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Nonlinear Analysis and Applications -Problem Sheet Two

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Problem 1. *Assume that $V_j \in L^p(\Omega, \mathbb{R}^d)$, $W_j \in L^0(\Omega, \mathbb{R}^d)$,*

$$V_j \overset{Y^p}{\to} \nu \text{ and } V_j - W_j \to 0 \text{ in measure.}$$

(a) Show that (V_j) is weakly convergent in $L^p(\Omega, \mathbb{R}^d)$ and express its weak limit V in terms of ν. What is the p-Young measure generated by the shifted sequence $(V_j - V)$?

Proof. Let $\varphi \in C(\bar{\Omega})$ and put $\Phi(x,z) = \varphi(x) z_i$ for $1 \le i \le d$), then $\Phi \in \mathbb{E}_p$. Since $V_j \overset{Y^p}{\to} \nu$, we have

$$\langle\langle \varepsilon_{V_j}, \Phi \rangle\rangle \to \langle\langle \nu, \Phi \rangle\rangle \text{ as } j \to \infty.$$

Here

$$\langle\langle \varepsilon_{V_j}, \Phi \rangle\rangle = \int_\Omega \varphi(x) V_j^i \, \mathrm{d}x$$

and

$$\langle\langle \nu, \Phi \rangle\rangle = \int_\Omega \varphi(x) \langle \nu_x, z_i \rangle \, \mathrm{d}x = \int_\Omega \varphi(x) \bar{\nu}_x^i \, \mathrm{d}x \text{ for each } 1 \le i \le d.$$

Thus

$$\int_\Omega \varphi(x) V_j^i \, \mathrm{d}x \to \int_\Omega \varphi(x) \bar{\nu}_x^i \, \mathrm{d}x \text{ as } j \to \infty.$$

Since $C(\bar{\Omega})$ is dense in $L^{p'}(\Omega)$ with $\frac{1}{p} + \frac{1}{p'} = 1$, and $\bar{\nu}_x \in L^p(\Omega, \mathbb{R}^d)$ by the moment condition, we can conclude that $V_j \rightharpoonup V$ weakly in $L^p(\Omega, \mathbb{R}^d)$ and $V(x) = \bar{\nu}_x$ a.e. (that is, $V = \bar{\nu}$).

Let $\mu = ((\mu_x)_{x \in \Omega}), \lambda_\mu, (\mu_x^\infty)_{x \in \bar{\Omega}})$ be the p-Young measure generated by the shifted sequence $(V_j - V)$.

Oscillation measure: For $\varphi \in C_c(\Omega), \psi \in C_c(\mathbb{R}^d)$, consider $\Phi(x,z) = \varphi(x)\psi(z)$, and $\tilde{\Phi}(x,z) = \varphi(x)\psi(z - V(x))$. Then we have

$$\langle\langle \varepsilon_{V_j - V}, \Phi \rangle\rangle = \int_\Omega \varphi\psi \circ (V_j - V) \to \int_\Omega \varphi \langle \mu_x, \psi \rangle \, \mathrm{d}x.$$

$$\left\langle\left\langle \varepsilon_{V_j}, \tilde{\Phi} \right\rangle\right\rangle = \int_\Omega \varphi\psi \circ (V_j - V) \to \int_\Omega \varphi \langle \nu_x, \psi(\cdot - V(x)) \rangle \, \mathrm{d}x.$$

This implies that

$$\int_\Omega \varphi \langle \mu_x, \psi \rangle \, \mathrm{d}x = \int_\Omega \varphi \langle \nu_x, \psi(\cdot - V(x)) \rangle \, \mathrm{d}x.$$

μ_x satisfies $\langle \mu_x, \psi \rangle = \langle \nu_x, \psi(\cdot - V) \rangle$. Then we have $\mu_x = \nu_x * (-V_{\#}\delta_x)$.

