Chapter I: Fundamentals

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1 Lecture 1 Matrix- Vector Multiplication (21/04/2018)

Exercise 1.1. *(a) Row operation $\to$ Multiply a matrix on the left of B;*
Column operation $\to$ Multiply a matrix on the right of B.

1. *double column* 1 *: Multiply on the right of B by* $R_1 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

2. *halve row* 3*: Multiply on the left of B by* $L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

3. *add row* 3 *to row* 1*: Multiply on the left of B by* $L_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

4. *interchange columns* 1 *and* 4*: Multiply on the right of B by* $R_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

5. *subtract row* 2 *from each of the other rows: Multiply on the left of B by* $L_3 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$.

6. *replace column 4 by column 3: Multiply on the right of B by* $R_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

7. *delete column 1 : Multiply on the right of B by* $R_4 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Thus the resulting matrix is

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} B \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b)

$$A = L_3 L_2 L_1 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1/2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 1/2 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$C = R_1 R_2 R_3 R_4 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Exercise 1.2. *(a) We know that*

$$f_1 = k_{12}(x_2 - x_1 - l_{12});$$

$$f_2 = k_{23}(x_3 - x_2 - l_{23});$$

$$f_3 = k_{34}(x_4 - x_3 - l_{34});$$

$$f_4 = 0.$$

By writing the above linear system as matrix-vector multiplication we have

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} -k_{12} & k_{12} & 0 & 0 \\ 0 & -k_{23} & k_{23} & 0 \\ 0 & 0 & -k_{34} & k_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} - \begin{bmatrix} k_{12}l_{12} \\ k_{23}l_{23} \\ k_{34}l_{34} \\ 0 \end{bmatrix}.$$

(b) $K = \begin{bmatrix} -k_{12} & k_{12} & 0 & 0 \\ 0 & -k_{23} & k_{23} & 0 \\ 0 & 0 & -k_{34} & k_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

The entries of K *are spring constants. By Hook's law, we know that force=spring constant* $\times$ *displacement. Thus, the entries of* K *are* $\frac{\textbf{force}}{\textbf{displacement}} = \frac{ma}{d} = \frac{\textbf{mass}}{\textbf{time}^2}$.

(c) The dimensions of $\det(K) = \left(\frac{\textbf{mass}}{\textbf{time}^2}\right)^4$.

(d) Since the dimensions of entry of K *only involves time and mass. The SI unit of time is second while the SI unit of mass is kilogram.* $1\textbf{kg} = 1000\textbf{g}$. *Thus* $K' = 1000K$ *and* $\det(K') = 1000^4 \det(K)$.

Exercise 1.3. *We first write* R *as*

$$R = \begin{bmatrix} r_{1,1} & r_{1,2} & r_{1,3} & \dots & r_{1,m} \\ 0 & r_{2,2} & r_{2,3} & \dots & r_{2,m} \\ 0 & 0 & r_{3,3} & \dots & r_{3,m} \\ \vdots & \ddots & & & \\ 0 & 0 & 0 & \dots & r_{m,m} \end{bmatrix} = \left[r_1 \middle| r_2 \middle| \cdots \middle| r_m \right]$$

.

Let e_j *denote the canonical unit vector with* 1 *in the* j*th entry. Then*

$$e_j = \sum_{i=1}^{m} z_{ij} r_i.$$

In particular

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = z_{1,1} \begin{bmatrix} r_{1,1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + z_{2,1} \begin{bmatrix} r_{1,2} \\ r_{2,2} \\ \vdots \\ 0 \end{bmatrix} + z_{3,1} \begin{bmatrix} r_{1,3} \\ r_{2,3} \\ \vdots \\ 0 \end{bmatrix} + \cdots + z_{m,1} \begin{bmatrix} r_{1,m} \\ r_{2,m} \\ \vdots \\ r_{m,m} \end{bmatrix}.$$

That is $z_{m,1} r_{m,m} = 0$ *which implies* $z_{m,1} = 0$. *This means that* e_1 *is spanned by* $\{r_1, r_2, \cdots r_{m-1}\}$ *only. More precisely,*

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = z_{1,1} \begin{bmatrix} r_{1,1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + z_{2,1} \begin{bmatrix} r_{1,2} \\ r_{2,2} \\ \vdots \\ 0 \end{bmatrix} + z_{3,1} \begin{bmatrix} r_{1,3} \\ r_{2,3} \\ \vdots \\ 0 \end{bmatrix} + \cdots + z_{m-1,1} \begin{bmatrix} r_{1,m-1} \\ r_{2,m-1} \\ \vdots \\ r_{m-1,m-1} \end{bmatrix}.$$

Then we can get $z_{m-1,1} r_{m-1,m-1} = 0$, *which implies that* $z_{m-1,1} = 0$. *Similarly, we can show that* $z_{m-2,1} = z_{m-2,1} = \cdots = z_{2,1} = 0$.

