$\mathfrak{cq}$:* Cached quotients for fast lookups

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Abstract

We present a protocol for checking the values of a committed polynomial $f(X) \in \mathbb{F}_{<N}[X]$ over a multiplicative subgroup $\mathbb{H} \subset \mathbb{F}$ of size n are contained in a table $\mathsf{t} \in \mathbb{F}^N$. After an $O(N \log N)$ preprocessing step, the prover algorithm runs in time $O(n \log n)$. Thus, we continue to improve upon the recent breakthrough sequence of results[ZBK+22, PK22, ?, ?] starting from Caulk [ZBK+22], which achieve sublinear complexity in the table size N. The two most recent works in this seqence [?, ?] achieved prover complexity $O(n \cdot \log^2 n)$.

Moreover, $\mathfrak{cq}$ has the following attractive features.

1. As in [ZBK+22, PK22, ?] our construction relies on homomorphic table commitments, which makes them amenable to vector lookups in the manner described in Section 4 of [GW20].
2. As opposed to [ZBK+22, PK22, ?, ?] the $\mathfrak{cq}$ verifier doesn't involve pairings with prover defined $\mathbb{G}_2$ points, which makes recursive aggregation of proofs more convenient.
3. The construction can be altered to a version we call $\mathfrak{cq}^*$ that loses the mentioned aggregatability, increase preprocessing time to $O(n \cdot N)$, and in return reduce prover complexity to a *linear* number of field and group operations!

1 Introduction

The *lookup problem* is fundamental to the efficiency of modern zk-SNARKs. Somewhat informally, it asks for a protocol to prove the values of a committed polynomial $\phi(X) \in \mathbb{F}_{<n}[X]$ are contained in a table T of size N of predefined legal values. When the table T corresponds to an operation without an efficient low-degree arithmetization in $\mathbb{F}$, such a protocol produces significant savings in proof construction time for programs containing the operation. Building on previous work of [BCG+18], **plookup** [GW20] was the first to explicitly describe a solution to this problem in the polynomial-IOP context.

*Pronounced "seek you".

plookup described a protocol with prover complexity quasilinear in both n and N. This left the intriguing question of whether the dependence on N could be made *sublinear* after performing a preprocessing step for the table T. Caulk [ZBK+22] answered this question in the affirmative by leveraging bi-linear pairings, achieving a run time of $O(n^2 + n \log N)$. Caulk+ [PK22] improved this to $O(n^2)$ getting rid of the dependence on table size completely.

However, the quadratic dependence on n of these works makes them impractical for a circuit with many lookup gates. We resolve this issue by giving a protocol called cq that is quasi-linear in n and has no dependence on N after the preprocessing step.

1.1 Comparison of results

Table 1: Scheme comparison. n = witness size, N = Table size, "Aggregatable"= All prover defined pairing arguments in $\mathbb{G}_1$

Scheme	Preprocessing	Proof size	Prover Work	Verifier Work	Homomorphic?	Aggregatable?
Caulk [ZBK+22]	O($N \log N$) $\mathbb{F}$,$\mathbb{G}_1$	14 $\mathbb{G}_1$, 1 $\mathbb{G}_2$, 4 $\mathbb{F}$	$3n + m - \ell$ $\mathbb{G}_1$ exp, n $\mathbb{G}_2$ exp	2 $\mathbb{G}_1$, 1 $\mathbb{G}_2$	✓	✗
Caulk+ [PK22]	$O(N \log N)$ $\mathbb{F}$,$\mathbb{G}_1$	7 $\mathbb{G}_1$, 1 $\mathbb{G}_2$, 2 $\mathbb{F}$	$18n$ $\mathbb{G}_1$ exp	4 $\mathbb{G}_1$, 2 $\mathbb{F}$	✓	✗
flookup [?]	$O(N \log^2 N)$ $\mathbb{F}$,$\mathbb{G}_1$	6 $\mathbb{G}_1$, 1 $\mathbb{G}_2$, 4 $\mathbb{F}$	$273n$ $\mathbb{G}_1$ exp	20 $\mathbb{G}_1$, 16 $\mathbb{F}$	✗	✗
ba*loo* [?]	$O(N \log N)$ $\mathbb{F}$,$\mathbb{G}_1$	12 $\mathbb{G}_1$, 1 $\mathbb{G}_2$, 4 $\mathbb{F}$	$8n$ $\mathbb{G}_1$ exp	6 $\mathbb{G}_1$, 4 $\mathbb{F}$	✓	✗
cq	$O(N \log N)$ $\mathbb{F}$,$\mathbb{G}_1$	8 $\mathbb{G}_1$, 4 $\mathbb{F}$	$11n + 11a$ $\mathbb{G}_1$ exp , $\approx 54(n+a)\log(n+a)$ $\mathbb{F}$ mul	7 $\mathbb{G}_1$, 6 $\mathbb{F}$	✓	✓
cq*	$O(N \cdot n)$ $\mathbb{F}$,$\mathbb{G}_1$	$n + a$ $\mathbb{G}_1$, 1 $\mathbb{G}_2$	$9n + 9a$ $\mathbb{G}_1$ exp , $\approx 54(n+a)\log(n+a)$ $\mathbb{F}$ mul	9 $\mathbb{G}_1$, 6 $\mathbb{F}$	✓	✗

