THE EQUIVARIANT ATIYAH CLASS

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ABSTRACT. Let X be a complex scheme acted on by an affine algebraic group G. We prove that the Atiyah class of a G-equivariant perfect complex on X, as constructed by Huybrechts and Thomas, is naturally G-equivariant in a precise sense. As an application, we show that the obstruction theory on the fine relative moduli space M/B of simple perfect complexes on a G-invariant smooth projective family Y/B (with G reductive) is G-equivariant. The results contained here are meant to suggest how to prove the equivariance of the natural obstruction theories on a wide variety of moduli spaces equipped with a torus action, arising in Donaldson–Thomas theory and Vafa–Witten theory.

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0. INTRODUCTION

Overview. The *Atiyah class* of a vector bundle V on a complex algebraic variety X, introduced in [1], is an extension class

$$\mathrm{At}_V \in \mathrm{Ext}^1_X(V, V \otimes \Omega_X)$$

whose vanishing is equivalent to the existence of an algebraic connection on V. The Atiyah class can be defined for every complex of sheaves on a scheme [14].

Let G be an algebraic group acting on a scheme X. The goal of this work is to make sense of, and prove, a rigorous version of the following slogan:

The Atiyah class of a G-equivariant perfect complex on X is G-equivariant.

Our main motivation is from Enumerative Geometry, in particular from the rich theory of moduli spaces of sheaves (or complexes) on algebraic 3-folds, such as Donaldson–Thomas theory, Pandharipande–Thomas theory, Vafa–Witten theory: the Atiyah class is a crucial ingredient in the construction of the *obstruction theory* [3, 19] on these moduli spaces. A powerful tool to compute the enumerative invariants defined via these obstruction theories is the technique of *virtual localisation* introduced in the context of torus actions by Graber and Pandharipande [7]. Their localisation formula requires as input an *equivariant* obstruction theory.

After confirming an equivariant version of *Verdier duality*, we show in a very general example that the equivariance of the Atiyah class ensures that the obstruction theory it induces is equivariant, at least for reductive groups; so, in the case of a torus action, the virtual localisation formula can be applied.

By the above slogan, in a typical situation a 'working mathematician' would only have to verify the equivariance of the universal sheaf (or complex) on the moduli space under consideration in order to apply the localisation theorem. In fact, in Section 4 we prove equivariance of the universal complex on the moduli space of simple perfect complexes on a smooth projective family. This suffices to establish equivariance of the associated obstruction theory.

Most sheaf-theoretic moduli problems should behave in a way that is entirely parallel to the one discussed here.

Main result. Let X be a separated noetherian scheme over $\mathbb{C}$, and let QCoh_X be the abelian category of quasi-coherent $\mathscr{O}_X$-modules. Let $E \in \operatorname{Perf} X \subset \mathbf{D}(\mathrm{QCoh}_X)$ be a perfect complex. Assuming X admits a closed embedding in a smooth variety, Huybrechts and Thomas defined the *truncated Atiyah class* of E as an element

$$\mathrm{At}_E \in \mathrm{Ext}^1_X(E, E \otimes \mathbb{L}_X),$$

where $\mathbb{L}_X \in \mathbf{D}^{[-1,0]}(\mathrm{QCoh}_X)$ is the truncated cotangent complex. If X carries an action of an algebraic group G, one can form the abelian category QCoh_X^G of G-equivariant quasi-coherent sheaves on X. Its derived category admits an exact functor $\Phi\colon \mathbf{D}(\mathrm{QCoh}_X^G) \to \mathbf{D}(\mathrm{QCoh}_X)$ forgetting the equivariant structure. We say that At_E is *G-equivariant* if the corresponding morphism $E \to E \otimes \mathbb{L}_X[1]$ in $\mathbf{D}(\mathrm{QCoh}_X)$ can be lifted to $\mathbf{D}(\mathrm{QCoh}_X^G)$ along Φ.

Our main result is the following.

Theorem A (Theorem 3.3)**.** *Let G be an affine algebraic group acting on a scheme X admitting a G-equivariant embedding in a smooth G-scheme. Fix a perfect complex $E \in \operatorname{Perf} X$. Then every lift of E to $\mathbf{D}(\mathrm{QCoh}_X^G)$ makes At_E canonically G-equivariant.*

Application to equivariant obstruction theories. As we briefly recall below, the Atiyah class is the main ingredient in the construction of an *obstruction theory* on moduli spaces of simple perfect complexes on a smooth projective family Y/B, see [12, Thm. 4.1]. An obstruction theory [3, Def. 4.4] on a scheme X is a morphism $\phi\colon \mathbb{E} \to \mathbb{L}_X$ in $\mathbf{D}(\mathrm{QCoh}_X)$ such that $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is a surjection. See [3, Sec. 7] for a relative version.

In case X is acted on by an algebraic group G, the complex $\mathbb{L}_X$ has a canonical lift to $\mathbf{D}(\mathrm{QCoh}_X^G)$ (see Section 3.1.1), and one has the following notion.

Definition 0.1 ([7, 4])**.** Let G be an algebraic group acting on X. Let $\phi\colon \mathbb{E} \to \mathbb{L}_X$ be an obstruction theory. Then ϕ is a *G-equivariant obstruction theory* if it can be lifted to a morphism in $\mathbf{D}(\mathrm{QCoh}_X^G)$.

To make a statement about equivariant obstruction theories, one needs to get a handle of Hom-sets in $\mathbf{D}(\mathrm{QCoh}_X^G)$. For this, we restrict to G reductive (in order to exploit a technical result, Lemma 2.21). For instance, the theory works for a torus $G = \mathbb{T} = \mathbb{G}_m^n$, which includes most applications we have in mind.

Here is the precise statement of our second main result.

Theorem B (Theorem 4.3)**.** *Let G be an affine reductive algebraic group. Let Y/B be a G-invariant smooth projective family of varieties. Let M/B be a fine relative separated moduli space of simple perfect complexes on the fibres of Y/B. Then the associated relative obstruction theory on M/B is naturally G-equivariant.*

We refer to [12, Sec. 4.1] (or our Section 4.1) for the precise assumptions on M/B.

We briefly outline here the role of the Atiyah class and of Verdier duality in the construction of the relative obstruction theory on M/B. The Atiyah class one has to consider is

$$\mathrm{At}_{E/Y} \in \mathrm{Ext}^1_X(E, E \otimes \pi_M^* \mathbb{L}_{M/B}), \quad X = Y \times_B M \xrightarrow{\pi_M} M,$$

where $E \in \mathrm{Perf}(Y \times_B M)$ is the universal perfect complex. Via the distinguished triangle

$$\mathbf{R}\mathscr{H}\mathrm{om}_X(E,E)_0 \to \mathbf{R}\mathscr{H}\mathrm{om}_X(E,E) \xrightarrow{\text{trace}} \mathscr{O}_X,$$

the Atiyah class projects onto an element of

$$\mathrm{Ext}^1_X(\mathbf{R}\mathscr{H}\mathrm{om}_X(E,E)_0, \pi_M^*\mathbb{L}_{M/B}) \cong \mathrm{Ext}^{1-d}_M(\mathbf{R}\pi_{M*}(\mathbf{R}\mathscr{H}\mathrm{om}_X(E,E)_0 \otimes \omega_{\pi_M}), \mathbb{L}_{M/B}), \tag{0.1}$$

where d is the relative dimension of Y/B and the isomorphism is given by Verdier duality. The image of $\mathrm{At}_{E/Y}$ along this journey is a morphism

$$\phi\colon \mathbf{R}\pi_{M*}(\mathbf{R}\mathscr{H}\mathrm{om}_X(E,E)_0 \otimes \omega_{\pi_M})[d-1] \to \mathbb{L}_{M/B},$$

and it is shown in [12, Thm. 4.1] that ϕ is a relative obstruction theory in the sense of [3, Sec. 7]. The following strategy will prove Theorem B:

(1) E is equivariant (Proposition 4.2).
(2) $\mathrm{At}_{E/Y}$ is G-invariant.
(3) The Verdier duality isomorphism (0.1) is G-equivariant (Corollary 2.28).
(4) ϕ lies in the G-invariant part of the right hand side of (0.1).
(5) G-invariant extensions correspond to morphisms in $\mathbf{D}(\mathrm{QCoh}^G_M)$.

Items (2) and (5) follow by reductivity of G.

Conventions. All schemes are noetherian and separated over $\mathbb{C}$. By an *algebraic group G* we mean a connected group scheme of finite type over $\mathbb{C}$ (often affine). We follow Olsson [25, Ch. 8] for conventions on algebraic stacks (in particular, we make no separation assumptions, just as in [30]). For an algebraic stack $\mathscr{X}$, we denote by $\mathrm{QCoh}_{\mathscr{X}}$ the abelian category of quasi-coherent sheaves on the lisse-étale site of $\mathscr{X}$ [25, Ch. 9], and $\mathbf{D}(\mathscr{X})$ will denote the unbounded derived category of the abelian category $\mathrm{Mod}_{\mathscr{O}_{\mathscr{X}}}$ of all $\mathscr{O}_{\mathscr{X}}$-modules.

1. Equivariant sheaves and complexes

1.1. **The category of equivariant sheaves.** Let X be a noetherian scheme over $\mathbb{C}$,[1] equipped with an action $\sigma\colon G \times X \to X$ of a group scheme G. We call such a pair (X,σ) a *G-scheme*. The abelian category $\mathrm{Mod}_{\mathscr{O}_X}$ of $\mathscr{O}_X$-modules contains the abelian subcategories QCoh_X (resp. Coh_X) of quasi-coherent (resp. coherent) $\mathscr{O}_X$-modules. We will mostly focus on QCoh_X in this paper.

[1] The theory works relatively to a fixed base scheme B (see Remark 1.5). This requires all relative operations (such as fibre products) to be performed over B, as well as the requirement that $G \to B$ be flat (this is needed e.g. in the construction of f_* and $\mathscr{H}\mathrm{om}_X$).

Denoting by $\mathsf{m}\colon G\times G\to G$ the group law of G, there is a commutative diagram

$$\begin{tikzcd} G\times G\times X \arrow[r, "\mathsf{m}\times\mathrm{id}_X"] \arrow[d, "\mathrm{id}_G\times\sigma"'] & G\times X \arrow[d, "\sigma"] \\ G\times X \arrow[r, "\sigma"] & X \end{tikzcd} \tag{1.1}$$

translating the condition $g\cdot(h\cdot x)=(gh)\cdot x$.

Let $p_i\colon G\times X\to X$ and $p_{ij}\colon G\times G\times X\to G\times X$ denote the projections onto the labeled factors.

Definition 1.1. A *G-equivariant quasi-coherent sheaf* on X is a pair $(\mathscr{F},\vartheta)$ where $\mathscr{F}\in\mathrm{QCoh}_X$ and $\vartheta\colon p_2^*\mathscr{F}\xrightarrow{\sim}\sigma^*\mathscr{F}$ is an isomorphism in $\mathrm{QCoh}_{G\times X}$ compatible with the diagram (1.1). In other words, ϑ is required to satisfy the cocycle condition

$$(\mathsf{m}\times\mathrm{id}_X)^*\vartheta=(\mathrm{id}_G\times\sigma)^*\vartheta\circ p_{23}^*\vartheta.$$

The isomorphism ϑ is called a *G-equivariant structure*.

The same definition can be given for objects $\mathscr{F}\in\mathrm{Mod}_{\mathscr{O}_X}$ as well as $\mathscr{F}\in\mathrm{Coh}_X$.

Explicitly, the cocycle condition means that the diagram of isomorphisms

$$\begin{tikzcd} (\mathsf{m}\times\mathrm{id}_X)^*p_2^*\mathscr{F} \arrow[rr, "(\mathsf{m}\times\mathrm{id}_X)^*\vartheta"] \arrow[d, equal] & & (\mathsf{m}\times\mathrm{id}_X)^*\sigma^*\mathscr{F} \arrow[d, equal] \\ p_{23}^*p_2^*\mathscr{F} \arrow[d, "p_{23}^*\vartheta"] & & (\mathrm{id}_G\times\sigma)^*\sigma^*\mathscr{F} \\ p_{23}^*\sigma^*\mathscr{F} \arrow[rr, equal] & & (\mathrm{id}_G\times\sigma)^*p_2^*\mathscr{F} \arrow[u, "(\mathrm{id}_G\times\sigma)^*\vartheta"'] \end{tikzcd}$$

commutes in $\mathrm{QCoh}_{G\times G\times X}$.

Remark 1.2. Let (X,σ) be a G-scheme. Then σ is flat. Indeed, it agrees with the composition

$$G\times X\xrightarrow{\gamma}G\times X\xrightarrow{p_2}X$$

where γ is the automorphism $(g,x)\mapsto(g,\sigma(g,x))$, having $(g,x)\mapsto(g,\sigma(g^{-1},x))$ as inverse.

