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FEMs for PDEs - Problem Sheet 4

1. Let $\Omega = (0,1)^2$. Consider the finite element space $V_h \subset H^1(\Omega)$ consisting of all continuous piecewise linear functions on a triangulation of Ω obtained from a uniform square mesh of size $h = 1/N$, $N \geq 2$, by subdividing each square into two triangles with the diagonal of negative slope. Assuming that $u \in H^2(\Omega)$ let $\mathcal{I}_h u$ denote its continuous piecewise linear interpolant from V_h. You may take it for granted that

$$\|u - \mathcal{I}_h u\|_{H^1(\Omega)} \leq K_1 h |u|_{H^2(\Omega)},$$

where K_1 is a positive constant, independent of u, u_h and h. [If you are really ambitious, you may try to prove this, but this is not compulsory.]

Now consider the elliptic boundary value problem

$$-\Delta u + u = f(x,y) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where $f \in L_2(\Omega)$.

Assuming that V_h is the finite element space introduced above and letting u_h denote the finite element approximation to u from V_h, show that

$$\|u - u_h\|_{H^1(\Omega)} \leq K_2 h |u|_{H^2(\Omega)}$$

where K_2 is a positive constant, independent of u, u_h and h.

Show further that

$$\|u - u_h\|_{L_2(\Omega)} \leq K_3 h^2 |u|_{H^2(\Omega)}$$

where K_3 is a positive constant, independent of u, u_h and h.

2. Suppose that Ω is a bounded open set in $\mathbb{R}^n$. Consider the elliptic boundary value problem (labelled (P)):

$$\begin{aligned} -\Delta u + c(x)u &= f(x), \quad x \in \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned}$$

with $c(x) \geq 0$, $x = (x_1, \ldots, x_n) \in \bar{\Omega}$ and assume that $c \in L_\infty(\Omega)$, $f \in L_2(\Omega)$.

a) Show that there exists a quadratic functional $J : H_0^1(\Omega) \to \mathbb{R}$ of the form

$$J(v) = \frac{1}{2}a(v,v) - l(v), \qquad v \in H_0^1(\Omega),$$

where $a(\cdot,\cdot)$ is a symmetric bilinear functional on $H_0^1(\Omega) \times H_0^1(\Omega)$ with the property $a(v,v) > 0$ for all $v \in H_0^1(\Omega) \setminus \{0\}$, and $l(\cdot)$ is a linear functional on $H_0^1(\Omega)$, such that the weak solution u to (P) satisfies

$$J(u) \leq J(v) \qquad \forall v \in H_0^1(\Omega).$$

Does J have other minimisers in $H_0^1(\Omega)$? (Justify your answer!)

b) Show that if u minimises $J(\cdot)$ over $H_0^1(\Omega)$ then u is the weak solution to problem (P).

c) Assume that $n = 1$, $\Omega = (0,1)$, $c(x) \equiv 1$ and $f \in L_2(0,1)$. Suppose further that V_h is a finite element subspace of $H_0^1(0,1)$ consisting of continuous piecewise linear functions on a uniform subdivision of $[0,1]$ into subintervals of length $h = 1/N$. Show that there exists a unique $u_h \in V_h$ such that $J(u_h) \leq J(v_h)$ for all $v_h \in V_h$. Show further that u_h can be found by solving a suitable system of linear algebraic equations with a tri-diagonal matrix A (called the global stiffness matrix) whose entries you should calculate.

FEMs for PDEs – Problem Sheet 4

Q1 $\Omega = (0,1)^2$

$$(P) \quad \begin{cases} -\Delta u + u = f(x,y) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad f \in L_2(\Omega).$$

Weak formulation: find $u \in H_0^1(\Omega)$ s.t $a(u,v) = \ell(v) \quad \forall\, v \in H_0^1(\Omega)$

where $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx\,dy + \int_\Omega u \cdot v \, dx\,dy$

$\ell(v) = \int_\Omega f \cdot v \, dx\,dy.$

Recall that $(u,v)_a = a(u,v)$

$\|u\|_a = (u,u)_a^{\frac{1}{2}} = [a(u,u)]^{\frac{1}{2}}.$

In our case $\|u\|_a = \left(\int_\Omega |\nabla u|^2 dx\,dy + \int_\Omega |u|^2 dx\,dy\right)^{\frac{1}{2}} = \|u\|_{H^1(\Omega)}.$

Consider the finite element space $V_h \subset H^1(\Omega)$ consisting of all continuous piecewise linear functions on a triangulation of Ω obtained from a uniform square mesh of size $h = 1/N$, $N \geq 2$, by subdividing each square into two triangles with the diagonal of negative slope.

