Algebraic Geometry

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[...] *Oscar Zariski bewitched me. When he spoke the words "algebraic variety", there was a certain resonance in his voice that said distinctly that he was looking into a secret garden. I immediately wanted to be able to do this too. It led me to 25 years of struggling to make this world tangible and visible.*

David Mumford

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0 Before we start

About this course

This is a 50 hours course (2.5 cycles for SISSA students).

The exam consists of an oral presentation by the student about a topic mutually agreed on, plus a few questions regarding the material covered in the course.

For UNITS students: the exam can only be scheduled within the official exam session, see the academic calendar.

Prerequisites

Familiarity with basic theory of commutative rings and modules is of great help, but not necessary. The relevant notions will be recalled as we need them. We have, however, included Appendix B to cover the basic commutative algebra constructions we will be referring to (and much more), and Appendix A to cover the basics of category theory as well.

Conventions

We list here a series of conventions that will be used throughout this text.

- The axiom of choice (or Zorn's Lemma) is assumed; so, for instance, every ring has a maximal ideal, and a poset $(P, \leq)$ in which every chain has an upper bound admits a maximal element.
- Given two sets A and B, the phrase '$A \subset B$' means that A is contained in B, *possibly equal* to B.
- A *ring* is a commutative, unitary ring. The zero ring (the one where $1 = 0$) is allowed (and in fact needed), but we always assume our rings are nonzero unless we explicitly mention it. Ring homomorphisms preserve the identity.
- By $\mathbf{k}$ we indicate an algebraically closed field, by $\mathbb{F}$ an arbitrary field.

- An open cover of a topological space U is the datum of a set I, and an open subset $U_i \subset U$ for every $i \in I$, such that $U = \bigcup_{i \in I} U_i$. We set $U_{ij} = U_i \cap U_j$. If $I = \emptyset$, then $U = \emptyset$.

- To say that Ω is an object a category $\mathscr{C}$ we simply write '$\Omega \in \mathscr{C}$' instead of $\Omega \in \mathrm{Ob}(\mathscr{C})$, with the exception of Appendix A, where a crash course on categories and functors is provided.

Main references

We list here a series of bibliographical references that integrate this text.

- Q. Liu, *Algebraic geometry and arithmetic curves* [10],

- R. Hartshorne, *Algebraic geometry* [7],

- R. Vakil, *The rising sea* [13],

- D. Eisenbud, *Commutative Algebra: With a View Toward Algebraic Geometry* [4],

- M. F. Atiyah and I. G. MacDonald, *Introduction to Commutative Algebra* [1],

1 Introduction

Algebraic Geometry deals with the study of *algebraic varieties*. At a first approximation, these are common zero loci of collections of polynomials, i.e. solutions to systems

$$\begin{cases} f_1(x_1,\ldots,x_n)=0 \\ \quad\vdots \\ f_r(x_1,\ldots,x_n)=0 \end{cases}$$

of polynomial equations. When $\deg f_j = 1$ for all $j = 1,\ldots,r$, this is the content of *Linear Algebra*, but the higher degree case poses nontrivial difficulties!

The concept of algebraic variety has been vastly generalised by Grothendieck's theory of *schemes*, introduced in [6].

Figure 1.1: Alexander Grothendieck (1928–2014).

This course is an introduction to schemes and to (part of) the massive dictionary, shared by all algebraic geometers, centered around schemes. Even though algebraic varieties are somewhat 'easier' objects, schemes are an incredibly useful and powerful tool to study them.

In this introduction, we briefly recap the key relation

$$\text{Algebra} \longleftrightarrow \text{Geometry}$$

in the land of *classical* algebraic varieties. We provide no proofs for now, but you shouldn't worry about this, because we will be proving more general results in the main body of these notes.

Let $\mathbf{k}$ be an algebraically closed field. Classical *affine n-space over* $\mathbf{k}$ is just

$$\mathbb{A}^n_{\mathbf{k}} = \{ (a_1,\dots,a_n) \mid a_i \in \mathbf{k} \text{ for } i=1,\dots,n \}.$$

We denote it $\mathbb{A}^n_{\mathbf{k}}$ and not $\mathbf{k}^n$ to emphasise that we view it as a set of points rather than a vector space over $\mathbf{k}$. For instance, $\mathbb{A}^1_{\mathbf{k}}$ is called the *affine line* over $\mathbf{k}$, and $\mathbb{A}^2_{\mathbf{k}}$ is called the *affine plane* over $\mathbf{k}$. Let

$$A = \mathbf{k}[x_1,\dots,x_n]$$

be the polynomial ring in n variables over the field $\mathbf{k}$. Each element $f \in A$ defines a function $\widetilde{f}\colon \mathbb{A}^n_{\mathbf{k}} \to \mathbf{k}$ sending $(a_1,\dots,a_n) \mapsto f(a_1,\dots,a_n)$, and since $\mathbf{k}$ is algebraically closed one has $f=g$ if and only if $\widetilde{f}=\widetilde{g}$.[1] Thus we shall just write f instead of $\widetilde{f}$.

Let $I=(f_1,\dots,f_r)\subset A$ be an arbitrary ideal (here we are using that every ideal in A is finitely generated, by Hilbert's basis theorem [8]). The 'vanishing locus'

$$\mathrm{V}(I) = \left\{ (a_1,\dots,a_n) \in \mathbb{A}^n_{\mathbf{k}} \,\middle|\, f_j(a_1,\dots,a_n)=0 \text{ for } j=1,\dots,r \right\} \subset \mathbb{A}^n_{\mathbf{k}}$$

is called an *algebraic set*. There is precisely one topology on $\mathbb{A}^n_{\mathbf{k}}$ having the algebraic sets as closed sets. It is called the *Zariski topology*.

Indeed, one has

- $\mathbb{A}^n_{\mathbf{k}} = \mathrm{V}(0)$,
- $\emptyset = \mathrm{V}(A)$,
- $\mathrm{V}(I)\cup\mathrm{V}(J) = \mathrm{V}(IJ)$,
- $\bigcap_{s\in S}\mathrm{V}(I_s) = \mathrm{V}\big(\sum_{s\in S} I_s\big)$ for any family of ideals $(I_s\subset A)_{s\in S}$.

Figure 1.2: Oscar Zariski (1899–1986).

Example 1.0.1. Every ideal in $\mathbf{k}[x]$ is principal, i.e. of the form (f) for some $f\in\mathbf{k}[x]$. Since $\mathbf{k}$ is algebraically closed, we have $f=\alpha(x-a_1)\cdots(x-a_d)$, for $\alpha,a_1,\dots,a_d\in\mathbf{k}$, and where $d=\deg f$. Thus, if $f\neq 0$, then $\mathrm{V}(f)=\{a_1,\dots,a_d\}\subset\mathbb{A}^1_{\mathbf{k}}$, proving that all proper closed subsets of $\mathbb{A}^1_{\mathbf{k}}$ are finite. In particular, all open sets are infinite (since $\mathbf{k}$ is algebraically closed, thus infinite).

[1] For instance, the field $\mathbb{F}_3=\{0,1,2\}$ is not algebraically closed, and the polynomials $f=x^2+1$ and $g=x^4+1$ are different, nevertheless one has $\widetilde{f}=\widetilde{g}$ as functions on the three point space $\mathbb{A}^1_{\mathbb{F}_3}$.

We have thus established an assignment

$$\{\text{ideals } I \subset \mathbf{k}[x_1,\dots,x_n]\} \xrightarrow{\mathrm{V}(-)} \{\text{algebraic sets in } \mathbb{A}^n_{\mathbf{k}}\}.$$

Conversely, given a subset $S \subset \mathbb{A}^n_{\mathbf{k}}$, the assignment

$$\mathrm{I}(S) = \left\{ f \in A \,\middle|\, f(p) = 0 \text{ for all } p \in S \right\} \subset A$$

defines a map the other way around, namely

$$\{\text{ideals } I \subset \mathbf{k}[x_1,\dots,x_n]\} \xleftarrow{\mathrm{I}(-)} \{\text{subsets } S \subset \mathbb{A}^n_{\mathbf{k}}\}.$$

The two maps are *not* inverse to each other, even if we restrict I(−) to algebraic sets. For instance, consider the ideal $(x^r) \subset \mathbf{k}[x]$ for $r > 1$. Then $\mathrm{V}(x^r) = \{0\}$, and thus $\mathrm{I}(\mathrm{V}(x^r)) = (x)$, which is strictly larger than (x^r). The next result says that this is what *always* happens.

THEOREM 1.0.2 (Hilbert's Nullstellensatz [9]). *Let $I \subset \mathbf{k}[x_1,\dots,x_n]$ be an ideal, where $\mathbf{k}$ is an algebraically closed field. Then, $\mathrm{I}(\mathrm{V}(I)) = \sqrt{I}$, i.e. $f \in \mathrm{I}(\mathrm{V}(I))$ if and only if $f^r \in I$ for some $r > 0$.*

See [10, Ch. 2, Corollary 1.15] for a modern proof of Hilbert's Nullstellensatz.

Composing our two assignments the other way around, we also find something larger than what we started with: consider for instance the complement $S \subset \mathbb{A}^1_{\mathbf{k}}$ of a finite set. Then $\mathrm{I}(S) = (0)$, since there are no nonzero polynomials with infinitely many zeroes. Thus $\mathrm{V}(\mathrm{I}(S)) = \mathbb{A}^1_{\mathbf{k}}$. In general, if S is an arbitrary subset of $\mathbb{A}^n_{\mathbf{k}}$, one can easily prove the identity

$$\mathrm{V}(\mathrm{I}(S)) = \overline{S},$$

where $\overline{S}$ is the closure of S in $\mathbb{A}^n_{\mathbf{k}}$ (with respect to the Zariski topology), namely the smallest algebraic set containing S. Thus in order to get $\mathrm{V}(\mathrm{I}(S)) = S$ we have to start with an algebraic set S (which is closed by definition).

Furthermore, one can prove that an algebraic set $Y \subset \mathbb{A}^n_{\mathbf{k}}$ is irreducible (i.e. it cannot be written as a union of two proper closed subsets) if and only if $\mathrm{I}(Y) \subset A$ is a prime ideal.

$$\boxed{\text{An irreducible algebraic set in } \mathbb{A}^n_{\mathbf{k}} \text{ is called an } \textit{affine variety in } \mathbb{A}^n_{\mathbf{k}}.}$$

Of course, an affine variety carries the induced Zariski topology by default. Combining these observations together, we obtain correspondences (with 'algebra' on the left, and 'geometry' on the right)

$$\begin{tikzcd}
\{\text{radical ideals in } \mathbf{k}[x_1,\dots,x_n]\} \arrow[r, shift left, "\mathrm{V}(-)"] & \{\text{algebraic sets in } \mathbb{A}^n_{\mathbf{k}}\} \arrow[l, shift left, "\mathrm{I}(-)"] \\
\{\text{prime ideals in } \mathbf{k}[x_1,\dots,x_n]\} \arrow[u, hook] \arrow[r, shift left, "\mathrm{V}(-)"] & \{\text{affine varieties in } \mathbb{A}^n_{\mathbf{k}}\} \arrow[u, hook] \arrow[l, shift left, "\mathrm{I}(-)"]
\end{tikzcd}$$

where an ideal $I \subset \mathbf{k}[x_1, \ldots, x_n]$ is *radical* if $I = \sqrt{I}$ (**??**).

Recall that, by definition, a *finitely generated* $\mathbf{k}$*-algebra* is a $\mathbf{k}$-algebra B isomorphic to a quotient $\mathbf{k}[x_1, \ldots, x_n]/I$ for some n and some ideal $I \subset \mathbf{k}[x_1, \ldots, x_n]$. Such a B is an integral domain (i.e. as a ring it has no nonzero zero-divisors) precisely when I is prime. Thus the bottom correspondence above can be rephrased as

$$\{\, \mathbf{k}[x_1, \ldots, x_n]/\mathfrak{p} \mid \mathfrak{p} \text{ is prime} \,\} \underset{\mathrm{I}(-)}{\overset{\mathrm{V}(-)}{\rightleftarrows}} \{\text{affine varieties in } \mathbb{A}^n_{\mathbf{k}}\,\}.$$

In the first part of this course, we will extend this correspondence to arbitrary *rings* on the left. What will be constructed on the right will be called an *affine scheme*, and what we shall establish is not just a bijection, but an equivalence of categories

$$\mathsf{Rings}^{\mathrm{op}} \cong \text{Affine schemes}.$$

Affine schemes are the basic building blocks for the construction of general *schemes*, in the same way as open subsets of $\mathbb{R}^m$ are the basic building blocks for m-dimensional smooth manifolds. As we shall see, a *scheme* is defined by the property that every point has an open neighborhood isomorphic to an affine scheme.

2 | Sheaves

Sheaves were defined by Leray (1906–1998), while he was a prisoner in Austria during World War II.

Sheaves are a key notion present in the toolbox of every mathematician keen to understand the "nature" of a *geometric space*. They incarnate one of the basic principles that will be unraveled throughout this course, which can be stated as the slogan

$$\boxed{\text{geometric spaces are determined by functions on them.}}$$

Even though there may be "few" functions on a space X, a complete knowledge of all functions on all open subsets of X allows one, in principle, to reconstruct X. This local-to-global principle is perfectly encoded in the notion of a sheaf.

2.1 Key example: smooth functions

Before diving into precise definitions, we explore a key example of sheaf.

Let X be a smooth manifold. For each open subset $U \subset X$, we have a ring (actually, an $\mathbb{R}$-algebra)

$$C^\infty(U,\mathbb{R}) = \{\text{smooth functions } U \to \mathbb{R}\}.$$

Indeed, smooth functions with the same source can naturally be added and multiplied exploiting the ring structure on $\mathbb{R}$. If $V \hookrightarrow U$ is an open subset, we have a restriction map

$$\rho_{UV} \colon C^\infty(U,\mathbb{R}) \to C^\infty(V,\mathbb{R}), \quad f \mapsto f|_V,$$

which is an $\mathbb{R}$-algebra homomorphism. One has $\rho_{UU} = \mathrm{id}_{C^\infty(U,\mathbb{R})}$, and if $W \hookrightarrow V \hookrightarrow U$ is a chain of open subsets of X, we have a commutative diagram

$$\begin{tikzcd} C^\infty(U,\mathbb{R}) \arrow[r, "\rho_{UV}"] \arrow[rr, bend right, "\rho_{UW}"'] & C^\infty(V,\mathbb{R}) \arrow[r, "\rho_{VW}"] & C^\infty(W,\mathbb{R}). \end{tikzcd}$$

So far, we have just observed that the assignment $U \mapsto C^\infty(U,\mathbb{R})$ is *functorial*, from open subsets of X (which form a category) to the category of $\mathbb{R}$-algebras. The two distinguished features of the assignment $U \mapsto C^\infty(U,\mathbb{R})$, which make it into a *sheaf* of $\mathbb{R}$-algebras on X, are the following:

(i) Fix an open subset $U \subset X$ and an open cover $U = \bigcup_{i \in I} U_i$. If $f, g \in C^\infty(U, \mathbb{R})$ are smooth functions such that $f|_{U_i} = g|_{U_i}$ for every $i \in I$, then $f = g$. In other words, a smooth function is determined by its restriction to the open subsets forming a covering. This is the *locality axiom*.

(ii) Fix an open subset $U \subset X$ and an open cover $U = \bigcup_{i \in I} U_i$. Given a smooth function $f_i \in C^\infty(U_i, \mathbb{R})$ on each U_i, such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for every $(i, j) \in I \times I$, there is a smooth function $f \in C^\infty(U, \mathbb{R})$ such that $f_i = f|_{U_i}$ for every $i \in I$. In other words, functions glue along an open cover. This is the *glueing axiom*.

A sheaf is an abstract notion formalising this "ability of glueing". The formal definition will be given in Important Definition 2.2.1. Note that the result of the glueing in Condition (ii) is *unique* by Condition (i).

Let us continue with our example. Let $x \in X$ be a point. Consider the ring

$$C^\infty_{X,x} = \left\{ (U, f) \,\middle|\, x \in U,\, f \in C^\infty(U, \mathbb{R}) \right\} \big/ \sim$$

where $(U, f) \sim (V, g)$ whenever there exists an open subset $W \subset U \cap V$, containing x, such that $f|_W = g|_W$. Note that $C^\infty_{X,x}$ is indeed a ring, with addition and multiplication

$$\begin{aligned} [U, f] + [U', f'] &= [U \cap U', f + f'] \\ [U, f] \cdot [U', f'] &= [U \cap U', f f']. \end{aligned}$$

This ring, which is in fact an $\mathbb{R}$-algebra via $c \mapsto [X, c]$, for all $c \in \mathbb{R}$, is called the *stalk* of the sheaf $C^\infty(-, \mathbb{R})$ at x (cf. Important Definition 2.4.1), and it receives a natural $\mathbb{R}$-algebra homomorphism from $C^\infty(U, \mathbb{R})$ for every open subset U of X such that $x \in U$, sending $f \mapsto [U, f]$. The image of f along this map is called the *germ of f at x*. The subset

$$\mathfrak{m}_x = \left\{ [U, f] \in C^\infty_{X,x} \,\middle|\, f(x) = 0 \right\} \subset C^\infty_{X,x}$$

forms an ideal, which is a *maximal* ideal, being the kernel of the (surjective) evaluation map

$$\begin{aligned} C^\infty_{X,x} &\twoheadrightarrow \mathbb{R} \\ [U, f] &\longmapsto f(x). \end{aligned}$$

In fact, $\mathfrak{m}_x$ is the *unique* maximal ideal of $C^\infty_{X,x}$. To see this, it is enough to check that every element of $C^\infty_{X,x} \setminus \mathfrak{m}_x$ is invertible. But this is true, since a smooth function that is nonzero in a neighbourhood of x is invertible there.

The upshot is, then, that the pair $(C^\infty_{X,x}, \mathfrak{m}_x)$ defines a *local ring* with residue field $\mathbb{R}$. The geometric spaces X one deals with in algebraic geometry, namely *schemes*, have precisely this property: they come with a sheaf of rings $\mathscr{O}_X$ such that each stalk $\mathscr{O}_{X,x}$ is a local ring. These spaces $(X, \mathscr{O}_X)$ actually form a larger category, that of locally ringed spaces (cf. **??**). Schemes are particular instances of locally ringed spaces.

2.2 Presheaves, sheaves, morphisms

Let $\mathscr{C}$ be a concrete category (Definition A.1.16) with a final object $0 \in \mathscr{C}$. The concreteness assumption means that part of the structure is the datum of a faithful functor $F: \mathscr{C} \to \mathsf{Sets}$, but we will (for the moment) ignore this datum. To fix ideas, $\mathscr{C}$ should be thought of as any of the following categories:

- $\mathscr{C} = \mathsf{Sets}$,
- $\mathscr{C} = \mathsf{Rings}$,
- $\mathscr{C} = \mathsf{Ab} = \mathsf{Mod}_{\mathbb{Z}}$,
- $\mathscr{C} = \mathsf{Mod}_R$, where R is a ring.

If X is a topological space, we denote by τ_X the category of open subsets of X. The set $\operatorname{Hom}_{\tau_X}(V, U)$ between two open sets $V, U \subset X$ is just the empty set if $V \not\subset U$, or the singleton $\{ V \hookrightarrow U \}$ in case V is contained in U. Thus the opposite category τ_X^{op} satisfies

$$\operatorname{Hom}_{\tau_X^{\mathrm{op}}}(U, V) = \begin{cases} \{ V \hookrightarrow U \} & \text{if } V \subset U \\ \emptyset & \text{if } V \not\subset U \end{cases}$$

and a functor $\mathcal{F}: \tau_X^{\mathrm{op}} \to \mathscr{C}$ (i.e. a contravariant functor $\tau_X \to \mathscr{C}$) determines a map

$$\operatorname{Hom}_{\tau_X^{\mathrm{op}}}(U, V) \to \operatorname{Hom}_{\mathscr{C}}(\mathcal{F}(U), \mathcal{F}(V)),$$

which is nothing but a choice of an element $\rho_{UV} \in \operatorname{Hom}_{\mathscr{C}}(\mathcal{F}(U), \mathcal{F}(V))$ for any inclusion of open subsets $V \subset U$.

Definition 2.2.1 (Presheaf, take I)**.** A *presheaf* on a topological space X, with values in $\mathscr{C}$, is a contravariant functor $\mathcal{F}$ from τ_X to $\mathscr{C}$, i.e. an object of the functor category $\mathsf{Fun}(\tau_X^{\mathrm{op}}, \mathscr{C})$.

For those who do not like the categorical definition, here is an equivalent definition, which just unravels the definition of a functor (cf. Definition A.1.6).

Definition 2.2.2 (Presheaf, take II)**.** A *presheaf* on a topological space X, with values in $\mathscr{C}$, is the assignment $U \mapsto \mathcal{F}(U)$ of an object $\mathcal{F}(U) \in \mathscr{C}$ for each open subset $U \subset X$, and of a morphism $\rho_{UV}: \mathcal{F}(U) \to \mathcal{F}(V)$ in $\mathscr{C}$ for each inclusion $V \hookrightarrow U$, such that

(1) $\rho_{UU} = \mathrm{id}_{\mathcal{F}(U)}$ for every $U \in \tau_X$, and

(2) $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$ for every chain of inclusions $W \hookrightarrow V \hookrightarrow U$.

Terminology 2.2.3. Elements of $\mathcal{F}(U)$ are often called 'sections of $\mathcal{F}$ over U', or (somewhat more vaguely) 'local sections' when $U \subsetneq X$. Elements of $\mathcal{F}(X)$ are called 'global sections', or just 'sections'. Possible alternative notations for $\mathcal{F}(U)$ are $\Gamma(U,\mathcal{F})$ and $\mathrm{H}^0(U,\mathcal{F})$. The maps ρ_{UV} are often called 'restriction maps' (from U to V, the larger set being U).

Notation 2.2.4. Motivated by Terminology 2.2.3, we shall often write $s|_V$ for the image of a section $s \in \mathcal{F}(U)$ along the restriction map ρ_{UV}.

Important Definition 2.2.1 (Sheaf, take I)**.** A *sheaf* on a topological space X, with values in $\mathscr{C}$, is a presheaf $\mathcal{F}$ such that the following two conditions hold:

(3) Fix an open subset $U \subset X$, an open cover $U = \bigcup_{i \in I} U_i$, and two sections $s, t \in \mathcal{F}(U)$ satisfying $s|_{U_i} = t|_{U_i}$ for all $i \in I$. Then $s = t$.

(4) Fix an open subset $U \subset X$, an open cover $U = \bigcup_{i \in I} U_i$ and a tuple $(s_i)_{i \in I}$ of sections $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $(i,j) \in I \times I$. Then there exists a section $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$.

Conditions (3) and (4) generalise the conditions (i) and (ii), respectively, anticipated with the example $\mathcal{F} = C^\infty(-,\mathbb{R})$ in Section 2.1.

Terminology 2.2.5. A presheaf $\mathcal{F}$ is called *separated* if Condition (3) holds. Sometimes this condition is called *locality axiom*. Condition (4), on the other hand, is called the *glueing axiom* (or *glueing condition*).

Remark 2.2.6. Let $\mathcal{F}$ be a sheaf. Then, the section $s \in \mathcal{F}(U)$ in the glueing condition (4) is necessarily unique because $\mathcal{F}$ is separated. In fact, the two sheaf conditions could be replaced by a single condition, identical to (4), but imposing uniqueness of s.

