Algebraic Geometry

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[...] *Oscar Zariski bewitched me. When he spoke the words "algebraic variety", there was a certain resonance in his voice that said distinctly that he was looking into a secret garden. I immediately wanted to be able to do this too. It led me to 25 years of struggling to make this world tangible and visible.*

David Mumford

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0 | Before we start

Conventions

We list here a series of conventions that will be used throughout this text.

- The axiom of choice (or Zorn's Lemma) is assumed.
- Given two sets A, B, the phrase '$A \subset B$' means that A is contained in B, *possibly equal* to B.
- A *ring* is a commutative, unitary ring. The zero ring (the one where $1 = 0$) is allowed (and in fact needed), but we always assume our rings are nonzero unless we explicitly mention it.
- By $\mathbf{k}$ we indicate an algebraically closed field.
- An open cover of a topological space U is the datum of a set I, and an open subset $U_i \subset U$ for every $i \in I$, such that $U = \bigcup_{i \in I} U_i$. If $I = \emptyset$, then $U = \emptyset$.
- To say that Ω is an object a category $\mathscr{C}$ we simply write '$\Omega \in \mathscr{C}$'.

Main references

We list here a series of bibliographical references that integrate this text.

- Q. Liu, *Algebraic geometry and arithmetic curves* [8],
- R. Hartshorne, *Algebraic geometry* [5],
- R. Vakil, *The rising sea* [10],
- D. Eisenbud, *Commutative Algebra: With a View Toward Algebraic Geometry* [2],
- M. F. Atiyah and I. G. MacDonald, *Introduction to Commutative Algebra* [1],

1 | Introduction

Algebraic Geometry is concerned with the study of *algebraic varieties*. At a first approximation, these are common zero loci of collections of polynomials, i.e. solutions to systems

$$\begin{cases} f_1(x_1,\ldots,x_n)=0 \\ \qquad\vdots \\ f_r(x_1,\ldots,x_n)=0 \end{cases}$$

of polynomial equations. When $\deg f_i = 1$ for $i = 1,\ldots,r$, this is the content of *Linear Algebra*, but the higher degree case poses nontrivial difficulties!

The concept of algebraic variety has been vastly generalised by Grothendieck's theory of *schemes*, introduced in [4]. This course is an introduction to schemes and to part of the massive dictionary, shared by all algebraic geometers, centered around schemes. Even though algebraic varieties are 'easier' objects, schemes are an incredibly useful and powerful tool to study them.

In this introduction, we briefly recap the key relation

$$\text{Algebra} \longleftrightarrow \text{Geometry}$$

in the land of *classical* algebraic varieties. We provide no proofs, but you shouldn't worry about this, because we will be proving more general results in the main body of these notes.

Let $\mathbf{k}$ be an algebraically closed field. Classical affine n-space over $\mathbf{k}$ is just

$$\mathbb{A}^n_{\mathbf{k}} = \{\,(a_1,\ldots,a_n)\mid a_i\in\mathbf{k} \text{ for } i=1,\ldots,n\,\}.$$

We denote it $\mathbb{A}^n_{\mathbf{k}}$ and not $\mathbf{k}^n$ to emphasise that we view it as a set of points rather than a vector space over $\mathbf{k}$. Set

$$A = \mathbf{k}[x_1,\ldots,x_n].$$

Each element $f \in A$ defines a function $\widetilde{f}\colon \mathbb{A}^n_{\mathbf{k}} \to \mathbf{k}$ sending $(a_1,\ldots,a_n)\mapsto f(a_1,\ldots,a_n)$, and since $\mathbf{k}$ is algebraically closed one has $f = g$ if and only if $\widetilde{f} = \widetilde{g}$. Thus we shall just write f instead of $\widetilde{f}$.

Let $I = (f_1, \dots, f_r) \subset A$ be an arbitrary ideal (here we are using that every ideal in A is finitely generated, by Hilbert's basis theorem [6]). The 'vanishing locus'

$$\mathrm{V}(I) = \left\{ (a_1, \dots, a_n) \in \mathbb{A}^n_{\mathbf{k}} \,\middle|\, f_j(a_1, \dots, a_n) = 0 \text{ for } j = 1, \dots, r \right\} \subset \mathbb{A}^n_{\mathbf{k}}$$

is called an *algebraic set*. There is precisely one topology on $\mathbb{A}^n_{\mathbf{k}}$ having the algebraic sets as closed sets. It is called the *Zariski topology*.

Example 1.0.1. Every ideal in $\mathbf{k}[x]$ is principal, i.e. of the form (f) for some $f \in \mathbf{k}[x]$. Since $\mathbf{k}$ is algebraically closed, we have $f = \alpha(x - a_1) \cdots (x - a_d)$, for $\alpha, a_1, \dots, a_d \in \mathbf{k}$, and where $d = \deg f$. Thus $\mathrm{V}(f) = \{a_1, \dots, a_d\} \subset \mathbb{A}^1_{\mathbf{k}}$, proving that all closed sets in $\mathbb{A}^1_{\mathbf{k}}$ are finite. In particular, all open sets are infinite (again, since $\mathbf{k}$ is algebraically closed).

We have thus established an assignment

$$\{\text{ideals } I \subset \mathbf{k}[x_1, \dots, x_n]\} \xrightarrow{\mathrm{V}(-)} \{\text{algebraic sets in } \mathbb{A}^n_{\mathbf{k}}\}.$$

Conversely, given a subset $S \subset \mathbb{A}^n_{\mathbf{k}}$, the assignment

$$\mathrm{I}(S) = \left\{ f \in A \,\middle|\, f(p) = 0 \text{ for all } p \in S \right\} \subset A$$

defines a map the other way around, namely

$$\{\text{ideals } I \subset \mathbf{k}[x_1, \dots, x_n]\} \xleftarrow{\mathrm{I}(-)} \{\text{algebraic sets in } \mathbb{A}^n_{\mathbf{k}}\}.$$

Unfortunately, the two maps are not inverse to each other. For instance, consider the ideal $(x^r) \subset \mathbf{k}[x]$ for $r > 1$. Then $\mathrm{V}(x^r) = \{0\}$, and thus $\mathrm{I}(\mathrm{V}(x^r)) = (x)$, which is strictly larger than (x^r). The next result says that this is what *always* happens.

THEOREM 1.0.2 (Hilbert's Nullstellensatz [7]). *Let $I \subset \mathbf{k}[x_1, \dots, x_n]$ be an ideal, where $\mathbf{k}$ is an algebraically closed field. Then,* $\mathrm{I}(\mathrm{V}(I)) = \sqrt{I}$, *i.e.* $f \in \mathrm{I}(\mathrm{V}(I))$ *if and only if* $f^r \in I$ *for some* $r > 0$.

See [8, Ch. 2, Corollary 1.15] for a modern proof of Hilbert's Nullstellensatz.

Composing our two assignments the other way around, we also find something larger than what we started with: if S is an arbitrary subset of $\mathbb{A}^n_{\mathbf{k}}$, one can easily prove the identity

$$\mathrm{V}(\mathrm{I}(S)) = \overline{S},$$

the closure of S in $\mathbb{A}^n_{\mathbf{k}}$ (with respect to the Zariski topology), namely the smallest algebraic set containing S. Thus in order to get $\mathrm{V}(\mathrm{I}(S)) = S$ we have to start with an algebraic set S (which is closed by definition).

Furthermore, one can prove that an algebraic set $Y \subset \mathbb{A}^n_{\mathbf{k}}$ is irreducible (i.e. it cannot be written as a union of two proper closed subsets) if and only if $\mathrm{I}(Y) \subset A$ is a prime ideal.

An irreducible algebraic set in $\mathbb{A}^n_{\mathbf{k}}$ is called an *affine variety in* $\mathbb{A}^n_{\mathbf{k}}$.

Combining these observations together, we obtain correspondences

$$\begin{tikzcd}
\{\text{radical ideals in } A\} \arrow[r, shift left, "\mathrm{V}(-)"] \arrow[d, hookleftarrow] & \{\text{algebraic sets in } \mathbb{A}^n_{\mathbf{k}}\} \arrow[l, shift left, "\mathrm{I}(-)"] \arrow[d, hookleftarrow] \\
\{\text{prime ideals in } A\} \arrow[r, shift left, "\mathrm{V}(-)"] & \{\text{affine varieties in } \mathbb{A}^n_{\mathbf{k}}\} \arrow[l, shift left, "\mathrm{I}(-)"]
\end{tikzcd}$$

where an ideal $I \subset A$ is *radical* if $I = \sqrt{I}$.

Recall that, by definition, a *finitely generated* $\mathbf{k}$*-algebra* is a $\mathbf{k}$-algebra B isomorphic to a quotient $\mathbf{k}[x_1, \dots, x_n]/I$ for some n and some ideal $I \subset \mathbf{k}[x_1, \dots, x_n]$. Such a B is an integral domain (i.e. as a ring it has no nonzero zero-divisors) precisely when I is prime. Thus the bottom correspondence above can be rephrased as

$$\{\,\mathbf{k}[x_1, \dots, x_n]/I \mid I \text{ is prime}\,\} \longleftrightarrow \{\,\text{algebraic varieties in } \mathbb{A}^n_{\mathbf{k}}\,\}.$$

In the first part of this course, we will be concerned with extending this correspondence to arbitrary *rings* on the left. What will be constructed on the right will be called an *affine scheme*, and what we shall establish is not just a bijection, but an equivalence of categories

$$\text{Rings}^{\text{op}} \cong \text{Affine schemes}.$$

Affine schemes are the basic building blocks for the construction of general *schemes*. Indeed, a scheme is defined by the property that every point has an open neighborhood isomorphic to an affine scheme.

2 | Sheaves

2.1 Key example: smooth functions

Sheaves were defined by Leray (1906–1998), while he was a prisoner in Austria during World War II.

Sheaves are a key notion present in the toolbox of every mathematician keen to understand the “nature” of a *geometric space*. They incarnate one of the basic principles that will be unraveled in this course, which can be stated as the slogan

$$\boxed{\text{geometric spaces are determined by functions on them.}}$$

Even though there may be “few” functions on a space X, a complete knowledge of all functions on all open subsets of X allows one, in principle, to reconstruct X. This local-to-global principle is perfectly encoded in the notion of a sheaf.

Before diving into precise definitions, we explore a key example. Let X be a smooth manifold. For each open subset $U \subset X$, we have a ring (actually, an $\mathbb{R}$-algebra)

$$C^\infty(U,\mathbb{R}) = \{\text{ smooth functions } U \to \mathbb{R}\,\}.$$

If $V \hookrightarrow U$ is an open subset, we have a restriction map

$$\rho_{UV} : C^\infty(U,\mathbb{R}) \to C^\infty(V,\mathbb{R}), \quad f \mapsto f|_V,$$

which is a ring homomorphism, and if $W \hookrightarrow V \hookrightarrow U$ is a chain of open subsets of X, we have a commutative diagram

$$\begin{tikzcd} C^\infty(U,\mathbb{R}) \arrow[r, "\rho_{UV}"] \arrow[rr, bend right, "\rho_{UW}"'] & C^\infty(V,\mathbb{R}) \arrow[r, "\rho_{VW}"] & C^\infty(W,\mathbb{R}). \end{tikzcd}$$

So far, we have just observed that the assignment $U \mapsto C^\infty(U,\mathbb{R})$ is *functorial*, from open subsets of X (which form a category) to the category of $\mathbb{R}$-algebras. The two distinguished features of the assignment $U \mapsto C^\infty(U,\mathbb{R})$, which make it into a *sheaf* of $\mathbb{R}$-algebras on X, are the following:

(i) If $f, g \in C^\infty(U, \mathbb{R})$ are smooth functions and $U = \bigcup_{i\in I} U_i$ is an open cover such that $f|_{U_i} = g|_{U_i}$ for each $i \in I$, then $f = g$. In other words, a function is determined by its restriction to the open subsets forming a covering.

(ii) If $U = \bigcup_{i\in I} U_i$ is an open cover of an open subset $U \subset X$, and one has a smooth function $f_i \in C^\infty(U_i, \mathbb{R})$ on each U_i, such that $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ for every $(i, j) \in I \times I$, then there is precisely one global function $f \in C^\infty(U, \mathbb{R})$ such that $f_i = f|_{U_i}$. In other words, functions glue uniquely along an open cover.

A sheaf is an abstract notion formalising this "ability of glueing" (cf. Definition 2.2.5).

Let us continue with our example. Let $x \in X$ be a point. Consider the ring

$$C^\infty_{X,x} = \left\{ (U, f) \,\middle|\, x \in U,\ f \in C^\infty(U, \mathbb{R}) \right\} \big/ \sim$$

where $(U, f) \sim (V, g)$ whenever there exists an open subset $W \subset U \cap V$, containing x, such that $f|_W = g|_W$. This ring is called the *stalk* of the sheaf $C^\infty(-, \mathbb{R})$ at x (cf. Definition 2.3.1), and it receives a natural map from $C^\infty(U, \mathbb{R})$ for every open subset U of X such that $x \in U$, sending $f \mapsto [U, f]$. The image of f along this map is called the *germ of f at x*. Note that $C^\infty_{X,x}$ is indeed a ring, with addition and multiplication

$$\begin{aligned} [U, f] + [U', f'] &= [U \cap U', f + f'] \\ [U, f] \cdot [U', f'] &= [U \cap U', f f']. \end{aligned}$$

The subset

$$\mathfrak{m}_x = \left\{ [U, f] \in \mathscr{O}_{X,x} \,\middle|\, f(x) = 0 \right\} \subset C^\infty_{X,x}$$

forms an ideal, which is a *maximal* ideal, being the kernel of the (surjective) evaluation map

$$C^\infty_{X,x} \to \mathbb{R}, \quad [U, f] \mapsto f(x).$$

In fact, $\mathfrak{m}_x$ is the *unique* maximal ideal of $C^\infty_{X,x}$. To see this, it is enough to check that every element of $\mathscr{O}_{X,x} \setminus \mathfrak{m}_x$ is invertible. But this is true, since a function that is nonzero in a neighborhood of x is invertible there.

Conclusion: $(C^\infty_{X,x}, \mathfrak{m}_x)$ is a *local ring* with residue field $\mathbb{R}$. The geometric spaces X one deals with in algebraic geometry, namely *schemes*, have precisely this property: they come with a sheaf of rings $\mathscr{A}_X$ such that each stalk $\mathscr{A}_{X,x}$ is a local ring. These spaces $(X, \mathscr{A}_X)$ actually form a larger category, that of locally ringed spaces (cf. **??**). Schemes are particular instances of locally ringed spaces.