Finite element approximation: find $u_h \in V_h$ s.t

$a(u_h, v_h) = \ell(v_h) \quad \forall \quad v_h \in V_h.$

Assuming that $u \in H^2(\Omega)$, let $I_h u$ denote its continuous piecewise linear interpolant from V_h. (i.e. $I_h u \in V_h$).

Then $\|u - u_h\|_a^2 = a(u-u_h, u-u_h)$

$= a(u-u_h, u - I_h u)$ (by Galerkin Orthogonality)

$= (u-u_h, u-I_h u)_a$

$\leq \|u-u_h\|_a \|u - I_h u\|_a$

$\Rightarrow \|u-u_h\|_a \leq \|u - I_h u\|_a$

$\Rightarrow \|u - u_h\|_{H^1(\Omega)} \leq \|u - I_h u\|_{H^1(\Omega)}$ (since $\|u\|_a = \|u\|_{H^1(\Omega)}$)

However, $\|u - I_h u\|_{H^1(\Omega)} \leq K_1 h |u|_{H^2(\Omega)}$ where K_1 is a positive constant, independent of u, u_h and h.

$\Rightarrow \|u-u_h\|_{H^1(\Omega)} \leq K_2 h |u|_{H^2(\Omega)}$ with $K_2 = K_1$ in this case.

Equivalently,

$$|u-u_h|_{H^1(\Omega)} \leq k_2 h |u|_{H^2(\Omega)}$$

$$\therefore \|\nabla u - \nabla u_h\|_{L_2(\Omega)} \leq k_2 h |u|_{H^2(\Omega)}.$$

Put $g = u - u_h \in H_0^1(\Omega) \subset L_2(\Omega)$.

Consider the auxiliary (dual) problem:

$$\begin{cases} -\Delta w_g + w_g = g \text{ in } \Omega \\ w_g = 0 \quad \text{on } \partial\Omega \end{cases}$$

$$\begin{aligned}
\|u-u_h\|^2_{L_2(\Omega)} &= (u-u_h, u-u_h) \\
&= (u-u_h, g) \\
&= (u-u_h, -\Delta w_g + w_g) \\
&= -\int_\Omega (u-u_h)\, \nabla\cdot(\nabla w_g)\, dx\,dy + \int_\Omega (u-u_h) w_g \, dx\,dy \\
&= \int_\Omega \nabla(u-u_h)\cdot\nabla w_g \, dx\,dy - \int_{\partial\Omega} (u-u_h)\, n\cdot\nabla w_g \, ds \\
&\quad + \int_\Omega (u-u_h) w_g \, dx\,dy
\end{aligned}$$

$$\begin{aligned}
\|u-u_h\|^2_{L_2(\Omega)} &= \int_\Omega \nabla(u-u_h)\cdot\nabla w_g\, dx\,dy + \int_\Omega (u-u_h) w_g \, dx\,dy \\
&= \int_\Omega \nabla(u-u_h)\cdot\nabla(w_g - I_h w_g)\, dx\,dy + \int_\Omega (u-w_g)(w_g - I_h w_g)\, dx\,dy
\end{aligned}$$

(Galerkin orthogonality)

$$\begin{aligned}
&\leq \|u-u_h\|_a \; \|w_g - I_h w_g\|_a \\
&= \|u-u_h\|_{H^1(\Omega)} \; \|w_g - I_h w_g\|_{H^1(\Omega)} \\
&\leq k_2 h |u|_{H^2(\Omega)} \; k_1 h |w_g|_{H^2(\Omega)} \\
&= k_2^2 h^2 |u|_{H^2(\Omega)} \cdot |w_g|_{H^2(\Omega)}
\end{aligned}$$