Example 2.2.7 (Trivial sheaf)**.** The presheaf defined by $U \mapsto 0$ for every U is a sheaf and is called the *trivial sheaf* (or sometimes the *zero sheaf*). It is simply denoted by '0'.

Example 2.2.8 (Restriction to an open)**.** Let $U \subset X$ be an open subset, $\mathcal{F}$ a presheaf on X. Then, setting $\mathcal{F}|_U(V) = \mathcal{F}(V)$ for V an open subset of U, defines a presheaf $\mathcal{F}|_U$ on U, which is a sheaf as soon as $\mathcal{F}$ is. It is called the *restriction of $\mathcal{F}$ to U*.

Definition 2.2.9 (Morphism of (pre)sheaves)**.** Let X be a topological space. A *morphism* between two presheaves $\mathcal{F}, \mathcal{G}$ on X is a natural transformation $\eta\colon \mathcal{F} \Rightarrow \mathcal{G}$, i.e. a morphism in the functor category $\mathsf{Fun}(\tau_X^{\mathrm{op}}, \mathscr{C})$. A morphism of sheaves is just a morphism between the underlying presheaves.

Let us unravel the definition of natural transformation (cf. Definition A.1.9), to translate Definition 2.2.9 in more concrete terms.

To give a morphism of (pre)sheaves, one has to assign a homomorphism

$$\eta_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$$

in $\mathscr{C}$ to each $U \in \tau_X$, such that for every inclusion $V \hookrightarrow U$ of open subsets of X, the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\eta_U"] \arrow[d, "\rho^{\mathcal{F}}_{UV}"'] & \mathcal{G}(U) \arrow[d, "\rho^{\mathcal{G}}_{UV}"] \\ \mathcal{F}(V) \arrow[r, "\eta_V"] & \mathcal{G}(V) \end{tikzcd} \tag{2.2.1}$$

commutes. For the sake of clarity, we have emphasised the relevant (pre)sheaf in the restriction maps notation, but we will not be doing that systematically.

Notation 2.2.10. It is clear that presheaves on X with values in $\mathscr{C}$ form a category $\mathsf{pSh}(X,\mathscr{C})$, tautologically defined as

$$\mathsf{pSh}(X,\mathscr{C}) = \mathsf{Fun}(\tau_X^{\mathrm{op}},\mathscr{C}).$$

By Definition 2.2.9, sheaves form a full subcategory, denoted $\mathsf{Sh}(X,\mathscr{C})$. We denote by

$$j_{X,\mathscr{C}} \colon \mathsf{Sh}(X,\mathscr{C}) \hookrightarrow \mathsf{pSh}(X,\mathscr{C}) \tag{2.2.2}$$

the (fully faithful) inclusion functor.

An isomorphism of (pre)sheaves is an isomorphism in $\mathsf{pSh}(X,\mathscr{C})$, i.e. a *natural isomorphism*, i.e. a natural transformation $\eta \colon \mathcal{F} \Rightarrow \mathcal{G}$ such that η_U is an isomorphism in $\mathscr{C}$ for every $U \in \tau_X$ (cf. Definition A.1.10).

Notation 2.2.11. Since (pre)sheaves form a genuine category, from now on we shall use the classical arrow notation '$\mathcal{F} \to \mathcal{G}$' (instead of $\mathcal{F} \Rightarrow \mathcal{G}$) to denote a morphism of (pre)sheaves.

The following definition makes sense, because $\mathscr{C}$ is assumed to be a concrete category.

Definition 2.2.12 (Injective map of presheaves)**.** A morphism of (pre)sheaves $\eta \colon \mathcal{F} \to \mathcal{G}$ is *injective* if η_U is injective for every U. We denote this by writing η as '$\mathcal{F} \hookrightarrow \mathcal{G}$' (or somewhat more informally '$\mathcal{F} \subset \mathcal{G}$'), and we say that $\mathcal{F}$ is a *sub(pre)sheaf* of $\mathcal{G}$.

We close this section with a few examples and exercises.

Example 2.2.13 (Smooth functions)**.** Let X be a smooth manifold. Then, sending $U \subset X$ to the set $C^\infty(U,\mathbb{R})$ of smooth functions $U \to \mathbb{R}$, defines a sheaf $C^\infty(-,\mathbb{R})$ with values in the category of $\mathbb{R}$-algebras.

Example 2.2.14 (Holomorphic functions)**.** Let X be a complex manifold. Then, sending an open subset $U \subset X$ to the set $\mathscr{O}_X^{\mathrm{h}}(U)$ of holomorphic functions on U, defines a sheaf $\mathscr{O}_X^{\mathrm{h}}$ with values in the category of $\mathbb{C}$-algebras. Sending U to the set $\mathscr{O}_X^{\mathrm{h},\times}(U)$ of nowhere zero holomorphic functions on U defines a sheaf of abelian groups on X (the group structure being given by pointwise multiplication of functions).

Example 2.2.15 (Continuous functions are a sheaf)**.** Let X, Y be topological spaces. For $U \subset X$ open, define

$$\mathcal{F}(U) = \{\, \text{continuous functions } U \to Y \,\}.$$

Then $\mathcal{F}$ is a sheaf of sets on X.

Example 2.2.16 (Separated presheaf, not a sheaf, take I)**.** Set $X = \mathbb{C}$. Then, sending $U \subset X$ to the subset

$$\mathcal{F}(U) = \left\{\, f \in \mathscr{O}_X^{\mathrm{h}}(U) \;\middle|\; f = g^2 \text{ for some } g \in \mathscr{O}_X^{\mathrm{h}}(U) \,\right\}$$

defines a (separated) presheaf. However, $\mathcal{F}$ is not a sheaf: the function $f(z) = z$ on the annulus

$$U = \{\, z \in \mathbb{C} \mid 1 - \varepsilon < |z| < 1 + \varepsilon \,\} \subset \mathbb{C}$$

has a square root in any neighbourhood of any point $x \in U$, but there is no global $g(z) = \sqrt{z}$ defined on the whole of U.

Exercise 2.2.17 (Separated presheaf, not a sheaf, take II)**.** Let $X = \mathbb{R}$, with the standard topology. Show that

$$U \mapsto \mathrm{B}(U) = \{\, \text{bounded continuous functions } U \to \mathbb{R} \,\}$$

is a separated presheaf on X, but not a sheaf (i.e. Condition (4) fails).

Example 2.2.18 (Constant presheaf)**.** Work with $\mathscr{C} = \mathsf{Ab} = \mathsf{Mod}_{\mathbb{Z}}$, the category of abelian groups, and fix $G \neq 0$ in this category. Fix a topological space X, and define

$$\underline{G}_X^{\mathrm{pre}}(U) = \begin{cases} G & \text{if } U \neq \emptyset, \\ 0 & \text{if } U = \emptyset. \end{cases}$$

As for the restriction maps, set $\rho_{UV} = \mathrm{id}_G$ if both U and V are nonempty. This is a presheaf, which happens to be a sheaf only in precise circumstances (cf. Exercise 2.2.20). For instance, suppose $X = U_1 \amalg U_2$ is a disjoint union of two nonempty open subsets. Then $\underline{G}_X^{\mathrm{pre}}(X) = G = \underline{G}_X^{\mathrm{pre}}(U_i)$ for $i = 1, 2$. Now, $X = U_1 \amalg U_2$ is an open cover. Pick two *distinct* sections $s_i \in G = \underline{G}_X^{\mathrm{pre}}(U_i)$ for $i = 1, 2$. Then, $s_1|_{U_1 \cap U_2} = s_1|_\emptyset = 0 = s_2|_\emptyset = s_2|_{U_1 \cap U_2}$, but there is no section $s \in \underline{G}_X^{\mathrm{pre}}(X) = G$ such that $s|_{U_i} = s_i$ since $\rho_{XU_i} = \mathrm{id}_G$ for $i = 1, 2$ and $s_1 \neq s_2$ by assumption. Hence Condition (4), i.e. the gluing axiom, fails (whereas Condition (3) is trivially satisfied). We will see in Example 2.5.3 that $\underline{G}_X^{\mathrm{pre}}$ can be "transformed" into a sheaf by a canonical procedure.

Exercise 2.2.19. Provide examples of presheaves which satisfy the glueing axiom but not the separation axiom.

Exercise 2.2.20. Show that the constant presheaf $\underline{G}_X^{\mathrm{pre}}$ of Example 2.2.18 is a sheaf if and only if every nonempty open subset $U \subset X$ is connected.

Exercise 2.2.21 (Preheaves kernel and cokernel)**.** Let $\mathscr{C}$ be an abelian category, so that every arrow has a kernel and a cokernel. Let $\eta\colon \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves with values in $\mathscr{C}$. Consider the assignments

$$
\begin{aligned}
U &\mapsto (\ker_{\mathrm{pre}} \eta)(U) = \ker(\eta_U) \\
U &\mapsto (\operatorname{coker}_{\mathrm{pre}} \eta)(U) = \operatorname{coker}(\eta_U) = \mathcal{G}(U)/\operatorname{im}(\eta_U).
\end{aligned}
$$

Show that

(i) both $\ker_{\mathrm{pre}} \eta$ and $\operatorname{coker}_{\mathrm{pre}} \eta$ are presheaves,

(ii) There is a morphism of presheaves $\ker_{\mathrm{pre}} \eta \to \mathcal{F}$ (resp. $\mathcal{G} \to \operatorname{coker}_{\mathrm{pre}} \eta$) which satisfies the universal property of the kernel (resp. the cokernel) in $\mathsf{pSh}(X, \mathscr{C})$,

(iii) $\ker_{\mathrm{pre}} \eta$ is a sheaf, denoted $\ker(\eta)$, as soon as η is a morphism of *sheaves*,

(iv) if η is a morphism of sheaves, then $\ker(\eta)$ satisfies the universal property of the kernel in $\mathsf{Sh}(X, \mathscr{C})$, and η is injective if and only if $\ker(\eta) = 0$.

Example 2.2.22 ($\operatorname{coker}_{\mathrm{pre}} \eta$ may not be a sheaf)**.** Let $X = \mathbb{C}$ and $\mathscr{C} = \mathsf{Ab}$. Consider the morphism of sheaves

$$
\exp\colon \mathscr{O}_X^{\mathrm{h}} \to \mathscr{O}_X^{\mathrm{h},\times}, \quad f \mapsto \exp(f),
$$

where $\mathscr{O}_X^{\mathrm{h},\times}$ is the sheaf of nowhere zero holomorphic functions (cf. Example 2.2.14). We have that the open subset $U = X \setminus \{0\} \subset X$ is covered by the two open subsets

$$
U_1 = X \setminus [0, +\infty] \subset X, \quad U_2 = X \setminus (-\infty, 0] \subset X.
$$

The function $g(z) = z$ viewed in $\mathscr{O}_X^{\mathrm{h},\times}(U)$ is not of the form $\exp(f)$ for any $f \in \mathscr{O}_X^{\mathrm{h}}(U)$. Thus the $\widetilde{g}$ image of g along

$$
\mathscr{O}_X^{\mathrm{h},\times}(U) \to \operatorname{coker}_{\mathrm{pre}}(\exp)(U)
$$

is nonzero. However, U_1 and U_2 are simply connected, thus every function $h_i \in \mathscr{O}_X^{\mathrm{h},\times}(U_i)$ is of the form $\exp(f_i)$ for some $f_i \in \mathscr{O}_X^{\mathrm{h}}(U_i)$. Thus $\operatorname{coker}_{\mathrm{pre}}(\exp)(U_i) = 0$ for $i = 1, 2$. In particular, the restrictions $g|_{U_i}$ have this property, namely they go (necessarily) to 0 in $\operatorname{coker}_{\mathrm{pre}}(\exp)(U_i)$. If $\operatorname{coker}_{\mathrm{pre}}(\exp)$ were a sheaf, the gluing axiom would force $\widetilde{g} = 0$, which is not true.

2.3 The sheaf condition via equalisers

We now present an alternative way to define sheaves. We will repeatedly use this reinterpretation throughout these notes.

Let $\mathscr{C}$ be a category with limits (cf. Definition B.3.1). In particular, $\mathscr{C}$ has products, equalisers, and a final object (cf. Appendix B.3.1 for full details). The reader may imagine $\mathscr{C}$ to be, for instance, any of the following categories: sets, groups, rings, algebras over a fixed ring, modules over a fixed ring.

Fix a presheaf $\mathcal{F}$ with values in $\mathscr{C}$ on a topological space X. Let $\{U_i\}_{i\in I}$ be a family of open subsets of X, and set $U = \bigcup_{i\in I} U_i$. Then, by our assumption on $\mathscr{C}$, one can consider the map

$$\rho\colon \mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i), \quad s \mapsto (s|_{U_i})_{i\in I},$$

as well as the family of maps

$$\mu_{ij}\colon \prod_{i\in I} \mathcal{F}(U_i) \to \mathcal{F}(U_i) \to \mathcal{F}(U_i\cap U_j), \qquad (s_i)_{i\in I} \mapsto s_i|_{U_i\cap U_j}$$
$$\nu_{ij}\colon \prod_{i\in I} \mathcal{F}(U_i) \to \mathcal{F}(U_j) \to \mathcal{F}(U_i\cap U_j), \qquad (s_i)_{i\in I} \mapsto s_j|_{U_i\cap U_j}$$

which, taking products over $(i,j)\in I\times I$, can be assembled into two maps

$$\prod_{i\in I} \mathcal{F}(U_i) \overset{\mu}{\underset{\nu}{\rightrightarrows}} \prod_{(i,j)\in I\times I} \mathcal{F}(U_i\cap U_j).$$

Definition 2.3.1 (Sheaf, take II)**.** Let $\mathscr{C}$ be a category with limits, X a topological space. A presheaf $\mathcal{F}\in \mathsf{pSh}(X,\mathscr{C})$ is a *sheaf* if for every family of open subsets $\{U_i\}_{i\in I}$, with $U = \bigcup_{i\in I} U_i$, the diagram

$$\mathcal{F}(U) \xrightarrow{\rho} \prod_{i\in I} \mathcal{F}(U_i) \overset{\mu}{\underset{\nu}{\rightrightarrows}} \prod_{(i,j)\in I\times I} \mathcal{F}(U_i\cap U_j)$$

is an equaliser diagram in $\mathscr{C}$.

Informally, being an equaliser means that ρ is injective and its image agrees with the set of tuples $(s_i)_{i\in I}$ such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all pairs (i,j).

Note that Definition 2.3.1 is *element-free*. However, let us check that it agrees with Important Definition 2.2.1 when $\mathscr{C}$ is concrete: in this case the injectivity of ρ, implied by the equaliser condition, coincides with separatedness; the fact that the set-theoretic image of ρ coincides with the collection of tuples of sections $(s_i)_{i\in I}$ such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ is precisely the glueing condition.

Remark 2.3.2. Let $\mathcal{F}$ be a sheaf. Then, one has $\mathcal{F}(\emptyset) = 0$, the final object in $\mathscr{C}$. This is sometimes listed as an axiom defining a (pre)sheaf, but it does in fact follow from our assumptions (cf. Example B.3.3).

Example 2.3.3. Let $\mathcal{F}$ be a sheaf on X. If $U = \coprod_{i\in I} U_i$ is a *disjoint* union of open subsets $U_i \subset U$, then ρ is an isomorphism, i.e. $s \mapsto (s|_{U_i})_{i\in I}$ defines an isomorphism

$$\rho : \mathcal{F}(U) \xrightarrow{\sim} \prod_{i\in I} \mathcal{F}(U_i).$$

Example 2.3.4. Let $\mathscr{C}$ be an abelian category. Then a presheaf $\mathcal{F} \in \mathsf{pSh}(X, \mathscr{C})$ is a sheaf if for every family of open subsets $\{U_i\}_{i\in I}$, with $U = \bigcup_{i\in I} U_i$, the sequence

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\rho} \prod_{i\in I} \mathcal{F}(U_i) \xrightarrow{\mu-\nu} \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \cap U_j)$$

is exact, where the map denoted $\mu - \nu$ sends $(s_i)_{i\in I} \mapsto (s_i|_{U_i\cap U_j} - s_j|_{U_i\cap U_j})_{i,j}$.

The following lemma applies, for instance, to categories of groups, rings, algebras over a ring, and modules over a ring. It allows one to check the sheaf conditions in the category of sets.

LEMMA 2.3.5 ([12, Tag 0073]). *Let $\mathscr{C}$ be a category, $F : \mathscr{C} \to \mathsf{Sets}$ a faithful functor such that $\mathscr{C}$ has limits and F commutes with them. Assume that F reflects isomorphisms. Then a presheaf $\mathcal{F} \in \mathsf{pSh}(X, \mathscr{C})$ is a sheaf of and only if the underlying presheaf of sets $F \circ \mathcal{F} : \tau_X^{\mathrm{op}} \to \mathsf{Sets}$ is a sheaf.*

At the beginning of this chapter we have defined (pre)sheaves of objects in an arbitrary concrete category $\mathscr{C}$. We still have to define a few things, though, e.g. stalks and sheafification. In order for everything to be well-defined and work well (but still be compatible with all we have discussed so far, including Definition 2.3.1), we need to add a few initial data. This is provided by the following definition.

Definition 2.3.6 ([12, Tag 007L])**.** A *type of algebraic structure* is a pair $(\mathscr{C}, F)$, where $\mathscr{C}$ is a category, $F : \mathscr{C} \to \mathsf{Sets}$ is a faithful functor, such that

1. $\mathscr{C}$ has limits and F commutes with them,
2. $\mathscr{C}$ has filtered colimits and F commutes with them,
3. F reflects isomorphisms (i.e. F is *conservative*).

A few remarks are in order, before we go on.

- Equipping a category $\mathscr{C}$ with a faithful functor $F : \mathscr{C} \to \mathsf{Sets}$ is like saying that $\mathscr{C}$ is a *concrete category*, which we had already assumed in Section 2.2.
- If we have a type of algebraic structure $(\mathscr{C}, F)$, then we can verify whether a presheaf is a sheaf in the category of sets, by Lemma 2.3.5.

- The condition that F be conservative implies that a bijective morphism in $\mathscr{C}$ is an isomorphism.
- For every type of algebraic structure $(\mathscr{C}, F)$, one has the following properties:
  - (i) $\mathscr{C}$ has a final object 0, and $F(0)$ is a final object in Sets (i.e. a singleton).
  - (ii) $\mathscr{C}$ has products, fibre products, and equalisers — this follows from the examples in Appendix B.3.1. Moreover, F commutes with all of them.
- Examples of categories $\mathscr{C}$ having the additional structure of Definition 2.3.6 are:
  - monoids,
  - groups,
  - abelian groups,
  - rings,
  - modules over a ring.

  In all these cases, we take as the functor F the obvious forgetful functor. The reader is encouraged to just think of $\mathscr{C}$ as one of these familiar categories, and not bother too much about Definition 2.3.6. As a counterexample, however, consider the category Top of topological spaces: the forgetful functor to Sets exists but does not reflect isomorphisms (a continuous bijection need not be a homeomorphism).

2.4 Stalks, and what they tell us

Fix a type of algebraic structure $(\mathscr{C}, F\colon \mathscr{C} \to \mathsf{Sets})$ as in Definition 2.3.6. Let X be a topological space, $x \in X$ a point. The collection of open subsets $U \subset X$ containing x forms a directed system (the partial order $\succeq$ being the inclusion relation, i.e. $V \succeq U$ if and only if $V \subset U$). Indeed, given two open neighbourhoods U and V of x, there is always a third open neighbourhood of x contained in both U and V, namely $U \cap V$ or any smaller open subset containing x. In fancier language, the subcategory

$$\iota_x\colon \mathsf{Ngb}_x = \{\, U \in \tau_X \mid x \in U \,\}^{\mathrm{op}} \hookrightarrow \tau_X^{\mathrm{op}}$$

is a filtered category (see Definition B.3.9).

Important Definition 2.4.1 (Stalks)**.** Let $x \in X$ be a point, $\mathcal{F}$ a presheaf. The *stalk of* $\mathcal{F}$ *at* x is the filtered colimit

$$\mathcal{F}_x = \varinjlim_{\mathsf{Ngb}_x} \mathcal{F} \circ \iota_x = \varinjlim_{U \ni x} \mathcal{F}(U) \in \mathscr{C}.$$

Because F commutes with colimits, the underlying *set* $F(\mathcal{F}_x) \in \mathsf{Sets}$, still denoted $\mathcal{F}_x$, is

$$\mathcal{F}_x = \{(U, s) \mid x \in U,\ s \in \mathcal{F}(U)\} / \sim$$

where $(U, s) \sim (V, t)$ whenever there is an open neighbourhood $W \subset U \cap V$ of x such that $s|_W = t|_W$. We denote by

$$s_x = [U, s] \in \mathcal{F}_x$$

the equivalence class of the pair (U, s). It is called the *germ of s at x*. By definition of direct limit, there are natural homomorphisms

$$\mathcal{F}(U) \to \mathcal{F}_x, \quad s \mapsto s_x,$$

in $\mathscr{C}$, for every open neighbourhood U of x. The diagram

$$\begin{tikzcd}
s \in \mathcal{F}(U) \arrow[dr] \arrow[dddr, bend right] & & \mathcal{F}(V) \ni t \arrow[dl] \arrow[dddl, bend left] \\
& \mathcal{F}(U \cap V) \arrow[d] & \\
& \mathcal{F}(W) \arrow[d] & \\
& \mathcal{F}_x &
\end{tikzcd}$$

illustrates the fact that two sections $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$ define the same element in the stalk $\mathcal{F}_x$ if and only if there is an intermediate open subset $W \subset U \cap V$ over which they agree.

Figure 2.1: A bunch of sheaves sitting in their natural habitat. The little tops of each leaf of corn are the stalks.

LEMMA 2.4.1. *If $\mathcal{F}$ is a separated presheaf of sets (e.g. a sheaf), then the natural map*

(2.4.1) $$\sigma_U^{\mathcal{F}} \colon \mathcal{F}(U) \to \prod_{x\in U} \mathcal{F}_x, \quad s \mapsto (s_x)_{x\in U}$$

is injective for every open subset U of X.

The lemma means, at an informal level, that sections are determined by their germs.

Proof. If s and t are sections in $\mathcal{F}(U)$ such that $s_x = t_x$ in $\mathcal{F}_x$ for every $x \in U$, then for every $x \in U$ there is an open neighbourhood $U_x \subset U$ such that $s|_{U_x} = t|_{U_x}$. But this holds for every $x \in U$, and $U = \bigcup_{x\in U} U_x$ is an open covering, thus by the separation axiom we deduce $s = t$, i.e. $\sigma_U^{\mathcal{F}}$ is injective. □

Consider the following property of a tuple $(s_x)_{x\in U} \in \prod_{x\in U} \mathcal{F}_x$, for $U \subset X$ an open subset:

(2.4.2) $$\begin{gathered}\text{for every } x \in U \text{ there exists a pair } (V_x, t^x),\\ \text{with } x \in V_x \subset U \text{ and } t^x \in \mathcal{F}(V_x),\\ \text{such that } t^x_y = s_y \text{ for all } y \in V_x.\end{gathered}$$

Definition 2.4.2 (Compatible germs)**.** Let $\mathcal{F}$ be a presheaf on X, and let $U \subset X$ be an open subset. We say that $(s_x)_{x\in U} \in \prod_{x\in U} \mathcal{F}_x$ is a *tuple of compatible germs* if Condition (2.4.2) is fulfilled.