Concentration measure: Let $S_j = V_j - V$, then $V_j, S_j \in L^p(\Omega, \mathbb{R}^d)$ and $V_j \overset{Y^p}{\to} \nu$, $S_j \overset{Y^p}{\to} \mu$. Note that $V_j - S_j = V$ is p-equi-integrable on Ω. Then we can use the theorem concerning L^p concentration from the lecture notes to conclude that $\lambda_\mu = \lambda_\nu$.

Concentration-angle measure: Similarly as the concentration measure, since $V_j - S_j = V$ is p-equi-integrable on Ω, then $\mu_x^\infty = \nu_x^\infty$ λ-a.e. $x \in \bar{\Omega}$ where $\lambda = \lambda_\mu = \lambda_\nu$. □

(b) *Let C be a closed subset of $\mathbb{R}^d$. Show that $V_j \in C$ in measure, meaning that*

$$\forall \varepsilon > 0\colon \mathcal{L}^n(\{x \in \Omega \colon V_j(x) \notin B_\varepsilon(C)\}) \to 0 \text{ as } j \to \infty,$$

if and only if the support $\operatorname{supp}(\nu_x) \subseteq C$ *for $\mathcal{L}^n$ almost all $x \in \Omega$. Here $B_\varepsilon(C) := C + B_\varepsilon(0)$ is the ε-tube around C and the support of a positive measure is the smallest closest whose complement has zero measure.*

Proof. Since $V_j \overset{Y^p}{\to} \nu$, for $\varphi \in C_c(\Omega)$, $\psi \in C_c(\mathbb{R}^d)$, we have

$$\int_\Omega \varphi\psi \circ V_j \, \mathrm{d}x \to \int_\Omega \varphi(x) \langle \nu_x, \psi \rangle \, \mathrm{d}x \text{ as } j \to \infty.$$

Consider $\psi \in C_c^C(\mathbb{R}^d) = \{g \in C_c^C(\mathbb{R}^d) \colon g_{|C} = 0\}$. It follows from the hypothesis that $V_j \to C$ in measure that $\psi \circ V_j \to 0$ in measure. Then

$$\int_\Omega \varphi(x) \langle \nu_x, \psi \rangle \, \mathrm{d}x = \lim_{j \to \infty} \int_\Omega \varphi\psi \circ V_j \, \mathrm{d}x = 0$$

for every $\varphi \in C_c(\Omega)$. Then we have for a.e. $x \in \Omega$,

$$\langle \nu_x, \psi \rangle = 0 \text{ for every } \psi \in C_c^C(\mathbb{R}^d).$$

That is, $\operatorname{supp}\nu_x \subseteq C$.

Now assume that $\text{supp}\nu_x \subseteq C$ for $\mathcal{L}^n$ a.e. $x \in \Omega$. For each $\varepsilon > 0$, $F(x,z) := 1_\Omega(x)\min(\text{dist}(z,C),\varepsilon)$. Then $F(x,z)$ is a bounded Carathéodory function and thus $\{F(\cdot,V_j)\}$ is equi-integrable. By a proposition from lecture notes (pg 21 of L3), we know that

$$\begin{aligned}\int_\Omega \min(\text{dist}(V_j,C),\varepsilon) &= \int_\Omega F(\cdot,V_j)\,\mathrm{d}x\\ &\to \int_\Omega \langle \nu_x, F(x,\cdot)\rangle\,\mathrm{d}x\\ &= \int_\Omega\int_{\mathbb{R}^d} \min(\text{dist}(z,C),\varepsilon)\mathrm{d}\nu_x(z)dx\\ &= 0\end{aligned}$$

since $\text{supp}\nu_x \subseteq C$ for $\mathcal{L}^n$ almost all $x \in \Omega$. Then by Vitali's convergence theorem, we have $\min(\text{dist}(V_j,C),\varepsilon) \to 0$ in measure. That is, $V_j \to C$ in measure. $\square$

Now assume that $V_j \to C$ in measure, where C is some closed subset of $\mathbb{R}^d$. Show that the weak limit V has essential range contained in the closed convex hull of C. Recall that the essential range R of V is defined as

$$R := \bigcap A,$$

where we take intersection over all closed subsets A of $\mathbb{R}^d$ for which $V(x) \in A$ for $\mathcal{L}^n$ a.e. $x \in \Omega$ and the closed convex hull of C is the intersection of all closed convex sets that contain C.

