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Nonlinear Analysis and Applications -Problem Sheet Four

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Throughout assume that $p \in (1, \infty)$ and that Ω is a proper, bounded Lipschitz domain in $\mathbb{R}^n$.

Problem 1. *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a continuous integrand satisfying the growth condition (G_p):*

$$|F(z)| \leq c(|z|+1)^p \ \ \forall z \in \mathbb{R}^{N\times n}.$$

Let $g \in W^{1,p}(\Omega, \mathbb{R}^N)$ and assume that the variational integral

$$I(u,\Omega) := \int_\Omega F(\nabla u(x))\,\mathrm{d}x$$

is swlsc on the Dirichlet class $W^{1,p}_g(\Omega, \mathbb{R}^N)$:

$$\liminf_{j\to\infty} I(u_j,\Omega) \geq I(u,\Omega)$$

holds whenever $u_j, u \in W^{1,p}_g(\Omega, \mathbb{R}^N)$ and $u_j \rightharpoonup u$ in $W^{1,p}$.

(a) *Prove that F is quasiconvex.*

Proof. We adapt a proof from Meyers [1]. Let x_0 be an arbitrary point in Ω, and let Q_h be the cube such that $x_0^i \leq x^i \leq x^i + \frac{1}{h}$ for each $i = 1, 2 \ldots n$. Let $\varphi = \varphi(x)$ be an arbitrary function in $W^{1,p}_0(Q_1, \mathbb{R}^N)$ and extend $\varphi(x)$ to all of $\mathbb{R}^n$ with period equal to 1 in each of the x^i. Then define

$$\varphi_{h,j}(x) = \begin{cases} \frac{1}{hj}\varphi(hj(x-x_0)+x_0) & \text{for } x \in Q_h, \\ 0 & \text{for } x \notin Q_h. \end{cases}$$

where $h, j = 1, 2 \ldots$. Note that $\varphi_{h,j} \in W^{1,p}_0(Q_h, \mathbb{R}^N)$ and is periodic in Q_h with Q_{hj} as a period cube. Number the period cubes in Q_h in some manner and denote them by $Q_{hj,k}(k = 1, 2 \dots, j^n)$ with $Q_{hj,1} = Q_{hj}$ and denote the corner of each $Q_{hj,k}$ nearest to x_0 by x_k. Let $u(x)$ be a function in $W_g^{1,p}(\Omega, \mathbb{R}^N)$ which is a smooth function in some neighbourhood of x_0 and for which $\nabla u(x_0) = z$ for an arbitrary constant matrix $z \in \mathbb{R}^{N\times n}$. For sufficiently large h such that $Q_h \subset \Omega$. Define $u_j(x) = u(x) + \varphi_{h,j}(x) \in W_g^{1,p}(\Omega, \mathbb{R}^N)$, then we know that $u_j \rightharpoonup u$ in $W^{1,p}$ since $\varphi_{hj}(x) \rightharpoonup 0$ in $W^{1,p}$ by Riemann-Lebesgue lemma. Note that

$$
\begin{aligned}
I(u_j) =& \sum_{k=1}^{j^n} \int_{Q_{hj,k}} F(\nabla u(x) + \nabla \varphi_{h,j}(x)) \, \mathrm{d}x \\
=& \sum_{k=1}^{j^n} \int_{Q_{hj,k}} F(\nabla u(x) + \nabla \varphi_{h,j}(x)) - F(\nabla u(x_k) + \nabla \varphi_{h,j}(x)) \, \mathrm{d}x \\
&+ \sum_{k=1}^{j^n} \int_{Q_{hj,k}} F(\nabla u(x_k) + \nabla \varphi_{h,j}(x)) \, \mathrm{d}x.
\end{aligned}
$$

For h sufficiently large, by continuity of ∇u and F, the first term converges to 0 as $j \to \infty$. By taking $y = hj(x - x_k) + x_0$ in the kth integral, the second term is reduced to

$$
\sum_{k=1}^{j^n} \int_{Q_{hj,k}} F(\nabla u(x_k) + \nabla \varphi_{h,j}(x)) \, \mathrm{d}x = \sum_{k=1}^{j^n} (hj)^{-n} \int_{Q_1} F(\nabla u(x_k) + \nabla \varphi(y)) \, \mathrm{d}y \quad (0.1)
$$

(0.1) being a Riemann sum, tends, as $j \to \infty$, to

$$
\lim_{j\to\infty} I(u_j; Q_h) = \int_{Q_h} \int_{Q_1} F(\nabla u(x) + \nabla \varphi(y)) \, \mathrm{d}y \, \mathrm{d}x. \quad (0.2)
$$

Since we are assuming sequentially weakly lower semi-continuity, we have

$$
\begin{aligned}
\liminf_{j\to\infty} I(u_j, \Omega) =& \liminf_{j\to\infty} I(u_j, Q_h) + I(u, \Omega \setminus Q_h) \\
\geq& I(u, Q_h) + I(u, \Omega \setminus Q_h).
\end{aligned}
$$