We always have inclusions

(2.4.3) $$\operatorname{im}(\sigma_U^{\mathcal{F}}) \subset \{\text{ tuples } (s_x)_{x\in U} \text{ of compatible germs }\} \subset \prod_{x\in U} \mathcal{F}_x$$

where the first inclusion is justified by taking $V_x = U$ and $t^x = s$ for every $x \in U$ as soon as $\sigma_U^{\mathcal{F}}(s) = (s_x)_{x\in U}$. If $\mathcal{F}$ is a sheaf, then tuples of compatible germs form precisely the image of the map (2.4.1), i.e. the first inclusion in (2.4.3) is an equality. Indeed, assume $(s_x)_{x\in U}$ consists of compatible germs. Let $\{(V_x, t^x) \mid x \in U\}$ be as in the displayed condition (2.4.2). By the compatibility condition, for every pair $(x, x') \in U \times U$ we have

$$t^x_y = t^{x'}_y, \quad y \in V_x \cap V_{x'}.$$

It follows from Lemma 2.4.1 that

(2.4.4) $$t^x\big|_{V_x\cap V_{x'}} = t^{x'}\big|_{V_x\cap V_{x'}}.$$

Now, we have an open cover $U = \bigcup_{x\in U} V_x$, so by the glueing axiom, applicable by (2.4.4), the sections $t^x \in \mathcal{F}(V_x)$ glue to a (unique) section $t \in \mathcal{F}(U)$ such that $t|_{V_x} = t^x$. But $t^x_y = s_y$ for $y \in V_x$, and this holds for every $x \in U$, so $\sigma_U^{\mathcal{F}}(t) = (s_x)_{x\in U}$.

Summing up, when $\mathcal{F}$ is a sheaf, we have a bijection

$$\sigma_U^{\mathcal{F}} \colon \mathcal{F}(U) \xrightarrow{\sim} \{\,\text{tuples } (s_x)_{x\in U} \text{ of compatible germs}\,\}.$$

This also shows that *sections of a sheaf can always be identified with 'nicely gluable' functions!* Indeed, tuples $(s_x)_{x\in U}$ correspond to particular functions $U \to \coprod_{x\in U} \mathcal{F}_x$, sending $x \in U$ inside the corresponding stalk, and doing so in a compatible way.

LEMMA 2.4.3. *Let $s, t \in \mathcal{F}(X)$ be two global sections of a sheaf $\mathcal{F}$, such that $s_x = t_x \in \mathcal{F}_x$ for every $x \in X$. Then $s = t$.*

Proof. This is just a special case of Lemma 2.4.1. □

Exercise 2.4.4. Let $\mathcal{F}$ be a sheaf on X, and let $s, t \in \mathcal{F}(X)$ be two global sections. Show that

$$\{\, x \in X \mid s_x = t_x \,\} \subset X$$

is an open subset of X.

A morphism of presheaves $\eta \colon \mathcal{F} \to \mathcal{G}$ induces a morphism $\eta_x \colon \mathcal{F}_x \to \mathcal{G}_x$ at the level of stalks for every $x \in X$, defined by

$$s_x = [U, s] \mapsto [U, \eta_U(s)] = (\eta_U(s))_x. \tag{2.4.5}$$

Exercise 2.4.5. Check that (2.4.5) is well-defined.

If $U \subset X$ is an open subset containing a point $x \in X$, then the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\eta_U"] \arrow[d] & \mathcal{G}(U) \arrow[d] \\ \mathcal{F}_x \arrow[r, "\eta_x"] & \mathcal{G}_x \end{tikzcd} \qquad \begin{tikzcd} s \arrow[r, mapsto, "\eta_U"] \arrow[d, mapsto] & \eta_U(s) \arrow[d, mapsto] \\ s_x \arrow[r, mapsto, "\eta_x"] & (\eta_U(s))_x \end{tikzcd}$$

commutes. What we have just said can be rephrased by saying that the association $\mathcal{F} \mapsto \mathcal{F}_x$ defines a functor

$$\mathsf{stalk}_x \colon \mathsf{pSh}(X, \mathscr{C}) \to \mathscr{C}. \tag{2.4.6}$$

We will see that in reasonable circumstances the restriction of this functor to the category of sheaves is *exact* (cf. Proposition 2.5.14).

Definition 2.4.6. A morphism of (pre)sheaves $\eta \colon \mathcal{F} \to \mathcal{G}$ is *surjective* if η_x is surjective for every $x \in X$.

Warning 2.4.7. You may have noticed that surjectivity of a map of sheaves (cf. Definition 2.4.6) is defined differently than injectivity (cf. Definition 2.2.12)!

Let $\eta\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then

$$\eta_x \text{ is surjective} \iff \begin{array}{c}\text{for every } t_x \in \mathcal{G}_x \text{ there exists an open neighbourhood}\\ U \text{ of } x \text{ and a section } s \in \mathcal{F}(U) \text{ such that } (\eta_U(s))_x = t_x.\end{array}$$

$$\eta \text{ is surjective} \iff \begin{array}{c}\text{for every open subset } U \subset X \text{ and for every}\\ t \in \mathcal{G}(U), \text{ there exists a covering } U = \bigcup_{i\in I} U_i\\ \text{such that } t|_{U_i} \text{ is in the image of } \eta_{U_i} \text{ for every } i.\end{array}$$

The second equivalence is obtained as follows.

Proof of '⇒'. Assume η is surjective, i.e. η_x is surjective for every $x \in X$. Fix $U \subset X$ open and a local section $t \in \mathcal{G}(U)$. For every $x \in U$, we have a commutative diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\eta_U"] \arrow[d] & \mathcal{G}(U) \arrow[d] & t \arrow[d, mapsto] \\ \mathcal{F}_x \arrow[r, two heads, "\eta_x"] & \mathcal{G}_x & t_x \end{tikzcd}$$

where $t_x \in \mathcal{G}_x$ can be lifted along η_x to an element $s_x \in \mathcal{F}_x$. Let (V_x, s) be a representative for s_x, so that in particular $s \in \mathcal{F}(V_x)$. The identity $\eta_x(s_x) = t_x$ implies that there is an open neighbourhood $x \in U_x \subset V_x \cap U$ such that

$$\eta_{U_x}(s|_{U_x}) = t|_{U_x}.$$

Now this holds for every $x \in U$, and the elements of $\{\, U_x \mid x \in U \,\}$ form a covering of U, thus we have proved the condition.

Proof of '⇐'. Conversely, assuming the condition, let us prove surjectivity of η. Fix $x \in X$ along with a germ $t_x \in \mathcal{G}_x$. We need to prove that t_x has a preimage in $\mathcal{F}_x$. Let (U, t) be a representative of t_x, so that $t \in \mathcal{G}(U)$. By the condition we are assuming, there exists a covering $U = \bigcup_{i\in I} U_i$ such that $t|_{U_i} = \eta_{U_i}(s_i)$ for some $s_i \in \mathcal{F}(U_i)$, for every $i \in I$. If $x \in U_i$, we have a commutative diagram

$$\begin{tikzcd} \mathcal{F}(U_i) \arrow[r, "\eta_{U_i}"] \arrow[d] & \mathcal{G}(U_i) \arrow[d] & s_i \arrow[r, mapsto] \arrow[d, mapsto] & t|_{U_i} \arrow[d, mapsto] \\ \mathcal{F}_x \arrow[r, "\eta_x"] & \mathcal{G}_x & \star \arrow[r, mapsto] & t_x \end{tikzcd}$$

so the element $\star \in \mathcal{F}_x$ is a preimage of t_x. The equivalence is proved.

The next result incarnates the local nature of sheaves.

LEMMA 2.4.8. *Let $\eta\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. The following are equivalent:*

(i) *η is an isomorphism,*

(ii) *η_x is an isomorphism for every $x \in X$,*

(iii) *η is injective and surjective.*

Proof. Recall that η is an isomorphism if and only if η_U is an isomorphism for every U.

Proof of (i) $\Rightarrow$ (ii). By functoriality of $\mathcal{F} \mapsto \mathcal{F}_x$, we have that if η is an isomorphism, then so is η_x for every $x \in X$.

Proof of (ii) $\Rightarrow$ (i). Suppose η_x is an isomorphism for every x. Let $U \subset X$ be an open subset: we need to show that η_U is an isomorphism.

To see that η_U is injective, pick $s, t \in \mathcal{F}(U)$ such that $\eta_U(s) = \eta_U(t) \in \mathcal{G}(U)$. Then, for any $x \in U$, one has

$$\eta_x(s_x) = (\eta_U(s))_x = (\eta_U(t))_x = \eta_x(t_x),$$

which implies $s_x = t_x$ by injectivity of η_x. This holds for every $x \in U$ by assumption, thus $s = t$ by Lemma 2.4.3. Therefore, η_U is injective for every U (i.e. η is injective).

To see that η_U is surjective, pick $t \in \mathcal{G}(U)$. By surjectivity of η (which we have by definition since η_x is surjective for every $x \in X$), we can find an open cover $U = \bigcup_{i \in I} U_i$ along with a collection of sections $s_i \in \mathcal{F}(U_i)$ such that $\eta_{U_i}(s_i) = t|_{U_i}$. But by the previous paragraph η is injective, so s_i and s_j agree on $U_i \cap U_j$. Therefore, since $\mathcal{F}$ is a sheaf, they glue to a section $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$. By construction, $\eta_U(s)|_{U_i} = \eta_{U_i}(s_i) = t|_{U_i}$, which implies $\eta_U(s) = t$ since $\mathcal{G}$ is a sheaf. Thus η_U is surjective.

Proof of (ii) $\Rightarrow$ (iii). The first paragraph of '(ii) $\Rightarrow$ (i)' already shows that if η_x is an isomorphism for every $x \in X$, then η_U is injective for all U, i.e. η is injective. Surjectivity follows from the definition.

Proof of (iii) $\Rightarrow$ (ii). We only need to show that if η_U is injective for every U, then η_x is injective for every $x \in X$. Consider $s_x = [U, s]$ and $s'_x = [U', s']$ two germs in $\mathcal{F}_x$ such that $\eta_x(s_x) = \eta_x(s'_x)$ in $\mathcal{G}_x$. Then there is an open subset $W \subset U \cap U'$ such that $\eta_U(s)|_W = \eta_{U'}(s')|_W$. But by compatibility of η_W with restrictions, this is equivalent to the identity $\eta_W(s|_W) = \eta_W(s'|_W)$, which by our assumption implies $s|_W = s'|_W$. But then $s_x = s'_x$. □

Warning 2.4.9. It is not true that two sheaves with isomorphic stalks are isomorphic: there may be no map between them! For instance, consider a topological space X consisting of two points x_0, x_1 where only x_0 is a closed point. Thus X and $U = X \setminus \{x_0\}$ are the only nonempty open subsets of X. Fix an abelian group $G \neq 0$ and define $\mathcal{F}(X) = G = \mathcal{F}(U)$. Then choose either $\rho_{XU} = \mathrm{id}_G$ or $\rho_{XU} = 0$ to define two distinct sheaves on X. They have the same stalks but they are not isomorphic.

Exercise 2.4.10. Show that Lemma 2.4.8 fails for presheaves.

Example 2.4.11 (Surjectivity is subtle)**.** Let $\mathcal{F} = \mathscr{O}_X^{\mathrm{h}}$ be the sheaf of holomorphic functions on $X = \mathbb{C} \setminus \{0\}$, and let $\mathcal{G} = \mathcal{F}^\times$ be the sheaf of invertible holomorphic functions on X. The map $\exp\colon \mathcal{F} \to \mathcal{G}$ is surjective, but $\exp_X\colon \mathcal{F}(X) \to \mathcal{G}(X)$ is not surjective, e.g. the function $f(z) = z$ in $\mathcal{G}(X)$ is not the exponential of a homolomorphic function (cf. Example 2.2.22).

Example 2.4.12 (Skyscraper sheaf)**.** Let X be a topological space, G a nontrivial abelian group, $x \in X$ a point. The assignment

$$U \mapsto G_x(U) = \begin{cases} G & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$$

defines a sheaf of abelian groups, choosing as restriction maps the identity of G or the zero map in the obvious way. This sheaf is called the *skyscraper sheaf* attached to (X, x, G). At the level of stalks, one has

$$(G_x)_y = \begin{cases} G & \text{if } y \in \overline{\{x\}} \\ 0 & \text{if } y \notin \overline{\{x\}}, \end{cases}$$

because if y is in the closure of x then every neighbourhood of y also contains x, whereas if y is *not* in the closure of x, then $U = X \setminus \overline{\{x\}}$ is the largest open neighbourhood of y and thus $(G_x)_y = 0$ since $G_x(U) = 0$. Thus G_x has only one nonzero stalk (at x) if and only if x is a closed point. This is the case where the name 'skyscraper sheaf' for G_x fits best.

Exercise 2.4.13. Let $\mathcal{F}$ be a presheaf, $\mathcal{G}$ a *sheaf*, and let $\eta_1, \eta_2 \colon \mathcal{F} \to \mathcal{G}$ be two morphisms of presheaves of sets such that $\eta_{1,x} = \eta_{2,x}$ for every $x \in X$. Show that $\eta_1 = \eta_2$. Show that it is in fact necessary to assume $\mathcal{G}$ to be a sheaf. This exercise will be needed in **??**.

2.5 Sheafification

Fix a type of algebraic structure $(\mathscr{C}, F \colon \mathscr{C} \to \mathsf{Sets})$. Friendly translation: fix $\mathscr{C}$ to be either of the following categories:

- monoids,
- groups,
- abelian groups,
- rings,
- modules over a ring.

Let X be a topological space. Let $\mathcal{F} \colon \tau_X^{\mathrm{op}} \to \mathscr{C}$ be a presheaf. We next define a *sheaf* $\mathcal{F}^\#$, called the sheafification of $\mathcal{F}$, via an explicit universal property, and having precisely the same stalks as the initial presheaf $\mathcal{F}$.

Definition 2.5.1 (Sheafification of a presheaf)**.** Let $\mathcal{F} \in \mathsf{pSh}(X, \mathscr{C})$ be a presheaf. A *sheafification* of $\mathcal{F}$ is a pair $(\mathcal{F}^\#, \theta)$, where $\mathcal{F}^\# \in \mathsf{Sh}(X, \mathscr{C})$ is a sheaf and $\theta \colon \mathcal{F} \to \mathcal{F}^\#$ is a morphism of presheaves, such that for every other pair $(\mathcal{G}, \alpha)$ where $\mathcal{G}$ is a sheaf and $\alpha\colon \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, there exists a unique morphism of sheaves $\widetilde{\alpha}\colon \mathcal{F}^\# \to \mathcal{G}$ such that $\alpha = \widetilde{\alpha} \circ \theta$.

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "\theta"] \arrow[dr, "\forall \alpha"'] & & \mathcal{F}^\# \arrow[dl, dashed, "\exists ! \widetilde{\alpha}"] \\ & \mathcal{G} & \end{tikzcd}$$

PROPOSITION 2.5.2. *Let $\mathcal{F} \in \mathsf{pSh}(X, \mathscr{C})$ be a presheaf. Then a sheafification $(\mathcal{F}^\#, \theta)$ exists, and the map $\theta_x : \mathcal{F}_x \to \mathcal{F}^\#_x$ is an isomorphism for every $x \in X$.*

What follows immediately from Proposition 2.5.2 is that $\mathcal{F}^\#$ is unique up to a unique isomorphism, and moreover the canonical map $\theta : \mathcal{F} \to \mathcal{F}^\#$ is an isomorphism precisely when $\mathcal{F}$ is already a sheaf.

Proof. Let $U \subset X$ be an open subset. Define

$$\mathcal{F}^\#(U) = \left\{ \text{functions } U \xrightarrow{f} \coprod_{x \in U} \mathcal{F}_x \;\middle|\; \begin{array}{c} \text{for every } x \in U,\ f(x) \in \mathcal{F}_x \text{ and there exist an} \\ \text{open neighbourhood } V \subset U \text{ of } x \text{ and } s \in \mathcal{F}(V) \\ \text{such that } f(y) = s_y \text{ for every } y \in V \end{array} \right\}.$$

Note that, since $\mathscr{C}$ has products, we can view a function f as above as a tuple

$$(f(x))_{x \in U} \in \prod_{x \in U} \mathcal{F}_x$$

and we can rephrase the definition of $\mathcal{F}^\#(U)$ by saying that

$$\mathcal{F}^\#(U) = \{ \text{tuples } (s_x)_{x \in U} \text{ of compatible germs} \}.$$

See Definition 2.4.2 for the definition of compatible germs. Functoriality of the assignment $U \mapsto \mathcal{F}^\#(U)$ is clear (functions restrict!), thus $\mathcal{F}^\#$ is a presheaf. The morphism $\theta_U : \mathcal{F}(U) \to \mathcal{F}^\#(U)$ defined by sending $s \in \mathcal{F}(U)$ to the function

$$f_s : U \to \coprod_{x \in U} \mathcal{F}_x, \quad x \mapsto s_x = [U, s] \in \mathcal{F}_x$$

determines a morphism of presheaves, being compatible with restrictions. It is just the function $\sigma^{\mathcal{F}}_U$ introduced in (2.4.1)!

The presheaf $\mathcal{F}^\#$ is a sheaf: Fix an open cover $U = \bigcup_{i \in I} U_i$ of some open subset $U \subset X$ and a collection of sections $f_i \in \mathcal{F}^\#(U_i)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for every i and j. We need to find a unique $f \in \mathcal{F}^\#(U)$ such that $f|_{U_i} = f_i$. Define

$$f \in \prod_{x \in U} \mathcal{F}_x = \mathrm{Hom}\left(U, \coprod_{x \in U} \mathcal{F}_x \right)$$

by the rule

$$f(x) = f_i(x) \in \mathcal{F}_x, \quad x \in U_i \subset U.$$

This is well-defined since, even though x can lie in more than one open U_i, by assumption we have $f_i(x) = f_j(x)$ as soon as $x \in U_i \cap U_j$. We need to check that f defines an element of the subset $\mathcal{F}^\#(U) \subset \prod_{x\in U} \mathcal{F}_x$. But for every $i \in I$ we know the following: for every $x \in U_i$ there exist an open neighbourhood $x \in V_i \subset U_i$ and a section $s_i \in \mathcal{F}(V_i)$ such that $f(y) = f_i(y) = (s_i)_y$ for all $y \in V_i$. But V_i is also open in U, so the condition defining $\mathcal{F}^\#(U)$ also holds for f. Thus $f \in \mathcal{F}^\#(U)$ satisfies $f|_{U_i} = f_i$, and is clearly unique with this property.

The pair $(\mathcal{F}^\#, \theta)$ is the sheafification. Assume we have a sheaf $\mathcal{G}$ and a morphism of presheaves $\alpha\colon \mathcal{F} \to \mathcal{G}$. We need to define a morphism $\widetilde{\alpha}\colon \mathcal{F}^\# \to \mathcal{G}$ of presheaves such that $\alpha = \widetilde{\alpha} \circ \theta$. For every U open in X, we need to define a morphism $\widetilde{\alpha}_U\colon \mathcal{F}^\#(U) \to \mathcal{G}(U)$ in such a way that $\alpha_U = \widetilde{\alpha}_U \circ \theta_U$. Fix $s = (s_x)_{x\in U} \in \mathcal{F}^\#(U)$. The composition

$$\begin{tikzcd}
U \arrow[r, "s"] \arrow[dr, "\widetilde{\alpha}_U(s)"'] & \coprod_{x\in U} \mathcal{F}_x \arrow[d, "\coprod_{x\in U} \alpha_x"] \\
& \coprod_{x\in U} \mathcal{G}_x
\end{tikzcd}$$

defines a tuple of compatible germs for $\mathcal{G}$ over U, hence an element $\widetilde{\alpha}_U(s) \in \mathcal{G}^\#(U) = \mathcal{G}(U)$, using that $\mathcal{G}$ is a sheaf for this identity. This is the required morphism $\widetilde{\alpha}\colon \mathcal{F}^\# \to \mathcal{G}$.

The map θ is an isomorphism on stalks. The map θ, at the level of stalks, is defined by

$$\theta_x[U, s] = [U, f_s].$$

Injectivity: Suppose $\theta_x[U, s] = \theta_x[V, t]$ for two classes $[U, s], [V, t] \in \mathcal{F}_x$, i.e. assume $[U, f_s] = [V, f_t]$ in $\mathcal{F}^\#_x$. Then, by definition of germ, there exists an open neighbourhood $W \subset U \cap V$ of x such that $f_s|_W = f_t|_W$. But this means, by definition of f_s and f_t, that $s_y = t_y$ for all $y \in W$. Thus, in particular, $s_x = t_x$. But this is just the equality $[U, s] = [V, t]$ we were after.

Surjectivity: Pick a class $[U, f] \in \mathcal{F}^\#_x$ for some $f \in \mathcal{F}^\#(U)$ and open neighbourhood U of x. Then, for every $z \in U$, there exist an open neighbourhood $V \subset U$ of z and a section $s \in \mathcal{F}(V)$ such that $f(y) = s_y$ in $\mathcal{F}_y$ for every $y \in V$. We claim that $[U, f] = \theta_x(s_x)$, where $s_x = [V, s]$. Indeed, $\theta_x(s_x) \in \mathcal{F}^\#_x$ is the equivalence class of the map

$$f_s\colon V \to \coprod_{y\in V} \mathcal{F}_y, \quad y \mapsto s_y.$$

But this map agrees with the restriction of f to $V \subset U$ (by the condition $f(y) = s_y$ recalled above), i.e. $f_s = f|_V \in \mathcal{F}^\#(V)$. Since V is also an open neighbourhood of x, it follows that $(f|_V)_x = (f_s)_x = [V, f_s] = \theta_x(s_x) \in \mathcal{F}^\#_x$, but of course $(f|_V)_x = [U, f]$. Thus θ_x is surjective. $\square$

Example 2.5.3 (Constant sheaf)**.** Let G be a nontrivial abelian group. The *constant sheaf* on a topological space X, with values in G, is the sheafification $\underline{G}_X$ of the presheaf $\underline{G}_X^{\text{pre}}$ defined in Example 2.2.18. This sheaf agrees with the sheaf whose sections over U are the locally constant functions $U \to G$. This, in turn, agrees with the following: endow G with the discrete topology and consider the assignment

$$U \mapsto \{\,\text{continuous maps } U \to G\,\},$$

which we know is a sheaf by Example 2.2.15. If $U \subset X$ is a connected open subset, then $\underline{G}_X(U) = G$. By Proposition 2.5.2, at the level of stalks we have $\underline{G}_{X,x} = G$ for every $x \in X$, since the stalks of the constant presheaf are manifestly all equal to G.

Exercise 2.5.4. Let X be a connected topological space, x a point, G a nontrivial abelian group. Under what condition(s) is the constant sheaf $\underline{G}_X$ equal to the skyscraper sheaf G_x (cf. Example 2.4.12)?

Exercise 2.5.5. Show that sending $\mathcal{F} \mapsto \mathcal{F}^\#$ defines a functor $(-)^\# \colon \mathsf{pSh}(X,\mathscr{C}) \to \mathsf{Sh}(X,\mathscr{C})$, and that the forgetful functor $j_{X,\mathscr{C}} \colon \mathsf{Sh}(X,\mathscr{C}) \hookrightarrow \mathsf{pSh}(X,\mathscr{C})$ is a right adjoint. This means (cf. Definition A.1.17) that are bifunctorial bijections

$$\psi_{\mathcal{F},\mathcal{G}} \colon \operatorname{Hom}_{\mathsf{Sh}(X,\mathscr{C})}(\mathcal{F}^\#,\mathcal{G}) \xrightarrow{\sim} \operatorname{Hom}_{\mathsf{pSh}(X,\mathscr{C})}(\mathcal{F},\mathcal{G}), \quad \widetilde{\alpha} \mapsto \widetilde{\alpha} \circ \theta$$

for any presheaf $\mathcal{F}$ and sheaf $\mathcal{G}$. (**Hint**: the universal property of the sheafification!).

