Linear Subspaces

Source: Linear subspaces (video) | Khan Academy

📌 Introduction to Subspaces

What is $\mathbb{R}^n$? 🧱

The symbol $\mathbb{R}^n$ represents the set of all n**-dimensional vectors** where each entry is a real number.

Formally:

$$\mathbb{R}^n = \left\{ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \mid x_i \in \mathbb{R}, \forall i \in \{1, 2, \ldots, n\} \right\}$$

It's like an **infinitely massive cloud** ☁️ of vectors, each with exactly n components (coordinates).

Vectors as Elements of $\mathbb{R}^n$ 🧭

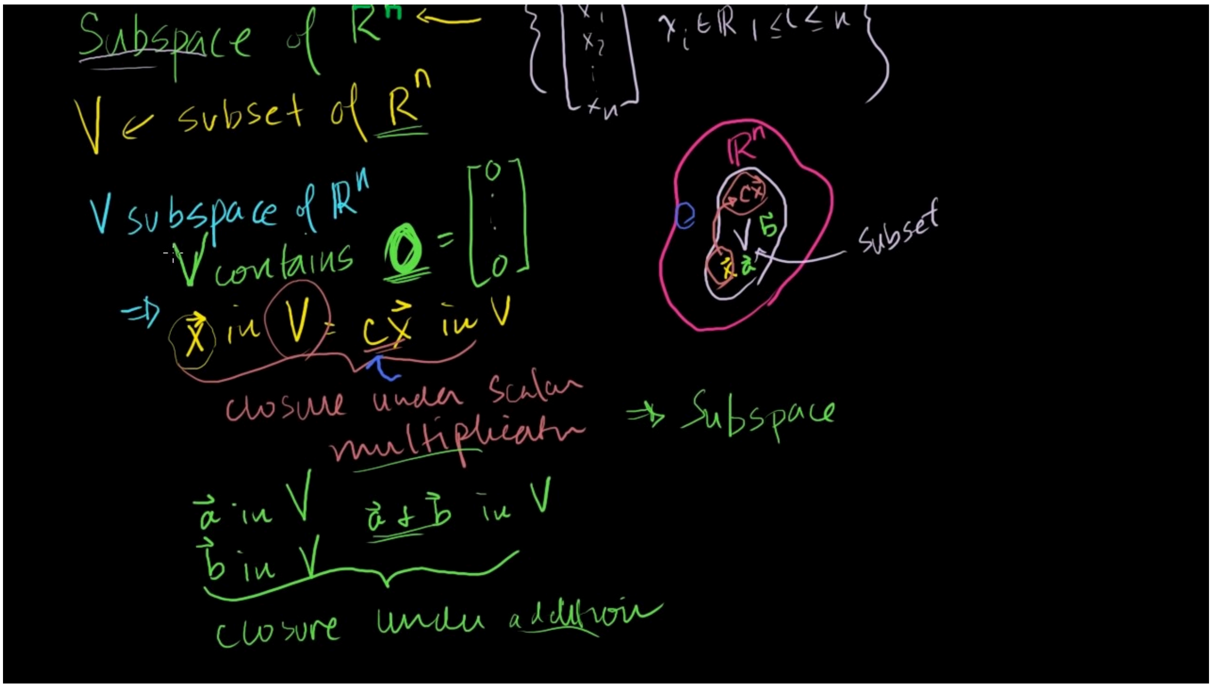
- Think of vectors as **arrows** ✏️ pointing from the origin in an n-dimensional space.
- Each vector is a **position**, a **direction**, or a **step**—depending on the context.
- For example:

- In $\mathbb{R}^2$: vectors are arrows on a flat plane 📄
- In $\mathbb{R}^3$: they float through 3D space like shooting stars 🌌💫

Visual vs Abstract 🌀

- **Visually**: We draw vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ as arrows.
- **Abstractly**: A vector is just an **ordered tuple** of real numbers.
- **Zoom out**: $\mathbb{R}^n$ is a space filled with vectors—a "vector universe"—in which patterns like lines, planes, and hyperplanes live.

🧾 Definition of a Subspace



Let $V \subseteq \mathbb{R}^n$ be a set of vectors.

We say that V is a **linear subspace** of $\mathbb{R}^n$ if it satisfies **three crucial properties**:

1. Contains the Zero Vector 🫧

$\vec{0} \in V$

The zero vector is:

$$\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n$$

> Analogy: Every party 🎉 needs the host! In vector space, the zero vector is that host—it must always be there in a subspace.

