



More on Linear Independence

Understanding when vectors are team players (dependent) or solo stars (independent)

Source: [More on linear independence \(video\)](#) | [Khan Academy](#)

🔥 What Does "Linearly Dependent" Mean? — Formal Definition

Linear Dependence

$$S = \{v_1, v_2, \dots, v_n\}$$

Linearly Dependent iff \iff

$$v_1 = a_2 v_2 + a_3 v_3 + \dots + a_n v_n$$
$$0 = \underbrace{(-1)}_{\text{non zero}} v_1 + a_2 v_2 + a_3 v_3 + \dots + a_n v_n$$

for some c_i 's
not all are zero
at least 1 is non-zero

Assume $c_1 \neq 0$

$$v_1 + \frac{c_2}{c_1} v_2 + \dots + \frac{c_n}{c_1} v_n = 0$$
$$-\frac{c_2}{c_1} v_2 + \dots - \frac{c_n}{c_1} v_n = +v_1$$

Let's define a set of vectors:

$$S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$$

This set is **linearly dependent** if and only if:

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n = \vec{0}$$

...for some scalars $c_1, c_2, \dots, c_n \in \mathbb{R}$ where **not all** $c_i = 0$!

- The condition $\vec{0}$ represents the **zero vector**, which is a vector where every component is 0, like $(0, 0)$, $(0, 0, 0)$, etc.
- If even **one scalar is non-zero**, 🎯 the set is linearly dependent.

☁ Think of this like cooking: If you can mix some ingredients (vectors) in just the right non-zero proportions to end up with something bland and flavorless (the zero vector), then the ingredients weren't unique — one or more could be made from the rest!

Equivalent Definition: Redundant Vector View

An alternate, equivalent definition:

A set of vectors is linearly dependent if at least one vector in the set can be written as a **linear combination** of the others.

Mathematically:

$$\vec{v}_1 = a_2 \vec{v}_2 + a_3 \vec{v}_3 + \cdots + a_n \vec{v}_n$$

Proof of Equivalence

Subtract \vec{v}_1 from both sides:

$$\vec{0} = -\vec{v}_1 + a_2 \vec{v}_2 + \cdots + a_n \vec{v}_n$$

We now have a linear combination that

✓ equals $\vec{0}$, and

✓ not all scalars are zero (-1)

→ matches the original definition ✓

Conversely (another way to prove this equivalent definition):

Suppose we have a linear dependence, then it is:

$$c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n = \vec{0}$$

💡 In a linearly dependent set, there must exist **at least one non-zero scalar** in the linear combination that yields the zero vector. For the purpose of analysis or proof, **it is not essential to identify exactly which scalar is non-zero**.

Therefore, any one may be chosen arbitrarily (e.g., $c_1 \neq 0$) to **demonstrate that a particular vector can be expressed as a linear combination** of the others.

Because it's a linear dependent, it has at least one non-zero constant → assume $c_1 \neq 0$.

Divide through by c_1 , rearrange, and boom 💥:

$$\vec{v}_1 = -\frac{c_2}{c_1} \vec{v}_2 - \cdots - \frac{c_n}{c_1} \vec{v}_n$$

Again, \vec{v}_1 is a combo of the others → equivalent definition confirmed ✓



Example 1

$\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$ linearly independent $\Rightarrow \text{span}(\cdot) = \mathbb{R}^2$

$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$2c_1 + 3c_2 = 0$
 $c_1 + 2c_2 = 0$
 $c_1 + \frac{3}{2}c_2 = 0$

$\frac{1}{2}c_2 = 0$
 $c_2 = 0$
 $c_1 = 0$

$c_1 \text{ or } c_2 \text{ nonzero} \Rightarrow \text{dependent}$
 $\underline{c_1 + c_2 \text{ both zero}} \Rightarrow \text{independent}$

Are $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ linearly dependent? 🤔

Let's solve:

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This leads to the system:

$$\begin{cases} 2c_1 + 3c_2 = 0 \\ 1c_1 + 2c_2 = 0 \end{cases}$$

Solve via elimination or substitution 🧮:

Multiply second equation by 2:

$$2c_1 + 4c_2 = 0$$

Subtract from first:

$$(2c_1 + 3c_2) - (2c_1 + 4c_2) = -c_2 = 0 \Rightarrow c_2 = 0 \Rightarrow c_1 = 0$$

🔴 Both coefficients must be 0 \Rightarrow **Vectors are linearly independent!**

✓ Neither vector can be made from the other.

