INTRODUCTION TO ENUMERATIVE GEOMETRY

— CLASSICAL AND VIRTUAL TECHNIQUES —

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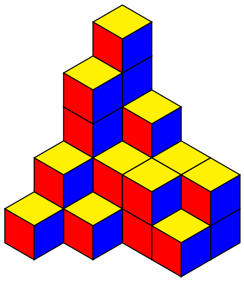
ABSTRACT. These are lecture notes for a PhD course held at SISSA in Fall 2019. Notes under construction.



CONTENTS

0. Why is Enumerative Geometry hard?

0.1. **Asking the right question.** Enumerative Geometry is a branch of Algebraic Geometry studying questions asking to count how many objects satisfy a given list of geometric conditions. The very nature of these questions, and the presence of this "list", make the subject tightly linked to Intersection Theory, which explains why we included Appendix A at the end of these lecture notes.

Examples of classical questions in the subject are the following:

(1) How many lines $\ell \subset \mathbb{P}^3$ intersect four general lines $\ell_1, \ell_2, \ell_3, \ell_4 \subset \mathbb{P}^3$?
(2) How many lines $\ell \subset \mathbb{P}^3$ lie on a smooth cubic surface $S \subset \mathbb{P}^3$?
(3) How many lines $\ell \subset \mathbb{P}^4$ lie on a generic quintic 3-fold $Y \subset \mathbb{P}^4$?
(4) How many flexes are there on a general genus 3 curve?
(5) How many smooth conics are tangent to five general plane conics?

The objects we want to count, say in the first three examples, are lines in some projective space. The geometric conditions are constraints we put on these lines, such as intersecting other lines or lying on a smooth cubic surface. We immediately see that one fundamental difficulty in the subjects is this:

> **D1**. How do we know how many constraints we should put on our objects in order to *expect* a finite answer? In other words, how do we ask the right question?

Here is a warm-up example to shape one's intuition. See Section 3 for a full treatment of the topic "expectations" in the case of lines on hypersurfaces. Problem (2) will be solved in Section **??**, problem (3) in Section **??**.

Exercise 0.1.1. Let $d > 0$ be an integer. Determine the number m_d having the following property: you expect finitely many smooth complex projective curves $C \subset \mathbb{P}^2$ of degree d passing through m_d general points in $\mathbb{P}^2$. (**Hint**: Start with small d. Then conjecture a formula for m_d).

0.2. **Counting the points on a moduli space.** The main idea to guide our geometric intuition in formulating and solving an enumerative problem should be the following recipe:

- construct a moduli[1] space $\mathcal{M}$ for the objects we are interested in,
- compactify $\mathcal{M}$ if necessary,
- impose $\dim \mathcal{M}$ conditions to expect a finite number of solutions, and
- count these solutions via Intersection Theory methods (exploiting compactness of $\mathcal{M}$).

None of these steps is a trivial one, in general.

Another difficulty in the subject is the following. Say we have a precise question, such as (2) above. Then, in the above recipe, as our $\mathcal{M}$ we should take the Grassmannian of lines in $\mathbb{P}^3$, which is a compact 4-dimensional complex manifold. Imagine we have found a sensible algebraic variety structure on the set $\mathcal{M}_S \subset \mathcal{M}$ of lines lying on the surface S. If we have done everything right, the space $\mathcal{M}_S$ consists of finitely

[1] The latin word *modulus* means *parameter*, and its plural is *moduli*. Thus a *moduli space* is to be thought of as a parameter space for objects of some kind.

many points, and now the only legal operation we can perform in order to get our answer is to take the degree of the (0-dimensional) fundamental class of $\mathcal{M}_S$. So here is the second problem we face:

> **D2**. How do we know this degree is the answer to our original question? In other words, how to ensure that our algebraic solution is actually *enumerative*?

Put in more technical terms, how do we make sure that each line $\ell \subset S$ appears as a point in the moduli space $\mathcal{M}_S$ with multiplicity one? The truth is that we cannot *always* be sure that this is the case. It will be, both for problem (2) and problem (3), but not in general. However, we should get used to the idea that this is not something to be worried about: if a solution comes with multiplicity bigger than one, there usually is a good geometric reason for this, and we should not disregard it (see Figure 4 for a simple example of a degenerate intersection where this phenomenon occurs).

Remark 0.2.1. Compactness of $\mathcal{M}$ (in the above example, the Grassmannian) is used in order to make sense of taking the *degree* of cycles. Intuitively, we need compactness in order to prevent the solutions of our enumerative problem to escape to infinity, like for instance it would occur if we were to intersect two *parallel* lines in $\mathbb{A}^2$.

Compatcness really is a non-negotiable condition we have to ask of our moduli space — with an important exception, that will be treated in later sections: the case when the moduli space has a torus action. In this case, if the torus-fixed locus $\mathcal{M}^{\mathbb{T}} \subset \mathcal{M}$ is compact, a sensible enumerative solution to a counting problem can be *defined* by means of the *localisation formula*. The original formula due to Atiyah and Bott will be proved in Theorem **??**. A virtual analogue due to Graber and Pandharipande [17] will be proved in Theorem **??**, and the latter will be applied to the study of 0-dimensional Donaldson–Thomas invariants of local Calabi–Yau 3-folds (arising from non-compact, but toric, moduli spaces).

A more fundamental difficulty is discussed in the next subsection, by means of an elementary example.

0.3. **Transversality, and counting lines through two points.** Consider the enumerative problem of counting the number of lines in $\mathbb{P}^2$ through two given points $p, q \in \mathbb{P}^2$. Let N_{pq} be this number. Then

$$\mathsf{N}_{pq} = 1, \quad \text{as long as } p \neq q.^{2}$$

However, the *true* answer would be ∞ when $p = q$, corresponding to the cardinality of the pencil $\mathbb{P}^1$ of lines through p (see Figure 1).

Now, the case $p = q$ is a degeneration of the case $p \neq q$, and we certainly want our enumerative answer not to depend on small perturbations of the geometry of the problem. It seems at first glance that the issue cannot be fixed. After all, there is an inevitable dimensional jump between the transverse case (yielding a dimension zero answer) and the non-transverse geometry (dimension one answer). However, the answer '1' can be recovered in the non-transverse setting (the picture on the right) by means of the *excess intersection formula*.

[2] For the sake of completeness, this will be proved in Section **??**.

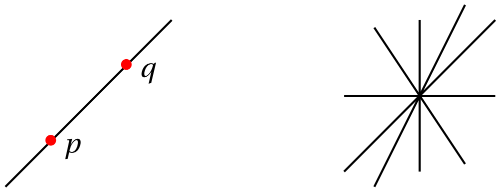


FIGURE 1. The unique line through two distinct points, and the infinitely many lines through one point in the plane.

The $\mathbb{P}^1$ of lines through p can be neatly seen as the exceptional divisor E in the blowup $B = \mathrm{Bl}_p\,\mathbb{P}^2$, cf. Figure 2.

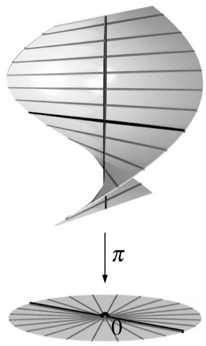


FIGURE 2. The blow-up of $\mathbb{P}^2$ at a point p. Picture stolen from Gathmann [15].

Recall that the *normal sheaf* of a closed embedding $X \hookrightarrow Y$ defined by an ideal $\mathscr{I} \subset \mathscr{O}_Y$ is the $\mathscr{O}_X$-module $N_{X/Y} = (\mathscr{I}/\mathscr{I}^2)^\vee = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{I}/\mathscr{I}^2, \mathscr{O}_X)$.

EXERCISE 0.3.1. Let $X \hookrightarrow Y$ be a closed embedding, $M \to Y$ a morphism, and let $g\colon P = X \times_Y M \to X$ be the induced map. Show that there is an inclusion $N_{P/M} \subset g^* N_{X/Y}$.

Looking at the Cartesian square

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & B \arrow[d, "\pi"] \\ p \arrow[r, hook] & \mathbb{P}^2 \end{tikzcd} \tag{0.3.1}$$

we know by Exercise 0.3.1 that there is an injection of vector bundles $N_{E/B} = \mathscr{O}_E(-1) \subset g^* N_{p/\mathbb{P}^2}$. The *excess bundle* (or *obstruction bundle*)

$$\mathrm{Ob} \to \mathbb{P}^1$$

of the fiber diagram (0.3.1) is defined as the quotient of these two bundles. But the short exact sequence

$$0 \to \mathscr{O}_E(-1) \to \mathscr{O}_E \otimes T_p\mathbb{P}^2 \to \mathrm{Ob} \to 0$$

is just the Euler sequence on $\mathbb{P}^1$ twisted by -1. Therefore

$$\mathrm{Ob} = T_{\mathbb{P}^1}(-1) = \mathscr{O}_{\mathbb{P}^1}(2-1) = \mathscr{O}_{\mathbb{P}^1}(1).$$

We have thus recovered '1' as the Euler number of the excess bundle, so that we can now write a universal formula for our counting problem: if $\mathcal{M}_{pq} = \pi^{-1}(q) \cap E$ is the "moduli space" of lines through p and q (this includes the case $p = q$), the *virtual number* of lines through p and q is

$$\int_{\mathcal{M}_{pq}} e(\mathrm{Ob}) = 1.$$

Note that the rank of the excess bundle is the difference between the actual dimension of the moduli space, and the expected one, and that $\mathrm{Ob} = 0$ unless $p = q$.

Unfortunately, in more complicated situations (but also not that complicated), we often do not even know whether our geometric setup is a degeneration of a transverse one. If it were, we would like to dispose of a technology allowing us to "count" in the transverse setup and argue that the number we obtain there equals the one we are after. This sounds like a reasonable wish, but it is way too optimistic. We should not aim at this: not only because counting is often difficult also in transverse situations, but mainly because we simply may not have enough algebraic deformations to pretend that the geometry of the problem is transverse.

Example 0.3.2. If we were to count self-intersections of a (-1)-curve on a surface,[3] there would be no way to deform these curves off themselves to make them intersect themselves transversely! See also Exercise 0.3.4 below.

This discussion leads us directly to another intrinsic difficulty in Enumerative Geometry. Suppose, just to dream for a second, that we are able to solve *all* enumerative problems in generic (transverse) situations, and we know that the answer does not change after a small perturbation of the initial data.

> **D3**. How do we reduce to a transverse situation when there is none available (e.g. in Example 0.3.2)?

The modern way around this is to use *virtual fundamental classes* (cf. Section **??** and Appendix **??**).

0.3.1. *Two more words on excess intersection.* Problem (5), known as "the five conics problem", is a typical example of an excess intersection problem. See [10] for a thorough analysis and solution of this problem. As we shall see in Section 2.5.1, a natural compact parameter space for plane conics is

$$\mathcal{M} = \mathbb{P}^5,$$

and the set of smooth conics is an open subvariety $U \subset \mathcal{M}$. The answer to Problem (5) is a certain finite subset of U. Let $C_1, \ldots, C_5$ be general plane conics. The conics that are tangent to a given conic C_i form a sextic hypersurface $Z_i \subset \mathcal{M}$, so we might be tempted to say that the answer to Problem (5) is the degree

$$\int_{\mathbb{P}^5} \alpha_1 \cdots \alpha_5 = 6^5,$$

[3] A (-1)-curve on a surface S is a curve $C \subset S$ such that $C.C = -1$, where the intersection number $C.C$ can be seen as the degree of the normal bundle $N_{C/S}$ to C in S.

where $\alpha_i = [Z_i] \in H^2(\mathbb{P}^5, \mathbb{Z})$ is the divisor class of a sextic.[4] However, the cycles Z_i share a common two-dimensional component, namely the Veronese surface $\mathbb{P}^2 \subset \mathbb{P}^5$ of double lines. Therefore their intersection is 2-dimensional, even though our intuition suggests that 5 hypersurfaces in $\mathbb{P}^5$ should intersect in a finite set. Note that this issue arose precisely "because" we insisted to work with a compact parameter space: double lines are singular, hence lie in the complement of U. But working with U directly is forbidden, because it is not compact!

The excess intersection formula is a tool that allows one to precisely compute (and hence get rid of) the enumerative contribution of the *excess locus*, namely the locus of non-transverse intersection among certain cycles — in this case the cycles $Z_1, \ldots, Z_5$. The way it works is precisely via blow-ups; often more than one is required to separate the common components of the non-transverse cycles. In the case of the five conics problem, only one blow-up is required.

In principle, blowing up the excess locus, checking that the proper transforms will be disjoint in the exceptional divisor, and blowing up again if necessary, one gets to the correct answer to the original question, but:

> **D4**. In practice it is often very hard to keep track of multiple blow-ups; the calculation becomes less and less intuitive and the modular meaning of the blow-ups appearing might be quite unclear.

In Exercise 0.3.4 you will compute an excess bundle for a more complicated problem than finding the number of lines through two points. Before tackling it, it is best to solve the following exercise.

EXERCISE 0.3.3. Show that the vector space V of homogeneous cubic polynomials in 3 variables is 10-dimensional. Identify

$$\mathbb{P}V = \mathbb{P}^9$$

with the space of degree 3 plane curves $C \subset \mathbb{P}^2$. Show that, for a given point $p \in \mathbb{P}^2$, the space of cubics passing through p forms a hyperplane

$$\mathbb{P}^8 \subset \mathbb{P}V.$$

EXERCISE 0.3.4. Let C_1 and C_2 be two plane cubics intersecting transversely in nine points $p_1, \ldots, p_9 \in \mathbb{P}^2$ (cf. Figure 3). Every cubic in the pencil $\mathbb{P}^1 \subset \mathbb{P}^9$ generated by C_1 and C_2 passes through $p_1, \ldots, p_9$. However, if the nine points were general, there would be a unique cubic passing through them. Find out where the answer '1' is hiding in this non-transverse geometry.

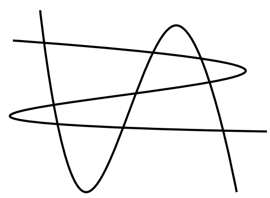


FIGURE 3. The nine intersection points $C_1 \cap C_2$.

[4]Recall that the Picard group $\operatorname{Pic} \mathbb{P}^r = H^2(\mathbb{P}^r, \mathbb{Z}) = \mathbb{Z}$ is generated by the hyperplane class h and the cohomology class of a degree d hypersurface in $\mathbb{P}^r$ corresponds to the class $d \cdot h$.

0.4. **Before and after the virtual class.** Here is a philosophical description of the field of Enumerative Geometry before and after the advent of *virtual fundamental classes*, introduced by Li–Tian [25] and Behrend–Fantechi [5].

Before: What is the answer?
After: What is the question?

Before virtual classes, there were a number of unanswered enumerative questions whose geometrical meaning was extremely clear. *After* the definition of virtual classes, many new invariants were defined through them, but the enumerative meaning of these invariants is often not very clear, so it fair to ask what integrals of the form

$$\int_{[\mathcal{M}]^{\mathrm{vir}}} \alpha \in \mathbb{Z}, \quad \alpha \in H^*(\mathcal{M}),$$

might be actually computing.

Virtual fundamental classes allow one to think that even a horrible moduli space $\mathcal{M}$, say a singular scheme of impure dimension (cf. Figure 6), has a well-defined *virtual dimension* vd at any point $p \in \mathcal{M}$, and this number is constant on p. It is given as the difference

$$\mathrm{vd} = \dim T_p\mathcal{M} - \dim \mathrm{Ob}|_p,$$

where both dimensions on the right may (and will) vary with p. The virtual fundamental class is a homology class

$$[\mathcal{M}]^{\mathrm{vir}} \in A_{\mathrm{vd}}\mathcal{M} \to H_{2\cdot\mathrm{vd}}(\mathcal{M}, \mathbb{Z})$$

that should be thought of as the fundamental class that $\mathcal{M}$ would have if it were of the form $\mathcal{M} = \{s = 0\}$ for s a regular section of a vector bundle (the bundle Ob) on a smooth variety.

As a matter of fact, many badly behaved moduli spaces turn out to have a virtual fundamental class. These include:

(i) the moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(X, \beta)$ to a smooth projective variety X,

(ii) the moduli space $M_Y^H(\alpha)$ of H-stable torsion free sheaves with Chern character α on a smooth 3-fold Y,

(iii) the moduli space $P_X^H(\alpha)$ of Pandharipande–Thomas pairs with Chern character α.