Now we consider a general canonical unit vector

$$e_j = \begin{bmatrix} 0 \\ \vdots \\ j \\ \vdots \end{bmatrix} = z_{1j} \begin{bmatrix} r_1 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + z_{2j} \begin{bmatrix} r_{12} \\ r_{22} \\ \vdots \\ 0 \end{bmatrix} + z_{3j} \begin{bmatrix} r_{13} \\ r_{23} \\ \vdots \\ 0 \end{bmatrix} + \cdots + z_{mj} \begin{bmatrix} r_{1m} \\ r_{2m} \\ \vdots \\ r_{mm} \end{bmatrix}.$$

Thus, $z_{m,j} = z_{m-1,j} = \cdots z_{j+1,j} = 0$ for every $j = 1, 2, \cdots, m$. That is, Z is also an upper-triangular matrix.
Note that $e_j = Rz_j$, so we have $I = RZ$. That is, $Z = R^{-1}$. This completes the proof.

Exercise 1.4. *(a) We can write*

$$\sum_{j=1}^{8} c_j f_j(i) = d_i \text{ for } i = 1, 2, \cdots 8$$

as matrix-vector multiplication:

$$d = Fc$$

where $d = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_8 \end{bmatrix}, c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_8 \end{bmatrix}$ while F is a 8×8 square matrix with the i, jth entry defined by $F_{ij} := f_j(i)$.
Since for each $d \in \mathbb{C}^8$, we can find $c \in \mathbb{C}^8$ such that $d = Fc$, the range of A is $\mathbb{C}^8$. By the equivalent theorem(Theorem 1.3 in the book), we know that $\textbf{rank}\,(F) = 8$. This implies that F has an inverse, and thus d determines c uniquely if we multiply the matrix-vector multiplication expression $d = Fc$ by F^{-1} on both sides.
(b) By definition, we have $Ad = c$. That is, $A = F^{-1}$. Thus $A^{-1} = F$. The ij entry of A^{-1} is simply the ijth entry of F, that is, $f_j(i)$.

2 Lecture 2 Orthogonal Vectors and Matrices (24/04/2018)

Exercise 2.1. *If A is upper-triangular, then A^{-1} is upper-triangular (by Exercise 1.3). Since A is also unitary, we know that $A^\star = A^{-1}$ is upper triangular . However, $A^\star$, as a transpose of an upper-triangular matrix, should be lower-triangular. Thus, A^{-1} must be diagonal. This implies that A is diagonal. The argument is the same when A is lower-triangular.*

Exercise 2.2. *(a)*

$$\|x_1 + x_2\|^2 = x_1^2 + x_2^2 + \cancel{2x_1 \cdot x_2} = x_1^2 + x_2^2$$

since x_1 is orthogonal to x_2.
(b) Assume the equality holds for sum up to $n - 1$, that is

$$\|\sum_{i=1}^{n-1} x_i\|^2 = \sum_{i=1}^{n-1} \|x_i\|^2.$$

Then

$$\begin{aligned}
\| \sum_{i=1}^{n} x_i \|^2 &= \| \sum_{i=1}^{n-1} x_i + x_n \|^2 \\
&= \| \sum_{i=1}^{n-1} x_i \|^2 + x_n^2 + \cancel{2 \sum_{i=1}^{n-1} x_n \cdot x_i} \\
&= \| \sum_{i=1}^{n} x_i \|^2
\end{aligned}$$

as x_n is orthogonal to x_i for $i = 1, 2, \cdots, n-1$.

