INTRODUCTION TO ENUMERATIVE GEOMETRY

— CLASSICAL AND VIRTUAL TECHNIQUES —

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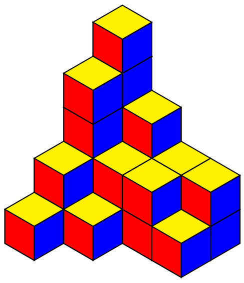
ABSTRACT. These are lecture notes for a PhD course held at SISSA in Fall 2019. Notes under construction.



CONTENTS

Modern enumerative geometry is not so much about numbers as it is about deeper properties of the moduli spaces that parametrize the geometric objects being enumerated.

Lectures on K-theoretic computations in enumerative geometry
A. Okounkov

0. Why is Enumerative Geometry hard?

0.1. **Asking the right question.** Enumerative Geometry is a branch of Algebraic Geometry studying questions asking to count how many objects satisfy a given list of geometric conditions. The very nature of these questions, and the presence of this "list", make the subject tightly linked to Intersection Theory, which explains why we included Appendix B at the end of these lecture notes.

Examples of classical questions in the subject are the following:

(1) How many lines $\ell \subset \mathbb{P}^3$ intersect four general lines $\ell_1, \ell_2, \ell_3, \ell_4 \subset \mathbb{P}^3$? (Answer in Section 8.4)
(2) How many lines $\ell \subset \mathbb{P}^3$ lie on a general cubic surface $S \subset \mathbb{P}^3$? (Answer in Section 8.2)
(3) How many lines $\ell \subset \mathbb{P}^4$ lie on a generic quintic 3-fold $Y \subset \mathbb{P}^4$? (Answer in Section 8.3)
(4) How many Weierstrass points are there on a general genus g curve?
(5) How many smooth conics are tangent to five general plane conics?

The objects we want to count, say in the first three examples, are lines in some projective space. The geometric conditions are constraints we put on these lines, such as intersecting other lines or lying on a smooth cubic surface. We immediately see that one fundamental difficulty in the subjects is this:

> **D1**. How do we know how many constraints we should put on our objects in order to *expect* a finite answer? In other words, how do we ask the right question?

See Section 4 for a full treatment of the topic "expectations" in the case of lines on hypersurfaces. Here is a warm-up example to shape one's intuition.

Exercise 0.1.1. Let $d > 0$ be an integer. Determine the number m_d having the following property: you expect finitely many smooth complex projective curves $C \subset \mathbb{P}^2$ of degree d passing through m_d general points in $\mathbb{P}^2$. (**Hint**: Start with small d. Then conjecture a formula for m_d).

0.2. **Counting the points on a moduli space.** The main idea to guide our geometric intuition in formulating and solving an enumerative problem should be the following recipe:

- construct a moduli[1] space $\mathcal{M}$ for the objects we are interested in,
- compactify $\mathcal{M}$ if necessary,
- impose $\dim \mathcal{M}$ conditions to expect a finite number of solutions, and
- count these solutions via Intersection Theory methods (exploiting compactness of $\mathcal{M}$).

None of these steps is a trivial one, in general.

Another difficulty in the subject is the following. Say we have a precise question, such as (2) above. Then, in the above recipe, as our $\mathcal{M}$ we should take the Grassmannian of lines in $\mathbb{P}^3$, which is a compact 4-dimensional complex manifold. Imagine we

[1] The latin word *modulus* means *parameter*, and its plural is *moduli*. Thus a *moduli space* is to be thought of as a parameter space for objects of some kind.

have found a sensible algebraic variety structure on the set $\mathcal{M}_S \subset \mathcal{M}$ of lines lying on the surface S. If we have done everything right, the space $\mathcal{M}_S$ consists of finitely many points, and now the only legal operation we can perform in order to get our answer is to take the degree of the (0-dimensional) fundamental class of $\mathcal{M}_S$. So here is the second problem we face:

> **D2**. How do we know this degree is the answer to our original question? In other words, how to ensure that our algebraic solution is actually *enumerative*?

Put in more technical terms, how do we make sure that each line $\ell \subset S$ appears as a point in the moduli space $\mathcal{M}_S$ with multiplicity one? The truth is that we cannot *always* be sure that this is the case. It will be, both for problem (2) and problem (3), but not in general. However, we should get used to the idea that this is not something to be worried about: if a solution comes with multiplicity bigger than one, there usually is a good geometric reason for this, and we should not disregard it (see Figure 4 for a simple example of a degenerate intersection where this phenomenon occurs).

Remark 0.2.1. Compactness of $\mathcal{M}$ (in the above example, the Grassmannian) is used in order to make sense of taking the *degree* of cycles. Intuitively, we need compactness in order to prevent the solutions of our enumerative problem to escape to infinity, like for instance it would occur if we were to intersect two *parallel* lines in $\mathbb{A}^2$.

Compatcness really is a non-negotiable condition we have to ask of our moduli space — with an important exception, that will be treated in later sections: the case when the moduli space has a torus action. In this case, if the torus-fixed locus $\mathcal{M}^{\mathbb{T}} \subset \mathcal{M}$ is compact, a sensible enumerative solution to a counting problem can be *defined* by means of the *localisation formula*. The original formula due to Atiyah and Bott will be proved in Theorem 7.5.1. A virtual analogue due to Graber and Pandharipande [32] will be proved in Theorem 11.2.5, and the latter will be applied to the study of 0-dimensional Donaldson–Thomas invariants of local Calabi–Yau 3-folds (arising from non-compact, but toric, moduli spaces).

A more fundamental difficulty is discussed in the next subsection, by means of an elementary example.

0.3. **Transversality, and counting lines through two points.** Consider the enumerative problem of counting the number of lines in $\mathbb{P}^2$ through two given points $p, q \in \mathbb{P}^2$. Let N_{pq} be this number. Then

$$\mathsf{N}_{pq} = 1, \quad \text{as long as } p \neq q.^{2}$$

However, the *true* answer would be ∞ when $p = q$, corresponding to the cardinality of the pencil $\mathbb{P}^1$ of lines through p (see Figure 1).

Now, the case $p = q$ is a degeneration of the case $p \neq q$, and we certainly want our enumerative answer not to depend on small perturbations of the geometry of the problem. It seems at first glance that the issue cannot be fixed. After all, there is an inevitable dimensional jump between the transverse case (yielding a dimension zero answer) and the non-transverse geometry (dimension one answer). However, the

[2] For the sake of completeness, this will be proved in Section 8.1.

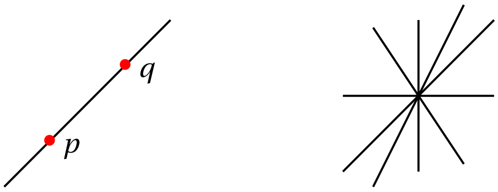


FIGURE 1. The unique line through two distinct points, and the infinitely many lines through one point in the plane.

answer '1' can be recovered in the non-transverse setting (the picture on the right) by means of the *excess intersection formula*.

The $\mathbb{P}^1$ of lines through p can be neatly seen as the exceptional divisor E in the blowup $B = \mathrm{Bl}_p\,\mathbb{P}^2$, cf. Figure 2.

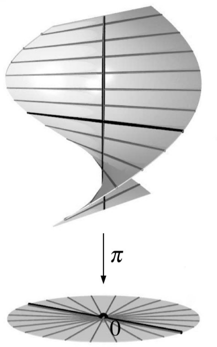


FIGURE 2. The blow-up of $\mathbb{P}^2$ at a point p. Picture stolen from Gathmann [27].

Recall that the *normal sheaf* of a closed embedding $X \hookrightarrow Y$ defined by an ideal $\mathscr{I} \subset \mathscr{O}_Y$ is the $\mathscr{O}_X$-module $N_{X/Y} = (\mathscr{I}/\mathscr{I}^2)^\vee = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{I}/\mathscr{I}^2, \mathscr{O}_X)$.

EXERCISE 0.3.1. Let $X \hookrightarrow Y$ be a closed embedding, $M \to Y$ a morphism, and let $g\colon P = X \times_Y M \to X$ be the induced map. Show that there is an inclusion $N_{P/M} \subset g^* N_{X/Y}$.

Looking at the Cartesian square

$$
\begin{tikzcd}
E \arrow[r, hook] \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & B \arrow[d, "\pi"] \\
p \arrow[r, hook] & \mathbb{P}^2
\end{tikzcd}
\tag{0.3.1}
$$

we know by Exercise 0.3.1 that there is an injection of vector bundles $N_{E/B} = \mathscr{O}_E(-1) \subset g^* N_{p/\mathbb{P}^2}$. The *excess bundle* (or *obstruction bundle*)

$$\mathrm{Ob} \to \mathbb{P}^1$$

of the fiber diagram (0.3.1) is defined as the quotient of these two bundles. But the short exact sequence

$$0 \to \mathscr{O}_E(-1) \to \mathscr{O}_E \otimes T_p\mathbb{P}^2 \to \mathrm{Ob} \to 0$$

is just the Euler sequence on $\mathbb{P}^1$ twisted by -1. Therefore

$$\mathrm{Ob} = T_{\mathbb{P}^1}(-1) = \mathscr{O}_{\mathbb{P}^1}(2-1) = \mathscr{O}_{\mathbb{P}^1}(1).$$

We have thus recovered '1' as the Euler number of the excess bundle, so that we can now write a universal formula for our counting problem: if $\mathcal{M}_{pq} = \pi^{-1}(q) \cap E$ is the "moduli space" of lines through p and q (this includes the case $p = q$), the *virtual number* of lines through p and q is

$$\int_{\mathcal{M}_{pq}} e(\mathrm{Ob}) = 1.$$

Note that the rank of the excess bundle is the difference between the actual dimension of the moduli space, and the expected one, and that $\mathrm{Ob} = 0$ unless $p = q$.

Unfortunately, in more complicated situations (but also not that complicated), we often do not even know whether our geometric setup is a degeneration of a transverse one. If it were, we would like to dispose of a technology allowing us to "count" in the transverse setup and argue that the number we obtain there equals the one we are after. This sounds like a reasonable wish, but it is way too optimistic. We should not aim at this: not only because counting is often difficult also in transverse situations, but mainly because we simply may not have enough algebraic deformations to pretend that the geometry of the problem is transverse.

Example 0.3.2. If we were to count self-intersections of a (-1)-curve on a surface,[3] there would be no way to deform these curves off themselves to make them intersect themselves transversely! See also Exercise 0.3.4 below.

This discussion leads us directly to another intrinsic difficulty in Enumerative Geometry. Suppose, just to dream for a second, that we are able to solve *all* enumerative problems in generic (transverse) situations, and we know that the answer does not change after a small perturbation of the initial data.

> **D3**. How do we "pretend" we can work in a transverse situation when there is none available (e.g. in Example 0.3.2)?

The modern way to do this is to use *virtual fundamental classes* (cf. Section 10.1 and Appendix **??**).

0.3.1. *Two more words on excess intersection.* Problem (5), known as "the five conics problem", is a typical example of an excess intersection problem. See [20] for a thorough analysis and solution of this problem. As we shall see in Section 3.5.1, a natural compact parameter space for plane conics is

$$\mathcal{M} = \mathbb{P}^5,$$

and the set of smooth conics is an open subvariety $U \subset \mathcal{M}$. The answer to Problem (5) is a certain finite subset of U. Let $C_1, \ldots, C_5$ be general plane conics. The conics

[3] A (-1)-curve on a surface S is a curve $C \subset S$ such that $C.C = -1$, where the intersection number $C.C$ can be seen as the degree of the normal bundle $N_{C/S}$ to C in S.

that are tangent to a given conic C_i form a sextic hypersurface $Z_i \subset \mathcal{M}$, so we might be tempted to say that the answer to Problem (5) is the degree

$$\int_{\mathbb{P}^5} \alpha_1 \cdots \alpha_5 = 6^5,$$

where $\alpha_i = [Z_i] \in H^2(\mathbb{P}^5, \mathbb{Z})$ is the divisor class of a sextic.[4] However, the cycles Z_i share a common two-dimensional component, namely the Veronese surface $\mathbb{P}^2 \subset \mathbb{P}^5$ of double lines. Therefore their intersection is 2-dimensional, even though our intuition suggests that 5 hypersurfaces in $\mathbb{P}^5$ should intersect in a finite set. Note that this issue arose precisely "because" we insisted to work with a compact parameter space: double lines are singular, hence lie in the complement of U. But working with U directly is forbidden, because it is not compact!

The excess intersection formula is a tool that allows one to precisely compute (and hence get rid of) the enumerative contribution of the *excess locus*, namely the locus of non-transverse intersection among certain cycles — in this case the cycles $Z_1, \ldots, Z_5$. The way it works is precisely via blow-ups; often more than one is required to separate the common components of the non-transverse cycles. In the case of the five conics problem, only one blow-up is required.

In principle, blowing up the excess locus, checking that the proper transforms will be disjoint in the exceptional divisor, and blowing up again if necessary, one gets to the correct answer to the original question, but:

> **D4**. In practice it is often very hard to keep track of multiple blow-ups; the calculation becomes less and less intuitive and the modular meaning of the blow-ups appearing might be quite unclear.

In Exercise 0.3.4 you will compute an excess bundle for a more complicated problem than finding the number of lines through two points. Before tackling it, it is best to solve the following exercise.

EXERCISE 0.3.3. Show that the vector space V of homogeneous cubic polynomials in 3 variables is 10-dimensional. Identify

$$\mathbb{P}V = \mathbb{P}^9$$

with the space of degree 3 plane curves $C \subset \mathbb{P}^2$. Show that, for a given point $p \in \mathbb{P}^2$, the space of cubics passing through p forms a hyperplane

$$\mathbb{P}^8 \subset \mathbb{P}V.$$

EXERCISE 0.3.4. Let C_1 and C_2 be two plane cubics intersecting transversely in nine points $p_1, \ldots, p_9 \in \mathbb{P}^2$ (cf. Figure 3). Every cubic in the pencil $\mathbb{P}^1 \subset \mathbb{P}^9$ generated by C_1 and C_2 passes through $p_1, \ldots, p_9$. However, if the nine points were general, there would be a unique cubic passing through them. Find out where the answer '1' is hiding in this non-transverse geometry. This example is also discussed in [61, § 0].

[4] Recall that the Picard group $\operatorname{Pic}\mathbb{P}^r = H^2(\mathbb{P}^r, \mathbb{Z}) = \mathbb{Z}$ is generated by the hyperplane class h and the cohomology class of a degree d hypersurface in $\mathbb{P}^r$ corresponds to the class $d \cdot h$.

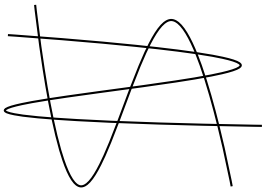


FIGURE 3. The nine intersection points $C_1 \cap C_2$.

0.4. **Before and after the virtual class.** Here is a philosophical description of the field of Enumerative Geometry before and after the advent of *virtual fundamental classes*, introduced by Li–Tian [46] and Behrend–Fantechi [8].

Before: What is the answer?
After: What is the question?

Before virtual classes, there were a number of unanswered enumerative questions whose geometrical meaning was extremely clear. *After* the definition of virtual classes, many new invariants were defined through them, but the enumerative meaning of these invariants is often not very clear, so it fair to ask what integrals of the form

$$\int_{[\mathcal{M}]^{\mathrm{vir}}} \alpha \in \mathbb{Z}, \quad \alpha \in H^*(\mathcal{M}),$$

might be actually computing.

Virtual fundamental classes allow one to think that even a horrible moduli space $\mathcal{M}$, say a singular scheme of impure dimension (cf. Figure 7), has a well-defined *virtual dimension* vd at any point $p \in \mathcal{M}$, and this number is constant on p. It is given as the difference

$$\mathrm{vd} = \dim T_p\mathcal{M} - \dim \mathrm{Ob}|_p,$$

where both dimensions on the right may (and will) vary with p. The virtual fundamental class is a homology class

$$[\mathcal{M}]^{\mathrm{vir}} \in A_{\mathrm{vd}}\mathcal{M} \to H_{2\cdot\mathrm{vd}}(\mathcal{M}, \mathbb{Z})$$

that should be thought of as the fundamental class that $\mathcal{M}$ would have if it were of the form $\mathcal{M} = \{s = 0\}$ for s a regular section of a vector bundle (the bundle Ob) on a smooth variety.

As a matter of fact, many badly behaved moduli spaces turn out to have a virtual fundamental class. These include:

(i) the moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(X,\beta)$ to a smooth projective variety X,
(ii) the moduli space $M_Y^H(\alpha)$ of H-stable torsion free sheaves with Chern character α on a smooth 3-fold Y,
(iii) the moduli space $P_X^H(\alpha)$ of Pandharipande–Thomas pairs with Chern character α.

All this richness gives rise to three amongst the most modern counting theories:

$$\begin{aligned} \text{Gromov–Witten theory} &\coloneqq \text{intersection theory on } \overline{\mathcal{M}}_{g,n}(X,\beta),\\ \text{Donaldson–Thomas theory} &\coloneqq \text{intersection theory on } M_Y^H(\alpha),\\ \text{Pandharipande–Thomas theory} &\coloneqq \text{intersection theory on } P_X^H(\alpha). \end{aligned}$$

All these theories can be seen as more complicated (virtual) versions of a well established theory:

$$\text{Schubert Calculus} \coloneqq \text{intersection theory on the Grassmannian } G(k,n).$$

No "virtualness" is arising in Schubert calculus, because — as already observed by Mumford [54] when he initiated the enumerative geometry of the moduli space of curves — the Grassmannian is the ideal moduli space one would like to work with: it is compact, smooth and, put in modern language, unobstructed. It does have a virtual fundamental class, but because of these properties it happens to coincide with its actual fundamental class.

0.5. **To the reader.** The reader might benefit from some familiarity with elementary aspects of scheme theory, basic theory of coherent sheaves on algebraic varieties, and intersection theory at the level of [36]. We shall, however, review some preliminaries in the next section. Here is a list of excellent references for the background material needed in these lecture notes (that we will refer to when necessary):

- for scheme theory at various levels, see [36, 19, 47, 73],
- for Intersection Theory, see [25, 20],
- for toric varieties, see [26, 17],
- for Deformation Theory, see [68, 37] and [22, Part 3].

1. Background material

1.1. **Varieties and schemes.** The notion of scheme used in this text is the standard one, see e.g. [47, Chapter 2]. The structure sheaf of a scheme X, its sheaf of regular functions, is denoted $\mathscr{O}_X$. A scheme X is *locally Noetherian* if every point $x \in X$ has a Zariski affine open neighborhood $x \in \operatorname{Spec} R \subset X$ such that R is a Noetherian ring. If X is locally Noetherian and quasi-compact, then it is called *Noetherian*. Any open or closed subscheme of a Noetherian scheme X is still Noetherian, and for every affine open subset $U \subset X$ the ring $\mathscr{O}_X(U)$ is Noetherian. An important property of Noetherian schemes is that they have a finite number of irreducible components, or, more generally, of associated points.

A morphism of schemes $f : X \to S$ is *quasi-compact* if the preimage of every affine open subset of S is quasi-compact. On the other hand, f is *locally of finite type* if for every $x \in X$ there exist Zariski open neighborhoods $x \in \operatorname{Spec} A \subset X$ and $f(x) \in \operatorname{Spec} B \subset S$ such that $f(\operatorname{Spec} A) \subset \operatorname{Spec} B$ and the induced map $B \to A$ is of finite type, i.e. A is isomorphic to a quotient of $B[x_1, \ldots, x_n]$ as a B-algebra. We say that f is *of finite type* if it is locally of finite type and quasi-compact.

Exercise 1.1.1. Let $f : X \to S$ be a morphism of schemes, with S (locally) Noetherian. If f is (locally) of finite type, then X is (locally) Noetherian.

For instance, a scheme of finite type over a field is Noetherian.

Notation 1.1.2. By k we will always mean an algebraically closed field. For most of the time, we will have $k = \mathbb{C}$.

Definition 1.1.3. A scheme X is *reduced* if for every point $p \in X$ the local ring $\mathscr{O}_{X,p}$ is reduced, i.e. it has no nilpotent elements besides zero.

The prototypical example of a nonreduced scheme is the curvilinear affine scheme

$$D_n = \operatorname{Spec} k[t]/t^n, \quad n > 1.$$

One can show that (quasi-compact) reduced schemes are precisely those schemes for which the regular functions on them are determined by their values on points. The function

$$\overline{t} \in k[t]/t^n$$

vanishes at the unique point of D_n, but it is not the zero function!

The case $n = 2$ is particularly important. For instance, the *Zariski tangent space* $T_x X$ of a k-scheme X at a point $x \in X$, which by definition is the k-vector space $(\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$, can be identified with

$$\operatorname{Hom}_x(D_2, X),$$

the space of k-morphisms $D_2 \to X$ such that the image of the closed point of D_2 is x.

Example 1.1.4 (D_2 as a limit of distinct points)**.** Consider the scheme

$$X_a = \operatorname{Spec} \mathbb{C}[x, y]/(y - x^2, y - a), \quad a \in \mathbb{C}.$$

For $a \neq 0$, this scheme consists of two reduced points, corresponding to the maximal ideals

$$(x \pm \sqrt{a}, y - a) \subset \mathbb{C}[x, y].$$

For $a = 0$, we get

$$X_0 = \operatorname{Spec} \mathbb{C}[x]/x^2 = D_2,$$

a point with multiplicity two. See Figure 4 for a visual explanation.

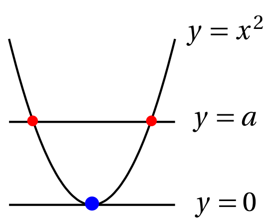


FIGURE 4. The intersection X_a of a parabola with the line $y = a$.

Definition 1.1.5. Let k be a field. An *algebraic variety* over k (or simply a k-variety) is a reduced, separated scheme of finite type over $\operatorname{Spec} k$, i.e. a reduced scheme X equipped with a finite type morphism $X \to \operatorname{Spec} k$, such that the diagonal map $\Delta_X \colon X \to X \times_k X$, sending $x \mapsto (x, x)$, is a closed immersion.

An *affine variety* is a k-scheme of the form $\operatorname{Spec} A$, where $A = k[x_1,\ldots,x_n]/I$ for some ideal I. An algebraic variety is *projective* if it admits a closed immersion into projective space $\mathbb{P}^n$ for some n. A variety is *quasi-projective* if it admits a locally closed immersion in some projective space, i.e. it is closed in an open subset of some $\mathbb{P}^n$. The same abstract scheme can of course be a (quasi-)projective variety in many different ways.

Example 1.1.6. The *rational normal curve* of degree d is the image of the closed embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^d$ defined by $(u:v) \mapsto (u^d : u^{d-1}v : \cdots : uv^{d-1} : v^d)$.

EXERCISE 1.1.7. Consider the algebraic variety $X = \operatorname{Spec} \mathbb{C}[x,y]/(xy,y^2)$, viewed as a subscheme of the affine plane $\mathbb{A}^2 = \operatorname{Spec} \mathbb{C}[x,y]$. Show that the origin $p = (0,0) \in X$ is the unique point such that $\mathscr{O}_{X,p}$ is not reduced.

The following definition will be relevant when we will discuss the Hilbert scheme of points in Section 5.

Definition 1.1.8. An algebraic k-variety X is *finite* if $\mathscr{O}_X(X)$ is a finite dimensional k-vector space. For any such X, the ring of functions is necessarily *Artinian*. In other words, X has dimension zero, and we say that $\mathscr{O}_X(X)$ is a finite dimensional k-algebra of *length*

$$\ell = \dim_k \mathscr{O}_X(X).$$

We also say that ℓ is the length of X.

EXERCISE 1.1.9. Show that an algebraic variety X is both affine and projective if and only if it is finite. Show that, for any ℓ, the only reduced finite k-variety of length ℓ is the disjoint union $\coprod_{1\le i\le \ell} \operatorname{Spec} k$.

EXERCISE 1.1.10. Classify all finite dimensional $\mathbb{C}$-algebras of length 2 and 3 up to isomorphism.

EXERCISE 1.1.11. Give an example of a scheme X whose underlying topological space consists of finitely many points, and yet is *not* finite.

1.2. **Some properties of morphisms.** We encountered separated morphisms in the definition of algebraic varieties (Definition 1.1.5). A morphism $f: X \to S$ is *separated* if the diagonal $X \to X \times_S X$ (which is always a locally closed immersion) is a closed immersion. A stronger notion is properness. A morphism $f: X \to S$ is *proper* if it is separated, of finite type, and universally closed. The *valuative criterion* for proper morphims says that f is proper if and only if for every valuation domain A with fraction field K there exists exactly one way to fill in the dotted arrow in a commutative square

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d, hook, "i"'] & X \arrow[d, "f"] \\
\operatorname{Spec} A \arrow[r] \arrow[ur, dotted] & S
\end{tikzcd}$$

in such a way that the resulting triangles are commutative. Such property can be rephrased by saying that for any A as above the map of sets

$$\mathrm{Hom}(\operatorname{Spec} A, X) \to \mathrm{Hom}(\operatorname{Spec} K, X) \times_{\mathrm{Hom}(\operatorname{Spec} K, S)} \mathrm{Hom}(\operatorname{Spec} A, S)$$

defined by $v \mapsto (v \circ i, f \circ v)$ is a bijection.

Let A and $\overline{A}$ be Artinian k-algebras with residue field k. We say that a surjection $u \colon \overline{A} \twoheadrightarrow A$ is a *square zero extension* if $(\ker u)^2 = 0$.

Definition 1.2.1. Let $f \colon X \to S$ be a locally of finite type morphism between k-schemes. Then f is *unramified* (resp. *smooth*, *étale*) if for any square zero extension $\overline{A} \twoheadrightarrow A$ and solid diagram

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\
\operatorname{Spec} \overline{A} \arrow[r] \arrow[ur, dotted] & S
\end{tikzcd}$$

there exists at most one (resp. at least one, exactly one) way to fill in the dotted arrow in such a way that the resulting triangles are commutative.

Example 1.2.2. The following are important features to keep in mind.

- A proper morphism which is injective on points and tangent spaces (i.e. a proper monomorphism) is a closed immersion.
- Let $f \colon X \to S$ be a morphism of smooth $\mathbb{C}$-varieties. If f induces isomorphism on tangent spaces, it is étale.
- A bijective morphism of smooth $\mathbb{C}$-varieties is an isomorphism.
- An étale injective (resp. bijective) morphism is an open immersion (resp. an isomorphism).

1.3. **Schemes with embedded points.** On a locally Noetherian scheme X there are a bunch of points that are more relevant than all other points, in the sense that they reveal part of the behavior of the structure sheaf: these points are the *associated points* of X.

Let R be a commutative ring with unity, and let M be an R-module. If $m \in M$, we let

$$\mathrm{Ann}_R(m) = \{\, r \in R \mid r \cdot m = 0 \,\} \subset R$$

denote its annihilator. A prime ideal $\mathfrak{p} \subset R$ is said to be *associated to* M if $\mathfrak{p} = \mathrm{Ann}_R(m)$ for some $m \in M$. The set of all associated primes is denoted

$$\mathrm{AP}_R(M) = \{\, \mathfrak{p} \mid \mathfrak{p} \text{ is associated to } M \,\}.$$

Lemma 1.3.1. *Let $\mathfrak{p}$ be a prime ideal of R. Then $\mathfrak{p} \in \mathrm{AP}_R(M)$ if and only if $R/\mathfrak{p}$ is an R-submodule of M.*

Proof. If $\mathfrak{p} = \mathrm{Ann}_R(m)$ for some $m \in M$, consider the map $\phi_m \colon R \to M$ defined by $\phi_m(r) = r \cdot m$. Since its kernel is by definition $\mathrm{Ann}_R(m)$, the quotient $R/\mathfrak{p}$ is an R-submodule of M. Conversely, given an R-linear inclusion $i \colon R/\mathfrak{p} \hookrightarrow M$, consider the composition $\phi \colon R \twoheadrightarrow R/\mathfrak{p} \hookrightarrow M$. Then $\phi = \phi_m$, where $m = i(1)$. □

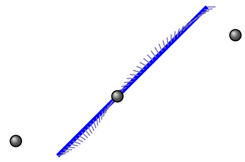


FIGURE 5. A thickened (Cohen–Macaulay) curve with an embedded point and two isolated (possibly fat) points.

Note that if $\mathfrak{p} \in \mathrm{AP}_R(M)$ then $\mathfrak{p}$ contains the annihilator of M, i.e. the ideal

$$\mathrm{Ann}_R(M) = \{ r \in R \mid r \cdot m = 0 \text{ for all } m \in M \} \subset R.$$

The minimal elements (with respect to inclusion) in the set

$$\{ \mathfrak{p} \subset R \mid \mathfrak{p} \supset \mathrm{Ann}_R(M) \}$$

are called *isolated primes* of M.

From now on we assume R is Noetherian and $M \neq 0$ is finitely generated. In this situation, M has a *composition series*, i.e. a filtration of R-submodules

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_s = M$$

such that $M_i/M_{i-1} = R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$. This series is not unique. However, for a prime ideal $\mathfrak{p} \subset R$, the number of times it occurs among the $\mathfrak{p}_i$ does not depend on the composition series. These primes are precisely the elements of $\mathrm{AP}_R(M)$. For $M = R/I$, elements of $\mathrm{AP}_R(R/I)$ are the radicals of the primary ideals in a *primary decomposition* of I.

EXERCISE 1.3.2. Let $R = k[x,y]$, $I = (xy, y^2)$ and $M = R/I$. Show that $\mathrm{AP}_R(M) = \{(y),(x,y)\}$.

Theorem 1.3.3 ([73, Theorem 5.5.10 (a)])**.** *Let R be a Noetherian ring, $M \neq 0$ a finitely generated R-module. Then $\mathrm{AP}_R(M)$ is a finite nonempty set containing all isolated primes.*

Definition 1.3.4. The non-isolated primes in $\mathrm{AP}_R(M)$ are called the *embedded primes* of M.

The most boring situation is when R is an integral domain, in which case the generic point $\xi \in \operatorname{Spec} R$ is the only associated prime. More generally, a reduced affine scheme $\operatorname{Spec} R$ has *no embedded point*, i.e. the only associated primes are the isolated (minimal) ones, corresponding to its irreducible components.

Fact 1.3.5. An algebraic curve has no embedded points if and only if it is Cohen–Macaulay. However, there can be nonreduced Cohen–Macaulay curves: those curves with a fat component, such as the affine plane curve $\operatorname{Spec} k[x,y]/x^2 \subset \mathbb{A}^2$. These objects often have moduli, i.e. deform (even quite mysteriously) in positive dimensional families.

Let R be an integral domain. For an ideal $I \subset R$, one often calls the associated primes of I the associated primes of R/I. The minimal primes above $I = \mathrm{Ann}_R(R/I)$

correspond to the irreducible components of the closed subscheme

$$\operatorname{Spec} R/I \subset \operatorname{Spec} R,$$

whereas for every embedded prime $\mathfrak{p} \subset R$ there exists a minimal prime $\mathfrak{p}'$ such that $\mathfrak{p}' \subset \mathfrak{p}$. Thus $\mathfrak{p}$ determines an *embedded component* — a subvariety $V(\mathfrak{p})$ embedded in an irreducible component $V(\mathfrak{p}')$. If the embedded prime $\mathfrak{p}$ is maximal, we talk about an *embedded point*.

Remark 1.3.6. An embedded component $V(\mathfrak{p})$, where $\mathfrak{p}$ is the radical of some primary ideal $\mathfrak{q}$ appearing in a primary decomposition $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_e$, is of course embedded in some irreducible component $V(\mathfrak{p}') \subset \operatorname{Spec} R/I$, but $V(\mathfrak{q})$ is not a *subscheme* of $V(\mathfrak{p}')$, because the fuzzyness caused by nilpotent behavior (i.e. the difference between $\mathfrak{q}$ and its radical $\mathfrak{p}$) makes the bigger scheme $V(\mathfrak{q}) \supset V(\mathfrak{p})$ "stick out" of $V(\mathfrak{p}') \subset \operatorname{Spec} R/I$.

Example 1.3.7. Consider $R = k[x,y]$ and $I = (xy, y^2)$. A primary decomposition of I is

$$I = (x,y)^2 \cap (y).$$

However, $\operatorname{Spec} R/(x,y)^2$ is not scheme-theoretically contained in $\operatorname{Spec} R/y$.

In general, a subscheme Z of scheme Y has an embedded component if there exists a dense open subset $U \subset Y$ such that $Z \cap U$ is dense in Z but the scheme-theoretic closure of $Z \cap U \subset Z$ does not equal Z scheme-theoretically. For instance, if Y is irreducible, we say that $p \in Y$ supports an embedded point of a closed subscheme $Z \subset Y$ if $\overline{Z \cap (Y \setminus p)} \neq Z$ as schemes. In the example above, where $Y = \mathbb{A}^2$ and $Z = \operatorname{Spec} k[x,y]/(xy, y^2)$, the scheme-theoretic closure of $Z \cap (\mathbb{A}^2 \setminus 0) \subset Z$ is not equal to Z.

1.4. **Sheaves and their support.**

Recall that a *coherent sheaf* on a (locally Noetherian) scheme X is an $\mathscr{O}_X$-module that is locally the cokernel of a map of free $\mathscr{O}_X$-modules of finite rank. Coherent sheaves form an abelian category, denoted

$$\operatorname{Coh} X.$$

For instance, if $\iota\colon Z \hookrightarrow X$ is a closed subscheme, both $\iota_*\mathscr{O}_Z$ and $\mathscr{I}_Z$ are coherent sheaves on X. The ideal sheaf, being a subsheaf of a free sheaf, is torsion free. In fact, ideal sheaves of subschemes of codimension at least two inside a scheme X are precisely the torsion free sheaves of rank one and trivial determinant. In codimension one, one has a bijection between the effective Cartier divisors on a scheme X and the invertible ideal sheaves $\mathscr{I} \subset \mathscr{O}_X$.

Definition 1.4.1. Let $X \to S$ be a finite type morphism of locally Noetherian schemes. A sheaf $F \in \operatorname{Coh} X$ is *flat over* S (or S-*flat*) if for every point $x \in X$, with image $s \in S$, the module F_x is flat over $\mathscr{O}_{S,s}$ via the ring map $\mathscr{O}_{S,s} \to \mathscr{O}_{X,x}$.

For instance, $\mathscr{O}_X$ is S-flat if and only if $X \to S$ is flat as a morphism of schemes.

The *support* of a coherent sheaf $F \in \operatorname{Coh} X$ is the following *closed subscheme* of X: consider the map $\mathscr{O}_X \to \mathscr{H}om_{\mathscr{O}_X}(F,F)$ defined on local sections by sending f to the $\mathscr{O}_X$-linear map $m \mapsto f \cdot m$. The kernel — the sheaf-theoretic annihilator ideal of F — defines the closed subscheme

$$\operatorname{Supp} F \subset X.$$

The support behaves well under pullback. However, the following remark is the origin of several issues around the existence of Hilbert–Chow morphisms.

Remark 1.4.2. Let $X \to S$ be a a finite type morphism of locally Noetherian schemes. It is not true that the support of an S-flat $\mathscr{O}_X$-module is flat over S.

EXERCISE 1.4.3. Give an example of the phenomenon described in Remark 1.4.2.

2. INFORMAL INTRODUCTION TO GRASSMANNIANS

We denote by $G(k,n)$ the *set* of k-dimensional subspaces of $\mathbb{C}^n$. This set is in fact a complex manifold, called the *Grassmannian* of k-planes in $\mathbb{C}^n$. It is naturally identified with the set of $(k-1)$-dimensional linear subspaces of $\mathbb{P}^{n-1}$, and when we think of it in this manner we denote it by $\mathbb{G}(k-1,n-1)$. For instance, $\mathbb{G}(0,n-1) = \mathbb{P}^{n-1}$.

We will give $G = G(k,n)$ the structure of a (smooth) projective variety of dimension $k(n-k)$, by describing a closed embedding in $\mathbb{P}^N$, where $N = \binom{n}{k} - 1$. We will see that G admits an affine stratification, which will enable us to explicitly describe the generators of its Chow ring A^*G. The affine strata will be called *Schubert cells*, while their closures will be called *Schubert cycles*. The classes of the Schubert cycles freely generate the Chow group of G. To determine the ring structure, one has to compute the products between these generators. These computations in A^*G go under the name of *Schubert Calculus*.

2.1. $G(k,n)$ as a projective variety.

Let us fix a point

$$[\mathcal{H}] \in G = G(k,n),$$

corresponding to a k-dimensional linear subspace $\mathcal{H} \subset V = \mathbb{C}^n$. If $v_1,\ldots,v_k$ is a basis of $\mathcal{H}$ then $v_1 \wedge \cdots \wedge v_k$ is the free generator of the line $\bigwedge^k \mathcal{H} \subset \bigwedge^k V \cong \mathbb{C}^{\binom{n}{k}} = \mathbb{C}^{N+1}$. So we get a map

$$\iota\colon G \to \mathbb{P}\left(\bigwedge^k V\right) = \mathbb{P}^N$$

sending $[\mathcal{H}] \mapsto [v_i \wedge \cdots \wedge v_k]$. Why is this well-defined? Let us view the point $[\mathcal{H}] \in G$ as (the space generated by the rows of) a full rank matrix $H = (a_{ij}) \in M_{k\times n}(\mathbb{C})$ and let us fix a basis $e_1,\ldots,e_n$ of V. Then a basis of $\bigwedge^k V$ is given by

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}_{1 \le i_1 < \cdots < i_k \le n}.$$

So when we view the element $v_1 \wedge \cdots \wedge v_k$ inside $\bigwedge^k V$ we can write it uniquely as

$$v_1 \wedge \cdots \wedge v_k = \sum_{1 \le i_1 < \cdots < i_k \le n} p_{i_1\cdots i_k}(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \sum_I p_I e_I,$$

where the coefficient $p_I = p_{i_1\cdots i_k}$ is the minor of the $(k\times k)$-matrix given by extracting from H the columns $i_1,\ldots,i_k$. Of course different choices of H may produce the same $\mathcal{H}$. But H is unique up to the left action of $\mathrm{GL}(k,\mathbb{C})$. Summing up, we have a commutative diagram

$$\begin{tikzcd}
\bigwedge^k \mathcal{H} \setminus 0 \arrow[r, hook] \arrow[d, "/\mathbb{C}^\times"'] & \bigwedge^k V \setminus 0 \arrow[d, "/\mathbb{C}^\times"] \\
\mathbb{P}\left(\bigwedge^k \mathcal{H}\right) \arrow[r, "\iota"] & \mathbb{P}\left(\bigwedge^k V\right)
\end{tikzcd}
\qquad
\begin{tikzcd}
v_1 \wedge \cdots \wedge v_k \arrow[r, mapsto] \arrow[d, mapsto] & \sum_I p_I e_I \arrow[d, mapsto] \\
{[v_1 \wedge \cdots \wedge v_k]} \arrow[r, mapsto] & (p_I)_I
\end{tikzcd}$$

Up to now, we have identified a point $[\mathcal{H}]\in G$ to the unique point of $\mathbb{P}(\bigwedge^k \mathcal{H})$ and we have defined a map $\iota\colon G\to\mathbb{P}^N$ by sending $[\mathcal{H}]$ to its *Plücker coordinates* $(p_I)_I$. Such a map is injective, and G can be identified with an irreducible algebraic set in $\mathbb{P}^N$, via a collection of homogeneous quadratic polynomials defining the *Plücker relations*. The (homogeneous prime) ideal of G is the kernel of the homomorphism

$$\mathbb{C}[p_{i_1\dots i_k}|1\le i_1<\dots<i_k\le n]\to\mathbb{C}[x_{lj}|1\le l\le k,\,1\le j\le n]$$

sending $p_{i_1\dots i_k}$ to the Plücker coordinate $\det(x_{lj})_{1\le l\le k,\,j=i_1,\dots,i_k}$.

Example 2.1.1. The Grassmannian $G(2,4)=\mathbb{G}(1,3)$ of lines in $\mathbb{P}^3$ is a smooth quadric hypersurface in $\mathbb{P}^5$, given by the single homogeneous polynomial

$$p_{12}p_{34}-p_{13}p_{24}+p_{14}p_{23}=0.$$

2.2. **Chow Ring of** $G(k,n)$**.** Let $G=G(k,n)$, and set $r=n-k$. We know that G is smooth and projective, so its Chow group is a ring and can be graded by codimension. Now we think of elements of G as linear subspaces of $\mathbb{P}^{n-1}$. So, let us fix a flag

$$\mathcal{F}\colon\ F_0\subset F_1\subset F_2\subset\dots\subset F_{n-1}=\mathbb{P}^{n-1}.$$

Let us look at the set of k-tuples

$$\mathscr{A}=\{(a_1,\dots,a_k)\,|\,r\ge a_1\ge\dots\ge a_k\ge 0\}.$$

For all $a=(a_1,\dots,a_k)\in\mathscr{A}$, define the closed subset of G

$$\Sigma_a(\mathcal{F})=\{\mathcal{H}\in G\,|\,\dim(\mathcal{H}\cap F_{r+i-1-a_i})\ge i-1\text{ for all } i=1,\dots,k\}.$$

These are called the *Schubert cycles* on G. They have a number of interesting properties, for instance:

(1) $c_a=\operatorname{codim}(\Sigma_a,G)=\sum_{1\le i\le k}a_i$. Hence $\sigma_a=[\Sigma_a]\in A^{c_a}G$.
(2) By defining $a\le b$ if and only if $a_i\le b_i$ for all $i=1,\dots,k$, one sees that $\Sigma_b\subseteq\Sigma_a\iff a\le b$.
(3) The Schubert cell $\tilde{\Sigma}_a=\Sigma_a\setminus(\bigcup_{a<b}\Sigma_b)\cong\mathbb{A}^{k(n-k)-c_a}$ and $G=\coprod_{a\in\mathscr{A}}\tilde{\Sigma}_a$ is an affine stratification of G, with closed strata the Schubert cycles.

By (3), the cycle classes σ_a freely generate the Chow group A^*G.

2.3. **The example of** $\mathbb{G}(1,3)$**.** Let $G=\mathbb{G}(1,3)$, so that $r=n-k=2$ and

$$\begin{aligned}\mathscr{A}&=\{(a_1,a_2)\,|\,2\ge a_1\ge a_2\ge 0\}\\&=\{(2,2),(1,1),(0,0),(2,1),(2,0),(1,0)\}.\end{aligned}$$

After fixing a flag of linear subspaces

$$\mathcal{F}\colon\ \{P\}\subset M\subset H\subset\mathbb{P}^3,$$

where $\{P\}=F_0$, $M=F_1$, $H=F_2$, and recalling that the Schubert cycles are given by

$$\Sigma_{a_1a_2}(\mathcal{F})=\{L\in G\,|\,\dim(L\cap F_{2-a_1})\ge 0,\ \dim(L\cap F_{3-a_2})\ge 1\},$$

we can write them all explicitly as follows:

$$\begin{aligned}
\Sigma_{22} &= \{L \in G \mid P \in L, \dim(L \cap M) = 1\} = \{M\} \\
\Sigma_{11} &= \{L \in G \mid L \cap M \neq \emptyset, L \subset H\} = \{L \in G \mid L \subset H\} \\
\Sigma_{00} &= \{L \in G \mid L \text{ meets a plane, and meets } \mathbb{P}^3 \text{ in a line}\} = G \\
\Sigma_{21} &= \{L \in G \mid P \in L \subset H\} \\
\Sigma_{20} &= \{L \in G \mid P \in L\} \\
\Sigma_{10} &= \{L \in G \mid L \cap M \neq \emptyset\}.
\end{aligned}$$

Remark 2.3.1. Of course, somewhere we used that L always intersects a plane in $\mathbb{P}^3$; also notice that two lines may not meet. Later on we will use that every L intersect a 3-plane in $\mathbb{P}^4$, and that two general 2-planes meet in a point.

Notation 2.3.2. Let us shorten $\Sigma_{a_1 0}$ to Σ_{a_1}. Later, $\sigma_{a_1 a_2}$ will denote $[\Sigma_{a_1 a_2}] \in A^{a_1+a_2} G$.

In order to calculate the Schubert cells $\tilde{\Sigma}_{a_1 a_2}$, it is useful to look at the following inclusions:

$$\begin{tikzcd}
 & & \Sigma_2 \arrow[dr, hook] & & \\
\Sigma_{22} \arrow[r, hook] & \Sigma_{21} \arrow[ur, hook] \arrow[dr, hook] & & \Sigma_1 \arrow[r, hook] & G \\
 & & \Sigma_{11} \arrow[ur, hook] & &
\end{tikzcd}$$

One can verify directly that $\tilde{\Sigma}_{a_1 a_2} \cong \mathbb{A}^{4-(a_1+a_2)}$. Let us concentrate on the problem of determining the ring structure of A^*G. For the moment, we have the free abelian group decomposition

$$A^*G = \underbrace{\mathbb{Z}[\sigma_{22}]}_{A^4 G} \oplus \underbrace{\mathbb{Z}[\sigma_{21}]}_{A^3 G} \oplus \underbrace{\mathbb{Z}[\sigma_{11}] \oplus \mathbb{Z}[\sigma_2]}_{A^2 G} \oplus \underbrace{\mathbb{Z}[\sigma_1]}_{A^1 G} \oplus \underbrace{\mathbb{Z}[\sigma_0]}_{A^0 G}.$$

In particular, any two points are linearly equivalent, and this is true in every Grassmannian.