Table with relative proof size, prover ops, verifier ops proof-size caulk caulk+ flookup baloo 12 $\mathbb{G}_1$, 1 $\mathbb{G}_2$, 4 $\mathbb{F}$ this work 6 G1, 1 G2

1.2 Technical Overview

The innovation of Caulk While [ZBK+22, PK22, ?, ?] use preprocessing and pairings to extract a subtable of witness size;

Our approach here we use preprocessing and pairings more directly to run an existing lookup protocol - mvlookup, in time independent from table size -logarithmic derivative method Let's review this protocol: It relies on the following lemma from [?] that says that $f|_{\mathbb{H}} \in \mathsf{t}$ if and only if for some $m \in \mathbb{F}^N$

$$\sum_{i \in [N]} \frac{m_i}{X + t_i} = \sum_{i \in [n]} \frac{1}{X + f_i}$$

Roughly, the protocol of [?] checks this identity on a random β, by sending polynomials A and B that agree on $\mathbb{V}$ with the rational function values of the LHS and RHS respectively. Given commitments to A, B we can check the equality holds via various sumcheck techniques, e.g. that descirbed in [BCR+19]. The RHS is not a problem because it is a sum of size n. Interpolating A, and computing its commitment is actually not a problem either, because the number of non-zero values is at most n. So if we precompute the commitments to the Lagrange base of $\mathbb{V}$ we're fine.

The main challenge, and innovation, is to convince the verifier $\mathbf{V}$ that A is correctly formed.

This protocol is amenable, because polynomials involved have sparsity depending on witness - For large table problem is computing A that agrees with $m/(\mathsf{t}+\beta)$ on $\mathbb{V}$

- Need way to compute A

2 Preliminaries

2.1 Notation:

$\mathbb{H}$- small space $\mathbb{V}$- big space Lagrange bases for big and small space

Lemma 2.1. *Assume the Q-DLOG for* $(\mathbb{G}_1, \mathbb{G}_2)$. *Given an algebraic adversary $\mathcal{A}$ participating in a protocol with a degree Q SRS, the probability of any real pairing check passing is larger by at most an additive* $\mathsf{negl}(\lambda)$ *factor than the probability the corresponding ideal check holds.*

AGM - real and ideal pairing checks, agm - real and ideal pairing KZG

2.2 log derivative method

Lemma from mvlookup

Lemma 2.2. *Given $f \in \mathbb{F}^n$, and $t \in \mathbb{F}^N$, we have $f \subset t$ as sets if and only if for some $m \in \mathbb{F}^N$ the following identity of rational functions holds*

$$\sum_{i\in[n]} \frac{1}{X+f_i} = \sum_{i\in[N]} \frac{m_i}{X+t_i}.$$

3 Cached quotients

Notation: In this section and the next we use the following conventions. $\mathbb{V} \subset \mathbb{F}$ denotes a mutliplicative subgroup of order N which is a power of two. We denote by $\mathbf{g}$ a generator of $\mathbb{V}$. Hence, $\mathbb{V} = \left\{\mathbf{g}, \mathbf{g}^2, \ldots, \mathbf{g}^N = 1\right\}$. Given $P \in \mathbb{F}[X]$ and integer $i \in [N]$, we denote $P_i := P(\mathbf{g}^i)$. For $i \in [N]$, we denote by $L_i \in \mathbb{F}_{<N}[X]$ the i'th Lagrange polynomial of $\mathbb{V}$. Thus, $(L_i)_i = 1$ and $(L_i)_j = 0$ for $i \neq j \in [N]$.

