INTRODUCTION TO ENUMERATIVE GEOMETRY

— CLASSICAL AND VIRTUAL TECHNIQUES —

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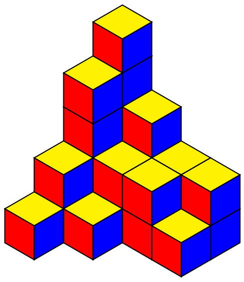
ABSTRACT. These are lecture notes for a PhD course held at SISSA in Fall 2019. Notes under construction.



CONTENTS

0. Informal introduction to Grassmannians

We denote by $G(k,n)$ the *set* of k-dimensional subspaces of $\mathbb{C}^n$. This set is in fact a complex manifold, called the *Grassmannian* of k-planes in $\mathbb{C}^n$. It is naturally identified with the set of $(k-1)$-dimensional linear subspaces of $\mathbb{P}^{n-1}$, and when we think of it in this manner we denote it by $\mathbb{G}(k-1,n-1)$. For instance, $\mathbb{G}(0,n-1)=\mathbb{P}^{n-1}$.

We will give $G=G(k,n)$ the structure of a (smooth) projective variety of dimension $k(n-k)$, by describing a closed embedding in $\mathbb{P}^N$, where $N=\binom{n}{k}-1$. We will see that G admits an affine stratification, which will enable us to explicitly describe the generators of its Chow ring A^*G. The affine strata will be called *Schubert cells*, while their closures will be called *Schubert cycles*. The classes of the Schubert cycles freely generate the Chow group of G. To determine the ring structure, one has to compute the products between these generators. These computations in A^*G go under the name of *Schubert Calculus*.

0.1. $G(k,n)$ as a projective variety.

Let us fix a point

$$[\mathcal{H}]\in G=G(k,n),$$

corresponding to a k-dimensional linear subspace $\mathcal{H}\subset V=\mathbb{C}^n$. If $v_1,\ldots,v_k$ is a basis of $\mathcal{H}$ then $v_1\wedge\cdots\wedge v_k$ is the free generator of the line $\bigwedge^k\mathcal{H}\subset\bigwedge^k V\cong\mathbb{C}^{\binom{n}{k}}=\mathbb{C}^{N+1}$. So we get a map

$$\iota\colon G\to\mathbb{P}\left(\bigwedge^k V\right)=\mathbb{P}^N$$

sending $[\mathcal{H}]\mapsto[v_i\wedge\cdots\wedge v_k]$. Why is this well-defined? Let us view the point $[\mathcal{H}]\in G$ as (the space generated by the rows of) a full rank matrix $H=(a_{ij})\in M_{k\times n}(\mathbb{C})$ and let us fix a basis $e_1,\ldots,e_n$ of V. Then a basis of $\bigwedge^k V$ is given by

$$\{\,e_{i_1}\wedge\cdots\wedge e_{i_k}\,\}_{1\le i_1<\cdots<i_k\le n}.$$

So when we view the element $v_1\wedge\cdots\wedge v_k$ inside $\bigwedge^k V$ we can write it uniquely as

$$v_1\wedge\cdots\wedge v_k=\sum_{1\le i_1<\cdots<i_k\le n}p_{i_1\cdots i_k}(e_{i_1}\wedge\cdots\wedge e_{i_k})=\sum_I p_I e_I,$$

where the coefficient $p_I=p_{i_1\cdots i_k}$ is the minor of the $(k\times k)$-matrix given by extracting from H the columns $i_1,\ldots,i_k$. Of course different choices of H may produce the same $\mathcal{H}$. But H is unique up to the left action of $\mathrm{GL}(k,\mathbb{C})$. Summing up, we have a commutative diagram

$$\begin{tikzcd}
\bigwedge^k\mathcal{H}\setminus 0 \arrow[r, hook] \arrow[d, "/\mathbb{C}^\times"'] & \bigwedge^k V\setminus 0 \arrow[d, "/\mathbb{C}^\times"] & v_1\wedge\cdots\wedge v_k \arrow[r, mapsto] \arrow[d, mapsto] & \sum_I p_I e_I \arrow[d, mapsto] \\
\mathbb{P}\left(\bigwedge^k\mathcal{H}\right) \arrow[r, "\iota"] & \mathbb{P}\left(\bigwedge^k V\right) & {[v_1\wedge\cdots\wedge v_k]} \arrow[r, mapsto] & (p_I)_I
\end{tikzcd}$$

Up to now, we have identified a point $[\mathcal{H}]\in G$ to the unique point of $\mathbb{P}(\bigwedge^k\mathcal{H})$ and we have defined a map $\iota\colon G\to\mathbb{P}^N$ by sending $[\mathcal{H}]$ to its *Plücker coordinates* $(p_I)_I$. Such a map is injective, and G can be identified with an irreducible algebraic set in $\mathbb{P}^N$, via a collection of homogeneous quadratic polynomials defining the *Plücker relations*. The (homogeneous prime) ideal of G is the kernel of the homomorphism

$$\mathbb{C}[p_{i_1\ldots i_k}\,|\,1\le i_1<\cdots<i_k\le n]\to\mathbb{C}[x_{lj}\,|\,1\le l\le k,\,1\le j\le n]$$

sending $p_{i_1\dots i_k}$ to the Plücker coordinate $\det(x_{lj})_{1\le l\le k,\, j=i_1,\dots,i_k}$.

Example 0.1.1. The Grassmannian $G(2,4)=\mathbb{G}(1,3)$ of lines in $\mathbb{P}^3$ is a smooth quadric hypersurface in $\mathbb{P}^5$, given by the single homogeneous polynomial

$$p_{12}p_{34}-p_{13}p_{24}+p_{14}p_{23}=0.$$

0.2. **Chow Ring of $G(k,n)$.** Let $G=G(k,n)$, and set $r=n-k$. We know that G is smooth and projective, so its Chow group is a ring and can be graded by codimension. Now we think of elements of G as linear subspaces of $\mathbb{P}^{n-1}$. So, let us fix a flag

$$\mathcal{F}\colon\ F_0\subset F_1\subset F_2\subset\cdots\subset F_{n-1}=\mathbb{P}^{n-1}.$$

Let us look at the set of k-tuples

$$\mathscr{A}=\{(a_1,\dots,a_k)\mid r\ge a_1\ge\cdots\ge a_k\ge 0\}.$$

For all $a=(a_1,\dots,a_k)\in\mathscr{A}$, define the closed subset of G

$$\Sigma_a(\mathcal{F})=\{\mathcal{H}\in G\mid \dim(\mathcal{H}\cap F_{r+i-1-a_i})\ge i-1 \text{ for all } i=1,\dots,k\}.$$

These are called the *Schubert cycles* on G. They have a number of interesting properties, for instance:

(1) $c_a=\operatorname{codim}(\Sigma_a,G)=\sum_{1\le i\le k}a_i$. Hence $\sigma_a=[\Sigma_a]\in A^{c_a}G$.
(2) By defining $a\le b$ if and only if $a_i\le b_i$ for all $i=1,\dots,k$, one sees that $\Sigma_b\subseteq\Sigma_a \iff a\le b$.
(3) The Schubert cell $\tilde{\Sigma}_a=\Sigma_a\setminus(\bigcup_{a<b}\Sigma_b)\cong\mathbb{A}^{k(n-k)-c_a}$ and $G=\coprod_{a\in\mathscr{A}}\tilde{\Sigma}_a$ is an affine stratification of G, with closed strata the Schubert cycles.

By (3), the cycle classes σ_a freely generate the Chow group A^*G.

0.3. **The example of $\mathbb{G}(1,3)$.** Let $G=\mathbb{G}(1,3)$, so that $r=n-k=2$ and

$$\begin{aligned}\mathscr{A}&=\{(a_1,a_2)\mid 2\ge a_1\ge a_2\ge 0\}\\&=\{(2,2),(1,1),(0,0),(2,1),(2,0),(1,0)\}.\end{aligned}$$

After fixing a flag of linear subspaces

$$\mathcal{F}\colon\ \{P\}\subset M\subset H\subset\mathbb{P}^3,$$

where $\{P\}=F_0$, $M=F_1$, $H=F_2$, and recalling that the Schubert cycles are given by

$$\Sigma_{a_1a_2}(\mathcal{F})=\{L\in G\mid \dim(L\cap F_{2-a_1})\ge 0,\ \dim(L\cap F_{3-a_2})\ge 1\},$$

we can write them all explicitly as follows:

$$\begin{aligned}
\Sigma_{22}&=\{L\in G\mid P\in L,\ \dim(L\cap M)=1\}=\{M\}\\
\Sigma_{11}&=\{L\in G\mid L\cap M\ne\emptyset,\ L\subset H\}=\{L\in G\mid L\subset H\}\\
\Sigma_{00}&=\{L\in G\mid L \text{ meets a plane, and meets } \mathbb{P}^3 \text{ in a line}\}=G\\
\Sigma_{21}&=\{L\in G\mid P\in L\subset H\}\\
\Sigma_{20}&=\{L\in G\mid P\in L\}\\
\Sigma_{10}&=\{L\in G\mid L\cap M\ne\emptyset\}.
\end{aligned}$$

Remark 0.3.1. Of course, somewhere we used that L always intersects a plane in $\mathbb{P}^3$; also notice that two lines may not meet. Later on we will use that every L intersect a 3-plane in $\mathbb{P}^4$, and that two general 2-planes meet in a point.

Notation 0.3.2. Let us shorten $\Sigma_{a_1 0}$ to Σ_{a_1}. Later, $\sigma_{a_1 a_2}$ will denote $[\Sigma_{a_1 a_2}] \in A^{a_1+a_2}G$.

In order to calculate the Schubert cells $\tilde{\Sigma}_{a_1 a_2}$, it is useful to look at the following inclusions:

$$
\begin{tikzcd}
 & & \Sigma_2 \arrow[dr, hook] & & \\
\Sigma_{22} \arrow[r, hook] & \Sigma_{21} \arrow[ur, hook] \arrow[dr, hook] & & \Sigma_1 \arrow[r, hook] & G \\
 & & \Sigma_{11} \arrow[ur, hook] & &
\end{tikzcd}
$$

One can verify directly that $\tilde{\Sigma}_{a_1 a_2} \cong \mathbb{A}^{4-(a_1+a_2)}$. Let us concentrate on the problem of determining the ring structure of A^*G. For the moment, we have the free abelian group decomposition

$$A^*G = \underbrace{\mathbb{Z}[\sigma_{22}]}_{A^4G} \oplus \underbrace{\mathbb{Z}[\sigma_{21}]}_{A^3G} \oplus \underbrace{\mathbb{Z}[\sigma_{11}] \oplus \mathbb{Z}[\sigma_2]}_{A^2G} \oplus \underbrace{\mathbb{Z}[\sigma_1]}_{A^1G} \oplus \underbrace{\mathbb{Z}[\sigma_0]}_{A^0G}.$$

In particular, any two points are linearly equivalent, and this is true in every Grassmannian.

Let us calculate the products in A^*G. It is crucial to work with two generically situated flags at the same time: so we will intersect a cycle taken from the first one with a cycle taken from the second one. These are generically transverse by Kleiman's Theorem (that we can apply because we are over $\mathbb{C}$). And the result only depends on the equivalence classes of the cycles we are intersecting, so what we find is the correct result. Now, let us fix two flags

$$\mathcal{F}\colon \; \{P\} \subset M \subset H \subset \mathbb{P}^3$$
$$\mathcal{F}'\colon \; \{P'\} \subset M' \subset H' \subset \mathbb{P}^3.$$

0.3.1. *Codimension* 4. We have to evaluate

$$\sigma_{11}^2,\ \sigma_2^2,\ \sigma_{11}\cdot\sigma_2,\ \sigma_1\cdot\sigma_{21} \in A^4G.$$

Let us start with the self-intersection σ_{11}^2. We have

$$|\Sigma_{11} \cap \Sigma_{11}'| = |\{L \in G \mid L \subset H \cap H'\}| = 1.$$

This unique line is of course $H \cap H'$. Hence

$$\sigma_{11}^2 = \sigma_{22}.$$

Similarly,

$$|\Sigma_2 \cap \Sigma_2'| = |\{L \in G \mid P \in L,\, P' \in L\}| = |\{\overline{PP'}\}| = 1.$$

Hence again

$$\sigma_2^2 = \sigma_{22}.$$

Since $P' \notin H$, we find

$$|\Sigma_{11} \cap \Sigma_2'| = |\{L \in G \mid P' \in L \subset H\}| = 0,$$

thus

$$\sigma_{11}\cdot\sigma_2 = 0.$$

The last calculation is

$$|\Sigma_1 \cap \Sigma'_{21}| = |\{L \in G \mid L \cap M \neq \emptyset,\, P' \in L \subset H'\}| = 1,$$

corresponding to this line: the one determined by P' and $M \cap H'$. Thus

$$\sigma_1 \cdot \sigma_{21} = \sigma_{22}.$$

0.3.2. *Codimension* 3. We have to evaluate

$$\sigma_1 \cdot \sigma_2,\, \sigma_{11} \cdot \sigma_1 \in A^3 G.$$

We see that

$$\Sigma_1 \cap \Sigma'_2 = \{L \in G \mid L \cap M \neq \emptyset,\, P' \in L\} = \Sigma''_{21}$$

with respect to the flag $\mathcal{F}''\colon \{P'\} \subset \ell \subset \langle P', M\rangle \subset \mathbb{P}^3$. Thus we get

$$\sigma_1 \cdot \sigma_2 = \sigma_{21}.$$

Similarly,

$$\Sigma_1 \cap \Sigma_{11} = \{L \in G \mid L \cap M \neq \emptyset,\, L \subset H'\} = \Sigma''_{21}$$

with respect to the flag $\mathcal{F}''\colon \{R\} \subset \ell \subset H' \subset \mathbb{P}^3$, where $R = M \cap H'$. Thus

$$\sigma_1 \cdot \sigma_{11} = \sigma_{21}.$$

0.3.3. *Codimension* 2. We have to evaluate $\sigma_1^2 \in A^2 G$. Here things get tricky because this product is not a Schubert cycle. What we know is that we can write $\sigma_1^2 = \alpha\sigma_{11} + \beta\sigma_2$ in $A^2 G$. We have to determine α and β. The strategy will be (now and in the future) to intersect both sides with cycles in complementary codimension so that one of the summands vanishes. Doing this twice allows us to recover α and β in two steps. So,

$$\sigma_1^2 \cdot \sigma_2 = (\alpha\sigma_{11} + \beta\sigma_2) \cdot \sigma_2$$

gives $(\sigma_{22}=)\sigma_1 \cdot \sigma_{21} = \beta\sigma_{22}$. Hence $\beta = 1$. In the same way,

$$\sigma_1^2 \cdot \sigma_{11} = (\alpha\sigma_{11} + \beta\sigma_2) \cdot \sigma_{11}$$

gives $(\sigma_{22}=)\sigma_1 \cdot \sigma_{21} = \alpha\sigma_{11}^2 = \alpha\sigma_{22}$. Hence $\alpha = 1$ and finally

$$\sigma_1^2 = \sigma_{11} + \sigma_2.$$

The next result has a precise enumerative meaning. It solves Problem (1) from the Introduction. We will also solve this problem via torus localisation in Section 9.4.

Proposition 0.3.3. *We have the identity*

$$\int_{G(2,4)} \sigma_1^4 = 2.$$

Proof. We can compute

$$\begin{aligned} \sigma_1^4 &= (\sigma_1^2)^2 \\ &= (\sigma_{11} + \sigma_2)^2 \\ &= \sigma_{11}^2 + 2\sigma_{11}\sigma_2 + \sigma_2^2 \\ &= \sigma_{22} + 0 + \sigma_{22}. \end{aligned}$$

The result follows from the fact that σ_{22} is the class of a point. □

1. Why is Enumerative Geometry hard?

1.1. **Asking the right question.** Enumerative Geometry is a branch of Algebraic Geometry studying questions asking to count how many objects satisfy a given list of geometric conditions. The very nature of these questions, and the presence of this "list", make the subject tightly linked to Intersection Theory, which explains why we included Appendix **??** at the end of these lecture notes.

Examples of classical questions in the subject are the following:

(1) How many lines $\ell \subset \mathbb{P}^3$ intersect four general lines $\ell_1, \ell_2, \ell_3, \ell_4 \subset \mathbb{P}^3$? (Answer in Section 9.4)
(2) How many lines $\ell \subset \mathbb{P}^3$ lie on a smooth cubic surface $S \subset \mathbb{P}^3$? (Answer in Section 9.2)
(3) How many lines $\ell \subset \mathbb{P}^4$ lie on a generic quintic 3-fold $Y \subset \mathbb{P}^4$? (Answer in Section 9.3)
(4) How many flexes are there on a general genus 3 curve?
(5) How many smooth conics are tangent to five general plane conics?

The objects we want to count, say in the first three examples, are lines in some projective space. The geometric conditions are constraints we put on these lines, such as intersecting other lines or lying on a smooth cubic surface. We immediately see that one fundamental difficulty in the subjects is this:

> **D1**. How do we know how many constraints we should put on our objects in order to *expect* a finite answer? In other words, how do we ask the right question?

Here is a warm-up example to shape one's intuition. See Section 5 for a full treatment of the topic "expectations" in the case of lines on hypersurfaces. Problem (2) will be solved in Section 9.2, problem (3) in Section 9.3.

Exercise 1.1.1. Let $d > 0$ be an integer. Determine the number m_d having the following property: you expect finitely many smooth complex projective curves $C \subset \mathbb{P}^2$ of degree d passing through m_d general points in $\mathbb{P}^2$. (**Hint**: Start with small d. Then conjecture a formula for m_d).

1.2. **Counting the points on a moduli space.** The main idea to guide our geometric intuition in formulating and solving an enumerative problem should be the following recipe:

- construct a moduli[1] space $\mathcal{M}$ for the objects we are interested in,
- compactify $\mathcal{M}$ if necessary,
- impose $\dim \mathcal{M}$ conditions to expect a finite number of solutions, and
- count these solutions via Intersection Theory methods (exploiting compactness of $\mathcal{M}$).

None of these steps is a trivial one, in general.

[1] The latin word *modulus* means *parameter*, and its plural is *moduli*. Thus a *moduli space* is to be thought of as a parameter space for objects of some kind.

Another difficulty in the subject is the following. Say we have a precise question, such as (2) above. Then, in the above recipe, as our $\mathcal{M}$ we should take the Grassmannian of lines in $\mathbb{P}^3$, which is a compact 4-dimensional complex manifold. Imagine we have found a sensible algebraic variety structure on the set $\mathcal{M}_S \subset \mathcal{M}$ of lines lying on the surface S. If we have done everything right, the space $\mathcal{M}_S$ consists of finitely many points, and now the only legal operation we can perform in order to get our answer is to take the degree of the (0-dimensional) fundamental class of $\mathcal{M}_S$. So here is the second problem we face:

> **D2**. How do we know this degree is the answer to our original question? In other words, how to ensure that our algebraic solution is actually *enumerative*?

Put in more technical terms, how do we make sure that each line $\ell \subset S$ appears as a point in the moduli space $\mathcal{M}_S$ with multiplicity one? The truth is that we cannot *always* be sure that this is the case. It will be, both for problem (2) and problem (3), but not in general. However, we should get used to the idea that this is not something to be worried about: if a solution comes with multiplicity bigger than one, there usually is a good geometric reason for this, and we should not disregard it (see Figure 4 for a simple example of a degenerate intersection where this phenomenon occurs).

Remark 1.2.1. Compactness of $\mathcal{M}$ (in the above example, the Grassmannian) is used in order to make sense of taking the *degree* of cycles. Intuitively, we need compactness in order to prevent the solutions of our enumerative problem to escape to infinity, like for instance it would occur if we were to intersect two *parallel* lines in $\mathbb{A}^2$.

Compatcness really is a non-negotiable condition we have to ask of our moduli space — with an important exception, that will be treated in later sections: the case when the moduli space has a torus action. In this case, if the torus-fixed locus $\mathcal{M}^{\mathbb{T}} \subset \mathcal{M}$ is compact, a sensible enumerative solution to a counting problem can be *defined* by means of the *localisation formula*. The original formula due to Atiyah and Bott will be proved in Theorem 8.5.1. A virtual analogue due to Graber and Pandharipande [30] will be proved in Theorem **??**, and the latter will be applied to the study of 0-dimensional Donaldson–Thomas invariants of local Calabi–Yau 3-folds (arising from non-compact, but toric, moduli spaces).

A more fundamental difficulty is discussed in the next subsection, by means of an elementary example.

1.3. **Transversality, and counting lines through two points.** Consider the enumerative problem of counting the number of lines in $\mathbb{P}^2$ through two given points $p, q \in \mathbb{P}^2$. Let N_{pq} be this number. Then

$$\mathsf{N}_{pq} = 1, \quad \text{as long as } p \neq q.^{2}$$

However, the *true* answer would be ∞ when $p = q$, corresponding to the cardinality of the pencil $\mathbb{P}^1$ of lines through p (see Figure 1).

Now, the case $p = q$ is a degeneration of the case $p \neq q$, and we certainly want our enumerative answer not to depend on small perturbations of the geometry of the

[2] For the sake of completeness, this will be proved in Section 9.1.

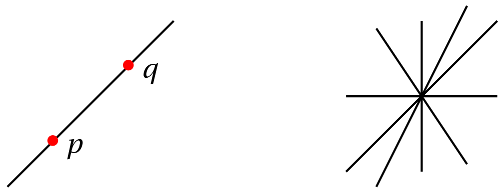


FIGURE 1. The unique line through two distinct points, and the infinitely many lines through one point in the plane.

problem. It seems at first glance that the issue cannot be fixed. After all, there is an inevitable dimensional jump between the transverse case (yielding a dimension zero answer) and the non-transverse geometry (dimension one answer). However, the answer '1' can be recovered in the non-transverse setting (the picture on the right) by means of the *excess intersection formula*.

The $\mathbb{P}^1$ of lines through p can be neatly seen as the exceptional divisor E in the blowup $B = \mathrm{Bl}_p\,\mathbb{P}^2$, cf. Figure 2.

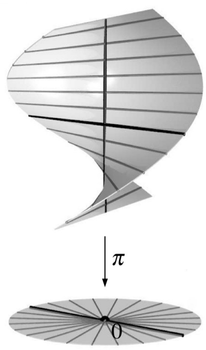


FIGURE 2. The blow-up of $\mathbb{P}^2$ at a point p. Picture stolen from Gathmann [26].

Recall that the *normal sheaf* of a closed embedding $X \hookrightarrow Y$ defined by an ideal $\mathscr{I} \subset \mathscr{O}_Y$ is the $\mathscr{O}_X$-module $N_{X/Y} = (\mathscr{I}/\mathscr{I}^2)^\vee = \mathscr{H}\mathrm{om}_{\mathscr{O}_X}(\mathscr{I}/\mathscr{I}^2, \mathscr{O}_X)$.

EXERCISE 1.3.1. Let $X \hookrightarrow Y$ be a closed embedding, $M \to Y$ a morphism, and let $g\colon P = X \times_Y M \to X$ be the induced map. Show that there is an inclusion $N_{P/M} \subset g^* N_{X/Y}$.

Looking at the Cartesian square

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & B \arrow[d, "\pi"] \\ p \arrow[r, hook] & \mathbb{P}^2 \end{tikzcd} \tag{1.3.1}$$

we know by Exercise 1.3.1 that there is an injection of vector bundles $N_{E/B} = \mathscr{O}_E(-1) \subset g^* N_{p/\mathbb{P}^2}$. The *excess bundle* (or *obstruction bundle*)

$$\mathrm{Ob} \to \mathbb{P}^1$$

of the fiber diagram (1.3.1) is defined as the quotient of these two bundles. But the short exact sequence

$$0 \to \mathscr{O}_E(-1) \to \mathscr{O}_E \otimes T_p\mathbb{P}^2 \to \mathrm{Ob} \to 0$$

is just the Euler sequence on $\mathbb{P}^1$ twisted by -1. Therefore

$$\mathrm{Ob} = T_{\mathbb{P}^1}(-1) = \mathscr{O}_{\mathbb{P}^1}(2-1) = \mathscr{O}_{\mathbb{P}^1}(1).$$

We have thus recovered '1' as the Euler number of the excess bundle, so that we can now write a universal formula for our counting problem: if $\mathcal{M}_{pq} = \pi^{-1}(q) \cap E$ is the "moduli space" of lines through p and q (this includes the case $p = q$), the *virtual number* of lines through p and q is

$$\int_{\mathcal{M}_{pq}} e(\mathrm{Ob}) = 1.$$

Note that the rank of the excess bundle is the difference between the actual dimension of the moduli space, and the expected one, and that $\mathrm{Ob} = 0$ unless $p = q$.

Unfortunately, in more complicated situations (but also not that complicated), we often do not even know whether our geometric setup is a degeneration of a transverse one. If it were, we would like to dispose of a technology allowing us to "count" in the transverse setup and argue that the number we obtain there equals the one we are after. This sounds like a reasonable wish, but it is way too optimistic. We should not aim at this: not only because counting is often difficult also in transverse situations, but mainly because we simply may not have enough algebraic deformations to pretend that the geometry of the problem is transverse.

Example 1.3.2. If we were to count self-intersections of a (-1)-curve on a surface,[3] there would be no way to deform these curves off themselves to make them intersect themselves transversely! See also Exercise 1.3.4 below.

This discussion leads us directly to another intrinsic difficulty in Enumerative Geometry. Suppose, just to dream for a second, that we are able to solve *all* enumerative problems in generic (transverse) situations, and we know that the answer does not change after a small perturbation of the initial data.

> **D3**. How do we reduce to a transverse situation when there is none available (e.g. in Example 1.3.2)?

The modern way around this is to use *virtual fundamental classes* (cf. Section **??** and Appendix **??**).

1.3.1. *Two more words on excess intersection.* Problem (5), known as "the five conics problem", is a typical example of an excess intersection problem. See [19] for a thorough analysis and solution of this problem. As we shall see in Section 4.5.1, a natural compact parameter space for plane conics is

$$\mathcal{M} = \mathbb{P}^5,$$

[3] A (-1)-curve on a surface S is a curve $C \subset S$ such that $C.C = -1$, where the intersection number $C.C$ can be seen as the degree of the normal bundle $N_{C/S}$ to C in S.

and the set of smooth conics is an open subvariety $U \subset \mathcal{M}$. The answer to Problem (5) is a certain finite subset of U. Let $C_1,\ldots,C_5$ be general plane conics. The conics that are tangent to a given conic C_i form a sextic hypersurface $Z_i \subset \mathcal{M}$, so we might be tempted to say that the answer to Problem (5) is the degree

$$\int_{\mathbb{P}^5} \alpha_1 \cdots \alpha_5 = 6^5,$$

where $\alpha_i = [Z_i] \in H^2(\mathbb{P}^5,\mathbb{Z})$ is the divisor class of a sextic.[4] However, the cycles Z_i share a common two-dimensional component, namely the Veronese surface $\mathbb{P}^2 \subset \mathbb{P}^5$ of double lines. Therefore their intersection is 2-dimensional, even though our intuition suggests that 5 hypersurfaces in $\mathbb{P}^5$ should intersect in a finite set. Note that this issue arose precisely "because" we insisted to work with a compact parameter space: double lines are singular, hence lie in the complement of U. But working with U directly is forbidden, because it is not compact!

The excess intersection formula is a tool that allows one to precisely compute (and hence get rid of) the enumerative contribution of the *excess locus*, namely the locus of non-transverse intersection among certain cycles — in this case the cycles $Z_1,\ldots,Z_5$. The way it works is precisely via blow-ups; often more than one is required to separate the common components of the non-transverse cycles. In the case of the five conics problem, only one blow-up is required.

In principle, blowing up the excess locus, checking that the proper transforms will be disjoint in the exceptional divisor, and blowing up again if necessary, one gets to the correct answer to the original question, but:

> **D4**. In practice it is often very hard to keep track of multiple blow-ups; the calculation becomes less and less intuitive and the modular meaning of the blow-ups appearing might be quite unclear.

In Exercise 1.3.4 you will compute an excess bundle for a more complicated problem than finding the number of lines through two points. Before tackling it, it is best to solve the following exercise.

EXERCISE 1.3.3. Show that the vector space V of homogeneous cubic polynomials in 3 variables is 10-dimensional. Identify

$$\mathbb{P}V = \mathbb{P}^9$$

with the space of degree 3 plane curves $C \subset \mathbb{P}^2$. Show that, for a given point $p \in \mathbb{P}^2$, the space of cubics passing through p forms a hyperplane

$$\mathbb{P}^8 \subset \mathbb{P}V.$$

EXERCISE 1.3.4. Let C_1 and C_2 be two plane cubics intersecting transversely in nine points $p_1,\ldots,p_9 \in \mathbb{P}^2$ (cf. Figure 3). Every cubic in the pencil $\mathbb{P}^1 \subset \mathbb{P}^9$ generated by C_1 and C_2 passes through $p_1,\ldots,p_9$. However, if the nine points were general, there would be a unique cubic passing through them. Find out where the answer '1' is hiding in this non-transverse geometry.

[4] Recall that the Picard group $\operatorname{Pic}\mathbb{P}^r = H^2(\mathbb{P}^r,\mathbb{Z}) = \mathbb{Z}$ is generated by the hyperplane class h and the cohomology class of a degree d hypersurface in $\mathbb{P}^r$ corresponds to the class $d\cdot h$.

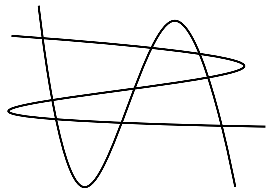


FIGURE 3. The nine intersection points $C_1 \cap C_2$.

1.4. **Before and after the virtual class.** Here is a philosophical description of the field of Enumerative Geometry before and after the advent of *virtual fundamental classes*, introduced by Li–Tian [40] and Behrend–Fantechi [8].

Before: What is the answer?
After: What is the question?

Before virtual classes, there were a number of unanswered enumerative questions whose geometrical meaning was extremely clear. *After* the definition of virtual classes, many new invariants were defined through them, but the enumerative meaning of these invariants is often not very clear, so it fair to ask what integrals of the form

$$\int_{[\mathcal{M}]^{\mathrm{vir}}} \alpha \in \mathbb{Z}, \quad \alpha \in H^*(\mathcal{M}),$$

might be actually computing.

