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Vector Dot Product and Vector Length

Source: [Vector dot product and vector length (video) | Khan Academy]

Vector Operations Recap 🔁

Let's start by refreshing our memory on two foundational vector operations:

Vector Addition ➕

Given two vectors:

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Their **sum** is:

$$\vec{a} + \vec{b} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

👉 Each component just adds up with the corresponding one from the other vector. Like pairing socks 🧦—one from each drawer!

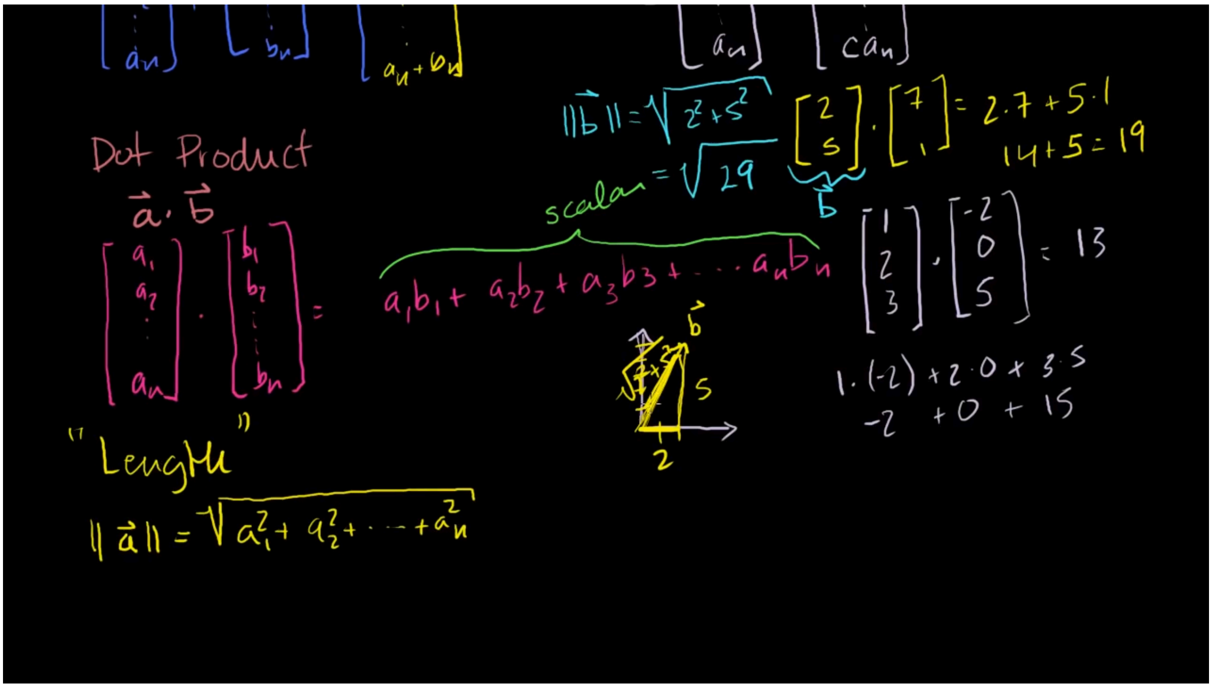
Scalar Multiplication 📏×📦

A scalar $c \in \mathbb{R}$ scales the vector:

$$c \cdot \vec{a} = \begin{bmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{bmatrix}$$

✨ Imagine stretching (or shrinking) the vector — like pulling on a rubber band.

The Dot Product 🧠🔘



Time for some real action—**multiplying two vectors**. But surprise: there are *two* kinds of multiplication. We're diving into the **dot product**!

Dot Product Definition

Given vectors

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Their dot product is:

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

⚠️ It's not another vector—it's a **scalar** (a single number)!

Examples 🧪

- $\begin{bmatrix} 2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 1 \end{bmatrix} = 2 \cdot 7 + 5 \cdot 1 = 14 + 5 = 19$
- $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix} = 1 \cdot (-2) + 2 \cdot 0 + 3 \cdot 5 = -2 + 0 + 15 = 13$

🔍 **Intuition**: It's like a compatibility score 👊🏻—the more aligned two vectors are in direction, the larger the dot product 💯.

Vector Length (a.k.a. Magnitude) 📐🧭

Let's give vectors a "size".

Definition of Length

For a vector

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Its **length** (also called **magnitude**) is:

$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

⚙️ This generalizes the **Pythagorean theorem** to any number of dimensions!

