DICHOTOMY THEOREM ON THE RATE OF CONVERGENCE

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Theorem 1. *([DH15] Theorem 5.4) Let X be a Banach space and $T\colon X \to X$ be a linear operator with $\|T\| \le 1$ and $T^n(x) \to 0$ for each $x \in X$, then exactly one of the following two statements hold:*

1. *there exists $n_1 \in \mathbb{N}$ such that $\|T^{n_1}\| < 1$, and there exists $\alpha \in [0,1)$ and $c \in \mathbb{R}$ such that $\|T^n\| \le c\alpha^n$ for each n.*

2. *$\|T^n\| = 1$ for each $n \in \mathbb{N}$, and for each $(r_n) \in c_0$, $r_n \in \mathbb{R}^+$, for all $x \in X$, $\|T^n x\| \ne O(r_n)$.*

Proof. For $T \in B(X)$, $\mathrm{Rad}(\sigma(T)) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} \le \lim_{n\to\infty} \|T\| \le 1$. This shows that the spectrum of T lie in the closed unit disc.

If $\mathrm{Rad}(\sigma(T)) < 1$, then there exists $n_1 \in \mathbb{N}$ such that $\rho := \|T^{n_1}\| < 1$.

If $\rho > 0$, we take $\alpha := \rho^{\frac{1}{n_1}}$, for any $n \in \mathbb{N}$, we can write $n = kn_1 + i$ for some non-negative integer k and $i \in \{0, 1, \cdots n_1 - 1\}$. Then $\|T^n\| = \|T^{kn_1+i}\| \le \|T^{n_1}\|^k = \rho^k = \alpha^{n_1 k} \le c\alpha^n$ where $c = \frac{1}{\alpha^{n_1-1}}$.

If $\rho = 0$, $\|T^n\| \le 2^{n_1}(\frac{1}{2})^n$ for each $n \in \mathbb{N}$, then we take $\alpha = \frac{1}{2}$ and $c = 2^{n_1}$.

If $\mathrm{Rad}(\sigma(T)) = 1$, then $\mathrm{Rad}(\sigma(T)) \le \|T\| \le 1$ implies $\|T\| = 1$. $\|T^n\| = 1$ for each $n \in \mathbb{N}$ since if $\|T^N\| < 1$ for some $N \in \mathbb{N}$, we have $\mathrm{Rad}(\sigma(T)) = lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} \le \lim_{n\to\infty} \|T^N\|^{\frac{1}{n}} < 1$ which yields a contradiction. We assume for contradiction that for each $x \in X$, there exists constant $c_x > 0$ such that $\|T^n x\| \le c_x r_n$ for each $n \in \mathbb{N}$. Since X is a Banach Space, *Uniform Boundedness Theorem* tells $\|T^n\| \le C r_n$ with C independent of x, then $\|T^n\| \to 0$ as $n \to \infty$. This contradicts $\|T^n\| = 1$. $\square$

Before applying the above dichotomy results to the rate of convergence of the cyclic projections in Hilbert space, we first introduce an important lemma which was proved by *Bauschke, Borwein and Lewis* in [BL97] by using regularities:

Lemma 2. *Let M_i for $1 \le i \le r$ be closed subspaces in the Hilbert space H, and $M := \bigcap_i^r M_i$. Let P_{M_i} and P_M be the orthogonal projections onto M_i and M respectively. Then the following are equivalent*

1. *$\sum_{i=1}^r M_i^\perp$ is closed;*

2. *$\|P_{M_r\cap M^\perp} P_{M_{r-1}\cap M^\perp} \cdots P_{M_1\cap M^\perp}\| < 1$;*

Now we can deduce the *Von Neumann-Halperin Dichotomy* from *Theorem*1 and the above lemma:

Theorem 3. *(*[DH15] *Theorem 9.5) Let M_i for $1 \leq i \leq r$ be closed subspaces in the Hilbert space H, and $M := \bigcap_i^r M_i$. Let $T = P_{M_r} P_{M_{r-1}} \cdots P_{M_1}$. Then exactly one of the two following statements holds:*

(1) *$\sum_{i=1}^r M_i^\perp$ is closed, then there exists $\alpha \in [0,1)$ and $c \in \mathbb{R}$ such that $\|T^n - P_M\| \leq c\alpha^n$ for each n.*

(2) *$\sum_{i=1}^r M_i^\perp$ is not closed, then for each $(r_n) \in c_0$, $r_n \in \mathbb{R}^+$, for all $x \in X$, $\|T^n x - P_M x\| \neq O(r_n)$.*

Proof. If $\sum_{i=1}^r M_i^\perp$ is closed, we have $\|P_{M_r \cap M^\perp} P_{M_{r-1} \cap M^\perp} \cdots P_{M_1 \cap M^\perp}\| < 1$ (by *Lemma 2*).

Note that

$$\begin{aligned} T^n - P_M &= T^n - T^n P_M \\ &= T^n (I - P_M) \\ &= T^n P_{M^\perp} \\ &= T^n P_{M^\perp}^n \text{ since } P_{M^\perp}^2 = P_{M^\perp} \\ &= (T P_{M^\perp})^n \text{ since } T \text{ commutes with } P_{M^\perp} \\ &= [(P_{M_r} P_{M^\perp})(P_{M_{r-1}} P_{M^\perp}) \cdots (P_{M_1} P_{M^\perp})]^n \\ &= [(P_{M_r \cap M^\perp})(P_{M_{r-1} \cap M^\perp}) \cdots (P_{M_1 \cap M^\perp})]^n. \end{aligned}$$

The second last equality follows from P_{M_i} commuting with $P_{M^\perp}$ and $P_{M^\perp}^2 = P_{M^\perp}$ while the last equality is true because $P_{M_i} P_{M^\perp} = P_{M_i \cap M^\perp}$.

By taking $\|P_{M_r \cap M^\perp} P_{M_{r-1} \cap M^\perp} \cdots P_{M_1 \cap M^\perp}\| = \alpha \in [0,1)$, we have

$$\begin{aligned} \|T^n - P_M\| &= \|[(P_{M_r \cap M^\perp})(P_{M_{r-1} \cap M^\perp}) \cdots (P_{M_1 \cap M^\perp})]^n\| \\ &\leq \|(P_{M_r \cap M^\perp})(P_{M_{r-1} \cap M^\perp}) \cdots (P_{M_1 \cap M^\perp})\|^n \\ &= \alpha^n. \end{aligned}$$

In this case, $c = 1$ and the result follows.

If $\sum_{i=1}^r M_i^\perp$ is not closed, by *Lemma* 2, $\|P_{M_r \cap M^\perp} P_{M_{r-1} \cap M^\perp} \cdots P_{M_1 \cap M^\perp}\| = 1$, and then $\|[P_{M_r \cap M^\perp} P_{M_{r-1} \cap M^\perp} \cdots P_{M_1 \cap M^\perp}]^n\| = 1$. By *Theorem 1(2)*, the result follows. $\square$

References

[BL97] H.H.Bauschke, J.M.Borwein, and A.S.Lewis, *The method of cyclic projections for closed convex sets in Hilbert space*, Contemporary Mathematics, Vol.204,(1997) 138.

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