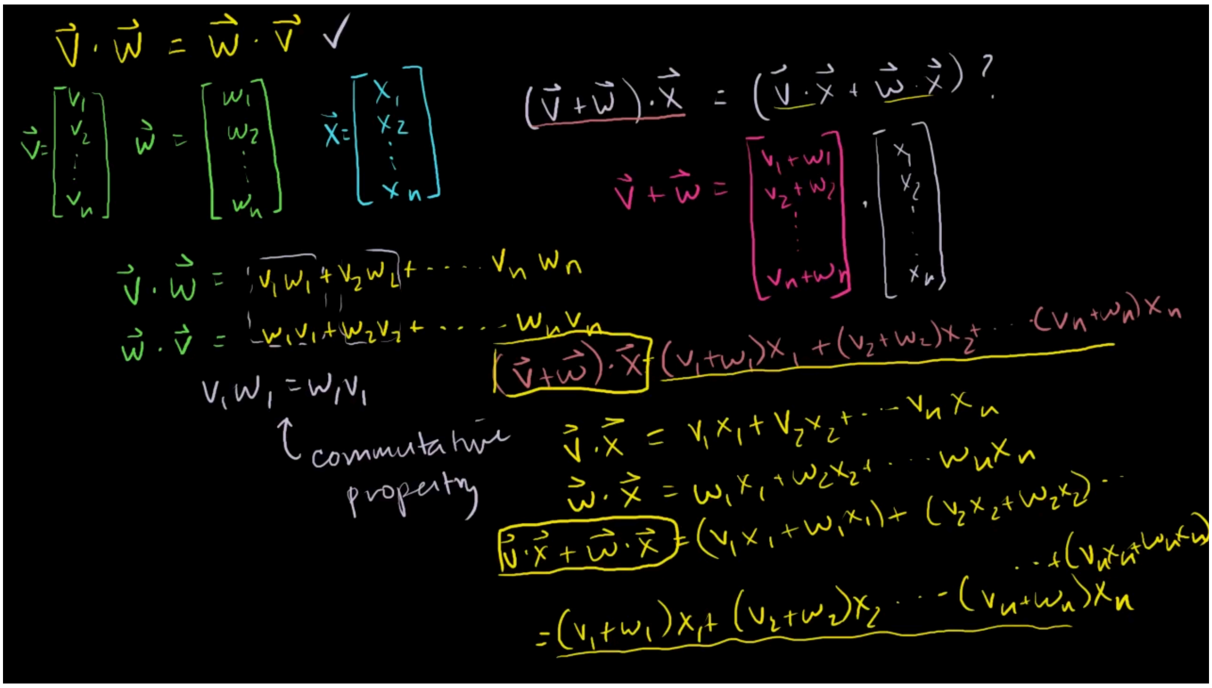
Proving Vector Dot Product Properties

Source: Proving vector dot product properties (video) | Khan Academy

Commutativity of the Dot Product 🔁



Problem Statement

We aim to prove that the **dot product is commutative**, i.e.,

$$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$$

Vector Definitions

Let

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

Compute Each Dot Product

- First, compute $\vec{v} \cdot \vec{w}$:

  $$v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

- Now compute $\vec{w} \cdot \vec{v}$:

  $$w_1 v_1 + w_2 v_2 + \cdots + w_n v_n$$

Why Are They Equal?

🔁 Because scalar multiplication of real numbers is **commutative**:

$$v_i w_i = w_i v_i \quad \text{for all } i$$

👉 Hence,

$$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$$

> 🖐🏻 Analogy: Think of two people high-fiving—doesn’t matter who raises their hand first, the clap happens either way!

Distributivity of the Dot Product Over Vector Addition ➕🟰

Property to Prove

We want to show:

$$(\vec{v} + \vec{w}) \cdot \vec{x} = \vec{v} \cdot \vec{x} + \vec{w} \cdot \vec{x}$$

Define All Vectors

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Step-by-Step Breakdown

1. Left-Hand Side

First compute $\vec{v} + \vec{w}$:

$$\vec{v} + \vec{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{bmatrix}$$

Then compute the dot product:

$$(\vec{v} + \vec{w}) \cdot \vec{x} = (v_1 + w_1)x_1 + (v_2 + w_2)x_2 + \cdots + (v_n + w_n)x_n$$

