SCHWARZ ALTERNATING METHOD

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We take Ω to be a bounded open domain in $\mathbb{R}^n$ (assume that it is smooth for simplicity), and decompose Ω into two subdomains Ω_1, Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$.

Consider the Neumann Problem

$$\begin{cases} -u'' + u = f \text{ in } \Omega, \\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \end{cases}$$

where n is the outward pointing normal.

Let $X = H^1(\Omega)$, and define

$$Y_1 := \overline{\{\phi \in C^\infty(\Omega) : \phi = 0 \text{ in the neighbourhood of } \Omega \setminus \Omega_1\}} = H_0^1(\Omega_1),$$

and

$$Y_2 := \overline{\{\phi \in C^\infty(\Omega) : \phi = 0 \text{ in the neighbourhood of } \Omega \setminus \Omega_2\}} = H_0^1(\Omega_2).$$

Fix $u_0 \in X$ and find u_1 by first solving

$$\begin{cases} -u_1'' + u_1 = f \text{ in } \Omega_1, \\ u_1 = u_0 \text{ on } \gamma_1 := \partial\Omega_1 \cap \Omega_2, \\ \frac{\partial u_1}{\partial n} = 0 \text{ on } \partial\Omega_1 \setminus \gamma_1 \end{cases}$$

and then extend by u_0 from Ω_1 to all of Ω. We repeat the procedure alternatingly to find $u_2, u_3, \cdots$ such that u_{2n+1} (for $n \geq 0$) solves

$$\begin{cases} -u_{2n+1}'' + u_{2n+1} = f \text{ in } \Omega_1, \\ u_{2n+1} = u_{2n} \text{ on } \gamma_1 := \partial\Omega_1 \cap \Omega_2, \\ \frac{\partial u_{2n+1}}{\partial n} = 0 \text{ on } \partial\Omega_1 \setminus \gamma_1 \end{cases}$$

while u_{2n} (for $n \geq 1$) is a solution of

$$\begin{cases} -u_{2n}'' + u_{2n} = f \text{ in } \Omega_2, \\ u_{2n} = u_{2n-1} \text{ on } \gamma_2 := \partial\Omega_2 \cap \Omega_1, \\ \frac{\partial u_{2n}}{\partial n} = 0 \text{ on } \partial\Omega_2 \setminus \gamma_2 \end{cases}$$

We may extend u_{2n+1} by u_{2n} and $u_{2n} = u_{2n-1}$ respectively to all of Ω.

Since $-(u_1 - u)'' + (u_1 - u) = 0$ in Ω_1,

$$\langle u_1 - u, \phi \rangle_{H^1} = \int_{\Omega_1} (u_1 - u)'\phi' + (u_1 - u)\phi dx = \int_{\Omega_1} -(u_1 - u)''\phi + (u_1 - u)\phi dx = 0$$

for all $\phi \in Y_1$. This shows that $u_1 - u \perp Y_1$. Let $M_i = Y_i^\perp$ for $i = 1, 2$, then $u_1 - u \in M_1$.

Note that

$$u_0 - u = (u_0 - u_1) + (u_1 - u)$$

where $u_0 - u_1 \in Y_1 = M_1^\perp$ and $u - u_1 \in M_1$, so $u_1 - u = P_{M_1}(u_0 - u)$ where P_{M_i} is the orthogonal projection onto M_i for $i = 1, 2$.

Similarly, $u_2 - u = P_{M_2}(u_1 - u)$ as

$$u_1 - u = (u_1 - u_2) + (u_2 - u)$$

where $u_2 - u \in M_2$ and $u_1 - u_2 \in M_2^\perp$.

Iteratively,

$$u_{2n+1} - u = P_{M_1}(u_{2n} - u)$$

and

$$u_{2n} - u = P_{M_2}(u_{2n-1} - u)$$

for $n \geq 1$.

