INTRODUCTION TO THE CALCULUS OF VARIATIONS

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Abstract. Using techniques from the calculus of variations, we explore necessary and sufficient conditions for the existence of local minima of functionals. We consider historical impetus for these methods, including applications to classical problems in the development of the subject, and the influence of this discipline on the advent of functional analysis in the 20th century.

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Introduction

These notes are based on the text *Calculus of Variations* by Gelfand and Fomin [3], with historical remarks due to papers of Kreyszig and Birkhoff: "On the Calculus of Variations and Its Major Influences on the Mathematics of the First Half of Our Century," [5] and "The Establishment of Functional Analysis" [1]. This slice of mathematics is often omitted from a typical undergraduate curriculum, but is nonetheless accessible to students with a strong footing in analysis. The calculus of variations has a fascinating history that is truly inseparable from the development of mathematics in the last three and a half centuries. Newton, Leibniz, the Bernoullis, Euler, Lagrange, Jacobi, Legendre, Weierstrass, and Carathéodory are just a few of the mathematicians who played important roles in its evolution. Today, the calculus of variations and its principles lend themselves to areas such as geometry, differential equations, optimization and optimal control, and mechanics.

For a modern tour of functional analysis and its relationship with the calculus of variations, the text *Functional Analysis, Calculus of Variations and Optimal Control* by Clarke [2] is another excellent reference. The text covers all the essential topics in a first course on

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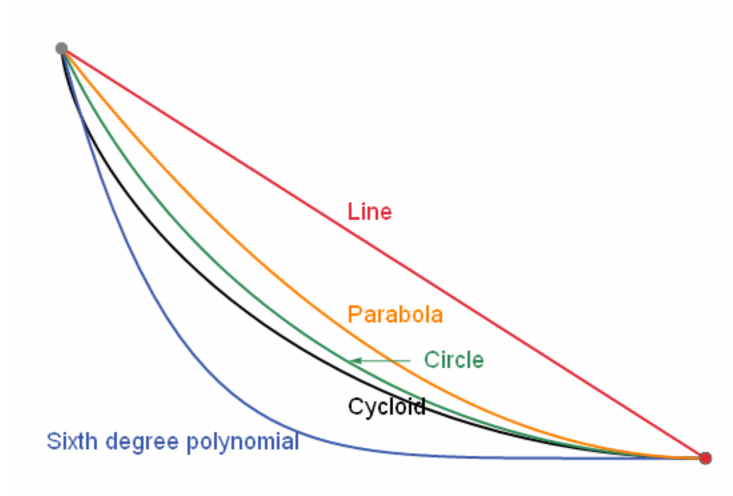

FIGURE 1. Various curves between two points [7]. The cycloid (brachistochrone) minimizes travel time under the influence of gravity.

functional analysis, with an emphasis on minimization and convexity in Banach and Hilbert spaces. From there it treats nonsmooth analysis and optimization before developing calculus of variations and optimal control. The use of functional analysis in this discipline runs (perhaps surprisingly) deep, and there is a wealth of theory and applications to explore for the motivated reader.

The First Variation and Euler's Equation

In calculus we are taught how to optimize sufficiently nice functions from $\mathbb{R}^n \to \mathbb{R}$, and there is a rich theory of optimization in finite-dimensional spaces for much more general classes of functions. As is often the case, the infinite-dimensional setting comes with many interesting problems to solve and just as many challenges to our intuition and existing techniques. The *calculus of variations* is concerned with finding extrema of functionals. The classical notion of a functional carries the meaning of a function mapping functions to real scalars. Thus the natural domains for these problems are function spaces, or more generally normed linear spaces.

Remark 0.1. Historically, the advent of the calculus of variations is attributed to the brachistochrone problem posed by Johann Bernoulli in 1696 [5]. The brachistochrone is defined to be the curve along which a particle falling between two points under the influence of gravity travels in the least amount time. See Figure 1 for reference. Solutions were delivered by Newton, Leibniz, and the Bernoulli brothers Jacob and Johann. As the story goes, Newton learned of the problem at 4pm on January 29th, 1697, and solved it by 4am [4]. It is of mild interest to note that perhaps the first nontrivial problem solved using calculus of variations predates the brachistochrone—Newton used such techniques as early as 1694 in determining the shape of a gun shell of minimal air resistance [5].

