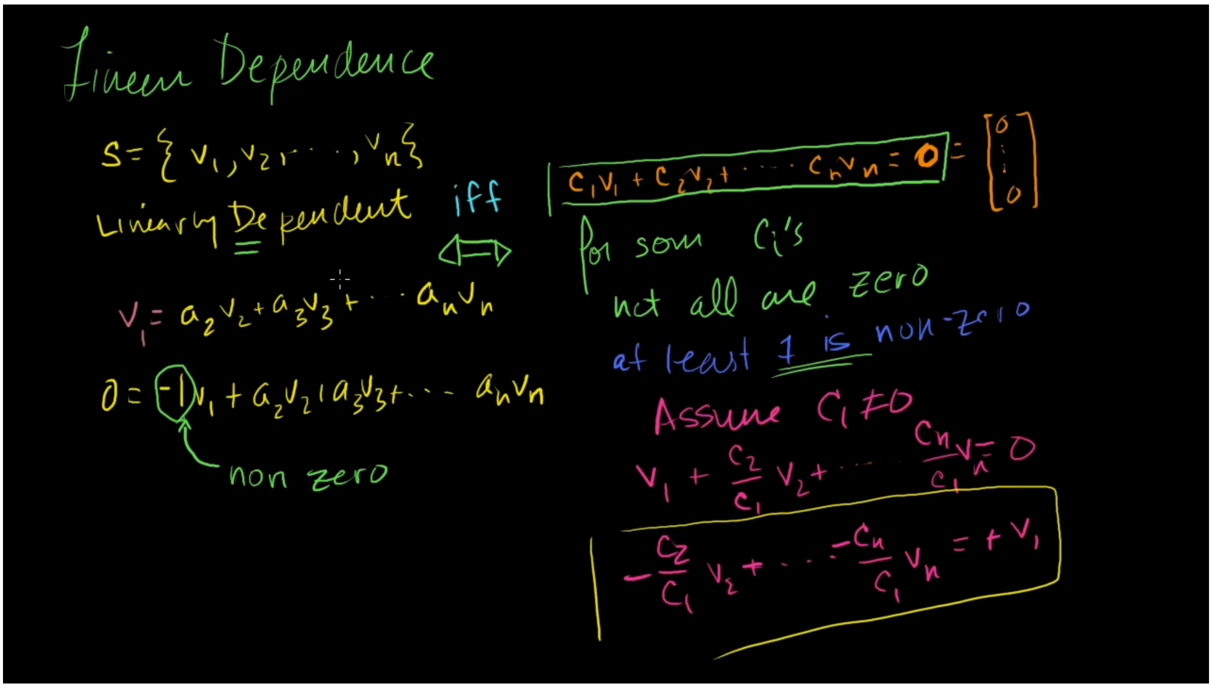
More on Linear Independence

Understanding when vectors are team players (dependent) or solo stars (independent)

Source: More on linear independence (video) | Khan Academy

What Does "Linearly Dependent" Mean? — Formal Definition



Let's define a set of vectors:

$$S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$$

This set is **linearly dependent** *if and only if*:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$$

...for some scalars $c_1, c_2, \ldots, c_n \in \mathbb{R}$ where **not all** $c_i = 0$ ❗

- The condition $\vec{0}$ represents the **zero vector**, which is a vector where every component is 0, like $(0, 0)$, $(0, 0, 0)$, etc.
- If even **one scalar is non-zero**, 🎯 the set is linearly dependent.

> 💭 Think of this like cooking: If you can mix some ingredients (vectors) in just the right non-zero proportions to end up with something bland and flavorless (the zero vector), then the ingredients weren't unique — one or more could be made from the rest!

🪢 Equivalent Definition: Redundant Vector View

An alternate, equivalent definition:

> A set of vectors is linearly dependent if at least one vector in the set can be written as a **linear combination** of the others.

Mathematically:

$$\vec{v}_1 = a_2\vec{v}_2 + a_3\vec{v}_3 + \cdots + a_n\vec{v}_n$$

Proof of Equivalence 📜

Subtract $\vec{v}_1$ from both sides:

$$\vec{0} = -\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$$

We now have a linear combination that

✔️ **equals** $\vec{0}$, and

✔️ **not all scalars are zero** (-1)

→ **matches the original definition** ✅

Conversely (another way to prove this equivalent definition)**:**

Suppose we have a linear dependence, then it is:

$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}$

> 💡 In a linearly dependent set, there must exist **at least one non-zero scalar** in the linear combination that yields the zero vector. For the purpose of analysis or proof, **it is not essential to identify exactly which scalar is non-zero**. Therefore, any one may be chosen arbitrarily (e.g., $c_1 \neq 0$) to **demonstrate that a particular vector can be expressed as a linear combination** of the others.