All this richness gives rise to three amongst the most modern counting theories:

$$\begin{aligned}
\text{Gromov–Witten theory} &\coloneqq \text{intersection theory on } \overline{\mathcal{M}}_{g,n}(X, \beta),\\
\text{Donaldson–Thomas theory} &\coloneqq \text{intersection theory on } M_Y^H(\alpha),\\
\text{Pandharipande–Thomas theory} &\coloneqq \text{intersection theory on } P_X^H(\alpha).
\end{aligned}$$

All these theories can be seen as more complicated (virtual) versions of a well established theory:

$$\text{Schubert Calculus} \coloneqq \text{intersection theory on the Grassmannian } G(k, n).$$

No "virtualness" is arising in Schubert calculus, because — as already observed by Mumford [28] when he initiated the enumerative geometry of the moduli space of curves — the Grassmannian is the ideal moduli space one would like to work with: it is compact, smooth and unobstructed. It does have a virtual fundamental class, but because of these properties it happens to coincide with its actual fundamental class.

0.5. **To the reader.** The reader might benefit from some familiarity with elementary aspects of scheme theory, basic theory of coherent sheaves on algebraic varieties, and intersection theory at the level of [20]. We shall, however, review some preliminaries in the next section. Here is a list of excellent references for the background material needed in these lecture notes (that we will refer to when necessary):

- for scheme theory at various levels, see [20, 9, 26, 40],
- for Intersection Theory, see [13, 10],
- for toric varieties, see [14, 7],
- for Deformation Theory, see [37, 21] and [11, Part 3].

1. Background material

1.1. **Varieties and schemes.** The notion of scheme used in this text is the standard one, see e.g. [26, Chapter 2]. The structure sheaf of a scheme X, its sheaf of regular functions, is denoted $\mathscr{O}_X$. A scheme X is *locally Noetherian* if every point $x \in X$ has a Zariski affine open neighborhood $x \in \operatorname{Spec} R \subset X$ such that R is a Noetherian ring. If X is locally Noetherian and quasi-compact, then it is called *Noetherian.* Any open or closed subscheme of a Noetherian scheme X is still Noetherian, and for every affine open subset $U \subset X$ the ring $\mathscr{O}_X(U)$ is Noetherian. An important property of Noetherian schemes is that they have a finite number of irreducible components, or, more generally, of associated points.

A morphism of schemes $f : X \to S$ is *quasi-compact* if the preimage of every affine open subset of S is quasi-compact. On the other hand, f is *locally of finite type* if for every $x \in X$ there exist Zariski open neighborhoods $x \in \operatorname{Spec} A \subset X$ and $f(x) \in \operatorname{Spec} B \subset S$ such that $f(\operatorname{Spec} A) \subset \operatorname{Spec} B$ and the induced map $B \to A$ is of finite type, i.e. A is isomorphic to a quotient of $B[x_1,\ldots,x_n]$ as a B-algebra. We say that f is *of finite type* if it is locally of finite type and quasi-compact.

Exercise 1.1.1. Let $f : X \to S$ be a morphism of schemes, with S (locally) Noetherian. If f is (locally) of finite type, then X is (locally) Noetherian.

For instance, a scheme of finite type over a field is Noetherian.

Notation 1.1.2. By k we will always mean an algebraically closed field. For most of the time, we will have $k = \mathbb{C}$.

Definition 1.1.3. A scheme X is *reduced* if for every point $p \in X$ the local ring $\mathscr{O}_{X,p}$ is reduced, i.e. it has no nilpotent elements besides zero.

The prototypical example of a nonreduced scheme is the curvilinear affine scheme

$$D_n = \operatorname{Spec} k[t]/t^n, \quad n > 1.$$

One can show that (quasi-compact) reduced schemes are precisely those schemes for which the regular functions on them are determined by their values on points.

The function

$$\overline{t} \in k[t]/t^n$$

vanishes at the unique point of D_n, but it is not the zero function!

The case $n = 2$ is particularly important. For instance, the *Zariski tangent space* $T_x X$ of a k-scheme X at a point $x \in X$, which by definition is the k-vector space $(\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$, can be identified with

$$\mathrm{Hom}_x(D_2, X),$$

the space of k-morphisms $D_2 \to X$ such that the image of the closed point of D_2 is x.

Example 1.1.4 (D_2 as a limit of distinct points)**.** Consider the scheme

$$X_a = \operatorname{Spec} \mathbb{C}[x,y]/(y - x^2, y - a), \quad a \in \mathbb{C}.$$

For $a \neq 0$, this scheme consists of two reduced points, corresponding to the maximal ideals

$$(x \pm \sqrt{a}, y - a) \subset \mathbb{C}[x,y].$$

For $a = 0$, we get

$$X_0 = \operatorname{Spec} \mathbb{C}[x]/x^2 = D_2,$$

a point with multiplicity two. See Figure 4 for a visual explanation.

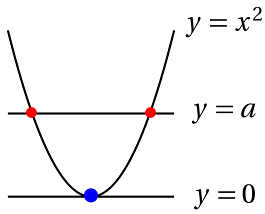


FIGURE 4. The intersection X_a of a parabola with the line $y = a$.

Definition 1.1.5. Let k be a field. An *algebraic variety* over k (or simply a k-variety) is a reduced, separated scheme of finite type over $\operatorname{Spec} k$, i.e. a reduced scheme X equipped with a finite type morphism $X \to \operatorname{Spec} k$, such that the diagonal map $\Delta_X : X \to X \times_k X$, sending $x \mapsto (x,x)$, is a closed immersion.

An *affine variety* is a k-scheme of the form $\operatorname{Spec} A$, where $A = k[x_1, \ldots, x_n]/I$ for some ideal I. An algebraic variety is *projective* if it admits a closed immersion into projective space $\mathbb{P}^n$ for some n. A variety is *quasi-projective* if it admits a locally closed immersion in some projective space, i.e. it is closed in an open subset of some $\mathbb{P}^n$. The same abstract scheme can of course be a (quasi-)projective variety in many different ways.

Example 1.1.6. The *rational normal curve* of degree d is the image of the closed embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^d$ defined by $(u : v) \mapsto (u^d : u^{d-1}v : \cdots : uv^{d-1} : v^d)$.

EXERCISE 1.1.7. Consider the algebraic variety $X = \operatorname{Spec} \mathbb{C}[x,y]/(xy, y^2)$, viewed as a subscheme of the affine plane $\mathbb{A}^2 = \operatorname{Spec} \mathbb{C}[x,y]$. Show that the origin $p = (0,0) \in X$ is the unique point such that $\mathscr{O}_{X,p}$ is not reduced.

The following definition will be relevant when we will discuss the Hilbert scheme of points in Section 4.

Definition 1.1.8. An algebraic k-variety X is *finite* if $\mathscr{O}_X(X)$ is a finite dimensional k-vector space. For any such X, the ring of functions is necessarily *Artinian*. In other words, X has dimension zero, and we say that $\mathscr{O}_X(X)$ is a finite dimensional k-algebra of *length*

$$\ell = \dim_k \mathscr{O}_X(X).$$

We also say that ℓ is the length of X.

EXERCISE 1.1.9. Show that an algebraic variety X is both affine and projective if and only if it is finite. Show that, for any ℓ, the only reduced finite k-variety of length ℓ is the disjoint union $\coprod_{1\leq i\leq \ell} \operatorname{Spec} k$.

EXERCISE 1.1.10. Classify all finite dimensional $\mathbb{C}$-algebras of length 2 and 3 up to isomorphism.

EXERCISE 1.1.11. Give an example of a scheme X whose underlying topological space consists of finitely many points, and yet is *not* finite.

1.2. **Some properties of morphisms.** We encountered separated morphisms in the definition of algebraic varieties (Definition 1.1.5). A morphism $f\colon X \to S$ is *separated* if the diagonal $X \to X \times_S X$ (which is always a locally closed immersion) is a closed immersion. A stronger notion is properness. A morphism $f\colon X \to S$ is *proper* if it is separated, of finite type, and universally closed. The *valuative criterion* for proper morphims says that f is proper if and only if for every valuation domain A with fraction field K there exists exactly one way to fill in the dotted arrow in a commutative square

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d, hook, "i"'] & X \arrow[d, "f"] \\
\operatorname{Spec} A \arrow[r] \arrow[ur, dotted] & S
\end{tikzcd}$$

in such a way that the resulting triangles are commutative. Such property can be rephrased by saying that for any A as above the map of sets

$$\operatorname{Hom}(\operatorname{Spec} A, X) \to \operatorname{Hom}(\operatorname{Spec} K, X) \times_{\operatorname{Hom}(\operatorname{Spec} K, S)} \operatorname{Hom}(\operatorname{Spec} A, S)$$

defined by $v \mapsto (v \circ i, f \circ v)$ is a bijection.

Let A and $\overline{A}$ be Artinian k-algebras with residue field k. We say that a surjection $u\colon \overline{A} \twoheadrightarrow A$ is a *square zero extension* if $(\ker u)^2 = 0$.

Definition 1.2.1. Let $f\colon X \to S$ be a locally of finite type morphism between k-schemes. Then f is *unramified* (resp. *smooth*, *étale*) if for any square zero extension $\overline{A} \twoheadrightarrow A$ and solid diagram

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\
\operatorname{Spec} \overline{A} \arrow[r] \arrow[ur, dotted] & S
\end{tikzcd}$$

there exists at most one (resp. at least one, exactly one) way to fill in the dotted arrow in such a way that the resulting triangles are commutative.

Example 1.2.2. The following are important features to keep in mind.

- Let $f: X \to S$ be a morphism of smooth k-varieties. If f induces isomorphism on tangent spaces, it is étale.
- A bijective morphism of smooth varieties is an isomorphism.
- If f is étale and injective, it is an open immersion.

1.3. **Schemes with embedded points.** On a locally Noetherian scheme X there are a bunch of points that are more relevant than all other points, in the sense that they reveal part of the behavior of the structure sheaf: these points are the *associated points* of X.

Let R be a commutative ring with unity, and let M be an R-module. If $m \in M$, we let

$$\operatorname{Ann}_R(m) = \{ r \in R \mid r \cdot m = 0 \} \subset R$$

denote its annihilator. A prime ideal $\mathfrak{p} \subset R$ is said to be *associated to M* if $\mathfrak{p} = \operatorname{Ann}_R(m)$ for some $m \in M$. The set of all associated primes is denoted

$$\operatorname{AP}_R(M) = \{ \mathfrak{p} \mid \mathfrak{p} \text{ is associated to } M \}.$$

Lemma 1.3.1. *Let $\mathfrak{p}$ be a prime ideal of R. Then $\mathfrak{p} \in \operatorname{AP}_R(M)$ if and only if $R/\mathfrak{p}$ is an R-submodule of M.*

Proof. If $\mathfrak{p} = \operatorname{Ann}_R(m)$ for some $m \in M$, consider the map $\phi_m: R \to M$ defined by $\phi_m(r) = r \cdot m$. Since its kernel is by definition $\operatorname{Ann}_R(m)$, the quotient $R/\mathfrak{p}$ is an R-submodule of M. Conversely, given an R-linear inclusion $i: R/\mathfrak{p} \hookrightarrow M$, consider the composition $\phi: R \twoheadrightarrow R/\mathfrak{p} \hookrightarrow M$. Then $\phi = \phi_m$, where $m = i(1)$. □

Note that if $\mathfrak{p} \in \operatorname{AP}_R(M)$ then $\mathfrak{p}$ contains the annihilator of M, i.e. the ideal

$$\operatorname{Ann}_R(M) = \{ r \in R \mid r \cdot m = 0 \text{ for all } m \in M \} \subset R.$$

The minimal elements (with respect to inclusion) in the set

$$\{ \mathfrak{p} \subset R \mid \mathfrak{p} \supset \operatorname{Ann}_R(M) \}$$

are called *isolated primes* of M.

From now on we assume R is Noetherian and $M \neq 0$ is finitely generated. In this situation, M has a *composition series*, i.e. a filtration of R-submodules

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_s = M$$

such that $M_i/M_{i-1} = R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$. This series is not unique. However, for a prime ideal $\mathfrak{p} \subset R$, the number of times it occurs among the $\mathfrak{p}_i$ does not depend on the composition series. These primes are precisely the elements of $\operatorname{AP}_R(M)$. For $M = R/I$, elements of $\operatorname{AP}_R(R/I)$ are the radicals of the primary ideals in a *primary decomposition* of I.

EXERCISE 1.3.2. Let $R = k[x, y]$, $I = (xy, y^2)$ and $M = R/I$. Show that $\operatorname{AP}_R(M) = \{(y), (x, y)\}$.

Theorem 1.3.3 ([40, Theorem 5.5.10 (a)])**.** *Let R be a Noetherian ring, $M \neq 0$ a finitely generated R-module. Then $\mathrm{AP}_R(M)$ is a finite nonempty set containing all isolated primes.*

Definition 1.3.4. The non-isolated primes in $\mathrm{AP}_R(M)$ are called the *embedded primes* of M.

The most boring situation is when R is an integral domain, in which case the generic point $\xi \in \operatorname{Spec} R$ is the only associated prime. More generally, a reduced affine scheme $\operatorname{Spec} R$ has *no embedded point*, i.e. the only associated primes are the isolated (minimal) ones, corresponding to its irreducible components.

Fact 1.3.5. An algebraic curve has no embedded points if and only if it is Cohen–Macaulay. However, there can be nonreduced Cohen–Macaulay curves: those curves with a fat component, such as the affine plane curve $\operatorname{Spec} k[x,y]/x^2 \subset \mathbb{A}^2$. These objects often have moduli, i.e. deform (even quite mysteriously) in positive dimensional families.

Let R be an integral domain. For an ideal $I \subset R$, one often calls the associated primes of I the associated primes of R/I. The minimal primes above $I = \operatorname{Ann}_R(R/I)$ correspond to the irreducible components of the closed subscheme

$$\operatorname{Spec} R/I \subset \operatorname{Spec} R,$$

whereas for every embedded prime $\mathfrak{p} \subset R$ there exists a minimal prime $\mathfrak{p}'$ such that $\mathfrak{p}' \subset \mathfrak{p}$. Thus $\mathfrak{p}$ determines an *embedded component* — a subvariety $V(\mathfrak{p})$ embedded in an irreducible component $V(\mathfrak{p}')$. If the embedded prime $\mathfrak{p}$ is maximal, we talk about an *embedded point*.

Remark 1.3.6. An embedded component $V(\mathfrak{p})$, where $\mathfrak{p}$ is the radical of some primary ideal $\mathfrak{q}$ appearing in a primary decomposition $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_e$, is of course embedded in some irreducible component $V(\mathfrak{p}') \subset \operatorname{Spec} R/I$, but $V(\mathfrak{q})$ is not a *subscheme* of $V(\mathfrak{p}')$, because the fuzzyness caused by nilpotent behavior (i.e. the difference between $\mathfrak{q}$ and its radical $\mathfrak{p}$) makes the bigger scheme $V(\mathfrak{q}) \supset V(\mathfrak{p})$ "stick out" of $V(\mathfrak{p}') \subset \operatorname{Spec} R/I$.

Example 1.3.7. Consider $R = k[x,y]$ and $I = (xy, y^2)$. A primary decomposition of I is

$$I = (x,y)^2 \cap (y).$$

However, $\operatorname{Spec} R/(x,y)^2$ is not scheme-theoretically contained in $\operatorname{Spec} R/y$.

In general, a subscheme Z of scheme Y has an embedded component if there exists a dense open subset $U \subset Y$ such that $Z \cap U$ is dense in Z but the scheme-theoretic closure of $Z \cap U \subset Z$ does not equal Z scheme-theoretically. For instance, if Y is irreducible, we say that $p \in Y$ supports an embedded point of a closed subscheme $Z \subset Y$ if $\overline{Z \cap (Y \setminus p)} \neq Z$ as schemes. In the example above, where $Y = \mathbb{A}^2$ and $Z = \operatorname{Spec} k[x,y]/(xy,y^2)$, the scheme-theoretic closure of $Z \cap (\mathbb{A}^2 \setminus 0) \subset Z$ is not equal to Z.

1.4. **Sheaves and their support.** Recall that a *coherent sheaf* on a (locally Noetherian) scheme X is an $\mathscr{O}_X$-module that is locally the cokernel of a map of free $\mathscr{O}_X$-modules of finite rank. Coherent sheaves form an abelian category

$$\operatorname{Coh} X.$$

For instance, if $\iota\colon Z \hookrightarrow X$ is a closed subscheme, both $\iota_*\mathscr{O}_Z$ and $\mathscr{I}_Z$ are coherent sheaves on X. The ideal sheaf, being a subsheaf of a free sheaf, is torsion free. In fact, ideal sheaves are precisely the torsion free sheaves of rank one and trivial determinant.

Definition 1.4.1. Let $X \to S$ be a finite type morphism of locally Noetherian schemes. A sheaf $F \in \operatorname{Coh} X$ is *flat over* S (or S-*flat*) if for every point $x \in X$, with image $s \in S$, the module F_x is flat over $\mathscr{O}_{S,s}$ via the ring map $\mathscr{O}_{S,s} \to \mathscr{O}_{X,x}$.

