Numerical Linear Algebra: Solutions for Sheet 5

1. We are trying to solve $Ax = b$ where we have $A \in \mathbb{R}^{n\times n}$ and $\mathrm{diag}(A) = I$. Using the usual splitting $A = M - N$ we have for the Jacobi iteration, that $M = I$ and $N = I - A$ so the iteration matrix J is $J = M^{-1}N = I - A$.

   Using the iteration $Mx^{(k)} = Nx^{(k-1)} + b$ we have that

   $$x - x^{(k)} = J(x - x^{(k-1)})$$

   and hence (by induction on k)

   $$x - x^{(k)} = (I - A)^k (x - x^{(0)})$$

   Writing this in the form required in the question, we have $(x - x^{(k)}) = p_k(A)(x - x^{(0)})$ where

   $$p_k(A) = (I - A)^k = \sum_{i=0}^{k} (-1)^i A^i \left(\begin{array}{c} k \\ i \end{array} \right)$$

   Returning to the iteration, we have $x^{(k)} = (I - A)x^{(k-1)} + b$ so

   $$x^{(1)} = (I - A)x^{(0)} + b$$

   and

   $$\begin{aligned} x^{(2)} &= (I - A)x^{(1)} + b \\ &= (I - A)^2 x^{(0)} + (I - A)b + b \end{aligned}$$

   and (by induction on k)

   $$x^{(k)} = (I - A)^k x^{(0)} + \sum_{i=0}^{k-1} (I - A)^i b$$

   If $x^{(0)} = 0$ then we can write this as

   $$x^{(k)} = q_k(A)b$$

   where we have

   $$q_k(A) = \sum_{i=0}^{k-1} (I - A)^i \equiv \sum_{i=0}^{k-1} (-1)^i A^i \left(\begin{array}{c} k \\ i+1 \end{array} \right)$$

2. For $A = M - N$ with $A, M, N \in \mathbb{R}^{n \times n}$ we are given the iteration

$$Mx^{(k)} = Nx^{(k-1)} + b$$

We have that

$$Mx = Nx + b$$

so, subtracting we have

$$M(x - x^{(k)}) = N(x - x^{(k-1)}) \quad \Rightarrow \quad x - x^{(k)} = M^{-1}N(x - x^{(k-1)})$$

and by induction

$$x - x^{(k)} = (M^{-1}N)^k (x - x^{(0)})$$

If $M^{-1}N$ is diagonalizable, then there exists a basis of eigenvectors $\{v_i\}$ so

$$x - x^{(0)} = \sum \alpha_i v_i \qquad \text{some } \alpha_i \in \mathbb{R}$$

so

$$\begin{aligned} x - x^{(k)} &= \sum \alpha_i (M^{-1}N)^k v_i \\ &= \sum \alpha_i \lambda_i^k v_i \qquad (1) \\ &\rightarrow 0 \text{ as } k \rightarrow \infty \text{ if } |\lambda_i| < 1 \;\; \forall\, i \end{aligned}$$

Conversely if $|\lambda_j| \geq 1$ then supposing $x^{(0)} = x + v_j$ we have that $x - x^{(0)} = v_j$ and $x - x^{(k)} = \lambda_j^k v_j$ which does not tend to zero as k increases.

If we have a basis of orthonormal eigenvectors $\{v_i\}$, we have $v_i^T v_j = \delta_{ij}$. Operating on the left hand side of equation (1) with v_i^T we have

$$\begin{aligned} v_i^T (x - x^{(k)}) &= \sum_{j=1}^{n} \alpha_i \lambda_i^k v_i^T v_i \\ &= \alpha_i \lambda_i \\ &= \lambda_i v_i^T (x - x^{(0)}) \end{aligned}$$