2. Closed Under Scalar Multiplication 🔁

If $\vec{x} \in V$ and $c \in \mathbb{R}$, then:

$c\vec{x} \in V$

> Meaning: Multiply any vector in V by any real number, and you should stay inside V.
>
> ⚠️ If multiplying takes you out of V, it's **not a subspace**!

3. Closed Under Addition ➕➕

If $\vec{a}, \vec{b} \in V$, then:

$\vec{a} + \vec{b} \in V$

> Meaning: Adding two vectors in the set must give you another vector still inside the set.
>
> If their sum escapes the set like a runaway balloon 🎈—nope, not a subspace.

🧠 Intuition Behind the Properties

What Does “Closure” Mean?

Closure is like a **forcefield** 🛡️—you can't leave the set no matter what you do inside it.

🔄 Closure under Scalar Multiplication

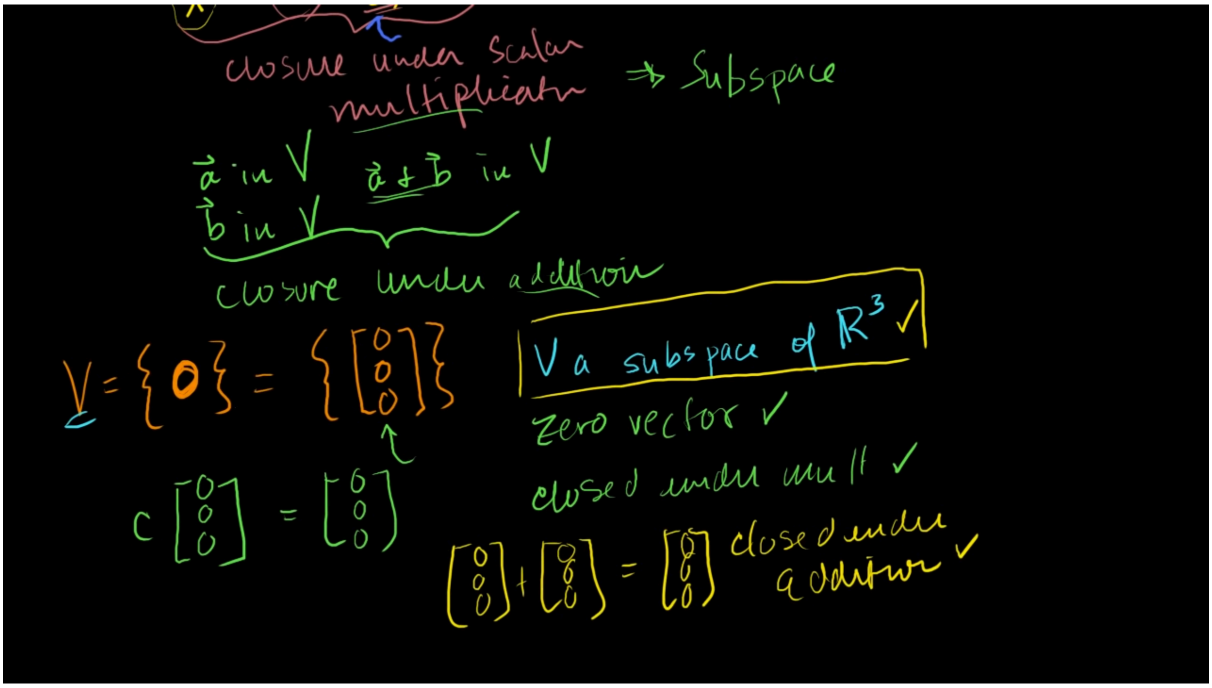
- Multiply a vector by any real number (stretch, flip, shrink it), and the result **must remain in the set**.

➕ Closure under Addition

- Add two vectors (like two moves in space), and you must **land back in the same set**.