For instance, $\mathscr{O}_X$ is S-flat if and only if $X \to S$ is flat as a morphism of schemes.

The *support* of a coherent sheaf $F \in \operatorname{Coh} X$ is the following *closed subscheme* of X: consider the map $\mathscr{O}_X \to \mathscr{H}om_{\mathscr{O}_X}(F, F)$ defined on local sections by sending f to the $\mathscr{O}_X$-linear map $m \mapsto f \cdot m$. The kernel — the sheaf-theoretic annihilator ideal of F — defines the closed subscheme

$$\operatorname{Supp} F \subset X.$$

The support behaves well under pullback. However, the following remark is the origin of several issues such as the existence of Hilbert–Chow morphisms.

Remark 1.4.2. Let $X \to S$ be a a finite type morphism of locally Noetherian schemes. It is not true that the support of an S-flat $\mathscr{O}_X$-module is flat over S.

EXERCISE 1.4.3. Give an example of the phenomenon described in Remark 1.4.2.

2. GRASSMANNIAN, QUOT, HILB

In this section we introduce the three most important examples of *fine moduli spaces* used in Algebraic Geometry: Grassmannians, Quot schemes and Hilbert schemes. As we will see, both Grassmannians and Hilbert schemes can be recovered as special instances of Quot schemes.

The technical way to define fine moduli spaces is via representable functors $\mathfrak{M}\colon \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Sets}$. The notion of representability will be introduced in Section 2.1, for the sake of completeness. More details and examples can be found, for instance, in [41].

The basic idea is as follows. First of all, every scheme $\mathcal{M}$ trivially represents its own functor of points, which is the functor $\mathsf{h}_{\mathcal{M}}$ sending

$$U \mapsto \operatorname{Hom}_{\mathsf{Sch}}(U, \mathcal{M}).$$

One would say that $\mathcal{M}$ is a "fine moduli space of things" if the functor $\mathfrak{M}$ assigning to a scheme U the set of "families of things" defined over U is isomorphic to $\operatorname{Hom}_{\mathsf{Sch}}(-, \mathcal{M})$.

A fine moduli space is special in this sense: its points have a "label", just as the items of a phone book. We know precisely each point's name and address, so we can always find it on the moduli space (see Figure 5). This is the power of *universal families.*

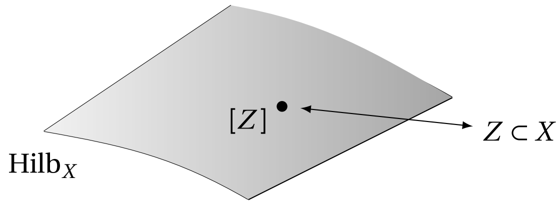


FIGURE 5. Each point of a fine moduli space (e.g. the Hilbert scheme) has a well precise label.

2.1. **Representable functors.** We start by making the following assumption.

Assumption 2.1.1. All categories are assumed to be *locally small*, i.e. we assume that $\mathrm{Hom}_{\mathcal{C}}(x,y)$ is a set for any pair of objects x and y.

Let $\mathcal{C}$ and $\mathcal{C}'$ be (locally small) categories.

Definition 2.1.2. A (covariant) functor $\mathsf{F}\colon \mathcal{C} \to \mathcal{C}'$ is called:

(i) *fully faithful* if for any two objects x, $y \in \mathcal{C}$ the map of sets

$$\mathrm{Hom}_{\mathcal{C}}(x,y) \to \mathrm{Hom}_{\mathcal{C}'}(\mathsf{F}(x),\mathsf{F}(y))$$

is a bijection.

(ii) *essentially surjective* if every object of $\mathcal{C}'$ is isomorphic to an object of the form $\mathsf{F}(x)$ for some $x \in \mathcal{C}$.

The following observation is quite useful.

Remark 2.1.3. A fully faithful functor $\mathsf{F}\colon \mathcal{C} \to \mathcal{C}'$ induces an equivalence of $\mathcal{C}$ with the essential image of F, namely the full subcategory of $\mathcal{C}'$ consisting of objects isomorphic to objects of the form $\mathsf{F}(x)$ for some $x \in \mathcal{C}$. Put differently, a functor is an equivalence if and only if it is fully faithful and essentially surjective.

Definition 2.1.4. A *natural transformation* $\eta\colon \mathsf{F} \Rightarrow \mathsf{G}$ between two functors $\mathsf{F}, \mathsf{G}\colon \mathcal{C} \to \mathcal{C}'$ is the datum, for every $x \in \mathcal{C}$, of a morphism $\eta_x\colon \mathsf{F}(x) \to \mathsf{G}(x)$ in $\mathcal{C}'$, such that for every $f \in \mathrm{Hom}_{\mathcal{C}}(x_1,x_2)$ the diagram

$$\begin{tikzcd}
\mathsf{F}(x_1) \arrow[r, "\eta_{x_1}"] \arrow[d, "\mathsf{F}(f)"'] & \mathsf{G}(x_1) \arrow[d, "\mathsf{G}(f)"] \\
\mathsf{F}(x_2) \arrow[r, "\eta_{x_2}"] & \mathsf{G}(x_2)
\end{tikzcd}$$

is commutative in $\mathcal{C}'$.

Definition 2.1.5. Let $\mathcal{C}$, $\mathcal{C}'$ be two categories. Let $\mathrm{Fun}(\mathcal{C},\mathcal{C}')$ be the category whose objects are functors $\mathcal{C} \to \mathcal{C}'$ and whose morphisms are the natural transformations. An isomorphism in the category $\mathrm{Fun}(\mathcal{C},\mathcal{C}')$ is called a *natural isomorphism*.

Let $\mathcal{C}$ be a (locally small) category. Its *opposite category* $\mathcal{C}^{\mathrm{op}}$, by definition, has the same objects of $\mathcal{C}$, and its morphisms are

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(x,y) = \mathrm{Hom}_{\mathcal{C}}(y,x), \quad x,y \in \mathcal{C}.$$

Consider the category of contravariant functors $\mathcal{C} \to \mathrm{Sets}$, i.e. the category

$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}).$$

For every object x of $\mathcal{C}$ there is a functor $\mathsf{h}_x : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$ defined by

$$u \mapsto \mathsf{h}_x(u) = \mathrm{Hom}_{\mathcal{C}}(u, x), \quad u \in \mathcal{C}.$$

A morphism $\phi \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(u, v) = \mathrm{Hom}_{\mathcal{C}}(v, u)$ gets sent to the map of sets

$$\mathsf{h}_x(\phi) : \mathsf{h}_x(u) \to \mathsf{h}_x(v), \quad \alpha \mapsto \alpha \circ \phi.$$

Consider the functor

$$\mathsf{h}_{\mathcal{C}} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}), \quad x \mapsto \mathsf{h}_x. \tag{2.1.1}$$

This is, indeed, a functor: for every arrow $f : x \to y$ in $\mathcal{C}$ and object u of $\mathcal{C}$ we can define a map of sets

$$\mathsf{h}_f u : \mathsf{h}_x(u) \to \mathsf{h}_y(u), \quad \alpha \mapsto f \circ \alpha,$$

with the property that for every morphism $\phi : v \to u$ in $\mathcal{C}$ there is a commutative diagram

$$\begin{tikzcd}
\mathsf{h}_x(u) \arrow[r, "\mathsf{h}_f u"] \arrow[d, "\mathsf{h}_x(\phi)"'] & \mathsf{h}_y(u) \arrow[d, "\mathsf{h}_y(\phi)"] \\
\mathsf{h}_x(v) \arrow[r, "\mathsf{h}_f v"] & \mathsf{h}_y(v)
\end{tikzcd}
\qquad
\begin{tikzcd}
u \xrightarrow{\alpha} x \arrow[r, mapsto] \arrow[d, mapsto] & u \xrightarrow{f \circ \alpha} y \arrow[d, mapsto] \\
v \xrightarrow{\alpha \circ \phi} x \arrow[r, mapsto] & u \xrightarrow{f \circ \alpha \circ \phi} y
\end{tikzcd}$$

defining a natural transformation

$$\mathsf{h}_f : \mathsf{h}_x \Rightarrow \mathsf{h}_y.$$

Lemma 2.1.6 (Weak Yoneda)**.** *The functor* $\mathsf{h}_{\mathcal{C}}$ *defined in* (2.1.1) *is fully faithful.*

Definition 2.1.7. A functor $\mathsf{F} \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets})$ is *representable* if it lies in the essential image of $\mathsf{h}_{\mathcal{C}}$, i.e. if it is isomorphic to a functor h_x for some $x \in \mathcal{C}$. In this case, we say that the object $x \in \mathcal{C}$ represents F.

Remark 2.1.8. By Lemma 2.1.6, if $x \in \mathcal{C}$ represents F, then x is unique up to a unique isomorphism. Indeed, suppose we have isomorphisms

$$a : \mathsf{h}_x \xrightarrow{\sim} F, \quad b : \mathsf{h}_y \xrightarrow{\sim} F$$

in the category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets})$. Then there exists a unique isomorphism $x \xrightarrow{\sim} y$ inducing $b^{-1} \circ a : \mathsf{h}_x \xrightarrow{\sim} \mathsf{h}_y$.

Let $\mathsf{F} \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets})$ be a functor, $x \in \mathcal{C}$ an object. One can construct a map of sets

$$g_x : \mathrm{Hom}(\mathsf{h}_x, \mathsf{F}) \to \mathsf{F}(x), \tag{2.1.2}$$

where the source is the hom-set in the category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets})$, which is indeed a set by Assumption 2.1.1.

To a natural transformation $\eta : \mathsf{h}_x \Rightarrow \mathsf{F}$ one can associate the element

$$g_x(\eta) = \eta_x(\mathrm{id}_x) \in \mathsf{F}(x),$$

the image of $\mathrm{id}_x \in \mathsf{h}_x(x)$ via the map $\eta_x\colon \mathsf{h}_x(x) \to \mathsf{F}(x)$.

Lemma 2.1.9 (Strong Yoneda)**.** *Let* $\mathsf{F} \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets})$ *be a functor,* $x \in \mathcal{C}$ *an object. Then the map* g_x *defined in* (2.1.2) *is bijective.*

Proof. The inverse of g_x is the map that assigns to an element $\xi \in \mathsf{F}(x)$ the natural transformation $\eta(x,\xi)\colon \mathsf{h}_x \Rightarrow \mathsf{F}$ defined as follows. For a given object $u \in \mathcal{C}$, we define

$$\eta(x,\xi)_u\colon \mathsf{h}_x(u) \to \mathsf{F}(u)$$

by sending a morphism $f\colon u \to x$ to the image of ξ under $\mathsf{F}(f)\colon \mathsf{F}(x) \to \mathsf{F}(u)$. □

EXERCISE 2.1.10. Show that Lemma 2.1.9 implies Lemma 2.1.6.

Definition 2.1.11. Let $\mathsf{F}\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$ be a functor. A *universal object* for F is a pair (x,ξ) where $\xi \in \mathsf{F}(x)$, such that for every pair (u,σ) with $\sigma \in \mathsf{F}(u)$, there exists a unique morphism $\alpha\colon u \to x$ such that $\mathsf{F}(\alpha)\colon \mathsf{F}(x) \to \mathsf{F}(u)$ sends ξ to σ.

EXERCISE 2.1.12. Show that a pair (x,ξ) is a universal object for a functor F if and only if the natural transformation $\eta(x,\xi)$ defined in Lemma 2.1.9 is a natural isomorphism. In particular, F is representable if and only if it has a universal object.

2.2. **Grassmannians.**

Fix integers $0 < k \le n$, a Noetherian scheme S and a coherent sheaf F on S. Let Sch_S be the category of locally Noetherian schemes over S. The *Grassmann functor*

$$\mathsf{G}(k,F)\colon \mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{Sets}$$

is defined by

$$(U \xrightarrow{g} S) \mapsto \left\{ \begin{array}{c} \text{equivalence classes of surjections } g^*F \twoheadrightarrow Q \\ \text{in } \mathrm{Coh}(U) \text{ with } Q \text{ locally free of rank } n-k \end{array} \right\} \tag{2.2.1}$$

where two quotients $p\colon g^*F \twoheadrightarrow Q$ and $p'\colon g^*F \twoheadrightarrow Q'$ are considered equivalent if there exists an $\mathscr{O}_U$-linear isomorphism $v\colon Q \xrightarrow{\sim} Q'$ such that $p' = v \circ p$.

Remark 2.2.1. When F is locally free, $G(k,F)$ is called the *Grassmann bundle* associated to F. Note that the kernel of a surjection between locally free sheaves is automatically locally free. Hence in this case $G(k,F)$ parameterises k-dimensional linear subspaces in the fibres of $F \to S$.

EXERCISE 2.2.2. Show that two quotients $p\colon g^*F \twoheadrightarrow Q$ and $p'\colon g^*F \twoheadrightarrow Q'$ are equivalent if and only if $\ker p = \ker p'$.

The functor (2.2.1) can be represented by an S-scheme

$$\rho\colon G(k,F) \to S.$$

The proof is an application of the general result that a Zariski sheaf that can be covered by representable subfunctors is representable [38, Tag 01JF, Lemma 25.15.4].

Example 2.2.3. Let $k = n-1$ and $F = \mathscr{O}_S^{\oplus n}$. Then we get the relative projective space

$$\mathbb{P}_S^{n-1} \to S.$$

We do know from the functorial description of projective space [20, Ch. II, Thm. 7.1] that an S-morphism $U \to \mathbb{P}^{n-1}_S$ is equivalent to the data

$$(\mathscr{L}; s_0, s_1, \ldots, s_{n-1})$$

where $\mathscr{L}$ is a line bundle on U and s_i are sections generating $\mathscr{L}$ — and moreover such tuple is considered equivalent to $(\mathscr{L}'; s'_0, s'_1, \ldots, s'_{n-1})$ if and only if there is an isomorphism of line bundles $\phi\colon \mathscr{L} \xrightarrow{\sim} \mathscr{L}'$ such that $\phi^* s'_i = s_i$. But this is precisely a U-valued point of $\mathsf{G}(n-1, \mathscr{O}_S^{\oplus n})$. Indeed, the functor prescribes the assignment of a surjection

$$\mathscr{O}_U^{\oplus n} \twoheadrightarrow \mathscr{L}$$

with $\mathscr{L}$ a line bundle. The equivalence class of this surjection is the same data as n generating sections of $\mathscr{L}$ up to isomorphism.

Example 2.2.4. If $S = \operatorname{Spec}\mathbb{C}$, we recover the usual Grassmannian

$$G(k,n) = G(k, \mathbb{C}^n) = \mathbb{G}(k-1, n-1)$$

of k-planes in $\mathbb{C}^n$ (or, equivalently, of projective linear subspaces $\mathbb{P}^{k-1} \hookrightarrow \mathbb{P}^{n-1}$), a smooth projective algebraic variety of dimension $k(n-k)$. When $k = n-1$ we obtain $G(n-1, n) = \mathbb{P}^{n-1}$.

By definition, representability of $\mathsf{G}(k, F)$ means that for every $g\colon U \to S$ there is a functorial bijection

$$\mathsf{G}(k,F)(g) \xrightarrow{\sim} \operatorname{Hom}_S(U, G(k,F)), \quad \alpha \mapsto \alpha_g. \tag{2.2.2}$$

Now take $U = G(k,F)$, $g = \rho$, and consider

$$\mathrm{id}_{G(k,F)} \in \operatorname{Hom}_S(G(k,F), G(k,F)).$$

The element in $\mathsf{G}(k,F)(\rho)$ mapping to $\mathrm{id}_{G(k,F)}$ via (2.2.2) is the *tautological exact sequence*

$$0 \to \mathscr{S} \to \rho^* F \to \mathscr{Q} \to 0 \tag{2.2.3}$$

over $G(k,F)$. Note that if F is locally free then $\mathscr{S}$ is locally free of rank k. The sequence (2.2.3) is called 'tautological' because of the following universal property: if $g\colon U \to S$ is any morphism and $\alpha \in \mathsf{G}(k,F)(g)$, then the equivalence class of the pullback surjection

$$\alpha_g^* \rho^* F \twoheadrightarrow \alpha_g^* \mathscr{Q}$$

coincides with α.