The Successive Over Relaxation method is: $x^{(0)}$ arbitrary

for $\quad k = 1, 2, ...$

for $i = 1, ..., n$

$$x_i^{(k)} = (1 - w)x_i^{(k-1)} + w \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)} \right) \Bigg/ a_{ii}$$

end

end

which is the componentwise form of

$$(D + wL)x^{(k)} = wb + [(1 - w)D - wU]x^{(k-1)}$$

so we have $wM = (D + wL)$ and $wN = (1 - w)D - wU$. We therefore have the SOR iteration matrix

$$T = M^{-1}N = (D + wL)^{-1}((1 - w)D - wU)$$

If $D = I$ then we have

$$\begin{aligned} \det(T) &= \det(I + wL)^{-1} \; \det[(1 - w)I - wU] \\ &= 1 \times (1 - w)^n \end{aligned}$$

since $I = wL$ and $(1-w)I - wU$ are both triangular with respectively 1's and $(1-w)$'s on the diagonal. So

$$\prod_{i=1}^{n} |\lambda_i| = |1-w|^n$$

where λ_i are the eigenvalues of T. Thus if $w \notin (0,2)$ then $|1-w| \geq 1$ and clearly $\max_i |\lambda_i| \geq |1-w| \geq 1$ so SOR cannot be convergent.

If $L^2 = 0$ then $(I + wL)(I - wL) = I$ so $(I + wL)^{-1} = (I - wL)$. Now λ is an eigenvalue of T if and only if:

$$\begin{aligned}
& T - \lambda I \text{ is singular} \\
\Leftrightarrow\ & (I + wL)^{-1}((1-w)I - wU) - \lambda I \text{ is singular} \\
\Leftrightarrow\ & (I - wL)[((1-w)I - wU) - \lambda(I + wL)] \text{ is singular} \\
\Leftrightarrow\ & (1-w)I - wU - \lambda(I + wL) \text{ is singular} \\
\Leftrightarrow\ & (1-\lambda)I - wI - wU - wL + wL - \lambda wL \text{ is singular} \\
\Leftrightarrow\ & (1-\lambda)I - wA + (1-\lambda)wL \text{ is singular} \\
\Leftrightarrow\ & (1-\lambda)(I + wL) - wA \text{ is singular} \\
\Leftrightarrow\ & (1-\lambda)I - w(I - wL)A \text{ is singular} \\
\Leftrightarrow\ & \frac{1-\lambda}{w} \text{ is an eigenvalue of } (1 - wL)A
\end{aligned}$$

and $\left(\frac{1-\lambda}{w}\right) \in \mathrm{B}\left(\frac{1}{w}, \frac{1}{w}\right) \Leftrightarrow (1-\lambda) \in \mathrm{B}(1,1) \Leftrightarrow \lambda \in \mathrm{B}(0,1)$ ie. if only if SOR is convergent.

3. Iteration converges if and only if the eigenvalues of the iteration matrix $M^{-1}N$ lie strictly inside the unit disk. We have for the Jacobi iteration that $M = diag(A)$.(Bookwork) Write $A = L + D + U$, where L is the lower part of A and U the upper part of A and D the main diagonal. Then

$$M^{-1}N = -D^{-1}(L + U)$$

has eigenvalues $\lambda \iff$

$$-D^{-1}(L + U)x = \lambda x$$

where $x \neq 0 \iff$

$$(L + U + \lambda D)x = 0 \tag{2}$$

with $x \neq 0$. Now if $|a_{ii}| > \sum_{i \neq j} |a_{ij}|$ holds than certainly for every λ with $|\lambda| > 1$ $a_{ii}| > \sum |a_{ij}|$ $(L+U+\lambda D)$ is also strictly row diagonal dominant (SRDD). We now use Gershgorin Theorem to prove that to prove that SRDD implies that the matrix is non-singular and hence we have a contradiction to equation 2. Suppose A is SRDD and that λ be an eigenvalue then $\exists x \neq 0$ with $Ax = \lambda x$. Take i such that $|x_i| \geq |x_j|$ for all j. We have $\sum a_{ij}x_j = \lambda x_i$ or $(a_{ii} - \lambda)x_i = -\sum_{j \neq i} a_{ij}x_j \implies |a_{ii} - \lambda| < -\sum_{j \neq i} |a_{ij}||\frac{x_j}{x_i}| \leq \sum_{j \neq i} |a_{ij}|$. So every eigenvalue lies in at least one disk centered on a_{ii}, of radius $\sum_{j \neq i} |a_{ij}|$. Therefore, SRDD implies that no disc include the origin and hence A is non-singular. This means that $(L + U + \lambda D)$ would be non-singular and λ is not an eigenvalue and therefore all eigenvalues lie inside the unit disk. For blockdiagonal D the above argument still holds. Writing in component form with the double index notation (as used extensively on this example in the lectures), $(L + U + \lambda D)u = 0$ is

$$-u_{jk-1} - u_{jk+1} - \lambda u_{j+1k} - \lambda u_{j-1k} + \lambda u_{jk} = 0 \qquad \forall j,k = 1,\ldots,n. \tag{3}$$

