Algebraic Geometry

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[...] *Oscar Zariski bewitched me. When he spoke the words "algebraic variety", there was a certain resonance in his voice that said distinctly that he was looking into a secret garden. I immediately wanted to be able to do this too. It led me to 25 years of struggling to make this world tangible and visible.*

David Mumford

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About this course

This is a 50 hours course (2.5 cycles for SISSA students).

The exam consists of an oral presentation by the student about a topic mutually agreed on, plus a few questions regarding the material covered in the course.

For UNITS students: the exam can only be scheduled within the official exam session, see the academic calendar.

Prerequisites

Familiarity with basic theory of commutative rings and modules is of great help, but not necessary. The relevant notions will be recalled as we need them. We have, however, included Appendix B to cover the basic commutative algebra constructions we will be referring to (and much more), and Appendix A to cover the basics of category theory as well.

Conventions

We list here a series of conventions that will be used throughout this text.

- The axiom of choice (or Zorn's Lemma) is assumed; so, for instance, every ring has a maximal ideal, and a poset $(P, \leq)$ in which every chain has an upper bound admits a maximal element.

- Given two sets A and B, the phrase '$A \subset B$' means that A is contained in B, *possibly equal* to B.

- A *ring* is a commutative, unitary ring. The zero ring (the one where $1 = 0$) is allowed (and in fact needed), but we always assume our rings are nonzero unless we explicitly mention it. Ring homomorphisms preserve the identity.

- By $\mathbf{k}$ we indicate an algebraically closed field, by $\mathbb{F}$ an arbitrary field.

- An open cover of a topological space U is the datum of a set I, and an open subset $U_i \subset U$ for every $i \in I$, such that $U = \bigcup_{i \in I} U_i$. We set $U_{ij} = U_i \cap U_j$. If $I = \emptyset$, then $U = \emptyset$.

- To say that Ω is an object a category $\mathscr{C}$ we simply write '$\Omega \in \mathscr{C}$' instead of $\Omega \in \mathrm{Ob}(\mathscr{C})$, with the exception of Appendix A, where a crash course on categories and functors is provided.

Main references

We list here a series of bibliographical references that integrate this text.

- Q. Liu, *Algebraic geometry and arithmetic curves* [12],

- R. Hartshorne, *Algebraic geometry* [8],

- R. Vakil, *The rising sea* [19],

- D. Eisenbud, *Commutative Algebra: With a View Toward Algebraic Geometry* [5],

- M. F. Atiyah and I. G. MacDonald, *Introduction to Commutative Algebra* [1],

1 | Introduction

Algebraic Geometry deals with the study of *algebraic varieties*. At a first approximation, these are common zero loci of collections of polynomials, i.e. solutions to systems

$$\begin{cases} f_1(x_1,\ldots,x_n) = 0 \\ \quad\quad\vdots \\ f_r(x_1,\ldots,x_n) = 0 \end{cases}$$

of polynomial equations. When $\deg f_j = 1$ for all $j = 1,\ldots,r$, this is the content of *Linear Algebra*, but the higher degree case poses nontrivial difficulties!

The concept of algebraic variety has been vastly generalised by Grothendieck's theory of *schemes*, introduced in [7].

Figure 1.1: Alexander Grothendieck (1928–2014).

This course is an introduction to schemes and to (part of) the massive dictionary, shared by all algebraic geometers, centered around schemes. Even though algebraic varieties are somewhat 'easier' objects, schemes are an incredibly useful and powerful tool to study them.

In this introduction, we briefly recap the key relation

$$\text{Algebra} \longleftrightarrow \text{Geometry}$$

in the land of *classical* algebraic varieties. We provide no proofs for now, but you shouldn't worry about this, because we will be proving more general results in the main body of these notes.

Let $\mathbf{k}$ be an algebraically closed field. Classical *affine n-space over* $\mathbf{k}$ is just

$$\mathbb{A}^n_{\mathbf{k}} = \{ (a_1, \ldots, a_n) \mid a_i \in \mathbf{k} \text{ for } i = 1, \ldots, n \}.$$

We denote it $\mathbb{A}^n_{\mathbf{k}}$ and not $\mathbf{k}^n$ to emphasise that we view it as a set of points rather than a vector space over $\mathbf{k}$. For instance, $\mathbb{A}^1_{\mathbf{k}}$ is called the *affine line* over $\mathbf{k}$, and $\mathbb{A}^2_{\mathbf{k}}$ is called the *affine plane* over $\mathbf{k}$. Let

$$A = \mathbf{k}[x_1, \ldots, x_n]$$

be the polynomial ring in n variables over the field $\mathbf{k}$. Each element $f \in A$ defines a function $\widetilde{f} \colon \mathbb{A}^n_{\mathbf{k}} \to \mathbf{k}$ sending $(a_1, \ldots, a_n) \mapsto f(a_1, \ldots, a_n)$, and since $\mathbf{k}$ is algebraically closed one has $f = g$ if and only if $\widetilde{f} = \widetilde{g}$.[1] Thus we shall just write f instead of $\widetilde{f}$.

Let $I = (f_1, \ldots, f_r) \subset A$ be an arbitrary ideal (here we are using that every ideal in A is finitely generated, by Hilbert's basis theorem [9]). The 'vanishing locus'

$$\mathrm{V}(I) = \left\{ (a_1, \ldots, a_n) \in \mathbb{A}^n_{\mathbf{k}} \;\middle|\; f_j(a_1, \ldots, a_n) = 0 \text{ for } j = 1, \ldots, r \right\} \subset \mathbb{A}^n_{\mathbf{k}}$$

is called an *algebraic set*. There is precisely one topology on $\mathbb{A}^n_{\mathbf{k}}$ having the algebraic sets as closed sets. It is called the *Zariski topology*.

Indeed, one has

- $\mathbb{A}^n_{\mathbf{k}} = \mathrm{V}(0)$,
- $\emptyset = \mathrm{V}(A)$,
- $\mathrm{V}(I) \cup \mathrm{V}(J) = \mathrm{V}(IJ)$,
- $\bigcap_{s \in S} \mathrm{V}(I_s) = \mathrm{V}\big(\sum_{s \in S} I_s\big)$ for any family of ideals $(I_s \subset A)_{s \in S}$.

Figure 1.2: Oscar Zariski (1899–1986).

Example 1.0.1. Every ideal in $\mathbf{k}[x]$ is principal, i.e. of the form (f) for some $f \in \mathbf{k}[x]$. Since $\mathbf{k}$ is algebraically closed, we have $f = \alpha(x - a_1) \cdots (x - a_d)$, for $\alpha, a_1, \ldots, a_d \in \mathbf{k}$, and where $d = \deg f$. Thus, if $f \neq 0$, then $\mathrm{V}(f) = \{a_1, \ldots, a_d\} \subset \mathbb{A}^1_{\mathbf{k}}$, proving that all proper closed subsets of $\mathbb{A}^1_{\mathbf{k}}$ are finite. In particular, all open sets are infinite (since $\mathbf{k}$ is algebraically closed, thus infinite).

[1] For instance, the field $\mathbb{F}_3 = \{0, 1, 2\}$ is not algebraically closed, and the polynomials $f = x^2 + 1$ and $g = x^4 + 1$ are different, nevertheless one has $\widetilde{f} = \widetilde{g}$ as functions on the three point space $\mathbb{A}^1_{\mathbb{F}_3}$.

We have thus established an assignment

$$\{\text{ideals } I \subset \mathbf{k}[x_1,\ldots,x_n]\} \xrightarrow{\mathrm{V}(-)} \{\text{algebraic sets in } \mathbb{A}^n_{\mathbf{k}}\}.$$

Conversely, given a subset $S \subset \mathbb{A}^n_{\mathbf{k}}$, the assignment

$$\mathrm{I}(S) = \left\{ f \in A \,\middle|\, f(p) = 0 \text{ for all } p \in S \right\} \subset A$$

defines a map the other way around, namely

$$\{\text{ideals } I \subset \mathbf{k}[x_1,\ldots,x_n]\} \xleftarrow{\mathrm{I}(-)} \{\text{subsets } S \subset \mathbb{A}^n_{\mathbf{k}}\}.$$

The two maps are *not* inverse to each other, even if we restrict I(−) to algebraic sets. For instance, consider the ideal $(x^r) \subset \mathbf{k}[x]$ for $r > 1$. Then $\mathrm{V}(x^r) = \{0\}$, and thus $\mathrm{I}(\mathrm{V}(x^r)) = (x)$, which is strictly larger than (x^r). The next result says that this is what *always* happens.

THEOREM 1.0.2 (Hilbert's Nullstellensatz [10]). *Let $I \subset \mathbf{k}[x_1,\ldots,x_n]$ be an ideal, where $\mathbf{k}$ is an algebraically closed field. Then,* $\mathrm{I}(\mathrm{V}(I)) = \sqrt{I}$*, i.e. $f \in \mathrm{I}(\mathrm{V}(I))$ if and only if $f^r \in I$ for some $r > 0$.*

See [12, Ch. 2, Corollary 1.15] for a modern proof of Hilbert's Nullstellensatz.

Composing our two assignments the other way around, we also find something larger than what we started with: consider for instance the complement $S \subset \mathbb{A}^1_{\mathbf{k}}$ of a finite set. Then $\mathrm{I}(S) = (0)$, since there are no nonzero polynomials with infinitely many zeroes. Thus $\mathrm{V}(\mathrm{I}(S)) = \mathbb{A}^1_{\mathbf{k}}$. In general, if S is an arbitrary subset of $\mathbb{A}^n_{\mathbf{k}}$, one can easily prove the identity

$$\mathrm{V}(\mathrm{I}(S)) = \overline{S},$$

where $\overline{S}$ is the closure of S in $\mathbb{A}^n_{\mathbf{k}}$ (with respect to the Zariski topology), namely the smallest algebraic set containing S. Thus in order to get $\mathrm{V}(\mathrm{I}(S)) = S$ we have to start with an algebraic set S (which is closed by definition).

Furthermore, one can prove that an algebraic set $Y \subset \mathbb{A}^n_{\mathbf{k}}$ is irreducible (i.e. it cannot be written as a union of two proper closed subsets) if and only if $\mathrm{I}(Y) \subset A$ is a prime ideal.

$$\boxed{\text{An irreducible algebraic set in } \mathbb{A}^n_{\mathbf{k}} \text{ is called an } \textit{affine variety in } \mathbb{A}^n_{\mathbf{k}}.}$$

Of course, an affine variety carries the induced Zariski topology by default. Combining these observations together, we obtain correspondences (with 'algebra' on the left, and 'geometry' on the right)

$$\begin{tikzcd}
\{\text{radical ideals in } \mathbf{k}[x_1,\ldots,x_n]\} \arrow[r, shift left, "\mathrm{V}(-)"] & \{\text{algebraic sets in } \mathbb{A}^n_{\mathbf{k}}\} \arrow[l, shift left, "\mathrm{I}(-)"] \\
\{\text{prime ideals in } \mathbf{k}[x_1,\ldots,x_n]\} \arrow[u, hook] \arrow[r, shift left, "\mathrm{V}(-)"] & \{\text{affine varieties in } \mathbb{A}^n_{\mathbf{k}}\} \arrow[u, hook] \arrow[l, shift left, "\mathrm{I}(-)"]
\end{tikzcd}$$

where an ideal $I \subset \mathbf{k}[x_1, \dots, x_n]$ is *radical* if $I = \sqrt{I}$ (Definition 3.1.2).

Recall that, by definition, a *finitely generated* $\mathbf{k}$-*algebra* is a $\mathbf{k}$-algebra B isomorphic to a quotient $\mathbf{k}[x_1, \dots, x_n]/I$ for some n and some ideal $I \subset \mathbf{k}[x_1, \dots, x_n]$. Such a B is an integral domain (i.e. as a ring it has no nonzero zero-divisors) precisely when I is prime. Thus the bottom correspondence above can be rephrased as

$$\{\, \mathbf{k}[x_1, \dots, x_n]/\mathfrak{p} \mid \mathfrak{p} \text{ is prime} \,\} \underset{\mathrm{I}(-)}{\overset{\mathrm{V}(-)}{\rightleftarrows}} \{\text{affine varieties in } \mathbb{A}^n_{\mathbf{k}}\}.$$

In the first part of this course, we will extend this correspondence to arbitrary *rings* on the left. What will be constructed on the right will be called an *affine scheme*, and what we shall establish is not just a bijection, but an equivalence of categories

$$\mathsf{Rings}^{\mathrm{op}} \cong \text{Affine schemes}.$$

Affine schemes are the basic building blocks for the construction of general *schemes*, in the same way as open subsets of $\mathbb{R}^m$ are the basic building blocks for m-dimensional smooth manifolds. As we shall see, a *scheme* is defined by the property that every point has an open neighborhood isomorphic to an affine scheme.

2 | Sheaves

Sheaves were defined by Leray (1906–1998), while he was a prisoner in Austria during World War II.

Sheaves are a key notion present in the toolbox of every mathematician keen to understand the "nature" of a *geometric space*. They incarnate one of the basic principles that will be unraveled throughout this course, which can be stated as the slogan

$$\boxed{\text{geometric spaces are determined by functions on them.}}$$

Even though there may be "few" functions on a space X, a complete knowledge of all functions on all open subsets of X allows one, in principle, to reconstruct X. This local-to-global principle is perfectly encoded in the notion of a sheaf.

2.1 Key example: smooth functions

Before diving into precise definitions, we explore a key example of sheaf.

Let X be a smooth manifold. For each open subset $U \subset X$, we have a ring (actually, an $\mathbb{R}$-algebra)

$$C^\infty(U,\mathbb{R}) = \{\text{smooth functions } U \to \mathbb{R}\}.$$

Indeed, smooth functions with the same source can naturally be added and multiplied exploiting the ring structure on $\mathbb{R}$. If $V \hookrightarrow U$ is an open subset, we have a restriction map

$$\rho_{UV}\colon C^\infty(U,\mathbb{R}) \to C^\infty(V,\mathbb{R}), \quad f \mapsto f|_V,$$

which is an $\mathbb{R}$-algebra homomorphism. One has $\rho_{UU} = \mathrm{id}_{C^\infty(U,\mathbb{R})}$, and if $W \hookrightarrow V \hookrightarrow U$ is a chain of open subsets of X, we have a commutative diagram

$$\begin{tikzcd} C^\infty(U,\mathbb{R}) \arrow[r, "\rho_{UV}"] \arrow[rr, bend right, "\rho_{UW}"'] & C^\infty(V,\mathbb{R}) \arrow[r, "\rho_{VW}"] & C^\infty(W,\mathbb{R}). \end{tikzcd}$$

So far, we have just observed that the assignment $U \mapsto C^\infty(U,\mathbb{R})$ is *functorial*, from open subsets of X (which form a category) to the category of $\mathbb{R}$-algebras. The two distinguished features of the assignment $U \mapsto C^\infty(U,\mathbb{R})$, which make it into a *sheaf* of $\mathbb{R}$-algebras on X, are the following:

(i) Fix an open subset $U \subset X$ and an open cover $U = \bigcup_{i\in I} U_i$. If $f, g \in C^\infty(U,\mathbb{R})$ are smooth functions such that $f|_{U_i} = g|_{U_i}$ for every $i \in I$, then $f = g$. In other words, a smooth function is determined by its restriction to the open subsets forming a covering. This is the *locality axiom*.

(ii) Fix an open subset $U \subset X$ and an open cover $U = \bigcup_{i\in I} U_i$. Given a smooth function $f_i \in C^\infty(U_i,\mathbb{R})$ on each U_i, such that $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ for every $(i, j) \in I \times I$, there is a smooth function $f \in C^\infty(U,\mathbb{R})$ such that $f_i = f|_{U_i}$ for every $i \in I$. In other words, functions glue along an open cover. This is the *glueing axiom*.

A sheaf is an abstract notion formalising this "ability of glueing". The formal definition will be given in Important Definition 2.2.1. Note that the result of the glueing in Condition (ii) is *unique* by Condition (i).

Let us continue with our example. Let $x \in X$ be a point. Consider the ring

$$C^\infty_{X,x} = \left\{ (U, f) \,\middle|\, x \in U,\ f \in C^\infty(U,\mathbb{R}) \right\} \big/ \sim$$

where $(U, f) \sim (V, g)$ whenever there exists an open subset $W \subset U \cap V$, containing x, such that $f|_W = g|_W$. Note that $C^\infty_{X,x}$ is indeed a ring, with addition and multiplication

$$\begin{aligned} [U, f]+[U', f'] &= [U \cap U', f + f'] \\ [U, f]\cdot[U', f'] &= [U \cap U', f f']. \end{aligned}$$

This ring, which is in fact an $\mathbb{R}$-algebra via $c \mapsto [X, c]$, for all $c \in \mathbb{R}$, is called the *stalk* of the sheaf $C^\infty(-,\mathbb{R})$ at x (cf. Important Definition 2.4.1), and it receives a natural $\mathbb{R}$-algebra homomorphism from $C^\infty(U,\mathbb{R})$ for every open subset U of X such that $x \in U$, sending $f \mapsto [U, f]$. The image of f along this map is called the *germ of f at x*. The subset

$$\mathfrak{m}_x = \left\{ [U, f] \in C^\infty_{X,x} \,\middle|\, f(x) = 0 \right\} \subset C^\infty_{X,x}$$

forms an ideal, which is a *maximal* ideal, being the kernel of the (surjective) evaluation map

$$\begin{aligned} C^\infty_{X,x} &\twoheadrightarrow \mathbb{R} \\ [U, f] &\longmapsto f(x). \end{aligned}$$

In fact, $\mathfrak{m}_x$ is the *unique* maximal ideal of $C^\infty_{X,x}$. To see this, it is enough to check that every element of $C^\infty_{X,x} \setminus \mathfrak{m}_x$ is invertible. But this is true, since a smooth function that is nonzero in a neighbourhood of x is invertible there.

The upshot is, then, that the pair $(C^\infty_{X,x}, \mathfrak{m}_x)$ defines a *local ring* with residue field $\mathbb{R}$. The geometric spaces X one deals with in algebraic geometry, namely *schemes*, have precisely this property: they come with a sheaf of rings $\mathscr{O}_X$ such that each stalk $\mathscr{O}_{X,x}$ is a local ring. These spaces $(X, \mathscr{O}_X)$ actually form a larger category, that of locally ringed spaces (cf. Section 2.10). Schemes are particular instances of locally ringed spaces.

2.2 Presheaves, sheaves, morphisms

Let $\mathscr{C}$ be a concrete category (Definition A.1.16) with a final object $0 \in \mathscr{C}$. The concreteness assumption means that part of the structure is the datum of a faithful functor $F: \mathscr{C} \to \mathsf{Sets}$, but we will (for the moment) ignore this datum. To fix ideas, $\mathscr{C}$ should be thought of as any of the following categories:

- $\mathscr{C} = \mathsf{Sets}$,
- $\mathscr{C} = \mathsf{Rings}$,
- $\mathscr{C} = \mathsf{Ab} = \mathsf{Mod}_{\mathbb{Z}}$,
- $\mathscr{C} = \mathsf{Mod}_R$, where R is a ring.

If X is a topological space, we denote by τ_X the category of open subsets of X. The set $\mathrm{Hom}_{\tau_X}(V, U)$ between two open sets $V, U \subset X$ is just the empty set if $V \not\subset U$, or the singleton $\{ V \hookrightarrow U \}$ in case V is contained in U. Thus the opposite category τ_X^{op} satisfies

$$\mathrm{Hom}_{\tau_X^{\mathrm{op}}}(U, V) = \begin{cases} \{ V \hookrightarrow U \} & \text{if } V \subset U \\ \emptyset & \text{if } V \not\subset U \end{cases}$$

and a functor $\mathcal{F}: \tau_X^{\mathrm{op}} \to \mathscr{C}$ (i.e. a contravariant functor $\tau_X \to \mathscr{C}$) determines a map

$$\mathrm{Hom}_{\tau_X^{\mathrm{op}}}(U, V) \to \mathrm{Hom}_{\mathscr{C}}(\mathcal{F}(U), \mathcal{F}(V)),$$

which is nothing but a choice of an element $\rho_{UV} \in \mathrm{Hom}_{\mathscr{C}}(\mathcal{F}(U), \mathcal{F}(V))$ for any inclusion of open subsets $V \subset U$.

Definition 2.2.1 (Presheaf, take I)**.** A *presheaf* on a topological space X, with values in $\mathscr{C}$, is a contravariant functor $\mathcal{F}$ from τ_X to $\mathscr{C}$, i.e. an object of the functor category $\mathsf{Fun}(\tau_X^{\mathrm{op}}, \mathscr{C})$.

For those who do not like the categorical definition, here is an equivalent definition, which just unravels the definition of a functor (cf. Definition A.1.6).

Definition 2.2.2 (Presheaf, take II)**.** A *presheaf* on a topological space X, with values in $\mathscr{C}$, is the assignment $U \mapsto \mathcal{F}(U)$ of an object $\mathcal{F}(U) \in \mathscr{C}$ for each open subset $U \subset X$, and of a morphism $\rho_{UV}: \mathcal{F}(U) \to \mathcal{F}(V)$ in $\mathscr{C}$ for each inclusion $V \hookrightarrow U$, such that

(1) $\rho_{UU} = \mathrm{id}_{\mathcal{F}(U)}$ for every $U \in \tau_X$, and

(2) $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$ for every chain of inclusions $W \hookrightarrow V \hookrightarrow U$.

Terminology 2.2.3. Elements of $\mathcal{F}(U)$ are often called 'sections of $\mathcal{F}$ over U', or (somewhat more vaguely) 'local sections' when $U \subsetneq X$. Elements of $\mathcal{F}(X)$ are called 'global sections', or just 'sections'. Possible alternative notations for $\mathcal{F}(U)$ are $\Gamma(U,\mathcal{F})$ and $\mathrm{H}^0(U,\mathcal{F})$. The maps ρ_{UV} are often called 'restriction maps' (from U to V, the larger set being U).

Notation 2.2.4. Motivated by Terminology 2.2.3, we shall often write $s|_V$ for the image of a section $s \in \mathcal{F}(U)$ along the restriction map ρ_{UV}.

Important Definition 2.2.1 (Sheaf, take I)**.** A *sheaf* on a topological space X, with values in $\mathscr{C}$, is a presheaf $\mathcal{F}$ such that the following two conditions hold:

(3) Fix an open subset $U \subset X$, an open cover $U = \bigcup_{i\in I} U_i$, and two sections $s, t \in \mathcal{F}(U)$ satisfying $s|_{U_i} = t|_{U_i}$ for all $i \in I$. Then $s = t$.

(4) Fix an open subset $U \subset X$, an open cover $U = \bigcup_{i\in I} U_i$ and a tuple $(s_i)_{i\in I}$ of sections $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all $(i, j) \in I \times I$. Then there exists a section $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$.

Conditions (3) and (4) generalise the conditions (i) and (ii), respectively, anticipated with the example $\mathcal{F} = C^\infty(-,\mathbb{R})$ in Section 2.1.

Terminology 2.2.5. A presheaf $\mathcal{F}$ is called *separated* if Condition (3) holds. Sometimes this condition is called *locality axiom*. Condition (4), on the other hand, is called the *glueing axiom* (or *glueing condition*).

Remark 2.2.6. Let $\mathcal{F}$ be a sheaf. Then, the section $s \in \mathcal{F}(U)$ in the glueing condition (4) is necessarily unique because $\mathcal{F}$ is separated. In fact, the two sheaf conditions could be replaced by a single condition, identical to (4), but imposing uniqueness of s.

Example 2.2.7 (Trivial sheaf)**.** The presheaf defined by $U \mapsto 0$ for every U is a sheaf and is called the *trivial sheaf* (or sometimes the *zero sheaf*). It is simply denoted by '0'.

Example 2.2.8 (Restriction to an open)**.** Let $U \subset X$ be an open subset, $\mathcal{F}$ a presheaf on X. Then, setting $\mathcal{F}|_U(V) = \mathcal{F}(V)$ for V an open subset of U, defines a presheaf $\mathcal{F}|_U$ on U, which is a sheaf as soon as $\mathcal{F}$ is. It is called the *restriction of $\mathcal{F}$ to U*.

Definition 2.2.9 (Morphism of (pre)sheaves)**.** Let X be a topological space. A *morphism* between two presheaves $\mathcal{F}, \mathcal{G}$ on X is a natural transformation $\eta\colon \mathcal{F} \Rightarrow \mathcal{G}$, i.e. a morphism in the functor category $\mathsf{Fun}(\tau_X^{\mathrm{op}}, \mathscr{C})$. A morphism of sheaves is just a morphism between the underlying presheaves.

Let us unravel the definition of natural transformation (cf. Definition A.1.9), to translate Definition 2.2.9 in more concrete terms.

To give a morphism of (pre)sheaves, one has to assign a homomorphism

$$\eta_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$$

in $\mathscr{C}$ to each $U \in \tau_X$, such that for every inclusion $V \hookrightarrow U$ of open subsets of X, the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\eta_U"] \arrow[d, "\rho^{\mathcal{F}}_{UV}"'] & \mathcal{G}(U) \arrow[d, "\rho^{\mathcal{G}}_{UV}"] \\ \mathcal{F}(V) \arrow[r, "\eta_V"] & \mathcal{G}(V) \end{tikzcd} \tag{2.2.1}$$

commutes. For the sake of clarity, we have emphasised the relevant (pre)sheaf in the restriction maps notation, but we will not be doing that systematically.

Notation 2.2.10. It is clear that presheaves on X with values in $\mathscr{C}$ form a category $\mathsf{pSh}(X,\mathscr{C})$, tautologically defined as

$$\mathsf{pSh}(X,\mathscr{C}) = \mathsf{Fun}(\tau_X^{\mathrm{op}},\mathscr{C}).$$

By Definition 2.2.9, sheaves form a full subcategory, denoted $\mathsf{Sh}(X,\mathscr{C})$. We denote by

$$j_{X,\mathscr{C}} \colon \mathsf{Sh}(X,\mathscr{C}) \hookrightarrow \mathsf{pSh}(X,\mathscr{C}) \tag{2.2.2}$$

the (fully faithful) inclusion functor.

An isomorphism of (pre)sheaves is an isomorphism in $\mathsf{pSh}(X,\mathscr{C})$, i.e. a *natural isomorphism*, i.e. a natural transformation $\eta \colon \mathcal{F} \Rightarrow \mathcal{G}$ such that η_U is an isomorphism in $\mathscr{C}$ for every $U \in \tau_X$ (cf. Definition A.1.10).

Notation 2.2.11. Since (pre)sheaves form a genuine category, from now on we shall use the classical arrow notation '$\mathcal{F} \to \mathcal{G}$' (instead of $\mathcal{F} \Rightarrow \mathcal{G}$) to denote a morphism of (pre)sheaves.

The following definition makes sense, because $\mathscr{C}$ is assumed to be a concrete category.

Definition 2.2.12 (Injective map of presheaves)**.** A morphism of (pre)sheaves $\eta \colon \mathcal{F} \to \mathcal{G}$ is *injective* if η_U is injective for every U. We denote this by writing η as '$\mathcal{F} \hookrightarrow \mathcal{G}$' (or somewhat more informally '$\mathcal{F} \subset \mathcal{G}$'), and we say that $\mathcal{F}$ is a *sub(pre)sheaf* of $\mathcal{G}$.

We close this section with a few examples and exercises.

Example 2.2.13 (Smooth functions)**.** Let X be a smooth manifold. Then, sending $U \subset X$ to the set $C^\infty(U,\mathbb{R})$ of smooth functions $U \to \mathbb{R}$, defines a sheaf $C^\infty(-,\mathbb{R})$ with values in the category of $\mathbb{R}$-algebras.

Example 2.2.14 (Holomorphic functions)**.** Let X be a complex manifold. Then, sending an open subset $U \subset X$ to the set $\mathscr{O}_X^{\mathrm{h}}(U)$ of holomorphic functions on U, defines a sheaf $\mathscr{O}_X^{\mathrm{h}}$ with values in the category of $\mathbb{C}$-algebras. Sending U to the set $\mathscr{O}_X^{\mathrm{h},\times}(U)$ of nowhere zero holomorphic functions on U defines a sheaf of abelian groups on X (the group structure being given by pointwise multiplication of functions).

Example 2.2.15 (Continuous functions are a sheaf)**.** Let X, Y be topological spaces. For $U \subset X$ open, define

$$\mathcal{F}(U) = \{\, \text{continuous functions } U \to Y \,\}.$$

Then $\mathcal{F}$ is a sheaf of sets on X.

Example 2.2.16 (Separated presheaf, not a sheaf, take I)**.** Set $X = \mathbb{C}$. Then, sending $U \subset X$ to the subset

$$\mathcal{F}(U) = \left\{ f \in \mathscr{O}_X^{\mathrm{h}}(U) \,\middle|\, f = g^2 \text{ for some } g \in \mathscr{O}_X^{\mathrm{h}}(U) \right\}$$

defines a (separated) presheaf. However, $\mathcal{F}$ is not a sheaf: the function $f(z) = z$ on the annulus

$$U = \{\, z \in \mathbb{C} \mid 1 - \varepsilon < |z| < 1 + \varepsilon \,\} \subset \mathbb{C}$$

has a square root in any neighbourhood of any point $x \in U$, but there is no global $g(z) = \sqrt{z}$ defined on the whole of U.

Exercise 2.2.17 (Separated presheaf, not a sheaf, take II)**.** Let $X = \mathbb{R}$, with the standard topology. Show that

$$U \mapsto \mathrm{B}(U) = \{\, \text{bounded continuous functions } U \to \mathbb{R} \,\}$$

is a separated presheaf on X, but not a sheaf (i.e. Condition (4) fails).

Example 2.2.18 (Constant presheaf)**.** Work with $\mathscr{C} = \mathsf{Ab} = \mathsf{Mod}_{\mathbb{Z}}$, the category of abelian groups, and fix $G \neq 0$ in this category. Fix a topological space X, and define

$$\underline{G}_X^{\mathrm{pre}}(U) = \begin{cases} G & \text{if } U \neq \emptyset, \\ 0 & \text{if } U = \emptyset. \end{cases}$$

As for the restriction maps, set $\rho_{UV} = \mathrm{id}_G$ if both U and V are nonempty. This is a presheaf, which happens to be a sheaf only in precise circumstances (cf. Exercise 2.2.20). For instance, suppose $X = U_1 \amalg U_2$ is a disjoint union of two nonempty open subsets. Then $\underline{G}_X^{\mathrm{pre}}(X) = G = \underline{G}_X^{\mathrm{pre}}(U_i)$ for $i = 1,2$. Now, $X = U_1 \amalg U_2$ is an open cover. Pick two *distinct* sections $s_i \in G = \underline{G}_X^{\mathrm{pre}}(U_i)$ for $i = 1,2$. Then, $s_1|_{U_1 \cap U_2} = s_1|_{\emptyset} = 0 = s_2|_{\emptyset} = s_2|_{U_1 \cap U_2}$, but there is no section $s \in \underline{G}_X^{\mathrm{pre}}(X) = G$ such that $s|_{U_i} = s_i$ since $\rho_{XU_i} = \mathrm{id}_G$ for $i = 1,2$ and $s_1 \neq s_2$ by assumption. Hence Condition (4), i.e. the gluing axiom, fails (whereas Condition (3) is trivially satisfied). We will see in Example 2.5.3 that $\underline{G}_X^{\mathrm{pre}}$ can be "transformed" into a sheaf by a canonical procedure.

Exercise 2.2.19. Provide examples of presheaves which satisfy the glueing axiom but not the separation axiom.

Exercise 2.2.20. Show that the constant presheaf $\underline{G}_X^{\mathrm{pre}}$ of Example 2.2.18 is a sheaf if and only if every nonempty open subset $U \subset X$ is connected.

Exercise 2.2.21 (Preheaves kernel and cokernel)**.** Let $\mathscr{C}$ be an abelian category, so that every arrow has a kernel and a cokernel. Let $\eta\colon \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves with values in $\mathscr{C}$. Consider the assignments

$$\begin{aligned} U &\mapsto (\ker_{\mathrm{pre}} \eta)(U) = \ker(\eta_U) \\ U &\mapsto (\operatorname{coker}_{\mathrm{pre}} \eta)(U) = \operatorname{coker}(\eta_U) = \mathcal{G}(U)/\operatorname{im}(\eta_U). \end{aligned}$$

Show that

(i) both $\ker_{\mathrm{pre}} \eta$ and $\operatorname{coker}_{\mathrm{pre}} \eta$ are presheaves,

(ii) There is a morphism of presheaves $\ker_{\mathrm{pre}} \eta \to \mathcal{F}$ (resp. $\mathcal{G} \to \operatorname{coker}_{\mathrm{pre}} \eta$) which satisfies the universal property of the kernel (resp. the cokernel) in $\mathsf{pSh}(X, \mathscr{C})$,

(iii) $\ker_{\mathrm{pre}} \eta$ is a sheaf, denoted $\ker(\eta)$, as soon as η is a morphism of *sheaves*,

(iv) if η is a morphism of sheaves, then $\ker(\eta)$ satisfies the universal property of the kernel in $\mathsf{Sh}(X, \mathscr{C})$, and η is injective if and only if $\ker(\eta) = 0$.

Example 2.2.22 ($\operatorname{coker}_{\mathrm{pre}} \eta$ may not be a sheaf)**.** Let $X = \mathbb{C}$ and $\mathscr{C} = \mathsf{Ab}$. Consider the morphism of sheaves

$$\exp\colon \mathscr{O}_X^{\mathrm{h}} \to \mathscr{O}_X^{\mathrm{h},\times}, \quad f \mapsto \exp\big(f\big),$$

where $\mathscr{O}_X^{\mathrm{h},\times}$ is the sheaf of nowhere zero holomorphic functions (cf. Example 2.2.14). We have that the open subset $U = X \setminus \{0\} \subset X$ is covered by the two open subsets

$$U_1 = X \setminus [0, +\infty] \subset X, \quad U_2 = X \setminus (-\infty, 0] \subset X.$$

The function $g(z) = z$ viewed in $\mathscr{O}_X^{\mathrm{h},\times}(U)$ is not of the form $\exp\big(f\big)$ for any $f \in \mathscr{O}_X^{\mathrm{h}}(U)$. Thus the $\widetilde{g}$ image of g along

$$\mathscr{O}_X^{\mathrm{h},\times}(U) \to \operatorname{coker}_{\mathrm{pre}}(\exp)(U)$$

is nonzero. However, U_1 and U_2 are simply connected, thus every function $h_i \in \mathscr{O}_X^{\mathrm{h},\times}(U_i)$ is of the form $\exp\big(f_i\big)$ for some $f_i \in \mathscr{O}_X^{\mathrm{h}}(U_i)$. Thus $\operatorname{coker}_{\mathrm{pre}}(\exp)(U_i) = 0$ for $i = 1, 2$. In particular, the restrictions $g|_{U_i}$ have this property, namely they go (necessarily) to 0 in $\operatorname{coker}_{\mathrm{pre}}(\exp)(U_i)$. If $\operatorname{coker}_{\mathrm{pre}}(\exp)$ were a sheaf, the gluing axiom would force $\widetilde{g} = 0$, which is not true.

2.3 The sheaf condition via equalisers

We now present an alternative way to define sheaves. We will repeatedly use this reinterpretation throughout these notes.

Let $\mathscr{C}$ be a category with limits (cf. Definition B.4.1). In particular, $\mathscr{C}$ has products, equalisers, and a final object (cf. Appendix B.4.1 for full details). The reader may imagine $\mathscr{C}$ to be, for instance, any of the following categories: sets, groups, rings, algebras over a fixed ring, modules over a fixed ring.

Fix a presheaf $\mathcal{F}$ with values in $\mathscr{C}$ on a topological space X. Let $\{U_i\}_{i\in I}$ be a family of open subsets of X, and set $U = \bigcup_{i\in I} U_i$. Then, by our assumption on $\mathscr{C}$, one can consider the map

$$\rho\colon \mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i), \quad s \mapsto (s|_{U_i})_{i\in I},$$

as well as the family of maps

$$\mu_{ij}\colon \prod_{i\in I} \mathcal{F}(U_i) \to \mathcal{F}(U_i) \to \mathcal{F}(U_i \cap U_j), \qquad (s_i)_{i\in I} \mapsto s_i|_{U_i\cap U_j}$$
$$\nu_{ij}\colon \prod_{i\in I} \mathcal{F}(U_i) \to \mathcal{F}(U_j) \to \mathcal{F}(U_i \cap U_j), \qquad (s_i)_{i\in I} \mapsto s_j|_{U_i\cap U_j}$$

which, taking products over $(i,j) \in I \times I$, can be assembled into two maps

$$\prod_{i\in I} \mathcal{F}(U_i) \underset{\nu}{\overset{\mu}{\rightrightarrows}} \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \cap U_j).$$

Definition 2.3.1 (Sheaf, take II)**.** Let $\mathscr{C}$ be a category with limits, X a topological space. A presheaf $\mathcal{F} \in \mathsf{pSh}(X, \mathscr{C})$ is a *sheaf* if for every family of open subsets $\{U_i\}_{i\in I}$, with $U = \bigcup_{i\in I} U_i$, the diagram

$$\mathcal{F}(U) \xrightarrow{\ \rho\ } \prod_{i\in I} \mathcal{F}(U_i) \underset{\nu}{\overset{\mu}{\rightrightarrows}} \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \cap U_j)$$

is an equaliser diagram in $\mathscr{C}$.

Informally, being an equaliser means that ρ is injective and its image agrees with the set of tuples $(s_i)_{i\in I}$ such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all pairs (i,j).

Note that Definition 2.3.1 is *element-free*. However, let us check that it agrees with Important Definition 2.2.1 when $\mathscr{C}$ is concrete: in this case the injectivity of ρ, implied by the equaliser condition, coincides with separatedness; the fact that the set-theoretic image of ρ coincides with the collection of tuples of sections $(s_i)_{i\in I}$ such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ is precisely the glueing condition.

Remark 2.3.2. Let $\mathcal{F}$ be a sheaf. Then, one has $\mathcal{F}(\emptyset) = 0$, the final object in $\mathscr{C}$. This is sometimes listed as an axiom defining a (pre)sheaf, but it does in fact follow from our assumptions (cf. Example B.4.3).

Example 2.3.3. Let $\mathcal{F}$ be a sheaf on X. If $U = \coprod_{i\in I} U_i$ is a *disjoint* union of open subsets $U_i \subset U$, then ρ is an isomorphism, i.e. $s \mapsto (s|_{U_i})_{i\in I}$ defines an isomorphism

$$\rho : \mathcal{F}(U) \xrightarrow{\sim} \prod_{i\in I} \mathcal{F}(U_i).$$

Example 2.3.4. Let $\mathscr{C}$ be an abelian category. Then a presheaf $\mathcal{F} \in \mathsf{pSh}(X, \mathscr{C})$ is a sheaf if for every family of open subsets $\{U_i\}_{i\in I}$, with $U = \bigcup_{i\in I} U_i$, the sequence

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\rho} \prod_{i\in I} \mathcal{F}(U_i) \xrightarrow{\mu-\nu} \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \cap U_j)$$

is exact, where the map denoted $\mu - \nu$ sends $(s_i)_{i\in I} \mapsto (s_i|_{U_i\cap U_j} - s_j|_{U_i\cap U_j})_{i,j}$.

The following lemma applies, for instance, to categories of groups, rings, algebras over a ring, and modules over a ring. It allows one to check the sheaf conditions in the category of sets.

LEMMA 2.3.5 ([16, Tag 0073]). *Let $\mathscr{C}$ be a category, $F : \mathscr{C} \to \mathsf{Sets}$ a faithful functor such that $\mathscr{C}$ has limits and F commutes with them. Assume that F reflects isomorphisms. Then a presheaf $\mathcal{F} \in \mathsf{pSh}(X, \mathscr{C})$ is a sheaf of and only if the underlying presheaf of sets $F \circ \mathcal{F} : \tau_X^{\mathrm{op}} \to \mathsf{Sets}$ is a sheaf.*

At the beginning of this chapter we have defined (pre)sheaves of objects in an arbitrary concrete category $\mathscr{C}$. We still have to define a few things, though, e.g. stalks and sheafification. In order for everything to be well-defined and work well (but still be compatible with all we have discussed so far, including Definition 2.3.1), we need to add a few initial data. This is provided by the following definition.

Definition 2.3.6 ([16, Tag 007L])**.** A *type of algebraic structure* is a pair $(\mathscr{C}, F)$, where $\mathscr{C}$ is a category, $F : \mathscr{C} \to \mathsf{Sets}$ is a faithful functor, such that

1. $\mathscr{C}$ has limits and F commutes with them,
2. $\mathscr{C}$ has filtered colimits and F commutes with them,
3. F reflects isomorphisms (i.e. F is *conservative*).

A few remarks are in order, before we go on.

- Equipping a category $\mathscr{C}$ with a faithful functor $F : \mathscr{C} \to \mathsf{Sets}$ is like saying that $\mathscr{C}$ is a *concrete category*, which we had already assumed in Section 2.2.
- If we have a type of algebraic structure $(\mathscr{C}, F)$, then we can verify whether a presheaf is a sheaf in the category of sets, by Lemma 2.3.5.

- The condition that F be conservative implies that a bijective morphism in $\mathscr{C}$ is an isomorphism.
- For every type of algebraic structure $(\mathscr{C}, F)$, one has the following properties:
  (i) $\mathscr{C}$ has a final object 0, and $F(0)$ is a final object in Sets (i.e. a singleton).
  (ii) $\mathscr{C}$ has products, fibre products, and equalisers — this follows from the examples in Appendix B.4.1. Moreover, F commutes with all of them.
- Examples of categories $\mathscr{C}$ having the additional structure of Definition 2.3.6 are:
  - monoids,
  - groups,
  - abelian groups,
  - rings,
  - modules over a ring.

  In all these cases, we take as the functor F the obvious forgetful functor. The reader is encouraged to just think of $\mathscr{C}$ as one of these familiar categories, and not bother too much about Definition 2.3.6. As a counterexample, however, consider the category Top of topological spaces: the forgetful functor to Sets exists but does not reflect isomorphisms (a continuous bijection need not be a homeomorphism).

2.4 Stalks, and what they tell us

Fix a type of algebraic structure $(\mathscr{C}, F\colon \mathscr{C} \to \mathsf{Sets})$ as in Definition 2.3.6. Let X be a topological space, $x \in X$ a point. The collection of open subsets $U \subset X$ containing x forms a directed system (the partial order $\succeq$ being the inclusion relation, i.e. $V \succeq U$ if and only if $V \subset U$). Indeed, given two open neighbourhoods U and V of x, there is always a third open neighbourhood of x contained in both U and V, namely $U \cap V$ or any smaller open subset containing x. In fancier language, the subcategory

$$\iota_x\colon \mathsf{Ngb}_x = \{\, U \in \tau_X \mid x \in U \,\}^{\mathrm{op}} \hookrightarrow \tau_X^{\mathrm{op}}$$

is a filtered category (see Definition B.4.9).

Important Definition 2.4.1 (Stalks)**.** Let $x \in X$ be a point, $\mathcal{F}$ a presheaf. The *stalk of $\mathcal{F}$ at x* is the filtered colimit

$$\mathcal{F}_x = \varinjlim_{\mathsf{Ngb}_x} \mathcal{F} \circ \iota_x = \varinjlim_{U \ni x} \mathcal{F}(U) \in \mathscr{C}.$$

Because F commutes with colimits, the underlying *set* $F(\mathcal{F}_x) \in \mathsf{Sets}$, still denoted $\mathcal{F}_x$, is

$$\mathcal{F}_x = \{(U,s) \mid x \in U,\ s \in \mathcal{F}(U)\}/\sim$$

where $(U,s) \sim (V,t)$ whenever there is an open neighbourhood $W \subset U \cap V$ of x such that $s|_W = t|_W$. We denote by

$$s_x = [U,s] \in \mathcal{F}_x$$

the equivalence class of the pair (U,s). It is called the *germ of* s *at* x. By definition of direct limit, there are natural homomorphisms

$$\mathcal{F}(U) \to \mathcal{F}_x, \quad s \mapsto s_x,$$

in $\mathscr{C}$, for every open neighbourhood U of x. The diagram

$$\begin{tikzcd}
s \in \mathcal{F}(U) \arrow[dr] \arrow[ddd, bend right, shift right] & & \mathcal{F}(V) \ni t \arrow[dl] \arrow[ddd, bend left, shift left] \\
& \mathcal{F}(U \cap V) \arrow[d] & \\
& \mathcal{F}(W) \arrow[d] & \\
& \mathcal{F}_x &
\end{tikzcd}$$

illustrates the fact that two sections $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$ define the same element in the stalk $\mathcal{F}_x$ if and only if there is an intermediate open subset $W \subset U \cap V$ over which they agree.

Figure 2.1: A bunch of sheaves sitting in their natural habitat. The little tops of each leaf of corn are the stalks.

LEMMA 2.4.1. *If $\mathcal{F}$ is a separated presheaf of sets (e.g. a sheaf), then the natural map*

$$\sigma_U^{\mathcal{F}} : \mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x, \quad s \mapsto (s_x)_{x\in U} \tag{2.4.1}$$

is injective for every open subset U of X.

The lemma means, at an informal level, that sections are determined by their germs.

Proof. If s and t are sections in $\mathcal{F}(U)$ such that $s_x = t_x$ in $\mathcal{F}_x$ for every $x \in U$, then for every $x \in U$ there is an open neighbourhood $U_x \subset U$ such that $s|_{U_x} = t|_{U_x}$. But this holds for every $x \in U$, and $U = \bigcup_{x\in U} U_x$ is an open covering, thus by the separation axiom we deduce $s = t$, i.e. $\sigma_U^{\mathcal{F}}$ is injective. □

Consider the following property of a tuple $(s_x)_{x\in U} \in \prod_{x\in U} \mathcal{F}_x$, for $U \subset X$ an open subset:

$$\begin{gathered} \text{for every } x \in U \text{ there exists a pair } (V_x, t^x), \\ \text{with } x \in V_x \subset U \text{ and } t^x \in \mathcal{F}(V_x), \\ \text{such that } t^x_y = s_y \text{ for all } y \in V_x. \end{gathered} \tag{2.4.2}$$

Definition 2.4.2 (Compatible germs)**.** Let $\mathcal{F}$ be a presheaf on X, and let $U \subset X$ be an open subset. We say that $(s_x)_{x\in U} \in \prod_{x\in U} \mathcal{F}_x$ is a *tuple of compatible germs* if Condition (2.4.2) is fulfilled.

We always have inclusions

$$\operatorname{im}(\sigma_U^{\mathcal{F}}) \subset \{ \text{tuples } (s_x)_{x\in U} \text{ of compatible germs} \} \subset \prod_{x\in U} \mathcal{F}_x \tag{2.4.3}$$

where the first inclusion is justified by taking $V_x = U$ and $t^x = s$ for every $x \in U$ as soon as $\sigma_U^{\mathcal{F}}(s) = (s_x)_{x\in U}$. If $\mathcal{F}$ is a sheaf, then tuples of compatible germs form precisely the image of the map (2.4.1), i.e. the first inclusion in (2.4.3) is an equality. Indeed, assume $(s_x)_{x\in U}$ consists of compatible germs. Let $\{(V_x, t^x) \mid x \in U\}$ be as in the displayed condition (2.4.2). By the compatibility condition, for every pair $(x, x') \in U \times U$ we have

$$t^x_y = t^{x'}_y, \quad y \in V_x \cap V_{x'}.$$

It follows from Lemma 2.4.1 that

$$t^x\big|_{V_x \cap V_{x'}} = t^{x'}\big|_{V_x \cap V_{x'}}. \tag{2.4.4}$$

Now, we have an open cover $U = \bigcup_{x\in U} V_x$, so by the glueing axiom, applicable by (2.4.4), the sections $t^x \in \mathcal{F}(V_x)$ glue to a (unique) section $t \in \mathcal{F}(U)$ such that $t|_{V_x} = t^x$. But $t^x_y = s_y$ for $y \in V_x$, and this holds for every $x \in U$, so $\sigma_U^{\mathcal{F}}(t) = (s_x)_{x\in U}$.

Summing up, when $\mathcal{F}$ is a sheaf, we have a bijection

$$\sigma_U^{\mathcal{F}} \colon \mathcal{F}(U) \xrightarrow{\sim} \{\text{tuples } (s_x)_{x \in U} \text{ of compatible germs}\}.$$

This also shows that *sections of a sheaf can always be identified with 'nicely gluable' functions!* Indeed, tuples $(s_x)_{x\in U}$ correspond to particular functions $U \to \coprod_{x\in U} \mathcal{F}_x$, sending $x \in U$ inside the corresponding stalk, and doing so in a compatible way.

LEMMA 2.4.3. *Let $s, t \in \mathcal{F}(X)$ be two global sections of a sheaf $\mathcal{F}$, such that $s_x = t_x \in \mathcal{F}_x$ for every $x \in X$. Then $s = t$.*

Proof. This is just a special case of Lemma 2.4.1. $\square$

Exercise 2.4.4. Let $\mathcal{F}$ be a sheaf on X, and let $s, t \in \mathcal{F}(X)$ be two global sections. Show that

$$\{x \in X \mid s_x = t_x\} \subset X$$

is an open subset of X.

A morphism of presheaves $\eta \colon \mathcal{F} \to \mathcal{G}$ induces a morphism $\eta_x \colon \mathcal{F}_x \to \mathcal{G}_x$ at the level of stalks for every $x \in X$, defined by

$$s_x = [U, s] \mapsto [U, \eta_U(s)] = (\eta_U(s))_x. \tag{2.4.5}$$

Exercise 2.4.5. Check that (2.4.5) is well-defined.

If $U \subset X$ is an open subset containing a point $x \in X$, then the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\eta_U"] \arrow[d] & \mathcal{G}(U) \arrow[d] \\ \mathcal{F}_x \arrow[r, "\eta_x"] & \mathcal{G}_x \end{tikzcd} \qquad \begin{tikzcd} s \arrow[r, mapsto, "\eta_U"] \arrow[d, mapsto] & \eta_U(s) \arrow[d, mapsto] \\ s_x \arrow[r, mapsto, "\eta_x"] & (\eta_U(s))_x \end{tikzcd}$$

commutes. What we have just said can be rephrased by saying that the association $\mathcal{F} \mapsto \mathcal{F}_x$ defines a functor

$$\mathsf{stalk}_x \colon \mathsf{pSh}(X, \mathscr{C}) \to \mathscr{C}. \tag{2.4.6}$$

We will see that in reasonable circumstances the restriction of this functor to the category of sheaves is *exact* (cf. Proposition 2.5.14).

Definition 2.4.6. A morphism of (pre)sheaves $\eta \colon \mathcal{F} \to \mathcal{G}$ is *surjective* if η_x is surjective for every $x \in X$.

Warning 2.4.7. You may have noticed that surjectivity of a map of sheaves (cf. Definition 2.4.6) is defined differently than injectivity (cf. Definition 2.2.12)!

Let $\eta\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then

$$\eta_x \text{ is surjective} \iff \begin{array}{c}\text{for every } t_x \in \mathcal{G}_x \text{ there exists an open neighbourhood}\\ U \text{ of } x \text{ and a section } s \in \mathcal{F}(U) \text{ such that } (\eta_U(s))_x = t_x.\end{array}$$

$$\eta \text{ is surjective} \iff \begin{array}{c}\text{for every open subset } U \subset X \text{ and for every}\\ t \in \mathcal{G}(U), \text{ there exists a covering } U = \bigcup_{i\in I} U_i\\ \text{such that } t|_{U_i} \text{ is in the image of } \eta_{U_i} \text{ for every } i.\end{array}$$

The second equivalence is obtained as follows.

Proof of '⇒'. Assume η is surjective, i.e. η_x is surjective for every $x \in X$. Fix $U \subset X$ open and a local section $t \in \mathcal{G}(U)$. For every $x \in U$, we have a commutative diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\eta_U"] \arrow[d] & \mathcal{G}(U) \arrow[d] & t \arrow[d, mapsto] \\ \mathcal{F}_x \arrow[r, two heads, "\eta_x"] & \mathcal{G}_x & t_x \end{tikzcd}$$

where $t_x \in \mathcal{G}_x$ can be lifted along η_x to an element $s_x \in \mathcal{F}_x$. Let (V_x, s) be a representative for s_x, so that in particular $s \in \mathcal{F}(V_x)$. The identity $\eta_x(s_x) = t_x$ implies that there is an open neighbourhood $x \in U_x \subset V_x \cap U$ such that

$$\eta_{U_x}(s|_{U_x}) = t|_{U_x}.$$

Now this holds for every $x \in U$, and the elements of $\{\, U_x \mid x \in U \,\}$ form a covering of U, thus we have proved the condition.

Proof of '⇐'. Conversely, assuming the condition, let us prove surjectivity of η. Fix $x \in X$ along with a germ $t_x \in \mathcal{G}_x$. We need to prove that t_x has a preimage in $\mathcal{F}_x$. Let (U, t) be a representative of t_x, so that $t \in \mathcal{G}(U)$. By the condition we are assuming, there exists a covering $U = \bigcup_{i\in I} U_i$ such that $t|_{U_i} = \eta_{U_i}(s_i)$ for some $s_i \in \mathcal{F}(U_i)$, for every $i \in I$. If $x \in U_i$, we have a commutative diagram

$$\begin{tikzcd} \mathcal{F}(U_i) \arrow[r, "\eta_{U_i}"] \arrow[d] & \mathcal{G}(U_i) \arrow[d] & s_i \arrow[r, mapsto] \arrow[d, mapsto] & t|_{U_i} \arrow[d, mapsto] \\ \mathcal{F}_x \arrow[r, "\eta_x"] & \mathcal{G}_x & \star \arrow[r, mapsto] & t_x \end{tikzcd}$$

so the element $\star \in \mathcal{F}_x$ is a preimage of t_x. The equivalence is proved.

The next result incarnates the local nature of sheaves.

LEMMA 2.4.8. *Let $\eta\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. The following are equivalent:*

(i) *η is an isomorphism,*

(ii) *η_x is an isomorphism for every $x \in X$,*

(iii) *η is injective and surjective.*

Proof. Recall that η is an isomorphism if and only if η_U is an isomorphism for every U.

Proof of (i) $\Rightarrow$ (ii). By functoriality of $\mathcal{F} \mapsto \mathcal{F}_x$, we have that if η is an isomorphism, then so is η_x for every $x \in X$.

Proof of (ii) $\Rightarrow$ (i). Suppose η_x is an isomorphism for every x. Let $U \subset X$ be an open subset: we need to show that η_U is an isomorphism.

To see that η_U is injective, pick $s, t \in \mathcal{F}(U)$ such that $\eta_U(s) = \eta_U(t) \in \mathcal{G}(U)$. Then, for any $x \in U$, one has

$$\eta_x(s_x) = (\eta_U(s))_x = (\eta_U(t))_x = \eta_x(t_x),$$

which implies $s_x = t_x$ by injectivity of η_x. This holds for every $x \in U$ by assumption, thus $s = t$ by Lemma 2.4.3. Therefore, η_U is injective for every U (i.e. η is injective).

To see that η_U is surjective, pick $t \in \mathcal{G}(U)$. By surjectivity of η (which we have by definition since η_x is surjective for every $x \in X$), we can find an open cover $U = \bigcup_{i\in I} U_i$ along with a collection of sections $s_i \in \mathcal{F}(U_i)$ such that $\eta_{U_i}(s_i) = t|_{U_i}$. But by the previous paragraph η is injective, so s_i and s_j agree on $U_i \cap U_j$. Therefore, since $\mathcal{F}$ is a sheaf, they glue to a section $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$. By construction, $\eta_U(s)|_{U_i} = \eta_{U_i}(s_i) = t|_{U_i}$, which implies $\eta_U(s) = t$ since $\mathcal{G}$ is a sheaf. Thus η_U is surjective.

Proof of (ii) $\Rightarrow$ (iii). The first paragraph of '(ii) $\Rightarrow$ (i)' already shows that if η_x is an isomorphism for every $x \in X$, then η_U is injective for all U, i.e. η is injective. Surjectivity follows from the definition.

Proof of (iii) $\Rightarrow$ (ii). We only need to show that if η_U is injective for every U, then η_x is injective for every $x \in X$. Consider $s_x = [U, s]$ and $s'_x = [U', s']$ two germs in $\mathcal{F}_x$ such that $\eta_x(s_x) = \eta_x(s'_x)$ in $\mathcal{G}_x$. Then there is an open subset $W \subset U \cap U'$ such that $\eta_U(s)|_W = \eta_{U'}(s')|_W$. But by compatibility of η_W with restrictions, this is equivalent to the identity $\eta_W(s|_W) = \eta_W(s'|_W)$, which by our assumption implies $s|_W = s'|_W$. But then $s_x = s'_x$. $\square$

Warning 2.4.9. It is not true that two sheaves with isomorphic stalks are isomorphic: there may be no map between them! For instance, consider a topological space X consisting of two points x_0, x_1 where only x_0 is a closed point. Thus X and $U = X \setminus \{x_0\}$ are the only nonempty open subsets of X. Fix an abelian group $G \neq 0$ and define $\mathcal{F}(X) = G = \mathcal{F}(U)$. Then choose either $\rho_{XU} = \mathrm{id}_G$ or $\rho_{XU} = 0$ to define two distinct sheaves on X. They have the same stalks but they are not isomorphic.

Exercise 2.4.10. Show that Lemma 2.4.8 fails for presheaves.

Example 2.4.11 (Surjectivity is subtle)**.** Let $\mathcal{F} = \mathcal{O}_X^{\mathrm{h}}$ be the sheaf of holomorphic functions on $X = \mathbb{C} \setminus \{0\}$, and let $\mathcal{G} = \mathcal{F}^\times$ be the sheaf of invertible holomorphic functions on X. The map $\exp\colon \mathcal{F} \to \mathcal{G}$ is surjective, but $\exp_X\colon \mathcal{F}(X) \to \mathcal{G}(X)$ is not surjective, e.g. the function $f(z) = z$ in $\mathcal{G}(X)$ is not the exponential of a homolomorphic function (cf. Example 2.2.22).

Example 2.4.12 (Skyscraper sheaf)**.** Let X be a topological space, G a nontrivial abelian group, $x \in X$ a point. The assignment

$$U \mapsto G_x(U) = \begin{cases} G & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$$

defines a sheaf of abelian groups, choosing as restriction maps the identity of G or the zero map in the obvious way. This sheaf is called the *skyscraper sheaf* attached to (X, x, G). At the level of stalks, one has

$$(G_x)_y = \begin{cases} G & \text{if } y \in \overline{\{x\}} \\ 0 & \text{if } y \notin \overline{\{x\}}, \end{cases}$$

because if y is in the closure of x then every neighbourhood of y also contains x, whereas if y is *not* in the closure of x, then $U = X \setminus \overline{\{x\}}$ is the largest open neighbourhood of y and thus $(G_x)_y = 0$ since $G_x(U) = 0$. Thus G_x has only one nonzero stalk (at x) if and only if x is a closed point. This is the case where the name 'skyscraper sheaf' for G_x fits best.

Exercise 2.4.13. Let $\mathcal{F}$ be a presheaf, $\mathcal{G}$ a *sheaf*, and let $\eta_1, \eta_2 \colon \mathcal{F} \to \mathcal{G}$ be two morphisms of presheaves of sets such that $\eta_{1,x} = \eta_{2,x}$ for every $x \in X$. Show that $\eta_1 = \eta_2$. Show that it is in fact necessary to assume $\mathcal{G}$ to be a sheaf. This exercise will be needed in Theorem 3.1.61.

2.5 Sheafification

Fix a type of algebraic structure $(\mathscr{C}, F \colon \mathscr{C} \to \mathsf{Sets})$. Friendly translation: fix $\mathscr{C}$ to be either of the following categories:

- monoids,
- groups,
- abelian groups,
- rings,
- modules over a ring.

Let X be a topological space. Let $\mathcal{F} \colon \tau_X^{\mathrm{op}} \to \mathscr{C}$ be a presheaf. We next define a *sheaf* $\mathcal{F}^{\#}$, called the sheafification of $\mathcal{F}$, via an explicit universal property, and having precisely the same stalks as the initial presheaf $\mathcal{F}$.

Definition 2.5.1 (Sheafification of a presheaf)**.** Let $\mathcal{F} \in \mathsf{pSh}(X, \mathscr{C})$ be a presheaf. A *sheafification* of $\mathcal{F}$ is a pair $(\mathcal{F}^{\#}, \theta)$, where $\mathcal{F}^{\#} \in \mathsf{Sh}(X, \mathscr{C})$ is a sheaf and $\theta \colon \mathcal{F} \to \mathcal{F}^{\#}$ is a morphism of presheaves, such that for every other pair $(\mathcal{G}, \alpha)$ where $\mathcal{G}$ is a sheaf and $\alpha \colon \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, there exists a unique morphism of sheaves $\widetilde{\alpha} \colon \mathcal{F}^{\#} \to \mathcal{G}$ such that $\alpha = \widetilde{\alpha} \circ \theta$.

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "\theta"] \arrow[dr, "\forall \alpha"'] & & \mathcal{F}^{\#} \arrow[dl, dashed, "\exists ! \widetilde{\alpha}"] \\ & \mathcal{G} & \end{tikzcd}$$

PROPOSITION 2.5.2. *Let $\mathcal{F} \in \mathsf{pSh}(X, \mathscr{C})$ be a presheaf. Then a sheafification $(\mathcal{F}^{\#}, \theta)$ exists, and the map $\theta_x \colon \mathcal{F}_x \to \mathcal{F}^{\#}_x$ is an isomorphism for every $x \in X$.*

What follows immediately from Proposition 2.5.2 is that $\mathcal{F}^{\#}$ is unique up to a unique isomorphism, and moreover the canonical map $\theta \colon \mathcal{F} \to \mathcal{F}^{\#}$ is an isomorphism precisely when $\mathcal{F}$ is already a sheaf.

Proof. Let $U \subset X$ be an open subset. Define

$$\mathcal{F}^{\#}(U) = \left\{ \text{functions } U \xrightarrow{f} \coprod_{x \in U} \mathcal{F}_x \;\middle|\; \begin{array}{c} \text{for every } x \in U,\ f(x) \in \mathcal{F}_x \text{ and there exist an} \\ \text{open neighbourhood } V \subset U \text{ of } x \text{ and } s \in \mathcal{F}(V) \\ \text{such that } f(y) = s_y \text{ for every } y \in V \end{array} \right\}.$$

Note that, since $\mathscr{C}$ has products, we can view a function f as above as a tuple

$$(f(x))_{x \in U} \in \prod_{x \in U} \mathcal{F}_x$$

and we can rephrase the definition of $\mathcal{F}^{\#}(U)$ by saying that

$$\mathcal{F}^{\#}(U) = \{ \text{tuples } (s_x)_{x \in U} \text{ of compatible germs} \}.$$

See Definition 2.4.2 for the definition of compatible germs. Functoriality of the assignment $U \mapsto \mathcal{F}^{\#}(U)$ is clear (functions restrict!), thus $\mathcal{F}^{\#}$ is a presheaf. The morphism $\theta_U \colon \mathcal{F}(U) \to \mathcal{F}^{\#}(U)$ defined by sending $s \in \mathcal{F}(U)$ to the function

$$f_s \colon U \to \coprod_{x \in U} \mathcal{F}_x, \quad x \mapsto s_x = [U, s] \in \mathcal{F}_x$$

determines a morphism of presheaves, being compatible with restrictions. It is just the function $\sigma_U^{\mathcal{F}}$ introduced in (2.4.1)!

The presheaf $\mathcal{F}^{\#}$ is a sheaf: Fix an open cover $U = \bigcup_{i \in I} U_i$ of some open subset $U \subset X$ and a collection of sections $f_i \in \mathcal{F}^{\#}(U_i)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for every i and j. We need to find a unique $f \in \mathcal{F}^{\#}(U)$ such that $f|_{U_i} = f_i$. Define

$$f \in \prod_{x \in U} \mathcal{F}_x = \operatorname{Hom}\left(U, \coprod_{x \in U} \mathcal{F}_x \right)$$

by the rule

$$f(x) = f_i(x) \in \mathcal{F}_x, \quad x \in U_i \subset U.$$

This is well-defined since, even though x can lie in more than one open U_i, by assumption we have $f_i(x) = f_j(x)$ as soon as $x \in U_i \cap U_j$. We need to check that f defines an element of the subset $\mathcal{F}^\#(U) \subset \prod_{x\in U} \mathcal{F}_x$. But for every $i \in I$ we know the following: for every $x \in U_i$ there exist an open neighbourhood $x \in V_i \subset U_i$ and a section $s_i \in \mathcal{F}(V_i)$ such that $f(y) = f_i(y) = (s_i)_y$ for all $y \in V_i$. But V_i is also open in U, so the condition defining $\mathcal{F}^\#(U)$ also holds for f. Thus $f \in \mathcal{F}^\#(U)$ satisfies $f|_{U_i} = f_i$, and is clearly unique with this property.

The pair $(\mathcal{F}^\#, \theta)$ *is the sheafification.* Assume we have a sheaf $\mathcal{G}$ and a morphism of presheaves $\alpha\colon \mathcal{F} \to \mathcal{G}$. We need to define a morphism $\widetilde{\alpha}\colon \mathcal{F}^\# \to \mathcal{G}$ of presheaves such that $\alpha = \widetilde{\alpha} \circ \theta$. For every U open in X, we need to define a morphism $\widetilde{\alpha}_U\colon \mathcal{F}^\#(U) \to \mathcal{G}(U)$ in such a way that $\alpha_U = \widetilde{\alpha}_U \circ \theta_U$. Fix $s = (s_x)_{x\in U} \in \mathcal{F}^\#(U)$. The composition

$$\begin{tikzcd} U \arrow[r, "s"] \arrow[dr, "\widetilde{\alpha}_U(s)"'] & \coprod_{x\in U} \mathcal{F}_x \arrow[d, "\coprod_{x\in U} \alpha_x"] \\ & \coprod_{x\in U} \mathcal{G}_x \end{tikzcd}$$

defines a tuple of compatible germs for $\mathcal{G}$ over U, hence an element $\widetilde{\alpha}_U(s) \in \mathcal{G}^\#(U) = \mathcal{G}(U)$, using that $\mathcal{G}$ is a sheaf for this identity. This is the required morphism $\widetilde{\alpha}\colon \mathcal{F}^\# \to \mathcal{G}$.

The map θ *is an isomorphism on stalks.* The map θ, at the level of stalks, is defined by

$$\theta_x[U,s] = [U, f_s].$$

Injectivity: Suppose $\theta_x[U,s] = \theta_x[V,t]$ for two classes $[U,s],[V,t] \in \mathcal{F}_x$, i.e. assume $[U,f_s] = [V,f_t]$ in $\mathcal{F}^\#_x$. Then, by definition of germ, there exists an open neighbourhood $W \subset U \cap V$ of x such that $f_s|_W = f_t|_W$. But this means, by definition of f_s and f_t, that $s_y = t_y$ for all $y \in W$. Thus, in particular, $s_x = t_x$. But this is just the equality $[U,s] = [V,t]$ we were after.

Surjectivity: Pick a class $[U,f] \in \mathcal{F}^\#_x$ for some $f \in \mathcal{F}^\#(U)$ and open neighbourhood U of x. Then, for every $z \in U$, there exist an open neighbourhood $V \subset U$ of z and a section $s \in \mathcal{F}(V)$ such that $f(y) = s_y$ in $\mathcal{F}_y$ for every $y \in V$. We claim that $[U,f] = \theta_x(s_x)$, where $s_x = [V,s]$. Indeed, $\theta_x(s_x) \in \mathcal{F}^\#_x$ is the equivalence class of the map

$$f_s\colon V \to \coprod_{y\in V} \mathcal{F}_y, \quad y \mapsto s_y.$$

But this map agrees with the restriction of f to $V \subset U$ (by the condition $f(y) = s_y$ recalled above), i.e. $f_s = f|_V \in \mathcal{F}^\#(V)$. Since V is also an open neighbourhood of x, it follows that $(f|_V)_x = (f_s)_x = [V, f_s] = \theta_x(s_x) \in \mathcal{F}^\#_x$, but of course $(f|_V)_x = [U,f]$. Thus θ_x is surjective. $\square$

Example 2.5.3 (Constant sheaf)**.** Let G be a nontrivial abelian group. The *constant sheaf* on a topological space X, with values in G, is the sheafification $\underline{G}_X$ of the presheaf $\underline{G}_X^{\mathrm{pre}}$ defined in Example 2.2.18. This sheaf agrees with the sheaf whose sections over U are the locally constant functions $U \to G$. This, in turn, agrees with the following: endow G with the discrete topology and consider the assignment

$$U \mapsto \{\,\text{continuous maps } U \to G\,\},$$

which we know is a sheaf by Example 2.2.15. If $U \subset X$ is a connected open subset, then $\underline{G}_X(U) = G$. By Proposition 2.5.2, at the level of stalks we have $\underline{G}_{X,x} = G$ for every $x \in X$, since the stalks of the constant presheaf are manifestly all equal to G.

Exercise 2.5.4. Let X be a connected topological space, x a point, G a nontrivial abelian group. Under what condition(s) is the constant sheaf $\underline{G}_X$ equal to the skyscraper sheaf G_x (cf. Example 2.4.12)?

Exercise 2.5.5. Show that sending $\mathcal{F} \mapsto \mathcal{F}^\#$ defines a functor $(-)^\# \colon \mathsf{pSh}(X,\mathscr{C}) \to \mathsf{Sh}(X,\mathscr{C})$, and that the forgetful functor $j_{X,\mathscr{C}} \colon \mathsf{Sh}(X,\mathscr{C}) \hookrightarrow \mathsf{pSh}(X,\mathscr{C})$ is a right adjoint. This means (cf. Definition A.1.17) that are bifunctorial bijections

$$\psi_{\mathcal{F},\mathcal{G}} \colon \mathrm{Hom}_{\mathsf{Sh}(X,\mathscr{C})}(\mathcal{F}^\#,\mathcal{G}) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathsf{pSh}(X,\mathscr{C})}(\mathcal{F},\mathcal{G}), \quad \widetilde{\alpha} \mapsto \widetilde{\alpha} \circ \theta$$

for any presheaf $\mathcal{F}$ and sheaf $\mathcal{G}$. (**Hint**: the universal property of the sheafification!).

2.5.1 Subsheaves, Quotient sheaves

We have essentially already proved the following general result.

Proposition 2.5.6 ([16, Tag 007S]). *Let X be a topological space. Let $\mathcal{F},\mathcal{G} \in \mathsf{Sh}(X,\mathsf{Sets})$ be sheaves of sets, $\eta \colon \mathcal{F} \to \mathcal{G}$ a morphism. Then, the following are equivalent:*

(a) *η is a monomorphism,*

(b) *$\eta_x \colon \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$,*

(c) *$\eta_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all open subsets $U \subset X$ (i.e. η is injective).*

Furthermore, the following are equivalent:

(i) *η is an epimorphism,*

(ii) *$\eta_x \colon \mathcal{F}_x \to \mathcal{G}_x$ is surjective for all $x \in X$ (i.e. η is surjective),*

and are implied (but not equivalent to, cf. Example 2.4.11!) by the condition

(iii) *$\eta_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective for all open subsets $U \subset X$.*

If $\mathscr{C}$ is an abelian category (e.g. Mod_A for a fixed ring A), then Proposition 2.5.6 holds replacing Sets with $\mathscr{C}$.

Definition 2.5.7 (Subsheaf, quotient sheaf)**.** If there exists a morphism of sheaves $\eta\colon \mathcal{F} \to \mathcal{G}$ such that either of the equivalent conditions (a), (b) or (c) holds, we say that $\mathcal{F}$ is a *subsheaf* of $\mathcal{G}$ (and we may denote this by '$\mathcal{F} \subset \mathcal{G}$'). If either of the equivalent conditions (i) or (ii) holds, we say that $\mathcal{G}$ is a *quotient sheaf* of $\mathcal{F}$.

Example 2.5.8 (Quotient by a subsheaf)**.** Let $\mathscr{C}$ be an abelian category. If $\mathcal{F} \subset \mathcal{G}$ is a subsheaf (with values in $\mathscr{C}$), then sending

$$U \mapsto \mathcal{G}(U)/\mathcal{F}(U) \tag{2.5.1}$$

is a presheaf on X, because the restriction maps respect the inclusions $\mathcal{F}(U) \hookrightarrow \mathcal{G}(U)$, and thus pass to the quotients. Its sheafification $\mathcal{G}/\mathcal{F}$ is called the *quotient sheaf of* $\mathcal{G}$ *by* $\mathcal{F}$. There is a natural morphism of sheaves $\mathcal{G} \to \mathcal{G}/\mathcal{F}$.

Definition 2.5.9 (Sheaf image, sheaf cokernel)**.** Let $\mathscr{C}$ be an abelian category, $\eta\colon \mathcal{F} \to \mathcal{G}$ a morphism of sheaves (with values in $\mathscr{C}$), so that $\ker(\eta) \hookrightarrow \mathcal{F}$ is a subsheaf by Exercise 2.2.21. The sheafification $\operatorname{im}(\eta)$ of the presheaf

$$U \mapsto \operatorname{im}_{\text{pre}}(U) = \operatorname{im}(\eta_U) = \mathcal{F}(U)/\ker(\eta_U)$$

is called the *image of* η. It is a special case of Example 2.5.8 and defines a subsheaf

$$\operatorname{im}(\eta) = \mathcal{F}/\ker(\eta) \subset \mathcal{G}.$$

The quotient sheaf

$$\operatorname{coker}(\eta) = \mathcal{G}/\operatorname{im}(\eta),$$

again a special case of Example 2.5.8, is called the *sheaf cokernel*.

Exercise 2.5.10. Let $\mathscr{C}$ be an abelian category. Let $\eta\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves with values in $\mathscr{C}$. Show that the composition

$$\mathcal{G} \to \operatorname{coker}_{\text{pre}} \eta \to \operatorname{coker}(\eta),$$

where the first morphism is given by the natural maps $\mathcal{G}(U) \twoheadrightarrow \mathcal{G}(U)/\operatorname{im}(\eta_U)$ and the last morphism is the sheafification, is a cokernel in the category $\mathsf{Sh}(X, \mathscr{C})$.

Remark 2.5.11. Set $\mathscr{C} = \mathsf{Mod}_A$ (or any Grothendieck abelian category so that, by definition, filtered colimits exist and are exact). Let $\mathcal{F} \subset \mathcal{G}$ be a subsheaf, $x \in X$ a point. Then

$$(\mathcal{G}/\mathcal{F})_x = \mathcal{G}_x/\mathcal{F}_x \tag{2.5.2}$$

in Mod_A. This follows from the fact that $(\mathcal{G}/\mathcal{F})_x$ agrees with the stalk of the *presheaf* (2.5.1), and from right exactness of filtered colimits. Moreover, if $\eta\colon \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves and $x \in X$ is a point, then

$$\begin{aligned} \ker(\eta)_x &= \ker(\eta_x) \\ \operatorname{im}(\eta)_x &= \operatorname{im}(\eta_x) \\ \operatorname{coker}(\eta)_x &= \operatorname{coker}(\eta_x). \end{aligned} \tag{2.5.3}$$

The first identity in (2.5.3) follows from the fact that filtered colimits are *also left exact* in Mod_A, thus

$$\begin{aligned} \ker\left(\mathcal{F}_x \xrightarrow{\eta_x} \mathcal{G}_x\right) &= \ker\left(\varinjlim_{U \ni x} \mathcal{F}(U) \to \varinjlim_{U \ni x} \mathcal{G}(U)\right) \\ &= \varinjlim_{U \ni x} \ker(\mathcal{F}(U) \to \mathcal{G}(U)) \\ &= \ker(\eta)_x. \end{aligned}$$

The last two identities in (2.5.3) are a special case of (2.5.2).

THEOREM 2.5.12 ([6, §10]). *If $\mathscr{C}$ is a Grothendieck abelian category, then $\mathsf{Sh}(X, \mathscr{C})$ is a Grothendieck abelian category.*

Definition 2.5.13. A *short exact sequence of sheaves* with values in a Grothendieck abelian category $\mathscr{C}$ is a short exact sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{\ \iota\ } \mathcal{G} \xrightarrow{\ \pi\ } \mathcal{H} \longrightarrow 0$$

of objects in the abelian category $\mathsf{Sh}(X, \mathscr{C})$. Explicitly, exactness means that ι is injective, π is surjective and $\operatorname{im}(\iota) = \ker(\pi)$.

PROPOSITION 2.5.14. *Let $\mathscr{C}$ be a Grothendieck abelian category. A sequence*

$$0 \longrightarrow \mathcal{F} \xrightarrow{\ \iota\ } \mathcal{G} \xrightarrow{\ \pi\ } \mathcal{H} \longrightarrow 0$$

of objects in $\mathsf{Sh}(X, \mathscr{C})$ is a short exact sequence if and only if

$$0 \longrightarrow \mathcal{F}_x \xrightarrow{\ \iota_x\ } \mathcal{G}_x \xrightarrow{\ \pi_x\ } \mathcal{H}_x \longrightarrow 0$$

is a short exact sequence in $\mathscr{C}$ for every $x \in X$.

Proof. Combine Remark 2.5.11 and Lemma 2.4.8 with one another. □

Exercise 2.5.15. Let $\eta\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of A-modules, for A a ring. Prove that there is an exact sequence of sheaves

$$0 \longrightarrow \ker(\eta) \longrightarrow \mathcal{F} \xrightarrow{\ \eta\ } \mathcal{G} \longrightarrow \operatorname{coker}(\eta) \longrightarrow 0.$$

In particular, if η is injective, this reduces to

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{F} \longrightarrow 0.$$

Exercise 2.5.16. Let A be a ring. For a nonempty open subset U of a topological space X, consider the functor $\Gamma(U,-)\colon \mathsf{Sh}(X,\mathsf{Mod}_A) \to \mathsf{Mod}_A$ sending $\mathcal{F} \mapsto \mathcal{F}(U)$. Show that it is left exact. That is, it transforms an exact sequence of sheaves $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ into an exact sequence of A-modules

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \longrightarrow \mathcal{H}(U).$$

When $U = X$, this functor takes $\mathcal{F} \mapsto \mathcal{F}(X)$ and is thus called the *global section functor*. Another notation used for it in the literature is $\mathrm{H}^0(X,-)$, cf. Terminology 2.2.3.

2.6 Supports

Let A be a ring. Let $\mathcal{F} \in \mathsf{Sh}(X,\mathsf{Mod}_A)$ be a sheaf of A-modules on a topological space X. Let $U \subset X$ be an open subset, and fix a section $s \in \mathcal{F}(U)$. We have two notions of support: the support of $\mathcal{F}$, and the support of s, defined respectively as

$$\begin{aligned} \operatorname{Supp}(\mathcal{F}) &= \{\, x \in X \mid \mathcal{F}_x \neq 0 \,\}, \\ \operatorname{Supp}(s) &= \{\, x \in U \mid s_x \neq 0 \text{ in } \mathcal{F}_x \,\}. \end{aligned} \tag{2.6.1}$$

If $s_x = 0$, then there is an open neighbourhood $x \in V \subset U$ such that $s|_V = 0 \in \mathcal{F}(V)$. Thus $V \subset U \setminus \operatorname{Supp}(s)$ and hence $\operatorname{Supp}(s) \subset U$ is closed. In fact, this follows from (or solves) Exercise 2.4.4. In general, however, $\operatorname{Supp}(\mathcal{F}) \subset X$ is *not* closed, as the two next examples show.

Example 2.6.1 (Supp($\mathcal{F}$) need not be closed, take I)**.** Let X be an irreducible topological space. This means that any two nonempty open subsets of X intersect. Fix a nontrivial abelian group G, a point $x \in X$, and for $U \in \tau_X$ define

$$\mathcal{F}(U) = \begin{cases} 0 & \text{if } U = \emptyset \text{ or } x \in U \\ G & \text{otherwise.} \end{cases}$$

Let $\rho_{UV} \in \{\mathrm{id}_G, 0\}$ be chosen in the obvious way for all $U, V \in \tau_X$. Then $\mathcal{F}$ is a sheaf of abelian groups on X, with stalks

$$\mathcal{F}_y = \begin{cases} 0 & \text{if } y \in \overline{\{x\}} \\ G & \text{otherwise.} \end{cases}$$

It follows that

$$\operatorname{Supp}(\mathcal{F}) = X \setminus \overline{\{x\}},$$

which is not closed in X as soon as $\overline{\{x\}} \hookrightarrow X$ is not open.

Example 2.6.2 (Supp($\mathcal{F}$) need not be closed, take II)**.** Let $j\colon U \hookrightarrow X$ be the inclusion of an open subset U of a topological space X. Let $\mathcal{F} \in \mathsf{Sh}(U,\mathscr{C})$ be a sheaf. Define $j_!\mathcal{F} \in \mathsf{Sh}(X,\mathscr{C})$ to be the sheafification of the presheaf $j_!^{\mathrm{pre}}\mathcal{F} \in \mathsf{pSh}(X,\mathscr{C})$ defined by

$$j_!^{\mathrm{pre}}\mathcal{F}(V) = \begin{cases} \mathcal{F}(V) & \text{if } V \subset U \\ 0 & \text{otherwise.} \end{cases}$$

One has

$$(j_!\mathcal{F})_x = \begin{cases} \mathcal{F}_x & \text{if } x \in U \\ 0 & \text{otherwise,} \end{cases}$$

so that $\mathrm{Supp}(j_!\mathcal{F}) = \mathrm{Supp}(\mathcal{F})$. Let now $\mathscr{C} = \mathsf{Ab} = \mathsf{Mod}_{\mathbb{Z}}$ be the category of abelian groups. Fix $G \neq 0$ in $\mathscr{C}$ and consider the constant sheaf on U (cf. Example 2.5.3). We have $\mathrm{Supp}(j_!\underline{G}_U) = \mathrm{Supp}(\underline{G}_U) = U$. In particular, $\mathrm{Supp}(j_!\underline{G}_U) \subset X$ is not closed as soon as U is not closed in X.

If $\mathcal{F}$ is a sheaf of rings, the notions of support defined in (2.6.1) still make sense, and one has $\mathrm{Supp}(\mathcal{F}) = \mathrm{Supp}(1)$, where $1 \in \mathcal{F}(X)$ is the ring identity (recall that the '0 ring' is the one where $1 = 0$). Thus $\mathrm{Supp}(\mathcal{F})$ *is* in fact closed in this case.

2.7 Sheaves = sheaves on a base

Fix a type of algebraic structure $(\mathscr{C}, F\colon \mathscr{C} \to \mathsf{Sets})$.

Definition 2.7.1 (Base of open sets)**.** Let X be a topological space. A *base of open sets* for X is a collection of open subsets $\mathcal{B} \subset \tau_X$ satisfying the following requirements:

(a) $\mathcal{B}$ is stable under finite intersections,

(b) every $U \in \tau_X$ can be written as a union of open sets belonging to $\mathcal{B}$.

Definition 2.7.2 ($\mathcal{B}$-sheaf)**.** A $\mathcal{B}$-*presheaf* (resp. $\mathcal{B}$-*sheaf*) is an assignment

$$U \longmapsto \mathcal{F}(U) \in \mathscr{C}, \text{ for each } U \in \mathcal{B},$$

such that the presheaf conditions (1)–(2) of Definition 2.2.1 (resp. the presheaf conditions (1)–(2) of Definition 2.2.1 and the sheaf conditions (3)–(4) of Important Definition 2.2.1) hold, considering only open sets belonging to $\mathcal{B}$.

Notation 2.7.3. We shall use the notation $\{\mathcal{F}(\mathcal{B}), \rho_{\mathcal{B}}\}$ to denote a $\mathcal{B}$-(pre)sheaf.

Note that restriction maps

$$\rho_{UV}\colon \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$$

are part of the data of a $\mathcal{B}$-(pre)sheaf whenever $V \subset U$ is an inclusion of open sets both belonging to $\mathcal{B}$. Note, also, that condition (a) in Definition 2.7.1 ensures that open subsets of the form $U \cap V$ belong to $\mathcal{B}$ for all $U, V \in \mathcal{B}$. In particular, as in Section 2.3, a $\mathcal{B}$-presheaf $\{\mathcal{F}(\mathcal{B}), \rho_{\mathcal{B}}\}$ is a sheaf precisely when the following condition is fulfilled: for every open subset $U \in \mathcal{B}$ and for any open cover $U = \bigcup_{i \in I} U_i$ with all $U_i \in \mathcal{B}$, the diagram

$$\mathcal{F}(U) \xrightarrow{\rho} \prod_{i \in I} \mathcal{F}(U_i) \overset{\mu}{\underset{\nu}{\rightrightarrows}} \prod_{(i,j) \in I \times I} \mathcal{F}(U_i \cap U_j)$$

is an equaliser diagram in $\mathscr{C}$.

Remark 2.7.4. Let $x \in X$ be a point. The collection of open neighbourhoods

$$\mathcal{B}_x = \{ U \in \mathcal{B} \mid x \in U \}^{\mathrm{op}} \subset \tau_X^{\mathrm{op}}$$

is a fundamental system of open neighbourhoods of x, also called a local basis at x (i.e. for any $W \in \mathsf{Ngb}_x$ there exists $U \in \mathcal{B}_x$ such that $U \subset W$). In more technical terms, one may say that the filtered categories Ngb_x and $\mathcal{B}_x$ are *cofinal*, i.e. the inclusion $\mathcal{B}_x \hookrightarrow \mathsf{Ngb}_x$ is a cofinal functor. We will not use this terminology.

By Remark 2.7.4, the stalk

$$\mathcal{F}_x = \varinjlim_{\mathcal{B}_x} \mathcal{F}\big|_{\mathcal{B}_x} = \varinjlim_{U \in \mathcal{B}_x} \mathcal{F}(U) \in \mathscr{C}$$

of a $\mathcal{B}$-(pre)sheaf $\{\mathcal{F}(\mathcal{B}), \rho_{\mathcal{B}}\}$ at a point $x \in X$ is well-defined as an object of $\mathscr{C}$. It receives, by definition of direct limit, canonical morphisms

$$\mathcal{F}(U) \to \mathcal{F}_x, \quad U \in \mathcal{B}_x.$$

We denote by $s_x \in \mathcal{F}_x$, as ever, the image of $s \in \mathcal{F}(U)$ under this morphism.

Moreover, if $U \in \mathcal{B}$ and $\{\mathcal{F}(\mathcal{B}), \rho_{\mathcal{B}}\}$ is a $\mathcal{B}$-sheaf, the natural map

$$\begin{aligned} \mathcal{F}(U) &\xrightarrow{\sigma_U^{\mathcal{F}}} \prod_{x \in U} \mathcal{F}_x \\ s &\longmapsto (s_x)_{x \in U} \end{aligned}$$

is injective (as in Lemma 2.4.1), and its image agrees with the collections of compatible germs; to be more precise, we should now call them '$\mathcal{B}$-compatible', for they are, by definition, those tuples

$$(s_x)_{x \in U} \in \prod_{x \in U} \mathcal{F}_x$$

such that for every $x \in U$ there is a pair (V_x, t^x), where $V_x \in \mathcal{B}_x$ satisfies $V_x \subset U$ and $t^x \in \mathcal{F}(V_x)$ satisfies $t^x_y = s_y$ for every $y \in V_x$.

Definition 2.7.5 (Morphism of $\mathcal{B}$-sheaves)**.** A *morphism* of $\mathcal{B}$-(pre)sheaves

$$\eta_{\mathcal{B}}\colon \left\{\mathcal{F}(\mathcal{B}), \rho^{\mathcal{F}}_{\mathcal{B}}\right\} \longrightarrow \left\{\mathcal{G}(\mathcal{B}), \rho^{\mathcal{G}}_{\mathcal{B}}\right\} \tag{2.7.1}$$

is the datum of a collection of maps $\eta_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$, one for each $U \in \mathcal{B}$, such that Diagram (2.2.1) commutes for all $U, V \in \mathcal{B}$ such that $V \subset U$.

With this definition, $\mathcal{B}$-sheaves form a category, denoted $\mathsf{Sh}_{\mathcal{B}}(X, \mathscr{C})$.[1]

Remark 2.7.6. Let X be a topological space, $\mathcal{B}$ a base of open subsets of X. A (pre)sheaf $\mathcal{F}$ on X is a $\mathcal{B}$-(pre)sheaf in a natural way. More precisely, there is (say, at the level of sheaves) a *restriction functor*

$$\mathsf{res}_{\mathcal{B}}(X, \mathscr{C})\colon \mathsf{Sh}(X, \mathscr{C}) \longrightarrow \mathsf{Sh}_{\mathcal{B}}(X, \mathscr{C}), \tag{2.7.2}$$

defined on objects in the obvious way. Its actual functoriality is just a consequence of the definition of morphism of $\mathcal{B}$-sheaves, and is an easy routine check.

LEMMA 2.7.7. *A $\mathcal{B}$-sheaf $\{\mathcal{F}(\mathcal{B}), \rho_{\mathcal{B}}\}$ uniquely extends to a sheaf $\overline{\mathcal{F}}$, such that $\overline{\mathcal{F}}(U) = \mathcal{F}(U)$ for all $U \in \mathcal{B}$.*

Proof. Let $U \in \tau_X$ be an arbitrary open set. Define

$$\overline{\mathcal{F}}(U) = \left\{\text{tuples } (s_x)_{x \in U} \text{ of } \mathcal{B}\text{-compatible germs}\right\} \subset \prod_{x \in U} \mathcal{F}_x.$$

This is manifestly a presheaf. It is also clear that the above definition agrees with $\mathcal{F}(U)$ whenever $U \in \mathcal{B}$, since the injective map $\sigma^{\mathcal{F}}_U$ hits precisely the tuples of $\mathcal{B}$-compatible germs; moreover, for the same reason, this definition is the *only* possible extension of the original $\mathcal{B}$-sheaf. The sheaf property is fulfilled by $\overline{\mathcal{F}}$ precisely for the same reason why it is fulfilled by the sheafification of a presheaf (see the proof of Proposition 2.5.2). □

In fact, the statement of Lemma 2.7.7 can be made functorial: one can prove that the restriction functor (2.7.2) is an equivalence. The inverse is given precisely by Lemma 2.7.7 above at the level of objects and by Proposition 2.7.9 below for morphisms.

Remark 2.7.8. We have that $\mathcal{F}_x = \overline{\mathcal{F}}_x$ for all $x \in X$. This follows directly from Remark 2.7.4.

The analogue of Lemma 2.7.7 for *morphisms* is the following.

PROPOSITION 2.7.9. *Let X be a topological space, $\mathcal{B} \subset \tau_X$ a base of open sets and $\mathcal{F}, \mathcal{G}$ two sheaves on X. Suppose given a morphism*

$$\eta_{\mathcal{B}}\colon \mathsf{res}_{\mathcal{B}}(X, \mathscr{C})(\mathcal{F}) \longrightarrow \mathsf{res}_{\mathcal{B}}(X, \mathscr{C})(\mathcal{G})$$

[1] Also $\mathcal{B}$-presheaves form a category, but it is not as well-behaved as $\mathsf{Sh}_{\mathcal{B}}(X, \mathscr{C})$, and we do not need it, so we shall ignore it.

between the underlying $\mathcal{B}$-sheaves. Then $\eta_{\mathcal{B}}$ extends uniquely to a sheaf homomorphism $\eta\colon \mathcal{F} \to \mathcal{G}$. Furthermore, if η_U is surjective (or injective, or an isomorphism) for every $U \in \mathcal{B}$, then so is η.

Exercise 2.7.10. Prove Proposition 2.7.9 and deduce that the restriction functor (2.7.2) is an equivalence.

2.8 Pushforward, inverse image

In this section we learn how to "move" sheaves from a topological space X to another topological space Y, in the presence of a continuous map between the two spaces.

$$\begin{tikzcd} \mathcal{F} \arrow[r, rightsquigarrow] \arrow[d, dash] & f_*\mathcal{F} \arrow[d, dash] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad\qquad \begin{tikzcd} f^{-1}\mathcal{G} \arrow[d, dash] & \mathcal{G} \arrow[l, leftsquigarrow] \arrow[d, dash] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

2.8.1 Pushforward (or direct image)

Let $f\colon X \to Y$ be a continuous map of topological spaces, and let $\mathcal{F}$ be a presheaf on X. The assignment

$$V \mapsto f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}V)$$

defines a presheaf $f_*\mathcal{F}$ on Y, called the *pushforward* (or *direct image*) of $\mathcal{F}$ by f. It is a sheaf as soon as $\mathcal{F}$ is, because if $V = \bigcup_{i\in I} V_i$ is an open covering of an open subset $V \subset Y$, then $f^{-1}V = \bigcup_{i\in I} f^{-1}(V_i)$ is an open covering of $f^{-1}V \subset X$.

Example 2.8.1. If X is arbitrary and $Y = \mathsf{pt}$, then $f_*\mathcal{F}(\mathsf{pt}) = \mathcal{F}(X)$, an object of $\mathscr{C}$. We will see in a minute that the direct image along any continuous map defines a functor. The direct image along the constant map $(X \to \mathsf{pt})_*\colon \mathsf{Sh}(X, \mathscr{C}) \to \mathscr{C}$ is also called the *global section functor*. If $\mathscr{C} = \mathsf{Mod}_A$, it is a left exact functor (you already proved a more general statement in Exercise 2.5.16).

Example 2.8.2. If $f\colon X \hookrightarrow Y$ is the inclusion of a subspace, then $f_*\mathcal{F}$ is defined, for any open subset $V \subset Y$, by

$$f_*\mathcal{F}(V) = \mathcal{F}(V \cap X).$$

Example 2.8.3 (Skyscraper sheaf as a pushforward)**.** Let $x \in X$ be a point, G a nontrivial abelian group. Consider the constant sheaf $\underline{G}_{\{x\}}$ on $\{x\}$. Let $i_x\colon \{x\} \hookrightarrow X$ be the inclusion. Then the skyscraper sheaf $G_x \in \mathsf{Sh}(X, \mathsf{Mod}_{\mathbb{Z}})$ defined in Example 2.4.12 can be described as

$$G_x = i_{x,*}\underline{G}_{\{x\}}.$$

Next, we observe that pushforward of sheaves is functorial, i.e. sending $\mathcal{F} \mapsto f_*\mathcal{F}$ defines functors

$$\begin{tikzcd} \mathsf{Sh}(X,\mathscr{C}) \arrow[r, "f_*"] \arrow[d, hook] & \mathsf{Sh}(Y,\mathscr{C}) \arrow[d, hook] \\ \mathsf{pSh}(X,\mathscr{C}) \arrow[r, "f_*"] & \mathsf{pSh}(Y,\mathscr{C}) \end{tikzcd}$$

where the vertical maps are the natural inclusions (2.2.2). Indeed, given a morphism of (pre)sheaves $\eta\colon \mathcal{F} \to \mathcal{G}$, we can construct a morphism of (pre)sheaves

$$f_*\eta\colon f_*\mathcal{F} \to f_*\mathcal{G}$$

simply by setting

$$(f_*\eta)_V = \eta_{f^{-1}V} : \mathcal{F}(f^{-1}V) \to \mathcal{G}(f^{-1}V)$$

for an open subset $V \subset Y$. The compatibility with restriction maps follows from those of η (and the obvious observation that if $V' \subset V$ then $f^{-1}V' \subset f^{-1}V$).

Moreover, $(-)_*$ is compatible with compositions of continuous maps, in the following sense: if $f: X \to Y$ and $g: Y \to Z$ are continuous maps of topological spaces, then, as functors, we have an equality $(g \circ f)_* = g_* \circ f_*$ on the nose (both for presheaves and for sheaves). In other words, the diagram

$$\begin{tikzcd} \mathsf{Sh}(X,\mathscr{C}) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \mathsf{Sh}(Y,\mathscr{C}) \arrow[d, "g_*"] \\ & \mathsf{Sh}(Z,\mathscr{C}) \end{tikzcd} \tag{2.8.1}$$

commutes. Indeed, if $\mathcal{F}$ is a (pre)sheaf on X, then for every open $W \subset Z$ one has

$$\begin{aligned} (g \circ f)_*\mathcal{F}(W) &= \mathcal{F}((g \circ f)^{-1}(W)) \\ &= \mathcal{F}(f^{-1}g^{-1}(W)) \\ &= f_*\mathcal{F}(g^{-1}(W)) \\ &= (g_*f_*\mathcal{F})(W) \\ &= (g_* \circ f_*)\mathcal{F}(W). \end{aligned}$$

Note that no identifications are made here: all equalities are actual equalities!

LEMMA 2.8.4. *Let $f: X \to Y$ be a continuous map of topological spaces, and fix a sheaf $\mathcal{F} \in \mathsf{Sh}(X,\mathscr{C})$. Let $x \in X$ be a point, and set $y = f(x)$. There is a canonical morphism*

$$(f_*\mathcal{F})_y \longrightarrow \mathcal{F}_x,$$

which is an isomorphism when f is the inclusion of a subspace $X \hookrightarrow Y$.

Proof. If $y \in V' \subset V \subset Y$, then $x \in f^{-1}V' \subset f^{-1}V \subset X$, and the commutative diagram

$$\begin{tikzcd}
f_*\mathcal{F}(V) \arrow[rr] \arrow[d, equal] \arrow[dr] & & f_*\mathcal{F}(V') \arrow[d, equal] \arrow[dl] \\
\mathcal{F}(f^{-1}V) \arrow[dr] & (f_*\mathcal{F})_y \arrow[d, dotted] & \mathcal{F}(f^{-1}V') \arrow[dl] \\
& \mathcal{F}_x &
\end{tikzcd}$$

induces, via the universal property of the stalk (cf. Definition B.4.6)

$$(f_*\mathcal{F})_y = \varinjlim_{V \ni y} \mathcal{F}(f^{-1}V),$$

a canonical morphism $(f_*\mathcal{F})_y \to \mathcal{F}_x$, as required.

Now, let us assume $f\colon X \hookrightarrow Y$ is the inclusion of a subspace, and let us take $y \in X$. Note that every neighbourhood $y \in U \subset X$ is of the form $U = V \cap X$ for some open neighbourhood $y \in V \subset Y$. Thus

$$(f_*\mathcal{F})_y = \varinjlim_{Y \supset V \ni y} \mathcal{F}(V \cap X) \xrightarrow{\sim} \varinjlim_{X \supset U \ni y} \mathcal{F}(U) = \mathcal{F}_y. \tag{2.8.2}$$

The proof is complete. □

Remark 2.8.5. We shall use Lemma 2.8.4 crucially with $\mathscr{C}$ = Rings, when defining morphisms of locally ringed spaces (cf. Remark 2.10.5).

Caution 2.8.6. Even if $f\colon X \hookrightarrow Y$ is the inclusion of a subspace, it is not true that $(f_*\mathcal{F})_y = 0$ for all $y \in Y \setminus X$. This is nevertheless true when f is the inclusion of a *closed* subspace, cf. Remark 2.8.7.

Remark 2.8.7. If $f\colon X \hookrightarrow Y$ is the inclusion of a *closed* subspace, and $\mathcal{F}$ is a sheaf on X, then

$$(f_*\mathcal{F})_y = \begin{cases} \mathcal{F}_y & \text{if } y \in X \\ 0 & \text{if } y \notin X. \end{cases} \tag{2.8.3}$$

The case $y \in X$ is the computation (2.8.2). As for the case $y \notin X$, we use the definition

$$(f_*\mathcal{F})_y = \varinjlim_{Y \supset V \ni y} f_*\mathcal{F}(V) = \varinjlim_{Y \supset V \ni y} \mathcal{F}(V \cap X),$$

and the observation that, since X is closed in Y, there are arbitrarily small neighbourhoods V of y which are disjoint from X. For these, we have $\mathcal{F}(V \cap X) = \mathcal{F}(\emptyset) = 0$ since $\mathcal{F}$ is a sheaf (Remark 2.3.2). This causes the colimit to vanish.

Exactness of pushforward

We set $\mathscr{C} = \mathsf{Mod}_A$ in this subsection (for A a fixed ring), and we fix a continuous map $f : X \to Y$. Consider the direct image functor

$$f_* \colon \mathsf{Sh}(X, \mathsf{Mod}_A) \to \mathsf{Sh}(Y, \mathsf{Mod}_A).$$

It is important to remember that

$$\boxed{f_* \text{ is always left exact, and it is exact if } f : X \hookrightarrow Y \text{ is a closed subspace.}}$$

Since f_* will turn out to be a right adjoint (Lemma 2.8.16), it is left exact by general category theory. However, we prove it directly here. Note that you have already proved the case $Y = \mathrm{pt}$ in Exercise 2.5.16. You will notice in the proof of the above slogan that this was essentially enough to handle the general case.

PROPOSITION 2.8.8. *Let A be a ring, $f : X \to Y$ a continuous map of topological spaces. The functor $f_* \colon \mathsf{Sh}(X, \mathsf{Mod}_A) \to \mathsf{Sh}(Y, \mathsf{Mod}_A)$ is left exact. If f is the inclusion of a closed subspace, then f_* is exact.*

Proof. Let us prove the first assertion. We have to show that an exact sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{\ \alpha\ } \mathcal{G} \xrightarrow{\ \beta\ } \mathcal{H}$$

in $\mathsf{Sh}(X, \mathsf{Mod}_A)$ induces an exact sequence

$$0 \longrightarrow f_*\mathcal{F} \xrightarrow{\ f_*\alpha\ } f_*\mathcal{G} \xrightarrow{\ f_*\beta\ } f_*\mathcal{H}$$

in $\mathsf{Sh}(Y, \mathsf{Mod}_A)$. We know by Exercise 2.5.16 that we have an exact sequence

$$0 \longrightarrow \mathcal{F}(f^{-1}V) \xrightarrow{\ \alpha_{f^{-1}V}\ } \mathcal{G}(f^{-1}V) \xrightarrow{\ \beta_{f^{-1}V}\ } \mathcal{H}(f^{-1}V) \tag{2.8.4}$$

for any open subset $V \subset Y$, by applying the functor $\Gamma(f^{-1}V, -)$ to the original sequence. In particular, $\alpha_{f^{-1}V} = (f_*\alpha)_V$ is injective for all V, which shows that $f_*\alpha$ is injective. There is an equality of presheaves

$$\mathrm{im}_{\mathrm{pre}}(f_*\alpha) = \ker(f_*\beta)$$

again thanks to exactness of (2.8.4) in the middle, ensuring precisely that $\mathrm{im}(\alpha_{f^{-1}V}) = \ker(\beta_{f^{-1}V})$. But $\ker(f_*\beta)$ is a sheaf, therefore we get exactness in the middle, i.e. $\mathrm{im}(f_*\alpha) = \ker(f_*\beta)$.

Let us show the second statement. Assume f is the inclusion of a closed subspace. By the first part of the proof, we only need to show that if $\eta \colon \mathcal{G} \twoheadrightarrow \mathcal{H}$ is surjective as a map of sheaves on X, then $f_*\mathcal{G} \twoheadrightarrow f_*\mathcal{H}$ is surjective as a map of sheaves on Y. We check this on stalks. If $y \in Y \setminus X$, then (using that X is closed, cf. Remark 2.8.7)

$$(f_*\mathcal{G})_y = 0 = (f_*\mathcal{H})_y, \tag{2.8.5}$$

so there is nothing to prove here. Assume $y \in X$. Since $\mathcal{G}$ surjects onto $\mathcal{H}$, in the commutative diagram

$$\begin{tikzcd}
(f_*\mathcal{G})_y \arrow[r, "(f_*\eta)_y"] \arrow[d, equal] & (f_*\mathcal{H})_y \arrow[d, equal] \\
\mathcal{G}_y \arrow[r, two heads, "\eta_y"] & \mathcal{H}_y
\end{tikzcd}$$

the bottom map is surjective. The vertical equalities are given by Remark 2.8.7. Thus the top map is surjective as well. Hence $f_*\eta$ is surjective on all stalks, hence it is surjective. □

2.8.2 Inverse image

Let $f: X \to Y$ be a continuous map of topological spaces. Let $\mathcal{G}$ be a presheaf on Y. Given $U \subset X$, the collection of open subsets $V \subset Y$ containing $f(U)$ form a directed set via reverse inclusions. Sending

$$U \mapsto (f_{\mathrm{pre}}^{-1}\mathcal{G})(U) = \varinjlim_{V \supset f(U)} \mathcal{G}(V)$$

defines a presheaf on X. Indeed, assume $U' \subset U$ is an open subset. Then there is an inclusion $f(U') \subset f(U)$, inducing a map of directed systems

$$\{\, V \in \tau_Y \mid V \supset f(U) \,\} \hookrightarrow \{\, V \in \tau_Y \mid V \supset f(U') \,\},$$

which in turn induces a morphism

$$(f_{\mathrm{pre}}^{-1}\mathcal{G})(U) = \varinjlim_{V \supset f(U)} \mathcal{G}(V) \longrightarrow \varinjlim_{V \supset f(U')} \mathcal{G}(V) = (f_{\mathrm{pre}}^{-1}\mathcal{G})(U').$$

This is the restriction morphism $\rho_{UU'}$ for $f_{\mathrm{pre}}^{-1}\mathcal{G}$.

Remark 2.8.9. If $f(U)$ is an open subset of Y, then

$$(f_{\mathrm{pre}}^{-1}\mathcal{G})(U) = \mathcal{G}(f(U)).$$

Now assume $\mathcal{G}$ is a sheaf. We define the *inverse image* of $\mathcal{G}$ by f to be the sheafification

$$f^{-1}\mathcal{G} = \left(f_{\mathrm{pre}}^{-1}\mathcal{G}\right)^{\#}.$$

By Proposition 2.5.2, the canonical map $f_{\mathrm{pre}}^{-1}\mathcal{G} \to f^{-1}\mathcal{G}$ of presheaves induces an isomorphism

$$(f_{\mathrm{pre}}^{-1}\mathcal{G})_x \xrightarrow{\sim} (f^{-1}\mathcal{G})_x$$

on all the stalks.

Exercise 2.8.10. Both f_{pre}^{-1} and f^{-1} are functors.

Example 2.8.11. Let $\iota_y : \{y\} \hookrightarrow Y$ be the inclusion of a point $y \in Y$, and let $\mathcal{G}$ be a sheaf on Y. Then $\iota_y^{-1}\mathcal{G} = \mathcal{G}_y$, since $\iota_y^{-1}\mathcal{G}(\{y\}) = \varinjlim_{V \ni y} \mathcal{G}(V) = \mathcal{G}_y$. Thus ι_y^{-1} agrees with the stalk functor

$$\mathsf{stalk}_y : \mathsf{Sh}(Y, \mathscr{C}) \to \mathscr{C}, \quad \mathcal{G} \mapsto \mathcal{G}_y.$$

Example 2.8.12. If $p : X \to \mathrm{pt}$ is the constant map, and $G \in \mathscr{C} \cong \mathsf{Sh}(\mathrm{pt}, \mathscr{C})$, then $p^{-1}G = \underline{G}_X$, the constant sheaf on X with values in the object G.

Example 2.8.13. Let $j : U \hookrightarrow Y$ be the inclusion of an open subset. Then $j_{\mathrm{pre}}^{-1}\mathcal{G} = \mathcal{G}|_U$ for any sheaf $\mathcal{G}$ on Y. The reason is that if U' is open in U, it is also open in Y, and thus

$$j_{\mathrm{pre}}^{-1}\mathcal{G}(U') = \varinjlim_{V \supset U'} \mathcal{G}(V) = \mathcal{G}(U') = \mathcal{G}|_U(U').$$

In particular, $j_{\mathrm{pre}}^{-1}\mathcal{G}$ is already a sheaf, and hence

$$j^{-1}\mathcal{G} = \mathcal{G}|_U, \quad U \subset Y \text{ open}.$$

Remark 2.8.14. Despite Example 2.8.13, sheafifying f_{pre}^{-1} is in general necessary: consider a constant map $f : X = \{\star, \bullet\} \to \{*\} = Y$ from a two point space, and fix a nontrivial abelian group G. The constant sheaf $\mathcal{G} = \underline{G}_Y$ has the property $f_{\mathrm{pre}}^{-1}\mathcal{G} = \underline{G}_X^{\mathrm{pre}}$, which is not a sheaf (cf. Example 2.2.18).

Functoriality (cf. Exercise 2.8.10) can be translated into a diagram of functors

$$\begin{tikzcd}
\mathsf{Sh}(Y, \mathscr{C}) \arrow[r, "f^{-1}"] \arrow[d, hook] & \mathsf{Sh}(X, \mathscr{C}) \arrow[d, hook] \\
\mathsf{pSh}(Y, \mathscr{C}) \arrow[r, "f_{\mathrm{pre}}^{-1}"] & \mathsf{pSh}(X, \mathscr{C})
\end{tikzcd}$$

where f^{-1} is obtained by applying $(-)^{\#} : \mathsf{pSh}(X, \mathscr{C}) \to \mathsf{Sh}(X, \mathscr{C})$ in the last step.

Exercise 2.8.15. Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps of topological spaces. Show that

$$f^{-1} \circ g^{-1} = (g \circ f)^{-1}$$

as functors $\mathsf{Sh}(Z, \mathscr{C}) \to \mathsf{Sh}(X, \mathscr{C})$.

LEMMA 2.8.16 (Unit and counit maps). *For any pair of presheaves* $\mathcal{F} \in \mathsf{pSh}(X, \mathscr{C})$ *and* $\mathcal{G} \in \mathsf{pSh}(Y, \mathscr{C})$ *there are canonical presheaf homomorphisms*

$$\mathcal{G} \xrightarrow{\ \mathsf{unit}\ } f_* f_{\mathrm{pre}}^{-1}\mathcal{G}, \qquad f_{\mathrm{pre}}^{-1} f_* \mathcal{F} \xrightarrow{\ \mathsf{counit}\ } \mathcal{F}.$$

Proof. We start with the unit map. The observation here is that there is, for any open subset $V \subset Y$, a canonical inclusion $f(f^{-1}V) \subset V$. Thus $\mathcal{G}(V)$ appears in the colimit

$$\varinjlim_{W \supset f(f^{-1}V)} \mathcal{G}(W).$$

This induces a canonical morphism

$$\mathsf{unit}_V : \mathcal{G}(V) \to \varinjlim_{W \supset f(f^{-1}V)} \mathcal{G}(W) = f_{\mathrm{pre}}^{-1}\mathcal{G}(f^{-1}V) = f_* f_{\mathrm{pre}}^{-1}\mathcal{G}(V)$$

which does define a natural transformation $\mathcal{G} \to f_* f_{\mathrm{pre}}^{-1}\mathcal{G}$ because if $V' \subset V$, then any open $W \subset Y$ containing $f(f^{-1}V)$ also contains $f(f^{-1}V')$, simply because $f(f^{-1}V') \subset f(f^{-1}V)$. Thus there is a natural morphism

$$\varinjlim_{W \supset f(f^{-1}V)} \mathcal{G}(W) \to \varinjlim_{W \supset f(f^{-1}V')} \mathcal{G}(W)$$

and the induced diagram

$$\begin{array}{ccccc}
\mathcal{G}(V) & \xrightarrow{\mathsf{unit}_V} & \varinjlim_{W \supset f(f^{-1}V)} \mathcal{G}(W) & = & f_* f_{\mathrm{pre}}^{-1}\mathcal{G}(V) \\
\downarrow & & \downarrow & & \\
\mathcal{G}(V') & \xrightarrow{\mathsf{unit}_{V'}} & \varinjlim_{W \supset f(f^{-1}V')} \mathcal{G}(W) & = & f_* f_{\mathrm{pre}}^{-1}\mathcal{G}(V')
\end{array}$$

commutes. This defines the map $\mathsf{unit} \colon \mathcal{G} \to f_* f_{\mathrm{pre}}^{-1}\mathcal{G}$ of presheaves.

