Numerical Linear Algebra: Solutions for Sheet 4

1. **Part (a)**

First, we need to verify the eigenvalues and eigenvectors of the Jacobi iteration matrix. Let $A = L + D + U$, where L is strictly lower triangular, D is diagonal, and U is upper triangular. The Jacobi splitting takes $M = D$ and thus $N = M - A = -(L + U)$. For this particular case, $A \in \mathbb{R}^{n^2 \times n^2}$ has the form

$$A = \begin{pmatrix} B & C & & \\ C & B & \ddots & \\ & \ddots & \ddots & C \\ & & C & B \end{pmatrix} \quad \text{with} \quad B = \begin{pmatrix} 4 & -1 & & \\ -1 & 4 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 4 \end{pmatrix} \in \mathbb{R}^{n\times n}, \quad C = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

Thus, the Jacobi iteration matrix $J = M^{-1}N \in \mathbb{R}^{n^2 \times n^2}$ takes the form

$$J = \begin{pmatrix} E & F & & \\ F & E & \ddots & \\ & \ddots & \ddots & F \\ & & F & E \end{pmatrix} \quad \text{with} \quad E = \begin{pmatrix} 0 & \frac14 & & \\ \frac14 & 0 & \ddots & \\ & \ddots & \ddots & \frac14 \\ & & \frac14 & 0 \end{pmatrix} \in \mathbb{R}^{n\times n}, \quad F = \begin{pmatrix} \frac14 & & & \\ & \frac14 & & \\ & & \ddots & \\ & & & \frac14 \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

Now, let λ^{rs} be an eigenvalue of J corresponding to the eigenvector u^{rs}. The eigenvalue–eigenvector equation is $Ju^{rs} = \lambda^{rs}u^{rs}$. Look at the "$(j,k)$th row" to this equation:

$$\tfrac14 \left(u^{rs}_{j+1,k} + u^{rs}_{j-1,k} + u^{rs}_{j,k+1} + u^{rs}_{j,k-1} \right) = \lambda^{rs} u^{rs}_{j,k}, \tag{1}$$

where we take $u^{rs}_{j,k} = 0$ on the boundary (if $j = 0$, $j = n+1$, $k = 0$, or $k = n+1$). Consider

$$\lambda^{rs} = \tfrac12 (\cos r\pi h + \cos s\pi h), \qquad u^{rs}_{j,k} = \sin rj\pi h \sin sk\pi h.$$

(λ^{rs}, u^{rs}) is an eigenvalue–eigenvector pair if they satisfy Equation (1). To see that this does hold, consider

$$\begin{aligned}
&\tfrac14 \left(u^{rs}_{j+1,k} + u^{rs}_{j-1,k} + u^{rs}_{j,k+1} + u^{rs}_{j,k-1} \right) - \lambda^{rs} u^{rs}_{j,k} \\
&= \tfrac14 (\sin r(j+1)\pi h)(\sin sk\pi h) + \tfrac14 (\sin r(j-1)\pi h)(\sin sk\pi h) \\
&\quad + \tfrac14 (\sin rj\pi h)(\sin s(k+1)\pi h) + \tfrac14 (\sin rj\pi h)(\sin s(k-1)\pi h) \\
&\quad - \tfrac12 (\cos r\pi h + \cos s\pi h)(\sin rj\pi h \sin sk\pi h) \\
&= (\sin sk\pi h) \left(\tfrac14 (\sin r(j+1)\pi h) + \tfrac14 (\sin r(j-1)\pi h) - \tfrac12 (\cos r\pi h \sin rj\pi h) \right) \\
&\quad + (\sin rj\pi h) \left(\tfrac14 (\sin s(k+1)\pi h) + \tfrac14 (\sin s(k-1)\pi h) - \tfrac12 (\cos s\pi h \sin sj\pi h) \right) \\
&= (\sin sk\pi h) \left(\tfrac12 \sin rj\pi h \cos r\pi h - \tfrac12 \cos r\pi h \sin rj\pi h \right) \\
&\quad + (\sin rj\pi h) \left(\tfrac12 \sin sk\pi h \cos s\pi h - \tfrac12 \cos s\pi h \sin sk\pi h \right) \\
&= 0,
\end{aligned}$$

where we have used the identity $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$. (Note that $u^{rs}_{j,k}$ is zero on the boundaries, i.e., if $j = 0$, $j = n+1$, $k = 0$, or $k = n+1$.)

