# EFM Fall 2014, Week 5: Portfolio Theory

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# Introduction

Core motivating question: How do agents allocate resources among different assets?

- In Econ classes, models generally ignore this problem to avoid complication
- Ex. New Keynesian DSGE, Neo-Classical, etc
- But portfolio theory is very relevant in that it models how agents optimize
- Can apply to: 1) Investors, 2) Firm investment decisions

# Early Beginnings and Assumptions

Wasn't formalized, but the very broad ideas of Modern Portfolio Theory (MPT) were followed

- MPT in formal literature has its beginnings in Markowitz (1952)
- A lot of assumptions go into this question, some of which may or may not be true. Examples:
- 1) All agents are rational and risk averse.
- 2) All agents are price takers.
- 3) All agents have perfect information
- Do you think these assumptions always hold?

# The Maximization Problem

Strong assumptions, but first we want to understand the simplest case. Taking the previous assumptions as true:

- We clearly want to build the best portfolio of assets
- How to do that? Maximize expected returns and minimize the variance of that portfolio.
- We'll get to math in a second, but where are we going intuitively?
- Intuitively (and from common sense): Diversification. But what is the optimal level of diversification?
- Now we need math, and the stat from last time

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# The Maximization Problem: Set Up

We can build this model two ways:

- 1) Assume something about the investor's utility function (which describes the investor's preferences over wealth)
- 2) Ignore the utility function
- Doing 1) leads to a more comprehensive analysis
- Doing 2) is a bit more intuitive, and probably has more relevance for you guys, and for the stuff we'll be doing later
- Stop me at any point if I lose you

# The Maximization Problem: Assets

We consider a market with $n$ assets, and the problem for the agent is to determine how to build his portfolio given those $n$ assets:

- $i = 1, \ldots, n$ represents a subscript denoting each asset
- $x_i \geq 0$ represents the weight given to that asset in your portfolio
- $\mu_i$ represents the mean expected return of that asset
- $\sigma_i^2$ represents the variance of the return on that asset
- $\sigma_{i,j}$ represents the covariance of the return on any two assets

# Short Digression: Covariance

- Let the return on each asset be $i$ for simplicity
- Technical formula for variance: $\sigma_i^2 = E[(i - E[i])^2]$
- Formula for covariance: $\sigma_{i,j} = E[(i - E[i])(j - E[j])]$
- Note that $E[i] = \mu_i$, or you can think of it as the mean.
- Variance tells you how much your data is spread out from the mean
- Covariance tells you if your variables move in the same direction
- Note the magnitude doesn't say too much, but we can normalize covariance to see how much they change linearly with each other (this is called the correlation coefficient)

# The Maximization Problem: Portfolio

- To maximize expected returns and minimize variance, we need to get our portfolio wide returns and variance given the assets mean returns and variances. So we have:
- $\sum_{i=1}^{n} x_i = 1$ - Asset weights must sum to 1
- $\mu = E[R] = \sum_{i=1}^{n} \mu_i x_i$ - Expected return of the portfolio must be a weighted average of the returns of each asset
- $\sigma^2 = Var[R] = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{i,j} x_i x_j$ - The variance of the portfolio must be the weighted average of the variances of each asset and how it varies with every other asset

# Full Maximization Problem

$$\min_{\{x_i\}_{i=1}^n} \sigma^2 - A\mu$$

$$\text{subject to } \sum_{i=1}^n x_i = 1$$

Expanded version:

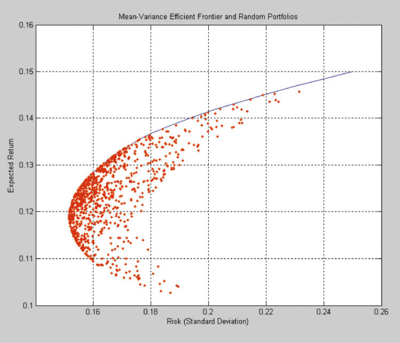
$$\min_{\{x_i\}_{i=1}^n} \sum_{i=1}^n \sum_{j=1}^n \sigma_{i,j} x_i x_j - A \sum_{i=1}^n \mu_i x_i$$

Explanation of variables:

- $A \in [0, \infty)$ = Risk Aversion Index. To see what this means: What happens if A $= 0$? What happens if A $= \infty$?
- Because of the set up of the maximization problem, the resulting portfolio is often called a Mean-Variance Efficient Portfolio.