Let us calculate the products in A^*G. It is crucial to work with two generically situated flags at the same time: so we will intersect a cycle taken from the first one with a cycle taken from the second one. These are generically transverse by Kleiman's Theorem (that we can apply because we are over $\mathbb{C}$). And the result only depends on the equivalence classes of the cycles we are intersecting, so what we find is the correct result. Now, let us fix two flags

$$\begin{aligned}
\mathcal{F}&\colon\ \{P\} \subset M \subset H \subset \mathbb{P}^3 \\
\mathcal{F}'&\colon\ \{P'\} \subset M' \subset H' \subset \mathbb{P}^3.
\end{aligned}$$

2.3.1. *Codimension* 4. We have to evaluate

$$\sigma_{11}^2,\ \sigma_2^2,\ \sigma_{11} \cdot \sigma_2,\ \sigma_1 \cdot \sigma_{21} \in A^4 G.$$

Let us start with the self-intersection σ_{11}^2. We have

$$|\Sigma_{11} \cap \Sigma'_{11}| = |\{L \in G \mid L \subset H \cap H'\}| = 1.$$

This unique line is of course $H\cap H'$. Hence

$$\sigma_{11}^2=\sigma_{22}.$$

Similarly,

$$|\Sigma_2\cap\Sigma_2'|=|\{\,L\in G\mid P\in L,\,P'\in L\,\}|=|\{\,\overline{PP'}\,\}|=1.$$

Hence again

$$\sigma_2^2=\sigma_{22}.$$

Since $P'\notin H$, we find

$$|\Sigma_{11}\cap\Sigma_2'|=|\{\,L\in G\mid P'\in L\subset H\,\}|=0,$$

thus

$$\sigma_{11}\cdot\sigma_2=0.$$

The last calculation is

$$|\Sigma_1\cap\Sigma_{21}'|=|\{\,L\in G\mid L\cap M\neq\emptyset,\,P'\in L\subset H'\,\}|=1,$$

corresponding to this line: the one determined by P' and $M\cap H'$. Thus

$$\sigma_1\cdot\sigma_{21}=\sigma_{22}.$$

2.3.2. *Codimension* 3. We have to evaluate

$$\sigma_1\cdot\sigma_2,\,\sigma_{11}\cdot\sigma_1\in A^3G.$$

We see that

$$\Sigma_1\cap\Sigma_2'=\{\,L\in G\mid L\cap M\neq\emptyset,\,P'\in L\,\}=\Sigma_{21}''$$

with respect to the flag $\mathcal{F}''\colon\{\,P'\,\}\subset\ell\subset\langle P',M\rangle\subset\mathbb{P}^3$. Thus we get

$$\sigma_1\cdot\sigma_2=\sigma_{21}.$$

Similarly,

$$\Sigma_1\cap\Sigma_{11}=\{\,L\in G\mid L\cap M\neq\emptyset,\,L\subset H'\,\}=\Sigma_{21}''$$

with respect to the flag $\mathcal{F}''\colon\{\,R\,\}\subset\ell\subset H'\subset\mathbb{P}^3$, where $R=M\cap H'$. Thus

$$\sigma_1\cdot\sigma_{11}=\sigma_{21}.$$

2.3.3. *Codimension* 2. We have to evaluate $\sigma_1^2\in A^2G$. Here things get tricky because this product is not a Schubert cycle. What we know is that we can write $\sigma_1^2=\alpha\sigma_{11}+\beta\sigma_2$ in A^2G. We have to determine α and β. The strategy will be (now and in the future) to intersect both sides with cycles in complementary codimension so that one of the summands vanishes. Doing this twice allows us to recover α and β in two steps. So,

$$\sigma_1^2\cdot\sigma_2=(\alpha\sigma_{11}+\beta\sigma_2)\cdot\sigma_2$$

gives $(\sigma_{22}=)\sigma_1\cdot\sigma_{21}=\beta\sigma_{22}$. Hence $\beta=1$. In the same way,

$$\sigma_1^2\cdot\sigma_{11}=(\alpha\sigma_{11}+\beta\sigma_2)\cdot\sigma_{11}$$

gives $(\sigma_{22}=)\sigma_1\cdot\sigma_{21}=\alpha\sigma_{11}^2=\alpha\sigma_{22}$. Hence $\alpha=1$ and finally

$$\sigma_1^2=\sigma_{11}+\sigma_2.$$

The next result has a precise enumerative meaning. It solves Problem (1) from the Introduction. We will also solve this problem via torus localisation in Section 8.4.

Proposition 2.3.3. *We have the identity*

$$\int_{G(2,4)} \sigma_1^4 = 2.$$

Proof. We can compute

$$\begin{aligned}\sigma_1^4 &= (\sigma_1^2)^2\\ &= (\sigma_{11}+\sigma_2)^2\\ &= \sigma_{11}^2 + 2\sigma_{11}\sigma_2 + \sigma_2^2\\ &= \sigma_{22} + 0 + \sigma_{22}.\end{aligned}$$

The result follows from the fact that σ_{22} is the class of a point. □

3. Grassmann bundles, Quot, Hilb

In this section we introduce the three most important examples of *fine moduli spaces* used in Algebraic Geometry: Grassmannians, Quot schemes and Hilbert schemes. As we will see, both Grassmannians and Hilbert schemes can be recovered as special instances of Quot schemes.

The technical way to define fine moduli spaces is via representable functors $\mathfrak{M}\colon \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Sets}$. The notion of representability will be introduced in Section 3.1, for the sake of completeness. More details and examples can be found, for instance, in [74].

The basic idea is as follows. First of all, every scheme $\mathcal{M}$ trivially represents its own functor of points, which is the functor $\mathsf{h}_{\mathcal{M}}$ sending

$$U \mapsto \mathrm{Hom}_{\mathrm{Sch}}(U,\mathcal{M}).$$

One would say that $\mathcal{M}$ is a "fine moduli space of things" if the functor $\mathfrak{M}$ assigning to a scheme U the set of "families of things" defined over U is isomorphic to $\mathrm{Hom}_{\mathrm{Sch}}(-,\mathcal{M})$.

A fine moduli space is special in this sense: its points have a "label", just as the items of a phone book. We know precisely each point's name and address, so we can always find it on the moduli space (see Figure 6). This is the power of *universal families.*

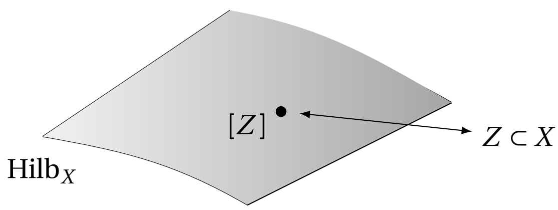


FIGURE 6. Each point of a fine moduli space (e.g. the Hilbert scheme) has a well precise label.

3.1. **Representable functors.** We start by making the following assumption.

Assumption 3.1.1. All categories are assumed to be *locally small*, i.e. we assume that $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is a set for any pair of objects x and y.

Let $\mathcal{C}$ and $\mathcal{C}'$ be (locally small) categories.

Definition 3.1.2. A (covariant) functor $\mathsf{F}\colon \mathcal{C} \to \mathcal{C}'$ is called:

(i) *fully faithful* if for any two objects $x, y \in \mathcal{C}$ the map of sets
$$\mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{C}'}(\mathsf{F}(x), \mathsf{F}(y))$$
is a bijection.

(ii) *essentially surjective* if every object of $\mathcal{C}'$ is isomorphic to an object of the form $\mathsf{F}(x)$ for some $x \in \mathcal{C}$.

The following observation is quite useful.

Remark 3.1.3. A fully faithful functor $\mathsf{F}\colon \mathcal{C} \to \mathcal{C}'$ induces an equivalence of $\mathcal{C}$ with the essential image of F, namely the full subcategory of $\mathcal{C}'$ consisting of objects isomorphic to objects of the form $\mathsf{F}(x)$ for some $x \in \mathcal{C}$. Put differently, a functor is an equivalence if and only if it is fully faithful and essentially surjective.

Definition 3.1.4. A *natural transformation* $\eta\colon \mathsf{F} \Rightarrow \mathsf{G}$ between two functors $\mathsf{F}, \mathsf{G}\colon \mathcal{C} \to \mathcal{C}'$ is the datum, for every $x \in \mathcal{C}$, of a morphism $\eta_x\colon \mathsf{F}(x) \to \mathsf{G}(x)$ in $\mathcal{C}'$, such that for every $f \in \mathrm{Hom}_{\mathcal{C}}(x_1, x_2)$ the diagram
$$\begin{tikzcd}
\mathsf{F}(x_1) \arrow[r, "\eta_{x_1}"] \arrow[d, "\mathsf{F}(f)"'] & \mathsf{G}(x_1) \arrow[d, "\mathsf{G}(f)"] \\
\mathsf{F}(x_2) \arrow[r, "\eta_{x_2}"] & \mathsf{G}(x_2)
\end{tikzcd}$$
is commutative in $\mathcal{C}'$.

Definition 3.1.5. Let $\mathcal{C}, \mathcal{C}'$ be two categories. Let $\mathrm{Fun}(\mathcal{C}, \mathcal{C}')$ be the category whose objects are functors $\mathcal{C} \to \mathcal{C}'$ and whose morphisms are the natural transformations. An isomorphism in the category $\mathrm{Fun}(\mathcal{C}, \mathcal{C}')$ is called a *natural isomorphism*.

Let $\mathcal{C}$ be a (locally small) category. Its *opposite category* $\mathcal{C}^{\mathrm{op}}$, by definition, has the same objects of $\mathcal{C}$, and its morphisms are
$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(x, y) = \mathrm{Hom}_{\mathcal{C}}(y, x), \quad x, y \in \mathcal{C}.$$

Consider the category of contravariant functors $\mathcal{C} \to \mathrm{Sets}$, i.e. the category
$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}).$$
For every object x of $\mathcal{C}$ there is a functor $\mathsf{h}_x\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$ defined by
$$u \mapsto \mathsf{h}_x(u) = \mathrm{Hom}_{\mathcal{C}}(u, x), \quad u \in \mathcal{C}.$$
A morphism $\phi \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(u, v) = \mathrm{Hom}_{\mathcal{C}}(v, u)$ gets sent to the map of sets
$$\mathsf{h}_x(\phi)\colon \mathsf{h}_x(u) \to \mathsf{h}_x(v), \quad \alpha \mapsto \alpha \circ \phi.$$

Consider the functor

$$\mathsf{h}_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}), \quad x \mapsto \mathsf{h}_x. \tag{3.1.1}$$

This is, indeed, a functor: for every arrow $f\colon x \to y$ in $\mathcal{C}$ and object u of $\mathcal{C}$ we can define a map of sets

$$\mathsf{h}_f\, u\colon \mathsf{h}_x(u) \to \mathsf{h}_y(u), \quad \alpha \mapsto f \circ \alpha,$$

with the property that for every morphism $\phi\colon v \to u$ in $\mathcal{C}$ there is a commutative diagram

$$\begin{tikzcd}
\mathsf{h}_x(u) \arrow[r, "\mathsf{h}_f u"] \arrow[d, "\mathsf{h}_x(\phi)"'] & \mathsf{h}_y(u) \arrow[d, "\mathsf{h}_y(\phi)"] & u \xrightarrow{\alpha} x \arrow[r, mapsto] \arrow[d, mapsto] & u \xrightarrow{f\circ\alpha} y \arrow[d, mapsto] \\
\mathsf{h}_x(v) \arrow[r, "\mathsf{h}_f v"] & \mathsf{h}_y(v) & v \xrightarrow{\alpha\circ\phi} x \arrow[r, mapsto] & u \xrightarrow{f\circ\alpha\circ\phi} y
\end{tikzcd}$$

defining a natural transformation

$$\mathsf{h}_f\colon \mathsf{h}_x \Rightarrow \mathsf{h}_y.$$

Lemma 3.1.6 (Weak Yoneda)**.** *The functor* $\mathsf{h}_{\mathcal{C}}$ *defined in* (3.1.1) *is fully faithful.*

Definition 3.1.7. A functor $\mathsf{F} \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets})$ is *representable* if it lies in the essential image of $\mathsf{h}_{\mathcal{C}}$, i.e. if it is isomorphic to a functor h_x for some $x \in \mathcal{C}$. In this case, we say that the object $x \in \mathcal{C}$ represents F.

Remark 3.1.8. By Lemma 3.1.6, if $x \in \mathcal{C}$ represents F, then x is unique up to a unique isomorphism. Indeed, suppose we have isomorphisms

$$a\colon \mathsf{h}_x \xrightarrow{\sim} F, \quad b\colon \mathsf{h}_y \xrightarrow{\sim} F$$

in the category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets})$. Then there exists a unique isomorphism $x \xrightarrow{\sim} y$ inducing $b^{-1} \circ a\colon \mathsf{h}_x \xrightarrow{\sim} \mathsf{h}_y$.

Let $\mathsf{F} \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets})$ be a functor, $x \in \mathcal{C}$ an object. One can construct a map of sets

$$g_x\colon \mathrm{Hom}(\mathsf{h}_x, \mathsf{F}) \to \mathsf{F}(x), \tag{3.1.2}$$

where the source is the hom-set in the category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets})$, which is indeed a set by Assumption 3.1.1.

To a natural transformation $\eta\colon \mathsf{h}_x \Rightarrow \mathsf{F}$ one can associate the element

$$g_x(\eta) = \eta_x(\mathrm{id}_x) \in \mathsf{F}(x),$$

the image of $\mathrm{id}_x \in \mathsf{h}_x(x)$ via the map $\eta_x\colon \mathsf{h}_x(x) \to \mathsf{F}(x)$.

Lemma 3.1.9 (Strong Yoneda)**.** *Let* $\mathsf{F} \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets})$ *be a functor,* $x \in \mathcal{C}$ *an object. Then the map* g_x *defined in* (3.1.2) *is bijective.*

Proof. The inverse of g_x is the map that assigns to an element $\xi \in \mathsf{F}(x)$ the natural transformation $\eta(x,\xi)\colon \mathsf{h}_x \Rightarrow \mathsf{F}$ defined as follows. For a given object $u \in \mathcal{C}$, we define

$$\eta(x,\xi)_u\colon \mathsf{h}_x(u) \to \mathsf{F}(u)$$

by sending a morphism $f\colon u \to x$ to the image of ξ under $\mathsf{F}(f)\colon \mathsf{F}(x) \to \mathsf{F}(u)$. □

EXERCISE 3.1.10. Show that Lemma 3.1.9 implies Lemma 3.1.6.

Definition 3.1.11. Let $\mathsf{F}\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$ be a functor. A *universal object* for F is a pair (x,ξ) where $\xi \in \mathsf{F}(x)$, such that for every pair (u,σ) with $\sigma \in \mathsf{F}(u)$, there exists a unique morphism $\alpha\colon u \to x$ such that $\mathsf{F}(\alpha)\colon \mathsf{F}(x) \to \mathsf{F}(u)$ sends ξ to σ.

EXERCISE 3.1.12. Show that a pair (x,ξ) is a universal object for a functor F if and only if the natural transformation $\eta(x,\xi)$ defined in Lemma 3.1.9 is a natural isomorphism. In particular, F is representable if and only if it has a universal object.

3.2. **Grassmannians.** Fix integers $0 < k \le n$, a Noetherian scheme S and a coherent sheaf F on S. Let Sch_S be the category of locally Noetherian schemes over S. The *Grassmann functor*

$$\mathsf{G}(k,F)\colon \mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{Sets}$$

is defined by

$$(U \xrightarrow{g} S) \mapsto \left\{ \begin{array}{c} \text{equivalence classes of surjections } g^*F \twoheadrightarrow Q \\ \text{in } \mathrm{Coh}(U) \text{ with } Q \text{ locally free of rank } n-k \end{array} \right\} \tag{3.2.1}$$

where two quotients $p\colon g^*F \twoheadrightarrow Q$ and $p'\colon g^*F \twoheadrightarrow Q'$ are considered equivalent if there exists an $\mathscr{O}_U$-linear isomorphism $v\colon Q \xrightarrow{\sim} Q'$ such that $p' = v \circ p$.

Remark 3.2.1. When F is locally free, $G(k,F)$ is called the *Grassmann bundle* associated to F. Note that the kernel of a surjection between locally free sheaves is automatically locally free. Hence in this case $G(k,F)$ parameterises k-dimensional linear subspaces in the fibres of $F \to S$.

EXERCISE 3.2.2. Show that two quotients $p\colon g^*F \twoheadrightarrow Q$ and $p'\colon g^*F \twoheadrightarrow Q'$ are equivalent if and only if $\ker p = \ker p'$.

The functor (3.2.1) can be represented by an S-scheme

$$\rho\colon G(k,F) \to S.$$

The proof is an application of the general result that a Zariski sheaf that can be covered by representable subfunctors is representable [69, Tag 01JF, Lemma 25.15.4].

Example 3.2.3. Let $k = n-1$ and $F = \mathscr{O}_S^{\oplus n}$. Then we get the relative projective space

$$\mathbb{P}_S^{n-1} \to S.$$

We do know from the functorial description of projective space [36, Ch. II, Thm. 7.1] that an S-morphism $U \to \mathbb{P}_S^{n-1}$ is equivalent to the data

$$(\mathscr{L}; s_0, s_1, \ldots, s_{n-1})$$

where $\mathscr{L}$ is a line bundle on U and s_i are sections generating $\mathscr{L}$ — and moreover such tuple is considered equivalent to $(\mathscr{L}'; s_0', s_1', \ldots, s_{n-1}')$ if and only if there is an isomorphism of line bundles $\phi\colon \mathscr{L} \xrightarrow{\sim} \mathscr{L}'$ such that $\phi^* s_i' = s_i$. But this is precisely a U-valued point of $\mathsf{G}(n-1, \mathscr{O}_S^{\oplus n})$. Indeed, the functor prescribes the assignment of a surjection

$$\mathscr{O}_U^{\oplus n} \twoheadrightarrow \mathscr{L}$$

with $\mathscr{L}$ a line bundle. The equivalence class of this surjection is the same data as n generating sections of $\mathscr{L}$ up to isomorphism.

Example 3.2.4. If $S = \operatorname{Spec}\mathbb{C}$, we recover the usual Grassmannian

$$G(k,n) = G(k,\mathbb{C}^n) = \mathbb{G}(k-1,n-1)$$

of k-planes in $\mathbb{C}^n$ (or, equivalently, of projective linear subspaces $\mathbb{P}^{k-1} \hookrightarrow \mathbb{P}^{n-1}$), a smooth projective algebraic variety of dimension $k(n-k)$. When $k = n-1$ we obtain $G(n-1,n) = \mathbb{P}^{n-1}$.

By definition, representability of $\mathsf{G}(k,F)$ means that for every $g\colon U \to S$ there is a functorial bijection

$$\mathsf{G}(k,F)(g) \xrightarrow{\sim} \operatorname{Hom}_S(U, G(k,F)), \quad \alpha \mapsto \alpha_g. \tag{3.2.2}$$

Now take $U = G(k,F)$, $g = \rho$, and consider

$$\operatorname{id}_{G(k,F)} \in \operatorname{Hom}_S(G(k,F), G(k,F)).$$

The element in $\mathsf{G}(k,F)(\rho)$ mapping to $\operatorname{id}_{G(k,F)}$ via (3.2.2) is the *tautological exact sequence*

$$0 \to \mathscr{S} \to \rho^*F \to \mathscr{Q} \to 0 \tag{3.2.3}$$

over $G(k,F)$. Note that if F is locally free then $\mathscr{S}$ is locally free of rank k. The sequence (3.2.3) is called 'tautological' because of the following universal property: if $g\colon U \to S$ is any morphism and $\alpha \in \mathsf{G}(k,F)(g)$, then the equivalence class of the pullback surjection

$$\alpha_g^*\rho^*F \twoheadrightarrow \alpha_g^*\mathscr{Q}$$

coincides with α.

Example 3.2.5. Let $F = \mathscr{O}_S^{\oplus n}$ be a free sheaf of rank n, and set $k = n-1$. Then we saw that

$$G(n-1,F) = \mathbb{P}_S^{n-1} = \operatorname{Proj}\operatorname{Sym}\mathscr{O}_S^{\oplus n},$$

and the tautological surjection is the familiar

$$\mathscr{O}_{\mathbb{P}_S^{n-1}}^{\oplus n} \twoheadrightarrow \mathscr{O}_{\mathbb{P}_S^{n-1}}(1).$$

EXERCISE 3.2.6. Let $S = \operatorname{Spec}\mathbb{C}$, and fix a point $[\Lambda] \in G(k,F)$. Show that the tangent space of $G(k,F)$ at $[\Lambda]$ is isomorphic to

$$\operatorname{Hom}_{\mathbb{C}}(\Lambda, F/\Lambda).$$

On $S = \operatorname{Spec}\mathbb{Z}$, the Grassmann bundle

$$\rho\colon G(k,\mathscr{O}_S^{\oplus n}) \to S$$

is *proper*. Moreover, there is a closed embedding

$$G(k,\mathscr{O}_S^{\oplus n}) \hookrightarrow \mathbb{P}_{\mathbb{Z}}^{N-1}, \quad N = \binom{n}{k}.$$

For general (Noetherian) scheme S and locally free sheaf F, the determinant

$$\mathscr{L} = \det\mathscr{Q}$$

of the universal quotient bundle is relatively very ample on $\rho\colon G(k,F)\to S$, so it gives a closed embedding

$$G(k,F)\hookrightarrow \mathbb{P}(\rho_*\mathscr{L})\hookrightarrow \mathbb{P}\left(\bigwedge^k F\right),$$

called the *Plücker embedding*.

3.3. **Quot and Hilbert schemes.** Let S be a Noetherian scheme and let $X\to S$ be a finite type morphism (so X is Noetherian by Exercise 1.1.1). Fix a coherent sheaf F on X. Denote by Sch_S the category of locally Noetherian schemes over S. Given such a scheme $U\to S$, define

$$\mathsf{Quot}_{X/S}(F)(U\to S)$$

to be the set of equivalence classes of pairs

$$(\mathscr{E},p)$$

where

- $\mathscr{E}$ is a coherent sheaf on $X\times_S U$, flat over U and with proper support over U,
- $p\colon F_U\twoheadrightarrow\mathscr{E}$ is an $\mathscr{O}_{X\times_S U}$-linear surjection, where F_U is the pullback of F along $X\times_S U\to X$, and finally
- two pairs $(\mathscr{E},p)$ and $(\mathscr{E}',p')$ are considered equivalent of $\ker\theta=\ker\theta'$.

EXERCISE 3.3.1. Show that $\mathsf{Quot}_{X/S}(F)$ defines a functor $\mathrm{Sch}_S^{\mathrm{op}}\to\mathrm{Sets}$, and that it generalises the Grassmann functor $\mathsf{G}(k,F)$ defined in (3.2.1).

Let k be a field. Fix a line bundle L over a k-scheme X. For a coherent sheaf E on X whose support is proper over k, the function

$$m\mapsto\chi(E\otimes_{\mathscr{O}_X}L^{\otimes m})$$

becomes polynomial for $m\gg 0$. It is called the *Hilbert polynomial* of E (with respect to L), and is denoted $P_L(E)$. If $\mathscr{E}$ is a flat family of coherent sheaves on $X\to S$, such that

$$\operatorname{Supp}\mathscr{E}\subset X\to S$$

is proper, then the function

$$s\mapsto P_L(\mathscr{E}_s)$$

is locally constant on S.

EXERCISE 3.3.2. Let $C\subset\mathbb{P}^n$ be a smooth curve of degree d and genus g. Compute the Hilbert polynomial of C with respect to $L=\mathscr{O}_{\mathbb{P}^n}(1)$.

Remark 3.3.3. It is not true that for fixed n there always exists a smooth curve $C\subset\mathbb{P}^n$ of degree d and genus g.

EXERCISE 3.3.4. What is the Hilbert polynomial of a conic in $\mathbb{P}^3$? What about a twisted cubic $C\subset\mathbb{P}^3$?

EXERCISE 3.3.5. Compute the Hilbert polynomial $P_{d,n}$ of a degree d hypersurface $Y \subset \mathbb{P}^n$. Show that there is a bijective morphism

$$\mathbb{P}^{N-1} \to \mathrm{Hilb}_{\mathbb{P}^n}^{P_{d,n},\mathscr{O}(1)}, \quad N = \binom{n+d}{d}.$$

EXERCISE 3.3.6. Interpret the Grassmannian

$$\mathbb{G}(k,n) = \left\{ \text{linear subvarieties } \mathbb{P}^k \hookrightarrow \mathbb{P}^n \right\}$$

as a Hilbert scheme, i.e. find the unique polynomial P such that $\mathbb{G}(k,n) = \mathrm{Hilb}_{\mathbb{P}^n}^{P,\mathscr{O}(1)}$.

The functor $\mathsf{Quot}_{X/S}(F)$ decomposes as a coproduct

$$\mathsf{Quot}_{X/S}(F) = \coprod_{P \in \mathbb{Q}[z]} \mathsf{Quot}_{X/S}^{P,L}(F)$$

where the component $\mathsf{Quot}_{X/S}^{P,L}(F)$ sends an S-scheme U to the set of equivalence classes of quotients $p\colon F_U \twoheadrightarrow \mathscr{E}$ such that for each $u \in U$ the Hilbert polynomial of $\mathscr{E}_u = \mathscr{E}|_{X_u}$ (whose support is a closed subscheme of X_u proper over $k(u)$ by definition!), calculated with respect to L_u (the pullback of L along $X_u \hookrightarrow X \times_S U \to X$), is equal to P.

Theorem 3.3.7 (Grothendieck [35])**.** *If $X \to S$ is projective, F is a coherent sheaf on X, L is a relatively very ample line bundle over X and $P \in \mathbb{Q}[z]$ is a polynomial, then the functor $\mathsf{Quot}_{X/S}^{P,L}(F)$ is representable by a projective S-scheme*

$$\mathrm{Quot}_{X/S}^{P,L}(F) \to S.$$

Remark 3.3.8. There are several notions of projectivity for a morphism $X \to S$. If S has an ample line bundle (e.g. when it is quasi-projective over an affine scheme), then these notions are all equivalent, see [22, Part 2, § 5.5.1]. Grothendieck's original definition [34, Def. 5.5.2], in general different from the one in [36, II, § 4], stated that $X \to S$ is *projective* if it factors as

$$X \xhookrightarrow{i} \mathbb{P}(E) \to S$$

where E is a coherent $\mathscr{O}_S$-module and i is a closed immersion. This can be rephrased by saying that $X \to S$ is proper and there exists an ample family of line bundles on X over S. This is the notion used in Theorem 3.3.7. Moreover, $X \to S$ is called *quasi-projective* if it factors as $X \hookrightarrow Y \to S$, with $X \hookrightarrow Y$ open and $Y \to S$ projective.

Remark 3.3.9. The Noetherian hypothesis in Theorem 3.3.7 could be removed by Altman and Kleiman [2], but they needed a stronger notion of (quasi-)projectivity, as well as a stronger assumption on F. The result is a (quasi-)projective S-scheme $\mathrm{Quot}_{X/S}^{P,L}(F) \to S$. As a consequence, one obtains the following: when $X \hookrightarrow \mathbb{P}_S^n$ is a closed subscheme, $L = \mathscr{O}_{\mathbb{P}_S^n}(1)|_X$ and F is a sheaf quotient of $L(m)^{\oplus \ell}$, the functor $\mathsf{Quot}_{X/S}^{P,L}(F)$ is representable by a scheme that can be embedded in $\mathbb{P}_S^N$ for some N.

Definition 3.3.10. Let $X \to S$ be a projective morphism, and set $F = \mathscr{O}_X$. Then

$$\mathrm{Hilb}_{X/S} = \mathrm{Quot}_{X/S}(\mathscr{O}_X)$$

is called the *Hilbert scheme* of X/S. When $S = \operatorname{Spec} k$, we omit it from the notation.

Definition 3.3.11. Let X be a quasi-projective k-scheme, and let n be an integer. The *Hilbert scheme of n points* on X is the component

$$\operatorname{Hilb}^n X \subset \operatorname{Hilb}_X$$

corresponding to the constant Hilbert polynomial $P = n$. Similarly, we let

$$\operatorname{Quot}_X(F, n) \subset \operatorname{Quot}_X(F)$$

denote the connected component parameterising quotients $F \twoheadrightarrow Q$ where Q is a finite dimensional sheaf of length n.

See Sections 5 and 9 for more information on $\operatorname{Hilb}^n X$. We will give an alternative definition of $\operatorname{Hilb}^n \mathbb{A}^d$ in Section 5.2.

A theorem of Vakil [72] asserts, roughly speaking, that arbitrarily bad singularities appear generically on some components of some Hilbert scheme.

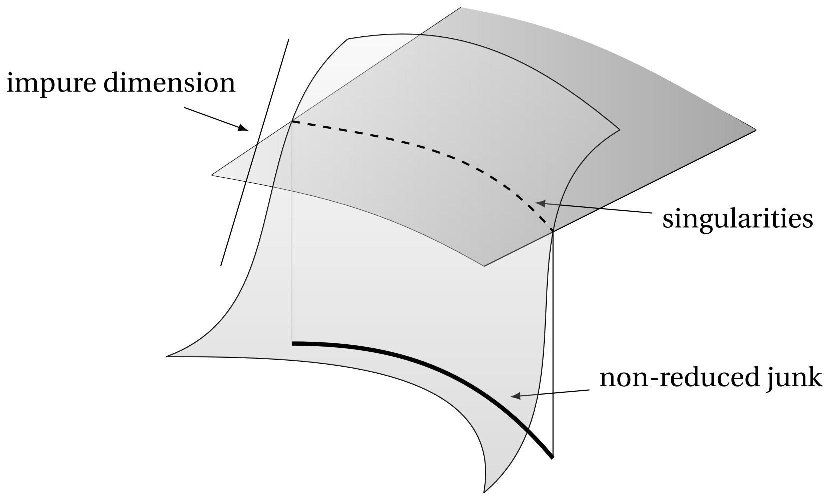


FIGURE 7. A nasty scheme. By Murphy's Law [72], it could be a Hilbert scheme component $H \subset \operatorname{Hilb}_X$ for some variety X.

However, despite its potentially horrible singularities, the Hilbert scheme has the great feature of representing a pretty explicit functor, so its functor of points is not that mysterious. In such a situation, the most important thing is to always keep in mind the *universal family* living over the representing scheme. In the case of the Hilbert scheme, this is a diagram

$$\begin{tikzcd} \mathcal{Z} \arrow[r, hook] \arrow[d, "\text{flat}"'] & X \times_S \operatorname{Hilb}_{X/S} \\ \operatorname{Hilb}_{X/S} & \end{tikzcd}$$

with the following property: for every S-scheme $g \colon U \to S$ along with a flat family of closed subschemes

$$\alpha \colon Z \subset X \times_S U \to U,$$

there exists precisely one S-morphism $\alpha_g \colon U \to \operatorname{Hilb}_{X/S}$ such that $Z = \alpha_g^* \mathcal{Z}$ as U-families of subschemes of X.

EXERCISE 3.3.12. Show that $\mathrm{Hilb}^1 X = X$. What is the universal family?

EXERCISE 3.3.13. Let C be a smooth curve embedded in a smooth 3-fold Y. Show that $\mathrm{Bl}_C Y \cong \mathrm{Quot}_Y(\mathscr{I}_C, 1)$.

3.4. **Tangent space to Quot.** Let X be a smooth projective variety over a field k. Let F be a coherent sheaf on Y. The Quot scheme

$$\mathrm{Quot}_X(F),$$

at a point $[F \twoheadrightarrow Q]$, has tangent space canonically isomorphic to

$$\mathrm{Hom}_{\mathscr{O}_X}(K, Q), \tag{3.4.1}$$

where $K = \ker(F \twoheadrightarrow Q)$. We already know this for the Grassmannian $G(k,n)$ by Exercise 3.2.6.

The case of the Hilbert scheme (i.e. when $F = \mathscr{O}_X$) is as follows. Let $p \in \mathrm{Hilb}_X$ be the point corresponding to a subscheme $Z \subset X$. Then, by definition,

$$T_p \mathrm{Hilb}_X = \mathrm{Hom}_p(\operatorname{Spec} k[t]/t^2, \mathrm{Hilb}_X),$$

and this is the set of all flat families

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Z} \arrow[r, hook] \arrow[d, "q"] & X \times_k D_2 \\ 0 \arrow[r, hook] & D_2 & \end{tikzcd}$$

such that the fibre of q over the closed point of $D_2 = \operatorname{Spec} k[t]/t^2$ equals Z. By definition, these are the *infinitesimal deformations* of the closed subscheme $Z \subset X$. It is shown in [37, Thm. 2.4] that these are classified by

$$\begin{aligned} \mathrm{Hom}_X(\mathscr{I}_Z, \mathscr{O}_Z) &= \mathrm{Hom}_Z(\mathscr{I}_Z/\mathscr{I}_Z^2, \mathscr{O}_Z) \\ &= H^0(Z, \mathscr{H}om(\mathscr{I}_Z/\mathscr{I}_Z^2, \mathscr{O}_Z)) \\ &= H^0(Z, N_{Z/X}), \end{aligned}$$

where $N_{Z/X}$ is the normal sheaf to Z in X.

EXERCISE 3.4.1. Show that $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^r_{\mathbb{A}^3}, 1)$ is smooth of dimension $r+2$. Show that $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^r_{\mathbb{A}^3}, r)$ is singular for all $r > 1$.

EXERCISE 3.4.2. Let $L \subset \mathbb{A}^3$ be a line. Compute the dimension of $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{I}_L, 2)$. Show that this Quot scheme is singular.

3.5. **Examples of Hilbert schemes.** The Hilbert scheme of points, i.e. the Hilbert scheme of zero-dimensional subschemes of a quasi-projective variety X, will be treated in later sections.

3.5.1. *Plane conics.* Let z_0, z_1 and z_2 be homogeneous coordinates on $\mathbb{P}^2$, and $\alpha_0, \ldots, \alpha_5$ be homogeneous coordinates on $\mathbb{P}^5$. Consider the closed subscheme

$$\mathcal{C} \subset \mathbb{P}^2 \times \mathbb{P}^5$$

cut out by the equation

$$\alpha_0 z_0^2 + \alpha_1 z_1^2 + \alpha_2 z_2^2 + \alpha_3 z_0 z_1 + \alpha_4 z_0 z_2 + \alpha_5 z_1 z_2 = 0.$$

Let π be the projection $\mathcal{C} \to \mathbb{P}^5$. Over a point $a = (a_0, \ldots, a_5) \in \mathbb{P}^5$, the fibre is the conic

$$\pi^{-1}(a) = \left\{ a_0 z_0^2 + a_1 z_1^2 + a_2 z_2^2 + a_3 z_0 z_1 + a_4 z_0 z_2 + a_5 z_1 z_2 = 0 \right\} \subset \mathbb{P}^2.$$

There is a set-theoretic bijection between $\mathbb{P}^5$ and $\mathrm{Hilb}^{2t+1}_{\mathbb{P}^2}$. By the universal property of projective space, we have the scheme-theoretic identity

$$\mathbb{P}^5 = \mathrm{Hilb}^{2t+1}_{\mathbb{P}^2},$$

and the map $\pi: \mathcal{C} \to \mathbb{P}^5$ is the universal family of the Hilbert scheme of plane conics.

EXERCISE 3.5.1. Make the last step precise and generalise the plane conics example to arbitrary hypersurfaces of $\mathbb{P}^n$. (**Hint**: Start out with the conclusion of Exercise 3.3.5 to write down the universal family).

Remark 3.5.2. Let X be a projective scheme. The universal family of the Hilbert scheme is always, *set-theoretically*, equal to

$$\mathcal{Z} = \{ (x, [Z]) \in X \times \mathrm{Hilb}_X \mid x \in Z \} \subset X \times \mathrm{Hilb}_X .$$

The problem is to determine the scheme structure on $\mathcal{Z}$. In the case of hypersurfaces of degree d in $\mathbb{P}^n$ (Exercise 3.5.1) this was easy precisely because $\mathcal{Z}$ is itself a hypersurface.

3.5.2. *Twisted cubics.* A twisted cubic is a smooth rational curve obtained as the image of the morphism

$$\mathbb{P}^1 \hookrightarrow \mathbb{P}^3, \quad (u, v) \mapsto (u^3, u^2 v, u v^2, v^3),$$

up to linear changes of coordinates of the codomain. The number of moduli of a twisted cubic is 12. Indeed, one has to specify four linearly independent degree 3 polynomials in two variables, up to $\mathbb{C}^\times$-scaling and automorphisms of $\mathbb{P}^1$. One then computes

$$4 \cdot h^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(3)) - 1 - \dim \mathrm{PGL}_2 = 16 - 1 - 3 = 12.$$

The Hilbert polynomial of a twisted cubic is $3t + 1$, cf. Exercise 3.3.2. There are other 1-dimensional subschemes $Z \subset \mathbb{P}^3$ with this Hilbert polynomial, e.g. a plane cubic union a point. This has 15 moduli: the choice of a plane $\mathbb{P}^2 \subset \mathbb{P}^3$ contributes $3 = \dim \mathbb{G}(2,3)$ moduli, a plane cubic $C \subset \mathbb{P}^2$ contributes 9 parameters, and the choice of a point $p \in \mathbb{P}^3$ accounts for the remaining three moduli.

The Hilbert scheme

$$\mathrm{Hilb}^{3t+1}_{\mathbb{P}^3}$$

was completely described in [62]. The two irreducible components we just described turn out to be the only ones. They are smooth, rational, of dimension 12 and 15 respectively, and they intersect along a smooth, rational 11-dimensional subvariety $V \subset \mathrm{Hilb}^{3t+1}_{\mathbb{P}^3}$ parameterising uninodal plane cubics with an embedded point at the node. In [36, § III, Ex. 9.8.4] a family of twisted cubics degenerating to a plane uninodal cubic with an embedded point is described. The total space of the family, in a local chart, is defined by the ideal

$$I = (a^2(x+1) - z^2, a x(x+1) - y z, x z - a y, y^2 - x^2(x+1)) \subset \mathbb{C}[a, x, y, z].$$

Letting $a = 0$ one obtains the special fibre given by

$$I_0 = (z^2, yz, xz, y^2 - x^2(x+1)) \subset \mathbb{C}[x, y, z],$$

and $p = (0,0,0)$ is a non-reduced point in $C_0 = \operatorname{Spec} \mathbb{C}[x, y, z]/I_0$. Note that C_0 is not scheme-theoretically contained in the plane $z = 0$, because the local ring $\mathscr{O}_{C_0,p}$ contains the nonzero nilpotent z (cf. Remark 1.3.6).

Remark 3.5.3. The *geometric genus* $p_g(X) = h^0(X, \omega_X)$ varies in flat families, as the twisted cubic example shows.

3.6. **A comment on fine moduli spaces and automorphisms.** Given a scheme S and a functor $\mathfrak{M}\colon \mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{Sets}$, an object $\mathcal{M}$ in Sch_S along with an isomorphism

$$\mathfrak{M} \cong \mathrm{Hom}_{\mathrm{Sch}_S}(-, \mathcal{M})$$

is a *fine moduli space* for the objects parameterised by $\mathfrak{M}$. It is common to hear that when the objects $\eta \in \mathfrak{M}(U/S)$ have automorphisms, the functor $\mathfrak{M}$ cannot be represented. This is, for instance, the case for the moduli functor of smooth (or stable) curves of genus g. Even though this is the correct *geometric* intuition to have, for a general functor the presence of automorphisms does not necessarily prevent the existence of a universal family, as the following exercise indicates.

EXERCISE 3.6.1. Construct the functor $\mathfrak{M}\colon \mathrm{Sets}^{\mathrm{op}} \to \mathrm{Sets}$ of isomorphism classes of finite sets. Show that it is representable (by what set?), even though every finite set has automorphisms.

In geometric situations, the presence of automorphisms prevents representability whenever one can construct a family of objects $\eta \in \mathfrak{M}(U/S)$ that is isotrivial (i.e. it becomes the trivial family after base change by an étale cover) but not globally trivial. This is for instance the case for families of curves: the moduli map $U \to \mathcal{M}_g$ associated to an isotrivial family $\mathcal{X} \to U$ of smooth curves of genus g, say with typical fibre C, would have to be constant for continuity reasons; but the same is of course happening for the trivial family $C \times U \to U$, so the functor of smooth curves of genus g cannot be represented.

4. LINES ON HYPERSURFACES: EXPECTATIONS

Let $Y \subset \mathbb{P}^n$ be a general hypersurface of degree d. We want to show the following:

We should expect a finite number of lines on Y if and only if $d = 2n - 3$. We should expect *no lines* on Y if $d > 2n - 3$. We should expect infinitely many lines on Y if $d < 2n - 3$.

To understand the condition

$$\ell \subset Y$$

for $\ell \subset \mathbb{P}^n$ and a hypersurface $Y \subset \mathbb{P}^n$, we give the following concrete example.

Example 4.0.1. Let $\ell \subset \mathbb{P}^3$ be the line cut out by $L_1 = L_2 = 0$, where $L_i = L_i(z_0, z_1, z_2, z_3)$ are linear forms on $\mathbb{P}^3$. To fix ideas, set $L_1 = z_0$ and $L_2 = z_0 + z_2 + z_3$. Let $Y \subset \mathbb{P}^3$ be defined by a homogeneous equation $f = 0$, for instance the cubic polynomial

$$f = z_0^3 + 3z_0 z_1^2 - z_2^2 z_3.$$

Then we see that plugging $L_1 = L_2 = 0$ into f does not give zero, i.e.

$$f|_\ell = 0 + 0 - z_2^2(-z_0 - z_2) = z_2^3.$$

This means that ℓ is not contained in Y. On the other hand, the line cut out by L_1 and $L_2' = z_3$ lies entirely on Y.

Let $Y \subset \mathbb{P}^n$ be the zero locus of a general homogeneous polynomial

$$f \in H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)).$$

As we anticipated in Example 4.0.1, a line $\ell \subset \mathbb{P}^n$ is contained in Y if and only if $f|_\ell = 0$. This condition can be rephrased by saying that the image of f under the restriction map

$$\mathsf{res}_\ell \colon H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)) \to H^0(\ell, \mathscr{O}_\ell(d)) \tag{4.0.1}$$

vanishes. We want to determine when we should expect Y to contain a finite number of lines. We set, informally,

$$\mathsf{N}_1(Y) = \text{expected number of lines in } Y.$$

Let us consider the Grassmannian

$$\mathbb{G} = \mathbb{G}(1, n) = \left\{ \text{Lines } \ell \subset \mathbb{P}^n \right\},$$

a smooth complex projective variety of dimension $2n-2$. Recall the universal structures living on $\mathbb{G}$. First of all, the tautological exact sequence

$$0 \longrightarrow \underset{\text{rank } 2}{\mathscr{S}} \longrightarrow \underset{\text{rank } n+1}{\mathscr{O}_{\mathbb{G}} \otimes_{\mathbb{C}} H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1))^\vee} \longrightarrow \underset{\text{rank } n-1}{\mathscr{Q}} \longrightarrow 0$$

where the fibre of $\mathscr{S}$ over a point $[\ell] \in \mathbb{G}$ is the 2-dimensional vector space $H^0(\ell, \mathscr{O}_\ell(1))^\vee$. Let, also,

$$\mathcal{L} = \left\{ (p, [\ell]) \in \mathbb{P}^n \times \mathbb{G} \,\middle|\, p \in \ell \right\} \subset \mathbb{P}^n \times \mathbb{G}$$

be the universal line. Consider the two projections

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{q} & \mathbb{P}^n \\ {\scriptstyle \pi}\big\downarrow & & \\ \mathbb{G} & & \end{array}$$

and the coherent sheaf

$$\mathscr{E}_d = \pi_* q^* \mathscr{O}_{\mathbb{P}^n}(d).$$

EXERCISE 4.0.2. Show that $\mathscr{E}_d$ is locally free of rank $d+1$. (**Hint**: use cohomology and base change, e.g. [20, Theorem B.5]).

In fact, one has an isomorphism of locally free sheaves

$$\mathscr{E}_d \cong \operatorname{Sym}^d \mathscr{S}^\vee,$$

where $\iota\colon \mathscr{S} \hookrightarrow \mathscr{O}_{\mathbb{G}} \otimes_{\mathbb{C}} H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1))^\vee$ is the universal subbundle. Dualising ι and applying Sym^d, we obtain a surjection

$$\mathscr{O}_{\mathbb{G}} \otimes_{\mathbb{C}} H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)) \twoheadrightarrow \operatorname{Sym}^d \mathscr{S}^\vee,$$

which is just a global version of (4.0.1). The association

$$\mathbb{G} \ni [\ell] \mapsto f|_\ell \in H^0(\ell, \mathscr{O}_\ell(d)) \cong \operatorname{Sym}^d H^0(\ell, \mathscr{O}_\ell(1))$$

defines a section τ_f of $\mathscr{E}_d \to \mathbb{G}$. The zero locus of $\tau_f = \pi_* q^* f$ is the locus of lines contained in Y.