Hence from (0.2), we have

$$
\int_{Q_h} \int_{Q_1} F(\nabla u(x) + \nabla \varphi(y)) \, \mathrm{d}y \, \mathrm{d}x \geq \int_{Q_h} F(\nabla u(x)) \, \mathrm{d}x. \quad (0.3)
$$

Multiplying both sides of (0.3) by h^n and letting $h \to \infty$, we have

$$
\int_{Q_1} F(\nabla u(x_0) + \nabla \varphi(y)) \, \mathrm{d}y \geq F(\nabla u(x_0)). \quad (0.4)
$$

That is,

$$
\int_{Q_1} F(z + \nabla \varphi(y)) \, \mathrm{d}y \geq F(z) \text{ for } z \in \mathbb{R}^{N\times n}, \varphi \in W_0^{1,p}(\Omega, \mathbb{R}^N).
$$

Since F satisfies (G_p), the above inequality implies that I is quasiconvex.

Fix $A \in \mathbb{R}^{N\times n}$. Let $Q \subset\subset \Omega$. Without loss of generality that we can assume Q is a unit cube. By changing g in a neighbourhood of Q, without loss of generality, we may assume that $g(x) = Ax$ for $x \in Q$. Take $\varphi \in C_0^\infty(Q, \mathbb{R}^N)$, extended it periodically. Define $\varphi_j(x) := 2^{-j}\varphi(2^j x)\mathbb{1}_Q$, then $\varphi_j \to 0$ in $W_0^{1,p}(Q, \mathbb{R}^N)$. By swlsc, we know that

$$\begin{aligned} 0 &\leq \liminf_{j\to\infty} \int_\Omega F(\nabla g + \nabla\varphi_j - F(\nabla g) \\ &= \liminf_{j\to\infty} \int_Q F(A + \nabla\varphi_j) - F(A). \end{aligned}$$

But

$$\int_Q F(A + \nabla\varphi_j)\,\mathrm{d}x = \int_Q F(A + \nabla\varphi(2^j x))\,\mathrm{d}x = \int_Q F(A + \nabla\varphi)\,\mathrm{d}x$$

by periodicity of φ. So we can see that

$$0 \leq \int_Q F(A + \nabla\varphi) - F(A).$$

□

(b) *Now assume in addition that F is pointwise p-coercive:$F(z) \geq c_1|z|^p + c_2$ holds for all $z \in \mathbb{R}^{N\times n}$, where $c_1 > 0$ and $c_2 \in \mathbb{R}$ are constants. Prove that $I(\cdot, \Omega)$ is swlsc on $W^{1,p}(\Omega, \mathbb{R}^N)$.*

Proof. Let v_j be a sequence in $W^{1,p}(\Omega, \mathbb{R}^N)$ such that $v_j \rightharpoonup v$ weakly in $W^{1,p}(\Omega, \mathbb{R}^N)$. Since F is quasiconvex and satisfies the p-growth condition, by the Morrey, Meyers, Fusco theorem from lecture notes, we have

$$\liminf_{j\to\infty} \int_\Omega \rho F(\nabla v_j)\mathrm{d}x \geq \int_\Omega \rho F(\nabla v)\,\mathrm{d}x \text{ for } \rho \in C_0(\Omega)^+.$$

Since F is pointwise p-coercive, we may assume without loss of generality that $F(z) \geq 0$. Taking $\rho \in C_0(\Omega)^+$ such that $0 \leq \rho \leq 1$, we have

$$\begin{aligned} \liminf_{j\to\infty} \int_\Omega F(\nabla v_j)\mathrm{d}x &\geq \liminf_{j\to\infty} \int_\Omega \rho F(\nabla v_j)\mathrm{d}x \\ &\geq \int_\Omega \rho F(\nabla v)\,\mathrm{d}x. \end{aligned}$$

Passing to the limit $\rho \to 1$, we have that

$$\liminf_{j\to\infty} \int_\Omega F(\nabla v_j)\mathrm{d}x \geq \int_\Omega F(\nabla v)\,\mathrm{d}x.$$

That is, $I(\cdot, \Omega)$ is sequentially weakly lower semicontinuous on $W^{1,p}(\Omega, \mathbb{R}^N)$. Take $\rho_k \in C_0^+(\Omega)$ such that $\rho_k \nearrow \mathbb{1}_\Omega$. Then send $k \nearrow \infty$ and apply MCT.

