MODULI SPACES OF SEMIORTHOGONAL DECOMPOSITIONS IN FAMILIES

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ABSTRACT. For a smooth projective family of schemes we prove that a semiorthogonal decomposition of the bounded derived category of coherent sheaves of the fiber of a point uniquely deforms over an étale neighborhood of the point. We do this using a comparison theorem between semiorthogonal decompositions and decomposition triangles of the structure sheaf of the diagonal. We then apply to the latter a deformation theory for morphisms with a fixed lift of the target, which is developed in the appendix.

Using this as a key ingredient we introduce a moduli space which classifies semiorthogonal decompositions of the category of perfect complexes of a smooth projective family. We show that this is an étale algebraic space over the base scheme of the family, which can be non-quasicompact and non-separated. This is done using Artin's criterion for a functor to be an étale algebraic space over the base.

We generalize this to families of geometric noncommutative schemes in the sense of Orlov. We also define the open subspace classifying non-trivial semiorthogonal decompositions only, which is used in a companion paper to study indecomposability of derived categories in various examples.

CONTENTS

1. INTRODUCTION

1.1. **Overview.** Triangulated categories and their enhancements are the subject of many recent developments in algebraic geometry, and other fields such as symplectic geometry or algebraic topology. Semiorthogonal decompositions provide a fundamental method to understand the structure of such categories, in particular in the setting of algebraic

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geometry and noncommutative algebra. This idea goes back to Beilinson's description of the derived category of $\mathbb{P}^n$ using an exceptional collection [11], and was generalised in [15, 17] to allow semiorthogonal decompositions with more complicated components.

The aim of this paper is to study the behavior of semiorthogonal decompositions in families, by constructing a moduli space for them and describing the geometric properties of this moduli space.

Before a brief survey on the role of semiorthogonal decompositions in algebraic geometry, we discuss a prototypical example which illustrates their expected behavior in families.

1.1.1. *The geometry of* -1*-curves.* By Castelnuovo's contraction theorem we have that a -1-curve on a surface is the exceptional curve of the blowup of another surface at a smooth point, and hence can be contracted. This fact constitutes the foundation of the birational classification of algebraic surfaces. A classical example in this setting is that of a cubic surface: it contains precisely 27 lines, and these are all the -1-curves on it.

If we consider a versal *family* $\mathcal{X} \to U$ of smooth cubic surfaces, we can upgrade the set of -1-curves to a *moduli space of lines in the fibres* of the family, or relative Fano scheme of lines, which will be denoted by

$$\mathcal{F} \to U. \tag{1.1}$$

This is a finite connected étale cover of schemes, of degree 27, reflecting the 27 lines on a cubic surface. A celebrated classical result is that the monodromy group of this covering space is the Weyl group of type $\mathbb{E}_6$, which is responsible for all the different realisations of a cubic surface as blowups of $\mathbb{P}^2$ in 6 points in general position.

As the structure sheaf of a -1-curve is an exceptional object of the bounded derived category of coherent sheaves of a surface, which hence induces a non-trivial semiorthogonal decomposition, the covering space (1.1) will be naturally included as a connected component of the moduli space of semiorthogonal decompositions associated to the family $\mathcal{X} \to U$. We will come back to this point in Section 9.

1.1.2. *Semiorthogonal decompositions in algebraic geometry.* The study of -1-curves is a precursor of the minimal model program, which is about the birational classification of higher-dimensional algebraic varieties. The DK-hypothesis asserts that there is a close relationship between operations in this program, and semiorthogonal decompositions of triangulated categories.

The DK-hypothesis claims that given a birational map $X \dashrightarrow Y$ between smooth projective varieties X and Y, and an equality $\mathrm{K}_X = \mathrm{K}_Y$ (resp. inequality $\mathrm{K}_X > \mathrm{K}_Y$) of canonical divisors, there is an equivalence of categories $\mathbf{D}^{\mathrm{b}}(X) \simeq \mathbf{D}^{\mathrm{b}}(Y)$ (resp. a semiorthogonal decomposition of $\mathbf{D}^{\mathrm{b}}(X)$ in terms of $\mathbf{D}^{\mathrm{b}}(Y)$ and some complement). A typical example of such a birational map is a divisorial contraction and a flip (resp. a flop). The paper [55] studied the case of smooth blowups and projective bundles, thereby initiating the whole subject, and [17] studied the behavior of derived categories for standard flips and flops. See [37] for a survey for the development after these works.

There also exist semiorthogonal decompositions which do not originate from the minimal model program. A typical example is the *Kuznetsov component* for a Fano variety of index i_X, which appears as the semiorthogonal complement of the exceptional collection $\mathcal{O}_X, \mathcal{O}_X(1), \ldots, \mathcal{O}_X(\mathrm{i}_X - 1)$ in $\mathbf{D}^{\mathrm{b}}(X)$.

In [17] the example of the intersection of two quadrics was studied, whose geometry is controlled by the geometry of the associated hyperelliptic curve. The derived category of this hyperelliptic curve is precisely the Kuznetsov category. Similarly, the relationship between the geometry of cubic fourfolds and the properties of its associated Kuznetsov category are a topic of current interest [6, 42]. Moreover, studies of analogous decompositions on "fake rational surface" resulted in the discovery of (quasi-)phantom categories. For an overview of these, and many other examples, one is referred to Kuznetsov's ICM address [45].

The notion of semiorthogonal decomposition also plays an important role in other areas, such as matrix factorisations [56], or Fukaya–Seidel categories. They also play an important role in mirror symmetry, by virtue of the generalisation of Dubrovin's conjecture to semiorthogonal decompositions given in [60].

1.2. **Results.** In this paper we will construct a moduli space for semiorthogonal decompositions and describe its geometry, for arbitrary families of smooth projective varieties (and more generally geometric noncommutative schemes as defined in [57]). This is a far-reaching generalisation of the moduli space of lines (1.1) for a family of cubic surfaces. The following statement is for semiorthogonal decompositions of length 2. Later in the introduction we will discuss variants for decompositions with more components.

Theorem A (Theorem 8.1 + Proposition 4.7). *Let U be an excellent scheme, and let $f\colon \mathcal{X} \to U$ be a smooth and proper morphism which admits an f-ample invertible sheaf on $\mathcal{X}$. Then there exists an algebraic space $\mathsf{SOD}_f \to U$, which is moreover étale over U, and which admits a functorial bijection*

$$\mathsf{SOD}_f(V \to U) \simeq \left\{ \begin{array}{c} V\text{-linear semiorthogonal} \\ \text{decompositions } \operatorname{Perf}\mathcal{X}_V = \langle \mathcal{A}, \mathcal{B} \rangle \end{array} \right\}$$

for any quasicompact and semiseparated U-scheme $V \to U$.

The V-linearity condition means that $\mathcal{A}$ and $\mathcal{B}$ are closed under tensor products with perfect complexes pulled back from V, and this is one of the essential conditions that ensure the existence of base change for semiorthogonal decompositions.

The proof of Theorem A consists of checking Artin's axioms for algebraic spaces which are étale over the base, which we take in the form of [27, Theorem 11.3] (see Criterion 8.2). The three axioms are:

(1) SOD_f is a sheaf on the big étale site of U (Theorem 4.2 and Lemma 2.25),
(2) SOD_f is locally of finite presentation (Theorem 6.7),
(3) let $(B, \mathfrak{m}, \mathbf{k})$ be a local noetherian ring that is $\mathfrak{m}$-adically complete; then, for any morphism $\operatorname{Spec} B \to U$, the natural map

$$\mathsf{SOD}_f(\operatorname{Spec} B) \to \mathsf{SOD}_f(\operatorname{Spec} B/\mathfrak{m})$$

is bijective (Theorem 7.1).

In this paper we give a proof for point (1) as an application of the descent results in [23]. This result is based on the linear reductivity of finite groups, so our proof works under the assumption that U is defined in characteristic 0. Independently of the current work, in a more general setting and using derived algebro-geometric methods, an fppf descent result for semiorthogonal decompositions for arbitrary families of enhanced triangulated categories was obtained in [1, Theorem 1.4]. This implies point (1) without the assumption on characteristic. We come back to the relationship between this result and ours in Remarks 4.5 and 8.9.

Strictly speaking, we first consider a functor of semiorthogonal decompositions on the big affine étale site $(\mathrm{Aff}_U)_{\mathrm{\acute{E}t}}$ and show that it is a sheaf (Theorem 4.2). Then we define SOD_f (cf. Definition 4.4) to be the corresponding sheaf on the big étale site $(\mathrm{Sch}_U)_{\mathrm{\acute{E}t}}$ under the equivalence of topoi (cf. Lemma 2.25) $\mathsf{Sh}(\mathrm{Aff}_U)_{\mathrm{\acute{E}t}} \simeq \mathsf{Sh}(\mathrm{Sch}_U)_{\mathrm{\acute{E}t}}$.

For point (2) we use a description of SOD_f in terms of decompositions of the structure sheaf of the diagonal, so that we can describe the projection functors in terms of Fourier–Mukai kernels. For this description the smoothness and properness is an important ingredient, to obtain base change for semiorthogonal decompositions. This description using kernels also allows us to study the problem via deformation theory, as required for point (3), and by the semiorthogonality of the base change we obtain that all obstructions vanish and that there is a unique lift.

It is important to remark that Theorem A does not work for the unbounded derived category at all. This is caused by the non-representability of the projection functors via kernels, which is crucial for our argument as discussed in the previous paragraph. We come back to this in Examples 8.10 and 8.11.

Remark 1.1. It is not clear whether or not SOD_f is in fact a scheme in general. As pointed out in Example 8.7 it can happen that the structure morphism $\mathsf{SOD}_f \to U$ is neither separated nor quasicompact, even restricted to connected components. Separatedness is a common ingredient in criteria such as [39, Corollary II.6.17] to show that algebraic spaces are in fact schemes. This is a new phenomenon, which is not present in the example (1.1) of -1-curves. In Question 8.8 we also ask whether the existence part of the valuative criterion is still satisfied for our moduli space, which is a weaker version of properness, holding for -1-curves in the context of (1.1).

As a corollary to the proof of Theorem A we obtain a generalisation of [30, Theorem 1.1] which shows that étale-locally in a family one can extend exceptional collections.

Corollary B (Corollary 7.4)**.** *Let $f\colon \mathcal{X} \to U$ be a smooth projective morphism of noetherian schemes. Let $X = f^{-1}(0)$ be the fibre over a closed point $0 \in U$. Assume*

$$\mathbf{D}^{\mathrm{b}}(\operatorname{coh} X) = \langle \mathcal{A}, \mathcal{B} \rangle \tag{1.2}$$

is a $\mathbf{k}(0)$-linear semiorthogonal decomposition. Then, possibly after shrinking U to an étale neighborhood of 0, there exists a unique U-linear semiorthogonal decomposition

$$\operatorname{Perf} \mathcal{X} = \langle \mathcal{A}_U, \mathcal{B}_U \rangle$$

whose base change to the closed point $0 \in U$ agrees with (1.2).

One can not take the neighborhood to be Zariski open in general, since in general there is a monodromy as we observed in the example (1.1). We explain this phenomenon with the easier example of the linear pencil of quadrics in Example 7.5.

1.3. **Relation to earlier works.** To explain the motivation behind our results, let us describe the history of similar results for the easiest semiorthogonal decompositions: full exceptional collections. Here all the subcategories are equivalent to $\mathbf{D}^{\mathrm{b}}(\mathbf{k})$. The deformation theory of exceptional objects goes back a long way, as it already provided the intuition and inspiration for the results in [18], using the expectation that exceptional objects uniquely lift to deformations.

This intuition was made precise for infinitesimal deformations of abelian categories (and their derived categories) in [51], and for formal deformations of abelian categories (and their derived categories) in [66]. In a strictly geometric context these liftability results are explained by the deformation theory results in [34, 48].

More recently, in [30, Theorem 1.1] and [26] it was shown that for smooth and proper families $\mathcal{X} \to U$, the existence of a full exceptional collection in a closed fibre extends to the existence of a full exceptional collection in an étale neighbourhood.

Theorem A (and its variants discussed below) generalise these results to the case of a semiorthogonal decomposition whose components are *not* necessarily built up of exceptional objects. The main reason for the extra level of complexity in the proofs is that we are no longer lifting exceptional *objects* in the derived category itself. Rather, we will encode semiorthogonal decompositions using triangles in the derived category of the product, and need to lift morphisms instead of just objects. For this we have developed a deformation of morphisms in Appendix A (joint with Wendy Lowen), similar to but different from the one in [51], as we do not fix a lift of the domain of the morphism. This is a far-generalisation of the deformation theory of Quot functors (see, say, [24]) and stable pairs ([47]), and hence would be of independent interest.

Another precursor of Corollary B, not just for exceptional collections, was studied in [25]. It takes as the base U the moduli stack $\mathcal{M}_g$ of genus g curves, and the family $\mathcal{X}$ is the relative moduli space of rank 2 vector bundles with odd determinant, and the point $0 \in U$ is any point in the hyperelliptic locus. The semiorthogonal decomposition of the fibre in 0 is the decomposition given by the universal vector bundle, which is shown to give a fully faithful functor $\mathbf{D}^{\mathrm{b}}(C) \hookrightarrow \mathbf{D}^{\mathrm{b}}(\mathrm{M}_C(2,\mathcal{L}))$ when C is hyperelliptic using an explicit geometric construction of the moduli space. Corollary B generalises this example to arbitrary families and arbitrary semiorthogonal decompositions where the components are not necessarily derived categories themselves.

1.4. **Amplifications.** Having constructed SOD_f it is possible to bootstrap, and construct

(1) the moduli space SOD_f^ℓ of semiorthogonal decompositions of length ℓ, so that $\mathsf{SOD}_f = \mathsf{SOD}_f^2$;
(2) the moduli space $\mathsf{SOD}_{\mathcal{B}}^\ell$ of semiorthogonal decompositions of a given U-linear admissible subcategory $\mathcal{B} \subseteq \operatorname{Perf}\mathcal{X}$, such that $\mathsf{SOD}_f^\ell = \mathsf{SOD}_{\operatorname{Perf}\mathcal{X}}^\ell$;
(3) the moduli space $\mathsf{ntSOD}_{\mathcal{B}}^\ell$ of nontrivial semiorthogonal decompositions (Definition 8.26): those where each component is required to be non-zero, as an open and closed subspace of $\mathsf{SOD}_{\mathcal{B}}^\ell$

We show in Theorem 8.13, Corollary 8.25 and Theorem 8.30 that these are all algebraic spaces, étale over U. The latter is the most interesting part, and could be empty if the fibers of f admit no non-trivial semiorthogonal decompositions. The other two are always non-empty.

These moduli spaces are the natural recipients of an action by the auto-equivalence group and the braid group (by mutation), as discussed in Subsection 8.4. These group actions reflect many deep questions on properties of semiorthogonal decompositions, such as the transitivity of the braid group action.

1.5. **A conjecture.** The generalisation of Theorem A to $\mathsf{SOD}^{\ell}_{\mathcal{B}}$ given in Corollary 8.25 suggests the following conjecture, in the spirit of [57].

Conjecture C. *Let $\mathcal{D}$ be a family of smooth and proper dg categories over an excellent scheme U. Then there exists an étale algebraic space $\mathsf{SOD}_{\mathcal{D}/U}$ over U with a functorial bijection*

$$\mathsf{SOD}_{\mathcal{D}/U}(\phi) \simeq \left\{ \begin{array}{c} V\text{-linear semiorthogonal} \\ \text{decompositions } \operatorname{Perf}(\mathcal{D}_V) = \langle \mathcal{A}, \mathcal{B} \rangle \end{array} \right\}$$

for quasicompact and semiseparated schemes $\phi\colon V \to U$.

The case where $\mathcal{D} = \operatorname{Perf}\mathcal{X}$ (with the appropriate enhancement) is Theorem A. Proving it will likely require some techniques from derived algebraic geometry, as in [1], and is related to the construction of moduli of objects in dg categories from [65].

More generally one would like to take the base U a sufficiently nice algebraic space, or even an algebraic stack.

1.6. **Applications.** In Sections 8 and 9 we discuss various examples, and revisit the geometry of -1-curves for families of cubic surfaces. A theoretical application is the following:

- In Remark 8.42 we deduce from Theorem A the fact that semiorthogonal decompositions are rigid under the actions of topologically trivial autoequivalences, which is first shown in [38]. The proof in Remark 8.42 is more conceptual than the original argument.

Other applications of the moduli spaces constructed here are deferred to the following companion papers:

- In [5] it will be explained that the moduli space being étale over the base can be used to show how being indecomposable is a condition which specialises to closed fibres, and how this can be used to give new indecomposability results for derived categories, extending those of [38]. We very briefly touch on this topic in Example 8.34.
- In [35] the example of $\mathbb{P}^1 \times \mathbb{P}^1$ and its degeneration to the second Hirzebruch surface will be discussed in detail, extending upon Example 8.7.

1.7. **Structure of the paper and conventions.** In Section 2 we collect various preliminaries on derived categories of sheaves and semiorthogonal decompositions for the reader's convenience. In Section 3 we discuss base change for semiorthogonal decompositions, generalising the results of [43]. An alternative discussion can be found in [6, Section 3.2].

The main object of interest, namely the functor of semiorthogonal decompositions, is introduced in Section 4. The main result in this section is that it is in fact a sheaf on the étale site.

To prove that this sheaf is an algebraic space, which is moreover étale over the base, we need an alternative description which makes the deformation theory of semiorthogonal decompositions more explicit. In Section 5 we introduce the presheaf of decompositions of the structure sheaf of the diagonal, and show that this presheaf is isomorphic to the presheaf of semiorthogonal decompositions (Theorem 5.9).

Using this alternative description we show in Section 6 that these functors are limit preserving. In Section 7 we study the deformation theory of semiorthogonal decompositions, and show that semiorthogonal decompositions for perfect complexes are unobstructed and have unique lifts.

In Section 8 we can then conclude from Artin's axioms that we indeed have an algebraic space which is étale over the base. We give various amplifications (to arbitrary length, relative to a fixed subcategory, and restricting to non-trivial semiorthogonal decompositions).

In Section 9 we discuss the relationship between the moduli space of lines in the family of cubic surfaces (1.1) and the moduli space of semiorthogonal decompositions in some detail.

In Appendix A we develop the deformation theory for a morphism in a derived category, with fixed lift of the codomain. This is written in the generality of deformations of abelian categories, and to not break the flow of the main argument we have isolated it in a self-contained appendix (joint with Wendy Lowen).

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2. Preliminaries

In this section we state the general conventions that will be adopted for the rest of the paper, and we collect most of the technical results that we will need.

2.1. **Derived categories of sheaves.** For a scheme X, let $\operatorname{Mod}\mathcal{O}_X$ be the category of $\mathcal{O}_X$-modules and $\operatorname{Qcoh}X$ be the subcategory of quasicoherent modules. When X is locally noetherian, also we let $\operatorname{coh}X$ be the subcategory of coherent $\mathcal{O}_X$-modules. All of them are known to be abelian categories. Let $\mathbf{D}(\mathcal{O}_X)=\mathbf{D}(\operatorname{Mod}\mathcal{O}_X)$ be the unbounded derived category of $\operatorname{Mod}\mathcal{O}_X$. For $*\in\{+,-,\mathrm{b},\emptyset,[a,b]\}$, we let $\mathbf{D}^*(\mathcal{O}_X)$ denote the full subcategories of $\mathbf{D}(\mathcal{O}_X)$, where the decoration $*$ indicates the location of the (possibly) non-vanishing cohomology sheaves. We use the same convention for $\mathbf{D}^*(\operatorname{Qcoh}X)$ and $\mathbf{D}^*(X):=\mathbf{D}^*(\operatorname{coh}X)$.

By $\mathbf{D}^*_{\mathrm{qcoh}}(\mathcal{O}_X)\subseteq\mathbf{D}^*(\mathcal{O}_X)$, resp. $\mathbf{D}^*_{\mathrm{coh}}(\mathcal{O}_X)\subseteq\mathbf{D}^*_{\mathrm{qcoh}}(\mathcal{O}_X)$, we denote the full triangulated subcategory of complexes with quasicoherent, resp. coherent cohomology.

Definition 2.1 ([64, Section 2]). A complex $E \in \mathbf{D}(\mathcal{O}_X)$ is called *perfect* (resp. *strictly perfect*) if it is locally (resp. globally) quasi-isomorphic to a bounded complex of locally free $\mathcal{O}_X$-modules of finite type.

We write $\operatorname{Perf} X \subseteq \mathbf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ for the full triangulated subcategory of perfect complexes on X.

We recapitulate some known results on the identification of various triangulated categories of sheaves:

- By [62, Tag 08DB], if X is quasicompact and semiseparated (i.e. has affine diagonal) the canonical functor
$$\mathbf{D}(\operatorname{Qcoh} X) \to \mathbf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$$
is an exact equivalence (see also [14, Corollary 5.5] for the proof in the case where X is quasicompact and separated).
- By [10, Corollary II.2.2.2.1], if X is noetherian, the canonical functor
$$\mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_X)$$
is an exact equivalence.
- By [62, Tag 0FDC], if X is noetherian and regular of finite dimension, the canonical functor
$$\operatorname{Perf} X \to \mathbf{D}^{\mathrm{b}}(X)$$
is an exact equivalence.

The general construction of derived functors in the context of unbounded complexes is carried out in [61], where all the usual compatibilities between these derived functors are also proved.

2.1.1. *Derived Nakayama–Azumaya–Krull.* For the sake of completeness, we include a derived version of Nakayama–Azumaya–Krull lemma.

Lemma 2.2 (Derived Nakayama–Azumaya–Krull, local version). *Let $(R, \mathfrak{m}, \mathbf{k})$ be a local ring. Let*

$$P^\bullet = \left[\cdots \to P^i \xrightarrow{d^i} P^{i+1} \to \cdots\right] \tag{2.1}$$

be a bounded complex of finitely generated free R-modules. Then

$$P^\bullet = 0 \in \operatorname{Perf}(R) \iff P^\bullet \otimes_R \mathbf{k} = 0 \in \operatorname{Perf}(\mathbf{k}).$$

Proof. The implication $\Rightarrow$ is obvious, so we show $\Leftarrow$. Since $(P^\bullet)^\vee$ also is a bounded complex of finitely generated free R-modules, by replacing $P^\bullet$ with it, we are led to show the implication

$$\operatorname{Hom}^\bullet_R(P^\bullet, \mathbf{k}) = 0 \Rightarrow P^\bullet = 0 \in \operatorname{Perf}(R).$$

In the rest, we show the contraposition of this assertion.

Let m be the maximal index for which $P^m \neq 0$. Without loss of generality we may assume that $\mathcal{H}^m(P^\bullet) \neq 0$. In fact, if d^{m-1} is surjective, since P^m is a free module over R, we can find a splitting $s\colon P^m \to P^{m-1}$ of d^{m-1}, so that $P^\bullet$ is quasi-isomorphic to the new complex of finitely generated free R-modules

$$\left[\cdots \to P^{m-3} \xrightarrow{d^{m-3}} P^{m-2} \to P^{m-1}/s(P^m) \to 0 \to \cdots\right],$$

which reduces m by 1. Since $P^\bullet$ is a bounded complex, we can repeat this process only finitely many times.

Since $0 \neq \mathcal{H}^m(P^\bullet)$ is a finitely generated R-module, the usual Nakayama–Azumaya–Krull lemma implies that $V := \mathcal{H}^m(P^\bullet) \otimes_R \mathbf{k} \neq 0$. Choosing any surjective $\mathbf{k}$-linear map $f\colon V \to \mathbf{k}$, we obtain a sequence of morphisms

$$P^\bullet \to \mathcal{H}^m(P^\bullet)[-m] \to V[-m] \xrightarrow{f[-m]} \mathbf{k}[-m] \tag{2.2}$$

in $\mathrm{Perf}(R)$. One can see that $\mathcal{H}^m$ of the composition of these morphisms is a surjective R-linear homomorphism $\mathcal{H}^m(P^\bullet) \to \mathbf{k}$. Thus $\mathcal{H}^m\left(\mathrm{Hom}^\bullet_R(P^\bullet, \mathbf{k})\right) \neq 0$. □

For a scheme X and a point $x \in X$, let

$$\iota_x\colon \operatorname{Spec} \mathbf{k}(x) \to X$$

be the standard morphism from the spectrum of the residue field at the point x.

Lemma 2.3 (Derived Nakayama–Azumaya–Krull, global version)**.** *Let X be a scheme and $F \in \mathrm{Perf}\,X$. Then*

$$F = 0 \in \mathrm{Perf}\,X \iff \mathbf{L}\iota_x^* F = 0 \in \mathrm{Perf}(\mathbf{k}(x)) \ \textit{for all } x \in X.$$

Proof. Note that ι_x has a factorisation

$$\operatorname{Spec} \mathbf{k}(x) \to \operatorname{Spec} \mathcal{O}_{X,x} \xrightarrow{j_x} X.$$

Since j_x is flat we have that

$$F = 0 \iff j_x^* F = 0 \in \mathrm{Perf}(\mathcal{O}_{X,x}) \ \text{for all } x \in X.$$

Thus we have reduced the assertion to Lemma 2.2. □

2.2. **A few notions from scheme theory.** For later use we recall some notions from scheme theory, which apply to not necessarily noetherian schemes.

Definition 2.4.
(1) A collection of line bundles $\{\mathcal{L}_i\}_{i\in I}$ on a quasicompact and quasiseparated scheme X is *an ample family of line bundles* if one can find a set of triples (i, n, s), where $i \in I, n \in \mathbb{N}, s \in \mathrm{H}^0(X, \mathcal{L}_i^{\otimes n})$, such that $X = \bigcup_{(i,n,s)} X_s$ is an affine covering of X (see [64, Definition 2.1.1]).
(2) An invertible sheaf $\mathcal{L}$ on a quasicompact and separated scheme X is *ample* if $\{\mathcal{L}\}$ is an ample family of line bundles (see [62, Tag 01RP]).
(3) For a quasicompact and separated morphism of schemes $f\colon X \to U$, an invertible sheaf $\mathcal{L}$ on X is *f-ample* if for every affine open subset $V \subset U$, the restriction $\mathcal{L}|_{f^{-1}(V)}$ is ample on $f^{-1}(V)$ (see [62, Tag 01VH]).
(4) A morphism of schemes $f\colon X \to U$ is *H-projective* if f factors through a closed immersion of U-schemes into $\mathbb{P}^N_U$ for some $N > 0$, and f is *locally projective* if there is an open cover $U = \bigcup_{\lambda\in\Lambda} U_\lambda$ such that for each $\lambda \in \Lambda$ the restriction $f^{-1}(U_\lambda) \to U_\lambda$ is H-projective (see [62, Tag 01W7]).

Proposition 2.5 ([10, §II.2.2], [64, Proposition 2.3.1(d)]). *Let X be a scheme with an ample family of line bundles. Then every perfect complex on X is strictly perfect. In particular, there is an inclusion*

$$\operatorname{Perf} X \hookrightarrow \mathbf{D}^{\mathrm{b}}_{\mathrm{qcoh}}(\mathcal{O}_X).$$

The following result will be important in translating semiorthogonal decomposition, and their deformation theory, into the deformation theory of perfect complexes (and morphisms between them).

Lemma 2.6. *Let $f\colon \mathcal{X} \to U$ be a smooth and separated morphism of schemes, and let $\iota_{\Delta_f}\colon \mathcal{X} \hookrightarrow \mathcal{X} \times_U \mathcal{X}$ be the inclusion of the diagonal. Then the structure sheaf of the diagonal Δ_f is a perfect complex on $\mathcal{X} \times_U \mathcal{X}$.*

Proof. The structure sheaf $\mathcal{O}_{\Delta_f}$ is by definition $\iota_{\Delta_f,*}\mathcal{O}_{\mathcal{X}} = \mathbf{R}\iota_{\Delta_f,*}\mathcal{O}_{\mathcal{X}}$. But ι_{Δ_f} is a closed immersion by the separatedness of f, and a locally complete intersection morphism by the smoothness of f, so in particular it is a proper perfect morphism. Thus by [49, Example 2.2] we know that $\mathbf{R}\iota_{\Delta_f *}\colon \mathbf{D}_{\mathrm{qcoh}}(\mathcal{O}_{\mathcal{X}}) \to \mathbf{D}_{\mathrm{qcoh}}(\mathcal{O}_{\mathcal{X}\times_U \mathcal{X}})$ preserves perfect complexes. ☐

Remark 2.7. When we drop the smoothness assumption on $f\colon \mathcal{X} \to U$, perfectness of $\mathcal{O}_{\Delta_f}$ fails. It would be interesting to see what can be said if we instead think of the semiorthogonal decompositions of $\mathbf{D}^{\mathrm{b}}(\operatorname{coh}\mathcal{X})$.

2.3. **Deformation theory in the derived category.** In this subsection we recall a few definitions and results on deformations of complexes from the literature. In Appendix A we develop the deformation theory of morphisms of complexes with fixed lift of the target, which is a generalisation of the deformation theory of complexes, as explained in Remark A.22.

Let $f\colon X \to S$ be a flat morphism.

Definition 2.8. A complex $E \in \mathbf{D}(\mathcal{O}_X)$ is S-*perfect* if it is locally quasi-isomorphic to a bounded complex of S-flat quasicoherent $\mathcal{O}_X$-modules. We let $\operatorname{Perf}(X/S)$ denote the category of such complexes.

Remark 2.9. The category $\operatorname{Perf}(X/S)$ is a full triangulated subcategory of $\mathbf{D}(\mathcal{O}_X)$. It plays a prominent role in Lieblich's work [48], where it is denoted $\mathbf{D}^{\mathrm{b}}_{\mathrm{p}}(X/S)$.

Since we will use Lieblich's work [48] on the deformation theory of complexes to deform Fourier–Mukai kernels for admissible embeddings, we will now spend some time reviewing the language necessary to study deformations of perfect complexes.

Definition 2.10. For a scheme S, an S-*ring* is a commutative ring A equipped with a morphism of schemes $\operatorname{Spec} A \to S$. The category of S-rings will be denoted by Alg_S.

Let $f\colon X \to S$ be a flat morphism of schemes. Let $A \twoheadrightarrow A_0$ be a square-zero extension of commutative noetherian S-rings. Form the cartesian diagrams

$$\begin{tikzcd}
X_{A_0} \arrow[r, hook, "\iota"] \arrow[d] \arrow[dr, phantom, "\square"] & X_A \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
\operatorname{Spec} A_0 \arrow[r, hook] & \operatorname{Spec} A \arrow[r] & S
\end{tikzcd}$$

and fix a complex

$$E_0 \in \mathrm{Perf}(X_{A_0}/A_0).$$

Definition 2.11. A *deformation* of E_0 over $\operatorname{Spec} A$ is a complex $E \in \mathbf{D}(\mathcal{O}_{X_A})$ along with an isomorphism $E \otimes_A^{\mathbf{L}} A_0 \xrightarrow{\sim} E_0$. If $E' \in \mathbf{D}(\mathcal{O}_{X_A})$ is another deformation of E_0, an *isomorphism of deformations* is an isomorphism $\phi : E \xrightarrow{\sim} E'$ in $\mathbf{D}(\mathcal{O}_{X_A})$ such that

$$\begin{tikzcd} E \otimes_A^{\mathbf{L}} A_0 \arrow[d, "\phi\otimes \mathrm{id}_{A_0}"'] \arrow[dr] & \\ E' \otimes_A^{\mathbf{L}} A_0 \arrow[r] & E_0 \end{tikzcd}$$

commutes.

In the definition, we have set $E \otimes_A^{\mathbf{L}} A_0 = \mathbf{L}\iota^* E$, and we will use similar tensor product notation throughout for the restriction functor along an arbitrary morphism.

2.3.1. *Formal deformations.* Let $(R, \mathfrak{m}, \mathbf{k})$ be a complete noetherian local ring, and let

$$f : X \to \operatorname{Spec} R$$

be a flat morphism. Set $R_n = R/\mathfrak{m}^{n+1}$ and $X_{R_n} = X \times_R R_n$ for $n \geq 0$, and consider the natural closed immersions

$$\iota_n : X_{R_n} \hookrightarrow X_{R_{n+1}}, \quad n \geq 0.$$

Definition 2.12. A *formal deformation* of a complex $E_0 \in \mathrm{Perf}(X_{R_0}/\mathbf{k})$ over R is a sequence of objects $E_n \in \mathbf{D}(\mathcal{O}_{X_{R_n}})$ with compatible isomorphisms $\mathbf{L}\iota_n^* E_{n+1} \xrightarrow{\sim} E_n$.

Remark 2.13. Let $\widehat{X}$ denote the formal completion of X along the fibre $X_0 = f^{-1}(0)$, where 0 is the unique closed point of $\operatorname{Spec} R$, corresponding to $\mathfrak{m}$. We have the cartesian diagrams

$$\begin{tikzcd} X_0 \arrow[r, hook, "j"] \arrow[d] \arrow[dr, phantom, "\square"] & \widehat{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ \operatorname{Spec} \mathbf{k} \arrow[r] & \operatorname{Spf} R \arrow[r] & \operatorname{Spec} R \end{tikzcd}$$

and to give a formal deformation of E_0 is equivalent to giving a complex $E \in \mathbf{D}(\mathcal{O}_{\widehat{X}})$, along with an isomorphism $\mathbf{L} j^* E \xrightarrow{\sim} E_0$.

2.4. **Semiorthogonal decompositions.** We will recall some important definitions from [43] in this section.

Let $\mathcal{T}$ be a triangulated category. Given a class of objects $\mathcal{A} \subseteq \mathcal{T}$, its *right* and *left orthogonal*

$$\mathcal{A}^{\perp} = \{\, T \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(\mathcal{A}[k], T) = 0 \text{ for all } k \in \mathbb{Z} \,\}$$

$$^{\perp}\mathcal{A} = \{\, T \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(T, \mathcal{A}[k]) = 0 \text{ for all } k \in \mathbb{Z} \,\}$$

define triangulated subcategories of $\mathcal{T}$, closed under taking direct summands.

Two classes of objects $\mathcal{A}, \mathcal{B} \subseteq \mathcal{T}$ are called *semiorthogonal* if $\mathcal{A} \subseteq \mathcal{B}^{\perp}$ (which is equivalent to $\mathcal{B} \subseteq {}^{\perp}\mathcal{A}$).

