ALGEBRAIC GEOMETRY

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Homework 3 — **Due date**: 9 December 2022

Exercise 1. Let A be a ring, $\mathfrak{p} \subset A$ a prime ideal. Describe an explicit isomorphism of fields $\operatorname{Frac} A/\mathfrak{p} \cong \kappa(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p} \cdot A_\mathfrak{p}$.

Exercise 2. Let $\mathbf{k}$ be an algebraically closed field of characteristic different from 2. Show that any non-degenerate conic $C \subset \mathbb{P}^2_\mathbf{k}$ is isomorphic to $\mathbb{P}^1$.

Exercise 3. Show that if $V \subset \mathbb{P}^n_\mathbf{k}$ is a hypersurface (the zero locus of a homogeneous polynomial $f \in \mathbf{k}[x_0, \ldots, x_n]$), then $\mathbb{P}^n_\mathbf{k} \setminus V$ is affine. (**Hint**: start solving the case $\deg f = 1$, then use the Veronese embedding).

Exercise 4. Let $Y \subset \mathbb{A}^n_\mathbf{k}$ be an irreducible affine variety of dimension r, and let $H \subset \mathbb{A}^n_\mathbf{k}$ be a hypersurface such that $Y \not\subset H$. Show that every irreducible component of $X = Y \cap H$ has dimension $r - 1$.

Exercise 5. Let $I \subset \mathbf{k}[x_1, \ldots, x_n]$ be an ideal which can be generated by r elements. Show that ever irreducible component of $\mathrm{V}(I) \subset \mathbb{A}^n_\mathbf{k}$ has dimension at least $n - r$.

Exercise 6. Let $X = \mathrm{D}(f_1) \cup \cdots \cup \mathrm{D}(f_r) \subset \mathbb{A}^n_\mathbf{k}$ be a finite union of principal open sets $\mathrm{D}(f_i) \subset \mathbb{A}^n_\mathbf{k}$. Show that $\mathscr{O}_{\mathbb{A}^n_\mathbf{k}}(X) = \mathbf{k}[x_1, \ldots, x_n]_f$, where $f = \gcd(f_1, \ldots, f_r)$. Show that that every affine open subset of $\mathbb{A}^n_\mathbf{k}$ is principal.

Exercise 7. Show that every irreducible closed subset $Z \subset \mathbb{P}^n_\mathbf{k}$ of dimension $n - 1$ is principal.

Exercise 8. Show that the complement $W = \mathbb{P}^n_\mathbf{k} \setminus U$ of an affine open subset $U \subset \mathbb{P}^n_\mathbf{k}$ has pure dimension $n - 1$, and that U is principal.