Algebraic Geometry

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[...] *Oscar Zariski bewitched me. When he spoke the words "algebraic variety", there was a certain resonance in his voice that said distinctly that he was looking into a secret garden. I immediately wanted to be able to do this too. It led me to 25 years of struggling to make this world tangible and visible.*

David Mumford

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0 | Before we start

Conventions

We list here a series of conventions that will be used throughout this text.

- The axiom of choice (or Zorn's Lemma) is assumed.
- Given two sets A, B, the phrase '$A \subset B$' means that A is contained in B, *possibly equal* to B.
- A *ring* is a commutative, unitary ring. The zero ring (the one where $1 = 0$) is allowed (and in fact needed), but we always assume our rings are nonzero unless we explicitly mention it.
- By $\mathbf{k}$ we indicate an algebraically closed field.
- An open cover of a topological space U is the datum of a set I, and an open subset $U_i \subset U$ for every $i \in I$, such that $U = \bigcup_{i \in I} U_i$. If $I = \emptyset$, then $U = \emptyset$.
- To say that Ω is an object a category $\mathscr{C}$ we simply write '$\Omega \in \mathscr{C}$'.

Main references

We list here a series of bibliographical references that integrate this text.

- Q. Liu, *Algebraic geometry and arithmetic curves* [8],
- R. Hartshorne, *Algebraic geometry* [5],
- R. Vakil, *The rising sea* [12],
- D. Eisenbud, *Commutative Algebra: With a View Toward Algebraic Geometry* [2],
- M. F. Atiyah and I. G. MacDonald, *Introduction to Commutative Algebra* [1],

1 Introduction

Algebraic Geometry is concerned with the study of *algebraic varieties*. At a first approximation, these are common zero loci of collections of polynomials, i.e. solutions to systems

$$\begin{cases} f_1(x_1,\dots,x_n)=0 \\ \qquad\vdots \\ f_r(x_1,\dots,x_n)=0 \end{cases}$$

of polynomial equations. When $\deg f_i = 1$ for $i = 1,\dots,r$, this is the content of *Linear Algebra*, but the higher degree case poses nontrivial difficulties!

The concept of algebraic variety has been vastly generalised by Grothendieck's theory of *schemes*, introduced in [4]. This course is an introduction to schemes and to part of the massive dictionary, shared by all algebraic geometers, centered around schemes. Even though algebraic varieties are 'easier' objects, schemes are an incredibly useful and powerful tool to study them.

In this introduction, we briefly recap the key relation

$$\text{Algebra} \longleftrightarrow \text{Geometry}$$

in the land of *classical* algebraic varieties. We provide no proofs, but you shouldn't worry about this, because we will be proving more general results in the main body of these notes.

Let $\mathbf{k}$ be an algebraically closed field. Classical affine n-space over $\mathbf{k}$ is just

$$\mathbb{A}^n_{\mathbf{k}} = \{\,(a_1,\dots,a_n) \mid a_i \in \mathbf{k} \text{ for } i = 1,\dots,n\,\}.$$

We denote it $\mathbb{A}^n_{\mathbf{k}}$ and not $\mathbf{k}^n$ to emphasise that we view it as a set of points rather than a vector space over $\mathbf{k}$. Set

$$A = \mathbf{k}[x_1,\dots,x_n].$$

Each element $f \in A$ defines a function $\widetilde{f} \colon \mathbb{A}^n_{\mathbf{k}} \to \mathbf{k}$ sending $(a_1,\dots,a_n) \mapsto f(a_1,\dots,a_n)$, and since $\mathbf{k}$ is algebraically closed one has $f = g$ if and only if $\widetilde{f} = \widetilde{g}$. Thus we shall just write f instead of $\widetilde{f}$.

Let $I=(f_1,\dots,f_r)\subset A$ be an arbitrary ideal (here we are using that every ideal in A is finitely generated, by Hilbert's basis theorem [6]). The 'vanishing locus'

$$\mathrm{V}(I)=\left\{\,(a_1,\dots,a_n)\in\mathbb{A}^n_{\mathbf{k}}\,\middle|\,f_j(a_1,\dots,a_n)=0\ \text{ for }\ j=1,\dots,r\,\right\}\subset\mathbb{A}^n_{\mathbf{k}}$$

is called an *algebraic set*. There is precisely one topology on $\mathbb{A}^n_{\mathbf{k}}$ having the algebraic sets as closed sets. It is called the *Zariski topology*.

Example 1.0.1. Every ideal in $\mathbf{k}[x]$ is principal, i.e. of the form (f) for some $f\in\mathbf{k}[x]$. Since $\mathbf{k}$ is algebraically closed, we have $f=\alpha(x-a_1)\cdots(x-a_d)$, for $\alpha,a_1,\dots,a_d\in\mathbf{k}$, and where $d=\deg f$. Thus $\mathrm{V}(f)=\{\,a_1,\dots,a_d\,\}\subset\mathbb{A}^1_{\mathbf{k}}$, proving that all closed sets in $\mathbb{A}^1_{\mathbf{k}}$ are finite. In particular, all open sets are infinite (again, since $\mathbf{k}$ is algebraically closed).

We have thus established an assignment

$$\{\,\text{ideals }\ I\subset\mathbf{k}[x_1,\dots,x_n]\,\}\xrightarrow{\mathrm{V}(-)}\{\,\text{algebraic sets in }\ \mathbb{A}^n_{\mathbf{k}}\,\}.$$

Conversely, given a subset $S\subset\mathbb{A}^n_{\mathbf{k}}$, the assignment

$$\mathrm{I}(S)=\left\{\,f\in A\,\middle|\,f(p)=0\text{ for all }p\in S\,\right\}\subset A$$

defines a map the other way around, namely

$$\{\,\text{ideals }\ I\subset\mathbf{k}[x_1,\dots,x_n]\,\}\xleftarrow{\mathrm{I}(-)}\{\,\text{algebraic sets in }\ \mathbb{A}^n_{\mathbf{k}}\,\}.$$

Unfortunately, the two maps are not inverse to each other. For instance, consider the ideal $(x^r)\subset\mathbf{k}[x]$ for $r>1$. Then $\mathrm{V}(x^r)=\{0\}$, and thus $\mathrm{I}(\mathrm{V}(x^r))=(x)$, which is strictly larger than (x^r). The next result says that this is what *always* happens.

THEOREM 1.0.2 (Hilbert's Nullstellensatz [7]). *Let $I\subset\mathbf{k}[x_1,\dots,x_n]$ be an ideal, where $\mathbf{k}$ is an algebraically closed field. Then,* $\mathrm{I}(\mathrm{V}(I))=\sqrt{I}$*, i.e.* $f\in\mathrm{I}(\mathrm{V}(I))$ *if and only if $f^r\in I$ for some $r>0$.*

See [8, Ch. 2, Corollary 1.15] for a modern proof of Hilbert's Nullstellensatz.

Composing our two assignments the other way around, we also find something larger than what we started with: if S is an arbitrary subset of $\mathbb{A}^n_{\mathbf{k}}$, one can easily prove the identity

$$\mathrm{V}(\mathrm{I}(S))=\overline{S},$$

the closure of S in $\mathbb{A}^n_{\mathbf{k}}$ (with respect to the Zariski topology), namely the smallest algebraic set containing S. Thus in order to get $\mathrm{V}(\mathrm{I}(S))=S$ we have to start with an algebraic set S (which is closed by definition).

Furthermore, one can prove that an algebraic set $Y\subset\mathbb{A}^n_{\mathbf{k}}$ is irreducible (i.e. it cannot be written as a union of two proper closed subsets) if and only if $\mathrm{I}(Y)\subset A$ is a prime ideal.

An irreducible algebraic set in $\mathbb{A}^n_{\mathbf{k}}$ is called an *affine variety in* $\mathbb{A}^n_{\mathbf{k}}$.

Of course, an affine variety is to be thought of as carrying the induced Zariski topology. Combining these observations together, we obtain correspondences

$$\begin{tikzcd}
\{\text{radical ideals in } A\} \arrow[r, shift left, "\mathrm{V}(-)"] & \{\text{algebraic sets in } \mathbb{A}^n_{\mathbf{k}}\} \arrow[l, shift left, "\mathrm{I}(-)"] \\
\{\text{prime ideals in } A\} \arrow[u, hook] \arrow[r, shift left, "\mathrm{V}(-)"] & \{\text{affine varieties in } \mathbb{A}^n_{\mathbf{k}}\} \arrow[l, shift left, "\mathrm{I}(-)"] \arrow[u, hook]
\end{tikzcd}$$

where an ideal $I \subset A$ is *radical* if $I = \sqrt{I}$.

Recall that, by definition, a *finitely generated* $\mathbf{k}$*-algebra* is a $\mathbf{k}$-algebra B isomorphic to a quotient $\mathbf{k}[x_1,\ldots,x_n]/I$ for some n and some ideal $I \subset \mathbf{k}[x_1,\ldots,x_n]$. Such a B is an integral domain (i.e. as a ring it has no nonzero zero-divisors) precisely when I is prime. Thus the bottom correspondence above can be rephrased as

$$\{\,\mathbf{k}[x_1,\ldots,x_n]/I \mid I \text{ is prime}\,\} \longleftrightarrow \{\,\text{algebraic varieties in } \mathbb{A}^n_{\mathbf{k}}\,\}.$$

In the first part of this course, we will be concerned with extending this correspondence to arbitrary *rings* on the left. What will be constructed on the right will be called an *affine scheme*, and what we shall establish is not just a bijection, but an equivalence of categories

$$\mathsf{Rings}^{\mathrm{op}} \cong \text{Affine schemes}.$$

Affine schemes are the basic building blocks for the construction of general *schemes*. Indeed, a scheme is defined by the property that every point has an open neighborhood isomorphic to an affine scheme.

2 | Sheaves

2.1 Key example: smooth functions

Sheaves were defined by Leray (1906–1998), while he was a prisoner in Austria during World War II.

Sheaves are a key notion present in the toolbox of every mathematician keen to understand the "nature" of a *geometric space*. They incarnate one of the basic principles that will be unraveled in this course, which can be stated as the slogan

$$\boxed{\text{geometric spaces are determined by functions on them.}}$$

Even though there may be "few" functions on a space X, a complete knowledge of all functions on all open subsets of X allows one, in principle, to reconstruct X. This local-to-global principle is perfectly encoded in the notion of a sheaf.

Before diving into precise definitions, we explore a key example. Let X be a smooth manifold. For each open subset $U \subset X$, we have a ring (actually, an $\mathbb{R}$-algebra)

$$C^{\infty}(U,\mathbb{R}) = \{\,\text{smooth functions } U \to \mathbb{R}\,\}.$$

If $V \hookrightarrow U$ is an open subset, we have a restriction map

$$\rho_{UV}\colon C^{\infty}(U,\mathbb{R}) \to C^{\infty}(V,\mathbb{R}), \quad f \mapsto f|_V,$$

which is a ring homomorphism, and if $W \hookrightarrow V \hookrightarrow U$ is a chain of open subsets of X, we have a commutative diagram

$$\begin{tikzcd} C^{\infty}(U,\mathbb{R}) \arrow[r, "\rho_{UV}"] \arrow[rr, bend right, "\rho_{UW}"'] & C^{\infty}(V,\mathbb{R}) \arrow[r, "\rho_{VW}"] & C^{\infty}(W,\mathbb{R}). \end{tikzcd}$$

So far, we have just observed that the assignment $U \mapsto C^{\infty}(U,\mathbb{R})$ is *functorial*, from open subsets of X (which form a category) to the category of $\mathbb{R}$-algebras. The two distinguished features of the assignment $U \mapsto C^{\infty}(U,\mathbb{R})$, which make it into a *sheaf* of $\mathbb{R}$-algebras on X, are the following:

(i) If $f, g \in C^\infty(U,\mathbb{R})$ are smooth functions and $U = \bigcup_{i\in I} U_i$ is an open cover such that $f|_{U_i} = g|_{U_i}$ for each $i \in I$, then $f = g$. In other words, a function is determined by its restriction to the open subsets forming a covering.

(ii) If $U = \bigcup_{i\in I} U_i$ is an open cover of an open subset $U \subset X$, and one has a smooth function $f_i \in C^\infty(U_i,\mathbb{R})$ on each U_i, such that $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ for every $(i, j) \in I \times I$, then there is precisely one global function $f \in C^\infty(U,\mathbb{R})$ such that $f_i = f|_{U_i}$. In other words, functions glue uniquely along an open cover.

A sheaf is an abstract notion formalising this "ability of glueing" (cf. Definition 2.2.5).

Let us continue with our example. Let $x \in X$ be a point. Consider the ring

$$C^\infty_{X,x} = \left\{ (U, f) \,\middle|\, x \in U,\, f \in C^\infty(U,\mathbb{R}) \right\} \big/ \sim$$

where $(U, f) \sim (V, g)$ whenever there exists an open subset $W \subset U \cap V$, containing x, such that $f|_W = g|_W$. This ring is called the *stalk* of the sheaf $C^\infty(-,\mathbb{R})$ at x (cf. Definition 2.3.1), and it receives a natural map from $C^\infty(U,\mathbb{R})$ for every open subset U of X such that $x \in U$, sending $f \mapsto [U, f]$. The image of f along this map is called the *germ of f at x*. Note that $C^\infty_{X,x}$ is indeed a ring, with addition and multiplication

$$\begin{aligned} [U, f] + [U', f'] &= [U \cap U', f + f'] \\ [U, f] \cdot [U', f'] &= [U \cap U', f f']. \end{aligned}$$

The subset

$$\mathfrak{m}_x = \left\{ [U, f] \in \mathscr{O}_{X,x} \,\middle|\, f(x) = 0 \right\} \subset C^\infty_{X,x}$$

forms an ideal, which is a *maximal* ideal, being the kernel of the (surjective) evaluation map

$$C^\infty_{X,x} \to \mathbb{R}, \quad [U, f] \mapsto f(x).$$

In fact, $\mathfrak{m}_x$ is the *unique* maximal ideal of $C^\infty_{X,x}$. To see this, it is enough to check that every element of $\mathscr{O}_{X,x} \setminus \mathfrak{m}_x$ is invertible. But this is true, since a function that is nonzero in a neighborhood of x is invertible there.

Conclusion: $(C^\infty_{X,x}, \mathfrak{m}_x)$ is a *local ring* with residue field $\mathbb{R}$. The geometric spaces X one deals with in algebraic geometry, namely *schemes*, have precisely this property: they come with a sheaf of rings $\mathscr{A}_X$ such that each stalk $\mathscr{A}_{X,x}$ is a local ring. These spaces $(X, \mathscr{A}_X)$ actually form a larger category, that of locally ringed spaces (cf. Section 3.1). Schemes are particular instances of locally ringed spaces.

2.2 Presheaves, sheaves, morphisms

Let $\mathscr{C}$ be a concrete category with a final object $0 \in \mathscr{C}$. The concreteness assumption means that part of the structure is the datum of a faithful functor $F : \mathscr{C} \to \mathsf{Sets}$, but we will (for the moment) ignore this datum. To fix ideas, $\mathscr{C}$ should be thought of as any of the following categories: the category $\mathscr{C} = \mathsf{Sets}$, the category $\mathscr{C} = \mathsf{Rings}$, the category $\mathscr{C} = \mathsf{Ab}$ of abelian groups ($\mathbb{Z}$-modules) or a more general abelian category.

If X is a topological space, we denote by τ_X the category of open subsets of X. The set $\operatorname{Hom}_{\tau_X}(V, U)$ between two open sets $V, U \subset X$ is just the empty set if $V \not\subset U$, or the singleton $\{ V \hookrightarrow U \}$ in case V is contained in U. Thus the opposite category τ_X^{op} satisfies

$$\operatorname{Hom}_{\tau_X^{\mathrm{op}}}(U, V) = \begin{cases} \{ V \hookrightarrow U \} & \text{if } V \subset U \\ \emptyset & \text{if } V \not\subset U \end{cases}$$

and a functor $\mathcal{F}\colon \tau_X^{\mathrm{op}} \to \mathscr{C}$ determines a map

$$\operatorname{Hom}_{\tau_X^{\mathrm{op}}}(U, V) \to \operatorname{Hom}_{\mathscr{C}}(\mathcal{F}(U), \mathcal{F}(V)),$$

which is nothing but a choice of an element $\rho_{UV} \in \operatorname{Hom}_{\mathscr{C}}(\mathcal{F}(U), \mathcal{F}(V))$ for any inclusion of open subsets $V \subset U$.

Definition 2.2.1 (Presheaf)**.** A *presheaf* on a topological space X, with values in $\mathscr{C}$, is a contravariant functor $\mathcal{F}$ from τ_X to $\mathscr{C}$, i.e. an object of the functor category $\mathsf{Fun}(\tau_X^{\mathrm{op}}, \mathscr{C})$. In other words, a presheaf $\mathcal{F}$ is the assignment $U \mapsto \mathcal{F}(U)$ of an object $\mathcal{F}(U) \in \mathscr{C}$ for each open subset $U \subset X$, and of a morphism $\rho_{UV}\colon \mathcal{F}(U) \to \mathcal{F}(V)$ in $\mathscr{C}$ for each inclusion $V \hookrightarrow U$, such that

(1) $\rho_{UU} = \mathrm{id}_{\mathcal{F}(U)}$ for every $U \in \tau_X$, and

(2) $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$ for every chain of inclusions $W \hookrightarrow V \hookrightarrow U$.

Terminology 2.2.2. Elements of $\mathcal{F}(U)$ are often called 'sections of $\mathcal{F}$ over U', or (somewhat more vaguely) 'local sections' when $U \subsetneq X$. Elements of $\mathcal{F}(X)$ are called 'global sections', or just 'sections'. Possible alternative notations for $\mathcal{F}(U)$ are $\Gamma(U, \mathcal{F})$ and $\mathrm{H}^0(U, \mathcal{F})$. The maps ρ_{UV} are often called 'restriction maps' (from U to V, the larger set being U).

Notation 2.2.3. Motivated by Terminology 2.2.2, we shall often write $s|_V$ for the image of a section $s \in \mathcal{F}(U)$ along ρ_{UV}.

Notation 2.2.4. The presheaf defined by $U \mapsto 0$ for every U is called the *trivial sheaf* (or sometimes the *zero sheaf*), and is simply denoted by '0'.

Definition 2.2.5 (Sheaf, take I)**.** A *sheaf* on a topological space X, with values in $\mathscr{C}$, is a presheaf $\mathcal{F}$ such that the following two conditions hold:

(3) Fix an open subset $U \subset X$, an open cover $U = \bigcup_{i \in I} U_i$, and two sections $s, t \in \mathcal{F}(U)$ satisfying $s|_{U_i} = t|_{U_i}$ for all $i \in I$. Then $s = t$.

(4) Fix an open subset $U \subset X$, an open cover $U = \bigcup_{i \in I} U_i$ and a tuple $(s_i)_{i \in I}$ of sections $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $(i, j) \in I \times I$. Then there exists a section $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$.

Conditions (3) and (4) generalise the conditions (i) and (ii), respectively, anticipated with the example $\mathcal{F} = C^\infty(-, \mathbb{R})$ in Section 2.1.

Terminology 2.2.6. A presheaf $\mathcal{F}$ is called *separated* if Condition (3) holds. Sometimes this condition is called *locality axiom*. Condition (4), on the other hand, is called the *glueing condition* (or *glueing axiom*).

Remark 2.2.7. Let $\mathcal{F}$ be a sheaf. Then, the section $s \in \mathcal{F}(U)$ in the glueing condition (4) is necessarily unique because $\mathcal{F}$ is separated. In fact, the two sheaf conditions could be replaced by a single condition, identical to (4), but imposing uniqueness of s.

Remark 2.2.8. Let $\mathcal{F}$ be a sheaf. Then, one has $\mathcal{F}(\emptyset) = 0$, the final object in $\mathscr{C}$. This is sometimes listed as an axiom defining a (pre)sheaf, but it does follow from our assumptions (cf. Example A.1.3).

Example 2.2.9 (Restriction to an open)**.** Let $U \subset X$ be an open subset, $\mathcal{F}$ a presheaf on X. Then, setting $\mathcal{F}|_U(V) = \mathcal{F}(V)$ for V an open subset of U, defines a presheaf $\mathcal{F}|_U$ on U, which is a sheaf as soon as $\mathcal{F}$ is. It is called the *restriction of $\mathcal{F}$ to U*.

Definition 2.2.10 (Morphism of (pre)sheaves)**.** A *morphism* between two presheaves $\mathcal{F}, \mathcal{G}$ on X is a natural transformation $\eta \colon \mathcal{F} \Rightarrow \mathcal{G}$. A morphism of sheaves is just a morphism between the underlying presheaves.

By definition, to give a morphism of (pre)sheaves, one has to assign a homomorphism $\eta_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ in $\mathscr{C}$ for each $U \in \tau_X$, such that for every inclusion $V \hookrightarrow U$ of open subsets of X, the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\eta_U"] \arrow[d, "\rho^{\mathcal{F}}_{UV}"'] & \mathcal{G}(U) \arrow[d, "\rho^{\mathcal{G}}_{UV}"] \\ \mathcal{F}(V) \arrow[r, "\eta_V"] & \mathcal{G}(V) \end{tikzcd} \tag{2.2.1}$$

commutes. For the sake of clarity, we have emphasised the relevant sheaf in the restriction maps notation, but we won't be doing that again.

Notation 2.2.11. It is clear that presheaves on X with values in $\mathscr{C}$ form a category $\mathsf{pSh}(X, \mathscr{C})$, tautologically defined as the functor category $\mathsf{Fun}(\tau_X^{\mathrm{op}}, \mathscr{C})$. Sheaves form a full subcategory, denoted $\mathsf{Sh}(X, \mathscr{C})$. We denote by $j_{X,\mathscr{C}} \colon \mathsf{Sh}(X, \mathscr{C}) \hookrightarrow \mathsf{pSh}(X, \mathscr{C})$ the inclusion functor.

An isomorphism of (pre)sheaves is an isomorphism in $\mathsf{pSh}(X, \mathscr{C})$, i.e. a natural equivalence, i.e. a natural transformation $\eta \colon \mathcal{F} \Rightarrow \mathcal{G}$ such that η_U is an isomorphism in $\mathscr{C}$ for every $U \in \tau_X$.

Notation 2.2.12. Since (pre)sheaves form a genuine category, we shall use the classical arrow notation '$\mathcal{F} \to \mathcal{G}$' (instead of $\mathcal{F} \Rightarrow \mathcal{G}$) to denote a morphism of (pre)sheaves.

The following definition makes sense, because $\mathscr{C}$ is assumed to be a concrete category.

Definition 2.2.13 (Injective map of presheaves)**.** A morphism of (pre)sheaves $\eta \colon \mathcal{F} \to \mathcal{G}$ is *injective* if η_U is injective for every U. We denote this by writing η as $\mathcal{F} \hookrightarrow \mathcal{G}$.

We close this section with a few examples and exercises.

Example 2.2.14 (Smooth functions)**.** Let X be a smooth manifold. Then, sending $U \subset X$ to the set $C^\infty(U, \mathbb{R})$ of smooth functions $U \to \mathbb{R}$, defines a sheaf $C^\infty(-, \mathbb{R})$ with values in the category of $\mathbb{R}$-algebras.

Example 2.2.15 (Holomorphic functions)**.** Let X be a complex manifold. Then, sending $U \subset X$ to the set $\mathscr{O}_X^{\mathrm{h}}(U)$ of holomorphic functions on U, defines a sheaf $\mathscr{O}_X^{\mathrm{h}}$ with values in the category of $\mathbb{C}$-algebras.

Example 2.2.16 (Separated presheaf, not a sheaf)**.** Set $X = \mathbb{C}$. Then, sending $U \subset X$ to the subset

$$\mathcal{F}(U) = \left\{ f \in \mathscr{O}_X^{\mathrm{h}}(U) \,\middle|\, f = g^2 \text{ for some } g \in \mathscr{O}_X^{\mathrm{h}}(U) \right\}$$

defines a (separated) presheaf. However, $\mathcal{F}$ is not a sheaf: the function $f(z) = z$ on the annulus

$$U = \{ z \in \mathbb{C} \mid 1 - \varepsilon < |z| < 1 + \varepsilon \} \subset \mathbb{C}$$

has a square root in any neighborhood of any point $x \in X$, but there is no global $\sqrt{z}$ defined on the whole of U.

Example 2.2.17 (Constant presheaf)**.** Work with $\mathscr{C} = \mathsf{Ab} = \mathsf{Mod}_{\mathbb{Z}}$, the category of abelian groups, and fix $G \neq 0$ in this category. Fix a topological space X, and define

$$\underline{G}_X^{\mathrm{pre}}(U) = \begin{cases} G & \text{if } U \neq \emptyset, \\ 0 & \text{if } U = \emptyset. \end{cases}$$

As for the restriction maps, set $\rho_{UV} = \mathrm{id}_G$ if both U and V are nonempty. This is a presheaf, which happens to be a sheaf only in precise circumstances (cf. Exercise 2.2.18). For instance, suppose $X = U_1 \amalg U_2$ is a disjoint union of two nonempty open subsets. Then $\underline{G}_X(X) = G = \underline{G}_X(U_i)$ for $i = 1, 2$. Now, $X = U_1 \amalg U_2$ is an open cover. Pick two *distinct* sections $s_i \in G = \underline{G}_X(U_i)$ for $i = 1,2$. Then, $s_1|_{U_1\cap U_2} = s_1|_\emptyset = 0 = s_2|_\emptyset = s_2|_{U_1\cap U_2}$, but there is no section $s \in \underline{G}_X(X) = G$ such that $s|_{U_i} = s_i$ since $\rho_{XU_i} = \mathrm{id}_G$ for $i = 1,2$ and $s_1 \neq s_2$ by assumption. Hence Condition (4) fails. We will see in Example 2.4.3 that $\underline{G}_X^{\mathrm{pre}}$ can be "transformed" into a sheaf by a canonical procedure.

Exercise 2.2.18. Show that the constant presheaf $\underline{G}_X^{\mathrm{pre}}$ of Example 2.2.17 is a sheaf if and only if every nonempty open subset $U \subset X$ is connected.

Exercise 2.2.19 (Preheaves kernel and cokernel)**.** Let $\mathscr{C}$ be an abelian category, so that every arrow has a kernel and a cokernel. Let $\eta\colon \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves with values in $\mathscr{C}$. Consider the assignments

$$
\begin{aligned}
U &\mapsto (\ker_{\mathrm{pre}} \eta)(U) = \ker(\eta_U) \\
U &\mapsto (\operatorname{coker}_{\mathrm{pre}} \eta)(U) = \operatorname{coker}(\eta_U) = \mathcal{G}(U)/\operatorname{im}(\eta_U).
\end{aligned}
$$

Show that

(i) both are presheaves,

(ii) $\ker_{\mathrm{pre}} \eta \to \mathcal{F}$ (resp. $\mathcal{G} \to \operatorname{coker}_{\mathrm{pre}} \eta$) satisfies the universal property of the kernel (resp. the cokernel) in $\mathsf{pSh}(X,\mathscr{C})$,

(iii) $\ker_{\mathrm{pre}} \eta$ is a sheaf, denoted $\ker(\eta)$, as soon as η is a morphism of *sheaves*,

(iv) if η is a morphism of sheaves, then $\ker(\eta)$ satisfies the universal property of the kernel in $\mathsf{Sh}(X,\mathscr{C})$, and η is injective if and only if $\ker(\eta) = 0$.

Exercise 2.2.20 (Bounded functions are not a sheaf)**.** Let $X = \mathbb{R}$, with the standard topology. Show that

$$U \mapsto \mathrm{B}(U) = \{\,\text{bounded continuous functions } U \to \mathbb{R}\,\}$$

is a separated presheaf on X, but not a sheaf (i.e. Condition (4) fails).

Exercise 2.2.21 (Continuous functions are a sheaf)**.** Let X, Y be topological spaces. For $U \subset X$ open, define

$$\mathcal{F}(U) = \{\,\text{continuous functions } U \to Y\,\}.$$

Show that $\mathcal{F}$ is a sheaf of sets (i.e. take $\mathscr{C} = \mathsf{Sets}$).

2.2.1 A translation of the sheaf conditions

We now present an alternative way to define sheaves.

Let $\mathscr{C}$ be a category with limits. In particular, $\mathscr{C}$ has products, equalisers, and a final object. Fix a presheaf $\mathcal{F}$ with values in $\mathscr{C}$ on a topological space X. Let $\{U_i\}_{i\in I}$ be a family of open subsets of X, and set $U = \bigcup_{i\in I} U_i$. Then, one can consider the map

$$\rho\colon \mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i), \quad s \mapsto (s|_{U_i})_{i\in I},$$

as well as the family of maps

$$\mu_{ij}\colon \prod_{i\in I} \mathcal{F}(U_i) \to \mathcal{F}(U_i) \to \mathcal{F}(U_i\cap U_j), \qquad (s_i)_{i\in I} \mapsto s_i|_{U_i\cap U_j}$$
$$\nu_{ij}\colon \prod_{i\in I} \mathcal{F}(U_i) \to \mathcal{F}(U_j) \to \mathcal{F}(U_i\cap U_j), \qquad (s_i)_{i\in I} \mapsto s_j|_{U_i\cap U_j}$$

which, taking products over $(i,j)\in I\times I$, can be assembled into two maps

$$\prod_{i\in I} \mathcal{F}(U_i) \underset{\nu}{\overset{\mu}{\rightrightarrows}} \prod_{(i,j)\in I\times I} \mathcal{F}(U_i\cap U_j).$$

Definition 2.2.22 (Sheaf, take II)**.** Let $\mathscr{C}$ be a category with limits, X a topological space. A presheaf $\mathcal{F}\in \mathsf{pSh}(X,\mathscr{C})$ is a *sheaf* if every family of open subsets $\{U_i\}_{i\in I}$, with $U = \bigcup_{i\in I} U_i$, the diagram

$$\mathcal{F}(U) \xrightarrow{\rho} \prod_{i\in I} \mathcal{F}(U_i) \underset{\nu}{\overset{\mu}{\rightrightarrows}} \prod_{(i,j)\in I\times I} \mathcal{F}(U_i\cap U_j)$$

is an equaliser diagram in $\mathscr{C}$.

Note that Definition 2.2.22 is *element-free*. However, let us check that it agrees with Definition 2.2.5 when $\mathscr{C}$ is concrete: injectivity of ρ, implied by the equaliser condition, coincides with separatedness; the fact that the set-theoretic image of ρ coincides with the collection of tuples of sections $(s_i)_{i\in I}$ such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ is precisely the glueing condition.

Example 2.2.23. Let $\mathcal{F}$ be a sheaf on X. If $U = \coprod_{i\in I} U_i$ is a *disjoint* union of open subsets $U_i \subset U$, then ρ is an isomorphism, i.e. $s \mapsto (s_{U_i})_{i\in I}$ defines an isomorphism

$$\rho\colon \mathcal{F}(U) \xrightarrow{\sim} \prod_{i\in I} \mathcal{F}(U_i).$$

Example 2.2.24. Let $\mathscr{C}$ be an abelian category. Then a presheaf $\mathcal{F}\in \mathsf{pSh}(X,\mathscr{C})$ is a sheaf if for every family of open subsets $\{U_i\}_{i\in I}$, with $U = \bigcup_{i\in I} U_i$, the sequence

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\rho} \prod_{i\in I} \mathcal{F}(U_i) \xrightarrow{\mu-\nu} \prod_{(i,j)\in I\times I} \mathcal{F}(U_i\cap U_j)$$

is exact, where the map denoted $\mu-\nu$ sends $(s_i)_{i\in I} \mapsto (s_i|_{U_i\cap U_j} - s_j|_{U_i\cap U_j})_{i,j}$.

The following lemma applies, for instance, to categories of groups, rings, algebras over a ring, and modules over a ring. It allows one to check the sheaf condition of Definition 2.2.22 in the category of sets.

LEMMA 2.2.25 ([11, Tag 0073]). *Let $\mathscr{C}$ be a category, $F\colon \mathscr{C} \to \mathsf{Sets}$ a faithful functor such that $\mathscr{C}$ has limits and F commutes with them. Assume that F reflects isomorphisms. Then a presheaf $\mathcal{F} \in \mathsf{pSh}(X, \mathscr{C})$ is a sheaf of and only if the underlying presheaf $F \circ \mathcal{F}\colon \tau_X \to \mathsf{Sets}$ is a sheaf.*

At the beginning of this chapter we have defined (pre)sheaves of objects in an arbitrary concrete category $\mathscr{C}$. We still have to define a few things, though, e.g. stalks and sheafification. In order for everything to be well-defined and work well (but still be compatible with all we have discussed so far, including Definition 2.2.22), we need to add a few initial data. This is provided by the following definition.

Definition 2.2.26 ([11, Tag 007L])**.** A *type of algebraic structure* is a pair $(\mathscr{C}, F)$, where $\mathscr{C}$ is a category, $F\colon \mathscr{C} \to \mathsf{Sets}$ is a faithful functor, such that

1. $\mathscr{C}$ has limits and F commutes with them,
2. $\mathscr{C}$ has filtered colimits and F commutes with them,
3. F reflects isomorphisms (i.e. F is *conservative*).

A few remarks are in order, before we go on.

- Equipping a category $\mathscr{C}$ with a faithful functor $F\colon \mathscr{C} \to \mathsf{Sets}$ is like saying that $\mathscr{C}$ is a *concrete category*, which we had already assumed in Section 2.2.
- If we have a type of algebraic structure $(\mathscr{C}, F)$, then we can verify whether a presheaf is a sheaf in the category of sets, by Lemma 2.2.25.
- The condition that F be conservative implies that a bijective morphism in $\mathscr{C}$ is an isomorphism.
- For every type of algebraic structure $(\mathscr{C}, F)$, one has the following properties:
  (i) $\mathscr{C}$ has a final object 0, and $F(0)$ is a final object in Sets (i.e. a singleton).
  (ii) $\mathscr{C}$ has products, fibre products, and equalisers — this follows by the examples in Appendix A.1.1. Moreover, F commutes with all of them.
- Examples of categories $\mathscr{C}$ having the additional structure of Definition 2.2.26 are: monoids, groups, abelian groups, rings, modules over a ring. In all these cases, we take as the functor F the obvious forgetful functor. As a counterexample, consider the category Top of topological spaces: the forgetful functor exists but does not reflect isomorphisms.

2.3 Stalks, and what they tell us

Fix a type of algebraic structure $(\mathscr{C}, F : \mathscr{C} \to \mathsf{Sets})$ as in Definition 2.2.26. Let X be a topological space, $x \in X$ a point. The collection of open subsets $U \subset X$ containing x forms a directed system (the partial order $\succeq$ being the inclusion relation, i.e. $V \succeq U$ if and only if $V \subset U$). Indeed, given two open neighborhoods U and V of x, there is always a third open neighborhood of x contained in both U and V, namely $U \cap V$ or any smaller open subset containing x. In fancier language, the subcategory

$$\mathrm{Nbg}_x = \{\, U \in \tau_X \mid x \in U \,\}^{\mathrm{op}} \subset \tau_X^{\mathrm{op}}$$

is a filtered category (see Definition A.1.9).

Definition 2.3.1 (Stalks)**.** Let $x \in X$ be a point, $\mathcal{F}$ a presheaf. The *stalk of* $\mathcal{F}$ *at* x is the filtered colimit

$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U),$$

which exists as an object of $\mathscr{C}$. In categorical language, it is the filtered colimit of the functor $\mathcal{F}|_{\mathrm{Nbg}_x} : \mathrm{Nbg}_x \to \mathscr{C}$.

Because F commutes with colimits, the underlying set $F(\mathcal{F}_x)$, still denoted $\mathcal{F}_x$, is

$$\mathcal{F}_x = \{(U, s) \mid x \in U,\ s \in \mathcal{F}(U)\} / \sim$$

where $(U, s) \sim (V, t)$ whenever there is an open neighborhood $W \subset U \cap V$ of x such that $s|_W = t|_W$. We denote by

$$s_x = [U, s] \in \mathcal{F}_x$$

the equivalence class of the pair (U, s). It is called the *germ of* s *at* x. By definition of direct limit, there are natural homomorphisms

$$\mathcal{F}(U) \to \mathcal{F}_x, \quad s \mapsto s_x,$$

in $\mathscr{C}$, for every open neighborhood U of x. The diagram

$$\begin{tikzcd}
s \in \mathcal{F}(U) \arrow[dr] \arrow[dddr, bend right] & & \mathcal{F}(V) \ni t \arrow[dl] \arrow[dddl, bend left] \\
& \mathcal{F}(U \cap V) \arrow[d] & \\
& \mathcal{F}(W) \arrow[d] & \\
& \mathcal{F}_x &
\end{tikzcd}$$

illustrates the fact that two sections $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$ define the same element in the stalk $\mathcal{F}_x$ if and only if there is an intermediate open $W \subset U \cap V$ over which they agree.

