GFFT on projective line

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FFT Reminder

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Requires $n | p - 1$, where $p = |\mathbb{F}|$.
Can we do something when $n | (p + 1)$ instead??

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The Projective line and fractional transformations

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claim: For the right choice of $\mathbf{a, b}$ $\boldsymbol{\sigma}$ makes a cycle over all of $\mathbb{P}$!

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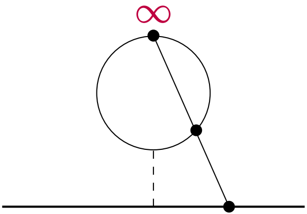
As a circle in the plane:



See Circle STARK paper [HLP24] for this approach

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Valuation rings in K, are also called "places" of K.

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Cool thing: $\mathbf{R}_\alpha / \mathbf{I}_\alpha = \mathbb{F}$*. And we can evaluate*
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This gives the same result as “normal” evaluation!

The infinity point in the algebraic representation

There is *one more* place of degree one in $\mathbf{K}$:
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$\mathbf{R}_\infty =$ "the set of functions that can be evaluated at infinity"

What does this last part have to do with FFT?

Like in regular FFT - we'll end up needing to represent $\mathbf{f(X)}$ as a combination of two functions $\mathbf{f_e(N(X)), f_o(N(X))}$ of half the "degree"; where $\mathbf{N}$ will be a degree two rational function.

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Working within the function field gives us convenient tools to construct the right bases for representing $\mathbf{f, f_e, f_o}$, and defining degree in the right way.