Proof. Since $V_j \to C$ in measure where C is some closed subset of $\mathbb{R}^d$, then $\text{supp}\nu_x \subseteq C$ for $\mathcal{L}^n$ almost all $x \in \Omega$. Note that

$$V(x) = \bar{\nu}_x = \int_{\mathbb{R}^d} z\,\mathrm{d}\nu_x(z).$$

If $V(x)$ for $\mathcal{L}^n$ a.e. $x \in \Omega$, then $V(x) \in B$ where B is a closed convex set that contains C. Then it follows that

$$R := \bigcap_{V(x)\in A \text{ closed}} A \subseteq \bigcap_{B \text{ closed,convex } C\subseteq B} B.$$

Assume for contradiction that $\bar{\nu}_x$ is not in the convex hull of C. Then $\{\bar{\nu}_x\}$ is a compact closed convex set, then by Hahn-Banach Separation Theorem, there exists linear functional such that

$$L(\bar{\nu}_x) < d < L(x) \text{ for all } x \text{ in the convex hull of } C.$$

Thus

$$d < \int_\Omega L(x)\,\mathrm{d}\nu_x = L(\bar{\nu}_x),$$

yielding a contradiction. $\square$

(c) *Show that for each bounded Carathéodory integrand* $F\colon \Omega\times\mathbb{R}^d \to \mathbb{R}$ *the representation*

$$\lim_{j\to\infty}\int_\Omega F(\cdot, W_j)\,\mathrm{d}x = \int_\Omega \langle \nu_x, F(x,\cdot)\rangle\;\mathrm{d}x$$

holds. [Use the Scorza-Dragoni theorem.]

Proof. Without loss of generality, we may assume that $F \geq 0$ (since F^+, F^- are Carathéodory and can be considered separately.) Since F is bounded, there exists a $M > 0$ such that $|F| \leq M$. By *Scorza-Dragoni theorem*, for each $\varepsilon > 0$, there exists a compact set $C = C_\varepsilon \subset \Omega$ with $\mathcal{L}^n(\Omega \setminus C) < \frac{\varepsilon}{M}$ such that $F\colon C \times \mathbb{R}^d \to \mathbb{R}$ is jointly continuous. Next by *Tietze extension theorem*, we can find $G\colon \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$ continuous such that $G = F$ on $C \times \mathbb{R}^d$ and $0 \leq G \leq M$. Clearly, $G \in \mathbb{E}_p(\Omega, \mathbb{R}^d)$ and $G^\infty = 0$, so

$$\lim_{j\to\infty}\int_\Omega G(\cdot, V_j)\,\mathrm{d}x = \int_\Omega \langle \nu_x, G(x,\cdot)\rangle\;\mathrm{d}x.$$

Note that

$$|\int_\Omega \langle \nu_x, G(x,\cdot)\rangle\;\mathrm{d}x - \langle \nu_x, F(x,\cdot)\rangle\;\mathrm{d}x| \leq \int_{\Omega\setminus C} \langle \nu_x, |G-F|(x,\cdot)\rangle\;\mathrm{d}x < \varepsilon.$$

Thus

$$\lim_{j\to\infty}\int_\Omega F(\cdot, V_j)\,\mathrm{d}x = \int_\Omega \langle \nu_x, F(x,\cdot)\rangle\;\mathrm{d}x.$$

