GFFT on the projective line

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Requires $n | p - 1$, where $p = |\mathbb{F}|$.
Can we do something when $n | (p + 1)$ instead??

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Can we find a set of size $p + 1$ *with a similar cyclical* σ*?*

The Projective line and fractional transformations

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Define: $\sigma(-b/a) = \infty, \sigma(\infty) = 0$.

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Define: $\boldsymbol{\sigma}(-\mathbf{b}/\mathbf{a}) = \infty, \boldsymbol{\sigma}(\infty) = \mathbf{0}$.

claim: For the right choice of $\mathbf{a}, \mathbf{b}$ $\boldsymbol{\sigma}$ makes a cycle over all of $\mathbb{P}$!

How do people formally represent the projective line?

Projective coordinates: Represent $a \in F$ by (c, d) with $a = c/d$ e.g. $(a, 1)$.

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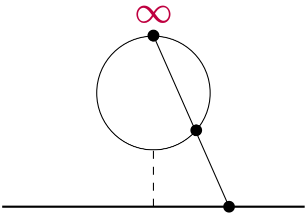
As a circle in the plane:



See Circle STARK paper [HLP24] for this approach

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Valuation rings in K, are also called "places" of K.

The infinity point in the algebraic representation

There is *one more* place of degree one in $\mathbf{K}$:
$\mathbf{R}_\infty = \{\mathbf{f}(\mathbf{X})/\mathbf{g}(\mathbf{X}) | \mathbf{deg}(\mathbf{f}) \leq \mathbf{deg}(\mathbf{g})\}$.

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$\mathbf{R}_\infty =$ "the set of functions that can be evaluated at infinity"

Regular FFT via GFFT

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How do these operations look in the framework of places?

Applying $\boldsymbol{\tau}$

1. Define $\boldsymbol{\tau}$ as operation on $\mathbf{K}$:
   $\boldsymbol{\tau}(\mathbf{r}(\mathbf{X})) := \mathbf{r}(-\mathbf{X})$.

Applying τ

1. Define τ as operation on K:
   $\tau(r(X)) := r(-X)$.
2. Apply τ on place R_a element-wise:
   $\tau(R_a) := \{\tau(r)\}_{r \in R_a}$
   $= \left\{ \frac{f(-X)}{g(-X)} | g(a) \neq 0 \right\} = R_{-a}$

Galois subfields

Look at the *subfield* of $\mathbf{K}$ fixed by $\boldsymbol{\tau}$ - $\{\mathbf{r}|\boldsymbol{\tau}(\mathbf{r}) = \mathbf{r}\}$.

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This is exactly $\mathbb{F}(\mathbf{X}^2)$.
e.g. $\boldsymbol{\tau}(\mathbf{X}^2) = (-\mathbf{X})^2 = \mathbf{X}^2$.

Places splitting in extensions

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For $r \in R'_b$

$r = f(X^2)/g(X^2)$ with $g(b) \neq 0$.
So $r(X) = f'(X)/g'(X)$ with $g'(a), g'(-a) \neq 0$.

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Implies $R'_b \subset R_a$ and $R'_b \subset R_{-a}$.

Places splitting in extensions

Let $\mathbf{b} = \mathbf{a}^2$. Look at place $\mathbf{R}'_{\mathbf{b}}$ of $\mathbb{F}(\mathbf{X}^2)$.

For $\mathbf{r} \in \mathbf{R}'_{\mathbf{b}}$

$\mathbf{r} = \mathbf{f}(\mathbf{X}^2)/\mathbf{g}(\mathbf{X}^2)$ with $\mathbf{g}(\mathbf{b}) \neq \mathbf{0}$.
So $\mathbf{r}(\mathbf{X}) = \mathbf{f}'(\mathbf{X})/\mathbf{g}'(\mathbf{X})$ with $\mathbf{g}'(\mathbf{a}), \mathbf{g}'(-\mathbf{a}) \neq \mathbf{0}$.

Implies $\mathbf{R}'_{\mathbf{b}} \subset \mathbf{R}_{\mathbf{a}}$ and $\mathbf{R}'_{\mathbf{b}} \subset \mathbf{R}_{-\mathbf{a}}$.

We say $\mathbf{R}'_{\mathbf{b}}$ *splits* in $\mathbf{K}$ into $\mathbf{R}_{\mathbf{a}}$ and $\mathbf{R}_{-\mathbf{a}}$.

For more details see:

FAST FOURIER TRANSFORM VIA AUTOMORPHISM GROUPS OF RATIONAL FUNCTION FIELDS

SONGSONG LI AND CHAOPING XING

ABSTRACT. The Fast Fourier Transform (FFT) over a finite field $\mathbb{F}_q$ computes evaluations of a given polynomial of degree less than n at a specifically chosen set of n distinct evaluation points in $\mathbb{F}_q$. If q or $q-1$ is a smooth number, then the divide-and-conquer approach leads to the fastest known FFT algorithms. Depending on the type of group that the set of evaluation points forms, these algorithms are classified as multiplicative (Math of Comp. 1965) and additive (FOCS 2014) FFT algorithms. In this work, we provide a unified framework for FFT algorithms that include both multiplicative and additive FFT algorithms as special cases, and beyond: our framework also works when $q+1$ is smooth, while all known results require q or $q-1$ to be smooth. For the new case where $q+1$ is smooth (this new case was not considered before in literature as far as we know), we show that if n is a divisor of $q+1$ that is B-smooth for a real $B>0$, then our FFT needs $O(Bn\log n)$ arithmetic operations in $\mathbb{F}_q$. Our unified framework is a natural consequence of introducing the algebraic function fields into the study of FFT.

1. INTRODUCTION