Introduction to Enumerative Geometry

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Modern enumerative geometry is not so much about numbers as it is about deeper properties of the moduli spaces that parametrize the geometric objects being enumerated.

Lectures on K-theoretic computations in enumerative geometry
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Preface

What is in this book. This book is an expanded version of the lecture notes I wrote for the PhD course *Introduction to enumerative geometry – Classical and virtual techniques* I gave at SISSA in the fall of 2019. The goal of the course was to provide a gentle introduction, aimed mainly at first year graduate students, to the subject of enumerative geometry, highlighting the differences and the analogies between the modern approach (using virtual fundamental classes) and the classical approach (without virtual fundamental classes). A natural bridge between the classical and the modern world is provided by the theory of *torus localisation*. In some sense, these lecture notes can be thought of as an introduction to equivariant cohomology and localisation. Indeed, Chapters 6–8 are entirely devoted to this subject in the classical setup, and Chapters 9–10 constitute un upgrade to the virtual setting.

Another possible way to read this book is as an introduction to the *Hilbert scheme of points* $\mathrm{Hilb}^n Y$ on a nonsingular variety Y. Indeed, after explaining in Chapter 5 a hands-on approach to the construction of $\mathrm{Hilb}^n \mathbb{A}^d$, we discuss special properties in the cases $d \leq 3$, proving in particular that $\mathrm{Hilb}^n \mathbb{A}^3$ is a *critical locus*, i.e. the zero locus of the differential of a regular function on a smooth variety. We present a series of calculations involving $\mathrm{Hilb}^n Y$, with special focus on $\dim Y = 3$. On the classical side, we compute the Euler characteristic of the Hilbert scheme of points in arbitrary dimension. On the virtual side, we explain the seminal calculation by Maulik–Nekrasov–Okounkov–Pandharipande [**135**] of the degree 0 Donaldson–Thomas invariants of a toric Calabi–Yau 3-fold Y, thus proving the relation

$$\mathsf{DT}_n^Y = (-1)^n \chi(\mathrm{Hilb}^n Y).$$

The calculation of DT_n^Y was chosen as a concrete way to see the virtual localisation formula in action. The reader is referred to Pandharipande–Thomas [**165**], Okounkov [**158**] or Szendrői [**189**] for beautiful surveys (technically more advanced than this text) on Donaldson–Thomas theory and other modern enumerative theories. We finally point out that we do not claim any originality in this text, except possibly for the exposition.

To the reader. This text is meant to be accessible without a strong background in algebraic geometry, although some familiarity with elementary aspects of scheme theory, basic theory of coherent sheaves and intersection theory might make the reading smoother. In any case, we shall review some preliminaries in Chapter 2. Also, Appendix A and Appendix B are short introductions to Deformation Theory and Intersection Theory respectively. In addition, here is a list of excellent references, that we will refer to in the text whenever necessary:

- for scheme theory at various levels, see [**96, 50, 129, 198**],
- for toric varieties, see [**68, 42**],
- for quotients in algebraic geometry (GIT), see [**148, 153, 149, 44**],
- for intersection theory, see [**67, 51**],
- for deformation theory, see [**180, 98**] and [**61**, Chapter 6],
- for derived categories and derived functors, see [**107**].

Finally, the reader will find a number of exercises disseminated along the text, of varying difficulty.

Acknowledgements.

CHAPTER 1

Counting in Algebraic Geometry

SUMMARY. In this introductory chapter we give a rather informal overview of the most prominent aspects of enumerative geometry, both in its classical and modern flavours. By *classical enumerative geometry* we mean, essentially, the subject of intersection theory on various classical moduli spaces (such as the Grassmannian), *before virtual fundamental classes were invented* — this is what the adjective 'classical' refers to. On the other hand, *modern* enumerative geometry was born with virtual fundamental classes, in the late 90's [**127, 19**].

The effect of having virtual classes at our disposal is not (just) that we have a more sophisticated way to answer old questions with advanced techniques. It means that we are given for free a cascade of invariants to compute. Moreover, such a machinery is so powerful that it allows us (or forces us) to cross the land of numbers: although it is beyond the scope of these notes, many invariants, more 'refined' than bare numbers, can be defined via virtual classes. This is for instance the point of view of Okounkov's lecture notes on K-theoretic enumerative geometry [**158**].

This first chapter, in which we work entirely over $\mathbb{C}$ for the sake of concreteness, is a slow motion tour through the unexpected difficulties one faces when trying to perform rigorously one of the most naive operations in mathematics: counting.

1.1. Asking the right question

The typical question in classical enumerative geometry asks how many objects satisfy a given list of geometric conditions. The presence of this 'list' makes the subject tightly linked to Intersection Theory, which explains why we included Appendix B at the end of these lecture notes.

Examples of classical problems in the subject are the following:

(1) How many lines $\ell \subset \mathbb{P}^{n+1}$ are incident to $2n$ general $(n-1)$-planes $\Lambda_1, \ldots, \Lambda_{2n} \subset \mathbb{P}^{n+1}$? (Answer in Section 8.4, preview in Section 3.4)
(2) How many lines $\ell \subset \mathbb{P}^3$ lie on a general cubic surface $S \subset \mathbb{P}^3$? (Answer in Section 8.2)
(3) How many lines $\ell \subset \mathbb{P}^4$ lie on a generic quintic 3-fold $Y \subset \mathbb{P}^4$? (Answer in Section 8.3)
(4) How many Weierstrass points are there on a general genus g curve? (Answer in Section 1.5)
(5) How many smooth conics are tangent to five general plane conics? (Answer and much more in the fantastic book [**51**]).

Let us consider the examples (1), (2) and (3) above: the objects we want to count are lines in some projective space, on which we impose some geometric constraints, such as intersecting other linear spaces or lying on a smooth hypersurface. Let us pause for a second: in (1), we impose incidence with $2n$ linear spaces. Why exactly $2n$? We immediately see that in order to even get started we have to ask ourselves the following:

> **Question 1**. How do we know how many constraints we should put on our objects in order to *expect* a finite answer? In other words, how do we ask the right question?

See Section 4.5 for a full treatment of the topic 'expectations' in the case of lines on hypersurfaces: we will confirm that the expected number of lines on a hypersurface $Y \subset \mathbb{P}^n$ of degree d is finite (neither zero nor infinity) precisely when $d = 2n-3$. Here is a warm-up example to shape one's intuition.

EXERCISE 1.1.1. Let $d > 0$ be an integer. Determine the number m_d having the following property: you expect finitely many smooth complex projective curves $C \subset \mathbb{P}^2$ of degree d passing through m_d general points in $\mathbb{P}^2$. (**Hint**: Start with small d. Then conjecture a formula for m_d).

1.2. Counting the points on a moduli space

The main idea to guide our geometric intuition in formulating and solving an enumerative problem should be the following recipe:

- construct a moduli[1] space $\mathcal{M}$ for the objects we are interested in,
- compactify $\mathcal{M}$ if necessary,
- impose $\dim \mathcal{M}$ conditions to expect a finite number of solutions, and
- count these solutions via Intersection Theory methods (exploiting compactness of $\mathcal{M}$).

None of these steps is a trivial one, in general. The last two, in a little more detail, would ideally go as follows: each 'condition' we impose is described by a cycle $Z_i \subset \mathcal{M}$ which is Poincaré dual to a Chow class $\alpha_i \in A^*\mathcal{M}$, and the intersection of these cycles is represented by the product $\alpha = \alpha_1 \cup \cdots \cup \alpha_r \in A^{\dim \mathcal{M}}\mathcal{M}$, where $\cup$ is the ring multiplication in the Chow ring $A^*\mathcal{M}$. The final step asks us to compute the degree

$$\int_{\mathcal{M}} \alpha = \deg_{\mathcal{M}}(\alpha \cap [\mathcal{M}]) \in \mathbb{Z},$$

where the integral sign (also written $\deg_{\mathcal{M}}$) stands for the pushforward of Chow groups $A_*\mathcal{M} \to A_*\mathrm{pt} = \mathbb{Z}$ along the structure morphism $\mathcal{M} \to \mathrm{pt}$.

Another important point in the subject is the following. Say we have a precise question, such as (2) above. Then, in the above recipe, as our $\mathcal{M}$ we should take the Grassmannian of lines in $\mathbb{P}^3$ (informally introduced in Chapter 3, more thoroughly in Section 4.1), which is a compact 4-dimensional smooth algebraic variety (or complex manifold). Imagine we have found a sensible algebraic variety structure on the set $\mathcal{M}_S \subset \mathcal{M}$ of lines lying on the surface S. If we have done everything right, the space $\mathcal{M}_S$ consists of finitely many points, and now we need to count these points. In mathematical terms, the only sensible operation we can perform is to take the *degree* (cf. Definition B.1.3) of the (0-dimensional) fundamental class of $\mathcal{M}_S$. But then it is natural to ask ourselves:

> **Question 2**. How do we know this degree is the answer to our original question? In other words, how to ensure that our algebraic solution is actually *enumerative*?

Put in more technical terms, how do we make sure that each line $\ell \subset S$ appears as a point in the moduli space $\mathcal{M}_S$ with multiplicity one? The truth is that we cannot *always* be sure that this is the case. It will be, both for problem (2) and problem (3) (by Lemma 4.5.4 and Theorem 8.3.3 respectively), but not in general. However, we should get used to the idea that this is not something to be worried about: if a solution comes with multiplicity bigger than one, there usually is a good geometric reason for this, and we should not disregard it (see Figure 2.1 for a simple example of a degenerate intersection where this phenomenon occurs). More precisely (but not too precisely), if a point on the moduli space is 'fat', i.e. nonreduced, then it means that the geometric object it corresponds to has nontrivial deformations. A short, incomplete introduction to Deformation Theory is given in Appendix A for the reader's convenience.

Remark 1.2.1. Compactness of $\mathcal{M}$ (in the above example, the Grassmannian) is used in order to make sense of taking the *degree* of cycles. Intuitively, we need compactness in order to prevent the solutions of our enumerative problem to escape to infinity, like for instance it would occur if we were to intersect two *parallel* lines in $\mathbb{A}^2$.

Compatcness really is a non-negotiable condition we have to ask of our moduli space — with an important exception, that will be treated in later sections: the case when the moduli space has a torus action. In this case, if the torus fixed locus $\mathcal{M}^{\mathbb{T}} \subset \mathcal{M}$ is compact, a sensible enumerative solution to a counting

[1] The latin word *modulus* means *parameter*, and its plural is *moduli*. Thus a *moduli space* is to be thought of as a parameter space for objects of some kind.

problem can be *defined* by means of the *localisation formula*, one of the most important tools in enumerative geometry (and in these notes). The original formula due to Atiyah and Bott will be proved in Theorem 7.5.1. A virtual analogue due to Graber and Pandharipande [**81**] will be proved in Theorem 9.3.8, and the latter will be applied to the study of 0-dimensional Donaldson–Thomas invariants of local Calabi–Yau 3-folds (arising from non-compact moduli spaces carrying a torus action).

One more fundamental notion in counting problems, *transversality*, is discussed in the next subsection, by means of an elementary example.

1.3. Transversality, and counting lines through two points

DEFINITION 1.3.1. Let Y be a quasiprojective variety. We say that subvarieties $Z_1,\ldots,Z_m \subset Y$ *intersect transversely* at a smooth point $y \in Y$ if y is a smooth point on each Z_i and $\operatorname{codim}(\bigcap_i T_y Z_i, T_y Y) = \sum_i (\operatorname{codim} T_y Z_i, T_y Y)$. Here $T_x X = (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ denotes the Zariski tangent space of X at a point $x \in X$.

Consider the enumerative problem of counting the number of lines in $\mathbb{P}^2$ through two given points $p, q \in \mathbb{P}^2$. Let N_{pq} be this number. Then, of course,

$$\mathsf{N}_{pq} = 1, \quad \text{as long as } p \neq q.^{2}$$

However, the *true* answer would be $\mathsf{N}_{pp} = \infty$ when $p = q$, corresponding to the cardinality of the pencil $Z_p \cong \mathbb{P}^1$ of lines through p (see Figure 1.1).

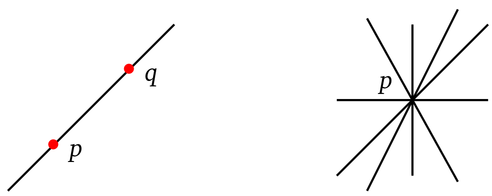

FIGURE 1.1. The unique line through two distinct points, and the infinitely many lines through one point in the plane.

The point we want to make in this section is that

$$\boxed{\text{the answer ‘1’ can be recovered in the degenerate setting } p = q.}$$

Remark 1.3.2 (Let us not cheat)**.** To obtain the answer relative to the picture on the left, we have to notice that the two cycles

$$Z_p = \left\{ \ell \subset \mathbb{P}^2 \,\middle|\, p \in \ell \right\} \subset \mathbb{P}^{2*}, \quad Z_q = \left\{ \ell \subset \mathbb{P}^2 \,\middle|\, q \in \ell \right\} \subset \mathbb{P}^{2*}$$

intersect transversely in the dual projective space $\mathbb{P}^{2*}$, and we can use the intersection product on $\mathbb{P}^{2*}$ to compute $Z_p \cdot Z_q = \#(Z_p \cap Z_q) = 1$. Now, using basic intersection theory, it is clear how to obtain the answer ‘1’ also in the case $p = q$. Since we are working in $\mathbb{P}^{2*} \cong \mathbb{P}^2$, we know that any $q \neq p$ yields a *homologous* cycle $Z_q \sim Z_p$, and again the intersection product yields $Z_p^2 = Z_p \cdot Z_q = 1$. But in general we will not be working in such a pleasant ambient space and thus we will not know whether algebraic deformations such as $Z_p \rightsquigarrow Z_q$, leading to a transverse setup, are available.

Now, the case $p = q$ is a ‘degeneration’ of the case $p \neq q$, and we certainly want our enumerative answer not to depend on small perturbations of the geometry of the problem. Why do we want that? Just because we are reasonable people: we were already taught how to be reasonable when our first Calculus teacher told us that a decent function is *at least* continuous.[3]

[2] For the sake of completeness, this will be proved in Section 8.1.

[3] Ultimately, we are going to study *invariants*, e.g. Donaldson–Thomas invariants. They deserve to be called that precisely because they don’t change if we slightly (but holomorphically) deform the variety they are attached to.

Next we explain how to get the 'correct' answer

$$\mathsf{N}_{pp}^{\text{corrected}} = 1$$

by means of the *excess intersection formula*, one of the most important tools in classical enumerative geometry. We mention it not only because it is a beautiful piece of intersection theory, but also because it lies at the very roots of *modern* enumerative geometry, lying right at the foundation of the idea of virtual classes.

Before we start, recall that the following important notion.

DEFINITION 1.3.3. The *conormal sheaf* of a closed immersion of schemes $X \hookrightarrow M$ defined by an ideal $\mathscr{I} \subset \mathscr{O}_M$ is the quasicoherent[4] $\mathscr{O}_X$-module

$$\mathscr{C}_{X/M} = \mathscr{I}/\mathscr{I}^2,$$

and the *normal sheaf* is its $\mathscr{O}_X$-linear dual,

$$\mathscr{N}_{X/M} = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{I}/\mathscr{I}^2, \mathscr{O}_X).$$

The sheaves $\mathscr{C}_{X/M}$ and $\mathscr{N}_{X/M}$ are locally free (of rank d) when $X \hookrightarrow M$ is a regular immersion (of codimension d).

Example 1.3.4. If $X \hookrightarrow M = \mathbb{P}^r$ is a hypersurface of degree d, then the ideal sheaf of X in $\mathbb{P}^r$ is the invertible sheaf $\mathscr{O}_{\mathbb{P}^r}(-d)$, so $\mathscr{N}_{X/\mathbb{P}^r} = \mathscr{O}_{\mathbb{P}^r}(d)|_X$.

DEFINITION 1.3.5. Let $X \hookrightarrow M$ be a closed immersion. We set $N_{X/M} = \operatorname{Spec}\operatorname{Sym}\mathscr{C}_{X/M}$. It is naturally a scheme over X. We will also refer to it as the *normal sheaf* to X in M, by abuse of terminology.

EXERCISE 1.3.6. Let $X \hookrightarrow M$ be a closed immersion, $\widetilde{M} \to M$ a morphism, set $\widetilde{X} = X \times_M \widetilde{M}$ and let $g: \widetilde{X} \to X$ be the induced map. Show that there is a natural injective map of sheaves $\mathscr{N}_{\widetilde{X}/\widetilde{M}} \hookrightarrow g^*\mathscr{N}_{X/M}$, which is an isomorphism whenever $\widetilde{M} \to M$ is flat. Deduce that there is a closed immersion $N_{\widetilde{X}/\widetilde{M}} \hookrightarrow g^*N_{X/M}$ of schemes over $\widetilde{X}$. (**Hint**: Try to construct a surjection $g^*\mathscr{C}_{X/M} \twoheadrightarrow \mathscr{C}_{\widetilde{X}/\widetilde{M}}$ involving the conormal sheaves. If in need of further hints, see [**185**, Tag 01R1]).

Also recall (see [**96**, II.8.13] for a reference) that on any projective space $\mathbb{P}^r$ we have the *Euler sequence*

$$0 \to \mathscr{O}_{\mathbb{P}^r} \to \mathscr{O}_{\mathbb{P}^r}(1)^{\oplus r+1} \to \mathscr{T}_{\mathbb{P}^r} \to 0, \tag{1.3.1}$$

where $\mathscr{O}_{\mathbb{P}^r}(1)$ is the hyperplane bundle and $\mathscr{T}_{\mathbb{P}^r}$ is the algebraic tangent bundle.

Now back to our problem. The $\mathbb{P}^1$ of lines through p can be neatly seen as the exceptional divisor E in the blowup $B = \operatorname{Bl}_p \mathbb{P}^2$, cf. Figure 1.2.

Looking at the fibre diagram

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & B \arrow[d, "\pi"] \\ p \arrow[r, hook] & \mathbb{P}^2 \end{tikzcd} \tag{1.3.2}$$

we know by Exercise 1.3.6 that there is an injection of locally free sheaves $\mathscr{N}_{E/B} = \mathscr{O}_E(-1) \subset g^*\mathscr{N}_{p/\mathbb{P}^2}$. The *excess bundle* (or *obstruction bundle*)

$$\mathrm{Ob}_{pp} \to \mathbb{P}^1$$

of the fibre diagram (1.3.2) is defined as the quotient of these two bundles [**67**, Section 6.3]. But the short exact sequence

$$0 \to \mathscr{O}_E(-1) \to \mathscr{O}_E \otimes T_p\mathbb{P}^2 \to \mathrm{Ob}_{pp} \to 0$$

[4] It is coherent as long as M is locally noetherian. Quasicoherent and coherent sheaves are recalled in Definition 2.2.2.

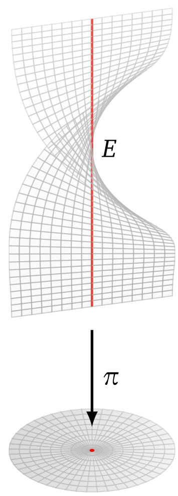

FIGURE 1.2. The blowup $\pi \colon B \to \mathbb{P}^2$ of the plane in one point.

is nothing but the Euler sequence (1.3.1) on $\mathbb{P}^1$ twisted by -1. Therefore

$$\mathrm{Ob}_{pp} = \mathscr{T}_{\mathbb{P}^1}(-1) = \mathscr{O}_{\mathbb{P}^1}(2-1) = \mathscr{O}_{\mathbb{P}^1}(1).$$

Note that we can repeat the process with $q \neq p$, which would yield $\pi^{-1}(q) = \mathrm{pt}$. In this case we get $\mathrm{Ob}_{pq} = 0$. We can now write a universal formula for our counting problem: if $\mathcal{M}_{pq} = \pi^{-1}(q)$ is the 'moduli space of lines' through p and q, the *virtual number* of lines through p and q is

$$\int_{\mathcal{M}_{pq}} e(\mathrm{Ob}_{pq}) = 1,$$

for all $(p,q) \in \mathbb{P}^2 \times \mathbb{P}^2$. The *Euler class* $e(V)$ of a vector bundle V is its top Chern class — see Section 2.3 or Appendix B.2. Note that the rank of Ob_{pq} is the difference between the actual dimension of the moduli space, and the expected one.

Remark 1.3.7. Note that, unlike in Remark 1.3.2, we have now obtained $\mathsf{N}_{pp}^{\text{corrected}} = 1$ as an intersection number on the *actual* moduli space $Z_p \cong \mathbb{P}^1 \cong E$.

Remark 1.3.8 (Usually it's worse)**.** Unfortunately, in more complicated situations (but also not that complicated), we often do not even know whether the geometric setup we are studying is a degeneration of a transverse one. If it were, we would like to dispose of a technology allowing us to 'count' in the transverse setup and argue that the number we obtain there equals the one we are after. This sounds like a reasonable wish, but it is way too optimistic. We should *not* aim at this: not only because counting is often difficult also in transverse situations, but mainly because we simply may not have enough algebraic deformations to pretend that the geometry of the problem is transverse.

Example 1.3.9 (You can't always achieve transversality)**.** If we were to count self-intersections of a (-1)-curve on a surface,[5] there would be no way to deform these curves off themselves to make them self-intersect transversely! See also Exercise 1.3.11 below.

[5] A (-1)-curve on a surface S is a curve $C \subset S$ such that $C.C = -1$, where the intersection number $C.C$ can be seen as the degree of the normal bundle $\mathscr{N}_{C/S} = \mathscr{O}_S(C)|_C$ to C in S.

This discussion allows us to formulate another intrinsic difficulty in enumerative geometry. Suppose, just to dream for a second, that we are able to solve *all* enumerative problems in generic (transverse) situations, and we know that the answer does not change after a small perturbation of the initial data.

> **Question 3**. How do we 'pretend' we can work in a transverse situation when there is none available (e.g. in Example 1.3.9)?

The modern way to do this is to use *virtual fundamental classes* (cf. Section 9.1 and Appendix C).

1.3.1. Two more words on excess intersection. Problem (5), known as 'the five conics problem', is a typical example of an excess intersection problem. See [**51**] for a thorough analysis and solution of this problem. As we shall see in Section 4.4.1, a natural compact parameter space for plane conics is

$$\mathcal{M} = \mathbb{P}^5,$$

and the set of smooth conics is an open subvariety $U \subset \mathcal{M}$. The answer to Problem (5) is a certain finite subset of U. Let $C_1, \ldots, C_5$ be general plane conics. The conics that are tangent to a given conic C_i form a sextic hypersurface $Z_i \subset \mathcal{M}$, so we might be tempted to say that the answer to Problem (5) is the degree

$$\int_{\mathbb{P}^5} \alpha_1 \cup \cdots \cup \alpha_5 = 6^5,$$

where $\alpha_i = [Z_i] \in \mathrm{H}^2(\mathbb{P}^5, \mathbb{Z})$ is the divisor class of a sextic.[6] However, the cycles Z_i share a common two-dimensional component, namely the Veronese surface $\mathbb{P}^2 \subset \mathbb{P}^5$ of double lines. Therefore their intersection is 2-dimensional (far from transverse!), even though our intuition suggests that 5 hypersurfaces in $\mathbb{P}^5$ should intersect in a finite set. Note that this issue arose precisely because we insisted to work with a compact parameter space: double lines are singular, hence lie in the complement of U. But working with U directly was forbidden, because it is not proper!

The excess intersection formula is a tool that allows one to precisely compute (and hence get rid of) the enumerative contribution of the *excess locus*, namely the locus of non-transverse intersection among certain cycles — in this case the cycles $Z_1, \ldots, Z_5$. The way it works is precisely via blow-ups; often more than one is required to separate the common components of the nontransverse cycles. In the case of the five conics problem, only one blow-up is required.

In principle, after blowing up the excess locus, checking that the proper transforms are disjoint in the exceptional divisor, and blowing up again if necessary, one gets to the correct answer to the original question, but in practice it is often very hard to keep track of multiple blow-ups; the calculation becomes less and less intuitive and the modular meaning of the blowups appearing might be quite unclear.

In Exercise 1.3.11 you will compute an excess bundle for a more complicated problem than finding the number of lines through two points. Before tackling it, it is best to solve the following exercise.

EXERCISE 1.3.10. Show that the vector space V of homogeneous cubic polynomials in 3 variables is 10-dimensional. Identify

$$\mathbb{P}V = \mathbb{P}^9$$

with the space of degree 3 plane curves $C \subset \mathbb{P}^2$. Show that, for a given point $p \in \mathbb{P}^2$, the space of cubics passing through p forms a hyperplane

$$\mathbb{P}^8 \subset \mathbb{P}V.$$

EXERCISE 1.3.11. Let C_1 and C_2 be two plane cubics intersecting transversely in nine points $p_1, \ldots, p_9 \in \mathbb{P}^2$ (cf. Figure 1.3). Every cubic in the pencil $\mathbb{P}^1 \subset \mathbb{P}^9$ generated by C_1 and C_2 passes through $p_1, \ldots, p_9$. However, if the nine points were general, there would be a unique cubic passing through them. Find out where the answer '1' is hiding in this non-transverse geometry. This example is also discussed in [**165**, Section 0].

[6] Recall that the Picard group $\operatorname{Pic} \mathbb{P}^r = \mathrm{H}^2(\mathbb{P}^r, \mathbb{Z}) = \mathbb{Z}$ is generated by the hyperplane class h corresponding to $\mathscr{O}_{\mathbb{P}^r}(1)$, and the cohomology class of a degree d hypersurface in $\mathbb{P}^r$ corresponds to the class $d \cdot h$.

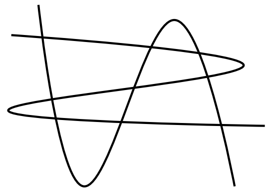

FIGURE 1.3. The nine intersection points $C_1 \cap C_2$.

1.4. Before and after the virtual class

Before virtual classes were invented (by Li–Tian [127] and Behrend–Fantechi [19]), excess intersection theory on a moduli space $\mathcal{M}$ was the way to go to solve enumerative problems regarding the objects parametrised by $\mathcal{M}$. Often, in practice, one needs several applications of the excess intersection formula in order to get to the final solution of the problem. One can see the virtual class as a way of packaging all the excess intersection theory in just *one* cycle class[7]

$$[\mathcal{M}]^{\mathrm{vir}} \in A_*\mathcal{M},$$

and appropriate integrals against this cycle[8] yield the enumerative answer we are after. Of course, integration against $[\mathcal{M}]^{\mathrm{vir}}$ is in general very hard, reflecting the difficulties arising from the classical methods.

Virtual fundamental classes allow one to think that even a horrible moduli space $\mathcal{M}$, say a singular scheme of impure dimension (cf. Figure 4.2 for a drawing of a nasty scheme), has a well-defined *virtual dimension* $\mathrm{d}^{\mathrm{vir}}$ at any point $p \in \mathcal{M}$, and this number is constant on p. It is given as the difference

$$\mathrm{d}^{\mathrm{vir}} = \dim T_p\mathcal{M} - \dim \mathrm{Ob}\,|_p,$$

where Ob is part of the data (a *perfect obstruction theory*) defining $[\mathcal{M}]^{\mathrm{vir}}$, and both dimensions on the right hand side may (and will) vary with p. The virtual fundamental class is a Chow (or homology) class

$$[\mathcal{M}]^{\mathrm{vir}} \in A_{\mathrm{d}^{\mathrm{vir}}}\mathcal{M} \to \mathrm{H}_{2\mathrm{d}^{\mathrm{vir}}}(\mathcal{M}, \mathbb{Q}),$$

that could be thought of as the fundamental class that $\mathcal{M}$ would have if it were of the form $\mathcal{M} = \{s = 0\}$ for s a *regular* section of a vector bundle (the bundle Ob) on a smooth variety.

There are just a handful of cases where integrating a cohomology class $\alpha \in A^*\mathcal{M} \to \mathrm{H}^{2*}(\mathcal{M}, \mathbb{Q})$ against $[\mathcal{M}]^{\mathrm{vir}}$ is accessible (at least in theory):

(1) The moduli space $\mathcal{M}$ is smooth. In this case $[\mathcal{M}]^{\mathrm{vir}} = e(\mathrm{Ob}) \cap [\mathcal{M}]$, hence

$$\int_{[\mathcal{M}]^{\mathrm{vir}}} \alpha = \deg_{\mathcal{M}}(e(\mathrm{Ob}) \cap \alpha), \quad \alpha \in \mathrm{H}^*(\mathcal{M}, \mathbb{Q}).$$

(2) The moduli space $\iota \colon \mathcal{M} \hookrightarrow A$ is the zero locus of a section of a vector bundle $V \to A$, where A is a smooth projective variety. In this case, we have $\iota_*[\mathcal{M}]^{\mathrm{vir}} = e(V) \cap [A]$, hence

$$\int_{[\mathcal{M}]^{\mathrm{vir}}} \iota^*\alpha = \deg_A(e(V) \cap \alpha), \quad \alpha \in \mathrm{H}^*(A, \mathbb{Q}).$$

(3) The moduli space $\mathcal{M}$ has a torus action. In this case, one applies the Graber–Pandharipande virtual localisation formula [81], that we review in Section 9.3.

It is clear that, since the machinery of the virtual class was invented, for any moduli space $\mathcal{M}$ we may want to study, we can ask the following question:

$$\boxed{\text{Does } \mathcal{M} \text{ have a virtual class?}}$$

[7]We refer to Appendix B.1 for the definition of the Chow group A_*X of a scheme X. As a sensible approximation, the reader can replace A_* with homology H_*.

[8]Recall that we assume $\mathcal{M}$ to be compact.

If the answer is yes, then we can define enumerative invariants attached to $\mathcal{M}$ (and often much more). What these invariants actually mean, geometrically, might be mysterious. For instance, they may be negative or rational. In any case, nowadays, defining a virtual class on a given space has become part of the thrill. In other words, a sensible portion of the enumerative geometer's effort is directed towards *defining* the problem, rather than stating it and solving it.

As a matter of fact, many (often badly behaved) moduli spaces turn out to have a virtual fundamental class. These include:

(i) the moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(X,\beta)$ to a smooth projective variety X (see [**15**]),
(ii) the moduli space $M_X^H(\alpha)$ of μ-stable torsion free sheaves with Chern character α on a smooth polarised 3-fold (X,H) such that $\mathrm{H}^0(X,K_X^{-1}) \neq 0$ (see [**191**, Theorem 3.30]),
(iii) the moduli space $P_X^H(\alpha)$ of stable pairs on a smooth polarised Calabi–Yau 3-fold (X,H), with Chern character α (see [**109**, Section 4]).

All this richness gives rise to three amongst the most modern counting theories (ordered from the oldest to the youngest):

$$\begin{aligned} \text{Gromov–Witten theory} &\coloneqq \text{intersection theory on } \overline{\mathcal{M}}_{g,n}(X,\beta), \\ \text{Donaldson–Thomas theory} &\coloneqq \text{intersection theory on } M_X^H(\alpha), \\ \text{Pandharipande–Thomas theory} &\coloneqq \text{intersection theory on } P_X^H(\alpha). \end{aligned}$$

All these theories can be seen as more complicated (virtual) versions of a well established theory:

$$\text{Schubert Calculus} \coloneqq \text{intersection theory on the Grassmannian } G(k,n).$$

No 'virtualness' is arising in Schubert Calculus, because — as already observed by Mumford [**147**] when he initiated the enumerative geometry of the moduli space of curves — the Grassmannian is the ideal moduli space one would like to work with: it is compact, smooth and, put in modern language, unobstructed. It does have a virtual fundamental class, but because of these properties it happens to coincide with its actual fundamental class.

1.5. Warming up: counting Weierstrass points

In this section we solve Question (4), following [**74**] verbatim. We use this as an example of the following toy model situation: the moduli space is the zero locus of a section of a vector bundle; in this case it has expected dimension 0, since the vector bundle is in fact a line bundle on a curve.

Let C be a smooth projective curve over $\mathbb{C}$. A *linear system* on C is a pair (L,V) where L is a line bundle on C and $V \subset \mathrm{H}^0(C,L)$ is a linear subspace. If $\deg L = d$ and $\dim_{\mathbb{C}} V = r+1$, then (L,V) is said to be a g_d^r on the curve C.

Let $v \in V \setminus 0$ be a section, $P \in C$ a point. One defines

$$\operatorname{ord}_P v = \dim_{\mathbb{C}} L_P / v_P \cdot L_P \in \mathbb{Z}_{\geq 0}$$

to be the *order of vanishing* of v at P.

Definition 1.5.1. Let (L,V) be a g_d^r. A point $P \in C$ is said to be a *ramification point* of (L,V) if there exists a section $v \in V \setminus 0$ such that $\operatorname{ord}_P v \geq r+1$. A ramification point of the canonical linear system $(K_C, \mathrm{H}^0(C,K_C))$ is called a *Weierstrass point*.

It is clear that $P \in C$ is a Weierstrass point if and only if $K_C(-gP)$ has a nonzero global section, where g is the genus of C. In the language of the previous sections, we are interested in 'counting'

$$\mathcal{M} = \{\, \text{Weierstrass points on } C \,\}.$$

Definition 1.5.1 can be phrased also in the following way, which was used for the first time by Laksov [120] to study ramification points of linear systems on curves in arbitrary characteristic. There exists a map

$$D^r\colon C\times V\to J^rL,\qquad (P,v)\mapsto D^rv(P),\tag{1.5.1}$$

where J^rL is the r-th jet bundle associated to L and $D^rv\in \mathrm{H}^0(C,J^rL)$ is a natural section defined by v, that is locally represented by the partial derivatives of order at most r of the local functions representing v. The vanishing of D^rv at P is equivalent to the condition $\mathrm{ord}_Pv\geq r+1$ of Definition 1.5.1. The map D^r is a map of vector bundles of the same rank $r+1$, so it is locally represented by an $(r+1)\times(r+1)$ matrix. The condition $D^rv(P)=0$ then says that (1.5.1) drops rank at P. This in turn means that P is a zero of the *Wronskian section*

$$\mathbb{W}_V=\det D^r\in \mathrm{H}^0\left(C,\wedge^{r+1}J^rL\right)$$

attached to (L,V). The *total ramification weight* of (L,V), namely the total number of ramification points (counted with multiplicities), is, by definition,

$$\mathrm{wt}_V=\int_C c_1\left(\wedge^{r+1}J^rL\right)=\deg_C\left(\wedge^{r+1}J^rL\right).$$

It can be computed by means of the classical short exact sequence

$$0\to L\otimes K_C^{\otimes r}\to J^rL\to J^{r-1}L\to 0$$

of jet bundles (see [74, Proposition 1.7] and the references therein for a proof). By induction, one obtains a canonical identification

$$\wedge^{r+1}J^rL=L^{\otimes r+1}\otimes K_C^{r(r+1)/2}.$$

Using that $\deg K_C=2g-2$, one finds the *Brill–Segre formula*

$$\mathrm{wt}_V=(r+1)d+(g-1)r(r+1)\tag{1.5.2}$$

attached to (L,V). For instance, since $h^0(C,K_C)=g$, the number of Weierstrass points (counted with multiplicities) is easily computed as

$$\mathrm{wt}_{K_C}=\int_C c_1\left(\wedge^gJ^{g-1}K_C\right)=\deg_C\left(\wedge^gJ^{g-1}K_C\right)=(g-1)g(g+1).\tag{1.5.3}$$

The zero locus of the Wronskian section $\mathbb{W}_V$ is reduced when C is general. Thus, going back to Question (4), we have found that there are $(g-1)g(g+1)$ Weierstrass points on a general curve of genus g.

Example 1.5.2. Formula (1.5.3) gives the 24 flexes on a plane quartic (a general curve of genus 3). Indeed, $K_C=\mathscr{O}_C(1)$ for such a curve, and a flex is by definition a ramification point of the linear system cut out by lines.

CHAPTER 2

Background material

SUMMARY. In this chapter we recall standard preliminaries from algebraic geometry, such as schemes, properties of morphisms of schemes and sheaves. We will also sketch the definition of algebraic Chern classes, and conclude the chapter with a brief recap on representable functors, that will be needed when we will define fine moduli spaces and universal families. By $\mathbf{k}$ we will always mean an algebraically closed field. Most of the time in later chapters, we will set $\mathbf{k} = \mathbb{C}$.

2.1. Varieties, schemes, morphisms

2.1.1. Schemes and their basic properties. The notion of *scheme* used in this text is the standard one (see e.g. [**129**, Chapter 2]), namely that of a locally ringed space $(X, \mathscr{O}_X)$ that is Zariski locally isomorphic to $(\operatorname{Spec} R, \mathscr{O}_{\operatorname{Spec} R})$, where R is a commutative unitary ring. The category of *affine schemes* is, by definition, dual to the category of such rings; more precisely, an affine scheme is a scheme isomorphic to $\operatorname{Spec} R$, for R a commutative ring. The global sections $\mathscr{O}_X(X) = \Gamma(X, \mathscr{O}_X)$ form the ring of *regular functions* on X. If $x \in X$ is a point, we denote by $(\mathscr{O}_{X,x}, \mathfrak{m}_x, \mathbf{k}(x))$ the local ring of germs of regular functions at x, and if $f : X \to Y$ is a morphism, we denote by $f^\# : \mathscr{O}_Y \to f_*\mathscr{O}_X$ the induced map on sheaves. Let S be a scheme; an S-scheme is a pair (X, f) where X is a scheme and $f : X \to S$ is a morphism. If $S = \operatorname{Spec} R$ for a ring R, we talk about R-schemes.

A topological space X is called *irreducible* if whenever one can write $X = V_1 \cup V_2$, with $V_i \subset X$ a closed subset, one has that either $V_1 = X$ or $V_2 = X$. A scheme is irreducible if its underlying topological space is irreducible. An important property of noetherian schemes is that they have a finite number of irreducible components (the maximal closed irreducible subsets), or, more generally, of associated points (cf. Section 2.1.3).

DEFINITION 2.1.1 (Dimension). The *dimension* $\dim X$ of a scheme X is the supremum of the lengths ℓ of the chains $V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_\ell \subset X$ of irreducible closed subsets. If X is a scheme, then $\dim X = \sup_i \dim X_i$, where $\{X_i\}_i$ are the irreducible components of X. If X is irreducible and $Y \subset X$ is a closed subset such that $\dim X = \dim Y$, then $X = Y$. We say that X is *of pure dimension* if all its irreducible components have the same dimension.

DEFINITION 2.1.2 (Finite type). A morphism of schemes $f : X \to S$ is *quasicompact* if the preimage of every affine open subset of S is quasicompact. We say that f is *locally of finite type* if for every $x \in X$ there exist Zariski open neighborhoods $x \in \operatorname{Spec} A \subset X$ and $f(x) \in \operatorname{Spec} B \subset S$ such that $f(\operatorname{Spec} A) \subset \operatorname{Spec} B$ and the induced map $B \to A$ is of finite type, i.e. A is isomorphic to a quotient of $B[x_1, \ldots, x_n]$ as a B-algebra, for some n. We say that f is *of finite type* if it is locally of finite type and quasicompact.

DEFINITION 2.1.3 (Locally noetherian). A scheme X is *locally noetherian* if every point $x \in X$ has an affine Zariski open neighborhood $x \in \operatorname{Spec} R \subset X$ such that R is a noetherian ring. If X is locally noetherian and quasicompact, then it is called *noetherian*.

EXERCISE 2.1.4. Let $f : X \to S$ be a morphism of schemes; show that if S is (locally) noetherian and f is (locally) of finite type, then X is (locally) noetherian. Show that a morphism from a noetherian scheme is quasicompact.

A scheme of finite type over a field $\mathbf{k}$ is then noetherian. On the other hand, even though for any noetherian ring R (e.g. a field), the affine scheme $\operatorname{Spec} R[\![x_1,\ldots,x_n]\!]$ is noetherian, the natural morphism $\operatorname{Spec} R[\![x_1,\ldots,x_n]\!] \to \operatorname{Spec} R$ is not locally of finite type. Also note that the ring $\mathscr{O}_X(X)$ need not be noetherian even if X is noetherian.

DEFINITION 2.1.5 (Immersions). A morphism of schemes $f : X \to S$ is a *closed immersion* (resp. *open immersion*) if f induces a homeomorphism between X and a closed subset (resp. an open subset) of S, and the induced local homomorphism $f_x^\# : \mathscr{O}_{S,f(x)} \to \mathscr{O}_{X,x}$ is surjective (resp. an isomorphism) for all $x \in X$. A closed (resp. open) subscheme of a scheme S is the image of a closed (resp. open) immersion. A morphism $X \to S$ is called an *immersion* if it can be factored as $X \to Y \to S$, where $X \to Y$ is an open immersion and $Y \to S$ is a closed immersion.

All immersions are *locally closed immersions*, i.e. can be factored as a closed immersion followed by an open immersion.[1] On the other hand, a locally closed immersion $X \to S$ is an immersion as long as S is locally noetherian, in which case it is also quasicompact. Also note that any open or closed subscheme of a noetherian scheme X is still noetherian, and for every affine open subset $U \subset X$ the ring $\mathscr{O}_X(U)$ is noetherian.

DEFINITION 2.1.6 (Reduced and integral). A scheme X is *reduced* if for every point $x \in X$ the local ring $\mathscr{O}_{X,x}$ is reduced, i.e. it has no nilpotent elements besides $0 \in \mathscr{O}_{X,x}$, the additive identity. A scheme is *integral* if it is reduced and irreducible.

EXERCISE 2.1.7. Let $R = \mathbb{C}[u,v]/(uv, v^2)$. Consider the affine scheme $X = \operatorname{Spec} R$. Show that the point $x \in X$ corresponding to the maximal ideal $(u,v) \subset R$ is the unique point such that $\mathscr{O}_{X,x}$ is not reduced.

For any scheme X, there is a unique reduced closed subscheme $i : X_{\mathrm{red}} \hookrightarrow X$ having the same topological space as X. In particular, the complement $X \setminus X_{\mathrm{red}}$ is empty. If X is quasicompact, then the kernel of the natural map $\mathscr{O}_X(X) \to \mathscr{O}_X(X_{\mathrm{red}})$ is the nilradical of $\mathscr{O}_X(X)$, usually denoted $\sqrt{0}$. See [**129**, Section 2.4.1] for more details.

Example 2.1.8. The prototypical example of a nonreduced scheme is the curvilinear[2] affine scheme

$$D_n = \operatorname{Spec} \mathbf{k}[t]/t^n, \quad n > 1.$$

One can show that quasicompact reduced schemes are precisely those schemes for which the regular functions on them are determined by their values on points. The function

$$0 \neq \bar{t} \in \mathbf{k}[t]/t^n,$$

i.e. the image of t under the canonical map $\mathbf{k}[t] \twoheadrightarrow \mathbf{k}[t]/t^n$, vanishes at the unique closed point of D_n, but it is not the zero function!

Let $f : X \to S$ be a morphism of schemes. The universal property of the fibre product, depicted in the diagram

$$\begin{tikzcd}
X \arrow[dr, dotted, "\Delta_f"] \arrow[drr, bend left, "\mathrm{id}"] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\
& X \times_S X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& X \arrow[r, "f"] & S
\end{tikzcd}$$

[1] Beware that some references treat ‘immersion’ and ‘locally closed immersion’ as synonyms, see e.g. [**185**, Tag 01IO]. Here we follow Liu’s terminology [**129**, p. 96].

[2] A $\mathbf{k}$-scheme is curvilinear if its embedding dimension is 1. The embedding dimension of a $\mathbf{k}$-scheme Y is the smallest integer e such that Y embeds as a closed subscheme inside a smooth $\mathbf{k}$-scheme of dimension e.

yields a canonical map $\Delta_f : X \to X \times_S X$, called the *diagonal*. It is a locally closed immersion [185, Tag 01KH]. The map f is said to be *separated* (resp. *quasiseparated*) if Δ_f is a closed immersion (resp. quasicompact). A $\mathbf{k}$-scheme X is said to be separated if the structure morphism $X \to \operatorname{Spec}\mathbf{k}$ is separated.

A morphism $f : X \to S$ is *affine* if the preimage of every affine open subscheme of S is affine. In fact, f is affine if and only if X is isomorphic (over S) to $\operatorname{Spec}_{\mathscr{O}_S} \mathscr{A}$, where $\mathscr{A}$ is a quasicoherent sheaf of $\mathscr{O}_S$-algebras [185, Tag 01S5], in which case one recovers $\mathscr{A} = f_* \mathscr{O}_X$, where $f : X \to S$ is the structure morphism. Usually, we will write Spec instead of $\operatorname{Spec}_{\mathscr{O}_S}$. See [185, Tag 01LQ] for the 'global Spec' construction; it is an example of representable functor, see Example 2.4.13.

2.1.2. Varieties, fat points and more morphisms. Let $\mathbf{k}$ be an algebraically closed field.

DEFINITION 2.1.9. An *algebraic variety* over $\mathbf{k}$ (or a $\mathbf{k}$-variety) is a separated scheme of finite type over $\operatorname{Spec}\mathbf{k}$.

Remark 2.1.10. Just for the reader to be aware, some people include reducedness (Definition 2.1.6) and/or irreducibility in the definition of algebraic variety. Other people (see e.g. [129, Section 2.3.4]) exclude separatedness. We will use a more classical definition of algebraic variety in Appendix B, since we will be following [67].

An *affine variety* is a $\mathbf{k}$-scheme of the form $\operatorname{Spec} A$, where $A = \mathbf{k}[x_1, \ldots, x_n]/I$ for some ideal I. Note that A is reduced if and only if I is a radical ideal. Thus a separated $\mathbf{k}$-scheme is an algebraic variety if it admits a finite covering by affine varieties. Note that, reassuringly, affine varieties are algebraic varieties, since affine morphisms are separated [185, Tag 01S5], and an algebraic variety X over $\mathbf{k}$ is affine if and only if the structure morphism $X \to \operatorname{Spec}\mathbf{k}$ is affine in the above sense.

An algebraic variety X is *projective* if it admits a closed immersion into projective space

$$\mathbb{P}^n = \mathbb{P}^n_{\mathbf{k}} = \operatorname{Proj}\mathbf{k}[x_0, x_1, \ldots, x_n],$$

for some n. In other words, X has to be of the form $V_+(J) = \operatorname{Proj}\mathbf{k}[x_0, x_1, \ldots, x_n]/J$ for J a homogeneous ideal in $\mathbf{k}[x_0, x_1, \ldots, x_n]$ — see [129, Section 2.3.3] for the notation $V_+(-)$. A variety is *quasiprojective* if it admits a locally closed immersion in some projective space, i.e. it is closed in an open subset of some $\mathbb{P}^n$. The same abstract scheme can of course be a (quasi)projective variety in many different ways.

Example 2.1.11. The *rational normal curve* of degree d is the image of the closed embedding $\iota_d : \mathbb{P}^1 \hookrightarrow \mathbb{P}^d$ defined by $(u : v) \mapsto (u^d : u^{d-1}v : \cdots : uv^{d-1} : v^d)$. This is the d-th Veronese embedding, determined by the complete linear series $(\mathscr{O}_{\mathbb{P}^1}(d), \mathrm{H}^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(d)))$. If $d = 3$, we get a *twisted cubic* in $\mathbb{P}^3$.

The following definition will be relevant when we will discuss the Hilbert scheme of points in Chapter 5.

DEFINITION 2.1.12 (Finite schemes). An algebraic $\mathbf{k}$-variety X is *finite* if $\mathscr{O}_X(X)$ is a finite dimensional $\mathbf{k}$-vector space. In this case, we say that $\ell = \dim_{\mathbf{k}} \mathscr{O}_X(X)$ is the *length* of X.

DEFINITION 2.1.13 (Fat points). A *fat point* (of length ℓ) over $\operatorname{Spec}\mathbf{k}$ is a $\mathbf{k}$-scheme of the form $X = \operatorname{Spec} A$, where A is a local artinian $\mathbf{k}$-algebra with residue field $\mathbf{k}$ (such that $\dim_{\mathbf{k}} A = \ell$).

The scheme D_n of Example 2.1.8 is an example of fat point of length n over $\operatorname{Spec}\mathbf{k}$. See Example 2.3.6 for an example of a very peculiar fat point. Fat points are important in Deformation Theory (and indeed they appear in Appendix A). The fat point D_2 is particularly important. Indeed, the *Zariski tangent space* $T_x X$ of a $\mathbf{k}$-scheme X at a point $x \in X$, which by definition is the $\mathbf{k}$-vector space $(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$, can be identified with

$$\operatorname{Hom}_x(D_2, X) = \{ D_2 \to X \mid \text{the closed point maps to } x \}.$$

Example 2.1.14 (D_2 as a limit of distinct points)**.** Consider the scheme

$$X_a = \operatorname{Spec}\mathbf{k}[x, y]/(y - x^2, y - a), \quad a \in \mathbf{k}.$$

For $a \neq 0$, this scheme consists of two reduced points, corresponding to the maximal ideals

$$(x \pm \sqrt{a}, y - a) \subset \mathbf{k}[x, y].$$

For $a = 0$, we get

$$X_0 = \operatorname{Spec} \mathbf{k}[x]/x^2 = D_2,$$

a point with multiplicity two. See Figure 2.1 for a visual explanation in the case $\mathbf{k} = \mathbb{C}$ (although only the real points are visible).

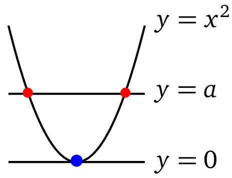

FIGURE 2.1. The intersection X_a of a parabola with the line $y = a$.

EXERCISE 2.1.15. Show that an algebraic variety X is both affine and projective if and only if it is finite. In particular, $X = \operatorname{Spec} \mathscr{O}_X(X)$, where $\mathscr{O}_X(X)$ is an artinian $\mathbf{k}$-algebra, and $\dim X = 0$. Show that, for any ℓ, the only reduced finite $\mathbf{k}$-variety of length ℓ is the disjoint union $\coprod_{1 \leq i \leq \ell} \operatorname{Spec} \mathbf{k}$, and more generally every finite $\mathbf{k}$-variety is a disjoint union of fat points. (In other words, a fat point over $\mathbf{k}$ could be defined as a finite $\mathbf{k}$-scheme $X \to \operatorname{Spec} \mathbf{k}$ such that $X_{\mathrm{red}} = \operatorname{Spec} \mathbf{k}$).

EXERCISE 2.1.16. Classify all finite dimensional $\mathbb{C}$-algebras of length 2 and 3 up to isomorphism.

EXERCISE 2.1.17. Give an example of a scheme X whose underlying topological space consists of finitely many points, and yet is *not* finite.

We finish this section recalling a few more properties of morphisms of schemes.

A stronger notion than separatedness is properness. A morphism $f : X \to S$ is *proper* if it is separated, of finite type, and universally closed. The latter means that for every base change $T \to S$, the induced map $X \times_S T \to T$ is topologically a closed map. The *valuative criterion* for proper morphisms says that a finite type morphism f is *proper* if and only if for every valuation domain A with fraction field K there exists exactly one way to fill in the dotted arrow in a commutative square

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d, hook, "i"'] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

in such a way that the resulting triangles are commutative. Such property can be rephrased by saying that for any A as above the map of sets

$$\operatorname{Hom}(\operatorname{Spec} A, X) \to \operatorname{Hom}(\operatorname{Spec} K, X) \times_{\operatorname{Hom}(\operatorname{Spec} K, S)} \operatorname{Hom}(\operatorname{Spec} A, S)$$

defined by $v \mapsto (v \circ i, f \circ v)$ is a bijection.

All closed immersions are proper, in particular of finite type (hence quasicompact).

DEFINITION 2.1.18 (Finite morphisms). A morphism is *finite* if it is both proper and affine.

Projective morphisms provide other examples of proper morphisms. We postpone them to Definition 2.2.8.

An important notion in moduli theory is *flatness*.

DEFINITION 2.1.19 (Flat morphisms). A morphism of schemes $X \to S$ is *flat at* $x \in X$ if the induced map of local rings $f_x^\#\colon \mathscr{O}_{S,f(x)} \to \mathscr{O}_{X,x}$ is flat. The morphism is *flat* if it is flat at every point $x \in X$. It is moreover called *faithfully flat* if it is flat and surjective.

All flat morphisms are open maps, in particular they have open image. Faithfully flat morphisms are epimorphisms in the category of schemes.

Let A and $\overline{A}$ be local artinian $\mathbf{k}$-algebras with residue field $\mathbf{k}$. We say that a surjection $u\colon \overline{A} \twoheadrightarrow A$ is a *square zero extension* if $(\ker u)^2 = 0$.

DEFINITION 2.1.20 (Unramified, smooth, étale). Let $f : X \to S$ be a locally of finite type morphism between $\mathbf{k}$-schemes. Then f is *unramified* (resp. *smooth*, *étale*) if for any square zero extension $\overline{A} \twoheadrightarrow A$ of fat points over $\mathbf{k}$ and for any solid diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \operatorname{Spec} \overline{A} \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

there exists at most one (resp. at least one, exactly one) way to fill in the dotted arrow in such a way that the resulting triangles are commutative.

Example 2.1.21. The following facts are often useful to keep in mind.

- A finite morphism $f : X \to S$ to a locally noetherian scheme is flat if and only if f is affine and $f_*\mathscr{O}_X$ is finite locally free. See [**185**, Tag 02KB].
- A morphism $X \to S$ of finite type $\mathbf{k}$-schemes is smooth (of relative dimension d) if and only if it is flat and the fibres $X_s \to \operatorname{Spec} \mathbf{k}(s)$ are smooth (and of pure dimension d) for every $s \in S$. See [**96**, III.10.2].
- A morphism $X \to S$ of nonsingular $\mathbf{k}$-varieties is smooth if and only if the tangent maps (see Notation 2.2.7 for the definition of the tangent sheaf) $\mathscr{T}_{X,x} \to \mathscr{T}_{Y,f(x)}$ are surjective for all $x \in X$. See [**96**, III.10.4].
- A morphism $X \to S$ of finite type $\mathbf{k}$-schemes, where X is Cohen–Macaulay (see Definition 2.2.35) and S is smooth, is flat whenever the fibres have the same dimension. This is *miracle flatness* — see [**185**, Tag 00R4] or the original reference [**134**, Theorem 23.1].
- A proper morphism which is injective on points and on tangent spaces is a closed immersion. In fact, closed immersions of schemes are precisely the proper monomorphisms.
- A morphism of smooth $\mathbb{C}$-varieties inducing an isomorphism on tangent spaces is étale.
- A bijective morphism of smooth $\mathbb{C}$-varieties of the same dimension is an isomorphism (this follows from Zariski Main Theorem).
- An étale injective (resp. bijective) morphism is an open immersion (resp. an isomorphism).

2.1.3. Schemes with embedded points. On a locally noetherian scheme X there are a bunch of points that are more relevant than all other points, in the sense that they reveal part of the behaviour of the structure sheaf: these points are the *associated points* of X.

Let R be a commutative ring with unity, and let M be an R-module. If $m \in M$, we let

$$\operatorname{Ann}_R(m) = \{\, r \in R \mid r \cdot m = 0 \,\} \subset R$$

denote its annihilator. A prime ideal $\mathfrak{p} \subset R$ is said to be *associated to* M if $\mathfrak{p} = \operatorname{Ann}_R(m)$ for some $m \in M$. The set of all associated primes is denoted[3]

$$\operatorname{AP}_R(M) = \{\, \mathfrak{p} \mid \mathfrak{p} \text{ is associated to } M \,\}.$$

Lemma 2.1.22. *Let $\mathfrak{p} \subset R$ be a prime ideal. Then $\mathfrak{p} \in \operatorname{AP}_R(M)$ if and only if $R/\mathfrak{p}$ is an R-submodule of M.*

[3] I personally have no problem with the more common notation $\operatorname{Ass}_R(M)$, but I chose to use $\operatorname{AP}_R(M)$ not to upset anyone.

PROOF. If $\mathfrak{p} = \mathrm{Ann}_R(m)$ for some $m \in M$, consider the map $\phi_m : R \to M$ defined by $\phi_m(r) = r \cdot m$. Since its kernel is by definition $\mathrm{Ann}_R(m)$, the quotient $R/\mathfrak{p}$ is an R-submodule of M. Conversely, given an R-linear inclusion $i : R/\mathfrak{p} \hookrightarrow M$, consider the composition $\phi : R \twoheadrightarrow R/\mathfrak{p} \hookrightarrow M$. Then $\phi_{i(1)}(r) = r \cdot i(1) = i(r+\mathfrak{p}) = \phi(r)$ for all $r \in R$, i.e. $\phi = \phi_{i(1)}$. □

Note that if $\mathfrak{p} \in \mathrm{AP}_R(M)$ then $\mathfrak{p}$ contains the annihilator of M, i.e. the ideal

$$\mathrm{Ann}_R(M) = \{\, r \in R \mid r \cdot m = 0 \text{ for all } m \in M \,\} \subset R.$$

DEFINITION 2.1.23. The minimal elements (with respect to inclusion) in the set

$$\{\, \mathfrak{p} \subset R \mid \mathfrak{p} \supset \mathrm{Ann}_R(M) \,\}$$

are called *isolated primes* of M.

From now on we assume R is noetherian and $M \neq 0$ is finitely generated. We have the following result.

Theorem 2.1.24 ([198, Theorem 5.5.10 (a)]). *Let R be a noetherian ring, $M \neq 0$ a finitely generated R-module. Then $\mathrm{AP}_R(M)$ is a finite nonempty set containing all isolated primes.*

DEFINITION 2.1.25. The non-isolated primes in $\mathrm{AP}_R(M)$ are called the *embedded primes* of M.

Moreover, we have the following facts:

- the R-module M has a *composition series*, i.e. a filtration by R-submodules

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_s = M$$

  such that $M_i/M_{i-1} = R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$. This series is not unique. However, for a prime ideal $\mathfrak{p} \subset R$, the number of times it occurs among the $\mathfrak{p}_i$ does not depend on the composition series. These primes are precisely the elements of $\mathrm{AP}_R(M)$.
- Any ideal $I \subset R$ has a *primary decomposition*, i.e. an expression as intersection

$$I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$$

  of primary ideals. A proper ideal $\mathfrak{q} \subsetneq R$ is called *primary* if whenever a product xy lies in $\mathfrak{q}$, either x or a power of y lies in $\mathfrak{q}$. Put differently, every zero-divisor in $R/\mathfrak{q}$ is nilpotent. One verifies that the radical of a primary ideal is prime, and one says that $\mathfrak{q}$ is $\mathfrak{p}$-primary if $\sqrt{\mathfrak{q}} = \mathfrak{p}$. One can always ensure that the decomposition is irredundant, i.e. removing any $\mathfrak{q}_i$ changes the intersection, and $\sqrt{\mathfrak{q}_i} \neq \sqrt{\mathfrak{q}_j}$ for all $i \neq j$.

EXERCISE 2.1.26. Let $I \subset R$ be an ideal. Show that the set

$$\{\, \sqrt{\mathfrak{q}_i} \,\}_i$$

is determined by I. Then show that elements of $\mathrm{AP}_R(R/I)$ are precisely the radicals of the primary ideals in a primary decomposition of I. In symbols,

$$\mathrm{AP}_R(R/I) = \{\, \sqrt{\mathfrak{q}_i} \,\}_i\,.$$

EXERCISE 2.1.27. Let $R = \mathbf{k}[x,y]$, $I = (xy, y^2)$ and $M = R/I$. Show that $\mathrm{AP}_R(M) = \{(y),(x,y)\}$.

The most boring situation is when R is an integral domain, in which case the generic point $\xi \in \operatorname{Spec} R$ is the only associated (and clearly isolated) prime. More generally, a reduced affine scheme $\operatorname{Spec} R$ has no embedded primes (in particular no embedded points, see below), i.e. the only associated primes are the isolated (minimal) ones, corresponding to its irreducible components.

Let R be an integral domain. For an ideal $I \subset R$, one often calls the associated primes of I the associated primes of R/I. The minimal primes above $I = \mathrm{Ann}_R(R/I)$ (i.e. containing I) correspond to the irreducible components of the closed subscheme

$$\operatorname{Spec} R/I \subset \operatorname{Spec} R,$$

whereas for every embedded prime $\mathfrak{p} \subset R$ there exists a minimal prime $\mathfrak{p}'$ such that $\mathfrak{p}' \subset \mathfrak{p}$. Thus $\mathfrak{p}$ determines an *embedded component* — a subvariety $V(\mathfrak{p})$ embedded in an irreducible component $V(\mathfrak{p}')$. If the embedded prime $\mathfrak{p}$ is maximal, we talk about an *embedded point*.

Fact 2.1.28. An algebraic curve (an algebraic variety of dimension 1) has no embedded points if and only if it is Cohen–Macaulay (the formal definition is given in Definition 2.2.35). However, there can be nonreduced Cohen–Macaulay curves: those curves with a fat component, such as the affine plane curve $\operatorname{Spec}\mathbf{k}[x,y]/x^2 \subset \mathbb{A}^2$. These objects often have moduli, i.e. deform (even quite mysteriously) in positive dimensional families. See [**46**, **47**, **199**, **200**] for generalities on multiple structures on schemes and [**34**] for a careful study of their moduli in some special cases.

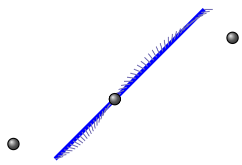

FIGURE 2.2. A thickened (Cohen–Macaulay) curve with an embedded point and two isolated (possibly fat) points.

Remark 2.1.29. An embedded component $V(\mathfrak{p})$, where $\mathfrak{p}$ is the radical of some primary ideal $\mathfrak{q}$ appearing in a primary decomposition $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_e$, is of course embedded in some irreducible component $V(\mathfrak{p}') \subset \operatorname{Spec} R/I$, but $V(\mathfrak{q})$ is not a *subscheme* of $V(\mathfrak{p}')$, because the fuzziness caused by nilpotent behavior (i.e. the difference between $\mathfrak{q}$ and its radical $\mathfrak{p}$) makes the bigger scheme $V(\mathfrak{q}) \supset V(\mathfrak{p})$ 'stick out' of $V(\mathfrak{p}') \subset \operatorname{Spec} R/I$.

Example 2.1.30. Consider $R = \mathbf{k}[x,y]$ and $I = (xy, y^2)$. A primary decomposition of I is

$$I = (x,y)^2 \cap (y).$$

However, $\operatorname{Spec} R/(x,y)^2$ is not scheme-theoretically contained in $\operatorname{Spec} R/y$.

In general, a subscheme Z of scheme Y has an embedded component if there exists a dense open subset $U \subset Y$ such that $Z \cap U$ is dense in Z but the scheme-theoretic closure of $Z \cap U \subset Z$ does not equal Z scheme-theoretically. For instance, if Y is irreducible, we say that $p \in Y$ supports an embedded point of a closed subscheme $Z \subset Y$ if $\overline{Z \cap (Y \setminus p)} \neq Z$ as schemes. In the example above, where $Y = \mathbb{A}^2$ and $Z = \operatorname{Spec}\mathbf{k}[x,y]/(xy,y^2)$, the scheme-theoretic closure of $Z \cap (\mathbb{A}^2 \setminus 0) \subset Z$ is not equal to Z.

2.2. Sheaves and supports

2.2.1. Coherent sheaves, projective morphisms. All sheaves on a scheme X in these notes will be sheaves of $\mathscr{O}_X$-modules, as treated in [**96**, II.5]. On any locally ringed space $(X, \mathscr{O}_X)$, the category $\operatorname{Mod}\mathscr{O}_X$ of sheaves of $\mathscr{O}_X$-modules is abelian [**185**, Tag 01AG].

DEFINITION 2.2.1. Let X be a scheme. We say that $F \in \operatorname{Mod}\mathscr{O}_X$ is *free* if it is isomorphic to $\mathscr{O}_X^{\oplus I}$, for a possibly infinite set I. A *free sheaf of rank* $r \in \mathbb{N}$ on X is a sheaf F of $\mathscr{O}_X$-modules isomorphic to $\mathscr{O}_X^{\oplus r}$ (i.e. take I of cardinality r). A *locally free sheaf* of rank r on X is a sheaf F such that there exists an open covering $X = \bigcup_i U_i$ for which $F|_{U_i}$ is free of rank r, for all i.

DEFINITION 2.2.2. A *quasicoherent sheaf* on a scheme X is an $\mathscr{O}_X$-module F such that every point $x \in X$ has an open neighborhood $U \subset X$ on which there is an exact sequence

$$\mathscr{O}_X^{\oplus I}\big|_U \to \mathscr{O}_X^{\oplus J}\big|_U \to F\big|_U \to 0$$

for some (possibly infinite) sets I and J. A quasicoherent sheaf F is *coherent* if it satisfies the following additional conditions:

(1) it is *finitely generated*, i.e. every point $x \in X$ has an open neighborhood $U \subset X$ such that there is a surjective morphism $\mathcal{O}_X^{\oplus n}|_U \twoheadrightarrow F|_U$ for some positive integer n, and
(2) for any open subset $U \subset X$, for any positive integer n, and any morphism $s\colon \mathcal{O}_X^{\oplus n}|_U \to F|_U$, the kernel of s is finitely generated.

Example 2.2.3 (Structure sheaf)**.** The structure sheaf $\mathcal{O}_X$ is always quasicoherent; it is coherent when X is locally noetherian. If X is locally noetherian, a quasicoherent sheaf is coherent if and only if it is finitely generated, if and only if $F(U)$ is a finitely generated $\mathcal{O}_X(U)$-module for every open affine subset $U \subset X$.

Example 2.2.4 (Relative cotangent sheaf)**.** Let $f : X \to S$ be a morphism of schemes. The diagonal $\Delta_f : X \hookrightarrow X \times_S X$ is a locally closed immersion, so we can find an open subscheme $U \subset X \times_S X$ and a closed immersion $i : X \hookrightarrow U$. Let $\mathscr{J} \subset \mathcal{O}_U$ be the corresponding sheaf of ideals. Then $\Omega_f = i^*(\mathscr{J}/\mathscr{J}^2) = i^*\mathscr{C}_{X/U}$ (also denoted $\Omega_{X/S}$) is the (quasicoherent) sheaf of relative differentials associated to f. It does not depend on the factorisation $X \hookrightarrow U \hookrightarrow X \times_S X$. It is coherent whenever f is of finite type and S is noetherian [**129**, Ch. 6, Proposition 1.20].

We will only consider coherent and quasicoherent sheaves on locally noetherian schemes, essentially because (quasi)coherent sheaves on a locally noetherian scheme X form an abelian category [**185**, Tag 01XY]. Such categories will be denoted

$$\operatorname{Coh} X \subset \operatorname{QCoh} X.$$

Example 2.2.5 (Coherent sheaves on a point)**.** If $X = \operatorname{Spec} \mathbf{k}$, where $\mathbf{k}$ is a field, then $\operatorname{Coh} X$ is equivalent to the category of finite dimensional $\mathbf{k}$-vector spaces.

Notation 2.2.6. Let $F^* = \mathscr{H}om_{\mathcal{O}_X}(F, \mathcal{O}_X)$ denote the $\mathcal{O}_X$-linear dual of a coherent sheaf $F \in \operatorname{Coh} X$ on a locally noetherian scheme X. Note that it is a coherent sheaf on X (a more general statement can be found in [**185**, Tag 01CM]). We save the notation $F^\vee = \mathbf{R}\mathscr{H}om_{\mathcal{O}_X}(F, \mathcal{O}_X)$ for the *derived dual* of a complex of $\mathcal{O}_X$-modules F, so that when F is a sheaf we have $F^* = h^0(F^\vee)$, where h^i denotes the i-th cohomology sheaf of a complex of $\mathcal{O}_X$-modules.

Recall the correspondence between locally free sheaves on X and algebraic vector bundles $V \to X$, sending F to the X-scheme

$$\pi\colon \mathbb{V}(F) = \operatorname{Spec}_{\mathcal{O}_X} \operatorname{Sym} F^* \to X.$$

Under this correspondence, the global sections $\mathrm{H}^0(X, \mathbb{V}(F))$ of $\mathbb{V}(F)$ (i.e. sections, or right inverses, of the map π) correspond to $\mathcal{O}_X$-linear homomorphisms $\mathcal{O}_X \to F$, i.e. F is the *sheaf of sections* of $\mathbb{V}(F)$, and the fibre $\pi^{-1}(x) \subset \mathbb{V}(F)$ over a point $x \in X$ is naturally identified to the sheaf-theoretic fibre $F(x) = F_x/\mathfrak{m}_x \cdot F_x = F_x \otimes_{\mathcal{O}_{X,x}} \mathbf{k}(x)$.

A locally free sheaf (resp. vector bundle) of rank 1 is called an *invertible sheaf* (resp. *line bundle*).

Notation 2.2.7. Let $f : X \to S$ be a morphism of schemes. We set $\mathcal{T}_f = \mathcal{T}_{X/S} = \mathscr{H}om_{\mathcal{O}_X}(\Omega_f, \mathcal{O}_X)$ and $\Omega_f^i = \wedge^i \Omega_f$. (See [**185**, Tag 01CF] for the construction of the sheaf-theoretic exterior algebra of an $\mathcal{O}_X$-module.) If X is smooth over $S = \operatorname{Spec} \mathbf{k}$ and $i = 1$, we denote by

$$T^*X = \mathbb{V}(\Omega_X), \quad TX = \mathbb{V}(\mathcal{T}_X)$$

the total spaces of the locally free sheaves Ω_X and $\mathcal{T}_X = \mathscr{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)$. These are respectively the cotangent bundle and the tangent bundle of X. So, for instance, a 1-form on X is an element of $\mathrm{H}^0(X, T^*X) = \operatorname{Hom}_X(\mathcal{O}_X, \Omega_X)$.

We now pause for a second to define the class of projective morphisms. We refer to [**185**, Tag 01NM] for the relative Proj construction, but note that $\operatorname{Proj} \operatorname{Sym}(-) = \mathbb{P}(-)$, where $\mathbb{P}(-)$ will be introduced in Notation 4.1.5.

DEFINITION 2.2.8 (Grothendieck [83, Definition 5.5.2]). A morphism of schemes $f: X \to S$ is said to be *projective* if there exists a quasicoherent sheaf F on S such that f factors as

$$X \xhookrightarrow{i} \operatorname{Proj} \operatorname{Sym} F \longrightarrow S$$

where i is a closed immersion. We say that f is *quasiprojective* if it factors as $X \hookrightarrow Y \to S$, where $X \hookrightarrow Y$ is an open immersion and $Y \to S$ is projective.

DEFINITION 2.2.9 (Relative (very) ampleness). We have the following notions:

- An invertible sheaf $\mathscr{L}$ on a scheme X is called *ample* if X is quasicompact and every point $y \in X$ has an affine open neighborhood of the form $X_s = \{ x \in X \mid s_x \neq 0 \}$, where $s \in \mathrm{H}^0(X, \mathscr{L}^{\otimes n})$ for some $n > 0$.
- Let $f: X \to S$ be a morphism. An invertible sheaf $\mathscr{L}$ on X is f*-ample* if f is quasicompact and for every open affine subset $U \subset S$ the line bundle $\mathscr{L}|_{f^{-1}(U)}$ is ample [185, Tag 01VH].
- Let $f: X \to S$ be a morphism. An invertible sheaf $\mathscr{L}$ on X is f*-very ample* is there is a quasicoherent $\mathscr{O}_S$-module F and a locally closed immersion $\iota: X \hookrightarrow \mathbb{P}(F)$ over S such that $\mathscr{L} \cong \iota^* \mathscr{O}_{\mathbb{P}(F)}(1)$, where $\mathscr{O}_{\mathbb{P}(F)}(1)$ is the universal quotient bundle on $\mathbb{P}(F)$ [185, Tag 01VL].

If f is quasicompact, then an f-very ample invertible sheaf is f-ample. A morphism $f: X \to S$ such that X admits an f-ample invertible sheaf is necessarily separated. By [185, Tag 01VL], if $\mathscr{L}$ is a line bundle on X, then it is f-very ample on a quasicompact morphism $f: X \to S$ if and only if f is quasiseparated, $f^* f_* \mathscr{L} \to \mathscr{L}$ is surjective and the associated map $X \to \mathbb{P}(f_* \mathscr{L})$ is a locally closed immersion.

Remark 2.2.10. A morphism $f: X \to S$ is projective if and only if f is proper and there is an f-very ample invertible sheaf on X. Note that the above notion of projectivity is in general different from the one found in Hartshorne [96, II.4].

2.2.2. Properties of sheaves: torsion free, pure, reflexive, flat. Recall that a module M over a commutative ring A is *torsion free* if whenever $am = 0 \in M$ one has that either a is a zero divisor or $m = 0$. To simplify matters, we will restrict to the case where A is an integral domain. Globally speaking, this means working over integral schemes.

DEFINITION 2.2.11. Let X be an integral locally noetherian scheme. A coherent sheaf $F \in \operatorname{Coh} X$ is *torsion free* if for every $x \in X$ the stalk F_x is torsion free as an $\mathscr{O}_{X,x}$-module, i.e. if for any $s \in \mathscr{O}_{X,x} \setminus 0$ the map $F_x \to F_x$ defined by $\tau \mapsto s\tau$ is injective.

Note that F is torsion free if and only if for every affine open subscheme $U \subset X$, the $\mathscr{O}_X(U)$-module $F(U)$ is torsion free.

EXERCISE 2.2.12. Let X be an integral locally noetherian scheme. Confirm that a subsheaf of a torsion free sheaf on X is torsion free. Prove that an extension of torsion free sheaves is torsion free.

Recall the dualisation functor $(-) = \mathscr{H}om_{\mathscr{O}_X}(-, \mathscr{O}_X)$. The natural transformation $\mathrm{id} \Rightarrow (-)^{**}$ of functors $\operatorname{Coh} X \rightrightarrows \operatorname{Coh} X$ induces a natural $\mathscr{O}_X$-linear map

$$\nu_F : F \to F^{**}.$$

Its kernel is the *torsion subsheaf* of F, i.e. the subsheaf $T(F) \subset F$ whose sections over $U \subset X$ are given by those elements $\tau \in F(U)$ such that there exists a nonzero $s \in \mathscr{O}_X(U)$ for which $s\tau = 0 \in F(U)$. Clearly then, F is torsion free if and only if $T(F) = 0$, if and only if ν_F is injective.

DEFINITION 2.2.13. Let X be an integral locally noetherian scheme. Then $F \in \operatorname{Coh} X$ is called *reflexive* if ν_F is an isomorphism.

As shown in [97, Corollary 1.2], F^* is reflexive, for all $F \in \operatorname{Coh} X$ on a noetherian integral scheme X.

Let F be a coherent sheaf on a smooth irreducible $\mathbf{k}$-variety X. In particular, for every $x \in X$, the stalk F_x is a finitely generated module over $\mathscr{O}_{X,x}$. It is free if and only if F_x has *homological dimension* 0, i.e. $\mathrm{Ext}^i_{\mathscr{O}_{X,x}}(F_x, N) = 0$ for all finitely generated $\mathscr{O}_{X,x}$-modules N and for all $i > 0$. There is a closed subset

$$\mathrm{S}(F) = \{\, x \in X \mid F_x \text{ is not free} \,\} \subset X$$

of codimension at least 1, called the *singularity set* of F.

Lemma 2.2.14 ([157, p. 76])**.** *Let F be a coherent sheaf on a smooth irreducible $\mathbf{k}$-variety. Then* $\operatorname{codim}(\mathrm{S}(F), X) \geq 2$ *if F is torsion free, and* $\operatorname{codim}(\mathrm{S}(F), X) \geq 3$ *if F is reflexive. In particular, a torsion free sheaf on a smooth curve is locally free, and so is a reflexive sheaf on a smooth surface.*

DEFINITION 2.2.15. The *rank* of a coherent sheaf F on a smooth irreducible $\mathbf{k}$-variety X is the integer

$$\operatorname{rk} F = \operatorname{rk} F\big|_{X \setminus \mathrm{S}(F)}.$$

EXERCISE 2.2.16. Show that the rank is additive on short exact sequences.

Lemma 2.2.17 ([157, Chapter 2, Lemma 1.1.15])**.** *A reflexive sheaf of rank* 1 *on a smooth irreducible $\mathbf{k}$-variety is a line bundle.*

DEFINITION 2.2.18. Let X be a smooth irreducible $\mathbf{k}$-variety. Let $F \in \operatorname{Coh} X$. The *determinant* of F is the line bundle

$$\det F = \left(\wedge^{\operatorname{rk} F} F\right)^{**}.$$

Example 2.2.19 (Ideal sheaves)**.** Let X be a smooth irreducible $\mathbf{k}$-variety. If $\iota\colon Z \hookrightarrow X$ is a closed subscheme, both $\iota_*\mathscr{O}_Z$ and $\mathscr{I}_Z$ are coherent sheaves on X. The ideal sheaf, being a subsheaf of a free sheaf, is torsion free. In fact, ideal sheaves of subschemes of codimension at least 2 inside a scheme X are precisely the torsion free sheaves of rank 1 and trivial determinant. In codimension 1, one has a bijection between the effective Cartier divisors on a scheme X and the invertible ideal sheaves $\mathscr{I} \subset \mathscr{O}_X$.

Remark 2.2.20. A closed immersion $\iota\colon X \hookrightarrow M$ of noetherian schemes has coherent sheaf of ideals

$$\mathscr{I} = \ker\left(\iota^\#\colon \mathscr{O}_M \twoheadrightarrow \iota_*\mathscr{O}_X\right) \subset \mathscr{O}_M,$$

and the pushforward functor ι_* induces an equivalence between $\operatorname{Coh} X$ and the category of coherent $\mathscr{O}_M$-modules annihilated by $\mathscr{I}$. See [185, Tag 01XY] for more details.

DEFINITION 2.2.21. Let $X \to S$ be a morphism. A sheaf $F \in \operatorname{QCoh} X$ is *flat over S* (or *S-flat*) if for every point $x \in X$, with image $s \in S$, the module F_x is flat over $\mathscr{O}_{S,s}$ via the ring map $f_x^\#\colon \mathscr{O}_{S,s} \to \mathscr{O}_{X,x}$. For instance, $\mathscr{O}_X$ is S-flat if and only if $X \to S$ is flat as a morphism of schemes (Definition 2.1.19). If we consider the identity map on $S = X$, we simply say that F is flat.

Example 2.2.22. A coherent sheaf $F \in \operatorname{Coh} X$ on a noetherian scheme is flat if and only if it is locally free [96, III.9.2 (e)]. A flat coherent sheaf on an integral scheme is torsion free.

Example 2.2.23. Let $f: X \to Y$ be a morphism over a base scheme S, with $Y \to S$ flat. Let $F \in \operatorname{QCoh} X$ be S-flat. Then, by transitivity of flatness, f_*F is S-flat. If f is affine, then the converse also holds, i.e. F is S-flat if and only if f_*F is S-flat [185, Tag 01U2].

Example 2.2.24. Let $f: X \to S$ be a projective morphism of locally noetherian schemes. Let $F \in \operatorname{Coh} X$. Then F is S-flat if and only if $f_*F(m)$ is locally free for $m \gg 0$.

Proposition 2.2.25 (Generic flatness [185, Tag 0529])**.** *Let $X \to S$ be a morphism of finite type with S reduced. Let $F \in \operatorname{Coh} X$. Then there is a dense open subset $U \subset S$ such that $f^{-1}(U) \to U$ is flat and $F|_{f^{-1}(U)}$ is flat over U .*

Lemma 2.2.26 ([**157**, Chapter 2, Lemma 1.1.2]). *A coherent sheaf on a smooth* $\mathbf{k}$*-variety is reflexive if and only if it is torsion free and normal, where 'normal' means that for every open subset* $U \subset X$ *and for every closed subset* $Z \subset U$ *of codimension at least* 2 *the restriction map* $F(U) \to F(U \setminus A)$ *is an isomorphism.*

2.2.3. Supports. The *support* of a coherent sheaf F on a locally noetherian scheme X is the following *closed subscheme* of X: consider the map $\alpha_F : \mathscr{O}_X \to \mathscr{H}\!om_{\mathscr{O}_X}(F, F)$ defined on local sections by sending f to the $\mathscr{O}_X$-linear map $m \mapsto fm$. The kernel of α_F — the sheaf-theoretic annihilator ideal of F — defines the closed subscheme

$$j \colon \operatorname{Supp} F \hookrightarrow X,$$

which can be viewed as the minimal closed subscheme of X over which the natural morphism $F \to j_* j^* F$ is an isomorphism. Set-theoretically, $\operatorname{Supp} F$ consists of those points $x \in X$ such that $F_x \neq 0$. This set is *not* closed for an arbitrary sheaf of $\mathscr{O}_X$-modules, but it is for coherent ones [**185**, Tag 01XY]. The support of a coherent sheaf on a proper S-scheme $X \to S$, where S is a base scheme, is proper over S.

DEFINITION 2.2.27 (Dimension of sheaves). The *dimension* of a coherent sheaf $F \in \operatorname{Coh} X$, denoted $\dim F$, is the dimension of $\operatorname{Supp} F \subset X$. We say that F is *pure* of dimension d if $\dim \operatorname{Supp} E = d$ for all subsheaves $0 \neq E \subseteq F$. Thus we see that a torsion free sheaf is pure of maximal dimension.

We thus have the following chains of implications for sheaves on locally noetherian schemes.

$$\begin{tikzcd}
& & \text{flat} \arrow[dr, Rightarrow] & & \\
\text{free} \arrow[r, Rightarrow] & \text{locally free} \arrow[ur, Rightarrow] \arrow[r, Rightarrow] & \text{reflexive} \arrow[l, dotted, bend left, "X \text{ smooth},\ \dim X \leq 2"] \arrow[r, Rightarrow] & \text{torsion free} \arrow[ul, dotted, bend right, "\text{over Dedekind domains}"'] \arrow[l, dotted, bend left, "\text{normal}"] \arrow[r, Rightarrow] & \text{pure} \arrow[l, dotted, bend left, "\text{max dimension}"]
\end{tikzcd}$$

The scheme-theoretic support behaves well under flat base change [**185**, Tag 089C]. However, the following remark is the origin of several issues around the existence of Hilbert–Chow type morphisms.

Remark 2.2.28. Let $X \to S$ be a a finite type morphism of locally noetherian schemes. It is not true that the support of an S-flat $\mathscr{O}_X$-module is flat over S, see e.g. [**105**].

EXERCISE 2.2.29. Find examples of the phenomenon described in Remark 2.2.28.

DEFINITION 2.2.30 (Support of sections). Let F be an $\mathscr{O}_X$-module on a scheme X. Let $s \in \mathrm{H}^0(X, F)$ be a global section. The set-theoretic support of $\operatorname{Supp}(s)$ is the set of points $x \in X$ such that the image of s along the map $\mathrm{H}^0(X, F) \to F_x$ is not zero.

EXERCISE 2.2.31. Let X be a scheme, $F \in \operatorname{Mod} \mathscr{O}_X$. Show that $\operatorname{Supp}(s) \subset X$ is closed for every section $s \in \mathrm{H}^0(X, F)$.

2.2.4. Derived category notation. This subsection is purely meant to set up some notation for later. We refer to [**107**] for a thorough guide to triangulated and derived categories and derived functors. In general, by $\mathbf{D}(\mathscr{A})$ we mean the (unbounded) derived category of an abelian category $\mathscr{A}$.

Let X be a locally noetherian scheme. For $* \in \{+, -, \mathrm{b}, \emptyset, [a, b]\}$, we let $\mathbf{D}^*(\operatorname{Mod} \mathscr{O}_X)$ denote the full subcategories of $\mathbf{D}(\operatorname{Mod} \mathscr{O}_X)$, where the decoration $*$ indicates the location of the (possibly) non-vanishing cohomology sheaves. The decoration 'b' stands for *bounded complexes*, and '$\emptyset$' stands for no decoration. We use the same convention for $\mathbf{D}^*(\operatorname{QCoh} X)$ and $\mathbf{D}^*(X) = \mathbf{D}^*(\operatorname{Coh} X)$.

By $\mathbf{D}^*_{\mathsf{qcoh}}(\operatorname{Mod} \mathscr{O}_X) \subset \mathbf{D}^*(\operatorname{Mod} \mathscr{O}_X)$, resp. $\mathbf{D}^*_{\mathsf{coh}}(\operatorname{Mod} \mathscr{O}_X) \subset \mathbf{D}^*_{\mathsf{qcoh}}(\operatorname{Mod} \mathscr{O}_X)$, we denote the full triangulated subcategory of complexes with quasicoherent, resp. coherent cohomology. The cohomology sheaves of an object $E \in \mathbf{D}(\operatorname{Mod} \mathscr{O}_X)$ are denoted $h^i(E) \in \operatorname{Mod} \mathscr{O}_X$.

DEFINITION 2.2.32 ([**192**, Section 2]). A complex $E \in \mathbf{D}(\operatorname{Mod} \mathscr{O}_X)$ is called *perfect* (resp. *strictly perfect*) if it is locally (resp. globally) quasi-isomorphic to a bounded complex of locally free $\mathscr{O}_X$-modules of finite type. We write $\mathrm{Perf}\,X \subset \mathbf{D}_{\mathsf{qcoh}}(\mathrm{Mod}\,\mathscr{O}_X)$ for the full triangulated subcategory of perfect complexes on X. Finally, $E \in \mathrm{Perf}\,X$ is said to be *of perfect amplitude in* $[a,b]$ if E is Zariski locally isomorphic to a complex $[E^a \to \cdots \to E^b]$ of locally free sheaves of finite rank.

A quasicompact scheme X is said to have an *ample family of line bundles* if there is a collection $\{\mathscr{L}_i\}_{i\in I}$ of invertible sheaves on X such that for every $x \in X$ there is an $i \in I$, an integer $n > 0$ and a section $s \in \mathrm{H}^0(X, \mathscr{L}_i^{\otimes n})$ such that $x \in X_s$ and X_s is affine [185, Tag 0FXQ]. If X has an ample family of line bundles, then every perfect complex on X is strictly perfect, see [22, II.2.2] or [192, Proposition 2.3.1(d)].

We recapitulate some known results on the identification of various triangulated categories of sheaves:

- By [185, Tag 08DB], if X is quasicompact and semiseparated (i.e. has affine diagonal) the canonical functor $\mathbf{D}(\mathrm{QCoh}\,X) \to \mathbf{D}_{\mathsf{qcoh}}(\mathrm{Mod}\,\mathscr{O}_X)$ is an exact equivalence (see also [26, Corollary 5.5] for the proof in the case where X is quasicompact and separated).
- By [22, Corollary II.2.2.2.1], if X is noetherian, the canonical functor $\mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}_{\mathsf{coh}}(\mathrm{Mod}\,\mathscr{O}_X)$ is an exact equivalence.
- By [185, Tag 0FDC], if X is noetherian and regular of finite dimension, the canonical functor $\mathrm{Perf}\,X \to \mathbf{D}^{\mathrm{b}}(X)$ is an exact equivalence.

2.2.5. Dualising complexes, Cohen–Macaulay and Gorenstein schemes. Let Y be a quasicompact and quasiseparated scheme, $f : X \to Y$ a proper flat morphism of finite presentation. Let $f^!$ be the right adjoint of $\mathbf{R}f_* : \mathbf{D}_{\mathsf{qcoh}}(\mathrm{Mod}\,\mathscr{O}_X) \to \mathbf{D}_{\mathsf{qcoh}}(\mathrm{Mod}\,\mathscr{O}_Y)$, which exists by [185, Tag 0A9E]. A standard reference for this statement — known as *Grothendieck duality* — is [151, Example 4.2]. It is also proved in [151, Section 6] that if f is a proper morphism of noetherian separated schemes, the natural morphism

$$\mathbf{R}f_*\,\mathbf{R}\mathscr{H}\!om_{\mathscr{O}_X}(F, f^!E) \to \mathbf{R}\mathscr{H}\!om_{\mathscr{O}_Y}(\mathbf{R}f_*F, E) \tag{2.2.1}$$

is an isomorphism in $\mathbf{D}(\mathrm{Mod}\,Y)$ for all $F \in \mathbf{D}(\mathrm{QCoh}\,X)$ and $E \in \mathbf{D}(\mathrm{QCoh}\,Y)$. A proof of the isomorphism (2.2.1) assuming f is a morphism essentially of finite type between noetherian separated schemes can be found in [113, Equation 1.6.1]. We refer the reader to Neeman [151] and Lipman [128] for very informative discussions around the history of Grothendieck duality, as well its more modern versions.

As a special case, if f as above is a smooth proper morphism of relative dimension d, where X and Y are of finite type and separated over a base scheme S, then $f^!$ agrees with the functor $\mathbf{L}f^*(-) \otimes^{\mathbf{L}}_{\mathscr{O}_X} \Omega^d_f[d]$. The object

$$\omega^\bullet_f = f^!\mathscr{O}_Y \in \mathbf{D}_{\mathsf{qcoh}}(\mathrm{Mod}\,\mathscr{O}_X)$$

is the *relative dualising complex* attached to f. One has functorial isomorphisms $f^!E = \mathbf{L}f^*E \otimes^{\mathbf{L}}_{\mathscr{O}_X} \omega^\bullet_f$ for all $E \in \mathbf{D}_{\mathsf{qcoh}}(\mathrm{Mod}\,\mathscr{O}_Y)$. One has a canonical morphism $\mathrm{Tr}_f : \mathbf{R}f_*\omega^\bullet_f \to \mathscr{O}_Y$, corresponding to the identity $\mathrm{id} \in \mathrm{Hom}_X(\omega^\bullet_f, \omega^\bullet_f)$ under the adjunction isomorphism

$$\mathrm{Hom}_X(\omega^\bullet_f, \omega^\bullet_f) \cong \mathrm{Hom}_Y(\mathbf{R}f_*\omega^\bullet_f, \mathscr{O}_Y).$$

Moreover, $\mathbf{R}f_*\omega^\bullet_f$ has vanishing cohomology in positive degrees, and the natural map $\mathscr{O}_X \to \mathbf{R}\mathscr{H}\!om_{\mathscr{O}_X}(\omega^\bullet_f, \omega^\bullet_f)$ is an isomorphism (see [185, Tag 0E4L]).

