ALGEBRAIC GEOMETRY

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Homework 1 — **Due date**: 25/10/2022

Exercise 1. Let $\eta\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of abelian groups on a topological space X. Show by a direct computation that, for any point $x \in X$, one has $(\ker \eta)_x = \ker(\eta_x)$, and $(\operatorname{im}\eta)_x = \operatorname{im}(\eta_x)$. Deduce that a sequence of sheaves

$$\cdots \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to \cdots$$

is exact if and only if the induced sequence

$$\cdots \to \mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x \to \cdots$$

is exact for every $x \in X$.

Exercise 2. Let A be a ring. For a nonempty open subset U of a topological space X, consider the functor $\Gamma(U,-)\colon \operatorname{Sh}(X,\operatorname{Mod}_A) \to \operatorname{Mod}_A$ sending $\mathcal{F} \mapsto \mathcal{F}(U)$. Show that it is left exact. That is, it transforms an exact sequence of sheaves $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ into an exact sequence of A-modules

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U).$$

Exercise 3. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of abelian groups on a topological space X. Show that, if U is an open subset of X, the set of homomorphisms $\operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ has a natural structure of abelian group. Show that the presheaf

$$U \mapsto \operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$$

is a sheaf of abelian groups. It is called the *sheaf of local morphisms* of $\mathcal{F}$ into $\mathcal{G}$.

Exercise 4. Let $\mathcal{F}$ be a presheaf, $\mathcal{G}$ a sheaf, $\eta_1, \eta_2\colon \mathcal{F} \to \mathcal{G}$ two morphisms of presheaves of sets such that $\eta_{1,x} = \eta_{2,x}$ for every $x \in X$. Show that $\eta_1 = \eta_2$.

Exercise 5. Let X be a topological space, $\mathcal{B} \subset \tau_X$ a base of open sets and $\mathcal{F}$, $\mathcal{G}$ two sheaves on X. Suppose given a morphism of $\mathcal{B}$-sheaves

$$\{\eta_U\colon \mathcal{F}(U) \to \mathcal{G}(U) \mid U \in \mathcal{B}\}.$$

Then, it extends uniquely to a sheaf homomorphism $\eta\colon \mathcal{F} \to \mathcal{G}$. Furthermore, if η_U is surjective (resp. injective, or an isomorphism) for every $U \in \mathcal{B}$, then so is η.