Virtual fundamental classes allow one to think that even a horrible moduli space $\mathcal{M}$, say a singular scheme of impure dimension (cf. Figure 6), has a well-defined *virtual dimension* vd at any point $p \in \mathcal{M}$, and this number is constant on p. It is given as the difference

$$\mathrm{vd} = \dim T_p\mathcal{M} - \dim \mathrm{Ob}|_p,$$

where both dimensions on the right may (and will) vary with p. The virtual fundamental class is a homology class

$$[\mathcal{M}]^{\mathrm{vir}} \in A_{\mathrm{vd}}\mathcal{M} \to H_{2\cdot\mathrm{vd}}(\mathcal{M}, \mathbb{Z})$$

that should be thought of as the fundamental class that $\mathcal{M}$ would have if it were of the form $\mathcal{M} = \{ s = 0 \}$ for s a regular section of a vector bundle (the bundle Ob) on a smooth variety.

As a matter of fact, many badly behaved moduli spaces turn out to have a virtual fundamental class. These include:

(i) the moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(X, \beta)$ to a smooth projective variety X,
(ii) the moduli space $M_Y^H(\alpha)$ of H-stable torsion free sheaves with Chern character α on a smooth 3-fold Y,
(iii) the moduli space $P_X^H(\alpha)$ of Pandharipande–Thomas pairs with Chern character α.

All this richness gives rise to three amongst the most modern counting theories:

$$\begin{aligned}\text{Gromov–Witten theory} &\coloneqq \text{intersection theory on } \overline{\mathcal{M}}_{g,n}(X,\beta),\\ \text{Donaldson–Thomas theory} &\coloneqq \text{intersection theory on } M_Y^H(\alpha),\\ \text{Pandharipande–Thomas theory} &\coloneqq \text{intersection theory on } P_X^H(\alpha).\end{aligned}$$

All these theories can be seen as more complicated (virtual) versions of a well established theory:

$$\text{Schubert Calculus} \coloneqq \text{intersection theory on the Grassmannian } G(k,n).$$

No "virtualness" is arising in Schubert calculus, because — as already observed by Mumford [47] when he initiated the enumerative geometry of the moduli space of curves — the Grassmannian is the ideal moduli space one would like to work with: it is compact, smooth and unobstructed. It does have a virtual fundamental class, but because of these properties it happens to coincide with its actual fundamental class.

1.5. **To the reader.** The reader might benefit from some familiarity with elementary aspects of scheme theory, basic theory of coherent sheaves on algebraic varieties, and intersection theory at the level of [33]. We shall, however, review some preliminaries in the next section. Here is a list of excellent references for the background material needed in these lecture notes (that we will refer to when necessary):

- for scheme theory at various levels, see [33, 18, 41, 61],
- for Intersection Theory, see [24, 19],
- for toric varieties, see [25, 16],
- for Deformation Theory, see [57, 34] and [21, Part 3].

2. Background material

2.1. **Varieties and schemes.** The notion of scheme used in this text is the standard one, see e.g. [41, Chapter 2]. The structure sheaf of a scheme X, its sheaf of regular functions, is denoted $\mathscr{O}_X$. A scheme X is *locally Noetherian* if every point $x\in X$ has a Zariski affine open neighborhood $x\in \operatorname{Spec} R\subset X$ such that R is a Noetherian ring. If X is locally Noetherian and quasi-compact, then it is called *Noetherian*. Any open or closed subscheme of a Noetherian scheme X is still Noetherian, and for every affine open subset $U\subset X$ the ring $\mathscr{O}_X(U)$ is Noetherian. An important property of Noetherian schemes is that they have a finite number of irreducible components, or, more generally, of associated points.

A morphism of schemes $f\colon X\to S$ is *quasi-compact* if the preimage of every affine open subset of S is quasi-compact. On the other hand, f is *locally of finite type* if for every $x\in X$ there exist Zariski open neighborhoods $x\in\operatorname{Spec} A\subset X$ and $f(x)\in \operatorname{Spec} B\subset S$ such that $f(\operatorname{Spec} A)\subset \operatorname{Spec} B$ and the induced map $B\to A$ is of finite type, i.e. A is isomorphic to a quotient of $B[x_1,\ldots,x_n]$ as a B-algebra. We say that f is *of finite type* if it is locally of finite type and quasi-compact.

EXERCISE 2.1.1. Let $f\colon X\to S$ be a morphism of schemes, with S (locally) Noetherian. If f is (locally) of finite type, then X is (locally) Noetherian.

For instance, a scheme of finite type over a field is Noetherian.

Notation 2.1.2. By k we will always mean an algebraically closed field. For most of the time, we will have $k = \mathbb{C}$.

Definition 2.1.3. A scheme X is *reduced* if for every point $p \in X$ the local ring $\mathscr{O}_{X,p}$ is reduced, i.e. it has no nilpotent elements besides zero.

The prototypical example of a nonreduced scheme is the curvilinear affine scheme

$$D_n = \operatorname{Spec} k[t]/t^n, \quad n > 1.$$

One can show that (quasi-compact) reduced schemes are precisely those schemes for which the regular functions on them are determined by their values on points. The function

$$\overline{t} \in k[t]/t^n$$

vanishes at the unique point of D_n, but it is not the zero function!

The case $n = 2$ is particularly important. For instance, the *Zariski tangent space* $T_x X$ of a k-scheme X at a point $x \in X$, which by definition is the k-vector space $(\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$, can be identified with

$$\operatorname{Hom}_x(D_2, X),$$

the space of k-morphisms $D_2 \to X$ such that the image of the closed point of D_2 is x.

Example 2.1.4 (D_2 as a limit of distinct points)**.** Consider the scheme

$$X_a = \operatorname{Spec} \mathbb{C}[x, y]/(y - x^2, y - a), \quad a \in \mathbb{C}.$$

For $a \neq 0$, this scheme consists of two reduced points, corresponding to the maximal ideals

$$(x \pm \sqrt{a}, y - a) \subset \mathbb{C}[x, y].$$

For $a = 0$, we get

$$X_0 = \operatorname{Spec} \mathbb{C}[x]/x^2 = D_2,$$

a point with multiplicity two. See Figure 4 for a visual explanation.

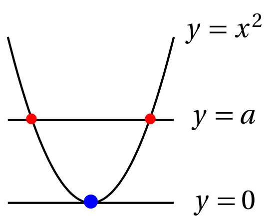


FIGURE 4. The intersection X_a of a parabola with the line $y = a$.

Definition 2.1.5. Let k be a field. An *algebraic variety* over k (or simply a k-variety) is a reduced, separated scheme of finite type over $\operatorname{Spec} k$, i.e. a reduced scheme X equipped with a finite type morphism $X \to \operatorname{Spec} k$, such that the diagonal map $\Delta_X : X \to X \times_k X$, sending $x \mapsto (x, x)$, is a closed immersion.

An *affine variety* is a k-scheme of the form $\operatorname{Spec} A$, where $A = k[x_1,\ldots,x_n]/I$ for some ideal I. An algebraic variety is *projective* if it admits a closed immersion into projective space $\mathbb{P}^n$ for some n. A variety is *quasi-projective* if it admits a locally closed immersion in some projective space, i.e. it is closed in an open subset of some $\mathbb{P}^n$. The same abstract scheme can of course be a (quasi-)projective variety in many different ways.

Example 2.1.6. The *rational normal curve* of degree d is the image of the closed embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^d$ defined by $(u:v) \mapsto (u^d : u^{d-1}v : \cdots : uv^{d-1} : v^d)$.

EXERCISE 2.1.7. Consider the algebraic variety $X = \operatorname{Spec} \mathbb{C}[x,y]/(xy, y^2)$, viewed as a subscheme of the affine plane $\mathbb{A}^2 = \operatorname{Spec} \mathbb{C}[x,y]$. Show that the origin $p = (0,0) \in X$ is the unique point such that $\mathscr{O}_{X,p}$ is not reduced.

The following definition will be relevant when we will discuss the Hilbert scheme of points in Section 6.

Definition 2.1.8. An algebraic k-variety X is *finite* if $\mathscr{O}_X(X)$ is a finite dimensional k-vector space. For any such X, the ring of functions is necessarily *Artinian*. In other words, X has dimension zero, and we say that $\mathscr{O}_X(X)$ is a finite dimensional k-algebra of *length*

$$\ell = \dim_k \mathscr{O}_X(X).$$

We also say that ℓ is the length of X.

EXERCISE 2.1.9. Show that an algebraic variety X is both affine and projective if and only if it is finite. Show that, for any ℓ, the only reduced finite k-variety of length ℓ is the disjoint union $\coprod_{1\le i\le \ell} \operatorname{Spec} k$.

EXERCISE 2.1.10. Classify all finite dimensional $\mathbb{C}$-algebras of length 2 and 3 up to isomorphism.

EXERCISE 2.1.11. Give an example of a scheme X whose underlying topological space consists of finitely many points, and yet is *not* finite.

2.2. **Some properties of morphisms.** We encountered separated morphisms in the definition of algebraic varieties (Definition 2.1.5). A morphism $f\colon X \to S$ is *separated* if the diagonal $X \to X \times_S X$ (which is always a locally closed immersion) is a closed immersion. A stronger notion is properness. A morphism $f\colon X \to S$ is *proper* if it is separated, of finite type, and universally closed. The *valuative criterion* for proper morphims says that f is proper if and only if for every valuation domain A with fraction field K there exists exactly one way to fill in the dotted arrow in a commutative square

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d, hook, "i"'] & X \arrow[d, "f"] \\
\operatorname{Spec} A \arrow[r] \arrow[ur, dotted] & S
\end{tikzcd}$$

in such a way that the resulting triangles are commutative. Such property can be rephrased by saying that for any A as above the map of sets

$$\operatorname{Hom}(\operatorname{Spec} A, X) \to \operatorname{Hom}(\operatorname{Spec} K, X) \times_{\operatorname{Hom}(\operatorname{Spec} K, S)} \operatorname{Hom}(\operatorname{Spec} A, S)$$

defined by $v \mapsto (v \circ i, f \circ v)$ is a bijection.

Let A and $\overline{A}$ be Artinian k-algebras with residue field k. We say that a surjection $u \colon \overline{A} \twoheadrightarrow A$ is a *square zero extension* if $(\ker u)^2 = 0$.

Definition 2.2.1. Let $f \colon X \to S$ be a locally of finite type morphism between k-schemes. Then f is *unramified* (resp. *smooth*, *étale*) if for any square zero extension $\overline{A} \twoheadrightarrow A$ and solid diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \operatorname{Spec} \overline{A} \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

there exists at most one (resp. at least one, exactly one) way to fill in the dotted arrow in such a way that the resulting triangles are commutative.

Example 2.2.2. The following are important features to keep in mind.

- Let $f \colon X \to S$ be a morphism of smooth k-varieties. If f induces isomorphism on tangent spaces, it is étale.
- A bijective morphism of smooth varieties is an isomorphism.
- If f is étale and injective, it is an open immersion.

2.3. **Schemes with embedded points.** On a locally Noetherian scheme X there are a bunch of points that are more relevant than all other points, in the sense that they reveal part of the behavior of the structure sheaf: these points are the *associated points* of X.

Let R be a commutative ring with unity, and let M be an R-module. If $m \in M$, we let

$$\operatorname{Ann}_R(m) = \{\, r \in R \mid r \cdot m = 0 \,\} \subset R$$

denote its annihilator. A prime ideal $\mathfrak{p} \subset R$ is said to be *associated to* M if $\mathfrak{p} = \operatorname{Ann}_R(m)$ for some $m \in M$. The set of all associated primes is denoted

$$\operatorname{AP}_R(M) = \{\, \mathfrak{p} \mid \mathfrak{p} \text{ is associated to } M \,\}.$$

Lemma 2.3.1. *Let $\mathfrak{p}$ be a prime ideal of R. Then $\mathfrak{p} \in \operatorname{AP}_R(M)$ if and only if $R/\mathfrak{p}$ is an R-submodule of M.*

Proof. If $\mathfrak{p} = \operatorname{Ann}_R(m)$ for some $m \in M$, consider the map $\phi_m \colon R \to M$ defined by $\phi_m(r) = r \cdot m$. Since its kernel is by definition $\operatorname{Ann}_R(m)$, the quotient $R/\mathfrak{p}$ is an R-submodule of M. Conversely, given an R-linear inclusion $i \colon R/\mathfrak{p} \hookrightarrow M$, consider the composition $\phi \colon R \twoheadrightarrow R/\mathfrak{p} \hookrightarrow M$. Then $\phi = \phi_m$, where $m = i(1)$. $\square$

Note that if $\mathfrak{p} \in \operatorname{AP}_R(M)$ then $\mathfrak{p}$ contains the annihilator of M, i.e. the ideal

$$\operatorname{Ann}_R(M) = \{\, r \in R \mid r \cdot m = 0 \text{ for all } m \in M \,\} \subset R.$$

The minimal elements (with respect to inclusion) in the set

$$\{\,\mathfrak{p}\subset R\,|\,\mathfrak{p}\supset \mathrm{Ann}_R(M)\,\}$$

are called *isolated primes* of M.

From now on we assume R is Noetherian and $M\neq 0$ is finitely generated. In this situation, M has a *composition series*, i.e. a filtration of R-submodules

$$0=M_0\subsetneq M_1\subsetneq\cdots\subsetneq M_s=M$$

such that $M_i/M_{i-1}=R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$. This series is not unique. However, for a prime ideal $\mathfrak{p}\subset R$, the number of times it occurs among the $\mathfrak{p}_i$ does not depend on the composition series. These primes are precisely the elements of $\mathrm{AP}_R(M)$. For $M=R/I$, elements of $\mathrm{AP}_R(R/I)$ are the radicals of the primary ideals in a *primary decomposition* of I.

EXERCISE 2.3.2. Let $R=k[x,y]$, $I=(xy,y^2)$ and $M=R/I$. Show that $\mathrm{AP}_R(M)=\{(y),(x,y)\}$.

Theorem 2.3.3 ([61, Theorem 5.5.10 (a)]). *Let R be a Noetherian ring, $M\neq 0$ a finitely generated R-module. Then $\mathrm{AP}_R(M)$ is a finite nonempty set containing all isolated primes.*

Definition 2.3.4. The non-isolated primes in $\mathrm{AP}_R(M)$ are called the *embedded primes* of M.

The most boring situation is when R is an integral domain, in which case the generic point $\xi\in\operatorname{Spec}R$ is the only associated prime. More generally, a reduced affine scheme $\operatorname{Spec}R$ has *no embedded point*, i.e. the only associated primes are the isolated (minimal) ones, corresponding to its irreducible components.

Fact 2.3.5. An algebraic curve has no embedded points if and only if it is Cohen–Macaulay. However, there can be nonreduced Cohen–Macaulay curves: those curves with a fat component, such as the affine plane curve $\operatorname{Spec}k[x,y]/x^2\subset\mathbb{A}^2$. These objects often have moduli, i.e. deform (even quite mysteriously) in positive dimensional families.

Let R be an integral domain. For an ideal $I\subset R$, one often calls the associated primes of I the associated primes of R/I. The minimal primes above $I=\mathrm{Ann}_R(R/I)$ correspond to the irreducible components of the closed subscheme

$$\operatorname{Spec}R/I\subset\operatorname{Spec}R,$$

whereas for every embedded prime $\mathfrak{p}\subset R$ there exists a minimal prime $\mathfrak{p}'$ such that $\mathfrak{p}'\subset\mathfrak{p}$. Thus $\mathfrak{p}$ determines an *embedded component* — a subvariety $V(\mathfrak{p})$ embedded in an irreducible component $V(\mathfrak{p}')$. If the embedded prime $\mathfrak{p}$ is maximal, we talk about an *embedded point*.

Remark 2.3.6. An embedded component $V(\mathfrak{p})$, where $\mathfrak{p}$ is the radical of some primary ideal $\mathfrak{q}$ appearing in a primary decomposition $I=\mathfrak{q}_1\cap\cdots\cap\mathfrak{q}_e$, is of course embedded in some irreducible component $V(\mathfrak{p}')\subset\operatorname{Spec}R/I$, but $V(\mathfrak{q})$ is not a *subscheme* of $V(\mathfrak{p}')$, because the fuzzyness caused by nilpotent behavior (i.e. the difference between $\mathfrak{q}$ and its radical $\mathfrak{p}$) makes the bigger scheme $V(\mathfrak{q}) \supset V(\mathfrak{p})$ "stick out" of $V(\mathfrak{p}') \subset \operatorname{Spec} R/I$.

Example 2.3.7. Consider $R = k[x,y]$ and $I = (xy, y^2)$. A primary decomposition of I is

$$I = (x,y)^2 \cap (y).$$

However, $\operatorname{Spec} R/(x,y)^2$ is not scheme-theoretically contained in $\operatorname{Spec} R/y$.

In general, a subscheme Z of scheme Y has an embedded component if there exists a dense open subset $U \subset Y$ such that $Z \cap U$ is dense in Z but the scheme-theoretic closure of $Z \cap U \subset Z$ does not equal Z scheme-theoretically. For instance, if Y is irreducible, we say that $p \in Y$ supports an embedded point of a closed subscheme $Z \subset Y$ if $\overline{Z \cap (Y \setminus p)} \neq Z$ as schemes. In the example above, where $Y = \mathbb{A}^2$ and $Z = \operatorname{Spec} k[x,y]/(xy, y^2)$, the scheme-theoretic closure of $Z \cap (\mathbb{A}^2 \setminus 0) \subset Z$ is not equal to Z.

2.4. **Sheaves and their support.** Recall that a *coherent sheaf* on a (locally Noetherian) scheme X is an $\mathscr{O}_X$-module that is locally the cokernel of a map of free $\mathscr{O}_X$-modules of finite rank. Coherent sheaves form an abelian category

$$\operatorname{Coh} X.$$

For instance, if $\iota \colon Z \hookrightarrow X$ is a closed subscheme, both $\iota_* \mathscr{O}_Z$ and $\mathscr{I}_Z$ are coherent sheaves on X. The ideal sheaf, being a subsheaf of a free sheaf, is torsion free. In fact, ideal sheaves are precisely the torsion free sheaves of rank one and trivial determinant.

Definition 2.4.1. Let $X \to S$ be a finite type morphism of locally Noetherian schemes. A sheaf $F \in \operatorname{Coh} X$ is *flat over* S (or S-*flat*) if for every point $x \in X$, with image $s \in S$, the module F_x is flat over $\mathscr{O}_{S,s}$ via the ring map $\mathscr{O}_{S,s} \to \mathscr{O}_{X,x}$.

For instance, $\mathscr{O}_X$ is S-flat if and only if $X \to S$ is flat as a morphism of schemes.

The *support* of a coherent sheaf $F \in \operatorname{Coh} X$ is the following *closed subscheme* of X: consider the map $\mathscr{O}_X \to \mathscr{H}om_{\mathscr{O}_X}(F,F)$ defined on local sections by sending f to the $\mathscr{O}_X$-linear map $m \mapsto f \cdot m$. The kernel — the sheaf-theoretic annihilator ideal of F — defines the closed subscheme

$$\operatorname{Supp} F \subset X.$$

The support behaves well under pullback. However, the following remark is the origin of several issues such as the existence of Hilbert–Chow morphisms.

Remark 2.4.2. Let $X \to S$ be a a finite type morphism of locally Noetherian schemes. It is not true that the support of an S-flat $\mathscr{O}_X$-module is flat over S.

EXERCISE 2.4.3. Give an example of the phenomenon described in Remark 2.4.2.

3. INFORMAL INTRODUCTION TO GRASSMANNIANS

We denote by $G(k,n)$ the *set* of k-dimensional subspaces of $\mathbb{C}^n$. This set is in fact a complex manifold, called the *Grassmannian* of k-planes in $\mathbb{C}^n$. It is naturally identified with the set of $(k-1)$-dimensional linear subspaces of $\mathbb{P}^{n-1}$, and when we think of it in this manner we denote it by $\mathbb{G}(k-1, n-1)$. For instance, $\mathbb{G}(0, n-1) = \mathbb{P}^{n-1}$.

We will give $G = G(k,n)$ the structure of a (smooth) projective variety of dimension $k(n-k)$, by describing a closed embedding in $\mathbb{P}^N$, where $N = \binom{n}{k} - 1$. We will see that G admits an affine stratification, which will enable us to explicitly describe the generators of its Chow ring A^*G. The affine strata will be called *Schubert cells*, while their closures will be called *Schubert cycles*. The classes of the Schubert cycles freely generate the Chow group of G. To determine the ring structure, one has to compute the products between these generators. These computations in A^*G go under the name of *Schubert Calculus*.

3.1. $G(k,n)$ **as a projective variety.** Let us fix a point

$$[\mathcal{H}] \in G = G(k,n),$$

corresponding to a k-dimensional linear subspace $\mathcal{H} \subset V = \mathbb{C}^n$. If $v_1, \ldots, v_k$ is a basis of $\mathcal{H}$ then $v_1 \wedge \cdots \wedge v_k$ is the free generator of the line $\bigwedge^k \mathcal{H} \subset \bigwedge^k V \cong \mathbb{C}^{\binom{n}{k}} = \mathbb{C}^{N+1}$. So we get a map

$$\iota \colon G \to \mathbb{P}\left(\bigwedge^k V\right) = \mathbb{P}^N$$

sending $[\mathcal{H}] \mapsto [v_i \wedge \cdots \wedge v_k]$. Why is this well-defined? Let us view the point $[\mathcal{H}] \in G$ as (the space generated by the rows of) a full rank matrix $H = (a_{ij}) \in M_{k\times n}(\mathbb{C})$ and let us fix a basis $e_1, \ldots, e_n$ of V. Then a basis of $\bigwedge^k V$ is given by

$$\{ e_{i_1} \wedge \cdots \wedge e_{i_k} \}_{1 \le i_1 < \cdots < i_k \le n}.$$

So when we view the element $v_1 \wedge \cdots \wedge v_k$ inside $\bigwedge^k V$ we can write it uniquely as

$$v_1 \wedge \cdots \wedge v_k = \sum_{1 \le i_1 < \cdots < i_k \le n} p_{i_1 \cdots i_k} (e_{i_1} \wedge \cdots \wedge e_{i_k}) = \sum_I p_I e_I,$$

where the coefficient $p_I = p_{i_1 \cdots i_k}$ is the minor of the $(k \times k)$-matrix given by extracting from H the columns $i_1, \ldots, i_k$. Of course different choices of H may produce the same $\mathcal{H}$. But H is unique up to the left action of $\mathrm{GL}(k,\mathbb{C})$. Summing up, we have a commutative diagram

$$\begin{tikzcd}
\bigwedge^k \mathcal{H} \setminus 0 \arrow[r, hook] \arrow[d, "/\mathbb{C}^\times"'] & \bigwedge^k V \setminus 0 \arrow[d, "/\mathbb{C}^\times"] \\
\mathbb{P}\left(\bigwedge^k \mathcal{H}\right) \arrow[r, "\iota"] & \mathbb{P}\left(\bigwedge^k V\right)
\end{tikzcd}
\qquad
\begin{tikzcd}
v_1 \wedge \cdots \wedge v_k \arrow[r, mapsto] \arrow[d, mapsto] & \sum_I p_I e_I \arrow[d, mapsto] \\
{[v_1 \wedge \cdots \wedge v_k]} \arrow[r, mapsto] & (p_I)_I
\end{tikzcd}$$

Up to now, we have identified a point $[\mathcal{H}] \in G$ to the unique point of $\mathbb{P}(\bigwedge^k \mathcal{H})$ and we have defined a map $\iota \colon G \to \mathbb{P}^N$ by sending $[\mathcal{H}]$ to its *Plücker coordinates* $(p_I)_I$. Such a map is injective, and G can be identified with an irreducible algebraic set in $\mathbb{P}^N$, via a collection of homogeneous quadratic polynomials defining the *Plücker relations*. The (homogeneous prime) ideal of G is the kernel of the homomorphism

$$\mathbb{C}[p_{i_1 \ldots i_k} | 1 \le i_1 < \cdots < i_k \le n] \to \mathbb{C}[x_{lj} | 1 \le l \le k, 1 \le j \le n]$$

sending $p_{i_1 \ldots i_k}$ to the Plücker coordinate $\det(x_{lj})_{1 \le l \le k,\, j = i_1, \ldots, i_k}$.

Example 3.1.1. The Grassmannian $G(2,4) = \mathbb{G}(1,3)$ of lines in $\mathbb{P}^3$ is a smooth quadric hypersurface in $\mathbb{P}^5$, given by the single homogeneous polynomial

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0.$$

3.2. **Chow Ring of $G(k,n)$.** Let $G = G(k,n)$, and set $r = n-k$. We know that G is smooth and projective, so its Chow group is a ring and can be graded by codimension. Now we think of elements of G as linear subspaces of $\mathbb{P}^{n-1}$. So, let us fix a flag

$$\mathcal{F}\colon\ F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} = \mathbb{P}^{n-1}.$$

Let us look at the set of k-tuples

$$\mathscr{A} = \{(a_1,\dots,a_k) \mid r \geq a_1 \geq \cdots \geq a_k \geq 0\}.$$

For all $a = (a_1,\dots,a_k) \in \mathscr{A}$, define the closed subset of G

$$\Sigma_a(\mathcal{F}) = \{\mathcal{H} \in G \mid \dim(\mathcal{H} \cap F_{r+i-1-a_i}) \geq i-1 \text{ for all } i = 1,\dots,k\}.$$

These are called the *Schubert cycles* on G. They have a number of interesting properties, for instance:

(1) $c_a = \operatorname{codim}(\Sigma_a, G) = \sum_{1\leq i\leq k} a_i$. Hence $\sigma_a = [\Sigma_a] \in A^{c_a}G$.
(2) By defining $a \leq b$ if and only if $a_i \leq b_i$ for all $i = 1,\dots,k$, one sees that $\Sigma_b \subseteq \Sigma_a \iff a \leq b$.
(3) The Schubert cell $\tilde{\Sigma}_a = \Sigma_a \setminus (\bigcup_{a<b} \Sigma_b) \cong \mathbb{A}^{k(n-k)-c_a}$ and $G = \coprod_{a\in\mathscr{A}} \tilde{\Sigma}_a$ is an affine stratification of G, with closed strata the Schubert cycles.

By (3), the cycle classes σ_a freely generate the Chow group A^*G.

3.3. **The example of $\mathbb{G}(1,3)$.** Let $G = \mathbb{G}(1,3)$, so that $r = n-k = 2$ and

$$\begin{aligned}\mathscr{A} &= \{(a_1,a_2) \mid 2 \geq a_1 \geq a_2 \geq 0\}\\ &= \{(2,2),(1,1),(0,0),(2,1),(2,0),(1,0)\}.\end{aligned}$$

After fixing a flag of linear subspaces

$$\mathcal{F}\colon\ \{P\} \subset M \subset H \subset \mathbb{P}^3,$$

where $\{P\} = F_0$, $M = F_1$, $H = F_2$, and recalling that the Schubert cycles are given by

$$\Sigma_{a_1a_2}(\mathcal{F}) = \{L \in G \mid \dim(L \cap F_{2-a_1}) \geq 0,\ \dim(L \cap F_{3-a_2}) \geq 1\},$$

we can write them all explicitly as follows:

$$\begin{aligned}
\Sigma_{22} &= \{L \in G \mid P \in L,\ \dim(L\cap M) = 1\} = \{M\}\\
\Sigma_{11} &= \{L \in G \mid L \cap M \neq \emptyset,\ L \subset H\} = \{L \in G \mid L \subset H\}\\
\Sigma_{00} &= \{L \in G \mid L \text{ meets a plane, and meets } \mathbb{P}^3 \text{ in a line}\} = G\\
\Sigma_{21} &= \{L \in G \mid P \in L \subset H\}\\
\Sigma_{20} &= \{L \in G \mid P \in L\}\\
\Sigma_{10} &= \{L \in G \mid L \cap M \neq \emptyset\}.
\end{aligned}$$

Remark 3.3.1. Of course, somewhere we used that L always intersects a plane in $\mathbb{P}^3$; also notice that two lines may not meet. Later on we will use that every L intersect a 3-plane in $\mathbb{P}^4$, and that two general 2-planes meet in a point.

Notation 3.3.2. Let us shorten Σ_{a_10} to Σ_{a_1}. Later, $\sigma_{a_1a_2}$ will denote $[\Sigma_{a_1a_2}] \in A^{a_1+a_2}G$.

In order to calculate the Schubert cells $\tilde{\Sigma}_{a_1a_2}$, it is useful to look at the following inclusions:

$$\begin{tikzcd}
 & & \Sigma_2 \arrow[dr, hook] & & \\
\Sigma_{22} \arrow[r, hook] & \Sigma_{21} \arrow[ur, hook] \arrow[dr, hook] & & \Sigma_1 \arrow[r, hook] & G \\
 & & \Sigma_{11} \arrow[ur, hook] & &
\end{tikzcd}$$

One can verify directly that $\tilde{\Sigma}_{a_1a_2} \cong \mathbb{A}^{4-(a_1+a_2)}$. Let us concentrate on the problem of determining the ring structure of A^*G. For the moment, we have the free abelian group decomposition

$$A^*G = \underbrace{\mathbb{Z}[\sigma_{22}]}_{A^4G} \oplus \underbrace{\mathbb{Z}[\sigma_{21}]}_{A^3G} \oplus \underbrace{\mathbb{Z}[\sigma_{11}] \oplus \mathbb{Z}[\sigma_2]}_{A^2G} \oplus \underbrace{\mathbb{Z}[\sigma_1]}_{A^1G} \oplus \underbrace{\mathbb{Z}[\sigma_0]}_{A^0G}.$$

In particular, any two points are linearly equivalent, and this is true in every Grassmannian.