□

(c) *Next, prove that $I(\cdot,\Omega)$ is weakly semicontinuous on $W^{1,p}(\Omega,\mathbb{R}^N)$.*

Proof. Assume for contradiction that there exist $\varepsilon > 0$ and $u \in W^{1,p}$ so that for every weakly open neighbourhood $\mathcal{V}$ of u in $W^{1,p}$, we can find $v \in \mathcal{V}$ with $I(v,\Omega) \leq I(u,\Omega) - \varepsilon$. Now suppose that $\{\varphi_j : j \in \mathbb{N}\}$ is a system of smooth maps that is dense in $L^{p'}(\Omega,\mathbb{R}^N)$ and so $\varphi_j = \mathbb{1}_\Omega e_j$ for $1 \leq j \leq N$ where (e_j) is the standard basis in $\mathbb{R}^n$. Then

$$\mathcal{V}_s := \bigcup_{j,k\leq s} \{v \in W^{1,p} \colon -\frac{1}{k} < \int_\Omega (v-u)\cdot\varphi_j \mathrm{d}x < \frac{1}{k}\}$$

is for each $s \in \mathbb{N}$ a weakly open neighbourhood of u in $W^{1,p}$. Now consider a sequence $(v_s)_{s=1}^\infty$ such that $v_s \in V_s$ for each $s \in \mathbb{N}$. Note that $I(v_s) = \int_\Omega F(\nabla v_s(x))\,\mathrm{d}x \geq \int_\Omega c_1|\nabla v_s|^p + c_2$ by pointwise p-coercivity of F. we know that $I(v_s) < \infty$, then we can deduce that ∇v_s is uniformly bounded in L^p. By Poincare's inequality, we know that $\|v_s - (v_s)_\Omega\|_{L^p(\Omega),\mathbb{R}^N)} \leq \|\nabla v_s\|_{L^p(\Omega)}$. Thus v_s is uniformly bounded in L^p. We have shown that $\sup_s \|v_s\|_{W^{1,p}(\Omega,\mathbb{R}^N} < \infty$. Then by Banach Alaoglu theorem, we know that there exists a subsequence $v_{s_k} \rightharpoonup v$ weakly in $W^{1,p}$. By definition of v_{s_k}, we know that $v_{s_k} \rightharpoonup u$ weakly in L^p. By uniqueness of the limit, we have that $v = u$ a.e. Since $I(\cdot,\Omega)$ is swlsc, we have

$$I(u,\Omega) \leq \liminf_{k\to\infty} I(v_{s_k},\Omega) \leq I(u,\Omega) - \varepsilon$$

which yields a contradiction.

Since F is coercive, we only need to check that lower semi-continuity on the intersection of balls with w-open sets. On bounded sets, w-topology is metrizable, so w-continuous if and only if sequentially w-continuous. Assume not, let $V_k := \bigcup_{j\leq k}\{v \in W^{1,p} \colon |\int_\Omega (v-u)\cdot\varphi_j| < \frac{1}{k}\}$. Then V_k is an w-open neighbourhood of u in $W^{1,p}$. Let $\varphi_i := \mathbb{1}_\Omega e_i$ for $1 \leq i \leq N$. $\{\varphi_i\}$ are dense in $L^{p'}$. Let $v_k \in V_k$ such that $I[u_k] \leq I[u] - \varepsilon$. By p-coercivity, $\int_\Omega |\nabla v_k|^p\,\mathrm{d}x \lesssim \int_\Omega |\nabla u|^p\,\mathrm{d}x < \infty$. Moreover, $\int_\Omega |v_k|^p\,\mathrm{d}x \lesssim \int_\Omega |v_k - (v_k)_\Omega|^p\,\mathrm{d}x + \int_\Omega |(v_k)_\Omega - (u)_\Omega|^p\,\mathrm{d}x + \int_\Omega |(u)_\Omega - u|^p\,\mathrm{d}x + \int_\Omega |u|^p\,\mathrm{d}x < \infty$ by Poincaré inequality. Hence v_k is bounded in $W^{1,p}$ so $v_k \rightharpoonup v$ in $W^{1,p}$. For all $1 \leq i \leq N$, $\langle v, \varphi_i\rangle = \lim_k \langle v_k, \varphi_j\rangle = \langle u, \varphi_j\rangle$. Since φ_i are dense, $u = v$ a.e. But then $I[u] \leq \liminf_k I[u_k] \leq I[u] - \varepsilon$. ☐

Problem 2. *(a) Let $\nu \in GY^n(\Omega,\mathbb{R}^{n\times n})$ be a gradient n-Young measure. Prove directly from the Decomposition Lemma and Problem 1 on Sheet 3 that*

$$\langle \nu_x, \det\rangle = \det\bar{\nu}_x \;\; \mathcal{L}^n \;\; a.e.\; x \in \Omega$$

and

$$\langle \nu_x^\infty, \det\rangle = 0 \;\; \lambda \;\; a.e.\; x \in \Omega. \tag{0.5}$$

Conclude from Tartar's example mentioned in Lecture 5 that (0.5) can fail at points $x \in \partial\Omega$.