Definition 1.3. A morphism $(\mathscr{F},\vartheta)\to(\mathscr{F}',\vartheta')$ of G-equivariant quasi-coherent sheaves is a morphism $\phi\colon\mathscr{F}\to\mathscr{F}'$ in QCoh_X such that the diagram

$$\begin{tikzcd} p_2^*\mathscr{F} \arrow[r, "p_2^*\phi"] \arrow[d, "\vartheta"'] & p_2^*\mathscr{F}' \arrow[d, "\vartheta'"] \\ \sigma^*\mathscr{F} \arrow[r, "\sigma^*\phi"] & \sigma^*\mathscr{F}' \end{tikzcd} \tag{1.2}$$

commutes in $\mathrm{QCoh}_{G\times X}$.

Notation 1.4. Let $\mathcal{C}$ be any of the categories $\mathrm{Mod}_{\mathscr{O}_X}$, QCoh_X or Coh_X. We let $\mathcal{C}^G$ denote the corresponding category of G-equivariant sheaves $(\mathscr{F},\vartheta)$ where $\mathscr{F}\in\mathcal{C}$. We mainly focus on $\mathcal{C}=\mathrm{QCoh}_X$. The category QCoh_X^G is a $\mathbb{C}$-linear abelian category (see also Lemma 2.12 for a stronger statement). Its (unbounded) derived category will be denoted $\mathbf{D}(\mathrm{QCoh}_X^G)$.

Every object $(\mathscr{F},\vartheta)\in \mathrm{QCoh}_X^G$ comes with a collection of isomorphisms

$$\vartheta_g\colon \mathscr{F} \xrightarrow{\sim} \rho_g^*\mathscr{F}, \quad g\in G,$$

satisfying $\vartheta_{hg}=\rho_g^*\vartheta_h\circ\vartheta_g$, where ρ_g is the composition

$$\rho_g\colon X \xrightarrow{\sim} \{g\}\times X \hookrightarrow G\times X \xrightarrow{\sigma} X.$$

The map ϑ_g is obtained by restricting ϑ along $X \xrightarrow{\sim} \{g\}\times X \subset G\times X$.

Remark 1.5 (Relative version)**.** In general, when working with a group scheme G/B acting on a scheme X/B, where B is a base scheme, a G-equivariant sheaf $(\mathscr{F},\vartheta)$ can be described in the following equivalent fashion. Some notation first. For every B-scheme T, set $X_T = T\times_B X = T\times_T X_T$ and let $\mathscr{F}_T$ denote the pullback of $\mathscr{F}$ along the projection $X_T\to X$. For every T-valued point $g\colon T\to G_T = T\times_B G$ of G one has an isomorphism

$$\rho_g\colon X_T \xrightarrow{g\times \mathrm{id}_{X_T}} G_T\times_T X_T \xrightarrow{\sigma_T} X_T, \quad (t,x)\mapsto (t,g(t)\cdot x).$$

The condition '$\mathscr{F}$ is G-equivariant' is equivalent to the following condition: for every T-valued point $g\in G_T(T)$ as above there is an isomorphism $\vartheta_g\colon \mathscr{F}_T \xrightarrow{\sim} \rho_g^*\mathscr{F}_T$ such that for every pair of T-valued points $g,\,h\in G_T(T)$ one has a commutative diagram of isomorphisms

$$\begin{tikzcd}
\rho_g^*\rho_h^*\mathscr{F}_T \arrow[d, equal] & \rho_g^*\mathscr{F}_T \arrow[l, "\rho_g^*\vartheta_h"'] \\
\rho_{hg}^*\mathscr{F}_T & \mathscr{F}_T \arrow[l, "\vartheta_{hg}"'] \arrow[u, "\vartheta_g"']
\end{tikzcd}$$

in QCoh_{X_T}. The correspondence between this condition and the one in Definition 1.1 is given by setting $\vartheta_g = (g\times \mathrm{id}_X)^*\vartheta$, which indeed realises an isomorphism

$$\mathscr{F}_T = (g\times\mathrm{id}_X)^*p_2^*\mathscr{F} \xrightarrow{\sim} (g\times \mathrm{id}_X)^*\sigma^*\mathscr{F} = \rho_g^*\mathscr{F}_T.$$

Example 1.6. Let (X,σ) be a G-scheme over a scheme B. Then the structure sheaf $\mathscr{O}_X$ is G-equivariant in a natural way. For a B-scheme T, set $X_T = T\times_B X$. Then the inverse of the natural isomorphisms $\rho_g^*\mathscr{O}_{X_T} \xrightarrow{\sim} \rho_g^*\rho_{g*}\mathscr{O}_{X_T} \xrightarrow{\sim} \mathscr{O}_{X_T}$ is a G-equivariant structure on $\mathscr{O}_X$.

Example 1.7. Let (X,σ) be a G-scheme over a scheme B. Then the sheaf $\Omega_{X/B}$ of relative differentials is G-equivariant in a natural way. Indeed, for an B-scheme T, consider the natural isomorphisms $\alpha_T\colon (\Omega_{X/B})_T \xrightarrow{\sim} \Omega_{X_T/T}$ and $\ell_g\colon \rho_g^*\Omega_{X_T/T}\xrightarrow{\sim}\Omega_{X_T/T}$. Then the composition

$$(\Omega_{X/B})_T \xrightarrow{\alpha_T} \Omega_{X_T/T} \xrightarrow{\ell_g^{-1}} \rho_g^*\Omega_{X_T/T} \xrightarrow{\rho_g^*\alpha_T^{-1}} \rho_g^*(\Omega_{X/B})_T$$

defines an equivariant structure on $\Omega_{X/B}$.

Notation. For an object $(\mathscr{F},\vartheta)\in \mathrm{QCoh}_X^G$, we will often somewhat sloppily omit the G-equivariant structure 'ϑ' from the notation. We will also write Hom_X instead of $\mathrm{Hom}_{\mathrm{QCoh}_X}$ or $\mathrm{Hom}_{\mathbf{D}(\mathrm{QCoh}_X)}$, and shorten $g^* = \rho_g^*$.

Remark 1.8. If $(\mathscr{F},\vartheta_{\mathscr{F}}),(\mathscr{F}',\vartheta_{\mathscr{F}'})\in\mathrm{QCoh}_X^G$, the $\mathbb{C}$-vector space

$$\mathrm{Hom}_X(\mathscr{F},\mathscr{F}')$$

is naturally a G-representation. Indeed, for a morphism $\phi\colon \mathscr{F} \to \mathscr{F}'$ in QCoh_X, one defines $g\cdot\phi$ by means of the composition

$$\begin{tikzcd} \mathscr{F} \arrow[r, dashed, "g\cdot\phi"] \arrow[d, "\vartheta_{\mathscr{F},g}"'] & \mathscr{F}' \\ g^*\mathscr{F} \arrow[r, "g^*\phi"] & g^*\mathscr{F}' \arrow[u, "\vartheta^{-1}_{\mathscr{F}',g}"'] \end{tikzcd} \tag{1.3}$$

exploiting the invertibility of $\vartheta_{\mathscr{F}',g}$. The structure of G-representation on $\mathrm{Hom}_X(\mathscr{F},\mathscr{F}')$ clearly depends on the chosen equivariant structures $\vartheta_{\mathscr{F}}$ and $\vartheta_{\mathscr{F}'}$.

Remark 1.9. It is immediate to see that, in QCoh_X^G, the morphisms are the G-invariant morphisms between the underlying quasi-coherent sheaves. In symbols,

$$\mathrm{Hom}_{\mathrm{QCoh}_X^G}((\mathscr{F},\vartheta_{\mathscr{F}}),(\mathscr{F}',\vartheta_{\mathscr{F}'})) = \mathrm{Hom}_X(\mathscr{F},\mathscr{F}')^G. \tag{1.4}$$

Indeed, the diagram (1.2) becomes precisely

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\phi"] \arrow[d, "\vartheta_{\mathscr{F},g}"'] & \mathscr{F}' \arrow[d, "\vartheta_{\mathscr{F}',g}"] \\ g^*\mathscr{F} \arrow[r, "g^*\phi"] & g^*\mathscr{F}' \end{tikzcd}$$

when restricted to $\{g\}\times X \cong X$. Again, in the right hand side of (1.4) the 'G-invariant part' depends on the G-structure on $\mathrm{Hom}_X(\mathscr{F},\mathscr{F}')$, which in turn is determined by the pair $(\vartheta_{\mathscr{F}},\vartheta_{\mathscr{F}'})$.

The following result is classical, and is key to our paper. It is proved in [18, Ex. 12.4.6], but see also [25, Exercise 9.H].

Theorem 1.10. *Let G be a smooth group scheme, X a G-scheme. There is a natural equivalence*

$$\mathrm{QCoh}_X^G \cong \mathrm{QCoh}_{[X/G]}.$$

Example 1.11. Let $X = \operatorname{Spec} k$, for a field k. Then $\mathrm{QCoh}^G_{\operatorname{Spec} k}$ is equivalent to the category $\mathrm{Rep}_k(G)$ of locally finite linear representations of G.

1.2. **Forgetful functor.** There is an exact functor $\Phi\colon \mathbf{D}(\mathrm{QCoh}_X^G) \to \mathbf{D}(\mathrm{QCoh}_X)$ that forgets the equivariant structure. This results in a commutative diagram

$$\begin{tikzcd} \mathrm{QCoh}_X^G \arrow[r, "\Phi"] \arrow[d, hook] & \mathrm{QCoh}_X \arrow[d, hook] \\ \mathbf{D}(\mathrm{QCoh}_X^G) \arrow[r, "\Phi"] & \mathbf{D}(\mathrm{QCoh}_X) \end{tikzcd}$$

where the vertical arrows are the inclusions of the standard hearts. More concretely, if $\mathsf{p}\colon X \to [X/G]$ is the standard smooth atlas, we can identify Φ as the composition

$$\mathbf{D}(\mathrm{QCoh}_X^G) \xrightarrow{\sim} \mathbf{D}(\mathrm{QCoh}_{[X/G]}) \xrightarrow{\mathsf{p}^*} \mathbf{D}(\mathrm{QCoh}_X), \tag{1.5}$$

where $\mathsf{p}^* = \mathbf{L}\mathsf{p}^*$ is the pullback functor as defined in [24, Sec. 7] and the first equivalence comes from Theorem 1.10.

Remark 1.12. The forgetful functor Φ reflects exactness: a sequence in QCoh_X^G that becomes exact in QCoh_X was already exact in QCoh_X^G.

1.3. **Geometric functors.** Fix two noetherian G-schemes (X,σ_X) and (Y,σ_Y). All morphims $X \to Y$ in this subsection are assumed to be G-equivariant. Since X and Y are noetherian, pushforward preserves quasi-coherence.

Let $(\mathscr{F},\vartheta)$ and $(\mathscr{F}',\vartheta')$ be two objects of QCoh_X^G. Then there is a canonical lift

$$(\mathscr{F}\otimes\mathscr{F}',\vartheta\otimes\vartheta')\in\mathrm{QCoh}_X^G$$

of the object $\mathscr{F}\otimes\mathscr{F}'\in\mathrm{QCoh}_X$. This gives a bi-functor

$$\mathrm{QCoh}_X^G\times\mathrm{QCoh}_X^G\xrightarrow{\otimes}\mathrm{QCoh}_X^G.$$

If $f\colon X\to Y$ is a morphism of G-schemes, there is a pullback functor

$$\mathrm{QCoh}_Y^G\xrightarrow{f^*}\mathrm{QCoh}_X^G,\quad (\mathscr{E},\vartheta)\mapsto(f^*\mathscr{E},(\mathrm{id}_G\times f)^*\vartheta),$$

and (exploiting flat base change along $p_2,\sigma_Y\colon G\times Y\rightrightarrows Y$) a pushforward functor

$$\mathrm{QCoh}_X^G\xrightarrow{f_*}\mathrm{QCoh}_Y^G,\quad (\mathscr{F},\vartheta)\mapsto(f_*\mathscr{F},(\mathrm{id}_G\times f)_*\vartheta),$$

such that (f^*,f_*) is an adjoint pair.