Thus, λ^{rs} and u^{rs} as defined above must be an eigenvalue–eigenvector pair for $r, s \in \{1, \dots, n\}$.

Part (b)

The Gauss–Seidel iteration matrix is $M^{-1}N = -(L + D)^{-1}(U)$. Suppose μ and v are an eigenvalue–eigenvector pair for $M^{-1}N$. This holds if and only if

$$-(L+D)^{-1}Uv = \mu v \iff -Uv = \mu(L+D)v \iff 0 = \mu(L+D)v + Uv \iff 0 = (\mu L + \mu D + U)v.$$

Since each step is if-and-only-if, we have shown that μ and v is an eigenvalue–eigenvector pair for $M^{-1}N$ if and only if $(\mu D + \mu L + U)v = 0$.

Now v^{rs}, defined by $v^{rs}_{j,k} = \mu^{(j+k)/2} \sin rj\pi h \sin sk\pi h$, is an eigenvector of the Gauss–Seidel iteration matrix $M^{-1}N$ if and only if $(\mu D + \mu L + U)v^{rs}_{j,k} = 0$ for some scalar μ. Consider one row of the expression $(\mu D + \mu L + U)v$:

$$\begin{aligned}
&\mu(4v^{rs}_{j,k} - v^{rs}_{j-1,k} - v^{rs}_{j,k-1}) - v^{rs}_{j+1,k} - v^{rs}_{j,k+1} \\
&= \mu(4\mu^{(j+k)/2}(\sin rj\pi h \sin sk\pi h)) \\
&\quad - \mu\left(\mu^{(j-1+k)/2}(\sin r(j-1)\pi h \sin sk\pi h) + \mu^{(j+k-1)/2}(\sin rj\pi h \sin s(k-1)\pi h)\right) \\
&\quad - \left(\mu^{(j+1+k)/2}(\sin r(j+1)\pi h \sin sk\pi h) + \mu^{(j+k+1)/2}(\sin rj\pi h \sin s(k+1)\pi h)\right) \\
&= 4\mu^{(j+1+k)/2}\Big(\mu^{1/2}(\sin rj\pi h) \\
&\qquad - \tfrac{1}{4}(\sin r(j-1)\pi h \sin sk\pi h) - \tfrac{1}{4}(\sin rj\pi h \sin s(k-1)\pi h) \\
&\qquad - \tfrac{1}{4}(\sin r(j+1)\pi h \sin sk\pi h) - \tfrac{1}{4}(\sin rj\pi h \sin s(k+1)\pi h)\Big) \\
&= 4\mu^{(j+1+k)/2}\left(\mu^{1/2}u^{rs}_{j,k} - \tfrac{1}{4}u^{rs}_{j-1,k} - \tfrac{1}{4}u^{rs}_{j,k-1} - \tfrac{1}{4}u^{rs}_{j+1,k} - \tfrac{1}{4}u^{rs}_{j,k-1}\right),
\end{aligned}$$

where u^{rs} is an eigenvector for the Jacobi iteration matrix. In Question 1(a), we illustrated that the term

$$\mu^{1/2}u^{rs}_{j,k} - \tfrac{1}{4}u^{rs}_{j-1,k} - \tfrac{1}{4}u^{rs}_{j,k-1} - \tfrac{1}{4}u^{rs}_{j+1,k} - \tfrac{1}{4}u^{rs}_{j,k-1}$$

is zero if $\mu^{1/2}$ is the eigenvalue corresponding to u^{rs}, i.e., if $\mu = (\lambda^{rs})^2$.