2. Right-Hand Side

- Compute each dot product:

$$\vec{v} \cdot \vec{x} = v_1x_1 + v_2x_2 + \cdots + v_nx_n$$

$$\vec{w} \cdot \vec{x} = w_1x_1 + w_2x_2 + \cdots + w_nx_n$$

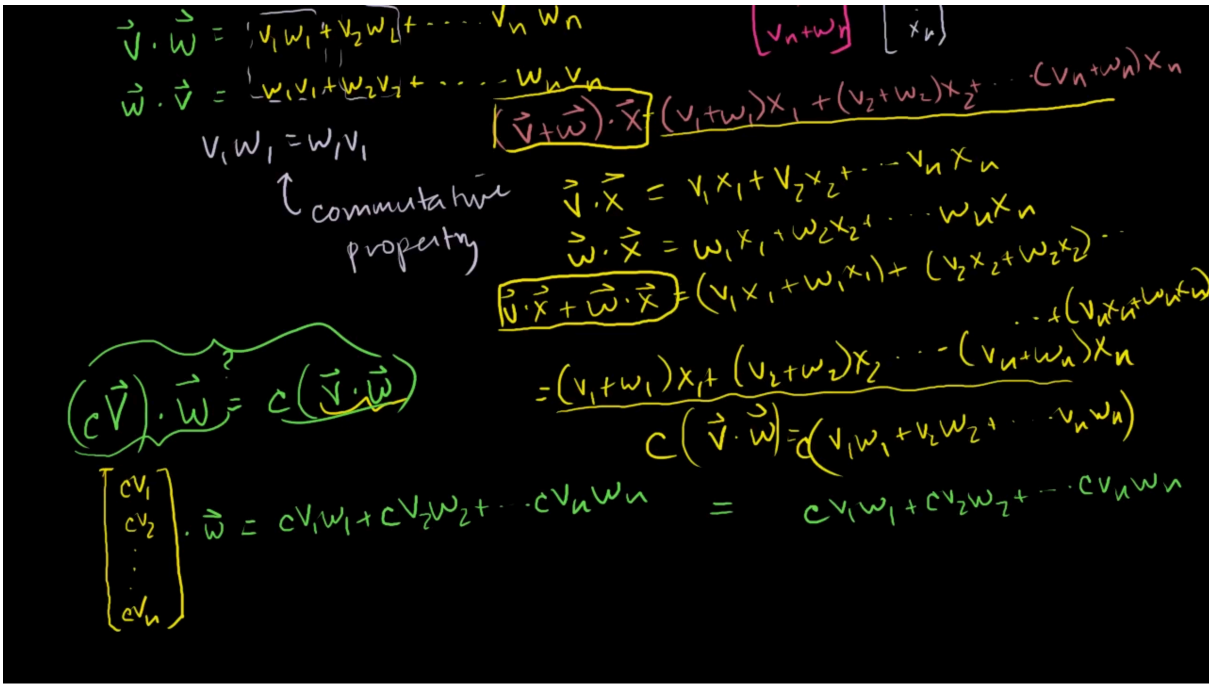
- Add the results:

$$\vec{v} \cdot \vec{x} + \vec{w} \cdot \vec{x} = (v_1 + w_1)x_1 + \cdots + (v_n + w_n)x_n$$

🎉 Voilà! Both sides are identical.

> 🍕 Analogy: Like splitting a pizza (dot product with x) between two friends (v and w). Distribute the slices equally—they still add up to the full pie!

Scalar Associativity with the Dot Product 🔗



Statement to Prove

Show that scalar multiplication associates with the dot product:

$(c\vec{v}) \cdot \vec{w} = c(\vec{v} \cdot \vec{w})$

Define Scalar and Vectors

Let c be a real number, and

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

Compute Both Sides

Left-Hand Side

- Compute scalar multiplication first:

  $$c\vec{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}$$

- Then take the dot product:

  $(cv_1)w_1 + (cv_2)w_2 + \cdots + (cv_n)w_n = c(v_1w_1 + v_2w_2 + \cdots + v_nw_n)$

Right-Hand Side

- Compute $\vec{v} \cdot \vec{w}$ first:

  $v_1w_1 + v_2w_2 + \cdots + v_nw_n$

- Then multiply by scalar:

$$c(v_1w_1 + v_2w_2 + \cdots + v_nw_n)$$

💯 Both sides yield the same result!

> 🧃 Analogy: Scaling a smoothie recipe (vector) before or after blending (dot product) gives the same delicious outcome—just multiplied by the scalar (c)!

Key Takeaways 🧠💥

- ✅ **Commutativity**:

  Order doesn't matter in dot product:

  $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$

- ✅ **Distributivity**:

  Dot product distributes over vector addition:

  $(\vec{v} + \vec{w}) \cdot \vec{x} = \vec{v} \cdot \vec{x} + \vec{w} \cdot \vec{x}$

- ✅ **Associativity with Scalars**:

  You can factor out the scalar:

  $(c\vec{v}) \cdot \vec{w} = c(\vec{v} \cdot \vec{w})$

- 📚 These properties make the dot product behave like normal multiplication in arithmetic—but with vectors!
- 🧱 We **can't just assume** these properties—they must be proven from first principles using component-wise definitions.

> 🧮 Feeling bored by the repetition? That's the price of mathematical rigor—it may feel mundane, but it builds intuition and confidence 💪