Try for an eigenvalue $u^{rs} = u^{rs}_{jk}$ with equation (3) $u^{rs}_{jk} = \sin\frac{rj\pi}{n+1} \sin\frac{sk\pi}{n+1}$ we have in

$$-2\sin\frac{rj\pi}{n+1}\sin\frac{sk\pi}{n+1}\left[\cos\frac{s\pi}{n+1} + \lambda^{rs}\cos\frac{r\pi}{n+1} - 2\lambda^{rs}\right] = 0$$

and so

$$\lambda^{rs}(2 - \cos\frac{r\pi}{n+1}) = \cos\frac{s\pi}{n+1}$$

and

$$\lambda^{rs} = \frac{\cos\frac{s\pi}{n+1}}{2-\cos\frac{r\pi}{n+1}} \qquad \forall r,s = 1,\ldots,n$$

are the n^2 eigenvalues. For large n the max eigenvalues is when $s=1$ (to maximise numerator) and $r=1$ (to minimise the denominator) so spectral radius is

$$\rho = \frac{\cos\frac{\pi}{n+1}}{2-\cos\frac{\pi}{n+1}}.$$

Now, $\cos\frac{\pi}{n+1} = 1 - \frac{\pi^2}{2(n+1)^2 + o(n^{-4})}$ and $\left(1+\frac{\pi^2}{2(n+1)^2}+o(n^{-4})\right)^{-1} = 1-\frac{\pi^2}{2(n+1)^2}+o(n^{-4})$ and so

$$\rho \sim (1-\frac{\pi^2}{2(n+1)^2}+o(n^{-4}))(1-\frac{\pi^2}{2(n+1)^2}+o(n^{-4}) = 1-\frac{\pi^2}{2(n+1)^2}+o(n^{-4}) \text{ for large n.}$$

4. Let the iteration matrix be

$$T = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}.$$

Let λ be an eigenvalue of the iteration matrix T. The characteristic polynomial for λ is given by

$$-\lambda\left(\lambda^2-\frac{1}{9}\right)-\frac{1}{3}\left(-\frac{\lambda}{3}-\frac{1}{9}\right)+\frac{1}{3}\left(\frac{\lambda}{3}+\frac{1}{9}\right).$$

This simplifies to

$$-\lambda^3+\frac{\lambda}{3}+\frac{2}{27}$$

Factorising yields

$$-\left(\lambda-\frac{2}{3}\right)\left(\lambda+\frac{1}{3}\right)\left(\lambda+\frac{1}{3}\right).$$

Therefore the eigenvalues are $\frac{2}{3}$, $-\frac{1}{3}$, $-\frac{1}{3}$ as required. A direct calculation shows that

$$\begin{aligned}\left(T-\frac{2}{3}I\right)\left(T+\frac{1}{3}I\right) &= \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{2}{9}+\frac{1}{9}+\frac{1}{9} & -\frac{2}{9}+\frac{1}{9}+\frac{1}{9} & -\frac{2}{9}+\frac{1}{9}+\frac{1}{9} \\ -\frac{2}{9}+\frac{1}{9}+\frac{1}{9} & -\frac{2}{9}+\frac{1}{9}+\frac{1}{9} & -\frac{2}{9}+\frac{1}{9}+\frac{1}{9} \\ -\frac{2}{9}+\frac{1}{9}+\frac{1}{9} & -\frac{2}{9}+\frac{1}{9}+\frac{1}{9} & -\frac{2}{9}+\frac{1}{9}+\frac{1}{9} \end{bmatrix} \\ &= 0 \end{aligned} \tag{4}$$

