Tipos y Términos

Las expresiones de tipos (o simplemente tipos) son:

$\sigma ::= Bool \mid Nat \mid \sigma \to \sigma$

Sea X un conjunto infinito enumerable de variables y $x \in X$. Los términos están dados por:

$M ::= x \mid \lambda x : \sigma.M \mid M\ M \mid true \mid false \mid if\ M\ then\ M\ else\ M \mid zero \mid succ(M) \mid pred(M) \mid isZero(M)$

Conjunto de Valores:

$V ::= true \mid false \mid \lambda x : \sigma.M \mid zero \mid succ(V)$

Axiomas y Reglas de Tipado

$$\frac{}{\Gamma \vdash \text{true} : \text{Bool}}\ ax_{\text{true}} \qquad \frac{}{\Gamma \vdash \text{false} : \text{Bool}}\ ax_{\text{false}} \qquad \frac{}{\Gamma \vdash \text{zero} : \text{Nat}}\ ax_{\text{zero}}$$

$$\frac{}{\Gamma, x : \sigma \vdash x : \sigma}\ ax_v \qquad \frac{\Gamma \vdash M : \text{Nat}}{\Gamma \vdash \text{succ}(M) : \text{Nat}}\ \text{succ} \qquad \frac{\Gamma \vdash M : \text{Nat}}{\Gamma \vdash \text{pred}(M) : \text{Nat}}\ \text{pred}$$

$$\frac{\Gamma \vdash M : \text{Bool} \quad \Gamma \vdash P : \sigma \quad \Gamma \vdash Q : \sigma}{\Gamma \vdash \text{if } M \text{ then } P \text{ else } Q : \sigma} \qquad \frac{\Gamma \vdash M : \text{Nat}}{\Gamma \vdash \text{isZero}(M) : \text{Bool}}\ \text{isZero}$$

$$\frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x : \sigma.M : \sigma \to \tau}\ \to_i \qquad \frac{\Gamma \vdash M : \sigma \to \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash M\ N : \tau}\ \to_e$$

Semántica operacional Small Step (CBV)

Evaluación un paso

Si $M_1 \to M_1'$, entonces $M_1\ M_2 \to M_1'\ M_2$ (appl o μ)

Si $M_2 \to M_2'$, entonces $V\ M_2 \to V\ M_2'$ (appr o ν)

* en CBN la regla ν no se aplica *

$(\lambda x : \sigma.M)V \to M\{x := V\}$ (β)

if true then M_2 else M3 → M2(if-t)

if false then M_2 else M3 → M3(if-f)

Si $M_1 \to M_1'$, entonces

if M_1 then M_2 else M_3 → if M_1' then M_2 else M_3 (if-c)

Nats

pred(succ(n)) → n (pred)

Opcional*: pred(zero) → zero (pred-0)

isZero(zero) → true (isZero-0)

isZero(succ(n)) → false (isZero-n)

Si M → N, entonces succ(M) → succ(N) (succ-c)

Si M → N, entonces pred(M) → pred(N) (pred-c)

Si M → N, entonces isZero(M) → isZero(N) (isZero-c)

Extensión μ

$$M ::= \ldots \mid \mu x : \tau.M$$

$$\frac{\Gamma, x : \sigma \vdash M : \sigma}{\Gamma \vdash \mu x : \sigma.M : \sigma}\ \mu$$

$$\mu x : \sigma.M \to M\{x := \mu x : \sigma.M\} \quad \text{(fix)}$$

Extensiones vistas

Pares

Reglas de Tipado

$$\frac{\Gamma \vdash M : \sigma \qquad \Gamma \vdash N : \tau}{\Gamma \vdash \langle M, N \rangle : \sigma \times \tau} \texttt{T-PAR}$$

$$\frac{\Gamma \vdash M : \sigma \times \tau}{\Gamma \vdash \pi_1(M) : \sigma} \texttt{T-}\pi_1 \qquad \frac{\Gamma \vdash M : \tau \times \sigma}{\Gamma \vdash \pi_2(M) : \sigma} \texttt{T-}\pi_2$$