Finally, for any two objects $(\mathscr{F},\vartheta)$ and $(\mathscr{F}',\vartheta')$ of QCoh_X^G, there is a canonical G-equivariant structure on the $\mathscr{O}_X$-module $\mathscr{H}om_X(\mathscr{F},\mathscr{F}')$,[2] given by $\mathscr{H}om_{G\times X}(\vartheta,\vartheta')$. More precisely, the isomorphism

$$\mathscr{H}om_{G\times X}(\vartheta,\vartheta')\colon \mathscr{H}om_{G\times X}(p_2^*\mathscr{F},p_2^*\mathscr{F}')\xrightarrow{\sim}\mathscr{H}om_{G\times X}(\sigma^*\mathscr{F},\sigma^*\mathscr{F}')$$

can be used to define the equivariant structure

$$\begin{tikzcd}
p_2^*\mathscr{H}om_X(\mathscr{F},\mathscr{F}') \arrow[r, dotted, "\vartheta_{\mathscr{H}om_X(\mathscr{F},\mathscr{F}')}"] \arrow[d, "\wr"] & \sigma^*\mathscr{H}om_X(\mathscr{F},\mathscr{F}') \\
\mathscr{H}om_{G\times X}(p_2^*\mathscr{F},p_2^*\mathscr{F}') \arrow[r, "\mathscr{H}om_{G\times X}(\vartheta,\vartheta')"] & \mathscr{H}om_{G\times X}(\sigma^*\mathscr{F},\sigma^*\mathscr{F}') \arrow[u, "\wr"']
\end{tikzcd}$$

where the vertical isomorphisms use the flatness of p_2 and σ. This construction defines a bi-functor

$$\mathrm{QCoh}_X^G\times\mathrm{QCoh}_X^G\xrightarrow{\mathscr{H}om_X(-,-)}\mathrm{Mod}_{\mathscr{O}_X}^G.$$

2. Equivariant derived functors and Verdier duality

2.1. **Equivariant derived categories.** Let G be an affine algebraic group over $\mathbb{C}$. In particular G is smooth by Cartier's theorem.

In this section we assume all G-schemes to be noetherian and separated over $\mathbb{C}$, so in particular all morphisms, which we always assume to be G-equivariant, are quasi-compact and separated. Moreover, we highlight the following crucial hypothesis:

(†) we assume all schemes to admit an ample family of G-equivariant line bundles.

Given a quasi-compact G-scheme X, condition (†) means that there exists a family $\{\mathscr{L}_i\}_i$ of G-equivariant line bundles such that, for every object $\mathscr{E}\in\mathrm{QCoh}_X$, the evaluation map yields a surjective morphism

$$\bigoplus_i\bigoplus_{n>0}\mathrm{H}^0(X,\mathscr{E}\otimes\mathscr{L}_i^{\otimes n})\otimes(\mathscr{L}_i^\vee)^{\otimes n}\twoheadrightarrow\mathscr{E}.$$

[2]Note that $\mathscr{H}om_X(\mathscr{F},\mathscr{F}')$ is quasi-coherent if $\mathscr{F}$ is finitely presented (i.e. coherent), but not in general.

Example 2.1. If X is a quasi-projective scheme with a linear G-action, then (†) holds. If X admits a G-equivariant embedding in a smooth scheme, then X satisfies (†). Also notice that, conversely, if X is quasi-projective and carries a G-equivariant line bundle, then X admits a G-equivariant embedding in a smooth scheme.

A few remarks on condition (†) are in order.

Remark 2.2 ([34, Rem. 1.5.4])**.** Condition (†) implies that for every $\mathscr{E} \in \mathrm{QCoh}_X^G$ there is a G-equivariant flat quasi-coherent sheaf $\mathscr{P}$ along with a surjection $q\colon \mathscr{P} \twoheadrightarrow \mathscr{E}$ in QCoh_X^G, with $(\mathscr{P}, q)$ depending functorially on $\mathscr{E}$. This is the analogue of the fact that if X is a quasi-compact quasi-separated scheme with an ample family of line bundles, then every quasi-coherent sheaf can be written as a quotient of a locally free sheaf.

Remark 2.3. By a result of Thomason [31], condition (†) implies that every G-equivariant *coherent* $\mathscr{O}_X$-module is the quotient of a G-equivariant locally free $\mathscr{O}_X$-module (this uses quasi-compactness of X). The latter condition can be rephrased by saying that the quotient stack $[X/G]$ has the *resolution property*. For this implication, we need G to be affine, as well as X to be noetherian. See also [10, Ex. 7.5 (3)] and [33, Thm. 2.1] for further references.

Remark 2.4. For a G-scheme X, Condition (†) is *not* equivalent to the resolution property for the stack $[X/G]$. In [33, Sec. 9] an example is given of a projective variety X (a nodal cubic curve) acted on by an algebraic group G (the torus $\mathbb{G}_m$), such that X does not admit a family of G-equivariant line bundles. However, the quotient stack $[X/G]$ does have the resolution property [33, Prop. 9.1].

2.2. **Quasi-coherent sheaves on quotient stacks.** For the sake of completeness, and for future reference, we record here a few properties of (quotient) stacks and their derived categories.

2.2.1. *Perfect complexes on schemes.* Let X be an arbitrary scheme.

Definition 2.5 ([32, Section 2])**.** A complex $E \in \mathbf{D}(X)$ is called *perfect* (resp. *strictly perfect*) if it is locally (resp. globally) quasi-isomorphic to a bounded complex of locally free $\mathscr{O}_X$-modules of finite type.

We let $\mathrm{Perf}\,X$ denote the category of perfect complexes on X. As long as X has an ample family of line bundles, there is no difference between perfect and strictly perfect. Since our schemes satisfy condition (†), every perfect complex will in fact be a *bounded* complex.

By [30, Tag 08DB], if X is quasi-compact and semi-separated (i.e. has affine diagonal), the canonical functor $\mathbf{D}(\mathrm{QCoh}_X) \to \mathbf{D}_{\mathsf{qc}}(X)$ is an exact equivalence. Here the decoration 'qc' means that the cohomology sheaves of our complexes lie in QCoh_X. All our schemes will be quasi-compact (in fact, noetherian) and separated, so all statements usually made about $\mathbf{D}_{\mathsf{qc}}(X)$ can, and will be rephrased here using $\mathbf{D}(\mathrm{QCoh}_X)$.

2.2.2. *Separation and noetherianity for algebraic stacks.* Let S be a scheme. Recall that a morphism of schemes $X \to S$ is *quasi-separated* if the diagonal $X \to X \times_S X$ is quasi-compact. On the other hand, an algebraic stack $\mathscr{X} \to S$ is *quasi-separated* if the diagonal $\mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$ is quasi-compact *and quasi-separated* (cf. [25, Def. 8.2.12] or [30, Tag 04YV]).

Let $G \to S$ be a smooth separated group scheme acting on an S-scheme X. An algebraic stack of the form $[X/G] \to S$ has representable, quasi-compact and separated diagonal, cf. [18, Ex. 4.6.1], therefore it is quasi-separated.

Definition 2.6. An algebraic stack is *noetherian* if it is quasi-compact, quasi-separated and admits a noetherian atlas.

For instance, if X is a noetherian scheme acted on by a smooth affine algebraic group, then $[X/G]$ is a noetherian algebraic stack. Indeed, $X \to [X/G]$ is an atlas; we already established quasi-separatedness, and quasi-compactness can be checked on an atlas [30, Tag 04YA].

2.2.3. *Compact generation for derived categories.* For an algebraic stack $\mathscr{X}$, the inclusion $\mathrm{QCoh}_{\mathscr{X}} \subset \mathrm{Mod}_{\mathscr{O}_{\mathscr{X}}}$ of the abelian category of quasi-coherent $\mathscr{O}_{\mathscr{X}}$-modules on the lisse-étale site of $\mathscr{X}$ (cf. [25, Def. 9.1.6]) inside the abelian category of all $\mathscr{O}_{\mathscr{X}}$-modules induces a canonical functor

$$\mathbf{D}(\mathrm{QCoh}_{\mathscr{X}}) \to \mathbf{D}_{\mathsf{qc}}(\mathscr{X}) \subset \mathbf{D}(\mathscr{X}).$$

We now briefly recall the notion of compact generation. It will be essential in the proof of equivariant Verdier duality (Theorem 2.25).

Definition 2.7 ([22, Def. 1.7])**.** A triangulated category $\mathcal{S}$ with small coproducts is said to be *compactly generated* if there is a set of objects $S \subset \mathcal{S}$ such that for every $s \in S$ the functor $\mathrm{Hom}_{\mathcal{S}}(s,-)$ commutes with coproducts, and if y is an object of $\mathcal{S}$ such that $\mathrm{Hom}_{\mathcal{S}}(s,y) = 0$ for all $s \in S$, then it follows that $y = 0$.

Proposition 2.8. *Let X be a noetherian scheme over $\mathbb{C}$, acted on by an affine algebraic group G and satisfying (†). The derived category $\mathbf{D}(\mathrm{QCoh}_{[X/G]})$ has small coproducts and is compactly generated.*

Proof. Set $\mathscr{X}_G = [X/G]$. First of all, $\mathbf{D}(\mathrm{QCoh}_{\mathscr{X}_G})$ has small coproducts because $\mathrm{QCoh}_{\mathscr{X}_G}$ is a Grothendieck abelian category [30, Tag 06WU].

Next, $\mathscr{X}_G$ is quasi-separated and noetherian: this was established in Section 2.2.2. Since G is affine, $\mathscr{X}_G$ has affine stabiliser groups at closed points. By [33, Prop. 1.3], a noetherian algebraic stack with affine stabiliser groups at closed points and having the resolution property, which $\mathscr{X}_G$ has by condition (†), has affine diagonal. Therefore $\mathscr{X}_G$ has affine diagonal.

Since $\mathbb{C}$ has characteristic zero and $\mathscr{X}_G$ has the resolution property, the category $\mathbf{D}_{\mathsf{qc}}(\mathscr{X}_G)$ is compactly generated by perfect complexes [10, Thm. B]. So $\mathscr{X}_G$ is a quasi-compact algebraic stack with affine diagonal and such that $\mathbf{D}_{\mathsf{qc}}(\mathscr{X}_G)$ is compactly generated: by [8, Thm. 1.2] this implies that the canonical functor $\mathbf{D}(\mathrm{QCoh}_{\mathscr{X}_G}) \to \mathbf{D}_{\mathsf{qc}}(\mathscr{X}_G)$ is an equivalence. Thus $\mathbf{D}(\mathrm{QCoh}_{\mathscr{X}_G})$ is compactly generated. $\square$

Corollary 2.9. *The category $\mathbf{D}(\mathrm{QCoh}_X^G)$ is compactly generated.*

Proof. This follows directly from Theorem 1.10. $\square$

Remark 2.10. Combining Theorem 1.10 with the proof of Proposition 2.8 shows that if X is a noetherian $\mathbb{C}$-scheme acted on by an affine algebraic group G, then we have equivalences

$$\mathbf{D}(\mathrm{QCoh}_X^G) \xrightarrow{\sim} \mathbf{D}(\mathrm{QCoh}_{[X/G]}) \xrightarrow{\sim} \mathbf{D}_{\mathsf{qc}}([X/G]). \tag{2.1}$$

The literature on derived functors for algebraic stacks usually refers to $\mathbf{D}_{\mathsf{qc}}(-)$, but given our assumptions, and because of these equivalences, we will state our results for $\mathbf{D}(\mathrm{QCoh}_X^G)$.

Example 2.11. Let G be a group scheme of finite type over a field k of characteristic zero. Then $\mathbf{D}_{\mathsf{qc}}(\mathrm{B}_k G)$ is compactly generated. Moreover, if G is affine, it is compactly generated by the irreducible k-representations of G, see [9, Thm. A]. Let $\mathrm{Rep}_k(G)$ be the abelian category of k-linear locally finite representations of G. Then there is a natural identification

$$\mathrm{QCoh}_{\mathrm{B}_k G} = \mathrm{Rep}_k(G),$$

and by [8, Thm. 1.2] the natural functor $\mathbf{D}(\mathrm{Rep}_k(G)) = \mathbf{D}(\mathrm{QCoh}_{\mathrm{B}_k G}) \to \mathbf{D}_{\mathsf{qc}}(\mathrm{B}_k G)$ is an equivalence.

2.3. **Equivariant derived functors.** In this section all schemes are noetherian, separated over $\operatorname{Spec}\mathbb{C}$, and satisfy (†). The group G is an affine complex algebraic group.

The following result lies at the foundations of the construction of equivariant versions of the geometric functors recalled in Section 1.3.

Lemma 2.12. *The category QCoh_X^G is a Grothendieck abelian category with enough injectives. Moreover, any complex of objects in QCoh_X^G has a K-injective resolution and a K-flat resolution.*

Proof. This is proved in Prop. 1.5.7 (a) and Prop. 1.5.6 (a) of [34]. The fact that QCoh_X^G is Grothendieck abelian also follows from Theorem 1.10 along with the fact that $\mathrm{QCoh}_{[X/G]}$ is Grothendieck abelian [30, Tag 06WU]. □

Remark 2.13. It is proved in [28, Thm. 3.13] that, in fact, unbounded complexes on any Grothendieck category admit K-injective resolutions.

Remark 2.14. The resolutions mentioned in Lemma 2.12 (which are carefully defined in [34, Def. 1.5.3 (c), (d)]) are precisely the G-equivariant analogues of those used by Spaltenstein to construct derived functors for unbounded derived categories in the non-equivariant case, see in particular Definitions 1.1 and 5.1 in [29].

K-flat and K-injective resolutions allow one to define equivariant derived functors, as we now recall following [34].

