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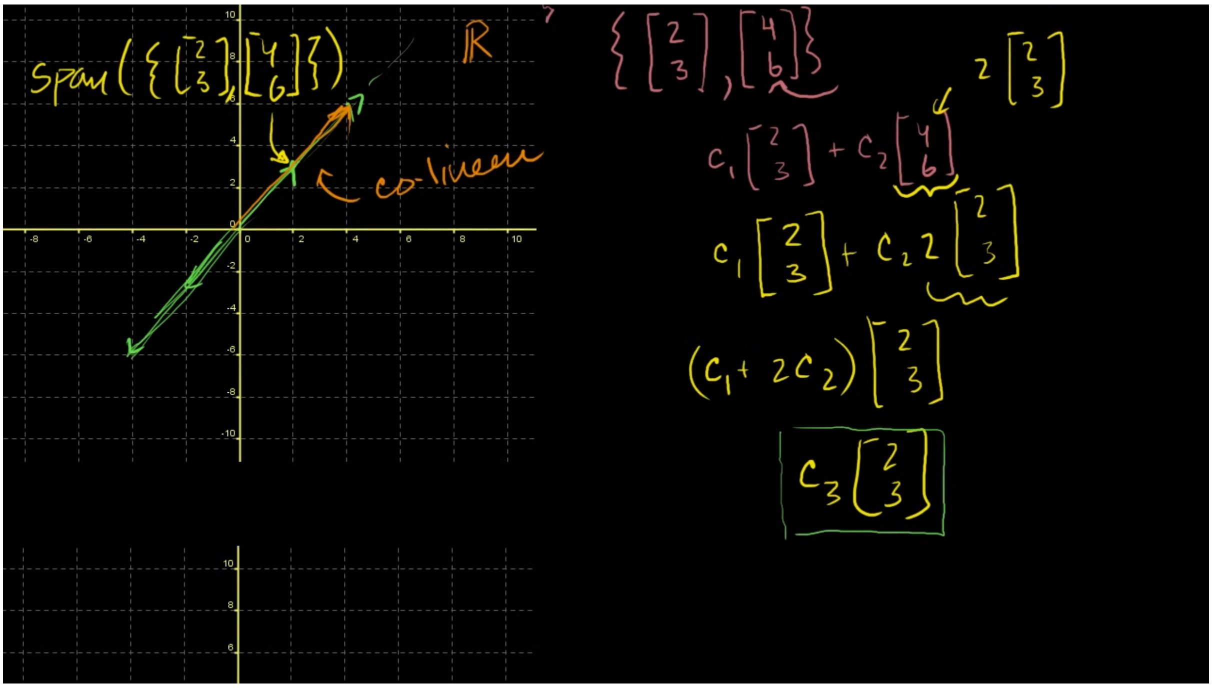
Introduction to Linear Independence

Source: [Introduction to linear independence (video) | Khan Academy]

🧮 Span of Vectors: What Directions Can We Reach?



Example: Vectors $\vec{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

- The **span** of a set of vectors is the set of **all vectors you can reach** by forming **linear combinations**:

$$c_1\vec{v}_1 + c_2\vec{v}_2$$

- Since $\vec{v}_2 = 2\vec{v}_1$, we can write:

  $$c_1\vec{v}_1 + c_2\vec{v}_2 = (c_1 + 2c_2)\vec{v}_1$$

  Let's call $c_3 = c_1 + 2c_2$. So:

  $$\text{All combinations } \Rightarrow \vec{v} = c_3\vec{v}_1$$

- **Visual Analogy** 🖼️: Every linear combination of $\vec{v}_1$ and $\vec{v}_2$ lies on a **straight line** through the origin—the line defined by $\vec{v}_1$ in both directions (positive and negative).

> Think of $\vec{v}_1$ as a rail, and $\vec{v}_2$ just another train car riding that same track 🚂.

🔗 Linear Dependence: The "No New Info" Scenario

Definition

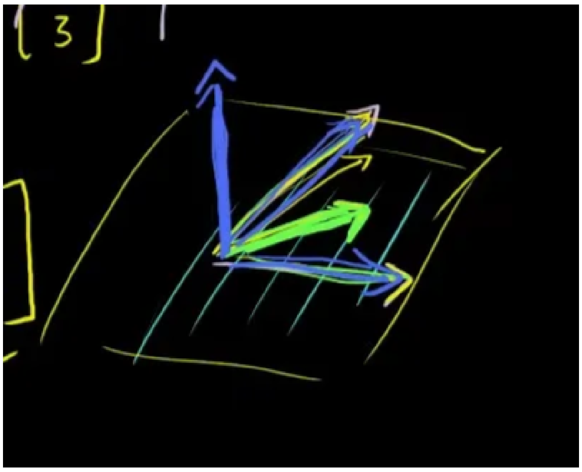
A set of vectors is **linearly dependent** if **one vector can be expressed as a combination of the others**.

Our Earlier Example

- $\vec{v}_2 = 2\vec{v}_1$, so $\vec{v}_2$ contributes **no new direction**.
- Span is still just a **1D line**, even with two vectors.
- **Conclusion**: The set $\{\vec{v}_1, \vec{v}_2\}$ is **linearly dependent**.

> 🔁 Imagine having two compasses, but both only ever point north. One isn't helping!