Valores y Reglas Semánticas

$$V ::= \ldots | \langle V, V \rangle$$

$$\frac{M \to M'}{\langle M, N \rangle \to \langle M', N \rangle} \texttt{E-PAR1} \qquad \frac{N \to N'}{\langle V, N \rangle \to \langle V, N' \rangle} \texttt{E-PAR2}$$

$$\frac{M \to M'}{\pi_1(M) \to \pi_1(M')} \texttt{E-}\pi_1 \qquad \frac{M \to M'}{\pi_2(M) \to \pi_2(M')} \texttt{E-}\pi_2$$

$$\frac{}{\pi_1(\langle V_1, V_2 \rangle) \to V_1} \texttt{E-}\pi_1(V) \qquad \frac{}{\pi_2(\langle V_1, V_2 \rangle) \to V_2} \texttt{E-}\pi_2(V)$$

Uniones Disjuntas

Reglas de Tipado

$$\frac{\Gamma \vdash M : \sigma}{\Gamma \vdash \mathsf{left}(M) : \sigma + \tau} \texttt{T-LEFT} \qquad \frac{\Gamma \vdash M : \tau}{\Gamma \vdash \mathsf{right}(M) : \sigma + \tau} \texttt{T-RIGHT}$$

$$\frac{\Gamma \vdash M : \sigma + \tau \qquad \Gamma\ x : \sigma \vdash M_1 : \rho \qquad \Gamma\ y : \tau \vdash M_2 : \rho}{\Gamma \vdash \mathsf{case}\ M\ \mathsf{of}\ \mathsf{left}(x) \hookrightarrow M_1 \parallel \mathsf{right}(y) \hookrightarrow M_2 : \rho} \texttt{T-CASE}$$

Valores y Reglas Semánticas

$$V ::= \ldots \mid \mathsf{left}(V) \mid \mathsf{right}(V)$$

$$\frac{M \to M'}{\mathsf{left}(M) \to \mathsf{left}(M')}\ \texttt{E-LEFT} \qquad \frac{M \to M'}{\mathsf{right}(M) \to \mathsf{right}(M')}\ \texttt{E-RIGHT}$$

$$\frac{M \to M'}{\mathsf{case}\ M\ \mathsf{of}\ \mathsf{left}(x) \hookrightarrow M_1 \parallel \mathsf{right}(y) \hookrightarrow M_2 \to \mathsf{case}\ M'\ \mathsf{of}\ \mathsf{left}(x) \hookrightarrow M_1 \parallel \mathsf{right}(y) \hookrightarrow M_2}\ \texttt{E-CASE}$$

$$\frac{}{\mathsf{case}\ \mathsf{left}(V)\ \mathsf{of}\ \mathsf{left}(x) \hookrightarrow M_1 \parallel \mathsf{right}(y) \hookrightarrow M_2 \to M_1\{x \leftarrow V\}}\ \texttt{E-CASE-L}$$

$$\frac{}{\mathsf{case}\ \mathsf{right}(V)\ \mathsf{of}\ \mathsf{left}(x) \hookrightarrow M_1 \parallel \mathsf{right}(y) \hookrightarrow M_2 \to M_2\{y \leftarrow V\}}\ \texttt{E-CASE-R}$$

Extensión con Árboles Binarios y Map

M ::= …, Nil_σ, AB (M, M, M), map (M, M) V :: = …, Nil_σ, AB (M, M, M)

σ ::= … , ABl_σ

$$\frac{}{\Gamma \vdash \text{Nil}_\sigma : \text{AB}_\sigma}\ (\text{T-Nil}) \qquad \frac{\Gamma \vdash M : \sigma \to \tau \quad \Gamma \vdash N : \text{AB}_\sigma}{\Gamma \vdash \mathit{map}(M, N) : \text{AB}_\tau}\ (\text{T-Map})$$

$$\frac{\Gamma \vdash M : \text{AB}_\sigma \quad \Gamma \vdash N : \sigma \quad \Gamma \vdash O : \text{AB}_\sigma}{\Gamma \vdash \mathit{Bin}(M, N, O) : \text{AB}_\sigma}\ (\text{T-Bin})$$