Figure 2.1: A bunch of sheaves sitting in their natural habitat. The little tops of each leaf of corn are the stalks.

LEMMA 2.3.2. *If $\mathcal{F}$ is a separated presheaf of sets (e.g. a sheaf), then the natural map*

$$\sigma_U^{\mathcal{F}}: \mathcal{F}(U) \to \prod_{x\in U} \mathcal{F}_x, \quad s \mapsto (s_x)_{x\in U} \tag{2.3.1}$$

is injective for every open subset U of X.

The lemma means, at an informal level, that sections are determined by their germs.

Proof. If s and t are sections in $\mathcal{F}(U)$ such that $s_x = t_x$ in $\mathcal{F}_x$ for every $x \in U$, then for every $x \in U$ there is an open neighborhood $U_x \subset U$ such that $s|_{U_x} = t|_{U_x}$. But this holds for every $x \in U$, and $U = \bigcup_{x\in U} U_x$ is an open covering, thus by the separation axiom we deduce $s = t$, i.e. $\sigma_U^{\mathcal{F}}$ is injective. □

Consider the following property of a tuple $(s_x)_{x\in U} \in \prod_{x\in U} \mathcal{F}_x$, for $U \subset X$ an open subset:

$$\begin{gathered}\text{for every } x \in U \text{ there exists a pair } (V_x, t_x),\\ \text{with } x \in V_x \subset U \text{ and } t_x \in \mathcal{F}(V_x),\\ \text{such that } (t_x)_y = s_y \text{ for all } y \in V_x.\end{gathered} \tag{2.3.2}$$

Definition 2.3.3 (Compatible germs)**.** Let $\mathcal{F}$ be a presheaf on X, and let $U \subset X$ be an open subset. When Condition (2.3.2) is fulfilled, we say that the tuple $(s_x)_{x\in U} \in \prod_{x\in U} \mathcal{F}_x$ consists of *compatible germs.*

We always have inclusions

$$\operatorname{im}(\sigma_U^{\mathcal{F}}) \subset \{\text{tuples } (s_x)_{x\in U} \text{ of compatible germs}\} \subset \prod_{x\in U} \mathcal{F}_x \tag{2.3.3}$$

where the first inclusion is justified by taking $V_x = U$ and $t_x = s$ for every $x \in U$ as soon as $\sigma_U^{\mathcal{F}}(s) = (s_x)_{x\in U}$. If $\mathcal{F}$ is a sheaf, then tuples of compatible germs form precisely the image of the map (2.3.1), i.e. the first inclusion in (2.3.3) is an equality. Indeed, assume $(s_x)_{x\in U}$ consists of compatible germs. Let $\{(V_x, t_x) \mid x \in U\}$ be as in the displayed condition (2.3.2). Then we have an open cover $U = \bigcup_{x\in U} V_x$, so by the glueing axiom the sections $t_x \in \mathcal{F}(V_x)$ glue to a (unique) section $t \in \mathcal{F}(U)$ such that $t|_{V_x} = t_x$. But $(t_x)_y = s_y$ for $y \in V_x$, and this holds for every $x \in U$, so $\sigma_U^{\mathcal{F}}(t) = (s_x)_{x\in U}$.

Summing up, when $\mathcal{F}$ is a sheaf, we have a bijection

$$\sigma_U^{\mathcal{F}} \colon \mathcal{F}(U) \xrightarrow{\ \sim\ } \{\text{tuples } (s_x)_{x\in U} \text{ of compatible germs}\}.$$

This also shows that *sections of a sheaf can always be identified with genuine functions!* Indeed, tuples $(s_x)_{x\in U}$ correspond to particular functions $U \to \coprod_{x\in U} \mathcal{F}_x$, sending $x \in U$ inside the corresponding stalk, and doing so in a compatible way.

LEMMA 2.3.4. *Let $s, t \in \mathcal{F}(X)$ be two global sections of a sheaf $\mathcal{F}$, such that $s_x = t_x \in \mathcal{F}_x$ for every $x \in X$. Then $s = t$.*

Proof. This is just a special case of Lemma 2.3.2. □

Exercise 2.3.5. Let $\mathcal{F}$ be a sheaf on X, and let $s, t \in \mathcal{F}(X)$ be two global sections. Show that

$$\{x \in X \mid s_x = t_x\} \subset X$$

is an open subset of X.

A morphism of presheaves $\eta \colon \mathcal{F} \to \mathcal{G}$ induces a morphism $\eta_x \colon \mathcal{F}_x \to \mathcal{G}_x$ at the level of stalks for every $x \in X$, defined by

$$s_x = [U, s] \mapsto [U, \eta_U(s)] = (\eta_U(s))_x. \tag{2.3.4}$$

Exercise 2.3.6. Check that (2.3.4) is well-defined.

If $U \subset X$ is an open subset containing a point $x \in X$, then the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\eta_U"] \arrow[d] & \mathcal{G}(U) \arrow[d] & s \arrow[r, mapsto, "\eta_U"] \arrow[d, mapsto] & \eta_U(s) \arrow[d, mapsto] \\ \mathcal{F}_x \arrow[r, "\eta_x"] & \mathcal{G}_x & s_x \arrow[r, mapsto, "\eta_x"] & (\eta_U(s))_x \end{tikzcd}$$

commutes. What we have just said can be rephrased by saying that the association $\mathcal{F} \mapsto \mathcal{F}_x$ defines a functor

$$\mathsf{stalk}_x \colon \mathsf{pSh}(X, \mathscr{C}) \to \mathscr{C}. \tag{2.3.5}$$

We will see that in reasonable circumstances this functor is *exact* (cf. Proposition 2.4.14).

Definition 2.3.7. A morphism of (pre)sheaves $\eta\colon \mathcal{F} \to \mathcal{G}$ is *surjective* if η_x is surjective for every $x \in X$.

Warning 2.3.8. You may have noticed that surjectivity of a map of sheaves (cf. Definition 2.3.7) is defined differently than injectivity (cf. Definition 2.2.13)!

Let $\eta\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then

$$\eta_x \text{ is surjective} \iff \begin{array}{c}\text{for every } t_x \in \mathcal{G}_x \text{ there exists an open neighborhood}\\ U \text{ of } x \text{ and a section } s \in \mathcal{F}(U) \text{ such that } (\eta_U(s))_x = t_x.\end{array}$$

$$\eta \text{ is surjective} \iff \begin{array}{c}\text{for every open subset } U \subset X \text{ and for every}\\ t \in \mathcal{G}(U), \text{ there exists a covering } U = \bigcup_{i\in I} U_i\\ \text{such that } t|_{U_i} \text{ is in the image of } \eta_{U_i} \text{ for every } i.\end{array}$$

The second equivalence is obtained as follows.

Proof of '⇒'. First of all assume η is surjective, i.e. η_x is surjective for every $x \in X$. Fix $U \subset X$ open and a local section $t \in \mathcal{G}(U)$. For every $x \in U$, we have a commutative diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\eta_U"] \arrow[d] & \mathcal{G}(U) \arrow[d] & t \arrow[d, mapsto] \\ \mathcal{F}_x \arrow[r, two heads, "\eta_x"] & \mathcal{G}_x & t_x \end{tikzcd}$$

where $t_x \in \mathcal{G}_x$ can be lifted along η_x to an element $s_x \in \mathcal{F}_x$. Let (V_x, s) be a representative for s_x, so that in particular $s \in \mathcal{F}(V_x)$. The identity $\eta_x(s_x) = t_x$ implies that there is an open neighborhood $x \in U_x \subset V_x \cap U$ such that the diagram

$$\begin{tikzcd} s \in \mathcal{F}(V_x) \arrow[r, "\rho^{\mathcal{F}}_{V_x U_x}"] & \mathcal{F}(U_x) \arrow[r, "\eta_{U_x}"] \arrow[d] & \mathcal{G}(U_x) \arrow[d] & s \arrow[r, mapsto] & s|_{U_x} \arrow[r, mapsto] \arrow[d, mapsto] & t|_{U_x} \arrow[d, mapsto] \\ & \mathcal{F}_x \arrow[r, two heads, "\eta_x"] & \mathcal{G}_x & & s_x \arrow[r, mapsto] & t_x \end{tikzcd}$$

commutes, in particular $\eta_{U_x}(s|_{U_x}) = t|_{U_x}$. Now this holds for every $x \in U$, so the elements of $\{\, U_x \mid x \in U \,\}$ form a covering of U, and we have proved the condition.

Proof of '⇐'. Conversely, assuming the condition, let us prove surjectivity of η. Fix $x \in X$ along with a germ $t_x \in \mathcal{G}_x$. We need to prove that t_x has a preimage in $\mathcal{F}_x$. Let (U, t) be a representative of t_x, so that $t \in \mathcal{G}(U)$. By the condition we are assuming, there exists a covering $U = \bigcup_{i\in I} U_i$ such that $t|_{U_i} = \eta_{U_i}(s_i)$ for some $s_i \in \mathcal{F}(U_i)$, for every $i \in I$. If $x \in U_i$, we have a commutative diagram

$$\begin{tikzcd} \mathcal{F}(U_i) \arrow[r, "\eta_{U_i}"] \arrow[d] & \mathcal{G}(U_i) \arrow[d] & s_i \arrow[r, mapsto] \arrow[d, mapsto] & t|_{U_i} \arrow[d, mapsto] \\ \mathcal{F}_x \arrow[r, "\eta_x"] & \mathcal{G}_x & \star \arrow[r, mapsto] & t_x \end{tikzcd}$$

so the element is a preimage of t_x. The equivalence is proved.

The next result incarnates the local nature of sheaves.

LEMMA 2.3.9. *Let $\eta\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. The following are equivalent:*

(i) η is an isomorphism,

(ii) η_x is an isomorphism for every $x \in X$,

(iii) η is injective and surjective.

Proof. Recall that η is an isomorphism if and only if η_U is an isomorphism for every U.

Proof of (i) $\Rightarrow$ (ii). By functoriality of $\mathcal{F} \mapsto \mathcal{F}_x$, we have that if η is an isomorphism, then so is η_x for every $x \in X$.

Proof of (ii) $\Rightarrow$ (i). Suppose η_x is an isomorphism for every x. Let $U \subset X$ be an open subset: we need to show that η_U is an isomorphism.

To see that η_U is injective, pick $s, t \in \mathcal{F}(U)$ such that $\eta_U(s) = \eta_U(t) \in \mathcal{G}(U)$. Then, for any $x \in U$, one has

$$\eta_x(s_x) = (\eta_U(s))_x = (\eta_U(t))_x = \eta_x(t_x),$$

which implies $s_x = t_x$ by injectivity of η_x. This holds for every $x \in U$ by assumption, thus $s = t$ by Lemma 2.3.4. Therefore, η_U is injective for every U (i.e. η is injective).

To see that η_U is surjective, pick $t \in \mathcal{G}(U)$. By surjectivity of η (which we have since η_x is surjective for every $x \in X$), we can find an open cover $U = \bigcup_{i\in I} U_i$ along with a collection of sections $s_i \in \mathcal{F}(U_i)$ such that $\eta_{U_i}(s_i) = t|_{U_i}$. But η is injective, so s_i and s_j agree on $U_i \cap U_j$. Therefore, since $\mathcal{F}$ is a sheaf, they glue to a section $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$. By construction, $\eta_U(s)|_{U_i} = \eta_{U_i}(s_i) = t|_{U_i}$, which implies $\eta_U(s) = t$ since $\mathcal{G}$ is a sheaf. Thus η_U is surjective.

Proof of (ii) $\Rightarrow$ (iii). The first paragraph of '(ii) $\Rightarrow$ (i)' already shows that if η_x is an isomorphism for every $x \in X$, then η_U is injective for all U, i.e. η is injective. Surjectivity follows from the definition.

Proof of (iii) $\Rightarrow$ (ii). We only need to show that if η_U is injective for every U, then η_x is injective for every $x \in X$. Consider $s_x = [U, s]$ and $s'_x = [U', s']$ two germs in $\mathcal{F}_x$ such that $\eta_x(s_x) = \eta_x(s'_x)$ in $\mathcal{G}_x$. Then there is an open subset $W \subset U \cap U'$ such that $\eta_U(s)|_W = \eta_{U'}(s')|_W$. But by compatibility of η_W with restrictions, this is equivalent to the identity $\eta_W(s|_W) = \eta_W(s'|_W)$, which by our assumption implies $s|_W = s'|_W$. But then $s_x = s'_x$. □

Warning 2.3.10. It is not true that two sheaves with isomorphic stalks are isomorphic: there may be no map between them! For instance, consider a topological space X consisting of two points x_0, x_1 where only x_0 is a closed point. Thus X and $U = X \setminus \{x_0\}$ are the only nonempty open subsets of X. Fix an abelian group $G \neq 0$ and define

$\mathcal{F}(X) = G = \mathcal{F}(U)$. Then choose either $\rho_{XU} = \mathrm{id}_G$ or $\rho_{XU} = 0$ to define two distinct sheaves on X. They have the same stalks but they are not isomorphic.

Exercise 2.3.11. Show that Lemma 2.3.9 fails for presheaves.

Example 2.3.12 (Surjectivity is subtle)**.** Let $\mathcal{F} = \mathscr{O}_X^{\mathrm{h}}$ be the sheaf of holomorphic functions on $X = \mathbb{C} \setminus \{0\}$, and let $\mathcal{G} = \mathcal{F}^\times$ be the sheaf of invertible holomorphic functions on X. The map $\exp\colon \mathcal{F} \to \mathcal{G}$ is surjective, but $\exp_X\colon \mathcal{F}(X) \to \mathcal{G}(X)$ is not surjective, e.g. the function $f(z) = z$ in $\mathcal{G}(X)$ is not the exponential of a homolomorphic function.

Example 2.3.13 (Skyscraper sheaf)**.** Let X be a topological space, G a nontrivial abelian group, $x \in X$ a point. The assignment

$$U \mapsto G_x(U) = \begin{cases} G & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$$

defines a sheaf, choosing as restriction maps the identity of G or the zero map in the obvious way. This sheaf is called the *skyscraper sheaf* attached to (X, x, G). At the level of stalks, one has

$$(G_x)_y = \begin{cases} G & \text{if } y \in \overline{\{x\}} \\ 0 & \text{if } y \notin \overline{\{x\}}, \end{cases}$$

because if y is in the closure of x then every neighborhood of y also contains x, whereas if y is *not* in the closure of x, then $U = X \setminus \{x\}$ is largest open neighborhoods of y and thus $(G_x)_y = 0$. Thus G_x has only one nonzero stalk (at x) if and only if x is a closed point. This is the case where the name 'skyscraper sheaf' for G_x fits best.

Exercise 2.3.14. Let $\mathcal{F}$ be a presheaf, $\mathcal{G}$ a sheaf, $\eta_1, \eta_2\colon \mathcal{F} \to \mathcal{G}$ two morphisms of presheaves of sets such that $\eta_{1,x} = \eta_{2,x}$ for every $x \in X$. Show that $\eta_1 = \eta_2$. Show that it is in fact necessary to assume $\mathcal{G}$ to be a sheaf. This exercise will be needed in Theorem 3.2.50.

2.3.1 Supports

Let A be a ring. Let $\mathcal{F}$ be a sheaf of A-modules on a topological space X. Let $s \in \mathcal{F}(U)$ be a local section. We have two notions of support: the support of $\mathcal{F}$, and the support of s, defined respectively as

$$\mathrm{Supp}(\mathcal{F}) = \{ x \in X \mid \mathcal{F}_x \neq 0 \},$$
$$\mathrm{Supp}(s) = \{ x \in U \mid s_x \neq 0 \}.$$

If $s_x = 0$, then there is an open neighborhood $x \in V \subset U$ such that $s|_V = 0 \in \mathcal{F}(V)$. Thus $V \subset U \setminus \mathrm{Supp}(s)$ and hence $\mathrm{Supp}(s) \subset U$ is closed. In fact, this follows from (or solves)

Exercise 2.3.5 as well. In general, however, $\mathrm{Supp}(\mathcal{F}) \subset X$ is *not* closed. If $\mathcal{F}$ is a sheaf of rings, these notions of support still make sense, and one has $\mathrm{Supp}(\mathcal{F}) = \mathrm{Supp}(1)$, where $1 \in \mathcal{F}(X)$ is the ring identity. Thus $\mathrm{Supp}(\mathcal{F})$ *is* closed in this special case.

2.4 Sheafification

Fix a type of algebraic structure $(\mathscr{C}, F : \mathscr{C} \to \mathsf{Sets})$.

Let X be a topological space. Let $\mathcal{F} \colon \tau_X^{\mathrm{op}} \to \mathscr{C}$ be a presheaf. We next define a *sheaf* $\mathcal{F}^\#$, called the sheafification of $\mathcal{F}$, via an explicit universal property, and having precisely the same stalks as the initial presheaf $\mathcal{F}$.

Definition 2.4.1. Let $\mathcal{F} \in \mathsf{pSh}(X, \mathscr{C})$ be a presheaf. A *sheafification* of $\mathcal{F}$ is a pair $(\mathcal{F}^\#, \theta)$, where $\mathcal{F}^\#$ is a sheaf and $\theta : \mathcal{F} \to \mathcal{F}^\#$ is a morphism of presheaves, such that for every other pair $(\mathcal{G}, j)$ where $\mathcal{G}$ is a sheaf and $\alpha \colon \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, there exists a unique morphism of sheaves $\widetilde{\alpha} \colon \mathcal{F}^\# \to \mathcal{G}$ such that $\alpha = \widetilde{\alpha} \circ \theta$.

$$
\begin{tikzcd}
\mathcal{F} \arrow[rr, "\theta"] \arrow[dr, "\forall\,\alpha"'] & & \mathcal{F}^\# \arrow[dl, dashed, "\exists!\,\widetilde{\alpha}"] \\
& \mathcal{G} &
\end{tikzcd}
$$

PROPOSITION 2.4.2. *Let $\mathcal{F} \in \mathsf{pSh}(X, \mathscr{C})$ be a presheaf. Then a sheafification $(\mathcal{F}^\#, \theta)$ exists, and the map $\theta_x : \mathcal{F}_x \to \mathcal{F}^\#_x$ is an isomorphism for every $x \in X$.*

What follows immediately from Proposition 2.4.2 is that $\mathcal{F}^\#$ is unique up to a unique isomorphism, and moreover the canonical map $\theta : \mathcal{F} \to \mathcal{F}^\#$ is an isomorphism precisely when $\mathcal{F}$ is already a sheaf.

Proof. Let $U \subset X$ be an open subset. Define

$$
\mathcal{F}^\#(U) = \left\{ \text{functions } U \xrightarrow{f} \coprod_{x \in U} \mathcal{F}_x \;\middle|\; \begin{array}{c} \text{for every } x \in U \text{ there exist an open} \\ \text{neighborhood } V \subset U \text{ of } x \text{ and } s \in \mathcal{F}(V) \\ \text{such that } f(y) = s_y \text{ for every } y \in V \end{array} \right\}.
$$

Note that, since $\mathscr{C}$ has products, we can view a function f as above as a tuple

$$
(f(x))_{x \in U} \in \prod_{x \in U} \mathcal{F}_x
$$

and we can rephrase the definition of $\mathcal{F}^\#(U)$ by saying that

$$
\mathcal{F}^\#(U) = \left\{ \text{tuples } (s_x)_{x \in U} \text{ of compatible germs} \right\}.
$$

See Definition 2.3.3 for the definition of compatible germs. Functoriality of the assignment $U \mapsto \mathcal{F}^\#(U)$ is clear (functions restrict!), thus $\mathcal{F}^\#$ is a presheaf. The morphism

$\theta_U\colon \mathcal{F}(U) \to \mathcal{F}^\#(U)$ defined by sending $s \in \mathcal{F}(U)$ to the function

$$f_s\colon U \to \coprod_{x\in U} \mathcal{F}_x, \quad x \mapsto s_x = [U,s] \in \mathcal{F}_x$$

determines a morphism of presheaves, being compatible with restrictions. It is just the function $\sigma_U^{\mathcal{F}}$ introduced in (2.3.1)!

The presheaf $\mathcal{F}^\#$ is a sheaf: Fix an open cover $U = \bigcup_{i\in I} U_i$ of some open subset $U \subset X$ and a collection of sections $f_i \in \mathcal{F}^\#(U_i)$ such that $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ for every i and j. We need to find a unique $f \in \mathcal{F}^\#(U)$ such that $f|_{U_i} = f_i$. Define

$$f(x) = f_i(x) \in \mathcal{F}_x, \quad x \in U_i \subset U.$$

This is well-defined since, even though x can lie in more than one open U_i, by assumption we have $f_i(x) = f_j(x)$ as soon as $x \in U_i \cap U_j$. We need to check that f defines an element of $\mathcal{F}^\#(U)$, not just of the full product $\prod_{x\in U}\mathcal{F}_x$. But for every i we know the following: for every $x \in U_i$ there exist an open neighborhood $x \in V_i \subset U_i$ and a section $s_i \in \mathcal{F}(V_i)$ such that $f(y) = f_i(y) = (s_i)_y$ for all $y \in V_i$. But V_i is also open in U, so the condition defining $\mathcal{F}^\#(U)$ also holds for f. Thus $f \in \mathcal{F}^\#(U)$ satisfies $f|_{U_i} = f_i$, and is clearly unique with this property.

The pair $(\mathcal{F}^\#, \theta)$ is the sheafification. Assume we have a sheaf $\mathcal{G}$ and a morphism of presheaves $\alpha\colon \mathcal{F} \to \mathcal{G}$. We need to define a morphism $\widetilde{\alpha}\colon \mathcal{F}^\# \to \mathcal{G}$ of presheaves such that $\alpha = \widetilde{\alpha} \circ \theta$. For every U open in X, we need to define a morphism $\widetilde{\alpha}_U\colon \mathcal{F}^\#(U) \to \mathcal{G}(U)$ in such a way that $\alpha_U = \widetilde{\alpha}_U \circ \theta_U$. Fix $s = (s_x)_{x\in U} \in \mathcal{F}^\#(U)$. Consider the composition

$$\begin{tikzcd} U \arrow[r, "s"] \arrow[dr, "\widetilde{\alpha}_U(s)"'] & \coprod_{x\in U}\mathcal{F}_x \arrow[d, "\coprod_{x\in U}\alpha_x"] \\ & \coprod_{x\in U}\mathcal{G}_x. \end{tikzcd}$$

It defines a tuple of compatible germs for $\mathcal{G}$ over U, hence an element $\widetilde{\alpha}_U(s) \in \mathcal{G}^\#(U) = \mathcal{G}(U)$, using that $\mathcal{G}$ is a sheaf for this identity. This is the required morphism $\widetilde{\alpha}\colon \mathcal{F}^\# \to \mathcal{G}$.

The map θ is an isomorphism on stalks. The map θ, at the level of stalks, is defined by

$$\theta_x[U,s] = [U, f_s].$$

Injectivity: Suppose $\theta_x[U,s] = \theta_x[V,t]$ for two classes $[U,s],[V,t] \in \mathcal{F}_x$, i.e. assume $[U,f_s] = [V,f_t]$ in $\mathcal{F}^\#_x$. Then, by definition of germ, there exists an open neighborhood $W \subset U \cap V$ of x such that $f_s|_W = f_t|_W$. But this means, by definition of f_s and f_t, that $s_y = t_y$ for all $y \in W$. Thus, in particular, $s_x = t_x$. But this is just the equality $[U,s] = [V,t]$ we were after.

Surjectivity: Pick a class $[U,f] \in \mathcal{F}^\#_x$ for some $f \in \mathcal{F}^\#(U)$ and open neighborhood U of

x. Then, for every $z \in U$, there exist an open neighborhood $V \subset U$ of z and a section $s \in \mathcal{F}(V)$ such that $f(y) = s_y$ in $\mathcal{F}_y$ for every $y \in V$. We claim that $[U, f] = \theta_x(s_x)$, where $s_x = [V, s]$. Indeed, $\theta_x(s_x) \in \mathcal{F}^\#_x$ is the equivalence class of the map

$$f_s \colon V \to \coprod_{y \in V} \mathcal{F}_y, \quad y \mapsto s_y.$$

But this map agrees with the restriction of f to $V \subset U$ (by the condition $f(y) = s_y$ recalled above), i.e. $f_s = f|_V \in \mathcal{F}^\#(V)$. Since V is also an open neighborhood of x, it follows that $(f|_V)_x = (f_s)_x = [V, f_s] = \theta_x(s_x) \in \mathcal{F}^\#_x$, but of course $(f|_V)_x = [U, f]$. Thus θ_x is surjective. □

Example 2.4.3 (Constant sheaf)**.** Let G be a nontrivial abelian group. The *constant sheaf* on a topological space X, with values in G, is the sheafification $\underline{G}_X$ of the presheaf $\underline{G}_X^{\mathrm{pre}}$ defined in Example 2.2.17. This sheaf agrees with the sheaf whose sections over U are the locally constant functions $U \to G$. This, in turn, agrees with the following: endow G with the discrete topology and consider the assignment

$$U \mapsto \{\, \text{continuous maps } U \to G \,\},$$

which we know is a sheaf by Exercise 2.2.21. If $U \subset X$ is a connected open subset, then $\underline{G}_X(U) = G$. By Proposition 2.4.2, at the level of stalks we have $\underline{G}_{X,x} = G$ for every $x \in X$, since the stalks of the constant presheaf are manifestly all equal to G.

Exercise 2.4.4. Let X be a connected topological space, x a point, G a nontrivial abelian group. Under what condition(s) is the constant sheaf $\underline{G}_X$ equal to the skyscraper sheaf G_x (cf. Example 2.3.13)?

Exercise 2.4.5. Show that sending $\mathcal{F} \mapsto \mathcal{F}^\#$ defines a functor $(-)^\# \colon \mathsf{pSh}(X, \mathscr{C}) \to \mathsf{Sh}(X, \mathscr{C})$, and that the forgetful functor $j_{X,\mathscr{C}} \colon \mathsf{Sh}(X, \mathscr{C}) \hookrightarrow \mathsf{pSh}(X, \mathscr{C})$ is a right adjoint. This means that are bifunctorial bijections

$$\psi_{\mathcal{F},\mathcal{G}} \colon \mathrm{Hom}_{\mathsf{Sh}(X,\mathscr{C})}(\mathcal{F}^\#, \mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{pSh}(X,\mathscr{C})}(\mathcal{F}, \mathcal{G}), \quad \tilde{\alpha} \mapsto \tilde{\alpha} \circ \theta$$

for any presheaf $\mathcal{F}$ and sheaf $\mathcal{G}$.

2.4.1 Subsheaves, Quotient sheaves

Proposition 2.4.6 ([11, Tag 007S]). *Let X be a topological space. Let $\mathcal{F}, \mathcal{G} \in \mathsf{Sh}(X, \mathsf{Sets})$ be sheaves of sets, $\eta \colon \mathcal{F} \to \mathcal{G}$ a morphism. Then, the following are equivalent:*

(a) η is a monomorphism,

(b) $\eta_x \colon \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$,

(c) *$\eta_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all open subsets $U \subset X$ (i.e. η is injective).*

Furthermore, the following are equivalent:

(i) *η is an epimorphism,*

(ii) *$\eta_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective for all $x \in X$ (i.e. η is surjective),*

and are implied (but not equivalent to, cf. Example 2.3.12!) by the condition

(iii) *$\eta_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective for all open subsets $U \subset X$.*

If $\mathscr{C}$ is an abelian category (e.g. Mod_A for a fixed ring A), then Proposition 2.4.6 holds replacing Sets with $\mathscr{C}$. For more general categories, such as Rings, one should replace 'injective' (resp. 'surjective') with 'monomorphism' (resp. 'epimorphism').

Definition 2.4.7 (Subsheaf, quotient sheaf)**.** If there exists a morphism of sheaves $\eta : \mathcal{F} \to \mathcal{G}$ such that either of the equivalent conditions (a), (b) or (c) holds, we say that $\mathcal{F}$ is a *subsheaf* of $\mathcal{G}$ (and we may denote this by '$\mathcal{F} \subset \mathcal{G}$'). If either of the equivalent conditions (i) or (ii) holds, we say that $\mathcal{G}$ is a *quotient sheaf* of $\mathcal{F}$.

Example 2.4.8 (Quotient sheaf)**.** Let $\mathscr{C}$ be an abelian category. If $\mathcal{F} \subset \mathcal{G}$ is a subsheaf (with values in $\mathscr{C}$), then sending

$$U \mapsto \mathcal{G}(U)/\mathcal{F}(U) \tag{2.4.1}$$

is a presheaf on X, because the restriction maps respect the inclusions $\mathcal{F}(U) \hookrightarrow \mathcal{G}(U)$, and thus pass to the quotients. Its sheafification $\mathcal{G}/\mathcal{F}$ is called the *quotient sheaf of $\mathcal{G}$ by $\mathcal{F}$*. There is a natural morphism of sheaves $\mathcal{G} \to \mathcal{G}/\mathcal{F}$.

Definition 2.4.9 (Sheaf image, sheaf cokernel)**.** Let $\mathscr{C}$ be an abelian category, $\eta : \mathcal{F} \to \mathcal{G}$ a morphism of sheaves (with values in $\mathscr{C}$), so that $\ker(\eta) \hookrightarrow \mathcal{F}$ is a subsheaf by Exercise 2.2.19. The sheafification $\operatorname{im}(\eta)$ of the presheaf

$$U \mapsto \operatorname{im}_{\text{pre}}(U) = \operatorname{im}(\eta_U) = \mathcal{F}(U)/\ker(\eta_U)$$

is called the *image of η*. It is a special case of Example 2.4.8 and defines a subsheaf

$$\operatorname{im}(\eta) = \mathcal{F}/\ker(\eta) \subset \mathcal{G}.$$

The quotient sheaf

$$\operatorname{coker}(\eta) = \mathcal{G}/\operatorname{im}(\eta),$$

again a special case of Example 2.4.8, is called the *sheaf cokernel.*

Exercise 2.4.10. Let $\mathscr{C}$ be an abelian category. Let $\eta\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves with values in $\mathscr{C}$. Show that the composition

$$\mathcal{G} \to \operatorname{coker}_{\mathrm{pre}} \eta \to \operatorname{coker}(\eta),$$

where the first morphism is given by the natural maps $\mathcal{G}(U) \twoheadrightarrow \mathcal{G}(U)/\operatorname{im}(\eta_U)$ and the last morphism is the sheafification, is a cokernel in the category $\mathsf{Sh}(X, \mathscr{C})$.

Remark 2.4.11. Set $\mathscr{C} = \mathsf{Mod}_A$ (or any Grothendieck abelian category in which, by definition, filtered colimits exist and are exact). Let $\mathcal{F} \subset \mathcal{G}$ be a subsheaf, $x \in X$ a point. Then

$$(\mathcal{G}/\mathcal{F})_x = \mathcal{G}_x/\mathcal{F}_x \tag{2.4.2}$$

in Mod_A. This follows from the fact that $(\mathcal{G}/\mathcal{F})_x$ agrees with the stalk of the *presheaf* (2.4.1), and from right exactness of filtered colimits. Moreover, if $\eta\colon \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves and $x \in X$ is a point, then

$$\begin{aligned}\ker(\eta)_x &= \ker(\eta_x)\\ \operatorname{im}(\eta)_x &= \operatorname{im}(\eta_x)\\ \operatorname{coker}(\eta)_x &= \operatorname{coker}(\eta_x).\end{aligned}$$

The first identity follows from the fact that filtered colimits are *also left exact* in Mod_A, thus

$$\begin{aligned}\ker\left(\mathcal{F}_x \xrightarrow{\eta_x} \mathcal{G}_x\right) &= \ker\left(\varinjlim_{U \ni x} \mathcal{F}(U) \to \varinjlim_{U \ni x} \mathcal{G}(U)\right)\\ &= \varinjlim_{U \ni x} \ker(\mathcal{F}(U) \to \mathcal{G}(U))\\ &= \ker(\eta)_x.\end{aligned}$$

The last two identities are a special case of (2.4.2).

THEOREM 2.4.12 ([3, §10]). *If $\mathscr{C}$ is a Grothendieck abelian category, then $\mathsf{Sh}(X, \mathscr{C})$ is a Grothendieck abelian category.*

Definition 2.4.13. A *short exact sequence of sheaves* with values in a Grothendieck abelian category $\mathscr{C}$ is a short exact sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{\iota} \mathcal{G} \xrightarrow{\theta} \mathcal{H} \longrightarrow 0$$

of objects in the abelian category $\mathsf{Sh}(X, \mathscr{C})$. Explicitly, exactness means that ι is injective, θ is surjective and $\operatorname{im}(\iota) = \ker(\theta)$.

PROPOSITION 2.4.14. *Let $\mathscr{C}$ be a Grothendieck abelian category. A sequence*

$$0 \longrightarrow \mathcal{F} \xrightarrow{\iota} \mathcal{G} \xrightarrow{\theta} \mathcal{H} \longrightarrow 0$$

of objects in $\mathsf{Sh}(X,\mathscr{C})$ is a short exact sequence if and only if

$$0 \longrightarrow \mathcal{F}_x \xrightarrow{\iota_x} \mathcal{G}_x \xrightarrow{\theta_x} \mathcal{H}_x \longrightarrow 0$$

is a short exact sequence in $\mathscr{C}$ for every $x \in X$.

Proof. Combine Remark 2.4.11 and Lemma 2.3.9 with one another. □

Exercise 2.4.15. Let $\eta\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of A-modules, for A a ring. Confirm the following.

- If η is injective, there is a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{F} \longrightarrow 0$$

- In general, there is an exact sequence of sheaves

$$0 \longrightarrow \ker(\eta) \longrightarrow \mathcal{F} \xrightarrow{\eta} \mathcal{G} \longrightarrow \operatorname{coker}(\eta) \longrightarrow 0.$$

Exercise 2.4.16. Let A be a ring. For a nonempty open subset U of a topological space X, consider the functor $\Gamma(U,-)\colon \mathsf{Sh}(X,\mathsf{Mod}_A) \to \mathsf{Mod}_A$ sending $\mathcal{F} \mapsto \mathcal{F}(U)$. Show that it is left exact. That is, it transforms an exact sequence of sheaves $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ into an exact sequence of A-modules

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U).$$

When $U = X$, this functor takes $\mathcal{F} \mapsto \mathcal{F}(X)$ and is thus called the *global section functor*. Another notation used for it in the literature is $\mathrm{H}^0(X,-)$, cf. Terminology 2.2.2.

2.5 Defining sheaves and morphisms on basic open sets

Fix a type of algebraic structure $(\mathscr{C}, F\colon \mathscr{C} \to \mathsf{Sets})$.

Definition 2.5.1 (Base of open sets)**.** Let X be a topological space. A *base of open sets* for X is a collection of open subsets $\mathcal{B} \subset \tau_X$ satisfying the following requirements:

(a) $\mathcal{B}$ is stable under finite intersections,

(b) every $U \in \tau_X$ can be written as a union of open sets belonging to $\mathcal{B}$.

Definition 2.5.2 ($\mathcal{B}$-sheaf)**.** A $\mathcal{B}$-*presheaf* (resp. $\mathcal{B}$-*sheaf*) is an assignment $U \mapsto \mathcal{F}(U) \in \mathscr{C}$ for each $U \in \mathcal{B}$ such that the presheaf conditions (1)–(2) of Definition 2.2.1 (resp. the presheaf conditions (1)–(2) of Definition 2.2.1 and the sheaf conditions (3)–(4) of Definition 2.2.5) hold, considering only open sets belonging to $\mathcal{B}$.

Notation 2.5.3. We shall use the notation $\{\mathcal{F}(\mathcal{B}), \rho_{\mathcal{B}}\}$ to denote a $\mathcal{B}$-(pre)sheaf.