Thus, v^{rs} is an eigenvector corresponding to the eigenvalue

$$\mu^{rs} = (\lambda^{rs})^2 = (\tfrac{1}{2}(\cos r\pi h + \cos s\pi h))^2.$$

Note that $|\lambda^{rs}| \leq 1$ for all $r, s \in \{1, \dots, n\}$ since $|\cos x| \leq 1$. Thus, the Jacobi iteration will converge. How does Gauss–Seidel compare? Compare the spectral radii of the iteration matrices:

$$\rho(-(L+D)^{-1}U) < \rho(-D^{-1}(L+U))$$

since $(\lambda^{rs})^2 \leq |\lambda^{rs}|$. Thus, Gauss–Seidel iteration will converge more rapidly than Jacobi.

Analysis of Gauss–Seidel as a smoother is more complicated. The eigenvectors no longer form an orthogonal basis for $\mathbb{R}^{n^2}$. Eigenvectors for which $r \approx 1$ and $s \approx n$, or $r \approx 1$ and $s \approx n$ point in nearly the same direction, and the reasoning we used to analyze Jacobi smoothing can now be misleading.

2. Let $u^e = (u_2, u_4, \dots, u_n)^{\mathrm{T}}$ be the coarse grid vector. Then consider the matrix-vector product

$$Pu^e = \begin{pmatrix} \frac{1}{2} & & & \\ 1 & & & \\ \frac{1}{2} & \frac{1}{2} & & \\ & 1 & \vdots & \\ & \frac{1}{2} & \vdots & \\ & & \frac{1}{2} & \frac{1}{2} \\ & & & 1 \\ & & & \frac{1}{2} \end{pmatrix} \begin{pmatrix} u_2 \\ u_4 \\ u_6 \\ \vdots \\ u_{n-3} \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}u_2 \\ u_2 \\ \frac{1}{2}(u_2 + u_4) \\ \vdots \\ \frac{1}{2}(u_{n-3} + u_{n-1}) \\ u_{n-1} \\ \frac{1}{2}u_{n-1} \end{pmatrix}.$$

This can be summarised as

$$(Pu^e)_j = \begin{cases} \frac{1}{2}(u_{j-1} + u_{j+1}) & j \text{ odd;} \\ u_j & j \text{ even.} \end{cases}$$

where $u_0 = u_{n+1} = 0$. Thus, P prolongates by performing linear interpolation.

Now consider $R = \frac{1}{2}P^{\mathrm{T}}$. Let $u^f = (u_1, u_2, \ldots, u_n)$ be the fine grid vector. Then consider the matrix-vector product

$$Ru^f = \frac{1}{2}\begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} & & & & & \\ & & \frac{1}{2} & 1 & \frac{1}{2} & & & \\ & & & \cdots & \cdots & & & \\ & & & & & \frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}.$$

This can be summarised as

$$(Ru^f)_j = \frac{1}{2}\left(\frac{1}{2}u_{2j-1} + u_{2j} + \frac{1}{2}u_{2j+1}\right).$$

The restriction operator sets coarse grid values equal to weighted averages of the fine grid values.

3. Let us define the following notations:

$$u_0 = u^{(i)}, \qquad u_k = u_s^{(i)}.$$

The smoothing iteration is

$$u_j = (M^{(-1)}\,N)\,u_{j-1} + M^{-1}\,f = (I - M^{-1}\,A)\,u_{j-1} + M^{-1}\,f$$

So

$$\begin{aligned} u_1 &= (M^{-1}\,N)\,u_0 + M^{-1}\,f \\ u_2 &= (M^{-1}\,N)\,u_1 + M^{-1}\,f \\ &= (M^{(-1)}\,N)^2\,u_0 + (I + M^{-1}\,N)\,M^{-1}\,f \\ u_3 &= (M^{(-1)}\,N)^3\,u_0 + (I + M^{-1}\,N + (M^{-1}\,N)^2)\,M^{-1}\,f \end{aligned}$$

and inductively

$$u_k = (M^{-1}\,N)^k\,u_0 + \left(I + M^{-1}\,N + (M^{-1}\,N)^2 + \cdots + (M^{-1}\,N)^{(k-1)}\right)\,M^{-1}\,f. \tag{2}$$

Similarly the exact solution satisfies

$$u = (M^{-1}\,N)\,u + M^{-1}\,f$$

hence

$$u = (M^{-1}\,N)^k\,u + \left(I + M^{-1}\,N + (M^{-1}\,N)^2 + \cdots + (M^{-1}\,N)^{(k-1)}\right)\,M^{-1}\,f. \tag{3}$$