2.2 Presheaves, sheaves, morphisms

Let $\mathscr{C}$ be a concrete category with a final object $0 \in \mathscr{C}$. The concreteness assumption means that part of the structure is the datum of a faithful functor $F : \mathscr{C} \to \text{Sets}$, but we will (for the moment) ignore this datum. To fix ideas, $\mathscr{C}$ should be thought of as any of the following categories: the category $\mathscr{C} = \mathrm{Sets}$, the category $\mathscr{C} = \mathrm{Rings}$, the category $\mathscr{C} = \mathrm{Ab}$ of abelian groups ($\mathbb{Z}$-modules) or a more general abelian category.

If X is a topological space, we denote by τ_X the category of open subsets of X. The set $\mathrm{Hom}_{\tau_X}(V, U)$ between two open sets $V, U \subset X$ is just the empty set if $V \not\subset U$, or the singleton $\{ V \hookrightarrow U \}$ in case V is contained in U. Thus the opposite category τ_X^{op} satisfies

$$\mathrm{Hom}_{\tau_X^{\mathrm{op}}}(U, V) = \begin{cases} \{ V \hookrightarrow U \} & \text{if } V \subset U \\ \emptyset & \text{if } V \not\subset U \end{cases}$$

and a functor $\mathcal{F} \colon \tau_X^{\mathrm{op}} \to \mathscr{C}$ determines a map

$$\mathrm{Hom}_{\tau_X^{\mathrm{op}}}(U, V) \to \mathrm{Hom}_{\mathscr{C}}(\mathcal{F}(U), \mathcal{F}(V)),$$

which is nothing but a choice of an element $\rho_{UV} \in \mathrm{Hom}_{\mathscr{C}}(\mathcal{F}(U), \mathcal{F}(V))$ for any inclusion of open subsets $V \subset U$.

Definition 2.2.1 (Presheaf)**.** A *presheaf* on a topological space X, with values in $\mathscr{C}$, is a contravariant functor $\mathcal{F}$ from τ_X to $\mathscr{C}$, i.e. an object of the functor category $\mathrm{Fun}(\tau_X^{\mathrm{op}}, \mathscr{C})$. In other words, a presheaf $\mathcal{F}$ is the assignment $U \mapsto \mathcal{F}(U)$ of an object $\mathcal{F}(U) \in \mathscr{C}$ for each open subset $U \subset X$, and of a morphism $\rho_{UV} \colon \mathcal{F}(U) \to \mathcal{F}(V)$ in $\mathscr{C}$ for each inclusion $V \hookrightarrow U$, such that

(1) $\rho_{UU} = \mathrm{id}_{\mathcal{F}(U)}$ for every $U \in \tau_X$, and

(2) $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$ for every chain of inclusions $W \hookrightarrow V \hookrightarrow U$.

Terminology 2.2.2. Elements of $\mathcal{F}(U)$ are often called 'sections of $\mathcal{F}$ over U', or (somewhat more vaguely) 'local sections' when $U \subsetneq X$. Elements of $\mathcal{F}(X)$ are called 'global sections', or just 'sections'. Possible alternative notations for $\mathcal{F}(U)$ are $\Gamma(U, \mathcal{F})$ and $\mathrm{H}^0(U, \mathcal{F})$. The maps ρ_{UV} are often called 'restriction maps' (from U to V, the larger set being U).

Notation 2.2.3. Motivated by Terminology 2.2.2, we shall often write $s|_V$ for the image of a section $s \in \mathcal{F}(U)$ along ρ_{UV}.

Notation 2.2.4. The presheaf defined by $U \mapsto 0$ for every U is called the *trivial sheaf* (or sometimes the *zero sheaf*), and is simply denoted by '0'.

Definition 2.2.5 (Sheaf, take I)**.** A *sheaf* on a topological space X, with values in $\mathscr{C}$, is a presheaf $\mathcal{F}$ such that the following two conditions hold:

(3) Fix an open subset $U \subset X$, an open cover $U = \bigcup_{i \in I} U_i$, and two sections $s, t \in \mathcal{F}(U)$ satisfying $s|_{U_i} = t|_{U_i}$ for all $i \in I$. Then $s = t$.

(4) Fix an open subset $U \subset X$, an open cover $U = \bigcup_{i\in I} U_i$ and a tuple $(s_i)_{i\in I}$ of sections $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all $(i,j) \in I\times I$. Then there exists a section $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$.

Conditions (3) and (4) generalise the conditions (i) and (ii), respectively, anticipated with the example $\mathcal{F} = C^\infty(-,\mathbb{R})$ in Section 2.1.

Terminology 2.2.6. A presheaf $\mathcal{F}$ is called *separated* if Condition (3) holds. Sometimes this condition is called *locality axiom.* Condition (4), on the other hand, is called the *glueing condition* (or *glueing axiom*).

Remark 2.2.7. Let $\mathcal{F}$ be a sheaf. Then, the section $s \in \mathcal{F}(U)$ in the glueing condition (4) is necessarily unique because $\mathcal{F}$ is separated. In fact, the two sheaf conditions could be replaced by a single condition, identical to (4), but imposing uniqueness of s.

Remark 2.2.8. Let $\mathcal{F}$ be a sheaf. Then, one has $\mathcal{F}(\emptyset) = 0$, the final object in $\mathscr{C}$. This is sometimes listed as an axiom defining a (pre)sheaf, but it does follow from our assumptions (cf. Example A.1.3).

Example 2.2.9 (Restriction to an open)**.** Let $U \subset X$ be an open subset, $\mathcal{F}$ a presheaf on X. Then, setting $\mathcal{F}|_U(V) = \mathcal{F}(V)$ for V an open subset of U, defines a presheaf $\mathcal{F}|_U$ on U, which is a sheaf as soon as $\mathcal{F}$ is. It is called the *restriction of $\mathcal{F}$ to U*.

Definition 2.2.10 (Morphism of (pre)sheaves)**.** A *morphism* between two presheaves $\mathcal{F}, \mathcal{G}$ on X is a natural transformation $\eta\colon \mathcal{F} \Rightarrow \mathcal{G}$. A morphism of sheaves is just a morphism between the underlying presheaves.

By definition, to give a morphism of (pre)sheaves, one has to assign a homomorphism $\eta_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ in $\mathscr{C}$ for each $U \in \tau_X$, such that for every inclusion $V \hookrightarrow U$ of open subsets of X, the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\eta_U"] \arrow[d, "\rho^{\mathcal{F}}_{UV}"'] & \mathcal{G}(U) \arrow[d, "\rho^{\mathcal{G}}_{UV}"] \\ \mathcal{F}(V) \arrow[r, "\eta_V"] & \mathcal{G}(V) \end{tikzcd} \tag{2.2.1}$$

commutes. For the sake of clarity, we have emphasised the relevant sheaf in the restriction maps notation, but we won't be doing that again.

Notation 2.2.11. It is clear that presheaves on X with values in $\mathscr{C}$ form a category $\mathrm{pSh}(X,\mathscr{C})$, tautologically defined as the functor category $\mathrm{Fun}(\tau_X^{\mathrm{op}},\mathscr{C})$. Sheaves form a full subcategory, denoted $\mathrm{Sh}(X,\mathscr{C})$. We denote by $j_{X,\mathscr{C}}\colon \mathrm{Sh}(X,\mathscr{C}) \hookrightarrow \mathrm{pSh}(X,\mathscr{C})$ the inclusion functor.

An isomorphism of (pre)sheaves is an isomorphism in $\mathrm{pSh}(X,\mathscr{C})$, i.e. a natural equivalence, i.e. a natural transformation $\eta\colon \mathcal{F} \Rightarrow \mathcal{G}$ such that η_U is an isomorphism in $\mathscr{C}$ for every $U \in \tau_X$.

Notation 2.2.12. Since (pre)sheaves form a genuine category, we shall use the classical arrow notation '$\mathcal{F} \to \mathcal{G}$' (instead of $\mathcal{F} \Rightarrow \mathcal{G}$) to denote a morphism of (pre)sheaves.

The following definition makes sense, because $\mathscr{C}$ is assumed to be a concrete category.

Definition 2.2.13 (Injective map of presheaves)**.** A morphism of (pre)sheaves $\eta\colon \mathcal{F} \to \mathcal{G}$ is *injective* if η_U is injective for every U. We denote this by writing η as $\mathcal{F} \hookrightarrow \mathcal{G}$.

We close this section with a few examples and exercises.

Example 2.2.14 (Smooth functions)**.** Let X be a smooth manifold. Then, sending $U \subset X$ to the set $C^\infty(U,\mathbb{R})$ of smooth functions $U \to \mathbb{R}$, defines a sheaf $C^\infty(-,\mathbb{R})$ with values in the category of $\mathbb{R}$-algebras.

Example 2.2.15 (Holomorphic functions)**.** Let X be a complex manifold. Then, sending $U \subset X$ to the set $\mathscr{O}_X^{\mathrm{h}}(U)$ of holomorphic functions on U, defines a sheaf $\mathscr{O}_X^{\mathrm{h}}$ with values in the category of $\mathbb{C}$-algebras.

Example 2.2.16 (Separated presheaf, not a sheaf)**.** Set $X = \mathbb{C}$. Then, sending $U \subset X$ to the subset

$$\mathcal{F}(U) = \left\{ f \in \mathscr{O}_X^{\mathrm{h}}(U) \,\middle|\, f = g^2 \text{ for some } g \in \mathscr{O}_X^{\mathrm{h}}(U) \right\}$$

defines a (separated) presheaf. However, $\mathcal{F}$ is not a sheaf: the function $f(z) = z$ on the annulus

$$U = \{ z \in \mathbb{C} \mid 1-\varepsilon < |z| < 1+\varepsilon \} \subset \mathbb{C}$$

has a square root in any neighborhood of any point $x \in X$, but there is no global $\sqrt{z}$ defined on the whole of U.

Example 2.2.17 (Constant presheaf)**.** Work with $\mathscr{C} = \mathrm{Ab} = \mathrm{Mod}_{\mathbb{Z}}$, the category of abelian groups, and fix $G \neq 0$ in this category. Fix a topological space X, and define

$$\underline{G}_X^{\mathrm{pre}}(U) = \begin{cases} G & \text{if } U \neq \emptyset, \\ 0 & \text{if } U = \emptyset. \end{cases}$$

As for the restriction maps, set $\rho_{UV} = \mathrm{id}_G$ if both U and V are nonempty. This is a presheaf, which happens to be a sheaf only in precise circumstances (cf. Exercise 2.2.18). For instance, suppose $X = U_1 \amalg U_2$ is a disjoint union of two nonempty open subsets. Then $\underline{G}_X(X) = G = \underline{G}_X(U_i)$ for $i = 1, 2$. Now, $X = U_1 \amalg U_2$ is an open cover. Pick two *distinct* sections $s_i \in G = \underline{G}_X(U_i)$ for $i = 1,2$. Then, $s_1|_{U_1\cap U_2} = s_1|_\emptyset = 0 = s_2|_\emptyset = s_2|_{U_1\cap U_2}$, but there is no section $s \in \underline{G}_X(X) = G$ such that $s|_{U_i} = s_i$ since $\rho_{XU_i} = \mathrm{id}_G$ for $i = 1,2$ and $s_1 \neq s_2$ by assumption. Hence Condition (4) fails. We will see in Example 2.4.3 that $\underline{G}_X^{\mathrm{pre}}$ can be "transformed" into a sheaf by a canonical procedure.

Exercise 2.2.18. Show that the constant presheaf $\underline{G}_X^{\mathrm{pre}}$ of Example 2.2.17 is a sheaf if and only if every nonempty open subset $U \subset X$ is connected.

Exercise 2.2.19 (Preheaves kernel and cokernel)**.** Let $\mathscr{C}$ be an abelian category, so that every arrow has a kernel and a cokernel. Let $\eta \colon \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves with values in $\mathscr{C}$. Consider the assignments

$$\begin{aligned} U &\mapsto (\ker_{\mathrm{pre}} \eta)(U) = \ker(\eta_U) \\ U &\mapsto (\operatorname{coker}_{\mathrm{pre}} \eta)(U) = \operatorname{coker}(\eta_U) = \mathcal{G}(U)/\operatorname{im}(\eta_U). \end{aligned}$$

Show that

(i) both are presheaves,

(ii) $\ker_{\mathrm{pre}} \eta \to \mathcal{F}$ (resp. $\mathcal{G} \to \operatorname{coker}_{\mathrm{pre}} \eta$) satisfies the universal property of the kernel (resp. the cokernel) in $\mathrm{pSh}(X, \mathscr{C})$,

(iii) $\ker_{\mathrm{pre}} \eta$ is a sheaf, denoted $\ker(\eta)$, as soon as η is a morphism of *sheaves*,

(iv) if η is a morphism of sheaves, then $\ker(\eta)$ satisfies the universal property of the kernel in $\mathrm{Sh}(X, \mathscr{C})$, and η is injective if and only if $\ker(\eta) = 0$.

Exercise 2.2.20 (Bounded functions are not a sheaf)**.** Let $X = \mathbb{R}$, with the standard topology. Show that

$$U \mapsto \mathrm{B}(U) = \{\,\text{bounded continuous functions } U \to \mathbb{R}\,\}$$

is a separated presheaf on X, but not a sheaf (i.e. Condition (4) fails).

Exercise 2.2.21 (Continuous functions are a sheaf)**.** Let X, Y be topological spaces. For $U \subset X$ open, define

$$\mathcal{F}(U) = \{\,\text{continuous functions } U \to Y\,\}.$$

Show that $\mathcal{F}$ is a sheaf of sets (i.e. take $\mathscr{C} = \mathrm{Sets}$).

2.2.1 A translation of the sheaf conditions

We now present an alternative way to define sheaves.

Let $\mathscr{C}$ be a category with limits. In particular, $\mathscr{C}$ has products, equalisers, and a final object. Fix a presheaf $\mathcal{F}$ with values in $\mathscr{C}$ on a topological space X. Let $\{U_i\}_{i\in I}$ be a family of open subsets of X, and set $U = \bigcup_{i\in I} U_i$. Then, one can consider the map

$$\rho\colon \mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i), \qquad s \mapsto (s|_{U_i})_{i\in I},$$

as well as the family of maps

$$\mu_{ij}\colon \prod_{i\in I} \mathcal{F}(U_i) \to \mathcal{F}(U_i) \to \mathcal{F}(U_i \cap U_j), \qquad (s_i)_{i\in I} \mapsto s_i|_{U_i\cap U_j}$$
$$\nu_{ij}\colon \prod_{i\in I} \mathcal{F}(U_i) \to \mathcal{F}(U_j) \to \mathcal{F}(U_i \cap U_j), \qquad (s_i)_{i\in I} \mapsto s_j|_{U_i\cap U_j}$$

which, taking products over $(i,j) \in I\times I$, can be assembled into two maps

$$\prod_{i\in I} \mathcal{F}(U_i) \overset{\mu}{\underset{\nu}{\rightrightarrows}} \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \cap U_j).$$

Definition 2.2.22 (Sheaf, take II)**.** Let $\mathscr{C}$ be a category with limits, X a topological space. A presheaf $\mathcal{F} \in \mathrm{pSh}(X, \mathscr{C})$ is a *sheaf* if every family of open subsets $\{U_i\}_{i\in I}$, with $U = \bigcup_{i\in I} U_i$, the diagram

$$\mathcal{F}(U) \xrightarrow{\ \rho\ } \prod_{i\in I} \mathcal{F}(U_i) \overset{\mu}{\underset{\nu}{\rightrightarrows}} \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \cap U_j)$$

is an equaliser diagram in $\mathscr{C}$.