Definition 2.14. A *semiorthogonal decomposition of length* ℓ of a triangulated category $\mathcal{T}$ is a finite sequence $\mathcal{A}^1,\dots,\mathcal{A}^\ell$ of strictly full triangulated subcategories of $\mathcal{T}$, such that

(1) $\mathcal{A}^i \subseteq \mathcal{A}^{j,\perp}$ for $i<j$, and
(2) for every object $T\in\mathcal{T}$ there exists a sequence of morphisms

$$0 = T_\ell \to T_{\ell-1} \to \cdots \to T_1 \to T_0 = T \tag{2.3}$$

such that $\mathrm{Cone}(T_i \to T_{i-1}) \in \mathcal{A}^i$ for all $i=1,\dots,\ell$.

Recall that a subcategory $\mathcal{A}\subset\mathcal{T}$ is *strictly full* if, whenever an object $a\in\mathcal{A}$ is isomorphic to an object $t\in\mathcal{T}$, one has that $t\in\mathcal{A}$ [62, Tag 001D].

A semiorthogonal decomposition of $\mathcal{T}$ as in Definition 2.14 is denoted by

$$\mathcal{T} = \left\langle \mathcal{A}^1,\dots,\mathcal{A}^\ell \right\rangle.$$

We use superscripts, rather than the more conventional subscripts, for compatibility with the notation in what follows.

The association $T\mapsto T_k$ is functorial. The corresponding functor $\alpha_k\colon\mathcal{T}\to\mathcal{T}$ is called the kth *projection functor* of the given semiorthogonal decomposition.

Definition 2.15. A full triangulated subcategory $\mathcal{A}\subseteq\mathcal{T}$ is called *right admissible* (resp. *left admissible*) if the inclusion functor $i\colon\mathcal{A}\hookrightarrow\mathcal{T}$ has a right adjoint $i^!\colon\mathcal{T}\to\mathcal{A}$ (resp. a left adjoint $i^*\colon\mathcal{T}\to\mathcal{A}$). It is called *admissible* if it is both left and right admissible.

Lemma 2.16 ([15, Lemma 3.1])**.** *If* $\mathcal{T}=\langle\mathcal{A},\mathcal{B}\rangle$ *is a semiorthogonal decomposition, then* $\mathcal{A}$ *is left admissible and* $\mathcal{B}$ *is right admissible. Conversely, if* $\mathcal{A}\subseteq\mathcal{T}$ *is left admissible, then* $\mathcal{T}=\langle\mathcal{A},{}^\perp\mathcal{A}\rangle$ *is a semiorthogonal decomposition, and if* $\mathcal{B}\subseteq\mathcal{T}$ *is right admissible then* $\mathcal{T}=\langle\mathcal{B}^\perp,\mathcal{B}\rangle$ *is a semiorthogonal decomposition.*

Definition 2.17. A semiorthogonal decomposition $\mathcal{T}=\langle\mathcal{A}^1,\dots,\mathcal{A}^\ell\rangle$ is called *strong* if $\mathcal{A}^i$ is admissible in $\langle\mathcal{A}^i,\dots,\mathcal{A}^\ell\rangle$, the smallest strictly full triangulated subcategory of $\mathcal{T}$ which contains all of $\mathcal{A}^i,\mathcal{A}^{i+1},\dots,\mathcal{A}^\ell$, for every i.

Definition 2.18. Given a morphism of schemes $f\colon\mathcal{X}\to U$, a semiorthogonal decomposition $\mathrm{Perf}\,\mathcal{X}=\langle\mathcal{A}^1,\dots,\mathcal{A}^\ell\rangle$ is said to be U*-linear* if

$$\mathcal{A}^i \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{X}} \mathbf{L}f^*(\mathrm{Perf}\,U) \subseteq \mathcal{A}^i$$

for all $i=1,\dots,\ell$.

We next observe that U-linear semirthogonal decompositions along a smooth and proper morphism are strong.

Lemma 2.19. *Let* $f\colon\mathcal{X}\to U$ *be a smooth and proper morphism. Then every* U*-linear semiorthogonal decomposition* $\mathrm{Perf}\,\mathcal{X}=\langle\mathcal{A},\mathcal{B}\rangle$ *is strong.*

Proof. Since f is smooth and proper, the category $\mathrm{Perf}\,\mathcal{X}$ is smooth and proper over U by [58, Lemma 4.9(6)]. Then, by [58, Lemma 4.15(4)], for every U-linear semiorthogonal decomposition $\mathrm{Perf}\,\mathcal{X}=\langle\mathcal{A},\mathcal{B}\rangle$, the inclusion $\mathcal{A}\hookrightarrow\mathrm{Perf}\,\mathcal{X}$ has a right adjoint. This means that the semiorthogonal decomposition is strong. $\square$

By induction on the length of the semiorthogonal decomposition we obtain the following corollary.

Corollary 2.20. *Let $f : \mathcal{X} \to U$ be a smooth and proper morphism. Then every U-linear semiorthogonal decomposition* $\operatorname{Perf}\mathcal{X} = \langle \mathcal{A}^1, \dots, \mathcal{A}^\ell \rangle$ *is strong.*

Next we recall the action of the braid group on the set of all (strong) semiorthogonal decompositions of a triangulated category $\mathcal{T}$. Here we will denote by Br_ℓ the braid group on ℓ strands.

Definition 2.21. Let $\mathcal{T} = \langle \mathcal{A}^1, \dots, \mathcal{A}^\ell \rangle$ be a strong semiorthogonal decomposition, which we will denote by $\mathcal{A}^\bullet$ for the purposes of this definition. The *right mutation* (resp. *left mutation*) at position i is the semiorthogonal decomposition

$$\mathcal{T} = \langle \mathcal{A}^1, \dots, \mathcal{A}^{i-2}, \mathcal{A}^i, \mathbb{R}_i(\mathcal{A}^\bullet), \mathcal{A}^{i+1}, \dots, \mathcal{A}^\ell \rangle$$

resp.

$$\mathcal{T} = \langle \mathcal{A}^1, \dots, \mathcal{A}^{i-1}, \mathbb{L}_i(\mathcal{A}^\bullet), \mathcal{A}^{i+1}, \dots, \mathcal{A}^\ell \rangle,$$

where

$$\mathbb{R}_i(\mathcal{A}^\bullet) := {}^\perp\langle \mathcal{A}^1, \dots, \mathcal{A}^{i-2}, \mathcal{A}^i \rangle \cap \langle \mathcal{A}^{i+1}, \dots, \mathcal{A}^\ell \rangle^\perp$$

resp.

$$\mathbb{L}_i(\mathcal{A}^\bullet) := {}^\perp\langle \mathcal{A}^1, \dots, \mathcal{A}^{i-1} \rangle \cap \langle \mathcal{A}^i, \mathcal{A}^{i+2}, \dots, \mathcal{A}^\ell \rangle^\perp.$$

Mutations are well-defined, i.e. these are in fact semiorthogonal decompositions, by [16, Proposition 4.9]. We have that $\mathbb{R}_i(\mathcal{A}^\bullet) \simeq \mathcal{A}^{i-1}$ and $\mathbb{L}_i(\mathcal{A}^\bullet) \simeq \mathcal{A}^i$. The left and right mutation operations moreover satisfy the defining relations of the braid group, so Br_ℓ acts on the set of strong semiorthogonal decompositions of length ℓ of $\mathcal{T}$.

To conclude this subsection, we record here a general result, that we will use in Lemma 3.13 and in Lemma 5.5.

Lemma 2.22. *Let $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ be a semiorthogonal decomposition of a triangulated category $\mathcal{T}$ and take an object $E \in \mathcal{T}$. Let*

$$b_1 \xrightarrow{s_1} E \xrightarrow{t_1} a_1 \xrightarrow{u_1} b_1[1]$$

$$b_2 \xrightarrow{s_2} E \xrightarrow{t_2} a_2 \xrightarrow{u_2} b_2[1]$$

be two distinguished triangles such that $a_i \in \mathcal{A}$ and $b_i \in \mathcal{B}$ for $i = 1, 2$. Then there exists exactly one isomorphism between these decompositions. Namely, there is exactly one pair of isomorphisms

$$\beta : b_1 \xrightarrow{\sim} b_2, \quad \alpha : a_1 \xrightarrow{\sim} a_2$$

which yields an isomorphism of distinguished triangles as follows.

$$\begin{tikzcd}
b_1 \arrow[r, "s_1"] \arrow[d, "\beta"] & E \arrow[r, "t_1"] \arrow[d, equal] & a_1 \arrow[r, "u_1"] \arrow[d, "\alpha"] & b_1[1] \arrow[d, "\beta[1]"] \\
b_2 \arrow[r, "s_2"] & E \arrow[r, "t_2"] & a_2 \arrow[r, "u_2"] & b_2[1]
\end{tikzcd}$$

Proof. This follows from the observation that the natural maps

$$\operatorname{Hom}(b_1, b_2) \xrightarrow{s_{2*}} \operatorname{Hom}(b_1, E)$$

$$\operatorname{Hom}(a_1, a_2) \xrightarrow{t_1^*} \operatorname{Hom}(E, a_2)$$

are isomorphisms. □

2.5. **Big étale sites.** We discuss here the sites and topoi that will be used in this paper.

Definition 2.23. The *big étale site* $(\mathrm{Sch}_U)_{\mathrm{Ét}}$ of a scheme U consists of:

(1) the category Sch_U of *arbitrary* schemes over U, equipped with
(2) the Grothendieck topology in which an *étale covering* of an object $(V \to U) \in \mathrm{Sch}_U$ is a collection of étale morphisms $(\pi_i \colon V_i \to V)_{i\in I}$ such that $V = \bigcup_{i\in I} \pi_i(V_i)$.

Set theoretic aspects in the definition of the big étale site are discussed in [62, Tag 021A].

For technical reasons, we will use the following auxiliary site, which is equivalent to $(\mathrm{Sch}_U)_{\mathrm{Ét}}$ on the level of topoi.

Definition 2.24 ([62, Tag 0241])**.** The *big affine étale site* $(\mathrm{Aff}_U)_{\mathrm{Ét}}$ of a scheme U consists of:

(1) the category Aff_U of arbitrary *affine* schemes over U, equipped with
(2) the Grothendieck topology consisting of standard étale coverings. A standard étale covering of an affine U-scheme $V \to U$ is an étale covering $(\pi_i \colon V_i \to V)_{i\in I}$ such that $|I| < \infty$ and each V_i is affine ([62, Tag 0241]).

The following isomorphism of topoi is technically quite important (see Remark 2.26 below) for the definition of functors we will be working with.

Lemma 2.25 ([62, Tag 021E])**.** *The natural functor* $(\mathrm{Aff}_U)_{\mathrm{Ét}} \to (\mathrm{Sch}_U)_{\mathrm{Ét}}$ *induces an equivalence of the categories of sheaves (topoi):*

$$\mathsf{Sh}(\mathrm{Aff}_U)_{\mathrm{Ét}} \xrightarrow{\sim} \mathsf{Sh}(\mathrm{Sch}_U)_{\mathrm{Ét}}$$

Remark 2.26. In Section 4 we will define a presheaf of semiorthogonal decompositions as a functor on Aff_U, and in Proposition 4.2 we will show that this presheaf defines in fact a *sheaf* on the big affine étale site $(\mathrm{Aff}_U)_{\mathrm{Ét}}$ of Definition 2.24. Using Lemma 2.25, this induces a sheaf on the big étale site $(\mathrm{Sch}_U)_{\mathrm{Ét}}$. This is the sheaf of semiorthogonal decompositions, which constitutes the main character of this paper. It will be shown in Theorem 8.1 that this sheaf is in fact an algebraic space étale over U, as stated in Theorem A.

Note that for any object $\phi \colon V \to U$ in $(\mathrm{Aff}_U)_{\mathrm{Ét}}$, the scheme $\mathcal{X}_V = \mathcal{X} \times_{f,U,\phi} V$ is quasi-compact and separated. It is convenient to discuss perfect complexes and the derived category of quasicoherent sheaves under these assumptions.

3. Base change for semiorthogonal decompositions

In this section we review Kuznetsov's base change theorem for semiorthogonal decompositions [43, Proposition 5.1]. For some background one is referred to Sections 2.3 and 5.1 in op. cit. and the references therein. In Subsection 3.1 we only recall the details required for our construction. Next we explain how it can be generalised to the setting which is required for this paper, dropping the quasi-projectivity assumptions: we will prove the analogue of the base change theorem for more general schemes in Subsection 3.2. A further generalisation of the base change theorem is proved in [6, Section 3.2].

3.1. **Kuznetsov's base change theorem.** Throughout this subsection, we fix a base field and all schemes will thus be *quasiprojective varieties.* Before we start, we introduce the following convenient (and standard) piece of notation.

Notation 3.1. If $\mathcal{A}$ and $\mathcal{B}$ are subcategories of $\mathbf{D}_{\mathrm{qcoh}}(\mathcal{O}_{\mathcal{X}})$ for a scheme $\mathcal{X}$, we will write

$$\mathbf{R}\mathscr{H}\mathrm{om}_{\mathcal{X}}(\mathcal{B},\mathcal{A})=0$$

to mean that $\mathbf{R}\mathscr{H}\mathrm{om}_{\mathcal{X}}(b,a)=0$ for all $a\in\mathcal{A}$ and $b\in\mathcal{B}$.

Let $f:\mathcal{X}\to U$ be a morphism between quasiprojective varieties. Similar to Definition 2.18 we define the following notion.

Definition 3.2. A triangulated subcategory $\mathcal{T}\subseteq\mathbf{D}_{\mathrm{qcoh}}(\mathcal{O}_{\mathcal{X}})$ is called U*-linear* if for every $t\in\mathcal{T}$ and $p\in\operatorname{Perf}U$ one has $t\otimes^{\mathbf{L}}_{\mathcal{O}_{\mathcal{X}}}\mathbf{L}f^*p\in\mathcal{T}$.

A semiorthogonal decomposition $\mathcal{T}=\langle\mathcal{A},\mathcal{B}\rangle$ is called U*-linear* if the triangulated subcategory $\mathcal{A}$ is U-linear. Note that this holds if and only if $\mathcal{B}$ is U-linear.

Lemma 3.3 ([43, Lemma 2.7])**.** *Two U-linear triangulated subcategories $\mathcal{A},\mathcal{B}\subseteq\mathbf{D}_{\mathrm{qcoh}}(\mathcal{O}_{\mathcal{X}})$ are semiorthogonal if and only if*

$$\mathbf{R}f_*\mathbf{R}\mathscr{H}\mathrm{om}_{\mathcal{X}}(\mathcal{B},\mathcal{A})=0.$$

Given a morphism $\phi:V\to U$, consider the fibre product:

(3.1)
$$\begin{tikzcd} \mathcal{X}_V \arrow[r, "\phi_{\mathcal{X}}"] \arrow[d, "f_V"'] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "f"] \\ V \arrow[r, "\phi"'] & U \end{tikzcd}$$

The map ϕ is said to be a *faithful base change* for f if the natural transformation $\phi^*f_*\Rightarrow f_{V*}\phi^*_{\mathcal{X}}$, where one suitably derives the functors, is an isomorphism of (derived) functors from $\mathbf{D}_{\mathrm{qcoh}}(\mathcal{O}_{\mathcal{X}})$ to $\mathbf{D}_{\mathrm{qcoh}}(\mathcal{O}_V)$. If this condition holds, we also say that the diagram (3.1) is *exact.*

Lemma 3.4 ([41, Corollary 2.21])**.** *If f is flat, any morphism $\phi:V\to U$ is a faithful base change for f.*

Fix a U-linear semiorthogonal decomposition

$$\operatorname{Perf}\mathcal{X}=\left\langle\mathcal{A}^1,\dots,\mathcal{A}^\ell\right\rangle.$$

For a morphism ϕ as above, Kuznetsov defines $\mathcal{A}^i_V$ to be the minimal triangulated subcategory of $\operatorname{Perf}\mathcal{X}_V$ closed under direct summands and containing all objects of the form $\mathbf{L}\phi^*_{\mathcal{X}}F\otimes^{\mathbf{L}}\mathbf{L}f^*_VG$, with $F\in\mathcal{A}^i$ and $G\in\operatorname{Perf}V$. He then proves the following.

Theorem 3.5 ([43, Proposition 5.1])**.** *If $\phi:V\to U$ is a faithful base change for f, then*

$$\operatorname{Perf}\mathcal{X}_V=\left\langle\mathcal{A}^1_V,\dots,\mathcal{A}^\ell_V\right\rangle$$

is a V-linear semiorthogonal decomposition.

3.2. **Generalised base change theorem.** The aim of this subsection is to confirm that the base change theorem of Kuznetsov (Theorem 3.5) for Perf works in a generality which is sufficient for the purpose of this paper. Kuznetsov's base change theorem was also generalised in [6, Section 3.2].

In [43], the schemes involved are assumed to be algebraic varieties over a base field. In this subsection we consider more general schemes.

3.2.1. *Defining generalised base change.* For notational ease we will implicitly derive functors appropriately.

Fix once and for all a cartesian square

$$\begin{tikzcd} \mathcal{X}_V \arrow[r, "\phi_{\mathcal{X}}"] \arrow[d, "f_V"'] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "f"] \\ V \arrow[r, "\phi"'] & U \end{tikzcd} \tag{3.2}$$

of schemes. Recall from Definition 2.18 the notion of U-linearity for semiorthogonal decompositions.

Definition 3.6. Given a U-linear semiorthogonal decomposition

$$\operatorname{Perf}\mathcal{X} = \left\langle \mathcal{A}^1, \ldots, \mathcal{A}^\ell \right\rangle, \tag{3.3}$$

we say that a V-linear semiorthogonal decomposition

$$\operatorname{Perf}\mathcal{X}_V = \left\langle \mathcal{A}^1_\phi, \ldots, \mathcal{A}^\ell_\phi \right\rangle$$

is a *base change* of (3.3) if $\phi^*_{\mathcal{X}}\mathcal{A}^i \subseteq \mathcal{A}^i_\phi$ for all $i = 1, \ldots, \ell$.

Definition 3.7. The fiber square (3.2) is said to be *exact* if the canonical natural transformation $\phi^* f_* \Rightarrow f_{V*}\phi^*_{\mathcal{X}}$ of (derived) functors from $\mathbf{D}_{\mathrm{qcoh}}(\mathcal{O}_{\mathcal{X}})$ to $\mathbf{D}_{\mathrm{qcoh}}(\mathcal{O}_V)$ is an isomorphism. We also say that $\phi\colon V \to U$ is *faithful* with respect to $f\colon \mathcal{X} \to U$ if (3.2) is exact.

Lemma 3.8 ([62, Tag 08IB])**.** *If f is flat, then any quasicompact and quasiseparated morphism $\phi\colon V \to U$ is faithful with respect to f.*

The following is a generalisation of [43, Lemma 2.7].

Lemma 3.9. *Suppose that U is quasicompact and quasiseparated. Then a pair of U-linear subcategories $\mathcal{A}, \mathcal{B} \subseteq \operatorname{Perf}\mathcal{X}$ is semiorthogonal if and only if $\mathbf{R}f_*\mathbf{R}\mathcal{H}om_{\mathcal{X}}(\mathcal{B}, \mathcal{A}) = 0$.*

Proof. From [19, Theorem 3.1.1] it follows that $\mathbf{D}_{\mathrm{qcoh}}(\mathcal{O}_U)$ is generated by a perfect complex. Then the proof of [43, Lemma 2.7] works verbatim. □

The following result is the generalisation of Kuznetsov's base change theorem (Theorem 3.5) that we need in this paper.

Proposition 3.10. *Let U be a quasiseparated scheme. Let $f\colon \mathcal{X} \to U$ be a smooth and proper morphism between schemes with an f-ample invertible sheaf, and $\phi\colon V \to U$ be a morphism from an affine scheme. Let*

$$\operatorname{Perf}\mathcal{X} = \left\langle \mathcal{A}^1, \ldots, \mathcal{A}^\ell \right\rangle \tag{3.4}$$

be a U-linear semiorthogonal decomposition. Let

$$\mathcal{A}^i_\phi \subseteq \operatorname{Perf}\mathcal{X}_V \tag{3.5}$$

be the smallest strictly full triangulated subcategory closed under taking direct summands, containing all objects of the form $f_V^*F \otimes \phi_{\mathcal{X}}^*E$, where $E \in \mathcal{A}^i$ and $F \in \operatorname{Perf} V$. Then

$$\operatorname{Perf}\mathcal{X}_V = \left\langle \mathcal{A}^1_\phi, \ldots, \mathcal{A}^\ell_\phi \right\rangle \tag{3.6}$$

is the unique base change of (3.4).

Note that ϕ is quasicompact and quasiseparated. Hence by Lemma 3.8, the morphism ϕ is automatically faithful with respect to f because by assumption f is smooth, in particular flat.

Proof. To show the uniqueness of the base change, suppose that $\operatorname{Perf}\mathcal{X}_V = \langle \mathcal{B}^1, \ldots, \mathcal{B}^\ell \rangle$ is another base change of (3.4). Then each $\mathcal{B}^i$ is closed under $\otimes f_V^*(\operatorname{Perf} V)$ and contains $\phi_{\mathcal{X}}^*\mathcal{A}^i$, so that $\mathcal{A}^i_\phi \subseteq \mathcal{B}^i$. Taking the semiorthogonality into account, and by the strict fullness condition, this immediately implies $\mathcal{A}^i_\phi = \mathcal{B}^i$.

To show that (3.6) is a base change in the sense of Definition 3.6, we point out how the original proof of [43, Proposition 5.1] is justified in the greater generality we are working in. As in the first part of the proof of [43, Proposition 5.1], the required semiorthogonality of the subcategories $\mathcal{A}^i_\phi, \mathcal{A}^j_\phi$ follows from Lemmas 3.8 and 3.9. To see this, note that since V is affine and f is proper, the scheme $\mathcal{X}_V$ is quasicompact and separated.

The assertion that the subcategories $\mathcal{A}^1_\phi, \ldots, \mathcal{A}^\ell_\phi$ generate the whole $\operatorname{Perf}\mathcal{X}_V$ follows, as explained in the latter half of the proof, from the following lemma. □

The following is a generalisation of [43, Lemma 5.2].

Lemma 3.11. *Under the assumptions of Proposition 3.10,* $\operatorname{Perf}\mathcal{X}_V$ *coincides with the minimal strictly full triangulated subcategory of* $\mathbf{D}_{\mathrm{qcoh}}(\mathcal{O}_{\mathcal{X}_V})$ *closed under taking direct summands and containing the objects* $f_V^* \operatorname{Perf} V \otimes \phi_{\mathcal{X}}^* \operatorname{Perf}\mathcal{X}$.

Proof. Consider the fibre square

$$\begin{tikzcd} \mathcal{X}_V \arrow[r, "\phi_{\mathcal{X}}"] \arrow[d, "f_V"'] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "f"] \\ V \arrow[r, "\phi"'] & U \end{tikzcd}$$

and let $\mathcal{O}_f(1)$ be an f-ample invertible sheaf on X. By [62, Tag 0893], relative ampleness is stable under base change, so the invertible sheaf $\mathcal{O}_{\mathcal{X}_V}(1) := \phi_{\mathcal{X}}^*\mathcal{O}_f(1)$ on $\mathcal{X}_V$ is f_V-ample. However, since V is affine, it is ample as an invertible sheaf on $\mathcal{X}_V$. By [62, Tag 01Q3], for each quasicoherent sheaf of finite type F on $\mathcal{X}_V$ there exist integers $n \geq 0$ and $k > 0$ and an epimorphism $\mathcal{O}_{\mathcal{X}_V}(-n)^{\oplus k} \to F$. This fact implies the analogue of the first part of the proof of [43, Lemma 5.2], i.e. we have shown that a given perfect complex on $\mathcal{X}_V$ admits a quasi-isomorphism from another perfect complex $P^\bullet$ such that for each $i \in \mathbb{Z}$ there are locally free sheaves F_i on $\mathcal{X}$ and G_i on V such that $P^i \simeq \phi_{\mathcal{X}}^* F_i \otimes f_V^* G_i$. In fact, one can even take $G_i = \mathcal{O}_V$ and $F_i = \mathcal{O}_f(-n)^{\oplus k}$ for some k, n chosen as above.

Next, by using the quasicompactness and the separatedness of $\mathcal{X}_V$ again, take a cover by finitely many affine schemes $\mathcal{X}_V = \bigcup_{j=1}^{D} W_j$ such that the intersections $W_{ij} := W_i \cap W_j$ are all affine. Then for any $F \in \operatorname{Qcoh} \mathcal{X}_V$ one has

$$\mathrm{H}^i(\mathcal{X}_V, F) = 0 \tag{3.7}$$

if $i < 0$ or $i > D$. In fact, this immediately follows if we compute it as a Čech cohomology with respect to the chosen covering.

Then the rest of the proof of [43, Lemma 5.2] works verbatim, once $\dim X$ in the proof is replaced by D. □

Lemma 3.12. *Suppose that $X = U \cup V$ is an open cover of a scheme X and fix two perfect complexes $\mathcal{E}, \mathcal{F} \in \operatorname{Perf} X$. Then there is a distinguished triangle*

$$\begin{aligned} \mathbf{R}\mathrm{Hom}_X(\mathcal{E},\mathcal{F}) \to \mathbf{R}\mathrm{Hom}_U(\mathcal{E}|_U,\mathcal{F}|_U) \oplus \mathbf{R}\mathrm{Hom}_V(\mathcal{E}|_V,\mathcal{F}|_V) \\ \to \mathbf{R}\mathrm{Hom}_{U\cap V}(\mathcal{E}|_{U\cap V},\mathcal{F}|_{U\cap V}) \xrightarrow{+1} \mathbf{R}\mathrm{Hom}_X(\mathcal{E},\mathcal{F})[1]. \end{aligned}$$

Proof. Let $j_U, j_V, j_{U\cap V}$ be the open immersions from $U, V, U\cap V$ to X. Then there exists a distinguished triangle

$$\mathcal{O}_X \to \mathbf{R}j_{U,*}\mathcal{O}_U \oplus \mathbf{R}j_{V,*}\mathcal{O}_V \to \mathbf{R}j_{U\cap V,*}\mathcal{O}_{U\cap V} \xrightarrow{+1} \mathcal{O}_X[1]. \tag{3.8}$$

Applying the derived functors $- \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathbf{R}\mathscr{H}\mathrm{om}_X(\mathcal{E},\mathcal{F})$ and $\mathbf{R}\Gamma(X,-)$, we obtain the conclusion. □

For later purposes, we extend the construction of the base change of semiorthogonal decompositions to more general schemes. We first consider a gluing lemma for semiorthogonal decompositions.

Lemma 3.13. *Keep the assumptions of Proposition 3.10 on the morphism $f\colon \mathcal{X} \to U$, and assume that U is quasicompact and semiseparated. Let $U = \bigcup_{i\in I} U_i$ be a finite affine Zariski open covering and suppose that for each $i \in I$ there exists a U_i-linear semiorthogonal decomposition*

$$\operatorname{Perf} \mathcal{X}_i = \langle \mathcal{A}_i, \mathcal{B}_i \rangle, \tag{3.9}$$

where $\mathcal{X}_i = f^{-1}(U_i)$, whose base changes to the intersection $U_{ij} := U_i \cap U_j$ (as in Proposition 3.10) coincide. Then there exists a unique U-linear semiorthogonal decomposition

$$\operatorname{Perf} \mathcal{X} = \langle \mathcal{A}, \mathcal{B} \rangle \tag{3.10}$$

whose base change by $U_i \hookrightarrow U$ (as in Proposition 3.10) coincides with (3.9). Moreover, the similar assertion holds for semiorthogonal decompositions of length $\ell > 2$ as well.

Note that since U is assumed to be semiseparated, the intersections U_{ij} are affine schemes.

Proof. Let us define the subcategory

$$\mathcal{A} := \left\{ E \in \operatorname{Perf} \mathcal{X} \mid E|_{\mathcal{X}_i} \in \mathcal{A}_i \text{ for all } i \in I \right\} \subseteq \operatorname{Perf} \mathcal{X}. \tag{3.11}$$

We also define $\mathcal{B}$ in the same way. We will show that $(\mathcal{A}, \mathcal{B})$ is the unique pair of subcategories having the desired properties, by an induction on the number $N = \# I < \infty$.

When $N = 1$, we have nothing to show. Suppose $I = \{1,2\}$, so that $N = 2$. The semiorthogonality $\mathcal{A} \subseteq \mathcal{B}^{\perp}$ immediately follows from Lemma 3.12. We next show that $\operatorname{Perf}\mathcal{X}$ is generated by $\mathcal{A}$ and $\mathcal{B}$. Take an arbitrary object $E \in \operatorname{Perf}\mathcal{X}$. For each $i = 1,2$, consider the distinguished triangle

$$b_i \xrightarrow{s_i} E|_{\mathcal{X}_i} \xrightarrow{t_i} a_i \xrightarrow{u_i} b_i[1] \tag{3.12}$$

induced uniquely by the semiorthogonal decomposition (3.9), which is unique up to unique isomorphism by Lemma 2.22. Again by Lemma 2.22, the assumption implies that there exists a unique pair of isomorphisms

$$\beta_{ij} \colon b_j|_{\mathcal{X}_{ij}} \xrightarrow{\sim} b_i|_{\mathcal{X}_{ij}}, \quad \alpha_{ij} \colon a_j|_{\mathcal{X}_{ij}} \xrightarrow{\sim} a_i|_{\mathcal{X}_{ij}}$$

both of which automatically satisfy the cocycle conditions with respect to the covering $U = U_1 \cup U_2$ and are compatible with the morphisms s_i, t_i, u_i in the sense that below is a(n iso)morphism of triangles.

$$\begin{tikzcd}
b_i|_{U_{12}} \arrow[r, "s_i|_{U_{12}}"] \arrow[d, "\beta_{ji}"] & E|_{U_{12}} \arrow[r, "t_i|_{U_{12}}"] \arrow[d, equal] & a_i|_{U_{12}} \arrow[r, "u_i|_{U_{12}}"] \arrow[d, "\alpha_{ji}"] & b_i|_{U_{12}}[1] \arrow[d, "\beta_{ji}[1]"] \\
b_j|_{U_{12}} \arrow[r, "s_j|_{U_{12}}"] & E|_{U_{12}} \arrow[r, "t_j|_{U_{12}}"] & a_j|_{U_{12}} \arrow[r, "u_j|_{U_{12}}"] & b_j|_{U_{12}}[1]
\end{tikzcd}$$

Then by [62, Tag 08DG] we obtain objects $b, a \in \operatorname{Perf}\mathcal{X}$ together with isomorphisms $\alpha_i \colon a|_{\mathcal{X}_i} \xrightarrow{\sim} a_i$ and $\beta_i \colon b|_{\mathcal{X}_i} \xrightarrow{\sim} b_i$, which are compatible with the isomorphisms α_{ij}, β_{ij} in the sense that on the intersection U_{ij} one has $\alpha_{ji} \circ \alpha_i = \alpha_j$ and $\beta_{ji} \circ \beta_i = \beta_j$. Note that, by definition, $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Set $s_i' := s_i \circ \alpha_i$ and $t_i' := \beta_i^{-1} \circ t_i$. The compatibility we just mentioned implies that $s_1'|_{U_{12}} - s_2'|_{U_{12}} = 0$ and $t_1'|_{U_{12}} - t_2'|_{U_{12}} = 0$. On the other hand we have the exact sequences

$$\operatorname{Hom}_{\mathcal{X}}(b, E) \to \operatorname{Hom}_{\mathcal{X}_1}\left(b|_{U_1}, E|_{U_1}\right) \oplus \operatorname{Hom}_{\mathcal{X}_2}\left(b|_{U_2}, E|_{U_2}\right) \to \operatorname{Hom}_{\mathcal{X}_{12}}\left(b|_{U_{12}}, E|_{U_{12}}\right),$$
$$\operatorname{Hom}_{\mathcal{X}}(E, a) \to \operatorname{Hom}_{\mathcal{X}_1}\left(E|_{U_1}, a|_{U_1}\right) \oplus \operatorname{Hom}_{\mathcal{X}_2}\left(E|_{U_2}, a|_{U_2}\right) \to \operatorname{Hom}_{\mathcal{X}_{12}}\left(E|_{U_{12}}, a|_{U_{12}}\right),$$

which are obtained as part of the long exact sequences associated to the distinguished triangles in Lemma 3.12. Hence there exist morphisms $s \colon b \to E, t \colon E \to a$ such that $s|_{U_i} = s_i'$ and $t|_{U_i} = t_i'$. Thus we obtain a sequence

$$b \xrightarrow{s} E \xrightarrow{t} a. \tag{3.13}$$

At this point, we can assume without loss of generality that

$$b_i = b|_{U_i}, \ a_i = a|_{U_i}, \ s_i = s|_{U_i}, \ t_i = t|_{U_i},$$
$$\alpha_i = \mathrm{id}_{a|_{U_i}}, \ \beta_i = \mathrm{id}_{b|_{U_i}}, \ \beta_{ji} = \mathrm{id}_{b|_{U_{12}}}, \ \alpha_{ji} = \mathrm{id}_{a|_{U_{12}}},$$

and

$$u_i|_{U_{12}} = u_j|_{U_{12}} \colon a|_{U_{12}} \to b|_{U_{12}}.$$

By arguments similar to those for the constructions of s and t, one obtains a morphism $u \colon a \to b[1]$ such that $u|_{U_i} = u_i$. Thus we have obtained a sequence

$$b \xrightarrow{s} E \xrightarrow{t} a \xrightarrow{u} b[1] \tag{3.14}$$

whose restriction to U_i is a distinguished triangle for all $i = 1,2$. Now consider the following distinguished triangle.

$$b \xrightarrow{s} E \xrightarrow{t'} a' \coloneqq \mathrm{Cone}(s) \xrightarrow{u'} b[1] \tag{3.15}$$

Since the restriction of the sequence (3.14) to U_i is distinguished, by an axiom of triangulated categories, one obtains an isomorphism $\gamma_i\colon a'|_{U_i} \xrightarrow{\sim} a|_{U_i}$ which yields an isomorphism of distinguished triangles between the restrictions of (3.14) and (3.15) to U_i. This immediately implies that $\mathrm{Cone}(s) \in \mathcal{A}$, which means that the object E is contained in the subcategory of $\mathrm{Perf}\,\mathcal{X}$ generated by $\mathcal{A}$ and $\mathcal{B}$. Thus we have completedthe proof for the case $N = 2$.