For a polynomial $A(X) \in \mathbb{F}_{<N}[X]$, we say it is *$n$-sparse* if $A_i \neq 0$ for at most n values $i \in [N]$. The *sparse representation* of such A consists of the (at most) n pairs (i, A_i) such that $A_i \neq 0$. We denote $\mathrm{supp}(A) := \{i \in [N] | A_i \neq 0\}$.

The main result of this section is a method to compute a commitment to a quotient polynomial - derived from a product with a preprocessed polynomial; in a number of operations depending only on the sparsity of the other polynomial in the product.

The result crucially relies on the following lemma derived from a result of Feist and Khovratovich[FK].

Lemma 3.1. *Fix $T \in \mathbb{F}_{<N}[X]$, and a subgroup $\mathbb{V} \subset \mathbb{F}$ of size N. There is an algorithm that given the $\mathbb{G}_1$ elements $\left\{\left[x^i\right]_1\right\}_{i\in\{0,\ldots,N\}}$ computes for $i \in [N]$, the elements $q_i := [Q_i(x)]_1$ where $Q_i(X) \in \mathbb{F}[X]$ is such that*

$$L_i(X) \cdot T(X) = T_i \cdot L_i(X) + Z_{\mathbb{V}}(X) \cdot Q_i(X)$$

in $O(N \cdot \log N)$ $\mathbb{G}_1$ operations.

Proof. Recall the definition of the Lagrange polynomial

$$L_i(X) = \frac{Z_{\mathbb{V}}(X)}{Z'_{\mathbb{V}}(\mathbf{g}^i)(X - \mathbf{g}^i)}.$$

Substituting this definition, we can write the quotient $Q_i(X)$ as

$$Q_i(X) = \frac{T(X) - T_i}{Z'_{\mathbb{V}}(\mathbf{g}^i)(X - \mathbf{g}^i)} = Z'_{\mathbb{V}}(\omega^i)^{-1} K_i(X),$$

for $K_i(X) := \frac{T(X)-T_i}{X-\mathbf{g}^i}$. Note that the values $\{[K_i(X)]_1\}_{i\in[N]}$ are exactly the KZG opening proofs of $T(X)$ at the elements of $\mathbb{V}$. Thus, the algorithm of Feist and Khovratovich [FK, Tom] can be used to compute commitments to all the proofs $[K_i(X)]_1$ in $O(N \log N)$ $\mathbb{G}_1$-operations. This works by writing the vector of $[K_i(X)]_1$ as a the product of a matrix with the vector of $\left[X^i\right]_1$. This matrix is a DFT matrix times a Toeplitz matrix, both of which have algorithms for evaluating matrix vector products in $O(N \log N)$ operations. Thus, all the KZG proofs can be computed in $O(N \log N)$ field operations and operations in $\mathbb{G}_1$.

Finally, the algorithm just needs to scale each $[K_i(X)]_1$ by $Z'_{\mathbb{V}}(\omega^i)$ to compute $[Q_i(X)]_1$. Conveniently, these values admit a very simple description when $Z_{\mathbb{V}}(X) = X^N - 1$ is a group of roots of unity.

$$Z'_{\mathbb{V}}(X)^{-1} = (NX^{N-1})^{-1} \equiv X/N \bmod Z_{\mathbb{V}}(X)$$

In total, the prover computes the coefficients of $T(X)$ in $O(N \log N)$ field operations, computes the KZG proofs for $T(\omega^i) = t_i$ in $O(N \log N)$ group operations, and then scales these proofs by ω^i/n in $O(N)$ group operations. In total, this takes $O(N \log N)$ field and group operations in $\mathbb{G}_1$. □

We're now ready to state the main theorem of this section.