Note that Definition 2.2.22 is *element-free*. However, let us check that it agrees with Definition 2.2.5 when $\mathscr{C}$ is concrete: injectivity of ρ, implied by the equaliser condition, coincides with separatedness; the fact that the set-theoretic image of ρ coincides with the collection of tuples of sections $(s_i)_{i\in I}$ such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ is precisely the glueing condition.

Example 2.2.23. Let $\mathcal{F}$ be a sheaf on X. If $U = \coprod_{i\in I} U_i$ is a *disjoint* union of open subsets $U_i \subset U$, then ρ is an isomorphism, i.e. $s \mapsto (s_{U_i})_{i\in I}$ defines an isomorphism

$$\rho\colon \mathcal{F}(U) \xrightarrow{\ \sim\ } \prod_{i\in I} \mathcal{F}(U_i).$$

Example 2.2.24. Let $\mathscr{C}$ be an abelian category. Then a presheaf $\mathcal{F} \in \mathrm{pSh}(X, \mathscr{C})$ is a sheaf if for every family of open subsets $\{U_i\}_{i\in I}$, with $U = \bigcup_{i\in I} U_i$, the sequence

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\ \rho\ } \prod_{i\in I} \mathcal{F}(U_i) \xrightarrow{\ \mu-\nu\ } \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \cap U_j)$$

is exact, where the map denoted $\mu - \nu$ sends $(s_i)_{i\in I} \mapsto (s_i|_{U_i\cap U_j} - s_j|_{U_i\cap U_j})_{i,j}$.

The following lemma applies, for instance, to categories of groups, rings, algebras over a ring, and modules over a ring. It allows one to check the sheaf condition of Definition 2.2.22 in the category of sets.

LEMMA 2.2.25 ([9, Tag 0073]). *Let $\mathscr{C}$ be a category, $F : \mathscr{C} \to \mathrm{Sets}$ a faithful functor such that $\mathscr{C}$ has limits and F commutes with them. Assume that F reflects isomorphisms. Then a presheaf $\mathcal{F} \in \mathrm{pSh}(X, \mathscr{C})$ is a sheaf of and only if the underlying presheaf $F \circ \mathcal{F} : \tau_X \to \mathrm{Sets}$ is a sheaf.*

At the beginning of this chapter we have defined (pre)sheaves of objects in an arbitrary concrete category $\mathscr{C}$. We still have to define a few things, though, e.g. stalks and sheafification. In order for everything to be well-defined and work well (but still be compatible with all we have discussed so far, including Definition 2.2.22), we need to add a few initial data. This is provided by the following definition.

Definition 2.2.26 ([9, Tag 007L])**.** A *type of algebraic structure* is a pair $(\mathscr{C}, F)$, where $\mathscr{C}$ is a category, $F : \mathscr{C} \to \mathrm{Sets}$ is a faithful functor, such that

1. $\mathscr{C}$ has limits and F commutes with them,
2. $\mathscr{C}$ has filtered colimits and F commutes with them,
3. F reflects isomorphisms (i.e. F is *conservative*).

A few remarks are in order, before we go on.

- Equipping a category $\mathscr{C}$ with a faithful functor $F : \mathscr{C} \to \mathrm{Sets}$ is like saying that $\mathscr{C}$ is a *concrete category*, which we had already assumed in Section 2.2.
- If we have a type of algebraic structure $(\mathscr{C}, F)$, then we can verify whether a presheaf is a sheaf in the category of sets, by Lemma 2.2.25.
- The condition that F be conservative implies that a bijective morphism in $\mathscr{C}$ is an isomorphism.
- For every type of algebraic structure $(\mathscr{C}, F)$, one has the following properties:
  (i) $\mathscr{C}$ has a final object 0, and $F(0)$ is a final object in Sets (i.e. a singleton).
  (ii) $\mathscr{C}$ has products, fibre products, and equalisers — this follows by the examples in Appendix A.1.1. Moreover, F commutes with all of them.
- Examples of categories $\mathscr{C}$ having the additional structure of Definition 2.2.26 are: monoids, groups, abelian groups, rings, modules over a ring. In all these cases, we take as the functor F the obvious forgetful functor. As a counterexample, consider the category Top of topological spaces: the forgetful functor exists but does not reflect isomorphisms.

2.3 Stalks, and what they tell us

Fix a type of algebraic structure $(\mathscr{C}, F\colon \mathscr{C} \to \mathrm{Sets})$ as in Definition 2.2.26. Let X be a topological space, $x \in X$ a point. The collection of open subsets $U \subset X$ containing x forms a directed system (the partial order $\succeq$ being the inclusion relation, i.e. $V \succeq U$ if and only if $V \subset U$). Indeed, given two open neighborhoods U and V of x, there is always a third open neighborhood of x contained in both U and V, namely $U \cap V$ or any smaller open subset containing x. In fancier language, the subcategory

$$\mathrm{Nbg}_x = \{\, U \in \tau_X \mid x \in U \,\}^{\mathrm{op}} \subset \tau_X^{\mathrm{op}}$$

is a filtered category (see Definition A.1.9).

Definition 2.3.1 (Stalks)**.** Let $x \in X$ be a point, $\mathcal{F}$ a presheaf. The *stalk of* $\mathcal{F}$ *at* x is the filtered colimit

$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U),$$

which exists as an object of $\mathscr{C}$. In categorical language, it is the filtered colimit of the functor $\mathcal{F}|_{\mathrm{Nbg}_x}\colon \mathrm{Nbg}_x \to \mathscr{C}$.

Because F commutes with colimits, the underlying set $F(\mathcal{F}_x)$, still denoted $\mathcal{F}_x$, is

$$\mathcal{F}_x = \{(U,s) \mid x \in U,\ s \in \mathcal{F}(U)\}/\sim$$

where $(U,s) \sim (V,t)$ whenever there is an open neighborhood $W \subset U \cap V$ of x such that $s|_W = t|_W$. We denote by

$$s_x = [U,s] \in \mathcal{F}_x$$

the equivalence class of the pair (U,s). It is called the *germ of* s *at* x. By definition of direct limit, there are natural homomorphisms

$$\mathcal{F}(U) \to \mathcal{F}_x, \quad s \mapsto s_x,$$

in $\mathscr{C}$, for every open neighborhood U of x. The diagram

$$\begin{tikzcd}
s \in \mathcal{F}(U) \arrow[dr] \arrow[ddd, bend right] & & \mathcal{F}(V) \ni t \arrow[dl] \arrow[ddd, bend left] \\
& \mathcal{F}(U \cap V) \arrow[d] & \\
& \mathcal{F}(W) \arrow[d] & \\
& \mathcal{F}_x &
\end{tikzcd}$$

illustrates the fact that two sections $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$ define the same element in the stalk $\mathcal{F}_x$ if and only if there is an intermediate open $W \subset U \cap V$ over which they agree.

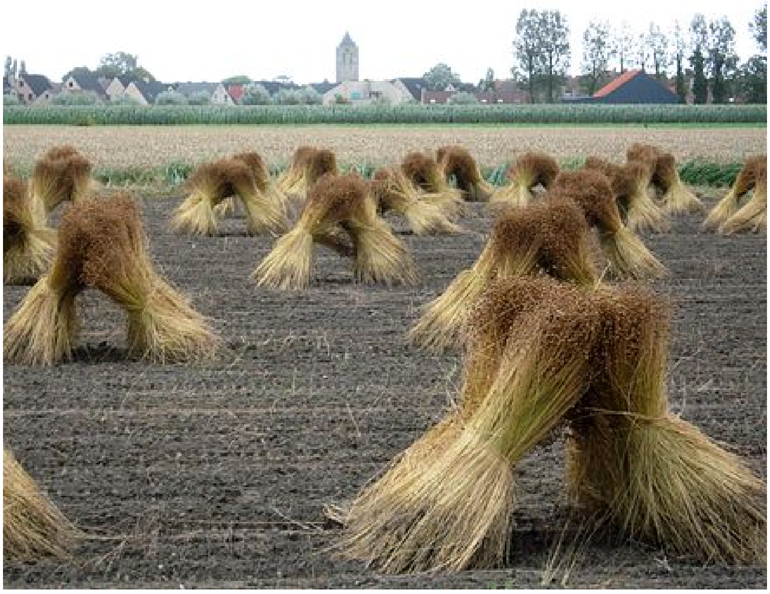

Figure 2.1: A bunch of sheaves sitting in their natural habitat. The little tops of each leaf of corn are the stalks.

LEMMA 2.3.2. *If $\mathcal{F}$ is a separated presheaf of sets (e.g. a sheaf), then the natural map*

$$\sigma_U^{\mathcal{F}} \colon \mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x, \quad s \mapsto (s_x)_{x \in U} \tag{2.3.1}$$

is injective for every open subset U of X.

The lemma means, at an informal level, that sections are determined by their germs.

Proof. If s and t are sections in $\mathcal{F}(U)$ such that $s_x = t_x$ in $\mathcal{F}_x$ for every $x \in U$, then for every $x \in U$ there is an open neighborhood $U_x \subset U$ such that $s|_{U_x} = t|_{U_x}$. But this holds for every $x \in U$, and $U = \bigcup_{x \in U} U_x$ is an open covering, thus by the separation axiom we deduce $s = t$, i.e. $\sigma_U^{\mathcal{F}}$ is injective. □

Consider the following property of a tuple $(s_x)_{x \in U} \in \prod_{x \in U} \mathcal{F}_x$, for $U \subset X$ an open subset:

$$\begin{gathered} \text{for every } x \in U \text{ there exists a pair } (V_x, t_x), \\ \text{with } x \in V_x \subset U \text{ and } t_x \in \mathcal{F}(V_x), \\ \text{such that } (t_x)_y = s_y \text{ for all } y \in V_x. \end{gathered} \tag{2.3.2}$$

Definition 2.3.3 (Compatible germs)**.** Let $\mathcal{F}$ be a presheaf on X, and let $U \subset X$ be an open subset. When Condition (2.3.2) is fulfilled, we say that the tuple $(s_x)_{x \in U} \in \prod_{x \in U} \mathcal{F}_x$ consists of *compatible germs.*

We always have inclusions

$$\operatorname{im}(\sigma_U^{\mathcal{F}}) \subset \{\text{tuples } (s_x)_{x \in U} \text{ of compatible germs}\} \subset \prod_{x \in U} \mathcal{F}_x \tag{2.3.3}$$

where the first inclusion is justified by taking $V_x = U$ and $t_x = s$ for every $x \in U$ as soon as $\sigma_U^{\mathcal{F}}(s) = (s_x)_{x\in U}$. If $\mathcal{F}$ is a sheaf, then tuples of compatible germs form precisely the image of the map (2.3.1), i.e. the first inclusion in (2.3.3) is an equality. Indeed, assume $(s_x)_{x\in U}$ consists of compatible germs. Let $\{(V_x, t_x) \mid x \in U\}$ be as in the displayed condition (2.3.2). Then we have an open cover $U = \bigcup_{x\in U} V_x$, so by the glueing axiom the sections $t_x \in \mathcal{F}(V_x)$ glue to a (unique) section $t \in \mathcal{F}(U)$ such that $t|_{V_x} = t_x$. But $(t_x)_y = s_y$ for $y \in V_x$, and this holds for every $x \in U$, so $\sigma_U^{\mathcal{F}}(t) = (s_x)_{x\in U}$.

Summing up, when $\mathcal{F}$ is a sheaf, we have a bijection

$$\sigma_U^{\mathcal{F}} \colon \mathcal{F}(U) \xrightarrow{\sim} \{\text{tuples } (s_x)_{x\in U} \text{ of compatible germs}\}.$$

This also shows that *sections of a sheaf can always be identified with genuine functions!* Indeed, tuples $(s_x)_{x\in U}$ correspond to particular functions $U \to \coprod_{x\in U} \mathcal{F}_x$, sending $x \in U$ inside the corresponding stalk, and doing so in a compatible way.

LEMMA 2.3.4. *Let $s, t \in \mathcal{F}(X)$ be two global sections of a sheaf $\mathcal{F}$, such that $s_x = t_x \in \mathcal{F}_x$ for every $x \in X$. Then $s = t$.*

Proof. This is just a special case of Lemma 2.3.2. □

Exercise 2.3.5. Let $\mathcal{F}$ be a sheaf on X, and let $s, t \in \mathcal{F}(X)$ be two global sections. Show that

$$\{x \in X \mid s_x = t_x\} \subset X$$

is an open subset of X.

A morphism of presheaves $\eta \colon \mathcal{F} \to \mathcal{G}$ induces a morphism $\eta_x \colon \mathcal{F}_x \to \mathcal{G}_x$ at the level of stalks for every $x \in X$, defined by

$$s_x = [U, s] \mapsto [U, \eta_U(s)] = (\eta_U(s))_x. \tag{2.3.4}$$

Exercise 2.3.6. Check that (2.3.4) is well-defined.

If $U \subset X$ is an open subset containing a point $x \in X$, then the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\eta_U"] \arrow[d] & \mathcal{G}(U) \arrow[d] \\ \mathcal{F}_x \arrow[r, "\eta_x"] & \mathcal{G}_x \end{tikzcd} \qquad \begin{tikzcd} s \arrow[r, mapsto, "\eta_U"] \arrow[d, mapsto] & \eta_U(s) \arrow[d, mapsto] \\ s_x \arrow[r, mapsto, "\eta_x"] & (\eta_U(s))_x \end{tikzcd}$$

commutes. What we have just said can be rephrased by saying that the association $\mathcal{F} \mapsto \mathcal{F}_x$ defines a functor

$$\mathrm{stalk}_x \colon \mathrm{pSh}(X, \mathscr{C}) \to \mathscr{C}. \tag{2.3.5}$$

We will see that in reasonable circumstances this functor is *exact* (cf. Proposition 2.4.14).

Definition 2.3.7. A morphism of (pre)sheaves $\eta\colon \mathcal{F} \to \mathcal{G}$ is *surjective* if η_x is surjective for every $x \in X$.

Warning 2.3.8. You may have noticed that surjectivity of a map of sheaves (cf. Definition 2.3.7) is defined differently than injectivity (cf. Definition 2.2.13)!

