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Nonlinear Analysis and Applications -Problem Sheet One

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Problem 1. *The Riemann-Lebesgue Lemma.*
Let $p \in [1, \infty]$ and suppose that $f : \mathbb{R} \to \mathbb{R}$ is 1-periodic and locally L^p.
Defining $\bar{f} := \int_0^1 f(x)\,\mathrm{d}x$ and $f_j(x) := f(jx), x \in (a, b)$, where $-\infty < a < b < \infty$, prove that

$$f_j \rightharpoonup \bar{f}1_{(a,b)} \text{ weakly in } L^p \text{ (weakly}^* \text{ if } p = \infty)$$

and that (f_j) is p-equi-integrable when $p < \infty$.
State the corresponding result in the vectorial case, that is, when $f : \mathbb{R}^n \to \mathbb{R}^d$ is R-periodic and locally L^p, where $R = (a_1, b_1] \times \ldots \times (a_n, b_n]$ is a rectangle. [No proof required for this]

Proof.

- For $p \in (1, \infty]$, $f_j \rightharpoonup \bar{f}1_{(a,b)}$ weakly in L^p (weakly* if $p = \infty$) if and only if
  (i) $\int_Q f_j\,\mathrm{d}x \to \int_Q \bar{f}1_{(a,b)}\,\mathrm{d}x$ as $j \to \infty$ for all bounded intervals $Q \subset \mathbb{R}$, and
  (ii) $\sup_j \|f_j\|_p < \infty$.

  Note that (ii) is true since that for each j and on any bounded interval (a, b), we have $\|f_j\|_{L^p(a,b)} = \left(\int_a^b |f(jx)|^p\,\mathrm{d}x\right)^{\frac{1}{p}} = \left(\int_{aj}^{bj} \frac{1}{j}|f(y)|^p\,\mathrm{d}y\right)^{\frac{1}{p}} \le \left(\frac{\lceil bj-aj\rceil}{j}\int_0^1 |f(y)|^p\right)^{\frac{1}{p}} < (b - a + 1)\|f\|_{L^p(0,1)} < \infty$ where $\lceil x\rceil$ is the smallest integer greater or equal to x.

  Now we verify (i). It suffices to calculate the limit of $\int_Q f_j$ when Q is an interval with rational endpoints since $\mathbb{Q}$ is dense in $\mathbb{R}$, which implies that 1_Q with rational endpoints for Q is dense in $L^{p'}$. For $Q = (\frac{p_1}{q_1}, \frac{p_2}{q_2})$ where $\frac{p_1}{q_1}, \frac{p_2}{q_2}$ are fractions of the simplest form. Then we have

$$\begin{aligned}
\int_{\frac{p_1}{q_1}}^{\frac{p_2}{q_2}} f_j(x)\,\mathrm{d}x =& \frac{1}{j}\int_{\frac{p_1 j}{q_1}}^{\frac{p_2 j}{q_2}} f(y)\,\mathrm{d}y \text{ for } y = jx \\
\xrightarrow[j\to\infty, j\to nq_1q_2]{}& \frac{1}{nq_1q_2}\int_{nq_2p_1}^{nq_1p_2} f(y)\,\mathrm{d}y \\
=& \frac{n(p_2q_1 - q_2p_1)}{nq_1q_2}\int_0^1 f(y)\,\mathrm{d}y \text{ (by periodicity)} \\
=& \left(\frac{p_2}{q_2} - \frac{p_1}{q_1}\right)\int_0^1 f(y)\,\mathrm{d}y \\
=& \int_{\frac{p_1}{q_1}}^{\frac{p_2}{q_2}}\int_0^1 f(x)\,\mathrm{d}x 1_{(a,b)}.
\end{aligned}$$

For $p = 1$, $f_j \rightharpoonup \bar{f}1_{(a,b)}$ weakly in L^1 if and only if

(i) (f_j) is equi-integrable, and

(ii) $\int_Q f_j \to \int_Q \bar{f}1_{(a,b)}$ for all bounded intervals $Q \subset \mathbb{R}$.

(ii) is true by the same proof as above. (i) follows from the following p-equi-integrability for $p < \infty$.

- Now we show that (f_j) is p-equi-integrable when $p < \infty$. Note that for any bounded interval (a, b), $t > 0$,

$$\sup_j \int_a^b |f_j|^p\,\mathrm{d}x = \sup_j \int_a^b |f_j|^p 1_{|\{f_j|^p \le t\}}\,\mathrm{d}x + \sup_j \int_a^b |f_j|^p 1_{\{|f_j|^p > t\}}\,\mathrm{d}x < \infty.$$

Since $|f_j|^p 1_{\{|f_j|^p \le t\}}$ is non-negative, increasing and converges to $|f_j|^p$ as $t \to \infty$. we have $\sup_j \int_a^b |f_j|^p 1_{|\{f_j|^p \le t\}}\,\mathrm{d}x \to \sup_j \int_a^b |f_j|^p\,\mathrm{d}x < \infty$ as $t \to \infty$ by Monotone Convergence Theorem. This implies that $\sup_j \int_a^b |f_j|^p 1_{|\{f_j|^p > t\}}\,\mathrm{d}x \to 0$ as $t \to \infty$.