Exercise 2.3. *(a)Assume that λ is an eigenvalue of A, then there exists non-zero x such that $Ax = \lambda x$. Then, using that $A^\star = A$.*

$$x^\star A x = x^\star (Ax) = x^\star (\lambda x) = \lambda (x^\star \cdot x),$$

$$x^\star A x = (Ax)^\star x = (\lambda x)^\star x = \lambda^\star (x^\star \cdot x).$$

Since $x \neq 0$, $(x^\star, x) > 0$. Thus, $\lambda^\star = \lambda$. That is, $\lambda \in \mathbb{R}$.
(b) Assume that $Ay = \lambda y, Ax = \mu x$, then using $A^\star = A$,

$$x^\star A y = x^\star (Ay) = x^\star \lambda y = \lambda x^\star y = \lambda (x \cdot y),$$
$$x^\star A y = (Ax)^\star y = (\mu x)^\star y = \mu^\star x^\star y = \mu (x \cdot y).$$

The last equality follows from the fact that all eigenvalues are real. Since $\lambda \neq \mu$, $x \cdot y = 0$. That is, x is orthogonal to y.

Exercise 2.4. *The eigenvalues of a unitary matrix has modulus 1. Using $A^{-1} = A^\star$, we have*

$$x^\star x = x^\star A^{-1} A x = (x^\star A^\star)(Ax) = (\lambda^\star x^\star (\lambda x) = |\lambda|^2 x^\star x.$$

Thus $|\lambda| = 1$.

Exercise 2.5. *(a) Similar to the proof in Exercise 2.3 (a), we have $\lambda = -\lambda^\star$ in this case. This implies that the eigenvalues of S are pure imaginary.*
(b) If there exists non-zero x such that $(I - S)x = 0$. Then $x = Sx$.

$$x^\star x = (Sx)^\star x = x^\star S^\star x = x^\star (-S) x = -x^\star x.$$

Thus $x \equiv 0$. That is, $\mathrm{Null}(I - S) = \{0\}$, so $I - S$ is non-singular.
(c)

$$Q^\star = (I+S)^\star ((I-S)^{-1})^\star = (S^\star + I)[(I-S)^\star]^{-1} = (I-S)(I+S)^{-1} = Q^{-1}.$$

Exercise 2.6. *If $u \equiv 0$, then it is trivial. Now assume that $u \neq 0$.*
If A is non-singular, we can find out A^{-1}. We write

$$A^{-1} = [x_1, x_2, \cdots, x_m]$$

where x_i represents the ith column of A^{-1}. Then

$$\begin{aligned} AA^{-1} =& (I + uv^\star)\,[x_1, x_2, \cdots, x_m] \\ =& [1 + uv^\star x_1, \cdots, x_m + uv^\star x_m] \\ =& I \\ =& [e_1, e_2, \cdots, e_m] \end{aligned}$$

This implies that $x_i + uv^\star x_i = e_i$ for each $1 \leq i \leq m$. Then we can write

$$A^{-1} = [e_1 - z_1 u, e_2 - z_2 u, \cdots, e_m - z_m u] = I - uz^\star.$$

$$\begin{aligned} I =& AA^{-1} \\ =& (I + uv^\star)(I - uz^\star) \\ =& I + uv^\star - uz^\star - uv^\star uz^\star \\ =& I + uv^\star - uz^\star - (v^\star u)uz^\star \\ =& I. \end{aligned}$$

This means that

$$uv^\star - uz^\star - (v^\star u)uz^\star = 0.$$

$$uz^\star(v^\star u + 1) = uv^\star.$$

$$uz^\star = \frac{uv^\star}{v^\star u + 1}.$$

This shows that

$$z^\star = \frac{v^\star}{v^\star u + 1}.$$

Thus, $A^{-1} = I - \frac{uv^\star}{v^\star u+1} = I + \alpha uv^\star$ where $\alpha = -\frac{1}{v^\star u+1}$.
Now we suppose that A is singular, then there exists a non-zero x such that $Ax = 0$, that is

$$(I + uv^\star)x = 0.$$

This implies that

$$x = -uv^\star x.$$

$$x = -u(v^\star x)$$

where $v^\star x$ is a scalar.

This shows that $x = \beta u$ for some scaler β. If we plug $x = \beta u$ into $Ax = 0$, we get

$$\begin{aligned}(I + uv^\star)\beta u =&\beta u + \beta u v^\star u\\ =&\beta u(1 + v^\star u)\\ =&0.\end{aligned}$$

Since $\beta u \neq 0$, we must have $v^\star u = -1$.
If A is singular, $\text{null}(A) = \{\beta u \colon \beta \in \mathbb{C}\}$.