Let $\eta\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then

$$\eta_x \text{ is surjective} \iff \begin{array}{c}\text{for every } t_x \in \mathcal{G}_x \text{ there exists an open neighborhood}\\ U \text{ of } x \text{ and a section } s \in \mathcal{F}(U) \text{ such that } (\eta_U(s))_x = t_x.\end{array}$$

$$\eta \text{ is surjective} \iff \begin{array}{c}\text{for every open subset } U \subset X \text{ and for every}\\ t \in \mathcal{G}(U), \text{ there exists a covering } U = \bigcup_{i \in I} U_i\\ \text{such that } t|_{U_i} \text{ is in the image of } \eta_{U_i} \text{ for every } i.\end{array}$$

The second equivalence is obtained as follows.

Proof of '⇒'. First of all assume η is surjective, i.e. η_x is surjective for every $x \in X$. Fix $U \subset X$ open and a local section $t \in \mathcal{G}(U)$. For every $x \in U$, we have a commutative diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\eta_U"] \arrow[d] & \mathcal{G}(U) \arrow[d] & t \arrow[d, mapsto] \\ \mathcal{F}_x \arrow[r, two heads, "\eta_x"] & \mathcal{G}_x & t_x \end{tikzcd}$$

where $t_x \in \mathcal{G}_x$ can be lifted along η_x to an element $s_x \in \mathcal{F}_x$. Let (V_x, s) be a representative for s_x, so that in particular $s \in \mathcal{F}(V_x)$. The identity $\eta_x(s_x) = t_x$ implies that there is an open neighborhood $x \in U_x \subset V_x \cap U$ such that the diagram

$$\begin{tikzcd} s \in \mathcal{F}(V_x) \arrow[r, "\rho^{\mathcal{F}}_{V_x U_x}"] & \mathcal{F}(U_x) \arrow[r, "\eta_{U_x}"] \arrow[d] & \mathcal{G}(U_x) \arrow[d] & s \arrow[r, mapsto] & s|_{U_x} \arrow[r, mapsto] \arrow[d, mapsto] & t|_{U_x} \arrow[d, mapsto] \\ & \mathcal{F}_x \arrow[r, two heads, "\eta_x"] & \mathcal{G}_x & & s_x \arrow[r, mapsto] & t_x \end{tikzcd}$$

commutes, in particular $\eta_{U_x}(s|_{U_x}) = t|_{U_x}$. Now this holds for every $x \in U$, so the elements of $\{\, U_x \mid x \in U \,\}$ form a covering of U, and we have proved the condition.

Proof of '⇐'. Conversely, assuming the condition, let us prove surjectivity of η. Fix $x \in X$ along with a germ $t_x \in \mathcal{G}_x$. We need to prove that t_x has a preimage in $\mathcal{F}_x$. Let (U, t) be a representative of t_x, so that $t \in \mathcal{G}(U)$. By the condition we are assuming, there exists a covering $U = \bigcup_{i \in I} U_i$ such that $t|_{U_i} = \eta_{U_i}(s_i)$ for some $s_i \in \mathcal{F}(U_i)$, for every $i \in I$. If $x \in U_i$, we have a commutative diagram

$$\begin{tikzcd} \mathcal{F}(U_i) \arrow[r, "\eta_{U_i}"] \arrow[d] & \mathcal{G}(U_i) \arrow[d] & s_i \arrow[r, mapsto] \arrow[d, mapsto] & t|_{U_i} \arrow[d, mapsto] \\ \mathcal{F}_x \arrow[r, "\eta_x"] & \mathcal{G}_x & \star \arrow[r, mapsto] & t_x \end{tikzcd}$$

so the element is a preimage of t_x. The equivalence is proved.

The next result incarnates the local nature of sheaves.

LEMMA 2.3.9. *Let $\eta\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. The following are equivalent:*

(i) η is an isomorphism,

(ii) η_x is an isomorphism for every $x \in X$,

(iii) η is injective and surjective.

Proof. Recall that η is an isomorphism if and only if η_U is an isomorphism for every U.

Proof of (i) ⇒ (ii). By functoriality of $\mathcal{F} \mapsto \mathcal{F}_x$, we have that if η is an isomorphism, then so is η_x for every $x \in X$.

Proof of (ii) ⇒ (i). Suppose η_x is an isomorphism for every x. Let $U \subset X$ be an open subset: we need to show that η_U is an isomorphism.

To see that η_U is injective, pick $s, t \in \mathcal{F}(U)$ such that $\eta_U(s) = \eta_U(t) \in \mathcal{G}(U)$. Then, for any $x \in U$, one has

$$\eta_x(s_x) = (\eta_U(s))_x = (\eta_U(t))_x = \eta_x(t_x),$$

which implies $s_x = t_x$ by injectivity of η_x. This holds for every $x \in U$ by assumption, thus $s = t$ by Lemma 2.3.4. Therefore, η_U is injective for every U (i.e. η is injective).

To see that η_U is surjective, pick $t \in \mathcal{G}(U)$. By surjectivity of η (which we have since η_x is surjective for every $x \in X$), we can find an open cover $U = \bigcup_{i \in I} U_i$ along with a collection of sections $s_i \in \mathcal{F}(U_i)$ such that $\eta_{U_i}(s_i) = t|_{U_i}$. But η is injective, so s_i and s_j agree on $U_i \cap U_j$. Therefore, since $\mathcal{F}$ is a sheaf, they glue to a section $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$. By construction, $\eta_U(s)|_{U_i} = \eta_{U_i}(s_i) = t|_{U_i}$, which implies $\eta_U(s) = t$ since $\mathcal{G}$ is a sheaf. Thus η_U is surjective.

Proof of (ii) ⇒ (iii). The first paragraph of '(ii) ⇒ (i)' already shows that if η_x is an isomorphism for every $x \in X$, then η_U is injective for all U, i.e. η is injective. Surjectivity follows from the definition.

Proof of (iii) ⇒ (ii). We only need to show that if η_U is injective for every U, then η_x is injective for every $x \in X$. Consider $s_x = [U, s]$ and $s'_x = [U', s']$ two germs in $\mathcal{F}_x$ such that $\eta_x(s_x) = \eta_x(s'_x)$ in $\mathcal{G}_x$. Then there is an open subset $W \subset U \cap U'$ such that $\eta_U(s)|_W = \eta_{U'}(s')|_W$. But by compatibility of η_W with restrictions, this is equivalent to the identity $\eta_W(s|_W) = \eta_W(s'|_W)$, which by our assumption implies $s|_W = s'|_W$. But then $s_x = s'_x$. □

Warning 2.3.10. It is not true that two sheaves with isomorphic stalks are isomorphic: there may be no map between them! For instance, consider a topological space X consisting of two points x_0, x_1 where only x_0 is a closed point. Thus X and $U = X \setminus \{x_0\}$ are the only nonempty open subsets of X. Fix an abelian group $G \neq 0$ and define

$\mathcal{F}(X) = G = \mathcal{F}(U)$. Then choose either $\rho_{XU} = \mathrm{id}_G$ or $\rho_{XU} = 0$ to define two distinct sheaves on X. They have the same stalks but they are not isomorphic.

Exercise 2.3.11. Show that Lemma 2.3.9 fails for presheaves.

Example 2.3.12 (Surjectivity is subtle)**.** Let $\mathcal{F} = \mathcal{O}_X^{\mathrm{h}}$ be the sheaf of holomorphic functions on $X = \mathbb{C} \setminus \{0\}$, and let $\mathcal{G} = \mathcal{F}^\times$ be the sheaf of invertible holomorphic functions on X. The map $\exp\colon \mathcal{F} \to \mathcal{G}$ is surjective, but $\exp_X\colon \mathcal{F}(X) \to \mathcal{G}(X)$ is not surjective, e.g. the function $f(z) = z$ in $\mathcal{G}(X)$ is not the exponential of a homolomorphic function.

Example 2.3.13 (Skyscraper sheaf)**.** Let X be a topological space, G a nontrivial abelian group, $x \in X$ a point. The assignment

$$U \mapsto G_x(U) = \begin{cases} G & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$$

defines a sheaf, choosing as restriction maps the identity of G or the zero map in the obvious way. This sheaf is called the *skyscraper sheaf* attached to (X, x, G). At the level of stalks, one has

$$(G_x)_y = \begin{cases} G & \text{if } y \in \overline{\{x\}} \\ 0 & \text{if } y \notin \overline{\{x\}}, \end{cases}$$

because if y is in the closure of x then every neighborhood of y also contains x, whereas if y is *not* in the closure of x, then $U = X \setminus \{x\}$ is largest open neighborhoods of y and thus $(G_x)_y = 0$. Thus G_x has only one nonzero stalk (at x) if and only if x is a closed point. This is the case where the name 'skyscraper sheaf' for G_x fits best.

Exercise 2.3.14. Let $\mathcal{F}$ be a presheaf, $\mathcal{G}$ a sheaf, $\eta_1, \eta_2\colon \mathcal{F} \to \mathcal{G}$ two morphisms of presheaves of sets such that $\eta_{1,x} = \eta_{2,x}$ for every $x \in X$. Show that $\eta_1 = \eta_2$. Show that it is in fact necessary to assume $\mathcal{G}$ to be a sheaf. This exercise will be needed in **??**.

2.3.1 Supports

Let A be a ring. Let $\mathcal{F}$ be a sheaf of A-modules on a topological space X. Let $s \in \mathcal{F}(U)$ be a local section. We have two notions of support: the support of $\mathcal{F}$, and the support of s, defined respectively as

$$\operatorname{Supp}(\mathcal{F}) = \{ x \in X \mid \mathcal{F}_x \neq 0 \},$$
$$\operatorname{Supp}(s) = \{ x \in U \mid s_x \neq 0 \}.$$

If $s_x = 0$, then there is an open neighborhood $x \in V \subset U$ such that $s|_V = 0 \in \mathcal{F}(V)$. Thus $V \subset U \setminus \operatorname{Supp}(s)$ and hence $\operatorname{Supp}(s) \subset U$ is closed. In fact, this follows from (or solves) Exercise 2.3.5 as well. In general, however, $\operatorname{Supp}(\mathcal{F}) \subset X$ is *not* closed. If $\mathcal{F}$ is a sheaf of rings, these notions of support still make sense, and one has $\operatorname{Supp}(\mathcal{F}) = \operatorname{Supp}(1)$, where $1 \in \mathcal{F}(X)$ is the ring identity. Thus $\operatorname{Supp}(\mathcal{F})$ *is* closed in this special case.

2.4 Sheafification

Fix a type of algebraic structure $(\mathscr{C}, F : \mathscr{C} \to \text{Sets})$.

Let X be a topological space. Let $\mathcal{F} \colon \tau_X^{\text{op}} \to \mathscr{C}$ be a presheaf. We next define a *sheaf* $\mathcal{F}^\#$, called the sheafification of $\mathcal{F}$, via an explicit universal property, and having precisely the same stalks as the initial presheaf $\mathcal{F}$.

Definition 2.4.1. Let $\mathcal{F} \in \text{pSh}(X, \mathscr{C})$ be a presheaf. A *sheafification* of $\mathcal{F}$ is a pair $(\mathcal{F}^\#, \theta)$, where $\mathcal{F}^\#$ is a sheaf and $\theta : \mathcal{F} \to \mathcal{F}^\#$ is a morphism of presheaves, such that for every other pair $(\mathcal{G}, j)$ where $\mathcal{G}$ is a sheaf and $\alpha \colon \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, there exists a unique morphism of sheaves $\widetilde{\alpha} \colon \mathcal{F}^\# \to \mathcal{G}$ such that $\alpha = \widetilde{\alpha} \circ \theta$.

$$\begin{tikzcd}
\mathcal{F} \arrow[rr, "\theta"] \arrow[dr, "\forall\,\alpha"'] & & \mathcal{F}^\# \arrow[dl, dashed, "\exists\,!\,\widetilde{\alpha}"] \\
& \mathcal{G} &
\end{tikzcd}$$

PROPOSITION 2.4.2. *Let $\mathcal{F} \in \text{pSh}(X, \mathscr{C})$ be a presheaf. Then a sheafification $(\mathcal{F}^\#, \theta)$ exists, and the map $\theta_x : \mathcal{F}_x \to \mathcal{F}^\#_x$ is an isomorphism for every $x \in X$.*

What follows immediately from Proposition 2.4.2 is that $\mathcal{F}^\#$ is unique up to a unique isomorphism, and moreover the canonical map $\theta : \mathcal{F} \to \mathcal{F}^\#$ is an isomorphism precisely when $\mathcal{F}$ is already a sheaf.

Proof. Let $U \subset X$ be an open subset. Define

$$\mathcal{F}^\#(U) = \left\{ \text{functions } U \xrightarrow{f} \coprod_{x \in U} \mathcal{F}_x \;\middle|\; \begin{array}{c} \text{for every } x \in U \text{ there exist an open} \\ \text{neighborhood } V \subset U \text{ of } x \text{ and } s \in \mathcal{F}(V) \\ \text{such that } f(y) = s_y \text{ for every } y \in V \end{array} \right\}.$$

Note that, since $\mathscr{C}$ has products, we can view a function f as above as a tuple

$$(f(x))_{x \in U} \in \prod_{x \in U} \mathcal{F}_x$$

and we can rephrase the definition of $\mathcal{F}^\#(U)$ by saying that

$$\mathcal{F}^\#(U) = \left\{ \text{tuples } (s_x)_{x \in U} \text{ of compatible germs} \right\}.$$

See Definition 2.3.3 for the definition of compatible germs. Functoriality of the assignment $U \mapsto \mathcal{F}^\#(U)$ is clear (functions restrict!), thus $\mathcal{F}^\#$ is a presheaf. The morphism $\theta_U : \mathcal{F}(U) \to \mathcal{F}^\#(U)$ defined by sending $s \in \mathcal{F}(U)$ to the function

$$f_s : U \to \coprod_{x \in U} \mathcal{F}_x, \quad x \mapsto s_x = [U, s] \in \mathcal{F}_x$$

determines a morphism of presheaves, being compatible with restrictions. It is just the function $\sigma_U^{\mathcal{F}}$ introduced in (2.3.1)!

The presheaf $\mathcal{F}^\#$ is a sheaf: Fix an open cover $U = \bigcup_{i\in I} U_i$ of some open subset $U \subset X$ and a collection of sections $f_i \in \mathcal{F}^\#(U_i)$ such that $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ for every i and j. We need to find a unique $f \in \mathcal{F}^\#(U)$ such that $f|_{U_i} = f_i$. Define

$$f(x) = f_i(x) \in \mathcal{F}_x, \quad x \in U_i \subset U.$$

This is well-defined since, even though x can lie in more than one open U_i, by assumption we have $f_i(x) = f_j(x)$ as soon as $x \in U_i \cap U_j$. We need to check that f defines an element of $\mathcal{F}^\#(U)$, not just of the full product $\prod_{x\in U} \mathcal{F}_x$. But for every i we know the following: for every $x \in U_i$ there exist an open neighborhood $x \in V_i \subset U_i$ and a section $s_i \in \mathcal{F}(V_i)$ such that $f(y) = f_i(y) = (s_i)_y$ for all $y \in V_i$. But V_i is also open in U, so the condition defining $\mathcal{F}^\#(U)$ also holds for f. Thus $f \in \mathcal{F}^\#(U)$ satisfies $f|_{U_i} = f_i$, and is clearly unique with this property.