Proof. Since $\nu \in GY^n(\Omega, \mathbb{R}^{n\times n})$ is a gradient n-Young measure, there exists a sequence $(v_j) \subset W^{1,n}(\Omega, \mathbb{R}^n)$ such that

$$v_j \rightharpoonup v \text{ in } W^{1,p}(\Omega, \mathbb{R}^n) \quad \text{and } \nabla v_j \xrightarrow{Y^n} \nu.$$

By the Decomposition Lemma, we know that there exist a subsequence, we call it v_j as well, and sequences (g_j) in $C_c^\infty(\Omega, \mathbb{R}^n)$, (b_j) in $W^{1,n}(\Omega, \mathbb{R}^n)$ with $g_j \overset{W^{1,n}}{\rightharpoonup} 0$, $b_j \overset{W^{1,n}}{\rightharpoonup} 0$, (∇g_j) is p-equi-integrable, $\nabla b_j \to 0$ in measure, and $v_j = v + g_j + b_j$. Consider $u_j := v + g_j$, then we know that $u_j \rightharpoonup v$ weakly in $W^{1,n}(\Omega, \mathbb{R}^n)$. In particular, $\nabla u_j \rightharpoonup \nabla v$ in $L^n(\Omega, \mathbb{R}^N)$ where $\nabla v = \bar{\nu}$ a.e. $x \in \Omega$. Note that $\nabla u_j = \nabla(v + g_j)$ carries the oscillation while (∇b_j) carries the L^n-concentration. That is, the n-Young measure generated by ∇u_j is $((\nu_x)_{x\in\Omega}, 0, n/a)$ while the n-Young measure generated by (∇b_j) is $(\delta_0, \lambda, (\nu_x^\infty)_{x\in\Omega})$. By Problem 1 on Sheet 3, we have

$$\int_\Omega \varphi \det \nabla v_j \mathrm{d}x \to \int_\Omega \varphi \det \bar{\nu}_x \,\mathrm{d}x \text{ as } j \to \infty.$$

Also

$$\int_\Omega \varphi \det \nabla v_j \mathrm{d}x = \int_\Omega \varphi \det \nabla u_j \mathrm{d}x + \int_\Omega \varphi \det \nabla b_j \mathrm{d}x.$$

Note that since $\nabla b_j \rightharpoonup 0$ weakly in $W^{1,n}(\Omega)$, we have $\int_\Omega \varphi \det \nabla b_j \mathrm{d}x \to 0$ as $j \to \infty$ by the second part of Problem 1 on Sheet 3. Since (∇u_j) only carries oscillation, we have

$$\int_\Omega \varphi \left\langle \nu_x, \det \right\rangle \mathrm{d}x = \int_\Omega \varphi \det \bar{\nu}_x \mathrm{d}x \text{ for all } \varphi \in C_0(\Omega).$$

This implies that

$$\langle \nu_x, \det \rangle = \det \bar{\nu}_x \ \ \mathcal{L}^n \ \ a.e.\ x \in \Omega.$$

We also proved in Problem 1 from Sheet 3 that

$$\langle \nu_x, \det \rangle + \langle \nu_x^\infty, \det \rangle \frac{d\lambda}{d\mathcal{L}^n}(x) = \det \bar{\nu}_x \ \ \mathcal{L}^n a.e.$$

and

$$\langle \nu_x^\infty, \det \rangle = 0 \ \ \lambda^s \ a.e.$$

Therefore, $\langle \nu_x^\infty, \det \rangle = 0 \ \ \lambda \ a.e.\ x \in \Omega$.

Take $n = 2$, $\Omega = (0, \delta)^2$ where $\delta \in (0, 1)$. Define $u_j(x, y) = \frac{(1-y)^j}{\sqrt{j}}(\sin(jx), \cos(jx))$. We know that $u_j \rightharpoonup 0$ in $W^{1,2}(\Omega, \mathbb{R}^2)$ since $\|u_j\|_\infty = \frac{1}{\sqrt{j}} \to 0$ and $\|\nabla u_j\|_2 < \sqrt{2}\delta$. Note that

$$\nabla u_j = \begin{bmatrix} \sqrt{j}(1-y)^j \cos(jx) & -\sqrt{j}(1-y)^j \sin(jx) \\ -\sqrt{j}(1-y)^{j-1} \sin(jx) & -\sqrt{j}(1-y)^{j-1} \cos(jx) \end{bmatrix}.$$

Then $\det(\nabla u_j) = -j(1-y)^{2j-1}$. This shows that

$$\int_\Omega \det(\nabla u_j) \mathrm{d}x \mathrm{d}y = -j\delta(\frac{1}{2j} - \frac{(1-\delta)^{2j}}{2j} \to -\frac{\delta}{2} < 0.$$

This implies that the sequence (∇u_j) is L^2 concentrates near the boundary $y = 0$. That is, (0.5) fails in this case. $\square$

