

DEFORMATION THEORY AND MODULI

SISSA ALGEBRAIC GEOMETRY SEMINAR — FALL 2023

1. THE TOPIC, AND THE GOAL OF THE SEMINAR

Deformation theory is an important subject in algebraic geometry, with a tight link to the study of **moduli spaces**. The easiest and most familiar instance of deformation is the following: for any point x on a variety X , the *tangent space* $T_{X,x}$ describes how x ‘deforms to first order’. Deformation theory studies deformations ‘of arbitrary order’ of geometric objects such as

- (i) abstract schemes,
- (ii) closed subschemes of a given scheme Y ,
- (iii) coherent sheaves on a variety X ,
- (iv) surjections $E \rightarrow Q$ of coherent sheaves from a fixed target E ,

and many more. A full understanding of all *infinitesimal deformations* of an object m is equivalent to understanding the moduli space M containing $m \in M$ as a point ‘arbitrarily near m ’. Knowing also the *obstructions* to extending all infinitesimal deformations ‘to higher order’ gives important local information, e.g. on the smoothness of M at m .

The goal of the seminar is twofold:

- understand the general machinery of deformation theory, which roughly means: **deformation functors** and their infinitesimal properties, such as *smoothness*.
- Master a series of **key examples**, namely (i), (ii), (iii), (iv), in the language of moduli spaces: as applications of the general machinery, we shall compute the tangent space to Hilbert schemes (situation (ii)), moduli spaces of stable sheaves (situation (iii)) and Quot schemes (situation (iv)).

2. SCHEDULE OF THE SEMINAR

We plan on having 8 talks by the participants before Christmas. After that, we shall regroup and decide what will come next. Here is the detailed program:

- **30 oct**: Deforming smooth varieties, take I: first chapter of Sernesi’s book [4]
- **6 nov**: Deforming smooth varieties, take II: first chapter of Sernesi’s book [4]
- **13 nov**: Interlude: Hilbert and Quot schemes [2], moduli of simple (or stable) sheaves [3]
- **20 nov**: Deformation functors and tangent obstruction theories, take I [2] (including formal and Jacobian criterion for smoothness [1, 5])
- **27 nov**: Deformation functors and tangent obstruction theories, take II (including tangent space to Quot) [2]
- **4 dec**: Deformation theory of simple sheaves, Atiyah classes and trace maps [3]
- **11 dec**: Deformation Theory and virtual classes (Andrea Ricolfi)
- **18 dec**: Groupoid valued deformation theory (Barbara Fantechi)

Every talk should contain a few detailed proofs and at least one serious calculation!

REFERENCES

1. Siegfried Bosch, *Algebraic geometry and commutative algebra*, 2nd edition ed., London: Springer, 2022.
2. Barbara Fantechi, Lothar Göttsche, Luc Illusie, Steven L. Kleiman, Nitin Nitsure, and Angelo Vistoli, *Fundamental algebraic geometry*, Mathematical Surveys and Monographs, vol. 123, American Mathematical Society, Providence, RI, 2005, Grothendieck’s FGA explained.
3. Daniel Huybrechts and Manfred Lehn, *The geometry of moduli spaces of sheaves*, Second Edition, Cambridge University Press, 2010, pp. xviii+325.
4. Edoardo Sernesi, *Deformations of algebraic schemes*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 334, Springer-Verlag, Berlin, 2006.
5. The Stacks Project Authors, *Stacks Project*, [stacks-project](https://stacks-project.org), 2020.