Example 🧠

Let

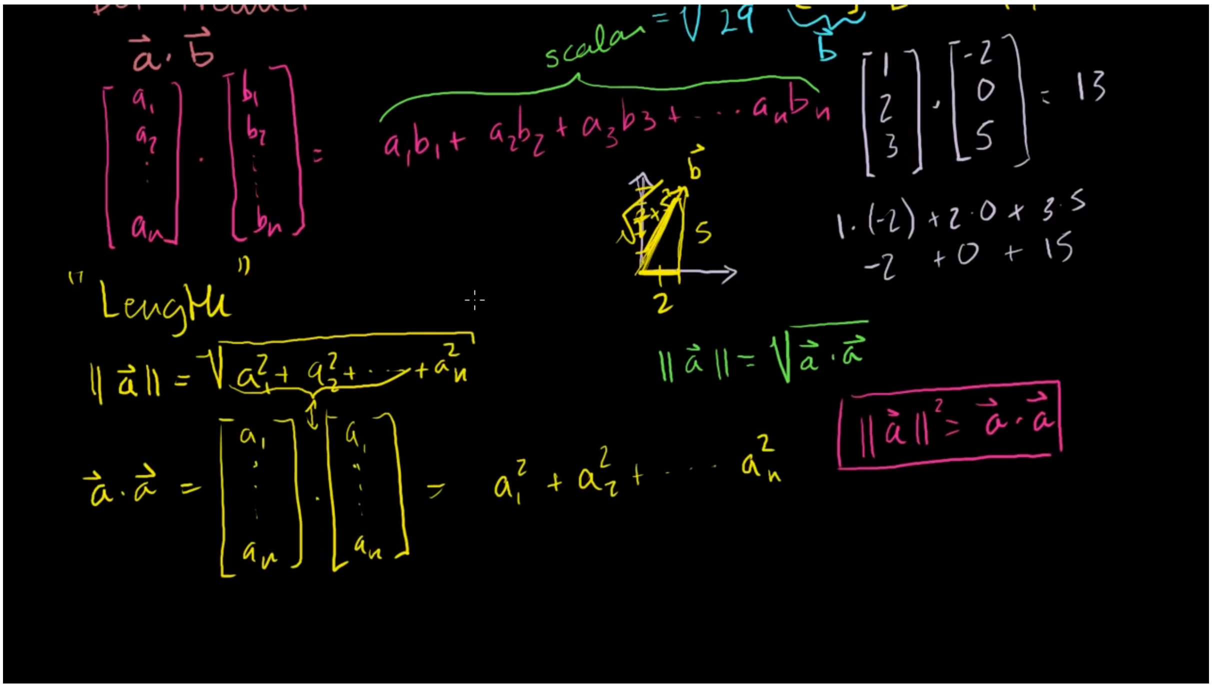
$$\vec{b} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

Then:

$$\left\|\vec{b}\right\| = \sqrt{2^2 + 5^2} = \sqrt{4 + 25} = \sqrt{29}$$

Visualized in 2D, you'd literally form a right triangle: base = 2, height = 5. 📏🧮

Dot Product & Length: The Secret Link 🤝



What happens if we **dot a vector with itself**?

$\vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + \cdots + a_n^2$

Whoa—familiar? That's exactly what's under the square root in the length formula!

Therefore

$\|\vec{a}\| = \sqrt{\vec{a} \cdot \vec{a}}$

💡 Or, equivalently:

$\|\vec{a}\|^2 = \vec{a} \cdot \vec{a}$

This duality between **geometry (length)** and **algebra (dot product)** is 🔥 powerful. It'll come in handy for projections, angles, orthogonality... and even machine learning!

Key Takeaways 🧠✨

- **Vector addition** combines component-wise: $\vec{a} + \vec{b}$.
- **Scalar multiplication** stretches/shrinks vectors: $c \cdot \vec{a}$.
- **Dot product** ($\vec{a} \cdot \vec{b}$) is a scalar, summing up pairwise component products.
- Dot product measures **alignment—**high when vectors point in similar directions.
- **Vector length** $||\vec{a}||$ is derived from dot product:
  $||\vec{a}|| = \sqrt{\vec{a} \cdot \vec{a}}$
- Works for **any dimension**, not just 2D/3D.
- 🧠 These concepts are foundational for **angles between vectors**, **projections**, **orthogonality**, and many AI/ML algorithms!