Example 2.2.33 ([185, Tag 0BQV])**.** Let Y be a noetherian scheme, $X = \mathbb{P}(\mathscr{E}) = \mathrm{Proj}\,\mathrm{Sym}\,\mathscr{E}$ where $\mathscr{E}$ is a locally free sheaf of rank $d+1$ over Y with determinant $\mathscr{L}$. Then, letting $f : X \to Y$ be the canonical morphism, one has an isomorphism

$$f^*\mathscr{L} \otimes \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-d-1)[d] \xrightarrow{\sim} \omega^\bullet_f.$$

So, for instance, the relative dualising complex of $\mathbb{P}^d_Y \to Y$ is the shifted line bundle $\mathscr{O}_{\mathbb{P}^d_Y}(-d-1)[d]$.

Example 2.2.34 ([185, Tag 0BRT])**.** If Y is a noetherian scheme and $f : X \to Y$ is a smooth proper morphism of relative dimension d, we have an isomorphism

$$\Omega^d_f[d] \xrightarrow{\sim} \omega^\bullet_f,$$

so in particular, as in the previous example, $\omega^\bullet_f$ sits entirely in cohomological degree $-d$.

DEFINITION 2.2.35. Let X be a locally noetherian scheme with dualising complex $\omega_X^\bullet$. Then X is called

- *Cohen–Macaulay* if $\omega_X^\bullet$ has only one nonvanishing cohomology sheaf;
- *Gorenstein* if $\omega_X^\bullet$ has only one nonvanishing cohomology sheaf, which is a line bundle.

A morphism $Y \to S$ with locally noetherian fibres is said to be a Cohen–Macaulay (resp. Gorenstein) morphism if it is flat and it has Cohen–Macaulay (resp. Gorenstein) fibres.

DEFINITION 2.2.36. Let $f: Y \to S$ be a Cohen–Macaulay morphism of relative dimension d (which means the fibres are pure of dimension d). Then

$$\omega_f = h^{-d}(f^!\mathscr{O}_S) = h^{-d}(\omega_f^\bullet)$$

is called the *relative dualising sheaf* attached to f.

Example 2.2.37. If $f: Y \to S$ is a smooth and proper morphism to a noetherian scheme S, then $\omega_f = \Omega_f^d$, which is a line bundle.

2.3. Degeneracy loci and Chern classes

Let X be an n-dimensional integral scheme of finite type over an algebraically closed field $\mathbf{k}$. If $s \in \mathrm{H}^0(X, F)$ is a section of a vector bundle $F = \operatorname{Spec}\operatorname{Sym}\mathscr{F}^* \to X$, corresponding to a $\mathscr{O}_X$-linear homomorphism $s\colon \mathscr{O}_X \to \mathscr{F}$ to a locally free sheaf $\mathscr{F}$, then the zero scheme $Z(s) \hookrightarrow X$ of s is, by definition, the closed subscheme corresponding to the ideal sheaf

$$\mathscr{I} = \operatorname{image}\big(\mathscr{F}^* \xrightarrow{s^\vee} \mathscr{O}_X\big) \subset \mathscr{O}_X.$$

More generally, let $\phi: E \to F$ be a homomorphism of vector bundles corresponding to a map of locally free sheaves $\phi: \mathscr{E} \to \mathscr{F}$ (denoted the same way) of ranks $e = \operatorname{rk}\mathscr{E}$, $f = \operatorname{rk}\mathscr{F}$. Then, for every $k \leq \min\{e, f\}$, one has a natural section

$$\wedge^{k+1}\phi \in \mathrm{H}^0\left(X, \wedge^{k+1}E^* \otimes \wedge^{k+1}F\right),$$

and the k-th *degeneracy locus* of ϕ is defined as the closed subscheme

$$\mathbb{D}_k(\phi) = Z\left(\wedge^{k+1}\phi\right) \xhookrightarrow{\iota} X. \tag{2.3.1}$$

It is supported on the locus of points $x \in X$ such that $\operatorname{rk}\phi(x) \leq k$, where $\phi(x)\colon E(x) \to F(x)$ is the $\mathbf{k}$-linear map induced by ϕ on the fibres over $x \in X$.

We use these notions to define *Chern classes* of a vector bundle $E = \operatorname{Spec}\operatorname{Sym}\mathscr{E}^* \to X$ (but see Appendix B.2 for a more systematic approach in the algebraic setup, or [**100**, Chapter 3] and [**143**] for the topological construction of Chern classes). Let e be the rank of E, and for simplicity assume E is globally generated, i.e. that there are global sections $\sigma_1, \ldots, \sigma_c \in \mathrm{H}^0(X, E)$ such that for all $x \in X$ the fibre $E(x) \cong \mathbf{k}^e$ is spanned by $\sigma_1(x), \ldots, \sigma_c(x)$. Of course we must have $c \geq e$ (and E is isomorphic to the trivial bundle $\mathbf{k}^e \times X$ if and only if $c = e$[4]). Let us choose e general sections $s_0, \ldots, s_{e-1}$, and for every $i = 0, \ldots, e$ let

$$\phi_{e-i+1}\colon \mathbf{k}^{e-i+1} \times X \to E$$

be the map determined by $s_0, \ldots, s_{e-i}$. The case $i = 0$ is apparently excluded since we only fixed e sections, so let us start with the case $i = 1$. The line bundle $\det E = \wedge^e E \to X$, corresponding to the invertible sheaf $\wedge^e\mathscr{E}^*$, is equipped with the canonical section $\wedge^e\phi_e \in \mathrm{H}^0(X, \det E) = \operatorname{Hom}_X(\mathscr{O}_X, \wedge^e\mathscr{E})$. This section has a well-defined zero locus

$$\mathbb{D}_{e-1}(\phi_e) = Z(\wedge^e\phi_e) \subset X,$$

supported on the locus where ϕ_e is not surjective (i.e. $s_0, \ldots, s_{e-1}$ fail to be linearly independent).

The identity[5]

$$c_1(\det E) \cap [X] = [Z(\wedge^e\phi_e)] \in A_{n-1}X \tag{2.3.2}$$

[4] Indeed, a map of locally free sheaves of the same rank is an isomorphism if and only if it is surjective [**185**, Tag 01C5].

[5] The definition of Chow groups A_*X will be recalled in Appendix B.1.

defines the first Chern class $c_1(\det E) \in A^1 X$.

DEFINITION 2.3.1. The *first Chern class* of E is $c_1(E) = c_1(\det E) \in A^1 X$.

More generally, consider the zero locus

$$\mathbb{D}_{e-i}(\phi_{e-i+1}) = Z\left(\wedge^{e-i+1}\phi_{e-i+1}\right) \hookrightarrow X.$$

DEFINITION 2.3.2. For all $i \geq 0$, one can define $c_i(E) \in A^i X$ via the identity

$$c_i(E) \cap [X] = \left[Z\left(\wedge^{e-i+1}\phi_{e-i+1}\right)\right] \in A_{n-i}X.$$

Thus we obtain the following slogan:

$c_i(E)$ is 'Poincaré dual' to the locus where $e-i+1$ general sections of E become linearly dependent.

Example 2.3.3 (Euler class). If $i = 0$, we see that we would be requiring $e+1 > e = \operatorname{rk} E$ general sections of E to be linearly dependent. This of course would happen over the whole of X, therefore it is natural to set $c_0(E) = 1$. At the other extreme, if $i = e$, we are forcing one section to be 'linearly dependent', i.e. we are forcing it to vanish. Thus the top Chern class $c_e(E)$, also known as the *Euler class* $e(E)$, represents the vanishing locus of a general section of E.

2.3.1. The Thom–Porteous formula. When nonempty, the closed subscheme (2.3.1) has codimension

$$\operatorname{codim}(\mathbb{D}_k(\phi), X) \leq (e-k)(f-k). \tag{2.3.3}$$

In fact, let m denote the expected dimension of this degeneracy locus, i.e. set $m = \dim X - (e-k)(f-k)$. Then Fulton [67, Chapter 14] constructs a Chow class

$$\mathbf{D}_k(\phi) \in A_m \mathbb{D}_k(\phi)$$

whose pushforward along the closed immersion $\iota\colon \mathbb{D}_k(\phi) \hookrightarrow X$ has a closed (determinantal) expression in terms of the Chern classes of E and F only. To start with, let us review the definition of $\mathbf{D}_k(\phi)$. Set $d = e-k$ and let $\rho\colon G(d,\mathscr{E}) \to X$ be the structure morphism of the Grassmann bundle $G(d,\mathscr{E})$ defined in Section 4.1 (see in particular Remark 4.1.7). Let $\mathbf{Z}(s_\phi) \in A_m Z(s_\phi)$ be the *refined Euler class* (cf. Appendix B.3.1) attached to the section

$$s_\phi \in \mathrm{H}^0(G(d,\mathscr{E}), \mathscr{S}^* \otimes \rho^*\mathscr{F})$$

corresponding to the map $s_\phi = \rho^*\phi|_{\mathscr{S}}\colon \mathscr{S} \hookrightarrow \rho^*\mathscr{E} \to \rho^*\mathscr{F}$, where the first homomorphism is the inclusion of the universal subbundle living on $G(d,\mathscr{E})$. Note that $\mathbf{Z}(s_\phi)$ is an element of the m-th Chow group of $Z(s_\phi)$ since $\dim G(d,\mathscr{E}) = \dim X + (e-k)k$, so that $\dim G(d,\mathscr{E}) - \operatorname{rk}(\mathscr{S}^* \otimes \rho^*\mathscr{F}) = \dim X + (e-k)k - (e-k)f = \dim X - (e-k)(f-k) = m$. Note, also, that ρ maps $Z(s_\phi) \subset G(d,\mathscr{E})$ onto $\mathbb{D}_k(\phi) \subset X$, i.e. we have a (proper) restricted morphism $\widetilde{\rho}\colon Z(s_\phi) \to \mathbb{D}_k(\phi)$. Then Fulton defines

$$\mathbf{D}_k(\phi) = \widetilde{\rho}_* \mathbf{Z}(s_\phi).$$

Now, for a general series $c = \sum_k c_k$ with $c_k \in A^* X$, let $\Delta_q^{(p)} c$ be the $p \times p$ determinant $|c_{q+j-i}|_{1 \leq i,j \leq p}$, i.e.

$$\Delta_q^{(p)} c = \begin{vmatrix} c_q & c_{q+1} & \cdots & c_{q+p-1} \\ c_{q-1} & c_q & \cdots & c_{q+p-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{q-p+1} & c_{q-p+2} & \cdots & c_q \end{vmatrix}.$$

Then, by [67, Theorem 14.4 (a)], one has

$$\iota_* \mathbf{D}_k(\phi) = \Delta_{f-k}^{(e-k)} c(E-F) \cap [X] \in A_m X.$$

This is the celebrated *Thom–Porteous formula*. The expression $c(E-F)$ is defined in Appendix B.2 in terms of the total Chern classes of E and F. It is defined formally by the identity

$$c(E-F)\cdot\sum_{i=0}^{f}c_i(F)=\sum_{i=0}^{e}c_i(E).$$

If X is Cohen–Macaulay and (2.3.3) is an equality, then

$$\mathbf{D}_k(\phi)=[\mathbb{D}_k(\phi)].$$

Throughout these notes we use the following terminology.

DEFINITION 2.3.4 (Critical locus). Let U be a smooth $\mathbf{k}$-scheme, $f\in \mathrm{H}^0(U,\mathscr{O}_U)$ a regular function. Then the vanishing scheme

$$Z(\mathrm{d}f)\subset U$$

of the exact 1-form $\mathrm{d}f\in \mathrm{H}^0(U,T^*U)=\mathrm{Hom}_U(\mathscr{O}_U,\Omega_U)$ is called the *critical locus* attached to (U,f).

Example 2.3.5. Let $U=\mathbb{A}^1=\operatorname{Spec}\mathbf{k}[t]$ and $n\geq 1$ an integer such that $n+1$ is not a multiple of $\operatorname{char}\mathbf{k}$. Then the fat point $D_n=\operatorname{Spec}\mathbf{k}[t]/t^n\subset U$ is a critical locus, being the vanishing locus of the differential of the function $t^{n+1}\in\mathbf{k}[t]=\mathrm{H}^0(U,\mathscr{O}_U)$.

Example 2.3.6. Let $A=\mathrm{H}^*(G(k,n),\mathbb{C})$ be the cohomology ring of the Grassmannian of k-planes in $\mathbb{C}^n$. Then A has a unique maximal ideal, in particular $Y=\operatorname{Spec}A$ is a fat point. In fact, Y is a critical locus. This is proved in [23, Proposition 5.9], we only give a sketch here. One starts with the presentation

$$A=\mathbb{C}[x_1,\ldots,x_k]/I,$$

where $x_i=c_i(\mathscr{S}^*)$, for $\mathscr{S}\hookrightarrow\mathscr{O}_{G(k,n)}\otimes_{\mathbb{C}}\mathbb{C}^n$ the universal rank k subbundle, and where I encodes the relations among $x_1,\ldots,x_k$ arising from the identity $c_t(\mathscr{S}^*)c_t(\mathscr{Q}^*)=1$, where c_t is the Chern polynomial. Define polynomials $W_i(x_1,\ldots,x_k)$ for $i\geq 0$ by a formal expansion

$$-\log c_t(\mathscr{S}^*)=\sum_{i\geq 0}W_i(x_1,\ldots,x_k)t^i.$$

Set $W=(-1)^{n+1}W_{n+1}$. This is the polynomial expressing the Newton polynomial

$$\frac{1}{n+1}\sum_{1\leq i\leq k}\alpha_i^{n+1}$$

in terms of $x_1,\ldots,x_k$, where $\alpha_1,\ldots,\alpha_k$ are the Chern roots of $\mathscr{S}^*$, defined through the identity $c_t(\mathscr{S}^*)=\prod_{1\leq i\leq k}(1+\alpha_i t)$. Then the ideal I is equal to the Jacobian ideal

$$\left\langle\frac{\partial W}{\partial x_1},\ldots,\frac{\partial W}{\partial x_k}\right\rangle.$$

So $Y=\operatorname{Spec}\mathrm{H}^*(G(k,n),\mathbb{C})$ is a critical locus in the sense of Definition 2.3.4.

2.4. Representable functors

In this section we study representable functors and recall the statement of the Yoneda Lemma. More details and examples can be found, for instance, in [202]. This material will be needed in Chapter 4.

We start by making the following assumption.

Assumption 2.4.1. All categories are assumed to be *locally small*, i.e. we assume that $\mathrm{Hom}_{\mathcal{C}}(x,y)$ is a set for any pair of objects x and y.

Let $\mathcal{C}$ and $\mathcal{C}'$ be (locally small) categories.

DEFINITION 2.4.2. A (covariant) functor $\mathsf{F}\colon\mathcal{C}\to\mathcal{C}'$ is called:

(i) *fully faithful* if for any two objects $x, y \in \mathcal{C}$ the map of sets
$$\mathrm{Hom}_{\mathcal{C}}(x,y) \to \mathrm{Hom}_{\mathcal{C}'}(\mathsf{F}(x),\mathsf{F}(y))$$
is a bijection.

(ii) *essentially surjective* if every object of $\mathcal{C}'$ is isomorphic to an object of the form $\mathsf{F}(x)$ for some $x \in \mathcal{C}$.

The following observation is quite useful.

Remark 2.4.3. A fully faithful functor $\mathsf{F}\colon \mathcal{C} \to \mathcal{C}'$ induces an equivalence of $\mathcal{C}$ with the essential image of F, namely the full subcategory of $\mathcal{C}'$ consisting of objects isomorphic to objects of the form $\mathsf{F}(x)$ for some $x \in \mathcal{C}$. Put differently, a functor is an equivalence if and only if it is fully faithful and essentially surjective.

DEFINITION 2.4.4. A *natural transformation* $\eta\colon \mathsf{F} \Rightarrow \mathsf{G}$ between two functors $\mathsf{F}, \mathsf{G}\colon \mathcal{C} \to \mathcal{C}'$ is the datum, for every $x \in \mathcal{C}$, of a morphism $\eta_x\colon \mathsf{F}(x) \to \mathsf{G}(x)$ in $\mathcal{C}'$, such that for every $f \in \mathrm{Hom}_{\mathcal{C}}(x_1,x_2)$ the diagram
$$\begin{tikzcd} \mathsf{F}(x_1) \arrow[r, "\eta_{x_1}"] \arrow[d, "\mathsf{F}(f)"'] & \mathsf{G}(x_1) \arrow[d, "\mathsf{G}(f)"] \\ \mathsf{F}(x_2) \arrow[r, "\eta_{x_2}"] & \mathsf{G}(x_2) \end{tikzcd}$$
is commutative in $\mathcal{C}'$.

DEFINITION 2.4.5. Let $\mathcal{C}, \mathcal{C}'$ be two categories. Let $\mathrm{Fun}(\mathcal{C},\mathcal{C}')$ be the category whose objects are functors $\mathcal{C} \to \mathcal{C}'$ and whose morphisms are the natural transformations. An isomorphism in the category $\mathrm{Fun}(\mathcal{C},\mathcal{C}')$ is called a *natural isomorphism*.

Let $\mathcal{C}$ be a (locally small) category. Its *opposite category* $\mathcal{C}^{\mathrm{op}}$, by definition, has the same objects of $\mathcal{C}$, and its morphisms are
$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(x,y) = \mathrm{Hom}_{\mathcal{C}}(y,x), \quad x,y \in \mathcal{C}.$$
Consider the category of contravariant functors $\mathcal{C} \to \mathrm{Sets}$, i.e. the category
$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}).$$
For every object x of $\mathcal{C}$ there is a functor $\mathsf{h}_x : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$ defined by
$$u \mapsto \mathsf{h}_x(u) = \mathrm{Hom}_{\mathcal{C}}(u,x), \quad u \in \mathcal{C}.$$
A morphism $\phi \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(u,v) = \mathrm{Hom}_{\mathcal{C}}(v,u)$ gets sent to the map of sets
$$\mathsf{h}_x(\phi)\colon \mathsf{h}_x(u) \to \mathsf{h}_x(v), \quad \alpha \mapsto \alpha \circ \phi.$$
Consider the functor
$$\mathsf{h}_{\mathcal{C}} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}), \quad x \mapsto \mathsf{h}_x. \tag{2.4.1}$$
This is, indeed, a functor: for every arrow $f : x \to y$ in $\mathcal{C}$ and object u of $\mathcal{C}$ we can define a map of sets
$$\mathsf{h}_f u\colon \mathsf{h}_x(u) \to \mathsf{h}_y(u), \quad \alpha \mapsto f \circ \alpha,$$
with the property that for every morphism $\phi : v \to u$ in $\mathcal{C}$ there is a commutative diagram
$$\begin{tikzcd} \mathsf{h}_x(u) \arrow[r, "\mathsf{h}_f u"] \arrow[d, "\mathsf{h}_x(\phi)"'] & \mathsf{h}_y(u) \arrow[d, "\mathsf{h}_y(\phi)"] \\ \mathsf{h}_x(v) \arrow[r, "\mathsf{h}_f v"] & \mathsf{h}_y(v) \end{tikzcd} \qquad \begin{tikzcd} u \xrightarrow{\alpha} x \arrow[r, mapsto] \arrow[d, mapsto] & u \xrightarrow{f\circ\alpha} y \arrow[d, mapsto] \\ v \xrightarrow{\alpha\circ\phi} x \arrow[r, mapsto] & u \xrightarrow{f\circ\alpha\circ\phi} y \end{tikzcd}$$
defining a natural transformation
$$\mathsf{h}_f : \mathsf{h}_x \Rightarrow \mathsf{h}_y.$$

Lemma 2.4.6 (Weak Yoneda)**.** *The functor* $\mathsf{h}_{\mathcal{C}}$ *defined in* (2.4.1) *is fully faithful.*

DEFINITION 2.4.7. A functor $\mathsf{F} \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets})$ is *representable* if it lies in the essential image of $\mathsf{h}_{\mathcal{C}}$, i.e. if it is isomorphic to a functor h_x for some $x \in \mathcal{C}$. In this case, we say that the object $x \in \mathcal{C}$ represents F.

Remark 2.4.8. By Lemma 2.4.6, if $x \in \mathcal{C}$ represents F, then x is unique up to a unique isomorphism. Indeed, suppose we have isomorphisms

$$a\colon \mathsf{h}_x \xrightarrow{\sim} F, \quad b\colon \mathsf{h}_y \xrightarrow{\sim} F$$

in the category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets})$. Then there exists a unique isomorphism $x \xrightarrow{\sim} y$ inducing $b^{-1} \circ a\colon \mathsf{h}_x \xrightarrow{\sim} \mathsf{h}_y$.

Let $\mathsf{F} \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets})$ be a functor, $x \in \mathcal{C}$ an object. One can construct a map of sets

$$g_x\colon \mathrm{Hom}(\mathsf{h}_x, \mathsf{F}) \to \mathsf{F}(x), \tag{2.4.2}$$

where the source is the hom-set in the category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets})$, which is indeed a set by Assumption 2.4.1.

To a natural transformation $\eta\colon \mathsf{h}_x \Rightarrow \mathsf{F}$ one can associate the element

$$g_x(\eta) = \eta_x(\mathrm{id}_x) \in \mathsf{F}(x),$$

the image of $\mathrm{id}_x \in \mathsf{h}_x(x)$ via the map $\eta_x\colon \mathsf{h}_x(x) \to \mathsf{F}(x)$.

Lemma 2.4.9 (Strong Yoneda)**.** *Let* $\mathsf{F} \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets})$ *be a functor,* $x \in \mathcal{C}$ *an object. Then the map* g_x *defined in* (2.4.2) *is bijective.*

PROOF. The inverse of g_x is the map that assigns to an element $\xi \in \mathsf{F}(x)$ the natural transformation $\eta(x, \xi)\colon \mathsf{h}_x \Rightarrow \mathsf{F}$ defined as follows. For a given object $u \in \mathcal{C}$, we define

$$\eta(x, \xi)_u\colon \mathsf{h}_x(u) \to \mathsf{F}(u)$$

by sending a morphism $f : u \to x$ to the image of ξ under $\mathsf{F}(f)\colon \mathsf{F}(x) \to \mathsf{F}(u)$. □

EXERCISE 2.4.10. Show that Lemma 2.4.9 implies Lemma 2.4.6.

DEFINITION 2.4.11. Let $\mathsf{F}\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$ be a functor. A *universal object* for F is a pair (x, ξ) where $\xi \in \mathsf{F}(x)$, such that for every pair (u, σ) with $\sigma \in \mathsf{F}(u)$, there exists a unique morphism $\alpha\colon u \to x$ with the property that $\mathsf{F}(\alpha)\colon \mathsf{F}(x) \to \mathsf{F}(u)$ sends ξ to σ.

EXERCISE 2.4.12. Show that a pair (x, ξ) is a universal object for a functor $\mathsf{F}\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$ if and only if the natural transformation $\eta(x, \xi)$ defined in the proof of Lemma 2.4.9 is a natural isomorphism. In particular, F is representable if and only if it has a universal object.

In classical moduli theory, one is interested in the category

$$\mathcal{C} = \mathrm{Sch}_S$$

of schemes over a fixed scheme S. Its objects are pairs (X, f), where X is a scheme and $f : X \to S$ is a morphism of schemes. A morphism $(X, f) \to (Y, g)$ is a morphism $p\colon X \to Y$ such that $g \circ p = f$.

Example 2.4.13 (Global Spec)**.** Let S be a scheme, $\mathscr{A}$ a quasicoherent $\mathscr{O}_S$-algebra. Then the S-scheme $\mathrm{Spec}_{\mathscr{O}_S} \mathscr{A} \to S$ represents the functor $\mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{Sets}$ sending

$$(U \xrightarrow{g} S) \;\mapsto\; \mathrm{Hom}_{\mathscr{O}_S\text{-alg}}(\mathscr{A}, g_* \mathscr{O}_U).$$

The notion introduced in the following definition will be needed in the proof of Theorem 4.1.4.

DEFINITION 2.4.14 ([**185**, Tag 01JI]). Let $\mathsf{F}\colon \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Sets}$ be a functor. A subfunctor $\mathsf{H} \subset \mathsf{F}$ is said to be *representable by open immersions* if for every $U \in \mathrm{Sch}$ and $\theta \in \mathsf{F}(U)$ there is an open subscheme $V_\theta \subset U$ such that a morphism $f : U' \to U$ factors through V_θ if and only if $\mathsf{F}(f)(\theta)$ lies in the subset $\mathsf{H}(U') \subset \mathsf{F}(U')$.

2.5. More notions of representability, and GIT quotients

2.5.1. Fine moduli spaces and automorphisms. Given a scheme S and a functor $\mathfrak{M}\colon \mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{Sets}$, an object $\mathcal{M}$ in Sch_S along with an isomorphism

$$\mathfrak{M} \cong \mathrm{Hom}_{\mathrm{Sch}_S}(-, \mathcal{M})$$

is a *fine moduli space* for the objects parametrised by $\mathfrak{M}$. It is common to hear that when the objects $\eta \in \mathfrak{M}(U/S)$ have automorphisms, the functor $\mathfrak{M}$ cannot be represented. This is, for instance, the case for the moduli functor of smooth (or stable) curves of genus g. Even though this is the correct *geometric* intuition to have, I learnt in [**72**] that for a general functor the presence of automorphisms does not necessarily prevent the existence of a universal family.

EXERCISE 2.5.1. Construct the functor $\mathfrak{M}\colon \mathrm{Sets}^{\mathrm{op}} \to \mathrm{Sets}$ of isomorphism classes of finite sets. Show that it is representable (by what set?), even though every finite set has automorphisms.

In geometric situations, the presence of automorphisms does prevent representability whenever one can construct a family of objects $\eta \in \mathfrak{M}(U/S)$ that is isotrivial (i.e. it becomes the trivial family after étale base change) but not globally trivial. This is for instance the case for families of curves: the moduli map $U \to \mathcal{M}_g$ associated to an isotrivial family $\mathcal{X} \to U$ of smooth curves of genus g, say with typical fibre C, would have to be constant for continuity reasons; but the same is of course happening for the trivial family $C \times U \to U$, so the functor of smooth curves of genus g cannot be represented.

2.5.2. More notions of moduli spaces. There are a couple of weaker notions than the notion of a fine moduli space encountered earlier.

Let us fix a base scheme S and a set-valued functor $\mathfrak{M}\colon \mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{Sets}$,

DEFINITION 2.5.2 (Moduli space). A pair (M, η), where $M \in \mathrm{Sch}_S$ and $\eta\colon \mathfrak{M} \to \mathsf{h}_M$ is a natural transformation of functors, is said to be a *moduli space* for $\mathfrak{M}$ if it is universal, i.e. for any other pair (M', η') as above there is a unique morphism $f : M \to M'$ in Sch_S such that $\mathsf{h}_f \circ \eta = \eta'$, as the diagram

$$\begin{tikzcd} \mathfrak{M} \arrow[r, "\eta"] \arrow[dr, "\eta'"'] & \mathsf{h}_M \arrow[d, dashed, "\mathsf{h}_f"] \\ & \mathsf{h}_{M'} \end{tikzcd}$$

shows. We also say, if this condition is satisfied, that M *corepresents* $\mathfrak{M}$.

DEFINITION 2.5.3. Set $S = \mathrm{Spec}\,\mathbf{k}$. A moduli space (M, η) for $\mathfrak{M}$ such that $\eta_{\mathbf{k}}$ is a bijection is called a *coarse moduli space* for $\mathfrak{M}$.

Clearly, one has the chain of implications

$$M \text{ is fine} \quad \Rightarrow \quad M \text{ is coarse} \quad \Rightarrow \quad M \text{ corepresents } \mathfrak{M}.$$

These are strictly inequivalent notions. For instance, if $X \to S$ is a smooth projective morphism of $\mathbf{k}$-schemes of finite type and L is a relatively very ample line bundle on X, for any polynomial P there is a functor $\mathfrak{M}_L^P\colon \mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{Sets}$ that assigns to $U \to S$ the set of isomorphism classes $[\mathscr{F}]$ of U-flat families of semistable sheaves $\mathscr{F} \in \mathrm{Coh}(X \times_S U)$ with Hilbert polynomial P in the fibres of $q\colon X \times_S U \to U$. Two such isomorphism classes $[\mathscr{F}]$ and $[\mathscr{F}']$ are further identified if $\mathscr{F} \cong \mathscr{F}' \otimes q^*\mathscr{L}$, where $\mathscr{L}$ is a line bundle on S. This functor is corepresented by a projective scheme M_L^P, as proved in [**108**, Theorem 4.3.7]. However, as soon as there is a strictly semistable sheaf (i.e. a semistable sheaf that is not stable), $\mathfrak{M}_L^P$ cannot be represented.

2.5.3. Quotients in Algebraic Geometry in a nutshell. Let G be an algebraic group over an algebraically closed field $\mathbf{k}$ of characteristic 0, i.e. a group object in the category of $\mathbf{k}$-varieties. This means that there is a factorisation $\mathsf{h}_G\colon \mathrm{Sch}_{\mathbf{k}}^{\mathrm{op}} \to \mathrm{Groups} \to \mathrm{Sets}$ of the functor of points of G, where the second arrow is the forgetful functor (so $G(B) = \mathrm{Hom}_{\mathbf{k}}(B,G)$ is naturally a group for every $\mathbf{k}$-scheme B).

If X is a $\mathbf{k}$-variety, a (left) G-action on X is a $\mathbf{k}$-morphism $\sigma\colon G\times_{\mathbf{k}}X \to X$ such that $\sigma(B)\colon G(B)\times X(B) \to X(B)$ is a group action on the set $X(B) = \mathrm{Hom}_{\mathbf{k}}(B,X)$, for every $\mathbf{k}$-scheme B. A pair (X,σ) as above will be simply called a *G-scheme*. If $x \in X$ is a $\mathbf{k}$-point, then its *orbit*, denoted $\mathsf{O}(x) \subset X$, is by definition the image of the morphism $\sigma_x\colon G \xrightarrow{\sim} G\times_{\mathbf{k}}\{x\} \hookrightarrow G\times_{\mathbf{k}}X \to X$. Note that $\mathsf{O}(x)$ is locally closed in X, and smooth (this uses flatness of σ, and the fact that algebraic groups in characteristic 0 are smooth by Cartier's theorem). The preimage of the point $x \in X$ along σ_x is denoted G_x, and is called the *stabiliser* of x.

DEFINITION 2.5.4 (G-equivariance). Let (X,σ_X) and (Y,σ_Y) be G-schemes. We say that a morphism $f: X \to Y$ is *G-equivariant* if the diagram

$$\begin{tikzcd} G\times_{\mathbf{k}}X \arrow[r, "\sigma_X"] \arrow[d, "\mathrm{id}_G\times f"'] & X \arrow[d, "f"] \\ G\times_{\mathbf{k}}Y \arrow[r, "\sigma_Y"] & Y \end{tikzcd}$$

commutes. When $\sigma_Y = \mathrm{pr}_2$, we say that f is *G-invariant*.

Let (X,σ) be a G-scheme. The first notion of quotient (the coarsest one) is that of a *categorical quotient*, which by definition is a pair (Y,π) such that $\pi\colon X \to Y$ is a G-invariant morphism, subject to the universal property that any G-invariant morphism $f: X \to Z$ factors as $\alpha_f \circ \pi\colon X \to Y \to Z$ for a unique morphism $\alpha_f: Y \to Z$.

EXERCISE 2.5.5. Show that a G-scheme (X,σ) admits a categorical quotient if and only if the 'functor of orbits' $\mathrm{Sch}_{\mathbf{k}}^{\mathrm{op}} \to \mathrm{Sets}$ sending $B \mapsto X(B)/G(B)$ is corepresentable. The corepresenting scheme *is* the target of the categorical quotient morphism $\pi\colon X \to Y$.

Categorical quotients often exist, but they are rarely related with the notion of quotient we would like to have, namely an orbit space. Note that also in differentiable category a quotient is not easily achieved: a free action is required in order to get a smooth manifold parametrising orbits. In the algebraic category, one has the following basic example: if $X = \mathrm{Spec}A$ is an affine $\mathbf{k}$-variety acted on by a *reductive*[6] algebraic group G, then the ring of invariants $A^G \hookrightarrow A$ is finitely generated, and the morphism $\mathrm{Spec}A \to \mathrm{Spec}A^G$ corresponding to the $\mathbf{k}$-algebra inclusion $A^G \hookrightarrow A$ is a categorical quotient. If $A = \mathbf{k}[x_1,\ldots,x_d]$ is acted on by $G = \mathbb{C}^\times$ via $t\cdot x_i = tx_i$, there is a unique fixed point (the origin in $\mathbb{A}^d = \mathrm{Spec}A$), and all orbits have this point in their closure, which forces $A^G = \mathbf{k}$.

More sophisticated notions of quotients are the following.

DEFINITION 2.5.6 (Good and geometric quotients). Assume G is affine, let X be a G-scheme and $\pi\colon X \to Y$ a surjective morphism. Then

(1) we say that π is a *good quotient* if π is affine, G-invariant and satisfies:
   - a subset $U \subset Y$ is open if and only if $\pi^{-1}(U) \subset X$ is open,
   - the natural morphism $\mathscr{O}_Y \to \pi_*\mathscr{O}_X$ is an isomorphism onto the subsheaf $(\pi_*\mathscr{O}_X)^G \subset \pi_*\mathscr{O}_X$,
   - π takes closed G-invariant subsets to closed subsets of Y, and if Z_1, Z_2 are disjoint closed G-invariant subsets of X, then $\pi(Z_1)\cap\pi(Z_2) = \emptyset$.

(2) we say that π is a *geometric quotient* if it is a good quotient and its geometric fibres are orbits of geometric points of X.

[6]Over a field of characteristic 0, reductive means that every rational G-representation is completely reducible. Another way to say this is: the functor $\mathrm{QCoh}^G(\mathrm{pt}) \to \mathrm{QCoh}(\mathrm{pt})$ taking a G-representation V to its invariant part $V^G \subset V$ is exact.

The morphism $\pi\colon \operatorname{Spec} A \to \operatorname{Spec} A^G$ described above is always a good quotient.

There are implications

$$\text{geometric} \quad \Rightarrow \quad \text{good} \quad \Rightarrow \quad \text{categorical}.$$

Let X be a projective $\mathbf{k}$-scheme with a fixed G-action. Let ϑ be a G-linearisation on an ample line bundle $\mathscr{L}$ over X. This notion is defined in Definition 9.2.1, where it is referred to as a G-equivariant structure (but in GIT, the terminology 'linearisation' is most common). The ring $R = \bigoplus_{n\geq 0} \mathrm{H}^0(X, \mathscr{L}^{\otimes n})$ contains the finitely generated $\mathbb{Z}$-graded algebra R^G and satisfies $X = \operatorname{Proj} R$. The inclusion $R^G \subset R$ defines a *rational* morphism

$$f_\vartheta \colon X \dashrightarrow \operatorname{Proj} R^G.$$

DEFINITION 2.5.7 (Semistable locus). The maximal open subset $X^{\vartheta\text{-ss}}(\mathscr{L}) \subset X$ on which f_ϑ is defined is called the *ϑ-semistable locus*.

Note that we have an explicit description of $X^{\vartheta\text{-ss}}(\mathscr{L}) \subset X$. This open subset is the complement of the closed subscheme $\operatorname{Proj} R/R^G_+ \cdot R \subset \operatorname{Proj} R$, where $R^G_+ \subset R^G$ is the irrelevant ideal induced by the grading on R^G. Put differently, a point $x \in X$ belongs to $X^{\vartheta\text{-ss}}(\mathscr{L})$ if and only if there is an $n \geq 1$ and a section $s \in \mathrm{H}^0(X, \mathscr{L}^{\otimes n})^G$ such that $s(x) \neq 0$.

DEFINITION 2.5.8 (Stable locus). The *ϑ-stable locus* is the open subscheme $X^{\vartheta\text{-st}}(\mathscr{L}) \subset X$ consisting of points $x \in X^{\vartheta\text{-ss}}(\mathscr{L})$ such that G_x is finite and $\mathsf{O}(x) \cap X^{\vartheta\text{-ss}}(\mathscr{L}) \subset X^{\vartheta\text{-ss}}(\mathscr{L})$ is closed.

Remark 2.5.9. The open subschemes

$$X^{\vartheta\text{-st}}(\mathscr{L}) \subset X^{\vartheta\text{-ss}}(\mathscr{L}) \subset X$$

change with ϑ, and they might be empty.

A fundamental result (proved e.g. in [148, 153]) in Geometric Invariant Theory is the following. Say G is a reductive group acting on a projective $\mathbf{k}$-scheme X. Let ϑ be a G-linearisation on an ample line bundle $\mathscr{L}$ over X. Then $\pi\colon X^{\vartheta\text{-ss}}(\mathscr{L}) \to \operatorname{Proj} R^G$ is a good quotient for the initial G-action. Moreover, there is an open subscheme $Y^{\mathrm{st}} \subset \operatorname{Proj} R^G$ such that $\pi^{-1}(Y^{\mathrm{st}}) = X^{\vartheta\text{-st}}(\mathscr{L})$, and the restricted morphism $\pi\colon X^{\vartheta\text{-st}}(\mathscr{L}) \to Y^{\mathrm{st}}$ is a geometric quotient. This can depicted in a diagram

$$\begin{tikzcd}
X^{\vartheta\text{-st}}(\mathscr{L}) \arrow[r, hook, "\text{open}"] \arrow[d, "\text{geometric}"'] & X^{\vartheta\text{-ss}}(\mathscr{L}) \arrow[r, hook, "\text{open}"] \arrow[d, "\text{good}"'] & X \arrow[dl, dashed, "f_\vartheta"] \\
Y^{\mathrm{st}} \arrow[r, hook, "\text{open}"] & \operatorname{Proj} R^G &
\end{tikzcd}$$

and since $\operatorname{Proj} R^G$ is projective, it be seen as a compactification of the geometric quotient Y^{st}.

Notation 2.5.10. Good quotients as above are often denoted

$$X /\!\!/_\vartheta G \ \text{ or } \ X^{\vartheta\text{-ss}}(\mathscr{L})/G$$

whereas geometric quotients are often denoted (somewhat sloppily) $X^{\vartheta\text{-st}}(\mathscr{L})/G$.

CHAPTER 3

Informal introduction to Grassmannians

SUMMARY. In this short chapter, where for simplicity we work over $\mathbb{C}$, we introduce via a hands-on approach, partially following [51], one of the most popular moduli spaces in algebraic geometry: the *Grassmannian* $G(k,n)$, classifying k-dimensional linear subspaces of the vector space $\mathbb{C}^n$.

We will see that $G(k,n)$ is a smooth projective algebraic variety of dimension $k(n-k)$, naturally embedded in $\mathbb{P}^N$, where $N = \binom{n}{k} - 1$. It is naturally identified with the set of $(k-1)$-dimensional linear subspaces of $\mathbb{P}^{n-1}$, and when we think of it in this manner we denote it by $\mathbb{G}(k-1,n-1)$. The variety $\mathbb{G} = G(k,n)$ admits an affine stratification, whose strata are parametrised by partitions λ contained in a $k \times (n-k)$ rectangle. The closures of the strata, called *Schubert varieties*, are usually denoted Ω_λ and generate the Chow group $A_*\mathbb{G}$. The *Schubert cycles* are the classes $\sigma_\lambda \in A^*\mathbb{G}$ determined by the relation $\sigma_\lambda \cap [\mathbb{G}] = [\Omega_\lambda]$. To determine the ring structure of $A^*\mathbb{G}$, one has to compute the products $\sigma_\lambda \cdot \sigma_\mu$ between these cycles. These computations in $A^*\mathbb{G}$ go under the name of *Schubert Calculus*. The simplest example is $\mathbb{G}(0,n-1) = G(1,n) = \mathbb{P}^{n-1}$, in which case $A^*\mathbb{P}^{n-1} = \mathbb{Z}[h]/h^n$, where h is the hyperplane class.

3.1. The Grassmannian as a projective variety

Fix integers $0 < k < n$ and let $V = \mathbb{C}^n$. Define $N = \binom{n}{k} - 1$. Form the *set*

$$\mathbb{G} = G(k,n) = G(k,V) = \{\, k\text{-dimensional linear subspaces } \mathcal{H} \subset V \,\}.$$

Pick an element

$$\mathcal{H} \in \mathbb{G}.$$

If $\{ v_1, \ldots, v_k \} \subset V$ is a basis of $\mathcal{H}$ then $v_1 \wedge \cdots \wedge v_k$ is the free generator of the line

$$\wedge^k \mathcal{H} \subset \wedge^k V \cong \mathbb{C}^{\binom{n}{k}} = \mathbb{C}^{N+1}.$$

So we get a set-theoretic map

$$\iota \colon \mathbb{G} \to \mathbb{P}\big(\wedge^k V\big) = \mathbb{P}^N$$

sending $\mathcal{H} \mapsto [v_1 \wedge \cdots \wedge v_k]$. Why is this map well-defined? Let us view $\mathcal{H}$ as (the space generated by the rows of) a full rank matrix $H = (a_{ij}) \in M_{k\times n}(\mathbb{C})$ and let us fix a basis $\{ e_1, \ldots, e_n \} \subset V$. Then a basis of $\wedge^k V$ is given by

$$\big\{ e_{i_1} \wedge \cdots \wedge e_{i_k} \big\}_{1 \le i_1 < \cdots < i_k \le n}.$$

So in this basis the element $v_1 \wedge \cdots \wedge v_k \in \wedge^k V$ writes uniquely as

$$v_1 \wedge \cdots \wedge v_k = \sum_{1 \le i_1 < \cdots < i_k \le n} p_{i_1 \cdots i_k} \cdot e_{i_1} \wedge \cdots \wedge e_{i_k} = \sum_I p_I \cdot e_I,$$

where the coefficient $p_I = p_{i_1 \cdots i_k} \in \mathbb{C}$ is the minor of the $(k \times k)$-matrix given by extracting from H the columns corresponding to $I = (i_1, \ldots, i_k)$. Of course different choices of H may produce the same $\mathcal{H}$. But H is unique up to the left action of GL_k. Summing up, we have a commutative diagram

$$\begin{tikzcd}
\wedge^k \mathcal{H} \setminus 0 \arrow[r, hook] \arrow[d, "/\mathbb{C}^\times"'] & \wedge^k V \setminus 0 \arrow[d, "/\mathbb{C}^\times"] & v_1 \wedge \cdots \wedge v_k \arrow[r, mapsto] \arrow[d, mapsto] & \sum_I p_I e_I \arrow[d, mapsto] \\
\mathbb{P}\big(\wedge^k \mathcal{H}\big) \arrow[r, "\iota"] & \mathbb{P}\big(\wedge^k V\big) & {[v_1 \wedge \cdots \wedge v_k]} \arrow[r, mapsto] & (p_I)_I
\end{tikzcd}$$

where the vertical maps are quotient maps. Up to now, we have identified a point $\mathcal{H} \in \mathbb{G}$ with the corresponding point (the unique point!) of $\mathbb{P}(\wedge^k \mathcal{H})$ and we have defined a map $\iota\colon \mathbb{G} \to \mathbb{P}^N$ by sending $\mathcal{H}$ to its *Plücker coordinates* $(p_I)_I$.

EXERCISE 3.1.1. Prove that the map ι is injective.

In fact, $\mathbb{G}$ can be identified with an irreducible algebraic set sitting inside $\mathbb{P}^N$, thanks to the following exercise.

EXERCISE 3.1.2. Show that the kernel of the homomorphism

$$\mathbb{C}\big[p_{i_1\dots i_k}\,\big|\,1 \le i_1 < \cdots < i_k \le n\big] \to \mathbb{C}\big[x_{lj}\,\big|\,1 \le l \le k,\, 1 \le j \le n\big]$$

sending $p_{i_1\dots i_k}$ to the *Plücker coordinate* $\det(x_{lj})_{1\le l\le k,\, j=i_1,\dots,i_k}$ is a homogeneous prime ideal J. Confirm that elements of $\mathbb{G}$ correspond bijectively to the points of the homogeneous vanishing locus $V_+(J)$.

A bit of work shows that the ideal J is generated by *quadratic* polynomials — see e.g. [**51**, Section 3.2.2]. These polynomials define the so-called *Plücker relations*. Then one can realise

$$\mathbb{G} = V_+(J) \subset \operatorname{Proj}\mathbb{C}\big[p_{i_1\dots i_k}\,\big|\,1 \le i_1 < \cdots < i_k \le n\big] = \mathbb{P}^N$$

as a closed subvariety of projective space, cut out by quadrics.

In fact, granting the existence of the universal rank k subbundle $\mathscr{S} \hookrightarrow V \otimes \mathscr{O}_{\mathbb{G}}$, one can show the following.

EXERCISE 3.1.3. Let $V = \mathrm{H}^0(\mathbb{P}^{n-1}, \mathscr{O}_{\mathbb{P}^{n-1}}(1))$, with basis $x_0, x_1, \dots, x_{n-1}$, the homogeneous coordinates on $\mathbb{P}^{n-1}$. Show that the embedding $\mathbb{G} \hookrightarrow \mathbb{P}^N$ is defined by the line bundle $\mathscr{O}_{\mathbb{G}}(1) = \wedge^k \mathscr{S}^*$, and that $\mathrm{H}^0(\mathscr{O}_{\mathbb{G}}(1)) \cong \wedge^k \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^{n-1}}(1))$. (So in this setting $x_{i_1} \wedge \cdots \wedge x_{i_k}$ are the $\binom{n}{k} = N+1$ homogeneous coordinates of $\mathbb{P}^N$).

EXERCISE 3.1.4. Show that, if Y is a homogeneous space (i.e. a variety equipped with a transitive action by an algebraic group K) defined over an algebraically closed field $\mathbf{k}$, and Y has a smooth point p, then Y is smooth everywhere. Deduce that $G(k,n)$ is a smooth variety (**Hint**: use $K = \mathrm{GL}(V) = \mathrm{GL}_n$).

Example 3.1.5. The Grassmannian $G(2,4) = \mathbb{G}(1,3)$ of lines in $\mathbb{P}^3$ is a smooth quadric hypersurface in $\mathbb{P}^5$, the vanishing locus of the single homogeneous polynomial

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0.$$

This is the smallest Grassmannian that is not a projective space.

3.2. Schubert cycles

We know that $\mathbb{G} = G(k,n)$ is smooth and projective, so its Chow group is a ring and can be graded by codimension

$$A^*\mathbb{G} = \bigoplus_{j=0}^{\dim\mathbb{G}} A^j\mathbb{G}, \qquad \dim\mathbb{G} = k(n-k).$$

Now we think of points $\mathcal{H} \in \mathbb{G} = \mathbb{G}(k-1, n-1)$ as linear subvarieties of $\mathbb{P}^{n-1}$. So, let us fix a flag

$$\mathcal{F}\colon\ \mathsf{pt} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} = \mathbb{P}^{n-1},$$

where each $F_i \cong \mathbb{P}^i$ is a linear subvariety of $\mathbb{P}^{n-1}$ of dimension i. Let us look at the set of k-tuples

$$\mathscr{A} = \{\, (a_1, \dots, a_k) \mid n-k \ge a_1 \ge \cdots \ge a_k \ge 0 \,\}.$$

For all $a = (a_1, \dots, a_k) \in \mathscr{A}$, define the closed subset

$$\Sigma_a(\mathcal{F}) = \Big\{\, \mathcal{H} \in \mathbb{G} \,\Big|\, \dim(\mathcal{H} \cap F_{n-k+i-1-a_i}) \ge i-1 \text{ for all } i = 1, \dots, k \,\Big\} \subset \mathbb{G}.$$

These are called the *Schubert subvarieties* of $\mathbb{G}$. Their classes in $A_*\mathbb{G}$ are independent of the choice of flag. Let us set

$$c_a = \operatorname{codim}(\Sigma_a(\mathcal{F}), \mathbb{G}).$$

The *Schubert cycles* $\sigma_a \in A^{c_a}\mathbb{G}$ are defined by $\sigma_a \cap [\mathbb{G}] = [\Sigma_a] \in A_*\mathbb{G}$. They have a number of interesting properties, for instance:

(1) Their codimension satisfies $c_a = \sum_{1\leq i\leq k} a_i$.
(2) By defining $a \leq b$ if and only if $a_i \leq b_i$ for all $i = 1,\ldots,k$, one sees that $\Sigma_b \subseteq \Sigma_a$ if and only if $a \leq b$.
(3) There is an isomorphism

$$\widetilde{\Sigma}_a = \Sigma_a \setminus \bigcup_{a<b} \Sigma_b \cong \mathbb{A}^{k(n-k)-c_a},$$

for all $a \in \mathscr{A}$. The locally closed subvarieties $\widetilde{\Sigma}_a \subset \mathbb{G}$ are called *Schubert cells*, and they form an affine stratification

$$\mathbb{G} = \coprod_{a\in\mathscr{A}} \widetilde{\Sigma}_a.$$

with closed strata given by the Schubert cycles.

Example 3.2.1. Let $a = (1,0,\ldots,0)$. Then $\Sigma_a = \Sigma_1$ is the locus of k-planes $\Lambda \subset \mathbb{C}^n$ meeting a given $(n-k)$-plane. It is a hyperplane section of $\mathbb{G} \hookrightarrow \mathbb{P}^N$.

Recall the following standard result in intersection theory.

Proposition 3.2.2 ([**196**])**.** *If a scheme Y has an affine stratification, then A_*Y is freely generated by the classes of the closures of the strata.*

EXERCISE 3.2.3. Using Proposition 3.2.2, show that

(1) $A_*\mathbb{G}$ is a free abelian group with basis parametrised by $\mathscr{A}$, the set of partitions of length at most k contained in a $k \times (n-k)$ rectangle,
(2) There is a $\mathbb{Z}$-module isomorphism $A_*\mathbb{G} \xrightarrow{\sim} \wedge^k \mathbb{Z}^n$ mapping $\Sigma_a \mapsto e_{1+a_k} \wedge \cdots \wedge e_{k+a_1}$, where $e_1,\ldots,e_n$ are the canonical basis elements of $\mathbb{Z}^n$. In particular, the fundamental class $[\mathbb{G}]$ is mapped to $e_1 \wedge \cdots \wedge e_k$.
(3) There is an isomorphism $A_{\dim \mathbb{G}-i}\mathbb{G} \cong (\wedge^k \mathbb{Z}^n)_i$, where the right hand side is the direct sum of the $\mathbb{Z}$-modules $\operatorname{Span}_{\mathbb{Z}}(e_{1+a_k} \wedge \cdots \wedge e_{k+a_1})$ such that $\sum_j a_j = i$.

The core of Schubert Calculus (intersection theory on Grassmannians) is the following: if we understand the intersection theory of projective space (which we do), then we understand that of every $G(k,n)$, by *taking exterior powers* and using the *Leibniz rule*. More precisely, let V be a finite dimensional vector space. The Chow group $A_*G(k,V)$ of the Grassmannian is naturally isomorphic to the k-th exterior power $\wedge^k A_*\mathbb{P}(V)$. The same formula holds for Grassmann bundles (introduced in Section 4.1): if F is a vector bundle over a connected noetherian scheme S, then $A_*G(k,F) \cong \wedge^k A_*\mathbb{P}(F)$, and the key point is that the Schubert cycle σ_i acts on the exterior algebra of $A_*\mathbb{P}(F)$ as a derivation of order i, see e.g. [**73, 76**].

3.3. The Chow ring of $\mathbb{G}(1,3)$

Let $\mathbb{G} = G(2,4) = \mathbb{G}(1,3)$ be the Grassmannian of lines $L \subset \mathbb{P}^3$, so that

$$\begin{aligned} \mathscr{A} &= \{ (a_1,a_2) \mid 2 \geq a_1 \geq a_2 \geq 0 \} \\ &= \{ (2,2),(1,1),(0,0),(2,1),(2,0),(1,0) \}. \end{aligned}$$

After fixing a flag of linear subvarieties

$$\mathcal{F}\colon\ \{P\} \subset M \subset H \subset \mathbb{P}^3,$$

where $\{P\} = F_0$, $M = F_1 \cong \mathbb{P}^1$, $H = F_2 \cong \mathbb{P}^2$, using the definition of Schubert cycles

$$\Sigma_{a_1a_2}(\mathcal{F}) = \{ L \in \mathbb{G} \mid \dim(L \cap F_{2-a_1}) \geq 0, \dim(L \cap F_{3-a_2}) \geq 1 \},$$

we can write them all explicitly as follows:

$$\begin{aligned}
\Sigma_{22} &= \{ L \in \mathbb{G} \mid P \in L, \dim(L \cap M) = 1 \} = \{ M \} \\
\Sigma_{11} &= \{ L \in \mathbb{G} \mid L \cap M \neq \emptyset, L \subset H \} = \{ L \in G \mid L \subset H \} \\
\Sigma_{00} &= \left\{ L \in \mathbb{G} \,\middle|\, L \text{ meets a plane, and meets } \mathbb{P}^3 \text{ in a line} \right\} = G \\
\Sigma_{21} &= \{ L \in \mathbb{G} \mid P \in L \subset H \} \\
\Sigma_{20} &= \{ L \in \mathbb{G} \mid P \in L \} \\
\Sigma_{10} &= \{ L \in \mathbb{G} \mid L \cap M \neq \emptyset \}.
\end{aligned}$$

Of course, we used that L always intersects a plane in $\mathbb{P}^3$.

The Chow classes of the Schubert varieties, in terms of the basis elements of $\wedge^2\mathbb{Z}^4$, are respectively

$$e_3 \wedge e_4, \; e_2 \wedge e_3, \; e_1 \wedge e_2, \; e_2 \wedge e_4, \; e_1 \wedge e_4, \; e_1 \wedge e_3.$$

Let us set $\Sigma_{a_10} = \Sigma_{a_1}$. In order to calculate the Schubert cells $\widetilde{\Sigma}_{a_1a_2}$, it is useful to look at the following inclusions:

$$\begin{tikzcd}
 & & \Sigma_2 \arrow[dr] & & \\
\mathrm{pt} = \Sigma_{22} \arrow[r, hook] & \Sigma_{21} \arrow[ur, hook] \arrow[dr] & & \Sigma_1 \arrow[r, hook] & \mathbb{G} \\
 & & \Sigma_{11} \arrow[ur, hook] & &
\end{tikzcd}$$

EXERCISE 3.3.1. Verify by direct calculation that $\widetilde{\Sigma}_{a_1a_2} \cong \mathbb{A}^{4-(a_1+a_2)}$.

Let us now focus on the problem of determining the ring structure of $A^*\mathbb{G}$. For the moment, we have the free abelian group decomposition

$$A^*\mathbb{G} = \underbrace{\mathbb{Z}[\sigma_{22}]}_{A^4\mathbb{G}} \oplus \underbrace{\mathbb{Z}[\sigma_{21}]}_{A^3\mathbb{G}} \oplus \underbrace{\mathbb{Z}[\sigma_{11}] \oplus \mathbb{Z}[\sigma_2]}_{A^2\mathbb{G}} \oplus \underbrace{\mathbb{Z}[\sigma_1]}_{A^1\mathbb{G}} \oplus \underbrace{\mathbb{Z}[\sigma_0]}_{A^0\mathbb{G}},$$

which induces the decomposition

$$A_*\mathbb{G} = \mathbb{Z} \cdot e_3 \wedge e_4 \oplus \mathbb{Z} \cdot e_2 \wedge e_3 \oplus \mathbb{Z} \cdot e_1 \wedge e_2 \oplus \mathbb{Z} \cdot e_2 \wedge e_4 \oplus \mathbb{Z} \cdot e_1 \wedge e_4 \oplus \mathbb{Z} \cdot e_1 \wedge e_3.$$

Remark 3.3.2. Any two points in $\mathbb{G}$ are rationally equivalent. This is true in every Grassmannian.

To calculate the products in $A^*\mathbb{G}$, we work with two generically situated flags at the same time: we will intersect a cycle taken from the first flag with a cycle taken from the second flag. Such cycles are generically transverse by the following transversality result (that we can apply because we are over $\mathbb{C}$).

Theorem 3.3.3 (Kleiman transversality [**51**, Theorem 1.7])**.** *Let G be an algebraic group acting transitively on a variety X over an algebraically closed field of characteristic* 0*. Let $Y \subset X$ be a subvariety. Then*

(a) *given another subvariety $Z \subset X$, there exists a dense open subset $U \subset G$ such that $g \cdot Y$ and Z are generically transverse for all $g \in U$.*
(b) *If G is affine, then $[g \cdot Y] = [Y]$ in the Chow group A_*X.*

Moreover, the result of the intersection of Schubert cycles coming from generically situated flags only depends on the equivalence classes of the cycles we are intersecting by the so-called Moving Lemma [**51**, Theorem 1.6]. Thus everything is well-defined. Now, let us fix two flags

$$\begin{aligned}
\mathcal{F}&: \; \{P\} \subset M \subset H \subset \mathbb{P}^3 \\
\mathcal{F}'&: \; \{P'\} \subset M' \subset H' \subset \mathbb{P}^3.
\end{aligned}$$

3.3.0.1. *Codimension* 4. We have to evaluate

$$\sigma_{11}^2,\ \sigma_2^2,\ \sigma_{11}\cdot\sigma_2,\ \sigma_1\cdot\sigma_{21}\in A^4\mathbb{G}.$$

Let us start with the self-intersection σ_{11}^2. We have

$$|\Sigma_{11}\cap\Sigma_{11}'| = |\{\,L\in\mathbb{G}\mid L\subset H\cap H'\,\}| = 1.$$

This unique line is of course $H\cap H'$. Hence

$$\sigma_{11}^2=\sigma_{22}.$$

Similarly,

$$|\Sigma_2\cap\Sigma_2'| = |\{\,L\in\mathbb{G}\mid P\in L,\ P'\in L\,\}| = |\{\,\overline{PP'}\,\}| = 1.$$

Hence again

$$\sigma_2^2=\sigma_{22}.$$

Since $P'\notin H$, we find

$$|\Sigma_{11}\cap\Sigma_2'| = |\{\,L\in\mathbb{G}\mid P'\in L\subset H\,\}| = 0,$$

thus

$$\sigma_{11}\cdot\sigma_2=0.$$

The last calculation is

$$|\Sigma_1\cap\Sigma_{21}'| = |\{\,L\in\mathbb{G}\mid L\cap M\neq\emptyset,\ P'\in L\subset H'\,\}| = 1,$$

corresponding to the line determined by P' and $M\cap H'$. Thus

$$\sigma_1\cdot\sigma_{21}=\sigma_{22}.$$

3.3.0.2. *Codimension* 3. We have to evaluate

$$\sigma_1\cdot\sigma_2,\ \sigma_{11}\cdot\sigma_1\in A^3\mathbb{G}.$$

We see that

$$\Sigma_1\cap\Sigma_2' = \{\,L\in\mathbb{G}\mid L\cap M\neq\emptyset,\ P'\in L\,\} = \Sigma_{21}''$$

with respect to the flag $\mathcal{F}''\colon\ \{\,P'\,\}\subset\ell\subset\langle P',M\rangle\subset\mathbb{P}^3$. Thus we get

$$\sigma_1\cdot\sigma_2=\sigma_{21}.$$

Similarly,

$$\Sigma_1\cap\Sigma_{11} = \{\,L\in\mathbb{G}\mid L\cap M\neq\emptyset,\ L\subset H'\,\} = \Sigma_{21}''$$

with respect to the flag $\mathcal{F}''\colon\ \{\,R\,\}\subset\ell\subset H'\subset\mathbb{P}^3$, where $R=M\cap H'$. Thus

$$\sigma_1\cdot\sigma_{11}=\sigma_{21}.$$

3.3.0.3. *Codimension* 2. We have to evaluate $\sigma_1^2\in A^2\mathbb{G}$. Here things get tricky because this product is not a Schubert cycle. What we know is that we can write $\sigma_1^2=\alpha\sigma_{11}+\beta\sigma_2$ in $A^2\mathbb{G}$. We have to determine α and β. The strategy is to intersect both sides with cycles in complementary codimension in such a way that one of the summands vanishes. Doing this twice allows us to recover α and β in two steps. So, using also relations previously obtained in codimension 4,

$$\begin{aligned}\sigma_1^2\cdot\sigma_2&=(\alpha\sigma_{11}+\beta\sigma_2)\cdot\sigma_2\\ \sigma_1^2\cdot\sigma_{11}&=(\alpha\sigma_{11}+\beta\sigma_2)\cdot\sigma_{11}\end{aligned}$$

yield, respectively,

$$\begin{aligned}\sigma_{22}&=\sigma_1\cdot\sigma_{21}=\beta\sigma_{22}\\ \sigma_{22}&=\sigma_1\cdot\sigma_{21}=\alpha\sigma_{11}^2=\alpha\sigma_{22},\end{aligned}$$

showing $\alpha=\beta=1$, and finally

$$\sigma_1^2=\sigma_{11}+\sigma_2.$$

The next result has a precise enumerative meaning. It solves Problem (1) from the Introduction. We will also solve this problem via torus localisation in Section 8.4.

Proposition 3.3.4. *There is an identity*

$$\int_{\mathbb{G}(1,3)} \sigma_1^4 = 2.$$

PROOF. We can compute

$$\begin{aligned}\sigma_1^4 &= (\sigma_1^2)^2\\ &= (\sigma_{11}+\sigma_2)^2\\ &= \sigma_{11}^2 + 2\sigma_{11}\sigma_2 + \sigma_2^2\\ &= \sigma_{22} + 0 + \sigma_{22}.\end{aligned}$$

The result follows from the fact that σ_{22} is the class of a point. □

EXERCISE 3.3.5. Use the intersections we computed above to show that there is a ring isomorphism

$$A^*\mathbb{G}(1,3) \cong \frac{\mathbb{Z}[\sigma_1,\sigma_2]}{(\sigma_1^3 - 2\sigma_1\sigma_2, \sigma_1^2\sigma_2 - \sigma_2^2)}.$$

3.4. The Leibniz rule and the degree of $\mathbb{G}(1,n+1)$

We start by reproving Proposition 3.3.4 using a slightly different formalism. We invite the reader to open the paper [73] and the book [75] to learn more about derivations on Grassmann algebras and their powerful enumerative applications.

The Chow ring $A^*\mathbb{G}(1,3)$ is generated by σ_1,σ_2 as a $\mathbb{Z}$-algebra. We let them act linearly on $\mathbb{Z}^4$ by setting $\sigma_i e_j = e_{i+j}$ if $i+j\le 4$ and 0 otherwise. Extend their action to $\wedge^2\mathbb{Z}^4 \cong A_*\mathbb{G}(1,3)$ by imposing the *Leibniz rule*

$$\begin{aligned}\sigma_1\cap[\Sigma_{a_1,a_2}] &= \sigma_1(e_{1+a_2}\wedge e_{2+a_1}) = \sigma_1 e_{1+a_2}\wedge e_{2+a_1} + e_{1+a_2}\wedge\sigma_1 e_{2+a_1}\\ \sigma_2\cap[\Sigma_{a_1,a_2}] &= \sigma_2(e_{1+a_2}\wedge e_{2+a_1}) = \sigma_2 e_{1+a_2}\wedge e_{2+a_1} + \sigma_1 e_{1+a_2}\wedge\sigma_1 e_{2+a_1} + e_{1+a_2}\wedge\sigma_2 e_{2+a_1}.\end{aligned}$$

Then σ_i maps $(\wedge^2\mathbb{Z}^4)_\ell$ to $(\wedge^2\mathbb{Z}^4)_{\ell+i}$, where $(\wedge^2\mathbb{Z}^4)_j = 0$ if $j>4$. It is easy to check by induction that

$$\sigma_1^j(u\wedge v) = \sum_{i=0}^{j}\binom{j}{i}\sigma_1^i u\wedge\sigma_1^{j-i}v.$$

Then Proposition 3.3.4 can be proved as follows: we have

$$\sigma_1^4\cap[\mathbb{G}(1,3)] = \sigma_1^4(e_1\wedge e_2) = \sum_{i=0}^{4}\binom{4}{i}\sigma_1^i e_1\wedge\sigma_1^{4-i}e_2,$$

and since σ_1^4 maps $e_1\wedge e_2$ to a multiple of $e_3\wedge e_4$ (the class of a point), the only two surviving terms are those corresponding to $j=2$ and $j=3$. Thus

$$\sigma_1^4\cap[\mathbb{G}(1,3)] = \binom{4}{2}e_3\wedge e_4 + \binom{4}{3}e_4\wedge e_3 = \left[\binom{4}{2}-\binom{4}{3}\right]e_3\wedge e_4 = 2\cdot e_3\wedge e_4.$$

This argument can obviously be generalised. For instance $A_*\mathbb{G}(1,n+1)\cong\wedge^2\mathbb{Z}^{2+n}$ and the degree of $\mathbb{G}(1,n+1)$ with respect to its Plücker embedding is obtained by intersecting $2n = \dim\mathbb{G}(1,n+1)$ hyperplane sections. The class of a hyperplane section is, by Exercise 3.1.3, the first Chern class of $\wedge^k\mathscr{S}^*$, and this equals σ_1 on every Grassmannian. The Plücker degree

$$\int_{\mathbb{G}(1,n+1)}\sigma_1^{2n} = \deg_{\mathbb{G}(1,n+1)}\left(\sigma_1^{2n}\cap[\mathbb{G}(1,n+1)]\right)$$

is the coefficient of $e_{n+1} \wedge e_{n+2}$ in the expansion of $\sigma_1^{2n}(e_1 \wedge e_2)$. This is

$$\sigma_1^{2n}(e_1 \wedge e_2) = \sum_{i=0}^{2n} \binom{2n}{i} e_{1+i} \wedge e_{2+2n-i} = \left[\binom{2n}{n} - \binom{2n}{n+1}\right] e_{n+1} \wedge e_{n+2},$$

so that

$$\int_{\mathbb{G}(1,n+1)} \sigma_1^{2n} = \frac{(2n)!}{(n+1)!n!}, \tag{3.4.1}$$

which are the well known Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$.

EXERCISE 3.4.1. Prove that

$$(\sigma_1^2 - \sigma_2) e_1 \wedge e_2 = e_2 \wedge e_3, \quad \sigma_1 \sigma_2 (e_1 \wedge e_2) = e_2 \wedge e_4, \quad \sigma_2^2 (e_1 \wedge e_2) = e_3 \wedge e_4.$$

We will see in Proposition 8.4.1 that (3.4.1) agrees with the number of lines $\ell \subset \mathbb{P}^{n+1}$ that are incident to $2n$ general $(n-1)$-planes $\Lambda_1, \ldots, \Lambda_{2n} \subset \mathbb{P}^{n+1}$.

CHAPTER 4

Relative Grassmannians, Quot, Hilb

SUMMARY. In this chapter we introduce three important examples of *fine moduli spaces* used in Algebraic Geometry: relative Grassmannians (including Grassmann bundles), Hilbert schemes and Quot schemes. We use the notion of (quasi)projectivity introduced in Definition 2.2.8 throughout.

The technical way to define fine moduli spaces is via representable functors $\mathfrak{M}\colon \mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{Sets}$, introduced in general in Section 2.4. The basic idea is as follows. First of all, every S-scheme $\mathcal{M}$ trivially 'represents' its own *functor of points*, which is the functor $\mathsf{h}_{\mathcal{M}}\colon \mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{Sets}$ sending

$$U \mapsto \mathsf{h}_{\mathcal{M}}(U) = \mathrm{Hom}_S(U, \mathcal{M}).$$

Let $\mathcal{T}$ be a class of... things. Suppose there is a notion of 'family of things' defined over any scheme $U \in \mathrm{Sch}_S$, and one can 'pullback' such families along arbitrary maps $V \to U$ in Sch_S.[1] Then one would say that an object $\mathcal{M} \in \mathrm{Sch}_S$ is a 'fine moduli space' for the objects in $\mathcal{T}$ if the functor $\mathfrak{M}$ assigning to a scheme U the set of 'families of objects in $\mathcal{T}$ defined over U' is isomorphic to $\mathsf{h}_{\mathcal{M}} = \mathrm{Hom}_S(-, \mathcal{M})$.

A *fine* moduli space is special in this sense: its points have a 'label', just as the items of a phone book. We know precisely each point's name and address, so we can always find it on the moduli space. This is, as we shall see, the power of *universal families*.

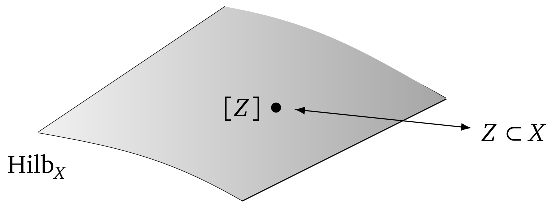

FIGURE 4.1. Each point of a fine moduli space has a well precise label. The figure depicts the *Hilbert scheme* of a scheme X, whose points correspond to closed subschemes $Z \subset X$.

4.1. Relative Grassmannians

4.1.1. The Grassmann functor and its representability. Fix a noetherian scheme S and a coherent sheaf F on S. Let Sch_S be the category of locally noetherian schemes over S. Recall from Section 2.2 that, for a locally noetherian scheme U, we denote by $\mathrm{Coh}\, U$ the abelian category of coherent sheaves on U. For any integer $d \geq 1$, the *Grassmann functor*

$$\mathsf{G}_d(F)\colon \mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{Sets}$$

is defined by

$$(U \xrightarrow{g} S) \mapsto \left\{ \begin{array}{c} \text{equivalence classes of surjections } g^*F \twoheadrightarrow Q \\ \text{in } \mathrm{Coh}\, U, \text{ with } Q \text{ locally free of rank } d \end{array} \right\} \tag{4.1.1}$$

[1] For instance, a family of smooth projective curves defined over U is a smooth projective morphism $\mathcal{C} \to U$ such that all geometric fibres $\mathcal{C}_u$ are smooth projective curves. If $V \to U$ is a morphism, then $\mathcal{C} \times_U V \to V$ is the pullback family.

where two quotients $p\colon g^*F \twoheadrightarrow Q$ and $p'\colon g^*F \twoheadrightarrow Q'$ are considered equivalent if there exists an $\mathscr{O}_U$-linear isomorphism $v\colon Q \xrightarrow{\sim} Q'$ such that $p' = v\circ p$. This functor is indeed set-valued by [**185**, Tag 01BC]. Note also that g^*F is coherent since both U and S are locally noetherian.[2]

Notation 4.1.1. When F is locally free of rank n, and $0<k\le n$, we set $\mathsf{G}(k,F) = \mathsf{G}_{n-k}(F)$.

EXERCISE 4.1.2. Show that two surjections $p\colon g^*F \twoheadrightarrow Q$ and $p'\colon g^*F \twoheadrightarrow Q'$ are equivalent if and only if $\ker p = \ker p'$.

EXERCISE 4.1.3. Show that $\mathsf{G}_d(F)$ is a Zariski sheaf. This means that for every S-scheme U and Zariski open covering $U=\bigcup_i U_i$, for any tuple $(\theta_i)_i$ with $\theta_i \in \mathsf{G}_d(F)(U_i\to S)$ such that $\theta_i|_{U_i\cap U_j} = \theta_j|_{U_i\cap U_j}$ there is a unique $\theta\in\mathsf{G}_d(F)(U\to S)$ such that $\theta|_{U_i} = \theta_i$.

We shall see in Theorem 4.1.4 that $\mathsf{G}_d(F)$ is representable. By definition, this means that there is an S-scheme $\rho\colon G_d(F)\to S$ such that for every $g\colon U\to S$ there is a functorial bijection

$$\mathsf{G}_d(F)(U \xrightarrow{g} S) \xrightarrow{\sim} \operatorname{Hom}_S(U, G_d(F)), \quad \alpha\mapsto \alpha_g. \tag{4.1.2}$$

Now take $U=G_d(F)$, $g=\rho$, and consider

$$\mathrm{id}_{G_d(F)} \in \operatorname{Hom}_S(G_d(F), G_d(F)).$$

The element in $\mathsf{G}_d(F)(\rho)$ mapping to $\mathrm{id}_{G_d(F)}$ via (4.1.2) is the *tautological exact sequence*

$$0\to \mathscr{S}\to \rho^*F\to \mathscr{Q}\to 0 \tag{4.1.3}$$

over $G_d(F)$. Note that if F is locally free of rank n then $\mathscr{S}$ is a locally free sheaf of rank $n-d$ over $G(n-d,F) = G_d(F)$. The sequence (4.1.3) is called 'tautological' because of the following universal property: if $g\colon U\to S$ is any morphism and $\alpha\in\mathsf{G}_d(F)(g)$, then the equivalence class of the pullback surjection

$$\alpha_g^*\rho^*F \twoheadrightarrow \alpha_g^*\mathscr{Q}$$

coincides with α.

The next result states that there is a *fine moduli space* associated to the Grassmann functor.

Theorem 4.1.4. *Let F be a coherent sheaf on a noetherian scheme S, and let $d\ge 1$ be an integer. The functor* (4.1.1) *can be represented by a projective S-scheme*

$$\rho\colon G_d(F)\to S.$$

The proof relies on the general result that a Zariski sheaf G, that can be covered by representable subfunctors G_i, with each inclusion $\mathsf{G}_i\hookrightarrow\mathsf{G}$ representable by open immersions (cf. Definition 2.4.14), is itself representable [**185**, Tag 01JF].

SKETCH OF PROOF. Note that the functor (4.1.1) makes sense for $(S,F) = (\operatorname{Spec}\mathbb{Z}, \mathscr{O}_{\operatorname{Spec}\mathbb{Z}}^{\oplus n})$, in which case for any $0<k\le n$ we denote $\mathsf{G}_{n-k}(\mathscr{O}_{\operatorname{Spec}\mathbb{Z}}^{\oplus n}) = \mathsf{G}(k,\mathscr{O}_{\operatorname{Spec}\mathbb{Z}}^{\oplus n})$ according to Notation 4.1.1. Let us grant the statement of the theorem in this case: for a proof of representability, relying on the above local criterion, we refer the reader to [**185**, Tag 089R]. On the other hand, the argument presented in [**155**, Section 5.1.6] shows that the resulting scheme

$$G(k,\mathscr{O}_{\operatorname{Spec}\mathbb{Z}}^{\oplus n})\to\operatorname{Spec}\mathbb{Z}$$

is projective.

Now, for a noetherian scheme S and the free sheaf $F=\mathscr{O}_S^{\oplus n}$, the statement of the theorem follows by base change along the unique morphism $S\to\operatorname{Spec}\mathbb{Z}$. If F is only *locally* free, we can choose an open covering $S=\bigcup_i S_i$ and confirm that $\mathsf{G}(k,F)$ is covered by open subfunctors isomorphic to $\mathsf{G}(k,\mathscr{O}_{S_i}^{\oplus n})$. But

[2] To conclude that the pullback of a coherent sheaf, along a morphism of locally noetherian schemes, is still coherent, use that the pullback always preserves quasicoherent modules [**185**, Tag 01BG] and finitely generated modules [**185**, Tag 01B6], and use Example 2.2.3.

these are representable, and $\mathrm{G}(k,F)$ is a Zariski sheaf by Exercise 4.1.3, thus the Grassmann functor is representable in the locally free case, too.

As for projectivity in the locally free case, let $\rho^*F \twoheadrightarrow \mathscr{Q}$ be the universal surjection. To see that $\rho\colon G(k,F)\to S$ is projective, one checks that the determinant

$$\mathscr{L}=\det\mathscr{Q}$$

of the universal quotient bundle is very ample relative to ρ, so it gives a closed embedding

$$\iota\colon G(k,F)\hookrightarrow \mathbb{P}(\rho_*\mathscr{L})\hookrightarrow \mathbb{P}\big(\wedge^k F\big). \tag{4.1.4}$$

Note that the target of ι is separated over S and a local calculation shows that $G(k,F)\to S$ is proper, thus the embedding ι is indeed proper, by [**185**, Tag 01W0].

Now for the actual statement: if F is a *coherent* sheaf, there is an open covering $S=\bigcup_i S_i$ such that $F_i = F|_{S_i}$ admits a surjection $E_i \twoheadrightarrow F_i$ from a locally free sheaf E_i. We get closed subfunctors $j_i\colon \mathrm{G}(k,F_i)\to\mathrm{G}(k,E_i)$ defined by sending

$$[F_i\twoheadrightarrow T]\longmapsto [E_i\twoheadrightarrow F_i\twoheadrightarrow T].$$

By the same argument as before, the open representable subfunctors $\mathrm{G}(k,F_i)\subset\mathrm{G}(k,F)$ cover $\mathrm{G}(k,F)$, which is then representable. Projectivity is a local calculation based on the closed embeddings j_i. □

Notation 4.1.5. When $d=1$ and $F\in\operatorname{Coh}S$, it is customary to denote $G_1(F)=\mathbb{P}(F)$, which by definition is $\operatorname{Proj}\operatorname{Sym}F$ [**185**, Tag 01OA]. Assume F is locally free of rank n. A closed point in the fibre $\mathbb{P}(F)_s$ over a geometric point $s\in S$ is represented by a surjective morphism $F_s\twoheadrightarrow\Lambda$, where $\dim_{\mathbf{k}(s)}\Lambda=1$. So in our notation, if V is a $\mathbf{k}$-vector space, by $\mathbb{P}(V)=\operatorname{Proj}\operatorname{Sym}V$ we mean the space of *hyperplanes in* V, whereas the space $(V\setminus 0)/\mathbf{k}^\times$ of *lines through the origin* $0\in V$ is naturally identified with $\mathbb{P}(V^*)$.

Remark 4.1.6. The embedding (4.1.4) is called the *Plücker embedding*. It generalises the embedding of Exercise 3.1.1. Note that if $F=\mathbb{C}^n$ is the trivial bundle over a point, then $\det\mathscr{S}^*\cong\det\mathscr{Q}$, which shows that the hyperplane class of the Plücker embedding is $\sigma_1=c_1(\mathscr{Q})=c_1(\mathscr{S}^*)$.

Remark 4.1.7. When F is locally free of rank n, and $0<k\le n$, the S-scheme $G(k,F)=G_{n-k}(F)$ is called the *Grassmann bundle* associated to F, and the structure morphism $\rho\colon G(k,F)\to S$ is smooth of relative dimension $k(n-k)$. Note that the kernel of a surjection between locally free sheaves is automatically locally free. Hence in this case, a closed point in the fibre $G(k,F)_s$ over a point $s\in S$ corresponds to a surjection $F(s)\twoheadrightarrow\Lambda$, where $\dim_{\mathbf{k}(s)}\Lambda=n-k$. More precisely, if $\mathbb{V}(F)=\operatorname{Spec}\operatorname{Sym}F^*\to S$ is the geometric vector bundle arising from the locally free sheaf F, then for every geometric point $s\in S$ one has

$$G(k,F)_s=G(k,F)\times_S\operatorname{Spec}\mathbf{k}(s)=G(k,\mathbb{V}(F)_s)=G(k,F(s)).$$

Example 4.1.8. Let $k=n-1$ and $F=\mathscr{O}_S^{\oplus n}$. Then the above construction yields the relative projective space

$$G(n-1,\mathscr{O}_S^{\oplus n})=\mathbb{P}_S^{n-1}=\operatorname{Proj}\operatorname{Sym}\mathscr{O}_S^{\oplus n}\xrightarrow{\rho}S.$$

The tautological surjection $\rho^*F\twoheadrightarrow\mathscr{Q}$ is the familiar

$$\mathscr{O}_{\mathbb{P}_S^{n-1}}^{\oplus n}\twoheadrightarrow\mathscr{O}_{\mathbb{P}_S^{n-1}}(1).$$

Indeed, we do know from the functorial description of projective space [**96**, II, Theorem 7.1] that an S-morphism $U\to\mathbb{P}_S^{n-1}$ is equivalent to the data

$$(\mathscr{L};s_0,s_1,\ldots,s_{n-1})$$

where $\mathscr{L}$ is a line bundle on U and s_i are sections generating $\mathscr{L}$ — and moreover such tuple is considered equivalent to $(\mathscr{L}';s_0',s_1',\ldots,s_{n-1}')$ if and only if there is an isomorphism of line bundles $\phi\colon\mathscr{L}\xrightarrow{\sim}\mathscr{L}'$ such that $\phi^*s_i'=s_i$. But this is the datum of a surjection $\mathscr{O}_U^{\oplus n}\twoheadrightarrow\mathscr{L}$, up to equivalence, which is precisely a U-valued point of $\mathrm{G}(n-1,\mathscr{O}_S^{\oplus n})$.

Example 4.1.9. If $S = \operatorname{Spec}\mathbf{k}$, we recover the usual Grassmannian (cf. Chapter 3 for $\mathbf{k} = \mathbb{C}$)

$$G(k,n) = \mathbb{G}(k-1,n-1)$$

of k-planes in $\mathbf{k}^n$ (or, equivalently, of projective linear subspaces $\mathbb{P}^{k-1}_{\mathbf{k}} \hookrightarrow \mathbb{P}^{n-1}_{\mathbf{k}}$), a smooth projective algebraic $\mathbf{k}$-variety of dimension $k(n-k)$.

EXERCISE 4.1.10. Let $S = \operatorname{Spec}\mathbb{C}$, so that $F = \mathbb{C}^n$, and fix a point $[\mathcal{H}] \in G(k,F)$. Show that the tangent space of $G(k,F) = G(k,n)$ at $[\mathcal{H}]$ is isomorphic to

$$\operatorname{Hom}_{\mathbb{C}}(\mathcal{H}, F/\mathcal{H}).$$

The relative statement is as follows. Let F be locally free. Then, the relative tangent bundle $T_{G(k,F)/S}$ is isomorphic to $\mathscr{H}\!om_{\mathcal{O}_{G(k,F)}}(\mathscr{S},\mathscr{Q}) = \mathscr{S}^* \otimes \mathscr{Q}$.

4.2. Quot and Hilbert schemes

4.2.1. The Quot functor and Grothendieck's theorem. Let S be a noetherian scheme and let $X \to S$ be a finite type morphism (so X is noetherian by Exercise 2.1.4). Fix a coherent sheaf F on X. Denote by Sch_S the category of locally noetherian schemes over S. Given such a scheme $U \to S$, define

$$\operatorname{Quot}_{X/S}(F)(U \to S)$$

to be the set of equivalence classes of pairs

$$(\mathscr{E}, p)$$

where

- $\mathscr{E}$ is a coherent sheaf on $X \times_S U$, flat over U and with proper support over U,
- $p\colon F_U \twoheadrightarrow \mathscr{E}$ is an $\mathcal{O}_{X\times_S U}$-linear surjection, where F_U denotes the pullback of F along $X \times_S U \to X$,
- two pairs $(\mathscr{E}, p)$ and $(\mathscr{E}', p')$ are considered equivalent if $\ker p = \ker p'$.

Flatness was defined in Definition 2.2.21. Properness of $\operatorname{Supp}\mathscr{E} \hookrightarrow X \times_S U \to U$ is automatic when $X \to S$ is proper, e.g. projective.

Let $\mathbf{k}$ be a field. Fix a very ample line bundle L over a $\mathbf{k}$-scheme X. For a coherent sheaf E on X whose support is proper over $\operatorname{Spec}\mathbf{k}$, the function

$$m \mapsto P_L(E,m) = \chi(E \otimes_{\mathcal{O}_X} L^{\otimes m})$$

becomes polynomial for $m \gg 0$, where $\chi(F) = \sum_{i\geq 0}(-1)^i \dim_{\mathbf{k}} \mathrm{H}^i(X,F)$, for a coherent sheaf F with proper support on a $\mathbf{k}$-scheme X, is called the *holomorphic Euler characteristic* of F. This is proved in [**185**, Tag 089X] for $\mathbb{P}^n$ and in [**116**, Theorem B.7] in general. This observation defines the *Hilbert polynomial* of E (with respect to L), which is denoted $P_L(E)$. If $\mathscr{E}$ is an S-flat family of coherent sheaves on $X \to S$, such that

$$\operatorname{Supp}\mathscr{E} \hookrightarrow X \to S$$

is proper, then the function

$$s \mapsto P_{L_s}(\mathscr{E}_s), \quad \mathscr{E}_s = \mathscr{E}|_{X_s},$$

is locally constant on S, where L_s denotes the restriction of L along $X_s \hookrightarrow X$. For the converse, one has that the constancy of the Hilbert polynomial along the fibres implies S-flatness of the sheaf if S is reduced.

For a fixed relatively very ample line bundle $L \in \operatorname{Pic}X$, the functor $\operatorname{Quot}_{X/S}(F)$ decomposes as a coproduct

$$\operatorname{Quot}_{X/S}(F) = \coprod_{P\in\mathbb{Q}[z]} \operatorname{Quot}^{P,L}_{X/S}(F)$$

where the component $\operatorname{Quot}^{P,L}_{X/S}(F)$ sends an S-scheme U to the set of equivalence classes of quotients $p\colon F_U \twoheadrightarrow \mathscr{E}$ such that for each $u \in U$ the Hilbert polynomial of $\mathscr{E}_u = \mathscr{E}|_{X_u}$ (whose support is a closed subscheme of X_u proper over $\operatorname{Spec}\mathbf{k}(u)$ by definition!), calculated with respect to L_u (the pullback of L along $X_u \hookrightarrow X \times_S U \to X$), is equal to P.

Theorem 4.2.1 (Grothendieck [**84**]). *Let S be a noetherian scheme. If $X \to S$ is a projective morphism, $F \in \operatorname{Coh} X$ is a coherent sheaf, L is a relatively very ample line bundle over X and $P \in \mathbb{Q}[z]$ is a polynomial, then $\mathrm{Quot}^{P,L}_{X/S}(F)$ is representable by a projective S-scheme*

$$\mathrm{Quot}^{P,L}_{X/S}(F) \to S.$$

Remark 4.2.2. The noetherian hypothesis in Theorem 4.2.1 could be removed by Altman and Kleiman [**3**], at the expense of using a stronger notion of projectivity, as well as a stronger assumption on F. The result is again a projective S-scheme $\mathrm{Quot}^{P,L}_{X/S}(F) \to S$ in this stronger sense. We mention the following consequence: when $X \hookrightarrow \mathbb{P}^n_S$ is a closed subscheme, $L = \mathscr{O}_{\mathbb{P}^n_S}(1)|_X$ and F is a sheaf quotient of $L(m)^{\oplus \ell}$ for some (m,ℓ), the functor $\mathrm{Quot}^{P,L}_{X/S}(F)$ is representable by an S-scheme that can be embedded in $\mathbb{P}^N_S$ for some N.

EXERCISE 4.2.3. Show that $\mathrm{Quot}_{X/S}(F)$ defines a functor $\mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{Sets}$, and that it generalises the Grassmann functor $\mathsf{G}_d(F)$ defined in (4.1.1). (**Hint**: To show that $\mathsf{G}_d(F)$ is a Quot scheme, pick $X = S \to S$ to be the identity and $P = d$. You may also need to have a look at [**185**, Tag 00NX] to link flatness and local freeness.)

4.2.2. The Hilbert scheme of a quasiprojective family.

We are ready to define an important character in these lecture notes.

DEFINITION 4.2.4. Let $X \to S$ be a projective morphism of noetherian schemes as in Theorem 4.2.1, and set $F = \mathscr{O}_X$. Then the S-scheme

$$\mathrm{Hilb}_{X/S} = \mathrm{Quot}_{X/S}(\mathscr{O}_X) \to S$$

is called the *Hilbert scheme* of $X \to S$, and is a disjoint union of projective connected components $\mathrm{Hilb}^{P,L}_{X/S}$.[3] When $S = \operatorname{Spec} \mathbf{k}$, we omit it from the notation and just write Hilb_X.

DEFINITION 4.2.5. Let $X \to S$ be a projective morphism of noetherian schemes as in Theorem 4.2.1, and let $n \geq 0$ be an integer. The *Quot scheme of points of* $F \in \operatorname{Coh} X$ relative to $X \to S$ is the component

$$\mathrm{Quot}_{X/S}(F,n) = \mathrm{Quot}^{n,L}_{X/S}(F) \subset \mathrm{Quot}_{X/S}(F),$$

where the choice of L is irrelevant since the Hilbert polynomial is a constant $P = n$. If $F = \mathscr{O}_X$, we set

$$\mathrm{Hilb}^n(X/S) = \mathrm{Quot}_{X/S}(\mathscr{O}_X, n) \subset \mathrm{Hilb}_{X/S}.$$

This connected component is the *Hilbert scheme of n points* relative to $X \to S$. When $S = \operatorname{Spec} \mathbf{k}$, it is omitted from the notation.

Remark 4.2.6. For any locally free sheaf $\mathscr{L}$ of rank 1 over a $\mathbf{k}$-variety X, there is an isomorphism $\mathrm{Quot}_X(\mathscr{L}, n) \cong \mathrm{Hilb}^n X$.