Let X be a normed linear space and $J\colon X \to \mathbb{R}$ a functional. Let $y \in X$ be fixed and suppose we perturb y slightly by adding $h \in X$ to obtain

$$\Delta J[h] := J[y+h] - J[h]$$

Remark 0.2. An important idea to keep in mind is that continuity of functionals is intrinsically related to our choice of X and its associated norm—we must be aware of what function spaces offer the right setting for solving our problem. For example, the arclength functional is continuous on $C^1[a,b]$ with the Sobolev norm, but not continuous (and sometimes not defined) on $C[a,b]$ with the usual norm.

Definition 0.3. Given $y \in X$, if there exists a bounded linear functional φ such that

$$\Delta J[h] = \varphi[h] + \varepsilon\|h\|$$

where $\varepsilon = \varepsilon(\|h\|) \to 0$ as $\|h\| \to 0$ we say that J is *differentiable* and we call φ the *variation* of $J[y]$, denoted by $\delta J[h] = \varphi[h]$.

This is the concept of a Fréchet derivative. It is unique if it exists, and satisfies many of the familiar derivative rules. Note that the variation and ΔJ depend on y but it is usually safe to omit this from our notation.

As we mentioned earlier, our goal is to analyze extrema so it is important that we define them carefully.

Definition 0.4. A functional J has a *relative extremum* at $\hat{y}$ if $J[y] - J[\hat{y}]$ maintains the same sign in a small neighbourhood of $\hat{y}$.

In what follows, we will focus on the function space $C^1[a,b]$. Note that we can endow this space with either the Sobolev norm or the supremum norm, and we can use these norms to qualify our extrema in a precise sense.

Definition 0.5. A point $\hat{y}$ is a *weak extremum* if there exists $\varepsilon > 0$ such that $J[y] - J[\hat{y}]$ maintains the same sign for all y satisfying $\|y-\hat{y}\|_1 < \varepsilon$ where $\|\cdot\|_1$ is the Sobolev norm. A *strong extremum* is characterized by the existence of $\varepsilon > 0$ such that $J[y] - J[\hat{y}]$ maintains the same sign for all y satisfying $\|y - \hat{y}\|_\infty < \varepsilon$.

Immediately we see that strong extrema are weak, but the converse need not hold. At this point we can formulate a simple necessary condition.

Theorem 0.6. *If $\hat{y}$ is an extremum of a differentiable functional J then $\delta J[h] = 0$ at $\hat{y}$ for all admissible h.*

Proof. The proof is analogous to the case of functions from $\mathbb{R} \to \mathbb{R}$. Suppose without loss of generality that $\hat{y}$ is a minimum, and for a contradiction suppose $\delta J[h_0] \neq 0$ for some admissible h_0. By definition of the variation, we have

$$\Delta J[h] = \delta J[h] + \varepsilon\|h\|$$

with $\varepsilon \to 0$ as $\|h\| \to 0$. It follows that for $\|h\|$ sufficiently small, $\Delta J[h]$ and $\delta J[h]$ have the same sign. By virtue of optimality of $\hat{y}$, for any $\alpha > 0$ sufficiently small we have $\Delta J[\pm\alpha h_0] \geq 0$. But since $\delta J[h_0] \neq 0$ we can use linearity of the variation to obtain

$$\delta J[\pm\alpha h_0] = \pm\alpha\delta J[h_0]$$

Thus $\Delta J[\pm\alpha h_0]$, which shares the sign of $\delta J[\pm\alpha h_0]$ for small α, can be made negative which contradicts our assumption. □

Remark 0.7. The word "admissible" will reappear, so we take a moment to define it. Even though our functionals are defined on linear spaces, it is often the case that the constraints of our problems are non-linear, so we refer to admissible functions in the given space as those which satisfy the relevant constraints, or satisfy different constraints such that $y + h$ satisfy the relevant constraints.

Already with just a bit of theory we can solve some interesting variational problems. Let $F \in C^2$ and consider the following problem:

$$\begin{aligned} \text{Find weak extrema of} \quad J[y] &:= \int_a^b F(x, y, y')dx \\ \text{s.t.} \quad y &\in C^1[a, b] \\ y(a) = A&, y(b) = B \end{aligned}$$

If we increment a given y by h that satisfies $h(a) = h(b) = 0$ then we preserve the boundary conditions, and Taylor's theorem shows that

$$\Delta J[h] = \int_a^b F_y(x, y, y')h + F_{y'}(x, y, y')h'dx + \cdots$$

where $\cdots$ denote terms of higher order in h, h' (i.e., $o(\|h\|)$). Then it follows by definition that

$$\delta J[h] = \int_a^b F_y(x, y, y')h + F_{y'}(x, y, y')h'dx$$

Our necessary condition tells us that for J to be extremized, we must have

$$0 = \delta J[h] = \int_a^b F_y h + F_{y'} h' dx \tag{1}$$

for all admissible h. This is reminiscent of *the du Bois-Reymond lemma*, also known as *the fundamental lemma of the calculus of variations*:

