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B+

FEMs for PDEs - Problem Sheet 5

1. Consider the two-point boundary value problem

$$-u'' + u = f(x), \quad x \in (0,1), \qquad u(0) = 0, \ u(1) = 0,$$

with $f \in C^2[0,1]$. State the piecewise linear finite element approximation to this problem on a non-uniform mesh $0 = x_0 < x_1 < \ldots < x_N = 1$, $N \geq 2$, with $h_i = x_i - x_{i-1}$, assuming that, for a continuous piecewise linear function $v_h(x)$,

$$\int_0^1 f(x) v_h(x)\,\mathrm{d}x$$

has been approximated by applying the trapezium rule on each subinterval $[x_{i-1}, x_i]$.

Verify that the following *a posteriori* bound holds for the error between $u(x)$ and its finite element approximation $u_h(x)$:

$$\begin{aligned}\|u - u_h\|_{L_2(0,1)} \ &\leq \ K_0 \left(\sum_{i=1}^{N} h_i^4 \|R(u_h)\|^2_{L_2(x_{i-1},x_i)} \right)^{1/2} \\ &\quad + K_1 \max_{1\leq i \leq N} h_i^2 \left(\|f''\|^2_{C[x_{i-1},x_i]} + 4\|f'\|^2_{C[x_{i-1},x_i]} \right)^{1/2},\end{aligned}$$

where $R(u_h) = f(x) - (-u_h''(x) + u_h(x))$ for $x \in (x_{i-1}, x_i)$, $i = 1, \ldots, N$, and K_0, K_1 are constants which you should specify.

How would you use this bound to compute u to within a specified tolerance $\texttt{TOL} > 0$?

2. Let $u(x,t)$ denote the solution to the initial-boundary-value problem

$$\begin{aligned} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad & 0 < x < 1, \quad 0 < t \leq T, \\ u(0,t) = 0, \quad u(1,t) = 0, \qquad & 0 \leq t \leq T, \\ u(x,0) = u_0(x), \qquad & 0 < x < 1, \end{aligned}$$

where $T > 0$ and $u_0 \in L_2(0,1)$. Construct a finite element method for the numerical solution of this problem, based on the backward Euler scheme with time step $\Delta t = T/M$, $M \geq 2$, and a piecewise linear approximation in x on a uniform subdivision of the interval $[0,1]$ of spacing $h = 1/N$, $N \geq 2$, denoting by u_h^m the finite element approximation to $u(\cdot, t^m)$ where $t^m = m\Delta t$, $0 \leq m \leq M$.

Show that, for $0 \leq m \leq M-1$,

$$\|u_h^{m+1}\|^2_{L_2(0,1)} + 2\Delta t |u_h^{m+1}|^2_{H^1(0,1)} \leq \|u_h^m\|^2_{L_2(0,1)}.$$

Deduce that the method is unconditionally stable in the L_2 norm in the sense that, for any Δt, independent of the choice of h,

$$\|u_h^m\|_{L_2(0,1)} \leq (1 + 4\Delta t)^{-m/2} \|u_h^0\|_{L_2(0,1)}, \quad 1 \leq m \leq M. \qquad (\star)$$

Show further that

$$|u_h^m|_{H^1(0,1)} \leq \frac{\alpha}{\sqrt{4\Delta t}} (1 + 4\Delta t)^{-m/2} \|u_h^0\|_{L_2(0,1)}, \quad 1 \leq m \leq M, \qquad (\star\star)$$

where $\alpha > 0$ is certain constant (independent of h and Δt, but possibly dependent on T) that you should specify. [The Poincare-Friedrichs inequality, $\|v\|^2_{L_2(0,1)} \leq \frac{1}{2}|v|^2_{H^1(0,1)}$ for $v \in H_0^1(0,1)$, may be used without proof.]

Observe that for Δt small the right-hand side in $(\star\star)$ is much larger than in $(\star)$. The aim of the final part of this exercise is to prove that this behaviour is *not* an artifact of a crude analysis, and the bound $(\star\star)$ is sharp. For this purpose, choose $u_h^0(x)$ as a suitable constant multiple of the finite element basis function $\phi_{i_0}(x) = (1 - |(x - i_0 h)/h|)_+$ where i_0 is a fixed integer, $1 \leq i_0 < N$, so that $\|u_h^0\|_{L_2(0,1)} = 1$; then show that, for $\Delta t = \frac{1}{6}h^2$,

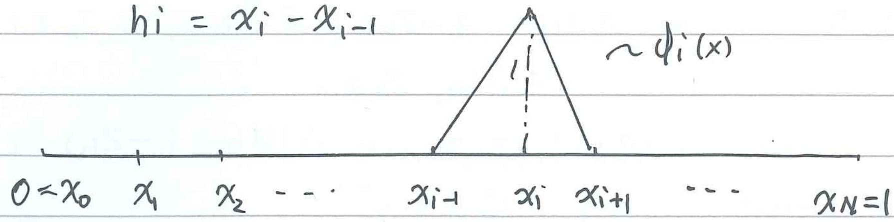
$$|u_h^1|_{H^1(0,1)} \geq \frac{\beta}{\sqrt{4\Delta t}} \geq \frac{\beta}{\sqrt{4\Delta t}} (1 + 4\Delta t)^{-1/2} \|u_h^0\|_{L_2(0,1)},$$

where β is a positive constant, $\beta < \alpha$.