Let $T = P_{M_2}P_{M_1}$. If $x_{2n} := u_{2n} - u$, $x_0 := u_0 - u$, then $x_{2n} = T^n x_0$. By *Von-Neumann Halperin Theorem*,

$$\lim_{n\to\infty} \|T^n x_0 - P_M x_0\|_{H^1} = 0$$

where $M = M_1 \cap M_2$. In this case, we have $X = \overline{M_1^\perp + M_2^\perp} = \overline{Y_1 + Y_2}(\star)$, so $M = M_1 \cap M_2 = \{0\}$, and $P_M x_0 = 0$. Then

$$\lim_{n\to\infty} \|x_{2n}\|_{H^1} = 0.$$

This implies that $u_{2n}(x)$ converges to $u(x)$ strongly.

Note that $x_{2n+1} = P_{M_1}x_{2n}$, then

$$\lim_{n\to\infty} \|x_{2n+1}\|_{H^1} = \lim_{n\to\infty} \|P_{M_1}x_{2n}\|_{H^1} = 0$$

by continuity of P_{M_1}. That is, u_{2n+1} converges to u strongly as well.

Thus, the *Schwarz Alternating Method* generates a sequence of solutions $\{u_n\}$ converging to the exact solution u strongly.

We formulate the proof of $(\star)$ in the following lemma:

Lemma 1. *For $X = H^1(\Omega)$,$Y_i = \overline{Z_i}$ where*

$$Z_i := \{u \in C^\infty(\Omega) \colon u = 0 \text{ in the neighbourhood of } \Omega \setminus \Omega_1\},$$

$Y = Y_1 + Y_2$ is a proper dense subspace of X.

Proof. For simplicity, we consider the two-dimensional domain Ω as shown below. We first show that $Y = Y_1 + Y_2$ is a proper subspace of X. Assume for contradiction that $X = Y_1 + Y_2$, then for each $u \in X$, we have $u = u_1 + u_2$ where $u_1 \in Y_1$, $u_2 \in Y_2$. Now we consider the trace defined as

$$\mathrm{Tr}\colon H^1 \to H^{\frac{1}{2}}(\partial\Omega).$$

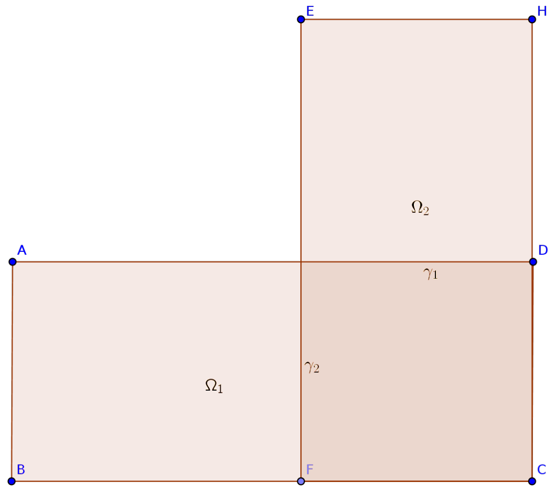
Since $u \equiv 1 \in H^1(\Omega)$ and u is continuous on $\overline{\Omega}$, then

$$1 = \mathrm{Tr}u = \mathrm{Tr}(u_1 + u_2) = \mathrm{Tr}u_1 + \mathrm{Tr}u_2.$$

We also have $\mathrm{Tr}u_1 = 0$ on $\partial\Omega_1$, and $\mathrm{Tr}u_2 = 0$ on $\partial\Omega_2$, so $\mathrm{Tr}u_2 = 1$ on γ_1 and $\mathrm{Tr}u_1 = 1$ on γ_2. This implies that $\mathrm{Tr}u_1 = \mathbb{1}_{\gamma_2}$ on $\gamma_1 \cup \gamma_2$, but $\mathbb{1}_{\gamma_2} \notin H^{\frac{1}{2}}(\gamma_1 \cup \gamma_2)$ as

$$\int_{(\gamma_1\cup\gamma_2)} \int_{(\gamma_1\cup\gamma_2)} \frac{|\mathbb{1}_{\gamma_2}(x) - \mathbb{1}_{\gamma_2}(y)|^2}{|x-y|} dxdy = \infty,$$

thus yields a contradiction.