Proposition 2.15 ([34, Prop. 1.5.6, 1.5.7])**.** *There is a left derived functor*

$$\overset{\mathbf{L}}{\otimes}\colon \mathbf{D}(\mathrm{QCoh}_X^G) \times \mathbf{D}(\mathrm{QCoh}_X^G) \to \mathbf{D}(\mathrm{QCoh}_X^G).$$

If $f\colon X \to Y$ is a morphism of G-schemes, there is a left derived functor

$$\mathbf{L}f^*\colon \mathbf{D}(\mathrm{QCoh}_Y^G) \to \mathbf{D}(\mathrm{QCoh}_X^G),$$

and a right derived functor

$$\mathbf{R}f_*\colon \mathbf{D}(\mathrm{QCoh}_X^G) \to \mathbf{D}(\mathrm{QCoh}_Y^G).$$

As explained in [34, Section 1.5], K-flat resolutions are needed to construct derived tensor product and derived pullback, whereas K-injective resolutions are used to construct derived pushforward.

To construct the derived sheaf Hom functor, we proceed as follows. We consider the quotient stack $\mathscr{X} = [X/G]$ and the canonical atlas $\mathsf{p}\colon X \to \mathscr{X}$. We use the derived functor (see [10, § 1.2] and the references therein)

$$\mathbf{R}\mathscr{H}om_{\mathscr{X}}(-,-)\colon \mathbf{D}_{\mathsf{qc}}(\mathscr{X}) \times \mathbf{D}_{\mathsf{qc}}(\mathscr{X})^{\mathrm{op}} \to \mathbf{D}(\mathscr{X}) \to \mathbf{D}_{\mathsf{qc}}(\mathscr{X})$$

where the last arrow is the the right adjoint to the inclusion $\mathbf{D}_{\mathrm{qc}}(\mathscr{X}) \subset \mathbf{D}(\mathscr{X})$, known as the *coherator*. Exploiting the equivalences (2.1) and the factorisation (1.5), we obtain a diagram

$$\begin{tikzcd}
\mathbf{D}(\mathrm{QCoh}_{\mathscr{X}}) \times \mathbf{D}(\mathrm{QCoh}_{\mathscr{X}})^{\mathrm{op}} \arrow[r, "\mathbf{R}\mathscr{H}\mathrm{om}_{\mathscr{X}}(-,-)"] \arrow[d, "\wr"] \arrow[dd, bend right=60, "\mathsf{p}^*\times\mathsf{p}^*"'] & \mathbf{D}(\mathrm{QCoh}_{\mathscr{X}}) \arrow[d, "\wr"] \arrow[dd, bend left=60, "\mathsf{p}^*"] \\
\mathbf{D}(\mathrm{QCoh}_X^G) \times \mathbf{D}(\mathrm{QCoh}_X^G)^{\mathrm{op}} \arrow[r, dotted, "\mathbf{R}\mathscr{H}\mathrm{om}_X(-,-)"] \arrow[d, "\Phi\times\Phi"] & \mathbf{D}(\mathrm{QCoh}_X^G) \arrow[d, "\Phi"] \\
\mathbf{D}(\mathrm{QCoh}_X) \times \mathbf{D}(\mathrm{QCoh}_X)^{\mathrm{op}} \arrow[r, "\mathbf{R}\mathscr{H}\mathrm{om}_X(-,-)"] & \mathbf{D}(\mathrm{QCoh}_X)
\end{tikzcd}$$

where the bottom row is the ordinary derived sheaf Hom functor, also defined via the coherator $\mathbf{D}(X) \to \mathbf{D}_{\mathrm{qc}}(X) = \mathbf{D}(\mathrm{QCoh}_X)$. The top square is used to *define* the G-equivariant $\mathbf{R}\mathscr{H}\mathrm{om}_X(-,-)$ in the middle row, so it commutes by default, whereas the commutativity of the whole diagram, which is nothing but the statement

$$\mathsf{p}^*\,\mathbf{R}\mathscr{H}\mathrm{om}_{\mathscr{X}}(\mathscr{E}^\bullet, \mathscr{F}^\bullet) = \mathbf{R}\mathscr{H}\mathrm{om}_X(\mathsf{p}^*\mathscr{E}^\bullet, \mathsf{p}^*\mathscr{F}^\bullet),$$

is used to observe that the lower square is also commutative: this means that the G-equivariant $\mathbf{R}\mathscr{H}\mathrm{om}_X(-,-)$ commutes with the forgetful functor Φ.

The derived functors listed above satisfy the usual compatibilities. Here are some of them: given a morphism $f\colon X \to Y$ of G-schemes,

(1) $\mathbf{L}f^*$ is left adjoint to $\mathbf{R}f_*$,
(2) $\mathbf{R}f_*$ preserves cohomologically bounded complexes,
(3) $\mathbf{L}f^*$ commutes with $\otimes^{\mathbf{L}}$ and with $\mathbf{R}\mathscr{H}\mathrm{om}_X(-,-)$,
(4) $\mathbf{L}f^*$, $\mathbf{R}f_*$, $\otimes^{\mathbf{L}}$ and $\mathbf{R}\mathscr{H}\mathrm{om}_X(-,-)$ commute with the forgetful functor Φ (cf. [34, Section 1.5.8]).
(5) The projection formula

$$\mathbf{R}f_*\mathscr{F}^\bullet \overset{\mathbf{L}}{\otimes} \mathscr{E}^\bullet = \mathbf{R}f_*\big(\mathscr{F}^\bullet \overset{\mathbf{L}}{\otimes} \mathbf{L}f^*\mathscr{E}^\bullet\big)$$

holds, for all $\mathscr{F}^\bullet \in \mathbf{D}(\mathrm{QCoh}_X^G)$ and $\mathscr{E}^\bullet \in \mathbf{D}(\mathrm{QCoh}_Y^G)$.

For us, the most important property is (4).

2.4. **Equivariant Ext groups.** For a G-scheme X with structure morphism $\pi\colon X \to \operatorname{Spec}\mathbb{C}$, we write $\mathbf{R}\Gamma_X = \mathbf{R}\pi_*$. Since the G-equivariant derived functors commute with the forgetful morphism, given $\mathscr{E}^\bullet, \mathscr{F}^\bullet \in \mathbf{D}(\mathrm{QCoh}_X^G)$, the complex

$$\mathbf{R}\Gamma_X\,\mathbf{R}\mathscr{H}\mathrm{om}_X(\mathscr{E}^\bullet, \mathscr{F}^\bullet) \in \mathbf{D}(\mathrm{QCoh}_{\mathrm{pt}}^G) = \mathbf{D}(\mathrm{Rep}_{\mathbb{C}}(G))$$

is a complex of G-representations with $\mathbf{R}\mathrm{Hom}_X(\Phi(\mathscr{E}^\bullet), \Phi(\mathscr{F}^\bullet))$ as underlying complex of vector spaces. We will often omit Φ from the notation.

Remark 2.16. The cohomology functors $h^i\colon \mathbf{D}(\mathrm{QCoh}_{\mathrm{pt}}^G) \to \mathrm{QCoh}_{\mathrm{pt}}^G$ also commute with the forgetful functor. In other words, for any object $V^\bullet \in \mathbf{D}(\mathrm{QCoh}_{\mathrm{pt}}^G)$, there is a natural structure of G-representation on the vector spaces $h^i(\Phi(V^\bullet))$. Thus, given $\mathscr{E}^\bullet, \mathscr{F}^\bullet \in \mathbf{D}(\mathrm{QCoh}_X^G)$, all Ext groups

$$\mathrm{Ext}_X^i(\mathscr{E}^\bullet, \mathscr{F}^\bullet) := h^i(\mathbf{R}\mathrm{Hom}_X(\Phi(\mathscr{E}^\bullet), \Phi(\mathscr{F}^\bullet))) \in \mathrm{QCoh}_{\mathrm{pt}} \tag{2.2}$$

have a natural structure of G-representations. Therefore the G-invariant part

$$\operatorname{Ext}^i_X(\mathscr{E}^\bullet,\mathscr{F}^\bullet)^G \subset \operatorname{Ext}^i_X(\mathscr{E}^\bullet,\mathscr{F}^\bullet)$$

is well-defined.

We now describe the G-representation structure on (2.2) explicitly.

Fix two objects $\mathscr{E}^\bullet, \mathscr{F}^\bullet \in \mathbf{D}(\operatorname{QCoh}^G_X)$ and a morphism $\alpha\colon \mathscr{E}^\bullet \to \mathscr{F}^\bullet$ in $\mathbf{D}(\operatorname{QCoh}_X)$. For simplicity, assume α is represented by a cochain map

$$\begin{tikzcd}
\cdots \arrow[r] & \mathscr{E}^i \arrow[r, "\mathrm{d}^i_{\mathscr{E}^\bullet}"] \arrow[d, "\alpha_i"'] & \mathscr{E}^{i+1} \arrow[r] \arrow[d, "\alpha_{i+1}"] & \cdots \\
\cdots \arrow[r] & \mathscr{F}^i \arrow[r, "\mathrm{d}^i_{\mathscr{F}^\bullet}"] & \mathscr{F}^{i+1} \arrow[r] & \cdots
\end{tikzcd} \tag{2.3}$$

where all arrows are in the category QCoh_X. However, the individual terms $\mathscr{E}^i$ (resp. $\mathscr{F}^i$) carry a G-equivariant structure $\vartheta_{\mathscr{E}^i}$ (resp. $\vartheta_{\mathscr{F}^i}$). We let $g\in G$ act on $\alpha=(\alpha_i)_{i\in\mathbb{Z}}$ by $g\cdot\alpha=(g\cdot\alpha_i)_{i\in\mathbb{Z}}$, where the element $g\cdot\alpha_i\in\operatorname{Hom}_X(\mathscr{E}^i,\mathscr{F}^i)$ has been defined via Diagram (1.3) exploiting the equivariant structures $\vartheta_{\mathscr{E}^i}$ and $\vartheta_{\mathscr{F}^i}$. For fixed $g\in G$ and $i\in\mathbb{Z}$, the diagram

$$\begin{tikzcd}
\mathscr{E}^i \arrow[r, "\mathrm{d}^i_{\mathscr{E}^\bullet}"] \arrow[d, "\vartheta_{\mathscr{E}^i,g}"] \arrow[ddd, bend right=60, "g\cdot\alpha_i"'] & \mathscr{E}^{i+1} \arrow[d, "\vartheta_{\mathscr{E}^{i+1},g}"] \arrow[ddd, bend left=60, "g\cdot\alpha_{i+1}"] \\
g^*\mathscr{E}^i \arrow[r, "g^*\mathrm{d}^i_{\mathscr{E}^\bullet}"] \arrow[d, "g^*\alpha_i"] & g^*\mathscr{E}^{i+1} \arrow[d, "g^*\alpha_{i+1}"] \\
g^*\mathscr{F}^i \arrow[r, "g^*\mathrm{d}^i_{\mathscr{F}^\bullet}"] \arrow[d, "\vartheta^{-1}_{\mathscr{F}^i,g}"] & g^*\mathscr{F}^{i+1} \arrow[d, "\vartheta^{-1}_{\mathscr{F}^{i+1},g}"] \\
\mathscr{F}^i \arrow[r, "\mathrm{d}^i_{\mathscr{F}^\bullet}"] & \mathscr{F}^{i+1}
\end{tikzcd} \tag{2.4}$$

in QCoh_X illustrates the situation: since $(\mathscr{E}^\bullet,\mathrm{d}_{\mathscr{E}^\bullet})$ and $(\mathscr{F}^\bullet,\mathrm{d}_{\mathscr{F}^\bullet})$ are objects of $\mathbf{D}(\operatorname{QCoh}^G_X)$, the morphisms $\mathrm{d}^i_{\mathscr{E}^\bullet}$ and $\mathrm{d}^i_{\mathscr{F}^\bullet}$, as soon as we view them in QCoh_X, satisfy

$$\begin{aligned}
\mathrm{d}^i_{\mathscr{E}^\bullet} &= g\cdot \mathrm{d}^i_{\mathscr{E}^\bullet} = \vartheta^{-1}_{\mathscr{E}^{i+1},g}\circ g^*\mathrm{d}^i_{\mathscr{E}^\bullet}\circ\vartheta_{\mathscr{E}^i,g}\\
\mathrm{d}^i_{\mathscr{F}^\bullet} &= g\cdot \mathrm{d}^i_{\mathscr{F}^\bullet} = \vartheta^{-1}_{\mathscr{F}^{i+1},g}\circ g^*\mathrm{d}^i_{\mathscr{F}^\bullet}\circ\vartheta_{\mathscr{F}^i,g}
\end{aligned}$$

respectively, for all $g\in G$ (cf. Remark 1.9). Therefore the top and bottom squares commute. So does the middle square, by the commutativity of (2.3). Therefore the outer square commutes, thus defining the morphism $g\cdot\alpha\in\operatorname{Hom}_X(\mathscr{E}^\bullet,\mathscr{F}^\bullet)$.

