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Basis of a Subspace

Source: Basis of a subspace (video) | Khan Academy

🧠 Conceptual Foundations

What's a Subspace, Really? 🤔

Imagine a subspace as a smaller "universe" inside a bigger vector space like $\mathbb{R}^n$.

Just like how a flat table lies inside a 3D room, a subspace lies within a vector space — but with some strict rules:

- It must include the **zero vector** $\vec{0}$
- It must be **closed under addition**: if $\vec{u}$ and $\vec{v}$ are in the subspace, so is $\vec{u} + \vec{v}$
- It must be **closed under scalar multiplication**: for any scalar c, $c\vec{v}$ must also be in the subspace

If a set satisfies these, congrats — it's a valid subspace! 🎉

What is Span? 🌈

The **span** of a set of vectors is the collection of **all possible linear combinations** you can make from them.

Formally:

$$\operatorname{span}\left(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}\right) = \left\{\sum_{i=1}^{n} c_i\vec{v}_i \mid c_i \in \mathbb{R}\right\}$$

🌿 Linearly Independent Sets

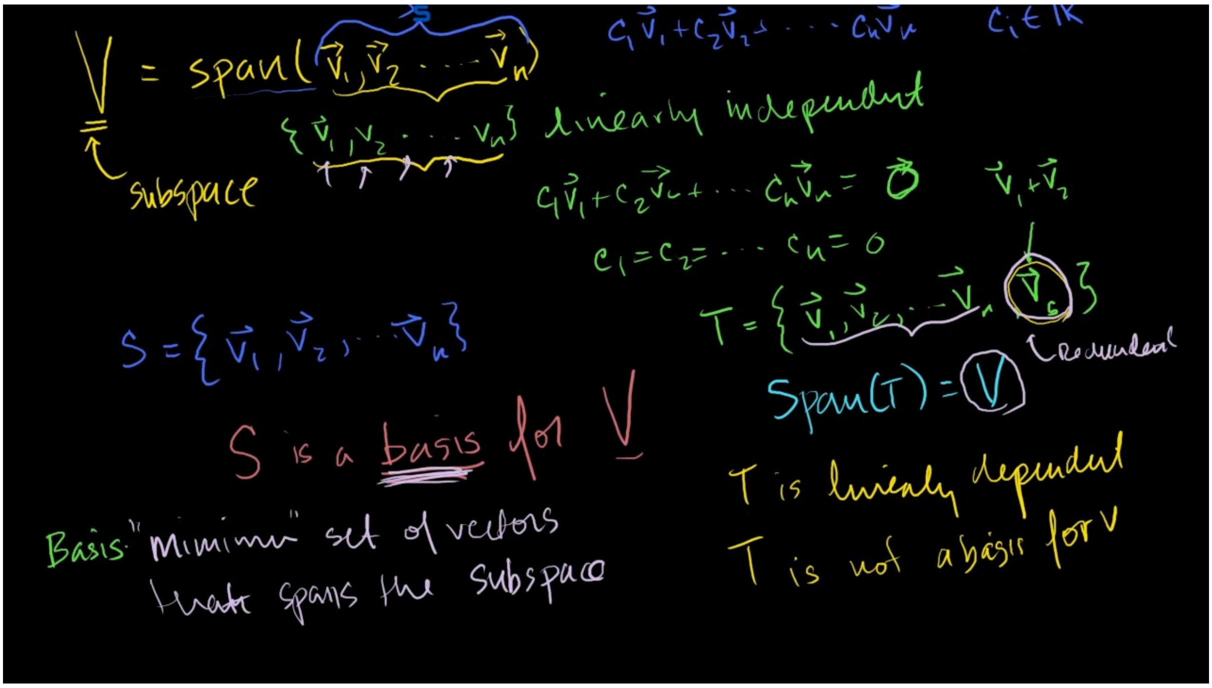
Definition 🌱

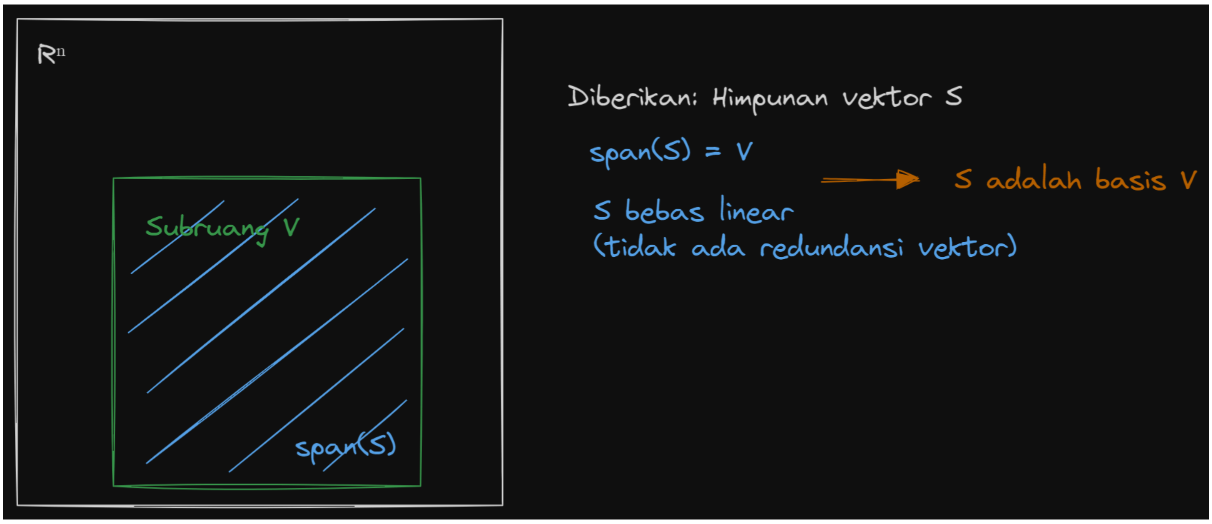
A set of vectors is **linearly independent** if **none of them is redundant** — you can't "fake" any vector by mixing the others.