Remark 4.2.7. The scheme $\mathrm{Quot}_{X/S}(F,n)$ exists also for *quasi*projective $X \to S$ because of the following observations:

(1) The support of a flat family of coherent sheaves of relative dimension 0 over an arbitrary locally noetherian scheme is proper over the base.
(2) One can compactify X to a proper S-scheme $\overline{X} \to S$, and extend F to a coherent sheaf $\overline{F} \in \operatorname{Coh} \overline{X}$ (see [**155**, Lemma 5.19] for a proof of this fact), in such a way that the functor $\mathrm{Quot}_{X/S}(F)$ is an open subfunctor of $\mathrm{Quot}_{\overline{X}/S}(\overline{F})$, hence representable.

See Chapter 5 and Section 8.5 for further discussions on $\mathrm{Hilb}^n X$. We will also give an alternative definition of $\mathrm{Hilb}^n \mathbb{A}^d$, and, in fact, of $\mathrm{Quot}_{\mathbb{A}^d}(\mathscr{O}^{\oplus r}, n)$, in Section 5.2.

[3] The fact that each component $\mathrm{Hilb}^{P,L}_{X/S}$ is connected is a theorem of Hartshorne [**95**].

4.2.3. Hilbert polynomials, universal families of Hilbert schemes. A theorem of Vakil [197] asserts, roughly speaking, that arbitrarily bad singularities appear generically on some component of some Hilbert scheme.

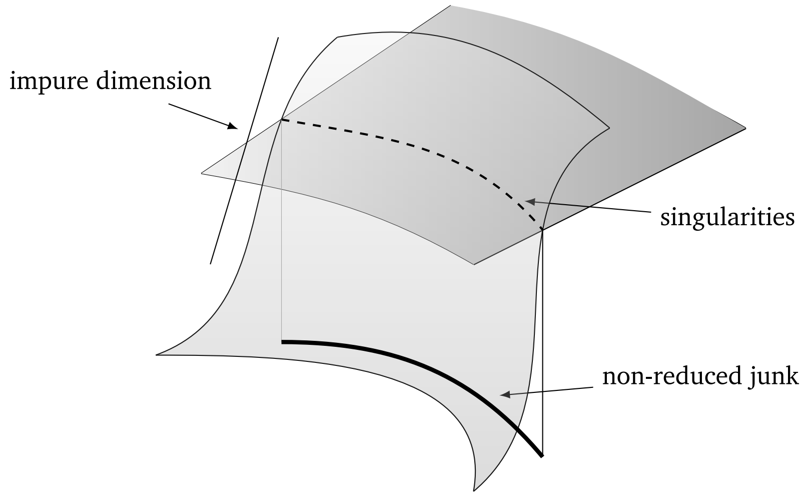

FIGURE 4.2. A nasty scheme. By Murphy's Law [197], it could be a Hilbert scheme component $H \subset \mathrm{Hilb}_X$ for some variety X.

However, despite its potentially horrible singularities, the Hilbert scheme has the great feature of representing a pretty explicit functor, so its functor of points is explicit. In such a situation, the most important thing is to always keep in mind the *universal family* living over the representing scheme. In the case of the Hilbert scheme, this is a diagram

$$\begin{tikzcd} \mathcal{Z} \arrow[r, hook, "\text{closed}"] \arrow[d, "\text{flat}"'] & X \times_S \mathrm{Hilb}_{X/S} \\ \mathrm{Hilb}_{X/S} & \end{tikzcd}$$

with the following property: for every S-scheme $g\colon U \to S$ along with a flat family of closed subschemes

$$\alpha\colon Z \subset X \times_S U \to U,$$

there exists precisely one S-morphism $\alpha_g\colon U \to \mathrm{Hilb}_{X/S}$ such that $Z = \alpha_g^* \mathcal{Z}$ as U-families of subschemes of X. If X is a smooth variety over $S = \operatorname{Spec} \mathbb{C}$, then $\mathcal{Z}$ is also flat over X, see [119, Theorem 1.1].

EXERCISE 4.2.8. Show that $\mathrm{Hilb}^1 X = X$. What is the universal family?

The rest of this section consists almost entirely of exercises, so for the sake of concreteness we may set $S = \operatorname{Spec} \mathbb{C}$. Let Y be a projective variety. The Hilbert polynomial of a subvariety $\iota\colon Z \hookrightarrow Y$ is, by definition, the Hilbert polynomial of its structure sheaf $\mathscr{O}_Z \in \operatorname{Coh} Y$, with respect to a given very ample line bundle L on Y. It can be computed by taking the holomorphic Euler characteristic of the short exact sequence

$$0 \to \mathscr{I}_{Z/Y}(m) \to \mathscr{O}_Y(m) \to \mathscr{O}_Z(m) \to 0$$

for $m \gg 0$, and using additivity of χ on short exact sequences [185, Tag 0BEI]. When dealing with the ambient variety $Y = \mathbb{P}^n$, we always fix the polarisation $L = \mathscr{O}_{\mathbb{P}^n}(1)$.

EXERCISE 4.2.9. Let $C \subset \mathbb{P}^n$ be a smooth curve of degree d and genus g. Compute the Hilbert polynomial of C with respect to $L = \mathscr{O}_{\mathbb{P}^n}(1)$.

Remark 4.2.10. It is not true that for fixed n there always exists a smooth curve $C \subset \mathbb{P}^n$ of degree d and genus g.

EXERCISE 4.2.11. Compute the Hilbert polynomial of a conic in $\mathbb{P}^3$, and that of a twisted cubic $C \subset \mathbb{P}^3$.

EXERCISE 4.2.12. Compute the Hilbert polynomial $P_{d,n}$ of a degree d hypersurface $Y \subset \mathbb{P}^n$.

EXERCISE 4.2.13. Let $P_{d,n}$ be the polynomial computed in Exercise 4.2.12. Use the universal property of the Hilbert scheme to prove that there is a bijective morphism

$$\mathbb{P}^{N-1} \to \mathrm{Hilb}_{\mathbb{P}^n}^{P_{d,n}}, \quad N = \binom{n+d}{d}.$$

EXERCISE 4.2.14. Interpret the Grassmannian

$$\mathbb{G}(k,n) = \left\{ \text{linear subvarieties } \mathbb{P}^k \hookrightarrow \mathbb{P}^n \right\}$$

as a Hilbert scheme, i.e. find the unique polynomial P such that $\mathbb{G}(k,n) = \mathrm{Hilb}_{\mathbb{P}^n}^P$.

EXERCISE 4.2.15. Let C be a smooth curve embedded in a smooth 3-fold Y. Show that there is an isomorphism of schemes $\mathrm{Bl}_C Y \cong \mathrm{Quot}_Y(\mathscr{I}_C, 1)$.

4.3. Tangent space to Hilb and Quot

Let X be a variety defined over an algebraically closed field $\mathbf{k}$. Let $p \in \mathrm{Hilb}_X(\mathbf{k})$ be the point corresponding to a closed subscheme $Z \subset X$. Then, by definition,

$$T_p \mathrm{Hilb}_X = \mathrm{Hom}_p(D_2, \mathrm{Hilb}_X),$$

where $\mathrm{Hom}_p(D_2, \mathrm{Hilb}_X)$ denotes the set of $\mathbf{k}$-morphisms $D_2 \to \mathrm{Hilb}_X$ sending the closed point of $D_2 = \mathrm{Spec}\,\mathbf{k}[t]/t^2$ to p. By representability of the Hilbert functor, this Hom set agrees with the set of all D_2-flat families

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Z} \arrow[r, hook] \arrow[d, "q"] & X \times_{\mathbf{k}} D_2 \\ 0 \arrow[r, hook] & D_2 & \end{tikzcd}$$

such that the fibre of q over the closed point of $D_2 = \mathrm{Spec}\,\mathbf{k}[t]/t^2$ equals Z. By definition, these are the *infinitesimal deformations* of the closed subscheme $Z \subset X$. It is shown in [98, Theorem 2.4] that these are classified by

$$\begin{aligned} \mathrm{Hom}_X(\mathscr{I}_Z, \mathscr{O}_Z) &= \mathrm{Hom}_Z(\mathscr{I}_Z/\mathscr{I}_Z^2, \mathscr{O}_Z) \\ &= \mathrm{H}^0(Z, \mathscr{H}om_{\mathscr{O}_Z}(\mathscr{I}_Z/\mathscr{I}_Z^2, \mathscr{O}_Z)) \\ &= \mathrm{H}^0(Z, \mathscr{N}_{Z/X}), \end{aligned}$$

where $\mathscr{N}_{Z/X}$ is the normal sheaf to Z in X (Definition 1.3.3). The local case, from which the identity $T_p \mathrm{Hilb}_X = \mathrm{H}^0(Z, \mathscr{N}_{Z/X})$ follows easily, is the following exercise. If in need of help, have a look at [98, Proposition 2.3].

EXERCISE 4.3.1. Let B be a $\mathbf{k}$-algebra of finite type, $I \subset B$ an ideal. Set $X = \mathrm{Spec}\,B$ and $Z = \mathrm{Spec}\,B/I \subset X$. Set $B' = B[t]/t^2$, and notice that $B'/tB' = B$. Show that $\mathrm{Hom}_B(I, B/I)$ is in bijective correspondence with ideals $I' \subset B'$ such that B'/I' is flat over $\mathbf{k}[t]/t^2$ and whose image in B equals I.

Here is a more general situation, upgrading the outcome of Exercise 4.1.10 for the Grassmannian.

Theorem 4.3.2 ([180, Proposition 4.4.4]). *Let F be a coherent sheaf on a projective $\mathbf{k}$-variety X. Let $p = [K \hookrightarrow F \twoheadrightarrow Q] \in \mathrm{Quot}_X(F)$ be a point. There is a canonical isomorphism*

$$T_p \mathrm{Quot}_X(F) \cong \mathrm{Hom}_X(K, Q). \tag{4.3.1}$$

If $\mathrm{Ext}_X^1(K,Q) = 0$, then $\mathrm{Quot}_X(F)$ is smooth at p.

EXERCISE 4.3.3. Fix $d, r \geq 1$. Show that $\mathrm{Quot}_{\mathbb{A}^d}(\mathscr{O}^{\oplus r}, 1)$ is smooth of dimension $d - 1 + r$.

EXERCISE 4.3.4. Fix $r \geq 1$ and $n \geq 0$. Ellingsrud and Lehn proved that $\mathrm{Quot}_{\mathbb{A}^2}(\mathscr{O}^{\oplus r}, n)$ is irreducible of dimension $(r+1)n$ [**54**]. Show that this Quot scheme is singular for instance when $r = n > 1$. (**Hint**: you may use Proposition 5.1.10, i.e. the nonsingularity in the $r = 1$ case.)

EXERCISE 4.3.5. Let $L \subset \mathbb{A}^3$ be a line. Compute the dimension of $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{I}_L, 2)$. Show that this Quot scheme is singular.

EXERCISE 4.3.6. Let $p \in \mathbb{A}^3$ be a point. Show that $\mathfrak{m}_p^2 \subset \mathscr{O}_{\mathbb{A}^3}$ defines a singular point of $\mathrm{Hilb}^4 \mathbb{A}^3$. Show that the singular locus of $\mathrm{Hilb}^4 \mathbb{A}^3$ is isomorphic to $\mathbb{A}^3$.

4.4. Examples of Hilbert schemes

4.4.1. Plane conics. Let z_0, z_1 and z_2 be homogeneous coordinates on $\mathbb{P}^2$, and $\alpha_0, \ldots, \alpha_5$ be homogeneous coordinates on $\mathbb{P}^5$. Consider the closed subscheme

$$\mathcal{C} \subset \mathbb{P}^2 \times \mathbb{P}^5$$

cut out by the bihomogeneous equation

$$\alpha_0 z_0^2 + \alpha_1 z_1^2 + \alpha_2 z_2^2 + \alpha_3 z_0 z_1 + \alpha_4 z_0 z_2 + \alpha_5 z_1 z_2 = 0.$$

Let π be the projection $\mathcal{C} \to \mathbb{P}^5$. Over a point $a = (a_0 : \cdots : a_5) \in \mathbb{P}^5$, the fibre is the conic

$$\pi^{-1}(a) = \left\{ a_0 z_0^2 + a_1 z_1^2 + a_2 z_2^2 + a_3 z_0 z_1 + a_4 z_0 z_2 + a_5 z_1 z_2 = 0 \right\} \subset \mathbb{P}^2.$$

There is a set-theoretic bijection between $\mathbb{P}^5$ and $\mathrm{Hilb}^{2t+1}_{\mathbb{P}^2}$. By the universal property of projective space, we have the scheme-theoretic identity

$$\mathbb{P}^5 = \mathrm{Hilb}^{2t+1}_{\mathbb{P}^2},$$

and the map $\pi \colon \mathcal{C} \to \mathbb{P}^5$ is the universal family of the Hilbert scheme of plane conics.

EXERCISE 4.4.1. Prove the last sentence rigorously and generalise the plane conics example to arbitrary hypersurfaces of $\mathbb{P}^n$. (**Hint**: Start out with the conclusion of Exercise 4.2.12 to write down the universal family).

Remark 4.4.2. Let X be a projective **k**-scheme. The universal family of the Hilbert scheme is always, *set-theoretically*, equal to

$$\mathcal{Z} = \{ (x, [Z]) \in X \times_{\mathbf{k}} \mathrm{Hilb}_X \mid x \in Z \} \subset X \times_{\mathbf{k}} \mathrm{Hilb}_X .$$

The problem is to determine the scheme structure on $\mathcal{Z}$. In the case of hypersurfaces of degree d in $\mathbb{P}^n$ (Exercise 4.4.1) this was easy precisely because $\mathcal{Z}$ is itself a hypersurface.

4.4.2. Curves in 3-space. In this subsection we describe a few Hilbert schemes of 1-dimensional subschemes of $\mathbb{P}^3$. We start with a pathological example, that of twisted cubics and their degenerations. By 'pathological' we just mean, here, that the Hilbert scheme has multiple components (of different dimensions), in particular it is not equal to the closure of the locus of smooth curves with the given Hilbert polynomial.

4.4.2.1. *Twisted cubics.* A twisted cubic is a smooth rational curve obtained as the image of the morphism[4]

$$\mathbb{P}^1 \hookrightarrow \mathbb{P}^3, \quad (u, v) \mapsto (u^3, u^2 v, u v^2, v^3),$$

up to linear changes of coordinates of the codomain. The number of moduli of a twisted cubic is 12. Indeed, one has to specify four linearly independent degree 3 polynomials in two variables, up to $\mathbb{C}^\times$-scaling and automorphisms of $\mathbb{P}^1$. One then computes

$$4 \cdot h^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(3)) - 1 - \dim \mathrm{PGL}_2 = 16 - 1 - 3 = 12.$$

[4]We already encounterd the twisted cubic in Example 2.1.11.

The Hilbert polynomial of a twisted cubic is $3t+1$, cf. Exercise 4.2.9. There are other 1-dimensional subschemes $Z \subset \mathbb{P}^3$ with this Hilbert polynomial, e.g. a plane cubic union a point. This has 15 moduli: the choice of a plane $\mathbb{P}^2 \subset \mathbb{P}^3$ contributes $3 = \dim \mathbb{G}(2,3)$ moduli, a plane cubic $C \subset \mathbb{P}^2$ contributes 9 parameters, and the choice of a point $p \in \mathbb{P}^3$ accounts for the remaining 3 moduli.

The Hilbert scheme

$$\mathrm{Hilb}^{3t+1}_{\mathbb{P}^3}$$

was completely described in [167]. The two irreducible components we just described turn out to be the only ones. They are smooth, rational, of dimension 12 and 15 respectively, and they intersect along a smooth, rational 11-dimensional subvariety $V \subset \mathrm{Hilb}^{3t+1}_{\mathbb{P}^3}$ parametrising uninodal plane cubics with an embedded point at the node. See Figure 4.3 for a full pictorial description of the degenerations occurring in $\mathrm{Hilb}^{3t+1}_{\mathbb{P}^3}$.

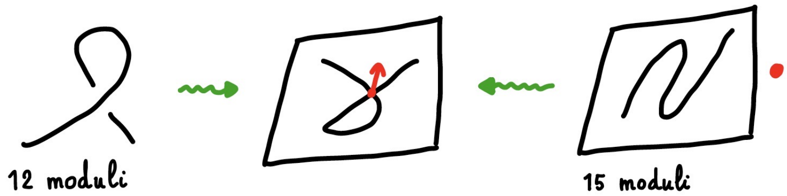

FIGURE 4.3. Description of $\mathrm{Hilb}^{3t+1}_{\mathbb{P}^3}$. The red arrow sticking out (cf. Remark 2.1.29 for this phenomenon) of the plane in the middle drawing represents the embedded point at the origin of the subscheme (4.4.1).

In [96, III, Example 9.8.4] a family of twisted cubics degenerating to a plane uninodal cubic with an embedded point is described. The total space of the family, in a local chart, is defined by the ideal

$$I = (a^2(x+1)-z^2, ax(x+1)-yz, xz-ay, y^2-x^2(x+1)) \subset \mathbb{C}[a,x,y,z].$$

Letting $a=0$ one obtains the special fibre given by

$$I_0 = (z^2, yz, xz, y^2-x^2(x+1)) \subset \mathbb{C}[x,y,z], \tag{4.4.1}$$

and $p=(0,0,0)$ is a non-reduced point in $C_0 = \operatorname{Spec} \mathbb{C}[x,y,z]/I_0$. Note that C_0 is not scheme-theoretically contained in the plane $z=0$, because the local ring $\mathscr{O}_{C_0,p}$ contains the nonzero nilpotent z (cf. Remark 2.1.29).

Remark 4.4.3. The *geometric genus* $p_g(X) = h^0(X, \omega_X)$ varies in flat families, as shown by the example of the twisted cubic degenerating to a nodal plane cubic.

4.4.2.2. *A line and a point.* Consider $\mathbb{P}^3$ and the Hilbert polynomial

$$p(z) = z+2.$$

The only subschemes $Z \subset \mathbb{P}^3$ with Hilbert polynomial p are those of Figure 4.4, i.e. Z must consist of a line and a point, either isolated or embedded.

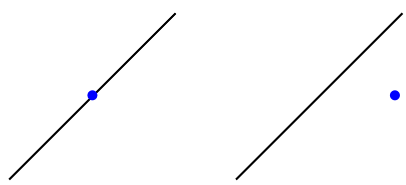

FIGURE 4.4. The only subschemes $Z \subset \mathbb{P}^3$ with Hilbert polynomial $z+2$.

A naive dimension count yields the prediction

$$\dim \mathrm{Hilb}^{z+2}_{\mathbb{P}^3} = 4+3 = 7,$$

where $4 = \dim \mathbb{G}(1,3)$ is the dimension of the Grassmannian of lines in $\mathbb{P}^3$ and $3 = \dim \mathbb{P}^3$ accounts for the choice of the extra point. The prediction turns out correct. There is an open subscheme $U \subset \mathrm{Hilb}^{z+2}_{\mathbb{P}^3}$ along with a diagram

$$\begin{tikzcd} \left\{ (p,[\ell]) \in \mathbb{P}^3 \times \mathbb{G}(1,3) \,\middle|\, p \notin \ell \right\} \arrow[r, hook] \arrow[d, "\wr"] & \mathbb{P}^3 \times \mathbb{G}(1,3) \\ U \arrow[r, hook, "\text{open}"] & \mathrm{Hilb}^{z+2}_{\mathbb{P}^3} \end{tikzcd}$$

where the map to the Hilbert scheme takes $(p,[\ell])$ to $\ell \cup p$. However, on the boundary, when the point degenerates on the line, a finer analysis is required. Let

$$\mathcal{L} = \left\{ (p,[\ell]) \in \mathbb{P}^3 \times \mathbb{G}(1,3) \,\middle|\, p \in \ell \right\} \subset \mathbb{P}^3 \times \mathbb{G}(1,3)$$

be the universal line.

EXERCISE 4.4.4. Show that there is a bijective morphism

$$f_1 \colon \mathrm{Bl}_{\mathcal{L}}(\mathbb{P}^3 \times \mathbb{G}(1,3)) \to \mathrm{Hilb}^{z+2}_{\mathbb{P}^3}.$$

Next, show that $\mathrm{Hilb}^{z+2}_{\mathbb{P}^3}$ is smooth. Conclude that f_1 is an isomorphism.

4.4.2.3. *A plane conic (and no point).* Consider again $\mathbb{P}^3$ and the Hilbert polynomial

$$p(z) = 2z+1.$$

This is the Hilbert polynomial of a plane conic (either smooth or singular, even possibly a double line), this time with no additional roaming points. In fact, one can verify that no other subscheme of $\mathbb{P}^3$ has this Hilbert polynomial. Since we are in $\mathbb{P}^3$, a naive count yields the expectation

$$\dim \mathrm{Hilb}^{2z+1}_{\mathbb{P}^3} = 3+5 = 8,$$

where $3 = \dim \mathbb{P}^{3*}$ accounts for the choice of a plane $H \subset \mathbb{P}^3$ and $5 = \dim \mathrm{Hilb}^{2z+1}_H$ accounts for the choice of a conic embedded in the chosen plane. Again, the expectation turns out correct.

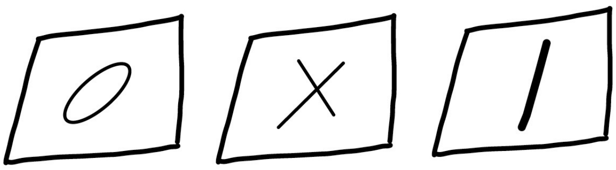

FIGURE 4.5. The subschemes parametrised by $\mathrm{Hilb}^{2z+1}_{\mathbb{P}^3}$.

The map

$$\pi \colon \mathrm{Hilb}^{2z+1}_{\mathbb{P}^3} \to \mathbb{P}^{3*}$$

sending a conic $C \subset \mathbb{P}^3$ to the plane $H \subset \mathbb{P}^3$ it lies on, is an algebraic morphism, whose fibre over a point $[H] \in \mathbb{P}^{3*}$ is the projective space $\mathbb{P}(\mathrm{Sym}^2 V^*)$, where V is the 3-dimensional vector space generated by the 'coordinates' of H (or $H = \mathbb{P}(V)$, in other words). Consider the tautological short exact sequence

$$0 \to \mathscr{S} \to \mathscr{O}^{\oplus 4}_{\mathbb{P}^{3*}} \to \mathscr{Q} \to 0$$

over $\mathbb{P}^{3*} = \mathbb{G}(2,3) = G(3,4)$, where the fibre of $\mathscr{S}$ over $[V] \in G(3,4)$ is precisely the 3-plane V corresponding to $[V]$. It follows that the source of the $\mathbb{P}^5$-bundle

$$\pi' \colon \mathbb{P}(\mathrm{Sym}^2 \mathscr{S}^*) \to \mathbb{P}^{3*}$$

parametrises pairs (H,γ), where $H \subset \mathbb{P}^3$ is a plane and $\gamma \in \mathbb{P}\mathrm{H}^0(H,\mathscr{O}_H(2))$.

One can prove that there is an isomorphism

$$\mathrm{Hilb}^{2z+1}_{\mathbb{P}^3} \cong \mathbb{P}(\mathrm{Sym}^2 \mathscr{S}^*)$$

over $\mathbb{P}^{3*}$. See [**124**, Section 3.4] for more details.

4.4.2.4. *A more general example.* Consider $\mathbb{P}^3$ and the Hilbert polynomial

$$p(z) = dz + 2 - g,$$

where $g = (d-1)(d-2)/2$ is the genus of a nonsingular plane curve of degree d. The Hilbert scheme

$$\mathrm{Hilb}^{dz+2-g}_{\mathbb{P}^3}$$

is in general very complicated. However, there is an irreducible component

$$\mathrm{H}_d \subset \mathrm{Hilb}^{dz+2-g}_{\mathbb{P}^3}$$

defined as the closure of the locus

$$\left\{\, C \cup p \;\middle|\; C \text{ is a degree } d \text{ plane curve, } p \in \mathbb{P}^3 \setminus C \,\right\}$$

that is easier to understand. We saw that $\mathrm{H}_1 = \mathrm{Hilb}^{z+2}_{\mathbb{P}^3}$ if $g = 0$ (i.e. there are no other 'extraneous' components for $(d,g) = (1,0)$), but this might be false for $d > 1$. In general, by a result of Chen–Nollet [**40**, Theorem 1.9], H_d is always smooth and there is an isomorphism

$$f_d \colon \mathrm{Bl}_\Sigma(\mathbb{P}^3 \times \mathrm{Hilb}^{dz+1-g}_{\mathbb{P}^3}) \xrightarrow{\sim} \mathrm{H}_d,$$

where Σ is the incidence correspondence consisting of pairs $(p,[C])$ such that $p \in C$.

It is also interesting to note the following result.

Proposition 4.4.5 ([**40**, Corollary 1.8])**.** *The Hilbert scheme* $\mathrm{Hilb}^{dz+1-h}_{\mathbb{P}^3}$ *is irreducible, as long as* $d \geq 6$ *and* $g-3 \leq h \leq g$*, where* $g = (d-1)(d-2)/2$.

4.4.3. The Hilbert scheme of a Jacobian. Let C be a smooth projective curve of genus g, defined over an algebraically closed field $\mathbf{k}$ of characteristic different from 2. Let $J = \mathrm{Pic}^0(C)$ be its Jacobian, an abelian variety of dimension g, principally polarised by the Theta divisor $\Theta \subset J$. It is well-known that the embedded deformations of the Abel–Jacobi map

$$C \hookrightarrow J$$

are unobstructed if and only if C is non-hyperelliptic [**121**]. In fact, if C is hyperelliptic, Griffiths [**82**] computed

$$\dim_{\mathbf{k}} \mathrm{H}^0(C,\mathscr{N}_{C/J}) = 2g-2,$$

as long as $g \geq 3$. Since the only local deformations of $C \hookrightarrow J$ are given by translations, it easily follows that the Hilbert scheme component

$$\mathrm{Hilb}_{C/J} \subset \mathrm{Hilb}_J$$

containing the Abel–Jacobi embedding as a point is isomorphic to J if and only if C is not hyperelliptic. (When treating these Hilbert schemes, we fix Θ as polarisation).

On the other hand, when C is hyperelliptic, the local deformations of $C \hookrightarrow J$ are still parametrised by translations, but by obstructedness, the Hilbert scheme component $\mathrm{Hilb}_{C/J}$ is everywhere nonreduced. Its underlying reduced variety is of course still equal to J. It is proved in [**175**] that, in the hyperelliptic case, there is a scheme-theoretic isomorphism

$$\mathrm{Hilb}_{C/J} \cong J \times \operatorname{Spec} \mathbf{k}[s_1,\ldots,s_{g-2}]/(s_1,\ldots,s_{g-2})^2.$$

Moreover, the artininan scheme $\operatorname{Spec}\mathbf{k}[s_1,\ldots,s_{g-2}]/(s_1,\ldots,s_{g-2})^2$ is precisely the scheme-theoretic fibre of the Torelli morphism

$$\tau \colon \mathcal{M}_g \to \mathcal{A}_g$$

over the hyperlliptic point $[J,\Theta]\in\mathcal{A}_g$. Note, also, that $\operatorname{Spec}\mathbf{k}[s_1,\ldots,s_{g-2}]/(s_1,\ldots,s_{g-2})^2$ is the *smallest* artinian scheme of embedding dimension (i.e. tangent space dimension at the unique closed point) $g-2=(2g-2)-g$, thus in this sense the nonreduced structure on the Hilbert scheme is the mildest possible.

Remark 4.4.6. If $g=3$, the above example exhibits an everywhere nonreduced Hilbert scheme component $\operatorname{Hilb}_{C/J}\subset\operatorname{Hilb}_J$ for a 3-fold J. In fact, the most famous example of this phenomenon is probably due to Mumford [**146**]. See also [**88, 53, 117**] for more examples.

4.5. Lines on hypersurfaces: expectations

Let $Y\subset\mathbb{P}^n$ be a general hypersurface of degree d. We want to show the following:

We should expect a finite number of lines on Y if and only if $d=2n-3$. We should expect *no lines* on Y if $d>2n-3$. We should expect infinitely many lines on Y if $d<2n-3$.

To understand the condition

$$\ell\subset Y$$

for a given line $\ell\subset\mathbb{P}^n$ and a hypersurface $Y\subset\mathbb{P}^n$, we give the following concrete example.

Example 4.5.1. Let $\ell\subset\mathbb{P}^3$ be the line cut out by $L_1=L_2=0$, where $L_i=L_i(z_0,z_1,z_2,z_3)$ are linear forms on $\mathbb{P}^3$. To fix ideas, set $L_1=z_0$ and $L_2=z_0+z_2+z_3$. Let $Y\subset\mathbb{P}^3$ be defined by a homogeneous equation $f=0$, for instance the cubic polynomial

$$f=z_0^3+3z_0z_1^2-z_2^2z_3.$$

Then we see that plugging $L_1=L_2=0$ into f does not give zero, for

$$f|_\ell=0+0-z_2^2(-z_0-z_2)=z_2^3.$$

This means that ℓ is not contained in Y. On the other hand, the line cut out by L_1 and $L_2'=z_3$ lies entirely on Y.

Let $Y\subset\mathbb{P}^n$ be the zero locus of a general homogeneous polynomial

$$f\in\mathrm{H}^0(\mathbb{P}^n,\mathscr{O}_{\mathbb{P}^n}(d)).$$

As we anticipated in Example 4.5.1, a line $\ell\subset\mathbb{P}^n$ is contained in Y if and only if $f|_\ell=0$. This condition can be rephrased by saying that the image of f under the restriction map

$$\mathsf{res}_\ell\colon\mathrm{H}^0(\mathbb{P}^n,\mathscr{O}_{\mathbb{P}^n}(d))\to\mathrm{H}^0(\ell,\mathscr{O}_\ell(d))\tag{4.5.1}$$

vanishes. We want to determine when we should expect Y to contain a finite number of lines. We set, informally,

$$\mathsf{N}_1(Y)=\text{expected number of lines in }Y.$$

Let us consider the Grassmannian

$$\mathbb{G}=\mathbb{G}(1,n)=\{\,\text{Lines }\ell\subset\mathbb{P}^n\,\},$$

a smooth complex projective variety of dimension $2n-2$. Recall the universal structures living on $\mathbb{G}$. First of all, the tautological exact sequence

$$0\longrightarrow\underbrace{\mathscr{S}}_{\text{rank }2}\longrightarrow\underbrace{\mathscr{O}_{\mathbb{G}}\otimes_{\mathbb{C}}\mathrm{H}^0(\mathbb{P}^n,\mathscr{O}_{\mathbb{P}^n}(1))^*}_{\text{rank }n+1}\longrightarrow\underbrace{\mathscr{Q}}_{\text{rank }n-1}\longrightarrow 0$$

where the fibre of $\mathscr{S}$ over a point $[\ell] \in \mathbb{G}$ is the 2-dimensional vector space $\mathrm{H}^0(\ell, \mathscr{O}_\ell(1))^*$. Let, also,

$$\mathcal{L} = \{ (p, [\ell]) \in \mathbb{P}^n \times \mathbb{G} \mid p \in \ell \} \subset \mathbb{P}^n \times \mathbb{G}$$

be the universal line. Consider the two projections

$$\begin{tikzcd} \mathcal{L} \arrow[r, "q"] \arrow[d, "\pi"'] & \mathbb{P}^n \\ \mathbb{G} & \end{tikzcd}$$

and the coherent sheaf

$$\mathscr{E}_d = \pi_* q^* \mathscr{O}_{\mathbb{P}^n}(d).$$

EXERCISE 4.5.2. Show that $\mathscr{E}_d$ is locally free of rank $d+1$. (**Hint**: use cohomology and base change, e.g. [**51**, Theorem B.5]).

In fact, one has an isomorphism of locally free sheaves

$$\mathscr{E}_d \cong \mathrm{Sym}^d \mathscr{S}^*,$$

where $\iota\colon \mathscr{S} \hookrightarrow \mathscr{O}_{\mathbb{G}} \otimes_{\mathbb{C}} \mathrm{H}^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1))^*$ is the universal subbundle. Dualising ι and applying Sym^d, we obtain a surjection

$$\mathscr{O}_{\mathbb{G}} \otimes_{\mathbb{C}} \mathrm{H}^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)) \twoheadrightarrow \mathrm{Sym}^d \mathscr{S}^*,$$

which is just a global version of (4.5.1). The association

$$\mathbb{G} \ni [\ell] \mapsto f|_\ell \in \mathrm{H}^0(\ell, \mathscr{O}_\ell(d)) \cong \mathrm{Sym}^d \mathrm{H}^0(\ell, \mathscr{O}_\ell(1))$$

defines a section τ_f of $\mathscr{E}_d \to \mathbb{G}$. The zero locus of $\tau_f = \pi_* q^* f \in \mathrm{H}^0(\mathbb{G}, \mathscr{E}_d)$ is the locus of lines contained in Y.

The following terminology is very common.

DEFINITION 4.5.3. Let $Y \subset \mathbb{P}^n$ be a hypersurface defined by $f = 0$. The zero scheme

$$F_1(Y) = Z(\tau_f) \hookrightarrow \mathbb{G} = \mathbb{G}(1, n)$$

is called the *Fano scheme of lines* in Y.

Since f is generic, $\tau_f \in \Gamma(\mathbb{G}, \mathscr{E}_d)$ is also generic. In this case, the fundamental class of the Fano scheme of lines in Y is Poincaré dual to the Euler class

$$e(\mathscr{E}_d) \in A^{d+1}\mathbb{G}.$$

Thus $[F_1(Y)] \in A_*\mathbb{G}$ is a 0-cycle if and only if $d+1 = 2n-2$, i.e.

$$d = 2n-3.$$

The degree of this 0-cycle is then

$$\mathsf{N}_1(Y) = \int_{\mathbb{G}} e(\mathscr{E}_d) = \int_{\mathbb{G}} c_{d+1}(\mathrm{Sym}^d \mathscr{S}^*).$$

This degree is the *actual* number of lines on Y whenever $\mathrm{H}^0(\ell, \mathscr{N}_{\ell/Y}) = 0$ for all $\ell \subset Y$. This condition means that the Fano scheme is reduced at all its points $[\ell]$, since $\mathrm{H}^0(\ell, \mathscr{N}_{\ell/Y})$ is its tangent space at the point $[\ell]$.

Lemma 4.5.4. *If $S \subset \mathbb{P}^3$ is a smooth cubic surface and $\ell \subset S$ is a line, then* $\mathrm{H}^0(\ell, \mathscr{N}_{\ell/S}) = 0$.

PROOF. It is enough to show that $\mathscr{N} = \mathscr{N}_{\ell/S}$, viewed as a line bundle on $\ell \cong \mathbb{P}^1$, has negative degree. By the adjunction formula,

$$K_\ell = K_S|_\ell \otimes_{\mathscr{O}_\ell} \mathscr{N}.$$

Using that $K_\ell = \mathscr{O}_\ell(-2)$ and $K_S = K_{\mathbb{P}^3}|_S \otimes_{\mathscr{O}_S} \mathscr{N}_{S/\mathbb{P}^3} = \mathscr{O}_S(d-4)$ for a surface of degree d in $\mathbb{P}^3$, by taking degrees we obtain

$$-2 = (3-4) + \deg \mathscr{N},$$

so that $\deg \mathscr{N} = -1 < 0$. $\square$

EXERCISE 4.5.5. Let $Y \subset \mathbb{P}^n$ be a general hypersurface of degree $d \leq 2n-3$. Show that $F_1(Y) \subset \mathbb{G}(1,n)$ is smooth of dimension $2n-3-d$.

CHAPTER 5

The Hilbert scheme of points

SUMMARY. In this chapter, for simplicity, we work over $\mathbb{C}$. Let X be a complex quasiprojective variety. We will study the Hilbert scheme of points

$$\operatorname{Hilb}^n X = \left\{ \mathscr{O}_X \twoheadrightarrow \mathscr{O}_Z \,\middle|\, \dim_{\mathbb{C}} \mathrm{H}^0(Z, \mathscr{O}_Z) = n \right\}$$

introduced in Definition 4.2.5. In Section 5.2 we focus on $X = \mathbb{A}^d$. The special cases of $d = 2, 3$ will be analysed carefully.

5.1. Subschemes and 0-cycles

5.1.1. The Hilbert–Chow morphism. Let X be a complex quasiprojective variety. Besides $\operatorname{Hilb}^n X$, there is a "coarser" way of parametrising *points with multiplicity* on X, one that does not distinguish between the possible scheme structures on a finite collection of points $\{ p_1, \ldots, p_n \} \subset X$. This is the content of the next definition.

DEFINITION 5.1.1. Let X be a quasiprojective variety. The n-th *symmetric product* of X is the quotient

$$\operatorname{Sym}^n X = X^n / \mathfrak{S}_n.$$

Remark 5.1.2. The quotient of a quasiprojective scheme by a finite group G always exists as a scheme, by a basic result of geometric invariant theory. If $X = \operatorname{Spec} A$ is affine, there is a categorical quotient $X \to \operatorname{Spec} A^G$ corresponding to the inclusion $A^G \hookrightarrow A$ (cf. Section 2.5.3). More generally, for a scheme X acted on by a finite group G, [**86**, Exposé V, Proposition 1.8] shows that X/G exists as a scheme if and only if every orbit is contained in an affine open subset of X, and this condition is clearly satisfied for quasiprojective X.

Remark 5.1.3. The scheme $\operatorname{Sym}^n X$ is also known as the *Chow scheme* of effective 0-cycles (of degree n) on X. In other words, $\operatorname{Sym}^n X$ represents the functor of families of effective 0-cycles. For higher dimensional cycles, however, the definition (and representability) of the Chow functor is a much subtler problem, carefully analysed in [**178**].

Each point $\xi \in \operatorname{Sym}^n X$ corresponds to a finite combination of points with multiplicity, i.e. it can be written as

$$\xi = \sum_i m_i \cdot p_i,$$

with $m_i \in \mathbb{Z}_{>0}$ and $p_i \in X$ pairwise distinct points. Clearly, one always has $\operatorname{Sym}^1 X = X = \operatorname{Hilb}^1 X$.

EXERCISE 5.1.4. Show that $\operatorname{Sym}^2 \mathbb{A}^2 \cong \mathbb{A}^2 \times C$, where C is the quadric cone $C = \operatorname{Spec} \mathbb{C}[x, y, z]/(xy - z^2)$.

EXERCISE 5.1.5. Use the previous exercise (if you want) to show that for any smooth surface X the singular locus of $\operatorname{Sym}^2 X$ is the image of the diagonal $X \hookrightarrow X^2$ under the quotient map $X^2 \to \operatorname{Sym}^2 X$.

EXERCISE 5.1.6. Let X be a smooth variety of dimension d. Show that the locus in $\operatorname{Hilb}^2 X$ of nonreduced subschemes $Z \subset X$ is isomorphic to $X \times \mathbb{P}^{d-1}$.

Note that the symmetric product $\operatorname{Sym}^n X$ does not see the scheme structure of fat points inside X. For instance, any of the double point schemes supported on a given point $p \in X$, parametrised by the

$\mathbb{P}^{d-1}$ of Exercise 5.1.6, has underlying cycle $2\cdot p$. The operation of 'forgetting the scheme structure' and only retaining the support can be made functorial. This means that there exists a well-defined algebraic morphism

$$\pi_X\colon \operatorname{Hilb}^n X \to \operatorname{Sym}^n X, \tag{5.1.1}$$

taking a subscheme $Z\subset X$ to its underlying effective 0-cycle. In symbols,

$$\pi_X[\mathscr{O}_X \twoheadrightarrow \mathscr{O}_Z] = \sum_{p\in \operatorname{Supp} Z} \operatorname{length} \mathscr{O}_{Z,p}\cdot p.$$

The morphism π_X is called the *Hilbert-to-Chow morphism*. In fact, as shown in [**87**, Section 6] (cf. also [**178**, Corollary 7.15]), for every coherent sheaf F on X there is a *Quot-to-Chow* morphism

$$\pi_F\colon \operatorname{Quot}_X(F,n) \to \operatorname{Sym}^n X \tag{5.1.2}$$

defined by sending $[F\twoheadrightarrow T]$ to the 0-cycle $[\operatorname{Supp} T] = \sum_{T_p\neq 0} \operatorname{length}_{\mathscr{O}_{X,p}} T_p\cdot p$.

Lemma 5.1.7. *Let X be a quasiprojective variety, F a coherent sheaf on X. The Quot-to-Chow morphism π_F is proper.*

Proof. Let X be projective, to start with. The symmetric product $\operatorname{Sym}^n X$ is separated, and $\operatorname{Quot}_X(F,n)$ is projective, hence proper. Then π_F is proper in this case. If X is only quasiprojective, choose a compactification $X\subset \overline{X}$. By [**155**, Lemma 5.19], the sheaf F extends to a coherent sheaf $\overline{F}$ on $\overline{X}$, and π_F is the pullback of $\pi_{\overline{F}}$ along the open immersion $\operatorname{Sym}^n X\subset \operatorname{Sym}^n \overline{X}$, and is thus proper because properness is stable under base change. □

Let X be a smooth quasiprojective variety. Consider the locus

$$U_n^\circ = \left\{\, x_1+\cdots+x_n \,\middle|\, x_i\neq x_j \text{ for all } 1\le i\neq j\le n \,\right\} \subset \operatorname{Sym}^n X.$$

Then π_X is an isomorphism when restricted to this locus. Indeed, there is a unique scheme structure on a set of n distinct points on a smooth variety. In fact, U_n° is the image under the quotient map $X^n\to \operatorname{Sym}^n X$ of the complement $X^n\setminus \Delta^{\mathrm{big}}$, where

$$\Delta^{\mathrm{big}} = \left\{\, (x_1,\ldots,x_n) \,\middle|\, x_i = x_j \text{ for some } i\neq j \,\right\} \subset X^n.$$

We have the following diagram of schemes

$$\begin{tikzcd}
\pi_X^{-1}(U_n^\circ) \arrow[r, "\sim"] \arrow[d, hook] \arrow[dr, phantom, "\square"] & U_n^\circ \arrow[d, hook] \arrow[dr, phantom, "\square"] & X^n\setminus\Delta^{\mathrm{big}} \arrow[l, "\sim"'] \arrow[d, hook] \\
\operatorname{Hilb}^n X \arrow[r, "\pi_X"] & \operatorname{Sym}^n X & X^n \arrow[l]
\end{tikzcd}$$

where the vertical inclusions are open immersions. The scheme $X^n\setminus\Delta^{\mathrm{big}}$, the complement of the 'big diagonal', is known in the literature as the n-th *configuration space* of X.

Definition 5.1.8. The *smoothable component* of $\operatorname{Hilb}^n X$ is the closure

$$W_{\mathrm{sm}} = \overline{\pi_X^{-1}(U_n^\circ)} \subset \operatorname{Hilb}^n X.$$

It has dimension $\dim U_n^\circ = n\cdot \dim X$.

Exercise 5.1.9. Show that if C is a smooth quasiprojective curve then $\operatorname{Hilb}^n C\cong \operatorname{Sym}^n C$ via π_C. Deduce that $\operatorname{Hilb}^n \mathbb{A}^1\cong \mathbb{A}^n$ and that $\operatorname{Hilb}^n\mathbb{P}^1\cong\mathbb{P}^n$.

The following result was first proved by Fogarty [**64**]. We present the proof given in [**61**, Theorem 7.2.3].

Proposition 5.1.10. *If X is a smooth quasiprojective surface, then* $\operatorname{Hilb}^n X$ *is smooth and irreducible of dimension $2n$. In particular, it is equal to the smoothable component.*

PROOF. We may assume X to be projective, since the general result will follow from this one via a compactification $X \hookrightarrow \overline{X}$, inducing an open immersion $\mathrm{Hilb}^n X \hookrightarrow \mathrm{Hilb}^n \overline{X}$.

Suppose $\dim T_p \mathrm{Hilb}^n X = 2n$ for any $p \in \mathrm{Hilb}^n X$, then $\mathrm{Hilb}^n X$ is smooth along the smoothable component, because $\pi_X^{-1}(U_n^\circ)$ is a smooth open subset of dimension $2n$. Since $\mathrm{Hilb}^n X$ is connected, the presence of any other irreducible component W would yield singularities along $W \cap W_{\mathrm{sm}}$. Thus we are reduced to showing that $\dim T_p \mathrm{Hilb}^n X = 2n$ for any $p \in \mathrm{Hilb}^n X$.

Let $Z \subset X$ be the closed subscheme corresponding to p. By Section 4.3, we have $T_p \mathrm{Hilb}^n X = \mathrm{Hom}_X(\mathscr{I}_Z, \mathscr{O}_Z)$, so we get an exact sequence

$$0 \to \mathrm{Hom}_X(\mathscr{O}_Z, \mathscr{O}_Z) \xrightarrow{e} \mathrm{Hom}_X(\mathscr{O}_X, \mathscr{O}_Z) \xrightarrow{u} T_p \mathrm{Hilb}^n X \xrightarrow{v} \mathrm{Ext}^1_X(\mathscr{O}_Z, \mathscr{O}_Z)$$

where e is an isomorphism between n-dimensional vector spaces, so that $u = 0$ and v is injective. We next show that $\mathrm{ext}^1_X(\mathscr{O}_Z, \mathscr{O}_Z) \le 2n$. Note that $\chi(\mathscr{O}_Z, \mathscr{O}_Z) = \mathrm{ext}^0(\mathscr{O}_Z, \mathscr{O}_Z) - \mathrm{ext}^1(\mathscr{O}_Z, \mathscr{O}_Z) + \mathrm{ext}^2(\mathscr{O}_Z, \mathscr{O}_Z) = 2n - \mathrm{ext}^1(\mathscr{O}_Z, \mathscr{O}_Z)$, where we used Serre duality to compute ext^2. Note that, for any locally free sheaf E on X, one has $\chi(E, \mathscr{O}_Z) = \dim \mathrm{H}^0(Z, \mathscr{O}_Z)^{\mathrm{rk}\, E} = n \cdot \mathrm{rk}\, E$. So, let $E^\bullet \to \mathscr{O}_Z$ be a resolution consisting of finitely many locally free sheaves E^i on X, so that $0 = \mathrm{rk}\, \mathscr{O}_Z = \mathrm{rk}\, E^\bullet = \sum_\ell (-1)^\ell \mathrm{rk}\, E^\ell$. This implies $\chi(\mathscr{O}_Z, \mathscr{O}_Z) = \sum_\ell (-1)^\ell \chi(E^\ell, \mathscr{O}_Z) = \sum_\ell (-1)^\ell n \cdot \mathrm{rk}\, E^\ell = 0$, proving $\mathrm{ext}^1(\mathscr{O}_Z, \mathscr{O}_Z) = 2n$. □

Corollary 5.1.11. *Let X be a smooth surface. The Hilbert–Chow map $\pi_X \colon \mathrm{Hilb}^n X \to \mathrm{Sym}^n X$ is a resolution of singularities.*

5.1.2. The punctual Hilbert scheme.

The easiest subvariety of the symmetric product $\mathrm{Sym}^n X$ is the *small diagonal*, which is just a copy of X embedded as

$$X \hookrightarrow \mathrm{Sym}^n X, \quad x \mapsto n \cdot x.$$

DEFINITION 5.1.12. Let X be a smooth quasiprojective variety, $x \in X$ a point. The *punctual Hilbert scheme* relative to (X, x) is the closed subscheme

$$\mathrm{Hilb}^n(X)_x = \pi_X^{-1}(n \cdot x) \subset \mathrm{Hilb}^n X$$

defined as the preimage of the cycle $n \cdot x$ via the Hilbert-to-Chow map (5.1.1). If F is a locally free sheaf of rank r, the *punctual Quot scheme* relative to (X, x, F) is the closed subscheme

$$\mathrm{Quot}_X(F, n)_x = \pi_F^{-1}(n \cdot x) \subset \mathrm{Quot}_X(F, n)$$

defined as the preimage of the cycle $n \cdot x$ via the Quot-to-Chow map (5.1.2).

EXERCISE 5.1.13. Let X be a smooth variety. Show that that $\mathrm{Hilb}^n(X)_x = \pi_X^{-1}(n \cdot x)$ does not depend on $x \in X$. Show that it does not depend on X either, but only on $\dim X$. More generally, prove that $\mathrm{Quot}_X(F, n)_x$ does not depend on (X, x, F) but only on $(\dim X, \mathrm{rk}\, F)$. (**Hint**: Use étale coordinates on X to construct an isomorphism $\mathrm{Hilb}^n(X)_x \cong \mathrm{Hilb}^n(\mathbb{A}^d)_0$, where $0 \in \mathbb{A}^d$ is the origin; in the higher rank case, trivialise F on an étale neighborhood of x and construct an isomorphism $\mathrm{Quot}_X(F, n)_x \cong \mathrm{Quot}_{\mathbb{A}^d}(\mathscr{O}^{\oplus r}, n)_0$).

Notation 5.1.14. If X is a smooth variety of dimension d, and F is a locally free sheaf of rank r, we will denote by

$$\mathrm{Hilb}^n(\mathbb{A}^d)_0 \subset \mathrm{Hilb}^n X, \quad \mathrm{Quot}_{\mathbb{A}^d}(\mathscr{O}^{\oplus r}, n)_0 \subset \mathrm{Quot}_X(F, n)$$

the punctual Hilbert scheme and the punctual Quot scheme of Definition 5.1.12. This makes sense by Exercise 5.1.13.

Remark 5.1.15. By Lemma 5.1.7, the punctual Quot scheme $\mathrm{Quot}_{\mathbb{A}^d}(\mathscr{O}^{\oplus r}, n)_0$ is proper.

EXERCISE 5.1.16. Let X be a smooth variety. Show that $\mathrm{Hilb}^2 X$ is isomorphic to the blowup of $\mathrm{Sym}^2 X$ along the diagonal.

EXERCISE 5.1.17. Show that if X is a smooth variety and $n \leq 3$ then $\mathrm{Hilb}^n X$ is smooth. (**Hint**: show that a finite planar subscheme $Z \subset X$ defines a smooth point of the Hilbert scheme. Then use your classification from Exercise 2.1.16 to conclude).

5.2. The Hilbert scheme of points on affine space

5.2.1. Equations for the Hilbert scheme of points. Let $d \geq 1$ and $n \geq 0$ be integers. In this section we give a description of the Hilbert scheme

$$\mathrm{Hilb}^n \mathbb{A}^d = \left\{ I \subset \mathbb{C}[x_1, \ldots, x_d] \;\middle|\; \dim_{\mathbb{C}} \mathbb{C}[x_1, \ldots, x_d]/I = n \right\}.$$

More precisely, we will provide equations cutting out the Hilbert scheme inside a smooth quasiprojective variety (the so-called *noncommutative Hilbert scheme*), cf. Theorem 5.2.4. This is easily generalised to the *Quot scheme of points* $\mathrm{Quot}_{\mathbb{A}^d}(\mathscr{O}^{\oplus r}, n)$, see Section 5.2.2.

Recall that not much is known about $\mathrm{Hilb}^n \mathbb{A}^d$ for $d \geq 3$. So the construction we are about to describe is quite remarkable, essentially because *it is there*.

Here is a recap on the well-known properties of Hilbert and Quot scheme of points:

- $\mathrm{Hilb}^n \mathbb{A}^d$ is smooth if and only if $d \leq 2$ or $n \leq 3$.
- $\mathrm{Hilb}^n \mathbb{A}^d$ is irreducible for all d and $n \leq 7$, see [**139**].
- $\mathrm{Hilb}^n \mathbb{A}^3$ is irreducible for $n \leq 11$, see [**101, 45**] and the references therein. See also [**203**] for $n = 9, 10$.
- $\mathrm{Hilb}^n \mathbb{A}^3$ is reducible for $n \geq 78$, see [**110**].
- If $r > 1$, then $\mathrm{Quot}_{\mathbb{A}^d}(\mathscr{O}^{\oplus r}, n)$ is smooth if and only if $n = 1$.
- $\mathrm{Quot}_{\mathbb{A}^2}(\mathscr{O}^{\oplus r}, n)$ is irreducible of dimension $(r+1)n$, see [**54**].

On the other hand, the following questions are open since a long time:

- What is the smallest n such that $\mathrm{Hilb}^n \mathbb{A}^3$ is reducible?
- If $d \geq 3$, what is an upper bound for the number of irreducible components of $\mathrm{Hilb}^n \mathbb{A}^d$?
- If $d \geq 3$, what is the maximal tangent space dimension at a point $p \in W_{\mathrm{sm}} \subset \mathrm{Hilb}^n \mathbb{A}^d$ in the smoothable component?
- If $d \geq 3$, is $\mathrm{Hilb}^n \mathbb{A}^d$ generically reduced? That is, given an irreducible component $W \subset \mathrm{H} = \mathrm{Hilb}^n \mathbb{A}^d$ with generic point ξ, is $\mathscr{O}_{\mathrm{H},\xi}$ reduced?

Let us get started with our description of $\mathrm{Hilb}^n \mathbb{A}^d$. An ideal $I \in \mathrm{Hilb}^n \mathbb{A}^d$ will be tacitly identified with the associated finite closed subscheme

$$\operatorname{Spec} \mathbb{C}[x_1, \ldots, x_d]/I \subset \mathbb{A}^d$$

of length n.

To ease notation, let us put $R_d = \mathbb{C}[x_1, \ldots, x_d]$. The condition defining a point $I \in \mathrm{Hilb}^n \mathbb{A}^d$ is that the $\mathbb{C}$-algebra quotient

$$R_d \twoheadrightarrow R_d/I$$

is a vector space of dimension n. Let us examine this condition in detail. To construct a point in the Hilbert scheme, we need:

(1) a vector space

$$V_n \cong \mathbb{C}^n,$$

(2) an R_d-module structure

$$\vartheta \colon R_d \to \mathrm{End}_{\mathbb{C}}(V_n)$$

with the property that

(3) such structure is induced by an R_d-linear surjection from R_d.

So let us fix an n-dimensional vector space V_n. Later we will have to remember that we made such a choice, and since all we wanted was "$\dim V_n = n$" we will have to quotient out all equivalent choices. Let us forget about this for the moment. In (2), we need ϑ to be a ring homomorphism, so we need to specify one endomorphism of V_n for each coordinate $x_i \in R_d$. All in all, ϑ gives us d matrices

$$A_1, A_2, \ldots, A_d \in \mathrm{End}_{\mathbb{C}}(V_n).$$

The matrix A_i will be responsible for the R_d-linear operator "multiplication by x_i" for the resulting module structure on V_n. Also in this step we should note a reminder for later: strictly speaking, what we have defined so far is a $\mathbb{C}\langle x_1, x_2, \ldots, x_d\rangle$-module structure on V_n. But in R_d the variables commute with one another. So we will have to impose the relations $[A_i, A_j] = 0$ for all $1 \le i < j \le d$.

Condition (3) is tricky. Let us reason backwards, assuming we already have an R_d-linear quotient $\phi : R_d \twoheadrightarrow V_n$. Then it is clear that the image of $1 \in R_d$ generates V_n as an R_d-module. In other words, every element $w \in V_n$ can be written as

$$w = A_1^{m_1} A_2^{m_2} \cdots A_d^{m_d} \cdot \phi(1),$$

for some $m_i \in \mathbb{Z}_{\ge 0}$. This tells us exactly what we should add to the picture to obtain Condition (3): for a fixed module structure, i.e. d-tuple of matrices $(A_1, A_2, \ldots, A_d)$, we need to specify a *cyclic vector*

$$v \in V_n,$$

i.e. a vector with the property that the $\mathbb{C}$-linear span of the set

$$\left\{ A_1^{m_1} A_2^{m_2} \cdots A_d^{m_d} \cdot v \;\middle|\; m_i \in \mathbb{Z}_{d \ge 0} \right\}$$

equals the whole V_n.

Let us consider the $(dn^2 + n)$-dimensional affine space

$$W_n = \mathrm{End}_{\mathbb{C}}(V_n)^{\oplus d} \oplus V_n. \tag{5.2.1}$$

EXERCISE 5.2.1. Show that the locus

$$U_n = \{\, (A_1, A_2, \ldots, A_d, v) \mid v \text{ is } (A_1, A_2, \ldots, A_d)\text{-cyclic} \,\} \subset W_n$$

is a Zariski open subset.

Consider the GL_n-action on W_n given by

$$g \cdot (A_1, A_2, \ldots, A_d, v) = (A_1^g, A_2^g, \ldots, A_d^g, gv) \tag{5.2.2}$$

where $M^g = g^{-1} M g$ is conjugation.

Lemma 5.2.2. *The* GL_n*-action* (5.2.2) *is free on* U_n*.*

PROOF. If $g \in \mathrm{GL}_n$ fixes a point $(A_1, A_2, \ldots, A_d, v) \in U_n$, then $v = gv$ lies in the invariant subspace $\ker(g - \mathrm{id}) \subset V_n$. But by definition of U_n, the smallest invariant subspace containing v is V_n itself, thus $g = \mathrm{id}$. □

DEFINITION 5.2.3. The (geometric) GIT quotient

$$\mathrm{ncHilb}_d^n = U_n / \mathrm{GL}_n$$

is called the *noncommutative Hilbert scheme*.

Note that ncHilb_d^n is smooth of dimension $(d-1)n^2 + n$.

The discussion carried out so far proves the following:

Theorem 5.2.4 ([**190**, **189**, **14**]). *There is a closed immersion*

$$\mathrm{Hilb}^n \mathbb{A}^d \hookrightarrow \mathrm{ncHilb}_d^n$$

cut out by the ideal of relations

$$[A_i, A_j] = 0 \textit{ for all } 1 \le i < j \le d. \tag{5.2.3}$$

EXERCISE 5.2.5. Let $d = 1$. Show that $\mathrm{ncHilb}_1^n = \mathbb{A}^n$.

Example 5.2.6. If $d = 1$ there is just one operator "x" so the Relations (5.2.3) are vacuous. We have

$$\mathrm{Hilb}^n \mathbb{A}^1 = \mathrm{ncHilb}_1^n = \mathbb{A}^n,$$

which thanks to Exercise 5.2.5 reproves the second part of Exercise 5.1.9.

Remark 5.2.7. If $d = 2$ the description of $\mathrm{Hilb}^n \mathbb{A}^2$ is equivalent to Nakajima's description [**150**, Theorem 1.14]. See also [**101**] for an equivalent description of $\mathrm{Hilb}^n \mathbb{A}^d$, in terms of *perfect extended monads*.

5.2.2. Equations for the Quot scheme of points. Fix $r \ge 1$, $d \ge 1$ and $n \ge 0$. A description of the Quot scheme

$$\mathrm{Quot}_{\mathbb{A}^d}(\mathscr{O}^{\oplus r}, n) \hookrightarrow \mathrm{ncQuot}_d^{n,r} \tag{5.2.4}$$

as a closed subscheme of a smooth variety $\mathrm{ncQuot}_d^{n,r}$ of 'matrices and vectors' is given in [**14**, Section 2], though limited to the case $d = 3$. The ambient scheme is the (geometric) GIT quotient

$$\mathrm{ncQuot}_d^{n,r} = \left\{ (A_1, \ldots, A_d, v_1, \ldots, v_r) \;\middle|\; \begin{array}{c} \text{the } \mathbb{C}\text{-linear span of all monomials in} \\ A_1, \ldots, A_d \text{ applied to } v_1, \ldots, v_r \text{ equals } V_n \end{array} \right\} \Big/ \mathrm{GL}_n \tag{5.2.5}$$

called the *noncommutative Quot scheme*. It is a smooth quasiprojective variety of dimension $(d-1)n^2 + rn$, by the (easily proved) higher rank analogue of Lemma 5.2.2. This space reduces to ncHilb_d^n when $r = 1$. The inclusion (5.2.4) is of course cut out by the relations $[A_i, A_j] = 0$. As proved in [**14**, Section 2], the noncommutative Quot scheme $\mathrm{ncQuot}_d^{n,r}$ is isomorphic to the moduli space of left R_d-submodules

$$K \hookrightarrow R_d^{\oplus r}$$

of colength n. In a little more detail, the only thing that changes from the $r = 1$ case is that now we have r 'spanning vectors' instead of just one cyclic vector. A map $R_d^{\oplus r} \to V_n$ is described by specifying the images $v_1, \ldots, v_r$ of the canonical basis vectors $e_1, \ldots, e_r$, and surjectivity is again the spanning condition appearing in (5.2.5). Thus we have seen how (5.2.4) presents the Quot scheme of points as a closed subvariety of an explicit smooth quasiprojective variety.

See Remark 5.4.3 for a description of $\mathrm{Quot}_{\mathbb{A}^d}(\mathscr{O}^{\oplus r}, n)$ in terms of framed sheaves on $\mathbb{P}^d$, available when $d \ge 3$.

5.2.3. Quot-to-Chow revisited. Fix $n \ge 0$ and $d, r \ge 1$. The Quot-to-Chow morphism

$$\pi_{\mathscr{O}^{\oplus r}} \colon \mathrm{Quot}_{\mathbb{A}^d}(\mathscr{O}^{\oplus r}, n) \to \mathrm{Sym}^n \mathbb{A}^d$$

introduced in (5.1.2) can be reinterpreted as follows. Pick a point

$$[A_1, \ldots, A_d, v_1, \ldots, v_r] \in \mathrm{Quot}_{\mathbb{A}^d}(\mathscr{O}^{\oplus r}, n) \hookrightarrow \mathrm{ncQuot}_d^{n,r}$$

and notice that since the matrices pairwise commute, they can be simultaneously made upper triangular. So, since the tuple is defined up to GL_n, we may assume they are in the form

$$A_\ell = \begin{pmatrix} a_{11}^{(\ell)} & * & * & \cdots & * \\ 0 & a_{22}^{(\ell)} & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn}^{(\ell)} \end{pmatrix}, \quad 1 \le \ell \le d.$$

Then $\pi_{\mathscr{O}^{\oplus r}}$ is given by

$$[A_1, \ldots, A_d, v_1, \ldots, v_r] \mapsto \sum_{1 \le i \le n} \left(a_{ii}^{(1)}, \ldots, a_{ii}^{(d)} \right).$$

When all the matrices are *nilpotent*, the corresponding quotient $\mathscr{O}^{\oplus r} \twoheadrightarrow T$ is entirely supported at the origin. In other words,

$$\mathrm{Quot}_{\mathbb{A}^d}(\mathscr{O}^{\oplus r}, n)_0 = \{ [A_1,\ldots,A_d,v_1,\ldots,v_r] \mid A_1,\ldots,A_d \text{ are nilpotent} \}$$

is a way to describe the punctual Quot scheme.

The following result is special to $\mathbb{A}^2$. No such description is currently available for higher dimensional spaces.

Theorem 5.2.8 ([29, Corollaire V.3.3]). *The punctual Hilbert scheme* $\mathrm{Hilb}^n(\mathbb{A}^2)_0$ *is irreducible of dimension* $n-1$.

5.2.4. Varieties of commuting matrices: what's known. Let $\mathbf{k}$ be an algebraically closed field of characteristic 0, and fix V to be an n-dimensional $\mathbf{k}$-vector space. Let

$$C_n = \left\{ (A,B) \in \mathrm{End}(V)^2 \;\middle|\; [A,B]=0 \right\} \subset \mathrm{End}(V)^2 \tag{5.2.6}$$

be the *commuting variety*. Letting GL_n act on C_n by simultaneous conjugation, one can form the quotient stack

$$\mathcal{C}(n) = [C_n / \mathrm{GL}_n],$$

which is equivalent to the stack $\mathrm{Coh}_n(\mathbb{A}^2)$ of finite coherent sheaves of length n on the affine plane over $\mathbf{k}$. Letting

$$\big[\mathcal{C}(n)\big] = \frac{\big[C_n\big]}{\big[\mathrm{GL}_n\big]} \in K_0(\mathrm{St}_{\mathbf{k}}) \tag{5.2.7}$$

be the motivic class of the stack $\mathcal{C}(n)$, let us form the generating series

$$\mathsf{C}(t) = \sum_{n\geq 0} \big[\mathcal{C}(n)\big] t^n \in K_0(\mathrm{St}_{\mathbf{k}})[\![t]\!], \tag{5.2.8}$$

where the Grothendieck ring of stacks $K_0(\mathrm{St}_{\mathbf{k}})$ is quickly introduced in Section 5.3.1. The next result is a formula essentially due to Feit and Fine, but also proven recently by Behrend–Bryan–Szendrői and Bryan–Morrison (all these proofs are over $\mathbb{C}$ but they can be made to work over $\mathbf{k}$).

Theorem 5.2.9 ([63, 18, 33]). *There is an identity*

$$\mathsf{C}(t) = \prod_{k\geq 1}\prod_{m\geq 1} \left(1-\mathbb{L}^{2-k}t^m\right)^{-1} = 1 + \frac{1}{\mathbb{L}-1}t + \left(\frac{1}{\mathsf{GL}_2} + \frac{\mathbb{L}+1}{\mathbb{L}(\mathbb{L}-1)}\right)t^2 + \cdots.$$

All $\mathbf{k}[x,y]$-modules of length $n \leq 4$ have been classified up to isomorphism in [144]. Such classification has been turned into a stratification of $\mathcal{C}(n)$ and has been used to recompute (or confirm!) the first few coefficients of the series (5.2.8), giving them a geometric meaning.

It has been known since a long time that the variety of pairs of commuting matrices C_n over $\mathbb{C}$ is irreducible [145, 170]. The same is true for the space $N_n \subset C_n$ of *nilpotent* commuting linear operators, see [11] for a proof in characteristic 0 and [12] for an extension to fields of characteristic bigger than $n/2$. Premet even showed irreducibility of N_n over *any* field [169].

However the situation is very different for 3 or more matrices. Let $C(d,n)$ be the space of d-tuples of pairwise commuting endomorphisms of an n-dimensional vector space, and let $N(d,n)$ be the space of nilpotent endomorphisms. Then $C(d,n)$ is irreducible for all n if $d \leq 2$. But it is reducible if d and n are both at least 4 [77], and the same is true for $N(d,n)$ [154]. For $d=3$ the situation is as follows: one has that $C(3,n)$ is reducible for $n \geq 30$ [103] and irreducible for $n \leq 10$ (in characteristic 0). Moreover, $N(3,n)$ is known to be irreducible for $n \leq 6$ [154], but $N(3,n)$ is reducible for $n \geq 13$ [161, Theorem 7.10.5].

5.3. The special case of $\operatorname{Hilb}^n \mathbb{A}^2$

An *irreducible holomorphic symplectic manifold* is a simply connected compact Kähler manifold Y such that $\mathrm{H}^0(Y,\Omega_Y^2)$ is spanned by a holomorphic symplectic 2-form. The Hilbert scheme of points $\operatorname{Hilb}^n S$, for S a projective nonsingular complex surface carrying a holomorphic symplectic 2-form (e.g. a K3 surface), is an irreducible holomorphic symplectic manifold, see Fujiki [**66**] for $n=2$ and Beauville [**13**] for the general case. Even though $\mathbb{A}^2$ is only quasiprojective, $\operatorname{Hilb}^n \mathbb{A}^2$ still carries a holomorphic symplectic form. In fact, it is an example of a *Nakajima quiver variety*. On a separate note, the *motivic class* of $\operatorname{Hilb}^n \mathbb{A}^2$ in the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ of complex varieties has been computed by Göttsche [**80**]. We review in this section the quiver variety description of $\operatorname{Hilb}^n \mathbb{A}^2$ and the formula for its Hodge–Deligne polynomial.

5.3.1. The motive of the Hilbert scheme. The content of this subsection is not needed elsewhere in the text, although there would be much to say about motivic invariants in enumerative geometry. We nevertheless want to motivate the study of the Grothendieck ring of varieties with one single example: the computation of the Hodge–Deligne polynomial of the Hilbert scheme of points on $\mathbb{A}^2$.

DEFINITION 5.3.1. Let S be a variety over $\mathbb{C}$.

(i) The *Grothendieck group of S-varieties* is the free abelian group $K_0(\mathrm{Var}_S)$ generated by isomorphism classes $[X]$ of S-varieties $X \to S$, modulo the scissor relations, namely the identities $[Y] = [X]+[Y\setminus X]$ whenever X is a closed S-subvariety of Y. The group $K_0(\mathrm{Var}_S)$ is a ring via $[Y]\cdot[Z]=[Y\times_S Z]$.

(ii) We denote by $\mathbb{L}=[\mathbb{A}^1_S]\in K_0(\mathrm{Var}_S)$ the *Lefschetz motive*, the class of the affine line over S.

The class $[X]\in K_0(\mathrm{Var}_{\mathbb{C}})$ of a complex variety X is called its *motive*, or universal Euler characteristic.

Example 5.3.2. We will let GL_d denote the class of GL_d in $K_0(\mathrm{Var}_{\mathbb{C}})$ throughout. As proved e.g. in [**31**, Lemma 2.6], one has

$$\mathsf{GL}_d=\prod_{i=0}^{d-1}(\mathbb{L}^d-\mathbb{L}^i)=\mathbb{L}^{\binom{d}{2}}\cdot\prod_{k=1}^{d}(\mathbb{L}^k-1).$$

Sometimes, one uses the shorthand $[d]_{\mathbb{L}}!=\prod_{k=1}^d(\mathbb{L}^k-1)$. Then, the motive of the Grassmannian can be computed as

$$\big[G(k,n)\big]=\frac{[n]_{\mathbb{L}}!}{[k]_{\mathbb{L}}![n-k]_{\mathbb{L}}!}\in K_0(\mathrm{Var}_{\mathbb{C}}).$$

One can also define Grothendieck rings of schemes and algebraic spaces. These are both isomorphic to $K_0(\mathrm{Var}_{\mathbb{C}})$ by [**31**, Lemma 2.12]. The situation is different with stacks. There is a Grothendieck ring of stacks

$$K_0(\mathrm{St}_{\mathbb{C}}),$$

generated by isomorphism classes of stacks of finite type over $\mathbb{C}$, having affine geometric stabilisers. We refer the reader to [**52**] or to [**31**, Definition 3.6] for the precise definition. Here we simply recall that $K_0(\mathrm{St}_{\mathbb{C}})$ can be obtained from $K_0(\mathrm{Var}_{\mathbb{C}})$ in the following equivalent ways:

- by localising at the classes of special algebraic groups,
- by localising at $\mathbb{L}$ and $\mathbb{L}^i-1$ for $i\geq 1$,
- by localising at the classes GL_d for $d\geq 1$.

The motivic class of a quotient stack U/G is the quotient $[U]/[G]$ when G is *special*, but not in general. See [**52**] or [**31**, Lemmas 3.8 and 3.9] for a proof of this fact.

Ring homomorphisms with source $K_0(\mathrm{Var}_{\mathbb{C}})$ are frequently called *motivic measures*, realizations, or generalized Euler characteristics. We recall some of them here.

The *E-polynomial* of a complex variety X is

$$E(X;u,v)=\sum_{p,q\geq 0}u^pv^q\sum_{i\geq 0}(-1)^ih^{p,q}\big(\mathrm{H}^i_c(X,\mathbb{Q})\big)\in\mathbb{Z}[u,v],$$

where H_c denotes cohomology with compact support. This defines a ring homomorphism $E\colon K_0(\mathrm{Var}_{\mathbb{C}}) \to \mathbb{Z}[u,v]$. Note that if X is smooth and projective one has $E(X;u,v) = \sum_{p,q\geq 0} h^{p,q}(X)(-u)^p(-v)^q$. Moreover $E(\mathbb{L}) = E(\mathbb{P}^1;u,v) - E(\mathrm{pt};u,v) = (1+uv)-1 = uv$. The map E defines a homomorphism of rings with *power structure* [92], which means that for any generator $w = [W]$ of $K_0(\mathrm{Var}_{\mathbb{C}})$, one has the identity $E((1-t)^{-w}) = (1-t)^{-E(w)}$, where $(1-t)^{-w} = 1+\sum_{n>0}[\mathrm{Sym}^n W]t^n$. Moreover, if $h(u,v) = \sum_{i,j} p_{ij}u^iv^j$ is a polynomial, the expression $(1-t)^{-h(u,v)}$ stands for $\prod_{i,j}(1-u^iv^jt)^{-p_{ij}}$.

Göttsche's theorem [80] proves the identity

$$\mathsf{Z}_{\mathbb{A}^2,0}(t) = \sum_{n\geq 0}\left[\mathrm{Hilb}^n(\mathbb{A}^2)_0\right]t^n = \prod_{m\geq 1}\left(1-\mathbb{L}^{m-1}t^m\right)^{-1} \tag{5.3.1}$$

in $K_0(\mathrm{Var}_{\mathbb{C}})[\![t]\!]$, and by the main result of [93] one has

$$\mathsf{Z}_{\mathbb{A}^2}(t) = \sum_{n\geq 0}\left[\mathrm{Hilb}^n \mathbb{A}^2\right]t^n = \mathsf{Z}_{\mathbb{A}^2,0}(t)^{\mathbb{L}^2}, \tag{5.3.2}$$

where the right hand side is defined in terms of the power structure on $K_0(\mathrm{Var}_{\mathbb{C}})$.[1]

Since $E(\mathbb{A}^2;u,v) = u^2v^2$ and the two series above are related by the power structure relation (5.3.2), one has

$$\begin{aligned}\sum_{n\geq 0} E(\mathrm{Hilb}^n \mathbb{A}^2;u,v)t^n &= \left(\sum_{n\geq 0} E(\mathrm{Hilb}^n(\mathbb{A}^2)_0;u,v)t^n\right)^{E(\mathbb{L}^2)} \\ &= \prod_{m\geq 1}\left(1-(uv)^{m-1}t^m\right)^{-u^2v^2} = \prod_{m\geq 1}\left(1-(uv)^{m+1}t^m\right)^{-1}.\end{aligned}$$

Setting $u = v = -z$, one gets the (compactly supported) Poincaré polynomial generating function

$$\sum_{n\geq 0} P_c(\mathrm{Hilb}^n \mathbb{A}^2, z)t^n = \prod_{m\geq 1}\left(1-z^{2m+2}t^m\right)^{-1} = 1+z^4t+\cdots.$$

Note that this differs from the ordinary Poincaré polynomial generating function

$$\sum_{n\geq 0} P(\mathrm{Hilb}^n \mathbb{A}^2, z)t^n = \prod_{m\geq 1}\left(1-z^{2m-2}t^m\right)^{-1} = 1+t+\cdots$$

determined in [79, 150]. In fact, in [79], a general formula is proved expressing the Poincaré polynomial of $\mathrm{Hilb}^n S$, where S is any smooth *projective* surface: such formula, namely

$$\sum_{n\geq 0} P(\mathrm{Hilb}^n S, z)t^n = \prod_{m\geq 1}\prod_{i=1}^{4}\left(1-z^{2k-2+i}t^k\right)^{(-1)^{i+1}b_i(S)}$$

shows that the generating series only depends on the Betti numbers of S.

Remark 5.3.3. The series (5.3.1) starts as

$$\begin{aligned}\mathsf{Z}_{\mathbb{A}^2,0}(t) = 1 + t &+ (1+\mathbb{L})t^2 + (1+\mathbb{L}+\mathbb{L}^2)t^3 \\ &+ (1+\mathbb{L}+2\mathbb{L}^2+\mathbb{L}^3)t^4 + (1+\mathbb{L}+2\mathbb{L}^2+2\mathbb{L}^3+\mathbb{L}^4)t^5+\cdots\end{aligned}$$

and in fact the n-th coefficient of $\mathsf{Z}_{\mathbb{A}^2,0}(t)$ always contains a summand of the form $(\mathbb{L}+1)\mathbb{L}^{n-2}$. This motive is the class of the *curvilinear locus*, an open subscheme $C_n^0 \subset \mathrm{Hilb}^n(\mathbb{A}^2)_0$ that Briançon proved to be *dense* [29, Théorème V.3.2] and fibred over $\mathbb{P}^1 = \mathbb{P}(\mathfrak{m}/\mathfrak{m}^2)$ (the space of double points at the origin $0 \in \mathbb{A}^2$), with fibre $\mathbb{A}^{n-2}$ [29, Proposition IV.1.1]. Here $\mathfrak{m} = (x,y)$ is the ideal of the origin. The remaining class is the class of its complement. For instance, if $n=3$, the complement has class equal to 1, corresponding to the single non-curvilinear ideal $\mathfrak{m}^2 \subset \mathbb{C}[x,y]$. For $n=4$, the complement has class $1+\mathbb{L}+\mathbb{L}^2$.

[1] See [174] for a generalisation of formula (5.3.2) to Quot schemes.

5.3.2. Nakajima quiver varieties. Let $Q = (Q_0, Q_1, h, t)$ be a quiver, i.e. a finite directed graph with vertex set Q_0, edge set Q_1, and head and tail maps $h, t\colon Q_1 \to Q_0$, meaning that

$$t(a) \bullet \xrightarrow{\quad a \quad} \bullet\, h(a)$$

denotes the *tail* and the *head* of an edge $a \in Q_1$. The following definition will be used in Section 5.4.2 too.

DEFINITION 5.3.4. Let $\mathbf{d} = (\mathbf{d}_i) \in \mathbb{N}^{Q_0}$ be a vector of nonnegative integers. Then a $\mathbf{d}$-dimensional *representation* ρ of Q is the datum of a $\mathbb{C}$-vector space of dimension $\mathbf{d}_i$ for each $i \in Q_0$, along with a linear map $\mathbb{C}^{\mathbf{d}_i} \to \mathbb{C}^{\mathbf{d}_j}$ for every edge $i \to j$ in Q_1. The space of such representations is the affine space

$$\operatorname{Rep}(Q, \mathbf{d}) = \prod_{a \in Q_1} \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}^{\mathbf{d}_{t(a)}}, \mathbb{C}^{\mathbf{d}_{h(a)}}).$$

Fix dimension vectors $\mathbf{v}, \mathbf{w} \in \mathbb{N}^{Q_0}$ and $\mathbb{C}$-vector spaces V_k, W_k of dimensions $\mathbf{v}_k$ and $\mathbf{w}_k$, respectively. The doubled quiver $\overline{Q}$ is the quiver with the same vertices as Q, and with vertex set $\overline{Q}_1 = Q_1 \amalg Q_1'$, where the Q_1' is another copy of Q_1, containing a new edge $a^*\colon j \to i$ for every edge $a\colon i \to j$ in the original Q_1. Consider the affine space

$$\begin{aligned} R(\mathbf{v}, \mathbf{w}) &= \prod_{a \in Q_1} \operatorname{Hom}_{\mathbb{C}}(V_{t(a)}, V_{h(a)}) \times \prod_{k \in Q_0} \operatorname{Hom}_{\mathbb{C}}(V_k, W_k) \\ &= \operatorname{Rep}(Q, \mathbf{v}) \times \prod_{k \in Q_0} \operatorname{Hom}_{\mathbb{C}}(V_k, W_k), \end{aligned}$$

where the second factor is a $\mathbf{w}$-dimensional *framing* on Q. The group $G = \operatorname{GL}(\mathbf{v}) = \prod_{k \in Q_0} \operatorname{GL}(\mathbf{v}_k)$ acts on $R(\mathbf{v}, \mathbf{w})$ by conjugation on the first product and ordinary product on the second factor. The Lie algebra $\mathfrak{g} = \operatorname{Lie}(G)$ can, and will be naturally identified with its dual via the trace pairing. The cotangent bundle to $R(\mathbf{v}, \mathbf{w})$ can be written as

$$T^* R(\mathbf{v}, \mathbf{w}) = \operatorname{Rep}(\overline{Q}, \mathbf{v}) \times \prod_{k \in Q_0} \operatorname{Hom}_{\mathbb{C}}(V_k, W_k) \times \operatorname{Hom}_{\mathbb{C}}(W_k, V_k).$$

A typical element of $T^*R(\mathbf{v}, \mathbf{w})$ will be of the form $(x_a, x_{a^*}, i_k, j_k)_{a,k}$, where $x_a \in \operatorname{Hom}(V_{t(a)}, V_{h(a)})$, $x_{a^*} \in \operatorname{Hom}(V_{h(a)}, V_{t(a)})$, $i_k \in \operatorname{Hom}(V_k, W_k)$ and $j_k \in \operatorname{Hom}(W_k, V_k)$. There is an associated G-equivariant moment map

$$\mu\colon T^* R(\mathbf{v}, \mathbf{w}) \to \mathfrak{g}^*,$$

sending

$$(x_a, x_{a^*}, i_k, j_k)_{a,k} \mapsto \sum_{a \in Q_1} (x_a x_{a^*} - x_{a^*} x_a) - \sum_{k \in Q_0} j_k i_k.$$

The space $\mathbb{C}^{Q_0}$ sits naturally inside $\mathfrak{g}^*$ (as the space of G-invariants), embedded as

$$\lambda = (\lambda_k)_{k \in Q_0} \mapsto \left((M_k)_{k \in Q_0} \mapsto \sum_{k \in Q_0} \lambda_k \operatorname{Tr}(M_k) \right).$$

Now consider, for $\theta \in \mathbb{Z}^{Q_0}$, the character

$$\chi_\theta\colon G \to \mathbb{C}^\times, \quad (g_k)_{k \in Q_0} \mapsto \prod_{k \in Q_0} (\det g_k)^{\theta_k}.$$

The *Nakajima quiver variety* attached to these data is the GIT quotient

$$\mathcal{M}_\lambda^\theta(Q, \mathbf{v}, \mathbf{w}) = \mu^{-1}(\lambda) /\!\!/_{\chi_\theta} G = \mu^{-1}(\lambda)^{\theta\text{-ss}}/G.$$

If $\pi\colon \mu^{-1}(\lambda)^{\theta\text{-ss}} \to X = \mathcal{M}_\lambda^\theta(Q, \mathbf{v}, \mathbf{w})$ denotes the quotient map, by general GIT we have an isomorphism $\mathscr{O}_X \xrightarrow{\sim} (\pi_* \mathscr{O}_{\mu^{-1}(\lambda)^{\theta\text{-ss}}})^G \hookrightarrow \pi_* \mathscr{O}_{\mu^{-1}(\lambda)^{\theta\text{-ss}}}$. The pullback $\pi^*\colon \operatorname{Pic} X \to \operatorname{Pic}(\mu^{-1}(\lambda)^{\theta\text{-ss}})$ is an isomorphism onto the equivariant Picard group $\operatorname{Pic}^G(\mu^{-1}(\lambda)^{\theta\text{-ss}})$, since π is a principal G-bundle.

The following definition will make more sense after Section 6.4.

DEFINITION 5.3.5. The *tautological bundles* on $\mathcal{M}^\theta_\lambda(Q,\mathbf{v},\mathbf{w})$ are the rank $\mathbf{v}_k$ bundles

$$\mathcal{V}_k = \mu^{-1}(\lambda)^{\theta\text{-ss}} \times^G V_k \to \mathcal{M}^\theta_\lambda(Q,\mathbf{v},\mathbf{w}), \quad k \in Q_0,$$

associated to the principal G-bundle π and the $\mathrm{GL}(\mathbf{v}_k)$-equivariant vector bundles (over a point!) $V_k \to \mathrm{pt}$. The Chern classes of $\mathcal{V}_k$ live in the equivariant cohomology

$$\mathrm{H}^{2\mathbf{v}_k}_G(\mathcal{M}^\theta_\lambda(Q,\mathbf{v},\mathbf{w}),\mathbb{C}).$$

5.3.3. The Hilbert scheme as a Nakajima quiver variety. Let us take Q to be the quiver with one loop, and set $\mathbf{v} = n$, $\mathbf{w} = 1$.

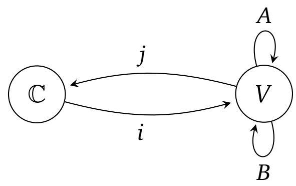

FIGURE 5.1. The quiver $\overline{Q}$ obtained by doubling the quiver Q with one vertex and one loop. The arrows represent a typical element in $T^*R(n,1)$.

Then an element of $T^*R(n,1)$ can be identified with a tuple $(A,B,i,j) \in \mathrm{End}(V)^{\oplus 2} \oplus V \oplus V^\vee$, where $V \cong \mathbb{C}^n$, and the moment map is $\mu(A,B,i,j) = [A,B] - ji$. Picking $\theta \in \mathbb{Z}_{<0}$, the corresponding θ-semistable points (which are all stable) in $T^*R(n,1)$ correspond to tuples (A,B,i,j) such that i is a cyclic vector generating the underlying representation (A,B) of the 2-loop quiver. Consider $\lambda = 0 \in \mathfrak{g}^*$ and take the fibre $\mu^{-1}(0)$. Here, we have the relation $[A,B] = ji$, but in the stable locus this implies $j = 0$ by [**150**, Proposition 2.7], so we are left with the relation $[A,B] = 0$.[2] Taking into account the cyclic vector corresponding to i, it is clear that after getting rid of the choice of V, i.e. quotienting out by GL_n, we have an identification

$$\mathcal{M}^\theta_0(Q,n,1) = \mu^{-1}(0) /\!\!/_{\chi_\theta} \mathrm{GL}_n = \mathrm{Hilb}^n \mathbb{A}^2.$$

The Hilbert-to-Chow map down to $\mathrm{Sym}^n \mathbb{A}^2$ is identified with the projection onto

$$\left\{ (A,B,i,j) \in \mathrm{End}(V)^{\oplus 2} \oplus V \oplus V^\vee \,\middle|\, [A,B] + ij = 0 \right\} \big/ \mathrm{GL}_n \cong \mathrm{Sym}^n \mathbb{A}^2,$$

the GIT quotient computed with respect to the trivial character. Note that this is yet another way of proving the properness of the Hilbert–Chow map. The tautological (rank n) bundle over $\mathrm{Hilb}^n \mathbb{A}^2$ is the pushdown

$$\mathcal{V} = q_* \mathcal{O}_{\mathcal{Z}}$$

of the structure sheaf of the universal subscheme $\mathcal{Z} \subset \mathbb{A}^2 \times \mathrm{Hilb}^n \mathbb{A}^2$ along the projection map $q\colon \mathbb{A}^2 \times \mathrm{Hilb}^n \mathbb{A}^2 \to \mathrm{Hilb}^n \mathbb{A}^2$.

EXERCISE 5.3.6. The Hilbert scheme $\mathrm{Hilb}^n \mathbb{A}^2$ is acted on by $\mathbb{T} = (\mathbb{C}^\times)^2$. Let $K_0^{\mathbb{T}}(\mathrm{pt})$ be the K-group of $\mathbb{T}$-representations. Show that, at a $\mathbb{T}$-fixed point $\mathscr{I}_Z \in \mathrm{Hilb}^n \mathbb{A}^2$, one can write the tangent space

$$T_{\mathscr{I}_Z} \mathrm{Hilb}^n \mathbb{A}^2 = \chi(\mathscr{O}_{\mathbb{A}^2}) - \chi(\mathscr{I}_Z, \mathscr{I}_Z) \in K_0^{\mathbb{T}}(\mathrm{pt}),$$

where $\chi(F,G) = \sum_i (-1)^i \mathrm{Ext}^i(F,G)$ for F, G two equivariant sheaves on $\mathbb{A}^2$, and $\chi(G) = \chi(\mathscr{O}_{\mathbb{A}^2}, G)$.

The character of the tangent space (i.e. its decomposition into weight spaces) at a fixed point can be computed combining Exercise 5.3.6 with the general formula (cf. Lemma 10.3.2)

$$\chi(F,G) = \frac{\overline{\chi(F)}\chi(G)}{\overline{\chi(\mathscr{O}_{\mathbb{A}^2})}}, \tag{5.3.3}$$

[2] The presence of the arrow j is crucial in establishing the symplectic structure of $\mathrm{Hilb}^n \mathbb{A}^2$.

that we take for granted for now. Here $\overline{(\cdot)}$ is the involution on $K_0^{\mathbb{T}}(\mathrm{pt})$ sending the equivariant line bundles t_i to t_i^{-1}. The action is conceived so that one has

$$\chi(\mathscr{O}_{\mathbb{A}^2}) = \sum_{\square \in \mathbb{Z}^2_{\geq 0}} t^{\square} = \frac{1}{(1-t_1)(1-t_2)}$$
$$\overline{\chi(\mathscr{O}_{\mathbb{A}^2})}^{-1} = (1-t_1^{-1})(1-t_2^{-1}) = \frac{(1-t_1)(1-t_2)}{t_1t_2}.$$

If $Z \subset \mathbb{A}^2$ defines a torus fixed point, it is determined by a monomial ideal $\mathscr{I}_\nu \subset \mathbb{C}[x_1, x_2]$, corresponding to a partition (Young diagram) ν. The character associated to the $\mathbb{T}$-module $\mathscr{O}_Z$ (the restriction of the tautological bundle to the point determined by Z) is

$$\mathcal{V}_\nu = \chi(\mathscr{O}_Z) = \sum_{\square \in \nu} t^{\square} = \sum_{(i,j) \in \nu} t_1^i t_2^j.$$

It follows from Formula (5.3.3) that one can compute

$$\begin{aligned}
T_{\mathscr{I}_Z} \operatorname{Hilb}^n(\mathbb{A}^2) &= \chi(\mathscr{O}_{\mathbb{A}^2}) - \chi(\mathscr{I}_\nu, \mathscr{I}_\nu) = \chi(\mathscr{O}_{\mathbb{A}^2}) - \chi(\mathscr{O}_{\mathbb{A}^2} - \mathscr{O}_Z, \mathscr{O}_{\mathbb{A}^2} - \mathscr{O}_Z) \\
&= \chi(\mathscr{O}_{\mathbb{A}^2}) - \overline{\chi(\mathscr{O}_{\mathbb{A}^2} - \mathscr{O}_Z)}\chi(\mathscr{O}_{\mathbb{A}^2} - \mathscr{O}_Z)\overline{\chi(\mathscr{O}_{\mathbb{A}^2})}^{-1} \\
&= \chi(\mathscr{O}_{\mathbb{A}^2}) - \overline{\mathcal{V}}_\nu \mathcal{V}_\nu \overline{\chi(\mathscr{O}_{\mathbb{A}^2})}^{-1} + \overline{\mathcal{V}}_\nu \chi(\mathscr{O}_{\mathbb{A}^2})\overline{\chi(\mathscr{O}_{\mathbb{A}^2})}^{-1} - \chi(\mathscr{O}_{\mathbb{A}^2}) + \mathcal{V}_\nu \\
&= \mathcal{V}_\nu + \frac{\overline{\mathcal{V}}_\nu}{t_1t_2} - \overline{\mathcal{V}}_\nu \mathcal{V}_\nu \frac{(1-t_1)(1-t_2)}{t_1t_2}.
\end{aligned} \tag{5.3.4}$$

For instance,

$$T_\square \operatorname{Hilb}^1 \mathbb{A}^2 = 1 + \frac{1}{t_1t_2} - \frac{(1-t_1)(1-t_2)}{t_1t_2} = t_1^{-1} + t_2^{-1},$$

in agreement with $\operatorname{Hilb}^1 \mathbb{A}^2 = \mathbb{A}^2$, and the fact that

$$T_0\mathbb{A}^2 = \frac{\partial}{\partial x_1} \cdot \mathbb{C} \oplus \frac{\partial}{\partial x_2} \cdot \mathbb{C}$$

as a $\mathbb{T}$-representation.