Exercise 2.7. *Note that*

$$H_{k+1}^{-1} = \begin{bmatrix}(2H_k)^{-1} & (2H_k)^{-1}\\ (2H_k)^{-1} & (-H_k)^{-1} + (2H_k)^{-1}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}H_k^{-1} & H_k^{-1}\\ H_k^{-1} & -H_k^{-1}\end{bmatrix},$$

$$H_{k+1}^T = \begin{bmatrix}H_k^T & H_k^T\\ H_k^T & -H_k^T\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\alpha_k H_k^{-1} & \alpha_k H_k^{-1}\\ \alpha_k H_k^{-1} & -\alpha_k H_k^{-1}\end{bmatrix} = 2\alpha_k H_{k+1}^{-1}.$$

By induction, the recursive description provides a Hadamard matrix for each $m = 2^k$.

3 Lecture 3 Norms (01/05/2018)

Exercise 3.1. *By definition,*

$$\|x\|_W = \|Wx\|.$$

1. $\|x\|_W \geq 0$, *and* $\|x\|_W = 0$ *only if $Wx = 0$ which implies that $x = 0$ since W is non-singular.*
2. $\|x + y\|_W = \|W(x + y)\| \leq \|Wx\| + \|Wy\| = \|x\|_W + \|y\|_W$.
3. $\|\alpha x\|_W = \|W(\alpha x)\| = \|\alpha W x\| = |\alpha|\|Wx\| = |\alpha|\|x\|_W$.

Thus $\|\cdot\|_W$ defines a vector norm for any arbitrary non-singular matrix W.

Exercise 3.2. *By definition, we have $\|A\| = \sup_{x\in\mathbb{C}^n, x\neq 0} \frac{\|Ax\|}{\|x\|}$. Since $\rho(A)$ is the largest absolute value $|\lambda|$ of an eigenvalue λ of A. There exists a non-zero $y \in \mathbb{C}^n$, such that $Ay = \lambda y$. That is, $\|Ay\| = |\lambda|\|y\|$. Thus,*

$$|\lambda| = \frac{\|Ay\|}{\|y\|} \leq \sup_{x\in\mathbb{C}^n, x\neq 0} \frac{\|Ax\|}{\|x\|} = \|A\|.$$

Exercise 3.3.

1. $\|x\|_\infty \leq \|x\|_2$ *can be easily proved:*

$$\|x\|_2 = \sqrt{\sum_{i=1}^m x_i^2} \geq |x_i| \text{ for every } 1 \leq i \leq n.$$

Take $x = e_1 = (1, 0, \cdots, 0)$, then $\|x\|_\infty = \|x\|_2$.

2. $\|x\|_2 \le \sqrt{m}\|x\|_\infty$ *can be proved in the following way:*

$$\|x\|_2 = \sqrt{\sum_{i=1}^{m} x_i^2} \le \sqrt{\sum_{i=1}^{m} \|x\|_\infty^2} = \sqrt{m\|x\|_\infty^2} = \sqrt{m}\|x\|_\infty.$$

Take $x = (1, 1, \cdots, 1)$. *Then* $\|x\|_2 = \sqrt{\sum_{i=1}^{m} 1^2} = \sqrt{m}$. $\sqrt{m}\|x\|_\infty = \sqrt{m} \cdot 1 = \sqrt{m}$.

3. *By part (1) and part (2),* $\|A\|_\infty = \sup_{x \in \mathbb{C}^n \setminus \{0\}} \frac{\|Ax\|_\infty}{\|x\|_\infty} \le \frac{\|Ax\|_2}{1/\sqrt{n}\|x\|_2} \le \sqrt{n}\|A\|_2$.

   Recall that $\|A\|_\infty$ *is the maximum row sum while* $\|A\|_2$ *is the maximum absolute value of the singular value (eigenvalue for square matrix). Take*

$$A = \begin{bmatrix} 1 & 1 \\ 0 & \sqrt{2} \end{bmatrix},$$

   then $\|A\|_\infty = 2, \|A\|_2 = \sqrt{2}, \sqrt{n} = \sqrt{2}$, *so the equality holds.*

4. *By part(1) and part (2),* $\|A\|_2 = \sup_{x \in \mathbb{C}^n \setminus \{0\}} \frac{\|Ax\|_2}{\|x\|_2} \le \frac{\sqrt{m}\|Ax\|_\infty}{\|x\|_\infty} \le \sqrt{m}\|A\|_\infty$. *Similar to (3), but we consider a rectangle matrix in this case. Take*

$$A = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

   then $\|A\|_2 = 1$ *(the largest singular value of* A*),* $\|A\|_\infty = 1$ *and* $m = 1$*, so* $\|A\|_2 = \sqrt{1}\|A\|_\infty$.