Lemma 0.8. *(du Bois-Reymond) Let $\alpha, \beta \in C[a, b]$. Then*

a) If $\int_a^b \alpha(x)h(x)dx = 0$ for all $h \in C_0[a, b]$ then $\alpha = 0$.
b) If $\int_a^b \alpha(x)h'(x)dx = 0$ for all $h \in C_0^1[a, b]$ then α is a constant function.
c) If $\int_a^b \alpha(x)h''(x)dx = 0$ for all $h \in C_0^2[a, b]$ then α is an affine function.

d) *If* $\int_a^b \alpha(x)h(x) + \beta(x)h'(x)dx = 0$ *for all* $h \in C_0^1[a,b]$ *then* β *is differentiable on* $[a,b]$ *with* $\beta' = \alpha$.

Proof. [3, Chapter 1, Lemmas 1, 2, 3, and 4] □

Apply part d) of Lemma 0.8 to (1) to deduce

$$F_y - \frac{d}{dx}F_{y'} = 0$$

This is known as *Euler's equation*, and it helps us characterize solutions to a wide class of variational problems. We summarize our work so far in the following theorem:

Theorem 0.9. *Let* J *be a functional of the form*

$$J[y] = \int_a^b F(x, y, y')dx$$

with $F \in C^2$, *defined on the subset of* $C^1[a,b]$ *satisfying the boundary conditions* $y(a) = A, y(b) = B$. *Then a necessary condition for* J *to have an extremum at* y *is that* y *satisfy Euler's equation*

$$F_y - \frac{d}{dx}F_{y'} = 0$$

Remark 0.10. Roughly fifty years after the statement of the brachistochrone problem, the year 1744 is usually credited as the birthyear of theory for the calculus of variations [5]. Euler published a book *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive solutio problematis isoperimetrici latissimo sensu accepti*, which translates to "A method for discovering curved lines that enjoy a maximum or minimum property, or the solution of the isoperimetric problem taken in its widest sense". The language used in this title is significant, as Euler speaks of a systematic "method of invention" in contrast to the "art of invention" that was frequently used by contemporaries of Leibniz in the late 17th century. Mathematics was in the midst of a shift towards systematization.

If F does not depend explicitly on x or y or y', we obtain three special cases in which Euler's equation is usually amenable to elementary techniques. For example, if F does not depend on x explicitly (F is said to be *autonomous* in this case) then Euler's equation yields

$$0 = F_y - \frac{d}{dx}F_{y'} = F_y - F_{y'y}y' - F_{y'y'}y''$$

Multiply by y' and we obtain

$$0 = F_y y' - F_{y'y}y'^2 - F_{y'y'}y'y'' = \frac{d}{dx}\left(F - y'F_{y'}\right)$$

Integrating gives us $F - y'F_{y'} = C$ for some constant C. This is sometimes referred to as the *Erdmann condition.* The reader should consult [3, Chapter 1] for the cases where F does not depend explicitly on y or y'.

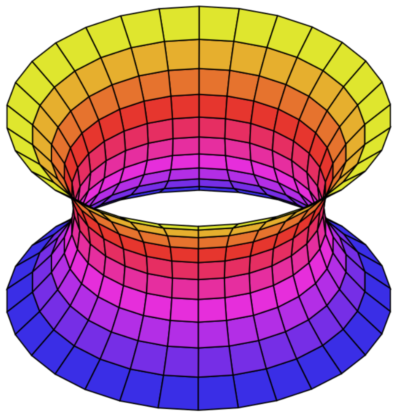

FIGURE 2. A catenoid [6].

Example 0.11. [3, Chapter 1, Example 2]

Among all the curves joining given points $(a, y_0), (b, y_1)$, determine the curve which generates the minimum surface area when rotated around the x-axis. We know from calculus that the surface area functional is given by

$$J[y] = 2\pi \int_a^b y\sqrt{1 + y'^2}dx$$

Since the integrand does not contain x explicitly we may apply the Erdmann condition from before to obtain

$$2\pi \left(y\sqrt{1 + y'^2} - \frac{yy'^2}{\sqrt{1 + y'^2}} \right) = C$$

This simplifies to

$$\frac{y}{\sqrt{1 + y'^2}} = B$$

for some other constant B. At this point it amounts to a separation of variables problem whose solution is

$$y = A\cosh\left(\frac{x + B}{A}\right)$$

for constants A, B determined by the boundary conditions. This is a catenary, and the corresponding minimal surface is a catenoid, depicted in Figure 2. The catenoid was the first nontrivial minimal surface, and was discovered by Euler using the calculus of variations [5]. The study of minimal surfaces grew out of this discovery and was augmented by Lagrange during the 18th century. By generalizing this example to functions of two variables and deriving a corresponding Euler equation, these techniques can be used to derive the zero mean curvature condition for explicitly parametrized minimal surfaces.

