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FEMs for PDEs - Problem Sheet 1

1. Draw the graph of the function ϕ defined by $\phi(x) = (1 - |x|)_+$ for $x \in [-2, 2]$. Is it true that $\phi \in C[-2,2] \cap C^1(-2,2)$?

   Calculate the first (weak) derivative $\phi' = D\phi$ of ϕ on the interval $[-2,2]$. Verify that ϕ, $\phi' \in L_p(-2,2)$ for all $p \in [1, \infty]$. Hence deduce that $\phi \in W_p^1(-2,2)$ for all $p \in [1,\infty]$.

2. Suppose that $u(x) = x^\alpha$, $x \in [0,1]$, where α is a fixed real number, $0 < \alpha < 1$. Show that $u \in C^\infty(0,1)$, but $u \notin W_p^1(0,1)$ for $p \geq (1-\alpha)^{-1}$.

   Let $\Omega = \{(x,y) \in \mathbf{R}^2 : x^2 + y^2 < \frac{1}{4}\}$ and consider the function w defined on Ω by $w(x,y) = \log|\log\sqrt{x^2+y^2}|$. Show that $w \in W_2^1(\Omega)(= H^1(\Omega))$ but $w \notin C(\Omega)$.

3. Given that (a,b) is an open interval of the real line, let $H^1_{E_0}(a,b) = \{v \in H^1(a,b) : v(a) = 0\}$.

   a) By writing
   $$v(x) = \int_a^x v'(\xi)\,\mathrm{d}\xi, \qquad a \leq x \leq b,$$
   for $v \in H^1_{E_0}(a,b)$, show that the following (Poincaré-Friedrichs) inequality holds for each $v \in H^1_{E_0}(a,b)$:
   $$\|v\|^2_{L_2(a,b)} \leq \frac{1}{2}(b-a)^2|v|^2_{H^1(a,b)}.$$

   b) By writing
   $$[v(x)]^2 = \int_a^x \frac{d}{d\xi}[v(\xi)]^2\,\mathrm{d}\xi = 2\int_a^x v(\xi)v'(\xi)\,\mathrm{d}\xi, \qquad a \leq x \leq b,$$
   for $v \in H^1_{E_0}(a,b)$, show that the following (Agmon's) inequality holds for each $v \in H^1_{E_0}(a,b)$:
   $$\max_{x\in[a,b]} |v(x)|^2 \leq 2\|v\|_{L_2(a,b)}|v|_{H^1(a,b)}.$$

4. Given that $f \in L_2(0,1)$, state the weak formulation of each of the following boundary value problems:

   a) $-u'' + u = f(x)$ for $x \in (0,1)$, $u(0) = 0$, $u(1) = 0$;

   b) $-u'' + u = f(x)$ for $x \in (0,1)$, $u(0) = 0$, $u'(1) = 0$;

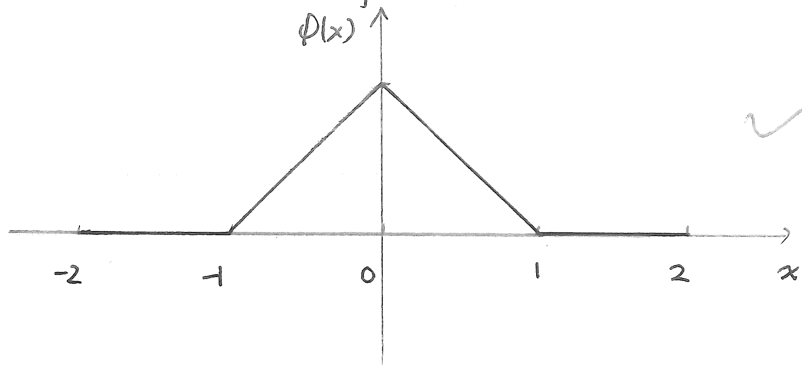
   c) $-u'' + u = f(x)$ for $x \in (0,1)$, $u(0) = 0$, $u(1) + u'(1) = 0$.

