Numerical Linear Algebra: Solutions for Sheet 3

1. Suppose $Ax = b$ and $(A+\delta A)(x+\delta x) = b$, which together imply $(A+\delta A)(x+\delta x) = Ax$. Multiplying this out gives $Ax + A\delta x + \delta A(x+\delta x) = Ax$ and consequently $A\delta x = -\delta A(x+\delta x)$. Assuming A is invertible, $\delta x = -A^{-1}(\delta A)(x+\delta x)$. We now apply norm inequalities to obtain

$$\|\delta x\| = \|A^{-1}(\delta A)(x+\delta x)\| \leq \|A^{-1}\|\,\|\delta A\|\,\|x+\delta x\| = \frac{\|A\|}{\|A\|}\|A^{-1}\|\,\|\delta A\|\,\|x+\delta x\|.$$

Dividing by $\|x+\delta x\|$ gives the desired result,

$$\frac{\|\delta x\|}{\|x+\delta x\|} \leq \|A\|\,\|A^{-1}\|\frac{\|\delta A\|}{\|A\|}.$$

This bounds the relative perturbation of the solution by the relative perturbation in the matrix, magnified by the condition number $\kappa(A) = \|A\|\,\|A^{-1}\|$. Small errors in A might lead to large errors in the computed solution.

2. We have

$$\kappa(Q) = \|Q\|\|Q^{-1}\| = \|Q\|\|Q^T\|$$

and since $\|Z\| = 1$ for an orthogonal matrix Z we get $\kappa(Q) = 1$. The Cholesky factorisation of A^TA is given by $A^TA = LL^T$ where L is a lower triangular matrix. Now consider the QR factorisation of A, i.e. $A = QR$. Then we get

$$A^TA = (QR)^TQR = R^TQ^TQR = R^TR.$$

If

$$R = \begin{bmatrix} \hat{R} \\ 0 \end{bmatrix}$$

with $\hat{R} \in \mathbb{R}^{n\times n}$ and $\hat{R}^T\hat{R} = R^TR$. So $\hat{R}^T$ could be L^T. In fact this precisely depends on the choices of sign of the diagonal entries in the Cholesky and QR algorithm.

3. Hilbert matrix

size	condition number
4	$1.5514e+04$
8	$1.5258e+10$
12	$1.8032e+16$

Result for $x = A\backslash b$ where $b = sum(A,2)$ with solution $x = [1,\ldots,1]^T$ gives

$$x = \begin{bmatrix} 0.999999976736699 \\ 1.000002910963504 \\ 0.999909292281920 \\ 1.001227445722276 \\ 0.991048902442979 \\ 1.039167445001900 \\ 0.891228720678228 \\ 1.196361444783960 \\ 0.770302875862095 \\ 1.167911583742747 \\ 0.930297943531136 \\ 1.012541473056602 \end{bmatrix}.$$

4. One Gauss–Seidel iteration applied to the system $Ax = b$ is

   for $i = 1, \ldots, n$

   $$x_i^{(k+1)} = \Big(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)}\Big) \Big/ a_{ii}.$$

   Since A is lower triangular, $a_{ij} = 0$ if $j > i$. Thus, the Gauss–Seidel iteration reduces to

   for $i = 1, \ldots, n$

   $$x_i^{(k+1)} = \Big(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)}\Big) \Big/ a_{ii}.$$

   Notice that this is identical to forward substitution:

   $$x_1 = b_1/a_{11}, \quad x_2 = (b_2 - x_1 a_{21})/a_{22}, \quad \ldots, \quad x_n = \Big(b_n - \sum_{j=1}^{n-1} a_{nj} x_j\Big) \Big/ a_{nn}$$

   Thus, Gauss–Seidel will solve a lower triangular matrix *exactly* in only one iteration.

   Suppose A is nearly upper triangular. Then we would want an iterative method for which one step is similar to backwards substitution. Such a method executes the Gauss–Seidel iteration in reverse order:

   for $i = n, \ldots, 1$

   $$x_i^{(k+1)} = \Big(b_i - \sum_{j=i+1}^{n} a_{ij} x_j^{(k+1)} - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)}\Big) \Big/ a_{ii}.$$

5. The *spectral radius* is defined as $\rho(A) := \max_{\lambda \in \Lambda(A)} |\lambda|$, where $\Lambda(A)$ is the spectrum of A. Let $\lambda_{\max}$ be an eigenvalue of A such that $\rho(A) = |\lambda_{\max}|$. Then there must be a non-trivial eigenvector v associated with $\lambda_{\max}$. As eigenvectors are not scale-dependent, we can pick v such that $\|v\| = 1$, where $\|\cdot\|$ is the vector norm which induces the operator norm. Then

   $$\rho(A) = |\lambda_{\max}| = |\lambda_{\max}|\,\|v\| = \|\lambda_{\max} v\| = \|Av\| \le \max_{\|x\|=1} \|Ax\| = \|A\|.$$

   Thus $\rho(A) \le \|A\|$ for any operator norm A.