The following definition is part of a very exciting story involving elliptic cohomology and elliptic stable envelopes [**1**], an upgrade of the notion defined by Maulik–Okounkov in equivariant cohomology [**137**]. See [**184**] for the explicit case of $\operatorname{Hilb}^n \mathbb{A}^2$ worked out in detail.

DEFINITION 5.3.7. Let X be a $\mathbb{T}$-variety. A *polarisation* of X is a class $T^{1/2}X \in K_0^{\mathbb{T}}(X)$ such that

$$TX = T^{1/2}X + \hbar\overline{(T^{1/2}X)},$$

where $\hbar$ is a character of $\mathbb{T}$.

EXERCISE 5.3.8. Show that, if X is a Nakajima quiver variety, then X has a canonical polarisation. Set $\mathbb{T} = (\mathbb{C}^\times)^2$ and $\hbar = t_1t_2$. What is this polarisation for $X = \operatorname{Hilb}^n \mathbb{A}^2$?

5.4. The special case of $\operatorname{Hilb}^n \mathbb{A}^3$

5.4.1. Critical locus description.

In this subsection we set $d = 3$. The Hilbert scheme of points

$$\operatorname{Hilb}^n \mathbb{A}^3$$

is singular as soon as $n \geq 4$, but as we shall see it carries a (symmetric) perfect obstruction theory.

Theorem 5.4.1. *There exists a smooth quasiprojective variety M_n along with a regular function $f_n : M_n \to \mathbb{A}^1$ such that*

$$\operatorname{Hilb}^n \mathbb{A}^3 \subset M_n$$

can be realised as the scheme-theoretic vanishing locus of the exact 1-form $\mathrm{d}f_n$.

PROOF. As M_n we can take the noncommutative Hilbert scheme ncHilb_3^n. By [**179**, Proposition 3.8], the commutator relations (5.2.3) agree *scheme-theoretically* with the single vanishing relation

$$\mathrm{d}f_n = 0,$$

where $f_n \colon \mathrm{ncHilb}_3^n \to \mathbb{A}^1$ is the function

$$[A_1, A_2, A_3, v] \mapsto \operatorname{Tr} A_1 [A_2, A_3]. \qquad \square$$

EXERCISE 5.4.2. Show that, for $\{i, j, k\} = \{1, 2, 3\}$, one has

$$[A_i, A_j] = 0 \quad \iff \quad \frac{\partial f_n}{\partial A_k} = 0,$$

at least set-theoretically.

The above critical locus description is special to $d = 3$. However, as [**14**, Theorem 2.6] shows, it is *not* special to $r = 1$. In fact, one can realise

$$\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n) \hookrightarrow \mathrm{ncQuot}_3^{n,r}$$

as the scheme-theoretically vanishing locus of the differential of the function

$$(A_1, A_2, A_3, v_1, \dots, v_r) \mapsto \operatorname{Tr} A_1 [A_2, A_3]$$

on $\mathrm{ncQuot}_3^{n,r}$. This is the starting point of the definition of the degree 0 *higher rank Donaldson–Thomas theory* of $\mathbb{A}^3$, see [**62**].

Remark 5.4.3. There is yet another description of $\mathrm{Quot}_{\mathbb{A}^d}(\mathscr{O}^{\oplus r}, n)$ available for all $d \geq 3$ and for all algebraically closed fields $\mathbf{k}$ of characteristic 0. Consider the moduli space $\mathrm{Fr}_{r,n}(\mathbb{P}^d)$ of D-framed sheaves on $\mathbb{P}^d$, where $D \subset \mathbb{P}^d$ is a fixed hyperplane. These are, by definition, pairs (E, α) where E is a torsion free sheaf on $\mathbb{P}^d$ with Chern character $(r, 0, \dots, 0, -n)$ and $\alpha \colon E|_D \xrightarrow{\sim} \mathscr{O}_D^{\oplus r}$ is a trivialisation. Then the main result of [**38**] is the existence of a natural isomorphism

$$\mathrm{Quot}_{\mathbb{A}^d}(\mathscr{O}^{\oplus r}, n) \xrightarrow{\sim} \mathrm{Fr}_{r,n}(\mathbb{P}^d),$$

defined on $\mathbf{k}$-points by the assignment

$$E \overset{\iota}{\hookrightarrow} \mathscr{O}_{\mathbb{P}^d}^{\oplus r} \twoheadrightarrow T \quad \longmapsto \quad (E, \iota|_D),$$

where the short exact sequence on the left has the quotient T supported on $\mathbb{A}^d \subset \mathbb{P}^d$.

5.4.2. A quiver description. Let $Q = (Q_0, Q_1, h, t)$ be a quiver, $\mathbf{d} = (\mathbf{d}_i) \in \mathbb{N}^{Q_0}$ a dimension vector. Recall from Definition 5.3.4 the space

$$\mathrm{Rep}(Q, \mathbf{d}) = \prod_{a \in Q_1} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\mathbf{d}_{t(a)}}, \mathbb{C}^{\mathbf{d}_{h(a)}}).$$

of $\mathbf{d}$-dimensional representations ρ of Q. We write $\underline{\dim}\ \rho = \mathbf{d}$. The moduli stack of $\mathbf{d}$-dimensional representations of Q is the quotient stack

$$\mathfrak{M}(Q, \mathbf{d}) = [\mathrm{Rep}(Q, \mathbf{d}) / \mathrm{GL}_{\mathbf{d}}],$$

where $\mathrm{GL}_{\mathbf{d}} = \prod_{i \in Q_0} \mathrm{GL}_{\mathbf{d}_i}$ acts on $\mathrm{Rep}(Q, \mathbf{d})$ by $(g_i)_i \cdot (\rho(a))_{a \in Q_1} = (g_{h(a)} \circ \rho(a) \circ g_{t(a)}^{-1})_{a \in Q_1}$.

DEFINITION 5.4.4. Let $n \geq 0$ be an integer. A *path of length n* in a quiver Q is a sequence of edges $a_n \cdots a_2 a_1$ such that $t(a_{i+1}) = h(a_i)$ for all i. The notation is by juxtaposition from right to left. The *path algebra* $\mathbb{C}Q$ of a quiver Q is defined as follows. As a $\mathbb{C}$-vector space, it is spanned by all paths of length $n \geq 0$, including a single trivial path e_i of length 0 for each vertex $i \in Q_0$. The product is given by concatenation (juxtaposition) of paths, and is defined to be 0 if two paths cannot be concatenated.

EXERCISE 5.4.5. Prove that $\mathbb{C}Q$ is an associative algebra with identity $1 = \sum_{i \in Q_0} e_i$.

EXERCISE 5.4.6. Prove that representations of a quiver Q form a category $\mathrm{Rep}(Q)$ (i.e. define a sensible notion of morphisms of representations). Show that $\mathrm{Rep}(Q)$ is equivalent to the category of left $\mathbb{C}Q$-modules, in particular it is abelian.

EXERCISE 5.4.7. Show that the path algebra of the quiver L_d with one vertex and d loops is isomorphic to $\mathbb{C}\langle x_1, x_2, \ldots, x_d\rangle$.

Let Q be a quiver. Let

$$\mathbb{H}_+ = \{ s\cdot \exp(i\pi\phi) \in \mathbb{C} \mid s \in \mathbb{R}_{>0},\ \phi \in (0,1] \}$$

be the upper half plane with the positive real axis removed. A *central charge* is a group homomorphism $Z\colon \mathbb{Z}^{Q_0} \to \mathbb{C}$ mapping $\mathbb{N}^{Q_0}\setminus 0$ inside $\mathbb{H}_+$. For every $\alpha \in \mathbb{N}^{Q_0}\setminus 0$ we let $\phi(\alpha)$ be the unique real number such that $Z(\alpha) = s\cdot\exp(i\pi\phi(\alpha))$. It is called the *phase* of α (with respect to Z). Every vector $\theta \in \mathbb{Q}^{Q_0}$ induces a central charge Z_θ via

$$Z_\theta(\alpha) = -\theta\cdot\alpha + i|\alpha|,$$

where $|\alpha| = \sum_{i\in Q_0}\alpha_i$. We let ϕ_θ denote the associated phase function, and we set

$$\phi_\theta(\rho) = \phi_\theta(\underline{\dim}\,\rho),$$

for every finite dimensional representation ρ of Q.

Fix $\theta \in \mathbb{Q}^{Q_0}$. We call θ a *stability condition*. A representation ρ is said to be θ-*semistable* if for every proper nontrivial subrepresentation $0 \neq \rho' \subset \rho$ one has

$$\phi_\theta(\rho') \leq \phi_\theta(\rho).$$

Strict inequality in the latter formula defines θ-*stability*, and θ is called $\mathbf{d}$-*generic* if for any $0 < \mathbf{e} < \mathbf{d}$ one has $\phi_\theta(\mathbf{e}) \neq \phi_\theta(\mathbf{d})$. This implies that every θ-semistable representation of dimension $\mathbf{d}$ is θ-stable. For any $\alpha \in \mathbb{N}^{Q_0}\setminus 0$ one can define its *slope* (with respect to θ) as the ratio

$$\mu_\theta(\alpha) = \frac{\theta\cdot\alpha}{|\alpha|} \in \mathbb{Q}.$$

Let us set $\mu_\theta(\rho) = \mu_\theta(\underline{\dim}\,\rho)$, for a representation ρ of Q. There is a chain of open subsets

$$\mathrm{Rep}^{\zeta\text{-st}}(Q,\mathbf{d}) \subset \mathrm{Rep}^{\zeta\text{-ss}}(Q,\mathbf{d}) \subset \mathrm{Rep}(Q,\mathbf{d}).$$

EXERCISE 5.4.8. Let ρ be a finite dimensional representation of a quiver Q, and let $\rho' \subset \rho$ be a subrepresentation. Show that $\phi_\theta(\rho') < \phi_\theta(\rho)$ if and only if $\mu_\theta(\rho') < \mu_\theta(\rho)$, so that stability can be checked using slopes instead of phases.

DEFINITION 5.4.9 (Framed quiver). Let Q be a quiver, $r \geq 1$ an integer, and let $0 \in Q_0$ be a distinguished vertex. The *r-framed quiver* $\widetilde{Q}$ is obtained by adding a new vertex ∞ to the vertices of Q, along with r arrows $\infty \to 0$. Thus a $(1,\mathbf{d})$-dimensional representation $\widetilde{\rho}$ of $\widetilde{Q}$ can be seen as an $(r+1)$-tuple $\widetilde{\rho} = (v_1,\ldots,v_r,\rho)$, where $v_i\colon V_\infty \to V_0$ is a linear map from the 1-dimensional vector space V_∞ (i.e. v is a vector in V_0) and ρ is a $\mathbf{d}$-dimensional representation of Q.

DEFINITION 5.4.10 (Framed stability). Fix $\theta \in \mathbb{Q}^{Q_0}$. A representation $\widetilde{\rho} = (v_1,\ldots,v_r,\rho)$ of $\widetilde{Q}$ is said to be θ-(semi)stable if it is (θ_∞,θ)-(semi)stable, where $\theta_\infty = -\theta_\infty\cdot\underline{\dim}\,\rho$.

We now consider the r-framed quiver $\widetilde{L}_d$, i.e. the d-loop quiver L_d equipped with one additional vertex ∞ and r additional arrows $\infty \to 0$ — see Figure 5.2.

For the dimension vector $\mathbf{d} = (1,n)$, we have

$$\mathrm{Rep}(\widetilde{L}_d,(1,n)) = \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)^{\oplus d} \oplus \mathrm{Hom}_{\mathbb{C}}(\mathbb{C},\mathbb{C}^n)^{\oplus r},$$

which for $r = 1$ reduces to the space W_n defined in (5.2.1).

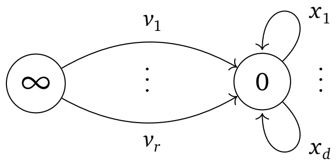

FIGURE 5.2. The r-framed d-loop quiver $\widetilde{L}_d$.

EXERCISE 5.4.11. Fix $\theta \in \mathbb{Q}_{<0}$ and $n \geq 0$. Prove that the open subset

$$U_{n,r} = \left\{ (A_1, \ldots, A_d, v_1, \ldots, v_r) \;\middle|\; \begin{array}{c} \text{the } \mathbb{C}\text{-linear span of all monomials in} \\ A_1, \ldots, A_d \text{ applied to } v_1, \ldots, v_r \text{ equals } \mathbb{C}^n \end{array} \right\} \subset \operatorname{Rep}(\widetilde{L}_d, (1,n))$$

appeared for the first time in (5.2.5) (this was the space $U_n \subset W_n$ of Exercise 5.2.1 in the $r = 1$ case) agrees with the set of θ-semistable framed representations of L_d (i.e. $(-n\theta, \theta)$-semistable representations of $\widetilde{L}_d$, according to Definition 5.4.10), and every such representation is θ-stable.

Thanks to the previous exercise, the noncommutative Quot scheme $\operatorname{ncQuot}_d^{n,r} = U_{n,r}/\operatorname{GL}_n$ can be viewed as a fine moduli space of quiver representations,

$$\operatorname{ncQuot}_d^{n,r} = \operatorname{Rep}^{\theta\text{-st}}(\widetilde{L}_d, (1,n))/\operatorname{GL}_n.$$

If $I \subset \mathbb{C}Q$ is a two-sided ideal, one can consider the full subcategory

$$\operatorname{Rep}_I(Q) \subset \operatorname{Rep}(Q)$$

of representations $(M_a)_{a \in Q_1}$ such that $M_{a_k} \cdots M_{a_2} M_{a_1} = 0$ for every element $a_k \cdots a_2 a_1 \in I$. The category of representations of the quotient algebra $\mathbb{C}Q/I$ is equivalent to $\operatorname{Rep}_I(Q)$. For instance, given the ideal $I \subset \mathbb{C}L_d = \mathbb{C}\langle x_1, \ldots, x_d \rangle$ spanned by the commutators

$$[x_i, x_j], \quad 1 \leq i < j \leq d,$$

one has $\operatorname{Rep}_I(L_d) = \operatorname{Rep} \mathbb{C}[x_1, \ldots, x_d]$.

A special class of ideals arises from *superpotentials*, i.e. elements $\mathsf{W} \in \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$ that can be represented by a finite linear combinations of cyclic paths

$$\sum_{c \text{ cycle in } Q} a_c c \in \mathbb{C}Q,$$

where $[\mathbb{C}Q, \mathbb{C}Q]$ is the vector space spanned by commutators in the path algebra $\mathbb{C}Q$. For a cyclic word w and an arrow $a \in Q_1$, one defines the non-commutative derivative

$$\frac{\partial w}{\partial a} = \sum_{\substack{w = cac' \\ c, c' \text{ paths in } Q}} c'c.$$

This rule extends to an operator $\frac{\partial}{\partial a} \colon \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q] \to \mathbb{C}Q$ acting on every superpotential W. Thus every superpotential $\mathsf{W} = \sum_c a_c c$ gives rise to a two-sided ideal

$$I_{\mathsf{W}} = \left\langle \frac{\partial \mathsf{W}}{\partial a} \;\middle|\; a \in Q_1 \right\rangle \subset \mathbb{C}Q$$

and to a regular function

$$\operatorname{Tr}(\mathsf{W})_{\mathbf{d}} \colon \operatorname{Rep}(Q, \mathbf{d}) \to \mathbb{A}^1, \qquad \rho \mapsto \sum_{c \text{ cycle in } Q} a_c \cdot \operatorname{Tr}(\rho(c)).$$

The quotient

$$J(Q, \mathsf{W}) = \mathbb{C}Q/I_{\mathsf{W}}$$

is called the *Jacobi algebra* of the quiver with potential (Q, W). By our discussion, we have an equivalence of abelian categories

$$\operatorname{Rep}_{I_{\mathsf{W}}}(Q) \cong \operatorname{Rep} J(Q, \mathsf{W}).$$

The proof of Theorem 5.4.1 then reveals that $\operatorname{Hilb}^n \mathbb{A}^3$ can be seen as the moduli space of *stable* 1-framed representations of the Jacobi algebra

$$\mathbb{C}\langle x_1, x_2, x_3\rangle / I_{\mathsf{W}} = \mathbb{C}[x_1, x_2, x_3],$$

for the potential $\mathsf{W} = x_1[x_2, x_3] \in \mathbb{C}L_3$. The same is true for $\operatorname{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)$, which is then a moduli space of stable r-framed representations of $\mathbb{C}\langle x_1, x_2, x_3\rangle / I_{\mathsf{W}}$ (for the same potential as in the rank 1 case). Quotienting out by I_{W}, where $\mathsf{W} = x_1[x_2, x_3]$, corresponds to considering the critical points of the function $\operatorname{Tr}(\mathsf{W})_{(1,n)} \colon \operatorname{ncQuot}_d^{n,r} \to \mathbb{A}^1$.

CHAPTER 6

Equivariant Cohomology

SUMMARY. Equivariant Cohomology was introduced by Borel in his seminar on trasformation groups [27]. The goal of this section is to introduce the framework necessary to state the Atiyah–Bott *localisation formula* [28, 9]. This will be done in Chapter 7. We shall see how the formula works via concrete examples in Chapter 8. See [4, 140] for the theory of localisation in the context of Algebraic Geometry, [195, 49] for extensions from cohomology to Chow, and see also [55] for the first appearance of equivariant cohomology in Enumerative Geometry.

6.1. Universal principal bundles and classifying spaces

6.1.1. Classifying spaces in topology. A topological group is a group object G in the category of topological spaces. Given a topological group G, there is a category of (left, or right) G-*spaces*, i.e. pairs (X,σ) where X is a topological space and σ is a (left, or right) continuous G-action. Morphisms in this category are the G-equivariant maps between the underlying topological spaces. For instance, a morphism of left G-spaces $(X,\sigma_X)\to(Y,\sigma_Y)$ is a continuous map $\alpha\colon X\to Y$ such that the diagram

$$\begin{tikzcd} G\times X \arrow[r, "\sigma_X"] \arrow[d, "\mathrm{id}\times\alpha"'] & X \arrow[d, "\alpha"] \\ G\times Y \arrow[r, "\sigma_Y"] & Y \end{tikzcd}$$

commutes. A G-*homotopy* between G-equivariant maps $\alpha,\beta\colon X\rightrightarrows Y$ is defined to be a G-equivariant homotopy $f\colon[0,1]\times X\to Y$, where G acts trivially on the interval.

Let S be a topological space, E another topological space equipped with a (fixed) right G-action and with a G-invariant map $p\colon E\to S$ (i.e. p is G-equivariant, with respect to the trivial G-action on S). We say that (E,p) is a *principal G-bundle* over S if p is furthermore G-equivariantly locally trivial, i.e. S has an open covering $\{S_i\}_i$ such that on each S_i there are G-equivariant homeomorphisms $p^{-1}(S_i)\xrightarrow{\sim}S_i\times G$ where G acts on $S_i\times G$ by $(s,h)\cdot g=(s,hg)$. This implies that the initial right G-action on E is *free*, and moreover there is a canonical homeomorphism $E/G\xrightarrow{\sim}S$, where E/G is the topological orbit space induced by the G-action $E\times G\to E$. Every fibre $p^{-1}(s)$ of a principal G-bundle $p\colon E\to S$ is in particular homeomorphic to the group G.

A morphism between principal G-bundles $E\to S$ and $F\to S$ is, by definition, a G-equivariant map between the total spaces, commuting with the projections down to S. Any morphism of principal G-bundles over the same base S is in fact an isomorphism (one way to rephrase this is by saying: the category of principal G-bundles over S is a groupoid), and a principal G-bundle $E\to S$ is isomorphic to the trivial principal G-bundle $\mathrm{pr}_1\colon S\times G\to S$ if and only if $E\to S$ has a section (i.e. a continuous right inverse).

If $E\to S$ is a principal G-bundle and $\alpha\colon S'\to S$ is a continuous map, then the pullback

$$\begin{tikzcd} \alpha^*E \arrow[r] \arrow[d, "p'"'] \arrow[dr, phantom, "\square"] & E \arrow[d, "p"] \\ S' \arrow[r, "\alpha"'] & S \end{tikzcd}$$

defines a principal G-bundle (α^*E,p'). Note that the pullback of (E,p) along p itself defines a trivial principal G-bundle $E\times_S E\to E$, since the diagonal provides a section. Given two homotopic maps $\alpha,\beta\colon S'\rightrightarrows S$, the two principal G-bundles $\alpha^*E\to S'$ and $\beta^*E\to S'$ are isomorphic. This fact has the following use. Let HoTop denote the *homotopy category* of Hausdorff, paracompact topological spaces, and let $G \in$ HoTop be a topological group. Then, sending a space $S \in$ HoTop to the set of isomorphism classes of principal G-bundles over S defines a *functor* $k_G\colon \mathrm{HoTop}^{\mathrm{op}} \to \mathrm{Sets}$. This functor is representable, as proved for instance in [**106**, Ch. 4.12], i.e. there exists an object $\mathrm{B}G \in$ HoTop, unique up to unique isomorphism in HoTop, such that

$$k_G(S) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{HoTop}}(S, \mathrm{B}G)$$

functorially in S.

Definition 6.1.1 (Homotopy equivalences). Let X and Y be two topological spaces.

- A continuous map $h\colon X \to Y$ is called a *weak homotopy equivalence* if it induces isomorphisms on all homotopy groups.
- A *homotopy equivalence* between X and Y is a pair of maps $h\colon X \to Y$ and $l\colon Y \to X$ such that $l \circ h$ is homotopy to id_X and $h \circ l$ is homotopic to id_Y. We say that X and Y are *homotopy equivalent*, or have the same *homotopy type*.
- A topological space E is called *weakly contractible* if the constant map $E \to \mathrm{pt}$ is a weak homotopy equivalence, which is equivalent to all the homotopy groups of E being trivial. It is called *contractible* if it is homotopy equivalent to a point.

Remark 6.1.2. If X and Y have the same homotopy type, then their cohomology groups are isomorphic.

Definition 6.1.3. A principal G-bundle $p\colon E \to S$ is called *universal* if its total space E is contractible.

Theorem 6.1.4 (Whitehead [**204**]). *Let $h\colon X \to Y$ be a weak homotopy equivalence of CW complexes. Then h induces a homotopy equivalence between X and Y. In particular, every weakly contractible CW complex is contractible.*

Definition 6.1.5. If $p\colon E \to S$ is a principal G-bundle over a topological space S, such that E is a weakly contractible CW complex, we say that S is a *classifying space* for G.

If $p\colon E \to S$ is as in Definition 6.1.5, then it is universal by Whitehead's theorem. Moreover, S can be taken to be a CW complex (by exploiting a weak equivalence $S' \to S$ where S' is a CW complex, cf. [**99**, Proposition 4.13]), and the classifying space S (resp. the total space E) is unique up to homotopy equivalence (resp. up to G-homotopy equivalence). In fact, such an S represents the functor k_G, with

$$\eta_G = [p\colon E \to S] \in k_G(S)$$

the tautological object. The existence of a principal G-bundle $p\colon E \to S$ as in Definition 6.1.5, for every topological group G, was proved by Milnor [**141**]. We stress that the total space E of such a bundle is contractible, by Whitehead's theorem. Therefore the equivalence class η_G contains a representative with contractible total space, that we will denote

$$\mathrm{E}G \to \mathrm{B}G.$$

The tautological object $\eta_G = [\mathrm{E}G \to \mathrm{B}G] \in k_G(\mathrm{B}G)$ corresponds to the identity $\mathrm{id} \in \mathrm{Hom}_{\mathrm{HoTop}}(\mathrm{B}G, \mathrm{B}G)$ under the isomorphism of functors $k_G \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{HoTop}}(-, \mathrm{B}G)$. For every CW complex S and for every principal G-bundle (F, p) over S, there exists a unique morphism

$$\varphi_{(F,p)} \in \mathrm{Hom}_{\mathrm{HoTop}}(S, \mathrm{B}G)$$

such that $\varphi^*_{(F,p)}\mathrm{E}G$ is isomorphic to $p\colon F \to S$.

Milnor's proof is usually referred to as the *join construction*, for in [**142**] he constructs $\mathrm{E}G$ as

$$\mathrm{E}G = \varinjlim_n G * G * \cdots * G$$

where $*$ denotes the topological join. If we identify G, up to homotopy, with a CW complex W, then $\mathrm{E}G$ is homotopy equivalent to a colimit of finite joins of W, and is therefore weakly contractible (which implies the above universal property). The fact that $\mathrm{E}G$ is contractible will be crucial, since then the product space $\mathrm{E}G \times X$ will have the same homotopy type as X, for every topological space X.

Finally, it is wise to keep in mind that $\mathrm{E}G$ and its free quotient $\mathrm{B}G = \mathrm{E}G/G$ are only determined up to homotopy.

6.1.2. First examples of classifying spaces. We now list a few classical examples of universal principal G-bundles $\mathrm{E}G \to \mathrm{B}G$. We insist that $\mathrm{E}G$ should be contractible.

(1) If $G = 1$, one can take $\mathrm{pt} \to \mathrm{pt}$ as $\mathrm{E}G \to \mathrm{B}G$.
(2) If G and K are two groups, then one can always take $\mathrm{E}(G \times K) = \mathrm{E}G \times \mathrm{E}K$.
(3) If $G = \mathbb{R}$, then one can take $\mathrm{E}\mathbb{R} = \mathbb{R}$ with the constant map $\mathbb{R} \to \mathrm{pt}$.
(4) If $G = \mathbb{Z}^n$, one can take $\mathbb{R}^n \to (S^1)^n$ to be the map

$$(y_1, \ldots, y_n) \mapsto (\exp(i\pi y_1), \ldots, \exp(i\pi y_n)).$$

(5) If $G = \mathbb{Z}/2$, one can take the double cover $S^\infty \to \mathbb{RP}^\infty$.
(6) If $G = \mathbb{C}^\times$, one can take $\mathbb{C}^\infty \setminus 0 \to \mathbb{P}^\infty$.
(7) If $G = S^1 \subset \mathbb{C}^\times$ is the circle, we can restrict the bundle from the previous example to get $S^\infty \to \mathbb{P}^\infty$.

Example (6) generalises in two different ways:

(6′) If $G = (\mathbb{C}^\times)^n$, one has $(\mathbb{C}^\infty \setminus 0)^n \to (\mathbb{P}^\infty)^n$. Similarly, replacing $\mathbb{C}^\times$ by S^1, example (7) becomes $(S^\infty)^n \to (\mathbb{P}^\infty)^n$, the direct limit of the $(S^1)^n$-bundles $(S^{2m+1})^n \to (\mathbb{P}^m)^n$.
(6″) If $G = \mathrm{GL}_n(\mathbb{C})$, one has $F(n, \mathbb{C}^\infty) \to G(n, \mathbb{C}^\infty)$ where $F(n, \mathbb{C}^\infty)$ denotes the space of n-frames in an infinite dimensional vector space and the map sends an n-frame to its span, viewed as a point of the infinite Grassmannian.

Remark 6.1.6. The infinite dimensional complex Grassmannian $G(n, \mathbb{C}^\infty) = \varinjlim_m G(n, \mathbb{C}^m)$ is also the classifying space for the unitary group $\mathrm{U}(n)$, for which one can take

$$\mathrm{EU}(n) = F^{\mathrm{orth}}(n, \mathbb{C}^\infty) = \varinjlim_m F^{\mathrm{orth}}(n, \mathbb{C}^m)$$

as universal principal bundle, where $F^{\mathrm{orth}}(n, \mathbb{C}^m)$ denotes the space of orthonormal n-frames in $\mathbb{C}^m$. Similarly, if $G = \mathrm{O}(n)$ is the real orthogonal group, one can take $F^{\mathrm{orth}}(n, \mathbb{R}^\infty) \to G(n, \mathbb{R}^\infty)$ as universal principal G-bundle, and if $G = \mathrm{GL}_n(\mathbb{R})$, one can take all n-frames, and end up with $F(n, \mathbb{R}^\infty) \to G(n, \mathbb{R}^\infty)$ as universal principal bundle.

Before defining equivariant cohomology, we pause for a tiny remark. We notice that in most examples above, the spaces involved are infinite dimensional. We want to answer the following questions:

(A) Why is that the case?
(B) Why is it not a problem?

Consider example (6), say. We have that $G = \mathbb{C}^\times$ acts on $\mathbb{C}^n$ by

$$\lambda \cdot (x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n),$$

and removing the origin we get a nice principal $\mathbb{C}^\times$-bundle

$$\mathbb{C}^n \setminus 0 \to \mathbb{P}^{n-1}.$$

But we cannot stop here and declare this to be our $\mathrm{E}G \to \mathrm{B}G$, since we want the total space to be contractible, and unfortunately

$$\pi_n(\mathbb{C}^n \setminus 0) \neq 0.$$

It turns out that taking the limit is what we need in order to kill all higher homotopy groups. Formally, we can use the $\mathbb{C}^\times$-equivariant maps

$$\mathbb{C}^n \setminus 0 \hookrightarrow \mathbb{C}^{n+1} \setminus 0, \quad (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0)$$

to get an inductive system of inclusions $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$. The result of the limit process

$$\begin{tikzcd} \mathbb{C}^n \setminus 0 \arrow[r, hook] \arrow[d] & \mathbb{C}^{n+1} \setminus 0 \arrow[d] & \overset{n\to\infty}{\longrightarrow} & \mathbb{C}^\infty \setminus 0 \arrow[d] \\ \mathbb{P}^{n-1} \arrow[r, hook] & \mathbb{P}^n & & \mathbb{P}^\infty \end{tikzcd}$$

has the desired property $\pi_n(\mathbb{C}^\infty \setminus 0) = 0$ for all n. This answers Question (A). As for Question (B), we will see in Section 6.3 that computations can be performed at the level of certain finite dimensional *approximation spaces*.

6.2. Definition of equivariant cohomology

Let X be any CW complex equipped with a *left* G-action. We call such an X a G-space. Choose a universal principal G-bundle $\mathrm{E}G \to \mathrm{B}G$, with a free right G-action on $\mathrm{E}G$. Form the quotient space

$$\mathrm{E}G \times^G X = \frac{\mathrm{E}G \times X}{(e \cdot g, x) \sim (e, g \cdot x)}. \tag{6.2.1}$$

This quotient exists as an 'infinite-dimensional smooth manifold' when the initial G-space X is a smooth manifold, since the diagonal action of G on $\mathrm{E}G \times X$ is *free*. Note that the product $\mathrm{E}G \times X$ is homotopy equivalent to X since $\mathrm{E}G$ is contractible. The quotient (6.2.1) is the right notion of orbit space (from a homotopy theory point of view), for it is much better behaved than the (topological) orbit space X/G.

DEFINITION 6.2.1. The *G-equivariant cohomology of X* is defined to be the singular cohomology of the quotient (6.2.1). The notation will be

$$\mathrm{H}_G^*(X) = \mathrm{H}^*(\mathrm{E}G \times^G X). \tag{6.2.2}$$

The definition works also for real and complex algebraic varieties, since any real or complex algebraic variety is homotopy equivalent to a CW complex [**102**]. The above definition is upgraded from cohomology to Chow in [**49**] for algebraic varieties.

CONVENTION 1. We work with $\mathbb{Q}$ coefficients, so $\mathrm{H}^*(V) = \mathrm{H}^*_{\mathrm{sing}}(V, \mathbb{Q})$ for any space V. Moreover, in the algebraic context (i.e. when X is a complex algebraic variety and G is an algebraic group), we implicitly assume all representations to be *complex representations*, and vector bundles to be *algebraic vector bundles*.

Example 6.2.2. When $X = \mathrm{pt}$, we set

$$\mathrm{H}_G^* = \mathrm{H}_G^*(\mathrm{pt}) = \mathrm{H}^*(\mathrm{B}G).$$

This can be viewed as the ring of characteristic classes of principal G-bundles. See [**195**] for the definition of the G-equivariant *Chow ring* $A^*(\mathrm{B}G)$, a special case of [**49**].

Remark 6.2.3. A G-space X can have nonzero equivariant cohomology in degrees higher than its dimension, unlike ordinary cohomology. This is already true for $X = \mathrm{pt}$.

Remark 6.2.4. Even though ordinary cohomology is a special case of equivariant cohomology (it is enough to set $G = 1$), it is not true that one can reconstruct ordinary cohomology knowing equivariant cohomology. For instance, setting $X = G = \mathbb{C}^\times$, with G acting on X by left complex multiplication, one finds

$$\mathrm{H}_G^1(X) = \mathrm{H}^1(\mathbb{C}^\infty \setminus 0 \times^{\mathbb{C}^\times} \mathbb{C}^\times) = \mathrm{H}^1(\mathbb{C}^\infty \setminus 0) = 0,$$

whereas $\mathrm{H}^1(\mathbb{C}^\times) = \mathrm{H}^1(S^1) = \mathbb{Z}$. More generally, by letting a compact connected Lie group G of rank ℓ act on $X = G$ by left multiplication, one has

$$\mathrm{H}_G^*(G, \mathbb{Q}) = \mathrm{H}^*(\mathrm{E}G, \mathbb{Q}) = \mathrm{H}^*(\mathrm{pt}, \mathbb{Q}) = \mathbb{Q},$$

whereas by a theorem of Hopf [**104**] one has

$$\mathrm{H}^*(G,\mathbb{Q}) = \bigwedge(a_1,\dots,a_\ell),$$

i.e. the rational cohomology ring of G is an exterior algebra on ℓ generators (of odd degree).

Remark 6.2.5. The cohomology of a topological orbit space X/G does not agree with the G-equivariant cohomology of X, unless the G-action on X is free (cf. Equation (6.2.8)). One example is $G = S^1$ acting on $X = S^2 \subset \mathbb{R}^3$ by rotation along the z-axis. There are two fixed points so the action is not free. The quotient X/G can be identified with a closed interval, which is contractible. Therefore the cohomology of the quotient is the cohomology of a point, and the action has manifestly been forgotten. But $\mathrm{H}^*_{S^1}(S^2)$ is more interesting (cf. Exercise 6.2.17).

EXERCISE 6.2.6. Let E be a right G-space, and let G act on itself by left multiplication. Then there is an isomorphism $E \times^G G \xrightarrow{\sim} E$.

EXERCISE 6.2.7. Let E be a right G-space, $K \subset G$ a closed subgroup. Then there is an isomorphism $E \times^G G/K \cong E/K$.

Lemma 6.2.8. *Let X be a CW complex with a left G-action. Then the definition of $\mathrm{H}^*_G(X)$ does not depend upon the choice of universal principal G-bundle.*

PROOF. Let $\mathrm{E}G \to \mathrm{B}G$ and $\mathrm{F}G \to \mathrm{B}G$ be two universal principal G-bundles. Consider the space $Y = (\mathrm{E}G \times \mathrm{F}G \times X)/G$. Then we have fibrations

$$\begin{tikzcd} & Y \arrow[dl, "p_E"'] \arrow[dr, "p_F"] & \\ \mathrm{E}G \times^G X & & \mathrm{F}G \times^G X \end{tikzcd}$$

with contractible fibers $\mathrm{F}G$ and $\mathrm{E}G$, respectively. Let us focus on p_E. For every positive integer n, we have the fibration long exact sequence

$$\cdots \to \pi_n(\mathrm{F}G) \to \pi_n(Y) \to \pi_n(\mathrm{E}G \times^G X) \to \pi_{n-1}(\mathrm{F}G) \to \cdots$$

but the homotopy groups of the fiber $\mathrm{F}G$ are trivial, therefore p_E induces a family of isomorphisms

$$\pi_n(Y) \xrightarrow{\sim} \pi_n(\mathrm{E}G \times^G X).$$

By Whitehead's theorem (cf. Theorem 6.1.4), p_E is in fact a homotopy equivalence, so it induces isomorphisms between cohomology groups. Repeating the process with p_F gives the result. □

Remark 6.2.9. There is a fibre bundle

(6.2.3)
$$\begin{tikzcd} X \arrow[r, hook] & \mathrm{E}G \times^G X \arrow[d, "p"] \\ & \mathrm{B}G \end{tikzcd}$$

with fibre X, induced by the G-equivariant projection $\mathrm{E}G \times X \to \mathrm{E}G$. The standard pullback in cohomology induces a canonical ring homomorphism

(6.2.4)
$$p^* \colon \mathrm{H}^*_G \to \mathrm{H}^*_G(X),$$

making $\mathrm{H}^*_G(X)$ into a H^*_G-module.

Example 6.2.10 (Equivariant formality)**.** When G acts trivially on X we have $\mathrm{E}G \times^G X = \mathrm{B}G \times X$, so that

(6.2.5)
$$\mathrm{H}^*_G(X) \cong \mathrm{H}^*_G \otimes \mathrm{H}^*(X).$$

For instance, given any action of G on X and letting X^G be the fixed locus, we have $\mathrm{H}^*_G(X^G) \cong \mathrm{H}^*_G \otimes \mathrm{H}^*(X^G)$. There are G-spaces X with nontrivial action such that the isomorphism (6.2.5) holds — see Exercise 6.2.17. When one has an isomorphism (6.2.5), we say that X (or its G-action) is *equivariantly formal*.

Example 6.2.11 (Contractible G-spaces)**.** Let X be a contractible space with a left G-action. Then (6.2.4) is an isomorphism.

Example 6.2.12 (Equivariant cohomology of a subgroup)**.** Let $K \subset G$ be a closed subgroup. Then $\mathrm{E}G/K$ exists and in fact $\mathrm{E}K = \mathrm{E}G \to \mathrm{E}G/K = \mathrm{B}K$ is a classifying space for K. Therefore,

$$\mathrm{H}^*_G(G/K) = \mathrm{H}^*(\mathrm{E}G \times^G (G/K)) = \mathrm{H}^*(\mathrm{E}G/K) = \mathrm{H}^*(\mathrm{B}K) = \mathrm{H}^*_K,$$

where the second equality follows by Exercise 6.2.7.

Example 6.2.13. Generalising the previous example, let $K \subset G$ be again a closed subgroup, acting on a space X on the left. We can also consider the diagonal action of K upon $G \times X$. The quotient

$$G \times^K X = (G \times X)/(gk, x) \sim (g, k \cdot x)$$

makes sense and we have

$$\mathrm{H}^*_G(G \times^K X) = \mathrm{H}^*(\mathrm{E}G \times^G G \times^K X) = \mathrm{H}^*(\mathrm{E}G \times^K X) = \mathrm{H}^*_K(X).$$

Again, we have used that $\mathrm{E}G \to \mathrm{E}G/K$ is a classifying space for $K \subset G$.

Theorem 6.2.14 ([**181**, I.5])**.** *Let $f : X \to S$ be a topological fibre bundle with contractible fibre. Then f is a homotopy equivalence. In particular, it induces an isomorphism* $\mathrm{H}^*(X, \mathbb{Z}) \cong \mathrm{H}^*(S, \mathbb{Z})$.

EXERCISE 6.2.15. Let $K \subset G$ be a closed subgroup of a topological group G such that G/K is contractible, and let X be a G-space. Then

$$\mathrm{H}^*_K(X) \cong \mathrm{H}^*_G(X).$$

(**Hint**: use that one can take $\mathrm{E}K = \mathrm{E}G$ to construct a map $\mathrm{E}K \times^K X = \mathrm{E}G \times^K X \to \mathrm{E}G \times^G X$. Show that it is a fibre bundle with fibre G/K. Conclude by Theorem 6.2.14).

Example 6.2.16. Let us revisit the examples (6′) and (6″) discussed above.

(A) In the case $\mathbb{T} = (\mathbb{C}^\times)^n$, we find

$$\mathrm{H}^*_{\mathbb{T}} = \mathrm{H}^*((\mathbb{P}^\infty)^n) = \mathbb{Z}[s_1, \ldots, s_n] \tag{6.2.6}$$

where $s_i \in \mathrm{H}^2_{\mathbb{T}}$ are the Chern classes obtained by pulling back to $\mathrm{B}\mathbb{T}$ the universal line bundles living over the individual spaces $\mathbb{P}^\infty$. In other words, if $\pi_i : \mathrm{B}\mathbb{T} \to \mathbb{P}^\infty$ is the i-th projection, the generators of $\mathrm{H}^*_{\mathbb{T}}$ are

$$s_i = c_1(\pi_i^* \mathscr{O}(-1)).$$

(B) In the case $G = \mathrm{GL}_n(\mathbb{C})$, we find

$$\mathrm{H}^*_G = \mathrm{H}^*(G(n, \mathbb{C}^\infty)) = \mathbb{Z}[e_1, \ldots, e_n] \tag{6.2.7}$$

where $e_i = c_i(\mathscr{S})$ are the Chern classes of the universal rank n bundle $\mathscr{S}$ living over the Grassmannian.

EXERCISE 6.2.17. Let $G = S^1$ act on the sphere $S^2 \subset \mathbb{R}^3$ by rotation along the z-axis. Show that S^2 is equivariantly formal for this S^1-action. In other words, show that

$$\mathrm{H}^*_{S^1}(S^2) \cong \mathrm{H}^*(\mathrm{B}S^1) \otimes \mathrm{H}^*(S^2),$$

even though the action has two fixed points. (**Hint**: Find a suitable open cover $S^2 = X_1 \cup X_2$, inducing an open cover of $\mathrm{E}S^1 \times^{S^1} S^2$. Apply Mayer–Vietoris to this open cover, and use that $\mathrm{H}^*_{S^1} = \mathbb{Z}[c]$ is one-dimensional, resp. zero-dimensional, in even degree, resp. odd degree).

6.2.1. Preview: how to calculate via equivariant cohomology. Let X be a smooth manifold acted on (smoothly) by a Lie group G. There is a commutative diagram

$$\begin{tikzcd} \mathrm{E}G \arrow[d] & \mathrm{E}G \times X \arrow[l] \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{B}G & \mathrm{E}G \times^G X \arrow[l, "p"'] \arrow[r, "\sigma"] & X/G \end{tikzcd}$$

where σ, unlike p, is in general not a fiber bundle, but has the property that

$$\sigma^{-1}(\operatorname{Orb} x) = \mathrm{E}G/G_x \cong \mathrm{B}G_x$$

for all $x \in X$, where $G_x \subset G$ is the stabiliser at x. When G is a compact Lie group acting (smoothly and) *freely* on X, the map σ induces a homotopy equivalence $\mathrm{E}G \times^G X \approx X/G$ yielding a natural isomorphism

$$\sigma^* \colon \mathrm{H}^*(X/G) \xrightarrow{\sim} \mathrm{H}^*_G(X). \tag{6.2.8}$$

Let $\iota \colon X \hookrightarrow \mathrm{E}G \times^G X$ denote the inclusion of X as a fibre of $p \colon \mathrm{E}G \times^G X \to \mathrm{B}G$ (after choosing a base point $\mathrm{pt} \in \mathrm{B}G$). Then ι induces a natural map (surjective in sufficiently nice situations)

$$\iota^* \colon \mathrm{H}^*_G(X) \to \mathrm{H}^*(X)$$

from equivariant to ordinary cohomology.

There is a pullback diagram (on the left) inducing, whenever X is compact and oriented, a commutative diagram of cohomology rings:

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[d, "\iota"'] \arrow[dr, phantom, "\square"] & \mathrm{pt} \arrow[d, "b"] \\ \mathrm{E}G \times^G X \arrow[r, "p"] & \mathrm{B}G \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathrm{H}^*(X) \arrow[r, "q_*"] & \mathbb{Z} \\ \mathrm{H}^*_G(X) \arrow[r, "p_*"] \arrow[u, "\iota^*"'] & \mathrm{H}^*_G \arrow[u, "b^*"'] \end{tikzcd}$$

In algebraic geometry, this compatibility makes sense and is useful when X is a proper variety, which ensures the existence of q_*. The *equivariant pushforward* p_*, on the other hand, will be defined in Section 7.2. Note that the above commutativity implies in particular that

$$\boxed{\textit{ordinary integrals can be performed in equivariant cohomology.}}$$

Indeed, if one wanted to compute

$$\int_X \alpha = q_*\alpha \in \mathbb{Z},$$

then one could proceed as follows: suppose one can pick a lift $\widetilde{\alpha}$ of α along ι^*. Then one can compute the above integral as

$$b^* p_* \widetilde{\alpha},$$

i.e. via equivariant cohomology. This strategy will be crucial for the applications discussed in Chapter 8.

Remark 6.2.18. We want to stress that, even though we often want to compute numbers, an equivariant integral takes values in H^*_G.

6.3. Approximation spaces

Let us now assume that X is a complex algebraic variety and G is an algebraic group. The fact that the spaces involved, like $\mathrm{E}G$ and $\mathrm{B}G$, are infinite-dimensional, is not quite an obstacle to the computation of the equivariant cohomology groups. This is the case because of the following "approximation" result.

Theorem 6.3.1. *Let $(E_m)_{m\geq 0}$ be a family of connected spaces on which G acts freely on the right. Let $\mathsf{k} \colon \mathbb{N} \to \mathbb{N}$ be a function such that $\pi_i(E_m) = 0$ for $0 < i < \mathsf{k}(m)$ and such that $\lim_{m\to\infty} \mathsf{k}(m) = \infty$. Then, for any left G-action on a space X, there are natural isomorphisms*

$$\mathrm{H}^i_G(X) \cong \mathrm{H}^i(E_m \times^G X), \quad i < \mathsf{k}(m).$$

PROOF. Set $E = \mathrm{E}G$. The diagonal action of G on $E \times E_m$ induces a commutative diagram

$$\begin{tikzcd} E_m \times X \arrow[d] & E \times E_m \times X \arrow[l, "E"'] \arrow[r, "E_m"] \arrow[d] & E \times X \arrow[d] \\ E_m \times^G X & (E \times E_m) \times^G X \arrow[l, "E"'] \arrow[r, "E_m"] & E \times^G X \end{tikzcd} \tag{6.3.1}$$

where the vertical maps are the (free) quotient maps, and the horizontal maps are locally trivial fibre bundles with fibre indicated on top of the corresponding arrow. As a consequence of the Leray–Hirsch Lemma, if a fibre bundle $Y \to B$ has $\mathrm{H}^i(\text{fibre}) = 0$ for $0 < i \le m$, then the pullback induces a natural isomorphism $\mathrm{H}^i(B) \xrightarrow{\sim} \mathrm{H}^i(Y)$ for $i \le m$. (See also Theorem 6.2.14). Since $\mathsf{k}(m)$ goes to infinity as m grows, we can apply the previous statement to $\mathsf{k}(m)$ directly, showing that for all $i < \mathsf{k}(m)$ we have isomorphisms

$$\mathrm{H}^i(E_m \times^G X) \xrightarrow{\sim} \mathrm{H}^i((E \times E_m) \times^G X) \xleftarrow{\sim} \mathrm{H}^i_G(X)$$

induced by the lower row of (6.3.1). □

We will refer to the E_m spaces as *approximation spaces*. The idea behind their existence can be traced back to [**27**, Remark XII.3.7].

Remark 6.3.2. In the smooth category, if G is a compact lie group, then $\mathrm{E}G \to \mathrm{B}G$ is a colimit of smooth principal G-bundles

$$E_m \to B_m,$$

where E_m is m-connected, i.e. $\pi_i(E_m) = 0$ for $1 \le i \le m$, i.e. if S^k is a sphere of dimension $k \le m$, then any continuous map $S^k \to E_m$ is homotopic to the constant map.

Example 6.3.3. Let us revisit once more the examples $\mathbb{T} = (\mathbb{C}^\times)^n$ and $G = \mathrm{GL}_n(\mathbb{C})$. For the case of the torus, one can take $E_m = (\mathbb{C}^m \setminus 0)^n \to B_m = (\mathbb{P}^{m-1})^n$ as approximations of $\mathrm{E}\mathbb{T} \to \mathrm{B}\mathbb{T}$. Since $\mathbb{C}^m \setminus 0$ has the same homotopy type as S^{2m-1}, the function $\mathsf{k}(m) = n(2m-1)$ will let us fall in the assumption $\pi_i(E_m) = 0$ of the theorem, where $0 < i < \mathsf{k}(m)$. Setting for instance $X = \mathrm{pt}$, we find isomorphisms

$$\mathrm{H}^i_{\mathbb{T}} \cong \mathrm{H}^i((\mathbb{P}^{m-1})^n), \quad i < n(2m-1).$$

This fully explains the assertion we made in (6.2.6). When $G = \mathrm{GL}_n(\mathbb{C})$, we can take E_m, for $m > n$, to be the set of $m \times n$ rank n matrices. The approximation spaces then look like the free quotients $E_m \to G(n, \mathbb{C}^m)$. The function $\mathsf{k}(m) = 2(m-n)$ will do the job again and we would find, again for $X = \mathrm{pt}$, isomorphisms

$$\mathrm{H}^i_{\mathrm{GL}_n} \cong \mathrm{H}^i(G(n, \mathbb{C}^m)), \quad i < 2(m-n).$$

This explains the assertion we made in (6.2.7).

Remark 6.3.4. If $Y \subset \mathbb{A}^N$ is a subvariety of codimension d, then

$$\pi_i(\mathbb{A}^N \setminus Y) = 0, \quad 0 < i \le 2d-2,$$

as proved in [**69**, Section A.4]. Assume $n < m$. By [**6**, Section II.2], the locus of matrices of rank at most k is irreducible and of codimension $(m-k)(n-k)$ in $\mathbb{A}^{mn}$ for every $k \le n$. Thus, setting $k = n-1$, we see that the complement of the open subset

$$\{\,\text{full rank matrices}\,\} \subset \mathrm{Mat}_{m\times n} = \mathbb{A}^{mn}$$

has codimension $m-n+1$, and the choice $\mathsf{k}(m) = 2(m-n)$ works.

Remark 6.3.5. It is clear that the method of approximation spaces allows one to compute H^*_G in all degrees, since $\mathsf{k}(m) \to \infty$ for $m \to \infty$.

6.4. Equivariant vector bundles

Let G be a group acting on X and H a group acting on Y. Suppose there are maps $\phi\colon G \to H$ and $f : X \to Y$. The condition

$$f(g \cdot x) = \phi(g) \cdot f(x) \tag{6.4.1}$$

for all $x \in X$ and $g \in G$ is the condition under which one can construct a natural map

$$\mathrm{E}G \times^G X \to \mathrm{E}H \times^H Y. \tag{6.4.2}$$

When (6.4.1) is satisfied, taking cohomology of the map (6.4.2) we obtain a natural *equivariant pullback* homomorphism

$$f^*\colon \mathrm{H}^*_H(Y) \to \mathrm{H}^*_G(X). \tag{6.4.3}$$

When $\phi = \mathrm{id}_G$ we say that f is *G-equivariant*, and when moreover G acts trivially on Y we say that f is *G-invariant*.

A crucial example is that of a G-equivariant vector bundle $\pi\colon E \to X$. Such an object is a geometric vector bundle together with a lift of the G-action on X to a G-action on E, as the diagram

$$\begin{tikzcd} G \times E \arrow[r] \arrow[d, "\mathrm{id} \times \pi"'] & E \arrow[d, "\pi"] \\ G \times X \arrow[r, "\text{action}"] & X \end{tikzcd}$$

illustrates. A choice of lift of the G-action is also called a *G-linearisation* on E. Given an equivariant vector bundle one can define a new vector bundle

$$V_E = \mathrm{E}G \times^G E \to \mathrm{E}G \times^G X \tag{6.4.4}$$

of the same rank as E.

Next, we define equivariant Chern classes. The definition is based on ordinary Chern classes, for which the reader is referred to [**100**, Chapter 3] or [**143**]. On the other hand, in Appendix B.2 we follow Fulton's abstract approach [**67**] to define Chern classes in the algebraic context.

DEFINITION 6.4.1 (Equivariant Chern classes). The *equivariant Chern classes* of a G-equivariant vector bundle $E \to X$ are the characteristic classes

$$c_i^G(E) = c_i(V_E) \in \mathrm{H}^{2i}_G(X).$$

These classes can clearly be computed through approximation spaces. Indeed, the vector bundle (6.4.4) can be approximated by vector bundles

$$V_{E,m} = E_m \times^G E \to E_m \times^G X$$

whose Chern classes $c_i(V_{E,m})$ live in $\mathrm{H}^{2i}(E_m \times^G X) \cong \mathrm{H}^{2i}_G(X)$ for $m \gg 0$.

Example 6.4.2 (Equivariant fundamental class)**.** Let X be a smooth algebraic variety acted on by a linear algebraic group G, and let $Y \subset X$ be a G-invariant closed subvariety of codimension d. The space $E_m \times^G X$ is smooth, and

$$E_m \times^G Y \hookrightarrow E_m \times^G X$$

is a closed subvariety of codimension d. Then

$$[Y]^G = [E_m \times^G Y] \in \mathrm{H}^{2d}_G(X) \tag{6.4.5}$$

is called the *equivariant fundamental class* of $Y \subset X$.

EXERCISE 6.4.3. Show that the classes $[Y]^G$ defined in (6.4.5) are compatible when m varies and are independent of the choice of $\mathrm{E}G$ and $(E_m)_{m \geq 0}$.

Example 6.4.4 (Weight of a character)**.** When X is a point, an equivariant vector bundle is a representation $\rho\colon G \to \mathrm{GL}(E)$. This still gives a vector bundle $V_\rho \to \mathrm{B}G$ whose Chern classes live in H^*_G. When E is one-dimensional, an equivariant line bundle is simply a *character* $\chi\colon G \to \mathbb{C}^\times$ and one defines the *weight* of χ to be the Chern class

$$w_\chi = c_1(V_\chi) \in \mathrm{H}^2_G.$$

When a complex representation $\rho\colon G \to \mathrm{GL}(E)$ of dimension r splits as a direct sum of characters χ_i (for example this is always the case when $G = \mathbb{T} = (\mathbb{C}^\times)^g$ is a torus), the *G-equivariant Euler class* splits as a product of weights,

$$e^G(E) = c^G_{\mathrm{top}}(E) = c_{\mathrm{top}}(V_\rho) = \prod_{i=1}^r w_i \in \mathrm{H}^{2r}_G,$$

where $w_i = c_1(V_{\chi_i}) \in \mathrm{H}^2_G$ is the weight of χ_i. More generally, the i-th equivariant Chern class of E is the i-th symmetric function in $w_1,\ldots,w_r$. See Example 6.4.6 for a detailed description of this fact.

Example 6.4.5. Let $\mathbb{T} = \mathbb{C}^\times$ be the one dimensional torus and $\rho_a\colon \mathbb{T} \to \mathbb{C}^\times = \mathrm{GL}(\mathbb{C}_a)$ the character $z \mapsto z^a$ for an integer $a \in \mathbb{Z}$. Here we are viewing the one dimensional vector space $\mathbb{C}_a$ as an equivariant vector bundle over $X = \mathrm{pt}$. The line bundle $V_{\rho_a} \to \mathrm{B}\mathbb{C}^\times$ is approximated by line bundles $V_{\rho_a,m} \to \mathbb{P}^{m-1}$. In fact, $V_{\rho_a,m} \cong \mathscr{O}_{\mathbb{P}^{m-1}}(-a)$ so that the weight of ρ_a is

$$c_1^{\mathbb{C}^\times}(\mathbb{C}_a) = c_1(V_{\rho_a,m}) = a\cdot s \in \mathrm{H}^2_{\mathbb{C}^\times}.$$

Notice that s corresponds to the case $a = 1$. This is called the *standard action* of $\mathbb{T}$ on $\mathbb{C}$. It generalises in the next example.

Example 6.4.6. Let $F = \mathbb{C}^n$ be an n-dimensional vector space and let the torus $\mathbb{T} = (\mathbb{C}^\times)^n$ act on F via the *standard action* $\theta\cdot(v_1,\ldots,v_n) = (\theta_1 v_1,\ldots,\theta_n v_n)$. The induced representation $\rho\colon \mathbb{T} \to \mathrm{GL}(F)$, defined by $\rho_\theta(v) = \theta\cdot v$ as above, gives a rank n vector bundle $V_\rho = \mathrm{E}\mathbb{T}\times^{\mathbb{T}} F \to \mathrm{B}\mathbb{T} = (\mathbb{P}^\infty)^n$. Now, there is a splitting $F = F_1 \oplus\cdots\oplus F_n$ where each summand corresponds to a character

$$\chi_i\colon \mathbb{T} \to \mathbb{C}^\times, \quad \theta \mapsto \theta_i$$

whose weight is just $s_i \in \mathrm{H}^2_{\mathbb{T}}$ by Example 6.4.5. In other words, for each i, $V_{\chi_i} = \mathrm{E}\mathbb{T}\times^{\mathbb{T}} F_i \to (\mathbb{P}^\infty)^n$ is the total space of the line bundle $\mathscr{O}_i(-1) = \pi_i^*\mathscr{O}(-1)$. We quickly verify that

$$c_i^{\mathbb{T}}(F) = e_i(s_1,\ldots,s_n) \in \mathrm{H}^{2i}_{\mathbb{T}}, \tag{6.4.6}$$

the i-th symmetric function in the Chern classes $s_i = c_1(\mathscr{O}_i(-1))$. We have a decomposition

$$V_\rho = \bigoplus_{i=1}^n \mathrm{E}\mathbb{T}\times^{\mathbb{T}} F_i = \bigoplus_{i=1}^n \mathscr{O}_i(-1).$$

Then V_ρ has the obvious filtration

$$0 \subset \mathscr{O}_1(-1) \subset \mathscr{O}_1(-1)\oplus\mathscr{O}_2(-1) \subset \cdots \subset V_\rho$$

with $\mathscr{O}_i(-1)$ as line bundle quotients. The identity (6.4.6) is then a straightforward property of Chern classes (cf. Appendix B.2). For instance, the top Chern class of the standard representation is given by $e^{\mathbb{T}}(F) = c_n^{\mathbb{T}}(V_\rho) = s_1\cdots s_n \in \mathrm{H}^{2n}_{\mathbb{T}}$.

6.5. Two computations on $\mathbb{P}^{n-1}$

In this section we compute the equivariant cohomology of projective space $\mathbb{P}^{n-1}$ and the weights of the tangent representation induced by the standard action. As a convention, we identify

$$\mathbb{P}^{n-1} = \mathbb{P}(F^*) = \{\,1\text{-dimensional quotients } F^* \twoheadrightarrow \Lambda\,\}$$

where $F = \mathbb{C}\cdot e_1 \oplus\cdots\oplus \mathbb{C}\cdot e_n$ is the standard representation, which means

$$\mathrm{weight}(\mathbb{C}\cdot e_i) = s_i \in \mathrm{H}^2_{\mathbb{T}}, \quad \mathbb{T} = (\mathbb{C}^\times)^n.$$

In other words, we are acting on $\mathbb{P}^{n-1}$ by $t\cdot(a_1,\ldots,a_n)=(t_1a_1,\ldots,t_na_n)$, and we have a natural identification $F=\mathrm{H}^0(\mathbb{P}^{n-1},\mathscr{O}_{\mathbb{P}^{n-1}}(1))$ so that e_i corresponds to the linear form x_i.

6.5.1. Equivariant cohomology of $\mathbb{P}^{n-1}$. We already know, thanks to Equation (6.4.6), that $c_i^{\mathbb{T}}(F)=c_i(V_{\rho,m})=e_i(s_1,\ldots,s_n)\in\mathrm{H}^{2i}_{\mathbb{T}}\cong\mathrm{H}^{2i}(B_m)$, where ρ is the standard representation and $E_m=(\mathbb{C}^m\setminus 0)^n\to B_m=(\mathbb{P}^{m-1})^n$ are the finite approximations of $\mathrm{E}\mathbb{T}\to\mathrm{B}\mathbb{T}$. The tautological short exact sequence

$$0\to\mathscr{S}\to F^*\otimes_{\mathbb{C}}\mathscr{O}_{\mathbb{P}^{n-1}}\to\mathscr{O}_{\mathbb{P}^{n-1}}(1)\to 0$$

defines a $\mathbb{T}$-equivariant structure on the line bundle $\mathscr{O}_{\mathbb{P}^{n-1}}(1)$ living on the $\mathbb{T}$-space $\mathbb{P}^{n-1}=\mathbb{P}(F^*)$. Let

$$W=\mathrm{Tot}_{\mathbb{P}^{n-1}}(\mathscr{O}_{\mathbb{P}^{n-1}}(1))\to\mathbb{P}^{n-1}$$

be its total space, and let ζ be the first Chern class of the induced line bundle $E_m\times^{\mathbb{T}}W$ living over the total space of the projective space bundle $\mathbb{P}(V_{\rho,m})$, i.e. look at

$$\begin{array}{c} E_m\times^{\mathbb{T}}W \\ \Big\downarrow{\scriptstyle \text{line bundle}} \\ E_m\times^{\mathbb{T}}\mathbb{P}^{n-1}\xrightarrow{\sim}\mathbb{P}(V_{\rho,m}). \end{array}$$

Then one has $\zeta=c_1(\mathscr{O}_{\mathbb{P}(V_{\rho,m})}(1))=c_1^{\mathbb{T}}(\mathscr{O}_{\mathbb{P}^{n-1}}(1))$. We can now compute

$$\begin{aligned}\mathrm{H}^*_{\mathbb{T}}(\mathbb{P}^{n-1}) &\cong \mathrm{H}^*(E_m\times^{\mathbb{T}}\mathbb{P}^{n-1})\\ &\cong \mathrm{H}^*(\mathbb{P}(V_{\rho,m}))\\ &\cong \mathrm{H}^*_{\mathbb{T}}[\zeta]\Big/\sum_{0\le i\le n}c_i(V_{\rho,m})\zeta^{n-i}=\mathrm{H}^*_{\mathbb{T}}[\zeta]\Big/\prod_{1\le i\le n}(\zeta+s_i).\end{aligned}$$

The last equality follows by the description $c_i(V_{\rho,m})=e_i(s_1,\ldots,s_n)$.

6.5.2. The tangent representation. In the tautological exact sequence

$$0\to\mathscr{S}\to F^*\otimes_{\mathbb{C}}\mathscr{O}_{\mathbb{P}^{n-1}}\to\mathscr{O}_{\mathbb{P}^{n-1}}(1)\to 0$$

we have, for each coordinate point $p_i\in\mathbb{P}^{n-1}$,

$$\mathscr{S}\big|_{p_i}=\bigoplus_{j\neq i}\mathbb{C}\cdot x_j^*,\quad \mathscr{O}_{\mathbb{P}^{n-1}}(1)\big|_{p_i}=\mathbb{C}\cdot x_i^*.$$

The tangent space at the fixed point p_i is then identified to

$$T_{p_i}\mathbb{P}^{n-1}=\mathscr{S}^*\big|_{p_i}\otimes\mathscr{O}_{\mathbb{P}^{n-1}}(1)\big|_{p_i}=\mathrm{Span}_{\mathbb{C}}\left\{x_j\otimes x_i^*\;\middle|\;j\neq i\right\}.$$

This says that the weights of $T_{p_i}\mathbb{P}^{n-1}$ are s_j-s_i for $j\neq i$, in particular the Euler class is computed as

$$e^{\mathbb{T}}(T_{p_i}\mathbb{P}^{n-1})=\prod_{j\neq i}(s_j-s_i).$$

CHAPTER 7

The Atiyah–Bott localisation formula

SUMMARY. In this chapter we prove a version of the Atiyah–Bott localisation formula (Section 7.5) in equivariant cohomology. We mainly follow [**140**, **4**]. We note that the results hold in equivariant Chow theory [**140**, **49**], but for the purpose of our applications we content ourselves with equivariant cohomology.

7.1. A glimpse on the self-intersection formula

Let X be a connected closed (i.e. compact with boundary) oriented manifold. The orientation defines a *fundamental class* $[X] \in \mathrm{H}_{\dim X}(X)$, inducing the Poincaré duality isomorphisms $\mathsf{pd}\colon \mathrm{H}^k(X) \xrightarrow{\sim} \mathrm{H}_{\dim X-k}(X)$ for every k. If $E \to X$ is an oriented vector bundle of rank r on X, the orientation

$$\eta \in \mathrm{H}^r(E, E \setminus X)$$

corresponds, by construction, to the identity element

$$\mathbb{1} = \mathsf{pd}^{-1}([X]) \in \mathrm{H}^0(X)$$

under the Thom isomorphism

$$\mathrm{H}^0(X) \xrightarrow{\ \sim\ } \mathrm{H}^r(E, E \setminus X),$$

where $X \hookrightarrow E$ is embedded as the zero section. The inclusions

$$(X, \emptyset) \hookrightarrow (E, \emptyset) \hookrightarrow (E, E \setminus X)$$

induce maps

$$\begin{tikzcd} \mathrm{H}^r(E, E \setminus X) \arrow[r, "u_E"] \arrow[rr, bend left, "\alpha"] & \mathrm{H}^r(E) \arrow[r] & \mathrm{H}^r(X) \end{tikzcd}$$

and the *Euler class* of E is, by definition, the image of η under this composition, i.e.

$$e(E) = \alpha(\eta) \in \mathrm{H}^r(X).$$

Let $f : X \to Y$ be a map of connected compact oriented manifolds. Set $n = \dim X$, $m = \dim Y$ and $d = m - n$. There are Gysin maps

$$f_*\colon \mathrm{H}^p(X) \to \mathrm{H}^{p+d}(Y),$$

defined via Poincaré duality through the diagram

$$\begin{tikzcd} \mathrm{H}^p(X) \arrow[r, dashed] \arrow[d, "\mathsf{pd}"', "\wr"] & \mathrm{H}^{p+d}(Y) \\ \mathrm{H}_{n-p}(X) \arrow[r, "f_*"] & \mathrm{H}_{n-p}(Y) \arrow[u, "\mathsf{pd}^{-1}"', "\wr"] \end{tikzcd} \tag{7.1.1}$$

where f_* is the pushforward in homology.

When $f = \iota$ is a closed embedding of codimension d, and the normal bundle

$$N_{X/Y} = T_Y|_X / T_X$$

is oriented compatibly with respect to f (this means that the canonical isomorphism of line bundles $\det T_X \otimes \det N_{X/Y} \xrightarrow{\sim} \det T_Y|_X$ preserves orientations), there is a factorisation of ι_* as

$$\mathrm{H}^p(X) \xrightarrow{\sim} \mathrm{H}^{p+d}(N_{X/Y}, N_{X/Y} \setminus X) \xrightarrow{\sim} \mathrm{H}^{p+d}(Y, Y \setminus X) \longrightarrow \mathrm{H}^{p+d}(Y),$$

where the last map comes from Mayer–Vietoris for the inclusion $\iota \colon X \hookrightarrow Y$, and the middle isomorphism is computed via excision by fixing a tubular neighborhood T of X in Y, so that

$$\mathrm{H}^*(Y, Y \setminus X) \cong \mathrm{H}^*(T, T \setminus X) \cong \mathrm{H}^*(N_{X/Y}, N_{X/Y} \setminus X).$$

Now set $p = 0$. Then there is a diagram

$$\begin{tikzcd}
\mathrm{H}^0(X) \arrow[r, "\sim"] \arrow[rrr, bend left, "\iota_*"] & \mathrm{H}^d(N_{X/Y}, N_{X/Y} \setminus X) \arrow[r, "\sim"] \arrow[d, "u_{N_{X/Y}}"] & \mathrm{H}^d(Y, Y \setminus X) \arrow[r] & \mathrm{H}^d(Y) \arrow[d, "\iota^*"] \\
& \mathrm{H}^d(N_{X/Y}) \arrow[rr] & & \mathrm{H}^d(X)
\end{tikzcd}$$

whose commutativity implies the content of the *self-intersection formula*.

Theorem 7.1.1 (Self-intersection formula). *Let $\iota \colon X \hookrightarrow Y$ be a closed embedding of connected compact oriented manifolds. the operator $\iota^*\iota_*$ agrees with multiplication by the Euler class of $N_{X/Y}$. In particular, there is an identity*

$$\iota^*\iota_*\mathbb{1} = e(N_{X/Y}) \in \mathrm{H}^r(X).$$

See Fulton [67] for the corresponding statement in Algebraic Geometry.

7.2. Equivariant pushforward

Let G be a compact Lie group. Let $f: X \to Y$ be a G-equivariant map of compact manifolds. Set

$$n = \dim X, \quad m = \dim Y, \quad d = m - n.$$

In order to construct an equivariant pushforward $f_*^G : \mathrm{H}^*_G(X) \to \mathrm{H}^*_G(Y)$ we cannot apply the same procedure leading to Diagram 7.1.1, because Poincaré duality is not available on infinite dimensional spaces such as $X_G = \mathrm{E}G \times^G X$ and $Y_G = \mathrm{E}G \times^G Y$.

We use approximation spaces to solve this issue. Fix a directed system of principal G-bundles

$$\{ E_i \to B_i \}_{i \geq 0}$$

whose limit recovers the classifying space $\mathrm{E}G \to \mathrm{B}G$ (cf. Remark 6.3.2). Since E_i are compact spaces, so are the Borel spaces

$$X_G^i = E_i \times^G X, \quad Y_G^i = E_i \times^G Y,$$

and the approximation result (Theorem 6.3.1) ensures that

$$\mathrm{H}^p_G(X) \cong \mathrm{H}^p(X_G^i), \quad \mathrm{H}^p_G(Y) \cong \mathrm{H}^p(Y_G^i), \quad p \leq i. \tag{7.2.1}$$

Recall from (6.2.3) that we have fibrations

$$\begin{tikzcd}
X \arrow[r, hook] & X_G^i \arrow[rr, "f^i"] \arrow[dr, "p_X"'] & & Y_G^i \arrow[dl, "p_Y"] & Y \arrow[l, hook'] \\
& & B_i & &
\end{tikzcd}$$

with fibre X and Y, respectively. It follows that

$$\dim X_G^i = \ell + n, \quad \dim Y_G^i = \ell + m,$$

where $\ell = \dim B_i$. Exploiting Poincaré duality on the compact (finite dimensional) spaces X_G^i and Y_G^i, we can replace the diagram (7.1.1) by a new diagram

$$\begin{tikzcd} \mathrm{H}^p(X_G^i) \arrow[r, dashed] \arrow[d, "\mathrm{pd}"', "\wr"] & \mathrm{H}^{p+d}(Y_G^i) \\ \mathrm{H}_{\ell+n-p}(X_G^i) \arrow[r, "f_*^i"] & \mathrm{H}_{\ell+n-p}(Y_G^i) \arrow[u, "\wr", "\mathrm{pd}^{-1}"'] \end{tikzcd}$$

which we can of course redraw as

$$\begin{tikzcd} \mathrm{H}_G^p(X) \arrow[r, dashed, "f_*^{G,p}"] \arrow[d, "\mathrm{pd}"', "\wr"] & \mathrm{H}_G^{p+d}(Y) \\ \mathrm{H}_{\ell+n-p}(X_G^i) \arrow[r, "f_*^i"] & \mathrm{H}_{\ell+n-p}(Y_G^i) \arrow[u, "\wr", "\mathrm{pd}^{-1}"'] \end{tikzcd}$$

by exploiting (7.2.1). This yields a system of maps

$$f_*^{G,p} \colon \mathrm{H}_G^p(X) \to \mathrm{H}_G^{p+d}(Y).$$

EXERCISE 7.2.1. Prove that the maps $f_*^{G,p}$ are compatible with the structure of inverse system of $\mathrm{H}^p(X_G^i)$ and $\mathrm{H}^p(Y_G^i)$ attached to the directed systems $(X_G^i)_i$ and $(Y_G^i)_i$ respectively.

By Exercise 7.2.1, we can construct the *equivariant pushforward*

$$f_*^G \colon \mathrm{H}_G^*(X) \to \mathrm{H}_G^*(Y).$$

The superscript 'G' will often be omitted.

7.3. Trivial torus actions

Let $\mathbb{T}$ be a torus acting *trivially* on a smooth complex algebraic variety X. Given a $\mathbb{T}$-equivariant vector bundle $E \to X$ of rank r, we get a canonical decomposition

$$E = \bigoplus_{\chi} E_\chi, \tag{7.3.1}$$

where $\mathbb{T}$ acts by the character χ on E_χ. The characters χ vary in the character group $\widehat{\mathbb{T}} = \mathbb{Z}^{\dim \mathbb{T}}$ of the torus. One should expect to be able to express the $\mathbb{T}$-equivariant Chern classes of E in terms of the $\mathbb{T}$-equivariant Chern classes of its subbundles $E_\chi \subset E$.

Let

$$V_E = \mathrm{E}\mathbb{T} \times^{\mathbb{T}} E \to \mathrm{E}\mathbb{T} \times^{\mathbb{T}} X = X \times \mathrm{B}\mathbb{T}$$

be the induced rank r bundle, cf. (6.4.4) (and (6.2.5) for the identification on the right hand side).

EXERCISE 7.3.1. Show that $V_{E_\chi} \cong E_\chi \boxtimes V_\chi$, where the box product refers to the canonical projections from $X \times \mathrm{B}\mathbb{T}$ and V_χ is the line bundle on $\mathrm{B}\mathbb{T}$ introduced in Example 6.4.4.

EXERCISE 7.3.2. Assume again $\mathbb{T}$ acts trivially on a smooth variety X, and let E be a $\mathbb{T}$-equivariant vector bundle. Show that

$$c_i^{\mathbb{T}}(E_\chi) = \sum_{k=0}^{i} \binom{r_\chi - k}{i-k} c_k(E_\chi)\chi^{i-k} \in \mathrm{H}_{\mathbb{T}}^{2i}(X),$$

where $r_\chi = \mathrm{rk}\, E_\chi$. (**Hint**: use the previous exercise and a standard property of Chern classes of tensor products, cf. [**67**, Example 3.2.2] or Example B.2.3 in these notes).

7.4. Torus fixed loci

Throughout this section, we let X be a smooth complex algebraic variety acted on by a torus $\mathbb{T}$. The scheme structure of the fixed locus

$$X^{\mathbb{T}} \subset X$$

is discussed in [**65**, Section 2].

Theorem 7.4.1 ([**112**, **65**]). *If X is a smooth $\mathbb{T}$-variety, the fixed locus $X^{\mathbb{T}}$ is smooth.*

EXERCISE 7.4.2. Let Y be a compact normal $\mathbb{T}$-variety. Show that if Y has a singular point, then it has a *torus-fixed* singular point.

Let N be the normal bundle of the inclusion $F \subset X$ of a component $F \subset X^{\mathbb{T}}$. Then N is $\mathbb{T}$-equivariant and for each $x \in F$ one has $(T_x X)^{\mathbb{T}} = T_x F$ by results of [**65**], so that the action of $\mathbb{T}$ on the normal space

$$N_x = T_x X / T_x F$$

is nontrivial, i.e. N_x has no trivial subrepresentations, i.e. $N_x^{\mathbb{T}} = 0$. It follows that the Euler class of the normal bundle N is nonzero, being a product of nonzero weights. In fact, $e^{\mathbb{T}}(N)$ becomes invertible in the ring $\mathrm{H}^*_{\mathbb{T}}(F)[\chi_i^{-1}]$, where χ_i are the characters that occur in the decomposition of N into eigenbundles.

Let E be a $\mathbb{T}$-equivariant vector bundle of rank r over X. Let $F \subset X^{\mathbb{T}}$ be a component of the fixed locus, so that $\mathrm{H}^*_{\mathbb{T}}(F) = \mathrm{H}^*(F) \otimes \mathrm{H}^*_{\mathbb{T}}$. According to Equation (7.3.1), the vector bundle $E|_F$ on F has a decomposition

$$E|_F = \bigoplus_{\chi} E_{F,\chi}$$

into eigen-subbundles. By Exercise 7.3.2, the component of

$$c_i^{\mathbb{T}}(E_{F,\chi}) \in \mathrm{H}^{2i}_{\mathbb{T}}(F) \tag{7.4.1}$$

in $\mathrm{H}^{2i}_{\mathbb{T}}$ is given by

$$\binom{\operatorname{rk} E_{F,\chi}}{i} \chi^i \in \mathrm{H}^{2i}_{\mathbb{T}}. \tag{7.4.2}$$

Here we are denoting simply by χ the weight $w_\chi \in \mathrm{H}^2_{\mathbb{T}}$, cf. Example 6.4.4. Since $\mathrm{H}^{2k}(F) = 0$ for $k > \dim F$, we have that for all $j > 0$, the classes in $\mathrm{H}^{2j}(F)$ are nilpotent in $\mathrm{H}^*_{\mathbb{T}}(F)$. Therefore the element (7.4.1) is invertible if and only if its component (7.4.2) is invertible. It follows that

$$c_i^{\mathbb{T}}(E_{F,\chi}) \text{ is invertible in the localised ring } \mathrm{H}^{2i}_{\mathbb{T}}(F)\big[\chi^{-1}\big]. \tag{7.4.3}$$

Proposition 7.4.3. *Let X be a smooth $\mathbb{T}$-variety, let $F \subset X^{\mathbb{T}}$ be a component of codimension d. Then there are finitely many characters $\chi_1, \ldots, \chi_s$ such that the Euler class*

$$e^{\mathbb{T}}(N_{F/X}) \in \mathrm{H}^{2d}_{\mathbb{T}}(F)$$

becomes invertible in the ring

$$\mathrm{H}^*_{\mathbb{T}}(F)\big[\chi_s^{-1}, \ldots, \chi_s^{-1}\big].$$

PROOF. Set $N = N_{F/X}$. We saw above that the action on the normal space N_x is nontrivial for all $x \in F$, so the characters appearing in the decomposition

$$N = \bigoplus_{i=1}^{s} N_{\chi_i}$$

are all nontrivial, and $e^{\mathbb{T}}(N)$ is a product of nonzero weights,

$$0 \neq e^{\mathbb{T}}(N) = \prod_{i=1}^{s} e^{\mathbb{T}}(N_{\chi_i}),$$

so the observation (7.4.3) implies the result. $\square$

7.5. The localisation formula

This subsection introduces the technique of localisation in Algebraic Geometry. Our purpose is to apply this powerful tool to solve enumerative problems, such as finding the number of lines on a general cubic surface $S \subset \mathbb{P}^3$, or on a general quintic 3-fold $Y \subset \mathbb{P}^4$.

Let $\iota\colon X^{\mathbb{T}} \hookrightarrow X$ be the inclusion of the fixed point locus. We have the equivariant pushforward

$$\iota_*\colon \mathrm{H}^*_{\mathbb{T}}(X^{\mathbb{T}}) \to \mathrm{H}^*_{\mathbb{T}}(X),$$

and the localisation theorem states that this map becomes an isomorphism after inverting finitely many nontrivial characters. In particular, it becomes an isomorphism after extending scalars to the field of fractions

$$\mathcal{H}_{\mathbb{T}} = \operatorname{Frac} \mathrm{H}^*_{\mathbb{T}}.$$

Notice that

$$\mathrm{H}^*_{\mathbb{T}}(X^{\mathbb{T}}) = \bigoplus_{\alpha} \mathrm{H}^*_{\mathbb{T}}(F_\alpha)$$

where F_α are the components of the fixed locus. The crucial point is that, if N_α is the normal bundle of the inclusion $\iota_\alpha\colon F_\alpha \hookrightarrow X$, then

$$\iota_\alpha^*\iota_{\alpha*}(-) = e^{\mathbb{T}}(N_\alpha) \cap -$$

and as we saw this Euler class is nonzero when restricted to any point $x \in F_\alpha$. This is enough for $e^{\mathbb{T}}(N_\alpha)$ to become invertible after a suitable localisation.

The statement of the localisation theorem for compact manifolds with torus action is the following. See also the work of Edidin–Graham [49] for the localisation formula in Chow.

Theorem 7.5.1 (Atiyah–Bott [9]). *Let M be a compact smooth manifold equipped with an action of a torus $\mathbb{T}$. Then the equivariant pushforward along $\iota\colon M^{\mathbb{T}} \hookrightarrow M$ induces an isomorphism*

$$\iota_*\colon \mathrm{H}^*_{\mathbb{T}}(M^{\mathbb{T}}) \otimes_{\mathrm{H}^*_{\mathbb{T}}} \mathcal{H}_{\mathbb{T}} \xrightarrow{\ \sim\ } \mathrm{H}^*_{\mathbb{T}}(M) \otimes_{\mathrm{H}^*_{\mathbb{T}}} \mathcal{H}_{\mathbb{T}}.$$

Its inverse is given by

$$\psi \mapsto \sum_\alpha \frac{\iota_\alpha^*\psi}{e^{\mathbb{T}}(N_\alpha)}.$$

Remark 7.5.2. In particular, every class $\psi \in \mathrm{H}^*_{\mathbb{T}}(M) \otimes_{\mathrm{H}^*_{\mathbb{T}}} \mathcal{H}_{\mathbb{T}}$ writes uniquely as

$$\psi = \sum_\alpha \iota_{\alpha*} \frac{\iota_\alpha^*\psi}{e^{\mathbb{T}}(N_\alpha)}. \tag{7.5.1}$$

Let M be a compact manifold with a $\mathbb{T}$-action and structure map $q\colon M \to \mathrm{pt}$. We have the equivariant pushforward $q_*\colon \mathrm{H}^*_{\mathbb{T}}(M) \to \mathrm{H}^*_{\mathbb{T}}$, which after tensoring by $\mathcal{H}_{\mathbb{T}}$ yields the integration map

$$\int_M \colon \mathrm{H}^*_{\mathbb{T}}(M) \otimes_{\mathrm{H}^*_{\mathbb{T}}} \mathcal{H}_{\mathbb{T}} \to \mathcal{H}_{\mathbb{T}}.$$

For any component $F_\alpha \subset M^{\mathbb{T}}$, the structure map $q_\alpha\colon F_\alpha \to \mathrm{pt}$ factors as $q \circ \iota_\alpha$ where $\iota_\alpha\colon F_\alpha \hookrightarrow M$ is the inclusion. The condition $q_{\alpha*} = q_* \circ \iota_{\alpha*}$ then allows us, simply by looking at (7.5.1), to deduce the following integration formula: for any equivariant class $\psi \in \mathrm{H}^*_{\mathbb{T}}(M) \otimes_{\mathrm{H}^*_{\mathbb{T}}} \mathcal{H}_{\mathbb{T}}$, one has

$$\int_M \psi = \sum_\alpha q_{\alpha*} \frac{\iota_\alpha^*\psi}{e^{\mathbb{T}}(N_\alpha)} = \sum_\alpha \int_{F_\alpha} \frac{\iota_\alpha^*\psi}{e^{\mathbb{T}}(N_\alpha)} \in \mathcal{H}_{\mathbb{T}}. \tag{7.5.2}$$

Let us go back to the algebraic setting (in the special case of finitely many fixed points).

Proposition 7.5.3. *Let X be a smooth $\mathbb{T}$-variety with finitely many fixed points. Set*

$$\mathsf{e} = \prod_{p \in X^{\mathbb{T}}} e^{\mathbb{T}}(T_pX) \in \mathrm{H}^*_{\mathbb{T}},$$

and fix a multiplicative subset $S \subseteq \mathrm{H}^*_{\mathbb{T}} \setminus 0$ containing e. Then

(1) *The pullback map* $S^{-1}\iota^*\colon S^{-1}\mathrm{H}^*_{\mathbb{T}}(X) \to S^{-1}\mathrm{H}^*_{\mathbb{T}}(X^{\mathbb{T}})$ *is onto, and*
(2) *if* $\mathrm{H}^*_{\mathbb{T}}(X)$ *is a free* $\mathrm{H}^*_{\mathbb{T}}$*-module of rank* $r \le |X^{\mathbb{T}}|$*, then* $r = |X^{\mathbb{T}}|$ *and* $S^{-1}\iota^*$ *is an isomorphism.*

PROOF. To prove (1), note that the composition

$$S^{-1}\mathrm{H}^*_{\mathbb{T}}(X^{\mathbb{T}}) \xrightarrow{S^{-1}\iota_*} S^{-1}\mathrm{H}^*_{\mathbb{T}}(X) \xrightarrow{S^{-1}\iota^*} S^{-1}\mathrm{H}^*_{\mathbb{T}}(X^{\mathbb{T}})$$

is surjective because it equals $S^{-1}(\iota^*\circ\iota_*)$, and the determinant of $\iota^*\circ\iota_*$ is precisely e because it is a diagonal map, equal to $e^{\mathbb{T}}(T_pX)$ on the component indexed by p. But e becomes invertible after localisation by Proposition 7.4.3, so (1) follows.

By Part (1), we have

$$r = \mathrm{rk}_{S^{-1}\mathrm{H}^*_{\mathbb{T}}} S^{-1}\mathrm{H}^*_{\mathbb{T}}(X) \ge \mathrm{rk}_{S^{-1}\mathrm{H}^*_{\mathbb{T}}} S^{-1}\mathrm{H}^*_{\mathbb{T}}(X^{\mathbb{T}}) = |X^{\mathbb{T}}| \ge r,$$

which implies $\mathrm{rk}_{S^{-1}\mathrm{H}^*_{\mathbb{T}}} S^{-1}\mathrm{H}^*_{\mathbb{T}}(X) = |X^{\mathbb{T}}|$. To prove Part (2), observe that $S^{-1}\mathrm{H}^*_{\mathbb{T}}$ is a noetherian ring (being a localisation of a noetherian ring), hence a surjective $S^{-1}\mathrm{H}^*_{\mathbb{T}}$-linear map of free modules of the same rank, such as $S^{-1}\iota^*$, is necessarily an isomorphism. □

Remark 7.5.4. Let X be a smooth complex projective variety, with an action by a torus $\mathbb{T}$ having finitely many fixed points $p_1,\ldots,p_s$. Then the Białynicki-Birula decomposition [24] yields s $\mathbb{T}$-invariant subvarieties

$$Y_1,\ldots,Y_s \subset X$$

with the property that the $\mathbb{T}$-equivariant cohomology classes (cf. Example 6.4.2)

$$[Y_\ell]^{\mathbb{T}} \in \mathrm{H}^*_{\mathbb{T}}(X)$$

form a *free* $\mathrm{H}^*_{\mathbb{T}}$*-basis* of the equivariant cohomology ring, restricting to a $\mathbb{Z}$-basis of the ordinary cohomology ring $\mathrm{H}^*(X)$. In other words,

$$\mathrm{H}^*_{\mathbb{T}}(X) \cong \bigoplus_{\ell=1}^{s} \mathrm{H}^*_{\mathbb{T}}\cdot [Y_\ell]^{\mathbb{T}}$$

is a free $\mathrm{H}^*_{\mathbb{T}}$-module of rank s.

The form of the localisation theorem that we will need is the following.

Corollary 7.5.5 (Integration Formula). *Let X be a smooth complex projective $\mathbb{T}$-variety with finitely many fixed points. Then for all $\psi \in \mathrm{H}^*_{\mathbb{T}}(X)$ there is an identity*

$$\int_X \psi = \sum_{q\in X^{\mathbb{T}}} \frac{i_q^*\psi}{e^{\mathbb{T}}(T_qX)}. \tag{7.5.3}$$

PROOF. Let $S \subseteq \mathrm{H}^*_{\mathbb{T}} \setminus 0$ be a multiplicative subset as in Theorem 7.5.3. By the surjectivity of

$$S^{-1}\iota_*\colon S^{-1}\mathrm{H}^*_{\mathbb{T}}(X^{\mathbb{T}}) \xrightarrow{\ \sim\ } S^{-1}\mathrm{H}^*_{\mathbb{T}}(X),$$

due to part (2) of Proposition 7.5.3 (which we may apply thanks to Remark 7.5.4), we may assume

$$\psi = \iota_{p*}\theta,$$

for $p\in X^{\mathbb{T}}$ and $\theta \in \mathrm{H}^*_{\mathbb{T}}$. Then clearly

$$\int_X \psi = \int_X \iota_{p*}\theta = \theta,$$

because $(\int_X)\circ\iota_{p*}$ is an isomorphism. On the other hand,

$$\sum_{q\in X^{\mathbb{T}}} \frac{i_q^*\iota_{p*}\theta}{e^{\mathbb{T}}(T_qX)} = \frac{\iota_p^*\iota_{p*}\theta}{e^{\mathbb{T}}(T_pX)} = \frac{e^{\mathbb{T}}(T_pX)}{e^{\mathbb{T}}(T_pX)} \cap \theta = \theta.$$ □

Remark 7.5.6. Formula (7.5.2) is true in the algebraic context as well (for X a smooth projective variety) if one sets $S = \mathrm{H}^*_{\mathbb{T}} \setminus 0$, but we will not deal with positive dimensional fixed loci in these notes. See e.g. [140].