Remark 2.17. Let $\mathscr{E}^\bullet$ be an object of $\mathbf{D}(\operatorname{QCoh}^G_X)$. Then

$$\mathrm{id}_{\mathscr{E}^\bullet}\in\operatorname{Hom}_X(\mathscr{E}^\bullet,\mathscr{E}^\bullet)^G\subset\operatorname{Hom}_X(\mathscr{E}^\bullet,\mathscr{E}^\bullet).$$

This is clear by looking at the diagram (2.4) where all $\alpha_i=\mathrm{id}_{\mathscr{E}^i}$.

The homological algebra developed so far immediately implies the following two lemmas.

Lemma 2.18. *Fix $\mathscr{F}^\bullet\in\mathbf{D}(\operatorname{QCoh}^G_X)$. A distinguished triangle $\mathscr{E}^\bullet_1\to\mathscr{E}^\bullet_2\to\mathscr{E}^\bullet_3\to\mathscr{E}^\bullet_1[1]$ in $\mathbf{D}(\operatorname{QCoh}^G_X)$ induces a long exact sequence*

$$\cdots\to\operatorname{Ext}^i_X(\mathscr{F}^\bullet,\mathscr{E}^\bullet_1)\to\operatorname{Ext}^i_X(\mathscr{F}^\bullet,\mathscr{E}^\bullet_2)\to\operatorname{Ext}^i_X(\mathscr{F}^\bullet,\mathscr{E}^\bullet_3)\to\cdots$$

of G-representations.

Proof. Apply $\mathbf{R}\Gamma_X \circ \mathbf{R}\mathscr{H}om_X(\mathscr{F}^\bullet, -)$ and then cohomology $h^\bullet\colon \mathbf{D}(\mathrm{Rep}_{\mathbb{C}}(G)) \to \mathrm{Rep}_{\mathbb{C}}(G)$ to the given distinguished triangle. Conclude by Remark 2.16. □

Lemma 2.19. *Fix $\mathscr{F}^\bullet \in \mathbf{D}(\mathrm{QCoh}_X^G)$. A morphism $i\colon \mathscr{E}_1^\bullet \to \mathscr{E}_2^\bullet$ in $\mathbf{D}(\mathrm{QCoh}_X^G)$ induces morphisms of G-representations*

$$\begin{aligned} i_*\colon \mathrm{Hom}_X(\mathscr{F}^\bullet, \mathscr{E}_1^\bullet) &\to \mathrm{Hom}_X(\mathscr{F}^\bullet, \mathscr{E}_2^\bullet) & \alpha &\mapsto i \circ \alpha \\ i^*\colon \mathrm{Hom}_X(\mathscr{E}_2^\bullet, \mathscr{F}^\bullet) &\to \mathrm{Hom}_X(\mathscr{E}_1^\bullet, \mathscr{F}^\bullet) & \beta &\mapsto \beta \circ i. \end{aligned}$$

Proof. The morphism i_* is a special case of Lemma 2.18. The morphism

$$\mathbf{R}\mathscr{H}om_X(i, \mathscr{F}^\bullet)\colon \mathbf{R}\mathscr{H}om_X(\mathscr{E}_2^\bullet, \mathscr{G}^\bullet) \to \mathbf{R}\mathscr{H}om_X(\mathscr{E}_1^\bullet, \mathscr{F}^\bullet)$$

lives in $\mathbf{D}(\mathrm{QCoh}_X^G)$, and i^* is obtained by applying $h^0 \circ \mathbf{R}\Gamma_X$ to it. □

We now give a definition that will be central in the next sections.

Definition 2.20. Let $\mathscr{E}^\bullet$ and $\mathscr{F}^\bullet$ be two objects of $\mathbf{D}(\mathrm{QCoh}_X^G)$. We say that an extension class

$$\alpha \in \mathrm{Ext}_X^i(\mathscr{E}^\bullet, \mathscr{F}^\bullet)$$

is G-*equivariant* if the corresponding morphism $\mathscr{E}^\bullet \to \mathscr{F}^\bullet[i]$ admits a lift to $\mathbf{D}(\mathrm{QCoh}_X^G)$, i.e. if it lies in the image of the natural morphism $\mathrm{Hom}_{\mathbf{D}(\mathrm{QCoh}_X^G)}(\mathscr{E}^\bullet, \mathscr{F}^\bullet[i]) \to \mathrm{Hom}_X(\mathscr{E}^\bullet, \mathscr{F}^\bullet[i])$ (recall that we omit Φ from the notation in the target Hom-set).

2.4.1. *The case of reductive groups.* Recall from [21, App. A] that, over a field of characteristic zero, a linear algebraic group G is *reductive* if and only if it is linearly reductive. This means that the functor

$$(-)^G\colon \mathrm{QCoh}_{\mathrm{pt}}^G \to \mathrm{QCoh}_{\mathrm{pt}}, \quad V \mapsto V^G,$$

taking a G-representation to its G-invariant part, is exact.

Reductivity has the following important property. Fix, as ever, a complex noetherian separated G-scheme X, and set $\mathscr{X}_G = [X/G]$.

Lemma 2.21 ([2, Lemma 2.2.8])**.** *Let G be a reductive algebraic group. Fix two objects $\mathscr{E}^\bullet$ and $\mathscr{F}^\bullet$ in $\mathbf{D}(\mathrm{QCoh}_X^G) = \mathbf{D}(\mathrm{QCoh}_{\mathscr{X}_G})$. Then there are natural isomorphisms*

$$\mathrm{Hom}_{\mathscr{X}_G}(\mathscr{E}^\bullet, \mathscr{F}^\bullet[i]) \xrightarrow{\sim} \mathrm{Hom}_X(\mathscr{E}^\bullet, \mathscr{F}^\bullet[i])^G.$$

The Hom-set on the left hand side is taken in the derived category of the abelian category $\mathrm{QCoh}_{\mathscr{X}_G}$, which we identify with QCoh_X^G via Theorem 1.10. The Hom-set on the right hand side is taken in $\mathbf{D}(\mathrm{QCoh}_X)$.

Remark 2.22. If G is reductive, then by Lemma 2.21 to say that an extension class α is G-equivariant (Definition 2.20) is the same as saying that it belongs to the G-invariant part of the corresponding Ext group.

2.5. **Equivariant Verdier duality.** Classical Verdier duality for a morphism of schemes f is the following assertion: the right derived functor $\mathbf{R}f_*$ has a right adjoint. Such adjoint is usually denoted $f^\times$, or $f^!$ if f is a proper morphism. We will stick to the $f^!$ notation.

The most general statement we are aware of is due to Neeman. Note that this is stated for $\mathbf{D}_{\mathrm{qc}}(-)$ in [22], but (as we observed in Section 2.2.1) under the given assumptions these categories are equivalent to $\mathbf{D}(\mathrm{QCoh})$.

Theorem 2.23 ([22]). *Let $f: X \to Y$ be a morphism of quasi-compact separated schemes. Then $\mathbf{R}f_*: \mathbf{D}(\mathrm{QCoh}_X) \to \mathbf{D}(\mathrm{QCoh}_Y)$ has a right adjoint $f^!$. If f is a proper morphism of noetherian separated schemes, the natural morphism*

$$\mathbf{R}f_* \mathbf{R}\mathscr{H}om_X(\mathscr{F}^\bullet, f^!\mathscr{E}^\bullet) \to \mathbf{R}\mathscr{H}om_Y(\mathbf{R}f_*\mathscr{F}^\bullet, \mathscr{E}^\bullet) \tag{2.5}$$

is an isomorphism in $\mathbf{D}(\mathrm{QCoh}_Y)$ for all $\mathscr{F}^\bullet \in \mathbf{D}(\mathrm{QCoh}_X)$ and $\mathscr{E}^\bullet \in \mathbf{D}(\mathrm{QCoh}_Y)$.

Proof. The first assertion is [22, Ex. 4.2]. The sheafified Verdier duality isomorphism (2.5) is obtained in [22, Sec. 6]. A proof of (2.5) assuming f a morphism essentially of finite type between noetherian separated schemes can be found in [16, Eq. 1.6.1]. □

We refer the reader to Neeman [22] and Lipman [20] for very informative discussions around the history of Verdier duality, as well its more modern versions.

In this section we prove a G-equivariant version of Theorem 2.23. We follow Neeman's strategy entirely. See also [10, Thm. 4.14 (1)] for a generalisation, proving the existence of a right adjoint of $\mathbf{R}h_*: \mathbf{D}_{\mathsf{qc}}(\mathscr{X}) \to \mathbf{D}_{\mathsf{qc}}(\mathscr{Y})$ for $h: \mathscr{X} \to \mathscr{Y}$ an arbitrary concentrated morphism (cf. [10, Def. 2.4]) of algebraic stacks.

The main tool used by Neeman is the following version of Brown's representability theorem.

Theorem 2.24 (Brown representability [22, Thm. 4.1]). *Let $\mathcal{S}$ be a compactly generated triangulated category, $\mathcal{T}$ any triangulated category. Let $F: \mathcal{S} \to \mathcal{T}$ be a triangulated functor respecting coproducts. Then F has a right adjoint.*

Theorem 2.25 (Equivariant Verdier duality). *Let $f: X \to Y$ be a morphism of noetherian separated G-schemes satisfying (†). Then $\mathbf{R}f_*: \mathbf{D}(\mathrm{QCoh}_X^G) \to \mathbf{D}(\mathrm{QCoh}_Y^G)$ has a right adjoint $f^!$.*

Proof. Recall that $\mathbf{D}(\mathrm{QCoh}_X^G)$ is compactly generated by Corollary 2.9. Set $\mathscr{X}_G = [X/G]$ and $\mathscr{Y}_G = [Y/G]$. The morphism $f: X \to Y$ induces a *representable* morphism of algebraic stacks

$$\overline{f}: \mathscr{X}_G \to \mathscr{Y}_G,$$

that by our assumptions on X and Y is quasi-compact and quasi-separated. In particular, by [10, Lemma 2.5], $\overline{f}$ is a *concentrated* morphism in the sense of [10, Def. 2.4]. Then, by [10, Thm. 2.6 (3)], the direct image $\mathbf{R}\overline{f}_*: \mathbf{D}_{\mathsf{qc}}(\mathscr{X}_G) \to \mathbf{D}_{\mathsf{qc}}(\mathscr{Y}_G)$ preserves coproducts. Under the equivalences (2.1), the functor $\mathbf{R}\overline{f}_*$ corresponds precisely to $\mathbf{R}f_*: \mathbf{D}(\mathrm{QCoh}_X^G) \to \mathbf{D}(\mathrm{QCoh}_Y^G)$. Thus the existence of $f^!: \mathbf{D}(\mathrm{QCoh}_Y^G) \to \mathbf{D}(\mathrm{QCoh}_X^G)$ follows by Theorem 2.24. □

Lemma 2.26 (Sheafified Verdier duality). *Let $f: X \to Y$ be a proper morphism of noetherian separated G-schemes satisfying (†). Fix objects $\mathscr{F}^\bullet \in \mathbf{D}(\mathrm{QCoh}_X^G)$ and $\mathscr{E}^\bullet \in \mathbf{D}(\mathrm{QCoh}_Y^G)$. Then there is a natural isomorphism*

$$\mathbf{R}f_* \mathbf{R}\mathscr{H}om_X(\mathscr{F}^\bullet, f^!\mathscr{E}^\bullet) \xrightarrow{\sim} \mathbf{R}\mathscr{H}om_Y(\mathbf{R}f_*\mathscr{F}^\bullet, \mathscr{E}^\bullet) \tag{2.6}$$

in $\mathbf{D}(\mathrm{QCoh}_Y^G)$.

Proof. This is a special case of [23, Lemma 5.3], which is actually proved without the properness assumption. □

So far we were not concerned with the question whether $f^!: \mathbf{D}(\mathrm{QCoh}_Y^G) \to \mathbf{D}(\mathrm{QCoh}_X^G)$ commutes with the forgetful functor. This question can be restated as follows. Given the

2-cartesian diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathsf{p}_X"] & Y \arrow[d, "\mathsf{p}_Y"] \\ \mathscr{X}_G \arrow[r, "\overline{f}"] & \mathscr{Y}_G \end{tikzcd}$$

we ask whether the natural transformation $\mathsf{p}_X^*\overline{f}^! \to f^!\mathsf{p}_Y^*$ an isomorphism of functors. This is answered in full generality in [23, Lemma 5.20]. For the purpose of this paper, we content ourselves with a special case of that result: the answer is positive when f is proper and $\mathscr{E}^\bullet \in \mathbf{D}^+_{\mathsf{qc}}(\mathscr{Y}_G)$ is bounded below. Under these assumptions one has

$$\Phi_X(f^!(\mathscr{E}^\bullet)) = f^!(\Phi_Y(\mathscr{E}^\bullet)) \tag{2.7}$$

where $f^!$ in the left hand side (resp. in the right hand side) is the G-equivariant right adjoint (resp. the ordinary right adjoint) of $\mathbf{R}f_*$.

