UNIVERSITY OF BRITISH COLUMBIA

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1 Schwarz Methods–Methods of Proof (By L. Halpern)

The sample problem we consider is

$$\eta u - \Delta u = f.$$

Laurence's talk consists of the following three Schwarz-type algorithms:

1.1 Schwarz Algorithm

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = U & \text{on } \partial\Omega. \end{cases}$$

We define

$$g := \sup U$$

and

$$k = \inf U.$$

Recall the *Maximal Principle*:

Theorem 1. *If $u \in \mathcal{C}^2(\bar{\Omega})$, harmonic in Ω, then* $\max u = \max U$.

- Step 1:

$$\begin{cases} -\Delta u_1 = 0 & \text{in } \Omega_1 \\ u_1 = U & \text{on } \Gamma_1^\circ \\ u_1 = k & \text{on } \Gamma_1 \end{cases}$$

By Maximum Principle, we have

$$k \le u_1 \le g.$$

$$\begin{cases} -\Delta u_2 = 0 & \text{in } \Omega_2 \\ u_2 = U & \text{on } \Gamma_2^\circ \\ u_2 = u_1 & \text{on } \Gamma_2 \end{cases}$$

Similarly, by Maximum Principle, we have

$$k \le u_2 \le g.$$

- Step 2: what is the value of $u_2 - u_1$ on Γ_1?

$$0 \le u_2 - u_1 = u_2 - k \le g - k := G$$

$$0 \le u_2 - u_1 \le G \text{ on } \Gamma_1.$$

- Step 3: now we compute u_3 on Ω_1 by solving

$$\begin{cases} -\Delta u_3 = 0 & \text{on } \Omega_1 \\ u_3 = U & \text{on } \Gamma_1^\circ \\ u_3 = u_2 & \text{on } \Gamma_1 \end{cases}$$

By Maximum Principle again,

$$k \le u_3 \le g.$$

$$u_3 - u_1 = \begin{cases} 0 & \text{on } \Gamma_1^\circ \\ u_2 - u_1 & \text{on } \Gamma_1 \end{cases}$$

Now we have

$$\begin{cases} -\Delta(u_3 - u_1) = 0 & \text{on } \Omega_1 \\ u_3 - u_1 = 0 & \text{on } \Gamma_1^\circ \\ 0 \le u_3 - u_1 \le G & \text{on } \Gamma_1 \end{cases}$$

Key Lemma (Schwarz):

Lemma 1. *For the following linear elliptic PDE*

$$\begin{cases} -\Delta v = 0 & \text{on } \Omega \\ v = 0 & \text{on } \Gamma_1^\circ \\ v = 1 & \text{on } \Gamma_1, \end{cases}$$

there exists $q < 1$ such that

$$0 \le \max v \le q \cdot 1 = q.$$

Thus, there exists q_1 such that

$$0 \le u_3 - u_1 \le Gq_1$$

on Γ_2.

$$0 \le u_4 - u_2 \le Gq_1 q_2.$$

Inductively, for u_{2k} in Ω_2, and u_{2k-1} in Ω_1, we have

$$0 \le (u_{2k+2} - u_{2k}) \mid_{\Gamma_1} \le G(q_1 q_2)^k \le \text{ in } \Omega_2,$$
$$0 \le (u_{2k+1} - u_{2k-1}) \mid_{\Gamma_2} \le G(q_1 q_2)^{k-1} q_1 \text{ in } \Omega_1.$$

We can then show that

$$u_{2k} = \sum_{p=1}^{k} (u_{2p} - u_{2p-2}) \to v$$
$$u_{2k-1} = \sum_{p=1}^{k-1} (u_{2p+1} - u_{2p-1}) \to w$$

where w and v are equal on some boundaries.

1.2 Schwarz-Robin

We are solving the same Poisson problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

where $\Omega = \Omega_1 \cup \Omega_2$ is a union of two non-overlapping sub-domains.
We call the interior common boundary Γ in this case.
Similarly, we solve

$$-\Delta u_1 = f \text{in } \Omega_1$$
$$-\Delta u_2 = f \text{in } \Omega_2$$

but on the boundary Γ, we impose that

$$\begin{cases} u_1 = u_2, \\ \frac{\partial u_1}{\partial n_1} + \frac{\partial u_2}{\partial n_2} = 0. \end{cases}$$

The problem at kth iteration we are solving is

$$\begin{cases} -\Delta u_j^k = f \text{in } \Omega_j \\ \frac{\partial u_j^k}{\partial n_j} + p u_j^k = \frac{\partial u_i^{k-1}}{\partial_j} + p u_i^{k-1}. \end{cases}$$

where $i = 1, j = 2$ or $j = 1, i = 2$ and p is a parameter.