2.5.1 Subsheaves, Quotient sheaves

We have essentially already proved the following general result.

PROPOSITION 2.5.6 ([12, Tag 007S]). *Let X be a topological space. Let $\mathcal{F},\mathcal{G} \in \mathsf{Sh}(X,\mathsf{Sets})$ be sheaves of sets, $\eta \colon \mathcal{F} \to \mathcal{G}$ a morphism. Then, the following are equivalent:*

(a) *η is a monomorphism,*

(b) *$\eta_x \colon \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$,*

(c) *$\eta_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all open subsets $U \subset X$ (i.e. η is injective).*

Furthermore, the following are equivalent:

(i) *η is an epimorphism,*

(ii) *$\eta_x \colon \mathcal{F}_x \to \mathcal{G}_x$ is surjective for all $x \in X$ (i.e. η is surjective),*

and are implied (but not equivalent to, cf. Example 2.4.11!) by the condition

(iii) *$\eta_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective for all open subsets $U \subset X$.*

If $\mathscr{C}$ is an abelian category (e.g. Mod_A for a fixed ring A), then Proposition 2.5.6 holds replacing Sets with $\mathscr{C}$.

Definition 2.5.7 (Subsheaf, quotient sheaf)**.** If there exists a morphism of sheaves $\eta\colon \mathcal{F} \to \mathcal{G}$ such that either of the equivalent conditions (a), (b) or (c) holds, we say that $\mathcal{F}$ is a *subsheaf* of $\mathcal{G}$ (and we may denote this by '$\mathcal{F} \subset \mathcal{G}$'). If either of the equivalent conditions (i) or (ii) holds, we say that $\mathcal{G}$ is a *quotient sheaf* of $\mathcal{F}$.

Example 2.5.8 (Quotient by a subsheaf)**.** Let $\mathscr{C}$ be an abelian category. If $\mathcal{F} \subset \mathcal{G}$ is a subsheaf (with values in $\mathscr{C}$), then sending

$$U \mapsto \mathcal{G}(U)/\mathcal{F}(U) \tag{2.5.1}$$

is a presheaf on X, because the restriction maps respect the inclusions $\mathcal{F}(U) \hookrightarrow \mathcal{G}(U)$, and thus pass to the quotients. Its sheafification $\mathcal{G}/\mathcal{F}$ is called the *quotient sheaf of* $\mathcal{G}$ *by* $\mathcal{F}$. There is a natural morphism of sheaves $\mathcal{G} \to \mathcal{G}/\mathcal{F}$.

Definition 2.5.9 (Sheaf image, sheaf cokernel)**.** Let $\mathscr{C}$ be an abelian category, $\eta\colon \mathcal{F} \to \mathcal{G}$ a morphism of sheaves (with values in $\mathscr{C}$), so that $\ker(\eta) \hookrightarrow \mathcal{F}$ is a subsheaf by Exercise 2.2.21. The sheafification $\operatorname{im}(\eta)$ of the presheaf

$$U \mapsto \operatorname{im}_{\mathrm{pre}}(U) = \operatorname{im}(\eta_U) = \mathcal{F}(U)/\ker(\eta_U)$$

is called the *image of* η. It is a special case of Example 2.5.8 and defines a subsheaf

$$\operatorname{im}(\eta) = \mathcal{F}/\ker(\eta) \subset \mathcal{G}.$$

The quotient sheaf

$$\operatorname{coker}(\eta) = \mathcal{G}/\operatorname{im}(\eta),$$

again a special case of Example 2.5.8, is called the *sheaf cokernel*.

Exercise 2.5.10. Let $\mathscr{C}$ be an abelian category. Let $\eta\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves with values in $\mathscr{C}$. Show that the composition

$$\mathcal{G} \to \operatorname{coker}_{\mathrm{pre}} \eta \to \operatorname{coker}(\eta),$$

where the first morphism is given by the natural maps $\mathcal{G}(U) \twoheadrightarrow \mathcal{G}(U)/\operatorname{im}(\eta_U)$ and the last morphism is the sheafification, is a cokernel in the category $\mathsf{Sh}(X, \mathscr{C})$.

Remark 2.5.11. Set $\mathscr{C} = \mathsf{Mod}_A$ (or any Grothendieck abelian category so that, by definition, filtered colimits exist and are exact). Let $\mathcal{F} \subset \mathcal{G}$ be a subsheaf, $x \in X$ a point. Then

$$(\mathcal{G}/\mathcal{F})_x = \mathcal{G}_x/\mathcal{F}_x \tag{2.5.2}$$

in Mod_A. This follows from the fact that $(\mathcal{G}/\mathcal{F})_x$ agrees with the stalk of the *presheaf* (2.5.1), and from right exactness of filtered colimits. Moreover, if $\eta\colon \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves and $x \in X$ is a point, then

$$\begin{aligned} \ker(\eta)_x &= \ker(\eta_x) \\ \operatorname{im}(\eta)_x &= \operatorname{im}(\eta_x) \\ \operatorname{coker}(\eta)_x &= \operatorname{coker}(\eta_x). \end{aligned} \tag{2.5.3}$$

The first identity in (2.5.3) follows from the fact that filtered colimits are *also left exact* in Mod_A, thus

$$\begin{aligned} \ker\left(\mathcal{F}_x \xrightarrow{\eta_x} \mathcal{G}_x\right) &= \ker\left(\varinjlim_{U \ni x} \mathcal{F}(U) \to \varinjlim_{U \ni x} \mathcal{G}(U)\right) \\ &= \varinjlim_{U \ni x} \ker(\mathcal{F}(U) \to \mathcal{G}(U)) \\ &= \ker(\eta)_x. \end{aligned}$$

The last two identities in (2.5.3) are a special case of (2.5.2).

THEOREM 2.5.12 ([5, §10]). *If $\mathscr{C}$ is a Grothendieck abelian category, then $\mathsf{Sh}(X,\mathscr{C})$ is a Grothendieck abelian category.*

Definition 2.5.13. A *short exact sequence of sheaves* with values in a Grothendieck abelian category $\mathscr{C}$ is a short exact sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{\iota} \mathcal{G} \xrightarrow{\pi} \mathcal{H} \longrightarrow 0$$

of objects in the abelian category $\mathsf{Sh}(X,\mathscr{C})$. Explicitly, exactness means that ι is injective, π is surjective and $\operatorname{im}(\iota) = \ker(\pi)$.

PROPOSITION 2.5.14. *Let $\mathscr{C}$ be a Grothendieck abelian category. A sequence*

$$0 \longrightarrow \mathcal{F} \xrightarrow{\iota} \mathcal{G} \xrightarrow{\pi} \mathcal{H} \longrightarrow 0$$

of objects in $\mathsf{Sh}(X,\mathscr{C})$ is a short exact sequence if and only if

$$0 \longrightarrow \mathcal{F}_x \xrightarrow{\iota_x} \mathcal{G}_x \xrightarrow{\pi_x} \mathcal{H}_x \longrightarrow 0$$

is a short exact sequence in $\mathscr{C}$ for every $x \in X$.

Proof. Combine Remark 2.5.11 and Lemma 2.4.8 with one another. □

Exercise 2.5.15. Let $\eta\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of A-modules, for A a ring. Prove that there is an exact sequence of sheaves

$$0 \longrightarrow \ker(\eta) \longrightarrow \mathcal{F} \xrightarrow{\eta} \mathcal{G} \longrightarrow \operatorname{coker}(\eta) \longrightarrow 0.$$

In particular, if η is injective, this reduces to

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{F} \longrightarrow 0.$$

Exercise 2.5.16. Let A be a ring. For a nonempty open subset U of a topological space X, consider the functor $\Gamma(U,-)\colon \mathsf{Sh}(X,\mathsf{Mod}_A) \to \mathsf{Mod}_A$ sending $\mathcal{F} \mapsto \mathcal{F}(U)$. Show that it is left exact. That is, it transforms an exact sequence of sheaves $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ into an exact sequence of A-modules

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U).$$

When $U = X$, this functor takes $\mathcal{F} \mapsto \mathcal{F}(X)$ and is thus called the *global section functor*. Another notation used for it in the literature is $\mathrm{H}^0(X,-)$, cf. Terminology 2.2.3.

2.6 Supports

Let A be a ring. Let $\mathcal{F} \in \mathsf{Sh}(X,\mathsf{Mod}_A)$ be a sheaf of A-modules on a topological space X. Let $U \subset X$ be an open subset, and fix a section $s \in \mathcal{F}(U)$. We have two notions of support: the support of $\mathcal{F}$, and the support of s, defined respectively as

$$\begin{aligned} \operatorname{Supp}(\mathcal{F}) &= \{\, x \in X \mid \mathcal{F}_x \neq 0 \,\}, \\ \operatorname{Supp}(s) &= \{\, x \in U \mid s_x \neq 0 \text{ in } \mathcal{F}_x \,\}. \end{aligned} \tag{2.6.1}$$

If $s_x = 0$, then there is an open neighbourhood $x \in V \subset U$ such that $s|_V = 0 \in \mathcal{F}(V)$. Thus $V \subset U \setminus \operatorname{Supp}(s)$ and hence $\operatorname{Supp}(s) \subset U$ is closed. In fact, this follows from (or solves) Exercise 2.4.4. In general, however, $\operatorname{Supp}(\mathcal{F}) \subset X$ is *not* closed, as the two next examples show.

Example 2.6.1 (Supp($\mathcal{F}$) need not be closed, take I)**.** Let X be an irreducible topological space. This means that any two nonempty open subsets of X intersect. Fix a nontrivial abelian group G, a point $x \in X$, and for $U \in \tau_X$ define

$$\mathcal{F}(U) = \begin{cases} 0 & \text{if } U = \emptyset \text{ or } x \in U \\ G & \text{otherwise.} \end{cases}$$

Let $\rho_{UV} \in \{\mathrm{id}_G, 0\}$ be chosen in the obvious way for all $U, V \in \tau_X$. Then $\mathcal{F}$ is a sheaf of abelian groups on X, with stalks

$$\mathcal{F}_y = \begin{cases} 0 & \text{if } y \in \overline{\{\, x \,\}} \\ G & \text{otherwise.} \end{cases}$$

It follows that

$$\operatorname{Supp}(\mathcal{F}) = X \setminus \overline{\{\, x \,\}},$$

which is not closed in X as soon as $\overline{\{\, x \,\}} \hookrightarrow X$ is not open.

Example 2.6.2 (Supp($\mathcal{F}$) need not be closed, take II)**.** Let $j: U \hookrightarrow X$ be the inclusion of an open subset U of a topological space X. Let $\mathcal{F} \in \mathsf{Sh}(U, \mathscr{C})$ be a sheaf. Define $j_! \mathcal{F} \in \mathsf{Sh}(X, \mathscr{C})$ to be the sheafification of the presheaf $j_!^{\mathrm{pre}} \mathcal{F} \in \mathsf{pSh}(X, \mathscr{C})$ defined by

$$j_!^{\mathrm{pre}} \mathcal{F}(V) = \begin{cases} \mathcal{F}(V) & \text{if } V \subset U \\ 0 & \text{otherwise.} \end{cases}$$

One has

$$(j_! \mathcal{F})_x = \begin{cases} \mathcal{F}_x & \text{if } x \in U \\ 0 & \text{otherwise,} \end{cases}$$

so that $\mathrm{Supp}(j_! \mathcal{F}) = \mathrm{Supp}(\mathcal{F})$. Let now $\mathscr{C} = \mathsf{Ab} = \mathsf{Mod}_{\mathbb{Z}}$ be the category of abelian groups. Fix $G \neq 0$ in $\mathscr{C}$ and consider the constant sheaf on U (cf. Example 2.5.3). We have $\mathrm{Supp}(j_! \underline{G}_U) = \mathrm{Supp}(\underline{G}_U) = U$. In particular, $\mathrm{Supp}(j_! \underline{G}_U) \subset X$ is not closed as soon as U is not closed in X.

If $\mathcal{F}$ is a sheaf of rings, the notions of support defined in (2.6.1) still make sense, and one has $\mathrm{Supp}(\mathcal{F}) = \mathrm{Supp}(1)$, where $1 \in \mathcal{F}(X)$ is the ring identity (recall that the '0 ring' is the one where $1 = 0$). Thus $\mathrm{Supp}(\mathcal{F})$ *is* in fact closed in this case.

2.7 Sheaves = sheaves on a base

Fix a type of algebraic structure $(\mathscr{C}, F: \mathscr{C} \to \mathsf{Sets})$.

Definition 2.7.1 (Base of open sets)**.** Let X be a topological space. A *base of open sets* for X is a collection of open subsets $\mathcal{B} \subset \tau_X$ satisfying the following requirements:

(a) $\mathcal{B}$ is stable under finite intersections,

(b) every $U \in \tau_X$ can be written as a union of open sets belonging to $\mathcal{B}$.

Definition 2.7.2 ($\mathcal{B}$-sheaf)**.** A $\mathcal{B}$-*presheaf* (resp. $\mathcal{B}$-*sheaf*) is an assignment

$$U \mapsto \mathcal{F}(U) \in \mathscr{C}, \text{ for each } U \in \mathcal{B},$$

such that the presheaf conditions (1)–(2) of Definition 2.2.1 (resp. the presheaf conditions (1)–(2) of Definition 2.2.1 and the sheaf conditions (3)–(4) of Important Definition 2.2.1) hold, considering only open sets belonging to $\mathcal{B}$.

Notation 2.7.3. We shall use the notation $\{\mathcal{F}(\mathcal{B}), \rho_{\mathcal{B}}\}$ to denote a $\mathcal{B}$-(pre)sheaf.

Note that restriction maps

$$\rho_{UV}: \mathcal{F}(U) \to \mathcal{F}(V)$$

are part of the data of a $\mathcal{B}$-(pre)sheaf whenever $V \subset U$ is an inclusion of open sets both belonging to $\mathcal{B}$. Note, also, that condition (a) in Definition 2.7.1 ensures that open subsets of the form $U \cap V$ belong to $\mathcal{B}$ for all $U, V \in \mathcal{B}$. In particular, as in Section 2.3, a $\mathcal{B}$-presheaf $\{\mathcal{F}(\mathcal{B}), \rho_{\mathcal{B}}\}$ is a sheaf precisely when the following condition is fulfilled: for every open subset $U \in \mathcal{B}$ and for any open cover $U = \bigcup_{i\in I} U_i$ with all $U_i \in \mathcal{B}$, the diagram

$$\mathcal{F}(U) \xrightarrow{\rho} \prod_{i\in I} \mathcal{F}(U_i) \overset{\mu}{\underset{\nu}{\rightrightarrows}} \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \cap U_j)$$

is an equaliser diagram in $\mathscr{C}$.

Remark 2.7.4. Let $x \in X$ be a point. The collection of open neighbourhoods

$$\mathcal{B}_x = \{\, U \in \mathcal{B} \mid x \in U \,\}^{\mathrm{op}} \subset \tau_X^{\mathrm{op}}$$

is a fundamental system of open neighbourhoods of x, also called a local basis at x (i.e. for any $W \in \mathsf{Ngb}_x$ there exists $U \in \mathcal{B}_x$ such that $U \subset W$). In more technical terms, one may say that the filtered categories Ngb_x and $\mathcal{B}_x$ are *cofinal*, i.e. the inclusion $\mathcal{B}_x \hookrightarrow \mathsf{Ngb}_x$ is a cofinal functor. We will not use this terminology.

By Remark 2.7.4, the stalk

$$\mathcal{F}_x = \varinjlim_{\mathcal{B}_x} \mathcal{F}\big|_{\mathcal{B}_x} = \varinjlim_{U\in\mathcal{B}_x} \mathcal{F}(U) \in \mathscr{C}$$

of a $\mathcal{B}$-(pre)sheaf $\{\mathcal{F}(\mathcal{B}), \rho_{\mathcal{B}}\}$ at a point $x \in X$ is well-defined as an object of $\mathscr{C}$. It receives, by definition of direct limit, canonical morphisms

$$\mathcal{F}(U) \to \mathcal{F}_x, \quad U \in \mathcal{B}_x.$$

We denote by $s_x \in \mathcal{F}_x$, as ever, the image of $s \in \mathcal{F}(U)$ under this morphism.

Moreover, if $U \in \mathcal{B}$ and $\{\mathcal{F}(\mathcal{B}), \rho_{\mathcal{B}}\}$ is a $\mathcal{B}$-sheaf, the natural map

$$\begin{aligned} \mathcal{F}(U) &\xrightarrow{\sigma_U^{\mathcal{F}}} \prod_{x\in U} \mathcal{F}_x \\ s &\longmapsto (s_x)_{x\in U} \end{aligned}$$

is injective (as in Lemma 2.4.1), and its image agrees with the collections of compatible germs; to be more precise, we should now call them '$\mathcal{B}$-compatible', for they are, by definition, those tuples

$$(s_x)_{x\in U} \in \prod_{x\in U} \mathcal{F}_x$$

such that for every $x \in U$ there is a pair (V_x, t^x), where $V_x \in \mathcal{B}_x$ satisfies $V_x \subset U$ and $t^x \in \mathcal{F}(V_x)$ satisfies $t^x_y = s_y$ for every $y \in V_x$.

Definition 2.7.5 (Morphism of $\mathcal{B}$-sheaves)**.** A *morphism* of $\mathcal{B}$-(pre)sheaves

$$\eta_{\mathcal{B}}\colon \left\{\mathcal{F}(\mathcal{B}), \rho_{\mathcal{B}}^{\mathcal{F}}\right\} \longrightarrow \left\{\mathcal{G}(\mathcal{B}), \rho_{\mathcal{B}}^{\mathcal{G}}\right\} \tag{2.7.1}$$

is the datum of a collection of maps $\eta_U : \mathcal{F}(U) \to \mathcal{G}(U)$, one for each $U \in \mathcal{B}$, such that Diagram (2.2.1) commutes for all $U, V \in \mathcal{B}$ such that $V \subset U$.

With this definition, $\mathcal{B}$-sheaves form a category, denoted $\mathsf{Sh}_{\mathcal{B}}(X, \mathscr{C})$.[1]

Remark 2.7.6. Let X be a topological space, $\mathcal{B}$ a base of open subsets of X. A (pre)sheaf $\mathcal{F}$ on X is a $\mathcal{B}$-(pre)sheaf in a natural way. More precisely, there is (say, at the level of sheaves) a *restriction functor*

$$\mathsf{res}_{\mathcal{B}}(X, \mathscr{C})\colon \mathsf{Sh}(X, \mathscr{C}) \longrightarrow \mathsf{Sh}_{\mathcal{B}}(X, \mathscr{C}), \tag{2.7.2}$$

defined on objects in the obvious way. Its actual functoriality is just a consequence of the definition of morphism of $\mathcal{B}$-sheaves, and is an easy routine check.

LEMMA 2.7.7. *A $\mathcal{B}$-sheaf $\{\mathcal{F}(\mathcal{B}), \rho_{\mathcal{B}}\}$ uniquely extends to a sheaf $\overline{\mathcal{F}}$, such that $\overline{\mathcal{F}}(U) = \mathcal{F}(U)$ for all $U \in \mathcal{B}$.*

Proof. Let $U \in \tau_X$ be an arbitrary open set. Define

$$\overline{\mathcal{F}}(U) = \left\{\text{tuples } (s_x)_{x\in U} \text{ of } \mathcal{B}\text{-compatible germs}\right\} \subset \prod_{x\in U} \mathcal{F}_x.$$

This is manifestly a presheaf. It is also clear that the above definition agrees with $\mathcal{F}(U)$ whenever $U \in \mathcal{B}$, since the injective map $\sigma_U^{\mathcal{F}}$ hits precisely the tuples of $\mathcal{B}$-compatible germs; moreover, for the same reason, this definition is the *only* possible extension of the original $\mathcal{B}$-sheaf. The sheaf property is fulfilled by $\overline{\mathcal{F}}$ precisely for the same reason why it is fulfilled by the sheafification of a presheaf (see the proof of Proposition 2.5.2). $\square$

In fact, the statement of Lemma 2.7.7 can be made functorial: one can prove that the restriction functor (2.7.2) is an equivalence. The inverse is given precisely by Lemma 2.7.7 above at the level of objects and by Proposition 2.7.9 below for morphisms.

Remark 2.7.8. We have that $\mathcal{F}_x = \overline{\mathcal{F}}_x$ for all $x \in X$. This follows directly from Remark 2.7.4.

The analogue of Lemma 2.7.7 for *morphisms* is the following.

PROPOSITION 2.7.9. *Let X be a topological space, $\mathcal{B} \subset \tau_X$ a base of open sets and $\mathcal{F}, \mathcal{G}$ two sheaves on X. Suppose given a morphism*

$$\eta_{\mathcal{B}}\colon \mathsf{res}_{\mathcal{B}}(X, \mathscr{C})(\mathcal{F}) \to \mathsf{res}_{\mathcal{B}}(X, \mathscr{C})(\mathcal{G})$$

[1] Also $\mathcal{B}$-presheaves form a category, but it is not as well-behaved as $\mathsf{Sh}_{\mathcal{B}}(X, \mathscr{C})$, and we do not need it, so we shall ignore it.

between the underlying $\mathcal{B}$-sheaves. Then $\eta_{\mathcal{B}}$ extends uniquely to a sheaf homomorphism $\eta\colon \mathcal{F} \to \mathcal{G}$. Furthermore, if η_U is surjective (or injective, or an isomorphism) for every $U \in \mathcal{B}$, then so is η.

Exercise 2.7.10. Prove Proposition 2.7.9 and deduce that the restriction functor (2.7.2) is an equivalence.

2.8 Pushforward, inverse image

In this section we learn how to "move" sheaves from a topological space X to another topological space Y, in the presence of a continuous map between the two spaces.

$$\begin{tikzcd} \mathcal{F} \arrow[r, rightsquigarrow] \arrow[d, dash] & f_*\mathcal{F} \arrow[d, dash] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad\qquad \begin{tikzcd} f^{-1}\mathcal{G} \arrow[d, dash] & \mathcal{G} \arrow[l, leftsquigarrow] \arrow[d, dash] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

2.8.1 Pushforward (or direct image)

Let $f\colon X \to Y$ be a continuous map of topological spaces, and let $\mathcal{F}$ be a presheaf on X. The assignment

$$V \mapsto f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}V)$$

defines a presheaf $f_*\mathcal{F}$ on Y, called the *pushforward* (or *direct image*) of $\mathcal{F}$ by f. It is a sheaf as soon as $\mathcal{F}$ is, because if $V = \bigcup_{i\in I} V_i$ is an open covering of an open subset $V \subset Y$, then $f^{-1}V = \bigcup_{i\in I} f^{-1}(V_i)$ is an open covering of $f^{-1}V \subset X$.

Example 2.8.1. If X is arbitrary and $Y = \mathsf{pt}$, then $f_*\mathcal{F}(\mathsf{pt}) = \mathcal{F}(X)$, an object of $\mathscr{C}$. We will see in a minute that the direct image along any continuous map defines a functor. The direct image along the constant map $(X \to \mathsf{pt})_*\colon \mathsf{Sh}(X, \mathscr{C}) \to \mathscr{C}$ is also called the *global section functor*. If $\mathscr{C} = \mathsf{Mod}_A$, it is a left exact functor (you already proved a more general statement in Exercise 2.5.16).