Since T is symmetric it has a full set of orthogonal eigenvectors. Let $T = V\Lambda V^T$, where $\Lambda = \mathrm{diag}\left\{\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right\}$ and V and V^T are non-singular eigenvectors. Then

$$\begin{aligned} T^2 &= \left(V\Lambda V^T\right)\left(V\Lambda V^T\right) \\ &= V\Lambda^2 V^T, \\ T^k &= V\Lambda^k V^T \qquad \text{where} \\ \Lambda^k &= \begin{bmatrix} \left(\frac{2}{3}\right)^k & 0 & 0 \\ 0 & \left(-\frac{1}{3}\right)^k & 0 \\ 0 & 0 & \left(-\frac{1}{3}\right)^k \end{bmatrix} \\ &= \mathrm{diag}\left\{\left(\frac{2}{3}\right)^k, \left(-\frac{1}{3}\right)^k, \left(-\frac{1}{3}\right)^k\right\} \end{aligned} \tag{5}$$

which is non-singular. Therefore $T^k \neq 0$ the zero matrix.
Clearly

$$\|T^k\|_2 = \|V\Lambda^k V^T\|_2 = \|\Lambda^k\|_2 = \left(\frac{2}{3}\right)^k \longrightarrow 0$$

as $k \longrightarrow \infty$. Therefore $T^k \longrightarrow 0$ as $k \longrightarrow \infty$.
Let us construct a polynomial that terminates in two iterations as follows:

$$P_2(x) = \left(x - \frac{2}{3}\right)\left(x + \frac{1}{3}\right).$$

In most cases we require that $P_2(1) = 1$. But from the definition of $P_2(x)$, $P_2(1) = \frac{4}{9}$. Therefore the required polynomial is given by

$$P_2(x) = \frac{9}{4}\left(x - \frac{2}{3}\right)\left(x + \frac{1}{3}\right).$$

$P_2(x)$ is the minimum polynomial of the iteration matrix T.

5. Let $t \geq 1$, and define $T_k(t) = \frac{1}{2^{k-1}} \cosh k\left(\cosh^{-1} t\right)$. Recall the hyperbolic identities:

$$\cosh(k \pm 1)\,\theta = \cosh\theta\cosh k\,\theta \pm \sinh\theta\sinh k\,\theta$$
$$\cosh(k+1)\,\theta + \cosh(k-1)\,\theta = 2\,\cosh\theta\cosh k\,\theta$$

¿From the definition of $T_k(t)$, we have for:
$k = 0$,

$$T_0(t) = 2\,\cosh\left(0(\cosh^{-1} t)\right) = 2,$$

For $k = 1$,

$$T_1(t) = 2\,\cosh\left(1(\cosh^{-1} t)\right) = t,$$

For $k = 2$,

$$\begin{aligned}
T_2(t) &= \frac{1}{2}\,\cosh\left(2(\cosh^{-1} t)\right) \\
&= \frac{1}{2}\,\left[\cosh^2\left(\cosh^{-1} t\right) + \sinh^2\left(\cosh^{-1} t\right)\right] \\
&= \frac{1}{2}\,\left[t^2 + \left(\cosh^2\left(\cosh^{-1} t\right) - 1\right)\right] \\
&= \frac{1}{2}\,\left[t^2 + (t^2 - 1)\right] \\
&= t^2 - \frac{1}{2}.
\end{aligned}$$

Let us define $\theta = \cosh^{-1} t$ then from

$$\begin{aligned}
\cosh(k+1)\,\theta + \cosh(k-1)\,\theta &= 2\,\cosh\theta\cosh k\,\theta \\
\frac{1}{2^k}\,\cosh(k+1)\,\theta + \frac{1}{2^k}\,\cosh(k-1)\,\theta &= \frac{1}{2^{k-1}}\,\cosh\theta\cosh k\,\theta \\
T_{k+1}(t) + \frac{4}{4}\,\frac{1}{2^k}\,\cosh(k-1)\,\theta &= T_1(t)\,T_k(t) \\
T_{k+1}(t) + \frac{1}{4}\,\frac{1}{2^{k-2}}\,\cosh(k-1)\,\theta &= T_1(t)\,T_k(t) \\
&\Longrightarrow T_{k+1}(t) = t\,T_k(t) - \frac{1}{4}\,T_{k-1}(t)
\end{aligned}$$

which is exactly the same recurrence obtained from the Chebyshev polynomial

$$T_k(x) = \frac{1}{2^{k-1}}\,\cos k\left(\cos^{-1} t\right)$$

for $|t| \leq 1$, with the same initial values. Therefore both functions must be the same.