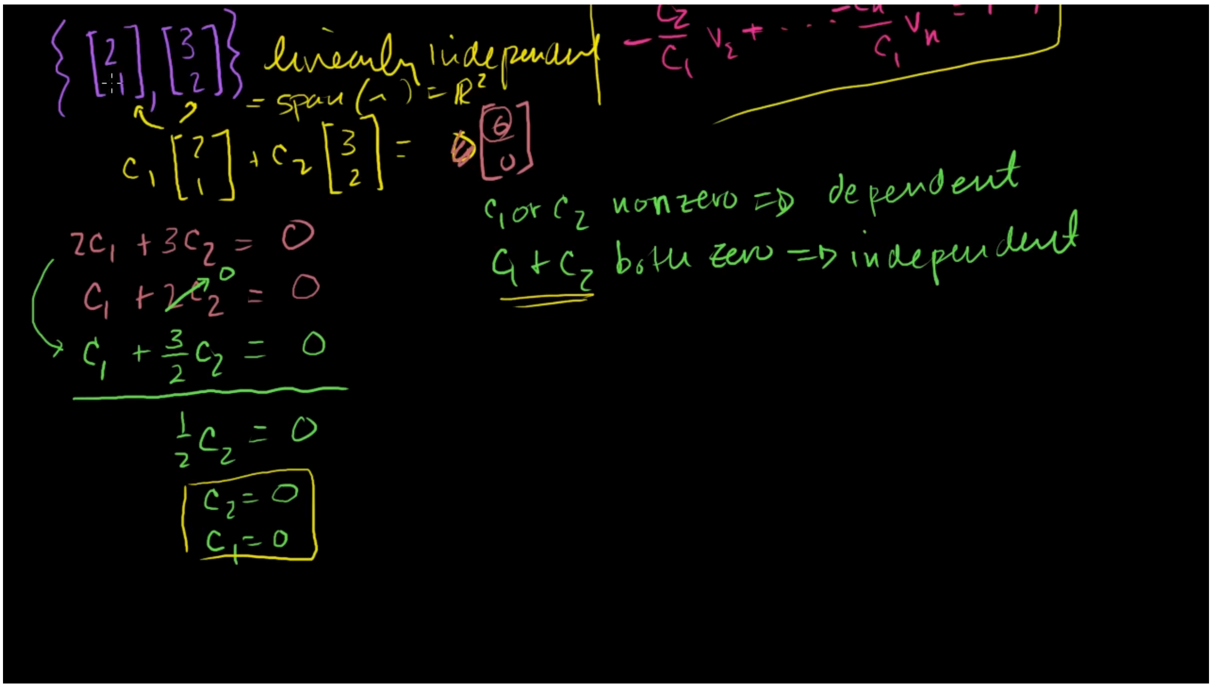
Because it's a linear dependent, it has at least one non-zero constant → assume $c_1 \neq 0$.

Divide through by c_1, rearrange, and boom 💥:

$\vec{v}_1 = -\dfrac{c_2}{c_1}\vec{v}_2 - \cdots - \dfrac{c_n}{c_1}\vec{v}_n$

Again, $\vec{v}_1$ is a combo of the others → equivalent definition confirmed ✅

🧪 Example 1



Are $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ linearly dependent? 🤔

Let's solve:

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This leads to the system:

$$\begin{cases} 2c_1 + 3c_2 = 0 \\ 1c_1 + 2c_2 = 0 \end{cases}$$

Solve via elimination or substitution 🧮:

Multiply second equation by 2:

$$2c_1 + 4c_2 = 0$$

Subtract from first:

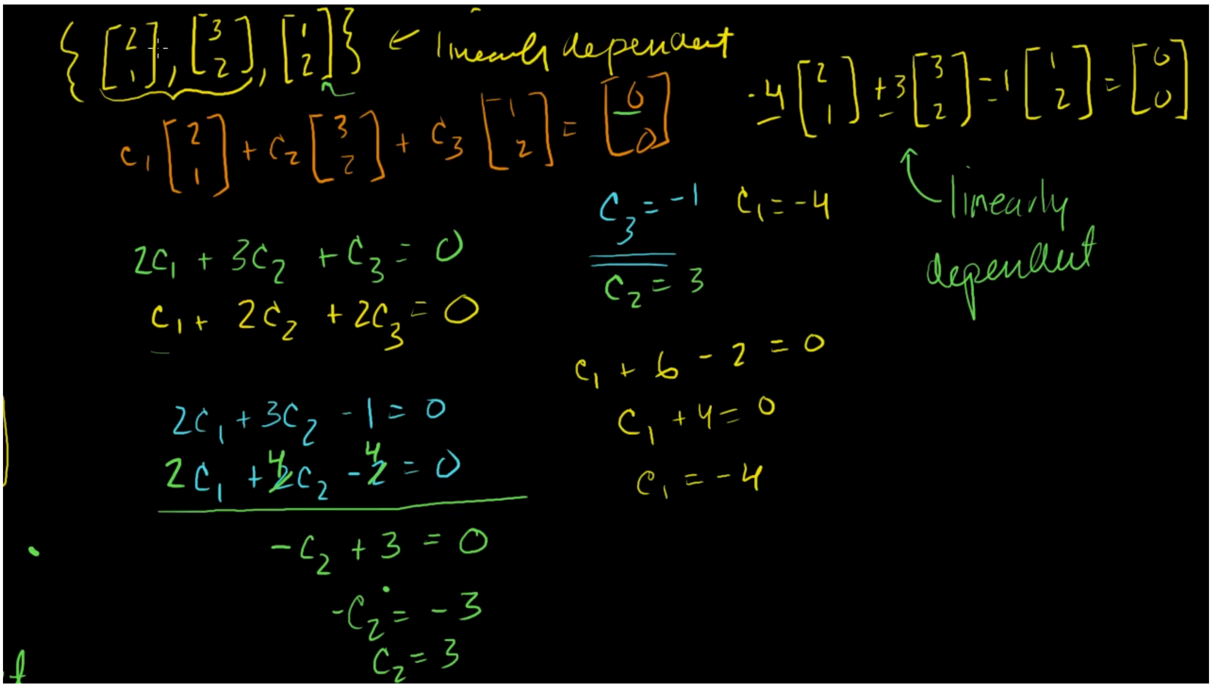
$$(2c_1 + 3c_2) - (2c_1 + 4c_2) = -c_2 = 0 \Rightarrow c_2 = 0 \Rightarrow c_1 = 0$$

🚨 Both coefficients must be 0 ⇒ **Vectors are linearly independent!**

✅ Neither vector can be made from the other.

✅ These vectors span $\mathbb{R}^2$ (the full 2D plane).

🧪 Example 2



Are $\vec{v}_1=\begin{bmatrix}2\\1\end{bmatrix}$, $\vec{v}_2=\begin{bmatrix}3\\2\end{bmatrix}$, $\vec{v}_3=\begin{bmatrix}1\\2\end{bmatrix}$ linearly dependent? 🧐

Now test:

$$c_1\vec{v}_1+c_2\vec{v}_2+c_3\vec{v}_3=\vec{0}$$

Leads to the system:

$$\begin{cases}2c_1+3c_2+1c_3=0\\1c_1+2c_2+2c_3=0\end{cases}$$

> 💡 This system has **more unknowns than equations** (3 variables, 2 equations), making it underdetermined. Hence, it **admits infinitely many solutions**. Infinite solutions mean we

> can assign an arbitrary value to any one variable (e.g., $c_3 = -1$) to **explore whether at least one non-trivial solution exists** among them.

> 🔍 Observation: 3 vectors in $\mathbb{R}^2$? Always dependent! It's overkill.
> Why? Because in a 2D space,
> **only 2 vectors max can be linearly independent**. Third one's always a combo 💥

Still, let's verify:

Pick $c_3 = -1$

Substitute into the equations:

$2c_1 + 3c_2 - 1 = 0 \Rightarrow 2c_1 + 3c_2 = 1$

$c_1 + 2c_2 - 2 = 0 \Rightarrow c_1 + 2c_2 = 2$

Now solve:

Multiply second equation by 2:

$2c_1 + 4c_2 = 4$

Subtract first:

$(2c_1 + 4c_2) - (2c_1 + 3c_2) = c_2 = 3 \Rightarrow c_1 = -4$

So:

$c_1 = -4, \quad c_2 = 3, \quad c_3 = -1$

✅ Not all zero → **Linearly dependent**.

→ Extra proof: $-4 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + (-1) \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -8 \\ -4 \end{bmatrix} + \begin{bmatrix} 9 \\ 6 \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$ ✔️

💡 We found a linear combo that makes zero! In fact, all 3 coefficients are non-zero 🔥

🧠✨ Key Takeaways

- A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is **linearly dependent** if:

  $$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0} \quad \text{for some } c_i \neq 0$$

  Equivalent view: One vector is a **linear combo of others** 🪢

- The **zero vector** can always be made by the zero-scalar combo (trivial solution) — not enough to claim dependence.
- If **only solution is all-zero** → linearly **independent** ✔️
- More vectors than dimensions (e.g. 3 vectors in $\mathbb{R}^2$) → 🚨 Guaranteed dependence
- **Independent vectors = no redundancy**. Each one adds new direction 📈
- **Dependent vectors = redundancy**. Some directions are recycled ♻️