Example 2.8.2. If $f\colon X \hookrightarrow Y$ is the inclusion of a subspace, then $f_*\mathcal{F}$ is defined, for any open subset $V \subset Y$, by

$$f_*\mathcal{F}(V) = \mathcal{F}(V \cap X).$$

Example 2.8.3 (Skyscraper sheaf as a pushforward)**.** Let $x \in X$ be a point, G a nontrivial abelian group. Consider the constant sheaf $\underline{G}_{\{x\}}$ on $\{x\}$. Let $i_x\colon \{x\} \hookrightarrow X$ be the inclusion. Then the skyscraper sheaf $G_x \in \mathsf{Sh}(X, \mathsf{Mod}_{\mathbb{Z}})$ defined in Example 2.4.12 can be described as

$$G_x = i_{x,*}\underline{G}_{\{x\}}.$$

Next, we observe that pushforward of sheaves is functorial, i.e. sending $\mathcal{F} \mapsto f_*\mathcal{F}$ defines functors

$$\begin{tikzcd} \mathsf{Sh}(X,\mathscr{C}) \arrow[r, "f_*"] \arrow[d, hook] & \mathsf{Sh}(Y,\mathscr{C}) \arrow[d, hook] \\ \mathsf{pSh}(X,\mathscr{C}) \arrow[r, "f_*"] & \mathsf{pSh}(Y,\mathscr{C}) \end{tikzcd}$$

where the vertical maps are the natural inclusions (2.2.2). Indeed, given a morphism of (pre)sheaves $\eta\colon \mathcal{F} \to \mathcal{G}$, we can construct a morphism of (pre)sheaves

$$f_*\eta\colon f_*\mathcal{F} \to f_*\mathcal{G}$$

simply by setting

$$(f_*\eta)_V = \eta_{f^{-1}V} \colon \mathcal{F}(f^{-1}V) \to \mathcal{G}(f^{-1}V)$$

for an open subset $V \subset Y$. The compatibility with restriction maps follows from those of η (and the obvious observation that if $V' \subset V$ then $f^{-1}V' \subset f^{-1}V$).

Moreover, $(-)_*$ is compatible with compositions of continuous maps, in the following sense: if $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous maps of topological spaces, then, as functors, we have an equality $(g \circ f)_* = g_* \circ f_*$ on the nose (both for presheaves and for sheaves). In other words, the diagram

(2.8.1)
$$\begin{tikzcd} \mathsf{Sh}(X,\mathscr{C}) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \mathsf{Sh}(Y,\mathscr{C}) \arrow[d, "g_*"] \\ & \mathsf{Sh}(Z,\mathscr{C}) \end{tikzcd}$$

commutes. Indeed, if $\mathcal{F}$ is a (pre)sheaf on X, then for every open $W \subset Z$ one has

$$\begin{aligned} (g \circ f)_*\mathcal{F}(W) &= \mathcal{F}((g \circ f)^{-1}(W)) \\ &= \mathcal{F}(f^{-1}g^{-1}(W)) \\ &= f_*\mathcal{F}(g^{-1}(W)) \\ &= (g_*f_*\mathcal{F})(W) \\ &= (g_* \circ f_*)\mathcal{F}(W). \end{aligned}$$

Note that no identifications are made here: all equalities are actual equalities!

LEMMA 2.8.4. *Let $f\colon X \to Y$ be a continuous map of topological spaces, and fix a sheaf $\mathcal{F} \in \mathsf{Sh}(X,\mathscr{C})$. Let $x \in X$ be a point, and set $y = f(x)$. There is a canonical morphism*

$$(f_*\mathcal{F})_y \longrightarrow \mathcal{F}_x,$$

which is an isomorphism when f is the inclusion of a subspace $X \hookrightarrow Y$.

Proof. If $y \in V' \subset V \subset Y$, then $x \in f^{-1}V' \subset f^{-1}V \subset X$, and the commutative diagram

$$\begin{tikzcd}
f_*\mathcal{F}(V) \arrow[rr] \arrow[d, equal] \arrow[dr] & & f_*\mathcal{F}(V') \arrow[d, equal] \arrow[dl] \\
\mathcal{F}(f^{-1}V) \arrow[dr] & (f_*\mathcal{F})_y \arrow[d, dotted] & \mathcal{F}(f^{-1}V') \arrow[dl] \\
& \mathcal{F}_x &
\end{tikzcd}$$

induces, via the universal property of the stalk (cf. Definition B.3.6)

$$(f_*\mathcal{F})_y = \varinjlim_{V \ni y} \mathcal{F}(f^{-1}V),$$

a canonical morphism $(f_*\mathcal{F})_y \to \mathcal{F}_x$, as required.

Now, let us assume $f : X \hookrightarrow Y$ is the inclusion of a subspace, and let us take $y \in X$. Note that every neighbourhood $y \in U \subset X$ is of the form $U = V \cap X$ for some open neighbourhood $y \in V \subset Y$. Thus

$$(f_*\mathcal{F})_y = \varinjlim_{Y \supset V \ni y} \mathcal{F}(V \cap X) \xrightarrow{\sim} \varinjlim_{X \supset U \ni y} \mathcal{F}(U) = \mathcal{F}_y. \tag{2.8.2}$$

The proof is complete. □

Remark 2.8.5. We shall use Lemma 2.8.4 crucially with $\mathscr{C}$ = Rings, when defining morphisms of locally ringed spaces (cf. **??**).

Caution 2.8.6. Even if $f : X \hookrightarrow Y$ is the inclusion of a subspace, it is not true that $(f_*\mathcal{F})_y = 0$ for all $y \in Y \setminus X$. This is nevertheless true when f is the inclusion of a *closed* subspace, cf. Remark 2.8.7.

Remark 2.8.7. If $f : X \hookrightarrow Y$ is the inclusion of a *closed* subspace, and $\mathcal{F}$ is a sheaf on X, then

$$(f_*\mathcal{F})_y = \begin{cases} \mathcal{F}_y & \text{if } y \in X \\ 0 & \text{if } y \notin X. \end{cases} \tag{2.8.3}$$

The case $y \in X$ is the computation (2.8.2). As for the case $y \notin X$, we use the definition

$$(f_*\mathcal{F})_y = \varinjlim_{Y \supset V \ni y} f_*\mathcal{F}(V) = \varinjlim_{Y \supset V \ni y} \mathcal{F}(V \cap X),$$

and the observation that, since X is closed in Y, there are arbitrarily small neighbourhoods V of y which are disjoint from X. For these, we have $\mathcal{F}(V \cap X) = \mathcal{F}(\emptyset) = 0$ since $\mathcal{F}$ is a sheaf (Remark 2.3.2). This causes the colimit to vanish.

Exactness of pushforward

We set $\mathscr{C} = \mathsf{Mod}_A$ in this subsection (for A a fixed ring), and we fix a continuous map $f: X \to Y$. Consider the direct image functor

$$f_*: \mathsf{Sh}(X, \mathsf{Mod}_A) \to \mathsf{Sh}(Y, \mathsf{Mod}_A).$$

It is important to remember that

$$\boxed{f_* \text{ is always left exact, and it is exact if } f: X \hookrightarrow Y \text{ is a closed subspace.}}$$

Since f_* will turn out to be a right adjoint (Lemma 2.8.16), it is left exact by general category theory. However, we prove it directly here. Note that you have already proved the case $Y = \mathrm{pt}$ in Exercise 2.5.16. You will notice in the proof of the above slogan that this was essentially enough to handle the general case.

PROPOSITION 2.8.8. *Let A be a ring, $f: X \to Y$ a continuous map of topological spaces. The functor $f_*: \mathsf{Sh}(X, \mathsf{Mod}_A) \to \mathsf{Sh}(Y, \mathsf{Mod}_A)$ is left exact. If f is the inclusion of a closed subspace, then f_* is exact.*

Proof. Let us prove the first assertion. We have to show that an exact sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}$$

in $\mathsf{Sh}(X, \mathsf{Mod}_A)$ induces an exact sequence

$$0 \longrightarrow f_*\mathcal{F} \xrightarrow{f_*\alpha} f_*\mathcal{G} \xrightarrow{f_*\beta} f_*\mathcal{H}$$

in $\mathsf{Sh}(Y, \mathsf{Mod}_A)$. We know by Exercise 2.5.16 that we have an exact sequence

$$0 \longrightarrow \mathcal{F}(f^{-1}V) \xrightarrow{\alpha_{f^{-1}V}} \mathcal{G}(f^{-1}V) \xrightarrow{\beta_{f^{-1}V}} \mathcal{H}(f^{-1}V) \tag{2.8.4}$$

for any open subset $V \subset Y$, by applying the functor $\Gamma(f^{-1}V, -)$ to the original sequence. In particular, $\alpha_{f^{-1}V} = (f_*\alpha)_V$ is injective for all V, which shows that $f_*\alpha$ is injective. There is an equality of presheaves

$$\mathrm{im}_{\mathrm{pre}}(f_*\alpha) = \ker(f_*\beta)$$

again thanks to exactness of (2.8.4) in the middle, ensuring precisely that $\mathrm{im}(\alpha_{f^{-1}V}) = \ker(\beta_{f^{-1}V})$. But $\ker(f_*\beta)$ is a sheaf, therefore we get exactness in the middle, i.e. $\mathrm{im}(f_*\alpha) = \ker(f_*\beta)$.

Let us show the second statement. Assume f is the inclusion of a closed subspace. By the first part of the proof, we only need to show that if $\eta: \mathcal{G} \twoheadrightarrow \mathcal{H}$ is surjective as a map of sheaves on X, then $f_*\mathcal{G} \twoheadrightarrow f_*\mathcal{H}$ is surjective as a map of sheaves on Y. We check this on stalks. If $y \in Y \setminus X$, then (using that X is closed, cf. Remark 2.8.7)

$$(f_*\mathcal{G})_y = 0 = (f_*\mathcal{H})_y, \tag{2.8.5}$$

so there is nothing to prove here. Assume $y \in X$. Since $\mathcal{G}$ surjects onto $\mathcal{H}$, in the commutative diagram

$$\begin{tikzcd} (f_*\mathcal{G})_y \arrow[r, "(f_*\eta)_y"] \arrow[d, equal] & (f_*\mathcal{H})_y \arrow[d, equal] \\ \mathcal{G}_y \arrow[r, two heads, "\eta_y"] & \mathcal{H}_y \end{tikzcd}$$

the bottom map is surjective. The vertical equalities are given by Remark 2.8.7. Thus the top map is surjective as well. Hence $f_*\eta$ is surjective on all stalks, hence it is surjective. $\square$

2.8.2 Inverse image

Let $f: X \to Y$ be a continuous map of topological spaces. Let $\mathcal{G}$ be a presheaf on Y. Given $U \subset X$, the collection of open subsets $V \subset Y$ containing $f(U)$ form a directed set via reverse inclusions. Sending

$$U \mapsto (f_{\text{pre}}^{-1}\mathcal{G})(U) = \varinjlim_{V \supset f(U)} \mathcal{G}(V)$$

defines a presheaf on X. Indeed, assume $U' \subset U$ is an open subset. Then there is an inclusion $f(U') \subset f(U)$, inducing a map of directed systems

$$\{V \in \tau_Y \mid V \supset f(U)\} \hookrightarrow \{V \in \tau_Y \mid V \supset f(U')\},$$

which in turn induces a morphism

$$(f_{\text{pre}}^{-1}\mathcal{G})(U) = \varinjlim_{V \supset f(U)} \mathcal{G}(V) \longrightarrow \varinjlim_{V \supset f(U')} \mathcal{G}(V) = (f_{\text{pre}}^{-1}\mathcal{G})(U').$$

This is the restriction morphism $\rho_{UU'}$ for $f_{\text{pre}}^{-1}\mathcal{G}$.

Remark 2.8.9. If $f(U)$ is an open subset of Y, then

$$(f_{\text{pre}}^{-1}\mathcal{G})(U) = \mathcal{G}(f(U)).$$

Now assume $\mathcal{G}$ is a sheaf. We define the *inverse image* of $\mathcal{G}$ by f to be the sheafification

$$f^{-1}\mathcal{G} = \left(f_{\text{pre}}^{-1}\mathcal{G}\right)^{\#}.$$

By Proposition 2.5.2, the canonical map $f_{\text{pre}}^{-1}\mathcal{G} \to f^{-1}\mathcal{G}$ of presheaves induces an isomorphism

$$(f_{\text{pre}}^{-1}\mathcal{G})_x \xrightarrow{\sim} (f^{-1}\mathcal{G})_x$$

on all the stalks.

Exercise 2.8.10. Both f_{pre}^{-1} and f^{-1} are functors.

Example 2.8.11. Let $\iota_y\colon \{y\} \hookrightarrow Y$ be the inclusion of a point $y \in Y$, and let $\mathcal{G}$ be a sheaf on Y. Then $\iota_y^{-1}\mathcal{G} = \mathcal{G}_y$, since $\iota_y^{-1}\mathcal{G}(\{y\}) = \varinjlim_{V \ni y} \mathcal{G}(V) = \mathcal{G}_y$. Thus ι_y^{-1} agrees with the stalk functor

$$\mathsf{stalk}_y\colon \mathsf{Sh}(Y,\mathscr{C}) \to \mathscr{C}, \quad \mathcal{G} \mapsto \mathcal{G}_y.$$

Example 2.8.12. If $p\colon X \to \mathrm{pt}$ is the constant map, and $G \in \mathscr{C} \cong \mathsf{Sh}(\mathrm{pt},\mathscr{C})$, then $p^{-1}G = \underline{G}_X$, the constant sheaf on X with values in the object G.

Example 2.8.13. Let $j\colon U \hookrightarrow Y$ be the inclusion of an open subset. Then $j_{\mathrm{pre}}^{-1}\mathcal{G} = \mathcal{G}|_U$ for any sheaf $\mathcal{G}$ on Y. The reason is that if U' is open in U, it is also open in Y, and thus

$$j_{\mathrm{pre}}^{-1}\mathcal{G}(U') = \varinjlim_{V \supset U'} \mathcal{G}(V) = \mathcal{G}(U') = \mathcal{G}|_U(U').$$

In particular, $j_{\mathrm{pre}}^{-1}\mathcal{G}$ is already a sheaf, and hence

$$j^{-1}\mathcal{G} = \mathcal{G}|_U, \quad U \subset Y \text{ open}.$$

Remark 2.8.14. Despite Example 2.8.13, sheafifying f_{pre}^{-1} is in general necessary: consider a constant map $f\colon X = \{\star, \bullet\} \to \{*\} = Y$ from a two point space, and fix a nontrivial abelian group G. The constant sheaf $\mathcal{G} = \underline{G}_Y$ has the property $f_{\mathrm{pre}}^{-1}\mathcal{G} = \underline{G}_X^{\mathrm{pre}}$, which is not a sheaf (cf. Example 2.2.18).

Functoriality (cf. Exercise 2.8.10) can be translated into a diagram of functors

$$\begin{tikzcd}
\mathsf{Sh}(Y,\mathscr{C}) \arrow[r, "f^{-1}"] \arrow[d, hook] & \mathsf{Sh}(X,\mathscr{C}) \arrow[d, hook] \\
\mathsf{pSh}(Y,\mathscr{C}) \arrow[r, "f_{\mathrm{pre}}^{-1}"] & \mathsf{pSh}(X,\mathscr{C})
\end{tikzcd}$$

where f^{-1} is obtained by applying $(-)^{\#}\colon \mathsf{pSh}(X,\mathscr{C}) \to \mathsf{Sh}(X,\mathscr{C})$ in the last step.

Exercise 2.8.15. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be continuous maps of topological spaces, and let $\mathcal{E}$ be a sheaf on Z. Show that

$$f^{-1}(g^{-1}\mathcal{E}) = (g \circ f)^{-1}\mathcal{E}.$$

LEMMA 2.8.16 (Unit and counit maps). *For any pair of presheaves $\mathcal{F} \in \mathsf{pSh}(X,\mathscr{C})$ and $\mathcal{G} \in \mathsf{pSh}(Y,\mathscr{C})$ there are canonical presheaf homomorphisms*

$$\mathcal{G} \xrightarrow{\ \mathsf{unit}\ } f_* f_{\mathrm{pre}}^{-1}\mathcal{G}, \qquad f_{\mathrm{pre}}^{-1} f_* \mathcal{F} \xrightarrow{\ \mathsf{counit}\ } \mathcal{F}.$$

Proof. We start with the unit map. The observation here is that there is, for any open subset $V \subset Y$, a canonical inclusion $f(f^{-1}V) \subset V$. Thus $\mathcal{G}(V)$ appears in the colimit

$$\varinjlim_{W \supset f(f^{-1}V)} \mathcal{G}(W).$$

This induces a canonical morphism

$$\mathsf{unit}_V : \mathcal{G}(V) \to \varinjlim_{W \supset f(f^{-1}V)} \mathcal{G}(W) = f_{\mathrm{pre}}^{-1}\mathcal{G}(f^{-1}V) = f_* f_{\mathrm{pre}}^{-1}\mathcal{G}(V)$$

which does define a natural transformation $\mathcal{G} \to f_* f_{\mathrm{pre}}^{-1}\mathcal{G}$ because if $V' \subset V$, then any open $W \subset Y$ containing $f(f^{-1}V)$ also contains $f(f^{-1}V')$, simply because $f(f^{-1}V') \subset f(f^{-1}V)$. Thus there is a natural morphism

$$\varinjlim_{W \supset f(f^{-1}V)} \mathcal{G}(W) \to \varinjlim_{W \supset f(f^{-1}V')} \mathcal{G}(W)$$

and the induced diagram

$$\begin{tikzcd}
\mathcal{G}(V) \arrow[r, "\mathsf{unit}_V"] \arrow[d] & \varinjlim_{W \supset f(f^{-1}V)} \mathcal{G}(W) \arrow[r, equal] \arrow[d] & f_* f_{\mathrm{pre}}^{-1}\mathcal{G}(V) \\
\mathcal{G}(V') \arrow[r, "\mathsf{unit}_{V'}"] & \varinjlim_{W \supset f(f^{-1}V')} \mathcal{G}(W) \arrow[r, equal] & f_* f_{\mathrm{pre}}^{-1}\mathcal{G}(V')
\end{tikzcd}$$

commutes. This defines the map $\mathsf{unit}\colon \mathcal{G} \to f_* f_{\mathrm{pre}}^{-1}\mathcal{G}$ of presheaves.

To construct the map counit, one observes that for any open subset $U \subset X$ there is (by the universal property of colimits, cf. Definition B.3.6) a canonical map

$$f_{\mathrm{pre}}^{-1} f_* \mathcal{F}(U) = \varinjlim_{V \supset f(U)} f_* \mathcal{F}(V) = \varinjlim_{V \supset f(U)} \mathcal{F}(f^{-1}V) \to \mathcal{F}(U),$$

since if $V \supset f(U)$ inside Y, then $U \subset f^{-1}f(U) \subset f^{-1}V$ inside X. This map is also functorial in $U' \subset U$, thus the map $\mathsf{counit}\colon f_{\mathrm{pre}}^{-1} f_* \mathcal{F} \to \mathcal{F}$ is defined. □

The usefulness of the homomorphisms unit and counit is that they make $(f_{\mathrm{pre}}^{-1}, f_*)$ into an adjoint pair of functors. More precisely, there are bijections

$$\varphi_{\mathcal{F},\mathcal{G}} \colon \mathrm{Hom}_{\mathsf{pSh}(Y,\mathscr{C})}(\mathcal{G}, f_*\mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{pSh}(X,\mathscr{C})}(f_{\mathrm{pre}}^{-1}\mathcal{G}, \mathcal{F}),$$

functorial in both $\mathcal{F}$ and $\mathcal{G}$. Specifically, $\varphi_{\mathcal{F},\mathcal{G}}$ sends $\eta\colon \mathcal{G} \to f_*\mathcal{F}$ to

$$f_{\mathrm{pre}}^{-1}\mathcal{G} \xrightarrow{f_{\mathrm{pre}}^{-1}\eta} f_{\mathrm{pre}}^{-1} f_* \mathcal{F} \xrightarrow{\mathsf{counit}} \mathcal{F}$$

with inverse sending $\zeta\colon f_{\mathrm{pre}}^{-1}\mathcal{G} \to \mathcal{F}$ to

$$\mathcal{G} \xrightarrow{\mathsf{unit}} f_* f_{\mathrm{pre}}^{-1}\mathcal{G} \xrightarrow{f_*\zeta} f_*\mathcal{F}.$$

Using the adjunction

$$\mathsf{pSh}(Y,\mathscr{C}) \underset{f_*}{\overset{f_{\mathrm{pre}}^{-1}}{\rightleftarrows}} \mathsf{pSh}(X,\mathscr{C}) \tag{2.8.6}$$

it is immediate to show that also

$$\mathsf{Sh}(Y,\mathscr{C}) \underset{f_*}{\overset{f^{-1}}{\rightleftarrows}} \mathsf{Sh}(X,\mathscr{C}) \tag{2.8.7}$$

is an adjoint pair of functors. Indeed, for any pair of sheaves $\mathcal{F} \in \mathsf{Sh}(X,\mathscr{C})$ and $\mathcal{G} \in \mathsf{Sh}(Y,\mathscr{C})$, we have

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Sh}(Y,\mathscr{C})}(\mathcal{G}, f_*\mathcal{F}) &= \mathrm{Hom}_{\mathsf{pSh}(Y,\mathscr{C})}(\mathcal{G}, f_*\mathcal{F}) && j_{Y,\mathscr{C}} \text{ is fully faithful} \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathsf{pSh}(X,\mathscr{C})}(f^{-1}_{\mathrm{pre}}\mathcal{G}, \mathcal{F}) && \text{adjunction (2.8.6)} \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Sh}(X,\mathscr{C})}(f^{-1}\mathcal{G}, \mathcal{F}) && \text{Exercise 2.5.5.} \end{aligned}$$

Remark 2.8.17. Fix $\mathcal{G} \in \mathsf{Sh}(Y,\mathscr{C})$. Once more, the adjunction (2.8.7) gives a canonical morphism $\mathcal{G} \to f_*f^{-1}\mathcal{G}$, corresponding to $\mathrm{id}_{f^{-1}\mathcal{G}}$ under

$$\mathrm{Hom}_{\mathsf{Sh}(Y,\mathscr{C})}(\mathcal{G}, f_*f^{-1}\mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Sh}(X,\mathscr{C})}(f^{-1}\mathcal{G}, f^{-1}\mathcal{G}).$$

Clearly sending $\mathcal{G} \mapsto f_*f^{-1}\mathcal{G}$ is a functor $f_*f^{-1}\colon \mathsf{Sh}(Y,\mathscr{C}) \to \mathsf{Sh}(Y,\mathscr{C})$, and the naturality of this operation yields a natural transformation

$$\mathsf{unit}\colon \mathrm{Id}_{\mathsf{Sh}(Y,\mathscr{C})} \Rightarrow f_*f^{-1}$$

of functors $\mathsf{Sh}(Y,\mathscr{C}) \to \mathsf{Sh}(Y,\mathscr{C})$, which is called the *unit* of the adjunction (2.8.7). Similarly, let $\mathcal{F} \in \mathsf{Sh}(X,\mathscr{C})$. There is a canonical morphism $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ corresponding to $\mathrm{id}_{f_*\mathcal{F}}$ under

$$\mathrm{Hom}_{\mathsf{Sh}(Y,\mathscr{C})}(f_*\mathcal{F}, f_*\mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Sh}(X,\mathscr{C})}(f^{-1}f_*\mathcal{F}, \mathcal{F}).$$

Clearly sending $\mathcal{F} \mapsto f^{-1}f_*\mathcal{F}$ is a functor $f^{-1}f_*\colon \mathsf{Sh}(X,\mathscr{C}) \to \mathsf{Sh}(X,\mathscr{C})$, and the naturality of this operation yields a natural transformation

$$\mathsf{counit}\colon f^{-1}f_* \Rightarrow \mathrm{Id}_{\mathsf{Sh}(X,\mathscr{C})}$$

of functors $\mathsf{Sh}(X,\mathscr{C}) \to \mathsf{Sh}(X,\mathscr{C})$, which is called the *counit* of the adjunction (2.8.7).