Theorem 3.2. *Fix integer parameters $0 \le n \le N$ such that n, N are powers of two. Fix $T \in \mathbb{F}_{<N}[X]$, and a subgroup $\mathbb{V} \subset \mathbb{F}$ of size N. Let $\mathsf{srs} = \left\{\left[x^i\right]_1\right\}_{i\in[0,\ldots,N]}$ for some $x \in \mathbb{F}$. There is an algorithm $\mathscr{A}$ that after a preprocessing step of $O(N \log N)$ $\mathbb{F}$- and $\mathbb{G}_1$-operations starting with srs does the following.*

Given input $A(X) \in \mathbb{F}_{<N}[X]$ that is n-sparse and given in sparse representation, $\mathscr{A}$ computes in $O(n)$ $\mathbb{F}$-operations and n $\mathbb{G}_1$-operations the element $\mathsf{cm} = [Q(x)]_1$ *where $Q \in \mathbb{F}_{<N}[X]$ is such that*

$$A(X) \cdot T(X) = Q(X) \cdot Z_{\mathbb{V}}(X) + R(X),$$

for $R(X) \in \mathbb{F}_{<N}[X]$.

Proof. The preprocessing step consists of computing the quotient commitments $[Q_i(X)]_1$ in $O(N \log N)$ operations, as described in Lemma 3.1. As stated in the lemma, for each $i \in [N]$ we have

$$L_i(X) \cdot T(X) = T_i \cdot L_i(X) + Z_{\mathbb{V}}(X) \cdot Q_i(X).$$

By assumption, the polynomial $A(X)$ can be written as a linear combination of at most n summands in the Lagrange basis of $\mathbb{V}$.

$$A(X) = \sum_{i \in \mathrm{supp}(A)} A_i \cdot L_i(X)$$

Substituting this into the product with $T(X)$, and substituting each of the products $L_i(X)T(X)$ with the appropriate cached quotient $Q_i(X)$ we find

$$\begin{aligned} A(X)T(X) = \sum_{i \in \mathrm{supp}(A)} A_i \cdot L_i(X)T(X) &= \sum_{i \in \mathrm{supp}(A)} A_i \cdot T_i L_i(X) + A_i \cdot Z_{\mathbb{V}}(X) Q_i(X) \\ &= \sum_{i \in \mathrm{supp}(A)} A_i \cdot T_i L_i(X) + Z_{\mathbb{V}}(X) \cdot \sum_{i \in \mathrm{supp}(A)} A_i \cdot Q_i(X). \end{aligned}$$

Observing that the terms of the first sum are all of degree smaller than N, we get that

$$\begin{aligned} Q(X) &= \sum_{i \in \mathrm{supp}(A)} A_i \cdot Q_i(X) \\ R(X) &= \sum_{i \in \mathrm{supp}(A)} A_i T_i \cdot L_i(X) \end{aligned}$$

Hence, commitments to both the quotient $Q(X)$ and remainder $R(X)$ can be computed in at most n group operations as

$$\begin{aligned} [Q(X)]_1 &= \sum_{i \in \mathrm{supp}(A)} A_i \cdot [Q_i(X)]_1 \\ [R(X)]_1 &= \sum_{i \in \mathrm{supp}(A)} A_i T_i \cdot [L_i(X)]_1 \end{aligned}$$

□

4 Main protocol

Definition 4.1. *$\mathcal{R}$ is all pairs (cm, f) such that cm is a commitment to f and $f|_{\mathbb{H}} \subset T$.*
..bla problem is relation is defined only after srs is chosen

4.1 Definitions

Definition 4.2. *Ad-hoc dfn of ks protocol for table lookup*
Relations dependent on srs. Tuple gen,$\mathsf{IsInTable}_{\mathbb{H}}$