# Maximization Solution: Graphical

- So what's the solution to the maximization problem? Something really messy
- Once you go above 3 assets, people get lazy and just represent the resulting portfolios graphically, with something like this:



# Maximization Solution: Analytical

But being at UChicago we're not really satisfied with that, so we want an analytical solution:

$$\sigma^2 = \frac{Z\mu^2 - 2X\mu + Y}{YZ - X^2}$$

$$X = \sum_1^n \sum_1^n \sigma_{i,j}^T \mu_j$$

$$Y = \sum_1^n \sum_1^n \sigma_{i,j}^T \mu_j \mu_i$$

$$Z = \sum_1^n \sum_1^n \sigma_{i,j}^T$$

So you can see that $\sigma$ is parabolic in $\mu$, and you get a hyperbola when you flip the axes. For a detailed derivation of the above, see Merton 1972. And as for where A went, I was too lazy in tracking exactly where it goes in these equations, but it's just constant so whatever.

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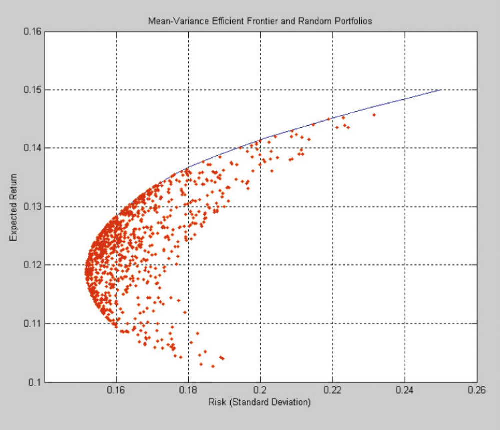
# Which Portfolio Would You Pick?

Going back to the previous graph (called the Efficient Frontier), which portfolio would you pick?:



# Maximization Solution: Graphical Interpretation

- So how do you interpret the graph? Each point is a portfolio
- Anything along the edge of the graph is an optimal portfolio to choose.
- How to see this? Consider two portfolios on the same horizontal line.
- Now consider two portfolios on the same vertical line.
- Or you could just note that clearly the top left of this graph is the best, and the bottom right is the worst.
- But without appealing to the investor's utility function, can we figure out what portfolio he would ideally choose?

# The Risk Free Asset

- If we make two simplifying assumptions, then yes we can easily find the desired portfolio:
- 1) Assume that there exists a riskless asset with a guaranteed rate of return.
- 2) Assume that the investor chooses between the riskless asset and every other risky asset.
- Is 1) a reasonable assumption?
- How about 2)?
- If you're unconvinced by 2), think of the "every other risky asset" as a portfolio of risky assets that have already been maximized over.

# Finding the New Optimal Portfolio

Our portfolio with the above conditions is:

$$\mu = (1-x)\mu_f + x\mu_r = \mu_f + x(\mu_r - \mu_f)$$
$$\sigma^2 = (1-x)^2\sigma_{f,f} + 2x(1-x)\sigma_{f,r} + x^2\sigma_{r,r} = x^2\sigma_r^2$$

Where f denotes the risk-free asset, r denotes the risky asset, and x is the share of the portfolio given to the risky asset. Now we can solve for x in the second equation and substitute:

$$\mu = \mu_f + \frac{\mu_r - \mu_f}{\sigma_r}\sigma$$

Interpretation? What happens when $\sigma = 0$? If the investor wants to maximize expected returns, what should he do?