Now consider the general case when $N \geq 3$ and suppose that the assertion is true up to $N-1$. Let $I = \{1, 2, \ldots, N\}$ and set $U' \coloneqq \bigcup_{i=1}^{N-1} U_i$. Applying the induction hypothesis to this covering, we can glue the semiorthogonal decompositions on $U_1, \ldots, U_{N-1}$ uniquely to produce a U'-linear semiorthogonal decomposition

$$\mathrm{Perf}\,\mathcal{X}_{U'} = \langle \mathcal{A}', \mathcal{B}' \rangle \tag{3.16}$$

By using the case $\# I = 2$, we can further uniquely glue (3.16) with the one on U_N to obtain the desired U-linear semiorthogonal decomposition.

The case of a semiorthogonal decomposition of arbitrary length can be proven by induction. Consider U_i-linear semiorthogonal decompositions

$$\mathrm{Perf}\,\mathcal{X}_i = \langle \mathcal{A}_i^1, \ldots, \mathcal{A}_i^{\ell+1} \rangle,$$

of length $\ell + 1$ which coincide over U_{ij}. Then we can reduce it to the statement for semiorthogonal decompositions of length ℓ by considering the semiorthogonal decompositions

$$\begin{aligned} \mathrm{Perf}\,\mathcal{X}_i &= \langle \mathcal{A}_i^1, \langle \mathcal{A}_i^2, \mathcal{A}_i^3 \rangle, \ldots, \mathcal{A}_i^{\ell+1} \rangle \\ &= \langle \langle \mathcal{A}_i^1, \mathcal{A}_i^2 \rangle, \mathcal{A}_i^3, \ldots, \mathcal{A}_i^{\ell+1} \rangle \end{aligned}$$

and glue these two semiorthogonal decompositions to semiorthogonal decompositions of $\mathrm{Perf}\,\mathcal{X}$. It then suffices to consider the intersection of the first components in the two gluings to find the required semiorthogonal decomposition of length $\ell + 1$. □

Corollary 3.14. *Keep the assumptions of Proposition 3.10 on the morphism $f\colon \mathcal{X} \to U$. For any morphism $\phi\colon V \to U$ from a quasicompact and semiseparated scheme V, and for any U-linear semiorthogonal decomposition*

$$\mathrm{Perf}\,\mathcal{X} = \left\langle \mathcal{A}^1, \ldots, \mathcal{A}^\ell \right\rangle, \tag{3.17}$$

there exists a unique base change

$$\mathrm{Perf}\,\mathcal{X}_V = \left\langle \mathcal{A}_\phi^1, \ldots, \mathcal{A}_\phi^\ell \right\rangle$$

of (3.17) by ϕ.

Proof. Note that V admits a finite Zariski affine open covering with affine intersections. Over each affine open subset of V one has the base change of (3.17) by Proposition 3.10, and they coincide on the intersections by the uniqueness part of Proposition 3.10. Hence by Lemma 3.13, they uniquely glue together to produce a V-linear semiorthogonal decomposition

$$\operatorname{Perf}\mathcal{X}_V = \left\langle \mathcal{A}^1_\phi, \dots, \mathcal{A}^\ell_\phi \right\rangle,$$

which is a base change of (3.17) by construction. Now let $\operatorname{Perf}\mathcal{X}_V = \langle \mathcal{B}^1_\phi, \dots, \mathcal{B}^\ell_\phi \rangle$ be another base change of (3.17). Then again by the uniqueness of Proposition 3.10 it follows that $\mathcal{A}^i_\phi \subseteq \mathcal{B}^i_\phi$ for all i, which immediately implies the equality for all i. Thus we obtain the uniqueness of the base change. □

Finally, we establish that whether or not two linear semiorthogonal decompositions coincide can be checked on geometric points.

Lemma 3.15. *Keep the assumptions of Proposition 3.10 on the morphism $f : \mathcal{X} \to U$. Suppose that $\operatorname{Perf}\mathcal{X} = \langle \mathcal{A}_1, \mathcal{B}_1 \rangle = \langle \mathcal{A}_2, \mathcal{B}_2 \rangle$ are U-linear semiorthogonal decompositions whose base change to all geometric points of U coincide. Then they coincide.*

Proof. Take an arbitrary object $E \in \mathcal{A}_1$, and consider the distinguished triangle $b \to E \to a \to b[1]$ with respect to the second semiorthogonal decomposition. The assumption implies that for each geometric point $u\colon \operatorname{Spec} K \to U$ from an algebraically closed field K, one has that $b|_{\mathcal{X}_u} = 0$. This implies for any point $x \in \mathcal{X}$ that $\mathbf{L}\iota_x^* b = 0$, where $\iota_x\colon \operatorname{Spec}\mathbf{k}(x) \to \mathcal{X}$ is the canonical morphism. Since b is perfect, Lemma 2.3 implies that $b = 0$. Thus we obtain $E \simeq a \in \mathcal{A}_2$, so that $\mathcal{A}_1 \subseteq \mathcal{A}_2$. Exchanging the roles of $\mathcal{A}_1$ and $\mathcal{A}_2$, we obtain the other inclusion. □

4. The functor SOD_f of semiorthogonal decompositions

4.1. **Definition of the functor.** Thanks to the results of Section 3.2 (and Proposition 3.10 in particular), we can now define a functor of semiorthogonal decompositions.

Throughout this section, we fix a smooth and proper morphism of quasiseparated schemes $f : \mathcal{X} \to U$ carrying an f-ample invertible sheaf $\mathcal{O}_f(1)$ (cf. Definition 2.4 (3)). We also assume U is defined over $\mathbb{Q}$, which is needed only in a single step of the proof of Theorem 4.2. The main result of [1] shows that such an assumption is not necessary, and that it can be avoided by using derived algebro-geometric techniques. We come back to this in Remark 4.5.

Definition 4.1. Given $f : \mathcal{X} \to U$ as above, we define a functor

$$\mathsf{SOD}_f\colon \mathrm{Aff}_U^{\mathrm{op}} \to \mathrm{Sets}$$

by sending an affine U-scheme $\phi : V \to U$ to the following set:

$$\mathsf{SOD}_f(\phi) = \left\{ (\mathcal{A}_\phi, \mathcal{B}_\phi) \;\middle|\; \begin{array}{c} \operatorname{Perf}\mathcal{X}_V = \langle \mathcal{A}_\phi, \mathcal{B}_\phi \rangle \text{ is a } V\text{-linear} \\ \text{semiorthogonal decomposition} \end{array} \right\}.$$

The pullback maps are given in Proposition 3.10.

Here we use tacitly that all our categories are essentially small, so that one does not run into set-theoretic issues. Alternatively, and this is the approach taken in Appendix A, one can use universes, see Remark A.6.

4.2. **The sheaf property.** In this subsection we prove that the functor defined in the previous section is an étale sheaf.

Theorem 4.2. *The functor*

$$\mathsf{SOD}_f\colon \mathrm{Aff}_U^{\mathrm{op}} \to \mathrm{Sets} \tag{4.1}$$

is a sheaf on the big affine étale site $(\mathrm{Aff}_U)_{\text{Ét}}$.

Proof. Let $u\colon U' \to U$ be an arbitrary affine U-scheme, and let $(V_i \to U')_{1\le i\le N}$ be an arbitrary standard étale covering. Let $\phi\colon V = \coprod_{i=1}^N V_i \to U'$ be the induced map. Consider the cartesian diagrams

$$\begin{tikzcd} V\times_{U'} V \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\square"] & V \arrow[d, "\phi"] \\ V \arrow[r, "\phi"'] & U' \end{tikzcd} \qquad \begin{tikzcd} \mathcal{X}_{U'} \arrow[r] \arrow[d, "f_{U'}"'] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "f"] \\ U' \arrow[r, "u"'] & U \end{tikzcd}$$

and let $\phi^2 \coloneqq \phi\times_{U'}\phi$ denote the canonical map $V\times_{U'} V \to U'$. We need to show that the sequence

$$\mathsf{SOD}_{f_{U'}}(\mathrm{id}_{U'}) \to \mathsf{SOD}_{f_{U'}}(\phi) \rightrightarrows \mathsf{SOD}_{f_{U'}}(\phi^2) \tag{4.2}$$

is an equaliser in the category Sets, where the arrows from the middle term to the last term denote the base changes by p_1 and p_2, respectively.

In the rest of the proof, to ease notation, we will let U and f denote the base scheme U' and the morphism $f_{U'}$, respectively. In particular, we are in a situation where U is affine. With this convention, what we need to show is that the sequence

$$\mathsf{SOD}_f(\mathrm{id}_U) \to \mathsf{SOD}_f(\phi) \rightrightarrows \mathsf{SOD}_f(\phi^2) \tag{4.3}$$

is an equaliser. Note that the second and the third term have canonical bijections

$$\mathsf{SOD}_f(\phi) \simeq \prod_{i=1}^N \mathsf{SOD}_f(V_i \to U)$$
$$\mathsf{SOD}_f(\phi^2) \simeq \prod_{i,j=1}^N \mathsf{SOD}_f(V_i\times_U V_j \to U),$$

since the construction of SOD_f takes disjoint unions of affine U-schemes to products in Sets.

Let us show that the first map of (4.3) is injective. Namely, let us show that if

$$\mathrm{Perf}\,\mathcal{X} = \langle \mathcal{A}_1, \mathcal{B}_1\rangle,$$
$$\mathrm{Perf}\,\mathcal{X} = \langle \mathcal{A}_2, \mathcal{B}_2\rangle,$$

are U-linear semiorthogonal decompositions of $\mathrm{Perf}\,\mathcal{X}$ such that

$$\phi_{\mathcal{X}}^*\mathcal{A}_1 = \phi_{\mathcal{X}}^*\mathcal{A}_2 \subseteq \mathrm{Perf}\,\mathcal{X}_V,$$

then $\mathcal{A}_1 = \mathcal{A}_2 \subseteq \mathrm{Perf}\,\mathcal{X}$.

Take any object $E\in\mathcal{A}_1$, and let $b\to E\to a \xrightarrow{+1} b[1]$ be the distinguished triangle such that $a\in\mathcal{A}_2$, $b\in\mathcal{B}_2$. Consider its pullback by $\phi_{\mathcal{X}}$, namely

$$\phi_{\mathcal{X}}^* b \to \phi_{\mathcal{X}}^* E \to \phi_{\mathcal{X}}^* a \xrightarrow{+1} \phi_{\mathcal{X}}^* b[1].$$

The assumption implies that $\phi_{\mathcal{X}}^* E, \phi_{\mathcal{X}}^* a \in \phi_{\mathcal{X}}^* \mathcal{A}_1 = \phi_{\mathcal{X}}^* \mathcal{A}_2$, so that $\phi_{\mathcal{X}}^* b = 0$. Since ϕ is faithfully flat, this implies that $b = 0$ and hence $E \in \mathcal{A}_2$, meaning $\mathcal{A}_1 \subseteq \mathcal{A}_2$. Exchanging the roles of 1 and 2, we obtain the equality $\mathcal{A}_1 = \mathcal{A}_2$.

It remains to show the following assertion: Fix a V-linear semiorthogonal decomposition

$$\operatorname{Perf} \mathcal{X}_V = \langle \mathcal{A}_V, \mathcal{B}_V \rangle \tag{4.4}$$

which satisfies the descent condition in the sense that $p_2^* \mathcal{A}_V = p_1^* \mathcal{A}_V \subseteq \operatorname{Perf} \mathcal{X}_{V \times_U V}$. Then there exists a U-linear semiorthogonal decomposition $\operatorname{Perf} \mathcal{X} = \langle \mathcal{A}, \mathcal{B} \rangle$ whose pullback by ϕ coincides with (4.4).

To start with, we may and will refine ϕ so that each V_i is connected. Fix i, and write $W = V_i$, $\mathcal{U} := \phi(W) \subseteq U$, and $\phi = \phi|_W : W \to \mathcal{U}$ to ease notation. Let

$$\operatorname{Perf} \mathcal{X}_W = \langle \mathcal{A}_W, \mathcal{B}_W \rangle \tag{4.5}$$

be the base change of (4.4) by $W \hookrightarrow V$.

Since $W \to U$ is an étale morphism between affine schemes, it (and also $W \to \mathcal{U}$) is quasi-finite and separated. Since $\mathcal{U}$ is quasicompact, by a version of the Zariski main theorem [53, Theorem 1.8], there is a factorisation

$$W \hookrightarrow \mathcal{W} \to \mathcal{U},$$

where the first morphism is an open immersion and the second one is a finite surjective morphism. Note that the surjectivity of the second map follows from that of the map $W \to \mathcal{U}$. Take the Galois closure $\widetilde{\mathcal{W}} \to \mathcal{U}$ of $\mathcal{W} \to \mathcal{U}$, whose Galois group will be denoted by G. Since $\mathcal{U}$ is a connected scheme, the existence of the Galois closure follows from the standard theory of Galois categories (see, say, [22, Theorem 5.10 and Proposition 3.3]). Let $\widetilde{W} \subseteq \widetilde{\mathcal{W}}$ be the inverse image of W by the morphism $\widetilde{\mathcal{W}} \to \mathcal{W}$, and let

$$\operatorname{Perf} \mathcal{X}_{\widetilde{W}} = \langle \mathcal{A}_{\widetilde{W}}, \mathcal{B}_{\widetilde{W}} \rangle \tag{4.6}$$

be the base change of (4.5) by $\widetilde{W} \to W$, which inherits the descent condition. Note that we have the finite affine Zariski open covering

$$\widetilde{\mathcal{W}} = \bigcup_{g \in G} g \widetilde{W}.$$

We let $\widetilde{\phi} : \widetilde{W} \to \mathcal{U}$ denote the composition of $\phi : W \to \mathcal{U}$ with $\widetilde{W} \to W$. The situation is summarised in the following commutative diagram.

$$\begin{tikzcd}
\widetilde{W} \arrow[r, hook] \arrow[d, two heads] \arrow[ddr, bend right=60, two heads, "\widetilde{\phi}"'] & \widetilde{\mathcal{W}} \arrow[d, two heads] & \\
W \arrow[r, hook] \arrow[dr, two heads] & \mathcal{W} \arrow[d, two heads] & \\
& \mathcal{U} \arrow[r, hook] & U
\end{tikzcd}$$

Now, for each $g \in G$ we have the $g\widetilde{W}$-linear semiorthogonal decomposition on $\mathcal{X}_{g\widetilde{W}}$ which is the base change of (4.6) by the isomorphism $g^{-1} : g\widetilde{W} \xrightarrow{\sim} \widetilde{W}$.

We pause for a second to prove the following auxiliary lemma.

Lemma 4.3. *Let $u\colon \operatorname{Spec} K \to \mathcal{U}$ be a geometric point from an algebraically closed field K, and let $w_1, w_2\colon \operatorname{Spec} K \to \widetilde{W}$ be lifts of u. Then the base change of* (4.5) *by w_1 and w_2, naturally regarded as semiorthogonal decompositions of $\operatorname{Perf}\mathcal{X}_u$, coincide.*

Proof of Lemma 4.3. The assumption $\widetilde{\phi} \circ w_1 = u = \widetilde{\phi} \circ w_2$ implies that there exists a (unique) morphism $\theta\colon \operatorname{Spec} K \to \widetilde{W} \times_{\mathcal{U}} \widetilde{W}$ such that $p_1 \circ \theta = w_1, p_2 \circ \theta = w_2$. Thus we see $w_1^* \mathcal{A}_{\widetilde{W}} = \theta^* p_1^* \mathcal{A}_{\widetilde{W}} = \theta^* p_2^* \mathcal{A}_{\widetilde{W}} = w_2^* \mathcal{A}_{\widetilde{W}}$. □

It follows from Lemma 4.3 that the semiorthogonal decompositions on $g\widetilde{W}$ and $g'\widetilde{W}$ we have constructed coincide at any geometric point of the intersection $g\widetilde{W} \cap g'\widetilde{W}$, so that base changes to the intersection coincide by Lemma 3.15. Hence they glue together uniquely to a semiorthogonal decomposition on $\widetilde{\mathcal{W}}$ by Lemma 3.13. Note that $\widetilde{W}$ (and hence also $g\widetilde{W}$) is affine, since it is finite over the affine scheme W. Moreover thus obtained semiorthogonal decomposition is invariant under the action of $G \curvearrowright \widetilde{\mathcal{W}}$ in the sense of [23, Theorem 8], again by Lemma 4.3 and Lemma 3.15.

Now since U is defined over a field of characteristic zero,[1] it follows that G is a linearly reductive group. Therefore, thus obtained semiorthogonal decomposition on $\widetilde{\mathcal{W}}$ descends uniquely to a semiorthogonal decomposition on $\mathcal{U}$ by the Elagin's descent [23, Theorem 8] (note that $\widetilde{\mathcal{W}} \to \mathcal{U}$ is étale, so that the action $G \curvearrowright \widetilde{\mathcal{W}}$ is free and hence we have isomorphisms $[\widetilde{\mathcal{W}}/G] \simeq \widetilde{\mathcal{W}}/G \simeq \mathcal{U}$).

Summing up, for each $i = 1, 2, \ldots, N$ we have obtained a semiorthogonal decomposition on the quasicompact open subset $\phi(V_i)(=\mathcal{U}) \subseteq U$. By the similar arguments as in the proof of Lemma 4.3, the descent condition on (3.6) (more precisely, the condition on $V_i \times_U V_j$) implies that at any geometric point of $\phi(V_i) \cap \phi(V_j)$, the base changes of the semiorthogonal decompositions on $\phi(V_i)$ and $\phi(V_j)$ coincide. Hence again by Lemma 3.15, their base changes to the intersection $\phi(V_i) \cap \phi(V_j)$ also coincide. Therefore they glue together uniquely to a U-linear semiorthogonal decomposition of $\operatorname{Perf}\mathcal{X}$ by Corollary 3.14. Once again by Lemma 3.15, one can check this is the descent of the V-linear semiorthogonal decomposition (3.6). □

Definition 4.4. By Lemma 2.25, the sheaf SOD_f on the big affine étale site $(\mathrm{Aff}_U)_{\text{Ét}}$, which we obtained in Theorem 4.2, uniquely defines a sheaf on the big étale site $(\mathrm{Sch}_U)_{\text{Ét}}$. It will be also denoted by SOD_f by abuse of notation.

Remark 4.5. The main result of [1] states that semiorthogonal decompositions satisfy fppf descent. We can obtain Theorem 4.2 as a special case of their main result.

Whilst it is not featured explicitly in the main body of their text, the étale sheaf SOD_f (in our notation) corresponds to what would be the fppf stack $\mathbf{Sod}_{[1]}^{\mathcal{P}\mathrm{erf}(X)}$ (in their notation). In op. cit. they consider descent for arbitrary poset-shaped semiorthogonal decompositions, and our definition is a special instance of theirs when

- the poset P is the totally ordered set $[1]$ on 2 elements, see [1, Example 2.7(i)];
- one takes the fibre of $\mathbf{Sod}_{[1]}$ at $\mathcal{C} = \mathcal{P}\mathrm{erf}(X)$, as in [1, Theorem 2.17].

[1]This is the only place where the existence of characteristic zero assumption is used. It can be avoided by appealing to the main result of [1], as explained in Remark 4.5.

The decoration $\mathcal{P}\mathrm{erf}(X)$ is not defined as such in op. cit., but it corresponds to taking the fibre of the "universal stack $\mathbf{Sod}_{[1]}$ of semiorthogonal decompositions with two components" at the point $\mathcal{C} = \mathcal{P}\mathrm{erf}(X)$, similar to what happens for the fppf stack of filtrations.

Finally, in [1] the formalism of derived algebraic geometry is used, and in particular the categories of perfect complexes are considered as enhanced categories. By [8, Theorem 1.2(2)] and the restriction to perfect complexes for a smooth and proper morphism $\mathcal{X} \to S$ as discussed in [9], we obtain that this agrees with the approach using Fourier–Mukai transforms taken in this paper.

We can conclude that the restriction to characteristic zero is therefore only a remnant of working with unenhanced triangulated categories, and our appeal to Elagin's descent results, but that it is not essential.

In what follows we will further amplify Theorem 4.2 in our specific setting, to show that SOD_f is an algebraic space which is étale over the base. As explained in Example 8.10 one cannot expect such a result in complete generality, despite semiorthogonal decompositions satisfying fppf descent.

4.3. **Values of the sheaf SOD_f at quasicompact and semiseparated U-schemes.** We confirm in Proposition 4.7 that the value of the sheaf $\mathsf{SOD}_f \in \mathsf{Sh}(\mathrm{Sch}_U)_{\mathrm{Ét}}$ at a quasicompact and semiseparated U-scheme coincides with what we naively expect, by spelling out the construction of the equivalence between topoi Lemma 2.25. Since the underlying functor $\mathrm{Aff}_U \to \mathrm{Sch}_U$ is fully faithful, it turns out to be much simpler than otherwise. To distinguish, only in this section, we let $\mathsf{SOD}'_f\colon \mathrm{Aff}_U^{\mathrm{op}} \to \mathrm{Sets}$ denote the sheaf on the big *affine* étale site.

For a U-scheme $(g\colon X \to U) \in \mathrm{Sch}_U$, let $g_*\colon \mathrm{Sch}_X \to \mathrm{Sch}_U$ be the obvious functor $(a\colon V \to X) \mapsto (g \circ a\colon V \to U)$. Then the value $\mathsf{SOD}_f(X)$ is the limit set

$$\mathsf{SOD}_f(X) = \lim_{\mathrm{Aff}_X^{\mathrm{op}}} \mathsf{SOD}'_f \circ g_*.$$

Lemma 4.6. *There is a canonical isomorphism of functors*

$$\mathsf{SOD}'_f \xrightarrow{\sim} \mathsf{SOD}_f\big|_{\mathrm{Aff}_U^{\mathrm{op}}}\colon \mathrm{Aff}_U^{\mathrm{op}} \to \mathrm{Sets}.$$

Proof. For $(X \to U) \in \mathrm{Aff}_U$, we have an obvious map

$$\mathsf{SOD}'_f(X) \to \lim_{\mathrm{Aff}_X^{\mathrm{op}}} \mathcal{S} = \mathsf{SOD}_f(X) \tag{4.7}$$

which yields a map of presheaves $\mathsf{SOD}'_f \to \mathsf{SOD}_f\big|_{\mathrm{Aff}_U^{\mathrm{op}}}$. Since the identity morphism $(\mathrm{id}_X\colon X \to X)$ is a final object of the category Aff_X, one can easily verify that (4.7) is a bijection. □

Combining Lemma 4.6 with Lemma 3.13 and the sheaf property of SOD_f, we obtain the following conclusion by taking a finite affine Zariski open cover of V.

Proposition 4.7. *For any U-scheme $(V \to U) \in \mathrm{Sch}_U$ which is both quasicompact and semiseparated as a scheme, we have*

$$\mathsf{SOD}_f(V \to U) = \left\{ \begin{array}{c} V\text{-linear semiorthogonal} \\ \text{decompositions } \mathrm{Perf}\,\mathcal{X}_V = \langle \mathcal{A}, \mathcal{B} \rangle \end{array} \right\}.$$

5. THE FUNCTOR DEC_{Δ_f} OF DECOMPOSITIONS OF THE DIAGONAL

The aim of this section is to introduce an alternative description of the functor SOD_f, denoted by DEC_{Δ_f}, to which we can apply the deformation theory established in Appendix A.

We start by setting up a few notational conventions. As always, all functors mean their derived versions.

Notation 5.1. Fix a smooth proper morphism $f\colon \mathcal{X} \to U$ which admits an f-ample invertible sheaf $\mathcal{O}_f(1)$ on $\mathcal{X}$. Set $\mathcal{Y} = \mathcal{X} \times_U \mathcal{X}$ and suppose given an object

$$K \in \operatorname{Perf}\mathcal{Y}.$$

We denote by $\Phi_K\colon \operatorname{Perf}\mathcal{X} \to \operatorname{Perf}\mathcal{X}$ the corresponding *Fourier–Mukai functor*, defined by

$$\Phi_K(E) = \mathrm{pr}_{2*}(\mathrm{pr}_1^*(E) \otimes K). \tag{5.1}$$

Here, pr_i are the projections $\mathcal{X} \times_U \mathcal{X} \to \mathcal{X}$. Note that we are implicitly using the fact that pr_{2*} preserves perfectness. To see this, note that pr_2 is smooth and hence perfect. Since it is also proper, it follows that pr_2 is quasi-perfect (see [49, Example 2.2] and [49, page 213]). We denote by

$$\mathcal{E}_K \subseteq \operatorname{Perf}\mathcal{X}$$

the essential image of the Fourier–Mukai functor Φ_K defined in (5.1). Moreover, if $\phi\colon V \to U$ is an arbitrary morphism, we write $\mathcal{X}_V = \mathcal{X} \times_U V$ for the base change and $f_V\colon \mathcal{X}_V \to V$ for the induced map. Finally, we set $\mathcal{Y}_V = \mathcal{X}_V \times_V \mathcal{X}_V$ and we let $\mathcal{O}_{\Delta_{f_V}} \in \operatorname{Perf}\mathcal{Y}_V$ be the structure sheaf of the diagonal attached to f_V. This is a perfect complex because f_V is smooth and separated (cf. Lemma 2.6).

Definition 5.2. We define a category fibered in groupoids

$$\mathcal{DEC}_{\Delta_f} \to \mathrm{Aff}_U$$

as follows.

- An object of $\mathcal{DEC}_{\Delta_f}$ is a pair (ϕ, ζ) where $\phi\colon V \to U$ is an affine U-scheme and
$$\zeta\colon \quad K_{\mathcal{B}} \to \mathcal{O}_{\Delta_{f_V}} \to K_{\mathcal{A}} \xrightarrow{+1} K_{\mathcal{B}}[1] \tag{5.2}$$
is a distinguished triangle in $\operatorname{Perf}\mathcal{Y}_V$ such that
$$\mathbf{R}f_{V,*}\mathbf{R}\mathscr{H}om_{\mathcal{X}_V}(\mathcal{E}_{K_{\mathcal{B}}}, \mathcal{E}_{K_{\mathcal{A}}}) = 0.$$
- A morphism from (ϕ, ζ) as above to a pair (ϕ', ζ') given by a morphism $\phi'\colon V' \to U$ and a distinguished triangle
$$\zeta'\colon \quad K'_{\mathcal{B}} \to \mathcal{O}_{\Delta_{f_{V'}}} \to K'_{\mathcal{A}} \xrightarrow{+1} K'_{\mathcal{B}}[1]$$
in $\operatorname{Perf}\mathcal{Y}_{V'}$, is a pair consisting of a morphism of U-schemes $\psi\colon V \to V'$ together with an isomorphism $\zeta \xrightarrow{\sim} \psi^*\zeta'$, where $\psi^*\zeta'$ denotes the distinguished triangle
$$\begin{tikzcd} \psi^* K'_{\mathcal{B}} \arrow[r] & \psi^*\mathcal{O}_{\Delta_{f_{V'}}} \arrow[r] \arrow[d, equal] & \psi^* K'_{\mathcal{A}} \arrow[r, "+1"] & \psi^* K'_{\mathcal{B}}[1] \\ & \mathcal{O}_{\Delta_{f_V}} & & \end{tikzcd}$$
in $\operatorname{Perf}\mathcal{Y}_V$. Here we somewhat sloppily wrote ψ^* for the pullback along $\mathcal{Y}_V \to \mathcal{Y}_{V'}$.

In other words, a morphism $(\phi,\zeta)\to(\phi',\zeta')$ is an isomorphism of distinguished triangles

$$\begin{tikzcd} K_{\mathcal{B}} \arrow[r] \arrow[d, "\wr"] & \mathcal{O}_{\Delta_{f_V}} \arrow[r] \arrow[d, equal] & K_{\mathcal{A}} \arrow[r, "+1"] \arrow[d, "\wr"] & K_{\mathcal{B}}[1] \arrow[d, "\wr"] \\ \psi^*K'_{\mathcal{B}} \arrow[r] & \mathcal{O}_{\Delta_{f_V}} \arrow[r] & \psi^*K'_{\mathcal{A}} \arrow[r, "+1"] & \psi^*K'_{\mathcal{B}}[1] \end{tikzcd}$$

where we insist that the second vertical isomorphism be the identity of $\mathcal{O}_{\Delta_{f_V}}$.

Remark 5.3. The subscripts $\mathcal{B}$ and $\mathcal{A}$ in the definition of $\mathcal{DEC}_{\Delta_f}$ are just (for now) for cosmetic reasons. They will help during the proof of Theorem 5.9.

Definition 5.4. The associated functor (i.e., π_0 of $\mathcal{DEC}_{\Delta_f}$) will be denoted by

$$\mathsf{DEC}_{\Delta_f}\colon \mathrm{Aff}_U^{\mathrm{op}}\to \mathrm{Sets}.$$

We now verify, in the next lemma, that $\mathcal{DEC}_{\Delta_f}$ is a category fibred in *setoids*. Therefore, after all, $\mathcal{DEC}_{\Delta_f}$ is equivalent to the category fibered in groupoids associated to the functor DEC_{Δ_f}.

Lemma 5.5. *For each affine U-scheme $\phi\colon V\to U$, the fiber category $\left(\mathcal{DEC}_{\Delta_f}\right)_\phi$ is a setoid. Namely, let*

$$\zeta_i=\left(K_i\to\mathcal{O}_{\Delta_{f_V}}\to L_i\xrightarrow{+1}K_i[1]\right)\in\left(\mathcal{DEC}_{\Delta_f}\right)_\phi$$

for $i=1,2$ be decompositions of $\mathcal{O}_{\Delta_{f_V}}$. Then there exists at most one isomorphism between them.

Proof. Recall that an isomorphism $\vartheta\colon\zeta_1\xrightarrow{\sim}\zeta_2$ is a pair of isomorphisms (α,β) which yields the following isomorphism of distinguished triangles.

$$\begin{tikzcd} K_1 \arrow[r] \arrow[d, "\beta"] & \mathcal{O}_{\Delta_{f_V}} \arrow[r] \arrow[d, equal] & L_1 \arrow[r, "+1"] \arrow[d, "\alpha"] & K_1[1] \arrow[d, "\beta[1]"] \\ K_2 \arrow[r] & \mathcal{O}_{\Delta_{f_V}} \arrow[r] & L_2 \arrow[r, "+1"] & K_2[1] \end{tikzcd}$$

It then follows from Lemma 2.22 that the pair (α,β) is unique. □

Recall that we fixed a smooth and proper morphism $f\colon\mathcal{X}\to U$ with an f-ample invertible sheaf on $\mathcal{X}$. Fix an object $(\phi\colon V\to U)\in\mathrm{Aff}_U$ and consider the following fibre products.

$$\begin{tikzcd} \mathcal{Y}_V \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\square"] & \mathcal{X}_V \arrow[r] \arrow[d, "f_V"] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{X}_V \arrow[r, "f_V"'] & V \arrow[r, "\phi"'] & U \end{tikzcd}$$

Lemma 5.6. *Consider a decomposition $\zeta\in\mathsf{DEC}_{\Delta_f}(\phi)$ as in (5.2). Then*

$$(\mathscr{E}_{K_{\mathcal{A}}},\mathscr{E}_{K_{\mathcal{B}}})\in\mathsf{SOD}_f(\phi).$$

Proof. It follows from the definition of a decomposition that $\mathrm{Hom}_{\mathcal{X}_V}(\mathscr{E}_{K_{\mathcal{B}}}, \mathscr{E}_{K_{\mathcal{A}}}) = 0$. Also, for any $F \in \mathrm{Perf}\,\mathcal{X}_V$, by applying the Fourier–Mukai functors to the terms of the distinguished triangle ζ as in (5.2), we see that $\mathrm{Perf}\,\mathcal{X}_V$ is generated by $\mathscr{E}_{K_{\mathcal{A}}}, \mathscr{E}_{K_{\mathcal{B}}}$. Thus we see that $(\mathscr{E}_{K_{\mathcal{A}}}, \mathscr{E}_{K_{\mathcal{B}}})$ gives a semiorthogonal decomposition of $\mathrm{Perf}\,\mathcal{X}_V$, which is V-linear by the definition of the subcategory $\mathscr{E}_\bullet$ for an object $\bullet$ (Notation 5.1). ☐

Given a pair $(\mathcal{A}_\phi, \mathcal{B}_\phi) \in \mathsf{SOD}_f(\phi)$, by Lemma 2.16, there are adjoints

$$\mathcal{B}_\phi \underset{i^!}{\overset{i}{\rightleftarrows}} \mathrm{Perf}\,\mathcal{X}_V \underset{j^*}{\overset{j}{\leftrightarrows}} \mathcal{A}_\phi$$

and one has

$$j \circ j^* = \alpha_1, \quad i \circ i^! = \alpha_2, \tag{5.3}$$

where α_1 (resp. α_2) is the projection functor attached to $\mathcal{A}_\phi$ (resp. $\mathcal{B}_\phi$). On the other hand, using faithfulness of f_V, form the $\mathcal{X}_V$-linear pullback semiorthogonal decomposition

$$\mathrm{Perf}\,\mathcal{Y}_V = \left\langle \mathrm{pr}_2^* \mathcal{A}_\phi, \mathrm{pr}_2^* \mathcal{B}_\phi \right\rangle$$

via Corollary 3.14 (using that $\mathcal{X}_V$ is quasicompact and separated). According to this, decompose the structure sheaf $\mathcal{O}_{\Delta_{f_V}} \in \mathrm{Perf}\,\mathcal{Y}_V$ to obtain the distinguished triangle

$$K_{\mathcal{B}_\phi} \to \mathcal{O}_{\Delta_{f_V}} \to K_{\mathcal{A}_\phi} \xrightarrow{+1} K_{\mathcal{B}_\phi}[1]. \tag{5.4}$$

Lemma 5.7. *One has that*

$$(\mathscr{E}_{K_{\mathcal{A}_\phi}}, \mathscr{E}_{K_{\mathcal{B}_\phi}}) = (\mathcal{A}_\phi, \mathcal{B}_\phi) \in \mathsf{SOD}_f(\phi).$$

In particular, the distinguished triangle (5.4) *defines an element of* $\in \mathsf{DEC}_{\Delta_f}(\phi)$.

Proof. By the explicit description of the base change given in Proposition 3.10, one observes that $\Phi_{K_{\mathcal{A}_\phi}}(E) \in \mathcal{A}_\phi$ for any $E \in \mathrm{Perf}\,\mathcal{X}_V$. In the same way one also sees $\Phi_{K_{\mathcal{B}_\phi}}(E) \in \mathcal{B}_\phi$. This already shows the second assertion.