Note that restriction map $\rho_{UV}\colon \mathcal{F}(U) \to \mathcal{F}(V)$ is part of the data of a $\mathcal{B}$-presheaf whenever $V \subset U$ is an inclusion of open sets belonging to $\mathcal{B}$. Note, also, that condition (a) in Definition 2.5.2 ensures that open sets of the form $U_i \cap U_j$, for $U_i, U_j \in \mathcal{B}$, still belong to $\mathcal{B}$. In particular, a $\mathcal{B}$-presheaf is a sheaf precisely when, for any open $U \in \mathcal{B}$ and any open cover $U = \bigcup_{i\in I} U_i$ with all $U_i \in \mathcal{B}$, the sequence

$$\mathcal{F}(U) \xrightarrow{\ \rho\ } \prod_{i\in I} \mathcal{F}(U_i) \overset{\mu}{\underset{\nu}{\rightrightarrows}} \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \cap U_j)$$

is an equaliser in $\mathscr{C}$.

Remark 2.5.4. The collection of open neighborhoods

$$\mathcal{B}_x = \{\, U \in \mathcal{B} \mid x \in U \,\}^{\mathrm{op}} \subset \tau_X^{\mathrm{op}}$$

is a fundamental system of open neighborhoods of x (i.e. for any $U \in \mathcal{B}_x$ there is $V \in \tau_X$ such that $x \in V \subset U$, and for any $W \in \tau_X$ such that $x \in W$ there exists $U \in \mathcal{B}_x$ such that $U \subset W$). In more technical terms, one may say that the filtered categories Nbg_x and $\mathcal{B}_x$ are *cofinal*, i.e. the inclusion $\mathcal{B}_x \hookrightarrow \mathrm{Nbg}_x$ is a cofinal functor.

By Remark 2.5.4, the stalk

$$\mathcal{F}_x = \varinjlim_{U\in\mathcal{B}_x} \mathcal{F}(U)$$

of a $\mathcal{B}$-(pre)sheaf $\{\mathcal{F}(\mathcal{B}), \rho_{\mathcal{B}}\}$ at a point $x \in X$ is well-defined as an object of $\mathscr{C}$.

Moreover, if $U \in \mathcal{B}$ and $\{\mathcal{F}(\mathcal{B}), \rho_{\mathcal{B}}\}$ is a $\mathcal{B}$-sheaf, the natural map

$$\sigma_U^{\mathcal{F}}\colon \mathcal{F}(U) \longrightarrow \textstyle\prod_{x\in U} \mathcal{F}_x$$

is injective (as in Lemma 2.3.2), and its image still agrees with the collections of compatible germs; to be more precise, we should now call them '$\mathcal{B}$-compatible', for they are, by definition, those tuples

$$(s_x)_{x\in U} \in \prod_{x\in U} \mathcal{F}_x$$

such that for every $x \in U$ there is a pair (V_x, t_x), where $V_x \in \mathcal{B}$ satisfies $x \in V_x \subset U$ and $t_x \in \mathcal{F}(V_x)$ satisfies $(t_x)_y = s_y$ for every $y \in V_x$.

Definition 2.5.5 (Morphism of $\mathcal{B}$-sheaves)**.** A *morphism* of $\mathcal{B}$-(pre)sheaves

$$\eta_{\mathcal{B}}\colon \left\{ \mathcal{F}(\mathcal{B}), \rho^{\mathcal{F}}_{\mathcal{B}} \right\} \longrightarrow \left\{ \mathcal{G}(\mathcal{B}), \rho^{\mathcal{G}}_{\mathcal{B}} \right\} \tag{2.5.1}$$

is the datum of a collection of maps $\eta_U : \mathcal{F}(U) \to \mathcal{G}(U)$, one for each $U \in \mathcal{B}$, such that Diagram (2.2.1) commutes for all $U, V \in \mathcal{B}$ such that $V \subset U$.

With this definition, $\mathcal{B}$-sheaves form a category, denoted $\mathsf{Sh}_{\mathcal{B}}(X, \mathscr{C})$.

Remark 2.5.6. Let $\mathcal{B}$ and X be as above. A (pre)sheaf $\mathcal{F}$ on X is a $\mathcal{B}$-(pre)sheaf in a natural way. More precisely, there is (say, at the level of sheaves) a *restriction functor*

$$\mathsf{res}_{\mathcal{B}}(X, \mathscr{C})\colon \mathsf{Sh}(X, \mathscr{C}) \longrightarrow \mathsf{Sh}_{\mathcal{B}}(X, \mathscr{C}), \tag{2.5.2}$$

defined on objects in the obvious way. Its actual functoriality is just a consequence of the definition of morphism of $\mathcal{B}$-sheaves, and we leave it to the reader to check all functoriality details.

LEMMA 2.5.7. *A $\mathcal{B}$-sheaf $\{ \mathcal{F}(\mathcal{B}), \rho_{\mathcal{B}} \}$ uniquely extends to a sheaf $\overline{\mathcal{F}}$, whose sections over $U \in \mathcal{B}$ agree with $\mathcal{F}(U)$.*

Proof. Let U be an open set in X. Define

$$\overline{\mathcal{F}}(U) = \left\{ \text{tuples } (s_x)_{x \in U} \text{ of } \mathcal{B}\text{-compatible germs} \right\} \subset \prod_{x \in U} \mathcal{F}_x.$$

This is manifestly a presheaf. It is also clear that the above definition agrees with $\mathcal{F}(U)$ whenever $U \in \mathcal{B}$, since the injective map $\sigma^{\mathcal{F}}_U$ hits precisely the tuple of $\mathcal{B}$-compatible germs; moreover, for the same reason, this definition is the *only* possible extension of the original $\mathcal{B}$-sheaf. The sheaf property is fulfilled by $\overline{\mathcal{F}}$ precisely for the same reason why it is fulfilled by the sheafification of a presheaf (see the proof of Proposition 2.4.2). □

In fact, the statement of Lemma 2.5.7 can be made functorial: one can prove that the restriction functor (2.5.2) is an equivalence. The inverse is given precisely by Lemma 2.5.7 above for objects and by Proposition 2.5.9 below for morphisms.

Remark 2.5.8. We have that $\mathcal{F}_x = \overline{\mathcal{F}}_x$ for all $x \in X$. This follows directly from Remark 2.5.4.

The analogue of Lemma 2.5.7 for *morphisms* is the following.

PROPOSITION 2.5.9. *Let X be a topological space, $\mathcal{B} \subset \tau_X$ a base of open sets and $\mathcal{F}, \mathcal{G}$ two sheaves on X. Suppose given a morphism $\eta_{\mathcal{B}}$ between the underlying $\mathcal{B}$-sheaves, as in* (2.5.1)*. Then, $\eta_{\mathcal{B}}$ extends uniquely to a sheaf homomorphism $\eta\colon \mathcal{F} \to \mathcal{G}$. Furthermore, if η_U is surjective (resp. injective, or an isomorphism) for every $U \in \mathcal{B}$, then so is η.*

Proof. Exercise. □

2.6 Pushforward, inverse image

In this section we learn how to "move" sheaves from a topological space X to another topological space Y, in the presence of a continuous map between the two spaces.

$$\begin{tikzcd} \mathcal{F} \arrow[r, rightsquigarrow] \arrow[d, dash] & f_*\mathcal{F} \arrow[d, dash] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} f^{-1}\mathcal{G} \arrow[d, dash] & \mathcal{G} \arrow[l, leftsquigarrow] \arrow[d, dash] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

2.6.1 Pushforward (or direct image)

Let $f: X \to Y$ be a continuous map of topological spaces, and let $\mathcal{F}$ be a presheaf on X. The assignment

$$V \mapsto f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$$

defines a presheaf $f_*\mathcal{F}$ on Y, called the *pushforward* (or *direct image*) of $\mathcal{F}$ by f. It is a sheaf as soon as $\mathcal{F}$ is, because if $V = \bigcup_{i\in I} V_i$ is an open covering of an open subset $V \subset Y$, then $f^{-1}(V) = \bigcup_{i\in I} f^{-1}(V_i)$ is an open covering of $f^{-1}(V) \subset X$.

Example 2.6.1. If X is arbitrary and $Y = \mathsf{pt}$, then $f_*\mathcal{F}(\mathsf{pt}) = \mathcal{F}(X)$, an object of $\mathscr{C}$. We will see in a minute that the direct image along any continuous map defines a functor. The direct image along the constant map $(X \to \mathsf{pt})_*: \mathsf{Sh}(X, \mathscr{C}) \to \mathscr{C}$ is also called the *global section functor*. If $\mathscr{C} = \mathsf{Mod}_A$, it is a left exact functor (you proved a more general statement in Exercise 2.4.16).

Example 2.6.2. If $f: X \hookrightarrow Y$ is the inclusion of a subspace, then $f_*\mathcal{F}$ is defined, for any open subset $V \subset Y$, by

$$f_*\mathcal{F}(V) = \mathcal{F}(V \cap X).$$

Example 2.6.3 (Skyscraper sheaf as a pushforward)**.** Let $x \in X$ be a point, G a nontrivial abelian group. Consider the constant sheaf $\underline{G}_{\{x\}}$ on $\{x\}$. Let $i_x: \{x\} \hookrightarrow X$ be the inclusion. Then the skyscraper sheaf $G_x \in \mathsf{Sh}(X, \mathsf{Mod}_{\mathbb{Z}})$ defined in Example 2.3.13 can be described as

$$G_x = i_{x,*}\underline{G}_{\{x\}}.$$

Next, we observe that pushforward of sheaves is functorial, i.e. sending $\mathcal{F} \mapsto f_*\mathcal{F}$ defines functors

$$\begin{tikzcd} \mathsf{Sh}(X,\mathscr{C}) \arrow[r, "f_*"] \arrow[d, hook] & \mathsf{Sh}(Y,\mathscr{C}) \arrow[d, hook] \\ \mathsf{pSh}(X,\mathscr{C}) \arrow[r, "f_*"] & \mathsf{pSh}(Y,\mathscr{C}) \end{tikzcd}$$

where the vertical maps are the natural inclusions. Indeed, given a morphism of (pre)sheaves $\eta: \mathcal{F} \to \mathcal{G}$, we can construct a morphism of (pre)sheaves

$$f_*\eta: f_*\mathcal{F} \to f_*\mathcal{G}$$

simply by setting

$$(f_*\eta)_V = \eta_{f^{-1}(V)} \colon \mathcal{F}(f^{-1}(V)) \to \mathcal{G}(f^{-1}(V))$$

for an open subset $V \subset Y$. The compatibility with restriction maps follows from those of η (and the obvious observation that if $V' \subset V$ then $f^{-1}V' \subset f^{-1}V$).

Moreover, $(-)_*$ is compatible with compositions of continuous maps, in the following sense: if $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous maps of topological spaces, then, as functors, we have an equality $(g \circ f)_* = g_* \circ f_*$ on the nose (both for presheaves and for sheaves).

(2.6.1)
$$\begin{tikzcd} \mathsf{Sh}(X,\mathscr{C}) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \mathsf{Sh}(Y,\mathscr{C}) \arrow[d, "g_*"] \\ & \mathsf{Sh}(Z,\mathscr{C}) \end{tikzcd}$$

Indeed, if $\mathcal{F}$ is a (pre)sheaf on X, then for every open $W \subset Z$ one has

$$\begin{aligned} (g \circ f)_*\mathcal{F}(W) &= \mathcal{F}((g \circ f)^{-1}(W)) \\ &= \mathcal{F}(f^{-1}g^{-1}(W)) \\ &= f_*\mathcal{F}(g^{-1}(W)) \\ &= (g_*f_*\mathcal{F})(W) \\ &= (g_* \circ f_*)\mathcal{F}(W). \end{aligned}$$

LEMMA 2.6.4. *Let $f\colon X \to Y$ be a continuous map of topological spaces, and fix a sheaf $\mathcal{F} \in \mathsf{Sh}(X,\mathscr{C})$. Set $y = f(x)$. There is a canonical morphism*

$$(f_*\mathcal{F})_y \longrightarrow \mathcal{F}_x.$$

This morphism is an isomorphism when f is the inclusion of a subspace $X \hookrightarrow Y$.

Proof. If $y \in V' \subset V \subset Y$, then $x \in f^{-1}V' \subset f^{-1}V \subset X$, and the commutative diagram

$$\begin{tikzcd} f_*\mathcal{F}(V) \arrow[rr] \arrow[d, equal] \arrow[dr] & & f_*\mathcal{F}(V') \arrow[d, equal] \arrow[dl] \\ \mathcal{F}(f^{-1}V) \arrow[dr] & (f_*\mathcal{F})_y \arrow[d, dotted] & \mathcal{F}(f^{-1}V') \arrow[dl] \\ & \mathcal{F}_x & \end{tikzcd}$$

induces, via the universal property of the stalk $(f_*\mathcal{F})_y$, a canonical morphism $(f_*\mathcal{F})_y \to \mathcal{F}_x$, as required.

Now, let us assume f is the inclusion of a subspace, and let us take $y \in X$. Note that every neighborhood $y \in U \subset X$ is of the form $U = V \cap X$ for some open neighborhood $y \in V \subset Y$. Thus

$$\begin{aligned}(f_*\mathcal{F})_y &= \varinjlim_{Y \supset V \ni y} \mathcal{F}(X \cap V) \\ &\xrightarrow{\sim} \varinjlim_{X \supset U \ni y} \mathcal{F}(U) \\ &= \mathcal{F}_y.\end{aligned}$$

□

We shall use Lemma 2.6.4 crucially with $\mathscr{C} = \mathsf{Rings}$.

We now say two (important) words on the exactness of direct image. What needs to be remembered is:

$$\boxed{f_* \text{ is always left exact, and it is exact if } f: X \hookrightarrow Y \text{ is a closed subspace.}}$$

We set $\mathscr{C} = \mathsf{Mod}_A$ for the rest of this subsection. Since f_* will turn out to be a right adjoint (Lemma 2.6.14), it is left exact by general category theory. However, we prove it directly here. Note that you have already proved the case $Y = \mathrm{pt}$ in Exercise 2.4.16. You will notice in the proof that this was essentially enough to handle the general case.

PROPOSITION 2.6.5. *Let $f: X \to Y$ be a continuous map of topological spaces. Then, $f_*: \mathsf{Sh}(X, \mathsf{Mod}_A) \to \mathsf{Sh}(Y, \mathsf{Mod}_A)$ is left exact.*

Proof. We have show that an exact sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}$$

in $\mathsf{Sh}(X, \mathsf{Mod}_A)$ induces an exact sequence

$$0 \longrightarrow f_*\mathcal{F} \xrightarrow{f_*\alpha} f_*\mathcal{G} \xrightarrow{f_*\beta} f_*\mathcal{H}$$

in $\mathsf{Sh}(Y, \mathsf{Mod}_A)$. We know by Exercise 2.4.16 that we have an exact sequence

$$0 \longrightarrow \mathcal{F}(f^{-1}V) \xrightarrow{\alpha_{f^{-1}V}} \mathcal{G}(f^{-1}V) \xrightarrow{\beta_{f^{-1}V}} \mathcal{H}(f^{-1}V) \tag{2.6.2}$$

for any open subset $V \subset Y$, by applying the functor $\Gamma(f^{-1}V, -)$ to the original sequence. In particular, $\alpha_{f^{-1}V} = (f_*\alpha)_V$ is injective for all V, which shows that $f_*\alpha$ is injective. There is an equality of presheaves

$$\mathrm{im}_{\mathrm{pre}}(f_*\alpha) = \ker(f_*\beta)$$

again thanks to exactness of (2.6.2) in the middle, ensuring precisely that $\mathrm{im}(\alpha_{f^{-1}V}) = \ker(\beta_{f^{-1}V})$. But $\ker(f_*\beta)$ is a sheaf, therefore we get exactness in the middle, i.e. $\mathrm{im}(f_*\alpha) = \ker(f_*\beta)$. □

PROPOSITION 2.6.6. *Let $f: X \hookrightarrow Y$ be the inclusion of a closed subspace. Then f_* is exact.*

Proof. By Proposition 2.6.5, we only need to show that if $\eta: \mathcal{G} \twoheadrightarrow \mathcal{H}$ is surjective as a map of sheaves on X, then $f_*\mathcal{G} \twoheadrightarrow f_*\mathcal{H}$ is surjective as a map of sheaves on Y. If $y \in Y \setminus X$, then (using that X is closed)

$$(f_*\mathcal{G})_y = 0 = (f_*\mathcal{H})_y.$$

Assume $y \in X$. Since $\mathcal{G}$ surjects onto $\mathcal{H}$, in the commutative diagram

$$\begin{tikzcd} (f_*\mathcal{G})_y \arrow[r, "(f_*\eta)_y"] \arrow[d, "\wr"] & (f_*\mathcal{H})_y \arrow[d, "\wr"] \\ \mathcal{G}_y \arrow[r, two heads, "\eta_y"] & \mathcal{H}_y \end{tikzcd}$$

the bottom map is surjective. The vertical maps are isomorphisms by the last assertion of Lemma 2.6.4. Thus the top map is surjective as well. Hence $f_*\eta$ is surjective on all stalks, hence it is surjective. □

2.6.2 Inverse image

Let $f: X \to Y$ be a continuous map of topological spaces. Let $\mathcal{G}$ be a presheaf on Y. Given $U \subset X$, the collection of open subsets $V \subset Y$ containing $f(U)$ form a directed set via reverse inclusions. Sending

$$U \mapsto (f^{-1}_{\text{pre}}\mathcal{G})(U) = \varinjlim_{V \supset f(U)} \mathcal{G}(V).$$

defines a presheaf on X. Indeed, assume $U' \subset U$ is an open subset. Then there is an inclusion $f(U') \subset f(U)$, inducing a map of directed systems

$$\{V \in \tau_Y \mid V \supset f(U)\} \hookrightarrow \{V \in \tau_Y \mid V \supset f(U')\},$$

and in turn a morphism

$$\varinjlim_{V \supset f(U)} \mathcal{G}(V) \to \varinjlim_{V \supset f(U')} \mathcal{G}(V).$$

Remark 2.6.7. Note that if $f(U)$ is an open subset of Y, then

$$(f^{-1}_{\text{pre}}\mathcal{G})(U) = \mathcal{G}(f(U)).$$

Now assume $\mathcal{G}$ is a sheaf. We define the *inverse image* of $\mathcal{G}$ by f to be the sheafification

$$f^{-1}\mathcal{G} = \left(f^{-1}_{\text{pre}}\mathcal{G}\right)^{\#}.$$

Note that there is a canonical map $f^{-1}_{\text{pre}}\mathcal{G} \to f^{-1}\mathcal{G}$ of presheaves inducing an isomorphism on all the stalks.

Exercise 2.6.8. Both f_{pre}^{-1} and f^{-1} are functors.

Example 2.6.9. Let $\iota_y \colon \{y\} \hookrightarrow Y$ be the inclusion of a point $y \in Y$, and let $\mathcal{G}$ be a sheaf on Y. Then $\iota_y^{-1}\mathcal{G} = \mathcal{G}_y$, since $\iota_y^{-1}\mathcal{G}(\{y\}) = \varinjlim_{V \ni y} \mathcal{G}(V) = \mathcal{G}_y$. Thus ι_y^{-1} agrees with the stalk functor

$$\mathsf{stalk}_y \colon \mathsf{Sh}(Y,\mathscr{C}) \to \mathscr{C}, \quad \mathcal{G} \mapsto \mathcal{G}_y.$$

Example 2.6.10. If $p\colon X \to \mathrm{pt}$ is the constant map, and $G \in \mathscr{C} \cong \mathsf{Sh}(\mathrm{pt},\mathscr{C})$, then $p^{-1}G = \underline{G}_X$, the constant sheaf on X with values in the object G.

Example 2.6.11. Let $j\colon U \hookrightarrow Y$ be the inclusion of an open subset. Then $j_{\mathrm{pre}}^{-1}\mathcal{G} = \mathcal{G}|_U$ for any sheaf $\mathcal{G}$ on Y. The reason is that if U' is open in U, it is also open in Y, and thus $j_{\mathrm{pre}}^{-1}\mathcal{G}(U') = \varinjlim_{V \supset U'} \mathcal{G}(V) = \mathcal{G}(U')$. In particular, $j_{\mathrm{pre}}^{-1}\mathcal{G}$ is already a sheaf, and hence

$$j^{-1}\mathcal{G} = \mathcal{G}|_U, \quad U \subset Y \text{ open}.$$

Remark 2.6.12. Sheafification is indeed necessary: consider a constant map $f\colon X = \{\star, \bullet\} \to \{*\} = Y$ from a two point set, and fix a nontrivial abelian group G. The constant sheaf $\mathcal{G} = \underline{G}_Y$ has the property $f_{\mathrm{pre}}^{-1}\mathcal{G} = \underline{G}_X^{\mathrm{pre}}$, which is not a sheaf (cf. Example 2.2.17).

Functoriality can be translated into a diagram of functors

$$\begin{tikzcd}
\mathsf{Sh}(Y,\mathscr{C}) \arrow[r, "f^{-1}"] \arrow[d, hook] & \mathsf{Sh}(X,\mathscr{C}) \arrow[d, hook] \\
\mathsf{pSh}(Y,\mathscr{C}) \arrow[r, "f_{\mathrm{pre}}^{-1}"] & \mathsf{pSh}(X,\mathscr{C})
\end{tikzcd}$$

where f^{-1} is obtained by applying $(-)^{\#}\colon \mathsf{pSh}(X,\mathscr{C}) \to \mathsf{Sh}(X,\mathscr{C})$ in the last step.

Exercise 2.6.13. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be continuous maps of topological spaces, and let $\mathcal{E}$ be a sheaf on Z. Show that

$$f^{-1}(g^{-1}\mathcal{E}) = (g \circ f)^{-1}\mathcal{E}.$$

LEMMA 2.6.14 (Unit and counit maps). *For any pair of presheaves $\mathcal{F} \in \mathsf{pSh}(X,\mathscr{C})$ and $\mathcal{G} \in \mathsf{pSh}(Y,\mathscr{C})$ there are canonical presheaf homomorphisms*

$$\mathcal{G} \xrightarrow{\ \mathsf{unit}\ } f_* f_{\mathrm{pre}}^{-1}\mathcal{G}, \qquad f_{\mathrm{pre}}^{-1} f_* \mathcal{F} \xrightarrow{\ \mathsf{counit}\ } \mathcal{F}.$$

Proof. We start with the unit map. The observation here is that there is, for any open subset $V \subset Y$, a canonical inclusion $f(f^{-1}V) \hookrightarrow V$. Thus $\mathcal{G}(V)$ appears in the colimit

$$\varinjlim_{W \supset f(f^{-1}V)} \mathcal{G}(W)$$

This induces a canonical morphism

$$\mathsf{unit}_V \colon \mathcal{G}(V) \to \varinjlim_{W \supset f(f^{-1}V)} \mathcal{G}(W) = f_{\mathrm{pre}}^{-1}\mathcal{G}(f^{-1}V) = f_* f_{\mathrm{pre}}^{-1}\mathcal{G}(V)$$

which does define a natural transformation $\mathcal{G} \to f_* f_{\mathrm{pre}}^{-1}\mathcal{G}$ because if $V' \subset V$, then any open $W \subset Y$ containing $f(f^{-1}V)$ also contains $f(f^{-1}V')$, simply because $f(f^{-1}V') \subset f(f^{-1}V)$. Thus there is a natural morphism

$$\varinjlim_{W \supset f(f^{-1}V)} \mathcal{G}(W) \to \varinjlim_{W \supset f(f^{-1}V')} \mathcal{G}(W)$$

and the induced diagram

$$\begin{tikzcd}
\mathcal{G}(V) \arrow[r, "\mathsf{unit}_V"] \arrow[d] & \varinjlim_{W \supset f(f^{-1}V)} \mathcal{G}(W) \arrow[d] \\
\mathcal{G}(V') \arrow[r, "\mathsf{unit}_{V'}"] & \varinjlim_{W \supset f(f^{-1}V')} \mathcal{G}(W)
\end{tikzcd}$$

commutes. This defines the map $\mathsf{unit}\colon \mathcal{G} \to f_* f_{\mathrm{pre}}^{-1}\mathcal{G}$ of presheaves.

To construct the counit map, one observes that for any open subset $U \subset X$ there is a canonical map

$$f_{\mathrm{pre}}^{-1} f_* \mathcal{F}(U) = \varinjlim_{V \supset f(U)} f_*\mathcal{F}(V) = \varinjlim_{V \supset f(U)} \mathcal{F}(f^{-1}V) \to \mathcal{F}(U),$$

since if $V \supset f(U)$ inside Y, then $U \subset f^{-1}f(U) \subset f^{-1}V$ inside X. This map is also functorial in $U' \subset U$, thus the map $\mathsf{counit}\colon f_{\mathrm{pre}}^{-1} f_*\mathcal{F} \to \mathcal{F}$ is defined. $\square$

The usefulness of the homomorphisms unit and counit is that they make

$$(f_{\mathrm{pre}}^{-1}, f_*)$$

into an adjoint pair of functors. More precisely, there are bijections

$$\varphi_{\mathcal{F},\mathcal{G}}\colon \mathrm{Hom}_{\mathsf{pSh}(Y,\mathscr{C})}(\mathcal{G}, f_*\mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{pSh}(X,\mathscr{C})}(f_{\mathrm{pre}}^{-1}\mathcal{G}, \mathcal{F}),$$

functorial in both $\mathcal{F}$ and $\mathcal{G}$. Specifically, $\varphi_{\mathcal{F},\mathcal{G}}$ sends

$$\mathcal{G} \xrightarrow{\eta} f_*\mathcal{F} \qquad \mapsto \qquad f_{\mathrm{pre}}^{-1}\mathcal{G} \xrightarrow{f_{\mathrm{pre}}^{-1}\eta} f_{\mathrm{pre}}^{-1} f_*\mathcal{F} \xrightarrow{\mathsf{counit}} \mathcal{F}$$

with inverse

$$f_{\mathrm{pre}}^{-1}\mathcal{G} \xrightarrow{\iota} \mathcal{F} \qquad \mapsto \qquad \mathcal{G} \xrightarrow{\mathsf{unit}} f_* f_{\mathrm{pre}}^{-1}\mathcal{G} \xrightarrow{f_*\iota} f_*\mathcal{F}.$$

Using Lemma 2.6.14, it is immediate to show that also

$$(f^{-1}, f_*)$$

is an adjoint pair $\mathsf{Sh}(Y,\mathscr{C}) \rightleftarrows \mathsf{Sh}(X,\mathscr{C})$ on *sheaves*. Indeed,

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{Sh}(Y,\mathscr{C})}(\mathcal{G}, f_*\mathcal{F}) &= \mathrm{Hom}_{\mathsf{pSh}(Y,\mathscr{C})}(\mathcal{G}, f_*\mathcal{F}) && j_{Y,\mathscr{C}} \text{ is full} \\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathsf{pSh}(X,\mathscr{C})}(f_{\mathrm{pre}}^{-1}\mathcal{G}, \mathcal{F}) && \text{Lemma 2.6.14} \\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Sh}(X,\mathscr{C})}(f^{-1}\mathcal{G}, \mathcal{F}) && \text{Exercise 2.4.5.}
\end{aligned}$$

Terminology 2.6.15. If $\mathcal{F} = f^{-1}\mathcal{G}$, the map $\mathcal{G} \to f_* f^{-1}\mathcal{G}$ corresponding to $\mathrm{id}_{f^{-1}\mathcal{G}}$ is called the *unit* of the adjunction. If $\mathcal{G} = f_*\mathcal{F}$, the map $f^{-1} f_*\mathcal{F} \to \mathcal{F}$ corresponding to $\mathrm{id}_{f_*\mathcal{F}}$ is called the *counit* of the adjunction.

The next lemma says that the stalk of the inverse image is somewhat easy to compute (unlike for the pushforward).

LEMMA 2.6.16 (Stalk of inverse image). *Let $f: X \to Y$ be a continuous map of topological spaces, $\mathcal{G}$ a sheaf on Y, and $x \in X$ a point. There is a canonical identification*

$$(f^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)}.$$

Proof. We have

$$\begin{aligned}
(f^{-1}\mathcal{G})_x &= (\{x\} \hookrightarrow X)^{-1}(f^{-1}\mathcal{G}) && \text{by Example 2.6.9} \\
&= (\{x\} \hookrightarrow X \to Y)^{-1}\mathcal{G} && \text{by Exercise 2.6.13} \\
&= (\{f(x)\} \hookrightarrow Y)^{-1}\mathcal{G} \\
&= \mathcal{G}_{f(x)}
\end{aligned}$$

where we have used Example 2.6.9 once more for the last identity. □

LEMMA 2.6.17. *Let $f: X \hookrightarrow Y$ be the inclusion of a closed subspace, $\mathcal{G}$ a sheaf on Y such that* $\mathrm{Supp}(\mathcal{G}) \subset X$ *(cf. Section 2.3.1). Then, the unit map $\mathcal{G} \to f_* f^{-1}\mathcal{G}$ is an isomorphism.*

Proof. It is enough to verify that $\mathsf{unit}_y : \mathcal{G}_y \to (f_* f^{-1}\mathcal{G})_y$ is an isomorphism for every $y \in Y$. We identify X with its image in Y, as usual. If $y \notin X$, then $\mathcal{G}_y = 0$ maps to

$$\begin{aligned}
(f_* f^{-1}\mathcal{G})_y &= \varinjlim_{V \ni y} (f_* f^{-1}\mathcal{G})(V) \\
&= \varinjlim_{V \ni y} f^{-1}\mathcal{G}(f^{-1}V) \\
&= 0,
\end{aligned}$$

where the vanishing is caused by $y \notin V \cap X = f^{-1}V$. Let us then assume $y \in \mathrm{Supp}(\mathcal{G})$, so that $\mathcal{G}_y \neq 0$. In this case,

$$\mathsf{unit}_y : \mathcal{G}_y = (f^{-1}\mathcal{G})_y \xrightarrow{\sim} (f_* f^{-1}\mathcal{G})_y$$

is the inverse of the canonical map of Lemma 2.6.4. □

Exercise 2.6.18. Find examples of maps f and sheaves $\mathcal{G}$ such that $\mathcal{G} \to f_* f^{-1}\mathcal{G}$ is not an isomorphism.

Remark 2.6.19. If $j : X \hookrightarrow Y$ is *open* and $\mathcal{G}$ is a sheaf on Y, then $j_* j^{-1}\mathcal{G}$ satisfies

$$j_* j^{-1}\mathcal{G}(V) = (j_*\mathcal{G}|_X)(V) = \mathcal{G}(V \cap X), \quad V \subset Y \text{ open.}$$

The natural map $\mathcal{G}(V) \to j_* j^{-1}\mathcal{G}(V)$ sends $s \mapsto s|_{V\cap X}$.

Exercise 2.6.20. Show that if $f : X \hookrightarrow Y$ is the inclusion of a subspace, then the counit

$$f^{-1} f_* \mathcal{F} \to \mathcal{F}$$

is an isomorphism for every $\mathcal{F} \in \mathsf{Sh}(X, \mathsf{Mod}_A)$.

PROPOSITION 2.6.21. *Let $\mathscr{C} = \mathsf{Mod}_A$, for a ring A. Then the inverse image functor f^{-1} is exact.*

Proof. Indeed, let

$$0 \to \mathcal{G} \to \mathcal{H} \to \mathcal{K} \to 0$$

be an exact sequence in $\mathsf{Sh}(Y, \mathsf{Mod}_A)$. Then,

$$0 \to \mathcal{G}_{f(x)} \to \mathcal{H}_{f(x)} \to \mathcal{K}_{f(x)} \to 0$$

is exact in Mod_A by Proposition 2.4.14, for every $x \in X$. But this is precisely the sequence

$$0 \to (f^{-1}\mathcal{G})_x \to (f^{-1}\mathcal{H})_x \to (f^{-1}\mathcal{K})_x \to 0.$$

Thus

$$0 \to f^{-1}\mathcal{G} \to f^{-1}\mathcal{H} \to f^{-1}\mathcal{K} \to 0$$

is exact, again by Proposition 2.4.14. $\square$

3 | The definition of schemes

3.1 Locally ringed spaces

The goal of this chapter is to introduce the category of *affine schemes* and the larger category of all *schemes*. We will define the category of locally ringed spaces first, and the category of schemes will be a full subcategory

$$\text{Affine Schemes} \subset \text{Schemes} \subset \text{Locally ringed spaces}.$$

Definition 3.1.1 (Locally ringed space)**.** A *locally ringed space* is a pair $(X, \mathscr{O}_X)$ where X is a topological space and $\mathscr{O}_X$ is a sheaf of commutative rings on X, such that the stalk $\mathscr{O}_{X,x}$ is a local ring for every $x \in X$. The sheaf $\mathscr{O}_X$ is called the *structure sheaf.*

Notation 3.1.2. Let $(X, \mathscr{O}_X)$ be a locally ringed space, $x \in X$ a point. We will write $\mathfrak{m}_x$ for the maximal ideal $\mathscr{O}_{X,x}$, and $\kappa(x) = \mathscr{O}_{X,x}/\mathfrak{m}_x$ for the corresponding *residue field.*

Recall that, given two local rings $(B, \mathfrak{m}_B)$ and $(A, \mathfrak{m}_A)$, a local homomorphism between them is a ring homomorphism $h \colon B \to A$ such that $h^{-1}(\mathfrak{m}_A) = \mathfrak{m}_B$, or, equivalently, $h(\mathfrak{m}_B) \subset \mathfrak{m}_A$.

Definition 3.1.3 (Morphism of locally ringed spaces)**.** A morphism of locally ringed spaces

$$(X, \mathscr{O}_X) \xrightarrow{(f, f^\#)} (Y, \mathscr{O}_Y) \tag{3.1.1}$$

is a pair $(f, f^\#)$ where $f \colon X \to Y$ is a continuous map of topological spaces and $f^\# \colon \mathscr{O}_Y \to f_* \mathscr{O}_X$ is a sheaf homomorphism on Y, such that $f^\#_x \colon \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is a local homomorphism of local rings for every $x \in X$.

Notation 3.1.4. In what follows, we shall often omit the sheaf of rings from the notation, and simply write X to denote the locally ringed space $(X, \mathscr{O}_X)$, or $f \colon X \to Y$ to denote a morphism of locally ringed spaces $(f, f^\#)$ as in (3.1.1).

Remark 3.1.5. Let $f \colon X \to Y$ be a morphism of locally ringed spaces. Set $y = f(x)$. The local homomorphism $f^\#_x \colon \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is the composition of the stalk map $f^\#_y \colon \mathscr{O}_{Y,y} \to (f_* \mathscr{O}_X)_y$ and the morphism $(f_* \mathscr{O}_X)_y \to \mathscr{O}_{X,x}$ of Lemma 2.6.4.

Example 3.1.6. Let $(X, \mathcal{O}_X)$ be a locally ringed space, $U \subset X$ an open subset. Then $(U, \mathcal{O}_X|_U)$ is a locally ringed space. We shall *always* take $\mathcal{O}_X|_U$ as the structure sheaf of an open subset $U \subset X$ of a locally ringed space X. We denote it by $\mathcal{O}_U$.

The composition of two morphisms of locally ringed spaces is defined in a straightforward way (but you need to know that pushforward commutes with composition, see (2.6.1)). Locally ringed spaces thus form a (large) category, where isomorphisms are simply the invertible morphisms (those that compose to the identity both ways).

Remark 3.1.7. A morphism of locally ringed spaces $(f, f^\#)$ as in (3.1.1) is an isomorphism if and only if f is a homeomorphism and $f^\#: \mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism of sheaves.

Definition 3.1.8 (Immersions)**.** Let $(f, f^\#): (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of locally ringed spaces. It is called an *open immersion* (resp. a *closed immersion*) if $f: X \to Y$ is a topological open immersion (resp. closed immersion) and $f^\#_x$ is an isomorphism (resp. surjective) for every $x \in X$.

It follows immediately that $(f, f^\#)$ as above is an open immersion if and only if there exists an open subset $V \subset Y$ such that $(f, f^\#)$ induces an isomorphism $(X, \mathcal{O}_X) \xrightarrow{\sim} (V, \mathcal{O}_Y|_V)$. It is also clear that the composition of two open (resp. closed) immersions is an open immersion (resp. a closed immersion).