Let us calculate the products in A^*G. It is crucial to work with two generically situated flags at the same time: so we will intersect a cycle taken from the first one with a cycle taken from the second one. These are generically transverse by Kleiman's Theorem (that we can apply because we are over $\mathbb{C}$). And the result only depends on the equivalence classes of the cycles we are intersecting, so what we find is the correct result. Now, let us fix two flags

$$\mathcal{F}\colon \ \{P\} \subset M \subset H \subset \mathbb{P}^3$$
$$\mathcal{F}'\colon \ \{P'\} \subset M' \subset H' \subset \mathbb{P}^3.$$

3.3.1. *Codimension* 4. We have to evaluate

$$\sigma_{11}^2,\ \sigma_2^2,\ \sigma_{11}\cdot\sigma_2,\ \sigma_1\cdot\sigma_{21} \in A^4G.$$

Let us start with the self-intersection σ_{11}^2. We have

$$|\Sigma_{11} \cap \Sigma'_{11}| = |\{L \in G \mid L \subset H \cap H'\}| = 1.$$

This unique line is of course $H \cap H'$. Hence

$$\sigma_{11}^2 = \sigma_{22}.$$

Similarly,

$$|\Sigma_2 \cap \Sigma'_2| = |\{L \in G \mid P \in L,\ P' \in L\}| = |\{\overline{PP'}\}| = 1.$$

Hence again

$$\sigma_2^2 = \sigma_{22}.$$

Since $P' \notin H$, we find

$$|\Sigma_{11} \cap \Sigma'_2| = |\{L \in G \mid P' \in L \subset H\}| = 0,$$

thus

$$\sigma_{11}\cdot\sigma_2 = 0.$$

The last calculation is

$$|\Sigma_1 \cap \Sigma'_{21}| = |\{L \in G \mid L \cap M \neq \emptyset,\ P' \in L \subset H'\}| = 1,$$

corresponding to this line: the one determined by P' and $M \cap H'$. Thus

$$\sigma_1\cdot\sigma_{21} = \sigma_{22}.$$

3.3.2. *Codimension* 3. We have to evaluate

$$\sigma_1 \cdot \sigma_2, \sigma_{11} \cdot \sigma_1 \in A^3 G.$$

We see that

$$\Sigma_1 \cap \Sigma_2' = \{ L \in G \mid L \cap M \neq \emptyset, P' \in L \} = \Sigma_{21}''$$

with respect to the flag $\mathcal{F}'' \colon \{P'\} \subset \ell \subset \langle P', M \rangle \subset \mathbb{P}^3$. Thus we get

$$\sigma_1 \cdot \sigma_2 = \sigma_{21}.$$

Similarly,

$$\Sigma_1 \cap \Sigma_{11} = \{ L \in G \mid L \cap M \neq \emptyset, L \subset H' \} = \Sigma_{21}''$$

with respect to the flag $\mathcal{F}'' \colon \{R\} \subset \ell \subset H' \subset \mathbb{P}^3$, where $R = M \cap H'$. Thus

$$\sigma_1 \cdot \sigma_{11} = \sigma_{21}.$$

3.3.3. *Codimension* 2. We have to evaluate $\sigma_1^2 \in A^2 G$. Here things get tricky because this product is not a Schubert cycle. What we know is that we can write $\sigma_1^2 = \alpha \sigma_{11} + \beta \sigma_2$ in $A^2 G$. We have to determine α and β. The strategy will be (now and in the future) to intersect both sides with cycles in complementary codimension so that one of the summands vanishes. Doing this twice allows us to recover α and β in two steps. So,

$$\sigma_1^2 \cdot \sigma_2 = (\alpha \sigma_{11} + \beta \sigma_2) \cdot \sigma_2$$

gives $(\sigma_{22} =) \sigma_1 \cdot \sigma_{21} = \beta \sigma_{22}$. Hence $\beta = 1$. In the same way,

$$\sigma_1^2 \cdot \sigma_{11} = (\alpha \sigma_{11} + \beta \sigma_2) \cdot \sigma_{11}$$

gives $(\sigma_{22} =) \sigma_1 \cdot \sigma_{21} = \alpha \sigma_{11}^2 = \alpha \sigma_{22}$. Hence $\alpha = 1$ and finally

$$\sigma_1^2 = \sigma_{11} + \sigma_2.$$

The next result has a precise enumerative meaning. It solves Problem (1) from the Introduction. We will also solve this problem via torus localisation in Section 9.4.

Proposition 3.3.3. *We have the identity*

$$\int_{G(2,4)} \sigma_1^4 = 2.$$

Proof. We can compute

$$\begin{aligned} \sigma_1^4 &= (\sigma_1^2)^2 \\ &= (\sigma_{11} + \sigma_2)^2 \\ &= \sigma_{11}^2 + 2\sigma_{11}\sigma_2 + \sigma_2^2 \\ &= \sigma_{22} + 0 + \sigma_{22}. \end{aligned}$$

The result follows from the fact that σ_{22} is the class of a point. □

4. Grassmann bundles, Quot, Hilb

In this section we introduce the three most important examples of *fine moduli spaces* used in Algebraic Geometry: Grassmannians, Quot schemes and Hilbert schemes. As we will see, both Grassmannians and Hilbert schemes can be recovered as special instances of Quot schemes.

The technical way to define fine moduli spaces is via representable functors $\mathfrak{M}\colon \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Sets}$. The notion of representability will be introduced in Section 4.1, for the sake of completeness. More details and examples can be found, for instance, in [62].

The basic idea is as follows. First of all, every scheme $\mathcal{M}$ trivially represents its own functor of points, which is the functor $\mathsf{h}_{\mathcal{M}}$ sending

$$U \mapsto \mathrm{Hom}_{\mathrm{Sch}}(U, \mathcal{M}).$$

One would say that $\mathcal{M}$ is a "fine moduli space of things" if the functor $\mathfrak{M}$ assigning to a scheme U the set of "families of things" defined over U is isomorphic to $\mathrm{Hom}_{\mathrm{Sch}}(-, \mathcal{M})$.

A fine moduli space is special in this sense: its points have a "label", just as the items of a phone book. We know precisely each point's name and address, so we can always find it on the moduli space (see Figure 5). This is the power of *universal families*.

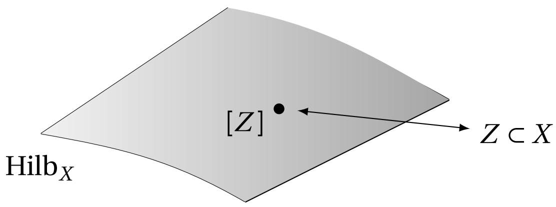


FIGURE 5. Each point of a fine moduli space (e.g. the Hilbert scheme) has a well precise label.

4.1. **Representable functors.** We start by making the following assumption.

Assumption 4.1.1. All categories are assumed to be *locally small*, i.e. we assume that $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is a set for any pair of objects x and y.

Let $\mathcal{C}$ and $\mathcal{C}'$ be (locally small) categories.

Definition 4.1.2. A (covariant) functor $\mathsf{F}\colon \mathcal{C} \to \mathcal{C}'$ is called:

(i) *fully faithful* if for any two objects $x, y \in \mathcal{C}$ the map of sets

$$\mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{C}'}(\mathsf{F}(x), \mathsf{F}(y))$$

is a bijection.

(ii) *essentially surjective* if every object of $\mathcal{C}'$ is isomorphic to an object of the form $\mathsf{F}(x)$ for some $x \in \mathcal{C}$.

The following observation is quite useful.

Remark 4.1.3. A fully faithful functor $\mathsf{F}\colon \mathcal{C} \to \mathcal{C}'$ induces an equivalence of $\mathcal{C}$ with the essential image of F, namely the full subcategory of $\mathcal{C}'$ consisting of objects isomorphic to objects of the form $\mathsf{F}(x)$ for some $x \in \mathcal{C}$. Put differently, a functor is an equivalence if and only if it is fully faithful and essentially surjective.

Definition 4.1.4. A *natural transformation* $\eta\colon \mathsf{F} \Rightarrow \mathsf{G}$ between two functors $\mathsf{F}, \mathsf{G}\colon \mathcal{C} \to \mathcal{C}'$ is the datum, for every $x \in \mathcal{C}$, of a morphism $\eta_x\colon \mathsf{F}(x) \to \mathsf{G}(x)$ in $\mathcal{C}'$, such that for every $f \in \mathrm{Hom}_{\mathcal{C}}(x_1, x_2)$ the diagram

$$\begin{tikzcd}
\mathsf{F}(x_1) \arrow[r, "\eta_{x_1}"] \arrow[d, "\mathsf{F}(f)"'] & \mathsf{G}(x_1) \arrow[d, "\mathsf{G}(f)"] \\
\mathsf{F}(x_2) \arrow[r, "\eta_{x_2}"] & \mathsf{G}(x_2)
\end{tikzcd}$$

is commutative in $\mathcal{C}'$.

Definition 4.1.5. Let $\mathcal{C}$, $\mathcal{C}'$ be two categories. Let $\mathrm{Fun}(\mathcal{C},\mathcal{C}')$ be the category whose objects are functors $\mathcal{C} \to \mathcal{C}'$ and whose morphisms are the natural transformations. An isomorphism in the category $\mathrm{Fun}(\mathcal{C},\mathcal{C}')$ is called a *natural isomorphism.*

Let $\mathcal{C}$ be a (locally small) category. Its *opposite category* $\mathcal{C}^{\mathrm{op}}$, by definition, has the same objects of $\mathcal{C}$, and its morphisms are

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(x, y) = \mathrm{Hom}_{\mathcal{C}}(y, x), \quad x, y \in \mathcal{C}.$$

Consider the category of contravariant functors $\mathcal{C} \to \mathrm{Sets}$, i.e. the category

$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}).$$

For every object x of $\mathcal{C}$ there is a functor $\mathsf{h}_x\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$ defined by

$$u \mapsto \mathsf{h}_x(u) = \mathrm{Hom}_{\mathcal{C}}(u, x), \quad u \in \mathcal{C}.$$

A morphism $\phi \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(u, v) = \mathrm{Hom}_{\mathcal{C}}(v, u)$ gets sent to the map of sets

$$\mathsf{h}_x(\phi)\colon \mathsf{h}_x(u) \to \mathsf{h}_x(v), \quad \alpha \mapsto \alpha \circ \phi.$$

Consider the functor

$$\mathsf{h}_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}), \quad x \mapsto \mathsf{h}_x. \tag{4.1.1}$$

This is, indeed, a functor: for every arrow $f\colon x \to y$ in $\mathcal{C}$ and object u of $\mathcal{C}$ we can define a map of sets

$$\mathsf{h}_f\, u\colon \mathsf{h}_x(u) \to \mathsf{h}_y(u), \quad \alpha \mapsto f \circ \alpha,$$

with the property that for every morphism $\phi\colon v \to u$ in $\mathcal{C}$ there is a commutative diagram

$$\begin{tikzcd}
\mathsf{h}_x(u) \arrow[r, "\mathsf{h}_f\, u"] \arrow[d, "\mathsf{h}_x(\phi)"'] & \mathsf{h}_y(u) \arrow[d, "\mathsf{h}_y(\phi)"] \\
\mathsf{h}_x(v) \arrow[r, "\mathsf{h}_f\, v"] & \mathsf{h}_y(v)
\end{tikzcd}
\qquad
\begin{tikzcd}
u \xrightarrow{\alpha} x \arrow[r, mapsto] \arrow[d, mapsto] & u \xrightarrow{f\circ\alpha} y \arrow[d, mapsto] \\
v \xrightarrow{\alpha\circ\phi} x \arrow[r, mapsto] & u \xrightarrow{f\circ\alpha\circ\phi} y
\end{tikzcd}$$

defining a natural transformation

$$\mathsf{h}_f\colon \mathsf{h}_x \Rightarrow \mathsf{h}_y.$$

Lemma 4.1.6 (Weak Yoneda). *The functor* $\mathsf{h}_{\mathcal{C}}$ *defined in* (4.1.1) *is fully faithful.*

Definition 4.1.7. A functor $\mathsf{F} \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets})$ is *representable* if it lies in the essential image of $\mathsf{h}_{\mathcal{C}}$, i.e. if it is isomorphic to a functor h_x for some $x \in \mathcal{C}$. In this case, we say that the object $x \in \mathcal{C}$ represents F.

Remark 4.1.8. By Lemma 4.1.6, if $x \in \mathcal{C}$ represents F, then x is unique up to a unique isomorphism. Indeed, suppose we have isomorphisms

$$a\colon \mathsf{h}_x \xrightarrow{\sim} F, \quad b\colon \mathsf{h}_y \xrightarrow{\sim} F$$

in the category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets})$. Then there exists a unique isomorphism $x \xrightarrow{\sim} y$ inducing $b^{-1} \circ a\colon \mathsf{h}_x \xrightarrow{\sim} \mathsf{h}_y$.

Let $\mathsf{F} \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets})$ be a functor, $x \in \mathcal{C}$ an object. One can construct a map of sets

$$g_x\colon \mathrm{Hom}(\mathsf{h}_x, \mathsf{F}) \to \mathsf{F}(x), \tag{4.1.2}$$

where the source is the hom-set in the category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets})$, which is indeed a set by Assumption 4.1.1.

To a natural transformation $\eta\colon \mathsf{h}_x \Rightarrow \mathsf{F}$ one can associate the element

$$g_x(\eta) = \eta_x(\mathrm{id}_x) \in \mathsf{F}(x),$$

the image of $\mathrm{id}_x \in \mathsf{h}_x(x)$ via the map $\eta_x\colon \mathsf{h}_x(x) \to \mathsf{F}(x)$.

Lemma 4.1.9 (Strong Yoneda). *Let* $\mathsf{F} \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets})$ *be a functor,* $x \in \mathcal{C}$ *an object. Then the map* g_x *defined in* (4.1.2) *is bijective.*

Proof. The inverse of g_x is the map that assigns to an element $\xi \in \mathsf{F}(x)$ the natural transformation $\eta(x,\xi)\colon \mathsf{h}_x \Rightarrow \mathsf{F}$ defined as follows. For a given object $u \in \mathcal{C}$, we define

$$\eta(x,\xi)_u\colon \mathsf{h}_x(u) \to \mathsf{F}(u)$$

by sending a morphism $f\colon u \to x$ to the image of ξ under $\mathsf{F}(f)\colon \mathsf{F}(x) \to \mathsf{F}(u)$. □

EXERCISE 4.1.10. Show that Lemma 4.1.9 implies Lemma 4.1.6.

Definition 4.1.11. Let $\mathsf{F}\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$ be a functor. A *universal object* for F is a pair (x,ξ) where $\xi \in \mathsf{F}(x)$, such that for every pair (u,σ) with $\sigma \in \mathsf{F}(u)$, there exists a unique morphism $\alpha\colon u \to x$ such that $\mathsf{F}(\alpha)\colon \mathsf{F}(x) \to \mathsf{F}(u)$ sends ξ to σ.

EXERCISE 4.1.12. Show that a pair (x,ξ) is a universal object for a functor F if and only if the natural transformation $\eta(x,\xi)$ defined in Lemma 4.1.9 is a natural isomorphism. In particular, F is representable if and only if it has a universal object.

4.2. Grassmannians.

Fix integers $0 < k \le n$, a Noetherian scheme S and a coherent sheaf F on S. Let Sch_S be the category of locally Noetherian schemes over S. The *Grassmann functor*

$$\mathsf{G}(k,F)\colon \mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{Sets}$$

is defined by

$$(U \xrightarrow{g} S) \mapsto \left\{ \begin{array}{c} \text{equivalence classes of surjections } g^*F \twoheadrightarrow Q \\ \text{in } \mathrm{Coh}(U) \text{ with } Q \text{ locally free of rank } n-k \end{array} \right\} \tag{4.2.1}$$

where two quotients $p\colon g^*F \twoheadrightarrow Q$ and $p'\colon g^*F \twoheadrightarrow Q'$ are considered equivalent if there exists an $\mathscr{O}_U$-linear isomorphism $v\colon Q \xrightarrow{\sim} Q'$ such that $p' = v\circ p$.

Remark 4.2.1. When F is locally free, $G(k,F)$ is called the *Grassmann bundle* associated to F. Note that the kernel of a surjection between locally free sheaves is automatically locally free. Hence in this case $G(k,F)$ parameterises k-dimensional linear subspaces in the fibres of $F \to S$.

EXERCISE 4.2.2. Show that two quotients $p\colon g^*F \twoheadrightarrow Q$ and $p'\colon g^*F \twoheadrightarrow Q'$ are equivalent if and only if $\ker p = \ker p'$.

The functor (4.2.1) can be represented by an S-scheme

$$\rho\colon G(k,F) \to S.$$

The proof is an application of the general result that a Zariski sheaf that can be covered by representable subfunctors is representable [58, Tag 01JF, Lemma 25.15.4].

Example 4.2.3. Let $k = n-1$ and $F = \mathscr{O}_S^{\oplus n}$. Then we get the relative projective space

$$\mathbb{P}^{n-1}_S \to S.$$

We do know from the functorial description of projective space [33, Ch. II, Thm. 7.1] that an S-morphism $U \to \mathbb{P}^{n-1}_S$ is equivalent to the data

$$(\mathscr{L}; s_0, s_1, \ldots, s_{n-1})$$

where $\mathscr{L}$ is a line bundle on U and s_i are sections generating $\mathscr{L}$ — and moreover such tuple is considered equivalent to $(\mathscr{L}'; s_0', s_1', \ldots, s_{n-1}')$ if and only if there is an isomorphism of line bundles $\phi\colon \mathscr{L} \xrightarrow{\sim} \mathscr{L}'$ such that $\phi^* s_i' = s_i$. But this is precisely a U-valued point of $\mathsf{G}(n-1, \mathscr{O}_S^{\oplus n})$. Indeed, the functor prescribes the assignment of a surjection

$$\mathscr{O}_U^{\oplus n} \twoheadrightarrow \mathscr{L}$$

with $\mathscr{L}$ a line bundle. The equivalence class of this surjection is the same data as n generating sections of $\mathscr{L}$ up to isomorphism.

Example 4.2.4. If $S = \operatorname{Spec}\mathbb{C}$, we recover the usual Grassmannian

$$G(k,n) = G(k,\mathbb{C}^n) = \mathbb{G}(k-1, n-1)$$

of k-planes in $\mathbb{C}^n$ (or, equivalently, of projective linear subspaces $\mathbb{P}^{k-1} \hookrightarrow \mathbb{P}^{n-1}$), a smooth projective algebraic variety of dimension $k(n-k)$. When $k = n-1$ we obtain $G(n-1,n) = \mathbb{P}^{n-1}$.

By definition, representability of $\mathsf{G}(k,F)$ means that for every $g\colon U \to S$ there is a functorial bijection

$$\mathsf{G}(k,F)(g) \xrightarrow{\sim} \operatorname{Hom}_S(U, G(k,F)), \quad \alpha \mapsto \alpha_g. \tag{4.2.2}$$

Now take $U = G(k,F)$, $g = \rho$, and consider

$$\operatorname{id}_{G(k,F)} \in \operatorname{Hom}_S(G(k,F), G(k,F)).$$

The element in $\mathsf{G}(k,F)(\rho)$ mapping to $\mathrm{id}_{G(k,F)}$ via (4.2.2) is the *tautological exact sequence*

$$0 \to \mathscr{S} \to \rho^* F \to \mathscr{Q} \to 0 \tag{4.2.3}$$

over $G(k,F)$. Note that if F is locally free then $\mathscr{S}$ is locally free of rank k. The sequence (4.2.3) is called 'tautological' because of the following universal property: if $g\colon U \to S$ is any morphism and $\alpha \in \mathsf{G}(k,F)(g)$, then the equivalence class of the pullback surjection

$$\alpha_g^* \rho^* F \twoheadrightarrow \alpha_g^* \mathscr{Q}$$

coincides with α.

Example 4.2.5. Let $F = \mathscr{O}_S^{\oplus n}$ be a free sheaf of rank n, and set $k = n-1$. Then we saw that

$$G(n-1,F) = \mathbb{P}_S^{n-1} = \operatorname{Proj}\operatorname{Sym}\mathscr{O}_S^{\oplus n},$$

and the tautological surjection is the familiar

$$\mathscr{O}_{\mathbb{P}_S^{n-1}}^{\oplus n} \twoheadrightarrow \mathscr{O}_{\mathbb{P}_S^{n-1}}(1).$$

EXERCISE 4.2.6. Let $S = \operatorname{Spec}\mathbb{C}$, and fix a point $[\Lambda] \in G(k,F)$. Show that the tangent space of $G(k,F)$ at $[\Lambda]$ is isomorphic to

$$\operatorname{Hom}_{\mathbb{C}}(\Lambda, F/\Lambda).$$

On $S = \operatorname{Spec}\mathbb{Z}$, the Grassmann bundle

$$\rho\colon G(k, \mathscr{O}_S^{\oplus n}) \to S$$

is *proper*. Moreover, there is a closed embedding

$$G(k,\mathscr{O}_S^{\oplus n}) \hookrightarrow \mathbb{P}_{\mathbb{Z}}^{N-1}, \quad N = \binom{n}{k}.$$

For general (Noetherian) scheme S and locally free sheaf F, the determinant

$$\mathscr{L} = \det \mathscr{Q}$$

of the universal quotient bundle is relatively very ample on $\rho\colon G(k,F) \to S$, so it gives a closed embedding

$$G(k,F) \hookrightarrow \mathbb{P}(\rho_* \mathscr{L}) \hookrightarrow \mathbb{P}\left(\bigwedge^k F\right),$$

called the *Plücker embedding*.

4.3. **Quot and Hilbert schemes.** Let S be a Noetherian scheme and let $X \to S$ be a finite type morphism (so X is Noetherian by Exercise 2.1.1). Fix a coherent sheaf F on X. Denote by Sch_S the category of locally Noetherian schemes over S. Given such a scheme $U \to S$, define

$$\mathsf{Quot}_{X/S}(F)(U \to S)$$

to be the set of equivalence classes of pairs

$$(\mathscr{E}, p)$$

where

- $\mathscr{E}$ is a coherent sheaf on $X \times_S U$, flat over U and with proper support over U,

- $p\colon F_U \twoheadrightarrow \mathscr{E}$ is an $\mathscr{O}_{X\times_S U}$-linear surjection, where F_U is the pullback of F along $X\times_S U \to X$, and finally
- two pairs $(\mathscr{E},p)$ and $(\mathscr{E}',p')$ are considered equivalent of $\ker\theta=\ker\theta'$.

EXERCISE 4.3.1. Show that $\mathsf{Quot}_{X/S}(F)$ defines a functor $\mathrm{Sch}_S^{\mathrm{op}}\to\mathrm{Sets}$, and that it generalises the Grassmann functor $\mathsf{G}(k,F)$ defined in (4.2.1).

Let k be a field. Fix a line bundle L over a k-scheme X. For a coherent sheaf E on X whose support is proper over k, the function

$$m\mapsto \chi(E\otimes_{\mathscr{O}_X} L^{\otimes m})$$

becomes polynomial for $m\gg 0$. It is called the *Hilbert polynomial* of E (with respect to L), and is denoted $P_L(E)$. If $\mathscr{E}$ is a flat family of coherent sheaves on $X\to S$, such that

$$\operatorname{Supp}\mathscr{E}\subset X\to S$$

is proper, then the function

$$s\mapsto P_L(\mathscr{E}_s)$$

is locally constant on S.

EXERCISE 4.3.2. Let $C\subset\mathbb{P}^n$ be a smooth curve of degree d and genus g. Compute the Hilbert polynomial of C with respect to $L=\mathscr{O}_{\mathbb{P}^n}(1)$.

Remark 4.3.3. It is not true that for fixed n there always exists a smooth curve $C\subset\mathbb{P}^n$ of degree d and genus g.

EXERCISE 4.3.4. What is the Hilbert polynomial of a conic in $\mathbb{P}^3$? What about a twisted cubic $C\subset\mathbb{P}^3$?

EXERCISE 4.3.5. Compute the Hilbert polynomial $P_{d,n}$ of a degree d hypersurface $Y\subset\mathbb{P}^n$. Show that there is a bijective morphism

$$\mathbb{P}^{N-1}\to \mathrm{Hilb}_{\mathbb{P}^n}^{P_{d,n},\mathscr{O}(1)},\qquad N=\binom{n+d}{d}.$$

EXERCISE 4.3.6. Interpret the Grassmannian

$$\mathbb{G}(k,n)=\left\{\text{linear subvarieties } \mathbb{P}^k\hookrightarrow\mathbb{P}^n\right\}$$

as a Hilbert scheme, i.e. find the unique polynomial P such that $\mathbb{G}(k,n)=\mathrm{Hilb}_{\mathbb{P}^n}^{P,\mathscr{O}(1)}$.

The functor $\mathsf{Quot}_{X/S}(F)$ decomposes as a coproduct

$$\mathsf{Quot}_{X/S}(F)=\coprod_{P\in\mathbb{Q}[z]}\mathsf{Quot}_{X/S}^{P,L}(F)$$

where the component $\mathsf{Quot}_{X/S}^{P,L}(F)$ sends an S-scheme U to the set of equivalence classes of quotients $p\colon F_U\twoheadrightarrow\mathscr{E}$ such that for each $u\in U$ the Hilbert polynomial of $\mathscr{E}_u=\mathscr{E}|_{X_u}$ (whose support is a closed subscheme of X_u proper over $k(u)$ by definition!), calculated with respect to L_u (the pullback of L along $X_u\hookrightarrow X\times_S U\to X$), is equal to P.

Theorem 4.3.7 (Grothendieck [32])**.** *If $X \to S$ is projective, F is a coherent sheaf on X, L is a relatively very ample line bundle over X and $P \in \mathbb{Q}[z]$ is a polynomial, then the functor $\mathrm{Quot}_{X/S}^{P,L}(F)$ is representable by a projective S-scheme*

$$\mathrm{Quot}_{X/S}^{P,L}(F) \to S.$$

Remark 4.3.8. There are several notions of projectivity for a morphism $X \to S$. If S has an ample line bundle (e.g. when it is quasi-projective over an affine scheme), then these notions are all equivalent, see [21, Part 2, § 5.5.1]. Grothendieck's original definition [31, Def. 5.5.2], in general different from the one in [33, II, § 4], stated that $X \to S$ is *projective* if it factors as

$$X \xhookrightarrow{i} \mathbb{P}(E) \to S$$

where E is a coherent $\mathscr{O}_S$-module and i is a closed immersion. This can be rephrased by saying that $X \to S$ is proper and there exists an ample family of line bundles on X over S. This is the notion used in Theorem 4.3.7. Moreover, $X \to S$ is called *quasi-projective* if it factors as $X \hookrightarrow Y \to S$, with $X \hookrightarrow Y$ open and $Y \to S$ projective.

Remark 4.3.9. The Noetherian hypothesis in Theorem 4.3.7 could be removed by Altman and Kleiman [2], but they needed a stronger notion of (quasi-)projectivity, as well as a stronger assumption on F. The result is a (quasi-)projective S-scheme $\mathrm{Quot}_{X/S}^{P,L}(F) \to S$. As a consequence, one obtains the following: when $X \hookrightarrow \mathbb{P}_S^n$ is a closed subscheme, $L = \mathscr{O}_{\mathbb{P}_S^n}(1)|_X$ and F is a sheaf quotient of $L(m)^{\oplus \ell}$, the functor $\mathrm{Quot}_{X/S}^{P,L}(F)$ is representable by a scheme that can be embedded in $\mathbb{P}_S^N$ for some N.

Definition 4.3.10. Let $X \to S$ be a projective morphism, and set $F = \mathscr{O}_X$. Then

$$\mathrm{Hilb}_{X/S} = \mathrm{Quot}_{X/S}(\mathscr{O}_X)$$

is called the *Hilbert scheme* of X/S. When $S = \operatorname{Spec} k$, we omit it from the notation.

Definition 4.3.11. Let X be a quasi-projective k-scheme, and let n be an integer. The *Hilbert scheme of n points* on X is the component

$$\mathrm{Hilb}^n X \subset \mathrm{Hilb}_X$$

corresponding to the constant Hilbert polynomial $P = n$. Similarly, we let

$$\mathrm{Quot}_X(F,n) \subset \mathrm{Quot}_X(F)$$

denote the connected component parameterising quotients $F \twoheadrightarrow Q$ where Q is a finite dimensional sheaf of length n.

See Sections 6 and 10 for more information on $\mathrm{Hilb}^n X$. We will give an alternative definition of $\mathrm{Hilb}^n \mathbb{A}^d$ in Section 6.2.

A theorem of Vakil [60] asserts, roughly speaking, that arbitrarily bad singularities appear generically on some components of some Hilbert scheme.

However, despite its potentially horrible singularities, the Hilbert scheme has the great feature of representing a pretty explicit functor, so its functor of points is not that mysterious. In such a situation, the most important thing is to always keep in

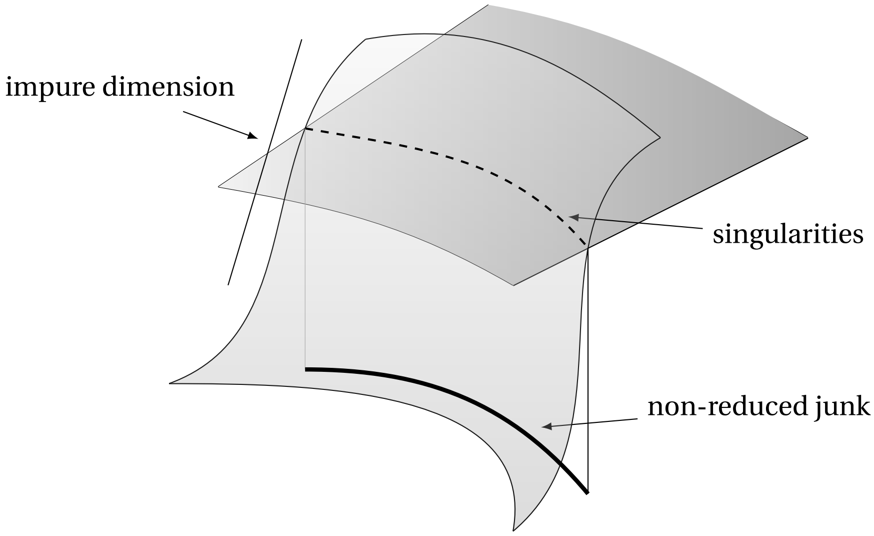


FIGURE 6. A nasty scheme. By Murphy's Law [60], it could be a Hilbert scheme component $H \subset \mathrm{Hilb}_X$ for some variety X.

mind the *universal family* living over the representing scheme. In the case of the Hilbert scheme, this is a diagram

$$\begin{tikzcd} \mathcal{Z} \arrow[r, hook] \arrow[d, "\text{flat}"'] & X \times_S \mathrm{Hilb}_{X/S} \\ \mathrm{Hilb}_{X/S} & \end{tikzcd}$$

with the following property: for every S-scheme $g: U \to S$ along with a flat family of closed subschemes

$$\alpha: Z \subset X \times_S U \to U,$$

there exists precisely one S-morphism $\alpha_g: U \to \mathrm{Hilb}_{X/S}$ such that $Z = \alpha_g^* \mathcal{Z}$ as U-families of subschemes of X.