Example 2.2.5. Let $F = \mathscr{O}_S^{\oplus n}$ be a free sheaf of rank n, and set $k = n-1$. Then we saw that

$$G(n-1, F) = \mathbb{P}^{n-1}_S = \operatorname{Proj}\operatorname{Sym}\mathscr{O}_S^{\oplus n},$$

and the tautological surjection is the familiar

$$\mathscr{O}^{\oplus n}_{\mathbb{P}^{n-1}_S} \twoheadrightarrow \mathscr{O}_{\mathbb{P}^{n-1}_S}(1).$$

EXERCISE 2.2.6. Let $S = \operatorname{Spec}\mathbb{C}$, and fix a point $[\Lambda] \in G(k,F)$. Show that the tangent space of $G(k,F)$ at $[\Lambda]$ is isomorphic to

$$\operatorname{Hom}_{\mathbb{C}}(\Lambda, F/\Lambda).$$

On $S = \operatorname{Spec}\mathbb{Z}$, the Grassmann bundle

$$\rho\colon G(k, \mathscr{O}_S^{\oplus n}) \to S$$

is *proper*. Moreover, there is a closed embedding

$$G(k, \mathscr{O}_S^{\oplus n}) \hookrightarrow \mathbb{P}^{N-1}_{\mathbb{Z}}, \quad N = \binom{n}{k}.$$

For general (Noetherian) scheme S and locally free sheaf F, the determinant

$$\mathscr{L} = \det \mathscr{Q}$$

of the universal quotient bundle is relatively very ample on $\rho\colon G(k,F) \to S$, so it gives a closed embedding

$$G(k,F) \hookrightarrow \mathbb{P}(\rho_*\mathscr{L}) \hookrightarrow \mathbb{P}\left(\bigwedge^k F\right),$$

called the *Plücker embedding*.

2.3. **Quot and Hilbert schemes.** Let S be a Noetherian scheme and let $X \to S$ be a finite type morphism (so X is Noetherian by Exercise 1.1.1). Fix a coherent sheaf F on X. Denote by Sch_S the category of locally Noetherian schemes over S. Given such a scheme $U \to S$, define

$$\mathsf{Quot}_{X/S}(F)(U \to S)$$

to be the set of equivalence classes of pairs

$$(\mathscr{E}, p)$$

where

- $\mathscr{E}$ is a coherent sheaf on $X \times_S U$, flat over U and with proper support over U,
- $p\colon F_U \twoheadrightarrow \mathscr{E}$ is an $\mathscr{O}_{X\times_S U}$-linear surjection, where F_U is the pullback of F along $X \times_S U \to X$, and finally
- two pairs $(\mathscr{E}, p)$ and $(\mathscr{E}', p')$ are considered equivalent of $\ker\theta = \ker\theta'$.

EXERCISE 2.3.1. Show that $\mathsf{Quot}_{X/S}(F)$ defines a functor $\mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{Sets}$, and that it generalises the Grassmann functor $\mathsf{G}(k,F)$ defined in (2.2.1).

Let k be a field. Fix a line bundle L over a k-scheme X. For a coherent sheaf E on X whose support is proper over k, the function

$$m \mapsto \chi(E \otimes_{\mathscr{O}_X} L^{\otimes m})$$

becomes polynomial for $m \gg 0$. It is called the *Hilbert polynomial* of E (with respect to L), and is denoted $P_L(E)$. If $\mathscr{E}$ is a flat family of coherent sheaves on $X \to S$, such that

$$\operatorname{Supp}\mathscr{E} \subset X \to S$$

is proper, then the function

$$s \mapsto P_L(\mathscr{E}_s)$$

is locally constant on S.

EXERCISE 2.3.2. Let $C \subset \mathbb{P}^n$ be a smooth curve of degree d and genus g. Compute the Hilbert polynomial of C with respect to $L = \mathscr{O}_{\mathbb{P}^n}(1)$.

EXERCISE 2.3.3. What is the Hilbert polynomial of a conic in $\mathbb{P}^3$? What about a twisted cubic $C \subset \mathbb{P}^3$?

EXERCISE 2.3.4. Compute the Hilbert polynomial $P_{d,n}$ of a degree d hypersurface $Y \subset \mathbb{P}^n$. Show that there is a bijection between $\mathrm{Hilb}_{\mathbb{P}^n}^{P_{d,n},\mathscr{O}(1)}$ and $\mathbb{P}^{N-1}$, where $N = \binom{n+d}{d}$.

EXERCISE 2.3.5. Interpret the Grassmannian

$$\mathbb{G}(k,n) = \left\{ \text{linear subvarieties } \mathbb{P}^k \hookrightarrow \mathbb{P}^n \right\}$$

as a Hilbert scheme, i.e. find the unique polynomial P such that $\mathbb{G}(k,n) = \mathrm{Hilb}_{\mathbb{P}^n}^{P,\mathscr{O}(1)}$.

The functor $\mathsf{Quot}_{X/S}(F)$ decomposes as a coproduct

$$\mathsf{Quot}_{X/S}(F) = \coprod_{P \in \mathbb{Q}[z]} \mathsf{Quot}_{X/S}^{P,L}(F)$$

where the component $\mathsf{Quot}_{X/S}^{P,L}(F)$ sends an S-scheme U to the set of equivalence classes of quotients $p\colon F_U \twoheadrightarrow \mathscr{E}$ such that for each $u \in U$ the Hilbert polynomial of $\mathscr{E}_u = \mathscr{E}|_{X_u}$ (whose support is a closed subscheme of X_u proper over $k(u)$ by definition!), calculated with respect to L_u (the pullback of L along $X_u \hookrightarrow X \times_S U \to X$), is equal to P.

Theorem 2.3.6 (Grothendieck [19])**.** *If $X \to S$ is projective, F is a coherent sheaf on X, L is a relatively very ample line bundle over X and $P \in \mathbb{Q}[z]$ is a polynomial, then the functor $\mathsf{Quot}_{X/S}^{P,L}(F)$ is representable by a projective S-scheme*

$$\mathrm{Quot}_{X/S}^{P,L}(F) \to S.$$

Remark 2.3.7. There are several notions of projectivity for a morphism $X \to S$. If S has an ample line bundle (e.g. when it is quasi-projective over an affine scheme), then these notions are all equivalent, see [11, Part 2, § 5.5.1]. Grothendieck's original definition [18, Def. 5.5.2], in general different from the one in [20, II, § 4], stated that $X \to S$ is *projective* if it factors as

$$X \xhookrightarrow{i} \mathbb{P}(E) \to S$$

where E is a coherent $\mathscr{O}_S$-module and i is a closed immersion. This can be rephrased by saying that $X \to S$ is proper and there exists an ample family of line bundles on X over S. This is the notion used in Theorem 2.3.6. Moreover, $X \to S$ is called *quasi-projective* if it factors as $X \hookrightarrow Y \to S$, with $X \hookrightarrow Y$ open and $Y \to S$ projective.

Remark 2.3.8. The Noetherian hypothesis in Theorem 2.3.6 could be removed by Altman and Kleiman [1], but they needed a stronger notion of (quasi-)projectivity, as well as a stronger assumption on F. The result is a (quasi-)projective S-scheme $\mathrm{Quot}_{X/S}^{P,L}(F) \to S$. As a consequence, one obtains the following: when $X \hookrightarrow \mathbb{P}_S^n$ is a closed subscheme, $L = \mathscr{O}_{\mathbb{P}_S^n}(1)|_X$ and F is a sheaf quotient of $L(m)^{\oplus \ell}$, the functor $\mathsf{Quot}_{X/S}^{P,L}(F)$ is representable by a scheme that can be embedded in $\mathbb{P}_S^N$ for some N.

Definition 2.3.9. Let $X \to S$ be a projective morphism, and set $F = \mathscr{O}_X$. Then

$$\mathrm{Hilb}_{X/S} = \mathrm{Quot}_{X/S}(\mathscr{O}_X)$$

is called the *Hilbert scheme* of X/S. When $S = \operatorname{Spec} k$, we omit it from the notation.

EXERCISE 2.3.10. Let $C \subset \mathbb{P}^n$ be a smooth curve of degree d and genus g. Compute the Hilbert polynomial of C with respect to $L = \mathscr{O}_{\mathbb{P}^n}(1)$.

Remark 2.3.11. It is not true that for fixed n there always exists a smooth curve $C \subset \mathbb{P}^n$ of degree d and genus g.

Definition 2.3.12. Let X be a quasi-projective k-scheme, and let n be an integer. The *Hilbert scheme of n points* on X is the component

$$\mathrm{Hilb}^n X \subset \mathrm{Hilb}_X$$

corresponding to the constant Hilbert polynomial $P = n$. Similarly, we let

$$\mathrm{Quot}_X(F, n) \subset \mathrm{Quot}_X(F)$$

denote the connected component parameterising quotients $F \twoheadrightarrow Q$ where Q is a finite dimensional sheaf of length n.

See Sections 4 and **??** for more information on $\mathrm{Hilb}^n X$. We will give an alternative definition of $\mathrm{Hilb}^n \mathbb{A}^d$ in Section 4.2.

A theorem of Vakil [39] asserts that arbitrarily bad singularities appear generically on some components of some Hilbert scheme.

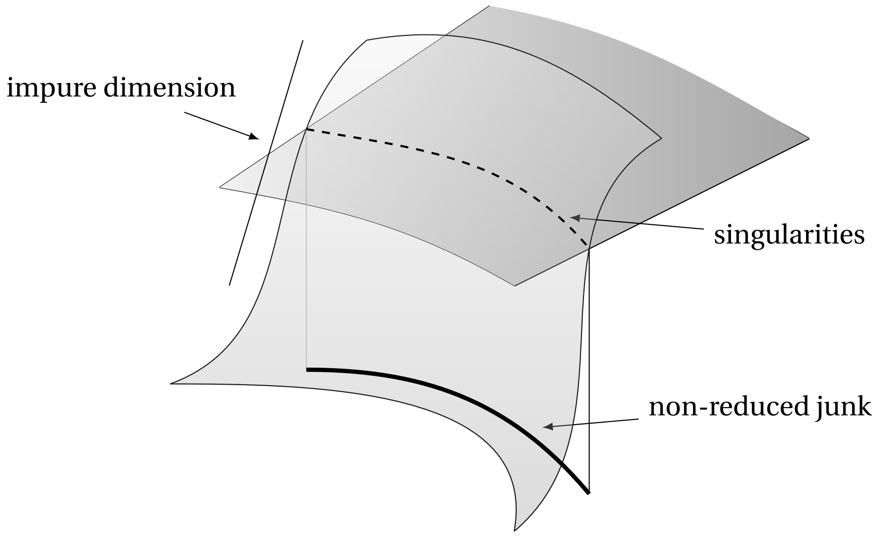


FIGURE 6. A nasty scheme. By Murphy's Law [39], it could be a Hilbert scheme component $H \subset \mathrm{Hilb}_X$ for some variety X.

However, despite its potentially horrible singularities, the Hilbert scheme has the great feature of representing a pretty explicit functor, so its functor of points is not that mysterious. In such a situation, the most important thing is to always keep in mind the *universal family* living over the representing scheme. In the case of the Hilbert scheme, this is a diagram

$$\begin{tikzcd} \mathcal{Z} \arrow[r, hook] \arrow[d, "\text{flat}"'] & X \times_S \mathrm{Hilb}_{X/S} \\ \mathrm{Hilb}_{X/S} & \end{tikzcd}$$

with the following property: for every S-scheme $g\colon U \to S$ along with a flat family of closed subschemes

$$\alpha\colon Z \subset X \times_S U \to U,$$

there exists precisely one S-morphism $\alpha_g\colon U \to \mathrm{Hilb}_{X/S}$ such that $Z = \alpha_g^* \mathcal{Z}$ as U-families of subschemes of X.

EXERCISE 2.3.13. Show that $\mathrm{Hilb}^1 X = X$. What is the universal family?

EXERCISE 2.3.14. Let C be a smooth curve embedded in a smooth 3-fold Y. Show that $\mathrm{Bl}_C Y \cong \mathrm{Quot}_Y(\mathscr{I}_C, 1)$.

2.4. **Tangent space to Quot.** Let X be a smooth projective variety over a field k. Let F be a coherent sheaf on Y. The Quot scheme

$$\mathrm{Quot}_X(F),$$

at a point $[F \twoheadrightarrow Q]$, has tangent space canonically isomorphic to

$$\mathrm{Hom}_{\mathscr{O}_X}(K, Q), \tag{2.4.1}$$

where $K = \ker(F \twoheadrightarrow Q)$. We already know this for the Grassmannian $G(k,n)$ by Exercise 2.2.6.

The case of the Hilbert scheme (i.e. when $F = \mathscr{O}_X$) is as follows. Let $p \in \mathrm{Hilb}_X$ be the point corresponding to a subscheme $Z \subset X$. Then, by definition,

$$T_p \,\mathrm{Hilb}_X = \mathrm{Hom}_p(\mathrm{Spec}\, k[t]/t^2, \mathrm{Hilb}_X),$$

and this is the set of all flat families

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Z} \arrow[r, hook] \arrow[d, "q"] & X \times_k D_2 \\ 0 \arrow[r, hook] & D_2 & \end{tikzcd}$$

such that the fibre of q over the closed point of $D_2 = \mathrm{Spec}\, k[t]/t^2$ equals Z. By definition, these are the *infinitesimal deformations* of the closed subscheme $Z \subset X$. It is shown in [21, Thm. 2.4] that these are classified by

$$\begin{aligned} \mathrm{Hom}_X(\mathscr{I}_Z, \mathscr{O}_Z) &= \mathrm{Hom}_Z(\mathscr{I}_Z/\mathscr{I}_Z^2, \mathscr{O}_Z) \\ &= H^0(Z, \mathscr{H}om(\mathscr{I}_Z/\mathscr{I}_Z^2, \mathscr{O}_Z)) \\ &= H^0(Z, N_{Z/X}), \end{aligned}$$

where $N_{Z/X}$ is the normal sheaf to Z in X.

EXERCISE 2.4.1. Show that $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}_{\mathbb{A}^3}^r, 1)$ is smooth of dimension $r+2$. Show that $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}_{\mathbb{A}^3}^r, r)$ is singular for all $r > 1$.

EXERCISE 2.4.2. Let $L \subset \mathbb{A}^3$ be a line. Compute the dimension of $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{I}_L, 2)$. Show that this Quot scheme is singular.

2.5. **Examples of Hilbert schemes.** The Hilbert scheme of points, i.e. the Hilbert scheme of zero-dimensional subschemes of a quasi-projective variety X, will be treated in later sections.

2.5.1. *Plane conics.* Let z_0, z_1 and z_2 be homogeneous coordinates on $\mathbb{P}^2$, and $\alpha_0,\ldots,\alpha_5$ be homogeneous coordinates on $\mathbb{P}^5$. Consider the closed subscheme

$$\mathcal{C}\subset\mathbb{P}^2\times\mathbb{P}^5$$

cut out by the equation

$$\alpha_0z_0^2+\alpha_1z_1^2+\alpha_2z_2^2+\alpha_3z_0z_1+\alpha_4z_0z_2+\alpha_5z_1z_2=0.$$

Let π be the projection $\mathcal{C}\to\mathbb{P}^5$. Over a point $a=(a_0,\ldots,a_5)\in\mathbb{P}^5$, the fibre is the conic

$$\pi^{-1}(a)=\left\{\,a_0z_0^2+a_1z_1^2+a_2z_2^2+a_3z_0z_1+a_4z_0z_2+a_5z_1z_2=0\,\right\}\subset\mathbb{P}^2.$$

There is a set-theoretic bijection between $\mathbb{P}^5$ and $\mathrm{Hilb}^{2t+1}_{\mathbb{P}^2}$. By the universal property of projective space, we have the scheme-theoretic identity

$$\mathbb{P}^5=\mathrm{Hilb}^{2t+1}_{\mathbb{P}^2},$$

and the map $\pi\colon\mathcal{C}\to\mathbb{P}^5$ is the universal family of the Hilbert scheme of plane conics.

EXERCISE 2.5.1. Make the last step precise and generalise the plane conics example to arbitrary hypersurfaces of $\mathbb{P}^n$. (**Hint**: Start out with the conclusion of Exercise 2.3.4 to write down the universal family).

Remark 2.5.2. Let X be a projective scheme. The universal family of the Hilbert scheme is always, *set-theoretically*, equal to

$$\mathcal{Z}=\{\,(x,[Z])\in X\times\mathrm{Hilb}_X\mid x\in Z\,\}\subset X\times\mathrm{Hilb}_X\,.$$

The problem is to determine the scheme structure on $\mathcal{Z}$. In the case of hypersurfaces of degree d in $\mathbb{P}^n$ (Exercise 2.5.1) this was easy precisely because $\mathcal{Z}$ is itself a hypersurface.