- Let Ω be a subset of $\mathbb{R}^n$ with $\mathcal{L}^n(\Omega) < \infty$. Similarly, we define $f_j(x) := f(jx), x \in \Omega$ $(j \in \mathbb{N})$. Then

$$f_j \rightharpoonup \fint_R f\,\mathrm{d}x 1_\Omega \text{ weakly in } L^p(\Omega, \mathbb{R}^d) \text{ (weakly}^* \text{ if } p = \infty),$$

where $\fint_R f\,\mathrm{d}x := \frac{1}{\mathcal{L}^n(R)}\int_R f\,\mathrm{d}x$.

□

Problem 2. *Continuity and lower semicontinuity in the unconstrained case.*
Let $\Phi\colon \mathbb{R}^d \to \mathbb{R}$ be a continuous function (an integrand) satisfying the p-growth condition

$$(G_p) \qquad |\Phi(z)| \le c(|z| + 1)^p \;\; \forall z \in \mathbb{R}^d$$

where $c > 0$ is a constant and $p \in [1, \infty)$. (We define ∞-growth condition (G_∞) to mean locally bounded, which is automatic when the integrand is continuous.) Assume $p \in [1, \infty)$, let Ω be a bounded open and proper subset of $\mathbb{R}^d$ and define the integral functional

$$I(u) := \int_\Omega \Phi(u(x))\,\mathrm{d}x, \;\; u \in L^p(\Omega, \mathbb{R}^d).$$

(a) Show that I is strongly continuous on L^p.

(b) Show that I is weakly sequentially continuous on L^p if and only if Φ is affine (so $\Phi(z) = z_0 \cdot z + c_0$ for constants $z_0 \in \mathbb{R}^d$, $c_0 \in \mathbb{R}$.) When is I weakly continuous on L^p?

(c) Show that I is weakly sequentially lower semicontinuous if and only if Φ is convex. When is I weakly lower semicontinuous?

Proof. (a)

$$\begin{aligned}
|I(u) - I(v)| \leq &\int_\Omega |\Phi(u) - \Phi(v)|\,\mathrm{d}x \\
= &\int_\Omega |\nabla\Phi(\xi)||u - v|\,\mathrm{d}x \text{ for some } \xi \in [u; v] \text{ (by mean value theorem)} \\
\leq &C \int_\Omega (|\xi| + 1)^{p-1}|u - v|\,\mathrm{d}x \\
\leq &C \left(\int_\Omega (|\xi| + 1)^{(p-1)p'}\,\mathrm{d}x\right)^{\frac{1}{p}} \left(\int_\Omega |u - v|^p\,\mathrm{d}x\right)^{\frac{1}{p}} \text{ with } p' = \frac{p}{p-1} \\
< &\tilde{C}\|u - v\|^p_{L^p(\Omega)} \text{ since } \xi \in L^p(\Omega, \mathbb{R}^d) \\
< &\varepsilon \text{ whenever } \|u - v\|_{L^p(\Omega)} < \delta \; (i.e.\ \delta = (\frac{\varepsilon}{\tilde{C}})^{\frac{1}{p}}).
\end{aligned}$$

This shows that I is strongly continuous in L^p.

(b) We first assume that Φ is affine, that is $\Phi(z) = z_0 \cdot z + c_0$ for constants $z_0 \in \mathbb{R}^d$. Assume that $(u_j)_{j=1}^\infty$ is a sequence such that $u_k \rightharpoonup u$ in L^p. Then

$$\begin{aligned}
I(u_k) - I(u) = &\int_\Omega \Phi(u_k) - \Phi(u)\,\mathrm{d}x \\
= &\int_\Omega z_0 \cdot (u_k - u)\,\mathrm{d}x \\
= &\int_\Omega z_0 \cdot u_k\,\mathrm{d}x - \int_\Omega z_0 \cdot u\,\mathrm{d}x \\
\to &0 \text{ as } u_k \rightharpoonup u \text{ in } L^p(\Omega) \text{ and } z_0 \in L^{p'}(\Omega).
\end{aligned}$$

The converse follows from applying the result in (*c*) to both Φ and $-\Phi$, which gives

$$\Phi(\lambda a + (1 - \lambda)b) = \lambda\Phi(a) + (1 - \lambda)\Phi(b)$$

for $a, b \in \mathbb{R}^d, \lambda \in [0, 1]$.