Semántica del Map

$$\frac{M \to M'}{\mathit{map}(M, N) \to \mathit{map}(M', N)}\ (\text{E-Map1}) \qquad \frac{N \to N'}{\mathit{map}(V, N) \to \mathit{map}(V, N')}\ (\text{E-Map2})$$

$$\frac{\vdash V : \sigma \to \tau}{\mathit{map}(V, \text{Nil}_\sigma) \to \text{Nil}_\tau}\ (\text{E-MapNil})$$

$$\frac{}{\mathit{map}(V, \text{Bin}(U, X, W)) \to \text{Bin}(\mathit{map}(V, U), V\ X, \mathit{map}(V, W))}\ (\text{E-MapBin})$$

Intérprete con estrategia Call By Name (CBN)

$$\frac{\Gamma' \vdash M \hookrightarrow V}{\Gamma, x = \langle M, \Gamma' \rangle, \Delta \vdash x \hookrightarrow V} \; x \notin \mathsf{D}(\Delta)$$

$$\frac{}{\Gamma \vdash \lambda x.M \hookrightarrow \langle x, M, \Gamma \rangle} \qquad \frac{\Gamma \vdash M \hookrightarrow \langle x, M', \Gamma' \rangle \quad \Gamma', x = \langle N, \Gamma \rangle \vdash M' \hookrightarrow V}{\Gamma \vdash MN \hookrightarrow V}$$

$$\frac{}{\Gamma \vdash \mathsf{True} \hookrightarrow \mathsf{True}} \qquad \frac{}{\Gamma \vdash \mathsf{False} \hookrightarrow \mathsf{False}}$$

$$\frac{\Gamma \vdash M \hookrightarrow \mathsf{True} \quad \Gamma \vdash N_1 \hookrightarrow V}{\Gamma \vdash \mathsf{if}\ M\ \mathsf{then}\ N_1\ \mathsf{else}\ N_2 \hookrightarrow V} \qquad \frac{\Gamma \vdash M \hookrightarrow \mathsf{False} \quad \Gamma \vdash N_2 \hookrightarrow V}{\Gamma \vdash \mathsf{if}\ M\ \mathsf{then}\ N_1\ \mathsf{else}\ N_2 \hookrightarrow V}$$

$$\frac{\Gamma, x = \langle \mu x.M, \Gamma \rangle \vdash M \hookrightarrow V}{\Gamma \vdash \mu x.M \hookrightarrow V}$$

Intérprete con estrategia Call By Value (CBV)

$$\frac{}{\Gamma, x = V, \Delta \vdash x \hookrightarrow V} \; x \notin \mathsf{D}(\Delta) \qquad \frac{\Gamma' \vdash \mu y.M \to V}{\Gamma, x = \langle \mu y.M, \Gamma' \rangle, \Delta \vdash x \hookrightarrow V} \; x \notin \mathsf{D}(\Delta)$$

$$\frac{}{\Gamma \vdash \lambda x.M \hookrightarrow \langle x, M, \Gamma \rangle} \qquad \frac{\Gamma \vdash N \hookrightarrow W \quad \Gamma \vdash M \hookrightarrow \langle x, M', \Gamma' \rangle \quad \Gamma', x = W \vdash M' \hookrightarrow V}{\Gamma \vdash MN \hookrightarrow V}$$

$$\frac{}{\Gamma \vdash \mathsf{True} \hookrightarrow \mathsf{True}} \qquad \frac{}{\Gamma \vdash \mathsf{False} \hookrightarrow \mathsf{False}}$$

$$\frac{\Gamma \vdash M \hookrightarrow \mathsf{True} \quad \Gamma \vdash N_1 \hookrightarrow V}{\Gamma \vdash \mathsf{if}\ M\ \mathsf{then}\ N_1\ \mathsf{else}\ N_2 \hookrightarrow V} \qquad \frac{\Gamma \vdash M \hookrightarrow \mathsf{False} \quad \Gamma \vdash N_2 \hookrightarrow V}{\Gamma \vdash \mathsf{if}\ M\ \mathsf{then}\ N_1\ \mathsf{else}\ N_2 \hookrightarrow V}$$

$$\frac{\Gamma, x = \langle \mu x.M, \Gamma \rangle \vdash M \hookrightarrow V}{\Gamma \vdash \mu x.M \hookrightarrow V}$$