3.1.1 Closed immersions = Ideal sheaves

Closed immersions are quite subtle. They are characterised by *ideal sheaves*, in a way that we now describe.

Fix a locally ringed space $(X, \mathcal{O}_X)$. Define an ideal sheaf (or a sheaf of ideals) to be a subsheaf $\mathcal{I} \subset \mathcal{O}_X$ (as abelian groups) such that $\mathcal{I}(U) \subset \mathcal{O}_X(U)$ is an ideal for every open subset $U \subset X$. Given an ideal sheaf $\mathcal{I}$, the subset

$$\mathrm{V}(\mathcal{I}) = \left\{ x \in X \,\middle|\, \mathcal{I}_x \neq \mathcal{O}_{X,x} \right\} \xhookrightarrow{j} X$$

is a closed subset. Indeed, for any $x \in X \setminus \mathrm{V}(\mathcal{I})$, i.e. for any x such that $\mathcal{I}_x = \mathcal{O}_{X,x}$, there is a neighborhood U of x and a section $f \in \mathcal{I}(U)$ such that $f_x = 1 \in \mathcal{O}_{X,x}$. But this means that $f|_V = 1 \in \mathcal{O}_X(V)$ for some open subset $V \subset U$. Thus $V \subset X \setminus \mathrm{V}(\mathcal{I})$, and thus $X \setminus \mathrm{V}(\mathcal{I})$ is open.

The quotient sheaf $\mathcal{O}_X/\mathcal{I}$ is a sheaf of rings (because, by definition, it is the sheafification of a sheaf of rings), not just abelian groups. The pair

$$(\mathrm{V}(\mathcal{I}), j^{-1}(\mathcal{O}_X/\mathcal{I}))$$

defines a locally ringed space (indeed, for any $x \in \mathrm{V}(\mathcal{I})$, the stalk $(j^{-1}(\mathcal{O}_X/\mathcal{I}))_x = (\mathcal{O}_X/\mathcal{I})_{j(x)} = \mathcal{O}_{X,j(x)}/\mathcal{I}_{j(x)}$ is a local ring: we have used Lemma 2.6.16 and Remark 2.4.11), and the canonical surjection

$$j^{\#}\colon \mathscr{O}_X \twoheadrightarrow \mathscr{O}_X/\mathscr{I} = j_* j^{-1}(\mathscr{O}_X/\mathscr{I})$$

turns $(j, j^{\#})$ into a closed immersion $\mathrm{V}(\mathscr{I}) \hookrightarrow X$. Note that we have used Lemma 2.6.17 for the identification $\mathscr{O}_X/\mathscr{I} = j_* j^{-1}(\mathscr{O}_X/\mathscr{I})$. So we have defined an assignment

$$\mathscr{O}_X \supset \mathscr{I} \qquad \longmapsto \qquad \mathrm{V}(\mathscr{I}) \hookrightarrow X.$$

Conversely, to any closed immersion $j\colon Z \hookrightarrow X$ one can associate an ideal sheaf, namely

$$\mathscr{I}_Z = \ker(\mathscr{O}_X \xrightarrow{j^{\#}} j_*\mathscr{O}_Z).$$

LEMMA 3.1.9. *The correspondence sending* $\mathscr{I} \mapsto \mathrm{V}(\mathscr{I})$ *is bijective, with inverse sending* $j\colon Z \hookrightarrow X$ *to* $\mathscr{I}_Z = \ker j^{\#}$. *Moreover, an inclusion of ideal sheaves* $\mathscr{I}_2 \hookrightarrow \mathscr{I}_1 \hookrightarrow \mathscr{O}_X$ *induces a chain of closed immersions* $\mathrm{V}(\mathscr{I}_1) \hookrightarrow \mathrm{V}(\mathscr{I}_2) \hookrightarrow X$.

3.2 Affine schemes

Let A be a nonzero ring (commutative, with unit). The set

$$\boxed{\operatorname{Spec} A = \{\, \mathfrak{p} \subset A \mid \mathfrak{p} \text{ is a prime ideal} \,\}}$$

is called the *prime spectrum* of A. For now, this is just a set. We will endow it with a topology and with a sheaf of rings having local rings as stalks, to obtain a locally ringed space. Such locally ringed space will be called an *affine scheme* (cf. Definition 3.2.26). General schemes can be obtained by glueing affine schemes, just as a smooth manifold is obtained by glueing open subsets of $\mathbb{R}^n$.

Notation 3.2.1. We introduce the following notation, that will be used throughout: given a ring B, the spectrum

$$\mathbb{A}^n_B = \operatorname{Spec} B[x_1, \ldots, x_n]$$

will be called *affine n-space* over B. If $n = 1$ (resp. $n = 2$), we speak of *affine line* (resp. *affine plane*) over B.

3.2.1 The Zariski topology on Spec A

For an arbitrary ideal $I \subset A$, set

$$\boxed{\mathrm{V}(I) = \{\, \mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \supset I \,\} \subset \operatorname{Spec} A.}$$

Note that there is a bijection

$$\mathrm{V}(I) \simeq \operatorname{Spec} A/I,$$

since (prime) ideals of A/I correspond precisely to (prime) ideals in A containing I.

If $I = (f) = fA \subset A$ (we will use both notations for principal ideals) for $f \in A$, simply write $\mathrm{V}(f)$ instead of $\mathrm{V}(I)$, and define

$$\mathrm{D}(f) = \operatorname{Spec} A \setminus \mathrm{V}(f) = \left\{ \mathfrak{p} \in \operatorname{Spec} A \,\middle|\, f \notin \mathfrak{p} \right\}.$$

Example 3.2.2. Let $\mathbf{k}$ be an algebraically closed field. If $f \in \mathbf{k}[x]$ is nonzero, then $\mathrm{D}(f)$ consists of those prime ideals $\mathfrak{p} \subset \mathbf{k}[x]$ such that $f \notin \mathfrak{p}$. One such ideal is the trivial ideal (0), and the other ideals $\mathfrak{p}$ with this property are all the ideals of the form $\mathfrak{p} = (x - a)$, for $a \in \mathbf{k}$, such that $f(a) \neq 0 \in \mathbf{k}$.

Note that, for any nonzero ring A, one has

$$\operatorname{Spec} A = \mathrm{D}(1), \quad \emptyset = \mathrm{D}(0).$$

Let $(I_\lambda)_{\lambda\in\Lambda}$ be an arbitrary family of ideals. Recall that $\sum_{\lambda\in\Lambda} I_\lambda$ is, by definition, the ideal generated by the union of the ideals (which is not an ideal in general). It can be described set-theoretically as

$$\sum_{\lambda\in\Lambda} I_\lambda = \left\{ a_1 i_{\lambda_1} + \cdots + a_p i_{\lambda_p} \,\middle|\, a_j \in A,\, i_{\lambda_j} \in I_{\lambda_j},\, p < \infty \right\}. \tag{3.2.1}$$

LEMMA 3.2.3. *Let A be a ring.*

(1) If $I, J \subset A$ are two ideals, then $\mathrm{V}(I) \cup \mathrm{V}(J) = \mathrm{V}(I \cap J)$.

(2) If $(I_\lambda)_{\lambda\in\Lambda}$ is an arbitrary family of ideals, then $\bigcap_{\lambda\in\Lambda} \mathrm{V}(I_\lambda) = \mathrm{V}\left(\sum_{\lambda\in\Lambda} I_\lambda\right)$.

(3) $\operatorname{Spec} A = \mathrm{V}(0)$ and $\emptyset = \mathrm{V}(1)$.

Proof. This is straightforward. However, here is the proof:

(1) If $\mathfrak{p} \subset A$ contains either I or J, then it contains the smaller ideal $I \cap J$, thus $\mathrm{V}(I) \cup \mathrm{V}(J) \subset \mathrm{V}(I \cap J)$. If $\mathfrak{p} \supset I \cap J$ but $\mathfrak{p} \not\supset I$, there is $h \in I$ such that $h \notin \mathfrak{p}$. If $j \in J$, then $jh \in I \cap J \subset \mathfrak{p}$, which implies $j \in \mathfrak{p}$ (because $\mathfrak{p}$ is prime), thus $J \subset \mathfrak{p}$. Therefore $\mathrm{V}(I \cap J) \subset \mathrm{V}(I) \cup \mathrm{V}(J)$.

(2) If $\mathfrak{p}$ contains the sum $\sum_\lambda I_\lambda$, then it contains each I_λ, therefore $V\left(\sum_\lambda I_\lambda\right) \subset \bigcap_\lambda \mathrm{V}(I_\lambda)$. On the other hand, assume $\mathfrak{p} \supset I_\lambda$ for every index λ. Let $h = a_1 i_{\lambda_1} + \cdots + a_p i_{\lambda_p}$ as in Equation (3.2.1). Then $a_j i_{\lambda_j} \in \mathfrak{p}$ by assumption, thus $h \in \mathfrak{p}$ as well, i.e. $\mathfrak{p} \supset \sum_\lambda I_\lambda$.

(3) Every prime $\mathfrak{p} \subset A$ contains $0 \in A$. No prime ideal $\mathfrak{p} \subset A$ contains $1 \in A$ (here we use that $1 \neq 0$). □

COROLLARY 3.2.4. *There exists a unique topology on $\operatorname{Spec} A$ whose closed sets are of the form $\mathrm{V}(I)$. Moreover, the sets $\mathrm{D}(f) \subset \operatorname{Spec} A$ form a base of open sets for this topology (according to Definition 2.5.1).*

Proof. The first statement is clear from the definition of a topology. The second one follows by these observations:

(i) $\mathrm{D}(f_1)\cap \mathrm{D}(f_2) = \mathrm{D}(f_1 f_2)$,

(ii) an open subset $\operatorname{Spec} A \setminus \mathrm{V}(I)$ can be written as $\bigcup_{f\in I} \mathrm{D}(f)$. □

For instance,

$$\operatorname{Spec} A = \mathrm{D}(1) = \operatorname{Spec} A \setminus \mathrm{V}(1) = \bigcup_{f\in A} \mathrm{D}(f). \tag{3.2.2}$$

Definition 3.2.5 (Zariski topology)**.** The topology on $\operatorname{Spec} A$ given by Corollary 3.2.4 is called the *Zariski topology*.

Terminology 3.2.6. We call $\mathrm{D}(f)$ a *principal open set* in $\operatorname{Spec} A$, and $\mathrm{V}(f)$ a *principal closed set* in $\operatorname{Spec} A$.

Convention 3.2.7. When thinking of $\operatorname{Spec} A$ as a topological space, it will *always* be endowed with the Zariski topology.

Consider the ideals $I_r = (x^r) \subset \mathbb{C}[x]$ for some $r > 0$. Then $\mathrm{V}(I_1) = \{\mathfrak{p} \subset \mathbb{C}[x] \mid \mathfrak{p} \supset (x)\} = \{(x)\} = \mathrm{V}(I_r)$ for every r. Thus we see that we may end up in the situation

$$\mathrm{V}(I) = \mathrm{V}(J), \quad I \neq J.$$

It is useful to characterise when this happens, in a precise manner. Recall that the *radical* of an ideal $I \subset A$ is the ideal

$$\sqrt{I} = \left\{ a \in A \,\middle|\, a^r \in I \text{ for some } r > 0 \right\} \subset A.$$

It clearly contains I, and satisfies

$$\mathrm{V}(I) = \mathrm{V}(\sqrt{I}) \subset \operatorname{Spec} A.$$

LEMMA 3.2.8. *Let $I, J \subset A$ be two ideals in a ring A. Then*

$$\mathrm{V}(I) \subset \mathrm{V}(J) \Longleftrightarrow J \subset \sqrt{I}.$$

Proof. This is a rephrasing of the identity $\sqrt{I} = \bigcap_{\mathfrak{p}\in \mathrm{V}(I)} \mathfrak{p}$. □

Example 3.2.9. Let $f, g \in A$. We have

$$\begin{aligned} \mathrm{D}(g) \subset \mathrm{D}(f) &\Longleftrightarrow \mathrm{V}(f) \subset \mathrm{V}(g) \\ &\Longleftrightarrow g \in \sqrt{(f)}. \end{aligned}$$

LEMMA 3.2.10 (Spec A is quasicompact). *Let K be an index set for a collection of elements $\{ f_k \mid k \in K \} \subset A$. Then* $\operatorname{Spec} A = \bigcup_{k\in K} \mathrm{D}(f_k)$ *if and only if there is a finite subset $F \subset K$ such that one can write* $1 = \sum_{k\in F} a_k f_k$ *for some $a_k \in A \setminus 0$. In particular,* Spec A *equipped with the Zariski topology is quasicompact.*

Proof. Let $(-)^{\mathrm{c}}$ denote the complement of a subset of Spec A. We have that

$$\begin{aligned}\bigcup_{k\in K} \mathrm{D}(f_k) &= \bigcup_{k\in K} \operatorname{Spec} A \setminus \mathrm{V}(f_k)\\ &= \left(\bigcap_{k\in K} \mathrm{V}(f_k)\right)^{\mathrm{c}}\\ &= \mathrm{V}\left(\sum_{k\in K} f_k A\right)^{\mathrm{c}}\end{aligned}$$

equals Spec A if and only if

$$\mathrm{V}\left(\sum_{k\in K} f_k A\right) = \emptyset = \mathrm{V}(1),$$

which by Lemma 3.2.8 happens if and only if $\sqrt{\sum_{k\in K} f_k A} = (1)$. The first assertion follows from the definition of sum of ideals. Quasicompactness follows from the first claim combined with the fact that principal open subsets of Spec A form a base of the Zariski topology. □

Remark 3.2.11. The proof of Lemma 3.2.10 also shows that any principal open subset $\mathrm{D}(f) \subset \operatorname{Spec} A$ is quasicompact, and that an open subset $U \subset \operatorname{Spec} A$ is quasicompact if and only if it is a finite union of principal opens.

Warning 3.2.12. Not *every* open subset $U \subset \operatorname{Spec} A$ is quasicompact! For instance, consider $A = \mathbf{k}[x_i | i \in \mathbb{N}]$, and let $U \subset \operatorname{Spec} A$ be the complement of the origin (the point corresponding to the maximal ideal $(x_i | i \in \mathbb{N}) \subset A$). Then the covering $U = \bigcup_{i\in\mathbb{N}} U \setminus \mathrm{V}(x_i)$ has no finite subcover.

Remark 3.2.13 (Closed points = maximal ideals)**.** Let $\mathfrak{p} \in \operatorname{Spec} A$ be a closed point, i.e. such that $\{\mathfrak{p}\} \subset \operatorname{Spec} A$ is closed. Then $\{\mathfrak{p}\} = \mathrm{V}(I) = \{\mathfrak{q} \mid \mathfrak{q} \supset I\}$ for an ideal $I \subset A$. This says that $\mathfrak{p}$ is the only prime ideal containing I. But any ideal sits inside a maximal ideal, and maximal ideals are prime. Thus $\mathfrak{p}$ is maximal. Conversely, if $\mathfrak{m} \subset A$ is maximal, then $\{\mathfrak{m}\} = \mathrm{V}(\mathfrak{m})$, in particular $\{\mathfrak{m}\} \subset \operatorname{Spec} A$ is closed, i.e. $\mathfrak{m} \in \operatorname{Spec} A$ is a closed point.

The previous remark can be generalised by the following lemma.

LEMMA 3.2.14. *Let $T \subset \operatorname{Spec} A$ be a subset, $\overline{T} \subset \operatorname{Spec} A$ its closure. Then*

$$\overline{T} = \mathrm{V}\left(\bigcap_{\mathfrak{p}\in T} \mathfrak{p}\right).$$

In particular, the closure of $\{\mathfrak{p}\} \subset \operatorname{Spec} A$ *is precisely* $\mathrm{V}(\mathfrak{p})$.

Proof. We have

$$\overline{T} = \bigcap_{\mathrm{V}(I)\supset T} \mathrm{V}(I) = \mathrm{V}\left(\sum_{\mathrm{V}(I)\supset T} I\right).$$

But by definition $\mathrm{V}(I) \supset T$ means that every $\mathfrak{p} \in T$ satisfies $\mathfrak{p} \supset I$, so the sum is over all $I \subset A$ such that $I \subset \bigcap_{\mathfrak{p}\in T} \mathfrak{p}$. The largest such ideal is precisely $\bigcap_{\mathfrak{p}\in T} \mathfrak{p}$, which concludes the proof. □

Remark 3.2.15. Combining the previous topological observations, we conclude that

$$\boxed{\text{the Zariski topology on } \operatorname{Spec} A \text{ is almost never Hausdorff, or even } \mathrm{T}_1.}$$

For instance, take an integral domain A that is not a field, so that $(0) \subset A$ is prime and the closure of the corresponding point $\xi \in \operatorname{Spec} A$ is equal to $\mathrm{V}(0) = \operatorname{Spec} A$ by Lemma 3.2.14. Then any two nonempty open subsets intersect (thus $\operatorname{Spec} A$ is not Hausdorff), and in fact every open neighborhood of a point $\mathfrak{p} \in \operatorname{Spec} A$ will also contain ξ (thus $\operatorname{Spec} A$ is not T_1).

3.2.2 Interlude: functions on Spec A

The following slogan is important (but it will take a while to digest):

$$\boxed{\text{elements of } A \text{ are functions on } \operatorname{Spec} A.}$$

The slogan is a bit premature, since by 'function on Spec A' we actually mean 'regular function on *the scheme* Spec A, and so far we only have a topological space, we haven't yet defined the sheaf of regular functions. However, it is worth explaining the slogan just to build some intuition.

Here is the explanation. To any $f \in A$, we can associate the map

$$\theta_f \colon \operatorname{Spec} A \to \coprod_{\mathfrak{p}\in\operatorname{Spec} A} A/\mathfrak{p}, \quad \mathfrak{p} \mapsto f \bmod \mathfrak{p}.$$

For instance, $f = 9 \in \mathbb{Z}$ takes the value $[1]$ in $\mathbb{Z}/2\mathbb{Z}$, and the value $[4]$ in $\mathbb{Z}/5\mathbb{Z}$. Its value in $\mathbb{Z}/0 = \mathbb{Z}$ is just... 9. Of course, the most confusing thing here is that the ring where the function takes values depends on the point on which the function is evaluated! Now obviously the function '9' vanishes on the point $(3) \in \operatorname{Spec}\mathbb{Z}$. In general,

$$\boxed{\theta_f(\mathfrak{p}) = 0 \in A/\mathfrak{p} \text{ if and only if } f \in \mathfrak{p}.}$$

Note also that addition and multiplication of 'functions' works as one might expect, i.e. $\theta_{f+g}(\mathfrak{p}) = f + g \bmod \mathfrak{p} = \theta_f(\mathfrak{p}) + \theta_g(\mathfrak{p})$, and similarly $\theta_{fg}(\mathfrak{p}) = fg \bmod \mathfrak{p} = \theta_f(\mathfrak{p})\theta_g(\mathfrak{p})$. This is just a rephrasing of the fact that $A \to A/\mathfrak{p}$ is a ring homomorphism!

One more example: consider $\mathbb{C}[x]$, and the 'function' $f(x) = 2x^2 - x + 3 \in \mathbb{C}[x]$. The prime ideals of $\mathbb{C}[x]$ are $(0) \subset \mathbb{C}[x]$, and the maximal ideals $\mathfrak{m}_a = (x - a) \subset \mathbb{C}[x]$ for $a \in \mathbb{C}$.

The value of f on the point $\mathfrak{m}_a \in \operatorname{Spec}\mathbb{C}[x]$ is just the evaluation of the polynomial $f(x)$ at $x = a$. Indeed,

$$\theta_f(\mathfrak{m}_a) = f \bmod \mathfrak{m}_a \in \mathbb{C}[x]/\mathfrak{m}_a,$$

corresponds to the element

$$f(a) = 2a^2 - a + 3 \in \mathbb{C} \cong \mathbb{C}[x]/\mathfrak{m}_a.$$

3.2.3 First examples of ring spectra

In this subsection we analyse the Zariski topology on $\operatorname{Spec} A$ for a few interesting rings A.

Example 3.2.16 (Spec $\mathbf{k}$ aka the point)**.** Let $\mathbf{k}$ be a field. The spectrum $\operatorname{Spec}\mathbf{k}$ consists of a single point corresponding to $(0) \subset \mathbf{k}$. Its 'functions' are just the constants $\mathbf{k}$, as expected. For now, this (merely topological and hence dry) description is enough. However, when

Figure 3.1: This is $\operatorname{Spec}\mathbf{k}$. Nothing more, nothing less.

$\operatorname{Spec}\mathbf{k}$ will be endowed with a scheme structure, things will change: for instance, as we shall see, it is not true that the only function $\operatorname{Spec}\mathbf{k} \to \operatorname{Spec}\mathbf{k}$ is the identity function!

Example 3.2.17 ($\mathbb{A}^1_{\mathbf{k}} = \operatorname{Spec}\mathbf{k}[x]$)**.** Let $\mathbf{k}$ be an algebraically closed field, such as $\mathbb{C}$. The ring $\mathbf{k}[x]$ is a principal ideal domain, whose prime ideals are (0) and $(x-a)$, one for each $a \in \mathbf{k}$. The spectrum

$$\mathbb{A}^1_{\mathbf{k}} = \operatorname{Spec}\mathbf{k}[x]$$

is called the *affine line* (over $\mathbf{k}$). Note that there is exactly one point, namely

$$\xi = (0) \in \mathbb{A}^1_{\mathbf{k}},$$

that is not closed. In fact, by Lemma 3.2.14, we have

$$\overline{\{\xi\}} = \mathrm{V}(0) = \mathbb{A}^1_{\mathbf{k}}.$$

This point was invisible in the land of *classical varieties*, where only closed points were allowed. It has a name: it is the *generic point* of the affine line. We will say a lot more about generic points later, but for now notice that the terminology is somewhat well chosen: if we think that $(x-a)$ corresponds to the 'classical' point $a \in \mathbb{C}$, then since $x - x = 0$ it is reasonable to think that the coordinate of this point has indeed stayed 'generic'. This is what an 'indeterminate' should be!

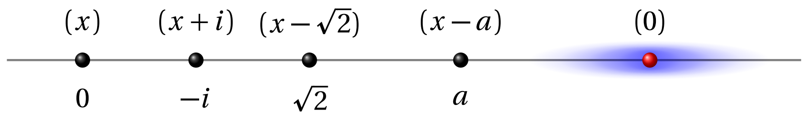

Figure 3.2: The topological space $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec}\mathbb{C}[x]$, with one closed point for every $a \in \mathbb{C}$. The generic point $\xi = (0)$ is 'dense', i.e. $\overline{\{\xi\}} = \mathbb{A}^1_{\mathbb{C}}$, since $0 \in \mathbb{C}[x]$ is in every prime ideal.

Note that, if $\mathbf{k}$ is not necessarily algebraically closed, $\mathbf{k}[x]$ is still a principal ideal domain, but now the closed points of $\mathbb{A}^1_{\mathbf{k}}$ correspond to those ideals (f) generated by irreducible polynomials of degree possibly larger than 1. We have a commutative diagram

$$\begin{tikzcd} \mathbf{k}[x] \arrow[r, "\ell"] \arrow[d, two heads] & \mathbf{k}[x]_{(f)} \arrow[d, two heads, "/(f)\mathbf{k}[x]_{(f)}"] \\ \mathbf{k}[x]/(f) \arrow[r, hook, "\iota"] & \kappa((f)) \end{tikzcd}$$

where ι is a finite extension of fields, of degree equal to $\deg f$ (cf. Example 3.2.19).

Example 3.2.18 (Spec $\mathbb{Z}$)**.** The spectrum $\operatorname{Spec}\mathbb{Z}$ is the arithmetic counterpart of $\operatorname{Spec}\mathbf{k}[x]$. It has one closed point for every nonzero prime ideal $(p) \subset \mathbb{Z}$, and, again, precisely one non-closed point $\xi = (0) \in \operatorname{Spec}\mathbb{Z}$ called the generic point.

Figure 3.3: The topological space $\operatorname{Spec}\mathbb{Z}$, with one closed point for every prime $p \in \mathbb{Z}$. The generic point $\xi = (0)$ is 'dense', i.e. $\overline{\{\xi\}} = \operatorname{Spec}\mathbb{Z}$, since $0 \in \mathbb{Z}$ is in every prime ideal.

Example 3.2.19 ($\mathbb{A}^1_{\mathbb{R}} = \operatorname{Spec}\mathbb{R}[x]$)**.** The ring $\mathbb{R}[x]$ is a principal ideal domain. Its prime ideals are

$$(0), \quad (x-a), \quad (x^2+bx+c),$$

where $a \in \mathbb{R}$ and x^2+bx+c is irreducible and satisfies $b^2-4c < 0$. The only prime ideal which is not maximal is, once more, $(0) \subset \mathbb{R}[x]$. However, we see here an important phenomenon arising when one considers fields that are not algebraically closed: we have, for the two types of *maximal* ideals,

$$\mathbb{R}[x]/(x-a) \cong \mathbb{R}, \quad \mathbb{R}[x]/(x^2+bx+c) \cong \mathbb{C}.$$

Of course, if f is an irreducible quadratic polynomial as above, the prime ideal $(f) \subset \mathbb{R}[x]$ defines one precise point in the spectrum, although we may want to think of it as the

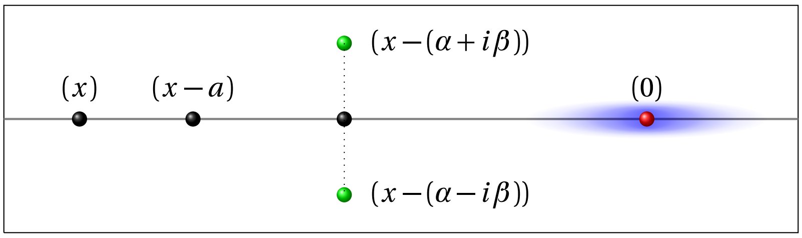

Figure 3.4: The real affine line $\mathbb{A}^1_{\mathbb{R}}$. Points $(f)\in\mathbb{A}^1_{\mathbb{R}}$ with $\mathbb{R}[x]/(f)\cong\mathbb{C}$ can be thought of pairs of conjugate *complex* points coming together.

identification of two complex conjugate points, just as $\pm i$ give rise to $x^2+1=(x-i)(x+i)$. See Example 3.2.64 for more on this.

Example 3.2.20 ($\mathbb{A}^2_{\mathbf{k}}=\operatorname{Spec}\mathbf{k}[x,y]$)**.** Let $\mathbf{k}$ be an algebraically closed field. The spectrum $\mathbb{A}^2_{\mathbf{k}}=\operatorname{Spec}\mathbf{k}[x,y]$ is called the *affine plane* (over $\mathbf{k}$). The prime ideals in $\mathbf{k}[x,y]$ are

$$(0),\quad (x-a,y-b),\quad (f)$$

where $(a,b)\in\mathbf{k}^2$ and $f=f(x,y)$ is an irreducible polynomial. Maximal ideals are those of the form $(x-a,y-b)$, and correspond indeed to the 'classical' points (a,b) of $\mathbf{k}^2$. These are then closed points of $\mathbb{A}^2_{\mathbf{k}}$. Given an irreducible polynomial $f\in\mathbf{k}[x,y]$, we have

$$\mathrm{V}(f)=\left\{\mathfrak{p}\in\mathbb{A}^2_{\mathbf{k}}\,\middle|\,f\in\mathfrak{p}\right\}=\left\{(x-a,y-b)\,\middle|\,f(a,b)=0\right\}\cup\left\{(f)\right\}.$$

Clearly $\mathrm{V}(f)$ is the closure of $\{(f)\}$. The ideal (f) is the generic point of $\mathrm{V}(f)$, because it corresponds to the trivial ideal in $\mathbf{k}[x,y]/(f)$, whereas the other points are closed points.

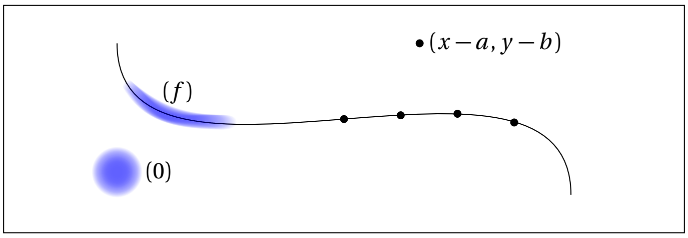

Figure 3.5: Sketchy picture of $\mathbb{A}^2_{\mathbf{k}}$.

Example 3.2.21 (Spec of a DVR)**.** Let A be a DVR, shorthand for 'discrete valuation ring'. Then, by definition, A is a principal ideal domain with exactly one maximal ideal $\mathfrak{m}\subset A$. This ideal is also prime, and there is precisely one other prime ideal, namely $(0)\subset A$. In other words,

$$\operatorname{Spec}A=\{\,\xi,\mathfrak{m}\,\}$$

consists of two points, where $\mathfrak{m}$ is closed (cf. Remark 3.2.13) and hence ξ, corresponding to $(0)\subset A$, is open. Note that the Zariski topology is not the discrete on two points topology here, for the point $\mathfrak{m}$ is a specialisation of the point ξ. We shall see later that, despite being a finite set, $\operatorname{Spec} A$ is a 1-dimensional scheme, simply because $0 \in \mathfrak{m}$. An example of DVR is given by the ring of formal power series $\mathbf{k}[\![t]\!]$, where $\mathbf{k}$ is a field. In this case, the maximal ideal is just the ideal generated by t.

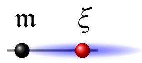

Figure 3.6: The black bullet represents the closed point $\mathfrak{m}$. The red point surrounded by the cloud, as usual, the generic point.

Example 3.2.22 ($\operatorname{Spec} \mathbf{k}[t]/t^2$)**.** First of all some terminology: the ring $A = \mathbf{k}[t]/t^2$ is called the *ring of dual numbers*[1] (over $\mathbf{k}$). Some people write $\mathbf{k}[\varepsilon]$ to denote this ring, being understood that $\varepsilon^2 = 0$. There is only one prime (and in fact maximal) ideal in A, namely

$$(\overline{t}) \subset A,$$

where $\overline{t}$ is the image of $t \in \mathbf{k}[t]$ under the projection $\mathbf{k}[t] \twoheadrightarrow A$. Thus, topologically, this space is the same as $\operatorname{Spec} \mathbf{k}$. However, it will be different (non-isomorphic to $\operatorname{Spec} \mathbf{k}$) as an affine scheme. A first strong indication of this fact is the following: if we see the element $\overline{t}$ as a function on $\operatorname{Spec} A$ as in the first paragraph of this section, we see that $\overline{t}$ evaluated on the point $(\overline{t})$ gives $0 \in A/\overline{t}$, i.e. the *nonzero* function $\overline{t} \in A$ *vanishes at every point* of $\operatorname{Spec} A$. Here we encounter for the first time one of the magic aspects of scheme theory:

$$\boxed{\text{functions are not determined by their values on points!}}$$

This is due to the presence of nilpotents, which were not part of the game with classical algebraic varieties. As mentioned, we will see that $\operatorname{Spec} A \neq \operatorname{Spec} \mathbf{k}$ *as affine schemes*, because their rings of functions are different: there is no ring isomorphism $\mathbf{k} \cong \mathbf{k}[t]/t^2$.

3.2.4 The sheaf of rings $\mathscr{O}_{\operatorname{Spec} A}$

Let A be a ring. Set $X = \operatorname{Spec} A$, equipped as always with the Zariski topology. We now define a sheaf of rings

$$\mathscr{O}_X.$$

By Lemma 2.5.7, to define a sheaf of rings on the topological space X, it is enough to define a $\mathcal{B}$-sheaf of rings where

$$\mathcal{B} = \left\{ \mathrm{D}(f) \,\middle|\, f \in A \right\} \subset \tau_X$$

[1] In case you care to know why they have this name, here is the answer directly from Wikipedia: Dual numbers were introduced in 1873 by William Clifford, and were used at the beginning of the twentieth century by the German mathematician Eduard Study, who used them to represent the dual angle which measures the relative position of two skew lines in space.

is the base of principal open sets in X (cf. Corollary 3.2.4). The working definition for such $\mathcal{B}$-sheaf will be

$$\mathrm{D}(f) \mapsto A_f = \left\{ \frac{a}{f^n} \,\middle|\, a \in A, n \geq 0 \right\}. \tag{3.2.3}$$

Note that (if we take $f = 1$) we are *defining*

$$\mathscr{O}_X(X) = A.$$

See Appendix A.2 for all you need to know about localisation. Sometimes we shall write af^{-n} or a/f^n for the element

$$\frac{a}{f^n} \in A_f.$$

We need to verify that (3.2.3) does indeed define a $\mathcal{B}$-sheaf of rings.

First of all, let us make sure this assignment is well-defined. We know (cf. Example 3.2.9) that $\mathrm{D}(g) \subset \mathrm{D}(f)$ is equivalent to $g \in \sqrt{(f)}$, which means $g^r = fb$ in A for some $b \in A$ and some $r > 0$. Thus

$$\frac{f}{1} \cdot \frac{b}{1} = \frac{fb}{1} = \frac{g^r}{1} \in A_g,$$

which is invertible in A_g. Therefore

$$\frac{f}{1} \in A_g$$

is also invertible in A_g, with inverse

$$\left(\frac{f}{1}\right)^{-1} = \frac{1}{g^r} \cdot \frac{b}{1} = \frac{b}{g^r} \in A_g.$$

By the universal property of localisation, we get a ring homomorphism

$$\rho_{\mathrm{D}(f)\mathrm{D}(g)} \colon A_f \to A_g, \qquad \frac{a}{f^n} \mapsto \frac{ab^n}{g^{nr}} \tag{3.2.4}$$

completing the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_g \\ A_f \arrow[ur, "\rho_{\mathrm{D}(f)\mathrm{D}(g)}"'] & \end{tikzcd}$$

This map is an isomorphism as soon as $\mathrm{D}(g) = \mathrm{D}(f)$, showing that (3.2.3) is well-defined.

Furthermore, the assignment (3.2.3) prescribes (cf. Remark A.2.5)

$$\emptyset = \mathrm{D}(0) \mapsto A_0 = 0,$$

and (3.2.4) is the identity as soon as $f = g$. The following lemma confirms that the maps just defined compose well, thus turning $\mathrm{D}(f) \mapsto A_f$ into a presheaf.

LEMMA 3.2.23. *Fix $f, g, h \in A$ giving rise to a chain of principal open subsets* $\mathrm{D}(h) \subset \mathrm{D}(g) \subset \mathrm{D}(f)$ *in* $\operatorname{Spec} A$*. Then*

$$\rho_{\mathrm{D}(g)\mathrm{D}(h)} \circ \rho_{\mathrm{D}(f)\mathrm{D}(g)} = \rho_{\mathrm{D}(f)\mathrm{D}(h)}$$

as maps $A_f \to A_h$.

Proof. First we write

$$g^r = fb, \quad h^s = gc,$$

for some $r, s > 0$ and $b, c \in A$. Then

$$h^{rs} = (h^s)^r = g^r c^r = fbc^r.$$

Then the map $\rho_{\mathrm{D}(f)\mathrm{D}(h)} \colon A_f \to A_h$ is given by

$$\frac{a}{f^n} \mapsto \frac{a(bc^r)^n}{h^{rsn}}.$$

On the other hand, we have to compose

$$\begin{array}{ccc} A_f & \xrightarrow{\rho_{\mathrm{D}(f)\mathrm{D}(g)}} & A_g \\ \dfrac{a}{f^n} & \longmapsto & \dfrac{ab^n}{g^{nr}} \end{array}$$

and

$$\begin{array}{ccc} A_g & \longrightarrow & A_h \\ \dfrac{a}{g^m} & \longmapsto & \dfrac{ac^m}{h^{ms}} \end{array}$$

with one another. The result of the composition is

$$\frac{a}{f^n} \mapsto \frac{ab^n}{g^{nr}} \mapsto \frac{ab^n c^{nr}}{h^{nrs}} = \frac{a(bc^r)^n}{h^{rsn}}.$$

This is what we wanted. □

THEOREM 3.2.24. *The following statements are true.*

(a) *The rule* (3.2.3) *defines a $\mathcal{B}$-sheaf of rings on $X = \operatorname{Spec} A$. The induced sheaf will be denoted $\mathcal{O}_X$.*

(b) *We have $\mathcal{O}_X(X) = A$.*

(c) *The stalk $\mathcal{O}_{X,x}$ of $\mathcal{O}_X$ at the point $x \in X$ corresponding to $\mathfrak{p} \subset A$ is isomorphic to $A_{\mathfrak{p}}$.*

Proof. We proceed step by step.