Lemma 7.5.7. *Let M be a smooth oriented compact manifold with a torus action having finitely many fixed points $p_1, \ldots, p_m$. Then*

$$\chi(M) = m.$$

PROOF. Recall that $\chi(M) = \int_M e(T_M)$ by Poincaré–Hopf. We have

$$\int_M e(T_M) = \int_M e^{\mathbb{T}}(T_M) = \sum_{1 \le i \le m} \frac{e^{\mathbb{T}}(T_M)|_{p_i}}{e^{\mathbb{T}}(N_{p_i/M})} = \sum_{1 \le i \le m} 1 = m. \qquad \square$$

In the algebraic context, we can use the localisation formula to prove the following.

Proposition 7.5.8. *Let $\mathbb{T}$ be a torus acting on a smooth complex projective variety Y. Then*

$$\chi(Y) = \chi(Y^{\mathbb{T}}).$$

PROOF. By the ($\mathbb{T}$-equivariant) normal bundle exact sequence

$$0 \to T_{Y^{\mathbb{T}}} \to T_Y|_{Y^{\mathbb{T}}} \to N_{Y^{\mathbb{T}}/Y} \to 0,$$

we have an identity

$$e^{\mathbb{T}}(T_Y|_{Y^{\mathbb{T}}}) = e^{\mathbb{T}}(T_{Y^{\mathbb{T}}}) \cdot e^{\mathbb{T}}(N_{Y^{\mathbb{T}}/Y}). \tag{7.5.4}$$

We compute

$$\begin{aligned}
\chi(Y) &= \int_Y e(T_Y) && \text{by Poincaré–Hopf} \\
&= \int_Y e^{\mathbb{T}}(T_Y) && \text{by equivariance of } T_Y \\
&= \int_{Y^{\mathbb{T}}} \frac{e^{\mathbb{T}}(T_Y|_{Y^{\mathbb{T}}})}{e^{\mathbb{T}}(N_{Y^{\mathbb{T}}/Y})} && \text{by localisation} \\
&= \int_{Y^{\mathbb{T}}} e^{\mathbb{T}}(T_{Y^{\mathbb{T}}}) && \text{by Equation (7.5.4)} \\
&= \int_{Y^{\mathbb{T}}} e(T_{Y^{\mathbb{T}}}) && \text{by equivariance of } T_{Y^{\mathbb{T}}} \\
&= \chi(Y^{\mathbb{T}}) && \text{by Poincaré–Hopf,}
\end{aligned}$$

as required. $\square$

The last result holds in greater generality:

Lemma 7.5.9 ([41, Proposition 2.5.1])**.** *Let $\mathbb{T}$ be a torus acting on a quasiprojective scheme Y of finite type over $\mathbb{C}$. Then*

$$\chi(Y) = \chi(Y^{\mathbb{T}}).$$

The following result is also very useful in computations.

Lemma 7.5.10. *Let Y be a variety with a $\mathbb{T}$-action. Suppose the fixed locus $Y^{\mathbb{T}}$ is finite. Then there is a 1-dimensional subtorus $\mathbb{G}_m \subset \mathbb{T}$ such that $Y^{\mathbb{T}} = Y^{\mathbb{G}_m}$.*

PROOF. Let $Y^{\mathbb{T}} = \{ y_1, \ldots, y_e \}$ be the fixed locus. For each $i = 1, \ldots, e$, the tangent space $T_{y_i}Y$, as a representation, splits as a sum of characters $\chi_1^{(i)}, \ldots, \chi_{r_i}^{(i)}$. Consider the set $\Upsilon = \{ \chi_j^{(i)} \mid 1 \le i \le e,\, 1 \le j \le r_i \}$. Then, for every 1-parameter subgroup $\lambda \colon \mathbb{G}_m \to \mathbb{T}$, we have that $Y^{\mathbb{G}_m} = Y^{\mathbb{T}}$ if and only if $\chi \circ \lambda \neq 0 \in \mathbb{Z} = \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m)$ for every $\chi \in \Upsilon$. So it is enough to pick $\lambda \in \mathrm{Hom}(\mathbb{G}_m, \mathbb{T}) \cong \mathbb{Z}^{\dim \mathbb{T}}$ away from $|\Upsilon|$ hyperplanes. $\square$

EXERCISE 7.5.11. Compute the Euler characteristic of the Grassmannian $G(k,n)$. (**Hint**: Lift the standard torus action of $\mathbb{T} = \mathbb{G}_m^n$ on $\mathbb{C}^n$ to the Grassmannian).

EXERCISE 7.5.12. Let $n > 0$ be an integer. Show that

$$\sum_{i=1}^{n} \frac{(-s_i)^k}{\prod\limits_{\substack{1\leq j\leq n\\ j\neq i}} (s_j - s_i)} = \begin{cases} 0 & \text{if } 0 \leq k < n-1 \\ 1 & \text{if } k = n-1. \end{cases}$$

(**Hint**: Apply Equation (7.5.3) to $\psi = c_1^{\mathbb{T}}(\mathscr{O}_{\mathbb{P}^{n-1}}(1))$ having fixed the standard action on $X = \mathbb{P}^{n-1}$).

CHAPTER 8

Applications of the localisation formula

SUMMARY. In this chapter we give four complete examples on how to use Theorem 7.5.1 to solve enumerative problems. The work of Meireles Araújo and Vainsencher [**140**] was of great inspiration for the first three sections in this chapter, and we take the opportunity to refer the reader to loc. cit. for more examples of application of the localisation formula (upgraded to equivariant Chow theory) in enumerative geometry.

We work over $\mathbb{C}$ throughout.

8.1. How not to compute the simplest intersection number

In this section we show how to use localisation to compute the number of intersection points between two general lines in $\mathbb{P}^2$. Note that this problem is (equivalent, and) dual to the problem of counting lines through two general points in $\mathbb{P}^2$. This explains why we work with the usual weights on $\mathbb{P}^2$ instead of the dual weights on $\mathbb{P}^{2*} = \mathbb{G}(1,2)$, which would have required us to start out with $\mathrm{H}^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1))^*$ instead of $\mathrm{H}^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1))$.

Let $\mathbb{T} = \mathbb{G}_m \subset \mathbb{G}_m^3$ be a one-parameter subgroup acting with weights w_0, w_1, w_2 on the vector space of linear forms

$$V = \mathrm{H}^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1)) = \mathrm{Span}_{\mathbb{C}}\{\, x_0, x_1, x_2 \,\}.$$

This means that $t \cdot x_i = t^{w_i} x_i$. We might also use the notation

$$V = \bigoplus_{i=0}^{2} \mathbb{C} \cdot x_i \otimes t^{w_i}$$

if we were to emphasise the equivariant splitting. Choosing the w_i's distinct from one another ensures that the induced action on $\mathbb{P}^2$ has the three fixed points $p_0 = (1,0,0), p_1 = (0,1,0), p_2 = (0,0,1)$. The intersection number we want to compute is

$$\int_{\mathbb{P}^2} c_1(\mathscr{O}_{\mathbb{P}^2}(1))^2 = \int_{\mathbb{P}^2} c_1^{\mathbb{T}}(\mathscr{O}_{\mathbb{P}^2}(1))^2 = \sum_{i=0}^{2} \frac{c_1^{\mathbb{T}}(\mathscr{O}_{\mathbb{P}^2}(1)|_{p_i})^2}{e^{\mathbb{T}}(T_{p_i}\mathbb{P}^2)}, \tag{8.1.1}$$

where the last identity is the application of the localisation formula. The universal exact sequence

$$0 \to \mathscr{S} \to V \otimes_{\mathbb{C}} \mathscr{O}_{\mathbb{P}^2} \to \mathscr{O}_{\mathbb{P}^2}(1) \to 0$$

restricts, at p_i, to a short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_{jk} & \longrightarrow & V & \longrightarrow & V/V_{jk} & \longrightarrow & 0 \\ & & & & & & \| & & \\ & & & & & & \mathbb{C} \cdot x_i & & \end{array}$$

where $V_{jk} \subset V$ is the span of x_j, x_k, i.e. the space of linear forms vanishing at p_i. The weights of the tangent representations

$$T_{p_i}\mathbb{P}^2 = V_{jk}^* \otimes V/V_{jk} = \mathrm{Span}_{\mathbb{C}}\{\, x_j^* \otimes x_i, x_k^* \otimes x_i \,\}$$

are simply $w_i - w_j, w_i - w_k$, where $\{\, i,j,k \,\} = \{\, 0,1,2 \,\}$. The line bundle $\mathscr{O}_{\mathbb{P}^2}(1)$ is the universal quotient bundle and clearly

$$\mathscr{O}_{\mathbb{P}^2}(1)|_{p_i} = V/V_{jk} = \mathbb{C} \cdot x_i$$

has weight $c_1^{\mathbb{T}}(\mathscr{O}_{\mathbb{P}^2}(1)|_{p_i}) = w_i$. The sum in the right hand side of (8.1.1) then equals correct signs

$$\frac{w_0^2}{(w_0-w_1)(w_0-w_2)}+\frac{w_1^2}{(w_1-w_0)(w_1-w_2)}+\frac{w_2^2}{(w_2-w_0)(w_2-w_1)}. \tag{8.1.2}$$

The latter equals 1 for *every* choice of (pairwise distinct) w_0, w_1, w_2.

EXERCISE 8.1.1. Compare the latter calculation to the one of Exercise 7.5.12.

Remark 8.1.2. The fact that the sum (8.1.2) equals 1 has the following interpretation.[1] The three fractions in Equation (8.1.2) can be seen as the residues of the differential form

$$\frac{z^2\,\mathrm{d}z}{(z-w_0)(z-w_1)(z-w_2)}.$$

However, there is yet another residue to compute: the one at ∞. This residue equals -1. The residue theorem then precisely states that

$$0=-1+\frac{w_0^2}{(w_0-w_1)(w_0-w_2)}+\frac{w_1^2}{(w_1-w_0)(w_1-w_2)}+\frac{w_2^2}{(w_2-w_0)(w_2-w_1)}.$$

8.2. The 27 lines on a smooth cubic surface

Let $\mathbb{T}=\mathbb{G}_m$ be a torus acting on $\mathbb{P}^3$ with distinct weights (w_0,w_1,w_2,w_3). This means, as ever,

$$t\cdot x_i = t^{w_i}x_i,\quad 0\le i\le 3.$$

This is also equivalent to considering the vector space

$$V=\mathrm{H}^0(\mathbb{P}^3,\mathscr{O}_{\mathbb{P}^3}(1))=\mathrm{Span}_{\mathbb{C}}\{\,x_0,x_1,x_2,x_3\,\}$$

as an equivariant vector bundle over a point, splitting into a sum of characters $t\mapsto t^{w_i}$. The torus action has four fixed points $p_0,\dots,p_3\in\mathbb{P}^3$ and six invariant lines $\ell_{ij}\subset\mathbb{P}^3$ which are the lines joining the fixed points (see Figure 8.1). These correspond to the fixed points of the Grassmannian $\mathbb{G}(1,3)$ under the lifted $\mathbb{T}$-action. You computed the Euler characteristic of the Grassmannian in Exercise 7.5.11.

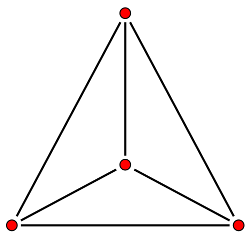

FIGURE 8.1. The toric polytope of $\mathbb{P}^3$.

Let S be a general cubic hypersurface in $\mathbb{P}^3$, defined by a general homogeneous cubic polynomial $f\in\mathbb{C}[x_0,x_1,x_2,x_3]_3\subset\mathrm{H}^0(\mathbb{P}^3,\mathscr{O}_{\mathbb{P}^3}(3))$. A line $\ell\subset\mathbb{P}^3$ is contained in S if and only if the image of f under the restriction map

$$\mathrm{H}^0(\mathbb{P}^3,\mathscr{O}_{\mathbb{P}^3}(3))\to\mathrm{H}^0(\ell,\mathscr{O}_\ell(3))$$

vanishes. We know from Section 4.5 that the cycle of lines in S is Poincaré dual to the top Chern class

$$e(\mathrm{Sym}^3\mathscr{S}^*)=c_4(\mathrm{Sym}^3\mathscr{S}^*)\in A^4\mathbb{G}(1,3),$$

where $\mathscr{S}^*$ is the dual of the tautological subbundle. By Lemma 4.5.4, we have

$$\#\{\,\text{lines in } S\,\}=\int_{\mathbb{G}(1,3)}e(\mathrm{Sym}^3\mathscr{S}^*).$$

[1]Thanks to Fran Globlek for pointing this out.

According to the strategy outlined in Section 6.2.1, we will compute the latter intersection number by computing instead the equivariant integral

$$\int_{\mathbb{G}(1,3)} e^{\mathbb{T}}(\operatorname{Sym}^3 \mathscr{S}^*) \in \mathcal{H}_{\mathbb{T}}$$

via the localisation formula, and specialising the equivariant parameters appropriately (i.e. avoiding the creation of poles in the localisation formula).

Restricting the tautological exact sequence

$$0 \to \mathscr{S} \to \mathscr{O}_{\mathbb{G}(1,3)} \otimes_{\mathbb{C}} V^* \to \mathscr{Q} \to 0 \tag{8.2.1}$$

to the point corresponding to the linear subspace

$$\ell_{ij} = \mathbb{C} \cdot x_i^* \oplus \mathbb{C} \cdot x_j^* \subset V^*,$$

we obtain the sequence of $\mathbb{T}$-representations

$$0 \to \mathbb{C} \cdot x_i^* \oplus \mathbb{C} \cdot x_j^* \to V^* \to \mathbb{C} \cdot x_h^* \oplus \mathbb{C} \cdot x_k^* \to 0$$

where $\{0,1,2,3\} = \{i,j,h,k\}$. Therefore

$$\mathscr{Q}\Big|_{\ell_{ij}} = \mathbb{C} \cdot x_h^* \oplus \mathbb{C} \cdot x_k^*$$

has weights $-w_h$, $-w_k$, and similarly

$$\mathscr{S}^*\Big|_{\ell_{ij}} = \mathbb{C} \cdot x_i \oplus \mathbb{C} \cdot x_j$$

has weights w_i and w_j. Since

$$T_{\ell_{ij}}\mathbb{G}(1,3) = \mathscr{S}^*|_{\ell_{ij}} \otimes \mathscr{Q}|_{\ell_{ij}} = \operatorname{Span}_{\mathbb{C}}\left\{ x_i \otimes x_h^*, x_j \otimes x_h^*, x_i \otimes x_k^*, x_j \otimes x_k^* \right\},$$

we obtain an identity

$$e^{\mathbb{T}}(T_{\ell_{ij}}\mathbb{G}(1,3)) = (w_i - w_h) \cdot (w_j - w_h) \cdot (w_i - w_k) \cdot (w_j - w_k) \in \mathrm{H}_{\mathbb{T}}^*. \tag{8.2.2}$$

This is one of the key ingredients in the localisation formula, which reads

$$\int_{\mathbb{G}(1,3)} e^{\mathbb{T}}(\operatorname{Sym}^3 \mathscr{S}^*) = \sum_{\ell_{ij}} \frac{e^{\mathbb{T}}(\operatorname{Sym}^3 \mathscr{S}^*)|_{\ell_{ij}}}{e^{\mathbb{T}}(T_{\ell_{ij}}\mathbb{G}(1,3))}. \tag{8.2.3}$$

The numerators in (8.2.3) are computed as follows. Note that $\operatorname{Sym}^3 \mathscr{S}^*|_{\ell_{ij}} = \operatorname{Sym}^3(\mathbb{C} \cdot x_i \oplus \mathbb{C} \cdot x_j)$ is the four dimensional vector space generated by the classes of $x_i^3, x_i^2 x_j, x_i x_j^2, x_j^3$. Using the weights w_i and w_j of $\mathscr{S}^*$ we find

$$e^{\mathbb{T}}(\operatorname{Sym}^3 \mathscr{S}^*)|_{\ell_{ij}} = (3w_i) \cdot (2w_i + w_j) \cdot (w_i + 2w_j) \cdot (3w_j) \in \mathrm{H}_{\mathbb{T}}^*.$$

We are now able to write down the right hand side of (8.2.3) as follows:

$$
\begin{aligned}
(8.2.4)\qquad \sum_{0\le i<j\le 3}\frac{e^{\mathbb{T}}(\mathrm{Sym}^3\mathscr{S}^*)|_{\ell_{ij}}}{e^{\mathbb{T}}(T_{\ell_{ij}}\mathbb{G}(1,3))} &= \sum_{0\le i<j\le 3}\frac{(3w_i)(2w_i+w_j)(w_i+2w_j)(3w_j)}{(w_i-w_h)(w_j-w_h)(w_i-w_k)(w_j-w_k)}\\
&= 9\frac{w_0(2w_0+w_1)(w_0+2w_1)w_1}{(w_0-w_2)(w_0-w_3)(w_1-w_2)(w_1-w_3)}\\
&+9\frac{w_0(2w_0+w_2)(w_0+2w_2)w_2}{(w_0-w_1)(w_0-w_3)(w_2-w_1)(w_2-w_3)}\\
&+9\frac{w_0(2w_0+w_3)(w_0+2w_3)w_3}{(w_0-w_1)(w_0-w_2)(w_3-w_1)(w_3-w_2)}\\
&+9\frac{w_1(2w_1+w_2)(w_1+2w_2)w_2}{(w_1-w_0)(w_1-w_3)(w_2-w_0)(w_2-w_3)}\\
&+9\frac{w_1(2w_1+w_3)(w_1+2w_3)w_3}{(w_1-w_0)(w_1-w_2)(w_3-w_0)(w_3-w_2)}\\
&+9\frac{w_2(2w_2+w_3)(w_2+2w_3)w_3}{(w_2-w_0)(w_2-w_1)(w_3-w_0)(w_3-w_1)}.
\end{aligned}
$$

This sum equals 27 for *every* choice of pairwise distinct w_i's. For instance, evaluating at $(w_0,w_1,w_2,w_3)=(0,2,-1,1)$ yields

$$\#\{\,\text{lines in } S\,\} = 9\left(0+0+0+0+\frac{40}{12}-\frac{1}{3}\right) = 30-3 = 27.$$

8.3. The 2875 ines on the quintic 3-fold

In this section we will prove the following result.

Theorem 8.3.1. *Let $Y\subset\mathbb{P}^4$ be a generic quintic 3-fold. Then Y contains exactly 2875 lines.*

Remark 8.3.2. Unlike the case of cubic surfaces, the statement is not true for *all* smooth quintic 3-folds. Indeed, by work of Albano–Katz [2], the Fermat quintic

$$\left\{\,x_0^5+x_1^5+x_2^5+x_3^5+x_4^5=0\,\right\}\subset\mathbb{P}^4$$

contains fifty 1-dimensional families of lines.

We will need the following auxiliary result.

Theorem 8.3.3 ([114, Theorem 2.1])**.** *Let Y be a generic quintic 3-fold. Then the open subscheme of the Hilbert scheme of Y parametrising smooth irreducible rational curves of degree d is finite, nonempty and reduced. If $C\subset Y$ is any such curve, then its normal bundle is $\mathscr{N}_{C/Y}=\mathscr{O}_C(-1)\oplus\mathscr{O}_C(-1)$.*

We should mention that the result on the normal bundle also holds for other local complete intersection Calabi–Yau 3-folds and various degrees $d\le 7$, see [114, Theorem 2.2] for the precise statement. See also the three appendices in Katz [115], where he works with $d\le 3$. There he computes the possible bidegrees of the normal bundle to a smooth rational curve of degree d inside a smooth quintic. In particular, he finds that the only admissible $\mathscr{N}_{C/Y}=\mathscr{O}_C(a)\oplus\mathscr{O}_C(-2-a)$ are

$$
\begin{aligned}
d=1 &\quad\Rightarrow\quad a\in\{-1,0,1\}\\
d=2 &\quad\Rightarrow\quad a\in\{-1,0,1,2\}\\
d=3 &\quad\Rightarrow\quad a\in\{-1,0,1,2,3,4,5\}.
\end{aligned}
$$

This variety of possible normal bundles is not available in the local (i.e. non compact) cases studied by Laufer, where the only possible normal bundles are $\mathscr{O}_C(a)\oplus\mathscr{O}_C(-2-a)$ with $a\in\{-1,0,1\}$ [122]. See also [32] for an explicit calculation of the normal bundle to *Laufer curves* in certain local Calabi–Yau 3-fold.

The role of Theorem 8.3.3, for us, is to make sure that the intersection number we compute via localisation really is the number we are after. The above theorem, in other words, plays the role of Lemma 4.5.4 that we needed for lines on a cubic surface. Indeed, all other possible normal bundles of smooth rational curves in Calabi–Yau 3-folds have a nonzero global section.

Back to counting lines on a generic quintic 3-fold. The ambient space we have to work in now is the 6-dimensional Grassmannian

$$G(2,5) = \mathbb{G}(1,4).$$

Let $\mathscr{S}$ be the rank 2 universal subbundle

$$\mathscr{S} \hookrightarrow \mathscr{O}_{\mathbb{G}(1,4)} \otimes_{\mathbb{C}} \mathrm{H}^0(\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}(1))^*.$$

Let $\mathbb{T} = \mathbb{G}_m^5$ be a torus acting as

$$t \cdot x_i = t_i \cdot x_i$$

on $\mathbb{P}^4$. This means, as ever, that $V = \mathrm{H}^0(\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}(1))$ is viewed as a $\mathbb{T}$-module with equivariant decomposition $\bigoplus_{0\le k\le 4} \mathbb{C} \cdot x_k \otimes t_k$. This action lifts to a $\mathbb{T}$-action on $\mathbb{G}(1,4)$, with 10 fixed points corresponding to the subsets

$$I \subset \{0,1,2,3,4\}, \quad |I| = 2.$$

Let $\ell_{ij} \subset V^*$ be the linear subspace spanned by x_i^* and x_j^*. The characters of the 2-dimensional $\mathbb{T}$-representation

$$\mathscr{S}\big|_{\ell_{ij}} = \mathbb{C} \cdot x_i^* \oplus \mathbb{C} \cdot x_j^*$$

are $\{-\chi_i, -\chi_j\}$, where $\chi_i \colon \mathbb{T} \to \mathbb{C}^\times$ sends $t \mapsto t_i$. Similarly, we have that

$$\text{characters of } \mathscr{Q}\big|_{\ell_{ij}} = \{-\chi_a, -\chi_b, -\chi_c\}$$

where $\{a,b,c,i,j\} = \{0,1,2,3,4\}$. Let $s_k \in \mathrm{H}^2_{\mathbb{T}}$ denote the weight of $-\chi_k$, for all k. The tangent space of $\mathbb{G}(1,4)$ at ℓ_{ij} is, as ever,

$$\mathscr{S}^*\big|_{\ell_{ij}} \otimes \mathscr{Q}\big|_{\ell_{ij}} = \mathrm{Span}_{\mathbb{C}} \left\{ x_i \otimes x_u^*, x_j \otimes x_u^* \,\middle|\, u \in \{a,b,c\} \right\},$$

so we obtain the Euler class

$$e^{\mathbb{T}}(T_{\ell_{ij}}\mathbb{G}(1,4)) = \prod_{k \notin I} (s_i - s_k)(s_j - s_k).$$

The rank 6 vector bundle

$$\begin{array}{ccc} \mathrm{Sym}^5(\mathscr{S}^*) & & \mathrm{Sym}^5(\mathscr{S}^*)|_{\ell_{ij}} \\ \downarrow & \rightsquigarrow & \downarrow \\ \mathbb{G}(1,4) & & \{\ell_{ij}\} \end{array}$$

inherits weights

$$5s_j, s_i + 4s_j, 2s_i + 3s_j, 3s_i + 2s_j, 4s_i + s_j, 5s_i.$$

Therefore, denoting i_1 and i_2 the generic elements of a subset $I \subset \{0,1,2,3,4\}$ of size 2, the localisation formula reads

$$\int_{\mathbb{G}(1,4)} e^{\mathbb{T}}(\mathrm{Sym}^5 \mathscr{S}^*) = \sum_{|I|=2} \frac{\prod_{h=0}^5 (hs_{i_1} + (5-h)s_{i_2})}{\prod_{i\in I} \prod_{k \notin I} (s_i - s_k)} = 2875. \tag{8.3.1}$$

If you do not believe the last identity, you can copy the following code[2] in Mathematica:

[2]Thanks to Matteo Gallet for providing the code.

```
Plus @@
(Product[h Subscript[t, #[[1]]] + (5 - h)
Subscript[t, #[[2]]], {h, 0, 5}]/
Product[Subscript[t, i] - Subscript[t, k], {i, #},
{k, DeleteCases[Range[0, 4], Alternatives @@ #]}]
& /@ Subsets[Range[0, 4], {2}])
// Together
```

Theorem 8.3.1 now follows from Theorem 8.3.3.

8.4. The degree of $\mathbb{G}(1,n+1)$ and its enumerative meaning

Set $n \geq 2$. We shall work with the $2n$-dimensional Grassmannian $\mathbb{G} = G(2,n+2) = G(2,V) = \mathbb{G}(1,n+1)$, where $V = \mathrm{H}^0(\mathbb{P}^{n+1},\mathscr{O}_{\mathbb{P}^{n+1}}(1))$. The goal of this section is to prove the following.

Proposition 8.4.1. *Set $n \geq 2$. There are exactly $C_n = \frac{1}{n+1}\binom{2n}{n}$ lines $\ell \subset \mathbb{P}^{n+1}$ that are incident to $2n$ general $(n-1)$-planes $\Lambda_1,\ldots,\Lambda_{2n} \subset \mathbb{P}^{n+1}$. For instance, there are exactly 2 lines $\ell \subset \mathbb{P}^3$ intersecting four general lines in projective 3-space.*

Let L be a general 2-dimensional linear subspace of V. The cycle

$$\Sigma_1(L) = \{\, \Lambda \subset V \mid \Lambda \cap L \neq 0 \,\} \subset \mathbb{G}$$

has codimension 1, and its cohomology class $\sigma_1 = [\Sigma_1(L)] \in A^1\mathbb{G}$ is independent of L. In fact, we have $\sigma_1 = c_1(\mathscr{S}^*)$ by Remark 4.1.6, where $\mathscr{S}$ is the universal subbundle. We wish to prove (via localisation) the identity

$$\int_{\mathbb{G}(1,n+1)} \sigma_1^{2n} = C_n, \tag{8.4.1}$$

which in fact was already sketched in Equation (3.4.1). Let $\mathbb{T} = (\mathbb{C}^\times)^{n+2}$ act with distinct weights $w_0,w_1,\ldots,w_{n+1}$ on

$$V = \mathrm{H}^0(\mathbb{P}^{n+1},\mathscr{O}_{\mathbb{P}^{n+1}}(1)) = \bigoplus_{0\leq j\leq n+1} \mathbb{C}\cdot x_j.$$

We have, as ever, $\mathscr{S}^*|_{\ell_{ij}} = \mathbb{C}\cdot x_i \oplus \mathbb{C}\cdot x_j$ carrying the weight w_i+w_j and $T_{\ell_{ij}}\mathbb{G} = \mathscr{S}^*|_{\ell_{ij}} \otimes \mathscr{Q}|_{\ell_{ij}}$, equivariantly isomorphic to

$$\mathrm{Span}_{\mathbb{C}}\left\{\, x_i \otimes x_u^*, x_j \otimes x_u^* \;\middle|\; u \in I \setminus \{\, i,j \,\}\,\right\},$$

where $I = \{\,0,1,\ldots,n+1\,\}$. We compute, by localisation,

$$\begin{aligned}\int_{\mathbb{G}(1,n+1)} \sigma_1^{2n} = \int_{\mathbb{G}(1,n+1)} c_1^{\mathbb{T}}(\mathscr{S}^*)^{2n} &= \sum_{\ell_{ij}} \frac{c_1^{\mathbb{T}}(\mathscr{S}^*)|_{\ell_{ij}}^{2n}}{e^{\mathbb{T}}(T_{\ell_{ij}}\mathbb{G}(1,n+1))} \\ &= \sum_{0\leq i<j\leq n+1} \frac{(w_i+w_j)^{2n}}{\prod_{u\in I\setminus\{i,j\}}(w_i-w_u)(w_j-w_u)}.\end{aligned}$$

Checking that the result agrees with with the n-th Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}$$

is left as a combinatorial exercise. However, as an example, set $n = 2$. We find

$$\begin{aligned}\int_{\mathbb{G}(1,3)} \sigma_1^4 &= \sum_{0\leq i<j\leq 3} \frac{(w_i+w_j)^4}{(w_i-w_h)(w_j-w_h)(w_i-w_k)(w_j-w_k)} \\ &= \frac{(w_2+w_3)^4}{(w_2-w_0)(w_2-w_1)(w_3-w_0)(w_3-w_1)} \\ &+ \frac{(w_1+w_3)^4}{(w_1-w_0)(w_1-w_2)(w_3-w_0)(w_3-w_2)} \\ &+ \frac{(w_1+w_2)^4}{(w_1-w_0)(w_1-w_3)(w_2-w_0)(w_2-w_3)} \\ &+ \frac{(w_0+w_3)^4}{(w_0-w_1)(w_0-w_2)(w_3-w_1)(w_3-w_2)} \\ &+ \frac{(w_0+w_2)^4}{(w_0-w_1)(w_0-w_3)(w_2-w_1)(w_2-w_3)} \\ &+ \frac{(w_0+w_1)^4}{(w_0-w_2)(w_0-w_3)(w_1-w_2)(w_1-w_3)}.\end{aligned}$$

Specialising the equivariant parameters at, say, $(w_0, w_1, w_2, w_3) = (-2, 1, -1, 2)$, we obtain

$$\begin{aligned}\int_{\mathbb{G}(1,3)} \sigma_1^4 &= \frac{1^4}{1\cdot(-2)\cdot 4\cdot 1} + \frac{3^4}{3\cdot 2\cdot 4\cdot 3} + 0 + 0 \\ &+ \frac{(-3)^4}{(-3)\cdot(-4)\cdot(-2)\cdot(-3)} + \frac{(-1)^4}{(-1)\cdot(-4)\cdot 2\cdot(-1)} \\ &= -\frac{1}{8} + \frac{81}{72} + \frac{81}{72} - \frac{1}{8} = \frac{-1+9+9-1}{8} = 2,\end{aligned} \tag{8.4.2}$$

in agreement with the calculation of Proposition 3.3.4.

In order to complete the proof of Proposition 8.4.1, we have to address a transversality issue. In other words, we need to exclude the following cases, which are not automatically ruled out by the computation we just completed:

(1) The answer is a number smaller than C_n, i.e. some lines come with multiplicity;
(2) The answer is ∞.

In other words, what we know so far is the following: *if* the number of lines through $2n$ general $(n-1)$-dimensional planes in $\mathbb{P}^{n+1}$ is finite, then it does not exceed C_n. We employ Kleiman transversality to rule out these two degenerate cases at the same time. More precisely, we apply Theorem 3.3.3 to $X = \mathbb{G}(1, n+1)$, acted on by the affine algebraic group $G = \mathrm{GL}(V) = \mathrm{GL}_{n+2}$. For general $(n-1)$-planes Λ_1, $\Lambda_2 \subset \mathbb{P}^{n+1}$, form the codimension 1 cycles

$$\Gamma_i = \left\{ \ell \subset \mathbb{P}^{n+1} \,\middle|\, \ell \cap \Lambda_i \neq \emptyset \right\}, \quad i = 1, 2.$$

Then, in the notation of Theorem 3.3.3, set $Y = \Gamma_1$ and $Z = \Gamma_2$ and apply the result repeatedly.

The proof of Proposition 8.4.1 is now complete.

EXERCISE 8.4.2. Let $C_1, \ldots, C_4 \subset \mathbb{P}^3$ be general translated of curves of degree $d_1, \ldots, d_4$. Compute the number of lines $\ell \subset \mathbb{P}^3$ passing through $C_1, \ldots, C_4$. (**Hint**: Start out by proving that the cycle $\Gamma_C \subset \mathbb{G}(1,3)$ of lines meeting a curve $C \subset \mathbb{P}^3$ of degree d is a divisor in the Grassmannian, with cohomology class $d \cdot \sigma_1$).

8.5. The Euler characteristic of the Hilbert scheme of points

In this chapter we again work with the Hilbert scheme

$$\mathrm{Hilb}^n \mathbb{A}^d.$$

We view it as a fine moduli space of ideals $I \subset \mathbb{C}[x_1,\dots,x_d]$ of colength n, i.e. such that the quotient algebra $\dim_{\mathbb{C}} \mathbb{C}[x_1,\dots,x_d]/I$ has dimension n as a complex vector space. Computing its Euler characteristic via torus localisation will be the basic building block for the computation of the topological Euler characteristic

$$\chi(\mathrm{Hilb}^n X),$$

where X is an arbitrary smooth quasiprojective variety of dimension d (cf. Theorem 8.5.15).

8.5.1. The torus action. Consider the d-dimensional torus

$$\mathbb{T} = \mathbb{G}_m^d$$

acting on $\mathbb{A}^d$ by

$$t \cdot (a_1,\dots,a_d) = (t_1^{-1}a_1,\dots,t_d^{-1}a_d). \tag{8.5.1}$$

EXERCISE 8.5.1. Show that the action (8.5.1) lifts to a $\mathbb{T}$-action $\mathbb{T} \times \mathrm{Hilb}^n \mathbb{A}^d \to \mathrm{Hilb}^n \mathbb{A}^d$. Upgrade this replacing $\mathbb{A}^d$ with any smooth toric d-fold (If in need of a hint, open [**177**, Section 4.2] or [**62**, Section 9.1]).

EXERCISE 8.5.2. Show that a $\mathbb{T}$-fixed subscheme $Z \subset \mathbb{A}^d$ is entirely supported at the origin $0 \in \mathbb{A}^d$. (**Hint**: Show that $\mathrm{Supp}(t \cdot [Z]) = t^{-1} \cdot \mathrm{Supp}(Z)$).

Proposition 8.5.3. *An ideal $I \in \mathrm{Hilb}^n \mathbb{A}^d$ is $\mathbb{T}$-fixed if and only if it is a monomial ideal.*

PROOF. Recall that the character lattice of the torus $\mathbb{T}^* = \mathrm{Hom}(\mathbb{T},\mathbb{G}_m)$ is isomorphic to $\mathbb{Z}^d$, since each character $\mathbb{T} \to \mathbb{G}_m$ is necessarily of the form

$$\chi_{\mathsf{m}} \colon (t_1,\dots,t_d) \mapsto t_1^{m_1}\cdots t_d^{m_d}$$

for some $\mathsf{m} = (m_1,\dots,m_d) \in \mathbb{Z}^d$. As an initial step, we notice that the geometric action (8.5.1) dualises to a $\mathbb{T}$- action on $\mathbb{C}[x_1,\dots,x_d]$ via the rule

$$t \cdot f = f \circ t. \tag{8.5.2}$$

This already shows that a monomial ideal is necessarily $\mathbb{T}$-fixed, so it remains to prove the converse.

We next show that the monomials

$$\mathbf{x}^{\mathsf{m}} = x_1^{m_1}\cdots x_d^{m_d}$$

form an eigenbasis of $\mathbb{C}[x_1,\dots,x_d]$ as a $\mathbb{T}$-representation. An *eigenvector* of a $\mathbb{T}$-representation V, in this context, is an element $v \in V$ for which there exists a character $\chi \in \mathbb{T}^*$ such that $t \cdot v = \chi(t)v$ for all $t \in \mathbb{T}$. (This χ plays the role of "classical" eigenvalues in linear algebra.) For us, $V = \mathbb{C}[x_1,\dots,x_d]$. Pick $v = \mathbf{x}^{\mathsf{m}}$. Then according to the rule (8.5.2) one has

$$t \cdot \mathbf{x}^{\mathsf{m}} = (t_1x_1)^{m_1}\cdots(t_dx_d)^{m_d} = (t_1^{m_1}\cdots t_d^{m_d})\cdot(x_1^{m_1}\cdots x_d^{m_d}) = \chi_{\mathsf{m}}(t)\cdot\mathbf{x}^{\mathsf{m}}.$$

So each monomial $\mathbf{x}^{\mathsf{m}}$ is an eigenvector with respect to the weight χ_{m}. In particular, each corresponds to a different weight, therefore all weight spaces

$$V_{\mathsf{m}} = \{\, f \in \mathbb{C}[x_1,\dots,x_d] \mid t \cdot f = \chi_{\mathsf{m}}(t)f \text{ for all } t \in \mathbb{T}\,\}$$

are 1-dimensional $\mathbb{T}$-subrepresentations (each spanned by $\mathbf{x}^{\mathsf{m}}$) and the action (8.5.2) is diagonalisable by monomials.

Now pick a $\mathbb{T}$-fixed ideal $I \subset \mathbb{C}[x_1,\dots,x_d]$. In particular, as a vector space, I is a $\mathbb{T}$-subrepresentation. But a $\mathbb{T}$-subrepresentation of a diagonalisable $\mathbb{T}$-representation is again diagonalisable (prove this!), so I has a basis of eigenvectors. But each eigenvector is of the form $f = \mathbf{x}^{\mathsf{m}} \cdot g$, where $g \in \mathbb{C}[u]$ satisfies $g(0) \neq 0$. Since $V(I)$ is entirely supported at the origin (cf. Exercise 8.5.2), the zero locus of I is disjoint from the set of $(x_1,\dots,x_d)$ such that $g(x_1\cdots x_d) = 0$. It follows from Hilbert's Nullstellensatz that the monomial $\mathbf{x}^{\mathsf{m}} \in I$. $\square$

Proposition 8.5.4 ([35, Proposition 4.15]). *All monomial ideals $I \subset \mathbb{C}[x_1,\ldots,x_d]$ of colength n lie in the smoothable component of* $\mathrm{Hilb}^n \mathbb{A}^d$.

DEFINITION 8.5.5. Let $d \geq 1$ and $n \geq 0$ be integers. A $(d-1)$-dimensional partition of n is a collection of n points $\mathcal{A} = \{\mathbf{a}_1,\ldots,\mathbf{a}_n\} \subset \mathbb{N}^d$ with the following property: if $\mathbf{a}_i = (a_{i1},\ldots,a_{id}) \in \mathcal{A}$, then whenever a point $\mathbf{y} = (y_1,\ldots,y_d) \in \mathbb{N}^d$ satisfies $0 \leq y_j \leq a_{ij}$ for all $j = 1,\ldots,d$, one has $\mathbf{y} \in \mathcal{A}$. We set

$$\mathsf{P}_{d-1}(n) = \big|\{\, (d-1)\text{-dimensional partitions of } n \,\}\big|.$$

EXERCISE 8.5.6. Show that $\mathsf{P}_0(n) = 1$ for all $n \geq 0$.

Notation 8.5.7. We indicate a classical (i.e. 1-dimensional) partition by

$$\alpha = (1^{\alpha_1} \cdots i^{\alpha_i} \cdots \ell^{\alpha_\ell}). \tag{8.5.3}$$

The notation means that there are α_i parts of length i, and we set

$$||\alpha|| = \sum_i \alpha_i, \quad |\alpha| = \sum_i i\alpha_i.$$

The latter is the *size* of α, the former is the number of *distinct parts* of α. The number $\ell = \ell(\alpha)$ is the *length* of α.

Example 8.5.8. Set $d = 2$. Then a 1-dimensional partition is the same thing as a Young diagram (see Figure 8.2). If $d = 3$, a *plane partition* of n is the same thing as a way of stacking n boxes in the corner of a room (assuming gravity points in the $(-1,-1,-1)$ direction!). See Figure 8.3 for a visual explanation.

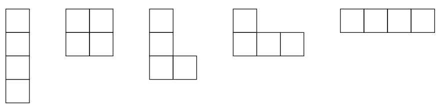

FIGURE 8.2. The five partitions $\alpha_1,\ldots,\alpha_5$ of 4. In the notation (8.5.3), $\alpha_1 = (4^1)$, $\alpha_2 = (2^2)$, $\alpha_3 = (1^1 3^1)$, $\alpha_4 = (1^2 2^1)$, $\alpha_5 = (1^4)$.

Note that, with our conventions, we have

$$||\alpha|| = \text{ number of columns in the corresponding Young diagram.}$$

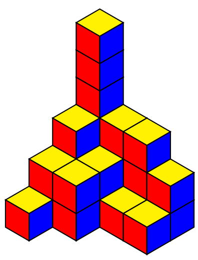

FIGURE 8.3. A plane partition.

The generating function of the numbers of partitions (for $d = 2$) is given by the following formula.

Theorem 8.5.9 (Euler [57, Chapter 16]). *There is an identity*

$$\sum_{n\geq 0} \mathsf{P}_1(n) q^n = \prod_{m\geq 1} (1-q^m)^{-1}.$$

PROOF. By inspection. For instance, say we want to compute $\mathsf{P}_1(4)$, that we know equals 5 (cf. Figure 8.2). Then expanding

$$\prod_{m\geq 1}(1-q^m)^{-1} = (1+q^1+q^{1+1}+q^{1+1+1}+q^{1+1+1+1}+\cdots)$$
$$\cdot(1+q^2+q^{2+2}+q^{2+2+2}+\cdots)\cdot(1+q^3+q^{3+3}+\cdots)\cdot(1+q^4+q^{4+4}+\cdots)\cdots$$

we see that to compute the coefficient of q^4 we have to sum the coefficients of

$$\begin{gathered} q^4 \\ q^1\cdot q^3 \\ q^{1+1}\cdot q^2 \\ q^{1+1+1+1} \\ q^{2+2}. \end{gathered}$$

These clearly correspond to partitions of 4. □

DEFINITION 8.5.10. The *MacMahon function* is the infinite product

$$\mathsf{M}(q) = \prod_{m\geq 1}(1-q^m)^{-m}.$$

Theorem 8.5.11 (MacMahon [131]). *There is an identity*

$$\sum_{n\geq 0}\mathsf{P}_2(n)q^n = \mathsf{M}(q).$$

Next, we will see that monomial ideals correspond to partitions. This is best explained via a picture — see Figure 8.4.

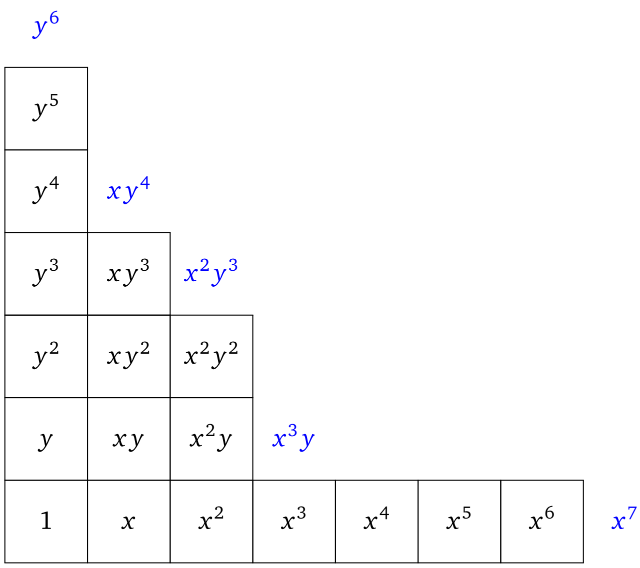

FIGURE 8.4. A 1-dimensional partition (Young diagram) draws a staircase and determines (and is determined by) a monomial ideal, in this case $I_\lambda = (y^6, xy^4, x^2y^3, x^3y, x^7)$. Note that the colength of I_λ is $|\lambda| = 17$.

EXERCISE 8.5.12. Show that there is a bijective correspondence between $\mathbb{T}$-fixed subschemes $Z\subset\mathbb{A}^d$ of length n and $(d-1)$-dimensional partitions of n.

8.5.2. Euler characteristic of Hilbert schemes. Let X be a smooth quasiprojective variety of dimension d. In this section we give a formula for the series

$$\mathsf{H}_X(q) = \sum_{n\geq 0} \chi(\operatorname{Hilb}^n X) q^n.$$

We use a combinatorial lemma, combined with a stratification argument. The following lemma will also be used repeatedly.

Lemma 8.5.13. *Let $g: X \to S$ be a morphism of algebraic varieties over $\mathbb{C}$ such that the Euler characteristic of the fibres $\chi(X_s) = m$ does not depend on $s \in S$. Then*

$$\chi(X) = \chi(S)\cdot m.$$

Proof. We expand the argument from [89, Section 2.1] for completeness. A consequence of [201, Corollaire 5.1] is that one can stratify S as a disjoint union of locally closed subvarieties $S_i \subset S$ such that $g^{-1}(S_i) \to S_i$ is Zariski locally trivial. Then we are left with proving the statement for a Zariski locally trivial fibration $X \to S$. In this case, one can cover S with open subvarieties $U_j \subset S$ such that $g^{-1}(U_j) \to U_j$ is isomorphic to the projection $U_j \times F \to U_j$, for some variety F. Thus $\chi(g^{-1}(U_j)) = \chi(U_j)\cdot\chi(F) = \chi(U_j)\cdot m$. The covering $S = \bigcup_j U_j$ can be refined to a locally closed stratification, with each stratum contained in some U_j. The result then follows from the additivity property $\chi(U \amalg V) = \chi(U) + \chi(V)$ of the Euler characteristic. $\square$

Lemma 8.5.14 ([186, p. 40])**.** *Let $P(q) = 1 + \sum_{n>0} p_n q^n \in \mathbb{Q}[\![q]\!]$ be a formal power series. If χ is an integer, then*

$$P(q)^\chi = 1 + \sum_\alpha \left(\prod_{j=0}^{\|\alpha\|-1} (\chi - j) \cdot \frac{\prod_i p_i^{\alpha_i}}{\prod_i \alpha_i!} \right) q^{|\alpha|}.$$

Theorem 8.5.15. *Let X be a smooth quasiprojective variety of dimension d. There is an identity*

$$\mathsf{H}_X(q) = \left(\sum_{n\geq 0} \mathsf{P}_{d-1}(n) q^n \right)^{\chi(X)}.$$

Proof. The case $X = \mathbb{A}^d$ follows from the observation that

$$\chi(\operatorname{Hilb}^n \mathbb{A}^d) = \left|(\operatorname{Hilb}^n \mathbb{A}^d)^{\mathbb{T}}\right| = \mathsf{P}_{d-1}(n) \tag{8.5.4}$$

along with $\chi(\mathbb{A}^d) = 1$. The identities (8.5.4) follow from Lemma 7.5.9 and Exercise 8.5.12.

For general X, we proceed as follows. First of all, notice that

$$\chi(\operatorname{Hilb}^n(\mathbb{A}^d)_0) = \mathsf{P}_{d-1}(n),$$

because the punctual Hilbert scheme is $\mathbb{T}$-invariant and contains the $\mathbb{T}$-fixed locus (Exercise 8.5.2). Let us stratify the symmetric product

$$\operatorname{Sym}^n X = \coprod_{\alpha \vdash n} \operatorname{Sym}^n_\alpha X$$

according to partitions of n. Each stratum dictates the multiplicity of the supporting points in a given zero-cycle. Set

$$\operatorname{Hilb}^n_\alpha X = \pi_X^{-1}(\operatorname{Sym}^n_\alpha X),$$

where π_X is the Hilbert–Chow morphism (5.1.1). On the deepest stratum, corresponding to the full partition (n), i.e. to the small diagonal $X \hookrightarrow \operatorname{Sym}^n X$, we have that

$$\operatorname{Hilb}^n_{(n)} X \to X$$

is Zariski locally trivial with fibre $\operatorname{Hilb}^n(\mathbb{A}^d)_0$. This follows easily from the local case, where in fact there is a global decomposition

$$\operatorname{Hilb}^n_{(n)} \mathbb{A}^d = \mathbb{A}^d \times \operatorname{Hilb}^n(\mathbb{A}^d)_0,$$

and $\pi_{\mathbb{A}^d}$ is identified with the first projection. For an arbitrary partition α, let

$$V_\alpha \subset \prod_i (\mathrm{Hilb}^i X)^{\alpha_i}$$

be the open locus of clusters with pairwise disjoint support.

EXERCISE 8.5.16. Use the infinitesimal criterion for étale maps to show that taking the 'union of points' defines an étale morphism

$$f_\alpha \colon V_\alpha \to \mathrm{Hilb}^n X.$$

Let U_α denote the image of the morphism f_α. Then U_α contains the stratum $\mathrm{Hilb}^n_\alpha X$ as a closed subscheme. We can form the fibre diagram

$$\begin{tikzcd} Z_\alpha \arrow[r, hook] \arrow[d, "f_\alpha"'] \arrow[dr, phantom, "\square"] & V_\alpha \arrow[r, hook] \arrow[d, "\text{étale}"] & \prod_i (\mathrm{Hilb}^i X)^{\alpha_i} \\ \mathrm{Hilb}^n_\alpha X \arrow[r, hook] & U_\alpha \arrow[r, hook] & \mathrm{Hilb}^n X \end{tikzcd}$$

defining the scheme Z_α. Now the map f_α on the left is a finite étale G_α-cover, where $G_\alpha = \prod_i \mathfrak{S}_{\alpha_i}$ is the automorphism group of the partition α. In other words, the only difference between Z_α and $\mathrm{Hilb}^n_\alpha X$ is the relabelling of points that appear with the same α-multiplicity. So we find

$$\begin{aligned} \chi(\mathrm{Hilb}^n X) &= \sum_{\alpha \vdash n} \chi(\mathrm{Hilb}^n_\alpha X) \\ &= \sum_{\alpha \vdash n} \frac{\chi(Z_\alpha)}{|G_\alpha|} \\ &= \sum_{\alpha \vdash n} \frac{\chi(Z_\alpha)}{\prod_i \alpha_i!}. \end{aligned}$$

On the other hand, it is easy to see that

$$Z_\alpha = \prod_i \mathrm{Hilb}^i_{(i)}(X)^{\alpha_i} \setminus \Delta^{\mathrm{big}}$$

also fits in a cartesian diagram

$$\begin{tikzcd} Z_\alpha \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \prod_i \mathrm{Hilb}^i_{(i)}(X)^{\alpha_i} \arrow[d, "p"] \\ X^{||\alpha||} \setminus \Delta^{\mathrm{big}} \arrow[r, hook] & X^{||\alpha||} \end{tikzcd}$$

where $||\alpha||$ is the number of distinct parts in the partition. Now p is a product of Zariski locally trivial fibrations with fibre $\mathrm{Hilb}^i(\mathbb{A}^d)_0$, therefore

$$\begin{aligned} \chi(Z_\alpha) &= \chi(X^{||\alpha||} \setminus \Delta^{\mathrm{big}}) \cdot \prod_i \chi(\mathrm{Hilb}^i(\mathbb{A}^d)_0)^{\alpha_i} \\ &= \prod_{j=0}^{||\alpha||-1} (\chi(X) - j) \cdot \prod_i \mathsf{P}_{d-1}(i)^{\alpha_i}. \end{aligned}$$

Putting everything together, we find

$$\mathsf{H}_X(q) = \sum_\alpha \left(\prod_{j=0}^{||\alpha||-1} (\chi(X) - j) \cdot \frac{\prod_i \mathsf{P}_{d-1}(i)^{\alpha_i}}{\prod_i \alpha_i!} \right) q^{|\alpha|}.$$

The result now follows from the combinatorial formula of Lemma 8.5.14. $\square$

The identity of Theorem 8.5.15 specialises to the following in low dimension.

Corollary 8.5.17. *Let X be a smooth quasiprojective variety of dimension $d = 1,2$ or 3. Then*

$$\sum_{n\geq 0}\chi(\mathrm{Hilb}^n X)q^n = \begin{cases} (1-q)^{-\chi(X)} & \text{if } d=1 \\ \prod_{m\geq 1}(1-q^m)^{-\chi(X)} & \text{if } d=2 \\ \prod_{m\geq 1}(1-q^m)^{-m\chi(X)} & \text{if } d=3 \end{cases}$$

PROOF. For the case $d = 2$ use Euler's formula, for the case $d = 3$ use MacMahon's result (Theorem 8.5.11). □

Example 8.5.18. For $X = \mathbb{P}^2$, we get

$$\chi(\mathrm{Hilb}^n \mathbb{P}^2) = \sum_{\substack{n_1+n_2+n_3=n \\ n_i\geq 0}} \mathsf{P}_1(n_1)\mathsf{P}_1(n_2)\mathsf{P}_1(n_3).$$

Remark 8.5.19. The fact that, if C is a smooth curve, one has

$$\sum_{n\geq 0}\chi(\mathrm{Sym}^n C)q^n = (1-q)^{-\chi(C)},$$

was shown directly by Macdonald [**130**]. See Göttsche [**79**] for the case of a surface and Cheah [**39**] for a 3-fold.

Fact 8.5.20. The relation $\mathsf{H}_X(q) = \mathsf{H}_{\mathbb{A}^d}(q)^{\chi(X)}$ is true for every smooth quasiprojective variety of dimension d, but there is no infinite product formula available for $\mathsf{H}_{\mathbb{A}^d}(q)$ if $d \geq 4$. There also is no closed formula for the number $\mathsf{P}_1(n)$ of Young diagrams of size n, or for the number $\mathsf{P}_2(n)$ of plane partitions of size n. However, for $d = 1,2$ there are recursive formulae

$$n\mathsf{P}_d(n) = \sum_{k=0}^{n}\sigma_d(k)\mathsf{P}_d(n-k),$$

where $\sigma_d(m) = \sum_{\ell|m}\ell^d$. The formula for $d = 1$ was proved by Hardy–Ramanujan [**94**] in 1918, and by Erdös [**56**] in 1942. For the $d = 2$ case, see Example 2 in [**5**, Chapter 6].

Remark 8.5.21. The stratification argument that we just described is quite standard. It has been used for instance in [**20**, **173**, **91**, **14**] for numerical invariants and in [**18**, **176**, **174**] for motivic invariants. In fact, the stratification argument applied to the Quot scheme

$$\mathrm{Quot}_X(F) = \{\text{ finite length quotients } F \twoheadrightarrow T\ \}$$

of a locally free sheaf F of rank r on a smooth quasiprojective variety X, yields the relation

$$\sum_{n\geq 0}\chi(\mathrm{Quot}_X(F,n))q^n = \left(\sum_{n\geq 0}\chi(\mathrm{Quot}_X(F,n)_0)q^n\right)^{\chi(X)}, \tag{8.5.5}$$

where $\mathrm{Quot}_X(F,n)_0$ is the *punctual Quot scheme*, i.e. the preimage of any cycle of the form $n\cdot x$ under the Quot-to-Chow map $\mathrm{Quot}_X(F,n) \to \mathrm{Sym}^n X$ (cf. **??**). It is easy to verify that the right hand side can be rewritten in terms of the rank 1 Quot schemes (i.e. Hilbert schemes).

EXERCISE 8.5.22. Show that, if $r = \mathrm{rk}\, F$, then

$$\sum_{n\geq 0}\chi(\mathrm{Quot}_X(F,n)_0)q^n = \left(\sum_{n\geq 0}\chi(\mathrm{Hilb}^n(\mathbb{A}^d)_0)q^n\right)^r.$$

(**Hint**: first prove that $\mathrm{Quot}_X(F,n)_0 \cong \mathrm{Quot}_X(\mathscr{O}^{\oplus r},n)_0$, then use the $(\mathbb{C}^\times)^r$-action on the latter punctual Quot scheme. You are allowed to use Bifet's theorem [**25**]

$$\mathrm{Quot}_X(\mathscr{O}^{\oplus r},n)_0^{(\mathbb{C}^\times)^r} \cong \coprod_{n_1+\cdots+n_r=n} \prod_{i=1}^{r} \mathrm{Hilb}^{n_i}(X)_0$$

to conclude).

For instance, for $\dim X = 3$, combining Exercise 8.5.22 and (8.5.5) we find

$$\sum_{n\geq 0} \chi(\mathrm{Quot}_X(F,n))q^n = \mathsf{M}(q)^{r\chi(X)}.$$

EXERCISE 8.5.23. Let Y be an abelian variety. Show directly (i.e. without using Corollary 8.5.17) that, if $n>0$, then $\chi(\mathrm{Hilb}^n Y) = 0$.

CHAPTER 9

The toy model for the virtual class and its localisation

SUMMARY. As we shall explain better in later sections, for a scheme X one has that

$$\boxed{\text{a perfect obstruction theory on } X \text{ induces a virtual fundamental class } [X]^{\mathrm{vir}}.}$$

The Chow class $[X]^{\mathrm{vir}} \in A_*X$ truly depends on the perfect obstruction theory it comes from (see Remark 9.1.15). This construction has historically two approaches: that of Li–Tian [**127**] and that of Behrend–Fantechi [**19**]. In this chapter we shall explicitly construct the perfect obstruction theory on a scheme of the form $X = Z(s) \subset Y$, where s is a section of a vector bundle on a smooth variety Y. Furthermore, we shall review in Section 9.3.3 the proof of the Graber–Pandharipande virtual localisation formula [**81**] in the case of $X = Z(s)$.

Even though in this chapter we do define perfect obstruction theories in general (see Definition 9.1.4), a slower path towards them will be taken later in Appendix C, following [**19, 60**].

9.1. Obstruction theories on vanishing loci of sections

In this section we only study the 'toy model' for a perfect obstruction theory, which is the situation where X is the vanishing locus of a section of a vector bundle on a smooth ambient space Y. In this case, the associated virtual fundamental class

$$[X]^{\mathrm{vir}} \in A_*X$$

can be constructed directly, without mentioning the perfect obstruction theory it comes from. The only technical tool needed to understand the definition of this particular virtual class is the fact that for a vector bundle $\pi \colon E \to S$ on a scheme S the flat pullback π^* on Chow groups is an isomorphism (cf. Theorem B.1.1).

Let Y be a smooth variety of dimension d and let $E = \operatorname{Spec}\operatorname{Sym}\mathscr{E}^*$ be the total space of a rank r vector bundle on Y. Let $s \in \mathrm{H}^0(Y,E) = \operatorname{Hom}_Y(\mathscr{O}_Y,\mathscr{E})$ be a section and let $\mathscr{I} = \operatorname{im}(s^\vee) \subset \mathscr{O}_Y$ be the ideal sheaf of the vanishing locus

$$X = Z(s) \subset Y.$$

We have the fibre diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d, "s"] \\ Y \arrow[r, "0"'] & E \end{tikzcd} \tag{9.1.1}$$

and since Y is smooth and E has rank r, we would naively be tempted to expect X to have dimension

$$\begin{aligned} \mathrm{d}^{\mathrm{vir}} &= (\text{number of variables}) - (\text{number of equations}) \\ &= d - r. \end{aligned}$$

This will indeed be the *virtual dimension* of the obstruction theory on X, and $[X]^{\mathrm{vir}}$ will be constructed as an element of $A_{d-r}X$.

The image of the cosection $s^\vee \colon \mathscr{E}^* \to \mathscr{O}_Y$ is precisely $\mathscr{I}$, thus restricting $s^\vee$ to X we obtain a surjective morphism

$$\sigma \colon \mathscr{E}^*\big|_X \twoheadrightarrow \mathscr{I}/\mathscr{I}^2.$$

Applying Spec Sym to this map, we obtain a closed immersion of cones (see Appendix C.1 for more details on cones)

$$N_{X/Y} \hookrightarrow E\big|_X,$$

where $N_{X/Y} = \operatorname{Spec}\operatorname{Sym}\mathscr{I}/\mathscr{I}^2 = \operatorname{Spec}\operatorname{Sym}\mathscr{N}^*_{X/Y}$ is the *normal sheaf* to X in Y (Definition 1.3.5). Composing with the closed immersion $C_{X/Y} \hookrightarrow N_{X/Y}$ of the normal cone inside the normal sheaf, we obtain a diagram

$$\begin{tikzcd} C_{X/Y} \arrow[r, hook] & N_{X/Y} \arrow[r, hook] & E\big|_X \arrow[d] \\ & & X \arrow[u, bend right, "0"'] \end{tikzcd} \tag{9.1.2}$$

realising $C_{X/Y}$ as a purely d-dimensional (cf. Remark C.1.1) subvariety of $E|_X$. It therefore determines a cycle class $[C_{X/Y}] \in A_d(E|_X)$. Let $0^*\colon A_d(E|_X) \xrightarrow{\sim} A_{d-r}X$ denote the inverse of the flat pullback.

DEFINITION 9.1.1. The *virtual fundamental class* of the vanishing locus $X = Z(s) \subset Y$ is the Chow class

$$[X]^{\mathrm{vir}} = 0^*[C_{X/Y}] \in A_{d-r}X. \tag{9.1.3}$$

Remark 9.1.2. Note that $[X]^{\mathrm{vir}}$ belongs to $A_{\mathrm{d}^{\mathrm{vir}}}X$ even if the section s cuts out a subscheme $X \subset Y$ of dimension bigger that $\mathrm{d}^{\mathrm{vir}} = d - r$, i.e. if s is not transverse to the zero section.

Remark 9.1.3. Note that $[X]^{\mathrm{vir}}$ agrees with Fulton's *localised top Chern class*, denoted $\mathbf{Z}(s)$ in [67]. Thus

$$[X]^{\mathrm{vir}} = \mathbf{Z}(s) = 0^![Y]$$

where $0^!\colon A_*Y \to A_*X$ is the refined Gysin homomorphism that we review in Appendix B.3. Therefore, if $\iota\colon X \hookrightarrow Y$ is the inclusion, then (cf. Remark B.3.8)

$$\iota_*[X]^{\mathrm{vir}} = c_r(E) \cap [Y] \in A_{d-r}Y.$$

The cotangent sheaf Ω_X of a scheme X can be seen as the 0-th cohomology sheaf of a canonical object

$$\mathbb{L}_X \in \mathbf{D}^{[-1,0]}(X),$$

the *truncated cotangent complex* (cf. Appendix C.2). If X is quasiprojective, in particular it can be embedded in a nonsingular scheme Y, and if $\mathscr{I} \subset \mathscr{O}_Y$ is the ideal sheaf of this inclusion, we can write

$$\mathbb{L}_X = \left[\mathscr{I}/\mathscr{I}^2 \xrightarrow{\mathrm{d}} \Omega_Y\big|_X\right]$$

where d is the exterior derivative. See again Appendix C.2 for the reason why the right hand side does not depend on the chosen embedding. If no embedding in a smooth variety is available,[1] $\mathbb{L}_X$ can still be defined as the truncation

$$\mathbb{L}_X = \tau_{\geq -1}L_X$$

of Illusie's cotangent complex $L_X \in \mathbf{D}^{(-\infty,0]}(X)$, cf. [111].

Recall (Definition 2.2.32) that a complex $\mathbb{E}$ of $\mathscr{O}_X$-modules is called *perfect of perfect amplitude in* $[a,b]$ if it is locally isomorphic to a complex $[E^a \to \cdots \to E^b]$ of locally free sheaves of finite rank.

DEFINITION 9.1.4 ([19]). A *perfect obstruction theory* on a scheme X is a pair $(\mathbb{E},\phi)$ where $\mathbb{E}$ is a perfect complex, of perfect amplitude in $[-1,0]$, and $\phi\colon \mathbb{E} \to \mathbb{L}_X$ is a morphism in $\mathbf{D}^{[-1,0]}(X)$ such that

(1) $h^0(\phi)$ is an isomorphism, and
(2) $h^{-1}(\phi)$ is surjective.

[1] We will never deal with such a situation in these notes.

The *virtual dimension* of an obstruction theory is its rank,

$$\mathrm{d}^{\mathrm{vir}} = \operatorname{rk}\mathbb{E}$$

which agrees with the difference $\operatorname{rk} E^0 - \operatorname{rk} E^{-1}$ if $\mathbb{E}$ is locally written $[E^{-1} \to E^0]$. Finally, the cohomology sheaf $\mathrm{Ob} = h^1(\mathbb{E}^\vee)$ is called the *obstruction sheaf* of $(\mathbb{E}, \phi)$.

DEFINITION 9.1.5 ([20]). A perfect obstruction theory $(\mathbb{E}, \phi)$ is *symmetric* if there exists an isomorphism

$$\theta \colon \mathbb{E} \xrightarrow{\sim} \mathbb{E}^\vee[1]$$

such that $\theta = \theta^\vee[1]$.

Remark 9.1.6 ([20, Proposition 1.14]). If $(\mathbb{E}, \phi)$ is symmetric, then $\mathrm{Ob} = \Omega_X$. Indeed, $\mathrm{Ob} = h^0(\mathbb{E}^\vee[1]) = h^0(\mathbb{E}) = h^0(\mathbb{L}_X) = \Omega_X$. In particular, $h^{-1}(\mathbb{E}) = \mathrm{Ob}^* = \mathcal{T}_X$.

Remark 9.1.7. Given a scheme X along with a perfect obstruction theory $(\mathbb{E}, \phi)$ on it, one can restrict it to any open subscheme $i \colon U \hookrightarrow X$. Indeed, the composition

$$i^*\mathbb{E} \xrightarrow{i^*\phi} i^*\mathbb{L}_X \xrightarrow{\sim} \mathbb{L}_U$$

is a perfect obstruction theory on U. Symmetry passes to open subschemes, too.

Remark 9.1.8. The perfect obstruction theory inducing the virtual class (9.1.3) is explicitly given by the map of complexes

$$\begin{tikzcd} \mathbb{E} \arrow[d, "\phi"'] & = & \big[\mathscr{E}^*\big|_X \arrow[rr, "\mathrm{d}\circ\sigma"] \arrow[d, "\sigma"] & & \Omega_Y\big|_X\big] \arrow[d, "\mathrm{id}"] \\ \mathbb{L}_X & = & \big[\mathscr{I}/\mathscr{I}^2 \arrow[rr, "\mathrm{d}"] & & \Omega_Y\big|_X\big] \end{tikzcd} \tag{9.1.4}$$

both concentrated in degrees $[-1, 0]$. The way $[X]^{\mathrm{vir}}$ is 'induced' by this diagram is straightforward in this case: indeed, the morphism of complexes (9.1.4) is just a way to rewrite the diagram (9.1.2).

EXERCISE 9.1.9. Let $X = Z(s)$ as above. Prove that the obstruction sheaf Ob of the obstruction theory depicted in Remark 9.1.8 agrees with the excess sheaf in the sense of Fulton [67, Section 6.3] (we computed our first excess sheaf a long time ago right after Diagram (1.3.2)), i.e. with the obstruction sheaf attached to Diagram (9.1.1). More precisely, prove that

$$\mathrm{Ob} = \operatorname{coker}\big(\mathscr{N}_{X/Y} \hookrightarrow \mathscr{E}\big|_X\big).$$

Example 9.1.10 (The symmetric obstruction theory on a critical locus). If we put $\mathscr{E} = \Omega_Y$ and $s = \mathrm{d}f \in \mathrm{H}^0(Y, T^*Y) = \mathrm{Hom}_Y(\mathscr{O}_Y, \Omega_Y)$ for f a regular function on a smooth variety Y, then, setting $X = Z(\mathrm{d}f) = \operatorname{crit}(f)$, one has

$$\mathbb{E} = \big[\mathrm{d}\circ\sigma \colon \mathcal{T}_Y\big|_X \to \Omega_Y\big|_X\big],$$

and the virtual class (9.1.3) on X is 0-dimensional. In this case, $\mathbb{E}$ can be identified with the *Hessian* of f, which is indeed a section

$$\operatorname{Hess}(f) \in \mathrm{H}^0\big(X, T^*Y\big|_X \otimes T^*Y\big|_X\big).$$

In fact, (9.1.4) in the case $s = \mathrm{d}f$ is the prototypical example of a *symmetric* perfect obstruction theory. Moreover, $[X]^{\mathrm{vir}}$ is intrinsic to X, i.e. it does not depend on the pair (Y, f) (in fact more is true, see Remark 9.1.14 below).

EXERCISE 9.1.11. Prove that the morphism of complexes (9.1.4) satisfies the axioms of a perfect obstruction theory (from Definition 9.1.4) on $X = Z(s) \subset Y$. Prove also that, if $s = \mathrm{d}f$, then (9.1.4) is symmetric. (The Hessian of a function is a symmetric matrix!).

Remark 9.1.12. All perfect obstruction theories are *locally* of the form (9.1.4). It is *not* true that all symmetric obstruction theories are locally given by $\mathsf{Hess}(f)$ for f a function on a smooth scheme, see [**166**] for a counterexample.

Example 9.1.13 (The symmetric obstruction theory on a smooth scheme)**.** Let Y be a smooth connected scheme. As a further specialisation of Example 9.1.10, consider the case $f = 0$, so that $\mathbb{E} = \mathsf{Hess}(0) = \left[0\colon \mathcal{T}_Y \to \Omega_Y\right]$. Then the induced virtual fundamental class on $X = Y$ is

$$[Y]^{\mathrm{vir}} = e(\Omega_Y) \cap [Y] \in A_0 Y.$$

Thus every smooth scheme can be seen as a 'virtually 0-dimensional' scheme. If Y is proper over $\operatorname{Spec}\mathbb{C}$, we have

$$\int_{[Y]^{\mathrm{vir}}} 1 = \int_Y e(\Omega_Y) = (-1)^{\dim Y} \chi(Y),$$

where $\chi(-)$ denotes the topological Euler characteristic.

Remark 9.1.14. If $\mathbb{E} \to \mathbb{L}_X$ is symmetric, then $\operatorname{rk}\mathbb{E} = \operatorname{rk}\mathbb{E}^\vee[1] = -\operatorname{rk}\mathbb{E}^\vee = -\operatorname{rk}\mathbb{E}$, so the virtual dimension is 0. Moreover, the K-theory class of $\mathbb{E}$ in $K_0(X) = K_0(\operatorname{Coh} X)$ is equal to $\Omega_X - \mathcal{T}_X$. By a result of Siebert [**183**, Theorem 4.6], the virtual fundamental class $[X]^{\mathrm{vir}}$ induced by a given perfect obstruction theory only depends on the K-theory class of the perfect obstruction theory. Therefore, all symmetric obstruction theories induce the same virtual class.

Remark 9.1.15. We should stress that the virtual fundamental class, in general, *does depend* on the chosen perfect obstruction theory. For instance, on a smooth connected scheme Y, the map

$$\mathbb{E} = [0 \to \Omega_Y] = \mathbb{L}_Y \xrightarrow{\mathrm{id}} \mathbb{L}_Y$$

defines a perfect obstruction theory of virtual dimension $\dim Y$. The associated virtual fundamental class is the usual fundamental class of Y, i.e.

$$[Y]^{\mathrm{vir}} = [Y] \in A_{\dim Y} Y.$$

This clearly differs from the one constructed in Example 9.1.13 as long as Y is not a point.

Conversely, if $\mathbb{E} \to \mathbb{L}_Y$ is a perfect obstruction theory on a scheme Y, such that $h^{-1}(\mathbb{E}) = 0$ and $h^0(\mathbb{E})$ is locally free, then Y is smooth, $\operatorname{rk}\mathbb{E} = \dim Y$ and the induced virtual class is $[Y] \in A_{\dim Y} Y$.

Remark 9.1.16 ([**20**, Proposition 1.20 + Corollary 1.21])**.** Let $(\mathbb{E}, \phi)$ be a perfect obstruction theory on a scheme Y. The $\mathrm{Ob} = h^1(\mathbb{E}^\vee)$ is locally free if and only if Y is a reduced local complete intersection. In particular, a reduced local complete intersection carrying a *symmetric* perfect obstruction theory is smooth.

We end this section with the observation that, by its critical locus description, the Hilbert scheme of points $\operatorname{Hilb}^n \mathbb{A}^3$ carries a symmetric perfect obstruction theory.

Corollary 9.1.17. *The Hilbert scheme* $\operatorname{Hilb}^n \mathbb{A}^3$ *carries a symmetric perfect obstruction theory.*

PROOF. Follows by combining Theorem 5.4.1 and Exercise 9.1.11. □

9.2. Basic theory of equivariant sheaves

In this section, we provide the background on equivariant sheaves needed for Section 9.3, following [**177**]. A standard reference for on equivariant sheaves, in the topological setup, is the book by Bernstein–Lunts [**21**].

9.2.1. The category of quasicoherent equivariant sheaves. Let X be a noetherian separated scheme over $\mathbb{C}$,[2] equipped with an action $\sigma\colon G\times X\to X$ of a complex group scheme G. Note that G is smooth by Cartier's theorem. We call such a pair (X,σ) a *G-scheme*. We define G-equivariant analogues of the abelian categories $\operatorname{Coh}X\subset\operatorname{QCoh}X\subset\operatorname{Mod}\mathscr{O}_X$.

Denoting by $\mathsf{m}\colon G\times G\to G$ the group law of G, there is a commutative diagram

$$\begin{tikzcd} G\times G\times X \arrow[r, "\mathsf{m}\times\mathrm{id}_X"] \arrow[d, "\mathrm{id}_G\times\sigma"'] & G\times X \arrow[d, "\sigma"] \\ G\times X \arrow[r, "\sigma"] & X \end{tikzcd} \tag{9.2.1}$$

translating the condition $g\cdot(h\cdot x)=(gh)\cdot x$.

Let $p_i\colon G\times X\to X$ and $p_{ij}\colon G\times G\times X\to G\times X$ denote the projections onto the labeled factors.

DEFINITION 9.2.1. A *G-equivariant quasicoherent sheaf* on X is a pair $(\mathscr{F},\vartheta)$ where $\mathscr{F}\in\operatorname{QCoh}X$ and $\vartheta\colon p_2^*\mathscr{F}\xrightarrow{\sim}\sigma^*\mathscr{F}$ is an isomorphism in $\operatorname{QCoh}(G\times X)$ compatible with the diagram (9.2.1). In other words, ϑ is required to satisfy the cocycle condition

$$(\mathsf{m}\times\mathrm{id}_X)^*\vartheta=(\mathrm{id}_G\times\sigma)^*\vartheta\circ p_{23}^*\vartheta. \tag{9.2.2}$$

The isomorphism ϑ is called a *G-equivariant structure* on $\mathscr{F}$.

The same definition can be given for objects $\mathscr{F}\in\operatorname{Mod}\mathscr{O}_X$ as well as $\mathscr{F}\in\operatorname{Coh}X$.

DEFINITION 9.2.2. A morphism $(\mathscr{F},\vartheta)\to(\mathscr{F}',\vartheta')$ of G-equivariant quasicoherent sheaves is a morphism $\phi\colon\mathscr{F}\to\mathscr{F}'$ in $\operatorname{QCoh}X$ such that the diagram

$$\begin{tikzcd} p_2^*\mathscr{F} \arrow[r, "p_2^*\phi"] \arrow[d, "\vartheta"'] & p_2^*\mathscr{F}' \arrow[d, "\vartheta'"] \\ \sigma^*\mathscr{F} \arrow[r, "\sigma^*\phi"] & \sigma^*\mathscr{F}' \end{tikzcd} \tag{9.2.3}$$

commutes in $\operatorname{QCoh}(G\times X)$.

Notation 9.2.3. Let $\operatorname{QCoh}^G X$ denote the category of G-equivariant quasicoherent sheaves $(\mathscr{F},\vartheta)$. It is a $\mathbb{C}$-linear Grothendieck abelian category and its (unbounded) derived category will be denoted $\mathbf{D}(\operatorname{QCoh}^G X)$.

Remark 9.2.4. Every object in $\mathbf{D}(\operatorname{QCoh}^G X)$ has a K-injective resolution and a K-flat resolution: this makes it possible to extend the theory of derived functors (pushforward, tensor product, local $\mathscr{H}om$) to the G-equivariant setting, see [**177**, Section 2.3] for more details.

Remark 9.2.5 (Relative version)**.** Let B be a noetherian $\mathbb{C}$-scheme. When working with a flat group scheme $G\to B$ acting on a scheme $X\to B$ with action $\sigma\colon G\times_B X\to X$, a G-equivariant sheaf $(\mathscr{F},\vartheta)$ can be described in the following fashion. Some notation first. For every B-scheme T, set $X_T=T\times_B X=T\times_T X_T$ and let $\mathscr{F}_T$ denote the pullback of $\mathscr{F}$ along the projection $X_T\to X$. For every T-valued point $g\colon T\to G_T=T\times_B G$ of G one has an isomorphism of T-schemes

$$\rho_g\colon X_T\xrightarrow{g\times\mathrm{id}_{X_T}}G_T\times_T X_T\xrightarrow{\sigma_T}X_T,\quad (t,x)\mapsto(t,\sigma_T(g(t),x)).$$

The condition '$\mathscr{F}$ is G-equivariant' is equivalent to the following condition: for every T-valued point $g\in G_T(T)$ as above there is an isomorphism $\vartheta_g\colon\mathscr{F}_T\xrightarrow{\sim}\rho_g^*\mathscr{F}_T$ such that for every pair of T-valued points $g,h\in G_T(T)$ one has a commutative diagram of isomorphisms

$$\begin{tikzcd} \rho_g^*\rho_h^*\mathscr{F}_T \arrow[d, equal] & \rho_g^*\mathscr{F}_T \arrow[l, "\rho_g^*\vartheta_h"'] \\ \rho_{hg}^*\mathscr{F}_T & \mathscr{F}_T \arrow[l, "\vartheta_{hg}"] \arrow[u, "\vartheta_g"'] \end{tikzcd} \tag{9.2.4}$$

[2]The theory works relatively to a fixed base scheme B. This requires all relative operations (such as fibre products) to be performed over B, as well as the structure morphism $G\to B$ be flat.

in $\operatorname{QCoh} X_T$.

Example 9.2.6. Let (X,σ) be a G-scheme over a scheme B. Then the structure sheaf $\mathscr{O}_X$ is G-equivariant in a natural way. For a B-scheme T, set $X_T = T \times_B X$. Then the inverse of the natural isomorphisms $\rho_g^*\mathscr{O}_{X_T} \xrightarrow{\sim} \rho_g^*\rho_{g*}\mathscr{O}_{X_T} \xrightarrow{\sim} \mathscr{O}_{X_T}$ is a G-equivariant structure on $\mathscr{O}_X$.

Example 9.2.7. Let (X,σ) be a G-scheme over a scheme B. Then the sheaf $\Omega_{X/B}$ of relative differentials is G-equivariant in a natural way. Indeed, for a B-scheme T, consider the natural isomorphisms $\alpha_T : (\Omega_{X/B})_T \xrightarrow{\sim} \Omega_{X_T/T}$ and $\ell_g : \rho_g^*\Omega_{X_T/T} \xrightarrow{\sim} \Omega_{X_T/T}$, where $g \in G_T(T)$. Then the composition

$$\vartheta_g : (\Omega_{X/B})_T \xrightarrow{\alpha_T} \Omega_{X_T/T} \xrightarrow{\ell_g^{-1}} \rho_g^*\Omega_{X_T/T} \xrightarrow{\rho_g^*\alpha_T^{-1}} \rho_g^*(\Omega_{X/B})_T$$

defines an equivariant structure on $\Omega_{X/B}$.

We may simply write g^* instead of ρ_g^*.

Remark 9.2.8. If $(\mathscr{F},\vartheta_{\mathscr{F}})$, $(\mathscr{F}',\vartheta_{\mathscr{F}'}) \in \operatorname{QCoh}^G X$, the $\mathbb{C}$-vector space

$$\operatorname{Hom}_X(\mathscr{F},\mathscr{F}')$$

is naturally a G-representation. Indeed, for a morphism $\phi : \mathscr{F} \to \mathscr{F}'$ in $\operatorname{QCoh} X$, one defines $g\cdot\phi$ by means of the composition

$$\begin{tikzcd} \mathscr{F} \arrow[r, dashed, "g\cdot\phi"] \arrow[d, "\vartheta_{\mathscr{F},g}"'] & \mathscr{F}' \\ g^*\mathscr{F} \arrow[r, "g^*\phi"] & g^*\mathscr{F}' \arrow[u, "\vartheta^{-1}_{\mathscr{F}',g}"'] \end{tikzcd} \tag{9.2.5}$$

exploiting the invertibility of $\vartheta_{\mathscr{F}',g}$. The structure of G-representation on $\operatorname{Hom}_X(\mathscr{F},\mathscr{F}')$ clearly depends on the chosen equivariant structures $\vartheta_{\mathscr{F}}$ and $\vartheta_{\mathscr{F}'}$.

Remark 9.2.9. It is immediate to see that, in $\operatorname{QCoh}^G X$, the morphisms are the G-invariant morphisms between the underlying quasicoherent sheaves. In symbols,

$$\operatorname{Hom}_{\operatorname{QCoh}^G X}((\mathscr{F},\vartheta_{\mathscr{F}}),(\mathscr{F}',\vartheta_{\mathscr{F}'})) = \operatorname{Hom}_X(\mathscr{F},\mathscr{F}')^G. \tag{9.2.6}$$

Example 9.2.10. If X is a G-scheme and $E = \operatorname{Spec}\operatorname{Sym}\mathscr{E}^*$ is a G-equivariant vector bundle (i.e. $\mathscr{E}$ is a G-equivariant locally free sheaf), then

$$\operatorname{Hom}_X(\mathscr{O}_X,\mathscr{E}) = \mathrm{H}^0(X,E)$$

is a G-representation, in particular it has a well-defined G-fixed part $\mathrm{H}^0(X,E)^G \subset \mathrm{H}^0(X,E)$.

Definition 9.2.11. Let X be a G-scheme. We define $K_0^G(X)$ to be the Grothendieck group of the abelian category $\operatorname{Coh}^G X$. For instance, $K_0^G(\mathrm{pt})$ can be identified with $\operatorname{Rep}_{\mathbb{C}}(G)$, the category of $\mathbb{C}$-linear finite dimensional G-representations. If G acts trivially on X, we have $K_0^G(X) \cong R(G) \otimes_{\mathbb{Z}} K_0(X)$, where $K_0(X) = K_0(\operatorname{Coh} X)$ is the usual K-theory of X.

9.2.2. Forgetful functor. The abelian category $\operatorname{QCoh}^G X$ has a forgetful functor to $\operatorname{QCoh} X$ sending $(\mathscr{F},\vartheta) \mapsto \mathscr{F}$, and extending to an exact functor

$$\Phi : \mathbf{D}(\operatorname{QCoh}^G X) \to \mathbf{D}(\operatorname{QCoh} X)$$

between the corresponding derived categories. This functor can be interpreted as follows. By [123, Exemple 12.4.6] (but see also [160, Exercise 9.H]), there is an equivalence of abelian categories

$$\operatorname{QCoh}^G X \cong \operatorname{QCoh}[X/G],$$

where $[X/G]$ is the *quotient stack* of X by G. If $\mathsf{p}\colon X \to [X/G]$ is the standard smooth atlas, we can view Φ as the composition

$$\mathbf{D}(\mathrm{QCoh}^G X) \xrightarrow{\sim} \mathbf{D}(\mathrm{QCoh}\,[X/G]) \xrightarrow{\mathsf{p}^*} \mathbf{D}(\mathrm{QCoh}\,X), \tag{9.2.7}$$

where $\mathsf{p}^* = \mathbf{L}\mathsf{p}^*$ is the pullback functor as defined in [**159**, Section 7].

Assume for simplicity that X admits a G-equivariant embedding in a nonsingular scheme Y. Then the truncated cotangent complex is naturally an object

$$\mathbb{L}_X \in \mathbf{D}^{[-1,0]}(\mathrm{QCoh}^G X),$$

see [**177**, Section 3.1] for more details.

DEFINITION 9.2.12. An object $E \in \mathbf{D}(\mathrm{QCoh}^G X)$ is called *perfect* (of perfect amplitude in $[a,b]$) if the underlying complex $\Phi(E) \in \mathbf{D}(\mathrm{QCoh}\,X)$ is perfect (of perfect amplitude in $[a,b]$).

9.3. Virtual localisation formula for the toy model

In this section we mainly follow [**81**], whose main result is a virtual analogue of the Atiyah–Bott localisation formula. We work in equivariant Chow groups, whose definition can be found in [**48**].

9.3.1. Equivariant obstruction theories. Let G be a complex algebraic group acting on a noetherian separated $\mathbb{C}$-scheme X.

DEFINITION 9.3.1 ([**81, 20**]). Suppose X is acted on by an algebraic group G. A perfect obstruction theory $\phi : \mathbb{E} \to \mathbb{L}_X$ is *G-equivariant* if it can be lifted to $\mathbf{D}^{[-1,0]}(\mathrm{QCoh}^G X)$. The G-structure on $\mathbb{L}_X$ is the one induced by the G-action on X.

DEFINITION 9.3.2 ([**20**]). A *G-equivariant symmetric perfect obstruction theory* is a pair

$$\left(\mathbb{E} \xrightarrow{\phi} \mathbb{L}_X, \mathbb{E} \xrightarrow{\theta} \mathbb{E}^\vee[1]\right)$$

of morphisms in $\mathbf{D}^{[-1,0]}(\mathrm{QCoh}^G X)$ such that $\Phi(\phi)$ is a perfect obstruction theory and θ satisfies $\theta^\vee[1] = \theta$.

9.3.2. Statement of the virtual localisation formula. In this section we state the Graber–Pandharipande virtual localisation formula. We set $G = \mathbb{G}_m$. The same formula holds for an arbitrary torus $\mathbb{G}_m^r$.

Let X be a $\mathbb{C}$-scheme equipped with a perfect obstruction theory, and carrying a closed embedding

$$X \hookrightarrow Y$$

into a nonsingular scheme Y, with ideal sheaf $\mathscr{I} \subset \mathscr{O}_Y$. The obstruction theory is given by a pair $(\mathbb{E}, \phi)$, where $\mathbb{E} \in \mathbf{D}^{[-1,0]}(X)$ is a perfect complex, of perfect amplitude contained in $[-1,0]$, and ϕ is a morphism from $\mathbb{E}$ to the truncated cotangent complex

$$\mathbb{L}_X = \left[\mathscr{I}/\mathscr{I}^2 \to \Omega_Y\big|_X\right].$$

Suppose Y carries a $\mathbb{G}_m$-action preserving X and the obstruction theory is $\mathbb{G}_m$-equivariant, i.e. ϕ can be lifted to a morphism in $\mathbf{D}^{[-1,0]}(\mathrm{QCoh}^{\mathbb{G}_m} X)$ (cf. Definition 9.3.1). We let

$$Y^{\mathrm{fix}} = Y^{\mathbb{G}_m} \subset Y, \quad X^{\mathrm{fix}} = X^{\mathbb{G}_m} \subset X$$

be the scheme-theoretic fixed point loci (see [**65**] for a description of such scheme structure). Recall that Y^{fix} is smooth by Theorem 7.4.1. We have the relation $X^{\mathrm{fix}} = X \cap Y^{\mathrm{fix}}$, and if $\{ Y_i \mid i \in I \}$ are the irreducible components of Y^{fix}, we set

$$X_i = X \cap Y_i. \tag{9.3.1}$$

Note that X_i might be reducible. Any $\mathbb{G}_m$-equivariant coherent sheaf $\mathcal{S} \in \mathrm{Coh}^{\mathbb{G}_m} X_i$ decomposes as a sum of eigensheaves

$$\mathcal{S} = \bigoplus_{k \in \mathbb{Z}} \mathcal{S}^k, \quad \mathcal{S}^k \in \mathrm{Coh}\, X_i,$$

where we are identitying the character group of the 1-dimensional torus $\mathbb{G}_m$ with $\mathbb{Z}$. The eigensheaves

$$\mathcal{S}^{\mathsf{fix}} = \mathcal{S}^0, \quad \mathcal{S}^{\mathsf{mov}} = \bigoplus_{k\neq 0} \mathcal{S}^k$$

are the *fixed part* and the *moving part* of $\mathcal{S}$, respectively. The construction of fixed and moving part of a sheaf extends to complexes in $\mathbf{D}(\mathrm{Coh}^{\mathbb{G}_m} X_i)$.

Lemma 9.3.3. *There is an identity*

$$\Omega_X\big|_{X_i}^{\mathsf{fix}} = \Omega_{X_i}.$$

PROOF. By Theorem 7.4.1, we have

$$\Omega_Y\big|_{Y_i}^{\mathsf{fix}} = \Omega_{Y_i}. \tag{9.3.2}$$

Then the claimed identity follows from (9.3.1). □

EXERCISE 9.3.4. Prove that a morphism of complexes $\psi\colon \mathbb{A} \to \mathbb{B}$ satisfies

(1) $h^0(\psi)$ is an isomorphism, and
(2) $h^{-1}(\psi)$ is surjective

if and only if the complex

$$A^{-1} \oplus B^{-2} \to A^0 \oplus B^{-1} \to B^0 \to 0$$

induced by the mapping cone construction is exact.

Notation 9.3.5. Define $\mathbb{E}_i$ to be the restriction of the complex $\mathbb{E}$ to the subvariety $X_i \subset X$.

DEFINITION 9.3.6. The *virtual normal bundle* to $X_i \hookrightarrow X$ is the moving part of the derived dual $\mathbb{E}_i^\vee$, i.e. it is the two-term complex

$$N_i^{\mathrm{vir}} = \mathbb{E}_i^{\vee,\mathsf{mov}} \in \mathbf{D}^{[0,1]}(\mathrm{Coh}^{\mathbb{G}_m} X_i).$$

Restricting ϕ to X_i, we have a composition

$$\mathbb{E}_i \xrightarrow{\phi_i} \mathbb{L}_X\big|_{X_i} \xrightarrow{\delta_i} \mathbb{L}_{X_i},$$

which after taking invariants becomes

$$\psi_i\colon \mathbb{E}_i^{\mathsf{fix}} \xrightarrow{\phi_i^{\mathsf{fix}}} \mathbb{L}_X\big|_{X_i}^{\mathsf{fix}} \xrightarrow{\delta_i^{\mathsf{fix}}} \mathbb{L}_{X_i}. \tag{9.3.3}$$

Lemma 9.3.7. *The composition* (9.3.3) *is a perfect obstruction theory on* X_i.