For each $j \in \mathbb{N}$,

$$\begin{aligned}
|\int_\Omega F(x,V_j) - F(x,W_j)| \leq& |\int_{\{|V_j-W_j|<\delta\}} F(x,V_j) - F(x,W_j)| \\
&+ \int_{\{|V_j-W_j|\geq\delta\}} |F(x,V_j)| + \int_{\{|V_j-W_j|\geq\delta\}} |F(x,W_j)| \\
\leq& |\int_{\{|V_j-W_j|<\delta\}} F(x,V_j) - F(x,W_j)| + 2M\mathcal{L}^n\{|V_j-W_j|\geq\delta\} \\
\to& 0.
\end{aligned}$$

$\mathcal{L}^n\{|V_j - W_j| \geq \delta\} \to 0$ since $V_j - W_j \to 0$ in measure. The first term converges to 0 by continuity of F. Then the desired result follows. ☐

Problem 2. *The recession cone of a subset* $A \subset \mathbb{R}^d$ *is defined as*

$$A^\infty := \{z \in \mathbb{R}^d\colon \exists z_j \to z, \exists t_j \to \infty \text{ so } t_j z_j \in A\}.$$

(a) *Show that* A^∞ *is a closed cone.* [It is a cone provided $tz \in A^\infty$ whenever $z \in A^\infty$ and $t > 0$.] *Show that* A^∞ *is convex if* A *is.*

Proof. For $z \in A^\infty$, there exists $z_j \to z, t_j \to \infty$ such that $t_j z_j \in A$. Now consider tz for $t > 0$, then $\tilde{z}_j = tz_j \to tz$ and $\tilde{t}_j = \frac{t_j}{t} \to \infty$, $\tilde{t}_j \tilde{z}_j = t_j z_j \in A$. This shows that $tz \in A^\infty$, so A^∞ is a cone. Now we show that A^∞ is closed. Consider a sequence $\{z^i\} \in A^\infty$ such that $z^i \to z$ as $i \to \infty$. We aim to show that $z \in A^\infty$. Note that for each $z^i \in A^\infty$, there exist $z^i_j \to z^i$, $t^i_j \to \infty$ such that $t^i_j z^i_j \in A$. Then we have $|z - z^i_j| \le |z - z^i| + |z^i - z^i_j| \to 0$ as $i, j \to \infty$. That is, we can find a sequence $z^i_j \to z$ and $t^i_j \to \infty$ such that $t^i_j z^i_j \in A$. Thus, $z \in A^\infty$.

Next, assume that A is convex, that is for $\alpha \in (0,1)$ $\alpha x + (1-\alpha)y \in A$ whenever $x, y \in A$. Consider $\alpha x + (1-\alpha)y$ for $x, y \in A^\infty$. Then there exist $x_j \to x$, $t_j \to \infty$ such that $t_j x_j \in A$. Similarly, then there exist $y_j \to y$, $s_j \to \infty$ such that $s_j y_j \in A$. Then for $z = \alpha x + (1-\alpha)y$, there exists $\tilde{z}_j = \alpha x_j + (1-\alpha)y_j \to z$, and there exists $\tilde{t}_j = t_j s_j \to \infty$ as $t_j, s_j \to \infty$ such that $\tilde{t}_j \tilde{z}_j = \alpha s_j t_j x_j + ((1-\alpha)t_j s_j y_j \in A$ by convexity of A. This shows that A^∞ is convex.

For $x, y \in A^\infty$, there exist $t_j, s_j \to \infty$, $x_j \to x$, $y_j \to y$ such that $s_j x_j \in A$, $t_j y_j \in A$. We aim to show that for $\lambda \in (0,1)$, $\lambda x + (1-\lambda)y \in A^\infty$. We want

$$\tau_j(\lambda x_j + (1-\lambda)y_j) = \frac{\tau_j \lambda}{s_j} s_j x_j + \frac{(1-\lambda)\tau_j}{t_j} t_j y_j \in A$$

for some $\tau_j \to \infty$. We can pick τ_j such that $\frac{\tau_j \lambda}{s_j} + \frac{(1-\lambda)\tau_j}{t_j} = 1$. Then by convexity of A, we can conclude that $\tau_j(\lambda x_j + (1-\lambda)y_j) \in A$, that is, A^∞ is convex. $\square$

