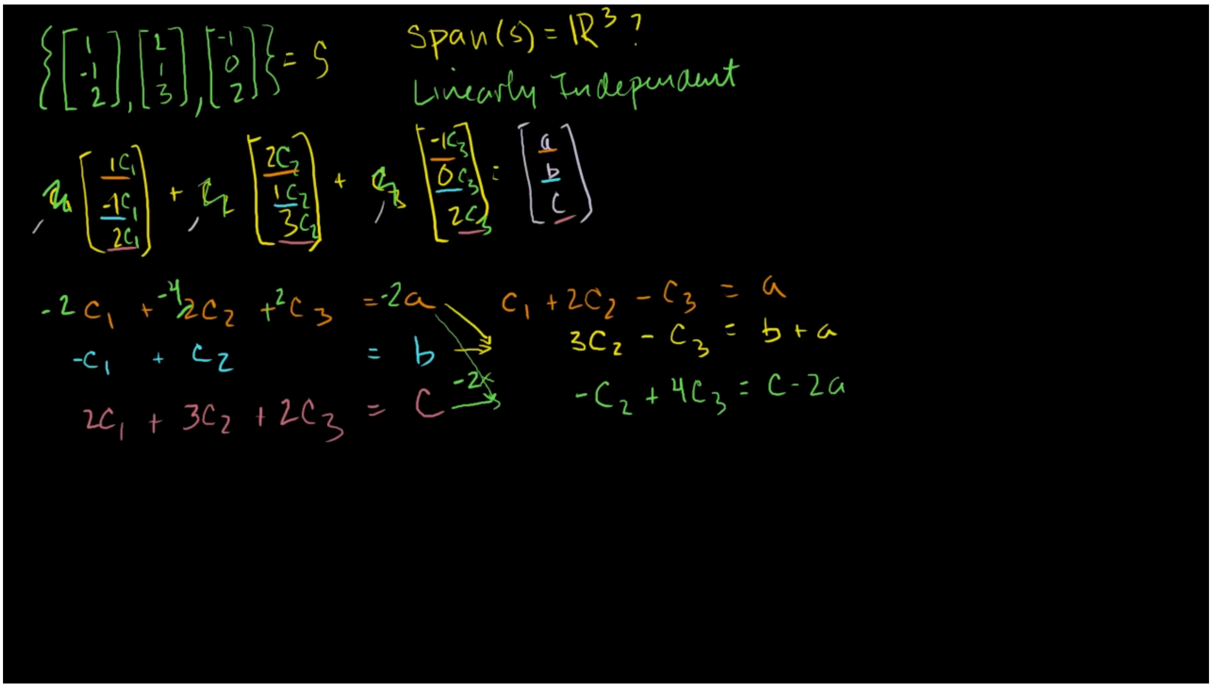
Span and Linear Independence Example

Source: [Span and linear independence example (video) | Khan Academy]

Problem Setup



We are given **three 3D vectors**:

- $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

- $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$
- $\vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

We're asked two questions:

1. Do these vectors **span** $\mathbb{R}^3$?
2. Are these vectors **linearly independent**?

Goal 1: Do the Vectors Span $\mathbb{R}^3$?

The Definition

A set of vectors **spans** $\mathbb{R}^3$ if **any vector** in $\mathbb{R}^3$ (say $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$) can be written as a **linear combination** of them. In other words, we can construct any linear combinations from that 3 vectors.