The next lemma says that the stalk of the inverse image is somewhat easy to compute (unlike for the pushforward).

LEMMA 2.8.18 (Stalk of inverse image). *Let $f\colon X \to Y$ be a continuous map of topological spaces, $\mathcal{G}$ a sheaf on Y, and $x \in X$ a point. There is a canonical identification*

$$(f^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)}.$$

Proof. We have

$$\begin{aligned}
(f^{-1}\mathcal{G})_x &= (\{x\} \hookrightarrow X)^{-1}(f^{-1}\mathcal{G}) && \text{by Example 2.8.11} \\
&= (\{x\} \hookrightarrow X \to Y)^{-1}\mathcal{G} && \text{by Exercise 2.8.15} \\
&= (\{f(x)\} \hookrightarrow Y)^{-1}\mathcal{G} \\
&= \mathcal{G}_{f(x)}
\end{aligned}$$

where we have used Example 2.8.11 once more for the last identity. □

Exercise 2.8.19. Show that if $f\colon X \hookrightarrow Y$ is the inclusion of a subspace, then the counit

$$f^{-1}f_*\mathcal{F} \to \mathcal{F}$$

is an isomorphism for every $\mathcal{F} \in \mathsf{Sh}(X, \mathscr{C})$. (**Hint**: check it on stalks).

PROPOSITION 2.8.20. *Let $f\colon X \hookrightarrow Y$ be the inclusion of a closed subspace.*

(1) *Let $\mathcal{G}$ be a sheaf on Y such that* $\operatorname{Supp}(\mathcal{G}) = X$. *Then the unit map*

$$\mathcal{G} \xrightarrow{\sim} f_*f^{-1}\mathcal{G}$$

is an isomorphism.

(2) *The functor f_* induces an equivalence of categories*

$$f_*\colon \mathsf{Sh}(X, \mathsf{Mod}_A) \xrightarrow{\sim} \mathsf{Sh}_X(Y, \mathsf{Mod}_A),$$

where $\mathsf{Sh}_X(Y, \mathsf{Mod}_A) \hookrightarrow \mathsf{Sh}(Y, \mathsf{Mod}_A)$ *is the full subcategory of sheaves on Y with support equal to X.*

Proof. To prove (1) it is enough to prove that the unit map is an isomorphism on all the stalks. If $y \in Y \setminus X$, we get $0 \xrightarrow{\sim} 0$, since $\mathcal{G}_y = 0$ by the assumption $\operatorname{Supp}(\mathcal{G}) = X$ and $(f_*f^{-1}\mathcal{G})_y = 0$ by (2.8.3). On the other hand, if $y \in X$, then $\mathcal{G}_y \to (f_*f^{-1}\mathcal{G})_y$ is nothing but the inverse of the isomorphism

$$(f_*f^{-1}\mathcal{G})_y \xrightarrow{\sim} (f^{-1}\mathcal{G})_y = \mathcal{G}_y$$

of Lemma 2.8.4.

To prove (2), observe first of all that f_* lands in the category $\mathsf{Sh}_X(Y, \mathsf{Mod}_A)$ by (2.8.3). Next, note that sending $\mathcal{G} \mapsto f^{-1}\mathcal{G}$ is an inverse to f_* by (1). In a little more detail, the equivalence (cf. Definition A.1.12) is set up by considering the pair of functors (f_*, f^{-1}) and exploiting the unit and counit *natural isomorphisms*

$$\mathsf{unit}\colon \mathrm{Id}_{\mathsf{Sh}(Y,\mathscr{C})} \Longrightarrow f_*f^{-1} \qquad \mathsf{counit}\colon f^{-1}f_* \Longrightarrow \mathrm{Id}_{\mathsf{Sh}(X,\mathscr{C})}$$

using (1) and Exercise 2.8.19. □

Exercise 2.8.21. Find examples of maps f and sheaves $\mathcal{G}$ such that $\mathcal{G} \to f_* f^{-1}\mathcal{G}$ is not an isomorphism.

Remark 2.8.22. If $j: X \hookrightarrow Y$ is *open* and $\mathcal{G}$ is a sheaf on Y, then $j_* j^{-1}\mathcal{G}$ satisfies

$$j_* j^{-1}\mathcal{G}(V) = (j_*\mathcal{G}|_X)(V) = \mathcal{G}(V \cap X), \quad V \subset Y \text{ open.}$$

The natural map $\mathcal{G}(V) \to j_* j^{-1}\mathcal{G}(V)$ sends $s \mapsto s|_{V\cap X}$.

PROPOSITION 2.8.23. *Let $\mathscr{C} = \mathsf{Mod}_A$, for a ring A. Then the inverse image functor*

$$f^{-1}: \mathsf{Sh}(Y, \mathsf{Mod}_A) \to \mathsf{Sh}(X, \mathsf{Mod}_A)$$

is exact.

Proof. Indeed, let

$$0 \to \mathcal{G} \to \mathcal{H} \to \mathcal{K} \to 0$$

be an exact sequence in $\mathsf{Sh}(Y, \mathsf{Mod}_A)$. Then,

$$0 \to \mathcal{G}_{f(x)} \to \mathcal{H}_{f(x)} \to \mathcal{K}_{f(x)} \to 0$$

is exact in Mod_A by Proposition 2.5.14, for every $x \in X$. But this is precisely the sequence

$$0 \to (f^{-1}\mathcal{G})_x \to (f^{-1}\mathcal{H})_x \to (f^{-1}\mathcal{K})_x \to 0.$$

Thus

$$0 \to f^{-1}\mathcal{G} \to f^{-1}\mathcal{H} \to f^{-1}\mathcal{K} \to 0$$

is exact, again by Proposition 2.5.14. □

2.9 Gluing sheaves

The purpose of this section is to prove the next theorem, which is of crucial importance (see e.g. the proof of **??**).

THEOREM 2.9.1 (Gluing sheaves). *Let X be a topological space, $X = \bigcup_{i\in I} U_i$ an open covering. Set $U_{ij} = U_i \cap U_j$ and similarly $U_{ijk} = U_{ij} \cap U_k$. Assume given a sheaf $\mathcal{F}_i$ on U_i for every $i \in I$, along with a collection of isomorphisms*

$$\varphi_{ij}: \mathcal{F}_i\big|_{U_{ij}} \xrightarrow{\sim} \mathcal{F}_j\big|_{U_{ij}}$$

such that $\varphi_{ii} = \mathrm{id}_{\mathcal{F}_i}$ for every i, and such that

$$\begin{tikzcd}
\mathcal{F}_i\big|_{U_{ijk}} \arrow[r, "\varphi_{ij}|_{U_{ijk}}"] \arrow[dr, "\varphi_{ik}|_{U_{ijk}}"'] & \mathcal{F}_j\big|_{U_{ijk}} \arrow[d, "\varphi_{jk}|_{U_{ijk}}"] \\
& \mathcal{F}_k\big|_{U_{ijk}}
\end{tikzcd}$$

commutes for every triple intersection. Then there is a unique sheaf $\mathcal{F}$ on X equipped with isomorphisms

$$\alpha_i : \mathcal{F}\big|_{U_i} \xrightarrow{\sim} \mathcal{F}_i$$

such that the diagrams

$$\begin{tikzcd}
\mathcal{F}\big|_{U_{ij}} \arrow[rr, "\alpha_j|_{U_{ij}}"] \arrow[dr, "\alpha_i|_{U_{ij}}"'] & & \mathcal{F}_j\big|_{U_{ij}} \\
& \mathcal{F}_i\big|_{U_{ij}} \arrow[ur, "\varphi_{ij}"'] &
\end{tikzcd}$$

commute for every $(i, j) \in I \times I$. The sheaf $\mathcal{F}$ is called the gluing of $(\mathcal{F}_i, \varphi_{ij})_{i,j}$ along the given covering.

A | Categories, functors, Yoneda Lemma

A.1 Minimal background on categories and functors

Definition A.1.1 (Category)**.** A *category* $\mathscr{C}$ is the datum of

(i) a class $\mathrm{Ob}(\mathscr{C})$ of 'objects',

(ii) a class $\mathrm{Hom}(\mathscr{C})$ of 'morphisms' (or 'arrows', or 'maps') between the objects,

(iii) class functions $d\colon \mathrm{Hom}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{C})$ and $t\colon \mathrm{Hom}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{C})$ specifying domain and target of every morphism,

(iv) for each pair of objects x and y, a subclass $\mathrm{Hom}_{\mathscr{C}}(x,y) \subset \mathrm{Hom}(\mathscr{C})$ of morphisms with domain x and target y,

(v) a binary operation

$$\mathrm{Hom}_{\mathscr{C}}(x,y) \times \mathrm{Hom}_{\mathscr{C}}(y,z) \xrightarrow{\circ} \mathrm{Hom}_{\mathscr{C}}(x,z),$$

called 'composition' of morphisms, for every triple of objects x, y and z.

Such data must fulfill the following axioms:

(CAT1) For every $x \in \mathrm{Ob}(\mathscr{C})$, there is an identity morphism $\mathrm{id}_x \in \mathrm{Hom}_{\mathscr{C}}(x,x)$ enjoying the properties

$$f \circ \mathrm{id}_x = f, \quad \mathrm{id}_y \circ g = g$$

for every morphism f with domain x, and for every morphism g with target y.

(CAT2) The associativity relation

$$(h \circ g) \circ f = h \circ (g \circ f)$$

holds for every triple (f,g,h) of composable morphisms.

Definition A.1.2 (Isomorphism)**.** Let $\mathscr{C}$ be a category, and fix two objects $x, y \in \mathrm{Ob}(\mathscr{C})$. An isomorphism between x and y is an invertible morphism $f \in \mathrm{Hom}_{\mathscr{C}}(x,y)$, i.e. a morphism $f\colon x \to y$ such that there exists a morphism $g\colon y \to x$ satisfying $f \circ g = \mathrm{id}_y$ and $g \circ f = \mathrm{id}_x$. Two objects $x, y \in \mathrm{Ob}(\mathscr{C})$ are said to be *isomorphic* when there is an isomorphism $x \to y$ (often denoted '$x \xrightarrow{\sim} y$').

Definition A.1.3 (Small and locally small)**.** A category $\mathscr{C}$ is *small* if both $\mathrm{Ob}(\mathscr{C})$ and $\mathrm{Hom}(\mathscr{C})$ are sets, and not proper classes. We say that $\mathscr{C}$ is *locally small* if $\mathrm{Hom}_{\mathscr{C}}(x, y)$ is a set, and not a proper class, for every pair of objects x and y. For a locally small category $\mathscr{C}$, the sets $\mathrm{Hom}_{\mathscr{C}}(x, y)$ are called *hom-sets*.

Example A.1.4. The following are familiar examples of categories:

- Sets, the category of sets with morphisms the functions between sets,
- Grp, the category of groups with morphisms the group homomorphisms,
- Ab, the category of abelian groups with morphisms the group homomorphisms,
- Rings, the category of rings with morphisms the ring homomorphisms,
- Fields, the category of fields with morphisms the field homomorphisms,
- $\mathsf{Vec}_{\mathbb{F}}$, the category of vector spaces over a field $\mathbb{F}$ with morphisms the $\mathbb{F}$-linear maps,
- Alg_R, the category of algebras over a ring R, with morphisms the R-algebra homomorphisms,
- Top, the category of topological spaces, with morphisms the continuous maps,
- Mod_R, the category of modules over a ring R, with morphisms the R-linear maps,
- Mfd, the category of smooth manifolds, with morphisms the C^∞ maps.

Remark A.1.5. The category Sets is locally small, but not small (Russell's Paradox). The same is true, by the same argument, for all the categories in Example A.1.4.

Definition A.1.6 (Functor)**.** Let $\mathscr{C}$ and $\mathscr{C}'$ be two categories. A functor from $\mathscr{C}$ to $\mathscr{C}'$, denoted $\mathsf{F}\colon \mathscr{C} \to \mathscr{C}'$, is the assignment of

- an object $\mathsf{F}(x) \in \mathrm{Ob}(\mathscr{C}')$ for every $x \in \mathrm{Ob}(\mathscr{C})$, and
- a morphism $\mathsf{F}(f) \in \mathrm{Hom}_{\mathscr{C}'}(\mathsf{F}(x), \mathsf{F}(y))$ for every morphism $f \in \mathrm{Hom}_{\mathscr{C}}(x, y)$,

subject to the following axioms:

(1) $\mathsf{F}(\mathrm{id}_x) = \mathrm{id}_{\mathsf{F}(x)}$ for every $x \in \mathrm{Ob}(\mathscr{C})$,

(2) $\mathsf{F}(g \circ f) = \mathsf{F}(g) \circ \mathsf{F}(f)$ for every pair (f, g) of composable arrows.

Remark A.1.7. By the axioms, functors preserve isomorphisms.

A functor as in Definition A.1.6 is said to be *covariant*. On the other hand, a *contravariant* functor $\mathsf{F}\colon \mathscr{C} \to \mathscr{C}'$ assigns a morphism $\mathsf{F}(f) \in \mathrm{Hom}_{\mathscr{C}'}(\mathsf{F}(y), \mathsf{F}(x))$ for every morphism $f \in \mathrm{Hom}_{\mathscr{C}}(x, y)$, and condition (2) becomes $\mathsf{F}(g \circ f) = \mathsf{F}(f) \circ \mathsf{F}(g)$. For instance, taking a K-vector space V to its dual $V^* = \mathrm{Hom}_K(V, K)$ is a contravariant functor.

Example A.1.8. Every category $\mathscr{C}$ admits an *identity functor* $\mathrm{Id}_{\mathscr{C}}\colon \mathscr{C} \to \mathscr{C}$, sending every object and every morphism to itself.

Define $\mathscr{C}^{\mathrm{op}}$ to be the category with objects $\mathrm{Ob}(\mathscr{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathscr{C})$ and with

$$\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(x, y) = \mathrm{Hom}_{\mathscr{C}}(y, x)$$

for every $x, y \in \mathrm{Ob}(\mathscr{C})$. Then a contravariant functor $\mathscr{C} \to \mathscr{C}'$ is the same as a covariant functor $\mathscr{C}^{\mathrm{op}} \to \mathscr{C}'$.

Definition A.1.9 (Natural transformation)**.** A *natural transformation* $\eta\colon \mathsf{F} \Rightarrow \mathsf{G}$ between two functors $\mathsf{F}, \mathsf{G}\colon \mathscr{C} \to \mathscr{C}'$ is the datum, for every $x \in \mathscr{C}$, of a morphism $\eta_x\colon \mathsf{F}(x) \to \mathsf{G}(x)$ in $\mathscr{C}'$, such that for every $f \in \mathrm{Hom}_{\mathscr{C}}(x_1, x_2)$ the diagram

$$\begin{tikzcd}
\mathsf{F}(x_1) \arrow[r, "\eta_{x_1}"] \arrow[d, "\mathsf{F}(f)"'] & \mathsf{G}(x_1) \arrow[d, "\mathsf{G}(f)"] \\
\mathsf{F}(x_2) \arrow[r, "\eta_{x_2}"] & \mathsf{G}(x_2)
\end{tikzcd}$$

is commutative in $\mathscr{C}'$.

Definition A.1.10 (Natural isomorphism)**.** Let $\mathscr{C}, \mathscr{C}'$ be two categories. Let $\mathsf{Fun}(\mathscr{C}, \mathscr{C}')$ be the category whose objects are functors $\mathscr{C} \to \mathscr{C}'$ and whose morphisms are the natural transformations. An isomorphism in the category $\mathsf{Fun}(\mathscr{C}, \mathscr{C}')$ is called a *natural isomorphism*.

Example A.1.11. Let K be a field, and $\mathscr{C}$ the category of finite dimensional K-vector spaces. Then we have two (covariant) functors $\mathscr{C} \to \mathscr{C}$, the former being the identity functor and the latter being the double dual functor, sending $V \mapsto V^{**}$. These two functors are naturally isomorphic.

Definition A.1.12 (Equivalence of categories)**.** Let $\mathscr{C}$ and $\mathscr{C}'$ be categories. An *equivalence* between them is a pair of functors

$$\mathsf{F}\colon \mathscr{C} \to \mathscr{C}', \quad \mathsf{G}\colon \mathscr{C}' \to \mathscr{C}$$

along with a pair of natural isomorphisms

$$\mathsf{F} \circ \mathsf{G} \xrightarrow{\sim} \mathrm{Id}_{\mathscr{C}'}, \quad \mathsf{G} \circ \mathsf{F} \xrightarrow{\sim} \mathrm{Id}_{\mathscr{C}}.$$

Terminology A.1.13. One often says that a functor $\mathsf{F}\colon \mathscr{C} \to \mathscr{C}'$ is an equivalence when there exists a functor $\mathsf{G}\colon \mathscr{C}' \to \mathscr{C}$ along with a pair of natural isomorphisms as in Definition A.1.12.

Definition A.1.14 (Fully faithful, essentially surjective)**.** A (covariant) functor $\mathsf{F}\colon \mathscr{C} \to \mathscr{C}'$ is called:

(i) *fully faithful* if for any two objects $x, y \in \mathscr{C}$ the map of sets

$$\mathrm{Hom}_{\mathscr{C}}(x, y) \to \mathrm{Hom}_{\mathscr{C}'}(\mathsf{F}(x), \mathsf{F}(y))$$

is a bijection.

(ii) *essentially surjective* if every object of $\mathscr{C}'$ is isomorphic to an object of the form $\mathsf{F}(x)$ for some $x \in \mathscr{C}$.

The following observation is quite useful.

Remark A.1.15. A fully faithful functor $\mathsf{F}\colon \mathscr{C} \to \mathscr{C}'$ induces an equivalence of $\mathscr{C}$ with the essential image of F, namely the full subcategory of $\mathscr{C}'$ consisting of objects isomorphic to objects of the form $\mathsf{F}(x)$ for some $x \in \mathscr{C}$. Put differently, a functor induces an equivalence if and only if it is fully faithful and essentially surjective.

Definition A.1.16 (Concrete category)**.** A *concrete category* is a category $\mathscr{C}$ that is equipped with a faithful functor $\mathsf{F}\colon \mathscr{C} \to \mathsf{Sets}$ to the category of sets.

Note that concreteness is not a property, but rather an additional structure present on the category.

Another notion that is rather important in category theory is that of an adjoint pair of functors.

Definition A.1.17 (Adjoint pair)**.** Let $\mathscr{C}$ and $\mathscr{D}$ be (locally small) categories. Let $\mathsf{F}\colon \mathscr{C} \to \mathscr{D}$ and $\mathsf{G}\colon \mathscr{D} \to \mathscr{C}$ be functors. We say that (F, G) is an *adjoint pair* of functors if for every pair of objects $(c, d) \in \mathrm{Ob}(\mathscr{C}) \times \mathrm{Ob}(\mathscr{D})$ one has a bijection of sets

$$\mathrm{Hom}_{\mathscr{D}}(\mathsf{F}(c), d) \cong \mathrm{Hom}_{\mathscr{C}}(c, \mathsf{G}(d)),$$

natural in both c and d. We say, more precisely, that F is a *left adjoint* to G and that G is a *right adjoint* to F.

Sometimes, one uses the pictorial description

$$\begin{tikzcd} \mathscr{C} \arrow[r, shift left, "\mathsf{F}"] & \mathscr{D} \arrow[l, shift left, "\mathsf{G}"] \end{tikzcd}$$

to say that (F, G) is an adjoint pair.

Example A.1.18. Here are some examples of adjunctions.

(a) Let $\mathsf{F}\colon \mathsf{Sets} \to \mathsf{Grp}$ be the functor sending a set S to the free group generated by the element of S. Let $\Phi\colon \mathsf{Grp} \to \mathsf{Sets}$ be the forgetful functor. Then (F, Φ) is an adjoint pair.
$$\mathsf{Sets} \underset{\Phi}{\overset{\mathsf{F}}{\rightleftarrows}} \mathsf{Grp}$$

(b) Let $\mathsf{j}\colon \mathsf{Ab} \hookrightarrow \mathsf{Grp}$ be the inclusion. It is right adjoint to the abelianisation functor $\mathsf{ab}\colon \mathsf{Grp} \to \mathsf{Ab}$ sending a group G to $G^{\mathrm{ab}} = G/[G,G]$. That is, $(\mathsf{ab}, \mathsf{j})$ is an adjoint pair.
$$\mathsf{Grp} \underset{\mathsf{j}}{\overset{\mathsf{ab}}{\rightleftarrows}} \mathsf{Ab}$$

(c) Let R be a ring. Consider the functor $\mathsf{sym}_R\colon \mathsf{Mod}_R \to \mathsf{Alg}_R$ sending $M \mapsto \mathsf{Sym}_R(M)$. Consider the forgetful functor $\Phi_R\colon \mathsf{Alg}_R \to \mathsf{Mod}_R$ sending an R-algebra to its underlying R-module. Then (sym_R, Φ_R) is an adjoint pair.
$$\mathsf{Mod}_R \underset{\Phi_R}{\overset{\mathsf{sym}_R}{\rightleftarrows}} \mathsf{Alg}_R$$

(d) Let $\alpha\colon R \to S$ be a ring homomorphism. Then every S-module is naturally an R-module, thus we have a forgetful functor $\Phi_\alpha\colon \mathsf{Mod}_S \to \mathsf{Mod}_R$. On the other hand, we have a functor (called extension of scalars) $-\otimes_R S\colon \mathsf{Mod}_R \to \mathsf{Mod}_S$ sending an R-module M to the S-module $M \otimes_R S$. Then $(-\otimes_R S, \Phi_\alpha)$ is an adjoint pair.
$$\mathsf{Mod}_R \underset{\Phi_\alpha}{\overset{-\otimes_R S}{\rightleftarrows}} \mathsf{Mod}_S$$

(e) Let ID be the category of integral domains (with morphisms the injective ring homomorphisms), Fields the category of fields. We have a functor $\mathsf{frac}\colon \mathsf{ID} \to \mathsf{Fields}$ sending a domain to its fraction field, and an inclusion functor $\mathsf{j}\colon \mathsf{Fields} \hookrightarrow \mathsf{ID}$. Then $(\mathsf{frac}, \mathsf{j})$ is an adjoint pair.
$$\mathsf{ID} \underset{\mathsf{j}}{\overset{\mathsf{frac}}{\rightleftarrows}} \mathsf{Fields}$$

(f) Let R be a ring, $M \in \mathsf{Mod}_R$ an R-module. Consider the endofunctors on the category Mod_R given by $-\otimes_R M$ and $\mathsf{h}^M = \mathrm{Hom}_{\mathsf{Mod}_R}(M,-)$. Then $(-\otimes_R M, \mathsf{h}^M)$ is an adjoint pair. The (natural) bijections
$$\mathrm{Hom}_{\mathsf{Mod}_R}(N \otimes_R M, P) \cong \mathrm{Hom}_{\mathsf{Mod}_R}(N, \mathrm{Hom}_{\mathsf{Mod}_R}(M,P))$$
induced by the adjunction
$$\mathsf{Mod}_R \underset{\mathsf{h}^M}{\overset{-\otimes_R M}{\rightleftarrows}} \mathsf{Mod}_R$$
are in fact isomorphisms of abelian groups (recall that Mod_R is abelian).