The pair $(\mathcal{F}^\#, \theta)$ is the sheafification. Assume we have a sheaf $\mathcal{G}$ and a morphism of presheaves $\alpha\colon \mathcal{F} \to \mathcal{G}$. We need to define a morphism $\widetilde{\alpha}\colon \mathcal{F}^\# \to \mathcal{G}$ of presheaves such that $\alpha = \widetilde{\alpha} \circ \theta$. For every U open in X, we need to define a morphism $\widetilde{\alpha}_U \colon \mathcal{F}^\#(U) \to \mathcal{G}(U)$ in such a way that $\alpha_U = \widetilde{\alpha}_U \circ \theta_U$. Fix $s = (s_x)_{x\in U} \in \mathcal{F}^\#(U)$. Consider the composition

$$\begin{tikzcd} U \arrow[r, "s"] \arrow[dr, "\widetilde{\alpha}_U(s)"'] & \coprod_{x\in U} \mathcal{F}_x \arrow[d, "\coprod_{x\in U}\alpha_x"] \\ & \coprod_{x\in U} \mathcal{G}_x. \end{tikzcd}$$

It defines a tuple of compatible germs for $\mathcal{G}$ over U, hence an element $\widetilde{\alpha}_U(s) \in \mathcal{G}^\#(U) = \mathcal{G}(U)$, using that $\mathcal{G}$ is a sheaf for this identity. This is the required morphism $\widetilde{\alpha}\colon \mathcal{F}^\# \to \mathcal{G}$.

The map θ is an isomorphism on stalks. The map θ, at the level of stalks, is defined by

$$\theta_x[U,s] = [U, f_s].$$

Injectivity: Suppose $\theta_x[U,s] = \theta_x[V,t]$ for two classes $[U,s],[V,t] \in \mathcal{F}_x$, i.e. assume $[U, f_s] = [V, f_t]$ in $\mathcal{F}^\#_x$. Then, by definition of germ, there exists an open neighborhood $W \subset U \cap V$ of x such that $f_s|_W = f_t|_W$. But this means, by definition of f_s and f_t, that $s_y = t_y$ for all $y \in W$. Thus, in particular, $s_x = t_x$. But this is just the equality $[U,s] = [V,t]$ we were after.

Surjectivity: Pick a class $[U, f] \in \mathcal{F}^\#_x$ for some $f \in \mathcal{F}^\#(U)$ and open neighborhood U of x. Then, for every $z \in U$, there exist an open neighborhood $V \subset U$ of z and a section $s \in \mathcal{F}(V)$ such that $f(y) = s_y$ in $\mathcal{F}_y$ for every $y \in V$. We claim that $[U, f] = \theta_x(s_x)$, where $s_x = [V, s]$. Indeed, $\theta_x(s_x) \in \mathcal{F}^\#_x$ is the equivalence class of the map

$$f_s\colon V \to \coprod_{y\in V} \mathcal{F}_y, \quad y \mapsto s_y.$$

But this map agrees with the restriction of f to $V \subset U$ (by the condition $f(y) = s_y$ recalled above), i.e. $f_s = f|_V \in \mathcal{F}^\#(V)$. Since V is also an open neighborhood of x, it follows that $(f|_V)_x = (f_s)_x = [V, f_s] = \theta_x(s_x) \in \mathcal{F}^\#_x$, but of course $(f|_V)_x = [U, f]$. Thus θ_x is surjective. $\square$

Example 2.4.3 (Constant sheaf)**.** Let G be a nontrivial abelian group. The *constant sheaf* on a topological space X, with values in G, is the sheafification $\underline{G}_X$ of the presheaf $\underline{G}_X^{\mathrm{pre}}$ defined in Example 2.2.17. This sheaf agrees with the sheaf whose sections over U are the locally constant functions $U \to G$. This, in turn, agrees with the following: endow G with the discrete topology and consider the assignment

$$U \mapsto \{\text{continuous maps } U \to G\},$$

which we know is a sheaf by Exercise 2.2.21. If $U \subset X$ is a connected open subset, then $\underline{G}_X(U) = G$. By Proposition 2.4.2, at the level of stalks we have $\underline{G}_{X,x} = G$ for every $x \in X$, since the stalks of the constant presheaf are manifestly all equal to G.

Exercise 2.4.4. Let X be a connected topological space, x a point, G a nontrivial abelian group. Under what condition(s) is the constant sheaf $\underline{G}_X$ equal to the skyscraper sheaf G_x (cf. Example 2.3.13)?

Exercise 2.4.5. Show that sending $\mathcal{F} \mapsto \mathcal{F}^\#$ defines a functor $(-)^\#\colon \mathrm{pSh}(X, \mathscr{C}) \to \mathrm{Sh}(X, \mathscr{C})$, and that the forgetful functor $j_{X,\mathscr{C}}\colon \mathrm{Sh}(X, \mathscr{C}) \hookrightarrow \mathrm{pSh}(X, \mathscr{C})$ is a right adjoint. This means that are bifunctorial bijections

$$\psi_{\mathcal{F},\mathcal{G}}\colon \mathrm{Hom}_{\mathrm{Sh}(X,\mathscr{C})}(\mathcal{F}^\#, \mathcal{G}) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathrm{pSh}(X,\mathscr{C})}(\mathcal{F}, \mathcal{G}), \quad \widetilde{\alpha} \mapsto \widetilde{\alpha} \circ \theta$$

for any presheaf $\mathcal{F}$ and sheaf $\mathcal{G}$.

2.4.1 Subsheaves, Quotient sheaves

PROPOSITION 2.4.6 ([9, Tag 007S]). *Let X be a topological space. Let $\mathcal{F}, \mathcal{G} \in \mathrm{Sh}(X, \mathrm{Sets})$ be sheaves of sets, $\eta\colon \mathcal{F} \to \mathcal{G}$ a morphism. Then, the following are equivalent:*

(a) η is a monomorphism,

(b) $\eta_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$,

(c) $\eta_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all open subsets $U \subset X$ (i.e. η is injective).

Furthermore, the following are equivalent:

(i) η is an epimorphism,

(ii) $\eta_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is surjective for all $x \in X$ (i.e. η is surjective),

and are implied (but not equivalent to, cf. Example 2.3.12!) by the condition

(iii) $\eta_U : \mathcal{F}(U) \to \mathcal{G}(U)$ *is surjective for all open subsets* $U \subset X$.

If $\mathscr{C}$ is an abelian category (e.g. Mod_A for a fixed ring A), then Proposition 2.4.6 holds replacing Sets with $\mathscr{C}$. For more general categories, such as Rings, one should replace 'injective' (resp. 'surjective') with 'monomorphism' (resp. 'epimorphism').

Definition 2.4.7 (Subsheaf, quotient sheaf)**.** If there exists a morphism of sheaves $\eta: \mathcal{F} \to \mathcal{G}$ such that either of the equivalent conditions (a), (b) or (c) holds, we say that $\mathcal{F}$ is a *subsheaf* of $\mathcal{G}$ (and we may denote this by '$\mathcal{F} \subset \mathcal{G}$'). If either of the equivalent conditions (i) or (ii) holds, we say that $\mathcal{G}$ is a *quotient sheaf* of $\mathcal{F}$.

Example 2.4.8 (Quotient sheaf)**.** Let $\mathscr{C}$ be an abelian category. If $\mathcal{F} \subset \mathcal{G}$ is a subsheaf (with values in $\mathscr{C}$), then sending

$$U \mapsto \mathcal{G}(U)/\mathcal{F}(U) \tag{2.4.1}$$

is a presheaf on X, because the restriction maps respect the inclusions $\mathcal{F}(U) \hookrightarrow \mathcal{G}(U)$, and thus pass to the quotients. Its sheafification $\mathcal{G}/\mathcal{F}$ is called the *quotient sheaf of* $\mathcal{G}$ *by* $\mathcal{F}$. There is a natural morphism of sheaves $\mathcal{G} \to \mathcal{G}/\mathcal{F}$.

Definition 2.4.9 (Sheaf image, sheaf cokernel)**.** Let $\mathscr{C}$ be an abelian category, $\eta: \mathcal{F} \to \mathcal{G}$ a morphism of sheaves (with values in $\mathscr{C}$), so that $\ker(\eta) \hookrightarrow \mathcal{F}$ is a subsheaf by Exercise 2.2.19. The sheafification $\mathrm{im}(\eta)$ of the presheaf

$$U \mapsto \mathrm{im}_{\mathrm{pre}}(U) = \mathrm{im}(\eta_U) = \mathcal{F}(U)/\ker(\eta_U)$$

is called the *image of* η. It is a special case of Example 2.4.8 and defines a subsheaf

$$\mathrm{im}(\eta) = \mathcal{F}/\ker(\eta) \subset \mathcal{G}.$$

The quotient sheaf

$$\mathrm{coker}(\eta) = \mathcal{G}/\mathrm{im}(\eta),$$

again a special case of Example 2.4.8, is called the *sheaf cokernel*.

Exercise 2.4.10. Let $\mathscr{C}$ be an abelian category. Let $\eta: \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves with values in $\mathscr{C}$. Show that the composition

$$\mathcal{G} \to \mathrm{coker}_{\mathrm{pre}}\,\eta \to \mathrm{coker}(\eta),$$

where the first morphism is given by the natural maps $\mathcal{G}(U) \twoheadrightarrow \mathcal{G}(U)/\mathrm{im}(\eta_U)$ and the last morphism is the sheafification, is a cokernel in the category $\mathrm{Sh}(X, \mathscr{C})$.

Remark 2.4.11. Set $\mathscr{C} = \mathrm{Mod}_A$ (or any Grothendieck abelian category in which, by definition, filtered colimits exist and are exact). Let $\mathcal{F} \subset \mathcal{G}$ be a subsheaf, $x \in X$ a point. Then

$$(\mathcal{G}/\mathcal{F})_x = \mathcal{G}_x/\mathcal{F}_x \tag{2.4.2}$$

in Mod_A. This follows from the fact that $(\mathcal{G}/\mathcal{F})_x$ agrees with the stalk of the *presheaf* (2.4.1), and from right exactness of filtered colimits. Moreover, if $\eta\colon \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves and $x \in X$ is a point, then

$$\begin{aligned} \ker(\eta)_x &= \ker(\eta_x) \\ \mathrm{im}(\eta)_x &= \mathrm{im}(\eta_x) \\ \mathrm{coker}(\eta)_x &= \mathrm{coker}(\eta_x). \end{aligned}$$

The first identity follows from the fact that filtered colimits are *also left exact* in Mod_A, thus

$$\begin{aligned} \ker\left(\mathcal{F}_x \xrightarrow{\eta_x} \mathcal{G}_x\right) &= \ker\left(\varinjlim_{U \ni x} \mathcal{F}(U) \to \varinjlim_{U \ni x} \mathcal{G}(U)\right) \\ &= \varinjlim_{U \ni x} \ker(\mathcal{F}(U) \to \mathcal{G}(U)) \\ &= \ker(\eta)_x. \end{aligned}$$

The last two identities are a special case of (2.4.2).

THEOREM 2.4.12 ([3, §10]). *If $\mathscr{C}$ is a Grothendieck abelian category, then $\mathrm{Sh}(X,\mathscr{C})$ is a Grothendieck abelian category.*

Definition 2.4.13. A *short exact sequence of sheaves* with values in a Grothendieck abelian category $\mathscr{C}$ is a short exact sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{\iota} \mathcal{G} \xrightarrow{\theta} \mathcal{H} \longrightarrow 0$$

of objects in the abelian category $\mathrm{Sh}(X,\mathscr{C})$. Explicitly, exactness means that ι is injective, θ is surjective and $\mathrm{im}(\iota) = \ker(\theta)$.

PROPOSITION 2.4.14. *Let $\mathscr{C}$ be a Grothendieck abelian category. A sequence*

$$0 \longrightarrow \mathcal{F} \xrightarrow{\iota} \mathcal{G} \xrightarrow{\theta} \mathcal{H} \longrightarrow 0$$

of objects in $\mathrm{Sh}(X,\mathscr{C})$ is a short exact sequence if and only if

$$0 \longrightarrow \mathcal{F}_x \xrightarrow{\iota_x} \mathcal{G}_x \xrightarrow{\theta_x} \mathcal{H}_x \longrightarrow 0$$

is a short exact sequence in $\mathscr{C}$ for every $x \in X$.

Proof. Combine Remark 2.4.11 and Lemma 2.3.9 with one another. □

Exercise 2.4.15. Let $\eta\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of A-modules, for A a ring. Confirm the following.

- If η is injective, there is a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{F} \longrightarrow 0$$

- In general, there is an exact sequence of sheaves

$$0 \longrightarrow \ker(\eta) \longrightarrow \mathcal{F} \xrightarrow{\ \eta\ } \mathcal{G} \longrightarrow \operatorname{coker}(\eta) \longrightarrow 0.$$

Exercise 2.4.16. Let A be a ring. For a nonempty open subset U of a topological space X, consider the functor $\Gamma(U,-)\colon \mathrm{Sh}(X,\mathrm{Mod}_A) \to \mathrm{Mod}_A$ sending $\mathcal{F} \mapsto \mathcal{F}(U)$. Show that it is left exact. That is, it transforms an exact sequence of sheaves $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ into an exact sequence of A-modules

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U).$$

When $U = X$, this functor takes $\mathcal{F} \mapsto \mathcal{F}(X)$ and is thus called the *global section functor*. Another notation used for it in the literature is $\mathrm{H}^0(X,-)$, cf. Terminology 2.2.2.

2.5 Defining sheaves and morphisms on basic open sets

Fix a type of algebraic structure $(\mathscr{C}, F\colon \mathscr{C} \to \mathrm{Sets})$.

Definition 2.5.1 (Base of open sets)**.** Let X be a topological space. A *base of open sets* for X is a collection of open subsets $\mathcal{B} \subset \tau_X$ satisfying the following requirements:

(a) $\mathcal{B}$ is stable under finite intersections,

(b) every $U \in \tau_X$ can be written as a union of open sets belonging to $\mathcal{B}$.