To construct the map counit, one observes that for any open subset $U \subset X$ there is (by the universal property of colimits, cf. Definition B.4.6) a canonical map

$$f_{\mathrm{pre}}^{-1} f_* \mathcal{F}(U) = \varinjlim_{V \supset f(U)} f_* \mathcal{F}(V) = \varinjlim_{V \supset f(U)} \mathcal{F}(f^{-1}V) \to \mathcal{F}(U),$$

since if $V \supset f(U)$ inside Y, then $U \subset f^{-1} f(U) \subset f^{-1}V$ inside X. This map is also functorial in $U' \subset U$, thus the map $\mathsf{counit} \colon f_{\mathrm{pre}}^{-1} f_* \mathcal{F} \to \mathcal{F}$ is defined. □

The usefulness of the homomorphisms unit and counit is that they make $(f_{\mathrm{pre}}^{-1}, f_*)$ into an adjoint pair of functors. More precisely, there are bijections

$$\varphi_{\mathcal{F},\mathcal{G}} \colon \mathrm{Hom}_{\mathsf{pSh}(Y,\mathscr{C})}(\mathcal{G}, f_*\mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{pSh}(X,\mathscr{C})}(f_{\mathrm{pre}}^{-1}\mathcal{G}, \mathcal{F}),$$

functorial in both $\mathcal{F}$ and $\mathcal{G}$. Specifically, $\varphi_{\mathcal{F},\mathcal{G}}$ sends $\eta \colon \mathcal{G} \to f_*\mathcal{F}$ to

$$f_{\mathrm{pre}}^{-1}\mathcal{G} \xrightarrow{f_{\mathrm{pre}}^{-1}\eta} f_{\mathrm{pre}}^{-1} f_* \mathcal{F} \xrightarrow{\mathsf{counit}} \mathcal{F}$$

with inverse sending $\zeta \colon f_{\mathrm{pre}}^{-1}\mathcal{G} \to \mathcal{F}$ to

$$\mathcal{G} \xrightarrow{\mathsf{unit}} f_* f_{\mathrm{pre}}^{-1}\mathcal{G} \xrightarrow{f_*\zeta} f_*\mathcal{F}.$$

Using the adjunction

$$\mathsf{pSh}(Y,\mathscr{C}) \underset{f_*}{\overset{f_{\mathrm{pre}}^{-1}}{\rightleftarrows}} \mathsf{pSh}(X,\mathscr{C}) \tag{2.8.6}$$

it is immediate to show that also

$$\mathsf{Sh}(Y,\mathscr{C}) \underset{f_*}{\overset{f^{-1}}{\rightleftarrows}} \mathsf{Sh}(X,\mathscr{C}) \tag{2.8.7}$$

is an adjoint pair of functors. Indeed, for any pair of sheaves $\mathcal{F} \in \mathsf{Sh}(X,\mathscr{C})$ and $\mathcal{G} \in \mathsf{Sh}(Y,\mathscr{C})$, we have

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Sh}(Y,\mathscr{C})}(\mathcal{G}, f_*\mathcal{F}) &= \mathrm{Hom}_{\mathsf{pSh}(Y,\mathscr{C})}(\mathcal{G}, f_*\mathcal{F}) && j_{Y,\mathscr{C}} \text{ is fully faithful} \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathsf{pSh}(X,\mathscr{C})}(f^{-1}_{\mathrm{pre}}\mathcal{G}, \mathcal{F}) && \text{adjunction (2.8.6)} \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Sh}(X,\mathscr{C})}(f^{-1}\mathcal{G}, \mathcal{F}) && \text{Exercise 2.5.5.} \end{aligned}$$

Remark 2.8.17. Fix $\mathcal{G} \in \mathsf{Sh}(Y,\mathscr{C})$. Once more, the adjunction (2.8.7) gives a canonical morphism $\mathcal{G} \to f_*f^{-1}\mathcal{G}$, corresponding to $\mathrm{id}_{f^{-1}\mathcal{G}}$ under

$$\mathrm{Hom}_{\mathsf{Sh}(Y,\mathscr{C})}(\mathcal{G}, f_*f^{-1}\mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Sh}(X,\mathscr{C})}(f^{-1}\mathcal{G}, f^{-1}\mathcal{G}).$$

Clearly sending $\mathcal{G} \mapsto f_*f^{-1}\mathcal{G}$ is a functor $f_*f^{-1}\colon \mathsf{Sh}(Y,\mathscr{C}) \to \mathsf{Sh}(Y,\mathscr{C})$, and the naturality of this operation yields a natural transformation

$$\mathsf{unit}\colon \mathrm{Id}_{\mathsf{Sh}(Y,\mathscr{C})} \Rightarrow f_*f^{-1}$$

of functors $\mathsf{Sh}(Y,\mathscr{C}) \to \mathsf{Sh}(Y,\mathscr{C})$, which is called the *unit* of the adjunction (2.8.7). Similarly, let $\mathcal{F} \in \mathsf{Sh}(X,\mathscr{C})$. There is a canonical morphism $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ corresponding to $\mathrm{id}_{f_*\mathcal{F}}$ under

$$\mathrm{Hom}_{\mathsf{Sh}(Y,\mathscr{C})}(f_*\mathcal{F}, f_*\mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Sh}(X,\mathscr{C})}(f^{-1}f_*\mathcal{F}, \mathcal{F}).$$

Clearly sending $\mathcal{F} \mapsto f^{-1}f_*\mathcal{F}$ is a functor $f^{-1}f_*\colon \mathsf{Sh}(X,\mathscr{C}) \to \mathsf{Sh}(X,\mathscr{C})$, and the naturality of this operation yields a natural transformation

$$\mathsf{counit}\colon f^{-1}f_* \Rightarrow \mathrm{Id}_{\mathsf{Sh}(X,\mathscr{C})}$$

of functors $\mathsf{Sh}(X,\mathscr{C}) \to \mathsf{Sh}(X,\mathscr{C})$, which is called the *counit* of the adjunction (2.8.7).

The next lemma says that the stalk of the inverse image is somewhat easy to compute (unlike for the pushforward).

LEMMA 2.8.18 (Stalk of inverse image). *Let $f\colon X \to Y$ be a continuous map of topological spaces, $\mathcal{G}$ a sheaf on Y, and $x \in X$ a point. There is a canonical identification*

$$(f^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)}.$$

Proof. We have

$$\begin{aligned}
(f^{-1}\mathcal{G})_x &= (\{x\} \hookrightarrow X)^{-1}(f^{-1}\mathcal{G}) && \text{by Example 2.8.11} \\
&= (\{x\} \hookrightarrow X \to Y)^{-1}\mathcal{G} && \text{by Exercise 2.8.15} \\
&= (\{f(x)\} \hookrightarrow Y)^{-1}\mathcal{G} \\
&= \mathcal{G}_{f(x)}
\end{aligned}$$

where we have used Example 2.8.11 once more for the last identity. □

Exercise 2.8.19. Show that if $f\colon X \hookrightarrow Y$ is the inclusion of a subspace, then the counit

$$f^{-1}f_*\mathcal{F} \to \mathcal{F}$$

is an isomorphism for every $\mathcal{F} \in \mathsf{Sh}(X, \mathscr{C})$. (**Hint**: check it on stalks).

PROPOSITION 2.8.20. *Let $f\colon X \hookrightarrow Y$ be the inclusion of a closed subspace.*

(1) *Let $\mathcal{G}$ be a sheaf on Y such that* $\operatorname{Supp}(\mathcal{G}) = X$. *Then the unit map*

$$\mathcal{G} \xrightarrow{\sim} f_*f^{-1}\mathcal{G}$$

is an isomorphism.

(2) *The functor f_* induces an equivalence of categories*

$$f_*\colon \mathsf{Sh}(X, \mathsf{Mod}_A) \xrightarrow{\sim} \mathsf{Sh}_X(Y, \mathsf{Mod}_A),$$

where $\mathsf{Sh}_X(Y, \mathsf{Mod}_A) \hookrightarrow \mathsf{Sh}(Y, \mathsf{Mod}_A)$ *is the full subcategory of sheaves on Y with support equal to X.*

Proof. To prove (1) it is enough to prove that the unit map is an isomorphism on all the stalks. If $y \in Y \setminus X$, we get $0 \xrightarrow{\sim} 0$, since $\mathcal{G}_y = 0$ by the assumption $\operatorname{Supp}(\mathcal{G}) = X$ and $(f_*f^{-1}\mathcal{G})_y = 0$ by (2.8.3). On the other hand, if $y \in X$, then $\mathcal{G}_y \to (f_*f^{-1}\mathcal{G})_y$ is nothing but the inverse of the isomorphism

$$(f_*f^{-1}\mathcal{G})_y \xrightarrow{\sim} (f^{-1}\mathcal{G})_y = \mathcal{G}_y$$

of Lemma 2.8.4.

To prove (2), observe first of all that f_* lands in the category $\mathsf{Sh}_X(Y, \mathsf{Mod}_A)$ by (2.8.3). Next, note that sending $\mathcal{G} \mapsto f^{-1}\mathcal{G}$ is an inverse to f_* by (1). In a little more detail, the equivalence (cf. Definition A.1.12) is set up by considering the pair of functors (f_*, f^{-1}) and exploiting the unit and counit *natural isomorphisms*

$$\text{unit}\colon \mathrm{Id}_{\mathsf{Sh}(Y,\mathscr{C})} \Longrightarrow f_*f^{-1} \qquad \text{counit}\colon f^{-1}f_* \Longrightarrow \mathrm{Id}_{\mathsf{Sh}(X,\mathscr{C})}$$

using (1) and Exercise 2.8.19. □

Exercise 2.8.21. Find examples of maps f and sheaves $\mathcal{G}$ such that $\mathcal{G} \to f_* f^{-1}\mathcal{G}$ is not an isomorphism.

Remark 2.8.22. If $j\colon X \hookrightarrow Y$ is *open* and $\mathcal{G}$ is a sheaf on Y, then $j_* j^{-1}\mathcal{G}$ satisfies

$$j_* j^{-1}\mathcal{G}(V) = (j_*\mathcal{G}|_X)(V) = \mathcal{G}(V \cap X), \qquad V \subset Y \text{ open.}$$

The natural map $\mathcal{G}(V) \to j_* j^{-1}\mathcal{G}(V)$ sends $s \mapsto s|_{V\cap X}$.

PROPOSITION 2.8.23. *Let $\mathscr{C} = \mathsf{Mod}_A$, for a ring A. Then the inverse image functor*

$$f^{-1}\colon \mathsf{Sh}(Y, \mathsf{Mod}_A) \to \mathsf{Sh}(X, \mathsf{Mod}_A)$$

is exact, for any continuous map $f\colon X \to Y$ of topological spaces.

Proof. Indeed, let

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow \mathcal{K} \longrightarrow 0$$

be an exact sequence in $\mathsf{Sh}(Y, \mathsf{Mod}_A)$. Then,

$$0 \longrightarrow \mathcal{G}_{f(x)} \longrightarrow \mathcal{H}_{f(x)} \longrightarrow \mathcal{K}_{f(x)} \longrightarrow 0$$

is exact in Mod_A by Proposition 2.5.14, for every $x \in X$. But by Lemma 2.8.18, this is precisely the sequence

$$0 \longrightarrow (f^{-1}\mathcal{G})_x \longrightarrow (f^{-1}\mathcal{H})_x \longrightarrow (f^{-1}\mathcal{K})_x \longrightarrow 0.$$

Thus

$$0 \longrightarrow f^{-1}\mathcal{G} \longrightarrow f^{-1}\mathcal{H} \longrightarrow f^{-1}\mathcal{K} \longrightarrow 0$$

is exact, again by Proposition 2.5.14. □

2.9 Gluing sheaves

The purpose of this section is to prove the next theorem, which is of crucial importance (see e.g. the proof of Theorem 3.2.13).

THEOREM 2.9.1 (Gluing sheaves). *Let X be a topological space, $X = \bigcup_{i\in I} U_i$ an open covering. Set $U_{ij} = U_i \cap U_j$ and similarly $U_{ijk} = U_{ij} \cap U_k$. Assume given a sheaf $\mathcal{F}_i$ on U_i for every $i \in I$, along with a collection of isomorphisms*

$$\varphi_{ij}\colon \mathcal{F}_i\big|_{U_{ij}} \xrightarrow{\sim} \mathcal{F}_j\big|_{U_{ij}}$$

such that $\varphi_{ii} = \mathrm{id}_{\mathcal{F}_i}$ *for every* i*, and such that*

$$\begin{tikzcd} \mathcal{F}_i\big|_{U_{ijk}} \arrow[r, "\varphi_{ij}|_{U_{ijk}}"] \arrow[dr, "\varphi_{ik}|_{U_{ijk}}"'] & \mathcal{F}_j\big|_{U_{ijk}} \arrow[d, "\varphi_{jk}|_{U_{ijk}}"] \\ & \mathcal{F}_k\big|_{U_{ijk}} \end{tikzcd}$$

commutes for every triple intersection. Then there is a unique sheaf $\mathcal{F}$ *on* X *equipped with isomorphisms*

$$\alpha_i\colon \mathcal{F}\big|_{U_i} \xrightarrow{\sim} \mathcal{F}_i$$

such that the diagrams

$$\begin{tikzcd} \mathcal{F}\big|_{U_{ij}} \arrow[rr, "\alpha_j|_{U_{ij}}"] \arrow[dr, "\alpha_i|_{U_{ij}}"'] & & \mathcal{F}_j\big|_{U_{ij}} \\ & \mathcal{F}_i\big|_{U_{ij}} \arrow[ur, "\varphi_{ij}"'] & \end{tikzcd}$$

commute for every $(i,j) \in I \times I$*. The sheaf* $\mathcal{F}$ *is called the gluing of* $(\mathcal{F}_i, \varphi_{ij})_{i,j}$ *along the given covering.*

2.10 Locally ringed spaces

Given our background on sheaves, we are ready for the definition of locally ringed space.

Definition 2.10.1 (Locally ringed space)**.** A *ringed space* is a pair $(X, \mathscr{O}_X)$ where X is a topological space and $\mathscr{O}_X$ is a sheaf of commutative rings on X. The sheaf $\mathscr{O}_X$ is called the *structure sheaf.* A *locally ringed space* is a ringed space such that the stalk $\mathscr{O}_{X,x}$ is a local ring for every $x \in X$.

Notation 2.10.2. Let $(X, \mathscr{O}_X)$ be a locally ringed space, $x \in X$ a point. We will write $\mathfrak{m}_x$ for the maximal ideal $\mathscr{O}_{X,x}$, and $\kappa(x) = \mathscr{O}_{X,x}/\mathfrak{m}_x$ for the corresponding *residue field.*

Recall that, given two local rings $(B, \mathfrak{m}_B)$ and $(A, \mathfrak{m}_A)$, a local homomorphism between them is a ring homomorphism $h\colon B \to A$ such that $h^{-1}(\mathfrak{m}_A) = \mathfrak{m}_B$, or, equivalently, $h(\mathfrak{m}_B) \subset \mathfrak{m}_A$.

Definition 2.10.3 (Morphism of locally ringed spaces)**.** A morphism of locally ringed spaces, denoted

$$(X, \mathscr{O}_X) \xrightarrow{(f, f^\#)} (Y, \mathscr{O}_Y), \tag{2.10.1}$$

is a pair $(f, f^\#)$ where $f\colon X \to Y$ is a continuous map between the underlying topological spaces and $f^\#\colon \mathscr{O}_Y \to f_*\mathscr{O}_X$ is a sheaf homomorphism on Y, such that $f^\#_x\colon \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is a local homomorphism of local rings for every $x \in X$.

Notation 2.10.4. In what follows, when there is no confusion possible, we shall omit the sheaf of rings from the notation, and simply write X to denote the locally ringed space $(X, \mathcal{O}_X)$, or $f : X \to Y$ to denote a morphism $(f, f^\#)$ of locally ringed spaces as in (2.10.1). When we want to emphasise the underlying topological space of $(X, \mathcal{O}_X)$, we write $|X|$.

Remark 2.10.5. Let $f : X \to Y$ be a morphism of locally ringed spaces. Let $x \in X$ be a point, and set $y = f(x)$. The local homomorphism $f_x^\# : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is the composition of the stalk map $f_y^\# : \mathcal{O}_{Y,y} \to (f_*\mathcal{O}_X)_y$ and the morphism $(f_*\mathcal{O}_X)_y \to \mathcal{O}_{X,x}$ of Lemma 2.8.4.

Example 2.10.6. Let $(X, \mathcal{O}_X)$ be a locally ringed space, $U \subset X$ an open subset. Then $(U, \mathcal{O}_X|_U)$ is a locally ringed space. We shall *always* take $\mathcal{O}_X|_U$ as the structure sheaf of an open subset $U \subset X$ of a locally ringed space X. We denote it by $\mathcal{O}_U$.

The composition of two morphisms of locally ringed spaces is defined in a straightforward way (but you need to know that pushforward commutes with composition, see (2.8.1)). In a little more detail, consider two morphisms

$$(X, \mathcal{O}_X) \xrightarrow{(f, f^\#)} (Y, \mathcal{O}_Y), \quad (Y, \mathcal{O}_Y) \xrightarrow{(g, g^\#)} (Z, \mathcal{O}_Z)$$

and define their composition to be the morphism

$$(X, \mathcal{O}_X) \xrightarrow{(g \circ f, (g \circ f)^\#)} (Z, \mathcal{O}_Z)$$

where the map on sheaves $(g \circ f)^\#$ is the composition

$$\mathcal{O}_Z \xrightarrow{g^\#} g_*\mathcal{O}_Y \xrightarrow{g_* f^\#} g_* f_* \mathcal{O}_X = (g \circ f)_* \mathcal{O}_X.$$

Locally ringed spaces thus form a (large) category, denoted

$$\mathsf{LRS},$$

where isomorphisms are simply the invertible morphisms (those admitting a morphism in the opposite direction such that compositions are the identity both ways).

Remark 2.10.7. A morphism of locally ringed spaces $(f, f^\#)$ as in (2.10.1) is an isomorphism if and only if

- $f : X \to Y$ is a topological homeomorphism, and
- $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism of sheaves.

Definition 2.10.8 (Immersions)**.** Let $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of locally ringed spaces. It is called an *open immersion* (resp. a *closed immersion*) if $f : X \to Y$ is a topological open immersion (resp. closed immersion) and $f_x^\#$ is an isomorphism (resp. surjective) for every $x \in X$. It is called an *immersion* (or a *locally closed immersion*) if it factors as a closed immersion followed by an open immersion.

Notation 2.10.9. We sometimes may, and will, denote an immersion by '$X \hookrightarrow Y$'.

It is clear that a morphism of locally ringed spaces $(f, f^{\#})\colon (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is an open immersion if and only if there exists an open subset $V \subset Y$ such that $(f, f^{\#})$ induces an isomorphism $(X, \mathscr{O}_X) \xrightarrow{\sim} (V, \mathscr{O}_Y|_V)$. It is also clear that the composition of two open (resp. closed) immersions is an open immersion (resp. a closed immersion).

2.10.1 Closed immersions and ideal sheaves

In this section we characteriste closed immersions up to isomorphism by means *ideal sheaves.*

Definition 2.10.10 (Ideal sheaf)**.** Fix a locally ringed space $(X, \mathscr{O}_X)$. An *ideal sheaf* (or a *sheaf of ideals*) is a subsheaf $\mathscr{I} \subset \mathscr{O}_X$ (as abelian groups) such that $\mathscr{I}(U) \subset \mathscr{O}_X(U)$ is an ideal for every open subset $U \subset X$.

Given an ideal sheaf $\mathscr{I}$, the subset

$$\mathrm{V}(\mathscr{I}) = \left\{ x \in X \,\middle|\, \mathscr{I}_x \neq \mathscr{O}_{X,x} \right\} \xhookrightarrow{j} X \tag{2.10.2}$$

is a closed subset. Indeed, for any $x \in X \setminus \mathrm{V}(\mathscr{I})$, i.e. for any x such that $\mathscr{I}_x = \mathscr{O}_{X,x}$, there is a neighbourhood U of x and a section $f \in \mathscr{I}(U)$ such that $f_x = 1 \in \mathscr{O}_{X,x}$. But this means that $f|_V = 1 \in \mathscr{O}_X(V)$ for some open subset $V \subset U$. Thus $V \subset X \setminus \mathrm{V}(\mathscr{I})$, and thus $X \setminus \mathrm{V}(\mathscr{I})$ is open.

The quotient sheaf $\mathscr{O}_X/\mathscr{I}$ is a sheaf of rings (because, by definition, it is the sheafification of a presheaf of rings), not just abelian groups. The pair

$$(\mathrm{V}(\mathscr{I}), j^{-1}(\mathscr{O}_X/\mathscr{I}))$$

defines a locally ringed space (indeed, for any $x \in \mathrm{V}(\mathscr{I})$, the stalk $(j^{-1}(\mathscr{O}_X/\mathscr{I}))_x = (\mathscr{O}_X/\mathscr{I})_{j(x)} = \mathscr{O}_{X,j(x)}/\mathscr{I}_{j(x)}$ is a local ring: we have used Lemma 2.8.18 and Remark 2.5.11), and the canonical surjection

$$j^{\#}\colon \mathscr{O}_X \twoheadrightarrow \mathscr{O}_X/\mathscr{I} = j_* j^{-1}(\mathscr{O}_X/\mathscr{I})$$

turns $(j, j^{\#})$ into a closed immersion

$$(\mathrm{V}(\mathscr{I}), j^{-1}(\mathscr{O}_X/\mathscr{I})) \xhookrightarrow{(j, j^{\#})} (X, \mathscr{O}_X). \tag{2.10.3}$$

Note that we have used Proposition 2.8.20 (1) for the identification $\mathscr{O}_X/\mathscr{I} = j_* j^{-1}(\mathscr{O}_X/\mathscr{I})$. So we have defined an assignment

$$\mathscr{O}_X \supset \mathscr{I} \quad \longmapsto \quad \mathrm{V}(\mathscr{I}) \hookrightarrow X.$$

Conversely, to any closed immersion $\iota\colon Y \hookrightarrow X$ one can associate an ideal sheaf, namely

$$\mathscr{I}_Y = \ker(\mathscr{O}_X \xrightarrow{\iota^\#} \iota_*\mathscr{O}_Y) \subset \mathscr{O}_X.$$

These two operations are inverse to each other "up to isomorphism", as the next proposition clarifies.

PROPOSITION 2.10.11. *Let* $(\iota,\iota^\#)\colon (Y,\mathscr{O}_Y) \hookrightarrow (X,\mathscr{O}_X)$ *be a closed immersion of locally ringed spaces. Set* $\mathscr{I}_Y = \ker\iota^\# \subset \mathscr{O}_X$ *and consider the associated closed immersion*

$$(\mathrm{V}(\mathscr{I}_Y), j^{-1}(\mathscr{O}_X/\mathscr{I}_Y)) \xhookrightarrow{(j,j^\#)} (X,\mathscr{O}_X).$$

as in (2.10.3)*. There is a unique isomorphism of locally ringed spaces*

$$(g,g^\#)\colon (Y,\mathscr{O}_Y) \xrightarrow{\sim} (\mathrm{V}(\mathscr{I}_Y), j^{-1}(\mathscr{O}_X/\mathscr{I}))$$

such that $(\iota,\iota^\#)=(j,j^\#)\circ(g,g^\#)$*.*

Moreover, an inclusion of ideal sheaves $\mathscr{I}_2 \hookrightarrow \mathscr{I}_1 \hookrightarrow \mathscr{O}_X$ *determines, and is determined by (in the above sense) a chain of closed immersions* $\mathrm{V}(\mathscr{I}_1) \hookrightarrow \mathrm{V}(\mathscr{I}_2) \hookrightarrow X$*.*

Proof. The last statement is immediate. We thus only prove the first.

Let us use the shorthand notation

$$(Z,\mathscr{O}_Z) = (\mathrm{V}(\mathscr{I}_Y), j^{-1}(\mathscr{O}_X/\mathscr{I}_Y)).$$

where $j\colon Z \hookrightarrow X$ is the topological closed embedding first appeared in (2.10.2). We need to find the isomorphism g as in the statement. By Remark 2.8.7, we have

$$(\iota_*\mathscr{O}_Y)_x = \begin{cases} \mathscr{O}_{Y,y} & \text{if } x=\iota(y) \\ 0 & \text{if } x \notin \iota(Y). \end{cases} \tag{2.10.4}$$

Combine the exact sequence

$$0 \to \mathscr{I}_Y \to \mathscr{O}_X \to \iota_*\mathscr{O}_Y \to 0$$

with Proposition 2.5.14 and Equation (2.10.4) to deduce that

$$\mathscr{I}_{Y,x} = \mathscr{O}_{X,x} \quad\Longleftrightarrow\quad x \notin \iota(Y).$$

This shows we have a homeomorphism $g\colon Y \xrightarrow{\sim} Z$. There is a factorisation

$$\iota\colon Y \xrightarrow{g} Z \xhookrightarrow{j} X$$

as topological maps, and

$$j_*\mathscr{O}_Z = j_*j^{-1}(\mathscr{O}_X/\mathscr{I}_Y) \cong \mathscr{O}_X/\mathscr{I}_Y = \iota_*\mathscr{O}_Y = j_*g_*\mathscr{O}_Y$$

as sheaves of rings, the last identity being a consequence of the above factorisation and Diagram (2.8.1). Therefore, we have

$$\mathcal{O}_Z \cong j^{-1} j_* \mathcal{O}_Z = j^{-1} j_* g_* \mathcal{O}_Y \cong g_* \mathcal{O}_Y .$$

This extends g to the desired isomorphism $(g, g^{\#})$. It is straightforward to verify the identity

$$(\iota, \iota^{\#}) = (j, j^{\#}) \circ (g, g^{\#})$$

as morphisms of locally ringed spaces. $\square$

Proposition 2.10.11 will be used to make sense of the definition of *closed subscheme* (cf. Definition 3.2.4).

3 | Schemes

The goal of this chapter is to introduce the category Aff of *affine schemes* and the larger category Sch of all *schemes*. They will arise as full subcategories

$$\mathsf{Aff} \subset \mathsf{Sch} \subset \mathsf{LRS}.$$

3.1 Affine schemes

Let $A \neq 0$ be a nonzero ring (commutative, with unit $1 \neq 0$). The set

$$\boxed{\operatorname{Spec} A = \{\, \mathfrak{p} \subset A \mid \mathfrak{p} \text{ is a prime ideal} \,\}}$$

is called the *prime spectrum* of A. For now, this is just a set. We will endow it with a topology (cf. Corollary 3.1.7) and with a sheaf of rings $\mathscr{O}_{\operatorname{Spec} A}$ having local rings as stalks (cf. Theorem 3.1.28), to obtain a locally ringed space. Such locally ringed space will be called an *affine scheme* (cf. Important Definition 3.1.2). General schemes are obtained by glueing affine schemes, just as a smooth manifold is obtained by glueing open subsets of $\mathbb{R}^m$.

Notation 3.1.1. We introduce the following notation, that will be used throughout: given a ring B, the spectrum

$$\mathbb{A}^n_B = \operatorname{Spec} B[x_1, \ldots, x_n]$$

will be called *affine n-space* over B. If $n = 1$ (resp. $n = 2, 3$), we speak of *affine line* (resp. *affine plane, affine space*) over B.

Before getting started, we recall a few basic tools from commutative algebra (already used in Chapter 1).

Radical of an ideal

Definition 3.1.2 (Radical of an ideal)**.** The *radical* of an ideal $I \subset A$ is the subset

$$\sqrt{I} = \left\{ a \in A \,\middle|\, a^r \in I \text{ for some } r > 0 \right\} \subset A.$$

An ideal I is *radical* if $I = \sqrt{I}$.

Clearly $\sqrt{I} \subset A$ is an ideal containing I, and satisfies

$$\sqrt{I} = \bigcap_{\substack{\mathfrak{p} \in \operatorname{Spec} A \\ \mathfrak{p} \supset I}} \mathfrak{p}. \tag{3.1.1}$$

This implies that a prime ideal contains I if and only if it contains $\sqrt{I}$.

Remark 3.1.3. A prime ideal $\mathfrak{p} \subset A$ is radical (reason: let $a \in A$ be an element such that $a^r \in \mathfrak{p}$, with r is minimal; if $r = 1$ we are done, otherwise either $a \in \mathfrak{p}$ or $a^{r-1} \in \mathfrak{p}$ but the latter is excluded by minimality of r, thus $a \in \mathfrak{p}$), but the converse is false. For instance if $A = \mathbb{Z}$ we have $\sqrt{m\mathbb{Z}} = n\mathbb{Z}$, where $n = \prod_{p \mid m} p$. Thus if m is a product of distinct primes then $m\mathbb{Z}$ is radical but not prime.

Definition 3.1.4 (Nilradical of a ring)**.** The *nilradical* of a ring A is the ideal of nilpotent elements (the radical of the trivial ideal), namely

$$\operatorname{Nil}(A) = \sqrt{0} = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p} \subset A.$$

A ring A is *reduced* if $\operatorname{Nil}(A) = 0$, i.e. if it contains no nontrivial nilpotents.

Operations on ideals

Let $(I_\lambda)_{\lambda \in \Lambda}$ be an arbitrary family of ideals in A. Then the intersection $\bigcap_{\lambda \in \Lambda} I_\lambda \subset A$ is easily checked to be an ideal. Recall that the sum of ideals $\sum_{\lambda \in \Lambda} I_\lambda$ is, by definition, the ideal generated by (i.e. the smallest ideal containing) the union of the ideals in the family (which is not an ideal in general). It can be described set-theoretically as

$$\sum_{\lambda \in \Lambda} I_\lambda = \left\{ \sum_{\lambda \in F} a_\lambda i_\lambda \;\middle|\; a_\lambda \in A,\, i_\lambda \in I_\lambda,\, |F| < \infty \right\}. \tag{3.1.2}$$

If we have *finitely many* ideals $I_1, I_2, \ldots, I_m \subset A$, their product is the ideal generated by the products of the form $i_1 i_2 \cdots i_m$, where $i_k \in I_k$ for $k = 1, \ldots, m$. In symbols,

$$I_1 I_2 \cdots I_m = \left\{ \sum_{1 \le j \le p} i_1^{(j)} i_2^{(j)} \cdots i_m^{(j)} \;\middle|\; i_k^{(j)} \in I_k,\, p < \infty \right\}.$$

In general, we have $I_1 I_2 \cdots I_m \subset I_1 \cap I_2 \cap \cdots \cap I_m$, with equality when $I_k + I_h = A$ for any pair (k, h) such that $k \neq h$ (if $I_k + I_h = A$ we say that I_k and I_h are *comaximal*).

3.1.1 The Zariski topology on Spec A

For an arbitrary ideal $I \subset A$, set

$$\boxed{\mathrm{V}(I) = \{\, \mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \supset I \,\} \subset \operatorname{Spec} A.}$$

Note that there is a bijection

$$\mathrm{V}(I) \simeq \operatorname{Spec} A/I,$$

since (prime) ideals of A/I correspond precisely to (prime) ideals in A containing I.

If $I = (f) = fA \subset A$ (we will use both notations for principal ideals) for $f \in A$, simply write $\mathrm{V}(f)$ instead of $\mathrm{V}(I)$, and define

$$\boxed{\mathrm{D}(f) = \operatorname{Spec} A \setminus \mathrm{V}(f) = \left\{ \mathfrak{p} \in \operatorname{Spec} A \,\middle|\, f \notin \mathfrak{p} \right\}.}$$

Example 3.1.5. Let $\mathbf{k}$ be an algebraically closed field. If $f \in \mathbf{k}[x]$ is nonzero, then $\mathrm{D}(f)$ consists of those prime ideals $\mathfrak{p} \subset \mathbf{k}[x]$ such that $f \notin \mathfrak{p}$. One such ideal is the trivial ideal (0), and the other ideals $\mathfrak{p}$ with this property are all the ideals of the form $\mathfrak{p} = (x - a)$, for $a \in \mathbf{k}$, such that $f(a) \neq 0 \in \mathbf{k}$.

Note that, for any ring A, one has

$$\operatorname{Spec} A = \mathrm{D}(1), \quad \emptyset = \mathrm{D}(0).$$

LEMMA 3.1.6. *Let A be a ring.*

(1) *If $I, J \subset A$ are two ideals, then* $\mathrm{V}(I) \cup \mathrm{V}(J) = \mathrm{V}(I \cap J)$.

(2) *If $(I_\lambda)_{\lambda \in \Lambda}$ is an arbitrary family of ideals, then* $\bigcap_{\lambda \in \Lambda} \mathrm{V}(I_\lambda) = \mathrm{V}\left(\sum_{\lambda \in \Lambda} I_\lambda\right)$.

(3) $\operatorname{Spec} A = \mathrm{V}(0)$ *and* $\emptyset = \mathrm{V}(1)$.

Proof. This is straightforward. However, here is the proof:

(1) If $\mathfrak{p} \subset A$ contains either I or J, then it contains the smaller ideal $I \cap J$, thus $\mathrm{V}(I) \cup \mathrm{V}(J) \subset \mathrm{V}(I \cap J)$. If $\mathfrak{p} \supset I \cap J$ but $\mathfrak{p} \not\supset I$, there is $i \in I$ such that $i \notin \mathfrak{p}$. If $j \in J$, then $ij \in I \cap J \subset \mathfrak{p}$, which implies $j \in \mathfrak{p}$ (because $\mathfrak{p}$ is prime), thus $J \subset \mathfrak{p}$. Therefore $\mathrm{V}(I \cap J) \subset \mathrm{V}(I) \cup \mathrm{V}(J)$.

(2) If $\mathfrak{p}$ contains the sum $\sum_\lambda I_\lambda$, then it contains each I_λ, therefore $\mathrm{V}\left(\sum_\lambda I_\lambda\right) \subset \bigcap_\lambda \mathrm{V}(I_\lambda)$. On the other hand, assume $\mathfrak{p} \supset I_\lambda$ for every index λ. Let $h = a_1 i_{\lambda_1} + \cdots + a_p i_{\lambda_p}$ as in Equation (3.1.2). Then $a_j i_{\lambda_j} \in \mathfrak{p}$ by assumption, thus $h \in \mathfrak{p}$ as well, i.e. $\mathfrak{p} \supset \sum_\lambda I_\lambda$.

(3) Every prime $\mathfrak{p} \subset A$ contains $0 \in A$. No prime ideal $\mathfrak{p} \subset A$ contains $1 \in A$ (here we use that $1 \neq 0$). □

COROLLARY 3.1.7. *There exists a unique topology on* $\operatorname{Spec} A$ *whose closed sets are of the form* $\mathrm{V}(I)$. *Moreover, the sets* $\mathrm{D}(f) \subset \operatorname{Spec} A$ *form a base of open sets for this topology (according to Definition 2.7.1).*

Proof. The first statement is clear from Lemma 3.1.6 and the definition of a topology. The second one follows from these observations:

(i) $\mathrm{D}(f_1) \cap \mathrm{D}(f_2) = \mathrm{D}(f_1 f_2)$ for all $f_1, f_2 \in A$, and

(ii) an open subset $\operatorname{Spec} A \setminus \mathrm{V}(I)$ can be written as $\bigcup_{f \in I} \mathrm{D}(f)$. □

For instance,

$$\operatorname{Spec} A = \mathrm{D}(1) = \operatorname{Spec} A \setminus \mathrm{V}(1) = \bigcup_{f \in A} \mathrm{D}(f). \tag{3.1.3}$$

Important Definition 3.1.1 (Zariski topology)**.** The topology on $\operatorname{Spec} A$ given by Corollary 3.1.7 is called the *Zariski topology*.

Terminology 3.1.8. We call $\mathrm{D}(f)$ a *principal open set* in $\operatorname{Spec} A$, and $\mathrm{V}(f)$ a *principal closed set* in $\operatorname{Spec} A$.

Convention 3.1.9. When thinking of $\operatorname{Spec} A$ as a topological space, it will *always* be endowed with the Zariski topology.

Let $\mathbb{F}$ be a field. Consider the ideals $I_r = (x^r) \subset \mathbb{F}[x]$ for all $r > 0$. Then $\mathrm{V}(I_1) = \{\mathfrak{p} \subset \mathbb{F}[x] \mid \mathfrak{p} \supset (x)\} = \{(x)\} = \mathrm{V}(I_r)$ for every r. Thus it may happen that

$$\mathrm{V}(I) = \mathrm{V}(J), \quad \text{with } I \neq J.$$

In general, by Equation (3.1.1), we have the set-theoretic identity

$$\mathrm{V}(I) = \mathrm{V}(\sqrt{I}) \subset \operatorname{Spec} A.$$

LEMMA 3.1.10. *Let $I, J \subset A$ be two ideals in a ring A. Then*

$$\mathrm{V}(I) \subset \mathrm{V}(J) \Longleftrightarrow J \subset \sqrt{I}.$$

Proof. This is a again a rephrasing of the identity $\sqrt{I} = \bigcap_{\mathfrak{p} \in \mathrm{V}(I)} \mathfrak{p}$, cf. Equation (3.1.1). □

Example 3.1.11. Let $f, g \in A$. We have

$$\begin{aligned} \mathrm{D}(g) \subset \mathrm{D}(f) &\Longleftrightarrow \mathrm{V}(f) \subset \mathrm{V}(g) \\ &\Longleftrightarrow g \in \sqrt{fA}. \end{aligned}$$

LEMMA 3.1.12 ($\operatorname{Spec} A$ is quasicompact). *Fix a subset $S = \{f_k \mid k \in K\} \subset A$. Then* $\operatorname{Spec} A = \bigcup_{k \in K} \mathrm{D}(f_k)$ *if and only if there is a finite subset $F \subset K$ such that one can write $1 = \sum_{k \in F} a_k f_k$ for some nonzero elements $a_k \in A$.*

In particular, $\operatorname{Spec} A$ *equipped with the Zariski topology is quasicompact.*

Proof. Let $(-)^{\mathrm{c}}$ denote the complement of a subset of $\operatorname{Spec} A$. The union

$$\bigcup_{k \in K} \mathrm{D}(f_k) = \bigcup_{k \in K} \operatorname{Spec} A \setminus \mathrm{V}(f_k) = \left(\bigcap_{k \in K} \mathrm{V}(f_k) \right)^{\mathrm{c}} = \mathrm{V}\left(\sum_{k \in K} f_k A \right)^{\mathrm{c}}$$

equals Spec A if and only if

$$\mathrm{V}\left(\sum_{k\in K} f_k A\right) = \emptyset = \mathrm{V}(1),$$

which by Lemma 3.1.10 happens if and only if $\sqrt{\sum_{k\in K} f_k A} = (1) = A$, which in turn means that $A = \sum_{k\in K} f_k A$. The first assertion then follows from the definition of sum of ideals. The last sentence in the statement follows from the first, setting $S = A$ and using (3.1.3). □

Remark 3.1.13. The proof of Lemma 3.1.12 also shows that any principal open subset $\mathrm{D}(f) \subset \operatorname{Spec} A$ is quasicompact, for

$$\mathrm{D}(f) = \mathrm{D}(f) \cap \operatorname{Spec} A = \mathrm{D}(f) \cap \bigcup_{k\in F} \mathrm{D}(f_k) = \bigcup_{k\in F} \mathrm{D}(f) \cap \mathrm{D}(f_k) = \bigcup_{k\in F} \mathrm{D}(f f_k).$$

Moreover, it shows that an open subset $U \subset \operatorname{Spec} A$ is quasicompact if and only if it is a finite union of principal opens.

Warning 3.1.14. Not *every* open subset $U \subset \operatorname{Spec} A$ is quasicompact! For instance, consider $A = \mathbf{k}[x_i \mid i \in \mathbb{N}]$, and let $U \subset \operatorname{Spec} A$ be the complement of the origin (the point corresponding to the maximal ideal $(x_i \mid i \in \mathbb{N}) \subset A$). Then the covering $U = \bigcup_{i\in\mathbb{N}} U \setminus \mathrm{V}(x_i)$ has no finite subcover.

Remark 3.1.15 (Closed points = maximal ideals)**.** Let $\mathfrak{p} \in \operatorname{Spec} A$ be a closed point, i.e. such that $\{\mathfrak{p}\} \subset \operatorname{Spec} A$ is closed. Then $\{\mathfrak{p}\} = \mathrm{V}(I) = \{\mathfrak{q} \mid \mathfrak{q} \supset I\}$ for an ideal $I \subset A$. This says that $\mathfrak{p}$ is the only prime ideal containing I. But any ideal sits inside a maximal ideal, and maximal ideals are prime. Thus $\mathfrak{p}$ is maximal. Conversely, if $\mathfrak{m} \subset A$ is maximal, then $\{\mathfrak{m}\} = \mathrm{V}(\mathfrak{m})$, in particular $\{\mathfrak{m}\} \subset \operatorname{Spec} A$ is closed, i.e. $\mathfrak{m} \in \operatorname{Spec} A$ is a closed point.

The previous remark can be generalised by the following lemma.

LEMMA 3.1.16. *Let $T \subset \operatorname{Spec} A$ be a subset, $\overline{T} \subset \operatorname{Spec} A$ its closure. Then*

$$\overline{T} = \mathrm{V}\left(\bigcap_{\mathfrak{p}\in T} \mathfrak{p}\right).$$

In particular, the closure of $\{\mathfrak{p}\} \subset \operatorname{Spec} A$ is precisely $\mathrm{V}(\mathfrak{p})$.

Proof. We have

$$\overline{T} = \bigcap_{\mathrm{V}(I)\supset T} \mathrm{V}(I) = \mathrm{V}\left(\sum_{\mathrm{V}(I)\supset T} I\right).$$

But by definition $\mathrm{V}(I) \supset T$ means that every $\mathfrak{p} \in T$ satisfies $\mathfrak{p} \supset I$, so the sum is over all $I \subset A$ such that $I \subset \bigcap_{\mathfrak{p}\in T} \mathfrak{p}$. The largest such ideal is precisely $\bigcap_{\mathfrak{p}\in T} \mathfrak{p}$, which concludes the proof. □

Remark 3.1.17. Combining the previous topological observations, we conclude that

$$\boxed{\text{the Zariski topology on } \operatorname{Spec} A \text{ is almost never Hausdorff, or even } \mathrm{T}_1.}$$

For instance, take an integral domain A that is not a field, so that $(0) \subset A$ is prime and the closure of the corresponding point $\xi \in \operatorname{Spec} A$ is equal to $\mathrm{V}(0) = \operatorname{Spec} A$ by Lemma 3.1.16. Then any two nonempty open subsets intersect (thus $\operatorname{Spec} A$ is not Hausdorff), and in fact every open neighbourhood of a point $\mathfrak{p} \in \operatorname{Spec} A$ will also contain ξ (thus $\operatorname{Spec} A$ is not T_1).

3.1.2 Interlude: functions on Spec A

The following slogan is important (and will be formalised in Theorem 3.1.28(b)):

$$\boxed{\text{elements of } A \text{ are functions on } \operatorname{Spec} A.}$$

The slogan is a bit premature, since by 'function on $\operatorname{Spec} A$' we actually mean 'regular function on *the scheme* $\operatorname{Spec} A$', and so far we only have a topological space, we haven't yet defined the sheaf of regular functions. However, it is worth explaining the slogan just to build some intuition.

Here is the explanation. To any $f \in A$, we can associate the map

$$\theta_f \colon \operatorname{Spec} A \to \coprod_{\mathfrak{p} \in \operatorname{Spec} A} A/\mathfrak{p}, \quad \mathfrak{p} \mapsto f \bmod \mathfrak{p}.$$

For instance, $f = 9 \in \mathbb{Z}$ takes the value $[1]$ in $\mathbb{Z}/2\mathbb{Z}$, and the value $[4]$ in $\mathbb{Z}/5\mathbb{Z}$. Its value in $\mathbb{Z}/0 = \mathbb{Z}$ is just… 9.[1] Of course, the most confusing thing here is that the ring where the function takes values depends on the point on which the function is evaluated! Now obviously the function '9' vanishes on the point $(3) \in \operatorname{Spec} \mathbb{Z}$. In general,

$$\boxed{\theta_f(\mathfrak{p}) = 0 \in A/\mathfrak{p} \text{ if and only if } f \in \mathfrak{p}.}$$

Note also that addition and multiplication of 'functions' works as one might expect, i.e. $\theta_{f+g}(\mathfrak{p}) = f + g \bmod \mathfrak{p} = \theta_f(\mathfrak{p}) + \theta_g(\mathfrak{p})$, and similarly $\theta_{fg}(\mathfrak{p}) = fg \bmod \mathfrak{p} = \theta_f(\mathfrak{p})\theta_g(\mathfrak{p})$. This is just a rephrasing of the fact that $A \to A/\mathfrak{p}$ is a ring homomorphism!

Example 3.1.18. Consider $\mathbb{C}[x]$, and the 'function' $f(x) = 2x^2 - x + 3 \in \mathbb{C}[x]$. The prime ideals of $\mathbb{C}[x]$ are $(0) \subset \mathbb{C}[x]$, and the maximal ideals $\mathfrak{m}_a = (x - a) \subset \mathbb{C}[x]$ for $a \in \mathbb{C}$. The value of f on the point $\mathfrak{m}_a \in \operatorname{Spec} \mathbb{C}[x]$ is just the evaluation of the polynomial $f(x)$ at $x = a$. Indeed,

$$\theta_f(\mathfrak{m}_a) = f \bmod \mathfrak{m}_a \in \mathbb{C}[x]/\mathfrak{m}_a,$$

corresponds to the element

$$f(a) = 2a^2 - a + 3 \in \mathbb{C} \cong \mathbb{C}[x]/\mathfrak{m}_a.$$

[1] Please come back here after reading about generic points in Section 3.1.6.

Example 3.1.19. Consider $A = \mathbf{k}[t]/t^2$, and let $\overline{t} \in A$ be the image of $t \in \mathbf{k}[t]$ in A. If we see $\overline{t}$ as a function on $\operatorname{Spec} \mathbf{k}[t]/t^2$, we see that $\theta_{\overline{t}}$ evaluated on the point $(\overline{t})$ gives $0 \in A/\overline{t}$, i.e. the *nonzero* element $\overline{t} \in A$ determines a function that *vanishes at every point* of $\operatorname{Spec} \mathbf{k}[t]/t^2$. Here we encounter for the first time one of the magic aspects of scheme theory:

$$\boxed{\text{functions are not determined by their values on points!}}$$

This is due to the presence of nilpotents, which were not part of the game with classical algebraic varieties. As mentioned, we will see that $\operatorname{Spec} \mathbf{k}[t]/t^2 \neq \operatorname{Spec} \mathbf{k}$ *as affine schemes*, because their rings of functions are different: there is no ring isomorphism $\mathbf{k} \cong \mathbf{k}[t]/t^2$!

3.1.3 First examples of ring spectra

In this subsection we analyse the Zariski topology on $\operatorname{Spec} A$ for a few interesting rings A.

Example 3.1.20 (Spec $\mathbb{F}$, aka the point)**.** Let $\mathbb{F}$ be a field. The spectrum $\operatorname{Spec} \mathbb{F}$ consists of a single point corresponding to $(0) \subset \mathbb{F}$. Its 'functions' are just the constants $\mathbb{F}$, as expected. For now, this (merely topological and hence dry) description is enough. However, when $\operatorname{Spec} \mathbb{F}$ will be endowed with a scheme structure, things will change: for instance, as we shall see (cf. Example 3.1.66), it is not true that the only morphism $\operatorname{Spec} \mathbb{F} \to \operatorname{Spec} \mathbb{F}$ is the identity! And it is also not true that there exists a morphism $\operatorname{Spec} \mathbb{F} \to \operatorname{Spec} \mathbb{F}'$ for any pair of fields $\mathbb{F}$ and $\mathbb{F}'$.

Figure 3.1: This is Spec $\mathbb{F}$. Nothing more, nothing less.

Example 3.1.21 ($\mathbb{A}^1_{\mathbf{k}} = \operatorname{Spec} \mathbf{k}[x]$)**.** Let $\mathbf{k}$ be an algebraically closed field, such as $\mathbb{C}$. The ring $\mathbf{k}[x]$ is a principal ideal domain, whose prime ideals are (0) and $(x-a)$, one for each $a \in \mathbf{k}$. The spectrum

$$\mathbb{A}^1_{\mathbf{k}} = \operatorname{Spec} \mathbf{k}[x]$$

is called the *affine line* (over $\mathbf{k}$). Note that there is exactly one point, namely

$$\xi = (0) \in \mathbb{A}^1_{\mathbf{k}},$$

that is not closed. In fact, by Lemma 3.1.16, we have

$$\overline{\{\xi\}} = \mathrm{V}(0) = \mathbb{A}^1_{\mathbf{k}}.$$

This point was invisible in the land of *classical varieties*, where only closed points were allowed. It has a name: it is the *generic point* of the affine line. We will say a lot more about generic points later (cf. Section 3.1.6), but for now notice that the terminology is somewhat well chosen: if we think that $(x-a)$ corresponds to the 'classical' point $a \in \mathbb{C}$, then since $x - x = 0$ it is reasonable to think that the coordinate of this point has indeed stayed 'generic'. This is what an 'indeterminate' should be!

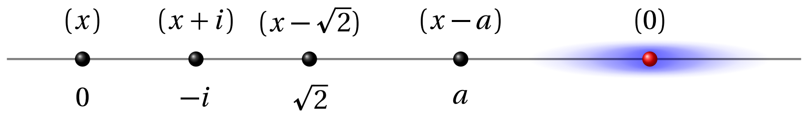

Figure 3.2: The topological space $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x]$, with one closed point for every $a \in \mathbb{C}$. The generic point $\xi = (0)$ is 'dense', i.e. $\overline{\{\xi\}} = \mathbb{A}^1_{\mathbb{C}}$, since $0 \in \mathbb{C}[x]$ is in every prime ideal.

Note that, if $\mathbb{F}$ is an arbitrary field, not necessarily algebraically closed, $\mathbb{F}[x]$ is still a principal ideal domain, but now the closed points of $\mathbb{A}^1_{\mathbb{F}}$ correspond to those maximal ideals (f) generated by irreducible polynomials of degree possibly larger than 1. The composition

$$\mathbb{F} \hookrightarrow \mathbb{F}[x] \twoheadrightarrow \mathbb{F}[x]/(f)$$

is a finite extension of fields, of degree equal to $\deg f$ (cf. Example 3.1.23).

Example 3.1.22 (Spec $\mathbb{Z}$)**.** The spectrum $\operatorname{Spec} \mathbb{Z}$ is the arithmetic counterpart of $\operatorname{Spec} \mathbf{k}[x]$. It has one closed point for every nonzero prime ideal $(p) \subset \mathbb{Z}$, and, again, precisely one non-closed point $\xi = (0) \in \operatorname{Spec} \mathbb{Z}$ called the generic point.

Figure 3.3: The topological space $\operatorname{Spec} \mathbb{Z}$, with one closed point for every prime $p \in \mathbb{Z}$. The generic point $\xi = (0)$ is 'dense', i.e. $\overline{\{\xi\}} = \operatorname{Spec} \mathbb{Z}$, since $0 \in \mathbb{Z}$ is in every prime ideal.

Example 3.1.23 ($\mathbb{A}^1_{\mathbb{R}} = \operatorname{Spec} \mathbb{R}[x]$)**.** The ring $\mathbb{R}[x]$ is a principal ideal domain. Its prime ideals are

$$(0), \quad (x-a), \quad (x^2 + bx + c),$$

where $a \in \mathbb{R}$ and $x^2 + bx + c$ is irreducible and satisfies $b^2 - 4c < 0$. The only prime ideal which is not maximal is, once more, $(0) \subset \mathbb{R}[x]$. However, we see here an important phenomenon arising when one considers fields that are not algebraically closed: we have, for the two types of *maximal* ideals,

$$\mathbb{R}[x]/(x-a) \cong \mathbb{R}, \quad \mathbb{R}[x]/(x^2 + bx + c) \cong \mathbb{C}.$$

Of course, if f is an irreducible quadratic polynomial as above, the prime ideal $(f) \subset \mathbb{R}[x]$ defines one precise point in the spectrum, although we may want to think of it as the identification of two complex conjugate points, just as $\pm i$ give rise to $x^2+1=(x-i)(x+i)$. See Example 3.1.81 for more on this.

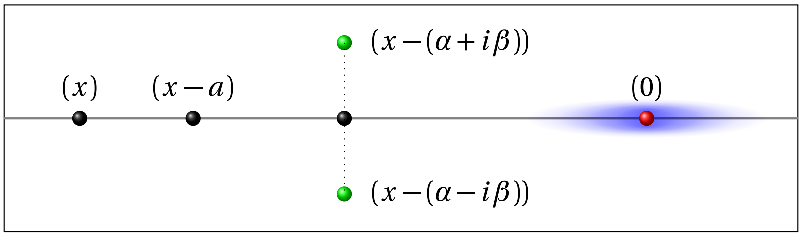

Figure 3.4: The real affine line $\mathbb{A}^1_{\mathbb{R}}$. Points $(f)\in\mathbb{A}^1_{\mathbb{R}}$ with $\mathbb{R}[x]/(f)\cong\mathbb{C}$ can be thought of pairs of conjugate *complex* points coming together.

Example 3.1.24 ($\mathbb{A}^2_{\mathbf{k}}=\operatorname{Spec}\mathbf{k}[x,y]$)**.** Let $\mathbf{k}$ be an algebraically closed field. The spectrum $\mathbb{A}^2_{\mathbf{k}}=\operatorname{Spec}\mathbf{k}[x,y]$ is called the *affine plane* (over $\mathbf{k}$). The prime ideals in $\mathbf{k}[x,y]$ are

$$(0),\quad (x-a,y-b),\quad (f)$$

where $(a,b)\in\mathbf{k}^2$ and $f=f(x,y)$ is an irreducible polynomial. Maximal ideals are those of the form $(x-a,y-b)$, and correspond indeed to the 'classical' points (a,b) of $\mathbf{k}^2$. These are then closed points of $\mathbb{A}^2_{\mathbf{k}}$. Given an irreducible polynomial $f\in\mathbf{k}[x,y]$, we have

$$\mathrm{V}(f)=\left\{\mathfrak{p}\in\mathbb{A}^2_{\mathbf{k}}\,\middle|\,f\in\mathfrak{p}\right\}=\left\{(x-a,y-b)\,\middle|\,f(a,b)=0\right\}\cup\left\{(f)\right\}.$$

Clearly $\mathrm{V}(f)$ is the closure of $\{(f)\}$. The ideal (f) is the generic point of $\mathrm{V}(f)$, because it corresponds to the trivial ideal in the integral domain $\mathbf{k}[x,y]/(f)$, whereas the other points are closed points.

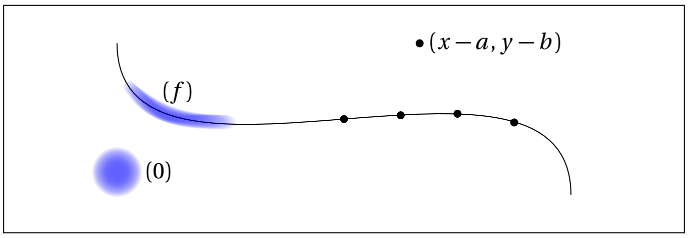

Figure 3.5: The affine plane $\mathbb{A}^2_{\mathbf{k}}$.

Example 3.1.25 (Spec of a DVR)**.** Let A be a DVR, shorthand for 'discrete valuation ring'. Then, by definition, A is a principal ideal domain with exactly one maximal ideal $\mathfrak{m}\subset A$. This ideal is also prime, and there is precisely one other prime ideal, namely $(0)\subset A$. In other words,

$$\operatorname{Spec}A=\{\xi,\mathfrak{m}\}$$

consists of two points, where $\mathfrak{m}$ is closed (cf. Remark 3.1.15) and hence ξ, corresponding to $(0)\subset A$, is open. Note that the Zariski topology is *not the discrete topology* on two points here, for the point $\mathfrak{m}$ is a *specialisation* of the point ξ, i.e. $\mathfrak{m}$ lies in the closure of ξ. We shall see later that, despite being a finite set, $\operatorname{Spec} A$ is a 1-dimensional scheme (or A is a 1-dimensional ring, cf. Definition 4.6.16), simply because $0 \in \mathfrak{m}$. An example of DVR is given by the ring of formal power series $\mathbf{k}[\![t]\!]$, where $\mathbf{k}$ is a field. In this case, the maximal ideal is just the ideal generated by t.

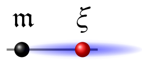

Figure 3.6: The black bullet represents the closed point $\mathfrak{m}$. The red point surrounded by the cloud, as usual, is the generic point.

Example 3.1.26 ($\operatorname{Spec}\mathbf{k}[t]/t^2$)**.** First of all some terminology: the ring $A = \mathbf{k}[t]/t^2$ is called the *ring of dual numbers*[2] (over $\mathbf{k}$). Some people write $\mathbf{k}[\varepsilon]$ to denote this ring, being understood that $\varepsilon^2 = 0$. There is only one prime (and in fact maximal) ideal in A, namely

$$(\overline{t}) \subset A,$$

where $\overline{t}$ is the image of $t \in \mathbf{k}[t]$ under the projection $\mathbf{k}[t] \twoheadrightarrow A$. Thus, topologically, this space is the same as $\operatorname{Spec}\mathbf{k}$. However, it will be different (i.e. *not isomorphic* to $\operatorname{Spec}\mathbf{k}$) as an affine scheme. A first strong indication of this fact was already given in Example 3.1.19.

3.1.4 The sheaf of rings $\mathscr{O}_{\operatorname{Spec} A}$

Let A be a ring. Set $X = \operatorname{Spec} A$, equipped as always with the Zariski topology. We now define a sheaf of rings

$$\mathscr{O}_X \in \mathsf{Sh}(X, \mathsf{Rings}),$$

that we will refer to as the *sheaf of regular functions* on $X = \operatorname{Spec} A$.

By Lemma 2.7.7, to define a sheaf of rings on the topological space X, it is enough to define a $\mathcal{B}$-sheaf of rings where

$$\mathcal{B} = \left\{ \mathrm{D}(f) \,\middle|\, f \in A \right\} \subset \tau_X$$

is the base of principal open sets in X (cf. Corollary 3.1.7). Our working definition for this $\mathcal{B}$-sheaf will be the assignment

$$\mathrm{D}(f) \longmapsto A_f = \left\{ \frac{a}{f^n} \,\middle|\, a \in A, n \geq 0 \right\}. \tag{3.1.4}$$

[2] In case you care to know why they have this name, here is the answer directly from Wikipedia: Dual numbers were introduced in 1873 by William Clifford, and were used at the beginning of the twentieth century by the German mathematician Eduard Study, who used them to represent the dual angle which measures the relative position of two skew lines in space.

Note that (if we take $f = 1$) we are *defining*

$$\mathscr{O}_X(X) = A.$$

See Appendix B.5 for all you need to know about localisation. Sometimes we shall write af^{-n} or a/f^n for the element

$$\frac{a}{f^n} \in A_f.$$

We need to verify that (3.1.4) does indeed define a $\mathcal{B}$-sheaf of rings.

First of all, let us make sure this assignment is well-defined. We know (cf. Example 3.1.11) that

$$\mathrm{D}(g) \subset \mathrm{D}(f) \iff g \in \sqrt{fA} \iff \frac{f}{1} \in A_g^{\times}.$$

For the second equivalence, write $g^r = fb$ in A for some $b \in A$ and some $r > 0$. Thus

$$\frac{f}{1} \cdot \frac{b}{1} = \frac{fb}{1} = \frac{g^r}{1} \in A_g,$$

which is invertible in A_g. Therefore

$$\frac{f}{1} \in A_g$$

is also invertible, with inverse

$$\left(\frac{f}{1}\right)^{-1} = \frac{1}{g^r} \cdot \frac{b}{1} = \frac{b}{g^r} \in A_g.$$

By the universal property of the localisation A_f, we get a canonical ring homomorphism

$$\begin{aligned} A_f &\xrightarrow{\rho_{\mathrm{D}(f)\mathrm{D}(g)}} A_g \\ \frac{a}{f^n} &\longmapsto \frac{ab^n}{g^{nr}} \end{aligned} \tag{3.1.5}$$

making the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_g \\ A_f \arrow[ur, "\rho_{\mathrm{D}(f)\mathrm{D}(g)}"'] & \end{tikzcd}$$

commute. This map is an isomorphism as soon as $\mathrm{D}(g) = \mathrm{D}(f)$, showing that (3.1.4) is well-defined.

Note that the assignment (3.1.4) prescribes (cf. Remark B.5.5)

$$\emptyset = \mathrm{D}(0) \mapsto A_0 = 0,$$

The following lemma confirms that the maps just defined compose well, thus turning $\mathrm{D}(f) \mapsto A_f$ into a $\mathcal{B}$-presheaf.

LEMMA 3.1.27. *Fix* $f, g, h \in A$.

(i) *We have* $\rho_{\mathrm{D}(f)\mathrm{D}(f)} = \mathrm{id}_{A_f}$.

(ii) *Given inclusions of principal open subsets*

$$\mathrm{D}(h) \subset \mathrm{D}(g) \subset \mathrm{D}(f)$$

in $\operatorname{Spec} A$, *we have an identity*

$$\rho_{\mathrm{D}(g)\mathrm{D}(h)} \circ \rho_{\mathrm{D}(f)\mathrm{D}(g)} = \rho_{\mathrm{D}(f)\mathrm{D}(h)}$$

as maps $A_f \to A_h$.

In particular, $\mathrm{D}(f) \mapsto A_f$ *defines a* $\mathcal{B}$*-presheaf on* $\operatorname{Spec} A$.

Proof. Condition (i) is clear, so we move to (ii). First we write

$$g^r = fb, \quad h^s = gc,$$

for some $r, s > 0$ and $b, c \in A$. Then

$$h^{rs} = (h^s)^r = g^r c^r = fbc^r.$$

Then, according to (3.1.5), the map $\rho_{\mathrm{D}(f)\mathrm{D}(h)} \colon A_f \to A_h$ is given by

$$\frac{a}{f^n} \mapsto \frac{a(bc^r)^n}{h^{rsn}}. \tag{3.1.6}$$

On the other hand, we have to compose

$$\begin{array}{ccc} A_f & \xrightarrow{\rho_{\mathrm{D}(f)\mathrm{D}(g)}} & A_g \\ \dfrac{a}{f^n} & \longmapsto & \dfrac{ab^n}{g^{nr}} \end{array}$$

and

$$\begin{array}{ccc} A_g & \xrightarrow{\rho_{\mathrm{D}(g)\mathrm{D}(h)}} & A_h \\ \dfrac{a}{g^m} & \longmapsto & \dfrac{ac^m}{h^{ms}} \end{array}$$

with one another. The result of the composition is

$$\frac{a}{f^n} \mapsto \frac{ab^n}{g^{nr}} \mapsto \frac{ab^n c^{nr}}{h^{nrs}} = \frac{a(bc^r)^n}{h^{rsn}},$$

which agrees with (3.1.6), as we wanted. $\square$

THEOREM 3.1.28. *Let* A *be a ring. Set* $X = \operatorname{Spec} A$.

(a) *The rule* (3.1.4) *defines a $\mathcal{B}$-sheaf of rings on X. The induced sheaf of rings will be denoted $\mathscr{O}_X$.*

(b) *We have $\mathscr{O}_X(X) = A$.*

(c) *The stalk $\mathscr{O}_{X,x}$ of $\mathscr{O}_X$ at the point $x \in X$ corresponding to $\mathfrak{p} \subset A$ is isomorphic to $A_{\mathfrak{p}}$.*

Proof. We proceed step by step.

(a) We know that (3.1.4) defines a $\mathcal{B}$-presheaf of rings by Lemma 3.1.27. We check the sheaf conditions (3)–(4) of Important Definition 2.2.1 on the open set $U = X = \mathrm{D}(1)$, the case of an arbitrary principal open $U = \mathrm{D}(h) \in \mathcal{B}$ being essentially identical. Recall from (3.1.3) that

$$\operatorname{Spec} A = \bigcup_{f \in A} \mathrm{D}(f).$$

By Lemma 3.1.12, this is equivalent to saying that there is a *finite* set F indexing a set of generators $\{ f_i \mid i \in F \} \subset A$ of the unit ideal $(1) = A$, so that in particular $1 \in \sum_{i \in F}(f_i)$. In what follows, set $U_i = \mathrm{D}(f_i)$ and $U_{ij} = U_i \cap U_j = \mathrm{D}(f_i f_j)$.

Sheaf axiom (3): Fix $s \in A_1 = A$ such that $s|_{U_i} = 0 \in A_{f_i}$. This means

$$\frac{s}{1} = \frac{0}{1} \in A_{f_i},$$

i.e. there exists $m > 0$ such that $f_i^m s = 0 \in A$. Since F is finite, we can pick a uniform m which works for every f_i. Since

$$X = \bigcup_{i \in F} \mathrm{D}(f_i) = \bigcup_{i \in F} \mathrm{D}(f_i^m),$$

as before we have

$$1 \in \sum_{i \in F} (f_i^m),$$

which implies

$$s \in \sum_{i \in F} (f_i^m s) = 0.$$

Hence $s = 0$, as required.

Sheaf axiom (4): By definition, $\mathscr{O}_X(U_{ij}) = A_{f_i f_j}$. Fix sections $s_i \in A_{f_i}$ such that $s_i|_{U_{ij}} = s_j|_{U_{ij}}$ for every i and j. That is, s_i and s_j have the same image along the maps

$$\begin{tikzcd} A_{f_i} \arrow[dr] & & A_{f_j} \arrow[dl] \\ & A_{f_i f_j} & \end{tikzcd}$$

Write (again for a uniform $m > 0$)

$$s_i = \frac{b_i}{f_i^m} \in A_{f_i}, \quad i \in F.$$

Now, $s_i|_{U_{ij}} = s_j|_{U_{ij}}$ means that there exists an integer $r > 0$ such that

$$(f_i f_j)^r (b_i f_j^m - b_j f_i^m) = 0 \in A \tag{3.1.7}$$

for all $i, j \in F$. As before,

$$1 \in \sum_{i\in F} (f_i^{m+r})$$

yields

$$1 = \sum_{j\in F} a_j f_j^{m+r}, \quad a_j \in A. \tag{3.1.8}$$

Define

$$s = \sum_{j\in F} a_j b_j f_j^r \in A, \tag{3.1.9}$$

so that the chain of identities

$$\begin{aligned} f_i^{m+r} s &= \sum_{j\in F} a_j b_j f_i^m (f_i f_j)^r && \text{by (3.1.9)} \\ &= \sum_{j\in F} a_j b_i f_j^m (f_i f_j)^r && \text{by (3.1.7)} \\ &= \sum_{j\in F} a_j f_j^{m+r} b_i f_i^r \\ &= b_i f_i^r && \text{by (3.1.8)} \end{aligned}$$

yields

$$f_i^r (b_i - f_i^m s) = 0.$$

But this in turn is equivalent to

$$\frac{s}{1} = \frac{b_i}{f_i^m} = s_i \in A_{f_i}.$$

So we have proved $s|_{U_i} = s_i$ for every $i \in F$.

(b) We have $\operatorname{Spec} A = \mathrm{D}(1)$, so this actually follows from the definition, using that $A_1 = A$ since $1 \in A$ is already invertible in A.

(c) Let $\mathfrak{p} \subset A$ be the ideal corresponding to $x \in X$. For every $f \notin \mathfrak{p}$, there is (by the universal property of A_f) a canonical map $A_f \to A_\mathfrak{p} = \{a/h \mid h \notin \mathfrak{p}\}$ because f is invertible in $A_\mathfrak{p}$. By the universal property of colimits, we get a canonical ring homomorphism

$$\mathscr{O}_{X,x} = \varinjlim_{f\notin\mathfrak{p}} A_f \xrightarrow{\ \alpha\ } A_\mathfrak{p}.$$

An element of the form $a/h \in A_\mathfrak{p}$ lies in the image of $A_h \to \mathscr{O}_{X,x} \to A_\mathfrak{p}$, therefore α is surjective. On the other hand, if $a/h^n \in A_h$ (for $h \notin \mathfrak{p}$ and some $n > 0$) maps to $0 = 0/1 \in A_{\mathfrak{p}}$, then by definition of localisation there exists $g \in A \setminus \mathfrak{p}$ such that $ga = 0 \in A$. Then the image of a/h^n in A_{gh} is

$$\frac{g^{n-1}ga}{(gh)^n} = 0 \in A_{gh},$$

so a/h^n goes to 0 in $\mathscr{O}_{X,x}$. We have confirmed that

$$\ker\big(A_h \to \mathscr{O}_{X,x} \to A_{\mathfrak{p}}\big) = \ker\big(A_h \to \mathscr{O}_{X,x}\big) \tag{3.1.10}$$

for every $h \in A \setminus \mathfrak{p}$, which is enough to conclude that α is injective. (Reason: Let $z \in \mathscr{O}_{X,x}$ be an element such that $\alpha(z) = 0 \in A_{\mathfrak{p}}$. There exist $a \in A$, $h \in A \setminus \mathfrak{p}$ and $n \geq 0$ such that $z = q(a/h^n)$ where $q \colon A_h \to \mathscr{O}_{X,x}$ is the canonical map. So $0 = \alpha(q(a/h^n))$ implies $a/h^n \in \ker(\alpha \circ q) = \ker q$, the identity of kernels being our assumption (3.1.10). It follows that $0 = q(a/h^n) = z$, as required). □

The following is now immediate from the definition of locally ringed space.

COROLLARY 3.1.29. *Let A be a ring. The pair* $(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$ *defines a locally ringed space. For every $\mathfrak{p} \in \operatorname{Spec} A$, the corresponding local ring at $\mathfrak{p}$ is the local ring* $(A_{\mathfrak{p}}, \mathfrak{p}A_{\mathfrak{p}})$.

The quotient $\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ is called the *residue field* at $\mathfrak{p}$.

Important Definition 3.1.2 (Affine scheme)**.** An *affine scheme* is a locally ringed space isomorphic (in the category of locally ringed spaces) to $(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$ for some ring A.

As an important class of examples of affine schemes, we have the notion of affine algebraic variety. We will see a different notion, that of projective variety, in Important Definition 3.2.1.

Important Definition 3.1.3 (Affine variety)**.** An *affine variety* over a field $\mathbb{F}$ (also called an *affine $\mathbb{F}$-variety*) is an affine scheme of the form $\operatorname{Spec} A$, where A is a finitely generated $\mathbb{F}$-algebra (i.e. $A = \mathbb{F}[x_1, \ldots, x_n]/I$ for some n and some ideal I).

Exercise 3.1.30. Let A be a ring, $\mathfrak{p} \subset A$ a prime ideal. Set $\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. Show that there is a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, two heads] & A_{\mathfrak{p}} \arrow[d, two heads] \\ A/\mathfrak{p} \arrow[r, hook] & \kappa(\mathfrak{p}) \end{tikzcd}$$

of rings, and that

$$\operatorname{Frac} A/\mathfrak{p} = \kappa(\mathfrak{p}).$$

In particular, if $\mathfrak{m} \subset A$ is maximal, then $\kappa(\mathfrak{m}) = A/\mathfrak{m}$.

Example 3.1.31. Let $\mathbb{F}$ be a field, $0 \in X = \mathbb{A}^1_{\mathbb{F}} = \operatorname{Spec}\mathbb{F}[x]$ the point corresponding to $(x) \subset \mathbb{F}[x]$. (You are allowed, and in fact encouraged, to call this point 'the origin' of the affine line). The local ring of X at 0 is

$$\mathscr{O}_{X,0} = \mathbb{F}[x]_{(x)} = \left\{ \frac{f(x)}{g(x)} \,\middle|\, g(x) \notin (x) \right\} = \left\{ \frac{f(x)}{g(x)} \,\middle|\, g(0) \neq 0 \right\} \subset \operatorname{Frac}\mathbb{F}[x] = \mathbb{F}(x),$$

and the residue field is

$$\kappa(0) = \mathscr{O}_{X,0}/\mathfrak{m}_0 = \mathbb{F}[x]_{(x)}/(x)\mathbb{F}[x]_{(x)} \cong \mathbb{F}[x]/(x) \cong \mathbb{F}.$$

For the second-last isomorphism we used Exercise 3.1.30 (see also Proposition B.5.10). The same chain of isomorphisms holds replacing 0 with any other closed point of the form $(x-a)$, with $a \in \mathbb{F}$. If $\xi = (0)$, we have

$$\kappa(\xi) = \mathscr{O}_{X,\xi} = \mathbb{F}(x).$$

Functions on $\operatorname{Spec} A$ revisited

We already saw, but we need to emphasise, that

$$\boxed{\mathscr{O}_{\operatorname{Spec}A}(\operatorname{Spec}A) = A.}$$

That is, regular functions on $\operatorname{Spec}A$ are precisely the elements of A.

We could have bypassed $\mathscr{B}$-sheaves and defined the sheaf of rings $\mathscr{O}_X$ on $X = \operatorname{Spec}A$ directly (as done in [8, Ch. 2]) by setting

$$\mathscr{O}_X(U) = \left\{ U \xrightarrow{s} \coprod_{\mathfrak{p}\in U} A_{\mathfrak{p}} \;\middle|\; \begin{array}{c} \text{for every } \mathfrak{p} \in U,\ s(\mathfrak{p}) \in A_{\mathfrak{p}} \text{ and there exist} \\ \text{an open neighbourhood } V \subset U \text{ of } \mathfrak{p} \\ \text{and } a, f \in A \text{ such that, for every } \mathfrak{q} \in V, \\ f \notin \mathfrak{q} \text{ and } s(\mathfrak{q}) = a/f \text{ in } A_{\mathfrak{q}} \end{array} \right\} \tag{3.1.11}$$

for every open subset $U \subset X$. The fact that $U \mapsto \mathscr{O}_X(U)$ is a sheaf (and coincides with the sheaf $\mathscr{O}_X$ defined in Theorem 3.1.28(a)) is clear once one realises that the very definition just rephrases the notion of compatible germs.

Let us focus on the case $U = X = \operatorname{Spec}A$. Consider the map $\psi\colon A \to \mathscr{O}_X(X)$ sending $a \in A$ to the function

$$s_a\colon X \to \coprod_{\mathfrak{p}\in X} A_{\mathfrak{p}}, \quad \mathfrak{p} \mapsto \text{image of } a \text{ along } A \to A_{\mathfrak{p}}.$$

This map ψ is injective. Indeed, assume $s_a = s_b$ for $a, b \in A$. This means that for every $\mathfrak{p} \in X$, the elements a and b have the same image in $A_{\mathfrak{p}}$. Hence there is an element $r \in A \setminus \mathfrak{p}$ such that $r(a-b) = 0$ in A. Set $J = \operatorname{Ann}(a-b)$, so that $r \in J$. Thus $J \not\subset \mathfrak{p}$ for every $\mathfrak{p}$. But then, in particular, J is not contained in any maximal ideal, hence $J = A$. Thus $a - b = 1 \cdot (a-b) = 0$, i.e. ψ is injective.

One may insist to call *regular function* on $X = \operatorname{Spec} A$ a map that is *field-valued*. This can be done as follows, starting from the definition in Equation (3.1.11). Let $a \in A = \mathscr{O}_X(X)$. Composing s_a with the quotient maps

$$A_\mathfrak{p} \twoheadrightarrow \kappa(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$$

one obtains the map

$$\widetilde{s}_a \colon X \longrightarrow \coprod_{\mathfrak{p}\in X} \kappa(\mathfrak{p}),$$

where the field $\kappa(\mathfrak{p})$ may (and will) vary from point to point (cf. Example 3.1.54).

Remark 3.1.32. The closed set $\mathrm{V}(I) = \{\, \mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \supset I \,\} \subset \operatorname{Spec} A$ can be reinterpreted as

$$\mathrm{V}(I) = \{\, \mathfrak{p} \in \operatorname{Spec} A \mid \widetilde{s}_a(\mathfrak{p}) = 0 \in \kappa(\mathfrak{p}) \text{ for all } a \in I \,\}. \tag{3.1.12}$$

This explains once more the letter ‘V’, standing for ‘vanishing’. But if you encounter the letter ‘Z’, it stands for ‘zero locus’!

3.1.5 The definition of schemes and first topological properties

We are ready for the definition of schemes.

Important Definition 3.1.4 (Scheme)**.** A *scheme* is a locally ringed space $(X, \mathscr{O}_X)$ in which every point $x \in X$ has an open neighbourhood $x \in U \subset X$ such that $(U, \mathscr{O}_X|_U)$ is an affine scheme.

Terminology 3.1.33. Let $(X, \mathscr{O}_X)$ be a scheme. The structure sheaf $\mathscr{O}_X$ is referred to as the *sheaf of regular functions* of the scheme. The ring $\mathscr{O}_X(X)$ is called the ring of *regular functions* on X. We keep the notation $(\mathscr{O}_{X,x}, \mathfrak{m}_x, \kappa(x))$ for the local ring at a point $x \in X$.

Definition 3.1.34 (Morphism of schemes)**.** A *morphism of schemes* is a morphism in the category of locally ringed spaces. In particular, an *isomorphism of schemes* $(X, \mathscr{O}_X) \xrightarrow{\sim} (Y, \mathscr{O}_Y)$ is a morphism $(f, f^\#)$ such that $f \colon X \xrightarrow{\sim} Y$ is a homeomorphism and $f^\# \colon \mathscr{O}_Y \xrightarrow{\sim} f_*\mathscr{O}_X$ is an isomorphism of sheaves of rings over Y.

Definition 3.1.35 (Immersions of schemes)**.** An open (resp. closed) immersion of schemes is an open (resp. closed) immersion in the category of locally ringed spaces (cf. Definition 2.10.8).

Definition 3.1.36 (Open subscheme)**.** An *open subscheme* of a scheme $(X, \mathscr{O}_X)$ is a scheme of the form $(U, \mathscr{O}_X|_U)$, where U is an open subset of X. We will often just write $\mathscr{O}_U$ instead of $\mathscr{O}_X|_U$.

Note that an open subscheme comes with an open immersion $U \hookrightarrow X$. We will see in Remark 3.1.59 that an open subset $U \subset X$ of a scheme X is naturally a scheme.

The definition of closed subscheme is more subtle.

Definition 3.1.37 (Closed subscheme)**.** Let X be a scheme. A *closed subscheme* of X is an equivalence class of closed immersions with target X, where two closed immersions $\iota\colon Z \hookrightarrow X$ and $\iota'\colon Z' \hookrightarrow X$ are isomorphic if there exists an isomorphism $\alpha\colon Z \xrightarrow{\sim} Z'$ such that $\iota' \circ \alpha = \iota$.

Notation 3.1.38. Affine schemes (resp. schemes) form a category, denoted Aff (resp. Sch), where morphisms are just the morphisms in the larger category of locally ringed spaces. We denote a scheme $(X, \mathscr{O}_X)$ simply by X, and a morphism $(f, f^\#)\colon (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ simply by $f\colon X \to Y$.

We have thus a chain of full inclusions of categories

$$\mathsf{Aff} \hookrightarrow \mathsf{Sch} \hookrightarrow \mathsf{LRS}.$$

Here are some purely topological properties of a scheme. Recall that a topological space X is *irreducible* if it cannot be written as a union $Z_1 \cup Z_2$ of two proper closed subsets $Z_i \subset X$. Equivalently, X is irreducible if and only if any two nonempty open subsets of X intersect. Moreover, any open subset U of an irreducible topological space is dense, i.e. $\overline{U} = X$.

Definition 3.1.39. A scheme $(X, \mathscr{O}_X)$ is said to be *quasicompact* (resp. *irreducible*, resp. *connected*) if the underlying topological space X is. A morphism of schemes $f\colon X \to Y$ is called *quasicompact* if the preimage of any affine open subset is quasicompact.

Translation: a scheme is quasicompact when it admits a finite open cover by affine schemes. For morphism, we have the following.

Exercise 3.1.40. A morphism $f\colon X \to Y$ is quasicompact if and only if Y has an affine open cover $Y = \bigcup_{i \in I} Y_i$ such that $f^{-1}(Y_i)$ is quasicompact for all i.

We already saw in Lemma 3.1.12 than an affine scheme is quasicompact. Any irreducible scheme is in particular connected. There are, however, connected schemes which are reducible (i.e. not irreducible), see Example 3.1.44.

Proposition 3.1.41. *If X is a quasicompact scheme, then X has a closed point.*

Proof. We present Schwede's proof [15, Prop. 4.1].

By quasicompactness of X, there is a finite open cover of X by affine schemes $U_i = \operatorname{Spec} A_i$, say $X = U_1 \cup \cdots \cup U_r$. Consider a closed point $x_1 \in U_1$. If x_1 is closed in X, we are done. Otherwise, pick a point $x_2 \in \overline{\{x_1\}}$, with $x_2 \neq x_1$. Then x_2 lies in some U_i, but not in

U_1 since $\overline{\{x_1\}} \cap U_1 = \{x_1\}$. Say $x_2 \in U_2 \setminus U_1$. If x_2 is closed in X, we are done. Otherwise, pick a point $x_3 \in \overline{\{x_2\}}$, with $x_3 \neq x_2$. But x_3 is also in the closure of x_1, thus $x_3 \notin U_1 \cup U_2$. Say $x_3 \in U_3$, and continue until the cover is exhausted: this process stops, so X must contain a closed point. □

Exercise 3.1.42. Prove that a scheme X is the spectrum of a local ring if and only if it is quasicompact and has a unique closed point.

3.1.6 Generic points, take I

We start here our discussion around generic points. We cover, for now, only the case of irreducible schemes. You will notice that most arguments are entirely of topological nature.

The next result tells us when an affine scheme, or a closed subset thereof, is irreducible.

PROPOSITION 3.1.43. *Let A be a ring, and set $X = \operatorname{Spec} A$. Then a closed subset* $\mathrm{V}(I) \subset X$ *is irreducible if and only if $\sqrt{I} \subset A$ is prime. In particular,*

(a) *X is irreducible if and only if $\sqrt{0} \subset A$ is prime.*

(b) *If A is an integral domain, then* $\operatorname{Spec} A$ *is irreducible.*

Proof. Assume $\mathrm{V}(I) = \{\mathfrak{p} \in X \mid \mathfrak{p} \supset I\} \subset X$ irreducible. To say that $\sqrt{I}$ is prime means that if $ab \in \sqrt{I}$ then either $a \in \sqrt{I}$ or $b \in \sqrt{I}$. Let us assume, by contradiction, that there are $a, b \in A \setminus \sqrt{I}$ such that $ab \in \sqrt{I}$. Set $X_a = \mathrm{V}(I) \cap \mathrm{V}(a)$ and $X_b = \mathrm{V}(I) \cap \mathrm{V}(b)$. Then $X_a \cup X_b \subset \mathrm{V}(I)$, and if $\mathfrak{p} \in \mathrm{V}(I) = \mathrm{V}(\sqrt{I})$ then $\mathfrak{p} \supset \sqrt{I} \ni ab$, so that either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$, which proves $X_a \cup X_b = \mathrm{V}(I)$. Since $X_a \neq \mathrm{V}(I) \neq X_b$, we contradict irreducibility of $\mathrm{V}(I)$.

Conversely, assume $\sqrt{I} \subset A$ is prime. If $\mathrm{V}(I) = \mathrm{V}(J_1) \cup \mathrm{V}(J_2) = \mathrm{V}(J_1 \cap J_2) = \mathrm{V}(J_1 J_2)$ then $\sqrt{I} = \sqrt{J_1 J_2} \supset J_1 J_2$, thus either $J_1 \subset \sqrt{I}$ or $J_2 \subset \sqrt{I}$. But if, say, $J_1 \subset \sqrt{I}$, it follows that $\mathrm{V}(I) = \mathrm{V}(\sqrt{I}) \subset \mathrm{V}(J_1)$, which yields $\mathrm{V}(J_1) = \mathrm{V}(I)$. Thus $\mathrm{V}(I)$ is irreducible. □

Example 3.1.44. The closed subset

$$\mathrm{V}(y - x^2) \subset \mathbb{A}^2_{\mathbf{k}}$$

is irreducible. On the other hand, the closed subset

$$\mathrm{V}(y^2 - x^2) \subset \mathbb{A}^2_{\mathbf{k}}$$

is reducible (but connected), being equal to $\mathrm{V}(x - y) \cup \mathrm{V}(x + y)$. The same conclusion holds for $\mathrm{V}(xy) \subset \mathbb{A}^2_{\mathbf{k}}$.

Caution 3.1.45 (Irreducibility depends on the base field)**.** Consider the polynomial $f = x^2+1 \in \mathbb{R}[x] \subset \mathbb{C}[x]$. Then $\mathrm{V}(f) \subset \mathbb{A}^1_{\mathbb{R}}$ is irreducible (a point), but $\mathrm{V}(f) \subset \mathbb{A}^1_{\mathbb{C}}$ is reducible, being equal to $\mathrm{V}(x-i) \cup \mathrm{V}(x+i)$.

LEMMA 3.1.46. *Let X be an irreducible scheme. Then, there exists a unique point $\xi \in X$ such that $X = \overline{\{\xi\}}$.*

Proof. Let us show uniqueness first. Let ξ_1, ξ_2 be two points such that $\overline{\{\xi_1\}} = X = \overline{\{\xi_2\}}$. Then any nonempty open $U \subset X$ contains both ξ_1 and ξ_2. Pick a nonempty affine open subset $U = \operatorname{Spec} A \subset X$. Let $\mathfrak{p}_i \subset A$ be the prime ideal corresponding to ξ_i. Since X is irreducible, U is irreducible and dense, hence

$$U = \overline{\{\xi_i\}} = \mathrm{V}(\mathfrak{p}_i), \quad i = 1,2,$$

where the closure is taken in U. So when we write $\xi_2 \in U = \mathrm{V}(\mathfrak{p}_1)$ we obtain $\mathfrak{p}_2 \supset \mathfrak{p}_1$, and when we write $\xi_1 \in U = \mathrm{V}(\mathfrak{p}_2)$ we obtain $\mathfrak{p}_1 \supset \mathfrak{p}_2$. Thus $\mathfrak{p}_1 = \mathfrak{p}_2$, i.e. $\xi_1 = \xi_2$.[3]

Now for existence. If $U = \operatorname{Spec} A \subset X$ is a nonempty open affine subset, then U is irreducible, i.e. $\mathfrak{p} = \sqrt{0} \subset A$ is prime by Proposition 3.1.43, and hence $U = \mathrm{V}(\sqrt{0}) = \mathrm{V}(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$. But since U is dense in X, the closure of $\mathfrak{p}$ in X is X itself. □

Definition 3.1.47 (Generic point, take I)**.** Let X be an irreducible scheme. The point $\xi \in X$ of Lemma 3.1.46 is called the *generic point* of X.

Remark 3.1.48. The proof of Lemma 3.1.46 actually works for an arbitrary irreducible closed subset X of an arbitrary scheme Y.

Remark 3.1.49. If A is a domain, then $(0) \subset A$ is the unique minimal prime, thus $\operatorname{Spec} A$ is irreducible (Proposition 3.1.43). For instance, if $\mathbb{F}$ is a field, the affine space $\mathbb{A}^n_{\mathbb{F}}$ is irreducible. However, the converse is false: for instance, $\operatorname{Spec} \mathbb{F}[t]/t^n$ is irreducible for every $n \geq 1$, but $\mathbb{F}[t]/t^n$ is not a domain as soon as $n > 1$.

Exercise 3.1.50. Prove that a scheme X is connected if and only if $\mathscr{O}_X(X)$ has only the trivial idempotents $0, 1$.

Exercise 3.1.51. Let A, A' be rings, $\mathbb{F}$ a field. Decide whether the following affine schemes are irreducible (resp. connected):

(i) $\operatorname{Spec} \mathbb{C}[x,y]/(y^2 - x^2(x+1))$,

(ii) $\operatorname{Spec} \mathbb{C}[x,y]/(y^2 - x^3)$

(iii) $\operatorname{Spec} \mathbb{Z}[x]/(2x)$,

(iv) $\operatorname{Spec} \mathbb{C}[x,y]/(xy, y^2)$,

[3] Alternatively, just note that $\mathfrak{p}_1 = \sqrt{\mathfrak{p}_1} = \sqrt{\mathfrak{p}_2} = \mathfrak{p}_2$.

(v) $\operatorname{Spec}\mathbb{F}[x,y]/(x^2,xy,y^3)$,

(vi) $\operatorname{Spec}(A\times A')$,

(vii) $\operatorname{Spec}\mathbb{C}[x,y,z]/(xy-z^2)$,

(viii) $\operatorname{Spec}\mathbb{C}[x,y]/(x^2+y^2-1)$.

It is clear from the last paragraph of the proof of Lemma 3.1.46 that if A is a domain then the generic point $\xi\in\operatorname{Spec}A$ is the point corresponding to $(0)\subset A$, which is manifestly the unique minimal prime. The following lemma clarifies the basic properties of the generic point in this special case.

LEMMA 3.1.52. *Let A be an integral domain with fraction field K. Let $\xi\in X=\operatorname{Spec}A$ be the point corresponding to $(0)\subset A$. Then*

(i) *We have $\mathscr{O}_{X,\xi}=K$.*

(ii) *ξ belongs to every nonempty open subset $U\subset X$, and $\mathscr{O}_X(U)\to\mathscr{O}_{X,\xi}$ is injective.*

(iii) *For every nonempty open subset $V\hookrightarrow U$, the map $\mathscr{O}_X(U)\to\mathscr{O}_X(V)$ is injective.*

Proof. We proceed step by step.

(i) This follows from Theorem 3.1.28(c) and the observation that the localisation of an integral domain at the prime ideal (0) is precisely the fraction field of the domain.

(ii) To say $\xi\in\operatorname{Spec}A\setminus \mathrm{V}(I)=\{\mathfrak{p}\in\operatorname{Spec}A\mid\mathfrak{p}\not\supset I\}$ for every $(0)\not\subset I\subset A$ means precisely that $(0)\not\supset I$ for all I, a tautology. Write $U=\bigcup_{i\in I}\mathrm{D}(f_i)$ and assume $s\in\mathscr{O}_X(U)$ goes to 0 in $K=\mathscr{O}_{X,\xi}$. Well, s goes to $s|_{\mathrm{D}(f_i)}\in\mathscr{O}_X(\mathrm{D}(f_i))=A_{f_i}$ first, and $A_{f_i}\hookrightarrow K$ is injective. Thus $s|_{\mathrm{D}(f_i)}=0$ for every $i\in I$, so $s=0$ by the sheaf conditions.

(iii) Follows immediately from (ii) and the factorisation $\mathscr{O}_X(U)\to\mathscr{O}_X(V)\to\mathscr{O}_{X,\xi}$. □

Example 3.1.53. Let $A=\mathbb{F}[x_1,\ldots,x_n]$, and consider the generic point $\xi\in\mathbb{A}^n_{\mathbb{F}}$, i.e. the point corresponding to the ideal $(0)\subset A$. Then

$$\kappa(\xi)=\mathbb{F}(x_1,\ldots,x_n).$$

If $f\in A\setminus 0$ is an irreducible polynomial, then the generic point $\xi_f\in\operatorname{Spec}A/(f)$ satisfies

$$\kappa(\xi_f)=\operatorname{Frac}A/(f).$$

Example 3.1.54. Let $A=\mathbb{Z}$. Every open subset $U\subset\operatorname{Spec}\mathbb{Z}$ is principal, i.e. of the form $U=\mathrm{D}(f)$ for some $f\in\mathbb{Z}$. We have $\mathscr{O}_{\operatorname{Spec}\mathbb{Z}}(\mathrm{D}(f))=\mathbb{Z}_f\subset\operatorname{Frac}(\mathbb{Z})=\mathbb{Q}$, and a rational number $a/b\in\mathbb{Q}$ (with a,b coprime) belongs to $\mathscr{O}_{\operatorname{Spec}\mathbb{Z}}(\mathrm{D}(f))$ if and only if every prime

p dividing b also divides f. As for the generic point $\xi = (0)$, we have $\kappa(\xi) = \mathscr{O}_{\operatorname{Spec}\mathbb{Z},\xi} = \mathbb{Q}$. If $x \in \operatorname{Spec}\mathbb{Z}$ corresponds to the maximal ideal $(p) \subset \mathbb{Z}$, then

$$\kappa(x) = \mathbb{Z}_{(p)}/(p)\cdot\mathbb{Z}_{(p)} \cong \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$$

by Exercise 3.1.30. Therefore the residue fields at different points of a scheme can have different characteristic!

3.1.7 Morphisms of affine schemes

Let $\phi\colon A \to B$ be a ring homomorphisms. Then we have a set-theoretic map

$$f_\phi\colon \operatorname{Spec} B \to \operatorname{Spec} A, \quad \mathfrak{q} \mapsto \phi^{-1}(\mathfrak{q}).$$

LEMMA 3.1.55. *Let $\phi\colon A \to B$ be a ring homomorphisms. Then*

(a) *f_ϕ is continuous.*

(b) *If ϕ is surjective, then f_ϕ induces a homeomorphism from* $\operatorname{Spec} B$ *onto the closed subset* $\mathrm{V}(\ker\phi) \subset \operatorname{Spec} A$.

(c) *If ϕ is a localisation $A \to S^{-1}A$, then f_ϕ induces a homeomorphism from* $\operatorname{Spec} S^{-1}A$ *onto the subspace $Y_S = \{\,\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \cap S = \emptyset\,\} \subset \operatorname{Spec} A$.*

Proof. We proceed step by step.

(a) We prove that the preimage of a closed subset $\mathrm{V}(I) \subset \operatorname{Spec} A$ is closed. We have

$$\begin{aligned} f_\phi^{-1}(\mathrm{V}(I)) &= \{\,\mathfrak{q} \in \operatorname{Spec} B \mid \phi^{-1}(\mathfrak{q}) \in \mathrm{V}(I)\,\} \\ &= \{\,\mathfrak{q} \in \operatorname{Spec} B \mid I \subset \phi^{-1}(\mathfrak{q})\,\} \\ &= \{\,\mathfrak{q} \in \operatorname{Spec} B \mid \phi(I) \subset \mathfrak{q}\,\} \\ &= \{\,\mathfrak{q} \in \operatorname{Spec} B \mid IB \subset \mathfrak{q}\,\} \\ &= \mathrm{V}(IB). \end{aligned}$$

(b) We have $B = A/\ker\phi$ by assumption, and we know that the prime ideals of B are in bijection with the prime ideals of A containing $\ker\phi$. By (a), and by definition of $\mathrm{V}(-)$, we then know that f_ϕ factors through a continuous bijection $\operatorname{Spec} B \to \mathrm{V}(\ker\phi)$, still denoted f_ϕ. To conclude it is a homeomorphism, it is enough to check the map is closed. Let then $J \subset B$ be an ideal. Then

$$\begin{aligned} f_\phi(\mathrm{V}(J)) &= \{\,\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} = \phi^{-1}(\mathfrak{q}) \text{ for some } \mathfrak{q} \supset J\,\} \\ &= \{\,\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \supset \phi^{-1}(J)\,\} \\ &= \mathrm{V}(\phi^{-1}(J)). \end{aligned}$$

(c) The existence of a continuous bijection $\operatorname{Spec} S^{-1}A \to Y_S \subset \operatorname{Spec} A$ is a combination of (a) with Lemma B.5.6. As before, to see that the map is closed, fix an ideal $J \subset S^{-1}A$. Then

$$\begin{aligned} f_\phi(\mathrm{V}(J)) &= \{ \mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \cap S = \emptyset \text{ and } \mathfrak{p} = \phi^{-1}(\mathfrak{q}) \text{ for some } \mathfrak{q} \supset J \} \\ &= Y_S \cap \mathrm{V}(\phi^{-1}(J)), \end{aligned}$$

which is closed in Y_S, as required. □

Exercise 3.1.56. Let $\mathbb{F}$ be a field. Let $\phi \colon A \to B$ be a homomorphism of finitely generated $\mathbb{F}$-algebras. Show that $f_\phi \colon \operatorname{Spec} B \to \operatorname{Spec} A$ maps closed points to closed points.

Remark 3.1.57. Note that if $S = \{ g^m \mid m \geq 0 \}$ for some $g \in A$, then

$$Y_S = \{ \mathfrak{p} \in \operatorname{Spec} A \mid g \notin \mathfrak{p} \} = \mathrm{D}(g) \subset \operatorname{Spec} A.$$

In particular, in this case the map $f_\ell \colon \operatorname{Spec} A_g \to \operatorname{Spec} A$ corresponding to the localisation $\ell \colon A \to A_g$ is a topological open embedding.

PROPOSITION 3.1.58. *Let $X = \operatorname{Spec} A$, and fix $g \in A$. The localisation $\ell \colon A \to A_g$ induces an isomorphism of locally ringed spaces*

$$(f_\ell, f_\ell^\#) \colon (\operatorname{Spec} A_g, \mathscr{O}_{\operatorname{Spec} A_g}) \xrightarrow{\sim} (\mathrm{D}(g), \mathscr{O}_X|_{\mathrm{D}(g)}).$$

In particular, $(\mathrm{D}(g), \mathscr{O}_X|_{\mathrm{D}(g)})$ *is an affine scheme.*

Proof. A topological open embedding $f_\ell \colon \operatorname{Spec} A_g \to \operatorname{Spec} A$ with image $\mathrm{D}(g)$ is provided by Lemma 3.1.55 (c), applied to the localisation $\ell \colon A \to A_g$. Let us denote by f the homeomorphism $\operatorname{Spec} A_g \to \mathrm{D}(g)$. We need to extend it to a morphism of locally ringed spaces and show the resulting map is an isomorphism.

To define a morphism of sheaves of rings

$$f^\# \colon \mathscr{O}_X|_{\mathrm{D}(g)} \to f_* \mathscr{O}_{\operatorname{Spec} A_g}$$

it is enough to define it on a base of open subsets by Proposition 2.7.9. Let $\mathrm{D}(h) \subset \mathrm{D}(g)$ be a principal open, for $h \in A$. Let $\overline{h} = h/1 \in A_g$ be the image of h in A_g. Then, canonically,

$$\begin{aligned} \mathscr{O}_X|_{\mathrm{D}(g)}(\mathrm{D}(h)) = \mathscr{O}_X(\mathrm{D}(h)) &= A_h \\ &\xrightarrow{\sim} (A_g)_{\overline{h}} \\ &= \mathscr{O}_{\operatorname{Spec} A_g}(\mathrm{D}(\overline{h})) \\ &= \mathscr{O}_{\operatorname{Spec} A_g}(f^{-1}\mathrm{D}(h)) \\ &= f_* \mathscr{O}_{\operatorname{Spec} A_g}(\mathrm{D}(h)). \end{aligned}$$

The isomorphism $A_h \xrightarrow{\sim} (A_g)_{\overline{h}}$ follows by directly checking that $(A_g)_{\overline{h}}$ satisfies the universal property of A_h. The above isomorphisms $\mathscr{O}_X|_{D(g)}(D(h)) \xrightarrow{\sim} f_* \mathscr{O}_{\operatorname{Spec} A_g}(D(h))$ are compatible with restrictions to smaller principal opens, thus they determine an isomorphism of $\mathcal{B}$-sheaves, which in turn uniquely determines an isomorphism of sheaves by Proposition 2.7.9. □

Remark 3.1.59. Let $(X, \mathscr{O}_X)$ be a scheme, $U \subset X$ an open subset. We know that $(U, \mathscr{O}_U)$ is a locally ringed space. (Recall that we set $\mathscr{O}_U = \mathscr{O}_X|_U$ in such a situation). We claim that it is in fact a scheme. That is, every open subset of a scheme has a natural scheme structure. To see this, cover X with open affine subsets $U_i = \operatorname{Spec} A_i \subset X$, so that $U = \bigcup_i U \cap U_i$. But $U \cap U_i \subset U_i$ is open in an affine scheme, therefore it is a union of principal open subsets $D(f_{ij}) \subset U_i$, for $f_{ij} \in A_i$. Thus U admits a covering $U = \bigcup_{i,j} D(f_{ij})$, and each $D(f_{ij})$ is an affine scheme by Proposition 3.1.58.

PROPOSITION 3.1.60. *Let $\phi: A \to B$ be a ring homomorphisms. Then the continuous map $f_\phi: \operatorname{Spec} B \to \operatorname{Spec} A$ of Lemma 3.1.55 extends to a morphism of affine schemes $(f_\phi, f_\phi^\#)$ such that $f_\phi^\#(\operatorname{Spec} A) = \phi$.*

Proof. Set $f = f_\phi$. First of all, we have to construct the sheaf homomorphism

$$f^\#: \mathscr{O}_{\operatorname{Spec} A} \to f_* \mathscr{O}_{\operatorname{Spec} B}.$$

For $g \in A$, the preimage of the principal open $D(g) \subset \operatorname{Spec} A$ is

$$\begin{aligned} f^{-1}(D(g)) &= \{\mathfrak{q} \in \operatorname{Spec} B \mid \phi^{-1}(\mathfrak{q}) \in D(g)\} \\ &= \{\mathfrak{q} \in \operatorname{Spec} B \mid g \notin \phi^{-1}(\mathfrak{q})\} \\ &= \{\mathfrak{q} \in \operatorname{Spec} B \mid \phi(g) \notin \mathfrak{q}\} \\ &= D(\phi(g)). \end{aligned}$$

There is an induced commutative diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d] & B \arrow[d] & \\ A_g \arrow[r, "\phi_g"] \arrow[d, equal] & B_{\phi(g)} \arrow[d, equal] & \frac{a}{g^r} \mapsto \frac{\phi(a)}{\phi(g)^r} \\ \mathscr{O}_{\operatorname{Spec} A}(D(g)) & \mathscr{O}_{\operatorname{Spec} B}(D(\phi(g))) \arrow[r, equal] & f_* \mathscr{O}_{\operatorname{Spec} B}(D(g)) \end{tikzcd}$$

allowing us to set $f^\#(D(g)) = \phi_g$. These morphisms are compatible with restrictions to smaller principal opens, thus they give rise to a morphism of $\mathcal{B}$-sheaves on $\operatorname{Spec} A$, which in turn uniquely determines a morphism of sheaves $f^\#$ by Proposition 2.7.9. If we take global sections of $f^\#$ (i.e. we evaluate it on $D(1) = \operatorname{Spec} A$), we get back our original map ϕ, by construction.

We are left with checking that $(f, f^\#)$ induces local homomorphisms of local rings at the level of stalks. Assume $y = f(x)$, where $x \in \operatorname{Spec} B$ corresponds to a prime ideal $\mathfrak{q} \subset B$ and $y \in \operatorname{Spec} A$ corresponds to $\mathfrak{p} = \phi^{-1}(\mathfrak{q}) \subset A$. Then the canonical map

$$A_{\mathfrak{p}} \longrightarrow B_{\mathfrak{q}}, \quad \frac{a}{s} \mapsto \frac{\phi(a)}{\phi(s)}$$

is a local homomorphism of local rings, which coincides with

$$f_x^\#\colon \mathscr{O}_{\operatorname{Spec} A, y} \to (f_* \mathscr{O}_{\operatorname{Spec} B})_y \to \mathscr{O}_{\operatorname{Spec} B, x}. \qquad \square$$

It is very easy to check that sending $A \mapsto \operatorname{Spec} A$ is a contravariant *functor* (cf. Definition A.1.6) from rings to affine schemes. Proposition 3.1.60 says 'what to do' on morphisms. The axioms defining a functor (identity goes to identity, and compositions are preserved) are elementary, and therefore left to the reader.

We can finally prove the main result of this chapter.

THEOREM 3.1.61. *The functor* Spec*, from rings to affine schemes, induces an equivalence*

$$\operatorname{Spec}\colon \mathsf{Rings}^{\mathrm{op}} \xrightarrow{\sim} \mathsf{Aff}, \quad A \mapsto \operatorname{Spec} A,$$

with inverse functor given by $X \mapsto \mathscr{O}_X(X)$. In particular, $\operatorname{Spec} \mathbb{Z}$ *is a final object in* Aff*.*

Proof. The Spec functor is essentially surjective, by definition of affine scheme. We need to show it is fully faithful (cf. Remark A.1.15). Set $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. We claim that the inverse of the mapping

$$\operatorname{Hom}_{\mathsf{Rings}^{\mathrm{op}}}(B, A) \to \operatorname{Hom}_{\mathsf{Aff}}(X, Y), \quad \phi \mapsto f_\phi.$$

is the map

$$\rho_{X,Y}\colon \operatorname{Hom}_{\mathsf{Aff}}(X, Y) \to \operatorname{Hom}_{\mathsf{Rings}}(A, B) \tag{3.1.13}$$

sending $f\colon X \to Y$ to $f^\#(Y)\colon A = \mathscr{O}_Y(Y) \to f_* \mathscr{O}_X(Y) = \mathscr{O}_X(X) = B$. We must show that (3.1.13) is bijective for any pair of affine schemes $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. Note that we already know that $\rho_{X,Y}$ is surjective thanks to Proposition 3.1.60.

Fix $f \in \operatorname{Hom}_{\mathsf{Aff}}(X, Y)$. Set $\phi = \rho_{X,Y}(f) = f^\#(Y)$. We know by Proposition 3.1.60 that ϕ gives rise to a morphism of affine schemes $f_\phi\colon X \to Y$ such that $\rho_{X,Y}(f_\phi) = \phi = \rho_{X,Y}(f)$. It is thus enough to show that $f = f_\phi$.

We need to show that f and f_ϕ are the same map set-theoretically, and, once we know this, that $f_x^\# = (f_\phi)_x^\#$ are the same as local homomorphisms of local rings, for every $x \in X$. This will imply that $f^\# = f_\phi^\#$ by Exercise 2.4.13. Let's start. Let $\mathfrak{q} \subset B$ be the prime ideal corresponding to $x \in X = \operatorname{Spec} B$, and let $\mathfrak{p} \subset A$ be the prime ideal corresponding to $f(x) \in Y = \operatorname{Spec} A$. We have two commutative diagrams

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\mathrm{loc}_{\mathfrak{p}}"'] & B \arrow[d, "\mathrm{loc}_{\mathfrak{q}}"] \\ A_{\mathfrak{p}} \arrow[r, "f_x^\#"] & B_{\mathfrak{q}} \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\mathrm{loc}_{\phi^{-1}(\mathfrak{q})}"'] & B \arrow[d, "\mathrm{loc}_{\mathfrak{q}}"] \\ A_{\phi^{-1}(\mathfrak{q})} \arrow[r, "(f_\phi)_x^\#"] & B_{\mathfrak{q}} \end{tikzcd}$$

that we need to show are the same. The very existence of $f_x^\#$ (fitting in the left diagram) implies that whenever $s \notin \mathfrak{p}$ one must have $\phi(s) \notin \mathfrak{q}$, i.e.

$$\phi(A \setminus \mathfrak{p}) \subset B \setminus \mathfrak{q},$$

which implies $\phi^{-1}(\mathfrak{q}) \subset \mathfrak{p}$. But the local condition $(f_x^\#)^{-1}(\mathfrak{q}B_{\mathfrak{q}}) = \mathfrak{p}A_{\mathfrak{p}}$ (combined with the classical correspondence between prime ideals in a ring and in a localisation of it, cf. Lemma B.5.6) implies $\phi^{-1}(\mathfrak{q}) = \phi^{-1}\mathrm{loc}_{\mathfrak{q}}^{-1}(\mathfrak{q}B_{\mathfrak{q}}) = \mathrm{loc}_{\mathfrak{p}}^{-1}(\mathfrak{p}A_{\mathfrak{p}}) = \mathfrak{p}$.

Thus $f = f_\phi$ set-theoretically. However, there is only one possible commutative diagram as above: the one where the bottom map sends $a/s \mapsto \phi(a)/\phi(s)$. Thus $f_x^\# = (f_\phi)_x^\#$ as wanted. This concludes the proof that $\rho_{X,Y}$ is bijective.

The final statement now follows, since a ring A is a $\mathbb{Z}$-algebra $\mathbb{Z} \to A$ in a unique way (or, equivalently, $\mathbb{Z}$ is an initial object in Rings). □

Remark 3.1.62. The map (3.1.13) is functorial in the following sense: for any morphism of affine schemes $g\colon Z = \operatorname{Spec} C \to \operatorname{Spec} B = X$ the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{Aff}}(X,Y) \arrow[r, "\rho_{X,Y}"] \arrow[d, "f \mapsto f\circ g"] & \mathrm{Hom}_{\mathsf{Rings}}(A,B) \arrow[d, "\phi \mapsto g^\#(X)\circ\phi"] \\ \mathrm{Hom}_{\mathsf{Aff}}(Z,Y) \arrow[r, "\rho_{Z,Y}"] & \mathrm{Hom}_{\mathsf{Rings}}(A,C) \end{tikzcd}$$

commutes. This is just a rephrasing of the fact that morphisms of locally ringed spaces can be composed! In a little more detail, fix $f \in \mathrm{Hom}_{\mathsf{Aff}}(X,Y)$. The upper journey takes f to the map $g^\#(X) \circ f^\#(Y) \in \mathrm{Hom}_{\mathsf{Rings}}(A,C)$, whereas the lower journey takes f to $(f \circ g)^\#(Y)$. These maps are clearly the same, since $(f \circ g)^\#\colon \mathscr{O}_Y \to f_* g_* \mathscr{O}_Z$ is nothing but the composition $f_* g^\# \circ f^\#\colon \mathscr{O}_Y \to f_* \mathscr{O}_X \to f_* g_* \mathscr{O}_Z$.

3.1.8 Examples of affine schemes and their morphisms

In this section we collect some examples of affine schemes (and morphisms between them), besides those already considered in Section 3.1.3 at a purely topological level.

Recall that open (resp. closed) immersion of schemes are just open (resp. closed) immersions in the category of locally ringed spaces (cf. Definition 2.10.8). The next two examples are very important.

Key Example 3.1.63 (Principal open immersions)**.** Let A be a ring, $f \in A$. It follows from Proposition 3.1.58 and Definition 2.10.8 that the canonical morphism

$$\operatorname{Spec} A_f \longrightarrow \operatorname{Spec} A$$

is an open immersion of affine schemes.

Key Example 3.1.64 (Closed immersions)**.** Let A be a ring, $I \subset A$ an ideal. Set $B = A/I$. The canonical surjection $\phi\colon A \twoheadrightarrow B$ determines, and is determined by, a morphism of affine schemes

$$i\colon \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$$

This morphism is a *closed immersion* according to Definition 2.10.8. Indeed, it is a homeomorphism onto $\mathrm{V}(I) \subset \operatorname{Spec} A$, and induces a surjective map of sheaves, because $i^\#(\mathrm{D}(g))$ is surjective for every $g \in A$ (it agrees with the canonical surjection $A_g \twoheadrightarrow B_{\phi(g)}$), and by Proposition 2.7.9 it is enough to check surjectivity on a base of open sets.

In what follows, we grant the following proposition, saying that *all* closed immersions into an affine scheme are as in Key Example 3.1.64.

PROPOSITION 3.1.65 ([12, Ch. 2, Prop. 3.20]). *Let $Y = \operatorname{Spec} A$ be an affine scheme, and let $\iota\colon Z \hookrightarrow Y$ be a closed immersion. Then Z is affine, and there is a unique ideal $I \subset A$ such that ι induces an isomorphism of schemes $Z \xrightarrow{\sim} \operatorname{Spec} A/I$.*

Example 3.1.66 (Many maps to the point!)**.** One is used to think that there is only one map $\bullet \to \bullet$. However, this is not necessarily true in algebraic geometry: think of the identity $\mathbb{C} \to \mathbb{C}$, which is different from complex conjugation $\mathbb{C} \to \mathbb{C}$. By Theorem 3.1.61, they give rise to different maps $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{C}$. In fact, there are *infinitely many* morphisms $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{C}$. The set $\operatorname{Hom}_{\mathsf{Rings}}(\mathbb{C},\mathbb{C})$ contains the Galois group of $\mathbb{C} \supset \mathbb{Q}$![4] Another example, in characteristic $p > 0$, is the *Frobenius morphism*, namely the map $\Phi_{\mathbb{F}}\colon \operatorname{Spec}\mathbb{F} \to \operatorname{Spec}\mathbb{F}$ induced by the field homomorphism $\phi_{\mathbb{F}}\colon \mathbb{F} \to \mathbb{F}$ sending $x \mapsto x^p$. A field $\mathbb{F}$ is *perfect* if and only if either $\mathbb{F}$ has characteristic 0 or $\phi_{\mathbb{F}}$ is surjective. In particular, the Frobenius morphism $\Phi_{\mathbb{F}}$ is an isomorphism if and only if $\mathbb{F}$ is perfect. For instance, all finite fields are perfect, but $\mathbb{F}_p(t)$ is not perfect, since t is (for degree reasons) not of the form $f(t)^p/g(t)^p$ for any two polynomials $f(t)$ and $g(t)$. In any case, $\phi_{\mathbb{F}} \neq \mathrm{id}_{\mathbb{F}}$, thus $\Phi_{\mathbb{F}} \neq \mathrm{id}_{\operatorname{Spec}\mathbb{F}}$.

Remark 3.1.67. Note that there is, on the other hand, *only one* morphism of schemes $\operatorname{Spec}\mathbb{R} \to \operatorname{Spec}\mathbb{R}$. This is because there is only one ring endomorphism $\mathbb{R} \to \mathbb{R}$, namely the identity. Can you prove it? (**Hint**: start by showing that a ring endomorphism $\mathbb{R} \to \mathbb{R}$ is increasing). Also, note that there is *no morphism* $\operatorname{Spec}\mathbb{R} \to \operatorname{Spec}\mathbb{C}$, since there is no ring homomorphism $\mathbb{C} \to \mathbb{R}$.