(b) Suppose that $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is a quasiconvex integrand satisfying growth condition (G_p) and let $\nu \in GY^p(\Omega, \mathbb{R}^{N\times n})$ be a gradient p-Young measure. Prove that for $\mathcal{L}^n$ almost every $x \in \Omega$ we have Jensen's inequality for F and ν_x:

$$\int_{\mathbb{R}^{N\times n}} F \,\mathrm{d}\nu_x \geq F(\bar{\nu}_x).$$

Proof. Since $\nu \in GY^p(\Omega, \mathbb{R}^{N\times n})$, there exists a sequence $(u_j) \subset W^{1,p}(\Omega, \mathbb{R}^N)$ such that $u_j \rightharpoonup u$ in $W^{1,p}(\Omega, \mathbb{R}^N)$ and $\nabla u_j \overset{Y^p}{\to} \nu$. By Problem 1 from Sheet 2, we know that $\nabla u(x) = \bar{\nu}_x$ a.e. $x \in \Omega$. Since $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is a quasiconvex integrand and F satisfies the p-growth condition (G_p), we have that

$$\liminf_{j\to\infty} \int_\Omega \rho F(\nabla u_j)\,\mathrm{d}x \geq \int_\Omega \rho F(\nabla u)\,\mathrm{d}x \text{ for all } \rho \in C_0(\Omega)^+.$$

By a similar argument as part (a), we can find a subsequence of (u_j), also call it u_j such that we can decompose $u_j = u + g_j + b_j$ where b_k and g_k are defined as before. Consider the sequence $v_j =: u + g_j$, then $v_j \rightharpoonup u$ weakly in $W^{1,p}(\Omega, \mathbb{R}^N)$, and the gradient p-Young measure generated by v_j is $((\nu_x)_{x\in\Omega}, 0, n/a)$. This implies that

$$\liminf_{j\to\infty} \int_\Omega \rho F(\nabla v_j)\,\mathrm{d}x = \int_\Omega \rho \langle \nu_x, F\rangle \,\mathrm{d}x \text{ for all } \rho \in C_0(\Omega)^+.$$

So we have

$$\int_\Omega \rho \langle \nu_x, F\rangle \,\mathrm{d}x \geq \int_\Omega \rho F(\nabla u)\,\mathrm{d}x = \int_\Omega \rho F(\bar{\nu}_x)\,\mathrm{d}x \text{ for all } \rho \in C_0(\Omega)^+.$$

As the above inequality is true for all $\rho \in C_0(\Omega)^+$, we can conclude that

$$\int_{\mathbb{R}^{N\times n}} F \,\mathrm{d}\nu_x = \langle \nu_x, F\rangle \geq F(\bar{\nu}_x).$$

F is quasiconvex, then rank-one convex, which in turn implies that F is continuous.

$\square$

Problem 3. *Assume that $F \in \mathbb{E}_p(\Omega, \mathbb{R}^{N\times n})$ and that the partial function $F(x, \cdot)$ is quasiconvex for each $x \in \Omega$. Using the localization principle for gradient p-Young measures prove that if $u_j \rightharpoonup u$ in $W^{1,p}(\Omega, \mathbb{R}^N)$, then*

$$\int_\Omega \rho F(\cdot, \nabla u)\,\mathrm{d}x \leq \liminf_{j\to\infty} \int_\Omega \rho F(\cdot, \nabla u_j)\,\mathrm{d}x$$

holds for each $\rho \in C_0(\Omega)^+$.

Proof. Considering a subsequence if necessary, we can assume that

$$\liminf_{j\to\infty}\int_\Omega \rho F(\cdot,\nabla u_j)\,\mathrm{d}x = \lim_{j\to\infty}\int_\Omega \rho F(\cdot,\nabla u_j)\,\mathrm{d}x.$$

Assume that $\nabla u_j \overset{Y^p}{\to} \nu$, then it follows from definition that

$$\lim_{j\to\infty}\int_\Omega \rho F(\cdot,\nabla u_j)\,\mathrm{d}x = \int_\Omega \rho\left\langle \nu_x, F\right\rangle\,\mathrm{d}x + \int_{\bar\Omega} \rho\left\langle \nu_x^\infty, F_p^\infty\right\rangle\,\mathrm{d}\lambda.$$

We may assume without loss of generality that $F\geq 0$ on $B(x_0,r)\subset\Omega$ for r small since quasiconvex integrand (thus rank-one convex integrand) is locally bounded (see Problem 4 on Sheet 3), then we have

$$\lim_{j\to\infty}\int_{B(x_0,r)} \rho F(x,\nabla u_j)\,\mathrm{d}x \geq \int_{B(x_0,r)} \rho\left\langle \nu_x, F\right\rangle\,\mathrm{d}x.$$

Recall from the localization principle that for $\nu = ((\nu_x)_{x\in\Omega},\lambda,(\nu_x^\infty)_{x\in\Omega})\in GY^p(\Omega,\mathbb{R}^{N\times n})$, for $\mathcal{L}^n$-a.e, $x_0\in\Omega$, $\nu_{x_0} = \left((\nu_{x_0})_{y\in B_1(0)}, \frac{\mathrm{d}\lambda}{\mathrm{d}\mathcal{L}^n}(x_0)\mathcal{L}^n\lfloor B_1(0), (\nu_x^\infty)_{y\in B_1(0)}\right)$ is a gradient p-Young measure, and its barycentre is $\bar\nu_{x_0}\mathbb{1}_{B_1(0)}$. Note that $F\in\mathbb{E}_p(\Omega,\mathbb{R}^{N\times n})$ if and only if

$$F\in C(\bar\Omega\times\mathbb{R}^{N\times n}) \text{ and } \sup_{(x,z)}\frac{|F(x,z)|}{(1+|z|)^p}<\infty.$$