   Does this hold if $\|\cdot\|$ is not an operator norm (e.g., the Frobenius norm)? Can you prove this?

6. Let λ be an eigenvalue of $A \in \mathbb{R}^{n\times n}$ such that $(A - \lambda I)x = 0$ for some $x \ne 0$. Take k such that $|x_k| = \max_i |x_i|$. Then the kth row of the equation $(A - \lambda I)x = 0$ gives

   $$(a_{kk} - \lambda)x_k = -\sum_{j=1, j\ne k}^{n} a_{kj} x_j.$$

   Thus,

   $$|a_{kk} - \lambda||x_k| = \Big|\sum_{j=1, j\ne k}^{n} a_{kj} x_j\Big| \le \sum_{j=1, j\ne k}^{n} |a_{kj}|\,|x_j|.$$

   Dividing through by $|x_k|$ (which must be greater than zero since $x \ne 0$), we have

   $$|a_{kk} - \lambda| \le \sum_{j=1, j\ne k}^{n} |a_{kj}| \frac{|x_j|}{|x_k|} \le \sum_{j=1, j\ne k}^{n} |a_{kj}|.$$

   Having proved Gerschgorin's Theorem, we will apply it to bound the spectral radius of the Jacobi iteration matrix. Recall that the Jacobi splitting of $A = L + D + U$ (L strictly lower triangular, D diagonal, U strictly upper triangular) yields the iteration matrix $M^{-1}N = D^{-1}(-L - U)$. Thus,

   $$A = \begin{pmatrix} 9 & 1 & 1 \\ 1 & 5 & 0 \\ 1 & 1 & 3 \end{pmatrix}, \quad M = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & 0 \\ -1 & -1 & 0 \end{pmatrix}.$$

The Jacobi iteration matrix is thus

$$M^{-1}N = \begin{pmatrix} 0 & -\frac{1}{9} & -\frac{1}{9} \\ -\frac{1}{5} & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & 0 \end{pmatrix}$$

Applying Gerschgorin's Theorem to each row yields three bounds:

$$|\lambda| \leq 2/9, \quad |\lambda| \leq 1/5, \quad |\lambda| \leq 2/3.$$

From this we deduce that $\rho(M^{-1}N) \leq 2/3$.

But we can get a tighter bound by using the result from Question 2. Calculate that $\|M^{-1}N\|_1 = 8/15 < 2/3$. Thus, $\rho(M^{-1}N) \leq 8/15$. In fact, we find $\rho(M^{-1}N) \approx 0.2910$.