The following terminology is very common.

Definition 4.0.3. Let $Y \subset \mathbb{P}^n$ be a hypersurface defined by $f = 0$. The zero scheme

$$F_1(Y) = Z(\tau_f) \hookrightarrow \mathbb{G} = \mathbb{G}(1,n)$$

is called the *Fano scheme of lines* in Y.

Since f is generic, $\tau_f \in \Gamma(\mathbb{G}, \mathscr{E}_d)$ is also generic. In this case, the fundamental class of the Fano scheme of lines in Y is Poincaré dual to the Euler class

$$e(\mathscr{E}_d) \in A^{d+1}\mathbb{G}.$$

Thus $[F_1(Y)] \in A_*\mathbb{G}$ is a zero-cycle if and only if $d+1 = 2n-2$, i.e.

$$d = 2n-3.$$

The degree of this zero-cycle is then

$$\mathsf{N}_1(Y) = \int_{\mathbb{G}} e(\mathscr{E}_d) = \int_{\mathbb{G}} c_{d+1}(\operatorname{Sym}^d \mathscr{S}^\vee).$$

This degree is the *actual* number of lines on Y whenever $H^0(\ell, N_{\ell/Y}) = 0$ for all $\ell \subset Y$. This condition means that the Fano scheme is reduced at all its points $[\ell]$, since $H^0(\ell, N_{\ell/Y})$ is its tangent space at the point $[\ell]$.

Lemma 4.0.4. *If $S \subset \mathbb{P}^3$ is a smooth cubic surface and $\ell \subset S$ is a line, then $H^0(\ell, N_{\ell/S}) = 0$.*

Proof. It has enough to show that $N = N_{\ell/S}$, viewed as a line bundle on $\ell \cong \mathbb{P}^1$, has negative degree. By the adjunction formula,

$$K_\ell = K_S|_\ell \otimes_{\mathscr{O}_\ell} N.$$

Using that $K_\ell = \mathscr{O}_\ell(-2)$ and $K_S = K_{\mathbb{P}^3}|_S \otimes_{\mathscr{O}_S} N_{S/\mathbb{P}^3} = \mathscr{O}_S(d-4)$ for a surface of degree d in $\mathbb{P}^3$, by taking degrees we obtain

$$-2 = (3-4) + \deg N,$$

so that $\deg N = -1 < 0$. □

EXERCISE 4.0.5. Let $Y \subset \mathbb{P}^n$ be a general hypersurface of degree $d \le 2n-3$. Show that $F_i(Y) \subset \mathbb{G}(1,n)$ is smooth of dimension $2n-3-d$.

5. The Hilbert scheme of points

5.1. **Subschemes and zero-cycles.** Let X be a complex quasi-projective variety. In Section 3.3 we encountered the Hilbert scheme of points X. Recall that, as a set,

$$\mathrm{Hilb}^n X$$

parameterises the finite subschemes $Z \subset X$ of length n. Or, equivalently, ideal sheaves $\mathscr{I}_Z \subset \mathscr{O}_X$ of *colength* n.

There is a "coarser" way of parameterising *points with multiplicity* on the variety X. This is the content of the next definition.

Definition 5.1.1. Let X be a quasi-projective variety. The n-th *symmetric product* of X is the quotient

$$\mathrm{Sym}^n X = X^n / \mathfrak{S}_n.$$

Remark 5.1.2. The scheme $\mathrm{Sym}^n X$ is the *Chow scheme* of effective zero-cycles (of degree n) on X. For higher dimensional cycles, the definition (and representability) of the Chow functor is a much subtler problem [66].

Each point $\xi \in \mathrm{Sym}^n X$ corresponds to a finite combination of points with multiplicity, i.e. it can be written as

$$\xi = \sum_i m_i \cdot p_i,$$

with $m_i \in \mathbb{Z}_{\geq 0}$ and $p_i \in X$.

Exercise 5.1.3. Let X be a smooth variety of dimension d. Show that the locus in $\mathrm{Hilb}^2 X$ of nonreduced subschemes $Z \subset X$ is isomorphic to $X \times \mathbb{P}^{d-1}$.

Note that the symmetric product $\mathrm{Sym}^n X$ does not deal with with the scheme structure of fat points inside X. For instance, any of the double point schemes supported on a given point $p \in X$, parameterised by the $\mathbb{P}^{d-1}$ of Exercise 5.1.3, has underlying cycle $2 \cdot p$. The operation of "forgetting the scheme structure" can be made functorial. This means that there exists a well-defined algebraic morphism

$$\pi_X \colon \mathrm{Hilb}^n X \to \mathrm{Sym}^n X, \tag{5.1.1}$$

taking a subscheme $Z \subset X$ to its underlying effective zero-cycle. In symbols,

$$\pi_X[Z] = \sum_{p \in \mathrm{Supp}\, Z} \mathrm{length}\, \mathscr{O}_{Z,p} \cdot p.$$

The morphism π_X is called the *Hilbert–Chow morphism.*

Lemma 5.1.4. *Let X be a smooth quasi-projective variety. The Hilbert–Chow morphism π_X is proper.*

Proof. Let X be projective, to start with. The symmetric product $\mathrm{Sym}^n X$ is separated, and $\mathrm{Hilb}^n X$ is projective, hence proper. Then π_X is proper in this case. If X is only quasi-projective, choose a compactification $X \subset \overline{X}$. Then π_X is the pullback of $\pi_{\overline{X}}$ along $\mathrm{Sym}^n X \subset \mathrm{Sym}^n \overline{X}$, and is thus proper because properness is stable under base change. □

EXERCISE 5.1.5. Show that if C is a smooth quasi-projective curve then $\operatorname{Hilb}^n C \cong \operatorname{Sym}^n C$ via π_C. Deduce that $\operatorname{Hilb}^n \mathbb{A}^1 \cong \mathbb{A}^n$ and that $\operatorname{Hilb}^n \mathbb{P}^1 \cong \mathbb{P}^n$. Note that this gives a new proof of the identity $\mathsf{P}_0(n) = 1$ that you proved in Exercise 9.1.5.

The easiest subvariety of the symmetric product $\operatorname{Sym}^n X$ is the *small diagonal*, which is just a copy of X embedded as

$$X \hookrightarrow \operatorname{Sym}^n X, \quad x \mapsto n \cdot x.$$

Definition 5.1.6. Let X be a smooth quasi-projective variety. The *punctual Hilbert scheme* is the closed subscheme

$$\operatorname{Hilb}^n(X)_x \subset \operatorname{Hilb}^n X$$

defined as the preimage of the cycle $n \cdot x$ via the Hilbert–Chow map (5.1.1).

EXERCISE 5.1.7. Let X be a smooth variety. Show that that $\pi_X^{-1}(n \cdot x)$ does not depend on $x \in X$. Show that it does not depend on X either, but only on $\dim X$.

Notation 5.1.8. If X is a smooth variety of dimension d, we will denote by

$$\operatorname{Hilb}^n(\mathbb{A}^d)_0 \subset \operatorname{Hilb}^n X$$

the punctual Hilbert scheme of Definition 5.1.6. This makes sense by Exercise 5.1.7.

Remark 5.1.9. By Lemma 5.1.4, the punctual Hilbert scheme $\operatorname{Hilb}^n(\mathbb{A}^d)_0$ is proper.

EXERCISE 5.1.10. Let X be a smooth variety. Show that $\operatorname{Hilb}^2 X$ is isomorphic to the blowup of $\operatorname{Sym}^2 X$ along the diagonal.

EXERCISE 5.1.11. Show that if X is a smooth variety and $n \leq 3$ then $\operatorname{Hilb}^n X$ is smooth. (**Hint**: show that a finite planar subscheme $Z \subset X$ defines a smooth point of the Hilbert scheme. Then use your classification from Exercise 1.1.10 to conclude).

5.2. **The Hilbert scheme of points on affine space.** Let $d \geq 1$ and $n \geq 0$ be integers. In this subsection we give a description of the Hilbert scheme

$$\operatorname{Hilb}^n \mathbb{A}^d = \{ I \subset \mathbb{C}[x_1, \ldots, x_d] \mid \dim_{\mathbb{C}} \mathbb{C}[x_1, \ldots, x_d]/I = n \}.$$

More precisely, we will provide equations cutting out the Hilbert scheme inside a smooth quasi-projective variety (the so-called *non-commutative Hilbert scheme*), cf. Theorem 5.2.4.

An ideal $I \in \operatorname{Hilb}^n \mathbb{A}^d$ will be tacitly identified with the associated finite closed subscheme

$$\operatorname{Spec} \mathbb{C}[x_1, \ldots, x_d]/I \subset \mathbb{A}^d$$

of length n.

The following properties are well-known:

- $\operatorname{Hilb}^n \mathbb{A}^d$ is smooth if and only if $d \leq 2$ or $n \leq 3$,
- $\operatorname{Hilb}^n \mathbb{A}^3$ is irreducible for $n \leq 11$ (see [38, 18] and the references therein),
- $\operatorname{Hilb}^n \mathbb{A}^3$ is reducible for $n \geq 78$, see [42].

On the other hand, the following questions are open since a long time:

- What is the smallest d such that $\operatorname{Hilb}^n \mathbb{A}^3$ is reducible?

- If $d \geq 3$, is $\operatorname{Hilb}^n \mathbb{A}^d$ generically reduced?

Let us get started with our description of $\operatorname{Hilb}^n \mathbb{A}^d$. To ease notation, let us put $R_d = \mathbb{C}[x_1,\ldots,x_d]$. The condition defining a point $I \in \operatorname{Hilb}^n \mathbb{A}^d$ is that the $\mathbb{C}$-algebra quotient

$$R_d \twoheadrightarrow R_d/I$$

is a vector space of dimension n. Let us examine this condition in detail. To construct a point in the Hilbert scheme, we need:

(1) a vector space

$$V_n \cong \mathbb{C}^n,$$

(2) an R_d-module structure

$$\vartheta\colon R_d \to \operatorname{End}_{\mathbb{C}}(V_n)$$

with the property that

(3) such structure is induced by an R_d-linear surjection from R_d.

So let us fix an n-dimensional vector space V_n. Later we will have to remember that we made such a choice, and since all we wanted was "$\dim V_n = n$" we will have to quotient out all equivalent choices. Let us forget about this for the moment. In (2), we need ϑ to be a ring homomorphism, so we need to specify one endomorphism of V_n for each coordinate $x_i \in R_d$. All in all, ϑ gives us d matrices

$$A_1, A_2, \ldots, A_d \in \operatorname{End}_{\mathbb{C}}(V_n).$$

The matrix A_i will be responsible for the R_d-linear operator "multiplication by x_i" for the resulting module structure on V_n. Also in this step we should note a reminder for later: strictly speaking, what we have defined so far is a $\mathbb{C}\langle x_1, x_2, \ldots, x_d\rangle$-module structure on V_n. But in R_d the variables commute with one another. So we will have to impose the relations $[A_i, A_j] = 0$ for all $1 \leq i < j \leq d$.

Condition (3) is tricky. Let us reason backwards, assuming we already have an R_d-linear quotient $\phi\colon R_d \twoheadrightarrow V_n$. Then it is clear that the image of $1 \in R_d$ generates V_n as an R_d-module. In other words, every element $w \in V_n$ can be written as

$$w = A_1^{m_1} A_2^{m_2} \cdots A_d^{m_d} \cdot \phi(1),$$

for some $m_i \in \mathbb{Z}_{\geq 0}$. This tells us exactly what we should add to the picture to obtain Condition (3): for a fixed module structure, i.e. d-tuple of matrices $(A_1, A_2, \ldots, A_d)$, we need to specify a *cyclic vector*

$$v \in V_n,$$

i.e. a vector with the property that the $\mathbb{C}$-linear span of the set

$$\left\{ A_1^{m_1} A_2^{m_2} \cdots A_d^{m_d} \cdot v \;\middle|\; m_i \in \mathbb{Z}_{d\geq 0} \right\}$$

equals the whole V_n.

Let us consider the $(dn^2 + n)$-dimensional affine space

$$W_n = \operatorname{End}_{\mathbb{C}}(V_n)^{\oplus d} \oplus V_n. \tag{5.2.1}$$

EXERCISE 5.2.1. Show that the locus

$$U_n = \left\{ (A_1, A_2, \ldots, A_d, v) \,\middle|\, v \text{ is } (A_1, A_2, \ldots, A_d)\text{-cyclic} \right\} \subset W_n$$

is a Zariski open subset.

Consider the GL_n-action on W_n given by

$$g \cdot (A_1, A_2, \ldots, A_d, v) = (A_1^g, A_2^g, \ldots, A_d^g, gv) \tag{5.2.2}$$

where $M^g = g^{-1}Mg$ is conjugation.

Lemma 5.2.2. *The* GL_n*-action* (5.2.2) *is free on* U_n*.*

Proof. If $g \in \mathrm{GL}_n$ fixes a point $(A_1, A_2, \ldots, A_d, v) \in U_n$, then $v = gv$ lies in the invariant subspace $\ker(g - \mathrm{id}) \subset V_n$. But by definition of U_n, the smallest invariant subspace containing v is V_n itself, thus $g = \mathrm{id}$. □

Definition 5.2.3. The GIT quotient

$$\mathrm{NCHilb}_d^n = U_n / \mathrm{GL}_n$$

is called the *non-commutative Hilbert scheme.*

The discussion carried out so far proves the following:

Theorem 5.2.4. *There is a closed immersion*

$$\mathrm{Hilb}^n \mathbb{A}^d \subset \mathrm{NCHilb}_d^n$$

cut out by the ideal of relations

$$[A_i, A_j] = 0 \text{ for all } 1 \le i < j \le d. \tag{5.2.3}$$

EXERCISE 5.2.5. Let $d = 1$. Show that $\mathrm{NCHilb}_1^n = \mathbb{A}^n$.

Example 5.2.6. If $d = 1$ there is just one operator "x" so the Relations (5.2.3) are vacuous. We have

$$\mathrm{Hilb}^n \mathbb{A}^1 = \mathrm{NCHilb}_1^n = \mathbb{A}^n,$$

which thanks to Exercise 5.2.5 reproves the second part of Exercise 5.1.5.

Remark 5.2.7. If $d = 2$ the description of $\mathrm{Hilb}^n \mathbb{A}^2$ is equivalent to Nakajima's description [55,]. See also [38] for another description of the Hilbert scheme of points, in terms of *perfect extended monads.*

5.3. **Hilbert–Chow revisited.** Fix $n \ge 0$ and $d > 0$. The Hilbert–Chow morphism

$$\pi \colon \mathrm{Hilb}^n \mathbb{A}^d \to \mathrm{Sym}^n \mathbb{A}^d$$

introduced in (5.1.1) can be reinterpreted as follows. Pick a point

$$[A_1, \ldots, A_d, v] \in \mathrm{Hilb}^n \mathbb{A}^d$$

and notice that since the matrices pairwise commute, they can be simultaneously triangularised. So, since the tuple is defined up to GL_n, we may assume they are in the form

$$A_\ell = \begin{pmatrix} a_{11}^{(\ell)} & * & * & \cdots & * \\ 0 & a_{22}^{(\ell)} & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn}^{(\ell)}. \end{pmatrix}$$

Then π is given by

$$[A_1,\ldots,A_d,v] \mapsto \sum_\ell \left(a_\ell^{(1)},\ldots,a_\ell^{(d)}\right).$$

When all the matrices are *nilpotent*, the corresponding subscheme $Z \subset \mathbb{A}^d$ is entirely supported at the origin. In other words,

$$\mathrm{Hilb}^n(\mathbb{A}^d)_0 = \{\,[A_1,\ldots,A_d,v] \mid A_1,\ldots,A_d \text{ are nilpotent}\,\}$$

is a way to describe the punctual Hilbert scheme.

5.4. **Varieties of commuting matrices: what's known.** Let V be an n-dimensional complex vector space and let

$$C_n = \left\{ (A,B) \in \mathrm{End}(V)^2 \,\middle|\, [A,B]=0 \right\} \subset \mathrm{End}(V)^2 \tag{5.4.1}$$

be the *commuting variety*. Letting GL_n act on C_n by simultaneous conjugation, one can form the quotient stack

$$\mathcal{C}(n) = C_n / \mathrm{GL}_n,$$

which is equivalent to the stack $\mathrm{Coh}_n(\mathbb{A}^2)$ of finite coherent sheaves of length n on the affine plane. Letting

$$\widetilde{c}_n = \left[\mathcal{C}(n)\right] = \frac{\left[C_n\right]}{\mathsf{GL}_n} \in K_0(\mathrm{St}_{\mathbb{C}}) \tag{5.4.2}$$

be the motivic class of the stack $\mathcal{C}(n)$, let us form the generating series

$$\mathsf{C}(t) = \sum_{n\geq 0} \widetilde{c}_n t^n \in K_0(\mathrm{St}_{\mathbb{C}})[\![t]\!].$$

The next result is a formula essentially due to Feit and Fine, but also proven recently by Behrend–Bryan–Szendrői and Bryan–Morrison.

Theorem 5.4.1 ([23, 7, 14])**.** *There is an identity*

$$\mathsf{C}(t) = \prod_{k\geq 1}\prod_{m\geq 1}(1-\mathbb{L}^{2-k}t^m)^{-1}.$$

It has been known since a long time that the variety of pairs of commuting matrices C_n is irreducible [53, 64]. The same is true for the space $N_n \subset C_n$ of *nilpotent* commuting linear operators, see [5] for a proof in characteristic zero and [6] for an extension to fields of characteristic bigger than $n/2$. Premet even showed irreducibility of N_n over *any* field [63].

However the situation is very different for 3 or more matrices. Let $C(d,n)$ be the space of d-tuples of pairwise commuting endomorphisms of an n-dimensional vector space, and let $N(d,n)$ be the space of nilpotent endomorphisms. Then $C(d,n)$ is irreducible for all n if $d \leq 3$. But it is reducible if d and n are both at least 4 [29], and the same is true for $N(d,n)$ [56]. For $d=3$ the situation is as follows. One has that $C(3,n)$ is reducible for $n \geq 30$ [39] and irreducible for $n \leq 10$ (in characteristic zero).

Moreover, $N(3,n)$ is known to be irreducible for $n \leq 6$ [56], but $N(3,n)$ is reducible for $n \geq 13$ [59, Thm. 7.10.5].

5.5. **The special case of** $\operatorname{Hilb}^n \mathbb{A}^3$**.** In this subsection we set $d = 3$. The Hilbert scheme of points

$$\operatorname{Hilb}^n \mathbb{A}^3$$

is singular as soon as $n \geq 4$, but as we shall see it is *virtually smooth*, i.e. it carries a (symmetric) perfect obstruction theory.

EXERCISE 5.5.1. Let $p \in \mathbb{A}^3$ be a point. Show that $\mathfrak{m}_p^2 \subset \mathscr{O}_{\mathbb{A}^3}$ defines a singular point of $\operatorname{Hilb}^4 \mathbb{A}^3$.

Theorem 5.5.2. *There exists a smooth quasi-projective variety M_n along with a regular function $f_n \colon M_n \to \mathbb{A}^1$ such that*

$$\operatorname{Hilb}^n \mathbb{A}^3 \subset M_n$$

can be realised as the scheme-theoretic zero locus of the exact 1-form $\mathrm{d} f_n$.

Proof. As M_n we can take the noncommutative Hilbert scheme $\operatorname{NCHilb}_3^n$. By [67, Prop. 3.8], the commutator relations (5.2.3) agree *scheme-theoretically* with the single vanishing relation

$$\mathrm{d} f_n = 0,$$

where $f_n \colon \operatorname{NCHilb}_3^n \to \mathbb{A}^1$ is the function

$$[A_1, A_2, A_3, v] \mapsto \operatorname{Tr} A_1 [A_2, A_3]. \qquad \square$$

The above description is special to $d = 3$.

EXERCISE 5.5.3. Show that, for $\{i, j, k\} = \{1, 2, 3\}$, one has

$$[A_i, A_j] = 0 \quad \Longleftrightarrow \quad \frac{\partial f_n}{\partial A_k} = 0,$$

at least set-theoretically.

By the construction of $[Z(\mathrm{d} f)]^{\mathrm{vir}}$ outlined in Section 10, we deduce from Theorem 5.5.2 the following:

Corollary 5.5.4. *The Hilbert scheme $\operatorname{Hilb}^n \mathbb{A}^3$ carries a symmetric perfect obstruction theory, giving rise to a zero-dimensional virtual fundamental class*

$$\left[\operatorname{Hilb}^n \mathbb{A}^3\right]^{\mathrm{vir}} \in A_0(\operatorname{Hilb}^n \mathbb{A}^3).$$

5.5.1. *A quiver description.* Recall that a *quiver* is a finite directed graph. If $Q = (Q_0, Q_1, s, t)$ is a quiver, the notation means Q_0 is the set of vertices, Q_1 is the set of arrows and $s, t \colon Q_1 \to Q_0$ are the source and tail maps respectively. Let $\mathbf{d} = (\mathbf{d}_i) \in \mathbb{N}^{Q_0}$ be a vector of nonnegative integers. Then a $\mathbf{d}$-dimensional *representation* ρ of Q is the datum of a $\mathbb{C}$-vector space of dimension $\mathbf{d}_i$ for each $i \in Q_0$, along with a linear map $\mathbb{C}^{\mathbf{d}_i} \to \mathbb{C}^{\mathbf{d}_j}$ for every arrow $i \to j$ in Q_1. We write $\underline{\dim}\, \rho = \mathbf{d}$. The space of such representations is the affine space

$$\operatorname{Rep}_{\mathbf{d}}(Q) = \prod_{a \in Q_1} \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}^{\mathbf{d}_{t(a)}}, \mathbb{C}^{\mathbf{d}_{s(a)}})$$

Definition 5.5.5. Let $n \geq 0$ be an integer. A *path of length n* in a quiver Q is a sequence of arrows $a_n \cdots a_2 a_1$ such that $s(a_{i+1}) = t(a_i)$ for all i. The notation is by juxtaposition from right to left. The *path algebra* $\mathbb{C}Q$ of a quiver Q is defined as follows. As a $\mathbb{C}$-vector space, it is spanned by all paths of length $n \geq 0$, including a single trivial path e_i of length 0 for each vertex $i \in Q_0$. The product is given by concatenation (juxtaposition) of paths, and is defined to be 0 if two paths cannot be concatenated.

EXERCISE 5.5.6. Prove that $\mathbb{C}Q$ is an associative algebra.

EXERCISE 5.5.7. Prove that representations of a quiver Q form a category (i.e. define a sensible notion of morphisms of representations). Show that this category is equivalent to the category of left $\mathbb{C}Q$-modules, in particular it is abelian.

EXERCISE 5.5.8. Show that the path algebra of the quiver L_d with one vertex and d loops is isomorphic to $\mathbb{C}\langle x_1, x_2, \ldots, x_d\rangle$.

Let Q be a quiver. Let

$$\mathbb{H}_+ = \left\{ r \cdot \exp(i\pi\phi) \in \mathbb{C} \,\middle|\, r \in \mathbb{R}_{>0},\, \phi \in (0,1] \right\}$$

be the upper half plane with the positive real axis removed. A *central charge* is a group homomorphism $Z \colon \mathbb{Z}^{Q_0} \to \mathbb{C}$ mapping $\mathbb{N}^{Q_0} \setminus 0$ inside $\mathbb{H}_+$. For every $\alpha \in \mathbb{Z}^{Q_0}$ we let $\phi(\alpha)$ be the unique real number such that $Z(\alpha) = r \cdot \exp(i\pi\phi(\alpha))$. It is called the *phase* of α (with respect to Z). Every vector $\theta \in \mathbb{Q}^{Q_0}$ induces a central charge Z_θ via

$$Z_\theta(\alpha) = -\theta \cdot \alpha + i|\alpha|,$$

where $|\alpha| = \sum_{i \in Q_0} \alpha_i$. We let ϕ_θ denote the associated phase function, and we set

$$\phi_\theta(\rho) = \phi_\theta(\underline{\dim}\, \rho),$$

for every finite dimensional representation ρ of Q.

Fix $\theta \in \mathbb{Q}^{Q_0}$. We call θ a *stability condition.* For any $\alpha \in \mathbb{N}^{Q_0} \setminus 0$ one can define its *slope* (with respect to θ) as the ratio

$$\mu_\theta(\alpha) = \frac{\theta \cdot \alpha}{|\alpha|} \in \mathbb{Q}.$$

Let us set $\mu_\theta(\rho) = \mu_\theta(\underline{\dim}\, \rho)$, for a representation ρ of Q. A representation ρ is said to be θ-*semistable* if for every proper nontrivial subrepresentation $0 \neq \rho' \subset \rho$ one has

$$\mu_\theta(\rho') \leq \mu_\theta(\rho).$$

Strict inequality in the latter formula defines θ-*stability*, and θ is called $\mathbf{d}$-*generic* if every θ-semistable representation of dimension $\mathbf{d}$ is θ-stable.

EXERCISE 5.5.9. Let ρ be a finite dimensional representation of a quiver Q, and let $\rho' \subset \rho$ be a subrepresentation. Show that $\phi_\theta(\rho') < \phi_\theta(\rho)$ if and only if $\mu_\theta(\rho') < \mu_\theta(\rho)$, so that stability can be checked using slopes instead of phases.

Let Q be a quiver, and let $0 \in Q_0$ be a distinguished vertex. The *framed quiver* $\widetilde{Q}$ is obtained by adding a new vertex ∞ to the vertices of Q, along with a new arrow $\infty \to 0$. Thus a $(\mathbf{d}, 1)$-dimensional representation $\widetilde{\rho}$ of $\widetilde{Q}$ can be seen as a pair (ρ, v), where ρ is a $\mathbf{d}$-dimensional representation of Q and $v\colon V_\infty \to V_0$ is a linear map from the 1-dimensional vector space V_∞. In other words, v is a vector in V_0.

Definition 5.5.10 (Framed stability)**.** Fix $\theta \in \mathbb{Q}^{Q_0}$. A representation $\widetilde{\rho} = (\rho, v)$ of $\widetilde{Q}$ is said to be θ-(semi)stable if it is (θ, θ_∞)-(semi)stable, where $\theta_\infty = -\theta_\infty \cdot \underline{\dim}\, \rho$.

We now consider the framed quiver $\widetilde{L}_d$, i.e. the d-loop quiver L_d equipped with one additional vertex ∞ and one additional arrow $\infty \to 0$ — see Figure 8.

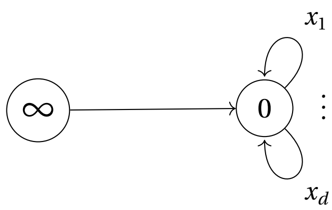


FIGURE 8. The framed d-loop quiver.

For the dimension vector $\mathbf{d} = (n, 1)$, we have

$$\mathrm{Rep}_{\mathbf{d}}(\widetilde{L}_d) = W_n,$$

where W_n was defined in (5.2.1).

EXERCISE 5.5.11. Find a ($\mathbf{d}$-generic) framed stability condition θ on L_d (in the sense of Definition 5.5.10) such that $U_n \subset W_n$ agrees with the set of θ-semistable framed representations of L_d. (**Hint**: consider $\theta = (\theta_1, \theta_2)$ with $\theta_1 \geq \theta_2$).

Thanks to the previous exercise, the noncommutative Hilbert scheme $\mathrm{NCHilb}_d^n = U_n/\mathrm{GL}_n$ can be viewed as a fine moduli space of quiver representations.

If $I \subset \mathbb{C}Q$ is a two-sided ideal, one can consider the full subcategory

$$\mathrm{Rep}(Q, I) \subset \mathrm{Rep}(Q)$$

of representations $(M_a)_{a \in Q_1}$ such that $M_{a_k} \cdots M_{a_2} M_{a_1} = 0$ for every element $a_k \cdots a_2 a_1 \in I$. The category of representations of the quotient algebra $\mathbb{C}Q/I$ is equivalent to $\mathrm{Rep}(Q, I)$. For instance, given the ideal $I \subset \mathbb{C}L_d = \mathbb{C}\langle x_1, \ldots, x_d \rangle$ spanned by the commutators

$$[x_i, x_j], \quad 1 \leq i < j \leq d,$$

one has $\mathrm{Rep}(L_d, I) = \mathrm{Rep}\, \mathbb{C}[x_1, \ldots, x_d]$.

A special class of ideals arises from *superpotentials*, i.e. formal linear combinations of cyclic paths

$$\mathsf{W} = \sum_{c \text{ cycle in } Q} a_c\, c \in \mathbb{C}Q$$

up to cyclic permutation.

For a cyclic word w and an arrow $a \in Q_1$, one defines the non-commutative derivative

$$\frac{\partial w}{\partial a} = \sum_{\substack{w = c a c' \\ c, c' \text{ paths in } Q}} c' c.$$

This rule extends to an operator $\frac{\partial}{\partial a}$ acting on every superpotential. Thus every superpotential $\mathsf{W} = \sum_c a_c c$ gives rise to a two-sided ideal

$$I_{\mathsf{W}} = \left\langle \frac{\partial \mathsf{W}}{\partial a} \,\middle|\, a \in Q_1 \right\rangle \subset \mathbb{C}Q$$

and to a regular function

$$\operatorname{Tr}\mathsf{W} \colon \operatorname{Rep}_{\mathbf{d}}(Q) \to \mathbb{A}^1$$

defined by sending

$$\rho \mapsto \sum_{c \text{ cycle in } Q} a_c \operatorname{Tr}(\rho(c)).$$

The quotient

$$J(Q,\mathsf{W}) = \mathbb{C}Q/I_{\mathsf{W}}$$

is called the *Jacobi algebra* of the quiver with potential (Q,W). By our discussion, we have an equivalence of abelian categories

$$\operatorname{Rep}(Q, I_{\mathsf{W}}) \cong \operatorname{Rep} J(Q,\mathsf{W}).$$

The proof of Theorem 5.5.2 then reveals that $\operatorname{Hilb}^n \mathbb{A}^3$ can be seen as the moduli space of *stable* framed representations of the Jacobi algebra

$$\mathbb{C}\langle x_1, x_2, x_3\rangle / I_{\mathsf{W}} = \mathbb{C}[x_1, x_2, x_3],$$

for the potential $\mathsf{W} = x_1[x_2, x_3]$.

6. Equivariant Cohomology

6.1. **Origins.** Equivariant Cohomology was introduced by Borel in his seminar on trasformation groups [11]. The goal of this section is to introduce the framework necessary to state the Atiyah–Bott *localisation formula* [12, 4]. This will be done in Section 7. We shall see how the formula works via concrete examples in Section 8. See [3, 50] for great expositions on the subject in the context of Algebraic Geometry, and [21] for the first appearance of equivariant cohomology in Enumerative Geometry. We will survey a special case of the *virtual localisation formula* by Graber–Pandharipande [31] in Section 10.

Let HTop denote the homotopy category of (Hausdorff, paracompact) topological spaces, and let G be a (Hausdorff, paracompact) topological group. The functor $\mathcal{P}_G \colon \mathrm{HTop}^{\mathrm{op}} \to \mathrm{Sets}$ sending a topological space S to the set isomorphism[5] classes of principal G-bundles over S is representable, i.e. there exists a topological space $\mathrm{B}G$ and an isomorphism of functors $\mathcal{P}_G \cong \operatorname{Hom}_{\mathrm{HTop}}(-, \mathrm{B}G)$. This is a consequence of the Brown representability theorem [13], which completely characterises the representable functors $\mathrm{HTop}^{\mathrm{op}} \to \mathrm{Sets}$.

Recall the following definitions and theorem.

Definition 6.1.1. A topological space is called *weakly contractible* if all its homotopy groups are trivial. It is called *contractible* if it is homotopy equivalent to a point.

[5]Any two principal bundles over the same base are isomorphic in all categories (topological, smooth, algebraic...). However, here two principal bundles are isomorphic if they are G-*homotopy equivalent*.

Definition 6.1.2. A principal G-bundle $E \to S$ is called *universal* if its total space E is contractible.

Theorem (Whitehead [75])**.** Every weakly contractible CW complex is contractible.

Going back to the representable functor $\mathcal{P}_G$, let $\eta_G \in \mathcal{P}_G(\mathrm{B}G)$ be the element corresponding to $\mathrm{id}_{\mathrm{B}G} \in \mathrm{Hom}_{\mathrm{HTop}}(\mathrm{B}G, \mathrm{B}G)$. Milnor has shown [51] that every Hausdorff topological group G admits a universal principal bundle $\mathrm{E}G \to \mathrm{B}G$, such that G acts freely on $\mathrm{E}G$. Therefore the equivalence class η_G contains one such representative, that we will denote

$$\mathrm{E}G \to \mathrm{B}G.$$

Every other principal G-bundle $F \to S$ is, up to isomorphism, pulled back from this one via a unique morphism $S \to \mathrm{B}G$ in the homotopy category.

Milnor's proof is usually referred to as the *join construction*, for in [52] he constructs $\mathrm{E}G$ as

$$\mathrm{E}G = \operatorname{colim} G * G * \cdots$$

where $*$ denotes the topological join. If we identify G, up to homotopy, with a CW complex W, then $\mathrm{E}G$ is homotopy equivalent to a colimit of finite joins of W, and is therefore weakly contractible (which implies the above universal property). By Whitehead's theorem, $\mathrm{E}G$ is in fact contractible. This will be crucial, since then the product space $\mathrm{E}G \times X$ will have the same homotopy type as X, for every topological space X.

However, it is wise to keep in mind that $\mathrm{E}G$ and its free quotient $\mathrm{B}G = \mathrm{E}G/G$ are only determined up to homotopy. After such a choice of representative is made, the space $\mathrm{B}G$, together with the map $\mathrm{E}G \to \mathrm{B}G$, is usually called the *classifying space* of principal G-bundles.

6.2. **Warming up: first examples.** We now list a few classical examples of classifying spaces $\mathrm{E}G \to \mathrm{B}G$. We insist that $\mathrm{E}G$ should be contractible.

(1) If $G = 1$, one can take $\mathrm{pt} \to \mathrm{pt}$ as classifying space.
(2) If G and K are two groups, then one can always take $\mathrm{E}(G \times K) = \mathrm{E}G \times \mathrm{E}K$.
(3) If $G = \mathbb{R}$, then one can take $\mathrm{E}\mathbb{R} = \mathbb{R}$ with the constant map $\mathbb{R} \to \mathrm{pt}$.
(4) If $G = \mathbb{Z}^n$, one can take $\mathbb{R}^n \to (S^1)^n$ to be the map

$$(y_1, \ldots, y_n) \mapsto (\exp(i\pi y_1), \ldots, \exp(i\pi y_n)).$$

(5) If $G = \mathbb{Z}/2$, one can take the double cover $S^\infty \to \mathbb{RP}^\infty$.
(6) If $G = \mathbb{C}^\times$, one can take $\mathbb{C}^\infty \setminus 0 \to \mathbb{P}^\infty$.
(7) If $G = S^1 \subset \mathbb{C}^\times$ is the circle, we can restrict the bundle from the previous example to get $S^\infty \to \mathbb{P}^\infty$.

Example (6) generalises in two different ways:

(6′) If $G = (\mathbb{C}^\times)^n$, one has $(\mathbb{C}^\infty \setminus 0)^n \to (\mathbb{P}^\infty)^n$. Similarly, replacing $\mathbb{C}^\times$ by S^1, example (7) becomes $(S^\infty)^n \to (\mathbb{P}^\infty)^n$, the direct limit of the $(S^1)^n$-bundles $(S^{2m+1})^n \to (\mathbb{P}^m)^n$.
(6″) If $G = \mathrm{GL}_n(\mathbb{C})$, one has $F(n, \mathbb{C}^\infty) \to G(n, \mathbb{C}^\infty)$ where $F(n, \mathbb{C}^\infty)$ denotes the space of orthonormal n-frames in an infinite dimensional vector space and

the map sends an n-frame to its span, viewed as a point of the infinite Grassmannian. The infinite dimensional Grassmannian $G(n,\mathbb{C}^\infty)$ is also the classifying space for the orthogonal group $O(n)$.

Before defining equivariant cohomology, we pause for a tiny remark. We notice that in most examples above, the spaces involved are infinite dimensional. We want to answer the following questions:

(A) Why is that the case?
(B) Why is it not a problem?

Consider example (6), say. We have that $G = \mathbb{C}^\times$ acts on $\mathbb{C}^n$ by

$$\lambda\cdot(x_1,\ldots,x_n) = (\lambda x_1,\ldots,\lambda x_n),$$

and removing the origin we get a nice principal $\mathbb{C}^\times$-bundle

$$\mathbb{C}^n\setminus 0 \to \mathbb{P}^{n-1}.$$

But we cannot stop here and declare this to be our $\mathrm{E}G \to \mathrm{B}G$, since we want the total space to be contractible, and unfortunately

$$\pi_n(\mathbb{C}^n\setminus 0)\neq 0.$$

It turns out that taking the limit is what we need in order to kill all higher homotopy groups. Formally, we can use the $\mathbb{C}^\times$-equivariant maps

$$\mathbb{C}^n\setminus 0 \hookrightarrow \mathbb{C}^{n+1}\setminus 0,\quad (x_1,\ldots,x_n)\mapsto (x_1,\ldots,x_n,0)$$

to get an inductive system of inclusions $\mathbb{P}^{n-1}\hookrightarrow\mathbb{P}^n$. The result of the limit process

$$\begin{tikzcd}
\mathbb{C}^n\setminus 0 \arrow[r, hook] \arrow[d] & \mathbb{C}^{n+1}\setminus 0 \arrow[d] & \arrow[r, "n\to\infty"] & {} & \mathbb{C}^\infty\setminus 0 \arrow[d] \\
\mathbb{P}^{n-1} \arrow[r, hook] & \mathbb{P}^n & & & \mathbb{P}^\infty
\end{tikzcd}$$

has the desired property $\pi_n(\mathbb{C}^\infty\setminus 0)=0$ for all n. This answers Question (A). As for Question (B), we will see in Section 6.4 that computations can be performed at the level of certain finite dimensional *approximation spaces*.

6.3. **Definition of equivariant cohomology.** Let X be any topological space equipped with a *left* G-action. We call such an X a G-space. Choose a universal principal G-bundle $\mathrm{E}G \to \mathrm{B}G$, with a free right G-action on $\mathrm{E}G$. Form the quotient space

$$\mathrm{E}G\times^G X = \frac{\mathrm{E}G\times X}{(e\cdot g,x)\sim(e,g\cdot x)}. \tag{6.3.1}$$

This quotient exists even in the smooth category when X is a smooth manifold, since the diagonal action of G on $\mathrm{E}G\times X$ is *free*. Note that the product $\mathrm{E}G\times X$ is homotopy equivalent to X since $\mathrm{E}G$ is contractible. The quotient (6.3.1) is the right notion of a G-space (from a homotopy theory point of view), for it is much better behaved than the (topological) orbit space X/G.

Definition 6.3.1. The G*-equivariant cohomology of* X is defined to be the singular cohomology of the quotient (6.3.1). The notation will be

$$H^*_G(X) = H^*(\mathrm{E}G\times^G X). \tag{6.3.2}$$

Convention. We work with $\mathbb{Z}$ coefficients.

Example 6.3.2. When $X = \mathsf{pt}$, we set

$$H_G^* = H_G^*(\mathsf{pt}) = H^*(\mathrm{B}G).$$

This can be viewed as the ring of characteristic classes on principal G-bundles.

Remark 6.3.3. A G-space X can have nonzero equivariant cohomology in degrees higher than its dimension, unlike ordinary cohomology. This is already true for $X = \mathsf{pt}$.

Remark 6.3.4. Even though ordinary cohomology is a special case of equivariant cohomology (it is enough to set $G = 1$), it is not true that one can reconstruct ordinary cohomology knowing equivariant cohomology. For instance, setting $X = G = \mathbb{C}^\times$, with G acting on X by left complex multiplication, one finds

$$H_G^1(X) = H^1(\mathbb{C}^\infty \setminus 0 \times^{\mathbb{C}^\times} \mathbb{C}^\times) = H^1(\mathbb{C}^\infty \setminus 0) = 0,$$

whereas $H^1(\mathbb{C}^\times) = H^1(S^1) = \mathbb{Z}$. More generally, by letting a compact connected Lie group G of dimension ℓ act on $X = G$ by left multiplication, one has

$$H_G^*(X, \mathbb{Q}) = H^*(\mathrm{E}G) = \mathbb{Q},$$

whereas by a theorem of Hopf [**?**] one has

$$H^*(X) = \wedge_{\mathbb{Q}}(a_1, \ldots, a_\ell).$$

Remark 6.3.5. The cohomology of a topological orbit space X/G does not agree with the G-equivariant cohomology of X, unless the G-action on X is free (cf. (6.3.7)). One example is $G = S^1$ acting on $X = S^2 \subset \mathbb{R}^3$ by rotation along the z-axis. There are two fixed points so the action is not free. The quotient X/G can be identified with a closed interval, which is contractible. Therefore the cohomology of the quotient is the cohomology of a point, and the action has manifestly been forgotten. But $H_{S^1}^*(S^2)$ is more interesting (cf. Exercise 6.3.15).

Lemma 6.3.6. *The definition of $H_G^*(X)$ does not depend upon the choice of universal principal G-bundle.*

Proof. Let $\mathrm{E}G \to \mathrm{B}G$ and $\mathrm{F}G \to \mathrm{B}G$ be two universal principal G-bundles. Consider the space $Y = (\mathrm{E}G \times \mathrm{F}G \times X)/G$. Then we have fibrations

$$\begin{tikzcd} & Y \arrow[dl, "p_E"'] \arrow[dr, "p_F"] & \\ \mathrm{E}G \times^G X & & \mathrm{F}G \times^G X \end{tikzcd}$$

with contractible fibers $\mathrm{F}G$ and $\mathrm{E}G$, respectively. Let us focus on p_E. For every positive integer n, we have the fibration long exact sequence

$$\cdots \longrightarrow \pi_n(\mathrm{F}G) \longrightarrow \pi_n(Y) \longrightarrow \pi_n(\mathrm{E}G \times^G X) \longrightarrow \pi_{n-1}(\mathrm{F}G) \longrightarrow \cdots$$

but the homotopy groups of the fiber $\mathrm{F}G$ are trivial, therefore we get a family of isomorphisms

$$\pi_n(Y) \xrightarrow{\sim} \pi_n(\mathrm{E}G \times^G X).$$

It follows that Y and $\mathrm{E}G \times^G X$ have the same homotopy type, and hence the same cohomology. Repeating the process with p_F gives the result. □

Remark 6.3.7. There is a fibre bundle

$$\begin{array}{c} \mathrm{E}G \times^G X \\ \big\downarrow p \\ \mathrm{B}G \end{array}$$

with fiber X, induced by the G-equivariant projection $\mathrm{E}G \times X \to \mathrm{E}G$. The standard pullback in cohomology induces a canonical ring homomorphism

$$p^*\colon H_G^* \to H_G^*(X), \tag{6.3.3}$$

making $H_G^*(X)$ into a H_G^*-module.

Example 6.3.8. When G acts trivially on X we have $\mathrm{E}G \times^G X = \mathrm{B}G \times X$, so that

$$H_G^*(X) \cong H_G^* \otimes H^*(X) \tag{6.3.4}$$

is a free H_G^*-module. For instance, given any action of G on X and letting X^G be the fixed locus, we have $H_G^*(X^G) = H_G^* \otimes H^*(X^G)$. There are G-spaces X with nontrivial action such that the isomorphism (6.3.4) holds — see Exercise 6.3.15. In this case the G-space X is called *equivariantly formal*.

Example 6.3.9. Let X be a contractible space with a left G-action. Then (6.3.3) is an isomorphism.

Example 6.3.10 (Equivariant cohomology of a subgroup)**.** Let $K \subset G$ be a closed subgroup. Then $\mathrm{E}G/K$ exists and in fact $\mathrm{E}K = \mathrm{E}G \to \mathrm{E}G/K = \mathrm{B}K$ is a classifying space for K. Therefore,

$$H_G^*(G/K) = H^*(\mathrm{E}G \times^G (G/K)) = H^*(\mathrm{E}G/K) = H^*(\mathrm{B}K) = H_K^*.$$

Example 6.3.11. Generalising the previous example, let $K \subset G$ be again a closed subgroup, acting on a space X on the left. We can also consider the diagonal action of K upon $G \times X$. The quotient

$$G \times^K X = (G \times X)/(gk, x) \sim (g, k \cdot x)$$

makes sense and we have

$$H_G^*(G \times^K X) = H^*(\mathrm{E}G \times^G G \times^K X) = H^*(\mathrm{E}G \times^K X) = H_K^*(X).$$

Again, we have used that $\mathrm{E}G \to \mathrm{E}G/K$ is a classifying space for $K \subset G$.