To obtain the first one, we need to show that $\mathcal{A}_\phi = \mathcal{B}_\phi^\perp \subseteq \mathscr{E}_{K_{\mathcal{A}_\phi}}$ and $\mathcal{B}_\phi = {}^\perp\mathcal{A}_\phi \subseteq \mathscr{E}_{K_{\mathcal{B}_\phi}}$. We show the first one, and the second one can be proved similarly.

Take any $F \in \mathcal{B}_\phi^\perp$, and take the distinguished triangle

$$B \to F \to A \xrightarrow{+1} B[1] \tag{5.5}$$

with $B \in \mathcal{B}_\phi$ and $A \in \mathcal{A}_\phi \subseteq \mathcal{B}_\phi^\perp$. Since $\mathcal{B}_\phi^\perp$ is closed under taking cones, it follows that $B \in \mathcal{B}_\phi^\perp$ as well. Combined with $B \in \mathcal{B}_\phi$, this implies that $0 = \mathrm{id}_B \in \mathrm{Hom}(B,B)$ and hence $B = 0$, meaning $F \simeq A \in \mathcal{A}_\phi$. ☐

Corollary 5.8. *The projection functors attached to the semiorthogonal decomposition* $(\mathscr{E}_{K_{\mathcal{A}_\phi}}, \mathscr{E}_{K_{\mathcal{B}_\phi}})$ *admit explicit descriptions* $\alpha_2 \simeq \Phi_{K_{\mathcal{B}_\phi}}$ *and* $\alpha_1 \simeq \Phi_{K_{\mathcal{A}_\phi}}$. *In this sense, the projection functors are of Fourier–Mukai type.*

Proof. For each object $F \in \mathrm{Perf}\,\mathcal{X}_V$, we have the following distinguished triangle:

$$\alpha_2(F) \to F \to \alpha_1(F) \xrightarrow{+1} \alpha_2(F)[1].$$

On the other hand, by applying the Fourier–Mukai functors whose kernels are the terms of the distinguished triangle (5.4), one also obtains the distinguished triangle

$$\Phi_{K_{\mathcal{B}_\phi}}(F) \to F \to \Phi_{K_{\mathcal{A}_\phi}}(F) \xrightarrow{+1} \Phi_{K_{\mathcal{B}_\phi}}(F)[1].$$

By Lemma 2.22, there is a unique pair of isomorphisms

$$\alpha_2(F) \xrightarrow{\sim} \Phi_{K_{\mathcal{B}_\phi}}(F), \quad \alpha_1(F) \xrightarrow{\sim} \Phi_{K_{\mathcal{A}_\phi}}(F),$$

which constitute the following isomorphism of distinguished triangles.

$$\begin{tikzcd}
\alpha_2(F) \arrow[r] \arrow[d, "\wr"] & F \arrow[r] \arrow[d, equal] & \alpha_1(F) \arrow[r] \arrow[d, "\wr"] & \alpha_2(F)[1] \arrow[d, "\wr"] \\
\Phi_{K_{\mathcal{B}_\phi}}(F) \arrow[r] & F \arrow[r] & \Phi_{K_{\mathcal{A}_\phi}}(F) \arrow[r] & \Phi_{K_{\mathcal{B}_\phi}}(F)[1]
\end{tikzcd}$$

Again by Lemma 2.22, the collection of these isomorphisms give the desired isomorphisms of functors. □

Now we start to compare the two functors SOD_f and DEC_{Δ_f}, as functors from $\mathrm{Aff}_U^{\mathrm{op}}$ to Sets. This allows us to study the functor SOD_f by applying the deformation theory of morphisms in the derived category, which is established in Appendix A, to the identical functor DEC_{Δ_f}.

Theorem 5.9. *Let $f\colon \mathcal{X} \to U$ be a smooth and proper morphism which admits an f-ample invertible sheaf, where U is quasiseparated. Then DEC_{Δ_f} and SOD_f, as presheaves on Aff_U, are isomorphic to each other. In particular, by Theorem 4.2, DEC_{Δ_f} is a sheaf on the big affine étale site $(\mathrm{Aff}_U)_{\text{Ét}}$.*

Proof. For each $(\phi\colon V \to U) \in \mathrm{Aff}_U^{\mathrm{op}}$, we want to construct maps $\mathsf{SOD}_f(\phi) \to \mathsf{DEC}_{\Delta_f}(\phi)$ and $\mathsf{DEC}_{\Delta_f}(\phi) \to \mathsf{SOD}_f(\phi)$. The first one is given in Lemma 5.7, and the second one is given by Lemma 5.6. One can see that these maps for various ϕ give natural transformations of functors.

It remains to show that the compositions $\mathsf{SOD}_f(\phi) \to \mathsf{DEC}_{\Delta_f}(\phi) \to \mathsf{SOD}_f(\phi)$ and $\mathsf{DEC}_{\Delta_f}(\phi) \to \mathsf{SOD}_f(\phi) \to \mathsf{DEC}_{\Delta_f}(\phi)$ are isomorphisms. The first one is shown in Lemma 5.7. To see the second one, take a decomposition ζ as in (5.2), and then consider the associated semiorthogonal decomposition $\operatorname{Perf}\mathcal{X}_V = \langle \mathscr{E}_{K_\mathcal{A}}, \mathscr{E}_{K_\mathcal{B}} \rangle$. All we need to show is that the decomposition induced by this semiorthogonal decomposition is nothing but the decomposition ζ we started with. This is equivalent to showing the following assertions:

$$K_\mathcal{A} \in \mathrm{pr}_2^* \mathscr{E}_{K_\mathcal{A}} = \left(\mathrm{pr}_2^* \mathscr{E}_{K_\mathcal{B}}\right)^\perp,$$
$$K_\mathcal{B} \in \mathrm{pr}_2^* \mathscr{E}_{K_\mathcal{B}} = {}^\perp\left(\mathrm{pr}_2^* \mathscr{E}_{K_\mathcal{A}}\right).$$

By the definition of the category $\mathrm{pr}_2^* \mathscr{E}_{K_\mathcal{B}}$, it is enough to show the vanishing

$$\mathbf{R}\mathrm{Hom}_{\mathcal{Y}_V}\left(g_V^* G \otimes \mathrm{pr}_2^* F, K_\mathcal{A}\right) = 0 \tag{5.6}$$

for all $F \in \mathscr{E}_{K_\mathcal{B}}$ and $G \in \operatorname{Perf}\mathcal{X}_V$, where $g_V\colon \mathcal{Y}_V \to V$ is the canonical morphism. On the other hand, for each such F there exists by definition a perfect complex $H \in \operatorname{Perf}\mathcal{X}_V$

such that $F \simeq \mathrm{pr}_{2*}\left(\mathrm{pr}_1^* H \otimes K_{\mathcal{B}}\right)$. Combining these, the left hand side of (5.6) is computed as follows:

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_{\mathcal{Y}_V}\left(g_V^* G \otimes \mathrm{pr}_2^* F, K_{\mathcal{A}}\right) &\simeq \mathbf{R}\mathrm{Hom}_{\mathcal{Y}_V}\left(g_V^* G \otimes \mathrm{pr}_2^* \mathrm{pr}_{2*}\left(\mathrm{pr}_1^* H \otimes K_{\mathcal{B}}\right), K_{\mathcal{A}}\right)\\ &\simeq \mathbf{R}\mathrm{Hom}_{\mathcal{Y}_V}\left(\mathrm{pr}_2^* \mathrm{pr}_{2*}\left(\mathrm{pr}_1^* H \otimes K_{\mathcal{B}}\right), g_V^* G^\vee \otimes K_{\mathcal{A}}\right)\\ &\simeq \mathbf{R}\mathrm{Hom}_{\mathcal{X}_V}\left(\mathrm{pr}_{2*}\left(\mathrm{pr}_1^* H \otimes K_{\mathcal{B}}\right), \mathrm{pr}_{2*}\left(g_V^* G^\vee \otimes K_{\mathcal{A}}\right)\right).\end{aligned}$$

By Lemma 5.6, the last term is 0. □

We next rephrase the functor DEC_{Δ_f} in a convenient way, using the notion of generator for triangulated categories.

Definition 5.10. ([19, § 2.1]) A collection of objects $\mathcal{E}$ *classically generates* a triangulated category $\mathcal{T}$ if $\mathcal{T}$ is the smallest strictly full triangulated subcategory of $\mathcal{T}$ which contains $\mathcal{E}$ and is closed under taking direct summands. An object E classically generates the category if $\mathcal{E} = \{E\}$ does.

Lemma 5.11. *Let $\mathcal{X}$ be a quasicompact and separated scheme. Then $\mathrm{Perf}\,\mathcal{X}$ admits a classical generator. Moreover, any right or left admissible triangulated subcategory $\mathcal{C} \subseteq \mathrm{Perf}\,\mathcal{X}$ is also classically generated by the image of the classical generator by the projection functor.*

Proof. See [38, Lemma 3.11] and its proof. □

This provides the following alternative description of the functor DEC_{Δ_f} from Definition 5.2, as it suffices to check vanishing of Ext's between generators to check semiorthogonality.

Proposition 5.12. *Let $f\colon \mathcal{X} \to U$ be a smooth and proper morphism with an f-ample invertible sheaf on $\mathcal{X}$. Let $\phi\colon V \to U$ be an affine U-scheme, and fix a classical generator $\Upsilon \in \mathrm{Perf}\,\mathcal{X}_V$. Then*

$$\mathsf{DEC}_{\Delta_f}(\phi) = \left\{ \begin{array}{c} \text{distinguished triangles } K_{\mathcal{B}} \to \mathcal{O}_{\Delta_{f_V}} \to K_{\mathcal{A}} \xrightarrow{+1} K_{\mathcal{B}}[1] \\ \text{in } \mathcal{Y}_V \text{ such that } \mathbf{R}f_{V,*}\mathbf{R}\mathscr{H}\mathrm{om}_{\mathcal{X}_V}(\Phi_{K_{\mathcal{B}}}(\Upsilon), \Phi_{K_{\mathcal{A}}}(\Upsilon)) = 0 \end{array} \right\} \Big/ \simeq,$$

where $\simeq$ is as in Definition 5.2.

Classical generators are compatible with base change in the following sense.

Lemma 5.13. *Suppose that $f\colon \mathcal{X} \to U$ is a smooth and proper morphism with an f-ample invertible sheaf, and that U is affine. Let $\Upsilon \in \mathrm{Perf}\,\mathcal{X}$ be a classical generator of $\mathrm{Perf}\,\mathcal{X}$. Let $V \to U$ be an affine U-scheme, and consider the following cartesian diagram.*

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "f"] \\ V \arrow[r, "g"'] & U \end{tikzcd}$$

Then $g^*\Upsilon := g'^*\Upsilon$ classically generates $\mathrm{Perf}\,\mathcal{X}'$.

Proof. Fix an f ample invertible sheaf $\mathcal{O}_f(1)$ on $\mathcal{X}$, and let $\mathcal{O}_{f'}(1) := g'^*\mathcal{O}_f(1)$. Since $\mathcal{X}'$ is quasicompact and separated, $\mathrm{Perf}\,\mathcal{X}'$ admits a single classical generator, say, Υ' ([19, Theorem 2.1.2 and Theorem 3.1.1]). We know that it is quasi-isomorphic to a bounded complex whose terms are finite direct sums of sheaves of the form $\mathcal{O}_{f'}(d)$ for $d \in \mathbb{Z}$ (see the proof of Lemma 3.11). In particular, the collection of objects $\{\mathcal{O}_{f'}(d)\}_{d\in\mathbb{Z}_{\geq 1}}$ classically generates $\operatorname{Perf}\mathcal{X}'$.

Since Υ classically generates $\operatorname{Perf}\mathcal{X}$, the objects $\mathcal{O}_f(d)$ are obtained by taking cones and direct summands for finitely many times, starting with Υ. This implies that the objects $\mathcal{O}_{f'}(d)$ are contained in the triangulated subcategory of $\operatorname{Perf}\mathcal{X}'$ obtained from $g'^*\Upsilon$ in the same way. Thus we see that $g'^*\Upsilon$ classically generates $\operatorname{Perf}\mathcal{X}'$. □

6. Local study of the functor SOD_f

The goal of this section is to prove that the functor SOD_f of Definition 4.4 is *locally of finite presentation*. This will then prove Criterion 8.2 (2), which appears in the proof of Theorem 8.1.

6.1. **An auxiliary functor.** Let R be a noetherian commutative ring, and let

$$g\colon \mathcal{Y} \to U = \operatorname{Spec} R$$

be a smooth and H-projective morphism (cf. Definition 2.4 (4)). In this subsection we study a class of functors

$$\mathsf{F}_{\mathcal{F}}\colon \mathrm{Sch}_U^{\mathrm{op}} \to \mathrm{Sets}, \quad \mathcal{F} \in \operatorname{Perf}\mathcal{Y},$$

where the one for $\mathcal{Y} = \mathcal{X} \times_U \mathcal{X}$ and $\mathcal{F} = \mathcal{O}_{\Delta_f}$ contains SOD_f as an open subfunctor:

$$\mathsf{F}_{\mathcal{O}_{\Delta_f}} \supseteq \mathsf{SOD}_f.$$

The aim of this subsection is to show that $\mathsf{F}_{\mathcal{F}}$ is locally of finite presentation (Theorem 6.6). The proof that SOD_f is itself locally of finite presentation will be given in Theorem 6.7.

Fix a factorisation

$$\mathcal{Y} \overset{\iota}{\hookrightarrow} \mathbb{P}_U^N \xrightarrow{p} U$$

of g, where ι is a closed immersion with ideal $\mathscr{I}_{\mathcal{Y}} \subseteq \mathcal{O}_{\mathbb{P}_U^N}$ and p is the canonical projection. Write

$$\mathcal{O}_g(1) = \mathcal{O}_{\mathbb{P}_U^N}(1)\big|_{\mathcal{Y}}.$$

By replacing $\mathcal{O}_g(1)$ with a higher multiple, if necessary, we may assume that the ideal sheaf $\mathscr{I}_{\mathcal{Y}}$ is 0-Castelnuovo–Mumford regular over U with respect to the projective embedding. Explicitly, this means that

$$\mathrm{H}^{\ell}(\mathbb{P}_U^N, \mathscr{I}_{\mathcal{Y}}(-\ell)) = 0$$

for all $\ell > 0$, and implies that

$$\mathrm{H}^1(\mathbb{P}_U^N, \mathscr{I}_{\mathcal{Y}}(n)) = 0 \tag{6.1}$$

for all $n \geq -1$. Since g and p are flat, these conditions continue to hold after arbitrary base change by an affine scheme $\phi\colon V \to U$. This is shown as follows.

Consider the cartesian diagram

$$\begin{tikzcd} \mathbb{P}_V^N \arrow[r, "\phi_{\mathbb{P}}"] \arrow[d, "p_V"'] \arrow[dr, phantom, "\square"] & \mathbb{P}_U^N \arrow[d, "p"] \\ V \arrow[r, "\phi"'] & U \end{tikzcd}$$

and note that the flatness of g, p implies the flatness of $\mathscr{I}_{\mathcal{Y}}$ over U, so that

$$\mathscr{I}_{\mathcal{Y}_V} \simeq \phi_{\mathbb{P}}^* \mathscr{I}_{\mathcal{Y}} \simeq \mathbf{L}\phi_{\mathbb{P}}^* \mathscr{I}_{\mathcal{Y}}.$$

Since both U and V are affine, the vanishing $\mathrm{H}^\ell(\mathbb{P}^N_V, \mathscr{I}_{\mathcal{Y}_V}(-\ell)) = 0$ that we want to prove can be rephrased as $\phi_* \mathbf{R}^\ell p_{V,*} \mathbf{L}\phi_{\mathbb{P}}^* \mathscr{I}_{\mathcal{Y}}(-\ell) = 0$. The left-hand side is computed as

$$\begin{aligned} \phi_* \mathbf{R}^\ell p_{V,*} \mathbf{L}\phi_{\mathbb{P}}^* \mathscr{I}_{\mathcal{Y}}(-\ell) &\simeq \mathcal{H}^\ell \left(\mathbf{R}\phi_* \mathbf{R}p_{V,*} \mathbf{L}\phi_{\mathbb{P}}^* \mathscr{I}_{\mathcal{Y}}(-\ell) \right) \\ &\simeq \mathcal{H}^\ell \left(\mathbf{R}\phi_* \mathbf{L}\phi^* \mathbf{R}p_* \mathscr{I}_{\mathcal{Y}}(-\ell) \right) \\ &\simeq \mathcal{H}^\ell \left(\mathbf{R}p_* \mathscr{I}_{\mathcal{Y}}(-\ell) \otimes^{\mathbf{L}}_{\mathcal{O}_U} \phi_* \mathcal{O}_V \right), \end{aligned}$$

where the last isomorphism is the projection formula along ϕ. Now for each ℓ, the assumption that $\mathscr{I}_{\mathcal{Y}}$ is 0-Castelnuovo–Mumford regular over U implies that

$$\mathbf{R}p_* \mathscr{I}_{\mathcal{Y}}(-\ell) \simeq \tau_{\le \ell-1} \mathbf{R}p_* \mathscr{I}_{\mathcal{Y}}(-\ell),$$

where $\tau_{\le \ell-1}$ is the canonical truncation functor at degree $\ell - 1$. This implies that the complex $\mathbf{R}p_* \mathscr{I}_{\mathcal{Y}}(-\ell) \otimes^{\mathbf{L}}_{\mathcal{O}_U} \phi_* \mathcal{O}_V$ is concentrated in degrees $< \ell$, so that its cohomology at degree ℓ vanishes.

Let

$$I \to \mathrm{Alg}_R, \quad i \mapsto A_i/R$$

be a functor from a filtered category I which admits a colimit

$$A = \varinjlim A_i.$$

Let us collect all the relevant base changes of g and ι in the following diagram

$$\begin{tikzcd} \mathbb{P}^N_A \arrow[r] & \mathbb{P}^N_{A_i} \arrow[r] & \mathbb{P}^N_U \\ \mathcal{Y}_\infty \arrow[r] \arrow[u, hook] \arrow[d, "g_\infty"'] \arrow[ur, phantom, "\square"] & \mathcal{Y}_i \arrow[r] \arrow[u, hook] \arrow[d, "g_i"] \arrow[ur, phantom, "\square"] & \mathcal{Y} \arrow[u, hook, "\iota"'] \arrow[d, "g"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, phantom, "\square"] & \operatorname{Spec} A_i \arrow[r] \arrow[ur, phantom, "\square"] & U \end{tikzcd} \tag{6.2}$$

that serves to fix our notation. We also let $\mathcal{O}_{g_i}(1)$ be the pullback of $\mathcal{O}_g(1)$ along $\mathcal{Y}_i \to \mathcal{Y}$, and similarly for $i = \infty$. Under these assumptions, we have the following result.

Lemma 6.1. *For all $n \in \mathbb{Z}$ the canonical map*

$$\varinjlim \mathrm{H}^0(\mathcal{Y}_i, \mathcal{O}_{g_i}(n)) \to \mathrm{H}^0(\mathcal{Y}_\infty, \mathcal{O}_{g_\infty}(n))$$

is an isomorphism.

Proof. We may and will assume $n \ge 0$, since otherwise both sides are 0. Let

$$0 \to \mathscr{I}_{\mathcal{Y}_i} \to \mathcal{O}_{\mathbb{P}^N_{A_i}} \to \mathcal{O}_{\mathcal{Y}_i} \to 0$$

be the ideal sheaf short exact sequence of $\mathcal{Y}_i \subseteq \mathbb{P}^N_{A_i}$. Twisting by $\mathcal{O}_{\mathbb{P}^N_{A_i}}(n)$ and taking cohomology yields a surjection

$$\mathrm{H}^0(\mathbb{P}^N_{A_i}, \mathcal{O}_{\mathbb{P}^N_{A_i}}(n)) \to \mathrm{H}^0(\mathcal{Y}_i, \mathcal{O}_{g_i}(n)) \to 0.$$

Since colimits preserve surjectivity, we obtain the commutative diagram

$$\begin{tikzcd} \varinjlim \mathrm{H}^0(\mathbb{P}^N_{A_i}, \mathcal{O}_{\mathbb{P}^N_{A_i}}(n)) \arrow[r] \arrow[d] & \varinjlim \mathrm{H}^0(\mathcal{Y}_i, \mathcal{O}_{g_i}(n)) \arrow[r] \arrow[d, "\vartheta"] & 0 \\ \mathrm{H}^0(\mathbb{P}^N_A, \mathcal{O}_{\mathbb{P}^N_A}(n)) \arrow[r] & \mathrm{H}^0(\mathcal{Y}_\infty, \mathcal{O}_{g_\infty}(n)) \arrow[r] & 0 \end{tikzcd}$$

where the leftmost vertical map is an isomorphism, both terms being naturally isomorphic to the degree n part of the polynomial ring $A[x_0,\dots,x_N]$. This implies that the right vertical map, ϑ, namely the map we are interested in, is surjective. To see that ϑ is also injective, suppose that $q_i \in \mathrm{H}^0(\mathcal{Y}_i, \mathcal{O}_{g_i}(n))$ is mapped to $0 \in \mathrm{H}^0(\mathcal{Y}_\infty, \mathcal{O}_{g_\infty}(n))$. Then the non-vanishing locus of q_i is an affine open subset $\operatorname{Spec} S_i \subseteq \mathcal{Y}_i$, on which the section q_i trivialises the invertible sheaf $\mathcal{O}_{g_i}(n)$. The assumption on q_i implies that $S_i \otimes_{A_i} A = 0$. On the other hand, since colimits commute with tensor products, we have

$$S_i \otimes_{A_i} A \simeq \varinjlim_{j \geq i} S_i \otimes_{A_i} A_j.$$

This implies that for some $j \geq i$ we have $S_i \otimes_{A_i} A_j = 0$. This means that

$$q_j := q_i \otimes_{A_i} A_j = 0 \in \mathrm{H}^0(\mathcal{Y}_j, \mathcal{O}_{g_j}(n)).$$

This concludes the proof. $\square$

For each $i \in \mathrm{Ob}(I) \cup \{\infty\}$, let us consider the category $\mathcal{C}_i$ defined as the enriched category over the strict monoidal category of bounded $\mathbb{Z}$-graded abelian groups. An object of $\mathcal{C}_i$ is a $\mathbb{Z}$-graded coherent sheaf $\mathcal{L} = \bigoplus_{d\in\mathbb{Z}} \mathcal{L}_d$ on $\mathcal{Y}_i$ such that $\mathcal{L}_d = 0$ except for finitely many $d \in \mathbb{Z}$, and each nonzero $\mathcal{L}_d$ is a direct sum of invertible sheaves $\mathcal{O}_{g_i}(n)$ for various $n \in \mathbb{Z}$. We shall use the notation $\mathsf{C}(-,-) = \mathrm{Hom}_{\mathsf{C}}(-,-)$ for homs in a category C. The graded abelian group of morphisms between two objects $\mathcal{L}$ and $\mathcal{L}'$ of $\mathcal{C}_i$ is given by

$$\mathcal{C}_i(\mathcal{L}, \mathcal{L}') = \bigoplus_{d\in\mathbb{Z}} \mathcal{C}_i^d(\mathcal{L}, \mathcal{L}'),$$

where we set

$$\mathcal{C}_i^d(\mathcal{L}, \mathcal{L}') = \prod_{e\in\mathbb{Z}} \operatorname{coh}(\mathcal{Y}_i)(\mathcal{L}_e, \mathcal{L}'_{e+d}).$$

The composition of morphisms in the category $\mathcal{C}_i$ is defined in the obvious way.

For a morphism $i \to j$ between objects in $\mathrm{Ob}(I) \cup \{\infty\}$, the corresponding morphism $\iota_{i,j} \colon \mathcal{Y}_j \to \mathcal{Y}_i$ induces a functor $\mathcal{C}_i \to \mathcal{C}_j$ defined by $\mathcal{L} \mapsto \iota_{i,j}^* \mathcal{L}$.

Objects of our interest are nicely encoded in the categories $\mathcal{C}_i$. A *standard perfect complex* on $\mathcal{Y}_i$ is a pair

$$(\mathcal{E}, \mathrm{d}_{\mathcal{E}})$$

of an object $\mathcal{E} \in \mathcal{C}_i$ and $\mathrm{d}_{\mathcal{E}} \in \mathcal{C}_i^1(\mathcal{E}, \mathcal{E})$, subject to the condition

$$\mathrm{d}_{\mathcal{E}}^2 = \mathrm{d}_{\mathcal{E}} \circ \mathrm{d}_{\mathcal{E}} = 0 \in \mathcal{C}_i^2(\mathcal{E}, \mathcal{E}).$$

A morphism between standard perfect complexes $(\mathcal{E}, \mathrm{d}_{\mathcal{E}})$ and $(\mathcal{E}', \mathrm{d}_{\mathcal{E}'})$ is an element

$$\psi \in \mathcal{C}_i^0(\mathcal{E}, \mathcal{E}')$$

such that $\mathrm{d}_{\mathcal{E}'} \psi - \psi\, \mathrm{d}_{\mathcal{E}} = 0$. Similarly, a *homotopy* between the same standard perfect complexes is an element

$$H \in \mathcal{C}_i^{-1}(\mathcal{E}, \mathcal{E}'),$$

and a morphism ψ as above is homotopic to 0 if there exists a homotopy H such that $\psi = \mathrm{d}_{\mathcal{E}'} H + H\,\mathrm{d}_{\mathcal{E}}$.

The following lemma is a direct consequence of Lemma 6.1.

Lemma 6.2. *There is a canonical isomorphism of enriched categories*

$$\varinjlim \mathcal{C}_i \xrightarrow{\sim} \mathcal{C}_\infty.$$

Recall the setup of this section. We have fixed a smooth and H-projective morphism $g\colon \mathcal{Y} \to U = \operatorname{Spec} R$, where R is a noetherian ring. Now fix a standard perfect complex

$$\mathcal{F} = (\mathcal{F}, \mathrm{d}_{\mathcal{F}}) \in \operatorname{Perf} \mathcal{Y}$$

whose terms are direct sums of invertible sheaves of the form $\mathcal{O}_g(n)$ for various $n \in \mathbb{Z}$.

Definition 6.3. The functor

$$\mathsf{F}_{\mathcal{F}}\colon \mathrm{Sch}_U^{\mathrm{op}} \to \mathrm{Sets} \tag{6.3}$$

assigns to a U-scheme $V \to U$ the set

$$\left\{ \text{morphisms } K \xrightarrow{s} \mathcal{F}_V \text{ in } \operatorname{Perf} \mathcal{Y}_V \right\} \Big/ \simeq$$

where $\mathcal{F}_V$ is the pullback of $\mathcal{F}$ along the projection $\mathcal{Y}_V \to \mathcal{Y}$ and an isomorphism between $s\colon K \to \mathcal{F}_V$ and $s'\colon K' \to \mathcal{F}_V$ is an isomorphism $\theta\colon K \to K'$ such that $s' \circ \theta = s$.

Let us recall the following definition.

Definition 6.4 ([2, Definition 1.5])**.** Let U be a scheme. A functor $\mathsf{F}\colon \mathrm{Alg}_U \to \mathrm{Sets}$ is said to be *locally of finite presentation* or *limit preserving* if for any functor

$$I \to \mathrm{Alg}_U, \quad i \mapsto A_i/U$$

from a filtered category I which admits a colimit $A = \varinjlim_{i\in I} A_i$, the canonical map

$$\varinjlim \mathsf{F}(A_i) \to \mathsf{F}(A)$$

is bijective. More generally, a functor $\mathsf{F}\colon \mathrm{Sch}_U^{\mathrm{op}} \to \mathrm{Sets}$ is said to be locally of finite presentation if its composition with $\operatorname{Spec}\colon \mathrm{Alg}_U \to \mathrm{Sch}_U^{\mathrm{op}}$ is.

Remark 6.5. In this remark, we freely identify a poset (partially ordered set) with the corresponding category. The notion of limit preservation given in [27, Definition 1.7] is, a priori, slightly different from the one we gave in Definition 6.4. In [27, Definition 1.7], they consider not-necessarily filtered posets for the index categories I. For the reader's convenience, here we include an explanation that these two definitions are in fact equivalent to each other. We will implicitly use the one from [27] in the application of Criterion 8.2.

Definition 6.4 is equivalent to the one where we only consider filtered *posets* as index categories I ([62, Tag 002Z]). Therefore, [27, Definition 1.7] implies Definition 6.4. To show the other implication, take an arbitrary poset for the index category I. Let $\overline{I}$ be the filtered poset $I \cup \{\infty\}$, where $i < \infty$ for any $i \in I$. By letting $A_\infty = A$ and $[i<\infty] \mapsto [A_i \to A]$, we obtain a functor $\overline{I} \to \mathrm{Alg}_U$ extending the given functor $I \to \mathrm{Alg}_U$. One can then easily show the canonical isomorphisms $\varinjlim_{i\in\overline{I}} A_i \simeq A$ and $\varinjlim_{i\in I} F(A_i) \xrightarrow{\sim} \varinjlim_{i\in\overline{I}} F(A_i)$. Therefore, if F is limit preserving in the sense of Definition 6.4, then it is so in the sense of [27, Definition 1.7].

We can now prove the following theorem.

Theorem 6.6. *The functor $\mathsf{F}_{\mathcal{F}}$ is locally of finite presentation.*

Proof. By Definition 6.4, we need to show that the composition

$$\mathsf{F}'_{\mathcal{F}} \colon \mathrm{Alg}_U \xrightarrow{\mathrm{Spec}} \mathrm{Sch}_U^{\mathrm{op}} \xrightarrow{\mathsf{F}_{\mathcal{F}}} \mathrm{Sets}$$

is locally of finite presentation. In other words, we need to show that for any filtered colimit $A = \varinjlim A_i$ of U-algebras, the natural map

$$\varphi \colon \varinjlim \mathsf{F}'_{\mathcal{F}}(A_i) \to \mathsf{F}'_{\mathcal{F}}(A)$$

is an isomorphism.

Let us first show the surjectivity of φ. Take an element

$$\left[K \xrightarrow{s} \mathcal{F}_\infty\right] \in \mathsf{F}'_{\mathcal{F}}(A).$$

By definition s is a morphism in $\mathrm{Perf}\,\mathcal{Y}_\infty$, where $\mathcal{Y}_\infty = \mathcal{Y} \times_U \mathrm{Spec}\, A$, just as in Diagram (6.2). Since g is a projective morphism, up to isomorphism we can replace s with a morphism of complexes of coherent sheaves on $\mathcal{Y}_\infty$,

$$(\mathcal{G}_\infty, \mathrm{d}_{\mathcal{G}_\infty}) \xrightarrow{u_\infty} \mathcal{F}_\infty,$$

where $\mathcal{G}_\infty \in \mathcal{C}_\infty$. In a little more detail, by the construction of the derived category, the morphism s can be represented by a diagram of the form

$$K \xleftarrow{\mathrm{qis}} \mathcal{G}' \xrightarrow{s'} \mathcal{F}_\infty,$$

where "qis" stands for quasi-isomorphism of bounded complexes of coherent sheaves and s' is a morphism of bounded complexes of coherent sheaves. Since we only care about isomorphism classes of morphisms, we can replace s with s'. Moreover, by a standard argument one can find a quasi-isomorphism of complexes

$$(\mathcal{G}_\infty, \mathrm{d}_{\mathcal{G}_\infty}) \xrightarrow{s''} \mathcal{G}'$$

from a standard perfect complex as above. Again by replacing s' with $s' \circ s''$ we obtain u_∞.

By Lemma 6.2, for some $i \in \mathrm{Ob}(I)$ one finds lifts

$$\mathrm{d}_{\mathcal{G}_i} \in \mathcal{C}_i^1(\mathcal{G}_i, \mathcal{G}_i), \quad u_i \in \mathcal{C}_i^0(\mathcal{G}_i, \mathcal{F}_i)$$

of $\mathrm{d}_{\mathcal{G}_\infty}$, u_∞ respectively. Moreover, the assumptions

$$\begin{aligned} \mathrm{d}_{\mathcal{G}_\infty}^2 &= 0 \\ u_\infty\, \mathrm{d}_{\mathcal{G}_\infty} - \mathrm{d}_{\mathcal{G}_\infty}\, u_\infty &= 0 \end{aligned}$$

imply that

$$\begin{aligned} \mathrm{d}_{\mathcal{G}_i}^2 &= 0 \\ u_i\, \mathrm{d}_{\mathcal{G}_i} - \mathrm{d}_{\mathcal{G}_i}\, u_i &= 0 \end{aligned}$$

after replacing i with a sufficiently larger one if necessary.

Let us show the injectivity of φ. By [48, Proposition 2.2.1], it is enough to consider the following assertion: Suppose for some $i \in \mathrm{Ob}(I)$ there exists a standard perfect complex $(\mathcal{E}_i, \mathrm{d}_{\mathcal{E}_i})$, with $\mathcal{E}_i \in \mathcal{C}_i$, along with two 0-cocycles (i.e. morphisms of complexes)

$$s_i,\ s_i' \in \mathrm{Z}^0(\mathcal{C}_i^\bullet((\mathcal{E}_i, \mathrm{d}_{\mathcal{E}_i}), (\mathcal{F}_i, \mathrm{d}_{\mathcal{F}_i}))),$$

both of which will be mapped to the same

$$s \in \mathrm{Hom}_{\mathrm{Perf}\,\mathcal{Y}_\infty}((\mathcal{E}_\infty, \mathrm{d}_{\mathcal{E}_\infty}), (\mathcal{F}_\infty, \mathrm{d}_{\mathcal{F}_\infty})).$$

Then

$$s_j = s_j' \in \mathrm{Hom}_{\mathrm{Perf}\,\mathcal{Y}_j}((\mathcal{E}_j, \mathrm{d}_{\mathcal{E}_j}), (\mathcal{F}_j, \mathrm{d}_{\mathcal{F}_j}))$$

for some $j \ge i$.