   Apply the Lax-Milgram lemma to show that each of the three weak formulations has a (corresponding) unique weak solution[1].

[1] Hint: You may wish to use the inequality from part b) of Question 3 when attempting to prove that the bilinear form, associated with the boundary value problem in part c), is bounded on $H^1_{E_0}(0,1)$.

FEMs for PDEs — Problem Sheet 1

Q1 $\phi(x) = (1-|x|)_+$ for $x \in [-2,2]$



From the plot, we know that $\phi \in C[-2,2]$. However, $\phi(x)$ is not differentiable at $x=0$, $x=\pm 1$ since

$$\phi'(0^-) = \lim_{\varepsilon \to 0} \frac{\phi(-\varepsilon)-\phi(0)}{-\varepsilon} = 1 \neq -1 = \lim_{\varepsilon\to 0} \frac{\phi(\varepsilon)-\phi(0)}{\varepsilon} = \phi'(0^+)$$

$$\phi'(-1^-) = \lim_{\varepsilon\to 0} \frac{\phi(-1-\varepsilon)-\phi(-1)}{-\varepsilon} = 0 \neq 1 = \lim_{\varepsilon \to 0} \frac{\phi(-1+\varepsilon)-\phi(-1)}{\varepsilon} = \phi'(-1^+)$$

$$\phi'(1^-) = \lim_{\varepsilon\to 0} \frac{\phi(1-\varepsilon)-\phi(1)}{-\varepsilon} = -1 \neq 0 = \lim_{\varepsilon\to 0} \frac{\phi(1+\varepsilon)-\phi(1)}{\varepsilon} = \phi'(1^+)$$

Thus, it is NOT true that $\phi \in C[-2,2] \cap C^1(-2,2)$.

For any $v \in C_0^\infty(-2,2)$,

$$\begin{aligned}
\int_{-\infty}^{+\infty} \phi(x) v'(x)\,dx &= \int_{-\infty}^{+\infty} (1-|x|)_+ v'(x)\,dx \\
&= \int_{-1}^{1} (1-|x|) v'(x)\,dx \\
&= \int_{-1}^{0} (1+x) v'(x)\,dx + \int_0^1 (1-x) v'(x)\,dx \\
&= (1+x)v(x)\Big|_{x=-1}^{x=0} - \int_{-1}^{0} v(x)\,dx \\
&\quad + (1-x)v(x)\Big|_{x=0}^{x=1} + \int_0^1 v(x)\,dx \\
&= \cancel{v(0)} - \int_{-1}^{0} v(x)\,dx - \cancel{v(0)} + \int_0^1 v(x)\,dx \\
&= \int_{-1}^{0} (-1)\, v(x)\,dx + \int_0^1 1\cdot v(x)\,dx \\
&= -\int_{-\infty}^{+\infty} w(x)\, v(x)\,dx
\end{aligned}$$

where

$$w(x) := \begin{cases} 1 & \text{if } x \in (-1,0) \\ -1 & \text{if } x \in (0,1) \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the first derivative $\phi' = D\phi = w$.

For $1 \le p < \infty$

$$\left(\int_{-2}^{2} |\phi|^p \, dx\right)^{\frac{1}{p}} = \left(\int_{-1}^{1} (1-|x|)^p \, dx\right)^{\frac{1}{p}}$$

$$\le \left(\int_{-1}^{1} 1^p \, dx\right)^{\frac{1}{p}} \quad \text{as } 1-|x| \le 1 \text{ for } -1 \le x \le 1$$

$$= 2^{\frac{1}{p}} < \infty$$

If $p = \infty$,

$$\operatorname*{ess\,sup}_{x \in (-2,2)} |\phi(x)| = 1 < \infty.$$

For $1 \le p < \infty$

$$\left(\int_{-2}^{2} |w|^p \, dx\right)^{\frac{1}{p}} = \left(\int_{-1}^{0} 1^p \, dx + \int_{0}^{1} |-1|^p \, dx\right)^{\frac{1}{p}}$$

$$= 2^{\frac{1}{p}} < \infty$$

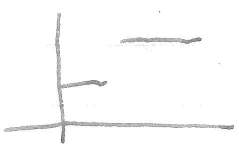
If $p = \infty$.