Replacing u_k by $u_s^{(i)}$ in the notation defined in the question and above we have that the 2-grid iteration becomes

$$\begin{array}{rll} \text{Smooth} & u^{(i)} \longrightarrow u_s^{(i)} & \\ \text{Calculate residual} & r_s \longleftarrow f - A\,u_s^{(i)} & [r_s := f - A\,u_s^{(i)}] \\ \text{Restrict residual} & \overline{r}_s \longleftarrow R\,r_s & [\overline{r}_s := R\,r_s] \\ \text{Solve} & \overline{A}\,\overline{e}_s = \overline{r}_s & [\overline{e}_s := \overline{A}^{-1}\,\overline{r}_s] \\ \text{Prolong} & e_s \longleftarrow P\,\overline{e}_s & [e_s := P\,\overline{e}_s] \\ \text{Update} & u^{(i+1)} \longleftarrow u_s^{(i)} + e_s & [u^{(i+1)} := u_s^{(i)} + e_s] \end{array}$$

By going in the upward direction we can recover the errors as follows:

$$\begin{aligned} e_s &= P\,\overline{e}_s \\ &= P\,(\overline{A}^{-1}\,\overline{r}_s) \\ &= P\,\overline{A}^{-1}\,(R\,r_s) \\ &= P\,\overline{A}^{-1}\,R\left(f - A\,u_s^{(i)}\right) \end{aligned}$$

Hence

$$u^{(i+1)} = u_s^{(i)} + P\,\overline{A}^{-1}\,R\,\left(f - A\,u_s^{(i)}\right). \tag{4}$$

Further $f = A\,u$ so $f - A\,u_s^{(i)} = A\,(u - u_k)$. But from (2) and (3)

$$u - u_s^{(i)} = u - u_k = (M^{-1}\,N)^k\,(u - u_0) = \left(I - M^{-1}\,A\right)^k (u - u_0) \tag{5}$$

So

$$f - A\,u_s^{(i)} = A\,\left(I - M^{-1}\,A\right)^k\,\left(u - u^{(i)}\right) \tag{6}$$

From equations (4), (5), and (6)

$$\begin{aligned} u - u^{(i+1)} &= u - u_s^{(i)} - \left(P\,\overline{A}^{-1}\,R\right)\,\left(f - A\,u_s^{(i)}\right) \\ &= \left(I - M^{-1}\,A\right)^k\,\left(u - u^{(i)}\right) - \left(P\,\overline{A}^{-1}\,R\right)\,A\,\left(I - M^{-1}\,A\right)^k\,\left(u - u^{(i)}\right) \end{aligned} \tag{7}$$

Therefore

$$u - u^{(i+1)} = \left(A^{-1} - P\,\overline{A}^{-1}\,R\right)\,A\,\left(I - M^{-1}\,A\right)^k\,\left(u - u^{(i)}\right) \tag{8}$$

as required.

4. If m steps of the post-smoothing iterations are employed then m steps of the smoothing iteration are applied to $u^{(i+1)}$ so that $u^{(i+1)}$ undergoes

$$u - u^{(i+1)} \longrightarrow \left(M^{-1}\,N\right)^m\,\left(u - u^{(i+1)}\right) = \left(I - M^{-1}\,A\right)^m\,\left(u - u^{(i+1)}\right) \tag{9}$$

so that from Question 2 (8)

$$u - u^{(i+1)} = \left(A^{-1} - P\,\overline{A}^{-1}\,R\right)\,A\,\left(I - M^{-1}\,A\right)^k\,\left(u - u^{(i)}\right). \tag{10}$$

Hence substituting (10) into (9)

$$u - u^{(i+1)} = \left(I - M^{-1}\,A\right)^m\,\left(A^{-1} - P\,\overline{A}^{-1}\,R\right)\,A\,\left(I - M^{-1}\,A\right)^k\,\left(u - u^{(i)}\right)$$

where $u^{(i+1)}$ is the $(i+1)^{\text{th}}$ 2-grid iterate after post-smoothing as well as coarse grid correction and pre-smoothing.