(b) *Assume that $V_j \xrightarrow{Y^p} \nu$ and that for some closed set C, we have $V_j(x) \in C$ $\mathcal{L}^n$ a.e. for all $j \in \mathbb{N}$. Show that the support* $\operatorname{supp}(\nu^\infty_x) \subseteq C^\infty$ *for λ a.e. $x \in \bar{\Omega}$.*

Proof. Consider the transformation map

$$T : \mathbb{R}^d \to \mathbb{B}^d$$

defined by

$$z \mapsto \hat{z} = \frac{z}{1 + |z|}.$$

Then T maps C^∞ to $\overline{T(C)} \cap \partial\mathbb{B}^d$. Let $\varphi \in C(\bar{\Omega})$, $\psi \in C(\partial\mathbb{B}^d \setminus C^\infty)$, and put for $k \in \mathbb{N}$,

$$\psi_k(z) := \begin{cases} ((|z| - k)^+)^p \psi(\frac{z}{|z|}), & z \neq 0, \\ 0 & z = 0. \end{cases}$$

We record that $\psi_k(z) = 0$ for $|z| \le k$ and that for $|z| = 1$, $\frac{\psi_k(tz)}{t^p} \to \psi(z)$ as $t \to \infty$. Also $\varphi \otimes \psi_k \in \mathbb{E}_p$ with $(\varphi \otimes \psi_k)^\infty_p = \varphi \otimes \psi_0$. Since $V_j \xrightarrow{Y^p} \nu$, we have

$$\int_\Omega \varphi\psi_k \circ V_j \, \mathrm{d}x \to \int_\Omega \varphi \langle \nu_x, \psi_k \rangle \, \mathrm{d}x + \int_{\bar{\Omega}} \varphi \langle \nu^\infty_x, \psi \rangle \, \mathrm{d}\lambda \text{ as } j \to \infty.$$

First note that

$$\int_\Omega \varphi \langle \nu_x, \psi_k \rangle \, \mathrm{d}x \to 0 \text{ as } k \to \infty.$$

Now we show that $\int_\Omega \varphi \psi_k \circ V_j = 0$. Note that $V_j(x) \in C$ $\mathcal{L}^n$ a.e for all j. By the result from Problem 1, we know that $V_j \rightharpoonup \bar{\nu}$ weakly in $L^p(\Omega, \mathbb{R}^d)$ and the limit $\bar{\nu} \in C^\infty$ $\mathcal{L}^n$ a.e. Then $\int_\Omega \varphi \psi_k \circ V_j = 0$ as $\psi \in C(\partial \mathbb{B}^d \setminus C^\infty)$.

Then we have

$$\int_\Omega \varphi \langle \nu_x^\infty, \psi \rangle \, \mathrm{d}\lambda = 0 \text{ for all } \varphi \in C(\bar{\Omega}).$$

For a.e. $x \in \Omega$,

$$\langle \nu_x^\infty, \psi \rangle = 0.$$

That is, $\operatorname{supp}(\nu_x^\infty) \subseteq C^\infty$.