Theorem 2. *"In some way", $u_j^k \to u_j$ as $k \to \infty (j = 1, 2)$.*

Proof.

$$-\int_{\Omega_1} u_1^k \Delta u_1^k = 0$$

$$\|\nabla u_1^k\|^2_{L^2(\Omega_1)} - \int_\Gamma \frac{\partial u_1^k}{\partial n_1} \cdot u_1^k dy = 0$$

By the identity $ab = \frac{1}{4p}[(a+bp)^2 - (a-bp)^2]$, we have that

$$\|\nabla u_1^k\|^2_{L^2(\Omega_1)} + \frac{1}{4p}\|\frac{\partial u_1^k}{\partial n_1} - pu_1^k\|^2_{L^2(\Gamma)} = \frac{1}{4p}\|\frac{\partial u_1^k}{\partial n_1} + pu_1^k\|^2_{L^2(\Gamma)}$$

$$\|\nabla u_2^k\|^2_{L^2(\Omega_2)} + \frac{1}{4p}\|\frac{\partial u_2^k}{\partial n_2} - pu_2^k\|^2_{L^2(\Gamma)} = \frac{1}{4p}\|\frac{\partial u_2^k}{\partial n_2} + pu_2^k\|^2_{L^2(\Gamma)}$$

By defining the L^2-norm to be E and using the fact that $\frac{\partial u_1^k}{\partial n_1} + pu_1^k = \frac{\partial u_2^{k-1}}{\partial n_1} + pu_2^{k-1}$ and $\frac{\partial u_2^k}{\partial n_2} + pu_2^k = \frac{\partial u_1^{k-1}}{\partial n_2} + pu_1^{k-1}$, we have

$$\begin{aligned}\sum_{k=1}^{K}[E(u_1^k) + E(u_2^k)] + \frac{1}{4p}\|\frac{\partial u_1^k}{\partial n_1} - pu_1^k\|^2_{L^2(\Gamma)} + \frac{1}{4p}\|\frac{\partial u_2^k}{\partial n_2} - pu_2^k\|^2_{L^2(\Gamma)} \\ = \frac{1}{4p}\|\frac{\partial u_1^k}{\partial n_1} - pu_1^k\|^2_{L^2(\Gamma)} + \frac{1}{4p}\|\frac{\partial u_2^0}{\partial n_2} - pu_2^0\|^2_{L^2(\Gamma)}.\end{aligned}$$

This implies that $E(u_1^k) + E(u_2^k) \to 0$ as $k \to \infty$. □

1.3 Optimised Schwarz

We solve

$$-u_1^k + \eta u_1^k = f \text{ in } \Omega_1$$

$$-u_2^k + \eta u_2^k = f \text{ in } \Omega_2$$

on two overlapping sub-domains Ω_1 and Ω_2 separately.
Without loss of generality, we may take $f = 0$. Consider a Fourier transform in y, we have

$$-\partial_{xx} u_\xi^k + (\xi^2 + \eta) u_\xi^k = 0.$$

$$u_1^k = \alpha_1^k(\xi) e^{+\sqrt{\eta+\xi^2}x}$$

$$u_2^k = \alpha_2^k(\xi) e^{-\sqrt{\eta+\xi^2}x}$$

where α_1 and α_2 satisfy

$$\alpha_1^k(\xi)(\xi + p)e^{\xi L} = \alpha_2^{k-1}(\xi)(-\xi + p)e^{-\xi L}.$$

$$\alpha_2^k(\xi)(\xi+p) = \alpha_1^{k-1}(\xi)(-\xi+p).$$

Thus

$$\alpha_1^k(\xi) = (\frac{-\xi+p}{\xi+p})^2 e^{-2\xi L}\alpha_1^{k-1}$$

where $(\frac{-\xi+p}{\xi+p})^2 e^{-2\xi L}$ is the convergence factor.
For high frequency, the convergence factor is close to 1.