[4] The cardinality of the automorphism group of $\mathbb{C}$ is 2^c, where $c = 2^{\aleph_0}$. More generally, the cardinality of $\mathrm{Aut}(\mathbf{k})$ for $\mathbf{k}$ an *algebraically closed* field, is $2^{\operatorname{card}\mathbf{k}}$, see [2].

Example 3.1.68 (Dual numbers)**.** Let A be a ring. Thanks to Theorem 3.1.61, we finally have made rigorous our claim (formulated with $A = \mathbf{k}$)

$$\operatorname{Spec} A \neq \operatorname{Spec} A[t]/t^2$$

from Example 3.1.26! See Example 3.1.69 and Example 3.1.70 for generalisations.

Example 3.1.69 (Curvilinear schemes)**.** Let $n > 0$ be an integer, and set $A_n = \mathbf{k}[t]/t^n$. There is a closed immersion

$$\operatorname{Spec} A_n \hookrightarrow \mathbb{A}^1_{\mathbf{k}}.$$

Note that $\dim_{\mathbf{k}} A_n = n$. The affine scheme $\operatorname{Spec} A_n$ is called a *curvilinear scheme of length* n. All these schemes admit a (bijective) closed immersion $\operatorname{Spec} \mathbf{k} \hookrightarrow \operatorname{Spec} A_n$, which is never an isomorphism (unless $n = 1$). Thus $\operatorname{Spec} A_n$ can be seen as a 'thickening' of the origin in $\mathbb{A}^1_{\mathbf{k}}$.

Example 3.1.70 (Fat points)**.** Let $\mathbf{k}$ be an algebraically closed field, $(A, \mathfrak{m})$ a local artinian $\mathbf{k}$-algebra with residue field $A/\mathfrak{m} = \mathbf{k}$. Then $\operatorname{Spec} A$ is topologically just one (closed) point, corresponding to the maximal ideal $\mathfrak{m} \subset A$. For instance, consider $A = \mathbf{k}[x, y]/(x^2, xy, y^2)$. These affine schemes are called *fat points*. Each fat point has a *length*, namely the number $\dim_{\mathbf{k}} A = \dim_{\mathbf{k}} \mathscr{O}_X(X)$. For instance, $\operatorname{Spec} \mathbf{k}[x, y]/(x^2, xy, y^2)$ has length 3, and is *not curvilinear*. The closed immersion $\operatorname{Spec} A/\mathfrak{m} \hookrightarrow \operatorname{Spec} A$ is a bijection (both schemes have precisely one point), but $\operatorname{Spec} A/\mathfrak{m} = \operatorname{Spec} \mathbf{k} \neq \operatorname{Spec} A$ as schemes. Fat points encode nontrivial information in their structure sheaves!

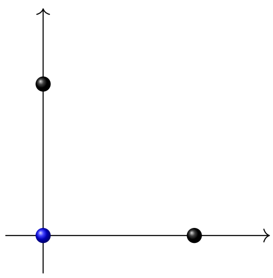

Figure 3.7: The length 3 fat point $\operatorname{Spec} \mathbf{k}[x, y]/(x^2, xy, y^2)$ arises as the 'collision' of two points (black bullets) running towards the origin. It can be seen as a degeneration of the product ideal $(x - a, y) \cdot (x, y - b)$ for $a, b \to 0$.

Exercise 3.1.71. Show that the only fat point of length 3 which is not curvilinear is, up to isomorphism, precisely $\operatorname{Spec} \mathbf{k}[x, y]/(x^2, xy, y^2)$.

Example 3.1.72 (A non-affine scheme)**.** Let $X = \operatorname{Spec} \mathbb{Z}[x]$, and $z \in X$ the closed point corresponding to (p, x), where $p \in \mathbb{Z}$ is a prime number. Then $U = X \setminus \{z\} = \mathrm{D}(p) \cup \mathrm{D}(x)$

is not affine. Indeed, by Lemma 3.1.52(iii), we have $\mathscr{O}_X(U) \subset \mathscr{O}_X(\mathrm{D}(p)) \cap \mathscr{O}_X(\mathrm{D}(x)) = \mathbb{Z}[x,1/p] \cap \mathbb{Z}[x,1/x] = \mathbb{Z}[x] = \mathscr{O}_X(X)$, which readily implies $\mathscr{O}_X(U) = \mathscr{O}_X(X)$. Note that this example also shows that the union of two affine schemes need not be affine.

Example 3.1.73 (A non-affine scheme)**.** Let $n > 1$ be an integer, $\mathbf{k}$ a field, $A = \mathbf{k}[x_1,\dots,x_n]$. Consider the origin of $X = \mathbb{A}^n_{\mathbf{k}} = \operatorname{Spec} A$, namely the point $0 \in X$ corresponding to the maximal ideal $(x_1,\dots,x_n) \subset A$. Form the open complement $U = X \setminus \{0\} \hookrightarrow X$. We now prove that the restriction map

$$\mathbf{k}[x_1,\dots,x_n] = \mathscr{O}_X(X) \to \mathscr{O}_X(U)$$

is the identity. This proves that U is not affine, since U is not isomorphic to $\mathbb{A}^n_{\mathbf{k}}$. As before, we have $U = \bigcup_{1\le i\le n} \mathrm{D}(x_i)$, so by Lemma 3.1.52(iii), we have

$$\mathscr{O}_X(U) \subset \bigcap_{1\le i\le n} \mathrm{D}(x_i) = A_{x_1} \cap \cdots \cap A_{x_n}.$$

This can be proven directly to be equal to A. However, it also follows from the algebraic version of Hartog's Lemma below, combined with the fact that height 1 primes (see Definition 4.6.15 for the definition of height of an ideal) in the (normal) domain A correspond to irreducible polynomials. The $n = 2$ case of this example can be seen as the geometric analogue of Example 3.1.72.

LEMMA 3.1.74 ([12, Ch. 4, Lemma 1.13]). *Let A be a normal noetherian ring of dimension at least 1. Then*

$$A = \bigcap_{\substack{\mathfrak{p}\in \operatorname{Spec} A \\ \operatorname{ht}(\mathfrak{p})=1}} A_{\mathfrak{p}},$$

the intersection being taken inside $\operatorname{Frac} A$*.*

Example 3.1.75 (Affine line minus one point)**.** If we take $n = 1$ in Example 3.1.73, we do in fact get an affine scheme $U = \mathbb{A}^1_{\mathbf{k}} \setminus \{0\} \hookrightarrow \mathbb{A}^1_{\mathbf{k}}$. Indeed, $U = \mathrm{D}(x) = \operatorname{Spec} \mathbf{k}[x]_x$. In fact, the ring $\mathbf{k}[x]_x = \{ f(x)/x^r \mid r \ge 0\}$ is isomorphic to the $\mathbf{k}$-algebra $\mathbf{k}[x,x^{-1}] = \mathbf{k}[x,y]/(xy-1)$, which yields a closed immersion $U \hookrightarrow \mathbb{A}^2_{\mathbf{k}}$.

Example 3.1.76 (Affine hypersurfaces)**.** Let $\mathbf{k}$ be a field, $f \in \mathbf{k}[x_1,\dots,x_n]$. Then

$$Y_f = \operatorname{Spec} \mathbf{k}[x_1,\dots,x_n]/(f)$$

is called an *affine hypersurface* in $\mathbb{A}^n_{\mathbf{k}}$. The surjection $\mathbf{k}[x_1,\dots,x_n] \twoheadrightarrow \mathbf{k}[x_1,\dots,x_n]/(f)$ canonically determines a closed immersion

$$Y_f \hookrightarrow \mathbb{A}^n_{\mathbf{k}} = \operatorname{Spec} \mathbf{k}[x_1,\dots,x_n].$$

Suppose (f) is a prime ideal in $\mathbf{k}[x_1,\dots,x_n]$, so that $\mathbf{k}[x_1,\dots,x_n]/(f)$ is an integral domain. Then (f) corresponds to the trivial (prime) ideal $(0) \subset \mathbf{k}[x_1,\dots,x_n]/(f)$. This is the generic point of Y_f.

Example 3.1.77. As a special case of Example 3.1.76, consider $f = xy - z^2 \in \mathbb{C}[x,y,z]$. Its vanishing scheme

$$Y_f = \operatorname{Spec}\mathbb{C}[x,y,z]/(xy-z^2) \hookrightarrow \mathbb{A}^3_{\mathbb{C}}$$

is called the *affine quadric cone*.

Example 3.1.78. Let R be a DVR with fraction field K, and set $X = \operatorname{Spec} R = \{x_0, \xi\}$ where x_0 is the closed point. Then $K = \mathscr{O}_{X,\xi}$. The open immersion $\{\xi\} = X \setminus \{x_0\} \hookrightarrow X$ corresponds to the canonical inclusion $R \hookrightarrow K$.

Example 3.1.79. Let $\mu_n = \operatorname{Spec}\mathbf{k}[x]/(x^n - 1)$ for some $n > 1$. This is the scheme-theoretic version of the group of n-th roots of unity. One can prove that it is a group object in the category of $\mathbf{k}$-schemes. Such objects are called *algebraic groups*. As for μ_n, it comes with a natural closed immersion inside the affine line $\mathbb{A}^1_{\mathbf{k}} = \operatorname{Spec}\mathbf{k}[x]$.

Example 3.1.80. Consider the morphism $f\colon \mathbb{A}^1_{\mathbf{k}} \to \mathbb{A}^1_{\mathbf{k}}$ defined by the ring homomorphism $\mathbf{k}[t] \to \mathbf{k}[t]$ sending $t \mapsto t^n$. This is the typical example of what we will call a *ramified* morphism. The intuition is the following: every point $x \in \mathbb{A}^1_{\mathbf{k}} \setminus 0$ in the target has precisely n preimages (because $\mathbf{k}$ is algebraically closed), but there is only one preimage over the origin $0 \in \mathbb{A}^1_{\mathbf{k}}$. Over this point, the morphism is 'fully ramified'. If we restrict f to $\mathbb{A}^1_{\mathbf{k}} \setminus 0 \to \mathbb{A}^1_{\mathbf{k}} \setminus 0$, it becomes *unramified*, and in fact *étale*. These notions are extremely important and will be treated in later chapters.

Example 3.1.81. The inclusion $\mathbb{R}[x] \hookrightarrow \mathbb{C}[x]$ induces a morphism of affine schemes

$$\mathbb{A}^1_{\mathbb{C}} \longrightarrow \mathbb{A}^1_{\mathbb{R}},$$

sending the generic point $(0) \subset \mathbb{C}[x]$ to the generic point $(0) \subset \mathbb{R}[x]$. For any $c \in \mathbb{R} \subset \mathbb{C}$, the maximal ideal $(x-c) \subset \mathbb{R}[x]$ is the preimage of the maximal ideal $(x-c) \subset \mathbb{C}[x]$, so $\mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{R}}$ sends the closed point $(x-c) \in \mathbb{A}^1_{\mathbb{C}}$ to the closed point $(x-c) \in \mathbb{A}^1_{\mathbb{R}}$. On the other hand, if $c = \alpha + i\beta \in \mathbb{C} \setminus \mathbb{R}$, then both ideals

$$\mathfrak{p}_1 = (x-c),\ \mathfrak{p}_2 = (x-\overline{c}) \subset \mathbb{C}[x]$$

viewed as closed points of $\mathbb{A}^1_{\mathbb{C}}$, map to the closed point

$$\mathfrak{q} = (f) \in \mathbb{A}^1_{\mathbb{R}}, \quad f = (x-c)(x-\overline{c}).$$

However, note that this closed point has 'degree 2', for

$$\kappa(\mathfrak{q}) = \frac{\mathbb{R}[x]_{(f)}}{(f)\mathbb{R}[x]_{(f)}} \cong \frac{\mathbb{R}[x]}{(f)} \cong \mathbb{C},$$

since $\deg f = 2$. This does not happen for the other points $(x-c) \in \mathbb{A}^1_{\mathbb{R}}$, in the sense that

$$\kappa(x-c) = \frac{\mathbb{R}[x]_{(x-c)}}{(x-c)\mathbb{R}[x]_{(x-c)}} \cong \frac{\mathbb{R}[x]}{(x-c)} \cong \mathbb{R}.$$

As for $\xi = (0) \in \operatorname{Spec}\mathbb{R}[x]$, we have

$$\kappa(\xi) = \frac{\mathbb{R}[x]_{(0)}}{(0)} = \mathbb{R}[x]_{(0)} = \operatorname{Frac}\mathbb{R}[x] = \mathbb{R}(x).$$

We have used Exercise 3.1.30 in the three last displayed equations. The elements of $\kappa(\xi)$ are 'rational functions' g/h, not defined everywhere but *almost everywhere*, away from the (finitely many) zeros of $h \in \mathbb{R}[x]$.

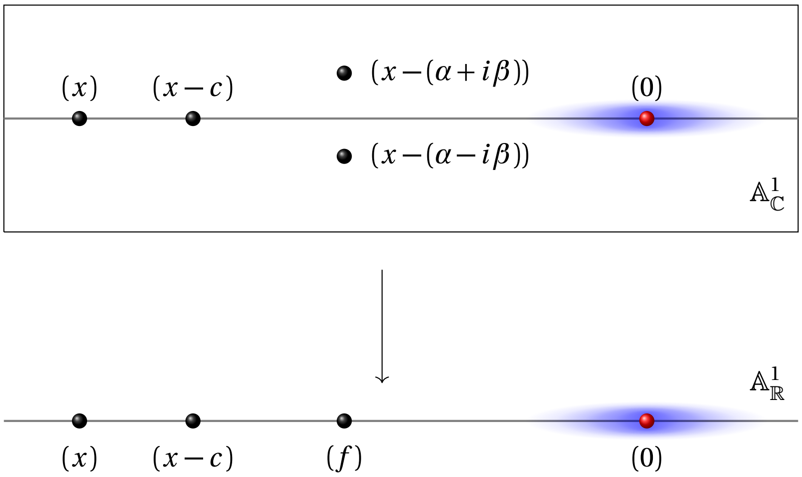

Figure 3.8: The morphism $\mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{R}}$ induced by $\mathbb{R}[x] \hookrightarrow \mathbb{C}[x]$.

Example 3.1.82. This example is the arithmetic analogue of Example 3.1.81. Consider the inclusion of rings

$$\phi : \mathbb{Z} \hookrightarrow \mathbb{Z}[i] = \mathbb{Z}[x]/(x^2+1), \quad i^2 = -1.$$

Here $\mathbb{Z}[i]$ is the ring of *Gaussian integers*, which is an euclidean domain, in particular a principal ideal domain. We will recall some basic number theory in this example, in order to study the induced morphism

$$f : \operatorname{Spec}\mathbb{Z}[i] \longrightarrow \operatorname{Spec}\mathbb{Z}.$$

The algebraic question is: what happens to a prime number $p \in \mathbb{Z}$ when one adds in the imaginary unit? More precisely, is the extension

$$(p) = p\mathbb{Z}[i] \subset \mathbb{Z}[i]$$

still a a prime ideal? If this happens we say that p is *inert*, otherwise that p *ramifies*. For sure $(p) \subset \mathbb{Z}[i]$ is still a principal ideal. By Fermat's theorem on sums of two squares, one has that $p > 2$ is a sum of squares if and only if $p \equiv 1 \bmod 4$. In this case, one can write $p = a^2 + b^2 = (a+ib)(a-ib)$ for some integers $a, b \in \mathbb{Z}$. Such primes then do ramify. On the other hand, if $p \equiv 3 \bmod 4$, then (p) *stays prime* in $\mathbb{Z}[i]$. Let us start with the smallest prime number: one has that $2=(1+i)(1-i)$, but $(1+i)=(1-i)$ as ideals in $\mathbb{Z}[i]$, since $i(1-i)=i+1$, thus $p=2$ ramifies. The next prime that ramifies is $5=(2+i)(2-i)$, followed by $13=(6+i)(6-i)$ (since 7 and 11 are inert). Primes (larger than 2) that ramify correspond, geometrically, to those points $(p)\in\operatorname{Spec}\mathbb{Z}$ having more than one preimage along f.

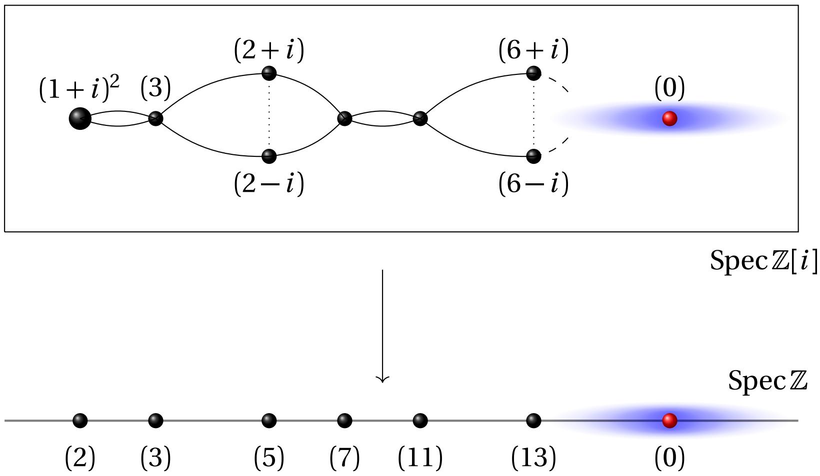

Figure 3.9: The morphism $f\colon \operatorname{Spec}\mathbb{Z}[i]\to\operatorname{Spec}\mathbb{Z}$ induced by the inclusion $\mathbb{Z}\hookrightarrow\mathbb{Z}[i]$.

Example 3.1.83 (The 'arithmetic surface')**.** Here is another arithmetic example. Consider the inclusion of rings $\mathbb{Z}\hookrightarrow\mathbb{Z}[x]$. We want to study the induced morphism

$$\operatorname{Spec}\mathbb{Z}[x] \longrightarrow \operatorname{Spec}\mathbb{Z}.$$

A pictorial description of $\operatorname{Spec}\mathbb{Z}[x]$ was given by Mumford [14], see Figure 3.10.

First let us list all prime ideals in $\mathbb{Z}[x]$.

- $(0)\subset\mathbb{Z}[x]$ is a prime ideal, since $\mathbb{Z}[x]$ is an integral domain. It corresponds to the generic point of $\operatorname{Spec}\mathbb{Z}[x]$. The residue field if $\kappa(0)=\operatorname{Frac}\mathbb{Z}[x]=\mathbb{Q}(x)$.

- $(p)\subset\mathbb{Z}[x]$ is a prime ideal, for any prime number $p\in\mathbb{Z}$, since the quotient

$$\mathbb{Z}[x]/(p)\cong\mathbb{F}_p[x]$$

  is an integral domain. These points are *not closed.* Note that each point (p) is precisely the generic point of the affine line

$$\mathbb{A}^1_{\mathbb{F}_p}=\operatorname{Spec}\mathbb{F}_p[x]=\operatorname{Spec}\mathbb{Z}[x]/(p)\hookrightarrow\operatorname{Spec}\mathbb{Z}[x].$$

  These lines are drawn as vertical lines in Figure 3.10, where they are denoted $\mathrm{V}(p)$. The residue field of $\operatorname{Spec}\mathbb{Z}[x]$ at these points is

$$\kappa(p)=\mathbb{Z}[x]_{(p)}/(p)\mathbb{Z}[x]_{(p)}=\operatorname{Frac}\mathbb{Z}[x]/(p)=\operatorname{Frac}\mathbb{F}_p[x]=\mathbb{F}_p(x).$$

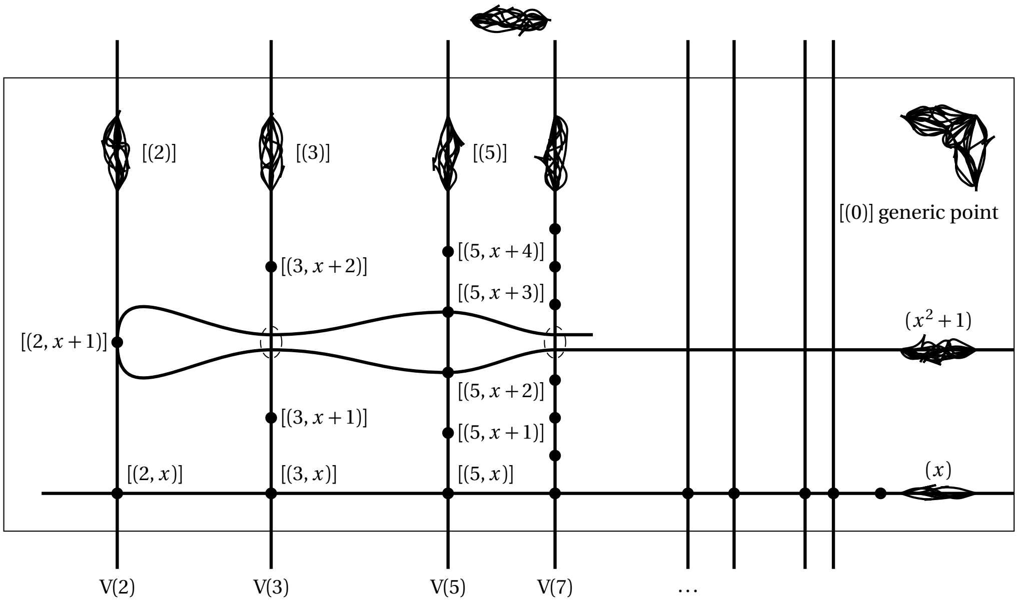

Figure 3.10: Picture's code is stolen from Pieter Belmans' website. This picture was originally drawn by David Mumford in [14], where he called $\operatorname{Spec}\mathbb{Z}[x]$ an *arithmetic surface*.

- $(f) \subset \mathbb{Z}[x]$, where $f \in \mathbb{Z}[x]$ is an irreducible polynomial (over $\mathbb{Z}$, hence over $\mathbb{Q}$ by Gauss' Lemma, hence one may even assume the gcd of its coefficients is equal to 1 after clearing denominators). Each such polynomial draws an "arithmetic curve"

$$\operatorname{Spec}\mathbb{Z}[x]/(f) \hookrightarrow \operatorname{Spec}\mathbb{Z}[x]$$

  depicted as a horizontal curve in Figure 3.10, where each such curve is denoted $\mathrm{V}(f)$. Clearly the point (f) is exactly the generic point of such arithmetic curve.

- $(p, f) \subset \mathbb{Z}[x]$, where p is a prime number and $f \in \mathbb{Z}[x]$ is an irreducible monic polynomial which stays irreducible over $\mathbb{F}_p$. These are all the closed points of $\operatorname{Spec}\mathbb{Z}[x]$. The residue fields of these points are finite extensions of $\mathbb{F}_p$.

Explicitly, one has

$$\begin{aligned}
\mathrm{V}(p) &= \left\{ (p), (p, f) \,\middle|\, f \text{ monic, irreducible over } \mathbb{Z} \text{ and } \mathbb{F}_p \right\} \\
\mathrm{V}(f) &= \left\{ (f), (p, g) \,\middle|\, g \text{ divides } f \text{ modulo } p \right\},
\end{aligned}$$

and the intersection between a horizontal curve and a vertical line is

$$\mathrm{V}(f) \cap \mathrm{V}(p) = \left\{ (p, g) \,\middle|\, g \text{ divides } f \text{ modulo } p \right\}.$$

One such arithmetic curve $\mathrm{V}(f)$ is for instance the one "cut out by $x = 0$", consisting of a copy of $\operatorname{Spec}\mathbb{Z}$ itself, for

$$\mathrm{V}(x) = \operatorname{Spec}\mathbb{Z}[x]/(x) = \operatorname{Spec}\mathbb{Z} \hookrightarrow \operatorname{Spec}\mathbb{Z}[x].$$

Another curve is

$$\mathrm{V}(x^2+1) = \operatorname{Spec}\mathbb{Z}[x]/(x^2+1) = \operatorname{Spec}\mathbb{Z}[i] \hookrightarrow \operatorname{Spec}\mathbb{Z}[x].$$

Let us now analyse Figure 3.10 carefully.

- V(2). Two "classical points" of this vertical line are the closed points $(2, x)$ and $(2, x+1)$, corresponding to the points with coordinates 0 and 1, respectively, in the affine line $\mathbb{A}^1_{\mathbb{F}_2} \subset \operatorname{Spec}\mathbb{Z}[x]$. These two points are drawn as black bullets.

  But this affine line also intersects the arithmetic curve $\mathrm{V}(x^2+1)$, since

  $$\mathrm{V}(x^2+1) \cap \mathrm{V}(2) = \{(2, x+1)\}.$$

  However, the point $(2, x+1)$ has 'multiplicity 2' since over $\mathbb{F}_2$ we have a splitting $x^2+1 = (x+1)(x+1)$. This is why the curve $\mathrm{V}(x^2+1)$ is depicted tangent to the affine line V(2).

  Of course there are many other curves $\mathrm{V}(f)$ meeting V(2). In other words, V(2) has many other points of the form $(2, f)$. They correspond to irreducible monic polynomials f which stay irreducible over $\mathbb{F}_2$. For instance,

  $$f = x^2 + x + 1$$

  has no roots over $\mathbb{F}_2$, and if we denote by α a root of f we have a splitting

  $$x^2 + x + 1 = (x+\alpha)(x+\alpha+1)$$

  over the larger field $\mathbb{F}_2[\alpha] = \{0, 1, \alpha, \alpha+1\} \supset \mathbb{F}_2$. We thus have two *different* residue fields

  $$\kappa(2, x+1) = \mathbb{F}_2$$
  $$\kappa(2, x^2+x+1) = \mathbb{F}_2[x]/(x^2+x+1) = \mathbb{F}_2[\alpha]$$

  for these two different types of points of $\mathbb{A}^1_{\mathbb{F}_2} \subset \operatorname{Spec}\mathbb{Z}[x]$.

- V(3). The polynomial x^2+1 is irreducible over $\mathbb{F}_3$ (having no roots), so the point $(3, x^2+1)$ is not a 'classical' point of $\mathbb{A}^1_{\mathbb{F}_3}$. Let α be a root of x^2+1. Then

  $$x^2 + 1 = (x-\alpha)(x-2\alpha)$$

  over $\mathbb{F}_3[\alpha] \supset \mathbb{F}_3$. In this larger field, the two points $(3, x-\alpha)$ and $(3, x-2\alpha)$ would be 'separated' and would be depicted as two classical points.

  The point $(3, x^2+1) = \mathrm{V}(3) \cap \mathrm{V}(x^2+1)$ is depicted as a small dotted circle. The curve $\mathrm{V}(x^2+1)$ passes through this circle, but in the picture the two branches of the curve remain separated: this reflects the fact that the ‘separation’ of the roots happens over the larger field $\mathbb{F}_3[\alpha]\supset\mathbb{F}_3$. The residue field of $(3, x^2+1)$ is

$$\kappa(3, x^2+1) = \mathbb{F}_3[x]/(x^2+1) = \mathbb{F}_3[\alpha],$$

a degree 2 extension of $\mathbb{F}_3$.

- V(5). The polynomial x^2+1 factors as $(x+2)(x+3)$ over $\mathbb{F}_5$, so we have two classical points $(5, x+2)$ and $(5, x+3)$, both with residue field equal to $\mathbb{F}_5$.
- V(7). The situation here is similar to that of V(3).

Exercise 3.1.84. We have, in the previous example, unconsciously confirmed that $\mathbb{A}^1_{\mathbb{F}_p} = \operatorname{Spec}\mathbb{F}_p[x]$ has way more that the ‘traditional’ p points $(x), (x-1), \ldots, (x-(p-1))$ corresponding to the coordinates $0, 1, \ldots, p-1 \in \mathbb{F}_p$. Show that this is always the case, by proving that $\mathbb{A}^1_{\mathbb{F}} = \operatorname{Spec}\mathbb{F}[x]$ is infinite for *any field* $\mathbb{F}$. (**Hint**: Euclid’s proof of the infinitude of prime numbers!)

Example 3.1.85 (Nodal cubic)**.** Let $C = \operatorname{Spec}\mathbb{C}[x, y]/(y^2 - x^2(x+1)) \hookrightarrow \mathbb{A}^2_{\mathbb{C}}$. Then the morphism

$$f_\phi \colon \mathbb{A}^1_{\mathbb{C}} \longrightarrow C$$

induced by the ring homomorphism $\phi\colon \mathbb{C}[x, y]/(y^2 - x^2(x+1)) \to \mathbb{C}[t]$ defined by $\phi(x) = t^2-1$ and $\phi(y) = t(t^2-1)$ is not an isomorphism. Indeed, the function $t = y/x$ is not regular at $(0,0)$, and as such it does not lie in the image of ϕ. There is no ring isomorphism $\mathbb{C}[t] \cong \mathbb{C}[x, y]/(y^2 - x^2(x+1))$. Note that f_ϕ is not even bijective on closed points: the origin $(0,0)\in C$ has two preimages, corresponding to $t = \pm 1$.

Example 3.1.86 (Cuspidal cubic)**.** Let $C = \operatorname{Spec}\mathbb{C}[x, y]/(y^2 - x^3) \hookrightarrow \mathbb{A}^2_{\mathbb{C}}$. Then the morphism

$$f_\phi \colon \mathbb{A}^1_{\mathbb{C}} \longrightarrow C$$

induced by the ring homomorphism $\phi\colon \mathbb{C}[x, y]/(y^2 - x^3) \to \mathbb{C}[t]$ defined by $\phi(x) = t^2$ and $\phi(y) = t^3$ is a bijective morphism, but not an isomorphism. It sends the closed point $(t-a)\in\mathbb{A}^1_{\mathbb{C}}$ to the point of $\mathbb{A}^2_{\mathbb{C}}$ with coordinates (a^2, a^3). The morphism f_ϕ is called a *rational parametrisation* of the plane curve $C \hookrightarrow \mathbb{A}^2_{\mathbb{C}}$.

We have learnt from several examples (including Examples 3.1.66, 3.1.68, 3.1.85 and 3.1.86) that

a bijective morphism of schemes need not be an isomorphism.

Example 3.1.87. Consider the ring homomorphism

$$\phi\colon \mathbf{k}[x, y] \to \mathbf{k}[x, y, z]/(xz - y)$$

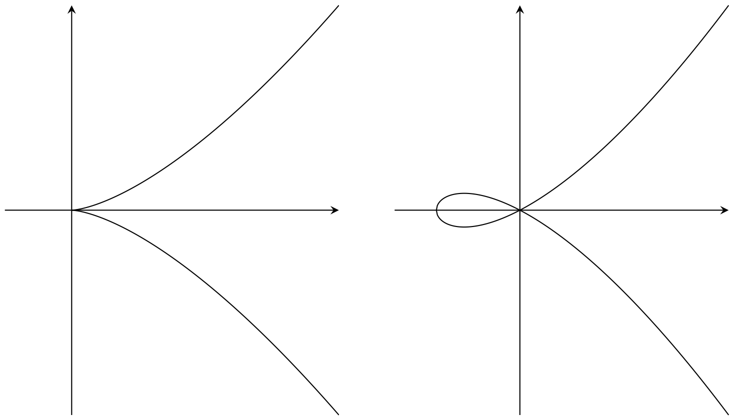

Figure 3.11: The (real points of the) cuspidal cubic curve $y^2 = x^3$ and the nodal cubic curve $y^2 = x^2(x+1)$.

sending $x \mapsto x$ and $y \mapsto y$. The corresponding morphism of affine schemes

$$f_\phi \colon \operatorname{Spec} \mathbf{k}[x,y,z]/(xz-y) \to \mathbb{A}^2_{\mathbf{k}}$$

sends $(a,ab,b) \mapsto (a,ab)$, and the generic point maps to generic point. The image of f_ϕ is $\mathrm{V}(x,y) \cup \mathrm{D}(x)$, which is neither open nor closed in $\mathbb{A}^2_{\mathbf{k}}$.

The image of the morphism f_ϕ in the previous example may look topologically weird. It is not that bad, though. Recall that a subset $T \subset X$ of a topological space X is *constructible* if it is a finite disjoint union of locally closed subsets. In general, a morphism of *algebraic varieties* (defined later in Important Definition 3.2.1 as those $\mathbf{k}$-schemes $X \to \operatorname{Spec} \mathbf{k}$ admitting an open cover by finitely many affine varieties) preserves constructible subsets.

THEOREM 3.1.88 (Chevalley). *Let $f \colon X \to Y$ be a morphism of algebraic varieties over $\mathbf{k}$. If $T \subset X$ is constructible, then $f(T) \subset Y$ is constructible. In particular, $f(X) \subset Y$ is constructible.*

3.2 Schemes

We already anticipated the definition of schemes in Important Definition 3.1.4, just because we could do so. Now we start with the general theory, but first we recall the definition verbatim.

Definition 3.2.1 (Scheme)**.** A *scheme* is a locally ringed space $(X, \mathscr{O}_X)$ in which every point $x \in X$ has an open neighbourhood $x \in U \subset X$ such that $(U, \mathscr{O}_X|_U)$ is an affine scheme.

Keep also Terminology 3.1.33 and Definition 3.1.34 in mind.

3.2.1 The category of S-schemes and algebraic varieties

As we saw (cf. Notation 3.1.38), schemes form a category, denoted Sch. For any scheme S, we can form the *category* Sch_S *of* S*-schemes*, whose objects are pairs (X, f), where X is a scheme and $f : X \to S$ is a morphism of schemes. Morphisms

$$(X_1, f_1) \to (X_2, f_2)$$

in Sch_S are morphisms of schemes $g : X_1 \to X_2$ such that $f_2 \circ g = f_1$. We often call them *morphisms over* S or S-morphisms (or A-morphisms if $S = \operatorname{Spec} A$).

$$\begin{tikzcd} X_1 \arrow[rr, "g"] \arrow[dr, "f_1"'] & & X_2 \arrow[dl, "f_2"] \\ & S & \end{tikzcd}$$

If $S = \operatorname{Spec} A$ is affine, we simply write Sch_A instead of $\mathsf{Sch}_{\operatorname{Spec} A}$. For instance, when B is an A-algebra via a ring homomorphism $A \to B$, one says that $\operatorname{Spec} B \to \operatorname{Spec} A$ is an A-scheme via the canonical scheme morphism attached to $A \to B$.

Important Definition 3.2.1 (Algebraic variety)**.** Let $\mathbb{F}$ be a field. An *algebraic variety* over a field $\mathbb{F}$ (or simply a $\mathbb{F}$-*variety*) is a $\mathbb{F}$-scheme which admits an open cover by finitely many affine varieties over $\mathbb{F}$.

Caution 3.2.2. Different authors give different definitions of algebraic variety. Other variants include: reduced scheme of finite type over a field, reduced separated[5] scheme of finite type over a field, integral scheme of finite type over a field. Note that, with our definitions, fat points (different from $\operatorname{Spec} \mathbf{k}$) are considered to be (affine) algebraic varieties, even though they are not reduced. On the other hand, the nodal cubic (Example 4.4.5) and the cuspidal cubic (Example 4.4.4) are algebraic varieties.

Caution 3.2.3. It is *not* true that if X is an algebraic variety, its ring of regular function $\mathscr{O}_X(X)$ is finitely generated! See [18].

Definition 3.2.4 (Closed subscheme)**.** A *closed subscheme* of a scheme X is an equivalence class of closed immersions $Z \hookrightarrow X$ of schemes into X. The equivalence relation says that $\iota : Z \hookrightarrow X$ is equivalent to $\iota' : Z' \hookrightarrow X$ if there is an isomorphism $\alpha : Z \xrightarrow{\sim} Z'$ such that $\iota' \circ \alpha = \iota$ (in other words, α is an isomorphism in Sch_X).

Note the crucial difference between open subscheme (cf. Definition 3.1.36) and closed subscheme: a given open *subset* $U \subset X$ is given by default a well precise structure sheaf (making into a scheme, cf. Remark 3.1.59), namely $\mathscr{O}_X|_U$, whereas on a closed *subset* $Z \hookrightarrow X$ there are a pletora of possible scheme structures. Finally, note that

[5]This notion will be introduced in Section 5.4.

we haven't defined a closed subscheme of X as a scheme Z together with a closed immersion: we have defined it to be an *equivalence class* of closed immersions, so that by Proposition 2.10.11 we have a precise correspondence between closed subschemes of a scheme X and ideal sheaves $\mathscr{I} \subset \mathscr{O}_X$. In the affine case, thanks to Proposition 3.1.65, we have the following: for any ring A, there is a bijection

$$\{\text{closed subschemes of } \operatorname{Spec} A\} \simeq \{\text{ideals } I \subset A\}. \tag{3.2.1}$$

Spoiler 3.2.5. We will see in Section 4.3 that for every scheme X there is a 'nicest' closed subscheme $X_{\mathsf{red}} \hookrightarrow X$, called the *reduction* of X, which is topologically the same as X and is the smallest with this property.

3.2.2 Morphisms to an affine scheme

A complete characterisation of morphisms *of affine schemes* was given, somewhat implicitly, in Theorem 3.1.61. Now we let X be an arbitrary scheme. Our goal is to characterise morphisms

$$X \to \operatorname{Spec} A.$$

We will show that the natural map

$$\rho_{X,Y} \colon \operatorname{Hom}_{\mathsf{Sch}}(X,Y) \to \operatorname{Hom}_{\mathsf{Rings}}(A, \mathscr{O}_X(X)), \tag{3.2.2}$$

already introduced in (3.1.13) in the affine case, is a bijection. The map works just as in the affine case: a morphism $f \colon X \to Y$ is sent to $f^\#(Y) \colon A = \mathscr{O}_Y(Y) \to f_*\mathscr{O}_X(Y) = \mathscr{O}_X(X)$. Functoriality also holds, i.e. the diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathsf{Sch}}(X,Y) \arrow[r, "\rho_{X,Y}"] \arrow[d, "f \mapsto f\circ g"] & \operatorname{Hom}_{\mathsf{Rings}}(A, \mathscr{O}_X(X)) \arrow[d, "\phi \mapsto g^\#(X)\circ\phi"] \\
\operatorname{Hom}_{\mathsf{Sch}}(Z,Y) \arrow[r, "\rho_{Z,Y}"] & \operatorname{Hom}_{\mathsf{Rings}}(A, \mathscr{O}_Z(Z))
\end{tikzcd}$$

commutes for any morphism $g \colon Z \to X$ (for the same reason as in Remark 3.1.62).

We need the following preliminary result.

LEMMA 3.2.6. *Let X, Y be schemes. Then sending*

$$U \mapsto \operatorname{Hom}_{\mathsf{Sch}}(U,Y) \in \mathsf{Sets}$$

for each open subset $U \subset X$ defines a sheaf of sets on X.

Proof. By Remark 2.2.6, we need to verify that given an open subset $U \subset X$, an open cover $U = \bigcup_{i\in I} U_i$ and a collection of morphisms $f_i \colon U_i \to Y$ such that $f_i|_{U_{ij}} = f_i|_{U_{ij}}$ (as morphisms of schemes!) for every $(i,j) \in I \times I$, there exists a unique $f \colon U \to Y$ such that

$f|_{U_i} = f_i$. (We have used the usual notation $U_{ij} = U_i \cap U_j$). At the level of topological spaces, it is clear that there is a unique continous map $f: U \to Y$ with the required property. We need to extend it (uniquely) to a morphism of *schemes*. So we need a well defined sheaf homomorphism $f^{\#}: \mathscr{O}_Y \to f_* \mathscr{O}_U$. Let $V \subset Y$ be an open subset. We define $f^{\#}(V): \mathscr{O}_Y(V) \to \mathscr{O}_U(f^{-1}V)$ as follows.

First of all, each $f_i: U_i \to Y$ induces a map $\mathscr{O}_Y(V) \to \mathscr{O}_{U_i}(f_i^{-1}V) = \mathscr{O}_U(f_i^{-1}V)$. Moreover, $f^{-1}V = \bigcup_{i\in I} f_i^{-1}V$ is an open covering, and since $\mathscr{O}_U$ is a sheaf we have a diagram

$$\begin{tikzcd}
 & \mathscr{O}_Y(V) \arrow[dl, dotted] \arrow[d, "\tau"] & \\
\mathscr{O}_U(f^{-1}V) \arrow[r] & \prod\limits_{i\in I} \mathscr{O}_U(f_i^{-1}V) \arrow[r, shift left, "\mu"] \arrow[r, shift right, "\nu"'] & \prod\limits_{(i,j)\in I\times I} \mathscr{O}_U(f_i^{-1}V \cap f_j^{-1}V)
\end{tikzcd}$$

where the bottom row is an equaliser sequence. Saying that $f_i|_{U_{ij}} = f_i|_{U_{ij}}$ is like saying that $\mu \circ \tau = \nu \circ \tau$, thus by the universal property of equalisers there is precisely one way to fill in the dotted arrow to $\mathscr{O}_U(f^{-1}V)$. This is the definition of $f^{\#}(V)$. □

Remark 3.2.7. The statement of Lemma 3.2.6 remains true for locally ringed spaces or, more generally, ringed spaces: we have not used the actual definition of schemes for its proof.

THEOREM 3.2.8. *Let X be a scheme, $Y = \operatorname{Spec} A$ an affine scheme. Then the canonical map* (3.2.2) *is bijective.*

Proof. Fix a covering $X = \bigcup_{i\in I} U_i$, where $\iota_i: U_i = \operatorname{Spec} B_i \hookrightarrow X$ is an affine open subset. Since $U \mapsto \operatorname{Hom}_{\mathsf{Sch}}(U, Y)$ is a sheaf on X (cf. Lemma 3.2.6), the natural map

$$\operatorname{Hom}_{\mathsf{Sch}}(X, Y) \xrightarrow{\alpha} \prod_{i\in I} \operatorname{Hom}_{\mathsf{Sch}}(U_i, Y)$$

is injective. We have a new diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathsf{Sch}}(X, Y) \arrow[r, "\rho_{X,Y}"] \arrow[d, hook, "\alpha"] & \operatorname{Hom}_{\mathsf{Rings}}(A, \mathscr{O}_X(X)) \arrow[d, "\phi \mapsto (\iota_i^{\#}(X)\circ\phi)_{i\in I}"] \\
\prod_{i\in I} \operatorname{Hom}_{\mathsf{Sch}}(U_i, Y) \arrow[r, "\beta"] & \prod_{i\in I} \operatorname{Hom}_{\mathsf{Rings}}(A, B_i)
\end{tikzcd}$$

where β is a bijection as confirmed during the proof of Theorem 3.1.61. It follows that $\rho_{X,Y}$ is injective. We are left to prove its surjectivity. Fix $\phi \in \operatorname{Hom}_{\mathsf{Rings}}(A, \mathscr{O}_X(X))$, and consider its image $(\phi_i)_{i\in I} \in \prod_{i\in I} \operatorname{Hom}_{\mathsf{Rings}}(A, B_i)$. This corresponds to a unique tuple of morphisms $(f_i: U_i \to Y)_{i\in I}$. These have the property that $f_i|_V = f_j|_V$ for every affine open subset $V \subset U_i \cap U_j$. To see this, notice that for any $i \in I$ we have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{Sch}}(U_i, Y) \arrow[r, "\sim"] \arrow[d] & \mathrm{Hom}_{\mathsf{Rings}}(A, B_i) \arrow[d, "\psi \mapsto j_i^{\#}(U_i)\circ\psi"] \\ \mathrm{Hom}_{\mathsf{Sch}}(V, Y) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathsf{Rings}}(A, \mathscr{O}_X(V)) \end{tikzcd}$$

where $j_i \colon V \hookrightarrow U_i$ is the open immersion. It is clear that the image of f_i in the set $\mathrm{Hom}_{\mathsf{Rings}}(A, \mathscr{O}_X(V))$ does not depend on i, being equal to the image of ϕ, namely its post-composition with $\mathscr{O}_X(X) \to \mathscr{O}_X(V)$. Therefore all the f_i map to the same element of $\mathrm{Hom}_{\mathsf{Sch}}(V, Y)$, which is what we wanted to confirm. It now follows from Lemma 3.2.6 that $(f_i \colon U_i \to Y)_{i\in I}$ glue to a (unique) morphism $f \colon X \to Y$, which by construction maps to ϕ via $\rho_{X,Y}$. Thus $\rho_{X,Y}$ is surjective. □

COROLLARY 3.2.9. *Let A be a ring. To give a scheme over* $\operatorname{Spec} A$ *is the same as to give a scheme $(X, \mathscr{O}_X)$ along with an A-algebra structure on $\mathscr{O}_X$.*

COROLLARY 3.2.10. *Let X be a scheme. There is a canonical morphism*

$$X \longrightarrow \operatorname{Spec} \mathscr{O}_X(X),$$

called the affinisation morphism for X. And sometimes $\operatorname{Spec} \mathscr{O}_X(X)$ *is called the affinisation of X.*

Proof. Take $Y = \operatorname{Spec} \mathscr{O}_X(X)$ and consider the morphism corresponding to the identity $\mathrm{id} \in \mathrm{Hom}_{\mathsf{Rings}}(\mathscr{O}_X(X), \mathscr{O}_X(X))$ under $\rho_{X,Y}$. □

Remark 3.2.11. A possible translation of Theorem 3.2.8 is the following: if $\Gamma(-)$ denotes the functor taking a scheme X to the ring of its regular functions $\mathscr{O}_X(X)$, then the pair of functors $(\Gamma(-), \operatorname{Spec})$ is an adjoint pair on

$$\begin{tikzcd} \mathsf{Sch} \arrow[r, shift left, "\Gamma(-)"] & \mathsf{Rings}^{\mathrm{op}} \arrow[l, shift left, "\mathrm{Spec}"] \end{tikzcd}$$

where of course Spec is now viewed as the composition $\mathsf{Rings}^{\mathrm{op}} \xrightarrow{\sim} \mathsf{Aff} \hookrightarrow \mathsf{Sch}$.

Exercise 3.2.12. Confirm that $\operatorname{Spec} \mathbb{Z}$ is a final object in the category of schemes, so that (in the notation of Section 3.2.1) in particular $\mathsf{Sch} = \mathsf{Sch}_{\mathbb{Z}}$.

3.2.3 Glueing schemes

You may have encountered interesting spaces such as *projective spaces* or *Grassmannians* before. For example, projective n-space over a field $\mathbb{F}$ can be defined as follows: consider the scaling action of $\mathbb{F}^\times$ on $\mathbb{F}^{n+1} \setminus 0$, sending $v \mapsto \lambda v$ for $\lambda \in \mathbb{F}^\times$, and set

$$\mathbb{P}^n(\mathbb{F}) = (\mathbb{F}^{n+1} \setminus 0)/\mathbb{F}^\times.$$

This is all good in the topological (or smooth) category, however we cannot make such a definition in algebraic geometry. Quotients exist (sometimes) and their theory has now become classical, but they are delicate to deal with.

We shall see *two ways* to define projective space in algebraic geometry. The first one is by glueing schemes (along open immersions). We now describe this procedure in full generality.

The input data are as follows:

(1) a scheme S,

(2) a family of S-schemes $\{ X_i \to S \mid i \in I \}$,

(3) open subschemes $X_{ij} \subset X_i$ for every $(i,j) \in I \times I$,

(4) isomorphisms $f_{ij} \colon X_{ij} \xrightarrow{\sim} X_{ji}$ over S for every $(i,j) \in I \times I$.

$$\begin{tikzcd}
X_{ij} \arrow[r, hook, "\text{open}"] \arrow[d, "\text{isomorphism } f_{ij}"'] & X_i \arrow[dr] & \\
 & & S \\
X_{ji} \arrow[r, hook, "\text{open}"] & X_j \arrow[ur] &
\end{tikzcd}$$

The assumptions on the input data are the following:

(i) $X_{ii} = X_i$ and $f_{ii} = \mathrm{id}_{X_i}$ for every $i \in I$,

(ii) $f_{ij}(X_{ij} \cap X_{ik}) = X_{jk} \cap X_{ji}$, for every $(i,j,k) \in I \times I \times I$

(iii) the *cocycle condition* holds: $f_{jk} \circ f_{ij} = f_{ik}$ on $X_{ij} \cap X_{ik}$.

The cocycle condition is the following compatibility:

$$\begin{tikzcd}
X_{ij} \cap X_{ik} \arrow[rr, hook] \arrow[drr, "f_{ij}"] \arrow[dd, "f_{ik}"'] & & X_{ij} \arrow[r, "f_{ij}"] & X_{ji} \\
 & & X_{jk} \cap X_{ji} \arrow[ur, hook] \arrow[r, hook] \arrow[dll, "f_{jk}"'] & X_{jk} \arrow[dl, "f_{jk}"] \\
X_{kj} \cap X_{ki} \arrow[rr, hook] & & X_{kj} &
\end{tikzcd}$$

THEOREM 3.2.13 (Glueing schemes). *Given the data (1)–(4) satisfying conditions (i)–(iii), there exists an S-scheme X (unique up to isomorphism), along with open immersions $\theta_i \colon X_i \hookrightarrow X$ over S such that $\theta_j|_{X_{ji}} \circ f_{ij} = \theta_i|_{X_{ij}}$ and $X = \bigcup_{i \in I} \theta_i(X_i)$. Moreover, $\theta_i(X_i) \cap \theta_j(X_j) = \theta_i(X_{ij})$.*

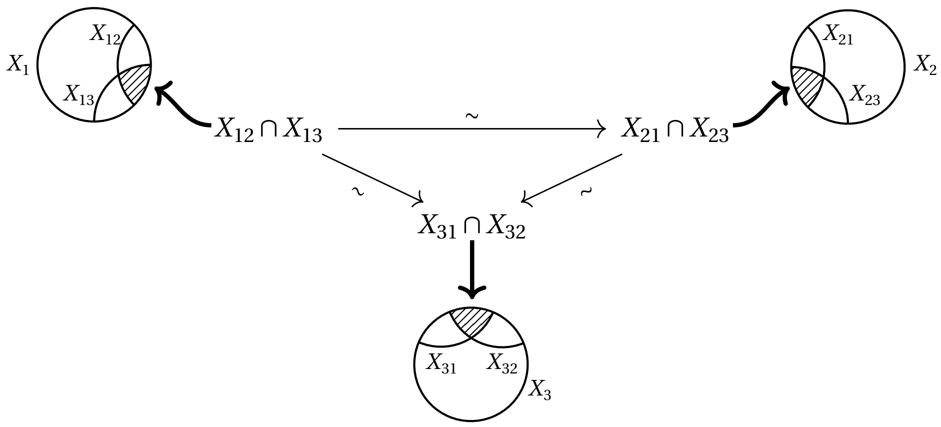

Figure 3.12: The glueing construction starting with 3 open subsets $X_1, X_2, X_3 \subset X$.

Proof. See [12, Ch. 2, Lemma 3.33]. □

Definition 3.2.14. The *disjoint union* of a family of S-schemes $\{X_i \to S\}_{i \in I}$ is the glueing of the family along $X_{ij} = \emptyset$ (and empty maps f_{ij}). It is denoted $\coprod_{i \in I} X_i$.

Next, we describe the construction of

$$\boxed{\mathbb{P}^n_A = \text{projective } n\text{-space over } A.}$$

Let A be a ring, $S = \operatorname{Spec} A$ the corresponding affine scheme, $n \geq 0$ an integer and $I = \{0, 1, \ldots, n\}$. Fix a variable x_i for every $i \in I$, and form the ring $R = A[x_0^{\pm}, x_1^{\pm}, \ldots, x_n^{\pm}]$. Consider the A-subalgebras

$$A_i = A\left[x_k x_i^{-1} \,\middle|\, 0 \leq k \leq n\right] \subset R, \quad i \in I.$$

Note that A_i is the homogeneous localisation of $A[x_0, x_1, \ldots, x_n]$ at the degree 1 element x_i, a polynomial ring in n variables (cf. Example 3.3.13). Each yields an A-scheme

$$X_i = \operatorname{Spec} A_i \to \operatorname{Spec} A, \quad i \in I.$$

Now, for each $j \neq i$, the scheme X_i contains the (principal) affine open subscheme

$$X_{ij} = \mathrm{D}(x_j x_i^{-1}) = \operatorname{Spec} (A_i)_{x_j x_i^{-1}} \subset X_i.$$

But

$$(A_i)_{x_j x_i^{-1}} = (A_j)_{x_i x_j^{-1}}$$

are *equal* as subrings of R, and thus we have canonical isomorphisms $f_{ij}\colon X_{ij} \xrightarrow{\sim} X_{ji}$. Explicitly, after the identifications

$$
\begin{aligned}
(A_i)_{x_j x_i^{-1}} &\cong A_i[t]/(t\cdot x_j x_i^{-1} - 1)\\
(A_j)_{x_i x_j^{-1}} &\cong A_j[u]/(u\cdot x_i x_j^{-1} - 1),
\end{aligned}
$$

we see that an isomorphism between the quotient rings on the right hand sides is given by sending

$$x_j x_i^{-1} \mapsto u, \quad t \mapsto x_i x_j^{-1}, \quad x_k x_i^{-1} \mapsto x_k x_i^{-1} \text{ for } k \neq i, j.$$

The hypotheses of Theorem 3.2.13 are satisfied by our glueing data. The resulting A-scheme is called *projective n-space over A*, and is denoted $\mathbb{P}^n_A$. It has an open cover by $n+1$ affine open subsets isomorphic to affine spaces over A, namely $X_i = \operatorname{Spec} A_i \cong \mathbb{A}^n_A$. Indeed, the variables $\{\, x_k x_i^{-1} \mid 0 \le k \le n \,\}$ are algebraically independent.

Remark 3.2.15. This construction shows that $\mathbb{P}^n_A$ is a quasicompact scheme. We shall see that it is not affine (unless $n = 0$).

Example 3.2.16 (Projective line)**.** The most explicit instance of the above construction of $\mathbb{P}^n_A$ arises for $n = 1$. In this case, our input data are simply two schemes $X_1 = \operatorname{Spec} A[t]$ and $X_2 = \operatorname{Spec} A[u]$, and the isomorphism $X_{12} = \mathrm{D}(t) \xrightarrow{\sim} \mathrm{D}(u) = X_{21}$ induced by the A-algebra isomorphism $A[u, u^{-1}] \xrightarrow{\sim} A[t, t^{-1}]$ sending $u \mapsto t^{-1}$. The glueing gives, by definition, the *projective line* $\mathbb{P}^1_A$. Similarly, Figure 3.12 can be seen as a pictorial construction of $\mathbb{P}^2_A$.

Example 3.2.17. Keep the notation of Example 3.2.16, but assume $A = \mathbf{k}$ is a field, for simplicity. Then, had we chosen the isomorphism $\mathbf{k}[u, u^{-1}] \xrightarrow{\sim} \mathbf{k}[t, t^{-1}]$ sending $u \mapsto t$, we would have of course identified the complements of the origin in the two affine lines, but we also would have 'kept the origin twice'. The result of the glueing is called an *affine line with double origin*. We shall come back to this scheme, for it is the prototypical example of a non-separated scheme. Separatedness is, as we shall see, the scheme-theoretic analogue of the Hausdorff property, which we have already given up on (cf. Remark 3.1.17). Since affine schemes are separated (cf. Proposition 5.4.16), this also gives another example (besides Example 3.1.72 and Example 3.1.73) of a non-affine scheme.

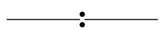

Figure 3.13: The affine line with two origins.

Since the schemes X_i form an open covering of the glued up scheme X, by Example 2.3.4 we have an exact sequence of abelian groups

$$0 \to \mathscr{O}_X(X) \to \prod_{i\in I} \mathscr{O}_X(X_i) \to \prod_{(i,j)\in I\times I} \mathscr{O}_X(X_{ij}) \tag{3.2.3}$$

where the first map is restriction and the second map sends $(f_i)_i \mapsto (f_i|_{X_{ij}} - f_j|_{X_{ij}})_{i,j}$.

Exercise 3.2.18. Use the sequence (3.2.3) to show that $\mathscr{O}_{\mathbb{P}^n_A}(\mathbb{P}^n_A) = A$. Observe, then, that $\mathbb{P}^n_A$ is not affine (unless $n = 0$)!

3.3 Projective schemes

In this section we define an important class of schemes, including *projective schemes*. These are the closed subschemes of $\mathbb{P}^n_A$ for some given ring A and some $n \geq 0$. They are the natural upgrade of classical *projective varieties* over a field. The general construction is somewhat analogous (though exhibiting many differences as well, see e.g. Remark 3.3.10, Caution 3.3.18 and Caution 3.3.28) to the construction of $\operatorname{Spec} A$ starting from a ring A. The main difference is that now we have to work with *graded rings*. The ubiquity of gradings and homogeneous ideals can be 'explained' at an informal level as follows: points $p = (a_0 : a_1 : \cdots : a_n)$ of classical projective space $\mathbb{P}^n(\mathbb{C})$ have *homogeneous coordinates*, meaning e.g. that $(1:2)$ is the same point as $(-3:-6)$ in $\mathbb{P}^1(\mathbb{C})$. As such, the evaluation of a polynomial $f \in \mathbb{C}[x_0, x_1, \ldots, x_n]$ at p is not well-defined. What *is* well-defined though, is the *vanishing* of f at p, so long as f is homogeneous, for

$$f(\lambda x_0, \lambda x_1, \ldots, \lambda x_n) = \lambda^d f(x_0, x_1, \ldots, x_n), \quad d = \deg f, \quad \lambda \in \mathbb{C}^\times.$$

We will see that the equation $f = 0$ defines a closed subscheme of $\mathbb{P}^n_{\mathbb{C}}$. This will be called a *hypersurface in* $\mathbb{P}^n_{\mathbb{C}}$.

3.3.1 Zariski topology on $\operatorname{Proj} B$

Let A be a ring. A *graded A-algebra* is an A-algebra $A \to B$ equipped with a decomposition

$$B = \bigoplus_{d \geq 0} B_d,$$

where $B_d \subset B$ are subgroups satisfying $B_d B_e \subset B_{d+e}$ for each $d, e \geq 0$, and such that the image of $A \to B$ is contained in B_0. In this situation, we have that

(1) $B_0 \subset B$ is a subring, so that B is naturally a B_0-algebra, and

(2) each graded piece B_d is a B_0-module.

Elements of B_d are called *homogeneous of degree d* (and $0 \in B$ is considered homogeneous of any degree). Every $f \in B$ has a unique decomposition $f = \sum_{0 \leq i \leq e} f_i$ into homogeneous elements f_i. An A-algebra homomorphism $\phi \colon B \to C$ is called a *graded homomorphism* if there exists an integer $e > 0$ such that $\phi(B_d) \subset C_{ed}$ for every $d \geq 0$. Graded A-algebras thus form a category. If B is a graded A-algebra, the ideal

$$B_+ = \bigoplus_{d > 0} B_d \subset B$$

is called the *irrelevant ideal*, for reasons that will become clear soon (cf. Remark 3.3.8).

Example 3.3.1. Let $B = A[x_0, x_1, \ldots, x_n]$ be the polynomial ring with A-coefficients and with x_i in degree 1 for all i. Elements of B_d are simply the homogeneous polynomials of degree d in the classical sense. The irrelevant ideal is $B_+ = (x_0, x_1, \ldots, x_n) \subset B$.

Let $I \subset B$ be an ideal. Then, the following are equivalent:

- $I \subset B$ is a graded submodule,
- I can be generated by homogeneous elements,
- $I = \bigoplus_{d \geq 0}(I \cap B_d)$,
- If $f \in I$ has homogeneous decomposition $f = f_0 + f_1 + \cdots + f_k$, then $f_e \in I$ for all e.

If any of these equivalent conditions is fulfilled, we say that I is *homogeneous*.

Remark 3.3.2. The class of homogeneous ideals in B is closed under sum, product, intersection, and radical.

Remark 3.3.3. Let $I \subset B$ be a homogeneous ideal. The quotient B/I is naturally a graded A-algebra via $(B/I)_d = B_d/(I \cap B_d)$.

Let $A \to B$ be a graded A-algebra. The localisation of B at a multiplicative subset $S \subset B$ inherits a grading as soon as S consists of homogeneous elements. If $\mathfrak{p} \subset B$ is a homogeneous prime ideal, we may localise B at

$$S(\mathfrak{p}) = \left\{ b \in B \setminus \mathfrak{p} \,\middle|\, b \text{ is homogeneous} \right\}.$$

This localisation, denoted $B_\mathfrak{p}$ with a slight abuse of notation (see also Warning B.5.9), contains as a subring its degree 0 piece, denoted $B_{(\mathfrak{p})}$. It is a local ring with maximal ideal

$$\mathfrak{m}_{(\mathfrak{p})} = \left\{ \frac{a}{h} \,\middle|\, a \in \mathfrak{p},\ h \in S(\mathfrak{p}),\ \deg a = \deg h \right\}.$$

The local ring $(B_{(\mathfrak{p})}, \mathfrak{m}_{(\mathfrak{p})})$ may be called the homogeneous localisation of B at $\mathfrak{p}$. Another key example of homogeneous localisation is the *homogeneous principal localisation.*

Construction 3.3.4 (Homogeneous principal localisation)**.** Let $A \to B$ be a graded A-algebra. If $f \in B$ is homogeneous of degree e, then $B_f = \bigoplus_{d \in \mathbb{Z}} (B_f)_d$, where

$$(B_f)_d = \left\{ \frac{a}{f^n} \,\middle|\, a \in B_{d+ne} \right\} \subset B_f.$$

Such graded rings are the only ones (that we consider) with negative graded pieces. We set

$$B_{(f)} = (B_f)_0 = \left\{ \frac{a}{f^n} \,\middle|\, a \in B_{ne} \right\}.$$

It is called the *homogeneous principal localisation* (or simply *homogeneous localisation*) of B at f. This is a ring by the condition (1), and in fact, it is an A-subalgebra of B_f, by our key assumption that $A \to B$ lands in B_0. This can be seen via the diagram

(3.3.1)
$$\begin{tikzcd} A \arrow[r] & B_0 \arrow[r, hook, "\text{subring}"] \arrow[d, dotted] & B \arrow[d, "\ell"] \\ & B_{(f)} \arrow[r, hook] & B_f \end{tikzcd}$$

and the fact that ℓ preserves the grading. We endow $B_{(f)}$ with the trivial grading. The inclusion $B_{(f)} \hookrightarrow B_f$ turns B_f into a graded $B_{(f)}$-algebra, and this gives a natural morphism of affine schemes

$$\operatorname{Spec} B_f \longrightarrow \operatorname{Spec} B_{(f)}.$$

To an arbitrary ideal $I \subset B$ we may associate a homogeneous ideal $I^{\mathrm{h}} = \bigoplus_{d \geq 0}(I \cap B_d)$. Note that $I^{\mathrm{h}} \subset I$, with equality if and only if I is homogeneous.

LEMMA 3.3.5. *Let $I \subset B$ be a homogeneous ideal. Then I is prime if and only if whenever $ab \in I$ for homogeneous elements $a, b \in B$, one has that either $a \in I$ or $b \in I$.*

Proof. Let $a = \sum_{1 \leq i \leq n} a_i$ and $b = \sum_{1 \leq j \leq m} b_j$ be the homogeneous decompositions of two elements $a, b \in B$ such that $ab \in I$. Since I is homogeneous, it must contain all the homogeneous components of ab. Assume, by contradiction, that $a \notin I$ and $b \notin I$. Then, there is a largest d such that $a_d \notin I$ and a largest e such that $b_e \notin I$. We have $(ab)_{d+e} = \sum_{i+j=d+e} a_i b_j$, but every pair $(i, j) \neq (d, e)$ appearing in the sum satisfies either $i > d$ or $j > e$. Thus $a_i b_j \in I$ for every such pair. But since $(ab)_{d+e} \in I$ as well, we must have $a_d b_e \in I$. Thus, by our assumption, either $a_d \in I$ or $b_e \in I$. Contradiction. □

LEMMA 3.3.6. *If $I \subset B$ is a prime ideal, then the homogeneous ideal $I^{\mathrm{h}} \subset B$ is prime.*

Proof. We exploit Lemma 3.3.5. Let $a, b \in B$ be two homogeneous elements, say $a \in B_d$ and $b \in B_e$, such that $ab \in I^{\mathrm{h}}$. In fact, $ab \in I^{\mathrm{h}}_{d+e} = I \cap B_{d+e} \subset I$. Then, since I is prime, we have either $a \in I$ or $b \in I$. Thus either $a \in I \cap B_d \subset I^{\mathrm{h}}$, or $b \in I \cap B_e \subset I^{\mathrm{h}}$. □

Let $A \to B$ be a graded A-algebra. Define the *projective spectrum* of B to be the set

$$\boxed{\operatorname{Proj} B = \left\{ \mathfrak{p} \subset B \,\middle|\, \begin{array}{c} \mathfrak{p} \text{ is a homogeneous prime} \\ \text{ideal such that } \mathfrak{p} \not\supset B_+ \end{array} \right\}.}$$

Our goal is to put a structure of A-scheme on $\operatorname{Proj} B$. By Corollary 3.2.9, this amounts to construct a scheme $(\operatorname{Proj} B, \mathscr{O}_{\operatorname{Proj} B})$ along with an A-algebra structure on $\mathscr{O}_{\operatorname{Proj} B}$.

As in the affine case, we start from the topology on $\operatorname{Proj} B$. For a homogeneous ideal $I \subset B$, we define

$$\mathrm{V}_+(I) = \left\{ \mathfrak{p} \in \operatorname{Proj} B \,\middle|\, \mathfrak{p} \supset I \right\} \subset \operatorname{Proj} B.$$

These sets satisfy the axioms of closed subsets for a topology on $\operatorname{Proj} B$. The properties

(1) $\mathrm{V}_+(I)\cup\mathrm{V}_+(J)=\mathrm{V}_+(I\cap J)=\mathrm{V}_+(IJ)$

(2) $\bigcap_{\lambda\in\Lambda}\mathrm{V}_+(I_\lambda)=\mathrm{V}_+\left(\sum_{\lambda\in\Lambda}I_\lambda\right)$

(3) $\mathrm{V}_+(B)=\emptyset$ and $\mathrm{V}_+(0)=\operatorname{Proj}B$

are proved in a similar fashion to the affine case (cf. Lemma 3.1.6). The induced topology on $\operatorname{Proj}B$ is called the *Zariski topology*. Note that one also has

$$\mathrm{V}_+(I)=\mathrm{V}_+(\sqrt{I})$$

for any homogeneous ideal $I\subset B$.

LEMMA 3.3.7. *Let $I,J\subset B$ be homogeneous ideals.*

(i) $\mathrm{V}_+(I)\subset\mathrm{V}_+(J)$ *if and only if* $J\cap B_+\subset\sqrt{I}$.

(ii) *One has* $\operatorname{Proj}B=\emptyset$ *if and only if* B_+ *is nilpotent.*

(iii) $\mathrm{V}_+(I)=\emptyset$ *if and only if* $B_+\subset\sqrt{I}$. *If* $\sqrt{I}=B_+$, *then* $\mathrm{V}_+(J)=\mathrm{V}_+(J\cap I)$.

Proof. We proceed step by step.

(i) Assume $J\cap B_+\subset\sqrt{I}$, and fix a prime $\mathfrak{p}\in\mathrm{V}_+(I)$. Then

$$\mathfrak{p}\supset\sqrt{I}\supset J\cap B_+\supset JB_+,$$

and since $\mathfrak{p}\not\supset B_+$ we must have $\mathfrak{p}\in\mathrm{V}_+(J)$.

Assume now that $\mathrm{V}_+(I)\subset\mathrm{V}_+(J)$. Recall that $\sqrt{I}=\bigcap_{\mathfrak{p}\in\mathrm{V}(I)}\mathfrak{p}$. Observe that any prime $\mathfrak{p}\in\mathrm{V}(I)$ satisfies (since I is homogeneous) $I=I^{\mathrm{h}}\subset\mathfrak{p}^{\mathrm{h}}$. Now there are two possibilities. First: if $\mathfrak{p}^{\mathrm{h}}\not\supset B_+$, then by Lemma 3.3.6 $\mathfrak{p}^{\mathrm{h}}$ belongs to $\mathrm{V}_+(I)$, thus by assumption $\mathfrak{p}\supset\mathfrak{p}^{\mathrm{h}}\supset J\supset J\cap B_+$, which implies $J\cap B_+\subset\sqrt{I}$. Second option: if $\mathfrak{p}^{\mathrm{h}}\supset B_+$, we still have $\mathfrak{p}\supset\mathfrak{p}^{\mathrm{h}}\supset B_+\supset J\cap B_+$. Thus, also in this case, we have $J\cap B_+\subset\sqrt{I}$.

(ii) We have $\operatorname{Proj}B=\emptyset$ if and only if every homogeneous prime ideal $\mathfrak{p}\subset B$ contains B_+, i.e. $\mathrm{V}_+(0)\subset\mathrm{V}_+(B_+)$, i.e. $B_+\subset\sqrt{0}$ by (i).

(iii) Follows from (i) applied to $\mathrm{V}_+(I)\subset\emptyset=\mathrm{V}_+(B_+)$. Finally, $\mathrm{V}_+(J\cap I)=\mathrm{V}_+(J)\cup\mathrm{V}_+(I)=\mathrm{V}_+(J)\cup\mathrm{V}_+(B_+)=\mathrm{V}_+(J)$. □

Remark 3.3.8. Condition (iii) explains why B_+ is called *irrelevant*: the operation $\mathrm{V}_+(-)$ sends it to the empty set, and so does to all radical ideals that contain it. One should keep in mind the case $B=\mathbf{k}[x_0,x_1,\dots,x_n]$, where $B_+=(x_0,x_1,\dots,x_n)$ should 'correspond to the origin'. But there is no origin in $\mathbb{P}^n(\mathbf{k})$!

Notation 3.3.9. Let $f \in B$ be a homogeneous element. We call $\mathrm{D}_+(f) = \operatorname{Proj} B \setminus \mathrm{V}_+(fB)$ a *principal open set* in $\operatorname{Proj} B$. We simply write $\mathrm{V}_+(f)$ instead of $\mathrm{V}_+(fB)$.

Note that, as in the affine case, we have the identity

$$\mathrm{D}_+(fg) = \mathrm{D}_+(f) \cap \mathrm{D}_+(g)$$

for any two homogeneous elements $f, g \in B_+$.

Remark 3.3.10. Note that $\operatorname{Proj} B$ need not be quasicompact. For instance,

$$\operatorname{Proj} \mathbb{Z}[x_1, x_2, \ldots] = \bigcup_{i \geq 1} \mathrm{D}_+(x_i)$$

is an open cover admitting no finite subcover. This may sound counterintuitive: projective things 'should' be compact, affine things should not. But the Zariski topology is funny. When we will have the correct notion of compactness, your intuition will get realigned.

There is a canonical inclusion

$$\varepsilon \colon \operatorname{Proj} B \hookrightarrow \operatorname{Spec} B$$

and for any $f \in B$ one has $\mathrm{V}(f) \cap \operatorname{Proj} B = \bigcap_{0 \leq i \leq e} \mathrm{V}_+(f_i)$ if $f = f_0 + f_1 + \cdots + f_e$ is the homogeneous decomposition of $f \in B$. Then $\mathrm{D}(f) \cap \operatorname{Proj} B = \bigcup_{0 \leq i \leq e} \mathrm{D}_+(f_i)$. This shows that the Zariski topology on $\operatorname{Proj} B$ is induced by the Zariski topology on $\operatorname{Spec} B$ (i.e. it agrees with the subspace topology), and moreover

$$\operatorname{Proj} B \setminus \mathrm{V}_+(I) = \bigcup_{\substack{f \in I \\ f \text{ homogeneous}}} \mathrm{D}_+(f)$$

for any homogeneous ideal $I \subset B$. In particular, the principal opens

$$\left\{ \mathrm{D}_+(f) \subset \operatorname{Proj} B \,\middle|\, f \text{ is homogeneous} \right\}$$

form a base for the Zariski topology. In fact, one can focus only on those $\mathrm{D}_+(f)$ where $f \in B_+$. The reason is the following: suppose $B_+ = (f_i \mid i \in I)$ with f_i homogeneous. Then,

$$\operatorname{Proj} B = \operatorname{Proj} B \setminus \emptyset = \operatorname{Proj} B \setminus \mathrm{V}_+(B_+) = \operatorname{Proj} B \setminus \bigcap_{i \in I} \mathrm{V}_+(f_i) = \bigcup_{i \in I} \mathrm{D}_+(f_i),$$

so that for any homogeneous $g \in B$ we have

$$\mathrm{D}_+(g) = \mathrm{D}_+(g) \cap \operatorname{Proj} B = \mathrm{D}_+(g) \cap \bigcup_{i \in I} \mathrm{D}_+(f_i) = \bigcup_{i \in I} \mathrm{D}_+(g) \cap \mathrm{D}_+(f_i) = \bigcup_{i \in I} \mathrm{D}_+(g f_i),$$

where of course $g f_i \in B_+$ for every $i \in I$. We will thus use

$$\mathscr{B} = \left\{ \mathrm{D}_+(f) \subset \operatorname{Proj} B \,\middle|\, f \in B_+ \text{ is homogeneous} \right\} \tag{3.3.2}$$

as a base of open sets for $\operatorname{Proj} B$.

3.3.2 Structure sheaf on Proj B

Let B be a graded A-algebra as in the previous section. We want to define a sheaf of A-algebras $\mathscr{O}_X$ on $X = \operatorname{Proj} B$, making $(X, \mathscr{O}_X)$ into an A-scheme. Our working definition will be

$$\mathrm{D}_+(f) \longmapsto B_{(f)}, \quad f \in B_+.$$

Here $B_{(f)}$ is the homogeneous principal localisation of Construction 3.3.4, which is an A-algebra by Diagram (3.3.1). In order to make sense of this and verify it is a $\mathcal{B}$-sheaf, we need some (algebraic) preparation.

LEMMA 3.3.11. *Let $f \in B_+$ be homogeneous of degree d. Set $B^{(d)} = \bigoplus_{e\geq 0} B_{de} \subset B$. Then, there is a ring isomorphism*

$$\alpha_f \colon B^{(d)}/(f-1)B^{(d)} \xrightarrow{\sim} B_{(f)}.$$

In particular, if $\deg f = 1$*, we have*

$$\alpha_f \colon B/(f-1)B \xrightarrow{\sim} B_{(f)}.$$

Proof. There is a surjective ring homomorphism $B^{(d)} \twoheadrightarrow B_{(f)}$ defined on homogeneous elements (and then extended additively) by sending $a \in B_{de}$ to a/f^e. This sends $f \in B_d = B_1^{(d)}$ to 1, so descends to a map α_f. The inverse is constructed as follows. Pick $w = z/f^n \in B_{(f)}$, so that z is homogeneous of degree dn. Send w to

$$\text{the image of } z \in B_{dn} \subset B^{(d)} \text{ along } B^{(d)} \twoheadrightarrow B^{(d)}/(f-1)B^{(d)}.$$

It is straightforward to check that this is well-defined, and is the inverse of α_f. □

Terminology 3.3.12. The ring $B^{(d)}$ is called the *d-th Veronese ring* attached to B. It is an A-subalgebra of B.

Example 3.3.13. Let $B = A[x_0, x_1, \ldots, x_n]$ and $f = x_i$, which has degree 1. Then, $B^{(1)} = B$ and Lemma 3.3.11 yields

$$A[x_0,\ldots,x_n]_{(x_i)} \cong A[x_0,\ldots,x_n]/(x_i - 1) \cong A[x_0,\ldots,\widehat{x}_i,\ldots,x_n].$$ [6]

Let $\mathcal{B}$ be the base of the Zariski topology on Proj B as in Equation (3.3.2). Our next goal is to construct a $\mathcal{B}$-sheaf of rings on $X = \operatorname{Proj} B$. By Lemma 2.7.7, this will uniquely extend to a sheaf, which will be denoted $\mathscr{O}_X$.

[6] Notational warning: do not confuse $A[x_0,\ldots,x_n]_{(x_i)}$ (homogeneous localisation) with the localisation of $A[x_0,\ldots,x_n]$ at the prime ideal $(x_i) = x_i A[x_0,\ldots,x_n]$. Same potential problem when $(f) = fB \subset B$ is a prime ideal.

If $f \in B_+$ is homogeneous, we have $\mathrm{D}_+(f) = \mathrm{D}(f) \cap \operatorname{Proj} B$. We next prove a few crucial properties of the composition

$$\begin{aligned} \theta\colon \mathrm{D}_+(f) \hookrightarrow \mathrm{D}(f) = \operatorname{Spec} B_f &\longrightarrow \operatorname{Spec} B_{(f)} \\ \mathfrak{p} &\longmapsto \mathfrak{p} B_f \cap B_{(f)}. \end{aligned}$$

LEMMA 3.3.14 ((De)homogenisation). *Let $f \in B_+$ be a homogeneous element.*

(i) $\theta\colon \mathrm{D}_+(f) \to \operatorname{Spec} B_{(f)}$ *is a homeomorphism.*

(ii) *If* $\mathrm{D}_+(g) \subset \mathrm{D}_+(f)$ *and* $\alpha = g^{\deg f}/f^{\deg g} \in B_{(f)}$, *then* $\theta(\mathrm{D}_+(g)) = \mathrm{D}(\alpha)$.

(iii) *If g and α are as in (ii), then there is a canonical homomorphism $B_{(f)} \to B_{(g)}$ inducing a ring isomorphism*

$$(B_{(f)})_\alpha \xrightarrow{\sim} B_{(g)}.$$

Proof. We proceed step by step.

(i)–(ii) The map θ is continuous, as we have already observed that the Zariski topology on $\operatorname{Proj} B$ is induced by that of $\operatorname{Spec} B$. We first need to prove it is bijective. Then, proving (ii) will show that it is open, hence a homeomorphism.

θ is injective: Suppose $\mathfrak{p} B_f \cap B_{(f)} = \mathfrak{p}' B_f \cap B_{(f)}$ for $\mathfrak{p}$, $\mathfrak{p}'$ two elements of $\mathrm{D}_+(f)$. Fix a homogeneous generator $b \in \mathfrak{p}$, so that $b^{\deg f}/f^{\deg b} \in \mathfrak{p} B_f \cap B_{(f)} \subset \mathfrak{p}' B_f$. Then $b^{\deg f} \in \mathfrak{p}'$, and since $\mathfrak{p}'$ is prime we deduce $b \in \mathfrak{p}'$. Hence $\mathfrak{p} \subset \mathfrak{p}'$. Exchanging the roles of $\mathfrak{p}$ and $\mathfrak{p}'$ we obtain equality.

θ is surjective: Fix $\mathfrak{q} \in \operatorname{Spec} B_{(f)}$. Define a homogeneous ideal $\mathfrak{p} \subset B$ by declaring that $x \in B_d$ lies in $\mathfrak{p}$ if and only if $x^{\deg f}/f^d \in \mathfrak{q} \subset B_{(f)}$. It is homogeneous by construction, and it is prime as well. Indeed, pick two homogeneous elements $x \in B_d$ and $y \in B_e$ such that $xy \in \mathfrak{p}_{d+e}$. This means that

$$\frac{(xy)^{\deg f}}{f^{d+e}} = \frac{x^{\deg f}}{f^d}\frac{y^{\deg f}}{f^e} \in \mathfrak{q}.$$

Then either $x^{\deg f}/f^d$ or $y^{\deg f}/f^e$ lies in $\mathfrak{q}$, because $\mathfrak{q}$ is prime. But by definition this means that either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. Moreover $f \notin \mathfrak{p}$, for otherwise we would have $1 \in \mathfrak{q}$. Finally, $\mathfrak{p} B_f \cap B_{(f)} = \mathfrak{q}$, which shows surjectivity.

(ii) holds: Fix $\mathfrak{p} \in \mathrm{D}_+(f)$. It is clear that $g \in \mathfrak{p}$ if and only if $\alpha \in \mathfrak{p} B_f \cap B_{(f)}$.

(iii) (sketch): If $\mathrm{D}_+(g) \subset \mathrm{D}_+(f)$, by Lemma 3.3.7(i) we have $gB = gB \cap B_+ \subset \sqrt{fB}$, i.e. $g^r = fb$ for some $b \in B$ and some $r > 0$. We may assume b is homogeneous by replacing it with its component of degree $r \cdot \deg g - \deg f$. Then $B_{(f)} \to B_{(g)}$ is defined by sending $a/f^n \mapsto ab^n/g^{rn}$. We are then in the situation

$$\begin{tikzcd} B_{(f)} \arrow[r, "\ell"] \arrow[d] & (B_{(f)})_\alpha \arrow[dl, dotted] \\ B_{(g)} & \end{tikzcd}$$

where ℓ is the (classical) localisation at $\alpha \in B_{(f)}$. Since α is invertible in $B_{(g)}$, the dotted arrow can be completed to a solid one. We leave it as an exercise to show that this map is an isomorphism. □

We are ready for the main theorem of this section.

THEOREM 3.3.15. *Let B be a graded A-algebra. Then $X = \operatorname{Proj} B$ is canonically an A-scheme, with the property that the principal open subset $\mathrm{D}_+(f) \subset X$ is affine and isomorphic to $\operatorname{Spec} B_{(f)}$, for any homogeneous element $f \in B_+$. Moreover, the local ring $\mathcal{O}_{X,x}$ at a point $x \in X$ corresponding to a homogeneous prime $\mathfrak{p} \subset B$ is canonically isomorphic to the homogeneous localisation $B_{(\mathfrak{p})}$.*

Proof. For $f \in B_+$ a homogeneous element, define

$$\mathcal{O}_X(\mathrm{D}_+(f)) = B_{(f)}. \tag{3.3.3}$$

We first confirm that this prescription defines a $\mathcal{B}$-presheaf on X. This, in fact, follows at once by Lemma 3.3.14(iii), which shows that a canonical restriction map $\mathcal{O}_X(\mathrm{D}_+(f)) \to \mathcal{O}_X(\mathrm{D}_+(g))$ exists whenever $\mathrm{D}_+(g) \subset \mathrm{D}_+(f)$, and that $B_{(f)}$ and $B_{(g)}$ are canonically isomorphic as soon as $\mathrm{D}_+(g) = \mathrm{D}_+(f)$.

In fact, (3.3.3) defines a $\mathcal{B}$-*sheaf* on X. This can be confirmed via the equaliser sequence. What we need to show is that for any $f \in B_+$ and any open cover $\mathrm{D}_+(f) = \bigcup_{i \in I} \mathrm{D}_+(f_i)$, the sequence

$$\mathcal{O}_X(\mathrm{D}_+(f)) \longrightarrow \prod_{i \in I} \mathcal{O}_X(\mathrm{D}_+(f_i)) \rightrightarrows \prod_{(i,j) \in I \times I} \mathcal{O}_X(\mathrm{D}_+(f_i f_j))$$

is an equaliser diagram in the category of A-algebras. This can be rewritten as the sequence

$$B_{(f)} \longrightarrow \prod_{i \in I} B_{(f_i)} \rightrightarrows \prod_{(i,j) \in I \times I} B_{(f_i f_j)}. \tag{3.3.4}$$

Let us define

$$\alpha_i = f_i^{\deg f} / f^{\deg f_i}, \quad \alpha_{ij} = (f_i f_j)^{\deg f} / f^{\deg f_i f_j}$$

in $B_{(f)}$. Then, we know that

$$\theta(\mathrm{D}_+(f_i)) = \mathrm{D}(\alpha_i), \quad \theta(\mathrm{D}_+(f_i f_j)) = \mathrm{D}(\alpha_{ij})$$

by Lemma 3.3.14(ii). Since $\theta\colon \mathrm{D}_+(f) \to \operatorname{Spec} B_{(f)}$ is a homeomorphism, we have an open covering

$$\operatorname{Spec} B_{(f)} = \bigcup_{i\in I} \theta(\mathrm{D}_+(f_i)) = \bigcup_{i\in I} \mathrm{D}(\alpha_i).$$

In particular, we have an equaliser sequence

$$B_{(f)} \longrightarrow \prod_{i\in I} (B_{(f)})_{\alpha_i} \rightrightarrows \prod_{(i,j)\in I\times I} (B_{(f)})_{\alpha_{ij}}$$

in the category of A-algebras, because $\mathscr{O}_{\operatorname{Spec} B_{(f)}}$ is a sheaf of A-algebras. But thanks to Lemma 3.3.14(iii) this is precisely the sequence (3.3.4). Therefore $\mathscr{O}_X$ is a $\mathcal{B}$-sheaf.

Let $\mathscr{O}_X$ denote the induced sheaf of A-algebras. The stalks are local rings. Indeed, if $x \in X$ corresponds to a homogeneous prime ideal $\mathfrak{p} \subset B$, one has a canonical isomorphism

$$\mathscr{O}_{X,x} = \varinjlim_{\substack{f \text{ homogeneous} \\ f\notin\mathfrak{p}}} B_{(f)} \xrightarrow{\sim} B_{(\mathfrak{p})}.$$

The proof is identical to the one we gave for Spec (cf. Theorem 3.1.28(c)).

It follows that the pair $(X, \mathscr{O}_X)$ defines a locally ringed space. Now, the homeomorphism $\theta\colon \mathrm{D}_+(f) \to \operatorname{Spec} B_{(f)}$ extends to an isomorphism of locally ringed spaces

$$(\theta, \theta^\#)\colon (\mathrm{D}_+(f), \mathscr{O}_X|_{\mathrm{D}_+(f)}) \xrightarrow{\sim} (\operatorname{Spec} B_{(f)}, \mathscr{O}_{\operatorname{Spec} B_{(f)}})$$

which shows that $(X, \mathscr{O}_X)$ is a scheme with the sought after property. In a little more detail, to construct $\theta^\#\colon \mathscr{O}_{\operatorname{Spec} B_{(f)}} \to \theta_*(\mathscr{O}_X|_{\mathrm{D}_+(f)})$, we take a principal open $\mathrm{D}(\alpha) \subset \operatorname{Spec} B_{(f)}$ and since $\alpha \in B_{(f)}$ we may write it as $g^r/f^{\deg g}$, where $r = \deg f$. Therefore we can apply Lemma 3.3.14(iii), which gives the isomorphism

$$(B_{(f)})_\alpha \xrightarrow{\sim} B_{(g)}.$$

This is our $\theta^\#(\mathrm{D}(\alpha))$, which makes sense since

$$\theta_*(\mathscr{O}_X|_{\mathrm{D}_+(f)})(\mathrm{D}(\alpha)) = \mathscr{O}_X|_{\mathrm{D}_+(f)}(\theta^{-1}\mathrm{D}(\alpha)) = \mathscr{O}_X|_{\mathrm{D}_+(f)}(\mathrm{D}_+(g)) = \mathscr{O}_X(\mathrm{D}_+(g)) = B_{(g)}.$$

Finally, the A-scheme structure of X is given by the fact that each $B_{(f)}$ is naturally an A-algebra, combined with Corollary 3.2.9. □

Example 3.3.16. Let $B = A[x_0, x_1, \ldots, x_n]$, with the usual grading ($\deg x_i = 1$ for all i). Then

$$\operatorname{Proj} A[x_0, x_1, \ldots, x_n] = \mathbb{P}^n_A$$

where projective n-space over A was defined via glueing in Section 3.2.3. The structure morphism $\mathbb{P}^n_A \to \operatorname{Spec} A$ allows one to think of $\mathbb{P}^n_A$ as a 'family of projective spaces' parametrised by the points of $\operatorname{Spec} A$.

Example 3.3.17. We have $\mathbb{P}^0_A = \operatorname{Proj} A[x] \xrightarrow{\sim} \operatorname{Spec} A$. Indeed,

$$\operatorname{Proj} A[x] = \mathrm{D}_+(x) = \operatorname{Spec} A[x]_{(x)} = \operatorname{Spec} A[x]/(x-1) \xrightarrow{\sim} \operatorname{Spec} A,$$

using Example 3.3.13 for the identification $A[x]_{(x)} = A[x]/(x-1)$.

3.3.3 Proj is not a functor

One may think that, in analogy with the case of the affine spectrum, sending $B \mapsto \operatorname{Proj} B$ could be a functor from graded A-algebras to schemes. This is not the case. In this section we discuss why this fails and to what extend it can be remedied.

Caution 3.3.18. Proj is *not a functor*! It is not true that a morphism of graded A-algebras $\phi\colon B \to C$ induces a morphism of A-schemes $\operatorname{Proj} C \to \operatorname{Proj} B$ sending $\mathfrak{p} \mapsto \phi^{-1}\mathfrak{p}$. The problem is that

$$\mathfrak{p} \not\supset C_+ \text{ does not imply } \phi^{-1}\mathfrak{p} \not\supset B_+$$

for a homogeneous prime ideal $\mathfrak{p} \subset C$. See, however, Proposition 3.3.20 for the closest to a functor one can get.

Example 3.3.19. If $\phi\colon B = \mathbf{k}[x_0, x_1] \hookrightarrow \mathbf{k}[x_0, x_1, x_2] = C$ is the natural inclusion, then $C_+ = (x_0, x_1, x_2) \not\subset \mathfrak{p} = (x_0, x_1) \in \operatorname{Proj} C$, but $\phi^{-1}\mathfrak{p} = (x_0, x_1) = B_+$. This is the only 'problematic' point.

PROPOSITION 3.3.20. *Let $\phi\colon B \to C$ be a graded morphism of graded A-algebras. Then there is a canonical morphism of schemes*

$$f\colon \operatorname{Proj} C \setminus \mathrm{V}_+(B_+C) \longrightarrow \operatorname{Proj} B, \quad \mathfrak{p} \mapsto \phi^{-1}\mathfrak{p}$$

such that for any homogeneous $h \in B_+$ we have $f^{-1}(\mathrm{D}_+(h)) = \mathrm{D}_+(\phi(h))$, and the induced morphism $\mathrm{D}_+(\phi(h)) \to \mathrm{D}_+(h)$ of affine schemes corresponds to the canonical restriction $B_{(h)} \to C_{(\phi(h))}$.

Proof. If $\mathfrak{p} \subset C$ is a homogeneous prime ideal, then $\phi^{-1}\mathfrak{p} \subset B$ is a homogeneous prime ideal. We have

$$B_+ \not\subset \phi^{-1}\mathfrak{p} \iff B_+C \not\subset \phi(\phi^{-1}\mathfrak{p}) \subset \mathfrak{p},$$

therefore the association $\mathfrak{p} \mapsto \phi^{-1}\mathfrak{p}$ is well-defined precisely on the subset $\operatorname{Proj} C \setminus \mathrm{V}_+(B_+C) \subset \operatorname{Proj} C$, which, being open, has a canonical scheme structure inherited from $\operatorname{Proj} C$. Note that f is continuous, since the Zariski topology is induced by that of the affine spectrum, and f is the restriction of the same map $\mathfrak{p} \mapsto \phi^{-1}\mathfrak{p}$ going from $\operatorname{Spec} C$ to $\operatorname{Spec} B$.

We thus only need to construct the morphism at the level of structure sheaves. Since morphisms to a fixed target form a sheaf (cf. Lemma 3.2.6), it is enough to define the morphism on a base of open subsets of $\operatorname{Proj} C$. Consider principal open subsets

$$\mathrm{D}_+(h) \subset \operatorname{Proj} B, \quad \mathrm{D}_+(\phi(h)) \subset \operatorname{Proj} C, \quad h \in B_+.$$

As h runs in B_+, the opens $\mathrm{D}_+(h) \subset \operatorname{Proj} B$ cover the target $\operatorname{Proj} B$, and $\mathrm{D}_+(\phi(h)) \subset \operatorname{Proj} C$ cover $\operatorname{Proj} C \setminus \mathrm{V}_+(B_+ C)$. In the commutative diagram

$$\begin{tikzcd}
B \arrow[r, "\phi"] \arrow[d, "\ell_h"'] & C \arrow[d, "\ell_{\phi(h)}"] \\
B_h \arrow[r, "\phi_h"] & C_{\phi(h)} \\
B_{(h)} \arrow[u, hook] \arrow[r, dotted] & C_{(\phi(h))} \arrow[u, hook]
\end{tikzcd}$$

the map ϕ_h is a graded morphism, therefore it preserves the degree 0 pieces, which induces $B_{(h)} \to C_{(\phi(h))}$. Taking Spec of this map recovers precisely the morphism of affine schemes $f_h \colon f^{-1}\mathrm{D}_+(h) = \mathrm{D}_+(\phi(h)) \to \mathrm{D}_+(h)$. These morphisms agree on the intersections (reason: the map $B_{hk} \to C_{\phi(hk)}$ agrees with both the localisation of $B_h \to C_{\phi(h)}$ and the localisation of $B_k \to C_{\phi(k)}$), and therefore glue to a global morphism f. □

Example 3.3.21 (Projection from a point)**.** In the situation of Example 3.3.19, the morphism we obtain applying Proposition 3.3.20 is

$$\mathbb{P}^2_{\mathbf{k}} \setminus \mathrm{V}_+(x_0, x_1) = \mathbb{P}^2_{\mathbf{k}} \setminus \{(0:0:1)\} \to \mathbb{P}^1_{\mathbf{k}}.$$

More generally, we have a morphism

$$\mathbb{P}^{n+1}_{\mathbf{k}} \setminus \{(0:\cdots:0:1)\} \to \mathbb{P}^n_{\mathbf{k}}$$

defined by the inclusion of graded $\mathbf{k}$-algebras $\mathbf{k}[x_0,\ldots,x_n] \hookrightarrow \mathbf{k}[x_0,\ldots,x_n,x_{n+1}]$. This is called *projection from a point*. It sends the closed point $(a_0:\cdots:a_n:a_{n+1})$ to the closed point $(a_0:\cdots:a_n)$.

Example 3.3.22. Consider the graded morphism $B = \mathbf{k}[x_0,x_1] \to \mathbf{k}[y_0,y_1] = C$ sending $x_i \mapsto y_i^n$ for $i = 0,1$. In this case, $\mathrm{V}_+(B_+C) = \mathrm{V}_+(y_0^n, y_1^n) = \mathrm{V}_+(y_0,y_1) = \emptyset$, so we get a well-defined morphism $\mathbb{P}^1_{\mathbf{k}} \to \mathbb{P}^1_{\mathbf{k}}$, which on closed points sends $(a_0:a_1) \mapsto (a_0^n : a_1^n)$.

Projective varieties

If $\phi\colon B \to C$ is a *surjective* graded morphism of graded A-algebras, we have $C_+ = \phi(B_+) = B_+C$, hence by Proposition 3.3.20 there is global A-morphism

$$f\colon \operatorname{Proj} C \to \operatorname{Proj} B,$$

locally given by the closed immersions

$$\operatorname{Spec} C_{(\phi(f))} \hookrightarrow \operatorname{Spec} B_{(f)}$$

induced by the natural surjections $B_{(f)} \twoheadrightarrow C_{(\phi(f))}$. Therefore f is a closed immersion of A-schemes. A special case of this will be recorded as the next corollary.

COROLLARY 3.3.23. *Let $I \subset B = A[x_0, x_1, \ldots, x_n]$ be a homogeneous ideal, and let $\phi : B \to B/I$ be the canonical surjection. Then $\phi(B_+) = (B/I)_+$. Therefore, the* Proj *construction yields a closed immersion*

$$\operatorname{Proj} B/I \longhookrightarrow \mathbb{P}^n_A = \operatorname{Proj} B$$

over $\operatorname{Spec} A$*, with image homeomorphic to* $\mathrm{V}_+(I) \subset \operatorname{Proj} B$.

Terminology 3.3.24. Algebras of the form $A[x_0, x_1, \ldots, x_n]/I$ as in Corollary 3.3.23 are called *homogeneous A-algebras*. A scheme that is isomorphic to a closed subscheme of $\mathbb{P}^n_A$, for some $n \geq 0$, is called a *projective scheme over A*.

The converse of Corollary 3.3.23 is the content of the following exercise.

Exercise 3.3.25. Let $X \hookrightarrow \mathbb{P}^n_A$ be a closed immersion. Show that there exists a homogeneous ideal $I \subset B = A[x_0, x_1, \ldots, x_n]$ such that X is isomorphic to $\operatorname{Proj} B/I$. Show, by exhibiting an example, that I is not unique with this property (and note the difference with the affine case, cf. Equation (3.2.1)).

Important Definition 3.3.1 (Projective variety)**.** A *projective variety* over a field $\mathbb{F}$ is a projective scheme over $\mathbb{F}$, i.e. a closed subscheme of $\mathbb{P}^n_{\mathbb{F}}$ for some n. An algebraic variety is called *quasiprojective* (resp. *quasiaffine*) if it admits an open immersion into a projective (resp. affine) variety.

A morphism $X \to Y = \operatorname{Spec} A$ is is said to be *projective* if it factors as a closed immersion $X \hookrightarrow \mathbb{P}^n_A$ followed by the canonical projection $\mathbb{P}^n_A \to Y$.

Remark 3.3.26. Sanity check: projective varieties are algebraic varieties in the sense of Important Definition 3.2.1, by the observation (cf. Remark 3.2.15) that $\mathbb{P}^n_{\mathbb{F}}$ is quasicompact.

Terminology 3.3.27. Let $B = \mathbf{k}[x_0, \ldots, x_n]$. Fix a homogeneous polynomial of degree d. The closed subscheme $X = \operatorname{Proj} B/(f) \hookrightarrow \mathbb{P}^n_{\mathbf{k}}$ is called a hypersurface of degree d. If $d = 1$ (resp. $d = 2, 3, 4, 5, \ldots$), we say X is a *hyperplane* (resp. a *quadric*, a *cubic*, a *quartic*, a *quintic*... hypersurface).

Caution 3.3.28. It is not true that an isomorphism of schemes $\operatorname{Proj} B \cong \operatorname{Proj} C$ yields an isomorphism of graded A-algebras. For instance, one has $\operatorname{Proj} B \cong \operatorname{Proj} B^{(d)}$ for any $d \geq 1$, but B is not isomorphic to $B^{(d)}$ if $d > 1$.

Exercise 3.3.29. Show that there is no nonconstant morphism $\mathbb{P}^n_{\mathbf{k}} \to \mathbb{P}^m_{\mathbf{k}}$ if $m < n$.

3.3.4 Examples of projective schemes

Let $\mathbf{k}$ be a field. First of all, let us clarify the relationship between the scheme $\mathbb{P}^n_{\mathbf{k}}$ and classical projective space $\mathbb{P}^n(\mathbf{k}) = (\mathbf{k}^{n+1} \setminus 0)/\mathbf{k}^\times$. There is a set-theoretic map

$$\mathbb{P}^n(\mathbf{k}) \longrightarrow \mathbb{P}^n_{\mathbf{k}} = \operatorname{Proj} \mathbf{k}[x_0, x_1, \ldots, x_n] \tag{3.3.5}$$

sending a point $(a_0 : a_1 : \cdots : a_n)$ to the homogeneous prime ideal

$$(a_i x_j - a_j x_i \mid 0 \le i, j \le n) \subset \mathbf{k}[x_0, x_1, \ldots, x_n]. \tag{3.3.6}$$

Note that such ideal can be viewed as generated by the 2-minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_n \\ a_0 & a_1 & \cdots & a_n \end{pmatrix}.$$

The map (3.3.5) is a bijection onto the set of closed points of $\mathbb{P}^n_{\mathbf{k}}$. The points of the form (3.3.6) will be referred to as the *classical points* of $\mathbb{P}^n_{\mathbf{k}}$.

Terminology 3.3.30. If we set $\mathbb{P}^n_{\mathbf{k}} = \operatorname{Proj} \mathbf{k}[x_0, x_1, \ldots, x_n]$, we call $(x_0, x_1, \ldots, x_n)$ the homogeneous coordinates of $\mathbb{P}^n_{\mathbf{k}}$. The closed point of $\mathbb{P}^n_{\mathbf{k}}$ corresponding to (3.3.6) is denoted $(a_0 : a_1 : \cdots : a_n)$. For instance, the 'classical point' $(0 : 0 : \cdots : 1) \in \mathbb{P}^n(\mathbf{k})$ corresponds to the ideal $(x_0, x_1, \ldots, x_{n-1}) \in \mathbb{P}^n_{\mathbf{k}}$.

Remark 3.3.31. In the case of $\mathbb{P}^1_{\mathbf{k}}$, just as for $\mathbb{A}^1_{\mathbf{k}}$, there is only one nonclassical point, namely the point ξ corresponding to the trivial ideal $(0) \subset \mathbf{k}[x_0, x_1]$. It is the generic point of $\mathbb{P}^1_{\mathbf{k}}$, and one has $\kappa(\xi) = \mathbf{k}(t)$.

Remark 3.3.32. If $\mathfrak{p} \in X = \operatorname{Proj} B$, the stalk of the structure sheaf $\mathscr{O}_X$ at $\mathfrak{p}$ is the homogeneous localisation $B_{(\mathfrak{p})}$. But, if $U = \operatorname{Spec} R \subset X$ is any affine open neighbourhood of $\mathfrak{p}$, we clearly have

$$R_{\mathfrak{p}} = \mathscr{O}_{X,\mathfrak{p}} = B_{(\mathfrak{p})}.$$

However, R is a different ring, so we have to understand what ideal $\mathfrak{p} \subset B$ becomes when viewed in the ring R. We explain this via an example. Consider for instance the (closed) 'coordinate point'

$$z_i = (0 : \cdots : 0 : 1 : 0 : \cdots : 0) \in \mathbb{P}^n_{\mathbf{k}},$$

with 1 sitting in the $(i+1)$-st slot, corresponding to the homogeneous prime ideal

$$\mathfrak{p}_i = (x_0, \ldots, \widehat{x}_i, \ldots, x_n) \subset \mathbf{k}[x_0, x_1, \ldots, x_n].$$

Then, we have $z_i \in \mathrm{D}_+(x_i) = \operatorname{Spec} \mathbf{k}[x_0, x_1, \ldots, x_n]_{(x_i)}$, and

$$\mathbf{k}[x_0, x_1, \ldots, x_n]_{(x_i)} \cong \mathbf{k}[x_0, \ldots, \widehat{x}_i, \ldots, x_n]$$

by Lemma 3.3.11. Under this identification, z_i corresponds to the origin in

$$\operatorname{Spec}\mathbf{k}[x_0,\dots,\widehat{x}_i,\dots,x_n],$$

which in turn corresponds to the ideal $\mathfrak{q}_i=(x_0,\dots,\widehat{x}_i,\dots,x_n)$. Therefore

$$\mathscr{O}_{\mathbb{P}^n_{\mathbf{k}},z_i}=\mathbf{k}[x_0,x_1,\dots,x_n]_{(\mathfrak{p}_i)}=\mathbf{k}[x_0,\dots,\widehat{x}_i,\dots,x_n]_{\mathfrak{q}_i},$$

which consists of fractions f/g of polynomials in n variables, where $g(0,\dots,0)\neq 0\in\mathbf{k}$.

Terminology 3.3.33. Let V be a $\mathbf{k}$-vector space of dimension $n+1$. The symmetric algebra $\operatorname{Sym}V^\vee$ is the polynomial ring $\mathbf{k}[x_0,x_1,\dots,x_n]$ (polynomial functions on V), and one defines

$$\mathbb{P}(V)=\operatorname{Proj}\operatorname{Sym}V^\vee.$$

This is the projective space attached to V, whose closed points correspond to lines in V. It is isomorphic to $\mathbb{P}^n_{\mathbf{k}}$.

Example 3.3.34 (Plane curves)**.** Consider the same polynomial $xy-z^2\in\mathbb{C}[x,y,z]$ of Example 3.1.77. Note that it is homogeneous. Now, its vanishing locus $\mathrm{V}_+(xy-z^2)$ is the topological image of a closed immersion into $\mathbb{P}^2_{\mathbb{C}}=\operatorname{Proj}\mathbb{C}[x,y,z]$, namely the morphism

$$\operatorname{Proj}\mathbb{C}[x,y,z]/(xy-z^2)\hookrightarrow\mathbb{P}^2_{\mathbb{C}}$$

induced by the surjection $\mathbb{C}[x,y,z]\twoheadrightarrow\mathbb{C}[x,y,z]/(xy-z^2)$. In general, the vanishing scheme of a degree 2 homogeneous polynomial $f\in\mathbb{C}[x,y,z]$ is called a *plane conic*. The vanishing scheme of an arbitrary homogeneous polynomial of degree d is called a *plane curve of degree d*.

Exercise 3.3.35. Show that all nondegenerate plane conics $\operatorname{Proj}\mathbb{C}[x,y,z]/(f)\hookrightarrow\mathbb{P}^2_{\mathbb{C}}$ are isomorphic to $\mathbb{P}^1_{\mathbb{C}}$ (**Hint**: Show that you can reduce to the normal form $f=xy-z^2$, or $f=x^2+y^2+z^2$ if you prefer; then show that this particular plane conic is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$). In fact, you may want to show this over an arbitrary algebraically closed field $\mathbf{k}$ of characteristic different from 2.

Example 3.3.36. Let B be a graded A-algebra. Fix $d>0$. Consider the Veronese ring $B^{(d)}=\bigoplus_{e\geq 0}B^{(d)}_e$, defined by $B^{(d)}_e=B_{de}$. We have an inclusion

$$\phi\colon B^{(d)}\hookrightarrow B,$$

which is a graded homomorphism of graded A-algebras. We claim that ϕ induces an A-morphism

$$v_d\colon\operatorname{Proj}B\longrightarrow\operatorname{Proj}B^{(d)}.$$

We have

$$B_+^{(d)} = \bigoplus_{e>0} B_{de}.$$

To prove that v_d is well-defined, fix a point $\mathfrak{p} \in \operatorname{Proj} B$. Assume, by contradiction, that $\phi^{-1}\mathfrak{p} \supset B_+^{(d)}$. Then $\mathfrak{p} \supset \phi(\phi^{-1}\mathfrak{p}) \supset \phi(B_+^{(d)}) = B_+ \cap \phi(B^{(d)})$. If $a \in B_+$, then $a^d \in B_+ \cap \phi(B^{(d)}) \subset \mathfrak{p}$, which implies $a \in \mathfrak{p}$, and in turn $B_+ \subset \mathfrak{p}$, whence a contradiction. Thus v_d is globally well-defined by Proposition 3.3.20.

PROPOSITION 3.3.37. *Let B be a graded A-algebra, and fix $d > 0$. Then the morphism*

$$v_d\colon \operatorname{Proj} B \longrightarrow \operatorname{Proj} B^{(d)}$$

is an isomorphism of A-schemes.

Proof. We first confirm that v_d, sending $\mathfrak{p} \mapsto \mathfrak{p} \cap B^{(d)}$, is a homeomorphism, and then we show that it is an isomorphism on an open cover of the source. To prove it is a homeomorphism, we prove it is a continuous and open bijection. Continuity is automatic given that v_d is in fact a morphism, by Example 3.3.36.

To see that v_d is injective, assume $\mathfrak{p} \cap B^{(d)} = \mathfrak{p}' \cap B^{(d)}$ for $\mathfrak{p}, \mathfrak{p}' \in \operatorname{Proj} B$. If $a \in B$ is a homogeneous element, we have $a \in \mathfrak{p}$ if and only if $a^d \in \mathfrak{p}$, if and only if $a^d \in \mathfrak{p}'$, if and only if $a \in \mathfrak{p}'$, thus $\mathfrak{p} = \mathfrak{p}'$.

To see that v_d is surjective, fix $\mathfrak{q} \in \operatorname{Proj} B^{(d)}$ and define

$$\mathfrak{p} = \bigoplus_{e \geq 0} \left\{ x \in B_e \,\middle|\, x^d \in \mathfrak{q} \right\} \subset B.$$

It is immediate to check that $\mathfrak{p}$ is a homogeneous prime ideal in B and that it satisfies $\mathfrak{p} \cap B^{(d)} = \mathfrak{q}$, thus v_d is bijective on points.

To see that v_d is a homeomorphism, it is now enough to observe that

$$v_d(\mathrm{D}_+(f)) = \mathrm{D}_+(f^d),$$

for any homogeneous element $f \in B_+$. This follows from the observation that, for any $\mathfrak{p} \in \operatorname{Proj} B$, one has $f \in \mathfrak{p}$ if and only if $f^d \in \mathfrak{p} \cap B^{(d)}$.

The homeomorphisms

$$\mathrm{D}_+(f) \longrightarrow \mathrm{D}_+(f^d)$$

extend to morphisms of affine schemes

$$\operatorname{Spec} B_{(f)} \longrightarrow \operatorname{Spec} B^{(d)}_{(f^d)},$$

each corresponding to the canonical A-algebra homomorphism

$$\psi_f\colon B^{(d)}_{(f^d)} \longrightarrow B_{(f)}. \tag{3.3.7}$$

Such morphism is an isomorphism. Indeed, it is clearly injective, and to check it is surjective we simply observe that if $a/f^n \in B_{(f)}$, then a^d/f^{nd} is a preimage along ψ_f. The isomorphisms (3.3.7) are compatible with restrictions to smaller principal opens, thus v_d is globally an isomorphism. □

Example 3.3.38 (Veronese embedding)**.** Fix a pair of positive integers n, d. Set $B = \mathbf{k}[x_0,\ldots,x_n]$. We construct a closed immersion

$$v_{d,n}\colon \mathbb{P}^n_{\mathbf{k}} \hookrightarrow \mathbb{P}^N_{\mathbf{k}}, \quad N = \binom{n+d}{d} - 1$$

as follows. Let $\{m_0,\ldots,m_N\} \subset B_d$ be linearly independent monomials of degree d in the variables $x_0,\ldots,x_n$. In other words, fix the standard monomial basis of B_d. Fix indeterminates $w_0,\ldots,w_N$. Define a $\mathbf{k}$-algebra homomorphism

$$\mathbf{k}[w_0,\ldots,w_N] \longrightarrow B, \quad w_i \mapsto m_i.$$

This morphism has image the Veronese subalgebra $B^{(d)} \hookrightarrow B$, therefore the factorisation

$$\mathbf{k}[w_0,\ldots,w_N] \twoheadrightarrow B^{(d)} \hookrightarrow B$$

induces

$$v_{d,n}\colon \mathbb{P}^n_{\mathbf{k}} \xrightarrow{\sim} \operatorname{Proj} B^{(d)} \hookrightarrow \mathbb{P}^N_{\mathbf{k}}.$$

This is the d-th *Veronese embedding* of $\mathbb{P}^n_{\mathbf{k}}$. It is also called the d-*uple embedding* of $\mathbb{P}^n_{\mathbf{k}}$. The next two examples are important special cases.

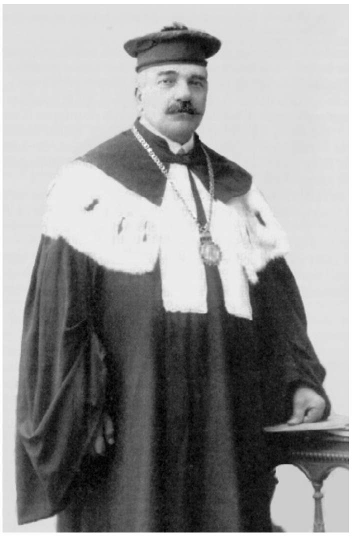

Figure 3.14: Giuseppe Veronese (1854–1917).

Example 3.3.39 (Rational normal curve)**.** Let $d > 0$ be an integer. The d-th Veronese embedding of $\mathbb{P}^1_{\mathbf{k}}$ is the map

$$\begin{aligned} \mathbb{P}^1_{\mathbf{k}} &\xhookrightarrow{\;\;v_{d,1}\;\;} \mathbb{P}^d_{\mathbf{k}} \\ (u:v) &\longmapsto (u^d : u^{d-1}v : \cdots : uv^{d-1} : v^d) \end{aligned}$$

defined by the map

$$\begin{aligned} \mathbf{k}[x_0, x_1, \ldots, x_d] &\longrightarrow \mathbf{k}[u,v] \\ x_i &\longmapsto u^{d-i}v^i. \end{aligned}$$

The image of this closed immersion is called the *rational normal curve* in $\mathbb{P}^d_{\mathbf{k}}$. If $d = 3$, the image of $\mathbb{P}^1_{\mathbf{k}} \hookrightarrow \mathbb{P}^3_{\mathbf{k}}$ is called a *twisted cubic in* $\mathbb{P}^3_{\mathbf{k}}$. Observe that the closed immersion defined above is cut out by the ideal J_d generated by the 2-minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_{d-1} \\ x_1 & x_2 & \cdots & x_d \end{pmatrix}.$$

For instance, the twisted cubic is cut out by the ideal

$$J_3 = (x_0x_3 - x_1x_2, x_1^2 - x_0x_2, x_2^2 - x_1x_3) \subset \mathbf{k}[x_0, x_1, x_2, x_3].$$

Example 3.3.40. Set $d = n = 2$. Then we have a factorisation

$$v_{2,2} \colon \mathbb{P}^2_{\mathbf{k}} \xrightarrow{\;\sim\;} S \hookrightarrow \mathbb{P}^5_{\mathbf{k}}$$

where S is called the *Veronese surface*. Consider the symmetric matrix

$$M = \begin{pmatrix} w_0 & w_1 & w_3 \\ w_1 & w_2 & w_4 \\ w_3 & w_4 & w_5 \end{pmatrix}.$$

One may show that, as schemes,

$$S = \operatorname{Proj} \mathbf{k}[w_0, \ldots, w_5]/J,$$

where J is the ideal generated by the 2-minors of M.