Extensión Nats (CBN y CBV)

$$\frac{}{\Gamma \vdash \mathsf{zero} \hookrightarrow \mathsf{zero}} \qquad \frac{\Gamma \vdash \mathsf{M} \hookrightarrow \mathsf{zero}}{\Gamma \vdash \mathsf{pred(M)} \hookrightarrow \mathsf{zero}} \qquad \frac{\Gamma \vdash \mathsf{M} \hookrightarrow \mathsf{zero}}{\Gamma \vdash \mathsf{isZero(M)} \hookrightarrow \mathsf{True}}$$

$$\frac{\Gamma \vdash \mathsf{M} \hookrightarrow \mathsf{V}}{\Gamma \vdash \mathsf{succ(M)} \hookrightarrow \mathsf{succ(V)}} \qquad \frac{\Gamma \vdash \mathsf{M} \hookrightarrow \mathsf{succ(V)}}{\Gamma \vdash \mathsf{pred(M)} \hookrightarrow \mathsf{V}} \qquad \frac{\Gamma \vdash \mathsf{M} \hookrightarrow \mathsf{succ(V)}}{\Gamma \vdash \mathsf{isZero(M)} \hookrightarrow \mathsf{False}}$$

Semántica Denotacional

$$[\![x]\!]_v = v(x)$$

$$[\![\lambda x{:}\tau.M]\!]_v = V^{[\![\tau]\!]} \mapsto [\![M]\!]_{v,x=V}$$

$$[\![MN]\!]_v = [\![M]\!]_v [\![N]\!]_v$$

$$[\![\mathsf{True}]\!]_v = \mathsf{true}$$

$$[\![\mathsf{False}]\!]_v = \mathsf{false}$$

$$[\![0]\!]_v = 0$$

$$[\![\mathsf{succ}(M)]\!]_v = [\![M]\!]_v + 1$$

$$[\![\mathsf{if}\ M\ \mathsf{then}\ N\ \mathsf{else}\ O]\!]_v = \begin{cases} [\![N]\!]_v & \text{si } [\![M]\!]_v = \mathsf{true} \\ [\![O]\!]_v & \text{si } [\![M]\!]_v = \mathsf{false} \\ \bot & \text{si } [\![M]\!]_v = \bot \end{cases}$$

$$[\![\mathsf{pred}(M)]\!]_v = \begin{cases} 0 & \text{si } [\![M]\!]_v = 0 \\ [\![M]\!]_v - 1 & \text{si no} \end{cases}$$

$$[\![\mathsf{isZero}(M)]\!]_v = \begin{cases} \mathsf{true} & \text{si } [\![M]\!]_v = 0 \\ \mathsf{false} & \text{si } [\![M]\!]_v = n \neq 0 \\ \bot & \text{si } [\![M]\!]_v = \bot \end{cases}$$

$$[\![\mu x{:}\tau.M]\!]_v = \mathsf{FIX}(V^{[\![\tau]\!]} \mapsto [\![M]\!]_{v,x=V})$$

donde FIX(f) es el mínimo punto fijo de f.