Now we aim to show that $|w_g|_{H^2(\Omega)} \leq 2\|g\|_{L_2(\Omega)}$

$$\begin{aligned}
2\|g\|^2_{L_2(\Omega)} &\geq \|\Delta w_g\|^2_{L_2(\Omega)} \\
&= \int_\Omega \left(\frac{\partial^2 w_g}{\partial x^2} + \frac{\partial^2 w_g}{\partial y^2}\right)^2 dx\,dy \\
&= \int_\Omega \left(\frac{\partial^2 w_g}{\partial x^2}\right)^2 + \underbrace{2\int_\Omega \frac{\partial^2 w_g}{\partial x^2}\cdot\frac{\partial^2 w_g}{\partial y^2}}_{= 2\int_\Omega \left(\frac{\partial^2 w_g}{\partial x \partial y}\right)^2} + \int_\Omega \left(\frac{\partial^2 w_g}{\partial y^2}\right)^2
\end{aligned}$$

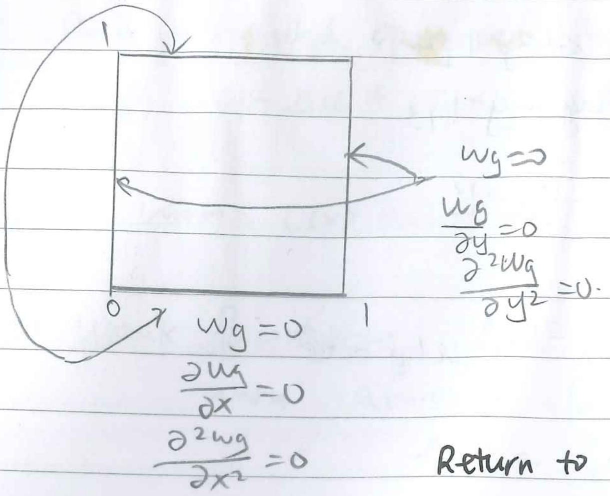
This is because if we test $-\Delta w_g + w_g = g$ against w_g.

$$\int_\Omega \nabla w_g \cdot \nabla w_g + w_g w_g = \int g w_g$$

$$\|w_g\|^2_{L^2(\Omega)} \leq \|g\|_{L^2(\Omega)} \|w_g\|_{L^2(\Omega)}$$

$$\|w_g\|_{L^2(\Omega)} \leq \|g\|_{L^2(\Omega)}$$

$$\|\Delta w_g\|_{L^2(\Omega)} \leq \|g\|_{L^2(\Omega)} + \|w_g\|_{L^2(\Omega)} \leq 2\|g\|_{L^2(\Omega)}$$



$$\therefore 2\|g\|^2_{L^2(\Omega)} \geq \int_\Omega \left(\frac{\partial^2 w_g}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w_g}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 w_g}{\partial y \partial x}\right)^2 + \left(\frac{\partial^2 w_g}{\partial y^2}\right)^2 dxdy$$

$$= |w_g|^2_{H^2(\Omega)}$$

Return to the previous inequality, we have

$$\|u-u_h\|^2_{L^2(\Omega)} \leq K_2^2 h^2 |u|_{H^2(\Omega)} |w_g|_{H^2(\Omega)}$$

$$\leq K_2^2 h^2 |u|_{H^2(\Omega)} 2\|g\|_{L^2(\Omega)}$$

$$= 2K_2^2 h^2 |u|_{H^2(\Omega)} \cancel{\|u-u_h\|_{L^2(\Omega)}}$$

$$\therefore \|u-u_h\|_{L^2(\Omega)} \leq 2K_2^2 h^2 |u|_{H^2(\Omega)}$$

$$= K_3 h^2 |u|_{H^2(\Omega)}$$

$K_3 = 2K_2^2$ is a positive constant, independent of u, u_h and h.