Remark 2.27. In a little more detail, properness of f implies properness of $\overline{f}$ (reason: $\overline{f}$ is separated by [30, Tag 04YV], universally closed by [30, Tag 0CL3] and of finite type by [30, Tag 06FR]); since $\mathscr{X}_G$ and $\mathscr{Y}_G$ are noetherian and $\overline{f}$ is representable, $\overline{f}$ is *quasi-proper*, which together with $\mathscr{E}^\bullet \in \mathbf{D}^+_{\mathsf{qc}}(\mathscr{Y}_G)$ is the assumption needed in [23, Lemma 5.20].

Finally, note that restricting attention to bounded below complexes does not affect the applications we have in mind, which involve perfect complexes: we already observed in Section 2.2.1 that by assumption (†) perfect complexes are *bounded*.

Corollary 2.28. *Let $f\colon X \to Y$ be a proper morphism of noetherian separated G-schemes satisfying* (†). *Given $\mathscr{F}^\bullet \in \mathbf{D}(\mathrm{QCoh}_X^G)$ and $\mathscr{E}^\bullet \in \mathbf{D}^+(\mathrm{QCoh}_Y^G)$, for all $i \in \mathbb{Z}$ there is a canonical isomorphism of G-representations*

$$\mathrm{Ext}^i_X(\mathscr{F}^\bullet, f^!\mathscr{E}^\bullet) \xrightarrow{\sim} \mathrm{Ext}^i_Y(\mathbf{R}f_*\mathscr{F}^\bullet, \mathscr{E}^\bullet).$$

Taking G-invariant parts, it restricts to an isomorphism of $\mathbb{C}$-vector spaces

$$\mathrm{Ext}^i_X(\mathscr{F}^\bullet, f^!\mathscr{E}^\bullet)^G \xrightarrow{\sim} \mathrm{Ext}^i_Y(\mathbf{R}f_*\mathscr{F}^\bullet, \mathscr{E}^\bullet)^G.$$

Proof. It is enough to apply $h^i \circ \mathbf{R}\Gamma_Y$ to the isomorphism (2.6) and to observe that all functors involved commute with the forgetful functor. For $f^!$, we exploit (2.7). □

Example 2.29. Keep the assumptions of Corollary 2.28. If G is reductive, by Lemma 2.21 we have a commutative diagram of isomorphisms

$$\begin{tikzcd} \mathrm{Ext}^i_X(\mathscr{F}^\bullet, f^!\mathscr{E}^\bullet)^G \arrow[r, "\sim"] & \mathrm{Ext}^i_Y(\mathbf{R}f_*\mathscr{F}^\bullet, \mathscr{E}^\bullet)^G \\ \mathrm{Hom}_{\mathbf{D}(\mathrm{QCoh}_X^G)}(\mathscr{F}^\bullet, f^!\mathscr{E}^\bullet[i]) \arrow[r, "\sim"] \arrow[u, "\wr"] & \mathrm{Hom}_{\mathbf{D}(\mathrm{QCoh}_Y^G)}(\mathbf{R}f_*\mathscr{F}^\bullet, \mathscr{E}^\bullet[i]) \arrow[u, "\wr"] \end{tikzcd}$$

where the bottom map is the adjunction isomorphism obtained via Theorem 2.25.

3. Equivariance of the truncated Atiyah class

3.1. **Truncated Atiyah classes after Huybrechts–Thomas.**

In this section all schemes are noetherian and separated over $\mathbb{C}$.

3.1.1. *The relative truncated cotangent complex.* The goal of this short subsection is to revisit the classical fact that the truncated cotangent complex, though defined through the choice of a smooth embedding, does not depend on this choice. We review this from [12, Sec. 2] in order to make clear the statement that the same feature occurs in the equivariant setting.

Let B be a scheme. Let $X\subset A_1$ be a closed embedding inside a smooth B-scheme A_1. Let $J_1\subset \mathscr{O}_{A_1}$ be the ideal sheaf of the embedding. Consider the exterior derivative

$$\mathrm{d}\colon J_1\hookrightarrow \mathscr{O}_{A_1}\to \Omega_{A_1/B}$$

and restrict it to X to obtain the (relative) *truncated cotangent complex*

$$\mathbb{L}_{X/B}=\left[\,J_1/J_1^2\to \Omega_{A_1/B}|_X\,\right]\in \mathbf{D}^{[-1,0]}(\mathrm{QCoh}_X). \tag{3.1}$$

Suppose X admits an embedding in another smooth B-scheme $A_2\to B$. Then the composition $X\hookrightarrow A_1\times_B A_2\to A_1$, where $X\hookrightarrow A_1\times_B A_2$ is the diagonal embedding defined by an ideal $J_{12}\subset \mathscr{O}_{A_1\times_B A_2}$, induces a quasi-isomorphism of two-term complexes

$$\begin{tikzcd}
J_1/J_1^2 \arrow[r]\arrow[d] & \Omega_{A_1/B}|_X \arrow[d]\\
J_{12}/J_{12}^2 \arrow[r]\arrow[d] & \Omega_{A_1/B}|_X\oplus\Omega_{A_2/B}|_X \arrow[d]\\
\Omega_{A_2/B}|_X \arrow[r, equal] & \Omega_{A_2/B}|_X
\end{tikzcd} \tag{3.2}$$

showing that replacing $X\subset A_1$ with $X\subset A_1\times_B A_2$ does not change the isomorphism class of $\mathbb{L}_{X/B}$ in the derived category.

We recalled this argument in order to make the following observation. Suppose $\iota_i\colon X\hookrightarrow A_i$ is a G-*equivariant* closed embedding, for $i=1,2$. Then

$$0\to J_i\to \mathscr{O}_{A_i}\to \iota_{i*}\mathscr{O}_X\to 0$$

is a G-equivariant short exact sequence, and similarly for $\iota_{12}\colon X\hookrightarrow A_1\times_B A_2$. Since the exterior derivative $\mathrm{d}\colon \mathscr{O}_{A_i}\to\Omega_{A_i/B}$ is also G-equivariant, the whole diagram (3.2) can be canonically lifted to QCoh_X^G. This yields a well-defined element

$$\mathbb{L}_{X/B}\in \mathbf{D}^{[-1,0]}(\mathrm{QCoh}_X^G), \tag{3.3}$$

whose isomorphism class again does not depend on the choice of equivariant embedding.

Alternatively, one can define the equivariant $\mathbb{L}_{X/B}$ as follows. Consider the B-relative atlas

$$\mathsf{p}\colon X\to\mathscr{X}=[X/G]$$

and the induced pullback functor $\mathsf{p}^*\colon \mathbf{D}(\mathrm{QCoh}_{\mathscr{X}})\to\mathbf{D}(\mathrm{QCoh}_X)$. Then p^* factors through $\mathbf{D}(\mathrm{QCoh}_{\mathscr{X}})\xrightarrow{\sim}\mathbf{D}(\mathrm{QCoh}_X^G)$, and we can take $\mathbb{L}_{X/B}$ to be the image of $\mathbb{L}_{\mathscr{X}/B}$ under this equivalence. In this sense, (3.3) is a canonical lift of the ordinary truncated cotangent complex (3.1). The equivariant truncated cotangent complex is also discussed by Illusie in [15, Ch. VII, § 2.2.5].

Lemma 3.1. *Let $f\colon X\to Y$ and $g\colon Y\to Z$ be G-equivariant morphisms. Then there is a sequence of morphisms*

$$\tau_{\geq -1}\mathbf{L}f^*L^\bullet_{Y/Z}\to\mathbb{L}_{X/Z}\to\mathbb{L}_{X/Y}$$

in $\mathbf{D}^{[-1,0]}(\mathrm{QCoh}_X^G)$.

Proof. Let us shorten $\mathscr{X} = [X/G]$, and similarly for Y and Z. The given G-equivariant morphisms induce 2-cartesian diagrams of algebraic stacks

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathsf{p}_X"] & Y \arrow[r, "g"] \arrow[d, "\mathsf{p}_Y"] & Z \arrow[d, "\mathsf{p}_Z"] \\ \mathscr{X} \arrow[r, "\overline{f}"] & \mathscr{Y} \arrow[r, "\overline{g}"] & \mathscr{Z} \end{tikzcd}$$

where the morphisms $\overline{f}$ and $\overline{g}$ are quasi-compact, quasi-separated and of Deligne–Mumford type [30, Tag 04YW]. Hence their cotangent complexes live in $\mathbf{D}^{\leq 0}(\mathrm{QCoh})$. By [24, Thm. 8.1], there is an exact triangle

$$\mathbf{L}\overline{f}^* L^\bullet_{\mathscr{Y}/\mathscr{Z}} \to L^\bullet_{\mathscr{X}/\mathscr{Z}} \to L^\bullet_{\mathscr{X}/\mathscr{Y}} \tag{3.4}$$

in $\mathbf{D}(\mathrm{QCoh}_{\mathscr{X}})$, where $L^\bullet$ denotes the full cotangent complex. If we applied the pullback functor

$$\mathsf{p}_X^*\colon \mathbf{D}(\mathrm{QCoh}_{\mathscr{X}}) \xrightarrow{\sim} \mathbf{D}(\mathrm{QCoh}_X^G) \xrightarrow{\Phi} \mathbf{D}(\mathrm{QCoh}_X)$$

to the triangle (3.4), we would get the usual triangle of full cotangent complexes

$$\mathbf{L}f^* L^\bullet_{Y/Z} \to L^\bullet_{X/Z} \to L^\bullet_{X/Y} \tag{3.5}$$

in $\mathbf{D}^{\leq 0}(\mathrm{QCoh}_X)$. Instead, we get a lift to $\mathbf{D}(\mathrm{QCoh}_X^G)$ of the triangle (3.5) by applying the exact equivalence $\mathbf{D}(\mathrm{QCoh}_{\mathscr{X}}) \xrightarrow{\sim} \mathbf{D}(\mathrm{QCoh}_X^G)$ to (3.4). Applying the truncation functor $\tau_{\geq -1}$ on $\mathbf{D}(\mathrm{QCoh}_X^G)$ yields the desired sequence of morphisms

$$\tau_{\geq -1}\mathbf{L}f^* L^\bullet_{Y/Z} \to \mathbb{L}_{X/Z} \to \mathbb{L}_{X/Y}$$

in $\mathbf{D}^{[-1,0]}(\mathrm{QCoh}_X^G)$, as required. □

3.1.2. *Absolute setting.* Let $X \hookrightarrow A$ be a closed immersion of a scheme X inside a smooth $\mathbb{C}$-scheme A. Let $J \subset \mathscr{O}_A$ be the corresponding sheaf of ideals. The (absolute) truncated cotangent complex is the two term complex

$$\mathbb{L}_X = \left[J/J^2 \to \Omega_A|_X\right] \in \mathbf{D}^{[-1,0]}(\mathrm{QCoh}_X). \tag{3.6}$$

Let $\mathscr{I}_A \subset \mathscr{O}_{A\times A}$ and $\mathscr{I}_X \subset \mathscr{O}_{X\times X}$ be the ideal sheaves of the diagonal embeddings

$$A \overset{i_{\Delta_A}}{\hookrightarrow} A\times A, \quad X \overset{i_{\Delta_X}}{\hookrightarrow} X\times X,$$

respectively. Huybrechts–Thomas [12, Sec. 2] show how to construct a canonical morphism

$$\alpha_X\colon \mathscr{O}_{\Delta_X} \to i_{\Delta_X *}\mathbb{L}_X[1]. \tag{3.7}$$

It is represented in degrees $[-2,0]$ by the morphism of complexes

$$\begin{tikzcd} i_{\Delta_X *}(J/J^2) \arrow[r] \arrow[d, equal] & \mathscr{I}_A\big|_{X\times X} \arrow[r] \arrow[d] & \mathscr{O}_{X\times X} \\ i_{\Delta_X *}(J/J^2) \arrow[r] & \mathscr{I}_A/\mathscr{I}_A^2\big|_{X\times X} & \end{tikzcd} \tag{3.8}$$

where the quasi-isomorphism between the top complex and $\mathscr{O}_{\Delta_X}$ is proved as a consequence of [12, Lemma 2.2]. The extension class

$$\alpha_X \in \mathrm{Ext}^1_{X\times X}(\mathscr{O}_{\Delta_X}, i_{\Delta_X *}\mathbb{L}_X)$$

corresponding to (3.7) is called the *truncated universal Atiyah class*. It does not depend on the choice of embedding $X \subset A$.

The main observation in [12], at this point, is that the map (3.7) can be seen as a map of Fourier–Mukai kernels. In particular, for a perfect complex E on X, one can view $\mathbf{R}\pi_{2*}(\pi_1^* E \otimes \alpha_X)$ as a canonical morphism

$$\mathrm{At}_E : E \to E \otimes \mathbb{L}_X[1]$$

in $\mathbf{D}(\mathrm{QCoh}_X)$, where $\pi_i : X \times X \to X$ are the projections. This is, by definition, the *truncated Atiyah class* of E introduced in [12, Def. 2.6]. It can of course be seen as an element

$$\mathrm{At}_E \in \mathrm{Ext}^1_X(E, E \otimes \mathbb{L}_X). \tag{3.9}$$

Under the canonical morphism $\mathbb{L}_X \to h^0(\mathbb{L}_X) = \Omega_X$, the class At_E projects onto the classical Atiyah class in $\mathrm{Ext}^1_X(E, E \otimes \Omega_X)$.