It is often called the '$\mathbb{G}_m$-fixed obstruction theory' on X_i.

PROOF. Clearly $\mathbb{E}_i$ is perfect in $[-1,0]$. To check the properties

(1) $h^0(\psi_i)$ is an isomorphism, and
(2) $h^{-1}(\psi_i)$ is surjective,

it is enough to check them for ϕ_i^{fix} and δ_i^{fix} separately.

Since $\phi\colon \mathbb{E} \to \mathbb{L}_X$ is an obstruction theory, and because the restriction $-\otimes_{\mathscr{O}_X} \mathscr{O}_{X_i}$ is right exact, Exercise 9.3.4 ensures that ϕ_i satisfies both (1) and (2). Since taking invariants is exact, the same holds for ϕ_i^{fix}.

Let $\mathscr{I} \subset \mathscr{O}_Y$ (resp, $\mathscr{I}_i \subset \mathscr{O}_{Y_i}$) be the ideal sheaf of $X \subset Y$ (resp. $X_i \subset Y_i$). Then one can write

$$\mathbb{L}_X = \left[\mathscr{I}/\mathscr{I}^2 \to \Omega_Y\big|_X\right] \in \mathbf{D}^{[-1,0]}(X), \qquad \mathbb{L}_{X_i} = \left[\mathscr{I}_i/\mathscr{I}_i^2 \to \Omega_{Y_i}\big|_{X_i}\right] \in \mathbf{D}^{[-1,0]}(X_i).$$

Of course, one has

$$h^0(\mathbb{L}_X) = \Omega_X, \quad h^0(\mathbb{L}_{X_i}) = \Omega_{X_i},$$

and $h^0(\delta_i^{\mathsf{fix}})$ is the identity map

$$h^0\left(\mathbb{L}_X\big|_{X_i}^{\mathsf{fix}}\right) = \Omega_X\big|_{X_i}^{\mathsf{fix}} = \Omega_{X_i} \xrightarrow{\mathrm{id}} \Omega_{X_i}.$$

So property (1) holds for δ_i^{fix}. As for property (2), one can represent $\delta_i^{\mathsf{fix}}\colon \mathbb{L}_X\big|_{X_i}^{\mathsf{fix}} \to \mathbb{L}_{X_i}$ as the map of complexes

$$\begin{tikzcd}
(\mathscr{I}/\mathscr{I}^2)\big|_{X_i}^{\mathsf{fix}} \arrow[r] \arrow[d, "\mathrm{d}^{-1}"'] & \Omega_Y\big|_{X_i}^{\mathsf{fix}} \arrow[d, "\wr"] \\
\mathscr{I}_i/\mathscr{I}_i^2 \arrow[r] & \Omega_{Y_i}\big|_{X_i}
\end{tikzcd}$$

concentrated in degrees $[-1,0]$. The vertical isomorphism is induced by (9.3.2), i.e. it is obtained by restriction to X_i and using exactness of $(-)^{\mathsf{fix}}$. Since $X_i = X \cap Y_i$ by definition, the natural map

$$(\mathscr{I}/\mathscr{I}^2)\big|_{X_i} \to \mathscr{I}_i/\mathscr{I}_i^2$$

is surjective (Exercise 1.3.6), which implies surjectivity of d^{-1}. Since $h^0(\delta_i^{\mathsf{fix}})$ is an isomorphism, it follows that $h^{-1}(\delta_i^{\mathsf{fix}})$ is surjective. □

The *virtual localisation formula* is the following relation.

Theorem 9.3.8 (Graber–Pandharipande [81]). *Let $\iota\colon X^{\mathsf{fix}} \hookrightarrow X$ be the inclusion of the fixed locus. Then there is an identity*

$$[X]^{\mathrm{vir}} = \iota_* \sum_i \frac{[X_i]^{\mathrm{vir}}}{e^{\mathbb{G}_m}(N_i^{\mathrm{vir}})}$$

in $A_*^{\mathbb{G}_m}(X) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t,t^{-1}]$.

9.3.3. Proof of the virtual localisation formula in the case of vanishing loci. Here is the setup in which we wish to prove (following [81, Section 2]) Theorem 9.3.8. We let Y be a smooth variety equipped with a $\mathbb{G}_m$-action. We fix a $\mathbb{G}_m$-equivariant vector bundle

$$\begin{tikzcd}
E = \operatorname{Spec}\operatorname{Sym}\mathscr{E}^* \arrow[d, shift right] \\
Y \arrow[u, bend right, "s"']
\end{tikzcd}$$

along with an equivariant section (cf. Example 9.2.10)

$$s \in \mathrm{H}^0(Y,\mathscr{E})^{\mathbb{G}_m}.$$

The vanishing scheme

$$X = Z(s) \hookrightarrow Y$$

then carries a natural $\mathbb{G}_m$-equivariant perfect obstruction theory

$$\mathbb{E} = \left[\mathscr{E}^*\big|_X \to \Omega_Y\big|_X\right] \xrightarrow{\phi} \mathbb{L}_X,$$

and we let

$$[X]^{\mathrm{vir}} = \mathbf{Z}(s) = 0^![Y] = 0^*[C_{X/Y}] \in A_*^{\mathbb{G}_m}(X)$$

be the associated virtual fundamental class.[3] The irreducible components $Y_i \hookrightarrow Y^{\mathsf{fix}}$ of the (nonsingular) fixed locus are nonsingular submanifolds, and for each i one has a decomposition

$$E_i = E\big|_{Y_i} = E_i^{\mathsf{fix}} \oplus E_i^{\mathsf{mov}}.$$

[3]We refer to [48, 140] for equivariant Chow groups; the formula for the virtual class is 'the same' as in the non-equivariant setting precisely because the operations 0^* and $0^!$ work equivariantly. The reader can safely replace $A_*^{\mathbb{G}_m}$ with $\mathbb{G}_m$-equivariant cohomology.

Since s is $\mathbb{G}_m$-equivariant, its pullback to Y_i determines a section $s_i \in \mathrm{H}^0(Y_i, E_i) = \mathrm{Hom}_{Y_i}(\mathscr{O}_{Y_i}, \mathscr{E}_i)$ that is entirely contained in E_i^{fix}, i.e. one can represent it as

$$\begin{tikzcd} & E_i^{\mathsf{fix}} \arrow[r, hook] \arrow[d, shift left] & E_i \arrow[dl] \\ X_i = Z(s_i) = Z(\widetilde{s}_i) \arrow[r, hook] & Y_i \arrow[u, bend left, "\widetilde{s}_i"] & \end{tikzcd}$$

and the vanishing locus of s_i is precisely $X_i = X \cap Y_i$.

Remark 9.3.9. At this point we have *two* virtual classes on X_i, one coming from (Y_i, E_i, s_i) and one coming from $(Y_i, E_i^{\mathsf{fix}}, \widetilde{s}_i)$. We shall use the notation $[X_i]^{\mathrm{vir}}$ only for $\mathbf{Z}(\widetilde{s}_i)$, i.e. the virtual class arising from the second triple. This is the one arising from the $\mathbb{G}_m$-fixed obstruction theory on X_i, see below. Clearly $[X_i]^{\mathrm{vir}} = \mathbf{Z}(\widetilde{s}_i)$ sits in dimension $\mathrm{d}_i = \dim Y_i - \mathrm{rk}\, E_i^{\mathsf{fix}}$, whereas $\mathbf{Z}(s_i)$ sits in dimension $\dim Y_i - \mathrm{rk}\, E = \mathrm{d}_i - \mathrm{rk}\, E_i^{\mathsf{mov}}$.

Consider the fibre diagram

$$\begin{tikzcd} X_i \arrow[r, hook, "\epsilon_i"] \arrow[d, hook, "\iota_i"'] \arrow[dr, phantom, "\square"] & Y_i \arrow[d, hook, "j_i"] \\ X \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\square"] & Y \arrow[d, hook, "s"] \\ Y \arrow[r, hook, "0"'] & E \end{tikzcd}$$

where we labeled the relevant inclusions for later convenience. Taking the fixed part of the complex

$$\mathbb{E}_i = \mathbb{E}\big|_{X_i} = \big[\mathscr{E}^*\big|_{X_i} \to \Omega_Y\big|_{X_i}\big]$$

yields the canonical '$\mathbb{G}_m$-fixed' perfect obstruction theory

$$\psi_i \colon \mathbb{E}_i^{\mathsf{fix}} \to \mathbb{L}_{X_i}$$

of Lemma 9.3.7. Recall that $\Omega_Y\big|_{X_i}^{\mathsf{fix}} \xrightarrow{\sim} \Omega_{Y_i}\big|_{X_i}$. Then under this identification one can represent ψ_i as

$$\begin{tikzcd} \mathbb{E}_i^{\mathsf{fix}} \arrow[d, "\psi_i"'] & = & \big[\mathscr{E}^*\big|_{X_i}^{\mathsf{fix}} \arrow[r, "\mathrm{d}\circ\sigma_i"] \arrow[d, "\sigma_i"'] & \Omega_{Y_i}\big|_{X_i}\big] \arrow[d, "\mathrm{id}"] \\ \mathbb{L}_{X_i} & = & \big[\mathscr{I}_i/\mathscr{I}_i^2 \arrow[r, "\mathrm{d}"] & \Omega_{Y_i}\big|_{X_i}\big] \end{tikzcd}$$

and this induces the identity

$$\mathbf{Z}(s_i) = \mathbf{Z}(\widetilde{s}_i) \cap e(\mathscr{E}_i^{\mathsf{mov}}) \in A_*^{\mathbb{G}_m}(X_i). \tag{9.3.4}$$

Lemma 9.3.10. *There is an identity*

$$N_i^{\mathrm{vir}} = \epsilon_i^*\left(\mathscr{N}_{Y_i/Y} - \mathscr{E}_i^{\mathsf{mov}}\right) \in K_0^{\mathbb{G}_m}(X_i).$$

PROOF. By definition, N_i^{vir} is the moving part of the complex

$$\mathbb{E}_i^{\vee} = \big[\mathscr{T}_Y\big|_{X_i} \to \mathscr{E}\big|_{X_i}\big] \in \mathbf{D}^{[0,1]}(X_i).$$

So, in K-theory, we have

$$
\begin{aligned}
N_i^{\mathrm{vir}} &= \mathbb{E}_i^{\vee} - \mathbb{E}_i^{\vee,\mathsf{fix}} \\
&= \mathcal{T}_Y\big|_{X_i} - \mathcal{E}\big|_{X_i} - \mathcal{T}_{Y_i}\big|_{X_i} + \mathcal{E}\big|_{X_i}^{\mathsf{fix}} \\
&= (\mathcal{T}_Y\big|_{Y_i} - \mathcal{T}_{Y_i})\big|_{X_i} - \mathcal{E}_i\big|_{X_i}^{\mathsf{mov}} \\
&= (\mathcal{N}_{Y_i/Y} - \mathcal{E}_i^{\mathsf{mov}})\big|_{X_i},
\end{aligned}
$$

as required. □

Therefore, by definition of Euler class of a complex (cf. Remark B.2.10), we obtain

$$
e^{\mathbb{G}_m}(N_i^{\mathrm{vir}}) = \epsilon_i^* \left(\frac{e^{\mathbb{G}_m}(\mathcal{N}_{Y_i/Y})}{e(\mathcal{E}_i^{\mathsf{mov}})} \right). \tag{9.3.5}
$$

The usual Atiyah–Bott localisation formula for Y states

$$
[Y] = \sum_i j_{i*} \frac{[Y_i]}{e^{\mathbb{G}_m}(\mathcal{N}_{Y_i/Y})} \in A_*^{\mathbb{G}_m}(Y) \otimes_{\mathbb{Q}} \mathbb{Q}[t, t^{-1}].
$$

Taking on both sides refined intersection with $[X]^{\mathrm{vir}} = \mathbf{Z}(s)$, and using standard properties of $0^!$ (cf. Appendix B.3.2 or directly [**67**, Theorem 6.2] for the second equality), yields

$$
\begin{aligned}
[X]^{\mathrm{vir}} &= \sum_i 0^! j_{i*} \frac{[Y_i]}{e^{\mathbb{G}_m}(\mathcal{N}_{Y_i/Y})} \\
&= \sum_i \iota_{i*} 0^! \frac{[Y_i]}{e^{\mathbb{G}_m}(\mathcal{N}_{Y_i/Y})} \\
&= \sum_i \iota_{i*} \frac{\mathbf{Z}(s_i)}{e^{\mathbb{G}_m}(\mathcal{N}_{Y_i/Y})|_{X_i}} \\
&= \sum_i \iota_{i*} \frac{\mathbf{Z}(\widetilde{s}_i) \cap e(\mathcal{E}_i^{\mathsf{mov}})}{e^{\mathbb{G}_m}(N_i^{\mathrm{vir}}) e(\mathcal{E}_i^{\mathsf{mov}})|_{X_i}} \\
&= \sum_i \iota_{i*} \frac{\mathbf{Z}(\widetilde{s}_i)}{e^{\mathbb{G}_m}(N_i^{\mathrm{vir}})} \\
&= \iota_* \sum_i \frac{[X_i]^{\mathrm{vir}}}{e^{\mathbb{G}_m}(N_i^{\mathrm{vir}})}.
\end{aligned}
$$

We have used the refined Euler classes commute with pullback in the third equality, and combined Equations (9.3.4) and (9.3.5) for the fourth equality. This chain of identities completes the proof Theorem 9.3.8 for $X = Z(s)$.

CHAPTER 10

Degree 0 DT invariants of a local Calabi–Yau 3-fold

SUMMARY. In this chapter we will see the virtual localisation formula (Theorem 9.3.8) in action. We will start out with a toric Calabi–Yau 3-fold X (necessarily nonprojective), and we will construct a torus equivariant obstruction theory on $\operatorname{Hilb}^n X$. We will exploit the virtual localisation formula to define the associated Donaldson–Thomas (DT in short) invariant DT_n^X, and we shall prove that it is related to the Euler characteristic of $\operatorname{Hilb}^n X$ via the relation

$$\mathsf{DT}_n^X = (-1)^n \chi(\operatorname{Hilb}^n X).$$

In other words, 'virtualness' contributes a sign to the naive topological count. This result, the first complete calculation in DT theory, was first proved in [**135**, **136**].

10.1. Preliminary tools

We introduce a couple of technical tools that will be needed in the construction of the perfect obstruction theory on the Hilbert scheme of points on a 3-fold.

10.1.1. The trace map of a perfect complex. Let X be a scheme. Recall (see e.g. [**108**]) that for any torsion free coherent sheaf E of positive rank $r > 0$ there is a trace map $\mathrm{tr}\colon \mathscr{H}om(E,E) \to \mathscr{O}_X$ sitting in a diagram

$$\begin{tikzcd} & \mathscr{H}om(E,E) \arrow[d] \arrow[dr, "\mathrm{tr}"] & \\ \mathscr{O}_X \arrow[ur, "f \mapsto f\cdot \mathrm{id}"] \arrow[r] & \mathbf{R}\mathscr{H}om(E,E) \arrow[r, "\mathrm{tr}"'] & \mathscr{O}_X \end{tikzcd}$$

where the lower row is *split*: the inclusion of the scalars $\mathscr{O}_X \to \mathbf{R}\mathscr{H}om(E,E)$ composes with tr to give $r \cdot \mathrm{id} \in \operatorname{Hom}(\mathscr{O}_X, \mathscr{O}_X)$. We denote by

$$\mathrm{tr}^i \colon \operatorname{Ext}^i(E,E) \to \mathrm{H}^i(X, \mathscr{O}_X)$$

the i-th cohomology of the map tr and we set

$$\operatorname{Ext}^i(E,E)_0 = \ker\left(\mathrm{tr}^i\right) \subset \operatorname{Ext}^i(E,E).$$

For instance, if E is a Gieseker stable sheaf, the map tr^1 can be identified with the differential of the determinant map $M \to \operatorname{Pic} X$, where M is the moduli space of Gieseker stable sheaves containing $[E]$ as a closed point. The maps tr^i are surjective as long as $r \neq 0$. See [**108**, Sections 4.5 and 10.1] for more details on the trace map. The splitting induces direct sum decompositions

$$\operatorname{Ext}^i(E,E) = \mathrm{H}^i(X, \mathscr{O}_X) \oplus \operatorname{Ext}^i(E,E)_0.$$

The *trace-free* Ext groups $\operatorname{Ext}^1(E,E)_0$ and $\operatorname{Ext}^2(E,E)_0$ encode tangents and obstructions for the deformation theory of E with fixed determinant. These groups globalise to a complex $\mathbf{R}\mathscr{H}om(E,E)_0$, defined as the cone of tr shifted by -1. This induces a distinguished triangle

$$\mathbf{R}\mathscr{H}om(E,E)_0 \longrightarrow \mathbf{R}\mathscr{H}om(E,E) \xrightarrow{\mathrm{tr}} \mathscr{O}_X. \tag{10.1.1}$$

EXERCISE 10.1.1. Let $\mathbb{E}$ be a bounded perfect complex on a scheme X. Show that there is a well-defined trace map $\mathrm{tr}\colon \mathbf{R}\mathscr{H}om(\mathbb{E},\mathbb{E}) \to \mathscr{O}_X$, so the triangle (10.1.1) generalises to this setting.

EXERCISE 10.1.2. Show that, if $\mathbb{E}$ is a bounded perfect complex, then in the exact triangle (10.1.1) all three objects are canonically self-dual.

10.1.2. The Huybrechts–Thomas Atiyah class of a perfect complex. Sheaves of principal parts were introduced in [85, Chapter 16.3]. Let $\pi\colon X \to S$ be a separated morphism of schemes, and let $\mathscr{I}$ be the ideal sheaf of the diagonal $\Delta\colon X \to X \times_S X$. Let p and q denote the projections $X \times_S X \to X$, and denote by $\Delta_k \subset X \times_S X$ the closed subscheme defined by $\mathscr{I}^{k+1}$, for every $k \geq 0$. Then, for any quasicoherent sheaf V on X, the sheaf

$$P^k_\pi(V) = p_*\left(q^*V \otimes \mathscr{O}_{\Delta_k}\right)$$

is quasicoherent and is called the k-th *sheaf of principal parts* associated to the pair (π, V). The following result is classical. A proof can be found e.g. in [74, Proposition 1.3].

Proposition 10.1.3. *Let $\pi\colon X \to S$ be a smooth morphism of schemes, V a quasicoherent sheaf on X. The sheaves of principal parts fit into short exact sequences*

$$V \otimes \operatorname{Sym}^k \Omega^1_\pi \to P^k_\pi(V) \to P^{k-1}_\pi(V) \to 0$$

for every $k \geq 1$. If V is locally free then the sequence is exact on the left, and $P^k_\pi(V)$ is locally free for all $k \geq 0$.

Example 10.1.4. Given a smooth morphism $\pi\colon X \to S$, there is a splitting $P^1_\pi(\mathscr{O}_X) = \mathscr{O}_X \oplus \Omega^1_\pi$. For an arbitrary vector bundle V, the splitting of $P^1_\pi(V)$ usually fails even when $S = \operatorname{Spec} \mathbf{k}$. In fact, in this case, the splitting is equivalent to the vanishing of the *Atiyah class* of V, which by definition is the extension class

$$A(V) \in \operatorname{Ext}^1_X(V, V \otimes \Omega_X)$$

attached to the short exact sequence of Proposition 10.1.3 taken with $k = 1$. But the vanishing of the Atiyah class is known to be equivalent to the existence of an algebraic connection on V, see e.g. [108, Section 10.1.5].

The Atiyah class exists in much broader generality than the one mentioned in Example 10.1.4, see for instance the more general construction in [185, Tag 09DF].

We now recall, following verbatim [177, Section 3], the construction due to Huybrechts and Thomas [109], which is relevant in the context of (possibly relative) obstruction theories.

Let $X \hookrightarrow A$ be a closed immersion of a scheme X inside a smooth $\mathbb{C}$-scheme A. Let $J \subset \mathscr{O}_A$ be the corresponding sheaf of ideals. The (absolute) truncated cotangent complex is the two term complex

$$\mathbb{L}_X = \left[J/J^2 \to \Omega_A|_X\right] \in \mathbf{D}^{[-1,0]}(\operatorname{QCoh} X). \tag{10.1.2}$$

Let $\mathscr{I}_A \subset \mathscr{O}_{A\times A}$ and $\mathscr{I}_X \subset \mathscr{O}_{X\times X}$ be the ideal sheaves of the diagonal embeddings

$$A \xhookrightarrow{i_{\Delta_A}} A \times A \quad \text{and} \quad X \xhookrightarrow{i_{\Delta_X}} X \times X,$$

respectively. Huybrechts–Thomas [109, Section 2] show how to construct a canonical morphism

$$\alpha_X : \mathscr{O}_{\Delta_X} \to i_{\Delta_X *}\mathbb{L}_X[1]. \tag{10.1.3}$$

It is represented in degrees $[-2, 0]$ by the morphism of complexes

$$\begin{tikzcd} i_{\Delta_X *}(J/J^2) \arrow[r] \arrow[d, equal] & \mathscr{I}_A\big|_{X\times X} \arrow[r] \arrow[d] & \mathscr{O}_{X\times X} \\ i_{\Delta_X *}(J/J^2) \arrow[r] & \mathscr{I}_A/\mathscr{I}_A^2\big|_{X\times X} & \end{tikzcd} \tag{10.1.4}$$

where the quasi-isomorphism between the top complex and $\mathscr{O}_{\Delta_X}$ is proved as a consequence of [109, Lemma 2.2]. The extension class

$$\alpha_X \in \operatorname{Ext}^1_{X\times X}(\mathscr{O}_{\Delta_X}, i_{\Delta_X *}\mathbb{L}_X)$$

corresponding to (10.1.3) is called the *truncated universal Atiyah class*. It does not depend on the choice of embedding $X \subset A$.

The main observation in [**109**], at this point, is that the map (10.1.3) can be seen as a map of Fourier–Mukai kernels. In particular, for a perfect complex $\mathbb{E}$ on X, one can view $\mathbf{R}\pi_{2*}(\pi_1^*\mathbb{E} \otimes \alpha_X)$ as a canonical morphism

$$A(\mathbb{E}) \colon \mathbb{E} \to \mathbb{E} \otimes \mathbb{L}_X[1]$$

in $\mathbf{D}(\mathrm{QCoh}\, X)$, where $\pi_i \colon X \times X \to X$ are the projections. This is, by definition, the *truncated Atiyah class* of $\mathbb{E}$ introduced in [**109**, Definition 2.6]. It can of course be seen as an element

$$A(\mathbb{E}) \in \mathrm{Ext}^1_X(\mathbb{E}, \mathbb{E} \otimes \mathbb{L}_X). \tag{10.1.5}$$

Under the canonical morphism $\mathbb{L}_X \to h^0(\mathbb{L}_X) = \Omega_X$, the extension $A(\mathbb{E})$ projects onto the classical Atiyah class in $\mathrm{Ext}^1_X(\mathbb{E}, \mathbb{E} \otimes \Omega_X)$.

10.1.3. Calabi–Yau 3-folds. Even though Donaldson–Thomas theory is an enumerative theory for sheaves (or complexes) on arbitrary 3-folds, specialising to *Calabi–Yau* 3-folds the theory becomes particularly rich. The main feature of the Calabi–Yau condition is that moduli spaces of sheaves (as well as moduli spaces of unpointed stable maps, of stable pairs...) have virtual dimension 0. This is a consequence of the fact that the *obstruction theory* on the moduli space of simple sheaves on a Calabi–Yau 3-fold is *symmetric* (cf. Definitions 9.1.4 and 9.1.5), which in turn is a global incarnation of the Serre duality isomorphism

$$\mathrm{Ext}^1(E,E) \cong \mathrm{Ext}^2(E,E)^\vee.$$

However, our main object of study in these notes is the Hilbert scheme of points $\mathrm{Hilb}^n X$ on a more general 3-fold X. We shall see that in this case the obstruction theory has virtual dimension 0 even if X is not Calabi–Yau.

DEFINITION 10.1.5. A *Calabi–Yau 3-fold* is a nonsingular quasiprojective variety X of dimension 3, satisfying $\omega_X = \wedge^3\Omega^1_X \cong \mathscr{O}_X$ and the vanishing $\mathrm{H}^i(X, \mathscr{O}_X) = 0$ for $0 < i < 3$.

Example 10.1.6. According to our definition, the following are examples of Calabi–Yau 3-folds:

(1) affine space $\mathbb{A}^3$,
(2) the total space $\mathrm{Tot}(\omega_S)$ of the canonical bundle of a smooth projective surface,
(3) the total space of the rank 2 bundle $\mathscr{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ over $\mathbb{P}^1$,
(4) more generally, the total space of $L_1 \oplus L_2$, where L_i are line bundles on a smooth projective curve C such that $L_1 \otimes_{\mathscr{O}_C} L_2 \cong \omega_C$,
(5) a smooth quintic hypersurface $Y \subset \mathbb{P}^4$,
(6) a general complete intersection of type $(2,4)$ or $(3,3)$ in $\mathbb{P}^5$,
(7) a general complete intersection of type $(2,2,3)$ in $\mathbb{P}^6$,
(8) a general complete intersection of type $(2,2,2,2)$ in $\mathbb{P}^7$.

Note that only the last four are projective.

EXERCISE 10.1.7. Show that a smooth infinitesimally rigid[1] rational curve $C \subset Y$ in a Calabi–Yau 3-fold Y has normal bundle isomorphic to $\mathscr{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$.

EXERCISE 10.1.8. Show that a smooth quintic $Y \subset \mathbb{P}^4$ satisfies $\chi(Y) = -200$. (**Hint**: Combine the Euler sequence (1.3.1) of $\mathbb{P}^4$ with the normal bundle exact sequence of $Y \subset \mathbb{P}^4$, and use the Whitney sum formula for Chern polynomials).

[1]This means that $\mathrm{H}^0(C, \mathscr{N}_{C/Y}) = 0$.

10.2. The perfect obstruction theory on $\mathrm{Hilb}^n X$

The goal of this section is to prove (most of) the following result.

Theorem 10.2.1. *Let X be a smooth complex 3-fold, and fix $n \geq 0$.*

(1) *If X is projective and is either Calabi–Yau or satisfies $\mathrm{H}^{>0}(X, \mathscr{O}_X) = 0$ (e.g. X is toric), then $\mathrm{Hilb}^n X$ has a 0-dimensional perfect obstruction theory.*
(2) *In the Calabi–Yau case, the obstruction theory is symmetric.*
(3) *If X is toric and quasiprojective (possibly Calabi–Yau) with dense torus $\mathbb{T} \subset X$, such obstruction theory is naturally $\mathbb{T}$-equivariant.*
(4) *If X is toric and Calabi–Yau, such obstruction theory is furthermore $\mathbb{T}_0$-equivariant symmetric (Definition 9.3.2), where $\mathbb{T}_0 \subset \mathbb{T}$ is the 2-dimensional subtorus obtained as the kernel of the character $(1, 1, 1)$.*

The only statement that we will not attempt to prove here is (4). We refer the reader to [**20**] for details. We also give precise references checking the axioms of a perfect obstruction theory. Our goal is mainly to show how the morphism to the cotangent complex is constructed, and this we do in full detail.

A few more remarks are in order.

Remark 10.2.2. If we prove Theorem 10.2.1 (1) for X toric projective, then it holds for X toric *quasiprojective*, too, therefore (3) makes sense as formulated. Indeed, it is enough to compactify $X \hookrightarrow \overline{X}$ to a toric variety[2] $\overline{X}$ and restrict the obstruction theory $\mathbb{E} \to \mathbb{L}_{\mathrm{Hilb}^n \overline{X}}$ obtained via (1) along the open immersion $\mathrm{Hilb}^n X \hookrightarrow \mathrm{Hilb}^n \overline{X}$, cf. Remark 9.1.7.

Remark 10.2.3. Symmetry of the obstruction theory will be proved in Section 10.2.4. Equivariance will be proved in Section 10.2.5. See also [**62**, Lemma 3.13] for an explicit proof of the $\mathbb{T}$-equivariance of the *critical* obstruction theory on $\mathrm{Hilb}^n \mathbb{A}^3$ (Corollary 9.1.17), and in fact on the Quot scheme $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)$.

Remark 10.2.4. The Hilbert scheme $\mathrm{Hilb}^n X$ carries a 0-dimensional perfect obstruction theory for *every* smooth quasiprojective 3-fold. To simplify the exposition, and because we do not need more for our application, we strengthened the assumptions a bit. Also, note that a toric Calabi–Yau variety cannot be projective (because a toric variety is rational and a projective Calabi–Yau variety has geometric genus 1), so in (3) and (4) we are referring to quasiprojective Calabi–Yau 3-folds that happen to be toric varieties. These are also called *local toric Calabi–Yau 3-folds*.

We take the opportunity to mention an open problem in the subject (see also [**38**, Conjecture 2.12] for a higher rank version).

Open Problem 1 ([**62**, Conjecture 9.9]). Consider the critical symmetric perfect obstruction theory on $\mathrm{Hilb}^n \mathbb{A}^3$ (see Corollary 9.1.17), defined by the Hessian of the function $f_n \colon M_n \to \mathbb{A}^1$ (Theorem 5.4.1), i.e. realising $\mathrm{Hilb}^n \mathbb{A}^3 = \mathrm{crit}(f_n) \subset M_n = \mathrm{ncQuot}_3^{n,1}$. Is this obstruction theory the same as the one of Theorem 10.2.1? Are they $\mathbb{T}$-equivariantly isomorphic (over the cotangent complex)?

We can start proving Theorem 10.2.1 (1). This will take some work — we will confirm the existence of the obstruction theory in Theorem 10.2.11.

10.2.1. Useful vanishings.

We fix a smooth projective 3-fold X satisfying[3]

$$\mathrm{H}^i(X, \mathscr{O}_X) = \mathrm{Ext}^i(\mathscr{O}_X, \mathscr{O}_X) = 0, \quad i > 0. \tag{10.2.1}$$

Thus $\mathrm{tr}^i = 0$ for $i > 0$, and in particular

$$\mathrm{Ext}^i(E, E)_0 = \mathrm{Ext}^i(E, E), \quad i > 0.$$

[2] This is done by adding a suitable number of cones in the toric fan of X, so to fill up the whole $\mathbb{R}^3$, see [**68, 42**] for details.

[3] Of course, for a Calabi–Yau 3-fold we have $\mathrm{H}^3(X, \mathscr{O}_X) = \mathrm{H}^0(X, \omega_X)^\vee = \mathbb{C} \neq 0$. We will point out to the reader why the Calabi–Yau assumption is still compatible with the argument presented here.

By Serre duality, for every $E, F \in \operatorname{Coh} X$, we have

$$\operatorname{Ext}^i(E,F) \cong \operatorname{Ext}^{3-i}(F, E \otimes \omega_X)^\vee. \tag{10.2.2}$$

EXERCISE 10.2.5. Let E be a locally free sheaf of rank r on a smooth variety Y, and let T be a skyscraper sheaf. Show that $T^{\oplus r} \cong E \otimes T$. So, given a line bundle $\mathscr{L}$ and a 0-dimensional sheaf T, we have $T \otimes \mathscr{L} \cong T$.

Recall that we view the Hilbert scheme as a particular case of Quot scheme. Set-theoretically, we can write

$$\mathrm{Hilb}^n X = \{\, [\mathscr{O}_X \twoheadrightarrow \mathscr{O}_Z] \mid Z \subset X \text{ is finite, } \chi(\mathscr{O}_Z) = n \,\}.$$

Let $\mathscr{I} \subset \mathscr{O}_X$ be the ideal sheaf of a finite subscheme $Z \subset X$ of length n. Applying $\operatorname{Hom}(\mathscr{I}, -)$ to the ideal sheaf exact sequence $\mathscr{I} \hookrightarrow \mathscr{O}_X \twoheadrightarrow \mathscr{O}_Z$ we obtain

$$\begin{aligned} 0 \to \operatorname{Hom}(\mathscr{I},\mathscr{I}) \to \operatorname{Hom}(\mathscr{I},\mathscr{O}_X) \xrightarrow{u} T_{\mathscr{I}} \mathrm{Hilb}^n X \to \operatorname{Ext}^1(\mathscr{I},\mathscr{I}) \\ \to \operatorname{Ext}^1(\mathscr{I},\mathscr{O}_X) \to \operatorname{Ext}^1(\mathscr{I},\mathscr{O}_Z) \to \operatorname{Ext}^2(\mathscr{I},\mathscr{I}). \end{aligned} \tag{10.2.3}$$

Since $\dim Z = 0$, Serre duality (10.2.2) (along with Exercise 10.2.5 taken with $E = \omega_X$) implies

$$\operatorname{Ext}^{3-i}(\mathscr{O}_Z, \mathscr{O}_X)^\vee \cong \operatorname{Ext}^i(\mathscr{O}_X, \mathscr{O}_Z) \cong \mathrm{H}^i(Z, \mathscr{O}_Z) = 0, \quad i > 0. \tag{10.2.4}$$

Lemma 10.2.6. *We have* $\operatorname{Ext}^1(\mathscr{I}, \mathscr{O}_X) = 0$.

PROOF. The Ext group we are looking at appears in a long exact sequence containing the exact piece

$$\begin{tikzcd} \operatorname{Ext}^1(\mathscr{O}_X,\mathscr{O}_X) \arrow[r] \arrow[d, equal, "(10.2.1)"'] & \operatorname{Ext}^1(\mathscr{I},\mathscr{O}_X) \arrow[r] & \operatorname{Ext}^2(\mathscr{O}_Z,\mathscr{O}_X) \arrow[d, equal, "(10.2.4)"] \\ 0 & & 0 \end{tikzcd}$$

and therefore it has to vanish. □

Lemma 10.2.7. *In the long exact sequence* (10.2.3)*, we have* $u = 0$.

PROOF. Applying $\operatorname{Hom}(-, \mathscr{O}_X)$ to the ideal sheaf sequence $\mathscr{I} \hookrightarrow \mathscr{O}_X \twoheadrightarrow \mathscr{O}_Z$ yields an exact piece

$$\begin{tikzcd} \operatorname{Hom}(\mathscr{O}_Z,\mathscr{O}_X) \arrow[r] \arrow[d, equal, "(10.2.4)"'] & \operatorname{Hom}(\mathscr{O}_X,\mathscr{O}_X) \arrow[r] \arrow[d, equal] & \operatorname{Hom}(\mathscr{I},\mathscr{O}_X) \arrow[r] & \operatorname{Ext}^1(\mathscr{O}_Z,\mathscr{O}_X). \arrow[d, equal, "(10.2.4)"] \\ 0 & \mathrm{H}^0(X,\mathscr{O}_X) & & 0 \end{tikzcd}$$

thus $\operatorname{Hom}(\mathscr{I}, \mathscr{O}_X) \cong \operatorname{Hom}(\mathscr{O}_X, \mathscr{O}_X) = \mathbb{C} \cdot \mathrm{id}$ is 1-dimensional. From the decomposition

$$\operatorname{Hom}(\mathscr{I},\mathscr{I}) = \underbrace{\mathrm{H}^0(X,\mathscr{O}_X)}_{\mathbb{C}} \oplus \operatorname{Hom}(\mathscr{I},\mathscr{I})_0$$

we deduce that $\operatorname{Hom}(\mathscr{I},\mathscr{I})$ is *at least* 1-dimensional. This shows that $\operatorname{Hom}(\mathscr{I},\mathscr{I}) \to \operatorname{Hom}(\mathscr{I},\mathscr{O}_X)$ in (10.2.3) is an isomorphism. It follows that $u = 0$. □

Remark 10.2.8. Another way to see that $\operatorname{Hom}(\mathscr{I},\mathscr{I}) = \mathbb{C}$ is to simply observe that $\mathscr{I}$ is a stable sheaf, in particular it is simple, which means its only endomorphisms are the constants. See e.g. [**108**, Corollary 1.2.28] for a proof that stable sheaves are simple.

Before interpreting the vanishing of the map u, we pause to prove an additional vanishing result.

Proposition 10.2.9. *We have* $\operatorname{Hom}(\mathscr{I},\mathscr{I})_0 = \operatorname{Ext}^3(\mathscr{I},\mathscr{I})_0 = 0$.

PROOF. The vanishing $\operatorname{Hom}(\mathscr{I},\mathscr{I})_0 = 0$ is a consequence of $\operatorname{Hom}(\mathscr{I},\mathscr{I}) \cong \mathbb{C}$. To prove that $\operatorname{Ext}^3(\mathscr{I},\mathscr{I})_0 = \operatorname{Ext}^3(\mathscr{I},\mathscr{I})$ vanishes we use the argument of [**135**]. By Serre duality, we have

$$\operatorname{Ext}^3(\mathscr{I},\mathscr{I}) \cong \operatorname{Hom}(\mathscr{I}, \mathscr{I} \otimes \omega_X)^\vee$$

and, setting $U = X \setminus Z$, the space $\mathrm{Hom}(\mathscr{I}, \mathscr{I} \otimes \omega_X)$ injects in

$$\mathrm{Hom}(\mathscr{I}|_U, \mathscr{I}|_U \otimes \omega_X|_U) = \mathrm{Hom}(\mathscr{O}_U, \omega_U) = \mathrm{H}^0(U, \omega_U)$$

because $\mathscr{I}$ is torsion free. However, since Z has codimension at least 2, by Hartog's Lemma we can extend sections of line bundles from the complement of Z to the whole of X, i.e.

$$\mathrm{H}^0(U, \omega_U) = \mathrm{H}^0(X, \omega_X) = \mathrm{H}^3(X, \mathscr{O}_X)^\vee = 0.$$

The required vanishing follows. □

In the Calabi–Yau case, $\mathrm{Ext}^3(\mathscr{I}, \mathscr{I})_0 \neq \mathrm{Ext}^3(\mathscr{I}, \mathscr{I})$, but $\mathbb{C} = \mathrm{Hom}(\mathscr{I}, \mathscr{I})^\vee = \mathrm{Ext}^3(\mathscr{I}, \mathscr{I})$ by Serre duality, and by surjectivity of the trace map to $\mathrm{H}^3(X, \mathscr{O}_X) \cong \mathbb{C}$ we still obtain $\mathrm{Ext}^3(\mathscr{I}, \mathscr{I})_0 = 0$.

Lemma 10.2.7 has the following consequence.

Theorem 10.2.10. *Let X be a smooth projective 3-fold such that $\mathrm{H}^i(X, \mathscr{O}_X) = 0$ for $i = 1, 2$. Then for every $n \geq 0$ there is a natural isomorphism*

$$\mathrm{Hilb}^n X \xrightarrow{\sim} M(1, 0, 0, -n),$$

where $M(1, 0, 0, -n)$ is the moduli space of stable torsion free sheaves with Chern character $(1, 0, 0, -n)$.

Proof. First of all, note that any torsion free sheaf $[E] \in M = M(1, 0, 0, -n)$ is of the form $E = \mathscr{I} \otimes \mathscr{L}$, where $\mathscr{I}$ is an ideal sheaf of colength n and $\mathscr{L}$ is a degree 0 line bundle. But by our assumption, the Picard variety of X is trivial, thus $\mathscr{L} = \mathscr{O}_X$ and M is a moduli space of *ideal sheaves*.

Define a map

$$f : \mathrm{Hilb}^n X \to M(1, 0, 0, -n)$$

by taking $\mathscr{O}_X \twoheadrightarrow \mathscr{O}_Z$ to the kernel $\mathscr{I}_Z$. Note that this works in families because flatness of $\mathscr{O}_Z$ implies flatness of $\mathscr{I}_Z$. Observe that such a map is bijective on $\mathbb{C}$-valued points. Therefore by the last bullet of Example 2.1.21 it is enough to show f is étale. This can be done infinitesimally.

The vanishing of the map $u : \mathrm{Hom}(\mathscr{I}, \mathscr{O}_X) \to T_{\mathscr{I}} \mathrm{Hilb}^n X$ coupled with Lemma 10.2.6 says that the long exact sequence (10.2.3) gives us the following two pieces of information:

$$\begin{aligned} T_{\mathscr{I}} \mathrm{Hilb}^n(X) &\xrightarrow{\sim} \mathrm{Ext}^1(\mathscr{I}, \mathscr{I}) \\ \mathrm{Ext}^1(\mathscr{I}, \mathscr{O}_Z) &\hookrightarrow \mathrm{Ext}^2(\mathscr{I}, \mathscr{I}). \end{aligned} \tag{10.2.5}$$

The left hand sides are tangents and obstructions (T_1 and T_2) for the natural tangent-obstruction theory[4] on the deformation functor $\mathsf{Def}_{[\mathscr{O}_X \twoheadrightarrow \mathscr{O}_Z]} = \mathsf{H}_{Z/X}$ attached to the point $[\mathscr{O}_X \twoheadrightarrow \mathscr{O}_Z] \in \mathrm{Hilb}^n X$ (see Example A.3.1). The right hand sides are the tangents and obstructions for the natural tangent-obstruction theory on the local deformation functor $\mathsf{Def}_{[\mathscr{I}]} = \mathsf{M}_{\mathscr{I}}$ attached to the ideal sheaf $\mathscr{I}$ (see Example A.3.1 again, and see Appendix A.3.2 for the 'Def' notation). As proved in [**38**, Proposition 2.2] (see Proposition A.3.3 for the precise statement[5]), the situation (10.2.5) ensures that f induces a natural isomorphism of deformation functors

$$\mathsf{Def}_{[\mathscr{O}_X \twoheadrightarrow \mathscr{O}_Z]} \xrightarrow{\sim} \mathsf{Def}_{[\mathscr{I}]}.$$

Then, a simple application of the formal criterion for étale maps shows that f is étale at $[\mathscr{O}_X \twoheadrightarrow \mathscr{O}_Z]$. □

See [**38**, Theorem 2.7] or [**176**, Proposition 2.8] for more details on the application of the formal étale criterion.

[4]see Appendix A.3 and the references therein for more details on tangent-obstruction theories.

[5]Note that we use that $\mathscr{I}$ is simple to conclude that $\mathsf{M}_{\mathscr{I}}$ is prorepresentable (cf. Example A.3.2), which is needed to apply Proposition A.3.3.

It is an important observation of Thomas [191] that, even though the standard tangent-obstruction theory on the Hilbert scheme, given by

$$T^i\Big|_{\mathscr{O}_X \twoheadrightarrow \mathscr{O}_Z} = \mathrm{Ext}^{i-1}(\mathscr{I}, \mathscr{O}_Z),$$

does not lead to a virtual fundamental class (because it is not two-term, due to the presence of higher Ext groups), the tangent-obstruction theory on the (isomorphic!) moduli space of ideal sheaves, given by

$$T^i\Big|_{\mathscr{I}} = \mathrm{Ext}^i(\mathscr{I}, \mathscr{I})_0,$$

does lead to a virtual fundamental class, thanks to the vanishings of Proposition 10.2.9.

10.2.2. Dimension and point-wise symmetry. In the perfect obstruction theory we want to build, the tangent space at a point

$$z = [\mathscr{O}_X \twoheadrightarrow \mathscr{O}_Z] \in \mathrm{Hilb}^n X$$

is $\mathrm{Ext}^1(\mathscr{I}, \mathscr{I}) = \mathrm{Hom}(\mathscr{I}, \mathscr{O}_Z)$, and the obstruction space is $\mathrm{Ext}^2(\mathscr{I}, \mathscr{I})$. The virtual dimension at z would then be

$$\mathrm{d}_z^{\mathrm{vir}} = \mathrm{ext}^1(\mathscr{I}, \mathscr{I}) - \mathrm{ext}^2(\mathscr{I}, \mathscr{I}). \tag{10.2.6}$$

At a given point in the moduli space, tangents will always be dual to obstructions. This is clear on the nose if X is Calabi–Yau, by Serre duality. In general, one can exploit the long exact sequence

$$\mathrm{Ext}^2(\mathscr{O}_X, \mathscr{I}) \to \mathrm{Ext}^2(\mathscr{I}, \mathscr{I}) \to \mathrm{Ext}^3(\mathscr{O}_Z, \mathscr{I}) \to \mathrm{Ext}^3(\mathscr{O}_X, \mathscr{I}).$$

The two outer terms vanish, because they sit in long exact sequences

$$\begin{tikzcd} \mathrm{Ext}^{i-1}(\mathscr{O}_X, \mathscr{O}_Z) \arrow[r] \arrow[d, equal, "(10.2.4)"'] & \mathrm{Ext}^i(\mathscr{O}_X, \mathscr{I}) \arrow[r] & \mathrm{Ext}^i(\mathscr{O}_X, \mathscr{O}_X) \arrow[d, equal, "(10.2.1)"] \\ 0 & & 0 \end{tikzcd}$$

for $i = 2, 3$. Therefore we obtain an isomorphism

$$\mathrm{Ext}^2(\mathscr{I}, \mathscr{I}) \xrightarrow{\sim} \mathrm{Ext}^3(\mathscr{O}_Z, \mathscr{I})$$

Dualising, this becomes an isomorphism

$$T_z \mathrm{Hilb}^n X = \mathrm{Hom}(\mathscr{I}, \mathscr{O}_Z) \xrightarrow{\sim} \mathrm{Ext}^2(\mathscr{I}, \mathscr{I})^\vee.$$

To sum up, if we manage to produce a perfect obstruction theory with $\mathrm{Ext}^1(\mathscr{I}, \mathscr{I})$, $\mathrm{Ext}^2(\mathscr{I}, \mathscr{I})$ as tangents and obstructions, it will be 0-dimensional and 'point-wise symmetric'. However, point-wise symmetry does not imply global symmetry, cf. Definition 9.1.5.

10.2.3. Construction of the obstruction theory. To ease notation, let us shorten

$$\mathrm{H} = \mathrm{Hilb}^n X,$$

where X is a smooth projective 3-fold that is either Calabi–Yau or satisfies $\mathrm{H}^{>0}(X, \mathscr{O}_X) = 0$. We point out that what follows is a very standard construction and works for all 3-folds with minor modifications — see for instance [109] for a general treatment and [176] for an adaptation to the case of Quot schemes of (certain) locally free sheaves on a 3-fold.

Consider the universal ideal sheaf exact sequence

$$0 \to \mathfrak{I} \to \mathscr{O}_{X \times \mathrm{H}} \to \mathscr{O}_{\mathcal{Z}} \to 0$$

living over $X \times \mathrm{H}$. The trace map induces a canonical split exact triangle

$$\mathbf{R}\mathscr{H}\!om(\mathfrak{I}, \mathfrak{I})_0 \longrightarrow \mathbf{R}\mathscr{H}\!om(\mathfrak{I}, \mathfrak{I}) \xrightarrow{\mathrm{tr}_{\mathfrak{I}}} \mathscr{O}_{X \times \mathrm{H}}. \tag{10.2.7}$$

Let $q\colon X\times \mathrm{H}\to X$ and $p\colon X\times\mathrm{H}\to\mathrm{H}$ be the projections. The truncated cotangent complex $\mathbb{L}_{X\times\mathrm{H}}$ splits as $p^*\mathbb{L}_{\mathrm{H}}\oplus q^*\mathbb{L}_X$, so the truncated Atiyah class (10.1.5)

$$A(\mathfrak{I})\in \operatorname{Ext}^1_{X\times\mathrm{H}}(\mathfrak{I},\mathfrak{I}\otimes\mathbb{L}_{X\times\mathrm{H}})$$

projects onto the factor

$$\begin{aligned}\operatorname{Ext}^1_{X\times\mathrm{H}}(\mathfrak{I},\mathfrak{I}\otimes p^*\mathbb{L}_{\mathrm{H}}) &= \operatorname{Ext}^1_{X\times\mathrm{H}}(\mathfrak{I}^\vee\overset{\mathbf{L}}{\otimes}\mathfrak{I},p^*\mathbb{L}_{\mathrm{H}})\\ &= \operatorname{Ext}^1_{X\times\mathrm{H}}(\mathbf{R}\mathscr{H}om(\mathfrak{I},\mathfrak{I}),p^*\mathbb{L}_{\mathrm{H}}),\end{aligned}$$

which by the splitting of (10.2.7) can be further projected onto

$$\operatorname{Ext}^1_{X\times\mathrm{H}}(\mathbf{R}\mathscr{H}om(\mathfrak{I},\mathfrak{I})_0,p^*\mathbb{L}_{\mathrm{H}}).$$

By Grothendieck duality [**151**] along the smooth proper 3-dimensional morphism p, one has

$$\mathbf{R}p_*\,\mathbf{R}\mathscr{H}om_{X\times\mathrm{H}}(\mathscr{F},p^*\mathscr{G}\otimes\omega_p[3]) \;\cong\; \mathbf{R}\mathscr{H}om_{\mathrm{H}}(\mathbf{R}p_*\mathscr{F},\mathscr{G}) \tag{10.2.8}$$

for $\mathscr{F}\in\mathbf{D}(\operatorname{QCoh}(X\times\mathrm{H}))$ and $\mathscr{G}\in\mathbf{D}(\operatorname{QCoh}\mathrm{H})$, where $\omega_p=q^*\omega_X$ is the relative dualising sheaf. We reviewed this isomorphism in general in Equation (2.2.1). Setting $\mathscr{F}=\mathbf{R}\mathscr{H}om(\mathfrak{I},\mathfrak{I})_0\otimes\omega_p$ and $\mathscr{G}=\mathbb{L}_{\mathrm{H}}$ in (10.2.8), we obtain

$$\mathbf{R}p_*\,\mathbf{R}\mathscr{H}om_{X\times\mathrm{H}}(\mathbf{R}\mathscr{H}om(\mathfrak{I},\mathfrak{I})_0\otimes\omega_p,p^*\mathbb{L}_{\mathrm{H}}\otimes\omega_p[3]) \;\cong\; \mathbf{R}\mathscr{H}om_{\mathrm{H}}(\mathbf{R}p_*(\mathbf{R}\mathscr{H}om(\mathfrak{I},\mathfrak{I})_0\otimes\omega_p),\mathbb{L}_{\mathrm{H}}),$$

which after applying $h^{-2}\circ\mathbf{R}\Gamma$ becomes

$$\begin{aligned}\operatorname{Ext}^1_{X\times\mathrm{H}}(\mathbf{R}\mathscr{H}om(\mathfrak{I},\mathfrak{I})_0,p^*\mathbb{L}_{\mathrm{H}}) &\cong \operatorname{Ext}^{-2}_{\mathrm{H}}(\mathbf{R}p_*(\mathbf{R}\mathscr{H}om(\mathfrak{I},\mathfrak{I})_0\otimes\omega_p),\mathbb{L}_{\mathrm{H}})\\ &=\operatorname{Hom}(\mathbb{E},\mathbb{L}_{\mathrm{H}}),\end{aligned}$$

where we have set

$$\mathbb{E}=\mathbf{R}p_*(\mathbf{R}\mathscr{H}om(\mathfrak{I},\mathfrak{I})_0\otimes\omega_p)[2].$$

So the truncated Atiyah class $A(\mathfrak{I})$ determines a morphism

$$\phi\colon\mathbb{E}\to\mathbb{L}_{\mathrm{H}}.$$

We now verify that $\mathbb{E}$ is of perfect amplitude contained in $[-1,0]$. Let us shorten $\mathbb{H}=\mathbf{R}\mathscr{H}om(\mathfrak{I},\mathfrak{I})_0$. Note that $\mathbb{H}$ is canonically self-dual, by Exercise 10.1.2. The complex $\mathbf{R}p_*\mathbb{H}$ is isomorphic in the derived category to a two-term complex of vector bundles $\mathcal{T}^\bullet=[\mathcal{T}^1\to\mathcal{T}^2]$ concentrated in degrees 1 and 2. More precisely, as in [**109**, Lemma 4.2], the identification $\mathbf{R}p_*\mathbb{H}=\mathcal{T}^\bullet$ follows from the vanishings

$$\operatorname{Ext}^i(\mathscr{I},\mathscr{I})_0=0,\quad i\neq 1,2,$$

that we proved in Proposition 10.2.9. On the other hand, we have

$$\begin{aligned}(\mathbf{R}p_*\mathbb{H})^\vee[-1] &= \mathbf{R}\mathscr{H}om_{\mathrm{H}}(\mathbf{R}p_*\mathbb{H},\mathscr{O}_{\mathrm{H}})[-1] &&\\ &= \mathbf{R}p_*\,\mathbf{R}\mathscr{H}om_{X\times\mathrm{H}}(\mathbb{H},\omega_p[3])[-1] && \text{Grothendieck duality}\\ &= \mathbf{R}p_*\,\mathbf{R}\mathscr{H}om_{X\times\mathrm{H}}(\mathbb{H},\omega_p)[2] && \text{shift}\\ &= \mathbf{R}p_*\,\mathbf{R}\mathscr{H}om_{X\times\mathrm{H}}(\mathbb{H}^\vee,\omega_p)[2] && \mathbb{H}=\mathbb{H}^\vee\\ &= \mathbf{R}p_*(\mathbb{H}\otimes\omega_p)[2] &&\\ &= \mathbb{E}. &&\end{aligned}$$

Therefore $\mathbb{E}$ is perfect in $[-1,0]$.

We can now prove Theorem 10.2.1 (1).

Theorem 10.2.11. *The morphism $\phi\colon\mathbb{E}\to\mathbb{L}_{\mathrm{H}}$ constructed above is a* 0*-dimensional perfect obstruction theory on* $\operatorname{Hilb}^n X$*.*

PROOF. For any point $z = [\mathscr{I} \hookrightarrow \mathscr{O}_X \twoheadrightarrow \mathscr{O}_Z]$, with inclusion $\iota_z : z \hookrightarrow \mathrm{H}$, one has

$$h^{i-1}(\mathbf{L}\iota_z^* \mathbb{E}^\vee) = h^i(\mathbf{L}\iota_z^* \mathbf{R}p_* \mathbb{H}) = \mathrm{Ext}^i(\mathscr{I}, \mathscr{I}), \quad i = 1, 2.$$

Therefore we have $\mathrm{d}_z^{\mathrm{vir}} = \mathrm{rk}\, \mathbb{E} = \mathrm{ext}^1(\mathscr{I}, \mathscr{I}) - \mathrm{ext}^2(\mathscr{I}, \mathscr{I}) = 0$, as observed in Section 10.2.2. The fact that $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is surjective is a special case of [**109**, Corollary 4.3], to which we refer for details. $\square$

A perfect obstruction theory induces a virtual fundamental class [**19**]. This was explained in Section 9.1 only for the local model of a perfect obstruction theory, but more details will be given in Appendix C.5. Thus we obtain the following result.

Corollary 10.2.12. *The perfect obstruction theory $\phi : \mathbb{E} \to \mathbb{L}_{\mathrm{H}}$ induces a virtual fundamental class*

$$\left[\mathrm{Hilb}^n X\right]^{\mathrm{vir}} \in A_0(\mathrm{Hilb}^n X).$$

We are ready to define the Donaldson–Thomas invariant of a projective 3-fold.

DEFINITION 10.2.13. The *degree* 0 *Donaldson–Thomas invariants* of the smooth projective 3-fold X are defined by taking the degree of the virtual fundamental class,

$$\mathsf{DT}_n^X = \int_{[\mathrm{Hilb}^n X]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

Remark 10.2.14. If one does not impose the vanishing $\mathrm{H}^i(X, \mathscr{O}_X) = 0$ for $i = 1, 2$ in the projective Calabi–Yau 3-fold case, then abelian 3-folds are still in the game. For an abelian 3-fold X, the obstruction theories on the various moduli spaces of stable sheaves, such as $M = M(1, 0, 0, -n)$, can still be constructed but will contain a trivial summand, so the associated virtual count vanishes. In fact, one has $M = \mathrm{Hilb}^n X \times \mathrm{Pic}^0 X$. The trivial summand corresponding to the factor $\mathrm{Pic}^0 X$ can be removed (i.e. the determinant can be fixed and an honest obstruction theory on $\mathrm{Hilb}^n X$ can be constructed), but the DT invariant *still vanishes*,

$$\mathsf{DT}_n^X = 0, \quad n > 0.$$[6]

A modified version of the obstruction theory has been constructed by Gulbrandsen in [**90**], where he showed that to get a nontrivial invariant one has to fix the *codeterminant*, too. This amounts to looking at the closed subscheme $K^n X \subset \mathrm{Hilb}^n X$, called the *generalised Kummer scheme*, consisting of subschemes $Z \subset X$ whose underlying 0-cycle sums up to $0 \in X$ in the group law of the abelian 3-fold X. (When a curve class is present in the fixed Chern character, the condition on the Hilbert scheme of curves and points that corresponds to fixing the codeterminant is a bit more complicated). The corresponding Donaldson–Thomas invariants have been computed in [**182, 91**].

10.2.4. Symmetry of the obstruction theory. Let us prove Theorem 10.2.1 (2), i.e. the symmetry of the obstruction theory $\phi : \mathbb{E} \to \mathbb{L}_{\mathrm{Hilb}^n X}$ in the Calabi–Yau case. Any trivialisation $\omega_X \xrightarrow{\sim} \mathscr{O}_X$ induces, by pullback along $q : X \times \mathrm{H} \to X$, a trivialisation $\omega_p \xrightarrow{\sim} \mathscr{O}_{X \times \mathrm{H}}$, that we can use to construct an isomorphism

$$\mathbb{E}[-2] \xrightarrow{\sim} \mathbf{R}p_* \mathbb{H}.$$

Dualising and shifting the last isomorphism, we get

$$\theta : \mathbb{E} = (\mathbf{R}p_* \mathbb{H})^\vee[-1] \xrightarrow{\sim} \mathbb{E}^\vee[1],$$

The symmetry condition $\theta^\vee[1] = \theta$ follows from [**20**, Lemma 1.23].

This completes the proof of Theorem 10.2.1 (2).

[6]You proved a similar vanishing in Exercise 8.5.23.

Remark 10.2.15. More generally, a perfect obstruction theory of virtual dimension 0 on the Quot scheme

$$\mathrm{Quot}_X(F,n)$$

is constructed in [**176**], for F either a simple, rigid locally free sheaf on a projective Calabi–Yau 3-fold (in which case we again get symmetry), or an exceptional locally free sheaf on a smooth projective 3-fold such that $\mathrm{H}^{>0}(X,\mathscr{O}_X)=0$. In the toric case, an equivariant structure on F induces a canonical equivariant structure on the perfect obstruction theory [**62**, Proposition 9.2].

10.2.5. Equivariance of the obstruction theory. The structure sheaf $\mathscr{O}_X$ of the toric 3-fold X is naturally $\mathbb{T}$-equivariant (Example 9.2.6) via the action $\sigma_X\colon \mathbb{T}\times X\to X$, i.e. we are given an isomorphism $\vartheta\colon p_2^*\mathscr{O}_X \xrightarrow{\sim} \sigma_X^*\mathscr{O}_X$. This induces a canonical $\mathbb{T}$-action on H (Exercise 8.5.1), denoted $\sigma_{\mathrm{H}}\colon \mathbb{T}\times \mathrm{H}\to \mathrm{H}$. Let

$$0\to \mathfrak{I}\to \mathscr{O}_{X\times \mathrm{H}} \xrightarrow{\xi} \mathscr{O}_{\mathcal{Z}}\to 0 \tag{10.2.9}$$

be the universal exact sequence on $X\times\mathrm{H}$. Construct the morphism

$$\varphi\colon X\times \mathrm{H}\times\mathbb{T}\to X\times \mathrm{H},\quad (x,z,t)\mapsto (\sigma_X(t,x),\sigma_{\mathrm{H}}(t^{-1},z)).$$

We view this as a $\mathbb{T}$-action on $X\times\mathrm{H}$. Note that $q\circ\varphi=\sigma_X\circ p_{31}$, where $p_{31}\colon X\times\mathrm{H}\times\mathbb{T}\to\mathbb{T}\times X$ is the projection. The moduli map $\mathrm{H}\times\mathbb{T}\to\mathrm{H}$ corresponding to the family of quotients

$$\varphi^*\xi\circ p_{31}^*\vartheta\colon \mathscr{O}_{X\times\mathrm{H}\times\mathbb{T}}=p_{31}^*p_2^*\mathscr{O}_X \xrightarrow{\sim} p_{31}^*\sigma_X^*\mathscr{O}_X=\varphi^*\mathscr{O}_{X\times\mathrm{H}} \xrightarrow{\varphi^*\xi} \varphi^*\mathscr{O}_{\mathcal{Z}}$$

is easily seen to agree with the first projection $p_1\colon \mathrm{H}\times\mathbb{T}\to\mathrm{H}$. Let then $\tau=\mathrm{id}_X\times p_1\colon X\times\mathrm{H}\times\mathbb{T}\to X\times\mathrm{H}$ be the projection. Since $\varphi^*\xi\circ p_{12}^*\vartheta$ corresponds to $p_1\colon \mathrm{H}\times\mathbb{T}\to\mathrm{H}$, by the universal property of (H,ξ) we obtain an isomorphism of surjections $\tau^*\xi\xrightarrow{\sim}\varphi^*\xi$ that extends to an isomorphism of short exact sequences

$$\begin{tikzcd}
\tau^*\mathfrak{I} \arrow[r, hook] \arrow[d, "\wr"] & \mathscr{O}_{X\times\mathrm{H}\times\mathbb{T}} \arrow[r, two heads, "\tau^*\xi"] \arrow[d, "\wr"] & \tau^*\mathscr{O}_{\mathcal{Z}} \arrow[d, "\wr"] \\
\varphi^*\mathfrak{I} \arrow[r, hook] & \varphi^*\mathscr{O}_{X\times\mathrm{H}} \arrow[r, two heads, "\varphi^*\xi"] & \varphi^*\mathscr{O}_{\mathcal{Z}}
\end{tikzcd}$$

on $X\times\mathrm{H}\times\mathbb{T}$, where the middle vertical isomorphism is $p_{31}^*\vartheta$ and is a $\mathbb{T}$-equivariant structure on $\mathscr{O}_{X\times\mathrm{H}}$ because ϑ is. The diagram confirms that the $\mathbb{T}$-equivariant structure ϑ on $\mathscr{O}_X$ induces a canonical $\mathbb{T}$-equivariant structure on the universal short exact sequence (10.2.9). In particular, $\mathfrak{I}$ is naturally $\mathbb{T}$-equivariant.

Now, recall that the perfect obstruction theory ϕ from Theorem 10.2.11 is obtained by projecting the truncated Atiyah class $A(\mathfrak{I})\in \mathrm{Ext}^1_{X\times\mathrm{H}}(\mathfrak{I},\mathfrak{I}\otimes\mathbb{L}_{X\times\mathrm{H}})$ onto the Ext group

$$\begin{aligned}
\mathrm{Ext}^1_{X\times\mathrm{H}}(\mathbf{R}\mathscr{H}\!om(\mathfrak{I},\mathfrak{I})_0,p^*\mathbb{L}_{\mathrm{H}}) &= \mathrm{Ext}^{-2}_{X\times\mathrm{H}}(\mathbf{R}\mathscr{H}\!om(\mathfrak{I},\mathfrak{I})_0\otimes\omega_p,p^*\mathbb{L}_{\mathrm{H}}\otimes\omega_p[3])\\
&= \mathrm{Ext}^{-2}_{X\times\mathrm{H}}(\mathbf{R}\mathscr{H}\!om(\mathfrak{I},\mathfrak{I})_0\otimes\omega_p,p^!\mathbb{L}_{\mathrm{H}})\\
&\cong\ \mathrm{Ext}^{-2}_{\mathrm{H}}(\mathbf{R}p_*(\mathbf{R}\mathscr{H}\!om(\mathfrak{I},\mathfrak{I})_0\otimes\omega_p),\mathbb{L}_{\mathrm{H}}).
\end{aligned}$$

The last isomorphism is Grothendieck duality along p. Now we need three ingredients to confirm that ϕ lifts to the equivariant derived category:

- The Atiyah class $A(\mathfrak{I})$ is a $\mathbb{T}$-invariant extension,
- Grothendieck duality preserves $\mathbb{T}$-invariant extensions, and
- $\mathbb{T}$-invariant extensions correspond to morphisms in the equivariant derived category.

These assertions are proved in [**177**].

The proof of Theorem 10.2.1 (3) is complete.

10.3. Virtual localisation for $\mathrm{Hilb}^n X$

10.3.1. K-theory notation. Let $\mathbb{T} = (\mathbb{C}^\times)^g$ be a torus, with character lattice $\widehat{\mathbb{T}} = \mathrm{Hom}(\mathbb{T}, \mathbb{C}^\times) \cong \mathbb{Z}^g$. Let $K_0^{\mathbb{T}}(\mathrm{pt})$ be the K-group of the category of finite dimensional $\mathbb{T}$-representations. We let $t_1, \ldots, t_g$ be the generators (also called coordinates) of $K_0^{\mathbb{T}}(\mathrm{pt})$, with t_j being the class of the representation $\mathbb{T} \to \mathbb{C}^\times$ sending $(\theta_1, \ldots, \theta_g) \mapsto \theta_j$. We know that $\mathrm{H}^*_{\mathbb{T}} = \mathbb{Q}[s_1, \ldots, s_g]$ is the equivariant cohomology ring of the torus (with each s_j sitting in degree 2 as ever), where we have set

$$s_j = c_1^{\mathbb{T}}(t_j), \quad j = 1, \ldots, g.$$

Any finite dimensional $\mathbb{T}$-representation V splits as a sum of 1-dimensional representations called the *weights* of V. Each weight corresponds to a character $\mu \in \widehat{\mathbb{T}}$, and in turn each character corresponds to a monomial $t^\mu = t_1^{\mu_1} \cdots t_g^{\mu_g}$ in the coordinates of $\mathbb{T}$. The map

$$\mathrm{tr} \colon K_0^{\mathbb{T}}(\mathrm{pt}) \to \mathbb{Z}\big[t^\mu \mid \mu \in \widehat{\mathbb{T}} \big] = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_g^{\pm 1}], \tag{10.3.1}$$

denoted $V \mapsto \mathrm{tr}(V)$, sending the class of a $\mathbb{T}$-module to its decomposition into weight spaces is a ring isomorphism, where tensor product on the left corresponds to the natural multiplication on the right. We will therefore identify a (virtual) $\mathbb{T}$-module with its trace.

Example 10.3.1. Let $\mathbb{T} = (\mathbb{C}^\times)^g$ act on $S_g = \mathbb{C}[x_1, \ldots, x_g]$ via the standard action

$$(t_1, \ldots, t_g) \cdot (x_1, \ldots, x_g) = (t_1 x_1, \ldots, t_g x_g). \tag{10.3.2}$$

Let $0 \in \mathbb{A}^g = \operatorname{Spec} S_g$ be the origin. Then the tangent space $T_0 \mathbb{A}^g = \bigoplus_{1 \le i \le g} \frac{\partial}{\partial x_i} \cdot \mathbb{C}$ is a g-dimensional $\mathbb{T}$-representation with the i-th summand weighted by t_i^{-1}. Thus

$$\mathrm{tr}(T_0 \mathbb{A}^g) = \sum_{1 \le i \le g} t_i^{-1} \in \mathbb{Z}\big[t_1^{\pm 1}, \ldots, t_g^{\pm 1}\big]$$
$$e^{\mathbb{T}}(T_0 \mathbb{A}^g) = (-1)^g s_1 \cdots s_g \in \mathcal{H}_{\mathbb{T}}$$

and in particular the canonical line bundle $K_{\mathbb{A}^g}$ is equivariantly nontrivial, for

$$K_{\mathbb{A}^g} = \mathscr{O}_{\mathbb{A}^g} \otimes t_1 \cdots t_g.$$

In fact, we can extend the target of (10.3.1) to the ring $\mathbb{Q}(\!(t_1, \ldots, t_g)\!)$, and consider the character of a possibly infinite dimensional $\mathbb{T}$-module as an element of this ring. For instance, via the standard $\mathbb{T}$-action (10.3.2), we view S_g as an infinite dimensional $\mathbb{T}$-module, and we can write

$$\mathrm{tr}(S_g) = \sum_{k_1, \ldots, k_g \in \mathbb{Z}_{\ge 0}} \prod_{i=1}^g t_i^{k_i} = \prod_{i=1}^g (1 - t_i)^{-1}. \tag{10.3.3}$$

We denote by $\overline{(\cdot)}$ the involution (the 'dual') of $\mathbb{Q}(\!(t_1, \ldots, t_g)\!)$ sending $t_i \mapsto t_i^{-1}$. Then note that we have an identity

$$\mathrm{tr}(S_g) = (-1)^g \frac{\overline{\mathrm{tr}(S_g)}}{t_1 \cdots t_g}. \tag{10.3.4}$$

Let F, G be two $\mathbb{T}$-equivariant sheaves on $\mathbb{A}^g$, so that $\mathrm{Ext}^\bullet(F, G)$ is a $\mathbb{T}$-representation. Define $\chi(F, G) = \sum_{i \ge 0} (-1)^i \mathrm{Ext}^i(F, G)$. If $F = \mathscr{O}_{\mathbb{A}^g}$, we simply write $\chi(G)$. Note that the trace of $\chi(\mathscr{O}_{\mathbb{A}^g})$ is precisely $\mathrm{tr}(S_g)$ as computed in Equation (10.3.3), and we shall make no distinction between the two.

The following result, that we take from [**158**, Formula (3.4.14)], establishes a very useful relation.

Lemma 10.3.2. *Let F, G be two $\mathbb{T}$-equivariant sheaves on $\mathbb{A}^g$. Then*

$$\chi(F, G) = \frac{\overline{\chi(F)} \chi(G)}{\overline{\chi(\mathscr{O}_{\mathbb{A}^g})}}. \tag{10.3.5}$$

PROOF. The origin $0 \in \mathbb{A}^g$ is the unique $\mathbb{T}$-fixed point of $\mathbb{A}^g$. The stalk $F \otimes \mathscr{O}_0$ defines a K-theory class denoted $\chi(F \otimes \mathscr{O}_0)$, and one has the relation

$$\chi(F) = \chi(F \otimes \mathscr{O}_0)\chi(\mathscr{O}_{\mathbb{A}^g}). \tag{10.3.6}$$

Moreover, since stalks and tensor products commute, we have

$$\chi((F \otimes F') \otimes \mathscr{O}_0) = \chi((F \otimes \mathscr{O}_0) \otimes (F' \otimes \mathscr{O}_0)) = \chi(F \otimes \mathscr{O}_0)\chi(F' \otimes \mathscr{O}_0). \tag{10.3.7}$$

Also note that Serre duality $(-1)^g\overline{\chi(F', F)} = \chi(F, F' \otimes K_{\mathbb{A}^g})$ specialises (when F' is trivial) to

$$(-1)^g\overline{\chi(F)} = \chi(F, K_{\mathbb{A}^g}) = \chi(F^* \otimes K_{\mathbb{A}^g}) = \chi(F^*)t_1 \cdots t_g. \tag{10.3.8}$$

So we can compute

$$\begin{aligned}
\chi(F,G) &= \chi(F^* \otimes G) \\
&= \chi((F^* \otimes G) \otimes \mathscr{O}_0)\chi(\mathscr{O}_{\mathbb{A}^g}) && \text{by (10.3.6)} \\
&= \chi(F^* \otimes \mathscr{O}_0)\chi(G \otimes \mathscr{O}_0)\chi(\mathscr{O}_{\mathbb{A}^g}) && \text{by (10.3.7)} \\
&= \chi(F^*)\chi(G \otimes \mathscr{O}_0) && \text{by (10.3.6)} \\
&= \frac{(-1)^g\overline{\chi(F)}}{t_1 \cdots t_g}\chi(G \otimes \mathscr{O}_0) && \text{by (10.3.8)} \\
&= \frac{\chi(\mathscr{O}_{\mathbb{A}^g})}{\overline{\chi(\mathscr{O}_{\mathbb{A}^g})}}\overline{\chi(F)}\chi(G \otimes \mathscr{O}_0) && \text{by (10.3.4)} \\
&= \frac{\overline{\chi(F)}\chi(G)}{\overline{\chi(\mathscr{O}_{\mathbb{A}^g})}} && \text{by (10.3.6)}
\end{aligned}$$

as required. □

10.3.2. Projective toric 3-folds. Let X be a smooth projective toric 3-fold with open dense torus $(\mathbb{C}^\times)^3 = \mathbb{T} \subset X$. Let $\Delta(X)$ be its Newton polytope (see Figure 10.1 for two examples) and

$$X^{\mathbb{T}} = \{\, p_\alpha \mid \alpha \in \Delta(X)\,\} \subset X$$

its fixed point locus, of cardinality $\chi(X)$.

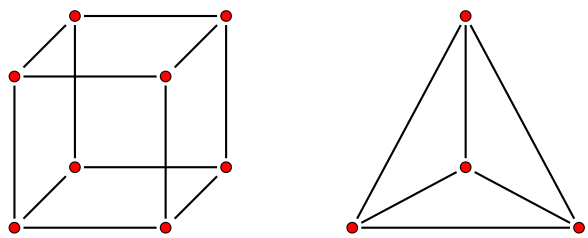

FIGURE 10.1. The Newton polytopes of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^3$.

For each α, there exists a a canonical $\mathbb{T}$-invariant open affine chart

$$U_\alpha = \operatorname{Spec} R_\alpha \subset X, \quad R_\alpha = \mathbb{C}[x_1, x_2, x_3],$$

such that the action of $\mathbb{T}$ on R_α is the standard one (10.3.2),

$$(t_1, t_2, t_3) \cdot (x_1, x_2, x_3) = (t_1x_1, t_2x_2, t_3x_3).$$

Let $K_0^{\mathbb{T}}(U_\alpha)$ be the Grothendieck ring of $\mathbb{T}$-equivariant coherent sheaves on U_α. We again have a ring homomorphism

$$\operatorname{tr}\colon K_0^{\mathbb{T}}(U_\alpha) \xrightarrow{\sim} \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}] \hookrightarrow \mathbb{Q}(\!(t_1, t_2, t_3)\!), \quad V \mapsto \operatorname{tr}(V),$$

sending a $\mathbb{T}$-equivariant coherent sheaf to its character.

Example 10.3.3. Let $\alpha \in \Delta(X)$. As in Equation (10.3.3), we have

$$\mathrm{tr}(R_\alpha) = \chi(\mathscr{O}_{U_\alpha}) = \sum_{\square \in \mathbb{Z}^3_{\geq 0}} t^\square = \sum_{(k_1,k_2,k_3)\in \mathbb{Z}^3_{\geq 0}} t_1^{k_1} t_2^{k_2} t_3^{k_3} = \frac{1}{(1-t_1)(1-t_2)(1-t_3)}. \tag{10.3.9}$$

Example 10.3.4. Let $\alpha \in \Delta(X)$. As we saw in Example 10.3.1, the tangent space $T_{p_\alpha}X = T_{p_\alpha}U_\alpha$ has character $t_1^{-1}+t_2^{-1}+t_3^{-1}$. Its equivariant Euler class is $e^{\mathbb{T}}(T_{p_\alpha}X) = -s_1 s_2 s_3 \in \mathcal{H}_{\mathbb{T}}$ and

$$K_{U_\alpha} = \det \Omega_{U_\alpha} = \mathscr{O}_{\mathbb{A}^3} \otimes t_1 t_2 t_3.$$

10.3.3. Donaldson–Thomas invariants via virtual localisation. In Theorem 10.2.1 we stated the existence of a 0-dimensional $\mathbb{T}$-equivariant perfect obstruction theory on $\mathrm{Hilb}^n X$, for any smooth quasiprojective toric 3-fold X with open torus $\mathbb{T} \subset X$. Granting for a second that the $\mathbb{T}$-fixed locus of $\mathrm{Hilb}^n X$ consists of finitely many fat points $\mathrm{S}(\mathscr{I}_Z)$, one for each $\mathbb{T}$-fixed ideal $\mathscr{I}_Z$, (something stronger will be proved below: they are all reduced!), the virtual localisation formula reads

$$\left[\mathrm{Hilb}^n X\right]^{\mathrm{vir}} = \iota_* \sum_{\mathscr{I}_Z} \frac{[\mathrm{S}(\mathscr{I}_Z)]^{\mathrm{vir}}}{e^{\mathbb{T}}(N_Z^{\mathrm{vir}})} \in A_0^{\mathbb{T}}(\mathrm{Hilb}^n X) \otimes_{\mathrm{H}^*_{\mathbb{T}}} \mathcal{H}_{\mathbb{T}}, \tag{10.3.10}$$

where

(i) $\iota\colon (\mathrm{Hilb}^n X)^{\mathbb{T}} \hookrightarrow \mathrm{Hilb}^n X$ is the inclusion of the $\mathbb{T}$-fixed locus,
(ii) the sum is over $\mathbb{T}$-fixed ideals $\mathscr{I}_Z \in (\mathrm{Hilb}^n X)^{\mathbb{T}}$,
(iii) $\mathrm{S}(\mathscr{I}_Z)$ is the largest closed subscheme of $(\mathrm{Hilb}^n X)^{\mathbb{T}}$ supported at the point $\mathscr{I}_Z$, and
(iv) $N_Z^{\mathrm{vir}} = \mathbb{E}^\vee\big|_{\mathscr{I}_Z}^{\mathsf{mov}} \in K_0^{\mathbb{T}}(\mathrm{pt})$ is the (K-theory class of the) virtual normal bundle to $\mathrm{S}(\mathscr{I}_Z) \hookrightarrow \mathrm{Hilb}^n X$, i.e. the moving part of the *virtual tangent space*[7]

$$T_Z^{\mathrm{vir}} = \mathbb{E}^\vee\big|_{\mathscr{I}_Z} = \mathrm{Ext}^1(\mathscr{I}_Z,\mathscr{I}_Z) - \mathrm{Ext}^2(\mathscr{I}_Z,\mathscr{I}_Z).$$

We have the following result.

Proposition 10.3.5. *Let X be a smooth quasiprojective toric 3-fold with open torus $\mathbb{T} \subset X$. Then*

(1) *There is an isomorphism*

$$(\mathrm{Hilb}^n X)^{\mathbb{T}} \cong \coprod_{\sum_\alpha n_\alpha = n} \prod_{\alpha \in \Delta(X)} (\mathrm{Hilb}^{n_\alpha} U_\alpha)^{\mathbb{T}}.$$

In particular, $(\mathrm{Hilb}^n X)^{\mathbb{T}}$ is finite, hence proper. By the correspondence realised in Exercise 8.5.12, it is parametrised by tuples of plane partitions $\{\, \pi_\alpha \subset \mathbb{Z}^3_{\geq 0} \mid \alpha \in \Delta(X)\,\}$.

(2) *At a $\mathbb{T}$-fixed ideal $\mathscr{I}_Z$, we have $\mathrm{Ext}^1(\mathscr{I}_Z,\mathscr{I}_Z)^{\mathbb{T}} = \mathrm{Ext}^2(\mathscr{I}_Z,\mathscr{I}_Z)^{\mathbb{T}} = 0$.*

(3) *If X is Calabi–Yau, at a $\mathbb{T}$-fixed ideal $\mathscr{I}_Z$, we have $\mathrm{Ext}^1(\mathscr{I}_Z,\mathscr{I}_Z)^{\mathbb{T}_0} = \mathrm{Ext}^2(\mathscr{I}_Z,\mathscr{I}_Z)^{\mathbb{T}_0} = 0$.*

PROOF. Let us prove (1). A $\mathbb{T}$-fixed ideal $\mathscr{I}_Z \in (\mathrm{Hilb}^n X)^{\mathbb{T}}$ corresponds to a finite subscheme $Z \subset X$ supported on the (finite) fixed locus $X^{\mathbb{T}} \subset X$. It is important to notice that each invariant chart $U_\alpha \subset X$ contains exactly *one* fixed point p_α, which corresponds to the origin of $\mathbb{A}^3$ under the identification $U_\alpha \cong \mathbb{A}^3$. Now, for each $\alpha \in \Delta(X)$, the restriction

$$I_\alpha = \mathscr{I}_Z\big|_{U_\alpha} \subset R_\alpha$$

is a monomial ideal, thus defining a point of $(\mathrm{Hilb}^{n_\alpha} \mathbb{A}^3)^{\mathbb{T}}$, where $n_\alpha = \chi(R_\alpha/I_\alpha)$. On the other hand, each collection of monomial ideals $(I_\alpha)_{\alpha\in\Delta(X)}$ whose colengths sum up to n gives rise to a $\mathbb{T}$-fixed ideal $\mathscr{I}_Z = \prod_\alpha I_\alpha$. It is straightforward to check that this correspondence works in families, too. More details on this can be found in [62, Lemma 9.4].

Item (2) is proved in [135], whereas (3) is proved in [20, Lemma 4.1 (b)] for $\mathbb{A}^3$, and the general case follows easily. □

Remark 10.3.6. Concretely, Proposition 10.3.5 (3) means that no power of $t_1 t_2 t_3$ is a weight of the representation $\mathrm{Ext}^i(\mathscr{I}_Z,\mathscr{I}_Z)$. This fact will be useful in Section 10.4.2.

[7] See Definition B.3.3 for the definition of the 'classical' virtual tangent bundle.

We have the following immediate consequence.

Corollary 10.3.7. *Let X be a smooth quasiprojective toric 3-fold. Let $\mathscr{I}_Z$ be a $\mathbb{T}$-fixed ideal. Then*

(1) $\mathsf{S}(\mathscr{I}_Z)$ *is reduced and carries the trivial $\mathbb{T}$-fixed obstruction theory. Thus $[\mathsf{S}(\mathscr{I}_Z)]^{\mathrm{vir}} = [\mathscr{I}_Z]$.*
(2) *the virtual tangent space T_Z^{vir} is entirely $\mathbb{T}$-movable, hence*

$$N_Z^{\mathrm{vir}} = T_Z^{\mathrm{vir}} = \mathrm{Ext}^1(\mathscr{I}_Z,\mathscr{I}_Z) - \mathrm{Ext}^2(\mathscr{I}_Z,\mathscr{I}_Z). \tag{10.3.11}$$

(3) *The same holds for the $\mathbb{T}_0$-equivariant symmetric obstruction theory in the Calabi–Yau case.*

In Definition 10.2.13 we defined $\mathsf{DT}_n^X \in \mathbb{Z}$ as the degree of the virtual class $[\mathrm{Hilb}^n X]^{\mathrm{vir}}$ in the case X is projective. In the toric *quasiprojective* case, we will define DT_n^X via equivariant residues (Definition 10.3.8 below), i.e. by integrating[8] the right hand side of the virtual localisation formula (10.3.10). Exploiting Corollary 10.3.7 (1)–(2), the right hand side of Equation (10.3.10) after $\mathbb{T}$-equivariant integration becomes

$$\sum_{\mathscr{I}_Z}\int_{[S(\mathscr{I}_Z)]^{\mathrm{vir}}} e^{\mathbb{T}}(-N_Z^{\mathrm{vir}}) = \sum_{\mathscr{I}_Z}\int_{\{\mathscr{I}_Z\}} \frac{e^{\mathbb{T}}(\mathrm{Ext}^2(\mathscr{I}_Z,\mathscr{I}_Z))}{e^{\mathbb{T}}(\mathrm{Ext}^1(\mathscr{I}_Z,\mathscr{I}_Z))} \in \mathcal{H}_{\mathbb{T}}. \tag{10.3.12}$$

Note that in the projective case we know this integration yields a *number*, namely DT_n^X.

Definition 10.3.8. If X is a smooth quasiprojective toric 3-fold, we define

$$\mathsf{DT}_n^X \in \mathcal{H}_{\mathbb{T}} = \mathbb{Q}(s_1,s_2,s_3)$$

to be the right hand side of (10.3.12).

Our goal is to compute

$$\mathsf{DT}_X(q) = \sum_{n\geq 0} \mathsf{DT}_n^X q^n$$

for X a toric Calabi–Yau 3-fold. As we shall see, the Calabi–Yau condition ensures that this series belongs to the subring $\mathbb{Z}[\![q]\!] \subset \mathcal{H}_{\mathbb{T}}[\![q]\!]$. We shall spend the rest of this chapter proving the following result.

Theorem 10.3.9. *Let X be a toric Calabi–Yau 3-fold. Then*

$$\mathsf{DT}_X(q) = \mathsf{M}(-q)^{\chi(X)} = \prod_{m\geq 1}(1-(-q)^m)^{-m\chi(X)}.$$

10.4. Evaluating the virtual localisation formula

10.4.1. General formula for the degree 0 DT vertex. Let X be a smooth quasiprojective toric 3-fold, and let $\alpha \in \Delta(X)$ be a vertex of its Newton polytope. Let $\mathscr{I}_Z = (I_\alpha \subset R_\alpha)_\alpha$ be a $\mathbb{T}$-fixed ideal in $\mathrm{Hilb}^n X$. The virtual tangent space (10.3.11) can be written

$$\begin{aligned} T_Z^{\mathrm{vir}} &= \chi(\mathscr{O}_X,\mathscr{O}_X) - \chi(\mathscr{I}_Z,\mathscr{I}_Z) \\ &= \bigoplus_{\alpha\in\Delta(X)}\left(\Gamma(U_\alpha) - \sum_i (-1)^i \Gamma(U_\alpha, \mathscr{E}xt^i(\mathscr{I}_Z,\mathscr{I}_Z))\right), \end{aligned}$$

so in order to compute each summand of (10.3.12) we have to evaluate the character of the virtual representation

$$T_\alpha^{\mathrm{vir}} = \chi(R_\alpha,R_\alpha) - \chi(I_\alpha,I_\alpha) \in K_0^{\mathbb{T}}(U_\alpha) \tag{10.4.1}$$

for each $\alpha \in \Delta(X)$. We do this via Lemma 10.3.2. We define

$$\mathsf{Q}_\alpha(t_1,t_2,t_3) = \mathrm{tr}(R_\alpha/I_\alpha) = \sum_{\square\in\pi_\alpha} t^{\square} = \sum_{(k_1,k_2,k_3)\in\pi_\alpha} t_1^{k_1}t_2^{k_2}t_3^{k_3},$$

the character of the representation R_α/I_α. We trained for this when making an identical calculation for $\mathrm{Hilb}^n \mathbb{A}^2$ in Equation (5.3.4).

[8]Integration will make sense thanks to the properness of $(\mathrm{Hilb}^n X)^{\mathbb{T}}$, see Proposition 10.3.5 (1).

Proposition 10.4.1. *There is an identity*

$$\operatorname{tr}(T_\alpha^{\mathrm{vir}}) = \mathsf{Q}_\alpha - \frac{\overline{\mathsf{Q}}_\alpha}{t_1t_2t_3} + \mathsf{Q}_\alpha\overline{\mathsf{Q}}_\alpha\frac{(1-t_1)(1-t_2)(1-t_3)}{t_1t_2t_3}. \tag{10.4.2}$$

PROOF. Simply write $\chi(R_\alpha)$ for $\operatorname{tr}(R_\alpha)$. We have

$$\begin{aligned}
\operatorname{tr}(T_\alpha^{\mathrm{vir}}) &= \chi(R_\alpha) - \chi(R_\alpha - R_\alpha/I_\alpha, R_\alpha - R_\alpha/I_\alpha)\\
&= \chi(R_\alpha) - \frac{(\chi(R_\alpha)-\mathsf{Q}_\alpha)(\overline{\chi(R_\alpha)}-\overline{\mathsf{Q}}_\alpha)}{\overline{\chi(R_\alpha)}}\\
&= \chi(R_\alpha) - \chi(R_\alpha) + \mathsf{Q}_\alpha + \frac{\chi(R_\alpha)\overline{\mathsf{Q}}_\alpha}{\overline{\chi(R_\alpha)}} - \frac{\mathsf{Q}_\alpha\overline{\mathsf{Q}}_\alpha}{\overline{\chi(R_\alpha)}}\\
&= \mathsf{Q}_\alpha - \frac{\overline{\mathsf{Q}}_\alpha}{t_1t_2t_3} + \mathsf{Q}_\alpha\overline{\mathsf{Q}}_\alpha\frac{(1-t_1)(1-t_2)(1-t_3)}{t_1t_2t_3},
\end{aligned}$$

where we used Equation (10.3.5) in the second identity and Equation (10.3.4) in the last identity. □

Setting $\mathsf{V}_\alpha = \operatorname{tr}(T_\alpha^{\mathrm{vir}})$ to be the vertex contribution at α, we obtain for each torus-fixed ideal $\mathscr{I}_Z$ the fundamental relation

$$\boxed{\operatorname{tr}(T_Z^{\mathrm{vir}}) = \textstyle\sum_\alpha \mathsf{V}_\alpha.}$$

This allows us to evaluate

$$\frac{e^{\mathbb{T}}(\operatorname{Ext}^2(\mathscr{I}_Z,\mathscr{I}_Z))}{e^{\mathbb{T}}(\operatorname{Ext}^1(\mathscr{I}_Z,\mathscr{I}_Z))} = \prod_\alpha e^{\mathbb{T}}(-\mathsf{V}_\alpha).$$

Using Proposition 10.4.1, we can split V_α as

$$\mathsf{V}_\alpha = \mathsf{V}_\alpha^+ + \mathsf{V}_\alpha^-,$$

where

$$\begin{aligned}
\mathsf{V}_\alpha^+ &= \mathsf{Q}_\alpha - \mathsf{Q}_\alpha\overline{\mathsf{Q}}_\alpha\frac{(1-t_1)(1-t_2)}{t_1t_2}\\
\mathsf{V}_\alpha^- &= -\frac{\overline{\mathsf{Q}}_\alpha}{t_1t_2t_3} + \mathsf{Q}_\alpha\overline{\mathsf{Q}}_\alpha\frac{(1-t_1)(1-t_2)}{t_1t_2t_3}.
\end{aligned} \tag{10.4.3}$$

10.4.2. Specialisation to local Calabi–Yau geometry. From now on we specialise to the *local* Calabi–Yau geometry of a nonsingular toric Calabi–Yau 3-fold X. There is a 2-dimensional subtorus

$$\mathbb{T}_0 \subset \mathbb{T}$$

defined by the condition $t_1t_2t_3 = 1$, i.e. $\mathbb{T}_0$ is the kernel of the character $(1,1,1)$. The subtorus $\mathbb{T}_0$ preserves the Calabi–Yau form on X, because it does so on each invariant open chart U_α — see Example 10.3.4. By Theorem 10.2.1 (4), together with Remark 10.3.6, we can evaluate the virtual localisation formula after specialising to $\mathbb{T}_0$. This is crucial to obtain the following remarkable identity.