Inferencia de Tipos
Machete

Paradigmas (de Lenguajes) de Programación

1. Algoritmo de inferencia

- $\mathbb{W}(x) \leadsto \{x : ?k\} \vdash x : ?k, \quad ?k$ incógnita fresca
- $\mathbb{W}(0) \leadsto \emptyset \vdash 0 : Nat$
- $\mathbb{W}(true) \leadsto \emptyset \vdash true : Bool$
- $\mathbb{W}(false) \leadsto \emptyset \vdash false : Bool$
- $\mathbb{W}(succ(U)) \leadsto S(\Gamma) \vdash S(succ(M)) : Nat$ donde
  - $\mathbb{W}(U) = \Gamma \vdash M : \tau$
  - $S = MGU\{\tau \stackrel{?}{=} Nat\}$
- $\mathbb{W}(pred(U)) \leadsto S(\Gamma) \vdash S(pred(M)) : Nat$ donde
  - $\mathbb{W}(U) = \Gamma \vdash M : \tau$
  - $S = MGU\{\tau \stackrel{?}{=} Nat\}$
- $\mathbb{W}(iszero(U)) \leadsto S(\Gamma) \vdash S(iszero(M)) : Bool$ donde
  - $\mathbb{W}(U) = \Gamma \vdash M : \tau$
  - $S = MGU\{\tau \stackrel{?}{=} Nat\}$
- $\mathbb{W}(if\ U\ then\ V\ else\ W) \leadsto S(\Gamma_1) \cup S(\Gamma_2) \cup S(\Gamma_3) \vdash S(if\ M\ then\ P\ else\ Q) : S(\sigma)$ donde
  - $\mathbb{W}(U) = \Gamma_1 \vdash M : \rho$
  - $\mathbb{W}(V) = \Gamma_2 \vdash P : \sigma$
  - $\mathbb{W}(W) = \Gamma_3 \vdash Q : \tau$
  - $S = MGU\{\sigma \stackrel{?}{=} \tau, \rho \stackrel{?}{=} Bool\} \cup \{\sigma_1 \stackrel{?}{=} \sigma_2 \mid x : \sigma_1 \in \Gamma_i, x : \sigma_2 \in \Gamma_j, i, j \in \{1,2,3\}\}$
- $\mathbb{W}(\lambda x.U) \leadsto \Gamma' \vdash \lambda x : \tau'.M : \tau' \to \rho$ donde
  - $\mathbb{W}(U) = \Gamma \vdash M : \rho$
  - $\tau' = \begin{cases} \alpha \text{ si } x : \alpha \in \Gamma \\ ?k \text{ con } ?k \text{ variable fresca en otro caso} \end{cases}$
  - $\Gamma' = \Gamma \ominus \{x\}$
- $\mathbb{W}(U\,V) \leadsto S(\Gamma_1) \cup S(\Gamma_2) \vdash S(M\,N) : S(?k)$ donde
  - $\mathbb{W}(U) = \Gamma_1 \vdash M : \tau$
  - $\mathbb{W}(V) = \Gamma_2 \vdash N : \rho$
  - $?k$ variable fresca
  - $S = MGU\{\tau \stackrel{?}{=} \rho \to ?k\} \cup \{\sigma_1 \stackrel{?}{=} \sigma_2 \mid x : \sigma_1 \in \Gamma_1, x : \sigma_2 \in \Gamma_2\}$

2. Algoritmo de unificación (Martelli-Montanari)

2.1. Reglas

Se enuncian las reglas para constructores de tipo C en general de cualquier aridad, y en particular para los constructores de tipo de λ^b

$$\sigma, \tau ::= Nat \mid Bool \mid \sigma \to \tau$$

1. **Descomposición**

   $\{\sigma_1 \to \sigma_2 \stackrel{?}{=} \tau_1 \to \tau_2\} \cup G \mapsto \{\sigma_1 \stackrel{?}{=} \tau_1, \sigma_2 \stackrel{?}{=} \tau_2\} \cup G$

   $\{Bool \stackrel{?}{=} Bool\} \cup G \mapsto G$

   $\{Nat \stackrel{?}{=} Nat\} \cup G \mapsto G$

   Caso general

   $\{C(\sigma_1, \ldots, \sigma_n) \stackrel{?}{=} C(\tau_1, \ldots, \tau_n)\} \cup G \mapsto \{\sigma_1 \stackrel{?}{=} \tau_1, \ldots, \sigma_n \stackrel{?}{=} \tau_n\} \cup G$