3.1.3. *Relative setting.* We will need the following setup, which we recall verbatim from [12, Sec. 2] and [13].

Let B be a scheme, $X \to B$ a B-scheme equipped with a closed immersion $X \hookrightarrow A$ with ideal $J \subset \mathscr{O}_A$. We assume we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr] & A_B \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & A \arrow[d] \\ & B \arrow[r, hook] & \widetilde{B} \end{tikzcd} \tag{3.10}$$

where $\widetilde{B}$ and $A \to \widetilde{B}$ are smooth and the square is cartesian. In particular, both A and $A_B \to B$ are smooth. Let $J_B \subset \mathscr{O}_{A_B}$ be the ideal sheaf of $X \subset A_B$. Then there is a natural morphism of chain complexes

$$\begin{tikzcd} J/J^2 \arrow[r] \arrow[d] & J_B/J_B^2 \arrow[d] & \\ \Omega_A|_X \arrow[r] & \Omega_{A/\widetilde{B}}|_X \arrow[r, equal] & \Omega_{A_B/B}|_X \end{tikzcd} \tag{3.11}$$

inducing a morphism

$$j : \mathbb{L}_X \to \mathbb{L}_{X/B}.$$

The *relative truncated Atiyah class* of a perfect complex $E \in \mathrm{Perf}\, X$ is, by definition, the composition

$$\mathrm{At}_{E/B} : E \xrightarrow{\mathrm{At}_E} E \otimes \mathbb{L}_X[1] \xrightarrow{\mathrm{id}_E \otimes j[1]} E \otimes \mathbb{L}_{X/B}[1].$$

It corresponds to the element

$$\mathrm{At}_{E/B} \in \mathrm{Ext}^1_X(E, E \otimes \mathbb{L}_{X/B})$$

obtained as the image of At_E under the map $j_* : \mathrm{Ext}^1_X(E, E \otimes \mathbb{L}_X) \to \mathrm{Ext}^1_X(E, E \otimes \mathbb{L}_{X/B})$.

3.2. **Adding in the group action.** In this section we prove Theorem A (which builds on the situation of Section 3.1.2), along with its relative analogue (which builds on the situation of Section 3.1.3).

3.2.1. *Absolute setting.* We first go back to the absolute setting of Section 3.1.2.

Let G be an affine algebraic group, and let

$$X \subset A$$

be a G-*equivariant embedding* of noetherian separated schemes, where A is smooth. Recall (cf. Example 2.1) that this situation is achieved if X is quasi-projective and has a G-equivariant line bundle. Under these assumptions, we have seen that the truncated cotangent complex is canonically G-equivariant, i.e. there is a canonical lift

$$\mathbb{L}_X \in \mathbf{D}^{[-1,0]}(\mathrm{QCoh}_X^G)$$

of the complex (3.6).

Let $i_{\Delta_X}\colon X \hookrightarrow X \times X$ be the diagonal embedding. The G-action on X determines a G-equivariant structure on the structure sheaf $\mathscr{O}_X$, therefore

$$\mathscr{O}_{\Delta_X} = i_{\Delta_X *}\mathscr{O}_X$$

is naturally G-equivariant, since i_{Δ_X} is an equivariant closed immersion.

Lemma 3.2. *The morphism*

$$\alpha_X\colon \mathscr{O}_{\Delta_X} \to i_{\Delta_X *}\mathbb{L}_X[1]$$

is naturally G-equivariant.

Proof. Since $X \subset A$ is a G-equivariant embedding, the diagram of closed immersions

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, hook, "i_{\Delta_X}"'] & A \arrow[d, hook, "i_{\Delta_A}"] \\ X \times X \arrow[r, hook] & A \times A \end{tikzcd}$$

along with its associated ideal sheaf short exact sequences, are also G-equivariant in a natural way. Therefore Diagram (3.8), which is built out of these equivariant short exact sequences through the G-equivariant geometric functors (cf. Section 1.3), inherits a G-equivariant structure. But Diagram (3.8) represents precisely α_X. The claim follows. □

We finally have all the tools to complete the proof of Theorem A.

Theorem 3.3. *Let G be an affine algebraic group acting on a scheme X having a G-equivariant embedding in a smooth scheme. Fix a perfect complex $E \in \mathrm{Perf}\,X$. Then every lift of E to $\mathbf{D}(\mathrm{QCoh}_X^G)$ makes At_E canonically G-equivariant.*

Recall (cf. Definition 2.20) that by 'At_E is G-equivariant' we mean that the corresponding morphism $E \to E \otimes \mathbb{L}_X[1]$ admits a lift to $\mathbf{D}(\mathrm{QCoh}_X^G)$.

Proof. Endow $X \times X$ with the diagonal action. Then the projections $\pi_i\colon X \times X \to X$ are G-equivariant. Since α_X is G-equivariant by Lemma 3.2, using equivariant pushforward $\mathbf{R}\pi_{2*}$, pullback π_1^* and tensor product $\otimes$ (cf. Section 2.3), we deduce that the morphism

$$\mathbf{R}\pi_{2*}(\pi_1^* E \otimes \alpha_X)\colon E \to E \otimes \mathbb{L}_X[1]$$

is canonically lifted to $\mathbf{D}(\mathrm{QCoh}_X^G)$, which proves the result. □

3.2.2. *Relative setting.* Suppose we are in the situation depicted in Diagram (3.10), and assume $X \hookrightarrow A_B \hookrightarrow A$ are G-equivariant embeddings. Then we obtain the following consequence of Theorem 3.3.

Corollary 3.4. *The relative truncated Atiyah class* $\mathrm{At}_{E/B}$ *is* G*-equivariant.*

Proof. The assumption that $X \hookrightarrow A_B \hookrightarrow A$ are G-equivariant implies that the morphism $j\colon \mathbb{L}_X \to \mathbb{L}_{X/B}$, induced by Diagram (3.11), is G-equivariant. Therefore

$$\mathrm{At}_{E/B}\colon E \to E \otimes \mathbb{L}_X[1] \xrightarrow{\mathrm{id}_E \otimes j[1]} E \otimes \mathbb{L}_{X/B}[1]$$

lives in $\mathbf{D}(\mathrm{QCoh}_X^G)$ entirely. □

Remark 3.5. By Lemma 2.19, the morphism $\mathrm{At}_{E/B}$ can be seen as an element of

$$\mathrm{Ext}^1(E, E \otimes \mathbb{L}_{X/B})^G \subset \mathrm{Ext}^1(E, E \otimes \mathbb{L}_{X/B}).$$

Indeed, since both At_E and $\mathrm{id}_E \otimes j$ are morphisms in $\mathbf{D}(\mathrm{QCoh}_X^G)$, the composition

$$\mathrm{Hom}_X(E,E) \xrightarrow{\mathrm{At}_{E*}} \mathrm{Hom}_X(E, E \otimes \mathbb{L}_X[1]) \xrightarrow{(\mathrm{id}_E \otimes j)_*} \mathrm{Hom}_X(E, E \otimes \mathbb{L}_{X/B}[1])$$

is a morphism of G-representations, and as such it preserves G-invariant parts. Therefore $\mathrm{id}_E \in \mathrm{Hom}_X(E,E)^G$ gets sent to $\mathrm{At}_{E/B} \in \mathrm{Ext}^1(E, E \otimes \mathbb{L}_{X/B})^G$.

4. Application to moduli spaces of perfect complexes

In this section we shall prove Theorem B, whose statement we recall below (Theorem 4.3) for the reader's convenience.

4.1. **Equivariance of the universal complex.** Fix an affine algebraic group G and a noetherian separated $\mathbb{C}$-scheme B carrying the trivial G-action $G \times B \to B$. Let $f\colon Y \to B$ be a smooth (connected) projective G-invariant morphism of relative dimension d, where the G-action on Y is denoted $\sigma_Y\colon G \times Y \to Y$. By assumption, G preserves the fibres of f.

As in [12, Sec. 4.1], let $M \to B$ be a relative fine separated moduli space of simple perfect complexes of rank $r \neq 0$ on the fibres of f, with fixed determinant $\mathcal{L} \in \operatorname{Pic} Y$ and fixed numerical invariants. Then M is an algebraic space, locally complete as a moduli space, and there is a universal perfect complex

$$\mathcal{E} \in \mathrm{Perf}(Y \times_B M).$$

Denote by $\iota_b\colon Y_b \hookrightarrow Y$ the inclusion of a fibre of f. If a point $m \in M$ sits over $b \in B$, let $i_m\colon Y_b \xrightarrow{\sim} Y_b \times \{m\} \hookrightarrow Y \times_B M$ denote the corresponding inclusion.

For a scheme S/B, the universal property of the pair $(M, \mathcal{E})$ translates into a bijection between

- morphisms $S \to M$ over B, and
- equivalence classes of complexes $F \in \mathrm{Perf}(Y \times_B S)$ such that for all $s \in S$ (say, sitting over $b \in B$) the derived restriction $F|_{Y_b}$ is isomorphic to $\mathbf{L}i_m^* \mathcal{E}$ for some $m \in M$ (sitting over b), and such that $\det F = \pi_S^* \mathcal{L}' \otimes \pi_Y^* \mathcal{L}$ for some $\mathcal{L}' \in \operatorname{Pic} S$ (where π_S and π_Y are the projections from $Y \times_B S$).

Two complexes F and F' are considered equivalent if there exists a line bundle $\mathcal{H} \in \operatorname{Pic} S$ such that $F = F' \otimes \pi_S^* \mathcal{H}$. The correspondence assigns to a B-morphism $h\colon S \to M$ the equivalence class of the perfect complex $(\mathrm{id}_Y \times h)^* \mathcal{E} \in \mathrm{Perf}(Y \times_B S)$.

We start with an auxiliary lemma, whose proof was indicated to us by R. Thomas.

Lemma 4.1. *We have* $\operatorname{Aut} E = \mathbb{C}^\times$*.*

Proof. Let $\pi_M : X = Y \times_B M \to M$ be the projection. The identity section

$$\mathscr{O}_M \to R^0\pi_{M*}\mathbf{R}\mathscr{H}\mathrm{om}_X(E, E)$$

is an injective morphism of sheaves. Since it becomes an isomorphism

$$k(m) \to R^0\pi_{M*}\mathbf{R}\mathscr{H}\mathrm{om}_X(E, E) \otimes k(m) \xrightarrow{\sim} \mathrm{End}_{Y_b}(\mathbf{L}i_m^* E, \mathbf{L}i_m^* E)$$

after restriction to all closed points ($m \in M$ sitting over b), it is globally an isomorphism. □

Proposition 4.2. *The universal complex* $E \in \operatorname{Perf} X$ *is naturally* G*-equivariant.*

Proof. First of all, we induce a G-action on M. Let $\sigma_Y : G \times Y \to Y$ denote the G-action on Y. Then pulling back E along

$$\sigma_Y \times \mathrm{id}_M : G \times Y \times_B M \to Y \times_B M$$

gives a family of perfect complexes parameterised by $G \times M$. By the universal property of (M, E), this in turn produces a B-morphism

$$\sigma_M : G \times M \to M,$$

which is a G-action on M. We have

$$(\mathrm{id}_Y \times \sigma_M)^* E \cong (\sigma_Y \times \mathrm{id}_M)^* E \otimes \pi_{G,M}^* \mathcal{H}$$

for some $\mathcal{H} \in \operatorname{Pic}(G \times M)$. We claim that $\mathcal{H}$ is the trivial line bundle. Consider the projection $\pi_2 : G \times M \to M$. Since G is smooth and affine, we have $\operatorname{Pic} G = 0$, thus $\mathcal{H} = \pi_2^* \mathcal{H}'$ for some $\mathcal{H}' \in \operatorname{Pic} M$. However, $\mathcal{H}|_{\{g\}\times M}$ is trivial for all $g \in G$, in particular for $g = e$, where $e \in G$ is the group identity. Thus $\mathcal{H}'$ is trivial. Then the previous isomorphism becomes

$$(\mathrm{id}_Y \times \sigma_M)^* E \cong (\sigma_Y \times \mathrm{id}_M)^* E.$$

Next, we have to make E equivariant. We choose the action

$$\tau : G \times Y \times_B M \to Y \times_B M, \quad (g, y, m) \mapsto (\sigma_Y(g, y), \sigma_M(g^{-1}, m))$$

on $X = Y \times_B M$. The pullback $\tau^* E$ corresponds to a B-morphism $\phi_\tau : G \times M \to M$. In fact, ϕ_τ is the second projection. Indeed,

$$\tau^* E|_{\{g\}\times Y_b \times \{m\}} = \mathbf{L}i_n^* E,$$

where $n = \sigma_M(g, \sigma_M(g^{-1}, m)) = \sigma_M(e, m) = m$. Thus $\tau^* E|_{\{g\}\times Y_b \times \{m\}} = \mathbf{L}i_m^* E$, and we obtain an isomorphism $\tau^* E \cong (\mathrm{id}_Y \times \phi_\tau)^* E \otimes \pi_{G,M}^* \mathcal{H}$ for some $\mathcal{H} \in \operatorname{Pic}(G \times M)$. For the same reason as before, $\mathcal{H}$ is trivial. Therefore, since ϕ_τ is the projection, we obtain an isomorphism

$$\vartheta : p_2^* E \xrightarrow{\sim} \tau^* E \tag{4.1}$$

of perfect complexes on $G \times X$, where $p_2 : G \times X \to X$ is the projection.