5. From Question 3 above and using

$$r^{(i+1)} = A\,\left(u - u^{(i+1)}\right) = f - A\,u^{(i+1)}$$

$$A\,\left(u - u^{(i+1)}\right) = A\,\left(I - M^{-1}\,A\right)^m\,\left(A^{-1} - P\,\overline{A}^{-1}\,R\right)\,A\,\left(I - M^{-1}\,A\right)^k\,\left(u - u^{(i)}\right) \tag{11}$$

but note that for some binomial coefficients α_j

$$\begin{aligned} A\,\left(I - M^{-1}\,A\right)^j &= A\,\left(I + \alpha_1\,M^{-1}\,A + \alpha_2\,(M^{-1}\,A)^2 + \cdots + (-1)^j\,(M^{-1}\,A)^j\right) \\ &= A + \alpha_1\,(A\,M^{-1})\,A + \alpha_2\,(A\,M^{-1})^2\,A + \cdots + (-1)^j\,(A\,M^{-1})^j\,A \\ &= \left(I + \alpha_1\,(A\,M^{-1}) + \alpha_2\,(A\,M^{-1})^2 + \cdots + (-1)^j\,(A\,M^{-1})^j\right)\,A \\ &= \left(I - A\,M^{-1}\right)^j\,A \end{aligned}$$

So equation (11) becomes

$$\begin{aligned} r^{(i+1)} &= A\,\left(u - u^{(i+1)}\right) \\ &= \left(I - A\,M^{-1}\right)^m\,A\,\left(A^{-1} - P\,\overline{A}^{-1}\,R\right)\,\left(I - A\,M^{-1}\right)^k\,A\,\left(u - u^{(i)}\right) \\ &= \left(I - A\,M^{-1}\right)^m\,A\,\left(A^{-1} - P\,\overline{A}^{-1}\,R\right)\,\left(I - A\,M^{-1}\right)^k\,r^{(i)} \end{aligned}$$

6. For $m = k$ the iteration matrix is given by (see Question 3)

$$\begin{aligned} &\left(I - M^{-1\,A}\right)^k \left(A^{-1} - \alpha\, P\, \overline{A}^{-1}\, P^T\right)\, A\, \left(I - M^{-1}\, A\right)^k \\ &= \left(I - M^{-1}\, A\right)^k \left(I - \alpha\, P\, \overline{A}^{-1}\, P^T\, A\right)\, \left(I - M^{-1}\, A\right)^k \end{aligned} \tag{12}$$

Recall that

$$(1-x)^n = \sum_{i=0}^{n} (-1)^i \begin{pmatrix} n \\ i \end{pmatrix} x^i \tag{13}$$

defines the binomial coefficients

$$\begin{pmatrix} n \\ i \end{pmatrix} = \frac{n!}{i!\,(n-i)!}$$

but note that

$$\begin{aligned} &\left(I - M^{-1}\, A\right)^n \\ &= I - \left(\begin{pmatrix} n \\ 1 \end{pmatrix} M^{-1} - \begin{pmatrix} n \\ 2 \end{pmatrix} M^{-1}\, A\, M^{-1} + \cdots + (-1)^{j-1} \begin{pmatrix} n \\ j \end{pmatrix} \left(M^{-1}\, A\right)^{j-1}\, M^{-1} \right. \\ &\left. + \cdots + (-1)^{n-1}\, \left(M^{-1}\, A\right)^{n-1}\, M^{-1}\right) A \end{aligned}$$

Let us denote by Q

$$\begin{aligned} Q = &\begin{pmatrix} n \\ 1 \end{pmatrix} M^{-1} - \begin{pmatrix} n \\ 2 \end{pmatrix} M^{-1}\, A\, M^{-1} + \cdots + (-1)^{j-1} \begin{pmatrix} n \\ j \end{pmatrix} \left(M^{-1}\, A\right)^{j-1}\, M^{-1} \\ &+ \cdots + (-1)^{n-1}\, \left(M^{-1}\, A\right)^{n-1}\, M^{-1} \end{aligned} \tag{14}$$

and we note that $Q = Q^T$ since M and A are symmetric so that $M^{-1}\, A\, M^{-1}\, A\, M^{-1}$ etc are all symmetric. Therefore $\left(I - M^{-1}\, A\right)^k = I - Q\, A$.