2.5.2. *Twisted cubics.* A twisted cubic is a smooth rational curve obtained as the image of the morphism

$$\mathbb{P}^1\hookrightarrow\mathbb{P}^3,\quad (u,v)\mapsto(u^3,u^2v,uv^2,v^3),$$

up to linear changes of coordinates of the codomain. The number of moduli of a twisted cubic is 12. Indeed, one has to specify four linearly independent degree 3 polynomials in two variables, up to $\mathbb{C}^\times$-scaling and automorphisms of $\mathbb{P}^1$. One then computes

$$4\cdot h^0(\mathbb{P}^1,\mathscr{O}_{\mathbb{P}^1}(3))-1-\dim\mathrm{PGL}_2=16-1-3=12.$$

The Hilbert polynomial of a twisted cubic is $3t+1$, cf. Exercise 2.3.2. There are other 1-dimensional subschemes $Z\subset\mathbb{P}^3$ with this Hilbert polynomial, e.g. a plane cubic union a point. This has 15 moduli: the choice of a plane $\mathbb{P}^2\subset\mathbb{P}^3$ contributes $3=\dim\mathbb{G}(2,3)$ moduli, a plane cubic $C\subset\mathbb{P}^2$ contributes 9 parameters, and the choice of a point $p\in\mathbb{P}^3$ accounts for the remaining three moduli.

The Hilbert scheme

$$\mathrm{Hilb}^{3t+1}_{\mathbb{P}^3}$$

was completely described in [32]. The two irreducible components we just described turn out to be the only ones. They are smooth, rational, of dimension 12 and 15 respectively, and they intersect along a smooth, rational 11-dimensional subvariety

$V \subset \mathrm{Hilb}^{3t+1}_{\mathbb{P}^3}$ parameterising uninodal plane cubics with an embedded point at the node. In [20, § III, Ex. 9.8.4] a family of twisted cubics degenerating to a plane uninodal cubic with an embedded point is described. The total space of the family, in a local chart, is defined by the ideal

$$I = (a^2(x+1) - z^2, ax(x+1) - yz, xz - ay, y^2 - x^2(x+1)) \subset \mathbb{C}[a,x,y,z].$$

Letting $a = 0$ one obtains the special fibre given by

$$I_0 = (z^2, yz, xz, y^2 - x^2(x+1)) \subset \mathbb{C}[x,y,z],$$

and $p = (0,0,0)$ is a non-reduced point in $C_0 = \operatorname{Spec}\mathbb{C}[x,y,z]/I_0$. Note that C_0 is not scheme-theoretically contained in the plane $z = 0$, because the local ring $\mathscr{O}_{C_0,p}$ contains the nonzero nilpotent z (cf. Remark 1.3.6).

Remark 2.5.3. The *geometric genus* $p_g(X) = h^0(X, \omega_X)$ varies in flat families, as the twisted cubic example shows.

2.6. **A comment on fine moduli spaces and automorphisms.** Given a scheme S and a functor $\mathfrak{M}\colon \mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{Sets}$, an object $\mathcal{M}$ in Sch_S along with an isomorphism

$$\mathfrak{M} \cong \mathrm{Hom}_{\mathrm{Sch}_S}(-, \mathcal{M})$$

is a *fine moduli space* for the objects parameterised by $\mathfrak{M}$. It is common to hear that when the objects $\eta \in \mathfrak{M}(U/S)$ have automorphisms, the functor $\mathfrak{M}$ cannot be represented. This is, for instance, the case for the moduli functor of smooth (or stable) curves of genus g. Even though this is the correct *geometric* intuition to have, for a general functor the presence of automorphisms does not necessarily prevent the existence of a universal family, as the following exercise indicates.

EXERCISE 2.6.1. Construct the functor $\mathfrak{M}\colon \mathrm{Sets}^{\mathrm{op}} \to \mathrm{Sets}$ of isomorphism classes of finite sets. Show that it is representable (by what set?), even though every finite set has automorphisms.

In geometric situations, the presence of automorphisms constitutes a problem whenever one can construct a family of objects $\eta \in \mathfrak{M}(U/S)$ that is isotrivial but not globally trivial. This is for instance the case for families of curves: the moduli map $U \to \mathcal{M}_g$ associated to an isotrivial family $\mathcal{X} \to U$ of smooth curves of genus g, say with typical fibre C, would have to be constant for continuity reasons; but the same is of course happening for the trivial family $C \times U \to U$, so the functor of smooth curves of genus g cannot be represented.

3. LINES ON HYPERSURFACES: EXPECTATIONS

Let $Y \subset \mathbb{P}^n$ be a general hypersurface of degree d. We want to show the following:

We should expect a finite number of lines on Y if and only if $d = 2n-3$. We should expect *no lines* on Y if $d > 2n-3$. We should expect infinitely many lines on Y if $d < 2n-3$.

To understand the condition
$$\ell \subset Y$$
for $\ell \subset \mathbb{P}^n$ and a hypersurface $Y \subset \mathbb{P}^n$, we give the following concrete example.

Example 3.0.1. Let $\ell \subset \mathbb{P}^3$ be the line cut out by $L_1 = L_2 = 0$, where $L_i = L_i(z_0, z_1, z_2, z_3)$ are linear forms on $\mathbb{P}^3$. To fix ideas, set $L_1 = z_0$ and $L_2 = z_0 + z_2 + z_3$. Let $Y \subset \mathbb{P}^3$ be defined by a homogeneous equation $f = 0$, for instance the cubic polynomial
$$f = z_0^3 + 3z_0z_1^2 - z_2^2z_3.$$
Then we see that plugging $L_1 = L_2 = 0$ into f does not give zero, i.e.
$$f|_\ell = 0 + 0 - z_2^2(-z_0 - z_2) = z_2^3.$$
This means that ℓ is not contained in Y. On the other hand, the line cut out by L_1 and $L_2' = z_3$ lies entirely on Y.

Let $Y \subset \mathbb{P}^n$ be the zero locus of a general homogeneous polynomial
$$f \in H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)).$$
As we anticipated in Example 3.0.1, a line $\ell \subset \mathbb{P}^n$ is contained in Y if and only if $f|_\ell = 0$. This condition can be rephrased by saying that the image of f under the restriction map
$$\mathsf{res}_\ell \colon H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)) \to H^0(\ell, \mathscr{O}_\ell(d)) \tag{3.0.1}$$
vanishes. We want to determine when we should expect Y to contain a finite number of lines. We set, informally,
$$\mathsf{N}_1(Y) = \text{expected number of lines in } Y.$$
Let us consider the Grassmannian
$$\mathbb{G} = \mathbb{G}(1,n) = \left\{ \text{Lines } \ell \subset \mathbb{P}^n \right\},$$
a smooth complex projective variety of dimension $2n-2$. Recall the universal structures living on $\mathbb{G}$. First of all, the tautological exact sequence
$$\begin{array}{ccccccccc} & & \text{rank } 2 & & \text{rank } n+1 & & \text{rank } n-1 & & \\ & & | & & | & & | & & \\ 0 & \longrightarrow & \mathscr{S} & \longrightarrow & \mathscr{O}_{\mathbb{G}} \otimes_{\mathbb{C}} H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1))^\vee & \longrightarrow & \mathscr{Q} & \longrightarrow & 0 \end{array}$$
where the fibre of $\mathscr{S}$ over a point $[\ell] \in \mathbb{G}$ is the 2-dimensional vector space $H^0(\ell, \mathscr{O}_\ell(1))^\vee$. Let, also,
$$\mathcal{L} = \left\{ (p, [\ell]) \in \mathbb{P}^n \times \mathbb{G} \,\middle|\, p \in \ell \right\} \subset \mathbb{P}^n \times \mathbb{G}$$
be the universal line. Consider the two projections
$$\begin{tikzcd} \mathcal{L} \arrow[r, "q"] \arrow[d, "\pi"'] & \mathbb{P}^n \\ \mathbb{G} & \end{tikzcd}$$
and the coherent sheaf
$$\mathscr{E}_d = \pi_* q^* \mathscr{O}_{\mathbb{P}^n}(d).$$

EXERCISE 3.0.2. Show that $\mathscr{E}_d$ is locally free of rank $d+1$. (**Hint**: use cohomology and base change, e.g. [10, Theorem B.5]).

In fact, one has an isomorphism of locally free sheaves

$$\mathscr{E}_d \cong \operatorname{Sym}^d \mathscr{S}^\vee,$$

where $\iota\colon \mathscr{S} \hookrightarrow \mathscr{O}_{\mathbb{G}} \otimes_{\mathbb{C}} H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1))^\vee$ is the universal subbundle. Dualising ι and applying Sym^d, we obtain a surjection

$$\mathscr{O}_{\mathbb{G}} \otimes_{\mathbb{C}} H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)) \twoheadrightarrow \operatorname{Sym}^d \mathscr{S}^\vee,$$

which is just a global version of (3.0.1). The association

$$\mathbb{G} \ni [\ell] \mapsto f|_\ell \in H^0(\ell, \mathscr{O}_\ell(d)) \cong \operatorname{Sym}^d H^0(\ell, \mathscr{O}_\ell(1))$$

defines a section τ_f of $\mathscr{E}_d \to \mathbb{G}$. The zero locus of $\tau_f = \pi_* q^* f$ is the locus of lines contained in Y.

The following terminology is very common.

Definition 3.0.3. Let $Y \subset \mathbb{P}^n$ be a hypersurface defined by $f = 0$. Then

$$F_1(Y) = Z(\tau_f)$$

is called the *Fano scheme of lines* in Y.

Since f is generic, $\tau_f \in \Gamma(\mathbb{G}, \mathscr{E}_d)$ is also generic. In this case, the fundamental class of the Fano scheme of lines in Y is Poincaré dual to the Euler class

$$e(\mathscr{E}_d) \in A^{d+1}\mathbb{G}.$$

Thus $[F_1(Y)] \in A_*\mathbb{G}$ is a zero-cycle if and only if $d+1 = 2n-2$, i.e.

$$d = 2n-3.$$

The degree of this zero-cycle is then

$$\mathsf{N}_1(Y) = \int_{\mathbb{G}} e(\mathscr{E}_d) = \int_{\mathbb{G}} c_{d+1}(\operatorname{Sym}^d \mathscr{S}^\vee).$$

This degree is the *actual* number of lines on Y whenever $H^0(\ell, N_{\ell/Y}) = 0$ for all $\ell \subset Y$. This condition means that the Fano scheme is reduced at all its points $[\ell]$, since $H^0(\ell, N_{\ell/Y})$ is its tangent space at the point $[\ell]$.

Lemma 3.0.4. *If $S \subset \mathbb{P}^3$ is a smooth cubic surface and $\ell \subset S$ is a line, then $H^0(\ell, N_{\ell/S}) = 0$.*

Proof. It has enough to show that $N = N_{\ell/S}$, viewed as a line bundle on $\ell \cong \mathbb{P}^1$, has negative degree. By the adjunction formula,

$$K_\ell = K_S|_\ell \otimes_{\mathscr{O}_\ell} N.$$

Using that $K_\ell = \mathscr{O}_\ell(-2)$ and $K_S = K_{\mathbb{P}^3}|_S \otimes_{\mathscr{O}_S} N_{S/\mathbb{P}^3} = \mathscr{O}_S(d-4)$ for a surface of degree d in $\mathbb{P}^3$, by taking degrees we obtain

$$-2 = (3-4) + \deg N,$$

so that $\deg N = -1 < 0$. $\square$

4. THE HILBERT SCHEME OF POINTS

4.1. **Subschemes and zero-cycles.** Let X be a complex quasi-projective variety. In Section 2.3 we encountered the Hilbert scheme of points X. Recall that, as a set,

$$\operatorname{Hilb}^n X$$

parameterises the finite subschemes $Z \subset X$ of length n. Or, equivalently, ideal sheaves $\mathscr{I}_Z \subset \mathscr{O}_X$ of *colength* n.

There is a "coarser" way of parameterising *points with multiplicity* on the variety X. This is the content of the next definition.

Definition 4.1.1. Let X be a quasi-projective variety. The n-th *symmetric product* (or *configuration space*) of X is the quotient

$$\operatorname{Sym}^n X = X^n/\mathfrak{S}_n.$$

Remark 4.1.2. The quotient $\operatorname{Sym}^n X$ is the *Chow scheme* of effective zero-cycles (of degree n) on X. For higher dimensional cycles, the definition (and representability) of the Chow functor is a much subtler problem [35].

Each point $\xi \in \operatorname{Sym}^n X$ corresponds to a finite combination of points with multiplicity, i.e. it can be written as

$$\xi = \sum_i m_i \cdot p_i,$$

with $m_i \in \mathbb{Z}_{\geq 0}$ and $p_i \in X$.

EXERCISE 4.1.3. Let X be a smooth variety of dimension d. Show that the locus in $\operatorname{Hilb}^2 X$ of nonreduced subschemes $Z \subset X$ is isomorphic to $X \times \mathbb{P}^{d-1}$.

Note that the symmetric product $\operatorname{Sym}^n X$ does not deal with with the scheme structure of fat points inside X. For instance, any of the double point schemes supported on a given point $p \in X$, parameterised by the $\mathbb{P}^{d-1}$ of Exercise 4.1.3, has underlying cycle $2 \cdot p$. The operation of "forgetting the scheme structure" can be made functorial. This means that there exists a well-defined algebraic morphism

$$\pi_X \colon \operatorname{Hilb}^n X \to \operatorname{Sym}^n X, \tag{4.1.1}$$

taking a subscheme $Z \subset X$ to its underlying effective zero-cycle. In symbols,

$$\pi_X[Z] = \sum_{p \in \operatorname{Supp} Z} \operatorname{length} \mathscr{O}_{Z,p} \cdot p.$$

The morphism π_X is called the *Hilbert–Chow morphism.*

EXERCISE 4.1.4. Show that if C is a smooth quasi-projective curve then $\operatorname{Hilb}^n C \cong \operatorname{Sym}^n C$ via π_C. Deduce that $\operatorname{Hilb}^n \mathbb{A}^1 \cong \mathbb{A}^n$.

The easiest subvariety of the symmetric product is the *small diagonal*, which is just a copy of X embedded as

$$X \hookrightarrow \operatorname{Sym}^n X, \quad x \mapsto n \cdot x.$$

Definition 4.1.5. Let X be a smooth quasi-projective variety. The *punctual Hilbert scheme* is the closed subscheme

$$\mathrm{Hilb}^n(X)_x \subset \mathrm{Hilb}^n X$$

defined as the preimage of the cycle $n\cdot x$ via the Hilbert–Chow map (4.1.1).

EXERCISE 4.1.6. Show that that $\pi_X^{-1}(n\cdot x)$ does not depend on $x\in X$. Show that it does not depend on X either, but only on $\dim X$, as long as X is smooth.

Notation 4.1.7. If X is a smooth variety of dimension d, we will denote by

$$\mathrm{Hilb}^n(\mathbb{A}^d)_0 \subset \mathrm{Hilb}^n X$$

the punctual Hilbert scheme of Definition 4.1.5. This makes sense by Exercise 4.1.6.

EXERCISE 4.1.8. Let X be a smooth variety. Show that $\mathrm{Hilb}^2 X$ is isomorphic to the blowup of $\mathrm{Sym}^2 X$ along the diagonal.

EXERCISE 4.1.9. Show that if X is a smooth variety and $n\le 3$ then $\mathrm{Hilb}^n X$ is smooth. (**Hint**: show that a finite planar subscheme $Z\subset X$ defines a smooth point of the Hilbert scheme. Then use your classification from Exercise 1.1.10 to conclude).

4.2. **The Hilbert scheme of points on affine space.** Let $d\ge 1$ and $n\ge 0$ be integers. In this subsection we give a description of the Hilbert scheme

$$\mathrm{Hilb}^n \mathbb{A}^d = \{\, I\subset \mathbb{C}[x_1,\ldots,x_d] \mid \dim_{\mathbb{C}} \mathbb{C}[x_1,\ldots,x_d]/I = n \,\}.$$

More precisely, we will provide equations cutting out the Hilbert scheme inside a smooth quasi-projective variety (the so-called *non-commutative Hilbert scheme*), cf. Theorem 4.2.4.

An ideal $I\in \mathrm{Hilb}^n \mathbb{A}^d$ will be tacitly identified with the associated finite closed subscheme

$$\mathrm{Spec}\, \mathbb{C}[x_1,\ldots,x_d]/I \subset \mathbb{A}^d$$

of length n.

The following properties are well-known:

- $\mathrm{Hilb}^n \mathbb{A}^d$ is smooth if and only if $d\le 2$ or $n\le 3$,
- $\mathrm{Hilb}^n \mathbb{A}^3$ is irreducible for $n\le 11$ (see [22, 8] and the references therein),
- $\mathrm{Hilb}^n \mathbb{A}^3$ is reducible for $n\ge 78$, see [24].

On the other hand, the following questions are open since a long time:

- What is the smallest d such that $\mathrm{Hilb}^n \mathbb{A}^3$ is reducible?
- If $d\ge 3$, is $\mathrm{Hilb}^n \mathbb{A}^d$ generically reduced?