Example 3.3.41. Let B be a graded A-algebra with irrelevant ideal $B_+ \subset B$. Then, sending $\mathfrak{p} \mapsto \mathfrak{p}^{\mathrm{h}}$ defines a morphism of schemes

$$\operatorname{Spec} B \setminus \mathrm{V}(B_+) \to \operatorname{Proj} B.$$

For instance, we get a 'projection'

$$\mathbb{A}^{n+1}_A \setminus \{0\} \to \mathbb{P}^n_A.$$

If $\mathbf{k}$ is a field, this morphism is precisely, on closed points, the *quotient* by the scaling action of $\mathbf{k}^\times$ on $\mathbf{k}^{n+1} \setminus \{0\}$.

Projective closure

If $A = \mathbf{k}[y_1,\ldots,y_n]$ and $B = \mathbf{k}[x_0,x_1,\ldots,x_n]$, we have (de)homogeneisation maps

$$\alpha\colon B^{\mathrm{h}} \longrightarrow A, \quad g \mapsto g(1,y_1,\ldots,y_n)$$

and

$$\beta\colon A \longrightarrow B^{\mathrm{h}}, \quad f \mapsto x_0^{\deg f} f(x_1/x_0,\ldots,x_n/x_0).$$

Clearly one can choose any other variable x_i and perform the same procedure. The open immersion

$$\mathbb{A}^n_{\mathbf{k}} = \mathrm{D}_+(x_0) \xhookrightarrow{\iota_0} \mathbb{P}^n_{\mathbf{k}}$$

allows one to turn an affine variety $Y = \operatorname{Spec} A/I \subset \mathbb{A}^n_{\mathbf{k}}$ into a projective variety $\overline{Y} \subset \mathbb{P}^n_{\mathbf{k}}$ by taking the closure along ι_0. It is a simple observation that $\overline{Y} = \operatorname{Proj} B/\overline{I}$, where $\overline{I} \subset B$ is nothing but the ideal generated by the image of I along the map β. However, it is *not* true that if $I = (f_1,\ldots,f_r)$ then $\overline{I} = (\beta(f_1),\ldots,\beta(f_r))$. For instance, consider the subvariety (isomorphic to $\mathbb{A}^1_{\mathbf{k}}$)

$$Y = \operatorname{Spec} \mathbf{k}[y_1,y_2,y_3]/(y_2 - y_1^2, y_3 - y_1^3) \subset \mathbb{A}^3_{\mathbf{k}}.$$

Its projective closure, i.e. its closure along the open immersion

$$\mathbb{A}^3_{\mathbf{k}} = \mathrm{D}_+(x_0) \hookrightarrow \mathbb{P}^3_{\mathbf{k}} = \operatorname{Proj} \mathbf{k}[x_0,x_1,x_2,x_3],$$

agrees with the closure of the image of

$$\mathbb{A}^1_{\mathbf{k}} \hookrightarrow \mathbb{P}^1_{\mathbf{k}} \xhookrightarrow{v} \mathbb{P}^3_{\mathbf{k}},$$

where the first map sends $t \mapsto (1:t)$ and the map v is the 3-uple embedding. In other words, $\overline{Y}$ is the twisted cubic in $\mathbb{P}^3_{\mathbf{k}}$. Thus $\overline{I} = (x_0x_3 - x_1x_2, x_1^2 - x_0x_2, x_2^2 - x_0x_3)$, whereas $(\beta(y_2 - y_1^2), \beta(y_3 - y_1^3)) = (x_1^2 - x_0x_2, x_1^3 - x_0x_3)$.

Example 3.3.42. Consider the cuspidal plane cubic (cf. Example 3.1.86)

$$C = \operatorname{Spec} \mathbf{k}[y_1,y_2]/(y_2^2 - y_1^3) \subset \mathbb{A}^2_{\mathbf{k}}.$$

Identify $\mathbb{A}^2_{\mathbf{k}}$ with $\mathbb{P}^2_{\mathbf{k}} \setminus \mathrm{V}_+(x_2) = \mathrm{D}_+(x_2) \subset \mathbb{P}^2_{\mathbf{k}}$ and let $\overline{C} \subset \mathbb{P}^2_{\mathbf{k}}$ be the closure of C in the projective plane. Then $\overline{C}$ is cut out by the homogeneisation of $f(y_1,y_2) = y_2^2 - y_1^3$ with respect to x_2, namely the polynomial

$$x_2^3 f(x_0/x_2, x_1/x_2) = x_2^3(x_1^2/x_2^2 - x_0^3/x_2^3) = x_1^2x_2 - x_0^3.$$

Example 3.3.43. Consider the twisted cubic $Y \subset \mathbb{P}^3_{\mathbf{k}} = \operatorname{Proj} \mathbf{k}[x,y,z,w]$ as in Example 3.3.39. That is, Y is the curve in $\mathbb{P}^3_{\mathbf{k}}$ given parametrically by

$$(x,y,z,w) = (u^3, u^2v, uv^2, v^3).$$

It can be defined by the homogeneous ideal

$$J_3 = (xw - yz, y^2 - xz, z^2 - yw) \subset \mathbf{k}[x, y, z, w].$$

The point $P = (0:0:1:0) \in \mathbb{P}^3_{\mathbf{k}}$ lies outside Y (look at the third equation). Consider the hyperplane cut out by z, namely

$$H = \operatorname{Proj} \mathbf{k}[x, y, w] = \operatorname{Proj} \mathbf{k}[x, y, z, w]/(z) \subset \mathbb{P}^3_{\mathbf{k}}.$$

Then, projection from P (cf. Example 3.3.21) is the morphism

$$\begin{aligned} \mathbb{P}^3_{\mathbf{k}} \setminus \{P\} &\xrightarrow{\mathsf{pr}_P} H \\ (a:b:c:d) &\longmapsto (a:b:d). \end{aligned}$$

We claim that pr_P sends $Y \subset \mathbb{P}^3_{\mathbf{k}} \setminus \{P\}$ to a *cuspidal plane cubic* $C \subset H$ (cf. Example 3.3.42). Indeed, $\mathsf{pr}_P \colon Y \to H$ sends $(u^3, u^2v, uv^2, v^3) \mapsto (x, y, w) = (u^3, u^2v, v^3)$. For instance, in the chart $v \neq 0$, we have

$$(u^2v)^3 = (u^3)^2 v^3 \xrightarrow{v \neq 0} (u^2)^3 = (u^3)^2 \xrightarrow{\mathrm{X}=u^2, \mathrm{Y}=u^3} \mathrm{X}^3 = \mathrm{Y}^2.$$

But $(u^2v)^3 = (u^3)^2 v^3$ holds globally, thus the image $\mathsf{pr}_P(Y)$ is exactly the plane curve $\mathrm{V}_+(y^3 - x^2w)$, which in turn, by Example 3.3.42, agrees with the closure of the affine cuspidal cubic along the open immersion $\mathrm{D}_+(w) \hookrightarrow \operatorname{Proj} \mathbf{k}[x, y, w] = H$. The morphism

$$Y \longrightarrow \operatorname{Proj} \mathbf{k}[x, y, w]/(y^3 - x^2w)$$

is not an isomorphism: it is the normalisation morphism.

4 | First properties of schemes

The word 'scheme' can be decorated with a large number of adjectives, each describing a particular property that a scheme may or may not have. The easiest properties are the purely *topological* ones: those that only concern the underlying topological space of a scheme, and do not involve the structure sheaf. Examples are quasicompactness, irreducibility and connectedness (cf. Definition 3.1.39).

In the next section, we discuss irreducibility in detail, and in particular the notion of *irreducible components* for schemes. After that, in Section 4.2, we will tackle *local properties*. Those involve the structure sheaf, not just the topology.

4.1 Irreducible components of schemes

Recall that a topological space X is called *irreducible* if it cannot be written as a union of two proper closed subsets. If it is not irreducible, it will be called *reducible*.

Example 4.1.1. The underlying topological space $\mathrm{V}(xy)$ of the closed subscheme

$$\operatorname{Spec}\mathbb{C}[x,y]/(xy) \hookrightarrow \mathbb{A}^2_{\mathbb{C}}$$

is reducible (but connected), cf. Figure 4.1.

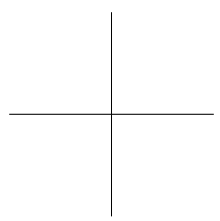

Figure 4.1: Pictorial description of $\mathrm{V}(xy) \subset \mathbb{A}^2_{\mathbb{C}}$.

Example 4.1.2. The underlying topological space $\mathrm{V}(xy-1)$ of the closed subscheme

$$\operatorname{Spec}\mathbb{C}[x,y]/(xy-1) \subset \mathbb{A}^2_{\mathbb{C}}$$

is irreducible (hence connected). It is isomorphic to $\operatorname{Spec}\mathbb{C}[x]_x = \mathbb{A}^1_{\mathbb{C}} \setminus 0$.

LEMMA 4.1.3. *Let X be a topological space. The following are equivalent:*

(a) *X is irreducible,*

(b) *every nonempty open subset $U \subset X$ is dense,*

(c) *any two nonempty open subsets intersect.*

In particular, every nonempty open subset of an irreducible space is irreducible.

Proof. We prove implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a).
(a) $\Rightarrow$ (b): Let $U \subset X$ be a nonempty open subset. Then $X = (X \setminus U) \cup \overline{U}$, which implies $\overline{U} = X$ by irreducibility.
(b) $\Rightarrow$ (c): Let $U, V \subset X$ be nonempty open subsets. If $U \cap V = \emptyset$, i.e. $U \subset X \setminus V$, then one would have $X = \overline{U} \subset X \setminus V$, i.e. $V = \emptyset$, contradiction.
(c) $\Rightarrow$ (a): Assume, by contradiction, we can write $X = X_1 \cup X_2$ where X_i are closed in X and $X_i \neq X$. Then $\emptyset \neq (X \setminus X_1) \cap (X \setminus X_2) = X \setminus (X_1 \cup X_2)$, contradiction. □

The following facts are easy to prove and are left as exercises.

Exercise 4.1.4. Let X be a topological space. Show the following:

(i) If $V \subset X$ is a subspace, it is irreducible if and only if $\overline{V}$ is irreducible.

(ii) If $V \subset X$ is irreducible and $f : X \to Y$ is a continuous map, then $f(V)$ is irreducible.

(iii) The product of two irreducible spaces is irreducible.

Definition 4.1.5 (Irreducible component)**.** Let X be a topological space. A maximal irreducible subspace $Z \subset X$ is called an *irreducible component* of X.

Maximality, here, is intended with respect to the inclusion relation. Note, then, that an irreducible component $Z \subset X$ is necessarily closed, because $\overline{Z} \supset Z$ is again irreducible by Exercise 4.1.4(i).

PROPOSITION 4.1.6. *Let X be a topological space. If $A \subset X$ is an irreducible subspace, then A is contained in an irreducible component of X. Moreover, X is the union of its irreducible components.*

Proof. Consider the set

$$\mathcal{S}_A = \{\text{irreducible subspaces } Y \subset X \mid Y \supset A\}.$$

Since A is irreducible, it belongs to $\mathcal{S}_A$, thus $\mathcal{S}_A \neq \emptyset$. Fix a chain $Y_1 \subset Y_2 \subset \cdots$ of elements in $\mathcal{S}_A$. We prove that $Z = \bigcup_{i \geq 1} Y_i$ is irreducible. Let $U, V \subset Z$ be nonempty open subsets. Then there are indices i and j such that $U \cap Y_i \neq \emptyset \neq V \cap Y_j$. We either have $Y_i \subset Y_j$ or

$Y_j \subset Y_i$. Say the former holds. Then $U \cap Y_j$ and $V \cap Y_j$ are nonempty open subsets of Y_j, which intersect by Lemma 4.1.3(c) since Y_j is irreducible. Thus $U \cap V \neq \emptyset$, hence Z is irreducible, again by Lemma 4.1.3(c).

It follows that any chain $Y_1 \subset Y_2 \subset \cdots$ as above has at least one extremal element (or upper bound) in $\mathcal{S}_A$, namely the union $\bigcup_{i \geq 1} Y_i$. It follows from Zorn's Lemma that $\mathcal{S}_A$ has a maximal element, which settles the first claim.

To see that X is the union of its irreducible components, it is enough to observe that for every $x \in X$ the singleton $\{x\}$ is irreducible, hence contained in an irreducible component by the previous claim. □

Exercise 4.1.7. If $U \subset X$ is a nonempty open subset, then $\{U \cap X_i\}_{i \in I}$ are the irreducible components of U, where $\{X_i\}_{i \in I}$ are the irreducible components of X that meet U.

Definition 4.1.8 (Noetherian topological space)**.** A topological space X is called *noetherian* if the descending chain condition holds for the closed subsets of X. Namely, any descending chain

$$Y_1 \supset Y_2 \supset Y_3 \supset \cdots$$

with $Y_i \subset X$ closed stabilises, i.e. there exists an index m such that $Y_m = Y_{m+j}$ for any $j \geq 0$.

PROPOSITION 4.1.9. *If X is a noetherian topological space, then any nonempty closed subset $Y \subset X$ can be written as a finite union $Y = Y_1 \cup \cdots \cup Y_r$, where Y_i are irreducible closed subsets of X, in such a way that $Y_i \not\subset Y_j$ for all $i \neq j$. These subspaces $Y_i \subset Y$ are the irreducible components of Y.*

In particular, a noetherian topological space has finitely many irreducible components.

Proof. Assume that the set

$$\mathfrak{S} = \left\{ \text{nonempty closed } Y \subset X \;\middle|\; \begin{array}{c} Y \text{ cannot be written as a finite union} \\ \text{of irreducible closed subsets of } X \end{array} \right\}$$

is nonempty. Since X is noetherian, there must be a minimal element $Y \in \mathfrak{S}$, and Y is necessarily reducible. Thus $Y = Y_1 \cup Y_2$, where $\emptyset \neq Y_i \subsetneq Y$ are closed. By minimality of Y, we have $Y_i \notin \mathfrak{S}$ for $i = 1, 2$, which means

$$Y_1 = Z_1 \cup \cdots \cup Z_\ell$$
$$Y_2 = V_1 \cup \cdots \cup V_k$$

for some irreducible closed subsets $Z_i \subset X$ and $V_j \subset X$. But then $Y = Y_1 \cup Y_2$ cannot belong to $\mathfrak{S}$, which shows that $\mathfrak{S} = \emptyset$. That is, a finite decomposition

$$Y = Y_1 \cup \cdots \cup Y_r$$

into irreducible closed subsets $Y_i \subset X$ exists for any nonempty closed subset $Y \subset X$. It is clear that any such decomposition can be made irredundant by throwing away a bunch of subsets appearing in the union. In other words, the condition $Y_i \not\subset Y_j$ for all $i \neq j$ can always be achieved.

We claim that, after a decomposition has been made minimal in the sense that $Y_i \not\subset Y_j$ for all $i \neq j$, it is in fact *unique* (which also proves that $Y_1, \ldots, Y_r$ are the irreducible components of Y). Let us assume Y is equal to

$$Y_1 \cup \cdots \cup Y_r = Z_1 \cup \cdots \cup Z_s$$

for $Y_i \subset X$ and $Z_j \subset X$ irreducible closed subsets. Then

$$Y_1 = Y \cap Y_1 = (Z_1 \cup \cdots \cup Z_s) \cap Y_1 = \bigcup_{1 \leq i \leq s} Z_i \cap Y_1.$$

But Y_1 is irreducible, so there is an index i such that $Z_i \cap Y_1 = Y_1$. Without loss of generality, we may assume $i = 1$, thus $Y_1 \subset Z_1$. But now the same reasoning applied to Z_1 shows that $Z_1 \subset Y_j$ for some j. Then $Y_1 \subset Y_j$, but this is impossible by irredundancy, unless $j = 1$. Thus $Z_1 \subset Y_1$, i.e. $Y_1 = Z_1$. Now the observation

$$Y_2 \cup \cdots \cup Y_r = \overline{Y \setminus Y_1} = \overline{Y \setminus Z_1} = Z_2 \cup \cdots \cup Z_s$$

allows us to apply an inductive argument, which finishes the proof. □

LEMMA 4.1.10. *Let A be a ring, and set $X = \operatorname{Spec} A$. Then*

(1) *If $\{\mathfrak{p}_i\}_i$ are the minimal prime ideals in A, then $\{\mathrm{V}(\mathfrak{p}_i)\}_i$ are the irreducible components of X.*

(2) *X is irreducible if and only if A has a unique minimal prime ideal.*

Proof. For (1), combine Proposition 3.1.43 with the definition of irreducible component. Then (2) follows immediately. □

Example 4.1.11 ([8, Ch. 1, Ex. 1.3]). Let $X = \operatorname{Spec} \mathbf{k}[x,y,z]/I \subset \mathbb{A}^3_{\mathbf{k}}$, where $I = (x^2 - yz, xz - x) \subset \mathbf{k}[x,y,z]$. Let us compute the irreducible components of X. We set $X_1 = \mathrm{V}(x^2 - yz)$ and $X_2 = \mathrm{V}(xz - x) = \mathrm{V}(x) \cup \mathrm{V}(z-1)$. The latter is a union of two affine planes. Then

$$X = X_1 \cap X_2 = (X_1 \cap \mathrm{V}(x)) \cup (X_1 \cap \mathrm{V}(z-1)).$$

The first bit further splits as $\mathrm{V}(x,y) \cup \mathrm{V}(x,z)$, whereas the second bit is just $\mathrm{V}(y^2 - x, z-1)$, a parabola lying on the plane $z = 1$. All in all, we get

$$X = \mathrm{V}(x,y) \cup \mathrm{V}(x,z) \cup \mathrm{V}(y^2 - x, z-1).$$

Note that there is no inclusion relation between these closed sets, and they are all irreducible, being homeomorphic to $\mathbb{A}^1_{\mathbf{k}}$. Thus they are the irreducible components of X.

Example 4.1.12. Let $X = \operatorname{Spec}\mathbf{k}[x,y,z]/I \subset \mathbb{A}^3_{\mathbf{k}}$, where $I = (x^2 - yz, xy - z) \subset \mathbf{k}[x,y,z]$. Let us compute the irreducible components of X. One can write

$$I = (x^2 - xy^2, xy - z) = (x(x - y^2), xy - z).$$

Thus

$$X = (\mathrm{V}(x) \cup \mathrm{V}(x - y^2)) \cap \mathrm{V}(xy - z) = (\mathrm{V}(x) \cap \mathrm{V}(xy - z)) \cup (\mathrm{V}(x - y^2) \cap \mathrm{V}(xy - z)).$$

The first bit is the y-axis $\mathrm{V}(x,z) \subset \mathbb{A}^3_{\mathbf{k}}$, the second bit is $\mathrm{V}(x - y^2, z - y^3)$, whose associated ring is

$$\mathbf{k}[x,y,z]/(x - y^2, z - y^3) = \mathbf{k}[y^2, y, y^3] = \mathbf{k}[y].$$

All in all, we get the y-axis union an affine twisted cubic, i.e. two irreducible components.

Exercise 4.1.13. Let $X = \operatorname{Spec}\mathbf{k}[x,y,z]/I \subset \mathbb{A}^3_{\mathbf{k}}$, where $I = (x^2 - yz, y^2 - xz) \subset \mathbf{k}[x,y,z]$. Compute the irreducible components of X.

4.2 Local properties and the locality lemma

Some properties of schemes, the so-called *affine-local properties* (that we will simply call *local properties*), are amongst the properties that involve the structure sheaf of the scheme, not just the underlying topological space. One reason we like them is:

$$\boxed{\text{local properties can be checked on an affine cover!}}$$

If your scheme is quasicompact, e.g. a quasiprojective variety, you can test your favourite *finite* open cover for the given local property you have in mind.

We start with a general definition.

Definition 4.2.1 (Local property)**.** Let $\mathbf{P}$ be a property of rings. We say that $\mathbf{P}$ is a *local property* if

(i) For any ring A, and any $f \in A$, if A has $\mathbf{P}$ then A_f has $\mathbf{P}$.

(ii) For any ring A, and any $f_1, \ldots, f_s \in A$ such that $A = (f_1, \ldots, f_s)$, if A_{f_i} has $\mathbf{P}$ for all $i = 1, \ldots, s$ then A has $\mathbf{P}$.

We also say that a scheme X is *locally* $\mathbf{P}$ if every $x \in X$ has an affine open neighbourhood U such that $\mathscr{O}_X(U)$ has $\mathbf{P}$.

LEMMA 4.2.2 (Locality Lemma). *Let $\mathbf{P}$ be a local property of rings. Let X be a scheme. The following conditions are equivalent.*

(1) *X is locally $\mathbf{P}$.*

(2) *For every open affine $U \subset X$, the ring $\mathscr{O}_X(U)$ has* **P**.

(3) *There is an affine open covering $X = \bigcup_{i\in I} U_i$ such that each $\mathscr{O}_X(U_i)$ has* **P**.

(4) *There is an open covering $X = \bigcup_{j\in J} V_j$ such that each V_j is locally* **P**.

Furthermore, if X is locally **P** *then so is every open subscheme $U \subset X$.*

Proof. We have (2) $\Rightarrow$ (1)$\Leftrightarrow$ (3) by definition.

Let us show (3) $\Rightarrow$ (2). Fix a covering $X = \bigcup_{i\in I} U_i$ where each $U_i = \operatorname{Spec} A_i$ is affine, with A_i satisfying **P**. Let $U = \operatorname{Spec} A \subset X$ be an affine open. We must show that A has **P**. By [16, Tag 01IX] there exist elements $f_1,\ldots,f_s \in A$ such that $U = \bigcup_{i\le j\le s} \mathrm{D}(f_j)$ where each A_{f_j} is of the form $(A_i)_{g_{ij}}$, for some $g_{ij} \in A_i$. But **P** is a local property, so condition (i) implies that A_{f_j} has **P**. Since $(f_1,\ldots,f_s) = A$, it follows that $\mathscr{O}_X(U) = A$ has **P** by (ii).

So far we have proved that the first 3 conditions are equivalent. This is enough to conclude that being locally **P** passes to open subschemes (e.g. by running (1) $\Leftrightarrow$ (2)). Thus (1) $\Rightarrow$ (4) $\Rightarrow$ (3) is immediate. □

We now provide three key examples of local properties, to which the Locality Lemma applies:

- **P** = reducedness (cf. also Proposition 4.3.8),
- **P** = normality (cf. also Proposition B.6.10),
- **P** = noetherianity (cf. also Proposition 4.5.5).

PROPOSITION 4.2.3. *Reducedness is a local property of rings.*

Proof. Any localisation of a reduced ring is reduced by Lemma B.5.15, so (i) is confirmed. Let us show (ii). Let A be a ring, and let $f_1,\ldots,f_s \in A$ be elements generating the unit ideal. Assume A_{f_i} is reduced for every $i = 1,\ldots,s$. Let $a \in A$ be an element such that $a^r = 0$ for some $r > 0$. Then $a/1 \in A_{f_i}$ is also nilpotent, and then by assumption $0 = 0/1 = a/1 \in A_{f_i}$, i.e. $af_i^{m_i} = 0 \in A$ for some $m_i \ge 0$. Pick a uniform m, so that $af_i^m = 0 \in A$ for all i. But $1 = \sum_{1\le i\le s} b_i f_i^m$ for some $b_1,\ldots,b_s \in A$. Thus $a = a\cdot 1 = ab_1f_1^m + \cdots + ab_sf_s^m = 0$ in A. Namely, A is reduced. □

Recall from Definition B.6.1 the definition of normality.

PROPOSITION 4.2.4. *Being normal is a local property of rings.*

Proof. We divide the proof in two steps.

Step 1: Being normal is a local property of *integral domains*.

We first check property (i) in Definition 4.2.1. Let A be a normal domain, $S \subset A$ a multiplicative subset. We verify that $B = S^{-1}A$ is a normal domain. We have an inclusion

$$\operatorname{Frac} A \subset \operatorname{Frac} B.$$

Let $u \in \operatorname{Frac} B$ be an element that is integral over B. We claim that $u \in B$. Let $p(t) = b_0 + b_1 t + \cdots + b_{d-1} t^{d-1} + t^d \in B[t]$ be a polynomial such that

$$b_0 + b_1 u + \cdots + b_{d-1} u^{d-1} + u^d = 0.$$

Choose $s \in S$ such that $s b_i \in A$ for all $i = 0, 1, \ldots, d-1$ (this is called 'clearing denominators'!). Then this expression is also 0 when multiplied by s^d, i.e.

$$s^d b_0 + s^{d-1} b_1 (su) + \cdots + s b_{d-1} (su)^{d-1} + (su)^d = 0,$$

which means that the polynomial

$$\widetilde{p}(t) = s^d b_0 + s^{d-1} b_1 t + \cdots + s b_{d-1} t^{d-1} + t^d \in A[t]$$

has $su \in \operatorname{Frac} A$ amongst its zeros. Since A is normal, we have $su \in A$. Thus $u \in B = S^{-1}A$.

In particular, all principal localisations A_f are normal.

We now have to check (ii) in Definition 4.2.1. Let A be a domain, and let $f_1, \ldots, f_s \in A$ be elements generating the unit ideal. Assume A_{f_i} is normal for every $i = 1, \ldots, s$. We claim that A is normal. Set $F = \operatorname{Frac} A$ and $F_i = \operatorname{Frac} A_{f_i}$. Let $u \in F$ be integral over A. Then u satisfies a monic equation over A. But $u \in F \subset F_i$ and $A[t] \subset A_{f_i}[t]$ for all i, so in fact u satisfies a monic equation over A_{f_i} for all i. Thus $u \in A_{f_i}$ for all i by our normality assumption. Thus for every i there is an exponent m_i such that $f_i^{m_i} u \in A$. Pick a uniform $m > 0$ such that $f_i^m u \in A$ for all i. Since $\operatorname{Spec} A = \bigcup_{1 \le i \le s} \mathrm{D}(f_i) = \bigcup_{1 \le i \le s} \mathrm{D}(f_i^m)$, we have $1 \in \sum_{1 \le i \le s} f_i^m A$, i.e.

$$1 = \sum_{1 \le i \le s} \alpha_i f_i^m, \quad \alpha_i \in A.$$

Thus

$$u = 1 \cdot u = \sum_{1 \le i \le s} \alpha_i f_i^m u \in A.$$

Step 2: Being normal is a local property of rings.

We first check property (i) in Definition 4.2.1. Let A be a normal ring, $f \in A$. We must prove that all prime localisations of A_f are normal domains. Let $\mathfrak{q} \subset A_f$ be a prime ideal. Let $\mathfrak{p} \subset A$ be its preimage along the localisation map $\ell \colon A \to A_f$. We have a diagram

$$\begin{tikzcd} A \arrow[r, "\ell"] \arrow[d] \arrow[dr, "\varphi"] & A_f \arrow[d] \\ A_{\mathfrak{p}} \arrow[r, dotted, "p_\varphi"'] & (A_f)_{\mathfrak{q}} \end{tikzcd}$$

where the dotted arrow can be completed to a solid arrow because every $s \in A \setminus \mathfrak{p}$ is sent by φ to an invertible element (reason: $\mathfrak{p} = \ell^{-1}(\mathfrak{q})$, so $s \in A \setminus \mathfrak{p}$ means $\ell(s) \notin \mathfrak{q}$, which implies $\varphi(s)$ is in the image of $A_f \setminus \mathfrak{q}$, thus invertible). But $A_{\mathfrak{p}}$ is a normal domain by assumption, the morphism p_φ is a localisation and thus its target is a normal domain by the proof exhibited in **Step 1**.

We now have to check (ii) in Definition 4.2.1. Let A be a ring, whose unit ideal is generated by elements $f_1, \ldots, f_s \in A$. Assume A_{f_i} is a normal ring, i.e. its prime localisations are normal domains, for every $i = 1, \ldots, s$. We must show that A is a normal ring. Let $\mathfrak{p} \in \operatorname{Spec} A$ be a prime ideal. Then $\square$ TBC

PROPOSITION 4.2.5. *Being noetherian is a local property of rings.*

Proof. A localisation of a noetherian ring is again a noetherian ring. Thus property (i) holds. To check (ii), fix a ring A and elements $f_1, \ldots, f_s \in A$ such that $A = (f_1, \ldots, f_s)$. Assume A_{f_i} is noetherian for all $i = 1, \ldots, s$. We need to show that A is noetherian.

Step 1. We prove an auxiliary result first, saying that if $\mathfrak{a} \subset A$ is an ideal, and $\ell_i : A \to A_{f_i}$ are the localisation morphisms (where $A = (f_1, \ldots, f_s)$ is assumed!), then

$$\mathfrak{a} = \bigcap_{i=1}^{s} \ell_i^{-1}(\mathfrak{a} \cdot A_{f_i}),$$

where $\mathfrak{a} \cdot A_{f_i} \subset A_{f_i}$ is the extension of $\mathfrak{a}$ along ℓ_i (i.e. the ideal generated by $\ell_i(\mathfrak{a}) \subset A_{f_i}$). Proving '$\subset$' is trivial, so we focus on the reverse inclusion. Fix an element

$$b \in \bigcap_{i=1}^{s} \ell_i^{-1}(\mathfrak{a} \cdot A_{f_i}) \subset A.$$

Then $\ell_i(b) \in \mathfrak{a} \cdot A_{f_i}$ for all $i = 1, \ldots, s$, i.e.

$$\ell_i(b) = \frac{a_i}{f_i^{n_i}}, \quad a_i \in \mathfrak{a}, \quad n_i \geq 0.$$

After possibly making n_i larger, and using that there are only finitely many of them, we can pick a uniform n such that $b/1 = \ell_i(b) = a_i / f_i^n$ in A_{f_i} for every $i = 1, \ldots, s$. Thus there is an $m_i \geq 0$ such that

$$f_i^{m_i}(b f_i^n - a_i) = 0.$$

Again, we pick a uniform m that works for all i, so to obtain

$$0 = f_i^m(b f_i^n - a_i) = f_i^{m+n} b - f_i^m a_i.$$

Thus, since $a_i \in \mathfrak{a}$, we get

$$f_i^{m+n} b = f_i^m a_i \in \mathfrak{a} \subset A.$$

Now, since $(f_1, \ldots, f_s) = A$, we have $(f_1^N, \ldots, f_s^N) = A$ for every $N > 0$. Taking $N = m + n$, we can write

$$1 = \sum_{i=1}^{s} c_i f_i^{m+n}, \quad c_i \in A.$$

Thus

$$b = \sum_{i=1}^{s} c_i f_i^{m+n} b \in \mathfrak{a},$$

which is what we wanted.

Step 2. Back to proving that A is noetherian. Fix an ascending chain $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots$ of ideals $\mathfrak{a}_i \subset A$. We must prove the chain stabilises. We have, for each i, an induced chain

$$\mathfrak{a}_1 \cdot A_{f_i} \subset \mathfrak{a}_2 \cdot A_{f_i} \subset \cdots$$

but A_{f_i} is noetherian by assumption, thus all such chains do stabilise. We can pick a uniform index M, working for all i, after which all the s induced chains in A_{f_i} stabilise, i.e.

$$\mathfrak{a}_M \cdot A_{f_i} = \mathfrak{a}_{M+1} \cdot A_{f_i} = \cdots$$

for all $i = 1, \ldots, s$. By Step 1, for all $k > 0$ we find

$$\mathfrak{a}_M = \bigcap_{i=1}^{s} \ell_i^{-1}(\mathfrak{a}_M \cdot A_{f_i}) = \bigcap_{i=1}^{s} \ell_i^{-1}(\mathfrak{a}_{M+k} \cdot A_{f_i}) = \mathfrak{a}_{M+k},$$

thus the initial chain $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots$ also stabilises. That is, A is noetherian. $\square$

Caution 4.2.6. Let A be an integral domain. Even though both being normal and being noetherian are local properties, there is a crucial difference:

$$A_{\mathfrak{p}} \text{ is normal for every } \mathfrak{p} \in \operatorname{Spec} A \implies A \text{ is normal},$$
$$A_{\mathfrak{p}} \text{ is noetherian for every } \mathfrak{p} \in \operatorname{Spec} A \not\Longrightarrow A \text{ is noetherian}.$$

For instance (cf. [17]), the ring

$$A = \mathbb{Z}\left[\frac{x}{p} \,\middle|\, p \text{ is a prime number}\right]$$

is an integral domain which is not noetherian, and yet for a prime ideal $\mathfrak{p} \in \operatorname{Spec} A$ one has two possibilities: either

- $\mathfrak{p} \cap \mathbb{Z} = 0$, in which case $A_{\mathfrak{p}}$ is a localisation of the noetherian ring $(\mathbb{Z} \setminus 0)^{-1} A = \mathbb{Q}[x]$, hence noetherian, or
- $\mathfrak{p} \cap \mathbb{Z} = q\mathbb{Z}$ for some prime q, in which case $A_{\mathfrak{p}}$ is a localisation of the noetherian ring $(\mathbb{Z} \setminus q\mathbb{Z})^{-1} A = \mathbb{Z}_{(q)}[x/q]$, hence again noetherian.

An example of such behaviour where A has nilpotent elements if provided, for instance, by the ring

$$A = \prod_{i \in \mathbb{N}} \mathbb{F}_2,$$

which is not noetherian, but has the property that all its prime ideal localisations are isomorphic to $\mathbb{F}_2$. The same works with B replaced by any infinite boolean ring.

Note that being 'stalk-local' is a stronger condition than being local. For instance, being reduced is stalk-local by definition, whereas being noetherian is local but not stalk-local. Make this precise!

4.3 Reduced schemes

Recall that a ring is called *reduced* if it has no nonzero nilpotent elements. For instance, an integral domain is reduced. Recall also, from Proposition 4.2.3, that reducedness is a local property of rings.

Definition 4.3.1 (Reduced scheme)**.** A scheme X is *reduced* if the local ring $\mathscr{O}_{X,x}$ is reduced for every $x \in X$.

Since the local ring at a point does not change passing to an open neighbourhood of the point, an open subscheme of a reduced scheme is still reduced.

Example 4.3.2. The schemes $\mathbb{P}^n_{\mathbb{F}}$ and $\mathbb{A}^n_{\mathbb{F}}$ are reduced.

LEMMA 4.3.3. *A scheme X is reduced if and only if $\mathscr{O}_X(U)$ is reduced for every open subset $U \subset X$.*

Proof. Assume $\mathscr{O}_{X,x}$ is reduced for every $x \in X$. Then, for any open subset $U \subset X$, there is an injective ring homomorphism

$$\mathscr{O}_X(U) \hookrightarrow \prod_{x \in U} \mathscr{O}_{X,x}$$

since $\mathscr{O}_X$ is a sheaf (cf. Lemma 2.4.1). But products of reduced rings are reduced, and subrings of reduced rings are reduced. Thus $\mathscr{O}_X(U)$ is reduced.

Conversely, assume $\mathscr{O}_X(U)$ is reduced for all open subsets $U \subset X$. Let $x \in X$ be a point. By definition of scheme, x lies in an affine open subset $U = \operatorname{Spec} A$, and A is reduced by assumption. We have $\mathscr{O}_{X,x} = \mathscr{O}_{U,x} = A_{\mathfrak{p}}$, where $\mathfrak{p} \subset A$ is the prime ideal corresponding to x. But localisations of reduced rings are reduced (cf. Lemma B.5.15). $\square$

COROLLARY 4.3.4. *Let A be a ring. Then* $\operatorname{Spec} A$ *is reduced if and only if A is reduced.*

Proof. If $X = \operatorname{Spec} A$ is reduced, then so is $\mathscr{O}_X(X) = A$ by Lemma 4.3.3. Conversely, if A is reduced, then so are all of its localisations (cf. Lemma B.5.15). $\square$

PROPOSITION 4.3.5. *Let X be a scheme.*

(i) *There is a unique reduced closed subscheme $X_{\mathsf{red}} \hookrightarrow X$ having the same underlying topological space as X.*

(ii) *If Y is a reduced scheme, then any morphism $Y \to X$ factors uniquely as the composition $Y \to X_{\mathsf{red}} \hookrightarrow X$.*

(iii) *If $Z \subset X$ is a closed subset, there is precisely one reduced closed subscheme structure on Z.*

Proof. Denote by $\mathrm{Nil}(A)$ the nilradical[1] of a ring A, i.e. the set (which forms an ideal) of nilpotent elements. Sending

$$U \mapsto \mathcal{N}_X(U) = \left\{ s \in \mathscr{O}_X(U) \,\middle|\, s_x \in \mathrm{Nil}(\mathscr{O}_{X,x}) \text{ for all } x \in U \right\}$$

defines a sheaf of ideals $\mathcal{N}_X \subset \mathscr{O}_X$. Define

$$(X_{\mathsf{red}}, \mathscr{O}_{X_{\mathsf{red}}}) = (X, \mathscr{O}_X/\mathcal{N}_X).$$

This is by construction a locally ringed space. To verify that it is a scheme, we need to check that every point has an open neighbourhood isomorphic to an affine scheme. Since this is a local question, we may assume $X = \operatorname{Spec} A$ is affine. The canonical surjection $\phi\colon A \twoheadrightarrow A/\mathrm{Nil}(A)$ determines a closed immersion of schemes

$$\iota\colon V = \operatorname{Spec} A/\mathrm{Nil}(A) \hookrightarrow X.$$

Evaluating $\iota^\#\colon \mathscr{O}_X \to \iota_*\mathscr{O}_V$ on a principal open subset $\mathrm{D}(g) \subset X$ yields

$$\iota^\#(\mathrm{D}(g))\colon A_g \twoheadrightarrow A_g/\mathrm{Nil}(A)_g,$$

whose kernel is

$$\mathrm{Nil}(A)_g = \mathrm{Nil}(A) \otimes_A A_g = \mathrm{Nil}(A_g) = \mathcal{N}_X(\mathrm{D}(g)).$$

The identity $\mathcal{N}_X = \ker \iota^\#$ follows, thus $(V, \mathscr{O}_V) = (X_{\mathsf{red}}, \mathscr{O}_{X_{\mathsf{red}}})$ and (i) and (ii) are proved.

Next we prove the uniqueness stated in (iii). If Z is an arbitrary closed subset of X, endowed with a scheme structure via a sheaf of rings $\mathscr{O}_Z$, and $U \subset X$ is an affine open subset of X, then $(Z \cap U, \mathscr{O}_{Z\cap U})$ is a scheme (it is open in Z, which is a scheme), and it is affine because it is closed in U, which is affine (see Proposition 3.1.65). Therefore there is precisely one ideal $I \subset \mathscr{O}_X(U)$ such that $Z \cap U = \operatorname{Spec} \mathscr{O}_X(U)/I$ as schemes. Now, if $(Z, \mathscr{O}_Z)$ is a reduced scheme, then I is *radical* by Lemma 4.3.3. Uniqueness follows.

Let us show the existence part of (iii). Let $X = \bigcup_i U_i$ be an open cover by affine schemes $U_i = \operatorname{Spec} A_i$. Consider the closed subsets $Z \cap U_i \subset U_i$. They can be written as $Z \cap U_i = \mathrm{V}(J_i)$ for ideals $J_i \subset A_i$. Define a scheme structure on each such intersection by putting $Z \cap U_i = \operatorname{Spec} A_i/\sqrt{J_i}$. In this way, $Z \cap U_i \subset U_i$ is a reduced closed subscheme of U_i for every i. The uniqueness of such a structure allows one to extend Z to a (reduced) closed subscheme of X. □

[1]We have denoted the nilradical by $\sqrt{0}$ elsewhere in these notes. For the sake of clarity, we use a different notation in this proof. Recall, also, that $\sqrt{0} = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p}$.

Terminology 4.3.6. Let X be a scheme, $Z \subset X$ a closed subset. The structure of (iii) is called the *induced reduced subscheme structure* on Z.

Definition 4.3.7 (Reduction)**.** Let X be a scheme. The closed subscheme

$$X_{\mathrm{red}} = \mathrm{V}(\mathscr{N}_X) \hookrightarrow X$$

defined by the ideal sheaf $\mathscr{N}_X \subset \mathscr{O}_X$ of Proposition 4.3.5 is called the *reduction* of X.

Informally, any scheme X has a 'smallest' closed subscheme homeomorphic to itself. This subscheme, namely X_{red}, is reduced, and no other closed subscheme homeomorphic to X can be reduced.

For a general scheme, the locality lemma gives us the following characterisations.

PROPOSITION 4.3.8. *Let X be a scheme. The following conditions are equivalent.*

(0) *X is reduced* (Definition 4.3.1)*.*

(1) *Every point $x \in X$ has an affine open neighbourhood U such that $\mathscr{O}_X(U)$ is reduced.*

(2) *For every open affine $U \subset X$, the ring $\mathscr{O}_X(U)$ is reduced.*

(3) *There is an affine open covering $X = \bigcup_{i \in I} U_i$ such that $\mathscr{O}_X(U_i)$ is reduced for all $i \in I$.*

(4) *There is an open covering $X = \bigcup_{j \in J} V_j$ such that V_j is reduced for all $j \in J$.*

Moreover, an open subscheme of a reduced scheme is reduced.

Proof. The last four conditions are equivalent by the Locality Lemma (cf. Lemma 4.2.2) applied to Proposition 4.2.3. Conditions (0) and (2) are equivalent by Lemma 4.3.3. □

Condition (1) would translate to 'X is locally reduced' in the sense of Definition 4.2.1, but the terminology 'locally reduced' is not used (since it is equivalent to 'reduced').

Definition 4.3.9 (Fat points, take II)**.** Let $\mathbf{k}$ be an algebraically closed field. A *fat point over* $\mathbf{k}$ is a $\mathbf{k}$-scheme X such that

$$\begin{tikzcd} X_{\mathrm{red}} \arrow[r, hook] \arrow[dr, equal] & X \arrow[d] \\ & \operatorname{Spec} \mathbf{k} \end{tikzcd}$$

commutes.

Exercise 4.3.10. Show that Definition 4.3.9 agrees with the definition given in Example 3.1.70.

Example 4.3.11. Consider $A_\alpha = \mathbb{C}[x,y]/(y-a, y-x^2)$ for $\alpha \in \mathbb{C}$. Then $X_\alpha = \operatorname{Spec} A_\alpha \hookrightarrow \mathbb{A}^2_{\mathbb{C}}$ is reduced for all $\alpha \in \mathbb{C}^\times$. Indeed, A_α is reduced, for

$$\begin{aligned}\mathbb{C}[x,y]/(y-\alpha, y-x^2) &\cong \mathbb{C}[x]/(x^2-\alpha)\\ &\cong \mathbb{C}[x]/(x-\sqrt{\alpha})(x+\sqrt{\alpha})\\ &\cong \mathbb{C}[x]/(x-\sqrt{\alpha}) \times \mathbb{C}[x]/(x+\sqrt{\alpha}),\end{aligned}$$

where the last identity is by the Chinese Remainder Theorem. This also shows that X_α is disconnected for $\alpha \in \mathbb{C}^\times$. On the other hand, $X_0 \cong \operatorname{Spec} \mathbb{C}[x]/x^2$ is nonreduced since $\mathbb{C}[x]/x^2$ is the ring of dual numbers.

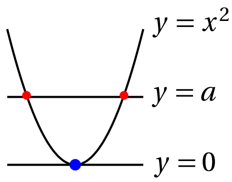

Figure 4.2: The (real points of the) intersection X_α of a parabola with the line $y = \alpha$.

4.4 Integral schemes

Definition 4.4.1 (Integral scheme)**.** A scheme X is integral if $\mathscr{O}_X(U)$ is an integral domain for every open subset $U \subset X$.

PROPOSITION 4.4.2. *A scheme X is integral if and only if it is reduced and irreducible. In particular, any open subset of an integral scheme is again an integral scheme.*

Proof. If X is integral, then it is reduced by Lemma 4.3.3. If X were reducible, by Lemma 4.1.3 (c) it would contain two disjoint, nonempty open subsets $U_1 \subset X$ and $U_2 \subset X$. This would imply $\mathscr{O}_X(U_1 \cup U_2) = \mathscr{O}_X(U_1) \times \mathscr{O}_X(U_2)$, but this ring is not integral, e.g. $(1,0)\cdot(0,1) = (0,0)$. This contradicts the assumption that X is integral.

Conversely, let us assume X is reduced and irreducible. Let $U \subset X$ be an open subset. We need to show that $\mathscr{O}_X(U)$ is an integral domain. Assume, by contradiction, that there exist $f, g \in \mathscr{O}_X(U) \setminus 0$ such that $fg = 0$. Consider

$$\begin{aligned}Z_1 &= \{\, x \in U \mid f_x \in \mathfrak{m}_x \,\},\\ Z_2 &= \{\, x \in U \mid g_x \in \mathfrak{m}_x \,\}.\end{aligned}$$

Then $Z_1, Z_2 \subset U$ are closed subsets. To see this, note that $\mathscr{O}^\times_{X,x} = \mathscr{O}_{X,x} \setminus \mathfrak{m}_x$ (an element of a local ring is invertible if and only if it lies outside the maximal ideal), thus the condition $f_x \notin \mathfrak{m}_x$ means that $f_x \in \mathscr{O}^\times_{X,x}$, i.e. there exist an open neighbourhood $x \in V \subset U$ and a section $h \in \mathscr{O}_X(V)$ such that $f_x h_x = 1 \in \mathscr{O}_{X,x}$. Then there exists an open neighbourhood $x \in W \subset V$ such that $f|_W h|_W = 1 \in \mathscr{O}_X(W)$. Thus $W \subset U \setminus Z_1$, and $U \setminus Z_1$ is open.

Moreover, we have $U = Z_1 \cup Z_2$. Since X is irreducible, so is U. Thus either $Z_1 = U$ or $Z_2 = U$. Let us say $Z_1 = U$, which means that $f_x \in \mathfrak{m}_x$ for every $x \in U$. In particular, for every affine open $\operatorname{Spec} A \hookrightarrow U$, the restriction $\mathscr{O}_X(U) \to A$ sends f to a nilpotent element $\overline{f} \in A$, since nilpotency of $\overline{f}$ is equivalent to $\mathrm{D}(\overline{f}) \subset \operatorname{Spec} A$ being empty. But by assumption A is reduced, thus $\overline{f} = 0 \in A$. Running this reasoning over an affine open covering of U, we get that $f = 0 \in \mathscr{O}_X(U)$, which is a contradiction. Therefore X is integral. □

On an affine scheme, we obtain the following characterisation.

COROLLARY 4.4.3. *Let A be a ring. Then* $\operatorname{Spec} A$ *is an integral scheme if and only if A is a domain.*

Proof. The scheme $X = \operatorname{Spec} A$ is reduced and irreducible if and only if A is reduced and $\sqrt{0}$ is prime. But the nilradical of a reduced ring is trivial. Thus X is integral if and only if $(0) \subset A$ is prime. □

Example 4.4.4. Let $A = \mathbb{C}[x,y]/(y^2 - x^3)$. Then $\operatorname{Spec} A$ is reduced and irreducible, i.e. integral.

Example 4.4.5. Let $A = \mathbb{C}[x,y]/(y^2 - x^2(x+1))$. Then $\operatorname{Spec} A$ is reduced and irreducible, i.e. integral.

Example 4.4.6. The schemes $\mathbb{P}^n_{\mathbf{k}}$ and $\mathbb{A}^n_{\mathbf{k}}$ are integral. Reducedness was settled in Example 4.3.2. It is enough to prove that $\mathbb{P}^n_{\mathbf{k}}$ is irreducible. Assume $\mathbb{P}^n_{\mathbf{k}} = \mathrm{V}_+(I) \cup \mathrm{V}_+(J)$ for two homogeneous ideals $I, J \subset \mathbf{k}[x_0, \ldots, x_n]$. Assume also $\mathrm{V}_+(I) \neq \mathbb{P}^n_{\mathbf{k}} \neq \mathrm{V}_+(J)$. Then $\mathbb{P}^n_{\mathbf{k}} = \mathrm{V}_+(IJ)$. Let $p \in I$ and $q \in J$ be nonconstant polynomials. Then $f = pq \in IJ$ is also nonconstant. Let $x \in \mathbb{P}^n_{\mathbf{k}}$ such that $f(x) \neq 0$ (there are infinitely many such points). Then $x \notin \mathrm{V}_+(IJ)$, which is a contradiction.

Caution 4.4.7. Integrality is *not* a property of 'local nature', for by Proposition 4.4.2 it is the combination of a local property (reducedness) and a global property (irreducibility). In fact, watch your intuition here: there exist affine connected schemes $X = \operatorname{Spec} A$ such that $A_{\mathfrak{p}}$ is a domain for every $\mathfrak{p} \in X$, and yet X is *not* integral [16, Tag 0568].

Example 4.4.8. We have the following generalisation of Example 4.4.6. Let A be a ring. Then $\mathbb{P}^n_A$ is integral as soon as A is a domain. The case $n = 0$ is covered by Corollary 4.4.3 since $\mathbb{P}^0_A = \operatorname{Spec} A$. Assume $n \geq 1$. The claim follows from the more general statement that

$$\boxed{\text{If } B \text{ is a graded domain with } B_+ \neq 0, \text{ then } \operatorname{Proj} B \text{ is integral.}}$$

We apply Proposition 4.4.2. Since (0) is prime and homogeneous, the scheme $\operatorname{Proj} B$ has a unique generic point, therefore it is irreducible. On the other hand, it is covered by affine schemes $\operatorname{Spec} B_{(f)}$ for nonzero elements $f \in B_+$. Each $B_{(f)}$ is a domain, in particular reduced. Therefore we conclude that $\operatorname{Proj} B$ is also reduced by Proposition 4.3.8.

Caution 4.4.9. It is not true that if $\operatorname{Proj} B$ is an integral scheme then B is an integral domain! Can you find examples?

Exercise 4.4.10. Show that $\operatorname{Spec} \mathbf{k}[x,y]/(xy,y^2)$ is irreducible, but not integral.

Exercise 4.4.11 (Integrality of hypersurfaces)**.** Let $f \in \mathbb{F}[x_1,\ldots,x_n]$ be a polynomial. Show that $\operatorname{Spec} \mathbb{F}[x_1,\ldots,x_n]/(f)$ is reduced (resp. irreducible, resp. integral) if and only if f is square-free (resp. admits only one irreducible factor, resp. is irreducible).

PROPOSITION 4.4.12. *Let X be an integral scheme with generic point ξ. Then the following properties hold.*

(1) *If $V = \operatorname{Spec} A \subset X$ is open and nonempty, then $A \to \mathscr{O}_{X,\xi}$ induces an isomorphism* $\operatorname{Frac} A \xrightarrow{\sim} \mathscr{O}_{X,\xi}$.

(2) *If $U \subset X$ is open and $x \in U$, then the canonical morphisms*
$$\mathscr{O}_X(U) \longrightarrow \mathscr{O}_{X,x}, \qquad \mathscr{O}_{X,x} \longrightarrow \mathscr{O}_{X,\xi}$$
are both injective.

(3) *If $U \subset X$ is open, we have*
$$\mathscr{O}_X(U) = \bigcap_{x \in U} \mathscr{O}_{X,x}$$
as subrings of $\mathscr{O}_{X,\xi}$.

Before giving the proof, we pause for a second to explain where the ring homomorphism
$$\mathscr{O}_{X,x} \longrightarrow \mathscr{O}_{X,\xi}$$
of (2) comes from. In general, when $x \in Z = \overline{\{\xi\}}$ for two points $x, \xi \in X$ on a scheme X, we can say the following. Let $\operatorname{Spec} R \subset X$ be an open subset containing ξ. The closed subset $Z \subset X$ satisfies $Z \cap \operatorname{Spec} R = \overline{\{\mathfrak{p}\}} = \mathrm{V}(\mathfrak{p}) \subset \operatorname{Spec} R$ for some prime ideal $\mathfrak{p} \subset R$. But if $x \in Z$ then every open neighbourhood of x also contains ξ. Pick any such affine open neighbourhood $\operatorname{Spec} R$ of x. Then we are saying $x \in Z \cap \operatorname{Spec} R = \overline{\{\mathfrak{p}\}} = \mathrm{V}(\mathfrak{p})$ as above, and therefore, when viewed as a prime ideal $\mathfrak{q} \subset R$, we must have $\mathfrak{q} \supset \mathfrak{p}$. This translates into an inclusion $R \setminus \mathfrak{q} \subset R \setminus \mathfrak{p}$ and hence into a canonical morphism $R_{\mathfrak{q}} \to R_{\mathfrak{p}}$. Alternatively, one may see the map $\mathscr{O}_{X,x} \to \mathscr{O}_{X,\xi}$ arising from the universal property of direct limits (see Definition B.4.6), since for every open subset $V \hookrightarrow X$ containing x,

the ring $\mathscr{O}_X(V)$ maps (injects, in fact) into $\mathscr{O}_{X,\xi}$ (recall that the generic point lies in *every* open subset).

We can now proceed with the proof of Proposition 4.4.12.

Proof. We proceed step by step.

(1) Since X is irreducible, so is V, which then has a unique generic point by Lemma 3.1.46. But the generic point $\xi \in X$ is also the generic point of V, thus $\operatorname{Frac} A = \mathscr{O}_{V,\xi} = \mathscr{O}_{X,\xi}$, the first identity being guaranteed by Lemma 3.1.52(i).

(2) Fix U and $x \in U$. Pick a regular function $f \in \mathscr{O}_X(U)$ such that $f_x = 0 \in \mathscr{O}_{X,x}$. Then $f|_W = 0 \in \mathscr{O}_X(W)$ for some affine open subset $W \subset U$ such that $x \in W$. If $W' \subset U$ is another (nonempty) affine open subset, then $W \cap W' \neq \emptyset$ (since U is irreducible), so the map

$$\mathscr{O}_{W'}(W') \to \mathscr{O}_{W'}(W \cap W')$$

is injective (see Lemma 3.1.52(iii) and note that $\mathscr{O}_{W'}(W') = \mathscr{O}_X(W')$ is an integral domain), in particular the commutative diagram

$$\begin{tikzcd} \mathscr{O}_X(U) \arrow[r] \arrow[d] & \mathscr{O}_X(W) \arrow[d] \\ \mathscr{O}_X(W') \arrow[r, hook] & \mathscr{O}_X(W \cap W') \end{tikzcd}$$

says that the identity $f|_{W\cap W'} = 0$ implies $f|_{W'} = 0$. We can repeat this by letting W' vary in affine open cover of U, so that by the glueing axiom we obtain $f = 0$, i.e. the map $\mathscr{O}_X(U) \to \mathscr{O}_{X,x}$ is injective.

As for the map $\mathscr{O}_{X,x} \to \mathscr{O}_{X,\xi}$, its injectivity follows from the injectivity of the maps $\mathscr{O}_X(V) \to \mathscr{O}_{X,\xi}$. Indeed, the universal property of colimits

$$\begin{tikzcd} & \mathscr{O}_X(V) \arrow[d] \arrow[dr, hook] & \\ \mathscr{O}_{X,x} \arrow[r, equal] & \varinjlim_{V \ni x} \mathscr{O}_X(V) \arrow[r, dotted] & \mathscr{O}_{X,\xi} \end{tikzcd}$$

realises precisely the map $\mathscr{O}_{X,x} \to \mathscr{O}_{X,\xi}$, which is then injective (e.g. because filtered colimits are exact in Ab, hence commute with kernels).

(3) The inclusion $\mathscr{O}_X(U) \subset \bigcap_{x \in U} \mathscr{O}_{X,x}$ is clear. To prove the converse, we may reduce to the case $U = \operatorname{Spec} A$, in which case $\mathscr{O}_{X,\xi} = \operatorname{Frac} A$. Suppose $f \in \operatorname{Frac} A$ belongs to $A_{\mathfrak{p}}$ for every $\mathfrak{p} \in \operatorname{Spec} A$. Consider the ideal

$$I = \left\{ g \in A \,\middle|\, fg \in A \right\} \subset A.$$

We prove that I is not contained in $\mathfrak{p}$ for any $\mathfrak{p} \in \operatorname{Spec} A$. Assume, by contradiction, that there exists a prime ideal $\mathfrak{p} \subset A$ such that $I \subset \mathfrak{p}$. We have

$$\frac{f}{1} = \frac{a}{g} \in A_{\mathfrak{p}} \subset \operatorname{Frac} A,$$

with $a \in A$ and $g \notin \mathfrak{p}$. But $a = fg \in A$, thus $g \in I \subset \mathfrak{p}$, contradiction. It follows that $I = A$, hence $f \in A$. □

Definition 4.4.13 (Field of rational functions)**.** Let X be an integral scheme. The field $K = \mathscr{O}_{X,\xi}$ is called the *field of rational functions of* X.

A generalisation of the field of rational functions which works on arbitrary schemes will be given in Definition 7.2.1.

4.5 Noetherian schemes

Definition 4.5.1 (Noetherian scheme)**.** A scheme X is called

- *locally noetherian* if every point $x \in X$ has an affine open neighbourhood U such that $\mathscr{O}_X(U)$ is noetherian,
- *noetherian* if it is locally noetherian and quasicompact.

Equivalently, X is noetherian if it can be covered by *finitely many* affine schemes $U_i = \operatorname{Spec} A_i \subset X$ such that each A_i is a noetherian ring.

Example 4.5.2 (Algebraic varieties are noetherian schemes)**.** Let $\mathbb{F}$ be a field. Then an algebraic variety over $\mathbb{F}$ is a noetherian scheme, since a finitely generated $\mathbb{F}$-algebra is a noetherian ring.

Example 4.5.3 (Projective space is a noetherian scheme)**.** If A is a noetherian ring, then $\mathbb{P}^n_A$ and $\mathbb{A}^n_A$ are noetherian schemes (reason: we know they are quasicompact, $\mathbb{P}^n_A$ has an open cover by affine schemes of the form $\mathbb{A}^n_A$, and finally $A[x_1,\ldots,x_n]$ is noetherian if A is, by Hilbert's basis theorem).

Caution 4.5.4. It is not true that the ring of regular functions $\mathscr{O}_X(X)$ of an algebraic variety X is finitely generated. See here. See also Example 4.5.9.

The Locality Lemma applied to noetherianity, which is a local property by Proposition 4.2.5, immediately implies the following.

PROPOSITION 4.5.5. *Let X be a scheme. The following are equivalent:*

(1) *X is locally noetherian.*

(2) *For every open affine $U \subset X$, the ring $\mathscr{O}_X(U)$ is noetherian.*

(3) *There is an affine open covering* $X = \bigcup_{i\in I} U_i$ *such that each* $\mathscr{O}_X(U_i)$ *is noetherian.*

(4) *There is an open covering* $X = \bigcup_{j\in J} V_j$ *such that each* V_j *is locally noetherian.*

An open subscheme of a locally noetherian scheme is locally noetherian.

Proof. Combine Proposition 4.2.5 and Lemma 4.2.2 with one another. □

Remark 4.5.6. An affine scheme $X = \operatorname{Spec} A$ is noetherian if and only if it is locally noetherian, if and only if A is noetherian (using Proposition 4.5.5(2)-(3)). However, $\operatorname{Spec} A$ can be noetherian *as a topological space* even if A is not noetherian. Take for instance $A = \mathbf{k}[x_1, x_2, \ldots]/\mathfrak{m}^2$, where $\mathfrak{m} = (x_1, x_2, \ldots)$. In this case, $\operatorname{Spec} A$ is reduced to a point, but the unique prime ideal $\mathfrak{m}/\mathfrak{m}^2 \subset A$ cannot be generated by finitely many elements.

PROPOSITION 4.5.7. *A noetherian scheme* X *has finitely many irreducible components and finitely many connected components, each a union of irreducible components.*

In particular, an algebraic variety over a field has finitely many irreducible components.

Proof. We show that the underlying topological space of a noetherian scheme is a noetherian topological space. Then the first claim follows by Proposition 4.1.9.

Since X is a noetherian scheme, it can be written as a union of finitely many open subschemes $U_i = \operatorname{Spec} A_i$, where A_i is a noetherian ring. But U_i is itself noetherian as a topological space, for any chain of closed subsets

$$\operatorname{Spec} A_i \supset \mathrm{V}(I_1) \supset \mathrm{V}(I_2) \supset \mathrm{V}(I_3) \supset \cdots$$

must stop (use that $\mathrm{V}(-)$ is inclusion-reversing and that A_i is noetherian). Finally, a finite union of noetherian topological spaces is a noetherian topological space [16, Tag 0053].

The second claim is proved as follows. Let $W_1, \ldots, W_s$ be the irreducible components of X. Pick W_1. If there is an index $i_1 \in \{2, \ldots, s\}$ such that $W_1 \cap W_{i_1} \neq \emptyset$, define $C_1 = W_1 \cup W_{i_1}$. Now check again: if there is an index $i_2 \in \{2, \ldots, s\} \setminus \{i_1\}$ such that $C_1 \cap W_{i_2} \neq \emptyset$, redefine C_1 as $C_1 \cup W_{i_2}$. The process will stop and will produce a closed and connected subset $C_1 \subset X$. Now repeat the process with any of the surviving irreducible components. Also this process will stop. Thus there are finitely many connected components, and each is a union of irreducible components. □

Example 4.5.8. Let $(F_i)_{i\in I}$ be an infinite family of fields, and set $A = \prod_{i\in I} F_i$. Then $X = \operatorname{Spec} A$ is quasicompact but not noetherian. Indeed the ideal

$$I = \{(a_i)_{i\in I} \mid a_i = 0 \text{ for almost all } i \in I\} \subset A$$

is not finitely generated. Each field gives rise to a connected component $C_i = \operatorname{Spec} F_i \subset X$.

Example 4.5.9. Let X be a noetherian scheme. It is not true that $\mathscr{O}_X(U)$ is noetherian for *every* open subset $U \subset X$. It is true for affine U. However, consider the following example, taken from here. Let $A, B \subset \mathbb{P}^3_{\mathbf{k}}$ be two planes meeting along a line $L \subset \mathbb{P}^3_{\mathbf{k}}$. Let $D \neq L$ be another line. Construct the noetherian scheme $X = (A \cup B) \setminus D$. Then $\mathscr{O}_X(X)$ consists of those polynomials $f \in \mathbf{k}[x, y]$ such that $f(x, 0)$ is constant. This is not a noetherian ring.

4.6 Dimension

4.6.1 Krull dimension of schemes

The *dimension* of a scheme X is a numerical invariant (of topological nature)

$$\dim X \in \mathbb{N} \cup \{\pm\infty\},$$

measuring the 'size' of X in a precise sense. If X is an arbitrary topological space, a chain of irreducible closed subsets of X of length ℓ is a filtration

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_\ell \subseteq X.$$

where each $Z_i \subset X$ is closed and irreducible. Recall that the empty set is considered *reducible* (algebro-geometric reason: the zero ring has no minimal primes!), so $\emptyset$ will never appear as the smallest closed subset in any chain as above. At the other side of the chain, $X = Z_\ell$ is allowed if X is irreducible.

We declare that

$$\dim X = -\infty \iff X = \emptyset.$$

Definition 4.6.1 (Krull dimension)**.** The *Krull dimension* of a nonempty topological space X is defined to be

$$\dim X = \sup_\ell \left\{ \ell \,\middle|\, X \text{ has a chain of irreducible closed subsets of length } \ell \right\} \in \mathbb{N} \cup \{\infty\}.$$

The *dimension* of a scheme $(X, \mathscr{O}_X)$ is the Krull dimension of the underlying topological space X.

Example 4.6.2. If X is discrete, then $\dim X = 0$. We also have that $\dim \mathbb{R}^n = 0$, if we endow $\mathbb{R}^n$ with the euclidean topology.

Remark 4.6.3. For any scheme X, we have $\dim X = \dim X_{\mathsf{red}}$, since the notion of dimension is purely topological. For instance, if X is a fat point over a field $\mathbb{F}$, then $\dim X = 0$.

Definition 4.6.4 (Pure dimension)**.** We say that a topological space (or a scheme) X has *pure dimension* d if all of its irreducible components have dimension d. We may also say that X is *equidimensional* (of dimension d).

Definition 4.6.5 (Local dimension)**.** Let $x \in X$ be a point on a topological space (or a scheme) X. The *local dimension of X at x* is defined to be

$$\dim_x X = \inf\left\{ \dim U \,\middle|\, U \text{ is an open neighbourhood of } x \right\}.$$

Example 4.6.6 (Two points, dimension 1)**.** Let R be a DVR (such as $\mathbf{k}[\![t]\!]$ for instance). Set $X = \operatorname{Spec} R$. Then X has only two points x_0, η, where η is open. We have a maximal chain $\{x_0\} \subset X$ of length 1, thus $\dim X = 1 = \dim_{x_0} X$, and $\dim_\eta X = 0 = \dim_\eta \{\eta\}$.

Example 4.6.7 (Spec of a PID)**.** Generalising Example 4.6.6, if A is any principal ideal domain which is not a field, then $X = \operatorname{Spec} A$ has dimension 1. For instance,

$$\begin{aligned} \dim \mathbb{A}^1_{\mathbb{F}} &= 1 \\ \dim \operatorname{Spec} \mathbb{Z} &= 1 \\ \dim \operatorname{Spec} \mathbb{Z}[i] &= 1. \end{aligned}$$

Example 4.6.8 (The arithmetic surface)**.** Consider the affine scheme $X = \operatorname{Spec} \mathbb{Z}[x]$. Then a maximal chain is $\mathrm{V}(p, f) \subset \mathrm{V}(p) \subset X$ (using that X is irreducible). Therefore $\dim X = 2$.

Example 4.6.9. The scheme $X = \operatorname{Spec} \mathbf{k}[x_i \mid i \in \mathbb{N}]$ has infinite Krull dimension. Note that X is not noetherian. However, there are examples of noetherian schemes of infinite Krull dimension [16, Tag 02JC].

LEMMA 4.6.10. *Let X be a topological space.*

(i) *If $Y \subset X$ is a subspace, then* $\dim Y \le \dim X$*.*

(ii) *If X is irreducible of finite dimension, and $Y \subset X$ is a closed subset, then $Y = X$ if and only if* $\dim Y = \dim X$*.*

(iii) *If $(W_i)_{i \in I}$ are the irreducible components of X, then*

$$\dim X = \sup_{i \in I} \dim W_i.$$

(iv) *There is an identity*

$$\dim X = \sup_{x \in X} \dim_x X.$$

Proof. We proceed step by step.

(i) Any chain of irreducible closed subsets of Y induces a chain of irreducible closed subsets of X. Indeed, if $Y_1 \subsetneq Y_2$ are closed irreducible in Y, then their closures in X satisfy $\overline{Y}_1 \subsetneq \overline{Y}_2$.

(ii) If $Y \subsetneq X$, then a chain $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_\ell \subseteq Y$ of length $\ell = \dim Y$ gives rise to a chain $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_\ell \subsetneq Z_{\ell+1} = X$ of length $\ell+1$. This contradicts $\dim Y = \dim X$.

(iii) Every Z_k appearing in a chain of closed irreducible subsets of X is contained in an irreducible component of X by Proposition 4.1.6.

(iv) For any $x \in X$, and for any open neighbourhood U of x, we have $\dim U \le \dim X$ by (i). Thus $\dim_x X \le \dim X$, which proves that $\sup_{x\in X} \dim_x X \le \dim X$. On the other hand, fix a chain of irreducible closed subsets

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_\ell \subseteq X.$$

Pick $x \in Z_0$, and an open neighbourhood U of x. Then

$$Z_0 \cap U \subsetneq Z_1 \cap U \subsetneq \cdots \subsetneq Z_\ell \cap U \subseteq U$$

is a chain of irreducible closed subsets of U. This shows that $\dim U \ge \ell$ and $\dim_x X \ge \ell$, as required. □

COROLLARY 4.6.11. *Let $\iota\colon Y \to X$ be a closed immersion of schemes, where X is integral of finite dimension, and* $\dim Y = \dim X$. *Then ι is an isomorphism.*

Proof. Combine Lemma 4.6.10(ii) and the properties of reduced schemes (Proposition 4.3.5). □

Remark 4.6.12 (Dimension is local)**.** Note that Lemma 4.6.10(iv) implies that the dimension is local: given an open cover $X = \bigcup_{i\in I} U_i$ one has

$$\dim X = \sup_{i\in I} \dim U_i.$$

For many assertions regarding the dimension of a scheme, we may then often be able to reduce to the case where X is affine. By Lemma 4.6.10(iii), in fact, a reduction to the irreducible case is possible. And since the dimension does not see any nonreduced structure (cf. Remark 4.6.3), the final reduction would be to *affine integral schemes.*

Definition 4.6.13 (Codimension)**.** Let $Y \subset X$ be an irreducible closed subset of a topological space X. The *codimension* of Y in X is defined as

$$\operatorname{codim}(Y,X) = \sup\left\{ \ell \;\middle|\; \begin{array}{c} \text{there is a chain of irreducible} \\ \text{closed subsets } Y = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_\ell \end{array} \right\}. \tag{4.6.1}$$

If Y is an arbitrary closed subset with irreducible components $(Y_i)_{i\in I}$, then we set

$$\operatorname{codim}(Y,X) = \inf_{i\in I} \operatorname{codim}(Y_i,X).$$

Example 4.6.14. If $Y = \{x\} \subset X$ for some point $x \in X$, then

$$\operatorname{codim}(\{x\}, X) = \operatorname{codim}(\overline{\{x\}}, X).$$

It follows easily from the definition that, for an arbitrary closed subset $Y \subset X$, one has the inequality

$$\operatorname{codim}(Y, X) + \dim Y \leq \dim X.$$

Equality holds in special cases (cf. Proposition 4.6.35). See Example 4.6.40 for an example where equality does *not* hold.

4.6.2 Krull dimension of rings

Let A be a ring, $\mathfrak{p} \subset A$ a prime ideal. We now consider strictly ascending chains

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_\ell = \mathfrak{p} \tag{4.6.2}$$

of prime ideals contained in $\mathfrak{p}$. Note that these correspond to filtrations

$$\mathrm{V}(\mathfrak{p}) = \mathrm{V}(\mathfrak{p}_\ell) \subsetneq \cdots \subsetneq \mathrm{V}(\mathfrak{p}_1) \subsetneq \mathrm{V}(\mathfrak{p}_0)$$

of closed irreducible subspaces of $\operatorname{Spec} A$, as in (4.6.1).

Definition 4.6.15 (Height of an ideal)**.** Let A be a ring, $\mathfrak{p} \subset A$ a prime ideal. The *height* of $\mathfrak{p}$, denoted $\mathsf{ht}(\mathfrak{p})$, is the supremum of the lengths of the strictly ascending chains of prime ideals contained in $\mathfrak{p}$, as in (4.6.2). The height of an arbitrary ideal $I \subset A$ is defined to be

$$\mathsf{ht}(I) = \inf_{\mathfrak{p} \in \mathrm{V}(I)} \mathsf{ht}(\mathfrak{p}).$$

Definition 4.6.16 (Krull dimension of a ring)**.** Let A be a nonzero ring. The *Krull dimension* of A is defined to be

$$\dim A = \sup_{\mathfrak{p} \in \operatorname{Spec} A} \mathsf{ht}(\mathfrak{p}).$$

We set $\dim(0) = -\infty$.

By Proposition 3.1.43, there is an inclusion reversing bijective correspondence $\mathrm{V}(I) \longleftrightarrow \sqrt{I}$ between irreducible closed subsets $Z \subset \operatorname{Spec} A$ and prime ideals $\mathfrak{p} \subset A$, therefore

$$\dim A = \dim \operatorname{Spec} A = \dim \operatorname{Spec} A/\sqrt{0} = \dim A/\sqrt{0},$$

where we have used Remark 4.6.3 for the middle equality.

Example 4.6.17. Let A be a ring. If $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_\ell$ is a chain of prime ideals in A, then we have a chain of prime ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_\ell \subsetneq \mathfrak{p}_\ell + (x)$$

in the ring $A[x]$. Therefore

$$\dim A[x] \geq 1 + \dim A.$$

In fact, equality holds if A is noetherian (cf. Theorem 4.6.24), but not in general. Thus, as expected,

$$\boxed{\dim \mathbb{A}^n_{\mathbb{F}} = n \text{ for any field } \mathbb{F}.}$$

LEMMA 4.6.18. *Let A be a ring, $\mathfrak{p} \subset A$ a prime ideal. Then*

$$\dim A_{\mathfrak{p}} = \operatorname{ht}(\mathfrak{p}) = \operatorname{codim}(\mathrm{V}(\mathfrak{p}), \operatorname{Spec} A).$$

Moreover,

$$\dim A = \sup_{\mathfrak{m} \subset A} \dim A_{\mathfrak{m}},$$

the supremum being taken over all maximal ideals of A.

Proof. The identity $\dim A_{\mathfrak{p}} = \operatorname{ht}(\mathfrak{p})$ is essentially by definition, after recalling that prime ideals in $A_{\mathfrak{p}}$ correspond, under an inclusion preserving bijection, to prime ideals $\mathfrak{q} \subset \mathfrak{p}$. The identity $\operatorname{ht}(\mathfrak{p}) = \operatorname{codim}(\mathrm{V}(\mathfrak{p}), \operatorname{Spec} A)$ again follows from the correspondence between ascending chains

$$\mathrm{V}(\mathfrak{p}) \subseteq Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_\ell$$

of irreducible closed subsets of $\operatorname{Spec} A$ and descending chains of prime ideals

$$\mathfrak{p} \supseteq \mathfrak{p}_0 \supsetneq \cdots \supsetneq \mathfrak{p}_\ell$$

contained in $\mathfrak{p}$.

The identity $\dim A = \sup_{\mathfrak{m} \subset A} \dim A_{\mathfrak{m}}$ follows by the same observation, coupled with the observation that every prime ideal of A is contained in a maximal ideal. In symbols,

$$\sup_{\mathfrak{m} \subset A} \dim A_{\mathfrak{m}} = \sup_{\mathfrak{p} \in \operatorname{Spec} A} \dim A_{\mathfrak{p}} = \sup_{\mathfrak{p} \in \operatorname{Spec} A} \operatorname{ht}(\mathfrak{p}) = \dim A. \qquad \square$$

LEMMA 4.6.19. *Let X be a scheme. Let $Y \subset X$ be an irreducible closed subset with generic point ξ. Then*

$$\operatorname{codim}(Y, X) = \dim \mathscr{O}_{X,\xi}.$$

If $x \in X$ is a point, then x is the generic point of an irreducible component of X if and only if $\dim \mathscr{O}_{X,x} = 0$.

Proof. To prove the first assertion, note that for any open subset $U \subset X$ such that $Y \cap U \neq \emptyset$ we have $\operatorname{codim}(Y, X) = \operatorname{codim}(Y \cap U, U)$. This follows from the fact that sending $V \mapsto \overline{V}$ (closure in X) defines a bijective inclusion preserving correspondence

$$\left\{ V \subset U \;\middle|\; \begin{array}{c} V \text{ is closed,} \\ \text{irreducible} \end{array} \right\} \xrightarrow{\sim} \left\{ W \subset X \;\middle|\; \begin{array}{c} W \text{ is closed, irreducible} \\ \text{and } W \cap U \neq \emptyset \end{array} \right\}.$$

We may then replace X with an affine open neighbourhood $U \subset X$ of ξ. Then

$$\operatorname{codim}(Y,X) = \operatorname{codim}(\mathrm{V}(\xi),U) = \dim \mathscr{O}_{U,\xi} = \dim \mathscr{O}_{X,\xi},$$

the second identity following from Lemma 4.6.18. The final assertion follows by maximality of irreducible components: they have codimension 0. □

THEOREM 4.6.20 (Krull). *Let A be a noetherian ring, $f \in A \setminus A^\times$, and $\mathfrak{p} \subset A$ a prime ideal which is minimal among those containing f. Then $\mathrm{ht}(\mathfrak{p}) \leq 1$.*

Proof. See [12, Ch. 2, Thm. 5.12]. □

In fact, combining Krull's theorem with an inductive argument, one can prove the following.

PROPOSITION 4.6.21 ([12, Ch. 2, Cor. 5.14]). *Let A be a noetherian ring.*

(a) *Let $I \subset A$ be an ideal that can be generated by r elements. Then every minimal prime $\mathfrak{p} \subset A$ above I has $\mathrm{ht}(\mathfrak{p}) \leq r$. In particular, $\mathrm{ht}(I) \leq r$.*

(b) *If A is local, with maximal ideal $\mathfrak{m} \subset A$, then $\dim A < \infty$ and*

$$\dim A \leq \dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2. \tag{4.6.3}$$

In fact, Equation (4.6.3) follows easily from (a). Indeed, by Nakayama's lemma, we have that $\mathfrak{m}$ can be generated by $e = \dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$ elements. On the other hand, since A is local, $\dim A = \dim A_{\mathfrak{m}} = \mathrm{ht}(\mathfrak{m})$ by Lemma 4.6.18. By (a), we conclude $\dim A \leq e$. Equation (4.6.3) has a geometric relevance: when $(A,\mathfrak{m})$ is the local ring of a point x on a noetherian scheme X, *equality* corresponds to X being *regular* at x.

PROPOSITION 4.6.22 ([12, Ch. 2, Thm. 5.15]). *Let $(A,\mathfrak{m})$ be a noetherian local ring. Fix $t \in \mathfrak{m}$. Then*

$$\dim A/tA \geq \dim A - 1,$$

with equality if t does not belong to any minimal prime ideal $\mathfrak{p} \subset A$.

Example 4.6.23. Let $(A,\mathfrak{m})$ be a *reduced* noetherian local ring. Assume $t \in \mathfrak{m}$ is not a zero-divisor. Then t does not belong to any minimal prime ideal $\mathfrak{p} \subset A$ therefore $\dim A/tA = \dim A - 1$.

THEOREM 4.6.24 ([12, Ch. 2, Cor. 5.17]). *If A is a noetherian ring, then $\dim A[x_1,\ldots,x_n] = n + \dim A$.*

In particular, we see that for any field $\mathbb{F}$ the formula

$$\dim \mathbb{A}^n_{\mathbb{F}} = n$$

holds true, and since $\mathbb{P}^n_{\mathbb{F}}$ has an affine open cover by n-dimensional affine spaces, by Remark 4.6.12 we have

$$\dim \mathbb{P}^n_{\mathbb{F}} = n.$$

4.6.3 Dimension of algebraic varieties

Recall that a field extension $K/\mathbb{F}$ is of finite transcendence degree equal to d if there is an algebraic extension $K \supset \mathbb{F}(x_1,\ldots,x_d)$. We write

$$\mathsf{trdeg}_{\mathbb{F}} K = d.$$

Definition 4.6.25 (Finite, integral)**.** Let $\phi\colon A \to B$ be a ring homomorphism. We say that ϕ is *integral*, or that B is integral over A, if for any $b \in B$ there exists a monic polynomial $p(T) = \alpha_0 + \alpha_1 T + \cdots + \alpha_{m-1} T^{m-1} + T^m \in A[T]$ such that

$$\phi(\alpha_0) + \phi(\alpha_1) b + \cdots + \phi(\alpha_{m-1}) b^{m-1} + b^m = 0$$

in B. We say that ϕ is *finite* if it makes B into a finitely generated A-module. Equivalently, ϕ is finite if it is integral and makes B into a finitely generated A-algebra.

Exercise 4.6.26. Given an injective integral ring homomorphism $A \hookrightarrow B$, one has that B is a field if and only if A is a field.

Exercise 4.6.27. Let $\phi\colon A \to B$ be an integral ring homomorphism.

(i) Show that $f_\phi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ maps closed points to closed points (cf. Exercise 3.1.56).

(ii) Let $\mathfrak{p} \in \operatorname{Spec} A$. Show that the canonical ring homomorphism $A_{\mathfrak{p}} \to B \otimes_A A_{\mathfrak{p}}$ is integral.

(iii) Let $\mathfrak{p} \in \operatorname{Spec} A$. Set $T = \phi(A \setminus \mathfrak{p})$. Assume ϕ is injective. Show that $T \subset B$ is a multiplicative subset of B and $B \otimes_A A_{\mathfrak{p}} = T^{-1} B \neq 0$.

Exercise 4.6.28. Let $\phi\colon A \to B$ be an integral ring homomorphism.

(i) Prove that $\dim B \le \dim A$.

(ii) Show that if ϕ is injective then $f_\phi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is surjective and, in fact, $\dim A = \dim B$. (**Hint**: Use Exercise 4.6.27(iii) to prove surjectivity of f_ϕ).

THEOREM 4.6.29 (Noether normalisation). *Let A be a nonzero finitely generated $\mathbb{F}$-algebra. Then, there exists a finite injective ring homomorphism*

$$\mathbb{F}[x_1,\ldots,x_d] \hookrightarrow A.$$

Proof. See [12, Ch. 2, Prop. 1.9]. □

Remark 4.6.30. The geometric content of Noether's normalisation theorem is that every affine variety $X = \operatorname{Spec} A$ admits a finite surjective morphism of schemes

$$\operatorname{Spec} A \longrightarrow \mathbb{A}^d_{\mathbb{F}},$$

where finiteness for a morphism will be defined in Definition 5.6.2.

Noether's normalisation theorem immediately implies that an integral affine $\mathbb{F}$-variety, i.e. an $\mathbb{F}$-scheme of the form $X = \operatorname{Spec} A$, where A if a finitely generated integral domain over $\mathbb{F}$, has its field of rational functions $K = \operatorname{Frac} A$ of finite transcendence degree over the base field $\mathbb{F}$.

$$\begin{tikzcd} A \arrow[r, hook] & K \arrow[d, dash, "\text{algebraic}"] \\ \mathbb{F}[x_1,\dots,x_d] \arrow[u, hook, "\text{finite}"] \arrow[r, hook] & \mathbb{F}(x_1,\dots,x_d) \arrow[d, dash] \\ & \mathbb{F} \end{tikzcd} \tag{4.6.4}$$

PROPOSITION 4.6.31. *If Y is an integral $\mathbb{F}$-variety and $U \subset Y$ is a nonempty open subset, one has*

$$\dim U = \dim Y = \operatorname{trdeg}_{\mathbb{F}} K(Y),$$

where $K(Y) = \mathscr{O}_{Y,\xi}$ is the field of rational functions on Y.

Proof. Granting the identity $\dim Y = \operatorname{trdeg}_{\mathbb{F}} \mathscr{O}_{Y,\xi}$ for a second, we obtain

$$\dim U = \operatorname{trdeg}_{\mathbb{F}} \mathscr{O}_{U,\xi} = \operatorname{trdeg}_{\mathbb{F}} \mathscr{O}_{Y,\xi} = \dim Y$$

since Y and U have the same generic point. As for the first identity, since the dimension is local, we may assume $Y = \operatorname{Spec} A$ where A is a finitely generated integral domain over $\mathbb{F}$. We find, if $d = \operatorname{trdeg}_{\mathbb{F}} K$ is as in (4.6.4),

$$\dim X = \dim A = \dim \mathbb{F}[x_1,\dots,x_d] = d = \operatorname{trdeg}_{\mathbb{F}} K.$$

The second identity uses the algebraic fact of Exercise 4.6.28(ii). □

COROLLARY 4.6.32. *Let B be a homogeneous $\mathbb{F}$-algebra. Then*

$$\dim \operatorname{Spec} B = 1 + \dim \operatorname{Proj} B.$$

Proof. We may assume B is integral, since all minimal primes $\mathfrak{p}$ of B are homogeneous, and it is enough to compare $\dim \mathrm{V}(\mathfrak{p})$ and $\dim \mathrm{V}_+(\mathfrak{p})$ by Lemma 4.6.10(iii).

Pick a degree 1 homogeneous element $f \in B_+$. Then, we have a canonical isomorphism

$$B_{(f)}[x, x^{-1}] \xrightarrow{\sim} B_f$$

sending $x \mapsto 1/f$. Thus

$$\begin{aligned} \dim \operatorname{Spec} B &= \dim \mathrm{D}(f) \\ &= \dim B_f \\ &= 1 + \dim B_{(f)} \\ &= 1 + \dim \mathrm{D}_+(f) \\ &= 1 + \dim \operatorname{Proj} B, \end{aligned}$$

as required. We have used Proposition 4.6.31 in the first and last equalities. □

Exercise 4.6.33. Show that the conclusion of Proposition 4.6.31 may fail even for affine $\mathbb{F}$-schemes $\operatorname{Spec} R$ where R is not finitely generated over $\mathbb{F}$.

LEMMA 4.6.34 ([12, Ch. 2, Lemma 5.22]). *Let A be a finitely generated integral domain over $\mathbb{F}$, and let $\mathfrak{p} \subset A$ be a prime ideal of height 1. Then*

$$\dim A/\mathfrak{p} = \dim A - 1.$$

PROPOSITION 4.6.35. *Let A be a finitely generated integral domain over $\mathbb{F}$, and let $\mathfrak{p} \subset A$ be a prime ideal. Then*

$$\mathsf{ht}(\mathfrak{p}) + \dim A/\mathfrak{p} = \dim A,$$

and if $\mathfrak{p}$ is maximal we have

$$\dim A_{\mathfrak{p}} = \dim A.$$

Proof. The second assertions follows from the first, combined with Lemma 4.6.18 and the fact that a field has dimension 0.

So let us prove the first assertion by induction on the height of $\mathfrak{p}$. If $\mathsf{ht}(\mathfrak{p}) = 0$, then $\mathfrak{p} = (0)$ and we are done. If $\mathsf{ht}(\mathfrak{p}) = d > 0$, then we have a chain of prime ideals

$$(0) = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_d = \mathfrak{p}$$

of length d. The prime ideal

$$\mathfrak{p}/\mathfrak{p}_1 \subset A/\mathfrak{p}_1$$

has height $d-1$, thus

$$\begin{aligned}\dim A/\mathfrak{p}_1 &= \mathrm{ht}(\mathfrak{p}/\mathfrak{p}_1) + \dim (A/\mathfrak{p}_1)/(\mathfrak{p}/\mathfrak{p}_1)\\ &= d-1+\dim A/\mathfrak{p}\\ &= \mathrm{ht}(\mathfrak{p})-1+\dim A/\mathfrak{p},\end{aligned}$$

where the first identity used the inductive step. Thus

$$\mathrm{ht}(\mathfrak{p})+\dim A/\mathfrak{p} = \dim A/\mathfrak{p}_1 + 1 = \dim A,$$

where we have used Lemma 4.6.34 and that $\mathfrak{p}_1$ has height 1 for the last identity. □

COROLLARY 4.6.36. *If X is an integral $\mathbb{F}$-variety and $Y \subset X$ is an integral subvariety, then*

$$\dim X = \dim Y + \operatorname{codim}(Y, X).$$

Proof. Since the dimension is local, we may reduce to the case where X (and therefore Y) is affine. The claim then becomes the content of Proposition 4.6.35. □

Remark 4.6.37. In fact, Corollary 4.6.36 holds in greater generality: it is enough to assume that X is a pure and locally of finite type[2] $\mathbb{F}$-scheme and Y is an irreducible closed subset. See [19, Thm. 11.2.9].

COROLLARY 4.6.38. *Let X be an irreducible algebraic variety over a field $\mathbb{F}$. Then, for any closed point $x \in X$, we have*

$$\dim \mathscr{O}_{X,x} = \dim X.$$

Proof. We may assume $X = \operatorname{Spec} A$ is affine and integral. Then a closed point x correspond to a maximal ideal $\mathfrak{m} \subset A$, so that $\dim X = \dim A = \dim A_{\mathfrak{m}} = \dim \mathscr{O}_{X,x}$, where Proposition 4.6.35 is used for the second identity. □

COROLLARY 4.6.39. *Let $X = \operatorname{Spec} A$ be an irreducible affine variety of dimension d over a field $\mathbb{F}$. Then, for any $f \in A \setminus \sqrt{0}$, the closed subset $\mathrm{V}(f) \subset X$ is of pure dimension $d-1$.*

Proof. If $\mathfrak{p} \subset A$ is a minimal prime above $(f) \subset A$, then $\mathrm{ht}(\mathfrak{p}) = 1$ by Theorem 4.6.20. The statement then follows from Lemma 4.6.34. □

Example 4.6.40. The conclusion of Corollary 4.6.36 can fail even for 'nice' rings. For instance, let $(R, \mathfrak{m})$ be a DVR and set $A = R[t]$. Then $\dim A = 2$. Let $\pi \in \mathfrak{m}$ be a uniformiser, and form the ideal $\mathfrak{p} = (\pi t - 1) \subset A$. Then $\mathfrak{p}$ is maximal, for

$$A/\mathfrak{p} = R[t]/(\pi t - 1) = R[1/\pi] = \operatorname{Frac} R.$$

In particular, $\dim A/\mathfrak{p} = 0$. But then $\mathrm{ht}(\mathfrak{p}) + \dim A/\mathfrak{p} = 1 + 0 < 2 = \dim A$.

[2] The notion 'locally of finite type' is defined in Definition 5.2.4.

4.6.4 0-dimensional schemes

Recall that a ring A is artinian if the descending chain condition holds for its ideals, i.e. every descending chain

$$I_0 \supset I_1 \supset \cdots \supset I_n \supset I_{n+1} \supset \cdots$$

of ideals $I_k \subset A$ eventually stabilises.

LEMMA 4.6.41. *Let $(A, \mathfrak{m}, \kappa)$ be a noetherian local ring. Then, the following are equivalent.*

(i) $\dim A = 0$,

(ii) $\mathfrak{m} = \sqrt{0}$,

(iii) $\mathfrak{m}^q = 0$ *for some* $q > 0$,

(iv) *A is artinian.*

Proof. We proceed as follows.

(i) $\Leftrightarrow$ (ii). Indeed, since A is local, being of dimension 0 means that there cannot be any other prime ideals. This, in turn, means that $\mathfrak{m} = \sqrt{0}$.

(ii) $\Rightarrow$ (iii). As A is noetherian, $\mathfrak{m}$ if finitely generated, say $\mathfrak{m} = (a_1, \ldots, a_s)$. These elements are nilpotent by assumption, so $a_i^e = 0$ for some $e > 0$, and for all $i = 1, \ldots, s$ at once. Thus $\mathfrak{m}^{es} = 0$.

(iii) $\Rightarrow$ (iv). Let us fix a descending chain

$$I_0 \supset I_1 \supset \cdots \supset I_n \supset I_{n+1} \supset \cdots$$

that we must prove to be stationary. Since A is noetherian, we have that $\mathfrak{m}^r/\mathfrak{m}^{r+1}$ is a *finite-dimensional* κ-vector space, for every $r \geq 0$. The inclusions

$$I_n \cap \mathfrak{m}^{r+1} \subset I_n \cap \mathfrak{m}^r$$

yield subquotients

$$\frac{\mathfrak{m}^r}{\mathfrak{m}^{r+1}} \supset \frac{I_0 \cap \mathfrak{m}^r}{I_0 \cap \mathfrak{m}^{r+1}} \supset \cdots \supset \frac{I_n \cap \mathfrak{m}^r}{I_n \cap \mathfrak{m}^{r+1}} \supset \frac{I_{n+1} \cap \mathfrak{m}^r}{I_{n+1} \cap \mathfrak{m}^{r+1}} \supset \cdots \tag{4.6.5}$$

with quotient in the n-th step isomorphic to

$$\frac{I_n \cap \mathfrak{m}^r}{I_{n+1} + (I_n \cap \mathfrak{m}^{r+1})}.$$

This descending filtration (4.6.5) must stop: there is an index n_0 such that for every $n \geq n_0$ and every $r \leq q$ one has

$$I_n \cap \mathfrak{m}^r \subseteq I_{n+1} + (I_n \cap \mathfrak{m}^{r+1}).$$

For $r = 0$ this yields

$$I_n \subseteq I_{n+1} + (I_n \cap \mathfrak{m}).$$

But then for $r = 1$ we get

$$I_n \cap \mathfrak{m} \subseteq I_{n+1} + (I_n \cap \mathfrak{m}^2).$$

Continuing, we obtain a chain

$$I_n \subseteq I_{n+1} + (I_n \cap \mathfrak{m}) \subseteq I_{n+1} + (I_n \cap \mathfrak{m}^2) \subseteq \cdots \subseteq I_{n+1} + (I_n \cap \mathfrak{m}^q) = I_{n+1},$$

which is what we wanted.

(iv) ⇒ (iii) ⇒ (ii). Since A is artinian, we have that

$$\mathfrak{m} \supset \mathfrak{m}^2 \supset \mathfrak{m}^3 \supset \cdots$$

must stop, i.e. there is an index q such that $\mathfrak{m}^q = \mathfrak{m}^{q+1}$. Then $\mathfrak{m}^q = 0$ by Nakayama's lemma, so (iii) is confirmed. But if $\mathfrak{m}^q = 0$, all the generators of $\mathfrak{m}$ are nilpotent, hence $\mathfrak{m} \subset \sqrt{0}$. Since $\mathfrak{m}$ is maximal, this is only possible with equality, thus (ii) is confirmed and the proof is complete. □

4.7 Fibre products of schemes and base change

4.7.1 The universal property, and statement of existence

Let S, X and Y be schemes. Given two morphisms

(4.7.1)
$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & S \end{tikzcd}$$

we want to construct their *fibre product* in Sch_S. In strictly categorical terms, the fibre product of the morphisms $f : X \to S$ and $g : Y \to S$ is a triple $(X \times_S Y, p, q)$ where $X \times_S Y$ is a scheme over S, $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ are morphisms making the diagram

$$\begin{tikzcd} X \times_S Y \arrow[r, "p"] \arrow[d, "q"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & S \end{tikzcd}$$

commutative, and the triple $(X \times_S Y, p, q)$ is universal with respect to this property. We present the following fundamental result without proof.

THEOREM 4.7.1 ([8, Ch. II, Thm. 3.3]). *The fibre product $(X \times_S Y, p, q)$ of any two morphisms as in (4.7.1) exists in Sch_S and is unique up to unique isomorphism.*

As ever, the uniqueness part follows from the universal property.

The diagram

$$\begin{tikzcd}
V \arrow[drr, bend left, "a"] \arrow[ddr, bend right, "b"'] \arrow[dr, dotted, "u"] & & \\
& X \times_S Y \arrow[r, "p"] \arrow[d, "q"] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"] & S
\end{tikzcd}$$

illustrates this universal property: a morphism of S-schemes $V \to X \times_S Y$ corresponds to pairs of morphisms

$$(a, b) \in \mathrm{Hom}_{\mathsf{Sch}_S}(V, X) \times \mathrm{Hom}_{\mathsf{Sch}_S}(V, Y)$$

such that $f \circ a = g \circ b$ as S-morphisms from V to S. Put differently, for any S-scheme $V \to S$ one has a bijection

$$\mathrm{Hom}_{\mathsf{Sch}_S}(V, X \times_S Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Sch}_S}(V, X) \times_{\mathrm{Hom}_{\mathsf{Sch}_S}(V,S)} \mathrm{Hom}_{\mathsf{Sch}_S}(V, Y). \tag{4.7.2}$$

Terminology 4.7.2. If Z is an S-scheme and T is another S-scheme, the elements of the set

$$\mathrm{Hom}_{\mathsf{Sch}_S}(T, Z)$$

are called the *T-valued points* of the S-scheme Z. Of course, if $T = \operatorname{Spec} A$ is affine, we talk about A-valued points. If $\pi \colon Z \to S$ is the structure morphism, and $T = S$, we set

$$Z(S) = \mathrm{Hom}_{\mathsf{Sch}_S}(S, Z) = \{\sigma : S \to Z \mid \pi \circ \sigma = \mathrm{id}_S\} = \{\text{sections of } \pi\}.$$

These are the S-valued points of $\pi \colon Z \to S$.

Setting $V = S$ in (4.7.2), we obtain a bijection

$$(X \times_S Y)(S) \xrightarrow{\sim} X(S) \times Y(S). \tag{4.7.3}$$

Notation 4.7.3. We use the notation

$$\begin{tikzcd}
Z \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
Y \arrow[r, "g"] & S
\end{tikzcd}$$

to indicate that (Z, p, q) is the fibre product of the maps f and g. We also say, interchangeably, that the square is *cartesian*, or a *fibre diagram*. When S is affine, say $S = \operatorname{Spec} R$, we use the shorthand notation

$$X \times_R Y = X \times_{\operatorname{Spec} R} Y.$$

When $Y = \operatorname{Spec} B$, we use the shorthand notation

$$X_B = X \times_S B = X \times_S \operatorname{Spec} B.$$

4.7.2 The affine case

Assume $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$ and $S = \operatorname{Spec} R$. Then to give (f, g) as in (4.7.1) is the same as to give two ring homomorphisms

$$\begin{tikzcd} & A \\ B & R \arrow[l] \arrow[u] \end{tikzcd}$$

i.e. two R-algebras. Their tensor product $A \otimes_R B$ satisfies the universal property of pushouts in this category, via the maps $A \to A \otimes_R B$ sending $a \mapsto a \otimes_R 1_B$ and $B \to A \otimes_R B$ sending $b \mapsto 1_A \otimes_R b$ respectively.

$$\begin{tikzcd} C & & \\ & A \otimes_R B \arrow[ul, dotted] & A \arrow[l] \arrow[ull, bend right] \\ & B \arrow[u] \arrow[uul, bend left] & R \arrow[l] \arrow[u] \end{tikzcd}$$

By applying Spec, which is a contravariant equivalence, we confirm that $\operatorname{Spec}(A \otimes_R B)$ satisfies the universal property of fibre products in the category of R-schemes. Exploiting Theorem 3.2.8, one can show that

$$\begin{tikzcd} \operatorname{Spec}(A \otimes_R B) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} A \arrow[d] \\ \operatorname{Spec} B \arrow[r] & \operatorname{Spec} R \end{tikzcd}$$

is a fibre diagram in the category of R-*schemes* (i.e. the universal property is satisfied by replacing $\operatorname{Spec} C$ with an arbitrary R-scheme). Indeed, if V is an arbitrary R-scheme, we find

$$\begin{aligned} \operatorname{Hom}_{\mathsf{Sch}_R}(V, \operatorname{Spec}(A \otimes_R B)) &= \operatorname{Hom}_{\mathsf{Alg}_R}(A \otimes_R B, \mathscr{O}_V(V)) \\ &= \operatorname{Hom}_{\mathsf{Alg}_R}(A, \mathscr{O}_V(V)) \times_{\operatorname{Hom}_{\mathsf{Alg}_R}(R, \mathscr{O}_V(V))} \operatorname{Hom}_{\mathsf{Alg}_R}(B, \mathscr{O}_V(V)) \\ &= \operatorname{Hom}_{\mathsf{Sch}_R}(V, \operatorname{Spec} A) \times_{\operatorname{Hom}_{\mathsf{Sch}_R}(V, \operatorname{Spec} R)} \operatorname{Hom}_{\mathsf{Sch}_R}(V, \operatorname{Spec} B). \end{aligned}$$

Example 4.7.4. Starting from a diagram

$$\begin{tikzcd} & R[x_1, \ldots, x_n] \\ B & R \arrow[l] \arrow[u] \end{tikzcd}$$

we obtain

$$\mathbb{A}^n_R \times_R B = \operatorname{Spec}(R[x_1, \ldots, x_n] \otimes_R B) = \operatorname{Spec} B[x_1, \ldots, x_n] = \mathbb{A}^n_B.$$

This can be seen as an *extension of scalars*, meaning that coefficients in R have been replaced, after base change, by coefficients in B. As a special case, if also B is a polynomial R-algebra, say in m variables, we get

$$\mathbb{A}^n_R \times_R \mathbb{A}^m_R = \mathbb{A}^{n+m}_R$$

canonically.

Remark 4.7.5. Setting $n = m = 1$ in Example 4.7.4 and $R = \mathbb{C}$ (for instance), we get

$$\mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}^1_{\mathbb{C}} = \mathbb{A}^2_{\mathbb{C}},$$

and the bijection (4.7.3) yields an identitification

$$\mathbb{A}^2_{\mathbb{C}}(\mathbb{C}) \xrightarrow{\sim} \mathbb{A}^1_{\mathbb{C}}(\mathbb{C}) \times \mathbb{A}^1_{\mathbb{C}}(\mathbb{C})$$

between the sets of $\mathbb{C}$-valued points (cf. Terminology 4.7.2), but we observe that the Zariski topology on the left hand side (the fibre product) is not the product topology: in $\mathbb{A}^2$ we have curves such as $\mathrm{V}(y - x^2)$, whereas in the product topology $\mathbb{A}^1_{\mathbb{C}} \times \mathbb{A}^1_{\mathbb{C}}$ we only have product of finite sets as closed subsets.

Remark 4.7.6. Let S be a scheme. Let $X \to S$ and $Y \to S$ be two S-schemes. The universal property of the fibre product in the category of topological spaces induces a canonical continuous map

$$\beta \colon |X \times_S Y| \longrightarrow |X| \times_{|S|} |Y|. \tag{4.7.4}$$

This map turns out to be surjective. However, it is not injective, in general. For instance, consider $S = \operatorname{Spec} \mathbb{R}$, and $X = Y = \operatorname{Spec} \mathbb{C}$ (and the obvious maps to S induced by $\mathbb{R} \hookrightarrow \mathbb{C}$). Then

$$\begin{aligned} X \times_S Y &= \operatorname{Spec} \mathbb{R}[x]/(x^2+1) \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec} \mathbb{C} \\ &= \operatorname{Spec} \mathbb{R}[x]/(x^2+1) \otimes_{\mathbb{R}} \mathbb{C} \\ &= \operatorname{Spec} \mathbb{C}[x]/(x^2+1) \\ &= \operatorname{Spec} \mathbb{C}[x]/(x-i) \oplus \mathbb{C}[x]/(x+i) \\ &= \operatorname{Spec} \mathbb{C} \oplus \mathbb{C} \\ &= \operatorname{Spec} \mathbb{C} \amalg \operatorname{Spec} \mathbb{C}, \end{aligned}$$

which consists of two points, unlike

$$|X| \times_{|S|} |Y|,$$

which consists of only one point.

Remark 4.7.7. The calculation also shows that

$$\boxed{\text{irreducibility is not preserved under base change!}}$$

Indeed, the $\mathbb{R}$-scheme $X = \operatorname{Spec}\mathbb{R}[x]/(x^2+1)$ is irreducible since $\mathbb{R}[x]/(x^2+1) = \mathbb{C}$ has prime nilradical, but its base change $X \times_{\mathbb{R}} \mathbb{C}$ is reducible.

Exercise 4.7.8. Let $\mathbb{F}$ be a field. Find examples of connected (resp. irreducible, resp. integral) $\mathbb{F}$-schemes X such that $X \times_{\mathbb{F}} \mathbb{E}$ is not connected (resp. irreducible, resp. integral) for some finite extension $\mathbb{E} \supset \mathbb{F}$.

Definition 4.7.9 (Geometrically BLA)**.** Let K be a field, X a K-scheme. We say that X is geometrically connected (resp. irreducible, resp. integral) if $X_{\overline{K}}$ is connected (resp. irreducible, resp. integral). Equivalently, if X_L is connected (resp. irreducible, resp. integral) for any finite extension L/K.

PROPOSITION 4.7.10. *Let $f : X \to S$ and $g : Y \to S$ be two S-schemes. Set $P = X \times_S Y$. There is a bijective between points of P and quadruples $(x, y, s, \mathfrak{p})$ where $f(x) = s = g(y)$ and $\mathfrak{p} \in \operatorname{Spec}\kappa(x) \otimes_{\kappa(s)} \kappa(y)$.*

In particular, the fibre of the map β in (4.7.4) over (x, y, s) is the set of prime ideals $\mathfrak{p}$ in the ring $\kappa(x) \otimes_{\kappa(s)} \kappa(y)$.

Proof. Fix a point $z \in P$, i.e. a morphism $\operatorname{Spec}\kappa(z) \to P$. The universal property of P depicted in

$$\begin{tikzcd}
\operatorname{Spec}\kappa(z) \arrow[dr, dotted] \arrow[drr, bend left, "a"] \arrow[ddr, bend right, "b"'] & & \\
& P \arrow[r, "p"] \arrow[d, "q"'] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"] & S
\end{tikzcd}$$

yields the two morphism a and b in the diagram, satisfying $f \circ a = g \circ b$. But a (resp. b) corresponds to a point $x \in X$ (resp. a point $y \in Y$), and the identity $f \circ a = g \circ b$ implies that $f(x) = g(y)$. Let s be this common point in the base. The same compatibility yields field homomorphisms $\kappa(x) \to \kappa(z)$ and $\kappa(y) \to \kappa(z)$ such that, via the universal property of pushouts

$$\begin{tikzcd}
\kappa(z) & & \\
& \kappa(x) \otimes_{\kappa(s)} \kappa(y) \arrow[ul, dotted, "\alpha"] & \kappa(x) \arrow[l] \arrow[ull, bend right] \\
& \kappa(y) \arrow[u] \arrow[uul, bend left] & \kappa(s) \arrow[l] \arrow[u]
\end{tikzcd}$$

one obtains a canonical ring homomorphism

$$\kappa(x) \otimes_{\kappa(s)} \kappa(y) \xrightarrow{\alpha} \kappa(z).$$

Define

$$\mathfrak{p} = \ker \alpha \subset \kappa(x) \otimes_{\kappa(s)} \kappa(y).$$

This is indeed a prime ideal, since $\kappa(z)$ is a field.

Conversely, assume a quadruple $(x, y, s, \mathfrak{p})$ is given. Then the relations $f(x) = s = g(y)$ induce a solid commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} \frac{\kappa(x)\otimes_{\kappa(s)}\kappa(y)}{\mathfrak{p}} \arrow[r] \arrow[d] & \operatorname{Spec} \kappa(y) \arrow[r] \arrow[d] & Y \arrow[dd, "g"] \\
\operatorname{Spec} \kappa(x) \arrow[r] \arrow[d] & \operatorname{Spec} \kappa(s) \arrow[dr] & \\
X \arrow[rr, "f"] & & S
\end{tikzcd}$$

and the universal property of the fibre product induces a canonical morphism

$$\operatorname{Spec} \frac{\kappa(x)\otimes_{\kappa(s)}\kappa(y)}{\mathfrak{p}} \xrightarrow{\omega} X \times_S Y = P.$$

We define $z \in P$ to be the image of the generic point under ω.

We leave it to the reader to verify that the two constructions are inverse to each other. $\square$

Note that to give the prime ideal

$$\mathfrak{p} \subset \kappa(x) \otimes_{\kappa(s)} \kappa(y) = R$$

is the same as to give a prime ideal

$$\mathfrak{q} \subset \mathscr{O}_{X,x} \otimes_{\mathscr{O}_{S,s}} \mathscr{O}_{Y,y}$$

restricting to $\mathfrak{m}_x \subset \mathscr{O}_{X,x}$, resp. to $\mathfrak{m}_y \subset \mathscr{O}_{Y,y}$ along $\mathscr{O}_{X,x} \to \mathscr{O}_{X,x} \otimes_{\mathscr{O}_{S,s}} \mathscr{O}_{Y,y}$, resp. along $\mathscr{O}_{Y,y} \to \mathscr{O}_{X,x} \otimes_{\mathscr{O}_{S,s}} \mathscr{O}_{Y,y}$. After this identification, and setting $z = (x, y, s, \mathfrak{p})$, one has

$$\begin{aligned} \mathscr{O}_{X\times_S Y,z} &= (\mathscr{O}_{X,x} \otimes_{\mathscr{O}_{S,s}} \mathscr{O}_{Y,y})_{\mathfrak{q}} \\ \kappa(z) &= \kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}. \end{aligned} \tag{4.7.5}$$

Exercise 4.7.11. Prove that if $X \to S$ is surjective and $T \to S$ is an arbitrary morphism, then $X \times_S T \to T$ is surjective. Prove that this fails with surjective replaced by injective or bijective.

4.7.3 Base change

Basic properties of fibre products

We start with some notation.

Notation 4.7.12. Let X be a scheme, and let $U \hookrightarrow X$ and $V \hookrightarrow X$ be immersions. We set $U \cap V = U \times_X V$, and we call it the *intersection* of U and V taken inside X. Moreover, if $f: X \to S$ is a morphism and $T \hookrightarrow S$ is an immersion, we denote by $f^{-1}(T)$ the fibre product $X \times_S T$.

PROPOSITION 4.7.13. *Let X, Y and Z be S-schemes. The following properties are true.*

(1) $X \times_S S = X$.

(2) $X \times_S Y = Y \times_S X$.

(3) $(X \times_S Y) \times_S Z = X \times_S (Y \times_S Z)$.

(4) *Given a diagram of schemes*

$$\begin{tikzcd} X'' \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ S'' \arrow[r] & S' \arrow[r] & S \end{tikzcd}$$

where the outer square is a fibre diagram, as well as the rightmost square, one has that the leftmost square is a fibre diagram as well, so that

$$X \times_S S'' = X'' = (X \times_S S') \times_{S'} S''.$$

(5) *If $i: U \to X$ and $j: V \to Y$ are morphisms of S-schemes, there is a canonical S-morphism*

$$i \times j : U \times_S V \longrightarrow X \times_S Y,$$

called the product of i and j, making the diagram

$$\begin{tikzcd} U \arrow[r, "i"] & X \\ U \times_S V \arrow[u, "p_U"] \arrow[r, "i \times j"] \arrow[d, "p_V"] & X \times_S Y \arrow[u, "p_X"] \arrow[d, "p_Y"] \\ V \arrow[r, "j"] & Y \end{tikzcd}$$

commutative.

(6) *In the situation of* (5)*, if i and j are open immersions, then $i \times j$ induces an isomorphism*

$$i \times j : U \times_S V \xrightarrow{\sim} p_X^{-1}(U) \cap p_Y^{-1}(V).$$

Proof. Exercise. □

Remark 4.7.14. Let $X \to S$ and $Y \to S$ be S-morphisms. Fix a morphism $S \to T$. Then there is a canonical morphism

$$X \times_S Y \longrightarrow X \times_T Y,$$

obtained by the universal property of the target, as the diagram

$$\begin{tikzcd}
X \times_S Y \arrow[dr, dotted] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_T Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\
& Y \arrow[r] & T
\end{tikzcd}$$

shows. The outer diagram is commutative, since the two maps $X \times_S Y \to T$ factor through S, and they already agree before post-composition with $S \to T$.

Base change of a projective scheme

Consider now a pushout diagram

$$\begin{tikzcd}
E & B \arrow[l, "\phi"'] \\
C \arrow[u] & A \arrow[l] \arrow[u]
\end{tikzcd}$$

where A is a ring, $B = \bigoplus_{d \geq 0} B_d$ is a graded A-algebra and $A \to C$ is a ring homomorphism. We can turn

$$E = B \otimes_A C$$

into a graded C-algebra by setting $E_d = B_d \otimes_A C$ for all $d \geq 0$. Then, we have the following.

PROPOSITION 4.7.15. *Let B be a graded A-algebra, C an A-algebra, and form the graded C-algebra $B \otimes_A C = \bigoplus_{d \geq 0} B_d \otimes_A C$. There is a canonical isomorphism of A-schemes*

$$h \colon \operatorname{Proj} B \otimes_A C \xrightarrow{\sim} \operatorname{Proj} B \times_A \operatorname{Spec} C.$$

Proof. We set $E = B \otimes_A C = \bigoplus_{d \geq 0} E_d$ as above. The graded C-algebra structure on $\operatorname{Proj} E$ gives a natural morphism $\operatorname{Proj} E \to \operatorname{Spec} C$, and the identity $B_+ E = E_+$ yields (cf. Proposition 3.3.20) a natural morphism $q \colon \operatorname{Proj} E \to \operatorname{Proj} B$. These morphisms are compatible with the morphisms from their targets down to $\operatorname{Spec} A$. The universal property of fibre product then yields a canonical morphism of A-schemes

$$\operatorname{Proj} E \xrightarrow{h} \operatorname{Proj} B \times_A \operatorname{Spec} C.$$

We check that h is an isomorphism by checking it is an isomorphism on an open cover. The target is covered by affine open subsets

$$Y_f = \operatorname{Spec} B_{(f)} \times_A \operatorname{Spec} C = \operatorname{Spec} B_{(f)} \otimes_A C,$$

with $f \in B_+$ homogeneous elements. The restriction

$$h_f \colon h^{-1}(Y_f) = q^{-1}(\mathrm{D}_+(f)) = \mathrm{D}_+(\phi(f)) \to Y_f$$

corresponds to the ring homomorphism

$$h_f^\# \colon B_{(f)} \otimes_A C \to E_{(\phi(f))}, \quad \frac{b}{f^n} \otimes_A c \mapsto \frac{b \otimes_A c}{\phi(f)^n},$$

which is manifestly surjective. Its injectivity follows from the factorisation

$$\begin{tikzcd} B_{(f)} \otimes_A C \arrow[r, "\varepsilon"] \arrow[d, "h_f^\#"'] & B_f \otimes_A C \arrow[d, equal] \\ E_{(\phi(f))} \arrow[r, hook] & E_{\phi(f)} \end{tikzcd}$$

combined with the injectivity of ε, which in turn follows from $B_{(f)}$ being not just a subring but in fact a direct summand of B_f. $\square$

See Example 4.7.16 for an easy consequence of this result.