$$\operatorname*{ess\,sup}_{x \in (-2,2)} |w(x)| = 1 < \infty.$$

Since $\phi \in L_p(-2,2)$ and $\phi' \in L_p(-2,2)$ for all $p \in [1,\infty]$,
$\phi \in W_p^1(-2,2)$ for all $p \in [1,\infty]$

$\int_{-2}^{2} \varphi \, Dv$

test function



has pointwise derivative.
But not weak derivative.

$\partial_{xy} f(x,y)$

$f(x,y) = g(x) + g(y)$

Q2 $\quad u(x) = x^{\alpha}$. $x \in [0,1]$ where α is a fixed real number, $0 < \alpha < 1$.

- $u \in C^{\infty}(0,1)$ with

$$u' = \alpha x^{\alpha-1}$$
$$u'' = \alpha(\alpha-1) x^{\alpha-2}$$
$$\vdots$$
$$u^{(k)} = \alpha(\alpha-1)\cdots(\alpha-k+1)\, x^{\alpha-k}$$

for all $k \in \mathbb{N}$.

- Since $u(x) = x^{\alpha}$ is smooth, its weak derivative coincides with its classical derivative $u' = \alpha x^{\alpha-1}$

$$\int_0^1 |u'|^p dx = \int_0^1 (\alpha x^{\alpha-1})^p \, dx$$
$$= \alpha^p \int_0^1 x^{(\alpha-1)p} \, dx$$

If $p \geq (1-\alpha)^{-1}$, then ~~$p(\alpha-1) \geq -1$~~ $p(\alpha-1) \leq -1$ (note $\alpha-1<0$)

$\Rightarrow \int_0^1 x^{(\alpha-1)p} dx = \infty$ if $p \geq (1-\alpha)^{-1}$.

$\Rightarrow u \notin W_p^1(0,1)$ for $p \geq (1-\alpha)^{-1}$.

$$\int_\Omega |w(x,y)|^2$$
$$= 2\pi \int_0^{1/2} (\log|\log r|)^2 r\, dr$$
$$\lim_{r\to 0} r \log|\log r|^2$$
$r = e^{-t}$
$$= \lim_{t\to\infty} e^{-t} \log|t|^2 = 0$$
L'Hôpital !

exponential goes to zero at ∞ faster than the growth of logarithm.

- $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < \frac{1}{4}\}$.

w is defined on Ω by $w(x,y) = \log|\log\sqrt{x^2+y^2}|$

Note that $w(0,0)$ is not defined. (i.e. $w(x,y) \to \infty$ as $(x,y)\to(0,0)$)

$\Rightarrow w(x,y) \notin C(\Omega)$.

$$\int_\Omega |w|^2 = \int_\Omega \left(\log|\log\sqrt{x^2+y^2}|\right)^2 dx\, dy$$
$$= \lim_{\varepsilon\to 0} \int_{B(0,\frac14)\setminus B(0,\varepsilon)} \left(\log|\log\sqrt{x^2+y^2}|\right)^2 dx dy$$
$$\leq \lim_{\varepsilon\to 0} \int_{B(0,\frac14)\setminus B(0,\varepsilon)} (\log|\log \varepsilon|)^2 dx dy$$
$$\leq \lim_{\varepsilon\to 0} C\cdot\varepsilon^2 (\log|\log\varepsilon|)^2$$
$$< \infty.$$

this is not obvious to me

$\nabla f(x,y) = \frac{d\hat{f}(r)}{dr}\hat{r}$ $\Rightarrow$ $w \in L_2(\Omega)$