(a) We have already seen in (3.2.4) that an inclusion $\mathrm{D}(g) \subset \mathrm{D}(f)$ induces a restriction map $\rho_{\mathrm{D}(f)\mathrm{D}(g)} \colon A_f \to A_g$. The presheaf condition (1) of Definition 2.2.1 has been checked before the proof. Condition (2), regarding compositions, is Lemma 3.2.23. We check the sheaf conditions (3)–(4) of Definition 2.2.5 on the open set $U = X = \mathrm{D}(1)$, the case of an arbitrary $U = \mathrm{D}(f) \in \mathcal{B}$ being essentially identical. Recall from (3.2.2) that

$$\operatorname{Spec} A = \bigcup_{f \in A} \mathrm{D}(f).$$

By Lemma 3.2.10, this is equivalent to saying that there is a *finite* set F indexing a set of generators $\{f_i \mid i \in F\} \subset A$ of the unit ideal $(1) = A$, so that in particular $1 \in \sum_{i \in F}(f_i)$. In what follows, set $U_i = \mathrm{D}(f_i)$ and $U_{ij} = U_i \cap U_j = \mathrm{D}(f_i f_j)$.

Sheaf axiom (3): Fix $s \in A_1 = A$ such that $s|_{U_i} = 0 \in A_{f_i}$. This means

$$\frac{s}{1} = \frac{0}{1} \in A_{f_i},$$

i.e. there exists $m > 0$ such that $f_i^m s = 0 \in A$. Since F is finite, we can pick a uniform m which works for every f_i. Since

$$X = \bigcup_{i \in F} \mathrm{D}(f_i) = \bigcup_{i \in F} \mathrm{D}(f_i^m),$$

as before we have

$$1 \in \sum_{i \in F} (f_i^m),$$

which implies

$$s \in \sum_{i \in F} (f_i^m s) = 0.$$

Hence $s = 0$, as required.

Sheaf axiom (4): By definition, $\mathcal{O}_X(U_{ij}) = A_{f_i f_j}$. Fix sections $s_i \in A_{f_i}$ such that $s_i|_{U_{ij}} = s_j|_{U_{ij}}$ for every i and j. That is, s_i and s_j have the same image along the maps

$$\begin{tikzcd} A_{f_i} \arrow[dr] & & A_{f_j} \arrow[dl] \\ & A_{f_i f_j} & \end{tikzcd}$$

Write (again for a uniform $m > 0$)

$$s_i = \frac{b_i}{f_i^m} \in A_{f_i}, \quad i \in F.$$

Now, $s_i|_{U_{ij}} = s_j|_{U_{ij}}$ means that there exists an integer $r > 0$ such that

$$(f_i f_j)^r (b_i f_j^m - b_j f_i^m) = 0 \in A \tag{3.2.5}$$

for all $i, j \in F$. As before,

$$1 \in \sum_{i \in F} (f_i^{m+r})$$

yields

$$1 = \sum_{j \in F} a_j f_j^{m+r}, \quad a_j \in A. \tag{3.2.6}$$

Define

$$s = \sum_{j \in F} a_j b_j f_j^r \in A, \tag{3.2.7}$$

so that the chain of identities

$$\begin{aligned} f_i^{m+r} s &= \sum_{j \in F} a_j b_j f_i^m (f_i f_j)^r && \text{by (3.2.7)} \\ &= \sum_{j \in F} a_j b_i f_j^m (f_i f_j)^r && \text{by (3.2.5)} \\ &= \sum_{j \in F} a_j f_j^{m+r} b_i f_i^r \\ &= b_i f_i^r && \text{by (3.2.6)} \end{aligned}$$

yields

$$f_i^r (b_i - f_i^m s) = 0.$$

But this in turn is equivalent to

$$\frac{s}{1} = \frac{b_i}{f_i^m} = s_i \in A_{f_i}.$$

So we have proved $s|_{U_i} = s_i$ for every $i \in F$.

(b) We have $\operatorname{Spec} A = \mathrm{D}(1)$, so this actually follows from the definition, using that $A_1 = A$ since $1 \in A$ is already invertible in A.

(c) Let $\mathfrak{p} \subset A$ be the ideal corresponding to $x \in X$. For every $f \notin \mathfrak{p}$, there is (by the universal property of A_f) a canonical map $A_f \to A_{\mathfrak{p}} = \{ a/h \mid h \notin \mathfrak{p} \}$ because f is invertible in $A_{\mathfrak{p}}$. By the universal property of colimits, we get a canonical ring homomorphism

$$\mathscr{O}_{X,x} = \varinjlim_{f \notin \mathfrak{p}} A_f \xrightarrow{\ \alpha\ } A_{\mathfrak{p}}.$$

An element of the form $a/h \in A_{\mathfrak{p}}$ lies in the image of $A_h \to \mathscr{O}_{X,x} \to A_{\mathfrak{p}}$, therefore α is surjective. On the other hand, if $a/h \in A_h$ maps to $0 = 0/1 \in A_{\mathfrak{p}}$, then by definition of localisation there exists $g \in A \setminus \mathfrak{p}$ such that $ga = 0 \in A$. Then a/h restricts to 0 in A_{gh} and is therefore 0 in $\mathscr{O}_{X,x}$. Thus α is injective. □

The following is now immediate from the definition of locally ringed space.

COROLLARY 3.2.25. *Let A be a ring. The pair* $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ *defines a locally ringed space.*

Definition 3.2.26 (Affine scheme)**.** An *affine scheme* is a locally ringed spaces isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some commutative ring A.

We are ready for the definition of a scheme. However, we will not deal with this general notion until Section 3.3.

Definition 3.2.27 (Scheme)**.** A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ in which every point $x \in X$ has an open neighborhood $x \in U \subset X$ such that $(U, \mathcal{O}_X|_U)$ is an affine scheme.

Terminology 3.2.28. Let $(X, \mathcal{O}_X)$ be a scheme. The sheaf $\mathcal{O}_X$ is called the *structure sheaf* of the scheme (as for any locally ringed space). The ring $\mathcal{O}_X(X)$ is called the ring of *regular functions* on X. We still keep using the notation $(\mathcal{O}_{X,x}, \mathfrak{m}_x, \kappa(x))$ for the local ring at a point $x \in X$.

Definition 3.2.29 (Morphism of schemes)**.** A morphism of (affine) schemes is a morphism in the category of locally ringed spaces. In particular, an isomorphism of schemes $X \xrightarrow{\sim} Y$ is a morphism $(f, f^\#)$ such that f is a homeomorphism and $f^\#$ is an isomorphism of sheaves.

Definition 3.2.30 (Immersions of schemes)**.** An open (resp. closed) immersion of schemes is an open (resp. closed) immersion as locally ringed spaces (cf. Definition 3.1.8).

Notation 3.2.31. Affine schemes (resp. schemes) from a category, denoted Aff (resp. Sch), where morphisms are, as expected, morphisms in the category of locally ringed spaces.

We have thus a chain of inclusions of categories

$$\text{Affine Schemes} \subset \text{Schemes} \subset \text{Locally ringed spaces}.$$

Here are some purely topological properties of a scheme. Recall that a topological space is *irreducible* if it cannot be written as a union of two proper closed subsets.

Definition 3.2.32. A scheme $(X, \mathcal{O}_X)$ is said to be *quasicompact* (resp. irreducible, resp. connected) if the underlying topological space X is. A morphism of schemes $f : X \to Y$ is called *quasicompact* if the preimage of any affine open subset is quasicompact.

We already saw in Lemma 3.2.10 than an affine scheme is quasicompact. Any irreducible scheme is in particular connected.

Exercise 3.2.33. Show that a quasicompact scheme has a closed point.

Example 3.2.34. Let $\mathbf{k}$ be a field, $\mathfrak{p} \in X = \mathbb{A}^1_{\mathbf{k}} = \operatorname{Spec} \mathbf{k}[x]$ the point corresponding to $(x) \subset \mathbf{k}[x]$. (You are allowed, and in fact encouraged, to call this point 'the origin' of the affine line, and to denote it by '0'). The local ring of X at 0 is

$$\mathscr{O}_{X,0} = \mathbf{k}[x]_{(x)} = \left\{ \frac{f(x)}{g(x)} \,\middle|\, g(x) \notin (x) \right\} = \left\{ \frac{f(x)}{g(x)} \,\middle|\, g(0) \neq 0 \right\} \subset \operatorname{Frac}(\mathbf{k}[x]) = \mathbf{k}(x),$$

and the residue field is

$$\kappa(0) = \mathscr{O}_{X,0}/\mathfrak{m}_0 = \mathbf{k}[x]_{(x)}/(x)\mathbf{k}[x]_{(x)} \cong \mathbf{k}[x]/(x) \cong \mathbf{k}.$$

For the second-last isomorphism we used Proposition A.2.9.

Back to functions on Spec A

We already saw, but we need to emphasise, that

$$\mathscr{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) = A.$$

That is, regular functions on Spec A are precisely the elements of A.

We could have bypassed $\mathcal{B}$-sheaves defining the sheaf of rings $\mathscr{O}_X$ on $X = \operatorname{Spec} A$ directly by setting (as done in [5, Chapter 2])

$$\mathscr{O}_X(U) = \left\{ U \xrightarrow{s} \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}} \,\middle|\, \begin{array}{c} \text{for every } \mathfrak{p} \in U,\ s(\mathfrak{p}) \in A_{\mathfrak{p}} \text{ and there exist} \\ \text{an open neighborhood } V \subset U \text{ of } \mathfrak{p} \\ \text{and } a, f \in A \text{ such that, for every } \mathfrak{q} \in V, \\ f \notin \mathfrak{q} \text{ and } s(\mathfrak{q}) = a/f \text{ in } A_{\mathfrak{q}} \end{array} \right\} \tag{3.2.8}$$

for every open subset $U \subset X$. The fact that $U \mapsto \mathscr{O}_X(U)$ is a sheaf is clear once one realises the very definition is traced along the same lines of the sheafification of a presheaf.

Let us focus on the case $U = \operatorname{Spec} A$. Consider the map $A \to \mathscr{O}_X(X)$ sending $a \in A$ to the function

$$s_a \colon X \to \coprod_{\mathfrak{p} \in X} A_{\mathfrak{p}}, \quad \mathfrak{p} \mapsto \text{image of } a \text{ along } A \to A_{\mathfrak{p}}.$$

Exercise 3.2.35. Show that the association $a \mapsto s_a$ is a ring isomorphism.

One may insist to call *regular function* a map that is field-valued. This can be done as follows, starting from the definition in Equation (3.2.8). Let $s \in \mathscr{O}_X(X)$. For each $\mathfrak{p} \in X$, composing with the quotient maps

$$A_{\mathfrak{p}} \twoheadrightarrow \kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p} \cdot A_{\mathfrak{p}}$$

one obtains the map

$$\widetilde{s} \colon X \to \coprod_{\mathfrak{p} \in X} \kappa(\mathfrak{p}),$$

where the field $\kappa(\mathfrak{p})$ may (and will) vary from point to point.

Note that for every prime ideal $\mathfrak{p} \subset A$ there is a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, two heads] & A_{\mathfrak{p}} \arrow[d, two heads] \\ A/\mathfrak{p} \arrow[r] & \kappa(\mathfrak{p}) \end{tikzcd}$$

of rings.

Remark 3.2.36. The closed set $\mathrm{V}(I) = \{\, \mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \supset I \,\} \subset \operatorname{Spec} A$ can be reinterpreted as

$$\mathrm{V}(I) = \{\, \mathfrak{p} \in \operatorname{Spec} A \mid \widetilde{s}_a(\mathfrak{p}) = 0 \in \kappa(\mathfrak{p}) \text{ for all } a \in I \,\} \tag{3.2.9}$$

This explains once more the letter 'V', standing for 'vanishing'. But if you encounter the letter 'Z', it stands for 'zero locus'!

A special case is the following. If $\mathfrak{m}$ is maximal (i.e. if $\mathfrak{m} \in \operatorname{Spec} A$ is a closed point, cf. Remark 3.2.13), the map $A/\mathfrak{m} \to \kappa(\mathfrak{m})$ is a map of *fields*, henceforth injective. In fact, it is an isomorphism

$$A/\mathfrak{m} \xrightarrow{\sim} \kappa(\mathfrak{m})$$

by **??** (or more generally by Proposition A.2.9). If we restrict Equation (3.2.9) to *maximal* ideals (closed points in $\operatorname{Spec} A$), we end up with the tautology

$$\{\, \mathfrak{m} \subset A \text{ maximal} \mid \mathfrak{m} \supset I \,\} = \{\, \mathfrak{m} \subset A \text{ maximal} \mid a \bmod \mathfrak{m} = 0 \in A/\mathfrak{m} \text{ for all } a \in I \,\}.$$

Exercise 3.2.37. Let A be a ring, $\mathfrak{p} \subset A$ a prime ideal. Show that

$$\operatorname{Frac}(A/\mathfrak{p}) = \kappa(\mathfrak{p}).$$

3.2.5 Generic points, take I

We start here a (temporary) discussion around generic points.

LEMMA 3.2.38. *Let X be an irreducible scheme. Then, there exists a unique point $\xi \in X$ such that $X = \overline{\{\, \xi \,\}}$.*

Proof. Let us show uniqueness first. Let ξ_1, ξ_2 be two points such that $\overline{\{\, \xi_1 \,\}} = X = \overline{\{\, \xi_2 \,\}}$. Let $U \subset X$ be an open subset. Note that any open $U \subset X$ contains both ξ_1 and ξ_2. Pick a nonempty affine open subset $U = \operatorname{Spec} A \subset X$. Let $\mathfrak{p}_i \subset A$ be the prime ideal corresponding to ξ_i. Since X is irreducible, U is irreducible and dense, hence

$$U = \overline{\{\, \xi_i \,\}} = \mathrm{V}(\mathfrak{p}_i), \quad i = 1, 2,$$

where the closure is taken in U. So when we write $\xi_2 \in U = \mathrm{V}(\mathfrak{p}_1)$ we obtain $\mathfrak{p}_2 \supset \mathfrak{p}_1$, and when we write $\xi_1 \in U = \mathrm{V}(\mathfrak{p}_2)$ we obtain $\mathfrak{p}_1 \supset \mathfrak{p}_2$. Thus $\mathfrak{p}_1 = \mathfrak{p}_2$, i.e. $\xi_1 = \xi_2$.

Now for existence. If $U = \operatorname{Spec} A \subset X$ is a nonempty open affine subset, then U is irreducible. We prove that $\sqrt{0} \subset A$ is prime. This will imply that A has precisely one minimal prime, namely $\mathfrak{p} = \sqrt{0}$, and hence $\operatorname{Spec} A = \mathrm{V}(\sqrt{0}) = \mathrm{V}(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$. But since U is dense in X, the closure of $\mathfrak{p}$ in X is X itself. So, if $\sqrt{0}$ were not prime, we could find $a, b \notin \sqrt{0}$ such that $ab \in \sqrt{0}$. We have $\mathrm{V}(ab) = \mathrm{V}(a) \cup \mathrm{V}(b) \subset \mathrm{V}(\sqrt{0}) = \operatorname{Spec} A$, but in fact for any $\mathfrak{p} \in \operatorname{Spec} A$ we have $\mathfrak{p} \supset \sqrt{0} \ni ab$, thus either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. So $\operatorname{Spec} A = \mathrm{V}(a) \cup \mathrm{V}(b)$. Since $\mathrm{V}(a) \neq \operatorname{Spec} A \neq \mathrm{V}(b)$ by assumption, we contradict irreducibility. □

Remark 3.2.39. The above proof actually works for an arbitrary irreducible closed subset X of an arbitrary scheme Y. See also **?? ??** for a generalisation of the argument for existence.

Definition 3.2.40 (Generic point, take I)**.** Let X be an irreducible scheme. The point $\xi \in X$ of Lemma 3.2.38 is called the *generic point* of X.

If A is an integral domain, then $X = \operatorname{Spec} A$ is irreducible. Indeed, if we had $X = \mathrm{V}(0) = \mathrm{V}(I_1) \cup \mathrm{V}(I_2) = \mathrm{V}(I_1 \cap I_2) = \mathrm{V}(I_1 I_2)$ with $\mathrm{V}(I_1) \neq X = \mathrm{V}(I_2)$, we would have $I_1 I_2 \subset \sqrt{I_1 I_2} = \sqrt{0} = (0)$, i.e. either $I_1 \subset (0)$ or $I_2 \subset (0)$ because $(0) \subset A$ is prime. But then either $\mathrm{V}(I_1) = X$ or $\mathrm{V}(I_2) = X$, contradiction.

It is also clear from the last paragraph of the proof of Lemma 3.2.38 that the generic point of $\operatorname{Spec} A$ is the point ξ corresponding to (0), which is manifestly the unique minimal prime. The following lemma clarifies the basic properties of the generic point in this special case.

LEMMA 3.2.41. *Let A be an integral domain with fraction field K. Let $\xi \in X = \operatorname{Spec} A$ be the point corresponding to $(0) \subset A$. Then*

(i) We have $\mathscr{O}_{X,\xi} = K$.

(ii) ξ belongs to every nonempty open subset $U \subset X$, and $\mathscr{O}_X(U) \to \mathscr{O}_{X,\xi}$ is injective.

(iii) For every open subset $V \hookrightarrow U$, the map $\mathscr{O}_X(U) \to \mathscr{O}_X(V)$ is injective.

Proof. We proceed step by step.

(i) This follows from Theorem 3.2.24 (c) and the observation that the localisation of an integral domain at the prime ideal (0) is precisely the fraction field of the domain.

(ii) To say $\xi \in \operatorname{Spec} A \setminus \mathrm{V}(I) = \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \not\supset I\}$ for every $(0) \not\subset I \subset A$ means precisely that $(0) \not\supset I$ for all I, a tautology. Write $U = \bigcup_{i \in I} \mathrm{D}(f_i)$ and assume $s \in \mathscr{O}_X(U)$ goes to 0 in $K = \mathscr{O}_{X,\xi}$. Well, s goes to $s|_{\mathrm{D}(f_i)} \in \mathscr{O}_X(\mathrm{D}(f_i)) = A_{f_i}$ first, and $A_{f_i} \hookrightarrow K$ is injective. Thus $s|_{\mathrm{D}(f_i)} = 0$ for every $i \in I$, so $s = 0$ by the sheaf conditions.

(iii) Follows immediately from (ii). □

Definition 3.2.42 (Field of rational functions)**.** Let A be an integral domain. The field $K = \mathscr{O}_{X,\xi}$ is called the *field of rational functions of* X.

Example 3.2.43. Let $A = \mathbf{k}[x_1,\ldots,x_n]$, and consider the generic point $\xi \in \mathbb{A}^n_{\mathbf{k}}$, i.e. the point corresponding to the ideal $(0) \subset A$. Then

$$\kappa(\xi) = \mathbf{k}(x_1,\ldots,x_n).$$

If $f \in A$ is an irreducible polynomial, then the generic point $\xi_f \in \operatorname{Spec} A/(f)$ satisfies

$$\kappa(\xi_f) = \operatorname{Frac} A/(f).$$

Example 3.2.44. Let $A = \mathbb{Z}$. Every open subset $U \subset \operatorname{Spec}\mathbb{Z}$ is principal, i.e. of the form $U = \mathrm{D}(f)$ for some $f \in \mathbb{Z}$. We have $\mathscr{O}_{\operatorname{Spec}\mathbb{Z}}(\mathrm{D}(f)) = \mathbb{Z}_f \subset \operatorname{Frac}(\mathbb{Z}) = \mathbb{Q}$, and a rational number $a/b \in \mathbb{Q}$ (with a, b coprime) belongs to $\mathscr{O}_{\operatorname{Spec}\mathbb{Z}}(\mathrm{D}(f))$ if and only if every prime p dividing b also divides f. As for the generic point, we have $\kappa((0)) = \mathscr{O}_{\operatorname{Spec}\mathbb{Z},(0)} = \mathbb{Q}$. If $x \in \operatorname{Spec}\mathbb{Z}$ corresponds to the maximal ideal $(p) \subset \mathbb{Z}$, then $\kappa(x) = \mathbb{Z}_{(p)}/(p)\cdot\mathbb{Z}_{(p)} \cong \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$. Therefore the residue fields at different points of a scheme can have different characteristic!

3.2.6 Morphisms of affine schemes

Let $\phi\colon A \to B$ be a ring homomorphisms. Then we have a set-theoretic map

$$f_\phi\colon \operatorname{Spec} B \to \operatorname{Spec} A, \quad \mathfrak{q} \mapsto \phi^{-1}(\mathfrak{q}).$$

LEMMA 3.2.45. *Let $\phi\colon A \to B$ be a ring homomorphisms. Then*

(a) f_ϕ is continuous.

(b) If ϕ is surjective, then f_ϕ induces a homeomorphism from $\operatorname{Spec} B$ *onto the closed subset* $\mathrm{V}(\ker\phi) \subset \operatorname{Spec} A$*.*

(c) If ϕ is a localisation $A \to S^{-1}A$, then f_ϕ induces a homeomorphism from $\operatorname{Spec} S^{-1}A$ *onto the subspace $Y_S = \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \cap S = \emptyset\} \subset \operatorname{Spec} A$.*

Proof. We proceed step by step.

(a) We prove that the preimage of a closed subset $\mathrm{V}(I) \subset \operatorname{Spec} A$ is closed. We have

$$\begin{aligned} f_\phi^{-1}(\mathrm{V}(I)) &= \{\mathfrak{q} \in \operatorname{Spec} B \mid \phi^{-1}(\mathfrak{q}) \in \mathrm{V}(I)\} \\ &= \{\mathfrak{q} \in \operatorname{Spec} B \mid I \subset \phi^{-1}(\mathfrak{q})\} \\ &= \{\mathfrak{q} \in \operatorname{Spec} B \mid \phi(I) \subset \mathfrak{q}\} \\ &= \{\mathfrak{q} \in \operatorname{Spec} B \mid IB \subset \mathfrak{q}\} \\ &= \mathrm{V}(IB). \end{aligned}$$

(b) We have $B = A/\ker\phi$ by assumption, and we know that the prime ideals of B are in bijection with the prime ideals of A containing $\ker\phi$. By (a), and by definition of $\mathrm{V}(-)$, we then know that f_ϕ factors through a continuous bijection $\operatorname{Spec} B \to \mathrm{V}(\ker\phi)$, still denoted f_ϕ. To conclude it is a homeomorphism, it is enough to check the map is closed. Let then $J \subset B$ be an ideal. Then

$$\begin{aligned} f_\phi(\mathrm{V}(J)) &= \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} = \phi^{-1}(\mathfrak{q}) \text{ for some } \mathfrak{q} \supset J\} \\ &= \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \supset \phi^{-1}(J)\} \\ &= \mathrm{V}(\phi^{-1}(J)). \end{aligned}$$

(c) The existence of a continuous bijection $\operatorname{Spec} S^{-1}A \to Y_S \subset \operatorname{Spec} A$ is a combination of (a) with Lemma A.2.6. As before, to see that the map is closed, fix an ideal $J \subset S^{-1}A$. Then

$$\begin{aligned} f_\phi(\mathrm{V}(J)) &= \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \cap S = \emptyset \text{ and } \mathfrak{p} = \phi^{-1}(\mathfrak{q}) \text{ for some } \mathfrak{q} \supset J\} \\ &= Y_S \cap \mathrm{V}(\phi^{-1}(J)), \end{aligned}$$

which is closed in Y_S, as required. □

Remark 3.2.46. Note that if $S = \{1, g, g^2, \dots\}$ for some $g \in A$, then

$$Y_S = \{\mathfrak{p} \in \operatorname{Spec} A \mid g \notin \mathfrak{p}\} = \mathrm{D}(g) \subset \operatorname{Spec} A.$$

In particular, in this case the map $f_\phi \colon \operatorname{Spec} A_g \to \operatorname{Spec} A$ is a topological open embedding.

PROPOSITION 3.2.47. *Let $X = \operatorname{Spec} A$, and fix $g \in A$. Then, the localisation $\ell \colon A \to A_g$ induces an isomorphism of locally ringed spaces*

$$(f_\ell, f_\ell^\#) \colon (\operatorname{Spec} A_g, \mathscr{O}_{\operatorname{Spec} A_g}) \xrightarrow{\sim} (\mathrm{D}(g), \mathscr{O}_X|_{\mathrm{D}(g)}).$$

In particular, $(\mathrm{D}(g), \mathscr{O}_X|_{\mathrm{D}(g)})$ *is an affine scheme.*

Proof. A topological open embedding $f_\ell \colon \operatorname{Spec} A_g \to \operatorname{Spec} A$ with image $\mathrm{D}(g)$ is provided by Lemma 3.2.45 (c), applied to the localisation $\ell \colon A \to A_g$. Let us denote by f the homeomorphism $\operatorname{Spec} A_g \to \mathrm{D}(g)$. We need to extend it to a morphism of locally ringed spaces and show the resulting map is an isomorphism.

To define a morphism of sheaves of rings

$$f^\# \colon \mathscr{O}_X|_{\mathrm{D}(g)} \to f_* \mathscr{O}_{\operatorname{Spec} A_g}$$

it is enough to define it on a base of open subsets by Proposition 2.5.9. Let $\mathrm{D}(h) \subset \mathrm{D}(g)$

be a principal open, for $h \in A$. Let $\overline{h}$ be the image of h in A_g. Then, canonically,

$$\begin{aligned}
\mathscr{O}_X|_{\mathrm{D}(g)}(\mathrm{D}(h)) = \mathscr{O}_X(\mathrm{D}(h)) &= A_h \\
&\xrightarrow{\sim} (A_g)_{\overline{h}} \\
&= \mathscr{O}_{\operatorname{Spec} A_g}(\mathrm{D}(\overline{h})) \\
&= \mathscr{O}_{\operatorname{Spec} A_g}(f^{-1}\mathrm{D}(h)) \\
&= f_*\mathscr{O}_{\operatorname{Spec} A_g}(\mathrm{D}(h)).
\end{aligned}$$

Note that the map $A_h \to (A_g)_{\overline{h}}$ induced by the universal property of A_h is defined explicitly by

$$\frac{a}{h^n} \mapsto \frac{\frac{a}{1}}{\overline{h}^n}.$$

The above map $\mathscr{O}_X|_{\mathrm{D}(g)}(\mathrm{D}(h)) \xrightarrow{\sim} f_*\mathscr{O}_{\operatorname{Spec} A_g}(\mathrm{D}(h))$ uniquely extends to an isomorphism of sheaves by Proposition 2.5.9. □

Exercise 3.2.48. Let $(X, \mathscr{O}_X)$ be a scheme, $U \subset X$ an open subset. Show that the locally ringed space $(U, \mathscr{O}_X|_U)$ is a scheme.

PROPOSITION 3.2.49. *Let $\phi : A \to B$ be a ring homomorphisms. Then the continuous map $f_\phi : \operatorname{Spec} B \to \operatorname{Spec} A$ of Lemma 3.2.45 extends to a morphism of affine schemes $(f_\phi, f_\phi^\#)$ such that $f_\phi^\#(\operatorname{Spec} A) = \phi$.*

Proof. Set $f = f_\phi$. First of all, we have to construct the sheaf homomorphism

$$f^\# : \mathscr{O}_{\operatorname{Spec} A} \to f_*\mathscr{O}_{\operatorname{Spec} B}.$$

For $g \in A$, the preimage of the principal open $\mathrm{D}(g) \subset \operatorname{Spec} A$ is

$$\begin{aligned}
f^{-1}(\mathrm{D}(g)) &= \{\mathfrak{q} \in \operatorname{Spec} B \mid \phi^{-1}(\mathfrak{q}) \in \mathrm{D}(g)\} \\
&= \{\mathfrak{q} \in \operatorname{Spec} B \mid g \notin \phi^{-1}(\mathfrak{q})\} \\
&= \{\mathfrak{q} \in \operatorname{Spec} B \mid \phi(g) \notin \mathfrak{q}\} \\
&= \mathrm{D}(\phi(g)).
\end{aligned}$$

There is an induced commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\phi"] \arrow[d] & B \arrow[d] & \\
A_g \arrow[r, "\phi_g"] \arrow[d, equal] & B_{\phi(g)} \arrow[d, equal] & \frac{a}{g^r} \mapsto \frac{\phi(a)}{\phi(g)^r} \\
\mathscr{O}_{\operatorname{Spec} A}(\mathrm{D}(g)) & \mathscr{O}_{\operatorname{Spec} B}(\mathrm{D}(\phi(g))) \arrow[r, equal] & f_*\mathscr{O}_{\operatorname{Spec} B}(\mathrm{D}(g))
\end{tikzcd}$$

allowing us to set $f^{\#}(\mathrm{D}(g)) = \phi_g$. These morphisms are compatible with restrictions to smaller principal opens, thus they uniquely determine a morphism of sheaves $f^{\#}$ by Proposition 2.5.9.

Assume $y = f(x)$, where $x \in \operatorname{Spec} B$ corresponds to a prime ideal $\mathfrak{q} \subset B$ and $y \in \operatorname{Spec} A$ corresponds to $\mathfrak{p} = \phi^{-1}(\mathfrak{q}) \subset A$. Then the canonical map

$$A_{\mathfrak{p}} \longrightarrow B_{\mathfrak{q}}, \quad \frac{a}{h} \mapsto \frac{\phi(a)}{\phi(h)}$$

is a local homomorphism of local rings, which coincides with

$$\mathscr{O}_{\operatorname{Spec} A, y} \to (f_* \mathscr{O}_{\operatorname{Spec} B})_y \to \mathscr{O}_{\operatorname{Spec} B, x}.$$

If we take global sections of $f^{\#}$ (i.e. we evaluate it on $\mathrm{D}(1) = \operatorname{Spec} A$), we get back our original map ϕ, by construction. □

We can finally prove the main result of this chapter.

THEOREM 3.2.50. *Sending $A \mapsto \operatorname{Spec} A$ turns 'Spec' into an equivalence*

$$\operatorname{Spec}\colon \mathsf{Rings}^{\mathrm{op}} \xrightarrow{\sim} \mathsf{Aff},$$

with inverse given by $X \mapsto \mathscr{O}_X(X)$. In particular, $\operatorname{Spec} \mathbb{Z}$ *is a final object in* Aff.

Proof. Say we have two affine schemes $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. Then we have a canonical map

$$\rho_{X,Y}\colon \operatorname{Hom}_{\mathsf{Aff}}(X, Y) \to \operatorname{Hom}_{\mathsf{Rings}}(A, B) \tag{3.2.10}$$

sending $f\colon X \to Y$ to $f^{\#}(Y)\colon A = \mathscr{O}_Y(Y) \to f_* \mathscr{O}_X(Y) = \mathscr{O}_X(X) = B$. Note that this map is functorial, in the following sense: for any morphism of affine schemes $g\colon Z = \operatorname{Spec} C \to \operatorname{Spec} B = X$ the diagram

$$\begin{tikzcd} \operatorname{Hom}_{\mathsf{Aff}}(X, Y) \arrow[r, "\rho_{X,Y}"] \arrow[d, "f \mapsto f \circ g"] & \operatorname{Hom}_{\mathsf{Rings}}(A, B) \arrow[d, "\phi \mapsto g^{\#}(X) \circ \phi"] \\ \operatorname{Hom}_{\mathsf{Aff}}(Z, Y) \arrow[r, "\rho_{Z,Y}"] & \operatorname{Hom}_{\mathsf{Rings}}(A, C) \end{tikzcd}$$

commutes. We must show that (3.2.10) is bijective for any pair of affine schemes $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$.

Fix $f \in \operatorname{Hom}_{\mathsf{Aff}}(X, Y)$. Set $\phi = \rho_{X,Y}(f) = f^{\#}(Y)$. We know by Proposition 3.2.49 that ϕ gives rise to a morphism of affine schemes $f_\phi\colon X \to Y$ such that $\rho_{X,Y}(f_\phi) = \phi = \rho_{X,Y}(f)$. It is thus enough to show that $f = f_\phi$.

We need to show that f and f_ϕ are the same map set-theoretically, and, once we know this, that $f^{\#}_x = (f_\phi)^{\#}_x$ are the same as local homomorphisms. This will imply that

$f^\# = f_\phi^\#$ by Exercise 2.3.14. Let's start. Let $\mathfrak{q} \subset B$ be the prime ideal corresponding to $x \in X = \operatorname{Spec} B$, and $\mathfrak{p} \subset A$ be the prime ideal corresponding to $f(x) \in Y = \operatorname{Spec} A$. We have two commutative diagrams

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\mathrm{loc}"'] & B \arrow[d, "\mathrm{loc}"] \\ A_{\mathfrak{p}} \arrow[r, "f_x^\#"] & B_{\mathfrak{q}} \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\mathrm{loc}_{\mathfrak{p}}"'] & B \arrow[d, "\mathrm{loc}_{\mathfrak{q}}"] \\ A_{\phi^{-1}(\mathfrak{q})} \arrow[r, "(f_\phi)_x^\#"] & B_{\mathfrak{q}} \end{tikzcd}$$

that we need to show are the same. The very existence of $f_x^\#$ implies that whenever $h \notin \mathfrak{p}$ one must have $\phi(h) \notin \mathfrak{q}$, i.e. $\phi(A \setminus \mathfrak{p}) \subset B \setminus \mathfrak{q}$, which implies $\phi^{-1}(\mathfrak{q}) \subset \mathfrak{p}$. But the local condition $(f_x^\#)^{-1}(\mathfrak{q}B_{\mathfrak{q}}) = \mathfrak{p}A_{\mathfrak{p}}$ (combined with the classical correspondence between prime ideals in a ring and in a localisation of it, cf. Lemma A.2.6) implies $\phi^{-1}(\mathfrak{q}) = \phi^{-1}\mathrm{loc}_{\mathfrak{q}}^{-1}(\mathfrak{q}) = \mathrm{loc}_{\mathfrak{p}}^{-1}(\mathfrak{p}A_{\mathfrak{p}}) = \mathfrak{p}$.

Thus $f = f_\phi$ set-theoretically. However, there is only one possible commutative diagram as above: the one where the bottom map sends $a/h \mapsto \phi(a)/\phi(h)$. Thus $f_x^\# = (f_\phi)_x^\#$ as wanted. This concludes the proof that ρ is bijective.

The final statement now follows, since a ring A is a $\mathbb{Z}$-algebra $\mathbb{Z} \to A$ in a unique way (or, equivalently, $\mathbb{Z}$ is an initial object in Rings). □

3.2.7 Examples of affine schemes and their morphisms

In this section we collect some examples of affine schemes (and morphisms between them), besides those already considered in Section 3.2.3 at a purely topological level.

Recall that open (resp. closed) immersion of schemes are just open (resp. closed) immersions in the category of locally ringed spaces (cf. Definition 3.1.8). The next two examples are very important.

Key Example 3.2.51 (Principal open immersions)**.** Let A be a ring, $f \in A$. It follows from Proposition 3.2.47 and Definition 3.1.8 that the canonical morphism

$$\operatorname{Spec} A_f \to \operatorname{Spec} A$$

is an open immersion of affine schemes.

Key Example 3.2.52 (Closed immersions)**.** Let A be a ring, $I \subset A$ an ideal. Set $B = A/I$. The canonical surjection $\phi \colon A \twoheadrightarrow B$ determines, and is determined by, a morphism of affine schemes

$$i \colon \operatorname{Spec} B \to \operatorname{Spec} A$$

This morphism is a *closed immersion* according to Definition 3.1.8. Indeed, it is a homeomorphism onto $\mathrm{V}(I) \subset \operatorname{Spec} A$, and induces a surjective map of sheaves, because $i^\#(\mathrm{D}(g))$ is surjective for every $g \in A$ (it agrees with the canonical surjection $A_g \twoheadrightarrow B_{\phi(g)}$). Indeed, by Proposition 2.5.9, it is enough to check surjectivity on a base of open sets.