# Solving the New Maximization Problem

Now let's maximize our portfolio for the previous problem. The two asset example from before is really trivial, so let's look at the general set up:

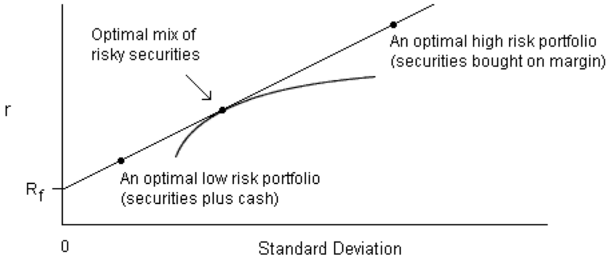
$$\max_{\{x_i\}_{i=1}^n} \frac{\mu_r - \mu_f}{\sigma_r}$$

$$\text{subject to } \sigma_r^2 = \sum_1^n \sum_1^n \sigma_{i,j} x_i x_j \text{ and } \sum_1^n x_i = 1$$

Why don't we maximize over $\mu = \mu_f + \frac{\mu_r - \mu_f}{\sigma_r}\sigma$?

# Results

Solving the problem gets you a very nice line with a very nice interpretation:




- The line is called the **Capital Allocation Line**. What does the line represent?
- The fact that you can mix portfolios to get to any other point on the line is a non-rigorous proof of the **Two Mutual Fund Theorem**.

# Slight Detour: The Sharpe Ratio

The slope of the Capital Allocation Line, which we found before was: $\frac{\mu_r - \mu_f}{\sigma_r}$ is generally known as the **Sharpe Ratio**, and is an important measure of how strong a portfolio is.

- Why? Consider what it measures. Remember that we are plotting things in the $\sigma, \mu$ space.
- The version above is the most basic one. It has been revised many times by people trying to improve it.
- Why does it need to be improved? Consider what aspects of markets aren't captured by it.
- A few major problems: What is the right risk free asset to use? Does it change over time?
- What about the distribution of returns? Does standard deviation fail to capture some crazy distributions?
- The elephant in the room: What would Sharpe ratios look like following a market readjustment (big or small?). As an extreme example, what happens to the ratio after the crash of 2008?

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# When Is This Used?

- In actuality? Never. Congratulations you learned something with no practical usage!
- Really though, the foundations of analysis and the general principles of this theory are most important.
- Analogy: Punch card computing. Useless today, but very, very important as a foundation.
- More advanced models build off of this framework, and usually make some simplifying assumptions

# Why Isn't This Used?

- But why do simplifying assumptions have to be made? This model doesn't seem particularly complex.
- Consider what pieces of information you need to know for a case of 3 assets:
- Expected Returns of the 3 assets
- Variance of the 3 assets
- Covariance of each asset with the others
- In total: $6 + \binom{3}{2} = 9$ pieces of information.
- In general: $2n + \binom{n}{2}$ pieces of information.
- For 50 assets, you need 5050 pieces of information.
- Getting covariance data in particular is really, really hard, and the models we will tackle later will attempt to solve this issue.

# Further Complications

- As you can also tell, this model is very simple. What else is missing?
- Complete vs Incomplete Markets
- Intertemporal Choice
- Heterogeneous Investors
- Stochasticity and Time-Varying Returns
- Labor Income (Hedging that risk)
- Behavioral Issues/Preference Shocks
- Equilibrium Changes and "Follow the Leader" Problems

# Next Week

To-Do

- Practical assignment for next week! You know the theory and the concepts, so apply them using Python
- If you get stuck, feel free to e-mail Jason/Allan with questions

Next Week

- We make our first stab at eliminating the need for covariances in our data and to finally sink our teeth into building a good portfolio.
- Primary topic will be the **Capital Asset Pricing Model (CAPM)**