Formally:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0} \Rightarrow c_1 = c_2 = \cdots = c_n = 0$$

🧱 Basis of a Subspace





Formal Definition 💎

A **basis** of a subspace V is a set of vectors that satisfies **two golden rules**:

1. **Spanning**: Their span equals the entire subspace
2. **Independence**: They are linearly independent

So, if:

- $\text{span}(\{\vec{v}_1, \ldots, \vec{v}_n\}) = V$
- and $\{\vec{v}_1, \ldots, \vec{v}_n\}$ are linearly independent

Then they form a **basis** for V.

Notation tip:

$$\text{If } S = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}, \text{ then } S \text{ is a basis of } V$$

Minimum Set That Spans 🚀

A basis is the **most efficient** team of vectors that span the space — no freeloaders allowed!

Think of it as your essential toolset 🔧: just enough to build anything in the subspace, but no extra clutter.

If you add **even one more vector** that can already be made from others, the set becomes **linearly dependent**, and hence, **not a basis**.

⚠️ Redundant Sets & Non-Bases

Extra Vectors? Bad News 😢

Suppose $S = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ is a basis of V.

Now add a new vector $\vec{v}_{\text{extra}} = \vec{v}_1 + \vec{v}_2$, and call the new set $T = S \cup \{\vec{v}_{\text{extra}}\}$.

You'll still span the same space (since $\vec{v}_{\text{extra}}$ was already "buildable"), but...