2. **Eliminación de par trivial**

   $\{?k \stackrel{?}{=} ?k\} \cup G \mapsto G$

3. **Swap**: si σ no es una variable

   $\{\sigma \stackrel{?}{=} ?k\} \cup G \mapsto \{?k \stackrel{?}{=} \sigma\} \cup G$

4. **Eliminación de variable**: si $?k \notin FV(\sigma)$

   $\{?k \stackrel{?}{=} \sigma\} \cup G \mapsto_{\{?k:=\sigma\}} G\{?k := \sigma\}$

5. **Colisión**

   $\{\sigma \stackrel{?}{=} \tau\} \cup G \mapsto$ **falla**, con $(\sigma, \tau) \in T \cup T^{-1}$ donde

   $T = \{(Bool, Nat), (Nat, \sigma_1 \to \sigma_2), (Bool, \sigma_1 \to \sigma_2)\}$ y T^{-1} representa invertir cada par

   Caso general: si $C \neq C'$ son constructores de tipo diferentes

   $\{C(\ldots) \stackrel{?}{=} C'(\ldots)\} \cup G \mapsto$ **falla**

6. **Occur check**: si $?k \neq \sigma$ y $?k \in FV(\sigma)$

   $\{?k \stackrel{?}{=} \sigma\} \cup G \mapsto$ **falla**

2.2. Ejemplos

2.2.1. Secuencia exitosa

$$\begin{array}{ll}
 & \{(Nat \to ?1) \to (?1 \to ?3) \stackrel{?}{=} ?2 \to (?4 \to ?4) \to ?2\} \\
\mapsto^1 & \{Nat \to ?1 \stackrel{?}{=} ?2, ?1 \to ?3 \stackrel{?}{=} (?4 \to ?4) \to ?2\} \\
\mapsto^3 & \{?2 \stackrel{?}{=} Nat \to ?1, ?1 \to ?3 \stackrel{?}{=} (?4 \to ?4) \to ?2\} \\
\mapsto^4_{\{?2:=Nat\to ?1\}} & \{?1 \to ?3 \stackrel{?}{=} (?4 \to ?4) \to (Nat \to ?1)\} \\
\mapsto^1 & \{?1 \stackrel{?}{=} ?4 \to ?4, ?3 \stackrel{?}{=} Nat \to ?1\} \\
\mapsto^4_{\{?1:=?4\to ?4\}} & \{?3 \stackrel{?}{=} Nat \to (?4 \to ?4)\} \\
\mapsto^4_{\{?3:=Nat\to(?4\to ?4)\}} & \emptyset
\end{array}$$

El MGU es

$$\begin{aligned}
&\{?3 := Nat \to (?4 \to ?4)\} \circ \{?1 := ?4 \to ?4\} \circ \{?2 := Nat \to ?1\} \\
= &\{?2 := Nat \to (?4 \to ?4),\ ?1 := ?4 \to ?4,\ ?3 := Nat \to (?4 \to ?4)\}
\end{aligned}$$

2.2.2. Secuencia fallida

$$
\begin{array}{ll}
 & \{?1 \to (?2 \to ?1) \stackrel{?}{=} ?2 \to ((?1 \to Nat) \to ?1)\} \\
\mapsto^{1} & \{?1 \stackrel{?}{=} ?2, ?2 \to ?1 \stackrel{?}{=} (?1 \to Nat) \to ?1\} \\
\mapsto^{4}_{\{?1:=?2\}} & \{?2 \to ?2 \stackrel{?}{=} (?2 \to Nat) \to ?2\} \\
\mapsto^{1} & \{?2 \stackrel{?}{=} ?2 \to Nat, ?2 \stackrel{?}{=} ?2\} \\
\mapsto^{6} & \textbf{falla}
\end{array}
$$