(c) Suppose that I is weakly sequentially lower semicontinuous. We adapt a proof from [1]. Let $a, b \in \mathbb{R}^d$ and $\lambda \in [0,1]$. Let Q be the unit cube $\{x \in \mathbb{R}^d \colon 0 \leq |x_i| \leq 1 \text{ for } i = 1, \ldots, d\}$ and define $u \in L^p(Q)$ by $u(x) = a$ if $x \in A_1$, $u(x) = b$ if $x \in A_2$ where $Q = A_1 \cup A_2$, $\mu(A_1) = \lambda, \mu(A_2) = 1 - \lambda$, and μ denotes d-dimensional Lebesgue measure. Tesselate $\mathbb{R}^d$ by disjoint congruent open cubes Q_j with centre x^j and side $\frac{1}{k}$. For $i = 1, 2$, let $E_{k,i} = \bigcup_j (x^j + \frac{1}{k} A_i)$. Define a sequence $u_k \in L^p(\Omega)\, k = 1, 2 \ldots$ by $u_k(x) = u(k(x - x^j))$ if $x \in Q_j \cap \Omega$. If $E \subset \Omega$ is measurable and $c \in \mathbb{R}^d$ then

$$\int_\Omega u_k \cdot c 1_E \, \mathrm{d}x = \int_E u_k \cdot c \, \mathrm{d}x = \mu(E \cap E_{k,1}) a \cdot c + \mu(E \cap E_{k,2}) b \cdot c,$$

which as $k \to \infty$ tends to

$$\int_\Omega [\lambda a + (1 - \lambda) b] \cdot c 1_E \, \mathrm{d}x = \mu(E) [\lambda a + (1 - \lambda) b] \cdot c.$$

Since simple functions are dense in $L^{p'}(\Omega)$ where p' satisfies $\frac{1}{p} + \frac{1}{p'} = 1$, it follow that $u_k \rightharpoonup \lambda a + (1 - \lambda) b$ weakly in $L^p(\Omega)$. Hence

$$\begin{aligned}
\Phi(\lambda a + (1 - \lambda) b) \leq & \liminf_{j \to \infty} \frac{1}{\mu(\Omega)} \int_\Omega \Phi(u_j(x)) \, \mathrm{d}x \\
= & \lim_{j \to \infty} \left[\frac{\mu(\Omega \cap E_{j,1})}{\mu(\Omega)} \Phi(a) + \frac{\mu(\Omega \cap E_{j,2})}{\mu(\Omega)} \Phi(b) \right] \\
= & \lambda \Phi(a) + (1 - \lambda) \Phi(b).
\end{aligned}$$

so that Φ is convex.

Conversely, let Φ be convex, so that in particular Φ is continuous. Suppose that u_k is a sequence in $L^p(\Omega)$ such that $u_k \rightharpoonup u$ weakly in $L^p(\Omega)$. By Banach Sacks theorem, there exists a subsequence, we also call it u_k such that

$$v^k = \frac{1}{k} \sum_{m=1}^{k} u_m \to u \text{ in } L^p(\Omega).$$

By convexity of Φ and strong continuity of I in $L^p(\Omega)$, we have

$$I(u) = \lim_{k \to \infty} I(v^k) \leq \lim_{k \to \infty} \frac{1}{k} \sum_{m=1}^{k} I(u_m) = \lim_{k \to \infty} I(u_k).$$

Thus $I(u) < \liminf_{k \to \infty} I(u_k)$.

□

Problem 3. *Examples of Young measure generation* *(a) Let*

$$g_j(x) = -\sqrt{j}1_{(-\frac{1}{j},0)}(x) + \sqrt{j}1_{(0,\frac{1}{j})}(x),\ \ x \in (-1,1).$$

Calculate the 2-Young measure generated by (g_j).

(b) Let $f\colon \mathbb{R} \to \mathbb{R}$ *be 1-periodic and locally in* L^2. *Put* $f_j(x) = f(jx), x \in (a,b)$. *Calculate the 2-Young measure generated by* (f_j). *If we take* $a = -1, b = 1$ *above, then what is the 2-Young measure generated by* $(g_j + f_j)$*?*

(c) Let

$$h_j(x) = \sum_{s=0}^{j-1} \sqrt{j}1_{[\frac{s}{j},\frac{s}{j}+\frac{1}{j^2}]}(x),\ \ x \in (0,1).$$

Calculate the 2-Young measure generated by (h_j).

(d) Define $u_j\colon (0,1) \to \mathbb{R}^2$ *by*

$$u_j(x) := \sum_{s=0}^{j-1} \sqrt{j}1_{[\frac{s}{j},\frac{s}{j}+\frac{1}{j^2}]}(x)(\cos(2\pi j^2 x), \sin(2\pi j^2 x))$$

Show that (u_j) *generates the 2-Young measure* ν *with* $\nu_x = \delta_0$, $\lambda = \mathcal{L}^1\lfloor[0,1]$ *and* $\nu_x^\infty = \mu$, *where*

$$\mu = \frac{1}{2\pi}\mathcal{H}^1\lfloor\mathbb{S}^1$$

that is, the normalized length measure on the unit circle.