7. Define A to be the upper triangular matrix with 1/2 on the main diagonal ones in all superdiagonal entries. This is an example of a *non-normal* matrix. All the eigenvalues are 1/2, so $\rho(A) = 1/2$. Thus, we must have $\|A^k\|_\infty \to 0$ as $k \to \infty$. But there is growth of A^k for transient values of k, and this can have wide-ranging implications. For example, it warns us that iterative methods may not converge very nicely, since $\|x^{(k)} - x\| \leq \|(M^{-1}N)^k\| \, \|x^{(0)} - x\|$.

```
A  = triu(ones(10)) - 1/2*eye(10);
Ak = A;
for k=1:50,
    normAk(k) = norm(Ak,'inf');
    Ak=Ak*A;
end

semilogy([1:50], normAk);
xlabel('power, k', 'fontsize', 14)
ylabel('||A||_\infty','fontsize', 14)
title('infinity norms of matrix powers', 'fontsize', 14)
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8. Let $Ax = b$, and let $A = D + L + U$. Define the SOR method by:

$$(D + \omega\, L)\; x^{(k+\frac{1}{2})} = \omega\, b + ((1-\omega)\, D - \omega\, U)\; x^{(k)}, \tag{1}$$

$$(D + \omega\, U)\; x^{(k+1)} = \omega\, b + ((1-\omega)\, D - \omega\, L)\; x^{(k+\frac{1}{2})} \tag{2}$$

¿From the 1st equation, we obtain that:

$$x^{(k+\frac{1}{2})} = \omega\; (D + \omega\, L)^{-1}\; b + (D + \omega\, L)^{-1}\; ((1-\omega)\, D - \omega\, U)\; x^{(k)}, \tag{3}$$

Substituting the value of $x^{(k+\frac{1}{2})}$ into the 2nd equation we have

$$x^{(k+1)} = (D + \omega\, U)^{-1}\; ((1-\omega)\, D - \omega\, L)\; \Big((D + \omega\, L)^{-1}\; ((1-\omega)\, D - \omega\, U)\Big)\; x^{(k)} \tag{4}$$

$$+ \,\omega\; (D + \omega\, U)^{-1}\; ((1-\omega)\, D - \omega\, L)\; (D + \omega\, L)^{-1}\; b + \omega\, (D + \omega\, U)^{-1}\, b \tag{5}$$

It is clear that

$$G = (D + \omega\, U)^{-1}\; ((1-\omega)\, D - \omega\, L)\; (D + \omega\, L)^{-1}\; ((1-\omega)\, D - \omega\, U)$$

is the iteration matrix. One can show that if

$$A = M - N \iff N = M - A \iff M^{-1}\, N = I - M^{-1}\, A \tag{6}$$

Hence, it is enough to show that $G = M^{-1}\,N$ as given above can be expressed as $G = I - M^{-1}\,A$ where M is the symmetric matrix to be determined. It follows that

$$(1-\omega)\,D - \omega\,L = -(D+\omega\,L) + (2-\omega)\,D$$

and similarly

$$(1-\omega)\,D - \omega\,U = -(D+\omega\,U) + (2-\omega)\,D$$

Then G can be transformed to:

$$\begin{aligned}
G &= (D+\omega\,U)^{-1}\,((1-\omega)\,D-\omega\,L)\,(D+\omega\,L)^{-1}\,((1-\omega)\,D-\omega\,U) && (7)\\
&= -(D+\omega\,U)^{-1}\,\big((D+\omega\,L)\,((D+\omega\,L)^{-1})\big)\,((1-\omega)\,D-\omega\,U) && (8)\\
&+ (2-\omega)\,(D+\omega\,U)^{-1}\,D\,(D+\omega\,L)^{-1}\,((1-\omega)\,D-\omega\,U) && (9)\\
&= (D+\omega\,U)^{-1}\,(D+\omega\,U) - (2-\omega)\,(D+\omega\,U)^{-1}\,D && (10)\\
&+ (2-\omega)\,(D+\omega\,U)^{-1}\,D\,(D+\omega\,L)^{-1}\,((1-\omega)\,D-\omega\,U) && (11)\\
&= I - (2-\omega)\,(D+\omega\,U)^{-1}\,D\,\big(I-(D+\omega\,L)^{-1}\,((1-\omega)\,D-\omega\,U)\big) && (12)\\
&= I - (2-\omega)\,(D+\omega\,U)^{-1}\,D\,(D+\omega\,L)^{-1}\,(D+\omega\,L-(1-\omega)\,D+\omega\,U) && (13)\\
&= I - \omega\,(2-\omega)\,(D+\omega\,U)^{-1}\,D\,(D+\omega\,L)^{-1}\,A && (14)
\end{aligned}$$

So M for this splitting is given by

$$M = \frac{1}{\omega\,(2-\omega)}\,(D+\omega\,L)\,D^{-1}\,(D+\omega\,U).$$

9. (or see solution for question 10) Writing $J = M^{-1}N$ the Jacobi iteration matrix, we have $M = \mathrm{diag}(A)$ and $N = M - A$ so

$$M = \begin{pmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} \frac{1}{a_{11}} & & & \\ & \frac{1}{a_{22}} & & \\ & & \ddots & \\ & & & \frac{1}{a_{nn}} \end{pmatrix}, \quad N = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ a_{21} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{nn-1} \\ a_{n1} & \cdots & a_{n-1n} & 0 \end{pmatrix}$$

and therefore

$$J = \begin{pmatrix} 0 & \frac{a_{12}}{a_{11}} & \cdots & \frac{a_{1n}}{a_{11}} \\ \frac{a_{21}}{a_{22}} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{a_{nn-1}}{a_{n-1n-1}} \\ \frac{a_{n1}}{a_{nn}} & \cdots & \frac{a_{n-1n}}{a_{nn}} & 0 \end{pmatrix}$$

We now apply Gershgorin's theorem as in question 6, and we have that for all eigenvalues λ there exists some i, such that

$$|\lambda| = |\lambda - 0| \leq \sum_{j=1, j\neq i}^{n} \left|\frac{a_{ij}}{a_{ii}}\right| < 1 \qquad (15)$$

by SRDD of A. Therefore all eigenvalues of J have modulus strictly less than one, and hence the Jacobi iteration converges.

10. We want to prove that if A is SRDD then both the Jacobi and Gauss-Seidel iterations converge. It is clear that if a matrix is SRDD then it is non-singular. By Gershgorin Circle Theorem its spectral radius does not contain the origin hence it is non-singular (see Question 6).

    Let λ be the eigenvalue for both the Jacobi and Gauss-Seidal iteration method. For the Jacobi iteration method it means that:

$$-D^{-1}\,(L+U)\,\underline{x} = \lambda\,\underline{x}$$

$$(L+U)\,\underline{x} = \lambda\,D\,\underline{x}$$

$$(L+U+\lambda\,D)\;\underline{x} = 0$$

    Similarly for the G-S iteration method:

$$(U+\lambda\,L+\lambda\,D)\;\underline{x} = 0.$$

    Now let us assume in both cases that there exists λ such that $|\lambda| \geq 1$. As λ is greater than equal to 1 then A becomes even more diagonally dominant as the term $\lambda\,D$ increases. Hence $(L+U+\lambda\,D)$ or $(U+\lambda\,L+\lambda\,D)$ are singular. But this contradicts the fact that A is non-singular. Therefore $|\lambda| < 1$. Therefore all eigenvalues of the Jacobi and Gauss-Seidal lie within the unit disc, hence the spectral radius is strictly less than 1,and both methods converge independent of b and $x^{(0)}$.