Scheme-theoretic fibre

Base change is a powerful technique. It can be used to define the scheme-theoretic fibre of a morphism $X \to S$ over a point $s \in S$. If $s \in S$, there is a canonical surjection $\mathscr{O}_{S,s} \twoheadrightarrow \kappa(s)$. Fix an affine open neighbourhood $T = \operatorname{Spec} R \subset S$ of s. We obtain

$$\operatorname{Spec} \kappa(s) \hookrightarrow \operatorname{Spec} \mathscr{O}_{S,s} = \operatorname{Spec} R_{\mathfrak{p}} \to \operatorname{Spec} R = T \hookrightarrow S,$$

where $\mathfrak{p} \subset R$ is the prime ideal corresponding to $s \in T$. The fibre product

$$\begin{tikzcd} X_s \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ \operatorname{Spec} \kappa(s) \arrow[r] & S \end{tikzcd}$$

defines the *scheme-theoretic fibre* of $X \to S$ over $s \in S$. We use, indeed, the notation X_s to denote $X \times_S \kappa(s) = X \times_S \operatorname{Spec} \kappa(s)$. By Equation (4.7.5), for every $x \in X_s$ we have

(4.7.6) $$\mathscr{O}_{X_s,x} = \mathscr{O}_{X,x} \otimes_{\mathscr{O}_{S,s}} \kappa(s) = \mathscr{O}_{X,x} / \mathfrak{m}_s \mathscr{O}_{X,x}.$$

Example 4.7.16. Let $S = \operatorname{Spec} A$ be an affine scheme, $s \in S$ a point corresponding to $\operatorname{Spec} \kappa(s) \to S$. Then

$$\mathbb{P}^n_A \times_A \kappa(s) = \operatorname{Proj} A[x_0, \ldots, x_n] \otimes_A \kappa(s) = \mathbb{P}^n_{\kappa(s)}$$

by Proposition 4.7.15. Thus $\mathbb{P}^n_A \to \operatorname{Spec} A$ is to be thought of as a family of projective spaces over (varying) fields.

Notation 4.7.17. So far we have defined relative projective space over an arbitrary affine scheme. If S is an arbitrary scheme, we set

$$\mathbb{P}^n_S = \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} S,$$

where we exploit the unique morphism of schemes from S to $\operatorname{Spec} \mathbb{Z}$ (cf. Exercise 3.2.12). It is naturally an S-scheme, via the second projection.

Exercise 4.7.18. Use Proposition 4.7.10 to confirm that $|X_s|$ agrees with the set-theoretic fibre $f^{-1}(s) \subset X$.

Permanence properties of morphisms

When dealing with a property **P** of morphisms of schemes, one may be interested in knowing

- whether **P** is preserved after base change along $T \to S$, for another S-scheme T,
- whether the composition of two morphisms having **P** still has **P**,
- whether **P** can be checked after base change along an open cover of the target.

These three notions have a name, which is the content of the next definitions.

Definition 4.7.19 (Stable under base change)**.** A property **P** of morphisms of schemes is said to be *stable under base change* if whenever $f : X \to S$ has **P**, for any base change $T \to S$ the morphism $X \times_S T \to T$ has **P** as well.

Definition 4.7.20 (Stable under composition)**.** A property **P** of morphisms of schemes is said to be *stable under composition* if whenever $f : X \to Y$ and $g : Y \to Z$ have **P**, the composition $g \circ f : X \to Z$ has **P** as well.

Definition 4.7.21 (Local on the target)**.** A property **P** of morphisms of schemes is said to be *local on the target* if a morphism $f : X \to S$ has **P** if and only if every point $s \in S$ has an affine open neighbourhood $s \in V \subset S$ such that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ has **P**.

As far as we know, the name 'affine communication lemma' for the next result is due to Vakil [19]. It is extremely useful in proving, for instance, that a property of morphisms of schemes is local on the target (see the proof of Proposition 4.7.23).

THEOREM 4.7.22 (Affine Communication Lemma). *Let* $\mathbf{P}$ *be a property enjoyed by some affine open subsets of a scheme* S*, such that*

(1) *if an affine open subset* $\operatorname{Spec} A \hookrightarrow S$ *has property* $\mathbf{P}$ *then for any* $f \in A$*, the principal open subscheme* $\operatorname{Spec} A_f \hookrightarrow S$ *does too.*

(2) *If* $f_1, \ldots, f_s \in A$ *generate the unit ideal and* $\operatorname{Spec} A_{f_i} \hookrightarrow S$ *have* $\mathbf{P}$ *for* $i = 1, \ldots, s$*, then* $\operatorname{Spec} A \hookrightarrow S$ *has* $\mathbf{P}$ *too.*

Suppose there is an open cover $S = \bigcup_{j \in J} \operatorname{Spec} A_j$ *such that* $\operatorname{Spec} A_j \hookrightarrow S$ *has* $\mathbf{P}$ *for all* $j \in J$*. Then every affine open* $\operatorname{Spec} A \hookrightarrow S$ *has* $\mathbf{P}$*.*

Proof. Let $\operatorname{Spec} A \hookrightarrow S$ be an affine open. We can cover $\operatorname{Spec} A$ with finitely many principal opens $\mathrm{D}(g_1), \ldots, \mathrm{D}(g_m)$ such that each $\mathrm{D}(g_i)$ is principal in some $\operatorname{Spec} A_j$. Then $A_{g_i} = (A_j)_{f_{ij}}$ for some $f_{ij} \in A_j$. But $\operatorname{Spec} A_j$ has $\mathbf{P}$ by assumption, therefore so does each A_{g_i} by (1). Since $(g_1, \ldots, g_m) = A$, we obtain the claim thanks to (2). □

PROPOSITION 4.7.23. *Open immersions, closed immersions and quasicompact morphisms are stable under composition, local on the target and stable under base change.*

Proof. We check each condition separately for the three classes of morphisms.

Step 1: Stability under composition. All three follow directly from the definitions. As for quasicompactness, let $f: X \to Y$ and $g: Y \to Z$ be quasicompact. Fix an affine open $U \subset Z$. Then $V = g^{-1}(U) \subset Y$ is quasicompact. Let $V = \bigcup_{1 \leq i \leq r} V_i$ be an affine open cover. Then $f^{-1}(V_i) \subset X$ is quasicompact for every $1 \leq i \leq r$. Thus $f^{-1}(V) = (g \circ f)^{-1}(U)$ is quasicompact as well.

Step 2: Local on the target. For quasicompactness, assume first that $f: X \to S$ is quasicompact. It is enough to prove that the base change of f along an open immersion stays quasicompact. So let $V \hookrightarrow S$ be an open immersion, and consider the base change $f_V: X_V \to V$, where $X_V = X \times_S V$. An open affine $W = \operatorname{Spec} R \subset V$ is in particular an open affine in S, and $f_V^{-1} W = f^{-1}(W)$ by Proposition 4.7.13(4). But $f^{-1}(W)$ is quasicompact because f is quasicompact. Conversely, let $f: X \to S$ be a morphism of schemes, and let us assume we have an open cover

$$S = \bigcup_{j \in J} S_j, \qquad S_j = \operatorname{Spec} A_j$$

such that $f_j = f|_{f^{-1} S_j}: f^{-1} S_j \to S_j$ is quasicompact for all $j \in J$. We must confirm that f is quasicompact. Consider the property $\mathbf{P}$ of open affines in S defined by

$$\operatorname{Spec} A \subset S \text{ has } \mathbf{P} \iff f^{-1}(\operatorname{Spec} A) \to \operatorname{Spec} A \text{ is quasicompact.}$$

By assumption, we have an affine open cover $S = \bigcup_{j \in J} S_j$ such that each S_j has **P**. Moreover, **P** satisfies the assumptions of the Affine Communication Lemma. Therefore

$$f^{-1}(\operatorname{Spec} A) \to \operatorname{Spec} A$$

is quasicompact for every affine open $\operatorname{Spec} A \subset S$. In particular $f^{-1}(\operatorname{Spec} A)$ is quasicompact for every affine open $\operatorname{Spec} A \subset S$, which proves that f is quasicompact, as claimed.

For closed immersions and open immersions, locality on the target is again (for half of the argument) an application of the Affine Communication Lemma. We explain this for closed immersions, the case of open immersions being identical.

Fix a morphism $f : X \to S$ and an affine open cover $S = \bigcup_{j \in J} \operatorname{Spec} A_j$ such that $f^{-1}(\operatorname{Spec} A_j) \to \operatorname{Spec} A_j$ is a closed immersion for every $j \in J$. To prove the first implication, we need to check that f is a closed immersion. Define **P** for affine open subsets $\operatorname{Spec} A \subset S$ via

$$\operatorname{Spec} A \subset S \text{ has } \mathbf{P} \iff f^{-1}(\operatorname{Spec} A) \to \operatorname{Spec} A \text{ is a closed immersion.}$$

Then **P** satisfies the assumptions of Theorem 4.7.22 (check this!). Thus, by the Affine Communication Lemma, $f^{-1}(\operatorname{Spec} A) \to \operatorname{Spec} A$ is a closed immersion for *every* affine open subset $\operatorname{Spec} A \subset S$. Thus $f^{\#}$ is surjective, and moreover $|X|$ is homeomorphic to a closed subset of $|S|$ via f, i.e. f is a closed immersion.

As for the converse, if $f : X \hookrightarrow S$ is a closed immersion, and $i : V \hookrightarrow S$ is an open subscheme, we form the cartesian diagram

$$\begin{tikzcd} V \times_S X \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\square"] & X \arrow[d, hook, "f"] \\ V \arrow[r, hook, "i"] & S \end{tikzcd}$$

and we claim that $p : V \times_S X \to V$ is a closed immersion (this implies our claim, namely that f is a closed immersion after restriction to an affine cover of the target). In this situation, the canonical continuous map

$$|V \times_S X| \to |V| \times_{|S|} |X| = i^{-1}|X|$$

is not only surjective, but also injective (cf. Proposition 4.7.10) and closed. Therefore $V \times_S X$ is homeomorphic under p to a closed subset of V. But $V \times_S X = f^{-1}(V)$ is an open subscheme of X, therefore the surjectivity of $p^{\#}$ is directly inherited by that of f.

Step 3: Stability under base change.

Open immersions. Let $f : X \to S$ be an open immersion, $T \to S$ a morphism. By Proposition 4.7.13(6), the product $X \times_S T \to S \times_S T = T$ of the S-morphisms f and $\mathrm{id}_T : T \to T$

induces an isomorphism from $X \times_S T$ onto $p^{-1}(X)$, where $p\colon S \times_S T \to S$ is the first projection. Therefore $X \times_S T \to T$ is an open immersion.

$$\begin{tikzcd} p^{-1}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, hook, "f"] \\ T \arrow[r] & S \end{tikzcd}$$

But $p^{-1}(X)$ is naturally an open subscheme of $T = S \times_S T$.

<u>Closed immersions</u>. Let $f\colon X \to S$ be a closed immersion. Since being a closed immersion is local on the target, we may assume $S = \operatorname{Spec} R$ is affine, as well as the new base scheme T, let us say $T = \operatorname{Spec} B$. By Proposition 3.1.65, X is also affine, isomorphic to $\operatorname{Spec} R/J$ for an ideal $J \subset R$. Then the fibre product $X \times_S T$ is also affine, isomorphic to $\operatorname{Spec} R/J \otimes_R B = \operatorname{Spec} B/JB$. Thus $X \times_S T \to T$ is a closed immersion.

<u>Quasicompact morphisms</u>. Let $f\colon X \to S$ be a quasicompact morphism of schemes. Let $g\colon T \to S$ be a morphism, and set $X_T = X \times_S T$. Let $q\colon X_T \to T$ be the base change of f. Since being quasicompact is local on the target, we need to prove that T has an affine open cover over which q is quasicompact.

Consider an open affine subset $U \subset S$. Cover $f^{-1}(U) \subset X$ by finitely many affine open subsets $W_1, \ldots, W_r \subset X$. Let $U' \subset g^{-1}(U) \subset T$ be an affine open subset. Then

$$\begin{aligned} q^{-1}(U') &= (X \times_S T) \times_T U' \\ &= X \times_S U' \\ &= (X \times_S U) \times_U U' \\ &= f^{-1}(U) \times_U U' = \bigcup_{1 \le i \le r} (W_i \times_U U'). \end{aligned}$$

We have used Proposition 4.7.13(4) for the second and third identities. Note that $W_i \times_U U'$ is affine since all three schemes involved are affine. Thus $q^{-1}(U')$ is quasicompact. Letting U vary we may form an open cover of S, which induces an open cover of T by affine open subsets. Letting U' vary among these open subsets, we have produced the desired covering of T, over which q is quasicompact. □

LEMMA 4.7.24. *An open immersion $X \to S$ with S locally noetherian is quasicompact.*

Proof. Quasicompact morphisms are local on the target, so we may assume $S = \operatorname{Spec} A$ is affine, i.e. A is noetherian. But then S is a noetherian topological space, which is equivalent to every open subspace being quasicompact. □

5 | Morphisms of schemes

5.1 Local properties of morphims

The aim of this section is to provide an analogue of the Locality Lemma (cf. Lemma 4.2.2) for morphisms of schemes. The following is the relative version of the notion of local property of rings (cf. Definition 4.2.1).

Definition 5.1.1 (Local property of ring homomorphisms)**.** Let $\mathbf{P}$ be a property of ring homomorphisms. We say that $\mathbf{P}$ is a *local property of ring homomorphisms* if

(1) For any ring homomorphism $R \to A$ and any $f \in R$, we have

$$\mathbf{P}(R \to A) \;\Rightarrow\; \mathbf{P}(R_f \to A_f).$$

Here A_f denotes the localisation of A (viewed as an R-module) at $f \in R$, which is the *ring* $A \otimes_R R_f$ by Lemma B.5.16.

(2) For any rings R, A with elements $f \in R$, $a \in A$ and for any ring homomorphism $R_f \to A$ we have

$$\mathbf{P}(R_f \to A) \;\Rightarrow\; \mathbf{P}(R \to A_a).$$

(3) For any ring homomorphism $R \to A$ and elements $a_1, \ldots, a_r \in A$ such that $A = (a_1, \ldots, a_r)$, we have the implication

$$\mathbf{P}(R \to A_{a_i}) \text{ for all } i = 1, \ldots, r \;\Rightarrow\; \mathbf{P}(R \to A).$$

Definition 5.1.2 (Stable under composition)**.** Let $\mathbf{P}$ be a property of ring homomorphisms. We say that $\mathbf{P}$ is a *stable under composition* if, given ring homomorphisms $A \to B$ and $B \to C$, one has the implication

$$\mathbf{P}(A \to B), \mathbf{P}(B \to C) \;\Rightarrow\; \mathbf{P}(A \to C).$$

Definition 5.1.3 (Stable under base change)**.** Let $\mathbf{P}$ be a property of ring homomorphisms. We say that $\mathbf{P}$ is a *stable under base change* if for any ring homomorphisms $R \to A$ and $R \to R'$, one has the implication

$$\mathbf{P}(R \to A) \;\Rightarrow\; \mathbf{P}(R' \to R' \otimes_R A).$$

Definition 5.1.4 (Morphism locally of type $\mathbf{P}$)**.** Let $\mathbf{P}$ be a property of ring homomorphisms. Let $f\colon X \to S$ be a morphism of schemes. We say that f is *locally of type* $\mathbf{P}$ if for any $x \in X$ there is an affine open neighbourhood $x \in \operatorname{Spec} A \subset X$ and an affine open subset $\operatorname{Spec} R \subset S$ such that $f(\operatorname{Spec} A) \subset \operatorname{Spec} R$ and $R \to A$ has property $\mathbf{P}$.

THEOREM 5.1.5 (Relative locality lemma). *Let* $\mathbf{P}$ *be a local property of ring homomorphisms. Let* $f\colon X \to S$ *be a morphism of schemes. The following conditions are equivalent.*

(i) *f is locally of type* $\mathbf{P}$*.*

(ii) *For every affine opens $U \subset X$ and $V \subset S$ such that $f(U) \subset V$, the ring homomorphism $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ has property* $\mathbf{P}$*.*

(iii) *There are affine open covers $S = \bigcup_{i\in I} \operatorname{Spec} R_i$ and $f^{-1}(\operatorname{Spec} R_i) = \bigcup_{j\in J_i} \operatorname{Spec} A_j$ such that the ring homomorphism $R_i \to A_j$ has property* $\mathbf{P}$ *for every i and $j \in J_i$.*

(iv) *There are open covers $S = \bigcup_{i\in I} V_i$ and $f^{-1}(V_i) = \bigcup_{j\in J_i} U_j$ such that the restriction $U_j \to V_i$ is locally of type* $\mathbf{P}$ *for every $i \in I$ and $j \in J_i$.*

Moreover, if f is locally of type $\mathbf{P}$*, for any open subschemes $U \subset X$ and $V \subset S$ such that $f(U) \subset V$ the morphism $U \to V$ is locally of type* $\mathbf{P}$*.*

Proof. It is enough to prove the following claim. Fix a morphism $f\colon X \to S$ and a property of ring homomorphisms $\mathbf{P}$. Let $U = \operatorname{Spec} A \subset X$ and $V = \operatorname{Spec} R \subset S$ be affine opens such that $f(U) \subset V$. Then, if f is locally of type $\mathbf{P}$ and $\mathbf{P}$ is local, then $R \to A$ has property $\mathbf{P}$. The proof of this claim is left as an exercise. See [16, Tag 01ST] if you need a hint. □

PROPOSITION 5.1.6. *Let* $\boldsymbol{P}$ *be a local property of ring homomorphisms. Assume that* $\boldsymbol{P}$ *is stable under base change (resp. stable under composition). Then being locally of type* $\boldsymbol{P}$ *is a property of morphisms of schemes that is stable under base change (resp. stable under composition).*

Proof. Follows at once from Theorem 5.1.5(ii). □

5.2 Finite type morphisms

Definition 5.2.1 (Finite type, finite presentation)**.** A ring homomorphism $R \to A$ is said to be

- *of finite type* if A is isomorphic, as an R-algebra, to an R-algebra of finite type (also called a finitely generated R-algebra), i.e. an algebra of the form $R[x_1,\dots,x_n]/I$ for some n and some ideal $I \subset R[x_1,\dots,x_n]$.

- *of finite presentation* if A is isomorphic, as an R-algebra, to an R-algebra of the form $R[x_1,\dots,x_n]/(f_1,\dots,f_r)$, for some n and a finite number of polynomials $f_1,\dots,f_r \in R[x_1,\dots,x_n]$.

Remark 5.2.2. Clearly finite presentation implies finite type, and the converse holds over noetherian R (as this implies $R[x_1,\dots,x_n]$ is noetherian by Hilbert's basis theorem, and thus every ideal in $R[x_1,\dots,x_n]$ is finitely generated).

$$\begin{tikzcd} R \to A \text{ of finite presentation} \arrow[r, Rightarrow] & R \to A \text{ of finite type} \arrow[l, Rightarrow, bend right, "R \text{ noetherian}"'] \end{tikzcd}$$

Example 5.2.3. If $f \in R$, then R_f is an R-algebra of finite presentation (and hence a fortiori of finite type), being isomorphic to $R[x]/(xf-1)$.

Following Definition 5.1.4, we have the following definition.

Definition 5.2.4 (Locally of finite type/presentation)**.** Let $f: X \to S$ be a morphism of schemes. Then, f is said to be *locally of finite type* (resp. *locally of finite presentation*) if for every point $x \in X$ there is an affine open neighborhood $x \in \operatorname{Spec} A \subset X$ and an affine open subset $\operatorname{Spec} R \subset S$ such that $f(\operatorname{Spec} A) \subset \operatorname{Spec} R$ and the induced map $R \to A$ is of finite type (resp. of finite presentation).

We say that f is *of finite type* if it is locally of finite type and quasicompact.

Remark 5.2.5. By Remark 5.2.2, a morphism which is locally of finite presentation is locally of finite type, and the converse holds if the target is locally noetherian.

Remark 5.2.6. A morphism of affine schemes, being quasicompact, is locally of finite type if and only if it is of finite type.

LEMMA 5.2.7. *Being of finite type, as a property of ring homomorphisms, is stable under base change and composition (cf. Definitions 4.7.19 and 4.7.20). Moreover, it is a local property in the sense of Definition 5.1.1.*

Proof. We proceed step by step.
Composition. Assume given a composition

$$A \to A[\underline{x}]/I = B \to B[\underline{z}]/J.$$

Consider $A[\underline{x},\underline{z}] \twoheadrightarrow B[\underline{z}] \twoheadrightarrow B[\underline{z}]/J$. This is a surjection, so we are done.
Base change. If $R \to A$ is a ring homomorphisms of finite type and $R \to R'$ is a ring homomorphism, then also $R' \to R' \otimes_R A$ is of finite type. Indeed, if $A = R[\underline{z}]/I$, then

$$R' \otimes_R A = R' \otimes_R R[\underline{z}]/I = R'[\underline{z}]/IR'[\underline{z}].$$

Locality. Condition (1) in Definition 5.1.1 follows from stability under base change, since the pushout of

$$\begin{tikzcd} & A \\ R_f & R \arrow[l] \arrow[u] \end{tikzcd}$$

is precisely $A \otimes_R R_f = A_f$. Condition (2) in Definition 5.1.1 follows from stability under composition: $R \to R_f$ and $A \to A_a$ are of finite type (with no assumptions), $R_f \to A$ is of finite type by assumption, thus $R \to R_f \to A \to A_a$ is of finite type. Condition (3) in Definition 5.1.1 follows from the following argument, taken from [16, Tag 00EP]. Write

$$\sum_{1 \le i \le r} h_i a_i = 1 \in A, \quad h_i \in A.$$

Remember that we are assuming $A = (a_1, \ldots, a_r)$, and we have a ring map $R \to A$ such that $R \to A_{a_i}$ is of finite type for every i, i.e. we have surjections

$$R[z_1, \ldots, z_{m_i}] \twoheadrightarrow A_{a_i}$$

for some $m_i \in \mathbb{Z}_{\geq 0}$. We fix (finitely many) generators

$$\frac{y_{ij}}{a_i^{n_{ij}}} \in A_{a_i}, \quad j = 1, \ldots, m_i,$$

where $y_{ij} \in A$ and $n_{ij} \in \mathbb{Z}_{\geq 0}$. Consider the (finite type) R-subalgebra

$$A' = R\big[a_i, h_i, y_{ij} \,|\, i = 1, \ldots, r,\ j = 1, \ldots, m_i\big] \hookrightarrow A.$$

We claim that this inclusion is an isomorphism. It is enough to prove it is an isomorphism after localising at a system of the generators of the unit ideal of A' by [16, Tag 00EO]. Note that $a_1, \ldots, a_r \in A'$ generate the unit ideal of A' because $h_1, \ldots, h_r \in A'$. Since localisation is exact, we have injections

$$A'_{a_i} \hookrightarrow A_{a_i}$$

These are in fact surjective as well by the presence of the elements y_{ij}, thus we have the required isomorphisms

$$A'_{a_i} \xrightarrow{\sim} A_{a_i}.$$

So, as anticipated, since $a_1, \ldots, a_r$ generate the unit ideal of A', the inclusion $A' \hookrightarrow A$ is in fact an R-linear isomorphism by [16, Tag 00EO]. Then $R \to A$ is of finite type. □

PROPOSITION 5.2.8 ([16, Tag 01T0]). *Let $f : X \to S$ be a morphism of schemes. The following conditions are equivalent.*

(a) *f is locally of finite type.*

(b) *For all affine open subsets* $U = \operatorname{Spec} A \subset X$ *and* $V = \operatorname{Spec} R \subset S$ *such that* $f(U) \subset V$, *the induced map* $R \to A$ *is of finite type.*

(c) *There are affine open covers* $S = \bigcup_{i \in I} \operatorname{Spec} R_i$ *and* $f^{-1}(\operatorname{Spec} R_i) = \bigcup_{j \in J_i} \operatorname{Spec} A_j$ *such that the ring homomorphism* $R_i \to A_j$ *is of finite type for every* $i \in I$ *and* $j \in J_i$.

(d) *There are open covers* $S = \bigcup_{i \in I} V_i$ *and* $f^{-1}(V_i) = \bigcup_{j \in J_i} U_j$ *such that the restriction* $U_j \to V_i$ *is locally of finite type for every* $i \in I$ *and* $j \in J_i$.

Moreover, if $X \to S$ *is locally of finite type and* $U \hookrightarrow X$ *is an open immersion, then* $U \to S$ *is locally of finite type.*

Proof. Combine Lemma 5.2.7 and Theorem 5.1.5 with one another. □

COROLLARY 5.2.9. *Morphisms (locally) of finite type are stable under composition and under base change.*

Proof. By Proposition 4.7.23, it is enough to prove the statement for morphisms locally of finite type. This follows combining Lemma 5.2.7 and Proposition 5.1.6 with one another. □

LEMMA 5.2.10. *Let* $f : X \to Y$ *be a morphism of* S*-schemes, with* $X \to S$ *locally of finite type. Then* f *is locally of finite type. In particular, if* f *is in addition quasicompact (which happens for instance when* X *is noetherian), then it is of finite type.*

Proof. Fix an open affine $\operatorname{Spec} C \subset S$, an open affine $\operatorname{Spec} B \subset g^{-1}(\operatorname{Spec} C) \subset Y$, and an open affine $\operatorname{Spec} A \subset f^{-1}(\operatorname{Spec} B) \subset X$. Set $h = g \circ f$.

(5.2.1)
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

Then, $\operatorname{Spec} A \subset h^{-1}(\operatorname{Spec} C)$, and at the level of rings (5.2.1) locally looks like

$$\begin{tikzcd} A & & B \arrow[ll, "f^\#"'] \\ & C \arrow[ul, "h^\#"] \arrow[ur, "g^\#"'] & \end{tikzcd}$$

where by assumption (and by Proposition 5.2.8 (b)) the map $C \to A$ is of finite type. We must prove that $B \to A$ is of finite type. If $a_1, \ldots, a_n \in A$ are generators of A as a C-algebra, we have a surjective homomorphism

$$C[x_1, \ldots, x_n] \twoheadrightarrow A, \quad x_i \mapsto a_i.$$

This factors as

$$C[x_1,\dots,x_n] \longrightarrow B[x_1,\dots,x_n] \longrightarrow A,$$

where the second map is obviously defined by $x_i \mapsto a_i$ again, which shows that

$$B[x_1,\dots,x_n] \twoheadrightarrow A, \quad x_i \mapsto a_i.$$

is surjective. Thus A is of finite type over B. Conclude by Proposition 5.2.8(b). $\square$

LEMMA 5.2.11. *If $X \to S$ is a finite type morphism to a (locally) noetherian scheme S, then X is a (locally) noetherian scheme.*

Proof. Since S is noetherian, it is quasicompact, hence it can be covered by finitely many open affines $V_i = \operatorname{Spec} R_i$, and moreover we can assume each R_i to be a noetherian ring. Since f is quasicompact, each $f^{-1}(V_i) \subset X$ is a finite union of affine schemes $\operatorname{Spec} A_{ij}$, thus in particular X is quasicompact because it is covered by the finitely many quasicompact open subsets $f^{-1}(V_i)$. But $R_i \to A_{ij}$ are of finite type by Proposition 5.2.8(b), i.e. we can write $A_{ij} = R_i[\underline{y}]/J_{ij}$ for a finite tuple $\underline{y}$ of variables, and for some ideal J_{ij}. Since R_i is noetherian, we have that $R_i[\underline{y}]$ is noetherian by Hilbert's basis theorem, and then A_{ij} is also noetherian. Thus X is locally noetherian. Then, being quasicompact, it is noetherian. $\square$

Example 5.2.12. Since open immersions are locally isomorphisms, they are locally of finite type (and in fact of finite type as soon as the target is locally noetherian, cf. Lemma 4.7.24). Any closed immersion is of finite type (by Proposition 3.1.65, a closed immersion is locally modeled on $\operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} A$). The open immersion $\operatorname{Spec} \mathbf{k}[x_i \mid i \in \mathbb{N}] \setminus \{0\} \hookrightarrow \operatorname{Spec} \mathbf{k}[x_i \mid i \in \mathbb{N}]$ is not quasicompact, hence a fortiori not of finite type.

Example 5.2.13. The absolute projective space $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ and the absolute affine space $\mathbb{A}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ are of finite type, as $\mathbb{Z} \to \mathbb{Z}[x_1,\dots,x_n]$ is of finite type. Note that the open immersion $\mathbb{A}^n_{\mathbb{Z}} \hookrightarrow \mathbb{P}^n_{\mathbb{Z}}$ is of finite type by Example 5.2.12, since $\mathbb{P}^n_{\mathbb{Z}}$ is locally noetherian.

Example 5.2.14. Let A be a ring. Then, $\operatorname{Spec} A[\![x_1,\dots,x_n]\!] \to \operatorname{Spec} A$ is quasicompact but not of finite type. The morphism

$$\coprod_{i \in \mathbb{N}} \operatorname{Spec} A \to \operatorname{Spec} A$$

is also not of finite type, though it is locally of finite type. The morphism

$$\operatorname{Spec} \mathbb{F}(x_1,\dots,x_n) \to \operatorname{Spec} \mathbb{F}[x_1,\dots,x_n]$$

induced by the inclusion $\mathbb{F}[x_1,\dots,x_n] \hookrightarrow \mathbb{F}(x_1,\dots,x_n)$ is not of finite type either.

Algebraic varieties revisited

We can now revisit the concept of algebraic variety with this new 'relative' terminology. Let $\mathbb{F}$ be a field. Combining Important Definition 3.2.1 and Definition 5.2.4 with one another, we notice that an algebraic variety is nothing but an $\mathbb{F}$-scheme X such that the structure morphism $X \to \operatorname{Spec}\mathbb{F}$ is of finite type. The slogan is:

$$\boxed{\text{A variety over } \mathbb{F} \text{ is a scheme of finite type over } \mathbb{F}.}$$

As a sanity check, let us verify directly the case of projective varieties, i.e. the closed subschemes $Y \subset \mathbb{P}^n_{\mathbb{F}}$. We have a diagram

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[dr] & \mathbb{P}^n_{\mathbb{F}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{P}^n_{\mathbb{Z}} \arrow[d, "\text{finite type}"] \\ & \operatorname{Spec}\mathbb{F} \arrow[r] & \operatorname{Spec}\mathbb{Z} \end{tikzcd}$$

where by base change property (cf. Corollary 5.2.9) we deduce that $\mathbb{P}^n_{\mathbb{F}} \to \operatorname{Spec}\mathbb{F}$ is of finite type, and by the stability under composition property we deduce that $Y \hookrightarrow \mathbb{P}^n_{\mathbb{F}} \to \operatorname{Spec}\mathbb{F}$ is of finite type. Of course we have used that the square is cartesian (cf. Example 4.7.16).

5.3 What is a point?

Let $(X, \mathscr{O}_X)$ be a scheme. There are the following notions of a *point* of X.

Definition 5.3.1 (Points)**.** Let $(X, \mathscr{O}_X)$ be a scheme.

- A *point* of $(X, \mathscr{O}_X)$ is a point x of the underlying topological space X.
- A *closed point* of $(X, \mathscr{O}_X)$ is a point x such that $\{x\} \subset X$ is closed.
- Let T be a scheme. A T*-valued point* of X is a morphism $T \to X$. The set of T-valued points of a scheme X is denoted $\mathsf{h}_X(T)$.
- Let $\mathbb{F}$ be a field, $X \to \operatorname{Spec}\mathbb{F}$ a scheme over $\mathbb{F}$. An $\mathbb{F}$-*rational point* (or simply a *rational point*) of X is a point $x \in X$ such that $\kappa(x) = \mathbb{F}$.
- A morphism $\overline{x} \colon \operatorname{Spec}\Omega \to X$ from an algebraically closed field Ω is called a *geometric point* of X. We say that $\overline{x}$ *lies over* x if $x \in X$ is the image of $\overline{x}$.

Let $\pi \colon X \to S$ be an S-scheme. Set

$$X(S) = \operatorname{Hom}_{\mathsf{Sch}_S}(S, X) = \left\{ S \xrightarrow{\sigma} X \,\middle|\, \pi \circ \sigma = \mathrm{id}_S \right\} \subset \mathsf{h}_X(S).$$

Elements in this set are called *sections* of π. If $S = \operatorname{Spec} A$, we write $X(A)$ instead of $X(\operatorname{Spec} A)$. The next lemma indicates a link between valued points and rational points, for schemes defined over a field.

LEMMA 5.3.2. *Let* $\mathbb{F}$ *be a field,* $X \to \operatorname{Spec}\mathbb{F}$ *a scheme over* $\mathbb{F}$*. Then*

$$\mathsf{h}_X(\mathbb{F}) \supset X(\mathbb{F}) = \{\, x \in X \mid \kappa(x) = \mathbb{F} \,\}.$$

Proof. Take $\sigma \in X(\mathbb{F})$, i.e.

$$\begin{tikzcd} \operatorname{Spec}\mathbb{F} \arrow[r, "\sigma"] \arrow[dr, equal] & X \arrow[d] \\ & \operatorname{Spec}\mathbb{F} \end{tikzcd}$$

commutes. Let $x \in X$ be the image of σ. The homomorphism

$$\sigma^{\#} \colon \mathscr{O}_{X,x} \longrightarrow \mathbb{F}$$

induces

$$\kappa(x) \longrightarrow \mathbb{F},$$

a map of fields. But $\kappa(x)$ is a $\mathbb{F}$-algebra, thus $\kappa(x) = \mathbb{F}$.

Conversely, let $x \in X$ be a point such that $\kappa(x) = \mathbb{F}$. We have a canonical surjection

$$\mathscr{O}_{X,x} \twoheadrightarrow \kappa(x) = \mathbb{F}.$$

Recall that we have a canonical map $\operatorname{Spec}\mathscr{O}_{X,x} \to X$ (reminder: take any affine open subset $U \subset X$ containing x, so that we have a ring homomorphism $\mathscr{O}_X(U) \to \mathscr{O}_{X,x}$, inducing $\operatorname{Spec}\mathscr{O}_{X,x} \to \operatorname{Spec}\mathscr{O}_X(U) = U \hookrightarrow X$). The composition

$$\operatorname{Spec}\mathbb{F} = \operatorname{Spec}\kappa(x) \hookrightarrow \operatorname{Spec}\mathscr{O}_{X,x} \to X$$

sends the unique point of $\operatorname{Spec}\mathbb{F}$ to x by construction: it is the required section of the structure morphism $X \to \operatorname{Spec}\mathbb{F}$. $\square$

PROPOSITION 5.3.3. *If* $\mathbf{k}$ *is an algebraically closed field and* $X \to \operatorname{Spec}\mathbf{k}$ *is a* $\mathbf{k}$*-scheme locally of finite type, then*

$$\boxed{\textit{the } \mathbf{k}\textit{-rational points are precisely the closed points}}$$

i.e. we have identities

$$\{\, x \in X \mid x \text{ is a closed point} \,\} = \{\, x \in X \mid \kappa(x) = \mathbf{k} \,\} = X(\mathbf{k}).$$

For instance, this holds for algebraic $\mathbf{k}$*-varieties.*

Proof. The second identity is Lemma 5.3.2. To prove the first identity, it is enough to prove that if $\mathbb{F}$ is an arbitrary field, A is a finitely generated $\mathbb{F}$-algebra and $\mathfrak{m} \subset A$ is a maximal ideal, the quotient $A/\mathfrak{m}$ is a finite extension of $\mathbb{F}$. This would mean that

$$\boxed{x \in X \text{ is closed if and only if } \mathbf{k} \hookrightarrow \kappa(x) \text{ is finite}}$$

but there are no finite extensions of an algebraically closed field, implying the first identity.

To prove the claim, we invoke Noether's normalisation lemma (cf. Theorem 4.6.29). Since $A/\mathfrak{m}$ is itself a finitely generated $\mathbb{F}$-algebra, we have a finite injective homomorphism $B = \mathbb{F}[t_1, \ldots, t_d] \hookrightarrow A/\mathfrak{m}$. We claim that $d = 0$. If d were positive, we could take t_1 and note that $\alpha = 1/t_1 \in A/\mathfrak{m}$ since $A/\mathfrak{m}$ is a field. Then α is integral over B (since finite implies integral), which means there is a monic polynomial $q(y) \in B[y]$ such that $q(\alpha) = 0$. If we write $q(y) = y^m + b_{m-1}y^{m-1} + \cdots + b_1 y + b_0$, then we have

$$0 = \alpha^m + b_{m-1}\alpha^{m-1} + \cdots + b_1\alpha + b_0$$

in $A/\mathfrak{m}$. But after multiplication by t_1^m, this becomes

$$0 = 1 + b_{m-1}t_1 + \cdots + b_1 t_1^{m-1} + b_0 t_1^m,$$

which says that t_1 is invertible in B, contradicting $d > 0$. □

Remark 5.3.4. The proof shows the following algebraic fact (which could be taken as a corollary of Noether's normalisation lemma): if R is a finitely generated $\mathbb{F}$-algebra which is a field, then R is a finite extension of $\mathbb{F}$. This is sometimes called *Zariski's Lemma*. Note that we could have finished the proof of Proposition 5.3.3 by invoking Exercise 4.6.26, whose proof is actually provided in the last paragraph.

Example 5.3.5. To see that Proposition 5.3.3 fails for schemes that are not locally of finite type over a field, consider

$$X = \operatorname{Spec} \mathbb{C}(x) \xrightarrow{\pi} \operatorname{Spec} \mathbb{C}.$$

Then X has a unique closed point, but π has many sections, each corresponding to a $\mathbb{C}$-algebra homomorphism $\mathbb{C}(x) \to \mathbb{C}$.

Caution 5.3.6. Mind the differences between the various notions of points. For instance, $X = \operatorname{Spec} \mathbb{C}$ admits infinitely many $\mathbb{C}$-valued points (there are (uncountably) many ring homomorphisms $\mathbb{C} \to \mathbb{C}$, cf. Example 3.1.66) and a fortiori infinitely many geometric points, but of course when viewed *as a $\mathbb{C}$-scheme* it has a unique $\mathbb{C}$-rational point (as there is *only one* $\mathbb{C}$-algebra homomorphism $\mathbb{C} \to \mathbb{C}$)!

Exercise 5.3.7. Let X be a $\mathbf{k}$-scheme locally of finite type. Show that the set of closed points is dense.

Aside 5.3.8. Mordell–Weil here

5.4 Quasiseparated, separated and affine morphisms

Fibre products allow one to define the *diagonal* of an arbitrary morphism of schemes. The diagonal plays a role in topology, where it is a classical fact that for X an arbitrary topological space,

$$\boxed{X \text{ is Hausdorff if and only if the diagonal } X \subset X \times X \text{ is closed.}}$$

We already know that Haurdoffness is meaningless in algebraic geometry, since open sets are too large. We shall see the parallel notion in algebraic geometry, namely *separatedness*.

Definition 5.4.1 (Diagonal)**.** Let $X \to S$ be an S-scheme. The *diagonal* of $X \to S$ is the morphism $\Delta_{X/S}\colon X \to X \times_S X$ corresponding to the pair $(\mathrm{id}_X, \mathrm{id}_X)$ under the bijection (4.7.2).

Definition 5.4.2 (Separated, quasiseparated)**.** A morphism $X \to S$ is *quasiseparated* (resp. *separated*) if $\Delta_{X/S}$ is a quasicompact morphism (resp. a closed immersion). We say that X is *(quasi)separated over* S. A scheme is (quasi)separated if its structure morphism $X \to \operatorname{Spec} \mathbb{Z}$ is (quasi)separated.

Remark 5.4.3. Since closed immersions are quasicompact, a separated morphism is quasiseparated.

Example 5.4.4. Let $X_1 = X_2 = \operatorname{Spec} \mathbf{k}[x_i \mid i \in \mathbb{N}]$, and consider the glueing $X = X_1 \cup X_2$ along the complement of the origin. Then X_i and X are naturally $\mathbf{k}$-schemes, $X_1 \times_{\mathbf{k}} X_2$ is an open affine subscheme of $X \times_{\mathbf{k}} X$, but $\Delta^{-1}_{X/\mathbf{k}}(X_1 \times_{\mathbf{k}} X_2) = \operatorname{Spec} \mathbf{k}[x_i \mid i \in \mathbb{N}] \setminus \{0\}$, which is not quasicompact (cf. Warning 3.1.14). Thus $X \to \operatorname{Spec} \mathbf{k}$ is not quasiseparated.

Example 5.4.5. If $X \to S$ is an open immersion or a closed immersion, then the diagonal $\Delta_{X/S}\colon X \to X \times_S X$ is an isomorphism (this is a general categorical statement: the diagonal associated to a monomorphism is always an isomorphism; you may prove as an exercise that immersions are monomorphisms!). Thus open immersions and closed immersions are separated.

PROPOSITION 5.4.6. *A morphism of affine schemes is separated. In particular, any affine scheme is separated.*

Proof. Let $f\colon X \to S$ be a morphism between $X = \operatorname{Spec} A$ and $S = \operatorname{Spec} B$. Then f corresponds to a ring homomorphism $B \to A$, and $X \times_S X = \operatorname{Spec}(A \otimes_B A)$, with the diagonal morphism $X \to X \times_S X$ corresponding to $A \otimes_B A \to A$ sending $a \otimes a' \mapsto a a'$. This morphism is surjective. □

Example 5.4.7. Let A be a ring. Affine n-space $\mathbb{A}^n_A = \operatorname{Spec} A[x_1, \ldots, x_n]$ is separated over $\operatorname{Spec} A$. Thus

$$\Delta_{\mathbb{A}^n_A/A} \colon \mathbb{A}^n_A \longrightarrow \mathbb{A}^n_A \times_A \mathbb{A}^n_A = \mathbb{A}^{2n}_A$$

is a closed immersion. Its ideal is generated by $(x_i - y_i \mid 1 \le i \le n)$, if we identify $\mathbb{A}^{2n}_A = \operatorname{Spec} A[x_1, \ldots, x_n, y_1, \ldots, y_n]$.

COROLLARY 5.4.8. *Let $f \colon X \to S$ be a morphism. Then $\Delta_{X/S}$ is an immersion.*

Proof. We need to check that there is an open subscheme $j \colon W \hookrightarrow X \times_S X$ and a closed immersion $i \colon X \hookrightarrow W$ such that $j \circ i = \Delta_{X/S}$.

Fix $U \subset X$ and $V \subset S$ open affines such that $f(U) \subset V$. By Proposition 4.7.13(6) we have an open immersion

$$U \times_V U \hookrightarrow X \times_S X.$$

Define W to be the union of these open subschemes of the form $U \times_V U$. Clearly $\Delta_{X/S}$ factors through W, so we do have a morphism $i \colon X \to W$. Now, closed immersions are local on the target, so to see that i a closed immersion it suffices to check that

$$\Delta^{-1}_{X/S}(U \times_V U) \longrightarrow U \times_V U$$

is a closed immersion. But $U \times_V U$ is affine and the above map agrees on the nose with $\Delta_{U/V} \colon U \to U \times_V U$, which is a closed immersion by Proposition 5.4.6. $\square$

COROLLARY 5.4.9. *A morphism of schemes $X \to S$ is separated if and only if the image of the diagonal morphism is closed in $X \times_S X$.*

Proof. If $X \to S$ is separated, the image of the diagonal is closed by definition of closed immersion.

Conversely, let us assume that $\Delta_{X/S} \colon X \to X \times_S X$ has closed image $\Delta_{X/S}(X) \subset X \times_S X$. First of all, the first projection $p \colon X \times_S X \to X$ satisfies $p \circ \Delta_{X/S} = \mathrm{id}_X$, so $\Delta_{X/S}$ defines a homeomorphism $X \to \Delta_{X/S}(X)$. (This also follows from Corollary 5.4.8). It remains to prove that

$$\Delta^{\#} \colon \mathscr{O}_{X \times_S X} \longrightarrow \Delta_* \mathscr{O}_X$$

is a surjective morphism of sheaves, where we have shortened $\Delta = \Delta_{X/S}$. The question is local. So for any point $x \in X$ we may choose an affine open $x \in U = \operatorname{Spec} A \subset X$, small enough to ensure that $f(U)$ is contained in an affine open subset $V = \operatorname{Spec} B$ of S. Then $U \times_V U = \operatorname{Spec}(A \otimes_B A)$ is an affine open neighborhood of $\Delta(x)$, and Δ restricted to this open subset is the diagonal $U \to U \times_V U$, which is a closed immersion by Proposition 5.4.6. Therefore $\Delta^{\#}$ is surjective in a neighborhood of $\Delta(x)$. This holds for any x, thus $\Delta^{\#}$ is globally surjective. $\square$

If X is any scheme and $U, V \subset X$ are open subsets, we have a canonical map

$$\mathscr{O}_X(U)\otimes_{\mathbb{Z}}\mathscr{O}_X(V) \xrightarrow{\psi_{UV}} \mathscr{O}_X(U\cap V), \quad f\otimes g \mapsto f|_{U\cap V}\cdot g|_{U\cap V}.$$

THEOREM 5.4.10. *Let X be a scheme. Then, the following conditions are equivalent:*

(i) *X is separated.*

(ii) *For every pair of affine open subsets $U, V \subset X$, their intersection $U\cap V$ is affine and ψ_{UV} is surjective.*

(iii) *There is an affine open cover $X = \bigcup_{i\in I} U_i$ such that $U_i\cap U_j$ is affine and $\psi_{U_iU_j}$ is surjective for all $(i,j)\in I\times I$.*

Proof. Let $U, V \subset X$ be two affine open subsets. Let us observe that the diagonal $\Delta = \Delta_{X/\mathbb{Z}}\colon X \to X\times_{\mathbb{Z}} X$ restricted to the (affine!) open subscheme $U\times_{\mathbb{Z}} V$ is

$$U\cap V = \Delta^{-1}(U\times_{\mathbb{Z}} V) \longrightarrow U\times_{\mathbb{Z}} V,$$

a morphism with affine target, whose associated ring homomorphism (global sections) is

$$\mathscr{O}_{X\times_{\mathbb{Z}}X}(U\times_{\mathbb{Z}} V) = \mathscr{O}_X(U)\otimes_{\mathbb{Z}}\mathscr{O}_X(V) \xrightarrow{\psi_{UV}} \mathscr{O}_X(U\cap V).$$

(i) $\Rightarrow$ (ii). If Δ is a closed immersion, then $U\cap V \to U\times_{\mathbb{Z}} V$ is a closed immersion by base change. Then $U\cap V$ is affine by Proposition 3.1.65, and ψ_{UV} is surjective.
(ii) $\Rightarrow$ (iii). Obvious.
(iii) $\Rightarrow$ (i). We are assuming

$$\mathscr{O}_X(U_i)\otimes_{\mathbb{Z}}\mathscr{O}_X(U_j) \xrightarrow{\psi_{U_iU_j}} \mathscr{O}_X(U_i\cap U_j)$$

surjective for all i, j. Since we are also assuming $U_i\cap U_j$ to be affine, $\psi_{U_iU_j}$ corresponds to the map $\Delta^{-1}(U_i\times_{\mathbb{Z}} U_j) \to U_i\times_{\mathbb{Z}} U_j$, which is then a closed immersion. But $X\times_{\mathbb{Z}} X = \bigcup_{i,j} U_i\times_{\mathbb{Z}} U_j$ is an open cover of the fibre product, therefore $\Delta^\#$ is surjective. $\square$

Example 5.4.11. Let $X_1 = X_2 = \operatorname{Spec}\mathbb{Z}$ with open subsets $X_{12} = \mathrm{D}(p) \subset X_1$ and $X_{21} = \mathrm{D}(p)\subset X_2$, where $p\in\mathbb{Z}$ is a prime number. We can perform the glueing of X_1 and X_2 along $\mathrm{id}\colon X_{12}\to X_{21}$. Call the resulting scheme X. Then, X is not separated, since the image of

$$\mathbb{Z} = \mathscr{O}_X(X_1)\otimes_{\mathbb{Z}}\mathscr{O}_X(X_2) \xrightarrow{\psi_{X_1X_2}} \mathscr{O}_X(X_1\cap X_2) = \mathscr{O}_X(X_{12}) = \mathbb{Z}[1/p]$$

is equal to $\mathbb{Z} \subsetneq \mathbb{Z}[1/p]$.

Example 5.4.12. Absolute projective space $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec}\mathbb{Z}$ is separated. Indeed, condition (iii) of Theorem 5.4.10 holds for the open cover $\mathbb{P}^n_{\mathbb{Z}} = U_0 \cup \cdots \cup U_n$ by the affine opens $U_i = \mathrm{D}_+(x_i)$, since

$$\mathbb{Z}[x_k x_i^{-1} \,|\, 0 \le k \le n] \otimes_{\mathbb{Z}} \mathbb{Z}[x_k x_j^{-1} \,|\, 0 \le k \le n] \to \mathbb{Z}[x_i x_j^{-1}, x_j x_i^{-1}, x_k x_i^{-1} \,|\, 0 \le k \le n]$$

is surjective for all $i, j \in \{0, 1, \ldots, n\}$.

Example 5.4.13. With our definition, it is not true that an arbitrary algebraic variety is separated (over the base field). However, this is true for (quasi)affine and (quasi)projective varieties.

Affine morphisms

Definition 5.4.14 (Affine morphism)**.** A morphism of schemes $X \to S$ is *affine* if the preimage of any affine open subset is affine.

Example 5.4.15. A closed immersion is an affine morphism. A morphism of affine schemes is affine.

PROPOSITION 5.4.16. *An affine morphism $f : X \to S$ is separated and quasicompact.*

Proof. Quasicompactness is clear, since affine schemes are quasicompact.

To show that $f : X \to S$ is separated, i.e. that $\Delta_{X/S} : X \times_S X$ is a closed immersion, we use that closed immersions are local on the target as follows. Cover S with affines $V_i = \operatorname{Spec} R_i$, and observe that

$$\begin{aligned} f^{-1}(V_i) \times_S f^{-1}(V_i) &= f^{-1}(V_i) \times_{V_i} f^{-1}(V_i) \\ &= \operatorname{Spec} B_i \times_{R_i} \operatorname{Spec} B_i \\ &= \operatorname{Spec}(B_i \otimes_{R_i} B_i) \end{aligned}$$

is again affine since f was affine. Now, $\Delta_{X/S}$ restricts, on these opens, to the diagonal of $f^{-1}(V_i) = \operatorname{Spec} B_i \to \operatorname{Spec} R_i = V_i$, which is a closed immersion by Proposition 5.4.6. Thus $\Delta_{X/S}$ is globally a closed immersion. □

5.5 Proper morphisms and the valuative criteria

Next we come to the important notion of proper morphisms. Such notion is the algebro-geometric analog of a proper map between complex analytic spaces. We shall see that projective morphisms, another important class of morphisms in algebraic geometry, are proper.

Definition 5.5.1 (Universally closed/open morphism)**.** A morphism $X \to S$ is *universally closed* (resp. *universally open*) if it is a closed map (resp. an open map), and the same holds true for the base change $X \times_S T \to T$ along an arbitrary map $T \to S$.

Note that being universally closed morphism is, by the very definition, stable under base change.

Definition 5.5.2 (Proper morphism)**.** A morphism $X \to S$ is *proper* if it is of finite type, separated and universally closed.

Example 5.5.3. Closed immersion are proper. They are clearly universally closed (e.g. by stability under base change), of finite type (cf. Example 5.2.12) and separated (e.g. because they are affine).

Example 5.5.4. Absolute projective space $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec}\mathbb{Z}$ is proper. Indeed, it is of finite type by Example 5.2.13, it is separated by Example 5.4.12, and finally it is universally closed by the following argument. Let Y be any scheme, and consider $\mathbb{P}^n_Y \to Y$. We need to show this map is closed. In fact, we may assume $Y = \operatorname{Spec} A$, and deal with $f : \mathbb{P}^n_A \to \operatorname{Spec} A$. Indeed, being closed is 'topologically local on the target', meaning that a subset Z of a topological space X is closed if and only for every (or for one) open covering $X = \bigcup_i U_i$, one has that $Z \cap U_i \subset U_i$ is closed for all i.

Let then $I \subset B = A[x_0, x_1, \ldots, x_n]$ be a homogeneous ideal, and set $Z = f(\mathrm{V}_+(I)) \subset \operatorname{Spec} A$. We claim that $Y \setminus Z \subset Y = \operatorname{Spec} A$ is open. For an arbitrary point $y \in Y$, denoting by $f^{-1}(y) = \operatorname{Proj}(B \otimes_A \kappa(y))$ the scheme-theoretic fibre of f over y, we have

$$\mathrm{V}_+(I) \cap f^{-1}(y) = \mathrm{V}_+(I \otimes_A \kappa(y)) \subset \mathbb{P}^n_A$$

by Proposition 4.7.15. Thus we have

$$\begin{aligned} y \in Y \setminus Z &\iff \mathrm{V}_+(I) \cap f^{-1}(y) = \emptyset \\ &\iff \mathrm{V}_+(I \otimes_A \kappa(y)) \subset \emptyset = \mathrm{V}_+((B \otimes_A \kappa(y))_+) = \mathrm{V}_+(B_+ \otimes_A \kappa(y)) \\ &\iff B_+ \otimes_A \kappa(y) \subset \sqrt{I \otimes_A \kappa(y)} \\ &\iff B_d \otimes_A \kappa(y) \subset I \otimes_A \kappa(y) \text{ for some } d > 0 \\ &\iff (B/I)_d \otimes_A \kappa(y) = 0 \text{ for some } d > 0. \end{aligned}$$

So, if $y \in Y \setminus Z$ and $d > 0$ realises the last vanishing, then by Nakayama's lemma (which we can apply since $(B/I)_d$ is a finitely generated A-module), we have

$$0 = (B/I)_d \otimes_A \mathscr{O}_{Y,y} = (B/I)_d \otimes_A \varinjlim_{\mathrm{D}(f) \ni y} A_f = \varinjlim_{\mathrm{D}(f) \ni y} (B/I)_d \otimes_A A_f.$$

But then there is an element $f \in A$ such that $y \in \mathrm{D}(f)$ and $(B/I)_d \otimes_A A_f = 0$, which proves that $\mathrm{D}(f) \subset Y \setminus Z$. Thus, $Y \setminus Z$ is open.

LEMMA 5.5.5. *(Quasi)separated and proper morphisms are stable under composition and base change.*

Proof. We start by proving stability under composition.
(Quasi)separated. Let $h\colon X \to Y$ and $Y \to Z$ be separated (resp. quasiseparated) morphisms of schemes. There is a commutative diagram (with cartesian square)

$$\begin{tikzcd} X \arrow[r, "\Delta_{X/Y}"] \arrow[rr, bend left, "\Delta_{X/Z}"] & X \times_Y X \arrow[r, "i"] \arrow[d] \arrow[dr, phantom, "\square"] & X \times_Z X \arrow[d, "h\times h"] \\ & Y \arrow[r, "\Delta_{Y/Z}"'] & Y \times_Z Y \end{tikzcd}$$

where $\Delta_{X/Y}$ is a closed immersion (resp. quasicompact) since $X \to Y$ is separated (resp. quasiseparated). But i is also a closed immersion (resp. quasicompact), since it is a base change of a morphism of the same kind (cf. Proposition 4.7.23) by our assumption on $Y \to Z$.
Proper. Stability under composition follows since finite type and universally closed morphisms have this property.

We proceed by proving stability under base change.
(Quasi)separated. Let $X \to S$ be a (quasi)separated morphism, $S' \to S$ another morphism. We need to check that $X' = X \times_S S' \to S'$ is (quasi)separated. The diagonal of $X' \to S'$ is the morphism

$$\Delta_{X'/S'}\colon X' \to X' \times_{S'} X'$$

appearing in the double cartesian diagram

$$\begin{tikzcd} & X' \arrow[r] \arrow[d, "\Delta_{X'/S'}"] \arrow[dr, phantom, "\square"] & X \arrow[d, "\Delta_{X/S}"] \\ X' \times_{S'} X' \arrow[r, equal] & (X \times_S X) \times_S S' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \times_S X \arrow[d] \\ & S' \arrow[r] & S \end{tikzcd}$$

therefore, since $\Delta_{X'/S'}$ is a base change of $\Delta_{X/S}$ (apply Proposition 4.7.13(4) here), what we need to prove follows once more from stability under base change of closed immersions and quasicompact morphisms (cf. Proposition 4.7.23).
Proper. Finite type morphisms are stable under base change (cf. Corollary 5.2.9), and so are universally closed morphisms. □

COROLLARY 5.5.6. *Let A be a ring. Any projective morphism $X \to \operatorname{Spec} A$ (Important Definition 3.3.1) is proper. In particular, this applies to projective varieties over a field.*

Proof. A closed immersion $X \hookrightarrow \mathbb{P}^n_A$ is proper, and $\mathbb{P}^n_A \to \operatorname{Spec} A$ is also proper (by base change along the proper morphism $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec}\mathbb{Z}$, cf. Example 5.5.4). Therefore the composition $X \hookrightarrow \mathbb{P}^n_A \to \operatorname{Spec} A$ is proper as well, by Lemma 5.5.5. □

Valuative criteria

There is an important criterion, often more useful in practice than the definition, to determine whether a morphism is separated (resp. proper). The rough idea is the following. A scheme, in order to be separated, should not contain anything like a line with a doubled origin or a variation thereof. This picture can be formalised by saying that whenever we are given a pair (C,x), where C is a curve and $x \in C$ is a point, along with a morphism

$$C \setminus \{x\} \longrightarrow X,$$

there must be at most one extension

$$C \longrightarrow X.$$

This for separatedness. Properness requires the existence of exactly one extension. However, this intuition has to be formalised correctly: we have work relatively over a base scheme S, and since the extension problem is of local nature, we need to replace (C,x) with the germ of $\mathscr{O}_C$ at x. If C were smooth at x, this would be a DVR. In order to keep the curve arbitrary, we need to consider an arbitrary valuation ring R, i.e. an arbitrary integral domain such that for any nonzero $\alpha \in \operatorname{Frac} R$, either α or α^{-1} is in R.

The criteria are as follows.

THEOREM 5.5.7 (Valuative criterion for separated maps). *Let $f: X \to S$ be a morphism of schemes, with X noetherian. Then, the following conditions are equivalent.*

- *f is separated, and*
- *For any valuation ring R with fraction field K, and for any commutative square diagram*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dotted, "u"] & S \end{tikzcd}$$

there is at most one arrow $u\colon \operatorname{Spec} R \to X$ making the whole diagram commutative.

THEOREM 5.5.8 (Valuative criterion for proper maps). *Let $f: X \to S$ be a morphism of schemes, with X noetherian. Assume f is of finite type. The following conditions are equivalent.*

- *f is proper, and*

- *For any valuation ring R with fraction field K, and for any commutative square diagram*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dotted, "u"] & S \end{tikzcd}$$

there is exactly one arrow $u\colon \operatorname{Spec} R \to X$ making the whole diagram commutative.

An often useful property is the following. It can be seen as the analog of the theorem in topology stating that the image of a continuous map from a compact space to a Hausdorff space is a closed subset.

PROPOSITION 5.5.9. *Let S be a scheme. Let $f: X \to Y$ be an S-morphism, where X and Y are noetherian, $g\colon Y \to S$ is separated and $g \circ f : X \to S$ is proper. Then f is proper.*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\text{proper}"'] & & Y \arrow[dl, "g \text{ separated}"] \\ & S & \end{tikzcd}$$

Proof. First note that f is quasicompact (since X is noetherian, which is equivalent to every subspace being quasicompact). Thus, since $g \circ f$ is of finite type, so is f by Lemma 5.2.10. Then we are in a position to exploit Theorem 5.5.8 to check properness of f. Consider the diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r, "\alpha"] \arrow[ur, dotted, "u"] \arrow[d, equal] & Y \arrow[d, "g"] \\ \operatorname{Spec} R \arrow[r, "g\circ\alpha"] & S \end{tikzcd}$$

where u exists since $X \to S$ is proper. Then $f \circ u$ and α are maps $\operatorname{Spec} R \to Y$, but they must agree thanks to Theorem 5.5.7, since $Y \to S$ is separated. □

The slogan to be rememberd here is:

$$\boxed{\text{proper to separated is proper.}}$$

THEOREM 5.5.10. *Let X be a proper integral $\mathbf{k}$-variety. Then*

$$\mathscr{O}_X(X) = \mathbf{k}.$$

Proof. We know by Theorem 3.2.8 that there is a canonical identification

$$\operatorname{Hom}_{\mathsf{Sch}_{\mathbf{k}}}(X, \mathbb{A}^1_{\mathbf{k}}) = \operatorname{Hom}_{\mathsf{Alg}_{\mathbf{k}}}(\mathbf{k}[t], \mathscr{O}_X(X)) = \mathscr{O}_X(X),$$

Let us pick a morphism $f: X \to \mathbb{A}^1_{\mathbf{k}}$. Consider the composition $i \circ f: X \to \mathbb{A}^1_{\mathbf{k}} \hookrightarrow \mathbb{P}^1_{\mathbf{k}}$. We are in the situation

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\text{proper}"'] & \mathbb{A}^1_{\mathbf{k}} \arrow[dl, "\text{separated}"] \\ \operatorname{Spec}\mathbf{k} & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "i\circ f"] \arrow[d, "\text{proper}"'] & \mathbb{P}^1_{\mathbf{k}} \arrow[dl, "\text{separated}"] \\ \operatorname{Spec}\mathbf{k} & \end{tikzcd}$$

which shows that f must be constant. Indeed, by Proposition 5.5.9, from the first diagram we get that f is proper, hence its image is either $\mathbb{A}^1_{\mathbf{k}}$ or a finite subset thereof. But the second diagram, by the same arguments, allows us to exclude the surjectivity of f. Now, since X is irreducible, the image of f is topologically just one point (cf. Exercise 4.1.4(ii)). However, since X is also reduced, by Proposition 4.3.5(ii) the image must be a reduced (closed) point. Since $\mathbf{k}$ is algebraically closed, the residue field of this point is just $\mathbf{k}$. Thus f is just an element of $\mathbf{k}$, i.e. a constant regular function. □

COROLLARY 5.5.11. *Let X be a proper, reduced $\mathbf{k}$-variety. Then $\mathscr{O}_X(X)$ is a finite-dimensional $\mathbf{k}$-vector space.*

Proof. Let $Z_1, \ldots, Z_m$ be the finitely many irreducible components of X, endowed with the reduced induced scheme structure. Then, by Lemma 5.5.5, each composition $Z_i \hookrightarrow X \to \operatorname{Spec}\mathbf{k}$ is proper. We thus have $\mathscr{O}_{Z_i}(Z_i) = \mathbf{k}$ for all i by Theorem 5.5.10. On the other hand, we have inclusions $\mathbf{k} \hookrightarrow \mathscr{O}_X(X) \hookrightarrow \bigoplus_{1\leq i\leq m} \mathscr{O}_{Z_i}(Z_i) = \mathbf{k}^{\oplus m}$. □

COROLLARY 5.5.12. *Let X be a proper, reduced $\mathbf{k}$-variety. If X is also affine, then X is the disjoint union of a finite set of (reduced) points.*

Proof. Write $X = \operatorname{Spec} A$. By Corollary 5.5.11, we have

$$A = \mathscr{O}_X(X) = \mathbf{k}^{\oplus d}$$

which is the coordinate ring of $\coprod_{1\leq i\leq d} \operatorname{Spec}\mathbf{k}$. □

Remark 5.5.13. A fortiori, the number d above is the number of connected components of X.

See Corollary 5.6.6 for a more general characterisation of proper affine morphisms.

5.6 Finite, integral and quasifinite morphisms

We give here more finiteness conditions on morphisms of schemes. There is a purely set-theoretic notion of finiteness that can be attached to a (finite type) morphism of schemes.

Definition 5.6.1 (Quasifinite morphism)**.** A morphism of schemes $f: X \to S$ is *quasifinite* if it is of finite type and has finite fibres.

Recall from Definition 4.6.25 the definition of finite and integral ring homomorphism.

Definition 5.6.2 (Finite, integral morphism)**.** A morphism of schemes $f: X \to S$ is *finite* (resp. *integral*) if it is affine and for any affine open subset $V = \operatorname{Spec} R \subset S$ with preimage $U = \operatorname{Spec} A \subset X$, the induced map $R \to A$ is finite (resp. integral).

Example 5.6.3. The blowup morphism $\mathrm{Bl}_0 \mathbb{A}^2 \to \mathbb{A}^2$ is not finite.

Remark 5.6.4. We make the following easy observations.

- Finite, integral and quasifinite morphisms are stable under base change and composition.
- A finite morphism is integral. An integral morphism locally of finite type is finite.
- Finite and integral morphisms are separated, quasicompact.
- A finite morphism is of finite type.

A nontrivial result is the following.

THEOREM 5.6.5 ([16, Tag 01WM]). *A morphism $X \to S$ is integral if and only if it is affine and universally closed.*

The following is a very useful criterion for finiteness.

COROLLARY 5.6.6. *A morphism $X \to S$ is finite if and only if it is proper and affine.*

Proof. If $X \to S$ is finite, it is affine by definition. We already observed that finite morphisms are separated and of finite type. Being integral, it is universally closed by Theorem 5.6.5. Hence $X \to S$ is proper. For the converse, we refer to [12, Ch. 3, Lemma 3.17]. □

The proof of the following result requires results we have not covered, but we give a reference.

PROPOSITION 5.6.7 ([16, Tag 02LS]). *A morphism $X \to S$ is finite if and only if it is proper and quasifinite.*

Here is a pictorial recap of the types of morphisms we have seen so far, and the various implications we have proved or mentioned. It would be great to have, for each implication, a few examples in mind and, most importantly, a few counterexamples to the failure of the converse implication!

$$
\begin{tikzcd}
 & \text{closed immersion} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & \text{proper monomorphism} \\
\begin{array}{c}\text{integral} = \text{affine} \\ + \text{universally closed}\end{array} \arrow[d, Rightarrow] & \begin{array}{c}\text{finite} = \\ \text{affine} + \text{proper}\end{array} \arrow[l, Rightarrow] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{quasifinite} \\
\text{affine} \arrow[d, Rightarrow] \arrow[ddr, Rightarrow] & \text{proper} \arrow[d, Rightarrow] & \text{projective} \arrow[l, Rightarrow] \\
\text{separated} \arrow[d, Rightarrow] & \text{finite type} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{locally finite type} \\
\text{quasiseparated} & \text{quasicompact} & \text{immersion} \arrow[u, Rightarrow]
\end{tikzcd}
$$

5.7 Rational maps and birational morphisms

First of all, a terminology warning: in 'common' mathematical language, people often say *map* to mean *morphism*. When one says *map* in algebraic geometry, it should mean a *wanna-be morphism*, namely a morphism defined only on a dense open subset of the source.

Let X be a scheme. Note that if $U, V \subset X$ are open and dense, then so is $U \cap V$. Let Y be another scheme. Declare two morphisms

$$f : U \to Y, \quad g : V \to Y$$

to be equivalent if there exists a dense open subset $W \subset U \cap V$ such that $f|_W = g|_W$. This is an equivalence relation.

Definition 5.7.1 (Rational map)**.** A *rational map from X to Y* is an equivalence class of morphisms $U \to Y$ with source an open dense subset of X.

We use the notation

$$\varphi : X \dashrightarrow Y$$

to denote a rational map from X to Y, with the dashed arrow reminding us that there might be points of X on which the map cannot be defined (i.e. there is no globally defined representative). Note that any honest morphism is a trivial example of a rational map.

We also have the following relative notion.

Definition 5.7.2 (S-Rational map)**.** Let S be a scheme, X and Y two S-schemes. An *S-rational map from X to Y* is a rational map $X \dashrightarrow Y$ as in Definition 5.7.1 such that there exists a representative $U \to Y$ which is an S-morphism.

Exercise 5.7.3. The affine scheme $\mathbb{A}^1_{\mathbb{Z}}$ is a ring object in Sch.

From Exercise 5.7.3 it immediately follows that the set

$$R(X) = \left\{ \varphi \colon X \dashrightarrow \mathbb{A}^1_{\mathbb{Z}} \,\middle|\, \varphi \text{ is a rational map} \right\}$$

has a natural ring structure. Its elements are called the *rational functions* on the scheme X.

Exercise 5.7.4. If X has finitely many irreducible components $Z_1, \ldots, Z_m$, with generic points $\xi_1, \ldots, \xi_m \in X$, then

$$R(X) = \prod_{1 \le i \le m} \mathscr{O}_{X,\xi_i}.$$

This equals $K(X) = \mathscr{O}_{X,\xi} = \kappa(\xi)$ when X is an integral scheme. If X is only reduced, then $\mathscr{O}_{X,\xi_i} = \kappa(\xi_i)$. A hint for the last part: if $A \neq 0$ is an arbitrary ring and $\mathfrak{p} \in \operatorname{Spec} A$ is a minimal prime, then $\mathfrak{p}A_{\mathfrak{p}}$ consists of nilpotent elements.

Definition 5.7.5 (Dominant rational map)**.** Let X, Y be irreducible schemes. A rational map $X \dashrightarrow Y$ is *dominant* if any representative $U \to Y$ is dominant (i.e. it has dense image).

Lemma 5.7.6. *Let $f \colon X \to Y$ be a morphism of irreducible schemes, with generic points $\xi_X \in X$ and $\xi_Y \in Y$ respectively. Then f is dominant if and only if $f(\xi_X) = \xi_Y$.*

Proof. Assume f is dominant, i.e. $\overline{f(X)} = Y$. For every subset $Z \subset X$, one has $f(\overline{Z}) \subset \overline{f(Z)}$, therefore

$$f(X) = f(\overline{\{\xi_X\}}) \subset \overline{\{f(\xi_X)\}},$$

which implies $Y = \overline{f(X)} = \overline{\{f(\xi_X)\}}$. But by Lemma 3.1.46 there is a unique point in Y, namely ξ_Y, whose closure is the whole of Y. Therefore $f(\xi_X) = \xi_Y$.

Conversely, if $f(\xi_X) = \xi_Y$, then $\overline{f(X)} \supset \overline{\{f(\xi_X)\}} = \overline{\{\xi_Y\}} = Y$. □

The advantage of dealing with dominant rational maps instead of all rational maps is that dominant ones can be composed (at least if we restrict our attention to irreducible schemes). Indeed, what happens if some $U \to Y$ is constantly equal to $y \in Y$ and y is not in the domain of any representative for some $Y \dashrightarrow Z$?

So, suppose given three irreducible schemes X, Y and Z, along with dominant rational maps

$$X \overset{\varphi}{\dashrightarrow} Y, \qquad Y \overset{\psi}{\dashrightarrow} Z.$$

Pick representatives

$$X \supset U \xrightarrow{f} Y, \qquad Y \supset V \xrightarrow{g} Z$$

and define $\psi \circ \varphi$ to be the class of the composition

$$U \cap f^{-1}(V) \xrightarrow{f} V \xrightarrow{g} Z.$$

We leave it to the reader to check that this is a well-defined rational map $X \dashrightarrow Z$.

This composition law allows us to define categories

$$\mathsf{Irr}^{\mathsf{dom}}, \quad \mathsf{Irr}_S^{\mathsf{dom}}$$

whose objects are irreducible schemes (or irreducible schemes over S for the second category), and whose morphisms are dominant rational maps (or dominant S-rational maps for the second category).

Definition 5.7.7 (Birational map)**.** Isomorphisms in $\mathsf{Irr}^{\mathsf{dom}}$ (resp. in $\mathsf{Irr}_S^{\mathsf{dom}}$) are called *birational maps* (resp. S-*birational maps*).

Fix two irreducible schemes X and Y.

(1) We say that X and Y are *birational* if there is a birational map $X \dashrightarrow Y$.

(2) If X and Y are defined over a common base scheme S, they are called S-*birational* if there is an S-birational map $X \dashrightarrow Y$.

(3) An irreducible algebraic variety X over a field $\mathbb{F}$ is called *rational* if it is $\mathbb{F}$-birational to $\mathbb{P}^d_{\mathbb{F}}$, where $d = \dim X$.

Example 5.7.8. Any open immersion into an irreducible scheme is a birational map. In particular, any nonempty open subset of $\mathbb{P}^n_{\mathbb{F}}$ is a rational variety.

We will soon see *birational morphisms.* Do not confuse these with the notions in Definition 5.7.7!

If $\varphi\colon X \dashrightarrow Y$ is a birational map and $\psi\colon Y \dashrightarrow X$ satisfies $\varphi \circ \psi = \mathrm{id}_Y$ and $\psi \circ \varphi = \mathrm{id}_X$ in the category $\mathsf{Irr}^{\mathsf{dom}}$, we say that ψ is a *rational inverse* for φ.

Example 5.7.9. Let us consider the inclusion $\mathbb{A}^1_{\mathbf{k}} \hookrightarrow \mathbb{P}^1_{\mathbf{k}}$. This is a morphism, hence it is a rational map. A rational inverse is given by $\mathbb{P}^1_{\mathbf{k}} \dashrightarrow \mathbb{A}^1_{\mathbf{k}}$, sending $(a:b) \mapsto a/b$. This is only defined on $\mathrm{D}_+(x_1)$.

PROPOSITION 5.7.10. *Fix irreducible schemes X and Y.*

(i) *X and Y are birational if and only if there are nonempty open subsets $U \subset X$ and $V \subset Y$ such that $U \cong V$.*

(ii) *If X and Y are defined over a common base scheme S, they are S-birational if and only if there are nonempty open subsets $U \subset X$ and $V \subset Y$ along with an isomorphism $U \cong V$ over S.*

Proof. Let us show (i) (the argument for (ii) is identical). In the situation

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, hook, "i"'] & V \arrow[d, hook, "j"] \\ X & Y \end{tikzcd}$$

where h is an isomorphism, we see that the equivalence class of $j \circ h \colon U \to Y$ (viewed as a rational map $X \dashrightarrow Y$) has rational inverse $i \circ h^{-1} \colon V \to X$ (viewed as a rational map $Y \dashrightarrow X$).

Conversely, assume X and Y are birational. Let

$$X \supset U \xrightarrow{f} Y, \qquad Y \supset V \xrightarrow{g} X$$

be morphisms representing rational inverses. We may shrink V enough to make it affine, and similarly we may replace U with an affine open subset of $f^{-1}V$. So now, by assumption, we have that

$$U \xrightarrow{f} V \xrightarrow{g} X$$

is the identity on a dense open subset of U. Thus, after shrinking U, we may assume that $U \to V \to X$ is the inclusion of U into X. Thus $U \to V$ is an immersion. Switching the roles of U and V, we obtain a nonempty affine open $j \colon V' \hookrightarrow V$ such that j factors as

$$V' \longrightarrow U \xrightarrow{f} V.$$

Then $V' \to U$ is an open immersion. Thus V' is isomorphic to an open of both X and Y, as required. $\square$

Example 5.7.11. The varieties $\mathbb{P}^2_{\mathbb{F}}$ and $\mathbb{P}^1_{\mathbb{F}} \times_{\mathbb{F}} \mathbb{P}^1_{\mathbb{F}}$ are birational to each other, since they share the common open subset $\mathbb{A}^2_{\mathbb{F}} \cong \mathbb{A}^1 \times_{\mathbb{F}} \mathbb{A}^1_{\mathbb{F}}$. Explicitly, consider the rational map

$$\mathbb{P}^2_{\mathbb{F}} \overset{\varphi}{\dashrightarrow} \mathbb{P}^1_{\mathbb{F}} \times_{\mathbb{F}} \mathbb{P}^1_{\mathbb{F}}$$

which is the class of the morphism

$$\mathrm{D}_+(t_2) \longrightarrow \mathrm{D}_+(x_1) \times_{\mathbf{k}} \mathrm{D}_+(y_1).$$

This is an $\mathbb{F}$-birational map.

Warning 5.7.12. Two irreducible schemes of different dimensions cannot be birational!

Note that a dominant rational map $\varphi\colon X \dashrightarrow Y$ between integral schemes induces a map of function fields

$$K(Y) \hookrightarrow K(X)$$

as follows. Say $f\colon U \to Y$ is a representative for φ. Then, by Lemma 5.7.6, f sends the generic point of X to the generic point of Y. This induces

$$K(Y) = \mathscr{O}_{Y,\xi_Y} \longrightarrow \mathscr{O}_{X,\xi_X} = K(X),$$

which is necessarily injective. The converse is false in general. For instance, consider $X = \operatorname{Spec} \mathbf{k}[x]$ and $Y = \operatorname{Spec} \mathbf{k}(x)$, which have the same function field, namely $\mathbf{k}(x)$. However, there is no dominant rational map $X \dashrightarrow Y$ since any such map would induce a dominant morphism

$$X \supset U \supset \mathrm{D}(f) \to \operatorname{Spec} \mathbf{k}(x)$$

and there are no injective ring homomorphisms $\mathbf{k}(x) \to \mathbf{k}[x]_f$ for any $f \in \mathbf{k}[x]$.

For integral algebraic varieties, however, the situation is different (and better). This is shown in the next result.

THEOREM 5.7.13. *Let $\mathbb{F}$ be a field. Sending $X \mapsto K(X)$ defines an equivalence of categories*

$$\left\{ \begin{array}{c} \text{integral separated } \mathbb{F}\text{-varieties} \\ \text{with dominant rational maps} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{finitely generated} \\ \text{field extensions } K/\mathbb{F} \end{array} \right\}.$$

In particular, two separated integral varieties are birational if and only if they have isomorphic function fields.

Proof. The assignment is clearly functorial. We need to show the functor $K(-)$ is fully faithful and essentially surjective. Essential surjectivity is the content of Lemma 5.7.14 below. As for fully faithfullness, □ to write

LEMMA 5.7.14. *Let K be a finitely generated field extension of a field $\mathbb{F}$. Then there is an integral domain A finitely generated over $\mathbb{F}$ such that $K = \operatorname{Frac} A$.*

Proof. Take generators $x_1,\ldots,x_n \in K$ of the extension $K/\mathbb{F}$. Construct the ring homomorphism $\phi\colon P = \mathbb{F}[t_1,\ldots,t_n] \to K$ sending $t_i \mapsto x_i$. Its kernel is a prime ideal since $A = P/\ker\phi$ is a subring of K, hence an integral domain. We have $K = \operatorname{Frac} A$, as required.

The proof is over, but we present an alternative proof in characteristic 0. We leave it to the reader to find a general argument. Let $x_1,\ldots,x_n \in K$ be algebraically independent elements. We then have field extensions

$$\mathbb{F} \subset M = \mathbb{F}(x_1,\ldots,x_n) \subset K, \quad [K:M] < \infty.$$

By the primitive element theorem, we have $K = M(z)$ for some algebraic element $z \in K$. Let $p(T) \in M[T]$ be its minimal polynomial, which is irreducible. Its coefficients are rational function in $x_1, \ldots, x_n$, with $\mathbb{F}$-coefficients. After clearing denominators, we find an irreducible polynomial $f \in \mathbb{F}[y_1, \ldots, y_n, y_{n+1}]$ such that $f(x_1, \ldots, x_n, z) = 0$. Define

$$A = \mathbb{F}[y_1, \ldots, y_n, y_{n+1}]/(f)$$

Then once more $K = \operatorname{Frac} A$, as required. □

Example 5.7.15. Consider the quadric cone $X = \operatorname{Spec} \mathbb{C}[x, y, z]/(x^2 - yz)$. It is an integral affine $\mathbb{C}$-scheme. Then X is birational to $\mathbb{A}^2_{\mathbb{C}}$ (hence a rational variety), since

$$K(X) = \operatorname{Frac} \mathbb{C}[x, y, z]/(x^2 - yz) = \mathbb{C}(x, y) = K(\mathbb{A}^2_{\mathbb{C}}).$$

COROLLARY 5.7.16. *Let X and Y be integral separated $\mathbb{F}$-varieties. The following conditions are equivalent.*

1. *X and Y are $\mathbb{F}$-birational,*
2. *$K(X)$ and $K(Y)$ are isomorphic,*
3. *there exist nonempty open subsets $U \subset X$ and $V \subset Y$ such that $U \cong V$,*

Proof. Combine Theorem 5.7.13 and Proposition 5.7.10 with one another. □

PROPOSITION 5.7.17. *Let X be an integral separated $\mathbb{F}$-variety of dimension d. Assume $\mathbb{F}$ has characteristic 0. Then X is $\mathbb{F}$-birational to an affine hypersurface $Y \subset \mathbb{A}^{d+1}_{\mathbb{F}}$.*

Proof. Saying $d = \dim X$ means that $K(X)$ is algebraic over $\mathbb{E} = \mathbb{F}(x_1, \ldots, x_d)$ for some transcendence basis $\{x_1, \ldots, x_d\} \subset K(X)$. Since $\mathbb{F}$ has characteristic 0, $K(X) \supset \mathbb{E}$ is separable. By the primitive element theorem, this extension is then simple, i.e. there exists $y \in K(X)$ such that $K(X) = \mathbb{E}(y)$. Let $f \in \mathbb{E}[t]$ be the minimal polynomial of y. In particular, $f(t)$ is monic and irreducible, and after clearing denominators we may assume that $f(t) \in \mathbb{F}[x_1, \ldots, x_d][t]$. By Gauss' Lemma, $f(t)$ is still irreducible. Let $Y \subset \mathbb{A}^{d+1}_{\mathbb{F}}$ be the hypersurface defined by $f \in \mathbb{F}[x_1, \ldots, x_d, t]$. Then by construction $K(Y) = \mathbb{E} = K(X)$, thus X and Y are $\mathbb{F}$-birational by Corollary 5.7.16. □

Birational morphisms

In this section all schemes are assumed to have a finite number of irreducible components.

Definition 5.7.18 (Birational morphism)**.** A morphism $f : X \to Y$ is birational if it induces a bijection between the generic points of X and the generic points of Y, and an isomorphism $\mathcal{O}_{Y,f(\eta)} \xrightarrow{\sim} \mathcal{O}_{X,\eta}$ for any generic point η of X.

A birational morphism is in particular a birational map. Any isomorphism is a birational morphism. Note that a birational morphism is always dominant, since all the generic points of the target are in the image (see [16, Tag 01RK] for more details).

Example 5.7.19. The morphism

$$\mathbb{A}^1_{\mathbb{Q}} \xrightarrow{f} X = \operatorname{Spec} \mathbb{Q}[x,y]/(y^2 - x^3)$$

induced by $x \mapsto y^2$, $y \mapsto t^3$ is birational (and, as we already know, not an isomorphism). A rational inverse can be found as follows. Construct the map sending $p \in X \subset \mathbb{A}^2_{\mathbb{Q}}$ to the slope m of the line connecting p and the origin $0 \in \mathbb{A}^2_{\mathbb{Q}}$. Such a line will be of the form $y = mx$ for some $m \in \mathbb{Q}$. This map $X \dashrightarrow \mathbb{A}^1_{\mathbb{Q}}$ cannot be extended over the origin by any means. The morphism f is the so-called *rational parametrisation* of the cubic curve X. If we substitute $y = mx$ into the equation $y^2 = x^3$ we find $0 = x^3 - m^2x^2 = x^2(x - m^2)$, which solving for x yields $(x,y) = (0,0)$ or $(x,y) = (m^2, m^3)$, as expected.

Example 5.7.20 (Blowup of the plane at the origin)**.** Consider the rational map $\mathbb{A}^2_{\mathbf{k}} \dashrightarrow \mathbb{P}^1_{\mathbf{k}}$ sending $(a,b) \mapsto (a:b)$. It is defined everywhere except at the origin 0. We can consider the closure of the graph of the morphism $f\colon \mathbb{A}^2_{\mathbf{k}} \setminus 0 \to \mathbb{P}^1_{\mathbf{k}}$ inside $\mathbb{A}^2_{\mathbf{k}} \times_{\mathbf{k}} \mathbb{P}^1_{\mathbf{k}}$. We have, indeed, immersions

$$\mathbb{A}^2_{\mathbf{k}} \setminus 0 \hookrightarrow \mathbb{A}^2_{\mathbf{k}} \setminus 0 \times_{\mathbf{k}} \mathbb{P}^1_{\mathbf{k}} \hookrightarrow \mathbb{A}^2_{\mathbf{k}} \times_{\mathbf{k}} \mathbb{P}^1_{\mathbf{k}}$$

where the first morphism is a closed immersion since $\mathbb{P}^1_{\mathbf{k}} \to \operatorname{Spec} \mathbf{k}$ is separated. The closure

$$\operatorname{Bl}_0 \mathbb{A}^2_{\mathbf{k}} = \overline{\mathbb{A}^2_{\mathbf{k}} \setminus 0} \hookrightarrow \mathbb{A}^2_{\mathbf{k}} \times_{\mathbf{k}} \mathbb{P}^1_{\mathbf{k}}$$

is called the *blowup of* $\mathbb{A}^2_{\mathbf{k}}$ *at* 0, and is an instance of a more general construction that we shall see in **??** Note that we have two natural morphisms

$$\begin{tikzcd} & \operatorname{Bl}_0 \mathbb{A}^2_{\mathbf{k}} \arrow[dl, "\pi"'] \arrow[dr, "\tilde{f}"] & \\ \mathbb{A}^2_{\mathbf{k}} & & \mathbb{P}^1_{\mathbf{k}} \end{tikzcd}$$

where $\tilde{f}$ can be thought of as a 'resolution' of f and π is the *blowup map* (cf. Figure 5.1). The preimage of the origin

$$E = \pi^{-1}(0) \subset \operatorname{Bl}_0 \mathbb{A}^2_{\mathbf{k}}$$

is called the *exceptional divisor* of the blowup. Away from this closed locus, the map π is an isomorphism, i.e. one has

$$\pi\colon \operatorname{Bl}_0 \mathbb{A}^2_{\mathbf{k}} \xrightarrow{\sim} \mathbb{A}^2_{\mathbf{k}} \setminus 0.$$

In particular, π is a birational morphism.

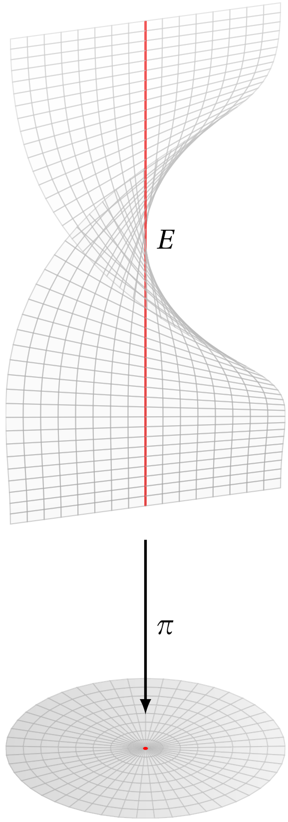

Figure 5.1: The blowup $\pi\colon \mathrm{Bl}_0 \mathbb{A}^2_{\mathbf{k}} \to \mathbb{A}^2_{\mathbf{k}}$ of the plane at the origin, with its exceptional divisor.

Caution 5.7.21. If $f\colon X \to Y$ is birational it does *not* follow that f restricts to an isomorphism onto an open subset of the target. Consider for instance the case of an affine morphism $f\colon \operatorname{Spec} S^{-1}A \to \operatorname{Spec} A$, where A is an integral domain and S is a multiplicative subset not containing 0. Since $A \to B = S^{-1}A$ is injective and both rings are domains, f sends generic point to generic point, and moreover $A \hookrightarrow B$ induces $\operatorname{Frac} A \xrightarrow{\sim} \operatorname{Frac} B$ (the fraction field is the same). If we had a cartesian diagram

$$\begin{tikzcd}
f^{-1}(\operatorname{Spec} A_f) \arrow[r, hook] \arrow[d, "\wr"] \arrow[dr, phantom, "\square"] & f^{-1}(U) \arrow[r, hook] \arrow[d, "\wr"] \arrow[dr, phantom, "\square"] & \operatorname{Spec} B \arrow[d, "f"] \\
\operatorname{Spec} A_f \arrow[r, hook] & U \arrow[r, hook] & \operatorname{Spec} A
\end{tikzcd}$$

for some open subset $U \subset \operatorname{Spec} A$, then we would have $A_f = A_f \otimes_A B$ for any $f \in A$ such that $\operatorname{Spec} A_f \subset U$. This would mean that every $s \in S$ is invertible in A_f. Considering $A = \mathbb{Z}$ and $S \subset \mathbb{Z}$ the set of odd integers then yields a counterexample.

6 | Infinitesimal properties

6.1 Regular schemes

Throughout this section, we work with *locally noetherian* schemes. Recall that, by Proposition 4.6.21(b), if $(A, \mathfrak{m})$ is a noetherian local ring, then A has finite Krull dimension, and

$$\dim A \leq \dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2. \tag{6.1.1}$$

Definition 6.1.1 ((Co)tangent space)**.** Let $x \in X$ be a point on a locally noetherian scheme X. The (finite-dimensional) $\kappa(x)$-vector space

$$\Omega_{X,x} = \mathfrak{m}_x/\mathfrak{m}_x^2 = \mathfrak{m}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$

is called the *cotangent space* of X at x. Its dual,

$$T_{X,x} = (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee = \mathrm{Hom}_{\kappa(x)}(\Omega_{X,x}, \kappa(x)),$$

is called the *tangent space* of X at x.

The inequality (6.1.1) gives

$$\dim \mathcal{O}_{X,x} \leq \dim_{\kappa(x)} \Omega_{X,x} = \dim_{\kappa(x)} T_{X,x},$$

for every $x \in X$. We are ready to define regular schemes.

Definition 6.1.2 (Regular scheme)**.** A noetherian local ring $(A, \mathfrak{m})$ is called *regular* if $\dim A = \dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$ (i.e. if $\mathfrak{m}$ is generated by $\dim A$ elements). A locally noetherian scheme X is called *regular* at a point $x \in X$ if the local ring $\mathcal{O}_{X,x}$ is regular, i.e. if $\dim \mathcal{O}_{X,x} = \dim_{\kappa(x)} T_{X,x}$. If X is regular at all of its points, we say that X is a *regular scheme.* A point x that is not regular is called *singular.* We let

$$X_{\mathsf{reg}} \subset X, \quad X_{\mathsf{sing}} = X \setminus X_{\mathsf{reg}} \subset X$$

denote the loci of regular and singular points in X.

If $f: X \to Y$ is a map of locally noetherian schemes, sending $x \mapsto y$, then since $f_x^\#: \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is a local homomorphism of local rings, the maximal ideal $\mathfrak{m}_y \subset \mathcal{O}_{Y,y}$ maps into the maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$, thus $f_x^\#$ descends to a $\kappa(y)$-linear map $\overline{f}_x^\#: \mathfrak{m}_y/\mathfrak{m}_y^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2$. This induces a $\kappa(x)$-linear *cotangent map*

$$\begin{tikzcd} \mathfrak{m}_y/\mathfrak{m}_y^2 \otimes_{\kappa(y)} \kappa(x) \arrow[r, "\Omega_{f,x}"] \arrow[d, equal] & \mathfrak{m}_x/\mathfrak{m}_x^2 \arrow[d, equal] \\ \Omega_{Y,y} \otimes_{\kappa(y)} \kappa(x) & \Omega_{X,x} \end{tikzcd} \tag{6.1.2}$$

defined by $h \otimes \alpha \mapsto \alpha \overline{f}_x^\#(h)$. Now since $\mathfrak{m}_y/\mathfrak{m}_y^2$ is finite-dimensional over $\kappa(y)$ (as Y is locally noetherian) we have

$$\begin{tikzcd} \operatorname{Hom}_{\kappa(x)}(\mathfrak{m}_y/\mathfrak{m}_y^2 \otimes_{\kappa(y)} \kappa(x), \kappa(x)) \arrow[r, "\sim"] \arrow[d, equal] & \operatorname{Hom}_{\kappa(y)}(\mathfrak{m}_y/\mathfrak{m}_y^2, \kappa(y)) \otimes_{\kappa(y)} \kappa(x) \arrow[d, equal] \\ (\Omega_{Y,y} \otimes_{\kappa(y)} \kappa(x))^\vee & T_{Y,y} \otimes_{\kappa(y)} \kappa(x) \end{tikzcd} \tag{6.1.3}$$

so that we can safely dualise (6.1.2) over $\kappa(x)$ to get the $\kappa(x)$-linear map

$$T_{f,x}: T_{X,x} \longrightarrow T_{Y,y} \otimes_{\kappa(y)} \kappa(x).$$

This is called the *tangent map* of f at x. Note that the dualisation (6.1.3) also works if we assume $\kappa(x) \supset \kappa(y)$ is a finite extension instead of assuming Y locally noetherian (which we do anyways for psychological reasons).

From now on we focus on the following (important) special case. Let $\mathbf{k}$ be an algebraically closed field. If $f: X \to Y$ is a morphism of $\mathbf{k}$-varieties, and $x \in X$ is a closed point with image $y \in Y$, then $\kappa(x) = \mathbf{k} = \kappa(y)$ (cf. Exercise 3.1.56 and Proposition 5.3.3). Thus the tangent map is in this case a $\mathbf{k}$-linear map

$$T_{f,x}: T_{X,x} \longrightarrow T_{Y,y}.$$

6.1.1 The tangent space to $\mathbb{A}^n_{\mathbf{k}}$

Let $Y = \mathbb{A}^n_{\mathbf{k}}$, with $\mathbf{k}$ an algebraically closed field as usual. Fix a closed point $y \in Y$ and let

$$\mathfrak{m} = (x_1 - \lambda_1, \ldots, x_n - \lambda_n) \subset A = \mathbf{k}[x_1, \ldots, x_n]$$

be the corresponding maximal ideal, so that

$$(\mathcal{O}_{Y,y}, \mathfrak{m}_y, \kappa(y)) = (A_\mathfrak{m}, \mathfrak{m}A_\mathfrak{m}, \mathbf{k}).$$

Set $E = \mathbf{k}^n$. Consider the $\mathbf{k}$-linear map

$$\mathrm{D}_y: \mathbf{k}[x_1, \ldots, x_n] \longrightarrow E^\vee = \operatorname{Hom}_{\mathbf{k}}(E, \mathbf{k})$$

sending a polynomial $P = P(x_1, \dots, x_n)$ to the linear functional

$$E \ni (t_1, \dots, t_n) \xmapsto{\mathrm{D}_y P} \sum_{1 \le i \le n} \frac{\partial P}{\partial x_i}(y) t_i \in \mathbf{k},$$

where, by definition,

$$\frac{\partial P}{\partial x_i}(y) = \frac{\partial P}{\partial x_i}(\lambda_1, \dots, \lambda_n) \in \mathbf{k}.$$

There is a commutative diagram

$$\begin{tikzcd}
\mathbf{k}[x_1, \dots, x_n] \arrow[r, "\mathrm{D}_y"] & E^\vee \\
\mathfrak{m} \arrow[u, hook] \arrow[r, two heads] & \mathfrak{m}/\mathfrak{m}^2 \arrow[u, "\overline{\mathrm{D}}_y \text{ (isomorphism)}"']
\end{tikzcd}$$

where the isomorphism is induced by truncating the Taylor expansion of P to first order around $(\lambda_1, \dots, \lambda_n)$. Note that, by dualising $\overline{\mathrm{D}}_y$, we also obtain an isomorphism

$$E \cong (\mathfrak{m}/\mathfrak{m}^2)^\vee \cong (\mathfrak{m}A_\mathfrak{m}/\mathfrak{m}^2 A_\mathfrak{m})^\vee = T_{Y,y}.$$

The middle isomorphism is a special case of Proposition B.5.10. See also [12, §4.2, Lemma 2.3], or try to prove it yourself.

6.1.2 The tangent space of an affine variety

Keeping the previous notation, note that any $\mathbf{k}$-linear subspace $F \subset \mathbf{k}^r$ induces a short exact sequence

$$0 \longrightarrow F^\perp \longrightarrow (\mathbf{k}^r)^\vee \longrightarrow F^\vee \longrightarrow 0. \tag{6.1.4}$$

Here, $F^\perp$ is the set of linear functionals $\phi \in (\mathbf{k}^r)^\vee$ such that $\phi(v) = 0 \in \mathbf{k}$ for all $v \in F$. For instance, if

$$I \subset A = \mathbf{k}[x_1, \dots, x_n]$$

is an ideal, we have a linear subspace $F = \mathrm{D}_y\, I \subset E^\vee$ for each closed point $y \in \mathbb{A}^n_\mathbf{k}$, and thus a short exact sequence

$$0 \longrightarrow (\mathrm{D}_y\, I)^\perp \longrightarrow E \longrightarrow (\mathrm{D}_y\, I)^\vee \longrightarrow 0. \tag{6.1.5}$$

Set $X = \operatorname{Spec} A/I$ and let

$$X \xhookrightarrow{\iota} Y = \mathbb{A}^n_\mathbf{k} = \operatorname{Spec} A$$

be the associated closed immersion. If $y \in X$ is a closed point, then there is a unique ideal $\mathfrak{m} \subset A$ corresponding to y, and the ideal $\mathfrak{n} = \mathfrak{m}/I \subset A/I = \mathscr{O}_X(X)$ is the maximal ideal corresponding to $y \in X$. As noted earlier, the tangent map in this case is a $\mathbf{k}$-linear map

$$T_{X,y} \xrightarrow{T_{\iota,y}} T_{Y,y} = E.$$

PROPOSITION 6.1.3. *The tangent map $T_{\iota,y}$ induces an isomorphism*

$$T_{X,y} \xrightarrow{\sim} (\mathrm{D}_y I)^{\perp} \subset E.$$

Explicitly, we have

$$T_{X,y} = \left\{ (t_1,\ldots,t_n) \in E \;\middle|\; \sum_{1\leq i\leq n} \frac{\partial P}{\partial x_i}(y) t_i = 0,\ P \in I \right\}.$$

Proof. We have an exact sequence of $\mathbf{k}$-vector spaces

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I/I\cap\mathfrak{m}^2 & \longrightarrow & \mathfrak{m}/\mathfrak{m}^2 & \longrightarrow & \mathfrak{n}/\mathfrak{n}^2 & \longrightarrow & 0 \\ & & & & \| & & \| & & \\ & & & & \Omega_{Y,y} & & \Omega_{X,y} & & \end{array}$$

Via $\overline{\mathrm{D}}_y : \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\sim} E^{\vee}$, we get an identification $I/I\cap\mathfrak{m}^2 \xrightarrow{\sim} \mathrm{D}_y I$. Therefore, dualising

$$0 \longrightarrow \mathrm{D}_y I \longrightarrow E^{\vee} \longrightarrow \mathfrak{n}/\mathfrak{n}^2 \longrightarrow 0$$

we obtain

$$0 \longrightarrow (\mathfrak{n}/\mathfrak{n}^2)^{\vee} \longrightarrow E \longrightarrow (\mathrm{D}_y I)^{\vee} \longrightarrow 0$$

and hence

$$(\mathfrak{n}/\mathfrak{n}^2)^{\vee} = (\mathrm{D}_y I)^{\perp},$$

by (6.1.5). We conclude

$$T_{X,y} = (\Omega_{X,y})^{\vee} = (\mathfrak{n}/\mathfrak{n}^2)^{\vee} = (\mathrm{D}_y I)^{\perp}.$$ □

Example 6.1.4 (The cusp)**.** Let $X = \operatorname{Spec}\mathbf{k}[x,y]/(y^2-x^3) \subset \mathbb{A}^2_{\mathbf{k}}$, an integral curve passing through the origin $0 \in \mathbb{A}^2_{\mathbf{k}}$. See Figure 3.11 for a (real) picture. Then, for a closed point $p \in X$ corresponding to the maximal ideal $(x-a, y-b) \subset \mathbf{k}[x,y]/(y^2-x^3)$, we have

$$T_{X,p} = \left\{ (t_1,t_2) \in \mathbf{k}^2 \;\middle|\; -3a^2 t_1 + 2b t_2 = 0 \right\} \subset \mathbf{k}^2,$$

which is 1-dimensional (resp. 2-dimensional) if $p \neq 0$ (resp. if $p = 0$).

Example 6.1.5 (The node)**.** Let $X = \operatorname{Spec}\mathbf{k}[x,y]/(xy) \subset \mathbb{A}^2_{\mathbf{k}}$, a reduced curve passing through the origin $0 \in \mathbb{A}^2_{\mathbf{k}}$. Then, for a closed point $p \in X$ corresponding to the maximal ideal $(x-a, y-b) \subset \mathbf{k}[x,y]/(xy)$, we have

$$T_{X,p} = \left\{ (t_1,t_2) \in \mathbf{k}^2 \;\middle|\; b t_1 + a t_2 = 0 \right\} \subset \mathbf{k}^2,$$

which is 1-dimensional (resp. 2-dimensional) if $p \neq 0$ (resp. if $p = 0$). The same result is achieved for the curve

$$X = \operatorname{Spec}\mathbf{k}[x,y]/(y^2 - x^2(x+1)),$$

depicted in Figure 3.11.

Remark 6.1.6. The tangent spaces to the cusp and the nodal curve at the origin are both 2-dimensional, but these two singularities are not isomorphic. Intuitively, the cusp exhibits a 'double tangent' at the origin, whereas the nodal curve exhibits two distinct tangents. Algebraically, the reason why these two singularities are not isomorphic is that the completions of the local rings at the origin are not isomorphic.

Exercise 6.1.7. Let X be an algebraic variety over $\mathbf{k}$. Let $D = \operatorname{Spec} \mathbf{k}[t]/t^2$ be the fat point associated to the ring of dual numbers. Let $x \in X$ be a closed point. Prove that the set

$$\operatorname{Hom}_x(D, X) = \left\{ f : D \to X \,\middle|\, f \text{ sends the closed point to } x \right\}$$

carries a structure of $\mathbf{k}$-vector space. Construct a $\mathbf{k}$-linear isomorphism

$$T_{X,x} \xrightarrow{\sim} \operatorname{Hom}_x(D, X).$$

6.1.3 Regularity and the Jacobian Criterion

In this section we prove a criterion to detect when a point on an algebraic variety is regular.

PROPOSITION 6.1.8 ([12, §4.2, Prop. 2.11]). *A regular noetherian local ring is an integral domain.*

We get the following consequences.

(1) Whenever two irreducible components of a scheme meet, the points in the intersection are singular. In particular, a regular scheme is irreducible if and only if it is connected.

(2) A regular scheme is reduced.

Being regular is a property of local nature, as the following result illustrates.

THEOREM 6.1.9 ([12, §4.2, Theorem 2.16]). *Let $(A, \mathfrak{m})$ be a regular noetherian local ring. Then $A_{\mathfrak{p}}$ is regular for all $\mathfrak{p} \in \operatorname{Spec} A$.*

This fact can be used to deduce the following result.

COROLLARY 6.1.10. *Let X be a noetherian scheme. Then X is regular (resp. reduced) if and only if it is regular (resp. reduced) at all of its closed points.*

Proof. Let $x \in X$ be an arbitrary point. Then, since X is noetherian, by Proposition 3.1.41 the closure $Z = \overline{\{x\}}$ (which is quasicompact) contains a closed point, say $z \in Z$. The canonical map $\mathscr{O}_{X,z} \to \mathscr{O}_{X,x}$ (see the discussion before the proof of Proposition 4.4.12 if you forgot how this map is defined) is a localisation at a prime ideal in $\mathscr{O}_{X,z}$, so by Theorem 6.1.9 (resp. Lemma B.5.15) it follows that $\mathscr{O}_{X,x}$ is regular (resp. reduced). $\square$

Example 6.1.11. The schemes $\mathbb{A}^n_{\mathbf{k}}$ and $\mathbb{P}^n_{\mathbf{k}}$ are regular, because they are noetherian and regular at all of their closed points.

THEOREM 6.1.12 (Jacobian Criterion). *Let $X = \operatorname{Spec} B \subset \mathbb{A}^n_{\mathbf{k}}$ be a subvariety, $y \in X$ a closed point, and assume that $B = \mathbf{k}[x_1, \ldots, x_n]/I$, with $I = (f_1, \ldots, f_r)$. Form the Jacobian matrix*

$$\operatorname{Jac}_y = \left(\frac{\partial f_i}{\partial x_j}(y)\right)_{i,j} \in M_{r\times n}(\mathbf{k}).$$

Then X is regular at y if and only if

$$\boxed{\operatorname{rank}\operatorname{Jac}_y = n - \dim \mathscr{O}_{X,y}.}$$

Proof. We have $\dim_{\mathbf{k}} T_{X,y} = \dim_{\mathbf{k}}(\mathrm{D}_y\, I)^{\perp} = n - \dim_{\mathbf{k}} \mathrm{D}_y\, I = n - \operatorname{rank}\operatorname{Jac}_y$. □

Example 6.1.13 (The cusp again)**.** If $I = (y^2 - x^3) \subset \mathbf{k}[x, y]$ and $p = (x-a, y-b)$ is a closed point of $\operatorname{Spec}\mathbf{k}[x, y]/I \subset \mathbb{A}^2_{\mathbf{k}}$, then $\operatorname{Jac}_p = (-3a^2, 2b)$, and this has rank $1 = 2 - \dim \mathscr{O}_{X,p}$ if and only if $(a, b) \neq (0,0)$.

Example 6.1.14 (Affine cone)**.** If $I = (x^2 - yz) \subset \mathbf{k}[x, y, z]$ and $p = (x-a, y-b, z-c)$ is a closed point of $\operatorname{Spec}\mathbf{k}[x, y, z]/I \subset \mathbb{A}^3_{\mathbf{k}}$, then $\operatorname{Jac}_p = (2a, -c, -b)$, which has rank $1 = 3 - \dim \mathscr{O}_{X,p}$ if and only if $p \neq 0$.

There is a projective version of Theorem 6.1.12, which goes as follows. Consider a homogeneous ideal $I \subset B = \mathbf{k}[x_0, x_1, \ldots, x_n]$ and a closed point $x \in X = \operatorname{Proj} B/I \subset \mathbb{P}^n_{\mathbf{k}}$. Set

$$\mathrm{P}\, T_{X,x} = \left\{ (t_0, t_1, \ldots, t_n) \in \mathbf{k}^{n+1} \,\middle|\, \sum_{0 \le j \le n} \frac{\partial f}{\partial x_j} t_j = 0,\ f \in I \right\}.$$

Then, one can prove that

$$\dim_{\mathbf{k}} \mathrm{P}\, T_{X,x} = \dim_{\mathbf{k}} T_{X,x}.$$

If $I = (f_1, \ldots, f_r)$ and $p \in X$ is a closed point, one defines the Jacobian matrix as in the affine case, namely

$$\operatorname{Jac}_p = \left(\frac{\partial f_i}{\partial x_j}(p)\right)_{i,j} \in M_{r\times(n+1)}(\mathbf{k}).$$

Then a closed point $p \in X$ is regular if and only if the relation

$$\boxed{\operatorname{rank}\operatorname{Jac}_p = n - \dim \mathscr{O}_{X,p}}$$

holds.

Exercise 6.1.15. Show that, for any positive integer $n \ge 1$, the projective variety $X = \operatorname{Proj}\mathbf{k}[x, y, z]/(x^n + y^n + z^n) \subset \mathbb{P}^2_{\mathbf{k}}$ is regular if and only if $\gcd(\operatorname{char}\mathbf{k}, n) = 1$.

6.1.4 Regularity is an open property

PROPOSITION 6.1.16 ([12, §4.2, Lemma 2.21]). *If X is a reduced algebraic variety over* $\mathbf{k}$*, then X has a regular closed point. In particular, $X_{\mathsf{reg}} \neq \emptyset$.*

THEOREM 6.1.17 ([12, §4.2, Prop. 2.24]). *Let X be an algebraic variety over* $\mathbf{k}$*. Then $X_{\mathsf{reg}} \subset X$ is open. If X is normal, then*

$$\operatorname{codim}(X_{\mathsf{sing}}, X) \geq 2.$$

Caution 6.1.18. If X is not reduced, the open locus $X_{\mathsf{reg}} \subset X$ might be empty. Consider for instance $X = \operatorname{Spec} \mathbf{k}[x, y]/x^2 \subset \mathbb{A}^2_{\mathbf{k}}$, a double line, which is everywhere nonreduced and hence everywhere singular by Proposition 6.1.8(2).

PROPOSITION 6.1.19. *Let X be an algebraic $\mathbf{k}$-variety, $x \in X$ a regular closed point. Let $\widehat{\mathscr{O}}_{X,x}$ be the $\mathfrak{m}_x$-adic completion of the local ring $(\mathscr{O}_{X,x}, \mathfrak{m}_x)$. Then there is a ring isomorphism*

$$\widehat{\mathscr{O}}_{X,x} \cong \mathbf{k}[\![t_1, \ldots, t_d]\!], \quad d = \dim \mathscr{O}_{X,x} = \dim_{\mathbf{k}} T_{X,x}.$$

6.2 Flat morphisms

The concept of flatness is a riddle that comes out of algebra,
but which technically is the answer to many prayers.

David Mumford [14, Ch. III]

We already saw that a morphism of schemes $X \to S$ can be seen as a family of schemes X_s parametrised by the points $s \in S$ of the target. However, several invariants of a scheme (such as the dimension, for instance) are not preserved. Flatness[1] is the algebraic notion ensuring that a family of schemes $X \to S$ has nice continuity properties. Consider, for instance, the morphisms

$$\operatorname{Spec} \mathbf{k}[x, y, t]/(xyt - t) \xrightarrow{f} \operatorname{Spec} \mathbf{k}[t], \quad \operatorname{Spec} \mathbf{k}[x, y, t]/(xy - t) \xrightarrow{g} \operatorname{Spec} \mathbf{k}[t],$$

both defined at the level of rings by $t \mapsto \overline{t}$. Both f and g have fibre isomorphic to $\operatorname{Spec} \mathbf{k}[x, y]/(xy - t)$ over $t \neq 0$, but $f^{-1}(0) = \operatorname{Spec} \mathbf{k}[x, y] = \mathbb{A}^2_{\mathbf{k}} \neq \operatorname{Spec} \mathbf{k}[x, y]/xy = g^{-1}(0)$. In fact, f is not a flat morphism of schemes: a plane is not a 'nice' degeneration of a family of curves. On the other hand, g is flat. It can be seen as a family of hyperbolas degenerating to the union of the coordinate axes in $\mathbb{A}^2_{\mathbf{k}}$. However, having constant fibre dimension (or even being bijective) is in general not enough to ensure flatness (cf. Caution 6.2.16). There is an important exception though, which follows from a theorem in commutative algebra known as *miracle flatness* (cf. Theorem 6.2.5).

[1] In French: *platitude*.

Flatness in algebra

Theorem-Definition 6.2.1 (Flat module)**.** Let A be a ring, M an A-module. Then M is *flat over* A (or A-*flat*) if any of the following equivalent conditions is satisfied:

(i) For any injective A-linear map $N \hookrightarrow N'$, the induced morphism of A-modules $N \otimes_A M \to N' \otimes_A M$ is injective.

(ii) The functor $- \otimes_A M$ is left exact.

(iii) For every ideal $I \subset A$, the induced map $I \otimes_A M \to A \otimes_A M = M$ is injective.

(iv) For every finitely generated ideal $I \subset A$, the induced map $I \otimes_A M \to A \otimes_A M = M$ is injective.

(v) For every ideal $I \subset A$ the induced map $I \otimes_A M \to IM$ is an isomorphism.

Moreover, a ring homomorphism $\phi: A \to B$ is said to be *flat* if B is A-flat when viewed as an A-module via ϕ.

Clearly one has (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (v). See [12, Ch. 1, Thm. 2.4] for the implication (v) $\Rightarrow$ (i).

Exercise 6.2.1. Show that the map $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is not flat as soon as $n \geq 2$.

Exercise 6.2.2. Prove that the algebra homomorphism $\mathbf{k}[t^2, t^3] \hookrightarrow \mathbf{k}[t]$ is not flat.

Let A be a ring, M an A-module. Recall that an element $x \in M$ is *torsion* if one has $ax = 0$ in M for some $a \in A \setminus 0$. If $M = A$, this reduces to the notion of zero-divisor. An A-module M is *torsion-free* if it has no nonzero torsion element. Recall, also, that an A-module M is *projective* if for any diagram

$$\begin{tikzcd} & N \arrow[d, two heads] \\ M \arrow[r] \arrow[ur, dotted] & N' \end{tikzcd}$$

of A-modules there is a way to fill in the dotted arrow.