Theorem 6.3.12. *Let $f\colon X \to S$ be a topological fibre bundle with contractible fibre. Then f induces an isomorphism $H^*(X, \mathbb{Z}) \cong H^*(S, \mathbb{Z})$.*

EXERCISE 6.3.13. Let $K \subset G$ be a closed subgroup of a topological group G such that G/K is contractible, and let X be a G-space. Then

$$H_K^*(X) \cong H_G^*(X).$$

(**Hint**: use that one can take $\mathrm{E}K = \mathrm{E}G$ to construct a map $\mathrm{E}K \times^K X = \mathrm{E}G \times^K X \to \mathrm{E}G \times^G X$. Show that it is a fibre bundle with fibre G/K. Conclude by Theorem 6.3.12).

Example 6.3.14. Let us revisit the examples (6′) and (6″) discussed above.

(A) In the case $\mathbb{T} = (\mathbb{C}^\times)^n$, we find

$$H^*_{\mathbb{T}} = H^*((\mathbb{P}^\infty)^n) = \mathbb{Z}[t_1,\ldots,t_n] \tag{6.3.5}$$

where $t_i \in H^2_{\mathbb{T}}$ are the Chern classes obtained by pulling back to $\mathrm{B}\mathbb{T}$ the universal line bundles living over the individual spaces $\mathbb{P}^\infty$. In other words, if $\pi_i\colon \mathrm{B}\mathbb{T} \to \mathbb{P}^\infty$ is the i-th projection, the generators of $H^*_{\mathbb{T}}$ are

$$t_i = c_1(\pi_i^*\mathscr{O}(-1)).$$

(B) In the case $G = \mathrm{GL}_n(\mathbb{C})$, we find

$$H^*_G = H^*(G(n,\mathbb{C}^\infty)) = \mathbb{Z}[e_1,\ldots,e_n] \tag{6.3.6}$$

where $e_i = c_i(\mathscr{S})$ are the Chern classes of the universal rank n bundle $\mathscr{S}$ living over the Grassmannian.

Exercise 6.3.15. Let $G = S^1$ act on the sphere $S^2 \subset \mathbb{R}^3$ by rotation along the z-axis. Show that S^2 is equivariantly formal for this S^1-action. In other words, show that

$$H^*_{S^1}(S^2) \cong H^*(\mathrm{B}S^1) \otimes H^*(S^2),$$

even though the action has two fixed points. (**Hint**: Find a suitable open cover $S^2 = X_1 \cup X_2$, inducing an open cover of $\mathrm{E}S^1 \times^{S^1} S^2$. Apply Mayer–Vietoris to this open cover, and use that $H^*_{S^1} = \mathbb{Z}[c]$ is one-dimensional, resp. zero-dimensional, in even degree, resp. odd degree).

Let X be a smooth manifold acted on (smoothly) by a Lie group G. There is a commutative diagram

$$\begin{tikzcd}
\mathrm{E}G \arrow[d] & \mathrm{E}G \times X \arrow[l] \arrow[r] \arrow[d] & X \arrow[d] \\
\mathrm{B}G & \mathrm{E}G \times^G X \arrow[l, "p"'] \arrow[r, "\sigma"] & X/G
\end{tikzcd}$$

where σ, unlike p, is in general not a fiber bundle, but has the property that

$$\sigma^{-1}(\mathrm{Orb}\,x) = \mathrm{E}G/G_x \cong \mathrm{B}G_x$$

for all $x \in X$. When G is a compact Lie group acting (smoothly and) *freely* on X, the map σ induces a homotopy equivalence $\mathrm{E}G \times^G X \approx X/G$ yielding a natural isomorphism

$$\sigma^*\colon H^*(X/G) \xrightarrow{\sim} H^*_G(X). \tag{6.3.7}$$

Let $\iota\colon X \hookrightarrow \mathrm{E}G \times^G X$ denote the inclusion of X as a fiber of $p\colon \mathrm{E}G \times^G X \to \mathrm{B}G$ (after choosing a base point $\mathrm{pt} \in \mathrm{B}G$). Then ι induces a natural map

$$\iota^*\colon H^*_G(X) \to H^*(X)$$

from equivariant to ordinary cohomology. There is a pullback diagram (on the left) inducing, whenever X is compact, a commutative diagram of cohomology rings:

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[d, "\iota"'] \arrow[dr, phantom, "\square"] & \mathrm{pt} \arrow[d, "b"] \\ \mathrm{E}G \times^G X \arrow[r, "p"] & \mathrm{B}G \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} H^*(X) \arrow[r, "q_*"] & \mathbb{Z} \\ H_G^*(X) \arrow[r, "p_*"] \arrow[u, "\iota^*"] & H_G^* \arrow[u, "b^*"'] \end{tikzcd}$$

In algebraic geometry, this compatibility makes sense and is useful when X is a proper variety, which ensures the existence of q_*. The *equivariant pushforward*, on the other hand, will be defined in Section 7.2. Note that the above commutativity implies in particular that

$$\boxed{\textit{ordinary integrals can be performed in equivariant cohomology.}}$$

An equivariant integral takes values in H_G^*, but evaluating along b^* always yields an integer.

6.3.1. *Preview of Section 8.* The following is the general setup we shall encounter in concrete calculations. Let X be a smooth projective complex variety with a torus action — for instance a Grassmannian. We will have a top cohomology class

$$\psi \in H^{2d}(X), \quad d = \dim_{\mathbb{C}} X,$$

and we will be interested in computing an ordinary integral

$$\int_X \psi = q_*(\psi \cap [X]) \in \mathbb{Z}.$$

To compute this number, we will lift the class ψ to an equivariant class $\psi^{\mathbb{T}} \in H_{\mathbb{T}}^*(X)$, and we will compute

$$\int_X \psi^{\mathbb{T}} = p_*(\psi^{\mathbb{T}} \cap [X]^{\mathbb{T}})$$

instead, via localisation. At this point, applying b^* means choosing a suitable[6] specialisation of the equivariant parameters to obtain a number — this is necessarily the number we want, by the commutativity of the above square!

6.4. **Approximation spaces.** Let us now assume that X is a complex algebraic variety and G is an algebraic group. The fact that the spaces involved, like $\mathrm{E}G$ and $\mathrm{B}G$, are infinite-dimensional, is not quite an obstacle to the computation of the equivariant cohomology groups. This is the case because of the following "approximation" result.

Theorem 6.4.1. *Let $(E_m)_{m\geq 0}$ be a family of connected spaces on which G acts freely on the right. Let $k\colon \mathbb{N} \to \mathbb{N}$ be a function such that $\pi_i(E_m) = 0$ for $0 < i < k(m)$. Then, for any left G-action on a space X, there are natural isomorphisms*

$$H_G^i(X) \cong H^i(E_m \times^G X), \quad i < k(m).$$

We will refer to the E_m spaces as *approximation spaces*. The idea behind their existence can be traced back to [11, Remark XII.3.7].

[6] One that does not produce poles in the localisation formula.

Remark 6.4.2. In the smooth category, if G is a compact lie group, then $\mathrm{E}G \to \mathrm{B}G$ is a colimit of smooth principal G-bundles

$$E_m \to B_m,$$

where E_m is m-connected, i.e. $\pi_i(E_m) = 0$ for $1 \le i \le m$, i.e. if S^k is a sphere of dimension $k \le m$, then any continuous map $S^k \to E_m$ is homotopic to the constant map.

Example 6.4.3. Let us revisit once more the examples $\mathbb{T} = (\mathbb{C}^\times)^n$ and $G = \mathrm{GL}_n(\mathbb{C})$. For the case of the torus, one can take $E_m = (\mathbb{C}^m \setminus 0)^n \to B_m = (\mathbb{P}^{m-1})^n$ as approximations of $\mathrm{E}\mathbb{T} \to \mathrm{B}\mathbb{T}$. Since $\mathbb{C}^m \setminus 0$ has the same homotopy type as S^{2m-1}, the function $k(m) = n(2m-1)$ will let us fall in the assumption $\pi_i(E_m) = 0$ of the theorem, where $0 < i < k(m)$. Setting for instance $X = \mathrm{pt}$, we find isomorphisms

$$H^i_{\mathbb{T}} \cong H^i((\mathbb{P}^{m-1})^n), \quad i < n(2m-1).$$

This fully explains the assertion we made in (6.3.5). When $G = \mathrm{GL}_n(\mathbb{C})$, we can take E_m, for $m > n$, to be the set of $m \times n$ rank n matrices. The approximations spaces then look like the free quotients $E_m \to G(n, \mathbb{C}^m)$. The function $k(m) = 2(m-n)$ will do the job again and we would find, again for $X = \mathrm{pt}$, isomorphisms

$$H^i_{\mathrm{GL}_n} \cong H^i(G(n, \mathbb{C}^m)), \quad i < 2(m-n).$$

This explains the assertion we made in (6.3.6).

Remark 6.4.4. If $Y \subset \mathbb{A}^N$ is a subvariety of codimension d, then

$$\pi_i(\mathbb{A}^N \setminus Y) = 0, \quad 0 < i \le 2d-2.$$

Since the complement of the open subset

$$\{\text{full rank matrices}\} \subset \mathrm{Mat}_{m\times n} = \mathbb{A}^{mn}$$

has codimension $(m-1)(n-1)$, the choice $k(m) = 2(m-n)$ works (in case $n < m$).

Remark 6.4.5. In the above examples, it is clear that the method of approximation spaces allows one to compute H^*_G in all degrees, since $k(m) \to \infty$ for $m \to \infty$.

6.5. **Equivariant vector bundles.** Let G be a group acting on X and H a group acting on Y. Suppose there are maps $\phi\colon G \to H$ and $f\colon X \to Y$. The condition

$$f(g\cdot x) = \phi(g)\cdot f(x) \tag{6.5.1}$$

for all $x \in X$ and $g \in G$ is the condition under which one can construct a natural map

$$\mathrm{E}G \times^G X \to \mathrm{E}H \times^H Y. \tag{6.5.2}$$

When (6.5.1) is satisfied, applying H^* to (6.5.2) yields a natural *equivariant pullback* homomorphism

$$f^*\colon H^*_H(Y) \to H^*_G(X). \tag{6.5.3}$$

When $\phi = \mathrm{id}_G$ we say that f is G-*equivariant*, and when moreover G acts trivially on Y we say that f is G-invariant.

A crucial example is that of an equivariant vector bundle $\pi\colon E \to X$. Such an object is a geometric vector bundle together with a lift of the G-action on X to a G-action on E:

$$\begin{tikzcd} G \times E \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ G \times X \arrow[r, "\text{action}"] & X \end{tikzcd}$$

Given an equivariant vector bundle one can define a new vector bundle

$$V_E = \mathrm{E}G \times^G E \to \mathrm{E}G \times^G X \tag{6.5.4}$$

of the same rank as E.

Definition 6.5.1 (Equivariant Chern classes)**.** The *equivariant Chern classes* of a G-equivariant vector bundle $E \to X$ are the characteristic classes

$$c_i^G(E) = c_i(V_E) \in H_G^{2i}(X).$$

These classes can clearly be computed through approximation spaces. Indeed, the vector bundle (6.5.4) can be approximated by vector bundles

$$V_{E,m} = E_m \times^G E \to E_m \times^G X$$

whose Chern classes $c_i(V_{E,m})$ live in $H^{2i}(E_m \times^G X) \cong H_G^{2i}(X)$ for $m \gg 0$.

Example 6.5.2 (Equivariant fundamental class)**.** Let X be a smooth algebraic variety acted on by a linear algebraic group G, and let $Y \subset X$ be a G-invariant closed subvariety of codimension d. The space $E_m \times^G X$ is smooth, and

$$E_m \times^G Y \hookrightarrow E_m \times^G X$$

is a closed subvariety of codimension d. Then

$$[Y]^G = [E_m \times^G Y] \in H_G^{2d}(X) \tag{6.5.5}$$

is called the *equivariant fundamental class* of $Y \subset X$.

EXERCISE 6.5.3. Show that the classes $[Y]^G$ defined in (6.5.5) are compatible when m varies and are independent of the choice of $\mathrm{E}G$ and $(E_m)_{m\geq 0}$.

Example 6.5.4 (Weight of a character)**.** When X is a point, an equivariant vector bundle is a representation $\rho\colon G \to \mathrm{GL}(E)$. This still gives a vector bundle $V_\rho \to \mathrm{B}G$ whose Chern classes live in H_G^*. When E is one-dimensional, an equivariant line bundle is simply a *character* $\chi\colon G \to \mathbb{C}^\times$ and one defines the *weight* of χ to be the Chern class

$$w_\chi = c_1(V_\chi) \in H_G^2.$$

When a representation $\rho\colon G \to \mathrm{GL}(E)$ of dimension r splits as a direct sum of characters χ_i (for example this is always the case when $G = \mathbb{T}$ is a torus), the G-*equivariant Euler class* splits as a product of weights,

$$e^G(E) = c_{\mathrm{top}}^G(E) = c_{\mathrm{top}}(V_\rho) = \prod_{i=1}^r w_i \in H_G^{2r},$$

where $w_i = c_1(V_{\chi_i}) \in H^2_G$ is the weight of χ_i. More generally, the i-th equivariant Chern class of E is the i-th symmetric function in $w_1,\ldots,w_r$. See Example 6.5.6 for a detailed description of this fact.

Example 6.5.5. Let $\mathbb{T}=\mathbb{C}^\times$ be the one dimensional torus and $\rho_a:\mathbb{T}\to\mathbb{C}^\times$ the character $z\mapsto z^a$ for an integer $a\in\mathbb{Z}$. Here we are viewing the one dimensional vector space $\mathbb{C}_a$ as an equivariant vector bundle over $X=\mathrm{pt}$. The line bundle $V_{\rho_a}\to\mathrm{B}\mathbb{C}^\times$ is approximated by line bundles $V_{\rho_a,m}\to\mathbb{P}^{m-1}$. In fact, $V_{\rho_a,m}\cong\mathscr{O}_{\mathbb{P}^{m-1}}(-a)$ so that the weight of ρ_a is

$$c_1^{\mathbb{C}^\times}(\mathbb{C}_a)=c_1(V_{\rho_a,m})=t\,a\in H^2_{\mathbb{C}^\times}.$$

Notice that t corresponds to the case $a=1$. This is called the *standard action* of $\mathbb{T}$ on $\mathbb{C}$. It generalises in the next example.

Example 6.5.6. Let $F=\mathbb{C}^n$ be an n-dimensional vector space and let the torus $\mathbb{T}=(\mathbb{C}^\times)^n$ act on F via the *standard action* $\theta\cdot(v_1,\ldots,v_n)=(\theta_1v_1,\ldots,\theta_nv_n)$. The induced representation $\rho:\mathbb{T}\to\mathrm{GL}(F)$, defined by $\rho_\theta(v)=\theta\cdot v$ as above, gives a rank n vector bundle $V_\rho=\mathrm{E}\mathbb{T}\times^{\mathbb{T}}F\to\mathrm{B}\mathbb{T}=(\mathbb{P}^\infty)^n$. Now, there is a splitting $F=F_1\oplus\cdots\oplus F_n$ where each summand corresponds to a character

$$\chi_i:\mathbb{T}\to\mathbb{C}^\times,\quad \theta\mapsto\theta_i$$

whose weight is just $t_i\in H^2_{\mathbb{T}}$ by Example 6.5.5. In other words, for each i, $V_{\chi_i}=\mathrm{E}\mathbb{T}\times^{\mathbb{T}}F_i\to(\mathbb{P}^\infty)^n$ is the total space of the line bundle $\mathscr{O}_i(-1)=\pi_i^*\mathscr{O}(-1)$. We quickly verify that

$$c_i^{\mathbb{T}}(F)=e_i(t_1,\ldots,t_n)\in H^*_{\mathbb{T}}, \tag{6.5.6}$$

the i-th symmetric function in the Chern classes $t_i=c_1(\mathscr{O}_i(-1))$. We have a decomposition

$$V_\rho=\bigoplus_{i=1}^n\mathrm{E}\mathbb{T}\times^{\mathbb{T}}F_i=\bigoplus_{i=1}^n\mathscr{O}_i(-1).$$

Then V_ρ has the obvious filtration

$$0\subset\mathscr{O}_1(-1)\subset\mathscr{O}_1(-1)\oplus\mathscr{O}_2(-1)\subset\cdots\subset V_\rho$$

with $\mathscr{O}_i(-1)$ as line bundle quotients. The conclusion (6.5.6) is then a straightforward property of Chern classes. For instance, the top Chern class of the standard representation is given by $c_n^{\mathbb{T}}(V_\rho)=\prod_i t_i$.

Example 6.5.7. Let $\mathbb{T}$ be a torus acting on an n-dimensional complex vector space F. Let $w_1,\ldots,w_n$ be the weights of the action. Again, this means that the corresponding representation $\rho:\mathbb{T}\to\mathrm{GL}(F)$ satisfies

$$\rho_\theta(v)=(\theta_1^{w_1}v_1,\ldots,\theta_n^{w_n}v_n).$$

So F splits as a sum of 1-dimensional representations, corresponding to characters $\chi_1,\ldots,\chi_n$, each given by $\theta\mapsto\theta_i^{w_i}$. We have

$$e^{\mathbb{T}}(F)=\prod_{i=1}^n s_i^{w_i}\in H^{2n}_{\mathbb{T}}.$$

The $\mathbb{T}$-*character* of F is given by

$$\mathrm{ch}^{\mathbb{T}}(F)=\sum_{i=1}^{n} w_i t_i, \quad t_i=\exp(s_i).$$

Assume the weights are *positive*. Then the number

$$\mathrm{ch}^{\mathbb{T}}(F)\big|_{t_i=1}=\sum_i w_i=\sum_{\text{distinct weights }\chi}\mathsf{n}_\chi \tag{6.5.7}$$

enumerates the sum of the multiplicities n_χ of the (distinct) characters χ appearing in the decomposition of F. This fact will be used in Equation (13.2.9), where we shall calculate the parity of the tangent space dimension at a torus fixed point of $\mathrm{Hilb}^n X$, for X a toric Calabi–Yau 3-fold.

6.6. **Two computations on** $\mathbb{P}^{n-1}$**.** In this section we compute the equivariant cohomology of projective space $\mathbb{P}^{n-1}$ and the weights of the tangent representation induced by the standard action.

6.6.1. *Equivariant cohomology of* $\mathbb{P}^{n-1}$. Let $\mathbb{T}=(\mathbb{C}^\times)^n$ act on the vector space $F=\mathbb{C}^n$ by the standard representation $\rho_\theta(v)=(\theta_1 v_1,\ldots,\theta_n v_n)$. We already know that $c_i^{\mathbb{T}}(F)=c_i(V_{\rho,m})=e_i(t_1,\ldots,t_n)\in H^{2i}_{\mathbb{T}}\cong H^{2i}(B_m)$, where $E_m=(\mathbb{C}^m\setminus 0)^n\to B_m=(\mathbb{P}^{m-1})^n$ are the finite approximations of $\mathrm{E}\mathbb{T}\to\mathrm{B}\mathbb{T}$. Now, the above action makes $\mathbb{P}^{n-1}=\mathbb{P}(F)$ into a $\mathbb{T}$-space, which comes naturally with the $\mathbb{T}$-linearisation $\mathscr{O}_{\mathbb{P}^{n-1}}(1)$. Let W be the total space of this line bundle on $\mathbb{P}^{n-1}$. Let ζ be the equivariant first Chern class of the line bundle $E_m\times^{\mathbb{T}} W$ living over the total space of the projective space bundle

$$\mathbb{P}(V_{\rho,m})\cong E_m\times^{\mathbb{T}}\mathbb{P}^{n-1}\to B_m.$$

Then one has $\zeta=c_1(\mathscr{O}_{\mathbb{P}(V_{\rho,m})}(1))$. We can now compute

$$\begin{aligned}
H^*_{\mathbb{T}}(\mathbb{P}^{n-1})&\cong H^*(E_m\times^{\mathbb{T}}\mathbb{P}^{n-1})\\
&\cong H^*(\mathbb{P}(V_{\rho,m}))\\
&\cong H^*_{\mathbb{T}}[\zeta]\Big/\sum_{i=0}^{n} c_i(V_{\rho,m})\zeta^{n-i}\\
&= H^*_{\mathbb{T}}[\zeta]\Big/\prod_{i=1}^{n}(\zeta+t_i).
\end{aligned}$$

The last equality follows by the description $c_i(V_{\rho,m})=e_i(t_1,\ldots,t_n)$.

6.6.2. *The tangent representation.* Let $V=H^0(\mathbb{P}^{n-1},\mathscr{O}_{\mathbb{P}^{n-1}}(1))$ be the n-dimensional vector space of linear forms on $\mathbb{P}^{n-1}$, along with the standard representation $\rho_\theta(v)=(\theta_1 v_1,\ldots,\theta_n v_n)$. The weights of the action are the Chern classes $t_i=c_1(\pi_i^*\mathscr{O}(-1))\in H^2_{\mathbb{T}}$. In the tautological exact sequence

$$0\to\mathscr{S}\to V\otimes_{\mathbb{C}}\mathscr{O}_{\mathbb{P}^{n-1}}\to\mathscr{O}_{\mathbb{P}^{n-1}}(1)\to 0$$

the bundle on the left evaluated at a point p is the rank $n-1$ vector space of linear forms vanishing at p. The tangent space at a fixed point p_i is identified to

$$T_{p_i}\mathbb{P}^{n-1}=\mathscr{S}_{p_i}^\vee\otimes(V/\mathscr{S}_{p_i})=\mathrm{Span}_{\mathbb{C}}\left\{\,x_j^\vee\otimes x_i\;\middle|\;j\neq i\,\right\}.$$

This says that the weights of $T_{p_i}\mathbb{P}^{n-1}$ are $t_i - t_j$ for $j \neq i$, in particular the Euler class is computed as

$$e^{\mathbb{T}}(T_{p_i}\mathbb{P}^{n-1}) = \prod_{j\neq i}(t_i - t_j).$$

7. The Atiyah–Bott localisation formula

7.1. **A glimpse on the self-intersection formula.** If $E \to X$ is an oriented vector bundle of rank r on a compact manifold X, the orientation

$$\eta \in H^r(E, E\setminus X)$$

corresponds, by construction, to the identity element

$$\mathbb{1} = \mathsf{pd}([X]) \in H^0(X)$$

under the Thom isomorphism

$$H^0(X) \xrightarrow{\sim} H^r(E, E\setminus X),$$

where $X \hookrightarrow E$ is embedded as the zero section. The inclusions

$$(X, \emptyset) \hookrightarrow (E, \emptyset) \hookrightarrow (E, E\setminus X)$$

induce maps

$$\begin{tikzcd} H^r(E, E\setminus X) \arrow[r] \arrow[rr, bend left, "\alpha"] & H^r(E) \arrow[r] & H^r(X) \end{tikzcd}$$

and the Euler class of E is, by definition, the image of η under this composition, i.e.

$$e(E) = \alpha(\eta) \in H^r(X).$$

The Mayer–Vietoris sequence for the inclusion $X \hookrightarrow E$ yields an exact piece

$$H^{r-1}(E\setminus X) \to H^r(E, E\setminus X) \xrightarrow{u} H^r(E).$$

Let $f: X \to Y$ be a map of compact smooth manifolds. Set $n = \dim X$, $m = \dim Y$ and $d = m - n$. There are Gysin maps

$$f_*: H^p(X) \to H^{p+d}(Y),$$

defined via Poincaré duality (cf. Diagram (7.2.1)).

When $f = \iota$ is a closed embedding of codimension d, and the normal bundle

$$N_{X/Y} = T_Y|_X / T_X$$

is oriented compatibly with respect to f, there is a factorisation of ι_* as

$$H^p(X) \xrightarrow{\sim} H^{p+d}(N_{X/Y}, N_{X/Y}\setminus X) \xrightarrow{\sim} H^{p+d}(Y, Y\setminus X) \to H^{p+d}(Y).$$

where the last map comes from Mayer–Vietoris for the inclusion $\iota: X \hookrightarrow Y$, and the middle isomorphism is computed via excision by fixing a tubular neighborhood T of X in Y, so that

$$H^*(Y, Y\setminus X) \cong H^*(T, T\setminus X) \cong H^*(N_{X/Y}, N_{X/Y}\setminus X).$$

Now set $p = 0$. Then there is a commutative diagram

$$\begin{tikzcd}
H^0(X) \arrow[r, "\sim"] \arrow[rrr, bend left, "\iota_*"] & H^d(N_{X/Y}, N_{X/Y} \setminus X) \arrow[r, "\sim"] \arrow[d, "u"] & H^d(Y, Y \setminus X) \arrow[r] & H^d(Y) \arrow[d, "\iota^*"] \\
& H^d(N_{X/Y}) \arrow[rr] & & H^d(X)
\end{tikzcd}$$

which shows the content of the self-intersection formula.

Theorem 7.1.1 (Self-intersection formula)**.** *Let $\iota\colon X \hookrightarrow Y$ be a closed embedding of compact manifolds. There is an identity*

$$\iota^*\iota_*\mathbb{1} = e(N_{X/Y}) \in H^r(X).$$

See Fulton [25] for the corresponding statement in Algebraic Geometry.

7.2. **Equivariant pushforward.** Let G be a compact Lie group. Let $f\colon X \to Y$ be a G-equivariant map of compact manifolds. Set

$$n = \dim X, \quad m = \dim Y, \quad q = n - m.$$

The equivariant pullback $f^*\colon H^*_G(Y) \to H^*_G(X)$ was defined in (6.5.3) by means of the map

$$f^G\colon \mathrm{E}G \times^G X \to \mathrm{E}G \times^G Y$$

i.e. the map (6.5.2) considered in the special case where $G \to H$ is the identity.

The ordinary pushforward $f_*\colon H^*(X) \to H^*(Y)$ can be constructed via Poincaré duality and its inverse through the diagrams

(7.2.1)
$$\begin{tikzcd}
H^p(X) \arrow[r, dashed] \arrow[d, "\wr"] & H^{p-q}(Y) \\
H_{n-p}(X) \arrow[r, "f_*"] & H_{n-p}(Y) \arrow[u, "\mathsf{pd}^{-1}"']
\end{tikzcd}$$

but in order to construct an equivariant pushforward $f^G_*\colon H^*_G(X) \to H^*_G(Y)$ we cannot apply the same procedure because Poincaré duality is not available on infinite dimensional spaces such as $X_G = \mathrm{E}G \times^G X$ and $Y_G = \mathrm{E}G \times^G Y$.

We use approximation spaces to solve this issue. Fix a directed system of principal G-bundles

$$\{\, E_i \to B_i \,\}_{i \geq 0}$$

whose limit recovers the classifying space $\mathrm{E}G \to \mathrm{B}G$ (cf. Remark 6.4.2). Since E_i are compact spaces, so are the Borel spaces

$$X^i_G = E_i \times^G X, \quad Y^i_G = E_i \times^G Y,$$

and the approximation result (Theorem 6.4.1) ensures that

(7.2.2)
$$H^p_G(X) \cong H^p(X^i_G), \quad H^p_G(Y) \cong H^p(Y^i_G), \quad p \leq i.$$

Recall that we have fibrations

$$\begin{tikzcd} X_G^i \arrow[d, "p_X"] & Y_G^i \arrow[d, "p_Y"] \\ B_i & B_i \end{tikzcd}$$

with fibre X and Y, respectively. It follows that

$$\dim X_G^i = \ell + n, \quad \dim Y_G^i = \ell + m,$$

where $\ell = \dim B_i$. Exploiting Poincaré duality on the compact (finite dimensional) spaces X_G^i and Y_G^i, we can replace the diagram (7.2.1) by a new diagram

$$\begin{tikzcd} H^p(X_G^i) \arrow[r, dashed] \arrow[d, "\wr"] & H^{p-q}(Y_G^i) \\ H_{\ell+n-p}(X_G^i) \arrow[r, "f_*^{G,p}"] & H_{\ell+n-p}(Y_G^i) \arrow[u, "\mathsf{pd}^{-1}"'] \end{tikzcd}$$

which we can of course redraw as

$$\begin{tikzcd} H_G^p(X) \arrow[r, dashed] \arrow[d, "\wr"] & H_G^{p-q}(Y) \\ H_{\ell+n-p}(X_G^i) \arrow[r, "f_*^{G,p}"] & H_{\ell+n-p}(Y_G^i) \arrow[u, "\mathsf{pd}^{-1}"'] \end{tikzcd}$$

after exploiting (7.2.2). This yields a system of maps

$$f_*^{G,p} \colon H_G^p(X) \to H_G^{p-q}(Y).$$

EXERCISE 7.2.1. Prove that the maps $f_*^{G,p}$ are compatible with the structure of inverse system of $H^p(X_G^i)$ and $H^p(Y_G^i)$ attached to the directed systems $(X_G^i)_i$ and $(Y_G^i)_i$ respectively.

By Exercise 7.2.1, we can construct the *equivariant pushforward*

$$f_*^G \colon H_G^*(X) \to H_G^*(Y).$$

7.3. **Trivial torus actions.** Let $\mathbb{T}$ be a torus acting *trivially* on a smooth complex algebraic variety X. Given a $\mathbb{T}$-equivariant vector bundle $E \to X$ of rank r, we get a canonical decomposition

$$E = \bigoplus_{\chi} E_\chi, \tag{7.3.1}$$

where $\mathbb{T}$ acts by the character χ on E_χ. The characters χ vary in the character group $\mathbb{T}^* = \mathbb{Z}^{\dim \mathbb{T}}$ of the torus. One should expect to be able to express the $\mathbb{T}$-equivariant Chern classes of E in terms of the $\mathbb{T}$-equivariant Chern classes of its subbundles $E_\chi \subset E$.

Let

$$V_E = \mathrm{E}\mathbb{T} \times^{\mathbb{T}} E \to \mathrm{E}\mathbb{T} \times^{\mathbb{T}} X = X \times \mathrm{B}\mathbb{T}$$

be the induced rank r bundle, cf. (6.5.4) (and (6.3.4) for the identification on the right hand side).

EXERCISE 7.3.1. Show that $V_{E_\chi} \cong E_\chi \boxtimes V_\chi$, where the box product refers to the canonical projections from $X \times \mathrm{B}\mathbb{T}$ and V_χ is the line bundle on $\mathrm{B}\mathbb{T}$ introduced in Example 6.5.4.

EXERCISE 7.3.2. Assume again $\mathbb{T}$ acts trivially on a smooth variety X, and let E be a $\mathbb{T}$-equivariant vector bundle. Show that

$$c_i^{\mathbb{T}}(E_\chi) = \sum_{k=0}^{i} \binom{r_\chi - k}{i-k} c_k(E_\chi)\chi^{i-k} \in H_{\mathbb{T}}^{2i}(X),$$

where $r_\chi = \operatorname{rk} E_\chi$. (**Hint**: use the previous exercise and a standard property of Chern classes of tensor products, cf. [25, Example 3.2.2] or Example B.2.4 in these notes).

7.4. **Torus fixed loci.** Throughout this section, we let X be a smooth complex algebraic variety acted on by a torus $\mathbb{T}$. The scheme structure of the fixed locus

$$X^{\mathbb{T}} \subset X$$

is discussed in [24, Section 2].

We have the following result.

Theorem 7.4.1 ([44, 24])**.** *If X is a smooth $\mathbb{T}$-variety, the fixed locus $X^{\mathbb{T}}$ is smooth.*

EXERCISE 7.4.2. Let Y be a compact normal $\mathbb{T}$-variety. Show that if Y has a singular point, then it has a *torus-fixed* singular point.

Let N be the normal bundle of the inclusion $F \subset X$ of a component $F \subset X^{\mathbb{T}}$. Then N is $\mathbb{T}$-equivariant and for each $x \in F$ one has $(T_x X)^{\mathbb{T}} = T_x F$ by results of [24], so that the action of $\mathbb{T}$ on the normal space

$$N_x = T_x X / T_x F$$

is nontrivial, i.e. N_x has no trivial subrepresentations, i.e. $N_x^{\mathbb{T}} = 0$. It follows that the Euler class of the normal bundle N is nonzero, being a product of nonzero weights. In fact, $e^{\mathbb{T}}(N)$ becomes invertible in the ring $H_{\mathbb{T}}^*(F)[\chi_i^{-1}]$, where χ_i are the characters that occur in the decomposition of N into eigenbundles.

Let E be a $\mathbb{T}$-equivariant vector bundle of rank r over X. Let $F \subset X^{\mathbb{T}}$ be a component of the fixed locus, so that $H_{\mathbb{T}}^*(F) = H^*(F) \otimes H_{\mathbb{T}}^*$. According to Equation (7.3.1), the vector bundle $E|_F$ on F has a decomposition

$$E|_F = \bigoplus_\chi E_{F,\chi}$$

into eigen-subbundles. By Exercise 7.3.2, the component of

$$c_i^{\mathbb{T}}(E_{F,\chi}) \in H_{\mathbb{T}}^{2i}(F) \tag{7.4.1}$$

in $H_{\mathbb{T}}^{2i}$ is given by

$$\binom{r}{i}\chi^i \in H_{\mathbb{T}}^{2i}. \tag{7.4.2}$$

Here we are denoting simply by χ the weight $w_\chi \in H_{\mathbb{T}}^2$, cf. Example 6.5.4. Since $H^{2k}(F) = 0$ for $k > \dim F$, we have that for all $j > 0$, the classes in $H^{2j}(F)$ are nilpotent in $H^*_{\mathbb{T}}(F)$. Therefore the element (7.4.1) is invertible if and only if its component (7.4.2) is invertible. It follows that

(7.4.3) $$c_i^{\mathbb{T}}(E_{F,\chi}) \text{ is invertible in the localised ring } H^{2i}_{\mathbb{T}}(F)\big[\chi^{-1}\big].$$

Proposition 7.4.3. *Let X be a smooth $\mathbb{T}$-variety, let $F \subset X^{\mathbb{T}}$ be a component of codimension d. Then there are finitely many characters $\chi_1,\ldots,\chi_s$ such that the Euler class*

$$e^{\mathbb{T}}(N_{F/X}) \in H^{2d}_{\mathbb{T}}(F)$$

becomes invertible in the ring

$$H^*_{\mathbb{T}}(F)\big[\chi_s^{-1},\ldots,\chi_s^{-1}\big].$$

Proof. Set $N = N_{F/X}$. We saw above that the action on the normal space N_x is nontrivial for all $x \in F$, so the characters appearing in the decomposition

$$N = \bigoplus_{i=1}^{s} N_{\chi_i}$$

are all nontrivial, and $e^{\mathbb{T}}(N)$ is a product of nonzero weights,

$$0 \neq e^{\mathbb{T}}(N) = \prod_{i=1}^{s} e^{\mathbb{T}}(N_{\chi_i}),$$

so the observation (7.4.3) implies the result. □

7.5. **The localisation formula.** This subsection introduces the technique of localisation in Algebraic Geometry. Our purpose is to apply this powerful tool to solve enumerative problems, such as finding the number of lines on a general cubic surface $S \subset \mathbb{P}^3$, or on a general quintic 3-fold $Y \subset \mathbb{P}^4$.

Let $\iota\colon X^{\mathbb{T}} \hookrightarrow X$ be the inclusion of the fixed point locus. We have the equivariant pushforward

$$\iota_*\colon H^*_{\mathbb{T}}(X^{\mathbb{T}}) \to H^*_{\mathbb{T}}(X),$$

and the localisation theorem states that this map becomes an isomorphism after inverting finitely many nontrivial characters. In particular, it becomes an isomorphism after extending scalars to the field of fractions

$$\mathcal{H}^*_{\mathbb{T}} = \operatorname{Frac} H^*_{\mathbb{T}}.$$

Notice that

$$H^*_{\mathbb{T}}(X^{\mathbb{T}}) = \bigoplus_{\alpha} H^*_{\mathbb{T}}(F_\alpha)$$

where F_α are the components of the fixed locus. The crucial point is that, if N_α is the normal bundle of the inclusion $\iota_\alpha\colon F_\alpha \hookrightarrow X$, then

$$\iota_\alpha^*\iota_{\alpha*}(-) = e^{\mathbb{T}}(N_\alpha) \cap -$$

and as we saw this Euler class is nonzero when restricted to any point $x \in F_\alpha$. This is enough for $e^{\mathbb{T}}(N_\alpha)$ to become invertible after a suitable localisation.

The statement of the localisation theorem for compact manifolds with torus action is the following.

Theorem 7.5.1 (Atiyah–Bott [4]). *Let M be a compact smooth manifold equipped with an action of a torus $\mathbb{T}$. Then the equivariant pushforward along $\iota\colon M^{\mathbb{T}} \hookrightarrow M$ induces an isomorphism*

$$\iota_*\colon H^*_{\mathbb{T}}(M^{\mathbb{T}}) \otimes_{H^*_{\mathbb{T}}} \mathcal{H}^*_{\mathbb{T}} \xrightarrow{\sim} H^*_{\mathbb{T}}(M) \otimes_{H^*_{\mathbb{T}}} \mathcal{H}^*_{\mathbb{T}}.$$

Its inverse is given by

$$\psi \mapsto \sum_{\alpha} \frac{\iota^*_\alpha \psi}{e^{\mathbb{T}}(N_\alpha)}.$$

Remark 7.5.2. In particular, every class $\psi \in H^*_{\mathbb{T}}(M) \otimes_{H^*_{\mathbb{T}}} \mathcal{H}^*_{\mathbb{T}}$ writes uniquely as

$$\psi = \sum_{\alpha} \iota_{\alpha*} \frac{\iota^*_\alpha \psi}{e^{\mathbb{T}}(N_\alpha)}. \tag{7.5.1}$$

Let M be a compact manifold with a $\mathbb{T}$-action and structure map $q\colon M \to \mathrm{pt}$. We have the equivariant pushforward $q_*\colon H^*_{\mathbb{T}}(M) \to H^*_{\mathbb{T}}$, which after tensoring by $\mathcal{H}^*_{\mathbb{T}}$ yields the integration map

$$\int_M \colon H^*_{\mathbb{T}}(M) \otimes_{H^*_{\mathbb{T}}} \mathcal{H}^*_{\mathbb{T}} \to \mathcal{H}^*_{\mathbb{T}}.$$

For any component $F_\alpha \subset M^{\mathbb{T}}$, the structure map $q_\alpha\colon F_\alpha \to \mathrm{pt}$ factors as $q \circ \iota_\alpha$ where $\iota_\alpha\colon F_\alpha \hookrightarrow M$ is the inclusion. The condition $q_{\alpha*} = q_* \circ \iota_{\alpha*}$ then allows us, simply by looking at (7.5.1), to deduce the following integration formula: for any equivariant class $\psi \in H^*_{\mathbb{T}}(M) \otimes_{H^*_{\mathbb{T}}} \mathcal{H}^*_{\mathbb{T}}$, one has

$$\int_X \psi = \sum_\alpha q_{\alpha*} \frac{\iota^*_\alpha \psi}{e^{\mathbb{T}}(N_\alpha)} = \sum_\alpha \int_{F_\alpha} \frac{\iota^*_\alpha \psi}{e^{\mathbb{T}}(N_\alpha)} \in \mathcal{H}^*_{\mathbb{T}}. \tag{7.5.2}$$

Let us go back to the algebraic setting (in the special case of finitely many fixed points).

Proposition 7.5.3. *Let X be a smooth $\mathbb{T}$-variety with finitely many fixed points. Set*

$$\mathsf{e} = \prod_{p \in X^{\mathbb{T}}} e^{\mathbb{T}}(T_p X) \in H^*_{\mathbb{T}},$$

and fix a multiplicative subset $S \subseteq H^*_{\mathbb{T}} \setminus 0$ containing e. Then

1. *The pullback map $S^{-1}\iota^*\colon S^{-1}H^*_{\mathbb{T}}(X) \to S^{-1}H^*_{\mathbb{T}}(X^{\mathbb{T}})$ is onto, and*
2. *if $H^*_{\mathbb{T}}(X)$ is a free $H^*_{\mathbb{T}}$-module of rank $r \leq |X^{\mathbb{T}}|$, then $r = |X^{\mathbb{T}}|$ and $S^{-1}\iota^*$ is an isomorphism.*

Proof. To prove (1), note that the composition

$$S^{-1}H^*_{\mathbb{T}}(X^{\mathbb{T}}) \xrightarrow{S^{-1}\iota_*} S^{-1}H^*_{\mathbb{T}}(X) \xrightarrow{S^{-1}\iota^*} S^{-1}H^*_{\mathbb{T}}(X^{\mathbb{T}})$$

is surjective because it equals $S^{-1}(\iota^* \circ \iota_*)$, and the determinant of $\iota^* \circ \iota_*$ is precisely e because it is a diagonal map, equal to $e^{\mathbb{T}}(T_p X)$ on the component indexed by p. But e becomes invertible after localisation by Proposition 7.4.3, so (1) follows.

By Part (1), we have

$$r = \mathrm{rk}_{S^{-1}H^*_{\mathbb{T}}} S^{-1}H^*_{\mathbb{T}}(X) \geq \mathrm{rk}_{S^{-1}H^*_{\mathbb{T}}} S^{-1}H^*_{\mathbb{T}}(X^{\mathbb{T}}) = |X^{\mathbb{T}}| \geq r,$$

which implies $\mathrm{rk}_{S^{-1}H^*_{\mathbb{T}}} S^{-1}H^*_{\mathbb{T}}(X) = |X^{\mathbb{T}}|$. To prove Part (2), observe that $S^{-1}H^*_{\mathbb{T}}$ is a Noetherian ring (being a localisation of a Noetherian ring), hence a surjective

$S^{-1}H_{\mathbb{T}}^*$-linear map of free modules of the same rank, such as $S^{-1}\iota^*$, is necessarily an isomorphism. □

Remark 7.5.4. Let X be a smooth complex projective variety, with an action by a torus $\mathbb{T}$ having finitely many fixed points $p_1,\ldots,p_s$. Then the Białynicki-Birula decomposition [10] yields s $\mathbb{T}$-invariant subvarieties

$$Y_1,\ldots,Y_s \subset X$$

with the property that the $\mathbb{T}$-equivariant cohomology classes (cf. Example 6.5.2)

$$[Y_\ell]^{\mathbb{T}} \in H_{\mathbb{T}}^*(X)$$

form a *free $H_{\mathbb{T}}^*$-basis* of the equivariant cohomology ring, restricting to a $\mathbb{Z}$-basis of the ordinary cohomology ring $H^*(X)$. In other words,

$$H_{\mathbb{T}}^*(X) \cong \bigoplus_{\ell=1}^{s} H_{\mathbb{T}}^* \cdot [Y_\ell]^{\mathbb{T}}$$

is a free $H_{\mathbb{T}}^*$-module of rank s.

The form of the localisation theorem that we will need is the following.

Corollary 7.5.5 (Integration Formula)**.** *Let X be a smooth complex projective $\mathbb{T}$-variety with finitely many fixed points. There is an identity*

$$\int_X \psi = \sum_{q\in X^{\mathbb{T}}} \frac{i_q^*\psi}{e^{\mathbb{T}}(T_qX)} \tag{7.5.3}$$

for all $\psi \in H_{\mathbb{T}}^*(X)$.

Proof. Let $S \subseteq H_{\mathbb{T}}^* \setminus 0$ be a multiplicative subset as in Theorem 7.5.3. By the surjectivity of

$$S^{-1}\iota_*\colon S^{-1}H_{\mathbb{T}}^*(X^{\mathbb{T}}) \xrightarrow{\sim} S^{-1}H_{\mathbb{T}}^*(X),$$

due to part (2) of Proposition 7.5.3, we may assume

$$\psi = \iota_{p*}\theta,$$

for $p \in X^{\mathbb{T}}$ and $\theta \in H_{\mathbb{T}}^*$. Then clearly

$$\int_X \psi = \int_X \iota_{p*}\theta = \theta,$$

because $(\int_X)\circ\iota_{p*}$ is an isomorphism. On the other hand,

$$\sum_{q\in X^{\mathbb{T}}} \frac{i_q^*\iota_{p*}\theta}{e^{\mathbb{T}}(T_qX)} = \frac{\iota_p^*\iota_{p*}\theta}{e^{\mathbb{T}}(T_pX)} = \frac{e^{\mathbb{T}}(T_pX)}{e^{\mathbb{T}}(T_pX)} \cap \theta = \theta.$$

□

Remark 7.5.6. Formula (7.5.2) is true in the algebraic context as well (for X a smooth projective variety) if one sets $S = H_{\mathbb{T}}^* \setminus 0$, but we will not need it in these notes.