Exercise 3.4. *(a) Similar to the step 7 in Exercise 1.1, if we multiply on the RHS of* A *by an identity matrix with* i*th column removed, the* i*th column of* A *is deleted. If we multiply* A *by an identity matrix with* j*th row removed on the LHS of* A*, the* j*th row of* A *is deleted. We can multiply* A *by such matrices to obtain* B*.*
(b) We know that $\|AB\|_p \le \|A\|_p\|B\|_p$ *for any* $m \times n$ *matrix* A*,* $n \times l$ *matrix* B *and for every* $1 \le p \le \infty$*. It is sufficient to show that* p*-norm of each matrix we used to multiply on the LHS and RHS of* A *is less than or equal to* 1*. This follows from direct computation.*

Exercise 3.5. *Since*

$$\frac{\|Ex\|_2}{\|x\|_2} = \frac{\|uv^\star x\|_2}{\|x\|_2} = \frac{\|u\|_2|v^\star x|}{\|x\|_2} \le \frac{\|u\|_2\|v\|_2\|x\|_2}{\|x\|_2} = \|u\|_2\|v\|_2,$$

$$\|E\|_2 = \sup_{x \in \mathbb{C}^n \setminus \{0\}} \frac{\|Ex\|_2}{\|x\|_2} \le \|u\|_2\|v\|_2.$$

The equality is achieved if we take $x = v$*. Thus,*

$$\|E\|_2 = \|u\|_2\|v\|_2.$$

Note that $E_{ij} = u_i v_j$*, thus*

$$\|E\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |E_{ij}|^2} = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |u_i|^2|v_j|^2} = \sqrt{\sum_{i=1}^{m} |u_i|^2}\sqrt{\sum_{j=1}^{n} |v_j|^2} = \|u|_F\|v\|_F.$$

Exercise 3.6. *(a)*

1. *By definition* $\|x\|' = \sup_{\|y\|=1} |y^\star x| \geq 0$ *and* $\|x\|' = 0$ *only if* $x = 0$.

2. $\|x+z\|' = \sup_{\|y\|=1} |y^\star(x+z)| \leq \sup_{\|y\|=1}(|y^\star x| + |y^\star z|) = \|x\|' + \|z\|'$.

3. $\|\alpha x\|' = \sup_{\|y\|=1} |y^\star(\alpha x)| = |\alpha| sup_{\|y\|=1} |y^\star x| = |\alpha|\|x\|'$.

   (b)

   We want to show that there exists $z \in \mathbb{C}^m$ *such that*

$$Bx = yz^\star x = (z^\star x)y = y.$$

So we need to construct z *such that* $z^\star x = 1$. *Following the hint, we know that for each given* x, *there exists a non-zero* $z_0 \in \mathbb{C}^m$ *such that* $|z_0^\star x| = \|z_0\|' \|x\|$.
Define $z := e^{i\theta} \frac{z_0}{\|z_0\|'}$ *where* $\theta = arg(z_0^\star x)$. *Then* $z^\star x = e^{i\theta} \frac{z_0^\star x}{\|z_0\|'} = \frac{|z_0^\star x}{\|z_0\|'} = \|x\| = 1$.
It follows that

$$\|B\| = \sup_{\|x\|=1} \|Bx\| = \sup_{\|x\|=1} \|yz^\star x\| = \sup_{\|x\|=1} |z^\star x| \|y\| = sup_{\|x\|=1} |x^\star z| = \|z\|' = 1.$$

4 The singular value decomposition (04/05/2018)

Exercise 4.1. *Since* $A = U\Sigma V^\star$,

$$AA^\star = U\Sigma(V^\star V)\Sigma^\star U^\star = U(\Sigma\Sigma^\star)U^\star.$$

That is,

$$AA^\star U = U(\Sigma\Sigma^\star).$$

Each column $\{u_j\}$ *of* U *is the eigenvector of* $AA^\star$ *corresponding to the eigenvalue* σ_j^2 *where* σ_j *is the* j*th singular value of* A.