Proof. (a) $g_j(x) = -\sqrt{j}1_{(-\frac{1}{j},0)}(x) + \sqrt{j}1_{(0,\frac{1}{j})}(x),\ \ x \in (-1,1)$.

- For $\varphi \in C_c(-1,1), \psi \in C_c(\{-1,1\})$, calculate

  $$\langle\langle \varepsilon_{g_j}, \varphi \otimes \psi \rangle\rangle = \int_{-1}^{1} \varphi\psi \circ g_j \,\mathrm{d}x \to \int_{-1}^{1} \varphi(x) \langle \nu_x, \psi \rangle \,\mathrm{d}x.$$

  Note that we can take $\varphi = 1_A$ for A to be a Borel subset of $(-1,1)$. Take $A = (-1+\varepsilon, 1-\varepsilon)$ for ε small, we have

  $$\int_{-1+\varepsilon}^{1-\varepsilon} \psi \circ g_j \,\mathrm{d}x = \int_{-\frac{1}{j}}^{0} \psi(-\sqrt{j}) \,\mathrm{d}x + \int_{0}^{\frac{1}{j}} \psi(\sqrt{j}) \,\mathrm{d}x + \psi(0) \to \psi(0) \text{ as } j \to \infty.$$

  This implies that $\nu_x = \delta_0$.

- For $\varphi \in C([-1,1])$, $\varphi \otimes |\cdot|^2 \in \mathbb{E}_2$ and we have

  $$\begin{aligned} \langle\langle \varepsilon_{g_j}, \varphi \otimes |\cdot|^2 \rangle\rangle &= \int_{-1}^{1} \varphi |g_j|^2 \,\mathrm{d}x \\ &\to \langle\langle v, \varphi \otimes |\cdot|^2 \rangle\rangle \\ &= \int_{-1}^{1} \varphi \langle v_x, |\cdot|^2 \rangle \,\mathrm{d}x + \int_{[-1,1]} \varphi \,\mathrm{d}\lambda \end{aligned}$$

So we have

$$\begin{aligned}\int_{[-1,1]} \varphi \, \mathrm{d}\lambda &= \lim_{j\to\infty} \int_{-1}^{1} \varphi |g_j|^2 \, \mathrm{d}x - \int_{-1}^{1} \varphi(x) \left\langle \delta_0, |\cdot|^2 \right\rangle \, \mathrm{d}x \\ &= \lim_{j\to\infty} \int_{-1}^{1} \varphi (j 1_{(-\frac{1}{j},0)}(x) + j 1_{(0,\frac{1}{j})}(x)) \, \mathrm{d}x \\ &= \lim_{j\to\infty} \int_{0}^{\frac{1}{j}} \varphi j \, \mathrm{d}x + \int_{-\frac{1}{j}}^{0} \varphi j \, \mathrm{d}x \\ &= \lim_{j\to\infty} \int_{0}^{1} \varphi(\frac{y}{j}) \, \mathrm{d}y + \lim_{j\to\infty} \int_{-1}^{0} \varphi(\frac{y}{j}) \, \mathrm{d}y \\ &= 2\varphi(0).\end{aligned}$$

This shows that $\lambda = 2\delta_0$.

- For $\varphi \in C([-1,1])$, $\psi \in C(\{-1,1\})$, calculate with $\tilde{\psi}(z) := \begin{cases} |z|^2 \psi(\frac{z}{|z|}) & z \neq 0; \\ 0 & z = 0 \end{cases}$ and $\varphi \otimes \tilde{\psi} \in \mathbb{E}_2$:

$$\begin{aligned}\left\langle \left\langle \varepsilon_{g_j}, \varphi \otimes \tilde{\psi} \right\rangle \right\rangle &= \int_{-1}^{1} \varphi |g_j|^2 \psi(\frac{g_j}{|g_j|}) \, \mathrm{d}x \\ &= \int_{0}^{\frac{1}{j}} \varphi j \psi(1) \, \mathrm{d}x + \int_{-\frac{1}{j}}^{0} \varphi j \psi(-1) \, \mathrm{d}x \\ &= \int_{0}^{1} \varphi(\frac{y}{j}) \psi(1) \, \mathrm{d}y + \int_{-1}^{0} \varphi(\frac{y}{j}) \psi(-1) \, \mathrm{d}y \\ &\to \phi(0)(\psi(1) + \psi(-1)) \text{ as } j \to \infty.\end{aligned}$$