Thus, $F(x,\cdot)$ satisfies the p-growth condition. We also have that $F(x,\cdot)$ is quasiconvex for each $x\in\Omega$, then the Jensen's inequality from Problem 2(b) reads

$$\langle v_x, F(x,\cdot)\rangle \geq F(x,\bar\nu_x) = F(x,\nabla u) \text{ for } \mathcal{L}^n a.e. x\in\Omega.$$

Then,

$$\lim_{j\to\infty}\int_{B(x_0,r)} \rho F(x,\nabla u_j)\,\mathrm{d}x \geq \int_{B(x_0,r)} \rho\left\langle \nu_x, F\right\rangle\,\mathrm{d}x \geq \int_{B(x_0,r)} \rho F(x,\nabla u(x))\,\mathrm{d}x$$

for each $x_0\in\Omega$ such that $B(x_0,r)\subset\Omega$. Therefore,

$$\liminf_{j\to\infty}\int_\Omega \rho F(\cdot,\nabla u_j)\,\mathrm{d}x \geq \int_\Omega \rho\left\langle \nu_x, F\right\rangle\,\mathrm{d}x \geq \int_\Omega \rho F(\cdot,\nabla u)\,\mathrm{d}x$$

for each $\rho\in C_0(\Omega)^+$.
Assume without loss of generality that $\liminf_{j\to\infty}\int_\Omega \rho F(\cdot,\nabla u_j)\,\mathrm{d}x = \lim_{j\to\infty}\int_\Omega \rho F(\cdot,\nabla u_j)\,\mathrm{d}x$. For $u_j \rightharpoonup u$ in $W^{1,p}$, ∇u_j generates $((\nu_x)_{x\in\Omega},\lambda,(\nu_x^\infty)_{x\in\Omega})$. By the localization principle, we know that for $\mathcal{L}^n$-a.e. $x\in\Omega$, $((\nu_x)_{y\in B},\frac{\mathrm{d}\lambda}{\mathrm{d}\mathcal{L}^n}(x)\mathrm{d}\mathcal{L}^n\lfloor B_1(0),(\nu_x^\infty)_{y\in B_1(0)})$ is a p-gradient Young measure. Consider the sequence $\bar\nu_x+\nabla\varphi_j$ that generates $((\nu_x)_{y\in B},\frac{\mathrm{d}\lambda}{\mathrm{d}\mathcal{L}^n}(x)\mathrm{d}\mathcal{L}^n\lfloor B_1(0),(\nu_x^\infty)_{y\in B_1(0)})$ with $\varphi_j\in C_c^\infty(B)$ such that $\varphi_j\overset{W^{1,p}}{\rightharpoonup}0$. Then we have

$$\lim_{j\to\infty}\int_B F(x,\bar\nu_x+\nabla\varphi_j)\,\mathrm{d}y \geq F(x,\bar\nu_x) = F(x,\nabla u(x))$$

by quasiconvexity of $F(x,\cdot)$. Fix x, we have

$$\int_B \langle \nu_x, F(x,\cdot)\rangle \, \mathrm{d}y + \int_B \langle \nu_x^\infty, F(x,\cdot)\rangle \frac{\mathrm{d}\lambda}{\mathrm{d}\mathcal{L}^n}(x)\mathrm{d}\mathcal{L}^n \geq F(x,\nabla u(x)) = \int_B F(x,\nabla u(x))\mathrm{d}y.$$

This shows that $\langle \nu_x, F(x,\cdot)\rangle + \langle \nu_x^\infty, F(x,\cdot)\rangle \, \mathrm{d}\lambda^a \geq F(x,\nabla u(x))$ *a.e.*$x \in \Omega$. Multiplying by $\rho(x)$ and integrating in x, we have

$$\int_\Omega \langle \nu_x, F(x,\cdot)\rangle \, \mathrm{d}x + \int_\Omega \rho(x) \langle \nu_x^\infty, F^\infty(x,\cdot)\rangle \, \mathrm{d}\lambda^a \geq \int_\Omega \rho(x) F(x,\nabla u(x)) \, \mathrm{d}x.$$