Example 6.2.3. Over an arbitrary ring, we have the implications

$$\text{free} \Rightarrow \text{projective} \Rightarrow \text{flat} \Rightarrow \text{torsion-free}.$$

Furthermore we have the following converses:

(1) Projective over a local ring or a principal ideal domain implies free.

(2) Flat and finitely generated over a noetherian ring implies projective.

(3) Torsion-free over a principal ideal domain implies free (cf. also Corollary 6.2.4).

(4) Flat and finitely generated over a local ring implies free (see [12, Ch. 1, Thm. 2.16]).

... And the following counterexamples:

(a) $\mathbb{Z}/2\mathbb{Z}$ is projective over $A = \mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, but not free.

(b) $\mathbb{Z} \hookrightarrow \mathbb{Q}$ and $\mathbb{F}[\![t]\!] \hookrightarrow \mathbb{F}(\!(t)\!)$ are flat, but not projective.

(c) $\mathbb{C}[t]$ is torsion-free over its subring $\mathbb{C}[t^2, t^3]$, but it is not flat over it (cf. Exercise 6.2.2).

Condition (v) in Theorem-Definition 6.2.1 is the most geometric one, in some sense. Indeed, say M is the module of sections of a vector bundle on $X = \operatorname{Spec} A$. An ideal $I \subset A$ uniquely defines a closed subscheme $\operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} A$. We have, naturally,

$$M/IM \cong M \otimes_A A/I,$$

and the isomorphism $I \otimes_A M \xrightarrow{\sim} IM$ tells us that

$$M \otimes_A A/I \cong M/I \otimes_A M.$$

This can be interpreted as M 'restricting nicely' to $\operatorname{Spec} A/I$, in the sense that what dies in the restriction (namely the module IM) is what one would naively expect, namely the *sub*module $I \otimes_A M \hookrightarrow M = A \otimes_A M$.

Granting Theorem-Definition 6.2.1, we get the following criterion for flatness over a PID.

Corollary 6.2.4. *A module M over a principal ideal domain A is flat if and only if it is torsion-free.*

Proof. Let $a \in A \setminus 0$, and set $I = aA$. Consider the canonical homomorphisms

$$t_a : A \xrightarrow{\ \sim\ } I, \qquad u_a : M \longrightarrow IM,$$

where $u_a(m) = am$ is multiplication by a, and similarly $t_a(b) = ab$. They fit in a commutative diagram

$$\begin{tikzcd} A \otimes_A M \arrow[r, "t_a \otimes_A \mathrm{id}_M"] \arrow[d, "u_a"'] & I \otimes_A M \arrow[dl, two heads, "\theta"] \\ IM & \end{tikzcd}$$

so we see that θ is an isomorphism if and only if u_a is an isomorphism, if and only if u_a is injective, if and only if $ax = 0 \in M$ implies $x = 0 \in M$. The statement follows. □

Here is the miracle flatness theorem announced earlier (a geometric version will be given in Theorem 6.2.13).

THEOREM 6.2.5 (Miracle flatness [16, Tag 00R4]). *Let $R \to S$ be a local homomorphism of noetherian local rings, where R is regular, S is Cohen–Macaulay and* $\dim S = \dim R + \dim S/\mathfrak{m}_R S$. *Then $R \to S$ is flat.*

Example 6.2.6. For curves, being Cohen–Macaulay (cf. Definition B.3.2) is equivalent to having no embedded points (cf. Appendix B.7). In particular, a reduced curve is always Cohen–Macaulay. Any nonconstant morphism $C \to C'$ from a reduced curve C to a regular curve C' always has 0-dimensional fibres, and is thus flat. See Exercise 6.2.2 for a counterexample.

We have the following general properties of flatness.

LEMMA 6.2.7 ([12, Ch. 1, Prop. 2.2]). *Let A be a ring.*

(i) *If M and N are flat over A, then $M \otimes_A N$ is flat over A. This is the product property.*

(ii) *If $A \to B$ is a ring homomorphism and M is A-flat, then $M \otimes_A B$ is B-flat. This is the base change property.*

(iii) *If $A \to B$ is a flat ring homomorphism and N is a flat B-module, then N is flat as an A-module. This is the transitivity property.*

(iv) *Let $S \subset A$ be a multiplicative subset. The localisation homomorphism $A \to S^{-1}A$ is flat.*

Proof. We proceed step by step.
Proof of (i).
Proof of (ii).
Proof of (iii).
Proof of (iv). Consider an A-linear injection $N' \hookrightarrow N$. Then we have

$$\begin{tikzcd} N' \otimes_A S^{-1}A \arrow[r, "i"] \arrow[d, "\wr"] & N \otimes_A S^{-1}A \arrow[d, "\wr"] \\ S^{-1}N' \arrow[r, hook] & S^{-1}N \end{tikzcd}$$

by Lemma B.5.16, from which it follows that i is injective. $\square$

LEMMA 6.2.8 ([12, Ch. 1, Prop. 2.13]). *The following conditions are equivalent for an A-module M.*

(1) *M is flat over A.*

(2) *$M_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}}$ for every* $\mathfrak{p} \in \operatorname{Spec} A$.

(3) *$M_{\mathfrak{m}}$ is flat over $A_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m} \subset A$.*

Proof. We proceed step by step.
(1) $\Rightarrow$ (2). We have $M_{\mathfrak{p}} = M \otimes_A A_{\mathfrak{p}}$ by Lemma B.5.16. This is flat over $A_{\mathfrak{p}}$ by base change of the flat A-module M along $A \to A_{\mathfrak{p}}$ (cf. Lemma 6.2.7(ii)).
(2) $\Rightarrow$ (3). Obvious.
(3) $\Rightarrow$ (1). Let $N' \hookrightarrow N$ be an A-linear injection. Consider the exact sequence of A-modules

$$0 \longrightarrow L \longrightarrow N' \otimes_A M \longrightarrow N \otimes_A M.$$

We claim that $L = 0$. It is enough to prove that $L_{\mathfrak{m}} = 0$ for every maximal ideal $\mathfrak{m} \subset A$. Localising at $\mathfrak{m}$ is exact and produces

$$\begin{tikzcd} 0 \arrow[r] & L \otimes_A A_{\mathfrak{m}} \arrow[r] \arrow[d, "\wr"] & (N' \otimes_A M) \otimes_A A_{\mathfrak{m}} \arrow[r] \arrow[d, "\wr"] & (N \otimes_A M) \otimes_A A_{\mathfrak{m}} \arrow[d, "\wr"] \\ & L_{\mathfrak{m}} \arrow[r] & N'_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} M_{\mathfrak{m}} \arrow[r, "j"] & N_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} M_{\mathfrak{m}} \end{tikzcd}$$

but j is injective since by assumption $M_{\mathfrak{m}}$ is flat over $A_{\mathfrak{m}}$ and $N' \hookrightarrow N$ is injective (therefore $N'_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is injective too). Thus $L_{\mathfrak{m}} = 0$ for all $\mathfrak{m}$, as required. □

COROLLARY 6.2.9 ([12, Ch. 1, Cor. 2.15]). *Given a ring homomorphism $\phi: A \to B$, with associated morphism $f: \operatorname{Spec} B \to \operatorname{Spec} A$, the following conditions are equivalent.*

(1) *$\phi: A \to B$ is flat.*

(2) *For every $\mathfrak{q} \in \operatorname{Spec} B$, the ring $B_{\mathfrak{q}}$ is flat over $A_{f(\mathfrak{q})}$.*

(3) *For every maximal ideal $\mathfrak{m} \subset B$, the ring $B_{\mathfrak{m}}$ is flat over $A_{f(\mathfrak{m})}$.*

Proof. We proceed step by step.
(1) $\Rightarrow$ (2). Let $\mathfrak{p} = f(\mathfrak{q}) = \phi^{-1}(\mathfrak{q})$. Then we are in the situation

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\mathrm{loc}"'] & B \arrow[d, "\mathrm{loc}"] \\ A_{\mathfrak{p}} \arrow[r] & B_{\mathfrak{q}} \end{tikzcd}$$

and since $B_{\mathfrak{q}}$ is a localisation of $B \otimes_A A_{\mathfrak{p}}$, it is flat over it by Lemma 6.2.7(iv). But $B \otimes_A A_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}}$ by base change along $A \to A_{\mathfrak{p}}$, since B is flat over A by assumption. By transitivity (cf. Lemma 6.2.7(iii)), $B_{\mathfrak{q}}$ is flat over $A_{\mathfrak{p}}$.
(2) $\Rightarrow$ (3). Obvious.
(3) $\Rightarrow$ (1). This is similar to Lemma 6.2.8((3) $\Rightarrow$ (1)). Let $N' \hookrightarrow N$ be an A-linear injection. Consider the exact sequence of B-modules

$$0 \longrightarrow L \longrightarrow N' \otimes_A B \longrightarrow N \otimes_A B.$$

We claim that $L=0$. Localising at $\mathfrak{m}$, the sequence becomes

$$\begin{tikzcd}
0 \arrow[r] & L\otimes_B B_{\mathfrak{m}} \arrow[r] \arrow[d, "\wr"] & (N'\otimes_A B)\otimes_B B_{\mathfrak{m}} \arrow[r] \arrow[d, "\wr"] & (N\otimes_A B)\otimes_B B_{\mathfrak{m}} \arrow[d, "\wr"] \\
& L_{\mathfrak{m}} \arrow[r] & N'_{f(\mathfrak{m})}\otimes_{A_{f(\mathfrak{m})}} B_{\mathfrak{m}} \arrow[r, "j"] & N_{f(\mathfrak{m})}\otimes_{A_{f(\mathfrak{m})}} B_{\mathfrak{m}}
\end{tikzcd}$$

and since $N'_{f(\mathfrak{m})} \to N_{f(\mathfrak{m})}$ is injective, condition (3) implies $L_{\mathfrak{m}}=0$, whence $L=0$, as claimed. □

More generally, we have the following result.

LEMMA 6.2.10 ([16, Tag 00H9]). *Let A be a ring, $S\subset A$ multiplicative subset. Then*

(a) *If M is a module over $S^{-1}A$, then M is flat over A if and only if it is flat over $S^{-1}A$.*

(b) *Let $A\to B$ be a ring homomorphism, M a B-module, $f_1,\ldots,f_r\in B$ generating the unit ideal $(1)=B$. Then M is flat over A if and only if M_{f_i} is flat over A for each $i=1,\ldots,r$.*

(c) *Let $A\to B$ be a ring homomorphism with induced map $f\colon \operatorname{Spec} B\to \operatorname{Spec} A$, and let M be a B-module. Then M is flat over A if and only if for every prime ideal $\mathfrak{q}\subset B$ the $B_{\mathfrak{q}}$-module $M_{\mathfrak{q}}$ is flat over $A_{f(\mathfrak{q})}$.*

(d) *Let $A\to B$ be a ring homomorphism with induced map $f\colon \operatorname{Spec} B\to \operatorname{Spec} A$, and let M be a B-module. Then M is flat over A if and only if for every maximal ideal $\mathfrak{m}\subset B$ the $B_{\mathfrak{m}}$-module $M_{\mathfrak{m}}$ is flat over $A_{f(\mathfrak{m})}$.*

Flatness in geometry

Definition 6.2.11 (Flat morphism)**.** A morphism of schemes $f\colon X\to S$ is *flat* if for every $x\in X$, with image $s\in S$, the ring homomorphism $f_x^{\#}\colon \mathscr{O}_{S,s}\to\mathscr{O}_{X,x}$ is flat (cf. Theorem-Definition 6.2.1). We say that f is *faithfully flat* if it is flat and surjective.

Exercise 6.2.12. Show that if $f\colon X\to S$ is faithfully flat the maps $f_x^{\#}\colon \mathscr{O}_{S,s}\to\mathscr{O}_{X,x}$ are injective.

Combining Lemma 6.2.7 and Corollary 6.2.9 with one another, we observe the following:

(i) Open immersions are flat.

(ii) Flatness is stable under base change and composition.

(iii) If $X\to Y$ and $X'\to Y'$ are flat S-morphisms, then so is $X\times_S X'\to Y\times_S Y'$.

(iv) $\operatorname{Spec} B\to\operatorname{Spec} A$ is flat if and only if $A\to B$ is flat.

THEOREM 6.2.13 (Miracle flatness). *A morphism $X \to S$ of locally noetherian schemes, where X is Cohen–Macaulay and S is regular, is flat as soon as its fibres have the same dimension.*

The following is another important property of flatness.

PROPOSITION 6.2.14 ([16, Tag 01UA]). *A flat morphism locally of finite presentation is universally open.*

Example 6.2.15. Consider the following examples.

(i) Every $\mathbb{F}$-algebra $\mathbb{F} \to A$ is flat. Every inclusion of a domain A into any localisation, e.g. $\operatorname{Frac} A$, is flat. Note that $\operatorname{Spec} \mathbb{F}(t) \to \operatorname{Spec} \mathbb{F}[t]$ is flat, but not open: it is not locally of finite presentation.

(ii) Consider the morphism

$$f \colon \operatorname{Spec} \mathbb{C}[x,y,t]/(y^2 - x(x-1)(x-t)) \to \operatorname{Spec} \mathbb{C}[t].$$

Then f is flat. Indeed, $A = \mathbb{C}[x,y,t]/(y^2 - x(x-1)(x-t))$ is torsion-free over the principal ideal domain $\mathbb{C}[t]$ via the action $t \mapsto \overline{t} \in A$.

(iii) Consider the ring of dual numbers $D = \mathbb{F}[t]/t^2$ and the exact sequence

$$0 \longrightarrow (\overline{t}) \longrightarrow D \longrightarrow \mathbb{F} \longrightarrow 0.$$

Then $\mathbb{F}$ is not D-flat (via the surjection), since tensoring $(\overline{t}) \hookrightarrow D$ with $\mathbb{F}$ yields $\mathbb{F} = (\overline{t}) \otimes_D \mathbb{F} \to \mathbb{F}$, the zero map, since its image is $(\overline{t})\mathbb{F} = 0$. See also (v) below.

(iv) Consider

$$\operatorname{Proj} \mathbb{C}[x,y,z]/(y^2 z - x(x-z)(x-\lambda z)) \xrightarrow{\ \pi\ } \mathbb{P}^1_{\mathbb{C}},$$

for $\lambda \in \mathbb{C} \setminus \{0,1\}$. The map sends

$$(a:b:c) \mapsto \begin{cases} (1:0) & \text{if } (a:b:c) = (0:1:0) \\ (a:c) & \text{otherwise.} \end{cases}$$

Then π is flat, since it is locally a projection $\mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$.

(v) Let A be a ring, $I \subset A$ a finitely generated ideal. Then the following conditions are equivalent:

(a) $A \twoheadrightarrow A/I$ is flat,

(b) $I = I^2$, and

(c) I is generated by an idempotent $\varepsilon \in A$.

This implies that a flat and locally finitely presented closed immersion $X \hookrightarrow Y$ is necessarily also an open immersion. Indeed, the third condition implies $A = \varepsilon A \oplus (1-\varepsilon)A$ so the complement of $\mathrm{V}(\varepsilon A) \subset \operatorname{Spec} A$ is closed. To get the global statement, just reduce to the affine case using that closed immersions are local on the target.

To see the equivalence, we proceed step by step.

(a) $\Rightarrow$ (b). If $A \twoheadrightarrow A/I$ is flat, we have

$$I/I^2 = I \otimes_A A/I \xrightarrow{\sim} I \cdot (A/I) = 0.$$

(b) $\Rightarrow$ (c). Assume $I = I^2$. Fix $\mathfrak{p} \in \mathrm{V}(I)$. Then $I_{\mathfrak{p}}$ is a finitely generated ideal of $A_{\mathfrak{p}}$, and so the relation $I_{\mathfrak{p}} = I_{\mathfrak{p}}^2$, by Nakayama's Lemma, shows that $I_{\mathfrak{p}} = 0$. So there is $s \in A \setminus \mathfrak{p}$ such that $sI = 0$. Let $J = \operatorname{Ann}(I) = \{\, t \in A \mid tI = 0 \,\}$. We have confirmed that, for any $\mathfrak{p} \in \mathrm{V}(I)$, we have $J \nsubseteq \mathfrak{p}$. This means that $I + J \nsubseteq \mathfrak{q}$ for any $\mathfrak{q} \in \operatorname{Spec} A$, i.e. $I + J = A$. Take $\varepsilon \in I$ and $j \in J$ such that $\varepsilon + j = 1$. If $i \in I$, we have $i = (\varepsilon + j)i = \varepsilon i + ji = \varepsilon i$ since $jI = 0$. Then $I = \varepsilon A$ and $\varepsilon = (\varepsilon + j)\varepsilon = \varepsilon^2$.

(c) $\Rightarrow$ (a). Assume $I = \varepsilon A$. We need to show that $B = A/I = A/\varepsilon A$ is A-flat. Let $p \colon N \hookrightarrow M$ be an injective A-linear map. We claim that

$$\begin{array}{ccc} N \otimes_A B & \xrightarrow{\;\overline{p}\;} & M \otimes_A B \\ \shortparallel & & \shortparallel \\ N/\varepsilon N & & M/\varepsilon M \end{array}$$

is injective. Assume $\overline{p}(n + \varepsilon N) = p(n) + \varepsilon M = 0 \in M/\varepsilon M$. This means that $p(n) = \varepsilon m$ for some $m \in M$. Then $p(\varepsilon n) = \varepsilon p(n) = \varepsilon^2 m = \varepsilon m = p(n)$. Since p is injective, we deduce $n = \varepsilon n$, so that in particular $n \in \varepsilon N$, therefore $n + \varepsilon N = 0 \in N/\varepsilon N$.

(vi) The blowup morphism $\mathrm{Bl}_0 \mathbb{A}^2_{\mathbf{k}} \to \mathbb{A}^2_{\mathbf{k}}$ is not flat. In fact, no blowup is flat (unless it is an isomorphism)!

(vii) Similarly, no normalisation $\widetilde{Y} \to Y$ is flat (unless it is an isomorphism)!

(viii) Consider

$$f \colon \operatorname{Spec} \mathbb{C}[x,y]/(y - x^2) \to \mathbb{A}^1_{\mathbb{C}}$$

corresponding to $\mathbb{C}[t] \to \mathbb{C}[x,y]/(y-x^2)$ defined by $t \mapsto y$ (i.e. we are projecting onto the y-axis). Then f is flat: the $\mathbb{C}[t]$-module $\mathbb{C}[x,y]/(y-x^2)$ is torsion-free!

(ix) The morphism $f \colon \mathbb{A}^1_{\mathbf{k}} \to \mathbb{A}^1_{\mathbf{k}}$ defined by $t \mapsto t^n$, for $n > 0$, is flat.

(x) The morphism $f \colon \operatorname{Spec} \mathbf{k}[x,y,t]/(y^2 - x^3 + t) \to \operatorname{Spec} \mathbf{k}[t]$ sending $t \mapsto \overline{t}$ is flat. It is a family of regular curves degenerating to a cuspidal cubic over the origin.

(xi) Let A be a ring. The morphism $f : \mathbb{A}^n_A \to \mathbb{A}^n_A$ defined by the n elementary symmetric functions in $x_1, \dots, x_n$ if faithfully flat (but not an isomorphism as soon as $n > 1$), and finite of degree $n!$.

Caution 6.2.16. Morphisms of schemes can be bijective without being flat. For instance, the morphism of schemes $\mathbb{A}^1_{\mathbf{k}} \to \operatorname{Spec} \mathbf{k}[x, y]/(y^2 - x^3)$ corresponding to the ring homomorphism $\mathbf{k}[x, y]/(y^2 - x^3) \to \mathbf{k}[t]$ sending $x \mapsto t^2$ and $y \mapsto t^3$ is bijective but not flat (cf. Exercise 6.2.2). Another example is of course Example 6.2.15(iii).

Terminology 6.2.17. Let $f : X \to Y$ be a morphism to an irreducible scheme. We say that a locally closed subscheme $Z \subset X$ dominates Y if $f(Z)$ is dense in Y.

LEMMA 6.2.18. *Let Y be an irreducible scheme, $f : X \to Y$ a flat morphism. Then, every nonempty open subset $U \subset X$ dominates Y.*

Proof. The question is local, so we may assume $Y = \operatorname{Spec} A$ (with prime nilradical) and $U = \operatorname{Spec} B$. Then $U \hookrightarrow X \to Y$ is flat, i.e. $A \to B$ is flat. Let $U_\xi \subset U$ be the generic fibre, i.e. the preimage of the generic point $\xi \in Y$ along $U \to Y$. Then, $U_\xi = \operatorname{Spec} B \otimes_A \kappa(\xi)$. Since $\kappa(\xi) = \operatorname{Frac} A/\operatorname{Nil}(A)$, we have $\mathcal{O}_{U_\xi}(U_\xi) = B \otimes_A \operatorname{Frac} A/\operatorname{Nil}(A)$. Now, B is flat over A, therefore the inclusion of the domain $A/\operatorname{Nil}(A)$ in its field of fraction induces an inclusion

$$B/\operatorname{Nil}(A)B = B \otimes_A A/\operatorname{Nil}(A) \hookrightarrow \mathcal{O}_{U_\xi}(U_\xi).$$

If the generic fibre U_ξ were empty, we would deduce $B = \operatorname{Nil}(A)B$. But $\operatorname{Nil}(A)B$ is nilpotent, so this would imply $\operatorname{Spec} B = U = \emptyset$, which is a contradiction. Therefore the generic fibre is nonempty, i.e. $f(U)$ is dense in Y, as claimed. □

COROLLARY 6.2.19. *Let Y be an irreducible scheme, $f : X \to Y$ a flat morphism. Then each irreducible component $Z \subset X$ dominates Y.*

Proof. If X has finitely many irreducible components $Z_1, \dots, Z_m$, then each Z_i contains a nonempty open subset of X, so we may apply Lemma 6.2.18 directly. Otherwise, replacing B in Lemma 6.2.18 with $\mathcal{O}_{X,\xi}$ for $\xi \in X$ a generic point we obtain the general statement. □

A partial converse of Lemma 6.2.18 holds over certain 1-dimensional schemes, called *Dedekind schemes*. These are integral noetherian schemes all of whose local rings are either discrete valuation rings or fields. This included regular curves! We prove here the version over a discrete valuation ring.

LEMMA 6.2.20. *Let X be a reduced scheme with finitely many irreducible components, and let $Y = \operatorname{Spec} A$ be the spectrum of a discrete valuation ring. Let $f : X \to Y$ be a morphism. Then f is flat if and only if every irreducible component of X dominates Y.*

Proof. If f is flat, just apply Corollary 6.2.19. Conversely, assume every irreducible component of X dominates Y. Let $x \in X$ be a point, with image $y \in Y$. We need to check that $\mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is flat. If y is the generic point, then $\mathscr{O}_{Y,y} = \operatorname{Frac} A$ is a field, therefore $\mathscr{O}_{X,x}$ is flat over it. If y is the closed point of Y, then we can pick a uniformiser $\pi \in \mathfrak{m}_y \subset \mathscr{O}_{Y,y}$. Let $\overline{\pi} \in \mathfrak{m}_x \subset \mathscr{O}_{X,x}$ be its image. Now, the minimal primes of $\mathscr{O}_{X,x}$ correspond to the irreducible components of X passing through x, and each of these dominates Y by assumption. Then $\overline{\pi}$ does not belong to any minimal prime of $\mathscr{O}_{X,x}$. But since this ring is reduced, this implies that $\overline{\pi}$ is not a zero-divisor in $\mathscr{O}_{X,x}$ (since the union of minimal primes in a reduced ring coincides with the set of zero-divisors, cf. Lemma B.1.3). Therefore $\mathscr{O}_{X,x}$ is torsion-free as an $\mathscr{O}_{Y,y}$-module, i.e. $\mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is flat (as $\mathscr{O}_{Y,y}$ is a PID, cf. Corollary 6.2.4). □

THEOREM 6.2.21 (Dimension of fibres). *Let $f\colon X \to Y$ be a morphism of locally noetherian schemes. Fix $x \in X$ and set $y = f(x)$. Then*

$$\dim \mathscr{O}_{X_y,x} \geq \dim \mathscr{O}_{X,x} - \dim \mathscr{O}_{Y,y}$$

and if f is flat then equality holds.

Proof. We start with the observation that after base change

$$\begin{tikzcd}
X_y \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & P_y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
\operatorname{Spec} \kappa(y) \arrow[r, hook] & \operatorname{Spec} \mathscr{O}_{Y,y} \arrow[r] & Y
\end{tikzcd}$$

we have identities

$$\begin{aligned}
\mathscr{O}_{\operatorname{Spec} \mathscr{O}_{Y,y}, \mathfrak{m}_y} &= (\mathscr{O}_{Y,y})_{\mathfrak{m}_y} = \mathscr{O}_{Y,y} \\
\mathscr{O}_{P_y,x} &= (\mathscr{O}_{X,x})_{\mathfrak{m}_x} = \mathscr{O}_{X,x}.
\end{aligned}$$

The first identity is by Exercise B.5.8, the second one is by Equation (4.7.5). Therefore, since this base change does not change the local rings involved in the statement, we may assume $Y = \operatorname{Spec} A$, where $(A, \mathfrak{m})$ is a noetherian local ring, and $y = \mathfrak{m} \in Y$ is its closed point.

We proceed by induction on $d = \dim Y$.

If $d = 0$, we have that $X_y \to X$ is a homeomorphism, thus $X_{\mathrm{red}} = (X_y)_{\mathrm{red}}$, thus $\dim \mathscr{O}_{X_y,x} = \dim \mathscr{O}_{X,x} = \dim \mathscr{O}_{X,x} - 0$ and the formula holds.

If $d > 0$, we make a further reduction: we may assume $Y = \operatorname{Spec} A$ is reduced. This is because after the base change

$$\begin{tikzcd}
X \times_Y Y_{\mathrm{red}} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
Y_{\mathrm{red}} \arrow[r, hook] & Y
\end{tikzcd}$$

we have

$$\dim \mathscr{O}_{X\times_Y Y_{\mathrm{red}},x} = \dim \mathscr{O}_{X,x}$$
$$\dim \mathscr{O}_{Y_{\mathrm{red}},y} = \dim \mathscr{O}_{Y,y}$$

and moreover flatness is stable under base change, so it passes from f to $X\times_Y Y_{\mathrm{red}} \to Y_{\mathrm{red}}$.

So we are under the assumption that $(A,\mathfrak{m})$ is a noetherian reduced local ring and $y \in \operatorname{Spec} A$ corresponds to $\mathfrak{m}$. Now, we pick an element $t \in A$ which is neither invertible nor a zero-divisor. Then by Example 4.6.23 we have

$$\dim A/tA = \dim A - 1. \tag{6.2.1}$$

Let $\overline{t}$ be the image of t along $f_x^\# \circ \ell_{\mathfrak{m}} \colon A \rightarrow A_{\mathfrak{m}} \to \mathscr{O}_{X,x}$. Set $B = \mathscr{O}_{X,x}$ for simplicity. We have

$$\dim B/\overline{t}B \geq \dim B - 1, \tag{6.2.2}$$

with equality if f is flat: indeed, in this case $A \to B$ is flat, therefore

$$(A \xhookrightarrow{\cdot t} A) \otimes_A B,$$

which is multiplication by $\overline{t}$, is injective and hence $\overline{t}$ is not a zero-divisor; this yields equality in (6.2.2).

Now we can set $Y' = \operatorname{Spec} A/tA$ and form the base change

$$\begin{tikzcd} X' \arrow[r, hook] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, hook] & Y \end{tikzcd}$$

to which we may apply the inductive hypothesis (since $\dim Y' = \dim Y - 1$): this yields

$$\dim \mathscr{O}_{X'_y,x} \geq \dim \mathscr{O}_{X',x} - \dim \mathscr{O}_{Y',y}$$

with equality as soon as f (and hence f') is flat. Now, to conclude, we combine the facts

$$X_y = X'_y$$
$$\dim \mathscr{O}_{Y',y} = \dim \mathscr{O}_{Y,y} - 1 \text{ by (6.2.1)}$$
$$\dim \mathscr{O}_{X',x} \geq \dim \mathscr{O}_{X,x} - 1, \text{ with equality if } f \text{ is flat, by (6.2.2).}$$

We obtain precisely

$$\dim \mathscr{O}_{X_y,x} \geq (\dim \mathscr{O}_{X,x} - 1) - (\dim \mathscr{O}_{Y,y} - 1) = \dim \mathscr{O}_{X,x} - \dim \mathscr{O}_{Y,y},$$

with equality as soon as f is flat, as required. □

COROLLARY 6.2.22. *Let $f : X \to Y$ be a faithfully flat morphism of $\mathbb{F}$-varieties, with Y irreducible and X of pure dimension. Then all fibres X_y are of pure dimension*

$$\dim X_y = \dim X - \dim Y.$$

Proof. Fix a closed point $x \in X_y$. Let $Z \subset X$ be an irreducible component passing through x. Then

$$\dim \mathcal{O}_{Z,x} = \dim Z - \dim \overline{\{x\}} = \dim X - \dim \overline{\{x\}}. \tag{6.2.3}$$

We used Proposition 4.6.35 for the first identity and purity of X for the second identity. The identities

$$\sup_{z\in Z} \dim \mathcal{O}_{Z,z} = \dim Z = \dim X = \sup_{p\in X} \dim \mathcal{O}_{X,p},$$

which use once more the purity of X, confirm that $\dim \mathcal{O}_{Z,x} = \dim \mathcal{O}_{X,x}$, so that using (6.2.3) we deduce

$$\dim \mathcal{O}_{X,x} = \dim X - \dim \overline{\{x\}}. \tag{6.2.4}$$

Next observe that x is the generic point of the irreducible subset $\overline{\{x\}} \subset X$, so

$$\dim \overline{\{x\}} = \operatorname{trdeg}_{\mathbb{F}} \kappa(x) = \operatorname{trdeg}_{\mathbb{F}} \kappa(y) = \dim \overline{\{y\}} = \dim Y - \dim \mathcal{O}_{Y,y}. \tag{6.2.5}$$

The first and third identities follow from Proposition 4.6.31. The second identity follows since $\kappa(x) \supset \kappa(y)$ is algebraic, as x is closed in X_y. The fourth identity is a version of Equation (6.2.3). Combining Equation (6.2.4) and Equation (6.2.5) with one another we deduce

$$\begin{aligned}
\dim X - \dim Y &= \dim \mathcal{O}_{X,x} + \dim \overline{\{x\}} - \dim \overline{\{x\}} - \dim \mathcal{O}_{Y,y} \\
&= \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{Y,y} \\
&= \dim \mathcal{O}_{X_y,x},
\end{aligned}$$

where the last identity follows from Theorem 6.2.21. Finally, X_y is pure because $\dim \mathcal{O}_{X_y,x} = \dim X - \dim Y$ for all closed points $x \in X_y$ (let x vary in the set of points belonging to only one irreducible component of X_y). The statement follows. □

The following result on the local dimension of fibres does not require any flatness assumption.

LEMMA 6.2.23. *Let $f : X \to Y$ be a morphism locally of finite type. Fix $x \in X$ and set $y = f(x)$. We have the equality*

$$\dim_x X_y = \dim \mathcal{O}_{X_y,x} + \operatorname{trdeg}_{\kappa(y)} \kappa(x). \tag{6.2.6}$$

Proof. Note that $\mathcal{O}_{X_y,x} = \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \kappa(y)$ as local rings by Equation (4.7.6). Moreover,

$$\dim_x X_y = \max \left\{ \dim Z \;\middle|\; \begin{array}{c} Z \subset X_y \text{ is an irreducible} \\ \text{component passing through } x \end{array} \right\}.$$

Consider the irreducible closed subset

$$V = \overline{\{x\}} \subset X_y.$$

Since $X_y \to \operatorname{Spec} \kappa(y)$ is locally of finite type, applying Remark 4.6.37 we find

$$\dim_x X_y = \dim V + \operatorname{codim}(V, X_y).$$

We have

$$\begin{aligned} \dim V &= \operatorname{trdeg}_{\kappa(y)} \kappa(x) \\ \operatorname{codim}(V, X_y) &= \dim \mathcal{O}_{X_y,x} \end{aligned}$$

therefore (6.2.6) is proved. □

6.3 Smooth, unramified, étale

We now introduce three types of morphism 'of infinitesimal nature'. These are smooth, unramified and étale morphisms. To explain how to think about these notions, we give a vague comparison with the complex geometry setup. This will become clearer (and precise) in later sections.

$$\begin{aligned} \text{smooth} &\sim \text{holomorphic submersion} \\ \text{unramified} &\sim \text{embedding} \\ \text{étale} &\sim \text{local isomorphism.} \end{aligned}$$

The idea to keep in mind (which is a theorem) is how these types of morphisms behave at the level of tangent spaces:

$$\begin{aligned} \text{smooth} &\sim \text{surjection on tangent spaces} \\ \text{unramified} &\sim \text{injection on tangent spaces} \\ \text{étale} &\sim \text{isomorphism on tangent spaces.} \end{aligned}$$

6.3.1 Infinitesimal language

We start with the purely infinitesimal picture, which will be equivalent to the classical notions, introduced later, under mild assumptions.

Definition 6.3.1 (First order thickening)**.** Let $T = \operatorname{Spec} B, \overline{T} = \operatorname{Spec} \overline{B}$ be two affine schemes. A closed immersion $\iota\colon T \hookrightarrow \overline{T}$ is called a *first order thickening* (or a *square-zero extension*) if the ideal $I \subset \overline{B}$ defining ι has square zero, i.e. $I^2 = 0$.

Definition 6.3.2 (Formally smooth, unramified, étale)**.** Let $f\colon X \to S$ be a morphism of schemes. Then, f is called *formally smooth* (resp. *formally unramified*, resp. *formally étale*) if for any commutative square diagram

$$\begin{tikzcd}
T \arrow[r] \arrow[d, hook, "\iota"'] & X \arrow[d, "f"] \\
\overline{T} \arrow[r] \arrow[ur, dotted, "u"] & S
\end{tikzcd}$$

where ι is a first order thickening of affine S-schemes, there is at least one (resp. at most one, resp. exactly one) morphism $u\colon \overline{T} \to X$ making the whole diagram commute.

We thus have, by definition, the slogan

$$\boxed{\text{formally étale} = \text{formally smooth} + \text{formally unramified.}}$$

Remark 6.3.3. As for the notion of formal étaleness, one could drop the assumption that T and $\overline{T}$ be affine, and obtain an equivalent definition with $T \subset \overline{T}$ an arbitrary first order thickening.

Remark 6.3.4. We have the following convenient set-theoretic reformulation of Definition 6.3.2 in terms of functor of points. Consider the fibre product of sets

$$\begin{tikzcd}
\operatorname{Hom}_{\mathsf{Sch}_S}(\overline{T},X) \arrow[dr, dotted, "\psi_{f,\iota}"] \arrow[drr, bend left, "(-)\circ\iota"] \arrow[ddr, bend right, "f\circ(-)"'] & & \\
& \operatorname{Hom}_{\mathsf{Sch}_S}(T,X) \times_{\operatorname{Hom}_{\mathsf{Sch}_S}(T,S)} \operatorname{Hom}_{\mathsf{Sch}_S}(\overline{T},S) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Hom}_{\mathsf{Sch}_S}(T,X) \arrow[d, "f\circ(-)"] \\
& \operatorname{Hom}_{\mathsf{Sch}_S}(\overline{T},S) \arrow[r, "(-)\circ\iota"'] & \operatorname{Hom}_{\mathsf{Sch}_S}(T,S)
\end{tikzcd}$$

attached to a pair $(f\colon X \to S, \iota\colon T \hookrightarrow \overline{T})$. Then Definition 6.3.2 can be rephrased by saying that $f\colon X \to S$ is *formally smooth* (resp. *formally unramified*, resp. *formally étale*) if for any first order thickening $\iota\colon T \hookrightarrow \overline{T}$ the map $\psi_{f,\iota}$ is *surjective* (resp. *injective*, resp. *bijective*).

Proposition 6.3.5. *We have the following properties.*

(1) *Closed immersions and open immersions are formally unramified.*

(2) *Open immersions are formally étale.*

(3) *For any localisation $A \to S^{-1}A$, the morphism $\operatorname{Spec} S^{-1}A \to \operatorname{Spec} A$ is formally étale.*

(4) *Formally smooth, formally unramified and formally étale morphisms are stable under composition and base change.*

Proof. Closed immersions and open immersions are categorical monomorphisms (left-cancellable), so this settles (1).

As for (2), write this

As for (3), consider the test diagram and its dual

$$
\begin{tikzcd}
T \arrow[r] \arrow[d, hook] & \operatorname{Spec} S^{-1}A \arrow[d, "f"] \\
\overline{T} \arrow[r, "\beta"'] \arrow[ur, dotted, "u"] & \operatorname{Spec} A
\end{tikzcd}
\qquad
\begin{tikzcd}
B & S^{-1}A \arrow[l] \arrow[dl, dotted, "u^{\#}"'] & \\
\overline{B} \arrow[u, two heads, "p"] & A \arrow[l, "\beta^{\#}"] \arrow[u] & S \arrow[l, hook'] \\
I \arrow[u, hook]
\end{tikzcd}
$$

where by assumption we know that $I^2 = 0$. By the universal property of localisations, we have $(p \circ \beta^{\#})(S) \subset B^{\times}$. Our claim is that $\beta^{\#}(S) \subset \overline{B}^{\times}$. This would give an extension $u^{\#} \colon S^{-1}A \to \overline{B}$, unique again by the universal property of localisations. By contradiction, take

$$\overline{b} = \beta^{\#}(s) \in \overline{B}, \qquad \overline{b} \notin \overline{B}^{\times}.$$

The image $b = p(\overline{b}) \in B$ is invertible by assumption, i.e. we have $bc = 1 \in B$ for some $c \in B$. Take a lift $\overline{c} \in \overline{B}$, so that $\overline{b}\,\overline{c} - 1 \in I$. Then, since $I^2 = 0$, we have

$$0 = (\overline{b}\,\overline{c} - 1)^2 = 1 + \overline{b}\,\overline{c}(\overline{b}\,\overline{c} - 2).$$

But this tells us that $\overline{b}$ is invertible, which is a contradiction.

As for (4), stability under composition is explained by the diagram

$$
\begin{tikzcd}
T \arrow[r] \arrow[dd, hook] & X \arrow[d] \\
& Y \arrow[d] \\
\overline{T} \arrow[r] \arrow[uur, dotted] \arrow[ur, dotted] & Z
\end{tikzcd}
$$

whereas stabilty under base change is explained by the diagram

$$
\begin{tikzcd}
T \arrow[r] \arrow[d, hook] & X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\
\overline{T} \arrow[r] \arrow[urr, dotted] & S' \arrow[r] & S
\end{tikzcd}
$$

where we get (at least one, at most one, exactly one extension $\overline{T} \to X'$ combining the assumption with the universal property of fibre products. $\square$

It is quite interesting to observe that a formally étale morphism of $\mathbf{k}$-varieties $f: X \to Y$ is bijective on tangent spaces at the level of closed points. Indeed, fix a closed point $x \in X$ and form a test diagram

$$\begin{tikzcd} \operatorname{Spec}\mathbf{k} \arrow[r, "x"] \arrow[d, hook] & X \arrow[d, "f"] \\ \operatorname{Spec}\mathbf{k}[t]/t^2 \arrow[r, "v"] \arrow[ur, dotted, "u"] & Y \end{tikzcd}$$

where of course $f \circ x$ corresponds to the closed point $f(x)$. By Exercise 6.1.7, the morphism v can be seen as a tangent vector

$$v \in T_{Y,f(x)},$$

and similarly the sought after extension $u: \operatorname{Spec}\mathbf{k}[t]/t^2 \to X$ can be seen as a tangent vector

$$u \in T_{X,x}.$$

The commutativity of the diagram translates precisely the condition that u maps to v under the tangent map $T_{f,x}$. Thus the condition defining formal étaleness translates into the required condition that $T_{f,x}: T_{X,x} \to T_{Y,f(x)}$ is bijective. See also Proposition 6.3.15 for the analogue statement for étale morphisms. Of course the same argument shows that a formally smooth (resp. formally unramified) morphism as above induces a surjection (resp. an injection) on tangent spaces at closed points.

Fact 6.3.6. If $\mathbb{F} \subset \kappa$ is a field extension in characteristic 0, then $\pi: \operatorname{Spec}\kappa \to \operatorname{Spec}\mathbb{F}$ is formally smooth. In positive characteristic, π is formally smooth if and only if separable [16, Tag 0321]. See also Exercise 6.3.9.

6.3.2 Classical language

We now introduce the 'classical' notions of smooth, unramified and étale morphisms.

Assumption 6.3.1. From now on, all morphisms are of finite type with locally noetherian target. This ensures that all such morphisms are locally of finite presentation (by Remark 5.2.5) and have locally noetherian source (by Lemma 5.2.11).

Definition 6.3.7 (Smoothness over a field)**.** Let $\pi: X \to \operatorname{Spec}\mathbb{F}$ be a variety. We say that

- π is smooth at $x \in X$ if all the points $z \in X_{\overline{\mathbb{F}}} = X \times_{\mathbb{F}} \overline{\mathbb{F}}$ lying above x are regular,
- π is smooth (or X is smooth over $\mathbb{F}$) if $X_{\overline{\mathbb{F}}}$ is regular.

Example 6.3.8. Let $\mathbb{F}$ be a field. The schemes $\mathbb{A}^n_{\mathbb{F}} \to \operatorname{Spec}\mathbb{F}$ and $\mathbb{P}^n_{\mathbb{F}} \to \operatorname{Spec}\mathbb{F}$ are smooth over $\mathbb{F}$ thanks to Example 6.1.11.

Recall that a finite extension $\mathbb{F} \subset \kappa$ is separable if for any $\alpha \in \kappa$ its minimal polynomial $m_\alpha(z) \in \mathbb{F}[z]$ is separable, i.e. it has no repeated roots in an algebraic closure of $\mathbb{F}$. An extension $\kappa \subset K$ of fields of characteristic $p > 0$ is *purely inseparable* if every $\alpha \in K$ is a root of a polynomial of the form $p(u) = u^q - b \in \kappa[u]$, where $q = p^e$ for some $e \geq 0$. If $\kappa \subset K$ is an algebraic extension of fields of characteristic $p > 0$, then it is purely inseparable if and only if for each $\alpha \in K$ there exists $e \geq 0$ such that $\alpha^{p^e} \in \kappa$, if and only if for each $\alpha \in K$, its minimal polynomial if of the form $p_{\min}(u) = u^{p^e} - b \in \kappa[u]$ for some $e \geq 0$. Note that a purely inseparable extension is necessarily algebraic.

Exercise 6.3.9. Let $\mathbb{F} \subset \kappa$ be a finite field extension. Then $\pi\colon \operatorname{Spec}\kappa \to \operatorname{Spec}\mathbb{F}$ is smooth if and only if $\mathbb{F} \subset \kappa$ is separable.

Solution. Set $X = \operatorname{Spec}\kappa$. Then $\pi\colon X \to \operatorname{Spec}\mathbb{F}$ is smooth if and only if $X_{\overline{\mathbb{F}}} = \operatorname{Spec}\kappa \otimes_{\mathbb{F}} \overline{\mathbb{F}}$ is regular, if and only if $\kappa \otimes_{\mathbb{F}} \overline{\mathbb{F}}$ is regular. Factor the extension $\mathbb{F} \subset \kappa$ as

$$\mathbb{F} \subset \mathbb{F}^{\mathrm{sep}} \subset \kappa$$

with $\mathbb{F} \subset \mathbb{F}^{\mathrm{sep}}$ finite separable and $\mathbb{F}^{\mathrm{sep}} \subset \kappa$ purely inseparable.[2] We have

$$\kappa \otimes_{\mathbb{F}} \overline{\mathbb{F}} = (\kappa \otimes_{\mathbb{F}^{\mathrm{sep}}} \mathbb{F}^{\mathrm{sep}}) \otimes_{\mathbb{F}} \overline{\mathbb{F}} = \kappa \otimes_{\mathbb{F}^{\mathrm{sep}}} (\mathbb{F}^{\mathrm{sep}} \otimes_{\mathbb{F}} \overline{\mathbb{F}}).$$

We have $\mathbb{F}^{\mathrm{sep}} = \mathbb{F}[t]/p(t)$ for some separable polynomial p, say of degree d, by the primitive element theorem. Therefore

$$\mathbb{F}^{\mathrm{sep}} \otimes_{\mathbb{F}} \overline{\mathbb{F}} = \mathbb{F}[t]/p(t) \otimes_{\mathbb{F}} \overline{\mathbb{F}} = \overline{\mathbb{F}}^{\oplus d}.$$

Therefore

$$\kappa \otimes_{\mathbb{F}} \overline{\mathbb{F}} = \kappa \otimes_{\mathbb{F}^{\mathrm{sep}}} \overline{\mathbb{F}}^{\oplus d} = (\kappa \otimes_{\mathbb{F}^{\mathrm{sep}}} \overline{\mathbb{F}})^{\oplus d}.$$

Now assume $\mathbb{F}^{\mathrm{sep}} \subset \kappa$ is nontrivial. Then $\kappa \otimes_{\mathbb{F}} \overline{\mathbb{F}}$ contains a nontrivial nilpotent element. Indeed, if $\alpha \in \kappa$ has minimal polynomial $u^{p^e} - b$ with $b \in \mathbb{F}^{\mathrm{sep}}$, then $(\alpha - b^{1/p^e})^{p^e} = 0 \in \kappa \otimes_{\mathbb{F}} \overline{\mathbb{F}}$, but $\alpha - b^{1/p^e} \neq 0$.

Notation 6.3.10. Let $f : X \to S$ be a morphism, $s \in S$ a point. Consider the residue field $\kappa(s)$ and its algebraic closure $\overline{\kappa(s)}$. We have a double fibre diagram

$$\begin{tikzcd}
X_{\overline{s}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X_s \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
\operatorname{Spec}\overline{\kappa(s)} \arrow[r] & \operatorname{Spec}\kappa(s) \arrow[r] & S
\end{tikzcd}$$

defining the usual fibre X_s and the *geometric fibre* $X_{\overline{s}}$ of f over s, respectively.

[2]This can be done in a unique way, see e.g. [16, Tag 030K].

Definition 6.3.11 (Smooth, unramified, étale)**.** Let $f: X \to S$ be a morphism of finite type between locally noetherian schemes. We say that

- f is *smooth at* $x \in X$ if it is flat at x and $X_s \to \operatorname{Spec} \kappa(s)$, where $s = f(x)$, is smooth at x (i.e. if $X_{\overline{s}}$ is regular at all the points lying above x, cf. Definition 6.3.7).
- f is *unramified at* $x \in X$ if under the canonical morphism $f_x^{\#}: \mathscr{O}_{S,s} \to \mathscr{O}_{X,x}$ one has $\mathfrak{m}_s \mathscr{O}_{X,x} = \mathfrak{m}_x$, where $s = f(x)$, and if $\kappa(s) \subset \kappa(x)$ is finite separable.
- f is *étale at* $x \in X$ if it is flat and unramified at x.

We say that $f: X \to S$ is *smooth* (resp. *unramified*, resp. *étale*) if it is smooth (resp. unramified, resp. étale) at all points.

We record here the following slogans:

$$\begin{aligned} \text{smooth} &= \text{flat} + \text{regular geometric fibres} \\ \text{étale} &= \text{flat} + \text{unramified.} \end{aligned}$$

Caution 6.3.12. Even though flat and unramified implies formally étale [16, Tag 04FF], in general,

$$\boxed{\text{flat} + \text{formally unramified } \textit{does not imply} \text{ formally étale.}}$$

See e.g. here.

THEOREM 6.3.13. *Let $f: X \to S$ be a morphism of finite type between locally noetherian schemes. The following conditions are equivalent.*

(i) *f is unramified, and*

(ii) *for every $s \in S$, the fibre X_s is finite and reduced and $\kappa(s) \subset \kappa(x)$ is finite separable for every $x \in X_s$.*

Proof. To prove that (i) implies (ii), observe that for every $x \in X_s$ one has, thanks to Equation (4.7.6),

$$\mathscr{O}_{X_s,x} = \mathscr{O}_{X,x} \otimes_{\mathscr{O}_{S,s}} \kappa(s) = \mathscr{O}_{X,x}/\mathfrak{m}_s \mathscr{O}_{X,x} = \mathscr{O}_{X,x}/\mathfrak{m}_x = \kappa(x),$$

which is a field. This proves that every $x \in X$ is closed in its fibre, and moreover that the fibre is reduced. Since $\kappa(s) \subset \kappa(x)$ is finite, Equation (6.2.6) yields

$$0 = \dim \mathscr{O}_{X_s,x} = \dim_x X_s$$

and thus, by Lemma 4.6.10(iv), we have

$$\dim X_s = 0.$$

The underlying space of a 0-dimensional scheme is discrete. But since $X_s \to \operatorname{Spec} \kappa(s)$ is of finite type by base change, we see that X_s is in fact finite (and reduced, as already observed).

Let us now prove that (ii) implies (i). We must show that for every $x \in X_s$ we have $\mathfrak{m}_s \mathscr{O}_{X,x} = \mathfrak{m}_x$. Our assumption is that X_s is finite and reduced. In particular, X_s is affine, say $X_s = \operatorname{Spec} R$, and

$$\mathscr{O}_{X,x}/\mathfrak{m}_s\mathscr{O}_{X,x} = \mathscr{O}_{X_s,x} = R_{\mathfrak{p}},$$

where $\mathfrak{p} \subset R$ is the prime (maximal, in fact) ideal corresponding to $x \in X_s$. But now $R_{\mathfrak{p}}$ is a local 0-dimensional ring, thus its maximal ideal is the unique *prime* ideal, which must then coincide with the nilradical. But R was reduced, therefore $R_{\mathfrak{p}}$ is reduced as well. Then it is a field. In other words, $\mathfrak{m}_s\mathscr{O}_{X,x} \subset \mathscr{O}_{X,x}$ is a maximal ideal. Therefore $\mathfrak{m}_s\mathscr{O}_{X,x} = \mathfrak{m}_x$, as required. □

Remark 6.3.14. By Corollary 6.2.22, a smooth surjective morphism of irreducible $\mathbb{F}$-varieties $X \to Y$ has a well-defined *relative dimension*, namely

$$d = \dim X - \dim Y.$$

It is the common dimension of the fibres. More generally, one may declare that a finite type morphism $X \to S$ between locally noetherian schemes is *smooth of relative dimension d* if it is smooth and all its nonempty fibres are of pure dimension d. With this definition, we find the slogan

$$\boxed{\text{étale} \ = \ \text{smooth of relative dimension } 0.}$$

Note that we have used Exercise 6.3.9 to confirm that étale implies smooth, whereas 0-dimensional fibres follows from unramified.

Proposition 6.3.15. *Let S be a locally noetherian scheme. Let $f: X \to S$ be a finite type morphism. If f is étale, then it is is quasifinite and the tangent map*

$$T_{f,x}\colon T_{X,x} \xrightarrow{\ \sim\ } T_{S,s} \otimes_{\kappa(s)} \kappa(x)$$

is an isomorphism for every $x \in X$ mapping to $s \in S$.

Proof. Since étale means smooth of relative dimension 0, f is quasifinite. Let us check that the cotangent map is an isomorphism. We have

$$\Omega_{S,s} \otimes_{\kappa(s)} \kappa(x) = \mathfrak{m}_s/\mathfrak{m}_s^2 \otimes_{\kappa(s)} \kappa(x) = (\mathfrak{m}_s \otimes_{\mathscr{O}_{S,s}} \kappa(s)) \otimes_{\kappa(s)} \kappa(x) = \mathfrak{m}_s \otimes_{\mathscr{O}_{S,s}} (\mathscr{O}_{X,x}/\mathfrak{m}_s\mathscr{O}_{X,x}).$$

Since $\mathscr{O}_{S,s} \to \mathscr{O}_{X,x}$ is flat and unramified, we have a natural isomorphism

$$\mathfrak{m}_s \otimes_{\mathscr{O}_{S,s}} \mathscr{O}_{X,x} \xrightarrow{\ \sim\ } \mathfrak{m}_s\mathscr{O}_{X,x} = \mathfrak{m}_x.$$

Therefore

$$\Omega_{S,s} \otimes_{\kappa(s)} \kappa(x) = (\mathfrak{m}_s \otimes_{\mathscr{O}_{S,s}} \mathscr{O}_{X,x})/\mathfrak{m}_s^2 \xrightarrow{\sim} \mathfrak{m}_x/\mathfrak{m}_x^2 = \Omega_{X,x},$$

as required. □

COROLLARY 6.3.16. *Let S be a locally noetherian scheme. Let $f: X \to S$ be a finite type morphism, étale at x. Then X is regular at x if and only if S is regular at $f(x)$.*

Proof. Recall that étale implies quasifinite. Then just apply the definition of regular point and use Proposition 6.3.15. □

PROPOSITION 6.3.17 ([12, Ch. 4, Prop. 3.26]). *Let S be a locally noetherian scheme. Let $f: X \to S$ be a finite type morphism. Fix a point $x \in X$, mapping to $s \in S$, such that $\kappa(x) = \kappa(s)$. Then f is étale at x if and only if the natural map between $\mathfrak{m}$-adic completions*

$$\widehat{\mathscr{O}}_{S,s} \longrightarrow \widehat{\mathscr{O}}_{X,x}$$

is an isomorphism.

For instance, the proposition applies to any morphism of $\mathbf{k}$-varieties $X \to S$ and to any closed point $x \in X$.

Example 6.3.18. Consider the following examples.

(i) The morphism $f: \mathbb{A}^1_{\mathbf{k}} \to \mathbb{A}^1_{\mathbf{k}}$ defined by $t \mapsto t^n$, for $n > 1$, is ramified at the origin $0 \in \mathbb{A}^1_{\mathbf{k}}$ (reason: the local homomorphism $f_0^\#: \mathbf{k}[t]_{(t)} \to \mathbf{k}[t]_{(t)}$ satisfies $f_0^\#(t\mathbf{k}[t]_{(t)}) = t^n\mathbf{k}[t]_{(t)}$), but it becomes unramified (in fact étale, since we already know it is flat) when restricted to $\mathbb{A}^1_{\mathbf{k}} \setminus 0$, as soon as n is invertible in $\mathbf{k}$. Note that the fibre over the origin $0 \in \mathbb{A}^1_{\mathbf{k}}$ is

$$\operatorname{Spec} \mathbf{k}[t] \otimes_{\mathbf{k}[t]} \kappa(0) = \operatorname{Spec} \mathbf{k}[t]/t^n,$$

which is nonreduced!

(ii) Given a finite field extension $\mathbb{F} \hookrightarrow \mathbb{E}$, the following conditions are equivalent:

(a) $\mathbb{F} \hookrightarrow \mathbb{E}$ is separable,

(b) $\operatorname{Spec} \mathbb{E} \to \operatorname{Spec} \mathbb{F}$ is étale,

(c) $\operatorname{Spec} \mathbb{E} \to \operatorname{Spec} \mathbb{F}$ is unramified,

(d) $\operatorname{Spec} \mathbb{E} \to \operatorname{Spec} \mathbb{F}$ is smooth.

If $\mathbb{F}$ has characteristic $p > 0$ and is *not perfect* (e.g. $\mathbb{F} = \mathbb{F}_p(t)$ is not perfect), by definition of perfectness there exists an inseparable algebraic extension $\mathbb{F} \hookrightarrow \mathbb{E}$, and thus a ramified morphism of schemes $\operatorname{Spec} \mathbb{E} \to \operatorname{Spec} \mathbb{F}$.

(iii) Closed immersions $\iota\colon X \hookrightarrow S$ of locally noetherian schemes are unramified: if $s = \iota(x)$, the local homomorphism $\iota_x^\#\colon \mathscr{O}_{S,s} \to \mathscr{O}_{X,x}$ is surjective, and since $\mathfrak{m}_s \supset \ker \iota_x^\#$, the extension of $\mathfrak{m}_s$ in $\mathscr{O}_{X,x}$ is maximal, hence equal to $\mathfrak{m}_x$. By Example 6.2.15(v), a closed immersion is étale of and only if it is also an open immersion.

(iv) The canonical morphism $\coprod_{1\le i\le n} S \to S$ is étale for any scheme S.

(v) $\operatorname{Spec}\mathbb{Z}[i] \to \operatorname{Spec}\mathbb{Z}$ is ramified (only) at $(1+i)$. It is flat, but not étale.

(vi) If X is an $\mathbb{F}$-variety, then the structure morphism $X \to \operatorname{Spec}\mathbb{F}$ is unramified (equivalently, étale, since it automatically flat) if and only if

$$X = \coprod_{i=1}^n \operatorname{Spec}\mathbb{E}_i = \operatorname{Spec}\prod_{i=1}^n \operatorname{Spec}\mathbb{E}_i, \quad \mathbb{F} \hookrightarrow \mathbb{E}_i \text{ finite separable.}$$

This results immediately from Theorem 6.3.13.

(vii) Consider $A = \mathbb{F}[x]/(h)$ for some monic polynomial h. Then $\operatorname{Spec} A \to \operatorname{Spec}\mathbb{F}$ is unramified if and only if h is separable, if and only if $(h, h') = \mathbb{F}[x]$, where h' denotes the derivative of h.

For instance, consider any field $\mathbb{F}$ of characteristic $p > 0$ and the polynomial

$$h = x^p - t \in \mathbb{F}[x], \quad t \in \mathbb{F}.$$

Since $h' = 0$, the morphism

$$\operatorname{Spec}\mathbb{F}[x]/(x^p - t) \to \operatorname{Spec}\mathbb{F}$$

is ramified (but of course flat).

(viii) A morphism which is flat and ramified (and is not (i) above): the projection of the parabola $\operatorname{Spec}\mathbb{C}[x,y]/(y-x^2)$ onto the y-axis (cf. Example 6.2.15 (viii)): the fibre over the origin is $\operatorname{Spec}\mathbb{C}[x]/x^2$, which is nonreduced.

(ix) A morphism which is unramified but not flat (and is not a closed immersion): the morphism

$$\mathbb{A}^1_{\mathbb{C}} \to \operatorname{Spec}\mathbb{C}[x,y]/(y^2 - x^2(x+1)), \quad t \mapsto (t^2-1, t(t^2-1)).$$

(x) A morphism which is neither flat nor unramified: the morphism $f\colon \mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}}$ sending $(a,b) \mapsto (ab, b)$. Its image is neither open nor closed, thus in particular not open, thus f is not flat. The fibre over $(0,0)$ is 1-dimensional, therefore it is not unramified.

(xi) The morphism $\mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{R}}$ of Example 3.1.81 is étale.

(xii) Let X and Y be proper integral 1-dimensional schemes over a field (i.e. proper curves). Then any nonconstant morphism $f: X \to Y$ is finite, and flat as soon as Y is regular. Indeed, $f(X)$ is closed and irreducible, hence either a point or the whole of Y for dimensional reasons. Then it must equal Y since f is nonconstant. It follows that f has finite fibres, i.e. it is quasifinite. Since it is proper by Proposition 5.5.9, it is in fact finite by Proposition 5.6.7. For flatness, combine (a slight upgrade of) Lemma 6.2.20 with the fact that Y is a Dedekind scheme (a noetherian integral scheme all of whose local rings are either discrete valuation rings or fields).

6.3.3 Comparison between regular and smooth

Regularity is an absolute property of a scheme, whereas smoothness is a relative property. In some cases, nevertheless, they do coincide: see Theorem 6.3.19 below. In general, given an arbitrary field $\mathbb{F}$ and an algebraic variety X over $\mathbb{F}$ (i.e. a scheme of finite type over $\operatorname{Spec}\mathbb{F}$), we have that if X is smooth at a closed point $x \in X$, then X is also regular at x, and the converse holds as long as $\kappa(x) \supset \mathbb{F}$ is separable (cf. [12, Ch. 4, Prop. 3.30]). In fact, one has that

$$\boxed{\text{for } \textit{any point } x \in X, \quad X \text{ smooth at } x \quad \Rightarrow \quad X \text{ regular at } x.}$$

as proven in [12, Ch. 4, Cor. 3.32]. The following result is important to keep in mind, along with the classical examples of perfect fields, e.g. finite fields and algebraically closed fields.

THEOREM 6.3.19 ([12, Ch. 4, Cor. 3.33]). *Let X be an algebraic variety over a perfect field $\mathbb{F}$. Then $X \to \operatorname{Spec}\mathbb{F}$ is smooth if and only X is regular.*

6.3.4 Étale coordinates: almost an implicit function theorem

It is important to keep in mind the following étale-local structure of smooth morphisms. It is known as *existence of étale coordinates*. Let $f: X \to S$ be a finite type morphism of locally noetherian schemes. Assume f is smooth of relative dimension d around x. Then for any affine open neighbourhood $V = \operatorname{Spec} R$ of $s = f(x)$, there exists an affine open neighbourhood $U = \operatorname{Spec} A$ of x mapping into V along with a commutative diagram

$$\begin{tikzcd} & U \arrow[r, hook] \arrow[dl, "\pi"'] \arrow[d] & X \arrow[d, "f"] \\ \mathbb{A}^d_R \arrow[r] & V \arrow[r, hook] & S \end{tikzcd}$$

where π is *étale*. This fact can be summarised in the following slogan:

$$\boxed{\text{smooth schemes are étale-locally like affine spaces.}}$$

An interpretation of the existence of étale coordinates is the following: it is the closest one can get to the implicit function theorem in algebraic geometry!

6.3.5 Infinitesimal lifting

We are ready to state the infinitesimal lifting criterion, which translates smoothness, unramifiedness and étaleness in terms of their formal counterparts (cf. Definition 6.3.2).

THEOREM 6.3.20 (Infinitesimal lifting criterion). *Let $f: X \to S$ be a morphism of finite type between locally noetherian schemes. Then*

(1) *f is smooth if and only if it is formally smooth* [16, Tag 02H6].

(2) *f is unramified if and only if it is formally unramified* [16, Tag 02HE].

(3) *f is étale if and only if it is formally étale* [16, Tag 02HM].

We illustrate the infinitesimal criterion for smoothness with a series of (singular) examples. This should help 'sensing' the singularity in a geometric way. The algebraic way of getting a feel of the singularities of a scheme is of course through Proposition 6.3.17.

Example 6.3.21 (The node)**.** Consider the $\mathbf{k}$-scheme $\pi: X = \operatorname{Spec}\mathbf{k}[x,y]/(xy) \to \operatorname{Spec}\mathbf{k}$. We wish to use the infinitesimal lifting criterion to confirm that π is not smooth. We already know it cannot be smooth, by combining Theorem 6.3.19 with the fact that the origin $0 \in X$ is a singular point (Example 6.1.5). First consider a commutative diagram

$$\begin{tikzcd} \operatorname{Spec}\mathbf{k} \arrow[r, "0"] \arrow[d, hook] & X \arrow[d, "\pi"] \\ \operatorname{Spec}\mathbf{k}[t]/t^2 \arrow[r] \arrow[ur, dotted, "u_1"] & \operatorname{Spec}\mathbf{k} \end{tikzcd}$$

where the top horizontal morphism corresponds to the origin $0 \in X$ and the left vertical morphism is the reduction morphism. This corresponds to a diagram of rings

$$\begin{tikzcd} \mathbf{k} & \mathbf{k}[x,y]/(xy) \arrow[l, "0 \mapsfrom \overline{x},\overline{y}"'] \arrow[dl, dotted, "u_1^\#"] \\ \mathbf{k}[t]/t^2 \arrow[u, two heads] & \mathbf{k} \arrow[l] \arrow[u] \end{tikzcd}$$

and we can define an extension $u_1^\#: \mathbf{k}[x,y]/(xy) \to \mathbf{k}[t]/t^2$ by sending

$$\overline{x} \mapsto at, \quad \overline{y} \mapsto bt.$$

We do not need to impose $u_1^\#(\overline{x})u_1^\#(\overline{y}) = 0$, since this is automatic: any pair (a,b) defines an extension. So we can always lift to first order! Now let us try to extend further, i.e. from first order to second order. Consider a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec}\mathbf{k}[t]/t^2 \arrow[r, "u_1"] \arrow[d, hook] & X \arrow[d, "\pi"] \\
\operatorname{Spec}\mathbf{k}[t]/t^3 \arrow[r] \arrow[ur, dotted, "u_2"] & \operatorname{Spec}\mathbf{k}
\end{tikzcd}$$

corresponding to the commutative diagram

$$\begin{tikzcd}
\mathbf{k}[t]/t^2 & \mathbf{k}[x,y]/(xy) \arrow[l, "{\substack{at \,\leftarrowtail\, \overline{x} \\ bt \,\leftarrowtail\, \overline{y}}}"'] \arrow[dl, dotted, "u_2^{\#}"'] \\
\mathbf{k}[t]/t^3 \arrow[u, two heads] & \mathbf{k} \arrow[l] \arrow[u]
\end{tikzcd}$$

of rings. Now a potential extension $u_2^{\#}\colon \mathbf{k}[x,y]/(xy) \to \mathbf{k}[t]/t^3$ would send

$$\overline{x} \mapsto at + a't^2, \quad \overline{y} \mapsto bt + b't^2.$$

This time we have to impose the relation

$$\mathbf{k}[t]/t^3 \ni 0 = (at + a't^2)(bt + b't^2) = abt^2.$$

Thus u_1 extends to u_2 if and only if $ab = 0$. Thus π is not smooth.

Example 6.3.22 (The cusp)**.** Consider the $\mathbf{k}$-scheme $\pi\colon X = \operatorname{Spec}\mathbf{k}[x,y]/(y^2 - x^3) \to \operatorname{Spec}\mathbf{k}$. We wish to use the infinitesimal lifting criterion to confirm that π is not smooth. First consider a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec}\mathbf{k} \arrow[r, "0"] \arrow[d, hook] & X \arrow[d, "\pi"] \\
\operatorname{Spec}\mathbf{k}[t]/t^2 \arrow[r] \arrow[ur, dotted, "u_1"] & \operatorname{Spec}\mathbf{k}
\end{tikzcd}$$

where the top horizontal morphism corresponds to the origin $0 \in X$ and the left vertical morphism is the reduction morphism. This corresponds to a diagram of rings

$$\begin{tikzcd}
\mathbf{k} & \mathbf{k}[x,y]/(y^2 - x^3) \arrow[l, "0 \,\leftarrowtail\, \overline{x}{,}\overline{y}"'] \arrow[dl, dotted, "u_1^{\#}"'] \\
\mathbf{k}[t]/t^2 \arrow[u, two heads] & \mathbf{k} \arrow[l] \arrow[u]
\end{tikzcd}$$

and we can define an extension $u_1^{\#}\colon \mathbf{k}[x,y]/(y^2 - x^3) \to \mathbf{k}[t]/t^2$ by sending

$$\overline{x} \mapsto at, \quad \overline{y} \mapsto bt.$$

Note that $u_1^{\#}(\overline{x}^3) = u_1^{\#}(\overline{y}^2)$ is just the identity $0 = 0$. Once more, any pair (a, b) defines an extension. Let us try to extend further, i.e. from first order to second order.

Consider a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec}\mathbf{k}[t]/t^2 \arrow[r, "u_1"] \arrow[d, hook] & X \arrow[d, "\pi"] \\
\operatorname{Spec}\mathbf{k}[t]/t^3 \arrow[r] \arrow[ur, dotted, "u_2"] & \operatorname{Spec}\mathbf{k}
\end{tikzcd}$$

corresponding to the commutative diagram

$$\begin{tikzcd}
\mathbf{k}[t]/t^2 & \mathbf{k}[x,y]/(y^2-x^3) \arrow[l, "\substack{at \leftarrow\!\shortmid \overline{x} \\ bt \leftarrow\!\shortmid \overline{y}}"'] \arrow[dl, dotted, "u_2^\#"] \\
\mathbf{k}[t]/t^3 \arrow[u, two heads] & \mathbf{k} \arrow[l] \arrow[u]
\end{tikzcd}$$

of rings. Now a potential extension $u_2^\#\colon \mathbf{k}[x,y]/(y^2-x^3) \to \mathbf{k}[t]/t^3$ would send

$$\overline{x} \mapsto at + a't^2, \quad \overline{y} \mapsto bt + b't^2.$$

This time we have to impose the relation

$$(at + a't^2)^3 = (bt + b't^2)^2$$

in $\mathbf{k}[t]/t^3$, which is equivalent to $b^2 = 0$. Thus u_1 extends to u_2 if and only if $b = 0$. Thus π is not smooth.

Example 6.3.23 (A union of 3 lines)**.** Consider the $\mathbf{k}$-scheme $\pi\colon X = \operatorname{Spec}\mathbf{k}[x,y]/(xy(x+y)) \to \operatorname{Spec}\mathbf{k}$. We wish to use the infinitesimal lifting criterion to confirm that π is not smooth. First consider a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec}\mathbf{k} \arrow[r, "0"] \arrow[d, hook] & X \arrow[d, "\pi"] \\
\operatorname{Spec}\mathbf{k}[t]/t^2 \arrow[r] \arrow[ur, dotted, "u_1"] & \operatorname{Spec}\mathbf{k}
\end{tikzcd}$$

where the top horizontal morphism corresponds to the origin $0 \in X$ and the left vertical morphism is the reduction morphism. This corresponds to a diagram of rings

$$\begin{tikzcd}
\mathbf{k} & \mathbf{k}[x,y]/(xy(x+y)) \arrow[l, "0 \leftarrow\!\shortmid \overline{x},\overline{y}"'] \arrow[dl, dotted, "u_1^\#"] \\
\mathbf{k}[t]/t^2 \arrow[u, two heads] & \mathbf{k} \arrow[l] \arrow[u]
\end{tikzcd}$$

and we can define an extension $u_1^\#\colon \mathbf{k}[x,y]/(xy(x+y)) \to \mathbf{k}[t]/t^2$ by sending

$$\overline{x} \mapsto at, \quad \overline{y} \mapsto bt.$$

Also in this case any pair $(a,b)\in\mathbf{k}^2$ defines an extension. Let us try to extend further, i.e. from first order to second order.

Consider a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec}\mathbf{k}[t]/t^2 \arrow[r, "u_1"] \arrow[d, hook] & X \arrow[d, "\pi"] \\
\operatorname{Spec}\mathbf{k}[t]/t^3 \arrow[r] \arrow[ur, dotted, "u_2"] & \operatorname{Spec}\mathbf{k}
\end{tikzcd}$$

corresponding to the commutative diagram

$$\begin{tikzcd}
\mathbf{k}[t]/t^2 & \mathbf{k}[x,y]/(xy(x+y)) \arrow[l, "{\substack{at \leftarrow\!\shortmid \overline{x} \\ bt \leftarrow\!\shortmid \overline{y}}}"'] \arrow[dl, dotted, "u_2^\#"'] \\
\mathbf{k}[t]/t^3 \arrow[u, two heads] & \mathbf{k} \arrow[l] \arrow[u]
\end{tikzcd}$$

of rings. Now a potential extension $u_2^\#\colon \mathbf{k}[x,y]/(xy(x+y))\to\mathbf{k}[t]/t^3$ would send

$$\overline{x}\mapsto at+a't^2,\quad \overline{y}\mapsto bt+b't^2.$$

This time we have to impose the relation

$$0=(at+a't^2)(bt+b't^2)(at+a't^2+bt+b't^2).$$

However this is satisfied for all $a,a',b,b'\in\mathbf{k}$. Therefore an extension to second order always exists. Let us try and extend further. Consider a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec}\mathbf{k}[t]/t^3 \arrow[r, "u_2"] \arrow[d, hook] & X \arrow[d, "\pi"] \\
\operatorname{Spec}\mathbf{k}[t]/t^4 \arrow[r] \arrow[ur, dotted, "u_3"] & \operatorname{Spec}\mathbf{k}
\end{tikzcd}$$

corresponding to the commutative diagram

$$\begin{tikzcd}
\mathbf{k}[t]/t^3 & \mathbf{k}[x,y]/(xy(x+y)) \arrow[l, "{\substack{at+a't^2 \leftarrow\!\shortmid \overline{x} \\ bt+b't^2 \leftarrow\!\shortmid \overline{y}}}"'] \arrow[dl, dotted, "u_3^\#"'] \\
\mathbf{k}[t]/t^4 \arrow[u, two heads] & \mathbf{k} \arrow[l] \arrow[u]
\end{tikzcd}$$

of rings. Now a potential extension $u_3^\#\colon \mathbf{k}[x,y]/(xy(x+y))\to\mathbf{k}[t]/t^4$ would send

$$\overline{x}\mapsto at+a't^2+a''t^3,\quad \overline{y}\mapsto bt+b't^2+b''t^3.$$

At this stage we have to impose the relation

$$0=(at+a't^2+a''t^3)(bt+b't^2+b''t^3)((a+b)t+(a'+b')t^2+(a''+b'')t^3).$$

Together with $t^4 = 0$, this leaves us with

$$ab(a+b)t^3 = 0.$$

Thus $\pi\colon \operatorname{Spec}\mathbf{k}[x,y]/(xy(x+y)) \to \operatorname{Spec}\mathbf{k}$ is not smooth.

Exercise 6.3.24 (The affine cone)**.** Perform the above analysis for the affine cone $\pi\colon X = \operatorname{Spec}\mathbf{k}[x,y,z]/(x^2 - yz) \to \operatorname{Spec}\mathbf{k}$.

7 | Coherent sheaves on noetherian schemes

7.1 $\mathscr{O}_X$-modules and (quasi)coherent sheaves

7.1.1 General definition of quasicoherent sheaves

Let $(X, \mathscr{O}_X)$ be a ringed space.

Definition 7.1.1 ($\mathscr{O}_X$-module)**.** An $\mathscr{O}_X$-*module* on the ringed space $(X, \mathscr{O}_X)$ is a sheaf of abelian groups $\mathcal{F}$ such that $\mathcal{F}(U)$ is an $\mathscr{O}_X(U)$-module for every open subset $U \subset X$, and such that for every inclusion $V \subset U$ of open subsets the diagram

$$\begin{tikzcd} \mathscr{O}_X(U) \times \mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{F}(U) \arrow[d] \\ \mathscr{O}_X(V) \times \mathcal{F}(V) \arrow[r] & \mathcal{F}(V) \end{tikzcd}$$

commutes. This means, explicitly, that $(a \cdot s)|_V = a|_V \cdot s|_V$ for every $a \in \mathscr{O}_X(U)$ and $s \in \mathcal{F}(U)$. A morphism of $\mathscr{O}_X$-modules is a morphism of sheaves that preserves the linear structures. Thus $\mathscr{O}_X$-modules form a category, denoted $\mathsf{Mod}_{\mathscr{O}_X}$.

Example 7.1.2. An ideal sheaf $\mathscr{I} \subset \mathscr{O}_X$ is an $\mathscr{O}_X$-module, as well as the quotient $\mathscr{O}_X/\mathscr{I}$.

It is easy to check that the category $\mathsf{Mod}_{\mathscr{O}_X}$ has arbitrary direct sums (categorical coproducts), kernels and cokernels. One can prove (cf. [16, Tag 01AG]) that $\mathsf{Mod}_{\mathscr{O}_X}$ is in fact an abelian category, in which a complex

$$\mathcal{F} \to \mathcal{G} \to \mathcal{H}$$

is exact in the middle if and only if

$$\mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x$$

is exact in the middle for every $x \in X$. In particular, a complex of $\mathscr{O}_X$-modules is exact if and only if it is exact as a complex of sheaves of abelian groups.

In $\mathsf{Mod}_{\mathscr{O}_X}$, the *tensor product* $\mathcal{F} \otimes_{\mathscr{O}_X} \mathcal{G}$ of two objects $\mathcal{F}$ and $\mathcal{G}$ is a new $\mathscr{O}_X$-module, defined to be the sheafification of the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathscr{O}_X(U)} \mathcal{G}(U)$. It satisfies

$$(\mathcal{F} \otimes_{\mathscr{O}_X} \mathcal{G})_x = \mathcal{F}_x \otimes_{\mathscr{O}_{X,x}} \mathcal{G}_x \tag{7.1.1}$$

for every $x \in X$. Moreover, the functor $- \otimes_{\mathscr{O}_X} \mathcal{G}$ is right exact for all $\mathcal{G} \in \mathsf{Mod}_{\mathscr{O}_X}$.

The restriction $\mathcal{F}|_U$ of an $\mathscr{O}_X$-module $\mathcal{F}$ to an open subset $U \subset X$ is naturally an $\mathscr{O}_U$-module, where, as ever, $\mathscr{O}_U = \mathscr{O}_X|_U$. Given $\mathcal{F}, \mathcal{G} \in \mathsf{Mod}_{\mathscr{O}_X}$, one also has a new $\mathscr{O}_X$-module

$$\mathscr{H}om_{\mathscr{O}_X}(\mathcal{F}, \mathcal{G})$$

called the *Hom sheaf*. Its sections over an open $U \subset X$ are given by

$$\mathscr{H}om_{\mathscr{O}_X}(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}_{\mathsf{Mod}_{\mathscr{O}_U}}(\mathcal{F}|_U, \mathcal{G}|_U).$$

Exercise 7.1.3. Show the following:

(i) $\mathscr{H}om_{\mathscr{O}_X}(\mathcal{F}, \mathcal{G})$ is a sheaf of $\mathscr{O}_X$-modules.

(ii) $\mathscr{H}om_{\mathscr{O}_X}(\mathcal{F}, -)$ is (covariant and) left exact, and $\mathscr{H}om_{\mathscr{O}_X}(-, \mathcal{G})$ is (contravariant and) left exact. (This is a shadow of a more general fact about Homs in abelian categories).

(iii) Given $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathsf{Mod}_{\mathscr{O}_X}$, there is an isomorphism of $\mathscr{O}_X$-modules

$$\mathscr{H}om_{\mathscr{O}_X}(\mathcal{F}, \mathscr{H}om_{\mathscr{O}_X}(\mathcal{G}, \mathcal{H})) \cong \mathscr{H}om_{\mathscr{O}_X}(\mathcal{F} \otimes_{\mathscr{O}_X} \mathcal{G}, \mathcal{H}).$$

Definition 7.1.4 (Free, globally generated)**.** Let I be a set. An $\mathscr{O}_X$-module $\mathcal{F}$ isomorphic to $\mathscr{O}_X^{\oplus I}$ is called *free*. If I is finite, we say that $\mathcal{F}$ is *free of finite rank*, the rank being the cardinality of I. We say that $\mathcal{F}$ is *globally generated* if it receives a surjection $\mathscr{O}_X^{\oplus I} \twoheadrightarrow \mathcal{F}$ for some set I.

Definition 7.1.5 (Quasicoherent sheaf)**.** An $\mathscr{O}_X$-module $\mathcal{F}$ is *quasicoherent* if it is locally the cokernel of a morphism of free sheaves, i.e. if every point $x \in X$ has an open neighbourhood $U \subset X$ such that there is an exact sequence

$$\mathscr{O}_X^{\oplus I}\big|_U \to \mathscr{O}_X^{\oplus J}\big|_U \to \mathcal{F}\big|_U \to 0$$

for some sets I and J. We denote by $\mathsf{QCoh}(X)$ the category of quasicoherent $\mathscr{O}_X$-modules on a ringed space $(X, \mathscr{O}_X)$.

Example 7.1.6. The structure sheaf $\mathscr{O}_X$ is quasicoherent.

7.1.2 Quasicoherent sheaves on affine schemes

Let us now move to the case of affine schemes. Let $X = \operatorname{Spec} A$, for a ring A, and let $\mathcal{B}$ be the standard basis of principal opens in X. Consider an A-module M. We define an $\mathscr{O}_X$-module $\widetilde{M}$ by defining a $\mathcal{B}$-sheaf via

$$\widetilde{M}(\mathrm{D}(f)) = M_f, \quad f \in A.$$

By Lemma 2.7.7, this uniquely extends to a sheaf, still denoted $\widetilde{M}$, which is an $\mathscr{O}_X$-module. It satisfies

$$\begin{aligned} \widetilde{M}(\operatorname{Spec} A) &= M \\ \widetilde{M}_{\mathfrak{p}} &= M_{\mathfrak{p}} \text{ for every } \mathfrak{p} \in \operatorname{Spec} A. \end{aligned}$$

The proof of all these assertions is identical to the corresponding proof for $M = A$ (cf. Theorem 3.1.28).

Given an open subset $U \subset X$ and an open cover $U = \bigcup_i \mathrm{D}(f_i)$ by principal opens $\mathrm{D}(f_i) \subset X$, there is an exact sequence

$$0 \to \widetilde{M}(U) \to \prod_i M_{f_i} \to \prod_{i,j} M_{f_i f_j}.$$

PROPOSITION 7.1.7. *Let A be a ring, and set $X = \operatorname{Spec} A$.*

(i) *If $(M_i)_i$ is a family of A-modules, then $(\bigoplus_i M_i)^\sim = \bigoplus_i \widetilde{M}_i$.*

(ii) *If $L \to M \to N$ is a sequence in Mod_A, then it is exact if and only if $\widetilde{L} \to \widetilde{M} \to \widetilde{N}$ is exact in $\mathsf{Mod}_{\mathscr{O}_X}$.*

(iii) *$\widetilde{M}$ is quasicoherent, for any A-module M.*

(iv) *If M, N are A-modules, then $(M \otimes_A N)^\sim = \widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{N}$.*

Proof. We proceed step by step.

Proof of (i). Fix $f \in A$. We have

$$\left(\bigoplus_i M_i\right)^{\sim}(\mathrm{D}(f)) = \left(\bigoplus_i M_i\right)_f = \bigoplus_i M_{i,f} = \bigoplus_i \widetilde{M}_i(\mathrm{D}(f)).$$

Proof of (ii). First assume $L \to M \to N$ is exact. Let $\mathfrak{p} \in \operatorname{Spec} A$. Then, since $A \to A_{\mathfrak{p}}$ is flat (cf. Lemma 6.2.7(iv)), the sequence of $A_{\mathfrak{p}}$-modules $L_{\mathfrak{p}} \to M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is exact. Then Proposition 2.5.14 implies that $\widetilde{L} \to \widetilde{M} \to \widetilde{N}$ is exact in $\mathsf{Mod}_{\mathscr{O}_X}$.

Conversely, assume $\widetilde{L} \to \widetilde{M} \to \widetilde{N}$ is exact in $\mathsf{Mod}_{\mathscr{O}_X}$. We have a commutative diagram

$$\begin{tikzcd} L \arrow[r, "a"] \arrow[d] & M \arrow[r, "b"] \arrow[d] & N \arrow[d] \\ L_{\mathfrak{p}} \arrow[r] & M_{\mathfrak{p}} \arrow[r] & N_{\mathfrak{p}} \end{tikzcd}$$

where

- the upper row is obtained by taking global sections in $\widetilde{L} \to \widetilde{M} \to \widetilde{N}$,
- the vertical arrows are localisations,
- the lower row is exact, being the stalk of the exact sequence $\widetilde{L} \to \widetilde{M} \to \widetilde{N}$.

It follows from the last bullet that $(\ker b / \operatorname{im} a)_{\mathfrak{p}} = 0$ and since this holds for every $\mathfrak{p} \in \operatorname{Spec} A$ we have globally $\ker b = \operatorname{im} a$, proving that $L \to M \to N$ is exact.

Proof of (iii). Every $M \in \mathsf{Mod}_A$ admits an exact sequence

$$A^{\oplus J} \longrightarrow A^{\oplus I} \longrightarrow M \longrightarrow 0.$$

By parts (i) and (ii) combined, we have a corresponding exact sequence

$$\begin{tikzcd} (A^{\oplus J})^{\sim} \arrow[r] \arrow[d, equal] & (A^{\oplus I})^{\sim} \arrow[r] \arrow[d, equal] & \widetilde{M} \arrow[r] & 0 \\ \mathscr{O}_X^{\oplus J} & \mathscr{O}_X^{\oplus I} & & \end{tikzcd}$$

therefore $\widetilde{M}$ is quasicoherent.

Proof of (iv). By Proposition 2.7.9 it is enough to confirm the isomorphism on the standard base of principal opens in X. So let us fix $f \in A$. We have

$$\begin{aligned} (M \otimes_A N)^{\sim}(\mathrm{D}(f)) &= (M \otimes_A N)_f \\ &= (M \otimes_A N) \otimes_A A_f \\ &= (M \otimes_A A_f) \otimes_{A_f} (N \otimes_A A_f) \\ &= M_f \otimes_{A_f} N_f \\ &= \widetilde{M}(\mathrm{D}(f)) \otimes_{\mathscr{O}_X(\mathrm{D}(f))} \widetilde{N}(\mathrm{D}(f)) \\ &= (\widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{N})(\mathrm{D}(f)). \end{aligned}$$

□

PROPOSITION 7.1.8 ([12, Ch. 5, Prop. 1.6]). *Let X be a noetherian scheme, or a separated quasicompact scheme. Let $\mathcal{F}$ be a quasicoherent sheaf. For $f \in \mathscr{O}_X(X)$, define*

$$X_f = \{\, x \in X \mid f_x \in \mathscr{O}_{X,x}^{\times} \,\}. \tag{7.1.2}$$

Then, $X_f \subset X$ is open, and there is a canonical isomorphism

$$\mathcal{F}(X)_f = \mathcal{F}(X) \otimes_{\mathscr{O}_X(X)} \mathscr{O}_X(X)_f \xrightarrow{\sim} \mathcal{F}(X_f).$$

COROLLARY 7.1.9 (Local nature of quasicoherence). *Let X be a scheme, $\mathcal{F} \in \mathsf{Mod}_{\mathscr{O}_X}$. Then $\mathcal{F}$ is quasicoherent if and only if for every open affine $U \subset X$ one has $\mathcal{F}|_U = \mathcal{F}(U)^{\sim}$.*

Proof. Assume $\mathcal{F}|_U = \mathcal{F}(U)^{\sim}$ for every open affine U. By Proposition 7.1.7 (iii), $\mathcal{F}|_U$ is then quasicoherent, and thus $\mathcal{F}$ is quasicoherent by definition.

Conversely, assume $\mathcal{F}$ is quasicoherent. Fix an affine open $U = \operatorname{Spec} A$. Note that U is separated and quasicompact. Then, by Proposition 7.1.8, for every $f \in \mathcal{O}_X(U)$ we have

$$\mathcal{F}(U)_f = \mathcal{F}(U) \otimes_A A_f \xrightarrow{\sim} \mathcal{F}(\mathrm{D}(f)),$$

because $U_f = \mathrm{D}(f)$ in the notation of Equation (7.1.2). Thus we have an isomorphism

$$\mathcal{F}(U)^{\sim} \xrightarrow{\sim} \mathcal{F}|_U$$

by Proposition 2.7.9. □

PROPOSITION 7.1.10 ([12, Ch. 5, Prop. 1.8]). *Let $X = \operatorname{Spec} A$ be an affine scheme. Let*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be an exact sequence of $\mathcal{O}_X$-modules, with $\mathcal{F} \in \mathsf{QCoh}(X)$. Then the sequence on global sections

$$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X) \to 0$$

is exact in Mod_A.

THEOREM 7.1.11. *Let A be a ring, and set $X = \operatorname{Spec} A$. Then sending $M \mapsto \widetilde{M}$ induces an equivalence of categories*

$$\Phi_A^{\mathsf{qcoh}} \colon \mathsf{Mod}_A \xrightarrow{\sim} \mathsf{QCoh}(X),$$

with inverse[1] *sending $\mathcal{F} \mapsto \mathcal{F}(X)$.*

In particular, quasicoherent sheaves on an affine scheme form an abelian category.

Proof. The assignment $\Phi_A^{\mathsf{qcoh}} \colon M \mapsto \widetilde{M}$ is functorial by construction and lands in $\mathsf{QCoh}(X)$ by Proposition 7.1.7(iii). Essential surjectivity follows from Corollary 7.1.9. As for fully faithfullness, we need to confirm that the natural map

$$\gamma \colon \operatorname{Hom}_{\mathsf{Mod}_A}(M, N) \longrightarrow \operatorname{Hom}_{\mathsf{QCoh}(X)}(\widetilde{M}, \widetilde{N})$$

sending $\phi \colon M \to N$ to the $\mathcal{O}_X$-linear map $\widetilde{\phi} \colon \widetilde{M} \to \widetilde{N}$ determined by $M_f \to N_f$ for $f \in A$ is bijective. Injectivity is clear (take $f = 1$). As for surjectivity, take $\alpha \in \operatorname{Hom}_{\mathsf{QCoh}(X)}(\widetilde{M}, \widetilde{N})$. Then $\alpha = \gamma(\widetilde{\alpha})$, as required. □

7.1.3 Coherent sheaves

Definition 7.1.12 (Finitely generated, coherent)**.** Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is *finitely generated* if every $x \in X$ has an open neighbourhood $U \subset X$ admitting a surjection $\mathcal{O}_X^{\oplus n}|_U \twoheadrightarrow \mathcal{F}|_U$ for some $n \geq 1$. We say that $\mathcal{F}$ is *coherent* if it is

[1] Abuse of language!

finitely generated and for every morphism $\theta\colon \mathcal{O}_X^{\oplus n}|_U \to \mathcal{F}|_U$ on an open $U \subset X$ one has that $\ker\theta$ is also finitely generated.

We denote by $\mathsf{Coh}(X)$ the category of coherent sheaves on a ringed space $(X, \mathcal{O}_X)$.

THEOREM 7.1.13. *Let X be a scheme, $\mathcal{F} \in \mathsf{QCoh}(X)$. Consider the conditions*

(1) *$\mathcal{F} \in \mathsf{Coh}(X)$.*

(2) *$\mathcal{F}$ is finitely generated.*

(3) *For any open affine subset $U = \operatorname{Spec} A \subset X$, the A-module $\mathcal{F}(U)$ is finitely generated.*

Then we always have (1) ⇒ (2) ⇒ (3) and if X is locally noetherian the three conditions are equivalent.

Proof. That (1) ⇒ (2) is true by definition.

Let us prove that (2) ⇒ (3). Assume $\mathcal{F} \in \mathsf{QCoh}(X)$ is finitely generated. Fix $U = \operatorname{Spec} A$ and a finite open cover $U = \bigcup_i U_i$ with $U_i = \mathrm{D}(f_i) = \operatorname{Spec} A_i$ a principal open, such that we have surjections

$$\mathcal{O}_X^{\oplus n_i}|_{U_i} \twoheadrightarrow \mathcal{F}|_{U_i}, \quad n_i \geq 1.$$

By Corollary 7.1.9, each such surjection is the same as a surjection

$$(\mathcal{O}_X^{\oplus n_i}|_{U_i}(U_i))^{\sim} = (\mathcal{O}_X(U_i)^{\oplus n_i})^{\sim} = (A_i^{\oplus n_i})^{\sim} \twoheadrightarrow \mathcal{F}(U_i)^{\sim}.$$

By Proposition 7.1.7(ii), we have surjections

$$A_i^{\oplus n_i} \twoheadrightarrow \mathcal{F}(U_i).$$

Thus $\mathcal{F}(U_i)$ is finitely generated over $\mathcal{O}_X(U_i) = A_i$ for every i. But

$$\mathcal{F}(U_i) = \mathcal{F}(U) \otimes_A A_i$$

since U_i is principal, thus there exists a finitely generated $\mathcal{O}_X(U)$-submodule $M_i \subset \mathcal{F}(U)$ such that

$$\mathcal{F}(U_i) = M_i \otimes_A A_i.$$

Since there are finitely many opens U_i, we may (if necessary) enlarge each M_i so to obtain a uniform M satisfying

$$\mathcal{F}(U_i) = M \otimes_A A_i$$

for all i. The sequences

$$\widetilde{M}|_{U_i} \longrightarrow \mathcal{F}|_{U_i} \longrightarrow 0$$

are then exact for every i, therefore

$$\widetilde{M} \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

is exact as well. It follows once more from Proposition 7.1.7(ii) that

$$\widetilde{M}(U) = M \longrightarrow \mathcal{F}(U) \longrightarrow 0$$

is exact, which implies that $\mathcal{F}(U)$ is finitely generated over $A = \mathcal{O}_X(U)$.

Finally, we assume X is locally noetherian and we prove (3) ⇒ (1). We fix an open subset $U \subset X$, a morphism $\theta: \mathcal{O}_X^{\oplus n}|_U \to \mathcal{F}|_U$, and we need to prove that $\ker\theta$ is finitely generated. We may assume $U = \operatorname{Spec} A$ is affine with A noetherian. This gives $\mathcal{F}|_U = \widetilde{M}$ for some $M \in \mathsf{Mod}_A$. It follows from Proposition 7.1.7(ii) that

$$\ker\theta = (\ker\theta_U)^{\sim}.$$

But A is noetherian therefore $\ker\theta_U \subset A^{\oplus n}$ is finitely generated. Therefore $\ker\theta$ is finitely generated. □

We have obtained the slogan

$$\boxed{\text{coherent on locally noetherian scheme} = \text{quasicoherent} + \text{finitely generated.}}$$

COROLLARY 7.1.14. *If A is a noetherian ring, the equivalence Φ_A^{qcoh} of* Theorem 7.1.11 *restricts to an equivalence*

$$\Phi_A^{\mathsf{coh}}: \mathsf{Mod}_A^{\mathsf{fg}} \xrightarrow{\sim} \mathsf{Coh}(\operatorname{Spec} A),$$

where $\mathsf{Mod}_A^{\mathsf{fg}}$ is the category of finitely generated A-modules.

PROPOSITION 7.1.15. *Let X be a locally noetherian scheme.*

(i) *A direct sum of quasicoherent sheaves is quasicoherent. A finite direct sum of coherent sheaves is coherent.*

(ii) *If $\mathcal{F}$ and $\mathcal{G}$ are (quasi)coherent, then so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$. If $U = \operatorname{Spec} A$ is open affine in X, then $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})(U) = \mathcal{F}(U) \otimes_A \mathcal{G}(U)$.*

(iii) *If $\mathcal{F}, \mathcal{G} \in \mathsf{Coh}(X)$, then $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \in \mathsf{Coh}(X)$.*

(iv) *If $\varphi: \mathcal{F} \to \mathcal{G}$ is a morphism of (quasi)coherent sheaves, then $\ker\varphi$ and $\operatorname{coker}\varphi$ are (quasi)coherent.*

(v) *Consider a short exact sequence*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

in $\mathsf{Mod}_{\mathcal{O}_X}$. If two of the sheaves are (quasi)coherent, then so is the third.

Remark 7.1.16. It is not true, for an arbitrary locally ringed space $(X, \mathscr{O}_X)$, that infinite direct sums of quasicoherent sheaves are again quasicoherent (cf. also Theorem 7.1.21 (i)). On the other hand, the statement is true for *finite* direct sums.

Example 7.1.17. Let X be a locally noetherian scheme, $\iota\colon Z \hookrightarrow X$ a closed immersion. Then the exact sequence

$$0 \to \mathscr{I}_Z \to \mathscr{O}_X \to \iota_*\mathscr{O}_Z \to 0$$

takes place in $\mathsf{Coh}(X)$.

An important example of coherent sheaf is given by the following notion.

Definition 7.1.18 (Locally free sheaf)**.** Let X be a scheme. An $\mathscr{O}_X$-module $\mathcal{F}$ is said to be *locally free of rank r* if X has an open cover $X = \bigcup_i U_i$ such that $\mathcal{F}|_{U_i}$ is free of rank r (i.e. isomorphic to $\mathscr{O}_{U_i}^{\oplus r}$) for every i. An *invertible sheaf* is a locally free $\mathscr{O}_X$-module of rank 1.

Notation 7.1.19. Let $\mathcal{F} \in \mathsf{Mod}_{\mathscr{O}_X}$. The *dual of* $\mathcal{F}$ is the sheaf $\mathcal{F}^\vee = \mathscr{H}om_{\mathscr{O}_X}(\mathcal{F}, \mathscr{O}_X)$.

LEMMA 7.1.20. *If $\mathcal{E}$ and $\mathcal{F}$ are $\mathscr{O}_X$-modules, with $\mathcal{E}$ locally free of finite rank, then*

$$\mathcal{E}^\vee \otimes_{\mathscr{O}_X} \mathcal{F} \cong \mathscr{H}om_{\mathscr{O}_X}(\mathcal{E}, \mathcal{F}).$$

Proof. For every $U \subset X$ open affine, there is an $\mathscr{O}_X(U)$-linear map

$$\begin{aligned} \mathrm{Hom}_{\mathscr{O}_X(U)}(\mathcal{E}(U), \mathscr{O}_X(U)) \otimes_{\mathscr{O}_X(U)} \mathcal{F}(U) &\xrightarrow{\beta_U} \mathrm{Hom}_{\mathscr{O}_X(U)}(\mathcal{E}(U), \mathcal{F}(U)) \\ \phi \otimes \sigma &\longmapsto (a \mapsto \phi(a)\cdot\sigma). \end{aligned}$$

Being compatible with localisation maps, it gives rise to a homomorphism

$$\mathcal{E}^\vee \otimes_{\mathscr{O}_X} \mathcal{F} \xrightarrow{\beta} \mathscr{H}om_{\mathscr{O}_X}(\mathcal{E}, \mathcal{F}).$$

We leave it to the reader to verify that β is an isomorphism. □

THEOREM 7.1.21. *Let $(X, \mathscr{O}_X)$ be a ringed space. The following are true.* add ref

(i) $\mathsf{Coh}(X)$ *is an abelian category, whereas* $\mathsf{QCoh}(X)$ *is not necessarily abelian.*

(ii) *If $(X, \mathscr{O}_X)$ is a scheme,* $\mathsf{QCoh}(X)$ *is an abelian category.*

LEMMA 7.1.22. *Let X be a noetherian scheme, $\mathcal{F} \in \mathsf{Coh}(X)$. Then*

(1) *Let $x \in X$. If $\mathcal{F}_x$ is free, then $\mathcal{F}|_U$ is free for some open neighbourhood U of x.*

(2) *$\mathcal{F}$ is locally free if and only if $\mathcal{F}_x$ is free for every x.*

(3) *$\mathcal{F}$ is invertible if and only if there exists $\mathcal{G} \in \mathsf{Coh}(X)$ such that $\mathcal{F} \otimes_{\mathscr{O}_X} \mathcal{G} \cong \mathscr{O}_X$.*

(4) *Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a surjective morphism of locally free sheaves, of the same (finite) rank. Then φ is an isomorphism.*

Proof. We proceed step by step.

(1) To be written.

(2) Fix $x \in X$. If $\mathcal{F}$ is locally free, then $\mathcal{F}|_U$ is free for some U containing x. The converse is just (1).

(3) If $\mathcal{F}$ is invertible (locally free of rank 1), then cover X with open subsets $U \subset X$ such that $\mathcal{F}|_U = \mathcal{O}_U$. We have $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F}) \cong \mathcal{O}_X$ via the map

$$\mathcal{O}_X(U) \to \mathrm{Hom}(\mathcal{F}|_U, \mathcal{F}|_U)$$

sending $f \in \mathcal{O}_X(U)$ to the map $f^*\colon \mathcal{F}|_U \to \mathcal{F}|_U$ defined by

$$f_V^*\colon \mathcal{F}(V) \to \mathcal{F}(V), \quad s \mapsto f|_V \cdot s.$$

By Lemma 7.1.20, we have $\mathcal{O}_X \cong \mathcal{F}^\vee \otimes_{\mathcal{O}_X} \mathcal{F}$.

Conversely, assume $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \cong \mathcal{O}_X$ for some $\mathcal{G} \in \mathsf{Coh}(X)$. finish

(4) To be written.

□

Caution 7.1.23. It is not true that an *injective* morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of locally free sheaves of the same rank is an isomorphism. Can you find examples?

Remark 7.1.24. Let X be a scheme. Let $\mathcal{F} \in \mathsf{Coh}(X)$ be locally free of rank r. Then $\mathcal{F}^\vee$ is also locally free of rank r. Let $\mathcal{G} \in \mathsf{Coh}(X)$ be locally free of rank s. The tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is locally free of rank rs. So the tensor product of two invertible sheaves is an invertible sheaf. The direct sum $\mathcal{F} \oplus \mathcal{G}$ is locally free of rank $r+s$.

LEMMA 7.1.25. *Let $\mathcal{L}$ be an invertible sheaf, and let $\mathcal{L}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ be its dual. Then there is a canonical isomorphism*

$$\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^\vee \cong \mathcal{O}_X.$$

Proof. Results from the proof of Lemma 7.1.22 (3). □

The set

$$\mathrm{Pic}(X) = \{\text{isomorphism classes of invertible sheaves over } X\}$$

is endowed with a natural abelian group structure

$$[\mathcal{L}]\cdot[\mathcal{L}'] = [\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}'], \quad [\mathcal{L}]^{-1} = [\mathcal{L}^\vee],$$

which makes sense by Remark 7.1.24 and Lemma 7.1.25.

Definition 7.1.26 (Picard group)**.** Let X be a scheme. The group $\mathrm{Pic}(X)$ is called the *Picard group of X*.

7.1.4 Functoriality of quasicoherent sheaves

Let $f: X \to Y$ be a morphism of ringed spaces, and consider the situation

$$\begin{tikzcd} \mathcal{F} \arrow[d, dash] & \mathcal{G} \arrow[d, dash] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where $\mathcal{F} \in \mathsf{Mod}_{\mathcal{O}_X}$ and $\mathcal{G} \in \mathsf{Mod}_{\mathcal{O}_Y}$. We have the two maps

$$\mathcal{O}_Y \to f_*\mathcal{O}_X, \quad f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$$

crucial for the adjoint pair (f^{-1}, f_*). We use them for the following constructions:

- Since $f_*\mathcal{F}$ is an $f_*\mathcal{O}_X$-module, it is naturally an $\mathcal{O}_Y$-module.
- Since $f^{-1}\mathcal{G}$ is an $f^{-1}\mathcal{O}_Y$-module, we can construct the $\mathcal{O}_X$-module

  $$f^*\mathcal{G} = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

  It is called the *pullback* of $\mathcal{G}$ along f.

Note that $f^*\mathcal{O}_Y = \mathcal{O}_X$ for every f. Moreover,

$$(f^*\mathcal{G})_x = \mathcal{G}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$$

combining (7.1.1) with Lemma 2.8.18.

PROPOSITION 7.1.27. *Let $f: X \to Y$ be a morphism of schemes. Fix $\mathcal{F} \in \mathsf{Mod}_{\mathcal{O}_X}$ and $\mathcal{G} \in \mathsf{Mod}_{\mathcal{O}_Y}$.*

(a) *If $\mathcal{G}$ is quasicoherent (resp. finitely generated, resp. locally free), then so is $f^*\mathcal{G}$. Therefore coherent sheaves pullback along morphisms between locally noetherian schemes.*[2]

(b) *Assume $\mathcal{F} \in \mathsf{QCoh}(X)$, and either X noetherian or f separated and quasicompact. Then $f_*\mathcal{F}$ is quasicoherent.*

(c) *If $\mathcal{F}$ is quasicoherent and finitely generated, and f is finite, then $f_*\mathcal{F}$ is quasicoherent and finitely generated.*

Proof. These statements are proved in [12, Ch. 5, Prop. 1.14]. For pullback of locally free sheaves in (a), see ******** □ ref

[2] But not along arbitrary morphisms!

If $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is a morphism of affine schemes and $\mathcal{G} = \widetilde{M} \in \mathsf{QCoh}(\operatorname{Spec} A)$, then

$$f^*\widetilde{M} = (M \otimes_A B)^{\sim}.$$

This shows quasicoherence of the pullback in the affine case.

THEOREM 7.1.28 ([12, Ch. 5, Prop. 1.15]). *Let $\iota\colon Z \hookrightarrow X$ be a closed immersion of schemes. Then the ideal sheaf $\ker \iota^\# \subset \mathscr{O}_X$ is quasicoherent. Thus Proposition 2.10.11 gives a 1-1 correspondence*

$$\{\, \text{closed subschemes } Z \hookrightarrow X \,\} \xrightarrow{\sim} \{\, \text{quasicoherent ideal sheaves } \mathscr{I} \subset \mathscr{O}_X \,\}.$$

Example 7.1.29. Consider the following examples.

(1) Consider the structure morphism $f\colon \mathbb{A}^n_{\mathbb{F}} \to \operatorname{Spec} \mathbb{F}$. Then $f_*\mathscr{O}_{\mathbb{A}^n_{\mathbb{F}}}$ is quasicoherent but not coherent, since $\mathbb{F}[x_1, \ldots, x_n]$ is not a finite dimensional $\mathbb{F}$-vector space.

(2) Let $\mathscr{I} \subset \mathscr{O}_{\mathbb{A}^2_{\mathbf{k}}}$ be the ideal sheaf of the origin $\mathfrak{m} \in \mathbb{A}^2_{\mathbf{k}}$ in the affine plane, i.e. the sheaf defined by

$$\mathscr{I}(U) = \left\{ f \in \mathscr{O}_{\mathbb{A}^2_{\mathbf{k}}}(U) \,\middle|\, f(\mathfrak{m}) = 0 \right\}.$$

Then $\mathscr{I}$ is coherent, and not locally free. The restriction $\mathscr{I}|_{\mathscr{O}_{\mathbb{A}^2_{\mathbf{k}}\setminus 0}}$ is free of rank 1. More generally, if $Z \hookrightarrow X$ is a closed immersion cut out by an ideal sheaf $\mathscr{J}$, then $\mathscr{J}|_{X\setminus Z} = \mathscr{O}_{X \setminus Z}$.

(3) Consider the morphism $f\colon \mathbb{A}^1_{\mathbf{k}} \to \mathbb{A}^1_{\mathbf{k}}$ defined at the level of rings by $\mathbf{k}[t] \to \mathbf{k}[x]$ sending $t \mapsto x^n$. Then $\mathbf{k}[x] = \mathbf{k}[x,t]/(t - x^n) = \mathbf{k}[t] \oplus \mathbf{k}[t] \cdot x \oplus \cdots \oplus \mathbf{k}[t] \cdot x^{n-1}$ as a $\mathbf{k}[t]$-module, therefore $f_*\mathscr{O}_{\mathbb{A}^1_{\mathbf{k}}} = \mathscr{O}_{\mathbb{A}^1_{\mathbf{k}}}^{\oplus n}$ is a free sheaf of rank n.

(4) Let $x \in X$ be a point on a variety X. Consider the skyscraper sheaf $\mathcal{F} = \iota_*\mathscr{O}_x$ where ι is the inclusion of x into X. Then $\mathcal{F}$ is coherent but never locally free.

If X is a regular curve, the ideal sheaf $\mathscr{I}_x = \ker(\mathscr{O}_X \twoheadrightarrow \iota_*\mathscr{O}_x)$ of a point $x \in X$ is locally free (of rank 1).

(5) Let $j\colon U = \mathbb{A}^1_{\mathbf{k}} \setminus 0 \hookrightarrow \mathbb{A}^1_{\mathbf{k}}$. Consider the sheaf $j_!\mathscr{O}_U \in \mathsf{Mod}_{\mathscr{O}_{\mathbb{A}^1_{\mathbf{k}}}}$ given by

$$j_!\mathscr{O}_U(V) = \begin{cases} \mathscr{O}_{\mathbb{A}^1_{\mathbf{k}}}(V) & \text{if } 0 \notin V \\ 0 & \text{if } 0 \in V. \end{cases}$$

If $j_!\mathscr{O}_U$ were quasicoherent, we would have $j_!\mathscr{O}_U = \Gamma(\mathbb{A}^1_{\mathbf{k}}, j_!\mathscr{O}_U)^{\sim} = 0$, but the sheaf is clearly nonzero.

7.1.5 Quasicoherent sheaves on projective schemes

Let $B = \bigoplus_{d\geq 0} B_d$ be a graded $\mathbb{Z}$-algebra, $M = \bigoplus_{n\in\mathbb{Z}} M_n$ a graded B-module, i.e. a B-module such that $B_d M_n \subset M_{d+n}$ for all $d \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}$. For $f \in B_+$ nonnilpotent, define the principal homogeneous localisation

$$M_{(f)} = \left\{ \frac{m}{f^e} \in M_f \,\middle|\, m \in M_{e \deg f} \right\}.$$

Consider the standard basis of principal opens $\mathcal{B}$ on $X = \operatorname{Proj} B$. Define a $\mathcal{B}$-sheaf $\widetilde{M}$ on X by the rule

$$\widetilde{M}(\mathrm{D}_+(f)) = M_{(f)}.$$

In other words,

$$\widetilde{M}\big|_{\operatorname{Spec} B_{(f)}} = \widetilde{M_{(f)}}.$$

This $\mathcal{B}$-sheaf uniquely extends to a quasicoherent sheaf on X, still denoted $\widetilde{M}$, which furthermore satisfies

$$\widetilde{M}_{\mathfrak{p}} = M_{(\mathfrak{p})} = \left\{ \text{degree 0 elements in } M_{\mathfrak{p}} \right\}.$$

for all $\mathfrak{p} \in X$.

Construction 7.1.30 (Serre's twisting sheaf)**.** Let A be a ring, B be a graded A-algebra. For $n \in \mathbb{Z}$, consider the graded B-module

$$B(n) = \bigoplus_{d\geq 0} B(n)_d, \quad B(n)_d = B_{n+d}.$$

Define, on $X = \operatorname{Proj} B$, the quasicoherent sheaf

$$\mathscr{O}_X(n) = \widetilde{B(n)}.$$

It is called Serre's twisting sheaf.

Remark 7.1.31. If $f \in B_+$ is homogeneous of degree 1, one has

$$B(n)_{(f)} = \left\{ \frac{b}{f^e} \in B(n)_f \,\middle|\, b \in B(n)_e \right\} = f^n B_{(f)},$$

by viewing $b/f^e = f^n b/f^{e+n}$.

This relation implies

$$\mathscr{O}_X(n)\big|_{\mathrm{D}_+(f)} = f^n \cdot \mathscr{O}_X\big|_{\mathrm{D}_+(f)} = f^n \cdot \mathscr{O}_{\mathrm{D}_+(f)}.$$

One also has

$$\mathscr{O}_X(n) \otimes_{\mathscr{O}_X} \mathscr{O}_X(m) = \mathscr{O}_X(n+m), \quad n, m \in \mathbb{Z}.$$

Definition 7.1.32. Let $\pi\colon \mathbb{P}^r_S \to \mathbb{P}^r_{\mathbb{Z}}$ be the canonical morphism. For $n \in \mathbb{Z}$, define

$$\mathscr{O}_{\mathbb{P}^r_S}(n) = \pi^* \mathscr{O}_{\mathbb{P}^r_{\mathbb{Z}}}(n).$$

This is Serre's twisting sheaf on an arbitrary projective space, and is an invertible sheaf.

Example 7.1.33. Let $X = \operatorname{Proj} \mathbf{k}[x_0, x_1] = \mathbb{P}^1_{\mathbf{k}}$. Then

$$\mathbf{k}[x_0, x_1](n)_{(x_0)} = x_0^n \mathbf{k}[x_1/x_0]$$

and

$$\mathscr{O}_X(n)\big|_{U_0} = x_0^n \mathscr{O}_{U_0}.$$

LEMMA 7.1.34. *Let $B = A[x_0, \dots, x_r]$. Then*

$$\Gamma(\mathbb{P}^r_A, \mathscr{O}_{\mathbb{P}^r_A}(n)) = \begin{cases} B_n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0. \end{cases}$$

In particular,

$$B = \bigoplus_{n \in \mathbb{Z}} \Gamma(\mathbb{P}^r_A, \mathscr{O}_{\mathbb{P}^r_A}(n)).$$

Definition 7.1.35 (Very ample line bundle)**.** Let $\iota\colon X \hookrightarrow \mathbb{P}^r_A$ be a locally closed immersion. The pullback

$$\mathscr{O}_X(n) = \iota^* \mathscr{O}_{\mathbb{P}^r_A}(n)$$

is an invertible sheaf depending on ι, and $\mathscr{O}_X(1)$ is called a *relatively very ample line bundle* on the A-scheme $X \to \operatorname{Spec} A$.