Now consider the relation $t > \cosh(\ln 2)$. The cosh functions are monotonically increasing functions in the region $t > \cosh(\ln 2)$. Let $\epsilon > 0$, such that $t = \cosh(\ln(2+\epsilon))$, then $\cosh^{-1} t = \ln(2+\epsilon)$. Then take

$$\begin{aligned}
T_k(t) &= \frac{1}{2^{k-1}} \cosh\left(k \cosh^{-1} t\right) \\
&= \frac{1}{2^{k-1}} \cosh\left(k \ln(2+\epsilon)\right) \\
&= \frac{1}{2^{k-1}} \cosh\left(\ln(2+\epsilon)^k\right) \\
&= \frac{1}{2^{k-1}} \frac{e^{\ln(2+\epsilon)^k} + e^{\ln(2+\epsilon)^{-k}}}{2} \\
&= \frac{1}{2^k} \left[(2+\epsilon)^k + (2+\epsilon)^{-k}\right] \\
&= \left(1 + \frac{\epsilon}{2}\right)^k \left(1 + \frac{\epsilon}{2}\right)^{-k} \\
&\longrightarrow \infty + 0 \longrightarrow \infty
\end{aligned}$$

as $k \longrightarrow \infty$. Here $\epsilon = \exp\left(\cosh^{-1} t\right) - 2 > 0$ when $t > \cosh(ln2) = \frac{5}{4}$.

6. Simple iteration $x^{(k+1)} = Hx^{(k)} + g$ converges when the spectral radius of H is less than one, $\rho(H) < 1$. In this problem we consider polynomial iteration, $y^{(k)} = \sum_{j=0}^{k} \beta_j^{(k)} x^{(j)}$, where $\sum_{j=0}^{k} \beta_j^{(k)} = 1$. If $x = Hx + g$ is the exact solution, then error can be expressed as

$$e^{(k)} \equiv y^{(k)} - x = p_k(H)(x^{(0)} - x), \qquad p_k(z) \equiv \sum_{j=0}^{k} \beta_j^{(k)} z^j \text{ with } p_k(1) = 1.$$

In particular, we let $p_k(z)$ be the shifted and scaled Chebyshev polynomials, $p_k(z) \equiv \widehat{T}_k(z) = T_k(\rho^{-1}z)/T_k(\rho^{-1})$. This polynomial is small on the interval $[-\rho, \rho]$, which is assumed to contain the spectrum of H. Note that it also satisfies the normalization condition, $\widehat{T}_k(1) = 1$.

Recall that the monic Chebyshev polynomials are given by:

$$T_0(z) = 1, \quad T_1(z) = z, \quad T_2(z) = z^2 - \frac{1}{2}; \qquad T_{k+1}(z) = zT_k(z) - \frac{1}{4}T_{k-1}(z) \text{ for } k \geq 2.$$

Suppose $k \geq 2$. Evaluating the Chebyshev recurrence at $\rho^{-1}z$ gives

$$T_{k+1}(\rho^{-1}z) = \frac{z}{\rho} T_k(\rho^{-1}z) - \frac{1}{4} T_{k-1}(\rho^{-1}z). \tag{6}$$

Rearranging our formula for $\widehat{T}_k$, we have $T_k(\rho^{-1}z) = T_k(\rho^{-1})\widehat{T}_k(z)$. Thus (6) is equivalent to

$$T_{k+1}(\rho^{-1})\widehat{T}_{k+1}(z) = \frac{1}{\rho} z T_k(\rho^{-1})\widehat{T}_k(z) - \frac{1}{4} T_{k-1}(\rho^{-1})\widehat{T}_{k-1}(z).$$

Substituting H for z yields

$$T_{k+1}(\rho^{-1})\widehat{T}_{k+1}(H) = \frac{1}{\rho} H T_k(\rho^{-1})\widehat{T}_k(H) - \frac{1}{4} T_{k-1}(\rho^{-1})\widehat{T}_{k-1}(H). \tag{7}$$