- $\mathsf{gen}(\mathsf{t}, N) \to \mathsf{srs}$
- $\mathsf{IsInTable}_{\mathbb{H}}$ *a protocol between* $\mathbf{P}$ *and* $\mathbf{V}$ *where* $\mathbf{P}$ *has input* $f \in \mathbb{F}_{<n}[X]$, $\mathbf{V}$ *has* $[f(x)]_1$. *Both have* t *and* srs. *such that*
  - *Completeness:If* $f|_{\mathbb{H}} \subset \mathsf{t}$ *then* $\mathbf{V}$ *outputs* acc *with probability one.*
  - *Knowledge soundness in the algebraic group model: For any* $\mathsf{t} \in \mathbb{F}^n$*, the probability of any algebraic* $\mathcal{A}$ *to win the following game is* $\mathsf{negl}(\lambda)$
    1. *Let* $\mathsf{srs} = \mathsf{gen}(\mathsf{t}, N)$.
    2. $\mathcal{A}$ *sends a message* cm *and values* $f_1, \ldots, f_n$ *such that* $\mathsf{cm} = \sum_{i\in[n]} f_i \cdot [L_i(x)]_1$.
    3. $\mathcal{A}$ *and* $\mathbf{V}$ *engage in the protocol* $\mathsf{IsInTable}_{\mathbb{H}}(\mathsf{t}, \mathsf{cm})$ *with* $\mathcal{A}$ *taking the role of* $\mathbf{P}$.
    4. $\mathcal{A}$ *wins if*
       * $\mathbf{V}$ *outputs* acc
       * $f|_{\mathbb{H}} \not\subset \mathsf{t}$.

Main protocol: Preprocessed inputs: $[Z_{\mathbb{V}}(x)]_2$, $[T(x)]_2$ Input (cm, f).

Round 1:Committing to the multiplicites vector

1. $\mathbf{P}$ computes poly $m \in \mathbb{F}_{<N}[X]$ such that m_i = number of times t_i appears in $f|_{\mathbb{H}}$
2. $\mathbf{P}$ sends $\mathsf{m} := [m(x)]_1$.

Round 2:Interpolating the rational identity at a random β; checking the identity for A using pairings, degree checks for A, B

1. $\mathbf{V}$ chooses and sends random $\alpha, \beta \in \mathbb{F}$.
2. $\mathbf{P}$ computes $A \in \mathbb{F}_{<N}[X]$ such that for $i \in [N]$, $A_i = m_i/(\mathsf{t}_i + \beta)$.
3. $\mathbf{P}$ sends $\mathsf{a} := [A(x)]_1$.

4. **P** computes $\mathsf{q_a} := [Q_A(x)]_2$ where $Q_A \in \mathbb{F}_{<N}[X]$ is such that

$$A(X)(T(X)+\beta) - m(X) = Q_A(X) \cdot Z_{\mathbb{V}}(X)$$

5. **P** computes $B \in \mathbb{F}_{<n}[X]$ such that for $i \in [n]$, $B_i = 1/(f_i + \beta)$.
6. **P** sends $\mathsf{q_b} := [B(x)]_1$.
7. **P** computes $Q_B(X)$ such that

$$B(X)(f(x)+\beta) - 1 = Q_B(X) \cdot Z_{\mathbb{H}}(X)$$

8. **P** computes and sends the value $a_0 := A(0)$.
9. **V** sets $b_0 := (N \cdot a_0)/n$.
10. **P** computes and sends $\mathsf{p} = [P(x)]_1$ where

$$P(X) := A(X) \cdot X^{d-N} + \alpha \cdot B(X) \cdot X^{d-n}.$$

11. **V** checks that A encodes the correct values:

$$e(\mathsf{a}, [T(x)]_2 + [\beta]_2) = e(\mathsf{q_a}, [Z_{\mathbb{V}}(x)]_2) \cdot e(\mathsf{m}, [1]_2)$$

12. **V** checks that A, B have the appropriate degrees:

$$e\left(\mathsf{a}, \left[x^{d-N}\right]_2\right) \cdot e\left(\alpha \cdot \mathsf{b}, \left[x^{d-n}\right]_2\right) = e(\mathsf{p}, [1]_2).$$

Round 3: Checking the identity for B at random $\gamma \in \mathbb{F}$

1. **V** sends random $\gamma, \eta, \zeta \in \mathbb{F}$.
2. **P** sends $b_\gamma := B(\gamma), Q_{b,\gamma} := Q_B(\gamma), f_\gamma := f(\gamma)$.
3. As part of checking the correctness of B, **V** computes $Z_{\mathbb{H}}(\gamma) = \gamma^n - 1$ and

$$Q_{b,\gamma} := \frac{b_\gamma \cdot (f_\gamma + \beta) - 1}{Z_{\mathbb{H}}(\gamma)}.$$

