Symmetric obstruction theories and Joyce's perverse sheaves

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Joyce (2013):

> *"The author and his collaborators tried for some time to construct perverse sheaves, and motivic Milnor fibres, from a scheme with symmetric obstruction theory, but failed, and the author now believes this is not possible."*

Joyce, A classical model for derived critical loci

Overview

We shall try to understand this quote, with a view towards applications to Donaldson–Thomas theory.

- Why perverse sheaves?
- Categorification of the DT invariant
- Why are symmetric obstruction theories bad?

I. Why perverse sheaves?

One can think of perverse sheaves as good systems of coefficients for cohomology. If $X = \{ d f = 0 \} \subset U$ is a critical locus for some function f on a smooth scheme U, we have the perverse sheaf of vanishing cycles Φ_f on X, whose cohomology

$$\mathbb{H}^*(X, \Phi_f)$$

measures how the cohomology $H^*(f^{-1}(t), \mathbb{Q})$ varies as t becomes a critical value of f.

The setup

Let Y be a projective Calabi–Yau 3-fold over $\mathbb{C}$. Let $\mathscr{M}$ be a (finite type) moduli scheme of simple coherent sheaves on Y.

Why would one want a "good system of coefficients" for cohomology of $\mathscr{M}$ in Donaldson–Thomas theory?

2000 Thomas defined a symmetric obstruction theory on $\mathscr{M}$, and proved that the number

$$\mathrm{DT}(\mathscr{M}) = \int_{[\mathscr{M}]^{\mathrm{vir}}} 1 \in \mathbb{Z}$$

is invariant under deformations of Y. This number is the *Donaldson–Thomas invariant*.

2005 Behrend proved that one can compute the DT invariant by

$$\mathrm{DT}(\mathscr{M}) = \chi(\mathscr{M}, \nu) = \sum_{r \in \mathbb{Z}} r \cdot \chi\big(\nu^{-1}(r)\big)$$

where $\nu : \mathscr{M} \to \mathbb{Z}$ is the… “Behrend function”. It treats the singularities in a special, mysterious way.

Moreover, $\nu_X = \chi(\Phi_f)$ when $X = Z(df) \subset U$ is a critical locus, so

$$\chi(X, \nu_X) = \sum_i (-1)^i \dim \mathbb{H}^i(X, \Phi_f).$$

Since $\mathscr{M}$ is locally of this form, this suggests that the DT invariant is just a *realization* of a cohomology theory on the moduli space.

2008 Joyce–Song proved that the moduli stack $\mathfrak{M}$ of coherent sheaves on Y is locally analytically isomorphic to a quotient of a critical locus by a Lie group.

Joyce:

> *"This is the complex analogue for $\mathfrak{M}$ of the fact that the moduli scheme $\mathscr{M}$ has a symmetric obstruction theory."*

It follows that $\mathscr{M}$ is locally analytically a critical locus.

This poses the *categorification problem*:

Find a canonical perverse sheaf $\mathscr{P}^{\cdot}$ *on* $\mathscr{M}$ *such that*

$$\mathrm{DT}(\mathscr{M}) = \sum_{i} (-1)^{i} \dim \mathbb{H}^{i}(\mathscr{M}, \mathscr{P}^{\cdot}).$$

The graded vector space $\mathbb{H}^{*}(\mathscr{M}, \mathscr{P}^{\cdot})$ would be called the *cohomological DT invariant*, $\mathscr{P}^{\cdot}$ would be called the *DT sheaf*.

1997 Behrend–Fantechi

A perfect obstruction theory on a scheme X is a two-term complex $E \in D(X)$, perfect in $[-1, 0]$, with a morphism $\phi : E \to L_X$ such that $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is onto.

We say (E, ϕ) is symmetric if there is an isomorphism $\theta : E^\vee[1] \xrightarrow{\sim} E$ such that $\theta^\vee[1] = \theta$.

A critical locus $X = Z(d f) \subset U$, with ideal $\mathscr{I} \subset \mathscr{O}_U$, carries the symmetric obstruction theory

$$\begin{array}{ccc} E_f & & \left[T_U|_X \xrightarrow{\partial^2(f)|_X} \Omega_U|_X \right] \\ \phi_f \big\downarrow & (d f)^\vee|_X \big\downarrow & \big\downarrow \mathrm{id} \\ L_X & & \left[\mathscr{I}/\mathscr{I}^2 \xrightarrow{\ d\ } \Omega_U|_X \right] \end{array}$$

where d is the usual differential.