We also know that

$$\begin{aligned}\left\langle \left\langle \varepsilon_{g_j}, \varphi \otimes \tilde{\psi} \right\rangle \right\rangle \to \left\langle \left\langle \nu, \varphi \otimes \tilde{\psi} \right\rangle \right\rangle &= \int_{-1}^{1} \varphi(x) \left\langle \delta_0, \tilde{\psi} \right\rangle \, \mathrm{d}x + \int_{[-1,1]} \varphi(x) \left\langle \nu_x^\infty, \psi \right\rangle \, \mathrm{d}\lambda \\ &= \int_{[-1,1]} \varphi(x) \left\langle \nu_x^\infty, \psi \right\rangle \, \mathrm{d}\lambda.\end{aligned}$$

Since $\lambda = 2\delta_0$, the above implies that $\nu_x^\infty = \begin{cases} n/a & \text{if } x \neq 0; \\ \frac{\delta_1 + \delta_{-1}}{2} & \text{if } x = 0. \end{cases}$

(b) $f_j(x) = f(jx), x \in (a,b)$ where $f \colon \mathbb{R} \to \mathbb{R}$ is 1-periodic and locally L^2.

- For $\varphi \in C_c(a,b), \psi \in C_c(\{-1,1\})$, calculate

$$\left\langle \left\langle \varepsilon_{f_j}, \varphi \otimes \psi \right\rangle \right\rangle = \int_{a}^{b} \varphi \psi \circ f_j \, \mathrm{d}x \to \int_{a}^{b} \varphi(x) \left\langle \nu_x, \psi \right\rangle \, \mathrm{d}x.$$

Using the result from problem 1, we have $\psi(f_j) \rightharpoonup \int_0^1 \psi(f)\mathrm{d}x$ weakly in L^2. This implies that ν_x is a measure satisfies $\langle \nu_x, \psi\rangle = \int_0^1 \psi(f(x))\,\mathrm{d}x$.

- For $\varphi \in C([-1,1])$, $\varphi \otimes |\cdot|^2 \in \mathbb{E}_2$ and we have

$$\begin{aligned}\langle\langle \varepsilon_{f_j}, \varphi \otimes |\cdot|^2\rangle\rangle &= \int_a^b \varphi |f_j|^2\,\mathrm{d}x\\ &\to \langle\langle v, \varphi \otimes |\cdot|^2\rangle\rangle\\ &= \int_a^b \varphi \left\langle v_x, |\cdot|^2\right\rangle\,\mathrm{d}x + \int_{[a,b]} \varphi\,\mathrm{d}\lambda\end{aligned}$$

So we have $\int_{[a,b]} \varphi\,\mathrm{d}\lambda = \lim_{j\to\infty} \int_a^b \varphi |f_j|^2\,\mathrm{d}x - \int_a^b \varphi(x)\left\langle \nu_x, |\cdot|^2\right\rangle\,\mathrm{d}x = 0$. This shows that $\lambda = 0$.

- Since $\delta = 0$, we can conclude that $\nu_x^\infty = n/a$.
- If we take $a = -1, b = -1$ above, then $f_j + g_j \xrightarrow{Y^2} \tilde{\nu}$ where $\tilde{\nu}_x = \delta_{\int_0^1 f(x)}$ and else are the same as (a).

(c) $h_j(x) = \sum_{s=0}^{j-1} \sqrt{j} 1_{[\frac{s}{j}, \frac{s}{j}+\frac{1}{j^2}]}$, $x \in (0,1)$.

- For $\varphi \in C_c(0,1), \psi \in C_c(\{-1,1\})$, calculate

$$\langle\langle \varepsilon_{h_j}, \varphi \otimes \psi\rangle\rangle = \int_0^1 \varphi\psi \circ h_j\,\mathrm{d}x \to \int_0^1 \varphi(x)\langle \nu_x, \psi\rangle\,\mathrm{d}x.$$

Note that we can take $\varphi = 1_{(0,\frac{1}{2})}$.

$$\int_0^{\frac{1}{2}} \psi \circ h_j\,\mathrm{d}x = \sum_{s=0}^{j-1} \int_{\frac{s}{j}}^{\frac{s}{j}+\frac{1}{j^2}} \psi(\sqrt{j})\,\mathrm{d}x + \sum_{s=0}^{j-1} \int_{\frac{s}{j}+\frac{1}{j}}^{\frac{s+1}{j}} \psi(0)\,\mathrm{d}x \to \psi(0) \text{ as } j \to \infty.$$

This implies that $\nu_x = \delta_0$.