EXERCISE 4.3.12. Show that $\mathrm{Hilb}^1 X = X$. What is the universal family?

EXERCISE 4.3.13. Let C be a smooth curve embedded in a smooth 3-fold Y. Show that $\mathrm{Bl}_C Y \cong \mathrm{Quot}_Y(\mathscr{I}_C, 1)$.

4.4. **Tangent space to Quot.** Let X be a smooth projective variety over a field k. Let F be a coherent sheaf on Y. The Quot scheme

$$\mathrm{Quot}_X(F),$$

at a point $[F \twoheadrightarrow Q]$, has tangent space canonically isomorphic to

$$\mathrm{Hom}_{\mathscr{O}_X}(K, Q), \tag{4.4.1}$$

where $K = \ker(F \twoheadrightarrow Q)$. We already know this for the Grassmannian $G(k, n)$ by Exercise 4.2.6.

The case of the Hilbert scheme (i.e. when $F = \mathscr{O}_X$) is as follows. Let $p \in \mathrm{Hilb}_X$ be the point corresponding to a subscheme $Z \subset X$. Then, by definition,

$$T_p \mathrm{Hilb}_X = \mathrm{Hom}_p(\mathrm{Spec}\, k[t]/t^2, \mathrm{Hilb}_X),$$

and this is the set of all flat families

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Z} \arrow[r, hook] \arrow[d, "q"] & X \times_k D_2 \\ 0 \arrow[r, hook] & D_2 & \end{tikzcd}$$

such that the fibre of q over the closed point of $D_2 = \operatorname{Spec} k[t]/t^2$ equals Z. By definition, these are the *infinitesimal deformations* of the closed subscheme $Z \subset X$. It is shown in [34, Thm. 2.4] that these are classified by

$$\begin{aligned} \operatorname{Hom}_X(\mathscr{I}_Z, \mathscr{O}_Z) &= \operatorname{Hom}_Z(\mathscr{I}_Z/\mathscr{I}_Z^2, \mathscr{O}_Z) \\ &= H^0(Z, \mathscr{H}om(\mathscr{I}_Z/\mathscr{I}_Z^2, \mathscr{O}_Z)) \\ &= H^0(Z, N_{Z/X}), \end{aligned}$$

where $N_{Z/X}$ is the normal sheaf to Z in X.

EXERCISE 4.4.1. Show that $\operatorname{Quot}_{\mathbb{A}^3}(\mathscr{O}_{\mathbb{A}^3}^r, 1)$ is smooth of dimension $r+2$. Show that $\operatorname{Quot}_{\mathbb{A}^3}(\mathscr{O}_{\mathbb{A}^3}^r, r)$ is singular for all $r > 1$.

EXERCISE 4.4.2. Let $L \subset \mathbb{A}^3$ be a line. Compute the dimension of $\operatorname{Quot}_{\mathbb{A}^3}(\mathscr{I}_L, 2)$. Show that this Quot scheme is singular.

4.5. **Examples of Hilbert schemes.** The Hilbert scheme of points, i.e. the Hilbert scheme of zero-dimensional subschemes of a quasi-projective variety X, will be treated in later sections.

4.5.1. *Plane conics.* Let z_0, z_1 and z_2 be homogeneous coordinates on $\mathbb{P}^2$, and $\alpha_0, \ldots, \alpha_5$ be homogeneous coordinates on $\mathbb{P}^5$. Consider the closed subscheme

$$\mathcal{C} \subset \mathbb{P}^2 \times \mathbb{P}^5$$

cut out by the equation

$$\alpha_0 z_0^2 + \alpha_1 z_1^2 + \alpha_2 z_2^2 + \alpha_3 z_0 z_1 + \alpha_4 z_0 z_2 + \alpha_5 z_1 z_2 = 0.$$

Let π be the projection $\mathcal{C} \to \mathbb{P}^5$. Over a point $a = (a_0, \ldots, a_5) \in \mathbb{P}^5$, the fibre is the conic

$$\pi^{-1}(a) = \left\{ a_0 z_0^2 + a_1 z_1^2 + a_2 z_2^2 + a_3 z_0 z_1 + a_4 z_0 z_2 + a_5 z_1 z_2 = 0 \right\} \subset \mathbb{P}^2.$$

There is a set-theoretic bijection between $\mathbb{P}^5$ and $\operatorname{Hilb}_{\mathbb{P}^2}^{2t+1}$. By the universal property of projective space, we have the scheme-theoretic identity

$$\mathbb{P}^5 = \operatorname{Hilb}_{\mathbb{P}^2}^{2t+1},$$

and the map $\pi \colon \mathcal{C} \to \mathbb{P}^5$ is the universal family of the Hilbert scheme of plane conics.

EXERCISE 4.5.1. Make the last step precise and generalise the plane conics example to arbitrary hypersurfaces of $\mathbb{P}^n$. (**Hint**: Start out with the conclusion of Exercise 4.3.5 to write down the universal family).

Remark 4.5.2. Let X be a projective scheme. The universal family of the Hilbert scheme is always, *set-theoretically*, equal to

$$\mathcal{Z} = \{ (x, [Z]) \in X \times \operatorname{Hilb}_X \mid x \in Z \} \subset X \times \operatorname{Hilb}_X .$$

The problem is to determine the scheme structure on $\mathcal{Z}$. In the case of hypersurfaces of degree d in $\mathbb{P}^n$ (Exercise 4.5.1) this was easy precisely because $\mathcal{Z}$ is itself a hypersurface.

4.5.2. *Twisted cubics.* A twisted cubic is a smooth rational curve obtained as the image of the morphism

$$\mathbb{P}^1 \hookrightarrow \mathbb{P}^3, \quad (u,v) \mapsto (u^3, u^2v, uv^2, v^3),$$

up to linear changes of coordinates of the codomain. The number of moduli of a twisted cubic is 12. Indeed, one has to specify four linearly independent degree 3 polynomials in two variables, up to $\mathbb{C}^\times$-scaling and automorphisms of $\mathbb{P}^1$. One then computes

$$4 \cdot h^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(3)) - 1 - \dim \mathrm{PGL}_2 = 16 - 1 - 3 = 12.$$

The Hilbert polynomial of a twisted cubic is $3t+1$, cf. Exercise 4.3.2. There are other 1-dimensional subschemes $Z \subset \mathbb{P}^3$ with this Hilbert polynomial, e.g. a plane cubic union a point. This has 15 moduli: the choice of a plane $\mathbb{P}^2 \subset \mathbb{P}^3$ contributes $3 = \dim \mathbb{G}(2,3)$ moduli, a plane cubic $C \subset \mathbb{P}^2$ contributes 9 parameters, and the choice of a point $p \in \mathbb{P}^3$ accounts for the remaining three moduli.

The Hilbert scheme

$$\mathrm{Hilb}^{3t+1}_{\mathbb{P}^3}$$

was completely described in [52]. The two irreducible components we just described turn out to be the only ones. They are smooth, rational, of dimension 12 and 15 respectively, and they intersect along a smooth, rational 11-dimensional subvariety $V \subset \mathrm{Hilb}^{3t+1}_{\mathbb{P}^3}$ parameterising uninodal plane cubics with an embedded point at the node. In [33, § III, Ex. 9.8.4] a family of twisted cubics degenerating to a plane uninodal cubic with an embedded point is described. The total space of the family, in a local chart, is defined by the ideal

$$I = (a^2(x+1) - z^2, ax(x+1) - yz, xz - ay, y^2 - x^2(x+1)) \subset \mathbb{C}[a,x,y,z].$$

Letting $a = 0$ one obtains the special fibre given by

$$I_0 = (z^2, yz, xz, y^2 - x^2(x+1)) \subset \mathbb{C}[x,y,z],$$

and $p = (0,0,0)$ is a non-reduced point in $C_0 = \operatorname{Spec} \mathbb{C}[x,y,z]/I_0$. Note that C_0 is not scheme-theoretically contained in the plane $z = 0$, because the local ring $\mathscr{O}_{C_0,p}$ contains the nonzero nilpotent z (cf. Remark 2.3.6).

Remark 4.5.3. The *geometric genus* $p_g(X) = h^0(X, \omega_X)$ varies in flat families, as the twisted cubic example shows.

4.6. **A comment on fine moduli spaces and automorphisms.** Given a scheme S and a functor $\mathfrak{M} \colon \mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{Sets}$, an object $\mathcal{M}$ in Sch_S along with an isomorphism

$$\mathfrak{M} \cong \mathrm{Hom}_{\mathrm{Sch}_S}(-, \mathcal{M})$$

is a *fine moduli space* for the objects parameterised by $\mathfrak{M}$. It is common to hear that when the objects $\eta \in \mathfrak{M}(U/S)$ have automorphisms, the functor $\mathfrak{M}$ cannot be represented. This is, for instance, the case for the moduli functor of smooth (or stable) curves of genus g. Even though this is the correct *geometric* intuition to have, for a general functor the presence of automorphisms does not necessarily prevent the existence of a universal family, as the following exercise indicates.

EXERCISE 4.6.1. Construct the functor $\mathfrak{M}\colon \mathrm{Sets}^{\mathrm{op}} \to \mathrm{Sets}$ of isomorphism classes of finite sets. Show that it is representable (by what set?), even though every finite set has automorphisms.

In geometric situations, the presence of automorphisms prevents representability whenever one can construct a family of objects $\eta \in \mathfrak{M}(U/S)$ that is isotrivial but not globally trivial. This is for instance the case for families of curves: the moduli map $U \to \mathcal{M}_g$ associated to an isotrivial family $\mathcal{X} \to U$ of smooth curves of genus g, say with typical fibre C, would have to be constant for continuity reasons; but the same is of course happening for the trivial family $C \times U \to U$, so the functor of smooth curves of genus g cannot be represented.

5. LINES ON HYPERSURFACES: EXPECTATIONS

Let $Y \subset \mathbb{P}^n$ be a general hypersurface of degree d. We want to show the following:

We should expect a finite number of lines on Y if and only if $d = 2n-3$.
We should expect *no lines* on Y if $d > 2n-3$.
We should expect infinitely many lines on Y if $d < 2n-3$.

To understand the condition

$$\ell \subset Y$$

for $\ell \subset \mathbb{P}^n$ and a hypersurface $Y \subset \mathbb{P}^n$, we give the following concrete example.

Example 5.0.1. Let $\ell \subset \mathbb{P}^3$ be the line cut out by $L_1 = L_2 = 0$, where $L_i = L_i(z_0, z_1, z_2, z_3)$ are linear forms on $\mathbb{P}^3$. To fix ideas, set $L_1 = z_0$ and $L_2 = z_0 + z_2 + z_3$. Let $Y \subset \mathbb{P}^3$ be defined by a homogeneous equation $f = 0$, for instance the cubic polynomial

$$f = z_0^3 + 3z_0 z_1^2 - z_2^2 z_3.$$

Then we see that plugging $L_1 = L_2 = 0$ into f does not give zero, i.e.

$$f|_\ell = 0 + 0 - z_2^2(-z_0 - z_2) = z_2^3.$$

This means that ℓ is not contained in Y. On the other hand, the line cut out by L_1 and $L_2' = z_3$ lies entirely on Y.

Let $Y \subset \mathbb{P}^n$ be the zero locus of a general homogeneous polynomial

$$f \in H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)).$$

As we anticipated in Example 5.0.1, a line $\ell \subset \mathbb{P}^n$ is contained in Y if and only if $f|_\ell = 0$. This condition can be rephrased by saying that the image of f under the restriction map

$$\mathsf{res}_\ell \colon H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)) \to H^0(\ell, \mathscr{O}_\ell(d)) \tag{5.0.1}$$

vanishes. We want to determine when we should expect Y to contain a finite number of lines. We set, informally,

$$\mathsf{N}_1(Y) = \text{expected number of lines in } Y.$$

Let us consider the Grassmannian

$$\mathbb{G}=\mathbb{G}(1,n)=\left\{\text{Lines } \ell\subset\mathbb{P}^n\right\},$$

a smooth complex projective variety of dimension $2n-2$. Recall the universal structures living on $\mathbb{G}$. First of all, the tautological exact sequence

$$\begin{tikzcd} & \text{rank } 2 \arrow[d, dash] & \text{rank } n+1 \arrow[d, dash] & \text{rank } n-1 \arrow[d, dash] & \\ 0 \arrow[r] & \mathscr{S} \arrow[r] & \mathscr{O}_{\mathbb{G}}\otimes_{\mathbb{C}} H^0(\mathbb{P}^n,\mathscr{O}_{\mathbb{P}^n}(1))^\vee \arrow[r] & \mathscr{Q} \arrow[r] & 0 \end{tikzcd}$$

where the fibre of $\mathscr{S}$ over a point $[\ell]\in\mathbb{G}$ is the 2-dimensional vector space $H^0(\ell,\mathscr{O}_\ell(1))^\vee$. Let, also,

$$\mathcal{L}=\left\{(p,[\ell])\in\mathbb{P}^n\times\mathbb{G}\,\middle|\,p\in\ell\right\}\subset\mathbb{P}^n\times\mathbb{G}$$

be the universal line. Consider the two projections

$$\begin{tikzcd} \mathcal{L} \arrow[r, "q"] \arrow[d, "\pi"'] & \mathbb{P}^n \\ \mathbb{G} & \end{tikzcd}$$

and the coherent sheaf

$$\mathscr{E}_d=\pi_*q^*\mathscr{O}_{\mathbb{P}^n}(d).$$

EXERCISE 5.0.2. Show that $\mathscr{E}_d$ is locally free of rank $d+1$. (**Hint**: use cohomology and base change, e.g. [19, Theorem B.5]).

In fact, one has an isomorphism of locally free sheaves

$$\mathscr{E}_d\cong\operatorname{Sym}^d\mathscr{S}^\vee,$$

where $\iota\colon\mathscr{S}\hookrightarrow\mathscr{O}_{\mathbb{G}}\otimes_{\mathbb{C}}H^0(\mathbb{P}^n,\mathscr{O}_{\mathbb{P}^n}(1))^\vee$ is the universal subbundle. Dualising ι and applying Sym^d, we obtain a surjection

$$\mathscr{O}_{\mathbb{G}}\otimes_{\mathbb{C}}H^0(\mathbb{P}^n,\mathscr{O}_{\mathbb{P}^n}(d))\twoheadrightarrow\operatorname{Sym}^d\mathscr{S}^\vee,$$

which is just a global version of (5.0.1). The association

$$\mathbb{G}\ni[\ell]\mapsto f|_\ell\in H^0(\ell,\mathscr{O}_\ell(d))\cong\operatorname{Sym}^d H^0(\ell,\mathscr{O}_\ell(1))$$

defines a section τ_f of $\mathscr{E}_d\to\mathbb{G}$. The zero locus of $\tau_f=\pi_*q^*f$ is the locus of lines contained in Y.

The following terminology is very common.

Definition 5.0.3. Let $Y\subset\mathbb{P}^n$ be a hypersurface defined by $f=0$. Then

$$F_1(Y)=Z(\tau_f)$$

is called the *Fano scheme of lines* in Y.

Since f is generic, $\tau_f\in\Gamma(\mathbb{G},\mathscr{E}_d)$ is also generic. In this case, the fundamental class of the Fano scheme of lines in Y is Poincaré dual to the Euler class

$$e(\mathscr{E}_d)\in A^{d+1}\mathbb{G}.$$

Thus $[F_1(Y)]\in A_*\mathbb{G}$ is a zero-cycle if and only if $d+1=2n-2$, i.e.

$$d=2n-3.$$

The degree of this zero-cycle is then

$$\mathsf{N}_1(Y)=\int_{\mathbb{G}} e(\mathscr{E}_d)=\int_{\mathbb{G}} c_{d+1}(\operatorname{Sym}^d \mathscr{S}^\vee).$$

This degree is the *actual* number of lines on Y whenever $H^0(\ell,N_{\ell/Y})=0$ for all $\ell\subset Y$. This condition means that the Fano scheme is reduced at all its points $[\ell]$, since $H^0(\ell,N_{\ell/Y})$ is its tangent space at the point $[\ell]$.

Lemma 5.0.4. *If $S\subset\mathbb{P}^3$ is a smooth cubic surface and $\ell\subset S$ is a line, then $H^0(\ell,N_{\ell/S})=0$.*

Proof. It has enough to show that $N=N_{\ell/S}$, viewed as a line bundle on $\ell\cong\mathbb{P}^1$, has negative degree. By the adjunction formula,

$$K_\ell = K_S|_\ell\otimes_{\mathscr{O}_\ell} N.$$

Using that $K_\ell=\mathscr{O}_\ell(-2)$ and $K_S=K_{\mathbb{P}^3}|_S\otimes_{\mathscr{O}_S} N_{S/\mathbb{P}^3}=\mathscr{O}_S(d-4)$ for a surface of degree d in $\mathbb{P}^3$, by taking degrees we obtain

$$-2=(3-4)+\deg N,$$

so that $\deg N=-1<0$. □

EXERCISE 5.0.5. Let $Y\subset\mathbb{P}^n$ be a general hypersurface of degree $d\le 2n-3$. Show that $F_i(Y)\subset\mathbb{G}(1,n)$ is smooth of dimension $2n-3-d$.

6. THE HILBERT SCHEME OF POINTS

6.1. **Subschemes and zero-cycles.** Let X be a complex quasi-projective variety. In Section 4.3 we encountered the Hilbert scheme of points X. Recall that, as a set,

$$\operatorname{Hilb}^n X$$

parameterises the finite subschemes $Z\subset X$ of length n. Or, equivalently, ideal sheaves $\mathscr{I}_Z\subset\mathscr{O}_X$ of *colength n*.

There is a "coarser" way of parameterising *points with multiplicity* on the variety X. This is the content of the next definition.

Definition 6.1.1. Let X be a quasi-projective variety. The n-th *symmetric product* (or *configuration space*) of X is the quotient

$$\operatorname{Sym}^n X = X^n/\mathfrak{S}_n.$$

Remark 6.1.2. The quotient $\operatorname{Sym}^n X$ is the *Chow scheme* of effective zero-cycles (of degree n) on X. For higher dimensional cycles, the definition (and representability) of the Chow functor is a much subtler problem [55].

Each point $\xi\in\operatorname{Sym}^n X$ corresponds to a finite combination of points with multiplicity, i.e. it can be written as

$$\xi=\sum_i m_i\cdot p_i,$$

with $m_i\in\mathbb{Z}_{\ge 0}$ and $p_i\in X$.

EXERCISE 6.1.3. Let X be a smooth variety of dimension d. Show that the locus in $\operatorname{Hilb}^2 X$ of nonreduced subschemes $Z \subset X$ is isomorphic to $X \times \mathbb{P}^{d-1}$.

Note that the symmetric product $\operatorname{Sym}^n X$ does not deal with with the scheme structure of fat points inside X. For instance, any of the double point schemes supported on a given point $p \in X$, parameterised by the $\mathbb{P}^{d-1}$ of Exercise 6.1.3, has underlying cycle $2 \cdot p$. The operation of "forgetting the scheme structure" can be made functorial. This means that there exists a well-defined algebraic morphism

$$\pi_X \colon \operatorname{Hilb}^n X \to \operatorname{Sym}^n X, \tag{6.1.1}$$

taking a subscheme $Z \subset X$ to its underlying effective zero-cycle. In symbols,

$$\pi_X[Z] = \sum_{p \in \operatorname{Supp} Z} \operatorname{length} \mathscr{O}_{Z,p} \cdot p.$$

The morphism π_X is called the *Hilbert–Chow morphism*.

EXERCISE 6.1.4. Show that if C is a smooth quasi-projective curve then $\operatorname{Hilb}^n C \cong \operatorname{Sym}^n C$ via π_C. Deduce that $\operatorname{Hilb}^n \mathbb{A}^1 \cong \mathbb{A}^n$.

The easiest subvariety of the symmetric product is the *small diagonal*, which is just a copy of X embedded as

$$X \hookrightarrow \operatorname{Sym}^n X, \quad x \mapsto n \cdot x.$$

Definition 6.1.5. Let X be a smooth quasi-projective variety. The *punctual Hilbert scheme* is the closed subscheme

$$\operatorname{Hilb}^n(X)_x \subset \operatorname{Hilb}^n X$$

defined as the preimage of the cycle $n \cdot x$ via the Hilbert–Chow map (6.1.1).

EXERCISE 6.1.6. Show that that $\pi_X^{-1}(n \cdot x)$ does not depend on $x \in X$. Show that it does not depend on X either, but only on $\dim X$, as long as X is smooth.

Notation 6.1.7. If X is a smooth variety of dimension d, we will denote by

$$\operatorname{Hilb}^n(\mathbb{A}^d)_0 \subset \operatorname{Hilb}^n X$$

the punctual Hilbert scheme of Definition 6.1.5. This makes sense by Exercise 6.1.6.

EXERCISE 6.1.8. Let X be a smooth variety. Show that $\operatorname{Hilb}^2 X$ is isomorphic to the blowup of $\operatorname{Sym}^2 X$ along the diagonal.

EXERCISE 6.1.9. Show that if X is a smooth variety and $n \le 3$ then $\operatorname{Hilb}^n X$ is smooth. (**Hint**: show that a finite planar subscheme $Z \subset X$ defines a smooth point of the Hilbert scheme. Then use your classification from Exercise 2.1.10 to conclude).

6.2. **The Hilbert scheme of points on affine space.** Let $d \ge 1$ and $n \ge 0$ be integers. In this subsection we give a description of the Hilbert scheme

$$\operatorname{Hilb}^n \mathbb{A}^d = \{ I \subset \mathbb{C}[x_1, \ldots, x_d] \mid \dim_{\mathbb{C}} \mathbb{C}[x_1, \ldots, x_d]/I = n \}.$$

More precisely, we will provide equations cutting out the Hilbert scheme inside a smooth quasi-projective variety (the so-called *non-commutative Hilbert scheme*), cf. Theorem 6.2.4.

An ideal $I \in \operatorname{Hilb}^n \mathbb{A}^d$ will be tacitly identified with the associated finite closed subscheme

$$\operatorname{Spec} \mathbb{C}[x_1,\ldots,x_d]/I \subset \mathbb{A}^d$$

of length n.

The following properties are well-known:

- $\operatorname{Hilb}^n \mathbb{A}^d$ is smooth if and only if $d \leq 2$ or $n \leq 3$,
- $\operatorname{Hilb}^n \mathbb{A}^3$ is irreducible for $n \leq 11$ (see [35, 17] and the references therein),
- $\operatorname{Hilb}^n \mathbb{A}^3$ is reducible for $n \geq 78$, see [37].

On the other hand, the following questions are open since a long time:

- What is the smallest d such that $\operatorname{Hilb}^n \mathbb{A}^3$ is reducible?
- If $d \geq 3$, is $\operatorname{Hilb}^n \mathbb{A}^d$ generically reduced?

Let us get started with our description of $\operatorname{Hilb}^n \mathbb{A}^d$. To ease notation, let us put $R_d = \mathbb{C}[x_1,\ldots,x_d]$. The condition defining a point $I \in \operatorname{Hilb}^n \mathbb{A}^d$ is that the $\mathbb{C}$-algebra quotient

$$R_d \twoheadrightarrow R_d/I$$

is a vector space of dimension n. Let us examine this condition in detail. To construct a point in the Hilbert scheme, we need:

(1) a vector space
$$V_n \cong \mathbb{C}^n,$$

(2) an R_d-module structure
$$\vartheta \colon R_d \to \operatorname{End}_{\mathbb{C}}(V_n)$$
with the property that

(3) such structure is induced by an R_d-linear surjection from R_d.

So let us fix an n-dimensional vector space V_n. Later we will have to remember that we made such a choice, and since all we wanted was "$\dim V_n = n$" we will have to quotient out all equivalent choices. Let us forget about this for the moment. In (2), we need θ to be a ring homomorphism, so we need to specify one endomorphism of V_n for each coordinate $x_i \in R_d$. All in all, ϑ gives us d matrices

$$A_1, A_2, \ldots, A_d \in \operatorname{End}_{\mathbb{C}}(V_n).$$

The matrix A_i will be responsible for the R_d-linear operator "multiplication by x_i" for the resulting module structure on V_n. Also in this step we should note a reminder for later: strictly speaking, what we have defined so far is a $\mathbb{C}\langle x_1, x_2, \ldots, x_d\rangle$-module structure on V_n. But in R_d the variables commute with one another. So we will have to impose the relations $[A_i, A_j] = 0$ for all $1 \leq i < j \leq d$.

Condition (3) is tricky. Let us reason backwards, assuming we already have an R_d-linear quotient $\phi \colon R_d \twoheadrightarrow V_n$. Then it is clear that the image of $1 \in R_d$ generates V_n

as an R_d-module. In other words, every element $w \in V_n$ can be written as

$$w = A_1^{m_1} A_2^{m_2} \cdots A_d^{m_d} \cdot \phi(1),$$

for some $m_i \in \mathbb{Z}_{\geq 0}$. This tells us exactly what we should add to the picture to obtain Condition (3): for a fixed module structure, i.e. d-tuple of matrices $(A_1, A_2, \ldots, A_d)$, we need to specify a *cyclic vector*

$$v \in V_n,$$

i.e. a vector with the property that the $\mathbb{C}$-linear span of the set

$$\left\{ A_1^{m_1} A_2^{m_2} \cdots A_d^{m_d} \cdot v \,\middle|\, m_i \in \mathbb{Z}_{d \geq 0} \right\}$$

equals the whole V_n.

Let us consider the $(dn^2 + n)$-dimensional affine space

$$W_n = \mathrm{End}_{\mathbb{C}}(V_n)^{\oplus d} \oplus V_n. \tag{6.2.1}$$

EXERCISE 6.2.1. Show that the locus

$$U_n = \left\{ (A_1, A_2, \ldots, A_d, v) \,\middle|\, v \text{ is } (A_1, A_2, \ldots, A_d)\text{-cyclic} \right\} \subset W_n$$

is a Zariski open subset.

Consider the GL_n-action on W_n given by

$$g \cdot (A_1, A_2, \ldots, A_d, v) = (A_1^g, A_2^g, \ldots, A_d^g, gv) \tag{6.2.2}$$

where $M^g = g^{-1} M g$ is conjugation.

Lemma 6.2.2. *The* GL_n*-action* (6.2.2) *is free on* U_n*.*

Proof. If $g \in \mathrm{GL}_n$ fixes a point $(A_1, A_2, \ldots, A_d, v) \in U_n$, then $v = gv$ lies in the invariant subspace $\ker(g - \mathrm{id}) \subset V_n$. But by definition of U_n, the smallest invariant subspace containing v is V_n itself, thus $g = \mathrm{id}$. □

Definition 6.2.3. The GIT quotient

$$\mathrm{NCHilb}_d^n = U_n / \mathrm{GL}_n$$

is called the *non-commutative Hilbert scheme.*

The discussion carried out so far proves the following:

Theorem 6.2.4. *There is a closed immersion*

$$\mathrm{Hilb}^n \mathbb{A}^d \subset \mathrm{NCHilb}_d^n$$

cut out by the ideal of relations

$$[A_i, A_j] = 0 \textit{ for all } 1 \leq i < j \leq d. \tag{6.2.3}$$

EXERCISE 6.2.5. Let $d = 1$. Show that $\mathrm{NCHilb}_1^n = \mathbb{A}^n$.

Example 6.2.6. If $d = 1$ there is just one operator "x" so the Relations (6.2.3) are vacuous. We have

$$\mathrm{Hilb}^n \mathbb{A}^1 = \mathrm{NCHilb}_1^n = \mathbb{A}^n,$$

which thanks to Exercise 6.2.5 reproves the second part of Exercise 6.1.4.

Remark 6.2.7. If $d = 2$ the description of $\mathrm{Hilb}^n \mathbb{A}^2$ is equivalent to Nakajima's description [48,]. See also [35] for another description of the Hilbert scheme of points, in terms of *perfect extended monads*.

6.3. **Hilbert–Chow revisited.** Fix $n \geq 0$ and $d > 0$. The Hilbert–Chow morphism

$$\pi\colon \mathrm{Hilb}^n \mathbb{A}^d \to \mathrm{Sym}^n \mathbb{A}^d$$

introduced in (6.1.1) can be reinterpreted as follows. Pick a point

$$[A_1, \ldots, A_d, v] \in \mathrm{Hilb}^n \mathbb{A}^d$$

and notice that since the matrices pairwise commute, they can be simultaneously triangularised. So, since the tuple is defined up to GL_n, we may assume they are in the form

$$A_\ell = \begin{pmatrix} a_{11}^{(\ell)} & * & * & \cdots & * \\ 0 & a_{22}^{(\ell)} & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn}^{(\ell)}. \end{pmatrix}$$

Then π is given by

$$[A_1, \ldots, A_d, v] \mapsto \sum_\ell \left(a_\ell^{(1)}, \ldots, a_\ell^{(d)}\right).$$

When all the matrices are *nilpotent*, the corresponding subscheme $Z \subset \mathbb{A}^d$ is entirely supported at the origin. In other words,

$$\mathrm{Hilb}^n(\mathbb{A}^d)_0 = \{\, [A_1, \ldots, A_d, v] \mid A_1, \ldots, A_d \text{ are nilpotent} \,\}$$

is a way to describe the punctual Hilbert scheme.

6.4. **Varieties of commuting matrices: what's known.** Let V be an n-dimensional complex vector space and let

$$C_n = \left\{ (A, B) \in \mathrm{End}(V)^2 \,\middle|\, [A, B] = 0 \right\} \subset \mathrm{End}(V)^2 \tag{6.4.1}$$

be the *commuting variety*. Letting GL_n act on C_n by simultaneous conjugation, one can form the quotient stack

$$\mathcal{C}(n) = C_n / \mathrm{GL}_n,$$

which is equivalent to the stack $\mathrm{Coh}_n(\mathbb{A}^2)$ of finite coherent sheaves of length n on the affine plane. Letting

$$\widetilde{c}_n = \left[\mathcal{C}(n)\right] = \frac{[C_n]}{\mathsf{GL}_n} \in K_0(\mathrm{St}_{\mathbb{C}}) \tag{6.4.2}$$

be the motivic class of the stack $\mathcal{C}(n)$, let us form the generating series

$$\mathsf{C}(t) = \sum_{n \geq 0} \widetilde{c}_n t^n \in K_0(\mathrm{St}_{\mathbb{C}})[\![t]\!].$$

The next result is a formula essentially due to Feit and Fine, but also proven recently by Behrend–Bryan–Szendrői and Bryan–Morrison.