It is important to remember the following properties:

- every equivalence of categories is an adjunction,
- every right adjoint (resp. left adjoint) functor between two abelian categories is left exact (resp. right exact),
- if a functor has two left (or right) adjoints, then they are naturally isomorphic.

A.2 Yoneda Lemma

In this section we study representable functors and recall the statement of the Yoneda Lemma. More details and examples can be found, for instance, in [16].

For simplicity, all categories are assumed to be *locally small* throughout.

Let $\mathscr{C}$ be a (locally small) category. Consider the category of contravariant functors $\mathscr{C} \to \mathsf{Sets}$, i.e. the *functor category*

$$\mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathsf{Sets}).$$

For every object x of $\mathscr{C}$ there is a functor $\mathsf{h}_x \colon \mathscr{C}^{\mathrm{op}} \to \mathsf{Sets}$ defined by

$$u \mapsto \mathsf{h}_x(u) = \mathrm{Hom}_{\mathscr{C}}(u, x), \quad u \in \mathscr{C}.$$

A morphism $\phi \in \mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(u, v) = \mathrm{Hom}_{\mathscr{C}}(v, u)$ gets sent to the map of sets

$$\mathsf{h}_x(\phi) \colon \mathsf{h}_x(u) \to \mathsf{h}_x(v), \quad \alpha \mapsto \alpha \circ \phi.$$

Consider the functor

$$\mathsf{h}_{\mathscr{C}} \colon \mathscr{C} \to \mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathsf{Sets}), \quad x \mapsto \mathsf{h}_x. \tag{A.2.1}$$

This is, indeed, a functor: for every arrow $f \colon x \to y$ in $\mathscr{C}$ and object u of $\mathscr{C}$ we can define a map of sets

$$\mathsf{h}_f u \colon \mathsf{h}_x(u) \to \mathsf{h}_y(u), \quad \alpha \mapsto f \circ \alpha,$$

with the property that for every morphism $\phi \colon v \to u$ in $\mathscr{C}$ there is a commutative diagram

$$\begin{tikzcd}
\mathsf{h}_x(u) \arrow[r, "\mathsf{h}_f u"] \arrow[d, "\mathsf{h}_x(\phi)"'] & \mathsf{h}_y(u) \arrow[d, "\mathsf{h}_y(\phi)"] & u \xrightarrow{\alpha} x \arrow[r, mapsto] \arrow[d, mapsto] & u \xrightarrow{f\circ\alpha} y \arrow[d, mapsto] \\
\mathsf{h}_x(v) \arrow[r, "\mathsf{h}_f v"] & \mathsf{h}_y(v) & v \xrightarrow{\alpha\circ\phi} x \arrow[r, mapsto] & u \xrightarrow{f\circ\alpha\circ\phi} y
\end{tikzcd}$$

defining a natural transformation

$$\mathsf{h}_f \colon \mathsf{h}_x \Rightarrow \mathsf{h}_y.$$

LEMMA A.2.1 (Weak Yoneda). *The functor* $\mathsf{h}_{\mathscr{C}}$ *defined in* (A.2.1) *is fully faithful.*

Definition A.2.2 (Representable functor)**.** A functor $\mathsf{F} \in \mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathsf{Sets})$ is *representable* if it lies in the essential image of $\mathsf{h}_{\mathscr{C}}$, i.e. if it is isomorphic to a functor h_x for some $x \in \mathscr{C}$. In this case, we say that the object $x \in \mathscr{C}$ represents F.

Remark A.2.3. By Lemma A.2.1, if $x \in \mathscr{C}$ represents F, then x is unique up to a unique isomorphism. Indeed, suppose we have isomorphisms

$$a \colon \mathsf{h}_x \xrightarrow{\sim} F, \quad b \colon \mathsf{h}_y \xrightarrow{\sim} F$$

in the category $\mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathsf{Sets})$. Then there exists a unique isomorphism $x \xrightarrow{\sim} y$ inducing $b^{-1} \circ a \colon \mathsf{h}_x \xrightarrow{\sim} \mathsf{h}_y$.

Let $\mathsf{F} \in \mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathsf{Sets})$ be a functor, $x \in \mathscr{C}$ an object. One can construct a map of sets

$$g_x \colon \mathrm{Hom}(\mathsf{h}_x, \mathsf{F}) \to \mathsf{F}(x). \tag{A.2.2}$$

To a natural transformation $\eta \colon \mathsf{h}_x \Rightarrow \mathsf{F}$ one can associate the element

$$g_x(\eta) = \eta_x(\mathrm{id}_x) \in \mathsf{F}(x),$$

the image of $\mathrm{id}_x \in \mathsf{h}_x(x)$ via the map $\eta_x \colon \mathsf{h}_x(x) \to \mathsf{F}(x)$.

LEMMA A.2.4 (Strong Yoneda). *Let* $\mathsf{F} \in \mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathsf{Sets})$ *be a functor,* $x \in \mathscr{C}$ *an object. Then the map* g_x *defined in* (A.2.2) *is bijective.*

Proof. The inverse of g_x is the map that assigns to an element $\xi \in \mathsf{F}(x)$ the natural transformation $\eta(x,\xi) \colon \mathsf{h}_x \Rightarrow \mathsf{F}$ defined as follows. For a given object $u \in \mathscr{C}$, we define

$$\eta(x,\xi)_u \colon \mathsf{h}_x(u) \to \mathsf{F}(u)$$

by sending a morphism $f \colon u \to x$ to the image of ξ under $\mathsf{F}(f) \colon \mathsf{F}(x) \to \mathsf{F}(u)$. □

Exercise A.2.5. Show that Lemma A.2.4 implies Lemma A.2.1.

Definition A.2.6 (Universal object)**.** Let $\mathsf{F} \colon \mathscr{C}^{\mathrm{op}} \to \mathsf{Sets}$ be a functor. A *universal object* for F is a pair (x, ξ) where $\xi \in \mathsf{F}(x)$, such that for every pair (u, σ) with $\sigma \in \mathsf{F}(u)$, there exists a unique morphism $\alpha \colon u \to x$ with the property that $\mathsf{F}(\alpha) \colon \mathsf{F}(x) \to \mathsf{F}(u)$ sends ξ to σ.

Exercise A.2.7. Show that a pair (x, ξ) is a universal object for a functor $\mathsf{F} \colon \mathscr{C}^{\mathrm{op}} \to \mathsf{Sets}$ if and only if the natural transformation $\eta(x,\xi)$ defined in the proof of Lemma A.2.4 is a natural isomorphism. In particular, F is representable if and only if it has a universal object.

A.3 Moduli spaces in algebraic geometry

In classical moduli theory, one is interested in the category

$$\mathscr{C} = \mathsf{Sch}_S$$

of schemes over a fixed base scheme S. Its objects are pairs (X, f), where X is a scheme and $f : X \to S$ is a morphism of schemes. Sometimes one just writes $(f : X \to S)$ to denote an object of Sch_S. A morphism $(X, f) \to (Y, g)$ in Sch_S is a morphism $p : X \to Y$ such that $g \circ p = f$. One has the following important notion in moduli theory.

Definition A.3.1 (Fine moduli space)**.** Let $\mathfrak{M} \colon \mathsf{Sch}_S^{\mathrm{op}} \to \mathsf{Sets}$ be a functor. If an S-scheme $M \to S$ represents $\mathfrak{M}$, then $M \to S$ is called a fine moduli space for the moduli problem defined by $\mathfrak{M}$.

To say that $M \to S$ is a fine moduli space for a functor $\mathfrak{M}$ in particular says that $M \to S$ is unique up to unique isomorphism, and by Exercise A.2.7 it has a universal object $\xi \in \mathfrak{M}(M \to S)$ in the sense of Definition A.2.6.

Example A.3.2. The existence of fibre products in the category of schemes $\mathsf{Sch} = \mathsf{Sch}_{\operatorname{Spec}\mathbb{Z}}$ amounts to the representability of the functor $\mathsf{Sch}^{\mathrm{op}} \to \mathsf{Sets}$ sending a scheme $A \in \mathsf{Sch}$ to the set

$$\operatorname{Hom}_{\mathsf{Sch}}(A, X) \times_{\operatorname{Hom}_{\mathsf{Sch}}(A,S)} \operatorname{Hom}_{\mathsf{Sch}}(A, Y).$$

Example A.3.3 (Global Spec)**.** Let S be a scheme, $\mathscr{A}$ a quasicoherent $\mathscr{O}_S$-algebra. Then the S-scheme $\operatorname{Spec}_{\mathscr{O}_S} \mathscr{A} \to S$ represents the functor $\mathsf{Sch}_S^{\mathrm{op}} \to \mathsf{Sets}$ sending

$$(U \xrightarrow{g} S) \;\mapsto\; \operatorname{Hom}_{\mathscr{O}_S\text{-alg}}(\mathscr{A}, g_*\mathscr{O}_U).$$

B | Commutative algebra

B.1 Frequently used theorems

LEMMA B.1.1 (Nakayama).

B.2 Tensor products

Definition B.2.1 (Tensor product of modules)**.** Let A be a ring, M and N two A-modules. The *tensor product* of M and N over A is defined to be a pair $(M \otimes_A N, p)$ where

- $M \otimes_A N$ is an A-module,
- $p\colon M \times N \to M \otimes_A N$ is a bilinear map,

such that the following universal property is satisfied: for every pair (E, q) where E is an A-module and $q\colon M \times N \to E$ is a bilinear map, there is exactly one A-linear homomorphism $\phi_q\colon M \otimes_A N \to E$ such that $q = \phi_q \circ p$.

The universal property of Definition B.2.1 can be depicted in the diagram

$$\begin{tikzcd} M \times N \arrow[rr, "p"] \arrow[dr, "\forall q"'] & & M \otimes_A N \arrow[dl, "\exists ! \phi_q"] \\ & E & \end{tikzcd}$$

and, more importantly, can be rephrased by saying that there is a bijection

$$\mathsf{Bil}_A(M \times N, E) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Mod}_A}(M \otimes N, E), \quad q \mapsto \phi_q,$$

functorial in E.

Regarding existence of an object $(M \otimes_A N, p)$ with the required universal property, one first considers the standard basis $\{ e_{m,n} \mid m \in M,\ n \in N \}$ of the direct sum $A^{\oplus M \times N}$. One then constructs the quotient module

$$M \otimes N = A^{\oplus M \times N} / T$$

where $T \subset A^{\oplus M\times N}$ is the submodule generated by elements of the form

$$\begin{aligned}
&e_{m_1+m_2,n}-e_{m_1,n}-e_{m_2,n},\\
&e_{m,n_1+n_2}-e_{m,n_1}-e_{m,n_2},\\
&e_{am,n}-e_{m,an},\\
&a\,e_{m,n}-e_{am,n}.
\end{aligned}$$

The map $p\colon M\times N \to M\otimes_A N$ is defined by sending $(m,n)\mapsto [e_{m,n}]$, where the square bracket means equivalence class. One sets

$$m\otimes n = [e_{m,n}].$$

This is standard notation. Note that not all elements of $M\otimes_A N$ are of the form $m\otimes n$ for elements $m\in M$ and $n\in N$. However, every element $u\in M\otimes_A N$ can be written (non-uniquely) as a finite sum

$$u=\sum_{k=1}^{r} m_k\otimes n_k, \quad r>0.$$

Granting that the above pair $(M\otimes_A N, p)$ satisfies the universal property of Definition B.2.1 (which is an easy exercise), one has automatically that such pair is unique. Note that one has the elementary identifications

$$\begin{aligned}
M\otimes_A A &= M,\\
M\otimes_A N &= N\otimes_A M,\\
(M\otimes_A N)\otimes_A P &= M\otimes_A(N\otimes_A P).
\end{aligned}$$

Exercise B.2.2. If $(M_i)_{i\in I}$ is a family of A-modules, one has a canonical isomorphism

$$\bigoplus_{i\in I}(M_i\otimes_A N) \xrightarrow{\sim} \Big(\bigoplus_{i\in I} M_i\Big)\otimes_A N$$

for any A-module N.

Exercise B.2.3. Let A be a ring, M an A-module. Prove that the functor

$$M\otimes_A -\colon \mathsf{Mod}_A \to \mathsf{Mod}_A, \quad N\mapsto M\otimes_A N$$

is right exact, i.e. that a surjection $N_1 \twoheadrightarrow N_2$ gets sent to a surjection $M\otimes_A N_1 \twoheadrightarrow M\otimes_A N_2$.

B.3 Universal constructions

B.3.1 Limits and colimits

Let $\mathscr{C}$ be a category, $\mathbf{I}$ a small category. Define an $\mathbf{I}$-diagram to be just a functor $M\colon \mathbf{I}\to\mathscr{C}$. Denote by M_i the object of $\mathscr{C}$ image of the object $i\in\mathbf{I}$ via M. If $f\colon i\to j$ is an arrow in $\mathbf{I}$, the induced arrow in $\mathscr{C}$ is denoted $M(f)\colon M_i\to M_j$.

Definition B.3.1 (Limit)**.** A *limit* of an **I**-diagram $M\colon \mathbf{I} \to \mathscr{C}$ is an object $\varprojlim_{i\in\mathbf{I}} M_i$ of $\mathscr{C}$ along with an arrow $p_i\colon \varprojlim_{i\in\mathbf{I}} M_i \to M_i$ for every $i \in \mathbf{I}$, such that for every arrow $f\colon i \to j$ in **I** one has $p_j = M(f)\circ p_i$, and satisfying the following universal property: given an object P along with morphisms $\pi_i\colon P \to M_i$ such that $\pi_j = M(f)\circ\pi_i$ for every $f\colon i \to j$ in **I**, there exists a unique arrow $\alpha\colon P \to \varprojlim_{i\in\mathbf{I}} M_i$ such that $\pi_i = p_i \circ \alpha$ for all $i \in \mathbf{I}$.

$$\begin{tikzcd}
P \arrow[d, dotted, "\exists!\alpha"'] \arrow[dr, "\pi_i"] \arrow[drr, bend left, "\pi_j"] & & \\
\varprojlim\limits_{i\in\mathbf{I}} M_i \arrow[r, "p_i"] \arrow[rr, bend right, "p_j"'] & M_i \arrow[r, "M(f)"] & M_j
\end{tikzcd}$$

Exercise B.3.2. The limit over the empty diagram satisfies the universal property of a final object of $\mathscr{C}$.

Example B.3.3 (Products are limits)**.** Let **I** be the category with two objects i, j and no morphisms between them. Then an **I**-diagram $M\colon \mathbf{I} \to \mathscr{C}$ is just the choice of two objects M_i, M_j of $\mathscr{C}$. The limit of M satisfies the universal property of the product $M_i \times M_j$.

$$\begin{tikzcd}
P \arrow[dr, dotted, "\alpha"] \arrow[drr, bend left, "\pi_i"] \arrow[ddr, bend right, "\pi_j"'] & & \\
& M_i \times M_j \arrow[r, "p_i"] \arrow[d, "p_j"] & M_i \\
& M_j &
\end{tikzcd}$$

Example B.3.4 (Equalisers are limits)**.** Let **I** be the category with two objects i, j and two arrows $i \rightrightarrows j$. Then an **I**-diagram $M\colon \mathbf{I} \to \mathscr{C}$ is just the choice of two parallel arrows $\phi, \psi\colon M_i \rightrightarrows M_j$ in $\mathscr{C}$. The limit of M satisfies the universal property of the equaliser of (ϕ, ψ).

Example B.3.5 (Kernels are limits)**.** This is because kernels are equalisers (in the previous example take $\psi = 0$).

Definition B.3.6 (Colimit)**.** A *colimit* of an **I**-diagram $M\colon \mathbf{I} \to \mathscr{C}$ is an object $\varinjlim_{i\in\mathbf{I}} M_i$ of $\mathscr{C}$ along with an arrow $s_i\colon M_i \to \varinjlim_{i\in\mathbf{I}} M_i$ for every $i \in \mathbf{I}$, such that for every arrow $f\colon i \to j$ in **I** one has $s_i = s_j \circ M(f)$, and satisfying the following universal property: given an object P along with morphisms $\sigma_i\colon M_i \to P$ such that $\sigma_i = \sigma_j \circ M(f)$ for every $f\colon i \to j$ in **I**, there exists a unique arrow $\alpha\colon \varinjlim_{i\in\mathbf{I}} M_i \to P$ such that $\sigma_i = \alpha \circ s_i$ for all $i \in \mathbf{I}$.

$$\begin{tikzcd}
P & & \\
\varinjlim\limits_{i\in\mathbf{I}} M_i \arrow[u, dotted, "\exists!\alpha"] & M_j \arrow[l, "s_j"'] \arrow[ul, "\sigma_j"'] & M_i \arrow[l, "M(f)"'] \arrow[ull, bend right, "\sigma_i"'] \arrow[ll, bend left, "s_i"]
\end{tikzcd}$$

Exercise B.3.7. The colimit over the empty diagram satisfies the universal property of an initial object of $\mathscr{C}$ (cf. Exercise B.3.2).

Exercise B.3.8. Convince yourself that coproducts, coequalisers and cokernels are examples of colimits, along the same lines of Examples B.3.3, B.3.4 and B.3.5.

Definition B.3.9 (Filtered category)**.** A nonempty category $\mathbf{I}$ is *filtered* if for every two objects $i, j \in \mathbf{I}$ the following are true:

- there exists an object $k \in \mathbf{I}$ along with two morphisms $i \to k$ and $j \to k$, and
- for any two morphisms $f, g \in \operatorname{Hom}_{\mathbf{I}}(i, j)$ there exists an object $k \in \mathbf{I}$ along with a morphism $h : j \to k$ such that $h \circ f = h \circ g$ in $\operatorname{Hom}_{\mathbf{I}}(i, k)$.

The colimit of an $\mathbf{I}$-diagram $M : \mathbf{I} \to \mathscr{C}$ where $\mathbf{I}$ is a filtered category is a *filtered colimit*.

Example B.3.10. In the definition of stalk of a presheaf $\mathcal{F} \in \mathsf{pSh}(X, \mathscr{C})$ at a point $x \in X$, we have been taking

$$\mathbf{I} = \{ U \in \tau_X \mid x \in U \}^{\mathrm{op}}$$
$$M(U) = \mathcal{F}(U).$$

B.4 Localisation

B.4.1 General construction for modules

Let A be a ring, M an A-module. Fix a *multiplicative subset* $S \subset A$, i.e. a subset containing the identity $1 \in A$ and such that $s_1 s_2 \in S$ whenever $s_1, s_2 \in S$.

Example B.4.1. The following are key examples of multiplicative subsets:

(i) $S = \{ f^n \mid n \geq 0 \}$ for some $f \in A$.

(ii) $S = A \setminus \mathfrak{p}$, where $\mathfrak{p} \subset A$ is a prime ideal.

(iii) $S = A \setminus 0$, if A is an integral domain.

(iv) $S = A \setminus \mathcal{Z}$, where $\mathcal{Z}$ is the set of all zero-divisors in A.

Consider the equivalence relation on $M \times S$ defined by

$$\boxed{(m, s) \sim (m', s') \Longleftrightarrow \text{there exists } u \in S \text{ such that } u(s'm - sm') = 0 \in M.}$$

We denote by m/s, or by $\frac{m}{s}$, the equivalence class of (m, s). The set of such equivalence classes

$$S^{-1}M = (M \times S)/\sim \tag{B.4.1}$$

is an abelian group via

$$\frac{m}{s}+\frac{m'}{s'}=\frac{sm'+s'm}{ss'},$$

and if $M=A$ then $S^{-1}A$ becomes a ring via

$$\frac{a}{s}\cdot\frac{a'}{s'}=\frac{aa'}{ss'}.$$

The $\mathbb{Z}$-module $S^{-1}M$ is an $S^{-1}A$-module via

$$\frac{a}{s}\cdot\frac{m}{s'}=\frac{am}{ss'}. \tag{B.4.2}$$

Here 'am' refers to the A-module structure on M.

Definition B.4.2 (Localisation of a module)**.** The localisation of M with respect to S is the $S^{-1}A$-module $S^{-1}M$, where the linear structure is given by Equation (B.4.2).

Localisation is functorial. If $\phi: N\to M$ is an A-linear map, there is an induced map

$$S^{-1}\phi: S^{-1}N\to S^{-1}M,\quad \frac{n}{s}\mapsto\frac{\phi(n)}{s}.$$

This map is $S^{-1}A$-linear, indeed if $a/t\in S^{-1}A$ then

$$S^{-1}\phi\left(\frac{a}{t}\cdot\frac{n}{s}\right)=S^{-1}\phi\left(\frac{an}{ts}\right)=\frac{\phi(an)}{ts}=\frac{a\cdot\phi(n)}{ts}=\frac{a}{t}\cdot\frac{\phi(n)}{s}.$$

Remark B.4.3. If $0\in S$, then $S^{-1}M=0$.

Notation B.4.4. If $S=\{f^n\mid n\geq 0\}$ as in Example B.4.1 (i) above, then we write M_f for the localisation. If $S=A\setminus\mathfrak{p}$ as in Example B.4.1 (ii) above, then we write $M_{\mathfrak{p}}$ for the localisation. Do not confuse M_f and $M_{(f)}$ when $(f)=fA\subset A$ is a prime ideal!

B.4.2 Localisation of a ring and its universal property

Set $M=A$. There is a canonical ring homomorphism

$$\ell: A\to S^{-1}A,\quad a\mapsto\frac{a}{1}$$

sending S inside the group of invertible elements of $S^{-1}A$ (the inverse of $s/1$ being $1/s$), and making the pair $(S^{-1}A,\ell)$ universal with this property: whenever one has a ring homomorphism $\phi: A\to B$ such that $\phi(S)\subset B^\times$, there is exactly one ring homomorphism $p: S^{-1}A\to B$ such that $\phi=p\circ\ell$.

$$\begin{tikzcd} A \arrow[r, "\ell"] \arrow[d, "\phi"'] & S^{-1}A \arrow[dl, "p"] \\ B & \end{tikzcd}$$

Explicitly, the map p is defined by $p(a/s)=\phi(a)\phi(s)^{-1}$.

Remark B.4.5. The localisations of the form A_f are crucial in algebraic geometry. In A_f, the equivalence relation defining the localisation reads

$$\frac{a}{f^n} = \frac{b}{f^m} \Longleftrightarrow \text{there exists } k \geq 0 \text{ such that } f^k(af^m - bf^n) = 0 \in A.$$

In particular, one has that $A_f = 0$ if and only if f is nilpotent.

The following lemma is of key importance to us.

