RATE OF CONVERGENCE

AILI SHAO

Definition 1. *Let H be a Hilbert space, and M_1, M_2 are two closed subspaces of H with intersection $M := M_1 \cap M_2$. The* Friedrichs angle *between M_1 and M_2 is defined to be the angle in $[0, 2\pi]$ whose cosine is given by*

$$c(M_1, M_2) = \sup\{| \langle x, y \rangle | : x \in M_1 \cap M^{\perp}, \|x\| \leq 1, y \in M_2 \cap M^{\perp}, \|y\| \leq 1\}.$$

Theorem 2. *Let H be a Hilbert space, and M_1, M_2 and M be defined as above. If P_1 and P_2 are the orthogonal projections onto M_1 and M_2 respectively, and P_M is the orthogonal projection onto M, then for each $n \in \mathbb{N}$, we have*

$$\|(P_2P_1)^n - P_M\| = c(M_1, M_2)^{2n-1}.$$

Before proving the main theorem, we introduce some fundamental results first.

Lemma 3. *Let $Q_i := P_i(I - P_M)$ for each $i = 1, 2$, then*

$$(P_2P_1)^n - P_M = (Q_2Q_1)^n.$$

Proof.

$$\begin{aligned}
(P_2P_1)^n - P_M &= (P_2P_1)^n - (P_2P_1)^n P_M \\
&= (P_2P_1)^n (I - P_M) \\
&= (P_2P_1)^n P_{M^\perp} \\
&= (P_2P_1)^n P_{M^\perp}^n \text{ as } P_{M^\perp}^2 = P_{M^\perp} \\
&= (P_2P_1P_{M^\perp})^n \text{ as } P_2P_1 \text{ commutes with } P_{M^\perp} \\
&= (P_2P_{M^\perp}P_1P_{M_\perp})^n \\
&= (Q_2Q_1)^n
\end{aligned}$$

where the second last inequality follows from P_i commuting with $P_{M^\perp}$. □

Lemma 4. *If $T \in B(H)$ with H being a Hilbert space is a self-adjoint linear operator, then for each $n \in \mathbb{N} \cup \{0\}$,*

$$\|T^n\| = \|T\|^n.$$

Proof. Note that if T is self-adjoint, we have $\|T^2\| = \|T\|^2$. (B4.2 Hilbert space lecture notes) Similarly, $\|T^4\| = \|T^2\|^2 = \|T\|^4$. By induction, the result is true for $n = 2^m$ with $m \in \mathbb{N} \cup \{0\}$.

For any $n \in \mathbb{N}$ not in this form, we can write $n = 2^m - r$ for some $m, r \in \mathbb{N} \cup \{0\}$, then $\|T\|^{n+r} = \|T^{n+r}\| \leq \|T^n\|\|T^r\| \leq \|T^n\|\|T\|^r$. This gives $\|T\|^n \leq \|T^n\|$, and thus $\|T^n\| = \|T\|^n$. □

Lemma 5. $c(M_1, M_2) = \|Q_2Q_1\| = \sqrt{\|Q_1Q_2Q_1\|}$.

Proof. By definition, we have

$$\begin{aligned}
c(M_1, M_2) &= \sup\{|\langle x, y\rangle| : x \in M_1 \cap M^\perp, \|x\| \le 1, y \in M_2 \cap M^\perp, \|y\| \le 1\}\\
&= \sup\{|\langle P_{M_1\cap M^\perp}x, P_{M_2\cap M^\perp}y\rangle| : \|x\| \le 1, \|y\| \le 1\}\\
&= \sup\{|\langle P_{M_2\cap M^\perp}P_{M_1\cap M^\perp}x, y\rangle| : \|x\| \le 1, \|y\| \le 1\}\\
&= \|P_{M_2\cap M^\perp}P_{M_1\cap M^\perp}\|\\
&= \|(P_{M_2}P_{M^\perp})(P_{M_1}P_{M^\perp})\| \text{ as } P_i \text{ commutes with } P_{M^\perp} \text{ for } i = 1, 2\\
&= \|Q_2Q_1\|.
\end{aligned}$$