Theorem 6.4.1 ([22, 7, 13])**.** *There is an identity*

$$\mathsf{C}(t) = \prod_{k \geq 1} \prod_{m \geq 1} (1 - \mathbb{L}^{2-k} t^m)^{-1}.$$

It has been known since a long time that the variety of pairs of commuting matrices C_n is irreducible [46, 54]. The same is true for the space $N_n \subset C_n$ of *nilpotent* commuting linear operators, see [5] for a proof in characteristic zero and [6] for an extension to fields of characteristic bigger than $n/2$. Premet even showed irreducibility of N_n over *any* field [53].

However the situation is very different for 3 or more matrices. Let $C(d,n)$ be the space of d-tuples of pairwise commuting endomorphisms of an n-dimensional vector space, and let $N(d,n)$ be the space of nilpotent endomorphisms. Then $C(d,n)$ is irreducible for all n if $d \le 3$. But it is reducible if d and n are both at least 4 [27], and the same is true for $N(d,n)$ [49]. For $d=3$ the situation is as follows. One has that $C(3,n)$ is reducible for $n \ge 30$ [36] and irreducible for $n \le 10$ (in characteristic zero). Moreover, $N(3,n)$ is known to be irreducible for $n \le 6$ [49], but $N(3,n)$ is reducible for $n \ge 13$ [51, Thm. 7.10.5].

6.5. **The special case of** $\operatorname{Hilb}^n \mathbb{A}^3$**.** In this subsection we set $d=3$. The Hilbert scheme of points

$$\operatorname{Hilb}^n \mathbb{A}^3$$

is singular as soon as $n \ge 4$, but as we shall see it is *virtually smooth*, i.e. it carries a (symmetric) perfect obstruction theory.

EXERCISE 6.5.1. Let $p \in \mathbb{A}^3$ be a point. Show that $\mathfrak{m}_p^2 \subset \mathscr{O}_{\mathbb{A}^3}$ defines a singular point of $\operatorname{Hilb}^4 \mathbb{A}^3$.

Theorem 6.5.2. *There exists a smooth quasi-projective variety M_n along with a regular function $f_n\colon M_n \to \mathbb{A}^1$ such that*

$$\operatorname{Hilb}^n \mathbb{A}^3 \subset M_n$$

can be realised as the scheme-theoretic zero locus of the exact 1-form $\mathrm{d}f_n$.

Proof. As M_n we can take the noncommutative Hilbert scheme $\operatorname{NCHilb}_3^n$. By [56, Prop. 3.8], the commutator relations (6.2.3) agree *scheme-theoretically* with the single vanishing relation

$$\mathrm{d}f_n = 0,$$

where $f_n\colon \operatorname{NCHilb}_3^n \to \mathbb{A}^1$ is the function

$$[A_1, A_2, A_3, v] \mapsto \operatorname{Tr} A_1[A_2, A_3]. \qquad \square$$

The above description is special to $d=3$.

EXERCISE 6.5.3. Show that, for $\{i,j,k\} = \{1,2,3\}$, one has

$$[A_i, A_j] = 0 \quad \iff \quad \frac{\partial f_n}{\partial A_k} = 0,$$

at least set-theoretically.

By the construction of $[Z(\mathrm{d}f)]^{\mathrm{vir}}$ outlined in Section **??**, we deduce from Theorem 6.5.2 the following:

Corollary 6.5.4. *The Hilbert scheme* $\mathrm{Hilb}^n \mathbb{A}^3$ *carries a symmetric perfect obstruction theory, giving rise to a zero-dimensional virtual fundamental class*

$$\left[\mathrm{Hilb}^n \mathbb{A}^3\right]^{\mathrm{vir}} \in A_0(\mathrm{Hilb}^n \mathbb{A}^3).$$

6.5.1. *A quiver description.* Recall that a *quiver* is a finite directed graph. If $Q = (Q_0, Q_1, s, t)$ is a quiver, the notation means Q_0 is the set of vertices, Q_1 is the set of arrows and $s,\, t\colon Q_1 \to Q_0$ are the source and tail maps respectively. Let $\mathbf{d} = (\mathbf{d}_i) \in \mathbb{N}^{Q_0}$ be a vector of nonnegative integers. Then a $\mathbf{d}$-dimensional *representation* ρ of Q is the datum of a $\mathbb{C}$-vector space of dimension $\mathbf{d}_i$ for each $i \in Q_0$, along with a linear map $\mathbb{C}^{\mathbf{d}_i} \to \mathbb{C}^{\mathbf{d}_j}$ for every arrow $i \to j$ in Q_1. We write $\underline{\dim}\, \rho = \mathbf{d}$. The space of such representations is the affine space

$$\mathrm{Rep}_{\mathbf{d}}(Q) = \prod_{a \in Q_1} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\mathbf{d}_{t(a)}}, \mathbb{C}^{\mathbf{d}_{s(a)}})$$

Definition 6.5.5. Let $n \geq 0$ be an integer. A *path of length n* in a quiver Q is a sequence of arrows $a_n \cdots a_2 a_1$ such that $s(a_{i+1}) = t(a_i)$ for all i. The notation is by juxtaposition from right to left. The *path algebra* $\mathbb{C}Q$ of a quiver Q is defined as follows. As a $\mathbb{C}$-vector space, it is spanned by all paths of length $n \geq 0$, including a single trivial path e_i of length 0 for each vertex $i \in Q_0$. The product is given by concatenation (juxtaposition) of paths, and is defined to be 0 if two paths cannot be concatenated.

EXERCISE 6.5.6. Prove that $\mathbb{C}Q$ is an associative algebra.

EXERCISE 6.5.7. Prove that representations of a quiver Q form a category (i.e. define a sensible notion of morphisms of representations). Show that this category is equivalent to the category of left $\mathbb{C}Q$-modules, in particular it is abelian.

EXERCISE 6.5.8. Show that the path algebra of the quiver L_d with one vertex and d loops is isomorphic to $\mathbb{C}\langle x_1, x_2, \ldots, x_d\rangle$.

Let Q be a quiver. Let

$$\mathbb{H}_+ = \left\{\, r \cdot \exp(i\pi\phi) \in \mathbb{C} \,\middle|\, r \in \mathbb{R}_{>0},\ \phi \in (0,1] \,\right\}$$

be the upper half plane with the positive real axis removed. A *central charge* is a group homomorphism $Z\colon \mathbb{Z}^{Q_0} \to \mathbb{C}$ mapping $\mathbb{N}^{Q_0} \setminus 0$ inside $\mathbb{H}_+$. For every $\alpha \in \mathbb{Z}^{Q_0}$ we let $\phi(\alpha)$ be the unique real number such that $Z(\alpha) = r \cdot \exp(i\pi\phi(\alpha))$. It is called the *phase* of α (with respect to Z). Every vector $\theta \in \mathbb{Q}^{Q_0}$ induces a central charge Z_θ via

$$Z_\theta(\alpha) = -\theta \cdot \alpha + i|\alpha|,$$

where $|\alpha| = \sum_{i \in Q_0} \alpha_i$. We let ϕ_θ denote the associated phase function, and we set

$$\phi_\theta(\rho) = \phi_\theta(\underline{\dim}\, \rho),$$

for every finite dimensional representation ρ of Q.

Fix $\theta \in \mathbb{Q}^{Q_0}$. We call θ a *stability condition.* For any $\alpha \in \mathbb{N}^{Q_0} \setminus 0$ one can define its *slope* (with respect to θ) as the ratio

$$\mu_\theta(\alpha) = \frac{\theta \cdot \alpha}{|\alpha|} \in \mathbb{Q}.$$

Let us set $\mu_\theta(\rho) = \mu_\theta(\underline{\dim}\,\rho)$, for a representation ρ of Q. A representation ρ is said to be θ-*semistable* if for every proper nontrivial subrepresentation $0 \neq \rho' \subset \rho$ one has

$$\mu_\theta(\rho') \leq \mu_\theta(\rho).$$

Strict inequality in the latter formula defines θ-*stability*, and θ is called $\mathbf{d}$-*generic* if every θ-semistable representation of dimension $\mathbf{d}$ is θ-stable.

EXERCISE 6.5.9. Let ρ be a finite dimensional representation of a quiver Q, and let $\rho' \subset \rho$ be a subrepresentation. Show that $\phi_\theta(\rho') < \phi_\theta(\rho)$ if and only if $\mu_\theta(\rho') < \mu_\theta(\rho)$, so that stability can be checked using slopes instead of phases.

Let Q be a quiver, and let $0 \in Q_0$ be a distinguished vertex. The *framed quiver* $\widetilde{Q}$ is obtained by adding a new vertex ∞ to the vertices of Q, along with a new arrow $\infty \to 0$. Thus a $(\mathbf{d}, 1)$-dimensional representation $\widetilde{\rho}$ of $\widetilde{Q}$ can be seen as a pair (ρ, v), where ρ is a $\mathbf{d}$-dimensional representation of Q and $v \colon V_\infty \to V_0$ is a linear map from the 1-dimensional vector space V_∞. In other words, v is a vector in V_0.

Definition 6.5.10 (Framed stability)**.** Fix $\theta \in \mathbb{Q}^{Q_0}$. A representation $\widetilde{\rho} = (\rho, v)$ of $\widetilde{Q}$ is said to be θ-(semi)stable if it is (θ, θ_∞)-(semi)stable, where $\theta_\infty = -\theta_\infty \cdot \underline{\dim}\,\rho$.

We now consider the framed quiver $\widetilde{L}_d$, i.e. the d-loop quiver L_d equipped with one additional vertex ∞ and one additional arrow $\infty \to 0$ — see Figure 7.

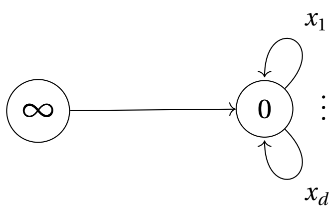


FIGURE 7. The framed d-loop quiver.

For the dimension vector $\mathbf{d} = (n, 1)$, we have

$$\mathrm{Rep}_{\mathbf{d}}(\widetilde{L}_d) = W_n,$$

where W_n was defined in (6.2.1).

EXERCISE 6.5.11. Find a ($\mathbf{d}$-generic) framed stability condition θ on L_d (in the sense of Definition 6.5.10) such that $U_n \subset W_n$ agrees with the set of θ-semistable framed representations of L_d. (**Hint**: consider $\theta = (\theta_1, \theta_2)$ with $\theta_1 \geq \theta_2$).

Thanks to the previous exercise, the noncommutative Hilbert scheme $\mathrm{NCHilb}_d^n = U_n / \mathrm{GL}_n$ can be viewed as a fine moduli space of quiver representations.

If $I \subset \mathbb{C}Q$ is a two-sided ideal, one can consider the full subcategory

$$\mathrm{Rep}(Q, I) \subset \mathrm{Rep}(Q)$$

of representations $(M_a)_{a \in Q_1}$ such that $M_{a_k} \cdots M_{a_2} M_{a_1} = 0$ for every element $a_k \cdots a_2 a_1 \in I$. The category of representations of the quotient algebra $\mathbb{C}Q/I$ is equivalent to

$\mathrm{Rep}(Q,I)$. For instance, given the ideal $I\subset \mathbb{C}L_d=\mathbb{C}\langle x_1,\ldots,x_d\rangle$ spanned by the commutators

$$[x_i,x_j],\quad 1\le i<j\le d,$$

one has $\mathrm{Rep}(L_d,I)=\mathrm{Rep}\,\mathbb{C}[x_1,\ldots,x_d]$.

A special class of ideals arises from *superpotentials*, i.e. formal linear combinations of cyclic paths

$$\mathsf{W}=\sum_{c \text{ cycle in } Q} a_c\, c\in \mathbb{C}Q$$

up to cyclic permutation.

For a cyclic word w and an arrow $a\in Q_1$, one defines the non-commutative derivative

$$\frac{\partial w}{\partial a}=\sum_{\substack{w=cac'\\ c,c' \text{ paths in } Q}} c'c.$$

This rule extends to an operator $\frac{\partial}{\partial a}$ acting on every superpotential. Thus every superpotential $\mathsf{W}=\sum_c a_c\, c$ gives rise to a two-sided ideal

$$I_{\mathsf{W}}=\left\langle \frac{\partial \mathsf{W}}{\partial a}\,\middle|\, a\in Q_1\right\rangle\subset \mathbb{C}Q$$

and to a regular function

$$\mathrm{Tr}\,\mathsf{W}\colon \mathrm{Rep}_{\mathbf{d}}(Q)\to \mathbb{A}^1$$

defined by sending

$$\rho\mapsto \sum_{c \text{ cycle in } Q} a_c\, \mathrm{Tr}(\rho(c)).$$

The quotient

$$J(Q,\mathsf{W})=\mathbb{C}Q/I_{\mathsf{W}}$$

is called the *Jacobi algebra* of the quiver with potential (Q,W). By our discussion, we have an equivalence of abelian categories

$$\mathrm{Rep}(Q,I_{\mathsf{W}})\cong \mathrm{Rep}\,J(Q,\mathsf{W}).$$

The proof of Theorem 6.5.2 then reveals that $\mathrm{Hilb}^n\mathbb{A}^3$ can be seen as the moduli space of *stable* framed representations of the Jacobi algebra

$$\mathbb{C}\langle x_1,x_2,x_3\rangle/I_{\mathsf{W}}=\mathbb{C}[x_1,x_2,x_3],$$

for the potential $\mathsf{W}=x_1[x_2,x_3]$.

7. Equivariant Cohomology

7.1. **Origins.** Equivariant Cohomology was introduced by Borel in his seminar on trasformation groups [10]. The goal of this section is to introduce the framework necessary to state the Atiyah–Bott *localisation formula* [11, 4]. This will be done in Section 8. We shall see how the formula works via concrete examples in Section 9. See [3, 43] for great expositions on the subject in the context of Algebraic Geometry, and [20] for the first appearance of equivariant cohomology in Enumerative Geometry. We will survey a special case of the *virtual localisation formula* by Graber–Pandharipande [29] in Section **??**.

Let HTop denote the homotopy category of (Hausdorff, paracompact) topological spaces, and let G be a (Hausdorff, paracompact) topological group. The functor $\mathcal{P}_G \colon \mathrm{HTop}^{\mathrm{op}} \to \mathrm{Sets}$ sending a topological space S to the set isomorphism[5] classes of principal G-bundles over S is representable, i.e. there exists a topological space $\mathrm{B}G$ and an isomorphism of functors $\mathcal{P}_G \cong \mathrm{Hom}_{\mathrm{HTop}}(-, \mathrm{B}G)$. This is a consequence of the Brown representability theorem [12], which completely characterises the representable functors $\mathrm{HTop}^{\mathrm{op}} \to \mathrm{Sets}$.

Recall the following definitions and theorem.

Definition 7.1.1. A topological space is called *weakly contractible* if all its homotopy groups are trivial. It is called *contractible* if it is homotopy equivalent to a point.

Definition 7.1.2. A principal G-bundle $E \to S$ is called *universal* if its total space E is contractible.

Theorem (Whitehead [63])**.** Every weakly contractible CW complex is contractible.

Going back to the representable functor $\mathcal{P}_G$, let $\eta_G \in \mathcal{P}_G(\mathrm{B}G)$ be the element corresponding to $\mathrm{id}_{\mathrm{B}G} \in \mathrm{Hom}_{\mathrm{HTop}}(\mathrm{B}G, \mathrm{B}G)$. Milnor has shown [44] that every Hausdorff topological group G admits a universal principal bundle $\mathrm{E}G \to \mathrm{B}G$, such that G acts freely on $\mathrm{E}G$. Therefore the equivalence class η_G contains one such representative, that we will denote

$$\mathrm{E}G \to \mathrm{B}G.$$

Every other principal G-bundle $F \to S$ is, up to isomorphism, pulled back from this one via a unique morphism $S \to \mathrm{B}G$ in the homotopy category.

Milnor's proof is usually referred to as the *join construction*, for in [45] he constructs $\mathrm{E}G$ as

$$\mathrm{E}G = \operatorname{colim} G * G * \cdots$$

where $*$ denotes the topological join. If we identify G, up to homotopy, with a CW complex W, then $\mathrm{E}G$ is homotopy equivalent to a colimit of finite joins of W, and is therefore weakly contractible (which implies the above universal property). By Whitehead's theorem, $\mathrm{E}G$ is in fact contractible. This will be crucial, since then the product space $\mathrm{E}G \times X$ will have the same homotopy type as X, for every topological space X.

However, it is wise to keep in mind that $\mathrm{E}G$ and its free quotient $\mathrm{B}G = \mathrm{E}G/G$ are only determined up to homotopy. After such a choice of representative is made, the space $\mathrm{B}G$, together with the map $\mathrm{E}G \to \mathrm{B}G$, is usually called the *classifying space* of principal G-bundles.

7.2. **Warming up: first examples.** We now list a few classical examples of classifying spaces $\mathrm{E}G \to \mathrm{B}G$. We insist that $\mathrm{E}G$ should be contractible.

(1) If $G = 1$, one can take $\mathrm{pt} \to \mathrm{pt}$ as classifying space.
(2) If G and K are two groups, then one can always take $\mathrm{E}(G \times K) = \mathrm{E}G \times \mathrm{E}K$.
(3) If $G = \mathbb{R}$, then one can take $\mathrm{E}\mathbb{R} = \mathbb{R}$ with the constant map $\mathbb{R} \to \mathrm{pt}$.

[5] Any two principal bundles over the same base are isomorphic in all categories (topological, smooth, algebraic…). However, here two principal bundles are isomorphic if they are G-*homotopy equivalent*.

(4) If $G=\mathbb{Z}^n$, one can take $\mathbb{R}^n \to (S^1)^n$ to be the map

$$(y_1,\ldots,y_n)\mapsto(\exp(i\pi y_1),\ldots,\exp(i\pi y_n)).$$

(5) If $G=\mathbb{Z}/2$, one can take the double cover $S^\infty \to \mathbb{RP}^\infty$.
(6) If $G=\mathbb{C}^\times$, one can take $\mathbb{C}^\infty\setminus 0\to\mathbb{P}^\infty$.
(7) If $G=S^1\subset\mathbb{C}^\times$ is the circle, we can restrict the bundle from the previous example to get $S^\infty\to\mathbb{P}^\infty$.

Example (6) generalises in two different ways:

(6′) If $G=(\mathbb{C}^\times)^n$, one has $(\mathbb{C}^\infty\setminus 0)^n\to(\mathbb{P}^\infty)^n$. Similarly, replacing $\mathbb{C}^\times$ by S^1, example (7) becomes $(S^\infty)^n\to(\mathbb{P}^\infty)^n$, the direct limit of the $(S^1)^n$-bundles $(S^{2m+1})^n\to(\mathbb{P}^m)^n$.
(6″) If $G=\mathrm{GL}_n(\mathbb{C})$, one has $F(n,\mathbb{C}^\infty)\to G(n,\mathbb{C}^\infty)$ where $F(n,\mathbb{C}^\infty)$ denotes the space of orthonormal n-frames in an infinite dimensional vector space and the map sends an n-frame to its span, viewed as a point of the infinite Grassmannian. The infinite dimensional Grassmannian $G(n,\mathbb{C}^\infty)$ is also the classifying space for the orthogonal group $O(n)$.

Before defining equivariant cohomology, we pause for a tiny remark. We notice that in most examples above, the spaces involved are infinite dimensional. We want to answer the following questions:

(A) Why is that the case?
(B) Why is it not a problem?

Consider example (6), say. We have that $G=\mathbb{C}^\times$ acts on $\mathbb{C}^n$ by

$$\lambda\cdot(x_1,\ldots,x_n)=(\lambda x_1,\ldots,\lambda x_n),$$

and removing the origin we get a nice principal $\mathbb{C}^\times$-bundle

$$\mathbb{C}^n\setminus 0\to\mathbb{P}^{n-1}.$$

But we cannot stop here and declare this to be our $\mathrm{E}G\to\mathrm{B}G$, since we want the total space to be contractible, and unfortunately

$$\pi_n(\mathbb{C}^n\setminus 0)\neq 0.$$

It turns out that taking the limit is what we need in order to kill all higher homotopy groups. Formally, we can use the $\mathbb{C}^\times$-equivariant maps

$$\mathbb{C}^n\setminus 0\hookrightarrow\mathbb{C}^{n+1}\setminus 0,\quad (x_1,\ldots,x_n)\mapsto(x_1,\ldots,x_n,0)$$

to get an inductive system of inclusions $\mathbb{P}^{n-1}\hookrightarrow\mathbb{P}^n$. The result of the limit process

$$\begin{array}{ccccc} \mathbb{C}^n\setminus 0 & \hookrightarrow & \mathbb{C}^{n+1}\setminus 0 & & \mathbb{C}^\infty\setminus 0 \\ \downarrow & & \downarrow & \xrightarrow{n\to\infty} & \downarrow \\ \mathbb{P}^{n-1} & \hookrightarrow & \mathbb{P}^n & & \mathbb{P}^\infty \end{array}$$

has the desired property $\pi_n(\mathbb{C}^\infty\setminus 0)=0$ for all n. This answers Question (A). As for Question (B), we will see in Section 7.4 that computations can be performed at the level of certain finite dimensional *approximation spaces*.

7.3. **Definition of equivariant cohomology.** Let X be any topological space equipped with a *left* G-action. We call such an X a G-space. Choose a universal principal G-bundle $\mathrm{E}G \to \mathrm{B}G$, with a free right G-action on $\mathrm{E}G$. Form the quotient space

$$\mathrm{E}G \times^G X = \frac{\mathrm{E}G \times X}{(e\cdot g, x) \sim (e, g\cdot x)}. \tag{7.3.1}$$

This quotient exists even in the smooth category when X is a smooth manifold, since the diagonal action of G on $\mathrm{E}G \times X$ is *free*. Note that the product $\mathrm{E}G \times X$ is homotopy equivalent to X since $\mathrm{E}G$ is contractible. The quotient (7.3.1) is the right notion of a G-space (from a homotopy theory point of view), for it is much better behaved than the (topological) orbit space X/G.

Definition 7.3.1. The *G-equivariant cohomology of X* is defined to be the singular cohomology of the quotient (7.3.1). The notation will be

$$H_G^*(X) = H^*(\mathrm{E}G \times^G X). \tag{7.3.2}$$

Convention. We work with $\mathbb{Z}$ coefficients.

Example 7.3.2. When $X = \mathrm{pt}$, we set

$$H_G^* = H_G^*(\mathrm{pt}) = H^*(\mathrm{B}G).$$

This can be viewed as the ring of characteristic classes on principal G-bundles.

Remark 7.3.3. A G-space X can have nonzero equivariant cohomology in degrees higher than its dimension, unlike ordinary cohomology. This is already true for $X = \mathrm{pt}$.

Remark 7.3.4. Even though ordinary cohomology is a special case of equivariant cohomology (it is enough to set $G = 1$), it is not true that one can reconstruct ordinary cohomology knowing equivariant cohomology. For instance, setting $X = G = \mathbb{C}^\times$, with G acting on X by left complex multiplication, one finds

$$H_G^1(X) = H^1(\mathbb{C}^\infty \setminus 0 \times^{\mathbb{C}^\times} \mathbb{C}^\times) = H^1(\mathbb{C}^\infty \setminus 0) = 0,$$

whereas $H^1(\mathbb{C}^\times) = H^1(S^1) = \mathbb{Z}$. More generally, by letting a compact connected Lie group G of dimension ℓ act on $X = G$ by left multiplication, one has

$$H_G^*(X, \mathbb{Q}) = H^*(\mathrm{E}G) = \mathbb{Q},$$

whereas by a theorem of Hopf [**?**] one has

$$H^*(X) = \wedge_{\mathbb{Q}}(a_1, \ldots, a_\ell).$$

Remark 7.3.5. The cohomology of a topological orbit space X/G does not agree with the G-equivariant cohomology of X, unless the G-action on X is free (cf. (7.3.7)). One example is $G = S^1$ acting on $X = S^2 \subset \mathbb{R}^3$ by rotation along the z-axis. There are two fixed points so the action is not free. The quotient X/G can be identified with a closed interval, which is contractible. Therefore the cohomology of the quotient is the cohomology of a point, and the action has manifestly been forgotten. But $H_{S^1}^*(S^2)$ is more interesting (cf. Exercise 7.3.15).

Lemma 7.3.6. *The definition of $H_G^*(X)$ does not depend upon the choice of universal principal G-bundle.*

Proof. Let $\mathrm{E}G \to \mathrm{B}G$ and $\mathrm{F}G \to \mathrm{B}G$ be two universal principal G-bundles. Consider the space $Y = (\mathrm{E}G \times \mathrm{F}G \times X)/G$. Then we have fibrations

$$\begin{tikzcd} & Y \arrow[dl, "p_E"'] \arrow[dr, "p_F"] & \\ \mathrm{E}G \times^G X & & \mathrm{F}G \times^G X \end{tikzcd}$$

with contractible fibers $\mathrm{F}G$ and $\mathrm{E}G$, respectively. Let us focus on p_E. For every positive integer n, we have the fibration long exact sequence

$$\cdots \longrightarrow \pi_n(\mathrm{F}G) \longrightarrow \pi_n(Y) \longrightarrow \pi_n(\mathrm{E}G \times^G X) \longrightarrow \pi_{n-1}(\mathrm{F}G) \longrightarrow \cdots$$

but the homotopy groups of the fiber $\mathrm{F}G$ are trivial, therefore we get a family of isomorphisms

$$\pi_n(Y) \xrightarrow{\sim} \pi_n(\mathrm{E}G \times^G X).$$

It follows that Y and $\mathrm{E}G \times^G X$ have the same homotopy type, and hence the same cohomology. Repeating the process with p_F gives the result. □

Remark 7.3.7. There is a fibre bundle

$$\begin{tikzcd} \mathrm{E}G \times^G X \arrow[d, "p"] \\ \mathrm{B}G \end{tikzcd}$$

with fiber X, induced by the G-equivariant projection $\mathrm{E}G \times X \to \mathrm{E}G$. The standard pullback in cohomology induces a canonical ring homomorphism

$$p^* \colon H_G^* \to H_G^*(X), \tag{7.3.3}$$

making $H_G^*(X)$ into a H_G^*-module.

Example 7.3.8. When G acts trivially on X we have $\mathrm{E}G \times^G X = \mathrm{B}G \times X$, so that

$$H_G^*(X) \cong H_G^* \otimes H^*(X) \tag{7.3.4}$$

is a free H_G^*-module. For instance, given any action of G on X and letting X^G be the fixed locus, we have $H_G^*(X^G) = H_G^* \otimes H^*(X^G)$. There are G-spaces X with nontrivial action such that the isomorphism (7.3.4) holds — see Exercise 7.3.15. In this case the G-space X is called *equivariantly formal*.

Example 7.3.9. Let X be a contractible space with a left G-action. Then (7.3.3) is an isomorphism.

Example 7.3.10 (Equivariant cohomology of a subgroup)**.** Let $K \subset G$ be a closed subgroup. Then $\mathrm{E}G/K$ exists and in fact $\mathrm{E}K = \mathrm{E}G \to \mathrm{E}G/K = \mathrm{B}K$ is a classifying space for K. Therefore,

$$H_G^*(G/K) = H^*(\mathrm{E}G \times^G (G/K)) = H^*(\mathrm{E}G/K) = H^*(\mathrm{B}K) = H_K^*.$$

Example 7.3.11. Generalising the previous example, let $K \subset G$ be again a closed subgroup, acting on a space X on the left. We can also consider the diagonal action of K upon $G \times X$. The quotient

$$G \times^K X = (G \times X)/(gk, x) \sim (g, k \cdot x)$$

makes sense and we have

$$H_G^*(G\times^K X) = H^*(\mathrm{E}G\times^G G\times^K X) = H^*(\mathrm{E}G\times^K X) = H_K^*(X).$$

Again, we have used that $\mathrm{E}G \to \mathrm{E}G/K$ is a classifying space for $K\subset G$.

Theorem 7.3.12. *Let $f\colon X\to S$ be a topological fibre bundle with contractible fibre. Then f induces an isomorphism $H^*(X,\mathbb{Z})\cong H^*(S,\mathbb{Z})$.*

EXERCISE 7.3.13. Let $K\subset G$ be a closed subgroup of a topological group G such that G/K is contractible, and let X be a G-space. Then

$$H_K^*(X)\cong H_G^*(X).$$

(**Hint**: use that one can take $\mathrm{E}K=\mathrm{E}G$ to construct a map $\mathrm{E}K\times^K X=\mathrm{E}G\times^K X\to \mathrm{E}G\times^G X$. Show that it is a fibre bundle with fibre G/K. Conclude by Theorem 7.3.12).

Example 7.3.14. Let us revisit the examples (6′) and (6″) discussed above.

(A) In the case $\mathbb{T}=(\mathbb{C}^\times)^n$, we find

$$H_{\mathbb{T}}^* = H^*((\mathbb{P}^\infty)^n) = \mathbb{Z}[t_1,\ldots,t_n] \tag{7.3.5}$$

where $t_i\in H_{\mathbb{T}}^2$ are the Chern classes obtained by pulling back to $\mathrm{B}\mathbb{T}$ the universal line bundles living over the individual spaces $\mathbb{P}^\infty$. In other words, if $\pi_i\colon \mathrm{B}\mathbb{T}\to\mathbb{P}^\infty$ is the i-th projection, the generators of $H_{\mathbb{T}}^*$ are

$$t_i = c_1(\pi_i^*\mathscr{O}(-1)).$$

(B) In the case $G=\mathrm{GL}_n(\mathbb{C})$, we find

$$H_G^* = H^*(G(n,\mathbb{C}^\infty)) = \mathbb{Z}[e_1,\ldots,e_n] \tag{7.3.6}$$

where $e_i=c_i(\mathscr{S})$ are the Chern classes of the universal rank n bundle $\mathscr{S}$ living over the Grassmannian.

EXERCISE 7.3.15. Let $G=S^1$ act on the sphere $S^2\subset\mathbb{R}^3$ by rotation along the z-axis. Show that S^2 is equivariantly formal for this S^1-action. In other words, show that

$$H_{S^1}^*(S^2)\cong H^*(\mathrm{B}S^1)\otimes H^*(S^2),$$

even though the action has two fixed points. (**Hint**: Find a suitable open cover $S^2=X_1\cup X_2$, inducing an open cover of $\mathrm{E}S^1\times^{S^1}S^2$. Apply Mayer–Vietoris to this open cover, and use that $H_{S^1}^*=\mathbb{Z}[c]$ is one-dimensional, resp. zero-dimensional, in even degree, resp. odd degree).