If one knew how to go

$$\text{symmetric obstruction theories} \quad \dashrightarrow \quad \text{perverse sheaves}$$

one could hope to be able to glue symmetric obstruction theories together, and $\mathscr{P}^{\cdot}$ (solution to the categorification problem) would be the perverse sheaf corresponding to the glueing.

Problems:

(i) Find the "right" global structure on $\mathscr{M}$ allowing you to pick compatible critical charts.

(ii) Glue the symmetric obstruction theories so to get $\mathscr{P}^{\cdot}$.

Upshot: (ii) is hopeless, (i) is not. In the paper containing the initial quote (2013),

Joyce defines d-critical loci,

solving problem (i).

II. Constructing $\mathscr{P}^{\cdot}$

2013 Pantev–Toën–Vaquié–Vezzosi define k-*shifted symplectic derived schemes* $(\mathbf{X}, \omega)$. Here $\mathbf{X}$ is a derived scheme, and

$$\omega = [(\omega^0, \omega^1, \dots)] \in H^k\left(\bigoplus_{i=0}^{\infty} \bigwedge^{2+i} L_{\mathbf{X}}[i], d + d_{dR}\right)$$

is a closed non-degenerate 2-form of degree k. Non-degenerate means that the induced 2-form $[\omega^0] \in H^k(\bigwedge^2 L_{\mathbf{X}}, d)$ is such that $\omega^0 \colon T_{\mathbf{X}} \to L_{\mathbf{X}}[k]$ is a quasi-isomorphism.

Important theorems:

2013 PTVV: The moduli scheme $\mathscr{M}$ has the structure of -1-shifted symplectic derived scheme.

2015 Brav–Bussi–Joyce: Let $(\mathbf{X}, \omega)$ be a -1-shifted symplectic derived scheme. Then the underlying scheme $X = t_0(\mathbf{X})$ has a natural structure of d-critical locus.

$\Rightarrow$ $\mathscr{M}$ is Zariski locally a critical locus.

Open patches in a d-critical locus (X, s) are critical charts, that is, tuples (R, U, f, i) where $R \subset X$ is Zariski open, f is a function on the smooth scheme U and $i : R \to U$ is a closed immersion such that $i(R) = Z(d\,f)$ as subschemes of U.

The structure includes the choice of a section $s \in H^0(\mathcal{S}_X^0)$ where $\mathcal{S}_X^0$ is a certain natural sheaf of $\mathbb{C}$-vector spaces. It prescribes how the critical charts should fit together.

Remember $\mathscr{M}$ is naturally a d-critical locus, and we want to construct $\mathscr{P}^{\cdot}$ on $\mathscr{M}$. So the question becomes:

> *Will the d-critical locus structure allow us to glue together the sheaves of vanishing cycles living on the critical charts?*

Answer: *almost*. When (X, s) is orientable one can glue.

In this sense, vanishing cycles are not better than symmetric obstruction theories: they both pose a glueing issue.

2015 Brav–Bussi–Dupont–Joyce–Szendrői solve this issue for vanishing cycles.

Example

Let $f : \mathbb{C}^n \to \mathbb{C}$ be $f(z_1, \ldots, z_n) = z_1^2 + \cdots + z_n^2$ for a fixed $n > 1$. Then $Z(d\,f) = \{0\}$ and

$$\Phi_f = H^{n-1}(MF_{f,0}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_0.$$

However, $MF_{f,0} = T^* S^{n-1}$, therefore we find

$$H^{n-1}(MF_{f,0}, \mathbb{Q}) = H^{n-1}(S^{n-1}, \mathbb{Q}) \cong \mathbb{Q},$$

where the last isomorphism corresponds to a choice of orientation for the sphere S^{n-1}.

Joyce proves that a d-critical locus (X, s) has a natural line bundle $K_{X,s}$ on X_{red}, called the *canonical line bundle*, and each critical chart $\gamma = (R, U, f, i)$ provides a natural isomorphism

$$\iota_\gamma : K_{X,s}\big|_{R_{red}} \xrightarrow{\sim} i^*(K_U^{\otimes 2})\big|_{R_{red}}.$$

Definition. An orientation on (X, s) is a choice of a square root of $K_{X,s}$.