Lemma 7.5.7. *Let M be a smooth oriented compact manifold with a torus action having finitely many fixed points $p_1,\ldots,p_s$. Then*

$$\chi(M) = s.$$

Proof. Recall that $\chi(M)=\int_M e(T_M)$ by Poincaré–Hopf. We have

$$\begin{aligned}\int_M e(T_M) &= \int_M e^{\mathbb{T}}(T_M)\\ &= \sum_{1\le i\le s} \frac{e^{\mathbb{T}}(T_M)|_{p_i}}{e^{\mathbb{T}}(N_{p_i/M})}\\ &= \sum_{1\le i\le s} 1\\ &= s.\end{aligned}$$ □

More generally, one can prove the following:

Lemma 7.5.8 ([16, Prop. 2.5.1])**.** *Let $\mathbb{T}$ be a torus acting on a quasi-projective scheme Y of finite type over $\mathbb{C}$. Then*

$$\chi(Y)=\chi(Y^{\mathbb{T}}).$$

The following result is also very useful in computations.

Lemma 7.5.9. *Let Y be a variety with a $\mathbb{T}$-action. Suppose the fixed locus $Y^{\mathbb{T}}$ is finite. Then there is a 1-dimensional subtorus $\mathbb{G}_m\subset\mathbb{T}$ such that $Y^{\mathbb{T}}=Y^{\mathbb{G}_m}$.*

EXERCISE 7.5.10. Compute the Euler characteristic of the Grassmannian $G(k,n)$. (**Hint**: Lift the standard torus action of $\mathbb{T}=\mathbb{G}_m^n$ on $\mathbb{C}^n$ to the Grassmannian).

EXERCISE 7.5.11. Let $n>0$ be an integer. Show that

$$\sum_{i=1}^n \frac{(-t_i)^k}{\displaystyle\prod_{\substack{1\le j\le n\\ j\neq i}}(t_i-t_j)} = \begin{cases} 0 & \text{if } 0\le k<n-1\\ 1 & \text{if } k=n-1.\end{cases}$$

(**Hint**: Consider the standard action on $X=\mathbb{P}^{n-1}$ and apply Equation (7.5.3) to $\psi=c_1^{\mathbb{T}}(\mathscr{O}_{\mathbb{P}^{n-1}}(1))$).

8. Applications of the localisation formula

In this section we give four complete examples on how to use Theorem 7.5.1 to solve enumerative problems.

8.1. **How not to compute the simplest intersection number.** In this section we show how to use localisation to compute the number of intersection points between two general lines in $\mathbb{P}^2$.

Let $\mathbb{T}=\mathbb{G}_m\subset\mathbb{G}_m^3$ be a one-parameter subgroup acting with weights w_0,w_1,w_2 on the vector space of linear forms

$$V=H^0(\mathbb{P}^2,\mathscr{O}_{\mathbb{P}^2}(1))=\mathrm{Span}_{\mathbb{C}}\{x_0,x_1,x_2\}.$$

This means that $t\cdot x_i=t^{w_i}x_i$. Choosing the w_i's distinct from one another ensures that the induced action on $\mathbb{P}^2$ has the three fixed points $p_0=(1,0,0)$, $p_1=(0,1,0)$, $p_2=(0,0,1)$. The intersection number we want to compute is

$$\int_{\mathbb{P}^2} c_1(\mathscr{O}_{\mathbb{P}^2}(1))^2=\int_{\mathbb{P}^2} c_1^{\mathbb{T}}(\mathscr{O}_{\mathbb{P}^2}(1))^2=\sum_{i=0}^2\frac{c_1^{\mathbb{T}}(\mathscr{O}_{\mathbb{P}^2}(1)|_{p_i})^2}{e^{\mathbb{T}}(T_{p_i}\mathbb{P}^2)}. \tag{8.1.1}$$

The universal exact sequence

$$0 \to \mathscr{S} \to V \otimes_{\mathbb{C}} \mathscr{O}_{\mathbb{P}^2} \to \mathscr{O}_{\mathbb{P}^2}(1) \to 0$$

restricts, at p_i, to a short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_{jk} & \longrightarrow & V & \longrightarrow & V/V_{jk} & \longrightarrow & 0 \\ & & & & & & \| & & \\ & & & & & & \mathbb{C}\cdot x_i & & \end{array}$$

where $V_{jk} \subset V$ is the span of x_j, x_k, i.e. the space of linear forms vanishing at p_i. We already know the weights of the tangent representations

$$T_{p_i}\mathbb{P}^2 = V_{jk}^\vee \otimes V/V_{jk} = \operatorname{Span}\{\, x_j^\vee \otimes x_i, x_k^\vee \otimes x_i \,\}.$$

They are simply $w_i - w_j, w_i - w_k$, where $\{\, i, j, k \,\} = \{0,1,2\}$. The line bundle $\mathscr{O}_{\mathbb{P}^2}(1)$ is the universal quotient bundle and clearly

$$\mathscr{O}_{\mathbb{P}^2}(1)|_{p_i} = V/V_{jk} = \mathbb{C}\cdot x_i$$

has weight $c_1^{\mathbb{T}}(\mathscr{O}_{\mathbb{P}^2}(1)|_{p_i}) = -w_i$ at p_i by Example 6.5.5. The sum in (8.1.1) then equals

$$\frac{(-w_0)^2}{(w_0-w_1)(w_0-w_2)} + \frac{(-w_1)^2}{(w_1-w_0)(w_1-w_2)} + \frac{(-w_2)^2}{(w_2-w_0)(w_2-w_1)}. \tag{8.1.2}$$

The latter equals 1 for *every* choice of (pairwise distinct) w_0, w_1, w_2.

EXERCISE 8.1.1. Compare the latter calculation to the one of Exercise 7.5.11.

Remark 8.1.2. The fact that the sum (8.1.2) equals 1 has the following interpretation.[7] The three fractions above can be seen as the residues of the differential form

$$\frac{z^2\,\mathrm{d}z}{(z-w_0)(z-w_1)(z-w_2)}.$$

However, there is another residue to compute: the one at ∞. This residue equals -1. The residue theorem then precisely states that

$$0 = -1 + \frac{w_0^2}{(w_0-w_1)(w_0-w_2)} + \frac{w_1^2}{(w_1-w_0)(w_1-w_2)} + \frac{w_2^2}{(w_2-w_0)(w_2-w_1)}.$$

8.2. **The 27 lines on a cubic surface.** Let $\mathbb{T} = \mathbb{G}_m$ be a torus acting on $\mathbb{P}^3$ with distinct weights (w_0, w_1, w_2, w_3). This means, as ever,

$$t \cdot x_i = t^{w_i} x_i, \quad 0 \le i \le 3.$$

This is also equivalent to considering the vector space

$$V = H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(1)) = \operatorname{Span}\{\, x_0, x_1, x_2, x_3 \,\}$$

as an equivariant vector bundle over $\operatorname{Spec}\mathbb{C}$, splitting into a sum of characters $t \mapsto t^{w_i}$. The torus action has four fixed points $p_0, \ldots, p_3 \in \mathbb{P}^3$ and six invariant lines $\ell_{ij} \subset \mathbb{P}^3$ which are the lines joining the fixed points (see Figure 9). These correspond to the fixed points of the Grassmannian $\mathbb{G}(1,3)$ under the same $\mathbb{T}$-action. You computed the Euler characteristic of the Grassmannian in Exercise 7.5.10.

[7] Thanks to Fran Globlek for pointing this out.

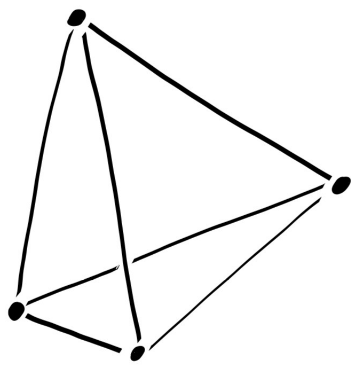


FIGURE 9. The toric polytope of $\mathbb{P}^3$.

Let S be a general cubic hypersurface in $\mathbb{P}^3$, defined by a homogeneous cubic polynomial $f \in \mathbb{C}[x_0, x_1, x_2, x_3]$. A line $\ell \subset \mathbb{P}^3$ is contained in S if and only if the image of f under the restriction map

$$H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(3)) \to H^0(\ell, \mathscr{O}_\ell(3))$$

vanishes. We know from Section 4 that the cycle of lines in S is Poincaré dual to the top Chern class

$$e(\operatorname{Sym}^3 \mathscr{S}^\vee) = c_4(\operatorname{Sym}^3 \mathscr{S}^\vee) \in A^4 \mathbb{G}(1,3),$$

where $\mathscr{S}^\vee$ is the dual of the tautological subbundle. By Lemma 4.0.4, we have

$$\#\{\text{lines in } S\} = \int_{\mathbb{G}(1,3)} e(\operatorname{Sym}^3 \mathscr{S}^\vee).$$

According to the strategy outlined in Section 6.3.1, we will compute the latter intersection number by computing instead the equivariant integral

$$\int_{\mathbb{G}(1,3)} e^{\mathbb{T}}(\operatorname{Sym}^3 \mathscr{S}^\vee) \in \mathcal{H}_{\mathbb{T}}$$

via the localisation formula, and specialising the equivariant parameters appropriately (i.e. avoiding the creation of poles in the localisation formula).

Restricting the tautological exact sequence

$$0 \to \mathscr{S} \to \mathscr{O}_{\mathbb{G}(1,3)} \otimes_{\mathbb{C}} V^\vee \to \mathscr{Q} \to 0 \tag{8.2.1}$$

to the point corresponding to the linear subspace

$$\ell_{ij} = \mathbb{C} \cdot x_i^\vee \oplus \mathbb{C} \cdot x_j^\vee \subset V^\vee,$$

we obtain the sequence of equivariant vector spaces

$$0 \to \mathbb{C} \cdot x_i^\vee \oplus \mathbb{C} \cdot x_j^\vee \to V^\vee \to \mathbb{C} \cdot x_h^\vee \oplus \mathbb{C} \cdot x_k^\vee \to 0$$

where $\{0,1,2,3\} = \{i,j,h,k\}$. Therefore

$$\mathscr{Q}|_{\ell_{ij}} = \mathbb{C} \cdot x_h^\vee \oplus \mathbb{C} \cdot x_k^\vee$$

has weights $-w_h, -w_k$, and similarly

$$\mathscr{S}^\vee|_{\ell_{ij}} = \mathbb{C} \cdot x_i \oplus \mathbb{C} \cdot x_j$$

has weights w_i and w_j. Since

$$T_{\ell_{ij}}\mathbb{G}(1,3) = \mathscr{S}^\vee|_{\ell_{ij}} \otimes \mathscr{Q}|_{\ell_{ij}} = \operatorname{Span}\left\{ x_i \otimes x_h^\vee, x_j \otimes x_h^\vee, x_i \otimes x_k^\vee, x_j \otimes x_k^\vee \right\},$$

we obtain an identity

$$e^{\mathbb{T}}(T_{\ell_{ij}}\mathbb{G}(1,3)) = (w_i - w_h)\cdot(w_j - w_h)\cdot(w_i - w_k)\cdot(w_j - w_k) \in H^*_{\mathbb{T}}. \tag{8.2.2}$$

This is one of the key ingredients in the localisation formula, which reads

$$\int_{\mathbb{G}(1,3)} e^{\mathbb{T}}(\operatorname{Sym}^3 \mathscr{S}^\vee) = \sum_{\ell_{ij}} \frac{e^{\mathbb{T}}(\operatorname{Sym}^3 \mathscr{S}^\vee)|_{\ell_{ij}}}{e^{\mathbb{T}}(T_{\ell_{ij}}\mathbb{G}(1,3))}. \tag{8.2.3}$$

The numerators in (8.2.3) are computed as follows. Note that $\operatorname{Sym}^3 \mathscr{S}^\vee|_{\ell_{ij}} = \operatorname{Sym}^3(\mathbb{C}\cdot x_i \oplus \mathbb{C}\cdot x_j)$ is the four dimensional vector space generated by the classes of $x_i^3, x_i^2 x_j, x_i x_j^2, x_j^3$. Using the weights w_i and w_j of $\mathscr{S}^\vee$ we find

$$e^{\mathbb{T}}(\operatorname{Sym}^3 \mathscr{S}^\vee)|_{\ell_{ij}} = (3w_i)\cdot(2w_i + w_j)\cdot(w_i + 2w_j)\cdot(3w_j) \in H^*_{\mathbb{T}}.$$

We are now able to write down the right hand side of (8.2.3) as follows:

$$\begin{aligned}
\sum_{0\le i<j\le 3} \frac{e^{\mathbb{T}}(\operatorname{Sym}^3 \mathscr{S}^\vee)|_{\ell_{ij}}}{e^{\mathbb{T}}(T_{\ell_{ij}}\mathbb{G}(1,3))} &= \sum_{0\le i<j\le 3} \frac{(3w_i)(2w_i + w_j)(w_i + 2w_j)(3w_j)}{(w_i - w_h)(w_j - w_h)(w_i - w_k)(w_j - w_k)} \\
&= 9\frac{w_0(2w_0 + w_1)(w_0 + 2w_1)w_1}{(w_0 - w_2)(w_0 - w_3)(w_1 - w_2)(w_1 - w_3)} \\
&\quad + 9\frac{w_0(2w_0 + w_2)(w_0 + 2w_2)w_2}{(w_0 - w_1)(w_0 - w_3)(w_2 - w_1)(w_2 - w_3)} \\
&\quad + 9\frac{w_0(2w_0 + w_3)(w_0 + 2w_3)w_3}{(w_0 - w_1)(w_0 - w_2)(w_3 - w_1)(w_3 - w_2)} \\
&\quad + 9\frac{w_1(2w_1 + w_2)(w_1 + 2w_2)w_2}{(w_1 - w_0)(w_1 - w_3)(w_2 - w_0)(w_2 - w_3)} \\
&\quad + 9\frac{w_1(2w_1 + w_3)(w_1 + 2w_3)w_3}{(w_1 - w_0)(w_1 - w_2)(w_3 - w_0)(w_3 - w_2)} \\
&\quad + 9\frac{w_2(2w_2 + w_3)(w_2 + 2w_3)w_3}{(w_2 - w_0)(w_2 - w_1)(w_3 - w_0)(w_3 - w_1)}.
\end{aligned} \tag{8.2.4}$$

Evaluating at $(w_0, w_1, w_2, w_3) = (0, 2, -1, 1)$ yields

$$\#\{\text{lines in } S\} = 9\left(0+0+0+0+\frac{40}{12} - \frac{1}{3}\right) = 30 - 3 = 27.$$

8.3. **Lines on the quintic 3-fold.** In this section we will prove the following result.

Theorem 8.3.1. *Let $Y \subset \mathbb{P}^4$ be a generic quintic 3-fold. Then Y contains exactly 2875 lines.*

Remark 8.3.2. The statement is not true for *all* quintic Calabi–Yau 3-folds. Indeed, by work of Albano–Katz [1], the Fermat quintic

$$\left\{ x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0 \right\} \subset \mathbb{P}^4$$

contains 50 one-dimensional families of lines.

We will need an auxiliary result by Katz.

Theorem 8.3.3 ([45, App. A]). *Let Y be a generic quintic 3-fold, $C \subset Y$ a smooth rational curve of degree $d \leq 3$. Then C has normal bundle $\mathscr{O}_C(-1)^{\oplus 2}$.*

The role of this theorem, for us, is to make sure that the intersection number we compute via localisation really is the number we are after. The above theorem, in other words, plays the role of Lemma 4.0.4 that we needed for lines on a cubic surface.

The ambient space we have to work in now is the 6-dimensional Grassmannian

$$G(2,5) = \mathbb{G}(1,4).$$

Let $\mathscr{S}$ be the rank 2 universal subbundle

$$\mathscr{S} \hookrightarrow \mathscr{O}_{\mathbb{G}(1,4)} \otimes_{\mathbb{C}} H^0(\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}(1))^\vee.$$

Let $\mathbb{T} = \mathbb{G}_m^5$ be a torus acting as

$$t \cdot x_i = t_i \cdot x_i$$

on $\mathbb{P}^4$. This action lifts to a $\mathbb{T}$-action on $\mathbb{G}(1,4)$, with 10 fixed points corresponding to the subsets

$$I \subset \{0,1,2,3,4\}, \quad |I| = 2.$$

Let $\ell_{ij} \subset V^\vee$ be the linear subspace spanned by $x_i^\vee$ and $x_j^\vee$. The characters of the 2-dimensional (equivariant) vector space

$$\mathscr{S}|_{\ell_{ij}} = \mathbb{C} \cdot x_i^\vee \oplus \mathbb{C} \cdot x_j^\vee$$

are $\{-\chi_i, -\chi_j\}$, where $\chi_i : \mathbb{T} \to \mathbb{C}^\times$ sends $t \mapsto t_i$. Similarly, the characters of $\mathscr{Q}|_{\ell_{ij}}$ are $\{-\chi_a, -\chi_b, -\chi_c\}$, where $\{a,b,c,i,j\} = \{0,1,2,3,4\}$. Let $t_k \in H^2_{\mathbb{T}}$ denote the weight of $-\chi_k$, for all k. The tangent space of $\mathbb{G}(1,4)$ at ℓ_{ij} is, as ever,

$$\mathscr{S}^\vee|_{\ell_{ij}} \otimes \mathscr{Q}|_{\ell_{ij}} = \operatorname{Span}\left\{ x_i \otimes x_u^\vee, x_j \otimes x_u^\vee \,\middle|\, u \in \{a,b,c\} \right\},$$

so we obtain the Euler class

$$e^{\mathbb{T}}(T_{\ell_{ij}} \mathbb{G}(1,4)) = \prod_{k \notin I} (t_i - t_k)(t_j - t_k).$$

The rank 6 vector bundle

$$\begin{array}{ccc} \operatorname{Sym}^5(\mathscr{S}^\vee) & & \operatorname{Sym}^5(\mathscr{S}^\vee)|_{\ell_{ij}} \\ \big\downarrow & \rightsquigarrow & \big\downarrow \\ \mathbb{G}(1,4) & & \mathsf{pt} \end{array}$$

inherits weights

$$5t_j, t_i + 4t_j, 2t_i + 3t_j, 3t_i + 2t_j, 4t_i + t_j, 5t_i.$$

Therefore, denoting i_1 and i_2 the generic elements of a subset I of size 2, the localisation formula reads

$$\int_{\mathbb{G}(1,4)} e^{\mathbb{T}}(\operatorname{Sym}^5(\mathscr{S}^\vee)) = \sum_{|I|=2} \frac{\prod_{h=0}^5 (h t_{i_1} + (5-h) t_{i_2})}{\prod_{i \in I} \prod_{k \notin I} (t_i - t_k)} = 2875. \tag{8.3.1}$$

If you do not believe the last identity, you can copy the following code[8] in Mathematica:

[8]Thanks to Matteo Gallet for providing the code.

```
Plus @@
(Product[h Subscript[t, #[[1]]] + (5 - h)
Subscript[t, #[[2]]], {h, 0, 5}]/
Product[Subscript[t, i] - Subscript[t, k], {i, #},
{k, DeleteCases[Range[0, 4], Alternatives @@ #]}]
& /@ Subsets[Range[0, 4], {2}])
// Together
```

Theorem 8.3.1 now follows from Theorem 8.3.3.

8.4. **The number of lines through 4 general lines in 3-space.** The goal of this section is to prove the following.

Proposition 8.4.1. *There are exactly two lines $\ell \subset \mathbb{P}^3$ intersecting four general lines.*

Let L be a general 2-dimensional linear subspace of $V = H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(1))$. Consider the cycle

$$\Sigma_1(L) = \{\Lambda \subset V \mid \Lambda \cap L \neq 0\}.$$

EXERCISE 8.4.2. Show that $\Sigma_1(L)$ is 3-dimensional, and its cohomology class

$$\sigma_1 = [\Sigma_1(L)] \in A^1 G(2,4)$$

is independent of L.

We need the following preliminary result.

Lemma 8.4.3. *The cycle $\Sigma_1 \subset \mathbb{G}(1,3)$ of lines meeting a general line $L \subset \mathbb{P}^3$ is Poincaré dual to the first Chern class*

$$c_1(\mathscr{S}^\vee).$$

The goal now becomes to compute the intersection number

$$\int_{\mathbb{G}(1,3)} \sigma_1^4 = \int_{\mathbb{G}(1,3)} c_1(\mathscr{S}^\vee)^4.$$

We already saw that for the $\mathbb{T}$-fixed linear subspace

$$\ell_{ij} = \mathbb{C} \cdot x_i^\vee \oplus \mathbb{C} \cdot x_j^\vee \subset H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(1))^\vee$$

the weights of the 2-dimensional representation

$$\mathscr{S}^\vee|_{\ell_{ij}} = \mathbb{C} \cdot x_i \oplus \mathbb{C} \cdot x_j$$

are w_i and w_j. Therefore

$$c_1^{\mathbb{T}}(\mathscr{S}^\vee)|_{\ell_{ij}} = w_i + w_j.$$

The weights of the tangent representation have been computed in (8.2.2). Putting everything together we can compute, by means of the localisation formula,

$$\begin{aligned}
\int_{\mathbb{G}(1,3)} c_1^{\mathbb{T}}(\mathscr{S}^\vee)^4 &= \sum_{\ell_{ij}} \frac{c_1^{\mathbb{T}}(\mathscr{S}^\vee)|_{\ell_{ij}}}{e^{\mathbb{T}}(T_{\ell_{ij}}\mathbb{G}(1,3))} \\
&= \sum_{0\le i<j\le 3} \frac{(w_i+w_j)^4}{(w_i-w_h)(w_j-w_h)(w_i-w_k)(w_j-w_k)} \\
&= \frac{(w_2+w_3)^4}{(w_2-w_0)(w_2-w_1)(w_3-w_0)(w_3-w_1)} \\
&+ \frac{(w_1+w_3)^4}{(w_1-w_0)(w_1-w_2)(w_3-w_0)(w_3-w_2)} \\
&+ \frac{(w_1+w_2)^4}{(w_1-w_0)(w_1-w_3)(w_2-w_0)(w_2-w_3)} \\
&+ \frac{(w_0+w_3)^4}{(w_0-w_1)(w_0-w_2)(w_3-w_1)(w_3-w_2)} \\
&+ \frac{(w_0+w_2)^4}{(w_0-w_1)(w_0-w_3)(w_2-w_1)(w_2-w_3)} \\
&+ \frac{(w_0+w_1)^4}{(w_0-w_2)(w_0-w_3)(w_1-w_2)(w_1-w_3)}.
\end{aligned}$$

Specialising the equivariant parameters at, say, $(w_0,w_1,w_2,w_3)=(-2,1,-1,2)$, we obtain

$$\begin{aligned}
\int_{\mathbb{G}(1,3)} \sigma_1^4 = \frac{(-1+2)^4}{1\cdot(-2)\cdot 4\cdot 1} + \frac{(1+2)^4}{3\cdot 2\cdot 4\cdot 3} + 0 + 0 + \frac{(-2-1)^4}{(-3)\cdot(-4)\cdot(-2)\cdot(-3)} + \frac{(-2+1)^4}{(-1)\cdot(-4)\cdot 2\cdot(-1)} \\
= -\frac{1}{8} + \frac{81}{72} + \frac{81}{72} - \frac{1}{8} = \frac{-1+9+9-1}{8} = 2,
\end{aligned}$$

in agreement with the calculation of Proposition 2.3.3.

In order to complete the proof of Proposition 8.4.1, we have to address a transversality issue. In other words, we need to exclude the following cases, which are not automatically ruled out by the computation we just completed:

(1) There is *one* line through 4 general lines, appearing with multiplicity two;
(2) There are infinitely many lines through 4 general lines.

In other words, what we know is the following: *if* the number of lines through four general lines is finite, then it is ≤ 2.

We employ the following powerful theorem of Kleiman to rule out these two degenerate cases at the same time.

Theorem 8.4.4 (Kleiman transversality)**.** *Let G be an algebraic group acting transitively on a variety X over an algebraically closed field of characteristic zero. Let $Y \subset X$ be a subvariety. Then*

(a) *given another subvariety $Z \subset X$, there exists a dense open subset $U \subset G$ such that $g\cdot Y$ and Z are generically transverse for all $g \in U$.*
(b) *If G is affine, then $[g\cdot Y]=[Y]$ in the Chow group A_*X.*

We apply Kleiman transversality to $X = \mathbb{G}(1,3)$, acted on by the affine algebraic group $G = \mathrm{GL}(V) = \mathrm{GL}_4$. For general lines ℓ_1 and ℓ_2 in $\mathbb{P}^3$, form the codimension one cycles

$$\Gamma_i = \left\{ \ell \subset \mathbb{P}^3 \,\middle|\, \ell \cap \ell_i \neq \emptyset \right\}, \quad i = 1,2.$$

Then set $Y = \Gamma_1$ and $Z = \Gamma_2$.

EXERCISE 8.4.5. Let $C_1, \ldots, C_4 \subset \mathbb{P}^3$ be general translated of curves of degree $d_1, \ldots, d_4$. Compute the number of lines $\ell \subset \mathbb{P}^3$ passing through $C_1, \ldots, C_4$. (**Hint**: Start out by proving that the cycle $\Gamma_C \subset \mathbb{G}(1,3)$ of lines meeting a curve $C \subset \mathbb{P}^3$ of degree d is a divisor in the Grassmannian, with cohomology class $d \cdot \sigma_1$).

9. TORUS ACTION ON THE HILBERT SCHEME OF POINTS

In this section we again work with the Hilbert scheme

$$\mathrm{Hilb}^n \mathbb{A}^d.$$

We view it as a fine moduli space of ideals

$$I \subset \mathbb{C}[x_1, \ldots, x_d]$$

of colength n.

9.1. **The torus action.**

Consider the d-dimensional torus

$$\mathbb{T} = \mathbb{G}_m^d$$

acting on $\mathbb{A}^d$ by rescaling the coordinates:

$$t \cdot (a_1, \ldots, a_d) = (t_1 a_1, \ldots, t_d a_d). \tag{9.1.1}$$

EXERCISE 9.1.1. Show that the action (9.1.1) lifts to a $\mathbb{T}$-action on $\mathrm{Hilb}^n \mathbb{A}^d$.

EXERCISE 9.1.2. Show that a $\mathbb{T}$-fixed subscheme $Z \subset \mathbb{A}^d$ is entirely supported at the origin $0 \in \mathbb{A}^d$. (**Hint**: Show that $\mathrm{Supp}(t \cdot [Z]) = t^{-1} \cdot \mathrm{Supp}(Z)$).

Proposition 9.1.3. *An ideal $I \in \mathrm{Hilb}^n \mathbb{A}^d$ is $\mathbb{T}$-fixed if and only if it is a monomial ideal.*

Proof. Recall that the character lattice of the torus $\mathbb{T}^* = \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ is isomorphic to $\mathbb{Z}^d$, since each character $\mathbb{T} \to \mathbb{G}_m$ is necessarily of the form

$$\chi_{\mathsf{m}} : (t_1, \ldots, t_d) \mapsto t_1^{m_1} \cdots t_d^{m_d}$$

for some $\mathsf{m} = (m_1, \ldots, m_d) \in \mathbb{Z}^d$. As an initial step, we notice that the geometric action (9.1.1) dualises to a $\mathbb{T}$- action on $\mathbb{C}[x_1, \ldots, x_d]$ via the rule

$$t \cdot f = f \circ t. \tag{9.1.2}$$

This already shows that a monomial ideal is necessarily $\mathbb{T}$-fixed, so it remains to prove the converse.

We next show that the monomials

$$\mathbf{x}^{\mathsf{m}} = x_1^{m_1} \cdots x_d^{m_d}$$

form an eigenbasis of $\mathbb{C}[x_1,\ldots,x_d]$ as a $\mathbb{T}$-representation. An *eigenvector* of a $\mathbb{T}$-representation V, in this context, is an element $v\in V$ for which there exists a character $\chi\in\mathbb{T}^*$ such that $t\cdot v=\chi(t)v$ for all $t\in\mathbb{T}$. (This χ plays the role of "classical" eigenvalues in linear algebra.) For us, $V=\mathbb{C}[x_1,\ldots,x_d]$. Pick $v=\mathbf{x}^{\mathsf{m}}$. Then according to the rule (9.1.2) one has

$$t\cdot\mathbf{x}^{\mathsf{m}}=(t_1x_1)^{m_1}\cdots(t_dx_d)^{m_d}=(t_1^{m_1}\cdots t_d^{m_d})\cdot(x_1^{m_1}\cdots x_d^{m_d})=\chi_{\mathsf{m}}(t)\cdot\mathbf{x}^{\mathsf{m}}.$$

So each monomial $\mathbf{x}^{\mathsf{m}}$ is an eigenvector with respect to the weight χ_{m}. In particular, each corresponds to a different weight, therefore all weight spaces

$$V_{\mathsf{m}}=\left\{\,f\in\mathbb{C}[x_1,\ldots,x_d]\,\middle|\,t\cdot f=\chi_{\mathsf{m}}(t)f\text{ for all }t\in\mathbb{T}\,\right\}$$

are 1-dimensional $\mathbb{T}$-subrepresentations (each spanned by $\mathbf{x}^{\mathsf{m}}$) and the action (9.1.2) is diagonalisable by monomials.

Now pick a $\mathbb{T}$-fixed ideal $I\subset\mathbb{C}[x_1,\ldots,x_d]$. In particular, as a vector space, I is a $\mathbb{T}$-subrepresentation. But a $\mathbb{T}$-subrepresentation of a diagonalisable $\mathbb{T}$-representation is again diagonalisable (prove this!), so I has a basis of eigenvectors. But each eigenvector is of the form $f=\mathbf{x}^{\mathsf{m}}\cdot g$, where $g\in\mathbb{C}[u]$ satisfies $g(0)\neq 0$. Since $V(I)$ is entirely supported at the origin (cf. Exercise 9.1.2), the zero locus of I is disjoint from the set of $(x_1,\ldots,x_d)$ such that $g(x_1\cdots x_d)=0$. It follows from Hilbert's Nullstellensatz that the monomial $\mathbf{x}^{\mathsf{m}}\in I$. $\square$

Definition 9.1.4. Let $d\geq 1$ and $n\geq 0$ be integers. A $(d-1)$-dimensional partition of n is a collection of n points $\mathcal{A}=\{\mathbf{a}_1,\ldots,\mathbf{a}_n\}\subset\mathbb{N}^d$ with the following property: if $\mathbf{a}_i=(a_{i1},\ldots,a_{id})\in\mathcal{A}$, then whenever a point $\mathbf{y}=(y_1,\ldots,y_d)\in\mathbb{N}^d$ satisfies $0\leq y_j\leq a_{ij}$ for all $j=1,\ldots,d$, one has $\mathbf{y}\in\mathcal{A}$. We set

$$\mathsf{P}_{d-1}(n)=\big|\{\,(d-1)\text{-dimensional partitions of }n\,\}\big|.$$

EXERCISE 9.1.5. Show that $P_0(n)=1$ for all $n\geq 0$.

Notation 9.1.6. We indicate a classical (i.e. 1-dimensional) partition by

$$\alpha=(1^{\alpha_1}\cdots i^{\alpha_i}\cdots\ell^{\alpha_\ell}).$$

The notation means that there are α_i parts of length i, and we set

$$\|\alpha\|=\sum_i\alpha_i,\quad |\alpha|=\sum_i i\alpha_i.$$

The latter is the *size* of α, the former is the number of *distinct parts* of α. The number $\ell=\ell(\alpha)$ is the *length* of α.

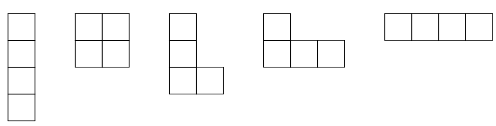


FIGURE 10. The five partitions $\alpha_1,\ldots,\alpha_5$ of 4.

Example 9.1.7. Set $d = 2$. Then a 1-dimensional partition is the same thing as a Young diagram. If $d = 3$, a *plane partition* of n is the same thing as a way of stacking n boxes in the corner of a room (assuming gravity points in the $(-1,-1,-1)$ direction!). See Figures 11 and 12 for a visual explanation.

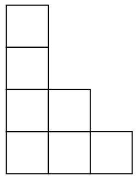


FIGURE 11. A Young diagram, representing the partition $\alpha = (1^1 2^1 4^1)$, and corresponding to the monomial ideal $I = (x^3, x^2y, xy^2, y^4)$ of colength 7.

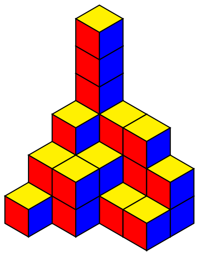


FIGURE 12. A plane partition.

The generating function of the numbers of partitions (for $d = 2$) is given by the following formula.

Theorem 9.1.8 (Euler)**.** *There is an identity*

$$\sum_{n\geq 0} \mathsf{P}_1(n) q^n = \prod_{m\geq 1} (1-q^m)^{-1}.$$

Proof. By inspection. For instance, say we want to compute $\mathsf{P}_1(4)$, that we know equals 5 (cf. Figure 10). Then expanding

$$\prod_{m\geq 1}(1-q^m)^{-1} = (1+q^1+q^{1+1}+q^{1+1+1}+q^{1+1+1+1}+\cdots)$$
$$\cdot(1+q^2+q^{2+2}+\cdots)\cdot(1+q^3+\cdots)\cdot(1+q^4+\cdots)\cdots$$

we see that to compute the coefficient of q^4 we have to sum the coefficients of

$$q^1 \cdot q^3$$
$$q^{1+1}\cdot q^2$$
$$q^{1+1+1+1}$$
$$q^{2+2}$$
$$q^4.$$

These clearly correspond to partitions of 4. □

Next, we will see that monomial ideals correspond to partitions. This is best explained via a picture — see Figure 13.

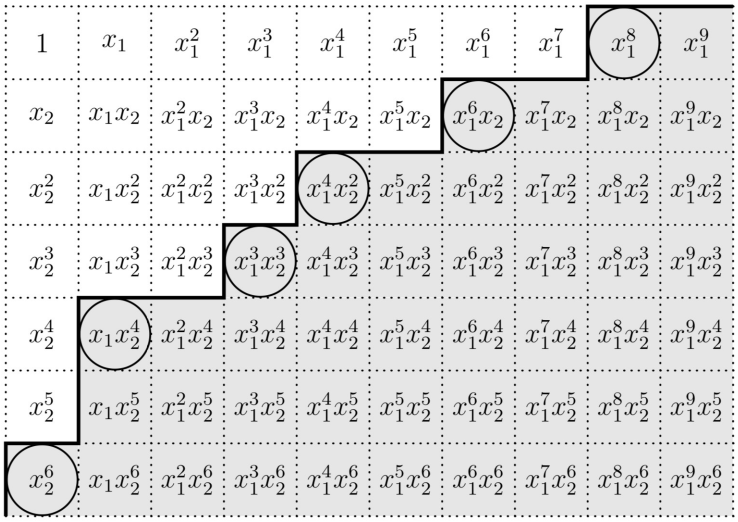


FIGURE 13. The staircase defined by the generators of a monomial ideal of colength n draws a 1-dimensional partition (Young tableau) of size n. Picture stolen from Okounkov [57].

EXERCISE 9.1.9. Show that there is a bijective correspondence between $\mathbb{T}$-fixed subschemes $Z \subset \mathbb{A}^d$ of length n and $(d-1)$-dimensional partitions of n.

9.2. **Euler characteristic of Hilbert schemes.** Let X be a smooth quasi-projective variety of dimension d. In this section we give a formula for the series

$$\mathsf{H}_X(q) = \sum_{n\geq 0} \chi(\operatorname{Hilb}^n X) q^n.$$

We use a combinatorial lemma, combined with a stratification argument. The following lemma will also be used repeatedly.

Lemma 9.2.1. *Let $X \to S$ be a morphism of algebraic varieties over $\mathbb{C}$ such that the Euler characteristic of the fibres $\chi(X_s) = m$ does not depend on $s \in S$. Then*

$$\chi(X) = \chi(S)\cdot m.$$

Lemma 9.2.2 ([70, p. 40])**.** *Let $P(q) = 1+\sum_{n>0} p_n q^n \in \mathbb{Q}[\![q]\!]$ be a formal power series. If χ is an integer, then*

$$P(q)^\chi = 1 + \sum_\alpha \left(\prod_{j=0}^{\|\alpha\|-1} (\chi - j)\cdot \frac{\prod_i p_i^{\alpha_i}}{\prod_i \alpha_i!}\right) q^{|\alpha|}.$$

Theorem 9.2.3. *Let X be a smooth quasi-projective variety of dimension d. There is an identity*

$$\mathsf{H}_X(q) = \left(\sum_{n\geq 0} \mathsf{P}_{d-1}(n) q^n\right)^{\chi(X)}.$$

Proof. The case $X = \mathbb{A}^d$ follows from the observation that

$$\chi(\operatorname{Hilb}^n \mathbb{A}^d) = \left|(\operatorname{Hilb}^n \mathbb{A}^d)^{\mathbb{T}}\right| = \mathsf{P}_{d-1}(n) \tag{9.2.1}$$

along with $\chi(\mathbb{A}^d) = 1$. The identities (9.2.1) follow from Lemma 7.5.8 and Exercise 9.1.9.

For general X, we proceed as follows. First of all, notice that

$$\chi(\operatorname{Hilb}^n(\mathbb{A}^d)_0) = \mathsf{P}_{d-1}(n),$$

because the punctual Hilbert scheme contains the $\mathbb{T}$-fixed locus (Exercise 9.1.2). Let us stratify the symmetric product

$$\operatorname{Sym}^n X = \coprod_{\alpha \vdash n} \operatorname{Sym}^n_\alpha X$$

according to partitions of n. Each stratum dictates the multiplicity of the supporting points in a given zero-cycle. Set

$$\operatorname{Hilb}^n_\alpha X = \pi^{-1}(\operatorname{Sym}^n_\alpha X),$$

where π is the Hilbert–Chow morphism (5.1.1). On the deepest stratum, corresponding to the full partition (n), i.e. to the small diagonal $X \hookrightarrow \operatorname{Sym}^n X$, we have that

$$\operatorname{Hilb}^n_{(n)} X \to X$$

is Zariski locally trivial with fibre $\operatorname{Hilb}^n(\mathbb{A}^d)_0$. This follows easily from the local case, where in fact there is a global decomposition

$$\operatorname{Hilb}^n_{(n)} \mathbb{A}^d = \mathbb{A}^d \times \operatorname{Hilb}^n(\mathbb{A}^d)_0,$$

and π is identified with the first projection. For an arbitrary partition α, let

$$V_\alpha \subset \prod_i (\operatorname{Hilb}^i X)^{\alpha_i}$$

be the open locus of clusters with pairwise disjoint support.

EXERCISE 9.2.4. Use the infinitesimal criterion for étale maps to show that taking the "union of points" defines an (étale) morphism

$$f_\alpha \colon V_\alpha \to \operatorname{Hilb}^n X.$$

Let U_α denote the image of the morphism f_α. Then U_α contains the stratum $\operatorname{Hilb}^n_\alpha X$ as a closed subscheme. We can form the fibre diagram

$$\begin{tikzcd} Z_\alpha \arrow[r, hook] \arrow[d, "f_\alpha"'] \arrow[dr, phantom, "\square"] & V_\alpha \arrow[r, hook] \arrow[d, "\text{étale}"] & \prod_i (\operatorname{Hilb}^i X)^{\alpha_i} \\ \operatorname{Hilb}^n_\alpha X \arrow[r, hook] & U_\alpha \arrow[r, hook] & \operatorname{Hilb}^n X \end{tikzcd}$$

defining the scheme Z_α. Now the map f_α on the left is a finite étale G_α-cover, where $G_\alpha = \prod_i \mathfrak{S}_{\alpha_i}$ is the automorphism group of the partition α. In other words, the only difference between Z_α and $\mathrm{Hilb}^n_\alpha X$ is the relabelling of points that appear with the same α-multiplicity. So we find

$$\begin{aligned}\chi(\mathrm{Hilb}^n X) &= \sum_\alpha \chi(\mathrm{Hilb}^n_\alpha X)\\ &= \sum_\alpha \frac{\chi(Z_\alpha)}{|G_\alpha|}\\ &= \sum_\alpha \frac{\chi(Z_\alpha)}{\prod_i \alpha_i!}.\end{aligned}$$

On the other hand, it is easy to see that

$$Z_\alpha = \prod_i \mathrm{Hilb}^i_{(i)}(X)^{\alpha_i} \setminus \Delta^{\mathrm{big}}$$

also fits in a cartesian diagram

$$\begin{tikzcd} Z_\alpha \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \prod_i \mathrm{Hilb}^i_{(i)}(X)^{\alpha_i} \arrow[d, "p"] \\ X^{\|\alpha\|} \setminus \Delta^{\mathrm{big}} \arrow[r, hook] & X^{\|\alpha\|} \end{tikzcd}$$

where $\|\alpha\|$ is the number of distinct parts in the partition. Now p is a product of Zariski locally trivial fibrations with fibre $\mathrm{Hilb}^i(\mathbb{A}^d)_0$, therefore

$$\begin{aligned}\chi(Z_\alpha) &= \chi(X^{\|\alpha\|} \setminus \Delta^{\mathrm{big}}) \cdot \prod_i \chi(\mathrm{Hilb}^i(\mathbb{A}^d)_0)^{\alpha_i}\\ &= \prod_{j=0}^{\|\alpha\|-1} \big(\chi(X) - j\big) \cdot \prod_i \mathsf{P}_{d-1}(i)^{\alpha_i}.\end{aligned}$$

Putting everything together, we find

$$\mathsf{H}_X(q) = \sum_\alpha \left(\prod_{j=0}^{\|\alpha\|-1} \big(\chi(X) - j\big) \cdot \frac{\prod_i \mathsf{P}_{d-1}(i)^{\alpha_i}}{\prod_i \alpha_i!} \right) q^{|\alpha|}.$$

The result now follows from the combinatorial formula of Lemma 9.2.2. □

Definition 9.2.5. The MacMahon function is the infinite product

$$\mathsf{M}(q) = \prod_{m \geq 1} (1 - q^m)^{-m}.$$

The identity of Theorem 9.2.3 specialises to the following in low dimension.

Theorem 9.2.6. *Let X be a smooth quasi-projective variety of dimension $d = 1, 2$ or 3. Then*

$$\sum_{n \geq 0} \chi(\mathrm{Hilb}^n X) q^n = \begin{cases} (1-q)^{-\chi(X)} & \text{if } d = 1 \\ \displaystyle\prod_{m \geq 1} (1 - q^m)^{-\chi(X)} & \text{if } d = 2 \\ \mathsf{M}(q)^{\chi(X)} & \text{if } d = 3 \end{cases}$$

Proof. The case $d = 1$ is due to Macdonald [48], the case $d = 2$ is due to Göttsche [30], the case $d = 3$ is due to Cheah [15]. □

Fact 9.2.7. There are no infinite product formulas for $\mathsf{H}_X(q)$ if $d = \dim X$ is at least 4.

10. Virtual fundamental class: the toy model

As we shall explain better in later sections, for a scheme X one has

$$\text{perfect obstruction theory on } X \quad \rightsquigarrow \quad \text{virtual fundamental class } [X]^{\mathrm{vir}}.$$

In words, a perfect obstruction theory induces a virtual fundamental class, and the constructed virtual fundamental class does depend on the perfect obstruction theory it comes from (see Example 10.1.9). This construction has historically two approaches: that of Li–Tian [46] and that of Behrend–Fantechi [8].

Remark 10.0.1. In this section we do define perfect obstruction theories in general (see Definition 10.1.3), but a slower and friendlier path towards them will be taken later in Appendix **??**.

10.1. **Obstruction theories on zero loci, or: the toy model.**

In this section we only study the "toy model" for a perfect obstruction theory, which is the situation where X is the zero locus of a section of a vector bundle on a smooth ambient space Y. In this case, the virtual fundamental class

$$[X]^{\mathrm{vir}} \in A_* X$$

can be constructed directly. The only essential tool needed is the knowledge that for a vector bundle $\pi: E \to S$ on a scheme S the flat pullback π^* on Chow groups is an isomorphism.

Let Y be a smooth variety of dimension d and let $E = \operatorname{Spec}\operatorname{Sym}\mathscr{E}^\vee$ be the total space of a rank r vector bundle on Y. Let $s \in H^0(Y, \mathscr{E}^\vee)$ be a section and let $I \subset \mathscr{O}_Y$ be the ideal sheaf of the zero locus

$$X = Z(s) \subset Y.$$

Since Y is smooth and E has rank r, we should expect X to have dimension

$$\begin{aligned} \mathrm{d}^{\mathrm{vir}} &= (\text{number of variables}) - (\text{number of equations}) \\ &= d - r. \end{aligned}$$

This will indeed be the *virtual dimension* of the obstruction theory, and $[X]^{\mathrm{vir}}$ will be constructed as an element of $A_{d-r}X$.