Example 3.2.53 (Many maps to the point!)**.** One is used to think that there is only one map $\bullet \to \bullet$. However, this is not true in algebraic geometry: think of the identity $\mathbb{C} \to \mathbb{C}$, which is different from complex conjugation $\mathbb{C} \to \mathbb{C}$. By Theorem 3.2.50, they give rise to different maps $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{C}$. Another example, in characteristic $p > 0$, is the *Frobenius morphism*, namely the map $\Phi_K \colon \operatorname{Spec} K \to \operatorname{Spec} K$ induced by the field homomorphism $K \to K$ sending $x \mapsto x^p$. Note that a field K is perfect if and only if either K has characteristic 0 or $x \mapsto x^p$ is surjective. In particular, the Frobenius morphism Φ_K is an isomorphism if and only if K is perfect. For instance, all finite fields are perfect, but $\mathbb{F}_p(t)$ is not perfect, since t is (for degree reasons) not of the form $f(t)^p/g(t)^p$ for any two polynomials $f(t)$ and $g(t)$.

Example 3.2.54 (Fat points)**.** Thanks to Theorem 3.2.50, we finally have made rigorous our claim

$$\operatorname{Spec}\mathbf{k} \neq \operatorname{Spec}\mathbf{k}[t]/t^2$$

from Example 3.2.22! The surjection $\mathbf{k}[t]/t^2 \twoheadrightarrow \mathbf{k}$ nevertheless induces a bijective closed immersion $\operatorname{Spec}\mathbf{k} \hookrightarrow \operatorname{Spec}\mathbf{k}[t]/t^2$. Note that the same applies to any local Artin $\mathbf{k}$-algebra $(A, \mathfrak{m})$ with residue field $\mathbf{k} = A/\mathfrak{m}$. Namely, the closed immersion $\operatorname{Spec} A/\mathfrak{m} \hookrightarrow \operatorname{Spec} A$ is a bijection (both schemes have precisely one point), but $\operatorname{Spec} A/\mathfrak{m} = \operatorname{Spec}\mathbf{k} \neq \operatorname{Spec} A$ as schemes. Such schemes are called *fat points*: they are topologically just one point, but they encode nontrivial information in their structure sheaves.

Example 3.2.55 (A non-affine scheme)**.** Let $X = \operatorname{Spec}\mathbb{Z}[x]$, and $z \in X$ the closed point corresponding to (p, x), where $p \in \mathbb{Z}$ is a prime number. Then $U = X \setminus \{z\} = \mathrm{D}(p) \cup \mathrm{D}(x)$ is not affine. Indeed, by Lemma 3.2.41(iii), we have $\mathscr{O}_X(U) \subset \mathscr{O}_X(\mathrm{D}(p)) \cap \mathscr{O}_X(\mathrm{D}(x)) = \mathbb{Z}[x, 1/p] \cap \mathbb{Z}[x, 1/x] = \mathbb{Z}[x] = \mathscr{O}_X(X)$, which readily implies $\mathscr{O}_X(U) = \mathscr{O}_X(X)$. Note that this example also shows that the union of two affine schemes is not necessarily affine.

Example 3.2.56 (A non-affine scheme)**.** Let $n > 1$ be an integer, $\mathbf{k}$ a field, $A = \mathbf{k}[x_1, \ldots, x_n]$. Consider the point $0 \in X = \mathbb{A}^n_{\mathbf{k}} = \operatorname{Spec} A$ corresponding to $(x_1, \ldots, x_n) \subset A$. Form the open complement $U = X \setminus \{0\} \hookrightarrow X$. We now prove that the restriction map

$$\mathbf{k}[x_1, \ldots, x_n] = \mathscr{O}_X(X) \to \mathscr{O}_X(U)$$

is the identity. This proves that U is not affine, since U is not isomorphic to $\mathbb{A}^n_{\mathbf{k}}$. As before, we have $U = \bigcup_{1 \le i \le n} \mathrm{D}(x_i)$, so by Lemma 3.2.41(iii), we have

$$\mathscr{O}_X(U) \subset \bigcap_{1 \le i \le n} \mathrm{D}(x_i) = A_{x_1} \cap \cdots \cap A_{x_n}.$$

This can be proven directly to be equal to A. However, it also follows from the algebraic version of Hartog's Lemma below, combined with the fact that height 1 primes in the (normal) domain A correspond to irreducible polynomials.

LEMMA 3.2.57. *Let A be a noetherian domain, $\overline{A} \subset \operatorname{Frac}(A)$ its normalisation. Then*

$$\overline{A} = \bigcap_{\operatorname{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}.$$

Example 3.2.58 (Affine line minus one point)**.** If we take $n = 1$ in Example 3.2.56, we do in fact get an affine scheme $U = \mathbb{A}^1_{\mathbf{k}} \setminus \{0\} \hookrightarrow \mathbb{A}^1_{\mathbf{k}}$. Indeed, $U = \mathrm{D}(x) = \operatorname{Spec} \mathbf{k}[x]_x$. In fact, the ring $\mathbf{k}[x]_x = \{ f(x)/x^r \mid r \geq 0 \}$ is isomorphic to the $\mathbf{k}$-algebra $\mathbf{k}[x, x^{-1}] = \mathbf{k}[x, y]/(xy-1)$, which realises U as a closed subscheme of $\mathbb{A}^2_{\mathbf{k}}$.

Example 3.2.59 (Affine hypersurfaces)**.** Let $\mathbf{k}$ be a field, $f \in \mathbf{k}[x_1, \ldots, x_n]$. Then

$$Y_f = \operatorname{Spec} \mathbf{k}[x_1, \ldots, x_n]/f$$

is called an *affine hypersurface* in $\mathbb{A}^n_{\mathbf{k}}$. The surjection $\mathbf{k}[x_1, \ldots, x_n] \twoheadrightarrow \mathbf{k}[x_1, \ldots, x_n]/f$ canonically determines a closed immersion

$$Y_f \hookrightarrow \mathbb{A}^n_{\mathbf{k}} = \operatorname{Spec} \mathbf{k}[x_1, \ldots, x_n].$$

Suppose (f) is a prime ideal in $\mathbf{k}[x_1, \ldots, x_n]$, so that $\mathbf{k}[x_1, \ldots, x_n]/f$ is an integral domain. Then (f) corresponds to the trivial (prime) ideal $(0) \subset \mathbf{k}[x_1, \ldots, x_n]/f$. This is the generic point of Y_f.

Example 3.2.60. As a special case of Example 3.2.59, consider $f = xy - z^2 \in \mathbb{C}[x, y, z]$. Its vanishing scheme

$$\operatorname{Spec} \mathbb{C}[x, y, z]/(xy - z^2) \hookrightarrow \mathbb{A}^3_{\mathbb{C}}$$

is called the *affine quadric cone*.

Example 3.2.61. Let R be a DVR with fraction field K, and set $X = \operatorname{Spec} R = \{x_0, \xi\}$ where x_0 is the closed point. Then $K = \mathscr{O}_{X,\xi}$. The open immersion $\{\xi\} = X \setminus \{x_0\} \hookrightarrow X$ corresponds to the canonical inclusion $R \hookrightarrow K$.

Example 3.2.62. Let $\mu_n = \operatorname{Spec} \mathbf{k}[x]/(x^n - 1)$ for some $n > 1$. This is the scheme-theoretic version of the group of n-th roots of unity. One can prove that it is a group object in the category of $\mathbf{k}$-schemes. Such objects are called *algebraic groups*. As for μ_n, it comes with a natural closed immersion inside the affine line $\mathbb{A}^1_{\mathbf{k}} = \operatorname{Spec} \mathbf{k}[x]$.

Example 3.2.63. Consider the morphism $f: \mathbb{A}^1_{\mathbf{k}} \to \mathbb{A}^1_{\mathbf{k}}$ defined by the ring homomorphism $\mathbf{k}[t] \to \mathbf{k}[t]$ sending $t \mapsto t^n$. This is the typical example of what we will call a *ramified* morphism. The intuition is the following: every point $x \in \mathbb{A}^1_{\mathbf{k}} \setminus 0$ in the target has precisely n preimages (because $\mathbf{k}$ is algebraically closed), but there is only one preimage over the origin $0 \in \mathbb{A}^1_{\mathbf{k}}$. Over this point, the morphism if 'fully ramified'. If we restrict f to $\mathbb{A}^1_{\mathbf{k}} \setminus 0 \to \mathbb{A}^1_{\mathbf{k}} \setminus 0$, it becomes *unramified*, and in fact *étale*. These notions are extremely important and will be treated in later chapters.

Example 3.2.64. The inclusion $\mathbb{R}[x] \hookrightarrow \mathbb{C}[x]$ induces a morphism of affine schemes

$$\mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{R}},$$

sending the generic point $(0) \subset \mathbb{C}[x]$ to the generic point $(0) \subset \mathbb{R}[x]$. For any $c \in \mathbb{R} \subset \mathbb{C}$, the maximal ideal $(x-c) \subset \mathbb{R}[x]$ is the preimage of the maximal ideal $(x-c) \subset \mathbb{C}[x]$, so $\mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{R}}$ sends the closed point $(x-c) \in \mathbb{A}^1_{\mathbb{C}}$ to the closed point $(x-c) \in \mathbb{A}^1_{\mathbb{R}}$. On the other hand, if $c = \alpha + i\beta \in \mathbb{C} \setminus \mathbb{R}$, then both ideals

$$\mathfrak{p}_1 = (x-c),\ \mathfrak{p}_2 = (x-\overline{c}) \subset \mathbb{C}[x]$$

viewed as closed points of $\mathbb{A}^1_{\mathbb{C}}$, map to the closed point

$$\mathfrak{q} = (f) \in \mathbb{A}^1_{\mathbb{R}}, \quad f = (x-c)(x-\overline{c}).$$

However, note that this closed point has 'degree 2', for

$$\kappa(\mathfrak{q}) = \frac{\mathbb{R}[x]_{(f)}}{(f)\mathbb{R}[x]_{(f)}} \cong \frac{\mathbb{R}[x]}{(f)} \cong \mathbb{C},$$

since $\deg f = 2$. On the other hand, this does not happen for the other points $(x-c) \in \mathbb{A}^1_{\mathbb{R}}$, in the sense that

$$\kappa(x-c) = \frac{\mathbb{R}[x]_{(x-c)}}{(x-c)\mathbb{R}[x]_{(x-c)}} \cong \frac{\mathbb{R}[x]}{(x-c)} \cong \mathbb{R}.$$

As for $\xi = (0) \in \operatorname{Spec} \mathbb{R}[x]$, we have

$$\kappa(\xi) = \frac{\mathbb{R}[x]_{(0)}}{(0)} = \mathbb{R}[x]_{(0)} = \operatorname{Frac}(\mathbb{R}[x]) = \mathbb{R}(x).$$

Its elements are 'rational functions' g/h, not defined everywhere but *almost everywhere*, away from the (finitely many) zeros of $h \in \mathbb{R}[x]$.

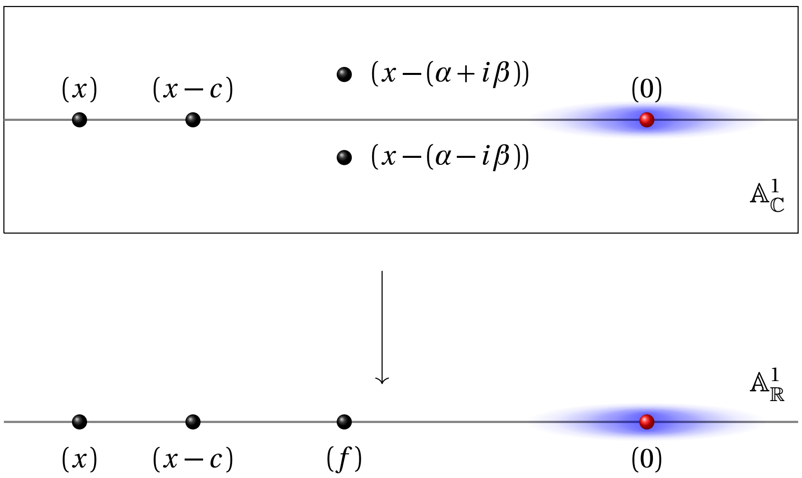

Figure 3.7: The morphism $\mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{R}}$ induced by $\mathbb{R}[x] \hookrightarrow \mathbb{C}[x]$.

Example 3.2.65. This example is the arithmetic analogue of Example 3.2.64. Consider the inclusion of rings

$$\phi\colon \mathbb{Z} \hookrightarrow \mathbb{Z}[i] = \mathbb{Z}[x]/(x^2+1), \quad i^2 = -1.$$

Here $\mathbb{Z}[i]$ is the ring of *Gaussian integers*, which is an euclidean domain, in particular a principal ideal domain. We will recall some basic number theory in this example, in order to study the induced morphism

$$f\colon \operatorname{Spec}\mathbb{Z}[i] \to \operatorname{Spec}\mathbb{Z}.$$

The algebraic question is: what happens to a prime number $p \in \mathbb{Z}$ when one adds in the imaginary unit? More precisely, is the extension

$$(p) \subset \mathbb{Z}[i]$$

still a a prime ideal? If this happens we say that p is inert, otherwise that p ramifies. For sure $(p) \subset \mathbb{Z}[i]$ is still a principal ideal (which helps). By Fermat's theorem on sums of two squares, one has that $p > 2$ is a sum of squares if and only if $p \equiv 1 \bmod 4$. In this case, one can write $p = a^2 + b^2 = (a+ib)(a-ib)$ for some integers $a, b \in \mathbb{Z}$. It follows that if $p \equiv 3 \bmod 4$, then (p) *stays prime* in $\mathbb{Z}[i]$. Let us start with the smallest prime number: one has that $2 = (1+i)(1-i)$, but $(1+i) = (1-i)$ as ideals in $\mathbb{Z}[i]$, since $i(1-i) = i+1$, thus $p = 2$ is inert. The first prime that ramifies is $5 = (2+i)(2-i)$, followed by $13 = (6+i)(6-i)$ (since 7 and 11 are inert). Primes that ramify correspond to those points $(p) \in \operatorname{Spec}\mathbb{Z}$ having more than one preimage along f.

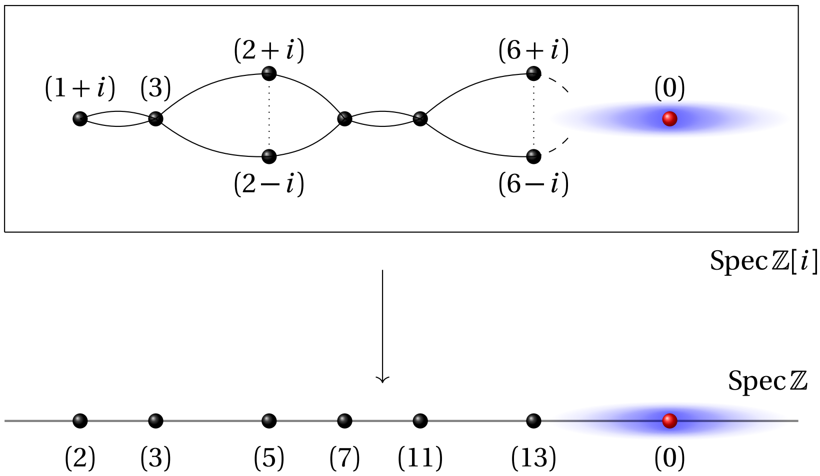

Figure 3.8: The morphism $f\colon \operatorname{Spec}\mathbb{Z}[i] \to \operatorname{Spec}\mathbb{Z}$ induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}[i]$.

Example 3.2.66. Here is another arithmetic example. Consider the inclusion of rings $\mathbb{Z} \hookrightarrow \mathbb{Z}[x]$. We want to study the induced morphism

$$\operatorname{Spec}\mathbb{Z}[x] \to \operatorname{Spec}\mathbb{Z}.$$

A pictorial description of Spec $\mathbb{Z}$ was given by Mumford [10], see Figure 3.9.

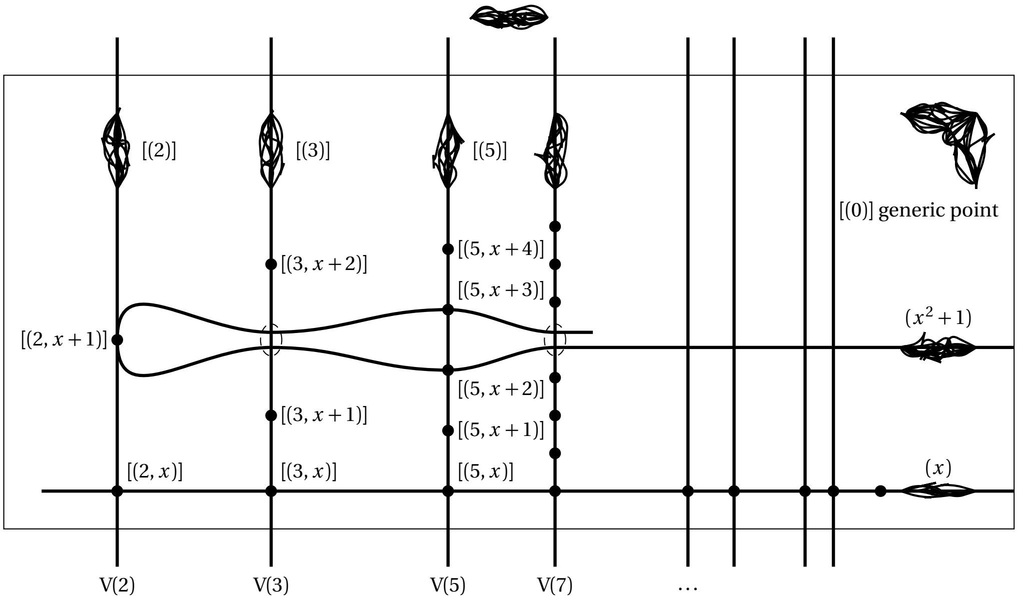

Figure 3.9: Picture's code is stolen from Pieter Belmans' website. This picture was originally drawn by David Mumford in [10], where he called Spec $\mathbb{Z}[x]$ an *arithmetic surface*.

First let us list all prime ideals in $\mathbb{Z}[x]$.

- $(0) \subset \mathbb{Z}[x]$ is a prime ideal, since $\mathbb{Z}[x]$ is an integral domain. It corresponds to the generic point of Spec $\mathbb{Z}[x]$. The residue field if $\kappa(0) = \operatorname{Frac}\mathbb{Z}[x]$.
- $(p) \subset \mathbb{Z}[x]$ is a prime ideal, for any prime number $p \in \mathbb{Z}$, since the quotient

$$\mathbb{Z}[x]/(p) \cong \mathbb{F}_p[x]$$

  is an integral domain. These points are *not closed.* Note that each point (p) is precisely the generic point of the affine line

$$\mathbb{A}^1_{\mathbb{F}_p} = \operatorname{Spec}\mathbb{F}_p[x] = \operatorname{Spec}\mathbb{Z}[x]/(p) \hookrightarrow \operatorname{Spec}\mathbb{Z}[x].$$

  These lines are drawn as vertical lines in Figure 3.9, where they are denoted V(p). The residue field of Spec $\mathbb{Z}[x]$ at these points is

$$\kappa(p) = \operatorname{Frac}\mathbb{F}_p[x] = \mathbb{F}_p(x).$$

- $(f) \subset \mathbb{Z}[x]$, where $f = a_0 + a_1 x + \cdots + a_d x^d \in \mathbb{Z}[x]$ is an irreducible polynomial (over $\mathbb{Z}$, hence over $\mathbb{Q}$) such that $\gcd(a_0, a_1, \ldots, a_d) = 1$. Each such polynomial draws an "arithmetic curve"

$$\operatorname{Spec}\mathbb{Z}[x]/(f) \hookrightarrow \operatorname{Spec}\mathbb{Z}[x]$$

depicted as a horizontal curve in Figure 3.9, where each such curve is denoted $\mathrm{V}(f)$. Clearly the point (f) is exactly the generic point of such arithmetic curve.

- $(p,f) \subset \mathbb{Z}[x]$, where p is a prime number and $f \in \mathbb{Z}[x]$ is an irreducible monic polynomial which stays irreducible over $\mathbb{F}_p$. These are closed points of $\operatorname{Spec}\mathbb{Z}[x]$. The residue fields of these points are finite extensions of $\mathbb{F}_p$.

Explicitly, one has

$$\mathrm{V}(p) = \left\{ (p),(p,f) \,\middle|\, f \text{ monic, irreducible over } \mathbb{Z} \text{ and } \mathbb{F}_p \right\}$$
$$\mathrm{V}(f) = \left\{ (f),(p,g) \,\middle|\, g \text{ divides } f \text{ modulo } p \right\},$$

and the intersection between a horizontal curve and a vertical line is

$$\mathrm{V}(f) \cap \mathrm{V}(p) = \left\{ (p,g) \,\middle|\, g \text{ divides } f \text{ modulo } p \right\}.$$

One such arithmetic curve $\mathrm{V}(f)$ is for instance the one "cut out by $x = 0$", consisting of a copy of $\operatorname{Spec}\mathbb{Z}$ itself, for

$$\mathrm{V}(x) = \operatorname{Spec}\mathbb{Z}[x]/(x) = \operatorname{Spec}\mathbb{Z} \hookrightarrow \operatorname{Spec}\mathbb{Z}[x].$$

Another curve is

$$\mathrm{V}(x^2+1) = \operatorname{Spec}\mathbb{Z}[x]/(x^2+1) = \operatorname{Spec}\mathbb{Z}[i] \hookrightarrow \operatorname{Spec}\mathbb{Z}[x].$$

Let us now analyse Figure 3.9 carefully.

- $\mathrm{V}(2)$. Two "classical points" of this vertical line are the closed points $(2,x)$ and $(2,x+1)$, corresponding to the points with coordinates 0 and 1, respectively, in the affine line $\mathbb{A}^1_{\mathbb{F}_2} \subset \operatorname{Spec}\mathbb{Z}[x]$. These two points are drawn as black bullets.

  But this affine line also intersects the arithmetic curve $\mathrm{V}(x^2+1)$, since

  $$\mathrm{V}(x^2+1) \cap \mathrm{V}(2) = \{(2,x+1)\}.$$

  However, the point $(2,x+1)$ has 'multiplicity 2' since over $\mathbb{F}_2$ we have a splitting $x^2+1 = (x+1)(x+1)$. This is why the curve $\mathrm{V}(x^2+1)$ is depicted tangent to the affine line $\mathrm{V}(2)$.

  Of course there are many other curves $\mathrm{V}(f)$ meeting $\mathrm{V}(2)$. In other words, $\mathrm{V}(2)$ has many other points of the form $(2,f)$. They correspond to irreducible monic polynomials f which stay irreducible over $\mathbb{F}_2$. For instance,

  $$f = x^2+x+1$$

  has no roots over $\mathbb{F}_2$, and if we denote by α a root of f we have a splitting

  $$x^2+x+1 = (x+\alpha)(x+\alpha+1)$$

over the larger field $\mathbb{F}_2[\alpha] = \{0, 1, \alpha, \alpha+1\} \supset \mathbb{F}_2$. We thus have two *different* residue fields

$$\kappa(2, x+1) = \mathbb{F}_2$$
$$\kappa(2, x^2+x+1) = \mathbb{F}_2[x]/(x^2+x+1) = \mathbb{F}_2[\alpha]$$

for these two different types of points of $\mathbb{A}^1_{\mathbb{F}_2} \subset \operatorname{Spec}\mathbb{Z}[x]$.

- V(3). The polynomial x^2+1 is irreducible over $\mathbb{F}_3$ (having no roots), so the point $(3, x^2+1)$ is not a 'classical' point of $\mathbb{A}^1_{\mathbb{F}_3}$. Let α be a root of x^2+1. Then

$$x^2+1 = (x-\alpha)(x-2\alpha)$$

over $\mathbb{F}_3[\alpha] \supset \mathbb{F}_3$. In this larger field, the two points $(3, x-\alpha)$ and $(3, x-2\alpha)$ would be 'separated' and would be depicted as two classical points.

The point $(3, x^2+1) = \mathrm{V}(3) \cap \mathrm{V}(x^2+1)$ is depicted as a small dotted circle. The curve $\mathrm{V}(x^2+1)$ passes through this circle, but in the picture the two branches of the curve remain separated: this reflects the fact that the 'separation' of the roots happens over the larger field $\mathbb{F}_3[\alpha] \supset \mathbb{F}_3$. The residue field of $(3, x^2+1)$ is

$$\kappa(3, x^2+1) = \mathbb{F}_3[x]/(x^2+1) = \mathbb{F}_3[\alpha],$$

a degree 2 extension of $\mathbb{F}_3$.

- V(5). The polynomial x^2+1 factors as $(x+2)(x+3)$ over $\mathbb{F}_5$, so we have two classical points $(5, x+2)$ and $(5, x+3)$, both with residue field equal to $\mathbb{F}_5$.

- V(7). The situation here is similar to that of V(3).

Exercise 3.2.67. We have, in the previous example, unconsciously confirmed that $\mathbb{A}^1_{\mathbb{F}_p} = \operatorname{Spec}\mathbb{F}_p[x]$ has way more that the 'traditional' p points $(x), (x-1), \ldots, (x-(p-1))$ corresponding to the coordinates $0, 1, \ldots, p-1 \in \mathbb{F}_p$. Show that this is always the case, by proving that $\mathbb{A}^1_K = \operatorname{Spec} K[x]$ is infinite for any field K.

Example 3.2.68 (Nodal cubic)**.** Let $C = \operatorname{Spec}\mathbb{C}[x,y]/(y^2 - x^2(x+1)) \hookrightarrow \mathbb{A}^2_{\mathbb{C}}$. Then the morphism

$$f_\phi \colon \mathbb{A}^1_{\mathbb{C}} \to C$$

induced by the ring homomorphism $\phi\colon \mathbb{C}[x,y]/(y^2 - x^2(x+1)) \to \mathbb{C}[t]$ defined by $\phi(x) = t^2-1$ and $\phi(y) = t(t^2-1)$ is a bijective morphism, but not an isomorphism. Indeed, the function $t = y/x$ is not regular at $(0,0)$, and as such it does not lie in the image of ϕ. There is no ring isomorphism $\mathbb{C}[t] \cong \mathbb{C}[x,y]/(y^2 - x^2(x+1))$. On the other hand, f_ϕ is a set-theoretic bijection, for it sends the generic point to the generic point, and the closed point $(t-a) \in \mathbb{A}^1_{\mathbb{C}}$ (for $a \in \mathbb{C}$) to the point of $\mathbb{A}^2_{\mathbb{C}}$ with coordinates $(a^2-1, a(a^2-1))$. The morphism f_ϕ is called a *rational parametrisation* of the plane curve $C \hookrightarrow \mathbb{A}^2_{\mathbb{C}}$.

Example 3.2.69 (Cuspidal cubic)**.** Let $C = \operatorname{Spec}\mathbb{C}[x,y]/(y^2-x^3) \hookrightarrow \mathbb{A}^2_{\mathbb{C}}$. Then the morphism

$$f_\phi : \mathbb{A}^1_{\mathbb{C}} \to C$$

induced by the ring homomorphism $\phi : \mathbb{C}[x,y]/(y^2-x^3) \to \mathbb{C}[t]$ defined by $\phi(x) = t^2$ and $\phi(y) = t^3$ is a bijective morphism, but not an isomorphism. It sends the closed point $(t-a)$ to the point of $\mathbb{A}^2_{\mathbb{C}}$ with coordinates (a^2, a^3). The morphism f_ϕ is called a *rational parametrisation* of the plane curve $C \hookrightarrow \mathbb{A}^2_{\mathbb{C}}$.

Example 3.2.70. Consider the ring homomorphism

$$\phi : \mathbf{k}[x,y] \to \mathbf{k}[x,y,z]/(xz-y)$$

sending $x \mapsto x$ and $y \mapsto y$. The corresponding morphism of affine schemes

$$f_\phi : \operatorname{Spec}\mathbf{k}[x,y,z]/(xz-y) \to \mathbb{A}^2_{\mathbf{k}}$$

sends $(a, ab, b) \mapsto (a, ab)$, and the generic point maps to generic point. The image of f_ϕ is $\mathrm{V}(x,y) \cup \mathrm{D}(x)$, which is neither open nor closed in $\mathbb{A}^2_{\mathbf{k}}$.

3.3 Schemes

We already anticipated the definition of schemes in Definition 3.2.27, just because we could do so. Now we start with the general theory, but first we recall the definition verbatim.

Definition 3.3.1 (Scheme)**.** A *scheme* is a locally ringed space $(X, \mathscr{O}_X)$ in which every point $x \in X$ has an open neighborhood $x \in U \subset X$ such that $(U, \mathscr{O}_X|_U)$ is an affine scheme. The sheaf $\mathscr{O}_X$ is called the *structure sheaf* of the scheme. The ring $\mathscr{O}_X(X)$ is called the ring of *regular functions* on X.

Definition 3.3.2 (Morphism of schemes)**.** A *morphism of schemes* is a morphism $(f, f^\#)$ in the category of locally ringed spaces (cf. Definition 3.1.3). An isomorphism of schemes is an invertible arrow in this category, i.e. a morphism $(f, f^\#)$ such that f is a homeomorphism and $f^\#$ is an isomorphism of sheaves.

Thus schemes also form a category, that will be denoted Sch. For any scheme S, we can form the *category* Sch_S *of* S*-schemes*, whose objects are pairs (X, f), where X is a scheme and $f : X \to S$ is a morphism of schemes. Morphisms $(X_1, f_1) \to (X_2, f_2)$ are morphisms of schemes $g : X_1 \to X_2$ such that $f_2 \circ g = f_1$. We often call them *morphisms over* S.

$$\begin{tikzcd} X_1 \arrow[rr, "g"] \arrow[dr, "f_1"'] & & X_2 \arrow[dl, "f_2"] \\ & S & \end{tikzcd}$$

We denote this category by Sch_S. If $S = \operatorname{Spec} A$ is affine, we simply write Sch_A instead of $\mathsf{Sch}_{\operatorname{Spec} A}$. For instance, when B is an A-algebra via a ring homomorphism $A \to B$, one says that $\operatorname{Spec} B \to \operatorname{Spec} A$ is an A-scheme via the canonical map attached to $A \to B$.

Exercise 3.3.3. Confirm that $\operatorname{Spec} \mathbb{Z}$ is a final object in the category of schemes, so that in particular $\mathsf{Sch} = \mathsf{Sch}_{\mathbb{Z}}$.

LEMMA 3.3.4. *Let X, Y be schemes. Then sending*

$$U \mapsto \operatorname{Hom}_{\mathsf{Sch}}(U, Y) \in \mathsf{Sets}$$

for each open subset $U \subset X$ defines a sheaf of sets on X.

Proof. By Remark 2.2.7, we need to verify that given an open subset $U \subset X$, an open cover $U = \bigcup_{i \in I} U_i$ and a collection of morphisms $f_i \colon U_i \to Y$ such that $f_i|_{U_{ij}} = f_i|_{U_{ij}}$ (as morphisms of schemes!) for every $(i, j) \in I \times I$, there exists a unique $f \colon U \to Y$ such that $f|_{U_i} = f_i$. (We have used the usual notation $U_{ij} = U_i \cap U_j$). At the level of topological spaces, it is clear that there is a unique continous map $f \colon U \to Y$ with the required property. We need to extend it (uniquely) to a morphism of *schemes*. So we need a well defined sheaf homomorphism $f^\# \colon \mathscr{O}_Y \to f_* \mathscr{O}_U$. Let $V \subset Y$ be an open subset. We define $f^\#(V) \colon \mathscr{O}_Y(V) \to \mathscr{O}_U(f^{-1}V)$ as follows.

First of all, each $f_i \colon U_i \to Y$ induces a map $\mathscr{O}_Y(V) \to \mathscr{O}_{U_i}(f_i^{-1}V) = \mathscr{O}_U(f_i^{-1}V)$. Moreover, $f^{-1}V = \bigcup_{i \in I} f_i^{-1}V$ is an open covering, and since $\mathscr{O}_U$ is a sheaf we have a diagram

$$\begin{tikzcd} & \mathscr{O}_Y(V) \arrow[dl, dotted] \arrow[d, "\tau"] & \\ \mathscr{O}_U(f^{-1}V) \arrow[r] & \prod_{i \in I} \mathscr{O}_U(f_i^{-1}V) \arrow[r, shift left, "\mu"] \arrow[r, shift right, "\nu"'] & \prod_{(i,j) \in I \times I} \mathscr{O}_U(f_i^{-1}V \cap f_j^{-1}V) \end{tikzcd}$$

where the bottom row is an equaliser sequence. Saying that $f_i|_{U_{ij}} = f_i|_{U_{ij}}$ is like saying that $\mu \circ \tau = \nu \circ \tau$, thus by the universal property of equalisers there is precisely one way to fill in the dotted arrow to $\mathscr{O}_U(f^{-1}V)$. This is the definition of $f^\#(V)$. □

Remark 3.3.5. The statement of Lemma 3.3.4 remains true for locally ringed space or, more generally, ringed spaces: we have not used the actual definition of schemes for its proof.

Definition 3.3.6 (Open subscheme)**.** An *open subscheme* of a scheme $(X, \mathscr{O}_X)$ is a scheme of the form $(U, \mathscr{O}_X|_U)$, where U is an open subset of X. We will often just write $\mathscr{O}_U$ instead of $\mathscr{O}_X|_U$. We say that $U \hookrightarrow X$ is an open immersion of schemes.

Definition 3.3.7 (Closed subscheme)**.** A *closed subscheme* of a scheme X is the datum of a closed subset $Z \subset X$ equipped with a scheme structure and a closed immersion $Z \hookrightarrow X$ of schemes.

Note the crucial difference between open and closed subscheme: a given open *subset* $U \subset X$ inherits a well precise structure sheaf, whereas on a closed *subset* $Z \hookrightarrow X$ there are a pletora of possible scheme structures. We will see, however, that for every scheme X there is a 'nicest' closed subscheme $X_{\mathrm{red}} \hookrightarrow X$, called the *reduction* of X, which is topologically the same as X and is the smallest with this property.

Example 3.3.8 (Curvilinear schemes)**.** Let $\mathbf{k}$ be a field, $n > 0$ an integer, $A_n = \mathbf{k}[t]/t^n$. Consider the origin $\{0\} \hookrightarrow \mathbb{A}^1_{\mathbf{k}} = \operatorname{Spec} \mathbf{k}[t]$, i.e. the closed point corresponding to $(t) \subset \mathbf{k}[t]$. Then for each n there is a closed immersion

$$\operatorname{Spec} A_n \hookrightarrow \mathbb{A}^1_{\mathbf{k}}.$$

Each $\operatorname{Spec} A_n$, called a *curvilinear scheme of length* n, is a scheme whose underlying topological space is $\{0\}$.

Example 3.3.9. Let $\mathbf{k}$ be an algebraically closed field, A be an artinian $\mathbf{k}$-algebra with residue field $\mathbf{k}$. Then $\operatorname{Spec} A$ is topologically just a (closed) point, corresponding to the maximal ideal $\mathfrak{m}_A \subset A$. For instance, consider $A = \mathbf{k}[x, y]/(x^2, xy, y^2)$. Such schemes are called *fat points*. Each fat point has a *length*, namely the number $\dim_{\mathbf{k}} A = \dim_{\mathbf{k}} \mathscr{O}_X(X)$. Such dimension is finite by definition of artinian algebra.

3.3.1 Morphisms to an affine scheme

A complete characterisation of morphisms *of affine schemes* was given, somewhat implicitly, in Theorem 3.2.50. Now we let X be an arbitrary scheme and we set $Y = \operatorname{Spec} A$. Consider the natural map

$$\rho_{X,Y} \colon \operatorname{Hom}_{\mathsf{Sch}}(X, Y) \to \operatorname{Hom}_{\mathsf{Rings}}(A, \mathscr{O}_X(X)) \tag{3.3.1}$$

already introduced in (3.2.10) in the affine case. It works just the same: a morphism $f \colon X \to Y$ is sent to $f^\#(Y) \colon A = \mathscr{O}_Y(Y) \to f_*\mathscr{O}_X(Y) = \mathscr{O}_X(X)$. Functoriality also holds, i.e. the diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathsf{Sch}}(X, Y) \arrow[r, "\rho_{X,Y}"] \arrow[d, "f \mapsto f \circ g"] & \operatorname{Hom}_{\mathsf{Rings}}(A, \mathscr{O}_X(X)) \arrow[d, "\phi \mapsto g^\#(X) \circ \phi"] \\
\operatorname{Hom}_{\mathsf{Sch}}(Z, Y) \arrow[r, "\rho_{Z,Y}"] & \operatorname{Hom}_{\mathsf{Rings}}(A, \mathscr{O}_Z(Z))
\end{tikzcd}$$

commutes for any morphism $g \colon Z \to X$.