7.1.6 Morphisms to projective space

Definition 7.1.36 ($\mathscr{O}_X$-module generated by sections)**.** An $\mathscr{O}_X$-module $\mathcal{F}$ on a scheme X is said to be *generated by sections* $s_0, \dots, s_r \in \mathcal{F}(X)$ if for every $x \in X$ one has

$$\mathcal{F}_x = \sum_{0 \leq i \leq r} s_{i,x} \mathscr{O}_{X,x}.$$

THEOREM 7.1.37. *Let A be a ring, X an A-scheme, and set $Y = \mathbb{P}^r_A = \operatorname{Proj} A[x_0, \dots, x_r]$.*

(i) *Let $f\colon X \to Y$ be a morphism of A-schemes. Then $f^*\mathscr{O}_Y(1)$ is an invertible sheaf, generated by $r+1$ global sections.*

(ii) *For any invertible sheaf $\mathscr{L}$ on X, generated by $r+1$ global sections $s_0, \dots, s_r$, there is an A-morphism $f\colon X \to Y$ such that $f^*\mathscr{O}_Y(1) \cong \mathscr{L}$ and $f^* x_i = s_i$ for $i = 0, \dots, r$.*

Proof. Let us prove (i). The sections $x_0,\dots,x_r$ generate $\mathscr{O}_Y(1)$ and induce canonical sections $s_0,\dots,s_r$ of the pullback $f^*\mathscr{O}_Y(1)$. Let $p\in X$ be a point, $q=f(p)\in Y$ its image. Then

$$(f^*\mathscr{O}_Y(1))_p=\mathscr{O}_Y(1)_q\otimes_{\mathscr{O}_{Y,q}}\mathscr{O}_{X,p}=\sum_{i=0}^r (x_i)_q\mathscr{O}_{Y,q}\otimes_{\mathscr{O}_{Y,q}}\mathscr{O}_{X,p}=\sum_{i=0}^r (s_i)_p\mathscr{O}_{X,p},$$

thus $f^*\mathscr{O}_Y(1)$ is generated by the global sections $s_0,\dots,s_r$.

Let us prove (ii). The scheme X is covered by the open subsets $X_{s_i}\subset X$. For each $i\le r$ we have the ring homomorphism

$$\mathscr{O}_Y(\mathrm{D}_+(x_i))\longrightarrow \mathscr{O}_X(X_{s_i})$$

sending $x_j/x_i\mapsto s_j/s_i$. This induces a morphism

$$X_{s_i}\xrightarrow{f_i}\mathrm{D}_+(x_i),$$

and the tuple $(f_i)_i$ glue to a unique morphism $f\colon X\to Y$. This morphism satisfies $f^*\mathscr{O}_Y(1)\cong\mathscr{L}$ and $f^*x_i=s_i$ for $i=0,\dots,r$. □

7.2 Divisors and line bundles

7.2.1 Cartier divisors

Let A be a ring. Let R_A be the set of *regular elements* of A, i.e.

$$\mathrm{R}_A=\{\,a\in A\mid a\text{ is not a zero divisor}\,\}\subset A.$$

Then there is an injective ring homomorphism

$$A\hookrightarrow \operatorname{Frac}A=\mathrm{R}_A^{-1}A$$

into the *total ring of fractions* of A. If X is a scheme, then sending

$$X\supset U\mapsto \mathcal{R}_X(U)=\{\,a\in\mathscr{O}_X(U)\mid a_x\in\mathrm{R}_{\mathscr{O}_{X,x}}\text{ for all }x\in U\,\}\subset\mathscr{O}_X(U)$$

defines a sheaf $\mathcal{R}_X$, satisfying

$$\mathcal{R}_X(\operatorname{Spec}A)=\mathrm{R}_{\mathscr{O}_X(\operatorname{Spec}A)}=\mathrm{R}_A \tag{7.2.1}$$

for every open affine $\operatorname{Spec}A\subset X$. The association

$$U\mapsto\mathcal{K}'_X(U)=\mathcal{R}_X(U)^{-1}\mathscr{O}_X(U)$$

defines a separated presheaf of rings on X, as $\mathcal{R}_X(U)\subset\mathrm{R}_{\mathscr{O}_X(U)}$. It satisfies

$$\mathcal{K}'_X(\operatorname{Spec}A)=\operatorname{Frac}A$$

for every open affine $\operatorname{Spec}A\subset X$, thanks to (7.2.1). By construction, we have an inclusion of presheaves $\mathscr{O}_X\hookrightarrow\mathcal{K}'_X$. The stalk $\mathcal{K}'_{X,x}$ is equal to $\operatorname{Frac}\mathscr{O}_{X,x}$ for every $x\in X$ as soon as X is locally noetherian. All these assertions are proved in [12, Ch. 7, Lemma 1.12].

Definition 7.2.1 (Sheaf of meromorphic functions)**.** The sheafification $\mathcal{K}_X$ of the presheaf $\mathcal{K}'_X$ is called the *sheaf of meromorphic functions on* X (or, also, the *sheaf of total quotient rings on* X). An element $f \in \mathrm{H}^0(X, \mathcal{K}_X)$ is called a *meromorphic function on* X.

Example 7.2.2. If X is an integral scheme, then $\mathcal{K}_X$ is the constant sheaf with values in the field of rational functions $K(X) = \mathcal{O}_{X,\xi}$, where ξ is, as ever, the generic point of X.

We have inclusions of presheaves

$$\mathcal{O}_X \hookrightarrow \mathcal{K}'_X \hookrightarrow \mathcal{K}_X,$$

therefore $\mathcal{O}_X$ is a subsheaf of $\mathcal{K}_X$. If X is locally noetherian (or reduced with finitely many irreducible components, e.g. integral) and $U = \operatorname{Spec} A \subset X$ is open, then

(7.2.2)
$$\operatorname{Frac} A = \mathcal{K}'_X(U) = \mathcal{K}_X(U).$$

Denote by $\mathcal{O}_X^\times \subset \mathcal{O}_X$ and by $\mathcal{K}_X^\times \subset \mathcal{K}_X$ the subsheaves of invertible regular functions and nonzero meromorphic functions respectively. One has a commutative diagram

$$\begin{tikzcd}
\mathcal{O}_X^\times \arrow[r, hook] \arrow[d, hook] & \mathcal{O}_X \arrow[d, hook] \\
\mathcal{K}_X^\times \arrow[r, hook] & \mathcal{K}_X
\end{tikzcd}$$

of subsheaves. All these sheaves are considered as sheaves of abelian groups, with respect to their multiplicative structure.

Caution 7.2.3. As underlined in Kleiman's paper *Misconceptions about* $\mathcal{K}_X$ [11], the following three statements about $\mathcal{K}_X$ correct some common false beliefs:

(1) $\mathcal{K}_X$ *cannot* be defined as the sheafification of the presheaf

$$U \mapsto \operatorname{Frac} \mathcal{O}_X(U) = \mathrm{R}^{-1}_{\mathcal{O}_X(U)} \mathcal{O}_X(U).$$

In fact, this may even fail to be a presheaf! There may be a regular element $f \in \mathrm{R}_{\mathcal{O}_X(X)} \subset \mathcal{O}_X(X)$ such that $f|_U \in \mathcal{O}_X(U) \setminus \mathrm{R}_{\mathcal{O}_X(U)}$ i.e. such that $f|_U$ is a zero-divisor.

(2) The identity $\mathcal{K}_{X,x} = \operatorname{Frac} \mathcal{O}_{X,x}$ fails in general!

(3) The identity $\mathcal{K}_X(\operatorname{Spec} A) = \operatorname{Frac} A$ in (7.2.2) fails in general!

Notation 7.2.4. We set, for convenience,

$$\operatorname{Div}(X) = \mathrm{H}^0(X, \mathcal{K}_X^\times / \mathcal{O}_X^\times).$$

We use additive notation for this abelian group.

Definition 7.2.5 (Cartier divisors)**.** Let X be a scheme. A *Cartier divisor* on X is a section $D \in \mathrm{Div}(X)$. A Cartier divisor is *principal* if it lies in the image of the natural group homomorphism

$$\mathrm{div}\colon \mathrm{H}^0(X, \mathcal{K}_X^\times) \to \mathrm{Div}(X).$$

Two Cartier divisors D_1, D_2 are *linearly equivalent* if $D_1 - D_2 = \mathrm{div}(f)$ for some element $f \in \mathrm{H}^0(X, \mathcal{K}_X^\times)$.

Definition 7.2.6 (Cartier divisor class group)**.** The group

$$\mathrm{CaCl}(X) = \mathrm{Div}(X)/\,\mathrm{im}(\mathrm{div})$$

of Cartier divisors modulo linear equivalence is called the *Cartier divisor class group*.

Remark 7.2.7. Let $D \in \mathrm{Div}(X)$ be a Cartier divisor. Then D can be represented by a tuple $(U_i, f_i)_{i\in I}$ where $X = \bigcup_{i \in I} U_i$ is an open cover, $f_i \in \mathrm{H}^0(U_i, \mathcal{K}_X^\times)$ and

$$\frac{f_i}{f_j} \in \mathscr{O}_X^\times(U_i \cap U_j) \tag{7.2.3}$$

for all $(i, j) \in I \times I$. Two tuples $(U_i, f_i)_{i\in I}$ and $(V_j, g_j)_{j\in J}$ define the same Cartier divisor precisely when

$$f_i g_j^{-1} \in \mathscr{O}_X^\times(U_i \cap V_j) \tag{7.2.4}$$

for every $(i, j) \in I \times J$. The (additive) group law on $\mathrm{Div}(X)$ can be understood as follows. If D_1 is represented by $(U_i, f_i)_{i\in I}$ and D_2 is represented by $(V_j, g_j)_{j\in J}$, then $D_1 + D_2 \in \mathrm{Div}(X)$ is represented by $(U_i \cap V_j, f_i g_j)_{I\times J}$.

Definition 7.2.8 (Effective Cartier divisor)**.** A Cartier divisor $D \in \mathrm{Div}(X)$ is *effective* when it can be represented as $(U_i, f_i)_{i\in I}$, where $f_i \in \mathrm{R}_{\mathscr{O}_X(U_i)}$ is a regular element for every $i \in I$.

We use the notation '$D \geq 0$' to denote an effective Cartier divisor.

Remark 7.2.9. A Cartier divisor $D \in \mathrm{Div}(X)$ is principal when it can be represented as (X, f), for some $f \in \mathrm{H}^0(X, \mathcal{K}_X^\times)$.

Definition 7.2.10 (Support of Cartier divisor)**.** The *support* of a Cartier divisor $D \in \mathrm{Div}(X)$ is the subset

$$\mathrm{Supp}\, D = \left\{\, x \in X \;\middle|\; D_x \neq 1 \text{ in } \mathcal{K}_{X,x}^\times / \mathscr{O}_{X,x}^\times \,\right\} \subset X.$$

Exercise 7.2.11. Show that $\mathrm{Supp}\, D \subset X$ is closed. (Hint: show it agrees with the set of points $x \in X$ such that $\mathscr{O}_X(D)_x \neq \mathscr{O}_{X,x}$.

Construction 7.2.12. Let $D \in \mathrm{Div}(X)$ be a Cartier divisor. Then, there is an associated invertible sheaf $\mathscr{O}_X(D)$ on X, defined as follows. If D is represented by $(U_i, f_i)_{i\in I}$, we set

$$\mathscr{O}_X(D)|_{U_i} = f_i^{-1} \cdot \mathscr{O}_X|_{U_i}, \quad i \in I.$$

This glues to a sheaf $\mathscr{O}_X(D)$ by Equation (7.2.3) and is well-defined, i.e. independent on the tuple $(U_i, f_i)_{i\in I}$, thanks to Equation (7.2.4). By design, there is an injection $\mathscr{O}_X(D) \hookrightarrow \mathcal{K}_X$, and a natural identification

$$\mathscr{O}_X(-D) = \mathscr{O}_X(D)^\vee = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X(D), \mathscr{O}_X),$$

sending a local generator f, say over the trivialising open $U \subset X$, to the section $\ell_f \in \operatorname{Hom}(\mathscr{O}_X(D)|_U, \mathscr{O}_U)$ sending $f^{-1} \mapsto 1$.

PROPOSITION 7.2.13. *Sending $D \mapsto \mathscr{O}_X(D)$ establishes a 1-1 correspondence between Cartier divisors and invertible subsheaves of $\mathcal{K}_X$. Furthermore, we have an isomorphism $\mathscr{O}_X(D_1-D_2) \cong \mathscr{O}_X(D_1)\otimes_{\mathscr{O}_X} \mathscr{O}_X(D_2)^\vee$, and two Cartier divisors D_1, D_2 are linearly equivalent if and only if $\mathscr{O}_X(D_1) \cong \mathscr{O}_X(D_2)$ as abstract sheaves.*

Proof. The inverse of $D \mapsto \mathscr{O}_X(D)$ is the following. Start from an invertible subsheaf $\mathscr{L} \subset \mathcal{K}_X$, fix an open covering $X = \bigcup_i U_i$ trivialising $\mathscr{L}$, and choose a local generator g_i of $\mathscr{L}|_{U_i}$. Define $f_i = g_i^{-1}$. Define D to be the Cartier divisor represented by $(U_i, f_i)_i$.

If D_1 is represented by $(U_i, f_i)_i$ and D_2 by $(V_j, g_j)_j$, then $f_i^{-1} g_j$ is a local generator for $\mathscr{O}_X(D_1 - D_2)$ on $U_i \cap V_j$. We have

$$\begin{aligned}(\mathscr{O}_X(D_1)\otimes_{\mathscr{O}_X} \mathscr{O}_X(D_2)^\vee)|_{U_i\cap V_j} &= \mathscr{O}_X(D_1)|_{U_i\cap V_j} \otimes_{\mathscr{O}_{U_i\cap V_j}} \mathscr{O}_X(D_2)^\vee|_{U_i\cap V_j} \\ &= f_i^{-1}|_{U_i\cap V_j}\cdot \mathscr{O}_{U_i\cap V_j} \otimes_{\mathscr{O}_{U_i\cap V_j}} g_j|_{U_i\cap V_j}\cdot \mathscr{O}_{U_i\cap V_j} \\ &= f_i^{-1}|_{U_i\cap V_j} g_j|_{U_i\cap V_j}\cdot \mathscr{O}_{U_i\cap V_j} \\ &= \mathscr{O}_X(D_1-D_2)|_{U_i\cap V_j},\end{aligned} \tag{7.2.5}$$

as required.

At this point, it is enough to prove that $D = D_1 - D_2$ is principal if and only if $\mathscr{O}_X(D) \cong \mathscr{O}_X$. If $D = \operatorname{div}(f)$ for some $f \in \mathrm{H}^0(X, \mathcal{K}_X^\times)$, then $\mathscr{O}_X(D)$ is *free*, generated by f^{-1}, so the mapping $1 \mapsto f^{-1}$ gives an isomorphism $\mathscr{O}_X \xrightarrow{\sim} \mathscr{O}_X(D)$. Conversely, given one such isomorphism $\eta\colon \mathscr{O}_X \xrightarrow{\sim} \mathscr{O}_X(D)$, the element $\eta_X(1) = g \in \mathrm{H}^0(X, \mathscr{O}_X(D)) \subset \mathrm{H}^0(X, \mathcal{K}_X^\times)$ yields $D = \operatorname{div}(g^{-1})$. □

Definition 7.2.14 (Locally principal subscheme)**.** Let X be a scheme, $Z \hookrightarrow X$ a closed subscheme. Then, we say that Z is locally principal if for every $x \in Z$ there is an affine open neighbourhood $x \in U = \operatorname{Spec} A \hookrightarrow X$ such that the closed immersion $U \cap Z \hookrightarrow U$ corresponds to $A \twoheadrightarrow A/fA$, where $f \in A$ is a regular element (i.e. not a zero divisor, i.e. $f \in \mathrm{R}_A$).

COROLLARY 7.2.15. *There are a 1-1 correspondences between the following objects:*

- *effective Cartier divisors,*
- *invertible ideal sheaves,*

- *locally principal closed subschemes of X.*

Proof. We have $D \geq 0$ if and only if $\mathscr{O}_X(-D) \subset \mathscr{O}_X$. This proves the correspondence between the first two items. On the other hand, D is effective if and only if the ideal sheaf cutting out the corresponding closed subscheme is locally generated by a regular element. □

Notation 7.2.16. If D is an effective Cartier divisor on X, we may abuse notation and write '$D \subset X$', which makes sense by Corollary 7.2.15.

THEOREM 7.2.17. *The association $D \mapsto \mathscr{O}_X(D)$ descends to an injective group homomorphism*

$$\rho_X \colon \mathrm{CaCl}(X) \hookrightarrow \mathrm{Pic}(X),$$

whose image consists of isomorphism classes of invertible sheaves on X that are subsheaves of $\mathcal{K}_X$.

Proof. Results immediately from Proposition 7.2.13. □

COROLLARY 7.2.18. *If X has finitely many irreducible components and is reduced, then ρ_X is an isomorphism.*

Proof. We need to verify that every invertible sheaf $\mathscr{L}$ on X is a subsheaf of $\mathcal{K}_X$. Let $\xi_1, \ldots, \xi_r \in X$ be the finitely many generic points of the irreducible components of X. Each ξ_j lies in an open subset $U_j \subset X$ such that

$$U_j \subset Z_j \setminus \bigcup_{k \neq j} Z_k,$$

where $Z_\ell = \overline{\{\xi_\ell\}}$ for all $\ell = 1, \ldots, r$. After shrinking each U_j is necessary, we may assume $\mathscr{L}|_{U_j} = \mathscr{O}_{U_j}$. Consider the open subset

$$U = \bigcup_{1 \leq j \leq r} U_j \xhookrightarrow{\iota} X.$$

Then, we have a morphism of sheaves

$$\mathscr{L} \xrightarrow{\alpha} \iota_*\iota^*\mathscr{L} = \iota_*\mathscr{O}_U \hookrightarrow \iota_*\mathcal{K}_U \xrightarrow{\sim} \mathcal{K}_X$$

where α is injective (see [12, Ch. 7, Lemma 1.9] for a proof), the identity follows by freeness of $\iota^*\mathscr{L} = \mathscr{L}|_U$, and the last isomorphism is a consequence of [12, Ch. 7, Prop. 1.15]. □

In fact, an easier proof is available for integral schemes.

COROLLARY 7.2.19. *If X is integral, then ρ_X is an isomorphism.*

Proof. We need to verify that every invertible sheaf is a subsheaf of $\mathcal{K}_X$. With the current assumptions, we have $\mathcal{K}_X = \underline{K}_X$, the constant sheaf on X with values in $K = K(X) = \mathcal{O}_{X,\xi}$. Let $\mathcal{L}$ be an invertible sheaf on X. The canonical map

$$\mathcal{L} \to \mathcal{L} \otimes_{\mathcal{O}_X} \underline{K}_X \tag{7.2.6}$$

is injective since $\mathcal{L}$ is locally free. On the other hand, let $U \subset X$ be any open subset such that $\mathcal{L}|_U = \mathcal{O}_U$. Then,

$$(\mathcal{L} \otimes_{\mathcal{O}_X} \underline{K}_X)|_U = \mathcal{L}|_U \otimes_{\mathcal{O}_U} \underline{K}_X|_U = \underline{K}_X|_U = \underline{K}_U.$$

But on an irreducible scheme, if a sheaf restricts to a constant sheaf on an open covering, then it is a constant sheaf. Thus $\mathcal{L} \otimes_{\mathcal{O}_X} K_X = K_X$, and $\mathcal{L}$ acquires the structure of a ref subsheaf of $\underline{K}_X$ via the injection (7.2.6). □

Exercise 7.2.20. Let $D \in \mathrm{Div}(X)$ be a Cartier divisor on a scheme X. Let $U \subset X$ be an open subset. Show that

$$\Gamma(U, \mathcal{O}_X(D)) = \left\{ f \in \mathrm{H}^0(U, \mathcal{K}_X^\times) \,\middle|\, \mathrm{div}(f) + D|_U \geq 0 \right\},$$

where $D|_U$ is the image of $D \in \mathrm{Div}(X)$ along the natural restriction map $\mathrm{Div}(X) \to \mathrm{H}^0(U, \mathcal{K}_X^\times / \mathcal{O}_X^\times)$.

7.2.2 Weil divisors

Throughout this section, we assume X is a noetherian integral scheme, and we denote by K its field of rational functions.

Schemes playing a special role in the theory of divisors are those defined as follows.

Definition 7.2.21 (Locally factorial, normal)**.** An integral scheme X is *locally factorial* (resp. *normal*) if $\mathcal{O}_{X,x}$ is a UFD (resp. integrally closed in its field of fractions) for every $x \in X$.

Definition 7.2.22 (Regular in codimension 1)**.** A point x on a scheme X is a *point of codimension d* if $\mathcal{O}_{X,x}$ has Krull dimension d. A scheme X is called *regular in codimension* 1 if every codimension 1 point $x \in X$ is regular.

Remark 7.2.23. A locally factorial scheme is normal. A normal scheme is regular in codimension 1. See Appendix B.6 for more details on normal schemes.

Definition 7.2.24 (Weil divisor)**.** Let X be a noetherian integral scheme. A *prime divisor* on X is a closed integral subscheme $Y \subset X$ of codimension 1. Form the free abelian group $\mathcal{Z}^1(X)$ generated by prime divisors. An element of $\mathcal{Z}^1(X)$ is called a *Weil divisor*, and is thus represented as a finite formal sum

$$D = \sum_{i=1}^{s} n_i Y_i$$

where $n_i \in \mathbb{Z}$ and $Y_i \subset X$ are prime divisors. A Weil divisor $D = \sum_i n_i Y_i$ is called *effective* (or *positive*) if $n_i \geq 0$ for all i.

We write '$D \geq 0$' to say that $D \in \mathcal{Z}^1(X)$ is effective, and we write '$D \geq E$' to say that $D - E \geq 0$.

Clearly, a prime divisor $Y \subset X$ corresponds to a codimension 1 point on X. Under this correspondence, $Y = \overline{\{\xi_Y\}}$.

Remark 7.2.25. Any Weil divisor D is the difference $D_1 - D_2$ of two effective Weil divisors: it is enough to separate the positive coefficients from the negative ones.

Example 7.2.26. Let $C \subset \mathbb{P}^2_{\mathbf{k}}$ be an irreducible plane curve. Then $C \in \mathcal{Z}^1(\mathbb{P}^2_{\mathbf{k}})$.

Example 7.2.27. Let $x_1, \ldots, x_s \in C$ be closed points on a curve C. Then $\sum_i n_i x_i \in \mathcal{Z}^1(C)$, for all $n_i \in \mathbb{Z}$.

Example 7.2.28. On $\mathbb{P}^n_{\mathbf{k}}$, a Weil divisor is of the form $\sum_i n_i \mathrm{V}_+(f_i)$, where $f_i \in \mathbf{k}[x_0, \ldots, x_n]$ are irreducible homogeneous polynomials.

Assumption 7.2.1. From now on, in this section we assume X is noetherian, integral, separated and regular in codimension 1. The key example is that of a smooth, connected, projective variety over an algebraically closed field.

Fix a prime divisor $Y \subset X$, corresponding to the codimension 1 point $\xi_Y \in X$. First of all, note that the local ring $(\mathscr{O}_{X,\xi_Y}, \mathfrak{m}_Y)$ is a DVR. Moreover, we have $K = \operatorname{Frac} \mathscr{O}_{X,\xi_Y}$ (on an arbitrary integral scheme, we have $K(X) = \operatorname{Frac} \mathscr{O}_{X,x}$ for every $x \in X$). By separatedness of X, the integral closed subscheme $Y \subset X$ determines and is determined by a discrete valuation

$$v_Y : K \to \mathbb{Z} \cup \{\infty\}, \quad f \mapsto v_Y(f),$$

with associated valuation ring precisely $\mathscr{O}_{X,\xi_Y}$. If $a \in \mathscr{O}_{X,\xi_Y} \setminus \{0\} \subset K^\times$, the number $v_Y(a)$ is 0 precisely when $a \in \mathscr{O}^\times_{X,\xi_Y}$, and is strictly positive when $a \in \mathfrak{m}_Y$ (the maximal ideal of $\mathscr{O}_{X,\xi_Y}$). More precisely, $v_Y(a) = n$ means that $a \in \mathfrak{m}_Y^n \setminus \mathfrak{m}_Y^{n+1}$, and we say that a has a *zero of order n* along Y. Equivalently (but more algebraically), one has

$$v_Y(a) = \operatorname{length}_R R/aR, \quad R = \mathscr{O}_{X,\xi_Y}.$$

For a rational function $f = a/b \in K^\times$, one has $v_Y(f) = v_Y(a) - v_Y(b)$. If $v_Y(f) < 0$, one says that f has a *pole of order* $-v_Y(f)$ along Y.

Example 7.2.29. Consider $X = \operatorname{Spec} \mathbb{Z}$, the prime divisor $Y = \mathrm{V}(5) \subset X$ and the meromorphic function $f = 45/4$. Then $v_Y(f) = v_Y(45) - v_Y(4) = 1 - 0 = 1$ (so f vanishes with order 1 along Y), and $v_Y(8/50) = 0 - 2 = -2$ (so 8/50 has a pole of order 2 along Y), whereas if $Y' = \mathrm{V}(3)$ we have $v_{Y'}(45/4) = 2 - 0 = 2$ since 3 appears in the prime factorisation of $45 = 3^2 \cdot 5$ with multiplicity 2.

LEMMA 7.2.30. *Let X be a scheme satisfying Assumption 7.2.1. Fix $f \in K^\times$. Then $v_Y(f) = 0$ for all but finitely many prime divisors $Y \subset X$.*

Proof. Fix $U = \operatorname{Spec} A \subset X$ such that f is regular on U. Then

$$\left\{ \text{prime divisors } Y \subset X \,\middle|\, v_Y(f) \neq 0 \right\}$$

is contained in the set of irreducible components of $X \setminus U$ and is therefore a finite set. □

We now introduce the notion of principal Weil divisor, analogously to the case of Cartier divisors. Let $f \in K^\times$ be a nonzero rational function on X. Define

$$(f) = \sum_Y v_Y(f) Y \in \mathcal{Z}^1(X).$$

This is the *principal Weil divisor* associated to f. By the properties of valuations, we have

$$(f/g) = (f) - (g),$$

thus sending $f \mapsto (f)$ defines a group homomorphism

$$\gamma_X \colon K^\times \to \mathcal{Z}^1(X),$$

to be thought of as the analogue of the homomorphism $\operatorname{div}\colon \mathrm{H}^0(X, \mathcal{K}_X^\times) \to \operatorname{Div}(X)$ for Cartier divisors. We have a subgroup

$$\{\text{principal Weil divisors}\} = \operatorname{im}(\gamma_X) \subset \mathcal{Z}^1(X).$$

Definition 7.2.31 (Divisor class group)**.** We say that two Weil divisors $D_1, D_2 \in \mathcal{Z}^1(X)$ are *linearly equivalent* if $D_1 - D_2 = (f)$ for some $f \in K^\times$. The quotient

$$\operatorname{Cl}(X) = \mathcal{Z}^1(X)/\operatorname{im}(\gamma_X)$$

is called the *divisor class group* of X.

In general, the group $\operatorname{Cl}(X)$ is hard to compute. In the case of projective space, it is just a copy of $\mathbb{Z}$, as we now prove.

Definition 7.2.32 (Degree of a Weil divisor on $\mathbb{P}^n_{\mathbf{k}}$)**.** Let $D = \sum_i n_i Y_i \in \mathcal{Z}^1(\mathbb{P}^n_{\mathbf{k}})$ be a Weil divisor on projective n-space over a field $\mathbf{k}$. Then we define the *degree* of D to be

$$\deg D = \sum_i n_i \deg Y_i.$$

This makes sense because each Y_i is a hypersurface, of a well precise degree.

Note that $\deg(D_1 + D_2) = \deg D_1 + \deg D_2$, i.e. this operation is additive, so we get a group homomorphism

$$\deg\colon \mathcal{Z}^1(\mathbb{P}^n_{\mathbf{k}}) \longrightarrow \mathbb{Z}.$$

Example 7.2.33. A hyperplane $H \subset \mathbb{P}^n_{\mathbf{k}}$, i.e. the zero locus of a linear form (e.g. a point on $\mathbb{P}^1_{\mathbf{k}}$, a line in $\mathbb{P}^2_{\mathbf{k}}$), has degree 1. A plane curve $C \subset \mathbb{P}^2_{\mathbf{k}}$ of degree d defines a Weil divisor of degree d.

THEOREM 7.2.34. *Fix a Weil divisor $D = \sum_i n_i Y_i \in \mathcal{Z}^1(\mathbb{P}^n_{\mathbf{k}})$ and the coordinate hyperplane $H = \mathrm{V}_+(x_0) \in \mathcal{Z}^1(\mathbb{P}^n_{\mathbf{k}})$.*

(i) *If $\deg D = d$, then D is linearly equivalent to dH.*

(ii) *If $f \in K^\times$, then $\deg(f) = 0$.*

(iii) *Sending $D \mapsto \deg D$ descends to a group isomorphism*

$$\deg\colon \mathrm{Cl}(\mathbb{P}^n_{\mathbf{k}}) \xrightarrow{\sim} \mathbb{Z}.$$

Proof. We proceed step by step.

(ii) If $g \in \mathbf{k}[x_0,\ldots,x_n]_d$ is a homogeneous polynomial of degree d, then there is a unique factorisation $g = g_1^{n_1}\cdots g_s^{n_s}$. Each g_i cuts out a hypersurface $Y_i \subset \mathbb{P}^n_{\mathbf{k}}$ and we define

$$(g) = \sum_{i=1}^s n_i Y_i \in \mathcal{Z}^1(\mathbb{P}^n_{\mathbf{k}}).$$

The degree of (g) is $\sum_i n_i(\deg g_i) = d$. If $f = g/h \in K^\times$, then g and h must have the same degree. Thus $\deg(f) = 0$.

(i) Write $D = D_1 - D_2$ for effective D_1 and D_2, according to Remark 7.2.25. Set $d_i = \deg D_i$. We may write $D_i = (g_i)$, for $i = 1,2$, for some homogeneous polynomial $g_i \in \mathbf{k}[x_0,\ldots,x_n]_{d_i}$. Clearly we have $d = d_1 - d_2$. And, in fact,

$$D - dH = D_1 - D_2 - dH = (g_1) - (g_2) - d(x_0) = (g_1/x_0^d g_2),$$

as required.

(iii) Follows by combining (i) and (ii) with one another. □

We have shown that, in $\mathbb{P}^n_{\mathbf{k}}$,

two Weil divisors are linearly equivalent if and only if they have the same degree.

A useful characterisation of the vanishing of the divisor class group is the following. Recall that a noetherian domain A is a UFD if and only if every height one prime $\mathfrak{p} \subset A$ is principal. Recall that an integral scheme is normal if all of its local rings $\mathscr{O}_{X,x}$ are integrally closed in their fields of fractions. ref

THEOREM 7.2.35 ([8, Ch. II, Prop. 6.2]). *Let A be a noetherian domain, and set $X = \operatorname{Spec} A$. Then A is a UFD if and only X is normal and $\mathrm{Cl}(X) = 0$.*

Example 7.2.36. The noetherian domain $A = \mathbf{k}[x_1, \ldots, x_n]$ is a UFD. Therefore $\mathrm{Cl}(\mathbb{A}^n_{\mathbf{k}}) = 0$.

PROPOSITION 7.2.37. *Let X be a scheme satisfying Assumption 7.2.1. Let $Z \subset X$ be a proper closed subset, $U = X \setminus Z$ the open complement.*

(a) *Sending $D = \sum_i n_i Y_i \mapsto D|_U = \sum_i n_i (Y_i \cap U)$, where one ignores those i such that $Y_i \cap U = \emptyset$, descends to a surjective group homomorphism*

$$\pi\colon \mathrm{Cl}(X) \twoheadrightarrow \mathrm{Cl}(U),$$

which is an isomorphism as soon as $\mathrm{codim}(Z, X) \geq 2$.

(b) *If Z is irreducible of codimension 1, then there is an exact sequence*

$$\mathbb{Z} \longrightarrow \mathrm{Cl}(X) \xrightarrow{\pi} \mathrm{Cl}(U) \longrightarrow 0$$

where the first map sends $1 \mapsto [Z]$.

Proof. We proceed step by step.

(a) If $Y \subset X$ is a prime divisor, then $Y \cap U$ is either empty or a prime divisor on U. For any $f \in K^\times = K(U)^\times$, the restriction of $(f) = \sum_i m_i Y_i$ to U is $\sum_i m_i (Y_i \cap U) = (f)$, where in the right hand side we view f as a rational function on U. Thus the mapping on Weil divisors preserves linear equivalence, and thus descends to $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$. The resulting map is surjective since every prime divisor $W \subset U$ satisfies $W = \overline{W} \cap U$. If Z has codimension at least 2, the map is an isomorphism since $\mathcal{Z}^1(X)$ and $\mathrm{Cl}(X)$ only depend on codimension 1 subsets of X.

(b) The kernel of π is the set of Weil divisors (classes) whose support is contained in Z. If Z is irreducible, $\ker \pi$ is generated by $1 \cdot [Z]$. □

Example 7.2.38. Let $C \subset \mathbb{P}^2_{\mathbf{k}}$ be an irreducible curve of degree d. Then, using Theorem 7.2.34 (iii), the exact sequence of Proposition 7.2.37 (b) becomes

$$\mathbb{Z} \longrightarrow \mathbb{Z} \xrightarrow{\pi} \mathrm{Cl}(\mathbb{P}^2_{\mathbf{k}} \setminus C) \longrightarrow 0$$

which yields $\mathrm{Cl}(\mathbb{P}^2_{\mathbf{k}} \setminus C) = \mathbb{Z}/d\mathbb{Z}$ since the first map sends $1 \mapsto d$.

Example 7.2.39 (Affine quadric cone)**.** Let $A = \mathbf{k}[x, y, z]/(xy - z^2)$ and set $X = \operatorname{Spec} A \subset \mathbb{A}^3_{\mathbf{k}}$. We now compute $\mathrm{Cl}(X) = \mathbb{Z}/2\mathbb{Z}$ assuming $\mathbf{k}$ does not have characteristic 2. Consider the prime divisor $Y \subset X$ cut out by $z = y = 0$. We have an exact sequence

$$\mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(X \setminus Y) \to 0.$$

We first show $\mathrm{Cl}(X \setminus Y) = 0$. Note that Y is cut out by $y = 0$ alone, *set-theoretically*. We then have

$$(y) = n \cdot Y$$

for some $n \geq 1$. Since $y = 0$ implies $z^2 = 0$, and the image of z generates the maximal ideal $\mathfrak{m}_Y \subset \mathscr{O}_{X,\xi_Y}$, we deduce that $n = 2$, i.e. $(y) = 2Y$ in $\mathscr{Z}^1(X)$. Moreover $X \setminus Y = X \setminus \mathrm{V}(y) = \operatorname{Spec} A_y$, where $A_y = \mathbf{k}[x, y^{\pm 1}, z]/(xy - z^2) = \mathbf{k}[y^{\pm 1}, z]$ since x can be eliminated via the relation $x = y^{-1}z^2$. It follows that A_y is a UFD, thus $\mathrm{Cl}(X \setminus Y) = 0$ by Theorem 7.2.35.

Thus we have a surjection $\mathbb{Z} \to \mathrm{Cl}(X)$ given by $1 \mapsto [Y]$, and we know $2[Y] = 0$ in $\mathrm{Cl}(X)$. To conclude that $\mathrm{Cl}(X) = \mathbb{Z}/2\mathbb{Z}$, it is enough to exclude $[Y] = 0$, i.e. that Y be principal. more det?

Lemma 7.2.40 ([8, Ch. II, Prop. 6.6]). *Let X be a scheme satisfying Assumption 7.2.1. Then $X \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}$ satisfies Assumption 7.2.1, and the association*

$$\sum_i n_i Y_i \mapsto \sum_i n_i \pi_1^{-1}(Y_i)$$

descends to an isomorphism

$$\pi_1^* \colon \mathrm{Cl}(X) \xrightarrow{\sim} \mathrm{Cl}(X \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}})$$

where $\pi_1 \colon X \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}} \to X$ is the first projection.

Example 7.2.41 (Quadric surface)**.** Consider $X = \operatorname{Proj} \mathbf{k}[x, y, z, w]/(xy - zw)$, the quadric surface in $\mathbb{P}^3_{\mathbf{k}}$. We know that $X \cong \mathbb{P}^1_{\mathbf{k}} \times_{\mathbf{k}} \mathbb{P}^1_{\mathbf{k}}$. We have, for $i = 1, 2$, projections $p_i \colon X \to \mathbb{P}^1_{\mathbf{k}}$, and corresponding pullback homomorphisms

$$p_i^* \colon \mathrm{Cl}(\mathbb{P}^1_{\mathbf{k}}) \to \mathrm{Cl}(X).$$

Consider $Z = \mathrm{pt} \times_{\mathbf{k}} \mathbb{P}^1_{\mathbf{k}}$ (the fibre of p_2 over $\mathrm{pt} \in \mathbb{P}^1_{\mathbf{k}}$), so that $X \setminus Z = \mathbb{A}^1 \times_{\mathbf{k}} \mathbb{P}^1_{\mathbf{k}}$. The composition

$$\mathrm{Cl}(\mathbb{P}^1_{\mathbf{k}}) \xrightarrow{p_2^*} \mathrm{Cl}(X) \xrightarrow{\pi} \mathrm{Cl}(X \setminus Z) = \mathrm{Cl}(\mathbb{A}^1_{\mathbf{k}} \times_{\mathbf{k}} \mathbb{P}^1_{\mathbf{k}})$$

is the isomorphism of Lemma 7.2.40, thus p_2^* is injective. The same is then true for p_1^*.

We also have an exact sequence

$$\mathbb{Z} \longrightarrow \mathrm{Cl}(X) \xrightarrow{\pi} \mathrm{Cl}(X \setminus Z) = \mathrm{Cl}(\mathbb{A}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}) \longrightarrow 0$$

where the first map sends $1 \mapsto [Z]$. Now, $\mathbb{Z}$ can be identified with $\mathrm{Cl}(\mathbb{P}^1_{\mathbf{k}})$ by sending $1 \mapsto [\mathrm{pt}]$. After this identification we see that the first map in the above sequence agrees with p_1^*. On the other hand, we have seen that the image of p_2^* maps isomorphically onto $\mathrm{Cl}(\mathbb{A}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}) \cong \mathrm{Cl}(\mathbb{P}^1_{\mathbf{k}}) \cong \mathbb{Z}$, therefore

$$\mathrm{Cl}(X) = \operatorname{im} p_1^* \oplus \operatorname{im} p_2^* = \mathbb{Z} \oplus \mathbb{Z}.$$

Exercise 7.2.42. Show that the ring $A = \mathbf{k}[x, y, z, w]/(xy - zw)$ is integrally closed, but not a UFD.

7.2.3 Relation with the Picard group

Recall that we have defined (cf. Definition 7.1.26) the *Picard group* $\operatorname{Pic}(X)$ of an arbitrary scheme X. It is the abelian group of isomorphism classes of invertible sheaves on X. The group operation is given by tensor product, the identity is the class of $\mathcal{O}_X$, and the inverse of (the class of) $\mathscr{L}$ is (the class of) $\mathscr{L}^\vee = \mathscr{H}om_{\mathcal{O}_X}(\mathscr{L}, \mathcal{O}_X)$. Indeed, there is a canonical isomorphism $\mathscr{L} \otimes_{\mathcal{O}_X} \mathscr{L}^\vee \cong \mathcal{O}_X$ by Lemma 7.1.25.

A noetherian ring A is regular in codimension 1 if all points of codimension at most 1 in $\operatorname{Spec} A$ are regular. In general, for a noetherian ring, we have the implications

$$\text{UFD} \;\Rightarrow\; \text{integrally closed} \;\Rightarrow\; \text{regular in codimension 1}.$$

PROPOSITION 7.2.43. *Let X be an integral, separated, noetherian, locally factorial scheme. Then there is an isomorphism*

$$\delta_X \colon \operatorname{Div}(X) \xrightarrow{\sim} \mathcal{Z}^1(X)$$

between Weil and Cartier divisors. Moreover, the notions of linear equivalence correspond to each other, i.e. the isomorphism δ_X preserves principal divisors. In particular, δ_X descends to an isomorphism

$$\overline{\delta}_X \colon \operatorname{CaCl}(X) \xrightarrow{\sim} \operatorname{Cl}(X)$$

Proof. We refer to [8, Ch. II, Prop. 6.11] for a complete proof. Here we just define the map δ_X. Consider a Cartier divisor $D \in \operatorname{Div}(X)$ represented by $(U_i, f_i)_{i\in I}$, with $f_i \in \Gamma(U_i, \mathcal{K}_X^\times) = K^\times$. For an arbitrary prime divisor $Y \subset X$, fix an integer i such that $Y \cap U_i \neq \emptyset$. Since $f_i/f_j \in \mathcal{O}_X^\times(U_i \cap U_j)$ for any other such index j, we have $0 = v_Y(f_i/f_j) = v_Y(f_i) - v_Y(f_j)$, thus the Weil divisor

$$\delta_X(D) = \sum_Y v_Y(f_i) Y$$

is well-defined. It is also clear that principal Cartier divisors get sent to principal Weil divisors, so that $\overline{\delta}_X$ is well-defined as well. □

COROLLARY 7.2.44. *Let X be an integral, separated, noetherian, locally factorial scheme. Then*

$$\operatorname{Cl}(X) \cong \operatorname{Pic}(X).$$

Proof. Combine Proposition 7.2.43 and Corollary 7.2.19 with one another. □

Example 7.2.45. The quadric cone X of Example 7.2.39 is not locally factorial. Indeed, $\operatorname{Cl}(X) = \mathbb{Z}/2\mathbb{Z}$, but $\operatorname{Pic}(X) = 0$.

Example 7.2.46. For $X = \mathbb{P}^n_{\mathbf{k}}$, we have

$$\mathbb{Z} \cong \mathrm{Cl}(\mathbb{P}^n_{\mathbf{k}}) \cong \mathrm{CaCl}(\mathbb{P}^n_{\mathbf{k}}) \cong \mathrm{Pic}(\mathbb{P}^n_{\mathbf{k}}).$$

This chain of isomorphisms makes the integer $d \in \mathbb{Z}$ correspond to the isomorphism class of the line bundle $\mathscr{O}_{\mathbb{P}^n_{\mathbf{k}}}(d)$, since a hyperplane $H \in \mathrm{Cl}(\mathbb{P}^n_{\mathbf{k}})$ corresponds to the class of $\mathscr{O}_{\mathbb{P}^n_{\mathbf{k}}}(1)$.

7.3 The sheaf of relative differentials

In this section we attach to any morphism of schemes $f: X \to Y$ a quasicoherent sheaf

$$\Omega_{X/Y} \in \mathsf{QCoh}(X),$$

sometimes also denoted Ω_f. It will be coherent as soon as f will be of finite type over a noetherian base scheme Y.

7.3.1 The affine case

We fix a ring A and an A-algebra $A \to B$. The reader should think of A as the analogue of real constants in first year Calculus.

Definition 7.3.1 (Derivation)**.** Let M be a B-module. An *A-derivation of B into M* is an A-linear map $\mathrm{d}\colon B \to M$ such that the Leibniz rule

$$\mathrm{d}(b_1 b_2) = b_1 \mathrm{d}b_2 + b_2 \mathrm{d}b_1$$

holds for all $b_1, b_2 \in B$, and that $\mathrm{d}(a) = 0$ for all $a \in A$ (by which we mean that d vanishes on the image of $A \to B$).

We set

$$\mathrm{Der}_A(B, M) = \{\, A\text{-derivations of } B \text{ into } M \,\}.$$

It is clearly an A-module via

$$a \cdot \mathrm{d}\colon B \to M, \quad b \mapsto a \cdot \mathrm{d}b.$$

Definition 7.3.2 (Module of differentials)**.** A *module of relative differentials of B over A* is a pair $(\Omega_{B/A}, \mathrm{d}_{B/A})$ where $\Omega_{B/A}$ is a B-module and $\mathrm{d}_{B/A} \in \mathrm{Der}_A(B, \Omega_{B/A})$ satisfies the following universal property: for any other pair (M, d) with $M \in \mathsf{Mod}_B$ and $\mathrm{d} \in \mathrm{Der}_A(B, M)$, there is a unique B-linear map $\phi_{(M,\mathrm{d})}\colon \Omega_{B/A} \to M$ such that the diagram

$$\begin{tikzcd} B \arrow[r, "\mathrm{d}"] \arrow[d, "\mathrm{d}_{B/A}"'] & M \\ \Omega_{B/A} \arrow[ur, "\phi_{(M,\mathrm{d})}"'] & \end{tikzcd}$$

commutes. We shall write ‘d’ instead of ‘$\mathrm{d}_{B/A}$’ most of the time.

In other words, a module of relative differentials of B over A as in the definition provides an isomorphism of A-modules

$$\mathrm{Hom}_{\mathsf{Mod}_B}(\Omega_{B/A}, M) \xrightarrow{\sim} \mathrm{Der}_A(B, M), \quad \phi \mapsto \phi \circ \mathrm{d}_{B/A},$$

functorially in $M \in \mathsf{Mod}_B$. As anything that is defined via a universal property, $(\Omega_{B/A}, \mathrm{d}_{B/A})$ either does not exist or is unique up to unique isomorphism.

Existence of the module of relative differentials of B over A is easy: just take the quotient of B-modules

$$\Omega_{B/A} = \frac{\bigoplus_{b\in B} B\cdot \mathrm{d}b}{\langle \mathrm{d}a,\, \mathrm{d}(b_1+b_2)-\mathrm{d}b_1-\mathrm{d}b_2,\, \mathrm{d}(b_1b_2)-b_1\mathrm{d}b_2-b_2\mathrm{d}b_1\rangle},$$

where '$\mathrm{d}b$' are pure symbols, along with the natural map $\mathrm{d}_{B/A}\colon B\to\Omega_{B/A}$ sending $b\in B$ to the class $[\mathrm{d}b]$. This is a derivation of B into $\Omega_{B/A}$ due to the defining relations.

An alternative description of $(\Omega_{B/A}, \mathrm{d}_{B/A})$ is as follows. Consider

$$\varphi\colon B\otimes_A B \twoheadrightarrow B, \qquad b_1\otimes_A b_2 \mapsto b_1b_2,$$

namely the surjection corresponding to the diagonal closed immersion

$$\Delta\colon \operatorname{Spec} B \hookrightarrow \operatorname{Spec} B \times_{\operatorname{Spec} A} \operatorname{Spec} B.$$

Let $I = \ker\varphi \subset B\otimes_A B$. The rule

$$b\cdot(b_1\otimes_A b_2) = bb_1\otimes_A b_2$$

sets up a B-module structure on $B\otimes_A B$. We have that

$$I/I^2 = I\otimes_{B\otimes_A B} B$$

is then a B-module as well, via

$$b\cdot(h+I^2) = (b\cdot h)+I^2, \qquad h\in I\subset B\otimes_A B.$$

The pair $(I/I^2, \mathrm{d})$ defined by the map

$$\begin{aligned} B &\xrightarrow{\quad\mathrm{d}\quad} I/I^2 \\ b &\longmapsto (1_B\otimes_A b - b\otimes_A 1_B)+I^2 \end{aligned}$$

satisfies the universal property of the module of differentials, and therefore $(I/I^2,\mathrm{d}) = (\Omega_{B/A}, \mathrm{d}_{B/A})$.

Example 7.3.3. If $B = A[x_1,\dots,x_n]$, then

$$\Omega_{B/A} = \bigoplus_{i=1}^n B\cdot \mathrm{d}x_i. \tag{7.3.1}$$

Indeed, for any A-derivation $\mathrm{d}'\colon B \to M$ into a B-module M, the Leibniz rule forces

$$\mathrm{d}' f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \mathrm{d}' x_i,$$

which shows that d' is completely determined by $\mathrm{d}' x_1, \ldots, \mathrm{d}' x_n$. In particular,

$$\mathrm{d}_{B/A}\colon B \to \bigoplus_{i=1}^{n} B \cdot \mathrm{d}x_i, \quad f \mapsto \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \mathrm{d}x_i$$

has the universal property of module of relative differentials, thus (7.3.1) follows.

The following are important properties of the module of differentials, in the form of exercises. See [12, Ch. 6, Prop. 1.8] if you need a hint.

Exercise 7.3.4. Prove that if $A \to B$ is either surjective or a localisation, then $\Omega_{B/A} = 0$.

Exercise 7.3.5 (Base change for Ω)**.** If $A \to A'$ is another A-algebra and $B' = B \otimes_A A'$, then $\Omega_{B'/A'} \cong \Omega_{B/A} \otimes_B B'$ canonically, as B'-modules.

Exercise 7.3.6. Let $\phi\colon B \to C$ be a morphism of A-algebras, so that we have a sequence of ring homomorphisms $A \to B \to C$. Use the universal property of the module of differentials to construct C-linear maps

$$\Omega_{C/A} \to \Omega_{C/B}, \qquad \Omega_{B/A} \otimes_B C \to \Omega_{C/A},$$

the second map sending $[\mathrm{d}_{B/A} b] \otimes c \mapsto c \cdot [\mathrm{d}_{C/A} \phi(b)]$. Prove that these maps fit into an exact sequence

$$\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0.$$

When $B \twoheadrightarrow C$ is surjective with kernel $I \subset B$, prove that there is an exact sequence of C-modules

$$I/I^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0,$$

where the first map sends $b + I^2 \mapsto [\mathrm{d}b] \otimes 1_C$.

Exercise 7.3.7 (Localisation for Ω)**.** Prove that if $S \subset B$ is a multiplicative subset, then $\Omega_{S^{-1}B/A} = S^{-1}\Omega_{B/A} = \Omega_{B/A} \otimes_B S^{-1}B$ as B-modules. (Hint: combine the first exact sequence of Exercise 7.3.6, taken with $C = S^{-1}B$, and Exercise 7.3.4 with one another).

Exercise 7.3.8. Suppose B is finitely generated as an A-algebra. Then $\Omega_{B/A}$ is a finitely generated B-module.

Example 7.3.9. Let $f \in B = A[x_1, \ldots, x_n]$, and set $C = B/fB$. Then

$$\Omega_{C/A} = \frac{\bigoplus_{1 \le i \le n} C \cdot \mathrm{d}x_i}{C \cdot \mathrm{d}f}, \qquad \mathrm{d}f = \sum_{1 \le i \le n} \frac{\partial f}{\partial x_i} \mathrm{d}x_i.$$

Example 7.3.10. Let A be a ring, and consider the localisation $B = A[t]_t = A[t, t^{-1}]$. Then $\Omega_{B/A} = (\Omega_{A[t]/A})_t = (A[t]\cdot \mathrm{d}t)_t = A[t]_t \cdot \mathrm{d}t$ thanks to Exercise 7.3.7 and Example 7.3.3. But also $\mathrm{d}t^{-1}$ is a free generator of $\Omega_{B/A}$. The Leibniz rule yields

$$0 = \mathrm{d}(1) = \mathrm{d}\big(t t^{-1}\big) = t\,\mathrm{d}t^{-1} + t^{-1}\mathrm{d}t,$$

which gives the relation

$$\mathrm{d}t = -t^2\mathrm{d}t^{-1}.$$

7.3.2 The global case

There are two ways to define the *sheaf of relative differentials*

$$\Omega_{X/Y} = \Omega_f \in \mathsf{QCoh}(X)$$

attached to a morphism of schemes $f : X \to Y$. The first way is to prove that the construction in the previous section sheafifies, namely that for any affine opens $V \subset Y$ and $U \subset X$, such that $f(U) \subset V$, the sheaves

$$(\Omega_{\mathscr{O}_X(U)/\mathscr{O}_Y(V)})^{\sim}$$

glue. We refer to [12, Ch. 6, Prop. 1.17] for a proof of this fact. With this approach, it is clear that $\Omega_{X/Y}$ is quasicoherent. Granting this for the moment, one can prove the following.

LEMMA 7.3.11. *If $f : X \to Y$ is of finite type over a noetherian scheme Y, then $\Omega_{X/Y}$ is coherent.*

Proof. Follows from Exercise 7.3.8. □

The derivations $\mathrm{d}_{B/A} : B \to \Omega_{B/A}$ glue to a global morphism of sheaves of abelian groups

$$\mathrm{d}_{X/Y} : \mathscr{O}_X \to \Omega_{X/Y},$$

which is a derivation on every stalk. Note that by Exercise 7.3.5, if one has a cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

then there is a relation

$$\Omega_{X'/Y'} \cong v^*\Omega_{X/Y}.$$

Alternatively, to construct $\Omega_{X/Y}$ we can do the following. The diagonal morphism $\Delta_f : X \to X \times_Y X$ is a locally closed immersion (a closed immersion if f is separated, which we may assume for simplicity), i.e. there is an open subscheme $i\colon U \hookrightarrow X \to X\times_Y X$ such that Δ_f factors as a closed immersion $X \hookrightarrow U$ followed by i. Let $\mathscr{I}\subset \mathscr{O}_U$ be the ideal sheaf of $X\hookrightarrow U$. Then define

$$\Omega_{X/Y} = \Delta_f^*(\mathscr{I}/\mathscr{I}^2).$$

Note that this is compatible, via the $\sim$ construction, with the previous approach. Indeed, fix affine opens $V = \operatorname{Spec} A \subset Y$ and $U = \operatorname{Spec} B \subset X$, such that $f(U)\subset V$, so that the sheaf $\Omega_{U/V}$ is well-defined, as well as the module $\Omega_{B/A}$. We have the affine open

$$U\times_V U = \operatorname{Spec}(B\otimes_A B)\subset X\times_Y X$$

and the closed subscheme $\Delta_f(X)\cap(U\times_V U)\subset U\times_V U$ corresponds to the surjection

$$B\otimes_A B \twoheadrightarrow B, \quad b_1\otimes b_2 \mapsto b_1 b_2,$$

whose kernel $I\subset B\otimes_A B$ is naturally a B-submodule of $B\otimes_A B$ (here, view $B\otimes_A B$ as a B-module via $b\cdot(b_1\otimes b_2) = bb_1\otimes b_2$). Thus $I/I^2 = I\otimes_{B\otimes_A B} B$ is a B-module as well. The map

$$\mathrm{d}_{B/A}\colon B\to I/I^2, \quad b\mapsto 1\otimes b - b\otimes 1 \ (\operatorname{mod} I^2)$$

makes the pair $(I/I^2, \mathrm{d}_{B/A})$ into a module of relative differentials for B over A. It follows that

$$\Omega_{U/V} = (I/I^2)^{\sim} = (\Omega_{B/A})^{\sim}.$$

Example 7.3.12. Let A be a ring, and consider $X=\mathbb{P}^1_A \to \operatorname{Spec} A = Y$. We now compute $\Omega_{X/Y}$. Consider the covering $X = \mathrm{D}_+(x_0)\cup \mathrm{D}_+(x_1)$ where $\mathrm{D}_+(x_0) = \operatorname{Spec} A[x_1/x_0]$ and $\mathrm{D}_+(x_1) = \operatorname{Spec} A[x_0/x_1]$. Set $t_i = x_j/x_i$, with $\{i,j\} = \{0,1\}$. We have the two free modules

$$\Omega_{X/Y}\big|_{\mathrm{D}_+(x_i)} = \Omega_{\mathrm{D}_+(x_i)/Y} = (A[t_i]\cdot \mathrm{d}t_i)^{\sim},$$

thanks to the combination of Exercise 7.3.5 and Example 7.3.3. But then these two modules are also free on the intersection $U = \mathrm{D}_+(x_0)\cap \mathrm{D}_+(x_1) = \operatorname{Spec} A[t_0,t_1] = \operatorname{Spec} A[t_0,t_0^{-1}]$, where the module $\Omega_{X/Y}|_U = \Omega_{U/Y} = \Omega_{A[t_0,t_0^{-1}]/A}$ is generated by $\mathrm{d}t_0$, as well as by $\mathrm{d}t_1$. By Example 7.3.10, we have

$$\mathrm{d}(t_0) = -t_0^2\mathrm{d}t_1.$$

This shows that

$$\Omega_{X/Y} = \mathscr{O}_X(-2).$$

See Corollary 7.3.18 for a generalisation to $\mathbb{P}^n_A$.

Notation 7.3.13. If X is a scheme over $Y=\operatorname{Spec} A$, we simply write $\Omega_{X/A}$ instead of $\Omega_{X/Y}$. If A is a field, we simply write Ω_X. If $p>0$ is an integer, we set $\Omega^p_{X/Y} = \wedge^p\Omega_{X/Y}$. This sheaf, called the sheaf of differential p-forms, is locally free as soon as $\Omega_{X/Y}$ is. In particular, in the locally free case, the sheaf $\det\Omega_{X/Y} = \wedge^n \Omega_{X/Y}$ is invertible, if $n = \operatorname{rk}\Omega_{X/Y}$. Finally, we set

$$\mathcal{T}_{X/Y} = \Omega_{X/Y}^\vee = \mathscr{H}om_{\mathscr{O}_X}(\Omega_{X/Y}, \mathscr{O}_X).$$

This is the *relative tangent sheaf* of $X \to Y$. We will write Ω_X and $\mathcal{T}_X$ when $Y = \operatorname{Spec}\mathbf{k}$.

It follows from Example 7.3.3 that

$$\Omega_{\mathbb{A}^n_Y/Y} \cong \mathscr{O}_{\mathbb{A}^n_Y}^{\oplus n} \cong \mathcal{T}_{\mathbb{A}^n_Y/Y}.$$

PROPOSITION 7.3.14 ([12, Ch. 6, Cor. 2.6]). *Let Y be a locally noetherian scheme, $f: X \to Y$ a flat finite type morphism with purely n-dimensional fibres. Then f is smooth if and only if Ω_f is locally free of rank n.*

Example 7.3.15. Let A be a noetherian ring, $X = \mathbb{P}^n_A$. Then $f: X \to \operatorname{Spec} A$ is smooth, hence $\Omega_{X/A}$ is locally free of rank n.

Definition 7.3.16 (Relative canonical bundle)**.** The *relative canonical bundle* of a smooth morphism of schemes $X \to Y$ as in Proposition 7.3.14 is the invertible sheaf

$$\omega_{X/Y} = \det\Omega_{X/Y} = \wedge^n \Omega_{X/Y},$$

where n is the relative dimension of $X \to Y$.

The following theorem is of crucial importance.

THEOREM 7.3.17 (Euler sequence). *Let A be a ring, $n > 0$ an integer. Then there is an exact sequence of locally free sheaves*

$$0 \to \Omega_{\mathbb{P}^n_A/A} \to \mathscr{O}_{\mathbb{P}^n_A}(-1)^{\oplus(n+1)} \to \mathscr{O}_{\mathbb{P}^n_A} \to 0.$$

Proof. See [8, II, Thm. 8.13]. □

In fact, it is easier to define the dual of the Euler sequence (also called the Euler sequence), namely

$$0 \to \mathscr{O}_{\mathbb{P}^n_A} \to \mathscr{O}_{\mathbb{P}^n_A}(1)^{\oplus(n+1)} \to \mathcal{T}_{\mathbb{P}^n_A/A} \to 0.$$

The first map sends

$$f \mapsto (f x_0, \ldots, f x_n)^{\mathrm{t}},$$

whereas the second map sends

$$(\ell_0, \ldots, \ell_n)^{\mathrm{t}} \mapsto \sum_{i=0}^n \ell_i \frac{\partial}{\partial x_i}.$$

We have the following important consequence.

COROLLARY 7.3.18. *Let A be a ring, $X = \mathbb{P}^n_A$. Then $\omega_{X/A} = \mathscr{O}_X(-n-1)$.*

Proof. By taking determinants along the Euler sequence, we find

$$\omega_{X/A} \otimes_{\mathscr{O}_X} \mathscr{O}_X = \mathscr{O}_X(-1)^{\otimes(n+1)} = \mathscr{O}_X(-n-1). \qquad \square$$

LEMMA 7.3.19. *Let $f: X \to Y$ and $g: Y \to S$ be morphisms. Then there is an exact sequence*

$$f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$$

of quasicoherent sheaves on X.

Proof. Follows from the first exact sequence of Exercise 7.3.6. $\square$

If $\iota: X \hookrightarrow Y$ is a closed immersion, then $\Omega_{X/Y} = 0$ by Exercise 7.3.4. So the above sequence becomes

$$\iota^*\Omega_{Y/S} \to \Omega_{X/S} \to 0.$$

What is the kernel? This is not always easy to answer. However, the following lemma is a step forward. And Theorem 7.3.21 is a full answer in the important special case of an immersion of smooth varieties.

LEMMA 7.3.20. *Let $g: Y \to S$ be a morphism, $\iota: X \hookrightarrow Y$ a closed subscheme cut out by an ideal sheaf $\mathscr{I} \subset \mathscr{O}_Y$. Then there is an exact sequence*

$$\mathscr{I}/\mathscr{I}^2 \to \iota^*\Omega_{Y/S} \to \Omega_{X/S} \to 0 \tag{7.3.2}$$

of quasicoherent sheaves on X.

Proof. Follows from the second exact sequence of Exercise 7.3.6. $\square$

Exactness of (7.3.2) on the left is a crucial theme in a branch of Algebraic Geometry called Deformation Theory. One of the most important theorems to remember is the following.

THEOREM 7.3.21. *Let Y be a smooth irreducible $\mathbf{k}$-variety. Let $\iota: X \hookrightarrow Y$ be a closed irreducible subscheme defined by the ideal $\mathscr{I} \subset \mathscr{O}_Y$. Then X is smooth if and only if Ω_X is locally free and the exact sequence (7.3.2) is exact on the left, i.e. we have a short exact sequence*

$$0 \to \mathscr{I}/\mathscr{I}^2 \to \iota^*\Omega_Y \to \Omega_X \to 0. \tag{7.3.3}$$

In this case, $\mathscr{I}/\mathscr{I}^2$ is locally free of rank $c = \operatorname{codim}(X, Y)$.

COROLLARY 7.3.22 (Adjunction Formula). *Let Y be a smooth irreducible $\mathbf{k}$-variety. Let $\iota: X \hookrightarrow Y$ be a closed, irreducible and smooth subscheme of codimension c defined by the ideal $\mathscr{I} \subset \mathscr{O}_Y$. Then*

$$\omega_X = \iota^*\omega_Y \otimes_{\mathscr{O}_X} \wedge^c(\mathscr{I}/\mathscr{I}^2)^\vee.$$

For instance, if $D \subset Y$ is a smooth (effective Cartier) divisor, then

$$\omega_D = (\omega_Y \otimes \mathscr{O}_Y(D))|_D.$$

Proof. Take determinants along (7.3.3). □

Terminology 7.3.23. If $X \hookrightarrow Y$ is a closed subscheme with ideal sheaf $\mathscr{I}$, then $\mathcal{C}_{X/Y} = \mathscr{I}/\mathscr{I}^2$ is called the *conormal sheaf* of the closed immersion. Its dual

$$\mathcal{N}_{X/Y} = \mathscr{H}om_{\mathscr{O}_X}(\mathcal{C}_{X/Y}, \mathscr{O}_X)$$

is called the *normal sheaf.* If both and X and Y are smooth irreducible $\mathbf{k}$-varieties, these are locally free by Theorem 7.3.21, and they are called *conormal bundle* and *normal bundle* respectively.

Exercise 7.3.24. Let $j\colon Y' \hookrightarrow Y$ be a closed immersion. Given a cartesian diagram

$$\begin{tikzcd} X' \arrow[r, hook, "i"] \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, hook, "j"'] & Y \end{tikzcd}$$

show that there is a canonical surjection

$$g^*\mathcal{C}_{Y'/Y} \twoheadrightarrow \mathcal{C}_{X'/X},$$

which is an isomorphism as soon as f is flat.

Example 7.3.25 (Normal bundle of a plane curve)**.** Let $C \subset \mathbb{P}^2_{\mathbf{k}}$ be a smooth plane curve of degree d. Then its ideal sheaf is $\mathscr{I} = \mathscr{O}_{\mathbb{P}^2_{\mathbf{k}}}(-d)$, thus

$$\mathcal{N}_{C/\mathbb{P}^2_{\mathbf{k}}} = (\mathscr{O}_{\mathbb{P}^2_{\mathbf{k}}}(-d)|_C)^\vee = (\mathscr{O}_{\mathbb{P}^2_{\mathbf{k}}}(-d)^\vee)|_C = \mathscr{O}_{\mathbb{P}^2_{\mathbf{k}}}(d)|_C.$$

Example 7.3.26 (Canonical bundle of a hypersurface)**.** Let $X \hookrightarrow \mathbb{P}^n_{\mathbf{k}}$ be a smooth hypersurface of degree d. Its ideal sheaf is $\mathscr{I} = \mathscr{O}_{\mathbb{P}^n_{\mathbf{k}}}(-d)$. Then we have an exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^n_{\mathbf{k}}}(-d)|_X \to \Omega_{\mathbb{P}^n_{\mathbf{k}}}|_X \to \Omega_X \to 0,$$

so that taking determinants we find

$$\omega_X = \omega_{\mathbb{P}^n_{\mathbf{k}}}|_X \otimes_{\mathscr{O}_X} \mathscr{O}_{\mathbb{P}^n_{\mathbf{k}}}(d)|_X = \mathscr{O}_{\mathbb{P}^n_{\mathbf{k}}}(d-n-1)|_X.$$

In particular, if $d = n+1$, then X has trivial canonical line bundle. Such varieties are said to be *of Calabi–Yau type.* First examples: a smooth cubic $C \subset \mathbb{P}^2_{\mathbf{k}}$, a smooth quartic $S \subset \mathbb{P}^3_{\mathbf{k}}$ (known as a K3 surface), a smooth quintic $X \subset \mathbb{P}^4_{\mathbf{k}}$. If $d-n-1<0$, i.e. if $d < n+1$, then the hypersurface is a *Fano variety.*

Example 7.3.27 (First order deformations of nodal curves)**.** Fix a field $\mathbf{k}$. Set $P = \mathbf{k}[x, y]$. Consider the polynomial $f = xy$, the principal ideal $I = (f) \subset P$ and the quotient ring $B = P/I$. Let $X = \operatorname{Spec} B \subset \operatorname{Spec} P = \mathbb{A}^2_{\mathbf{k}}$ be the nodal curve. The conormal exact sequence is

$$\begin{tikzcd} I/I^2 \arrow[r, "\mathrm{d}"] & \Omega_{P/\mathbf{k}} \otimes_P B \arrow[r] \arrow[d, equal] & \Omega_{B/\mathbf{k}} \arrow[r] & 0 \\ & B \cdot \mathrm{d}x \oplus B \cdot \mathrm{d}y & & \end{tikzcd} \tag{7.3.4}$$

and dualising it, i.e. applying $\operatorname{Hom}_B(-, B)$, we get

$$\begin{tikzcd} 0 \arrow[r] & T_{B/\mathbf{k}} \arrow[r] & T_{P/\mathbf{k}} \otimes_P B \arrow[r, "\mathrm{d}^\vee"] \arrow[d, equal] & \operatorname{Hom}_B(I/I^2, B) \\ & & B \cdot \frac{\partial}{\partial x} \oplus B \cdot \frac{\partial}{\partial y} & \end{tikzcd}$$

The vector space

$$\operatorname{coker} \mathrm{d}^\vee = \operatorname{Ext}^1_B(\Omega_{B/\mathbf{k}}, B)$$

is called the *space of first order deformations of X inside* $\mathbb{A}^2$.[3] We want to compute it. Its 'size' will be a sort of measure of the singularity present on X. Note that $I/I^2 \cong \overline{f} \cdot B$ is a principal module generated by $\overline{f} = f + I^2$ and, in fact, the conormal sequence (7.3.5) is exact on the left too. (Why does this not contradict Theorem 7.3.21?). Now, the map d sends

$$\overline{f} \mapsto y\mathrm{d}x + x\mathrm{d}y.$$

Thus its dual is determined by

$$\begin{aligned} \mathrm{d}^\vee\left(\frac{\partial}{\partial x}\right)(\overline{f}) &= \frac{\partial f}{\partial x} = y \\ \mathrm{d}^\vee\left(\frac{\partial}{\partial y}\right)(\overline{f}) &= \frac{\partial f}{\partial y} = x. \end{aligned}$$

Using that I/I^2 is a principal module, we get, as $\mathbf{k}$-vector spaces,

$$\operatorname{coker} \mathrm{d}^\vee = \operatorname{Hom}_B(I/I^2, B)/\operatorname{im} \mathrm{d}^\vee = B/(x, y) = \mathbf{k}.$$

Example 7.3.28 (First order deformations of cuspidal plane cubics)**.** Fix a field $\mathbf{k}$ of characteristic $p \neq 2, 3$. Set $P = \mathbf{k}[x, y]$. Consider the polynomial $f = y^2 - x^3$, the principal ideal $I = (f) \subset P$ and the quotient ring $B = P/I$. Let $X = \operatorname{Spec} B \subset \operatorname{Spec} P = \mathbb{A}^2_{\mathbf{k}}$ be the cuspidal plane cubic. The conormal exact sequence is

$$\begin{tikzcd} I/I^2 \arrow[r, "\mathrm{d}"] & \Omega_{P/\mathbf{k}} \otimes_P B \arrow[r] \arrow[d, equal] & \Omega_{B/\mathbf{k}} \arrow[r] & 0 \\ & B \cdot \mathrm{d}x \oplus B \cdot \mathrm{d}y & & \end{tikzcd} \tag{7.3.5}$$

[3] Ext groups will be introduced in Section 8.4.

and dualising it, i.e. applying $\mathrm{Hom}_B(-,B)$, we get

$$\begin{tikzcd} 0 \arrow[r] & T_{B/\mathbf{k}} \arrow[r] & T_{P/\mathbf{k}} \otimes_P B \arrow[r, "\mathrm{d}^\vee"] \arrow[d, equal] & \mathrm{Hom}_B(I/I^2, B) \\ & & B \cdot \frac{\partial}{\partial x} \oplus B \cdot \frac{\partial}{\partial y} & \end{tikzcd}$$

The vector space

$$\operatorname{coker} \mathrm{d}^\vee = \mathrm{Ext}^1_B(\Omega_{B/\mathbf{k}}, B)$$

is called the *space of first order deformations of X inside* $\mathbb{A}^2$. We want to compute it. Its 'size' will be a sort of measure of the singularity present on X. Note that $I/I^2 \cong \overline{f} \cdot B$ is a principal module generated by $\overline{f} = f + I^2$ and, in fact, the conormal sequence (7.3.5) is exact on the left too. Now, the map d sends

$$\overline{f} \mapsto -3x^2 \mathrm{d}x + 2y\,\mathrm{d}y.$$

Thus its dual is determined by

$$\begin{aligned} \mathrm{d}^\vee\left(\frac{\partial}{\partial x}\right)(\overline{f}) &= \frac{\partial f}{\partial x} = -3x^2 \\ \mathrm{d}^\vee\left(\frac{\partial}{\partial y}\right)(\overline{f}) &= \frac{\partial f}{\partial y} = 2y. \end{aligned}$$

Using that I/I^2 is a principal module, we get, as $\mathbf{k}$-vector spaces,

$$\operatorname{coker} \mathrm{d}^\vee = \mathrm{Hom}_B(I/I^2, B)/\operatorname{im} \mathrm{d}^\vee = B/(-3x^2, y) = B/(x^2, y) = \mathbf{k} \oplus \mathbf{k}.$$

Caution 7.3.29. Consider the assumptions of Theorem 7.3.21, but assume X to be *singular*. The conormal sequence (7.3.3) *can be* exact on the left in this case, too. However, Ω_X won't be locally free, by Proposition 7.3.14. We shall see that this implies

$$\mathscr{E}xt^1(\Omega_X, \mathscr{O}_X) \neq 0,$$

so that in particular the *dual* of the conormal sequence

$$0 \to \mathscr{T}_X \to \iota^* \mathscr{T}_Y \to \mathscr{N}_{X/Y}$$

won't be exact on the right (cf. Remark 8.4.5). The failure of this surjectivity is precisely what we computed in the two previous examples. We found a 1-dimensional obstruction to surjectivity in the case of the nodal cubic, and a 2-dimensional obstruction in case of the cuspidal cubic. This, intuitively, reflects the fact that a nodal curve $xy = 0$ has one extra tangent, whereas $x^3 = y^2$ has a double tangent. This calculation also confirms that two singularities are not isomorphic in any reasonable sense (cf. Remark 6.1.6).

7.4 Vector bundles and locally free sheaves

To be written.

8 | Cohomology of coherent sheaves

8.1 Derived functors in a nutshell

Let $\mathscr{A}$ be an abelian category, such as one of the following:

- the category Mod_R of modules over a ring R,
- the category $\mathsf{Ab}(X) = \mathsf{Sh}(X, \mathsf{Ab}) = \mathsf{Mod}_{\underline{\mathbb{Z}}}$ of abelian sheaves on a topological space X,
- the category $\mathsf{Mod}_{\mathscr{O}_X}$ of $\mathscr{O}_X$-modules for a ringed space $(X, \mathscr{O}_X)$,
- the category $\mathsf{QCoh}(X)$ of quasicoherent sheaves on a scheme X,
- the category $\mathsf{Coh}(X)$ of coherent sheaves on a scheme X (cf. Theorem 7.1.21).

Let $\mathrm{Com}(\mathscr{A})$ be the abelian category of complexes of objects in $\mathscr{A}$. Objects are sequences

$$A^\bullet\colon \quad \cdots \longrightarrow A^{i-1} \xrightarrow{\mathrm{d}^{i-1}} A^i \xrightarrow{\mathrm{d}^i} A^{i+1} \xrightarrow{\mathrm{d}^{i+1}} \cdots$$

of objects $A^i \in \mathscr{A}$, indexed by $\mathbb{Z}$, and maps satisfying $\mathrm{d}^i \circ \mathrm{d}^{i-1} = 0$ for every $i \in \mathbb{Z}$. We call $(\mathrm{d}^i)_i$ the *differentials* of the complex $A^\bullet$. A morphism $\phi\colon A^\bullet \to B^\bullet$ in $\mathrm{Com}(\mathscr{A})$ is a collection of maps $(\phi^i\colon A^i \to B^i)_i$ commuting with the differentials in each degree. The i-th *cohomology* of a complex $A^\bullet$ is the object

$$h^i(A^\bullet) = \ker \mathrm{d}^i / \operatorname{im} \mathrm{d}^{i-1} \in \mathscr{A}.$$

Every morphism $\phi\colon A^\bullet \to B^\bullet$ in $\mathrm{Com}(\mathscr{A})$ induces a canonical map between cohomology objects $h^i(\phi)\colon h^i(A^\bullet) \to h^i(B^\bullet)$, and every short exact sequence

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

in $\mathrm{Com}(\mathscr{A})$ induces a long exact sequence

$$\cdots \to h^i(A^\bullet) \to h^i(B^\bullet) \to h^i(C^\bullet) \to h^{i+1}(A^\bullet) \to \cdots$$

in $\mathscr{A}$. The homomorphism $h^i(C^\bullet) \to h^{i+1}(A^\bullet)$ is called the i-th *connecting homomorphism.*

A *homotopy* between two morphisms $\phi\colon A^\bullet \to B^\bullet$ and $\psi\colon A^\bullet \to B^\bullet$ in $\mathrm{Com}(\mathscr{A})$ is a collection of homomorphisms

$$\theta^i\colon A^i \to B^{i-1}, \quad i \in \mathbb{Z},$$

such that in each diagram

$$\begin{tikzcd}
A^{i-1} \arrow[r, "\mathrm{d}^{i-1}_{A^\bullet}"] \arrow[d, shift right, "\psi^{i-1}"'] \arrow[d, shift left, "\phi^{i-1}"] & A^i \arrow[r, "\mathrm{d}^{i}_{A^\bullet}"] \arrow[dl, "\theta^i"'] \arrow[d, shift right, "\psi^{i}"'] \arrow[d, shift left, "\phi^{i}"] & A^{i+1} \arrow[dl, "\theta^{i+1}"'] \arrow[d, shift right, "\psi^{i+1}"'] \arrow[d, shift left, "\phi^{i+1}"] \\
B^{i-1} \arrow[r, "\mathrm{d}^{i-1}_{B^\bullet}"'] & B^i \arrow[r, "\mathrm{d}^{i}_{B^\bullet}"'] & B^{i+1}
\end{tikzcd}$$

one has the relation

$$\phi^i - \psi^i = \mathrm{d}^{i-1}_{B^\bullet} \circ \theta^i - \theta^{i+1} \circ \mathrm{d}^i_{A^\bullet} \in \mathrm{Hom}_{\mathscr{A}}(A^i, B^i).$$

We say that ϕ and ψ are homotopy equivalent, written $\phi \sim \psi$, when there is a homotopy between them. A standard result states that

$$\boxed{\text{If } \phi \sim \psi \text{ then } h^i(\phi) = h^i(\psi) \text{ for all } i \in \mathbb{Z}.}$$

From now on we assume the following.

Assumption 8.1.1. All functors $F\colon \mathscr{A} \to \mathscr{B}$ of abelian categories are additive.

Recall that a covariant functor $F\colon \mathscr{A} \to \mathscr{B}$ between abelian categories is *left exact* if a short exact sequence

$$0 \to A' \to A \to A'' \to 0 \tag{8.1.1}$$

in $\mathscr{A}$ gets sent to an exact sequence

$$0 \to FA' \to FA \to FA''$$

in $\mathscr{B}$. It is said to be *right exact* if (8.1.1) gets sent to

$$FA' \to FA \to FA'' \to 0,$$

and it is *exact* if it is both left and right exact. If F is contravariant, it is left exact when a short exact sequence (8.1.1) gets sent to

$$0 \to FA'' \to FA \to FA'$$

in $\mathscr{B}$. And similarly for right exact and exact.

For instance, for any object $A \in \mathscr{A}$, the functor $\mathrm{Hom}_{\mathscr{A}}(-, A)\colon \mathscr{A} \to \mathsf{Ab}$ is contravariant and left exact, whereas $\mathrm{Hom}_{\mathscr{A}}(A, -)\colon \mathscr{A} \to \mathsf{Ab}$ is covariant and left exact.

Definition 8.1.1 (Injective object)**.** Let $\mathscr{A}$ be an abelian category. An object $I \in \mathscr{A}$ is called injective if the functor $\mathrm{Hom}_{\mathscr{A}}(-, I)$ is exact.

Definition 8.1.2 (Enough injectives)**.** Let $\mathscr{A}$ be an abelian category. Then $\mathscr{A}$ has *enough injectives* if for every object $A \in \mathscr{A}$ there exists a monomorphism $A \hookrightarrow I$ where I is an injective object.

When an abelian category $\mathscr{A}$ has enough injectives, every object $A \in \mathscr{A}$ admits an *injective resolution*, i.e. a pair $(I^\bullet, \epsilon)$ where

$$I^\bullet : \quad I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

is a complex of injective objects indexed by $\mathbb{Z}_{\geq 0}$ and $\epsilon\colon A \to I^0$ is a morphism such that the complex

$$0 \longrightarrow A \xrightarrow{\ \epsilon\ } I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

is exact. This means

$$A = \ker(I^0 \to I^1), \qquad \ker(I^k \to I^{k+1}) = \mathrm{im}(I^{k-1} \to I^k), \quad k > 0.$$

Notation 8.1.3. We shall sometimes denote by $A \xrightarrow{\sim} I^\bullet$ the injective resolution $(I^\bullet, \epsilon)$, with a slight abuse of notation.

Example 8.1.4. Let R be a ring. Then Mod_R has enough injectives.

THEOREM 8.1.5. *Let $\mathscr{A}$ be an abelian category with enough injectives, let $\mathscr{B}$ be another abelian category, and let $F\colon \mathscr{A} \to \mathscr{B}$ be a covariant left exact functor. For each $A \in \mathscr{A}$, pick an injective resolution $(I^\bullet, \epsilon)$, and for each $i \geq 0$ set*

$$\mathbf{R}^i F(A) = h^i(F I^\bullet).$$

(1) *The rule $A \mapsto \mathbf{R}^i F(A)$ defines an additive functor $\mathscr{A} \to \mathscr{B}$, independent of the choice of injective resolution chosen for each object of $\mathscr{A}$.*

(2) *One has $F \cong \mathbf{R}^0 F$.*

(3) *Each short exact sequence (8.1.1) gives rise to a long exact sequence*

$$\cdots \to \mathbf{R}^i F(A') \to \mathbf{R}^i F(A) \to \mathbf{R}^i F(A'') \to \mathbf{R}^{i+1} F(A') \to \mathbf{R}^{i+1} F(A) \to \cdots$$

in $\mathscr{B}$.

(4) *If $I \in \mathscr{A}$ is injective, then $\mathbf{R}^i F(I) = 0$ for all $i > 0$.*

(5) *If F is exact, then $\mathbf{R}^{>0} F = 0$.*

Proof. We give a sketch of proof.

A key fact needed to prove (1) is the well-known fact that any two injective resolutions

$$A \to I^\bullet, \qquad A \to J^\bullet$$

are homotopy equivalent, in the sense that there are homomorphisms

$$\sigma \colon I^\bullet \to J^\bullet, \qquad \tau \colon J^\bullet \to I^\bullet$$

such that $\sigma \circ \tau \sim \mathrm{id}_{J^\bullet}$ and $\tau \circ \sigma \sim \mathrm{id}_{I^\bullet}$.

To prove (2), just use left exactness of F in the third equality of

$$\mathbf{R}^0 F(A) = h^0(F I^\bullet) = \ker(F I^0 \to F I^1) = F(\ker(I^0 \to I^1)) = FA.$$

To prove (4), note that if I is injective then

$$0 \to I \xrightarrow{\mathrm{id}} I \to 0$$

is a resolution. Then $h^{>0}(FI \to FI \to 0) = 0$.

To prove (5), note that ☐ complete

Definition 8.1.6 (Right derived functor)**.** The functor $\mathbf{R}^i F$ is called the i-th *right derived functor* of F.

Definition 8.1.7 (F-acyclic objects)**.** Let $F \colon \mathscr{A} \to \mathscr{B}$ be a functor as in Theorem 8.1.5. An object $J \in \mathscr{A}$ is *F-acyclic* if $\mathbf{R}^i F(J) = 0$ for all $i > 0$.

Remark 8.1.8 (Injectives are acyclic)**.** Let $F \colon \mathscr{A} \to \mathscr{B}$ be a functor as in Theorem 8.1.5. Any injective object $J \in \mathscr{A}$ is *F-acyclic* by Theorem 8.1.5 (4). This means that injective objects are well suited to be used to form acyclic resolutions *for any functor* as in Theorem 8.1.5. However, if one has only a specific functor F in mind to right-derive, it may be more convenient to use a different type of resolution, as long as it is F-acyclic. This is what will happen next with $F = \Gamma(X, -)$, see Definition 8.1.10 below.

LEMMA 8.1.9. *Let $(X, \mathscr{O}_X)$ be a ringed space. Then $\mathsf{Mod}_{\mathscr{O}_X}$ has enough injectives. In particular, $\mathsf{Ab}(X)$ has enough injectives.*

Proof. The second statement follows from the first after observing that $\mathsf{Ab}(X) = \mathsf{Mod}_{\underline{\mathbb{Z}}_X}$.

So let us pick an arbitrary object $\mathcal{F} \in \mathsf{Mod}_{\mathscr{O}_X}$. For any point $x \in X$, we may choose an injective $\mathscr{O}_{X,x}$-module I_x along with a monomorphism

$$\ell_x \colon \mathcal{F}_x \hookrightarrow I_x$$

in $\mathsf{Mod}_{\mathscr{O}_{X,x}}$, thanks to Example 8.1.4. Let $\iota_x \colon \{x\} \hookrightarrow X$ be the inclusion. Define

$$\mathcal{I} = \prod_{x \in X} \iota_{x*} I_x \in \mathsf{Mod}_{\mathscr{O}_X}.$$

Then, we have

$$\begin{aligned}\mathrm{Hom}_{\mathscr{O}_X}(\mathcal{F},\mathcal{I}) &= \prod_{x\in X}\mathrm{Hom}_{\mathscr{O}_X}(\mathcal{F},\iota_{x*}I_x)\\ &= \prod_{x\in X}\mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathcal{F}_x,I_x)\end{aligned}$$

where we used the adjunction $(\iota_x^{-1},\iota_{x*})$ in the last equality. The collection $(\ell_x)_{x\in X}$ then determines an injective morphism

$$\ell\colon \mathcal{F}\hookrightarrow\mathcal{I}$$

in $\mathsf{Mod}_{\mathscr{O}_X}$. It remains to observe that $\mathcal{I}$ is injective. but this is true, since

$$\mathrm{Hom}_{\mathscr{O}_X}(-,\mathcal{I}) = \prod_{x\in X}\mathrm{Hom}_{\mathscr{O}_{X,x}}(-,I_x)\circ\iota_x^{-1}(-)$$

as functors, and both ι_x^{-1} and $\mathrm{Hom}_{\mathscr{O}_{X,x}}(-,I_x)$ are exact, for all $x\in X$. □

Definition 8.1.10 (Cohomology functors)**.** Let X be a topological space. Define $\mathrm{H}^i(X,-)$, for $i\geq 0$, to be the right derived functors of

$$\Gamma(X,-)\colon \mathsf{Ab}(X)\to\mathsf{Ab},\quad \mathcal{F}\mapsto\Gamma(X,\mathcal{F})=\mathcal{F}(X).$$

In symbols, the functor

$$\mathrm{H}^i(X,-)=\mathbf{R}^i\Gamma(X,-)$$

takes an abelian sheaf $\mathcal{F}\in\mathsf{Ab}(X)$ to the abelian group $\mathrm{H}^i(X,\mathcal{F})$, called the i-th *cohomology group* of $\mathcal{F}$.

We shall silently identify $\mathcal{F}(X)$ with $\mathrm{H}^0(X,\mathcal{F})$ throughout.

Definition 8.1.11 (Higher direct image)**.** Let $\pi\colon X\to Y$ be a morphism of ringed spaces. Consider $\pi_*\colon \mathsf{Mod}_{\mathscr{O}_X}\to\mathsf{Mod}_{\mathscr{O}_Y}$. The i-th right derived functor of π_* is the functor

$$\mathbf{R}^i\pi_*\colon \mathsf{Mod}_{\mathscr{O}_X}\to\mathsf{Mod}_{\mathscr{O}_Y}$$

sending $\mathcal{F}\mapsto h^i(\pi_*\mathcal{J}^\bullet)$, where $\mathcal{J}^\bullet$ is any injective (or π_*-acyclic) resolution of $\mathcal{F}$.

Definition 8.1.12 (Flasque sheaf)**.** Let X be a topological space. A sheaf $\mathcal{F}\in\mathsf{Ab}(X)$ is *flasque* if the restriction map $\mathcal{F}(U)\to\mathcal{F}(V)$ is surjective for every inclusion $V\subset U$ of open subsets of X.

Clearly, being flasque is equivalent to the surjectivity of the maps $\mathcal{F}(X)\to\mathcal{F}(U)$ for every $U\subset X$.

We formulate a few key properties of flasque sheaves in a series of (easy) exercises (cf. [8, Ch. II, Ex. 1.16]).

Exercise 8.1.13. Show that a constant sheaf on an irreducible topological space is flasque.

Exercise 8.1.14. Let X be a topological space. Let $\mathcal{F} \in \mathsf{Ab}(X)$ be a flasque sheaf sitting in a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0.$$

Show that

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U) \to 0$$

is exact for every open subset $U \subset X$.

Exercise 8.1.15. A quotient of flasque sheaves is flasque.

Exercise 8.1.16. The pushforward $f_*\mathcal{F}$ of a flasque sheaf $\mathcal{F}$ along a continuous map $f : X \to Y$ is a flasque sheaf on Y.

Construction 8.1.17. Let X be a topological space, $j : U \hookrightarrow X$ an open subset. If $\mathcal{F} \in \mathsf{Ab}(U)$ is a sheaf *on* U, then we define $j_!\mathcal{F} \in \mathsf{Ab}(X)$ to be the sheafification of the presheaf

$$X \supset W \mapsto \begin{cases} \mathcal{F}(W) & \text{if } W \subset U \\ 0 & \text{if } W \not\subset U. \end{cases}$$

One can show that $j^{-1}j_!\mathcal{F} = \mathcal{F}$, that

$$(j_!\mathcal{F})_x = \begin{cases} \mathcal{F}_x & \text{if } x \in U \\ 0 & \text{if } x \notin U, \end{cases}$$

and that $j_!\mathcal{F}$ is the only sheaf on X with these properties. Such sheaf is called *the extension of* $\mathcal{F}$ *by zero* outside U. Moreover, if $Z = X \setminus U$ and $\iota : Z \hookrightarrow X$ is the inclusion, then for any $\mathcal{G} \in \mathsf{Sh}(X)$ there is an exact sequence

$$0 \to j_!j^{-1}\mathcal{G} \to \mathcal{G} \to \iota_*\iota^{-1}\mathcal{G} \to 0. \tag{8.1.2}$$

This short exact sequence will be used in the proof of Theorem 8.2.1.

LEMMA 8.1.18. *Let* $(X, \mathscr{O}_X)$ *be a ringed space. An injective object* $\mathcal{I} \in \mathsf{Mod}_{\mathscr{O}_X}$ *is flasque.*

Proof. For an arbitrary open subset $j_U : U \hookrightarrow X$, consider $j_{U!}\mathscr{O}_U \in \mathsf{Mod}_{\mathscr{O}_X}$ as in Construction 8.1.17. It is the sheafification of the presheaf

$$X \supset W \mapsto \begin{cases} \mathscr{O}_U(W) & \text{if } W \subset U \\ 0 & \text{if } W \not\subset U. \end{cases}$$

Then, for every inclusion $V \subset U$ of open subsets, we have an injection

$$0 \to j_{V!}\mathscr{O}_V \to j_{U!}\mathscr{O}_U.$$

Applying $\mathrm{Hom}_{\mathscr{O}_X}(-,\mathcal{I})$ and using injectivity of $\mathcal{I}$, we obtain

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathscr{O}_X}(j_{U!}\mathscr{O}_U,\mathcal{I}) & \longrightarrow & \mathrm{Hom}_{\mathscr{O}_X}(j_{V!}\mathscr{O}_V,\mathcal{I}) & \longrightarrow & 0 \\
\| & & \| & & \\
\mathcal{I}(U) & & \mathcal{I}(V) & &
\end{array}$$

as required. □

LEMMA 8.1.19. *Let $\mathcal{F} \in \mathsf{Ab}(X)$ be a flasque sheaf on a topological space X. Then $\mathcal{F}$ is $\Gamma(X,-)$-acyclic, i.e. $\mathrm{H}^i(X,\mathcal{F}) = 0$ for all $i > 0$.*

Proof. Embed $\mathcal{F}$ in an injective object $\mathcal{I} \in \mathsf{Mod}_{\mathscr{O}_X}$, thanks to Lemma 8.1.9. Form the short exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{H} \to 0.$$

By Lemma 8.1.18, $\mathcal{I}$ is flasque. It follows that $\mathcal{H}$ is flasque as well. Indeed, by Exercise 8.1.14, for each inclusion of open subsets $V \subset U$, we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F}(U) & \longrightarrow & \mathcal{I}(U) & \longrightarrow & \mathcal{H}(U) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}(V) & \longrightarrow & \mathcal{I}(V) & \longrightarrow & \mathcal{H}(V) & \longrightarrow & 0
\end{array}$$

with exact rows. By the Snake Lemma, the rightmost vertical map is surjective, confirming that $\mathcal{H}$ is flasque. (Note that this solves Exercise 8.1.15).

Now, since $\mathcal{I}$ is injective, it is $\Gamma(X,-)$-acyclic, i.e. $\mathrm{H}^{>0}(X,\mathcal{I}) = 0$. On the other hand, $\mathcal{F}$ is flasque by assumption, therefore $\mathrm{H}^1(X,\mathcal{F}) = 0$ thanks to Exercise 8.1.14. Thus the long exact sequence of cohomology takes the form

$$\begin{aligned}
0 \to \mathcal{F}(X) \to \mathcal{I}(X) \to \mathcal{H}(X) \to 0 \to 0 \to \mathrm{H}^1(X,\mathcal{H}) \\
\to \mathrm{H}^2(X,\mathcal{F}) \to 0 \to \mathrm{H}^2(X,\mathcal{H}) \to \mathrm{H}^3(X,\mathcal{F}) \to \cdots
\end{aligned}$$

which shows that $\mathrm{H}^i(X,\mathcal{F}) \cong \mathrm{H}^{i-1}(X,\mathcal{H})$ canonically for every $i > 1$. By induction on i, which we can apply since $\mathcal{H}$ is flasque, we obtain the result. □

Consider the following slogan:

$$\boxed{\text{Any } F\text{-acyclic resolution computes } \mathbf{R}^{\geq 0}F.}$$

This is made precise by the following theorem.

THEOREM 8.1.20. *Let $\mathscr{A}$ be an abelian category with enough injectives, let $\mathscr{B}$ be another abelian category, and let $F : \mathscr{A} \to \mathscr{B}$ be a covariant left exact functor. Let $A \in \mathscr{A}$ be an object and let $A \xrightarrow{\sim} J^\bullet$ be an F-acyclic resolution. Then*

$$\mathbf{R}^i F(A) \cong h^i(FJ^\bullet), \quad i \geq 0.$$

The key observation needed to prove the theorem is the following lemma.

LEMMA 8.1.21. *Let $\mathscr{A}$ be an abelian category with enough injectives, let $\mathscr{B}$ be another abelian category, and let $F : \mathscr{A} \to \mathscr{B}$ be a covariant left exact functor. Let $A \in \mathscr{A}$ be an object sitting in a short exact sequence*

$$0 \to A \to J \to K \to 0 \tag{8.1.3}$$

where $J \in \mathscr{A}$ is F-acyclic. Then

$$\mathbf{R}^1 F(A) = \operatorname{coker}(FJ \to FK),$$

and

$$\mathbf{R}^i F(K) \cong \mathbf{R}^{i+1} F(A), \quad i > 0.$$

Proof. The long exact sequence in cohomology applied to (8.1.3) yields

$$0 \to FA \to FJ \to FK \to \mathbf{R}^1 F(A) \to 0 \to \mathbf{R}^1 F(K) \to \mathbf{R}^2 F(A) \to 0 \to \cdots$$

from which both statements follow at once. □

Proof of Theorem 8.1.20. Let us start with $i = 0$. We have

$$h^0(FJ^\bullet) = h^0(FJ^0 \to FJ^1 \to \cdots) = \ker(FJ^0 \to FJ^1) = F(\ker(J^0 \to J^1)) = FA = \mathbf{R}^0 F(A),$$

since $J^\bullet$ is a resolution of A and F is left exact.

Let us continue with $i = 1$. Define $K^1 = J^0/A$, so that we have an exact sequence

$$0 \to A \to J^0 \to K^1 \to 0.$$

By Lemma 8.1.21 we have

$$\mathbf{R}^1 F(A) = \operatorname{coker}(FJ^0 \to FK^1) = \frac{FK^1}{\operatorname{im}(FJ^0 \to FK^1)}. \tag{8.1.4}$$

Note that there is a canonical monomorphism $K^1 \hookrightarrow J^1$, and thus

$$\operatorname{im}(FJ^0 \to FJ^1) = \operatorname{im}(FJ^0 \to FK^1).$$

We have

$$h^1(FJ^\bullet) = \frac{\ker(FJ^1 \to FJ^2)}{\operatorname{im}(FJ^0 \to FJ^1)} = \frac{F(\ker(J^1 \to J^2))}{\operatorname{im}(FJ^0 \to FK^1)} = \frac{F(\operatorname{im}(J^0 \to J^1))}{\operatorname{im}(FJ^0 \to FK^1)} = \frac{FK^1}{\operatorname{im}(FJ^0 \to FK^1)}.$$

Combining this with (8.1.4) we obtain the statement for $i = 1$.

Let us continue with $i = 2$. We define $K^2 = J^1/K^1$, so that we have an exact sequence

$$0 \to K^1 \to J^1 \to K^2 \to 0.$$

By Lemma 8.1.21 (both equations) we have

$$\mathbf{R}^2F(A)=\mathbf{R}^1F(K^1)=\operatorname{coker}(FJ^1\to FK^2)=\frac{FK^2}{\operatorname{im}(FJ^1\to FK^2)}. \tag{8.1.5}$$

Thanks to the canonical monomorphism $K^2\hookrightarrow J^2$, we have $\operatorname{im}(FJ^1\to FJ^2)=\operatorname{im}(FJ^1\to FK^2)$ and

$$\frac{J^1}{\ker(J^1\to J^2)}=\frac{J^1}{\ker(J^1\to K^2)}=\frac{J^1}{K^1}=K^2.$$

We then find

$$h^2(FJ^\bullet)=\frac{\ker(FJ^2\to FJ^3)}{\operatorname{im}(FJ^1\to FJ^2)}=\frac{F(\ker(J^2\to J^3))}{\operatorname{im}(FJ^1\to FK^2)}=\frac{F(\operatorname{im}(J^1\to J^2))}{\operatorname{im}(FJ^1\to FK^2)}=\frac{FK^2}{\operatorname{im}(FJ^1\to FK^2)}.$$

Combining this with (8.1.5) we obtain the statement for $i=2$.

Clearly the strategy can be pursued for all higher $i>2$. □

So, in the case of cohomology of abelian sheaves, we get the following recipe. Consider an abelian sheaf $\mathcal{F}$, and a resolution $\mathcal{F}\xrightarrow{\sim}\mathcal{A}^\bullet$ where each $\mathcal{A}^i$ is $\Gamma(X,-)$-acyclic, for all $i\geq 0$. Then, form the complex

$$\Gamma(X,\mathcal{A}^\bullet):\quad \Gamma(X,\mathcal{A}^0)\to\Gamma(X,\mathcal{A}^1)\to\Gamma(X,\mathcal{A}^2)\to\cdots$$

and consider the i-th cohomology object

$$h^i(\Gamma(X,\mathcal{A}^\bullet))\in\mathsf{Ab}.$$

We have

$$h^i(\Gamma(X,\mathcal{A}^\bullet))\cong\mathrm{H}^i(X,\mathcal{F}),\quad i\geq 0.$$

Remark 8.1.22. To recap, we have

$$\text{injective}\ \Rightarrow\ \text{flasque}\ \Rightarrow\ \Gamma(X,-)\text{-acyclic},$$

and *any* $\Gamma(X,-)$-acyclic resolution can be used to compute $\mathrm{H}^i(X,-)$, even though the original definition was given in terms of injective resolutions. In particular, we may compute cohomology $\mathrm{H}^i(X,\mathcal{F})$ via flasque resolutions, if we want to. This has the following consequence. Consider the covariant left exact functor

$$\Gamma(X,-)\colon \mathsf{Mod}_{\mathscr{O}_X}\to\mathsf{Ab}$$

for a ringed space $(X,\mathscr{O}_X)$. In principle, its derived functors $\mathbf{R}^i\Gamma(X,-)$ should be computed via injective resolutions. However, injective resolutions are flasque resolutions, in particular $\Gamma(X,-)$-acyclic. Thus $\mathbf{R}^i\Gamma(X,\mathcal{F})$ agrees with $\mathrm{H}^i(X,\mathcal{F})$, where the latter is obtained by first viewing $\mathcal{F}$ as an object of $\mathsf{Ab}(X)$ and then applying Definition 8.1.10.

8.2 Vanishing cohomology above Krull dimension

In this section we shall prove the following important theorem.

THEOREM 8.2.1 (Grothendieck). *Let X be an n-dimensional noetherian topological space. Then $\mathrm{H}^i(X,\mathcal{F})=0$ for every $\mathcal{F}\in\mathsf{Ab}(X)$ and for every $i>n$.*

The direct limit of a family $(\mathcal{F}_\alpha)_{\alpha\in A}$ of abelian sheaves on a topological space X can be defined via the usual universal property (cf. Definition B.4.6), although this procedure, as usual, does not guarantee the existence of the representing object. In a more direct way, one can show that the sheafification of the presheaf

$$U\mapsto \varinjlim \mathcal{F}_\alpha(U) \tag{8.2.1}$$

satisfies the universal property of the direct limit.

LEMMA 8.2.2. *Let X be a noetherian topological space. Let $(\mathcal{F}_\alpha)_{\alpha\in A}$ be a directed system of flasque sheaves $\mathcal{F}_\alpha\in\mathsf{Ab}(X)$. Then $\varinjlim\mathcal{F}_\alpha$ is flasque.*

Proof. Since X is noetherian, (8.2.1) already defines a sheaf. Therefore

$$\varinjlim\mathcal{F}_\alpha(W)=(\varinjlim\mathcal{F}_\alpha)(W),\qquad W\subset X.$$

Since Ab is a Grothendieck abelian category, the direct limit functor is exact. Then, from the surjections $\mathcal{F}_\alpha(U)\to\mathcal{F}_\alpha(V)\to 0$ one deduces

$$\begin{tikzcd}
\varinjlim\mathcal{F}_\alpha(U) \arrow[r] \arrow[d, equal] & \varinjlim\mathcal{F}_\alpha(V) \arrow[r] \arrow[d, equal] & 0 \\
(\varinjlim\mathcal{F}_\alpha)(U) & (\varinjlim\mathcal{F}_\alpha)(V) &
\end{tikzcd}$$

for any inclusion $V\subset U$ of open subsets. □

LEMMA 8.2.3 ([8,]). *Let X be a noetherian topological space. Let $(\mathcal{F}_\alpha)_{\alpha\in A}$ be a directed system of sheaves $\mathcal{F}_\alpha\in\mathsf{Ab}(X)$. Then there is a natural isomorphism*

$$\varinjlim\mathrm{H}^i(X,\mathcal{F}_\alpha)\xrightarrow{\sim}\mathrm{H}^i(X,\varinjlim\mathcal{F}_\alpha),$$

for each $i\geq 0$.

LEMMA 8.2.4. *Let X be a noetherian topological space, $\iota\colon Y\hookrightarrow X$ a closed subset. Then, for any abelian sheaf $\mathcal{F}\in\mathsf{Ab}(Y)$, one has*

$$\mathrm{H}^i(Y,\mathcal{F})=\mathrm{H}^i(X,\iota_*\mathcal{F}),\qquad i\geq 0.$$

Proof. The case $i=0$ is just the definition of the direct image functor. For the general case, pick a flasque resolution $\mathcal{F} \xrightarrow{\sim} J^\bullet$, so that $\iota_*\mathcal{F} \xrightarrow{\sim} \iota_* J^\bullet$ is a flasque resolution, too, thanks to Exercise 8.1.16. It follows from the $i=0$ case that $\Gamma(Y, J^k) = \Gamma(X, \iota_* J^k)$ canonically. Then

$$\begin{aligned} \mathrm{H}^i(Y,\mathcal{F}) &= \mathbf{R}^i\Gamma(Y,\mathcal{F}) && \text{by definition} \\ &= h^i(\Gamma(Y,J^\bullet)) && \text{by Theorem 8.1.20} \\ &= h^i(\Gamma(X,\iota_*J^\bullet)) && \text{by } i=0 \text{ case} \\ &= \mathbf{R}^i\Gamma(X,\iota_*\mathcal{F}) && \text{by Theorem 8.1.20} \\ &= \mathrm{H}^i(X,\iota_*\mathcal{F}) && \text{by definition} \end{aligned}$$

so the lemma is proved. □

Before giving the proof of Theorem 8.2.1, let us fix some notation. Let X be a topological space. If $\iota\colon Z \hookrightarrow X$ is a closed subset with open complement $j\colon U \hookrightarrow X$, then for any $\mathcal{F} \in \mathsf{Ab}(X)$ we set $\mathcal{F}_U = j_! j^{-1}\mathcal{F} = j_!(\mathcal{F}|_U)$ and $\mathcal{F}_Z = \iota_*\iota^{-1}\mathcal{F} = \iota_*(\mathcal{F}|_Z)$, so that (8.1.2) becomes the short exact sequence

$$0 \to \mathcal{F}_U \to \mathcal{F} \to \mathcal{F}_Z \to 0. \tag{8.2.2}$$

Proof of Theorem 8.2.1. The proof will be by induction on $n = \dim X$, after a first reduction.

Step I. *Reduction to the irreducible case.* Assume X is reducible. Since it is noetherian, it has finitely many irreducible components (cf. Proposition 4.5.7). Let $Z \subset X$ be one of them, and let $U \subset X$ be the open complement. Then we have the short exact sequence (8.2.2) in $\mathsf{Ab}(X)$. By the long exact sequence in cohomology

$$\begin{aligned} 0 \to \mathrm{H}^0(X,\mathcal{F}_U) \to \mathrm{H}^0(X,\mathcal{F}) \to \mathrm{H}^0(X,\mathcal{F}_Z) \to \mathrm{H}^1(X,\mathcal{F}_U) \\ \to \mathrm{H}^1(X,\mathcal{F}) \to \mathrm{H}^1(X,\mathcal{F}_Z) \to \mathrm{H}^2(X,\mathcal{F}_U) \to \cdots \end{aligned}$$

it is enough to prove that $\mathrm{H}^i(X,\mathcal{F}_U) = \mathrm{H}^i(X,\mathcal{F}_Z) = 0$ for all $i > n$. As for the second group, we have $\mathrm{H}^i(X,\mathcal{F}_Z) = \mathrm{H}^i(Z,\mathcal{F}|_Z)$ by Lemma 8.2.4, and this vanishes because we are granting the irreducible case and because $\dim Y \le n$. As for the first group, note that the closed subset $c\colon \overline{U} \hookrightarrow X$ has one fewer irreducible components than X. So by induction on the number of irreducible components, we have

$$\begin{aligned} 0 &= \mathrm{H}^i(\overline{U}, c^{-1}\mathcal{F}_U) \\ &= \mathrm{H}^i(X, c_*c^{-1}\mathcal{F}_U) \\ &= \mathrm{H}^i(X,\mathcal{F}_U), \end{aligned}$$

where the second identity follows by Lemma 8.2.4, and the third one follows from the fact that the canonical surjection $\mathcal{F}_U \to c_*c^{-1}\mathcal{F}_U$ is an isomorphism. Indeed, by the short exact sequence (8.2.2) its kernel is $\omega_!\omega^{-1}\mathcal{F}_U$, where $\omega\colon X\setminus\overline{U}\hookrightarrow X$ is the open inclusion. But for any $x\in X\setminus\overline{U}\subset X\setminus U$ we have

$$(\omega_!\omega^{-1}\mathcal{F}_U)_x=(\mathcal{F}_U)_x=0.$$

Step II. *The 0-dimensional case.* Let us assume X is 0-dimensional and irreducible. Then, by definition of Krull dimension, we have $\tau_X=\{\emptyset,X\}$. This says that the global section functor $\Gamma(X,-)\colon \mathsf{Ab}(X)\to\mathsf{Ab}$ is an equivalence of categories, in particular exact. Then $0=\mathbf{R}^i\Gamma(X,-)=\mathrm{H}^i(X,-)$ for $i>0=\dim X$, as required.

Step III. *The general case.* Assume X is irreducible of dimension n. Fix $\mathcal{F}\in\mathsf{Ab}(X)$. The set

$$A=\left\{\text{finite subsets }\alpha\subset\bigcup_{U\in\tau_X}\mathcal{F}(U)\right\}$$

is directed via inclusion, and for each $\alpha\in A$, define $\mathcal{F}_\alpha$ to be the sheaf generated by the sections belonging to α. Note that these may be defined over different opens, but this is not an issue. Simply define $\mathcal{F}_\alpha$ as follows: say $\alpha=\{s_1,\ldots,s_\ell\}$, with $s_k\in\mathcal{F}(U_k)$, and denote in the same way the corresponding sheaf homomorphism $s_k\colon\underline{\mathbb{Z}}_{U_k}\to\mathcal{F}$. Consider their direct sum $\phi_\alpha\colon\bigoplus_{1\le k\le\ell}\underline{\mathbb{Z}}_{U_k}\to\mathcal{F}$ and set $\mathcal{F}_\alpha=\mathrm{im}(\phi_\alpha)$. One has

$$\mathcal{F}=\varinjlim\mathcal{F}_\alpha,$$

and thus, by Lemma 8.2.3,

$$\mathrm{H}^i(X,\mathcal{F})=\varinjlim\mathrm{H}^i(X,\mathcal{F}_\alpha),\quad i\ge 0.$$

We claim that

$$\mathrm{H}^{>n}(X,\mathcal{F}_\alpha)=0,\quad \alpha\in A.$$

This claim can be proved by induction on $|\alpha|$, thanks to the long exact sequence in cohomology, coupled with the following observation: any inclusion $\alpha'\subset\alpha$ induces a short exact sequence

$$0\to\mathcal{F}_{\alpha'}\to\mathcal{F}_\alpha\to\mathcal{G}_{\alpha/\alpha'}\to 0,$$

where $\mathcal{G}_{\alpha/\alpha'}$ is generated by $|\alpha\setminus\alpha'|$ sections. We may then assume that $\mathcal{F}$ is generated by a single section, say defined over a given $U\in\tau_X$. Such sheaves are precisely the quotients of $\underline{\mathbb{Z}}_U=j_!j^{-1}\underline{\mathbb{Z}}$, where $j\colon U\hookrightarrow X$ denotes the inclusion and $\underline{\mathbb{Z}}$ is the constant sheaf on X with values in $\mathbb{Z}$. We therefore have for free the exact sequence

$$0\to\mathcal{R}\to\underline{\mathbb{Z}}_U\to\mathcal{F}\to 0,\tag{8.2.3}$$

where $\mathcal{R}$ is defined to be the kernel. By the long exact sequence in cohomology, we are reduced to proving

$$\mathrm{H}^i(X,\mathcal{R})=0=\mathrm{H}^i(X,\underline{\mathbb{Z}}_U),\quad i>n.$$

Step III.1 We prove that $\mathrm{H}^i(X,\mathcal{R})=0$ for $i>n$. In this situation, for each $x\in U$ we have a subgroup

$$\mathcal{R}_x \subset \underline{\mathbb{Z}}_{U,x}=\mathbb{Z},$$

and now there are two possibilities: either $\mathcal{R}_x=0$ for all $x\in U$, or $\mathcal{R}\neq 0$. In the first case, the short exact sequence (8.2.3) reduces to an isomorphism $\underline{\mathbb{Z}}_U \xrightarrow{\sim} \mathcal{F}$, which would complete the proof by **Step III.2** below. So let us assume $\mathcal{R}\neq 0$. Let

$$d=\inf_{x\in U}\{\, e\in\mathbb{Z} \mid e\in\mathcal{R}_x \,\}.$$

Then

$$\mathcal{R}|_V = d\cdot\underline{\mathbb{Z}}|_V$$

as subsheaves of $\underline{\mathbb{Z}}|_V = j_V^{-1}\underline{\mathbb{Z}}$, for a suitable open subset $\emptyset\neq V\subset U$. Then

$$\mathcal{R}_V = j_{V!}\mathcal{R}|_V = j_{V!}(d\cdot\underline{\mathbb{Z}}|_V)=\underline{\mathbb{Z}}_V,$$

so that we get a short exact sequence

$$0\to\underline{\mathbb{Z}}_V\to\mathcal{R}\to\mathcal{R}/\underline{\mathbb{Z}}_V\to 0.$$

But $\mathcal{R}/\underline{\mathbb{Z}}_V$ is supported on $Z=(U\setminus V)^-\subset X$, which is a closed subset of dimension $\dim Z<n$ since X is irreducible. Thus combining the induction hypothesis with Lemma 8.2.4 we obtain $\mathrm{H}^{\geq n}(X,\mathcal{R}/\underline{\mathbb{Z}}_V)=0$. The next step takes care of $\underline{\mathbb{Z}}_V$, and the long exact sequence in cohomology does the rest.

Step III.2 We prove that $\mathrm{H}^i(X,\underline{\mathbb{Z}}_U)=0$ for every $U\in\tau_X$ and $i>n$. Let $Y=X\setminus U$ be the closed complement. Then we have a short exact sequence

$$0\to\underline{\mathbb{Z}}_U\to\underline{\mathbb{Z}}\to\underline{\mathbb{Z}}_Y\to 0.$$

But $\underline{\mathbb{Z}}_Y$ is by definition a pushforward from Y, and $\dim Y<\dim X=n$ since X is irreducible (and Y is closed). Thus, by induction and Lemma 8.2.4, we have $\mathrm{H}^i(X,\underline{\mathbb{Z}}_Y)=\mathrm{H}^i(Y,\underline{\mathbb{Z}}|_Y)=0$. On the other hand, $\underline{\mathbb{Z}}$ is constant on an irreducible space, hence flasque by Exercise 8.1.13. Thus $\mathrm{H}^{>0}(X,\underline{\mathbb{Z}})=0$. Thus $\mathrm{H}^{>n}(X,\underline{\mathbb{Z}}_U)=0$ by the long exact sequence in cohomology. □

8.3 Cohomology of coherent sheaves on projective schemes

In this section we state a series of very important theorems (mainly by Serre) regarding the cohomology of coherent sheaves on projective schemes. The main application is, of course, to the case of projective varieties over a field.

Definition 8.3.1 (Relatively very ample invertible sheaf)**.** Let $f: X \to Y$ be a morphism of schemes, $\mathscr{L}$ an invertible sheaf on X. Then $\mathscr{L}$ is said to be f*-very ample* (or *relatively very ample* with respect to f) if there exists a locally closed immersion $\iota: X \to \mathbb{P}^r_Y$ over Y for some r, such that $\iota^*\mathscr{O}_{\mathbb{P}^r_Y}(1) = \mathscr{L}$ (cf. Definition 7.1.35).