Let X be a smooth manifold acted on (smoothly) by a Lie group G. There is a commutative diagram

$$\begin{tikzcd}
\mathrm{E}G \arrow[d] & \mathrm{E}G\times X \arrow[l] \arrow[r] \arrow[d] & X \arrow[d] \\
\mathrm{B}G & \mathrm{E}G\times^G X \arrow[l, "p"'] \arrow[r, "\sigma"] & X/G
\end{tikzcd}$$

where σ, unlike p, is in general not a fiber bundle, but has the property that

$$\sigma^{-1}(\mathrm{Orb}\,x) = \mathrm{E}G/G_x \cong \mathrm{B}G_x$$

for all $x \in X$. When G is a compact Lie group acting (smoothly and) *freely* on X, the map σ induces a homotopy equivalence $\mathrm{E}G \times^G X \approx X/G$ yielding a natural isomorphism

$$\sigma^* \colon H^*(X/G) \xrightarrow{\sim} H^*_G(X). \tag{7.3.7}$$

Let $\iota\colon X \hookrightarrow \mathrm{E}G \times^G X$ denote the inclusion of X as a fiber of $p\colon \mathrm{E}G \times^G X \to \mathrm{B}G$ (after choosing a base point $\mathrm{pt} \in \mathrm{B}G$). Then ι induces a natural map

$$\iota^* \colon H^*_G(X) \to H^*(X)$$

from equivariant to ordinary cohomology. There is a pullback diagram (on the left) inducing, whenever X is compact, a commutative diagram of cohomology rings:

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[d, "\iota"'] \arrow[dr, phantom, "\square"] & \mathrm{pt} \arrow[d, "b"] \\ \mathrm{E}G \times^G X \arrow[r, "p"] & \mathrm{B}G \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} H^*(X) \arrow[r, "q_*"] & \mathbb{Z} \\ H^*_G(X) \arrow[r, "p_*"] \arrow[u, "\iota^*"] & H^*_G \arrow[u, "b^*"'] \end{tikzcd}$$

In algebraic geometry, this compatibility makes sense and is useful when X is a proper variety, which ensures the existence of q_*. The *equivariant pushforward*, on the other hand, will be defined in Section 8.2. Note that the above commutativity implies in particular that

$$\boxed{\textit{ordinary integrals can be performed in equivariant cohomology.}}$$

An equivariant integral takes values in H^*_G, but evaluating along b^* always yields an integer.

7.3.1. *Preview of Section 9.* The following is the general setup we shall encounter in concrete calculations. Let X be a smooth projective complex variety with a torus action — for instance a Grassmannian. We will have a top cohomology class

$$\psi \in H^{2d}(X), \quad d = \dim_{\mathbb{C}} X,$$

and we will be interested in computing an ordinary integral

$$\int_X \psi = q_*(\psi \cap [X]) \in \mathbb{Z}.$$

To compute this number, we will lift the class ψ to an equivariant class $\psi^{\mathbb{T}} \in H^*_{\mathbb{T}}(X)$, and we will compute

$$\int_X \psi^{\mathbb{T}} = p_*(\psi^{\mathbb{T}} \cap [X]^{\mathbb{T}})$$

instead, via localisation. At this point, applying b^* means choosing a suitable[6] specialisation of the equivariant parameters to obtain a number — this is necessarily the number we want, by the commutativity of the above square!

[6]One that does not produce poles in the localisation formula.

7.4. **Approximation spaces.** Let us now assume that X is a complex algebraic variety and G is an algebraic group. The fact that the spaces involved, like $\mathrm{E}G$ and $\mathrm{B}G$, are infinite-dimensional, is not quite an obstacle to the computation of the equivariant cohomology groups. This is the case because of the following "approximation" result.

Theorem 7.4.1. *Let $(E_m)_{m\geq 0}$ be a family of connected spaces on which G acts freely on the right. Let $k\colon \mathbb{N}\to\mathbb{N}$ be a function such that $\pi_i(E_m)=0$ for $0<i<k(m)$. Then, for any left G-action on a space X, there are natural isomorphisms*

$$H^i_G(X)\cong H^i(E_m\times^G X),\quad i<k(m).$$

We will refer to the E_m spaces as *approximation spaces*. The idea behind their existence can be traced back to [10, Remark XII.3.7].

Remark 7.4.2. In the smooth category, if G is a compact lie group, then $\mathrm{E}G\to\mathrm{B}G$ is a colimit of smooth principal G-bundles

$$E_m\to B_m,$$

where E_m is m-connected, i.e. $\pi_i(E_m)=0$ for $1\leq i\leq m$, i.e. if S^k is a sphere of dimension $k\leq m$, then any continuous map $S^k\to E_m$ is homotopic to the constant map.

Example 7.4.3. Let us revisit once more the examples $\mathbb{T}=(\mathbb{C}^\times)^n$ and $G=\mathrm{GL}_n(\mathbb{C})$. For the case of the torus, one can take $E_m=(\mathbb{C}^m\setminus 0)^n\to B_m=(\mathbb{P}^{m-1})^n$ as approximations of $\mathrm{E}\mathbb{T}\to\mathrm{B}\mathbb{T}$. Since $\mathbb{C}^m\setminus 0$ has the same homotopy type as S^{2m-1}, the function $k(m)=n(2m-1)$ will let us fall in the assumption $\pi_i(E_m)=0$ of the theorem, where $0<i<k(m)$. Setting for instance $X=\mathsf{pt}$, we find isomorphisms

$$H^i_{\mathbb{T}}\cong H^i((\mathbb{P}^{m-1})^n),\quad i<n(2m-1).$$

This fully explains the assertion we made in (7.3.5). When $G=\mathrm{GL}_n(\mathbb{C})$, we can take E_m, for $m>n$, to be the set of $m\times n$ rank n matrices. The approximations spaces then look like the free quotients $E_m\to G(n,\mathbb{C}^m)$. The function $k(m)=2(m-n)$ will do the job again and we would find, again for $X=\mathsf{pt}$, isomorphisms

$$H^i_{\mathrm{GL}_n}\cong H^i(G(n,\mathbb{C}^m)),\quad i<2(m-n).$$

This explains the assertion we made in (7.3.6).

Remark 7.4.4. If $Y\subset\mathbb{A}^N$ is a subvariety of codimension d, then

$$\pi_i(\mathbb{A}^N\setminus Y)=0,\quad 0<i\leq 2d-2.$$

Since the complement of the open subset

$$\{\text{full rank matrices}\}\subset\mathrm{Mat}_{m\times n}=\mathbb{A}^{mn}$$

has codimension $(m-1)(n-1)$, the choice $k(m)=2(m-n)$ works (in case $n<m$).

Remark 7.4.5. In the above examples, it is clear that the method of approximation spaces allows one to compute H^*_G in all degrees, since $k(m)\to\infty$ for $m\to\infty$.

7.5. **Equivariant vector bundles.** Let G be a group acting on X and H a group acting on Y. Suppose there are maps $\phi\colon G \to H$ and $f\colon X \to Y$. The condition

$$f(g\cdot x) = \phi(g)\cdot f(x) \tag{7.5.1}$$

for all $x \in X$ and $g \in G$ is the condition under which one can construct a natural map

$$\mathrm{E}G \times^G X \to \mathrm{E}H \times^H Y. \tag{7.5.2}$$

When (7.5.1) is satisfied, applying H^* to (7.5.2) yields a natural *equivariant pullback* homomorphism

$$f^*\colon H^*_H(Y) \to H^*_G(X). \tag{7.5.3}$$

When $\phi = \mathrm{id}_G$ we say that f is G-*equivariant*, and when moreover G acts trivially on Y we say that f is G-invariant.

A crucial example is that of an equivariant vector bundle $\pi\colon E \to X$. Such an object is a geometric vector bundle together with a lift of the G-action on X to a G-action on E:

$$\begin{tikzcd} G\times E \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ G\times X \arrow[r, "\text{action}"] & X \end{tikzcd}$$

Given an equivariant vector bundle one can define a new vector bundle

$$V_E = \mathrm{E}G \times^G E \to \mathrm{E}G \times^G X \tag{7.5.4}$$

of the same rank as E.

Definition 7.5.1 (Equivariant Chern classes)**.** The *equivariant Chern classes* of a G-equivariant vector bundle $E \to X$ are the characteristic classes

$$c_i^G(E) = c_i(V_E) \in H^{2i}_G(X).$$

These classes can clearly be computed through approximation spaces. Indeed, the vector bundle (7.5.4) can be approximated by vector bundles

$$V_{E,m} = E_m \times^G E \to E_m \times^G X$$

whose Chern classes $c_i(V_{E,m})$ live in $H^{2i}(E_m \times^G X) \cong H^{2i}_G(X)$ for $m \gg 0$.

Example 7.5.2 (Equivariant fundamental class)**.** Let X be a smooth algebraic variety acted on by a linear algebraic group G, and let $Y \subset X$ be a G-invariant closed subvariety of codimension d. The space $E_m \times^G X$ is smooth, and

$$E_m \times^G Y \hookrightarrow E_m \times^G X$$

is a closed subvariety of codimension d. Then

$$[Y]^G = [E_m \times^G Y] \in H^{2d}_G(X) \tag{7.5.5}$$

is called the *equivariant fundamental class* of $Y \subset X$.

EXERCISE 7.5.3. Show that the classes $[Y]^G$ defined in (7.5.5) are compatible when m varies and are independent of the choice of $\mathrm{E}G$ and $(E_m)_{m\geq 0}$.

Example 7.5.4 (Weight of a character)**.** When X is a point, an equivariant vector bundle is a representation $\rho\colon G\to \mathrm{GL}(E)$. This still gives a vector bundle $V_\rho\to \mathrm{B}G$ whose Chern classes live in H^*_G. When E is one-dimensional, an equivariant line bundle is simply a *character* $\chi\colon G\to\mathbb{C}^\times$ and one defines the *weight* of χ to be the Chern class

$$w_\chi = c_1(V_\chi)\in H^2_G.$$

When a representation $\rho\colon G\to \mathrm{GL}(E)$ of dimension r splits as a direct sum of characters χ_i (for example this is always the case when $G=\mathbb{T}$ is a torus), the G*-equivariant Euler class* splits as a product of weights,

$$e^G(E)=c^G_{\mathrm{top}}(E)=c_{\mathrm{top}}(V_\rho)=\prod_{i=1}^r w_i\in H^{2r}_G,$$

where $w_i=c_1(V_{\chi_i})\in H^2_G$ is the weight of χ_i. More generally, the i-th equivariant Chern class of E is the i-th symmetric function in $w_1,\ldots,w_r$. See Example 7.5.6 for a detailed description of this fact.

Example 7.5.5. Let $\mathbb{T}=\mathbb{C}^\times$ be the one dimensional torus and $\rho_a\colon\mathbb{T}\to\mathbb{C}^\times$ the character $z\mapsto z^a$ for an integer $a\in\mathbb{Z}$. Here we are viewing the one dimensional vector space $\mathbb{C}_a$ as an equivariant vector bundle over $X=\mathrm{pt}$. The line bundle $V_{\rho_a}\to \mathrm{B}\mathbb{C}^\times$ is approximated by line bundles $V_{\rho_a,m}\to\mathbb{P}^{m-1}$. In fact, $V_{\rho_a,m}\cong\mathscr{O}_{\mathbb{P}^{m-1}}(-a)$ so that the weight of ρ_a is

$$c_1^{\mathbb{C}^\times}(\mathbb{C}_a)=c_1(V_{\rho_a,m})=t\,a\in H^2_{\mathbb{C}^\times}.$$

Notice that t corresponds to the case $a=1$. This is called the *standard action* of $\mathbb{T}$ on $\mathbb{C}$. It generalises in the next example.

Example 7.5.6. Let $F=\mathbb{C}^n$ be an n-dimensional vector space and let the torus $\mathbb{T}=(\mathbb{C}^\times)^n$ act on F via the *standard action* $\theta\cdot(v_1,\ldots,v_n)=(\theta_1v_1,\ldots,\theta_nv_n)$. The induced representation $\rho\colon\mathbb{T}\to\mathrm{GL}(F)$, defined by $\rho_\theta(v)=\theta\cdot v$ as above, gives a rank n vector bundle $V_\rho=\mathrm{E}\mathbb{T}\times^{\mathbb{T}}F\to\mathrm{B}\mathbb{T}=(\mathbb{P}^\infty)^n$. Now, there is a splitting $F=F_1\oplus\cdots\oplus F_n$ where each summand corresponds to a character

$$\chi_i\colon\mathbb{T}\to\mathbb{C}^\times,\quad \theta\mapsto\theta_i$$

whose weight is just $t_i\in H^2_{\mathbb{T}}$ by Example 7.5.5. In other words, for each i, $V_{\chi_i}=\mathrm{E}\mathbb{T}\times^{\mathbb{T}}F_i\to(\mathbb{P}^\infty)^n$ is the total space of the line bundle $\mathscr{O}_i(-1)=\pi_i^*\mathscr{O}(-1)$. We quickly verify that

$$c_i^{\mathbb{T}}(F)=e_i(t_1,\ldots,t_n)\in H^*_{\mathbb{T}},\tag{7.5.6}$$

the i-th symmetric function in the Chern classes $t_i=c_1(\mathscr{O}_i(-1))$. We have a decomposition

$$V_\rho=\bigoplus_{i=1}^n\mathrm{E}\mathbb{T}\times^{\mathbb{T}}F_i=\bigoplus_{i=1}^n\mathscr{O}_i(-1).$$

Then V_ρ has the obvious filtration

$$0\subset\mathscr{O}_1(-1)\subset\mathscr{O}_1(-1)\oplus\mathscr{O}_2(-1)\subset\cdots\subset V_\rho$$

with $\mathscr{O}_i(-1)$ as line bundle quotients. The conclusion (7.5.6) is then a straightforward property of Chern classes. For instance, the top Chern class of the standard representation is given by $c_n^{\mathbb{T}}(V_\rho)=\prod_i t_i$.

Example 7.5.7. Let $\mathbb{T}$ be a torus acting on an n-dimensional complex vector space F. Let $w_1,\ldots,w_n$ be the weights of the action. Again, this means that the corresponding representation $\rho:\mathbb{T}\to\mathrm{GL}(F)$ satisfies

$$\rho_\theta(v)=(\theta_1^{w_1}v_1,\ldots,\theta_n^{w_n}v_n).$$

So F splits as a sum of 1-dimensional representations, corresponding to characters $\chi_1,\ldots,\chi_n$, each given by $\theta\mapsto\theta_i^{w_i}$. We have

$$e^{\mathbb{T}}(F)=\prod_{i=1}^n s_i^{w_i}\in H_{\mathbb{T}}^{2n}.$$

The $\mathbb{T}$-*character* of F is given by

$$\mathrm{ch}^{\mathbb{T}}(F)=\sum_{i=1}^n w_i t_i,\quad t_i=\exp(s_i).$$

Assume the weights are *positive*. Then the number

$$\mathrm{ch}^{\mathbb{T}}(F)\big|_{t_i=1}=\sum_i w_i=\sum_{\text{distinct weights }\chi}\mathsf{n}_\chi \tag{7.5.7}$$

enumerates the sum of the multiplicities n_χ of the (distinct) characters χ appearing in the decomposition of F. This fact will be used in Equation (**??**), where we shall calculate the parity of the tangent space dimension at a torus fixed point of $\mathrm{Hilb}^n X$, for X a toric Calabi–Yau 3-fold.

7.6. **Two computations on $\mathbb{P}^{n-1}$.** In this section we compute the equivariant cohomology of projective space $\mathbb{P}^{n-1}$ and the weights of the tangent representation induced by the standard action.

7.6.1. *Equivariant cohomology of* $\mathbb{P}^{n-1}$. Let $\mathbb{T}=(\mathbb{C}^\times)^n$ act on the vector space $F=\mathbb{C}^n$ by the standard representation $\rho_\theta(v)=(\theta_1v_1,\ldots,\theta_nv_n)$. We already know that $c_i^{\mathbb{T}}(F)=c_i(V_{\rho,m})=e_i(t_1,\ldots,t_n)\in H_{\mathbb{T}}^{2i}\cong H^{2i}(B_m)$, where $E_m=(\mathbb{C}^m\setminus 0)^n\to B_m=(\mathbb{P}^{m-1})^n$ are the finite approximations of $\mathrm{E}\mathbb{T}\to\mathrm{B}\mathbb{T}$. Now, the above action makes $\mathbb{P}^{n-1}=\mathbb{P}(F)$ into a $\mathbb{T}$-space, which comes naturally with the $\mathbb{T}$-linearisation $\mathscr{O}_{\mathbb{P}^{n-1}}(1)$. Let W be the total space of this line bundle on $\mathbb{P}^{n-1}$. Let ζ be the equivariant first Chern class of the line bundle $E_m\times^{\mathbb{T}}W$ living over the total space of the projective space bundle

$$\mathbb{P}(V_{\rho,m})\cong E_m\times^{\mathbb{T}}\mathbb{P}^{n-1}\to B_m.$$

Then one has $\zeta=c_1(\mathscr{O}_{\mathbb{P}(V_{\rho,m})}(1))$. We can now compute

$$\begin{aligned}H_{\mathbb{T}}^*(\mathbb{P}^{n-1})&\cong H^*(E_m\times^{\mathbb{T}}\mathbb{P}^{n-1})\\&\cong H^*(\mathbb{P}(V_{\rho,m}))\\&\cong H_{\mathbb{T}}^*[\zeta]\Big/\sum_{i=0}^n c_i(V_{\rho,m})\zeta^{n-i}\\&=H_{\mathbb{T}}^*[\zeta]\Big/\prod_{i=1}^n(\zeta+t_i).\end{aligned}$$

The last equality follows by the description $c_i(V_{\rho,m})=e_i(t_1,\ldots,t_n)$.

7.6.2. *The tangent representation.* Let $V = H^0(\mathbb{P}^{n-1}, \mathscr{O}_{\mathbb{P}^{n-1}}(1))$ be the n-dimensional vector space of linear forms on $\mathbb{P}^{n-1}$, along with the standard representation $\rho_\theta(v) = (\theta_1 v_1, \ldots, \theta_n v_n)$. The weights of the action are the Chern classes $t_i = c_1(\pi_i^* \mathscr{O}(-1)) \in H^2_{\mathbb{T}}$. In the tautological exact sequence

$$0 \to \mathscr{S} \to V \otimes_{\mathbb{C}} \mathscr{O}_{\mathbb{P}^{n-1}} \to \mathscr{O}_{\mathbb{P}^{n-1}}(1) \to 0$$

the bundle on the left evaluated at a point p is the rank $n-1$ vector space of linear forms vanishing at p. The tangent space at a fixed point p_i is identified to

$$T_{p_i}\mathbb{P}^{n-1} = \mathscr{S}_{p_i}^\vee \otimes (V/\mathscr{S}_{p_i}) = \operatorname{Span}_{\mathbb{C}} \left\{ x_j^\vee \otimes x_i \,\middle|\, j \neq i \right\}.$$

This says that the weights of $T_{p_i}\mathbb{P}^{n-1}$ are $t_i - t_j$ for $j \neq i$, in particular the Euler class is computed as

$$e^{\mathbb{T}}(T_{p_i}\mathbb{P}^{n-1}) = \prod_{j \neq i} (t_i - t_j).$$

8. The Atiyah–Bott localisation formula

8.1. **Self-intersection formula.** If $E \to X$ is an oriented vector bundle of rank r on a compact manifold X, the orientation

$$\eta \in H^r(E, E \setminus X)$$

corresponds, by construction, to the identity element

$$\mathbb{1} = \mathsf{pd}([X]) \in H^0(X)$$

under the Thom isomorphism

$$H^0(X) \xrightarrow{\sim} H^r(E, E \setminus X),$$

where $X \hookrightarrow E$ is embedded as the zero section. The inclusions

$$(X, \emptyset) \hookrightarrow (E, \emptyset) \hookrightarrow (E, E \setminus X)$$

induce maps

$$\begin{tikzcd} H^r(E, E \setminus X) \arrow[r] \arrow[rr, bend left, "\alpha"] & H^r(E) \arrow[r] & H^r(X) \end{tikzcd}$$

and the Euler class of E is, by definition, the image of η under this composition, i.e.

$$e(E) = \alpha(\eta) \in H^r(X).$$

The Mayer–Vietoris sequence for the inclusion $X \hookrightarrow E$ yields an exact piece

$$H^{r-1}(E \setminus X) \to H^r(E, E \setminus X) \xrightarrow{u} H^r(E).$$

Let $f : X \to Y$ be a map of compact smooth manifolds. Set $n = \dim X$, $m = \dim Y$ and $d = m - n$. There are Gysin maps

$$f_* : H^p(X) \to H^{p+d}(Y),$$

defined via Poincaré duality (cf. Diagram (8.2.1)).

When $f = \iota$ is a closed embedding of codimension d, and the normal bundle

$$N_{X/Y} = T_Y|_X / T_X$$

is oriented compatibly with respect to f, there is a factorisation of ι_* as

$$H^p(X) \xrightarrow{\sim} H^{p+d}(N_{X/Y}, N_{X/Y} \setminus X) \xrightarrow{\sim} H^{p+d}(Y, Y \setminus X) \to H^{p+d}(Y).$$

where the last map comes from Mayer–Vietoris for the inclusion $\iota \colon X \hookrightarrow Y$, and the middle isomorphism is computed via excision by fixing a tubular neighborhood T of X in Y, so that

$$H^*(Y, Y \setminus X) \cong H^*(T, T \setminus X) \cong H^*(N_{X/Y}, N_{X/Y} \setminus X).$$

Now set $p = 0$. Then there is a commutative diagram

$$\begin{tikzcd}
H^0(X) \arrow[r, "\sim"] \arrow[rrr, bend left, "\iota_*"] & H^d(N_{X/Y}, N_{X/Y} \setminus X) \arrow[r, "\sim"] \arrow[d, "u"] & H^d(Y, Y \setminus X) \arrow[r] & H^d(Y) \arrow[d, "\iota^*"] \\
& H^d(N_{X/Y}) \arrow[rr] & & H^d(X)
\end{tikzcd}$$

which shows the content of the self-intersection formula.

Theorem 8.1.1 (Self-intersection formula)**.** *Let $\iota \colon X \hookrightarrow Y$ be a closed embedding of compact manifolds. There is an identity*

$$\iota^* \iota_* \mathbb{1} = e(N_{X/Y}) \in H^r(X).$$

See Fulton [24] for the corresponding statement in Algebraic Geometry.

8.2. **Equivariant pushforward.** Let G be a compact Lie group. Let $f \colon X \to Y$ be a G-equivariant map of compact manifolds. Set

$$n = \dim X, \quad m = \dim Y, \quad q = n - m.$$

The equivariant pullback $f^* \colon H_G^*(Y) \to H_G^*(X)$ was defined in (7.5.3) by means of the map

$$f^G \colon \mathrm{E}G \times^G X \to \mathrm{E}G \times^G Y$$

i.e. the map (7.5.2) considered in the special case where $G \to H$ is the identity.

The ordinary pushforward $f_* \colon H^*(X) \to H^*(Y)$ can be constructed via Poincaré duality and its inverse through the diagrams

(8.2.1)
$$\begin{tikzcd}
H^p(X) \arrow[r, dashed] \arrow[d, "\wr"] & H^{p-q}(Y) \\
H_{n-p}(X) \arrow[r, "f_*"] & H_{n-p}(Y) \arrow[u, "\mathsf{pd}^{-1}"']
\end{tikzcd}$$

but in order to construct an equivariant pushforward $f_*^G \colon H_G^*(X) \to H_G^*(Y)$ we cannot apply the same procedure because Poincaré duality is not available on infinite dimensional spaces such as $X_G = \mathrm{E}G \times^G X$ and $Y_G = \mathrm{E}G \times^G Y$.

We use approximation spaces to solve this issue. Fix a directed system of principal G-bundles

$$\{ E_i \to B_i \}_{i \ge 0}$$

whose limit recovers the classifying space $\mathrm{E}G \to \mathrm{B}G$ (cf. Remark 7.4.2). Since E_i are compact spaces, so are the Borel spaces

$$X_G^i = E_i \times^G X, \quad Y_G^i = E_i \times^G Y,$$

and the approximation result (Theorem 7.4.1) ensures that

$$H_G^p(X) \cong H^p(X_G^i), \quad H_G^p(Y) \cong H^p(Y_G^i), \quad p \le i. \tag{8.2.2}$$

Recall that we have fibrations

$$\begin{tikzcd} X_G^i \arrow[d, "p_X"] & Y_G^i \arrow[d, "p_Y"] \\ B_i & B_i \end{tikzcd}$$

with fibre X and Y, respectively. It follows that

$$\dim X_G^i = \ell + n, \quad \dim Y_G^i = \ell + m,$$

where $\ell = \dim B_i$. Exploiting Poincaré duality on the compact (finite dimensional) spaces X_G^i and Y_G^i, we can replace the diagram (8.2.1) by a new diagram

$$\begin{tikzcd} H^p(X_G^i) \arrow[r, dashed] \arrow[d, "\wr"] & H^{p-q}(Y_G^i) \\ H_{\ell+n-p}(X_G^i) \arrow[r, "f_*^{G,p}"] & H_{\ell+n-p}(Y_G^i) \arrow[u, "\mathsf{pd}^{-1}"'] \end{tikzcd}$$

which we can of course redraw as

$$\begin{tikzcd} H_G^p(X) \arrow[r, dashed] \arrow[d, "\wr"] & H_G^{p-q}(Y) \\ H_{\ell+n-p}(X_G^i) \arrow[r, "f_*^{G,p}"] & H_{\ell+n-p}(Y_G^i) \arrow[u, "\mathsf{pd}^{-1}"'] \end{tikzcd}$$

after exploiting (8.2.2). This yields a system of maps

$$f_*^{G,p} \colon H_G^p(X) \to H_G^{p-q}(Y).$$

EXERCISE 8.2.1. Prove that the maps $f_*^{G,p}$ are compatible with the structure of inverse system of $H^p(X_G^i)$ and $H^p(Y_G^i)$ attached to the directed systems $(X_G^i)_i$ and $(Y_G^i)_i$ respectively.

By Exercise 8.2.1, we can construct the *equivariant pushforward*

$$f_*^G \colon H_G^*(X) \to H_G^*(Y).$$

8.3. **Trivial torus actions.** Let $\mathbb{T}$ be a torus acting *trivially* on a smooth complex algebraic variety X. Given a $\mathbb{T}$-equivariant vector bundle $E \to X$ of rank r, we get a canonical decomposition

$$E = \bigoplus_\chi E_\chi, \tag{8.3.1}$$

where $\mathbb{T}$ acts by the character χ on E_χ. The characters χ vary in the character group $\mathbb{T}^* = \mathbb{Z}^{\dim \mathbb{T}}$ of the torus. One should expect to be able to express the $\mathbb{T}$-equivariant Chern classes of E in terms of the $\mathbb{T}$-equivariant Chern classes of its subbundles $E_\chi \subset E$.

Let

$$V_E = \mathrm{E}\mathbb{T} \times^{\mathbb{T}} E \to \mathrm{E}\mathbb{T} \times^{\mathbb{T}} X = X \times \mathrm{B}\mathbb{T}$$

be the induced rank r bundle, cf. (7.5.4) (and (7.3.4) for the identification on the right hand side).

EXERCISE 8.3.1. Show that $V_{E_\chi} \cong E_\chi \boxtimes V_\chi$, where the box product refers to the canonical projections from $X \times \mathrm{B}\mathbb{T}$ and V_χ is the line bundle on $\mathrm{B}\mathbb{T}$ introduced in Example 7.5.4.

EXERCISE 8.3.2. Assume again $\mathbb{T}$ acts trivially on a smooth variety X, and let E be a $\mathbb{T}$-equivariant vector bundle. Show that

$$c_i^{\mathbb{T}}(E_\chi) = \sum_{k=0}^{i} \binom{r_\chi - k}{i-k} c_k(E_\chi)\chi^{i-k} \in H_{\mathbb{T}}^{2i}(X),$$

where $r_\chi = \operatorname{rk} E_\chi$. (**Hint**: use the previous exercise and a standard property of Chern classes of tensor products, cf. [24, Example 3.2.2] or Example **??** in these notes).

8.4. **Torus fixed loci.** Throughout this section, we let X be a smooth complex algebraic variety acted on by a torus $\mathbb{T}$. The scheme structure of the fixed locus

$$X^{\mathbb{T}} \subset X$$

is discussed in [23, Section 2].

We have the following result.

Theorem 8.4.1 ([38, 23])**.** *If X is a smooth $\mathbb{T}$-variety, the fixed locus $X^{\mathbb{T}}$ is smooth.*

EXERCISE 8.4.2. Let Y be a compact normal $\mathbb{T}$-variety. Show that if Y has a singular point, then it has a *torus-fixed* singular point.

Let N be the normal bundle of the inclusion $F \subset X$ of a component $F \subset X^{\mathbb{T}}$. Then N is $\mathbb{T}$-equivariant and for each $x \in F$ one has $(T_xX)^{\mathbb{T}} = T_xF$ by results of [23], so that the action of $\mathbb{T}$ on the normal space

$$N_x = T_xX/T_xF$$

is nontrivial, i.e. N_x has no trivial subrepresentations, i.e. $N_x^{\mathbb{T}} = 0$. It follows that the Euler class of the normal bundle N is nonzero, being a product of nonzero weights. In fact, $e^{\mathbb{T}}(N)$ becomes invertible in the ring $H_{\mathbb{T}}^*(F)[\chi_i^{-1}]$, where χ_i are the characters that occur in the decomposition of N into eigenbundles.