If we view $(\mathscr{M}, s)$ as a d-critical locus, its canonical line bundle is

$$\mathsf{K}_{\mathscr{M},s} = \det \mathscr{E}\big|_{\mathscr{M}_{\mathrm{red}}},$$

where $\mathscr{E} \to \mathsf{L}_{\mathscr{M}}$ is the natural symmetric obstruction theory constructed by Thomas or Huybrechts–Thomas.

If $\mathscr{M}$ is smooth, then $\mathsf{K}_{\mathscr{M},s} = \mathsf{K}_{\mathscr{M}}^{\otimes 2}$. Moreover, $(\mathscr{M}, s)$ is always orientable, although there exist d-critical loci that have no orientation.

Choose an orientation $K_{X,s}^{1/2}$ on (X, s)

$$\downarrow$$

Given a critical chart $\gamma = (R, U, f, i)$, consider the principal $\mathbb{Z}_2$-bundle $Q_\gamma \to R$ parametrizing local isomorphisms

$$\alpha : K_{X,s}^{1/2}\big|_{R_{\mathrm{red}}} \to i^*(K_U)\big|_{R_{\mathrm{red}}} \text{ such that } \alpha \otimes \alpha = \iota_\gamma.$$

2015 Brav–Bussi–Dupont–Joyce–Szendrői

Let $K_{X,s}^{1/2}$ be an orientation on (X, s). Then there exists a natural $\mathbb{Q}$-perverse sheaf $\mathscr{P}^{\cdot}$ on X, such that if $\gamma = (R, U, f, i)$ is a critical chart, there is a natural isomorphism

$$\omega_\gamma : \mathscr{P}^{\cdot}\big|_R \xrightarrow{\cong} i^*\Phi_f \otimes_{\mathbb{Z}_2} Q_\gamma.$$

Warning: $\mathscr{P}^{\cdot}$ depends on $K_{X,s}^{1/2}$.

Idea of the proof

Instead of glueing the sheaves of vanishing cycles together, glue the twists

$$i^* \Phi_f \otimes_{\mathbb{Z}_2} Q_\gamma. \tag{1}$$

The d-critical locus structure allows one to do this.

Crucial steps:

(i) Perverse sheaves glue uniquely (and (1) is perverse)

(ii) when X is a critical locus in two different ways, the sheaves of vanishing cycles differ by a principal $\mathbb{Z}_2$-bundle.

Explaining (ii)

Let U, V be smooth, and let $f : U \to \mathbb{A}^1$, $g : V \to \mathbb{A}^1$ be regular functions, with

$$X = Z(\mathrm{d}\, f) \subset U, \qquad Y = Z(\mathrm{d}\, g) \subset V.$$

Suppose you have a closed immersion $j : U \to V$ such that the restriction $h = j|_X$ is an isomorphism

$$h : X \xrightarrow{\sim} Y.$$

We wish to compare Φ_f and Φ_g. It is not true in general that $\Phi_f = h^* \Phi_g$.

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BBDJS prove the existence of a natural quadratic form $q_j \in H^0(X, S^2 N_j^\vee|_X)$ inducing an isomorphism of line bundles

$$J_j : K_U^{\otimes 2}\big|_{X_{red}} \xrightarrow{\sim} h\big|_{X_{red}}^* \left(K_V^{\otimes 2}\right)$$

Let $P_j \to X$ be the principal $\mathbb{Z}_2$-bundle parametrizing square roots of J_j on X_{red} (roughly: square roots of q_j). Then

$$\Phi_f = h^* \Phi_g \otimes_{\mathbb{Z}_2} P_j.$$

The twist by P_j disappears when "$\det q_j = 1$". But one can also have $\det q_j = -1$.

Remarks

- $\mathscr{P}^{\cdot}$ comes with Verdier duality and monodromy isomorphisms which, together with the isomorphisms ω_γ, characterize it uniquely.
- If X is a scheme equipped with an oriented d-critical locus structure, one has

$$\chi(X, \nu_X) = \sum_i (-1)^i \dim \mathbb{H}^i(X, \mathscr{P}^{\cdot}).$$

Example (Szendrői 2015).

Let Y contain a divisor $E \subset Y$ admitting a $\mathbb{P}^1$-fibration $\pi : E \to C$, where C is a smooth proper curve of genus g, and assume there is a contraction $Y \to \overline{Y}$ to a singular projective Calabi–Yau 3-fold, which contracts E and is an isomorphism on its complement.