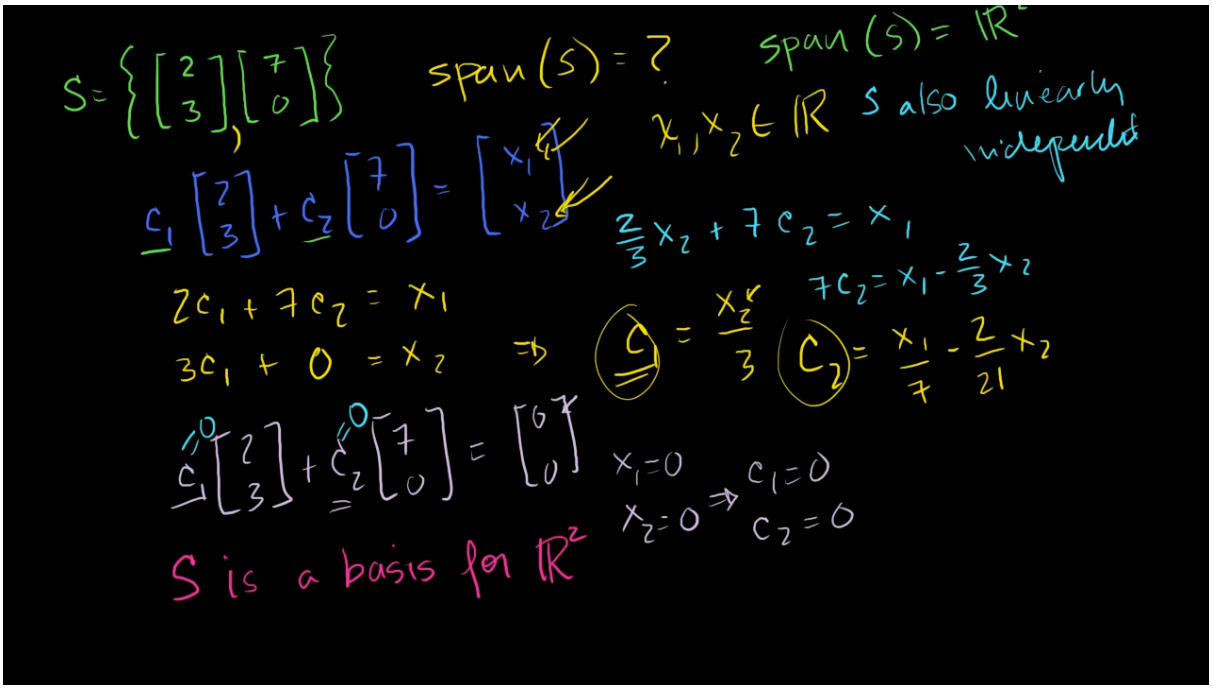
- ❌ T **is linearly dependent**
- ❌ T **is not a basis anymore**

💡 **Key idea**: A basis has **no redundancy**.

Each vector must be truly **necessary** to reach all parts of the subspace.

✨ Example in $\mathbb{R}^2$

1️⃣ Set $S = \{(2,3),(7,0)\}$



Let's check whether this set is a basis for $\mathbb{R}^2$.

🧪 Step 1: Spanning Test

We want to see if this set can reach **any** vector ($\vec{x} = (x_1, x_2) \in \mathbb{R}^2$) using a linear combination:

$$c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 7 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Break into component equations:

- $2c_1 + 7c_2 = x_1$ (1)
- $3c_1 = x_2$ (2)

From (2):

$$c_1 = \frac{x_2}{3}$$

Substitute into (1):

$$2 \cdot \frac{x_2}{3} + 7c_2 = x_1 \Rightarrow \frac{2x_2}{3} + 7c_2 = x_1 \Rightarrow c_2 = \frac{x_1 - \frac{2x_2}{3}}{7} = \frac{x_1}{7} - \frac{2x_2}{21}$$

✅ For **any** $x_1, x_2 \in \mathbb{R}$, you can compute c_1, c_2.

So yes, S spans all of $\mathbb{R}^2$!

🧪 Step 2: Linear Independence Test

We now solve:

$$c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 7 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Same logic:

- From the second component: $3c_1 = 0 \Rightarrow c_1 = 0$
- Plug into the first: $2(0) + 7c_2 = 0 \Rightarrow c_2 = 0$

✅ Only solution is $c_1 = c_2 = 0$ ⇒ The vectors are linearly independent!