🌌 R³ Example: The Plane vs 3D Space



Visualize

- Two non-parallel vectors in $\mathbb{R}^3$ define a **plane** 🪂.
- To span the full 3D space ($\mathbb{R}^3$), we need a third vector that **isn't** stuck in that plane.

Scenario

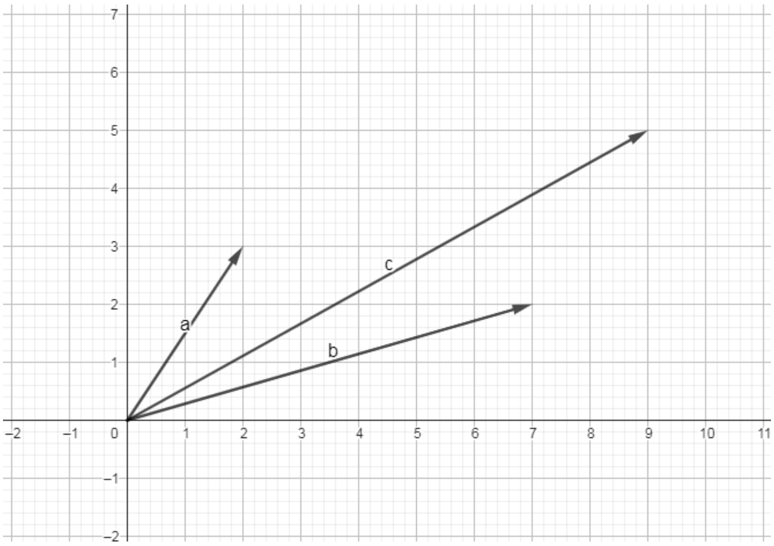
- If vector 3 lies **in** the plane formed by vectors 1 and 2 → it's **redundant** (still just define a plane).
- If vector 3 shoots **out of** the plane → it adds a **new dimension** and the set becomes linearly **independent**.

> ✈️ Picture a paper plane lying on a desk (2 vectors); adding a pencil standing upright is the 3rd vector breaking out into full 3D.

🧠 2D Trick Question: Three Vectors in $\mathbb{R}^2$

Vectors:

$$\vec{a} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}, \quad \vec{c} = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$$



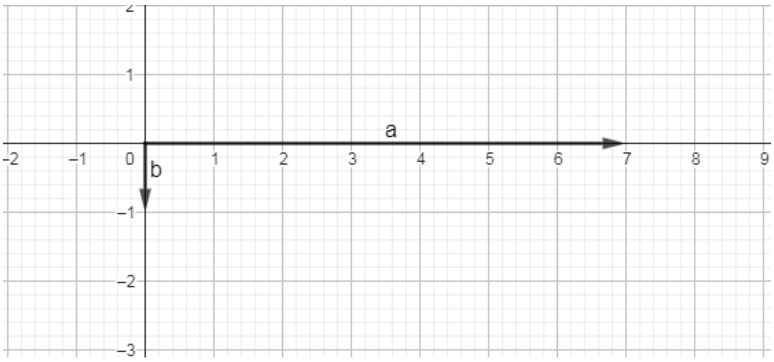
- At first glance, none are scalar multiples… so maybe they're independent?
- But 👀: $\vec{c} = \vec{a} + \vec{b}$
- So $\vec{c}$ is **dependent** on the first two!
- Hence, the set is **linearly dependent** despite appearing unique at first glance.

> 🧩 It's like solving a puzzle only to realize one piece was made by taping two others together.

📐 Pure Independence: Vectors That Go Their Own Way

Example:

$$\vec{v}_1 = \begin{bmatrix} 7 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$



- Can we get one from the other? Nope.
- No scalar of $\vec{v}_1$ gives $\vec{v}_2$, and vice versa.
- 💥 They point in **perpendicular directions**.
- Together, their span = $\mathbb{R}^2$, the entire 2D space.

> 🎨 Like mixing red and blue—suddenly you can make purple and everything else. Perfect independence!

🧱 Redundant Sets Still Span the Same Space

Even with dependent sets, the **span** can be the same:

- In the example above, the span of $\{\vec{a}, \vec{b}, \vec{c}\}$ is still $\mathbb{R}^2$.
- $\vec{c}$ is just **extra baggage**.
- The **most efficient** set that spans the space is called a **basis** (formal definition to come).

> ✂️ Why carry three pens when two are enough to draw the whole map?

🌐 Independence in 3D: The Axes Squad

Vectors:

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}$$

- None of these can be written as a combo of the others.
- Each adds a **completely new direction**.
- They're just scaled versions of the standard unit vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$.
- This is the **perfect linearly independent set** in $\mathbb{R}^3$.

> 🧊 Like x, y, and z axes—each pointing into its own dimension without overlap.

🔑 Key Takeaways

- 🧮 **Span**: All vectors that can be built from linear combinations of a given set.
- 🔗 **Linear dependence**: At least one vector is **redundant**—can be formed from the others.
- 🧊 **Linear independence**: No vector in the set can be built from others—each adds **new directionality**.
- 📐 In $\mathbb{R}^2$, any more than 2 vectors will be linearly dependent.
- 🌐 In $\mathbb{R}^3$, 3 **non-coplanar** vectors can span the entire space.
- 🎯 A **basis** is a **minimal set of linearly independent vectors** that span the entire space.
- 🎓 More vectors ≠ better span. Efficiency matters!