FEMs for PDEs — Problem Sheet 5

Q1 $-u''+u=f(x) \quad x\in(0,1) \quad u(0)=0,\ u(1)=0$ with $f\in C^2[0,1]$.

$h_i = x_i - x_{i-1}$



$$\phi_i(x) = \begin{cases} 0 & \text{if } x\le x_{i-1}, \\ \dfrac{x-x_i}{h_i} & \text{if } x_{i-1}\le x\le x_i, \\ \dfrac{x_{i+1}-x}{h_{i+1}} & \text{if } x_i \le x\le x_{i+1} \\ 0 & \text{if } x_{i+1}\le x \end{cases}$$

for $i=1,2,\dots,N$.

$V_h = \mathrm{span}\{\phi_1,\dots,\phi_{N-1}\} \subseteq H_0^1(0,1)$

Let $a(u,v)=\int_0^1 u'v'+uv\,dx \qquad \ell(v)=\int_0^1 fv\,dx$

Find $u\in H_0^1(0,1)$ such that

(P) $\quad a(u,v)=\ell(v) \quad \forall\, v\in H_0^1(0,1)$.

The finite element approximation of this problem is:

Find $u_h\in V_h$ such that

$(P_h)\quad a(u_h,v_h)=\ell_h(v_h) \quad \forall\, v_h\in V_h$

where $\ell_h(v_h) = \sum_{i=1}^{N}\frac{h_i}{2}\left[(fv_h)(x_{i-1})+(fv_h)(x_i)\right]$

for all $v_h\in V_h$. [i.e. trapezium rule].

Note that $R(u_h) = f(x)-(-u_h''(x)+u_h(x))$ for $x\in(x_{i-1},x_i)$ $i=1,2,\dots N$.

Consider the dual problem

$$\begin{cases} -z''+z = u-u_h & x\in(0,1) \\ z(0)=0,\ z(1)=0. \end{cases}$$

Then
$$\begin{aligned}\|u-u_h\|^2_{L_2(0,1)} &= (u-u_h,\,u-u_h) \\ &= (u-u_h,\,-z''+z) \\ &= \int_0^1 (u-u_h)\cdot(-z''+z)\,dx \\ &= \int_0^1 (u-u_h)'z' + (u-u_h)z\,dx \\ &= a(u-u_h, z)\end{aligned}$$

$\therefore\ \|u-u_h\|^2_{L_2(0,1)} = a(u-u_h,z).$

Let $z_h := I_h z \in V_h$.

$$a(u-u_h, z) = a(u-u_h, z-I_h z) + a(u-u_h, I_h z)$$
$$= a(u, z-I_h z) - a(u_h, z-I_h z) + a(u, I_h z) - a(u_h, I_h z)$$
$$= a(u, z-z_h) - a(u_h, z-z_h) + \ell(z_h) - \ell_h(z_h).$$

First note that

$$a(u_h, z-z_h) = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} u_h'(z-z_h)'\,dx + \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} u_h(z-z_h)\,dx$$

$$= \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (-u_h'' + u_h)(z-z_h)\,dx$$

since $(z-z_h)(x_i) = 0$ for $i=1,2,\dots N$

I say that $u_h''=0$ due to linear piecewise elements

$$= \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (f(x) - R(u_h))(z-z_h)\,dx$$

$$= \ell(z-z_h) - \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} R(u_h)(z-z_h)\,dx$$

$$\Rightarrow a(u-u_h, z) = \underbrace{\sum_{i=1}^{4} \int_{x_{i-1}}^{x_i} R(u_h)(z-z_h)\,dx}_{\mathrm{I}} + \underbrace{\ell(z_h) - \ell_h(z_h)}_{\mathrm{II}}.$$

$$\mathrm{I} = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} R(u_h)(z-z_h)\,dx$$

$$\leq \sum_{i=1}^{N} \cancel{\int_{x_{i-1}}^{x_i}} \|R(u_h)\|_{L_2(x_{i-1}, x_i)} \|z-z_h\|_{L_2(x_{i-1}, x_i)}$$

$$= \sum_{i=1}^{N} \cancel{\int_{x_{i-1}}^{x_i}} \|R(u_h)\|_{L_2(x_{i-1}, x_i)} \|z-I_h z\|_{L_2(x_{i-1}, x_i)}$$