1. $A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$, $AA^\star = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$. *Thus,* $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ *and* $V = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

2. $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $AA^\star = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$. *Thus,* $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ *and* $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

3. $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $AA^\star = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. *Thus,* $U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ *and* $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

4. $A = \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$, $AA^\star = \begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}$. *Thus,* $U = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$, $\Sigma = \begin{bmatrix}\sqrt{2} & 0\\0 & 0\end{bmatrix}$ *and* $V = \begin{bmatrix}\sqrt{2}/2 & -\sqrt{2}/2\\\sqrt{2}/2 & \sqrt{2}/2\end{bmatrix}$.

5. $A = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$, $AA^\star = \begin{bmatrix}2 & 2\\2 & 2\end{bmatrix}$. $\Sigma\Sigma^\star = \begin{bmatrix}4 & 0\\0 & 0\end{bmatrix}$. $U = \begin{bmatrix}\sqrt{2}/2 & \sqrt{2}/2\\\sqrt{2}/2 & -\sqrt{2}/2\end{bmatrix}$, $\Sigma = \begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}$ *and* $V = \begin{bmatrix}\sqrt{2}/2 & \sqrt{2}/2\\\sqrt{2}/2 & -\sqrt{2}/2\end{bmatrix}$.

Exercise 4.2. *The answer is Yes!*

By definition, $BQ = A^\star$ *where* $Q = \begin{bmatrix}0 & \cdots & 0 & 1\\0 & \cdots & 1 & 0\\\vdots & \ddots & 0 & 0\\1 & \cdots & 0 & 0\end{bmatrix}$ *is a unitary matrix.*

If $A = U\Sigma V^\star$, *then* $A^\star = V\Sigma^\star U^\star$. *Thus* $B = V\Sigma U^\star Q^\star = V\Sigma(UQ)^\star$ *Therefore,* A *and* B *has the same set of singluar values as* UQ, *the product two unitary matrices is a unitary matrix.*

Exercise 4.3. *The Matlab programme is summarized in the following function:*

```
function h=svdplot(A)
% This function is designed to plot the unit circle and the right singular
% vectors of a 2 by 2 matrix A and its corresponding ellipse and left
% singulat vectors on image plane.
%SVD of A
[U,S,V]=svd(A);
% Plot the unit circle and the right singular vectors on pre-image plane.
figure(1);
axis([-1.2 1.2 -1.2 1.2]);
theta=0:0.01:2*pi;
x1=cos(theta);
y1=sin(theta);
plot(x1, y1);
hold on;
t = 0:0.01:1;
x11 = V(1,1)*t; y11= V(2,1)*t; plot(x11, y11, 'r');
hold on;
x12=V(1,2)*t;y12=V(2,2)*t; plot(x12,y12,'r');
grid on;
title('On pre-image plane');

% Plot the ellipse and the scaled left signualr vectors on image plane.
figure(2);
```

```
axis([-3 3 -3 3]);
x=[x1;y1];
w=A*x;
x2=w(1,:);y2=w(2,:);
 plot(x2, y2);
hold on;
w1=S(1,1)*U(:,1); w2=S(2,2)*U(:,2)
t = 0:0.01:1;
x21 = w1(1)*t; y21= w1(2)*t;
plot(x21, y21, 'r');
hold on;
x22=w2(1)*t; y22=w2(2)*t;
plot(x22,y22,'r');
grid on;
title('On image plane');
end
```

Exercise 4.4. *If A and B are unitarily equivalent, then A and B have the same singular values.*
Assume that $B = U\Sigma V^\star$. Since $A = QBQ^\star$,

$$A = QU\Sigma V^\star Q^\star = (QU)\Sigma(QV)^\star$$

is a singular value decomposition of A. By uniqueness of SVD, A and B have the same singular values.
However the converse is not necessarily true.
Conside $A = \begin{bmatrix} 1 & 1 \\ 0 & \end{bmatrix}$ and $B = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}$ Then A and B have the same singular values but not unitarily equivalent (i.e. A is non-symmetric but B is symmetric).

Exercise 4.5. *Recall that calculating the SVD consists of finding the eigenvalues and eigenvectors of $AA^\star$ and $A^\star A$. The eigenvectors of $A^\star A$ make up the columns of V, the eigenvectors of $AA^\star$ make up the columns of U. Also, the singular values in Σ are square roots of eigenvalues from $AA^\star$ or $A^\star A$. The singular values are the diagonal entries of the Σ matrix and are arranged in descending order. The singular values are always real numbers since the eigenvalues of $A^\star A$ or $AA^\star$ are non-negative. If the matrix A is a real matrix, $AA^\star$ and $A^\star A$ are real and symmetric, which implies that U and V are also real.*