Let E be a $\mathbb{T}$-equivariant vector bundle of rank r over X. Let $F \subset X^{\mathbb{T}}$ be a component of the fixed locus, so that $H_{\mathbb{T}}^*(F) = H^*(F) \otimes H_{\mathbb{T}}^*$. According to Equation (8.3.1), the vector bundle $E|_F$ on F has a decomposition

$$E|_F = \bigoplus_\chi E_{F,\chi}$$

into eigen-subbundles. By Exercise 8.3.2, the component of

$$c_i^{\mathbb{T}}(E_{F,\chi}) \in H_{\mathbb{T}}^{2i}(F) \tag{8.4.1}$$

in $H^{2i}_{\mathbb{T}}$ is given by

$$\binom{r}{i}\chi^i \in H^{2i}_{\mathbb{T}}. \tag{8.4.2}$$

Here we are denoting simply by χ the weight $w_\chi \in H^2_{\mathbb{T}}$, cf. Example 7.5.4. Since $H^{2k}(F)=0$ for $k>\dim F$, we have that for all $j>0$, the classes in $H^{2j}(F)$ are nilpotent in $H^*_{\mathbb{T}}(F)$. Therefore the element (8.4.1) is invertible if and only if its component (8.4.2) is invertible. It follows that

$$c_i^{\mathbb{T}}(E_{F,\chi}) \text{ is invertible in the localised ring } H^{2i}_{\mathbb{T}}(F)\big[\chi^{-1}\big]. \tag{8.4.3}$$

Proposition 8.4.3. *Let X be a smooth $\mathbb{T}$-variety, let $F\subset X^{\mathbb{T}}$ be a component of codimension d. Then there are finitely many characters $\chi_1,\ldots,\chi_s$ such that the Euler class*

$$e^{\mathbb{T}}(N_{F/X})\in H^{2d}_{\mathbb{T}}(F)$$

becomes invertible in the ring

$$H^*_{\mathbb{T}}(F)\big[\chi_s^{-1},\ldots,\chi_s^{-1}\big].$$

Proof. Set $N=N_{F/X}$. We saw above that the action on the normal space N_x is nontrivial for all $x\in F$, so the characters appearing in the decomposition

$$N=\bigoplus_{i=1}^{s} N_{\chi_i}$$

are all nontrivial, and $e^{\mathbb{T}}(N)$ is a product of nonzero weights,

$$0\neq e^{\mathbb{T}}(N)=\prod_{i=1}^{s} e^{\mathbb{T}}(N_{\chi_i}),$$

so the observation (8.4.3) implies the result. □

8.5. **The localisation formula.** This subsection introduces the technique of localisation in Algebraic Geometry. Our purpose is to apply this powerful tool to solve enumerative problems, such as finding the number of lines on a general cubic surface $S\subset\mathbb{P}^3$, or on a general quintic 3-fold $Y\subset\mathbb{P}^4$.

Let $\iota\colon X^{\mathbb{T}}\hookrightarrow X$ be the inclusion of the fixed point locus. We have the equivariant pushforward

$$\iota_*\colon H^*_{\mathbb{T}}(X^{\mathbb{T}})\to H^*_{\mathbb{T}}(X),$$

and the localisation theorem states that this map becomes an isomorphism after inverting finitely many nontrivial characters. In particular, it becomes an isomorphism after extending scalars to the field of fractions

$$\mathcal{H}^*_{\mathbb{T}}=\operatorname{Frac} H^*_{\mathbb{T}}.$$

Notice that

$$H^*_{\mathbb{T}}(X^{\mathbb{T}})=\bigoplus_{\alpha} H^*_{\mathbb{T}}(F_\alpha)$$

where F_α are the components of the fixed locus. The crucial point is that, if N_α is the normal bundle of the inclusion $\iota_\alpha\colon F_\alpha\hookrightarrow X$, then

$$\iota_\alpha^*\iota_{\alpha*}(-)=e^{\mathbb{T}}(N_\alpha)\cap -$$

and as we saw this Euler class is nonzero when restricted to any point $x \in F_\alpha$. This is enough for $e^{\mathbb{T}}(N_\alpha)$ to become invertible after a suitable localisation.

The statement of the localisation theorem for compact manifolds with torus action is the following.

Theorem 8.5.1 (Atiyah–Bott [4])**.** *Let M be a compact smooth manifold equipped with an action of a torus $\mathbb{T}$. Then the equivariant pushforward along $\iota\colon M^{\mathbb{T}} \hookrightarrow M$ induces an isomorphism*

$$\iota_*\colon H^*_{\mathbb{T}}(M^{\mathbb{T}}) \otimes_{H^*_{\mathbb{T}}} \mathcal{H}^*_{\mathbb{T}} \xrightarrow{\sim} H^*_{\mathbb{T}}(M) \otimes_{H^*_{\mathbb{T}}} \mathcal{H}^*_{\mathbb{T}}.$$

Its inverse is given by

$$\psi \mapsto \sum_\alpha \frac{\iota_\alpha^* \psi}{e^{\mathbb{T}}(N_\alpha)}.$$

Remark 8.5.2. In particular, every class $\psi \in H^*_{\mathbb{T}}(M) \otimes_{H^*_{\mathbb{T}}} \mathcal{H}^*_{\mathbb{T}}$ writes uniquely as

$$\psi = \sum_\alpha \iota_{\alpha *} \frac{\iota_\alpha^* \psi}{e^{\mathbb{T}}(N_\alpha)}. \tag{8.5.1}$$

Let M be a compact manifold with a $\mathbb{T}$-action and structure map $q\colon M \to \mathrm{pt}$. We have the equivariant pushforward $q_*\colon H^*_{\mathbb{T}}(M) \to H^*_{\mathbb{T}}$, which after tensoring by $\mathcal{H}^*_{\mathbb{T}}$ yields the integration map

$$\int_M \colon H^*_{\mathbb{T}}(M) \otimes_{H^*_{\mathbb{T}}} \mathcal{H}^*_{\mathbb{T}} \to \mathcal{H}^*_{\mathbb{T}}.$$

For any component $F_\alpha \subset M^{\mathbb{T}}$, the structure map $q_\alpha\colon F_\alpha \to \mathrm{pt}$ factors as $q \circ \iota_\alpha$ where $\iota_\alpha\colon F_\alpha \hookrightarrow M$ is the inclusion. The condition $q_{\alpha *} = q_* \circ \iota_{\alpha *}$ then allows us, simply by looking at (8.5.1), to deduce the following integration formula: for any equivariant class $\psi \in H^*_{\mathbb{T}}(M) \otimes_{H^*_{\mathbb{T}}} \mathcal{H}^*_{\mathbb{T}}$, one has

$$\int_X \psi = \sum_\alpha q_{\alpha *} \frac{\iota_\alpha^* \psi}{e^{\mathbb{T}}(N_\alpha)} = \sum_\alpha \int_{F_\alpha} \frac{\iota_\alpha^* \psi}{e^{\mathbb{T}}(N_\alpha)} \in \mathcal{H}^*_{\mathbb{T}}. \tag{8.5.2}$$

Let us go back to the algebraic setting (in the special case of finitely many fixed points).

Proposition 8.5.3. *Let X be a smooth $\mathbb{T}$-variety with finitely many fixed points. Set*

$$\mathsf{e} = \prod_{p \in X^{\mathbb{T}}} e^{\mathbb{T}}(T_p X) \in H^*_{\mathbb{T}},$$

and fix a multiplicative subset $S \subseteq H^*_{\mathbb{T}} \setminus 0$ containing e. Then

1. *The pullback map $S^{-1}\iota^*\colon S^{-1}H^*_{\mathbb{T}}(X) \to S^{-1}H^*_{\mathbb{T}}(X^{\mathbb{T}})$ is onto, and*
2. *if $H^*_{\mathbb{T}}(X)$ is a free $H^*_{\mathbb{T}}$-module of rank $r \le |X^{\mathbb{T}}|$, then $r = |X^{\mathbb{T}}|$ and $S^{-1}\iota^*$ is an isomorphism.*

Proof. To prove (1), note that the composition

$$S^{-1}H^*_{\mathbb{T}}(X^{\mathbb{T}}) \xrightarrow{S^{-1}\iota_*} S^{-1}H^*_{\mathbb{T}}(X) \xrightarrow{S^{-1}\iota^*} S^{-1}H^*_{\mathbb{T}}(X^{\mathbb{T}})$$

is surjective because it equals $S^{-1}(\iota^* \circ \iota_*)$, and the determinant of $\iota^* \circ \iota_*$ is precisely e because it is a diagonal map, equal to $e^{\mathbb{T}}(T_p X)$ on the component indexed by p. But e becomes invertible after localisation by Proposition 8.4.3, so (1) follows.

By Part (1), we have

$$r = \mathrm{rk}_{S^{-1}H^*_{\mathbb{T}}} S^{-1}H^*_{\mathbb{T}}(X) \geq \mathrm{rk}_{S^{-1}H^*_{\mathbb{T}}} S^{-1}H^*_{\mathbb{T}}(X^{\mathbb{T}}) = |X^{\mathbb{T}}| \geq r,$$

which implies $\mathrm{rk}_{S^{-1}H^*_{\mathbb{T}}} S^{-1}H^*_{\mathbb{T}}(X) = |X^{\mathbb{T}}|$. To prove Part (2), observe that $S^{-1}H^*_{\mathbb{T}}$ is a Noetherian ring (being a localisation of a Noetherian ring), hence a surjective $S^{-1}H^*_{\mathbb{T}}$-linear map of free modules of the same rank, such as $S^{-1}\iota^*$, is necessarily an isomorphism. □

Remark 8.5.4. Let X be a smooth complex projective variety, with an action by a torus $\mathbb{T}$ having finitely many fixed points $p_1,\ldots,p_s$. Then the Białynicki-Birula decomposition [9] yields s $\mathbb{T}$-invariant subvarieties

$$Y_1,\ldots,Y_s \subset X$$

with the property that the $\mathbb{T}$-equivariant cohomology classes (cf. Example 7.5.2)

$$[Y_\ell]^{\mathbb{T}} \in H^*_{\mathbb{T}}(X)$$

form a *free $H^*_{\mathbb{T}}$-basis* of the equivariant cohomology ring, restricting to a $\mathbb{Z}$-basis of the ordinary cohomology ring $H^*(X)$. In other words,

$$H^*_{\mathbb{T}}(X) \cong \bigoplus_{\ell=1}^{s} H^*_{\mathbb{T}} \cdot [Y_\ell]^{\mathbb{T}}$$

is a free $H^*_{\mathbb{T}}$-module of rank s.

The form of the localisation theorem that we will need is the following.

Corollary 8.5.5 (Integration Formula)**.** *Let X be a smooth complex projective $\mathbb{T}$-variety with finitely many fixed points. There is an identity*

$$\int_X \psi = \sum_{q\in X^{\mathbb{T}}} \frac{i_q^*\psi}{e^{\mathbb{T}}(T_qX)} \tag{8.5.3}$$

for all $\psi \in H^*_{\mathbb{T}}(X)$.

Proof. Let $S \subseteq H^*_{\mathbb{T}} \setminus 0$ be a multiplicative subset as in Theorem 8.5.3. By the surjectivity of

$$S^{-1}\iota_*\colon S^{-1}H^*_{\mathbb{T}}(X^{\mathbb{T}}) \xrightarrow{\sim} S^{-1}H^*_{\mathbb{T}}(X),$$

due to part (2) of Proposition 8.5.3, we may assume

$$\psi = \iota_{p*}\theta,$$

for $p \in X^{\mathbb{T}}$ and $\theta \in H^*_{\mathbb{T}}$. Then clearly

$$\int_X \psi = \int_X \iota_{p*}\theta = \theta,$$

because $(\int_X)\circ \iota_{p*}$ is an isomorphism. On the other hand,

$$\sum_{q\in X^{\mathbb{T}}} \frac{i_q^*\iota_{p*}\theta}{e^{\mathbb{T}}(T_qX)} = \frac{\iota_p^*\iota_{p*}\theta}{e^{\mathbb{T}}(T_pX)} = \frac{e^{\mathbb{T}}(T_pX)}{e^{\mathbb{T}}(T_pX)} \cap \theta = \theta. \qquad \square$$

Remark 8.5.6. Formula (8.5.2) is true in the algebraic context as well (for X a smooth projective variety) if one sets $S = H^*_{\mathbb{T}} \setminus 0$, but we will not need it in these notes.

Lemma 8.5.7. *Let M be a smooth oriented compact manifold with a torus action having finitely many fixed points $p_1,\ldots,p_s$. Then*

$$\chi(M)=s.$$

Proof. Recall that $\chi(M)=\int_M e(T_M)$ by Poincaré–Hopf. We have

$$\begin{aligned}\int_M e(T_M) &= \int_M e^{\mathbb{T}}(T_M)\\ &= \sum_{1\le i\le s}\frac{e^{\mathbb{T}}(T_M)|_{p_i}}{e^{\mathbb{T}}(N_{p_i/M})}\\ &= \sum_{1\le i\le s} 1\\ &= s.\end{aligned}$$ ☐

More generally, one can prove the following:

Lemma 8.5.8 ([15, Prop. 2.5.1])**.** *Let $\mathbb{T}$ be a torus acting on a quasi-projective scheme Y of finite type over $\mathbb{C}$. Then*

$$\chi(Y)=\chi(Y^{\mathbb{T}}).$$

The following result is also very useful in computations.

Lemma 8.5.9. *Let Y be a variety with a $\mathbb{T}$-action. Suppose the fixed locus $Y^{\mathbb{T}}$ is finite. Then there is a 1-dimensional subtorus $\mathbb{G}_m\subset\mathbb{T}$ such that $Y^{\mathbb{T}}=Y^{\mathbb{G}_m}$.*

EXERCISE 8.5.10. Compute the Euler characteristic of the Grassmannian $G(k,n)$. (**Hint**: Lift the standard torus action of $\mathbb{T}=\mathbb{G}_m^n$ on $\mathbb{C}^n$ to the Grassmannian).

EXERCISE 8.5.11. Let $n>0$ be an integer. Show that

$$\sum_{i=1}^n\frac{(-t_i)^k}{\displaystyle\prod_{\substack{1\le j\le n\\ j\ne i}}(t_i-t_j)}=\begin{cases}0 & \text{if } 0\le k<n-1\\ 1 & \text{if } k=n-1.\end{cases}$$

(**Hint**: Consider the standard action on $X=\mathbb{P}^{n-1}$ and apply Equation (8.5.3) to $\psi=c_1^{\mathbb{T}}(\mathscr{O}_{\mathbb{P}^{n-1}}(1))$).

9. APPLICATIONS OF THE LOCALISATION FORMULA

In this section we give four complete examples on how to use Theorem 8.5.1 to solve enumerative problems.

9.1. **How not to compute the simplest intersection number.** In this section we show how to use localisation to compute the number of intersection points between two general lines in $\mathbb{P}^2$.

Let $\mathbb{T}=\mathbb{G}_m\subset\mathbb{G}_m^3$ be a one-parameter subgroup acting with weights w_0,w_1,w_2 on the vector space of linear forms

$$V=H^0(\mathbb{P}^2,\mathscr{O}_{\mathbb{P}^2}(1))=\operatorname{Span}_{\mathbb{C}}\{\,x_0,x_1,x_2\,\}.$$

This means that $t\cdot x_i = t^{w_i} x_i$. Choosing the w_i's distinct from one another ensures that the induced action on $\mathbb{P}^2$ has the three fixed points $p_0 = (1,0,0)$, $p_1 = (0,1,0)$, $p_2 = (0,0,1)$. The intersection number we want to compute is

$$\int_{\mathbb{P}^2} c_1(\mathscr{O}_{\mathbb{P}^2}(1))^2 = \int_{\mathbb{P}^2} c_1^{\mathbb{T}}(\mathscr{O}_{\mathbb{P}^2}(1))^2 = \sum_{i=0}^{2} \frac{c_1^{\mathbb{T}}(\mathscr{O}_{\mathbb{P}^2}(1)|_{p_i})^2}{e^{\mathbb{T}}(T_{p_i}\mathbb{P}^2)}. \tag{9.1.1}$$

The universal exact sequence

$$0 \to \mathscr{S} \to V\otimes_{\mathbb{C}} \mathscr{O}_{\mathbb{P}^2} \to \mathscr{O}_{\mathbb{P}^2}(1) \to 0$$

restricts, at p_i, to a short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_{jk} & \longrightarrow & V & \longrightarrow & V/V_{jk} & \longrightarrow & 0 \\ & & & & & & \| & & \\ & & & & & & \mathbb{C}\cdot x_i & & \end{array}$$

where $V_{jk}\subset V$ is the span of x_j, x_k, i.e. the space of linear forms vanishing at p_i. We already know the weights of the tangent representations

$$T_{p_i}\mathbb{P}^2 = V_{jk}^{\vee}\otimes V/V_{jk} = \operatorname{Span}\{\, x_j^{\vee}\otimes x_i,\, x_k^{\vee}\otimes x_i \,\}.$$

They are simply $w_i - w_j$, $w_i - w_k$, where $\{i,j,k\} = \{0,1,2\}$. The line bundle $\mathscr{O}_{\mathbb{P}^2}(1)$ is the universal quotient bundle and clearly

$$\mathscr{O}_{\mathbb{P}^2}(1)|_{p_i} = V/V_{jk} = \mathbb{C}\cdot x_i$$

has weight $c_1^{\mathbb{T}}(\mathscr{O}_{\mathbb{P}^2}(1)|_{p_i}) = -w_i$ at p_i by Example 7.5.5. The sum in (9.1.1) then equals

$$\frac{(-w_0)^2}{(w_0-w_1)(w_0-w_2)} + \frac{(-w_1)^2}{(w_1-w_0)(w_1-w_2)} + \frac{(-w_2)^2}{(w_2-w_0)(w_2-w_1)}. \tag{9.1.2}$$

The latter equals 1 for *every* choice of (pairwise distinct) w_0, w_1, w_2.

EXERCISE 9.1.1. Compare the latter calculation to the one of Exercise 8.5.11.

Remark 9.1.2. The fact that the sum (9.1.2) equals 1 has the following interpretation.[7] The three fractions above can be seen as the residues of the differential form

$$\frac{z^2\,\mathrm{d}z}{(z-w_0)(z-w_1)(z-w_2)}.$$

However, there is another residue to compute: the one at ∞. This residue equals -1. The residue theorem then precisely states that

$$0 = -1 + \frac{w_0^2}{(w_0-w_1)(w_0-w_2)} + \frac{w_1^2}{(w_1-w_0)(w_1-w_2)} + \frac{w_2^2}{(w_2-w_0)(w_2-w_1)}.$$

[7] Thanks to Fran Globlek for pointing this out.

9.2. **The 27 lines on a cubic surface.** Let $\mathbb{T} = \mathbb{G}_m$ be a torus acting on $\mathbb{P}^3$ with distinct weights (w_0, w_1, w_2, w_3). This means, as ever,

$$t \cdot x_i = t^{w_i} x_i, \quad 0 \le i \le 3.$$

This is also equivalent to considering the vector space

$$V = H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(1)) = \mathrm{Span}\{x_0, x_1, x_2, x_3\}$$

as an equivariant vector bundle over $\mathrm{Spec}\,\mathbb{C}$, splitting into a sum of characters $t \mapsto t^{w_i}$. The torus action has four fixed points $p_0, \ldots, p_3 \in \mathbb{P}^3$ and six invariant lines $\ell_{ij} \subset \mathbb{P}^3$ which are the lines joining the fixed points (see Figure 8). These correspond to the fixed points of the Grassmannian $\mathbb{G}(1,3)$ under the same $\mathbb{T}$-action. You computed the Euler characteristic of the Grassmannian in Exercise 8.5.10.

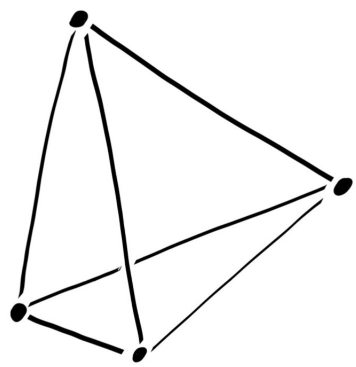


FIGURE 8. The toric polytope of $\mathbb{P}^3$.

Let S be a general cubic hypersurface in $\mathbb{P}^3$, defined by a homogeneous cubic polynomial $f \in \mathbb{C}[x_0, x_1, x_2, x_3]$. A line $\ell \subset \mathbb{P}^3$ is contained in S if and only if the image of f under the restriction map

$$H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(3)) \to H^0(\ell, \mathscr{O}_\ell(3))$$

vanishes. We know from Section 5 that the cycle of lines in S is Poincaré dual to the top Chern class

$$e(\mathrm{Sym}^3 \mathscr{S}^\vee) = c_4(\mathrm{Sym}^3 \mathscr{S}^\vee) \in A^4\mathbb{G}(1,3),$$

where $\mathscr{S}^\vee$ is the dual of the tautological subbundle. By Lemma 5.0.4, we have

$$\#\{\text{lines in } S\} = \int_{\mathbb{G}(1,3)} e(\mathrm{Sym}^3 \mathscr{S}^\vee).$$

According to the strategy outlined in Section 7.3.1, we will compute the latter intersection number by computing instead the equivariant integral

$$\int_{\mathbb{G}(1,3)} e^{\mathbb{T}}(\mathrm{Sym}^3 \mathscr{S}^\vee) \in \mathcal{H}_{\mathbb{T}}$$

via the localisation formula, and specialising the equivariant parameters appropriately (i.e. avoiding the creation of poles in the localisation formula).

Restricting the tautological exact sequence

$$0 \to \mathscr{S} \to \mathscr{O}_{\mathbb{G}(1,3)} \otimes_{\mathbb{C}} V^\vee \to \mathscr{Q} \to 0 \tag{9.2.1}$$

to the point corresponding to the linear subspace

$$\ell_{ij} = \mathbb{C} \cdot x_i^\vee \oplus \mathbb{C} \cdot x_j^\vee \subset V^\vee,$$

we obtain the sequence of equivariant vector spaces

$$0 \to \mathbb{C} \cdot x_i^\vee \oplus \mathbb{C} \cdot x_j^\vee \to V^\vee \to \mathbb{C} \cdot x_h^\vee \oplus \mathbb{C} \cdot x_k^\vee \to 0$$

where $\{0,1,2,3\} = \{i, j, h, k\}$. Therefore

$$\mathscr{Q}|_{\ell_{ij}} = \mathbb{C} \cdot x_h^\vee \oplus \mathbb{C} \cdot x_k^\vee$$

has weights $-w_h, -w_k$, and similarly

$$\mathscr{S}^\vee|_{\ell_{ij}} = \mathbb{C} \cdot x_i \oplus \mathbb{C} \cdot x_j$$

has weights w_i and w_j. Since

$$T_{\ell_{ij}} \mathbb{G}(1,3) = \mathscr{S}^\vee|_{\ell_{ij}} \otimes \mathscr{Q}|_{\ell_{ij}} = \operatorname{Span}\left\{ x_i \otimes x_h^\vee, x_j \otimes x_h^\vee, x_i \otimes x_k^\vee, x_j \otimes x_k^\vee \right\},$$

we obtain an identity

$$e^{\mathbb{T}}(T_{\ell_{ij}} \mathbb{G}(1,3)) = (w_i - w_h) \cdot (w_j - w_h) \cdot (w_i - w_k) \cdot (w_j - w_k) \in H_{\mathbb{T}}^*. \tag{9.2.2}$$

This is one of the key ingredients in the localisation formula, which reads

$$\int_{\mathbb{G}(1,3)} e^{\mathbb{T}}(\operatorname{Sym}^3 \mathscr{S}^\vee) = \sum_{\ell_{ij}} \frac{e^{\mathbb{T}}(\operatorname{Sym}^3 \mathscr{S}^\vee)|_{\ell_{ij}}}{e^{\mathbb{T}}(T_{\ell_{ij}} \mathbb{G}(1,3))}. \tag{9.2.3}$$

The numerators in (9.2.3) are computed as follows. Note that $\operatorname{Sym}^3 \mathscr{S}^\vee|_{\ell_{ij}} = \operatorname{Sym}^3(\mathbb{C} \cdot x_i \oplus \mathbb{C} \cdot x_j)$ is the four dimensional vector space generated by the classes of $x_i^3, x_i^2 x_j, x_i x_j^2, x_j^3$. Using the weights w_i and w_j of $\mathscr{S}^\vee$ we find

$$e^{\mathbb{T}}(\operatorname{Sym}^3 \mathscr{S}^\vee)|_{\ell_{ij}} = (3w_i) \cdot (2w_i + w_j) \cdot (w_i + 2w_j) \cdot (3w_j) \in H_{\mathbb{T}}^*.$$

We are now able to write down the right hand side of (9.2.3) as follows:

$$\begin{aligned}
\sum_{0 \le i < j \le 3} \frac{e^{\mathbb{T}}(\operatorname{Sym}^3 \mathscr{S}^\vee)|_{\ell_{ij}}}{e^{\mathbb{T}}(T_{\ell_{ij}} \mathbb{G}(1,3))} &= \sum_{0 \le i < j \le 3} \frac{(3w_i)(2w_i + w_j)(w_i + 2w_j)(3w_j)}{(w_i - w_h)(w_j - w_h)(w_i - w_k)(w_j - w_k)} \\
&= 9 \frac{w_0(2w_0 + w_1)(w_0 + 2w_1)w_1}{(w_0 - w_2)(w_0 - w_3)(w_1 - w_2)(w_1 - w_3)} \\
&\quad + 9 \frac{w_0(2w_0 + w_2)(w_0 + 2w_2)w_2}{(w_0 - w_1)(w_0 - w_3)(w_2 - w_1)(w_2 - w_3)} \\
&\quad + 9 \frac{w_0(2w_0 + w_3)(w_0 + 2w_3)w_3}{(w_0 - w_1)(w_0 - w_2)(w_3 - w_1)(w_3 - w_2)} \\
&\quad + 9 \frac{w_1(2w_1 + w_2)(w_1 + 2w_2)w_2}{(w_1 - w_0)(w_1 - w_3)(w_2 - w_0)(w_2 - w_3)} \\
&\quad + 9 \frac{w_1(2w_1 + w_3)(w_1 + 2w_3)w_3}{(w_1 - w_0)(w_1 - w_2)(w_3 - w_0)(w_3 - w_2)} \\
&\quad + 9 \frac{w_2(2w_2 + w_3)(w_2 + 2w_3)w_3}{(w_2 - w_0)(w_2 - w_1)(w_3 - w_0)(w_3 - w_1)}.
\end{aligned} \tag{9.2.4}$$

Evaluating at $(w_0, w_1, w_2, w_3) = (0, 2, -1, 1)$ yields

$$\#\{\text{lines in } S\} = 9\left(0+0+0+0+\frac{40}{12}-\frac{1}{3}\right) = 30-3 = 27.$$

9.3. **Lines on the quintic 3-fold.** In this section we will prove the following result.

Theorem 9.3.1. *Let $Y \subset \mathbb{P}^4$ be a generic quintic 3-fold. Then Y contains exactly 2875 lines.*

Remark 9.3.2. The statement is not true for *all* quintic Calabi–Yau 3-folds. Indeed, by work of Albano–Katz [1], the Fermat quintic

$$\left\{ x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0 \right\} \subset \mathbb{P}^4$$

contains 50 one-dimensional families of lines.

We will need an auxiliary result by Katz.

Theorem 9.3.3 ([39, App. A])**.** *Let Y be a generic quintic 3-fold, $C \subset Y$ a smooth rational curve of degree $d \le 3$. Then C has normal bundle $\mathscr{O}_C(-1)^{\oplus 2}$.*

The role of this theorem, for us, is to make sure that the intersection number we compute via localisation really is the number we are after. The above theorem, in other words, plays the role of Lemma 5.0.4 that we needed for lines on a cubic surface.

The ambient space we have to work in now is the 6-dimensional Grassmannian

$$G(2,5) = \mathbb{G}(1,4).$$

Let $\mathscr{S}$ be the rank 2 universal subbundle

$$\mathscr{S} \hookrightarrow \mathscr{O}_{\mathbb{G}(1,4)} \otimes_{\mathbb{C}} H^0(\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}(1))^\vee.$$

Let $\mathbb{T} = \mathbb{G}_m^5$ be a torus acting as

$$t \cdot x_i = t_i \cdot x_i$$

on $\mathbb{P}^4$. This action lifts to a $\mathbb{T}$-action on $\mathbb{G}(1,4)$, with 10 fixed points corresponding to the subsets

$$I \subset \{0,1,2,3,4\}, \quad |I| = 2.$$

Let $\ell_{ij} \subset V^\vee$ be the linear subspace spanned by $x_i^\vee$ and $x_j^\vee$. The characters of the 2-dimensional (equivariant) vector space

$$\mathscr{S}|_{\ell_{ij}} = \mathbb{C}\cdot x_i^\vee \oplus \mathbb{C}\cdot x_j^\vee$$

are $\{-\chi_i, -\chi_j\}$, where $\chi_i\colon \mathbb{T} \to \mathbb{C}^\times$ sends $t \mapsto t_i$. Similarly, the characters of $\mathscr{Q}|_{\ell_{ij}}$ are $\{-\chi_a, -\chi_b, -\chi_c\}$, where $\{a,b,c,i,j\} = \{0,1,2,3,4\}$. Let $t_k \in H^2_{\mathbb{T}}$ denote the weight of $-\chi_k$, for all k. The tangent space of $\mathbb{G}(1,4)$ at ℓ_{ij} is, as ever,

$$\mathscr{S}^\vee|_{\ell_{ij}} \otimes \mathscr{Q}|_{\ell_{ij}} = \operatorname{Span}\left\{ x_i \otimes x_u^\vee, x_j \otimes x_u^\vee \,\middle|\, u \in \{a,b,c\} \right\},$$

so we obtain the Euler class

$$e^{\mathbb{T}}(T_{\ell_{ij}}\mathbb{G}(1,4)) = \prod_{k \notin I} (t_i - t_k)(t_j - t_k).$$

The rank 6 vector bundle

$$\begin{tikzcd} \mathrm{Sym}^5(\mathscr{S}^\vee) \arrow[d] & \rightsquigarrow & \mathrm{Sym}^5(\mathscr{S}^\vee)|_{\ell_{ij}} \arrow[d] \\ \mathbb{G}(1,4) & & \mathsf{pt} \end{tikzcd}$$

inherits weights

$$5t_j, t_i+4t_j, 2t_i+3t_j, 3t_i+2t_j, 4t_i+t_j, 5t_i.$$

Therefore, denoting i_1 and i_2 the generic elements of a subset I of size 2, the localisation formula reads

$$\int_{\mathbb{G}(1,4)} e^{\mathbb{T}}(\mathrm{Sym}^5(\mathscr{S}^\vee)) = \sum_{|I|=2} \frac{\prod_{h=0}^{5}(ht_{i_1}+(5-h)t_{i_2})}{\prod_{i\in I}\prod_{k\notin I}(t_i-t_k)} = 2875. \tag{9.3.1}$$

If you do not believe the last identity, you can copy the following code[8] in Mathematica:

```
Plus @@
(Product[h Subscript[t, #[[1]]] + (5 - h)
Subscript[t, #[[2]]], {h, 0, 5}]/
Product[Subscript[t, i] - Subscript[t, k], {i, #},
{k, DeleteCases[Range[0, 4], Alternatives @@ #]}]
& /@ Subsets[Range[0, 4], {2}])
// Together
```

Theorem 9.3.1 now follows from Theorem 9.3.3.