Definition 2.5.2 ($\mathcal{B}$-sheaf)**.** A $\mathcal{B}$*-presheaf* (resp. $\mathcal{B}$*-sheaf*) is an assignment $U \mapsto \mathcal{F}(U) \in \mathscr{C}$ for each $U \in \mathcal{B}$ such that the presheaf conditions (1)–(2) of Definition 2.2.1 (resp. the presheaf conditions (1)–(2) of Definition 2.2.1 and the sheaf conditions (3)–(4) of Definition 2.2.5) hold, considering only open sets belonging to $\mathcal{B}$.

Notation 2.5.3. We shall use the notation $\{\mathcal{F}(\mathcal{B}), \rho_{\mathcal{B}}\}$ to denote a $\mathcal{B}$-(pre)sheaf.

Note that restriction map $\rho_{UV}\colon \mathcal{F}(U) \to \mathcal{F}(V)$ is part of the data of a $\mathcal{B}$-presheaf whenever $V \subset U$ is an inclusion of open sets belonging to $\mathcal{B}$. Note, also, that condition (a) in Definition 2.5.2 ensures that open sets of the form $U_i \cap U_j$, for $U_i, U_j \in \mathcal{B}$, still belong to $\mathcal{B}$. In particular, a $\mathcal{B}$-presheaf is a sheaf precisely when, for any open $U \in \mathcal{B}$ and any open cover $U = \bigcup_{i\in I} U_i$ with all $U_i \in \mathcal{B}$, the sequence

$$\mathcal{F}(U) \xrightarrow{\ \rho\ } \prod_{i\in I} \mathcal{F}(U_i) \overset{\mu}{\underset{\nu}{\rightrightarrows}} \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \cap U_j)$$

is an equaliser in $\mathscr{C}$.

Remark 2.5.4. The collection of open neighborhoods

$$\mathcal{B}_x = \{\, U \in \mathcal{B} \mid x \in U \,\}^{\mathrm{op}} \subset \tau_X^{\mathrm{op}}$$

is a fundamental system of open neighborhoods of x (i.e. for any $U \in \mathcal{B}_x$ there is $V \in \tau_X$ such that $x \in V \subset U$, and for any $W \in \tau_X$ such that $x \in W$ there exists $U \in \mathcal{B}_x$ such that $U \subset W$). In more technical terms, one may say that the filtered categories Nbg_x and $\mathcal{B}_x$ are *cofinal*, i.e. the inclusion $\mathcal{B}_x \hookrightarrow \mathrm{Nbg}_x$ is a cofinal functor.

By Remark 2.5.4, the stalk

$$\mathcal{F}_x = \varinjlim_{U\in\mathcal{B}_x} \mathcal{F}(U)$$

of a $\mathcal{B}$-(pre)sheaf $\{\mathcal{F}(\mathcal{B}), \rho_{\mathcal{B}}\}$ at a point $x \in X$ is well-defined as an object of $\mathscr{C}$.

Moreover, if $U \in \mathcal{B}$ and $\{\mathcal{F}(\mathcal{B}), \rho_{\mathcal{B}}\}$ is a $\mathcal{B}$-sheaf, the natural map

$$\sigma_U^{\mathcal{F}} \colon \mathcal{F}(U) \longrightarrow \textstyle\prod_{x\in U} \mathcal{F}_x$$

is injective (as in Lemma 2.3.2), and its image still agrees with the collections of compatible germs; to be more precise, we should now call them '$\mathcal{B}$-compatible', for they are, by definition, those tuples

$$(s_x)_{x\in U} \in \prod_{x\in U} \mathcal{F}_x$$

such that for every $x \in U$ there is a pair (V_x, t_x), where $V_x \in \mathcal{B}$ satisfies $x \in V_x \subset U$ and $t_x \in \mathcal{F}(V_x)$ satisfies $(t_x)_y = s_y$ for every $y \in V_x$.

Definition 2.5.5 (Morphism of $\mathcal{B}$-sheaves)**.** A *morphism* of $\mathcal{B}$-(pre)sheaves

$$\eta_{\mathcal{B}} \colon \left\{\mathcal{F}(\mathcal{B}), \rho_{\mathcal{B}}^{\mathcal{F}}\right\} \longrightarrow \left\{\mathcal{G}(\mathcal{B}), \rho_{\mathcal{B}}^{\mathcal{G}}\right\} \tag{2.5.1}$$

is the datum of a collection of maps $\eta_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$, one for each $U \in \mathcal{B}$, such that Diagram (2.2.1) commutes for all $U, V \in \mathcal{B}$ such that $V \subset U$.

With this definition, $\mathcal{B}$-sheaves form a category, denoted $\mathrm{Sh}_{\mathcal{B}}(X, \mathscr{C})$.

Remark 2.5.6. Let $\mathcal{B}$ and X be as above. A (pre)sheaf $\mathcal{F}$ on X is a $\mathcal{B}$-(pre)sheaf in a natural way. More precisely, there is (say, at the level of sheaves) a *restriction functor*

$$\mathsf{res}_{\mathcal{B}}(X, \mathscr{C}) \colon \mathrm{Sh}(X, \mathscr{C}) \longrightarrow \mathrm{Sh}_{\mathcal{B}}(X, \mathscr{C}), \tag{2.5.2}$$

defined on objects in the obvious way. Its actual functoriality is just a consequence of the definition of morphism of $\mathcal{B}$-sheaves, and we leave it to the reader to check all functoriality details.

LEMMA 2.5.7. *A $\mathcal{B}$-sheaf $\{\mathcal{F}(\mathcal{B}),\rho_{\mathcal{B}}\}$ uniquely extends to a sheaf $\overline{\mathcal{F}}$, whose sections over $U \in \mathcal{B}$ agree with $\mathcal{F}(U)$.*

Proof. Let U be an open set in X. Define

$$\overline{\mathcal{F}}(U) = \left\{ \text{tuples } (s_x)_{x\in U} \text{ of } \mathcal{B}\text{-compatible germs} \right\} \subset \prod_{x\in U} \mathcal{F}_x.$$

This is manifestly a presheaf. It is also clear that the above definition agrees with $\mathcal{F}(U)$ whenever $U \in \mathcal{B}$, since the injective map $\sigma_U^{\mathcal{F}}$ hits precisely the tuple of $\mathcal{B}$-compatible germs; moreover, for the same reason, this definition is the *only* possible extension of the original $\mathcal{B}$-sheaf. The sheaf property is fulfilled by $\overline{\mathcal{F}}$ precisely for the same reason why it is fulfilled by the sheafification of a presheaf (see the proof of Proposition 2.4.2). □

In fact, the statement of Lemma 2.5.7 can be made functorial: one can prove that the restriction functor (2.5.2) is an equivalence. The inverse is given precisely by Lemma 2.5.7 above for objects and by Proposition 2.5.9 below for morphisms.

Remark 2.5.8. We have that $\mathcal{F}_x = \overline{\mathcal{F}}_x$ for all $x \in X$. This follows directly from Remark 2.5.4.

The analogue of Lemma 2.5.7 for *morphisms* is the following.

PROPOSITION 2.5.9. *Let X be a topological space, $\mathcal{B} \subset \tau_X$ a base of open sets and $\mathcal{F}, \mathcal{G}$ two sheaves on X. Suppose given a morphism $\eta_{\mathcal{B}}$ between the underlying $\mathcal{B}$-sheaves, as in (2.5.1). Then, $\eta_{\mathcal{B}}$ extends uniquely to a sheaf homomorphism $\eta : \mathcal{F} \to \mathcal{G}$. Furthermore, if η_U is surjective (resp. injective, or an isomorphism) for every $U \in \mathcal{B}$, then so is η.*

Proof. Exercise. □

2.6 Pushforward, inverse image

In this section we learn how to "move" sheaves from a topological space X to another topological space Y, in the presence of a continuous map between the two spaces.

$$\begin{tikzcd} \mathcal{F} \arrow[r, squiggly] \arrow[d, dash] & f_*\mathcal{F} \arrow[d, dash] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} f^{-1}\mathcal{G} \arrow[d, dash] & \mathcal{G} \arrow[l, squiggly] \arrow[d, dash] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

2.6.1 Pushforward (or direct image)

Let $f: X \to Y$ be a continuous map of topological spaces, and let $\mathcal{F}$ be a presheaf on X. The assignment

$$V \mapsto f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$$

defines a presheaf $f_*\mathcal{F}$ on Y, called the *pushforward* (or *direct image*) of $\mathcal{F}$ by f. It is a sheaf as soon as $\mathcal{F}$ is, because if $V = \bigcup_{i\in I} V_i$ is an open covering of an open subset $V \subset Y$, then $f^{-1}(V) = \bigcup_{i\in I} f^{-1}(V_i)$ is an open covering of $f^{-1}(V) \subset X$.

Example 2.6.1. If X is arbitrary and $Y = \mathrm{pt}$, then $f_*\mathcal{F}(\mathrm{pt}) = \mathcal{F}(X)$, an object of $\mathscr{C}$. We will see in a minute that the direct image along any continuous map defines a functor. The direct image along the constant map $(X \to \mathrm{pt})_*\colon \mathrm{Sh}(X,\mathscr{C}) \to \mathscr{C}$ is also called the *global section functor*. If $\mathscr{C} = \mathrm{Mod}_A$, it is a left exact functor (you proved a more general statement in Exercise 2.4.16).

Example 2.6.2. If $f: X \hookrightarrow Y$ is the inclusion of a subspace, then $f_*\mathcal{F}$ is defined, for any open subset $V \subset Y$, by

$$f_*\mathcal{F}(V) = \mathcal{F}(V \cap X).$$

Example 2.6.3 (Skyscraper sheaf as a pushforward)**.** Let $x \in X$ be a point, G a nontrivial abelian group. Consider the constant sheaf $\underline{G}_{\{x\}}$ on $\{x\}$. Let $i_x\colon \{x\} \hookrightarrow X$ be the inclusion. Then the skyscraper sheaf $G_x \in \mathrm{Sh}(X, \mathrm{Mod}_{\mathbb{Z}})$ defined in Example 2.3.13 can be described as

$$G_x = i_{x,*}\underline{G}_{\{x\}}.$$

Next, we observe that pushforward of sheaves is functorial, i.e. sending $\mathcal{F} \mapsto f_*\mathcal{F}$ defines functors

$$\begin{tikzcd} \mathrm{Sh}(X,\mathscr{C}) \arrow[r, "f_*"] \arrow[d, hook] & \mathrm{Sh}(Y,\mathscr{C}) \arrow[d, hook] \\ \mathrm{pSh}(X,\mathscr{C}) \arrow[r, "f_*"] & \mathrm{pSh}(Y,\mathscr{C}) \end{tikzcd}$$

where the vertical maps are the natural inclusions. Indeed, given a morphism of (pre)sheaves $\eta\colon \mathcal{F} \to \mathcal{G}$, we can construct a morphism of (pre)sheaves

$$f_*\eta\colon f_*\mathcal{F} \to f_*\mathcal{G}$$

simply by setting

$$(f_*\eta)_V = \eta_{f^{-1}(V)}\colon \mathcal{F}(f^{-1}(V)) \to \mathcal{G}(f^{-1}(V))$$

for an open subset $V \subset Y$. The compatibility with restriction maps follows from those of η (and the obvious observation that if $V' \subset V$ then $f^{-1}V' \subset f^{-1}V$).

Moreover, $(-)_*$ is compatible with compositions of continuous maps, in the following sense: if $f: X \to Y$ and $g: Y \to Z$ are continuous maps of topological spaces, then, as functors, we have an equality $(g\circ f)_* = g_*\circ f_*$ on the nose (both for presheaves and for sheaves).

$$\begin{tikzcd} \mathrm{Sh}(X,\mathscr{C}) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \mathrm{Sh}(Y,\mathscr{C}) \arrow[d, "g_*"] \\ & \mathrm{Sh}(Z,\mathscr{C}) \end{tikzcd}$$

Indeed, if $\mathcal{F}$ is a (pre)sheaf on X, then for every open $W\subset Z$ one has

$$\begin{aligned} (g\circ f)_*\mathcal{F}(W) &= \mathcal{F}((g\circ f)^{-1}(W)) \\ &= \mathcal{F}(f^{-1}g^{-1}(W)) \\ &= f_*\mathcal{F}(g^{-1}(W)) \\ &= (g_*f_*\mathcal{F})(W) \\ &= (g_*\circ f_*)\mathcal{F}(W). \end{aligned}$$

LEMMA 2.6.4. *Let $f\colon X\to Y$ be a continuous map of topological spaces, and fix a sheaf $\mathcal{F}\in \mathrm{Sh}(X,\mathscr{C})$. Set $y=f(x)$. There is a canonical morphism*

$$(f_*\mathcal{F})_y \longrightarrow \mathcal{F}_x.$$

This morphism is an isomorphism when f is the inclusion of a subspace $X\hookrightarrow Y$.

Proof. If $y\in V'\subset V\subset Y$, then $x\in f^{-1}V'\subset f^{-1}V\subset X$, and the commutative diagram

$$\begin{tikzcd} f_*\mathcal{F}(V) \arrow[rr] \arrow[d, equal] \arrow[dr] & & f_*\mathcal{F}(V') \arrow[d, equal] \arrow[dl] \\ \mathcal{F}(f^{-1}V) \arrow[dr] & (f_*\mathcal{F})_y \arrow[d, dotted] & \mathcal{F}(f^{-1}V') \arrow[dl] \\ & \mathcal{F}_x & \end{tikzcd}$$

induces, via the universal property of the stalk $(f_*\mathcal{F})_y$, a canonical morphism $(f_*\mathcal{F})_y\to \mathcal{F}_x$, as required.

Now, let us assume f is the inclusion of a subspace, and let us take $y\in X$. Note that every neighborhood $y\in U\subset X$ is of the form $U=V\cap X$ for some open neighborhood $y\in V\subset Y$. Thus

$$\begin{aligned} (f_*\mathcal{F})_y &= \varinjlim_{Y\supset V\ni y} \mathcal{F}(X\cap V) \\ &\xrightarrow{\sim} \varinjlim_{X\supset U\ni y} \mathcal{F}(U) \\ &= \mathcal{F}_y. \end{aligned}$$

□

We shall use Lemma 2.6.4 crucially with $\mathscr{C} = \mathrm{Rings}$.

We now say two (important) words on the exactness of direct image. What needs to be remembered is:

$$\boxed{f_* \text{ is always left exact, and it is exact if } f\colon X \hookrightarrow Y \text{ is a closed subspace.}}$$

We set $\mathscr{C} = \mathrm{Mod}_A$ for the rest of this subsection. Since f_* will turn out to be a right adjoint (Lemma 2.6.14), it is left exact by general category theory. However, we prove it directly here. Note that you have already proved the case $Y = \mathrm{pt}$ in Exercise 2.4.16. You will notice in the proof that this was essentially enough to handle the general case.