The ~~pointf~~ pointwise derivatives of w ~~is~~ are

$\int_\Omega |\nabla w|^2 = 2\pi \int_0^{1/2} |w'(r)|^2 r\,dr$ $\quad w_x = \frac{1}{|\log\sqrt{x^2+y^2}|} \cdot \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2x$

$= 2\pi \int_0^{1/2} |w'(r)|^2 r\,dr$ $\quad = \frac{x}{|\log\sqrt{x^2+y^2}|(x^2+y^2)}$

$= 2\pi \int_0^{1/2} \frac{1}{r \log r^2}\,dr$

$< \infty$

$\frac{1}{r\log r^2} \to 0$ as $r \to 0$

$$w_y = \frac{y}{|\log\sqrt{x^2+y^2}|(x^2+y^2)}$$

We need to check that they are weak derivatives across 0.
For any $\phi \in C_c^\infty(\Omega)$,

$$\int_\Omega w\phi_x = \lim_{\varepsilon\to 0^+} \int_{\frac{1}{4} > x^2+y^2 > \varepsilon^2} w\phi_x\,dx$$

$$= \lim_{\varepsilon\to 0^+} \int_{\frac{1}{4} > x^2+y^2 > \varepsilon^2} -w_x\phi\,dx$$

$$= \lim_{\varepsilon\to 0^+} \int_{\frac{1}{4} > x^2+y^2 > \varepsilon^2} -\frac{x}{|\log\sqrt{x^2+y^2}|(x^2+y^2)}\phi\,dx$$

$$+ \int_{x^2+y^2=\varepsilon^2} w\phi\cdot n\,ds$$

$$+ \int_{x^2+y^2=\frac{1}{4}} w\phi\cdot n\,ds$$

$$= \lim_{\varepsilon\to 0^+} \int_{\frac{1}{4} > x^2+y^2 > \varepsilon^2} -\frac{x}{|\log\sqrt{x^2+y^2}|(x^2+y^2)}\phi\,dx.$$

The first surface integral $\to 0$ as $\varepsilon \to 0$. and the 2nd surface integral $\to 0$ since $\phi \in C_c^\infty(\Omega)$.

$$\int_\Omega |\nabla w|^2\,dx\,dy = \int_\Omega \frac{x^2+y^2}{|\log\sqrt{x^2+y^2}|^2 (x^2+y^2)^2}\,dx\,dy$$

$$= \int_\Omega \frac{1}{|\log\sqrt{x^2+y^2}|^2 (x^2+y^2)}\,dx\,dy$$

again not obvious $\quad < \infty$

$\Rightarrow$ $w \in W_2^1(\Omega) = H^1(\Omega)$.

Q3. $H^1_{E_0}(a,b) = \{ v \in H^1(a,b) : v(a) = 0 \}$

(a) Claim: $\| v \|^2_{L^2(a,b)} \le \frac{1}{2}(b-a)^2 |v|^2_{H^1(a,b)}$

Proof: $v(x) = \int_a^x v'(\xi)\, d\xi$, $a \le x \le b$, for $v \in H^1_{E_0}(a,b)$.

$$\int_a^b |v(x)|^2 dx = \int_a^b \left| \int_a^x v'(\xi)\, d\xi \right|^2 dx$$

$$(\text{Hölder inequality}) \le \int_a^b (x-a) \int_a^x |v'(\xi)|^2 d\xi \, dx$$

$$(\text{Tonelli Theorem}) = \int_a^b (x-a)\, dx \int_a^x |v'(\xi)|^2 d\xi$$

$$\le \frac{1}{2}(b-a)^2 \int_a^b |v'(\xi)|^2 d\xi$$

$$= \frac{1}{2}(b-a)^2 |v|^2_{H^1(a,b)}$$

(b) Claim: (Agmon's inequality).