$$\begin{cases} -\Delta u + u = f, & \Omega = (0,1)^2 \\ u = 0 & \partial\Omega \end{cases} \qquad V_h \subset H^1(\Omega), \quad u \in H^2(\Omega)$$

$$\|u - I_h u\|_{H^1} \leq K_1 h |u|_{H^2(\Omega)}$$

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v + uv$$

$$\ell_f(v) = \int_\Omega f v$$

$$\|u-u_h\|_{L^2(\Omega)} = \sup_{g \in L^2(\Omega)} \frac{\int_\Omega (u-u_h) g}{\|g\|_{L^2(\Omega)}}$$

we know that for any $g \in L^2(\Omega)$ $\exists!$ $w_g \in H_0^1(\Omega)$

$$a(v, w_g) = \ell_g(v) \qquad \forall v \in H_0^1(\Omega)$$

$$\|u-u_h\|_{L^2(\Omega)} \leq \sup_{g \in L^2} \frac{a(u-u_h, w_g - w_{g_h})}{\|g\|_{L^2}}$$ Galerkin orthogonality

$$\leq \|u-u_h\|_{H^1} \sup_{g \in L^2} \frac{1}{\|g\|_{L^2}} \inf_{w_h \in V_h} \|w_g - w_{g_h}\|_{H^1}$$

$$\leq C |u|_{H^2} h^2 \sup_{g \in L^2} \frac{|w_g|_{H^2}}{\|g\|_{L^2}}$$

$$|w_g|_{H^2} = \|\Delta w_g\|_{L^2}$$
$$= \|w_g - g\|_{L^2}$$
$$\leq 2\|g\|_{L^2}$$

$$\Rightarrow \|u-u_h\| \leq \tilde{C} h^2 |u|_{H^2}$$

Q2.

$$\begin{cases} -\Delta u + c(x)u = f & \Omega \quad c \geq 0 \\ u = 0 & \partial\Omega \quad c \in L^\infty(\Omega) \end{cases}$$

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v + cuv \qquad \ell(v) = \int_\Omega fv.$$

$$J(v) = \frac{1}{2} a(v,v) - \ell(v)$$

$$J: H_0^1(\Omega) \to \mathbb{R}$$

$$v \mapsto |v|_X^2 \qquad J: V \to W$$

$$\lim_{v\to 0} \frac{\| J(u+v) - J(u) - Tv \|_W}{\|v\|_V} = 0$$

$$T = J'(u)$$

$$|u+v|_X^2 - |u|_X^2 = (u+v, u+v)_X - (u,u)_X$$
$$= 2(u,v) + (v,v)$$

$$I'(u)v = 2(u,v)$$

$$J'(u)v = a(u,v) - \ell(v)$$

If $u \in H_0^1(\Omega)$ is a minimizer of J $\Rightarrow J'(u) = 0 \iff a(u,v) = \ell(v) \ \forall v \in H_0^1(\Omega)$

2nd derivative
$$\lim_{v\to 0} \frac{\| J(u+v) - J(u) - J'(u)v - \frac{1}{2} J''(u)(v,v) \|_W}{\|v\|_V} = 0$$

$$J': H_0^1(\Omega) \to \mathcal{L}(H_0^1(\Omega); \mathbb{R}) = H^{-1}(\Omega)$$

Prove $J''(u)(v,w) = a(v,w)$
$J''(u)(v,v) \geq 0$ for $v \neq 0$.

Q2 Ω is a bounded open set in $\mathbb{R}^n$

$$(P) \begin{cases} -\Delta u + c(x)\, u = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $c(x) \geq 0$. $x = (x_1 .. x_n) \in \Omega$. and $c \in L_\infty(\Omega)$. $f \in L_2(\Omega)$

(a) <u>Weak formulation</u> find $u \in H_0^1(\Omega)$ s.t $a(u,v) = \ell(v) \quad \forall\ v \in H_0^1(\Omega)$

where $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega c\, u \cdot v \, dx.$

$\ell(v) = \int_\Omega f(x) \cdot v \, dx.$

$a(\cdot,\cdot)$ is a symmetric bilinear functional on $H_0^1(\Omega) \times H_0^1(\Omega)$ with the property $a(v,v) > 0 \quad \forall\ v \in H_0^1(\Omega) \setminus \{0\}$.

$\ell(\cdot)$ is a linear functional on $H_0^1(\Omega)$.

Let u be the (unique) weak solution to (P) and let $v \in H_0^1$.

