The GKR method

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Overview

- Mutlilinear functions and sumcheck basics
- GKR - motivation and example.

Multilinear polynomials

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"Multilinear Lagranges": $L_x(Y) = eq(x, Y)$ for some $x \in \{0, 1\}^n$.

We have $L_x(x) = 1$ and $L_x(y) = 0$ for any $y \neq x$ in $\{0, 1\}^n$.

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Reduction doesn't require $\mathcal{P}$ to do FFT's or commit to other polynomials

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- Define $f'(X) := eq(\beta, X) f(X)$.

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- $\mathcal{V}$ chooses random $\boldsymbol{\beta} \in \mathbb{F}$.
- Define $\mathbf{f}'(\mathbf{X}) := \boldsymbol{eq}(\boldsymbol{\beta}, \mathbf{X})\mathbf{f}(\mathbf{X})$.
- $\mathcal{P}$ shows using sumcheck protocol that $\sum_{\boldsymbol{x} \in \{\mathbf{0},\mathbf{1}\}^{\boldsymbol{n}}} \mathbf{f}'(\boldsymbol{x}) = \mathbf{0}$. This implies desired claim on $\mathbf{f}$ w.h.p.

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State of the art: Basefold, Binius, Brakedown, Gemini,Zeromorph,...

Zero Testing - typical example

$\mathcal{P}$ has multilinears $\mathbf{f_1, f_2, f_3}$. $\mathcal{V}$ has $\mathbf{cm}(\mathbf{f_1}), \mathbf{cm}(\mathbf{f_2}), \mathbf{cm}(\mathbf{f_3})$.

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$\mathcal{P}$ wants to prove to $\mathcal{V}$ that

$$\forall x \in \{\mathbf{0}, \mathbf{1}\}^n : \mathbf{f_1}(x)\mathbf{f_2}(x) - \mathbf{f_3}(x) = \mathbf{0}.$$

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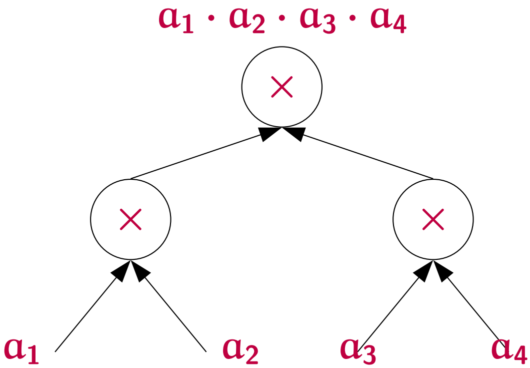
Committing to polynomials is expensive. Can we use sumcheck for polynomials we **don't** *have a commitment to?*

GKR idea - iterative sumcheck

When we don't have a commitment to the polynomial we're summing, reduce the random evaluation at the end to *another* sumcheck over a different polynomial

$$\textbf{sum} \xrightarrow{\text{sumcheck}} \textbf{randeval} \xrightarrow{\text{reduction}} \textbf{sum} \xrightarrow{\text{sumcheck}} \ldots$$

Example from [Thaler13]



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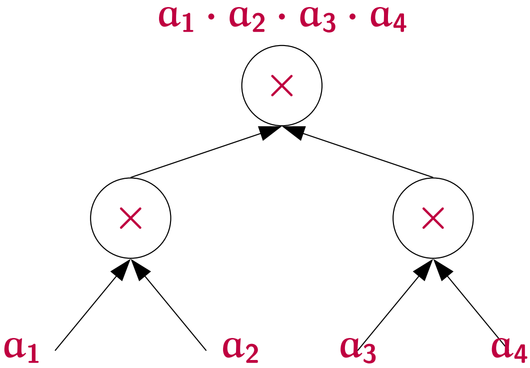

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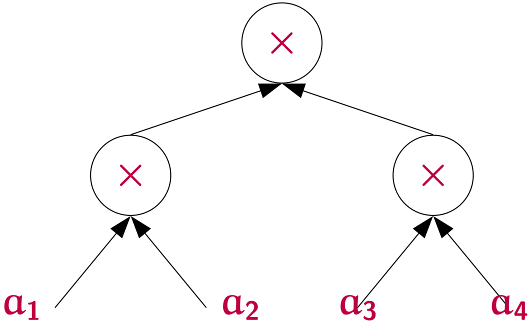
$$a_1 \cdot a_2 \cdot a_3 \cdot a_4$$



$\mathcal{P}$ has $f(Y_1, Y_2)$. $\mathcal{V}$ has $\mathbf{cm}(f)$

Wants to prove to $\mathcal{V}$ correctness of
$u := f(0,0) \cdot f(0,1) \cdot f(1,0) \cdot f(1,1)$.

Define multilinear "Intermediate layer function" $\mathbf{g}$:

$\mathbf{g}(\mathbf{0}) := \mathbf{f}(\mathbf{0}, \mathbf{0}) \cdot \mathbf{f}(\mathbf{0}, \mathbf{1})$

$\mathbf{g}(\mathbf{1}) := \mathbf{f}(\mathbf{1}, \mathbf{0}) \cdot \mathbf{f}(\mathbf{1}, \mathbf{1})$.

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Main goal: Avoid needing to compute $\mathbf{cm}(\mathbf{g})$ as “traditional SNARKs” would do!

Interlude: Representing mutlilinear functions via $\mathbf{eq}$

Recall

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Recall

$$eq(x, y) = \prod_{i \in [n]} (x_i y_i + (1 - x_i)(1 - y_i))$$

Claim: When h is multilinear, we have for any r

$$h(r) = \sum_{x \in \{0,1\}^n} eq(r, x) h(x)$$

Heart of GKR - representing $\mathbf{g}(\mathbf{r})$ as sum over $\mathbf{f}$

$$\mathbf{g}(\mathbf{r}) = \mathbf{eq}(\mathbf{r}, \mathbf{0})\mathbf{f}(\mathbf{0}, \mathbf{0})\mathbf{f}(\mathbf{0}, \mathbf{1}) + \mathbf{eq}(\mathbf{r}, \mathbf{1})\mathbf{f}(\mathbf{1}, \mathbf{0})\mathbf{f}(\mathbf{1}, \mathbf{1}$$

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$$g(r) = eq(r, 0)f(0, 0)f(0, 1) + eq(r, 1)f(1, 0)f(1, 1$$

$$= \sum_{x \in \{0,1\}} f'(x)$$

where $f'(X) := eq(r, X)f(X, 0)f(X, 1)$.

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$$= \sum_{x \in \{0,1\}} \mathbf{f}'(x)$$

where $\mathbf{f}'(\mathbf{X}) := \mathbf{eq(r,X)f(X,0)f(X,1)}$.

Thus, using SCP can reduce evaluating $\mathbf{g(r)}$ to evaluating $\mathbf{f}'(\mathbf{r_2})$ for a random $\mathbf{r_2} \in \mathbb{F}$.

Evaluating $\mathbf{f'(r_2)}$

$$\mathbf{f'(r_2) = eq(r, r_2) f(r_2, 0) f(r_2, 1)}$$

$\mathcal{V}$ can evaluate $\mathbf{eq(r, r_2)}$ itself.

Since it has $\mathbf{cm(f)}$ it can ask $\mathcal{P}$ for $\mathbf{f(r_2, 0), f(r_2, 1)}$ with proofs of correctness.