So we ask:

Can we find scalars $c_1, c_2, c_3 \in \mathbb{R}$ such that:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}?$$

Set Up the System

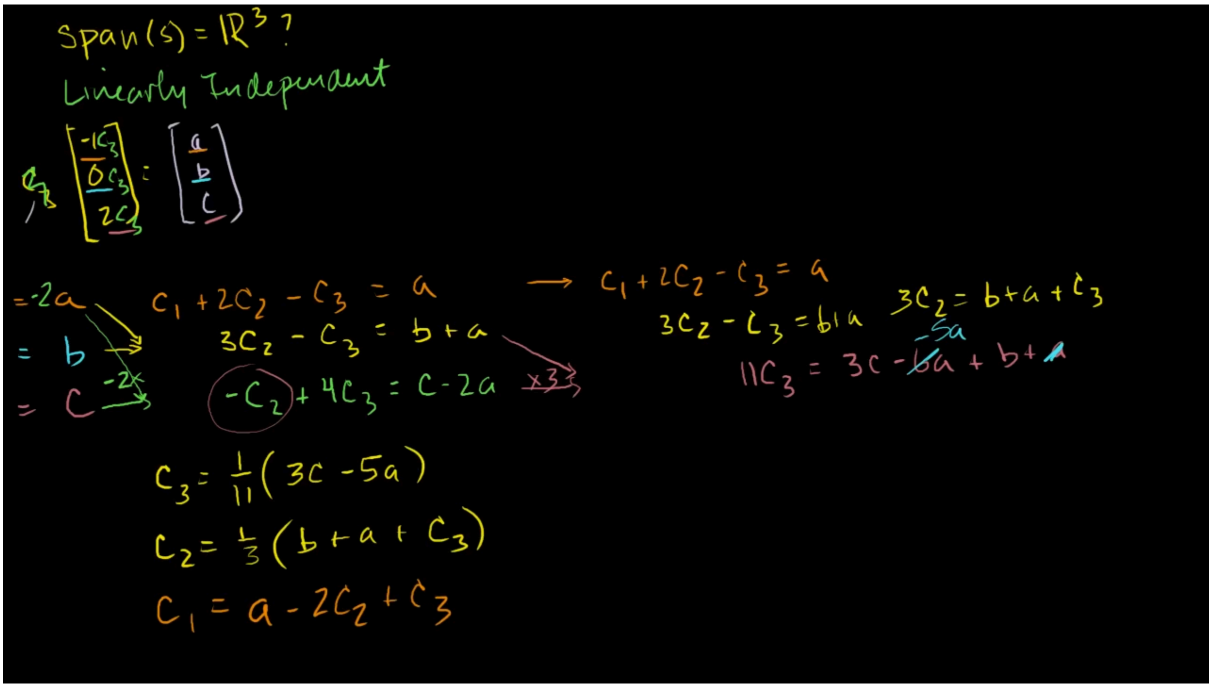
Expanding the linear combination:

$$c_1 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Which gives the system:

- $c_1 + 2c_2 - c_3 = a$
- $-c_1 + c_2 = b$
- $2c_1 + 3c_2 + 2c_3 = c$

🔧 Elimination Method (Step-by-Step)



Correction: c_3 = 1/11 (3c - 5a + b)

We solve this using elimination:

Step 1: Eliminate c_1 from 2nd Equation

Add the first and second equations:

$$(-c_1 + c_2) + (c_1 + 2c_2 - c_3) = b + a \Rightarrow 3c_2 - c_3 = a + b$$

Step 2: Eliminate c_1 from 3rd Equation

Subtract $2\times$ first equation from third:

$(2c_1 + 3c_2 + 2c_3) - 2(c_1 + 2c_2 - c_3) = c - 2a$

$\Rightarrow -c_2 + 4c_3 = c - 2a$

Step 3: Eliminate c_2

Multiply the new equation from Step 2 by 3 and add to the Step 1 result:

$3(-c_2 + 4c_3) + (3c_2 - c_3) = 3(c - 2a) + (a + b)$

$\Rightarrow 11c_3 = 3c - 5a + b \Rightarrow c_3 = \dfrac{3c - 5a + b}{11}$

Step 4: Back-substitute for c_2 and c_1

From $3c_2 - c_3 = a + b$:

$c_2 = \dfrac{a + b + c_3}{3}$

Then use:

$c_1 = a - 2c_2 + c_3$

🎉 So for **any** $a, b, c \in \mathbb{R}$ (any linear combinations), we can always solve for c_1, c_2, c_3.