Now we aim to show that $\int_\Omega \rho(x) \langle \nu_x^\infty, F^\infty\rangle \, \mathrm{d}\lambda^s \geq 0$. For $\tau \in \mathrm{Tan}(\lambda^s, x)$, $((\delta_0)_{y\in B}, \tau, (\nu_x^\infty)_{y\in B}) \in GY^p(B)$, you can choose $\tau(B) > 0$. For $B_2 \subset\subset B$, $\tau(B_2) > 0$, but $\tau(\partial \bar{B}_2) = 0$ so that $\tau(B_2) = \tau(\bar{B}_2)$. Note that $((\delta_0)_{y\in B_2}, \tau \mathbb{1}_{B_2}, (\nu_x^\infty)_{y\in B_2})$ is a gradient p-Young measure generated by $\varphi_k \in C_c^\infty(B_2)$. For each fixed x, we know that $\lim_{k\to\infty} \int_{B_2} F^\infty(x,\nabla\varphi_k)\mathrm{d}y \geq |B_2| F^\infty(x,0) = 0$. We also know that $\lim_{k\to\infty} \int_{B_2} F^\infty(x,\nabla\varphi_k) \, \mathrm{d}y = \int_{B_2} \langle \delta_0, F^\infty(x)\rangle \, \mathrm{d}y + \int_{B_2} \langle \nu_x^\infty, F^\infty(x,\cdot)\rangle \, \mathrm{d}\tau = \tau(B_2) \langle \nu_x^\infty, F^\infty(x)\rangle \geq 0$. This implies that $\langle \nu_x^\infty, F^\infty(x,\cdot)\rangle \geq 0 \; \lambda^s$ a.e. □

Problem 4. *Let $D := \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk in $\mathbb{C}$ and $\mathcal{H}$ the space of all Lipschitz continuous and holomorphic functions $f\colon D \to \mathbb{C}$.*

(a) Writing u^1 and u^2 for the real and imaginary parts of f, respectively, and denoting $u = (u^1, u^2)^T$, explain why

$$(\nabla u)^- = 0 \quad \textit{and} \quad 2\det \nabla u = |(\nabla u)^+|^2 . a.e.$$

hold whenever $f \in \mathcal{H}$. What can you say about the map $v = (u^1, -u^2)^T$?

Proof. Since f is holomorphic, we have the Cauchy Riemman Equations

$$\partial_x u^1 = \partial_y u^2, \partial_y u^1 = -\partial_x u^2.$$

So $\nabla u = \begin{bmatrix} \partial_x u^1 & \partial_y u^1 \\ \partial_x u^2 & \partial_y u^2 \end{bmatrix} = \begin{bmatrix} \partial_x u^1 & -\partial_x u^2 \\ \partial_x u^2 & \partial_x u^1 \end{bmatrix}$, and $\mathrm{cof}\nabla u = \begin{bmatrix} \partial_x u^1 & -\partial_x u^2 \\ \partial_x u^2 & \partial_x u^1 \end{bmatrix}$. Then we have

$$(\nabla u)^- = \frac{1}{2}(\nabla u - \mathrm{cof}\nabla u) = 0.$$

Note that

$$2\det \nabla u = 2(\partial_x u^1 \cdot \partial_x u^1 - (-\partial_x u^2)\cdot \partial_x u^2) = 2[(\partial_x u^1)^2 + (\partial_x u^2)^2] = |\nabla u|^2 = |(\nabla u)^+|^2$$

since $(\nabla u)^- = 0$. Now consider $v = (u^1, -u^2)^T$. Then we have

$\nabla v = \begin{bmatrix} \partial_x v^1 & \partial_y v^1 \\ \partial_x v^2 & \partial_y v^2 \end{bmatrix} = \begin{bmatrix} \partial_x u^1 & -\partial_x u^2 \\ -\partial_x u^2 & -\partial_x u^1 \end{bmatrix}$, and $\mathrm{cof}\nabla v = \begin{bmatrix} -\partial_x u^1 & \partial_x u^2 \\ \partial_x u^2 & \partial_x u^1 \end{bmatrix}$. Then we have

$$(\nabla v)^+ = \frac{1}{2}(\nabla v + \mathrm{cof}\nabla v) = 0.$$

$$2\det \nabla v = 2(\partial_x u^1 \cdot (-\partial_x u^1) - (-\partial_x u^2)\cdot(-\partial_x u^2)) = -2[(\partial_x u^1)^2 + (\partial_x u^2)^2] = -|\nabla v|^2 = -|(\nabla v)^-|^2$$

since $(\nabla v)^+ = 0$. □

Prove that

$$I(u,D) := \int_D \det \nabla u \, \mathrm{d}x$$

is not weakly continuous on* $W^{1,\infty}(D,\mathbb{R}^2)$.

Proof. First note by writing u^1 and u^2 for the real and imaginary parts of Lipschitz continuous and holomorphic function f, we know that $u = (u^1,u^2)^T \in W^{1,\infty}(D,\mathbb{R}^2)$ since u is Lipschitz continuous if and only if $u \in W^{1,\infty}(D,\mathbb{R}^2)$. We can decompose $W^{1,\infty}(D,\mathbb{R}^2) := \mathcal{H} \oplus \mathcal{N}$ where $\mathcal{H}$ is the infinite-dimensional function space of Lipschitz and holomorphic functions. Let $\mathcal{V}$ be a weakly* open neighbourhood of 0 in $W^{1,\infty}(D,\mathbb{R}^2)$, then $\mathcal{V}$ contains a subspace $\mathcal{U}$ of $W^{1,\infty}(D,\mathbb{R}^2)$ such that $W^{1,\infty}(D,\mathbb{R}^2)/\mathcal{U}$ has finite dimension. Then we have $\mathcal{H} \cap \mathcal{U} \neq \{0\}$ (I am not sure about how to prove this part), that is, we can find $u \in \mathcal{U} \cap \mathcal{H}$ such that $I(u,D) \neq 0$ by the calculation from part (a).