Returning to (12) we have that the 2-grid iteration matrix is

$$(I - Q\, A)\left(I - \alpha\, P\, \overline{A}^{-1}\, P^T\, A\right)(I - Q\, A)$$

expanding

$$\begin{aligned} (I - Q\, A)\left(I - \alpha\, P\, \overline{A}^{-1}\, P^T\, A\right)(I - Q\, A) &= \left(I - Q\, A - \alpha\, P\, \overline{A}^{-1}\, P^T\, A + \alpha Q A\, P\, \overline{A}^{-1}\, P^T\, A\right)(I - Q\, A) \\ &= I - Q\, A - \alpha\, P\, \overline{A}^{-1}\, P^T\, A + \alpha Q\, A\, P\, \overline{A}^{-1}\, P^T\, A - Q\, A \\ &+ Q\, A\, Q\, A + \alpha\, P\, \overline{A}^{-1}\, P^T\, A\, Q\, A - \alpha\, Q\, A\, P\, \overline{A}^{-1}\, P^T\, A\, Q\, A \\ &= I - (M_{MG})^{-1}\, A \end{aligned} \tag{15}$$

where

$$M_{MG}^{-1} = 2Q + \alpha\, P\, \overline{A}^{-1}\, P^T - \alpha Q\, A\, P\, \overline{A}^{-1}\, P^T - Q\, A\, Q - \alpha\, P\, \overline{A}^{-1}\, P^T\, A\, Q + \alpha\, Q\, A\, P\, \overline{A}^{-1}\, P^T\, A\, Q$$

all the terms of which are symmetric except for $\alpha Q\, A\, P\, \overline{A}^{-1}\, P^T$ and $\alpha\, P\, \overline{A}^{-1}\, P^T\, A\, Q$ the sum of which is symmetric since they are the transpose of each other. Finally note that M_{MG}^{-1} is symmetric therefore M_{MG} is also symmetric and

$$M_{MG}^{-1}\, N = M_{MG}^{-1}(M_{MG} - A) = I - M_{MG}^{-1}\, A$$

with $A = M_{MG} - N$.

7. Post-smoothing in this case is $(I - M_{MG}^{-T} A)^k$ which is

$$
\begin{aligned}
& I - \left(\begin{pmatrix} n \\ 1 \end{pmatrix} M^{-T} - \begin{pmatrix} n \\ 2 \end{pmatrix} M^{-T} A M^{-T} + \cdots + (-1)^{j-1} \begin{pmatrix} n \\ j \end{pmatrix} \left(M^{-T} A\right)^{j-1} M^{-T} \right. \\
& \left. + \cdots + (-1)^{n-1} \left(M^{-T} A\right)^{n-1} M^{-T} \right) A \\
& = I - Q^T A
\end{aligned}
$$

where Q is exactly as in the solution to Question 5 above.
Note that e.g.

$$\left(M^{-1} A M^{-1} A M^{-1}\right)^T = M^{-T} A M^{-T} A M^{-T}$$

where A is symmetric.
Hence the iteration matrix is

$$(I - Q^T A) \left(I - \alpha P \overline{A}^{-1} P^T A\right) (I - Q A)$$

with non-symmetric Q in this case as M is non-symmetric.

Expanding as in Question 5:

$$
\begin{aligned}
& (I - Q^T A) \left(I - \alpha P \overline{A}^{-1} P^T A\right) (I - Q A) \\
& = I - Q^T A - \alpha P \overline{A}^{-1} P^T A + \alpha Q^T A P \overline{A}^{-1} P^T A - QA + Q^T AQA + \alpha P \overline{A}^{-1} P^T AQA - \alpha Q^T A P \overline{A}^{-1} P^T A Q A \\
& = I - \left(Q^T + Q + \alpha P \overline{A}^{-1} P^T - \alpha Q^T A P \overline{A}^{-1} P^T - Q^T A Q - \alpha P \overline{A}^{-1} P^T A Q + \alpha Q^T A P \overline{A}^{-1} P^T A Q \right) A
\end{aligned}
$$

and the matrix

$$Q^T + Q + \alpha P \overline{A}^{-1} P^T - \alpha Q^T A P \overline{A}^{-1} P^T - Q^T A Q - \alpha P \overline{A}^{-1} P^T A Q + \alpha Q^T A P \overline{A}^{-1} P^T A Q$$

is symmetric since $(Q^T + Q)$ and $\alpha Q^T A P \overline{A}^{-1} P^T + \alpha P \overline{A}^{-1} P^T A Q$ are symmetric and all of the other terms are by themselves symmetric.