If the set is **not closed**, then it’s like a holey box 🕳️—you can fall out! Not allowed in subspace club. 🚫

🧪 Example 1: Trivial Subspace



The Set

$$V = \left\{\vec{0}\right\} \subseteq \mathbb{R}^3$$

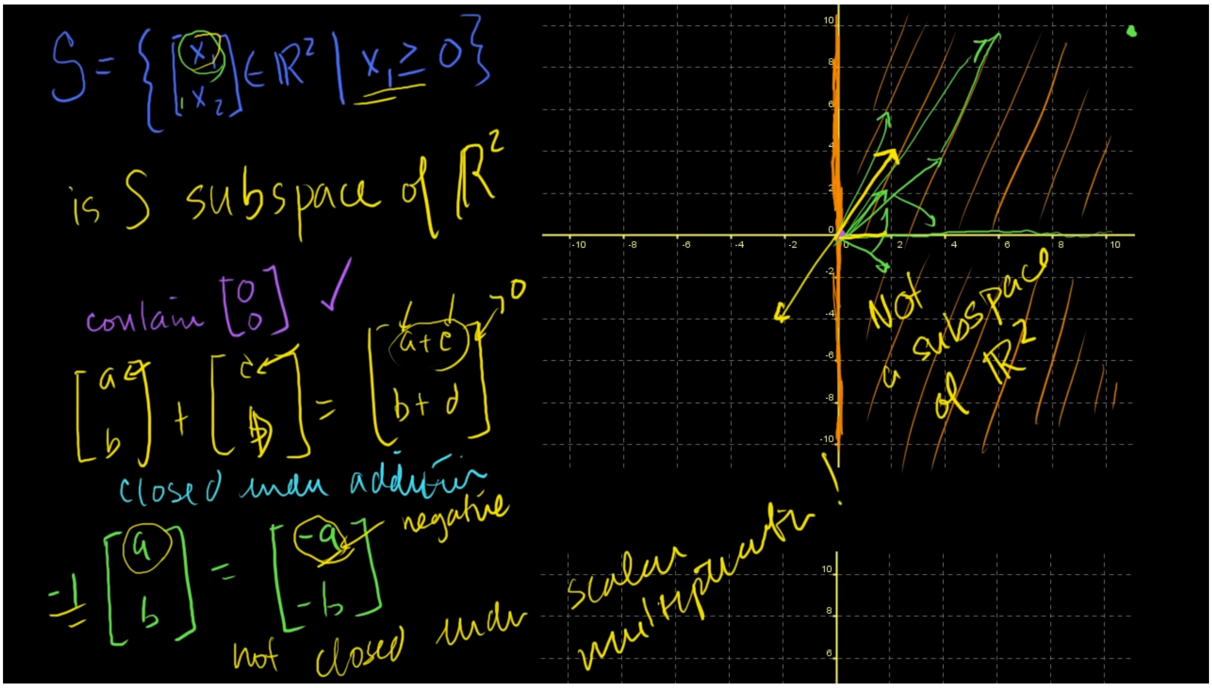
Only one element—the zero vector! So simple, it's kinda cute 🐣.

Check the Subspace Properties

- ✅ **Zero vector included**: It's the only member!
- ✅ **Closed under scalar multiplication**:
  - $c \cdot \vec{0} = \vec{0}$ for any $c \in \mathbb{R}$
- ✅ **Closed under addition**:
  - $\vec{0} + \vec{0} = \vec{0}$

> Even though it's "trivial," it's a valid subspace—just a super boring one. It's the vector-space version of silent mode 😶.

🚫 Example 2: Set That Is Not a Subspace



The Set

$S = \{\vec{x} \in \mathbb{R}^2 \mid x_1 \geq 0\}$

This includes **all vectors** in the plane whose **first component** is non-negative.

Visual 📉

- Includes the **right half** of the 2D plane (first and fourth quadrans).
- Includes the **y-axis** (where $x_1 = 0$)
- Excludes all vectors pointing left (where $x_1 < 0$)