THEOREM 3.3.10. *Let X be a scheme, $Y = \operatorname{Spec} A$ an affine scheme. Then the canonical map* (3.3.1) *is bijective.*

Proof. Fix a covering $X = \bigcup_{i \in I} U_i$, where $\iota_i \colon U_i = \operatorname{Spec} B_i \hookrightarrow X$ is an affine open subset. Since $U \mapsto \operatorname{Hom}_{\mathsf{Sch}}(U, Y)$ is a sheaf on X (cf. Lemma 3.3.4), the natural map

$$\alpha \colon \operatorname{Hom}_{\mathsf{Sch}}(X, Y) \to \prod_{i \in I} \operatorname{Hom}_{\mathsf{Sch}}(U_i, Y)$$

is injective. We have a new diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{Sch}}(X,Y) \arrow[r, "\rho_{X,Y}"] \arrow[d, hook, "\alpha"] & \mathrm{Hom}_{\mathsf{Rings}}(A,\mathscr{O}_X(X)) \arrow[d, "\phi\mapsto(\iota_i^\#(X)\circ\phi)_{i\in I}"] \\
\prod_{i\in I}\mathrm{Hom}_{\mathsf{Sch}}(U_i,Y) \arrow[r, "\beta"] & \prod_{i\in I}\mathrm{Hom}_{\mathsf{Rings}}(A,B_i)
\end{tikzcd}$$

where β is a bijection by Theorem 3.2.50 (or its proof). It follows that $\rho_{X,Y}$ is injective. We are left to prove its surjectivity. Fix $\phi\in\mathrm{Hom}_{\mathsf{Rings}}(A,\mathscr{O}_X(X))$, and consider its image $(\phi_i)_{i\in I}\in\prod_{i\in I}\mathrm{Hom}_{\mathsf{Rings}}(A,B_i)$. This corresponds to a unique tuple of morphisms $(f_i\colon U_i\to Y)_{i\in I}$. These have the property that $f_i|_V=f_j|_V$ for every affine open subset $V\subset U_i\cap U_j$. To see this, notice that for any $i\in I$ we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{Sch}}(U_i,Y) \arrow[r, "\sim"] \arrow[d] & \mathrm{Hom}_{\mathsf{Rings}}(A,B_i) \arrow[d, "\psi\mapsto j_i^\#(U_i)\circ\psi"] \\
\mathrm{Hom}_{\mathsf{Sch}}(V,Y) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathsf{Rings}}(A,\mathscr{O}_X(V))
\end{tikzcd}$$

where $j_i\colon V\hookrightarrow U_i$ is the open immersion. It is clear that the image of f_i in the set $\mathrm{Hom}_{\mathsf{Rings}}(A,\mathscr{O}_X(V))$ does not depend on i, being equal to the image of ϕ. Therefore all the f_i map to the same element of $\mathrm{Hom}_{\mathsf{Sch}}(V,Y)$, which is what we wanted to confirm. It now follows from Lemma 3.3.4 that $(f_i\colon U\to Y)_{i\in I}$ glue to a (unique) morphism $f\colon X\to Y$, which by construction maps to ϕ via $\rho_{X,Y}$. Thus $\rho_{X,Y}$ is surjective. □

COROLLARY 3.3.11. *Let X be a scheme. Then, there is a canonical morphism $X\to\operatorname{Spec}\mathscr{O}_X(X)$.*

Proof. Take $Y=\operatorname{Spec}\mathscr{O}_X(X)$ and consider the morphism corresponding to the identity $\mathrm{id}\in\mathrm{Hom}_{\mathsf{Rings}}(\mathscr{O}_X(X),\mathscr{O}_X(X))$ under $\rho_{X,Y}$. □

Remark 3.3.12. A possible translation of Theorem 3.3.10 is the following: if $\Gamma(-)$ denotes the functor taking a scheme X to the ring of its regular functions $\mathscr{O}_X(X)$, then the pair of functors $(\Gamma(-),\operatorname{Spec})$ is an adjoint pair on $\mathsf{Sch}\rightleftarrows\mathsf{Rings}^{\mathrm{op}}$, where of course Spec is now viewed as the composition $\mathsf{Rings}^{\mathrm{op}}\xrightarrow{\sim}\mathsf{Aff}\hookrightarrow\mathsf{Sch}$.

3.3.2 Glueing schemes

You may have encountered interesting spaces such as *projective spaces* or *Grassmannians* before. For example, projective n-space over a field $\mathbf{k}$ can be defined as follows: take an n-dimensional $\mathbf{k}$-vector space V, consider the scaling action of $\mathbf{k}^\times$ sending $v\mapsto\lambda v$ for $\lambda\in\mathbf{k}^\times$, and set

$$\mathbb{P}^n(\mathbf{k})=(V\setminus 0)/\mathbf{k}^\times.$$

This is all good in the topological (or smooth) category, however we cannot make such a definition in algebraic geometry. Quotients exist (sometimes) and their theory has now become classical, but they are delicate to deal with.

We shall see *two ways* to define projective space in algebraic geometry. The first one is by glueing schemes (along open immersions). We now describe this procedure.

The input data are as follows:

1. a scheme S,
2. a family of S-schemes $\{ X_i \to S \mid i \in I \}$,
3. open subschemes $X_{ij} \subset X_i$ for every $(i, j) \in I \times I$,
4. isomorphisms $f_{ij} \colon X_{ij} \xrightarrow{\sim} X_{ji}$ over S.

$$\begin{tikzcd}
X_{ij} \arrow[r, hookrightarrow, "\text{open}"] \arrow[dd, "\text{isomorphism } f_{ij}"'] & X_i \arrow[dr] & \\
 & & S \\
X_{ji} \arrow[r, hookrightarrow, "\text{open}"] & X_j \arrow[ur] &
\end{tikzcd}$$

The assumptions on the input data are the following:

- $X_{ii} = X_i$ and $f_{ii} = \mathrm{id}_{X_i}$ for every $i \in I$,
- $f_{ij}(X_{ij} \cap X_{ik}) = X_{jk} \cap X_{ji}$, for every $(i, j, k) \in I \times I \times I$
- the *cocycle condition* holds: $f_{jk} \circ f_{ij} = f_{ik}$ on $X_{ij} \cap X_{ik}$.

The cocycle condition is the following compatibility:

$$\begin{tikzcd}
X_{ij} \cap X_{ik} \arrow[r, hookrightarrow] \arrow[dr, "f_{ij}"] \arrow[dd, "f_{ik}"] & X_{ij} \arrow[r, "f_{ij}"] & X_{ji} \\
 & X_{jk} \cap X_{ji} \arrow[ur, hookrightarrow] \arrow[r, hookrightarrow] \arrow[dl, "f_{jk}"] & X_{jk} \arrow[dl, "f_{jk}"] \\
X_{kj} \cap X_{ki} \arrow[r, hookrightarrow] & X_{kj} &
\end{tikzcd}$$

THEOREM 3.3.13 (Glueing schemes). *With the above data and assumptions, there exists an S-scheme X (unique up to isomorphism), with open immersions $\theta_i \colon X_i \hookrightarrow X$ over S such that $\theta_j|_{X_{ji}} \circ f_{ij} = \theta_i|_{X_{ij}}$ and $X = \bigcup_{i \in I} \theta_i(X_i)$. Moreover, $\theta_i(X_i) \cap \theta_j(X_j) = \theta_i(X_{ij})$.*

Proof. See [8, Ch. 2, Lemma 3.33]. $\square$

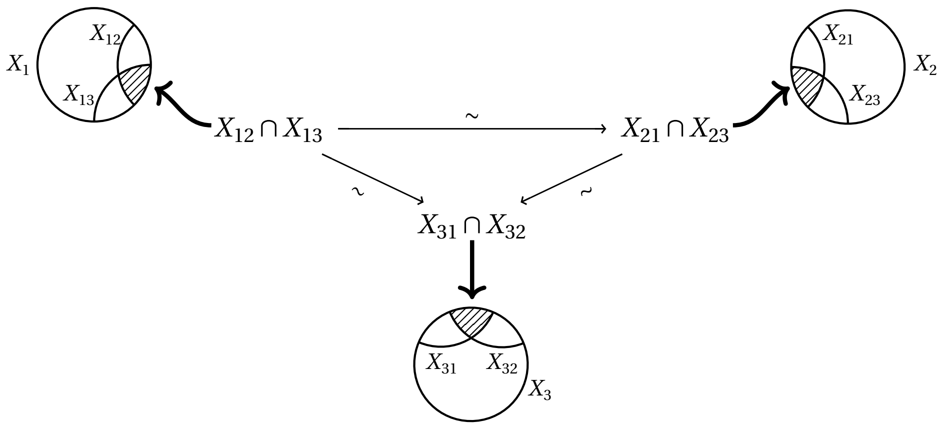

Figure 3.10: The glueing construction starting with 3 open subsets $X_1, X_2, X_3 \subset X$.

Definition 3.3.14. The *disjoint union* of a family of S-schemes $\{X_i \to S\}_{i\in I}$ is the glueing of the family along $X_{ij} = \emptyset$ (and empty maps f_{ij}). It is denoted $\coprod_{i\in I} X_i$.

Next, we describe the construction of

$$\mathbb{P}^n_A = \text{projective } n\text{-space over } A.$$

Let A be a ring, $S = \operatorname{Spec} A$ the corresponding affine scheme, $n \geq 0$ an integer and $I = \{0, 1, \ldots, n\}$. Fix a variable x_i for every $i \in I$, and form the ring $R = A[x_0^{\pm}, x_1^{\pm}, \ldots, x_n^{\pm}]$. Consider the A-subalgebras

$$A_i = A\left[x_k x_i^{-1} \,\middle|\, 0 \leq k \leq n\right] \subset R, \quad i \in I.$$

Note that A_i is the homogeneous localisation of $A[x_0, x_1, \ldots, x_n]$ at the degree 1 element x_i. Each yields an A-scheme

$$X_i = \operatorname{Spec} A_i \to \operatorname{Spec} A, \quad i \in I.$$

Now, for each $j \neq i$, the scheme X_i contains the (principal) affine open subscheme

$$X_{ij} = \mathrm{D}(x_j x_i^{-1}) = \operatorname{Spec} (A_i)_{x_j x_i^{-1}} \subset X_i.$$

But

$$(A_i)_{x_j x_i^{-1}} = (A_j)_{x_i x_j^{-1}}$$

are *equal* as subrings of R, and thus we have canonical isomorphisms $f_{ij}\colon X_{ij} \xrightarrow{\sim} X_{ji}$. Explicitly, after the identifications

$$(A_i)_{x_j x_i^{-1}} \cong A_i[t]/(t \cdot x_j x_i^{-1} - 1)$$
$$(A_j)_{x_i x_j^{-1}} \cong A_j[u]/(u \cdot x_i x_j^{-1} - 1),$$

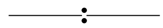

Figure 3.11: The affine line with two origins.

we see that an isomorphism between the quotient rings on the right hand sides is given by sending

$$x_j x_i^{-1} \mapsto u, \quad t \mapsto x_i x_j^{-1}, \quad x_k x_i^{-1} \mapsto x_k x_i^{-1} \text{ for } k \neq i, j.$$

The hypotheses of Theorem 3.3.13 are satisfied by our glueing data. The resulting A-scheme is called *projective n-space over A*, and is denoted $\mathbb{P}^n_A$. It has an open cover by $n+1$ affine open subsets isomorphic to affine spaces over A, namely $X_i = \operatorname{Spec} A_i \cong \mathbb{A}^n_A$. Indeed, the variables $\{ x_k/x_i \mid 0 \le k \le n \}$ are algebraically independent.

Remark 3.3.15. This construction shows that $\mathbb{P}^n_A$ is a quasicompact scheme. We shall see that it is not affine (unless $n = 0$).

Example 3.3.16 (Projective line)**.** The most explicit instance of the above construction of $\mathbb{P}^n_A$ arises for $n = 1$. In this case, our input data are simply two schemes $X_1 = \operatorname{Spec} A[t]$ and $X_2 = \operatorname{Spec} A[u]$, and the isomorphism $X_{12} = \mathrm{D}(t) \xrightarrow{\sim} \mathrm{D}(u) = X_{21}$ induced by the A-algebra isomorphism $A[u, u^{-1}] \xrightarrow{\sim} A[t, t^{-1}]$ sending $u \mapsto t^{-1}$. The glueing gives, by definition, the *projective line* $\mathbb{P}^1_A$.

Example 3.3.17. Keep the notation of Example 3.3.16, but assume $A = \mathbf{k}$ is a field, for simplicity. Then, had we chosen the isomorphism $\mathbf{k}[u, u^{-1}] \xrightarrow{\sim} \mathbf{k}[t, t^{-1}]$ sending $u \mapsto t$, we would have of course identified the complements of the origin in the two affine lines, but we also would have 'kept the origin twice'. The result of the glueing is called an *affine line with double origin*. We shall come back to this scheme, for it is the prototypical example of a non-separated scheme. Separatedness is, as we shall see, the scheme-theoretic analogue of the Hausdorff property, which we have already given up on (cf. Remark 3.2.15). Since affine schemes are separated (cf. **??**, this also gives another example (besides Example 3.2.55 and Example 3.2.56) of a non-affine scheme.

Since the schemes X_i form an open covering of the glued up scheme X, by Example 2.2.24 we have an exact sequence of abelian groups

$$0 \to \mathscr{O}_X(X) \to \prod_{i \in I} \mathscr{O}_X(X_i) \to \prod_{(i,j) \in I \times I} \mathscr{O}_X(X_{ij}) \tag{3.3.2}$$

where the first map is restriction and the second map sends $(f_i)_i \mapsto (f_i|_{X_{ij}} - f_j|_{X_{ij}})_{i,j}$.

Exercise 3.3.18. Use the sequence (3.3.2) to show that $\mathscr{O}_{\mathbb{P}^n_A}(\mathbb{P}^n_A) = A$. Observe, then, that $\mathbb{P}^n_A$ is not affine (unless $n = 0$)!

3.4 Projective schemes

In this section we define an important class of schemes, including *projective schemes*. These are the closed subschemes of $\mathbb{P}^n_A$ for some given ring A and some $n \geq 0$. They are the natural upgrade of classical *projective varieties* over a field. The general construction is somewhat analogous (though exhibiting many differences as well, see e.g. Caution 3.4.18 and Caution 3.4.27) to the construction of $\operatorname{Spec} A$ starting from a ring A. The main difference is that now we have to work with *graded rings*. The ubiquity of gradings and homogeneous ideals can be 'explained' at an informal level as follows: points $p = (a_0 : a_1 : \cdots : a_n)$ of classical projective space $\mathbb{P}^n(\mathbb{C})$ have *homogeneous coordinates*, meaning e.g. that $(1:2)$ is the same point as $(-1,-2)$ in $\mathbb{P}^1(\mathbb{C})$. As such, the evaluation of a polynomial $f \in \mathbb{C}[x_0, x_1, \ldots, x_n]$ at p is not well-defined. What *is* well-defined though, is the *vanishing* of f at p, so long as f is homogeneous, for

$$f(\lambda x_0, \lambda x_1, \ldots, \lambda x_n) = \lambda^d f(x_0, x_1, \ldots, x_n), \quad d = \deg f, \quad \lambda \in \mathbb{C}^\times.$$

We will see that the equation $f = 0$ defines a closed subscheme of $\mathbb{P}^n_{\mathbb{C}}$. This will be called a *hypersurface in* $\mathbb{P}^n_{\mathbb{C}}$.

3.4.1 Zariski topology on Proj B

Let A be a ring. A *graded A-algebra* is an A-algebra $A \to B$ equipped with a decomposition

$$B = \bigoplus_{d \geq 0} B_d,$$

where $B_d \subset B$ are subgroups satisfying $B_d B_e \subset B_{d+e}$ for each $d, e \geq 0$, and such that the image of $A \to B$ is contained in B_0. In this situation, $B_0 \subset B$ is a subring, B is naturally a B_0-algebra and each B_d is a B_0-module. Elements of B_d are called *homogeneous of degree d*. The ideal

$$B_+ = \bigoplus_{d > 0} B_d \subset B$$

is called the *irrelevant ideal*, for reasons that will become clear soon (cf. Remark 3.4.7).

Example 3.4.1. Let $B = A[x_0, x_1, \ldots, x_n]$ be the polynomial ring with A-coefficients and with x_i in degree 1 for all i. Elements of B_d are simply the homogeneous polynomials of degree d in the classical sense. The irrelevant ideal is $B_+ = (x_0, x_1, \ldots, x_n)$.

Let $I \subset B$ be an ideal. Then, the following are equivalent:

- $I \subset B$ is a graded submodule,
- I is generated by homogeneous elements,
- $I = \bigoplus_{d \geq 0} (I \cap B_d)$,

- If $f \in I \subset B$ decomposes into homogeneous parts as $f = f_0 + f_1 + \cdots + f_k$, then $f_e \in I$ for all e.

If this happens, we say that I is *homogeneous*.

Remark 3.4.2. The class of homogeneous ideals in B is closed under sum, product, intersection, and radical. The localisation of B at a multiplicative subset $S \subset B$ inherits a grading as soon as S consists of homogeneous elements.

To an arbitrary ideal $I \subset B$ we may associate a homogeneous ideal $I^{\mathrm{h}} = \bigoplus_{d \geq 0}(I \cap B_d)$. Note that $I^{\mathrm{h}} \subset I$ for any ideal I, with equality if and only if I is homogeneous.

Remark 3.4.3. Let $I \subset B$ be a homogeneous ideal. The quotient B/I is naturally a graded A-algebra via $(B/I)_d = B_d/(I \cap B_d)$.

LEMMA 3.4.4. *Let $I \subset B$ be a homogeneous ideal. Then I is prime if and only if whenever $ab \in I$ for homogeneous elements $a, b \in B$, one has that either $a \in I$ or $b \in I$.*

Proof. Let $a = \sum_{1 \leq i \leq n} a_i$ and $b = \sum_{1 \leq j \leq m} b_j$ be the homogeneous decompositions of two elements $a, b \in B$ such that $ab \in I$. Since I is homogeneous, it must contain all the homogeneous components of ab. Assume, by contradiction, that $a \notin I$ and $b \notin I$. Then, there is a largest d such that $a_d \notin I$ and a largest e such that $b_e \notin I$. We have $(ab)_{d+e} = \sum_{i+j=d+e} a_i b_j$, but every pair $(i, j) \neq (d, e)$ appearing in the sum satisfies either $i > d$ or $j > e$. Thus $a_i b_j \in I$ for every such pair. But since $(ab)_{d+e} \in I$ as well, we must have $a_d b_e \in I$. Thus, by our assumption, either $a_d \in I$ or $b_e \in I$. Contradiction. □

LEMMA 3.4.5. *If $I \subset B$ is a prime ideal, then $I^{\mathrm{h}} \subset B$ is a prime ideal.*

Proof. We exploit Lemma 3.4.4. Let $a, b \in B$ be two homogeneous elements, say $a \in B_d$ and $b \in B_e$, such that $ab \in I^{\mathrm{h}}$. In fact, $ab \in I \cap B_{d+e} \subset I$. Then, since I is prime, we have either $a \in I$ or $b \in I$. Thus either $a \in I \cap B_d \subset I^{\mathrm{h}}$, or $b \in I \cap B_e \subset I^{\mathrm{h}}$. □

Define

$$\operatorname{Proj} B = \left\{ \mathfrak{p} \subset B \;\middle|\; \begin{array}{c} \mathfrak{p} \text{ is a homogeneous prime} \\ \text{ideal such that } \mathfrak{p} \not\supset B_+ \end{array} \right\}.$$

Our goal is to put a structure of A-scheme on $\operatorname{Proj} B$. For a homogeneous ideal $I \subset B$, we define

$$\mathrm{V}_+(I) = \left\{ \mathfrak{p} \in \operatorname{Proj} B \;\middle|\; \mathfrak{p} \supset I \right\} \subset \operatorname{Proj} B.$$

These sets satisfy the axioms of closed subsets for a topology on $\operatorname{Proj} B$. The properties

- $\mathrm{V}_+(I) \cup \mathrm{V}_+(J) = \mathrm{V}_+(I \cap J)$
- $\bigcap_{\lambda \in \Lambda} \mathrm{V}_+(I_\lambda) = \mathrm{V}_+\left(\sum_{\lambda \in \Lambda} I_\lambda\right)$

- $V_+(1) = \emptyset$ and $V_+(0) = \operatorname{Proj} B$

are proved in a similar fashion to the affine case (cf. Lemma 3.2.3). The induced topology on $\operatorname{Proj} B$ is called the *Zariski topology*.

LEMMA 3.4.6. *We have the following.*

(i) If I, J are homogeneous ideals, then $V_+(I) \subset V_+(J)$ if and only if $J \cap B_+ \subset \sqrt{I}$.

(ii) One has $\operatorname{Proj} B = \emptyset$ if and only if B_+ is nilpotent.

Proof. We proceed step by step.

(i) Assume $J \cap B_+ \subset \sqrt{I}$, and fix a prime $\mathfrak{p} \in V_+(I)$. Then

$$\mathfrak{p} \supset \sqrt{I} \supset J \cap B_+ \supset J B_+,$$

and since $\mathfrak{p}$ is a prime not containing B_+ we must have $\mathfrak{p} \in V_+(J)$.

Assume now that $V_+(I) \subset V_+(J)$. Recall that $\sqrt{I} = \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p}$. Observe that any prime $\mathfrak{p} \in V(I)$ satisfies (since I is homogeneous) $I = I^{\mathrm{h}} \subset \mathfrak{p}^{\mathrm{h}}$. Now there are two possibilities. First: if $\mathfrak{p}^{\mathrm{h}} \not\supset B_+$, then by Lemma 3.4.5 $\mathfrak{p}^{\mathrm{h}}$ belongs to $V_+(I)$, thus by assumption $\mathfrak{p} \supset \mathfrak{p}^{\mathrm{h}} \supset J \supset J \cap B_+$, which implies $J \cap B_+ \subset \sqrt{I}$. Second option: if $\mathfrak{p}^{\mathrm{h}} \supset B_+$, we still have $\mathfrak{p} \supset \mathfrak{p}^{\mathrm{h}} \supset B_+ \supset J \cap B_+$. Thus, also in this case, we have $J \cap B_+ \subset \sqrt{I}$.

(ii) We have $\operatorname{Proj} B = \emptyset$ if and only if every homogeneous prime ideal $\mathfrak{p} \subset B$ contains B_+, i.e. $V_+(0) \subset V_+(B_+)$, i.e. $B_+ \subset \sqrt{0}$ by (i). □

Remark 3.4.7. More generally, one can prove that $V_+(I) = \emptyset$ if and only if $B_+ \subset \sqrt{I}$. This is why B_+ is called *irrelevant*: the operation $V_+(-)$ sends it to the empty set, and so does to all radical ideals that contain it. One should keep in mind the case $B = \mathbf{k}[x_0, x_1, \ldots, x_n]$, where $B_+ = (x_0, x_1, \ldots, x_n)$ should 'correspond to the origin'. But there is no origin in $\mathbb{P}^n(\mathbf{k})$.

Notation 3.4.8. Let $f \in B$ be a homogeneous element. We call $D_+(f) = \operatorname{Proj} B \setminus V_+(fB)$ a *principal open set* in $\operatorname{Proj} B$. We simply write $V_+(f)$ instead of $V_+(fB)$.

Remark 3.4.9. Note that $\operatorname{Proj} B$ need not be quasicompact. For instance,

$$\operatorname{Proj} \mathbb{Z}[x_1, x_2, \ldots] = \bigcup_{i \geq 1} D_+(x_i)$$

is an open cover admitting no finite subcover.

There is a canonical inclusion

$$\varepsilon \colon \operatorname{Proj} B \hookrightarrow \operatorname{Spec} B$$

and for any $f \in B$ one has $\mathrm{V}(f) \cap \operatorname{Proj} B = \bigcap_{0 \le i \le e} \mathrm{V}_+(f_i)$ if $f = f_0 + f_1 + \cdots + f_e$ is the homogeneous decomposition of $f \in B$. Then $\mathrm{D}(f) \cap \operatorname{Proj} B = \bigcup_{0 \le i \le e} \mathrm{D}_+(f_i)$. This shows that the Zariski topology on $\operatorname{Proj} B$ is induced by the Zariski topology on $\operatorname{Spec} B$ (i.e. it agrees with the subspace topology), and moreover

$$\operatorname{Proj} B \setminus \mathrm{V}_+(I) = \bigcup_{\substack{f \in I \\ f \text{ homogeneous}}} \mathrm{D}_+(f)$$

for any homogeneous ideal $I \subset B$. In particular, the principal opens

$$\left\{ \mathrm{D}_+(f) \subset \operatorname{Proj} B \;\middle|\; f \text{ is homogeneous} \right\}$$

form a base for the Zariski topology. In fact, one can focus only on those $\mathrm{D}_+(f)$ where $f \in B_+$. The reason is the following: suppose $B_+ = (f_i \mid i \in I)$ with f_i homogeneous. Then,

$$\begin{aligned} \operatorname{Proj} B &= \operatorname{Proj} B \setminus \emptyset \\ &= \operatorname{Proj} B \setminus \mathrm{V}_+(B_+) \\ &= \operatorname{Proj} B \setminus \bigcap_{i \in I} \mathrm{V}_+(f_i) \\ &= \bigcup_{i \in I} \mathrm{D}_+(f_i), \end{aligned}$$

so that for any homogeneous $g \in B$ we have

$$\begin{aligned} \mathrm{D}_+(g) &= \mathrm{D}_+(g) \cap \operatorname{Proj} B \\ &= \mathrm{D}_+(g) \cap \bigcup_{i \in I} \mathrm{D}_+(f_i) \\ &= \bigcup_{i \in I} \mathrm{D}_+(g) \cap \mathrm{D}_+(f_i) \\ &= \bigcup_{i \in I} \mathrm{D}_+(g f_i), \end{aligned}$$

where of course $g f_i \in B_+$ for every $i \in I$. We will thus use

$$\mathscr{B} = \left\{ \mathrm{D}_+(f) \subset \operatorname{Proj} B \;\middle|\; f \in B_+ \text{ is homogeneous} \right\} \tag{3.4.1}$$

as a base of open sets for $\operatorname{Proj} B$.

3.4.2 Structure sheaf on $\operatorname{Proj} B$

Let B be a graded A-algebra as in the previous section. For a homogeneous element $f \in B$, define the subring

$$B_{(f)} = \left\{ z \in B_f \;\middle|\; \deg(z) = 0 \right\} \subset B_f.$$

This is, in fact, an A-subalgebra, since $A \to B$ lands in B_0. It carries the trivial grading. It is called the *homogeneous localisation* of B at f. Its defining condition means the following:

an element $a/f^n \in B_f$ has degree 0 if $a \in B$ is homogeneous of degree $n \cdot \deg f = \deg f^n$. Note that B_f is, indeed, naturally a graded $B_{(f)}$-algebra via

$$(B_f)_d = \left\{ \frac{a}{f^n} \in B_f \;\middle|\; \deg a = d + n \cdot \deg f \right\}.$$

LEMMA 3.4.10. *Let $f \in B_+$ be homogeneous of degree d. Set $B^{(d)} = \bigoplus_{e \geq 0} B_{de} \subset B$. Then, there is a ring isomorphism*

$$\alpha_f \colon B^{(d)}/(f-1)B^{(d)} \xrightarrow{\sim} B_{(f)}.$$

In particular, if $\deg f = 1$*, we have*

$$\alpha_f \colon B/(f-1)B \xrightarrow{\sim} B_{(f)}.$$

Proof. There is a surjective ring homomorphism $B^{(d)} \twoheadrightarrow B_{(f)}$ defined on homogeneous elements (and then extended additively) by sending $a \in B_{de}$ to a/f^e. This sends $f \in B_d = B^{(d)}_1$ to 1, so descends to a map α_f. The inverse is constructed as follows. Pick $w = z/f^n \in B_{(f)}$, so that z is homogeneous of degree nd. Send w to

$$\text{the image of } z \in B_{nd} \subset B^{(d)} \text{ along } B^{(d)} \twoheadrightarrow B^{(d)}/(f-1)B^{(d)}.$$

It is straightforward to check that this is well-defined, and is the inverse of α_f. □

Terminology 3.4.11. The ring $B^{(d)}$ is called the *d-th Veronese ring* attached to B.

Example 3.4.12. Let $B = A[x_0, x_1, \ldots, x_n]$ and $f = x_i$, which has degree 1. Then, $B^{(1)} = B$ and Lemma 3.4.10 yields

$$A[x_0, \ldots, x_n]_{(x_i)} \cong A[x_0, \ldots, x_n]/(x_i - 1) \cong A[x_0, \ldots, \widehat{x}_i, \ldots, x_n].$$[2]

Let $\mathcal{B}$ be the base of the Zariski topology on $\operatorname{Proj} B$ as in Equation (3.4.1). Our next goal is to construct a $\mathcal{B}$-sheaf of rings on $X = \operatorname{Proj} B$. By Lemma 2.5.7, this will uniquely extend to a sheaf, which will be denoted $\mathscr{O}_X$.

If $f \in B_+$ is homogeneous, we have $\mathrm{D}_+(f) = \mathrm{D}(f) \cap \operatorname{Spec} B$. We next sketch the proof of a few crucial properties about the composition

$$\begin{array}{rcl} \theta\colon \mathrm{D}_+(f) \hookrightarrow \mathrm{D}(f) = \operatorname{Spec} B_f & \longrightarrow & \operatorname{Spec} B_{(f)} \\ \mathfrak{p} & \longmapsto & \mathfrak{p} B_f \cap B_{(f)}. \end{array}$$

LEMMA 3.4.13 ((De)homogenisation). *Let $f \in B_+$ be a homogeneous element.*

(i) $\theta\colon \mathrm{D}_+(f) \to \operatorname{Spec} B_{(f)}$ is a homeomorphism.

[2] Notational warning: do not confuse $A[x_0, \ldots, x_n]_{(x_i)}$ (homogeneous localisation) with the localisation of $A[x_0, \ldots, x_n]$ at the prime ideal $(x_i) = x_i A[x_0, \ldots, x_n]$. Same potential problem when $(f) = fB \subset B$ is a prime ideal.

(ii) If $\mathrm{D}_+(g) \subset \mathrm{D}_+(f)$ *and* $\alpha = g^{\deg f}/f^{\deg g} \in B_{(f)}$*, then* $\theta(\mathrm{D}_+(g)) = \mathrm{D}(\alpha)$*.*

(iii) If g *is as in (ii), then there is a canonical homomorphism* $B_{(f)} \to B_{(g)}$ *inducing a ring isomorphism* $(B_{(f)})_\alpha \xrightarrow{\sim} B_{(g)}$*.*

Proof. We proceed step by step.

(i)–(ii) The map θ is continuous, as we have already observed that the Zariski topology on $\operatorname{Proj} B$ is induced by that of $\operatorname{Spec} B$. We first need to prove it is bijective. Then, proving (ii) will show that it is open, hence a homeomorphism.

θ is injective: Suppose $\mathfrak{p}B_f \cap B_{(f)} = \mathfrak{p}'B_f \cap B_{(f)}$ for $\mathfrak{p}$, $\mathfrak{p}'$ two elements of $\mathrm{D}_+(f)$. Fix a homogeneous generator $b \in \mathfrak{p}$, so that $b^{\deg f}/f^{\deg b} \in \mathfrak{p}B_f \cap B_{(f)} \subset \mathfrak{p}'B_f$. Then $b^{\deg f} \in \mathfrak{p}'$, and since $\mathfrak{p}'$ is prime we deduce $b \in \mathfrak{p}'$. Hence $\mathfrak{p} \subset \mathfrak{p}'$. Exchanging the roles of $\mathfrak{p}$ and $\mathfrak{p}'$ we obtain equality.

θ is surjective: Fix $\mathfrak{q} \in \operatorname{Spec} B_{(f)}$. Define a homogeneous ideal $\mathfrak{p} \subset B$ by declaring that $x \in B_d$ lies in $\mathfrak{p}$ if and only if $x^{\deg f}/f^d \in \mathfrak{q} \subset B_{(f)}$. It is homogeneous by construction, and it is prime as well. Indeed, pick two homogeneous elements $x \in B_d$ and $y \in B_e$ such that $xy \in \mathfrak{p}_{d+e}$. This means that

$$\frac{(xy)^{\deg f}}{f^{d+e}} = \frac{x^{\deg f}}{f^d}\frac{y^{\deg f}}{f^e} \in \mathfrak{q}.$$

Then either $x^{\deg f}/f^d$ or $y^{\deg f}/f^e$ lies in $\mathfrak{q}$, because $\mathfrak{q}$ is prime. But by definition this means that either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. Moreover $f \notin \mathfrak{p}$, for otherwise we would have $1 \in \mathfrak{q}$. Finally, $\mathfrak{p}B_f \cap B_{(f)} = \mathfrak{q}$, which shows surjectivity.

(ii) holds: Fix $\mathfrak{p} \in \mathrm{D}_+(f)$. It is clear that $g \in \mathfrak{p}$ if and only if $\alpha \in \mathfrak{p}B_f \cap B_{(f)}$.

(iii) (sketch): If $\mathrm{D}_+(g) \subset \mathrm{D}_+(f)$, by Lemma 3.4.6 (i) we have $gB = gB \cap B_+ \subset \sqrt{fB}$, i.e. $g^k = fb$ for some $b \in B$ and some $k > 0$. We may assume b is homogeneous by replacing it with its component of degree $k\deg g - \deg f$. Then $B_{(f)} \to B_{(g)}$ is defined by sending $a/f^n \mapsto ab^n/g^{kn}$. It does have the required properties, whose confirmation is left as an exercise. □

Remark 3.4.14.

THEOREM 3.4.15. *Let* B *be a graded* A*-algebra. Then* $X = \operatorname{Proj} B$ *is canonically an* A*-scheme, with the property that the principal open subset* $\mathrm{D}_+(f) \subset X$ *is affine and isomorphic to* $\operatorname{Spec} B_{(f)}$*, for any homogeneous element* $f \in B_+$*.*

Proof. For $f \in B_+$ a homogeneous element, define

$$\mathscr{O}_X(\mathrm{D}_+(f)) = B_{(f)}. \tag{3.4.2}$$

We first confirm that this prescription defines a $\mathcal{B}$-presheaf on X. This, in fact, follows at once by Lemma 3.4.13 (iii), which shows that a canonical restriction map $\mathscr{O}_X(\mathrm{D}_+(f)) \to \mathscr{O}_X(\mathrm{D}_+(g))$ exists whenever $\mathrm{D}_+(g) \subset \mathrm{D}_+(f)$, and that $B_{(f)}$ and $B_{(g)}$ are canonically isomorphic as soon as $\mathrm{D}_+(g) = \mathrm{D}_+(f)$.