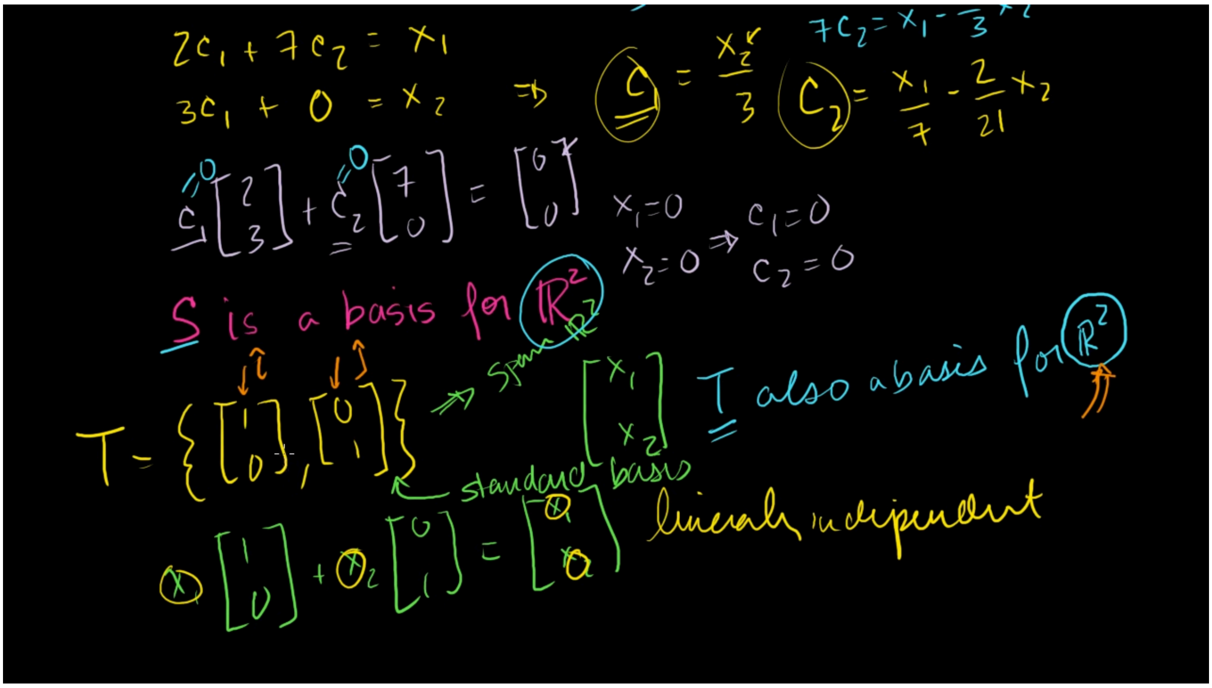
🏁 Conclusion

Since:

- S spans $\mathbb{R}^2$, and
- S is linearly independent,

⇒ S is a **basis** for $\mathbb{R}^2$ 🔥

2️⃣ Set $T = \{(1, 0), (0, 1)\}$



This one's a classic. Let's investigate whether this set is also a **basis** for $\mathbb{R}^2$.

🧪 Step 1: Spanning Check

Can we write any vector $\vec{x} = (x_1, x_2) \in \mathbb{R}^2$ as:

$$\vec{x} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

✅ That's literally the definition of component-wise vector representation in 2D!

So yes — it **spans** $\mathbb{R}^2$

🧪 Step 2: Independence Check

Let's solve:

$$c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \end{cases}$$

✅ Only the trivial solution ⇒ **linearly independent**

🏁 Conclusion

$T = \{(1, 0), (0, 1)\} \Rightarrow \text{Basis for } \mathbb{R}^2$

And not just any basis — this is the **standard basis**, also known as:

- $\hat{i} = (1, 0)$
- $\hat{j} = (0, 1)$

These are the **unit vectors** you've seen in physics and calculus — the MVPs of vector notation! 🏆

∞ Implication: Infinite Possibilities for Bases

Subspaces can have **infinite** different bases!

✅ As long as:

- The vectors **span the space**, and
- They are **linearly independent**

  ⇒ They form a valid basis.