Recall that a locally closed immersion of schemes is a closed immersion followed by an open immersion (cf. Definition 2.10.8).

Definition 8.3.2 (Projective morphism)**.** Let Y be a noetherian scheme. We say that a morphism $f: X \to Y$ is *projective* if it is proper and there is an f-very ample invertible sheaf on X.

Fix an affine scheme $Y = \operatorname{Spec} A$, for a noetherian ring A, and a morphism $f: X \to Y$. Then, by Theorem 7.1.37, to say that an invertible sheaf $\mathscr{L}$ on X is f-very ample is to say that $\mathscr{L}$ has $r+1$ global sections such that the associated morphism to $\mathbb{P}^r_A$ is an immersion.

Notation 8.3.3. In what follows, the n-th tensor power of a line bundle $\mathscr{L}$ will be denoted simply by $\mathscr{L}^n$ for any $n \geq 0$. Also, if $n \in \mathbb{Z}_{<0}$, by $\mathscr{L}^n$ we shall mean the tensor product $(\mathscr{L}^\vee)^{-n}$.

THEOREM 8.3.4 (Serre). *Let A be a noetherian ring, and set $Y = \operatorname{Spec} A$. Let $f: X \to Y$ be a projective morphism, and let $\mathscr{L}$ be an f-very ample invertible sheaf on X. Then, for every $\mathcal{F} \in \mathsf{Coh}(X)$, there is an integer $n_0 = n_0(\mathcal{F})$ such that $\mathcal{F} \otimes_{\mathscr{O}_X} \mathscr{L}^n$ is generated by finitely many global sections for every $n \geq n_0$.*

Notation 8.3.5. It is customary to denote an f-very ample invertible sheaf by $\mathscr{O}_X(1)$ (and by $\mathscr{O}_X(-1)$ its dual). When we do this, we will use the ubiquitous notation $\mathcal{F}(n)$ to denote the sheaf $\mathcal{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X(1)^{\otimes n}$ for any $n \geq 0$.

A consequence of Serre's theorem is the following.

COROLLARY 8.3.6. *Let A be a noetherian ring, and set $Y = \operatorname{Spec} A$. Let $f: X \to Y$ be a projective morphism, and let $\mathscr{O}_X(1)$ be an f-very ample invertible sheaf on X. Then any $\mathcal{F} \in \mathsf{Coh}(X)$ is a quotient of a sheaf of the form*

$$\mathcal{E} = \mathscr{O}_X(-n)^{\oplus d},$$

for some $n \in \mathbb{Z}$ and some $d \in \mathbb{N}$.

Proof. Fix $n \geq n_0(\mathcal{F})$ as in Theorem 8.3.4, so that $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X(n)$ is generated by finitely many global sections. Then we have a surjection

$$\mathscr{O}_X^{\oplus d} \twoheadrightarrow \mathcal{F}(n)$$

for some $d > 0$. Twisting with $\mathscr{O}_X(-n) = \mathscr{O}_X(n)^\vee$, we obtain

$$\mathscr{O}_X(-n)^{\oplus d} \twoheadrightarrow \mathscr{F},$$

as required. □

An absolute notion, related to very ampleness (which is a relative notion) is the following.

Definition 8.3.7 (Ample invertible sheaf)**.** Let X be a noetherian scheme, $\mathscr{L}$ an invertible sheaf on X. Then $\mathscr{L}$ is *ample* if for every $\mathscr{F} \in \mathsf{Coh}(X)$, there is a positive integer $n_0 = n_0(\mathscr{F}) > 0$ such that for every $n \geq n_0$ the sheaf $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^n$ is globally generated.

Example 8.3.8. Any coherent sheaf on an affine scheme is globally generated. Thus any invertible sheaf is ample.

Example 8.3.9. Let A be a noetherian ring, and set $Y = \operatorname{Spec} A$. Let $f : X \to Y$ be a projective morphism, and let $\mathscr{L}$ be an f-very ample invertible sheaf on X. Then $\mathscr{L}$ is ample by Theorem 8.3.4.

PROPOSITION 8.3.10. *Let X be a noetherian scheme, $\mathscr{L}$ an invertible sheaf on X. The following are equivalent:*

(1) *$\mathscr{L}$ is ample.*

(2) *$\mathscr{L}^m$ is ample for every $m > 0$.*

(3) *$\mathscr{L}^m$ is ample for some $m > 0$.*

Proof. The implications (1) ⇒ (2) ⇒ (3) are obvious. Let us prove (3) ⇒ (1). Fix a coherent sheaf $\mathscr{F} \in \mathsf{Coh}(X)$. Because $\mathscr{L}^m$ is ample, for each of the sheaves

$$\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^k, \quad k = 0, 1, \ldots, m-1,$$

we can find an integer $\ell_k > 0$ such that for each $n \geq \ell_k$ the sheaf $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^k \otimes_{\mathscr{O}_X} (\mathscr{L}^m)^n$ is globally generated. Then, setting $n_0 = m \cdot \max\{\ell_0, \ell_1, \ldots, \ell_{m-1}\} > 0$, the ampleness condition gets fulfilled by $\mathscr{L}$. □

THEOREM 8.3.11 ([8, Ch. II, Thm. 7.6]). *Let A be a noetherian ring, $f : X \to \operatorname{Spec} A$ a morphism of finite type, $\mathscr{L}$ an invertible sheaf on X. Then $\mathscr{L}$ is ample if and only if $\mathscr{L}^m$ is f-very ample for some $m > 0$.*

Example 8.3.12. Consider the structure morphism $f : \mathbb{P}^n_{\mathbf{k}} \to \operatorname{Spec} \mathbf{k}$. Then $\mathscr{L} = \mathscr{O}_{\mathbb{P}^n_{\mathbf{k}}}(1)$ is f-very ample (we may just say 'very ample' when the target of f is a point). Let $d > 0$. Since $\mathscr{O}_{\mathbb{P}^n_{\mathbf{k}}}(d) = \mathscr{L}^d$ gives rise to the Veronese embedding, also $\mathscr{O}_{\mathbb{P}^n_{\mathbf{k}}}(d)$ is very ample. That is, $\mathscr{O}_{\mathbb{P}^n_{\mathbf{k}}}(d)$ is ample. On the other hand, $\mathrm{H}^0(\mathbb{P}^n_{\mathbf{k}}, \mathscr{O}_{\mathbb{P}^n_{\mathbf{k}}}(k)) = 0$ if $k < 0$, thus

$$\boxed{\mathscr{O}_{\mathbb{P}^n_{\mathbf{k}}}(d) \text{ ample} \iff \mathscr{O}_{\mathbb{P}^n_{\mathbf{k}}}(d) \text{ very ample} \iff d > 0.}$$

Here is another celebrated theorem by Serre.

THEOREM 8.3.13 (Serre). *Let X be a noetherian scheme. The following are equivalent:*

(1) *X is affine.*

(2) $\mathrm{H}^{>0}(X,\mathcal{F})=0$ *for every* $\mathcal{F}\in\mathsf{QCoh}(X)$.

(3) $\mathrm{H}^1(X,\mathscr{I})=0$ *for every coherent sheaf of ideals* $\mathscr{I}$.

Proof. The implication (2) $\Rightarrow$ (3) is obvious. The implication (3) $\Rightarrow$ (1) is hard, and can be found in [8, Ch. III, Thm. 3.7]. The implication (1) $\Rightarrow$ (2) goes as follows. Take $\mathcal{F}\in\mathsf{QCoh}(X)$, where $X=\operatorname{Spec}A$ is affine. Since A is noetherian, $\mathcal{F}=\widetilde{M}$ for an A-module M, that is recovered as $M=\Gamma(X,\mathcal{F})$. Let $M\xrightarrow{\sim} I^\bullet$ be an injective resolution in Mod_A. Then a key technical result states that each sheaf $\widetilde{I}^i$ is flasque [8, Ch. III, Thm. 3.4]. It follows that $\mathcal{F}\xrightarrow{\sim}\widetilde{I}^\bullet$ is a flasque resolution of $\mathcal{F}$, therefore we can use it to compute the cohomology of $\mathcal{F}$ by Theorem 8.1.20. We find, for each $i>0$,

$$\mathrm{H}^i(X,\mathcal{F})=h^i(\Gamma(X,\widetilde{I}^\bullet))=h^i(I^\bullet)=0$$

where the last identity follows from exactness of $I^\bullet$. $\square$

The following result computes the cohomology of line bundles on projective space.

THEOREM 8.3.14 (Cohomology of projective space). *Let A be a noetherian ring, $r>0$ an integer. Set $B=A[x_0,x_1,\ldots,x_r]$ and $X=\operatorname{Proj}B=\mathbb{P}^r_A$.*

(i) *There is a B-linear graded isomorphism* $B\xrightarrow{\sim}\bigoplus_{n\in\mathbb{Z}}\mathrm{H}^0(X,\mathscr{O}_X(n))$.

(ii) $\mathrm{H}^i(X,\mathscr{O}_X(n))=0$ *for* $0<i<r$ *and* $n\in\mathbb{Z}$.

(iii) $\mathrm{H}^r(X,\mathscr{O}_X(-r-1))=A$.

(iv) $\mathrm{H}^0(X,\mathscr{O}_X(n))\cong\mathrm{H}^r(X,\mathscr{O}_X(-n-r-1))^\vee$ *via a canonical perfect pairing.*

The above theorem is useful to prove a fundamental theorem in algebraic geometry: the cohomology of coherent sheaves on a projective variety is finite dimensional.

THEOREM 8.3.15 (Serre). *Let A be a noetherian ring. Let $f\colon X\to\operatorname{Spec}A$ be a projective morphism, $\mathscr{O}_X(1)$ an f-very ample invertible sheaf, and let $\mathcal{F}\in\mathsf{Coh}(X)$.*

(a) $\mathrm{H}^i(X,\mathcal{F})$ *is a finitely generated A-module for every $i\geq 0$.*

(b) *There exists an integer $n_0(\mathcal{F})$ such that for every $i>0$ and for every $n\geq n_0(\mathcal{F})$ we have* $\mathrm{H}^i(X,\mathcal{F}(n))=0$.

Proof. We proceed step by step.

Proof of (a). By f-very ampleness, there is an integer $r > 0$ and a closed immersion

$$\iota\colon X \hookrightarrow \mathbb{P}^r_A,$$

such that $\mathscr{O}_X(1) = \iota^*\mathscr{O}_{\mathbb{P}^r_A}(1)$. Since ι is finite, $\iota_*\mathcal{F} \in \mathsf{Coh}(\mathbb{P}^r_A)$ by Proposition 7.1.27 (c), and one has

$$\mathrm{H}^i(X,\mathcal{F}) = \mathrm{H}^i(\mathbb{P}^r_A, \iota_*\mathcal{F})$$

by Lemma 8.2.4. This reduces the problem to the case of $X = \mathbb{P}^r_A$. By Theorem 8.3.14, the claim holds for any sheaf of the form $\mathscr{O}_X(q)$, for $q \in \mathbb{Z}$. By addititivity of $\mathrm{H}^i(X,-)$, the same holds true for all finite direct sums of sheaves of this form.

We prove the theorem by descending induction on i, exploiting the vanishing for $i > r + \dim A$ (cf. Theorem 8.2.1).[1] By Corollary 8.3.6, any $\mathcal{F} \in \mathsf{Coh}(X)$ is a quotient of a sheaf $\mathcal{E}$ of the form $\mathcal{E} = \bigoplus_i \mathscr{O}_X(q_i)$. We then have an exact sequence

$$0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{F} \to 0,$$

in $\mathsf{Coh}(X)$, inducing a long exact sequence

$$\cdots \to \mathrm{H}^i(X,\mathcal{E}) \to \mathrm{H}^i(X,\mathcal{F}) \to \mathrm{H}^{i+1}(X,\mathcal{K}) \to \cdots$$

in which the last term is finitely generated by the inductive hypothesis, and the first one is finitely generated since we already confirmed the statements holds for sheaves of the form $\mathcal{E} = \bigoplus_i \mathscr{O}_X(q_i)$. Since A is noetherian, the middle term is also finitely generated.

Proof of (b). Twisting by $n \gg 0$ the above short exact sequence and taking cohomology yields

$$\cdots \to \mathrm{H}^i(X,\mathcal{E}(n)) \to \mathrm{H}^i(X,\mathcal{F}(n)) \to \mathrm{H}^{i+1}(X,\mathcal{K}(n)) \to \cdots$$

where, again, the last term vanishes by the inductive hypothesis and the first one vanishes because $\mathcal{E} = \bigoplus_i \mathscr{O}_X(q_i)$. This shows that $\mathrm{H}^i(X,\mathcal{F}(n)) = 0$ for $n \gg 0$. Since there are finitely many i for which the statement is formulated, one can find an n_0 for each individual i and then define $n_0(\mathcal{F})$ to be their maximum. □

PROPOSITION 8.3.16. *Let A be a noetherian ring, $f\colon X \to \operatorname{Spec} A$ a proper morphism, $\mathscr{L}$ an invertible sheaf on X. The following conditions are equivalent:*

(i) *$\mathscr{L}$ is ample, and*

(ii) *for each $\mathcal{F} \in \mathsf{Coh}(X)$, there is an integer $n = n_0(\mathcal{F})$ such that for each $i > 0$ and each $n \geq n_0$ we have $\mathrm{H}^i(X, \mathcal{F} \otimes_{\mathscr{O}_X} \mathscr{L}^n) = 0$.*

[1] In fact, cohomology vanishes for $i > r$.

Proof. The implication (i) $\Rightarrow$ (ii) goes as follows. Note that f is of finite type, thus Theorem 8.3.11 applies: since $\mathscr{L}$ is ample, $\mathscr{L}^m$ is f-very ample for some $m > 0$. Since f is already proper, it is then projective, just by Definition 8.3.2. Now apply Theorem 8.3.15 (b) to the sheaves

$$\mathcal{F}, \mathcal{F} \otimes_{\mathscr{O}_X} \mathscr{L}, \ldots, \mathcal{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{m-1}.$$

Serre's theorem gives, for each $k = 0, 1, \ldots, m-1$, an integer ℓ_k such that

$$\mathrm{H}^{>0}(X, \mathcal{F} \otimes_{\mathscr{O}_X} \mathscr{L}^k \otimes_{\mathscr{O}_X} (\mathscr{L}^m)^n) = 0, \quad n \geq \ell_k.$$

Then fixing $n_0 \geq m \cdot \max\{\ell_0, \ell_1, \ldots, \ell_{m-1}\}$ yields the sought after implication. For the converse, see [8, Ch. III, Prop. 5.3]. $\square$

8.3.1 Application: the genus-degree formula

We now see a famous classical theorem that can be proven in an elementary fashion using cohomology.

We first need to introduce some terminology.

Let $\mathbf{k}$ be a field, X a projective $\mathbf{k}$-scheme. Define (thanks to Theorem 8.3.15), for any $\mathcal{F} \in \mathsf{Coh}(X)$, the integer

$$\chi(\mathcal{F}) = \sum_{i \geq 0} (-1)^i h^i(X, \mathcal{F}),$$

where $h^i(X, \mathcal{F}) = \dim_{\mathbf{k}} \mathrm{H}^i(X, \mathcal{F})$.

LEMMA 8.3.17. *Let* $\mathbf{k}$ *be a field,* X *a projective* $\mathbf{k}$*-scheme. Let*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

be an exact sequence in $\mathsf{Coh}(X)$*. Then*

$$\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'').$$

Proof. This follows from the long exact cohomology sequence. $\square$

Definition 8.3.18 (Arithmetic genus)**.** Let X be a projective $\mathbf{k}$-scheme of dimension r. The *arithmetic genus* of X is the integer

$$p_a(X) = (-1)^r(\chi(\mathscr{O}_X) - 1).$$

Note that if X is integral then $h^0(X, \mathscr{O}_X) = 1$, so

$$p_a(X) = \sum_{1 \leq i \leq r} (-1)^{r+i} h^i(X, \mathscr{O}_X) = \sum_{0 \leq i < r} (-1)^i h^{r-i}(X, \mathscr{O}_X).$$

In particular, for any integral curve X, we have

$$p_a(X) = h^1(X, \mathscr{O}_X). \tag{8.3.1}$$

For instance,

$$p_a(\mathbb{P}^1_{\mathbf{k}}) = h^1(\mathbb{P}^1_{\mathbf{k}}, \mathscr{O}_{\mathbb{P}^1_{\mathbf{k}}}) = 0. \tag{8.3.2}$$

THEOREM 8.3.19 (Genus-degree formula). *Let $C \subset \mathbb{P}^2_{\mathbf{k}}$ be a smooth plane curve of degree d. Then*

$$p_a(C) = \frac{(d-1)(d-2)}{2}.$$

Proof. The formula is equivalent to

$$\chi(\mathscr{O}_C) = 1 - \frac{(d-1)(d-2)}{2}$$

by the relation $p_a(C) = 1 - \chi(\mathscr{O}_C)$. We proceed in several steps.

Step 1. Let S be a smooth surface, $X, Y \subset S$ two linearly equivalent effective divisors. Then $\chi(\mathscr{O}_S(X)) = \chi(\mathscr{O}_S(Y))$. This follows from Proposition 7.2.13, since linearly equivalent divisors give rise to isomorphic line bundles.

Step 2. Let S be a smooth surface, $X, Y \subset S$ two curves intersecting transversely in e points. Then

$$\chi(\mathscr{O}_{X+Y}) = \chi(\mathscr{O}_X) + \chi(\mathscr{O}_Y) - e.$$

This follows from the short exact sequence

$$0 \to \mathscr{O}_{X+Y} \to \mathscr{O}_X \oplus \mathscr{O}_Y \to \mathscr{O}_{X\cap Y} \to 0$$

and the additivity of χ (cf. Lemma 8.3.17), which yields

$$\chi(\mathscr{O}_X) + \chi(\mathscr{O}_Y) = \chi(\mathscr{O}_X \oplus \mathscr{O}_Y) = \chi(\mathscr{O}_{X+Y}) + e,$$

where $\chi(\mathscr{O}_{X\cap Y}) = e$ follows from transversality.

Step 3. We start the proof of the formula now. We proceed by induction on d, the case $d = 1$ being settled by (8.3.2). Note that the formula also holds for $d = 2$, since any smooth conic is isomorphic to $\mathbb{P}^1_{\mathbf{k}}$. Let us assume $d \geq 3$. We have that C is linearly equivalent to a divisor of the form $X + L$, where X is a smooth curve of degree $d-1$ and L is a line, such that X and L intersect transversely, necessarily in $d-1$ points. We obtain

$$\begin{aligned}
\chi(\mathscr{O}_C) &= \chi(\mathscr{O}_{X+L}) && \text{by Step 1}\\
&= \chi(\mathscr{O}_X) + \chi(\mathscr{O}_L) - (d-1) && \text{by Step 2}\\
&= 1 - \frac{(d-2)(d-3)}{2} + 1 - d + 1 && \text{by induction}\\
&= 1 - \frac{(d-2)(d-3+2)}{2}\\
&= 1 - \frac{(d-1)(d-2)}{2}.
\end{aligned}$$

The *degree-genus formula* for plane curves is proved. □

8.4 Local and global Ext

Let $(X,\mathscr{O}_X)$ be a ringed space, $\mathcal{F}$ an $\mathscr{O}_X$-module. We have covariant left exact functors

$$\mathsf{Mod}_{\mathscr{O}_X} \xrightarrow{\mathrm{Hom}(\mathcal{F},-)} \mathsf{Ab} \qquad\qquad \mathsf{Mod}_{\mathscr{O}_X} \xrightarrow{\mathscr{H}om(\mathcal{F},-)} \mathsf{Mod}_{\mathscr{O}_X}.$$

Note that we are dropping the subscript $\mathscr{O}_X$ everywhere. Of course, by definition of local Hom, one has $\mathrm{Hom}(\mathcal{F},-)=\Gamma(X,\mathscr{H}om(\mathcal{F},-))$. Since $\mathsf{Mod}_{\mathscr{O}_X}$ has enough injectives (cf. Lemma 8.1.9), we can define

$$\mathrm{Ext}^i(\mathcal{F},-)=\mathbf{R}^i\mathrm{Hom}(\mathcal{F},-), \quad \mathscr{E}xt^i(\mathcal{F},-)=\mathbf{R}^i\mathscr{H}om(\mathcal{F},-)$$

to be the right derived functors of $\mathrm{Hom}(\mathcal{F},-)$ and $\mathscr{H}om(\mathcal{F},-)$ respectively, for all $i\geq 0$. They are called *global Ext* and *local Ext*, respectively.

Remark 8.4.1. We have the following properties:

(i) $\mathrm{Ext}^0(\mathcal{F},\mathcal{G})=\mathrm{Hom}(\mathcal{F},\mathcal{G})$ and $\mathscr{E}xt^0(\mathscr{O}_X,\mathcal{G})=\mathscr{H}om(\mathscr{O}_X,\mathcal{G})=\mathcal{G}$ for all $\mathcal{G}\in\mathsf{Mod}_{\mathscr{O}_X}$. Since $\mathscr{H}om(\mathscr{O}_X,-)=\mathrm{id}$, which is exact, we have $\mathscr{E}xt^{>0}(\mathscr{O}_X,-)=0$. And since $\mathrm{Hom}(\mathscr{O}_X,-)=\Gamma(X,-)$, we have $\mathrm{Ext}^i(\mathscr{O}_X,-)=\mathrm{H}^i(X,-)$ for all $i\geq 0$.

(ii) $\mathrm{Ext}^{>0}(\mathcal{F},\mathcal{I})=0$ and $\mathscr{E}xt^{>0}(\mathcal{F},\mathcal{I})=0$ for all injective $\mathcal{I}\in\mathsf{Mod}_{\mathscr{O}_X}$.

(iii) For any open subset $U\subset X$, we have

$$\mathscr{E}xt^i(\mathcal{F},\mathcal{G})|_U=\mathscr{E}xt^i(\mathcal{F}|_U,\mathcal{G}|_U).$$

This follows from the fact that the restriction of an injective sheaf to an open subset is still injective. See [8, Ch. III, Prop. 6.2] for more details.

(iv) Given $\mathcal{G}\in\mathsf{Mod}_{\mathscr{O}_X}$ and a short exact sequence

$$0\to\mathcal{F}'\to\mathcal{F}\to\mathcal{F}''\to 0 \tag{8.4.1}$$

in $\mathsf{Mod}_{\mathscr{O}_X}$, one has a long exact sequence

$$\begin{aligned}0\to\mathrm{Hom}(\mathcal{F}'',\mathcal{G})\to\mathrm{Hom}(\mathcal{F},\mathcal{G})\to\mathrm{Hom}(\mathcal{F}',\mathcal{G})\to\mathrm{Ext}^1(\mathcal{F}'',\mathcal{G})&\\ \to\mathrm{Ext}^1(\mathcal{F},\mathcal{G})\to\mathrm{Ext}^1(\mathcal{F}',\mathcal{G})\to\mathrm{Ext}^2(\mathcal{F}'',\mathcal{G})\to\mathrm{Ext}^2(\mathcal{F},\mathcal{G})\to\cdots&\end{aligned}$$

in Ab, as well as a long exact sequence

$$\begin{aligned}0\to\mathscr{H}om(\mathcal{F}'',\mathcal{G})\to\mathscr{H}om(\mathcal{F},\mathcal{G})\to\mathscr{H}om(\mathcal{F}',\mathcal{G})\to\mathscr{E}xt^1(\mathcal{F}'',\mathcal{G})&\\ \to\mathscr{E}xt^1(\mathcal{F},\mathcal{G})\to\mathscr{E}xt^1(\mathcal{F}',\mathcal{G})\to\mathscr{E}xt^2(\mathcal{F}'',\mathcal{G})\to\mathscr{E}xt^2(\mathcal{F},\mathcal{G})\to\cdots&\end{aligned}$$

in $\mathsf{Mod}_{\mathscr{O}_X}$. Indeed, replacing $\mathcal{G}$ with an injective resolution $\mathcal{I}^\bullet$ and applying to (8.4.1) the exact functors $\operatorname{Hom}(-,\mathcal{I}^k)\colon \mathsf{Mod}_{\mathscr{O}_X} \to \mathsf{Ab}$, we get a short exact sequence

$$0 \to \operatorname{Hom}(\mathcal{F}'',\mathcal{I}^\bullet) \to \operatorname{Hom}(\mathcal{F},\mathcal{I}^\bullet) \to \operatorname{Hom}(\mathcal{F}',\mathcal{I}^\bullet) \to 0$$

in $\operatorname{Com}(\mathsf{Ab})$. The associated long exact sequence of cohomology groups gives the first long exact sequence. The second one is the long exact sequence in cohomology arising from the short exact sequence

$$0 \to \mathscr{H}om(\mathcal{F}'',\mathcal{I}^\bullet) \to \mathscr{H}om(\mathcal{F},\mathcal{I}^\bullet) \to \mathscr{H}om(\mathcal{F}',\mathcal{I}^\bullet) \to 0$$

in $\operatorname{Com}(\mathsf{Mod}_{\mathscr{O}_X})$, obtained by applying the exact functors $\mathscr{H}om(-,\mathcal{I}^k)\colon \mathsf{Mod}_{\mathscr{O}_X} \to \mathsf{Mod}_{\mathscr{O}_X}$ to (8.4.1).

One can also prove that local Exts, a priori defined via injective resolutions in the second variable, can in fact be computed via *locally free resolutions* in the first variable. That does *not* mean that $\mathscr{E}xt^i(-,\mathcal{G})$ makes sense as the right derived functor of $\mathscr{H}om(-,\mathcal{G})$, since $\mathsf{Mod}_{\mathscr{O}_X}$ does not have enough projective objects. What *is* true (see [8, Ch. III, Prop. 6.5]) is that if you have an exact sequence

$$\cdots \to \mathscr{L}_1 \to \mathscr{L}_0 \to \mathcal{F} \to 0$$

in $\mathsf{Mod}_{\mathscr{O}_X}$, where $\mathscr{L}_k$ are all locally free of finite rank, then

$$\mathscr{E}xt^i(\mathcal{F},\mathcal{G}) \cong h^i(\mathscr{H}om(\mathscr{L}_\bullet,\mathcal{G}))$$

for any $\mathcal{G} \in \mathsf{Mod}_{\mathscr{O}_X}$ and for any $i \geq 0$. Note that this way of computing local Exts is, in fact, quite useful: any projective scheme over a noetherian ring has enough locally frees, i.e. any coherent sheaf is a quotient of a locally free sheaf (cf. Corollary 8.3.6), thus locally free resolutions always exist in this case (although they might be infinite).

LEMMA 8.4.2 ([8, Ch. III, Prop. 6.7]). *Let $\mathcal{E} \in \mathsf{Mod}_{\mathscr{O}_X}$ be locally free of finite rank. Fix $\mathcal{F},\mathcal{G} \in \mathsf{Mod}_{\mathscr{O}_X}$. Then for each $i \geq 0$ one has*

$$\begin{aligned}
&\operatorname{Ext}^i(\mathcal{F} \otimes_{\mathscr{O}_X} \mathcal{E},\mathcal{G}) = \operatorname{Ext}^i(\mathcal{F},\mathcal{E}^\vee \otimes_{\mathscr{O}_X} \mathcal{G}),\\
&\mathscr{E}xt^i(\mathcal{F} \otimes_{\mathscr{O}_X} \mathcal{E},\mathcal{G}) = \mathscr{E}xt^i(\mathcal{F},\mathcal{E}^\vee \otimes_{\mathscr{O}_X} \mathcal{G}) = \mathscr{E}xt^i(\mathcal{F},\mathcal{G}) \otimes_{\mathscr{O}_X} \mathcal{E}^\vee,
\end{aligned}$$

where $\mathcal{E}^\vee = \mathscr{H}om(\mathcal{E},\mathscr{O}_X)$ is the $\mathscr{O}_X$-linear dual of $\mathcal{E}$.

PROPOSITION 8.4.3 ([8, Ch. III, Prop. 6.9]). *Let A be a noetherian ring. Let $X \to \operatorname{Spec} A$ be a projective scheme with a relatively very ample invertible sheaf $\mathscr{O}_X(1)$. Fix $\mathcal{F},\mathcal{G} \in \mathsf{Coh}(X)$ and $i \geq 0$. Then there is an integer $n_0 = n_0(\mathcal{F},\mathcal{G},i) > 0$ such that for $n \geq n_0$ we have*

$$\operatorname{Ext}^i(\mathcal{F},\mathcal{G}(n)) \cong \Gamma(X,\mathscr{E}xt^i(\mathcal{F},\mathcal{G}(n))).$$

Proof. The statement is true for $i=0$ with no assumption on $\mathcal{F}$, $\mathcal{G}$ or n.

Let us assume $i>0$. If $\mathcal{F}=\mathcal{O}_X$, by Remark 8.4.1 (i) we have

$$\mathrm{Ext}^i(\mathcal{O}_X,\mathcal{G}(n))=\mathrm{H}^i(X,\mathcal{G}(n)).$$

But $\mathrm{H}^i(X,\mathcal{G}(n))=0$ for $n\gg 0$ by Theorem 8.3.15 (b). On the other hand, $\mathscr{E}xt^{>0}(\mathcal{O}_X,-)=0$ again by Remark 8.4.1 (i). The result is then true for $\mathcal{O}_X$. From this we deduce the case where $\mathcal{F}$ is locally free. Indeed, in this case, by Lemma 8.4.2, we have

$$\mathrm{Ext}^i(\mathcal{F},\mathcal{G}(n))=\mathrm{Ext}^i(\mathcal{O}_X,\mathcal{F}^\vee\otimes_{\mathcal{O}_X}\mathcal{G}(n))$$
$$\mathscr{E}xt^i(\mathcal{F},\mathcal{G}(n))=\mathscr{E}xt^i(\mathcal{O}_X,\mathcal{F}^\vee\otimes_{\mathcal{O}_X}\mathcal{G}(n)).$$

For the general case, use Corollary 8.3.6 to write down a short exact sequence

$$0\to\mathcal{K}\to\mathcal{E}\to\mathcal{F}\to 0$$

in $\mathsf{Coh}(X)$, where $\mathcal{E}$ is finite locally free (and $\mathcal{K}$ is coherent since $\mathsf{Coh}(X)$ is abelian). Since $\mathcal{E}$ is locally free, we have

$$\mathscr{E}xt^1(\mathcal{E},\mathcal{G}(n))=\mathscr{E}xt^1(\mathcal{O}_X,\mathcal{E}^\vee\otimes_{\mathcal{O}_X}\mathcal{G}(n))=0$$

for all $n\in\mathbb{Z}$, once more by Remark 8.4.1 (i). So the long exact sequence for local Homs (cf. Remark 8.4.1 (iv)) is, for $n\gg 0$,

$$0\to\mathrm{Hom}(\mathcal{F},\mathcal{G}(n))\to\mathrm{Hom}(\mathcal{E},\mathcal{G}(n))\to\mathrm{Hom}(\mathcal{K},\mathcal{G}(n))\to\mathrm{Ext}^1(\mathcal{F},\mathcal{G}(n))\to 0 \tag{8.4.2}$$

where the 0 on the right is due to Theorem 8.3.15 (b). The sequence continues with isomorphisms

$$\mathrm{Ext}^i(\mathcal{K},\mathcal{G}(n))\xrightarrow{\sim}\mathrm{Ext}^{i+1}(\mathcal{F},\mathcal{G}(n)),\quad i>0. \tag{8.4.3}$$

Similarly, we have

$$0\to\mathscr{H}om(\mathcal{F},\mathcal{G}(n))\to\mathscr{H}om(\mathcal{E},\mathcal{G}(n))\to\mathscr{H}om(\mathcal{K},\mathcal{G}(n))\to\mathscr{E}xt^1(\mathcal{F},\mathcal{G}(n))\to 0 \tag{8.4.4}$$

for $n\gg 0$, and isomorphisms

$$\mathscr{E}xt^i(\mathcal{K},\mathcal{G}(n))\xrightarrow{\sim}\mathscr{E}xt^{i+1}(\mathcal{F},\mathcal{G}(n)),\quad i>0. \tag{8.4.5}$$

By [8, Ch. III, Ex. 5.10], taking global sections $\Gamma(X,-)$ in (8.4.4), we get an exact sequence

$$0\to\mathrm{Hom}(\mathcal{F},\mathcal{G}(n))\to\mathrm{Hom}(\mathcal{E},\mathcal{G}(n))\to\mathrm{Hom}(\mathcal{K},\mathcal{G}(n))\to\Gamma(X,\mathscr{E}xt^1(\mathcal{F},\mathcal{G}(n)))\to 0,$$

with n possibly a little larger than in (8.4.2). This sequence must agree with (8.4.2), which settles the $i=1$ case. The case $i>1$ follows by induction and the displayed isomorphisms (8.4.3) and (8.4.5). $\square$

A remark on the notation ‘Ext’. A short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \tag{8.4.6}$$

in $\mathsf{Mod}_{\mathscr{O}_X}$ is called an *extension of* $\mathcal{F}''$ *by* $\mathcal{F}'$. One can consider

$$\mathrm{Extensions}(\mathcal{F}'', \mathcal{F}') = \left\{ \text{isomorphism classes of extensions of } \mathcal{F}'' \text{ by } \mathcal{F}' \right\},$$

where an isomorphism of extensions is an isomorphism of short exact sequences inducing identities on the leftmost and rightmost objects. The long exact sequence induced by the functor $\mathrm{Hom}(\mathcal{F}'', -)$ is

$$0 \to \mathrm{Hom}(\mathcal{F}'', \mathcal{F}') \to \mathrm{Hom}(\mathcal{F}'', \mathcal{F}) \to \mathrm{Hom}(\mathcal{F}'', \mathcal{F}'') \xrightarrow{\delta^1} \mathrm{Ext}^1(\mathcal{F}'', \mathcal{F}') \to \cdots$$

and one can show that

$$\mathrm{Extensions}(\mathcal{F}'', \mathcal{F}') = \mathrm{Ext}^1(\mathcal{F}'', \mathcal{F}').$$

The correspondence sends the isomorphism class of an extension (8.4.6) to the element $\delta^1(\mathrm{id}_{\mathcal{F}''})$.

Exercise 8.4.4. Let X be a noetherian scheme having enough locally frees. Define the homological dimension of $\mathcal{F} \in \mathsf{Coh}(X)$ to be

$$\mathsf{hd}(\mathcal{F}) = \inf\left\{ d \in \mathbb{N} \;\middle|\; \mathcal{F} \text{ has a locally free resolution of length } d \right\},$$

or ∞ if no finite locally free resolution exists. Show that

- $\mathscr{E}xt^1(\mathcal{F}, -) = 0$ on the whole of $\mathsf{Mod}_{\mathscr{O}_X}$ if and only if $\mathcal{F}$ is locally free, and
- $\mathsf{hd}(\mathcal{F}) \le n$ if and only if $\mathscr{E}xt^{>n}(\mathcal{F}, -) = 0$ on the whole of $\mathsf{Mod}_{\mathscr{O}_X}$.

Remark 8.4.5. As a consequence of Exercise 8.4.4, if we dualise the sequence of Theorem 7.3.21 arising from an inclusion of smooth irreducible $\mathbf{k}$-varieties $\iota\colon X \hookrightarrow Y$, we get a short exact sequence

$$0 \to \mathcal{T}_X \to \iota^*\mathcal{T}_Y \to \mathcal{N}_{X/Y} \to 0,$$

since $\mathscr{E}xt^1(\Omega_X, \mathscr{O}_X) = 0$ by local freeness of Ω_X.

8.5 Serre duality on projective varieties

To state Serre duality, we have to introduce the *canonical sheaf*. We start from a smooth integral $\mathbf{k}$-variety X of dimension d. In this case, we set (cf. Definition 7.3.16)

$$\omega_X = \det \Omega_X = \wedge^d \Omega_X.$$

This is an invertible sheaf, for the wedge construction $\wedge^p$ applied to a locally free sheaf of rank d (such as Ω_X, cf. Proposition 7.3.14) produces a locally free sheaf of rank $\binom{d}{p}$.

PROPOSITION 8.5.1. *If* $X=\mathbb{P}^n_{\mathbf{k}}$, *then* $\omega_X=\mathscr{O}_X(-n-1)$. *In particular, thanks to* Theorem 8.3.14, *we have* $\mathrm{H}^n(X,\omega_X)\cong \mathrm{H}^0(X,\mathscr{O}_X)^\vee=\mathbf{k}$.

Proof. This has been proved more generally in Corollary 7.3.18. □

THEOREM 8.5.2. *Let* $X=\mathbb{P}^n_{\mathbf{k}}$. *Then the natural pairing*

$$\mathrm{Hom}(\mathcal{F},\omega_X)\times \mathrm{H}^n(X,\mathcal{F})\longrightarrow \mathrm{H}^n(X,\omega_X)\xrightarrow{\sim}\mathbf{k}$$

is a perfect pairing of finite dimensional $\mathbf{k}$*-vector spaces, for any* $\mathcal{F}\in\mathsf{Coh}(X)$. *Moreover, we have natural isomorphisms*

$$\mathrm{Ext}^i(\mathcal{F},\omega_X)\xrightarrow{\sim}\mathrm{H}^{n-i}(X,\mathcal{F})^\vee,\quad i\geq 0. \tag{8.5.1}$$

Proof. Given $\phi\in\mathrm{Hom}(\mathcal{F},\omega_X)$, one has an induced morphism $\mathrm{H}^n(X,\mathcal{F})\to\mathrm{H}^n(X,\omega_X)$ by functoriality of cohomology. This defines the pairing. If $\mathcal{F}=\mathscr{O}_X(q)$ for $q\in\mathbb{Z}$, then

$$\mathrm{Hom}(\mathcal{F},\omega_X)=\mathrm{Hom}(\mathscr{O}_X,\omega_X(-q))=\mathrm{H}^0(X,\mathscr{O}_X(-n-1-q))=\mathrm{H}^n(X,\mathcal{F})^\vee$$

by Lemma 8.4.2 (first equality), Proposition 8.5.1 (second equality) and Theorem 8.3.14 (iv) (third equality). By additivity of cohomology, the result follows also for sheaves of the form $\mathcal{F}=\bigoplus_{1\leq i\leq s}\mathscr{O}_X(q_i)$. If $\mathcal{F}$ is an arbitrary coherent sheaf, then it fits into an exact sequence

$$\mathcal{G}_1\to\mathcal{G}_0\to\mathcal{F}\to 0,$$

where both $\mathcal{G}_0$ and $\mathcal{G}_1$ are direct sums of line bundles. The functors $\mathrm{Hom}(-,\omega_X)$ and $\mathrm{H}^n(X,-)^\vee$ are both contravariant and left exact. Thus we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \mathrm{Hom}(\mathcal{F},\omega_X) \arrow[r] \arrow[d, "\alpha"] & \mathrm{Hom}(\mathcal{G}_0,\omega_X) \arrow[r] \arrow[d, "\wr"] & \mathrm{Hom}(\mathcal{G}_1,\omega_X) \arrow[d, "\wr"] \\
0 \arrow[r] & \mathrm{H}^n(X,\mathcal{F})^\vee \arrow[r] & \mathrm{H}^n(X,\mathcal{G}_0)^\vee \arrow[r] & \mathrm{H}^n(X,\mathcal{G}_1)^\vee
\end{tikzcd}$$

with exact rows. By the 4-Lemma, α is surjective. However, it is also clearly injective, which settles the case $i=0$. For the proof of the isomorphisms (8.5.1) in the case $i>0$, see [8, Ch. III, Thm. 7.1]. □

Definition 8.5.3 (Dualising sheaf)**.** Let X be a proper n-dimensional $\mathbf{k}$-scheme. A *dualising sheaf* for X is a pair (ω_X°,t) where $\omega_X^\circ\in\mathsf{Coh}(X)$ and $t\colon\mathrm{H}^n(X,\omega_X^\circ)\to\mathbf{k}$ is a $\mathbf{k}$-linear map (called a *trace morphism*) such that for any $\mathcal{F}\in\mathsf{Coh}(X)$ the pairing

$$\mathrm{Hom}(\mathcal{F},\omega_X^\circ)\times\mathrm{H}^n(X,\mathcal{F})\to\mathrm{H}^n(X,\omega_X^\circ)\xrightarrow{t}\mathbf{k}$$

is perfect, i.e. it induces an isomorphism

$$\mathrm{Hom}(\mathcal{F},\omega_X^\circ)\xrightarrow{\sim}\mathrm{H}^n(X,\mathcal{F})^\vee.$$

A dualising sheaf either does not exist or is unique un to a unique isomorphism. This can be seen as follows. Indeed, by definition, a dualising sheaf (ω_X°, t) represents the functor $\mathsf{Coh}(X) \to \mathsf{Mod}_{\mathbf{k}}$ sending $\mathscr{F} \mapsto \mathrm{H}^n(X,\mathscr{F})^\vee$.

What about *existence* of a dualising sheaf? We know by Theorem 8.5.2 that, on $\mathbb{P}^n_{\mathbf{k}}$, the invertible sheaf

$$\omega^\circ_{\mathbb{P}^n_{\mathbf{k}}} = \mathscr{O}_{\mathbb{P}^n_{\mathbf{k}}}(-n-1) = \omega_{\mathbb{P}^n_{\mathbf{k}}}$$

works. If $\iota\colon X \hookrightarrow \mathbb{P}^n_{\mathbf{k}}$ is a closed immersion, where $\dim X = r$, and $\mathscr{F} \in \mathsf{Coh}(X)$, we know that

$$\mathrm{H}^r(X,\mathscr{F})^\vee = \mathrm{H}^r(\mathbb{P}^n_{\mathbf{k}}, \iota_*\mathscr{F})^\vee = \mathrm{Ext}^{n-r}(\iota_*\mathscr{F}, \omega_{\mathbb{P}^n_{\mathbf{k}}}),$$

where the last identity is again by by Theorem 8.5.2. Now, we want this $\mathbf{k}$-vector space to be isomorphic to $\mathrm{Hom}(\mathscr{F}, \omega_X^\circ)$ (functorially). The most naive thing that comes to mind is to turn the sought after isomorphism

$$\mathrm{Hom}(\mathscr{F}, \omega_X^\circ) \cong \mathrm{Ext}^{n-r}(\iota_*\mathscr{F}, \omega_{\mathbb{P}^n_{\mathbf{k}}})$$

into a sheaf-theoretic one and set $\mathscr{F} = \mathscr{O}_X$. This yields the hypothetical identification

$$\iota_*\omega_X^\circ = \iota_*\mathscr{H}om(\mathscr{O}_X, \omega_X^\circ) \overset{?}{=} \mathscr{E}xt^{n-r}(\iota_*\mathscr{O}_X, \omega_{\mathbb{P}^n_{\mathbf{k}}}).$$

From this, one can make the guess

$$\iota^*\iota_*\omega_X^\circ = \omega_X^\circ \overset{?}{=} \iota^*\mathscr{E}xt^{n-r}(\iota_*\mathscr{O}_X, \omega_{\mathbb{P}^n_{\mathbf{k}}}).$$

This turns out correct!

One can prove that a dualising sheaf exists on *any* proper $\mathbf{k}$-scheme. If $\iota\colon X \hookrightarrow \mathbb{P}^n_{\mathbf{k}}$ is a closed subscheme of codimension c (in particular X is *projective*), set

$$\omega_X^\circ = \iota^*\mathscr{E}xt^c(\iota_*\mathscr{O}_X, \omega_{\mathbb{P}^n_{\mathbf{k}}}). \tag{8.5.2}$$

This is a dualising sheaf on X. The duality theorem for *smooth*[2] projective schemes over a field is the following.

THEOREM 8.5.4 ([8, Ch. III, Thm. 7.6]). *Let X be an r-dimensional smooth projective variety over a field $\mathbf{k}$. Let ω_X° be a dualising sheaf. Then, for all $i \geq 0$ and $\mathscr{F} \in \mathsf{Coh}(X)$, there are functorial isomorphisms*

$$\mathrm{Ext}^i(\mathscr{F}, \omega_X^\circ) \xrightarrow{\ \sim\ } \mathrm{H}^{r-i}(X,\mathscr{F})^\vee.$$

The dualising sheaf ω_X° appearing in the statement of Theorem 8.5.4 comes directly from the one of the ambient projective space $\mathbb{P}^n_{\mathbf{k}}$, via Equation (8.5.2). However, one can in fact prove, with some work, that

$$\omega_X^\circ = \iota^*\omega_{\mathbb{P}^n_{\mathbf{k}}} \otimes_{\mathscr{O}_X} \wedge^c(\mathscr{I}/\mathscr{I}^2)^\vee,$$

[2] In fact, Cohen–Macaulay is enough. But we are not going into Cohen–Macaulay schemes in this course.

where $\mathscr{I} \subset \mathscr{O}_{\mathbb{P}^n_{\mathbf{k}}}$ is the ideal sheaf of $\iota\colon X \hookrightarrow \mathbb{P}^n_{\mathbf{k}}$ [8, Ch. 3, Thm. 7.11]. By Theorem 7.3.21, we then have that

$$\boxed{\text{if } X \text{ is smooth, then } \omega_X^\circ \text{ is an invertible sheaf.}}$$

Moreover, by the adjunction formula (cf. Corollary 7.3.22), we see that the canonical line bundle ω_X agrees with the dualising sheaf, i.e. that

$$\wedge^r \Omega_X = \omega_X = \omega_X^\circ = \iota^* \omega_{\mathbb{P}^n_{\mathbf{k}}} \otimes_{\mathscr{O}_X} \wedge^c (\mathscr{I}/\mathscr{I}^2)^\vee.$$

In particular, for a smooth r-dimensional $\mathbf{k}$-variety X, we have an isomorphism

$$\mathrm{H}^r(X, \omega_X) \cong \mathbf{k},$$

which was not a priori obvious from the definition of ω_X.

Example 8.5.5. Let $\iota\colon X \hookrightarrow \mathbb{P}^n_{\mathbf{k}}$ be a smooth hypersurface of degree d. By Example 7.3.26, we have

$$\iota^* \mathscr{O}_{\mathbb{P}^n_{\mathbf{k}}}(d-n-1) = \omega_X = \omega_X^\circ.$$

Notation 8.5.6. A slight and hence tolerable abuse of notation is the following. If $X \hookrightarrow \mathbb{P}^n_{\mathbf{k}}$ is a hypersurface, then one sets

$$\mathscr{O}_X(m) = \mathscr{O}_{\mathbb{P}^n_{\mathbf{k}}}(m)|_X.$$

Example 8.5.7. Let $C \subset \mathbb{P}^2_{\mathbf{k}}$ be a smooth plane curve of genus $g = h^1(C, \mathscr{O}_C)$. Then, by Serre duality,

$$\mathrm{H}^1(C, \mathscr{O}_C)^\vee = \mathrm{Ext}^0(\mathscr{O}_C, \omega_C) = \mathrm{H}^0(C, \omega_C) = \mathrm{H}^0(C, \mathscr{O}_C(d-3)).$$

In particular, $h^0(C, \omega_C) = g$.

Example 8.5.8. Let $C \subset \mathbb{P}^2_{\mathbf{k}}$ be a smooth plane curve of genus $g = h^1(C, \mathscr{O}_C)$. We can, thanks to the adjunction formula, compute the degree of ω_C, as follows. We have

$$\omega_C = \mathscr{O}_{\mathbb{P}^2_{\mathbf{k}}}(d-3)|_C$$

by Example 8.5.5. Then, taking degrees, we find

$$\deg \omega_C = (d-3) \cdot \deg \mathscr{O}_{\mathbb{P}^2_{\mathbf{k}}}(1)|_C = (d-3)d.$$

The degree-genus formula (cf. Theorem 8.3.19) is equivalent to

$$g = 1 + \frac{(d-3)d}{2},$$

which yields the important relation

$$\deg \omega_C = 2g-2.$$

This formula holds for *any* smooth curve of genus g (cf. **??**).

Terminology 8.5.9. Any divisor K_X such that $\mathscr{O}_X(K_X) \cong \omega_X$ is called a *canonical divisor*. In other words, 'the' canonical divisor is only defined up to linear equivalence.

Example 8.5.10. Let $C \subset \mathbb{P}^2$ be a smooth quartic curve (hence of genus 3). Then $\deg \omega_C = 4$ and, in fact, $\omega_C = \mathscr{O}_C(1)$. Therefore a canonical divisor K_C consists of 4 *aligned points* on C.

A | Categories, functors, Yoneda Lemma

A.1 Minimal background on categories and functors

Definition A.1.1 (Category)**.** A *category* $\mathscr{C}$ is the datum of

(i) a class $\mathrm{Ob}(\mathscr{C})$ of 'objects',

(ii) a class $\mathrm{Hom}(\mathscr{C})$ of 'morphisms' (or 'arrows', or 'maps') between the objects,

(iii) class functions $d\colon \mathrm{Hom}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{C})$ and $t\colon \mathrm{Hom}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{C})$ specifying domain and target of every morphism,

(iv) for each pair of objects x and y, a subclass $\mathrm{Hom}_{\mathscr{C}}(x,y) \subset \mathrm{Hom}(\mathscr{C})$ of morphisms with domain x and target y,

(v) a binary operation

$$\mathrm{Hom}_{\mathscr{C}}(x,y) \times \mathrm{Hom}_{\mathscr{C}}(y,z) \xrightarrow{\circ} \mathrm{Hom}_{\mathscr{C}}(x,z),$$

called 'composition' of morphisms, for every triple of objects x, y and z.

Such data must fulfill the following axioms:

(CAT1) For every $x \in \mathrm{Ob}(\mathscr{C})$, there is an identity morphism $\mathrm{id}_x \in \mathrm{Hom}_{\mathscr{C}}(x,x)$ enjoying the properties

$$f \circ \mathrm{id}_x = f, \quad \mathrm{id}_y \circ g = g$$

for every morphism f with domain x, and for every morphism g with target y.

(CAT2) The associativity relation

$$(h \circ g) \circ f = h \circ (g \circ f)$$

holds for every triple (f,g,h) of composable morphisms.

Definition A.1.2 (Isomorphism)**.** Let $\mathscr{C}$ be a category, and fix two objects $x, y \in \mathrm{Ob}(\mathscr{C})$. An isomorphism between x and y is an invertible morphism $f \in \mathrm{Hom}_{\mathscr{C}}(x,y)$, i.e. a morphism $f\colon x \to y$ such that there exists a morphism $g\colon y \to x$ satisfying $f \circ g = \mathrm{id}_y$ and $g \circ f = \mathrm{id}_x$. Two objects $x, y \in \mathrm{Ob}(\mathscr{C})$ are said to be *isomorphic* when there is an isomorphism $x \to y$ (often denoted '$x \xrightarrow{\sim} y$').

Definition A.1.3 (Small and locally small)**.** A category $\mathscr{C}$ is *small* if both $\mathrm{Ob}(\mathscr{C})$ and $\mathrm{Hom}(\mathscr{C})$ are sets, and not proper classes. We say that $\mathscr{C}$ is *locally small* if $\mathrm{Hom}_{\mathscr{C}}(x, y)$ is a set, and not a proper class, for every pair of objects x and y. For a locally small category $\mathscr{C}$, the sets $\mathrm{Hom}_{\mathscr{C}}(x, y)$ are called *hom-sets*.

Example A.1.4. The following are familiar examples of categories:

- Sets, the category of sets with morphisms the functions between sets,
- Grp, the category of groups with morphisms the group homomorphisms,
- Ab, the category of abelian groups with morphisms the group homomorphisms,
- Rings, the category of rings with morphisms the ring homomorphisms,
- Fields, the category of fields with morphisms the field homomorphisms,
- $\mathsf{Vec}_{\mathbb{F}}$, the category of vector spaces over a field $\mathbb{F}$ with morphisms the $\mathbb{F}$-linear maps,
- Alg_R, the category of algebras over a ring R, with morphisms the R-algebra homomorphisms,
- Top, the category of topological spaces, with morphisms the continuous maps,
- Mod_R, the category of modules over a ring R, with morphisms the R-linear maps,
- Mfd, the category of smooth manifolds, with morphisms the C^∞ maps.

Remark A.1.5. The category Sets is locally small, but not small (Russell's Paradox). The same is true, by the same argument, for all the categories in Example A.1.4.

Definition A.1.6 (Functor)**.** Let $\mathscr{C}$ and $\mathscr{C}'$ be two categories. A functor from $\mathscr{C}$ to $\mathscr{C}'$, denoted $\mathsf{F}\colon \mathscr{C} \to \mathscr{C}'$, is the assignment of

- an object $\mathsf{F}(x) \in \mathrm{Ob}(\mathscr{C}')$ for every $x \in \mathrm{Ob}(\mathscr{C})$, and
- a morphism $\mathsf{F}(f) \in \mathrm{Hom}_{\mathscr{C}'}(\mathsf{F}(x), \mathsf{F}(y))$ for every morphism $f \in \mathrm{Hom}_{\mathscr{C}}(x, y)$,

subject to the following axioms:

(1) $\mathsf{F}(\mathrm{id}_x) = \mathrm{id}_{\mathsf{F}(x)}$ for every $x \in \mathrm{Ob}(\mathscr{C})$,

(2) $\mathsf{F}(g \circ f) = \mathsf{F}(g) \circ \mathsf{F}(f)$ for every pair (f, g) of composable arrows.

Remark A.1.7. By the axioms, functors preserve isomorphisms.

A functor as in Definition A.1.6 is said to be *covariant*. On the other hand, a *contravariant* functor $\mathsf{F}\colon \mathscr{C} \to \mathscr{C}'$ assigns a morphism $\mathsf{F}(f) \in \mathrm{Hom}_{\mathscr{C}'}(\mathsf{F}(y), \mathsf{F}(x))$ for every morphism $f \in \mathrm{Hom}_{\mathscr{C}}(x, y)$, and condition (2) becomes $\mathsf{F}(g \circ f) = \mathsf{F}(f) \circ \mathsf{F}(g)$. For instance, taking a K-vector space V to its dual $V^* = \mathrm{Hom}_K(V, K)$ is a contravariant functor.

Example A.1.8. Every category $\mathscr{C}$ admits an *identity functor* $\mathrm{Id}_{\mathscr{C}}\colon \mathscr{C} \to \mathscr{C}$, sending every object and every morphism to itself.

Define $\mathscr{C}^{\mathrm{op}}$ to be the category with objects $\mathrm{Ob}(\mathscr{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathscr{C})$ and with

$$\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(x, y) = \mathrm{Hom}_{\mathscr{C}}(y, x)$$

for every $x, y \in \mathrm{Ob}(\mathscr{C})$. Then a contravariant functor $\mathscr{C} \to \mathscr{C}'$ is the same as a covariant functor $\mathscr{C}^{\mathrm{op}} \to \mathscr{C}'$.

Definition A.1.9 (Natural transformation)**.** A *natural transformation* $\eta\colon \mathsf{F} \Rightarrow \mathsf{G}$ between two functors $\mathsf{F}, \mathsf{G}\colon \mathscr{C} \to \mathscr{C}'$ is the datum, for every $x \in \mathscr{C}$, of a morphism $\eta_x\colon \mathsf{F}(x) \to \mathsf{G}(x)$ in $\mathscr{C}'$, such that for every $f \in \mathrm{Hom}_{\mathscr{C}}(x_1, x_2)$ the diagram

$$\begin{tikzcd}
\mathsf{F}(x_1) \arrow[r, "\eta_{x_1}"] \arrow[d, "\mathsf{F}(f)"'] & \mathsf{G}(x_1) \arrow[d, "\mathsf{G}(f)"] \\
\mathsf{F}(x_2) \arrow[r, "\eta_{x_2}"] & \mathsf{G}(x_2)
\end{tikzcd}$$

is commutative in $\mathscr{C}'$.

Definition A.1.10 (Natural isomorphism)**.** Let $\mathscr{C}, \mathscr{C}'$ be two categories. Let $\mathsf{Fun}(\mathscr{C}, \mathscr{C}')$ be the category whose objects are functors $\mathscr{C} \to \mathscr{C}'$ and whose morphisms are the natural transformations. An isomorphism in the category $\mathsf{Fun}(\mathscr{C}, \mathscr{C}')$ is called a *natural isomorphism*.

Example A.1.11. Let K be a field, and $\mathscr{C}$ the category of finite dimensional K-vector spaces. Then we have two (covariant) functors $\mathscr{C} \to \mathscr{C}$, the former being the identity functor and the latter being the double dual functor, sending $V \mapsto V^{**}$. These two functors are naturally isomorphic.

Definition A.1.12 (Equivalence of categories)**.** Let $\mathscr{C}$ and $\mathscr{C}'$ be categories. An *equivalence* between them is a pair of functors

$$\mathsf{F}\colon \mathscr{C} \to \mathscr{C}', \quad \mathsf{G}\colon \mathscr{C}' \to \mathscr{C}$$

along with a pair of natural isomorphisms

$$\mathsf{F} \circ \mathsf{G} \xrightarrow{\sim} \mathrm{Id}_{\mathscr{C}'}, \quad \mathsf{G} \circ \mathsf{F} \xrightarrow{\sim} \mathrm{Id}_{\mathscr{C}}.$$

Terminology A.1.13. One often says that a functor $\mathsf{F}\colon \mathscr{C} \to \mathscr{C}'$ is an equivalence when there exists a functor $\mathsf{G}\colon \mathscr{C}' \to \mathscr{C}$ along with a pair of natural isomorphisms as in Definition A.1.12.

Definition A.1.14 (Fully faithful, essentially surjective)**.** A (covariant) functor $\mathsf{F}\colon \mathscr{C} \to \mathscr{C}'$ is called:

(i) *fully faithful* if for any two objects $x, y \in \mathscr{C}$ the map of sets

$$\mathrm{Hom}_{\mathscr{C}}(x, y) \to \mathrm{Hom}_{\mathscr{C}'}(\mathsf{F}(x), \mathsf{F}(y))$$

is a bijection.

(ii) *essentially surjective* if every object of $\mathscr{C}'$ is isomorphic to an object of the form $\mathsf{F}(x)$ for some $x \in \mathscr{C}$.

The following observation is quite useful.

Remark A.1.15. A fully faithful functor $\mathsf{F}\colon \mathscr{C} \to \mathscr{C}'$ induces an equivalence of $\mathscr{C}$ with the essential image of F, namely the full subcategory of $\mathscr{C}'$ consisting of objects isomorphic to objects of the form $\mathsf{F}(x)$ for some $x \in \mathscr{C}$. Put differently, a functor induces an equivalence if and only if it is fully faithful and essentially surjective.

Definition A.1.16 (Concrete category)**.** A *concrete category* is a category $\mathscr{C}$ that is equipped with a faithful functor $\mathsf{F}\colon \mathscr{C} \to \mathsf{Sets}$ to the category of sets.

Note that concreteness is not a property, but rather an additional structure present on the category.

Another notion that is rather important in category theory is that of an adjoint pair of functors.

Definition A.1.17 (Adjoint pair)**.** Let $\mathscr{C}$ and $\mathscr{D}$ be (locally small) categories. Let $\mathsf{F}\colon \mathscr{C} \to \mathscr{D}$ and $\mathsf{G}\colon \mathscr{D} \to \mathscr{C}$ be functors. We say that (F, G) is an *adjoint pair* of functors if for every pair of objects $(c, d) \in \mathrm{Ob}(\mathscr{C}) \times \mathrm{Ob}(\mathscr{D})$ one has a bijection of sets

$$\mathrm{Hom}_{\mathscr{D}}(\mathsf{F}(c), d) \cong \mathrm{Hom}_{\mathscr{C}}(c, \mathsf{G}(d)),$$

natural in both c and d. We say, more precisely, that F is a *left adjoint* to G and that G is a *right adjoint* to F.

Sometimes, one uses the pictorial description

$$\begin{tikzcd} \mathscr{C} \arrow[r, shift left, "\mathsf{F}"] & \mathscr{D} \arrow[l, shift left, "\mathsf{G}"] \end{tikzcd}$$

to say that (F, G) is an adjoint pair.

Example A.1.18. Here are some examples of adjunctions.

(a) Let $\mathsf{F}\colon \mathsf{Sets} \to \mathsf{Grp}$ be the functor sending a set S to the free group generated by the element of S. Let $\Phi\colon \mathsf{Grp} \to \mathsf{Sets}$ be the forgetful functor. Then (F, Φ) is an adjoint pair.

$$\mathsf{Sets} \underset{\Phi}{\overset{\mathsf{F}}{\rightleftarrows}} \mathsf{Grp}$$

(b) Let $\mathsf{j}\colon \mathsf{Ab} \hookrightarrow \mathsf{Grp}$ be the inclusion. It is right adjoint to the abelianisation functor $\mathsf{ab}\colon \mathsf{Grp} \to \mathsf{Ab}$ sending a group G to $G^{\mathrm{ab}} = G/[G,G]$. That is, $(\mathsf{ab}, \mathsf{j})$ is an adjoint pair.

$$\mathsf{Grp} \underset{\mathsf{j}}{\overset{\mathsf{ab}}{\rightleftarrows}} \mathsf{Ab}$$

(c) Let R be a ring. Consider the functor $\mathsf{sym}_R\colon \mathsf{Mod}_R \to \mathsf{Alg}_R$ sending $M \mapsto \mathsf{Sym}_R(M)$. Consider the forgetful functor $\Phi_R\colon \mathsf{Alg}_R \to \mathsf{Mod}_R$ sending an R-algebra to its underlying R-module. Then (sym_R, Φ_R) is an adjoint pair.

$$\mathsf{Mod}_R \underset{\Phi_R}{\overset{\mathsf{sym}_R}{\rightleftarrows}} \mathsf{Alg}_R$$

(d) Let $\alpha\colon R \to S$ be a ring homomorphism. Then every S-module is naturally an R-module, thus we have a forgetful functor $\Phi_\alpha\colon \mathsf{Mod}_S \to \mathsf{Mod}_R$. On the other hand, we have a functor (called extension of scalars) $-\otimes_R S\colon \mathsf{Mod}_R \to \mathsf{Mod}_S$ sending an R-module M to the S-module $M \otimes_R S$. Then $(-\otimes_R S, \Phi_\alpha)$ is an adjoint pair.

$$\mathsf{Mod}_R \underset{\Phi_\alpha}{\overset{-\otimes_R S}{\rightleftarrows}} \mathsf{Mod}_S$$

(e) Let ID be the category of integral domains (with morphisms the injective ring homomorphisms), Fields the category of fields. We have a functor $\mathsf{frac}\colon \mathsf{ID} \to \mathsf{Fields}$ sending a domain to its fraction field, and an inclusion functor $\mathsf{j}\colon \mathsf{Fields} \hookrightarrow \mathsf{ID}$. Then $(\mathsf{frac}, \mathsf{j})$ is an adjoint pair.

$$\mathsf{ID} \underset{\mathsf{j}}{\overset{\mathsf{frac}}{\rightleftarrows}} \mathsf{Fields}$$

(f) Let R be a ring, $M \in \mathsf{Mod}_R$ an R-module. Consider the endofunctors on the category Mod_R given by $-\otimes_R M$ and $\mathsf{h}^M = \mathrm{Hom}_{\mathsf{Mod}_R}(M, -)$. Then $(-\otimes_R M, \mathsf{h}^M)$ is an adjoint pair. The (natural) bijections

$$\mathrm{Hom}_{\mathsf{Mod}_R}(N \otimes_R M, P) \cong \mathrm{Hom}_{\mathsf{Mod}_R}(N, \mathrm{Hom}_{\mathsf{Mod}_R}(M, P))$$

induced by the adjunction

$$\mathsf{Mod}_R \underset{\mathsf{h}^M}{\overset{-\otimes_R M}{\rightleftarrows}} \mathsf{Mod}_R$$

are in fact isomorphisms of abelian groups (recall that Mod_R is abelian).

It is important to remember the following properties:

- every equivalence of categories is an adjunction,
- every right adjoint (resp. left adjoint) functor between two abelian categories is left exact (resp. right exact),
- if a functor has two left (or right) adjoints, then they are naturally isomorphic.

A.2 Yoneda Lemma

In this section we study representable functors and recall the statement of the Yoneda Lemma. More details and examples can be found, for instance, in [22].

For simplicity, all categories are assumed to be *locally small* throughout.

Let $\mathscr{C}$ be a (locally small) category. Consider the category of contravariant functors $\mathscr{C} \to \mathsf{Sets}$, i.e. the *functor category*

$$\mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathsf{Sets}).$$

For every object x of $\mathscr{C}$ there is a functor $\mathsf{h}_x : \mathscr{C}^{\mathrm{op}} \to \mathsf{Sets}$ defined by

$$u \mapsto \mathsf{h}_x(u) = \mathrm{Hom}_{\mathscr{C}}(u, x), \quad u \in \mathscr{C}.$$

A morphism $\phi \in \mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(u, v) = \mathrm{Hom}_{\mathscr{C}}(v, u)$ gets sent to the map of sets

$$\mathsf{h}_x(\phi) : \mathsf{h}_x(u) \to \mathsf{h}_x(v), \quad \alpha \mapsto \alpha \circ \phi.$$

Consider the functor

$$\mathsf{h}_{\mathscr{C}} : \mathscr{C} \to \mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathsf{Sets}), \quad x \mapsto \mathsf{h}_x. \tag{A.2.1}$$

This is, indeed, a functor: for every arrow $f : x \to y$ in $\mathscr{C}$ and object u of $\mathscr{C}$ we can define a map of sets

$$\mathsf{h}_f\, u : \mathsf{h}_x(u) \to \mathsf{h}_y(u), \quad \alpha \mapsto f \circ \alpha,$$

with the property that for every morphism $\phi : v \to u$ in $\mathscr{C}$ there is a commutative diagram

$$\begin{tikzcd}
\mathsf{h}_x(u) \arrow[r, "\mathsf{h}_f\, u"] \arrow[d, "\mathsf{h}_x(\phi)"'] & \mathsf{h}_y(u) \arrow[d, "\mathsf{h}_y(\phi)"] & & u \xrightarrow{\alpha} x \arrow[r, mapsto] \arrow[d, mapsto] & u \xrightarrow{f\circ\alpha} y \arrow[d, mapsto] \\
\mathsf{h}_x(v) \arrow[r, "\mathsf{h}_f\, v"] & \mathsf{h}_y(v) & & v \xrightarrow{\alpha\circ\phi} x \arrow[r, mapsto] & u \xrightarrow{f\circ\alpha\circ\phi} y
\end{tikzcd}$$

defining a natural transformation

$$\mathsf{h}_f : \mathsf{h}_x \Rightarrow \mathsf{h}_y.$$

LEMMA A.2.1 (Weak Yoneda). *The functor* $\mathsf{h}_{\mathscr{C}}$ *defined in* (A.2.1) *is fully faithful.*

Definition A.2.2 (Representable functor)**.** A functor $\mathsf{F} \in \mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathsf{Sets})$ is *representable* if it lies in the essential image of $\mathsf{h}_{\mathscr{C}}$, i.e. if it is isomorphic to a functor h_x for some $x \in \mathscr{C}$. In this case, we say that the object $x \in \mathscr{C}$ represents F.

Remark A.2.3. By Lemma A.2.1, if $x \in \mathscr{C}$ represents F, then x is unique up to a unique isomorphism. Indeed, suppose we have isomorphisms

$$a : \mathsf{h}_x \xrightarrow{\sim} F, \quad b : \mathsf{h}_y \xrightarrow{\sim} F$$

in the category $\mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathsf{Sets})$. Then there exists a unique isomorphism $x \xrightarrow{\sim} y$ inducing $b^{-1} \circ a : \mathsf{h}_x \xrightarrow{\sim} \mathsf{h}_y$.

Let $\mathsf{F} \in \mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathsf{Sets})$ be a functor, $x \in \mathscr{C}$ an object. One can construct a map of sets

$$g_x : \mathrm{Hom}(\mathsf{h}_x, \mathsf{F}) \to \mathsf{F}(x). \tag{A.2.2}$$

To a natural transformation $\eta : \mathsf{h}_x \Rightarrow \mathsf{F}$ one can associate the element

$$g_x(\eta) = \eta_x(\mathrm{id}_x) \in \mathsf{F}(x),$$

the image of $\mathrm{id}_x \in \mathsf{h}_x(x)$ via the map $\eta_x : \mathsf{h}_x(x) \to \mathsf{F}(x)$.

LEMMA A.2.4 (Strong Yoneda). *Let* $\mathsf{F} \in \mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathsf{Sets})$ *be a functor,* $x \in \mathscr{C}$ *an object. Then the map* g_x *defined in* (A.2.2) *is bijective.*

Proof. The inverse of g_x is the map that assigns to an element $\xi \in \mathsf{F}(x)$ the natural transformation $\eta(x, \xi) : \mathsf{h}_x \Rightarrow \mathsf{F}$ defined as follows. For a given object $u \in \mathscr{C}$, we define

$$\eta(x, \xi)_u : \mathsf{h}_x(u) \to \mathsf{F}(u)$$

by sending a morphism $f : u \to x$ to the image of ξ under $\mathsf{F}(f) : \mathsf{F}(x) \to \mathsf{F}(u)$. □

Exercise A.2.5. Show that Lemma A.2.4 implies Lemma A.2.1.

Definition A.2.6 (Universal object)**.** Let $\mathsf{F} : \mathscr{C}^{\mathrm{op}} \to \mathsf{Sets}$ be a functor. A *universal object* for F is a pair (x, ξ) where $\xi \in \mathsf{F}(x)$, such that for every pair (u, σ) with $\sigma \in \mathsf{F}(u)$, there exists a unique morphism $\alpha : u \to x$ with the property that $\mathsf{F}(\alpha) : \mathsf{F}(x) \to \mathsf{F}(u)$ sends ξ to σ.

Exercise A.2.7. Show that a pair (x, ξ) is a universal object for a functor $\mathsf{F} : \mathscr{C}^{\mathrm{op}} \to \mathsf{Sets}$ if and only if the natural transformation $\eta(x, \xi)$ defined in the proof of Lemma A.2.4 is a natural isomorphism. In particular, F is representable if and only if it has a universal object.

A.3 Moduli spaces in algebraic geometry

In classical moduli theory, one is interested in the category

$$\mathscr{C} = \mathsf{Sch}_S$$

of schemes over a fixed base scheme S. Its objects are pairs (X, f), where X is a scheme and $f: X \to S$ is a morphism of schemes. Sometimes one just writes $(f: X \to S)$ to denote an object of Sch_S. A morphism $(X, f) \to (Y, g)$ in Sch_S is a morphism $p: X \to Y$ such that $g \circ p = f$. One has the following important notion in moduli theory.

Definition A.3.1 (Fine moduli space)**.** Let $\mathfrak{M}: \mathsf{Sch}_S^{\mathrm{op}} \to \mathsf{Sets}$ be a functor. If an S-scheme $M \to S$ represents $\mathfrak{M}$, then $M \to S$ is called a fine moduli space for the moduli problem defined by $\mathfrak{M}$.

To say that $M \to S$ is a fine moduli space for a functor $\mathfrak{M}$ in particular says that $M \to S$ is unique up to unique isomorphism, and by Exercise A.2.7 it has a universal object $\xi \in \mathfrak{M}(M \to S)$ in the sense of Definition A.2.6.

Example A.3.2. The existence of fibre products in the category of schemes $\mathsf{Sch} = \mathsf{Sch}_{\mathrm{Spec}\,\mathbb{Z}}$ amounts to the representability of the functor $\mathsf{Sch}^{\mathrm{op}} \to \mathsf{Sets}$ sending a scheme $A \in \mathsf{Sch}$ to the set

$$\mathrm{Hom}_{\mathsf{Sch}}(A, X) \times_{\mathrm{Hom}_{\mathsf{Sch}}(A,S)} \mathrm{Hom}_{\mathsf{Sch}}(A, Y).$$

Example A.3.3 (Global Spec)**.** Let S be a scheme, $\mathscr{A}$ a quasicoherent $\mathscr{O}_S$-algebra. Then the S-scheme $\mathrm{Spec}_{\mathscr{O}_S} \mathscr{A} \to S$ represents the functor $\mathsf{Sch}_S^{\mathrm{op}} \to \mathsf{Sets}$ sending

$$(U \xrightarrow{g} S) \mapsto \mathrm{Hom}_{\mathscr{O}_S\text{-alg}}(\mathscr{A}, g_* \mathscr{O}_U).$$

B | Commutative algebra

B.1 Frequently used theorems

LEMMA B.1.1 (Nakayama). *Let A be a ring, $I \subset A$ an ideal, M a finitely generated A-module. If $M = IM$, then there exists an element $r \in A$, with $1 - r \in I$, such that $rM = 0$. Equivalently, there exists $i \in I$ such that $im = m$ for every $m \in M$.*

The following weaker statements are also referred to as 'Nakayama's Lemma'.

LEMMA B.1.2. *Let A be a ring, $J_A \subset A$ its Jacobson radical (the intersection of all maximal ideals), M a finitely generated A-module. If $J_A M = 0$, then $M = 0$.*

In particular, if M is a finitely generated A-module and $N \subset M$ is a submodule such that $M = N + J_A$, then $M = N$.

LEMMA B.1.3. *Let A be a reduced ring. The union of the minimal prime ideals in A coincides with the set of zero-divisors.*

LEMMA B.1.4. *Let $(A, \mathfrak{m}, \mathbb{F})$ be a local ring, M a finitely generated A-module. Let $m_1, \ldots, m_n \in M$ be elements such that their classes $\overline{m}_1, \ldots, \overline{m}_n \in M/\mathfrak{m}M$ generate the $\mathbb{F}$-vector space $M/\mathfrak{m}M$. Then $m_1, \ldots, m_n$ generate M as an A-module.*

B.2 Tensor products

Definition B.2.1 (Tensor product of modules)**.** Let A be a ring, M and N two A-modules. The *tensor product* of M and N over A is defined to be a pair $(M \otimes_A N, p)$ where

- $M \otimes_A N$ is an A-module,
- $p \colon M \times N \to M \otimes_A N$ is a bilinear map,

such that the following universal property is satisfied: for every pair (E, q) where E is an A-module and $q \colon M \times N \to E$ is a bilinear map, there is exactly one A-linear homomorphism $\phi_q \colon M \otimes_A N \to E$ such that $q = \phi_q \circ p$.

The universal property of Definition B.2.1 can be depicted in the diagram

$$\begin{tikzcd} M \times N \arrow[rr, "p"] \arrow[dr, "\forall q"'] & & M \otimes_A N \arrow[dl, "\exists ! \phi_q"] \\ & E & \end{tikzcd}$$

and, more importantly, can be rephrased by saying that there is a bijection

$$\mathsf{Bil}_A(M \times N, E) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Mod}_A}(M \otimes N, E), \quad q \mapsto \phi_q,$$

functorial in E.

Regarding existence of an object $(M \otimes_A N, p)$ with the required universal property, one first considers the standard basis $\{ e_{m,n} \mid m \in M, n \in N \}$ of the direct sum $A^{\oplus M \times N}$. One then constructs the quotient module

$$M \otimes N = A^{\oplus M \times N} / T$$

where $T \subset A^{\oplus M \times N}$ is the submodule generated by elements of the form

$$\begin{aligned} & e_{m_1+m_2,n} - e_{m_1,n} - e_{m_2,n}, \\ & e_{m,n_1+n_2} - e_{m,n_1} - e_{m,n_2}, \\ & e_{am,n} - e_{m,an}, \\ & a e_{m,n} - e_{am,n}. \end{aligned}$$

The map $p \colon M \times N \to M \otimes_A N$ is defined by sending $(m, n) \mapsto [e_{m,n}]$, where the square bracket means equivalence class. One sets

$$m \otimes n = [e_{m,n}].$$

This is standard notation. Note that not all elements of $M \otimes_A N$ are of the form $m \otimes n$ for elements $m \in M$ and $n \in N$. However, every element $u \in M \otimes_A N$ can be written (non-uniquely) as a finite sum

$$u = \sum_{k=1}^{r} m_k \otimes n_k, \quad r > 0.$$

Granting that the above pair $(M \otimes_A N, p)$ satisfies the universal property of Definition B.2.1 (which is an easy exercise), one has automatically that such pair is unique. Note that one has the elementary identifications

$$\begin{aligned} M \otimes_A A &= M, \\ M \otimes_A N &= N \otimes_A M, \\ (M \otimes_A N) \otimes_A P &= M \otimes_A (N \otimes_A P). \end{aligned}$$

Exercise B.2.2. If $(M_i)_{i\in I}$ is a family of A-modules, one has a canonical isomorphism

$$\bigoplus_{i\in I}(M_i \otimes_A N) \xrightarrow{\sim} \left(\bigoplus_{i\in I} M_i\right) \otimes_A N$$

for any A-module N.

Exercise B.2.3. Let A be a ring, M an A-module. Prove that the functor

$$M \otimes_A -\colon \mathsf{Mod}_A \to \mathsf{Mod}_A, \quad N \mapsto M \otimes_A N$$

is right exact, i.e. that a surjection $N_1 \twoheadrightarrow N_2$ gets sent to a surjection $M \otimes_A N_1 \twoheadrightarrow M \otimes_A N_2$.

B.3 Cohen–Macaulay modules

Definition B.3.1. A finitely generated module over a noetherian local ring R is said to be *Cohen–Macaulay* if $\dim(\operatorname{Supp} M) = \operatorname{depth} M$. A noetherian local ring R is said to be *Cohen–Macaulay* if it is Cohen–Macaulay as a module over itself.

Definition B.3.2. Let X be a locally noetherian scheme. We say that X is *Cohen–Macaulay* if $\mathscr{O}_{X,x}$ is Cohen–Macaulay for every $x \in X$.

B.4 Universal constructions

B.4.1 Limits and colimits

Let $\mathscr{C}$ be a category, $\mathbf{I}$ a small category. Define an $\mathbf{I}$-diagram to be just a functor $M\colon \mathbf{I} \to \mathscr{C}$. Denote by M_i the object of $\mathscr{C}$ image of the object $i \in \mathbf{I}$ via M. If $f\colon i \to j$ is an arrow in $\mathbf{I}$, the induced arrow in $\mathscr{C}$ is denoted $M(f)\colon M_i \to M_j$.

Definition B.4.1 (Limit)**.** A *limit* of an $\mathbf{I}$-diagram $M\colon \mathbf{I} \to \mathscr{C}$ is an object $\varprojlim_{i\in\mathbf{I}} M_i$ of $\mathscr{C}$ along with an arrow $p_i\colon \varprojlim_{i\in\mathbf{I}} M_i \to M_i$ for every $i \in \mathbf{I}$, such that for every arrow $f\colon i \to j$ in $\mathbf{I}$ one has $p_j = M(f)\circ p_i$, and satisfying the following universal property: given an object P along with morphisms $\pi_i\colon P \to M_i$ such that $\pi_j = M(f)\circ \pi_i$ for every $f\colon i \to j$ in $\mathbf{I}$, there exists a unique arrow $\alpha\colon P \to \varprojlim_{i\in\mathbf{I}} M_i$ such that $\pi_i = p_i \circ \alpha$ for all $i \in \mathbf{I}$.

$$\begin{tikzcd}
P \arrow[d, dotted, "\exists!\alpha"'] \arrow[dr, "\pi_i"] \arrow[drr, bend left, "\pi_j"] & & \\
\varprojlim_{i\in\mathbf{I}} M_i \arrow[r, "p_i"] \arrow[rr, bend right, "p_j"'] & M_i \arrow[r, "M(f)"] & M_j
\end{tikzcd}$$

Exercise B.4.2. The limit over the empty diagram satisfies the universal property of a final object of $\mathscr{C}$.

Example B.4.3 (Products are limits)**.** Let $\mathbf{I}$ be the category with two objects i, j and no morphisms between them. Then an $\mathbf{I}$-diagram $M: \mathbf{I} \to \mathscr{C}$ is just the choice of two objects M_i, M_j of $\mathscr{C}$. The limit of M satisfies the universal property of the product $M_i \times M_j$.

$$\begin{tikzcd}
P \arrow[dr, dotted, "\alpha"] \arrow[drr, bend left, "\pi_i"] \arrow[ddr, bend right, "\pi_j"'] & & \\
& M_i \times M_j \arrow[r, "p_i"] \arrow[d, "p_j"] & M_i \\
& M_j &
\end{tikzcd}$$

Example B.4.4 (Equalisers are limits)**.** Let $\mathbf{I}$ be the category with two objects i, j and two arrows $i \rightrightarrows j$. Then an $\mathbf{I}$-diagram $M: \mathbf{I} \to \mathscr{C}$ is just the choice of two parallel arrows $\phi, \psi: M_i \rightrightarrows M_j$ in $\mathscr{C}$. The limit of M satisfies the universal property of the equaliser of (ϕ, ψ).

Example B.4.5 (Kernels are limits)**.** This is because kernels are equalisers (in the previous example take $\psi = 0$).

Definition B.4.6 (Colimit)**.** A *colimit* of an $\mathbf{I}$-diagram $M: \mathbf{I} \to \mathscr{C}$ is an object $\varinjlim_{i\in\mathbf{I}} M_i$ of $\mathscr{C}$ along with an arrow $s_i: M_i \to \varinjlim_{i\in\mathbf{I}} M_i$ for every $i \in \mathbf{I}$, such that for every arrow $f: i \to j$ in $\mathbf{I}$ one has $s_i = s_j \circ M(f)$, and satisfying the following universal property: given an object P along with morphisms $\sigma_i: M_i \to P$ such that $\sigma_i = \sigma_j \circ M(f)$ for every $f: i \to j$ in $\mathbf{I}$, there exists a unique arrow $\alpha: \varinjlim_{i\in\mathbf{I}} M_i \to P$ such that $\sigma_i = \alpha \circ s_i$ for all $i \in \mathbf{I}$.

$$\begin{tikzcd}
P & & \\
\varinjlim_{i\in\mathbf{I}} M_i \arrow[u, dotted, "\exists!\alpha"] & M_j \arrow[l, "s_j"'] \arrow[ul, "\sigma_j"'] & M_i \arrow[l, "M(f)"'] \arrow[ull, bend right, "\sigma_i"'] \arrow[ll, bend left, "s_i"]
\end{tikzcd}$$

Exercise B.4.7. The colimit over the empty diagram satisfies the universal property of an initial object of $\mathscr{C}$ (cf. Exercise B.4.2).

Exercise B.4.8. Convince yourself that coproducts, coequalisers and cokernels are examples of colimits, along the same lines of Examples B.4.3, B.4.4 and B.4.5.

Definition B.4.9 (Filtered category)**.** A nonempty category $\mathbf{I}$ is *filtered* if for every two objects $i, j \in \mathbf{I}$ the following are true:

- there exists an object $k \in \mathbf{I}$ along with two morphisms $i \to k$ and $j \to k$, and
- for any two morphisms $f, g \in \mathrm{Hom}_{\mathbf{I}}(i, j)$ there exists an object $k \in \mathbf{I}$ along with a morphism $h: j \to k$ such that $h \circ f = h \circ g$ in $\mathrm{Hom}_{\mathbf{I}}(i, k)$.

The colimit of an $\mathbf{I}$-diagram $M:\mathbf{I}\to\mathscr{C}$ where $\mathbf{I}$ is a filtered category is a *filtered colimit*.

Example B.4.10. In the definition of stalk of a presheaf $\mathcal{F}\in\mathsf{pSh}(X,\mathscr{C})$ at a point $x\in X$, we have been taking

$$\mathbf{I}=\{\,U\in\tau_X\mid x\in U\,\}^{\mathrm{op}}$$
$$M(U)=\mathcal{F}(U).$$

B.5 Localisation

B.5.1 General construction for modules

Let A be a ring, M an A-module. Fix a *multiplicative subset* $S\subset A$, i.e. a subset containing the identity $1\in A$ and such that $s_1s_2\in S$ whenever $s_1,s_2\in S$.

Example B.5.1. The following are key examples of multiplicative subsets:

(i) $S=\{\,f^n\mid n\geq 0\}$ for some $f\in A$.

(ii) $S=A\setminus\mathfrak{p}$, where $\mathfrak{p}\subset A$ is a prime ideal.

(iii) $S=A\setminus 0$, if A is an integral domain.

(iv) $S=A\setminus\mathcal{Z}$, where $\mathcal{Z}$ is the set of all zero-divisors in A.

Consider the equivalence relation on $M\times S$ defined by

$$\boxed{(m,s)\sim(m',s')\iff \text{there exists } u\in S \text{ such that } u(s'm-sm')=0\in M.}$$

We denote by m/s, or by $\frac{m}{s}$, the equivalence class of (m,s). The set of such equivalence classes

$$S^{-1}M=(M\times S)/\sim \tag{B.5.1}$$

is an abelian group via

$$\frac{m}{s}+\frac{m'}{s'}=\frac{sm'+s'm}{ss'},$$

and if $M=A$ then $S^{-1}A$ becomes a ring via

$$\frac{a}{s}\cdot\frac{a'}{s'}=\frac{aa'}{ss'}.$$

The $\mathbb{Z}$-module $S^{-1}M$ is an $S^{-1}A$-module via

$$\frac{a}{s}\cdot\frac{m}{s'}=\frac{am}{ss'}. \tag{B.5.2}$$

Here 'am' refers to the A-module structure on M.

Definition B.5.2 (Localisation of a module)**.** The localisation of M with respect to S is the $S^{-1}A$-module $S^{-1}M$, where the linear structure is given by Equation (B.5.2).

Localisation is functorial: if $\phi\colon N\to M$ is an A-linear map, there is an induced map

$$S^{-1}\phi\colon S^{-1}N \longrightarrow S^{-1}M, \qquad \frac{n}{s}\mapsto\frac{\phi(n)}{s}.$$

This map is $S^{-1}A$-linear, indeed if $a/t\in S^{-1}A$ then

$$S^{-1}\phi\left(\frac{a}{t}\cdot\frac{n}{s}\right)=S^{-1}\phi\left(\frac{an}{ts}\right)=\frac{\phi(an)}{ts}=\frac{a\cdot\phi(n)}{ts}=\frac{a}{t}\cdot\frac{\phi(n)}{s}=\frac{a}{t}\cdot S^{-1}\phi\left(\frac{n}{s}\right).$$

Remark B.5.3. If $0\in S$, then $S^{-1}M=0$.

Notation B.5.4. If $S=\{f^n\mid n\geq 0\}$ as in Example B.5.1 (i) above, then we write M_f for the localisation. If $S=A\setminus\mathfrak{p}$ as in Example B.5.1 (ii) above, then we write $M_{\mathfrak{p}}$ for the localisation. Do not confuse M_f and $M_{(f)}$ when $(f)=fA\subset A$ is a prime ideal!

B.5.2 Localisation of a ring and its universal property

Set $M=A$. There is a canonical ring homomorphism

$$\ell\colon A\to S^{-1}A,\quad a\mapsto\frac{a}{1}$$

sending S inside the group of invertible elements of $S^{-1}A$ (the inverse of $s/1$ being $1/s$), and making the pair $(S^{-1}A,\ell)$ universal with this property: whenever one has a ring homomorphism $\phi\colon A\to B$ such that $\phi(S)\subset B^\times$, there is exactly one ring homomorphism $p\colon S^{-1}A\to B$ such that $\phi=p\circ\ell$.

$$\begin{tikzcd} A \arrow[r, "\ell"] \arrow[d, "\phi"'] & S^{-1}A \arrow[dl, "p"] \\ B & \end{tikzcd}$$

Explicitly, the map p is defined by $p(a/s)=\phi(a)\phi(s)^{-1}$.

Remark B.5.5. The localisations of the form A_f are crucial in algebraic geometry. In A_f, the equivalence relation defining the localisation reads

$$\frac{a}{f^n}=\frac{b}{f^m}\iff\text{there exists } k\geq 0 \text{ such that } f^k(af^m-bf^n)=0\in A.$$

In particular, one has that $A_f=0$ if and only if f is nilpotent, and

$$A_f\ni 0=\frac{0}{1}=\frac{a}{f^n}\qquad\iff\qquad\text{there exists } k\geq 0 \text{ such that } f^ka=0\in A.$$

The following lemma is of key importance to us.

LEMMA B.5.6. *Let A be a ring, and $\ell\colon A\to S^{-1}A$ a localisation. Sending $\mathfrak{r}\mapsto \ell^{-1}(\mathfrak{r})$ establishes a bijection*

$$\begin{tikzcd} \{\text{prime ideals } \mathfrak{r}\subset S^{-1}A\} \arrow[r, "\simeq"] \arrow[d, equal] & \{\text{prime ideals } \mathfrak{q}\subset A \text{ such that } \mathfrak{q}\cap S=\emptyset\} \arrow[d, hook] \\ \operatorname{Spec} S^{-1}A & \operatorname{Spec} A \end{tikzcd}$$

having as inverse the extension operation, sending

$$\mathfrak{q}\mapsto \mathfrak{q}\cdot S^{-1}A=\left\{\frac{a}{f}\,\middle|\, a\in\mathfrak{q},\, f\in S\right\}\subset S^{-1}A.$$

COROLLARY B.5.7. *For any prime ideal $\mathfrak{p}\subset A$ the ring*

$$A_{\mathfrak{p}}=\left\{\frac{a}{f}\,\middle|\, a\in A,\, f\notin\mathfrak{p}\right\}$$

is local, with maximal ideal

$$\mathfrak{p}\cdot A_{\mathfrak{p}}=\left\{\frac{a}{f}\,\middle|\, a\in\mathfrak{p},\, f\notin\mathfrak{p}\right\}\subset A_{\mathfrak{p}}.$$

Proof. Indeed, the correspondence of Lemma B.5.6 becomes, in the case $S=A\setminus\mathfrak{p}$,

$$\{\text{prime ideals } \mathfrak{r}\subset A_{\mathfrak{p}}\}\xrightarrow{\ \simeq\ }\{\text{prime ideals } \mathfrak{q}\subset A \text{ such that } \mathfrak{q}\subset\mathfrak{p}\}$$

and since its inverse (extension along $A\to A_{\mathfrak{p}}$) is inclusion-preserving it follows that every prime ideal $\mathfrak{r}\subset A_{\mathfrak{p}}$ must be contained in $\mathfrak{p}\cdot A_{\mathfrak{p}}$. This means that $\mathfrak{p}\cdot A_{\mathfrak{p}}$ is the unique maximal ideal. □

Exercise B.5.8. If $(A,\mathfrak{m})$ is a local ring, then $A=A_{\mathfrak{m}}$.

Warning B.5.9. In the case when B is a graded ring and $\mathfrak{p}$ is a homogeneous prime ideal, we use the notation $B_{\mathfrak{p}}$ for the localisation of B at the multiplicative subset consisting of *homogeneous* elements that are not in $\mathfrak{p}$.

PROPOSITION B.5.10 ([13, Prop. 5.8]). *If $\mathfrak{m}\subset A$ is a maximal ideal and $k>0$ is an integer, there is a natural ring isomorphism*

$$A/\mathfrak{m}^k\xrightarrow{\ \sim\ } A_{\mathfrak{m}}/(\mathfrak{m}\cdot A_{\mathfrak{m}})^k.$$

It induces isomorphisms

$$\mathfrak{m}^h/\mathfrak{m}^k\xrightarrow{\ \sim\ }(\mathfrak{m}\cdot A_{\mathfrak{m}})^h/(\mathfrak{m}\cdot A_{\mathfrak{m}})^k$$

for every $h\le k$.

LEMMA B.5.11. *Let A be a ring, $S\subset A$ a multiplicative subset. Then $\ell\colon A\to S^{-1}A$ is injective if and only if S contains no zero divisors.*

Proof. Suppose $a/1 = 0/1$ in $S^{-1}A$. Then there is $u \in S$ such that $au = 0$. But u is not a zero divisor, thus $a = 0$. $\square$

Example B.5.12. Let A be an integral domain, which means that $(0) \subset A$ is prime. Then the localisation

$$A_{(0)} = \left\{ \frac{a}{b} \,\middle|\, a \in A,\, b \in A \setminus 0 \right\}$$

is a field, called the *fraction field* of A, that we denote by $\mathrm{Frac}(A)$. The canonical map $\ell \colon A \to \mathrm{Frac}(A)$ is injective by Lemma B.5.11.

Example B.5.13. Let A be a ring. Consider $S = A \setminus \mathcal{Z}$ as in Example B.5.1 (iv). The localisation $S^{-1}A$ is called the *total ring of fractions* of A. By Lemma B.5.11, $S = A \setminus \mathcal{Z}$ is the largest multiplicative set such that $\ell \colon A \to S^{-1}A$ is injective.

Example B.5.14. Let $A = \mathbb{Z}$. Fix a prime number $p \in \mathbb{Z}$. Then the localisation map

$$\mathbb{Z} \to \mathbb{Z}_{(p)} = \left\{ \frac{n}{m} \,\middle|\, n \in \mathbb{Z},\, p \nmid m \right\}$$

is injective, and so is the localisation map

$$\mathbb{Z} \to \mathbb{Z}_p = \left\{ \frac{n}{p^k} \,\middle|\, n \in \mathbb{Z},\, k \geq 0 \right\}.$$

LEMMA B.5.15. *If A is reduced and $S \subset A$ is a multiplicative subset, then $S^{-1}A$ is also reduced.*

Proof. Assume there exists $a \in A$, $s \in S$ and $r \in \mathbb{Z}_{>0}$ such that $0/1 = (a/s)^r = a^r/s^r \in S^{-1}A$. Then there exists $u \in S$ such that $ua^r = 0 \in A$, thus $(ua)^r = 0$, and hence $ua = 0$ by assumption. But this means $0/1 = a/1 \in S^{-1}A$, and thus $0/1 = (a/1)(1/s) = a/s \in S^{-1}A$. $\square$

B.5.3 Exactness of localisation

LEMMA B.5.16. *Let A be a ring, $S \subset A$ a multiplicative subset, M an A-module. Then, there is a canonical isomorphism of $S^{-1}A$-modules*

$$\phi \colon S^{-1}M \xrightarrow{\sim} M \otimes_A S^{-1}A.$$

Proof. First of all, the $S^{-1}A$-module structure on $M \otimes_A S^{-1}A$ si defined (on generators) by

$$\frac{a}{t} \cdot \left(m \otimes \frac{b}{s} \right) = m \otimes \frac{ab}{ts}.$$

The map ϕ is defined by

$$\phi\left(\frac{m}{s}\right) = m \otimes \frac{1}{s}.$$

It is $S^{-1}A$-linear, since

$$\begin{aligned}\phi\left(\frac{a}{t}\cdot\frac{m}{s}\right) &= \phi\left(\frac{am}{ts}\right)\\ &= am\otimes\frac{1}{ts}\\ &= m\otimes\frac{a}{ts}\\ &= \frac{a}{t}\cdot\left(m\otimes\frac{1}{s}\right)\\ &= \frac{a}{t}\cdot\phi\left(\frac{m}{s}\right).\end{aligned}$$

Its inverse is given by $m\otimes(a/s)\mapsto(am)/s$. □

PROPOSITION B.5.17. *Let A be a ring, $S\subset A$ a multiplicative subset. Then, sending $M\mapsto S^{-1}M$ defines an exact functor from A-modules to $S^{-1}A$-modules.*

Proof. Fix a short exact sequence

$$0\longrightarrow M\xrightarrow{\ \iota\ } N\xrightarrow{\ \pi\ } P\longrightarrow 0$$

of A-modules. We already know that

$$S^{-1}M\xrightarrow{S^{-1}\iota} S^{-1}N\xrightarrow{S^{-1}\pi} S^{-1}P\longrightarrow 0$$

is exact, since this sequence is isomorphic to

$$M\otimes_A S^{-1}A\longrightarrow N\otimes_A S^{-1}A\longrightarrow P\otimes_A S^{-1}A\longrightarrow 0$$

by Lemma B.5.16, and tensor product (by any A-module, e.g. $S^{-1}A$) is a right exact functor. So we only need to show that

$$S^{-1}M\xrightarrow{S^{-1}\iota} S^{-1}N$$

is injective. Assume there is an element $m/s\in S^{-1}M$ such that $0=0/1=S^{-1}\iota(m/s)=\iota(m)/s\in S^{-1}N$. Then there exists $u\in S$ such that $0=u\iota(m)=\iota(um)$ in N. This implies $um=0\in M$, hence $m/s=um/us=0/us=0$. □

B.6 Normalisation

Normal schemes are either regular or 'mildly singular' schemes. For instance, a key property of normal schemes is that singularities only occur in codimension 2 or higher. We now give precise definitions. First recall that an integral domain A, with fraction field K, is said to be *integrally closed* if whenever $\alpha\in K$ is a root of a monic polynomial $p(t)\in A[t]$, one has that $\alpha\in A$. All regular local rings are integrally closed. We have implications

$$\text{Euclidean domain}\Rightarrow\text{PID}\Rightarrow\text{UFD}\Rightarrow\text{integrally closed.}$$

Definition B.6.1 (Normality)**.** We say that

(i) an integral domain A is *normal* if it is integrally closed,

(ii) a ring is *normal* if all its local rings are normal domains,

(iii) a scheme is *normal* if $\mathscr{O}_{X,x}$ is a normal integral domain for every $x \in X$.

Remark B.6.2. A scheme is normal if and only if it is 'locally normal' in the sense of Definition 4.2.1. The terminology 'locally normal' is never used though.

Remark B.6.3. By definition, a ring A is normal precisely when $\operatorname{Spec} A$ is a normal scheme.

Example B.6.4. A regular scheme is normal. A normal scheme is reduced. To see the latter, it is enough to observe that for any open subset $U \subset X$ there is an injective ring homomorphism

$$\mathscr{O}_X(U) \hookrightarrow \prod_{x\in U} \mathscr{O}_{X,x}$$

since $\mathscr{O}_X$ is a sheaf (cf. Lemma 2.4.1), where $\mathscr{O}_{X,x}$ is reduced for every $x \in X$, since it is a domain.

Example B.6.5 (Locally factorial schemes are normal)**.** A scheme is *locally factorial* if $\mathscr{O}_{X,x}$ is a UFD for every $x \in X$. A UFD is normal, so a locally factorial scheme is normal.

Example B.6.6. Let A be a normal domain. Then $S^{-1}A$ is a normal domain for any multiplicative subset $S \subset A$ (see Proposition 4.2.4 for a proof). Thus, $\operatorname{Spec} A$ is normal, and so is any principal open $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A$.

Caution B.6.7. It is not true that if $\operatorname{Spec} A$ is normal, then A is an integral domain: for instance, if $\mathbb{F}$ is a field, then

$$\operatorname{Spec} \mathbb{F} \amalg \operatorname{Spec} \mathbb{F} = \operatorname{Spec} \mathbb{F} \times \mathbb{F} = \operatorname{Spec} \mathbb{F}[x]/(x(x-1))$$

is a normal scheme, but $\mathbb{F}[x]/(x(x-1))$ is not a domain.

Exercise B.6.8. Show that $\operatorname{Spec} \mathbb{C}[x,y,z]/(x^2+y^2-z^2)$ is normal but not locally factorial.

Exercise B.6.9. Let $\mathbb{F}$ be a field, with $\operatorname{char} \mathbb{F} \neq 2$. Show that the following schemes are normal.

- $\operatorname{Spec} \mathbb{Z}[x]/(x^2-n)$, where $n \in \mathbb{Z}$ is square-free and congruent to 3 modulo 4.
- $\operatorname{Spec} \mathbb{F}[x_1,\ldots,x_n]/(x_1^2+\cdots+x_m^2)$, where $n \geq m \geq 3$.
- $\operatorname{Spec} \mathbb{F}[x,y,z,w]/(xy-zw)$.

PROPOSITION B.6.10. *Let X be a scheme.*

(A) *The following conditions are equivalent:*

(1) *X is normal.*

(2) *$\mathscr{O}_X(U)$ is a normal ring for every affine open $U \subset X$.*

(3) *There is an affine open covering $X = \bigcup_{i\in I} U_i$ such that $\mathscr{O}_X(U_i)$ is a normal ring for every $i \in I$.*

(4) *There is an open covering $X = \bigcup_{j\in J} V_j$ such that V_j is normal for every $j \in I$.*

Moreover, every open subscheme of a normal scheme X is normal.

(B) *If X is quasicompact, the above conditions are equivalent to*

(5) *$\mathscr{O}_{X,x}$ is a normal domain for every closed point $x \in X$.*

(C) *If X is integral, the above conditions are equivalent to*

(6) *$\mathscr{O}_X(U)$ is a normal domain for every affine open $U \subset X$.*

Proof. To prove (A), combine the Locality Lemma (cf. Lemma 4.2.2), Remark B.6.2 and Proposition 4.2.4 with one another.

To prove (B), argue as in the proof of Corollary 6.1.10.

To prove (C), it is enough to use the definition of integral scheme (cf. Definition 4.4.1) and point (A). □

Remark B.6.11. By the above proof, the first two conditions are equivalent even without assuming quasicompactness.

LEMMA B.6.12 ([12, Ch. 4, Lemma 1.13]). *Let A be a normal noetherian ring of dimension at least 1. Then*

$$A = \bigcap_{\substack{\mathfrak{p}\in \operatorname{Spec} A \\ \operatorname{ht}(\mathfrak{p})=1}} A_{\mathfrak{p}},$$

the intersection being taken inside Frac A.

COROLLARY B.6.13. *Let X be a normal locally noetherian scheme. Let $Z \subset X$ be a closed subset of codimension at least 2. Then the natural map*

$$\mathscr{O}_X(X) \to \mathscr{O}_X(X \setminus Z)$$

is an isomorphism.

Proof. □

Example B.6.14. Note that Corollary B.6.13 reproves the content of Example 3.1.73, i.e. the identity

$$\mathscr{O}_{\mathbb{A}^n_{\mathbf{k}}}(\mathbb{A}^n_{\mathbf{k}}) = \mathscr{O}_{\mathbb{A}^n_{\mathbf{k}}}(\mathbb{A}^n_{\mathbf{k}} \setminus \{0\})$$

for any $n \geq 2$.

There is a procedure, called *normalisation*, which does the following. Given, as input, an integral scheme X, one constructs a pair $(\widetilde{X}, \pi)$ where $\widetilde{X}$ is a normal scheme and $\pi\colon \widetilde{X} \to X$ is a morphism of schemes which is universal in the following sense: for every pair (Y, f) where Y is a normal scheme and $f\colon Y \to X$ is normal, there exists exactly one morphism $\alpha_f\colon Y \to \widetilde{X}$ such that $\pi \circ \alpha_f = f$.

Remark B.6.15. The normalisation of an integral scheme, if it exists (which it does, see Theorem B.6.17 below), is unique up to unique isomorphism.[1] Moreover, the universal property also shows that if $\pi\colon \widetilde{X} \to X$ is the normalisation and $U \subset X$ is open, then the base change map $\pi^{-1}(U) \to U$ is the normalisation of U.

In the affine case, the normalisation is easy to construct, as the following lemma shows.

LEMMA B.6.16. *Let A be an integral domain. Let $\widetilde{A} \subset \operatorname{Frac} A$ be the integral closure of A. Then the morphism*

$$\operatorname{Spec} \widetilde{A} \to \operatorname{Spec} A$$

induced by the inclusion $A \hookrightarrow \widetilde{A}$ is the normalisation of $\operatorname{Spec} A$*.*

Proof. □

THEOREM B.6.17. *Let X be an integral scheme. Then there exists a (unique) normalisation $(\widetilde{X}, \pi)$. If X is an integral algebraic $\mathbf{k}$-variety, then the normalisation morphism $\pi\colon \widetilde{X} \to X$ is finite; in particular, $\widetilde{X}$ is an algebraic $\mathbf{k}$-variety.*

Proof. □

PROPOSITION B.6.18 ([12, Ch. 4, Cor. 1.30]). *Let X be an integral algebraic variety. Then the set of points $x \in X$ such that $\mathscr{O}_{X,x}$ is normal is open.*

Example B.6.19 (Nodal cubic)**.** Let $A = \mathbf{k}[x, y]/(y^2 - x^2(x+1))$. Then A is not normal. Let us determine its normalisation.

Example B.6.20 (Cuspidal cubic)**.** Let $A = \mathbf{k}[x, y]/(y^2 - x^3)$. Then A is not normal. Let us determine its normalisation.

[1] The normalisation being defined as a *pair*, by isomorphism we mean an isomorphism in the category Sch_X of X-schemes.

B.7 Embedded components

On a locally noetherian scheme X there are a bunch of points that are more relevant than all other points, in the sense that they reveal part of the behaviour of the structure sheaf: these points are the *associated points* of X. Some of these points are already familiar: they are the generic points, i.e. the points corresponding to the irreducible components of X. The other associated points correspond to the so-called *embedded components* of X. If X is reduced, it has no embedded components.

Let R be a commutative ring with unity, and let M be an R-module. If $m \in M$, we let

$$\mathrm{Ann}_R(m) = \{ r \in R \mid r \cdot m = 0 \} \subset R$$

denote its annihilator. A prime ideal $\mathfrak{p} \subset R$ is said to be *associated to* M if $\mathfrak{p} = \mathrm{Ann}_R(m)$ for some $m \in M$. The set of all associated primes is denoted

$$\mathsf{Ass}_R(M) = \{ \mathfrak{p} \mid \mathfrak{p} \text{ is associated to } M \}.$$

LEMMA B.7.1. *Let $\mathfrak{p} \subset R$ be a prime ideal. Then $\mathfrak{p} \in \mathsf{Ass}_R(M)$ if and only if $R/\mathfrak{p}$ is an R-submodule of M.*

Proof. If $\mathfrak{p} = \mathrm{Ann}_R(m)$ for some $m \in M$, consider the map $\phi_m \colon R \to M$ defined by $\phi_m(r) = r \cdot m$. Since its kernel is by definition $\mathrm{Ann}_R(m)$, the quotient $R/\mathfrak{p}$ is an R-submodule of M. Conversely, given an R-linear inclusion $i \colon R/\mathfrak{p} \hookrightarrow M$, consider the composition $\phi \colon R \twoheadrightarrow R/\mathfrak{p} \hookrightarrow M$. Then $\phi_{i(1)}(r) = r \cdot i(1) = i(r+\mathfrak{p}) = \phi(r)$ for all $r \in R$, i.e. $\phi = \phi_{i(1)}$. □

Note that if $\mathfrak{p} \in \mathsf{Ass}_R(M)$ then $\mathfrak{p}$ contains the annihilator of M, i.e. the ideal

$$\mathrm{Ann}_R(M) = \{ r \in R \mid r \cdot m = 0 \text{ for all } m \in M \} \subset R.$$

Definition B.7.2 (Isolated primes)**.** The minimal elements (with respect to inclusion) in the set

$$\{ \mathfrak{p} \subset R \mid \mathfrak{p} \supset \mathrm{Ann}_R(M) \}$$

are called *isolated primes* of M.

From now on we assume R is noetherian and $M \neq 0$ is finitely generated. We have the following result.

THEOREM B.7.3 ([19, Theorem 5.5.10 (a)]). *Let R be a noetherian ring, $M \neq 0$ a finitely generated R-module. Then $\mathsf{Ass}_R(M)$ is a finite nonempty set containing all isolated primes.*

Definition B.7.4 (Embedded primes)**.** The non-isolated primes in $\mathsf{Ass}_R(M)$ are called the *embedded primes* of M.

Moreover, we have the following facts:

- the R-module M has a *composition series*, i.e. a filtration by R-submodules

  $$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_s = M$$

  such that $M_i/M_{i-1} = R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$. This series is not unique. However, for a prime ideal $\mathfrak{p} \subset R$, the number of times it occurs among the $\mathfrak{p}_i$ does not depend on the composition series. These primes are precisely the elements of $\mathsf{Ass}_R(M)$.

- Any ideal $I \subset R$ has a *primary decomposition*, i.e. an expression as intersection

  $$I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$$

  of primary ideals. A proper ideal $\mathfrak{q} \subsetneq R$ is called *primary* if whenever a product xy lies in $\mathfrak{q}$, either x or a power of y lies in $\mathfrak{q}$. Put differently, every zero-divisor in $R/\mathfrak{q}$ is nilpotent. One verifies that the radical of a primary ideal is prime, and one says that $\mathfrak{q}$ is $\mathfrak{p}$-primary if $\sqrt{\mathfrak{q}} = \mathfrak{p}$. One can always ensure that the decomposition is irredundant, i.e. removing any $\mathfrak{q}_i$ changes the intersection, and $\sqrt{\mathfrak{q}_i} \neq \sqrt{\mathfrak{q}_j}$ for all $i \neq j$.

Exercise B.7.5. Let $I \subset R$ be an ideal. Show that the set

$$\{\sqrt{\mathfrak{q}_i}\}_i$$

is determined by I. Then show that elements of $\mathsf{Ass}_R(R/I)$ are precisely the radicals of the primary ideals in a primary decomposition of I. In symbols,

$$\mathsf{Ass}_R(R/I) = \{\sqrt{\mathfrak{q}_i}\}_i.$$

Exercise B.7.6. Let $R = \mathbf{k}[x,y]$, $I = (xy, y^2)$ and $M = R/I$. Show that $\mathsf{Ass}_R(M) = \{(y),(x,y)\}$.

The most boring situation is when R is an integral domain, in which case the generic point $\xi \in \operatorname{Spec} R$ is the only associated (and clearly isolated) prime. More generally, a reduced affine scheme $\operatorname{Spec} R$ has no embedded primes (in particular no embedded points, see below), i.e. the only associated primes are the isolated (minimal) ones, corresponding to its irreducible components.

Let R be an integral domain. For an ideal $I \subset R$, one often calls the associated primes of I the associated primes of R/I. The minimal primes above $I = \operatorname{Ann}_R(R/I)$ (i.e. containing I) correspond to the irreducible components of the closed subscheme

$$\operatorname{Spec} R/I \subset \operatorname{Spec} R,$$

whereas for every embedded prime $\mathfrak{p} \subset R$ there exists a minimal prime $\mathfrak{p}'$ such that $\mathfrak{p}' \subset \mathfrak{p}$. Thus $\mathfrak{p}$ determines an *embedded component* — a subvariety $\mathrm{V}(\mathfrak{p})$ embedded in an irreducible component $\mathrm{V}(\mathfrak{p}')$. If the embedded prime $\mathfrak{p}$ is maximal, we talk about an *embedded point*.

Fact B.7.7. An algebraic curve (an algebraic variety of dimension 1) has no embedded points if and only if it is Cohen–Macaulay (the formal definition is given in Definition B.3.2). However, there can be nonreduced Cohen–Macaulay curves: those curves with a fat component, such as the affine plane curve $\operatorname{Spec} \mathbf{k}[x, y]/x^2 \subset \mathbb{A}^2$. These objects often have moduli, i.e. deform (even quite mysteriously) in positive dimensional families. See [3, 4, 20, 21] for generalities on multiple structures on schemes.

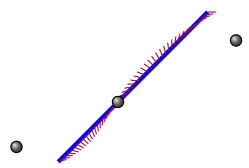

Figure B.1: A thickened (Cohen–Macaulay) curve with an embedded point and two isolated (possibly fat) points.

Remark B.7.8. An embedded component $\mathrm{V}(\mathfrak{p})$, where $\mathfrak{p}$ is the radical of some primary ideal $\mathfrak{q}$ appearing in a primary decomposition $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_e$, is of course embedded in some irreducible component $\mathrm{V}(\mathfrak{p}') \subset \operatorname{Spec} R/I$, but $\mathrm{V}(\mathfrak{q})$ is not a *subscheme* of $\mathrm{V}(\mathfrak{p}')$, because the fuzziness caused by nilpotent behavior (i.e. the difference between $\mathfrak{q}$ and its radical $\mathfrak{p}$) makes the bigger scheme $\mathrm{V}(\mathfrak{q}) \supset \mathrm{V}(\mathfrak{p})$ 'stick out' of $\mathrm{V}(\mathfrak{p}') \subset \operatorname{Spec} R/I$.

Example B.7.9. Consider $R = \mathbf{k}[x, y]$ and $I = (xy, y^2)$. A primary decomposition of I is

$$I = (x, y)^2 \cap (y).$$

However, $\operatorname{Spec} R/(x, y)^2$ is not scheme-theoretically contained in $\operatorname{Spec} R/y$.

In general, a subscheme Z of scheme Y has an embedded component if there exists a dense open subset $U \subset Y$ such that $Z \cap U$ is dense in Z but the scheme-theoretic closure of $Z \cap U \subset Z$ does not equal Z scheme-theoretically. For instance, if Y is irreducible, we say that $p \in Y$ supports an embedded point of a closed subscheme $Z \subset Y$ if $\overline{Z \cap (Y \setminus p)} \neq Z$ as schemes. In the example above, where $Y = \mathbb{A}^2$ and $Z = \operatorname{Spec} \mathbf{k}[x, y]/(xy, y^2)$, the scheme-theoretic closure of $Z \cap (\mathbb{A}^2 \setminus 0) \subset Z$ is not equal to Z.

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