Let $\sigma_i = s_i - s_i' \in \mathrm{Z}^0(\mathcal{C}_i^\bullet((\mathcal{E}_i, \mathrm{d}_{\mathcal{E}_i}), (\mathcal{F}_i, \mathrm{d}_{\mathcal{F}_i})))$. The assumption implies that σ_∞ is 0 as a morphism in $\mathrm{Perf}\,\mathcal{Y}_\infty$, which again by construction of the derived category means that there exists a quasi-isomorphism of bounded complexes of coherent sheaves

$$\mathcal{G}' \xrightarrow{t'} (\mathcal{E}_\infty, \mathrm{d}_{\mathcal{E}_\infty}),$$

such that $\sigma_\infty t'$ is homotopic to 0. Also, as above, one can find a standard perfect complex $(\mathcal{G}_\infty, \mathrm{d}_{\mathcal{G}_\infty})$ with $\mathcal{G}_\infty \in \mathcal{C}_\infty$ together with a quasi-isomorphism of complexes

$$(\mathcal{G}_\infty, \mathrm{d}_{\mathcal{G}_\infty}) \xrightarrow{t''} \mathcal{G}'.$$

Thus we obtain an element

$$t_\infty := t' t'' \in \mathrm{Z}^0(\mathcal{C}_\infty^\bullet((\mathcal{G}_\infty, \mathrm{d}_{\mathcal{G}_\infty}), (\mathcal{E}_\infty, \mathrm{d}_{\mathcal{E}_\infty})))$$

that is a quasi-isomorphism of complexes and is such that $\sigma_\infty t_\infty$ is a 0-coboundary. In other words, there exists a homotopy $H_\infty \in \mathcal{C}_\infty^{-1}(\mathcal{G}_\infty, \mathcal{F}_\infty)$ such that

$$\sigma_\infty t_\infty = \mathrm{d}_{\mathcal{F}_\infty} H_\infty + H_\infty \mathrm{d}_{\mathcal{G}_\infty}.$$

By similar arguments as above, for some $j \in \mathrm{Ob}(I)$ with $j \ge i$, one finds a lift $(\mathcal{G}_j, \mathrm{d}_{\mathcal{G}_j})$ of $(\mathcal{G}_\infty, \mathrm{d}_{\mathcal{G}_\infty})$ with $\mathrm{d}_{\mathcal{G}_j}^2 = 0 \in \mathcal{C}_j^2(\mathcal{G}_j, \mathcal{G}_j)$ and also a lift $H_j \in \mathcal{C}_j^{-1}(\mathcal{G}_j, \mathcal{F}_j)$ of H_∞ and a lift $t_j \in \mathrm{Z}^0(\mathcal{C}_j^\bullet((\mathcal{G}_j, \mathrm{d}_{\mathcal{G}_j}), (\mathcal{E}_j, \mathrm{d}_{\mathcal{E}_j})))$ such that

$$\sigma_j t_j = \mathrm{d}_{\mathcal{F}_j} H_j + H_j \mathrm{d}_{\mathcal{G}_j}.$$

Now all we need to confirm is that t_j, after further replacing j if necessary, is a quasi-isomorphism of complexes. This is equivalent to showing that the cone $\mathcal{P}_j = (\mathcal{P}_j, \mathrm{d}_{\mathcal{P}_j})$ of the morphism of complexes t_j is isomorphic to 0 in $\mathrm{Perf}\,\mathcal{Y}_j$. Note that, by the standard explicit construction of the cone, one sees that $\mathcal{P}_j \in \mathcal{C}_j$.

Fix an affine open cover

$$\mathcal{Y} = \bigcup_{\lambda=0}^{m} \mathcal{Y}_\lambda$$

of $\mathcal{Y}$ such that $\mathcal{O}_g(1)$ is trivial on each $\mathcal{Y}_\lambda$. For $i \in \mathrm{Ob}(I) \cup \{\infty\}$, let $\mathcal{C}_{\lambda,i}$ be the enriched category defined in the same way as $\mathcal{C}_i$, where $\mathcal{Y}_i$ is replaced by $\mathcal{Y}_{\lambda,i} = \mathcal{Y}_\lambda \times_{\mathcal{Y}} \mathcal{Y}_i$. It is enough to show that for each λ there exists $j \in I_{\ge i}$ such that $\mathcal{P}_j|_{\mathcal{Y}_{\lambda,j}} = 0 \in \mathrm{Perf}\,\mathcal{Y}_{\lambda,j}$. However, since each term of $\mathcal{P}_j|_{\mathcal{Y}_{\lambda,j}}$ is a free (hence projective) $\mathcal{O}_{\mathcal{Y}_{\lambda,j}}$-module, this is equivalent to the existence of a homotopy $H_{\lambda,j} \in \mathcal{C}_{\lambda,j}^{-1}(\mathcal{P}_j|_{\mathcal{Y}_{\lambda,j}}, \mathcal{P}_j|_{\mathcal{Y}_{\lambda,j}})$ such that

$$\mathrm{id}_{\mathcal{P}_j|_{\mathcal{Y}_{\lambda,j}}} = H_{\lambda,j} \mathrm{d}_{\mathcal{P}_j|_{\mathcal{Y}_{\lambda,j}}} + \mathrm{d}_{\mathcal{P}_j|_{\mathcal{Y}_{\lambda,j}}} H_{\lambda,j}. \tag{6.4}$$

On the other hand, for the cone $\mathcal{P}_\infty$ of the morphism of complexes t_∞ we know

$$\mathcal{P}_\infty = 0 \in \operatorname{Perf}\mathcal{Y}_\infty.$$

This implies the existence of a homotopy $H_{\lambda,\infty} \in \mathcal{C}^{-1}_{\lambda,\infty}(\mathcal{P}_\infty|_{\mathcal{Y}_{\lambda,\infty}}, \mathcal{P}_\infty|_{\mathcal{Y}_{\lambda,\infty}})$ such that $\mathrm{id}_{\mathcal{P}_\infty|_{\mathcal{Y}_{\lambda,\infty}}} = H_{\lambda,\infty}\,\mathrm{d}_{\mathcal{P}_\infty|_{\mathcal{Y}_{\lambda,\infty}}} + \mathrm{d}_{\mathcal{P}_\infty|_{\mathcal{Y}_{\lambda,\infty}}} H_{\lambda,\infty}$. By the same arguments as above, we can find a lift $H_{\lambda,j}$ of $H_{\lambda,\infty}$ (for sufficiently large j) which satisfies (6.4). Thus we conclude the proof. □

6.2. Locally finite presentation of SOD_f.

The next result corresponds to Criterion 8.2 (2), which appears in the proof of Theorem 8.1.

Theorem 6.7. *Let U be a locally noetherian scheme, and let $f\colon \mathcal{X} \to U$ be a smooth and proper morphism with an f-ample invertible sheaf on $\mathcal{X}$. Then the functor SOD_f of Definition 4.1 is locally of finite presentation.*

We begin with some reduction.

Lemma 6.8. *Theorem 6.7 is equivalent to the same assertion with extra assumptions that U is affine and f is H-projective.*

Proof. The assumptions of Theorem 6.7 imply that f is locally H-projective. Let $U = \bigcup_{i\in I} U_i$ be an affine open covering of U such that f is H-projective over U_i. Since SOD_f is a sheaf on the big étale site $(\mathrm{Sch}_U)_{\text{Ét}}$, it follows from [62, Tag 049P] that if the restriction $\mathsf{SOD}_f|_{(\mathrm{Sch}_{U_i})_{\text{Ét}}}$ is locally of finite presentation for all $i \in I$, then so is SOD_f. In fact [62, Tag 049P] is an assertion for the fppf topology, but the proof works for étale topology as well.

Let f_i be the base change of f by $U_i \hookrightarrow U$. It remains to show the isomorphism of sheaves $\mathsf{SOD}_f|_{(\mathrm{Sch}_{U_i})_{\text{Ét}}} \simeq \mathsf{SOD}_{f_i} \in \mathsf{Sh}\big(\mathrm{Sch}_{U_i}\big)_{\text{Ét}}$. By the sheaf properties, it amounts to constructing functorially a bijection $\mathsf{SOD}_f(V/U) \xrightarrow{\sim} \mathsf{SOD}_{f_i}(V/U_i)$ for each *affine* U_i-scheme $V \to U_i$.

Since $(V \to U_i \hookrightarrow U) \in \mathrm{Aff}_U$, we have

$$\mathsf{SOD}_f(V \to U) \simeq \left\{ \begin{array}{c} V\text{-linear semiorthogonal} \\ \text{decompositions } \operatorname{Perf}\mathcal{X}_V = \langle \mathcal{A}, \mathcal{B} \rangle \end{array} \right\}.$$

By similar reasons, this set is naturally identified with $\mathsf{SOD}_{f_i}(V/U_i)$. □

From now on, we assume that $U = \operatorname{Spec} R$ is affine and f is H-projective. Set $\mathcal{Y} = \mathcal{X} \times_U \mathcal{X}$ and let

$$g\colon \mathcal{Y} \to U$$

be the canonical morphism. Since f is H-projective, so is g.

Fix the perfect complex

$$\mathcal{O}_{\Delta_f} \in \operatorname{Perf}\mathcal{Y}$$

so that the functor $\mathsf{F}_{\mathcal{O}_{\Delta_f}}$ from (6.3) is well-defined.

Proof of Theorem 6.7. By Lemma 6.8, we can and will assume that $U = \operatorname{Spec} R$ for some commutative noetherian ring R and f is H-projective. By Definition 6.4, we need to show that the composition

$$\mathrm{Alg}_U = \mathrm{Alg}_R \xrightarrow{\mathrm{Spec}} \mathrm{Sch}_U^{\mathrm{op}} \xrightarrow{\mathsf{SOD}_f} \mathrm{Sets}$$

is locally of finite presentation. Let $A = \varinjlim A_i$ be a direct limit of R-algebras, and let

$$\phi\colon \operatorname{Spec} A \to U$$

be the induced map, with base change $f_A\colon \mathcal{X}_A \to \operatorname{Spec} A$.

The functor DEC_{Δ_f} is a subfunctor of the functor $\mathsf{F}_{\mathcal{O}_{\Delta_f}}$, which is locally of finite presentation by Theorem 6.6. Then we have the following diagram of sets

$$\begin{tikzcd} \varinjlim \mathsf{DEC}_{\Delta_f}(A_i) \arrow[r, "\tau"] \arrow[d, hook] & \mathsf{DEC}_{\Delta_f}(A) \arrow[d, hook] \\ \varinjlim \mathsf{F}_{\mathcal{O}_{\Delta_f}}(A_i) \arrow[r, "\sim"] & \mathsf{F}_{\mathcal{O}_{\Delta_f}}(A) \end{tikzcd} \tag{6.5}$$

where the map τ is necessarily injective. We need to show it is onto. We will use Proposition 5.12 for this purpose.

Pick an element

$$\zeta = \left[K \xrightarrow{s} \mathcal{O}_{\Delta_{f_A}} \to L \xrightarrow{+1} K[1]\right] \in \mathsf{DEC}_{\Delta_f}(A).$$

We know, by definition, that

$$\mathbf{R} f_{A,*} \mathbf{R}\mathscr{H}\mathrm{om}_{\mathcal{X}_A}(\mathscr{E}_K, \mathscr{E}_L) = 0.$$

Consider the diagram

$$\begin{tikzcd} \mathcal{Y}_A \arrow[r] \arrow[d] & \mathcal{Y}_i \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\ \mathcal{X}_A \arrow[r, "\rho_i"] \arrow[d, "f_A"] & \mathcal{X}_i \arrow[r] \arrow[d, "f_i"] & \mathcal{X} \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r, "j_i"] \arrow[rr, bend right, "\phi"'] & \operatorname{Spec} A_i \arrow[r] & U \end{tikzcd}$$

and use Diagram (6.5) to lift ζ to an element

$$\zeta_i = \left[K_i \xrightarrow{s_i} \mathcal{O}_{\Delta_{f_i}} \to L_i \xrightarrow{+1} K_i[1]\right] \in \mathsf{F}_{\mathcal{O}_{\Delta_f}}(A_i)$$

for some i. To show that ζ_i belongs to the subset $\mathsf{DEC}_{\Delta_f}(A_i)$, fix a classical generator Υ_{A_i} of $\operatorname{Perf} \mathcal{Y}_i$ (note that $\mathcal{Y}_i$ is quasicompact and separated), and define two objects

$$k_i := \Phi_{K_i}(\Upsilon_{A_i}) \in \mathscr{E}_{K_i} \subseteq \operatorname{Perf} \mathcal{X}_i, \quad l_i := \Phi_{L_i}(\Upsilon_{A_i}) \in \mathscr{E}_{L_i} \subseteq \operatorname{Perf} \mathcal{X}_i$$

and consider their pullbacks

$$\mathbf{L}\rho_i^* k_i \in \mathscr{E}_K \subseteq \operatorname{Perf} \mathcal{X}_A, \quad \mathbf{L}\rho_i^* l_i \in \mathscr{E}_L \subseteq \operatorname{Perf} \mathcal{X}_A.$$

Then we know

$$\begin{aligned} 0 &= \mathbf{R} f_{A,*} \mathbf{R}\mathscr{H}\mathrm{om}_{\mathcal{X}_A}(\mathbf{L}\rho_i^* k_i, \mathbf{L}\rho_i^* l_i) \\ &= \mathbf{R} f_{A,*} \mathbf{L}\rho_i^* \mathbf{R}\mathscr{H}\mathrm{om}_{\mathcal{X}_i}(k_i, l_i) \\ &= \mathbf{L} j_i^* \mathbf{R} f_{i,*} \mathbf{R}\mathscr{H}\mathrm{om}_{\mathcal{X}_i}(k_i, l_i). \end{aligned}$$

Note that $P \coloneqq \mathbf{R}f_{i,*}\mathbf{R}\mathscr{H}om_{\mathcal{X}_i}(k_i, l_i) \in \operatorname{Perf} A_i$, since $\mathbf{R}f_{i,*}$ preserves perfect complexes (see the proof of Lemma 2.6). Hence by applying Lemma 6.9 to the perfect complex P above, it follows that

$$0 = \mathbf{R}f_{i,*}\mathbf{R}\mathscr{H}om_{\mathcal{X}_i}(k_i, l_i) \otimes^{\mathbf{L}}_{A_i} A_j \tag{6.6}$$

for some $j \geq i$.

Let $\rho_{i,j} \colon \mathcal{X}_j \to \mathcal{X}_i$ be the base change of $j_{i,j} \colon \operatorname{Spec} A_j \to \operatorname{Spec} A_i$. Then the right-hand side of (6.6) can be computed as follows, by repeatedly using the base change isomorphisms

$$\mathbf{R}f_{j,*}\mathbf{R}\mathscr{H}om_{\mathcal{X}_j}\left(\mathbf{L}\rho^*_{i,j}\Phi_{K_i}(\Upsilon_{A_i}), \mathbf{L}\rho^*_{i,j}\Phi_{L_i}(\Upsilon_{A_i})\right) \simeq \mathbf{R}f_{j,*}\mathbf{R}\mathscr{H}om_{\mathcal{X}_j}\left(\Phi_{K_j}(\Upsilon_{A_j}), \Phi_{L_j}(\Upsilon_{A_j})\right).$$

Here $K_j = K_i \otimes_{A_i} A_j$ and $L_j = L_i \otimes_{A_i} A_j$, respectively. By Lemma 5.13, this, in turn, implies that $\zeta_i \otimes_{A_i} A_j \in \mathsf{DEC}_{\Delta_f}(A_j)$. □

Lemma 6.9. *If a perfect complex $P \in \operatorname{Perf} A_i$ has the property that $P \otimes_{A_i} A$ is homotopy equivalent to the zero complex, then so is $P \otimes_{A_i} A_j$ for some $j \geq i$.*

Proof. This can be shown by the arguments similar to the one which we used to show the injectivity of φ in the proof of Theorem 6.6. □

7. Deformations of semiorthogonal decompositions

Throughout this section we fix a smooth projective morphism $f \colon \mathcal{X} \to U$ as in Section 4, and we let $X = f^{-1}(0)$ be the fibre over a closed point $0 \in U$. The following result requires an application of the deformation theory developed in Appendix A. We will apply the results from the appendix to deformations of the product $X \times X$, induced by deformations of X, because we want to stay in the framework of functors representable as Fourier–Mukai functors.

The next result corresponds to Criterion 8.2(3), which appears in the proof of Theorem 8.1.

Theorem 7.1. *Let $(R, \mathfrak{m}, \mathbf{k})$ be a local noetherian ring that is $\mathfrak{m}$-adically complete; then, for any morphism $\operatorname{Spec} R \to U$, the natural map*

$$\vartheta \colon \mathsf{SOD}_f(\operatorname{Spec} R) \to \mathsf{SOD}_f(\operatorname{Spec} R/\mathfrak{m})$$

is bijective.

Proof. By replacing f with its base change by $\operatorname{Spec} R \to U$, without loss of generality, we may assume that $U = \operatorname{Spec} R$. Since f is proper and admits an f-ample invertible sheaf, it is locally H-projective. Since R is a local ring, it follows that f is H-projective, so that $\mathcal{X}$ is identified with a closed subscheme of $\mathbb{P}^N_U$ for some $N \in \mathbb{N}$, as a U-scheme.

Let us prove the surjectivity of ϑ. An element

$$\xi_{\mathfrak{m}} \in \mathsf{SOD}_f(\operatorname{Spec} R/\mathfrak{m})$$

corresponds to a semiorthogonal decomposition

$$\mathbf{D}^{\mathrm{b}}(X) = \operatorname{Perf} X = \langle \mathcal{A}, \mathcal{B} \rangle$$

on the $\mathbf{k}$-scheme $X = \mathcal{X} \times_U \operatorname{Spec} R/\mathfrak{m}$. This determines a morphism

$$s_0 \colon K_{\mathcal{B}} \to \mathcal{O}_{\Delta_X}$$

in $\mathbf{D}^{\mathrm{b}}(X \times_{\mathbf{k}} X) \simeq \operatorname{Perf}(X \times_{\mathbf{k}} X)$. For $n \in \mathbb{Z}_{\geq 0}$ set $R_n = R/\mathfrak{m}^{n+1}$. We consider the base change diagram

$$\begin{tikzcd}
\mathcal{Y}_{R_n} \arrow[r, hook] \arrow[d] \arrow[dd, bend right=60, "g_{R_n}"'] & \mathcal{Y}_R \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \arrow[dd, bend left=60, "g"] \\
\mathcal{X}_{R_n} \arrow[r, hook] \arrow[d, "f_n"] & \mathcal{X}_R \arrow[r] \arrow[d, "f_R"] & \mathcal{X} \arrow[d, "f"] \\
\operatorname{Spec} R_n \arrow[r, hook] & \operatorname{Spec} R \arrow[r] & U
\end{tikzcd}$$

along the structure morphism $\operatorname{Spec} R \to U$ and the closed immersion $\operatorname{Spec} R_n \hookrightarrow \operatorname{Spec} R$. Set $X \times_{\mathbf{k}} X = Y_0$. Since we know

$$\operatorname{Ext}^{\bullet}_{Y_0}(K_{\mathcal{B}}, K_{\mathcal{A}}) = 0$$

by the semi-orthogonality, starting with $n = 0$ we can inductively apply Corollary A.20 (see also Remark A.21) to

$$(\overline{A} \twoheadrightarrow A \twoheadrightarrow A_0) = (R_{n+1} \twoheadrightarrow R_n \twoheadrightarrow R_0 = \mathbf{k})$$

to obtain the unique lift s_{n+1} of s_n. Here the role played by $\overline{G}$ in Corollary A.20 is played (at the nth step) by the complex

$$\mathcal{O}_{\Delta_{n+1}} = \mathcal{O}_{\Delta_f}\big|_{R_{n+1}} = \mathcal{O}_{\Delta_{f_{n+1}}} \in \operatorname{Perf} \mathcal{Y}_{R_{n+1}}.$$

Thus for each n we obtain a morphism $s_n \colon K_n \to \mathcal{O}_{\Delta_n}$ which is a deformation of s_0. In particular, we obtain the sequence of deformations K_n of $K_{\mathcal{B}}$. Since $\mathcal{Y} \to U$ is smooth and $K_{\mathcal{B}}$ is perfect, so is K_n by [48, Lemma 3.2.4]. Hence by [48, Proposition 3.6.1], there exists a perfect complex $K_{\mathcal{B}_R}$ on $\mathcal{Y}_R$ with compatible isomorphisms $K_{\mathcal{B}_R}|_{\mathcal{Y}_{R_n}} \xrightarrow{\sim} K_n$.

Here the sequence (K_n) defines a formal deformation of $K_{\mathcal{B}}$ in the sense of Definition 2.12 — in other words, a deformation of $K_{\mathcal{B}}$ parameterised by the formal spectrum $\operatorname{Spf} R$ (see Remark 2.13). In [48, Proposition 3.6.1] it is shown that the associated complex on $\widehat{\mathcal{Y}}_R$ (the completion of $\mathcal{Y}_R$ along Y_0) is relatively perfect over $\operatorname{Spf} R$. Grothendieck existence theorem can be applied to this complex in order to induce a deformation $K_{\mathcal{B}_R} \in \mathbf{D}(\mathcal{O}_{\mathcal{Y}_R})$ over $\operatorname{Spec} R$.

We also need to lift s_0 to a morphism

$$s \colon K_{\mathcal{B}_R} \to \mathcal{O}_{\Delta_R}, \tag{7.1}$$

where the target is the structure sheaf of the diagonal $\mathcal{X}_R \hookrightarrow \mathcal{Y}_R$. For this purpose, consider the complex

$$\mathcal{H} = \mathbf{R}\mathscr{H}\mathrm{om}_{\mathcal{Y}_R}(K_{\mathcal{B}_R}, \mathcal{O}_{\Delta_R}) \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{Y}_R).$$

By [33, Remark 8.2.3(a)], the *comparison theorem* in formal geometry (see Theorem 8.2.2 in op. cit.) can be applied to $\mathcal{H}$. Since R is a complete noetherian local ring, we can identify $\operatorname{coh}(\operatorname{Spf} R)$ with $\operatorname{coh}(\operatorname{Spec} R)$. After this identification is made, the comparison theorem reads as an isomorphism

$$\mathrm{R}^0 g_{R,*} \mathcal{H} \xrightarrow{\sim} \varprojlim \mathrm{R}^0 g_{R_n,*} \mathcal{H}_n, \tag{7.2}$$

where
$$\mathcal{H}_n = \mathcal{H}\big|_{Y_{R_n}} = \mathbf{R}\mathscr{H}om_{\mathcal{Y}_{R_n}}(K_n, \mathcal{O}_{\Delta_n}).$$
Taking global sections in (7.2) yields
$$\begin{aligned} \mathrm{Hom}_{\mathcal{Y}_R}(K_{\mathcal{B}_R}, \mathcal{O}_{\Delta_R}) &\simeq \Gamma(\varprojlim \mathrm{R}^0 g_{R_n,*}\mathcal{H}_n) \\ &\simeq \varprojlim \Gamma(\mathrm{R}^0 g_{R_n,*}\mathbf{R}\mathscr{H}om_{\mathcal{Y}_{R_n}}(K_n, \mathcal{O}_{\Delta_n})) \\ &\simeq \varprojlim \mathrm{Hom}_{\mathcal{Y}_{R_n}}(K_n, \mathcal{O}_{\Delta_n}), \end{aligned}$$
where we have used that limits commute with taking global sections. Therefore the sequence (s_n) constructed above yields a morphism s as in (7.1).

Let $K_{\mathcal{A}_R}$ be the cone of s, so we get a distinguished triangle
$$K_{\mathcal{B}_R} \xrightarrow{s} \mathcal{O}_{\Delta_R} \to K_{\mathcal{A}_R} \xrightarrow{+1} K_{\mathcal{B}_R}[1]. \tag{7.3}$$
It then follows that
$$\mathbf{R}f_{R,*}\mathbf{R}\mathscr{H}om_{\mathcal{X}_R}(\mathscr{E}_{K_{\mathcal{B}_R}}, \mathscr{E}_{K_{\mathcal{A}_R}}) = 0. \tag{7.4}$$
To see this, note by Lemma 2.2 that it is enough to show the vanishing
$$\mathbf{R}f_{R,*}\mathbf{R}\mathscr{H}om_{\mathcal{X}_R}(\mathscr{E}_{K_{\mathcal{B}_R}}, \mathscr{E}_{K_{\mathcal{A}_R}}) \otimes^{\mathbf{L}}_R \mathbf{k} = 0.$$
Since f_R is flat, the cartesian diagram
$$\begin{tikzcd} \mathcal{X}_{\mathbf{k}} \arrow[r, "\iota"] \arrow[d, "f_{\mathbf{k}}"'] \arrow[dr, phantom, "\square"] & \mathcal{X}_R \arrow[d, "f_R"] \\ \mathrm{Spec}\,\mathbf{k} \arrow[r] & \mathrm{Spec}\,R \end{tikzcd}$$
is exact. Hence we can compute
$$\begin{aligned} \mathbf{R}f_{R,*}\mathbf{R}\mathscr{H}om_{\mathcal{X}_R}(\mathscr{E}_{K_{\mathcal{B}_R}}, \mathscr{E}_{K_{\mathcal{A}_R}}) \otimes^{\mathbf{L}}_R \mathbf{k} &\simeq \mathbf{R}f_{\mathbf{k},*}\iota^*\mathbf{R}\mathscr{H}om_{\mathcal{X}_R}(\mathscr{E}_{K_{\mathcal{B}_R}}, \mathscr{E}_{K_{\mathcal{A}_R}}) \\ &\simeq \mathbf{R}\mathrm{Hom}_{\mathcal{X}_{\mathbf{k}}}\left(\mathscr{E}_{K_{\mathcal{B}_R}}\big|_{\mathcal{X}_{\mathbf{k}}}, \mathscr{E}_{K_{\mathcal{A}_R}}\big|_{\mathcal{X}_{\mathbf{k}}}\right). \end{aligned}$$
Again by using the exact cartesian diagram above, one can confirm that
$$\mathscr{E}_{K_{\mathcal{B}_R}}\big|_{\mathcal{X}_{\mathbf{k}}} \subseteq \mathscr{E}_{K_{\mathcal{B}}} = \mathcal{B}, \quad \mathscr{E}_{K_{\mathcal{A}_R}}\big|_{\mathcal{X}_{\mathbf{k}}} \subseteq \mathscr{E}_{K_{\mathcal{A}}} = \mathcal{A}. \tag{7.5}$$
Thus we obtain the desired vanishing (7.4).

Therefore the distinguished triangle (7.3) represents an R-valued point of the functor DEC_{Δ_f}. This functor is isomorphic to the functor SOD_f by Theorem 5.9. In other words, (7.3) defines an element
$$\overline{\xi} \in \mathsf{SOD}_f(\mathrm{Spec}\,R \to U).$$
Its uniqueness, i.e. the injectivity of ϑ, follows from the vanishing $\mathrm{Hom}_X(\mathcal{B}, \mathcal{A}) = 0$. □

Via Artin approximation, which we will recall in Theorem 7.3, the locally finite presentation of SOD_f we obtained above will be used in the proof of Corollary 7.4 to deform a semiorthogonal decomposition on $X = f^{-1}(0)$ to an étale neighborhood of $0 \in U$.

We fix a scheme U, which is of finite type over a field $\mathbf{k}$.

Definition 7.2. An *elementary étale neighborhood* of a point $0 \in U$ is a scheme U' along with an étale map $U' \to U$ and a factorisation $\mathrm{Spec}\,\mathbf{k}(0) \to U' \to U$.

Let us recall the following theorem by Artin. Though more general versions exist, for our purpose (namely proving Corollary 7.4) this one will suffice.

Theorem 7.3 (Artin approximation [2, Corollary 2.2])**.** *Let* $\mathsf{F}\colon \mathrm{Sch}_U^{\mathrm{op}} \to \mathrm{Sets}$ *be a functor which is locally of finite presentation. Fix a point* $0 \in U$*, an element* $\overline{\xi} \in \mathsf{F}(\operatorname{Spec} \widehat{\mathcal{O}}_{U,0})$*, and* $c \in \mathbb{Z}_{>0}$*. Then there exists an elementary étale neighborhood* $U' \to U$ *of* 0 *and an element* $\xi' \in \mathsf{F}(U')$ *such that*

$$\xi' \equiv \overline{\xi} \pmod{\mathfrak{m}_{U,0}^c}. \tag{7.6}$$

Let us explain the statement of Theorem 7.3. For a ring A, set $\mathsf{F}(A) = \mathsf{F}(\operatorname{Spec} A)$, for brevity. Then (using that F is contravariant) an elementary étale neighborhood $\operatorname{Spec} \mathbf{k}(0) \to U' \to U$ induces a commutative diagram of sets

$$\begin{tikzcd}
\mathsf{F}(\widehat{\mathcal{O}}_{U,0}/\mathfrak{m}_{U,0}^c) & \mathsf{F}(\widehat{\mathcal{O}}_{U,0}) \arrow[l] \arrow[d, "\wr"] & \mathsf{F}(\mathcal{O}_{U,0}) \arrow[l] \arrow[d] & \mathsf{F}(U) \arrow[l] \arrow[d] \\
& \mathsf{F}(\widehat{\mathcal{O}}_{U',u'}) & \mathsf{F}(\mathcal{O}_{U',u'}) \arrow[l] & \mathsf{F}(U') \arrow[l]
\end{tikzcd}$$

for every $c \geq 1$, where $u' \in U'$ is the image of $\operatorname{Spec} \mathbf{k}(0) \to U'$. The condition (7.6) means that $\overline{\xi} \in \mathsf{F}(\widehat{\mathcal{O}}_{U,0})$ and $\xi' \in \mathsf{F}(U')$ go to the same element in $\mathsf{F}(\widehat{\mathcal{O}}_{U,0}/\mathfrak{m}_{U,0}^c)$ under the given maps.

We can now prove that a semiorthogonal decomposition always uniquely deforms to the nearby fibres in the following sense.

Corollary 7.4. *Keep the assumptions of Theorem 7.1 on* U *and* f*. Assume that the derived category of the central fibre* $X = \mathcal{X}_0$ *over the fixed* $\mathbf{k}$*-valued point* $0 \in U(\mathbf{k})$ *admits a* $\mathbf{k}$*-linear semiorthogonal decomposition*

$$\mathbf{D}^{\mathrm{b}}(\operatorname{coh} X) = \langle \mathcal{A}, \mathcal{B} \rangle. \tag{7.7}$$

Then, shrinking U *to an elementary étale neighborhood of* $0 \in U$ *if necessary, one finds a unique* U*-linear semiorthogonal decomposition*

$$\operatorname{Perf} \mathcal{X} = \langle \mathcal{A}_U, \mathcal{B}_U \rangle$$

whose base change to the closed point $\{0\} \hookrightarrow U$ *is the initial semiorthogonal decomposition (7.7).*

Proof. Let $(R, \mathfrak{m}, \mathbf{k})$ be the completion of the local ring $\mathcal{O}_{U,0}$, and consider the closed immersion

$$\operatorname{Spec} R/\mathfrak{m} \hookrightarrow \operatorname{Spec} R.$$

By the theorem, we know that there is a bijection

$$\mathsf{SOD}_f(\operatorname{Spec} R) \xrightarrow{\sim} \mathsf{SOD}_f(\operatorname{Spec} R/\mathfrak{m}).$$

Let

$$\xi_{\mathfrak{m}} \in \mathsf{SOD}_f(\operatorname{Spec} R/\mathfrak{m})$$

be the element corresponding to (7.7). Now, SOD_f is locally of finite presentation by Theorem 6.7. Therefore, by Artin approximation (fix $c = 1$ in Theorem 7.3), there exists an elementary étale neighborhood $U' \to U$ of the point $0 \in U(\mathbf{k})$ and a U'-linear semiorthogonal decomposition

$$\xi' \in \mathsf{SOD}_f(U' \to U)$$

deforming the semiorthogonal decomposition (7.7) on the central fibre. □

As we explain in the following concrete example, in general one can not take U' to be a Zariski neighborhood in Corollary 7.4.

Example 7.5. Take $U = \operatorname{Spec} \mathbf{k}[a, a^{-1}]$, and consider the following family of smooth quadric surfaces

$$f : \mathcal{X} = \left(xy + z^2 - aw^2 = 0\right) \subset \mathbb{P}^3_{x:y:z:w} \times U \xrightarrow{\mathrm{pr}_2} U. \tag{7.8}$$

Consider the fiber $\mathcal{X}_1$ at $a = 1$, and the divisor $D_1 \coloneqq (x = z - w = 0) \subset \mathcal{X}_1$. If the exceptional object $\mathcal{O}_{\mathcal{X}_1}(D_1)$ extends to an object of $\operatorname{Perf}\mathcal{X}$ which is exceptional relative to U, then it has to be a line bundle on $\mathcal{X}$, since any geometric fiber of f is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and any exceptional object of rank 1 on it is known to be a line bundle ([40]). However, no line bundle on $\mathcal{X}$ restricts to $\mathcal{O}_{\mathcal{X}_1}(D_1)$. To see this, consider the *irreducible* divisor $D = \mathcal{X} \cap (x = 0) \subset \mathcal{X}$. As the divisor class group of $\mathcal{X} \setminus D \simeq \mathbb{A}^2 \times U$ is trivial, it follows that any line bundle on $\mathcal{X}$ is proportional to $\mathcal{O}_{\mathcal{X}}(D)$. But no power of $\mathcal{O}_{\mathcal{X}}(D)|_{\mathcal{X}_1}$ is isomorphic to $\mathcal{O}_{\mathcal{X}_1}(D_1)$. Geometrically speaking, this originates from the fact that the monodromy around the puncture of U exchanges the two rulings of the surface $\mathcal{X}_1 \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

On the other hand, base-changing the family by the étale double cover given by $a = b^2$, one can kill the monodromy and obtain the line bundle on $\mathcal{X}$ given by the divisor $(x = z - bw = 0) \subset \mathcal{X}$, which restricts to $\mathcal{O}_{\mathcal{X}_1}(D_1)$.

8. Moduli spaces of semiorthogonal decompositions

In this section we combine the results on the deformation theory of semiorthogonal decompositions and the functor SOD_f to prove Theorem A, and give some examples of its geometric properties. Having done this, we bootstrap to construct various closely related moduli spaces.

8.1. Checking Artin's axioms.

Let us recall the statement of Theorem A.

Theorem 8.1. *Let U be an excellent scheme, and $f : \mathcal{X} \to U$ be a smooth and proper morphism which admits an f-ample invertible sheaf on $\mathcal{X}$. Let*

$$\mathsf{SOD}_f : \mathrm{Sch}_U^{\mathrm{op}} \to \mathrm{Sets}$$

be the sheaf on the big étale site $(\mathrm{Sch}_U)_{\text{Ét}}$ introduced in Definition 4.4. Then it is a (non-empty) algebraic space which is étale over U.