Remark 0.12. Observe the fundamental dependence of the calculus of variations upon the underlying spaces of functions, the norms that they are endowed with, and the functionals on these spaces. As the term "functional analysis" suggests, the discipline was borne out of the study of functionals and motivated by problems in the calculus of variations. Volterra published several notes in 1887 on special classes of functionals, and this is widely regarded as marking the birth of functional analysis [5]. Hadamard himself described the calculus of variations as "a first chapter of functional analysis, whose development will without doubt be one of the first tasks in the analysis of the future" [5]. In fact, this book by Hadamard on the calculus of the variations was the first to make use of the word "functional" as a noun in mathematics [1].

The Second Variation: Legendre's Condition

Our efforts thus far emphasized necessary conditions for optimality, but of course it is desirable to have sufficient conditions as well. By analogy with calculus, a vanishing variation is a necessary condition for optimality akin to the first derivative test. Further necessary conditions, and our first sufficient condition, make use of the so-called second variation and were studied by Legendre and Jacobi in the late 18th to early 19th centuries [5].

Given a bilinear functional $B[x, y]$ defined from $X \times X$ to $\mathbb{R}$, we obtain a *quadratic functional* $A[x] := B[x, x]$. The functional A is said to be *positive definite* if $A > 0$ unless $x = 0$. With this definition at hand we can introduce the second variation.

Definition 0.13. If there exists a bounded linear functional φ_1 and a bounded quadratic functional φ_2 such that

$$\Delta J[h] = \varphi_1[h] + \varphi_2[h] + \varepsilon\|h\|^2$$

where $\varepsilon = \varepsilon(\|h\|) \to 0$ as $\|h\| \to 0$ we say that J is *twice differentiable* and we call φ_2 the *second variation* of $J[y]$, denoted by $\delta^2 J[h] = \varphi[h]$.

Note that φ_1 in the definition above is the first variation. As with the first variation, the second variation is unique if it exists. Henceforth we assume all functionals to be twice differentiable. We now state and prove the corresponding necessary condition.

Theorem 0.14. *If $\hat{y}$ is a minimum for the functional J then $\delta^2 J[h] \geq 0$ at $\hat{y}$ for all admissible h. A symmetric statement with the inequality reversed holds in the case of a maximum.*

Proof. Directly from the definition, we can expand

$$\Delta J[h] = \delta J[h] + \delta^2 J[h] + \varepsilon\|h\|^2$$

The first variation $\delta J[h]$ vanishes because we are at an extremum, so we have

$$\Delta J[h] = \delta^2 J[h] + \varepsilon \|h\|^2$$

As in the proof of Theorem 0.6, this implies $\Delta J[h]$ and $\delta^2 J[h]$ have the same sign for $\|h\|$ sufficiently small. Suppose for a contradiction that $\delta^2 J[h_0] < 0$ for some admissible h_0. Taking $\alpha > 0$ sufficiently small we have

$$\delta^2 J[\alpha h_0] = \alpha^2 J[h_0] < 0$$

which contradicts the fact that $\Delta J[h] \geq 0$ by optimality of $\hat{y}$. □

We can impose a bounded below constraint on the second variation to obtain our first sufficient condition. We say that a quadratic functional φ is *coercive* (also called *strongly positive*) if there exists $c > 0$ such that $\varphi[h] \geq c\|h\|^2$.

