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Linear Combinations and Span

Source: Linear combinations and span (video) | Khan Academy

What Is a Linear Combination? 🤔

Imagine vectors as arrows in space—each pointing in a direction with a certain magnitude. Now, a **linear combination** is just a way of **mixing** these arrows using:

- **Scaling** (multiplying each vector by a real number),
- **Adding** them together.

Definition 📘

Given vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in \mathbb{R}^n$, a **linear combination** looks like this:

$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n,$

where each $c_i \in \mathbb{R}$.

🧪 **You're just stretching (scaling) and stacking (adding)** the vectors.

Visualizing Linear Combinations with Examples 🧠✨

Example 1: Two Vectors in $\mathbb{R}^2$

Let:

- $\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
- $\vec{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

🎯 A **linear combination** of $\vec{a}$ and $\vec{b}$ is any vector of the form:

$c_1\vec{a} + c_2\vec{b}$

Try this:

- $c_1 = 3$, $c_2 = -2$

$$3\vec{a} + (-2)\vec{b} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} - \begin{bmatrix} 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

🚀 So $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ is one possible result—a linear combination!

Can We Generate *Every* Vector in $\mathbb{R}^2$? 🌐

YES—if the two vectors are **not collinear** (not scalar multiples of each other), their combinations fill the whole plane!

Span: The Universe of Reachable Vectors 🌌

The **span** of a set of vectors is the **entire collection of all their linear combinations**.

Formal Definition 📏

$$\text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n) = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \mid c_i \in \mathbb{R}\}$$

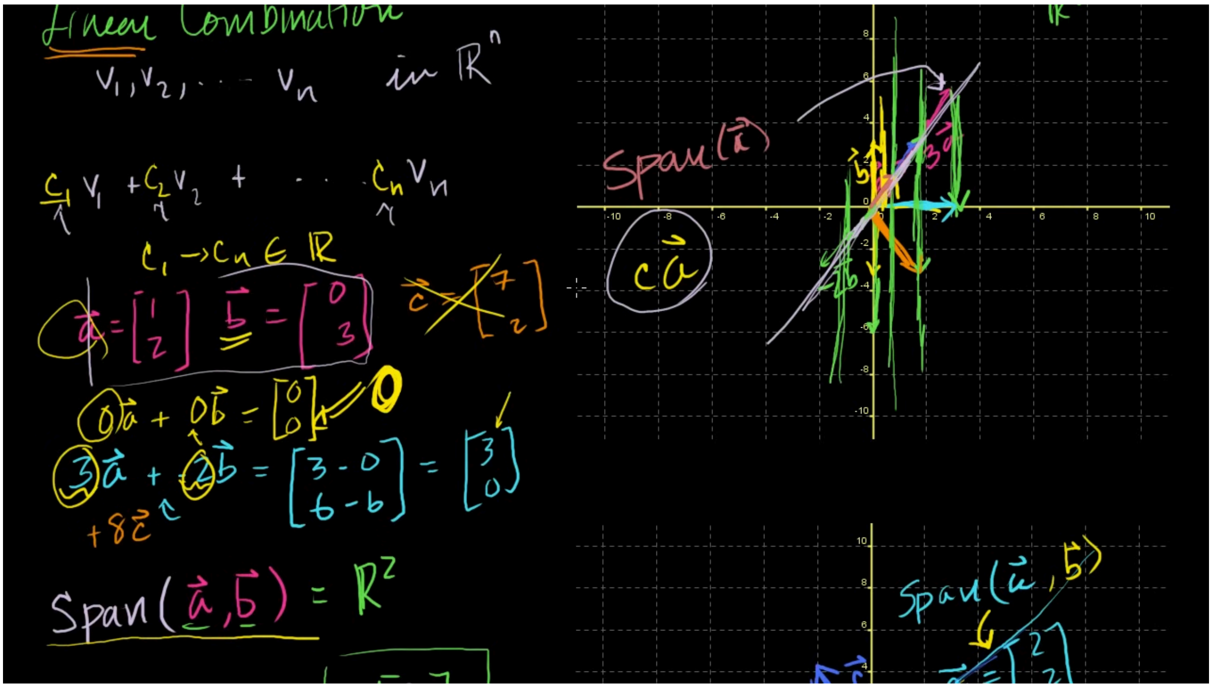
Think of **span** as the **territory** these vectors can explore if they work together 💪.

🧠 Analogy

You're on a grid. $\vec{a}$ lets you move diagonally. $\vec{b}$ sends you straight up. If you can stack movements using both, you can walk to any square. That's span!

Visual Interpretations 🎨📐

Case 1: Vectors That Span $\mathbb{R}^2$



Given $\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$:

- These are **not** multiples of each other.
- Their span = **entire plane** $\mathbb{R}^2$ ✅
- You can reach *any* point in the 2D world using combinations of these two!

Case 2: Vectors That Don't Span $\mathbb{R}^2$

Let:

- $\vec{a} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

🚫 These are **collinear**—they lie on the same line!

- Their span = a **line** in $\mathbb{R}^2$❌
- You can only move along that line. Try to escape → You're stuck.

Case 3: The Zero Vector

If all you have is the **zero vector**:

- $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- Every combination is still $\vec{0}$
- So the span is just $\{\vec{0}\}$ ❌

Sad, limited world. No adventure 😢

Span of a Single Vector ➡️ A Line

If $\vec{v} \in \mathbb{R}^2$, then:

$$\text{span}(\vec{v}) = \{c\vec{v} \mid c \in \mathbb{R}\}$$

📌 That's just a **line through the origin** in the direction of $\vec{v}$.

The Famous Duo: Unit Vectors i & j ⚔️

These guys are your **standard base** for 2D space.

- $\hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ → unit step along x-axis
- $\hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ → unit step along y-axis

💥 Any vector in $\mathbb{R}^2$ can be built with them:

$$\vec{x} = x_1\hat{i} + x_2\hat{j}$$

So they **span** $\mathbb{R}^2$ and also form a **basis** (fancy term for later 🎓).

Proving It Algebraically 🧮✏️

Let's prove that $\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ can reach any point $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$.

We want:

$$c_1\vec{a} + c_2\vec{b} = \vec{x} \Rightarrow c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Break it into equations:

- $c_1 = x_1$
- $2c_1 + 3c_2 = x_2$

Substitute $c_1 = x_1$ into the second:

$$2x_1 + 3c_2 = x_2 \Rightarrow c_2 = \frac{x_2 - 2x_1}{3}$$

📌 So for **any** (x_1, x_2), we have a solution ⇒ $\vec{a}$ and $\vec{b}$ span $\mathbb{R}^2$!

Quick Example

Target vector: $\vec{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

- $c_1 = 2$
- $c_2 = \dfrac{2 - 2(2)}{3} = \dfrac{-2}{3}$

So:

$\vec{x} = 2\vec{a} - \frac{2}{3}\vec{b}$

→ $2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ ✅

Boom. Landed at $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ via linear combo!

Key Takeaways 💡🧠

- A **linear combination** is a weighted sum of vectors using real coefficients.
- The **span** of vectors is the set of *all possible linear combinations*—aka the "territory" they can cover.
- Two vectors **span** $\mathbb{R}^2$ if they're not collinear.
- The **span of a single vector** is a **line**, and the **span of the zero vector** is just the zero vector.
- Vectors like $\hat{i}$ and $\hat{j}$ form a **basis**—they both span $\mathbb{R}^2$ and are orthogonal (90° apart).

- You can **algebraically prove** whether a set of vectors spans a space by solving systems of equations.