Let $\mathcal{N}$ be a weakly* open neighbourhood of 0 in $W^{1,\infty}(D,\mathbb{R}^2)$, that is, $I(\mathcal{N}) \subset (-\varepsilon,\varepsilon)$ for ε small. If $u \in \mathcal{N}$, then $\lambda u \in \mathcal{N}$, for all $\lambda \in \mathbb{R}$. Note that $I(\lambda u) = \lambda^2 I(u)$, so we must have $I(u) = 0$. We aim to show that there exists $u \in \mathcal{N}$ but $I(u) \neq 0$ to get contradiction. Recall that the kernel of a linear functional in a Banach space has co-dimension 1. Since any weakly* open neighbourhood of 0 in $W^{1,\infty}(D,\mathbb{R}^2)$ contains a subspace of finite co-dimension. We may find $u \in \bigcap_{i=1}^k \ker \varphi_i \subset \mathcal{N}$. Write $u = \sum_{j=1}^{k+1} c_j z^j$ for $z \in D$, then we can find $(c_j) \neq 0$ such that $\sum_{j=1}^{k+1} c_k \langle \varphi_i, z^j \rangle = 0$ for all $1 \le i \le k$. Since u is polynomial of z, it is holomorphic. □

Problem 5. *Let* $F\colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ *be a Carathéodory integrand such that for* $\mathcal{L}^n$ *almost all* $x \in \Omega$*, the partial function* $F(x,\cdot)$ *is quasiconvex. Suppose* $p\colon \Omega \to \mathbb{R}$ *is a Borel function satisfying* $p(x) \ge q$ *for* $\mathcal{L}^n$ *almost all* $x \in \Omega$*, where* $q > 1$ *is a constant.*
Show that if for some constant $c \ge 1$ *the coercivity-growth condition*

$$|z|^{p(x)} - c \le F(x,z) \le c(|z|^{p(x)} + 1)$$

holds for all $(x,z) \in \Omega \times \mathbb{R}^{N\times n}$*, then*

$$\liminf_{j\to\infty} \int_\Omega F(\cdot,\nabla u_j)\,\mathrm{d}x \ge \int_\Omega F(\cdot,\nabla u)\,\mathrm{d}x$$

holds whenever $u_j \rightharpoonup u$ *in* $W^{1,q}(\Omega,\mathbb{R}^N)$.

Proof. Assume that $\nu = ((\nu_x)_{x\in\Omega}, 0, n/a)$ is generated by ∇u_j where $u_j \rightharpoonup u$ in $W^{1,q}(\Omega,\mathbb{R}^N)$. Since $F\colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is a Carathédory integrand, we have

$$\liminf_{j\to\infty} \int_\Omega F(\cdot,\nabla u_j)\,\mathrm{d}x \ge \int_\Omega \langle \nu_x, F\rangle \,\mathrm{d}x.$$

We assume that $\liminf_{j\to\infty} \int_\Omega F(\cdot, \nabla u_j)\,\mathrm{d}x < \infty$, or otherwise we are done. Now we aim to show that $\langle \nu_x, F\rangle \geq F(x, \bar{\nu}_x)$ where $\bar{\nu}_x = \nabla u$. Note that $\nu_x \in GY^q$. Using the coercivity condition, we have

$$\int_\Omega \left\langle \nu_x, |z|^{p(x)} \right\rangle \mathrm{d}x \leq \int_\Omega \langle \nu_x, F\rangle \,\mathrm{d}x < \infty \;\; a.e. x \in \Omega.$$

So for each fixed x, $\nu_x \in GY^{p(x)}$. We also know that $F(x, \cdot)$ satisfies the $p(x)$ growth condition and is quasiconvex, then by Problem 2 (b), we have $\langle \nu_x, F\rangle \geq F(x, \bar{\nu}_x)$. To justify that $v_x \in GY^{p(x)}$, we use localization principle. That is, we pick x such that $\int_\Omega \left\langle \nu_x, |z|^{p(x)} \right\rangle \mathrm{d}x < \infty$, we have that $\mu = \nu_x$ is homogeneous q-gradient young measure. Take $p(x) = r$, then we have $\int_\Omega \langle \mu_y, |z|^r\rangle \,\mathrm{d}y < \infty$ where $((\mu_y)_{y\in B}, 0, n/a)$ is a $p(x)$-gradient Young measure. □

References

[1] N.G. Meyers Quasi-Convexity and Lower Semi-Continuity of Multiple Variational Integrals of Any Order *AMS* , 1965