Note that there is $\varepsilon > 0$ such that $B_\varepsilon(C^\infty) \cap \operatorname{supp}\psi = \emptyset$. Let k be sufficiently large so that $z \in B_k \setminus C$, which in turn implies that $\frac{z}{|z|} \in B_\varepsilon(C^\infty)$. Then $\varphi_k \circ V_j = 0$ for such k. Note that $\int_\Omega \varphi \langle \nu_x, \psi_k \rangle \to 0$ as $k \to \infty$ by DCT. □

Problem 3. *Given an example of a sequence (V_j) and a p-admissible integrand Φ so $V_j \overset{Y^p}{\to} \nu$ and*

$$\forall \varphi \in C(\bar{\Omega}) \colon \int_\Omega \varphi \Phi(\cdot, V_j) \, \mathrm{d}x \to \int_\Omega \varphi \langle \nu_x, \Phi(x, \cdot) \rangle \, \mathrm{d}x,$$

but the sequence $(\Phi(\cdot, V_j))$ is not equi-integrable on Ω.
On the other hand, show that $(\Phi(\cdot, V_j))$ is equi-integrable on Ω if

$$\int_O \Phi(\cdot, V_j) \, \mathrm{d}x \to \int_O \langle \nu_x, \Phi(x, \cdot) \rangle \, \mathrm{d}x$$

holds for all open subsets O of Ω, [See the proof of Vitali-Hahn-Saks.]

Proof. Let $\Omega = (-1, 1)$, $V_j = -j^{\frac{1}{p}} 1_{(-\frac{1}{j}, 0)} + j^{\frac{1}{p}} 1_{(0, \frac{1}{j})}$ and define $\Phi(x, z) = |z|^{p-1} z$. Then $(\Phi(\cdot, V_j))$ is not equi-integrable since

$$\int_{\Omega \cap |\{\Phi(\cdot, V_j)| > \frac{j}{2}\}} |\Phi(\cdot, V_j)| \, \mathrm{d}x = \int_{-\frac{1}{j}}^0 j \, \mathrm{d}x + \int_0^{\frac{1}{j}} j \, \mathrm{d}x$$

does not tend to zero as $j \to \infty$.

But we know that $\Phi \in \mathbb{E}_p$ and $V_j \xrightarrow{Yp} \nu$. Note that for $\varphi \in C(\bar{\Omega})$,

$$\begin{aligned}
\int_{-1}^{1} \varphi \Phi(\cdot, V_j)\,\mathrm{d}x &= \int_{-1}^{1} \varphi |V_j|^{p-1} V_j \,\mathrm{d}x \\
&= - \int_{-\frac{1}{j}}^{0} \varphi j \,\mathrm{d}x + \int_{0}^{\frac{1}{j}} \varphi j \,\mathrm{d}x \\
&= - \int_{-1}^{0} \varphi(\frac{y}{j})\,\mathrm{d}y + \int_{0}^{1} \varphi(\frac{y}{j})\,\mathrm{d}y \\
&\to - \varphi(0) + \varphi(0) \text{ as } j \to \infty \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\int_{-1}^{1} \varphi(x) \left\langle \nu_x, \Phi(x, \cdot) \right\rangle \,\mathrm{d}x &= \int_{-1}^{1} \varphi \left\langle \nu_x, |\cdot|^{p-1}(\cdot) \right\rangle \,\mathrm{d}x \\
&= 0
\end{aligned}$$

since we know that $\nu_x = \delta_0$ from the lecture notes. $\square$

Problem 4. *(Principle of convergence of energies)*

(a) Assume that $F\colon \mathbb{R}^d \to \mathbb{R}$ is strictly convex:

$$F(\lambda z_1 + (1-\lambda) z_0) < \lambda F(z_1) + (1-\lambda) F(z_0)$$

holds whenever $z_0 \neq z_1$ and $\lambda \in (0,1)$. Show that then the strict form of Jensen's inequality holds for F and any non-trivial probability measure ν:

$$\int_{\mathbb{R}^d} F \,\mathrm{d}\mu > F(\bar{\nu})$$

for probability measures μ on $\mathbb{R}^d$ with centre of mass $\bar{\mu}$ and so $\mu \neq \delta_{\bar{\mu}}$.