Lemma 10.4.2. *There is an identity*

$$\overline{\mathsf{V}_\alpha^+}\big|_{\mathbb{T}_0} = -\mathsf{V}_\alpha^-\big|_{\mathbb{T}_0}. \tag{10.4.4}$$

PROOF. It is enough to compare the specialisations of the fractions appearing in (10.4.3). For the left hand side, we find

$$\begin{aligned}
\frac{(1-t_1^{-1})(1-t_2^{-1})}{t_1^{-1}t_2^{-1}}\bigg|_{\mathbb{T}_0} &= \frac{(1-t_1^{-1})(1-t_2^{-1})}{t_3}\bigg|_{\mathbb{T}_0}\\
&= (1-t_1^{-1}-t_2^{-1}+t_1^{-1}t_2^{-1})t_3^{-1}\big|_{\mathbb{T}_0}\\
&= t_3^{-1} - t_2 - t_1 + 1.
\end{aligned}$$

This is clearly the same as $(1-t_1)(1-t_2)\big|_{\mathbb{T}_0}$. □

To finish the calculation we need the following observations:

(1) It follows from Lemma 10.4.2 that

$$\begin{aligned}
\mathsf{V}_\alpha\big|_{\mathbb{T}_0} &= \mathsf{V}_\alpha^+\big|_{\mathbb{T}_0} + \mathsf{V}_\alpha^-\big|_{\mathbb{T}_0}\\
&= \mathsf{V}_\alpha^+\big|_{\mathbb{T}_0} - \overline{\mathsf{V}_\alpha^+}\big|_{\mathbb{T}_0}\\
&= \mathsf{V}_\alpha^+\big|_{\mathbb{T}_0}^{\mathsf{mov}} + \mathsf{V}_\alpha^+\big|_{\mathbb{T}_0}^{\mathsf{fix}} - \overline{\mathsf{V}_\alpha^+}\big|_{\mathbb{T}_0}^{\mathsf{mov}} - \overline{\mathsf{V}_\alpha^+}\big|_{\mathbb{T}_0}^{\mathsf{fix}}\\
&= \mathsf{V}_\alpha^+\big|_{\mathbb{T}_0}^{\mathsf{mov}} + \mathsf{V}_\alpha^+\big|_{\mathbb{T}_0}^{\mathsf{fix}} - \overline{\mathsf{V}_\alpha^+}\big|_{\mathbb{T}_0}^{\mathsf{mov}} - \mathsf{V}_\alpha^+\big|_{\mathbb{T}_0}^{\mathsf{fix}}\\
&= \mathsf{V}_\alpha^+\big|_{\mathbb{T}_0}^{\mathsf{mov}} - \overline{\mathsf{V}_\alpha^+}\big|_{\mathbb{T}_0}^{\mathsf{mov}}.
\end{aligned}$$

(2) The constant term of $\mathsf{V}_\alpha^+\big|_{\mathbb{T}_0}$ is *even*. This is proved e.g. in [**135**, Lemma 10], and it implies that

$$(-1)^{\operatorname{rk}\mathsf{V}_\alpha^+|_{\mathbb{T}_0}} = (-1)^{\operatorname{rk}\mathsf{V}_\alpha^+|_{\mathbb{T}_0}^{\mathsf{mov}}},$$

reducing the identity $\operatorname{rk}\mathsf{V}_\alpha^+\big|_{\mathbb{T}_0} = \operatorname{rk}\mathsf{V}_\alpha^+\big|_{\mathbb{T}_0}^{\mathsf{mov}} + \operatorname{rk}\mathsf{V}_\alpha^+\big|_{\mathbb{T}_0}^{\mathsf{fix}}$ modulo 2.

(3) We have identities

$$\begin{aligned}
\operatorname{rk}\mathsf{V}_\alpha^+\big|_{\mathbb{T}_0} &= \mathsf{V}_\alpha^+\big|_{t_1=t_2=t_3=1}\\
&= \mathsf{Q}_\alpha(1,1,1)\\
&= |\pi_\alpha|.
\end{aligned}$$

We can now finish evaluating the virtual localisation formula. We find

$$\begin{aligned}
\prod_\alpha e^{\mathbb{T}_0}\left(-\mathsf{V}_\alpha\big|_{\mathbb{T}_0}\right) &= \prod_\alpha e^{\mathbb{T}_0}\left(\overline{\mathsf{V}_\alpha^+}\big|_{\mathbb{T}_0}^{\mathsf{mov}} - \mathsf{V}_\alpha^+\big|_{\mathbb{T}_0}^{\mathsf{mov}}\right)\\
&= \prod_\alpha \frac{e^{\mathbb{T}_0}\left(\overline{\mathsf{V}_\alpha^+}\big|_{\mathbb{T}_0}^{\mathsf{mov}}\right)}{e^{\mathbb{T}_0}\left(\mathsf{V}_\alpha^+\big|_{\mathbb{T}_0}^{\mathsf{mov}}\right)}\\
&= \prod_\alpha (-1)^{\operatorname{rk}\mathsf{V}_\alpha^+|_{\mathbb{T}_0}^{\mathsf{mov}}}\\
&= \prod_\alpha (-1)^{\operatorname{rk}\mathsf{V}_\alpha^+|_{\mathbb{T}_0}}\\
&= \prod_\alpha (-1)^{|\pi_\alpha|}\\
&= (-1)^{\sum_\alpha |\pi_\alpha|}\\
&= (-1)^n.
\end{aligned} \tag{10.4.5}$$

It follows that

$$\begin{aligned}
\mathsf{DT}_X(q) &= \sum_{n\geq 0} q^n \sum_{\mathscr{I}_Z \in (\operatorname{Hilb}^n X)^{\mathbb{T}}} \frac{e^{\mathbb{T}}(\operatorname{Ext}^2(\mathscr{I}_Z,\mathscr{I}_Z))}{e^{\mathbb{T}}(\operatorname{Ext}^1(\mathscr{I}_Z,\mathscr{I}_Z))}\\
&= \sum_{n\geq 0} q^n \sum_{\mathscr{I}_Z \in (\operatorname{Hilb}^n X)^{\mathbb{T}}} \prod_\alpha e^{\mathbb{T}_0}\left(-\mathsf{V}_\alpha\big|_{\mathbb{T}_0}\right)\\
&= \sum_{n\geq 0} \chi(\operatorname{Hilb}^n X)(-q)^n\\
&= \mathsf{M}(-q)^{\chi(X)},
\end{aligned} \tag{10.4.6}$$

where $\mathsf{M}(q)$ is the MacMahon function, cf. Definition 8.5.10. We saw in Theorem 8.5.17 that $\mathsf{M}(q)$ determines the Euler characteristics of $\operatorname{Hilb}^n X$ on *any* smooth 3-fold, which is what we used in the last equality displayed in (10.4.6).

We have thus proved Theorem 10.3.9.

10.5. Two words on some refinements

If X is a projective 3-fold (not necessarily toric), then by [**126**, **125**] one has

$$\mathsf{DT}_X(q) = \mathsf{M}(-q)^{\int_X c_3(T_X \otimes \omega_X)}.$$

Recall that, in the toric case, the coefficients DT_n^X of this series were obtained by integrating $e^{\mathbb{T}}(-T_Z^{\mathrm{vir}})$ and summing over $\mathbb{T}$-fixed ideals $\mathscr{I}_Z$. In fact, there is a more refined identity taking place in the power series ring $\mathbb{Q}(\!(s_1, s_2, s_3)\!)[\![q]\!]$, namely [**136**, Theorem 1]

$$\mathsf{DT}_{\mathbb{A}^3}^{\mathsf{coh}}(q) = \sum_{n \geq 0} q^n \sum_{\mathscr{I}_Z} e^{\mathbb{T}}(-T_Z^{\mathrm{vir}}) = \mathsf{M}(-q)^{-\frac{(s_1+s_2)(s_1+s_3)(s_2+s_3)}{s_1 s_2 s_3}}. \tag{10.5.1}$$

See [**62**] for a higher rank version of these formulae, solving Szabo's conjecture [**188**, Conjecture 4.10].

See also [**158**] for a further refinement to equivariant K-theory of Formula (10.5.1). Okounkov's theorem [**158**, Theorem 3.3.6] states that the degree 0 K-theoretic DT invariants of $\mathbb{A}^3$ satisfy the plethystic relation known as *Nekrasov's formula* [**152**]. A higher rank K-theoretic generalisation was proved in [**62**], solving the Awata–Kanno conjecture [**10**].

The interested reader can have a look at [**18**] for a motivic refinement of the degree 0 DT invariants studied here. A higher rank (motivic) version can be found in [**37**, **176**, **171**, **36**].

CHAPTER 11

A glimpse on the DT/PT correspondence

SUMMARY. In this chapter we survey an instance of the celebrated *DT/PT correspondence*, a relation between two types of virtual enumerative invariants: Donaldson–Thomas invariants and Pandharipande–Thomas invariants. This relation (Theorem 11.1.1) was proved by Bridgeland and Toda. The classical setup, summarised in the next section, involves a *projective* Calabi–Yau 3-fold. In Section 11.2 we will exploit virtual localisation and the results of Behrend–Bryan [17] to compute by hand the first few DT coefficients confirming the DT/PT correspondence for one of the most interesting *quasiprojective* Calabi–Yau 3-folds of all: the so called *resolved conifold*.

11.1. DT/PT for a projective Calabi–Yau 3-fold

Let Y be a projective Calabi–Yau 3-fold over $\mathbb{C}$. Fix the discrete invariants

$$m \in \mathbb{Z}, \quad \beta \in \mathrm{H}_2(Y,\mathbb{Z}).$$

Then one can form two moduli spaces:

- The DT moduli space, i.e. the Hilbert scheme of 1-dimensional closed subschemes $Z \subset Y$ with
$$\chi(\mathscr{O}_Z) = m, \quad [Z] = \beta.$$
This scheme is denoted $I_m(Y,\beta)$, and is studied in [**135, 136**]. Note that setting $\beta = 0$ one recovers $I_m(Y,0) = \mathrm{Hilb}^m Y$.
- The PT moduli space, i.e. the moduli space of *stable pairs* (F,s), where $F \in \mathrm{Coh}\, Y$ is a purely 1-dimensional sheaf, $s \colon \mathscr{O}_Y \to F$ is a section with 0-dimensional cokernel and
$$\chi(F) = m, \quad [\mathrm{Supp}\, F] = \beta.$$
This scheme is denoted $P_m(Y,\beta)$ and is studied in [**163, 164**].

Both $I_m(Y,\beta)$ and $P_m(Y,\beta)$ carry a symmetric perfect obstruction theory [**109**, Section 4], so we have the associated virtual invariants

$$\mathsf{DT}_{m,\beta} = \int_{[I_m(Y,\beta)]^{\mathrm{vir}}} 1, \quad \mathsf{PT}_{m,\beta} = \int_{[P_m(Y,\beta)]^{\mathrm{vir}}} 1 \tag{11.1.1}$$

and the corresponding generating functions

$$\mathsf{DT}_\beta(q) = \sum_{m\in\mathbb{Z}} \mathsf{DT}_{m,\beta} q^m, \quad \mathsf{PT}_\beta(q) = \sum_{m\in\mathbb{Z}} \mathsf{PT}_{m,\beta} q^m.$$

These series are related by a *wall-crossing formula*.[1] The 'wall-crossing factor' is the series

$$\mathsf{DT}_0(q) = \sum_{m\geq 0} \mathsf{DT}_{m,0} q^m = \mathsf{M}(-q)^{\chi(Y)} \tag{11.1.2}$$

where the right hand side appeared in Theorem 10.3.9. The second identity in (11.1.2) is proved in [**20**].[2]

[1] The reader who wants to know more about the 'wall-crossing' behavior of the DT invariant is referred to [**163**, Section 3.3], or the very instructive [**187**, Section 4]

[2] See also [**126, 125**] for the proof of $\mathsf{DT}_0(q) = \mathsf{M}(-q)^{c_3(T_Y \otimes \omega_Y)}$ for arbitrary smooth projective 3-folds Y, and [**176, 62**] for a higher rank version.

Theorem 11.1.1 (DT/PT correspondence [**30, 193**])**.** *Fix $\beta \in \mathrm{H}_2(Y,\mathbb{Z})$. There is a relation*

$$\mathsf{DT}_\beta(q) = \mathsf{DT}_0(q)\cdot \mathsf{PT}_\beta(q). \tag{11.1.3}$$

It is the point of view of [**163**, Section 3.3] that both invariants (11.1.1) enumerate stable objects in the derived category. The DT number virtually counts extensions

$$Q[-1] \to \mathscr{I}_Z \to \mathscr{I}_C$$

in $\mathbf{D}^{\mathrm{b}}(Y)$, where $C \subset Z$ is the maximal purely 1-dimensional subscheme of Z, defined by the torsion filtration of $\mathscr{O}_Z$, i.e. by the sequence

$$0 \to Q \to \mathscr{O}_Z \to \mathscr{O}_C \to 0,$$

where Q is the maximal 0-dimensional subsheaf of $\mathscr{O}_Z$ and $\mathscr{O}_C$ is defined to be the cokernel. It is of course determined by $Q = \mathscr{I}_C/\mathscr{I}_Z$. On the other hand, the PT number virtually enumerates complexes $J^\bullet = [\mathscr{O}_Y \to F]$, which can be viewed as extensions

$$\mathscr{I}_C \to J^\bullet \to Q[-1]$$

where $\mathscr{I}_C$ is the kernel of the section $s\colon \mathscr{O}_Y \to F$ (and thus C is Cohen–Macaulay since F is pure), and Q is its cokernel. Thus, as explained in [**163**], DT and PT invariants enumerate extensions of objects of the same type (ideal sheaves of pure subschemes, and shifted 0-dimensional sheaves), but in opposite directions. This is interpreted as a change of stability condition on $\mathbf{D}^{\mathrm{b}}(Y)$, even though on 3-folds making this precise requires some care.

Remark 11.1.2. The reader interested in DT/PT type relations can look up [**172, 173, 156**] for a version of the DT/PT correspondence centered at a fixed curve $C \subset Y$ and [**43**] for a motivic refinement. Higher rank versions of Theorem 11.1.1 are proved in [**194, 14**].

11.2. DT/PT on the resolved conifold

From now on, we will be concerned with the *resolved conifold*, also known as *local* $\mathbb{P}^1$, which is defined as

$$Y = \mathrm{Tot}_{\mathbb{P}^1}(\mathscr{O}_{\mathbb{P}^1}(-1)\oplus \mathscr{O}_{\mathbb{P}^1}(-1)) \to \mathbb{P}^1.$$

In fact, Y is the crepant resolution of the conifold singularity

$$X_{\mathrm{con}} = \operatorname{Spec}\mathbb{C}[x,y,z,w]/(xy-zw) \subset \mathbb{A}^4.$$

It is a smooth quasiprojective Calabi–Yau 3-fold. The preimage of the origin $0 \in X_{\mathrm{con}}$ under the birational contraction $Y \to X_{\mathrm{con}}$ is isomorphic to the zero section

$$C_0 \cong \mathbb{P}^1$$

of $Y \to \mathbb{P}^1$. This is the only proper curve contained in Y, and is *super-rigid*, meaning it is embedded with normal bundle $\mathscr{N}_{C_0/Y} = \mathscr{O}_{\mathbb{P}^1}(-1)\oplus\mathscr{O}_{\mathbb{P}^1}(-1)$. Remarkably, the DT/PT correspondence (11.1.3) holds for the resolved conifold as well — the result can be extracted from the formulae proved in [**17**], which we now briefly revisit. We let

$$\beta = [C_0] \in \mathrm{H}_2(Y,\mathbb{Z})$$

denote the homology class of the exceptional curve. The only allowed curve classes are those of the form $d\beta$, for $d \geq 1$.

The DT invariants of Y have been defined and computed in [**17**] via the *Behrend function*, as we now explain. The Behrend function is a canonical constructible function $\nu_Z\colon Z(\mathbb{C}) \to \mathbb{Z}$ defined on any $\mathbb{C}$-scheme Z, see [**16**]. One defines the ν-weighted Euler characteristic of a scheme Z as

$$\chi(Z,\nu_Z) = \int_Z \nu_Z\,\mathrm{d}\chi = \sum_{n\in\mathbb{Z}} n\chi(\nu_Z^{-1}(n)).$$

When Z is proper and carries a symmetric perfect obstruction theory, Behrend's theorem [16] asserts that

$$\chi(Z,\nu_Z)=\int_{[Z]^{\mathrm{vir}}}1.$$

However, the moduli space $I_m(Y,d\beta)$ is not proper, so one cannot take the degree of the virtual fundamental class. Nevertheless, one can still define

$$\mathsf{DT}_{m,d}=\chi(I_m(Y,d\beta),\nu).$$

Behrend–Bryan computed this number by working equivariantly, more precisely by considering the $\mathbb{T}_0$-equivariant symmetric perfect obstruction theory on the Hilbert scheme. Here $\mathbb{T}_0\subset(\mathbb{C}^\times)^3$ is the subtorus defined by $t_1t_2t_3=1$. The Hilbert scheme $I_m(Y,d\beta)$ has a finite number of $\mathbb{T}_0$-fixed points, all isolated with the trivial obstruction theory (i.e. both $\mathbb{T}_0$-fixed tangent space and $\mathbb{T}_0$-fixed obstruction space vanish at those points). By exploiting the main result of [20], the DT invariant is then computed as

$$\mathsf{DT}_{m,d}=(-1)^{m-d}\chi(I_m(Y,d\beta)), \tag{11.2.1}$$

where $(-1)^{m-d}=(-1)^{\dim T_xI_m(Y,d\beta)}$ is the parity of the tangent space dimension at a fixed point $x\in I_m(Y,d\beta)^{\mathbb{T}_0}$ — this is the precise analogue of the $(-1)^n$ sign that we computed for Hilb^nX in (10.4.5). The complete formula is as follows.

Theorem 11.2.1 ([17, Lemma 2.9])**.** *For each $d\geq 1$ there is an identity*

$$\sum_{m\in\mathbb{Z}}\chi(I_m(Y,d\beta))q^m=\mathsf{M}(q)^2\mathrm{P}_d(q),$$

where $\mathrm{P}_d(q)$ is the d-th coefficient of the expansion

$$\sum_{d\geq 0}\mathrm{P}_d(q)v^d=\prod_{m\geq 1}(1+q^mv)^m.$$

By (11.2.1), one then has

$$\sum_{m\in\mathbb{Z}}\mathsf{DT}_{m,d}q^m=(-1)^d\mathsf{M}(-q)^2\mathrm{P}_d(-q). \tag{11.2.2}$$

In fact, it is observed in [17, Remark 2.10] that

$$\mathrm{P}_d(q)=q^d\sum_{\lambda\vdash d}s_\lambda(q)s_{\lambda^t}(q),$$

where the Schur functions s_λ for a Young diagram $\lambda\vdash d$ and s_{λ^t} for its transposed λ^t are determined by the identities

$$s_{\lambda^t}(q)=q^{\binom{\lambda}{2}}\prod_{\square\in\lambda^t}\left(1-q^{h(\square)}\right)^{-1}=q^{\binom{\lambda}{2}-\binom{\lambda^t}{2}}s_\lambda(q).$$

Here, we have set $\binom{\lambda}{2}=\sum_i\binom{\lambda_i}{2}$, and $h(\square)$ is the *hook length* of a box $\square\in\lambda^t$.

Example 11.2.2. Set $d=1$. Then

$$\begin{aligned}\mathrm{P}_1(q)&=qs_\square(q)^2\\&=q(q^0(1-q)^{-1})^2\\&=q(1-q)^{-2}.\end{aligned} \tag{11.2.3}$$

It follows that, again for $d=1$, we obtain

$$\mathsf{DT}_\beta(q)=\sum_{m\in\mathbb{Z}}\mathsf{DT}_{m,1}q^m=-\mathsf{M}(-q)^2\frac{-q}{(1+q)^2}=\mathsf{M}(-q)^2\frac{q}{(1+q)^2}. \tag{11.2.4}$$

Our purpose here is just to confirm, for $d=1$, the DT/PT correspondence

$$\mathsf{DT}_\beta(q) = \mathsf{M}(-q)^{\chi(Y)} \cdot \mathsf{PT}_\beta(q) \tag{11.2.5}$$

by computing, via virtual localisation, the first few coefficients of both sides. The fact that $\mathsf{PT}_\beta(q) = q(1+q)^{-2}$ will be confirmed in Section 11.2.2. However, from now on, we pretend we are not aware of Equation (11.2.2) and of how it can be rewritten as Equation (11.2.5) for $d=1$.

11.2.1. Point contribution. To compute $\mathsf{DT}_0(q)$, it is enough to observe that $\chi(Y)=2$, since Y is an affine fibration over $\mathbb{P}^1$. Thus

$$\begin{aligned} \mathsf{DT}_0(q) &= \mathsf{M}(-q)^2 \\ &= (1-q+3q^2-6q^3+13q^4-\cdots)^2 \\ &= 1-2q+7q^2-18q^3+\cdots \end{aligned} \tag{11.2.6}$$

11.2.2. PT side. Since C_0 is the only curve living in the homology class β, all stable pairs on Y are of the form

$$0 \to \mathscr{I}_{C_0} \to \mathscr{O}_Y \to F \to Q \to 0.$$

In fact, since C_0 is smooth, a foundational result on stable pairs [164, Proposition B.5] implies that $\mathscr{O}_Y \to F$ is necessarily of the form

$$s\colon \mathscr{O}_Y \twoheadrightarrow \iota_*\mathscr{O}_{C_0} \xrightarrow{\iota_* s_D} \iota_*\mathscr{O}_{C_0}(D),$$

for $D \subset C_0$ an effective 0-cycle of degree $\chi(F)-1$. Here $\iota\colon C_0 \hookrightarrow Y$ is the inclusion, and s_D is the canonical section attached to $D \subset C_0$. Thus

$$P_m(Y,\beta) \cong \operatorname{Sym}^{m-1} C_0 \cong \mathbb{P}^{m-1},$$

and the moduli space is empty for $m \le 0$. The virtual class is the 0-cycle class

$$\big[P_m(Y,\beta)\big]^{\mathrm{vir}} = e(\mathrm{Ob}) \cap [P_m(Y,\beta)] \in A_0(P_m(Y,\beta)),$$

and, by symmetry of the obstruction theory, Ob is the cotangent bundle on the moduli space. The corresponding PT invariant is therefore

$$\begin{aligned} \mathsf{PT}_{m,\beta} &= \int_{[P_m(Y,\beta)]} e(\mathrm{Ob}) = \int_{\mathbb{P}^{m-1}} e(\Omega_{\mathbb{P}^{m-1}}) \\ &= (-1)^{m-1}\chi(\mathbb{P}^{m-1}) \\ &= (-1)^{m-1} m. \end{aligned} \tag{11.2.7}$$

The generating series is given by Macdonald's formula (cf. Corollary 8.5.17), i.e.

$$\begin{aligned} \mathsf{PT}_\beta(q) &= \sum_{m\ge 1} (-1)^{m-1} m q^m = \sum_{m \ge 0} (-1)^m (m+1) q^{m+1} \\ &= q(1+q)^{-2} = q-2q^2+3q^3-4q^4+\cdots \end{aligned} \tag{11.2.8}$$

Thus, multiplying (11.2.6) and (11.2.8) together we get

$$q-4q^2+14q^3-42q^4+\cdots \tag{11.2.9}$$

which is our prediction for the first few DT invariants of Y in the homology class β.[3]

[3] Of course, we do know this precisely computes $\mathsf{DT}_\beta(q)$, because of Equation (11.2.4), but we will make the DT calculation explicit, only using Equation (11.2.1).

11.2.3. DT side. First of all, we observe that Y has two canonical charts

$$\mathbb{A}^3_0,\ \mathbb{A}^3_\infty \ \subset \ Y,$$

obtained as the preimages of $\mathbb{P}^1 \setminus 0$ and $\mathbb{P}^1 \setminus \infty$ along the projection $Y \to \mathbb{P}^1$. We represent these two affine charts as two 3D 'vertices' as in Figure 11.1 below,[4] where the line of boxes represents the zero section $C_0 \subset Y$.

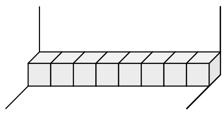

FIGURE 11.1. The DT vertex for Y, in the class $\beta = [C_0]$.

Now, the crucial observation is that

$$\boxed{\begin{array}{c}\chi(I_m(Y,\beta)) \text{ is the number of ways to stack} \\ m-1 \text{ boxes in the corners of Figure 11.1.}\end{array}}$$

This is because a torus fixed point corresponds to an ideal sheaf $\mathscr{I}_Z \subset \mathscr{O}_Y$ that becomes monomial when restricted to the two charts $\mathbb{A}^3_0$ and $\mathbb{A}^3_\infty$.

The reason for the number of boxes being $m-1$ is the following. A point in $I_m(Y,\beta)$ is a short exact sequence

$$0 \to \mathscr{I}_Z \to \mathscr{O}_Y \to \mathscr{O}_Z \to 0$$

where $\chi(\mathscr{O}_Z) = m$, $Z \supset C_0$, and moreover the maximal 0-dimensional subsheaf $Q \subset \mathscr{O}_Z$ is equal to $\mathscr{I}_{C_0}/\mathscr{I}_Z$. Thus $\chi(Q) = \chi(\mathscr{I}_{C_0}) - \chi(\mathscr{I}_Z) = -1-(-m) = m-1$. So, we may write

$$\mathsf{DT}_\beta(q) = \sum_{m\geq 1}(-1)^{m-1}\chi(I_m(Y,\beta))q^m = q\sum_{n\geq 0}(-1)^n\chi(I_{n+1}(Y,\beta))q^n, \tag{11.2.10}$$

where now $\chi(I_{n+1}(Y,\beta))$ is the number of ways to stack n boxes in the corners of Figure 11.1. Clearly, the series on the right hand side starts with 1. So we have to start computing $\chi(I_{n+1}(Y,\beta))$ for $n \geq 1$. The configurations determined by just 1 box are depicted in Figure 11.2.

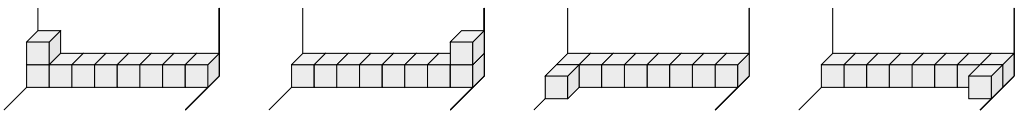

FIGURE 11.2. The 4 torus fixed points in $I_2(Y,\beta)^{\mathbb{T}_0} \subset I_2(Y,\beta)$.

Let us continue with $n = 2$ boxes. The result is shown in Figure 11.3.

We can now write down the first few coefficients of $\mathsf{DT}_\beta(q)$. Taking the sign $(-1)^{m-1}$ into account, Equation (11.2.10) yields

$$q(1-4q+14q^2-\cdots),$$

which agrees with the product computed in Equation (11.2.9). We have thus confirmed, via a purely combinatorial argument, the first few coefficients of the DT/PT correspondence for the resolved conifold (whose general statement for $\beta = [C_0]$ is just Equation (11.2.4), a consequence of the result Theorem 11.2.1 of Behrend–Bryan).

EXERCISE 11.2.3 (For the combinatorially minded). Compute (i.e. draw) the 42 fixed points of $I_4(Y,\beta)$ predicted by Equation (11.2.9).

[4]Thanks to Aurelio Carlucci for providing the code for these figures.

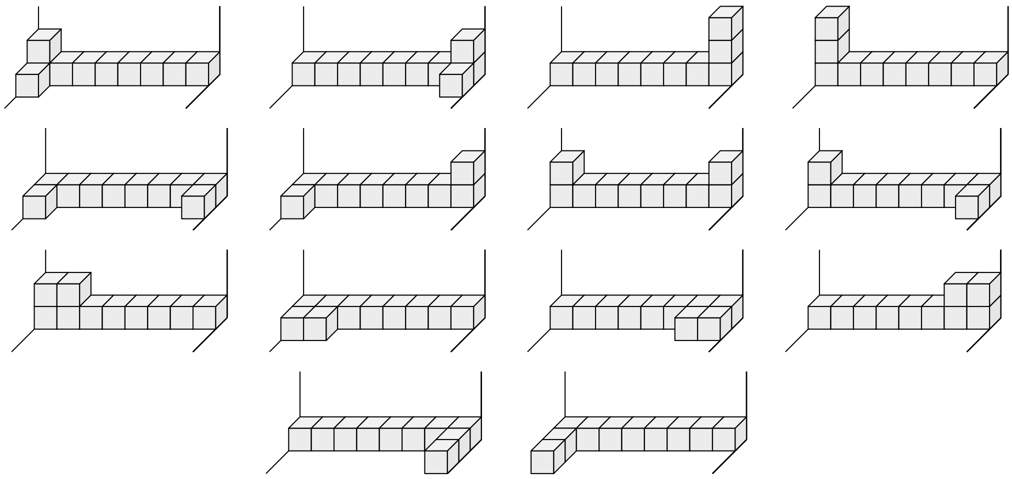

FIGURE 11.3. The 14 torus fixed points in $I_3(Y,\beta)^{\mathbb{T}_0} \subset I_3(Y,\beta)$.

11.3. Relation with multiple covers in Gromov–Witten theory

11.3.1. Moduli of stable maps. Let X be a smooth projective variety. We briefly introduce the moduli space of stable maps with target X, the main character in Gromov–Witten theory.

A *prestable* curve is a reduced connected projective curve C with at worst nodal singularities. A choice of n ordered smooth points $p_1,\ldots,p_n \in C$ defines a prestable n-pointed curve $(C,p_1,\ldots,p_n)$. A *special point* of a prestable n-pointed curve is a point $p \in C$ that is either a singular point or one of the markings.

DEFINITION 11.3.1. Let $(C,p_1,\ldots,p_n)$ be a prestable n-pointed curve and let $\mu\colon C \to X$ be a regular map. This data defines an *n-pointed stable map* if the following condition is met. Suppose μ contracts a component $E \subset C$. Then,

(1) if E is rational, then E has at least three special points, and
(2) if E is elliptic, then E has at least one special point.

Two n-pointed stable maps (C,p,μ) and (C',p',μ') are isomorphic if there is an isomorphism $\varphi : (C,p) \to (C',p')$ of prestable n-pointed curves such that $\mu' \circ \varphi = \mu$. The discrete data of a stable map $\mu\colon C \to X$ are the arithmetic genus g of the source curve C and the homology class $\beta = \mu_*[C] \in \mathrm{H}_2(X,\mathbb{Z})$ hit by the map.

The two conditions of Definition 11.3.1 say precisely that the automorphism group of the data (C,p,μ) is finite. Note that the conditions imply that there is no genus 1 unpointed stable map hitting $\beta = 0$, although this is a bit of a pathology, for condition (2) is always satisfied for $(g,n,\beta) \neq (1,0,0)$.

If $X = \operatorname{Spec}\mathbf{k}$, then a stable map $C \to \operatorname{Spec}\mathbf{k}$ from a prestable n-pointed curve $(C,p_1,\ldots,p_n)$ is called an *n-pointed stable curve*; the moduli space of stable curves is nothing but the Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space $\mathcal{M}_{g,n}$ of smooth n-pointed curves of genus g.

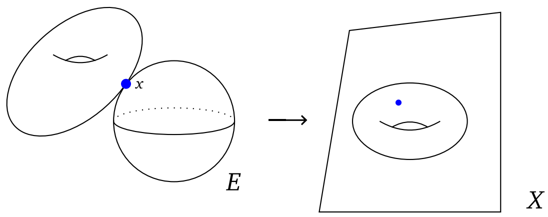

FIGURE 11.4. Example of a map which is not stable. The rational component E, which is contracted to the blue marking, has the node x as unique special point.

There exists a proper (often singular) Deligne–Mumford stack[5]

$$\overline{\mathcal{M}}_{g,n}(X,\beta)$$

parametrising n-pointed stable maps of genus g, hitting the homology class β. It is empty unless β is an effective class. It admits a projective coarse moduli space. When $n=0$, we omit it from the notation. We have a universal stable map (π,σ,f) which we represent as

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,n+1}(X,\beta) \arrow[r, "f"] \arrow[d, "\pi"] & X \\ \overline{\mathcal{M}}_{g,n}(X,\beta) \arrow[u, bend left, "\sigma_i"] & \end{tikzcd} \tag{11.3.1}$$

meaning that the universal curve π forgetting the last marking carries n sections $\sigma=(\sigma_1,\ldots,\sigma_n)$. For existence and construction of these moduli spaces we refer the reader to [**71**]. There exists a moduli space $\mathcal{M}_{g,n}(X,\beta)$ parametrising stable maps from smooth domain curves, but since it can be empty, we cannot consider $\overline{\mathcal{M}}_{g,n}(X,\beta)$ as a compactification in the same way as $\overline{\mathcal{M}}_{g,n}$ is a compactification of $\mathcal{M}_{g,n}$.

Example 11.3.2 (Back to stable curves)**.** Assume $2g-2+n>0$. Then we have $\overline{\mathcal{M}}_{g,n}(X,0)$ canonically identified with $\overline{\mathcal{M}}_{g,n}\times X$. We have $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r,1)=\mathbb{G}(1,r)$ and the map forgetting all the markings

$$\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,1)\longrightarrow\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r,1)$$

is a Zariski locally trivial fibration with fiber $\mathbb{P}^1[n]$, the Fulton–MacPherson configuration space [**70**].

Example 11.3.3 (Complete conics)**.** The coarse moduli space of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2,2)$ is a smooth projective scheme, known as the *moduli space of complete conics*, isomorphic to the blow-up of $\mathbb{P}^5$ along the Veronese surface of double lines.

Example 11.3.4 (Impure dimension)**.** Consider the moduli space $\overline{\mathcal{M}}_1(\mathbb{P}^2,3)$. It contains three irreducible components, whose general points are depicted in Figure 11.5. The first contains all smooth plane cubics and has the expected dimension, namely 9. The second has as general point the stable map from a smooth plane cubic union $\mathbb{P}^1$, contracting the cubic and mapping $\mathbb{P}^1$ with degree 3 onto a line. This component has dimension $8+1+1=10$, where 8 moduli account for the choice of a rational cubic, 1 modulus is for the elliptic curve and 1 modulus for its point of attachment on $\mathbb{P}^1$. The last component has general point an elliptic curve with two rational twigs attached, and the map contracts the elliptic component. The image is a conic union a line. This component has dimension $2+5+2=9$, where 2 moduli are the choice of a line, 5 moduli are the choice of a conic and 2 are the points of attachment.

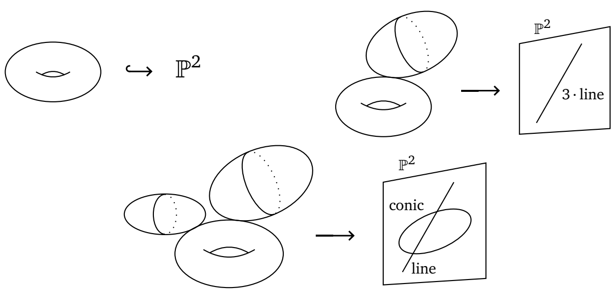

FIGURE 11.5. The generic points of $\overline{\mathcal{M}}_1(\mathbb{P}^2,3)$.

[5]The definition of Deligne–Mumford stack is recalled in Definition C.4.1.

As we shall see in Appendix C.6.1, the stack $\overline{\mathcal{M}}_{g,n}(X,\beta)$ has a perfect obstruction theory of virtual dimension

$$\mathrm{vd} = \int_\beta c_1(X) + (1-g)(\dim X - 3) + n$$

inducing a virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}} \in A_{\mathrm{vd}}(\overline{\mathcal{M}}_{g,n}(X,\beta))$. *Gromov–Witten invariants*, in modern language, are integrals against this cycle class. When X is a Calabi–Yau 3-fold and $n=0$, we have $\mathrm{vd}=0$, so the Gromov–Witten invariants of a Calabi–Yau 3-fold X are simply

$$\mathsf{N}_{g,\beta}(X) = \int_{[\overline{\mathcal{M}}_g(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Q}.$$

By the general formula for the virtual dimension, if $g>0$ and $d>0$ we have

$$\mathrm{vd}\,\overline{\mathcal{M}}_g(\mathbb{P}^1,d) = 2d+2g-2, \tag{11.3.2}$$

where 'd' stands for $d[\mathrm{pt}] \in \mathrm{H}_2(\mathbb{P}^1,\mathbb{Z})$.

11.3.2. The problem of multiple covers. In Gromov–Witten theory there are two main problems which make the theory in general non-enumerative. These are *degenerate contributions* and *multiple covers*. Let us focus on the latter. The basic problem is this: given a stable map $\mu\colon C \to X$ of degree β/k, we can consider all degree k covers $h: C' \to C$. These contribute to the degree β invariants through the compositions $\mu \circ h$, for *any* fixed μ. Computing these contributions is very hard in general. We will next deal with an example which is relevant for 'local' Gromov–Witten theory of Calabi–Yau 3-folds.

Let d be a positive integer. The moduli space $\overline{\mathcal{M}}_g(\mathbb{P}^1,d)$ is related to curve counting on Calabi–Yau 3-folds, as we now explain.

Let $Y \to \mathbb{P}^1$ be the resolved conifold. Recall that $C_0 = \mathbb{P}^1 \subset Y$ is a compact curve (and the only one, in the strong sense that it is infinitesimally rigid) inside Y, so in fact the moduli spaces of stable maps hitting $\beta = [\mathbb{P}^1]$ and its multiples can be defined. We have an isomorphism of moduli stacks

$$\overline{\mathcal{M}}_g(Y, d[\mathbb{P}^1]) \cong \overline{\mathcal{M}}_g(\mathbb{P}^1,d),$$

roughly because all stable maps $C \to Y$ factor through the zero section. This shows that the left moduli space is in fact *proper* even though Y is not. However, the obstruction theories are very different. The virtual dimension of $\overline{\mathcal{M}}_g(Y,d[\mathbb{P}^1])$ vanishes since Y is a Calabi–Yau 3-fold, while the virtual dimension of $\overline{\mathcal{M}}_g(\mathbb{P}^1,d)$ is given by (11.3.2) and is positive. Obstructions to deforming a stable map $\mu\colon C \to Y$ off the zero section lie in $\mathrm{H}^1(C,\mu^*N)$, where $N = \mathscr{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ is the normal bundle to $C_0 \subset Y$. Moreover, a quick Riemann–Roch calculation shows that the vector bundle $\mathbf{R}^1\pi_*f^*N$ has rank $2d+2g-2$. These observations yield the relation

$$\big[\overline{\mathcal{M}}_g(Y,d[\mathbb{P}^1])\big]^{\mathrm{vir}} = e(\mathbf{R}^1\pi_*f^*N) \cap \big[\overline{\mathcal{M}}_g(\mathbb{P}^1,d)\big]^{\mathrm{vir}},$$

where π and f are as in (11.3.1). So, the Gromov–Witten invariants of the resolved conifold can be expressed as integrals over the moduli space $\overline{\mathcal{M}}_g(\mathbb{P}^1,d)$. More precisely, we have

$$C(g,d) = \int_{[\overline{\mathcal{M}}_g(Y,d[\mathbb{P}^1])]^{\mathrm{vir}}} 1 = \int_{[\overline{\mathcal{M}}_g(\mathbb{P}^1,d)]^{\mathrm{vir}}} e(\mathbf{R}^1\pi_*f^*N). \tag{11.3.3}$$

The generating series of these Gromov–Witten invariants was computed for all d by Faber–Pandharipande [58] (see also the corresponding M-theory calculations [78, 133]). The result is

$$\sum_{g\geq 0} C(g,d)u^{2g-2} = \frac{1}{d}\left(2\sin\frac{du}{2}\right)^{-2}. \tag{11.3.4}$$

The genus 0 and 1 contributions had been computed before [8, 81]

$$C(0,d) = 1/d^3, \qquad C(1,d) = 1/12d.$$

The way to interpret the numbers (11.3.3) is as the multiple cover contribution of a fixed *rigid* rational curve C on a Calabi–Yau 3-fold. The series (11.3.4) should be interpreted as the full multiple cover contribution to degree d Gromov–Witten invariants.

11.3.3. Relation with Gopakumar–Vafa invariants. The multiple cover formula (11.3.4) motivated Gopakumar–Vafa [**78**] to conjecture a formula relating Gromov–Witten invariants of a Calabi–Yau 3-fold X to BPS state counts in physics. This formula was the original[6] *definition* of the so called *BPS numbers* $n_{g,\beta}(X)$ — see (11.3.7) below for the precise formula. The BPS numbers, also called Gopakumar–Vafa invariants, are so important because they seem to bypass the problem of multiple covers. Moreover, they agree with naive curve counting in many cases (though not all). They are conjectured to capture the integrality properties of Gromov–Witten invariants, in a possibly deeper way than Donaldson–Thomas invariants.

For a fixed Calabi–Yau 3-fold X, denote its genus g Gromov–Witten invariants by

$$\mathsf{N}_{g,\beta}(X) = \int_{[\overline{\mathcal{M}}_g(X,\beta)]^{\mathrm{vir}}} 1.$$

Introduce, for $\beta \in \mathrm{H}_2(X,\mathbb{Z})$, formal variables v^β satisfying $v^\beta v^{\beta'} = v^{\beta+\beta'}$. The *connected* Gromov–Witten series of X is

$$\begin{aligned} \mathsf{F}_{\mathrm{GW}}(X;u,v) &= \sum_{g\geq 0} u^{2g-2} \sum_{\beta\neq 0} \mathsf{N}_{g,\beta}(X) v^\beta \\ &= 1 + \sum_{\beta\neq 0} \mathsf{F}_{\mathrm{GW}}(X;u)_\beta v^\beta. \end{aligned} \tag{11.3.5}$$

For instance, in the case of the resolved conifold Y, we have

$$\mathsf{F}_{\mathrm{GW}}(Y;u,v) = \sum_{d>0} \frac{1}{d} \left(2\sin\frac{du}{2}\right)^{-2} v^{d[\mathbb{P}^1]}, \tag{11.3.6}$$

by Equation (11.3.4), and because the only admissible curve classes are those of the form $d[\mathbb{P}^1]$.

The BPS numbers mentioned above are the a priori rational numbers $n_{g,\beta}(X)$ defined by the Gopakumar–Vafa formula

$$\mathsf{F}_{\mathrm{GW}}(X;u,v) = \sum_{\substack{g\geq 0\\ \beta\neq 0}} n_{g,\beta}(X) \sum_{d>0} \frac{1}{d} \left(2\sin\frac{du}{2}\right)^{2g-2} v^{d\beta}. \tag{11.3.7}$$

If Y is the resolved conifold, (11.3.7) is equivalent to the Faber–Pandharipande multiple cover formula (11.3.4). Indeed, we have $\mathsf{N}_{g,d[\mathbb{P}^1]}(Y) = C(g,d)$, and by Equation (11.3.6) we deduce

$$n_{g,\beta}(Y) = \begin{cases} 1 & \text{if } g = 0, \beta = [\mathbb{P}^1], \\ 0 & \text{otherwise.} \end{cases}$$

11.3.4. Gromov–Witten/Pairs correspondence. The *disconnected series* in Gromov–Witten theory is, by definition,

$$\mathsf{Z}_{\mathrm{GW}}(X;u,v) = \exp \mathsf{F}_{\mathrm{GW}}(X;u,v) = 1 + \sum_{\beta\neq 0} \mathsf{Z}_{\mathrm{GW}}(X;u)_\beta v^\beta.$$

It counts stable maps to X excluding constant contributions. Indeed, the coefficients are the *disconnected Gromov–Witten invariants*, obtained after allowing the moduli space to contain maps from disconnected

[6] A new mathematical definition of Gopakumar–Vafa invariants was given by Maulik–Toda, in the language of perverse sheaves [**138**]. According to their definition, the new GV invariants $n_{g,\beta}(X)$ are manifestly *integers*, which was not obvious (it was the BPS integrality conjecture!) looking at the definition arising from (11.3.7). However, the fact that the two definitions agree is nontrivial, and still needs to be proven in full generality.

curves, but forbidding the contraction of any connected component. The original 'GW/DT correspondence conjecture' of Maulik–Nekrasov–Okounkov–Pandharipande [**135**, Conjecture 3] was the identity

$$\mathsf{Z}_{\mathsf{GW}}(X;u)_\beta = \mathsf{DT}'_\beta(X,-e^{iu}), \tag{11.3.8}$$

where $\mathsf{DT}'_\beta(X,q) = \mathsf{DT}_\beta(q)/\mathsf{DT}_0(q)$ and the change of variables makes sense because of [**135**, Conjecture 2], asserting that $\mathsf{DT}'_\beta(X,q)$ is the Laurent expansion of a rational function invariant under $q \longleftrightarrow 1/q$. However, we now know that [**135**, Conjecture 2] is true, since

$$\mathsf{DT}'_\beta(X,q) = \mathsf{PT}_\beta(q),$$

and the right hand side is the expansion of a rational function [**30**]. In particular, 'composing' with the established DT/PT correspondence stated in Theorem 11.1.1, the original GW/DT conjecture can be turned into a GW/Pairs conjecture, which to date has been proved in great generality in [**162**].

Let us perform the change of variable $q = -e^{iu}$ on the example of the resolved conifold Y with the reduced curve class $[\mathbb{P}^1]$. The Gromov–Witten side is

$$\mathsf{Z}_{\mathsf{GW}}(Y;u,v) = 1 + \mathsf{F}_{\mathsf{GW}}(X;u)_{[\mathbb{P}^1]}v^{[\mathbb{P}^1]} + \mathsf{Z}_{\mathsf{GW}}(X;u)_{2[\mathbb{P}^1]}v^{2[\mathbb{P}^1]}\cdots$$

and we know that

$$\begin{aligned}
\mathsf{F}_{\mathsf{GW}}(X;u)_{[\mathbb{P}^1]} &= \sum_{g\geq 0} C(g,1)u^{2g-2} = \left(2\sin\frac{u}{2}\right)^{-2} \\
&= \left(2\frac{e^{iu/2}-e^{-iu/2}}{2\sqrt{-1}}\right)^{-2} \\
&= \frac{-q}{q(q^{1/2}-q^{-1/2})^2} && \text{after } q = e^{iu} \\
&= \frac{-q}{(1-q)^2} \\
&= \frac{q}{(1+q)^2} && \text{after } q \mapsto -q \\
&= \mathsf{PT}_{[\mathbb{P}^1]}(q),
\end{aligned}$$

where the last identity is (11.2.8). Therefore we have confirmed (11.3.8) by hand in degree 1 for the resolved conifold.

APPENDIX A

Deformation Theory

This appendix is partially based on [61, Chapter 6]. We work over an algebraically closed field $\mathbf{k}$.

A.1. The general problem

Let $\mathrm{Art}_{\mathbf{k}}$ denote the category of local Artin $\mathbf{k}$-algebras $(A, \mathfrak{m}_A)$ with residue field $\mathbf{k}$. Its opposite category is equivalent to the category of *fat points* (Definition 2.1.13), i.e. the category of $\mathbf{k}$-schemes S such that the structure morphism $S_{\mathrm{red}} \to \operatorname{Spec} \mathbf{k}$ is an isomorphism.

DEFINITION A.1.1. A *deformation functor* is a covariant functor $\mathrm{D}\colon \mathrm{Art}_{\mathbf{k}} \to \mathrm{Sets}$ such that $\mathrm{D}(\mathbf{k})$ is a singleton. We denote by $\star$ the unique element of $\mathrm{D}(\mathbf{k})$.

Let D be a deformation functor. On morphisms, we have the following picture:

$$\begin{array}{ccc} B & & \mathrm{D}(B) \\ \phi\big\downarrow & \overset{\mathrm{D}}{\longmapsto} & \big\downarrow \phi_*=\mathrm{D}(\phi) \\ A & & \mathrm{D}(A) \end{array}$$

If $\phi_*\beta = \alpha$, then $\beta \in \mathrm{D}(B)$ is called a *lift* of $\alpha \in \mathrm{D}(A)$.

Remark A.1.2. Let $(B, \mathfrak{m}_B)$ be an object in $\mathrm{Art}_{\mathbf{k}}$ and let D be a deformation functor. Then $\mathrm{D}(B)$ is not empty. Indeed, look at the diagram

$$\begin{tikzcd} \mathbf{k} \arrow[r, "\phi"] \arrow[d, equal] & B \arrow[dl] \\ B/\mathfrak{m}_B & \end{tikzcd} \quad \overset{\mathrm{D}}{\longmapsto} \quad \begin{tikzcd} \{\star\} \arrow[r, "\phi_*"] \arrow[d, equal] & \mathrm{D}(B) \arrow[dl] \\ \{\star\} & \end{tikzcd}$$

$\phi_*(\star)$ is a canonical element of $\mathrm{D}(B)$.

Here are some examples of deformation functors.

Example A.1.3. Consider the following:

(1) Let Y be a projective $\mathbf{k}$-variety, $\iota\colon X \hookrightarrow Y$ a closed $\mathbf{k}$-subvariety. Then

$$A \mapsto \left\{ \begin{array}{c} \text{closed subschemes } i\colon Z \hookrightarrow Y \times_{\mathbf{k}} \operatorname{Spec} A \\ \text{such that } Z \text{ is } A\text{-flat and } i \otimes_A \mathbf{k} = \iota \end{array} \right\}$$

defines a deformation functor $\mathsf{H}_{X/Y}\colon \mathrm{Art}_{\mathbf{k}} \to \mathrm{Sets}$.

(2) Let E be a coherent sheaf on a noetherian $\mathbf{k}$-scheme Y. Then

$$A \mapsto \left\{ \begin{array}{c} \text{coherent sheaves } \mathscr{E} \in \operatorname{Coh}(Y \times_{\mathbf{k}} \operatorname{Spec} A) \\ \text{flat over } A \text{ such that } \mathscr{E} \otimes_A \mathbf{k} = E \end{array} \right\}$$

defines a deformation functor $\mathsf{M}_E\colon \mathrm{Art}_{\mathbf{k}} \to \mathrm{Sets}$.

(3) Let E be a coherent sheaf on a noetherian $\mathbf{k}$-scheme Y. Let $\vartheta\colon E \twoheadrightarrow Q$ be a surjection in $\operatorname{Coh} Y$. Then

$$A \mapsto \left\{ \begin{array}{c} \text{surjections } \alpha\colon E_A \twoheadrightarrow \mathscr{Q} \text{ in } \operatorname{Coh}(Y \times_{\mathbf{k}} \operatorname{Spec} A) \\ \text{such that } \mathscr{Q} \text{ is } A\text{-flat and } \alpha \otimes_A \mathbf{k} = \vartheta \end{array} \right\}$$

defines a deformation functor $\mathsf{Q}_\vartheta\colon \mathrm{Art}_{\mathbf{k}} \to \mathrm{Sets}$. The notation E_A indicates the pullback of E under the canonical projection $Y \times_{\mathbf{k}} \operatorname{Spec} A \to Y$.

In Definition A.1.1, one should think of $\star$ as the geometric object, defined over $\operatorname{Spec}\mathbf{k}$, that one wants to deform. Similarly, elements of $\mathrm{D}(A)$ should be interpreted as deformations of $\star$ parametrised by $\operatorname{Spec}A$. In Example (1) (resp. (2), (3)) we took $\star$ to be the closed immersion $\iota\colon X \hookrightarrow Y$ (resp. the sheaf E, the quotient $\vartheta\colon E \twoheadrightarrow Q$). Keeping this in mind, the basic questions of deformation theory are the following: given a surjection $B \twoheadrightarrow A$ in $\mathrm{Art}_{\mathbf{k}}$,

$Q1$. Which elements of $\mathrm{D}(A)$ lift? In other words, what is the image of $\mathrm{D}(B) \to \mathrm{D}(A)$?
$Q2$. If α is in the image of ϕ_*, how big is its preimage in $\mathrm{D}(B)$? In other words, 'how many' lifts does α have?

We care about *surjections* $B \twoheadrightarrow A$ because they corresponds to closed embeddings of schemes. Here is an illustrative picture:

$$\begin{tikzcd}
\star \arrow[r, hook] \arrow[d, no head] & ? \arrow[r, hook] \arrow[d, no head] & ?? \arrow[d, no head] \\
\operatorname{Spec}\mathbf{k} \arrow[r, hook] & \operatorname{Spec}A \arrow[r, hook] & \operatorname{Spec}B
\end{tikzcd}$$

where '?' (resp. '??') refers to $Q1$ (resp. $Q2$). Of course, this picture should be taken with a grain of salt. For instance, the meaning of the upper part of the diagram strictly depends upon the nature of the object $\star$ to be deformed (it could be a subscheme, a coherent sheaf, a surjection of coherent sheaves...).

In fact, one can restrict attention to those surjections called *small extensions*.

DEFINITION A.1.4. A surjection $B \twoheadrightarrow A$ in $\mathrm{Art}_{\mathbf{k}}$ with kernel I is called

(i) a *square zero extension* if $I^2 = 0$,
(ii) a *semi-small extension* if $I \cdot \mathfrak{m}_B = 0$,
(iii) a *small extension* if it is semi-small and $\dim_{\mathbf{k}} I = 1$.

We use the same notation for the induced closed immersion $\operatorname{Spec}A \hookrightarrow \operatorname{Spec}B$.

Clearly, one has

$$\text{small} \quad \Rightarrow \quad \text{semismall} \quad \Rightarrow \quad \text{square zero.}$$

EXERCISE A.1.5. Prove that every surjection $B \twoheadrightarrow A$ in $\mathrm{Art}_{\mathbf{k}}$ factors as a composition of finitely many small extensions.

DEFINITION A.1.6. Let $\mathrm{Loc}_{\mathbf{k}}$ denote the category of local noetherian $\mathbf{k}$-algebras $(R, \mathfrak{m}_R)$ with residue field $\mathbf{k}$ and such that $T_R = (\mathfrak{m}_R/\mathfrak{m}_R^2)^*$ is a finite dimensional vector space. Let $\mathrm{CLoc}_{\mathbf{k}} \hookrightarrow \mathrm{Loc}_{\mathbf{k}}$ be the subcategory of *complete* $\mathbf{k}$-algebras.

Every object R of $\mathrm{Loc}_{\mathbf{k}}$ defines a deformation functor

$$\mathsf{h}_R\colon \mathrm{Art}_{\mathbf{k}} \to \mathrm{Sets}, \qquad \mathsf{h}_R(A) = \mathrm{Hom}_{\mathbf{k}}(R, A).$$

Letting $\mathrm{Def}_{\mathbf{k}}$ be the category of deformation functors (with arrows being natural transformations between functors), we get a well defined functor

$$\mathsf{h}\colon \mathrm{Loc}_{\mathbf{k}} \to \mathrm{Def}_{\mathbf{k}}, \qquad R \mapsto \mathsf{h}_R.$$

The functor h restricts to a fully faithful functor $\mathrm{CLoc}_{\mathbf{k}} \to \mathrm{Def}_{\mathbf{k}}$. In particular, by Remark 2.4.3, there is an equivalence

$$\mathrm{CLoc}_{\mathbf{k}} \cong \mathsf{h}(\mathrm{CLoc}_{\mathbf{k}}).$$

A.2. Liftings

Let us suppose we are in the following situation:

$$
\begin{tikzcd}
 & 0 \arrow[d] \\
 & I \arrow[d] \\
R \arrow[r] \arrow[d] & B \arrow[d] \\
P \arrow[r] \arrow[ur, dashed, "\alpha"] & A \arrow[d] \\
 & 0
\end{tikzcd}
\qquad
\begin{array}{c}
B \twoheadrightarrow A \text{ is a square zero extension in } \mathrm{Art}_{\mathbf{k}}, \\
\text{the square is a commutative square} \\
\text{of } \mathbf{k}\text{-algebras homomorphisms.}
\end{array}
\tag{A.2.1}
$$

In what follows, we will simply say that the dotted arrow α 'makes the diagram commute' if the two inner triangles commute. Note that, since $I^2 = 0$, the ideal I has a natural A-module (and hence P-module) structure.

Let $z: R \to P$ be a ring homomorphism, and let N be a P-module. Recall that an R-derivation of P into N is an additive map $\mathrm{d}: P \to N$ such that $\mathrm{d}(fg) = f\,\mathrm{d}(g) + g\mathrm{d}(f)$ and $\mathrm{d}(z(r)) = 0$ for all $r \in R$. The set of R-derivations of P into N is denoted $\mathrm{Der}_R(P, N)$. By the universal property of the P-module of relative differentials $\Omega_{P/R}$, there is a natural identification $\mathrm{Der}_R(P, N) = \mathrm{Hom}_P(\Omega_{P/R}, N)$ for every P-module N.

The proofs of the following two propositions are left as exercises.

Proposition A.2.1. *In situation* (A.2.1)*, the set $\mathcal{S}$ of $\alpha \in \mathrm{Hom}_{\mathbf{k}}(P, B)$ such that the diagram commutes is either empty, or a torsor under*

$$\mathrm{Der}_R(P, I) = \mathrm{Hom}_P(\Omega_{P/R}, I).$$

The next lemma will be needed in the scheme-theoretic version of the previous result.

Lemma A.2.2. *Let $\tau: M_1 \to M_2$ be a homomorphism of abelian groups, and let $\mathcal{S}$ be a torsor under M_1, admitting an M_1-equivariant map $\theta: \mathcal{S} \to M_2$. Let*

$$M_1 \xrightarrow{\tau} M_2 \xrightarrow{q} K \to 0$$

be the cokernel exact sequence. Then

- *The image of $q(\theta(\mathcal{S}))$ is a single element $c_0 \in K$.*
- *An element $m \in M_2$ is in the image of θ if and only if $q(m) = c_0$.*
- *Any $m \in M_2$ is in the image of θ if and only if $c_0 = 0$.*

Next, we globalise Proposition A.2.1.

Proposition A.2.3. *Let $f: X \to Y$ be a morphism of $\mathbf{k}$-schemes with a factorisation $\pi \circ i: X \hookrightarrow M \to Y$, where $i: X \hookrightarrow M$ is a closed immersion with ideal $J \subset \mathscr{O}_M$ and $\pi: M \to Y$ is smooth. Let $S \hookrightarrow T$ be a semismall extension of fat points with ideal I. Given a commutative square*

$$
\begin{tikzcd}
S \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\
T \arrow[r] & Y
\end{tikzcd}
$$

and letting $\iota_x: x \hookrightarrow X$ be the image of the closed point of S in X, there is a natural obstruction to the existence of $\mathbf{k}$-morphisms $T \to X$ making the diagram commute. Such obstruction lives in the cokernel of the natural map of $\mathbf{k}$-vector spaces

$$\mathrm{Hom}_{\mathbf{k}}(\iota_x^* i^* \Omega_\pi, I) \to \mathrm{Hom}_{\mathbf{k}}(\iota_x^* J/J^2, I).$$

PROOF. The statement is local, meaning that it does not change if we replace X and Y with affine open neighborhoods of x and $f(x)$ respectively. So we may assume X and $Y = \operatorname{Spec} R$ are both affine. Moreover, the morphism

$$J/J^2 \to i^*\Omega_\pi$$

has the very pleasant property that its cokernel does not depend on the choice of factorisation, the cokernel being Ω_f. Hence we may assume that M is affine space over Y, i.e. $M = \operatorname{Spec} P$, where $P = R[t_1, \dots, t_n]$. We can then write $X = \operatorname{Spec} P/J$ for an ideal J. Let us compare the scheme situation and the algebra situation:

$$\begin{tikzcd} S \arrow[r] \arrow[ddd, hook] & X \arrow[d, hook, "i"] \\ & M \arrow[dd, "\pi"] \\ & \\ T \arrow[r] \arrow[uuur, dotted, "\alpha"] \arrow[uur, dotted, "\beta"'] & Y \end{tikzcd} \qquad \begin{tikzcd} A & P/J \arrow[l] \arrow[dddl, dotted, "\alpha"'] \\ & P \arrow[u] \arrow[ddl, dotted, "\beta"] \\ & \\ B \arrow[uuu, two heads] & R \arrow[l] \arrow[uu] \end{tikzcd}$$

Since i is a closed immersion the set of α's making the diagram (any of them) commutative injects in the set of β's making the diagram commutative. Letting $\mathcal{S}$ be the set of $\beta \in \operatorname{Hom}_{\mathbf{k}}(P, B)$ such that the diagram (on the right) commutes, we know $\mathcal{S} \neq \emptyset$ because P is a polynomial ring; therefore by Proposition A.2.1 it is a torsor under

$$\operatorname{Hom}_P(\Omega_{P/R}, I) \cong \operatorname{Hom}_A(\Omega_{P/R} \otimes_P A, I) \cong \operatorname{Hom}_{\mathbf{k}}(\Omega_{P/R} \otimes_P \mathbf{k}, I).$$

Using the identifications

$$\operatorname{Hom}_P(J, I) = \operatorname{Hom}_{P/J}(J/J^2, I) = \operatorname{Hom}_{\mathbf{k}}(\iota_x^* J/J^2, I),$$

we see that we are in the situation

$$\begin{tikzcd} & \mathcal{S} \arrow[d, "\theta"] & & \\ \operatorname{Hom}_{\mathbf{k}}(\Omega_{P/R} \otimes_P \mathbf{k}, I) \arrow[r, "\tau"] & \operatorname{Hom}_{\mathbf{k}}(\iota_x^* J/J^2, I) \arrow[r] & K \arrow[r] & 0 \end{tikzcd}$$

where the map τ is the action by translation coming from (pulling back along ι_x) the morphism $J/J^2 \to \Omega_{P/R} \otimes_P P/J$ (the local version of $J/J^2 \to i^*\Omega_\pi$), and θ is the $\operatorname{Hom}_{\mathbf{k}}(\Omega_{P/R} \otimes_P \mathbf{k}, I)$-equivariant map sending $\beta\colon P \to B$ to its restriction to J. By Lemma A.2.2, the torsor condition tells us that for any $\beta \in \mathcal{S}$, its image in $K = \operatorname{coker} \tau$ does not depend on β. By the same lemma, there exists an α making the diagram commutative if and only if for all $\beta \in \mathcal{S}$, the image of β in K is zero. □

A.3. Tangent-obstruction theories

A detailed exposition on tangent-obstruction theories can be found in [**61**], where the reader is referred to for further details.

A.3.1. Definitions and main examples. A *tangent-obstruction theory* on a deformation functor $\mathrm{D}\colon \mathrm{Art}_{\mathbf{k}} \to \mathrm{Sets}$ is defined to be a pair (T_1, T_2) of finite dimensional $\mathbf{k}$-vector spaces such that for any small extension $I \hookrightarrow B \twoheadrightarrow A$ in $\mathrm{Art}_{\mathbf{k}}$ there is an 'exact sequence of sets'

$$T_1 \otimes_{\mathbf{k}} I \longrightarrow \mathrm{D}(B) \longrightarrow \mathrm{D}(A) \xrightarrow{\mathrm{ob}} T_2 \otimes_{\mathbf{k}} I, \tag{A.3.1}$$

which would be decorated with an additional '0' on the left whenever $A = \mathbf{k}$, and is moreover functorial in small extensions in a precise sense [**61**, Def. 6.1.21]. We spell out here what exactness of a short exact sequence of sets such as (A.3.1) means. Exactness at $\mathrm{D}(A)$ means that an element $\alpha \in \mathrm{D}(A)$ lifts to $\mathrm{D}(B)$ if and only if $\mathrm{ob}(\alpha) = 0$. Exactness at $\mathrm{D}(B)$ means that, if there is a lift, then $T_1 \otimes_{\mathbf{k}} I$ acts transitively on the set of lifts. If the sequence started with a '0', it would mean that lifts form an affine space under $T_1 \otimes_{\mathbf{k}} I$.

The tangent space of the tangent-obstruction theory is T_1, and is canonically determined by the deformation functor as $T_1 = \mathrm{D}(\mathbf{k}[t]/t^2)$. A deformation functor is *prorepresentable* if it is isomorphic to

$\mathsf{h}_R = \mathrm{Hom}_{\mathbf{k}}(R, -)$ for some local $\mathbf{k}$-algebra $R \in \mathrm{Loc}_{\mathbf{k}}$. A tangent-obstruction theory on a prorepresentable deformation functor always has a '0' on the left in the sequences (A.3.1), which means that lifts of a given $\alpha \in \mathrm{D}(A)$, when they exist, form an affine space over $T_1 \otimes_{\mathbf{k}} I$.

If D is prorepresentable by R, then $R = S/J$ for an ideal $J \subset (y_1, \ldots, y_d)^2 \subset S = \mathbf{k}[\![y_1, \ldots, y_d]\!]$. In this case, setting $T_1(R) = T_R = (\mathfrak{m}_R/\mathfrak{m}_R^2)^*$ and $T_2(R) = (J/\mathfrak{m}_S \cdot J)^*$, one has the canonical tangent-obstruction theory $(T_1(R), T_2(R))$. Any other tangent-obstruction theory is of the form (T_1, T_2) with $T_1 = T_1(R)$ and $T_2 \supset T_2(R)$.

Here are some examples of tangent-obstruction theories.

Example A.3.1. We revisit the three examples from Example A.1.3.

(1) If $X \hookrightarrow Y$ is a local complete intersection subscheme inside a projective $\mathbf{k}$-variety Y, then

$$T_i = \mathrm{H}^{i-1}(X, \mathscr{N}_{X/Y}), \quad i = 1, 2,$$

is a tangent-obstruction theory on $\mathsf{H}_{X/Y}$. If $X \hookrightarrow Y$ is an arbitrary closed subscheme, $\mathsf{H}_{X/Y}$ has the tangent-obstruction theory

$$T_i = \mathrm{Ext}_Y^{i-1}(\mathscr{I}_X, \mathscr{O}_X), \quad i = 1, 2.$$

(2) If E is a coherent sheaf on a projective $\mathbf{k}$-variety Y, then

$$T_i = \mathrm{Ext}_Y^{i}(E, E), \quad i = 1, 2,$$

is a tangent-obstruction theory on M_E.

(3) If $\vartheta\colon E \twoheadrightarrow Q$ is a surjection in $\mathrm{Coh}\, Y$ with kernel S, then Q_ϑ has the tangent-obstruction theory

$$T_i = \mathrm{Ext}_Y^{i-1}(S, Q), \quad i = 1, 2.$$

Example A.3.2. Let Y be a projective $\mathbf{k}$-variety, and fix $E \in \mathrm{Coh}\, Y$. Let M_E be the deformation functor from Example A.1.3. If E is simple, i.e. if $\mathrm{Hom}_Y(E, E) \cong \mathbf{k}$, then M_E is prorepresentable [**180**, Theorem 19.2].

The following result is often useful. For instance, we used it in Theorem 10.2.10

Proposition A.3.3 ([**38**, Proposition 2.2])**.** *Let* D, D′ *be two prorepresentable deformation functors carrying tangent-obstruction theories* (T_1, T_2) *and* (T_1', T_2')*, respectively. Let* $\eta\colon \mathrm{D} \to \mathrm{D}'$ *be a morphism inducing an isomorphism* $h\colon T_1 \xrightarrow{\sim} T_1'$ *and a linear embedding* $T_2 \hookrightarrow T_2'$*. Then* η *is an isomorphism.*

A.3.2. Moduli Situation. We now focus on examples related to real life: moduli spaces.

Let $\mathfrak{M}\colon \mathrm{Sch}_{\mathbf{k}}^{\mathrm{op}} \to \mathrm{Sets}$ be a functor represented by a scheme M. Fix $p \in M(\mathbf{k}) = \mathfrak{M}(\mathrm{Spec}\, \mathbf{k})$. Consider the subfunctor

$$\mathsf{Def}_p \subset \mathfrak{M}\big|_{\mathrm{Art}_{\mathbf{k}}}$$

defined by

$$\mathsf{Def}_p(A) = \{\, \eta \in \mathfrak{M}(\mathrm{Spec} A) \mid \eta_{\mathbf{k}} = p \,\}$$

where restriction to $\mathrm{Spec}\, \mathbf{k} \subset \mathrm{Spec} A$ is the map $\mathfrak{M}(\mathrm{Spec} A) \to \mathfrak{M}(\mathrm{Spec}\, \mathbf{k})$.

In this situation, Def_p is prorepresentable by $R = \widehat{\mathscr{O}}_{M,p}$ and as such it has a tangent-obstruction theory $(T_1, T_2 \supset T_2(R))$ with tangent space

$$T_1 = \mathsf{Def}_p(\mathbf{k}[t]/t^2) = \mathsf{h}_R(\mathbf{k}[t]/t^2) = T_p M.$$

One always has the inequalities

$$\dim T_p M \geq \dim \mathscr{O}_{M,p} \geq \dim T_p M - \dim T_2$$

showing that if $T_2 = 0$ then M is smooth at p. The converse is not true (see Example A.3.7).

For instance, let Y be a projective $\mathbf{k}$-scheme and let $p = [\vartheta\colon E \twoheadrightarrow Q] \in \mathrm{Quot}_{Y/\mathbf{k}}(E)$ be a closed point of the Quot scheme of $E \in \mathrm{Coh}\, Y$. Set $S = \ker(E \twoheadrightarrow Q)$. Consider the subfunctor

$$\mathsf{Def}_p \subset \mathsf{Quot}_{Y/\mathbf{k}}(E)\big|_{\mathrm{Art}_{\mathbf{k}}}$$

that evaluated on a fat point A consists of surjections $f : E \otimes_{\mathbf{k}} A \twoheadrightarrow \mathcal{Q}$ in $\mathrm{Coh}(Y \times_{\mathbf{k}} A)$ such that $f \otimes_A \mathbf{k} = p$. Then

$$\mathsf{Def}_p = \mathsf{h}_{\mathscr{O}_{\mathrm{Quot}_{Y/\mathbf{k}}(E),p}}$$

is prorepresentable, and agrees with the functor Q_ϑ of Example A.1.3. We have inequalities

$$\dim_{\mathbf{k}} \mathrm{Hom}_Y(S,Q) \geq \dim \mathscr{O}_{\mathrm{Quot}_{Y/\mathbf{k}}(E),p} \geq \dim_{\mathbf{k}} \mathrm{Hom}_Y(S,Q) - \dim_{\mathbf{k}} \mathrm{Ext}^1_Y(S,Q).$$

Now set $E = \mathscr{O}_Y$. Let $X \hookrightarrow Y$ be a local complete intersection subscheme, defining a point $q = [\mathscr{O}_Y \twoheadrightarrow \mathscr{O}_X] \in \mathrm{Hilb}_{Y/\mathbf{k}}$. Then we have inequalities

$$\dim_{\mathbf{k}} \mathrm{H}^0(X, \mathscr{N}_{X/Y}) \geq \dim \mathscr{O}_{\mathrm{Hilb}_{Y/\mathbf{k}},q} \geq \dim_{\mathbf{k}} \mathrm{H}^0(X, \mathscr{N}_{X/Y}) - \dim_{\mathbf{k}} \mathrm{H}^1(X, \mathscr{N}_{X/Y}).$$

For a general closed subscheme $X \hookrightarrow Y$, we cannot use $\mathrm{H}^1(X, \mathscr{N}_{X/Y})$ as obstruction space (see Example A.3.5), but we still have

$$\dim_{\mathbf{k}} \mathrm{H}^0(X, \mathscr{N}_{X/Y}) \geq \dim \mathscr{O}_{\mathrm{Hilb}_{Y/\mathbf{k}},q} \geq \dim_{\mathbf{k}} \mathrm{H}^0(X, \mathscr{N}_{X/Y}) - \dim_{\mathbf{k}} \mathrm{Ext}^1_Y(\mathscr{I}_X, \mathscr{O}_X).$$

Example A.3.4. Let $C \subset \mathbb{P}^2$ be a curve of degree d. Then $\mathscr{I}_C = \mathscr{O}(-d)$ so $\mathscr{N}_{C/\mathbb{P}^2} = \mathscr{O}_C(d)$ and $\mathrm{H}^1(C, \mathscr{O}_C(d)) = \mathrm{H}^0(C, K_C \otimes \mathscr{O}_C(-d))^\vee = \mathrm{H}^0(C, \mathscr{O}_C(d-3-d))^\vee = 0$. Indeed $\mathbb{P}\mathrm{H}^0(\mathbb{P}^2, \mathscr{O}(d))$ is smooth.

Example A.3.5. Let $M = \mathrm{Hilb}^4 \mathbb{A}^3$, which is irreducible of dimension 12, and consider $\mathscr{I}_X = (x,y,z)^2$ giving rise to $p = [\mathscr{O} \twoheadrightarrow \mathscr{O}_X]$. Then $\mathrm{H}^1(X, \mathscr{N}_{X/\mathbb{A}^3}) = 0$ because $\dim X = 0$, but

$$\dim T_p M = \dim \mathrm{H}^0(X, \mathscr{N}_{X/\mathbb{A}^3}) = 18 > 12,$$

so $p \in M$ is singular, and $\mathrm{Ext}^1_{\mathbb{A}^3}(\mathscr{I}_X, \mathscr{O}_X) \neq 0$.

Remark A.3.6. Let X be a closed subscheme of $Y = \mathbb{P}^n$. If $\mathrm{H}^1(X, \mathscr{N}_{X/Y}) = 0$ and X is either

- nonsingular, or
- Cohen–Macaulay in codimension 2, or
- Gorenstein in codimension 3,

then $[\mathscr{O}_Y \twoheadrightarrow \mathscr{O}_X] \in \mathrm{Hilb}_Y$ is a smooth point. For instance, $\mathscr{O}_{\mathbb{P}^3}/\mathfrak{m}^2$, for $\mathfrak{m}$ the ideal of $(0:0:0:1) \in \mathbb{P}^3$, defines a subscheme $X \subset \mathbb{P}^3$ of length 4 that is not a Gorenstein scheme.

Example A.3.7. Let C be a smooth projective curve of genus $g \geq 3$ over a field $\mathbf{k}$ of characteristic not equal to 2, let $J = \mathrm{Pic}^0 C$ be its Jacobian, and let $C \hookrightarrow J$ be an Abel–Jacobi embedding. Let $H_C \subset \mathrm{Hilb}_J$ be the Hilbert scheme component containing $p = [\mathscr{O}_J \twoheadrightarrow \mathscr{O}_C]$, where J is polarised by the Theta divisor. Then C is hyperelliptic if and only if H_C is not smooth at p. In fact, for $g = 2$ the Hilbert scheme H_C is smooth (and equals J), as well as Def_p. For C hyperelliptic of genus $g \geq 2$ one has

$$\dim_{\mathbf{k}} \mathrm{H}^0(C, \mathscr{N}_{C/J}) = 2g-2, \quad \dim_{\mathbf{k}} \mathrm{H}^1(C, \mathscr{N}_{C/J}) = g^2 - 2g + 1,$$

so if $g = 2$ we get an example where Def_p is smooth but $T_2 = \mathrm{H}^1(C, \mathscr{N}_{C/J}) = \mathbf{k}$ does not vanish. If C is nonhyperelliptic genus at least 3 then again $H_C \cong J$ is smooth, and $\mathrm{H}^1(C, \mathscr{N}_{C/J}) \neq 0$. See [175] for the computation of the scheme structure of H_C in the hyperelliptic case.

APPENDIX B

Intersection theory

This appendix covers some of the foundational material in [**67**]. The important construction of *refined Gysin homomorphisms* [**67**, Section 6.2] is covered in Section B.3. Throughout, all schemes are of finite type over an algebraically closed field $\mathbf{k}$. Varieties are integral (i.e. reduced and irreducible) schemes. A subvariety V of a scheme X is a closed subscheme which is a variety.

B.1. Chow groups: pushforward, pullack, degree

Let X be an n-dimensional $\mathbf{k}$-scheme. A d-dimensional cycle on X (or simply a d-cycle) is a finite formal sum

$$\sum_i m_i \cdot V_i$$

where $V_i \subset X$ are (closed irreducible) subvarieties of dimension d and $m_i \in \mathbb{Z}$. The free abelian group generated by d-cycles is denoted $Z_d X$, and we set

$$Z_* X = \bigoplus_{d=0}^{n} Z_d X.$$

Elements of $Z_{n-1}X$ are called *Weil divisors*. The fundamental class of X is the (possibly inhomogeneous) cycle

$$[X] \in Z_* X$$

determined by the irreducible components $V_i \subset X$ and their geometric multiplicities $m_i = \operatorname{length}_{\mathscr{O}_{X,\xi_i}} \mathscr{O}_{X,\xi_i}$, where ξ_i is the generic point of V_i and $\mathscr{O}_{X,\xi_i}$ is a local Artin ring.

If $r \in \mathbf{k}(X)^\times$ is a nonzero rational function and $V \subset X$ is a codimension 1 subvariety with generic point ξ_V, pick a and b in $A = \mathscr{O}_{X,\xi_V}$ such that $r = a/b$ and set

$$\operatorname{ord}_V(r) = \operatorname{length}_A(A/a) - \operatorname{length}_A(A/b).$$

This is the *order of vanishing* of r along V. Note that $\operatorname{ord}_V(r \cdot r') = \operatorname{ord}_V(r) + \operatorname{ord}_V(r')$ for $r, r' \in \mathbf{k}(X)^\times$. A rational function r as above defines a Weil divisor

$$\operatorname{div}(r) = \sum_{\substack{V \subset X \\ \operatorname{codim}(V,X)=1}} \operatorname{ord}_V(r) \cdot V \in Z_{n-1}X.$$

A d-cycle α is said to be *rationally equivalent to* 0 if it belongs to the subgroup $R_d X \subset Z_d X$ generated by cycles of the form $\operatorname{div}(r)$, where r is a nonzero rational function on a $(d+1)$-dimensional subvariety of X. Form the direct sum $R_* X = \bigoplus_{d=0}^n R_d X$. The quotient

$$A_* X = Z_* X / R_* X = \bigoplus_{d=0}^{n} A_d X$$

is the *Chow group* of X, where we have set $A_d X = Z_d X / R_d X$. If X is pure, then $Z_n X = A_n X$ is freely generated by the classes of the irreducible components of X.

Chow groups are covariant for proper morphisms and contravariant for flat morphisms, as we now recall.

Let $f\colon X \to Y$ be a proper morphism of schemes. Then there is a *pushforward* map

$$f_*\colon A_* X \to A_* Y$$

defined on generators by sending a d-cycle class $[V] \in A_d X$ to 0 if $\dim f(V) < \dim V$, and to the cycle

$$e_V \cdot [f(V)] \in A_d Y$$

if $\dim V = \dim f(V)$. Here e_V is the degree of the field extension $\mathbf{k}(f(V)) \subset \mathbf{k}(V)$.

Let $f : X \to Y$ be a flat morphism of schemes. Then there is a *pullback* map

$$f^* : A_* Y \to A_* X$$

defined on generators by sending a d-cycle class $[W] \in A_* Y$ to the cycle class

$$[f^{-1}(W)] \in A_{d+s} X$$

where s is the relative dimension of f.

Theorem B.1.1 ([67, Theorem 3.3]). *Let $\pi : E \to Y$ be a vector bundle on a scheme Y. Then π^* is an isomorphism.*

Notation B.1.2. Let $\pi : E \to Y$ be a vector bundle. We denote by $0^* : A_* E \xrightarrow{\sim} A_* Y$ (or sometimes by 0^*_E) the inverse of π^*. Even though 0^* is hard to describe in general, we informally describe it as 'intersecting with the zero section'.

Definition B.1.3. Let $f : X \to \operatorname{Spec} \mathbf{k}$ be the structure morphism of a proper $\mathbf{k}$-scheme X. The *degree map* is by definition the proper pushforward $f_* : A_* X \to \mathbb{Z}$. It takes the value 0 on cycle classes of positive dimension.

B.2. Operations on bundles: formularium

Let E be a vector bundle of rank r on a scheme X, and let $p : \mathbb{P}(E) \to X$ be the projective bundle of lines in the fibres of $E \to X$. Note that p is both flat and proper. Let $\mathscr{O}_E(1)$ be the dual of the tautological line bundle $\mathscr{O}_E(-1) \subset p^* E$ on $\mathbb{P}(E)$. The *Segre classes* $s_i(E)$ can be seen as operators $A_k X \to A_{k-i} X$ defined by

$$s_i(E) \cap \alpha = p_*(\xi^{r-1+i} \cap p^* \alpha),$$

where $\xi = c_1(\mathscr{O}_E(1))$. The first Chern class of a line bundle was defined in Equation (2.3.2). Such operation is the identity for $i = 0$ and identically vanishes for $i < 0$. If L is a line bundle, then

$$s_k(E \otimes L) = \sum_{i=0}^{k} (-1)^{k-i} \binom{r-1+k}{r-1+i} s_i(E) c_1(L)^{k-i}.$$

Definition B.2.1. One defines the following objects:

- The *Segre series* of E is the formal power series
$$s_t(E) = 1 + \sum_{i>0} s_i(E) t^i.$$
- The *Chern polynomial* of E is
$$c_t(E) = s_t(E)^{-1} = 1 + \sum_{i>0} c_i(E) t^i.$$
It is indeed a polynomial, for $c_i(E) = 0$ for all $i > r = \operatorname{rk} E$.
- The *total Segre class* and *total Chern class* of E are
$$s(E) = 1 + s_1(E) + s_2(E) + \cdots$$
$$c(E) = 1 + c_1(E) + \cdots + c_r(E).$$

Let $E \to X$ be a vector bundle, $f : Y \to X$ a proper morphism. One has the *projection formula*

$$f_*(c_i(f^*E) \cap \alpha) = c_i(E) \cap f_*\alpha, \qquad \alpha \in A_*Y.$$

If $f : Y \to X$ is a flat morphism, on the other hand, one has

$$c_i(f^*E) \cap f^*\beta = f^*(c_i(E) \cap \beta), \qquad \beta \in A_*X.$$

Given a short exact sequence

$$0 \to E \to F \to G \to 0 \tag{B.2.1}$$

of vector bundles on X, one has Whitney's formula

$$c_t(F) = c_t(E) \cdot c_t(G). \tag{B.2.2}$$

The *splitting construction* says that if E is a vector bundle of rank r on a scheme X, there exists a flat morphism $f : Y \to X$ such that the flat pullback $f^* : A_*X \to A_*Y$ is injective and f^*E has a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{r-1} \subset E_r = f^*E$$

with line bundle quotients

$$L_i = E_i/E_{i-1}, \quad i = 1, \ldots, r.$$

Set $\alpha_i = c_1(L_i)$. Then each short exact sequence

$$0 \to E_{i-1} \to E_i \to L_i \to 0$$

gives, via Equation (B.2.2), an identity

$$(1 + \alpha_i t) \cdot c_t(E_{i-1}) = c_t(E_i).$$

So we have

$$\begin{aligned} f^*c_t(E) &= c_t(f^*E) \\ &= (1 + \alpha_r t) \cdot c_t(E_{r-1}) \\ &= (1 + \alpha_r t) \cdot (1 + \alpha_{r-1} t) \cdot c_t(E_{r-2}) \\ &= (1 + \alpha_1 t) \cdots (1 + \alpha_r t). \end{aligned}$$

By injectivity of f^*, we may view the latter product as a formal expression for $c_t(E)$. In other words, we can always pretend that E is filtered by $0 = E_0 \subset E_1 \subset \cdots \subset E_{r-1} \subset E_r = E$ with line bundle quotients L_i, and

$$c_t(E) = \prod_{i=1}^{r} (1 + \alpha_i t), \tag{B.2.3}$$

where $\alpha_i = c_1(L_i)$. In fact, one should regard (B.2.3) as a formal expression defining $\alpha_1, \ldots, \alpha_r$. These are called the *Chern roots* of E, and they satisfy the fundamental relation

$$c_i(E) = \sigma_i(\alpha_1, \ldots, \alpha_r), \quad i = 0, \ldots, r$$

where σ_i denotes the i-th symmetric function. For instance, if $\operatorname{rk} E = r$, one would have

$$c_0(E) = 1, \qquad c_1(E) = \alpha_1 + \cdots + \alpha_r, \qquad c_r(E) = \alpha_1 \cdots \alpha_r.$$

Example B.2.2 (Dual bundles). The Chern roots of the dual bundle $E^\vee$ are $-\alpha_1, \ldots, -\alpha_r$, so $c(E^\vee) = \prod_{1 \le i \le \operatorname{rk} E}(1 - \alpha_i)$. Thus

$$c_i(E^\vee) = (-1)^i c_i(E).$$

Example B.2.3 (Tensor products)**.** If F is a vector bundle of rank s, the Chern roots of $E \otimes F$ are $\alpha_i + \beta_j$, where $i = 1, \ldots, r$ and $j = 1, \ldots, s$. So $c_k(E \otimes F)$ is the k elementary symmetric function of $\alpha_1 + \beta_1, \ldots, \alpha_r + \beta_s$. For instance, if $s = 1$,

$$c_t(E \otimes L) = \sum_{i=0}^{r} c_t(L)^{r-i} c_i(E) t^i.$$

Term by term, this can be reformulated as

$$c_k(E \otimes L) = \sum_{i=0}^{k} \binom{r-i}{k-i} c_i(E) c_1(L)^{k-i}.$$

Example B.2.4 (Exterior product)**.** For the exterior power $\wedge^p E$ we have

$$c_t(\wedge^p E) = \prod_{i_1 < \cdots < i_p} (1 + (\alpha_{i_1} + \cdots + \alpha_{i_p}) t),$$

so that for instance we recover $c_1(\det E) = c_1(E)$, which was Definition 2.3.1.

DEFINITION B.2.5. The *Chern character* of a vector bundle E with Chern roots $\alpha_1, \ldots, \alpha_r$ is the expression

$$\mathrm{ch}(E) = \sum_{i=1}^{r} \exp(\alpha_i).$$

One has

$$\mathrm{ch}(E) = r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + \cdots$$

where $c_i = c_i(E)$, and moreover $\mathrm{ch}(-)$ satisfies the relation

$$\mathrm{ch}(F) = \mathrm{ch}(E) + \mathrm{ch}(G)$$

for any short exact sequence as in (B.2.1). One also has an identity

$$\mathrm{ch}(E \otimes E') = \mathrm{ch}(E) \cdot \mathrm{ch}(E').$$

All in all, the Chern character defines a ring homomorphism $\mathrm{ch}\colon K^0(X) \to A^*X$, and if X is a nonsingular projective variety, $\mathrm{ch} \otimes \mathbb{Q}$ is an isomorphism.

DEFINITION B.2.6. The *Todd class* of a line bundle L with $\eta = c_1(L)$ is the formal expression

$$\mathrm{Td}(L) = \frac{\eta}{1 - e^{-\eta}} = 1 + \frac{1}{2}\eta + \sum_{i \geq 1} \frac{B_{2i}}{(2i)!} \eta^{2i}$$

where B_k are the Bernoulli numbers.

One may also set

$$\mathrm{Td}^\vee(L) = \mathrm{Td}(L^\vee) = \frac{-\eta}{1 - e^{\eta}} = \frac{\eta}{e^\eta - 1} = 1 - \frac{1}{2}\eta + \sum_{i \geq 1} \frac{B_{2i}}{(2i)!} \eta^{2i},$$

satisfying the relation $\mathrm{Td}(L) = e^\eta \cdot \mathrm{Td}(L^\vee)$. For a vector bundle E with Chern roots $\alpha_1, \ldots, \alpha_r$, we set

$$\mathrm{Td}(E) = \prod_{i=1}^{r} \frac{\alpha_i}{1 - e^{-\alpha_i}}$$

and it is easy to compute

$$\mathrm{Td}(E) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \frac{1}{720}(-c_1^4 + 4c_1^2c_2 + 3c_2^2 + c_1c_3 - c_4) + \cdots$$

where $c_i = c_i(E)$. The multiplicativity property

$$\mathrm{Td}(F) = \mathrm{Td}(E) \cdot \mathrm{Td}(G)$$

holds for every short exact sequence of vector bundles as in (B.2.1).

DEFINITION B.2.7. For two vector bundles E, F on a scheme X, define

$$c(F-E) = \frac{c(F)}{c(E)} = 1 + c_1(F-E) + c_2(F-E) + \cdots \tag{B.2.4}$$

Example B.2.8. If $F = 0$, we have $c(-E) = s(E)$.