Check the Subspace Properties

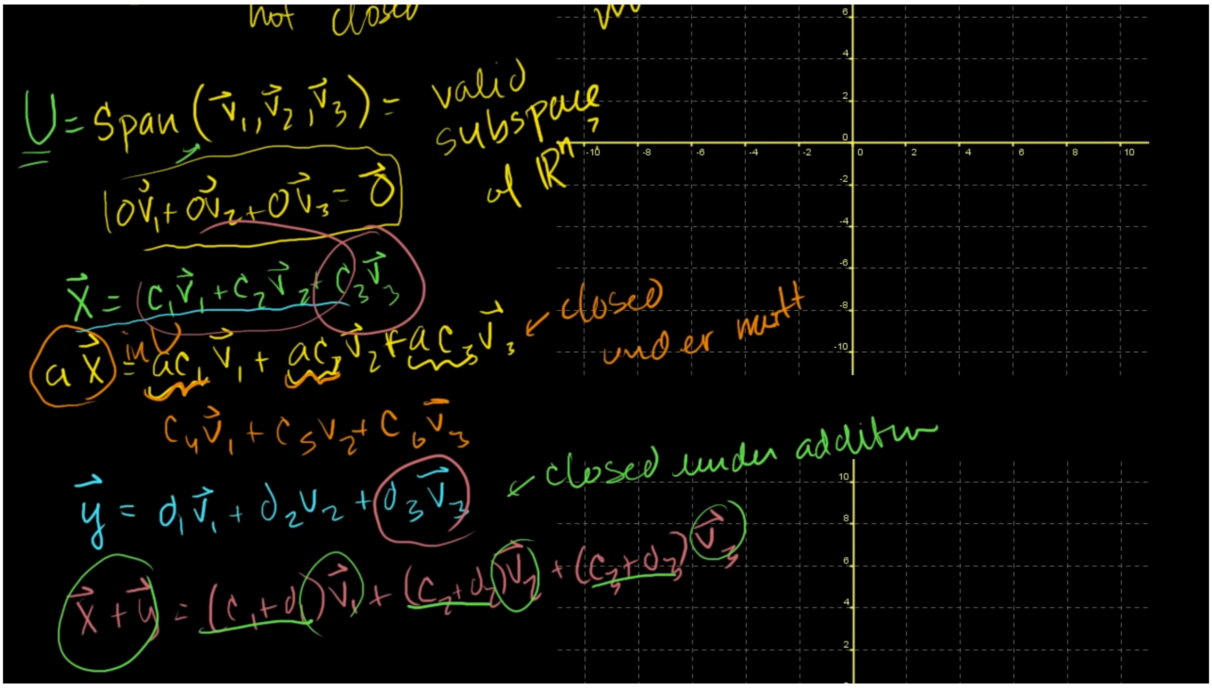
- ✅ **Zero vector included**:

$$\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in S$$

- ✅ **Closed under addition**:
  - Two vectors with $x_1 \geq 0$ added together still have $x_1 \geq 0$
- ❌ **Not closed under scalar multiplication**:
  - Pick $\vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \in S$
  - Multiply by -1: $\begin{bmatrix} -1 \\ -3 \end{bmatrix} \notin S$
  - $x_1 = -1 < 0 \Rightarrow$ **falls outside the set**

> So this is not a subspace. It's a leaky space—you can fly out just by flipping a vector. Subspace rules are strict, no loopholes! 📜🟥

🌈 Example 3: Span of Vectors as a Subspace



What's a Span? 🌌

The **span** of a set of vectors is the collection of **all linear combinations** of those vectors.

Let's say you've got:

$\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^n$

Then their span is:

$\text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \{c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 \mid c_1, c_2, c_3 \in \mathbb{R}\}$

It's like:

> Building a city out of building blocks. Each scalar c_i is a dial you turn to get different structures built from the same base pieces 🧱🏙️.

Prove: The Span Is a Subspace ✔️

Let $U = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$

✅ Contains the Zero Vector

Just set all coefficients to zero:

$$0 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + 0 \cdot \vec{v}_3 = \vec{0} \Rightarrow \vec{0} \in U$$

Boom. Zero vector's in.

🔁 Closed Under Scalar Multiplication

Take $\vec{x} \in U$, meaning:

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$$

Multiply by any scalar $a \in \mathbb{R}$:

$$a\vec{x} = a(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) = (ac_1)\vec{v}_1 + (ac_2)\vec{v}_2 + (ac_3)\vec{v}_3 \Rightarrow c_4\vec{v}_1 + c_5\vec{v}_2 + c_6\vec{v}_3$$

That's still a linear combination! Still in the span! ✅

➕ Closed Under Addition

Take two vectors in the span:

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 \quad \text{and} \quad \vec{y} = d_1\vec{v}_1 + d_2\vec{v}_2 + d_3\vec{v}_3$$