Proof. Note that if $F = \infty$, the inequality is trivially true, so we assume that $F < \infty$. Since F is strictly convex, by Hahn-Banach Separation theorem, we have $x^* \in \mathbb{R}^d$ with

$$\langle z - x, x^* \rangle + F(x) < F(z).$$

That is, there exist constants $a \in \mathbb{R}^d$, $b \in \mathbb{R}$ such that

$$a \cdot z + b < F(z) \text{ for all } z \in \mathbb{R}^d$$

and

$$a \cdot \bar{\mu} + b = F(\bar{\mu}).$$

Then, we have

$$\begin{aligned}\int_{\mathbb{R}^d} F(z)\,\mathrm{d}\mu(z) >& \int_{\Omega} a\cdot z + b\,\mathrm{d}\mu(z) \\ =& a\int_{\Omega} z\,\mathrm{d}\mu(z) + b \\ =& a\bar{\mu} + b \\ =& F(\bar{\mu}).\end{aligned}$$

Note that for all $z_1 \neq z_0 \in \mathbb{R}^n$, there exists $v_0 \in \mathbb{R}^n$, such that

$$F(z_1) - F(z_0) \geq v_0 \cdot (z_1 - z_0).$$

We claim that only strict inequality holds in this case. Assume for contradiction that

$$F(z_1) - F(z_0) = v_0 \cdot (z_1 - z_0)$$

and define

$$z_\lambda = \lambda z_1 + (1-\lambda) z_0.$$

Then

$$\begin{aligned}\lambda v_0 \cdot (z_1 - z_0) =& v_0 \cdot (z_\lambda - z_0) \\ \leq& F(z_\lambda) - F(z_0) \\ <& \lambda F(z_1) + (1-\lambda)F(z_0) - F(z_0) \text{ (by strict convexity)} \\ =& \lambda v_0 \cdot (z_1 - z_0) + \lambda F(z_0) + (1-\lambda)F(z_0) - F(z_0) \\ =& \lambda v_0 \cdot (z_1 - z_0)\end{aligned}$$

yields a contradiction.

Let $z_0 = \bar{\mu}$, then

$$\begin{aligned}\int_{\mathbb{R}^d} F\,\mathrm{d}\mu &= \int_{z_0} F\,\mathrm{d}\mu + \int_{\mathbb{R}^d\setminus\{z_0\}} F\,\mathrm{d}\mu \\ &> \int_{\{z_0\}} F\,\mathrm{d}\mu + \int_{\mathbb{R}^d\setminus\{z_0\}} v_0(z - z_0)\,\mathrm{d}\mu + F(z_0) \\ &= \int_{\{z_0\}} F\,\mathrm{d}\mu + F(z_0)\end{aligned}$$

where the middle term vanishes because $\int_{\mathbb{R}^d\setminus\{z_0\}} z\,\mathrm{d}\mu = \bar{\mu} = z_0$. Thus,

$$\int_{\mathbb{R}^d} F(z)\,\mathrm{d}\mu(z) > F(z_0) = F(\bar{\mu}).$$

□

(b) *Assume that $V_j \rightharpoonup V$ weakly in $L^p(\Omega, \mathbb{R}^d)$ and that (with F as above)*

$$\int_\Omega F(V_j)\,\mathrm{d}x \to \int_\Omega F(V)\,\mathrm{d}x \in \mathbb{R}.$$

Show that $F(V_j) \to F(V)$ in $L^1(\Omega)$ and $V_j \to V$ in $L^q(\Omega, \mathbb{R}^d)$ for each $q < p$.