Combining [27, Theorem 11.3] with the fact that a presheaf is a sheaf if and only if the associated category fibered in setoids is a stack [62, Tag 0432], we obtain the following criterion for a presheaf to be an algebraic space which is moreover *étale over the base.* If it were moreover quasicompact over U we could alternatively interpret it as a constructible sheaf, see [3, Proposition IX.2.7].

Criterion 8.2 (Algebraic space version of [27, Theorem 11.3])**.** Let U be an excellent scheme. Let P be a presheaf over Sch_U. Then P is an algebraic space étale (in particular, locally of finite presentation) over U if and only if it satisfies the following conditions.

(1) P is a sheaf on the big étale site $(\mathrm{Sch}_U)_{\text{Ét}}$.
(2) P is locally of finite presentation in the sense of Definition 6.4.
(3) Let $(B,\mathfrak{m})$ be a local noetherian ring that is $\mathfrak{m}$-adically complete, with structure morphism $\operatorname{Spec} B \to U$ such that the induced morphism $\operatorname{Spec} B/\mathfrak{m} \to U$ is of finite type. Then the map

$$\mathsf{P}(\operatorname{Spec} B) \to \mathsf{P}(\operatorname{Spec} B/\mathfrak{m})$$

is bijective.

Remark 8.3. Criterion 8.2 stays valid if the condition that $\operatorname{Spec} B/\mathfrak{m} \to U$ be of finite type in (3) is removed, as it is immediate to check applying the criterion to the base change $\mathsf{P} \times_U \operatorname{Spec} B \to \operatorname{Spec} B$.

Proof of Theorem 8.1. Let us verify the conditions in Criterion 8.2.

By construction (but see also Theorem 4.2 for the main step), SOD_f satisfies Condition (1). Condition (2) is Theorem 6.7. As for condition (3), choose a complete local noetherian U-ring B, and let

$$\vartheta\colon \mathsf{SOD}_f(\operatorname{Spec} B \to U) \to \mathsf{SOD}_f(\operatorname{Spec} B/\mathfrak{m} \hookrightarrow \operatorname{Spec} B \to U)$$

be the natural map. The surjectivity of ϑ follows by an application of our main deformation result, Theorem 7.1. The injectivity of ϑ follows by the uniqueness statement in Theorem 7.1, or alternatively from Lemma 3.15.

Finally, the algebraic space is non-empty because we always have the trivial linear semiorthogonal decompositions where either $\mathcal{A}_\phi$ or $\mathcal{B}_\phi$ in the definition of $\mathsf{SOD}_f(\phi)$ is the zero subcategory. □

8.2. **First examples, questions and properties.** We now elaborate on the geometric properties of $\mathsf{SOD}_f \to U$ by exhibiting some examples of its potentially complicated behavior.

The first thing to observe is that though SOD_f is shown to be an algebraic space over U in an abstract way, it turns out to be something tame over the generic points of U, or put differently, when U is the spectrum of a field.

Proposition 8.4. *When $U = \operatorname{Spec} \mathbf{k}$ for a field $\mathbf{k}$, SOD_f is a separated scheme of the form*

$$\coprod_{i \in I} \operatorname{Spec} L_i$$

for a (possibly infinite) collection of finite field extensions $\mathbf{k} \subset L_i$.

Proof. The assertion that SOD_f is a scheme under the assumption follows from a general result about algebraic spaces étale over a field. See, say, [62, Tag 03KX].

One can then easily show that $\mathsf{SOD}_f \to \operatorname{Spec} \mathbf{k}$, when restricted to each connected component, is affine. The second assertion then is a standard fact on field extensions. □

Next thing to point out is that there are always two sections $U \rightrightarrows \mathsf{SOD}_f$ of $\mathsf{SOD}_f \to U$, corresponding to the obvious trivial semiorthogonal decompositions $\operatorname{Perf} \mathcal{X} = \langle \operatorname{Perf} \mathcal{X}, 0\rangle$ and $\operatorname{Perf} \mathcal{X} = \langle 0, \operatorname{Perf} \mathcal{X}\rangle$. The images of the sections are disjoint and give two connected components $U \sqcup U \subset \mathsf{SOD}_f$. In §8.3.3 we discuss how to remove these trivial components from the algebraic space SOD_f by defining a subspace which takes into account only non-trivial semiorthogonal decompositions.

Example 8.5. Let $f\colon X \to \operatorname{Spec}\mathbf{k}$ be a smooth and projective variety whose derived category admits no non-trivial semiorthogonal decompositions, a typical example of which is a Calabi–Yau variety. Such a non-existence can be rephrased as the equality

$$\mathsf{SOD}_f = \operatorname{Spec}\mathbf{k} \sqcup \operatorname{Spec}\mathbf{k},$$

where the right hand side are the trivial components we just discussed. More generally for a family $\mathcal{X} \to U$ of such varieties we have

$$\mathsf{SOD}_f \simeq U \sqcup U.$$

Needless to say, things become more interesting when X admits non-trivial semiorthogonal decompositions. Below is the first non-trivial example.

Example 8.6. Let $f\colon \mathbb{P}^1_{\mathbf{k}} \to \operatorname{Spec}\mathbf{k}$. For $\mathbf{D}^{\mathrm{b}}(\mathbb{P}^1_{\mathbf{k}})$ it is known that all nontrivial semiorthogonal decompositions are given as exceptional collections of the form $\langle \mathcal{O}_{\mathbb{P}^1_{\mathbf{k}}}(i), \mathcal{O}_{\mathbb{P}^1_{\mathbf{k}}}(i+1)\rangle$ for $i \in \mathbb{Z}$ (up to shift, which is invisible when considering the subcategory generated by the exceptional object).

This implies that

$$\mathsf{SOD}_f = \overbrace{(\operatorname{Spec}\mathbf{k} \sqcup \operatorname{Spec}\mathbf{k})}^{\text{trivial components}} \sqcup \coprod_{i\in\mathbb{Z}} \operatorname{Spec}\mathbf{k}.$$

Thus we see that already in the simplest of cases there is no chance of the space SOD_f being quasicompact.

In Subsection 8.4 we suggest how SOD_f should be equipped with an action of the braid group, induced by mutation of semiorthogonal decompositions. By transitivity of the braid group action on the set of exceptional collections of $\mathbf{D}^{\mathrm{b}}(\mathbb{P}^1_{\mathbf{k}})$ this will imply that the quotient reduces to a single point (after we remove the trivial components).

As the following example[2] shows, in general the structure morphism $\mathsf{SOD}_f \to U$ is not separated, not of finite type, let alone quasi-finite, even when restricted to connected components.

Example 8.7. Set $U = \mathbb{A}^1$. There exists a smooth projective morphism $f\colon \mathcal{X} \to U$ of schemes such that

- $f^{-1}(0) \simeq \Sigma_2$ and
- $f^{-1}(\mathbb{G}_m) \simeq \Sigma_0 \times \mathbb{G}_m$ over $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\} \subseteq \mathbb{A}^1$,

where $\Sigma_d := \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-d))$ is the Hirzebruch surface of degree d.

By [54, Claim 3.1], there are infinitely many exceptional sheaves on Σ_2 all of which have the same class in $\mathrm{K}_0(\operatorname{Perf}\Sigma_2)$. By the deformation theory of exceptional objects, we see that they extend uniquely to f-exceptional objects on $\mathcal{X}$. On the other hand, since Σ_0 is a del Pezzo surface, exceptional objects (which are vector bundles up to shifts, in fact) on it are uniquely determined by their classes in $\mathrm{K}_0(\Sigma_0)$. Hence the restrictions of the f-exceptional objects constructed above to $f^{-1}(\mathbb{G}_m)$ are the same objects (vector bundles). In this way one can find f-linear semiorthogonal decompositions yielding an

[2]This example will be thoroughly investigated in [35].

irreducible component of SOD_f which is neither separated nor quasicompact over U. In particular, this means that SOD_f is not necessarily universally closed over U by [62, Tag 04XU].

As we already pointed out in Remark 1.1, this example makes it difficult to answer the question whether SOD_f is always a scheme or not.

We can also ask the following question, motivated by the fact that a morphism is universally closed if and only if it is quasicompact and satisfies the existence part of the valuative criterion ([62, Tag 01KF and Tag 04XU]). If answered affirmatively, then it would imply that for any connected component $Z \subset \mathsf{SOD}_f$, the restriction of the structure morphism $Z \to U$ is surjective.

Question 8.8. Let $f\colon \mathcal{X} \to U$ be a smooth and proper morphism between quasiseparated schemes which admits an f-ample invertible sheaf, and assume U is excellent. Does $\mathsf{SOD}_f \to U$ satisfy the existence part of the valuative criterion ([62, Tag 01KD])? Namely, let R be an arbitrary discrete valuation ring and Q its field of fractions. For any commutative diagram as follows, is there at least one (more than one in general) dotted arrow which makes the two little triangles commutative?

$$\begin{tikzcd} \operatorname{Spec} Q \arrow[r] \arrow[d] & \mathsf{SOD}_f \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dotted] & U \end{tikzcd}$$

This question is related to the conjectural existence of a specialisation morphism for Grothendieck rings of categories à la Bondal–Larsen–Lunts [20], as discussed in [31].

Remark 8.9. In Remark 4.5 we have explained how the main result of [1] can be used to get rid of the characteristic assumption in our paper. But the statement of Theorem 8.1 is strictly stronger than what is contained in op. cit., for our specific setting. Their results apply to more general families, but they do not give the algebraicity result that SOD_f is in fact an algebraic space, which is moreover étale over the base.

Whilst we certainly expect this result to be true in a more general setting than we have shown it (see Conjecture C for the precise formulation), it will not hold in the complete generality of [1] as the following example, suggested to us by Isamu Iwanari, and the one after show.

Example 8.10 (Non-rigidity)**.** The uniqueness of the deformation of semiorthogonal decompositions does not hold for semiorthogonal decompositions of the unbounded derived category $\mathbf{D}(\operatorname{Qcoh}\mathcal{X})$, essentially for any morphism $\mathcal{X} \to S$. In particular, it is not possible to obtain an algebraic space (or stack) which is étale over the base.

For example, let X be an arbitrary variety over a field $\mathbf{k}$ of positive dimension and $Z \subseteq X$ be a closed subset. Then there is the semiorthogonal decomposition

$$\mathbf{D}(\operatorname{Qcoh} X) = \langle \mathbf{D}(\operatorname{Qcoh} U), \mathbf{D}_Z(\operatorname{Qcoh} X)\rangle, \tag{8.1}$$

where $U = X \setminus Z$ and $\mathbf{D}_Z(\operatorname{Qcoh} X)$ is the subcategory of $\mathbf{D}(\operatorname{Qcoh} X)$ consisting of those complexes whose cohomology is supported in Z (see for instance [59, Section 6.2.1]). In general Z can be moved arbitrarily inside X, and for different Z the subcategories

$\mathbf{D}_Z(\mathrm{Qcoh}\,X) \hookrightarrow \mathbf{D}(\mathrm{Qcoh}\,X)$ are also different (for example, think of the set of skyscraper sheaves of points contained in Z).

One of the reasons why this example fails is that the semiorthogonal decomposition (8.1) is not strong. Hence no convenient way of representing it using Fourier–Mukai functors is available [43, Remark 7.2], and we have used this presentation of the semiorthogonal decomposition (see Section 5) to study its deformation theory. Another way of phrasing the problem is the following. The base change results for semiorthogonal decompositions, obtained in Section 3 and [6] do not apply in this setting, as the inclusion of $\mathbf{D}(\mathrm{Qcoh}\,U)$ into $\mathbf{D}(\mathrm{Qcoh}\,X)$ does not restrict to perfect complexes, as $\mathbf{R}j_*$ for $j\colon U \hookrightarrow X$ does not preserve perfect complexes. This is required to appeal to the machinery of base change for semiorthogonal decompositions so that one obtains a decomposition triangle in the derived category of the product.

Remark that [1] also imposes strongness, see Definition 3.15 of op. cit. So it is not even clear how one obtains an fppf stack of semiorthogonal decompositions of $\mathbf{D}(\mathrm{Qcoh}\,X)$ which contains (8.1).

On the contrary, there is an example where semiorthogonal decompositions on the central fiber does not seem to deform to nearby fibers.

Example 8.11 (Non-deformability)**.** Let $f\colon X \to Y$ be a resolution of rational singularities, so that $\mathbf{R}f_*\mathcal{O}_X \simeq \mathcal{O}_Y$. On the level of unbounded categories we have that $\mathbf{D}(\mathrm{Qcoh}\,Y)$ is a full subcategory of $\mathbf{D}(\mathrm{Qcoh}\,X)$ via the functor $\mathbf{L}f^*$, as $\mathbf{R}f_*\mathbf{L}f^* \simeq \mathrm{id}_{\mathbf{D}(\mathrm{Qcoh}\,Y)}$ by the assumption on the singularities and the projection formula. It is *right* admissible, via the projection functor $\mathbf{R}f_*$, but not left admissible, as by [4, Theorem 1.7] this is equivalent to $\mathbf{R}f_*$ preserving compact objects. The image of a skyscraper in the exceptional locus of the morphism is the skyscraper of a singular point, which is not a perfect complex. Nevertheless, we can consider the semiorthogonal decomposition

$$\mathbf{D}(\mathrm{Qcoh}\,X) = \langle \mathbf{L}f^*\mathbf{D}(\mathrm{Qcoh}\,Y)^{\perp}, \mathbf{L}f^*\mathbf{D}(\mathrm{Qcoh}\,Y)\rangle. \tag{8.2}$$

In general, this semiorthogonal decomposition does *not* deform to nearby fibres in a family. For example, consider an algebraic family $\mathcal{X} \to U$ of K3 surfaces, where some members have a −2-curve but the general member does not. For a concrete example, over the field of complex numbers $\mathbb{C}$, one can take a linear pencil of quartic surfaces whose central fiber is the Fermat quartic and a very general member is of Picard number one. Any of the lines on the central fiber is a −2-curve on it. Then the semiorthogonal decomposition of the form (8.2) obtained by contracting a −2-curve of the central fiber does not seem to deform to nearby fibers.

8.3. **Amplifications.** In this subsection we can bootstrap from the definition of SOD_f to define closely related moduli spaces, and show that these are also algebraic spaces, étale over U. These amplifications, as anticipated in § 1.4, happen in three steps:

(1) we explain how to deal with semiorthogonal decompositions with more than two components;
(2) we explain how to work relatively to a fixed subcategory $\mathcal{B}$ in a U-linear semiorthogonal decomposition
$$\mathrm{Perf}\,\mathcal{X} = \langle \mathcal{A}, \mathcal{B}\rangle,$$
thereby making the theory valid for geometric subcategories, in the spirit of [57];

(3) we explain how to restrict to an open and closed subfunctor of SOD_f involving only non-trivial semiorthogonal decompositions (i.e. the zero subcategory is not allowed as a component of the decomposition).

8.3.1. *Arbitrary length.* Similar to Definition 4.1 we have the following definition.

Definition 8.12. Fix an integer $\ell \geq 1$. Given a smooth proper morphism $f \colon \mathcal{X} \to U$ of quasiseparated schemes with an f-ample invertible sheaf, we define a functor

$$\mathsf{SOD}_f^\ell \colon \mathrm{Aff}_U^{\mathrm{op}} \to \mathrm{Sets}$$

by sending an affine U-scheme $\phi \colon V \to U$ to the set

$$\mathsf{SOD}_f^\ell(\phi) = \left\{ \left(\mathcal{A}_\phi^1, \ldots, \mathcal{A}_\phi^\ell\right) \;\middle|\; \begin{array}{c} \mathrm{Perf}\,\mathcal{X}_V = \langle \mathcal{A}_\phi^1, \ldots, \mathcal{A}_\phi^\ell \rangle \text{ is a } V\text{-linear} \\ \text{semiorthogonal decomposition} \end{array} \right\}.$$

With this definition we have

$$\mathsf{SOD}_f^2 = \mathsf{SOD}_f,$$

whilst we have

$$\mathsf{SOD}_f^1 = U.$$

The following theorem generalises Theorem 8.1 to an arbitrary $\ell \geq 1$.

Theorem 8.13. *For each $\ell \geq 1$, the presheaf SOD_f^ℓ is a sheaf on the big affine étale site* $(\mathrm{Aff}_U)_{\text{Ét}}$. *The corresponding sheaf on the big étale site* $(\mathrm{Sch}_U)_{\text{Ét}}$, *which will be denoted by the same symbol SOD_f^ℓ, is an étale algebraic space over U.*

The proof of Theorem 8.13 will be an induction on the length ℓ, where the case $\ell = 2$ is already established in Theorem 8.1. For this purpose, we define various natural transformations between the $\mathsf{SOD}_f^\bullet$ functors.

Definition 8.14. The category Δ is the category whose objects are totally ordered finite sets $[\ell] = \{0, 1, \ldots, \ell\}$ for $\ell = 0, 1, 2, \ldots$ and the morphisms are defined as

$$\Delta([\ell], [k]) \coloneqq \left\{ \varphi \colon [\ell] \to [k] \mid i \leq j \Rightarrow \varphi(i) \leq \varphi(j) \right\}.$$

Definition 8.15. We have the *amalgamation functor*

$$\mathsf{am} \coloneqq \mathsf{am}_f \colon \Delta \to \mathsf{PSh}(\mathrm{Aff}_U)$$

as the cosimplicial presheaf, given by

$$[\ell] \mapsto \mathsf{SOD}_f^{\ell+1}$$

and an order-preserving morphism $\varphi \colon [\ell] \to [k]$ is sent to the natural transformation $\mathsf{am}(\varphi) \colon \mathsf{SOD}_f^{\ell+1} \to \mathsf{SOD}_f^{k+1}$ sending a semiorthogonal decomposition $\langle \mathcal{A}_\phi^1, \ldots, \mathcal{A}_\phi^\ell, \mathcal{A}_\phi^{\ell+1} \rangle$ of length $\ell + 1$ to the semiorthogonal decomposition $\langle \mathcal{B}_\phi^1, \ldots, \mathcal{B}_\phi^k, \mathcal{B}_\phi^{k+1} \rangle$ of length $k + 1$ defined by

$$\mathcal{B}_\phi^i = \begin{cases} \langle \mathcal{A}_\phi^j \mid j \in \varphi^{-1}(i) \rangle & \varphi^{-1}(i) \neq \emptyset, \\ 0 & \varphi^{-1}(i) = \emptyset, \end{cases}$$

where $\langle \mathcal{A}_\phi^j \mid j \in \varphi^{-1}(i) \rangle$ is the smallest strictly full triangulated subcategory generated by those subcategories $\mathcal{A}_\phi^j$ such that $\varphi(j) = i$.

Consider the degeneracy morphisms

$$\sigma^{\ell,i}\colon [\ell]\to[\ell-1],\quad i=1,\dots,\ell,$$

which hits $i-1$ twice (by $i-1$ and i). Next we consider the composition

$$\Sigma := \sigma^{2,2}\circ\cdots\circ\sigma^{\ell-1,\ell-1}\colon [\ell-1]\to[1]$$

which collapses the elements $\{1,\dots,\ell-1\}$ to 1, sending 0 to 0.

Proof of Theorem 8.13. Suppose that for each $i=2,3,\dots,\ell-1$, it is already shown that SOD^i_f is a sheaf on the affine étale site $(\mathrm{Aff}_U)_{\text{Ét}}$ and that the corresponding sheaf on $(\mathrm{Sch}_U)_{\text{Ét}}$ is an étale algebraic space.

Let SOD^ℓ_f be the presheaf on Aff_U defined in Definition 8.12. We first show that it is a sheaf on $(\mathrm{Aff}_U)_{\text{Ét}}$. For this purpose, consider the following monomorphism[3]

$$\Theta := \mathsf{am}(\sigma^{\ell-1,1})\times\mathsf{am}(\Sigma)\colon \mathsf{SOD}^\ell_f=\mathsf{am}([\ell-1])\to\mathsf{SOD}^{\ell-1}_f\times\mathsf{SOD}^2_f \tag{8.3}$$

of presheaves on Aff_U. Concretely, for each affine U-scheme $\phi\colon V\to U$, the map $\Theta(\phi)$ sends a V-linear semiorthogonal decomposition $\left(\mathcal{A}^1_\phi,\mathcal{A}^2_\phi,\mathcal{A}^3_\phi,\dots,\mathcal{A}^\ell_\phi\right)$ to the pair of semiorthogonal decompositions

$$\left(\left(\langle\mathcal{A}^1_\phi,\mathcal{A}^2_\phi\rangle,\mathcal{A}^3_\phi,\dots,\mathcal{A}^\ell_\phi\right),\left(\mathcal{A}^1_\phi,\langle\mathcal{A}^2_\phi,\mathcal{A}^3_\phi,\dots,\mathcal{A}^\ell_\phi\rangle\right)\right)$$

of length $\ell-1$ and of length 2 respectively. The following assertion is easy to observe, by noting the equality

$$\mathcal{A}^2_\phi=\langle\mathcal{A}^1_\phi,\mathcal{A}^2_\phi\rangle\cap\langle\mathcal{A}^2_\phi,\mathcal{A}^3_\phi,\dots,\mathcal{A}^\ell_\phi\rangle.$$

Claim 8.16. The map $\Theta(\phi)$ is an injection, and the image consists of those pairs

$$\left(\left(\mathcal{B}^1_\phi,\dots,\mathcal{B}^{\ell-1}_\phi\right),\left(\mathcal{C}^1_\phi,\mathcal{C}^2_\phi\right)\right)$$

satisfying the condition

$$\mathcal{C}^1_\phi\subset\mathcal{B}^1_\phi \iff \mathcal{C}^1_\phi\subset\left(\mathcal{B}^i_\phi\right)^\perp \quad \text{for all } i=2,\dots,\ell-1. \tag{8.4}$$

By induction we know that both $\mathsf{SOD}^{\ell-1}_f,\mathsf{SOD}^2_f$ are sheaves on $(\mathrm{Aff}_U)_{\text{Ét}}$. Hence so is their product presheaf $\mathsf{SOD}^{\ell-1}_f\times\mathsf{SOD}^2_f$. This already implies that the subpresheaf SOD^ℓ_f satisfies the first sheaf condition (sections are defined locally). The second sheaf condition (descent) follows from the fact that the right hand side of (8.4) is a local condition on V by Lemma 3.9. Thus we have checked that SOD^ℓ_f is a sheaf on $(\mathrm{Aff}_U)_{\text{Ét}}$. By abuse of notation, we use the same symbol for the corresponding sheaf on $(\mathrm{Sch}_U)_{\text{Ét}}$. As before, one can easily check that the values on affine U-schemes remain the same.

Next, let us show the condition (2) (i.e. local finite presentation) for the sheaf SOD^ℓ_f. Consider a filtered colimit of U-rings $A=\varinjlim A_i$. We need to show that the natural map

$$\varinjlim_{i\in I}\mathsf{SOD}^\ell_f(A_i)\to\mathsf{SOD}^\ell_f(A)$$

is a bijection.

From the injection Θ and the induction hypothesis, we see that this is an injection. Again by the induction hypothesis, to prove the surjectivity amounts to showing the

[3] As we explain in Remark 8.21.

following assertion: Suppose we are given a pair of A-linear semiorthogonal decompositions satisfying (8.4). By the induction hypothesis, we can assume that both of the semiorthogonal decompositions are defined over A_i for some $i \in I$. Hence it remains to show that, after replacing i with j with $i \leq j$ if necessary, the semiorthogonality (8.4) holds over A_j. Here we can use similar arguments as in the proof of Theorem 6.7 (the argument which uses Lemma 6.9).

Next, let us show the condition (3) for SOD^{ℓ}_{f}. Namely, let $(B, \mathfrak{m})$ be a local noetherian ring that is $\mathfrak{m}$-adically complete, with structure morphism $\operatorname{Spec} B \to U$ such that the induced morphism $\operatorname{Spec} \mathbf{k} = \operatorname{Spec} B/\mathfrak{m} \to U$ is of finite type. Then the map

$$\mathsf{SOD}^{\ell}_{f}(\operatorname{Spec} B) \to \mathsf{SOD}^{\ell}_{f}(\operatorname{Spec} B/\mathfrak{m})$$

is required to be bijective.

The injectivity is obvious from the injectivity of Θ and the induction hypothesis. To see the surjectivity, take an element of $\mathsf{SOD}^{\ell}_{f}(\mathbf{k})$, or equivalently, a pair of $\mathbf{k}$-linear semiorthogonal decompositions satisfying the semiorthogonality (8.4). By the induction hypothesis, the pair of semiorthogonal decompositions lift uniquely to a pair of B-linear semiorthogonal decompositions. Now since B is local, the lift inherits the semiorthogonality (8.4) of the central fiber by Lemma 3.9 and Lemma 2.2. □

We can also show the following generalisation for SOD^{ℓ}_{f} of Proposition 4.7 describing the values of the functor on not necessarily affine schemes. It follows by combining the Zariski gluing from Lemma 3.13 with the identification of the restriction of the sheaf on the big étale site to the affine étale site as in Theorem 8.13.

Theorem 8.17. *For any U-scheme $\phi\colon V \to U$ which is quasicompact and semiseparated, there is a natural bijection*

$$\mathsf{SOD}^{\ell}_{f}(\phi) \simeq \left\{ \left(\mathcal{A}^{1}_{\phi}, \ldots, \mathcal{A}^{\ell}_{\phi}\right) \;\middle|\; \begin{array}{c} \operatorname{Perf} \mathcal{X}_V = \langle \mathcal{A}^{1}_{\phi}, \ldots, \mathcal{A}^{\ell}_{\phi} \rangle \text{ is a } V\text{-linear} \\ \text{semiorthogonal decomposition} \end{array} \right\}.$$

By Theorem 8.13, the cosimplicial presheaf am is naturally extended to a cosimplicial étale algebraic space over U. Therefore the morphisms between the algebraic spaces $\mathsf{am}([k]), \mathsf{am}([\ell])$ have the following nice properties.

Proposition 8.18. *For each morphism $\varphi\colon [k] \to [\ell]$ in Δ, the associated morphism*

$$\mathsf{am}(\varphi)\colon \mathsf{SOD}^{k+1}_{f} = \mathsf{am}([k]) \to \mathsf{am}([\ell]) = \mathsf{SOD}^{\ell+1}_{f}$$

of algebraic spaces is étale. When φ is injective, it is an open and closed immersion.

Proof. As $\mathsf{am}(\varphi)$ is a morphism of étale algebraic spaces over the scheme U, the first assertion is an immediate consequence of [62, Tag 05W3], which in turn is an immediate consequence of the fact that the property of morphisms between algebraic spaces being étale has the 2-out-of-3 property [62, Tag 03FV].

If φ is injective, then $\mathsf{am}(\varphi)$ is universally injective. In fact, by [62, Tag 040X and Tag 03MV], it is equivalent to the injectivity of the map $\mathsf{am}(\varphi)(\operatorname{Spec} K)$ for any U-field K; i.e., for any morphism $\operatorname{Spec} K \to U$ with K a field. As we know the values of the algebraic spaces SOD^{k}_{f} at affine U-schemes by Theorem 8.17, this is checked immediately. Hence in this case $\mathsf{am}(\varphi)$ is an open immersion by [62, Tag 05W5], which asserts that an étale and universally injective morphism between algebraic spaces is an open immersion (and vice versa).

Finally we show that $\mathsf{am}(\varphi)$ is proper (when φ is injective), to conclude that it is closed. Since everything is étale over the excellent scheme U, it follows that everything is locally noetherian and that the open immersion $\mathsf{am}(\varphi)$ is of finite type and quasi-separated. Hence the noetherian valuative criterion for properness for algebraic spaces [62, Tag 0ARK] reduces the problem to the following Lemma 8.19. □

The following lemma is an application of a result due to Rouquier about the schematic support of bounded complexes of coherent sheaves on a noetherian scheme. The noetherian assumption is crucial, as it is based on the Artin–Rees theorem.

Lemma 8.19. *Let $(R,\mathfrak{m})$ be a discrete valuation ring, and $U = \operatorname{Spec} R$. Suppose that $f\colon \mathcal{X} \to U$ is a smooth projective morphism of schemes, and let $\operatorname{Perf}\mathcal{X} = \langle \mathcal{A}, \mathcal{B}\rangle$ be a U-linear semiorthogonal decomposition. If $\mathcal{A} \otimes_R Q = 0$, where Q is the field of fractions of R, then $\mathcal{A} = 0$.*

Proof. The assumption implies that $K_{\mathcal{A}}|_{\mathcal{X}_Q} = 0$. Hence $K_{\mathcal{A}}$ should be supported in the central fiber.

By [59, Lemma 7.40], there is some $m > 0$ such that if one lets

$$\iota\colon X_m = \mathcal{X} \times_U \operatorname{Spec} R/\mathfrak{m}^{m+1} \hookrightarrow \mathcal{X}$$

be the inclusion of the mth thickening of the central fiber, then there is a bounded complex of coherent sheaves E on $X_m \times_{\operatorname{Spec} R/\mathfrak{m}^{m+1}} X_m$ such that $K_{\mathcal{A}} \xrightarrow{\sim} (\iota \times \iota)_* E$. This implies that

$$(\iota \times \iota)^* K_{\mathcal{A}} \simeq (\iota \times \iota)^* (\iota \times \iota)_* E \simeq E \oplus E[1],$$

so that

$$K_{\mathcal{A}} \simeq K_{\mathcal{A}}|_{X_0 \times X_0} \simeq ((\iota \times \iota)^* K_{\mathcal{A}})|_{X_0 \times X_0} \simeq E|_{X_0 \times X_0} \oplus E|_{X_0 \times X_0}[1].$$

This means that any non-zero object of $\mathcal{A}$ has more than one direct summands. Combining it with the fact that $\mathcal{A}$ is closed under taking direct summands, one can show that $\mathcal{A} = 0$. In fact, suppose that there exists a non-zero object $a \in \mathcal{A}$. By the assumption on f, we see that the coherent R-module $0 \neq \operatorname{Hom}(a,a)$ is a direct sum of finitely many indecomposable R-modules. Note that the number of summands $N \geq 1$ is an invariant of a, as R is a discrete valuation ring. On the other hand, we can write a as a direct sum of more than N objects. This is a contradiction. □

For later use we strengthen this result slightly, where R is replaced with an arbitrary noetherian local ring.

Lemma 8.20. *Let $(S,\mathfrak{n})$ be a local noetherian ring, and $U = \operatorname{Spec} S$. Suppose that $f\colon \mathcal{X} \to U$ is a smooth projective morphism of schemes, and let $\operatorname{Perf}\mathcal{X} = \langle \mathcal{A}, \mathcal{B}\rangle$ be a U-linear semiorthogonal decomposition. If $\mathcal{A} \otimes_R Q = 0$, where Q is the field of fractions of S, then $\mathcal{A} = 0$.*

Proof. There is a discrete valuation ring $(R,\mathfrak{m})$ whose field of fractions is Q and dominating $(S,\mathfrak{n})$. Concretely, it is obtained by normalising the localisation at a prime exceptional divisor of the blowup of $\operatorname{Spec} S$ at $\mathfrak{n}$, and then again localising at a maximal ideal. The fact that the normalisation is noetherian follows from the Krull–Akizuki theorem (see, say, [29, Chapter II, Exercise 4.11(a)]).

By taking the base change by $\operatorname{Spec} R \to \operatorname{Spec} S$ and then applying Lemma 8.19, it follows that $\mathcal{A}_R = 0$. Then it implies $\mathcal{A}_{R/\mathfrak{m}} = \left(\mathcal{A}_{S/\mathfrak{n}}\right)_{R/\mathfrak{m}} = 0$, hence $\mathcal{A}_{S/\mathfrak{n}} = 0$ by Lemma 8.27 below. As S is local, Lemma 2.3 immediately implies that $\mathcal{A} = 0$. □

As before, we have that SOD^{ℓ}_{f} is non-empty, as it always contains "trivial" semiorthogonal decompositions, where one component is the whole category and the others are zero.

Remark 8.21. The morphism Θ from (8.3) is an open immersion, similar to how we proved Proposition 8.18, as by Claim 8.16 we have that Θ is universally injective, and it is moreover automatically étale.

8.3.2. *Relative to a fixed subcategory.* Having the formalism for semiorthogonal decompositions of arbitrary length in place, we can bootstrap to not just semiorthogonal decompositions of perfect complexes on schemes, but to all categories of geometric nature. This is to be interpreted in the sense that they are components of a semiorthogonal decomposition of perfect complexes on a scheme, similar to the notion of a geometric dg category in [57].

Definition 8.22. Fix $\ell \geq 1$ and a U-linear semiorthogonal decomposition

$$\operatorname{Perf} \mathcal{X} = \langle \mathcal{A}, \mathcal{B} \rangle, \tag{8.5}$$

and consider the presheaf $\mathsf{SOD}^{\ell}_{\mathcal{B}} \in \mathsf{PSh}(\mathrm{Aff}_U)$ which sends an affine U-scheme $\phi \colon V \to U$ to the set

$$\mathsf{SOD}^{\ell}_{\mathcal{B}}(\phi) = \left\{ \mathcal{B}_{\phi} = \langle \mathcal{A}^{1}_{\phi}, \ldots, \mathcal{A}^{\ell}_{\phi} \rangle \;\middle|\; \begin{array}{c} V\text{-linear semiorthogonal} \\ \text{decomposition of length } \ell \end{array} \right\},$$

where $\mathcal{B}_{\phi}$ is the base change of $\mathcal{B}$.

Note that the semiorthogonal decomposition (8.5) we fixed corresponds to a section $s \colon U \to \mathsf{SOD}^{2}_{f}$. This choice allows us to prove the following proposition.

Proposition 8.23. *There exists a natural isomorphism of presheaves*

$$\mathsf{SOD}^{\ell}_{\mathcal{B}} \xrightarrow{\sim} \mathsf{SOD}^{\ell+1}_{f} \times_{\mathsf{am}(\Sigma),\, \mathsf{SOD}_{f},\, s} U \tag{8.6}$$

on Aff_U.