- For $\varphi \in C([-1,1])$, $\varphi \otimes |\cdot|^2 \in \mathbb{E}_2$ and we have

$$\begin{aligned}\langle\langle \varepsilon_{h_j}, \varphi \otimes |\cdot|^2\rangle\rangle &= \int_0^1 \varphi |h_j|^2\,\mathrm{d}x\\ &\to \langle\langle v, \varphi \otimes |\cdot|^2\rangle\rangle\\ &= \int_0^1 \varphi \left\langle v_x, |\cdot|^2\right\rangle\,\mathrm{d}x + \int_{[0,1]} \varphi\,\mathrm{d}\lambda\end{aligned}$$

So we have

$$\begin{aligned}\int_{[0,1]} \varphi \,\mathrm{d}\lambda &= \lim_{j\to\infty} \int_0^1 \varphi |h_j|^2 \,\mathrm{d}x - \int_0^1 \varphi(x) \left\langle \delta_0, |\cdot|^2 \right\rangle \,\mathrm{d}x \\ &= \lim_{j\to\infty} \int_0^1 \varphi |h_j|^2 \,\mathrm{d}x \\ &= \lim_{j\to\infty} \sum_{s=0}^{j-1} \int_{\frac{s}{j}}^{\frac{s}{j}+\frac{1}{j^2}} \varphi(j) j \mathrm{d}x \\ &= \int_0^1 \varphi(x) \,\mathrm{d}x.\end{aligned}$$

This shows that $\lambda = \mathcal{L}^1 \lfloor (0,1)$.

- For $\varphi \in C([0,1])$, $\psi \in C(\{-1,1\})$, calculate with $\tilde{\psi}(z) := \begin{cases} |z|^2 \psi(\frac{z}{|z|}) & z \neq 0; \\ 0 & z = 0 \end{cases}$ and $\varphi \otimes \tilde{\psi} \in \mathbb{E}_2$:

$$\begin{aligned}\left\langle\left\langle \varepsilon_{h_j}, \varphi \otimes \tilde{\psi} \right\rangle\right\rangle &= \int_0^1 |h_j|^2 \psi(\frac{h_j}{|h_j|}) \,\mathrm{d}x \\ &= \sum_{s=0}^{j-1} \int_{\frac{s}{j}}^{\frac{s}{j}+\frac{1}{j^2}} \varphi j \psi(1) \,\mathrm{d}x \\ &\to \int_0^1 \varphi(x) \psi(1) \,\mathrm{d}x \text{ as } j \to \infty.\end{aligned}$$

We also know that

$$\begin{aligned}\left\langle\left\langle \varepsilon_{h_j}, \varphi \otimes \tilde{\psi} \right\rangle\right\rangle &\to \left\langle\left\langle \nu, \varphi \otimes \tilde{\psi} \right\rangle\right\rangle \\ &= \int_0^1 \varphi(x) \left\langle \delta_0, \tilde{\psi} \right\rangle \,\mathrm{d}x + \int_{[0,1]} \varphi(x) \left\langle \nu_x^\infty, \psi \right\rangle \,\mathrm{d}\lambda \\ &= \int_{[0,1]} \varphi(x) \left\langle \nu_x^\infty, \psi \right\rangle \,\mathrm{d}\lambda.\end{aligned}$$

Since $\lambda = \mathcal{L}\lfloor(0,1)$, the above implies that $\nu_x^\infty = \delta_1$.

(d) $u_j(x) = \sum_{s=0}^{j-1} \sqrt{j} 1_{[\frac{s}{j}, \frac{s}{j}+\frac{1}{j^2}]}(x)(\cos(2\pi j^2 x), \sin(2\pi j^2 x))$, $x \in (0,1)$.

- For $\varphi \in C_c(0,1), \psi \in C_c(\mathbb{R}^2)$, calculate

$$\left\langle\left\langle \varepsilon_{u_j}, \varphi \otimes \psi \right\rangle\right\rangle = \int_0^1 \varphi \psi \circ u_j \,\mathrm{d}x \to \int_0^1 \varphi(x) \left\langle \nu_x, \psi \right\rangle \,\mathrm{d}x.$$

Note that we can take $\varphi = 1_A$ for any Borel subset A of $(0,1)$. Take $\varphi = 1_{(0,1-\varepsilon)}$ for ε small.

$$\begin{aligned}\int_0^{1-\varepsilon} \psi \circ u_j \,\mathrm{d}x = &\sum_{s=0}^{j-1} \int_{\frac{s}{j}}^{\frac{s}{j}+\frac{1}{j^2}} \psi(\sqrt{j}(\cos(2\pi j^2 x), \sin(2\pi j^2 x)))\,\mathrm{d}x \\ &+ \sum_{s=0}^{j-1} \int_{\frac{s}{j}+\frac{1}{j}}^{\frac{s+1}{j}} \psi(0,0)\,\mathrm{d}x \to \psi(0,0) \text{ as } j \to \infty.\end{aligned}$$

This implies that $\nu_x = \delta_0$.