LEMMA B.4.6. *Sending* $\mathfrak{r} \mapsto \ell^{-1}(\mathfrak{r})$ *establishes a bijection*

$$\{\textit{prime ideals } \mathfrak{r} \subset S^{-1}A\} \xrightarrow{\simeq} \{\textit{prime ideals } \mathfrak{q} \subset A \textit{ such that } \mathfrak{q} \cap S = \emptyset\}$$

having as inverse the extension operation, sending by definition

$$\mathfrak{q} \mapsto \mathfrak{q} \cdot S^{-1}A = \left\{ \frac{a}{f} \,\middle|\, a \in \mathfrak{q},\, f \in S \right\}.$$

COROLLARY B.4.7. *For any prime ideal* $\mathfrak{p} \subset A$ *the ring*

$$A_{\mathfrak{p}} = \left\{ \frac{a}{f} \,\middle|\, a \in A,\, f \notin \mathfrak{p} \right\}$$

is local, with maximal ideal

$$\mathfrak{p} \cdot A_{\mathfrak{p}} = \left\{ \frac{a}{f} \,\middle|\, a \in \mathfrak{p},\, f \notin \mathfrak{p} \right\} \subset A_{\mathfrak{p}}.$$

Proof. Indeed, the correspondence of Lemma B.4.6 becomes, in the case $S = A \setminus \mathfrak{p}$,

$$\{\text{prime ideals } \mathfrak{r} \subset A_{\mathfrak{p}}\} \xrightarrow{\simeq} \{\text{prime ideals } \mathfrak{q} \subset A \text{ such that } \mathfrak{q} \subset \mathfrak{p}\}$$

and since its inverse (extension along $A \to A_{\mathfrak{p}}$) is inclusion-preserving it follows that every prime ideal $\mathfrak{r} \subset A_{\mathfrak{p}}$ must be contained in $\mathfrak{p} \cdot A_{\mathfrak{p}}$. This means that $\mathfrak{p} \cdot A_{\mathfrak{p}}$ is the unique maximal ideal. □

Exercise B.4.8. If $(A, \mathfrak{m})$ is a local ring, then $A = A_{\mathfrak{m}}$.

Warning B.4.9. In the case when B is a graded ring and $\mathfrak{p}$ is a homogeneous prime ideal, we use the notation $B_{\mathfrak{p}}$ for the localisation of B at the multiplicative subset consisting of *homogeneous* elements that are not in $\mathfrak{p}$.

PROPOSITION B.4.10 ([11, Prop. 5.8]). *If* $\mathfrak{m} \subset A$ *is a maximal ideal and* $k > 0$ *is an integer, there is a natural ring isomorphism*

$$A/\mathfrak{m}^k \xrightarrow{\sim} A_{\mathfrak{m}}/(\mathfrak{m} \cdot A_{\mathfrak{m}})^k.$$

It induces isomorphisms

$$\mathfrak{m}^h/\mathfrak{m}^k \xrightarrow{\sim} (\mathfrak{m} \cdot A_{\mathfrak{m}})^h/(\mathfrak{m} \cdot A_{\mathfrak{m}})^k$$

for every $h \leq k$.

LEMMA B.4.11. *Let A be a ring, $S \subset A$ a multiplicative subset. Then $\ell: A \to S^{-1}A$ is injective if and only if S contains no zero divisors.*

Proof. Suppose $a/1 = 0/1$ in $S^{-1}A$. Then there is $u \in S$ such that $au = 0$. But u is not a zero divisor, thus $a = 0$. □

Example B.4.12. Let A be an integral domain, which means that $(0) \subset A$ is prime. Then the localisation

$$A_{(0)} = \left\{ \frac{a}{b} \,\middle|\, a \in A,\, b \in A \setminus 0 \right\}$$

is a field, called the *fraction field* of A, that we denote by $\mathrm{Frac}(A)$. The canonical map $\ell: A \to \mathrm{Frac}(A)$ is injective by Lemma B.4.11.

Example B.4.13. Let A be a ring. Consider $S = A \setminus \mathcal{Z}$ as in Example B.4.1 (iv). The localisation $S^{-1}A$ is called the *total ring of fractions* of A. By Lemma B.4.11, $S = A \setminus \mathcal{Z}$ is the largest multiplicative set such that $\ell: A \to S^{-1}A$ is injective.

Example B.4.14. Let $A = \mathbb{Z}$. Fix a prime number $p \in \mathbb{Z}$. Then the localisation map

$$\mathbb{Z} \to \mathbb{Z}_{(p)} = \left\{ \frac{n}{m} \,\middle|\, n \in \mathbb{Z},\, p \nmid m \right\}$$

is injective. Also the localisation map

$$\mathbb{Z} \to \mathbb{Z}_p = \left\{ \frac{n}{p^k} \,\middle|\, n \in \mathbb{Z},\, k \geq 0 \right\}$$

is injective.

LEMMA B.4.15. *If A is reduced and $S \subset A$ is a multiplicative subset, then $S^{-1}A$ is also reduced.*

Proof. Assume there exists $a \in A$, $s \in S$ and $r \in \mathbb{Z}_{>0}$ such that $0/1 = (a/s)^r = a^r/s^r \in S^{-1}A$. Then there exists $u \in S$ such that $ua^r = 0 \in A$, thus $(ua)^r = 0$, and hence $ua = 0$ by assumption. But this means $0/1 = a/1 \in S^{-1}A$. □

B.4.3 Exactness of localisation

LEMMA B.4.16. *Let A be a ring, $S \subset A$ a multiplicative subset, M an A-module. Then, there is a canonical isomorphism of $S^{-1}A$-modules*

$$\phi: S^{-1}M \xrightarrow{\sim} M \otimes_A S^{-1}A.$$

Proof. First of all, the $S^{-1}A$-module structure on $M \otimes_A S^{-1}A$ si defined by

$$\frac{a}{t} \cdot \left(m \otimes \frac{b}{s} \right) = m \otimes \frac{ab}{ts}.$$

The map ϕ is defined by

$$\phi\left(\frac{m}{s}\right) = m \otimes \frac{1}{s}.$$

It is $S^{-1}A$-linear, since

$$\begin{aligned}\phi\left(\frac{a}{t}\cdot\frac{m}{s}\right) &= \phi\left(\frac{am}{ts}\right)\\ &= am \otimes \frac{1}{ts}\\ &= m \otimes \frac{a}{ts}\\ &= \frac{a}{t}\cdot\left(m \otimes \frac{1}{s}\right)\\ &= \frac{a}{t}\cdot\phi\left(\frac{m}{s}\right).\end{aligned}$$

Its inverse is given by $m \otimes (a/s) \mapsto (am)/s$. □

PROPOSITION B.4.17. *Let A be a ring, $S \subset A$ a multiplicative subset. Then, sending $M \mapsto S^{-1}M$ defines an exact functor from A-modules to $S^{-1}A$-modules.*

Proof. Fix a short exact sequence

$$0 \longrightarrow M \xrightarrow{\iota} N \xrightarrow{\pi} P \longrightarrow 0$$

of A-modules. We already know that

$$S^{-1}M \to S^{-1}N \to S^{-1}P \to 0$$

is exact, since this sequence is isomorphic to

$$M \otimes_A S^{-1}A \to N \otimes_A S^{-1}A \to P \otimes_A S^{-1}A \to 0$$

by Lemma B.4.16, and tensor product (by any A-module, e.g. $S^{-1}A$) is a right exact functor. So we only need to show that

$$S^{-1}\iota : S^{-1}M \to S^{-1}N$$

is injective. Assume there is an element $m/s \in S^{-1}M$ such that $0 = 0/1 = S^{-1}\iota(m/s) = \iota(m)/s \in S^{-1}N$. Then there exists $u \in S$ such that $0 = u\iota(m) = \iota(um)$ in N. This implies $um = 0 \in M$, hence $m/s = um/us = 0/us = 0$. □

B.5 Normalisation

Normal schemes are either regular or 'mildly singular' schemes. For instance, a key property of normal schemes is that singularities only occur in codimension 2 or higher. We now give precise definitions.

Definition B.5.1 (Normality)**.** We say that

(i) an integral domain A is *normal* if it is integrally closed in $\operatorname{Frac} A$,

(ii) a ring is *normal* if all its local rings are normal domains,

(iii) a scheme is *normal* if $\mathscr{O}_{X,x}$ is a normal integral domain for every $x \in X$.

Remark B.5.2. A scheme is normal if and only if it is 'locally normal' in the sense of **??**. The terminology 'locally normal' is never used though.

Remark B.5.3. By definition, a ring A is normal precisely when $\operatorname{Spec} A$ is a normal scheme.

Example B.5.4. A regular scheme is normal. A normal scheme is reduced. To see the latter, it is enough to observe that for any open subset $U \subset X$ there is an injective ring homomorphism

$$\mathscr{O}_X(U) \hookrightarrow \prod_{x \in U} \mathscr{O}_{X,x}$$

since $\mathscr{O}_X$ is a sheaf (cf. Lemma 2.4.1), where $\mathscr{O}_{X,x}$ is reduced for every $x \in X$, since it is a domain.

Example B.5.5 (Locally factorial schemes are normal)**.** A scheme is *locally factorial* if $\mathscr{O}_{X,x}$ is a UFD for every $x \in X$. A UFD is normal, so a locally factorial scheme is normal.

Example B.5.6. Let A be a normal domain. Then $S^{-1}A$ is a normal domain for any multiplicative subset $S \subset A$ (see **??** for a proof). Thus, $\operatorname{Spec} A$ is normal, and so is any principal open $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A$.

Caution B.5.7. It is not true that if $\operatorname{Spec} A$ is normal, then A is an integral domain: for instance, if $\mathbb{F}$ is a field, then

$$\operatorname{Spec} \mathbb{F} \amalg \operatorname{Spec} \mathbb{F} = \operatorname{Spec} \mathbb{F} \times \mathbb{F} = \operatorname{Spec} \mathbb{F}[x]/(x(x-1))$$

is a normal scheme, but $\mathbb{F}[x]/(x(x-1))$ is not a domain.

Exercise B.5.8. Show that $\operatorname{Spec} \mathbb{C}[x,y,z]/(x^2+y^2-z^2)$ is normal but not locally factorial.

Exercise B.5.9. Let $\mathbb{F}$ be a field, with $\operatorname{char} \mathbb{F} \neq 2$. Show that the following schemes are normal.

- $\operatorname{Spec} \mathbb{Z}[x]/(x^2-n)$, where $n \in \mathbb{Z}$ is square-free and congruent to 3 modulo 4.
- $\operatorname{Spec} \mathbb{F}[x_1,\ldots,x_n]/(x_1^2+\cdots+x_m^2)$, where $n \geq m \geq 3$.
- $\operatorname{Spec} \mathbb{F}[x,y,z,w]/(xy-zw)$.

PROPOSITION B.5.10. *Let X be a scheme.*

(A) *The following conditions are equivalent:*

(1) *X is normal.*

(2) *$\mathscr{O}_X(U)$ is a normal ring for every affine open $U \subset X$.*

(3) *There is an affine open covering $X = \bigcup_{i \in I} U_i$ such that $\mathscr{O}_X(U_i)$ is a normal ring for every $i \in I$.*

(4) *There is an open covering $X = \bigcup_{j \in J} V_j$ such that V_j is normal for every $j \in I$.*

Moreover, every open subscheme of a normal scheme X is normal.

(B) *If X is quasicompact, the above conditions are equivalent to*

(5) *$\mathscr{O}_{X,x}$ is a normal domain for every closed point $x \in X$.*

(C) *If X is integral, the above conditions are equivalent to*

(6) *$\mathscr{O}_X(U)$ is a normal domain for every affine open $U \subset X$.*

Proof. To prove (A), combine the Locality Lemma (cf. **??**), Remark B.5.2 and **??** with one another.

To prove (B), argue as in the proof of **??**.

To prove (C), it is enough to use the definition of integral scheme (cf. **??**) and point (A). □

Remark B.5.11. By the above proof, the first two conditions are equivalent even without assuming quasicompactness.

LEMMA B.5.12 ([10, Ch. 4, Lemma 1.13]). *Let A be a normal noetherian ring of dimension at least 1. Then*

$$A = \bigcap_{\substack{\mathfrak{p} \in \operatorname{Spec} A \\ \operatorname{ht}(\mathfrak{p})=1}} A_{\mathfrak{p}},$$

the intersection being taken inside $\operatorname{Frac} A$.

COROLLARY B.5.13. *Let X be a normal locally noetherian scheme. Let $Z \subset X$ be a closed subset of codimension at least 2. Then the natural map*

$$\mathscr{O}_X(X) \to \mathscr{O}_X(X \setminus Z)$$

is an isomorphism.

Proof. □

Example B.5.14. Note that Corollary B.5.13 reproves the content of **??**, i.e. the identity

$$\mathscr{O}_{\mathbb{A}^n_{\mathbf{k}}}(\mathbb{A}^n_{\mathbf{k}}) = \mathscr{O}_{\mathbb{A}^n_{\mathbf{k}}}(\mathbb{A}^n_{\mathbf{k}} \setminus \{0\})$$

for any $n \geq 2$.

There is a procedure, called *normalisation*, which does the following. Given, as input, an integral scheme X, one constructs a pair $(\widetilde{X}, \pi)$ where $\widetilde{X}$ is a normal scheme and $\pi\colon \widetilde{X} \to X$ is a morphism of schemes which is universal in the following sense: for every pair (Y, f) where Y is a normal scheme and $f\colon Y \to X$ is normal, there exists exactly one morphism $\alpha_f\colon Y \to \widetilde{X}$ such that $\pi \circ \alpha_f = f$.

Remark B.5.15. The normalisation of an integral scheme, if it exists (which it does, see Theorem B.5.17 below), is unique up to unique isomorphism.[1] Moreover, the universal property also shows that if $\pi\colon \widetilde{X} \to X$ is the normalisation and $U \subset X$ is open, then the base change map $\pi^{-1}(U) \to U$ is the normalisation of U.

In the affine case, the normalisation is easy to construct, as the following lemma shows.

LEMMA B.5.16. *Let A be an integral domain. Let $\widetilde{A} \subset \operatorname{Frac} A$ be the integral closure of A. Then the morphism*

$$\operatorname{Spec} \widetilde{A} \to \operatorname{Spec} A$$

induced by the inclusion $A \hookrightarrow \widetilde{A}$ is the normalisation of $\operatorname{Spec} A$.

Proof. □

THEOREM B.5.17. *Let X be an integral scheme. Then there exists a (unique) normalisation $(\widetilde{X}, \pi)$. If X is an integral algebraic $\mathbf{k}$-variety, then the normalisation morphism $\pi\colon \widetilde{X} \to X$ is finite; in particular, $\widetilde{X}$ is an algebraic $\mathbf{k}$-variety.*

Proof. □

PROPOSITION B.5.18 ([10, Ch. 4, Cor. 1.30]). *Let X be an integral algebraic variety. Then the set of points $x \in X$ such that $\mathscr{O}_{X,x}$ is normal is open.*

Example B.5.19 (Nodal cubic)**.** Let $A = \mathbf{k}[x, y]/(y^2 - x^2(x+1))$. Then A is not normal. Let us determine its normalisation.

Example B.5.20 (Cuspidal cubic)**.** Let $A = \mathbf{k}[x, y]/(y^2 - x^3)$. Then A is not normal. Let us determine its normalisation.

[1] The normalisation being defined as a *pair*, by isomorphism we mean an isomorphism in the category Sch_X of X-schemes.

B.6 Embedded components

On a locally noetherian scheme X there are a bunch of points that are more relevant than all other points, in the sense that they reveal part of the behaviour of the structure sheaf: these points are the *associated points* of X. Some of these points are already familiar: they are the generic points, i.e. the points corresponding to the irreducible components of X. The other associated points correspond to the so-called *embedded components* of X. If X is reduced, it has no embedded components.

Let R be a commutative ring with unity, and let M be an R-module. If $m \in M$, we let

$$\mathrm{Ann}_R(m) = \{ r \in R \mid r \cdot m = 0 \} \subset R$$

denote its annihilator. A prime ideal $\mathfrak{p} \subset R$ is said to be *associated to* M if $\mathfrak{p} = \mathrm{Ann}_R(m)$ for some $m \in M$. The set of all associated primes is denoted

$$\mathsf{Ass}_R(M) = \{ \mathfrak{p} \mid \mathfrak{p} \text{ is associated to } M \}.$$

LEMMA B.6.1. *Let $\mathfrak{p} \subset R$ be a prime ideal. Then $\mathfrak{p} \in \mathsf{Ass}_R(M)$ if and only if $R/\mathfrak{p}$ is an R-submodule of M.*

Proof. If $\mathfrak{p} = \mathrm{Ann}_R(m)$ for some $m \in M$, consider the map $\phi_m \colon R \to M$ defined by $\phi_m(r) = r \cdot m$. Since its kernel is by definition $\mathrm{Ann}_R(m)$, the quotient $R/\mathfrak{p}$ is an R-submodule of M. Conversely, given an R-linear inclusion $i \colon R/\mathfrak{p} \hookrightarrow M$, consider the composition $\phi \colon R \twoheadrightarrow R/\mathfrak{p} \hookrightarrow M$. Then $\phi_{i(1)}(r) = r \cdot i(1) = i(r+\mathfrak{p}) = \phi(r)$ for all $r \in R$, i.e. $\phi = \phi_{i(1)}$. □

Note that if $\mathfrak{p} \in \mathsf{Ass}_R(M)$ then $\mathfrak{p}$ contains the annihilator of M, i.e. the ideal

$$\mathrm{Ann}_R(M) = \{ r \in R \mid r \cdot m = 0 \text{ for all } m \in M \} \subset R.$$

Definition B.6.2 (Isolated primes)**.** The minimal elements (with respect to inclusion) in the set

$$\{ \mathfrak{p} \subset R \mid \mathfrak{p} \supset \mathrm{Ann}_R(M) \}$$

are called *isolated primes* of M.

From now on we assume R is noetherian and $M \neq 0$ is finitely generated. We have the following result.

THEOREM B.6.3 ([13, Theorem 5.5.10 (a)]). *Let R be a noetherian ring, $M \neq 0$ a finitely generated R-module. Then $\mathsf{Ass}_R(M)$ is a finite nonempty set containing all isolated primes.*

Definition B.6.4 (Embedded primes)**.** The non-isolated primes in $\mathsf{Ass}_R(M)$ are called the *embedded primes* of M.

Moreover, we have the following facts:

- the R-module M has a *composition series*, i.e. a filtration by R-submodules

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_s = M$$

such that $M_i/M_{i-1} = R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$. This series is not unique. However, for a prime ideal $\mathfrak{p} \subset R$, the number of times it occurs among the $\mathfrak{p}_i$ does not depend on the composition series. These primes are precisely the elements of $\mathsf{Ass}_R(M)$.

- Any ideal $I \subset R$ has a *primary decomposition*, i.e. an expression as intersection

$$I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$$

of primary ideals. A proper ideal $\mathfrak{q} \subsetneq R$ is called *primary* if whenever a product xy lies in $\mathfrak{q}$, either x or a power of y lies in $\mathfrak{q}$. Put differently, every zero-divisor in $R/\mathfrak{q}$ is nilpotent. One verifies that the radical of a primary ideal is prime, and one says that $\mathfrak{q}$ is $\mathfrak{p}$-primary if $\sqrt{\mathfrak{q}} = \mathfrak{p}$. One can always ensure that the decomposition is irredundant, i.e. removing any $\mathfrak{q}_i$ changes the intersection, and $\sqrt{\mathfrak{q}_i} \neq \sqrt{\mathfrak{q}_j}$ for all $i \neq j$.

Exercise B.6.5. Let $I \subset R$ be an ideal. Show that the set

$$\{ \sqrt{\mathfrak{q}_i} \}_i$$

is determined by I. Then show that elements of $\mathsf{Ass}_R(R/I)$ are precisely the radicals of the primary ideals in a primary decomposition of I. In symbols,

$$\mathsf{Ass}_R(R/I) = \{ \sqrt{\mathfrak{q}_i} \}_i .$$

Exercise B.6.6. Let $R = \mathbf{k}[x,y]$, $I = (xy, y^2)$ and $M = R/I$. Show that $\mathsf{Ass}_R(M) = \{(y),(x,y)\}$.

The most boring situation is when R is an integral domain, in which case the generic point $\xi \in \operatorname{Spec} R$ is the only associated (and clearly isolated) prime. More generally, a reduced affine scheme $\operatorname{Spec} R$ has no embedded primes (in particular no embedded points, see below), i.e. the only associated primes are the isolated (minimal) ones, corresponding to its irreducible components.

Let R be an integral domain. For an ideal $I \subset R$, one often calls the associated primes of I the associated primes of R/I. The minimal primes above $I = \operatorname{Ann}_R(R/I)$ (i.e. containing I) correspond to the irreducible components of the closed subscheme

$$\operatorname{Spec} R/I \subset \operatorname{Spec} R,$$

whereas for every embedded prime $\mathfrak{p} \subset R$ there exists a minimal prime $\mathfrak{p}'$ such that $\mathfrak{p}' \subset \mathfrak{p}$. Thus $\mathfrak{p}$ determines an *embedded component* — a subvariety $\mathrm{V}(\mathfrak{p})$ embedded in an irreducible component $\mathrm{V}(\mathfrak{p}')$. If the embedded prime $\mathfrak{p}$ is maximal, we talk about an *embedded point*.

Fact B.6.7. An algebraic curve (an algebraic variety of dimension 1) has no embedded points if and only if it is Cohen–Macaulay (the formal definition is given in **??**). However, there can be nonreduced Cohen–Macaulay curves: those curves with a fat component, such as the affine plane curve $\operatorname{Spec} \mathbf{k}[x,y]/x^2 \subset \mathbb{A}^2$. These objects often have moduli, i.e. deform (even quite mysteriously) in positive dimensional families. See [2, 3, 14, 15] for generalities on multiple structures on schemes.

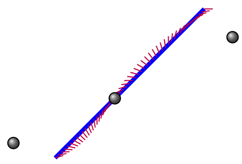

Figure B.1: A thickened (Cohen–Macaulay) curve with an embedded point and two isolated (possibly fat) points.

Remark B.6.8. An embedded component $\mathrm{V}(\mathfrak{p})$, where $\mathfrak{p}$ is the radical of some primary ideal $\mathfrak{q}$ appearing in a primary decomposition $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_e$, is of course embedded in some irreducible component $\mathrm{V}(\mathfrak{p}') \subset \operatorname{Spec} R/I$, but $\mathrm{V}(\mathfrak{q})$ is not a *subscheme* of $\mathrm{V}(\mathfrak{p}')$, because the fuzziness caused by nilpotent behavior (i.e. the difference between $\mathfrak{q}$ and its radical $\mathfrak{p}$) makes the bigger scheme $\mathrm{V}(\mathfrak{q}) \supset \mathrm{V}(\mathfrak{p})$ 'stick out' of $\mathrm{V}(\mathfrak{p}') \subset \operatorname{Spec} R/I$.

Example B.6.9. Consider $R = \mathbf{k}[x,y]$ and $I = (xy, y^2)$. A primary decomposition of I is

$$I = (x,y)^2 \cap (y).$$

However, $\operatorname{Spec} R/(x,y)^2$ is not scheme-theoretically contained in $\operatorname{Spec} R/y$.

In general, a subscheme Z of scheme Y has an embedded component if there exists a dense open subset $U \subset Y$ such that $Z \cap U$ is dense in Z but the scheme-theoretic closure of $Z \cap U \subset Z$ does not equal Z scheme-theoretically. For instance, if Y is irreducible, we say that $p \in Y$ supports an embedded point of a closed subscheme $Z \subset Y$ if $\overline{Z \cap (Y \setminus p)} \neq Z$ as schemes. In the example above, where $Y = \mathbb{A}^2$ and $Z = \operatorname{Spec} \mathbf{k}[x,y]/(xy, y^2)$, the scheme-theoretic closure of $Z \cap (\mathbb{A}^2 \setminus 0) \subset Z$ is not equal to Z.

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