Let us get started with our description of $\mathrm{Hilb}^n \mathbb{A}^d$. To ease notation, let us put $R_d = \mathbb{C}[x_1,\ldots,x_d]$. The condition defining a point $I\in \mathrm{Hilb}^n \mathbb{A}^d$ is that the $\mathbb{C}$-algebra quotient

$$R_d \twoheadrightarrow R_d/I$$

is a vector space of dimension n. Let us examine this condition in detail. To construct a point in the Hilbert scheme, we need:

(1) a vector space
$$V_n \cong \mathbb{C}^n,$$

(2) an R_d-module structure
$$\vartheta\colon R_d \to \operatorname{End}_{\mathbb{C}}(V_n)$$
with the property that

(3) such structure is induced by an R_d-linear surjection from R_d.

So let us fix an n-dimensional vector space V_n. Later we will have to remember that we made such a choice, and since all we wanted was "dim $V_n = n$" we will have to quotient out all equivalent choices. Let us forget about this for the moment. In (2), we need θ to be a ring homomorphism, so we need to specify one endomorphism of V_n for each coordinate $x_i \in R_d$. All in all, ϑ gives us d matrices
$$A_1, A_2, \ldots, A_d \in \operatorname{End}_{\mathbb{C}}(V_n).$$
The matrix A_i will be responsible for the R_d-linear operator "multiplication by x_i" for the resulting module structure on V_n. Also in this step we should note a reminder for later: strictly speaking, what we have defined so far is a $\mathbb{C}\langle x_1, x_2, \ldots, x_d\rangle$-module structure on V_n. But in R_d the variables commute with one another. So we will have to impose the relations $[A_i, A_j] = 0$ for all $1 \le i < j \le d$.

Condition (3) is tricky. Let us reason backwards, assuming we already have an R_d-linear quotient $\phi\colon R_d \twoheadrightarrow V_n$. Then it is clear that the image of $1 \in R_d$ generates V_n as an R_d-module. In other words, every element $w \in V_n$ can be written as
$$w = A_1^{m_1} A_2^{m_2} \cdots A_d^{m_d} \cdot \phi(1),$$
for some $m_i \in \mathbb{Z}_{\ge 0}$. This tells us exactly what we should add to the picture to obtain Condition (3): for a fixed module structure, i.e. d-tuple of matrices $(A_1, A_2, \ldots, A_d)$, we need to specify a *cyclic vector*
$$v \in V_n,$$
i.e. a vector with the property that the $\mathbb{C}$-linear span of the set
$$\left\{ A_1^{m_1} A_2^{m_2} \cdots A_d^{m_d} \cdot v \,\middle|\, m_i \in \mathbb{Z}_{d \ge 0} \right\}$$
equals the whole V_n.

Let us consider the $(dn^2 + n)$-dimensional affine space
$$W_n = \operatorname{End}_{\mathbb{C}}(V_n)^{\oplus d} \oplus V_n. \tag{4.2.1}$$

EXERCISE 4.2.1. Show that the locus
$$U_n = \left\{ (A_1, A_2, \ldots, A_d, v) \,\middle|\, v \text{ is } (A_1, A_2, \ldots, A_d)\text{-cyclic} \right\} \subset W_n$$
is a Zariski open subset.

Consider the GL_n-action on W_n given by
$$g \cdot (A_1, A_2, \ldots, A_d, v) = (A_1^g, A_2^g, \ldots, A_d^g, g v) \tag{4.2.2}$$
where $M^g = g^{-1} M g$ is conjugation.

Lemma 4.2.2. *The* GL_n*-action* (4.2.2) *is free on* U_n*.*

Proof. If $g \in \mathrm{GL}_n$ fixes a point $(A_1, A_2, \ldots, A_d, v) \in U_n$, then $v = gv$ lies in the invariant subspace $\ker(g - \mathrm{id}) \subset V_n$. But by definition of U_n, the smallest invariant subspace containing v is V_n itself, thus $g = \mathrm{id}$. □

Definition 4.2.3. The GIT quotient

$$\mathrm{NCHilb}_d^n = U_n / \mathrm{GL}_n$$

is called the *non-commutative Hilbert scheme*.

The discussion carried out so far proves the following:

Theorem 4.2.4. *There is a closed immersion*

$$\mathrm{Hilb}^n \mathbb{A}^d \subset \mathrm{NCHilb}_d^n$$

cut out by the ideal of relations

$$[A_i, A_j] = 0 \textit{ for all } 1 \le i < j \le d. \tag{4.2.3}$$

EXERCISE 4.2.5. Let $d = 1$. Show that $\mathrm{NCHilb}_1^n = \mathbb{A}^n$.

Example 4.2.6. If $d = 1$ there is just one operator "x" so the Relations (4.2.3) are vacuous. We have

$$\mathrm{Hilb}^n \mathbb{A}^1 = \mathrm{NCHilb}_1^n = \mathbb{A}^n,$$

which thanks to Exercise 4.2.5 reproves the second part of Exercise 4.1.4.

Remark 4.2.7. If $d = 2$ the description of $\mathrm{Hilb}^n \mathbb{A}^2$ is equivalent to Nakajima's description [29,]. See also [22] for another description of the Hilbert scheme of points, in terms of *perfect extended monads*.

4.3. **Hilbert–Chow revisited.** Fix $n \ge 0$ and $d > 0$. The Hilbert–Chow morphism

$$\pi \colon \mathrm{Hilb}^n \mathbb{A}^d \to \mathrm{Sym}^n \mathbb{A}^d$$

introduced in (4.1.1) can be reinterpreted as follows. Pick a point

$$[A_1, \ldots, A_d, v] \in \mathrm{Hilb}^n \mathbb{A}^d$$

and notice that since the matrices pairwise commute, they can be simultaneously triangularised. So, since the tuple is defined up to GL_n, we may assume they are in the form

$$A_\ell = \begin{pmatrix} a_{11}^{(\ell)} & * & * & \cdots & * \\ 0 & a_{22}^{(\ell)} & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn}^{(\ell)} \end{pmatrix}.$$

Then π is given by

$$[A_1, \ldots, A_d, v] \mapsto \sum_\ell \left(a_\ell^{(1)}, \ldots, a_\ell^{(d)} \right).$$

When all the matrices are *nilpotent*, the corresponding subscheme $Z \subset \mathbb{A}^d$ is entirely supported at the origin. In other words,

$$\mathrm{Hilb}^n(\mathbb{A}^d)_0 = \{ [A_1, \ldots, A_d, v] \mid A_1, \ldots, A_d \text{ are nilpotent} \}$$

is a way to describe the punctual Hilbert scheme.

4.4. **Varieties of commuting matrices: what's known.** Let V be an n-dimensional complex vector space and let

$$C_n = \left\{ (A,B) \in \mathrm{End}(V)^2 \,\middle|\, [A,B]=0 \right\} \subset \mathrm{End}(V)^2 \tag{4.4.1}$$

be the *commuting variety*. Letting GL_n act on C_n by simultaneous conjugation, one can form the quotient stack

$$\mathcal{C}(n) = C_n / \mathrm{GL}_n,$$

which is equivalent to the stack $\mathrm{Coh}_n(\mathbb{A}^2)$ of finite coherent sheaves of length n on the affine plane. Letting

$$\widetilde{c}_n = \left[\mathcal{C}(n)\right] = \frac{\left[C_n\right]}{\mathsf{GL}_n} \in K_0(\mathrm{St}_{\mathbb{C}}) \tag{4.4.2}$$

be the motivic class of the stack $\mathcal{C}(n)$, let us form the generating series

$$\mathsf{C}(t) = \sum_{n\geq 0} \widetilde{c}_n t^n \in K_0(\mathrm{St}_{\mathbb{C}})[\![t]\!].$$

The next result is a formula essentially due to Feit and Fine, but also proven recently by Behrend–Bryan–Szendrői and Bryan–Morrison.

Theorem 4.4.1 ([12, 4, 6])**.** *There is an identity*

$$\mathsf{C}(t) = \prod_{k\geq 1}\prod_{m\geq 1}(1-\mathbb{L}^{2-k}t^m)^{-1}.$$

It has been known since a long time that the variety of pairs of commuting matrices C_n is irreducible [27, 34]. The same is true for the space $N_n \subset C_n$ of *nilpotent* commuting linear operators, see [2] for a proof in characteristic zero and [3] for an extension to fields of characteristic bigger than $n/2$. Premet even showed irreducibility of N_n over *any* field [33].

However the situation is very different for 3 or more matrices. Let $C(d,n)$ be the space of d-tuples of pairwise commuting endomorphisms of an n-dimensional vector space, and let $N(d,n)$ be the space of nilpotent endomorphisms. Then $C(d,n)$ is irreducible for all n if $d \leq 3$. But it is reducible if d and n are both at least 4 [16], and the same is true for $N(d,n)$ [30]. For $d=3$ the situation is as follows. One has that $C(3,n)$ is reducible for $n \geq 30$ [23] and irreducible for $n \leq 10$ (in characteristic zero). Moreover, $N(3,n)$ is known to be irreducible for $n \leq 6$ [30], but $N(3,n)$ is reducible for $n \geq 13$ [31, Thm. 7.10.5].

4.5. **The special case of** $\mathrm{Hilb}^n \mathbb{A}^3$**.** In this subsection we set $d=3$. The Hilbert scheme of points

$$\mathrm{Hilb}^n \mathbb{A}^3$$

is singular as soon as $n \geq 4$, but as we shall see it is *virtually smooth*, i.e. it carries a (symmetric) perfect obstruction theory.

EXERCISE 4.5.1. Let $p \in \mathbb{A}^3$ be a point. Show that $\mathfrak{m}_p^2 \subset \mathscr{O}_{\mathbb{A}^3}$ defines a singular point of $\mathrm{Hilb}^4 \mathbb{A}^3$.

Theorem 4.5.2. *There exists a smooth quasi-projective variety M_n along with a regular function $f_n\colon M_n \to \mathbb{A}^1$ such that*

$$\operatorname{Hilb}^n \mathbb{A}^3 \subset M_n$$

can be realised as the scheme-theoretic zero locus of the exact 1-form $\mathrm{d}f_n$.

Proof. As M_n we can take the noncommutative Hilbert scheme $\operatorname{NCHilb}_3^n$. By [36, Prop. 3.8], the commutator relations (4.2.3) agree *scheme-theoretically* with the single vanishing relation

$$\mathrm{d}f_n = 0,$$

where $f_n\colon \operatorname{NCHilb}_3^n \to \mathbb{A}^1$ is the function

$$[A_1, A_2, A_3, v] \mapsto \operatorname{Tr} A_1[A_2, A_3]. \qquad \square$$

The above description is special to $d = 3$.

EXERCISE 4.5.3. Show that, for $\{i, j, k\} = \{1, 2, 3\}$, one has

$$[A_i, A_j] = 0 \quad \Longleftrightarrow \quad \frac{\partial f_n}{\partial A_k} = 0,$$

at least set-theoretically.

By the construction of $[Z(\mathrm{d}f)]^{\mathrm{vir}}$ outlined in Section **??**, we deduce from Theorem 4.5.2 the following:

Corollary 4.5.4. *The Hilbert scheme $\operatorname{Hilb}^n \mathbb{A}^3$ carries a symmetric perfect obstruction theory, giving rise to a zero-dimensional virtual fundamental class*

$$\left[\operatorname{Hilb}^n \mathbb{A}^3\right]^{\mathrm{vir}} \in A_0(\operatorname{Hilb}^n \mathbb{A}^3).$$

4.5.1. *A quiver description.* Recall that a *quiver* is a finite directed graph. If $Q = (Q_0, Q_1, s, t)$ is a quiver, the notation means Q_0 is the set of vertices, Q_1 is the set of arrows and $s, t\colon Q_1 \to Q_0$ are the source and tail maps respectively. Let $\mathbf{d} = (\mathbf{d}_i) \in \mathbb{N}^{Q_0}$ be a vector of nonnegative integers. Then a $\mathbf{d}$-dimensional *representation* ρ of Q is the datum of a $\mathbb{C}$-vector space of dimension $\mathbf{d}_i$ for each $i \in Q_0$, along with a linear map $\mathbb{C}^{\mathbf{d}_i} \to \mathbb{C}^{\mathbf{d}_j}$ for every arrow $i \to j$ in Q_1. We write $\underline{\dim}\, \rho = \mathbf{d}$. The space of such representations is the affine space

$$\operatorname{Rep}_{\mathbf{d}}(Q) = \prod_{a \in Q_1} \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}^{\mathbf{d}_{t(a)}}, \mathbb{C}^{\mathbf{d}_{s(a)}})$$

Definition 4.5.5. Let $n \geq 0$ be an integer. A *path of length n* in a quiver Q is a sequence of arrows $a_n \cdots a_2 a_1$ such that $s(a_{i+1}) = t(a_i)$ for all i. The notation is by juxtaposition from right to left. The *path algebra* $\mathbb{C}Q$ of a quiver Q is defined as follows. As a $\mathbb{C}$-vector space, it is spanned by all paths of length $n \geq 0$, including a single trivial path e_i of length 0 for each vertex $i \in Q_0$. The product is given by concatenation (juxtaposition) of paths, and is defined to be 0 if two paths cannot be concatenated.

EXERCISE 4.5.6. Prove that $\mathbb{C}Q$ is an associative algebra.

EXERCISE 4.5.7. Prove that representations of a quiver Q form a category (i.e. define a sensible notion of morphisms of representations). Show that this category is equivalent to the category of left $\mathbb{C}Q$-modules, in particular it is abelian.

EXERCISE 4.5.8. Show that the path algebra of the quiver L_d with one vertex and d loops is isomorphic to $\mathbb{C}\langle x_1, x_2, \ldots, x_d\rangle$.

Let Q be a quiver. Let

$$\mathbb{H}_+ = \left\{ r \cdot \exp(i\pi\phi) \in \mathbb{C} \,\middle|\, r \in \mathbb{R}_{>0},\, \phi \in (0,1] \right\}$$

be the upper half plane with the positive real axis removed. A *central charge* is a group homomorphism $Z\colon \mathbb{Z}^{Q_0} \to \mathbb{C}$ mapping $\mathbb{N}^{Q_0} \setminus 0$ inside $\mathbb{H}_+$. For every $\alpha \in \mathbb{Z}^{Q_0}$ we let $\phi(\alpha)$ be the unique real number such that $Z(\alpha) = r \cdot \exp(i\pi\phi(\alpha))$. It is called the *phase* of α (with respect to Z). Every vector $\theta \in \mathbb{Q}^{Q_0}$ induces a central charge Z_θ via

$$Z_\theta(\alpha) = -\theta \cdot \alpha + i|\alpha|,$$

where $|\alpha| = \sum_{i \in Q_0} \alpha_i$. We let ϕ_θ denote the associated phase function, and we set

$$\phi_\theta(\rho) = \phi_\theta(\underline{\dim}\,\rho),$$

for every finite dimensional representation ρ of Q.

Fix $\theta \in \mathbb{Q}^{Q_0}$. We sometimes call θ a *stability condition.* For any $\alpha \in \mathbb{N}^{Q_0} \setminus 0$ one can define its *slope* (with respect to θ) as the ratio

$$\mu_\theta(\alpha) = \frac{\theta \cdot \alpha}{|\alpha|} \in \mathbb{Q}.$$

Let us set $\mu_\theta(\rho) = \mu_\theta(\underline{\dim}\,\rho)$, for a representation ρ of Q. A representation ρ is said to be θ-*semistable* if for every proper nontrivial subrepresentation $0 \neq \rho' \subset \rho$ one has

$$\mu_\theta(\rho') \leq \mu_\theta(\rho).$$

Strict inequality in the latter formula defines θ-*stability*, and θ is called $\mathbf{d}$-*generic* if every θ-semistable representation of dimension $\mathbf{d}$ is θ-stable.

EXERCISE 4.5.9. Let ρ be a finite dimensional representation of a quiver Q, and let $\rho' \subset \rho$ be a subrepresentation. Show that $\phi_\theta(\rho') < \phi_\theta(\rho)$ if and only if $\mu_\theta(\rho') < \mu_\theta(\rho)$, so that stability can be checked using slopes instead of phases.

Let Q be a quiver, and let $0 \in Q_0$ be a distinguished vertex. The *framed quiver* $\widetilde{Q}$ is obtained by adding a new vertex ∞ to the vertices of Q, along with a new arrow $\infty \to 0$. Thus a $(\mathbf{d},1)$-dimensional representation $\widetilde{\rho}$ of $\widetilde{Q}$ can be seen as a pair (ρ, v), where ρ is a $\mathbf{d}$-dimensional representation of Q and $v\colon V_\infty \to V_0$ is a linear map from the 1-dimensional vector space V_∞. In other words, v is a vector in V_0.