9.4. **The number of lines through 4 general lines in 3-space.** The goal of this section is to prove the following.

Proposition 9.4.1. *There are exactly two lines $\ell \subset \mathbb{P}^3$ intersecting four general lines.*

Let L be a general 2-dimensional linear subspace of $V = H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(1))$. Consider the cycle

$$\Sigma_1(L) = \{\Lambda \subset V \mid \Lambda \cap L \neq 0\}.$$

EXERCISE 9.4.2. Show that $\Sigma_1(L)$ is 3-dimensional, and its cohomology class

$$\sigma_1 = [\Sigma_1(L)] \in A^1 G(2,4)$$

is independent of L.

We need the following preliminary result.

Lemma 9.4.3. *The cycle $\Sigma_1 \subset \mathbb{G}(1,3)$ of lines meeting a general line $L \subset \mathbb{P}^3$ is Poincaré dual to the first Chern class*

$$c_1(\mathscr{S}^\vee).$$

The goal now becomes to compute the intersection number

$$\int_{\mathbb{G}(1,3)} \sigma_1^4 = \int_{\mathbb{G}(1,3)} c_1(\mathscr{S}^\vee)^4.$$

[8] Thanks to Matteo Gallet for providing the code.

We already saw that for the $\mathbb{T}$-fixed linear subspace

$$\ell_{ij} = \mathbb{C}\cdot x_i^\vee \oplus \mathbb{C}\cdot x_j^\vee \subset H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(1))^\vee$$

the weights of the 2-dimensional representation

$$\mathscr{S}^\vee|_{\ell_{ij}} = \mathbb{C}\cdot x_i \oplus \mathbb{C}\cdot x_j$$

are w_i and w_j. Therefore

$$c_1^{\mathbb{T}}(\mathscr{S}^\vee)|_{\ell_{ij}} = w_i + w_j.$$

The weights of the tangent representation have been computed in (9.2.2). Putting everything together we can compute, by means of the localisation formula,

$$\begin{aligned}
\int_{\mathbb{G}(1,3)} c_1^{\mathbb{T}}(\mathscr{S}^\vee)^4 &= \sum_{\ell_{ij}} \frac{c_1^{\mathbb{T}}(\mathscr{S}^\vee)|_{\ell_{ij}}}{e^{\mathbb{T}}(T_{\ell_{ij}}\mathbb{G}(1,3))} \\
&= \sum_{0\le i<j\le 3} \frac{(w_i+w_j)^4}{(w_i-w_h)(w_j-w_h)(w_i-w_k)(w_j-w_k)} \\
&= \frac{(w_2+w_3)^4}{(w_2-w_0)(w_2-w_1)(w_3-w_0)(w_3-w_1)} \\
&\quad + \frac{(w_1+w_3)^4}{(w_1-w_0)(w_1-w_2)(w_3-w_0)(w_3-w_2)} \\
&\quad + \frac{(w_1+w_2)^4}{(w_1-w_0)(w_1-w_3)(w_2-w_0)(w_2-w_3)} \\
&\quad + \frac{(w_0+w_3)^4}{(w_0-w_1)(w_0-w_2)(w_3-w_1)(w_3-w_2)} \\
&\quad + \frac{(w_0+w_2)^4}{(w_0-w_1)(w_0-w_3)(w_2-w_1)(w_2-w_3)} \\
&\quad + \frac{(w_0+w_1)^4}{(w_0-w_2)(w_0-w_3)(w_1-w_2)(w_1-w_3)}.
\end{aligned}$$

Specialising the equivariant parameters at, say, $(w_0,w_1,w_2,w_3)=(-2,1,-1,2)$, we obtain

$$\begin{aligned}
\int_{\mathbb{G}(1,3)} \sigma_1^4 &= \frac{(-1+2)^4}{1\cdot(-2)\cdot 4\cdot 1} + \frac{(1+2)^4}{3\cdot 2\cdot 4\cdot 3} + 0 + 0 + \frac{(-2-1)^4}{(-3)\cdot(-4)\cdot(-2)\cdot(-3)} + \frac{(-2+1)^4}{(-1)\cdot(-4)\cdot 2\cdot(-1)} \\
&= -\frac{1}{8} + \frac{81}{72} + \frac{81}{72} - \frac{1}{8} = \frac{-1+9+9-1}{8} = 2,
\end{aligned}$$

in agreement with the calculation of Proposition 3.3.3.

In order to complete the proof of Proposition 9.4.1, we have to address a transversality issue. In other words, we need to exclude the following cases, which are not automatically ruled out by the computation we just completed:

(1) There is *one* line through 4 general lines, appearing with multiplicity two;
(2) There are infinitely many lines through 4 general lines.

In other words, what we know is the following: *if* the number of lines through four general lines is finite, then it is ≤ 2.

We employ the following powerful theorem of Kleiman to rule out these two degenerate cases at the same time.

Theorem 9.4.4 (Kleiman transversality)**.** *Let G be an algebraic group acting transitively on a variety X over an algebraically closed field of characteristic zero. Let $Y \subset X$ be a subvariety. Then*

(a) *given another subvariety $Z \subset X$, there exists a dense open subset $U \subset G$ such that $g \cdot Y$ and Z are generically transverse for all $g \in U$.*
(b) *If G is affine, then $[g \cdot Y] = [Y]$ in the Chow group $A_* X$.*

We apply Kleiman transversality to $X = \mathbb{G}(1,3)$, acted on by the affine algebraic group $G = \mathrm{GL}(V) = \mathrm{GL}_4$. For general lines ℓ_1 and ℓ_2 in $\mathbb{P}^3$, form the codimension one cycles

$$\Gamma_i = \left\{ \ell \subset \mathbb{P}^3 \,\middle|\, \ell \cap \ell_i \neq \emptyset \right\}, \quad i = 1,2.$$

Then set $Y = \Gamma_1$ and $Z = \Gamma_2$.

EXERCISE 9.4.5. Let $C_1, \ldots, C_4 \subset \mathbb{P}^3$ be general translated of curves of degree $d_1, \ldots, d_4$. Compute the number of lines $\ell \subset \mathbb{P}^3$ passing through $C_1, \ldots, C_4$. (**Hint**: Start out by proving that the cycle $\Gamma_C \subset \mathbb{G}(1,3)$ of lines meeting a curve $C \subset \mathbb{P}^3$ of degree d is a divisor in the Grassmannian, with cohomology class $d \cdot \sigma_1$).

10. TORUS ACTION ON THE HILBERT SCHEME OF POINTS

In this section we again work with the Hilbert scheme

$$\mathrm{Hilb}^n \mathbb{A}^d.$$

10.1. **The torus action.** Consider the d-dimensional torus

$$\mathbb{T} = \mathbb{G}_m^d$$

acting on $\mathbb{A}^d$ by rescaling the coordinates:

$$t \cdot (x_1, \ldots, x_d) = (t_1 x_1, \ldots, t_d x_d). \tag{10.1.1}$$

EXERCISE 10.1.1. Show that the action (10.1.1) lifts to a $\mathbb{T}$-action on $\mathrm{Hilb}^n \mathbb{A}^d$.

EXERCISE 10.1.2. Show that a $\mathbb{T}$-fixed subscheme $Z \subset \mathbb{A}^d$ is entirely supported at the origin $0 \in \mathbb{A}^d$.

EXERCISE 10.1.3. Show that $I \in \mathrm{Hilb}^n \mathbb{A}^d$ is $\mathbb{T}$-fixed if and only if it is a monomial ideal.

Solution. Recall that the character lattice of the torus $\mathbb{T}^* = \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ is isomorphic to $\mathbb{Z}^d$, since each character $\mathbb{T} \to \mathbb{G}_m$ is necessarily of the form

$$\chi_{\mathsf{m}} : (t_1, \ldots, t_d) \mapsto t_1^{m_1} \cdots t_d^{m_d}$$

for some $\mathsf{m} = (m_1, \ldots, m_d) \in \mathbb{Z}^d$. As an initial step, we notice that the geometric action (10.1.1) dualises to a $\mathbb{T}$- action on $\mathbb{C}[x_1, \ldots, x_d]$ via the rule

$$t \cdot f = f \circ t. \tag{10.1.2}$$

This already shows that a monomial ideal is necessarily $\mathbb{T}$-fixed, so it remains to prove the converse.

We next show that monomials

$$\mathbf{x}^{\mathsf{m}} = x_1^{m_1} \cdots x_d^{m_d}$$

form an eigenbasis of $\mathbb{C}[x_1,\dots,x_d]$ as a $\mathbb{T}$-representation. An *eigenvector* of a $\mathbb{T}$-representation V, in this context, is an element $v \in V$ for which there exists a character $\chi \in \mathbb{T}^*$ such that $t \cdot v = \chi(t)v$ for all $t \in \mathbb{T}$. (This χ plays the role of "classical" eigenvalues in linear algebra.) For us, $V = \mathbb{C}[x_1,\dots,x_d]$. Pick $v = \mathbf{x}^{\mathsf{m}}$. Then according to the rule (10.1.2) one has

$$t \cdot \mathbf{x}^{\mathsf{m}} = (t_1 x_1)^{m_1} \cdots (t_d x_d)^{m_d} = (t_1^{m_1} \cdots t_d^{m_d}) \cdot (x_1^{m_1} \cdots x_d^{m_d}) = \chi_{\mathsf{m}}(t) \cdot \mathbf{x}^{\mathsf{m}}.$$

So each monomial $\mathbf{x}^{\mathsf{m}}$ is an eigenvalue with respect to the weight χ_{m}. In particular, each corresponds to a different weight, therefore all weight spaces

$$V_{\mathsf{m}} = \left\{ f \in \mathbb{C}[x_1,\dots,x_d] \,\middle|\, t \cdot f = \chi_{\mathsf{m}}(t) f \text{ for all } t \in \mathbb{T} \right\}$$

are 1-dimensional $\mathbb{T}$-subrepresentations (each spanned by $\mathbf{x}^{\mathsf{m}}$) and the action (10.1.2) is diagonalisable by monomials.

Now pick a $\mathbb{T}$-fixed ideal $I \subset \mathbb{C}[x_1,\dots,x_d]$. In particular, as a vector space, I is a $\mathbb{T}$-subrepresentation. But a $\mathbb{T}$-subrepresentation of a diagonalisable $\mathbb{T}$-representation is again diagonalisable (prove this!), so I has a basis of eigenvectors. Since the weight spaces were 1-dimensional, I must be spanned by monomials.

Definition 10.1.4. Let $d \geq 1$ and $n \geq 0$ be integers. A $(d-1)$-dimensional partition of n is . We set

fill in

$$\mathsf{P}_{d-1}(n) = \left| \{ (d-1)\text{-dimensional partitions of } n \} \right|.$$

Example 10.1.5. If $d = 2$, a 1-dimensional partition is the same thing as a Young diagram. If $d = 3$, a *plane partition* of n is the same thing as a way of stacking n boxes in the corner of a room (assuming gravity points in the $(-1,-1,-1)$ direction!). See Figures 9 and 10 for a visual explanation.

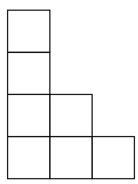


FIGURE 9. A Young diagram, corresponding to the monomial ideal $I = (x^3, x^2y, xy^2, y^4)$ of colength 7.

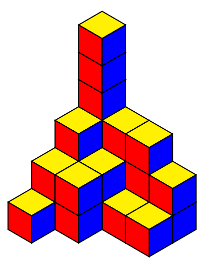


FIGURE 10. A plane partition.

Next, we will see that monomial ideals correspond to partitions. This is best explained via a picture — see Figure 11.

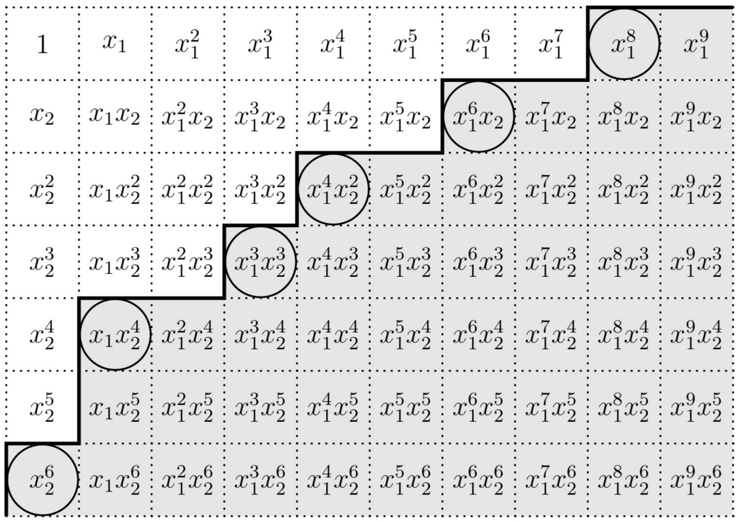


FIGURE 11. The staircase defined by the generators of a monomial ideal of colength n draws a 1-dimensional partition (Young tableau) of size n. Picture stolen from Okounkov [50]

EXERCISE 10.1.6. Show that there is a bijective correspondence between $\mathbb{T}$-fixed subschemes $Z \subset \mathbb{A}^d$ of length n and $(d-1)$-dimensional partitions of n.

10.2. **Euler characteristic of Hilbert schemes.** Let X be a smooth quasi-projective variety of dimension d. In this section we give a formula for the series

$$\mathsf{H}_X(q) = \sum_{n\geq 0} \chi(\mathrm{Hilb}^n X) q^n.$$

We use a combinatorial lemma, combined with a stratification argument.

Notation 10.2.1. We indicate a classical partition by $\alpha = (1^{\alpha_1} \cdots i^{\alpha_i} \cdots \ell^{\alpha_\ell})$, meaning that there α_i parts of length i, and we set

$$\|\alpha\| = \sum_i \alpha_i, \quad |\alpha| = \sum_i i\alpha_i.$$

The latter is the *size* of α, the former is the number of *distinct parts* of α.

Lemma 10.2.2 ([59, p. 40])**.** *Let $P(q) = 1 + \sum_{n>0} p_n q^n \in \mathbb{Q}[\![q]\!]$ be a formal power series. If χ is an integer, then*

$$P(q)^\chi = 1 + \sum_\alpha \left(\prod_{j=0}^{\|\alpha\|-1} (\chi - j) \cdot \frac{\prod_i p_i^{\alpha_i}}{\prod_i \alpha_i!} \right) q^{|\alpha|}.$$

Theorem 10.2.3. *Let X be a smooth quasi-projective variety of dimension d. There is an identity*

$$\mathsf{H}_X(q) = \left(\sum_{n\geq 0} \mathsf{P}_{d-1}(n) q^n \right)^{\chi(X)}.$$

Proof. The case $X = \mathbb{A}^d$ follows from the observation that

$$\chi(\mathrm{Hilb}^n \mathbb{A}^d) = \left|(\mathrm{Hilb}^n \mathbb{A}^d)^{\mathbb{T}}\right| = \mathsf{P}_{d-1}(n) \tag{10.2.1}$$

along with $\chi(\mathbb{A}^d)=1$. The identities (10.2.1) follow from Lemma 8.5.8 and Exercise 10.1.6.

For general X, we proceed as follows. First of all, notice that

$$\chi(\mathrm{Hilb}^n(\mathbb{A}^d)_0)=\mathsf{P}_{d-1}(n),$$

because the punctual Hilbert scheme contains the $\mathbb{T}$-fixed locus (Exercise 10.1.2). Let us stratify the symmetric product

$$\mathrm{Sym}^n X=\coprod_{\alpha\vdash n}\mathrm{Sym}^n_\alpha X$$

according to partitions of n. Each stratum dictates the multiplicity of the supporting points in a given zero-cycle. Set

$$\mathrm{Hilb}^n_\alpha X=\pi^{-1}(\mathrm{Sym}^n_\alpha X),$$

where π is the Hilbert–Chow morphism (6.1.1). On the deepest stratum, corresponding to the full partition (n), i.e. to the small diagonal $X\hookrightarrow \mathrm{Sym}^n X$, we have that

$$\mathrm{Hilb}^n_{(n)} X\to X$$

is Zariski locally trivial with fibre $\mathrm{Hilb}^n(\mathbb{A}^d)_0$. This follows easily from the local case, where in fact there is a global decomposition

$$\mathrm{Hilb}^n_{(n)}\mathbb{A}^d=\mathbb{A}^d\times\mathrm{Hilb}^n(\mathbb{A}^d)_0,$$

and π is identified with the first projection. For an arbitrary partition α, let

$$V_\alpha\subset\prod_i(\mathrm{Hilb}^i X)^{\alpha_i}$$

be the open locus of clusters with pairwise disjoint support. The "union of points" defines an (étale) map

$$V_\alpha\to\mathrm{Hilb}^n X,$$

and we let U_α denote its image. Then U_α contains the stratum $\mathrm{Hilb}^n_\alpha X$ as a closed subscheme. We can form the fibre diagram

$$\begin{tikzcd}
Z_\alpha \arrow[r, hook] \arrow[d, "f_\alpha"'] \arrow[dr, phantom, "\square"] & V_\alpha \arrow[r, hook] \arrow[d, "\text{étale}"] & \prod_i(\mathrm{Hilb}^i X)^{\alpha_i} \\
\mathrm{Hilb}^n_\alpha X \arrow[r, hook] & U_\alpha \arrow[r, hook] & \mathrm{Hilb}^n X
\end{tikzcd}$$

defining the scheme Z_α. Now the map f_α on the left is a finite étale G_α-cover, where $G_\alpha=\prod_i\mathfrak{S}_{\alpha_i}$ is the automorphism group of the partition α. In other words, the only difference between Z_α and $\mathrm{Hilb}^n_\alpha X$ is the relabelling of points that appear with the same α-multiplicity. So we find

$$\begin{aligned}
\chi(\mathrm{Hilb}^n X)&=\sum_\alpha\chi(\mathrm{Hilb}^n_\alpha X)\\
&=\sum_\alpha\frac{\chi(Z_\alpha)}{|G_\alpha|}\\
&=\sum_\alpha\frac{\chi(Z_\alpha)}{\prod_i\alpha_i!}.
\end{aligned}$$

On the other hand, it is easy to see that

$$Z_\alpha = \prod_i \mathrm{Hilb}^i_{(i)}(X)^{\alpha_i} \setminus \Delta^{\mathrm{big}}$$

also fits in a cartesian diagram

$$\begin{tikzcd} Z_\alpha \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \prod_i \mathrm{Hilb}^i_{(i)}(X)^{\alpha_i} \arrow[d, "p"] \\ X^{\|\alpha\|} \setminus \Delta^{\mathrm{big}} \arrow[r, hook] & X^{\|\alpha\|} \end{tikzcd}$$

where $\|\alpha\|$ is the number of distinct parts in the partition. Now p is a product of Zariski locally trivial fibrations with fibre $\mathrm{Hilb}^i(\mathbb{A}^d)_0$, therefore

$$\begin{aligned} \chi(Z_\alpha) &= \chi(X^{\|\alpha\|} \setminus \Delta^{\mathrm{big}}) \cdot \prod_i \chi(\mathrm{Hilb}^i(\mathbb{A}^d)_0)^{\alpha_i} \\ &= \prod_{j=0}^{\|\alpha\|-1} \big(\chi(X) - j\big) \cdot \prod_i \mathsf{P}_{d-1}(i)^{\alpha_i}. \end{aligned}$$

Putting everything together, we find

$$\mathsf{H}_X(q) = \sum_\alpha \left(\prod_{j=0}^{\|\alpha\|-1} \big(\chi(X) - j\big) \cdot \frac{\prod_i \mathsf{P}_{d-1}(i)^{\alpha_i}}{\prod_i \alpha_i!} \right) q^{|\alpha|}.$$

The result now follows from the combinatorial formula of Lemma 10.2.2. □

Definition 10.2.4. The MacMahon function is the infinite product

$$\mathsf{M}(q) = \prod_{m \geq 1} (1 - q^m)^{-m}.$$

The identity of Theorem 10.2.3 specialises to the following in low dimension.

Theorem 10.2.5. *Let X be a smooth quasi-projective variety of dimension $d = 1, 2$ or 3. Then*

$$\sum_{n \geq 0} \chi(\mathrm{Hilb}^n X) q^n = \begin{cases} (1-q)^{-\chi(X)} & \text{if } d = 1 \\ \displaystyle\prod_{m \geq 1} (1 - q^m)^{-\chi(X)} & \text{if } d = 2 \\ \mathsf{M}(q)^{\chi(X)} & \text{if } d = 3 \end{cases}$$

Proof. The case $d = 1$ is due to Macdonald [42], the case $d = 2$ is due to Göttsche [28], the case $d = 3$ is due to Cheah [14]. □

Fact 10.2.6. There are no infinite product formulas for $\mathsf{H}_X(q)$ if $d = \dim X$ is at least 4.

REFERENCES

1. Alberto Albano and Sheldon Katz, *Lines on the Fermat quintic threefold and the infinitesimal generalized Hodge conjecture*, Trans. Amer. Math. Soc. **324** (1991), no. 1, 353–368.
2. Allen B. Altman and Steven L. Kleiman, *Compactifying the Picard scheme*, Adv. in Math. **35** (1980), no. 1, 50–112.
3. Dave Anderson, *Introduction to equivariant cohomology in algebraic geometry*, Contributions to algebraic geometry, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2012, pp. 71–92.
4. M. F. Atiyah and R. Bott, *The moment map and equivariant cohomology*, Topology **23** (1984), no. 1, 1–28.
5. V. Baranovsky, *The variety of pairs of commuting nilpotent matrices is irreducible*, Transform. Groups **6** (2001), no. 1, 3–8.
6. Roberta Basili, *On the irreducibility of commuting varieties of nilpotent matrices*, J. Algebra **268** (2003), no. 1, 58–80.
7. Kai Behrend, Jim Bryan, and Balázs Szendrői, *Motivic degree zero Donaldson–Thomas invariants*, Invent. Math. **192** (2013), no. 1, 111–160.
8. Kai Behrend and Barbara Fantechi, *The intrinsic normal cone*, Inventiones Mathematicae **128** (1997), no. 1, 45–88.
9. A. Biał ynicki Birula, *Some theorems on actions of algebraic groups*, Ann. of Math. (2) **98** (1973), 480–497.
10. Armand Borel, *Seminar on transformation groups*, With contributions by G. Bredon, E. E. Floyd, D. Montgomery, R. Palais. Annals of Mathematics Studies, No. 46, Princeton University Press, Princeton, N.J., 1960.
11. Raoul Bott, *A residue formula for holomorphic vector-fields*, J. Differential Geometry **1** (1967), 311–330.
12. Edgar H. Brown, *Cohomology theories*, Ann. of Math. (2) **75** (1962), 467–484.
13. Jim Bryan and Andrew Morrison, *Motivic classes of commuting varieties via power structures*, J. Algebraic Geom. **24** (2015), no. 1, 183–199.
14. Jan Cheah, *On the cohomology of Hilbert schemes of points*, J. Alg. Geom. **5** (1996), no. 3, 479–511.
15. Neil Chriss and Victor Ginzburg, *Representation theory and complex geometry*, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2010, Reprint of the 1997 edition.
16. David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011.
17. Theodosios Douvropoulos, Joachim Jelisiejew, Bernt Ivar Utstø l Nø dland, and Zach Teitler, *The Hilbert scheme of 11 points in $\mathbb{A}^3$ is irreducible*, Combinatorial algebraic geometry, Fields Inst. Commun., vol. 80, Fields Inst. Res. Math. Sci., Toronto, ON, 2017, pp. 321–352.
18. David Eisenbud and Joe Harris, *The geometry of schemes*, Graduate Texts in Mathematics, vol. 197, Springer-Verlag, New York, 2000.
19. ______, *3264 and all that – a second course in Algebraic Geometry*, Cambridge University Press, Cambridge, 2016.
20. Geir Ellingsrud and Stein Arild Strø mme, *Bott's formula and enumerative geometry*, J. Amer. Math. Soc. **9** (1996), no. 1, 175–193.
21. Barbara Fantechi et al., *Fundamental Algebraic Geometry: Grothendieck's FGA Explained*, Mathematical surveys and monographs, American Mathematical Society, 2005.
22. Walter Feit and Nathan J. Fine, *Pairs of commuting matrices over a finite field*, Duke Math. J **27** (1960), 91–94.
23. John Fogarty, *Fixed point schemes*, Amer. J. Math. **95** (1973), 35–51.
24. William Fulton, *Intersection theory*, Springer, 1984.
25. William Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993, The William H. Roever Lectures in Geometry.
26. Andreas Gathmann, *Algebraic Geometry*, https://www.google.com, 2014.
27. M. Gerstenhaber, *On dominance and varieties of commuting matrices*, Ann. Math. (2) **73** (1961), 324–348.

28. Lothar Göttsche, *The Betti numbers of the Hilbert scheme of points on a smooth projective surface*, Math. Ann. **286** (1990), 193–207.
29. Tom Graber and Rahul Pandharipande, *Localization of virtual classes*, Invent. Math. **135** (1999), no. 2, 487–518.
30. ______, *Constructions of nontautological classes on moduli spaces of curves*, Michigan Math. J. **51** (2003), no. 1, 93–109.
31. Alexander Grothendieck, *Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes*, Inst. Hautes Études Sci. Publ. Math. (1961), no. 8, 222.
32. ______, *Fondements de la géométrie algébrique. [Extraits du Séminaire Bourbaki, 1957–1962.]*, Secrétariat mathématique, Paris, 1962.
33. Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52.
34. ______, *Deformation theory*, Graduate Texts in Mathematics, vol. 257, Springer, New York, 2010.
35. Abdelmoubine A. Henni and Marcos Jardim, *Commuting matrices and the Hilbert scheme of points on affine spaces*, Adv. Geom. **18** (2018), no. 4, 467–482.
36. John Holbrook and Matjaž Omladič, *Approximating commuting operators*, Linear Algebra Appl. **327** (2001), no. 1-3, 131–149.
37. A. Iarrobino, *Reducibility of the families of* 0*-dimensional schemes on a variety*, Invent. Math. **15** (1972), 72–77.
38. Birger Iversen, *A fixed point formula for action of tori on algebraic varieties*, Invent. Math. **16** (1972), 229–236.
39. Sheldon Katz, *On the finiteness of rational curves on quintic threefolds*, Compositio Math. **60** (1986), no. 2, 151–162.
40. Jun Li and Gang Tian, *Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties*, J. Amer. Math. Soc. **11** (1998), no. 1, 119–174.
41. Qing Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, vol. 6, Oxford University Press, Oxford, 2002, Translated from the French by Reinie Erné, Oxford Science Publications. MR 1917232
42. I. G. Macdonald, *Symmetric products of an algebraic curve*, Topology **1** (1962), 319–343.
43. André L. Meireles Araújo and Israel Vainsencher, *Equivariant intersection theory and Bott's residue formula*, vol. 20, 2001, 16th School of Algebra, Part I (Portuguese) (Brasília, 2000), pp. 1–70.
44. John Milnor, *Construction of universal bundles. I*, Ann. of Math. (2) **63** (1956), 272–284.
45. ______, *Construction of universal bundles. II*, Ann. of Math. (2) **63** (1956), 430–436.
46. T. S. Motzkin and Olga Taussky, *Pairs of matrices with property* L*. II*, Trans. Am. Math. Soc. **80** (1955), 387–401.
47. David Mumford, *Towards an enumerative geometry of the moduli space of curves*, pp. 271–328, Birkhäuser Boston, Boston, MA, 1983.
48. Hiraku Nakajima, *Lectures on Hilbert schemes of points on surfaces*, University Lecture Series, vol. 18, American Mathematical Society, Providence, RI, 1999.
49. Nham V. Ngo and Klemen Šivic, *On varieties of commuting nilpotent matrices*, Linear Algebra Appl. **452** (2014), 237–262.
50. Andrei Okounkov, *Random surfaces enumerating algebraic curves*, European Congress of Mathematics, Eur. Math. Soc., Zürich, 2005, pp. 751–768.
51. J. and Vinsonhaler C. I. O'Meara, K. C. and Clark, *Advanced topics in linear algebra: weaving matrix problems through the Weyr form*, Oxford University Press, New York, 2011.
52. Ragni Piene and Michael Schlessinger, *On the Hilbert scheme compactification of the space of twisted cubics*, Amer. J. Math. **107** (1985), no. 4, 761–774. MR 796901
53. Alexander Premet, *Nilpotent commuting varieties of reductive Lie algebras*, Invent. Math. **154** (2003), no. 3, 653–683.
54. R. W. Richardson, *Commuting varieties of semisimple Lie algebras and algebraic groups*, Compos. Math. **38** (1979), 311–327.
55. David Rydh, *Families of cycles and the Chow scheme*, Ph.D. thesis, KTH, Stockholm, 2008.

56. Ed Segal, *The A_∞ Deformation Theory of a Point and the Derived Categories of Local Calabi-Yaus*, J. Algebra **320** (2008), no. 8, 3232–3268.
57. Edoardo Sernesi, *Deformations of algebraic schemes*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 334, Springer-Verlag, Berlin, 2006.
58. The Stacks Project Authors, *Stacks Project*, `http://stacks.math.columbia.edu`, 2016.
59. Richard P. Stanley, *Enumerative combinatorics*, vol. 2, Cambridge Univ. Press, Cambridge, 1999.
60. Ravi Vakil, *Murphy's law in algebraic geometry: badly-behaved deformation spaces*, Invent. Math. **164** (2006), no. 3, 569–590.
61. ______, *Foundations of Algebraic Geometry*, `http://math.stanford.edu/~vakil/216blog/FOAGnov1817public.pdf`, 2017.
62. Angelo Vistoli, *Grothendieck topologies, fibered categories and descent theory*, Fundamental algebraic geometry, Math. Surveys Monogr., vol. 123, Amer. Math. Soc., Providence, RI, 2005, pp. 1–104.
63. J. H. C. Whitehead, *Combinatorial homotopy. I*, Bull. Amer. Math. Soc. **55** (1949), 213–245.