Now we proceed to the density argument. Note that for each $u \in Y = Y_1 + Y_2$, for $\varepsilon > 0$, there exists $\varphi_i \in Z_i$ such that

$$\|u - \varphi_1 - \varphi_2\|_{H^1} < \varepsilon.$$

Define $Z := \{\varphi \in C^\infty(\Omega) \colon \varphi = 0 \text{ near the problematic point}\}$, then $Z = Z_1 + Z_2$. It suffices to show that for each $u \in C^\infty(\Omega)$, for all $\varepsilon > 0$, there exists $\varphi \in Z$ such that $\|u - \varphi\|_{H^1} < \varepsilon$. Assume that z is the probematic point, and consider the function $\varphi_\varepsilon \colon \Omega \to \mathbb{C}$ defined as

$$\varphi_\varepsilon(x,y) = \begin{cases} \left(\frac{|(x,y)-z|}{\varepsilon}\right)^\varepsilon, & \text{for } r = |(x,y) - z| \in (0, \varepsilon), \\ 1, & \text{otherwise.} \end{cases}$$

For each $u \in C^\infty(\Omega)$, we have $u_\varepsilon := u \cdot \varphi_\varepsilon \in \tilde{Z} := \{\varphi \in C^1(\Omega), \varphi = 0 \text{ near the problematic point}\}$. Since Z is dense in $\tilde{Z}$, it suffices to show that $\|u - u_\varepsilon\|_{H^1} \to 0$ as $\varepsilon \to \infty$. Note that

$$\|u - u_\varepsilon\|_{L^2} \to 0 \text{ as } \varepsilon \to 0,$$

and

$$\begin{aligned} \| \bigtriangledown u - \bigtriangledown u_\varepsilon\|_{L^2} &= \| \bigtriangledown u(1 - \varphi_\varepsilon) - u \bigtriangledown \varphi_\varepsilon\|_{L^2} \\ &\le C(\|1 - \varphi_\varepsilon\|_{L^2} + \| \bigtriangledown \varphi_\varepsilon\|_{L^2}) \end{aligned}$$

for some positive constant C. It is obvious that $\|1-\varphi_\varepsilon\|_{L^2} \to 0$ as $\varepsilon \to 0$. Also,

$$\begin{aligned}
\| \bigtriangledown \varphi_\varepsilon(x,y)\|_{L^2} &= \| \bigtriangledown \varphi_\varepsilon(r)\|_{L^2} \\
&= \| \left(\frac{r}{\varepsilon}\right)^{\varepsilon-1} (\cos\theta, \sin\theta)\|_{L^2} \\
&= \left(\int_0^\varepsilon \int_0^{2\pi} \left(\frac{r}{\varepsilon}\right)^{2\varepsilon-2} r dr d\theta\right)^{\frac{1}{2}} \\
&= \sqrt{2\pi}\left(\int_0^\varepsilon \left(\frac{r}{\varepsilon}\right)^{2\varepsilon-2} r dr\right)^{\frac{1}{2}} \\
&= \sqrt{2\pi}\sqrt{\frac{\varepsilon}{2}} \\
&= \sqrt{\pi\varepsilon}
\end{aligned}$$

tends to 0 as $\varepsilon \to 0$. Thus, $\|u-u_\varepsilon\|_{H^1} = \|u-u_\varepsilon\|_{L^2} + \| \bigtriangledown u - \bigtriangledown u_\varepsilon\|_{L^2} \to 0$ as $\varepsilon \to 0$. $\square$

Remark 2. *In this case, $M_1^\perp + M_2^\perp$ is not closed, then for each $(r_n) \in c_0$, $r_n \in \mathbb{R}^+$, for all $x \in X$, $\|T^n x - P_M x\| = \|T^n x\| \neq O(r_n)$. This implies that the convergence rate is not exponetially fast. (Proof is given in the* dichotomy results*).*