- For $\varphi \in C([0,1])$, $\varphi \otimes |\cdot|^2 \in \mathbb{E}_2$ and we have

$$\begin{aligned}\langle\langle \varepsilon_{u_j}, \varphi \otimes |\cdot|^2 \rangle\rangle &= \int_0^1 \varphi |u_j|^2 \,\mathrm{d}x \\ &\to \langle\langle v, \varphi \otimes |\cdot|^2 \rangle\rangle \\ &= \int_0^1 \varphi \left\langle v_x, |\cdot|^2 \right\rangle \,\mathrm{d}x + \int_{[0,1]} \varphi \,\mathrm{d}\lambda\end{aligned}$$

So we have

$$\begin{aligned}\int_{[0,1]} \varphi \,\mathrm{d}\lambda &= \lim_{j\to\infty} \int_0^1 \varphi |u_j|^2 \,\mathrm{d}x - \int_0^1 \varphi(x) \left\langle \delta_0, |\cdot|^2 \right\rangle \,\mathrm{d}x \\ &= \lim_{j\to\infty} \int_0^1 \varphi |u_j|^2 \,\mathrm{d}x \\ &= \lim_{j\to\infty} \sum_{s=0}^{j-1} \int_{\frac{s}{j}}^{\frac{s}{j}+\frac{1}{j^2}} \varphi(j) j \mathrm{d}x \\ &= \int_0^1 \varphi(x)\,\mathrm{d}x.\end{aligned}$$

This shows that $\lambda = \mathcal{L}^1 \lfloor (0,1)$.

- For $\varphi \in C([0,1])$, $\psi \in C(\mathbb{S})$, calculate with $\tilde{\psi}(z) := \begin{cases} |z|^2 \psi(\frac{z}{|z|}) & z \neq 0; \\ 0 & z = 0 \end{cases}$ and

$\varphi \otimes \tilde{\psi} \in \mathbb{E}_2$:

$$
\begin{aligned}
\left\langle\left\langle \varepsilon_{u_j}, \varphi \otimes \tilde{\psi} \right\rangle\right\rangle &= \int_0^1 \varphi |u_j|^2 \psi(\frac{h_j}{|h_j|})\,\mathrm{d}x \\
&= \sum_{s=0}^{j-1} \int_{\frac{s}{j}}^{\frac{s}{j}+\frac{1}{j^2}} \varphi j \psi(\cos(2\pi j^2 x), \sin(2\pi j^2 x)))\,\mathrm{d}x \\
&= \sum_{s=0}^{j-1} \int_{2\pi j s}^{2\pi j s+2\pi} \frac{1}{2\pi j} \varphi(\frac{y}{2\pi j^2}) \psi(\cos(y), \sin(y))\,\mathrm{d}y \\
&= \sum_{s=0}^{j-1} \varphi(\xi_j) \int_{2\pi j s}^{2\pi j s+2\pi} \frac{1}{2\pi j} \psi(\cos(y), \sin(y))\,\mathrm{d}y \text{ with } \xi_j \in (\frac{s}{j}, \frac{s}{j}+\frac{1}{j^2}).
\end{aligned}
$$

We also know that

$$
\begin{aligned}
\left\langle\left\langle \varepsilon_{g_j}, \varphi \otimes \tilde{\psi} \right\rangle\right\rangle \to & \left\langle\left\langle \nu, \varphi \otimes \tilde{\psi} \right\rangle\right\rangle \\
&= \int_0^1 \varphi(x) \left\langle \delta_0, \tilde{\psi} \right\rangle \mathrm{d}x + \int_{[0,1]} \varphi(x) \left\langle \nu_x^\infty, \psi \right\rangle \mathrm{d}\lambda \\
&= \int_{[0,1]} \varphi(x) \left\langle \nu_x^\infty, \psi \right\rangle \mathrm{d}\lambda.
\end{aligned}
$$

By taking limit $j \to \infty$, we have $\nu_x^\infty = \mu = \frac{1}{2\pi} \mathcal{H}^1 \lfloor \mathbb{S}^1$.

$\square$

Problem 4. *Jensen's Inequality for convex integrands*
Let $F\colon \mathbb{R}^d \to (-\infty, \infty]$ be a proper (so not identically ∞), lower semicontinuous and convex integrand. Show, for instance by use of Hahn-Banach in $\mathbb{R}^d$, that

$$
\int_{\mathbb{R}^n} F(z)\,\mathrm{d}\nu(z) \geq F(\bar{\nu})
$$

holds for all probability measures ν on $\mathbb{R}^d$ with a centre of mass $\bar{\nu}$.
Recall that ν is said to have the centre of mass $\bar{\nu}$ if it has finite first moment

$$
\int_{\mathbb{R}^d} |z|\,\mathrm{d}\nu(z) < \infty
$$

and

$$
\bar{\nu} = \int_{\mathbb{R}^d} z\,\mathrm{d}\nu(z).
$$

Proof. Note that if $F = \infty$, the inequality is trivially true, so we assume that $F < \infty$. Since F is convex and lower semicontinuous, by Hahn-Banach Separation theorem, we have $x^* \in \mathbb{R}^d$ with

$$
\langle z - x, x^* \rangle + F(x) \leq F(z).
$$

That is, there exist constants $a \in \mathbb{R}^d$, $b \in \mathbb{R}$ such that

$$a \cdot z + b \leq F(z) \text{ for all } z \in \mathbb{R}^d$$

and

$$a \cdot \bar{\nu} + b = F(\bar{\nu}).$$

Then, we have

$$\begin{aligned} \int_{\mathbb{R}^d} F(z) \, \mathrm{d}\nu(z) \geq & \int_\Omega a \cdot z + b \, \mathrm{d}\nu(z) \\ =& a \int_\Omega z \, \mathrm{d}\nu(z) + b \\ =& a\bar{\nu} + b \\ =& F(\bar{\nu}). \end{aligned}$$