Proof. Assume that $V_j \overset{Y^p}{\to} \nu$. We first show that $V_j \to V$ in measure. Recall from lecture notes that $V_j \to V$ in measure if and only if $V(x) = \bar{\nu}_x$ a.e. and $\nu_x = \delta_{\bar{\nu}_x}$ a.e. We have already known from Problem 1 that $V(x) = \bar{\nu}_x$ a.e., so it is sufficient to show that $\nu_x = \delta_{\bar{\nu}_x}$. Note that

$$\begin{aligned}\int_\Omega F(V)\,\mathrm{d}x &= \lim_{j\to\infty}\int_\Omega F(V_j)\,\mathrm{d}x \text{ (by assumption)}\\ &\overset{*}{=} \int_\Omega \langle \nu_x, F\rangle \,\mathrm{d}x.\end{aligned}$$

Now we need to verify $*$. We define $\tilde{F}(x,z) = 1_\Omega F(z)$. Then for $\mathcal{L}^n$ a.e. $x \in \Omega$, $z \mapsto \tilde{F}(x,z)$ is continuous by convexity of F. For all $z \in \mathbb{R}^d$, $x \mapsto \tilde{F}(x,z)$ is measurable. This shows that $\tilde{F}$ is a $\mathcal{L}^n$-Carathéodory integrand. Together with the fact that $\tilde{F}(\cdot, V_j)$ is equi-integrable (I am not sure about this part), we can show that

$$\lim_{j\to\infty}\int_\Omega \tilde{F}(\cdot, V_j)\,\mathrm{d}x = \int_\Omega \left\langle \nu_x, \tilde{F}(x,\cdot)\right\rangle \,\mathrm{d}x.$$

which implies $*$.

Since $V(x) = \bar{\nu}_x$ a.e., then $\nu_x = \delta_{\bar{\nu}_x}$. $V_j \to V$ in measure. Now we have

$$\begin{aligned}&\int_\Omega |F(V_j) - F(V)|\,\mathrm{d}x\\ =&\int_{\{|V_j - V|<\delta\}} |F(V_j) - F(V)|\,\mathrm{d}x + \int_{\{|V_j - V|\geq\delta\}} |F(V_j) - F(V)|\,ddx\\ &\to 0.\end{aligned}$$

The first term tends to 0 because of the continuity (which follows from convexity of) F while the second term converges to 0 since $V_j \to V$ in measure. Then we can conclude that $F(V_j) \to F(V)$ in $L^1(\Omega, \mathbb{R}^d)$.

For the second part of this question, we know that $V_j \to V$ in measure. If we can show that (V_j) is q-equi-integrable for $q < p$, then the strong convergence follows from *Vitali's Convergence Theorem.* The q-equi-integrability of (V_j) follows from the fact that

$$\sup_j \int_{\Omega\cap\{|V_j|^q > t\}} |V_j|^q\,\mathrm{d}x \leq \sup_j \int_{\Omega\cap\{|V_j|^q > t\}} |V_j|^p\,\mathrm{d}x |\Omega|^{1-\frac{q}{p}} \to 0 \text{ as } t\to\infty.$$

If $F \geq 0$, since F is a normal integrand, we have

$$\liminf_{j\to\infty} \int_\Omega F(V_j)\,\mathrm{d}x \geq \int_\Omega \langle \nu_x, F\rangle \,\mathrm{d}x.$$

Note that $\bar{\nu}(x) = V(x)$, then

$$\langle \nu_x, F\rangle > F(V(x))$$

if $\nu_x \neq \delta_{\nu(x)}$. For a.e $x \in \Omega$, $\langle \nu_x, F\rangle = F(V(x))$ implies that for a.e. $x \in \Omega$, $\nu_x = \delta_{\nu(x)}$. By weak convergence, we know that

$$\liminf_{j\to\infty} \int_\Omega F(V_j)\,\mathrm{d}x = \int_\Omega F(V(x))\,\mathrm{d}x.$$

Thus for a.e. $x \in \Omega$, $\nu_x = \delta_{\nu(x)}$. This together with $V(x) = \bar{\nu}_x$ imply that $V_j \to V$ in measure. Since F is continuous, $F(V_j) \to F(V)$ in measure, and the result follows.

If we do not have the extra condition that $F \geq 0$, we can subtract a linear functional L such that $F - L \geq 0$ because F is strictly convex. The above argument works for $F - L$. Then the convergence result holds for F since L is affine. $\square$