🔁 It's like expressing the same idea in different languages. All valid, just different voices. You can rotate, scale, or skew — if you still cover the whole space without overlap, you're golden. 🥇

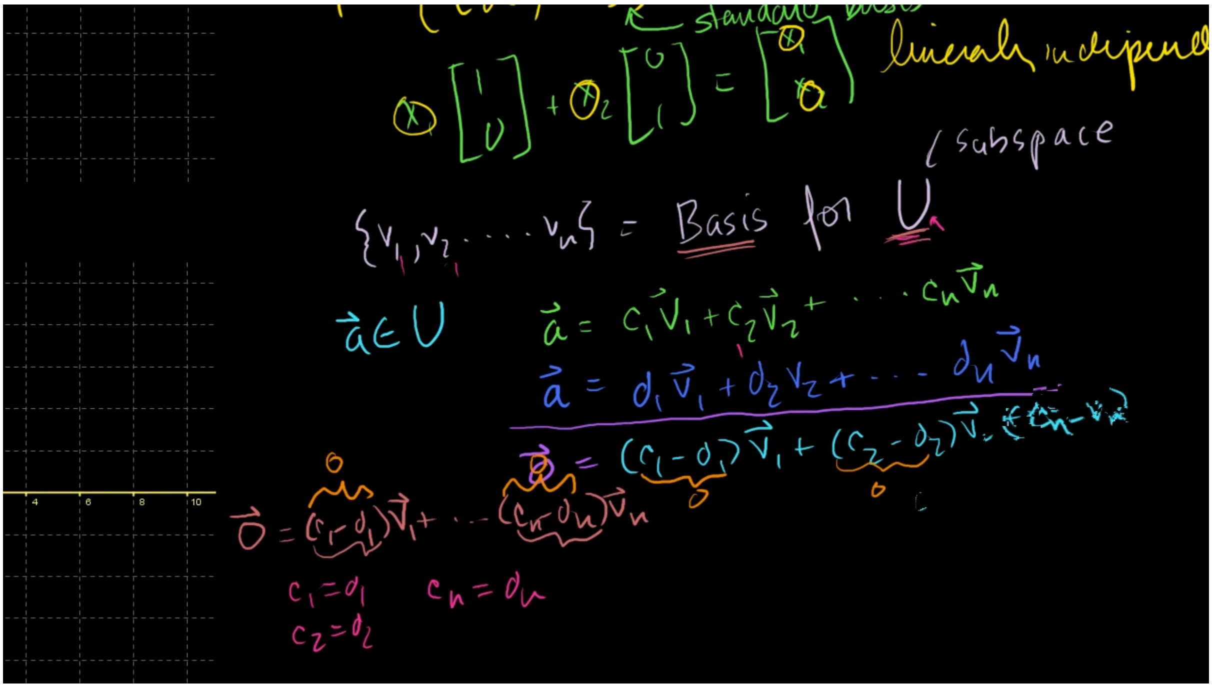
🌈 Example:

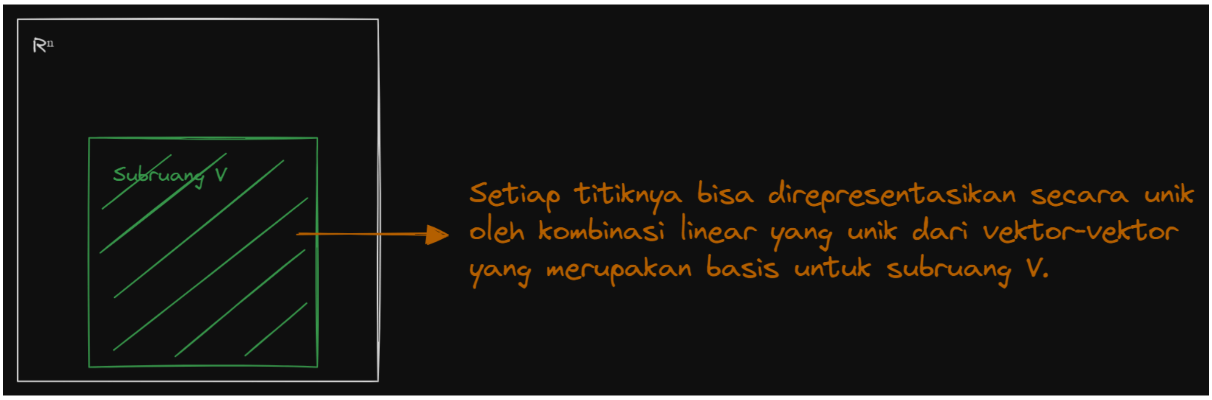
All of these are valid bases for $\mathbb{R}^2$:

- $\{(1, 0),\ (0, 1)\}$ ← standard (as in Example 2)
- $\{(1, 1),\ (-1, 2)\}$ ← tilted
- $\{(2, 3),\ (7, 0)\}$ ← weird but works (as in Example 1)

Each pair is ✅ linearly independent and ✅ spans the whole space.