✅ **Therefore, the three vectors span $\mathbb{R}^3$.**

🧠 Goal 2: Are the Vectors Linearly Independent? 🎭

🧪 The Definition

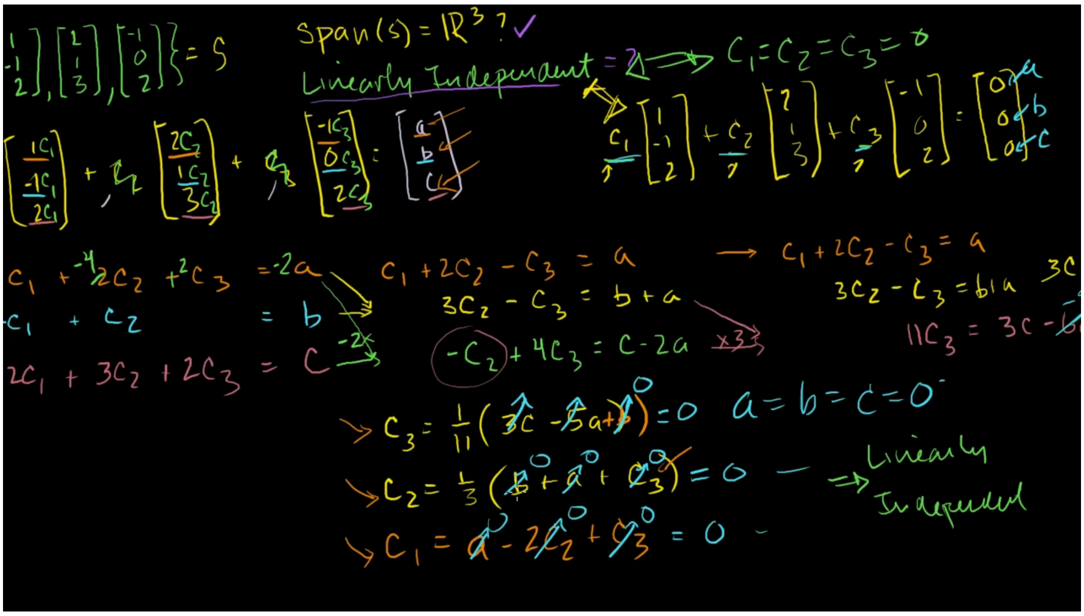
Vectors are **linearly independent** if the **only** solution to:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

is:

$$c_1 = c_2 = c_3 = 0$$

🎯 Test the Zero Vector



Now set:

- $a = b = c = 0$

Our previously derived formulas become:

- $c_3 = \dfrac{3(0) - 5(0) + 0}{11} = 0$ ✔
- $c_2 = \dfrac{0 + 0 + 0}{3} = 0$ ✔
- $c_1 = 0 - 2(0) + 0 = 0$ ✔

✅ So the **only** solution to the zero vector combination is the trivial one.

🎯 **Therefore, the vectors are linearly independent.**

🔒 Geometric Insight 🧭

- We have **exactly 3 vectors** in $\mathbb{R}^3$
- They span $\mathbb{R}^3$
- They are linearly independent

💥 This means they form a **basis** for $\mathbb{R}^3$. Like the ultimate trio—each one contributes **a unique direction**. There's **zero redundancy**, and you can build anything in 3D from just them. 😎🧱

📌 Key Takeaways 🧠✨

- **Span**: The three vectors can be linearly combined to form *any* vector in $\mathbb{R}^3$.
- **Linear independence**: The only way to get the zero vector is by setting all coefficients to zero.
- These vectors form a **basis** for $\mathbb{R}^3$.
- Geometrically, they're like 3 arrows shooting off in different directions—together, they fill the whole space. 🌌
- If you have exactly 3 vectors that span $\mathbb{R}^3$, they **must** be linearly independent (and vice versa, if they're linearly independent, they span it!).