Also, $\|Q_2Q_1\|^2 = \|(Q_2Q_1)^*Q_2Q_1\| = \|Q_1Q_2Q_2Q_1\| = \|Q_1Q_2Q_1\|$, then the second inequality follows. □

Now we are ready to prove Theorem 2:

Proof. By Lemma 3, $\|(P_2P_1)^n - P_M\| = \|(Q_2Q_1)^n\|$. Since $((Q_2Q_1)^n)^* = (Q_1Q_2)^n$, we have

$$\|(Q_2Q_1)^n\|^2 = \|(Q_1Q_2)^n(Q_2Q_1)^n\| = \|(Q_1Q_2Q_1)^{2n-1}\|.$$

As the operator $Q_1Q_2Q_1$ is self-adjoint, it follows from Lemma 4 that

$$\|(Q_1Q_2Q_1)^{2n-1}\| = \|Q_1Q_2Q_1\|^{2n-1}.$$

By applying Lemma 5, the result then follows. □

Therefore, $\|(P_2P_1)^n - P_M\|$ converges to 0 expoentially fast if and only if $c(M_1, M_2) < 1$.

Theorem 6. *Let M_i for $1 \le i \le r$ be closed subspaces in the Hilbert space H, and $M := \bigcap_i^r M_i$. Let P_{M_i} and P_M be the orthogonal projections onto M_i and M respectively. If $T = P_rP_{r-1}\cdots P_1$, then $\|T^n - P_M\|$ converges to 0 exponetially fast if and only if* $\mathrm{Im}(I - T)$ *is closed.*

Proof. Since $M := \bigcap_i^r M_i$ is closed, $H = M \oplus M^\perp$. We have proved that $M = \mathrm{Ker}(I - T^*)$, it follows that $M^\perp = \overline{\mathrm{Im}(I - T)}$. Let $Y = \mathrm{Im}(I - T)$, and $Z = \overline{Y}$.

In the proof of the *dichotomy results*, we have showed that the convergence is exponentially fast if and only if $r(S) < 1$ where $S := T\,|_Z = TP_{M^\perp}$. Then $I - S : Z \to Z$ has trivial kernel since if $(I - S)x = 0$ for some $x \in Z$, we have $x = Sx = Tx$, that is, $x \in M^\perp \cap M = \{0\}$. We also have $\mathrm{Im}(I - S) = Y$. For each $y \in \mathrm{Im}(I - S)$, there exists $x \in Z$ such that

$$\begin{aligned}
y &= (I - S)x\\
&= x - Sx\\
&= x - TP_{M^\perp}x\\
&= x - Tx \text{ as } x \in Z = M^\perp\\
&= (I - T)x.
\end{aligned}$$

This implies that $\mathrm{Im}(I-S) \subseteq Y$. If $y \in Y = \mathrm{Im}(I-T)$, there exists $x \in H$ such that

$$\begin{aligned}
y &= P_{M^\perp} y \text{ as } y \in Y \subseteq M^\perp \\
&= P_{M^\perp}(I-T)x \\
&= P_{M^\perp}x - P_{M^\perp}Tx \\
&= P_{M^\perp}x - P_{M^\perp}^2 Tx \\
&= P_{M^\perp}(I - P_{M^\perp}T)x \\
&= P_{M^\perp}(I - TP_{M^\perp})x \text{ as } T \text{ commutes with } P_{M^\perp} \\
&= P_{M^\perp}(I-S)x \\
&= (I-S)x.
\end{aligned}$$

This shows that $Y \subseteq \mathrm{Im}(I-S)$. So $I-S$ is a bounded bijiection from Z onto Y. By *Inverse Mapping Theorem*, $I-S$ is invertible if and only if $Y = Z$. So $1 \in \sigma(S)$ if and only if $Y \neq Z$. That is $r(S) < 1$ if and only if $Y = Z$. $\square$