Then
$$\begin{aligned} J(v) - J(u) &= \tfrac{1}{2} a(v,v) - \ell(v) - \tfrac{1}{2} a(u,u) + \ell(u) \\ &= \tfrac{1}{2} a(v,v) - \tfrac{1}{2} a(u,u) - \ell(v-u) \\ &= \tfrac{1}{2} a(v,v) - \tfrac{1}{2} a(u,u) - a(u, v-u) \\ &= \tfrac{1}{2} [a(v,v) - 2a(u,v) + a(u,u)] \\ &= \tfrac{1}{2} [a(v,v) - a(u,v) - a(v,u) + a(u,u)] \\ &= \tfrac{1}{2} [a(v-u,v) - a(v-u,u)] \\ &= \tfrac{1}{2} a(v-u, v-u). \end{aligned}$$

Note that
$$\begin{aligned} a(w,w) &= \int_\Omega |\nabla w|^2 dx + \int_\Omega c|w|^2 dx \\ &\geq \int_\Omega |\nabla w|^2 dx \\ &= \tfrac{1}{2} \int_\Omega |\nabla w|^2 dx + \tfrac{1}{2} \int_\Omega |\nabla w|^2 dx \end{aligned}$$

(By Poincaré's inequality) $\geq \frac{1}{2} \int_\Omega |\nabla w|^2 dx + \frac{c^*(\Omega)}{2} \int_\Omega |w|^2 dx$

$\geq c_0 \|w\|^2_{H^1(\Omega)}$

where $c_0 = \min\left(\frac{1}{2}, \frac{c^*(\Omega)}{2}\right)$.

Then ($\ast$) $J(v) - J(u) \geq \frac{1}{2} c_0 \|v - u\|^2_{H^1(\Omega)} \quad \forall\ v \in H_0^1(\Omega)$

Therefore ($\ast$) $J(v) \geq J(u) \quad \forall\ v \in H_0^1(\Omega)$

J has No other minimisers in $H_0^1(\Omega)$.

Assume for contradiction that $\tilde{u}$ is also a minimiser for $J(\cdot)$

Then $(**)$ $J(v) \geq J(\tilde{u}) \quad \forall \ v \in H_0^1(\Omega)$.

Now $(*)$ gives : $J(\tilde{u}) \geq J(u)$.

$(**)$ gives : $J(u) \geq J(\tilde{u})$.

$\Rightarrow \quad J(u) = J(\tilde{u}) \quad \Rightarrow \quad u = \tilde{u}$ by $(*)$.

(b) Let $\lambda \in \mathbb{R}_+$. $v \in H_0^1(\Omega)$.

Since u minimises $J(\cdot)$ over $H_0^1(\Omega)$.

$J(u+\lambda v) \geq J(u)$.

$\therefore \quad J(u+\lambda v) - J(u) \geq 0$.

$\therefore \ \frac{1}{2} a(u+\lambda v, u+\lambda v) - \ell(u+\lambda v)$

$- \frac{1}{2} a(u,u) + \ell(u) \geq 0$.

$\therefore \quad \lambda [a(u,v) - \ell(v)] + \frac{\lambda^2}{2} a(v,v) \geq 0$

$\therefore \quad a(u,v) - \ell(v) \geq -\frac{\lambda}{2} a(v,v)$

let $\lambda \to 0+$, we have

$a(u,v) - \ell(v) \geq 0 \quad \forall \ v \in H_0^1(\Omega) \quad (\heartsuit)$

Replacing v by $-v$ in $(\heartsuit)$. we have

$a(u,v) - \ell(v) \leq 0 \quad \forall \ v \in H_0^1(\Omega)$.

$\Rightarrow \quad a(u,v) = \ell(v) \quad \forall \ v \in H_0^1(\Omega)$

That is, u is the weak solution to problem (P).

(c). $n=1$. $\Omega = (0,1)$ $\quad c(x) \equiv 1$. $\quad f \in L_2(0,1)$.

$$\begin{cases} -u'' + u = f(x) & x \in (0,1) \\ u(0) = u(1) = 0. \end{cases}$$

The weak formulation of the 1-D problem becomes:

find $u \in H_0^1(0,1)$ s.t $a(u,v) = \ell(v) \quad \forall \ v \in H_0^1(0,1)$

where $a(u,v) = \int_0^1 u'v' \, dx + \int_0^1 uv \, dx$

and

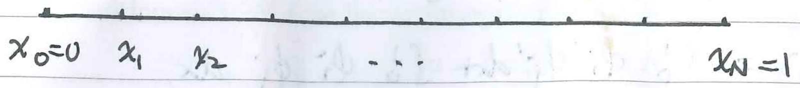
$\ell(v) = \int_0^1 fv \, dx$.

