ALGEBRAIC GEOMETRY

A. Ricolfi

Homework 2 — **Due date**: 8 November 2022

Exercise 1. Let $(X, \mathscr{O}_X)$ be a scheme, $U \subset X$ an open subset. Show that $(U, \mathscr{O}_X|_U)$ is a scheme.

Exercise 2 (Irreducibility)**.** Let X be a topological space.

1. Show that the following are equivalent:
   - (i) X is irreducible,
   - (ii) every nonempty open subset $U \subset X$ is dense,
   - (iii) any two nonempty open subsets of X intersect.
2. Show the following:
   - (a) If $V \subset X$ is a subspace, it is irreducible if and only if $\overline{V}$ is irreducible.
   - (b) If $V \subset X$ is irreducible and $f \colon X \to Y$ is a continuous map, then $f(V)$ is irreducible.
   - (c) The product of two irreducible spaces is irreducible.

Exercise 3. If A is an integral domain, then $X = \operatorname{Spec} A$ is irreducible.

Exercise 4. Let X be an irreducible scheme. Show that there exists a unique point $\xi \in X$ such that $X = \overline{\{\xi\}}$. We call it the *generic point of* X.

Exercise 5. Let $A \hookrightarrow \mathbf{k}[t]$ be the $\mathbf{k}$-subalgebra generated by t^2 and t^3.

1. Show that the $\mathbf{k}$-algebra homomorphism
$$\mathbf{k}[x,y] \xrightarrow{\pi} A, \quad x \mapsto t^2, \;\; y \mapsto t^3$$
is surjective and induces an isomorphism $\mathbf{k}[x,y]/(x^3 - y^2) \xrightarrow{\sim} A$.
2. Consider the inclusion $\phi \colon A \hookrightarrow \mathbf{k}[t]$. Show that the induced map on prime spectra $f_\phi \colon \mathbb{A}^1_{\mathbf{k}} \to \operatorname{Spec} A$ is bijective on points, but not an isomorphism of schemes.

Exercise 6. Exhibit an isomorphism of affine schemes $\operatorname{Spec} \mathbf{k}[x,y]/(y - x^2) \cong \mathbb{A}^1_{\mathbf{k}}$. Prove that there *cannot* be an isomorphism of affine schemes between the ‘circle’ $\operatorname{Spec} \mathbf{k}[x,y]/(y^2 + x^2 - 1)$ and $\mathbb{A}^1_{\mathbf{k}}$.