$ .

*Proof.* Let  $\mathcal{U}$  be an open cover of  $f(X)$  in  $Y$ ; that is,  $\mathcal{U}$  is a family of open sets on  $Y$  such that  $f(X) \subseteq \bigcup_{U \in \mathcal{U}} U$ . Then

$$X \subseteq f^{-1}\left(\bigcup_{U \in \mathcal{U}} U\right) = \bigcup_{U \in \mathcal{U}} f^{-1}(U)$$

with each  $f^{-1}(U)$  open in  $X$  (since  $f$  is continuous). Thus  $\{f^{-1}(U) : U \in \mathcal{U}\}$  is an open cover of  $X$ . By the compactness of  $X$ ,  $\exists U_1, \dots, U_n \in \mathcal{U}$  such that  $X = \bigcup_{k=1}^n f^{-1}(U_k)$ . Hence,

$$f(X) = f\left(\bigcup_{k=1}^n f^{-1}(U_k)\right) = \bigcup_{k=1}^n f(f^{-1}(U_k)) \subseteq \bigcup_{k=1}^n U_k.$$

So,  $\mathcal{U}$  has a finite subcover of  $f(X)$ . Therefore,  $f(X)$  is compact.  $\square$