Theorem 0.15. *Let J be a functional and $\hat{y} \in X$ such that $\delta J[h]$ is zero at $\hat{y}$ for all admissible h. If $\delta^2 J$ is coercive at $\hat{y}$ then $\hat{y}$ is a minimizer of J.*

Proof. By definition, we have

$$\Delta J[h] = \delta J[h] + \delta^2 J[h] + \varepsilon \|h\|^2$$

The first variation $\delta J[h]$ vanishes by assumption, so we have

$$\Delta J[h] = \delta^2 J[h] + \varepsilon \|h\|^2$$

Use the coercivity assumption to obtain

$$\Delta J[h] \geq (c + \varepsilon)\|h\|^2$$

for some $c > 0$. Now $\varepsilon \to 0$ as $\|h\| \to 0$ so take $\|h\|$ sufficiently small so that $|\varepsilon| < c/2$. It follows that

$$\Delta J[h] \geq (c + \varepsilon)\|h\|^2 > (c/2)\|h\|^2 > 0$$

for all h sufficiently small, and so $\hat{y}$ is a minimizer of J as desired. □

In what follows we will maintain our specific interest in functionals of the form

$$J[y] = \int_a^b F(x, y, y')dx$$

defined on curves y satisfying the boundary conditions $y(a) = A, y(b) = B$. We will assume moreover that F is C^3.

As we did in the case of the first variation, we will perturb y with a function h satisfying $h(a) = h(b) = 0$ and appeal to Taylor's theorem to determine the second variation. This time we will write explicitly some of the higher-order terms:

$$\Delta J[h] = \int_a^b F_y(x, y, y')h + F_{y'}(x, y, y')h'dx + \frac{1}{2}\int_a^b \overline{F}_{yy}h^2 + 2\overline{F}_{yy'}hh' + \overline{F}_{y'y'}h'^2 dx$$

Here the overline indicates evaluation of the derivatives at points along intermediate curves, as is often seen in the multivariate version of Taylor's theorem with mean value remainder.

For example, $\overline{F}_{yy} = F_{yy}(x, y + \lambda h, y' + \lambda h')$ for some $\lambda \in (0, 1)$. If we replace these mean values with the functions $F_{yy}, F_{yy'}, F_{y'y'}$ evaluated at the base point $(x, y(x), y'(x))$ then we can rewrite the equation above with an error term instead:

$$\Delta J[h] = \int_a^b F_y(x, y, y')h + F_{y'}(x, y, y')h' dx + \frac{1}{2}\int_a^b F_{yy}h^2 + 2F_{yy'}hh' + F_{y'y'}h'^2 dx + \varepsilon \quad (2)$$

Here ε takes the form

$$\int_a^b \varepsilon_1 h^2 + \varepsilon_2 hh' + \varepsilon_3 h'^2 dx \quad (3)$$

Using the fact that $F \in C^3$ we can deduce that $\varepsilon = o(\|h\|_1^2)$. In particular, since the first term of (2) is the first variation and the second term is a quadratic functional in h, we can conclude that

$$\delta^2 J[h] = \frac{1}{2}\int_a^b F_{yy}h^2 + 2F_{yy'}hh' + F_{y'y'}h'^2 dx$$

Now use integration by parts and the boundary conditions on h to realize

$$\int_a^b 2F_{yy'}hh' dx = -\int_a^b \frac{d}{dx}\left(F_{yy'}\right) h^2 dx$$

Set $P(x) = \frac{1}{2}F_{y'y'}$ and $Q(x) = \frac{1}{2}\left(F_{yy'} - \frac{d}{dx}\left(F_{yy'}\right)\right)$. Then we can rewrite the second variation as

$$\delta^2 J[h] = \int_a^b Ph'^2 + Qh^2 dx$$

The quadratic functional on the right-hand side is central to our study of sufficiency, and we can study it even without regard for its origins as a second variation. Our first result of this nature is as follows:

Theorem 0.16. *If the quadratic functional*

$$\delta^2 J[h] = \int_a^b Ph'^2 + Qh^2 dx$$

defined on $C_0^1[a, b]$ *is non-negative, then* P *is non-negative on* $[a, b]$.

For the sake of brevity we will skip the proof but sketch the idea. Assuming that P is negative at some point x_0, we aim to construct a function h such that $\delta^2 J[h]$ is negative. The relationship between h and h' is asymmetric in the sense that h' being small on $[a, b]$ combined with $h(a) = 0$ implies that h is also small on $[a, b]$, but h' can be large even if h is small. This suggests the creation of a function h whose derivative will be large enough that Ph'^2 dominates over Qh^2 in the integrand. It is not difficult to create such a function using a squared sinusoid, and simple estimates show that $\delta^2 J[h]$ is negative as a result. A complete proof can be found at [3, p. 103].