4. To perform a batched KZG check for the correctness of the values $a_\gamma, b_\gamma, f_\gamma$
   (a) **V** sends random $\eta \in \mathbb{F}$. **P** and **V** separately compute

   $$v := b_\gamma + \eta \cdot f_\gamma + \eta^2 \cdot Q_{b,\gamma}.$$

   (b) **P** computes $\pi_\gamma := [h(x)]_1$ for

   $$h(X) := \frac{B(X) + \eta \cdot f(X) + \eta^2 \cdot Q_B(X) - v}{X - \gamma}$$

(c) **V** computes

$$\mathsf{c} := \mathsf{b} + \eta \cdot \mathsf{f} + \eta^2 \cdot \mathsf{q_b}$$

and checks that

$$e(\mathsf{c} - [v]_1 + \gamma \cdot \pi_\gamma, [1]_2) = e(\pi_\gamma, [x]_2)$$

5. To perform a batched KZG check for the correctness of the values a_0, b_0

(a) **P** and **V** separately compute

$$u := a_0 + \zeta \cdot b_0.$$

(b) **P** computes and sends $\pi_0 := [h_0(x)]_1$ for

$$h_0(X) := \frac{A(X) + \zeta \cdot B(X)}{X}$$

(c) **V** computes

$$\mathsf{c_0} := \mathsf{a} + \zeta \mathsf{b}$$

and checks that

$$e(\mathsf{c_0} - [u]_1, [1]_2) = e(\pi_0, [x]_2)$$

Stats: verifier pairings:5 - pair a with random combination of T and $\left[x^{d-N}\right]_2$, pair $\mathsf{q_a}$ with $Z_{\mathbb{V}}$. pair b with $[d-n]_2$ for degree check. Proof size - 8 $\mathbb{G}_1$- a,b,p,m,$\mathsf{q_a}$,$\mathsf{q_b}$ π_γ,π_0 4 $\mathbb{F}$- b_γ,$Q_{b,\gamma}$,f_γ,a_0

Note that we can split p to two proofs and that reduces a verifier pairing

Lemma 4.3. *The element q_A in Step 4 can be computed in $n \log n$ $\mathbb{G}_2$-operations and $O(n \log n)$ $\mathbb{F}$-operations*

Lemma 4.4. *The elements π_0, π_γ can be computed in $2 \cdot n \log n$ $\mathbb{G}_1$-operations and $O(n \log n)$ $\mathbb{F}$-operations*

Knowledge soundness proof: Let $\mathcal{A}$ be an efficient algebraic adversary participating in the Knowledge Soundness game from Definition 4.2. We show its probability of winning the game is $\mathsf{negl}(\lambda)$. Let $f \in \mathbb{F}_{<d}[X]$ be the polynomial sent by $\mathcal{A}$ in the first step of the game such that $\mathsf{cm} = [f(x)]_1$. As $\mathcal{A}$ is algebraic, when sending the commitments m,a,b,p,$\mathsf{q_a}$,$\mathsf{q_b}$,π_γ,π_0 during protocol execution it also sends polynomials $m(X), A(X), B(X), P(X), Q_A(X), Q_B(X), h(X), h_0(X) \in \mathbb{F}_{<d}[X]$ such that the former are their corresponding commitments. Let E be the event **V** outputs acc. Note that the event that $\mathcal{A}$ wins the game is contained in E. E implies all pairing checks passed. Let $E' \subset E$ be the event that one of the corresponding ideal pairing checks didn't pass. According to Lemma 2.1, $\Pr(E' = \mathsf{negl}(\lambda))$. We'll show that the event that $\mathcal{A}$ wins is contained in E'. Let $E_1 = E \setminus E'$. Given E_1 we have

$$A(X)(T(X) + \beta) - M(X) = Q_A(X) \cdot Z_{\mathbb{V}}(X)$$

Which means that for all $i \in [N]$,

$$A_i = \frac{M_i}{T_i + \beta)}$$

As $\mathcal{A}$ is algebraic we can assume Look at the following events

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