Definition 4.5.10 (Framed stability)**.** Fix $\theta \in \mathbb{Q}^{Q_0}$. A representation $\widetilde{\rho} = (\rho, v)$ of $\widetilde{Q}$ is said to be θ-(semi)stable if it is (θ, θ_∞)-(semi)stable, where $\theta_\infty = -\theta_\infty \cdot \underline{\dim}\,\rho$.

We now consider the framed quiver $\widetilde{L}_d$, i.e. the d-loop quiver L_d equipped with one additional vertex ∞ and one additional arrow $\infty \to 0$ — see Figure 7.

For the dimension vector $\mathbf{d} = (n,1)$, we have

$$\operatorname{Rep}_{\mathbf{d}}(\widetilde{L}_d) = W_n,$$

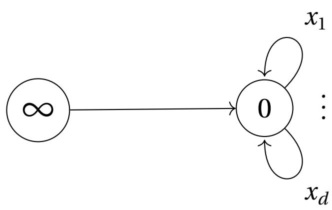


FIGURE 7. The framed d-loop quiver.

where W_n was defined in (4.2.1).

EXERCISE 4.5.11. Find a ($\mathbf{d}$-generic) framed stability condition θ on L_d (in the sense of Definition 4.5.10) such that $U_n \subset W_n$ agrees with the set of θ-semistable framed representations of L_d. (**Hint**: consider $\theta = (\theta_1, \theta_2)$ with $\theta_1 \geq \theta_2$).

Thanks to the previous exercise, the noncommutative Hilbert scheme $\mathrm{NCHilb}_d^n = U_n / \mathrm{GL}_n$ can be viewed as a fine moduli space of quiver representations.

If $I \subset \mathbb{C}Q$ is a two-sided ideal, one can consider the full subcategory

$$\mathrm{Rep}(Q, I) \subset \mathrm{Rep}(Q)$$

of representations $(M_a)_{a \in Q_1}$ such that $M_{a_k} \cdots M_{a_2} M_{a_1} = 0$ for every element $a_k \cdots a_2 a_1 \in I$. The category of representations of the quotient algebra $\mathbb{C}Q/I$ is equivalent to $\mathrm{Rep}(Q, I)$. For instance, given the ideal $I \subset \mathbb{C}L_d = \mathbb{C}\langle x_1, \ldots, x_d \rangle$ spanned by the commutators

$$[x_i, x_j], \quad 1 \leq i < j \leq d,$$

one has $\mathrm{Rep}(L_d, I) = \mathrm{Rep}\, \mathbb{C}[x_1, \ldots, x_d]$.

A special class of ideals arises from *superpotentials*, i.e. formal linear combinations of cyclic paths

$$\mathsf{W} = \sum_{c \text{ cycle in } Q} a_c c \in \mathbb{C}Q$$

up to cyclic permutation. Any superpotential W gives rise to an ideal

$$I_{\mathsf{W}} = \left\langle \frac{\partial \mathsf{W}}{\partial a} \,\middle|\, a \in Q_1 \right\rangle \subset \mathbb{C}Q$$

and to a regular function

$$\mathrm{Tr}\, \mathsf{W} \colon \mathrm{Rep}_{\mathbf{d}}(Q) \to \mathbb{A}^1$$

defined by sending

$$\rho \mapsto \sum_{c \text{ cycle in } Q} a_c \,\mathrm{Tr}(\rho(c)).$$

The quotient

$$J(Q, \mathsf{W}) = \mathbb{C}Q / I_{\mathsf{W}}$$

is called the *Jacobi algebra* of the quiver with potential (Q, W). By our discussion, we have

$$\mathrm{Rep}(Q, I_{\mathsf{W}}) \cong \mathrm{Rep}\, J(Q, \mathsf{W}).$$

The case of $\mathbb{A}^3$ described in the proof of Theorem 4.5.2 then reveals that $\mathrm{Hilb}^n \mathbb{A}^3$ is the moduli space of *stable* framed representations of the Jacobi algebra

$$\mathbb{C}\langle x_1, x_2, x_3 \rangle / I_{\mathsf{W}} = \mathbb{C}[x_1, x_2, x_3],$$

for the potential $\mathsf{W} = x_1[x_2, x_3]$.

APPENDIX A. INTERSECTION THEORY

A.1. **Chow groups, pushforward and pullack.** In this subsection, all schemes are of finite type over an algebraically closed field k. Varieties are integral schemes. A subvariety V of a scheme X is a closed subscheme which is a variety.

Let X be an n-dimensional scheme. A d-dimensional cycle on X (or simply a d-cycle) is a finite formal sum

$$\sum_i m_i \cdot V_i$$

where $V_i \subset X$ are (closed irreducible) subvarieties of dimension d and $m_i \in \mathbb{Z}$. The free abelian group generated by d-cycles is denoted $Z_d X$, and we set

$$Z_* X = \bigoplus_{d=0}^{n} Z_d X.$$

The fundamental class of X is the (possibly inhomogeneous) cycle

$$[X] \in Z_* X$$

determined by the irreducible components $V_i \subset X$ and their geometric multiplicities $m_i = \operatorname{length}_{\mathscr{O}_{X,\xi_i}} \mathscr{O}_{X,\xi_i}$, where ξ_i is the generic point of V_i. If X is pure, then $Z_n X = A_n X$ is freely generated by the classes of the irreducible components of X.

If $r \in k(X)$ is a nonzero rational function and $V \subset X$ is a codimension one subvariety, pick a and b in $A = \mathscr{O}_{X,\xi_V}$ such that $r = a/b$ and set

$$\operatorname{ord}_V(r) = \operatorname{length}_A(A/a) - \operatorname{length}_A(A/b).$$

This is the *order of vanishing* of r along V. Note that $\operatorname{ord}_V(r \cdot r') = \operatorname{ord}_V(r) + \operatorname{ord}_V(r')$ for $r, r' \in k(X)$. A rational function r as above defines a divisor

$$\operatorname{div}(r) = \sum_{\substack{V \subset X \\ \operatorname{codim}_X V = 1}} \operatorname{ord}_V(r) \cdot V \in Z_{n-1} X.$$

A d-cycle α is said to be *rationally equivalent to* 0 if it belongs to the subgroup $R_d X \subset Z_d X$ generated by cycles of the form $\operatorname{div}(r)$, where r is a nonzero rational function on a $(d+1)$-dimensional subvariety of X. Form the direct sum $R_* X = \oplus_{d=0}^{n} R_d X$. The quotient

$$A_* X = Z_* X / R_* X$$

is the *Chow group* of X.

Let $f : X \to Y$ be a proper morphism of schemes. Then there is a *pushforward* map

$$f_* : A_* X \to A_* Y$$

defined on generators by sending a d-cycle class $[V] \in A_d X$ to 0 if $\dim f(V) < \dim V$, and to the cycle

$$e_V \cdot [f(V)] \in A_d Y$$

if $\dim V = \dim f(V)$. Here e_V is the degree of the field extension $k(f(V)) \subset k(V)$.

Let $f : X \to Y$ be a flat morphism of schemes. Then there is a *pullback* map

$$f^* : A_* Y \to A_* X$$

defined on generators by sending a d-cycle class $[W] \in A_* Y$ to the cycle class

$$[f^{-1}(W)] \in A_{d+s} X$$

where s is the relative dimension of f.

Theorem A.1.1. *Let $p\colon E \to X$ be a vector bundle. Then p^* is an isomorphism.*

Proof. Combine [13, Prop. 1.9] and [13, Thm. 3.3] with one another. $\square$

Convention 1. Let $p\colon E \to X$ be a vector bundle. We denote by $0^*\colon A_*E \xrightarrow{\sim} A_*X$ the inverse of p^*.

Definition A.1.2. Let $p\colon X \to \operatorname{Spec} k$ be the structure morphism of a proper k-scheme X. The *degree map* is by definition the proper pushforward p_*. It is denoted

$$A_*X \xrightarrow{\int_X} \mathbb{Z}$$

and is zero on cycle classes of positive dimension.

A.2. **Operations on bundles.** Let E be a vector bundle of rank r on a scheme X, and let $p\colon \mathbb{P}(E) \to X$ be the projective bundle of lines in the fibres of $E \to X$. Let $\mathscr{O}_E(1)$ be the dual of the tautological line bundle $\mathscr{O}_E(-1) \subset p^*E$ on $\mathbb{P}(E)$. The *Segre classes* $s_i(E)$ can be seen as operators $A_kX \to A_{k-i}X$ defined by

$$s_i(E) \cap \alpha = p_*(\xi^{r-1+i} \cap p^*\alpha),$$

where $\xi = c_1(\mathscr{O}_E(1))$. Such operation is the identity for $i = 0$ and identically vanishes for $i < 0$. If L is a line bundle, then

$$s_p(E \otimes L) = \sum_{i=0}^{p} (-1)^{p-i} \binom{r-1+p}{r-1+i} s_i(E) c_1(L)^{p-i}.$$

Definition A.2.1. We define the following objects:

- The *Segre series* of E is the formal power series
$$s_t(E) = 1 + \sum_{i>0} s_i(E) t^i.$$
- The *Chern polynomial* of E is
$$c_t(E) = s_t(E)^{-1} = 1 + \sum_{i>0} c_i(E) t^i.$$
It is indeed a polynomial, for $c_i(E) = 0$ for all $i > \operatorname{rk} E$.
- The *total Chern class* of E is the finite sum
$$c(E) = 1 + c_1(E) + \cdots + c_r(E), \quad r = \operatorname{rk} E.$$

Example A.2.2. For a line bundle L, we have $c_t(L) = 1 + c_1(L)t$.

Let $E \to Y$ be a vector bundle. The projection formula

$$f_*(c_i(f^*E) \cap \alpha) = c_i(E) \cap f_*\alpha$$

holds for all proper morphisms $f\colon X \to Y$ and cycles $\alpha \in A_*X$. If f is a flat morphism, on the other hand, one has

$$c_i(f^*E) \cap f^*\beta = f^*(c_i(E) \cap \beta)$$

for all cycles $\beta \in A_*Y$. Given a short exact sequence

$$0 \to E \to F \to G \to 0 \tag{A.2.1}$$

of vector bundles on X, one has Whitney's formula

$$c_t(F) = c_t(E) \cdot c_t(G).$$

The *splitting construction* says that if E is a vector bundle of rank r on a scheme X, there exists a flat morphism $f: Y \to X$ such that the flat pullback $f^*: A_*X \to A_*Y$ is injective and the pullback f^*E has a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{r-1} \subset E_r = f^*E$$

with line bundle quotients

$$L_i = E_i/E_{i-1}, \quad i = 1, \dots, r.$$

Set $\alpha_i = c_1(L_i)$. Then each short exact sequence

$$0 \to E_{i-1} \to E_i \to L_i \to 0$$

gives an identity

$$(1+\alpha_i t) \cdot c_t(E_{i-1}) = c_t(E_i).$$

So we have

$$\begin{aligned} f^*c_t(E) &= c_t(f^*E) \\ &= (1+\alpha_r t) \cdot c_t(E_{r-1}) \\ &= (1+\alpha_r t) \cdot (1+\alpha_{r-1} t) \cdot c_t(E_{r-2}) \\ &= (1+\alpha_1 t) \cdots (1+\alpha_r t) \end{aligned}$$

By injectivity of f^*, we may view the latter product as a formal expression for $c_t(E)$. In other words, we can always pretend that E is filtered by $0 = E_0 \subset E_1 \subset \cdots \subset E_{r-1} \subset E_r = E$ with line bundle quotients L_i, and

$$c_t(E) = \prod_{i=1}^{r} (1+\alpha_i t), \tag{A.2.2}$$

where $\alpha_i = c_1(L_i)$. In fact, one should regard (A.2.2) as a formal expression defining $\alpha_1, \dots, \alpha_r$. These are called the *Chern roots* of E, and they satisfy

$$c_i(E) = \sigma_i(\alpha_1, \dots, \alpha_r), \quad i = 0, \dots, r$$

where σ_i denotes the i-th symmetric function.

Example A.2.3 (Dual bundles)**.** One has the formula

$$c_i(E^\vee) = (-1)^i c_i(E).$$

The Chern roots of the dual bundle $E^\vee$ are $-\alpha_1, \dots, -\alpha_r$.

Example A.2.4 (Tensor products)**.** If F is a vector bundle of rank s, the Chern roots of $E \otimes F$ are $\alpha_i + \beta_j$, where $i = 1, \dots, r$ and $j = 1, \dots, s$. So $c_k(E \otimes F)$ is the k elementary symmetric function of $\alpha_1 + \beta_1, \dots, \alpha_r + \beta_s$. For instance, if $s = 1$,

$$c_t(E \otimes L) = \sum_{i=0}^{r} c_t(L)^{r-i} c_i(E) t^i.$$

Term by term, this can be reformulated as

$$c_k(E\otimes L)=\sum_{i=0}^{k}\binom{r-i}{k-i}c_i(E)c_1(L)^{k-i}.$$

Example A.2.5 (Exterior product)**.** For the exterior power $\wedge^p E$ we have

$$c_t(\wedge^p E)=\prod_{i_1<\cdots<i_p}(1+(\alpha_{i_1}+\cdots+\alpha_{i_p})t),$$

so that for instance we have $c_1(\det E)=c_i(E)$.

Definition A.2.6. The *Chern character* of a vector bundle E with Chern roots $\alpha_1,\ldots,\alpha_r$ is the expression

$$\mathrm{ch}(E)=\sum_{i=1}^{r}\exp(\alpha_i).$$

One has

$$\mathrm{ch}(E)=r+c_1+\frac{1}{2}(c_1^2-2c_2)+\frac{1}{6}(c_1^3-3c_1c_2+3c_3)+\cdots$$

and moreover ch(–) satisfies the crucial relation

$$\mathrm{ch}(F)=\mathrm{ch}(E)+\mathrm{ch}(G)$$

for any short exact sequence as in (A.2.1). One also has

$$\mathrm{ch}(E\otimes E')=\mathrm{ch}(E)\cdot\mathrm{ch}(E').$$

Definition A.2.7. The *Todd class* of a line bundle L with $\eta=c_1(L)$ is the formal expression

$$\mathrm{Td}(L)=\frac{\eta}{1-e^{-\eta}}=1+\frac{1}{2}\eta+\sum_{i\geq 1}\frac{B_{2i}}{(2i)!}\eta^{2i}$$

where B_k are the Bernoulli numbers.

One may also set

$$\mathrm{Td}^{\vee}(L)=\mathrm{Td}(L^{\vee})=\frac{-\eta}{1-e^{\eta}}=\frac{\eta}{e^{\eta}-1}=1-\frac{1}{2}\eta+\sum_{i\geq 1}\frac{B_{2i}}{(2i)!}\eta^{2i}.$$

Note that we have the relation

$$\mathrm{Td}(L)=e^{\eta}\cdot\mathrm{Td}(L^{\vee}).$$

For a vector bundle E with Chern roots $\alpha_1,\ldots,\alpha_r$, we set

$$\mathrm{Td}(E)=\prod_{i=1}^{r}\frac{\alpha_i}{1-e^{-\alpha_i}}$$

and it is easy to compute

$$\begin{aligned}\mathrm{Td}(E)=1+\frac{1}{2}c_1+\frac{1}{12}(c_1^2+c_2)+\frac{1}{24}c_1c_2&\\ &+\frac{1}{720}(-c_1^4+4c_1^2c_2+3c_2^2+c_1c_3-c_4)+\cdots\end{aligned}$$

where $c_i=c_i(E)$. The multiplicativity property

$$\mathrm{Td}(F)=\mathrm{Td}(E)\cdot\mathrm{Td}(G)$$

holds for every sequence of vector bundles as in (A.2.1).

Definition A.2.8. For two vector bundles E, F on a scheme X, define

$$c(F-E) = \frac{c(F)}{c(E)} = 1 + c_1(F-E) + c_2(F-E) + \cdots \tag{A.2.3}$$

Example A.2.9. If $F=0$, we have

$$c(-E) = s(E) = 1 + \sum_{i>0} s_i(E)t^i.$$

Example A.2.10. The first few terms of the expansion (A.2.3) are

$$\begin{aligned} c_0(F-E) &= 1 \\ c_1(F-E) &= c_1(F) - c_1(E) \\ c_2(F-E) &= c_2(F) - c_1(F)c_1(E) + c_1(E)^2 - c_2(E). \end{aligned}$$

Remark A.2.11. If $\mathsf{F} = \sum_j [F_j]$ and $\mathsf{E} = \sum_i [E_i]$ are elements of the Grothendieck group $K^\circ X$ of vector bundles on a scheme X, then the Chern class of $\mathsf{F}-\mathsf{E}$ is defined as

$$c(\mathsf{F}-\mathsf{E}) = \frac{\prod_j c(F_j)}{\prod_i c(E_i)}.$$

Clearly, one has $c([F]-[E]) = c(F-E)$ for two vector bundles E, F. Similarly, the power series

$$c_t(\mathsf{F}-\mathsf{E}) = \frac{\prod_j c_t(F_j)}{\prod_i c_t(E_i)}$$

takes the role of the Chern polynomial in K-theory.