Hence by exactly the same argument as in Question 5 the 2-grid iteration is this case corresponds to a splitting $A = M_{MG} - N$ where M_{MG} is symmetric

8. Let

$$G_{\text{pre}} G_{\text{post}} = (I - M^{-1} A)^m \left(A^{-1} - \alpha P \overline{A}^{-1} P^T\right) A \left(A^{-1} - \alpha P \overline{A}^{-1} P^T\right) A (I - M^{-1} A)^k$$

We look at the term

$$
\begin{aligned}
\left(A^{-1} - \alpha P \overline{A}^{-1} P^T\right) A \left(A^{-1} - \alpha P \overline{A}^{-1} P^T\right) &= \left(A^{-1} - \alpha P \overline{A}^{-1} P^T\right) \left(I - \alpha A P \overline{A}^{-1} P^T\right) \\
&= A^{-1} - \alpha P \overline{A}^{-1} P^T - \alpha P \overline{A}^{-1} P^T + \alpha^2 P \overline{A}^{-1} P^T A P \overline{A}^{-1} P^T
\end{aligned}
$$

But $\alpha P A P^T = \overline{A}$ so we have

$$
\begin{aligned}
\left(A^{-1} - \alpha P \overline{A}^{-1} P^T\right) A \left(A^{-1} - \alpha P \overline{A}^{-1} P^T\right) &= A^{-1} - 2 \alpha P \overline{A}^{-1} P^T + \alpha P \overline{A}^{-1} \overline{A} \overline{A}^{-1} P^T \\
&= A^{-1} - \alpha P \overline{A}^{-1} P^T
\end{aligned}
$$

Hence

$$G_{\text{pre}} G_{\text{post}} = (I - M^{-1} A)^m \left(A^{-1} - \alpha P \overline{A}^{-1} P^T\right) A (I - M^{-1} A)^k$$

is the standard 2-grid iteration matrix with k pre-smoothing and m post-smoothing steps.

9. For the relaxation Jacobi iteration the iteration matrix is given by

$$J_\theta = (1-\theta)\, I - \theta\, D^{-1}(L+U)$$

where D is the diagonal of matrix A, L is the lower triangular part of $A-D$ and U is the upper triangular part of $A-D$.

So using the result of Exercise sheet 3, Question 5(a), the eigenvalues of the J_θ are

$$1-\theta+\frac{\theta}{2}\left(\cos\frac{r\,\pi}{n+1}+\cos\frac{s\,\pi}{n+1}\right)$$

for $r, s = 1, \cdots, n$. So now for $r > \frac{n}{2}$ or $s > \frac{n}{2}$ we have

$$-1 < \cos\frac{r\,\pi}{n+1} \leq 0 \quad \text{or} \quad -1 < \cos\frac{s\,\pi}{n+1} \leq 0$$

hence the eigenvalues of J_θ for $r > \frac{n}{2}$ or $s > \frac{n}{2}$ lie in the interval $(1-\theta-\theta, 1-\theta+\frac{\theta}{2}]$ (since clearly $-1 \leq \cos\phi \leq 1$) which is symmetric about the origin when

$$1-2\,\theta = -(1-\theta+\frac{\theta}{2}) = -(1-\frac{\theta}{2})$$

if and only if $\theta = \frac{4}{5}$.

For which choice of relaxation of parameter we have that the eigenvalues with $r > \frac{n}{2}$ or $s > \frac{n}{2}$ which correspond to high frequency eigenvalues in at least one of the coordinate directions lie in the interval $(-\frac{3}{5}, \frac{3}{5}]$.