Proof. This follows from the observation that for each U-scheme $\phi \colon V \to U$, a semiorthogonal decomposition of $\mathcal{B}_{\phi}$ of length ℓ is nothing but a semiorthogonal decomposition of $\operatorname{Perf} \mathcal{X}_V$ of length $\ell + 1$ whose first component is $\mathcal{A}_{\phi}$. □

Definition 8.24. We let $\mathsf{SOD}^{\ell}_{\mathcal{B}}$ denote the sheaf on the big étale site $(\mathrm{Sch}_U)_{\text{Ét}}$ corresponding to the sheaf on $(\mathrm{Aff}_U)_{\text{Ét}}$ denoted by the same symbol in Proposition 8.23.

Corollary 8.25. *The sheaf* $\mathsf{SOD}^{\ell}_{\mathcal{B}}$*, as defined in Definition 8.24, is a (non-empty) étale algebraic space over* U*.*

Proof. The description of $\mathsf{SOD}^\ell_{\mathcal{B}}$ in (8.6) as a fibre product implies that it also is a sheaf on the big affine étale site $(\mathrm{Aff}_U)_{\mathrm{Ét}}$, as the right hand side is a fiber product of presheaves all of which are known to be sheaves. Hence $\mathsf{SOD}^\ell_{\mathcal{B}}$ uniquely corresponds to a sheaf on the big étale site $(\mathrm{Sch}_U)_{\mathrm{Ét}}$, and it still satisfies the isomorphism of Proposition 8.6, under the equivalence of topoi. Now Theorem 8.13 immediately implies that $\mathsf{SOD}^\ell_{\mathcal{B}}$ is an étale algebraic space over U, as the right hand side of Proposition 8.6 is a fiber prouct of étale algebraic spaces over U. □

8.3.3. *Restricting to non-trivial semiorthogonal decompositions.* We next introduce the open subspace of SOD^ℓ_f which only parametrise *non-trivial* semiorthogonal decompositions. We can continue with the setup of §8.3.2, and consider a fixed U-linear semiorthogonal decomposition

$$\operatorname{Perf}\mathcal{X} = \langle \mathcal{A}, \mathcal{B}\rangle. \tag{8.7}$$

If $\mathcal{B}$ is taken to be all of $\operatorname{Perf}\mathcal{X}$ one can modify the notation accordingly to reflect the dependence on the morphism $f\colon \mathcal{X}\to U$ only.

Definition 8.26. The presheaf of non-trivial semiorthogonal decompositions of $\mathcal{B}$ on Aff_U, denoted $\mathsf{ntSOD}^\ell_{\mathcal{B}} \subseteq \mathsf{SOD}^\ell_{\mathcal{B}}$ is defined by taking

$$\mathsf{ntSOD}^\ell_{\mathcal{B}}(V \xrightarrow{\phi} U) := \left\{ \langle \mathcal{A}^1_\phi,\dots,\mathcal{A}^\ell_\phi\rangle \in \mathsf{SOD}^\ell_{\mathcal{B}}(\phi) \;\middle|\; \begin{array}{c} \mathcal{A}^i_\phi\big|_x \neq 0 \text{ for all } i \text{ and for all} \\ \text{geometric point } x \text{ of } V \end{array} \right\}. \tag{8.8}$$

Recall that a geometric point is a morphism of the form $x\colon \operatorname{Spec}\Omega \to V$, where Ω is an algebraically closed field. We will freely use the following descent lemma.

Lemma 8.27. *Let $f\colon \mathcal{X}\to \operatorname{Spec} K$ be a smooth projective morphism with K a field, and take an object $E \in \operatorname{Perf}\mathcal{X}$. Then for any field extension $K \subset L$, $E = 0$ if and only if $E\otimes_K L = 0 \in \operatorname{Perf}\mathcal{X}_L$. Similarly, for any subcategory $\mathcal{A}\subset \operatorname{Perf}\mathcal{X}$, $\mathcal{A} = 0$ if and only if $\mathcal{A}_L = 0$.*

Lemma 8.28. *The presheaf $\mathsf{ntSOD}^\ell_{\mathcal{B}}$ defined in Definition 8.26 is a sheaf on the big affine étale site $(\mathrm{Aff}_U)_{\mathrm{Ét}}$.*

Proof. As $\mathsf{ntSOD}^\ell_{\mathcal{B}}$ is a subpresheaf of the sheaf $\mathsf{SOD}^\ell_{\mathcal{B}}$, the first sheaf condition is automatically satisfied. The second sheaf condition is also satisfied, as the condition (8.8) is easily seen to be étale local on the base. □

Definition 8.29. Let $\mathsf{ntSOD}^\ell_{\mathcal{B}}$ be the sheaf on the big étale site $(\mathrm{Sch}_U)_{\mathrm{Ét}}$ which corresponds to the sheaf on the big affine étale site denoted by the same symbol in Lemma 8.28.

Theorem 8.30. *The sheaf $\mathsf{ntSOD}^\ell_{\mathcal{B}}$ on $(\mathrm{Sch}_U)_{\mathrm{Ét}}$ defined in Definition 8.29 is an étale algebraic space.*

Proof. The proof is similar to that of Theorem 8.13. We know that it is a sheaf, so it remains to confirm the conditions (2) and (3). But (2) follows from Lemma 8.27, and (3) follows from Lemma 8.20. □

Proposition 8.31. *For each quasicompact and semiseparated U-scheme $\phi\colon V \to U$, there is a natural bijection*

$$\mathsf{ntSOD}^\ell_{\mathcal{B}}(\phi) \simeq \left\{ \langle \mathcal{A}^1_\phi, \dots, \mathcal{A}^\ell_\phi \rangle \in \mathsf{SOD}^\ell_{\mathcal{B}}(\phi) \,\middle|\, \mathcal{A}^i_\phi\big|_x \neq 0 \;\forall i, \forall x \in V \right\}.$$

Proof. Similar to the proof of Proposition 4.7, taking into account that the condition of (8.8) is étale local on V. □

Proposition 8.32. *The natural morphism*

$$\tau\colon \mathsf{ntSOD}^\ell_{\mathcal{B}} \to \mathsf{SOD}^\ell_{\mathcal{B}} \tag{8.9}$$

is an open and closed immersion.

Proof. This is similar to the proof of Proposition 8.18. □

An important corollary of Theorem 8.30 is the following result.

Theorem 8.33. *Suppose that $f\colon \mathcal{X} \to U$ is a smooth projective family of schemes, and let $\mathcal{B} \subseteq \operatorname{Perf} \mathcal{X}$ be a U-linear admissible subcategory. Then the set of points $u \in U$ such that $\mathcal{B}|_u$ admits a non-trivial semiorthogonal decomposition is Zariski open in U.*

Proof. The set of points in the assertion is nothing but the image of the étale morphism $\mathsf{ntSOD}^2_{\mathcal{B}} \to U$, which is open in U, by [62, Tag 042S]. □

In other words, having a semiorthogonal decomposition is a *Zariski open condition* in (smooth projective) families. As explained in the introduction, this result has a rich history in (noncommutative) algebraic geometry and deformation theory of abelian categories, mostly from the point of view of exceptional objects. The theorem extends this to semiorthogonal decompositions with more complicated components, and assumes nothing about the properties of the components.

In the companion paper [5] we will revisit this theorem, and discuss applications to the *indecomposability* of derived categories in families. Let us quote an important example from op. cit.

Example 8.34. Let C be a smooth projective curve of genus g, and let $n = 1, \dots, \lfloor \frac{g+3}{2} \rfloor - 1$. Then $\mathbf{D}^{\mathrm{b}}(\operatorname{Sym}^n C)$ is indecomposable. The starting point for this is the indecomposability results of [38], which are used in [12] to show that for a *generic* curve the derived category of $\operatorname{Sym}^n C$ is indecomposable. One can then extend this to all curves. More details and more examples are discussed in [5].

Remark 8.35. The discussion above can be encoded in the combinatorial properties of the moduli spaces we have constructed, as it turns out that $\mathsf{SOD}^\ell_{\mathcal{B}}$ admits a convenient decomposition into open and closed subspaces. To describe this, consider the set of morphisms

$$\mathcal{I}^\ell = \left\{ \varphi \in \Delta([k],[\ell]) \mid k < \ell, \varphi \text{ is injective} \right\}$$

in the category Δ. We can upgrade Proposition 8.18 from SOD^ℓ_f to $\mathsf{SOD}^\ell_{\mathcal{B}}$ by replacing a morphism $\varphi\colon [k] \to [\ell]$ in $\mathcal{I}^\ell$ by the extension $\varphi'\colon [k+1] \to [\ell+1]$ which sends $k+1$ to $\ell+1$ which encodes the complement of the category $\mathcal{B}$, and using the same proof. We obtain that the morphisms of algebraic spaces $\mathsf{am}_{\mathcal{B}}(\varphi)\colon \mathsf{SOD}^{k+1}_{\mathcal{B}} \to \mathsf{SOD}^{\ell+1}_{\mathcal{B}}$ corresponding to $\varphi \in \mathcal{I}^\ell$ are open and closed immersions.

By composing these with the open and closed immersion $\tau\colon \mathsf{ntSOD}^{\ell}_{\mathcal{B}} \to \mathsf{SOD}^{\ell}_{\mathcal{B}}$ from (8.9) we obtain the following description, by specifying which components in a semiorthogonal decomposition are nonzero.

Proposition 8.36. *There is a decomposition into open and closed algebraic subspaces*

$$\mathsf{SOD}^{\ell}_{\mathcal{B}} = \mathsf{ntSOD}^{\ell}_{\mathcal{B}} \sqcup \left(\bigsqcup_{\varphi \in \mathcal{I}^{\ell}} \mathsf{Im}\left(\mathsf{am}_{\mathcal{B}} \circ \tau(\varphi)\right) \right).$$

8.4. **Group actions.** There are two interesting groups acting on the moduli space SOD^{ℓ}_{f} (and $\mathsf{ntSOD}^{\ell}_{f}$):

(1) the braid group, by mutation of semiorthogonal decompositions;
(2) the auto-equivalence group.

We will now introduce these two group actions, as we expect them to play an important role in the study of the geometry of SOD^{ℓ}_{f} and $\mathsf{ntSOD}^{\ell}_{f}$.

The braid group action for a single semiorthogonal decomposition was already introduced in Section 2.4, and will be upgraded to an action on SOD^{ℓ}_{f} and $\mathsf{ntSOD}^{\ell}_{f}$ now.

Proposition 8.37. *The action of the braid group on the set of semiorthogonal decompositions lifts to an action on* SOD^{ℓ}_{f} *and* $\mathsf{ntSOD}^{\ell}_{f}$*.*

Proof. Recall that the base change of U-linear semiorthogonal decompositions for $\operatorname{Perf}\mathcal{X}$ was defined in Proposition 3.10 as

$$\mathcal{A}^{i}_{\phi} := \langle f_V^* F \otimes \phi_{\mathcal{X}}^* E \mid E \in \mathcal{A}^{i}, F \in \operatorname{Perf} V \rangle$$

where $\phi\colon V \to U$ is a morphism from an affine scheme V. Let us check how base change is compatible with mutation, in the case of a right mutation, the case of a left mutation being similar. For this it suffices to check that

$$\mathbb{R}_i(\mathcal{A}^{\bullet})_{\phi} = \mathbb{R}_i(\mathcal{A}^{\bullet}_{\phi})$$

where $\mathcal{A}^{\bullet}_{\phi}$ denotes the base change along ϕ of the semiorthogonal decomposition $\mathcal{A}^{\bullet}$. But this follows from the equivalence of $\mathcal{A}^{i-1}$ and $\mathbb{R}_i(\mathcal{A}^{\bullet})$, so that both are identified with $\mathcal{A}^{i-1}_{\phi}$.

The action preserves the subspace $\mathsf{ntSOD}^{\ell}_{f}$ as mutation induces an equivalence of the components, as used above. □

An interesting and important question is to understand when this action is transitive. For full exceptional collections, this is a famous question [18] and known for del Pezzo surfaces [40] and the Hirzebruch surface of degree 2 [35]. For semiorthogonal decompositions in general, there are counter-examples [13], [44].

It is also possible to improve the somewhat pathological geometric properties of SOD^{ℓ}, in particular its nonseparatedness, by quotienting out the braid group action, as the following example shows.

Example 8.38. Let f be the morphism as in Example 8.7. Let $\mathsf{Exc}^{4}_{f} \hookrightarrow \mathsf{SOD}^{4}_{f}$ be the open subspace of semiorthogonal decompositions given by full f-exceptional collections. It is shown in [35] that $\mathsf{Exc}^{4}_{f}/\mathrm{Br}_4 \simeq U = \mathbb{A}^1$. Therefore, by taking the quotient, the annoying natures pointed out in Example 8.7 goes away.

For the second group action, let us recall the following definition.

Definition 8.39. An autoequivalence Φ of $\operatorname{Perf}\mathcal{X}$ is said to be f*-linear* if there exists a natural isomorphism

$$\left(f^*\mathcal{F}\otimes-\right)\circ\Phi\simeq\Phi\circ\left(f^*\mathcal{F}\otimes-\right)$$

for all $\mathcal{F}\in\operatorname{Perf}U$. The group of f-linear autoequivalences will be denoted by $\operatorname{Auteq}(f)$.

Remark 8.40. One would like to upgrade this to a sheaf of groups, by considering f_V-linear autoequivalences, for $V\to U$ étale. In [65, Corollary 3.24] this is shown to be a group algebraic space, locally of finite type, when U is affine and assuming f cannot be written as a disjoint union of morphisms. To check condition (1) from op. cit., one uses that Hochschild cohomology of schemes vanishes in negative degrees, by the Hochschild–Kostant–Rosenberg decomposition for smooth morphisms from [21, §0].

In light of Conjecture C it would be interesting to show that over an arbitrary base U, this sheaf of groups is a locally algebraic group space, and that it acts on the moduli space of semiorthogonal decompositions. The vanishing of negative Hochschild cohomology does not hold for smooth and proper dg categories in general, it suffices to consider the gluing of k to k along the dg bimodule $k\oplus k[-n]$.

For now we will restrict ourselves to the action of the global f-linear autoequivalences, and prove the following easy assertion.

Proposition 8.41. *The group* $\operatorname{Auteq}(f)$ *naturally acts on* SOD_f^ℓ*, and preserves the subspace* ntSOD_f^ℓ*.*

Proof. Let Φ be an f-linear autoequivalence, and let

$$\operatorname{Perf}\mathcal{X}=\langle\mathcal{A}^1,\dots,\mathcal{A}^\ell\rangle$$

be a semiorthogonal decomposition, which we will denote $\mathcal{A}^\bullet$. Then we define $\Phi^*\mathcal{A}^\bullet$ in the obvious way as the semiorthogonal decomposition

$$\operatorname{Perf}\mathcal{X}=\langle\Phi(\mathcal{A}^1),\dots,\Phi(\mathcal{A}^\ell)\rangle$$

The image of $\mathcal{A}^i$ under an autoequivalence is again an admissible subcategory by composing the inclusion and projection functors accordingly, and autoequivalences preserve the semiorthogonality. Finally, the subcategories $\Phi(\mathcal{A}^i)$ still generate $\operatorname{Perf}\mathcal{X}$ by considering the decomposition sequence of (2.3) for $\Phi^{-1}(T)$.

The image of a non-zero subcategory under an autoequivalence is again non-zero, hence the action restricts to an action on the subspace. □

Remark 8.42. This group action, together with the fact that SOD_f is étale over U leads to a conceptual proof of the fact that topologically trivial autoequivalences act trivially on semiorthogonal decompositions, as in [38, Corollary 3.15]. We will sketch it here.

Consider a smooth projective variety $g\colon X\to\operatorname{Spec}\mathbf{k}$ where $\mathbf{k}$ is an algebraically closed field. Consider the connected component of the identity $\operatorname{Auteq}^0(g)$ (which, in fact, is known to be isomorphic to $\operatorname{Pic}^0_{X/\mathbf{k}}\times\operatorname{Aut}^0_{X/\mathbf{k}}$), which we will denote by U. Then we can consider the smooth projective family $f=\operatorname{pr}_U\colon X\times U\to U$. There exists a U-linear autoequivalence $\Phi\colon\operatorname{Perf}(X\times U)\to\operatorname{Perf}(X\times U)$ which, over each point of $\operatorname{Auteq}^0(g)$, restricts to the autoequivalence of $\operatorname{Perf}X$ represented by the point. Let us fix a semiorthogonal decomposition $\mathbf{D}^{\mathrm{b}}(X) = \langle \mathcal{A}, \mathcal{B} \rangle$. We wish to show that $\mathcal{A} = \Phi(\mathcal{A}) \subset \operatorname{Perf} X$, so that any topologically trivial autoequivalence of $\operatorname{Perf} X$ preserves $\mathcal{A}$ as a subcategory of $\operatorname{Perf} X$.

The two U-linear semiorthogonal decompositions

$$\operatorname{Perf}(X \times U) = \langle \operatorname{pr}_X^* \mathcal{A}, \operatorname{pr}_X^* \mathcal{B} \rangle = \langle \Phi(\operatorname{pr}_X^* \mathcal{A}), \Phi(\operatorname{pr}_X^* \mathcal{B}) \rangle$$

correspond to two sections $U \rightrightarrows \mathsf{SOD}_f$, and they coincide at the origin of U. Now since $\mathsf{SOD}_f \to U$ is étale, it follows that they coincide on a Zariski open subset $V \subset U$ containing the origin.

Note that we have not yet proved the equality $V = U$, since $\mathsf{SOD}_f \to U$ may not be separated. However, note that V is actually a *subgroup scheme* of U, as it is a stabiliser subgroup of the semiorthogonal decomposition. As V is open dense in U, we can conclude $U = V$.

Let us for good measure show that topologically non-trivial autoequivalences can act non-trivially.

Example 8.43. Consider the case $U = \operatorname{Spec} \mathbf{k}$. To see that tensoring with line bundles can act non-trivially it suffices to consider $\mathbb{P}^1$ and the exceptional collection given by $\langle \mathcal{O}_{\mathbb{P}^1}(i), \mathcal{O}_{\mathbb{P}^1}(i+1) \rangle$. To see that automorphisms can act non-trivially, consider the surface $\mathbb{P}^1 \times \mathbb{P}^1$. Then any involution exchanging the two components, which is topologically non-trivial, sends the exceptional line bundle $\mathcal{O}(1,0)$ to $\mathcal{O}(0,1)$. In particular, such an involution does not preserve the semiorthogonal decomposition induced by $\mathcal{O}(1,0)$.

We will finally show that these two group actions commute. We expect the same result to hold for the action of the algebraic group action suggested in Remark 8.40. In the absolute case this result is given (without proof) as [46, Lemma 2.2(ii)]. For the reader's sake we fill in some of the details in the relative case.

Proposition 8.44. *The actions of* Br_ℓ *and* $\operatorname{Auteq}(f)$ *commute.*

Proof. It suffices to show that the mutation $\mathbb{R}_i(\mathcal{A}^\bullet)$ commutes with autoequivalences, the proof for the left mutation being dual. Moreover, we can reduce the statement and the notation to the right mutation of the f-linear semiorthogonal decomposition $\operatorname{Perf} \mathcal{X} = \langle \mathcal{A}, \mathcal{B} \rangle$, because the other components are not modified in the mutation.

It then suffices to observe that for the right mutation of $\mathcal{A}$ is ${}^\perp\mathcal{B}$, and we have that

$$f({}^\perp\mathcal{B}) = \{ f(T) \mid \operatorname{Hom}_{\operatorname{Perf} \mathcal{X}}(T, \mathcal{B}) = 0 \} = \{ T \mid \operatorname{Hom}_{\operatorname{Perf} \mathcal{X}}(f^{-1}(T), \mathcal{B}) = 0 \}$$

agrees with

$${}^\perp f(\mathcal{B}) = \{ T \mid \operatorname{Hom}_{\operatorname{Perf} \mathcal{X}}(T, f(\mathcal{B}) = 0 \}$$

and we are done. □

9. Example: families of cubic surfaces

In this final section we briefly discuss the precise relationship between the moduli spaces SOD_f^ℓ associated to a family of cubic surfaces and the moduli of lines, the motivating example from the introduction.

Let $f : \mathcal{X} \to U$ be a versal family of cubic surfaces, with U connected. One can associate to f the following étale morphisms over U:

(1) The moduli space of non-trivial semiorthogonal decompositions $\mathsf{ntSOD}^2_f \to U$ of length 2.
(2) The moduli space of (-1)-curves (or relative Fano scheme of lines) in the fibers as in (1.1), which will be denoted by $t\colon \mathcal{F} \to U$. It is a finite étale morphism of degree 27.

Let us explain how these two spaces are related. The base change

$$t^* f\colon \mathcal{X}_t := \mathcal{X} \times_{f,U,t} \mathcal{F} \to \mathcal{F}$$

admits a tautological closed immersion $e\colon \mathcal{L} \hookrightarrow \mathcal{X}_t$ over $\mathcal{F}$ by the universal line $\mathcal{L}$, such that for each $x \in \mathcal{F}$, the image over x is the line in $(\mathcal{X}_t)_x = \mathcal{X}_{t(x)}$ corresponding to the point x.

Firstly, we can construct two open immersions

$$\mathcal{F} \hookrightarrow \mathsf{ntSOD}^2_f \tag{9.1}$$

of algebraic spaces over U. To do this, we consider the f-linear semiorthogonal decomposition induced by the t^*f-exceptional object $\mathcal{O}_{e(\mathcal{L})}$ and its complement. The choice between the left and right orthogonal gives two different open immersions. They are related by the action of the braid group Br_2.

Next, let $\tilde{t}\colon \tilde{\mathcal{F}} \to U$ be the Galois closure of t. It is well known that the Galois group of $\tilde{t}$ is the Weyl group $\mathrm{W}(\mathrm{E}_6)$ of order 51840. See [28, page 716] for a proof, and the book [36] by Camille Jordan for the first treatment of the topic. The base change

$$\tilde{t}^* f\colon \mathcal{X}_{\tilde{t}} := \mathcal{X} \times_{f,U,\tilde{t}} \tilde{\mathcal{F}} \to \tilde{\mathcal{F}}$$

admits 6 closed immersions $e_1, \ldots, e_6\colon \mathcal{L} \hookrightarrow \mathcal{X}_{\tilde{t}}$ over $\tilde{\mathcal{F}}$ such that for each $x \in \tilde{\mathcal{F}}$, the images over x are mutually disjoint (-1)-curves of the fiber $(\mathcal{X}_{\tilde{t}})_x = \mathcal{X}_{\tilde{t}(x)}$. Now let

$$\iota\colon \tilde{\mathcal{F}} \hookrightarrow \mathsf{ntSOD}^9_f \tag{9.2}$$

be the open immersion of algebraic spaces over U corresponding to the $\tilde{t}^*f$-exceptional collection obtained by the f-exceptional collection

$$\mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}}(H), \mathcal{O}_{\mathcal{X}}(2H), \mathcal{O}_{e_1(\mathcal{L})}, \ldots, \mathcal{O}_{e_6(\mathcal{L})}$$

obtained from Orlov's blowup formula [55, Theorem 4.3].

Let $W \le \mathrm{Br}_9$ be the subgroup of elements which preserves the connected component $\iota(\tilde{\mathcal{F}})$. It acts as covering transformations, so there exists a natural homomorphism

$$W \to \mathrm{W}(\mathrm{E}_6). \tag{9.3}$$

Note that it has a big kernel, since no element of W has finite order.

Proposition 9.1. *The natural morphism* (9.3) *is surjective.*

Proof. It is enough to show that the action of W on a fiber of $\tilde{t}$ is transitive. For this purpose, fix a point $u \in U$ and let S be the corresponding cubic surface. Then the points of the fiber $\tilde{t}^{-1}(u)$ bijectively correspond to markings on S. Let $\ell_1, \ldots, \ell_6$ and $\ell'_1, \ldots, \ell'_6$ be two sequences of six disjoint lines on S. By [40], the action of Br_9 on the set of full exceptional collections of $\mathbf{D}^{\mathrm{b}}(S)$ is transitive (up to shifts). Hence there exists $b \in \mathrm{Br}_9$ such that

$$b\left(\mathcal{O}_S, \mathcal{O}_S(H), \mathcal{O}_S(2H), \mathcal{O}_{\ell_1}, \ldots, \mathcal{O}_{\ell_6}\right) = \left(\mathcal{O}_S, \mathcal{O}_S(H'), \mathcal{O}_S(2H'), \mathcal{O}_{\ell'_1}, \ldots, \mathcal{O}_{\ell'_6}\right),$$

where H and H' are the pull-backs of the classes of lines of $\mathbb{P}^2$ obtained by contracting the sets of 6 lines $\ell_1,\dots,\ell_6$ and $\ell'_1,\dots,\ell'_6$, respectively. As the braid group action exchanges the connected components of ntSOD^9_f, it follows that $b(\iota(\tilde{\mathcal{F}}))=\iota(\tilde{\mathcal{F}})$, i.e., $b\in W$. $\square$

PROBLEM 9.2. Identify the group W and the kernel of (9.3).

APPENDIX A. DEFORMATION THEORY FOR A MORPHISM IN THE DERIVED CATEGORY WITH FIXED LIFT OF THE CODOMAIN

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In this appendix we prove a deformation-theoretic result of independent interest which is a key ingredient in the deformation theory and moduli theory of semiorthogonal decompositions.

It is a generalisation of the deformation-obstruction theory for objects and morphisms in deformations of abelian categories from [51]. For this reason we will use the notation and terminology from op. cit. In particular, we fix a sequence of surjections of noetherian rings

$$\overline{R} \twoheadrightarrow R \twoheadrightarrow R_0 \tag{A.1}$$

where we denote[4] $\ker(\overline{R} \twoheadrightarrow R_0) = \mathfrak{a}$ and $\ker(\overline{R} \to R) = \mathfrak{b}$. We moreover assume that $\overline{R} \twoheadrightarrow R$ is a *small extension* relative to $R \twoheadrightarrow R_0$, i.e. that $\mathfrak{a}\mathfrak{b} = 0$. This implies that $\mathfrak{b}^2 = 0$ and that $\mathfrak{b}$ has the structure of an R_0-module, which will be useful later on.

Setup We consider an R_0-linear abelian category $\mathcal{C}_0$, which is assumed to be flat in the sense of [52, Definition 3.2]. Depending on the situation we are working in, we will often moreover assume it is either Grothendieck or co-Grothendieck.

Then we consider a flat abelian deformation $\mathcal{C}$ of $\mathcal{C}_0$ over R, and a further flat abelian deformation $\overline{\mathcal{C}}$ over $\overline{R}$. The functors $\mathcal{C}_0 \to \mathcal{C}$ and $\mathcal{C} \to \overline{\mathcal{C}}$ have *left* and *right adjoint restriction functors* given by the tensor product (resp. Hom functor). We can summarise the situation in the following diagram

$$\begin{tikzcd} \overline{R}: & \overline{\mathcal{C}} \arrow[d, bend right=60, "R\otimes_{\overline{R}}-"'] \arrow[d, bend left=60, "\mathrm{Hom}_{\overline{R}}(R,-)"] \\ R: & \mathcal{C} \arrow[u] \arrow[d, bend right=60, "R_0\otimes_R-"'] \arrow[d, bend left=60, "\mathrm{Hom}_R(R_0,-)"] \\ R_0: & \mathcal{C}_0 \arrow[u] \end{tikzcd} \tag{A.2}$$

The following prototypical example will be of our interest.

Example A.1. Let $\overline{X} \to \operatorname{Spec}\overline{R}$ be a flat morphism of quasicompact and separated schemes, and $X \to \operatorname{Spec} R$, $X_0 \to \operatorname{Spec} R_0$ be its base changes. Then

$$\mathcal{C}_0 \coloneqq \operatorname{Qcoh} X_0 \hookrightarrow \mathcal{C} \coloneqq \operatorname{Qcoh} X \hookrightarrow \overline{\mathcal{C}} \coloneqq \operatorname{Qcoh} \overline{X} \tag{A.3}$$

are flat abelian categories linear over $R_0, R, \overline{R}$, respectively, where the inclusion functors are given by pushforwards of quasicoherent sheaves. The *left* restriction functors $R \otimes_{\overline{R}} -, R_0 \otimes_R -$ in this context are nothing but the pull-backs of quasicoherent sheaves along the inclusions.

Remark A.2. As one sees in Example A.1, the restriction functors which appear naturally in algebraic geometry are the *left* ones, rather than the right ones. The left restriction functors preserve *projective objects*, rather than injective objects. As Qcoh X typically

[4]These are I and J in [51]. We use different symbols so that the reader should not mix them up with injective objects.

does not have enough projectives, one has to work in its Pro-completion $\mathrm{Pro}(\mathrm{Qcoh}\, X)$ to work with the left restriction functors. This turns out to be a co-Grothendieck category, the dual notion of the Grothendieck category, which is more familiar in algebraic geometry. In particular it has enough projectives.

If one is interested in the right restriction functor, then $\mathrm{Qcoh}\, X$ is a Grothendieck category and hence one does not need to take an Ind-completion.

The paper [51] mainly discusses the right restriction functors and hence works with injective objects. By duality, the results of [51] simultaneously imply the dual statements for the left restrictions.

Similar to the approach in [51] we will first prove results for homotopy categories of linear categories and their deformations, and then show how we can restrict to appropriate subcategories which decribe the derived categories of our interest. The first theorem we will prove is the following deformation-obstruction result. It is phrased for the setting of Grothendieck categories and *restriction along the right adjoint*, which is also the setting used in op. cit (but it is *dual* to the setting we are interested in, as we explained in Remark A.2).

The assumption that $\mathcal{C}_0$ is Grothendieck implies that $\mathcal{C}_0$ has enough injectives, and because the restriction functor $\mathrm{Hom}_R(R_0,-)$ is a right adjoint, injective objects are automatically preserved.

In what follows, the notation $\mathbf{K}(-)$ denotes the homotopy category of complexes of the additive category $-$ and also the functor induced between the homotopy categories.

Theorem A.3. *Let $\mathcal{C}_0$ be a Grothendieck abelian category.*

Consider an exact triangle

$$F_0 \xrightarrow{s_0} G_0 \to H_0 \to F_0[1] \tag{A.4}$$

in $\mathbf{K}(\mathrm{Inj}\,\mathcal{C}_0)$, and let

$$F \xrightarrow{s} G \tag{A.5}$$

be a morphism in $\mathbf{K}(\mathrm{Inj}\,\mathcal{C})$ which restricts to s_0, i.e. $\mathbf{K}(\mathrm{Hom}_R(R_0,s)) = s_0$.

Let $\overline{G}$ be a lift of G in $\mathbf{K}(\mathrm{Inj}\,\mathcal{C})$ to $\mathbf{K}(\mathrm{Inj}\,\overline{\mathcal{C}})$ along $\mathbf{K}(\mathrm{Hom}_{\overline{R}}(R,-))$. Then

(1) there exists an obstruction class

$$\mathfrak{o}(s) \in \mathrm{H}^1(\mathbf{R}\mathrm{Hom}_{\mathcal{C}_0}(\mathbf{R}\mathrm{Hom}_{R_0}(\mathfrak{b},F_0),H_0)) \tag{A.6}$$

such that $\mathfrak{o}(s)=0$ if and only if s lifts to a morphism $\overline{s}\colon \overline{F}\to\overline{G}$ in $\mathbf{K}(\mathrm{Inj}\,\overline{\mathcal{C}})$;

(2) assume that $\mathrm{H}^{-1}(\mathbf{R}\mathrm{Hom}_{\mathcal{C}_0}(\mathbf{R}\mathrm{Hom}_{R_0}(\mathfrak{b},F_0),H_0)) = 0$, if $\mathfrak{o}(s)=0$ then the set of isomorphism classes of such lifts is a torsor under

$$\mathrm{H}^0(\mathbf{R}\mathrm{Hom}_{\mathcal{C}_0}(\mathbf{R}\mathrm{Hom}_{R_0}(\mathfrak{b},F_0),H_0)). \tag{A.7}$$

We will explicitly prove this version (rather than its dual, which we will need for our application) to stay as close as possible to the exposition of [51]. Note that similar to op. cit. it is possible to deduce from this statement various other statements for subcategories, but we will refrain from doing so as systematically as in op. cit.

The H^{-1}-vanishing is a shadow of a derived deformation theory involving higher structures, and is also present in [51]. We refrain from developing it here because we don't need it for our application (and likely our tools are not quite adequate to deal with it).

Remark A.4. This result recovers the deformation theory of objects from [51, Theorem B] (in the version for the homotopy category of injectives), by considering the triangle

$$C_0 \to 0 \to C_0[1] \xrightarrow[\mathrm{id}_{C_0[1]}]{+1} C_0[1] \tag{A.8}$$

and where the fixed lift of the zero object is the zero object.

For our purposes we will need the following dual result, using restriction along the *left adjoint*, i.e. along the tensor product. This is the restriction functor which is used in algebraic geometry. The assumption that $\mathcal{C}_0$ is co-Grothendieck implies that $\mathcal{C}_0$ has enough projectives, and because the restriction functor $R_0 \otimes_R -$ is a left adjoint, these are automatically preserved.

Theorem A.5 (Dual version)**.** *Let $\mathcal{C}_0$ be a co-Grothendieck abelian category.*

Consider an exact triangle

$$F_0 \xrightarrow{s_0} G_0 \to H_0 \to F_0[1] \tag{A.9}$$

in $\mathbf{K}(\operatorname{Proj}\mathcal{C}_0)$, and let

$$F \xrightarrow{s} G \tag{A.10}$$

be a morphism in $\mathbf{K}(\operatorname{Proj}\mathcal{C})$ which restricts to s_0, i.e. $\mathbf{K}(R_0 \otimes_R s) = s_0$.

