ALGEBRAIC GEOMETRY — EXERCISE SHEET 2
DUE ON 10/11/2024

Exercise 0.1. Let $X = \operatorname{Spec} \mathbf{k}[x, y, z]/I \subset \mathbb{A}^3_{\mathbf{k}}$, where $I = (x^2 - yz, y^2 - xz) \subset \mathbf{k}[x, y, z]$. Compute the irreducible components of X.

Exercise 0.2. Let X be a scheme. Let U, V be affine opens of X, and let $x \in U \cap V$. Prove that there exists an affine open neighbourhood W of x such that W is a principal open of both U and V.

Exercise 0.3. A morphism of schemes $f : X \to Y$ is called *quasicompact* if the preimage of any affine open subset is quasicompact. Prove that $f : X \to Y$ is quasicompact if and only if Y has an affine open cover $Y = \bigcup_{i \in I} Y_i$ such that $f^{-1}(Y_i)$ is quasicompact for all i.

Exercise 0.4. Assume $\mathbf{k}$ is an algebraically closed field of characteristic different from 2. Let $Y \subset \mathbb{P}^2_{\mathbf{k}}$ be a nondegenerate conic. Show that Y is isomorphic to $\mathbb{P}^1_{\mathbf{k}}$.

Exercise 0.5. Let $\mathbf{k}$ be an algebraically closed field and $f \in \mathbf{k}[x_1, \ldots, x_n]$ a nonzero polynomial. Show that $\operatorname{Spec} \mathbf{k}[x_1, \ldots, x_n]/(f)$ is reduced (resp. irreducible, resp. integral) if and only if f is square-free (resp. admits only one irreducible factor, resp. is irreducible). Explain why the assumption that $\mathbf{k}$ be algebraically closed is crucial.

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