2.2.3. Constructores en general

Se usan los constructores de tipos de listas,

$$\sigma ::= \ldots \mid [\sigma]$$

$$
\begin{array}{ll}
 & \{(?3 \to ?4 \to ?4) \to ?4 \to [?3] \to ?4 \stackrel{?}{=} ((?1 \to ?2) \to [?1] \to [?2]) \to ?5\} \\
\mapsto^{1} & \{?3 \to ?4 \to ?4 \stackrel{?}{=} (?1 \to ?2) \to [?1] \to [?2], ?4 \to [?3] \to ?4 \stackrel{?}{=} ?5\} \\
\mapsto^{3} & \{?3 \to ?4 \to ?4 \stackrel{?}{=} (?1 \to ?2) \to [?1] \to [?2],\ ?5 \stackrel{?}{=} ?4 \to [?3] \to ?4\} \\
\mapsto^{4}_{\{?5:=?4 \to [?3] \to ?4\}} & \{?3 \to ?4 \to ?4 \stackrel{?}{=} (?1 \to ?2) \to [?1] \to [?2]\} \\
\mapsto^{1} & \{?3 \stackrel{?}{=} ?1 \to ?2,\ ?4 \to ?4 \stackrel{?}{=} [?1] \to [?2]\} \\
\mapsto^{4}_{\{?3:=?1 \to ?2\}} & \{?4 \to ?4 \stackrel{?}{=} [?1] \to [?2]\} \\
\mapsto^{1} & \{?4 \stackrel{?}{=} [?1],\ ?4 \stackrel{?}{=} [?2]\} \\
\mapsto^{4}_{\{?4:=[?1]\}} & \{[?1] \stackrel{?}{=} [?2]\} \\
\mapsto^{1} & \{?1 \stackrel{?}{=} ?2\} \\
\mapsto^{4}_{\{?1:=?2\}} & \emptyset
\end{array}
$$

El MGU es

$$
\begin{array}{c}
\{?1 := ?2\} \circ \{?4 := [?1]\} \circ \{?3 := ?1 \to ?2\} \circ \{?5 := ?4 \to [?3] \to ?4\} \\
= \{?5 := ?[?2] \to [?2 \to ?2] \to ?[?2],\ ?3 := ?2 \to ?2,\ ?4 := [?2],\ ?1 := ?2\}
\end{array}
$$

$$\frac{}{\Gamma, \tau \vdash \tau}\ \mathsf{ax}$$

$$\frac{\Gamma \vdash \tau \quad \Gamma \vdash \sigma}{\Gamma \vdash \tau \wedge \sigma}\ \wedge_i \qquad \frac{\Gamma \vdash \tau \wedge \sigma}{\Gamma \vdash \tau}\ \wedge_{e_1} \qquad \frac{\Gamma \vdash \tau \wedge \sigma}{\Gamma \vdash \sigma}\ \wedge_{e_2}$$

$$\frac{\Gamma, \tau \vdash \sigma}{\Gamma \vdash \tau \Rightarrow \sigma}\ \Rightarrow_i \qquad \frac{\Gamma \vdash \tau \Rightarrow \sigma \quad \Gamma \vdash \tau}{\Gamma \vdash \sigma}\ \Rightarrow_e$$

$$\frac{\Gamma \vdash \tau}{\Gamma \vdash \tau \vee \sigma}\ \vee_{i_1} \qquad \frac{\Gamma \vdash \sigma}{\Gamma \vdash \tau \vee \sigma}\ \vee_{i_2} \qquad \frac{\Gamma \vdash \tau \vee \sigma \quad \Gamma, \tau \vdash \rho \quad \Gamma, \sigma \vdash \rho}{\Gamma \vdash \rho}\ \vee_e$$

$$\frac{\Gamma, \tau \vdash \bot}{\Gamma \vdash \neg\tau}\ \neg_i \qquad \frac{\Gamma \vdash \tau \quad \Gamma \vdash \neg\tau}{\Gamma \vdash \bot}\ \neg_e$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \tau}\ \bot_e$$