The image of the cosection $s^\vee: \mathscr{E} \to \mathscr{O}_Y$ is precisely I, thus restricting $s^\vee$ to X we obtain a surjective morphism

$$\sigma: \mathscr{E}\big|_X \twoheadrightarrow I/I^2.$$

Applying Spec Sym to this map, we obtain a closed immersion of cones (see Appendix **??** for more details on cones)

$$N_{X/Y} \hookrightarrow E^\vee\big|_X,$$

where $N_{X/Y} = \operatorname{Spec}\operatorname{Sym} I/I^2$ is the *normal sheaf* to X in Y. Composing with the closed immersion $C_{X/Y} \hookrightarrow N_{X/Y}$ of the normal cone inside the normal sheaf, we obtain a diagram

$$\begin{tikzcd} C_{X/Y} \arrow[r, hook] & N_{X/Y} \arrow[r, hook] & E^\vee|_X \arrow[d] \\ & & X \arrow[u, bend right, "0"'] \end{tikzcd}$$

realising $C_{X/Y}$ as a purely d-dimensional (cf. Remark **??**) subvariety of $E^\vee|_X$. It therefore determines a cycle class $[C_{X/Y}] \in A_d(E^\vee|_X)$. Let $0^*\colon A_d(E^\vee|_X) \xrightarrow{\sim} A_{d-r}X$ be the inverse of the flat pullback.

Definition 10.1.1. The *virtual fundamental class* of the zero locus $X = Z(s) \subset Y$ is the Chow class

$$[X]^{\mathrm{vir}} = 0^*[C_{X/Y}] \in A_{d-r}X. \tag{10.1.1}$$

Remark 10.1.2. Note that $[X]^{\mathrm{vir}}$ belongs to $A_{\mathrm{d}^{\mathrm{vir}}}X$ even if the section s cuts out a subscheme $X \subset Y$ of dimension bigger that $\mathrm{d}^{\mathrm{vir}} = d - r$.

If $\iota\colon X \hookrightarrow Y$ is the inclusion, then it is easy to see (cf. Remark B.3.8) that

$$\iota_*[X]^{\mathrm{vir}} = c_r(E) \cap [Y] \in A_{d-r}Y.$$

In fact, $[X]^{\mathrm{vir}}$ is Fulton's *localised top Chern class*, denoted $\mathbf{Z}(s)$ in [25]. Thus

$$[X]^{\mathrm{vir}} = \mathbf{Z}(s) = 0^![Y]$$

where $0^!\colon A_*Y \to A_*X$ is the refined Gysin homomorphism defined in Section B.3.

The cotangent sheaf Ω_X of a scheme X can be seen as the zero-th cohomology sheaf of a canonical object

$$\mathbb{L}_X \in \mathrm{D}^{[-1,0]}(X),$$

the *truncated cotangent complex*. If X is quasi-projective, in particular it can be embedded in a nonsingular scheme Y. If $I \subset \mathscr{O}_Y$ is the ideal sheaf of this inclusion, then we can write

$$\mathbb{L}_X = \left[I/I^2 \xrightarrow{\mathrm{d}} \Omega_Y\big|_X \right]$$

where d is the exterior derivative. See Section **??** for the reason why the right hand side does not depend on the chosen embedding. If no embedding in a smooth variety is available,[9] $\mathbb{L}_X$ can still be defined as the truncation

$$\mathbb{L}_X = \tau_{\geq -1} L_X$$

of Illusie's cotangent complex $L_X \in \mathrm{D}^{(-\infty,0]}(X)$, cf. [43].

Definition 10.1.3 ([8]). A *perfect obstruction theory* on a scheme X is a pair $(\mathbb{E}, \phi)$ where $\mathbb{E}$ is a perfect complex, of perfect amplitude concentrated in degrees $[-1,0]$, and $\phi\colon \mathbb{E} \to \mathbb{L}_X$ is a morphism in $\mathrm{D}^{[-1,0]}(X)$ such that

(1) $h^0(\phi)$ is an isomorphism, and
(2) $h^{-1}(\phi)$ is surjective.

[9] We will never deal with such a situation in these notes.

The *virtual dimension* of an obstruction theory is the difference

$$\mathrm{vd}(X,\phi) = \mathrm{rk}\, E^0 - \mathrm{rk}\, E^{-1}$$

if $\mathbb{E}$ is locally written $[E^{-1} \to E^0]$.

Definition 10.1.4 ([9])**.** A perfect obstruction theory $(\mathbb{E},\phi)$ is *symmetric* if there exists an isomorphism $\theta\colon \mathbb{E} \xrightarrow{\sim} \mathbb{E}^\vee[1]$ such that $\theta = \theta^\vee[1]$.

Remark 10.1.5. Given a scheme X along with a perfect obstruction theory $(\mathbb{E},\phi)$ on it, one can restrict it to any open subscheme $i\colon U \hookrightarrow X$. Indeed,

$$i^*\mathbb{E} \xrightarrow{i^*\phi} i^*\mathbb{L}_X \xrightarrow{\sim} \mathbb{L}_U$$

is a perfect obstruction theory on U.

Remark 10.1.6. The perfect obstruction theory giving rise to the virtual class (10.1.1) is explicitly given by the map of complexes

$$\begin{array}{ccc} \mathbb{E} & = & \left[\mathscr{E}\big|_X \xrightarrow{\ \mathrm{d}\circ\sigma\ } \Omega_Y\big|_X\right] \\ {\scriptstyle\phi}\big\downarrow & & \quad {\scriptstyle\sigma}\big\downarrow \qquad\qquad\qquad \big\downarrow{\scriptstyle\mathrm{id}} \\ \mathbb{L}_X & = & \left[I/I^2 \xrightarrow{\ \mathrm{d}\ } \Omega_Y\big|_X\right] \end{array}$$

both concentrated in degrees $[-1,0]$. All perfect obstruction theories are *locally* of this form.

Example 10.1.7. If we put $\mathscr{E} = T_Y$ and $s = \mathrm{d}f \in H^0(Y,\Omega_Y)$ for f a regular function on a smooth variety Y, then the virtual class (10.1.1) is zero-dimensional because $\dim Y = \mathrm{rk}\,\Omega_Y$. In this case, $\mathrm{d}\circ\sigma\colon T_Y|_X \to \Omega_Y|_X$ can be identified with the Hessian of f. This is the prototypical example of a *symmetric* perfect obstruction theory. Moreover, $[X]^{\mathrm{vir}}$ is intrinsic to X, i.e. it does not depend on (Y,f). It is *not* true that all symmetric obstruction theories are locally of this form, see [60] for a counterexample.

Example 10.1.8. As a further specialisation of Example 10.1.7, consider the case $f = 0$. Then the induced virtual fundamental class on $X = Y$ is

$$[Y]^{\mathrm{vir}} = e(\Omega_Y) \cap [Y] \in A_0 Y.$$

Thus every smooth scheme can be seen as a "virtually zero-dimensional" scheme.

Remark 10.1.9. We should stress that the virtual fundamental class *depends* on the chosen perfect obstruction theory. For instance, on a smooth scheme X, the complex

$$\mathbb{E} = [0 \to \Omega_X]$$

defines a perfect obstruction theory of virtual dimension $\dim X$. The associated virtual fundamental class is the usual fundamental class of X, i.e.

$$[X]^{\mathrm{vir}} = [X] \in A_{\dim X} X.$$

This clearly differs from the one constructed in Example 10.1.8 as long as X is not a point.

11. Virtual localisation formula for the toy model

In this subsection we mainly follow [32]. The main result of *loc. cit.* is a virtual analogue of the Atiyah–Bott localisation formula.

11.1. Equivariant obstruction theories.

11.1. Equivariant obstruction theories. Let $\mathbb{T}$ be an algebraic torus acting on a scheme X. The abelian category of $\mathbb{T}$-equivariant coherent sheaves $(\operatorname{Coh} X)^{\mathbb{T}}$ has a forgetful functor to $\operatorname{Coh} X$, which extends to an exact functor

$$\rho\colon \mathrm{D}(X)^{\mathbb{T}} \to \mathrm{D}(X)$$

between the corresponding derived categories. Assume for simplicity that X admits a $\mathbb{T}$-equivariant embedding in a nonsingular scheme Y. Then the truncated cotangent complex is naturally an object $\mathbb{L}_X$ of $\mathrm{D}(X)^{\mathbb{T}}$. An object $E \in \mathrm{D}(X)^{\mathbb{T}}$ is called perfect (of perfect amplitude in $[a,b]$) if the underlying complex $\rho(E)$ is perfect (of perfect amplitude in $[a,b]$).

Definition 11.1.1 ([31, 9])**.** Suppose X is acted on by a torus $\mathbb{T}$. A $\mathbb{T}$-equivariant perfect obstruction theory is a morphism

$$\phi\colon \mathbb{E} \to \mathbb{L}_X$$

in $\mathrm{D}(X)^{\mathbb{T}}$ whose image in $\mathrm{D}(X)$ is a perfect obstruction theory. The $\mathbb{T}$-action on $\mathbb{L}_X$ is the one induced by the $\mathbb{T}$-action on X.

Definition 11.1.2 ([9])**.** A $\mathbb{T}$-equivariant symmetric perfect obstruction theory is a pair

$$\left(\mathbb{E} \xrightarrow{\phi} \mathbb{L}_X, \mathbb{E} \xrightarrow{\theta} \mathbb{E}^\vee[1]\right)$$

of morphisms in $\mathrm{D}(X)^{\mathbb{T}}$ such that ϕ is a $\mathbb{T}$-equivariant perfect obstruction theory and θ satisfies $\theta^\vee[1] = \theta$.

11.2. Statement of the virtual localisation formula. Let X be a $\mathbb{C}$-scheme equipped with a perfect obstruction theory, and carrying a closed embedding

$$X \hookrightarrow Y$$

into a nonsingular scheme Y. The obstruction theory is the data consisting of a pair $(\mathbb{E}, \phi)$, where $\mathbb{E} \in \mathrm{D}^{[-1,0]}(X)$ is a perfect complex, of perfect amplitude contained in $[-1,0]$, and ϕ is a morphism from $\mathbb{E}$ to the truncated cotangent complex

$$\mathbb{L}_X = \left[I/I^2 \to \Omega_Y\big|_X \right].$$

Suppose Y carries a $\mathbb{G}_m$-action and the obstruction theory is $\mathbb{G}_m$-equivariant, i.e. the complex $\mathbb{E}$ carries a $\mathbb{G}_m$-action such that the morphism

$$\phi\colon \mathbb{E} \to \mathbb{L}_X$$

can be lifted to a morphism in the derived category of $\mathbb{G}_m$-equivariant sheaves on X (cf. Definition 11.1.1). We let

$$Y^{\mathsf{fix}} \subset Y, \quad X^{\mathsf{fix}} \subset X$$

be the scheme-theoretic fixed point loci (see [24] for a description of such scheme structure). Recall that Y^{fix} is smooth by Theorem 7.4.1. We have the relation $X^{\mathsf{fix}} = X \cap Y^{\mathsf{fix}}$, and if $\{ Y_i \mid i \in I \}$ are the irreducible components of Y^{fix}, we set

$$X_i = X \cap Y_i. \tag{11.2.1}$$

Note that X_i might be reducible. Any $\mathbb{G}_m$-equivariant coherent sheaf $\mathcal{S} \in \operatorname{Coh}(X_i)$ decomposes as a sum of eigensheaves

$$\mathcal{S} = \bigoplus_{k \in \mathbb{Z}} \mathcal{S}^k, \quad \mathcal{S}^k \in \operatorname{Coh} X_i,$$

where we identity $\mathbb{Z}$ with the character group of the 1-dimensional torus $\mathbb{G}_m$. The eigensheaves

$$\mathcal{S}^{\mathsf{fix}} = \mathcal{S}^0, \quad \mathcal{S}^{\mathsf{mov}} = \bigoplus_{k \neq 0} \mathcal{S}^k$$

are the *fixed part* and the *moving part* of S, respectively. The construction of fixed and moving part of a sheaf extends to complexes.

Lemma 11.2.1. *There is an identity*

$$\Omega_X \big|_{X_i}^{\mathsf{fix}} = \Omega_{X_i}$$

Proof. By Theorem 7.4.1, we have

$$\Omega_Y \big|_{Y_i}^{\mathsf{fix}} = \Omega_{Y_i}. \tag{11.2.2}$$

Then the claimed identity follows from (11.2.1). □

Lemma 11.2.2. *A morphism of complexes $\psi \colon \mathbb{A} \to \mathbb{B}$ satisfies*

(1) *$h^0(\psi)$ is an isomorphism, and*
(2) *$h^{-1}(\psi)$ is surjective*

if and only if the complex

$$A^{-1} \oplus B^{-2} \to A^0 \oplus B^{-1} \to B^0 \to 0$$

induced by the mapping cone construction is exact.

Proof. □

Define $\mathbb{E}_i$ to be the restriction of the complex $\mathbb{E}$ to the subvariety $X_i \subset X$.

Definition 11.2.3. The *virtual normal bundle* to $X_i \hookrightarrow X$ is the moving part of $\mathbb{E}_i^\vee$, i.e. it is the two-term complex

$$N_i^{\mathsf{vir}} = \mathbb{E}_i^{\vee,\mathsf{mov}} \in \mathrm{D}^{[0,1]}(X_i).$$

Restricting ϕ to X_i, we have a composition

$$\mathbb{E}_i \xrightarrow{\phi_i} \mathbb{L}_X \big|_{X_i} \xrightarrow{\delta_i} \mathbb{L}_{X_i},$$

which after taking invariants becomes

$$\psi_i \colon \mathbb{E}_i^{\mathsf{fix}} \xrightarrow{\phi_i^{\mathsf{fix}}} \mathbb{L}_X \big|_{X_i}^{\mathsf{fix}} \xrightarrow{\delta_i^{\mathsf{fix}}} \mathbb{L}_{X_i}^{\mathsf{fix}} \subseteq \mathbb{L}_{X_i}. \tag{11.2.3}$$

Lemma 11.2.4. *The composition (11.2.3) is an obstruction theory on X_i.*

It is often called the "$\mathbb{G}_m$-fixed obstruction theory" on X_i.

Proof. To check the properties

(1) $h^0(\psi_i)$ is an isomorphism, and
(2) $h^{-1}(\psi_i)$ is surjective

it is enough to check them for ϕ_i^{fix} and δ_i^{fix} separately.

Since $\phi\colon \mathbb{E}\to\mathbb{L}_X$ is an obstruction theory, and because the restriction $-\otimes_{\mathscr{O}_X}\mathscr{O}_{X_i}$ is right exact, Lemma 11.2.2 ensures that ϕ_i satisfies both (1) and (2). Since taking invariants is exact, the same holds for ϕ_i^{fix}.

Let $I\subset\mathscr{O}_Y$ (resp, $I_i\subset\mathscr{O}_{Y_i}$) be the ideal sheaf of $X\subset Y$ (resp. $X_i\subset Y_i$). Then one can represent

$$\mathbb{L}_X=[I/I^2\to\Omega_Y|_X]$$
$$\mathbb{L}_{X_i}=[I_i/I_i^2\to\Omega_{Y_i}|_{X_i}],$$

in $\mathrm{D}^{[-1,0]}(X)$ and $\mathrm{D}^{[-1,0]}(X_i)$ respectively. Of course, one has

$$h^0(\mathbb{L}_X)=\Omega_X,\quad h^0(\mathbb{L}_{X_i})=\Omega_{X_i},$$

and $h^0(\delta_i^{\mathsf{fix}})$ is the identity map

$$h^0(\mathbb{L}_X|_{X_i}^{\mathsf{fix}})=\Omega_X\big|_{X_i}^{\mathsf{fix}}=\Omega_{X_i}\xrightarrow{\mathrm{id}}\Omega_{X_i}.$$

So property (1) holds for δ_i^{fix}. As for property (2), one can represent $\delta_i^{\mathsf{fix}}\colon\mathbb{L}_X|_{X_i}^{\mathsf{fix}}\to\mathbb{L}_{X_i}$ as the map of complexes

$$\begin{tikzcd}
(I/I^2)|_{X_i}^{\mathsf{fix}} \arrow[r] \arrow[d, "\mathrm{d}^{-1}"'] & \Omega_Y|_{X_i}^{\mathsf{fix}} \arrow[d, "\wr"] \\
I_i/I_i^2 \arrow[r] & \Omega_{Y_i}|_{X_i}
\end{tikzcd}$$

concentrated in degrees $[-1,0]$. The vertical isomorphism is induced by (11.2.2), i.e. it is obtained by restriction to X_i and using exactness of $(-)^{\mathsf{fix}}$. Because of the definition (11.2.1) of X_i, the natural map

$$(I/I^2)|_{X_i}\to I_i/I_i^2$$

is surjective, which implies surjectivity of d^{-1}. Since $h^0(\delta_i^{\mathsf{fix}})$ is an isomorphism, it follows that $h^{-1}(\delta_i^{\mathsf{fix}})$ is surjective. □

Let $\iota\colon X^{\mathsf{fix}}\hookrightarrow X$ be the inclusion of the fixed locus. The *virtual localisation formula* is the following relation.

Theorem 11.2.5 (Graber–Pandharipande [32]). *There is an identity*

$$[X]^{\mathrm{vir}}=\iota_*\sum_i\frac{[X_i]^{\mathrm{vir}}}{e^{\mathbb{G}_m}(N_i^{\mathrm{vir}})}$$

in $A_*^{\mathbb{G}_m}(X)\otimes_{\mathbb{Q}[t]}\mathbb{Q}[t,t^{-1}]$.

11.3. **Proof in the case of zero loci.** Here is the setup in which we wish to prove (following [32, Section 2]) Theorem 11.2.5. We let Y be a smooth variety equipped with a $\mathbb{G}_m$-action. We fix a $\mathbb{G}_m$-equivariant vector bundle

$$\begin{array}{c}\operatorname{Spec}\operatorname{Sym}V\\ \big\downarrow\\ Y\end{array}$$

($V = \mathscr{E}^\vee$ in the language of the previous section), along with an equivariant section

$$s \in H^0(Y,V)^{\mathbb{G}_m}.$$

The zero scheme

$$X = Z(s) \hookrightarrow Y$$

then carries a natural $\mathbb{G}_m$-equivariant perfect obstruction theory

$$\mathbb{E} = [V^\vee|_X \to \Omega_Y|_X] \xrightarrow{\phi} \mathbb{L}_X,$$

and we let

$$[X]^{\mathrm{vir}} = \mathbf{Z}(s) = 0^![Y] = 0^*[C_{X/Y}] \in A_*^{\mathbb{G}_m}X$$

be the associated virtual fundamental class. The irreducible components $Y_i \hookrightarrow Y^{\mathsf{fix}}$ of the (nonsingular) fixed locus are nonsingular submanifolds, and for each i one has decompositions

$$V_i = V|_{Y_i} = V_i^{\mathsf{fix}} \oplus V_i^{\mathsf{mov}}.$$

Since s is $\mathbb{G}_m$-equivariant, its pullback to Y_i determines a section $s_i \in H^0(Y_i, V_i)$ that is entirely contained in V_i^{fix}, i.e. one can represent it as

$$\begin{tikzcd} V_i^{\mathsf{fix}} \arrow[r, hook] \arrow[d] & V_i \arrow[dl] \\ Y_i \arrow[u, bend left, "\widetilde{s}_i"] & \end{tikzcd}$$

and the zero locus of s_i is precisely $X_i = X \cap Y_i$. Let

$$\begin{tikzcd} X_i \arrow[r, hook, "\iota_i"] \arrow[d, hook, "\epsilon_i"'] \arrow[dr, phantom, "\square"] & X \arrow[d, hook] \\ Y_i \arrow[r, hook, "j_i"'] & Y \end{tikzcd}$$

be the associated fibre diagram, that we use to label the relevant inclusions. Taking the fixed part of the complex

$$\mathbb{E}_i = \mathbb{E}|_{X_i} = [V^\vee|_{X_i} \to \Omega_Y|_{X_i}]$$

yields the canonical perfect obstruction theory

$$\psi_i : \mathbb{E}_i^{\mathsf{fix}} \to \mathbb{L}_{X_i}$$

of Lemma 11.2.4. We now express the induced virtual fundamental class

$$[X_i]^{\mathrm{vir}}$$

as a refined Euler class. Recall that $\Omega_Y|_{X_i}^{\mathrm{fix}} \xrightarrow{\sim} \Omega_{Y_i}|_{X_i}$. Then under this identification one has

$$\begin{tikzcd}
\mathbb{E}_i^{\mathrm{fix}} \arrow[d, "\psi_i"'] & = & \big[V^\vee|^{\mathrm{fix}}_{X_i} \arrow[r, "\mathrm{d}\circ\sigma_i"] \arrow[d, "\sigma_i"'] & \Omega_{Y_i}|_{X_i} \big] \arrow[d, "\mathrm{id}"] \\
\mathbb{L}_{X_i} & = & \big[I_i/I_i^2 \arrow[r, "\mathrm{d}"] & \Omega_{Y_i}|_{X_i} \big]
\end{tikzcd}$$

and this induces the identity

$$[X_i]^{\mathrm{vir}} = \mathbf{Z}(\widetilde{s}_i) = \mathbf{Z}(s_i) \cap e(V_i^{\mathrm{mov}}) \in A_*^{\mathbb{G}_m}(X_i).$$

Now consider

$$\mathbb{E}_i^\vee = [T_Y|_{X_i} \to V_i|_{X_i}] \in \mathrm{D}^{[0,1]}(X_i).$$

Its moving part, i.e. the normal bundle N_i^{vir}, is (in K-theory) the restriction along $\epsilon_i \colon X_i \hookrightarrow Y_i$ of the complex

$$N_{Y_i/Y} - V_i^{\mathrm{mov}}.$$

Therefore, by definition of Euler class of a complex, we obtain

$$e(N_i^{\mathrm{vir}}) = \frac{e^{\mathbb{G}_m}(N_{Y_i/Y})}{e(V_i^{\mathrm{mov}})}.$$

The usual localisation formula for Y states

$$[Y] = \sum_i j_{i*} \frac{[Y_i]}{e^{\mathbb{G}_m}(N_{Y_i/Y})}.$$

to be fixed

taking on both sides refined intersection with $[X]^{\mathrm{vir}} = \mathbf{Z}(s)$ yields

$$\begin{aligned}
[X]^{\mathrm{vir}} &= \sum_i \iota_{i*} \frac{j_i^![X]^{\mathrm{vir}}}{e^{\mathbb{G}_m}(N_{Y_i/Y})} \\
&= \sum_i \iota_{i*} \frac{j_i^! \mathbf{Z}(s)}{e^{\mathbb{G}_m}(N_{Y_i/Y})} \\
&= \sum_i \iota_{i*} \frac{\mathbf{Z}(s_i)}{e^{\mathbb{G}_m}(N_{Y_i/Y})} \\
&= \sum_i \iota_{i*} \frac{\mathbf{Z}(\widetilde{s}_i) \cap e(V_i^{\mathrm{mov}})}{e^{\mathbb{G}_m}(N_{Y_i/Y})} \\
&= \sum_i \iota_{i*} \frac{[X_i]^{\mathrm{vir}}}{e^{\mathbb{G}_m}(N_i^{\mathrm{vir}})} \\
&= \iota_* \sum_i \frac{[X_i]^{\mathrm{vir}}}{e^{\mathbb{G}_m}(N_i^{\mathrm{vir}})}.
\end{aligned}$$

This proves Theorem 11.2.5 for $X = Z(s)$.

12. THE PERFECT OBSTRUCTION THEORY ON $\mathrm{Hilb}^n X$

12.1. **Calabi–Yau varieties.** Even though Donaldson–Thomas theory is a sheaf counting theory defined for arbitrary 3-folds, specialising to *Calabi–Yau* 3-folds the theory becomes particularly rich. The main feature of the Calabi–Yau condition is that moduli spaces of sheaves (as well as moduli spaces of unpointed stable maps, of stable pairs...) have virtual dimension zero. In the Calabi–Yau case, this is a consequence of the fact the *obstruction theory* (cf. Definition 10.1.3) on the moduli space of simple sheaves on a Calabi–Yau 3-fold is *symmetric* (cf. Definition 10.1.4). However, our main object of study in these notes is the Hilbert scheme of points $\mathrm{Hilb}^n X$ on a 3-fold X. We shall see that in this case the obstruction theory has rank zero even if X is not Calabi–Yau.

Definition 12.1.1. A *Calabi–Yau d-fold* is a nonsingular quasi-projective variety X of dimension d, satisfying $\omega_X \cong \mathscr{O}_X$.

Remark 12.1.2. Sometimes the condition

$$H^i(X, \mathscr{O}_X) = 0, \quad 0 < i < \dim X$$

is included in the definition of Calabi–Yau variety. We will refer to Calabi–Yau varieties satisfying this additional condition as *strict* Calabi–Yau d-folds.

We will be only interested in the case $d = 3$.

Example 12.1.3. According to our definition, the following are examples of Calabi–Yau 3-folds:

(1) the affine space $\mathbb{A}^3$,
(2) the total space $\mathrm{Tot}(\omega_S)$ of the canonical bundle of a smooth projective surface,
(3) the total space of the line bundle $\mathscr{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ over $\mathbb{P}^1$,
(4) more generally, $\mathrm{Tot}(L_1 \oplus L_2)$, where L_i are line bundles on a smooth projective curve such that $L_1 \otimes_{\mathscr{O}_C} L_2 \cong \omega_C$,
(5) a smooth quintic hypersurface $Y \subset \mathbb{P}^4$.

EXERCISE 12.1.4. Show that a smooth quintic $Y \subset \mathbb{P}^4$ satisfies

$$\chi(Y) = -200.$$

(**Hint**: Combine the Euler sequence of $\mathbb{P}^4$ with the normal bundle exact sequence of $Y \subset \mathbb{P}^4$, and use the Whitney sum formula for Chern polynomials).

EXERCISE 12.1.5. Show that a toric Calabi–Yau variety cannot be projective.

12.2. **A couple of tools.** We introduce a couple of technical tools that will be needed in the construction of the virtual fundamental class of the Hilbert scheme of points on a 3-fold.

12.2.1. *Two words on the trace map.* Let X be a scheme. Recall (see e.g. [40]) that for any torsion free coherent sheaf E of positive rank r there is a trace map

$$\mathrm{tr}\colon \mathbf{R}\mathscr{H}\mathrm{om}(E,E) \to \mathscr{O}_X.$$

This map is *split*: the inclusion of the scalars $\mathscr{O}_X \to \mathbf{R}\mathscr{H}\mathrm{om}(E,E)$ composes with tr to give $r \cdot \mathrm{id} \in \mathrm{Hom}(\mathscr{O}_X, \mathscr{O}_X)$. We denote by

$$\mathrm{tr}^i \colon \mathrm{Ext}^i(E,E) \to H^i(X,\mathscr{O}_X)$$

the i-th cohomology of the map tr and we set

$$\mathrm{Ext}^i(E,E)_0 = \ker\left(\mathrm{tr}^i\right).$$

For instance, if E is a stable sheaf, the map tr^1 can be identified with the differential of the determinant map $M \to \mathrm{Pic}\, X$, where M is the moduli space of sheaves containing $[E]$ as a closed point. The maps tr^i are surjective as long as $\mathrm{rk}\, E \neq 0$. The splitting induces direct sum decompositions

$$\mathrm{Ext}^i(E,E) = H^i(X,\mathscr{O}_X) \oplus \mathrm{Ext}^i(E,E)_0.$$

The *trace-free* Ext groups $\mathrm{Ext}^1(E,E)_0$ and $\mathrm{Ext}^2(E,E)_0$ encode tangents and obstructions for the deformation theory of E with fixed determinant. See [40, § 4.5, § 10.1] for more details on the trace map.

12.2.2. *Two words on the Atiyah class.* Sheaves of principal parts were introduced in [33, Ch. 16.3]. Let $\pi \colon X \to S$ be a morphism of schemes, and let $\mathscr{I}$ be the ideal sheaf of the diagonal $\Delta \colon X \to X \times_S X$. Let p and q denote the projections $X \times_S X \to X$, and denote by $\Delta_k \subset X \times_S X$ the closed subscheme defined by $\mathscr{I}^{k+1}$, for every $k \geq 0$. Then, for any quasi-coherent sheaf V on X, the sheaf

$$P_\pi^k(V) := p_*\left(q^* V \otimes \mathscr{O}_{\Delta_k}\right)$$

is quasi-coherent and is called the k-th *sheaf of principal parts* associated to the pair (π, V).

Proposition 12.2.1. *Let $\pi \colon X \to S$ be a flat family of Cohen–Macaulay schemes, V a quasi-coherent sheaf on X. The sheaves of principal parts fit into short exact sequences*

$$V \otimes \mathrm{Sym}^k\, \Omega_\pi^1 \to P_\pi^k(V) \to P_\pi^{k-1}(V) \to 0$$

for every $k \geq 1$. If V is locally free then the sequence is exact on the left. If, in addition, π is smooth, then $P_\pi^k(V)$ is locally free for all $k \geq 0$.

This is standard. We repeat here the proof given in [28, Prop. 1.3].

Proof. Consider the short exact sequence

$$0 \to \mathscr{I}^k/\mathscr{I}^{k+1} \to \mathscr{O}_{\Delta_k} \to \mathscr{O}_{\Delta_{k-1}} \to 0.$$

Tensoring it with q^*V gives a right exact piece

$$q^*V \otimes \mathscr{I}^k/\mathscr{I}^{k+1} \to q^*V \otimes \mathscr{O}_{\Delta_k} \to q^*V \otimes \mathscr{O}_{\Delta_{k-1}} \to 0 \tag{12.2.1}$$

so the first assertion follows by applying p_* and observing that

$$\begin{aligned} p_*\left(q^*V \otimes \mathscr{I}^k/\mathscr{I}^{k+1}\right) &= \Delta^*\left(q^*V \otimes \mathscr{I}^k/\mathscr{I}^{k+1}\right) \\ &= \Delta^* q^* V \otimes \Delta^*\left(\mathscr{I}^k/\mathscr{I}^{k+1}\right) \\ &= V \otimes \Delta^* \mathrm{Sym}^k\left(\mathscr{I}/\mathscr{I}^2\right) \\ &= V \otimes \mathrm{Sym}^k\, \Omega_\pi^1. \end{aligned}$$

We also use that $R^i p_*$ applied to a sheaf supported on Δ vanishes. Note that we used the Cohen–Macaulay hypothesis in the identification $\mathscr{I}^k/\mathscr{I}^{k+1} = \operatorname{Sym}^k(\mathscr{I}/\mathscr{I}^2)$. If V is locally free, then (12.2.1) is exact on the left. If π is in addition smooth, then local freeness of $P^k_\pi(V)$ follows by induction exploiting the short exact sequence, the base case being given by $P^0_\pi(V) = V$. □

Example 12.2.2. Suppose $\pi\colon X \to S$ is smooth. Then there is a splitting $P^1_\pi(\mathscr{O}_X) = \mathscr{O}_X \oplus \Omega^1_\pi$. For an arbitrary vector bundle V, the splitting of the first bundle of principal parts usually fails even when S is a point. In fact, in this case, the splitting is equivalent to the vanishing of the *Atiyah class* of V, which by definition is the extension class

$$A(V) \in \operatorname{Ext}^1_X(V, V \otimes \Omega_X)$$

attached to the short exact sequence of Proposition 12.2.1 taken with $k = 1$. But the vanishing of the Atiyah class is known to be equivalent to the existence of an algebraic connection on V, see e.g. [40, § 10.1.5].

The Atiyah class exists in much broader generality than the one mentioned in Example 12.2.2, see for instance [41] or the more general construction in [69, Section 89.18].

12.3. **The setup.** The goal for the rest of this section is to sketch the main construction that underlies the following important result.

Theorem 12.3.1. *Let X be a smooth quasi-projective 3-fold. Then the Hilbert scheme* $\operatorname{Hilb}^n X$ *carries a 0-dimensional perfect obstruction theory. Moreover,*

- *this obstruction theory is symmetric if X is Calabi–Yau, and*
- *this obstruction theory is $\mathbb{T}_0$-equivariant symmetric (cf. Definition 11.1.2) if X is toric Calabi–Yau. Here $\mathbb{T}_0 \subset (\mathbb{C}^*)^3$ is the two-dimensional subtorus obtained as the kernel of cheracter* $(1,1,1)$.

We take the opportunity to mention an interesting open problem in the subject.

Open Problem 1. Consider the symmetric perfect obstruction theory on $\operatorname{Hilb}^n \mathbb{A}^3$, defined by the Hessian of the function f_n having this Hilbert scheme as critical locus. Is this obstruction theory the same as the one of Theorem 12.3.1?

To simplify the exposition, we make the assumption that X is projective and satisfies

$$H^i(X, \mathscr{O}_X) = 0, \quad i > 0. \tag{12.3.1}$$

For instance, X could be toric.

Remark 12.3.2. If we prove the statement for X projective, it will be automatically true for X quasi-projective. Indeed, one can compactify $X \hookrightarrow \overline{X}$ and restrict the obstruction theory $\mathbb{E} \to \mathbb{L}_{\operatorname{Hilb}^n \overline{X}}$ along the open immersion $\operatorname{Hilb}^n X \hookrightarrow \operatorname{Hilb}^n \overline{X}$, cf. Remark 10.1.5.

Since by this assumption line bundles on X do not deform, we have $\operatorname{tr}^i = 0$ for $i > 0$, and in particular

$$\operatorname{Ext}^i(E,E)_0 = \operatorname{Ext}^i(E,E), \quad i > 0.$$

Recall that we view the Hilbert scheme as a particular case of Quot scheme. As a set, we can write

$$\operatorname{Hilb}^n X = \left\{ \mathscr{O}_X \twoheadrightarrow \mathscr{O}_Z \,\middle|\, Z \subset X \text{ is finite, } \chi(\mathscr{O}_Z) = n \right\}.$$

12.4. **Useful vanishings.** A tool that we will use repeatedly is Serre duality on X, which asserts that

$$\operatorname{Ext}^i(E,F) \cong \operatorname{Ext}^{3-i}(F, E \otimes \omega_X)^\vee.$$

EXERCISE 12.4.1. Let E be a vector bundle of rank r on a smooth variety Y, and let T be a skyscraper sheaf. Show that $T^{\oplus r} \cong E \otimes T$.

Let $\mathscr{I} \subset \mathscr{O}_X$ be the ideal sheaf of a finite subscheme $Z \subset X$ of length n. Applying $\operatorname{Hom}(\mathscr{I}, -)$ to the ideal sheaf exact sequence we obtain

$$\begin{aligned} 0 \to \operatorname{Hom}(\mathscr{I},\mathscr{I}) \to \operatorname{Hom}(\mathscr{I},\mathscr{O}_X) \xrightarrow{u} T_{\mathscr{I}} \operatorname{Hilb}(X) \to \operatorname{Ext}^1(\mathscr{I},\mathscr{I}) \\ \to \operatorname{Ext}^1(\mathscr{I},\mathscr{O}_X) \to \operatorname{Ext}^1(\mathscr{I},\mathscr{O}_Z) \to \operatorname{Ext}^2(\mathscr{I},\mathscr{I}). \end{aligned} \tag{12.4.1}$$

Observe first of all that, since $\dim Z = 0$, Serre duality (along with Exercise 12.4.1 taken with $E = \omega_X$) implies

$$\operatorname{Ext}^{3-i}(\mathscr{O}_Z,\mathscr{O}_X)^* \cong \operatorname{Ext}^i(\mathscr{O}_X,\mathscr{O}_Z) \cong H^i(Z,\mathscr{O}_Z) = 0, \quad i > 0. \tag{12.4.2}$$

Lemma 12.4.2. *We have the vanishing*

$$\operatorname{Ext}^1(\mathscr{I},\mathscr{O}_X) = 0.$$

Proof. The Ext group we are looking at appears in a long exact sequence containing the exact piece

$$\begin{tikzcd} \operatorname{Ext}^1(\mathscr{O}_X,\mathscr{O}_X) \arrow[r] \arrow[d, equal, "(12.3.1)"'] & \operatorname{Ext}^1(\mathscr{I},\mathscr{O}_X) \arrow[r] & \operatorname{Ext}^2(\mathscr{O}_Z,\mathscr{O}_X) \arrow[d, equal, "(12.4.2)"] \\ 0 & & 0 \end{tikzcd}$$

and therefore it has to vanish. □

Lemma 12.4.3. *In the long exact sequence* (12.4.1)*, we have* $u = 0$*.*

Proof. Applying $\operatorname{Hom}(-,\mathscr{O}_X)$ to the ideal sheaf sequence yields an exact piece

$$\begin{tikzcd} \operatorname{Hom}(\mathscr{O}_Z,\mathscr{O}_X) \arrow[r] \arrow[d, equal] & \operatorname{Hom}(\mathscr{O}_X,\mathscr{O}_X) \arrow[r] \arrow[d, equal] & \operatorname{Hom}(\mathscr{I},\mathscr{O}_X) \arrow[r] & \operatorname{Ext}^1(\mathscr{O}_Z,\mathscr{O}_X) \arrow[d, equal] \\ 0 & H^0(X,\mathscr{O}_X) & & 0 \end{tikzcd}$$

where the two vanishings follow by (12.4.2). It follows that $\operatorname{Hom}(\mathscr{I},\mathscr{O}_X) \cong \operatorname{Hom}(\mathscr{O}_X,\mathscr{O}_X)$ is one-dimensional, and since the (cohomology of the) trace map $\operatorname{tr}\colon \mathscr{H}om(\mathscr{I},\mathscr{I}) \to \mathscr{O}_X$ splits, yielding a decomposition

$$\operatorname{Hom}(\mathscr{I},\mathscr{I}) = \underbrace{H^0(X,\mathscr{O}_X)}_{\mathbb{C}} \oplus \operatorname{Hom}(\mathscr{I},\mathscr{I})_0,$$

we see that $\operatorname{Hom}(\mathscr{I},\mathscr{I})$ is *at least* one-dimensional, which shows that the first nonzero map in (12.4.1) is an isomorphism. It follows that $u = 0$. □

Remark 12.4.4. Another way of seeing that $\operatorname{Hom}(\mathscr{I},\mathscr{I}) = \mathbb{C}$ is to simply observe that $\mathscr{I}$ is a stable sheaf, in particular it is simple (see e.g. [40, Cor. 1.2.28] for a proof that stable sheaves are simple), which means its only endomorphisms are the constants.

Before interpreting the vanishing of the map p u, we pause to prove an additional vanishing result.

Proposition 12.4.5. *We have the vanishings*

$$\operatorname{Hom}(\mathscr{I},\mathscr{I})_0 = \operatorname{Ext}^3(\mathscr{I},\mathscr{I})_0 = 0.$$

Proof. The vanishing $\operatorname{Hom}(\mathscr{I},\mathscr{I})_0 = 0$ has already been established when we proved that $\operatorname{Hom}(\mathscr{I},\mathscr{I}) \xrightarrow{\sim} \operatorname{Hom}(\mathscr{I},\mathscr{O}_X) = \mathbb{C}$ is an isomorphism. To prove that $\operatorname{Ext}^3(\mathscr{I},\mathscr{I})_0 = 0$ we use the argument of [49]. By Serre duality, we have

$$\operatorname{Ext}^3(\mathscr{I},\mathscr{I}) \cong \operatorname{Hom}(\mathscr{I},\mathscr{I}\otimes\omega_X)^\vee$$

and, setting $U = X\setminus Z$, the latter space injects in

$$\operatorname{Hom}(\mathscr{I}|_U,\mathscr{I}|_U\otimes\omega_X|_U) = \operatorname{Hom}(\mathscr{O}_U,\omega_U) = H^0(U,\omega_U)$$

because $\mathscr{I}$ is torsion free. However, since Z has codimension at least 2, by Hartog's Lemma we can extend sections of line bundles from the complement of Z to the whole of X, i.e.

$$H^0(U,\omega_U) = H^0(X,\omega_X) = H^3(X,\mathscr{O}_X)^\vee = 0.$$

The required vanishing follows. $\square$

The vanishing of the map $\operatorname{Hom}(\mathscr{I},\mathscr{O}_X) \to T_{\mathscr{I}}\operatorname{Hilb}(X)$ implies that all in all the long exact sequence (12.4.1) gives us two cohomological pieces of information:

$$\begin{aligned} T_{\mathscr{I}}\operatorname{Hilb}(X) &\xrightarrow{\sim} \operatorname{Ext}^1(\mathscr{I},\mathscr{I}) \\ \operatorname{Ext}^1(\mathscr{I},\mathscr{O}_Z) &\hookrightarrow \operatorname{Ext}^2(\mathscr{I},\mathscr{I}). \end{aligned} \tag{12.4.3}$$

The left hand sides are tangents and obstructions (T^1 and T^2) for the natural tangent-obstruction theory on the Hilbert scheme at its point $[\mathscr{O}_X \twoheadrightarrow \mathscr{O}_Z] \in \operatorname{Hilb}^n X$. The right hand sides are the tangents and obstructions for the natural tangent-obstruction theory on the ideal sheaf $\mathscr{I}$. As proved in [65, Prop. A.1], the situation (12.4.3) ensures that the deformation functors

$$\mathsf{Def}_{\mathscr{O}_X\twoheadrightarrow\mathscr{O}_Z} \ \text{ and } \ \mathsf{Def}_{\mathscr{I}}$$

are naturally isomorphic. It is an important observation of Thomas [71] that, even though the standard tangent-obstruction theory on the Hilbert scheme, given by

$$T^i|_{\mathscr{O}_X\twoheadrightarrow\mathscr{O}_Z} = \operatorname{Ext}^{i-1}(\mathscr{I},\mathscr{O}_Z),$$

does not lead to a virtual fundamental class (because it is not two-term, due to the presence of higher Ext groups), the tangent-obstruction theory on the moduli space of ideal sheaves, given by

$$T^i|_{\mathscr{I}} = \operatorname{Ext}^i(\mathscr{I},\mathscr{I}),$$

does lead to a virtual fundamental class.

12.4.1. *Dimension and point-wise symmetry.* In the perfect obstruction theory we want to build, the tangent space at a point

$$z = [\mathscr{O}_X \twoheadrightarrow \mathscr{O}_Z] \in \operatorname{Hilb}^n X$$

is $\operatorname{Ext}^1(\mathscr{I},\mathscr{I}) = \operatorname{Hom}(\mathscr{I},\mathscr{O}_Z)$, and the obstruction space is $\operatorname{Ext}^2(\mathscr{I},\mathscr{I})$. Its virtual dimension at z would then be

$$\operatorname{vd}_z = \operatorname{ext}^1(\mathscr{I},\mathscr{I}) - \operatorname{ext}^2(\mathscr{I},\mathscr{I}). \tag{12.4.4}$$

At a given point in the moduli space, tangents are always dual to obstructions. Indeed, one can exploit the long exact sequence

$$\operatorname{Ext}^2(\mathscr{O}_X,\mathscr{I}) \to \operatorname{Ext}^2(\mathscr{I},\mathscr{I}) \to \operatorname{Ext}^3(\mathscr{O}_Z,\mathscr{I}) \to \operatorname{Ext}^3(\mathscr{O}_X,\mathscr{I}).$$

The two outer terms vanish, because they sit in long exact sequences

$$\begin{tikzcd}
\operatorname{Ext}^{i-1}(\mathscr{O}_X,\mathscr{O}_Z) \arrow[r] \arrow[d, equal] & \operatorname{Ext}^i(\mathscr{O}_X,\mathscr{I}) \arrow[r] & \operatorname{Ext}^i(\mathscr{O}_X,\mathscr{O}_X) \arrow[d, equal] \\
0 & & H^i(X,\mathscr{O}_X)
\end{tikzcd}$$

for $i = 2,3$. Therefore we obtain an isomorphism

$$\operatorname{Ext}^2(\mathscr{I},\mathscr{I}) \xrightarrow{\sim} \operatorname{Ext}^3(\mathscr{O}_Z,\mathscr{I})$$

Dualising, this becomes an isomorphism

$$\operatorname{Hom}(\mathscr{I},\mathscr{O}_Z) \xrightarrow{\sim} \operatorname{Ext}^2(\mathscr{I},\mathscr{I})^*.$$

To sum up, if we manage to produce a perfect obstruction theory with $\operatorname{Ext}^1(\mathscr{I},\mathscr{I})$, $\operatorname{Ext}^2(\mathscr{I},\mathscr{I})$ as tangents and obstructions, it will be zero-dimensional and "point-wise symmetric". However, point-wise symmetry does not imply global symmetry, cf. Definition 10.1.4.

12.5. **Construction of the obstruction theory.** For ease of notation, let us shorten

$$\mathrm{H} = \operatorname{Hilb}^n X,$$

where X is a smooth projective 3-fold. We point out that what follows is a very standard construction and works in much greater generality, see for instance [41] for a general treatment and [65] for an adaptation to the case of Quot schemes of locally free sheaves on a 3-fold.