Combining Theorem 0.15 and Theorem 0.16 we obtain *Legendre's necessary condition.*

Theorem 0.17. *(Legendre) If the functional*

$$J[y] = \int_a^b F(x, y, y')dx$$

defined on the subset of $C^1[a, b]$ *satisfying boundary conditions* $y(a) = A, y(b) = B$ *has a minimum at* y *then* $F_{y'y'} \geq 0$ *at every point along the curve* y*.*

Remark 0.18. Legendre attempted to prove that the *strengthened Legendre condition* obtained by making the inequality in Theorem 0.17 strict is sufficient to guarantee that y is a minimizer of J. However, there was a serious flaw in his proof (which Legendre himself first showed) and he was criticized heavily by Lagrange for his unjustified assumption. It turns out that despite the flaw, the ideas presented in the proof are fruitful and can be employed in alternative results as we shall see [3],[5]. Clarke remarks in his book that despite Lagrange's harsh treatment of Legendre, ultimately both of their names are engraved on the Eiffel Tower [2].

We conclude this section with a lemma on the structure of $\Delta J[h]$ at extrema.

Lemma 0.19. *Suppose that* $F \in C^3$ *and the functional*

$$J[y] = \int_a^b F(x, y, y')dx$$

is defined on the subset of $C^1[a, b]$ *satisfying boundary conditions* $y(a) = A, y(b) = B$*. Suppose* y *satisfies the corresponding Euler equation. Then for* $\Delta J[h]$ *centered at* y*, there exist* μ_1, μ_2 *defined on* $[a, b]$ *such that for* $h \in C_0^1[a, b]$*:*

$$\Delta J[h] = \int_a^b Ph'^2 + Qh^2 dx + \int_a^b \mu_1 h'^2 + \mu_2 h^2 dx$$

and $\mu_1, \mu_2 \to 0$ *as* $\|h\|_1 \to 0$*.*

Proof. By equation (2) we can write

$$\Delta J[h] = \int_a^b F_y(x, y, y')h + F_{y'}(x, y, y')h'dx + \int_a^b Ph'^2 + Qh^2 dx + \varepsilon$$

where ε takes the form

$$\int_a^b \varepsilon_1 h^2 + \varepsilon_2 hh' + \varepsilon_3 h'^2 dx$$

and $\varepsilon_1, \varepsilon_2, \varepsilon_3 \to 0$ as $\|h\|_1 \to 0$. Since y satisfies Euler's equation, the first term of the expression for $\Delta J[h]$ vanishes and we are left with

$$\Delta J[h] = \int_a^b Ph'^2 + Qh^2 dx + \int_a^b \varepsilon_1 h^2 + \varepsilon_2 hh' + \varepsilon_3 h'^2 dx$$

Applying integration by parts to the term $\varepsilon_2 hh'$ and using the zero boundary conditions brings us to the form

$$\Delta J[h] = \int_a^b Ph'^2 + Qh^2 dx + \int_a^b \varepsilon_3 h'^2 + \left(\varepsilon_1 - \frac{1}{2}\varepsilon_2'\right) h^2 dx$$

Taking $\mu_1 = \varepsilon_3$ and $\mu_2 = \varepsilon_1 - \frac{1}{2}\varepsilon_2'$ we can use the fact that $F \in C^3$ to verify that $\mu_1, \mu_2 \to 0$ as $\|h\|_1 \to 0$. □

A Special Quadratic Functional

If we treat the quadratic functional

$$A[h] = \int_a^b Ph'^2 + Qh^2 dx$$

for its own merit, we can write its Euler equation as

$$-\frac{d}{dx}\left(Ph'\right) + Qh = 0 \tag{4}$$

with boundary conditions $h(a) = h(c) = 0$ for a fixed $c \in (a, b]$ (slight generalization of our previous boundary condition). Equation (4) is called *Jacobi's equation.* It has a trivial solution in the constant function $h = 0$, but we are interested in nontrivial solutions.

Definition 0.20. A point $a_0 \in (a, b]$ is *conjugate* to a if (4) has a nontrivial solution that vanishes at a_0 and a.

Conjugate points turn out to be an obstacle to sufficiency, so we aim to characterize the absence of conjugate points. The proof of the next result realizes the ideas of Legendre's original "proof" involving the strengthened Legendre condition.

Theorem 0.21. *If $P(x) > 0$ on $[a, b]$ and there are no points conjugate to a in $[a, b]$ then the quadratic functional $\int_a^b Ph'^2 + Qh^2 dx$ is positive definite for all h satisfying the zero boundary conditions.*

Proof. The idea is to find a differentiable function w defined on $[a, b]$ such that

$$\int_a^b Ph'^2 + Qh^2 dx = \int_a^b P\left(h' + \frac{w}{P}h\right)^2 dx \tag{5}$$

This implies the desired result because the square under the integrand is not identically zero unless $h = 0$ on $[a, b]$, and if the integral is equal to zero then

$$h' + \frac{w}{P}h = 0$$

The boundary condition $h(a) = 0$ combined with the uniqueness theorem for ODEs implies $h = 0$, and so the functional is positive definite.