□

Problem 5. *Let $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$ be the open unit ball in $\mathbb{R}^n$ and $p \in [1, \infty)$.*

(a) By suitably identifying $W^{1,p}(\Omega)$ with the subspace

$$\big\{(u, \partial_1 u, \ldots, \partial_n u) : u \in W^{1,p}(\Omega)\big\}$$

of $L^p(\Omega)^{n+1}$, show that a sequence (u_j) converges weakly to u in $W^{1,p}(\Omega)$ if and only if

$$\begin{cases} u_j \to u \textit{ strongly in } L^p(\Omega) \textit{ and} \\ \partial_i u_j \rightharpoonup \partial_i u \textit{ weakly in } L^p(\Omega) \textit{ for each } i \in \{1, \ldots n\}. \end{cases}$$

(b) Using that a closed subspace of a reflexive space is reflexive show that $W^{1,p}(\Omega)$ is reflexive when $p \in (1, \infty)$.

(c) Generalize the above results to higher order Sobolev spaces $W^{k,p}(\Omega, \mathbb{R}^N)$ of $\mathbb{R}^N$-valued maps, where $k, N \in \mathbb{N}$ and $p \in [1, \infty)$. [No proofs required.]

Proof. (a) If

$$\begin{cases} u_j \to u \text{ strongly in } L^p(\Omega) \text{ and} \\ \partial_i u_j \rightharpoonup \partial_i u \text{ weakly in } L^p(\Omega) \text{ for each } i \in \{1, \ldots n\}, \end{cases}$$

then

$$\begin{cases} u_j \rightharpoonup u \text{ weakly in } L^p(\Omega) \text{ and} \\ \partial_i u_j \rightharpoonup \partial_i u \text{ weakly in } L^p(\Omega) \text{ for each } i \in \{1, \ldots n\} \end{cases}$$

since strong convergence implies weak convergence. It then follows that (u_j) converges weakly to u in $W^{1,p}(\Omega)$.

Now we suppose that (u_j) converges weakly to u in $W^{1,p}(\Omega)$. Then we know that

$$\begin{cases} u_j \rightharpoonup u \text{ weakly in } L^p(\Omega) \text{ and} \\ \partial_i u_j \rightharpoonup \partial_i u \text{ weakly in } L^p(\Omega) \text{ for each } i \in \{1, \dots n\}. \end{cases}$$

Note that Ω is an open bounded domain with smooth boundary, then we can apply Sobolev embedding to conclude that $W^{1,p}(\Omega)$ is compactly embedded into $L^p(\Omega)$. That is, for every weak convergence sequence $u_j \rightharpoonup u$ in $W^{1,p}(\Omega)$, there is a strong convergence subsequence $u_{j_k} \to v$ in $L^p(\Omega)$. By uniqueness of limit, $v = u$. Then we can conclude that $u_j \to u$ strongly in $L^p(\Omega)$.

(b) Note that for $p \in (1, \infty)$, $L^p(\Omega)$ is a reflexive Banach space. Since a finite product of reflexive spaces is reflexive, $L^p(\Omega)^{n+1}$ is reflexive. We also know that $W^{1,p}(\Omega)$ is a subspace of $L^p(\Omega)^{n+1}$ and it is a Banach space, then it is a closed subspace of the reflexive space $L^p(\Omega)^{n+1}$. It then follows that $W^{1,p}(\Omega)$ is reflexive.

(c)
- (u_j) converges weakly to u in $W^{k,p}(\Omega, \mathbb{R}^N)$ if and only if

$$\begin{cases} D^\alpha u_j \to D^\alpha u \text{ strongly in } L^p(\Omega) \text{ for } |\alpha| \le k-1, \\ D^\alpha u_j \rightharpoonup D^\alpha u \text{ weakly in } L^p(\Omega) \text{ for } |\alpha| = k. \end{cases}$$

- $W^{k,p}(\Omega)$ is a reflexive space for $k \in \mathbb{N}$, and $p \in [1, \infty)$.

□

References

[1] J.M.Ball On the calculus of variations and sequentially weakly continuous maps, Ordinary and Partial Differential Equations *Springer* , 1976