✓ These vectors span \mathbb{R}^2 (the full 2D plane).



Example 2

$$\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \leftarrow \text{linearly dependent}$$

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-4 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 2 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} 2c_1 + 3c_2 + c_3 &= 0 \\ c_1 + 2c_2 + 2c_3 &= 0 \end{aligned}$$

$$\begin{aligned} 2c_1 + 3c_2 - 1 &= 0 \\ 2c_1 + 4c_2 - 2 &= 0 \end{aligned}$$

$$\begin{aligned} -c_2 + 3 &= 0 \\ -c_2 &= -3 \\ c_2 &= 3 \end{aligned}$$

$$\begin{aligned} c_3 &= -1 \\ c_1 &= -4 \end{aligned}$$

$$c_1 + 6 - 2 = 0$$

$$c_1 + 4 = 0$$

$$c_1 = -4$$

linearly dependent

Are $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ linearly dependent? 🤔

Now test:

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}$$

Leads to the system:

$$\begin{cases} 2c_1 + 3c_2 + 1c_3 = 0 \\ 1c_1 + 2c_2 + 2c_3 = 0 \end{cases}$$

💡 This system has **more unknowns than equations** (3 variables, 2 equations), making it underdetermined. Hence, it **admits infinitely many solutions**. Infinite solutions mean we

can assign an arbitrary value to any one variable (e.g., $c_3 = -1$) to **explore whether at least one non-trivial solution exists** among them.

🔍 Observation: 3 vectors in \mathbb{R}^2 ? Always dependent! It's overkill.
Why? Because in a 2D space, **only 2 vectors max can be linearly independent**. Third one's always a combo 🌟

Still, let's verify:

Pick $c_3 = -1$

Substitute into the equations:

$$2c_1 + 3c_2 - 1 = 0 \Rightarrow 2c_1 + 3c_2 = 1$$

$$c_1 + 2c_2 - 2 = 0 \Rightarrow c_1 + 2c_2 = 2$$

Now solve:

Multiply second equation by 2:

$$2c_1 + 4c_2 = 4$$

Subtract first:

$$(2c_1 + 4c_2) - (2c_1 + 3c_2) = c_2 = 3 \Rightarrow c_1 = -4$$

So:




$$c_1 = -4, \quad c_2 = 3, \quad c_3 = -1$$

✅ Not all zero → **Linearly dependent**.

$$\rightarrow \text{Extra proof: } -4 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + (-1) \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -8 \\ -4 \end{bmatrix} + \begin{bmatrix} 9 \\ 6 \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0} \quad \checkmark$$

💡 We found a linear combo that makes zero! In fact, all 3 coefficients are non-zero 🔥

✨ Key Takeaways

- A set of vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ is **linearly dependent** if:
$$c_1\vec{v}_1 + \dots + c_n\vec{v}_n = \vec{0} \quad \text{for some } c_i \neq 0$$
- Equivalent view: One vector is a **linear combo of others** 
- The **zero vector** can always be made by the zero-scalar combo (trivial solution) — not enough to claim dependence.
- If **only solution is all-zero** \rightarrow linearly **independent** ✓
- More vectors than dimensions (e.g. 3 vectors in \mathbb{R}^2) \rightarrow  Guaranteed dependence
- **Independent vectors = no redundancy.** Each one adds new direction 
- **Dependent vectors = redundancy.** Some directions are recycled 