For each $v \in H^1_{E_0}(a,b)$,

$$\max_{x \in [a,b]} |v(x)|^2 \le 2 \|v\|_{L_2(a,b)} |v|_{H^1(a,b)}.$$

Proof: $$[v(x)]^2 = \int_a^x \frac{d}{d\xi}[v(\xi)]^2 d\xi$$

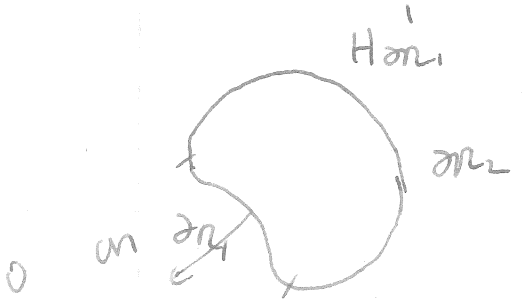
$$= 2\int_a^x v(\xi)\, v'(\xi)\, d\xi, \quad a \le x \le b$$

$$\max_{x\in[a,b]} |v(x)|^2 = \max_{x \in [a,b]} 2 \int_a^x v(\xi)\, v'(\xi)\, d\xi$$

$$\le \max_{x\in[a,b]} 2 \cdot \left(\int_a^x |v(\xi)|^2 d\xi \right)^{\frac{1}{2}} \left(\int_a^x (v'(\xi))^2 d\xi \right)^{\frac{1}{2}}$$

$$\le 2 \left(\int_a^b |v(\xi)|^2 d\xi \right)^{\frac{1}{2}} \left(\int_a^b |v'(\xi)|^2 d\xi \right)^{\frac{1}{2}}$$

$$= 2 \|v\|_{L_2(a,b)} |v|_{H^1(a,b)}.$$



→ poincaré inequality still applies

Q4 $f \in L_2(0,1)$

(a) $-u'' + u = f(x)$ for $x \in (0,1)$ $u(0)=0$, $u(1)=0$.

- weak formulation: Find $u \in H_0^1(0,1)$ s.t
$$\int_0^1 u'v'\,dx + \int_0^1 uv\,dx = \int_0^1 fv\,dx \quad \text{for all } v \in H_0^1(0,1)$$

- Existence of unique weak solution.

$H_0^1(0,1)$ is a closed subspace of the Hilbert space $H^1(0,1)$.
then $H_0^1(0,1)$ is a Hilbert space equipped with norm $\|\cdot\|_{H^1(0,1)}$ defined as $\|v\|_{H^1(0,1)} = \left(\int_0^1 |v|^2dx + \int_0^1 |v'(x)|^2dx\right)^{\frac{1}{2}}$

$a(u,v) = \int_0^1 u'v'\,dx + \int_0^1 uv\,dx$ is a bilinear form on $H_0^1(0,1) \times H_0^1(0,1)$ s.t

(a) $\exists\ C_0 > 0$ s.t $\forall\ v \in H_0^1(0,1)$ $a(v,v) \geq C_0\|v\|_{H^1(0,1)}$

ie. $a(v,v) = \int_0^1 |v'|^2 + \int_0^1 |v|^2 = \|v\|^2_{H^1(0,1)}$ ($C_0=1$)

(b) $\exists\ C_1 > 0\ \forall\ v,w \in H_0^1(0,1)$ $|a(v,w)| \leq C_1\|v\|_{H^1(0,1)}\|w\|_{H^1(0,1)}$

$$|a(v,w)| = \left|\int_0^1 v'w'\,dx + \int_0^1 vw\,dx\right|$$
$$\leq \left(\int_0^1 |v'|^2\right)^{\frac{1}{2}}\left(\int_0^1 |w'|^2dx\right)^{\frac{1}{2}} + \left(\int_0^1 |v|^2dx\right)^{\frac{1}{2}}\left(\int_0^1 |w|^2dx\right)^{\frac{1}{2}}$$

can get bound of 1

$$\leq 2\|v\|_{H^1(0,1)}\|w\|_{H^1(0,1)}$$

(c) $\ell(\cdot)$ defined as $\ell(v) = \int_0^1 fv\,dx$ is a linear functional on $H_0^1(0,1)$ s.t