Lógica intuicionista

$$\frac{\Gamma \vdash \neg\neg\tau}{\Gamma \vdash \tau}\ \neg\neg_e$$

Lógica clásica

Reglas intuicionistas

$$\frac{\Gamma \vdash \tau}{\Gamma \vdash \neg\neg\tau}\ \neg\neg_i \qquad \frac{\Gamma \vdash \tau \Rightarrow \sigma \quad \Gamma \vdash \neg\sigma}{\Gamma \vdash \neg\tau}\ \mathsf{MT}$$

Reglas clásicas

$$\frac{\Gamma, \neg\tau \vdash \bot}{\Gamma \vdash \tau}\ \mathsf{PBC} \qquad \frac{}{\Gamma \vdash \tau \vee \neg\tau}\ \mathsf{LEM}$$

Extra

Extensión μ Algoritmo de Inferencia

Si $W(U) = \Gamma \vdash M : \sigma$ entonces $W(\mu x.\ U) \rightsquigarrow$

$S(\Gamma) \ominus \{x\} \vdash S(\mu x{:}\sigma.\ M) : S(\sigma)$

con $S = mgu(\{ \sigma \stackrel{?}{=} \tau \})$

y con $\tau = \begin{cases} \Gamma(x) & \text{si } x \in \Gamma \\ ?k & \text{incógnita fresca si no} \end{cases}$

Extensión Denotacional Pares

$[\![\langle M, N\rangle]\!]_v = ([\![M]\!]_v , [\![N]\!]_v)$

$[\![\pi_1 (M)]\!]_v = ([\![M]\!]_v)_1$

$[\![\pi_2 (M)]\!]_v = ([\![M]\!]_v)_1$

Formulas Validas Deducción Natural:

Modus ponens relativizado: $(\rho \Rightarrow \sigma \Rightarrow \tau) \Rightarrow (\rho \Rightarrow \sigma) \Rightarrow \rho \Rightarrow \tau$

Reducción al absurdo: $(\rho \Rightarrow \bot) \Rightarrow \neg\rho$

Introducción de la doble negación: $\rho \Rightarrow \neg\neg\rho$

Eliminación de la triple negación: $\neg\neg\neg\rho \Rightarrow \neg\rho$

Contraposición: $(\rho \Rightarrow \sigma) \Rightarrow (\neg\sigma \Rightarrow \neg\rho)$

Adjunción: $((\rho \wedge \sigma) \Rightarrow \tau) \Leftrightarrow (\rho \Rightarrow \sigma \Rightarrow \tau)$

de Morgan (I): $\neg(\rho \vee \sigma) \Leftrightarrow (\neg\rho \wedge \neg\sigma)$

de Morgan (II): $\neg(\rho \wedge \sigma) \Leftrightarrow (\neg\rho \vee \neg\sigma)$

Conmutatividad ($\wedge$): $(\rho \wedge \sigma) \Rightarrow (\sigma \wedge \rho)$

Asociatividad ($\wedge$): $((\rho \wedge \sigma) \wedge \tau) \Leftrightarrow (\rho \wedge (\sigma \wedge \tau))$

Conmutatividad ($\vee$): $(\rho \vee \sigma) \Rightarrow (\sigma \vee \rho)$

Asociatividad ($\vee$): $((\rho \vee \sigma) \vee \tau) \Leftrightarrow (\rho \vee (\sigma \vee \tau))$

Absurdo clásico: $(\neg\tau \Rightarrow \bot) \Rightarrow \tau$

Ley de Peirce: $((\tau \Rightarrow \rho) \Rightarrow \tau) \Rightarrow \tau$

Tercero excluido: $\tau \vee \neg\tau$

Consecuencia milagrosa: $(\neg\tau \Rightarrow \tau) \Rightarrow \tau$

Contraposición clásica: $(\neg\rho \Rightarrow \neg\tau) \Rightarrow (\tau \Rightarrow \rho)$

Análisis de casos: $(\tau \Rightarrow \rho) \Rightarrow (\neg\tau \Rightarrow \rho) \Rightarrow \rho$

Implicación vs. disyunción: $(\tau \Rightarrow \rho) \Leftrightarrow (\neg\tau \vee \rho)$