SCHWARZ ALTERNATING METHOD FOR ONE-DIMENSIONAL DOMAIN

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We take $\Omega = (0,1) \subset \mathbb{R}$, and decompose Ω into two subdomains $\Omega_1 := (0,r)$, $\Omega_2 := (s,1)$ with $s \leq r$ such that $\Omega = \Omega_1 \cup \Omega_2$.

Consider the Neumann Problem

$$\begin{cases} -u'' + u = f \text{ in } \Omega, \\ u'(0) = u'(1) = 0 \end{cases}$$

where $\Omega = \Omega_1 \cup \Omega_2$.

Let $X = H^1(0,1)$. Since the one-dimensional Sobolev space $H^1(0,1)$ is embedded in the space $C([0,1])$ of continuous functions, we can identify each element of $H^1(0,1)$ with its corresponding representative in $C([0,1])$. Now define

$$Y_1 := \overline{\{\phi \in C^\infty(0,1) : \phi = 0 \text{ in the neighbourhood of } [r,1)\}} = H_0^1(\Omega_1),$$

and

$$Y_2 := \overline{\{\phi \in C^\infty(0,1) : \phi = 0 \text{ in the neighbourhood of } (0,s]\}} = H_0^1(\Omega_2).$$

Fix $u_0 \in X$ and find u_1 by first solving

$$\begin{cases} -u_1'' + u_1 = f \text{ in } \Omega_1, \\ u_1(r) = u_0(r), \\ u_1'(0) = 0, \end{cases}$$

and then extend by u_0 from Ω_1 to all of Ω. We repeat the procedure alternatingly to find $u_2, u_3, \cdots$ such that u_{2n+1} (for $n \geq 0$) solves

$$\begin{cases} -u_{2n+1}'' + u_{2n+1} = f \text{ in } \Omega_1, \\ u_{2n+1}(r) = u_{2n}(r), \\ u_{2n+1}'(0) = 0, \end{cases}$$

while u_{2n} (for $n \geq 1$) is a solution of

$$\begin{cases} -u_{2n}'' + u_{2n} = f \text{ in } \Omega_2, \\ u_{2n}(r) = u_{2n-1}(s), \\ u_{2n}'(1) = 0. \end{cases}$$

We may extend u_{2n+1} by u_{2n} and $u_{2n} = u_{2n-1}$ respectively to all of Ω.

Since $-(u_1 - u)'' + (u_1 - u) = 0$ in Ω_1, $\langle u_1 - u, \phi \rangle_{H^1} = \int_{\Omega_1} (u_1 - u)'\phi' + (u_1 - u)\phi dx = \int_{\Omega_1} -(u_1 - u)''\phi + (u_1 - u)\phi dx = 0$ for all $\phi \in Y_1$. This shows that $u_1 - u \perp Y_1$. Let $M_i = Y_i^\perp$ for $i = 1, 2$, then $u_1 - u \in M_1$.

If we take $w = u_1 - u$, then w satisfies

$$\begin{cases} -w'' + w = 0 \text{ in } \Omega_1, \\ w'(0) = 0. \end{cases}$$

By solving the above second order ordinary differential equation (ODE), we have $w = C(e^x + e^{-x})$ for some constant C. So $M_1 = span\{e^x + e^{-x}\}$.

Similarly, $v := u_2 - u \in M_2$ and satisfies

$$\begin{cases} -v'' + v = 0 \text{ in } \Omega_2, \\ v'(1) = 0. \end{cases}$$

The above ODE is solved by $v = D(e^x + e^{2-x})$ for some constant D, so $M_2 = span\{e^x + e^{2-x}\}$.

Note that $u_0 - u = (u_0 - u_1) + (u_1 - u)$ where $u_0 - u_1 \in Y_1 = M_1^{\perp}$ and $u - u_1 \in M_1$, so $u_1 - u = P_{M_1}(u_1 - u)$ where P_{M_i} is the orthogonal projection onto M_i for $i = 1, 2$. Similarly, $u_2 - u = P_{M_2}(u_1 - u)$ as $u_1 - u = (u_1 - u_2) + (u_2 - u)$ where $u_2 - u \in M_2$ and $u_1 - u_2 \in M_2^{\perp}$.

Iteratively, $u_{2n+1} - u = P_{M_1}(u_{2n} - u)$ and $u_{2n} - u = P_{M_2}(u_{2n-1} - u)$ for $n \geq 1$.

Let $T = P_{M_2}P_{M_1}$. If $x_{2n} := u_{2n} - u$, $x_0 := u_0 - u$, then $x_{2n} = T^n x_0$. By *Von-Neumann Halperin Theorem*, $\lim_{n\to\infty} \|T^n x_0 - P_M x_0\| = 0$ where $M = M_1 \cap M_2$. In this case $M = M_1 \cap M_2 = \{0\}$, so $P_M x_0 = 0$. Then $\lim_{n\to\infty} \|x_{2n}\| = 0$, that is, u_{2n} converges to u. Note that $x_{2n+1} = u_{2n+1} - u = P_{M_1}(u_{2n} - u) = P_{M_1}T^n x_0$. By definition of orthogonal projections, $T^n x_0 \in M_1$, so $P_{M_1}T^n x_0 = T^n x_0$. It follows that $\lim_{n\to\infty} \|x_{2n+1} - x_0\| \leq \lim_{n\to\infty}(\|x_{2n+1} - x_{2n}\| + \|x_{2n} - x_0\|) = 0$. That is, u_{2n+1} converges to u as well.

Thus, the *Schwarz Alternating Method* generates a sequence of solutions $\{u_n\}$ converging to the real solution u.

$$\begin{aligned} u(r) &= \int_0^1 uv_r + u'v_r{}'dx \\ &= \int_0^1 uv_r dx + \left[uv_r{}'\right]_0^1 - \int_0^1 uv_r{}''dx \\ &= \int_0^1 u(v_r - v_r{}'')dx + \left[uv_r{}'\right]_{r^+}^1 + \left[uv_r{}'\right]_0^{r^-} \end{aligned}$$