The GKR method

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Overview

- Mutlilinear functions and sumcheck basics
- GKR - motivation and example.

Multilinear polynomials

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"Multilinear Lagranges": $L_x(Y) = eq(x, Y)$ for some $x \in \{0, 1\}^n$.

We have $L_x(x) = 1$ and $L_x(y) = 0$ for any $y \neq x$ in $\{0, 1\}^n$.

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Reduction doesn't require $\mathcal{P}$ to do FFT's or commit to other polynomials

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- $\mathcal{V}$ chooses random $\beta \in \mathbb{F}^n$.
- Define $f'(X) := eq(\beta, X) f(X)$.
- $\mathcal{P}$ shows using sumcheck protocol that $\sum_{x \in \{0,1\}^n} f'(x) = 0$. This implies desired claim on f w.h.p.

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State of the art: Basefold, Binius, Brakedown, Gemini,Zeromorph,...

Zero Testing - typical example

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$\mathcal{P}$ wants to prove to $\mathcal{V}$ that

$$\forall x \in \{\mathbf{0}, \mathbf{1}\}^n : \mathbf{f_1}(x)\mathbf{f_2}(x) - \mathbf{f_3}(x) = \mathbf{0}.$$

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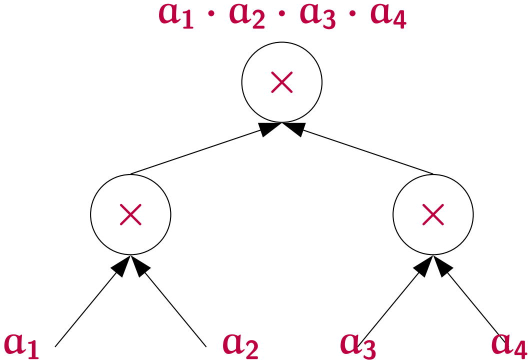
Committing to polynomials is expensive. Can we use sumcheck for polynomials we **don't** *have a commitment to?*

GKR idea - iterative sumcheck

When we don't have a commitment to the polynomial we're summing, reduce the random evaluation at the end to *another* sumcheck over a different polynomial

$$\textbf{sum} \xrightarrow{\text{sumcheck}} \textbf{randeval} \xrightarrow{\text{reduction}} \textbf{sum} \xrightarrow{\text{sumcheck}} \dots$$

Example from [Thaler13]



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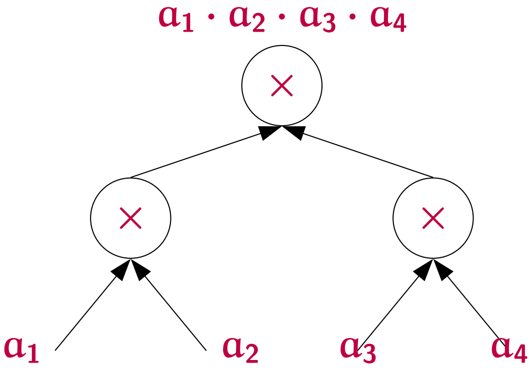

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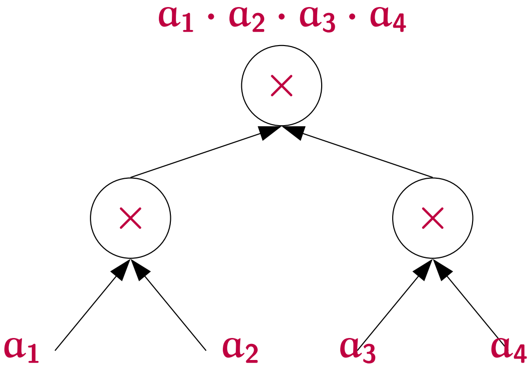

$\mathcal{P}$ has $f(Y_1, Y_2)$. $\mathcal{V}$ has $\mathbf{cm}(f)$

Wants to prove to $\mathcal{V}$ correctness of
$u := f(0,0) \cdot f(0,1) \cdot f(1,0) \cdot f(1,1)$.

Define multilinear "Intermediate layer function" $\mathbf{g}$:

$\mathbf{g}(\mathbf{0}) := \mathbf{f}(\mathbf{0}, \mathbf{0}) \cdot \mathbf{f}(\mathbf{0}, \mathbf{1})$

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Exercise: Can reduce this to evaluating $\mathbf{g}(\mathbf{r})$ for one random $\mathbf{r} \in \mathbb{F}$.

Main goal: Avoid needing to compute $\mathbf{cm}(\mathbf{g})$ as "traditional SNARKs" would do!

Interlude: Representing mutlilinear functions via $\mathbf{eq}$

Recall

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Interlude: Representing mutlilinear functions via eq

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$$eq(x, y) = \prod_{i \in [n]} (x_i y_i + (1 - x_i)(1 - y_i))$$

Claim: When h is multilinear, we have for any r

$$h(r) = \sum_{x \in \{0,1\}^n} eq(r, x) h(x)$$

Heart of GKR - representing $g(r)$ as sum over f

$$g(r) = eq(r,0)f(0,0)f(0,1) + eq(r,1)f(1,0)f(1,1$$

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$$= \sum_{x \in \{\mathbf{0},\mathbf{1}\}} \mathbf{f}'(x)$$

where $\mathbf{f}'(\mathbf{X}) := \mathbf{eq}(\mathbf{r},\mathbf{X})\mathbf{f}(\mathbf{X},\mathbf{0})\mathbf{f}(\mathbf{X},\mathbf{1})$.

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Thus, using SCP can reduce evaluating $\mathbf{g}(\mathbf{r})$ to evaluating $\mathbf{f}'(\mathbf{r_2})$ for a random $\mathbf{r_2} \in \mathbb{F}$.

Evaluating $\mathbf{f'(r_2)}$

$$\mathbf{f'(r_2) = eq(r, r_2) f(r_2, 0) f(r_2, 1)}$$

$\mathcal{V}$ can evaluate $\mathbf{eq(r, r_2)}$ itself.

Since it has $\mathbf{cm(f)}$ it can ask $\mathcal{P}$ for $\mathbf{f(r_2, 0), f(r_2, 1)}$ with proofs of correctness.