$\exists\ C_2 > 0\ \forall\ v \in H_0^1(0,1)$ s.t $|\ell(v)| \leq C_2\|v\|_{H^1(0,1)}$

$$|\ell(v)| = \left|\int_0^1 fv\right|dx$$
$$\leq \|f\|_{L^2(0,1)}\|v\|_{L^2(0,1)}$$
$$\leq \|f\|_{L^2(0,1)}\|v\|_{H^1(0,1)}$$

By ~~Max th~~ Lax-Milgram Theorem. $\exists!\ u \in H_0^1(0,1)$ s.t
$$a(u,v) = \ell(v) \quad \forall\ v \in H_0^1(0,1).$$

(b) $-u'' + u = f(x)$ for $x \in (0,1)$, $u(0)=0$, $u'(1)=0$.

- weak formulation: Find $u \in H^1_{E_0}(0,1) := \{v \in H^1(0,1),\ v(0)=0\}$

s.t.
$$\int_0^1 u'v'\,dx + \int_0^1 uv\,dx = \int_0^1 fv\,dx \quad \forall\ v \in H^1_{E_0}(0,1)$$

[$\int_0^1 u'v'\,dx + \int_0^1 uv\,dx + [-u'v]_{x=0}^{x=1} = \int_0^1 fv\,dx$

$\int_0^1 u'v'\,dx + \int_0^1 uv\,dx + u'(0)v(0) = \int_0^1 fv\,dx \Rightarrow$ chose $v \in H^1_{E_0}(0,1)$ so that $u'(0)v(0)=0$]

- Existence of weak solution:

since the bilinear ~~form~~ form on $H_{E_0}^1(0,1) \times H_{E_0}^1(0,1)$

~~is the sum~~ $a(u,v) = \int_0^1 u'v'\,dx + \int_0^1 uv\,dx$

and the linear functional on $H_{E_0}^1(0,1)$

$$\ell(v) = \int_0^1 fv\,dx$$

are the same as (a). it is sufficient to show that $H_{E_0}^1(0,1)$ is a closed subspace of $H^1(0,1)$.

① $H_{E_0}^1(0,1)$ is a subspace of $H^1(0,1)$ since

$\forall\ u, v \in H_{E_0}^1(0,1)$. $\alpha \in \mathbb{R}$.

$\alpha u + v \in H^1(0,1)$ and $\alpha u(0) + v(0) = 0 + 0$.

$\Rightarrow \alpha u + v \in H_{E_0}^1(0,1)$.

② let u_n be a sequence in $H_{E_0}^1(0,1)$ s.t

$u_n \to u$ in $H^1(0,1)$. then $u_n(0) = 0$

$v \in H^1(0,1) \mapsto v(0) \in \mathbb{R}$

$|v(0)| \le \max_{x\in(0,1)} |v(x)| \le C\|v\|_{H^1}$ $H^1(0,1) \subset C(0,1)$

$|u(0)| = |u_n(0) - u(0)| \le \max_{x\in(0,1)} |u_n(x) - u(x)| \le C\|u_n - u\|_{H^1(0,1)}$

$\to 0$ as $n \to \infty$

$\Rightarrow u(0) = 0$ i.e. $u \in H_{E_0}^1(0,1)$.

$\Rightarrow H_{E_0}^1(0,1)$ is a Hilbert space. with the H^1 norm.

Alternative, we can write the weak formulation as

find $u \in H^1(0,1)$ s.t

$$\int_0^1 u'v'\,dx + \int_0^1 uv\,dx = \int_0^1 fv - u'(0)v(0).$$

for all $v \in H^1(0,1)$.

$a(u,v) = \int_0^1 u'v'\,dx + \int_0^1 uv\,dx$ as before.

but $\ell(v) = \int_0^1 fv - u'(0)v(0)$

$|\ell(v)| \le \|f\|_{L^2(0,1)} \|v\|_{H^1(0,1)} + |u'(0)||v(0)|$

$\le C\|v\|_{H^1(0,1)}$.