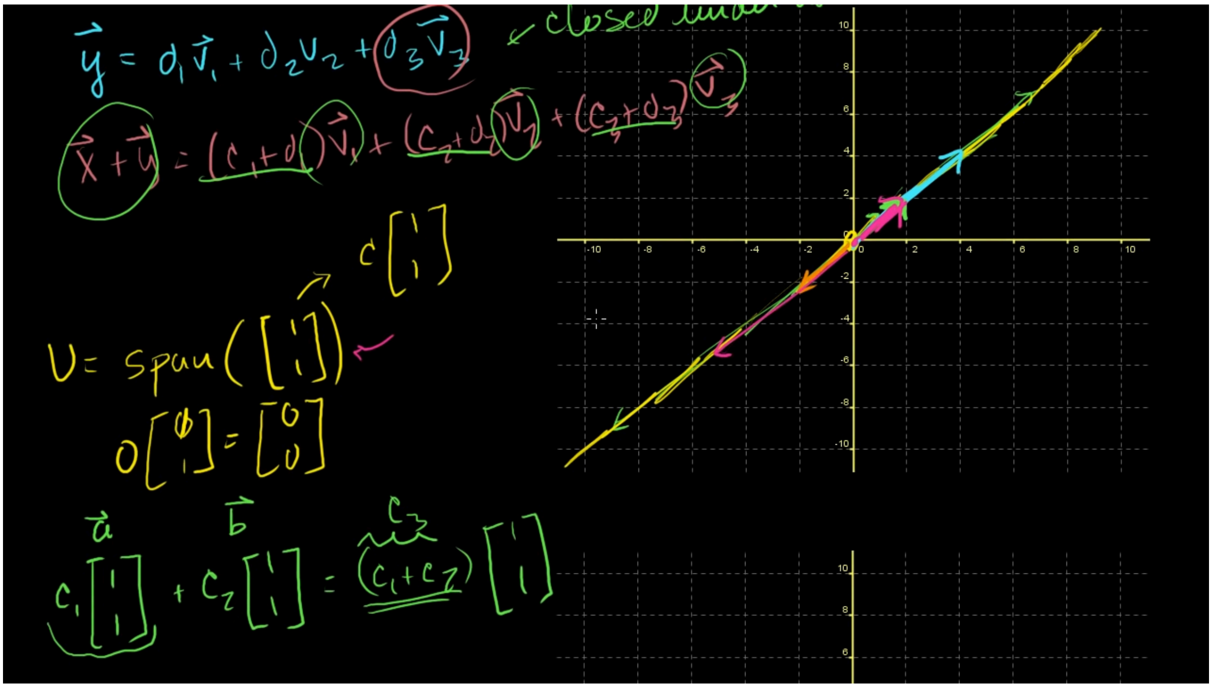
Add them:

$$\vec{x} + \vec{y} = (c_1 + d_1)\vec{v}_1 + (c_2 + d_2)\vec{v}_2 + (c_3 + d_3)\vec{v}_3$$

Again: linear combination → still in the span. ✅

> No matter how you scale or add, you can't escape the span. It's a mathematical gravity well 🔄🧲.

🎯 Special Case: Span of One Vector



Let's look at:

$$U = \text{span}(\vec{v}) = \{c \cdot \vec{v} \mid c \in \mathbb{R}\}$$

Say $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Visual 📏

- This creates a **line through the origin** in the direction of $\vec{v}$
- Any scaled version—up, down, flipped—stays on that line

Properties Check

- ✅ Contains $\vec{0}$ via $c = 0$
- ✅ Closed under scalar multiplication:

  $$c \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in U$$

- ✅ Closed under addition:

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (c_1 + c_2) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow c_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in U$$

> A span of one vector is just a 1D line living in $\mathbb{R}^n$. Minimalistic, but fully subspace-certified 🧼✨

🧠 General Theorem: All Spans Are Subspaces

Let $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$. Then:

$$\text{span}(\vec{v}_1, \ldots, \vec{v}_k) \quad \text{is a subspace of} \quad \mathbb{R}^n$$

✅ Always contains the zero vector

✅ Always closed under scalar multiplication

✅ Always closed under addition

> Any combination of real coefficients → still inside. No weird exceptions, no broken corners. Spans = Subspace machines 🏭💡

🔑 Key Takeaways

- A **subspace** of $\mathbb{R}^n$ must:
  - Include the **zero vector**
  - Be **closed under addition**

- Be **closed under scalar multiplication**

- **Closure** means you can stretch, shrink, flip, or combine, and **never fall out** of the set.
- Common subspaces:
  - The **trivial subspace** $\{\vec{0}\}$
  - **Lines or planes through the origin**
  - Any **span** of vectors in $\mathbb{R}^n$
- **Not all subsets** are subspaces!
  - Watch out for subsets that **fail closure**—even if they look nice.

> TL;DR: Subspaces are the "legal zones" of vector operations in $\mathbb{R}^n$—stable, closed, and centered at zero. If a set leaks when you add or scale, it's outta the subspace club. 🚷