PROPOSITION 2.6.5. *Let $f\colon X \to Y$ be a continuous map of topological spaces. Then, $f_*\colon \mathrm{Sh}(X,\mathrm{Mod}_A) \to \mathrm{Sh}(Y,\mathrm{Mod}_A)$ is left exact.*

Proof. We have show that an exact sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}$$

in $\mathrm{Sh}(X,\mathrm{Mod}_A)$ induces an exact sequence

$$0 \longrightarrow f_*\mathcal{F} \xrightarrow{f_*\alpha} f_*\mathcal{G} \xrightarrow{f_*\beta} f_*\mathcal{H}$$

in $\mathrm{Sh}(Y,\mathrm{Mod}_A)$. We know by Exercise 2.4.16 that we have an exact sequence

$$0 \longrightarrow \mathcal{F}(f^{-1}V) \xrightarrow{\alpha_{f^{-1}V}} \mathcal{G}(f^{-1}V) \xrightarrow{\beta_{f^{-1}V}} \mathcal{H}(f^{-1}V) \tag{2.6.1}$$

for any open subset $V \subset Y$, by applying the functor $\Gamma(f^{-1}V,-)$ to the original sequence. In particular, $\alpha_{f^{-1}V} = (f_*\alpha)_V$ is injective for all V, which shows that $f_*\alpha$ is injective. There is an equality of presheaves

$$\mathrm{im}_{\mathrm{pre}}(f_*\alpha) = \ker(f_*\beta)$$

again thanks to exactness of (2.6.1) in the middle, ensuring precisely that $\mathrm{im}(\alpha_{f^{-1}V}) = \ker(\beta_{f^{-1}V})$. But $\ker(f_*\beta)$ is a sheaf, therefore we get exactness in the middle, i.e. $\mathrm{im}(f_*\alpha) = \ker(f_*\beta)$. □

PROPOSITION 2.6.6. *Let $f\colon X \hookrightarrow Y$ be the inclusion of a closed subspace. Then f_* is exact.*

Proof. By Proposition 2.6.5, we only need to show that if $\eta\colon \mathcal{G} \twoheadrightarrow \mathcal{H}$ is surjective as a map of sheaves on X, then $f_*\mathcal{G} \twoheadrightarrow f_*\mathcal{H}$ is surjective as a map of sheaves on Y. If $y \in Y \setminus X$, then (using that X is closed)

$$(f_*\mathcal{G})_y = 0 = (f_*\mathcal{H})_y.$$

Assume $y \in X$. Since $\mathcal{G}$ surjects onto $\mathcal{H}$, in the commutative diagram

$$\begin{tikzcd} (f_*\mathcal{G})_y \arrow[r, "(f_*\eta)_y"] \arrow[d, "\wr"] & (f_*\mathcal{H})_y \arrow[d, "\wr"] \\ \mathcal{G}_y \arrow[r, two heads, "\eta_y"] & \mathcal{H}_y \end{tikzcd}$$

the bottom map is surjective. The vertical maps are isomorphisms by the last assertion of Lemma 2.6.4. Thus the top map is surjective as well. Hence $f_*\eta$ is surjective on all stalks, hence it is surjective. □

2.6.2 Inverse image

Let $f\colon X \to Y$ be a continuous map of topological spaces. Let $\mathcal{G}$ be a presheaf on Y. Given $U \subset X$, the collection of open subsets $V \subset Y$ containing $f(U)$ form a directed set via reverse inclusions. Sending

$$U \mapsto (f_{\mathrm{pre}}^{-1}\mathcal{G})(U) = \varinjlim_{V \supset f(U)} \mathcal{G}(V).$$

defines a presheaf on X. Indeed, assume $U' \subset U$ is an open subset. Then there is an inclusion $f(U') \subset f(U)$, inducing a map of directed systems

$$\{\, V \in \tau_Y \mid V \supset f(U)\,\} \hookrightarrow \{\, V \in \tau_Y \mid V \supset f(U')\,\},$$

and in turn a morphism

$$\varinjlim_{V \supset f(U)} \mathcal{G}(V) \to \varinjlim_{V \supset f(U')} \mathcal{G}(V).$$

Remark 2.6.7. Note that if $f(U)$ is an open subset of Y, then

$$(f_{\mathrm{pre}}^{-1}\mathcal{G})(U) = \mathcal{G}(f(U)).$$

Now assume $\mathcal{G}$ is a sheaf. We define the *inverse image* of $\mathcal{G}$ by f to be the sheafification

$$f^{-1}\mathcal{G} = \left(f_{\mathrm{pre}}^{-1}\mathcal{G}\right)^{\#}.$$

Note that there is a canonical map $f_{\mathrm{pre}}^{-1}\mathcal{G} \to f^{-1}\mathcal{G}$ of presheaves inducing an isomorphism on all the stalks.

Exercise 2.6.8. Both f_{pre}^{-1} and f^{-1} are functors.

Example 2.6.9. Let $\iota_y\colon \{\, y\,\} \hookrightarrow Y$ be the inclusion of a point $y \in Y$, and let $\mathcal{G}$ be a sheaf on Y. Then $\iota_y^{-1}\mathcal{G} = \mathcal{G}_y$, since $\iota_y^{-1}\mathcal{G}(\{\, y\,\}) = \varinjlim_{V \ni y} \mathcal{G}(V) = \mathcal{G}_y$. Thus ι_y^{-1} agrees with the stalk functor

$$\mathsf{stalk}_y\colon \mathrm{Sh}(Y, \mathscr{C}) \to \mathscr{C}, \quad \mathcal{G} \mapsto \mathcal{G}_y.$$

Example 2.6.10. If $p\colon X \to \mathrm{pt}$ is the constant map, and $G \in \mathscr{C} \cong \mathrm{Sh}(\mathrm{pt}, \mathscr{C})$, then $p^{-1}G = \underline{G}_X$, the constant sheaf on X with values in the object G.

Example 2.6.11. Let $j\colon U \hookrightarrow Y$ be the inclusion of an open subset. Then $j_{\mathrm{pre}}^{-1}\mathcal{G} = \mathcal{G}|_U$ for any sheaf $\mathcal{G}$ on Y. The reason is that if U' is open in U, it is also open in Y, and thus $j_{\mathrm{pre}}^{-1}\mathcal{G}(U') = \varinjlim_{V \supset U'} \mathcal{G}(V) = \mathcal{G}(U')$. In particular, $j_{\mathrm{pre}}^{-1}\mathcal{G}$ is already a sheaf, and hence

$$j^{-1}\mathcal{G} = \mathcal{G}|_U, \quad U \subset Y \text{ open}.$$

Remark 2.6.12. Sheafification is indeed necessary: consider a constant map $f\colon X = \{\star,\bullet\} \to \{*\} = Y$ from a two point set, and fix a nontrivial abelian group G. The constant sheaf $\mathcal{G} = \underline{G}_Y$ has the property $f_{\mathrm{pre}}^{-1}\mathcal{G} = \underline{G}_X^{\mathrm{pre}}$, which is not a sheaf (cf. Example 2.2.17).

Functoriality can be translated into a diagram of functors

$$\begin{tikzcd}
\mathrm{Sh}(Y,\mathscr{C}) \arrow[r, "f^{-1}"] \arrow[d, hook] & \mathrm{Sh}(X,\mathscr{C}) \arrow[d, hook] \\
\mathrm{pSh}(Y,\mathscr{C}) \arrow[r, "f_{\mathrm{pre}}^{-1}"] & \mathrm{pSh}(X,\mathscr{C})
\end{tikzcd}$$

where f^{-1} is obtained by applying $(-)^{\#}\colon \mathrm{pSh}(X,\mathscr{C}) \to \mathrm{Sh}(X,\mathscr{C})$ in the last step.

Exercise 2.6.13. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be continuous maps of topological spaces, and let $\mathcal{E}$ be a sheaf on Z. Show that

$$f^{-1}(g^{-1}\mathcal{E}) = (g\circ f)^{-1}\mathcal{E}.$$

LEMMA 2.6.14 (Unit and counit maps). *For any pair of presheaves $\mathcal{F} \in \mathrm{pSh}(X,\mathscr{C})$ and $\mathcal{G} \in \mathrm{pSh}(Y,\mathscr{C})$ there are canonical presheaf homomorphisms*

$$\mathcal{G} \xrightarrow{\ \mathsf{unit}\ } f_* f_{\mathrm{pre}}^{-1}\mathcal{G}, \qquad f_{\mathrm{pre}}^{-1} f_*\mathcal{F} \xrightarrow{\ \mathsf{counit}\ } \mathcal{F}.$$

Proof. We start with the unit map. The observation here is that there is, for any open subset $V \subset Y$, a canonical inclusion $f(f^{-1}V) \hookrightarrow V$. Thus $\mathcal{G}(V)$ appears in the colimit

$$\varinjlim_{W \supset f(f^{-1}V)} \mathcal{G}(W)$$

This induces a canonical morphism

$$\mathsf{unit}_V\colon \mathcal{G}(V) \to \varinjlim_{W \supset f(f^{-1}V)} \mathcal{G}(W) = f_{\mathrm{pre}}^{-1}\mathcal{G}(f^{-1}V) = f_* f_{\mathrm{pre}}^{-1}\mathcal{G}(V)$$

which does define a natural transformation $\mathcal{G} \to f_* f_{\mathrm{pre}}^{-1}\mathcal{G}$ because if $V' \subset V$, then any open $W \subset Y$ containing $f(f^{-1}V)$ also contains $f(f^{-1}V')$, simply because $f(f^{-1}V') \subset f(f^{-1}V)$. Thus there is a natural morphism

$$\varinjlim_{W \supset f(f^{-1}V)} \mathcal{G}(W) \to \varinjlim_{W \supset f(f^{-1}V')} \mathcal{G}(W)$$

and the induced diagram

$$\begin{tikzcd}
\mathcal{G}(V) \arrow[r, "\mathsf{unit}_V"] \arrow[d] & \varinjlim_{W \supset f(f^{-1}V)} \mathcal{G}(W) \arrow[d] \\
\mathcal{G}(V') \arrow[r, "\mathsf{unit}_{V'}"] & \varinjlim_{W \supset f(f^{-1}V')} \mathcal{G}(W)
\end{tikzcd}$$

commutes. This defines the map unit: $\mathcal{G} \to f_* f^{-1}_{\mathrm{pre}} \mathcal{G}$ of presheaves.

To construct the counit map, one observes that for any open subset $U \subset X$ there is a canonical map

$$f^{-1}_{\mathrm{pre}} f_* \mathcal{F}(U) = \varinjlim_{V \supset f(U)} f_* \mathcal{F}(V) = \varinjlim_{V \supset f(U)} \mathcal{F}(f^{-1} V) \to \mathcal{F}(U),$$

since if $V \supset f(U)$ inside Y, then $U \subset f^{-1} f(U) \subset f^{-1} V$ inside X. This map is also functorial in $U' \subset U$, thus the map counit: $f^{-1}_{\mathrm{pre}} f_* \mathcal{F} \to \mathcal{F}$ is defined. $\square$

The usefulness of the homomorphisms unit and counit is that they make

$$(f^{-1}_{\mathrm{pre}}, f_*)$$

into an adjoint pair of functors. More precisely, there are bijections

$$\varphi_{\mathcal{F},\mathcal{G}} \colon \mathrm{Hom}_{\mathrm{pSh}(Y,\mathscr{C})}(\mathcal{G}, f_* \mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{pSh}(X,\mathscr{C})}(f^{-1}_{\mathrm{pre}} \mathcal{G}, \mathcal{F}),$$

functorial in both $\mathcal{F}$ and $\mathcal{G}$. Specifically, $\varphi_{\mathcal{F},\mathcal{G}}$ sends

$$\mathcal{G} \xrightarrow{\eta} f_* \mathcal{F} \qquad \mapsto \qquad f^{-1}_{\mathrm{pre}} \mathcal{G} \xrightarrow{f^{-1}_{\mathrm{pre}} \eta} f^{-1}_{\mathrm{pre}} f_* \mathcal{F} \xrightarrow{\text{counit}} \mathcal{F}$$

with inverse

$$f^{-1}_{\mathrm{pre}} \mathcal{G} \xrightarrow{\iota} \mathcal{F} \qquad \mapsto \qquad \mathcal{G} \xrightarrow{\text{unit}} f_* f^{-1}_{\mathrm{pre}} \mathcal{G} \xrightarrow{f_* \iota} f_* \mathcal{F}.$$

Using Lemma 2.6.14, it is immediate to show that also

$$(f^{-1}, f_*)$$

is an adjoint pair $\mathrm{Sh}(Y, \mathscr{C}) \rightleftarrows \mathrm{Sh}(X, \mathscr{C})$ on *sheaves*. Indeed,

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Sh}(Y,\mathscr{C})}(\mathcal{G}, f_* \mathcal{F}) &= \mathrm{Hom}_{\mathrm{pSh}(Y,\mathscr{C})}(\mathcal{G}, f_* \mathcal{F}) && j_{Y,\mathscr{C}} \text{ is full} \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{pSh}(X,\mathscr{C})}(f^{-1}_{\mathrm{pre}} \mathcal{G}, \mathcal{F}) && \text{Lemma 2.6.14} \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Sh}(X,\mathscr{C})}(f^{-1} \mathcal{G}, \mathcal{F}) && \text{Exercise 2.4.5.} \end{aligned}$$

Terminology 2.6.15. If $\mathcal{F} = f^{-1} \mathcal{G}$, the map $\mathcal{G} \to f_* f^{-1} \mathcal{G}$ corresponding to $\mathrm{id}_{f^{-1}\mathcal{G}}$ is called the *unit* of the adjunction. If $\mathcal{G} = f_* \mathcal{F}$, the map $f^{-1} f_* \mathcal{F} \to \mathcal{F}$ corresponding to $\mathrm{id}_{f_* \mathcal{F}}$ is called the *counit* of the adjunction.

The next lemma says that the stalk of the inverse image is somewhat easy to compute (unlike for the pushforward).