Note that for such a function w we have

$$\int_a^b Ph'^2 + Qh^2 dx = \int_a^b Ph'^2 + Qh^2 + \frac{d}{dx}(wh^2)dx = \int_a^b Ph'^2 + 2whh' + (Q + w')h^2 dx$$

by virtue of our boundary conditions. Since P is positive on $[a, b]$ we can write this as

$$\int_a^b Ph'^2 + Qh^2 dx = \int_a^b P\left(h'^2 + \frac{2whh'}{P} + \frac{(Q + w')h^2}{P}\right) dx$$

Taking for granted that w is a solution to the equation

$$Q + w' = \frac{w^2}{P} \tag{6}$$

then this simplifies to

$$\int_a^b Ph'^2 + Qh^2 dx = \int_a^b P\left(h'^2 + \frac{2whh'}{P} + \frac{w^2h^2}{P^2}\right) dx$$

and the right-hand side contains a perfect square that brings it to the form (5).

It remains to justify the existence of a differentiable solution to (6) on $[a, b]$, and in fact this was Legendre's error [3]. This is where we must invoke our conjugate point assumption. By making the change of variables $w = -\frac{u'}{u}P$ for a new function u, equation (6) is equivalent to

$$-\frac{d}{dx}(Pu') + Qu = 0$$

which is precisely the Jacobi equation of the original quadratic functional. Then by definition of the absence of points conjugate to a, there exists a solution to this equation that is non-zero on $[a, b]$ (see [3, Footnote 9, p. 108] for details), and hence $w = -\frac{u'}{u}P$ is differentiable, defined on $[a, b]$, and solves the equation necessary to make the rest of the proof go through. □

The converse is also true, and completes our characterization of conjugate points in terms of positive definiteness.

Theorem 0.22. *If $P(x) > 0$ on $[a, b]$ and the quadratic functional $\int_a^b Ph'^2 + Qh^2 dx$ is positive definite for h satisfying the zero boundary conditions then there are no points conjugate to a in $[a, b]$.*

To keep the development focused, we will only sketch the proof. For $t \in [0, 1]$ we can define a family of quadratic functionals

$$A_t[h] \int_a^b t(Ph'^2 + Qh^2) + (1 - t)h'^2 dx$$

Notice that for $t = 0$ the functional is $A_0[h] = \int_a^b h'^2 dx$ for which it is straightforward to verify the absence of points conjugate to a in $[a, b]$. The rest of the proof amounts to showing that as t tends to 1, points conjugate to a cannot arise. As t varies, we obtain a family of Jacobi equations indexed by t and so we can interpret their solutions as functions $h(x,t)$ simultaneously. By assuming the existence of a conjugate point a_0 and defining $C = \{(x,t) \in [a,b] \times [0,1] \mid h(x,t) = 0\}$ one can show that C is a curve in the plane, and geometric case analysis shows that such a curve cannot exist, which is the desired contradiction. The reader should consult [3, Chapter 5, Theorem 2] for further details.

We can weaken the positive definiteness assumption to non-negativity and obtain a corresponding result on interior conjugate points.

Theorem 0.23. *If $P(x) > 0$ on $[a,b]$ and the quadratic functional $\int_a^b Ph'^2 + Qh^2 dx$ is non-negative for h satisfying the zero boundary conditions then there are no points conjugate to a in (a,b).*

The same proof as for Theorem 0.22 works, with the exception of a simple lemma relying on positive definiteness that excludes the possibility of b being conjugate to a. Theorem 0.21 and Theorem 0.22 provide us with the following characterization of positive definiteness in terms of conjugate points.

Theorem 0.24. *If $P(x) > 0$ on $[a,b]$ then the quadratic functional $\int_a^b Ph'^2 + Qh^2 dx$ is positive definite for h satisfying the zero boundary conditions if and only if there are no points conjugate to a in $[a,b]$.*

Jacobi's Necessary Condition and a Sufficient Condition

Returning to the concrete study of our integral functional, we consider again the problem of determining weak minima for

$$J[y] = \int_a^b F(x,y,y')dx, \quad y(a) = A, y(b) = B$$

Theorem 0.25. *(Jacobi) If y minimizes the functional $J[y] = \int_a^b F(x,y,y')dx$ and $F_{y'y'} > 0$ at all points of the curve y, then there are no points conjugate to a in (a,b).*

Proof. Theorem 0.15 implies that $\delta^2 J[h]$ is non-negative, and then $F_{y'y'} > 0$ implies $P > 0$ and so Theorem 0.23 completes the argument. □

We have seen three necessary conditions for minima, in the form of Euler's equation, Legendre's condition, and Jacobi's condition. It turns out that taken together and strengthened mildly, all three of these conditions are sufficient for a minimum. The following theorem ties together all of our work thus far.