🔁 Uniqueness of Representation





💡 The Statement

Every vector in a subspace has exactly one expression in terms of a basis.

🧠 It means if you have a **basis** for a subspace, then **every vector** in that subspace can be written as a **unique combination** of those basis vectors.

Let:

- $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis for subspace U
- $\vec{a} \in U$

Then there exists a **unique** set of scalars $\{c_1, \ldots, c_n\}$ such that:

$$\vec{a} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

🍱 **Analogy:** A basis gives every vector its **own secret recipe**, and because the ingredients (basis vectors) are perfectly unique, **no vector can be cooked in more than one way** 💎.

🔢 Example in $\mathbb{R}^2$

Basis $B = \{(1,0),(0,1)\}$. Pick a random vector: $(3,5)$.

$(3,5) = 3(1,0) + 5(0,1)$

✅ No other combo works — basis makes the coordinates (the scalars in the linear combo) **unique**.

🔬 Proof (By Contradiction)

Contradiction: Suppose you had two different ways to express $\vec{a}$:

$\vec{a} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = d_1\vec{v}_1 + \cdots + d_n\vec{v}_n$

Subtract both sides:

$\vec{0} = (c_1 - d_1)\vec{v}_1 + \cdots + (c_n - d_n)\vec{v}_n$

But your basis (the set) is **linearly independent** — the only way to get $\vec{0}$ is if:

$c_1 - d_1 = 0, \quad c_2 - d_2 = 0, \quad \ldots, \quad c_n - d_n = 0 \Rightarrow c_i = d_i \ \forall i$

✨ Which means: The two combinations weren't actually different — they were secretly the same!

Conclusion:

🚫 No duplicates allowed. **Each vector has a one-and-only-one coordinate combo** in its basis. Pure loyalty 💍.

🚫 Counter-Example: Redundant Vector

Let's break things for a moment 👀

Start with the valid basis:

$S = \{(2,3), (7,0)\}$

Now add $(1,0)$ to the set → $S' = \{(2,3), (7,0), (1,0)\}$

That third vector $(1,0)$ is in $\mathbb{R}^2$, so it can already be written as a combo of the first two. This means:

- ❌ S' is not linearly independent
- ❌ S' is not a basis (even though it still spans $\mathbb{R}^2$)

💡 If you can remove a vector and still span the space, then that vector was just freeloading 😤

🧩 Bonus: Standard Basis & Physics ⚙️

Why We Love $\hat{i}, \hat{j}$

In physics and engineering, the standard basis makes your life smooth and sweet 🍯:

- $\hat{i} = (1,0)$ = x-direction
- $\hat{j} = (0,1)$ = y-direction

You can represent any 2D vector as:

$\vec{v} = x \cdot \hat{i} + y \cdot \hat{j}$

This is literally how you break down forces, velocities, and fields. It's the **default language of vectors** in real-world applications.

🔌 When you plug a basis into physics, it becomes the grid reality lives on.

✅ Key Takeaways

- A **subspace** is a “mini-space” inside a vector space, defined by closure and zero inclusion.
- The **span** of vectors is all the linear combinations you can build with them.
- A **basis** is a set of vectors that:
    1. **Spans** the space
    2. Is **linearly independent**
- Bases are the **minimum** teams needed to construct the whole space — no redundancy allowed.
- A basis gives you a **unique coordinate system** for the subspace.
- Subspaces can have **infinitely many** different bases.
- The **standard basis** $\hat{\imath}, \hat{\jmath}$ in $\mathbb{R}^2$ is a classic and super useful in physics/math.
- Adding extra, dependent vectors turns a basis into a **non-basis** — beware of freeloaders ❗