In fact, (3.4.2) defines a $\mathcal{B}$-*sheaf* on X. This can be confirmed via the equaliser sequence. What we need to show is that for any $f \in B_+$ and any open cover $\mathrm{D}_+(f) = \bigcup_{i\in I} \mathrm{D}_+(f_i)$, the sequence

$$\mathscr{O}_X(\mathrm{D}_+(f)) \longrightarrow \prod_{i\in I} \mathscr{O}_X(\mathrm{D}_+(f_i)) \rightrightarrows \prod_{(i,j)\in I\times I} \mathscr{O}_X(\mathrm{D}_+(f_i f_j))$$

is an equaliser diagram in the category of A-algebras. This can be rewritten as the sequence

$$B_{(f)} \longrightarrow \prod_{i\in I} B_{(f_i)} \rightrightarrows \prod_{(i,j)\in I\times I} B_{(f_i f_j)}. \tag{3.4.3}$$

Let us define

$$\alpha_i = f_i^{\deg f} / f^{\deg f_i}, \quad \alpha_{ij} = (f_i f_j)^{\deg f} / f^{\deg f_i f_j}$$

in $B_{(f)}$. Then, we know that

$$\theta(\mathrm{D}_+(f_i)) = \mathrm{D}(\alpha_i), \quad \theta(\mathrm{D}_+(f_i f_j)) = \mathrm{D}(\alpha_{ij})$$

by Lemma 3.4.13 (ii). Since $\theta \colon \mathrm{D}_+(f) \to \operatorname{Spec} B_{(f)}$ is a homeomorphism, we have an open covering

$$\operatorname{Spec} B_{(f)} = \bigcup_{i\in I} \theta(\mathrm{D}_+(f_i)) = \bigcup_{i\in I} \mathrm{D}(\alpha_i).$$

In particular, we have an equaliser sequence

$$B_{(f)} \longrightarrow \prod_{i\in I} (B_{(f)})_{\alpha_i} \rightrightarrows \prod_{(i,j)\in I\times I} (B_{(f)})_{\alpha_{ij}}$$

in the category of A-algebras, because $\mathscr{O}_{\operatorname{Spec} B_{(f)}}$ is a sheaf of A-algebras. But thanks to Lemma 3.4.13 (iii) this is precisely the sequence (3.4.3). Therefore $\mathscr{O}_X$ is a $\mathcal{B}$-sheaf.

Let $\mathscr{O}_X$ denote the induced sheaf of A-algebras. The stalks are local rings. Indeed, if $x \in X$ corresponds to a homogeneous prime ideal $\mathfrak{p} \subset B$, one has a canonical isomorphism

$$\mathscr{O}_{X,x} = \varinjlim_{\substack{f \text{ homogeneous} \\ f \notin \mathfrak{p}}} B_{(f)} \xrightarrow{\sim} B_{(\mathfrak{p})}$$

where $B_{(\mathfrak{p})} \subset B_{\mathfrak{p}}$ denotes the homogeneous localisation at $\mathfrak{p}$, namely the set of degree 0 elements in the localisation $B_{\mathfrak{p}}$ (**warning**: this localisation $B_{\mathfrak{p}}$ is, by definition, the localisation at the multiplicative subset of B consisting of *homogeneous* elements that are not in $\mathfrak{p}$, cf. Warning A.2.8). The proof is identical to the one we gave for Spec (cf. Theorem 3.2.24 (c)).

It follows that the pair $(X, \mathscr{O}_X)$ defines a locally ringed space. Now, the homeomorphism $\theta\colon \mathrm{D}_+(f) \to \operatorname{Spec} B_{(f)}$ extends to an isomorphism of locally ringed spaces

$$(\theta, \theta^\#)\colon (\mathrm{D}_+(f), \mathscr{O}_X|_{\mathrm{D}_+(f)}) \xrightarrow{\sim} (\operatorname{Spec} B_{(f)}, \mathscr{O}_{\operatorname{Spec} B_{(f)}})$$

which shows that $(X, \mathscr{O}_X)$ is a scheme with the sought after property. In a little more detail, to construct $\theta^\#\colon \mathscr{O}_{\operatorname{Spec} B_{(f)}} \to \theta_*(\mathscr{O}_X|_{\mathrm{D}_+(f)})$, we take a principal open $\mathrm{D}(\alpha) \subset \operatorname{Spec} B_{(f)}$ and since $\alpha \in B_{(f)}$ we may write it as $g^r/f^{\deg g}$, where $r = \deg f$. Therefore we can apply Lemma 3.4.13 (iii), which gives the isomorphism

$$(B_{(f)})_\alpha \xrightarrow{\sim} B_{(g)}.$$

This is our $\theta^\#(\mathrm{D}(\alpha))$, which makes sense since

$$\theta_*(\mathscr{O}_X|_{\mathrm{D}_+(f)})(\mathrm{D}(\alpha)) = \mathscr{O}_X|_{\mathrm{D}_+(f)}(\theta^{-1}\mathrm{D}(\alpha)) = \mathscr{O}_X|_{\mathrm{D}_+(f)}(\mathrm{D}_+(g)) = \mathscr{O}_X(\mathrm{D}_+(g)) = B_{(g)}.$$

Finally, the A-scheme structure of X is given by the fact that each $B_{(f)}$ is naturally an A-algebra (since $A \to B$ has image inside B_0). □

Example 3.4.16. Let $B = A[x_0, x_1, \ldots, x_n]$, with the usual grading. Then

$$\operatorname{Proj} A[x_0, x_1, \ldots, x_n] = \mathbb{P}^n_A$$

where projective n-space over A was defined via glueing in Section 3.3.2. The glueing construction of the principal open subsets

$$\mathrm{D}_+(x_i) = \operatorname{Spec} A[x_k x_i^{-1} \mid 0 \le k \le n] = \mathbb{A}^n_A$$

along the open subschemes

$$\mathrm{D}_+(x_i x_j) = \operatorname{Spec} A[x_k x_i^{-1} \mid 0 \le k \le n]_{x_i x_j} \subset \mathrm{D}_+(x_i)$$

is nothing but the translation of the fact that sending $\mathrm{D}_+(f) \mapsto B_{(f)}$ is a sheaf.

Example 3.4.17. We have $\mathbb{P}^0_A = \operatorname{Proj} A[x] = \operatorname{Spec} A$.

Caution 3.4.18. Proj is *not a functor*! It is not true that a morphism of graded A-algebras $\phi\colon B \to C$ induces a morphism of A-schemes $\operatorname{Proj} C \to \operatorname{Proj} B$. The problem is that

$$\mathfrak{q} \not\supset C_+ \text{ does not imply } \phi^{-1}(\mathfrak{q}) \not\supset B_+$$

See, however, Proposition 3.4.20 for the closest to a functor one can get.

Example 3.4.19. If $\phi\colon B = \mathbf{k}[x_0, x_1] \hookrightarrow \mathbf{k}[x_0, x_1, x_2] = C$ is the natural inclusion, then $\mathfrak{q} = (x_0, x_1) \in \operatorname{Proj} C$, but $\phi^{-1}(\mathfrak{q}) = (x_0, x_1) = B_+$. This is the only 'problematic' point.

PROPOSITION 3.4.20. *Let $\phi\colon B \to C$ be a graded morphism of graded A-algebras. Then there is a canonical morphism*

$$f\colon \operatorname{Proj} C \setminus \mathrm{V}_+(B_+C) \longrightarrow \operatorname{Proj} B, \quad \mathfrak{p} \mapsto \phi^{-1}\mathfrak{p}$$

such that for any homogeneous $h \in B_+$ we have $f^{-1}(\mathrm{D}_+(h)) = \mathrm{D}_+(\phi(h))$, and the induced morphism $\mathrm{D}_+(\phi(h)) \to \mathrm{D}_+(h)$ of affine schemes corresponds to the canonical restriction $B_{(h)} \to C_{(\phi(h))}$.

Proof. If $\mathfrak{p} \subset C$ is a homogeneous prime ideal, then $\phi^{-1}\mathfrak{p} \subset B$ is a homogeneous prime ideal. We have $B_+ \not\subset \phi^{-1}\mathfrak{p}$ if and only if $B_+C \not\subset \phi(\phi^{-1}\mathfrak{p}) \subset \mathfrak{p}$. The property $f^{-1}(\mathrm{D}_+(h)) = \mathrm{D}_+(\phi(h))$ is easily confirmed just by using the definitions (as in the affine case), and note that in the commutative diagram

$$\begin{tikzcd}
B \arrow[r, "\phi"] \arrow[d] & C \arrow[d] \\
B_h \arrow[r, "\phi_h"] & C_{(\phi(h))} \\
B_{(h)} \arrow[u, hook] \arrow[r, dashed] & C_{(\phi(h))} \arrow[u, hook]
\end{tikzcd}$$

the map ϕ_h is a graded morphism, therefore it preserves the degree 0 pieces, which induces $B_{(h)} \to C_{(\phi(h))}$. Taking Spec of this map recovers precisely the morphism of affine schemes $f|_{\mathrm{D}_+(\phi(h))}\colon \mathrm{D}_+(\phi(h)) \to \mathrm{D}_+(h)$. $\square$

Example 3.4.21. In the situation of Example 3.4.19, the morphism we obtain applying Proposition 3.4.20 is

$$\mathbb{P}^2_{\mathbf{k}} \setminus \{(0:0:1)\} \to \mathbb{P}^1_{\mathbf{k}}.$$

This is called *projection from a point*.

Example 3.4.22. Consider the graded morphism $B = \mathbf{k}[x_0, x_1] \to \mathbf{k}[y_0, y_1] = C$ sending $x_i \mapsto y_i^2$ for $i = 0, 1$. In this case, $\mathrm{V}(B_+C) = \mathrm{V}(y_0^2, y_1^2) = \mathrm{V}(y_0, y_1) = \emptyset$, so we get a well-defined morphism $\mathbb{P}^1_{\mathbf{k}} \to \mathbb{P}^1_{\mathbf{k}}$, which on closed points sends $(a_0 : a_1) \mapsto (a_0^2 : a_1^2)$.

Exercise 3.4.23. Show that there is no nonconstant morphism $\mathbb{P}^n_{\mathbf{k}} \to \mathbb{P}^m_{\mathbf{k}}$ if $m < n$.

Remark 3.4.24. If $\phi\colon B \to C$ is a *surjective* graded morphism of graded A-algebras, we have $C_+ = \phi(B_+)$, hence there is global A-morphism

$$f\colon \operatorname{Proj} C \to \operatorname{Proj} B,$$

locally given by the closed immersions

$$\operatorname{Spec} C_{(\phi(f))} \hookrightarrow \operatorname{Spec} B_{(f)}$$

induced by the natural surjections $B_{(f)} \twoheadrightarrow C_{(\phi(f))}$. Therefore f is a closed immersion of A-schemes.

COROLLARY 3.4.25. *Let $I \subset B = A[x_0, x_1, \dots, x_n]$ be a homogeneous ideal, and let $\phi\colon B \to B/I$ be the canonical surjection. Then $\phi(B_+) = (B/I)_+$. Therefore, the* Proj *construction yields a closed immersion*

$$\operatorname{Proj} B/I \hookrightarrow \mathbb{P}^n_A = \operatorname{Proj} B$$

over $\operatorname{Spec} A$*, with image homeomorphic to* $\mathrm{V}_+(I) \subset \operatorname{Proj} B$*.*

Terminology 3.4.26. Algebras of the form $A[x_0, x_1, \dots, x_n]/I$ as in Corollary 3.4.25 are called *homogeneous A-algebras*. A scheme that is isomorphic to a closed subscheme of $\mathbb{P}^n_A$, for some $n \geq 0$, is called a *projective scheme over A*.

Caution 3.4.27. It is not true that an isomorphism of schemes $\operatorname{Proj} B \cong \operatorname{Proj} C$ yields an isomorphism of graded A-algebras. For instance, one has $\operatorname{Proj} B = \operatorname{Proj} B^{(d)}$ for any $d \geq 1$, but B is not isomorphic to $B^{(d)}$ if $d > 1$.

3.4.3 Examples of projective schemes

Let $\mathbf{k}$ be a field. First of all, let us clarify the relationship between the scheme $\mathbb{P}^n_{\mathbf{k}}$ and classical projective space $\mathbb{P}^n(\mathbf{k}) = (\mathbf{k}^{n+1} \setminus 0)/\mathbf{k}^\times$. There is a set-theoretic map

$$\mathbb{P}^n(\mathbf{k}) \to \mathbb{P}^n_{\mathbf{k}} = \operatorname{Proj} \mathbf{k}[x_0, x_1, \dots, x_n] \tag{3.4.4}$$

sending a point $(a_0 : a_1 : \cdots : a_n)$ to the homogeneous prime ideal

$$(a_i x_j - a_j x_i \mid 0 \leq i, j \leq n) \subset \mathbf{k}[x_0, x_1, \dots, x_n]. \tag{3.4.5}$$

Note that such ideal can be viewed as generated by the minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_n \\ a_0 & a_1 & \cdots & a_n \end{pmatrix}.$$

The map (**??**) is a bijection onto the set of closed points of $\mathbb{P}^n_{\mathbf{k}}$.

Terminology 3.4.28. If we set $\mathbb{P}^n_{\mathbf{k}} = \operatorname{Proj} \mathbf{k}[x_0, x_1, \dots, x_n]$, we call $(x_0, x_1, \dots, x_n)$ the homogeneous coordinates of $\mathbb{P}^n_{\mathbf{k}}$. The closed point of $\mathbb{P}^n_{\mathbf{k}}$ corresponding to (3.4.4) is denoted $(a_0 : a_1 : \cdots : a_n)$.

Remark 3.4.29. In the case of $\mathbb{P}^1_{\mathbf{k}}$, there is only one non-closed point, namely the point ξ corresponding to the trivial ideal $(0) \subset \mathbf{k}[x_0, x_1]$. It is the generic point of $\mathbb{P}^1_{\mathbf{k}}$, and one has $\kappa(\xi) = \mathbf{k}(t)$.

Remark 3.4.30. If $\mathfrak{p} \in X = \operatorname{Proj} B$, the stalk of the structure sheaf $\mathscr{O}_X$ at $\mathfrak{p}$ is the homogeneous localisation $B_{(\mathfrak{p})}$. But, if $U = \operatorname{Spec} R \subset X$ is any affine open neighborhood of $\mathfrak{p}$, we clearly have

$$R_{\mathfrak{p}} = \mathscr{O}_{X,\mathfrak{p}} = B_{(\mathfrak{p})}.$$

However, R is a different ring, so we have to understand what ideal $\mathfrak{p} \subset B$ becomes when viewed in the ring R. We explain this via an example. Consider for instance the (closed) 'coordinate point'

$$p_i = (0 : \cdots : 0 : 1 : 0 : \cdots : 0) \in \mathbb{P}^n_{\mathbf{k}},$$

with 1 sitting in the $(i+1)$-st slot, corresponding to the homogeneous prime ideal

$$\mathfrak{p}_i = (x_0, \ldots, \widehat{x}_i, \ldots, x_n) \subset \mathbf{k}[x_0, x_1, \ldots, x_n].$$

Then, we have $p_i \in \mathrm{D}_+(x_i) = \operatorname{Spec} \mathbf{k}[x_0, x_1, \ldots, x_n]_{(x_i)}$, and

$$\mathbf{k}[x_0, x_1, \ldots, x_n]_{(x_i)} \cong \mathbf{k}[x_0, \ldots, \widehat{x}_i, \ldots, x_n]$$

by Lemma 3.4.10. Under this identification, p_i corresponds to the origin in

$$\operatorname{Spec} \mathbf{k}[x_0, \ldots, \widehat{x}_i, \ldots, x_n],$$

which in turn corresponds to the ideal $\mathfrak{q}_i = (x_0, \ldots, \widehat{x}_i, \ldots, x_n)$. Therefore

$$\mathscr{O}_{\mathbb{P}^n_{\mathbf{k}}, p_i} = \mathbf{k}[x_0, x_1, \ldots, x_n]_{(\mathfrak{p}_i)} = \mathbf{k}[x_0, \ldots, \widehat{x}_i, \ldots, x_n]_{\mathfrak{q}_i},$$

which consists of fractions f/g of polynomials in n variables, where $g(0, \ldots, 0) \neq 0 \in \mathbf{k}$.

Terminology 3.4.31. Let V be a $\mathbf{k}$-vector space of dimension $n+1$. The symmetric algebra $\operatorname{Sym} V^\vee$ is the polynomial ring $\mathbf{k}[x_0, x_1, \ldots, x_n]$ (polynomial functions on V), and one defines

$$\mathbb{P}(V) = \operatorname{Proj} \operatorname{Sym} V^\vee.$$

This is the projective space attached to V, whose closed points correspond to lines in V.

Example 3.4.32 (Plane curves)**.** Consider the same polynomial $xy - z^2 \in \mathbb{C}[x, y, z]$ of Example 3.2.60. Note that it is homogeneous. Now, its vanishing scheme $\mathrm{V}_+(xy - z^2)$ is the topological image of a closed immersion into $\mathbb{P}^2_{\mathbb{C}} = \operatorname{Proj} \mathbb{C}[x, y, z]$, namely the morphism

$$\operatorname{Proj} \mathbb{C}[x, y, z]/(xy - z^2) \hookrightarrow \mathbb{P}^2_{\mathbb{C}}$$

induced by the surjection $\mathbb{C}[x, y, z] \twoheadrightarrow \mathbb{C}[x, y, z]/(xy - z^2)$. In general, the vanishing scheme of a degree 2 homogeneous polynomial $f \in \mathbb{C}[x, y, z]$ is called a *plane conic*. The vanishing scheme of an arbitrary homogeneous polynomial of degree d is called a *plane curve of degree d*.

Exercise 3.4.33. Show that all irreducible plane conics $\operatorname{Proj} \mathbb{C}[x, y, z]/f \hookrightarrow \mathbb{P}^2_{\mathbb{C}}$ are isomorphic to each other (**Hint**: Show that you can reduce to the normal form $f = xy - z^2$, or $f = x^2 + y^2 + z^2$ if you prefer; then show that this particular plane conic is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$). In fact, you may want to show this over an arbitrary algebraically closed field $\mathbf{k}$ of characteristic different from 2.

Example 3.4.34 (Rational normal curve)**.** Let $d > 0$ be an integer. There is a closed immersion

$$\begin{aligned} \mathbb{P}^1_{\mathbf{k}} &\hookrightarrow \mathbb{P}^d_{\mathbf{k}} \\ (u:v) &\longmapsto (u^d : u^{d-1}v : \cdots : uv^{d-1} : v^d) \end{aligned}$$

Defined by the surjection

$$\begin{aligned} \mathbf{k}[x_0, x_1, \ldots, x_d] &\twoheadrightarrow \mathbf{k}[u,v] \\ x_i &\longmapsto u^{d-i}v^i. \end{aligned}$$

The image of this closed immersion is called the *rational normal curve* in $\mathbb{P}^d_{\mathbf{k}}$. If $d = 3$, the image of $\mathbb{P}^1_{\mathbf{k}} \hookrightarrow \mathbb{P}^3_{\mathbf{k}}$ is called a *twisted cubic in* $\mathbb{P}^3_{\mathbf{k}}$.

Example 3.4.35 (Veronese embedding)**.**

Example 3.4.36 (Segre embedding)**.**

Example 3.4.37.

Example 3.4.38.

A | Commutative algebra

A.1 Universal constructions

A.1.1 Limits and colimits

Let $\mathscr{C}$ be a category, $\mathbf{I}$ a small category. Define an $\mathbf{I}$-diagram to be just a functor $M\colon \mathbf{I} \to \mathscr{C}$. Denote by M_i the object of $\mathscr{C}$ image of the object $i \in \mathbf{I}$ via M. If $f\colon i \to j$ is an arrow in $\mathbf{I}$, the induced arrow in $\mathscr{C}$ is denoted $M(f)\colon M_i \to M_j$.

Definition A.1.1 (Limit)**.** A *limit* of an $\mathbf{I}$-diagram $M\colon \mathbf{I} \to \mathscr{C}$ is an object $\varprojlim_{i\in\mathbf{I}} M_i$ of $\mathscr{C}$ along with an arrow $p_i\colon \varprojlim_{i\in\mathbf{I}} M_i \to M_i$ for every $i \in \mathbf{I}$, such that for every arrow $f\colon i \to j$ in $\mathbf{I}$ one has $p_j = M(f) \circ p_i$, and satisfying the following universal property: given an object P along with morphisms $\pi_i\colon P \to M_i$ such that $\pi_j = M(f) \circ \pi_i$ for every $f\colon i \to j$ in $\mathbf{I}$, there exists a unique arrow $\alpha\colon P \to \varprojlim_{i\in\mathbf{I}} M_i$ such that $\pi_i = p_i \circ \alpha$ for all $i \in \mathbf{I}$.

$$\begin{tikzcd}
P \arrow[d, dashed, "\exists!\alpha"'] \arrow[dr, "\pi_i"] \arrow[drr, bend left, "\pi_j"] & & \\
\varprojlim_{i\in\mathbf{I}} M_i \arrow[r, "p_i"] \arrow[rr, bend right, "p_j"'] & M_i \arrow[r, "M(f)"] & M_j
\end{tikzcd}$$

Exercise A.1.2. The limit over the empty diagram satisfies the universal property of a final object of $\mathscr{C}$.

Example A.1.3 (Products are limits)**.** Let $\mathbf{I}$ be the category with two objects i, j and no morphisms between them. Then an $\mathbf{I}$-diagram $M\colon \mathbf{I} \to \mathscr{C}$ is just the choice of two objects M_i, M_j of $\mathscr{C}$. The limit of M satisfies the universal property of the product $M_i \times M_j$.

$$\begin{tikzcd}
P \arrow[dr, dotted, "\alpha"] \arrow[drr, bend left, "\pi_i"] \arrow[ddr, bend right, "\pi_j"'] & & \\
& M_i \times M_j \arrow[r, "p_i"] \arrow[d, "p_j"] & M_i \\
& M_j &
\end{tikzcd}$$

A special case is $\mathbf{I} = \emptyset$, which recovers the universal property of a final object in $\mathscr{C}$. In other words, the product over the empty diagram is a final object.

Example A.1.4 (Equalisers are limits)**.** Let $\mathbf{I}$ be the category with two objects i, j and two arrows $i \rightrightarrows j$. Then an $\mathbf{I}$-diagram $M\colon \mathbf{I} \to \mathscr{C}$ is just the choice of two parallel arrows $\phi, \psi\colon M_i \rightrightarrows M_j$ in $\mathscr{C}$. The limit of M satisfies the universal property of the equaliser of (ϕ, ψ).

Example A.1.5 (Kernels are limits)**.** This is because kernels are equalisers (in the previous example take $\psi = 0$).

Definition A.1.6 (Colimit)**.** A *colimit* of an $\mathbf{I}$-diagram $M\colon \mathbf{I} \to \mathscr{C}$ is an object $\varinjlim_{i\in\mathbf{I}} M_i$ of $\mathscr{C}$ along with an arrow $s_i\colon M_i \to \varinjlim_{i\in\mathbf{I}} M_i$ for every $i \in \mathbf{I}$, such that for every arrow $f\colon i \to j$ in $\mathbf{I}$ one has $s_i = s_j \circ M(f)$, and satisfying the following universal property: given an object P along with morphisms $\sigma_i\colon M_i \to P$ such that $\sigma_i = \sigma_j \circ M(f)$ for every $f\colon i \to j$ in $\mathbf{I}$, there exists a unique arrow $\alpha\colon \varinjlim_{i\in\mathbf{I}} M_i \to P$ such that $\sigma_i = \alpha \circ s_i$ for all $i \in \mathbf{I}$.

$$\begin{tikzcd}
P & & \\
\varinjlim_{i\in\mathbf{I}} M_i \arrow[u, dotted, "\exists!\alpha"] & M_j \arrow[l, "s_j"'] \arrow[ul, "\sigma_j"'] & M_i \arrow[l, "M(f)"'] \arrow[ull, bend right, "\sigma_i"'] \arrow[ll, bend left, "s_i"]
\end{tikzcd}$$

Exercise A.1.7. The colimit over the empty diagram satisfies the universal property of an initial object of $\mathscr{C}$.

Exercise A.1.8. Convince yourself that coproducts, coequalisers and cokernels are examples of colimits, along the same lines of Examples A.1.3, A.1.4 and A.1.5.

Definition A.1.9 (Filtered category)**.** A nonempty category $\mathbf{I}$ is *filtered* if for every two objects $i, j \in \mathbf{I}$ the following are true:

- there exists $k \in \mathbf{I}$ and morphisms $i \to k$ and $j \to k$, and
- for any two morphisms $f, g \in \mathrm{Hom}_{\mathbf{I}}(i, j)$ there exists an object $k \in \mathbf{I}$ along with a morphism $h\colon j \to k$ such that $h \circ f = h \circ g$ in $\mathrm{Hom}_{\mathbf{I}}(i, k)$.

A colimit of an $\mathbf{I}$-diagram $M\colon \mathbf{I} \to \mathscr{C}$ where $\mathbf{I}$ is a filtered category is a *filtered colimit*.

In the definition of stalk of a presheaf $\mathcal{F} \in \mathsf{pSh}(X, \mathscr{C})$ at a point $x \in X$, we have been taking

$$\begin{aligned}
\mathbf{I} &= \{\, U \in \tau_X \mid x \in U \,\}^{\mathrm{op}} \\
M(U) &= \mathcal{F}(U).
\end{aligned}$$

A.2 Localisation

A.2.1 General construction for modules

Let A be a ring, M an A-module. Fix a *multiplicative subset* $S \subset A$, i.e. a subset containing the identity $1 \in A$ and such that $s_1 s_2 \in S$ whenever $s_1, s_2 \in S$.

Example A.2.1. The following are key examples of multiplicative subsets:

(i) $S = \{ f^n \mid n \geq 0 \}$ for some $f \in A$.

(ii) $S = A \setminus \mathfrak{p}$, where $\mathfrak{p} \subset A$ is a prime ideal.

(iii) $S = A \setminus 0$, if A is an integral domain.

(iv) $S = A \setminus \mathcal{Z}$, where $\mathcal{Z}$ is the set of all zero-divisors in A.

Consider the equivalence relation on $M \times S$ defined by

$$\boxed{(m, s) \sim (m', s') \Longleftrightarrow \text{there exists } u \in S \text{ such that } u(s'm - sm') = 0 \in M.}$$

We denote by m/s, or by $\frac{m}{s}$, the equivalence class of (m, s). The set of such equivalence classes

$$S^{-1}M = (M \times S)/\sim \tag{A.2.1}$$

is an abelian group via

$$\frac{m}{s} + \frac{m'}{s'} = \frac{sm' + s'm}{ss'},$$

and if $M = A$ then $S^{-1}A$ becomes a ring via

$$\frac{a}{s} \cdot \frac{a'}{s'} = \frac{aa'}{ss'}.$$

The $\mathbb{Z}$-module $S^{-1}M$ is an $S^{-1}A$-module via

$$\frac{a}{s} \cdot \frac{m}{s'} = \frac{am}{ss'}. \tag{A.2.2}$$

Here 'am' refers to the A-module structure on M.

Definition A.2.2. The localisation of M with respect to S is the $S^{-1}A$-module $S^{-1}M$, where the linear structure is given by Equation (A.2.2).

Localisation is functorial. If $\phi \colon N \to M$ is an A-linear map, there is an induced map

$$S^{-1}\phi \colon S^{-1}N \to S^{-1}M, \quad \frac{n}{s} \mapsto \frac{\phi(n)}{s}.$$

This map is $S^{-1}A$-linear, indeed if $a/t \in S^{-1}A$ then

$$S^{-1}\phi\left(\frac{a}{t} \cdot \frac{n}{s}\right) = S^{-1}\phi\left(\frac{an}{ts}\right) = \frac{\phi(an)}{ts} = \frac{a \cdot \phi(n)}{ts} = \frac{a}{t} \cdot \frac{\phi(n)}{s}.$$

Remark A.2.3. If $0 \in S$, then $S^{-1}M = 0$.

Notation A.2.4. If $S = \{ f^n \mid n \geq 0 \}$ as in Example A.2.1 (i) above, then we write M_f for the localisation. If $S = A \setminus \mathfrak{p}$ as in Example A.2.1 (ii) above, then we write $M_{\mathfrak{p}}$ for the localisation. Do not confuse M_f and $M_{(f)}$ when $(f) = fA \subset A$ is a prime ideal!

A.2.2 Localisation of a ring and its universal property

Set $M = A$. There is a canonical ring homomorphism

$$\ell \colon A \to S^{-1}A, \quad a \mapsto \frac{a}{1}$$

sending S inside the group of invertible elements of $S^{-1}A$ (the inverse of $s/1$ being $1/s$), and making the pair $(S^{-1}A, \ell)$ universal with this property: whenever one has a ring homomorphism $\phi \colon A \to B$ such that $\phi(S) \subset B^{\times}$, there is exactly one ring homomorphism $p \colon S^{-1}A \to B$ such that $\phi = p \circ \ell$.

$$\begin{tikzcd} A \arrow[r, "\ell"] \arrow[d, "\phi"'] & S^{-1}A \arrow[dl, "p"] \\ B & \end{tikzcd}$$

Explicitly, the map p is defined by $p(a/s) = \phi(a)\phi(s)^{-1}$.

Remark A.2.5. The localisations of the form A_f are crucial in algebraic geometry. In A_f, the equivalence relation defining the localisation reads

$$\frac{a}{f^n} = \frac{b}{f^m} \iff \text{there exists } k \geq 0 \text{ such that } f^k(af^m - bf^n) = 0 \in A.$$

In particular, one has that $A_f = 0$ if and only if f is nilpotent.

The following lemma is of key importance to us.

LEMMA A.2.6. *Sending* $\mathfrak{r} \mapsto \ell^{-1}(\mathfrak{r})$ *establishes a bijection*

$$\{\textit{prime ideals } \mathfrak{r} \subset S^{-1}A\} \xrightarrow{\simeq} \{\textit{prime ideals } \mathfrak{q} \subset A \textit{ such that } \mathfrak{q} \cap S = \emptyset\}$$

having as inverse the extension operation, sending by definition

$$\mathfrak{q} \mapsto \mathfrak{q} \cdot S^{-1}A = \left\{ \frac{a}{f} \,\middle|\, a \in \mathfrak{q},\, f \in S \right\}.$$

COROLLARY A.2.7. *For any prime ideal* $\mathfrak{p} \subset A$ *the ring*

$$A_{\mathfrak{p}} = \left\{ \frac{a}{f} \,\middle|\, a \in A,\, f \notin \mathfrak{p} \right\}$$

is local, with maximal ideal

$$\mathfrak{p} \cdot A_{\mathfrak{p}} = \left\{ \frac{a}{f} \,\middle|\, a \in \mathfrak{p},\, f \notin \mathfrak{p} \right\} \subset A_{\mathfrak{p}}.$$

Proof. Indeed, the correspondence of Lemma A.2.6 becomes, in the case $S = A \setminus \mathfrak{p}$,

$$\{\text{prime ideals } \mathfrak{r} \subset A_{\mathfrak{p}}\} \xrightarrow{\simeq} \{\text{prime ideals } \mathfrak{q} \subset A \text{ such that } \mathfrak{q} \subset \mathfrak{p}\}$$

and since its inverse (extension along $A \to A_{\mathfrak{p}}$) is inclusion-preserving it follows that every prime ideal $\mathfrak{r} \subset A_{\mathfrak{p}}$ must be contained in $\mathfrak{p} \cdot A_{\mathfrak{p}}$. This means that $\mathfrak{p} \cdot A_{\mathfrak{p}}$ is the unique maximal ideal. □

Warning A.2.8. In the case when B is a graded ring and $\mathfrak{p}$ is a homogeneous prime ideal, we use the notation $B_{\mathfrak{p}}$ for the localisation of B at the multiplicative subset consisting of *homogeneous* elements that are not in $\mathfrak{p}$.

PROPOSITION A.2.9 ([9, Prop. 5.8]). *If $\mathfrak{m} \subset A$ is a maximal ideal and $k > 0$ is an integer, there is a natural ring isomorphism*

$$A/\mathfrak{m}^k \xrightarrow{\sim} A_{\mathfrak{m}}/(\mathfrak{m} \cdot A_{\mathfrak{m}})^k.$$

It induces isomorphisms

$$\mathfrak{m}^h/\mathfrak{m}^k \xrightarrow{\sim} (\mathfrak{m} \cdot A_{\mathfrak{m}})^h/(\mathfrak{m} \cdot A_{\mathfrak{m}})^k$$

for every $h \le k$.

LEMMA A.2.10. *Let A be a ring, $S \subset A$ a multiplicative subset. Then, S contains no nontrivial zero divisors if and only if $\ell \colon A \to S^{-1}A$ is injective.*

Proof. Suppose $a/1 = 0/1$ in $S^{-1}A$. Then there is $u \in S$ such that $au = 0$. But u is not a zero divisor, thus $a = 0$. □

Example A.2.11. Let A be an integral domain, which means that $(0) \subset A$ is prime. Then the localisation

$$A_{(0)} = \left\{ \frac{a}{b} \,\middle|\, a \in A,\, b \in A \setminus 0 \right\}$$

is a field, called the *fraction field* of A, that we denote by $\mathrm{Frac}(A)$. The canonical map $\ell \colon A \to \mathrm{Frac}(A)$ is injective by Lemma A.2.10.

Example A.2.12. Let A be a ring. Consider $S = A \setminus \mathscr{Z}$ as in Example A.2.1 (iv). The localisation $S^{-1}A$ is called the *total ring of fractions* of A. By Lemma A.2.10, $S = A \setminus \mathscr{Z}$ is the largest multiplicative set such that $\ell \colon A \to S^{-1}A$ is injective.

Example A.2.13. Let $A = \mathbb{Z}$. Fix a prime number $p \in \mathbb{Z}$. Then the localisation map

$$\mathbb{Z} \to \mathbb{Z}_{(p)} = \left\{ \frac{n}{m} \,\middle|\, n \in \mathbb{Z},\, p \nmid m \right\}$$

is injective. Also the localisation map

$$\mathbb{Z} \to \mathbb{Z}_p = \left\{ \frac{n}{p^k} \,\middle|\, n \in \mathbb{Z},\, k \ge 0 \right\}$$

is injective.

LEMMA A.2.14. *If A is reduced and $S \subset A$ is a multiplicative subset, then $S^{-1}A$ is also reduced.*

Proof. Assume there exists $a \in A$, $s \in S$ and $r \in \mathbb{Z}_{>0}$ such that $0 = (a/s)^r = a^r/s^r$. Then $a^r = 1 \cdot a^r = 0 \cdot s^r = 0$, thus $a = 0$. □

A.2.3 Exactness of localisation

LEMMA A.2.15. *Let A be a ring, $S \subset A$ a multiplicative subset, M an A-module. Then, there is a canonical isomorphism of $S^{-1}A$-modules*

$$\phi : S^{-1}M \xrightarrow{\sim} M \otimes_A S^{-1}A.$$

Proof. First of all, the $S^{-1}A$-module structure on $M \otimes_A S^{-1}A$ si defined by

$$\frac{a}{t} \cdot \left(m \otimes \frac{b}{s} \right) = m \otimes \frac{ab}{ts}.$$

The map ϕ is defined by

$$\phi\left(\frac{m}{s}\right) = m \otimes \frac{1}{s}.$$

It is $S^{-1}A$-linear, since

$$\begin{aligned} \phi\left(\frac{a}{t} \cdot \frac{m}{s}\right) &= \phi\left(\frac{am}{ts}\right) \\ &= am \otimes \frac{1}{ts} \\ &= m \otimes \frac{a}{ts} \\ &= \frac{a}{t} \cdot \left(m \otimes \frac{1}{s}\right) \\ &= \frac{a}{t} \cdot \phi\left(\frac{m}{s}\right). \end{aligned}$$

Its inverse is given by $m \otimes (a/s) \mapsto (am)/s$. □

PROPOSITION A.2.16. *Let A be a ring, $S \subset A$ a multiplicative subset. Then, sending $M \mapsto S^{-1}M$ defines an exact functor from A-modules to $S^{-1}A$-modules.*

Proof. Fix a short exact sequence

$$0 \longrightarrow M \xrightarrow{\iota} N \xrightarrow{\pi} P \longrightarrow 0$$

of A-modules. We already know that

$$S^{-1}M \to S^{-1}N \to S^{-1}P \to 0$$

is exact, since this sequence is isomorphic to

$$M \otimes_A S^{-1}A \to N \otimes_A S^{-1}A \to P \otimes_A S^{-1}A \to 0$$

by Lemma A.2.15, and tensor product (by any A-module, e.g. $S^{-1}A$) is a right exact functor. So we only need to show that

$$S^{-1}\iota : S^{-1}M \to S^{-1}N$$

is injective. Assume there is an element $m/s \in S^{-1}M$ such that $0 = 0/1 = S^{-1}\iota(m/s) = \iota(m)/s \in S^{-1}N$. Then there exists $u \in S$ such that $0 = u\iota(m) = \iota(um)$ in N. This implies $um = 0 \in M$, hence $m/s = um/us = 0/us = 0$. $\square$

B | Normalisation

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