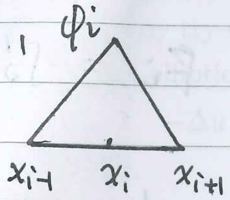
The weak solution is characterised by

$$J(u) = \min_{v \in H_0^1(\Omega)} J(v)$$

where $J(v) = \frac{1}{2} \int_0^1 |v'|^2 dx + \frac{1}{2} \int_0^1 |v|^2 dx - \int_0^1 f v \, dx$.

Finite element approximation:





$$\phi_i = \left(1 - \left|\frac{x - x_i}{h}\right|\right)_+ \quad \text{for } i = 1, \dots N-1.$$

$V_h = \text{span}\{\phi_1 \dots \phi_{N-1}\}$. $\quad v_h \in V_h \Rightarrow v_h = \sum_{i=1}^{N-1} V_i \phi_i(x)$.

The finite element weak formulation is:

find $u_h \in V_h$ s.t $a(u_h, v_h) = \ell(v_h) \quad \forall v_h \in V_h$. (★)

Using an argument similar to the proof in (a) with $H_0^1(\Omega)$ replaced by ~~H~~ V_h, we can show that the weak solution u_h to (★) satisfies

(V) $J(u_h) \le J(v_h)$ for all $v_h \in V_h$.

Assume for contradiction that $\hat{u}_h$ is also a minimiser for $J(\cdot)$ in the space V_h. Then (VV) $J(\hat{u}_h) \le J(v_h) \quad \forall v_h \in V_h$.

Now (V) gives $J(\hat{u}_h) \ge J(u_h)$

(VV) gives $J(u_h) \ge J(\hat{u}_h)$

$\Rightarrow J(u_h) = J(\hat{u}_h)$

~~and then $u_h = \hat{u}_h$~~

since as $0 = J(u_h) - J(\hat{u}_h) \ge \frac{1}{2} c_0 \| u_h - \hat{u}_h \|^2_{H^1(\Omega)}$.

$\Rightarrow \quad u_h = \hat{u}_h$.

This shows that there exists a unique $u_h \in V_h$ s.t $J(u_h) \le J(v_h)$ for all $v_h \in V_h$.

Note that $J(v_h) = \frac{1}{2} \int_0^1 |v_h'|^2 dx + \frac{1}{2} \int_0^1 |v_h|^2 dx$

$- \int_0^1 f v_h \, dx$

$= \frac{1}{2} \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} V_i V_j \int_0^1 \phi_i' \phi_j' dx + \frac{1}{2} \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} V_i V_j \int_0^1 \phi_i \phi_j dx$

$- \sum_{i=1}^{N-1} V_i \int_0^1 f \phi_i \, dx$.

$$= \frac{1}{2}(V_1, \cdots V_{N-1})\, A \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_{N-1} \end{pmatrix} - (V_1, \cdots V_{N-1}) F$$

where $A = (a_{ij})_{1 \le i,j \le N-1}$

with $a_{ij} = \int_0^1 \phi_i' \phi_j' dx + \int_0^1 \phi_i \phi_j \, dx$

and $F = (F_i)_{1 \le i \le N-1}$ with $F_i = \int_0^1 f \phi_i(x) dx$

Since $\phi_i = \left(1 - \frac{|x - x_i|}{h}\right)_+$. by a similar calculation to the exercise in problem sheet 2. we have

$$a_{ij} = \int_0^1 \phi_i'(x) \phi_j'(x) dx + \int_0^1 \phi_i \phi_j \, dx$$

$$= \begin{cases} \frac{2}{h} + \frac{4h}{6} & \text{if } i = j \\ -\frac{1}{h} + \frac{h}{6} & \text{if } |i-j| = 1 \\ 0 & \text{if } |i-j| > 1 \end{cases}$$

for $i, j = 1, \ldots N-1$.

Then the problem can be written as

Find $V \in \mathbb{R}^{N-1}$ s.t

$$\frac{1}{2} V^T A V - V^T F \quad \text{is minimum.}$$

with A and F stated as above.

This is equivalent to solve for

$$AU = F$$

where U is the coefficient vector of u_h.

s.t $u_h = \sum_{i=1}^{N-1} U_i \phi_i$.