Consider the universal ideal sheaf exact sequence

$$0 \to \mathfrak{I} \to \mathscr{O}_{X\times \mathrm{H}} \to \mathscr{O}_{\mathcal{Z}} \to 0$$

living over $X \times \mathrm{H}$. The trace map

$$\operatorname{tr}_{\mathfrak{I}} \colon \mathbf{R}\mathscr{H}om(\mathfrak{I},\mathfrak{I}) \to \mathscr{O}_{X\times\mathrm{H}}$$

has a canonical splitting, and we denote its kernel by

$$\mathbf{R}\mathscr{H}om(\mathfrak{I},\mathfrak{I})_0.$$

The truncated cotangent complex $\mathbb{L}_{X\times\mathrm{H}}$ splits as $p^*\mathbb{L}_{\mathrm{H}} \oplus q^*\mathbb{L}_X$, so the *truncated Atiyah class* (cf. [41, Def. 2.6])

$$A(\mathfrak{I}) \in \operatorname{Ext}^1(\mathfrak{I},\mathfrak{I}\otimes \mathbb{L}_{X\times\mathrm{H}})$$

projects onto the factor

$$\begin{aligned}\mathrm{Ext}^1(\mathfrak{I},\mathfrak{I}\otimes p^*\mathbb{L}_{\mathrm{H}}) &= \mathrm{Ext}^1(\mathfrak{I}^\vee \overset{\mathbf{L}}{\otimes} \mathfrak{I}, p^*\mathbb{L}_{\mathrm{H}}) \\ &= \mathrm{Ext}^1(\mathbf{R}\mathscr{H}\mathrm{om}(\mathfrak{I},\mathfrak{I}), p^*\mathbb{L}_{\mathrm{H}}),\end{aligned}$$

which by the splitting of $\mathrm{tr}_{\mathfrak{I}}$ can be further projected onto

$$\mathrm{Ext}^1(\mathbf{R}\mathscr{H}\mathrm{om}(\mathfrak{I},\mathfrak{I})_0, p^*\mathbb{L}_{\mathrm{H}}).$$

By Verdier duality along the smooth, proper 3-dimensional morphism p, one has

$$\mathbf{R}p_*\mathbf{R}\mathscr{H}\mathrm{om}(\mathscr{F}, p^*\mathscr{G}\otimes\omega_p[3]) = \mathbf{R}\mathscr{H}\mathrm{om}(\mathbf{R}p_*\mathscr{F},\mathscr{G}) \tag{12.5.1}$$

for $\mathscr{F}\in\mathrm{D}^b(X\times\mathrm{H})$ and $\mathscr{G}\in\mathrm{D}^b(\mathrm{H})$, where $\omega_p = q^*\omega_Y$ is the relative dualising sheaf. Setting $\mathscr{F} = \mathbf{R}\mathscr{H}\mathrm{om}(\mathfrak{I},\mathfrak{I})_0\otimes\omega_p$ and $\mathscr{G} = \mathbb{L}_{\mathrm{H}}$ in (12.5.1), we obtain

$$\begin{aligned}&\mathbf{R}p_*\mathbf{R}\mathscr{H}\mathrm{om}(\mathbf{R}\mathscr{H}\mathrm{om}(\mathfrak{I},\mathfrak{I})_0\otimes\omega_p, p^*\mathbb{L}_{\mathrm{H}}\otimes\omega_p[3]) \\ &\qquad\qquad = \mathbf{R}\mathscr{H}\mathrm{om}(\mathbf{R}p_*(\mathbf{R}\mathscr{H}\mathrm{om}(\mathfrak{I},\mathfrak{I})_0\otimes\omega_p),\mathbb{L}_{\mathrm{H}}),\end{aligned}$$

which after applying $h^{-2}\circ\mathbf{R}\Gamma$ becomes

$$\begin{aligned}\mathrm{Ext}^1(\mathbf{R}\mathscr{H}\mathrm{om}(\mathfrak{I},\mathfrak{I})_0, p^*\mathbb{L}_{\mathrm{H}}) &= \mathrm{Ext}^{-2}(\mathbf{R}p_*(\mathbf{R}\mathscr{H}\mathrm{om}(\mathfrak{I},\mathfrak{I})_0\otimes\omega_p),\mathbb{L}_{\mathrm{H}}) \\ &= \mathrm{Hom}(\mathbb{E},\mathbb{L}_{\mathrm{H}}),\end{aligned}$$

where we have set

$$\mathbb{E} = \mathbf{R}p_*(\mathbf{R}\mathscr{H}\mathrm{om}(\mathfrak{I},\mathfrak{I})_0\otimes\omega_p)[2].$$

Under the above identifications, the truncated Atiyah class $A(\mathfrak{I})$ determines a morphism

$$\phi\colon\mathbb{E}\to\mathbb{L}_{\mathrm{H}}.$$

Theorem 12.5.1. *The morphism $\phi\colon\mathbb{E}\to\mathbb{L}_{\mathrm{H}}$ constructed above is a perfect obstruction theory on* $\mathrm{Hilb}^n X$.

Granting the theorem, we can easily prove that $(\mathbb{E},\phi)$ is symmetric in the Calabi–Yau case.

Let us shorten $\mathbb{H} = \mathbf{R}\mathscr{H}\mathrm{om}(\mathfrak{I},\mathfrak{I})_0$. Note that $\mathbb{H}$ is canonically self-dual. The complex $\mathbf{R}p_*\mathbb{H}$ is isomorphic in the derived category to a two-term complex of vector bundles $\mathcal{T}^\bullet = [\mathcal{T}^1\to\mathcal{T}^2]$ concentrated in degrees 1 and 2. More precisely, as in [41, Lemma. 4.2], the identification $\mathbf{R}p_*\mathbb{H} = \mathcal{T}^\bullet$ follows from the vanishings

$$\mathrm{Ext}^i(\mathscr{I},\mathscr{I})_0 = 0,\quad i\neq 1,2,$$

that we proved in Proposition 12.4.5. On the other hand, we have

$$\begin{aligned}(\mathbf{R}p_*\mathbb{H})^\vee[-1] &= \mathbf{R}\mathscr{H}\mathrm{om}(\mathbf{R}p_*\mathbb{H},\mathscr{O}_{\mathrm{H}})[-1] & \\ &= \mathbf{R}p_*\mathbf{R}\mathscr{H}\mathrm{om}(\mathbb{H},\omega_p[3])[-1] & \text{Verdier duality} \\ &= \mathbf{R}p_*\mathbf{R}\mathscr{H}\mathrm{om}(\mathbb{H},\omega_p)[2] & \text{shift} \\ &= \mathbf{R}p_*\mathbf{R}\mathscr{H}\mathrm{om}(\mathbb{H}^\vee,\omega_p)[2] & \mathbb{H}=\mathbb{H}^\vee \\ &= \mathbf{R}p_*(\mathbb{H}\otimes\omega_p)[2] & \\ &= \mathbb{E}. &\end{aligned}$$

Therefore $\mathbb{E}$ is perfect in $[-1,0]$, i.e. ϕ is perfect.

For any point $z = [\mathscr{I} \hookrightarrow \mathscr{O}_X \twoheadrightarrow \mathscr{O}_Z]$, with inclusion $\iota_z \colon z \hookrightarrow \mathrm{H}$, one has

$$h^{i-1}(\mathbf{L}\iota_z^*\mathbb{E}^\vee) = \mathrm{Ext}^i(\mathscr{I},\mathscr{I}), \quad i = 1,2.$$

Therefore we have $\mathrm{vd}_z = \mathrm{rk}\,\mathbb{E} = \mathrm{ext}^1(\mathscr{I},\mathscr{I}) - \mathrm{ext}^2(\mathscr{I},\mathscr{I}) = 0$, as observed in Section 12.4.1.

12.5.1. *Symmetry of the obstruction theory.* Let us prove symmetry in the Calabi–Yau case. The argument is standard — see for instance [9]. Any trivialisation $\omega_X \xrightarrow{\sim} \mathscr{O}_X$ induces, by pullback along $X \times \mathrm{H} \to X$, a trivialisation $\omega_p \xrightarrow{\sim} \mathscr{O}_{X\times\mathrm{H}}$, that we can use to construct an isomorphism

$$\mathbb{E}[-2] \xrightarrow{\sim} \mathbf{R}p_*\mathbb{H}.$$

Dualising and shifting the last isomorphism, we get

$$\theta \colon (\mathbf{R}p_*\mathbb{H})^\vee[-1] \xrightarrow{\sim} \mathbb{E}^\vee[1],$$

where the source is canonically identified with $\mathbb{E}$. The symmetry condition $\theta^\vee[1] = \theta$ follows from [9, Lemma 1.23].

13. Zero-dimensional DT invariants of a local Calabi–Yau 3-fold

13.1. **Virtual localisation for the Hilbert scheme of points.** Let X be a smooth projective toric 3-fold with open dense torus $\mathbb{T} \subset X$. Let $\Delta(X)$ be its Newton polytope (see Figure 14 for two examples) and

$$X^{\mathbb{T}} = \{ p_\alpha \mid \alpha \in \Delta(X) \}$$

its fixed point locus, of cardinality $\chi(X)$.

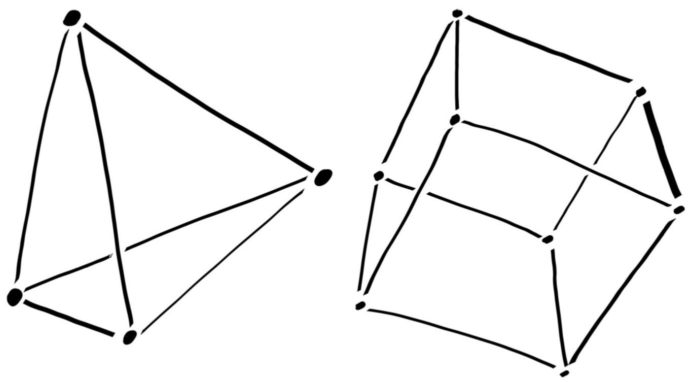


FIGURE 14. The Newton polytopes of $\mathbb{P}^3$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Picture stolen from Okounkov [58, § 6.2].

For each α, there exists a a canonical $\mathbb{T}$-invariant open affine chart

$$U_\alpha = \mathrm{Spec}\, R_\alpha, \quad R_\alpha = \mathbb{C}[x_1, x_2, x_3],$$

such that the action of $\mathbb{T}$ on U_α is the standard one, namely

$$(t_1, t_2, t_3) \cdot (x_1, x_2, x_3) = (t_1 x_1, t_2 x_2, t_3 x_3). \tag{13.1.1}$$

Notation 13.1.1. We let t_1, t_2, t_3 be the generators of the K-theoretic representation ring of $\mathbb{T}$, namely

$$K_0^{\mathbb{T}}(\mathrm{pt}) = K_0\big(\{\text{abelian category of } \mathbb{T}\text{-representations}\}\big).$$

In other words, t_j can be identified with the pullback of $\mathscr{O}(-1)$ along the j-th projection $\mathrm{pr}_j \colon (\mathbb{P}^\infty)^3 \to \mathbb{P}^\infty$. Then $H_{\mathbb{T}}^* = \mathbb{Q}[s_1, s_2, s_3]$ is the equivariant cohomology ring of the torus (with each s_i sitting in degree 2), where we have set

$$s_j = c_1(t_j), \quad j = 1, 2, 3.$$

A way to interpret t_j is as the Chern character of $\mathrm{pr}_j^* \mathscr{O}_{\mathbb{P}^\infty}(1)$, i.e.

$$t_j = \mathrm{ch}(\mathrm{pr}_j^* \mathscr{O}_{\mathbb{P}^\infty}(1)) = \exp(-s_j). \tag{13.1.2}$$

If $P(t_1, t_2, t_3) \in \mathbb{Q}((t_1, t_2, t_3))$, we set

$$\overline{P}(t_1, t_2, t_3) = P(t_1^{-1}, t_2^{-1}, t_3^{-1}).$$

Let $K_0^{\mathbb{T}}(U_\alpha) \cong \mathbb{Q}[t_1^{\pm}, t_2^{\pm}, t_3^{\pm}]$ be the Grothendieck ring of $\mathbb{T}$-equivariant vector bundles on U_α. We have a ring homomorphism

$$\mathrm{tr} \colon K_0^{\mathbb{T}}(U_\alpha) \to \mathbb{Q}((t_1, t_2, t_3)), \quad V \mapsto \mathrm{tr}_V,$$

sending a $\mathbb{T}$-equivariant vector bundle to its *character*, i.e. its decomposition into weight spaces. For instance, we have

$$\mathrm{tr}_{R_\alpha} = \sum_{(k_1,k_2,k_3) \in \mathbb{Z}_{\geq 0}^3} t_1^{k_1} t_2^{k_2} t_3^{k_3} = \frac{1}{(1-t_1)(1-t_2)(1-t_3)}. \tag{13.1.3}$$

Example 13.1.2. Let $\alpha \in \Delta(X)$. The tangent space $T_{p_\alpha} X = T_{p_\alpha} U_\alpha$ has character

$$\mathrm{tr}_{T_{p_\alpha} X} = t_1^{-1} + t_2^{-1} + t_3^{-1}.$$

Indeed, via the action (13.1.1), the weight of the $\mathbb{T}$-action on the i-th summand of

$$T_{p_\alpha} U_\alpha = \frac{\partial}{\partial x_1} \cdot \mathbb{C} \oplus \frac{\partial}{\partial x_2} \cdot \mathbb{C} \oplus \frac{\partial}{\partial x_3} \cdot \mathbb{C}$$

is precisely t_i^{-1}. Therefore the equivariant Euler class of $T_{p_\alpha} X$ is given by

$$e^{\mathbb{T}}(T_{p_\alpha} X) = (s_1 s_2 s_3)^{-1} \in \mathcal{H}_{\mathbb{T}}.$$

Remark 13.1.3. The canonical bundle of U_α is trivial as a line bundle, $K_{U_\alpha} = \mathscr{O}_{\mathbb{A}^3}$. Nevertheless, by Example 13.1.2 it is not *equivariantly* trivial: one has

$$K_{U_\alpha} = \det \Omega_{U_\alpha} = \mathscr{O}_{\mathbb{A}^3} \otimes t_1 t_2 t_3. \tag{13.1.4}$$

Recall Theorem 12.3.1, where we stated existence (and properties) of a 0-dimensional perfect obstruction theory on $\mathrm{Hilb}^n X$, for any smooth quasi-projective 3-fold X.

In the *projective* case, the Hilbert scheme is proper, hence we may define the associated Donaldson–Thomas invariant by simply taking the degree of the virtual fundamental class,

$$\mathsf{DT}_n^X = \int_{[\mathrm{Hilb}^n X]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

In the toric *quasi*-projective case, we will define DT_n^X via equivariant residues, i.e. by integrating the right hand side of the virtual localisation formula expressing the equivariant class

$$[\mathrm{Hilb}^n X]^{\mathrm{vir}} \in H_0(\mathrm{Hilb}^n X) \otimes_{H^*_{\mathbb{T}}} \mathcal{H}_{\mathbb{T}}.$$

This uses heavily the fact the torus fixed locus in the Hilbert scheme is proper.

Let us go back to the case where X is projective. A $\mathbb{T}$-fixed ideal

$$\mathscr{I}_Z \in \mathrm{Hilb}^n X$$

corresponds to a finite subscheme $Z \subset X$ supported on the fixed locus $X^{\mathbb{T}}$. For each $\alpha \in \Delta(X)$, the restriction

$$I_\alpha = \mathscr{I}_Z\big|_{U_\alpha} \subset R_\alpha$$

is a monomial ideal, and each collection of monomial ideals $(I_\alpha)_{\alpha\in\Delta(X)}$ gives rise to a $\mathbb{T}$-fixed ideal $\mathscr{I}_Z = \prod_\alpha I_\alpha$. In other words, each point of the torus fixed locus

$$\iota\colon (\mathrm{Hilb}^n X)^{\mathbb{T}} \hookrightarrow \mathrm{Hilb}^n X$$

determines a tuple of (finite) plane partitions

$$\{\pi_\alpha \mid \alpha \in \Delta(X)\}, \quad \pi_\alpha \subset \mathbb{Z}^3_{\geq 0}.$$

Remark 13.1.4. Let $Z \subset X$ be a torus fixed subscheme of length n. We have $n = \chi(\mathscr{O}_Z) = \sum_\alpha |\pi_\alpha|$.

The virtual localisation formula reads as the identity

$$[\mathrm{Hilb}^n X]^{\mathrm{vir}} = \iota_* \sum_{\mathscr{I}_Z} \frac{[S(\mathscr{I}_Z)]^{\mathrm{vir}}}{e^{\mathbb{T}}(T_Z^{\mathrm{vir}})}$$

in $A_0(\mathrm{Hilb}^n X) \otimes_{H^*_{\mathbb{T}}} \mathcal{H}^*_{\mathbb{T}}$, where the sum is over $\mathbb{T}$-fixed ideals, $S(\mathscr{I}_Z) \subset (\mathrm{Hilb}^n X)^{\mathbb{T}}$ is the $\mathbb{T}$-fixed subscheme supported at the point $\mathscr{I}_Z$, and

$$T_Z^{\mathrm{vir}} = \mathbb{E}^\vee\big|_{\mathscr{I}_Z}^{\mathsf{mov}} \in K_0^{\mathbb{T}}(\mathsf{pt})$$

is the virtual tangent space of $\mathrm{Hilb}^n X$ at $\mathscr{I}_Z$. Thus the virtual localisation formula after $\mathbb{T}$-equivariant integration reads

$$\int_{[\mathrm{Hilb}^n X]^{\mathrm{vir}}} 1 = \sum_{\mathscr{I}_Z} \int_{[S(\mathscr{I}_Z)]^{\mathrm{vir}}} e^{\mathbb{T}}(-T_Z^{\mathrm{vir}}) \in \mathcal{H}^*_{\mathbb{T}}. \tag{13.1.5}$$

We have to mention the following important technical step before we continue the calculation.

Theorem 13.1.5 ([49]). *At a fixed ideal $\mathscr{I}_Z$, we have*

$$\mathrm{Ext}^1(\mathscr{I}_Z, \mathscr{I}_Z)^{\mathbb{T}} = \mathrm{Ext}^2(\mathscr{I}_Z, \mathscr{I}_Z)^{\mathbb{T}} = 0.$$

In particular, $S(\mathscr{I}_Z)$ is a reduced point with the trivial perfect obstruction theory, i.e. $[S(\mathscr{I}_Z)]^{\mathrm{vir}} = [\mathscr{I}_Z]$ is the class of a point.

The last result implies that the tangent space is all movable, hence

$$T_Z^{\mathrm{vir}} = \mathrm{Ext}^1(\mathscr{I}_Z, \mathscr{I}_Z) - \mathrm{Ext}^2(\mathscr{I}_Z, \mathscr{I}_Z) \tag{13.1.6}$$

and in particular

$$e^{\mathbb{T}}(-T_Z^{\mathrm{vir}}) = \frac{e^{\mathbb{T}}(\mathrm{Ext}^2(\mathscr{I}_Z,\mathscr{I}_Z))}{e^{\mathbb{T}}(\mathrm{Ext}^1(\mathscr{I}_Z,\mathscr{I}_Z))}.$$

Equation (13.1.5) becomes

$$\int_{[\mathrm{Hilb}^n X]^{\mathrm{vir}}} 1 = \sum_{\mathscr{I}_Z} \int_{\{\mathscr{I}_Z\}} \frac{e^{\mathbb{T}}(\mathrm{Ext}^2(\mathscr{I}_Z,\mathscr{I}_Z))}{e^{\mathbb{T}}(\mathrm{Ext}^1(\mathscr{I}_Z,\mathscr{I}_Z))} \in \mathscr{H}_{\mathbb{T}}^*. \tag{13.1.7}$$

Note that in the projective case we know this integration yields a *number*, and this number is of course DT_n^X. If X is a quasi-projective toric 3-fold, we *define* DT_n^X as the right hand side of (13.1.7).

Our goal is to compute

$$\mathsf{DT}_X(q) = \sum_{n\geq 0} \mathsf{DT}_n^X q^n$$

for X a toric Calabi–Yau 3-fold.

The virtual tangent space (13.1.6) can be written

$$\begin{aligned} T_Z^{\mathrm{vir}} &= \chi(\mathscr{O}_X,\mathscr{O}_X) - \chi(\mathscr{I}_Z,\mathscr{I}_Z) \\ &= \bigoplus_\alpha \left(\Gamma(U_\alpha) - \sum_i (-1)^i \Gamma(U_\alpha, \mathscr{E}xt^i(\mathscr{I}_Z,\mathscr{I}_Z))\right), \end{aligned}$$

so in order to compute each summand of (13.1.7) we have to evaluate the character of the virtual representation

$$T_\alpha^{\mathrm{vir}} = \chi(R_\alpha,R_\alpha) - \chi(I_\alpha,I_\alpha) \in K_0^{\mathbb{T}}(U_\alpha) \tag{13.1.8}$$

for each $\alpha \in \Delta(X)$.

13.2. **The calculation at the vertex.** Let X be a smooth projective toric 3-fold, and let $\alpha \in \Delta(X)$ be a vertex of its Newton polytope. Let $I_\alpha \subset R_\alpha$ be a monomial ideal arising from a $\mathbb{T}$-fixed point of $\mathrm{Hilb}^n X$. We define

$$\mathsf{Q}_\alpha(t_1,t_2,t_3) = \mathrm{tr}_{R_\alpha/I_\alpha} = \sum_{(k_1,k_2,k_3)\in\pi_\alpha} t_1^{k_1} t_2^{k_2} t_3^{k_3},$$

the character of the representation R_α/I_α.

The argument for the next proof was shown to me by Sergej Monavari.

Lemma 13.2.1. *Let* $\mathsf{P}_\alpha(t_1,t_2,t_3)$ *be the Poincaré polynomial of* I_α. *Then there is an identity*

$$\mathsf{Q}_\alpha(t_1,t_2,t_3) = \frac{1+\mathsf{P}_\alpha(t_1,t_2,t_3)}{(1-t_1)(1-t_2)(1-t_3)}.$$

Proof. The claim is equivalent to the statement

$$-\mathrm{tr}_{I_\alpha} = \frac{\mathsf{P}_\alpha(t_1,t_2,t_3)}{(1-t_1)(1-t_2)(1-t_3)}. \tag{13.2.1}$$

Let us prove (13.2.1) for a free module of rank one

$$M_{abc} = x_1^a x_2^b x_3^c \cdot R_\alpha.$$

The Poincaré polynomial $P(M)$ of an arbitrary module M is computed using a free $\mathbb{T}$-equivariant resolution

$$0 \to F_s \to F_{s-1} \to \cdots \to F_2 \to F_1 \to M \to 0.$$

by means of the formula

$$P(M) = \sum_{i=1}^{s} (-1)^{i-1} P(F_i).$$

Thus in the case of the module M_{abc} above one has a 1-step resolution, yielding

$$P(M_{abc}) = -t_1^a t_2^b t_3^c.$$

On the other hand, we have

$$-\mathrm{tr}_{M_{abc}} = -\sum_{i\ge a}\sum_{j\ge b}\sum_{k\ge c} t_1^i t_2^j t_3^k = \frac{-t_1^a t_2^b t_3^c}{(1-t_1)(1-t_2)(1-t_3)},$$

thus proving (13.2.1) for the module M_{abc}.

For $M = I_\alpha$, pick a resolution as above, where each F_i is free,

$$F_i = \bigoplus_{j=1}^{r_i} R_\alpha(d_{ij}), \quad d_{ij} \in \mathbb{Z}^3.$$

In particular, each summand satisfies

$$\begin{aligned} P(R_\alpha(d_{ij})) &= -t_1^{d_{ij}(1)} t_2^{d_{ij}(2)} t_3^{d_{ij}(3)} \\ \mathrm{tr}_{R_\alpha(d_{ij})} &= \frac{t_1^{d_{ij}(1)} t_2^{d_{ij}(2)} t_3^{d_{ij}(3)}}{(1-t_1)(1-t_2)(1-t_3)}. \end{aligned} \tag{13.2.2}$$

We can write

$$\begin{aligned} P(I_\alpha) &= \sum_{i=1}^{s} (-1)^{i-1} P(F_i) \\ &= -\sum_{i=1}^{s} (-1)^i \sum_{j=1}^{r_i} P(R_\alpha(d_{ij})) \\ &= \sum_{i=1}^{s} (-1)^i \sum_{j=1}^{r_i} t_1^{d_{ij}(1)} t_2^{d_{ij}(2)} t_3^{d_{ij}(3)}. \end{aligned}$$

On the other hand,

$$\begin{aligned} -\mathrm{tr}_{I_\alpha} &= \sum_{i=1}^{s} (-1)^i \,\mathrm{tr}_{F_i} \\ &= \sum_{i=1}^{s} (-1)^i \sum_{j=1}^{r_i} \mathrm{tr}_{R_\alpha(d_{ij})} \\ &= \sum_{i=1}^{s} (-1)^i \sum_{j=1}^{r_i} \frac{t_1^{d_{ij}(1)} t_2^{d_{ij}(2)} t_3^{d_{ij}(3)}}{(1-t_1)(1-t_2)(1-t_3)} \\ &= \frac{P(I_\alpha)}{(1-t_1)(1-t_2)(1-t_3)}. \end{aligned}$$

The proof is complete. □

Lemma 13.2.2. *There is an identity*

$$\mathrm{tr}_{\chi(I_\alpha, I_\alpha)} = \frac{\mathsf{P}_\alpha \overline{\mathsf{P}}_\alpha}{(1-t_1)(1-t_2)(1-t_3)}.$$

Proof. The virtual representation $\chi(I_\alpha, I_\alpha)$ is given by the formula

$$\begin{aligned}\chi(I_\alpha, I_\alpha) &= \sum_{i,j,k,l} (-1)^{i+k} \operatorname{Hom}_{R_\alpha}(R_\alpha(d_{ij}), R_\alpha(d_{kl})) \\ &= \sum_{i,j,k,l} (-1)^{i+k} R_\alpha(d_{kl} - d_{ij}).\end{aligned}$$

Its trace is given by

$$\frac{\sum_{i,k}(-1)^i(-1)^k \sum_{j,l} t_1^{d_{kl}(1)-d_{ij}(1)} t_2^{d_{kl}(2)-d_{ij}(2)} t_3^{d_{kl}(3)-d_{ij}(3)}}{(1-t_1)(1-t_2)(1-t_3)} = \frac{\mathsf{P}_\alpha \overline{\mathsf{P}}_\alpha}{(1-t_1)(1-t_2)(1-t_3)}, \tag{13.2.3}$$

where we used the second equation in (13.2.2). □

Putting all these calculations together we can now evaluate the character of (13.1.8) as follows.

Proposition 13.2.3. *There is an identity*

$$\operatorname{tr}_{\chi(R_\alpha,R_\alpha)-\chi(I_\alpha,I_\alpha)} = \mathsf{Q}_\alpha - \frac{\overline{\mathsf{Q}}_\alpha}{t_1 t_2 t_3} + \mathsf{Q}_\alpha \overline{\mathsf{Q}}_\alpha \frac{(1-t_1)(1-t_2)(1-t_3)}{t_1 t_2 t_3} \tag{13.2.4}$$

Proof. We start by observing that

$$\begin{aligned}\frac{-1-\overline{\mathsf{P}}_\alpha}{(1-t_1)(1-t_2)(1-t_3)} &= \frac{1+\overline{\mathsf{P}}_\alpha}{t_1 t_2 t_3 (1-t_1^{-1})(1-t_2^{-1})(1-t_3^{-1})} \\ &= \frac{\overline{\mathsf{Q}}_\alpha}{t_1 t_2 t_3},\end{aligned} \tag{13.2.5}$$

where the last identity follows from Lemma 13.2.1. We have

$$\begin{aligned}\operatorname{tr}_{\chi(R_\alpha,R_\alpha)-\chi(I_\alpha,I_\alpha)} &= \operatorname{tr}_{\chi(R_\alpha,R_\alpha)} - \operatorname{tr}_{\chi(I_\alpha,I_\alpha)} \\ &= \frac{1-\mathsf{P}_\alpha \overline{\mathsf{P}}_\alpha}{(1-t_1)(1-t_2)(1-t_3)} && \text{by (13.1.3) and Lemma 13.2.2} \\ &= \frac{1+\mathsf{P}_\alpha - \mathsf{P}_\alpha - \mathsf{P}_\alpha \overline{\mathsf{P}}_\alpha}{(1-t_1)(1-t_2)(1-t_3)} \\ &= \mathsf{Q}_\alpha - \frac{\mathsf{P}_\alpha(1+\overline{\mathsf{P}}_\alpha)}{(1-t_1)(1-t_2)(1-t_3)} && \text{by Lemma 13.2.1} \\ &= \mathsf{Q}_\alpha + \mathsf{P}_\alpha \frac{\overline{\mathsf{Q}}_\alpha}{t_1 t_2 t_3} && \text{by (13.2.5)} \\ &= \mathsf{Q}_\alpha + (\mathsf{Q}_\alpha \cdot (1-t_1)(1-t_2)(1-t_3) - 1) \frac{\overline{\mathsf{Q}}_\alpha}{t_1 t_2 t_3}, && \text{by Lemma 13.2.1}\end{aligned}$$

which is what we had to prove. □

Setting

$$\mathsf{V}_\alpha = \operatorname{tr}_{T_\alpha^{\mathrm{vir}}} \operatorname{tr}_{\chi(R_\alpha,R_\alpha)-\chi(I_\alpha,I_\alpha)}$$

to be the vertex contribution at α, we obtain for each torus-fixed ideal $\mathscr{I}_Z$ the fundamental relation

$$\boxed{\mathrm{tr}_{T_Z^{\mathrm{vir}}} = \sum_\alpha \mathsf{V}_\alpha.}$$

This allows us to evaluate

$$\frac{e^{\mathbb{T}}(\mathrm{Ext}^2(\mathscr{I}_Z,\mathscr{I}_Z))}{e^{\mathbb{T}}(\mathrm{Ext}^1(\mathscr{I}_Z,\mathscr{I}_Z))} = \prod_\alpha e^{\mathbb{T}}(-\mathsf{V}_\alpha).$$

Next we decompose V_α in a positive part and a negative part, so to isolate the monomials appearing with a positive (resp. negative) sign in the K-theoretic virtual representation T_Z^{vir}. Using Proposition 13.2.3, we can split V_α as

$$\mathsf{V}_\alpha = \mathsf{V}_\alpha^+ + \mathsf{V}_\alpha^-,$$

where

$$\begin{aligned} \mathsf{V}_\alpha^+ &= \mathsf{Q}_\alpha - \mathsf{Q}_\alpha\overline{\mathsf{Q}}_\alpha\frac{(1-t_1)(1-t_2)}{t_1t_2} \\ \mathsf{V}_\alpha^- &= -\frac{\overline{\mathsf{Q}}_\alpha}{t_1t_2t_3} + \mathsf{Q}_\alpha\overline{\mathsf{Q}}_\alpha\frac{(1-t_1)(1-t_2)}{t_1t_2t_3}. \end{aligned} \tag{13.2.6}$$

From now on we specialise to the *local* Calabi–Yau geometry of a nonsingular toric Calabi–Yau 3-fold X (in particular X is not projective). There is a 2-dimensional subtorus

$$\mathbb{T}_0 \subset \mathbb{T}$$

defined by the condition $t_1t_2t_3 = 1$, i.e. $\mathbb{T}_0$ is the kernel of the character $(1,1,1)$. The subtorus $\mathbb{T}_0$ preserves the Calabi–Yau form on X, because it does so on each invariant open chart U_α — see Remark 13.1.4. Since, for each α, no trivial weight appears in V_α after restriction to $\mathbb{T}_0$, we can evaluate the virtual localisation formula after specialising to $\mathbb{T}_0$.

Restricting to this "Calabi–Yau torus" $\mathbb{T}_0$ yields the following remarkable identity.

Lemma 13.2.4. *There is an identity*

$$\overline{\mathsf{V}_\alpha^+}\big|_{t_1t_2t_3=1} = -\mathsf{V}_\alpha^-\big|_{t_1t_2t_3=1}. \tag{13.2.7}$$

Proof. It is enough to compare the specialisations of the fractions appearing in (13.2.6). For the left hand side, we find

$$\begin{aligned} \frac{(1-t_1^{-1})(1-t_2^{-1})}{t_1^{-1}t_2^{-1}}\bigg|_{t_1t_2t_3=1} &= \frac{(1-t_1^{-1})(1-t_2^{-1})}{t_3}\bigg|_{t_1t_2t_3=1} \\ &= (1-t_1^{-1}-t_2^{-1}+t_1^{-1}t_2^{-1})t_3^{-1}\big|_{t_1t_2t_3=1} \\ &= t_3^{-1}-t_2-t_1+1. \end{aligned}$$

This is clearly the same as $(1-t_1)(1-t_2)\big|_{t_1t_2t_3=1}$. □

Consider the transformation rule

$$\sum_i n_i e^{w_i} \mapsto \prod_i w_i^{n_i} \tag{13.2.8}$$

for weights $w_i \in H_{\mathbb{T}}^* = \mathbb{Q}[s_1,s_2,s_3]$. The right hand side can be considered as a function of t_1, t_2 and t_3 just as in Equation (13.1.2). Let $(w_{i,\alpha})_i$ be the weigths of the tangent representation of I_α. So $w_{i,\alpha} = w_{i,\alpha}(s_1, s_2, s_3)$ are functions of the equivariant parameters of the torus. We have the basic identity

$$\mathsf{V}_\alpha^+\big|_{t_1 t_2 t_3=1} = \sum_i n_{i,\alpha} e^{w_{i,\alpha}(s_1,s_2,s_3)}\big|_{s_1+s_2+s_3=0},$$

where $n_{i,\alpha} > 0$ is the multiplicity of $w_{i,\alpha}$.

It follows from the previous lemma that

$$\begin{aligned}
\mathsf{V}_\alpha\big|_{t_1 t_2 t_3=1} &= \mathsf{V}_\alpha^+\big|_{t_1 t_2 t_3=1} + \mathsf{V}_\alpha^-\big|_{t_1 t_2 t_3=1} \\
&= \mathsf{V}_\alpha^+\big|_{t_1 t_2 t_3=1} - \overline{\mathsf{V}_\alpha^+}\big|_{t_1 t_2 t_3=1} \\
&= \sum_i n_{i,\alpha} e^{w_{i,\alpha}(s_1,s_2,s_3)}\big|_{s_1+s_2+s_3=0} - \sum_i n_{i,\alpha} e^{-w_{i,\alpha}(s_1,s_2,s_3)}\big|_{s_1+s_2+s_3=0},
\end{aligned}$$

which under the transformation (13.2.8) corresponds to

$$\frac{\prod_i w_{i,\alpha}(s_1,s_2,s_3)^{n_{i,\alpha}}}{\prod_i (-w_{i,\alpha}(s_1,s_2,s_3))^{n_{i,\alpha}}}\Bigg|_{s_1+s_2+s_3=0}.$$

As we saw in Equation (6.5.7), turning off the weights (i.e. setting $s_1 = s_2 = s_3 = 0$, or equivalently $t_1 = t_2 = t_3 = 1$) computes the sum of the multiplicities $n_{i,\alpha}$ for fixed α. In symbols, we have

$$\sum_i n_{i,\alpha} = \mathsf{V}_\alpha^+\big|_{t_1=t_2=t_3=1}. \tag{13.2.9}$$

On the other hand, we have

$$\mathsf{V}_\alpha^+\big|_{t_1=t_2=t_3=1} = \mathsf{Q}_\alpha(1,1,1) = |\pi_\alpha|,$$

which yields the identity

$$\sum_i n_{i,\alpha} = |\pi_\alpha|.$$

We can now finish evaluating the virtual localisation formula. We find

$$\begin{aligned}
\prod_\alpha e^{\mathbb{T}}\left(-\mathsf{V}_\alpha\big|_{\mathbb{T}_0}\right) &= \prod_\alpha \frac{\prod_i (-w_{i,\alpha})^{n_{i,\alpha}}}{\prod_i w_{i,\alpha}^{n_{i,\alpha}}} \\
&= \prod_\alpha \prod_i (-1)^{n_{i,\alpha}} \\
&= \prod_\alpha (-1)^{\sum_i n_{i,\alpha}} \\
&= \prod_\alpha (-1)^{|\pi_\alpha|} \\
&= (-1)^{\sum_\alpha |\pi_\alpha|} \\
&= (-1)^n.
\end{aligned}$$

It follows that

$$
\begin{aligned}
\mathsf{DT}_X(q) &= \sum_{n\geq 0} q^n \sum_{\mathscr{I}_Z\in(\mathrm{Hilb}^n X)^{\mathbb{T}}} \frac{e^{\mathbb{T}}(\mathrm{Ext}^2(\mathscr{I}_Z,\mathscr{I}_Z))}{e^{\mathbb{T}}(\mathrm{Ext}^1(\mathscr{I}_Z,\mathscr{I}_Z))}\\
&= \sum_{n\geq 0} q^n \sum_{\mathscr{I}_Z\in(\mathrm{Hilb}^n X)^{\mathbb{T}}} \prod_{\alpha} e\left(-\mathsf{V}_\alpha\big|_{\mathbb{T}_0}\right)\\
&= \sum_{n\geq 0} \chi(\mathrm{Hilb}^n X)(-q)^n\\
&= \mathsf{H}_X(-q)\\
&= \mathsf{M}(-q)^{\chi(X)},
\end{aligned}
\tag{13.2.10}
$$

where $\mathsf{M}(q)$ is the MacMahon function, defined (cf. Definition 9.2.5) as

$$
\mathsf{M}(q) = \prod_{m\geq 1}(1-q^m)^{-m}.
$$

We saw in Theorem 9.2.6 that $\mathsf{M}(q)$ determines the Euler characteristics of $\mathrm{Hilb}^n X$ on *any* smooth 3-fold, which is what we used in the last equality displayed in (13.2.10).

We have thus proved:

Theorem 13.2.5. *Let X be a toric Calabi–Yau 3-fold. Then*

$$
\mathsf{DT}_X(q) = \prod_{m\geq 1}(1-(-q)^m)^{-m\chi(X)}.
$$

Appendix A. Deformation Theory

This section is based on the material covered in [22].

A.1. The general problem.

Let k be an algebraically closed field. Let Art_k denote the category of local Artin k-algebras $(A, \mathfrak{m}_A)$ with residue field k. Its opposite category is equivalent to the category of *fat points*, i.e. the category of k-schemes S such that the structure morphism $S_{\mathrm{red}} \to \operatorname{Spec} k$ is an isomorphism.

Definition A.1.1. A *deformation functor* is a covariant functor $\mathrm{D}\colon \mathrm{Art}_k \to \mathrm{Sets}$ such that $\mathrm{D}(k) = \{\star\}$ is a singleton.

Let D be a deformation functor. On morphisms, we have the following picture:

$$\begin{array}{ccc} B & & \mathrm{D}(B) \\ \phi\big\downarrow & \xmapsto{\mathrm{D}} & \big\downarrow \phi_* = \mathrm{D}\phi \\ A & & \mathrm{D}(A) \end{array}$$

If $\phi_*\beta = \alpha$, then β is called a *lift* of $\alpha \in \mathrm{D}(A)$.

Remark A.1.2. Let B be an object in Art_k and let D be a deformation functor. Then $\mathrm{D}(B)$ is not empty. Indeed, look at the diagram

$$\begin{tikzcd} k \arrow[r, "\phi"] \arrow[d, equal] & B \arrow[dl] \\ k & \end{tikzcd} \quad \xmapsto{\mathrm{D}} \quad \begin{tikzcd} \{\star\} \arrow[r, "\phi_*"] \arrow[d, equal] & \mathrm{D}(B) \arrow[dl] \\ \{\star\} & \end{tikzcd}$$

$\phi_*(\star)$ is a canonical element of $\mathrm{D}(B)$.

Here are some examples of deformation functors.

Example A.1.3. Consider the following:

(1) Let Y be a projective k-variety, $\iota\colon X \hookrightarrow Y$ a closed k-subvariety. Then

$$A \mapsto \left\{ \begin{array}{c} \text{closed subschemes } i\colon Z \hookrightarrow Y \times_k \operatorname{Spec} A \\ \text{such that } Z \text{ is } A\text{-flat and } i \otimes_A k = \iota \end{array} \right\}$$

defines a deformation functor $\mathsf{H}_{X/Y}\colon \mathrm{Art}_k \to \mathrm{Sets}$.

(2) Let E be a coherent sheaf on a Noetherian scheme Y. Then

$$A \mapsto \left\{ \begin{array}{c} \text{coherent sheaves } \mathscr{E} \in \operatorname{Coh}(Y \times_k \operatorname{Spec} A) \\ \text{flat over } A \text{ such that } \mathscr{E} \otimes_A k = E \end{array} \right\}$$

defines a deformation functor $\mathsf{M}_E\colon \mathrm{Art}_k \to \mathrm{Sets}$.

(3) Let E be a coherent sheaf on a Noetherian scheme Y. Let $\vartheta\colon E \twoheadrightarrow Q$ be a surjection in $\operatorname{Coh}(Y)$. Then

$$A \mapsto \left\{ \begin{array}{c} \text{surjections } \alpha\colon E_A \twoheadrightarrow \mathcal{Q} \text{ in } \operatorname{Coh}(Y \times_k \operatorname{Spec} A) \\ \text{such that } \mathcal{Q} \text{ is } A\text{-flat and } \alpha \otimes_A k = \vartheta \end{array} \right\}$$

defines a deformation functor $\mathsf{Q}_\vartheta\colon \mathrm{Art}_k \to \mathrm{Sets}$. The notation E_A indicates the pullback of E under the canonical projection $Y \times_k \operatorname{Spec} A \to Y$.

In Definition A.1.1, one should think of $\star$ as the geometric object, defined over $\operatorname{Spec} k$, that one wants to deform. Similarly, elements of $\mathrm{D}(A)$ should be interpreted as deformations of $\star$ parameterised by $\operatorname{Spec} A$. In Example (1) (resp. (2), (3)) we took $\star$ to be the closed immersion $\iota\colon X \hookrightarrow Y$ (resp. the sheaf E, the quotient $E \twoheadrightarrow Q$).

Keeping this in mind, the basic questions of deformation theory are the following: given a surjection $B \twoheadrightarrow A$ in Art_k,

- $Q1$. Which elements of $\mathrm{D}(A)$ lift? In other words, what is the image of $\mathrm{D}(B) \to \mathrm{D}(A)$?
- $Q2$. If α is in the image of ϕ_*, how big is its preimage in $\mathrm{D}(B)$? In other words, how many lifts does α have?

We care about *surjections* $B \twoheadrightarrow A$ because they corresponds to closed embeddings of schemes. Here is an illustrative picture:

$$\begin{tikzcd} \star \arrow[r, hook] \arrow[d, no head] & ? \arrow[r, hook] \arrow[d, no head] & ?? \arrow[d, no head] \\ \operatorname{Spec} k \arrow[r, hook] & \operatorname{Spec} A \arrow[r, hook] & \operatorname{Spec} B. \end{tikzcd}$$

Of course, this picture should be taken with a grain of salt. For instance, the meaning of the upper part of the diagram strictly depends upon the nature of the object $\star$ to be deformed (it could be a subscheme, a coherent sheaf, a surjection of coherent sheaves...).

In fact, one can restrict attention to those surjections called *small extensions*.

Definition A.1.4. A surjection $B \twoheadrightarrow A$ in Art_k with kernel I is called

- (i) a *square zero extension* if $I^2 = 0$,
- (ii) a *semi-small extension* if $I \cdot \mathfrak{m}_B = 0$,
- (iii) a *small extension* if it is semi-small and $\dim_k I = 1$.

Clearly, one has

$$\text{small} \quad \Rightarrow \quad \text{semismall} \quad \Rightarrow \quad \text{square zero.}$$

Lemma A.1.5. *Every surjection $B \twoheadrightarrow A$ in Art_k factors as a composition of finitely many small extensions.*

Definition A.1.6. Let Loc_k denote the category of local k-algebras $(R, \mathfrak{m}_R)$ with residue field k and such that $T_R = (\mathfrak{m}_R/\mathfrak{m}_R^2)^\vee$ is a finite dimensional vector space. Let CLoc_k be the subcategory of *complete* k-algebras.

Every object R of Loc_k defines a deformation functor

$$h_R \colon \mathrm{Art}_k \to \mathrm{Sets}, \qquad h_R(A) = \mathrm{Hom}_k(R, A).$$

Letting Def_k be the category of deformation functors (with arrows being natural transformations between functors), we get a well defined functor

$$h \colon \mathrm{Loc}_k \to \mathrm{Def}_k, \qquad R \mapsto h_R.$$

The functor h restricts to a fully faithful functor $\mathrm{CLoc}_k \to \mathrm{Def}_k$. In particular, there is an equivalence

$$\mathrm{CLoc}_k \cong h(\mathrm{CLoc}_k).$$

A.2. **Liftings.** Let us suppose we are in the following situation:

$$
\begin{tikzcd}
 & 0 \arrow[d] \\
 & I \arrow[d] \\
R \arrow[r] \arrow[d] & B \arrow[d] \\
P \arrow[r] \arrow[ur, dashed, "\alpha"] & A \arrow[d] \\
 & 0
\end{tikzcd}
\tag{A.2.1}
$$

$B \twoheadrightarrow A$ is a square zero extension in Art_k,

the square is a commutative square

of k-algebras homomorphisms.