Let $\overline{G}$ be a lift of G in $\mathbf{K}(\operatorname{Proj}\mathcal{C})$ to $\mathbf{K}(\operatorname{Proj}\overline{\mathcal{C}})$ along $\mathbf{K}(R \otimes_{\overline{R}} -)$. Then

(1) there exists an obstruction class

$$\mathfrak{o}(s) \in \mathrm{H}^1(\mathbf{R}\mathrm{Hom}_{\mathcal{C}_0}(F_0, \mathfrak{b} \otimes^{\mathbf{L}}_{R_0} H_0)) \tag{A.11}$$

such that $\mathfrak{o}(s) = 0$ if and only if s lifts to a morphism $\overline{s} \colon \overline{F} \to \overline{G}$ in $\mathbf{K}(\operatorname{Proj}\overline{\mathcal{C}})$;

(2) assume that $\mathrm{H}^{-1}(\mathbf{R}\mathrm{Hom}_{\mathcal{C}_0}(F_0, \mathfrak{b} \otimes^{\mathbf{L}}_{R_0} H_0)) = 0$. Then, if $\mathfrak{o}(s) = 0$ the set of isomorphism classes of such lifts is a torsor under

$$\mathrm{H}^0(\mathbf{R}\mathrm{Hom}_{\mathcal{C}_0}(F_0, \mathfrak{b} \otimes^{\mathbf{L}}_{R_0} H_0)). \tag{A.12}$$

Recall that a co-Grothendieck category is an abelian category which satisfies axiom (AB5*), i.e. it has all products (and limits) and filtered limits of exact sequences are exact, and it moreover has a cogenerator. This notion is dual to the notion of a Grothendieck category, and by taking the opposite of a Grothendieck category we get a co-Grothendieck category (and vice versa).

The condition that $\mathcal{C}_0$ is co-Grothendieck is very restrictive, and such categories do not arise often in algebra or geometry. But to set up the theory one needs enough projectives (or flats) when restricting along the tensor product. For our application we will therefore need to appeal to the comparison machinery of [51, §6] which explains how lift groupoids for subcategories are related to lift groupoids for the ambient categories.

We can always construct a co-Grothendieck category from an (essentially small) abelian category by taking the category of Pro-objects (similar to how we obtain a Grothendieck category from an abelian category by taking its Ind-completion), and after having done so we need to show how the deformation problems restrict to various smaller settings. We will apply the comparison results for sets of lifts as introduced in §A.3 to

(1) go from $\mathbf{K}(\operatorname{Proj}\operatorname{Pro}\mathcal{C})$ to $\mathbf{D}^{-}(\operatorname{Pro}\mathcal{C})$.

Then the next (and final) step in the geometric setting is to

(2) go to perfect complexes.

Remark A.6. As in [51] we will tacitly use universes, so that by enlarging the universe we can assume that all our categories are small. See [52, §2.1] for more details in the context of deformation theory of abelian categories. We will use the same setup.

A.1. **Deformation theory for morphisms in the homotopy category of injectives.** In this subsection we will prove Theorem A.3. We will need a convenient way of representing the morphism $s\colon F \to G$. We do this by using the "coderived model structure" on the category of cochain complexes $\operatorname{Ch}(\mathcal{C})$, as introduced in [7], whose homotopy category is $\mathbf{K}(\operatorname{Inj}\mathcal{C})$. The existence of this model structure is given in [63, Proposition 6.9] under a locally finitely presentability hypothesis, which can be removed [50, Proposition 1.18].

The upshot is that we can describe the homotopy category of injectives $\mathbf{K}(\operatorname{Inj}\mathcal{C})$ of a Grothendieck category $\mathcal{C}$ as a cofibrantly generated abelian model structure. All objects are cofibrant, the fibrant objects are the graded-injective complexes and weakly trivial objects are left orthogonal to the fibrant objects (or so-called coacyclic), so that

(1) cofibrations are degree-wise monomorphisms;
(2) fibrations are degree-wise epimorphisms with graded-injective kernel;
(3) weak equivalences are in particular homotopy equivalences (as the class of coacyclic complexes is contained in the class of acyclic complexes).

The following lemma then follows from standard model category techniques by taking the appropriate cofibrant replacements and factorisations.

Lemma A.7. *Let $\mathcal{C}$ be a Grothendieck category, and $s\colon F \to G$ a morphism in $\mathbf{K}(\operatorname{Inj}\mathcal{C})$. Then there exists a commutative diagram*

$$\begin{tikzcd} F \arrow[r, "s"] \arrow[d] \arrow[dr] & G \\ F' \arrow[ddr, "c"'] & G_1 \arrow[u] \arrow[d] \\ & G' \end{tikzcd} \tag{A.13}$$

where

(1) the vertical arrows are homotopy equivalences;
(2) F' and G' are complexes of injectives;
(3) the map c is termwise injective.

Hence we can write our morphism $s\colon F \to G$ as

$$s\colon J^{\bullet} \hookrightarrow I^{\bullet}, \tag{A.14}$$

with s termwise injective, and $J^{\bullet}$ (resp. $I^{\bullet}$) complexes of injectives.

We will repeatedly use the following lemma for lifting injective objects along the functor $\operatorname{Hom}_{\overline{R}}(R,-)$.

Lemma A.8. *Any object $I \in \operatorname{Inj}\mathcal{C}$ admits an object $\overline{I} \in \operatorname{Inj}\overline{\mathcal{C}}$ unique up to isomorphisms such that $\operatorname{Hom}_{\overline{R}}(R, \overline{I}) \simeq I$.*

Proof. As $\overline{\mathcal{C}}$ is flat over $\overline{R}$, [52, Proposition 3.4] implies that any injective object of $\overline{\mathcal{C}}$ is coflat (see, say, [52, Definition 2.5] for the definition of coflat objects). Hence it admits a lift to a coflat object of $\overline{\mathcal{C}}$, which is unique up to isomorphisms by the deformation theory [51, Theorem 6.11] and the injectivity of I.

Now it remains to show that there is at least one lift $\overline{I} \in \operatorname{Inj}(\overline{\mathcal{C}})$ of I, which has to be the coflat lift. This is shown in the proof of [51, Proposition 5.5]. □

Lemma A.9. *Suppose that there exists an object $\overline{G} \in \mathbf{K}(\operatorname{Inj}(\overline{\mathcal{C}}))$ such that*

$$\mathbf{K}\big(\operatorname{Hom}_{\overline{R}}(R, -)\big)(\overline{G}) \simeq J^\bullet \in \mathbf{K}(\operatorname{Inj}(\mathcal{C})).$$

Then there exists a complex $\overline{J}^\bullet$ of objects in $\operatorname{Inj}(\overline{\mathcal{C}})$ such that $\overline{G} \simeq \overline{J}^\bullet \in \mathbf{K}(\operatorname{Inj}(\overline{\mathcal{C}}))$ and for each $i \in \mathbb{Z}$ one has $\operatorname{Hom}_{\overline{R}}(R, \overline{J}^i) \simeq J^i \in \operatorname{Inj}(\mathcal{C})$.

Proof. For each $i \in \mathbb{Z}$, let $\overline{J}^i \in \operatorname{Inj}(\overline{\mathcal{C}})$ be the lift of J^i whose existence is shown in Lemma A.8. Now the assertion follows from the crude lifting lemma [51, Proposition 4.3]. □

From now on, we will fix a lift $\overline{I}^\bullet$ of $I^\bullet$ as in Lemma A.9 once for all. Also, for each $i \in \mathbb{Z}$ consider the lift $\overline{J}^i$ of J^i. Using the injectivity of I^i, it follows from the deformation theory [51, Theorem 6.12] that $s^i \colon J^i \hookrightarrow I^i$ admits *at least one lift*[5] $\overline{s}^i \colon \overline{J}^i \to \overline{I}^i$. Note that $\overline{s}^i$ is split injective. In fact, the injectivity of I^i implies that s^i admits a left inverse $t^i \colon I^i \to J^i$. By the same deformation theory as before, it admits a lift $\overline{t}^i \colon \overline{I}^i \to \overline{J}^i$ and we know that $n := \overline{t}^i \circ \overline{s}^i - \mathrm{id}_{\overline{J}^i} \in \operatorname{End}_{\overline{\mathcal{C}}}(\overline{J}^i)$ is actually coming from its subspace $\operatorname{Hom}_{\overline{\mathcal{C}}}\big(\operatorname{Hom}_{\overline{R}}(\mathfrak{b}, \overline{J}^i), \overline{J}^i\big)$.

As explained in [51, Proposition 5.4(i)] we have that $\ker(\operatorname{Hom}_{\overline{R}}(R, -))^2$ from the standing assumption on the ring extensions which implies that $\mathfrak{b}^2 = 0$, using the notion introduced in [51, §3.2]. [6]

We have that $n \in \ker(\operatorname{Hom}_{\overline{R}}(R, -))$, because $\operatorname{Hom}_{\overline{R}}(R, n) = t^i \circ s^i - \mathrm{id}_{J^i} = 0$. Hence $n \circ n = 0$ and $(\mathrm{id}_{\overline{J}^i} - n) \circ \overline{t}^i$ is a left inverse to $\overline{s}^i$. Now we fix for each $i \in \mathbb{Z}$ such a lift of s^i and name it $\overline{s}^i_0$. We also fix a left inverse $\overline{t}^i_0$ to $\overline{s}^i_0$. This will give us, for each $i \in \mathbb{Z}$, the direct sum decomposition

$$\overline{I}^i = \overline{J}^i \oplus \overline{K}^i, \tag{A.15}$$

where $\overline{J}^i = \overline{s}^i_0(\overline{J}^i)$ and $\overline{K}^i = \ker(\overline{t}^i_0)$. This brings us roughly in a situation like that of [32, §2.A.7], and we will combine the ideas from op. cit. with the tools from [51] to prove the result.

We consider the different (graded) lifts $\overline{s}$ of s up to the action of the infinitesimal automorphisms of $\overline{J}^\bullet$. These allow us to normalise the components of $\overline{s}$ such that they

[5]In fact, the zeroth cohomology in [51, Theorem 6.12(2)] is nonzero as soon as J^i is nonzero. Hence the lift is not unique in general.

[6]For convenience of the reader, we add a few words to the proof of [51, Proposition 5.4(i)], using the notation of the paper. The decomposition of the morphism f in the 3rd line of the proof follows from the assumption $J^2 = 0$, as this implies that $J\overline{C} \in \mathscr{C}$ and hence any map $J\overline{C} \to \overline{D}$ factors through the adjunction counit map $\iota \operatorname{Hom}_{\overline{R}}(R, \overline{D}) \to \overline{D}$ for the inclusion functor $\iota \colon \mathcal{C} \to \overline{\mathcal{C}}$. For $f \colon \overline{C} \to \overline{D}, g \colon \overline{D} \to \overline{E} \in \ker(\operatorname{Hom}_{\overline{R}}(R, -))$, by describing both of the morphisms as indicated, we see that the composition $g \circ f$ factors through the composition $\operatorname{Hom}_{\overline{R}}(R, \overline{D}) \to \overline{D} \to J\overline{D}$, which is independent of f, g and obviously 0.

are of the form

$$\overline{s}^i = \mathrm{id}_{\overline{J}^i} \oplus \beta^i_{\overline{s}}$$

for some

$$\beta^i_{\overline{s}} \colon \mathrm{Hom}_{\overline{R}}(\mathfrak{b}, \overline{J}^i) \to \overline{K}^i$$

under the decomposition (A.15). Via this normalisation, infinitesimal automorphisms between lifts are forced to be the identity on the component $\overline{J}^\bullet$.

By combining the morphisms $\overline{s}^i$ with the embeddings $\overline{K}^i \hookrightarrow \overline{I}^i$ we can identify $\overline{s}$ with an automorphism $b_{\overline{s}}$ of $\overline{I}^\bullet$ whose components $b^i_{\overline{s}}$ are of the form

$$b^i_{\overline{s}} = \begin{pmatrix} \mathrm{id}_{\overline{J}^i} & 0 \\ \beta^i_{\overline{s}} & \mathrm{id}_{\overline{K}^i} \end{pmatrix} \tag{A.16}$$

The differential $d_{\overline{I}^\bullet}$ can be decomposed using (A.15) into $\left(\begin{smallmatrix} d_{1,1} & d_{1,2} \\ d_{2,1} & d_{2,2} \end{smallmatrix}\right)$, and using $b_{\overline{s}}$ it can be conjugated into $\tilde{d}_{\overline{s}} = b^{-1}_{\overline{s}} \circ d_{\overline{I}^\bullet} \circ b_{\overline{s}}$ whose entry in position $(2,1)$ under the decomposition (A.15) we will denote by $\varphi_{\overline{s}}$. More explicitly, we have that

$$\varphi_{\overline{s}} = -\beta_{\overline{s}} \circ d_{1,1} + d_{2,1} + d_{2,2} \circ \beta_{\overline{s}}, \tag{A.17}$$

as $\beta_{\overline{s}} \circ d_{1,2} \circ \beta_{\overline{s}} = 0$, because $\beta_{\overline{s}} \in \ker(\mathrm{Hom}_{\overline{R}}(R,-))$ and we have that $\ker(\mathrm{Hom}_{\overline{R}}(R,-))^2 = 0$ as before. This discussion proves the following lemma.

Lemma A.10. *The map $\overline{s}$ embeds $\overline{J}^\bullet$ into $\overline{I}^\bullet$ as a subcomplex if and only if $\varphi_{\overline{s}} = 0$.*

Our next step is to analyse the differential $\tilde{d}_{\overline{s}}$ as an endomorphism of degree 1 of $\overline{I}^\bullet$. For this we consider the short exact sequence of complexes of vector spaces

$$0 \to \mathrm{Hom}^\bullet_-(\mathrm{Hom}_{\overline{R}}(\mathfrak{b}, \overline{I}^\bullet), \overline{I}^\bullet) \to \mathrm{Hom}^\bullet(\mathrm{Hom}_{\overline{R}}(\mathfrak{b}, \overline{I}^\bullet), \overline{I}^\bullet) \to \mathrm{Hom}^\bullet_+(\mathrm{Hom}_{\overline{R}}(\mathfrak{b}, \overline{I}^\bullet), \overline{I}^\bullet) \to 0 \tag{A.18}$$

where we define $\mathrm{Hom}^\bullet_-(\mathrm{Hom}_{\overline{R}}(\mathfrak{b}, \overline{I}^\bullet), \overline{I}^\bullet)$ as the subcomplex of morphisms $\overline{I}^i \to \overline{I}^{i+j}$ of degree j which are zero on $\overline{J}^i$ after reduction to R.

By the construction we have the following lemma.

Lemma A.11. *The differential $\tilde{d}_{\overline{s}}$ defines a 1-cocycle $\left(\begin{smallmatrix} 0 & 0 \\ \varphi & 0 \end{smallmatrix}\right)$ of $\mathrm{Hom}^\bullet_+(\mathrm{Hom}_{\overline{R}}(\mathfrak{b}, \overline{I}^\bullet), \overline{I}^\bullet)$.*

We are now in the position to define the obstruction class.

Definition A.12. The *obstruction class* for $s \colon F \to G$ is

$$\mathfrak{o}(s) := [\tilde{d}_{\overline{s}}] \in \mathrm{H}^1(\mathrm{Hom}^\bullet_+(\mathrm{Hom}_{\overline{R}}(\mathfrak{b}, \overline{I}^\bullet), \overline{I}^\bullet)). \tag{A.19}$$

In the definition of the obstruction class there were choices involved, but by the proof of Proposition A.14 these do not influence the cohomology class.

We now check that it really serves the role of an obstruction class.

Proposition A.13. *The class $\mathfrak{o}(s)$ is the obstruction to the lifting of s as a morphism in the homotopy category of injectives. Namely, $\mathfrak{o}(s) = 0$ if and only if there exists a morphism of complexes $\overline{s} \colon \overline{J}^\bullet \to \overline{I}^\bullet$ such that $\mathbf{K}(\mathrm{Hom}_{\overline{R}}(R,-))(\overline{s}) = s$.*

Proof. Assume that a lift $\overline{s}$ (as a morphism of cochain complexes) exists. By Lemma A.10 we have that $\varphi_{\overline{s}} = 0$, as then $\overline{J}^\bullet$ is a subcomplex of $\overline{I}^\bullet$ via $\overline{s}$.

Conversely, if $\mathfrak{o}(s)=0$, then there exists a morphism

$$\psi = \begin{pmatrix} 0 & 0 \\ \psi_{2,1} & 0 \end{pmatrix} \tag{A.20}$$

such that

$$[\tilde{d_{\overline{s}}}] = \delta([\psi]), \tag{A.21}$$

where δ is the differential in the complex $\mathrm{Hom}^\bullet_+(\mathrm{Hom}_{\overline{R}}(\mathfrak{b}, \overline{I}^\bullet), \overline{I}^\bullet)$. This condition can be rephrased as

$$\varphi = d_{2,2} \circ \psi_{2,1} - \psi_{2,1} \circ d_{1,1}. \tag{A.22}$$

Then we claim that $\overline{J}^\bullet$ becomes a subcomplex of $\overline{I}^\bullet$ via

$$\overline{s} := \mathrm{id}_{\overline{J}^\bullet} \oplus (\beta - \psi_{2,1}). \tag{A.23}$$

To prove this claim, it suffices by Lemma A.10 to compute the component $\varphi_{\overline{s}}$ of $\tilde{d}$ associated to $\beta - \psi_{2,1}$. Starting from the description in (A.17) we can regroup and apply (A.22) to see that

$$\begin{aligned} &-(\beta - \psi_{2,1}) \circ d_{1,1} + d_{2,1} + d_{2,2} \circ (\beta - \psi_{2,1}) \\ &= (-\beta \circ d_{1,1} + d_{2,1} + d_{2,2} \circ \beta) - (d_{2,2} \circ \psi_{2,1} - \psi_{2,1} \circ d_{1,1}) \\ &= 0. \end{aligned} \tag{A.24}$$

This completes the proof. $\square$

This doesn't quite prove part (1) of Theorem A.3 yet, as the obstruction class lives in an a priori different cohomology space. This will be remedied in Proposition A.15. Before doing so, we first prove how the set of lifts is a torsor.

Proposition A.14. *Assume that* $\mathrm{H}^{-1}(\mathbf{R}\mathrm{Hom}_{C_0}(\mathbf{R}\mathrm{Hom}_{R_0}(\mathfrak{b}, F_0), H_0)) = 0$*. If* $\mathfrak{o}(s)=0$*, then the set of isomorphism classes of lifts is a torsor under* $\mathrm{H}^0(\mathrm{Hom}^\bullet_+(\mathrm{Hom}_{\overline{R}}(\mathfrak{b}, \overline{I}^\bullet), \overline{I}^\bullet)$*, where* H^0 *denotes the* 0*-th cohomology of the complex.*

Proof. Let $\beta := \beta_{\overline{s}}$ and $\beta' := \beta_{\overline{s}'}$ be associated to different lifts $\overline{s}$ and $\overline{s}'$ of s. By Lemma A.10 this means that the associated $\varphi_{\overline{s}}$ and $\varphi_{\overline{s}'}$ are zero, i.e.

$$-\beta \circ d_{1,1} + d_{2,1} + d_{2,2} \circ \beta = 0 \tag{A.25}$$

and

$$-\beta' \circ d_{1,1} + d_{2,1} + d_{2,2} \circ \beta' = 0 \tag{A.26}$$

But then $\beta - \beta'$ is a 0-cocycle in $\mathrm{Hom}^\bullet_+(\mathrm{Hom}_{\overline{R}}(\mathfrak{b}, \overline{I}^\bullet), \overline{I}^\bullet)$, which is analogous to what happened in (A.22).

Conversely, if $\beta := \beta_{\overline{s}}$ is associated to a lift $\overline{s}$ of s, and ξ is a 0-cocycle in $\mathrm{Hom}_+(\mathrm{Hom}_{\overline{R}}(\mathfrak{b}, \overline{I}^\bullet), \overline{I}^\bullet)$, then $\beta + \xi$ defines another lift of s, because

$$\begin{aligned} &-(\beta + \xi) \circ d_{1,1} + d_{2,1} + d_{2,2} \circ (\beta + \xi) \\ &= (-\beta \circ d_{1,1} + d_{2,1} + d_{2,2} \circ \beta) + \delta(\xi) \\ &= 0 \end{aligned} \tag{A.27}$$

Finally, note that β and β' define the same morphism from $\overline{J}^\bullet$ to $\overline{I}^\bullet$ if and only if $b_\beta = b_{\beta'}$ (recall (A.16)) as endomorphisms of $\overline{I}^\bullet$. But $b_\beta - b_{\beta'}$ is a 0-coboundary of $\operatorname{Hom}^\bullet(\overline{I}^\bullet, \overline{I}^\bullet)$ if and only $b_\beta - b_{\beta'}$ is a 0-coboundary of $\operatorname{Hom}^\bullet_+(\operatorname{Hom}_{\overline{R}}(\mathfrak{b}, \overline{I}^\bullet), \overline{I}^\bullet)$. But this is the case if and only if there exists $\eta \in \operatorname{Hom}^{-1}(\overline{J}^\bullet, \overline{K}^\bullet)$ such that $\beta - \beta' = d_{2,2} \circ \eta - \eta \circ d_{1,1}$. ☐

Finally we prove the following identification, finishing the proof of Theorem A.3.

Proposition A.15. *There exist isomorphisms of complexes*

$$\operatorname{Hom}^\bullet_+(\operatorname{Hom}_{\overline{R}}(\mathfrak{b}, \overline{I}^\bullet), \overline{I}^\bullet) \simeq \operatorname{Hom}^\bullet_+(\operatorname{Hom}_{R_0}(\mathfrak{b}, I_0^\bullet), I_0^\bullet) \simeq \operatorname{Hom}^\bullet(\operatorname{Hom}_{R_0}(\mathfrak{b}, K_0^\bullet), J_0^\bullet) \tag{A.28}$$

where as before $K_0^\bullet := I_0^\bullet / J_0^\bullet$.

Proof. The first isomorphism follows from the fact that $\overline{R} \to R$ is small relative to $R \to R_0$, i.e. that $\mathfrak{a}\mathfrak{b} = 0$. We will denote the image of an element $(s^i)_i$ by $(t^i)_i$.

Now define

$$\mu^j \colon \operatorname{Hom}^j_+(\operatorname{Hom}_{R_0}(\mathfrak{b}, I_0^\bullet), I_0^\bullet) = \frac{\operatorname{Hom}^j(\operatorname{Hom}_{R_0}(\mathfrak{b}, I_0^\bullet), I_0^\bullet)}{\operatorname{Hom}^j_-(\operatorname{Hom}_{R_0}(\mathfrak{b}, I_0^\bullet), I_0^\bullet)} \to \operatorname{Hom}^j(\operatorname{Hom}_{R_0}(\mathfrak{b}, K_0^\bullet), J_0^\bullet)$$

by sending t^j to the morphism u^j defined by the composition

$$\begin{tikzcd}
\operatorname{Hom}_{R_0}(\mathfrak{b}, J_0^i) \arrow[r, "u^j"] \arrow[d, hook] & K_0^{i+j} \\
\operatorname{Hom}_{R_0}(\mathfrak{b}, I_0^i) \arrow[r, "t^j"] & I_0^{i+j} \arrow[u, two heads]
\end{tikzcd}$$

giving the second isomorphism. ☐

Proof of Theorem A.3. To prove (1) and (2) it suffices to apply the identification from Proposition A.15 to Propositions A.13 and A.14. ☐

A.2. **Deformation theory for morphisms in the homotopy category of projectives.** In this subsection we will comment on the proof of Theorem A.5, and how it is dual to that of Theorem A.3 and the discussion in Subsection A.1. This modification is standard, and left implicit in [51].

Indeed, it suffices to consider the opposite categories of $\mathcal{C}_0$, $\mathcal{C}$ and $\overline{\mathcal{C}}$ in (A.2), using [52, Proposition 8.7(3)]. This exchanges left and right adjoints, and turns co-Grothendieck categories into Grothendieck categories. It is again a diagram of deformations, and shows how to describe the restriction along the tensor product for a deformation of co-Grothendieck categories as the restriction along the Hom-functor for a deformation of Grothendieck categories. Hence Theorem A.3 is truly dual to Theorem A.5.

A.3. **Restriction to the derived category.** In order to apply the deformation theory from Subsection A.2 to the setting of interest to us we need a comparison result as in [51, §6.3]. After all, we are not so much interested in deforming morphisms in $\mathbf{K}(\operatorname{Proj}\operatorname{Pro}\mathcal{C})$, but rather in derived categories which avoid the Pro-construction.

We need the analogue of [51, Proposition 6.2]. Let us first recall the following definition from [51, Definition 6.1].

Definition A.16. A diagram of functors

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "H"] & \mathcal{C}' \arrow[d, "H'"] \\
\mathcal{D} \arrow[r, "G"] & \mathcal{D}'
\end{tikzcd} \tag{A.29}$$

satisfies (L) if the following conditions are satisfied:

(1) the diagram is commutative up to natural isomorphism;
(2) F and G are fully faithful;
(3) if $H'(C') \simeq G(D)$ for some $C' \in \mathcal{C}'$ and $D \in \mathcal{D}$, then there exists an object $C \in \mathcal{C}$ such that $C' \simeq F(C)$.

Here H and H' are to be interpreted as the restriction functors for two deformations. Then let $f\colon D_1 \to D_2$ be a morphism in $\mathcal{D}_2$, and let $d\colon D_2 \xrightarrow{\sim} H(\overline{D}_2)$ be a lift of D_2. Then the *set of lifts of f* will be denoted by $\mathrm{L}_H(f \mid d)$ (compare to [51, Definition 3.1(2)]). More precisely, $\mathrm{L}_H(f \mid d)$ is the set of isomorphism classes of the groupoid whose object is a pair of a lift $d_1\colon D_1 \xrightarrow{\sim} H(\overline{D}_1)$ of D_1 and a morphism $\overline{f}\colon \overline{D}_1 \to \overline{D}_2$ such that $H(\overline{f})d_1 = df$, and an isomorphism from such a pair to another one $(d_1'\colon D_1 \xrightarrow{\sim} H(\overline{D}_1'), \overline{f}'\colon \overline{D}_1' \to \overline{D}_2)$ is an isomorphism $\varphi\colon \overline{D}_1 \xrightarrow{\sim} \overline{D}_1'$ such that $\overline{f}'\varphi\overline{f} = \overline{f}$ and $H(\varphi)d_1 = d_1'$.

Suppose a diagram of the form (A.29) satisfies (L) from Definition A.16. Let $f\colon D_1 \to D_2$ be a morphism in $\mathcal{D}$, $c_2\colon D_2 \xrightarrow{\sim} H(C_2) \in \mathrm{L}_H(D_2)$ and $C_2' \coloneqq F(C_2)$. Let $g \coloneqq G(f)$, and

$$c_2' \coloneqq G(c_2)\colon G(D_2) \xrightarrow{\sim} F(C_2').$$

Here we used the isomorphism $GH \simeq H'F$. Consider the canonical map

$$\mathrm{L}_H(f \mid c_2) \to \mathrm{L}_{H'}(g \mid c_2') \tag{A.30}$$

which sends a lift $[c_1\colon D_1 \xrightarrow{\sim} H(C_1), \quad \overline{f}\colon C_1 \to C_2]$ of f to the following lift of g.

$$G(c_1)\colon G(D_1) \xrightarrow{\sim} GH(C_1) \simeq H'F(C_1), \quad F(\overline{f})\colon F(C_1) \to F(C_2).$$

Analogous to [51, Proposition 6.2] one obtains the following proposition from the property (L).

Proposition A.17. *The map* (A.30) *is a bijection.*

We wish to apply this to the dual setting of [51, §6.3]. For this, let us consider the setting of flat abelian deformations of (A.2), without any restriction on the category.

Then the dual of [51, Proposition 6.5] in our lifting problem for morphisms is the following.

Proposition A.18. *There is a diagram*

$$\begin{tikzcd}
\mathbf{D}^-(\mathrm{Pro}\,\overline{\mathcal{C}}) \arrow[d] & \mathbf{K}^-(\mathrm{Proj}\,\mathrm{Pro}\,\overline{\mathcal{C}}) \arrow[l, "\simeq"'] \arrow[r, hook] \arrow[d] & \mathbf{K}(\mathrm{Proj}\,\mathrm{Pro}\,\overline{\mathcal{C}}) \arrow[d] \\
\mathbf{D}^-(\mathrm{Pro}\,\mathcal{C}) & \mathbf{K}^-(\mathrm{Proj}\,\mathrm{Pro}\,\mathcal{C}) \arrow[l, "\simeq"'] \arrow[r, hook] & \mathbf{K}(\mathrm{Proj}\,\mathrm{Pro}\,\mathcal{C})
\end{tikzcd} \tag{A.31}$$

where the vertical arrows are the appropriately defined restriction functors. Both squares satisfy (L) (see Definition A.16), hence the deformation theory for morphisms with fixed lift of the target from Theorem A.5 restricts to lifting morphisms in the category $\mathbf{D}^-(\mathcal{C})$.

For our purposes we mostly care about perfect complexes on schemes, so we consider the setting of Example A.1. This requires a further restriction step, which is given by the following proposition. We will use the notion of Tor-dimension and pseudo-coherent complexes, introduced in [62, Tag 08CG] and [62, Tag 08CB].

Proposition A.19. *Let $\overline{X}, X$ be as in Example A.1. There is a diagram*

$$\begin{tikzcd}
\operatorname{Perf}\overline{X} \arrow[r, hook] \arrow[d] & \mathbf{D}^-_{\mathrm{fTd}}(\operatorname{Qcoh}\overline{X}) \arrow[r, "\cong"] \arrow[d] & \mathbf{D}^-_{\mathrm{fTd},\operatorname{Qcoh}\overline{X}}(\operatorname{Pro}\operatorname{Qcoh}\overline{X}) \arrow[r, hook] \arrow[d] & \mathbf{D}^-(\operatorname{Pro}\operatorname{Qcoh}\overline{X}) \arrow[d] \\
\operatorname{Perf}X \arrow[r, hook] & \mathbf{D}^-_{\mathrm{fTd}}(\operatorname{Qcoh}X) \arrow[r, "\cong"] & \mathbf{D}^-_{\mathrm{fTd},\operatorname{Qcoh}X}(\operatorname{Pro}\operatorname{Qcoh}X) \arrow[r, hook] & \mathbf{D}^-(\operatorname{Pro}\operatorname{Qcoh}X)
\end{tikzcd}$$

where the vertical arrows are the appropriately defined restriction functors. All three squares satisfy (L) (see Definition A.16).

Proof. The fact that the middle and the right square satisfy (L) is dual to [51, Proposition 6.9].

For the left square we use the characterization of perfect complexes as pseudo-coherent complexes of finite Tor dimension, from [62, Tag 08CQ]. To prove that the left square satisfies (L), we consider an object $\overline{E}$ in $\mathbf{D}^-_{\mathrm{fTd}}(\operatorname{Qcoh}\overline{X})$ such that its restriction $R\otimes^{\mathbf{L}}_{\overline{R}}\overline{E}$ lies in the essential image from the inclusion of $\operatorname{Perf}X$. It suffices to prove that $\overline{E}$ is pseudocoherent.

To see this we use the 2-out-of-3 property for pseudo-coherence from [62, Tag 08CD], and consider the triangle

$$\mathfrak{b}\otimes^{\mathbf{L}}_{\overline{R}}\overline{E}\to\overline{E}\to R\otimes_{\overline{R}}\overline{E}\to\mathfrak{b}\otimes^{\mathbf{L}}_{\overline{R}}\overline{E}[1] \tag{A.32}$$

induced by taking the derived tensor product the short exact sequence $0\to\mathfrak{b}\to\overline{R}\to R\to 0$ with $\overline{E}$. By assumption we have that $R\otimes_{\overline{R}}\overline{E}$ is perfect, hence pseudocoherent.

To see that $\mathfrak{b}\otimes^{\mathbf{L}}_{\overline{R}}\overline{E}$ is pseudocoherent, we consider the isomorphism

$$\mathfrak{b}\otimes^{\mathbf{L}}_{\overline{R}}\overline{E}\cong\mathfrak{b}\otimes^{\mathbf{L}}_{R}(R\otimes^{\mathbf{L}}_{\overline{R}}\overline{E}) \tag{A.33}$$

using the fact that $\mathfrak{b}^2=0$ to conclude that $\overline{E}$ is itself pseudocoherent, hence it lies in the essential image of the inclusion of $\operatorname{Perf}\overline{X}$ in $\mathbf{D}^-_{\mathrm{fTd}}(\operatorname{Qcoh}\overline{X})$. □

We obtain the following corollary.

Corollary A.20. *Consider the situation of Example A.1. Consider an exact triangle*

$$F_0\xrightarrow{s_0}G_0\to H_0\to F_0[1] \tag{A.34}$$

in $\operatorname{Perf}X_0$, and let

$$F\xrightarrow{s}G \tag{A.35}$$

be a morphism in $\operatorname{Perf}X$ which restricts to s_0, i.e. $R_0\otimes^{\mathbf{L}}_{R}s=s_0$.

Let $\overline{G}$ be a lift of G in $\operatorname{Perf}X$ to $\operatorname{Perf}\overline{X}$ along $R\otimes^{\mathbf{L}}_{\overline{R}}-$; i.e., suppose that the derived pull-back of $\overline{G}$ by the inclusion morphism $X\hookrightarrow\overline{X}$ is isomorphic to G. Then

(1) there exists an obstruction class

$$\mathfrak{o}(s)\in\operatorname{Ext}^1_{X_0}(F_0,\mathfrak{b}\otimes^{\mathbf{L}}_{R_0}H_0) \tag{A.36}$$

such that $\mathfrak{o}(s)=0$ if and only if s lifts to a morphism $\overline{s}\colon\overline{F}\to\overline{G}$ in $\operatorname{Perf}\overline{X}$;

(2) *assume that* $\mathrm{Ext}^{-1}_{X_0}(F_0, \mathfrak{b} \otimes^{\mathbf{L}}_{R_0} H_0) = 0$*, if* $\mathfrak{o}(s) = 0$ *then the set of isomorphism classes of such lifts is a torsor under*

$$\mathrm{Ext}^0_{X_0}(F_0, \mathfrak{b} \otimes^{\mathbf{L}}_{R_0} H_0). \tag{A.37}$$

We end this appendix with the following remarks.

Remark A.21. In the setting where R_0 is semisimple, we can make the following simplification: we have an isomorphism

$$\mathbf{R}\mathrm{Hom}_{X_0}(F_0, \mathfrak{b} \otimes^{\mathbf{L}}_{R_0} H_0) \simeq \mathbf{R}\mathrm{Hom}_{X_0}(F_0, H_0) \otimes_{R_0} \mathfrak{b}, \tag{A.38}$$

and the computations reduce to determining the usual Ext-groups of the objects which are involved.

Remark A.22. As explained in Remark A.4 we can recover the deformation theory of complexes from our deformation theory of complexes with fixed lift of the target. Namely we recover [48, Theorem 3.1.1] (for perfect complexes) as in Remark A.4 by taking $G_0 = 0$ and using the zero object as the fixed lift of the target. Then condition in Corollary A.20(2) translates to the gluability condition in [48].

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