Then we can apply the Lax-Milgram Theorem to conclude the existence of unique weak solution.

(c) $-u'' + u = f(x)$ for $x \in (0,1)$. $u(0)=0$, $u(1) + u'(1) = 0$

· weak formulation:

$$\int_0^1 u'v'\,dx + \int_0^1 uv\,dx + [-u'v]_{x=0}^{x=1} = \int_0^1 fv\,dx$$

$$\Leftrightarrow \int_0^1 u'v'\,dx + \int_0^1 uv\,dx + u'(0)v(0) - u'(1)v(1) = \int_0^1 fv\,dx$$

Consider the function space $H_{E_0}^1(0,1)$. ~~then~~ if $v \in H_{E_0}^1(0,1)$
$u'(0)v(0) = 0$.

Find $u \in H_{E_0}^1(0,1)$ s.t

$$\int_0^1 u'v'\,dx + \int_0^1 uv\,dx + v(1)u(1) = \int_0^1 fv\,dx$$

Existence of unique weak solutions.

$H_{E_0}^1(0,1)$ is a Hilbert space with the H^1-norm.

$a(u,v) := \int_0^1 u'v'\,dx + \int_0^1 uv\,dx + v(1)u(1)$

is a bilinear form on $H_{E_0}^1(0,1) \times H_{E_0}^1(0,1)$, s.t it is

(a) coercive since $a(v,v) = \int_0^1 |v'|^2 dx + \int_0^1 |v|^2 + |v(1)|^2$

$$\geq \|v\|^2_{H^1(0,1)}$$

and (b) bounded since

$$|a(u,v)| = \left|\int_0^1 u'v'\,dx + \int_0^1 uv\,dx + v(1)u(1)\right|$$

$$\leq \left|\int_0^1 u'v'\,dx + \int_0^1 uv\,dx\right| + |v(1)u(1)|$$

by (a) proof in part

$$\leq 2\|u\|_{H^1(0,1)} \|v\|_{H^1(0,1)} + |v(1)u(1)|$$

$$\leq 2\|u\|_{H^1(0,1)} \|v\|_{H^1(0,1)} + \max_{x \in (0,1)} |v(x)| \max_{x \in (0,1)} |u(x)|$$

$$\leq 2\|u\|_{H^1(0,1)} \|v\|_{H^1(0,1)} + \sqrt{2\|v\|_{L^2(0,1)}|v|_{H^1(0,1)}} \sqrt{2\|u\|_{L^2(0,1)}|u|_{H^1(0,1)}}$$

$$\leq 2\|u\|_{H^1(0,1)} \|v\|_{H^1(0,1)} + \sqrt{2}\|v\|_{H^1(0,1)} \sqrt{2}\|u\|_{H^1(0,1)}$$

can get 3

$$\leq 4\|u\|_{H^1(0,1)} \|v\|_{H^1(0,1)}.$$

nice!

$\ell(v) := \int_0^1 fv\,dx$ is the linear functional defined on $H_{E_0}^1(0,1)$

and $|\ell(v)| \leq \|f\|_{L^2(0,1)} \|v\|_{H^1(0,1)}$.

Then by Lax-Milgram Theorem $\exists!\ u \in H_{E_0}^1(0,1)$ s.t

$a(u,v) = \ell(v)$ $\forall v \in H_{E_0}^1(0,1)$.

$\|v\|_{L^2} \leq \frac{1}{\sqrt{2}} |v|_{H^1}$ poincaré

$$\max_{x \in [0,1]} |v(x)|^2 \leq 2 \|v\| |v|_{H^1}$$
$$\leq \sqrt{2} \cancel{\|v\|} |v|_{H^1}$$
$$\leq \sqrt{2} \|v\|_{H^1}$$

(b) $C = 1 + \sqrt{2}$