Theorem 0.26. *Suppose that the functional*

$$J[y] = \int_a^b F(x,y,y')dx$$

is defined on the subset of $C^1[a,b]$ satisfying boundary conditions $y(a) = A, y(b) = B$, and that y is an admissible curve such that the following conditions are satisfied:

(1) The curve y satisfies Euler's equation, $F_y - \frac{d}{dx}F_{y'} = 0$
(2) Along the curve y, $P(x) = \frac{1}{2}F_{y'y'}(x, y(x), y'(x)) > 0$ (strengthened Legendre condition)
(3) The interval $[a, b]$ does not contain any points conjugate to a (strengthened Jacobi condition).

Then J admits a weak minimum at y.

Proof. Observe that we can use continuity of P to obtain a slightly larger interval $[a, b+\varepsilon_1]$ on which $P(x) > 0$, and we can use continuity of the solution to Jacobi's equation to guarantee the absence of conjugate points to a in a slightly larger interval $[a, b + \varepsilon_2]$. Taking $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$ we obtain $[a, b + \varepsilon] \supsetneq [a, b]$ on which both conditions hold.

Define a quadratic functional

$$A[h] = \int_a^b Ph'^2 + Qh^2 dx - \alpha^2 \int_a^b h'^2 dx$$

Its Euler equation is

$$-\frac{d}{dx}\Big((P - \alpha^2)h'\Big) + Qh = 0$$

Note that $\inf_{x\in[a,b+\varepsilon]} P(x) > 0$. Moreover, the solution to this Euler equation satisfying the boundary conditions $h(a) = 0, h'(a) = 1$ is continuous as a function of α for α small. It follows that for sufficiently small α we have $P(x) - \alpha^2 > 0$ on $[a, b]$ and the solution to

$$-\frac{d}{dx}\Big((P - \alpha^2)h'\Big) + Qh = 0, \quad h(a) = 0, h'(a) = 1$$

is non-zero on $(a, b]$. These are precisely the conditions of Theorem 0.21, from which we deduce the positive definiteness of A. In particular, we can find $c = \alpha_0^2 > 0$ small such that for $h \neq 0$:

$$\int_a^b Ph'^2 + Qh^2 dx > c\|h'\|_2^2 dx$$

Now use the fact that y satisfies Euler's equation and apply Lemma 0.19 to write

$$\Delta J[h] = \int_a^b Ph'^2 + Qh^2 dx + \int_a^b \mu_1 h'^2 + \mu_2 h^2 dx$$

where $\mu_1, \mu_2 \to 0$ on $[a, b]$ as $\|h\|_1 \to 0$. Using Cauchy-Schwarz and our boundary conditions on h we estimate

$$h^2(x) = \left(\int_a^x h'(x)dx\right)^2 \leq (x - a)\int_a^x (h'(x))^2 dx \leq (x - a)\|h'\|_2^2$$

Integrating over $[a, b]$ yields

$$\|h\|_2^2 \leq \frac{(b - a)^2}{2}\|h'\|_2^2$$

Using this inequality we deduce

$$\left|\int_a^b \mu_1 h'^2 + \mu_2 h^2 dx\right| \leq \int_a^b |\mu_1| h'^2 + |\mu_2| h^2 dx \leq \varepsilon \left(1 + \frac{(b-a)^2}{2}\right) \|h'\|_2^2$$

provided $|\mu_i| \leq \varepsilon$, $i = 1, 2$ on $[a, b]$. Combining our estimates we obtain

$$\begin{aligned}
\Delta J[h] &= \int_a^b Ph'^2 + Qh^2 dx + \int_a^b \mu_1 h'^2 + \mu_2 h^2 dx \\
&> c\|h'\|_2^2 - \varepsilon \left(1 + \frac{(b-a)^2}{2}\right) \|h'\|_2^2 \\
&= \|h'\|_2^2 \left(c - \varepsilon \left(1 + \frac{(b-a)^2}{2}\right)\right) \\
&> 0
\end{aligned}$$

where the final inequality holds for $\varepsilon > 0$ sufficiently small, obtained by taking $\|h\|_1$ close to zero. This is precisely the definition of y being a weak minimizer of J.

□

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