Let β be the class of a fibre of π, and let $\mathscr{M}$ be the moduli space of ideal sheaves $\mathscr{I}_Z \subset \mathscr{O}_Y$ with Chern character $(1, 0, -\beta, -n)$, where n is the lowest possible.

Then $\mathscr{M} = C$.

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The numerical DT invariant is $DT(\mathscr{M}) = -\chi(C) = 2g - 2$. There is only one d-critical locus structure $(C, 0)$, and $K_{C,0} = K_C^{\otimes 2}$. Every 2-torsion line bundle $L \in \operatorname{Pic} C$ gives an orientation

$$K_C \otimes L.$$

If $L = \mathscr{O}_C$, we get DT sheaf $\mathbb{Q}_C[1]$. If L is nontrivial, let $\mathscr{L}$ be the rank one local system on C corresponding to the étale double cover $\widetilde{C} \to C$. Then

$$\text{DT sheaf} = \mathscr{L}[1].$$

III. Symmetric obstruction theories

The next example will show why obstruction theories are non-local. Incidentally, we will also note that they carry global information that the d-critical locus is not always able to see.

Example (Joyce 2013)

Let $t : U \to \mathbb{A}^1$ be a family of K3 surfaces, and set $X = t^{-1}(0)$. Let $\beta \in H^1(T_X)$ be the deformation corresponding to t. We compare the critical loci of

$$t^2 : U \to \mathbb{A}^1 \text{ and } 0 : X \to \mathbb{A}^1$$

They are the same scheme X, and they also agree as d-critical loci (because X is smooth).

The associated symmetric obstruction theories are

$$E = \left[\partial^2(t^2)\big|_X : T_U\big|_X \to \Omega_U\big|_X\right]$$
$$F = \left[0 : T_X \to \Omega_X\right]$$

and we wish to show these might not be isomorphic in $D(X)$.

We have $\tau_{<0}E = T_X[1]$ and $\tau_{\geqslant 0}E = \Omega_X$, so there is an exact triangle

$$\Omega_X[-1] \xrightarrow{\alpha} T_X[1] \longrightarrow E \longrightarrow \Omega_X,$$

realizing E as the cone of a morphism

$$\alpha \in \mathrm{Hom}(\Omega_X[-1], T_X[1]) = \mathrm{Ext}^2(\Omega_X, T_X).$$

We must find an example where $\alpha \neq 0$, as this is equivalent to $E \not\cong F$ in $D(X)$.

Notice that $N_{X/U} = t^*(T_0\mathbb{A}^1) = \mathscr{O}_X$ so we have

$$0 \to T_X \to T_U\big|_X \to \mathscr{O}_X \to 0$$

$$0 \to \mathscr{O}_X \to \Omega_U\big|_X \to \Omega_X \to 0$$

corresponding to

$$\begin{array}{ccccc} \operatorname{Ext}^1(\mathscr{O}_X, T_X) & \xrightarrow{\sim} & H^1(T_X) & \xleftarrow{\sim} & \operatorname{Ext}^1(\Omega_X, \mathscr{O}_X) \\ \beta' & \longrightarrow & \beta & \longleftarrow & \beta'' \end{array}$$

and such that $\alpha = \beta' \circ \beta'' \in \operatorname{Ext}^2(\Omega_X, T_X) = H^2(T_X \otimes T_X)$.

Under the projection

$$H^2(T_X \otimes T_X) \to H^2\left(\wedge^2 T_X\right) = \mathbb{C}$$

the class α maps to β^2. it is enough to pick our initial deformation $U \to \mathbb{A}^1$ such that the corresponding class $\beta \in H^1(T_X)$ satisfies $\beta^2 \neq 0 \in H^2(\mathcal{O}_X)$.

The example show that symmetric obstruction theories do not glue.

It also shows that the class

$$\alpha \in \mathrm{Ext}^2(\Omega_X, T_X)$$

induced by E is a global datum, which as Joyce observes, is locally trivial – restricting α to an affine open $A \subset X$ one has $\alpha|_A = 0$.

The (symmetric) obstruction theory remembers global, non-local information which is forgotten by the algebraic d-critical locus.

Joyce, A classical model for derived critical loci