Finally, we need to verify that ϑ satisfies the cocycle condition. We follow [17, Prop. 4.4]. A theorem of Rosenlicht [27], whose proof is sketched in [5, Rem. 7.1], says that if Z and Z' are irreducible varieties over an algebraically closed field, the natural homomorphism $\mathscr{O}(Z)^\times \otimes \mathscr{O}(Z')^\times \to \mathscr{O}(Z \times Z')^\times$ is surjective. In fact, [5, Rem. 7.1] shows more: one can write every function $\phi \in \mathscr{O}(Z \times Z')^\times$ as $\phi = c \boxtimes c'$ for $c \in \mathscr{O}(Z)^\times$ and $c' \in \mathscr{O}(Z')^\times$. We apply this to

$Z = Z' = G$, which is irreducible since it is smooth and connected. Let us normalise ϑ, if necessary, to achieve $\vartheta_e = \mathrm{id}_E$. We need to show that the function

$$F\colon G \times G \to \operatorname{Aut} E = \mathbb{C}^\times, \quad (g,h) \mapsto \rho_g^* \vartheta_h \circ \vartheta_g \circ \vartheta_{hg}^{-1}$$

is the constant 1, where, as in Section 1.1, ρ_g denotes the inclusion $X \xrightarrow{\sim} \{g\} \times X \hookrightarrow G \times X$. We have used Lemma 4.1 to confirm that $\operatorname{Aut} E = \mathbb{C}^\times$. We can view F as a function in $\mathscr{O}(G\times G)^\times$, thus by [5, Rem. 7.1] we can write $F(g,h) = F_1(g)\cdot F_2(h)$ for $F_1, F_2 \in \mathscr{O}(G)^\times$. Since $F(g,1) = 1 = F(1,h)$ for all closed points $g, h \in G$, the claim follows. □

From now on, we assume M carries the G-action constructed during the proof of Proposition 4.2, and we endow E with the G-equivariant structure produced in (4.1).

We assume, from now on, that M admits a G-equivariant embedding inside a smooth B-scheme. Our goal is to prove the following result, which is Theorem B from the Introduction.

Theorem 4.3. *Assume G is reductive. Then the relative obstruction theory on M/B is naturally G-equivariant.*

We recalled in the Introduction how the relative obstruction theory is obtained via the Atiyah class of the universal complex. We review this below, directly in the equivariant setting.

Since by assumption M admits a smooth embedding over B, the same is true for the B-scheme

$$X = Y \times_B M.$$

Also note that the second projection $\pi_M\colon X \to M$ is a smooth projective morphism of G-schemes satisfying condition (†), in particular all the machinery of Section 2, including equivariant Verdier duality, applies to π_M.

The complex E has has a well-defined truncated Atiyah class (3.9)

$$\mathrm{At}_E \in \operatorname{Ext}^1_X(E, E\otimes \mathbb{L}_X),$$

and our task is now to prove its equivariance. The proof of Theorem 4.3 will follow by equivariant Verdier duality. In fact, the equivariance of the Atiyah class is now an easy corollary of Proposition 4.2 and the main result of the paper.

Corollary 4.4. *The Atiyah class At_E is naturally G-equivariant.*

Proof. Follows by combining Proposition 4.2 with Theorem 3.3. □

The composition of At_E with $\mathrm{id}_E \otimes j[1]$ gives the relative Atiyah class

$$\mathrm{At}_{E/Y}\colon E \to E\otimes \mathbb{L}_X[1] \to E \otimes \mathbb{L}_{X/Y}[1].$$

Corollary 4.5. *The relative Atiyah class $\mathrm{At}_{E/Y}$ is naturally G-equivariant.*

Proof. The morphism $j\colon \mathbb{L}_X \to \mathbb{L}_{X/Y}$ has a canonical lift to $\mathbf{D}(\mathrm{QCoh}_X^G)$ by Lemma 3.1 applied to the G-equivariant maps $\pi_Y\colon X \to Y$ and $Y \to \operatorname{Spec}\mathbb{C}$. The statement follows by combining this fact with Corollary 4.4. □

4.2. **Proof of Theorem B.** From now on, we assume G to be *reductive* (cf. Section 2.4.1). Then, combining Corollary 4.5 with Lemma 2.21, we can identify the relative truncated Atiyah class of the perfect complex $E \in \mathrm{Perf}(Y\times_B M)$ with an element

$$\mathrm{At}_{E/Y} \in \operatorname{Ext}^1_X(E, E\otimes \mathbb{L}_{X/Y})^G = \operatorname{Ext}^1_X(E, E\otimes \pi_M^*\mathbb{L}_{M/B})^G,$$

where E has the G-equivariant structure obtained in Proposition 4.2, and we have observed that $\mathbb{L}_{X/Y} = \pi_M^* \mathbb{L}_{M/B}$, cf. [30, Tag 09DJ]. We now exploit the splitting

$$\mathbf{R}\mathscr{H}\mathrm{om}_X(E,E) = \mathscr{O}_X \oplus \mathbf{R}\mathscr{H}\mathrm{om}_X(E,E)_0, \tag{4.2}$$

which we wish to prove to be G-equivariant. Recall (see e.g. [11, Sec. 10.1] for more details on this construction) how (4.2) is obtained in the non-equivariant setup: the trace map $\mathrm{tr}\colon \mathbf{R}\mathscr{H}\mathrm{om}_X(E,E) \to \mathscr{O}_X$ splits the identity homomorphism $\mathrm{id}\colon \mathscr{O}_X \to \mathbf{R}\mathscr{H}\mathrm{om}_X(E,E)$, and the composition $\mathrm{tr}\circ\mathrm{id}$ is multiplication by the rank r (which we assumed nonzero). We now show that the induced distinguished triangle

$$\mathbf{R}\mathscr{H}\mathrm{om}_X(E,E)_0 \to \mathbf{R}\mathscr{H}\mathrm{om}_X(E,E) \xrightarrow{\mathrm{tr}} \mathscr{O}_X, \tag{4.3}$$

defining the *traceless* $\mathbf{R}\mathscr{H}\mathrm{om}$, is naturally lifted to $\mathbf{D}(\mathrm{QCoh}_X^G)$.

Consider the element

$$\mathrm{id}_E \in \mathrm{Hom}_X(E,E) = \mathrm{Hom}_X(\mathscr{O}_X, \mathbf{R}\mathscr{H}\mathrm{om}_X(E,E)).$$

By Remark 2.17 and Lemma 2.21, we know that

$$\mathrm{id}_E \in \mathrm{Hom}_X(\mathscr{O}_X, \mathbf{R}\mathscr{H}\mathrm{om}_X(E,E))^G \cong \mathrm{Hom}_{\mathbf{D}(\mathrm{QCoh}_X^G)}(\mathscr{O}_X, \mathbf{R}\mathscr{H}\mathrm{om}_X(E,E)).$$

Then take $\mathscr{F}^\bullet = \mathscr{O}_X$ and $i = \mathrm{id}_E$ in Lemma 2.19 to observe that

$$\mathrm{id}_E^*\colon \mathrm{Hom}_X(\mathbf{R}\mathscr{H}\mathrm{om}_X(E,E), \mathscr{O}_X) \xrightarrow{\sim} \mathrm{Hom}_X(\mathscr{O}_X, \mathscr{O}_X)$$

is G-equivariant. In particular, it restricts to an isomorphism on the G-invariant parts. Since the trace map $\mathrm{tr} \in \mathrm{Hom}_X(\mathbf{R}\mathscr{H}\mathrm{om}_X(E,E), \mathscr{O}_X)$ gets sent to $r\cdot \mathrm{id}_{\mathscr{O}_X}$, which is G-invariant in virtue of Remark 2.17, it follows that tr must be G-invariant, too. In other words,

$$\mathrm{tr} \in \mathrm{Hom}_X(\mathbf{R}\mathscr{H}\mathrm{om}_X(E,E), \mathscr{O}_X)^G \cong \mathrm{Hom}_{\mathbf{D}(\mathrm{QCoh}_X^G)}(\mathbf{R}\mathscr{H}\mathrm{om}_X(E,E), \mathscr{O}_X).$$

We can then take the shifted cone of tr in $\mathbf{D}(\mathrm{QCoh}_X^G)$ to obtain a distinguished triangle

$$\mathbf{R}\mathscr{H}\mathrm{om}_X(E,E)_0 \to \mathbf{R}\mathscr{H}\mathrm{om}_X(E,E) \xrightarrow{\mathrm{tr}} \mathscr{O}_X$$

in $\mathbf{D}(\mathrm{QCoh}_X^G)$, lifting (4.3).

We have proved that the splitting (4.2) is G-equivariant. This in particular implies that the projection

$$\mathsf{q}\colon \mathrm{Ext}_X^1(E, E\otimes \pi_M^*\mathbb{L}_{M/B}) \to \mathrm{Ext}_X^1(\mathbf{R}\mathscr{H}\mathrm{om}_X(E,E)_0, \pi_M^*\mathbb{L}_{M/B})$$

from the full Ext group containing the element $\mathrm{At}_{E/Y}$, is a morphism of G-representations, in particular it preserves G-invariant parts. Therefore, $\mathrm{At}_{E/Y}$ maps to an element

$$\mathsf{q}(\mathrm{At}_{E/Y}) \in \mathrm{Ext}_X^1(\mathbf{R}\mathscr{H}\mathrm{om}_X(E,E)_0, \pi_M^*\mathbb{L}_{M/B})^G.$$

Note that $\omega_{\pi_M} = \pi_Y^*\omega_{Y/B}$ is naturally G-equivariant. By equivariant Verdier duality along the proper morphism π_M (cf. Corollary 2.28), the latter group is canonically isomorphic to

$$\begin{aligned}
&\mathrm{Ext}_X^1(\mathbf{R}\mathscr{H}\mathrm{om}_X(E,E)_0\otimes\omega_{\pi_M}[d], \pi_M^*\mathbb{L}_{M/B}\otimes\omega_{\pi_M}[d])^G\\
&\qquad = \mathrm{Ext}_X^{1-d}(\mathbf{R}\mathscr{H}\mathrm{om}_X(E,E)_0\otimes\omega_{\pi_M}, \pi_M^!\mathbb{L}_{M/B})^G\\
&\qquad \cong \mathrm{Ext}_M^{1-d}(\mathbf{R}\pi_{M*}(\mathbf{R}\mathscr{H}\mathrm{om}_X(E,E)_0\otimes\omega_{\pi_M}), \mathbb{L}_{M/B})^G\\
&\qquad = \mathrm{Hom}_M(\mathbb{E}, \mathbb{L}_{M/B})^G\\
&\qquad \cong \mathrm{Hom}_{\mathbf{D}(\mathrm{QCoh}_M^G)}(\mathbb{E}, \mathbb{L}_{M/B})
\end{aligned}$$

where we have set

$$\mathbb{E} = \mathbf{R}\pi_{M*}(\mathbf{R}\mathscr{H}\mathrm{om}_X(E,E)_0 \otimes \omega_{\pi_M})[d-1],$$

and we have used again that G is reductive for the last isomorphism. The morphism $\phi \in \mathrm{Hom}_M(\mathbb{E},\mathbb{L}_{M/B})$ determined as the image of the truncated Atiyah class $\mathrm{At}_{E/Y}$ is a relative obstruction theory on M/B by [12, Thm. 4.1]. Therefore we have shown its equivariance in the sense of Definition 0.1.

The proof of Theorem B is complete.

Remark 4.6. Let Y be a smooth complex projective toric 3-fold. Let $G = \mathbb{G}_m^3 \subset Y$ be the open torus. The above result proves G-equivariance of the obstruction theory on the following classical moduli spaces:

(1) the Hilbert scheme of points $\mathrm{Hilb}^n Y$,
(2) the moduli space $I_m(Y,\beta)$ of ideal sheaves $\mathscr{I}$ with $\mathrm{ch}\,\mathscr{I} = (1,0,-\beta,-m)$,
(3) the moduli space $P_m(Y,\beta)$ of stable pairs $(\mathcal{F},s)$ with $\chi(\mathcal{F}) = m$ and $[\mathcal{F}] = \beta$,
(4) the Quot scheme $\mathrm{Quot}_Y(F,n)$ of length n quotients of a G-equivariant exceptional locally free sheaf F, as in [26].

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