5 More on the SVD (11/05/2018)

Exercise 5.1.

$$A^\star A = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix}.$$

The eigenvalues of $A^\star A$ satisfies $\lambda^2 - 9\lambda + 4 = 0$. Thus $\lambda = \frac{1}{2}(9 \pm \sqrt{65})$. This implies that

$$\sigma_{max}(A) = \sqrt{\frac{1}{2}(9+\sqrt{65})},\ \textit{and}\ \sigma_{min}(A) = \sqrt{\frac{1}{2}(9-\sqrt{65})}.$$

Exercise 5.2. *Suppose $A = U\Sigma V^\star$. Define $A_\varepsilon = U(\Sigma + \varepsilon I_{m\times n})V^\star$ for positive ε. We claim that A_ε is of full rank for each ε, and that $\|A - A_\varepsilon\|_2 = \varepsilon$.*
Since the diagonal entries of Σ are non-negative, the diagonal entries of $\Sigma + \varepsilon I_{m\times n}$ are positive. That is, A_ε is of full rank for each ε.

$$\|A - A_\varepsilon\|_2 = \|U(\varepsilon I_{m\times n}V^\star\| = \varepsilon\|I_{m\times n}\|_2 = \varepsilon.$$

This shows that any matrix $A \in \mathbb{C}^{m\times n}$ can be approximated by a sequence of full rank matrices $\{A_\varepsilon\}$.

Exercise 5.3. *(a)*

$$AA^\star = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}\begin{bmatrix} -2 & -10 \\ 11 & 5 \end{bmatrix} = \begin{bmatrix} 125 & 75 \\ 75 & 125 \end{bmatrix} = 25\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}.$$

So the eigenvalues of $AA^\star$ satisfies

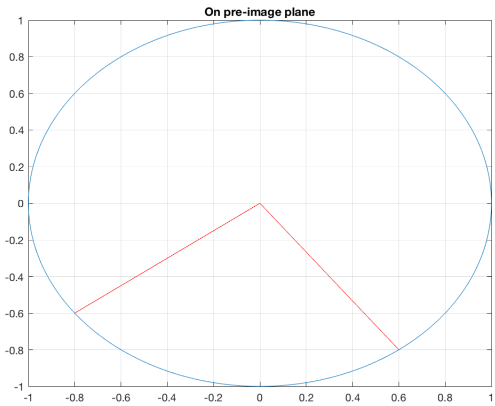
$$(\lambda/25 - 5)^2 - 9 = 0.$$

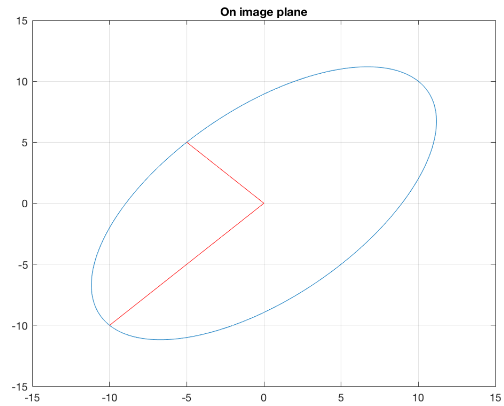
That is, $\lambda = 25*2 = 50$ or $\lambda = 25*8 = 200$. The corresponding eigenvectors are $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. This implies that $\sigma_{max}(A) = 10\sqrt{2}$, $\sigma_{min}(A) = 5\sqrt{2}$,

$$\Sigma = \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix}, U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

$$V = A^\star U\Sigma^{-1} = \begin{bmatrix} -2 & -10 \\ 11 & 5 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{10\sqrt{2}} & 0 \\ 0 & \frac{1}{5\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}.$$

Since we want U and V to have minimal number of negative signs, take $V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $U = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$.
(b) The singular values are $5\sqrt{2}$ and $10\sqrt{2}$.
The left singular vectors of A are $u_1 = \pm\frac{1}{5}\begin{bmatrix} -3 \\ 4 \end{bmatrix}$, $u_2 = \pm\frac{1}{5}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$.
The right singular vectors of A are $v_1 = \pm\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v_2 = \pm\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Using the MATLAB function in previous exercise, we get the following plot:





(c)

$$\|A\|_1 = \max_{1\le j\le 2} \|a_j\|_1 = 16.$$

$$\|A\|_2 = \sigma_{max}(A) = 10\sqrt{2}.$$

$$\|A\|_\infty = \max_{1\le i\le 2} \|a_i^\star\|_1 = 15.$$

$$\|A\|_F = \sqrt{\sigma_{max}^2 + \sigma_{min}^2} = \sqrt{250} = 5\sqrt{10}.$$

(d)

$$A^{-1} = (V^\star)^{-1}\Sigma^{-1}U^{-1} = V\Sigma U^\star.$$

Thus,

$$A^{-1} = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} \frac{1}{10\sqrt{2}} & 0 \\ 0 & \frac{1}{5\sqrt{2}} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{100}\begin{bmatrix} 5 & -11 \\ 10 & -2 \end{bmatrix}.$$

(e) The eigenvalues of A satisfies

$$\det\left(\begin{bmatrix} -2-\lambda & 11 \\ -10 & 5-\lambda \end{bmatrix}\right) = 0.$$

That is $(-2-\lambda)(5-\lambda) + 110 = \lambda^2 + 3\lambda + 100 = 0.$
Thus $\lambda_{1,2} = \frac{1}{2}(-3 \pm \sqrt{391}i).$
(f) $\lambda_1\lambda_2 = \frac{1}{4}(3-\sqrt{391}i)(3+\sqrt{391}i) = 100.$ $\det(A) = -2(5) - 11(-10) = 100.$ $\sigma_1\sigma_2 = 5\sqrt{2}\cdot 10\sqrt{2} = 100.$ $|\det(A)| = 100.$
(g)
The area of the ellipsoid onto which A maps the unit ball is $\det(A)\cdot$ *area of the unit ball in* $\mathbb{R}^2 = 100\pi$.

Exercise 5.4.

$$\begin{aligned}\begin{bmatrix}0 & A^\star\\ A & 0\end{bmatrix} &= \begin{bmatrix}0 & I\\ I & 0\end{bmatrix}\begin{bmatrix}U\Sigma V^\star & 0\\ 0 & V\Sigma U^\star\end{bmatrix}\\ &= \begin{bmatrix}0 & I\\ I & 0\end{bmatrix}\begin{bmatrix}U & 0\\ 0 & V\end{bmatrix}\begin{bmatrix}\Sigma & 0\\ 0 & \Sigma\end{bmatrix}\begin{bmatrix}V^\star & 0\\ 0 & U^\star\end{bmatrix}.\end{aligned}$$

Note that the inverse of $\begin{bmatrix}0 & I\\ I & 0\end{bmatrix}\begin{bmatrix}U & 0\\ 0 & V\end{bmatrix}$ *is* $\begin{bmatrix}U^\star & 0\\ 0 & V^\star\end{bmatrix} = \begin{bmatrix}0 & I\\ I & 0\end{bmatrix}\begin{bmatrix}V^\star & 0\\ 0 & U^\star\end{bmatrix}$. *The inverse of* $\begin{bmatrix}0 & I\\ I & 0\end{bmatrix}$ *is itself.*

Thus, $\begin{bmatrix}0 & A^\star\\ A & 0\end{bmatrix} = \begin{bmatrix}0 & I\\ I & 0\end{bmatrix}\begin{bmatrix}U & 0\\ 0 & V\end{bmatrix}\left(\begin{bmatrix}\Sigma & 0\\ 0 & \Sigma\end{bmatrix}\begin{bmatrix}0 & I\\ I & 0\end{bmatrix}\right)\left(\begin{bmatrix}0 & I\\ I & 0\end{bmatrix}\begin{bmatrix}U & 0\\ 0 & V\end{bmatrix}\right)^{-1}$.

Note that $\begin{bmatrix}\Sigma & 0\\ 0 & \Sigma\end{bmatrix}\begin{bmatrix}0 & I\\ I & 0\end{bmatrix} = \begin{bmatrix}0 & \Sigma\\ \Sigma & 0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}I & I\\ I & -I\end{bmatrix}\begin{bmatrix}\Sigma & 0\\ 0 & \Sigma\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}I & I\\ I & -I\end{bmatrix}$.

If $X = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & I\\ I & 0\end{bmatrix}\begin{bmatrix}U & 0\\ 0 & V\end{bmatrix}\begin{bmatrix}I & I\\ I & -I\end{bmatrix}$, $\begin{bmatrix}0 & A^\star\\ A & 0\end{bmatrix} = X\begin{bmatrix}\Sigma & 0\\ 0 & -\Sigma\end{bmatrix}X^{-1}$.