Example B.2.9. The first few terms of the expansion (B.2.4) are

$$\begin{aligned} c_0(F-E) &= 1 \\ c_1(F-E) &= c_1(F) - c_1(E) \\ c_2(F-E) &= c_2(F) - c_1(F)c_1(E) + c_1(E)^2 - c_2(E). \end{aligned}$$

Remark B.2.10. If $\mathsf{F} = \sum_j [F_j]$ and $\mathsf{E} = \sum_i [E_i]$ are elements of the Grothendieck group $K^0(X)$ of vector bundles on a scheme X, then the Chern class of $\mathsf{F} - \mathsf{E}$ is defined as

$$c(\mathsf{F}-\mathsf{E}) = \frac{\prod_j c(F_j)}{\prod_i c(E_i)}.$$

Clearly, one has $c([F]-[E]) = c(F-E)$ for two vector bundles E, F. Similarly, the power series

$$c_t(\mathsf{F}-\mathsf{E}) = \frac{\prod_j c_t(F_j)}{\prod_i c_t(E_i)}$$

takes the role of the Chern polynomial in K-theory.

The Chern character of a K-theory element F is related to its Chern classes via the formula

$$c_d(\mathsf{F}) = \left[\exp\left(\sum_{s\geq 1} (-1)^{s-1}(s-1)!\mathrm{ch}_s(\mathsf{F}) \right) \right]_d$$

where $[\,\cdot\,]_d$ means taking the degree d part.

B.3. Refined Gysin homomorphisms

The material in this section covers (and expands a bit) the construction of [67, Section 6.2].

DEFINITION B.3.1. We say that a morphism of schemes $f : X \to Y$ admits a *factorisation* if there is a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\ & M \arrow[ur, "\pi"'] & \end{tikzcd} \tag{B.3.1}$$

where i is a closed embedding and π is smooth.

Example B.3.2. If X, Y are quasiprojective, any morphism $X \to Y$ admits a factorisation. A factorisation always exists locally on Y.

If $f : X \to Y$ admits two factorisations

$$X \xrightarrow{i_1} M_1 \xrightarrow{\pi_1} Y, \qquad X \xrightarrow{i_2} M_2 \xrightarrow{\pi_2} Y,$$

there is a third one,

$$X \xrightarrow{i} M \xrightarrow{\pi} Y,$$

dominating both:

$$\begin{tikzcd} & M_1 \arrow[dr, "\pi_1"] & \\ X \arrow[ur, "i_1"] \arrow[r, "i"] \arrow[dr, "i_2"'] & M \arrow[u, dashed, "\exists"'] \arrow[d, dashed, "\exists"] \arrow[r, "\pi"] & Y \\ & M_2 \arrow[ur, "\pi_2"'] & \end{tikzcd}$$

It is enough to take $M = M_1 \times_Y M_2$ and use that smooth morphisms and closed immersions are stable under base change and composition.

DEFINITION B.3.3. A morphism $f: X \to Y$ is a *local complete intersection* (lci, for short) if it has a factorisation $X \hookrightarrow M \to Y$ where $X \hookrightarrow M$ is a regular closed embedding. If this is only true locally on Y, we say that f is *locally lci*. If f is lci, then the K-theory class

$$T_{M/Y}\big|_X - N_{X/M} \in K^0(X)$$

is independent on the factorisation. It is called the *virtual tangent bundle* of f.

Remark B.3.4. If $f: X \to Y$ is lci, *all* of its factorisations $X \hookrightarrow M \to Y$ satisfy the property that $X \hookrightarrow M$ is regular.

Remark B.3.5. An lci morphism $f: X \to Y$ has a well-defined *relative dimension*: given a factorisation (B.3.1), it is the integer

$$r = \operatorname{rk} T_{M/Y} - \operatorname{codim}(X, M) \in \mathbb{Z}.$$

For instance, a regular closed immersion of codimension d is an lci morphism of relative dimension $-d$.

By [**?**, EGA44], a composition of regular closed immersions $X \hookrightarrow Y$ and $Y \hookrightarrow Z$ is a regular closed immersion, and there is an exact sequence of vector bundles

$$0 \to N_{X/Y} \to N_{X/Z} \to N_{Y/Z}\big|_X \to 0.$$

Let $f: X \to Y$ be an lci morphism of relative dimension r, factoring as a regular immersion $i: X \to M$ followed by a smooth morphism $\pi: M \to Y$ of relative dimension s. Given any morphism $g: \widetilde{Y} \to Y$, consider the double fibre square

$$\begin{tikzcd} \widetilde{X} \arrow[r, hook] \arrow[d, "\widetilde{g}"'] \arrow[dr, phantom, "\square"] & \widetilde{M} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \widetilde{Y} \arrow[d, "g"] \\ X \arrow[r, hook, "i"] & M \arrow[r, "\pi"] & Y. \end{tikzcd}$$

For any $k \geq 0$, we will construct a group homomorphism

$$f^!: Z_k\widetilde{Y} \longrightarrow Z_{k+s}\widetilde{M} \xrightarrow{\sigma} Z_{k+s}C_{\widetilde{X}/\widetilde{M}} \xrightarrow{\phi} A_{k+r}\widetilde{X}.$$

The first arrow is just flat pullback: it is not a problem to pull back cycles from $\widetilde{Y}$ to $\widetilde{M}$ since π is smooth. The arrow σ, called *specialisation to the normal cone* in [**67**, Section 5.2], is given as follows. For any $(k+s)$-dimensional subvariety $V \hookrightarrow \widetilde{M}$, consider the intersection

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d, hook, "j"'] \arrow[dr, phantom, "\square"] & V \arrow[d, hook] \\ \widetilde{X} \arrow[r, hook] & \widetilde{M} \end{tikzcd}$$

and the normal cone $C_{W/V}$, which is purely of dimension $k+s$ (cf. Remark C.1.1). It defines a closed subcone

$$\ell: C_{W/V} \hookrightarrow j^*C_{\widetilde{X}/\widetilde{M}} = C_{\widetilde{X}/\widetilde{M}} \times_{\widetilde{X}} W \hookrightarrow C_{\widetilde{X}/\widetilde{M}}.$$

We define

$$\sigma[V] = \ell_*[C_{W/V}] = [C_{V\cap\widetilde{X}/V}] \in Z_{k+s}C_{\widetilde{X}/\widetilde{M}}.$$

As for the map ϕ, we observe from the cartesian diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r, hook] \arrow[d, "\widetilde{g}"'] \arrow[dr, phantom, "\square"] & \widetilde{M} \arrow[d] \\ X \arrow[r, hook, "i"'] & M \end{tikzcd}$$

that we have a closed subcone

(B.3.2) $$C_{\widetilde{X}/\widetilde{M}} \subset \widetilde{g}^* C_{X/M}.$$

Since i is *regular*, $C_{X/M} = N_{X/M}$ is a vector bundle on $\widetilde{X}$, so that

$$E = \widetilde{g}^* C_{X/M}$$

is a vector bundle too. Its rank is easily computed as

$$\operatorname{rk} E = \operatorname{rk} N_{X/M} = \operatorname{codim}(X, M) = s - r.$$

Let B be a subvariety of $C_{\widetilde{X}/\widetilde{M}} \subset E$ of dimension $k+s$. Then define

$$\phi[B] = 0^*[B],$$

where

$$0^* : A_{k+s}E \xrightarrow{\sim} A_{k+s-(s-r)}\widetilde{X} = A_{k+r}\widetilde{X}$$

is the inverse of the flat pullback on $E \to \widetilde{X}$ (cf. Notation B.1.2).

The morphism we have just constructed descends to rational equivalence, to give a morphism

(B.3.3) $$f^! : A_k\widetilde{Y} \to A_{k+r}\widetilde{X}$$

that Fulton calls *refined Gysin homomorphism*.

We have the following facts:

(a) The homomorphism $f^!$ agress with flat pullback when $\widetilde{Y} = Y \to Y$ is the identity and $f : X \to Y$ is flat.
(b) The homomorphism $f^!$ is called *refined Gysin pullback* when $\widetilde{Y} \to Y$ is a closed embedding.
(c) The homomorphism $f^!$ does not depend on the choice of the factorisation.
(d) Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms. If f is a regular embedding and both g and $g \circ f$ are flat, then

$$f^! \circ g^* = (g \circ f)^*.$$

Moreover, if f and $g \circ f$ are regular embeddings, and g is *smooth*, then

(B.3.4) $$f^! \circ g^* = (g \circ f)^!.$$

This is basically [**67**, Proposition 6.5]. However, (B.3.4) is false in general if g is just flat. See the functoriality property (D) in the next subsection for the general (lci) case.

B.3.1. An example: Localised top Chern class. Let X be a variety, $E = \operatorname{Spec}_{\mathscr{O}_X} \operatorname{Sym} \mathscr{E}^* \to X$ a vector bundle and let

(B.3.5) $$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "i"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "s"] \\ X \arrow[r, "0"'] & E \end{tikzcd}$$

be the fibre diagram defining the zero locus Z of a section $s \in \mathrm{H}^0(X, E) = \operatorname{Hom}(\mathscr{O}_X, \mathscr{E})$. Then $0 : X \to E$ is regular of codimension $e = \operatorname{rk} E$, and $E = N_{X/E}$ as vector bundles over X. For each k we get refined Gysin homomorphisms

$$0^! : A_k X \to A_{k-e} Z.$$

Suppose X is purely n-dimensional. Then the *localised top Chern class* of (E,s) is

$$\mathbf{Z}(s) = 0^![X] \in A_{n-e}Z,$$

where $[X]$ is the fundamental class of X. In the language of the previous section, we can rewrite

$$0^![X] = 0^*_{E|_Z}[C_{Z/X}]$$

where the right hand side is the image of the fundamental class of the n-dimensional cone

$$C_{Z/X} \subset N_{Z/X} \subset E\big|_Z$$

under the map $0^*_{E|_Z} : A_n(E|_Z) \xrightarrow{\sim} A_{n-e}Z$ (cf. Notation B.1.2). The closed embedding

$$N_{Z/X} \subset E\big|_Z \tag{B.3.6}$$

comes directly from the diagram (B.3.5): the dual section $s^\vee : \mathscr{E}^* \to \mathscr{O}_X$ hits the ideal sheaf $\mathscr{I} \subset \mathscr{O}_X$ of $Z \subset X$, and (B.3.6) is the result of applying $\operatorname{Spec}_{\mathscr{O}_Z} \operatorname{Sym}$ to the natural surjection

$$s^\vee\big|_Z : \mathscr{E}^*\big|_Z \twoheadrightarrow \mathscr{I}/\mathscr{I}^2.$$

Remark B.3.6 (Toy model for the virtual class)**.** In Definition 9.1.1, the class

$$\mathbf{Z}(s) = 0^![X] = 0^*_{E|_Z}[C_{Z/X}] \in A_{n-\operatorname{rk}E}Z$$

was defined to be the *virtual fundamental class* of $Z = Z(s)$.

Remark B.3.7 (Relation with the deformation to the normal cone)**.** The deformation to the normal cone [67, Chapter 5] enters the picture as follows: we have embeddings $\lambda s : Z \to E$ for all $\lambda \in \mathbb{A}^1$. Letting $\lambda \to \infty$ turns these embeddings into exactly $C_{Z/X} \subset E|_Z$. More explicitly, consider the graph of $\lambda s : X \to E$ as a line in $E \oplus \mathscr{O}_X$, to get an embedding

$$X \times \mathbb{A}^1 \hookrightarrow \mathbb{P}(E \oplus \mathscr{O}_X) \times \mathbb{P}^1, \quad (x,\lambda) \mapsto ((x, \lambda s(x)), (\lambda : 1)).$$

Then the deformation space of the deformation to the normal cone construction turns out to be the closure

$$M = \overline{X \times \mathbb{A}^1} \subset \mathbb{P}(E \oplus \mathscr{O}_X) \times \mathbb{P}^1,$$

and the embeddings $\lambda s : X \subset E$ deform to $X \subset C_{X/E} \subset N_{X/E} = E$. Restricting to Z gives $C_{Z/X} \subset E|_Z$.

Remark B.3.8. The localised top Chern class $\mathbf{Z}(s)$ is also called the *refined Euler class* of E, because

$$\begin{aligned} i_*\mathbf{Z}(s) &= i_*0^![X] \\ &= 0^*s_*[X] \\ &= s^*s_*[X] \\ &= c_e(E) \cap [X]. \end{aligned}$$

We have used that if $\pi : E \to X$ is any vector bundle then any $s \in \mathrm{H}^0(X,E)$ is a regular embedding and $s^! = s^* : A_kE \to A_{k-e}X$ is the inverse of flat pullback π^*. In particular s^* does not depend on s, so $s^* = 0^*$. The last equality is a special case of the self-intersection formula, using also that $E = N_{X/E}$ as vector bundles over X. Moreover, it can be interesting to notice that $\mathbf{Z}(s) = [Z]$ when s is a regular section (which happens for instance when X is Cohen–Macaulay and $\dim Z(s) = \dim X - e$, as we saw in Section 2.3.1).

Example B.3.9. If $Z = \operatorname{crit}(f)$ is the zero locus of $\mathrm{d}f$, an exact 1-form on a nonsingular scheme Y, then

$$[Z]^{\mathrm{vir}} = \mathbf{Z}(\mathrm{d}f) = 0^![Y] = 0^*_{\Omega_Y|_Z}[C_{Z/Y}] \in A_0Z$$

is the virtual fundamental class of Example 9.1.10. As we saw in Remark 9.1.6, the *obstruction sheaf* is completely intrinsic to Z. Without using the critical obstruction theory, Ob could be defined (cf. Exercise 9.1.9) to be the excess bundle of the fibre diagram

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "i"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "\mathrm{d}f"] \\ Y \arrow[r, "0"'] & \Omega_Y \end{tikzcd}$$

i.e. as the cokernel

$$\mathscr{N}_{Z/Y} \to i^*\mathscr{N}_{Y/\Omega_Y} \to \mathrm{Ob} \to 0.$$

But this sequence is of course nothing but

$$\mathscr{I}/\mathscr{I}^2 \to \Omega_Y\big|_Z \to \Omega_Z \to 0.$$

In other words, $\mathrm{Ob} = \Omega_Z$, in agreement with Remark 9.1.6 .

B.3.2. More properties of $f^!$ and relation with bivariant classes. We now quickly discuss the main properties of refined Gysin homomorpisms. We refer to [**67**, Chapter 6] for complete proofs.

B.3.3. Compatibilities of refined Gysin homomorphisms. Let $f: X \to Y$ be an lci morphism. Then we have a map $f^!$ as in (B.3.3) for all $\widetilde{Y} \to Y$ and for all $k \geq 0$. This trivial observation, together with the compatibilities we are about to describe, states precisely that any lci morphism $f: X \to Y$ of relative dimension r defines a *bivariant class*

$$[f^!] \in A^{-r}(X \xrightarrow{f} Y),$$

as described in [**67**, Chapter 17].

Let $f: X \to Y$ be an lci morphism of relative dimension r. We first state the properties of $f^!$ informally, and then we explain what they mean.

(A) Refined Gysin homomorphisms commute with proper pushforward and flat pullback.
(B) Refined Gysin homomorphisms are compatible with each other.
(C) Refined Gysin homomorphisms commute with each other.
(D) Refined Gysin homomorphisms are functorial.

Here is what the above statements mean. Fix once and for all an integer $k \geq 0$.

(A) For any double fibre diagram

(B.3.7)
$$\begin{tikzcd} X' \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\square"] & Y' \arrow[d, "h"] \\ \widetilde{X} \arrow[r, "\tilde{f}"] \arrow[d] \arrow[dr, phantom, "\square"] & \widetilde{Y} \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

one has the following:

(P) If h is proper, then for all $\alpha \in A_k Y'$ one has

$$f^!(h_*\alpha) = q_*(f^!\alpha) \in A_{k+r}\widetilde{X}.$$

(F) If h is flat of relative dimension n, then for all $\alpha \in A_k \widetilde{Y}$ one has

$$f^!(h^*\alpha) = q^*(f^!\alpha) \in A_{k+r+n}X'.$$

(B) In situation (B.3.7), if $\tilde{f}: \widetilde{X} \to \widetilde{Y}$ is also lci of relative dimension r, then for all $\alpha \in A_k Y'$ one has

$$f^!\alpha = \tilde{f}^!\alpha \in A_{k+r}X'.$$

(C) Let $j: S \to T$ be a regular embedding of codimension e. Given morphisms $\widetilde{Y} \to Y$ and $\widetilde{Y} \to T$, form the fibre square

(B.3.8)
$$\begin{tikzcd} X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S \arrow[d, "j"] \\ \widetilde{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \widetilde{Y} \arrow[r] \arrow[d] & T \\ X \arrow[r, "f"] & Y & \end{tikzcd}$$

and fix $\alpha \in A_k\widetilde{Y}$. Then one has

$$j^!(f^!\alpha) = f^!(j^!\alpha) \in A_{k+r-e}X'.$$

(D) Let $f : X \to Y$ and $g : Y \to Z$ be lci morphisms of relative dimensions r and s respectively. Then, for all morphisms $\tilde{Z} \to Z$, one has the identity

$$(g \circ f)^!(\alpha) = f^!(g^!\alpha) \in A_{k+r+s}(X \times_Z \tilde{Z}).$$

B.3.3.1. *Bivariant classes.* Let $f : X \to Y$ be any morphism. Suppose f has the property that when we let morphisms $g : \widetilde{Y} \to Y$ and integers $k \geq 0$ vary arbitrarily, we are able to construct homomorphisms

$$c_g^{(k)} : A_k\widetilde{Y} \to A_{k-p}\widetilde{X}, \quad \widetilde{X} = X \times_Y \widetilde{Y},$$

for some $p \in \mathbb{Z}$. Then the collection c of these homomorphisms is said to define a *bivariant class*

$$c \in A^p(X \xrightarrow{f} Y)$$

if compatibilities like the ones described in (A) and (C) in the previous section are satisfied. Here are precise requirements.

(A)′ In any double fibre square situation like (B.3.7), one has the following:

(P) If h is proper, then for all $\alpha \in A_kY'$ one has the identity

$$c_g^{(k)}(h_*\alpha) = q_*(c_{gh}^{(k)}\alpha) \in A_{k-p}\widetilde{X}.$$

(F) If h is flat of relative dimension n, then for all $\alpha \in A_k\widetilde{Y}$ one has the identity

$$c_{gh}^{(k+n)}(h^*\alpha) = q^*(c_g^{(k)}\alpha) \in A_{k+n-p}X'.$$

(C)′ In situation (B.3.8), for all $\alpha \in A_k\widetilde{Y}$ one has

$$j^!(c_g^{(k)}\alpha) = c_{gi}^{(k-e)}(j^!\alpha) \in A_{k-p-e}X'.$$

Conclusion. Any lci morphism $f : X \to Y$ of relative dimension r defines a bivariant class

$$[f^!] \in A^{-r}(X \xrightarrow{f} Y).$$

For instance, if $f = i$ is a regular immersion of codimension d, this class is

$$[i^!] \in A^d(X \xrightarrow{f} Y).$$

APPENDIX C

Perfect obstruction theories and virtual classes

SUMMARY. In this appendix we cover part of the material in the work of Behrend–Fantechi [19]. We aim at giving the minimal tools in order to properly define virtual fundamental classes and study their properties in reasonable generality. Our exposition follows closely Fantechi's course [60]. In this section all schemes and stacks are of finite type over a field $\mathbf{k}$.

C.1. Cones

C.1.1. Definition of cones. If $X \hookrightarrow M$ is a closed immersion of schemes with ideal $\mathscr{I} \subset \mathscr{O}_M$, the *normal cone* and the *normal sheaf* relative to this immersion are, respectively, the X-schemes

$$C_{X/M} = \operatorname{Spec} \bigoplus_{n \geq 0} \mathscr{I}^n/\mathscr{I}^{n+1},$$

$$N_{X/M} = \operatorname{Spec} \operatorname{Sym} \mathscr{I}/\mathscr{I}^2.$$

Recall our references [185, Tag 01LQ] or [96, II.5] for the *global Spec* construction. There is always a closed immersion (over X) of $C_{X/M}$ inside $N_{X/M}$, induced by the natural surjection

$$\operatorname{Sym} \mathscr{I}/\mathscr{I}^2 \twoheadrightarrow \bigoplus_{n \geq 0} \mathscr{I}^n/\mathscr{I}^{n+1}.$$

The schemes $C_{X/M}$ and $N_{X/M}$ are, as we shall see in a minute, particular examples of *cones*.

Remark C.1.1. Let $X \hookrightarrow M$ be a closed immersion with ideal $\mathscr{I}$, where M is of pure dimension. The normal cone $C_{X/M}$ has pure dimension $\dim M$. Indeed, let $B = \mathrm{Bl}_X M$ be the blowup of M along X. Then $\dim B = \dim M$ and the exceptional divisor is

$$E_X M = \operatorname{Proj} \bigoplus_{n \geq 0} \mathscr{I}^n/\mathscr{I}^{n+1},$$

so that $\dim C_{X/M} = \dim E_X M + 1 = \dim M$.

Remark C.1.2. If $X \hookrightarrow M$ is a closed immersion and M is purely d-dimensional, then so is $C_{X/M}$. Moreover, if $X = X_1 \cup \cdots \cup X_s$ is the decomposition of X into irreducible components, the number of irreducible components of $C_{X/M}$ is always bigger or equal to s.

Recall that a closed immersion $i\colon X \hookrightarrow M$ is *regular of codimension* d if for every point $x \in X$ there is an affine open neighborhood $i(x) \in \operatorname{Spec} A \subset M$ such that the ideal $I \subset A$ defining $X \cap \operatorname{Spec} A \hookrightarrow \operatorname{Spec} A$ is generated by a regular sequence of length d.

EXERCISE C.1.3. Let $X \hookrightarrow M$ be a closed immersion with ideal $\mathscr{I}$. We have that $N_{X/M}$ is a vector bundle if and only if $\mathscr{I}/\mathscr{I}^2$ is locally free. Prove the following:

(i) $C_{X/M}$ is a vector bundle if and only if the natural projection to X is smooth.
(ii) If $X \hookrightarrow M$ is regular then $N_{X/M} = C_{X/M}$, and it is a vector bundle.

Before giving the general definition of cones, here are a few concrete examples.

Example C.1.4. Let $Y \subset \mathbb{P}^{n-1}$ be a projective scheme and let $M \subset \mathbb{A}^n$ be the *affine cone* over Y. If $X = \{0\} \subset M$ is the origin, then $C_{X/M} \cong M$, and if Y is nondegenerate then $N_{X/M} = \mathbb{A}^n$, in which case the normal cone is strictly included in the normal sheaf.

Example C.1.5. Let $M = \mathbb{A}^2$ and $X = \operatorname{Spec} R \hookrightarrow M$, where $R = \mathbf{k}[x,y]/(x^2, xy, y^2)$. Then we have the closed immersion over X,

$$C_{X/M} = \operatorname{Spec} R[a,b,c]/(ab-c^2) \subset N_{X/M} = \operatorname{Spec} R[a,b,c].$$

We observe that $N_{X/M}$ is a vector bundle but $C_{X/M}$ is not: the immersion $X \hookrightarrow M$ is not regular.

Example C.1.6. Let $M = \mathbb{A}^2$ and $X = \operatorname{Spec} R$ where $R = \mathbf{k}[x,y]/(xy, y^2)$. Then $C_{X/M} = N_{X/M}$ has two irreducible components and is not a vector bundle. Again, $X \hookrightarrow M$ is not regular, since X is the x-axis with a vertical 'fuzz' at the origin.

Example C.1.7. Let $M = \mathbb{A}^3$ and $X = \operatorname{Spec} R$ where $R = \mathbf{k}[x,y,z]/(xz, yz)$. Then $C_{X/M} = N_{X/M} = \operatorname{Spec} R[a,b]/(a^2b^2 - ab)$ has two irreducible components and is not a vector bundle.

We now define cones in general. Let $\mathscr{A} = \bigoplus_{n\geq 0} \mathscr{A}_n$ be a quasicoherent sheaf of graded $\mathscr{O}_X$-algebras. Consider the following condition:

(†) The canonical map $\mathscr{A}_0 \to \mathscr{O}_X$ is an isomorphism, $\mathscr{A}_1$ is coherent and generates $\mathscr{A}$ over $\mathscr{A}_0$.

DEFINITION C.1.8. A *cone* over a scheme X is an X-scheme of the form

$$\pi\colon \operatorname{Spec} \mathscr{A} \to X,$$

where $\mathscr{A}$ satisfies (†).

Example C.1.9. A *vector bundle* over X is a cone of the form

$$E = \operatorname{Spec} \operatorname{Sym} \mathscr{E} \to X,$$

where $\mathscr{E}$ is a locally free sheaf of finite rank (and $\mathscr{E}^*$ is the sheaf of sections of E). Clearly, $\operatorname{Sym} \mathscr{E}$ satisfies (†).

C.1.2. A short digression on gradings and $\mathbb{A}^1$-actions. Let $\pi\colon C = \operatorname{Spec} \mathscr{A} \to X$ be a cone. The sheaf $\mathscr{A}$ is recovered from the structural morphism as $\mathscr{A} = \pi_* \mathscr{O}_C$, but there are other important elements characterising the *cone structure*:

(1) the grading on $\mathscr{A}$;
(2) the canonical surjection $\mathscr{A} \twoheadrightarrow \mathscr{A}_0 \cong \mathscr{O}_X$ ($a_i \mapsto 0$ if $i > 0$) endows C with a natural X-morphism $0_C : X \to C$, called the *zero section* of C. The image of the zero section is called the *vertex* of the cone.

Let us focus on (1). To give a grading on $\mathscr{A}$ is the same as to give a $\mathbb{G}_m$-action on $C = \operatorname{Spec} \mathscr{A}$. In one direction, given an action

$$\mathbb{G}_m \times C \to C$$

one obtains a grading on $\mathscr{A} = \pi_* \mathscr{O}_C$ by letting $\mathscr{A}_i$ be the eigensheaf corresponding to the character $t \mapsto t^i$. In the other direction, given a grading $\mathscr{A} = \bigoplus_{i\geq 0} \mathscr{A}_i$ one obtains an action

$$\mathbb{G}_m \times \operatorname{Spec} \mathscr{A} = \operatorname{Spec} \mathscr{A}[t, t^{-1}] \to \operatorname{Spec} \mathscr{A}$$

by applying Spec to the natural morphism $\mathscr{A} \to \mathscr{A}[t,t^{-1}]$ sending $a_0 + \cdots + a_n \mapsto \sum_i a_i t^i$. Since no negative power of t occurs, we can in fact replace $\mathbb{G}_m$ by $\mathbb{A}^1$. To sum up, applying Spec to the augmentation morphism $\mathscr{A} \to \mathscr{A}[t]$ (sending $a_i \in \mathscr{A}_i$ to $a_i t^i$), we see that every cone carries a canonical $\mathbb{A}^1$-action

$$a\colon \mathbb{A}^1 \times C \to C$$

where the word *action* refers, as expected, to the commutativity of the following diagrams:

$$\begin{tikzcd} C \arrow[r, "{(1,\mathrm{id}_C)}"] \arrow[dr, "\mathrm{id}_C"'] & \mathbb{A}^1 \times C \arrow[d, "a"] \\ & C \end{tikzcd} \qquad 1 \cdot c = c,$$

$$\begin{tikzcd} C \arrow[r, "(0,\mathrm{id}_C)"] \arrow[dr, "\mathrm{id}_C"'] & \mathbb{A}^1 \times C \arrow[d, "a"] \\ & C \end{tikzcd} \qquad 0 \cdot c = 0,$$

$$\begin{tikzcd} \mathbb{A}^1 \times \mathbb{A}^1 \times C \arrow[r, "\mathrm{id}\times a"] \arrow[d, "m\times \mathrm{id}_C"'] & \mathbb{A}^1 \times C \arrow[d, "a"] \\ \mathbb{A}^1 \times C \arrow[r, "a"] & C \end{tikzcd} \qquad (xy)\cdot c = x\cdot(y\cdot c).$$

C.1.3. Abelian cones and hulls. Particular cones are those of the form

$$C = \operatorname{Spec} \operatorname{Sym} \mathscr{F} \to X,$$

where $\mathscr{F}$ is any coherent sheaf on X. These are called *abelian cones*, and they are to be thought of as the geometric incarnation of coherent sheaves. For instance, the coherent module Ω_X corresponds to the abelian cone

$$T_X = \operatorname{Spec} \operatorname{Sym} \Omega_X.$$

One can also characterise abelian cones as those cones $\operatorname{Spec} \mathscr{A} \to X$ such that the natural surjection $\operatorname{Sym} \mathscr{A}_1 \twoheadrightarrow \mathscr{A}$ is an isomorphism.

Lemma C.1.10. *Let $C \to X$ be a cone. Then C is a vector bundle if and only if C is smooth over X. An abelian cone is a vector bundle if and only if it is flat over X.*

Every cone $C = \operatorname{Spec} \mathscr{A}$ embeds as a closed subscheme in its *abelian hull*, which is the abelian cone

$$A(C) = \operatorname{Spec} \operatorname{Sym} \mathscr{A}_1.$$

Therefore abelian cones are the cones for which the natural closed immersion $C \to A(C)$ is an isomorphism. Clearly, we have inclusions

$$\{\text{ vector bundles }\} \subset \{\text{ abelian cones }\} \subset \{\text{ cones }\}.$$

Example C.1.11. Let $X \hookrightarrow M$ be a closed immersion. The abelian hull of $C_{X/M}$ is $N_{X/M}$.

DEFINITION C.1.12. A *morphism of cones* over X is an X-morphism $\operatorname{Spec} \mathscr{A}' \to \operatorname{Spec} \mathscr{A}$ induced by a graded morphism $\mathscr{A} \to \mathscr{A}'$ of $\mathscr{O}_X$-algebras.

In a little more detail, a morphism of cones over X is an $\mathbb{A}^1$-equivariant X-morphism $f : C \to D$ between the total spaces, respecting the zero section. The commutativity of the diagram

$$\begin{tikzcd} \mathbb{A}^1 \times C \arrow[r] \arrow[d, "1\times f"] & C \arrow[d, "f"] \arrow[dr] & \\ \mathbb{A}^1 \times D \arrow[r] & D \arrow[r] & X \arrow[ul, bend right, "0_C"'] \arrow[l, bend left, "0_D"] \end{tikzcd}$$

illustrates the condition.

For example, the inclusion of a cone $C \hookrightarrow A(C)$ in its abelian hull is a morphism of cones. More generally, a *closed subcone* is the image of a closed immersion of cones. Let Cones_X (resp. $\mathrm{AbCones}_X$) denote the category of cones (resp. abelian cones) over X.

Let $\mathrm{Alg}^{\dagger}(X)$ denote, for short, the category of quasicoherent graded $\mathscr{O}_X$-algebras satisfying (†), with morphisms the graded morphisms of $\mathscr{O}_X$-algebras. There is an equivalence of categories

$$\operatorname{Spec}\colon \mathrm{Alg}^{\dagger}(X) \xrightarrow{\sim} \mathrm{Cones}_X^{\mathrm{op}}. \tag{C.1.1}$$

The following commutative diagram of functors completely describes the situation.

$$\begin{tikzcd}
\mathrm{Alg}^{\dagger}(X) \arrow[rr, "\sim"] & & \mathrm{Cones}_X^{\mathrm{op}} \\
\{\text{Locally free coherent sheaves on } X\} \arrow[u, "\mathrm{Sym}"] \arrow[r, "\sim"] \arrow[d, hook] & \{\text{Vector bundles over } X\}^{\mathrm{op}} \arrow[ur, hook, "\text{full}"'] \arrow[dr, hook] & \\
\mathrm{Coh}\, X \arrow[rr, "\sim"] & & \mathrm{AbCones}_X^{\mathrm{op}}.
\end{tikzcd}$$

The top equivalence is just (C.1.1), the bottom equivalence is Spec Sym.

Remark C.1.13 (A naive comparison)**.** The relation between cones and abelian cones is the same existing between groups and abelian groups, or between vector spaces and commutative algebras: the abelianisation functor, as well as the Sym functor (from groups to abelian groups and from vector spaces to commutative algebras, respectively) has the forgetful morphism as a right adjoint. The same happens in this context: the functor $C \mapsto A(C)$ has the forgetful functor (which is fully faithful from abelian cones to cones) as a right adjoint. In other words, for every abelian cone A and every cone C, one has a natural bijection

$$\mathrm{Hom}_{\mathrm{AbCones}_X}(A(C), A) \cong \mathrm{Hom}_{\mathrm{Cones}_X}(C, A). \tag{C.1.2}$$

Remark C.1.14. We will eventually use the following rephrasing of (C.1.2) in the particular case where A is a vector bundle: a morphism $C \to A$ of cones is determined by the corresponding morphism of abelian cones.

Remark C.1.15. Any morphism $h \colon \widetilde{X} \to X$ induces a functor

$$h^* \colon \mathrm{Cones}_X \to \mathrm{Cones}_{\widetilde{X}}, \qquad C \mapsto C \times_X \widetilde{X},$$

preserving abelian cones and vector bundles.

Definition C.1.16 ([**19**, Definition 1.2]). An *exact sequence of cones* is a sequence of morphisms of cones

$$0 \to E \to C \to C_1 \to 0$$

such that E is a vector bundle, C and C_1 are cones and, Zariski locally on X, there exists a splitting $C_1 \to C$ inducing (again locally) an isomorphism $C \cong C_1 \times_X E$.

We know how to construct a normal cone given a closed immersion $i \colon X \hookrightarrow M$. What happens if we have another closed immersion $j \colon X \hookrightarrow P$ such that $i = \rho \circ j$ for some smooth morphism $\rho \colon P \to M$? Let $\mathscr{I} \subset \mathscr{O}_M$ and $\mathscr{J} \subset \mathscr{O}_P$ be the corresponding ideal sheaves.

$$\begin{tikzcd}
 & P \arrow[d, "\rho\,(\text{smooth})"] \\
X \arrow[ur, hook, "j"] \arrow[r, hook, "i"'] & M
\end{tikzcd}$$

The pullback $\rho^*\mathscr{I} \hookrightarrow \rho^*\mathscr{O}_M = \mathscr{O}_P$ (which is an ideal because ρ is flat) surjects onto $\mathscr{J}$, and this induces an injective map $j^*\rho^*\mathscr{I} \hookrightarrow j^*\mathscr{J}$, hence a short exact sequence of sheaves on X

$$0 \to \mathscr{I}/\mathscr{I}^2 \to \mathscr{J}/\mathscr{J}^2 \to j^*\Omega_{P/M} \to 0, \tag{C.1.3}$$

where we have noted that $j^*\rho^*\mathscr{I} = i^*\mathscr{I} = \mathscr{I}/\mathscr{I}^2$ and $j^*\mathscr{J} = \mathscr{J}/\mathscr{J}^2$. Applying Spec Sym to (C.1.3) yields an exact sequence of (abelian) cones

$$0 \to j^*T_{P/M} \to N_{X/P} \to N_{X/M} \to 0. \tag{C.1.4}$$

The kernel being a vector bundle, the above sequence is *locally split*, i.e. there is (locally on X) a morphism $N_{X/M} \to N_{X/P}$ inducing an isomorphism $N_{X/P} \cong N_{X/M} \times_X j^*T_{P/M}$. This tells us that embedding X into a larger scheme produces a normal sheaf that is essentially unique, 'up to a vector bundle'. Now, by Remark C.1.14, the sequence (C.1.4) is equivalent to the exact sequence of cones

$$0 \to j^* T_{P/M} \to C_{X/P} \to C_{X/M} \to 0. \tag{C.1.5}$$

By [**22**, VIII.1.3], i is regular if and only if j is regular. In this case, (C.1.4) is an exact sequences of vector bundles and agrees with (C.1.5) by Exercise C.1.3.

C.2. The truncated cotangent complex

Let $f: X \to Y$ be a morphism admitting factorisations $\pi \circ i: X \hookrightarrow M \to Y$ and $\pi' \circ j: X \hookrightarrow P \to Y$. We can 'compare' these two factorisations of f whenever we have a commutative diagram

$$\begin{tikzcd} & & P \arrow[d, "\rho"] \arrow[drr, "\pi'"] & & \\ X \arrow[urr, hook, "j"] \arrow[rr, hook, "i"] & & M \arrow[rr, "\pi"] & & Y \end{tikzcd} \tag{C.2.1}$$

where ρ is smooth. In this case, we also have a commutative diagram with exact rows and columns

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & & \\ 0 \arrow[r] & \ker\alpha \arrow[r] \arrow[d] & \mathscr{I}/\mathscr{I}^2 \arrow[r, "\alpha"] \arrow[d] & i^*\Omega_\pi \arrow[r] \arrow[d] & \Omega_f \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \ker\beta \arrow[r] & \mathscr{J}/\mathscr{J}^2 \arrow[r, "\beta"] \arrow[d] & j^*\Omega_{\pi'} \arrow[r] \arrow[d] & \Omega_f \arrow[r] & 0 \\ & & j^*\Omega_\rho \arrow[d] & j^*\Omega_\rho \arrow[d] & & \\ & & 0 & 0 & & \end{tikzcd}$$

inducing an isomorphism $\ker\alpha \xrightarrow{\sim} \ker\beta$. Since α and β have isomorphic kernels and cokernels, the middle part of the diagram gives a morphism of two-term perfect complexes

$$\begin{tikzcd} \big[\mathscr{I}/\mathscr{I}^2 \arrow[r] \arrow[d] & i^*\Omega_\pi\big] \arrow[d] \\ \big[\mathscr{J}/\mathscr{J}^2 \arrow[r] & j^*\Omega_{\pi'}\big] \end{tikzcd}$$

which is a quasi-isomorphism. In other words, the complex

$$\mathbb{L}_f = \big[\mathscr{I}/\mathscr{I}^2 \xrightarrow{\alpha} i^*\Omega_\pi\big] \in \mathbf{D}^{[-1,0]}(X)$$

is unique up to canonical isomorphism, in particular it does not depend on the chosen factorisation of f. It is an important character for defining perfect obstruction theories and virtual classes.

Let $L_f \in \mathbf{D}^{(-\infty,0]}(X)$ be the (relative) cotangent complex associated to f, as defined in [**111**]. Then $\mathbb{L}_f$ coincides with the truncation, or cutoff at -1, of the relative cotangent complex,

$$\mathbb{L}_f = \tau_{\geq -1} L_f. \tag{C.2.2}$$

Remark C.2.1. The relative cotangent complex L_f exists even if f has no factorisation, and hence so does $\mathbb{L}_f$, via Equation (C.2.2). For algebraic stacks (also called Artin stacks: we recall this notion in Definition C.4.1), the situation is a little subtler. For any concentrated (i.e. quasicompact and quasiseparated) 1-morphism $f : \mathscr{X} \to \mathscr{Y}$ of Artin stacks there exists a relative cotangent complex

$$L_f \in \mathbf{D}^{(-\infty,1]}_{\mathrm{coh}}(\mathscr{X}_{\text{lis-ét}}),$$

see e.g. [**168**, Section 2.6] and the references therein. Existence is good, but the fact that the cotangent complex trespasses to positive degree forces one to pay more attention when performing the cutoff. If f has Deligne–Mumford fibres (equivalently, if the diagonal is unramified: such morphisms are called *of Deligne–Mumford type*), then this problem goes away, in the sense that $L_f \in \mathbf{D}^{(-\infty,0]}_{\mathrm{coh}}(\mathscr{X}_{\text{lis-ét}})$.

We record here, without proof, two more important properties of the cotangent complex:

(1) $L_f \cong \Omega_f$ if and only if f is smooth with unramified diagonal.
(2) If f is of Deligne–Mumford type and has a factorisation $\pi \circ i : \mathscr{X} \hookrightarrow \mathscr{M} \to \mathscr{Y}$ with $\pi : \mathscr{M} \to \mathscr{Y}$ smooth, then we still have the formula

$$\mathbb{L}_f = \tau_{\geq -1} L_f = \left[\mathscr{I}/\mathscr{I}^2 \to i^*\Omega_\pi\right] \in \mathbf{D}^{[-1,0]}_{\mathrm{coh}}(\mathscr{X}_{\text{lis-ét}})$$

for its cutoff. Moreover, if $f = \pi \circ i$ is lci, then $\mathbb{L}_f$ is perfect of perfect amplitude contained in $[-1,0]$. This means that locally it is isomorphic (in the derived category) to a complex of locally free sheaves (of finite rank). Since $\mathscr{I}/\mathscr{I}^2$ and $i^*\Omega_\pi$ are both locally free, this is even true *globally* for an lci morphism.
(3) Given concentrated morphisms $g : \mathscr{Y}' \to \mathscr{Y}$ and $f : \mathscr{X} \to \mathscr{Y}$ of algebraic stacks, and letting $\mathscr{X}'$ be their fibred product, with projections $h : \mathscr{X}' \to \mathscr{X}$ and $k : \mathscr{X}' \to \mathscr{Y}'$, there is a canonical morphism $\mathbf{L}h^*L_f \to L_k$, which is an isomorphism whenever either f or g is flat.
(4) the existence of the *transitivity triangle*: given two 1-morphisms of Artin stacks $f : \mathscr{X} \to \mathscr{Y}$ and $g : \mathscr{Y} \to \mathscr{Z}$, there is an exact trangle

$$\mathbf{L}f^*L_g \to L_{g\circ f} \to L_f \to \mathbf{L}f^*L_g[1]. \tag{C.2.3}$$

See also [**168**, Section 2.6].

A canonical reference for the cotangent complex of a concentrated morphism of algebraic stacks is [**123**]. However, in loc. cit. a complication due to the lack of functoriality of the lisse-étale topos has been overlooked. The theory has been settled completely in Olsson's work [**159**, Section 8].

C.3. The idea behind obstruction theories

For any base change $\widetilde{Y} \to Y$ of an lci morphism $f : X \to Y$ coming with an explicit factorisation $X \hookrightarrow M \to Y$, we are able to construct the refined Gysin homomorphism

$$f^! : A_k\widetilde{Y} \to A_{k+r}\widetilde{X},$$

where $\widetilde{X} = X \times_Y \widetilde{Y}$ and $r \in \mathbb{Z}$ is the relative dimension of f.

Recall that for $p = k+s$, specialisation to the normal cone assigns to a cycle class $[V] \in A_p\widetilde{M}$ the cycle class $[C_{W/V}] \in A_pC_{\widetilde{X}/\widetilde{M}}$, where $W = \widetilde{X} \cap V$. Specialisation to the normal cone states the existence of a flat family of closed embeddings

$$\begin{tikzcd} \widetilde{X} \times \mathbb{A}^1 \arrow[r, hook, "\iota"] \arrow[dr] & \mathcal{Z} \arrow[d, "\text{flat}"] \\ & \mathbb{A}^1 \end{tikzcd}$$

such that $\mathcal{Z}|_{\mathbb{A}^1\setminus 0} = \widetilde{M} \times (\mathbb{A}^1 \setminus 0)$ (meaning that ι_t is just the closed immersion $\widetilde{X} \hookrightarrow \widetilde{M}$ for $t \neq 0$), and over $0 \in \mathbb{A}^1$ one recovers the embedding

$$\widetilde{X} \subset \mathcal{Z}_0 = C_{\widetilde{X}/\widetilde{M}}.$$

The normal cone to W in V embeds as

$$C_{W/V} = \mathcal{Z}_0 \cap \overline{V \times (\mathbb{A}^1 \setminus 0)} \subset C_{\widetilde{X}/\widetilde{M}}.$$

Fulton teaches us that if $i\colon X \hookrightarrow M$ is *any* closed embedding and E is a rank r vector bundle on X admitting a closed embedding of cones

$$C_{X/M} \subset E,$$

then for any fiber square

$$\begin{tikzcd} \widetilde{X} \arrow[r, hook] \arrow[d, "\tilde{g}"'] \arrow[dr, phantom, "\square"] & \widetilde{M} \arrow[d] \\ X \arrow[r, hook, "i"] & M \end{tikzcd}$$

one can define a group homomorphism

$$i^!_E\colon A_k\widetilde{M} \to A_{k+r}\widetilde{X}, \tag{C.3.1}$$

which *depends on* E. On generators, it simply sends a cycle class $[V]$ to $0^*[C_{W/V}]$, where $W = \widetilde{X} \cap V$ and $0^*\colon A_k(E|_{\widetilde{X}}) \to A_{k+r}\widetilde{X}$ is the inverse of the flat pullback isomorphism attached to $E|_{\widetilde{X}} \to \widetilde{X}$. Here the basic observation is that we have closed embeddings of cones

$$C_{W/V} \subset C_{\widetilde{X}/\widetilde{M}} \subset E|_{\widetilde{X}}.$$

Finally, corresponding to (C.3.1) there is a bivariant class

$$[i^!_E] \in A^{-r}(X \to M).$$

In the situation where we were to define $f^!$, the closed immersion was a regular embedding, so we did get for free the closed embedding of cones (B.3.2), namely

$$C_{\widetilde{X}/\widetilde{M}} \subset E = \tilde{g}^* C_{X/M}.$$

This then allowed us to define the maps

$$i^!_E\colon A_k\widetilde{M} \to A_{k+\operatorname{rk} E}\widetilde{X}.$$

We see that we are are facing two parallel situations:

(A) Given a *regular* embedding $i\colon X \hookrightarrow M$ of codimension d, composing with a smooth morphism $\pi\colon M \to Y$ yields an lci map $f\colon X \to Y$ and a bivariant class

$$[f^!] \in A^{-r}(X \to Y), \quad r = \operatorname{rk} T_\pi - d,$$

only depending on f (that is, we get the same class choosing another factorisation). For example, if $\pi = \mathrm{id}_M$, we get the class corresponding to the refined Gysin pullback $[i^!] \in A^d(X \to M)$.

(B) Given an *arbitrary* closed embedding $i\colon X \hookrightarrow M$, adding the datum of a closed embedding

$$C_{X/M} \subset E$$

of a cone in a vector bundle (on X), we still get a bivariant class $[i^!_E] \in A^r(X \to M)$, only depending on E.

Clearly, both situations can be seen as generalisation of the following association:

$$i\colon X \hookrightarrow M \text{ regular of codimension } r \quad \rightsquigarrow \quad [i^!] \in A^r(X \to M).$$

So we may ask the following question.

$$\boxed{\text{Can we generalise (A) and (B) at the same time?}}$$

Situation (A) lacks the embedding $C_{X/M} \subset E$, while situation (B) does not deal with more general morphisms than closed immersions $X \hookrightarrow M$. So, as Fantechi neatly explains in [60], the question becomes the following. Let $f\colon X \to Y$ be a fixed morphism. What additional structure should we put on f so that for any factorisation

$$f = \pi \circ i\colon X \xrightarrow{\text{closed}} M \xrightarrow{\text{smooth}} Y$$

there is an embedding

$$C_{X/M} \subset E$$

such that $i_E^! \circ \pi^!$ only depends on this additional structure and not on the chosen factorisation? We would like to have a bundle for *any* factorisation, *not depending* on the factorisation! Whatever the resulting structure could be, it is something we should be able to attach to f without mentioning any factorisation of f. As it turns out, the answer to our quest is:

An obstruction theory!

The crucial remark here is that to give a closed embedding of cones $C_{X/M} \subset E$ is the same as to give a closed embedding of their abelian hulls,

$$N_{X/M} \subset E,$$

and the latter corresponds to a surjection

$$\mathscr{E} \twoheadrightarrow \mathscr{I}/\mathscr{I}^2,$$

where $\mathscr{I}$ is the ideal of X in M.

Recall that comparing factorisations through a diagram as in (C.2.1), one gets the short exact sequence of cones (C.1.5)

$$0 \to j^* T_{P/M} \to C_{X/P} \to C_{X/M} \to 0.$$

Suppose we are able to embed

$$C_{X/M} \subset E_i, \quad C_{X/P} \subset E_j.$$

In other words, we have surjections

$$\mathscr{E}_i \twoheadrightarrow \mathscr{I}/\mathscr{I}^2, \quad \mathscr{E}_j \twoheadrightarrow \mathscr{J}/\mathscr{J}^2.$$

How do we make these surjections become independent upon the factorisation? In fact, put in these terms, this is not quite possible. Unless we move to the derived category.

To rephrase our desire, we would like a surjection $\mathscr{E}_i \twoheadrightarrow \mathscr{I}/\mathscr{I}^2$ for any factorisation

$$X \xrightarrow{i} M \xrightarrow{\pi} Y,$$

such that whenever we are able to compare two factorisations through a diagram (C.2.1), we also get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{E}_i \arrow[r] \arrow[d] & \mathscr{E}_j \arrow[r] \arrow[d] & j^*\Omega_{P/M} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathscr{I}/\mathscr{I}^2 \arrow[r] & \mathscr{J}/\mathscr{J}^2 \arrow[r] & j^*\Omega_{P/M} \arrow[r] & 0 \end{tikzcd} \tag{C.3.2}$$

where, since $j^*\Omega_{P/M}$ is locally free, both short sequences are locally split; thus, given (C.3.2), $\mathscr{E}_i \to \mathscr{I}/\mathscr{I}^2$ is surjective if and only if $\mathscr{E}_j \to \mathscr{J}/\mathscr{J}^2$ is. Recall that we found an object, *well-defined* up to canonical isomorphism in the derived category of X,

$$\mathbb{L}_f = \left[\mathscr{I}/\mathscr{I}^2 \to i^*\Omega_\pi\right] \in \mathbf{D}^{[-1,0]}(X),$$

i.e. totally independent on the chosen factorisation. Then the correct way to formalise the solution to our problem is this: we need to look at morphisms $\mathbb{E} \to \mathbb{L}_f$ in the the derived category, with specific properties: as suggested by Diagram (C.3.2), we should require $h^{-1}(\phi)$ to be a surjection, and $h^0(\phi)$ to be an isomorphism. This is precisely the definition of an obstruction theory on $X \to Y$ (cf. Definition C.5.3 below).

C.4. The intrinsic normal cone

C.4.1. The absolute case. We recall the definition of *algebraic stack* (sometimes called an *Artin stack*) defined over a $\mathbf{k}$-scheme B, where $\mathbf{k}$ is a fixed algebraically closed field.

DEFINITION C.4.1. Let B be an object of the étale site $(\mathrm{Sch}_{\mathbf{k}})_{\text{ét}}$.

- A *B-stack* (or stack over B) is a category fibred in groupoids $X \to (\mathrm{Sch}_B)_{\text{ét}}$ such that the isomorphism functors are étale sheaves and all descent data are effective. See [**123, 59, 202**] for the precise meaning of these conditions.
- An *algebraic stack* (also called an *Artin stack*) over B is a B-stack $X \to (\mathrm{Sch}_B)_{\text{ét}}$ such that the diagonal 1-morphism $X \to X \times_B X$ is representable (by algebraic spaces), separated and quasicompact, and there is an object $U \in (\mathrm{Sch}_B)_{\text{ét}}$ with a smooth surjective 1-morphism $U \to X$, called an *atlas*.
- If an atlas $U \to X$ of an algebraic stack can be chosen to be étale, then X is called a *Deligne–Mumford stack*.

Assumption C.4.2. We assume that all 1-morphisms of algebraic stacks are *concentrated*, i.e. quasicompact and quasiseparated.

Sheaves on an algebraic stack X are considered with respect to the lisse-étale site: in this way, all familiar properties of the derived category and derived functors from the theory of schemes also hold in the land of stacks. See [**123, 159, 160**] for foundational material on the lisse-étale site.

In this subsection, which follows [**19**], X is a scheme or a Deligne–Mumford stack of finite type over $B = \operatorname{Spec} \mathbf{k}$. We still have the notion of cone, abelian cone and abelian hull on Deligne–Mumford stacks. In particular, normal cones and normal sheaves of closed embeddings are still defined.

Construction. Let X be a Deligne–Mumford stack, C a cone over X and E a vector bundle. Let $d\colon E \to C$ be a map of cones over X. Then the map $\tilde{d}\colon A(E) = E \to A(C)$ induces an action of E on the abelian hull $A(C)$:

$$\sigma_d\colon E \times A(C) \to A(C), \quad (e,x) \mapsto \tilde{d}(e) + x.$$

DEFINITION C.4.3. A map of cones $d\colon E \to C$ as above makes C into an *E-cone* in case $C \hookrightarrow A(C)$ is preserved by the action σ_d.

As E is a group over X, if C is an E-cone we can consider the quotient stack $[C/E]$, a relative stack over X. This is an Artin stack, not necessarily of Deligne–Mumford type.

Example C.4.4. Let $f: X \hookrightarrow Y$ be a closed immersion with ideal $\mathscr{I}$, and Y smooth. Then $f^*T_Y = T_Y|_X$ is a vector bundle and $N_{X/Y} = A(C_{X/Y})$ is an abelian cone. The map

$$d\colon T_Y|_X \to N_{X/Y}$$

(with $\tilde{d} = d$) makes $C_{X/Y}$ into a $T_Y|_X$-cone. We then have the relative stack

$$[C_{X/Y}/T_Y|_X] \to X.$$

Remark C.4.5. Let C be an E-cone. The morphism $\pi\colon C \to [C/E]$ is smooth and surjective, of relative dimension $\operatorname{rk} E$. Moreover, $[C/E]$ carries a zero section and an $\mathbb{A}^1$-action.

Let C, D be cones and E, F be vector bundles over X. Let us assume C is an E-cone and D is an F-cone, via two given maps $d\colon E \to C$ and $d'\colon F \to D$. A morphism $\Phi\colon (E,C,d) \to (F,D,d')$ between such objects is a commutative diagram

$$\begin{tikzcd} E \arrow[r, "d"] \arrow[d, "\phi"] & C \arrow[d, "\psi"] & E \times C \arrow[l, "\sigma_d"'] \arrow[d, "1\times\psi"] \\ F \arrow[r, "d'"] & D & F \times D \arrow[l, "\sigma_{d'}"'] \end{tikzcd} \tag{C.4.1}$$

where ψ (resp. ϕ) is a morphism of cones (resp. vector bundles). Any such morphism Φ induces a morphism $[C/E] \to [D/F]$.

Proposition C.4.6 ([19, Proposition 1.7])**.** *Let Φ be as above. If the left square in the diagram (C.4.1) is cartesian and the map $F \times C \to D$ sending $(\mu, \gamma) \mapsto d'(\mu) + \psi(\gamma)$ is surjective, then $[C/E] \to [D/F]$ is an isomorphism of X-stacks with $\mathbb{A}^1$-action.*

DEFINITION C.4.7. A stack $\mathcal{C}$ over X is called a *cone stack* if:

(1) it has a zero section and a compatible $\mathbb{A}^1$-action;
(2) locally with respect to the étale topology of X, there is a cone C over X and a smooth surjective $\mathbb{A}^1$-equivariant map $C \to \mathcal{C}$. (Such map is called a *local presentation*.)

Example C.4.8. The stack $\mathcal{C} = [C/E] \to X$ is a cone stack with local presentation $\pi \colon C \to \mathcal{C}$.

DEFINITION C.4.9. Let $\mathcal{C}$ be a cone stack over X. If the local presentation C is an abelian cone (resp. vector bundle), then $\mathcal{C}$ is called an *abelian cone* (resp. *vector bundle*) *stack*.

DEFINITION C.4.10. Let X be a Deligne–Mumford stack. A *local embedding* for X is a diagram

$$\begin{tikzcd} U \arrow[r, hook, "f"] \arrow[d, "i"] & M \\ X & \end{tikzcd}$$

where U, M are affine $\mathbf{k}$-schemes of finite type, M is smooth, f is a closed immersion and i is étale.

For any local embedding as above, we know that $f^*T_M = T_M|_U \to N_{U/M}$ makes $C_{U/M}$ into an f^*T_M-cone. Let us give a name to the corresponding quotient stack, namely

$$\mathfrak{C}_X|_U = [C_{U/M}/f^*T_M]. \tag{C.4.2}$$

Glueing these stacks along compatible local embeddings will give us a well-defined cone stack $\mathfrak{C}_X$, which Behrend–Fantechi call the *intrinsic normal cone* of X.

GLUING. Suppose we have a map $\phi \colon (U, M, f, i) \to (V, L, g, j)$ between two local embeddings, which can be represented as a commutative diagram

$$\begin{tikzcd} & U \arrow[dl, "i"'] \arrow[r, hook, "f"] \arrow[d, "\phi_U"] & M \arrow[d, "\phi_M"] \\ X & V \arrow[l, "j"] \arrow[r, hook, "g"] & L \end{tikzcd}$$

with ϕ_U étale and ϕ_M smooth. We have the cone stacks

$$\mathfrak{C}_X|_U = [C_{U/M}/f^*T_M], \quad \mathfrak{C}_X|_V = [C_{V/L}/g^*T_L],$$

and a commutative diagram on U

$$\begin{tikzcd} (g^*T_L)|_U \arrow[d] & f^*T_M \arrow[l] \arrow[d] & f^*T_{M/L} \arrow[l] \arrow[dl] \\ C_{V/L}|_U & C_{U/M} \arrow[l, "\alpha"'] & \end{tikzcd}$$

where the square is cartesian and α is surjective. Hence, by Proposition C.4.6, we get an isomorphism of quotient stacks

$$[C_{U/M}/f^*T_M] \cong [C_{V/L}/g^*T_L]. \tag{C.4.3}$$

DEFINITION C.4.11 ([**19**, Definition 3.10]). The *intrinsic normal cone* of X is the cone stack $\mathfrak{C}_X$ defined by gluing the stacks (C.4.2) along the isomorphisms (C.4.3). Similarly, the *intrinsic normal sheaf* of X to be the cone stack $\mathfrak{N}_X$ defined locally by

$$\mathfrak{N}_X|_U = [N_{U/M}/f^*T_M].$$

Proposition C.4.12. *The following properties hold.*

(1) *The intrinsic normal cone* $\mathfrak{C}_X$ *is of pure dimension* 0 *and its abelian hull is* $\mathfrak{N}_X$*.*
(2) X *is lci if and only if* $\mathfrak{N}_X = \mathfrak{C}_X$*, if and only if* $\mathfrak{C}_X$ *is a vector bundle stack.*
(3) *If* X *is smooth, then* $\mathfrak{C}_X = \mathfrak{N}_X = \mathrm{B}T_X$*.*
(4) *If* X, Y *are Deligne–Mumford stacks of finite type over* $\mathbf{k}$*, then* $\mathfrak{C}_{X\times Y} = \mathfrak{C}_X \times \mathfrak{C}_Y$ *and* $\mathfrak{N}_{X\times Y} = \mathfrak{N}_X \times \mathfrak{N}_Y$*.*

PROOF. The first property is [**19**, Theorem 3.11], the second is [**19**, Proposition 3.12], the third is a special case of the second, the fourth is [**19**, Proposition 3.13]. □

C.4.2. The relative case. Let Y be a pure dimensional algebraic $\mathbf{k}$-stack, $X \to Y$ a morphism of algebraic stacks with unramified diagonal $X \to X \times_Y X$ (which implies that the fibres are Deligne–Mumford stacks, see [**185**, Tag 04YW] — indeed such morphisms are called *of Deligne–Mumford type*). Then the cohomology of the relative cotangent complex $L_{X/Y}$ vanishes in positive degrees, and $L_{X/Y}$ is an object of $\mathbf{D}^{(-\infty,0]}(X_{\text{ét}})$. One can construct the relative intrinsic normal cone $\mathfrak{C}_{X/Y}$ (of pure dimension $\dim Y$) by the same gluing procedure as above, considering local embeddings of X inside schemes M smooth over Y. Similarly for the relative intrinsic normal sheaf $\mathfrak{N}_{X/Y}$, which in fact can also be defined as $h^1/h^0(L_{X/Y}^\vee)$. As in the absolute case, there is a closed embedding of cones $\mathfrak{C}_{X/Y} \hookrightarrow \mathfrak{N}_{X/Y}$, which is the inclusion of $\mathfrak{C}_{X/Y}$ in its abelian hull.

C.5. Virtual fundamental classes

C.5.1. The absolute case. Let X be a Deligne–Mumford stack locally of finite type over a field $\mathbf{k}$. Given a two-term complex $\mathbb{E} = [E^0 \to E^1]$ of abelian sheaves on X, we can consider the stack quotient

$$h^1/h^0(\mathbb{E}) = [E^1/E^0],$$

which is a Picard stack [**7**, Exposé XVIII]. In fact the operation h^1/h^0 is functorial; in particular, if $\phi\colon \mathbb{E} \to \mathbb{F}$ is a morphism of two-term complexes of abelian sheaves which induces isomorphisms on kernels and cokernels, then the induced morphism

$$h^1/h^0(\phi)\colon h^1/h^0(\mathbb{E}) \to h^1/h^0(\mathbb{F})$$

is an isomorphism of Picard stacks. This is proved in [**19**, Proposition 2.1].

The following assumption will soon be crucial.

Assumption C.5.1. An object $\mathscr{L} \in \mathbf{D}(\mathrm{Mod}\,\mathscr{O}_{X_{\text{ét}}})$ is said to satisfy condition $(*)$ if $h^i(\mathscr{L}) = 0$ for all $i > 0$ and $h^i(\mathscr{L})$ is coherent for $i = 0, -1$.

Proposition C.5.2 ([**19**, Proposition 2.4])**.** *Let* $\mathscr{L}$ *be an object in* $\mathbf{D}(\mathrm{Mod}\,\mathscr{O}_{X_{\text{ét}}})$ *satisfying condition* $(*)$*. Then the* X*-stack* $h^1/h^0(\mathscr{L}^\vee)$ *is an abelian cone stack over* X*. When* $\mathscr{L}$ *is of perfect amplitude contained in* $[-1,0]$*,* $h^1/h^0(\mathscr{L}^\vee)$ *is a vector bundle stack.*

The cotangent complex $L_X \in \mathbf{D}(\mathrm{Mod}\,\mathscr{O}_{X_{\text{ét}}})$ of $X \to \operatorname{Spec}\mathbf{k}$ satisfies condition $(*)$, and one has

$$\mathfrak{N}_X = h^1/h^0(L_X^\vee).$$

In fact, this is the original *definition* of the intrinsic normal sheaf, cf. [**19**, Definition 3.6].

Suppose $\phi : \mathbb{E} \to \mathscr{L}$ is a morphism in $\mathbf{D}(\mathrm{Mod}\, \mathscr{O}_{X_{\acute{e}t}})$, where both $\mathbb{E}$ and $\mathscr{L}$ satisfy condition $(*)$. Then the induced morphism of algebraic stacks

$$\phi^\vee : h^1/h^0(\mathscr{L}^\vee) \to h^1/h^0(\mathbb{E}^\vee)$$

is a morphism of abelian cone stacks. It is a closed immersion if and only if $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is onto. A proof of this assertion can be found in [**19**, Theorem 4.5].

Definition C.5.3. Let $\mathbb{E}$ be an object in $\mathbf{D}(\mathrm{Mod}\, \mathscr{O}_{X_{\acute{e}t}})$ satisfying condition $(*)$. Then a homomorphism $\phi : \mathbb{E} \to L_X$ is an *obstruction theory* on X if $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is onto. Moreover, ϕ is called *perfect* when $\mathbb{E}$ is of perfect amplitude contained in $[-1,0]$, in which case ϕ is equivalent to a morphism $\mathbb{E} \to \mathbb{L}_X = \tau_{\geq -1} L_X$ to the truncated cotangent complex.

If $\phi : \mathbb{E} \to L_X$ is an obstruction theory on X, we have a closed immersion of abelian cone stacks

$$\phi^\vee : \mathfrak{N}_X \to h^1/h^0(\mathbb{E}^\vee). \tag{C.5.1}$$

Definition C.5.4. Let $\phi : \mathbb{E} \to L_X$ be an obstruction theory. The image of the intrinsic normal cone $\mathfrak{C}_X \subset \mathfrak{N}_X$ under $\phi^\vee$ is called the *obstruction cone* of the obstruction theory ϕ and is denoted $\mathfrak{C}_\phi$. The sheaf $\mathrm{Ob} = h^1(\mathbb{E}^\vee)$ is called the *obstruction sheaf* of the obstruction theory.

Definition C.5.5. If $\phi : \mathbb{E} \to L_X$ is a perfect obstruction theory, and $\mathbb{E}$ is locally written as $[E^{-1} \to E^0]$, we define the *virtual dimension* of X with respect to ϕ as the difference

$$\mathrm{d}^{\mathrm{vir}} = \mathrm{rk}\, \mathbb{E} = \mathrm{rk}\, E^0 - \mathrm{rk}\, E^{-1}.$$

Definition C.5.6. Let $\phi : \mathbb{E} \to L_X$ be a perfect obstruction theory with obstruction cone $\mathfrak{C}_\phi \subset h^1/h^0(\mathbb{E}^\vee)$. The *virtual fundamental class* of X with respect to ϕ is defined to be the class

$$[X,\phi] = 0^*[\mathfrak{C}_\phi] \in A_{\mathrm{d}^{\mathrm{vir}}} X$$

where 0^* is the inverse of the flat pullback for the vector bundle stack $h^1/h^0(\mathbb{E}^\vee) \to X$.[1]

Example C.5.7. If X is a local complete intersection, L_X is of perfect amplitude contained in $[-1,0]$. The perfect obstruction theory $\mathrm{id} : L_X \to L_X$ induces the usual fundamental class $[X]$.

Proposition C.5.8 ([**19**, Propositions 5.5, 5.6])**.** *Let $\mathbb{E} \to L_X$ be a perfect obstruction theory with obstruction sheaf* $\mathrm{Ob} = h^1(\mathbb{E}^\vee)$.

(i) *If $h^0(\mathbb{E})$ is locally free and* $\mathrm{Ob} = 0$ *then X is smooth,* $\mathrm{d}^{\mathrm{vir}} = \dim X = d$ *and* $[X,\phi] = [X] \in A_d X$ *is the usual fundamental class.*

(ii) *If X is smooth and* Ob *is locally free of rank r, then*

$$[X,\phi] = c_r(\mathrm{Ob}) \cap [X].$$

C.5.2. The relative case. Let $f : X \to B$ be a morphism of Deligne–Mumford type, as defined in Appendix C.4.2.

Definition C.5.9. An *obstruction theory* relative to $f : X \to B$ is a morphism $\phi : \mathbb{E} \to L_f$ in $\mathbf{D}^{(-\infty,0]}(X)$, such that $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is a surjection. It is called *perfect* if $\mathbb{E} \in \mathbf{D}^{[-1,0]}_{\mathrm{coh}}(X)$ is of perfect amplitude contained in $[-1,0]$.

Assume X is of Deligne–Mumford type. Given a perfect obstruction theory $\phi : \mathbb{E} \to L_f$ as above, intersecting the obstruction cone $\mathfrak{C}_\phi \hookrightarrow \mathscr{E} = h^1/h^0(\mathbb{E}^\vee)$ with the zero section of the vector bundle stack $\mathscr{E} \to X$ we obtain a virtual fundamental class

$$[X,\phi] = 0^*_{\mathscr{E}}[\mathfrak{C}_\phi] \in A_{\mathrm{rk}\,\mathbb{E} + \dim B} X.$$

[1] The original definition by Behrend–Fantechi used the existence of global resolutions for the complex $\mathbb{E}$. By work of Kresch [**118**], we have Chow groups for Artin stacks at our disposal, so that the operation 0^* is well-defined.

C.5.2.1. *Pullback of obstruction theories.* Consider a cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "u"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, "v"'] & Y \end{tikzcd}$$

and a perfect obstruction theory $\phi : \mathbb{E} \to L_f$, where

- Y and Y' are smooth and pure dimensional,
- v has finite unramified diagonal,
- X and X' are separated Deligne–Mumford stacks.

Then, by [19, Proposition 7.2], the morphism $\phi' : u^*\mathbb{E} \to u^*L_f \to L_{f'}$ is a perfect obstruction theory on f', and there is an identity

$$v^![X,\phi] = [X',\phi'] \in A_*X'.$$

Example C.5.10 (Fibre of a morphism of stacks)**.** Let $f : X \to Y$ be a morphism of Deligne–Mumford type to a smooth stack Y, and let $y : \operatorname{Spec}\mathbf{k} \to Y$ be a point (i.e. we set $Y' = \operatorname{Spec}\mathbf{k}$). Then a perfect obstruction theory on f induces a perfect obstruction theory on the fibre product $X_y = X \times_{f,Y,y} \operatorname{Spec}\mathbf{k}$.

C.5.2.2. *Compatibility, take I.* Now consider a cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "u"] \arrow[d, "g"'] \arrow[rr, bend left, "f'"] \arrow[dr, phantom, "\square"] & X \arrow[r, "f"] \arrow[d] & Y \\ Z' \arrow[r, "v"'] & Z \arrow[ur] & \end{tikzcd}$$

of Y-stacks, where Z, Z' have finite unramified diagonal over Y and v is lci. Two relative perfect obstruction theories $\phi : \mathbb{E} \to L_f$ and $\phi' : \mathbb{E}' \to L_{f'}$ are said to be *v-compatible* if there is a morphism of exact triangles

$$\begin{tikzcd} u^*\mathbb{E} \arrow[r] \arrow[d] & \mathbb{E}' \arrow[r] \arrow[d] & g^*L_v \arrow[r] \arrow[d] & u^*\mathbb{E}[1] \arrow[d] \\ u^*L_f \arrow[r] & L_{f'} \arrow[r] & L_u \arrow[r] & u^*L_f[1] \end{tikzcd}$$

in $\mathbf{D}^{[-1,0]}(X'_{\text{lis-ét}})$, where the lower exact triangle is the transitivity triangle (C.2.3). By [19, Proposition 7.5], one has an identity

$$v^![X,\phi] = [X',\phi']$$

whenever v is smooth or both Z and Z' are smooth over Y.

C.5.2.3. *Compatibility, take II.* Let $f : F \to G$ be a Deligne–Mumford type morphism of algebraic stacks defined over a smooth algebraic stack Y. Suppose we are given relative obstruction theories $\mathbb{E}_{F/Y} \to L_{F/Y}$ and $\mathbb{E}_{G/Y} \to L_{G/Y}$ along with a solid diagram

$$\begin{tikzcd} f^*\mathbb{E}_{G/Y} \arrow[r, "\vartheta"] \arrow[d] & \mathbb{E}_{F/Y} \arrow[r, dashed] \arrow[d] & \mathbb{E}_{F/G} \arrow[r, dashed] \arrow[d, dashed] & f^*\mathbb{E}_{G/Y}[1] \arrow[d] \\ f^*L_{G/Y} \arrow[r] & L_{F/Y} \arrow[r] & L_{F/G} \arrow[r] & f^*L_{G/Y}[1] \end{tikzcd}$$

where the lower exact triangle is the transitivity triangle (C.2.3) and ϑ is also part of the data. Then the dotted arrows can be filled in, i.e. one can complete the diagram to a morphism of exact triangles, where indeed $\mathbb{E}_{F/G}$ is defined as the cone of ϑ. This is just an axiom of triangulated categories.

EXERCISE C.5.11. Prove that $\mathbb{E}_{F/G} \to L_{F/G}$ is a relative obstruction theory. (**Hint**: write down the long exact sequences of cohomology sheaves attached to the morphism of exact triangles; further hints are found in [132, Section 3.2]).

EXERCISE C.5.12. Prove that if $G \to Y$ is smooth and $\mathbb{E}_{G/Y} = L_{G/Y}$ with the identity map down to $L_{G/Y}$, then $\mathbb{E}_{F/G}$ is perfect in $[-1,0]$.

Remark C.5.13. Note that the morphism $\mathbb{E}_{F/G} \to L_{F/G}$ need not be unique. In fact, it is unique as long as $\mathrm{Hom}(f^*\mathbb{E}_{G/Y}[1], L_{F/G}) = 0$.

DEFINITION C.5.14 ([**132**, Definition 4.5]). A triple of perfect obstruction theories $(\mathbb{E}_{F/G}, \mathbb{E}_{G/Y}, \mathbb{E}_{F/Y})$ is said to be a *compatible triple* if they fit in a morphism of exact triangles

$$\begin{tikzcd} f^*\mathbb{E}_{G/Y} \arrow[r, "\vartheta"] \arrow[d] & \mathbb{E}_{F/Y} \arrow[r] \arrow[d] & \mathbb{E}_{F/G} \arrow[r] \arrow[d] & f^*\mathbb{E}_{G/Y}[1] \arrow[d] \\ f^*L_{G/Y} \arrow[r] & L_{F/Y} \arrow[r] & L_{F/G} \arrow[r] & f^*L_{G/Y}[1] \end{tikzcd}$$

in the derived category $\mathbf{D}^{[-1,0]}(F)$.

We shall use (with $Y = \operatorname{Spec}\mathbf{k}$) in Appendix C.6.3 that, as before, given a morphism $\mathbb{E}_{F/G} \to f^*\mathbb{E}_{G/Y}[1]$ such that

$$\begin{tikzcd} \mathbb{E}_{F/G} \arrow[r] \arrow[d] & f^*\mathbb{E}_{G/Y}[1] \arrow[d] \\ L_{F/G} \arrow[r] & f^*L_{G/Y}[1] \end{tikzcd}$$

commutes, one obtains a relative obstruction theory $\mathbb{E}_{F/Y} \to L_{F/Y}$.

C.6. The example of stable maps

C.6.1. Virtual dimension. We wish to compute the expected dimension of the moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(X,\beta)$ introduced in Section 11.3.1, where X is a smooth projective variety and $\beta \in \mathrm{H}_2(X,\beta)$ is an effective curve class. The answer is in Lemma C.6.3 below. We start with an easy example.

Example C.6.1. Let us perform a naive moduli count for genus 0 maps to $\mathbb{P}^r$. So to give a map

$$[\mathbb{P}^1 \to \mathbb{P}^r] \in \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$$

we have to specify $r+1$ degree d polynomials in two variables. Hence, after rescaling the polynomials at once and taking into account the automorphisms of $\mathbb{P}^1$, we find

$$(r+1)(d+1)-1-3 = (r+1)d+(r-3)$$

which is a special case of the general formula of Lemma C.6.3. It is special to genus 0 maps to convex varieties (like projective spaces, homogeneous spaces, flag varieties) that the expected dimension agrees with the actual dimension.

Now for the general case. At a point $\mu\colon (C,p_1,\ldots,p_n) \to X$, the moduli stack $\mathrm{M} = \overline{\mathcal{M}}_{g,n}(X,\beta)$ has tangent space equal to the hyperext group

$$\mathrm{Ext}^1([\mu^*\Omega_X \to \Omega_C(P)], \mathscr{O}_C) \tag{C.6.1}$$

where we have set $P = \sum p_i$. Obstructions live in the next hyperext group

$$\mathrm{Ext}^2([\mu^*\Omega_X \to \Omega_C(P)], \mathscr{O}_C). \tag{C.6.2}$$

Although the individual dimension of these spaces is usually impossible to control, their difference is constant and is called the *virtual dimension* of M. We denote it $\mathrm{vd}_{g,n}(X,\beta)$.

Remark C.6.2. A marked nodal curve $(C,p_1,\ldots,p_n)$ is stable if and only if it has no infinitesimal automorphisms, that is,

$$\mathrm{Ext}^0(\Omega_C(P), \mathscr{O}_C) = 0.$$

More generally, a map $\mu\colon C \to X$ from a marked nodal curve is stable if and only if first the map has no infinitesimal automorphisms, that is,

$$\mathrm{Ext}^0([\mu^*\Omega_X \to \Omega_C(P)], \mathscr{O}_C) = 0.$$

Lemma C.6.3. *We have* $\mathrm{vd}_{g,n}(X,\beta) = \int_\beta c_1(X) + (1-g)(\dim X - 3) + n.$

Proof. We have a long exact sequence

$$\begin{aligned} \mathrm{Ext}^{-1}(\mu^*\Omega_X, \mathscr{O}_C) &\to \mathrm{Ext}^0([\mu^*\Omega_X \to \Omega_C(P)], \mathscr{O}_C) \to \mathrm{Ext}^0(\Omega_C(P), \mathscr{O}_C) \\ \to \mathrm{Ext}^0(\mu^*\Omega_X, \mathscr{O}_C) &\to \mathrm{Ext}^1([\mu^*\Omega_X \to \Omega_C(P)], \mathscr{O}_C) \to \mathrm{Ext}^1(\Omega_C(P), \mathscr{O}_C) \\ \to \mathrm{Ext}^1(\mu^*\Omega_X, \mathscr{O}_C) &\to \mathrm{Ext}^2([\mu^*\Omega_X \to \Omega_C(P)], \mathscr{O}_C) \to 0. \end{aligned}$$

Each column can be interpreted as infinitesimal automorphisms, deformations and obstructions of a very precise geometric object. The first column on the left deals with deformations of the map $\mu\colon C \to X$ with *fixed domain*. The middle column deals with the stable map μ itself, the last column deals with the marked curved $(C, p_1, \ldots, p_n)$. We immediately see that we can simplify the sequence since the first two Ext groups vanish (the second one by Remark C.6.2). Moreover, we have isomorphisms $\mathrm{Ext}^i(\mu^*\Omega_X, \mathscr{O}_C) \cong \mathrm{H}^i(C, \mu^*T_X)$. The sequence becomes

$$\begin{aligned} 0 \to \mathrm{Ext}^0(\Omega_C(P), \mathscr{O}_C) \to \mathrm{H}^0(C,\, \mu^*T_X) \to \mathsf{Def}(\mu) \\ \to \mathrm{Ext}^1(\Omega_C(P), \mathscr{O}_C) \to \mathrm{H}^1(C, \mu^*T_X) \to \mathsf{Ob}(\mu) \to 0. \end{aligned}$$

Therefore we can compute $\mathrm{vd}_{g,n}(X,\beta) = \dim \mathsf{Def}(\mu) - \dim \mathsf{Ob}(\mu)$ as

$$\begin{aligned} \mathrm{vd}_{g,n}(X,\beta) &= \chi(\mu^*T_X) + \mathrm{Ext}^1(\Omega_C(P), \mathscr{O}_C) - \mathrm{Ext}^0(\Omega_C(P), \mathscr{O}_C) \\ &= \int_C \mathrm{ch}(\mu^*T_X)\cdot \mathrm{Td}(C) + \dim_{(C,p)} \mathfrak{M}_{g,n} \\ &= \int_C (\dim X, c_1(X), \ldots)\cdot(1, 1-g) + (3g-3+n) \\ &= \int_\beta c_1(X) + (1-g)\dim X + (3g-3+n) \\ &= \int_\beta c_1(X) + (1-g)(\dim X - 3) + n. \end{aligned}$$

The proof is complete. ◻

Remark C.6.4. The space $\mathrm{Ext}^0(\Omega_C(P), \mathscr{O}_C)$ does not vanish in general, because (C,p) need not be stable.

Remark C.6.5. If $\dim X = 3$, then $\mathrm{vd}_{g,n}(X,\beta)$ does not depend on the genus g of the source curves. If X is also Calabi–Yau, and $n = 0$, then $\mathrm{vd}_{g,0}(X,\beta) = 0$ for all β.

C.6.2. The scheme of morphisms. Let C and X be projective $\mathbf{k}$-schemes, with C Gorenstein of dimension e. Let $\mathrm{H} = \underline{\mathrm{Hom}}(C,X)$ be the scheme of morphisms from C to X.[2] Using the universal diagram

$$\begin{tikzcd} C \times \mathrm{H} \arrow[r, "f"] \arrow[d, "\pi"'] & X \\ \mathrm{H} \end{tikzcd}$$

together with the isomorphism $\pi^*L_{\mathrm{H}} \xrightarrow{\sim} L_{C\times \mathrm{H}/C}$ and the sequence[3]

$$f^*L_X \to L_{C\times \mathrm{H}} \to L_{C\times\mathrm{H}/C},$$

[2] A scheme structure on H is given by realising it as an open subscheme of the Hilbert scheme of the projective scheme $C \times X$, by identifying a map $h\colon C \to X$ with its graph $\Gamma_h \subset C \times X$, see [**185**, Tag 0D1B] for details.

[3] All pullbacks are derived, but we omit writing '$\mathbf{L}$'.

we get a morphism

$$\rho \colon f^*L_X \to \pi^* L_{\mathrm{H}}.$$

The morphism[4] $\rho \otimes \omega_\pi[e] \colon f^*L_X \otimes \omega_\pi[e] \to \pi^! L_{\mathrm{H}}$ corresponds by adjunction $\mathbf{R}\pi_* \dashv \pi^!$ to a morphism

$$\pi_*(\rho \otimes \omega_\pi[e]) \colon \mathbf{R}\pi_*(f^*L_X \otimes \omega_\pi[e]) \to L_{\mathrm{H}},$$

and by Grothendieck duality we have

$$\mathbf{R}\pi_*(f^*L_X \otimes \omega_\pi[e]) = (\mathbf{R}\pi_* f^* T_X)^\vee.$$

We have constructed a morphism

$$\phi \colon (\mathbf{R}\pi_* f^* T_X)^\vee \to L_{\mathrm{H}}.$$

The following result is proven in [**19**, Proposition 6.3].

Proposition C.6.6. *When C is a curve and X is smooth, ϕ is a perfect obstruction theory on* H.

C.6.3. Obstruction theory on moduli of stable maps. We assume X is a smooth projective variety and $(g,n,\beta) \neq (1,0,0)$. Set $\mathrm{M} = \overline{\mathcal{M}}_{g,n}(X,\beta)$ and let $\mathfrak{M} = \mathfrak{M}_{g,n}$ be the Artin stack of n-pointed prestable curves (which is smooth of dimension $3g-3+n$). We follow [**81**, **15**] throughout. Let us consider the morphism

$$\tau \colon \mathrm{M} \to \mathfrak{M}$$

which forgets the map but does not stabilise. The stack M is an open substack of the stack of morphisms

$$\widetilde{\mathrm{H}} = \{ C \to X \mid [C] \in \mathfrak{M} \} \to \mathfrak{M}$$

whose fibre over a moduli point $[C] \in \mathfrak{M}$ is the stack $\mathrm{H} = \underline{\mathrm{Hom}}(C,X)$ described in Appendix C.6.2. The morphism τ has an intrinsic normal cone $\mathfrak{C}_\tau$ which embeds as a closed subcone inside the intrinsic normal sheaf $\mathfrak{N}_\tau = h^1/h^0(L_\tau^\vee)$. Let (π, f) in the diagram

$$\begin{array}{ccc} \mathrm{C} & \xrightarrow{f} & X \\ {\scriptstyle \pi}\big\downarrow & & \\ \mathrm{M} & & \end{array}$$

be the universal curve and universal stable map over the moduli stack M, and let

$$\tau^* L_{\mathfrak{M}} \to L_{\mathrm{M}} \to L_\tau \to \tau^* L_{\mathfrak{M}}[1]$$

be the transitivity triangle attached to $\mathrm{M} \to \mathfrak{M} \to \operatorname{Spec} \mathbf{k}$ defining the relative cotangent complex L_τ. The tangent-obstruction complex attached to the space of 'maps with fixed domain' is the complex $\mathbf{R}\pi_* f^* T_X$ and there is a relative perfect obstruction theory

$$\varphi \colon \mathbb{E}_\tau = (\mathbf{R}\pi_* f^* T_X)^\vee \to L_\tau$$

obtained as in Appendix C.6.2. This means $h^0(\varphi)$ is an isomorphism and $h^{-1}(\varphi)$ is onto. In other words, recalling that $h^1/h^0(L_\tau^\vee) = \mathfrak{N}_\tau$, the functor h^1/h^0 induces a closed immersion of abelian cone stacks

$$\varphi^\vee \colon \mathfrak{N}_\tau \hookrightarrow \mathscr{E} = h^1/h^0(\mathbf{R}\pi_* f^* T_X).$$

Note that the relative dimension of $\tau \colon \mathrm{M} \to \mathfrak{M}$ is given by the dimension of this relative obstruction theory, namely

$$\operatorname{rk} \mathbb{E}_\tau = \operatorname{rk} \mathbf{R}^0 \pi_* f^* T_X - \operatorname{rk} \mathbf{R}^1 \pi_* f^* T_X = \chi(f^* T_X).$$

[4] We are not deriving the tensor product since ω_π is locally free, by the Gorenstein assumption on C.

Remark C.6.7. We can already define the virtual fundamental class of M, bypassing the construction of the *absolute* perfect obstruction theory $\mathbb{E} \to L_{\mathrm{M}}$ giving rise to it. One can intersect the subcone $\mathfrak{C}_\tau \subset \mathfrak{N}_\tau$ with the zero section of the vector bundle stack $\mathscr{E}$ to get the class

$$\left[\mathrm{M}\right]^{\mathrm{vir}} = \left[\mathrm{M}, \varphi\right] = 0_{\mathscr{E}}^{*}\left[\mathfrak{C}_\tau\right] \in A_d \mathrm{M}. \tag{C.6.3}$$

It lives in dimension

$$d = \operatorname{rk}\mathbb{E}_\tau + \dim \mathfrak{C}_\tau = \chi(f^* T_X) + (3g - 3 + n) = \operatorname{vd}\mathrm{M}.$$

The class (C.6.3) *is* the virtual fundamental class of M.

We want to explicitly construct the absolute perfect obstruction theory defining $[\mathrm{M}]^{\mathrm{vir}}$. So far we have the following diagram in the derived category

$$\begin{tikzcd} & & \mathbb{E}_\tau \arrow[d, "\varphi"] & \\ \tau^* L_{\mathfrak{M}} \arrow[r] & L_{\mathrm{M}} \arrow[r] & L_\tau \arrow[r] & \tau^* L_{\mathfrak{M}}[1] \end{tikzcd} \tag{C.6.4}$$

which we wish to complete to a morphism of exact triangles in order to 'add in' deformations of domain curves for stable maps. Consider the composition

$$\gamma \colon f^*\Omega_X \to \Omega_\pi \to \Omega_\pi(D),$$

where $D \subset \mathrm{C}$ is the divisor given by the union of the n canonical sections of π. We immediately obtain the dual morphism

$$\gamma^\vee \colon \mathbf{R}\mathscr{H}om(\Omega_\pi(D), \mathscr{O}_{\mathrm{C}}) \to \mathbf{R}\mathscr{H}om(f^*\Omega_X, \mathscr{O}_{\mathrm{C}})$$

and applying $\mathbf{R}\pi_*$ yields

$$\mathbf{R}\pi_*\gamma^\vee \colon \mathbf{R}\pi_* \mathbf{R}\mathscr{H}om(\Omega_\pi(D), \mathscr{O}_{\mathrm{C}}) \to \mathbf{R}\pi_* f^* T_X.$$

The composition

$$\mathbb{E}_\tau \xrightarrow{(\mathbf{R}\pi_*\gamma^\vee)^\vee} \left(\mathbf{R}\pi_* \mathbf{R}\mathscr{H}om(\Omega_\pi(D), \mathscr{O}_{\mathrm{C}})\right)^\vee \xrightarrow{\sim} \tau^* L_{\mathfrak{M}}[1]$$

is a morphism $\tilde{\gamma} \colon \mathbb{E}_\tau \to \tau^* L_{\mathfrak{M}}[1]$ which is compatible with the maps down to the cotangent complexes, so as explained in **??** C.5.2.3 we can use it to complete Diagram (C.6.4) to a morphism of exact triangles

$$\begin{tikzcd} \tau^* L_{\mathfrak{M}} \arrow[r] \arrow[d] & \mathbb{E} \arrow[r] \arrow[d, "\psi"] & \mathbb{E}_\tau \arrow[r, "\tilde{\gamma}"] \arrow[d, "\varphi"] & \tau^* L_{\mathfrak{M}}[1] \arrow[d] \\ \tau^* L_{\mathfrak{M}} \arrow[r] & L_{\mathrm{M}} \arrow[r] & L_\tau \arrow[r] & \tau^* L_{\mathfrak{M}}[1] \end{tikzcd} \tag{C.6.5}$$

in which $\mathbb{E}$ is a relative obstruction theory. We now show its perfectness. The morphism $\tilde{\gamma}$ can be represented as a morphism of two term complexes

$$\tilde{\gamma} \colon \mathbb{B} \to \mathbb{A}[1]$$

where $\mathbb{B} = [B^{-1} \to B^0]$ and $\mathbb{A} = [A^0 \to A^1]$ are two term locally free resolutions of $\mathbb{E}_\tau$ and $\tau^* L_{\mathfrak{M}}$ respectively. This gives a *three* term complex

$$\mathbb{E} = \left[B^{-1} \to B^0 \oplus A^0 \to A^1\right]$$

of perfect amplitude contained in $[-1, 1]$. By construction we have an exact triangle

$$\mathbb{B} \xrightarrow{\tilde{\gamma}} \mathbb{A}[1] \to \mathbb{E}[1] \to \mathbb{B}[1]$$

and in fact a morphism of exact triangles

$$\begin{tikzcd} \mathbb{B} \arrow[r, "\tilde{\gamma}"] \arrow[d, "\varphi"] & \mathbb{A}[1] \arrow[r] \arrow[d] & \mathbb{E}[1] \arrow[r] \arrow[d] & \mathbb{B}[1] \arrow[d] \\ L_\tau \arrow[r] & \tau^* L_{\mathfrak{M}}[1] \arrow[r] & L_{\mathrm{M}}[1] \arrow[r] & L_\tau[1] \end{tikzcd}$$

which can be shifted back by -1 and thus can be rewritten as (C.6.5) by the axioms of triangulated categories. In fact, $\mathbb{E}$ can be represented as a *two* term complex $[E^{-1} \to E^0]$, since one can show that $h^1(\mathbb{E}) = 0$ by the stability condition. The morphism

$$\psi \colon \mathbb{E} \to L_{\mathrm{M}}$$

appearing in (C.6.5) is a perfect obstruction theory for M. Let $\mathfrak{C} \subset \mathscr{E} = h^1/h^0(\mathbb{E}^\vee)$ be its obstruction cone. The virtual fundamental class $[\mathrm{M}]^{\mathrm{vir}}$ associated to ψ is the class $0^*_{\mathscr{E}}[\mathfrak{C}] \in A_d\mathrm{M}$, which agrees with (C.6.3).

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