

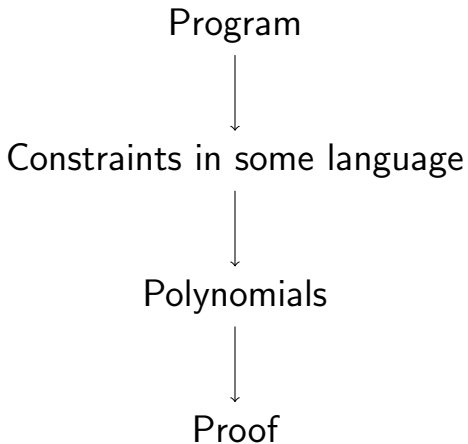
Ranged Polynomial Protocols

Ariel Gabizon

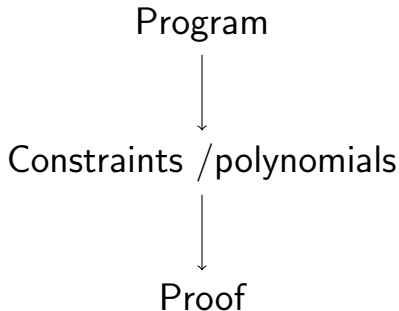
Aztec

(Based on work with Zachary J. Williamson)

“traditional” approach (QAP/r1cs/..)



Recently..¹ (similar in spirit to [..,BCGGHJ17,Arya,..]):



¹<https://ethresear.ch/t/using-polynomial-commitments-to-replace-state-roots/7095>,plookup

Ranged polynomials protocols

Preprocessing/inputs: Predefined polynomials

$$g_1, \dots, g_t \in \mathbb{F}_{<d}[\mathbf{X}]$$

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2. At end, \mathcal{V} asks \mathbf{I} if some identity holds between $\{f_1, \dots, f_\ell, g_1, \dots, g_t\}$ **on** \mathbf{H} .

$\mathbf{D} :=$ max degree of identity \mathbf{C} checked in exec with honest \mathcal{P} .

$$\mathfrak{d}(\mathbf{P}) := \left(\sum_{i \in [t]} \deg(\mathbf{f}_i) + 1 \right) + \mathbf{D} - |\mathbf{H}|.$$

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Thm:² Can compile to “real” protocol in Algebraic Group Model, where prover complexity $\mathfrak{d}(\mathbf{P})$.

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proof sketch: Use [KZG] polynomial commitment scheme. \mathcal{P} commits to all polys and \mathbf{C}/\mathbf{Z}_H . \mathcal{V} checks identity at random challenge point.

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Multiset equality check

Given $\mathbf{a}, \mathbf{b} \in \mathbb{F}^3$, want to check

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$$(\mathbf{a}_1 + \gamma)(\mathbf{a}_2 + \gamma)(\mathbf{a}_3 + \gamma) \stackrel{?}{=} (\mathbf{b}_1 + \gamma)(\mathbf{b}_2 + \gamma)(\mathbf{b}_3 + \gamma)$$

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Multiset equality check - polynomial version

Given $\mathbf{f}, \mathbf{g} \in \mathbb{F}_{<d}[\mathbf{X}]$, want to check $\{\mathbf{f}(\mathbf{x})\}_{\mathbf{x} \in \mathbf{H}} \stackrel{?}{=} \{\mathbf{g}(\mathbf{x})\}_{\mathbf{x} \in \mathbf{H}}$ as multisets

Multiplicative subgroups:

$$\mathbf{H} = \{ \alpha, \alpha^2, \dots, \alpha^n = 1 \}.$$

L_i is i 'th lagrange poly of \mathbf{H} :

$$L_i(\alpha^i) = 1, L_i(\alpha^j) = 0, j \neq i$$

Reduces to:

$$\mathbf{H} = \{\alpha, \alpha^2, \dots, \alpha^n\}.$$

\mathcal{P} has sent $\mathbf{f}, \mathbf{g} \in \mathbb{F}_{\langle n \rangle}[\mathbf{X}]$.

Wants to prove:

$$\prod_{i \in [n]} \mathbf{f}(\alpha^i) = \prod_{i \in [n]} \mathbf{g}(\alpha^i)$$

Checking products with \mathbf{H} -ranged protocols [GWC19]

1. \mathcal{P} computes \mathbf{Z} with
 $\mathbf{Z}(\alpha) = 1, \mathbf{Z}(\alpha^i) = \prod_{j \prec i} \mathbf{f}(\alpha^j) / \mathbf{g}(\alpha^j).$
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3. \mathcal{V} checks following identities on \mathbf{H} .
 - 3.1 $\mathbf{L}_1(\mathbf{X})(\mathbf{Z}(\mathbf{X}) - 1) = 0$
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We get $\mathfrak{d}(\mathbf{P}) = \mathfrak{n} + 2\mathfrak{n} - |\mathbf{H}| = 2\mathfrak{n}.$

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(most?) common SNARK operation

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with $\{s(x)\}_{x \in H} = \{f(x)\}_{x \in H}$, $s(\alpha^i) < s(\alpha^{i+1})$.

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