In what follows, we will simply say that the dotted arrow α "makes the diagram commute" if the two inner triangles commute.

Proposition A.2.1. *In situation* (A.2.1)*, the set $\mathcal{S}$ of $\alpha \in \mathrm{Hom}_k(P \to B)$ such that the diagram commutes is either empty, or a torsor under*

$$\mathrm{Der}_R(P, I) = \mathrm{Hom}_P(\Omega^1_{P/R}, I).$$

The next lemma will be needed in the scheme-theoretic version of the previous result.

Lemma A.2.2. *Let $\tau\colon M_1 \to M_2$ be a homomorphism of Abelian groups, and let $\mathcal{S}$ be a torsor under M_1, admitting an M_1-equivariant map $\theta\colon \mathcal{S} \to M_2$. Let*

$$M_1 \xrightarrow{\tau} M_2 \xrightarrow{q} K \to 0$$

be the cokernel exact sequence. Then

- *The image of $q(\theta(\mathcal{S}))$ is a single element $c_0 \in K$.*
- *An element $m \in M_2$ is in the image of θ if and only if $q(m) = c_0$.*
- *Any $m \in M_2$ is in the image of θ if and only if $c_0 = 0$.*

The global version of Proposition A.2.1 is the following:

Proposition A.2.3. *Let $f\colon X \to Y$ be a morphism of k-schemes with a factorisation $\pi \circ i\colon X \to M \to Y$. Let $J \subset \mathscr{O}_M$ be the ideal of X in M. Let $S \to T$ be a semismall extension of fat points with ideal I. Given a commutative square*

$$
\begin{tikzcd}
S \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\
T \arrow[r] & Y
\end{tikzcd}
$$

and letting x be the image of the closed point of S in X, there is a natural obstruction to the existence of k-morphisms $T \to X$ making the diagram commute. Such obstruction lives in the cokernel of the natural map of k-vector spaces

$$\mathrm{Hom}_k(x^* i^* \Omega^1_\pi, I) \to \mathrm{Hom}_k(x^* J/J^2, I).$$

Proof. The statement is local, meaning that it does not change if we replace X and Y with affine open neighborhoods of x and $f(x)$ respectively. So we may assume X

and $Y = \operatorname{Spec} R$ are both affine. Moreover, the morphism

$$J/J^2 \to i^*\Omega^1_\pi$$

has the very pleasant propert that its cokernel does not depend on the choice of factorisation. Hence we may very well assume that M is affine space over Y, i.e. $M = \operatorname{Spec} P$, where $P = R[t_1, \dots, t_n]$. We can then write $X = \operatorname{Spec} R/J$ for an ideal J. Let us compare the scheme situation and the algebra situation:

$$\begin{tikzcd}
S \arrow[r] \arrow[dd, hook] & X \arrow[d, "i"] \\
& M \arrow[d, "\pi"] \\
T \arrow[r] \arrow[uur, dotted, "\alpha"] \arrow[ur, dotted, "\beta"'] & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A & P/J \arrow[l] \arrow[ddl, dotted, "\alpha"'] \\
& P \arrow[u] \arrow[dl, dotted, "\beta"] \\
B \arrow[uu, two heads] & R \arrow[l] \arrow[u]
\end{tikzcd}$$

Since i is a closed immersion the set of α's making the diagram (any of them) commutative injects in the set of β's making the diagram commutative. Letting $\mathcal{S}$ be the set of $\beta \in \operatorname{Hom}_k(P, B)$ such that the diagram (on the right) commutes, we know $\mathcal{S}$ is not empty because P is a polynomial ring; therefore by Proposition A.2.1 it is a torson under

$$\operatorname{Hom}_P(\Omega^1_{P/R}, I) \cong \operatorname{Hom}_A(\Omega^1_{P/R} \otimes_P A, I) \cong \operatorname{Hom}_k(\Omega^1_{P/R} \otimes_P k, I).$$

Using the identifications

$$\operatorname{Hom}_P(J, I) = \operatorname{Hom}_{P/J}(J/J^2, I) = \operatorname{Hom}_k(x^* J/J^2, I),$$

we see that we are in the situation

$$\begin{tikzcd}
& \mathcal{S} \arrow[d, "\theta"] & & \\
\operatorname{Hom}_k(\Omega^1_{P/R} \otimes_P k, I) \arrow[r, "\tau"] & \operatorname{Hom}_k(x^* J/J^2, I) \arrow[r] & K \arrow[r] & 0
\end{tikzcd}$$

where the map τ is the action by translation coming from (pulling back alon x) the morphism $J/J^2 \to \Omega^1_{P/R} \otimes_P P/J$ (the local version of $J/J^2 \to i^*\Omega^1_{M/Y}$), and θ is the $\operatorname{Hom}_k(\Omega^1_{P/R} \otimes_P k, I)$-equivariant map sending $\beta\colon P \to B$ to its restriction to J. By the previous lemma the torsor condition tells us that for any $\beta \in \mathcal{S}$, its image in $K = \operatorname{coker} \tau$, does not depend on β. By the same lemma, there exists an α making the diagram commutative if and only if for all $\beta \in \mathcal{S}$, the image of β in K is zero. $\square$

A.3. **Tangent-obstruction theories.** A detailed exposition on tangent-obstruction theories can be found in [22], where the reader is referred to for further details.

Let k be an algebraically closed field and let Art_k denote the category of local Artinian k-algebras with residue field k. Recall that a *deformation functor* is a covariant functor $\mathrm{D}\colon \operatorname{Art}_k \to \mathrm{Sets}$ such that $\mathrm{D}(k)$ is a singleton. A *tangent-obstruction theory* on a deformation functor D is a pair (T_1, T_2) of finite dimensional k-vector spaces such that for any small extension $I \hookrightarrow B \twoheadrightarrow A$ in Art_k one has an exact sequence of sets

$$T_1 \otimes_k I \to \mathrm{D}(B) \to \mathrm{D}(A) \xrightarrow{\mathrm{ob}} T_2 \otimes_k I \tag{A.3.1}$$

that acquires a zero on the left whenever $A = k$, and is moreover functorial in small extensions. See [22] for the precise meaning of "exact sequence of sets" and functoriality in small extensions.

The tangent space of the tangent-obstruction theory is T_1, and is canonically determined by the deformation functor as $T_1 = \mathrm{D}(k[t]/t^2)$. A deformation functor is *pro-representable* if it is isomorphic to $\mathrm{Hom}_{k\text{-alg}}(R,-)$ for some local k-algebra R with residue field k. A tangent-obstruction theory on a pro-representable deformation functor always has a zero on the left in the sequences (A.3.1), which means that lifts of a given $\alpha \in \mathrm{D}(A)$, when they exist, form an affine space over $T_1 \otimes_k I$.

Here are some examples of tangent-obstruction theories.

Example A.3.1.

The following result is often useful.

Proposition A.3.2. *Let* D, D' *be two pro-representable deformation functors carrying tangent-obstruction theories* (T_1, T_2) *and* (T_1', T_2')*, respectively. Let* $\eta\colon \mathrm{D} \to \mathrm{D}'$ *be a morphism inducing an isomorphism* $h\colon T_1 \xrightarrow{\sim} T_1'$ *and a linear embedding* $T_2 \hookrightarrow T_2'$*. Then* η *is an isomorphism.*

Proof. The statement that $\eta_B\colon \mathrm{D}(B) \to \mathrm{D}'(B)$ is bijective is clear when $B = k$ and when $B = k[t]/t^2$, by assumption. So we proceed by induction on the length of the Artinian rings $A \in \mathrm{Art}_k$. Fix a small extension $I \hookrightarrow B \twoheadrightarrow A$ in Art_k and consider the associated commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & T_1 \otimes I \arrow[r] \arrow[d, "\wr"] & \mathrm{D}(B) \arrow[r] \arrow[d, "\eta_B"] & \mathrm{D}(A) \arrow[r, "\mathrm{ob}"] \arrow[d, "\wr"] & T_2 \otimes I \arrow[d, hook] \\
0 \arrow[r] & T_1' \otimes I \arrow[r] & \mathrm{D}'(B) \arrow[r] & \mathrm{D}'(A) \arrow[r] & T_2' \otimes I
\end{tikzcd}$$

where we have to show that η_B is bijective. For injectivity, pick two elements β_1, $\beta_2 \in \mathrm{D}(B)$ with images $\beta_i' = \eta_B(\beta_i) \in \mathrm{D}'(B)$. Assume $\beta_1' \neq \beta_2'$. We may assume that the images of β_1 and β_2 in $\mathrm{D}(A)$ agree, because if they differed, we would have $\beta_1 \neq \beta_2$ and we would be done with injectivity. By exactness of the bottom exact sequence of sets, we know that $\beta_2' = v' \cdot \beta_1'$ for a *unique* $v' \in T_1' \otimes I$ (here we are using pro-representability of D). So if $\beta_1' \neq \beta_2'$ we must have $v' \neq 0$. Above, we have $\beta_2 = v \cdot \beta_1$ where $v = (h \otimes \mathrm{id}_I)^{-1}(v') \in T_1 \otimes I$, but $v' \neq 0$ implies $v \neq 0$ and thus (by pro-representability of D) $\beta_1 \neq \beta_2$.

As for surjectivity, pick an element $\beta' \in \mathrm{D}(B)$. It certainly maps to zero in $T_2' \otimes I$. Its image α' in $\mathrm{D}'(A)$ lifts uniquely to an element $\alpha \in \mathrm{D}(A)$ such that $\mathrm{ob}(\alpha)$ goes to zero in $T_2' \otimes I$. But by the injectivity assumption, we must have $\mathrm{ob}(\alpha) = 0$, i.e. α lifts to some $\beta \in \mathrm{D}(B)$. For sure, $\eta_B(\beta)$ is a lift of $\alpha' \in \mathrm{D}(A)$, so $\beta' = v' \cdot \eta_B(\beta)$ for a unique v', as above. Then, if $v \in T_1 \otimes I$ is the preimage of v', we see that $v \cdot \beta \in \mathrm{D}(B)$ is a preimage of β' under η_B. □

We now focus on examples related to real life: moduli spaces.

Moduli Situation. Let $\mathfrak{M}\colon \mathrm{Sch}_k^{\mathrm{op}} \longrightarrow \mathrm{Sets}$ be a functor represented by a scheme M. Fix $p \in M(k) = \mathfrak{M}(\operatorname{Spec} k)$. Consider the subfunctor

$$\mathsf{Def}_p \subset \mathfrak{M}\big|_{\mathrm{Art}_k}$$

defined by

$$\mathsf{Def}_p(A) = \left\{ \eta \in \mathfrak{M}(\operatorname{Spec} A) \colon \eta \,\big|\, {}_k = p \right\}$$

where restriction to $\operatorname{Spec} k \subset \operatorname{Spec} A$ is the map $\mathfrak{M}(\operatorname{Spec} A) \longrightarrow \mathfrak{M}(\operatorname{Spec} k)$.

In this situation, Def_p is prorepresentable by $R = \mathscr{O}_{M,p}$ and as such it has a tangent/obstruction theory $(T_1, T_2 \supset T_2(R))$ with tangent space

$$T_1 = \mathsf{Def}_p(k[t]/t^2) = h_R(k[t]/t^2) = T_p M.$$

One always has the inequalities

$$\dim T_p M \geq \dim \mathscr{O}_{M,p} \geq \dim T_p M - \dim T_2$$

showing that if $T_2 = 0$ then M is smooth at p. The converse is not true.

Let $p = [F \twoheadrightarrow E] \in \mathrm{Quot}_X(F)$ be a closed point, and set $I = \ker(F \twoheadrightarrow E)$. Consider the subfunctor

$$\mathsf{Def}_p \subset \mathrm{Quot}_X(F)\big|_{\mathrm{Art}_k}$$

that evaluated on a fat point A consists of surjections $f\colon F \otimes_k A \twoheadrightarrow \mathscr{E}$ in $\mathrm{Coh}(X \times_k A)$ such that $f \otimes_A k = p$. Then

$$\mathsf{Def}_p = h_{\mathscr{O}_{\mathrm{Quot}_X(F)},p}$$

is prorepresentable. This functor has a tangent/obstruction theory given by

$$T_1 = \mathrm{Hom}_X(I, E), \quad T_2 = \mathrm{Ext}^1_X(I, E), \tag{A.3.2}$$

and by what we said above we have

$$T_1 = T_p \,\mathrm{Quot}_X(F).$$

Example A.3.3. Let $C \subset \mathbb{P}^2$ be a curve of degree d. Then $\mathscr{I}_C = (-d)$ so $N = (\mathscr{I}_C/\mathscr{I}_C^2)^\vee = \mathscr{O}_C(d)$ and $H^1(C, \mathscr{O}_C(d)) = H^0(C, K_C \otimes \mathscr{O}_C(-d))^\vee = H^0(C, \mathscr{O}_C(d-3-d)^\vee = 0$. Indeed $\mathbb{P}H^0(\mathbb{P}^2, (d))$ is smooth.

Example A.3.4. Let C be a smooth curve of genus 2, and fix an Abel–Jacobi map $C \hookrightarrow J = \operatorname{Jac} C$. Then the Hilbert scheme component $H \subset \mathrm{Hilb}(J)$ containing $[\mathscr{O}_J \twoheadrightarrow O_C]$ is smooth (and equals J). However, one can see that $T_2 = H^1(C, N_{C/J}) = k$.

Example A.3.5. Let $M = \mathrm{Hilb}^4(\mathbb{A}^3)$, and consider $\mathscr{I}_Z = (x,y,z)^2$ giving rise to $p = [\mathscr{O} \twoheadrightarrow \mathscr{O}_Z]$. Then $H^1(Z, N_{Z/\mathbb{A}^3}) = 0$ but

$$\dim T_p M = \dim H^0(Z, N_{Z/\mathbb{A}^3}) = 18 > 12,$$

so $p \in M$ is singular.

Remark A.3.6. Let Z be a closed subscheme of $X = \mathbb{P}^n$. If $H^1(Z, N_{Z/X}) = 0$ and Z is either

- nonsingular, or
- Cohen–Macaulay in codimension 2, or
- Gorenstein in codimension 3,

then $[\mathscr{O}_X \twoheadrightarrow \mathscr{O}_Z] \in \mathrm{Hilb}(X)$ is a smooth point.

Example A.3.7. Let C be a smooth curve of genus $g \geq 3$, and let $\iota\colon C \hookrightarrow J$ be an Abel–Jacobi embedding. Let $H_C \subset \operatorname{Hilb}(J)$ be the Hilbert scheme component containing $p = [\iota^\#\colon \mathscr{O}_J \twoheadrightarrow \mathscr{O}_C]$. Then

$$C \text{ is hyperelliptic} \iff \mathsf{Def}_p \text{ is not smooth} \iff H_C \text{ is not smooth at } p.$$

In fact, for $g = 2$ the Hilbert scheme H_C is smooth (and equals J), as well as Def_p. For C hyperelliptic of genus $g \geq 2$ one has

$$h^0(C, N_{C/J}) = 2g-2, \quad h^1(C, N_{C/J}) = g^2 - 2g + 1,$$

so if $g = 2$ we get an example where Def_p is smooth but $T_2 = H^1(C, N_{C/J}) = k$ does not vanish. See [**?**] for the computation of the scheme structure of H_C in the hyperelliptic case.

Appendix B. Intersection theory

B.1. **Chow groups, pushforward and pullack.** This appendix covers the basic material of [25]. The important construction of *refined Gysin homomorphisms* [25, § 6.2] is covered in Section B.3.

In this subsection, all schemes are of finite type over an algebraically closed field k. Varieties are integral schemes. A subvariety V of a scheme X is a closed subscheme which is a variety.

Let X be an n-dimensional scheme. A d-dimensional cycle on X (or simply a d-cycle) is a finite formal sum

$$\sum_i m_i \cdot V_i$$

where $V_i \subset X$ are (closed irreducible) subvarieties of dimension d and $m_i \in \mathbb{Z}$. The free abelian group generated by d-cycles is denoted $Z_d X$, and we set

$$Z_* X = \bigoplus_{d=0}^{n} Z_d X.$$

The fundamental class of X is the (possibly inhomogeneous) cycle

$$[X] \in Z_* X$$

determined by the irreducible components $V_i \subset X$ and their geometric multiplicities $m_i = \operatorname{length}_{\mathscr{O}_{X,\xi_i}} \mathscr{O}_{X,\xi_i}$, where ξ_i is the generic point of V_i. If X is pure, then $Z_n X = A_n X$ is freely generated by the classes of the irreducible components of X.

If $r \in k(X)$ is a nonzero rational function and $V \subset X$ is a codimension one subvariety, pick a and b in $A = \mathscr{O}_{X,\xi_V}$ such that $r = a/b$ and set

$$\operatorname{ord}_V(r) = \operatorname{length}_A(A/a) - \operatorname{length}_A(A/b).$$

This is the *order of vanishing* of r along V. Note that $\operatorname{ord}_V(r \cdot r') = \operatorname{ord}_V(r) + \operatorname{ord}_V(r')$ for r, $r' \in k(X)$. A rational function r as above defines a divisor

$$\operatorname{div}(r) = \sum_{\substack{V \subset X \\ \operatorname{codim}_X V = 1}} \operatorname{ord}_V(r) \cdot V \in Z_{n-1} X.$$

A d-cycle α is said to be *rationally equivalent to* 0 if it belongs to the subgroup $R_d X \subset Z_d X$ generated by cycles of the form $\operatorname{div}(r)$, where r is a nonzero rational function on a $(d+1)$-dimensional subvariety of X. Form the direct sum $R_*X = \oplus_{d=0}^n R_d X$. The quotient

$$A_*X = Z_*X/R_*X = \bigoplus_{d=0}^n A_d X$$

is the *Chow group* of X, where we have set $A_d X = Z_d X/R_d X$.

Let $f: X \to Y$ be a proper morphism of schemes. Then there is a *pushforward* map

$$f_*: A_*X \to A_*Y$$

defined on generators by sending a d-cycle class $[V] \in A_d X$ to 0 if $\dim f(V) < \dim V$, and to the cycle

$$e_V \cdot [f(V)] \in A_d Y$$

if $\dim V = \dim f(V)$. Here e_V is the degree of the field extension $k(f(V)) \subset k(V)$.

Let $f: X \to Y$ be a flat morphism of schemes. Then there is a *pullback* map

$$f^*: A_*Y \to A_*X$$

defined on generators by sending a d-cycle class $[W] \in A_*Y$ to the cycle class

$$[f^{-1}(W)] \in A_{d+s}X$$

where s is the relative dimension of f.

Theorem B.1.1. *Let $p: E \to X$ be a vector bundle. Then p^* is an isomorphism.*

Proof. Combine [25, Prop. 1.9] and [25, Thm. 3.3] with one another. □

Convention 1. Let $p: E \to X$ be a vector bundle. We denote by $0^*: A_*E \xrightarrow{\sim} A_*X$ the inverse of p^*.

Note that 0^* is much harder to describe than its inverse p^*.

Definition B.1.2. Let $\pi: X \to \operatorname{Spec} k$ be the structure morphism of a proper k-scheme X. The *degree map* is by definition the proper pushforward π_*. It is denoted

$$A_*X \xrightarrow{\int_X} \mathbb{Z}$$

and is zero on cycle classes of positive dimension.

B.2. **Operations on bundles.** Let E be a vector bundle of rank r on a scheme X, and let $p: \mathbb{P}(E) \to X$ be the projective bundle of lines in the fibres of $E \to X$. Let $\mathscr{O}_E(1)$ be the dual of the tautological line bundle $\mathscr{O}_E(-1) \subset p^*E$ on $\mathbb{P}(E)$. The *Segre classes* $s_i(E)$ can be seen as operators $A_k X \to A_{k-i} X$ defined by

$$s_i(E) \cap \alpha = p_*(\xi^{r-1+i} \cap p^*\alpha),$$

where $\xi = c_1(\mathscr{O}_E(1))$. Such operation is the identity for $i = 0$ and identically vanishes for $i < 0$. If L is a line bundle, then

$$s_p(E \otimes L) = \sum_{i=0}^p (-1)^{p-i} \binom{r-1+p}{r-1+i} s_i(E) c_1(L)^{p-i}.$$

Definition B.2.1. We define the following objects:

- The *Segre series* of E is the formal power series
$$s_t(E) = 1 + \sum_{i>0} s_i(E) t^i.$$
- The *Chern polynomial* of E is
$$c_t(E) = s_t(E)^{-1} = 1 + \sum_{i>0} c_i(E) t^i.$$
It is indeed a polynomial, for $c_i(E) = 0$ for all $i > \operatorname{rk} E$.
- The *total Chern class* of E is the finite sum
$$c(E) = 1 + c_1(E) + \cdots + c_r(E), \quad r = \operatorname{rk} E.$$

Example B.2.2. For a line bundle L, we have $c_t(L) = 1 + c_1(L)t$.

Let $E \to Y$ be a vector bundle. The *projection formula*
$$f_*(c_i(f^*E) \cap \alpha) = c_i(E) \cap f_*\alpha$$
holds for all proper morphisms $f: X \to Y$ and cycles $\alpha \in A_*X$. If f is a flat morphism, on the other hand, one has
$$c_i(f^*E) \cap f^*\beta = f^*(c_i(E) \cap \beta)$$
for all cycles $\beta \in A_*Y$. Given a short exact sequence
$$0 \to E \to F \to G \to 0 \tag{B.2.1}$$
of vector bundles on X, one has Whitney's formula
$$c_t(F) = c_t(E) \cdot c_t(G).$$
The *splitting construction* says that if E is a vector bundle of rank r on a scheme X, there exists a flat morphism $f: Y \to X$ such that the flat pullback $f^*: A_*X \to A_*Y$ is injective and the pullback f^*E has a filtration
$$0 = E_0 \subset E_1 \subset \cdots \subset E_{r-1} \subset E_r = f^*E$$
with line bundle quotients
$$L_i = E_i / E_{i-1}, \quad i = 1, \ldots, r.$$
Set $\alpha_i = c_1(L_i)$. Then each short exact sequence
$$0 \to E_{i-1} \to E_i \to L_i \to 0$$
gives an identity
$$(1 + \alpha_i t) \cdot c_t(E_{i-1}) = c_t(E_i).$$
So we have
$$\begin{aligned} f^*c_t(E) &= c_t(f^*E) \\ &= (1 + \alpha_r t) \cdot c_t(E_{r-1}) \\ &= (1 + \alpha_r t) \cdot (1 + \alpha_{r-1} t) \cdot c_t(E_{r-2}) \\ &= (1 + \alpha_1 t) \cdots (1 + \alpha_r t) \end{aligned}$$

By injectivity of f^*, we may view the latter product as a formal expression for $c_t(E)$. In other words, we can always pretend that E is filtered by $0=E_0\subset E_1\subset\cdots\subset E_{r-1}\subset E_r=E$ with line bundle quotients L_i, and

$$c_t(E)=\prod_{i=1}^{r}(1+\alpha_i t), \tag{B.2.2}$$

where $\alpha_i=c_1(L_i)$. In fact, one should regard (B.2.2) as a formal expression defining $\alpha_1,\ldots,\alpha_r$. These are called the *Chern roots* of E, and they satisfy

$$c_i(E)=\sigma_i(\alpha_1,\ldots,\alpha_r),\quad i=0,\ldots,r$$

where σ_i denotes the i-th symmetric function. For instance, if $\operatorname{rk} E=r$, one would have

$$\begin{aligned}c_1(E)&=\alpha_1+\cdots+\alpha_r\\ c_r(E)&=\alpha_1\cdots\alpha_r.\end{aligned}$$

Example B.2.3 (Dual bundles)**.** One has the formula

$$c_i(E^\vee)=(-1)^i c_i(E).$$

The Chern roots of the dual bundle $E^\vee$ are $-\alpha_1,\ldots,-\alpha_r$.

Example B.2.4 (Tensor products)**.** If F is a vector bundle of rank s, the Chern roots of $E\otimes F$ are $\alpha_i+\beta_j$, where $i=1,\ldots,r$ and $j=1,\ldots,s$. So $c_k(E\otimes F)$ is the k elementary symmetric function of $\alpha_1+\beta_1,\ldots,\alpha_r+\beta_s$. For instance, if $s=1$,

$$c_t(E\otimes L)=\sum_{i=0}^{r}c_t(L)^{r-i}c_i(E)t^i.$$

Term by term, this can be reformulated as

$$c_k(E\otimes L)=\sum_{i=0}^{k}\binom{r-i}{k-i}c_i(E)c_1(L)^{k-i}.$$

Example B.2.5 (Exterior product)**.** For the exterior power $\wedge^p E$ we have

$$c_t(\wedge^p E)=\prod_{i_1<\cdots<i_p}(1+(\alpha_{i_1}+\cdots+\alpha_{i_p})t),$$

so that for instance we have $c_1(\det E)=c_1(E)$.

Definition B.2.6. The *Chern character* of a vector bundle E with Chern roots $\alpha_1,\ldots,\alpha_r$ is the expression

$$\operatorname{ch}(E)=\sum_{i=1}^{r}\exp(\alpha_i).$$

One has

$$\operatorname{ch}(E)=r+c_1+\frac{1}{2}(c_1^2-2c_2)+\frac{1}{6}(c_1^3-3c_1c_2+3c_3)+\cdots$$

and moreover $\operatorname{ch}(-)$ satisfies the crucial relation

$$\operatorname{ch}(F)=\operatorname{ch}(E)+\operatorname{ch}(G)$$

for any short exact sequence as in (B.2.1). One also has

$$\operatorname{ch}(E\otimes E')=\operatorname{ch}(E)\cdot\operatorname{ch}(E').$$

Definition B.2.7. The *Todd class* of a line bundle L with $\eta = c_1(L)$ is the formal expression

$$\mathrm{Td}(L) = \frac{\eta}{1-e^{-\eta}} = 1 + \frac{1}{2}\eta + \sum_{i\geq 1} \frac{B_{2i}}{(2i)!}\eta^{2i}$$

where B_k are the Bernoulli numbers.

One may also set

$$\mathrm{Td}^{\vee}(L) = \mathrm{Td}(L^{\vee}) = \frac{-\eta}{1-e^{\eta}} = \frac{\eta}{e^{\eta}-1} = 1 - \frac{1}{2}\eta + \sum_{i\geq 1} \frac{B_{2i}}{(2i)!}\eta^{2i}.$$

Note that we have the relation

$$\mathrm{Td}(L) = e^{\eta}\cdot \mathrm{Td}(L^{\vee}).$$

For a vector bundle E with Chern roots $\alpha_1,\ldots,\alpha_r$, we set

$$\mathrm{Td}(E) = \prod_{i=1}^{r} \frac{\alpha_i}{1-e^{-\alpha_i}}$$

and it is easy to compute

$$\mathrm{Td}(E) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \frac{1}{720}(-c_1^4 + 4c_1^2c_2 + 3c_2^2 + c_1c_3 - c_4) + \cdots$$

where $c_i = c_i(E)$. The multiplicativity property

$$\mathrm{Td}(F) = \mathrm{Td}(E)\cdot \mathrm{Td}(G)$$

holds for every short exact sequence of vector bundles as in (B.2.1).

Definition B.2.8. For two vector bundles E, F on a scheme X, define

$$c(F-E) = \frac{c(F)}{c(E)} = 1 + c_1(F-E) + c_2(F-E) + \cdots \tag{B.2.3}$$

Example B.2.9. If $F = 0$, we have

$$c(-E) = s(E) = 1 + \sum_{i>0} s_i(E)t^i.$$

Example B.2.10. The first few terms of the expansion (B.2.3) are

$$\begin{aligned}
&c_0(F-E) = 1\\
&c_1(F-E) = c_1(F) - c_1(E)\\
&c_2(F-E) = c_2(F) - c_1(F)c_1(E) + c_1(E)^2 - c_2(E).
\end{aligned}$$

Remark B.2.11. If $\mathsf{F} = \sum_j [F_j]$ and $\mathsf{E} = \sum_i [E_i]$ are elements of the Grothendieck group $K^{\circ}X$ of vector bundles on a scheme X, then the Chern class of $\mathsf{F}-\mathsf{E}$ is defined as

$$c(\mathsf{F}-\mathsf{E}) = \frac{\prod_j c(F_j)}{\prod_i c(E_i)}.$$

Clearly, one has $c([F]-[E]) = c(F-E)$ for two vector bundles E, F. Similarly, the power series

$$c_t(\mathsf{F}-\mathsf{E}) = \frac{\prod_j c_t(F_j)}{\prod_i c_t(E_i)}$$

takes the role of the Chern polynomial in K-theory.

B.3. **Refined Gysin homomorphisms.** The material in this subsection covers the construction of [25, § 6.2].

Definition B.3.1. We say that a morphism of schemes $f\colon X \to Y$ admits a *factorisation* if there is a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, hook, "i"'] & & Y \\ & M \arrow[ur, "\pi"'] & \end{tikzcd} \tag{B.3.1}$$

where i is a closed embedding and π is smooth.

Example B.3.2. If X, Y are quasiprojective, any morphism $X \to Y$ admits a factorisation. A factorisation always exists locally on Y.

If $f\colon X \to Y$ admits two factorisations

$$X \xrightarrow{i_1} M_1 \xrightarrow{\pi_1} Y, \qquad X \xrightarrow{i_2} M_2 \xrightarrow{\pi_2} Y,$$

there is a third one,

$$X \xrightarrow{i} M \xrightarrow{\pi} Y,$$

dominating both:

$$\begin{tikzcd} & M_1 \arrow[dr, "\pi_1"] & \\ X \arrow[ur, "i_1"] \arrow[r, "i"] \arrow[dr, "i_2"'] & M \arrow[u, dashed, "\exists"'] \arrow[d, dashed, "\exists"] \arrow[r, "\pi"] & Y \\ & M_2 \arrow[ur, "\pi_2"'] & \end{tikzcd}$$

It is enough to take $M = M_1 \times_Y M_2$ and use that smooth morphisms and closed immersions are stable under base change and composition.

Definition B.3.3. A morphism $f\colon X \to Y$ is a *local complete intersection* (lci, for short) if it has a factorisation $X \hookrightarrow M \to Y$ where $X \hookrightarrow M$ is a regular closed embedding. If this is only true locally on Y, we say that f is *locally lci.*

Remark B.3.4. If $f\colon X \to Y$ is lci, *all* of its factorisations $X \hookrightarrow M \to Y$ satisfy the property that $X \hookrightarrow M$ is regular.

Remark B.3.5. An lci morphism $f\colon X \to Y$ has a well-defined *relative dimension*: given a factorisation (B.3.1), it is the integer

$$r = \operatorname{rk} T_{M/Y} - \operatorname{codim}(X, M) \in \mathbb{Z}.$$

For instance, a regular closed embedding of codimension d is an lci morphism of relative dimension $-d$.

Let $f\colon X \to Y$ be an lci morphism of relative dimension r, factoring as a regular immersion $i\colon X \to M$ followed by a smooth morphism $\pi\colon M \to Y$ of relative dimension

s. Given any morphism $g\colon \widetilde{Y} \to Y$, consider the double fiber square

$$\begin{tikzcd} \widetilde{X} \arrow[r, hook] \arrow[d, "\tilde{g}"'] \arrow[dr, phantom, "\square"] & \widetilde{M} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \widetilde{Y} \arrow[d, "g"] \\ X \arrow[r, hook, "i"] & M \arrow[r, "\pi"] & Y. \end{tikzcd}$$

For any $k \geq 0$, we will construct a group homomorphism

$$f^!\colon Z_k\widetilde{Y} \longrightarrow Z_{k+s}\widetilde{M} \xrightarrow{\sigma} Z_{k+s}C_{\widetilde{X}/\widetilde{M}} \xrightarrow{\phi} A_{k+r}\widetilde{X}.$$

The first arrow is just flat pullback: it is not a problem to pull back cycles from $\widetilde{Y}$ to $\widetilde{M}$. The arrow σ, called *specialisation to the normal cone* in [25, § 5.2], is given as follows. For any $(k+s)$-dimensional subvariety $V \subset \widetilde{M}$, consider the intersection

$$\begin{tikzcd} W \arrow[r] \arrow[d, "j"'] \arrow[dr, phantom, "\square"] & V \arrow[d] \\ \widetilde{X} \arrow[r] & \widetilde{M} \end{tikzcd}$$

and the normal cone $C_{W/V}$, which is purely of dimension $k+s$ (cf. Remark **??**). It lives naturally as a closed subcone

(B.3.2) $$C_{W/V} \subset j^*C_{\widetilde{X}/\widetilde{M}} \subset C_{\widetilde{X}/\widetilde{M}}.$$

If ℓ denotes the closed immersion (B.3.2), we define

$$\sigma[V] = \ell_*[C_{W/V}] = [C_{V\cap\widetilde{X}/V}],$$

the proper pushforward of the fundamental class $[C_{W/V}]$ of the normal cone.

As for the map ϕ, we observe from the Cartesian diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d, "\tilde{g}"'] \arrow[dr, phantom, "\square"] & \widetilde{M} \arrow[d] \\ X \arrow[r, "i"] & M \end{tikzcd}$$

that we have a closed subcone

(B.3.3) $$C_{\widetilde{X}/\widetilde{M}} \subset \tilde{g}^*C_{X/M}.$$

Since i is *regular*, $C_{X/M} = N_{X/M}$ is a vector bundle, so that

$$E = \tilde{g}^*C_{X/M}$$

is a vector bundle too. Its rank is easily computed as

$$\operatorname{rk} E = \operatorname{rk} N_{X/M} = \operatorname{codim}(X, M) = s - r.$$

Let B be a subvariety of $C_{\widetilde{X}/\widetilde{M}} \subset E$ of dimension $k+s$. Then define

$$\phi[B] = 0^*[B],$$

where

$$0^*\colon A_{k+s}E \to A_{k+s-(s-r)}\widetilde{X} = A_{k+r}\widetilde{X}$$

is the inverse of the flat pullback on $E \to \widetilde{X}$ (cf. Convention 1).

The morphism we have just constructed descends to rational equivalence, to give a morphism

$$f^!\colon A_k\widetilde{Y} \to A_{k+r}\widetilde{X} \tag{B.3.4}$$

that Fulton calls *refined Gysin homomorphism.*

We have the following facts:

(a) The homomorphism $f^!$ agress with Gysin pullback (flat pullback) when $\widetilde{Y} = Y \to Y$ is the identity and f is flat. This case is already interesting in its own: it is the intersection theory of X!
(b) The homomorphism $f^!$ is called *refined Gysin pullback* when $\widetilde{Y} \to Y$ is a closed embedding.
(c) The homomorphism $f^!$ does not depend on the choice of the factorisation.
(d) Working in $\mathrm{D}^b(X)$, one can perform the above construction even when *no factorisation* is actually available. (This is the generality one works in to construct virtual classes.)
(e) Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two morphisms. If f is a regular embedding and both g and $g\circ f$ are flat, then

$$f^!\circ g^* = (g\circ f)^*.$$

Moreover, if f and $g\circ f$ are regular embeddings, and g is *smooth*, then

$$f^!\circ g^* = (g\circ f)^!. \tag{B.3.5}$$

This is basically [25, Prop. 6.5]. However, (B.3.5) is false in general if g is just flat. See the functoriality property (D) in the next subsection for the general (lci) case.

B.3.1. *An example: Localized Top Chern Class.* Let X be a variety and let

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "i"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "s"] \\ X \arrow[r, "0"] & E \end{tikzcd} \tag{B.3.6}$$

be the fiber diagram defining the zero locus Z of a section $s \in H^0(X,E)$. Then $0\colon X \to E$ is regular of codimension $e = \operatorname{rk} E$, and $N_{X/E} = E$. We get refined Gysin homomorphisms

$$0^!\colon A_kX \to A_{k-e}Z.$$

Suppose X is purely n-dimensional. Then we can define the *localized top Chern class* of E as

$$\mathbf{Z}(s) = 0^![X] \in A_{n-e}Z,$$

where $[X]$ is the fundamental class of X and $0^!\colon A_nX \to A_{n-e}Z$. In the language of the previous section, the class $0^![X]$ is nothing but the intersection

$$0^*_{E|_Z}[C_{Z/X}]$$

of the cone

$$C_{Z/X} \subset N_{Z/X} \subset E|_Z$$

with the zero section of the rank e bundle $E|_Z \to Z$. The closed embedding

$$N_{Z/X} \subset E|_Z \tag{B.3.7}$$

comes directly from the diagram (B.3.6): the dual section $s^\vee\colon E^\vee \to \mathscr{O}_X$ hits the ideal sheaf $\mathscr{I} \subset \mathscr{O}_X$ of $Z \subset X$, and (B.3.7) is the result of applying Spec Sym to the natural restriction map

$$s^\vee|_Z\colon E^\vee|_Z \twoheadrightarrow \mathscr{I}/\mathscr{I}^2.$$

Remark B.3.6 (First example of perfect obstruction theory)**.** Let X, E, Z be as above. The class $0^![X] = 0^*_{E|_Z}[C_{Z/X}] \in A_{n-e}Z$ is our first example of a *virtual fundamental class*. For the time being, we can pretend the normal cone $C_{Z/X}$ is completely *intrinsic* to Z. This is not entirely false. Now look at (B.3.7): embedding $C_{Z/X}$ in a vector bundle $E|_Z$ is the choice of what is called a *perfect obstruction theory* [8].

Remark B.3.7 (Relation with the deformation to the normal cone)**.** The deformation to the normal cone [25, Chapter 5] enters the picture as follows: we have embeddings $\lambda s\colon Z \to E$ for all $\lambda \in \mathbb{A}^1$. Letting $\lambda \to \infty$ turns these embeddings into exactly $C_{Z/X} \subset E|_Z$. More explicitly, consider the graph of $\lambda s\colon X \to E$ as a line in $E \oplus \mathbf{1}$, to get an embedding

$$X \times \mathbb{A}^1 \hookrightarrow P(E \oplus \mathbf{1}) \times \mathbb{P}^1, \quad (x, \lambda) \mapsto ((x, \lambda s(x)), (\lambda : 1)).$$

Then the deformation space of the deformation to the normal cone construction turns out to be the closure

$$M = \overline{X \times \mathbb{A}^1} \subset P(E \oplus \mathbf{1}) \times \mathbb{P}^1,$$

and the embeddings $\lambda s\colon X \subset E$ deform to $X \subset C_{X/E} \subset N_{X/E} = E$. Restricting to Z gives $C_{Z/X} \subset E|_Z$.

Remark B.3.8. The localised top Chern class $\mathbf{Z}(s)$ is also called the *refined Euler class* of E, because

$$\begin{aligned} i_*\mathbf{Z}(s) &= i_* 0^![X] \\ &= 0^* s_*[X] \\ &= s^* s_*[X] \\ &= c_e(E) \cap [X]. \end{aligned}$$

We have used that if $\pi\colon E \to X$ is any vector bundle then any $s \in H^0(X, E)$ is a regular embedding and $s^! = s^*\colon A_k E \to A_{k-e} X$ is the inverse of flat pullback π^*. In particular s^* does not depend on s, so $s^* = 0^*$. The last equality is a special case of the self-intersection formula, using also that $E = N_{X/E}$. Moreover, it can be interesting to notice that $\mathbf{Z}(s) = [Z]$ when s is a regular section.

Example B.3.9. This example is relevant in Donaldson–Thomas theory. Let $E = \Omega_U$ be the cotangent bundle on a smooth scheme U. Let $f\colon U \to \mathbb{C}$ be a holomorphic function, giving a section $\mathrm{d}f$ of Ω_U. The above construction can be summarized by

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "i"'] \arrow[dr, phantom, "\square"] & U \arrow[d, "\mathrm{d}f"] \\ U \arrow[r, "0"] & \Omega_U \end{tikzcd} \qquad \rightsquigarrow \qquad [Z]^{\mathrm{vir}} = 0^![U] = 0^*_{\Omega_U|_Z}[C_{Z/U}] \in A_0 Z.$$

Notice that in this case the obstruction *sheaf* is completely intrinsic to Z. It is defined as the cokernel

$$N_{Z/U} \to i^* N_{U/\Omega_U} \to \mathrm{Ob} \to 0.$$

But this is of course the sequence

$$\mathscr{I}/\mathscr{I}^2 \to \Omega^1_U|_Z \to \Omega_Z \to 0.$$

In other words, $\mathrm{Ob} = \Omega^1_Z$.

B.3.2. *More properties of $f^!$ and relation with bivariant classes.* We now quickly discuss the main properties of refined Gysin homomorpisms. We refer to [25, Ch. 6] for complete proofs.

First of all, notice that we have a $f^!$ for all $\widetilde{Y} \to Y$ and for all $k \geq 0$. This trivial observation, together with the compatibilities we are about to describe, states precisely that any lci morphism $f : X \to Y$ of relative dimension r defines a *bivariant class*

$$[f^!] \in A^{-r}(X \xrightarrow{f} Y),$$

as described in [25, Ch. 17].

Let $f : X \to Y$ be an lci morphism of codimension r. We first state the properties of $f^!$ informally, and then we explain what they mean.

(A) Refined Gysin homomorphisms commute with proper pushforward and flat pullback.
(B) Refined Gysin homomorphisms are compatible with each other.
(C) Refined Gysin homomorphisms commute with each other.
(D) Refined Gysin homomorphisms are functorial.

Here is what the above statements mean. Fix once and for all an integer $k \geq 0$.

(A) For any double fiber square situation

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\square"] & Y' \arrow[d, "h"] \\ \widetilde{X} \arrow[r, "\tilde{f}"] \arrow[d] \arrow[dr, phantom, "\square"] & \widetilde{Y} \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{B.3.8}$$

one has the following:

(P) If h is proper, then for all $\alpha \in A_k Y'$ one has

$$f^!(h_*\alpha) = q_*(f^!\alpha) \in A_{k+r}\widetilde{X}.$$

(F) If h is flat of relative dimension n, then for all $\alpha \in A_k \widetilde{Y}$ one has

$$f^!(h^*\alpha) = q^*(f^!\alpha) \in A_{k+r+n} X'.$$

(B) In situation (B.3.8), if $\tilde{f} : \widetilde{X} \to \widetilde{Y}$ is also lci of relative dimension r, then for all $\alpha \in A_k Y'$ one has

$$f^!\alpha = \tilde{f}^!\alpha \in A_{k+r} X'.$$

(C) Let $j : S \to T$ be a regular embedding of codimension e. Given morphisms $\widetilde{Y} \to Y$ and $\widetilde{Y} \to T$, form the fiber square

(B.3.9)
$$\begin{tikzcd}
X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S \arrow[d, "j"] \\
\widetilde{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \widetilde{Y} \arrow[r] \arrow[d] & T \\
X \arrow[r, "f"] & Y &
\end{tikzcd}$$

and fix $\alpha \in A_k \widetilde{Y}$. Then one has

$$j^!(f^!\alpha) = f^!(j^!\alpha) \in A_{k+r-e} X'.$$

(D) Let $f : X \to Y$ and $g : Y \to Z$ be lci morphisms of relative dimensions r and s respectively. Then, for all morphisms $\tilde{Z} \to Z$, one has the identity

$$(g \circ f)^!(\alpha) = f^!(g^!\alpha) \in A_{k+r+s}(X \times_Z \tilde{Z}).$$

B.3.3. *Bivariant classes.* Let $f : X \to Y$ be any morphism. Suppose f has the property that when we let morphisms $g : \widetilde{Y} \to Y$ and integers $k \geq 0$ vary arbitrarily, we are able to construct homomorphisms

$$c_g^{(k)} : A_k \widetilde{Y} \to A_{k-p} \widetilde{X}, \quad \widetilde{X} = X \times_Y \widetilde{Y},$$

for some $p \in \mathbb{Z}$. Then the collection c of these homomorphisms is said to define a *bivariant class*

$$c \in A^p(X \xrightarrow{f} Y)$$

if compatibilities like the ones described in (A) and (C) in the previous section are satisfied. Here are precise requirements.

(A)′ In any double fiber square situation like (B.3.8), one has the following:

(P) If h is proper, then for all $\alpha \in A_k Y'$ one has the identity

$$c_g^{(k)}(h_*\alpha) = q_*(c_{gh}^{(k)}\alpha) \in A_{k-p} \widetilde{X}.$$

(F) If h is flat of relative dimension n, then for all $\alpha \in A_k \widetilde{Y}$ one has the identity

$$c_{gh}^{k+n}(h^*\alpha) = q^*(c_g^{(k)}\alpha) \in A_{k+n-p} X'.$$

(C)′ In situation (B.3.9), for all $\alpha \in A_k \widetilde{Y}$ one has

$$j^!(c_g^{(k)}\alpha) = c_{gi}^{(k-e)}(j^!\alpha) \in A_{k-p-e} X'.$$

Conclusion. Any lci morphism $f\colon X \to Y$ of codimension r defines a bivariant class

$$[f^!] \in A^{-r}(X \xrightarrow{f} Y).$$

For instance, if $f = i$ is a regular immersion of codimension d, this class is

$$[i^!] \in A^{d}(X \xrightarrow{f} Y).$$

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