A.3. **Refined Gysin homomorphisms.**

Definition A.3.1. We say that a morphism of schemes $f\colon X \to Y$ admits a *factorisation* if there is a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, hook, "i"'] & & Y \\ & M \arrow[ur, "\pi"'] & \end{tikzcd} \tag{A.3.1}$$

where i is a closed embedding and π is smooth.

Example A.3.2. If X, Y are quasiprojective, any morphism $X \to Y$ admits a factorisation. A factorisation always exists locally on Y.

If $f\colon X \to Y$ admits two factorisations

$$X \xrightarrow{i_1} M_1 \xrightarrow{\pi_1} Y, \qquad X \xrightarrow{i_2} M_2 \xrightarrow{\pi_2} Y,$$

there is a third one,

$$X \xrightarrow{i} M \xrightarrow{\pi} Y,$$

dominating both:

$$\begin{tikzcd} & M_1 \arrow[dr, "\pi_1"] & \\ X \arrow[ur, "i_1"] \arrow[r, "i"] \arrow[dr, "i_2"'] & M \arrow[u, dashed, "\exists"'] \arrow[d, dashed, "\exists"] \arrow[r, "\pi"] & Y \\ & M_2 \arrow[ur, "\pi_2"'] & \end{tikzcd}$$

It is enough to take $M = M_1 \times_Y M_2$ and use that smooth morphisms and closed immersions are stable under base change and composition.

Definition A.3.3. A morphism $f\colon X \to Y$ is a *local complete intersection* (lci, for short) if it has a factorisation $X \to M \to Y$ where $X \to M$ is a regular closed embedding. If this is only true locally on Y, we say that f is *locally lci*.

Remark A.3.4. If f is lci, *all* of its factorisations have the closed embedding regular.

Remark A.3.5. An lci morphism $f\colon X \to Y$ has a well-defined *relative dimension*: given a factorisation (A.3.1), it is the integer

$$r = \operatorname{rk} T_{M/Y} - \operatorname{codim}(X, M) \in \mathbb{Z}.$$

For instance, a regular closed embedding of codimension d is an lci morphism of relative dimension $-d$.

Let $f\colon X \to Y$ be an lci morphism of relative dimension r, factoring as a regular immersion $i\colon X \to M$ followed by a smooth morphism $\pi\colon M \to Y$ of relative dimension s. Given any morphism $g\colon \tilde{Y} \to Y$, consider the double fiber square

$$\begin{tikzcd} \tilde{X} \arrow[r, hook] \arrow[d, "\tilde{g}"'] \arrow[dr, phantom, "\square"] & \tilde{M} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \tilde{Y} \arrow[d, "g"] \\ X \arrow[r, hook, "i"] & M \arrow[r, "\pi"] & Y. \end{tikzcd}$$

For any $k \geq 0$, we will construct a group homomorphism

$$f^!\colon Z_k \tilde{Y} \longrightarrow Z_{k+s} \tilde{M} \xrightarrow{\sigma} Z_{k+s} C_{\tilde{X}/\tilde{M}} \xrightarrow{\phi} A_{k+r} \tilde{X}.$$

The first arrow is just flat pullback: it is not a problem to pull back cycles from $\tilde{Y}$ to $\tilde{M}$. The arrow σ, called *specialisation to the normal cone* in [13], is given as follows. For any $(k+s)$-dimensional subvariety $V \subset \tilde{M}$, consider the intersection

$$\begin{tikzcd} W \arrow[r] \arrow[d, "j"'] \arrow[dr, phantom, "\square"] & V \arrow[d] \\ \tilde{X} \arrow[r] & \tilde{M} \end{tikzcd}$$

and the normal cone $C_{W/V}$, which is purely of dimension $k+s$ (cf. Remark **??**). It lives naturally as a closed subcone

$$C_{W/V} \subset j^* C_{\tilde{X}/\tilde{M}} \subset C_{\tilde{X}/\tilde{M}}. \tag{A.3.2}$$

If ℓ denotes the closed immersion (A.3.2), we define

$$\sigma[V] = \ell_*[C_{W/V}] = [C_{V \cap \tilde{X}/V}],$$

the proper pushforward of the fundamental class $[C_{W/V}]$ of the normal cone.

As for the map ϕ, we observe from the Cartesian diagram

$$\begin{tikzcd} \tilde{X} \arrow[r] \arrow[d, "\tilde{g}"'] \arrow[dr, phantom, "\square"] & \tilde{M} \arrow[d] \\ X \arrow[r, "i"] & M \end{tikzcd}$$

that we have a closed subcone

$$C_{\tilde{X}/\tilde{M}} \subset \tilde{g}^* C_{X/M}. \tag{A.3.3}$$

Since i is *regular*, $C_{X/M} = N_{X/M}$ is a vector bundle, so that

$$E = \tilde{g}^* C_{X/M}$$

is a vector bundle too. Its rank is easily computed as

$$\operatorname{rk} E = \operatorname{rk} N_{X/M} = \operatorname{codim}(X, M) = s - r.$$

Let B be a subvariety of $C_{\tilde{X}/\tilde{M}} \subset E$ of dimension $k+s$. Then define

$$\phi[B] = 0^*[B],$$

where

$$0^* \colon A_{k+s} E \to A_{k+s-(s-r)} \tilde{X} = A_{k+r} \tilde{X}$$

is the inverse of the flat pullback on $E \to \tilde{X}$.

The morphism we have just constructed descends to rational equivalence, to give a morphism

$$f^! \colon A_k \tilde{Y} \to A_{k+r} \tilde{X} \tag{A.3.4}$$

that Fulton calls *refined Gysin homomorphism*.

We have the following facts:

(a) The homomorphism $f^!$ agress with Gysin pullback (flat pullback) when $\tilde{Y} = Y \to Y$ is the identity and f is flat. This case is already interesting in its own: it is the intersection theory of X!
(b) The homomorphism $f^!$ is called *refined Gysin pullback* when $\tilde{Y} \to Y$ is a closed embedding.
(c) The homomorphism $f^!$ does not depend on the choice of the factorisation.
(d) Working in $D^b(X)$, one can perform the above construction even when *no factorisation* is actually available. (This is the generality one works in to construct virtual classes.)
(e) Let $f \colon X \to Y$ and $g \colon Y \to Z$ be two morphisms. If f is a regular embedding and both g and $g \circ f$ are flat, then

$$f^! \circ g^* = (g \circ f)^*.$$

Moreover, if f and $g \circ f$ are regular embeddings, and g is *smooth*, then

$$f^! \circ g^* = (g \circ f)^!. \tag{A.3.5}$$

This is basically [13, Prop. 6.5]. However, (A.3.5) is false in general if g is just flat. See the functoriality property (D) in the next subsection for the general (lci) case.

A.3.1. *An example: Localized Top Chern Class.* Let X be a variety and let

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "i"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "s"] \\ X \arrow[r, "0"] & E \end{tikzcd} \tag{A.3.6}$$

be the fiber diagram defining the zero locus Z of a section $s \in H^0(X,E)$. Then $0\colon X \to E$ is regular of codimension $e = \operatorname{rk} E$, and $N_{X/E} = E$. We get refined Gysin homomorphisms

$$0^!\colon A_k X \to A_{k-e} Z.$$

Suppose X is purely n-dimensional. Then we can define the *localized top Chern class* of E as

$$\mathbb{Z}(s) = 0^![X] \in A_{n-e} Z,$$

where $[X]$ is the fundamental class of X and $0^!\colon A_n X \to A_{n-e} Z$. In the language of the previous section, the class $0^![X]$ is nothing but the intersection

$$0^*_{E|_Z}[C_{Z/X}]$$

of the cone

$$C_{Z/X} \subset N_{Z/X} \subset E|_Z$$

with the zero section of the rank e bundle $E|_Z \to Z$. The closed embedding

$$N_{Z/X} \subset E|_Z \tag{A.3.7}$$

comes directly from the diagram (A.3.6): the dual section $s^\vee\colon E^\vee \to \mathscr{O}_X$ hits the ideal sheaf $\mathscr{I} \subset \mathscr{O}_X$ of $Z \subset X$, and (A.3.7) is the result of applying Spec Sym to the natural restriction map

$$s^\vee|_Z\colon E^\vee|_Z \twoheadrightarrow \mathscr{I}/\mathscr{I}^2.$$

Remark A.3.6 (First example of perfect obstruction theory)**.** Let X, E, Z be as above. The class $0^![X] = 0^*_{E|_Z}[C_{Z/X}] \in A_{n-e} Z$ is our first example of a *virtual fundamental class*. For the time being, we can pretend the normal cone $C_{Z/X}$ is completely *intrinsic* to Z. This is not entirely false. Now look at (A.3.7): embedding $C_{Z/X}$ in a vector bundle $E|_Z$ is the choice of what is called a *perfect obstruction theory* [5].

Remark A.3.7 (Relation with the deformation to the normal cone)**.** The deformation to the normal cone [13, Chapter 5] enters the picture as follows: we have embeddings $\lambda s\colon Z \to E$ for all $\lambda \in \mathbb{A}^1$. Letting $\lambda \to \infty$ turns these embeddings into exactly $C_{Z/X} \subset E|_Z$. More explicitly, consider the graph of $\lambda s\colon X \to E$ as a line in $E \oplus \mathbf{1}$, to get an embedding

$$X \times \mathbb{A}^1 \hookrightarrow P(E \oplus \mathbf{1}) \times \mathbb{P}^1, \quad (x,\lambda) \mapsto ((x, \lambda s(x)), (\lambda : 1)).$$

Then the deformation space of the deformation to the normal cone construction turns out to be the closure

$$M = \overline{X \times \mathbb{A}^1} \subset P(E \oplus \mathbf{1}) \times \mathbb{P}^1,$$

and the embeddings $\lambda s\colon X \subset E$ deform to $X \subset C_{X/E} \subset N_{X/E} = E$. Restricting to Z gives $C_{Z/X} \subset E|_Z$.

Remark A.3.8. The class $\mathbb{Z}(s)$ is also called the *refined Euler class* of E, because

$$\begin{aligned} i_*\mathbb{Z}(s) &= i_* 0^![X] \\ &= 0^* s_*[X] \\ &= s^* s_*[X] \\ &= c_e(E) \cap [X]. \end{aligned}$$

We have used that if $\pi\colon E \to X$ is any vector bundle then any $s \in H^0(X,E)$ is a regular embedding and $s^! = s^*\colon A_k E \to A_{k-e} X$ is the inverse of flat pullback π^*. In particular s^* does not depend on s, so $s^* = 0^*$. The last equality is a special case of the self-intersection formula, using also that $E = N_{X/E}$. Moreover, it can be interesting to notice that $\mathbb{Z}(s) = [Z]$ when s is a regular section.

Example A.3.9. This example is relevant in Donaldson–Thomas theory. Let $E = \Omega_U$ be the cotangent bundle on a smooth scheme U. Let $f\colon U \to \mathbb{C}$ be a holomorphic function, giving a section $\mathrm{d}f$ of Ω_U. The above construction can be summarized by

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "i"'] \arrow[dr, phantom, "\square"] & U \arrow[d, "\mathrm{d}f"] \\ U \arrow[r, "0"] & \Omega_U \end{tikzcd} \qquad \rightsquigarrow \qquad [Z]^{\mathrm{vir}} = 0^![U] = 0^*_{\Omega_U|_Z}[C_{Z/U}] \in A_0 Z.$$

Notice that in this case the obstruction *sheaf* is completely intrinsic to Z. It is defined as the cokernel

$$N_{Z/U} \to i^* N_{U/\Omega_U} \to \mathrm{Ob} \to 0.$$

But this is of course the sequence

$$\mathscr{I}/\mathscr{I}^2 \to \Omega^1_U|_Z \to \Omega_Z \to 0.$$

In other words, $\mathrm{Ob} = \Omega^1_Z$.

A.3.2. *More properties of $f^!$ and relation with bivariant classes.* We now quickly discuss the main properties of refined Gysin homomorpisms. We refer to [13, Ch. 6] for complete proofs.

First of all, notice that we have a $f^!$ for all $\tilde{Y} \to Y$ and for all $k \geq 0$. This trivial observation, together with the compatibilities we are about to describe, states precisely that any lci morphism $f\colon X \to Y$ of relative dimension r defines a *bivariant class*

$$[f^!] \in A^{-r}(X \xrightarrow{f} Y),$$

as described in [13, Ch. 17].

Let $f\colon X \to Y$ be an lci morphism of codimension r. We first state the properties of $f^!$ informally, and then we explain what they mean.

(A) Refined Gysin homomorphisms commute with proper pushforward and flat pullback.
(B) Refined Gysin homomorphisms are compatible with each other.
(C) Refined Gysin homomorphisms commute with each other.
(D) Refined Gysin homomorphisms are functorial.

Here is what the above statements mean. Fix once and for all an integer $k \geq 0$.

(A) For any double fiber square situation

$$
\begin{tikzcd}
X' \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\square"] & Y' \arrow[d, "h"] \\
\tilde{X} \arrow[r, "\tilde{f}"] \arrow[d] \arrow[dr, phantom, "\square"] & \tilde{Y} \arrow[d, "g"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}
\tag{A.3.8}
$$

one has the following:

(P) If h is proper, then for all $\alpha \in A_k Y'$ one has

$$f^!(h_*\alpha) = q_*(f^!\alpha) \in A_{k+r}\tilde{X}.$$

(F) If h is flat of relative dimension n, then for all $\alpha \in A_k \tilde{Y}$ one has

$$f^!(h^*\alpha) = q^*(f^!\alpha) \in A_{k+r+n}X'.$$

(B) In situation (A.3.8), if $\tilde{f}\colon \tilde{X} \to \tilde{Y}$ is also lci of relative dimension r, then for all $\alpha \in A_k Y'$ one has

$$f^!\alpha = \tilde{f}^!\alpha \in A_{k+r}X'.$$

(C) Let $j\colon S \to T$ be a regular embedding of codimension e. Given morphisms $\tilde{Y} \to Y$ and $\tilde{Y} \to T$, form the fiber square

$$
\begin{tikzcd}
X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S \arrow[d, "j"] \\
\tilde{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \tilde{Y} \arrow[r] \arrow[d] & T \\
X \arrow[r, "f"] & Y &
\end{tikzcd}
\tag{A.3.9}
$$

and fix $\alpha \in A_k \tilde{Y}$. Then one has

$$j^!(f^!\alpha) = f^!(j^!\alpha) \in A_{k+r-e}X'.$$

(D) Let $f\colon X \to Y$ and $g\colon Y \to Z$ be lci morphisms of relative dimensions r and s respectively. Then, for all morphisms $\tilde{Z} \to Z$, one has the identity

$$(g \circ f)^!(\alpha) = f^!(g^!\alpha) \in A_{k+r+s}(X \times_Z \tilde{Z}).$$

A.3.3. *Bivariant classes.* Let $f\colon X \to Y$ be any morphism. Suppose f has the property that when we let morphisms $g\colon \tilde{Y} \to Y$ and integers $k \geq 0$ vary arbitrarily, we are able to construct homomorphisms

$$c_g^{(k)}\colon A_k \tilde{Y} \to A_{k-p}\tilde{X}, \quad \tilde{X} = X \times_Y \tilde{Y},$$

for some $p \in \mathbb{Z}$. Then the collection c of these homomorphisms is said to define a *bivariant class*

$$c \in A^p(X \xrightarrow{f} Y)$$

if compatibilities like the ones described in (A) and (C) in the previous section are satisfied. Here are precise requirements.

(A)′ In any double fiber square situation like (A.3.8), one has the following:

(P) If h is proper, then for all $\alpha \in A_k Y'$ one has the identity

$$c_g^{(k)}(h_*\alpha) = q_*(c_{gh}^{(k)}\alpha) \in A_{k-p}\tilde{X}.$$

(F) If h is flat of relative dimension n, then for all $\alpha \in A_k \tilde{Y}$ one has the identity

$$c_{gh}^{k+n}(h^*\alpha) = q^*(c_g^{(k)}\alpha) \in A_{k+n-p}X'.$$

(C)′ In situation (A.3.9), for all $\alpha \in A_k \tilde{Y}$ one has

$$j^!(c_g^{(k)}\alpha) = c_{gi}^{(k-e)}(j^!\alpha) \in A_{k-p-e}X'.$$

Conclusion. Any lci morphism $f\colon X \to Y$ of codimension r defines a bivariant class

$$[f^!] \in A^{-r}(X \xrightarrow{f} Y).$$

For instance, if $f = i$ is a regular immersion of codimension d, this class is

$$[i^!] \in A^d(X \xrightarrow{f} Y).$$

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