LEMMA 2.6.16 (Stalk of inverse image). *Let $f \colon X \to Y$ be a continuous map of topological spaces, $\mathcal{G}$ a sheaf on Y, and $x \in X$ a point. There is a canonical identification*

$$(f^{-1} \mathcal{G})_x = \mathcal{G}_{f(x)}.$$

Proof. We have

$$\begin{aligned}
(f^{-1}\mathcal{G})_x &= (\{x\} \hookrightarrow X)^{-1}(f^{-1}\mathcal{G}) && \text{by Example 2.6.9} \\
&= (\{x\} \hookrightarrow X \to Y)^{-1}\mathcal{G} && \text{by Exercise 2.6.13} \\
&= (\{f(x)\} \hookrightarrow Y)^{-1}\mathcal{G} \\
&= \mathcal{G}_{f(x)}
\end{aligned}$$

where we have used Example 2.6.9 once more for the last identity. □

LEMMA 2.6.17. *Let $f : X \hookrightarrow Y$ be the inclusion of a closed subspace, $\mathcal{G}$ a sheaf on Y such that* $\operatorname{Supp}(\mathcal{G}) \subset X$ *(cf. Section 2.3.1). Then, the unit map $\mathcal{G} \to f_* f^{-1}\mathcal{G}$ is an isomorphism.*

Proof. It is enough to verify that $\mathsf{unit}_y : \mathcal{G}_y \to (f_* f^{-1}\mathcal{G})_y$ is an isomorphism for every $y \in Y$. We identify X with its image in Y, as usual. If $y \notin X$, then $\mathcal{G}_y = 0$ maps to

$$\begin{aligned}
(f_* f^{-1}\mathcal{G})_y &= \varinjlim_{V \ni y} (f_* f^{-1}\mathcal{G})(V) \\
&= \varinjlim_{V \ni y} f^{-1}\mathcal{G}(f^{-1}V) \\
&= 0,
\end{aligned}$$

where the vanishing is caused by $y \notin V \cap X = f^{-1}V$. Let us then assume $y \in \operatorname{Supp}(\mathcal{G})$, so that $\mathcal{G}_y \neq 0$. In this case,

$$\mathsf{unit}_y : \mathcal{G}_y = (f^{-1}\mathcal{G})_y \xrightarrow{\sim} (f_* f^{-1}\mathcal{G})_y$$

is the inverse of the canonical map of Lemma 2.6.4. □

Exercise 2.6.18. Find examples of maps f and sheaves $\mathcal{G}$ such that $\mathcal{G} \to f_* f^{-1}\mathcal{G}$ is not an isomorphism.

Remark 2.6.19. If $j : X \hookrightarrow Y$ is *open* and $\mathcal{G}$ is a sheaf on Y, then $j_* j^{-1}\mathcal{G}$ satisfies

$$j_* j^{-1}\mathcal{G}(V) = (j_*\mathcal{G}|_X)(V) = \mathcal{G}(V \cap X), \qquad V \subset Y \text{ open.}$$

The natural map $\mathcal{G}(V) \to j_* j^{-1}\mathcal{G}(V)$ sends $s \mapsto s|_{V\cap X}$.

Exercise 2.6.20. Show that if $f : X \hookrightarrow Y$ is the inclusion of a subspace, then the counit

$$f^{-1} f_* \mathcal{F} \to \mathcal{F}$$

is an isomorphism for every $\mathcal{F} \in \operatorname{Sh}(X, \operatorname{Mod}_A)$.

PROPOSITION 2.6.21. *Let $\mathscr{C} = \operatorname{Mod}_A$, for a ring A. Then the inverse image functor f^{-1} is exact.*

Proof. Indeed, let

$$0 \to \mathcal{G} \to \mathcal{H} \to \mathcal{K} \to 0$$

be an exact sequence in $\mathrm{Sh}(Y, \mathrm{Mod}_A)$. Then,

$$0 \to \mathcal{G}_{f(x)} \to \mathcal{H}_{f(x)} \to \mathcal{K}_{f(x)} \to 0$$

is exact in Mod_A by Proposition 2.4.14, for every $x \in X$. But this is precisely the sequence

$$0 \to (f^{-1}\mathcal{G})_x \to (f^{-1}\mathcal{H})_x \to (f^{-1}\mathcal{K})_x \to 0.$$

Thus

$$0 \to f^{-1}\mathcal{G} \to f^{-1}\mathcal{H} \to f^{-1}\mathcal{K} \to 0$$

is exact, again by Proposition 2.4.14. □

A | Commutative algebra

A.1 Universal constructions

A.1.1 Limits and colimits

Let $\mathscr{C}$ be a category, $\mathbf{I}$ a small category. Define an $\mathbf{I}$-diagram to be just a functor $M : \mathbf{I} \to \mathscr{C}$. Denote by M_i the object of $\mathscr{C}$ image of the object $i \in \mathbf{I}$ via M. If $f : i \to j$ is an arrow in $\mathbf{I}$, the induced arrow in $\mathscr{C}$ is denoted $M(f) \colon M_i \to M_j$.

Definition A.1.1 (Limit)**.** A *limit* of an $\mathbf{I}$-diagram $M : \mathbf{I} \to \mathscr{C}$ is an object $\varprojlim_{i\in\mathbf{I}} M_i$ of $\mathscr{C}$ along with an arrow $p_i \colon \varprojlim_{i\in\mathbf{I}} M_i \to M_i$ for every $i \in \mathbf{I}$, such that for every arrow $f : i \to j$ in $\mathbf{I}$ one has $p_j = M(f) \circ p_i$, and satisfying the following universal property: given an object P along with morphisms $\pi_i \colon P \to M_i$ such that $\pi_j = M(f) \circ \pi_i$ for every $f : i \to j$ in $\mathbf{I}$, there exists a unique arrow $\alpha \colon P \to \varprojlim_{i\in\mathbf{I}} M_i$ such that $\pi_i = p_i \circ \alpha$ for all $i \in \mathbf{I}$.

$$\begin{tikzcd}
P \arrow[d, dotted, "\exists!\alpha"'] \arrow[dr, "\pi_i"] \arrow[drr, bend left, "\pi_j"] & & \\
\varprojlim_{i\in\mathbf{I}} M_i \arrow[r, "p_i"] \arrow[rr, bend right, "p_j"'] & M_i \arrow[r, "M(f)"] & M_j
\end{tikzcd}$$

Exercise A.1.2. The limit over the empty diagram satisfies the universal property of a final object of $\mathscr{C}$.

Example A.1.3 (Products are limits)**.** Let $\mathbf{I}$ be the category with two objects i, j and no morphisms between them. Then an $\mathbf{I}$-diagram $M : \mathbf{I} \to \mathscr{C}$ is just the choice of two objects M_i, M_j of $\mathscr{C}$. The limit of M satisfies the universal property of the product $M_i \times M_j$.

$$\begin{tikzcd}
P \arrow[dr, dotted, "\alpha"] \arrow[drr, bend left, "\pi_i"] \arrow[ddr, bend right, "\pi_j"'] & & \\
& M_i \times M_j \arrow[r, "p_i"] \arrow[d, "p_j"] & M_i \\
& M_j &
\end{tikzcd}$$

A special case is $\mathbf{I} = \emptyset$, which recovers the universal property of a final object in $\mathscr{C}$. In other words, the product over the empty diagram is a final object.

Example A.1.4 (Equalisers are limits)**.** Let $\mathbf{I}$ be the category with two objects i, j and two arrows $i \rightrightarrows j$. Then an $\mathbf{I}$-diagram $M : \mathbf{I} \to \mathscr{C}$ is just the choice of two parallel arrows $\phi, \psi : M_i \rightrightarrows M_j$ in $\mathscr{C}$. The limit of M satisfies the universal property of the equaliser of (ϕ, ψ).

Example A.1.5 (Kernels are limits)**.** This is because kernels are equalisers (in the previous example take $\psi = 0$).

Definition A.1.6 (Colimit)**.** A *colimit* of an $\mathbf{I}$-diagram $M : \mathbf{I} \to \mathscr{C}$ is an object $\varinjlim_{i \in \mathbf{I}} M_i$ of $\mathscr{C}$ along with an arrow $s_i : M_i \to \varinjlim_{i \in \mathbf{I}} M_i$ for every $i \in \mathbf{I}$, such that for every arrow $f : i \to j$ in $\mathbf{I}$ one has $s_i = s_j \circ M(f)$, and satisfying the following universal property: given an object P along with morphisms $\sigma_i : M_i \to P$ such that $\sigma_i = \sigma_j \circ M(f)$ for every $f : i \to j$ in $\mathbf{I}$, there exists a unique arrow $\alpha : \varinjlim_{i \in \mathbf{I}} M_i \to P$ such that $\sigma_i = \alpha \circ s_i$ for all $i \in \mathbf{I}$.

$$
\begin{tikzcd}
P & & \\
\varinjlim_{i \in \mathbf{I}} M_i \arrow[u, dotted, "\exists!\alpha"] & M_j \arrow[l, "s_j"'] \arrow[ul, "\sigma_j"'] & M_i \arrow[l, "M(f)"'] \arrow[ll, bend left, "s_i"] \arrow[ull, bend right, "\sigma_i"']
\end{tikzcd}
$$

Exercise A.1.7. The colimit over the empty diagram satisfies the universal property of an initial object of $\mathscr{C}$.

Exercise A.1.8. Convince yourself that coproducts, coequalisers and cokernels are examples of colimits, along the same lines of Examples A.1.3, A.1.4 and A.1.5.

Definition A.1.9 (Filtered category)**.** A nonempty category $\mathbf{I}$ is *filtered* if for every two objects $i, j \in \mathbf{I}$ the following are true:

- there exists $k \in \mathbf{I}$ and morphisms $i \to k$ and $j \to k$, and
- for any two morphisms $f, g \in \operatorname{Hom}_{\mathbf{I}}(i, j)$ there exists an object $k \in \mathbf{I}$ along with a morphism $h : j \to k$ such that $h \circ f = h \circ g$ in $\operatorname{Hom}_{\mathbf{I}}(i, k)$.

A colimit of an $\mathbf{I}$-diagram $M : \mathbf{I} \to \mathscr{C}$ where $\mathbf{I}$ is a filtered category is a *filtered colimit*.

In the definition of stalk of a presheaf $\mathcal{F} \in \operatorname{pSh}(X, \mathscr{C})$ at a point $x \in X$, we have been taking

$$
\begin{aligned}
\mathbf{I} &= \{\, U \in \tau_X \mid x \in U \,\}^{\mathrm{op}} \\
M(U) &= \mathcal{F}(U).
\end{aligned}
$$

A.2 Localisation

Let A be a ring, M an A-module. Fix a *multiplicative subset* $S \subset A$, i.e. a subset containing the identity $1 \in A$ and such that $s_1 s_2 \in S$ whenever $s_1, s_2 \in S$.

Example A.2.1. The following are key examples of multiplicative subsets:

(i) $S = \{1, f, f^2, \dots\}$ for some $f \in A$.

(ii) $S = A \setminus \mathfrak{p}$, where $\mathfrak{p} \subset A$ is a prime ideal.

(iii) $S = A \setminus 0$, if A is an integral domain.

Consider the equivalence relation on $M \times S$ defined by

$$(m, s) \sim (m', s') \Longleftrightarrow \text{there exists } u \in S \text{ such that } u(s'm - sm') = 0 \in M.$$

We denote by m/s, or by $\frac{m}{s}$, the equivalence class of (m, s). The set

$$S^{-1}M = (M \times S)/\sim \tag{A.2.1}$$

is an abelian group via

$$\frac{m}{s} + \frac{m'}{s'} = \frac{sm' + s'm}{ss'},$$

and if $M = A$ then $S^{-1}A$ becomes a ring via

$$\frac{a}{s} \cdot \frac{a'}{s'} = \frac{aa'}{ss'}.$$

The $\mathbb{Z}$-module $S^{-1}M$ is an $S^{-1}A$-module via

$$\frac{a}{s} \cdot \frac{m}{s'} = \frac{am}{ss'}.$$

Example A.2.2. In A_f, the equivalence relation defining the localisation reads

$$\frac{a}{f^n} = \frac{b}{f^m} \Longleftrightarrow \text{there exists } k \geq 0 \text{ such that } f^k(af^m - bf^n) = 0 \in A.$$

Definition A.2.3. The localisation of M with respect to S is the $S^{-1}A$-module $S^{-1}M$.

Remark A.2.4. If $0 \in S$, then $S^{-1}M = 0$.

Notation A.2.5. If $S = \{1, f, f^2, \dots\}$ as in Example A.2.1 (i) above, then we write M_f for the localisation. If $S = A \setminus \mathfrak{p}$ as in Example A.2.1 (ii) above, then we write $M_{\mathfrak{p}}$ for the localisation.

Remark A.2.6. If $f \in A$ is nilpotent, then $A_f = 0$ by Remark A.2.4.

Set $M = A$. There is a canonical ring homomorphism

$$\ell\colon A \to S^{-1}A, \quad a \mapsto \frac{a}{1}$$

sending S inside the group of invertible elements of $S^{-1}A$ (the inverse of $s/1$ being $1/s$), and making the pair $(S^{-1}A, \ell)$ universal with this property: whenever one has a ring homomorphism $\phi\colon A \to B$ such that $\phi(S) \subset B^\times$, there is exactly one ring homomorphism $p\colon S^{-1}A \to B$ such that $\phi = p \circ \ell$.

$$\begin{tikzcd} A \arrow[r, "\ell"] \arrow[d, "\phi"'] & S^{-1}A \arrow[dl, "p"] \\ B & \end{tikzcd}$$

Explicitly, the map p is defined by $p(a/s) = \phi(a)\phi(s)^{-1}$.

The following lemma is of key importance to us.

LEMMA A.2.7. *Sending* $\mathfrak{q} \mapsto \ell^{-1}(\mathfrak{q})$ *establishes a bijection*

$$\begin{tikzcd} \{\textit{prime ideals}\ \mathfrak{q} \subset S^{-1}A\} \arrow[r, "\simeq"] \arrow[d, equal] & \{\textit{prime ideals}\ \mathfrak{p} \subset A\ \textit{such that}\ \mathfrak{p} \cap S = \emptyset\} \arrow[d, hook] \\ \operatorname{Spec} S^{-1}A & \operatorname{Spec} A \end{tikzcd}$$

having as inverse the extension operation, sending by definition $\mathfrak{p} \mapsto \mathfrak{p} \cdot S^{-1}A$.

LEMMA A.2.8. *Let A be a ring, $S \subset A$ a multiplicative subset containing no zero divisors. Then $\ell\colon A \to S^{-1}A$ is injective.*

Proof. Suppose $a/1 = 0/1$ in $S^{-1}A$. Then there is $u \in S$ such that $au = 0$. But u is not a zero divisor, thus $a = 0$. □

Example A.2.9. Let A be an integral domain, which means that $(0) \subset A$ is prime. Then the localisation

$$A_{(0)} = \left\{ \frac{a}{b} \,\middle|\, a \in A,\ b \in A \setminus 0 \right\}$$

is a field, called the *fraction field* of A, that we denote by $\mathrm{Frac}(A)$. The canonical map $\ell\colon A \to \mathrm{Frac}(A)$ is injective by Lemma A.2.8.

LEMMA A.2.10. *If A is reduced and $S \subset A$ is a multiplicative subset, then $S^{-1}A$ is also reduced.*

Proof. Assume there exists $a \in A$, $s \in S$ and $r \in \mathbb{Z}_{>0}$ such that $0 = (a/s)^r = a^r/s^r$. Then $a^r = 1 \cdot a^r = 0 \cdot s^r = 0$, thus $a = 0$. □

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