$Y_1 + Y_2$ IS A DENSE SUBSPACE OF X

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We first consider the Dirichlet problem where $X = H_0^1(\Omega)$ and $\Omega = \Omega_1 \cup \Omega_2$. If $Y_1 = H_0^1(\Omega_1)$ and $Y_2 = H_0^1(\Omega_2)$, then $Y_1 + Y_2$ is a dense subspace of X.

Proof. We want to prove that $\overline{Y_1 + Y_2} = X$. It suffices to show that for each $u \in C_c^\infty(\Omega)$, $u = u_1 + u_2$ where $u_1 \in C_c^\infty(\Omega_1)$, $u_2 \in C_c^\infty(\Omega_2)$ since $C_c^\infty(\Omega_i)$ is dense in $H_0^1(\Omega_i)$ for each $i = 1, 2$. Suppose for $u \in C_c^\infty(\Omega)$,supp$(u) = K$ where K is a compact subset of Ω. There exists $\varepsilon > 0$ such that $K \subset (\Omega_1)_\varepsilon \cup (\Omega_2)_\varepsilon$ with $(\Omega_i)_\varepsilon = \{x \in \Omega_i, \mathrm{dist}(x, \Omega_i^c) > \varepsilon\}$. Let φ_i be the mollification of $\mathbb{1}_{(\Omega_i)_\varepsilon \cap K}$ for $i = 1, 2$, then $\varphi_i \cdot u \in C_c^\infty(\Omega_i)$, and

$$\begin{aligned}
\|u - (\varphi_1 u + \varphi_2 u)\|_{H_0^1} &= \| \bigtriangledown u - \bigtriangledown(\varphi_1 u + \varphi_2 u)\|_{L^2} \\
&\le \| \bigtriangledown u(1 - (\varphi_1 + \varphi_2))\| + \|u \bigtriangledown (\varphi_1 + \varphi_2)\|_{L^2} \\
&\le C(\|1 - (\varphi_1 + \varphi_2))\|_{L^2} + \| \bigtriangledown (\varphi_1 + \varphi_2)\|_{L^2})
\end{aligned}$$

for some positive constant C. The right hand side of the inequality tends to 0 as $\varepsilon \to 0$, so the result follows. $\square$

Now we prove the above statemnet for the Neumann problem with $H_0^1(\Omega)$ replaced by $H^1(\Omega)$, and $Y_i = \overline{Z_i}$ where

$$Z_i := \{u \in C^\infty(\Omega) \colon u = 0 \text{ in the neighbourhood of } \Omega \setminus \Omega_1\}.$$

In this case, $Y = Y_1 + Y_2$ is a proper dense subspace of X.

Proof.
(1) We first show that $Y = Y_1 + Y_2$ is a proper subspace of X. Assume for contradiction that $X = Y_1 + Y_2$, then for each $u \in X$, we have $u = u_1 + u_2$ where $u_1 \in Y_1$, $u_2 \in Y_2$. Now we consider the trace defined as
$$\mathrm{Tr} \colon H^1 \to H^{\frac{1}{2}}(\partial\Omega).$$
Since $u \equiv 1 \in H^1(\Omega)$ and u is continuous on $\overline{\Omega}$, then
$$1 = \mathrm{Tr}u = \mathrm{Tr}(u_1 + u_2) = \mathrm{Tr}u_1 + \mathrm{Tr}u_2.$$
We also have $\mathrm{Tr}u_1 = 0$ on $\partial\Omega_1$, and $\mathrm{Tr}u_2 = 0$ on $\partial\Omega_2$, so $\mathrm{Tr}u_2 = 1$ on γ_1 and $\mathrm{Tr}u_1 = 1$ on γ_2. This implies that $\mathrm{Tr}u_1 = \mathbb{1}_{\gamma_1}$ on $\gamma_1 \cup \gamma_2$, but $\mathbb{1}_{\gamma_1} \notin H^{\frac{1}{2}}\gamma_1 \cup \gamma_2)$, thus yields a contradiction.

(2) Now we proceed to the density argument. Note that for each $u \in Y = Y_1 + Y_2$, for $\varepsilon > 0$, there exists $\varphi_i \in Z_i$ such that
$$\|u - \varphi_1 - \varphi_2\|_{H^1} < \varepsilon.$$
Define $Z : \{\varphi \in C^\infty(\Omega) \colon \varphi = 0 \text{ near the problematic points}\}$, then $Z = Z_1 + Z_2$. It suffices to show that for each $u \in C^\infty(\Omega)$, for all $\varepsilon > 0$, there exists $\varphi \in Z$ such that $\|u - \varphi\|_{H^1} < \varepsilon$. We now simplify

our proof by considering the two dimensional domain. Assume that z is the probematic point, and consider the function $\varphi_\varepsilon \colon \Omega \to \mathbb{C}$ defined as

$$\varphi_\varepsilon(x,y) = \begin{cases} \left(\frac{|(x,y)-z|}{\varepsilon}\right)^\varepsilon, \text{ for } r = |(x,y)-z| \in (0,\varepsilon), \\ 1, \text{ otherwise.} \end{cases}$$

For each $u \in C^\infty(\Omega)$, $u_\varepsilon := u \cdot \varphi_\varepsilon \in Z$, then

$$\|u - u_\varepsilon\|_{L^2} \to 0 \text{ as } \varepsilon \to 0,$$

and

$$\begin{aligned} \| \bigtriangledown u - \bigtriangledown u_\varepsilon\|_{L^2} &= \| \bigtriangledown u(1-\varphi_\varepsilon) - u \bigtriangledown \varphi_\varepsilon\|_{L^2} \\ &\le C(\|1-\varphi_\varepsilon\|_{L^2} + \| \bigtriangledown \varphi_\varepsilon\|_{L^2}) \end{aligned}$$

for some positive constant C. As both $\|1-\varphi_\varepsilon\|_{L^2}$ and $\| \bigtriangledown \varphi_\varepsilon\|_{L^2}$ tend to 0 as $\varepsilon \to 0$, $\|u - u_\varepsilon\|_{H^1} = \|u - u_\varepsilon\|_{L^2} + \| \bigtriangledown u - \bigtriangledown u_\varepsilon\|_{L^2} \to 0$ as $\varepsilon \to 0$.

$\square$