Note that $e^{(k)} \equiv y^{(k)} - x = \widehat{T}_k(H)(x^{(0)} - x) = \widehat{T}_k(H)e^{(0)}$. Thus multiplying (7) on the right by $e^{(0)}$ yields

$$T_{k+1}(\rho^{-1})e^{(k+1)} = \frac{1}{\rho} H T_k(\rho^{-1})e^{(k)} - \frac{1}{4} T_{k-1}(\rho^{-1})e^{(k-1)}. \tag{8}$$

For the second part of the problem, note that $x = Hx + g$ and define $\omega_{k+1} = \rho^{-1}T_k(\rho^{-1})/T_{k+1}(\rho^{-1}) = 1 + \frac{1}{4}T_{k-1}(\rho^{-1})/T_{k+1}(\rho^{-1})$. Then divide (8) by $T_{k+1}(\rho^{-1})$ and adding x to both sides yields

$$\begin{aligned}
y^{(k+1)} &= \frac{1}{\rho}H\frac{T_k(\rho^{-1})}{T_{k+1}(\rho^{-1})}(y^{(k)} - x) - \frac{1}{4}\frac{T_{k-1}(\rho^{-1})}{T_{k+1}(\rho^{-1})}(y^{(k-1)} - x) + x \\
&= \omega_{k+1}(Hy^{(k)} - Hx) + \left(1 + \frac{1}{4}\frac{T_{k-1}(\rho^{-1})}{T_{k+1}(\rho^{-1})}\right)x - \frac{1}{4}\frac{T_{k-1}(\rho^{-1})}{T_{k+1}(\rho^{-1})}y^{(k-1)} \\
&= \omega_{k+1}(Hy^{(k)} + g - x) + \omega_{k+1}x + (1 - \omega_{k+1})y^{(k-1)} \\
&= \omega_{k+1}(Hy^{(k)} + g - y^{(k-1)}) + y^{(k-1)}
\end{aligned}$$

as required.

7. Consider $M = \sigma I + D$ for A being the matrix associated to the Laplacian with Dirichlet boundary conditions. We are interested in the eigenvalues of $M^{-1}N$ which becomes for the 5 point-finite difference $M^{-1}N = -(\sigma I + D)^{-1}(L + U)$, where $-(\sigma I + D)^{-1} = diag(1/(\sigma + 4))$. In more detail,

$$M^{-1}N = \begin{bmatrix} E & F & & \\ F & E & \ddots & \\ & \ddots & \ddots & F \\ & & F & E \end{bmatrix}$$

where $F = \frac{-1}{\sigma+4}I$ and

$$E = \begin{bmatrix} 0 & \frac{-1}{\sigma+4} & & \\ \frac{-1}{\sigma+4} & 0 & \ddots & \\ & \ddots & \ddots & \frac{-1}{\sigma+4} \\ & & \frac{-1}{\sigma+4} & 0 \end{bmatrix}.$$

Now using Gershgorin's theorem (see classes) and we get that $|\lambda| \leq \frac{4}{\sigma+4}$ since we have at most 4 times $\frac{-1}{\sigma+4}$. For $\sigma = \frac{1}{2}$ we know the eigenvalues are between, $a = -\frac{8}{9}$ and $b = \frac{8}{9}$, i.e.$\cosh\theta = \frac{9}{8}$ and therefore $\theta = \ln(\frac{9}{8} + \sqrt{\frac{17}{64}}) \Rightarrow \cosh k\theta > \frac{1}{2}(\frac{9}{8} + \sqrt{\frac{17}{64}})^k$. This means that

$$\max_{r\in[-\frac{8}{9},\frac{8}{9}]} |\hat{T}(r)| < \frac{2}{(\frac{9}{8} + \sqrt{\frac{17}{64}})^k} \simeq \frac{2}{(1.64)^k}$$

so

$$\frac{\|x - y^{(k)}\|_2}{\|x - x^{(0)}\|_2} \leq \frac{2}{(1.64)^k}$$

which is nearly a reduction by the factor 2 at each step of the iteration.