(by thm 3 from pg 48)
$$\leq \sum_{i=1}^{N} \cancel{\int_{x_{i-1}}^{x_i}} \|R(u_h)\|_{L_2(x_{i-1}, x_i)} \left(\frac{h_i}{\pi}\right)^2 \|z''\|_{L_2(x_{i-1}, x_i)}$$

$$\leq \left(\sum_{i=1}^{N} \frac{h_i^4}{\pi^4} \|R(u_h)\|^2_{L_2(x_{i-1}, x_i)}\right)^{1/2} \times \left(\sum_{i=1}^{N} \|z''\|^2_{L_2(x_{i-1}, x_i)}\right)^{\frac{1}{2}}$$

$$= \left(\sum_{i=1}^{N} \frac{h_i^4}{\pi^4} \|R(u_h)\|^2_{L_2(x_{i-1}, x_i)}\right)^{\frac{1}{2}} \|z''\|_{L_2(0,1)}$$

Now we try to eliminate $\|z''\|_{L_2(0,1)}$

$$\begin{cases} -z'' + z = u - u_h, \\ z(0) = 0, \quad z(1) = 0. \end{cases}$$

Multiplying by z and integrating, we obtain

$$\int_0^1 -z''\cdot z\,dx + \int_0^1 z^2\,dx = \int_0^1 z(u-u_n)$$

$$\Rightarrow \|z'\|^2_{L_2(0,1)} + \|z\|^2_{L_2(0,1)} \le \|z\|_{L_2(0,1)}\,\|u-u_n\|_{L_2(0,1)}$$

$$\Rightarrow \|z'\|^2_{L_2(0,1)} \le \|z\|_{L_2(0,1)}\,\|u-u_n\|_{L_2(0,1)}$$

Poincaré - Friedrichs inequality tells us that

Poincaré bound

$$\|z\|^2_{L_2(0,1)} \le \frac{1}{\pi^2}\|z'\|^2_{L_2(0,1)}$$

z' by z'' not other way around!

$$\Rightarrow \|z\|_{L_2(0,1)} \le \frac{1}{\pi^2}\pi^2\,\|u-u_n\|_{L_2(0,1)}$$

wrong way $\|z''\|_2 \geq \pi^2\|z'\|_2$

$$\|z'\|_{L_2(0,1)} \le \cancel{\tfrac{1}{\pi}}\pi\,\|u-u_n\|_{L_2(0,1)}$$

but can actually show $\|z''\|_2 \le \|u-u_n\|_2$ if you are careful

$$\|z''\|_{L_2(0,1)} \le \|z\|_{L_2(0,1)} + \|u-u_n\|_{L_2(0,1)}$$
$$\le (1+\cancel{\tfrac{1}{\pi^2}})\,1\,\|u-u_n\|_{L_2(0,1)}.$$

$$\Rightarrow I \le \frac{(1+\frac{1}{\pi^2})}{\pi^2}\left(\sum_{i=1}^N h_i^4\,\|R(u_h)\|^2_{L_2(x_{i-1},x_i)}\right)^{1/2}\|u-u_n\|_{L_2(0,1)}$$

$$= K_0\left(\sum_{i=1}^N h_i^4\,\|R(u_h)\|^2_{L_2(x_{i-1},x_i)}\right)^{1/2}\|u-u_n\|_{L_2(0,1)}$$

where $K_0 = (1+\frac{1}{\pi^2})/\pi^2$. & you get $K_0 = 1/\pi^2$.

$$II = \ell(z_n) - \ell_n(z_n)$$
$$= \sum_{i=1}^N \left\{ \int_{x_{i-1}}^{x_i} f z_n\,dx - \frac{h_i}{2}\left[(fz_n)(x_{i-1}) + (fz_n)(x_i)\right]\right\}$$

$$= \sum_{i=1}^N \int_{x_{i-1}}^{x_i} f z_n\,dx - \int_{x_{i-1}}^{x_i} I_h(f(z_n))(x)\,dx$$

$$= \sum_{i=1}^N \int_{x_{i-1}}^{x_i} (fz_n - I_h(fz_n))\,dx$$

$$|\ell(z_n) - \ell_n(z_n)| \le \sum_{i=1}^N \int_{x_{i-1}}^{x_i} |fz_n - I_h(fz_n)|\,dx$$

$$\le \sum_{i=1}^N \sqrt{h_i}\left(\int_{x_{i-1}}^{x_i} |fz_n - I_h(f(z_n))|^2\,dx\right)^{\frac{1}{2}}$$

$$\le \left(\sum_{i=1}^N h_i\right)^{1/2}\left(\sum_{i=1}^N \int_{x_{i-1}}^{x_i} |fz_n - I_h(fz_n)|^2\,dx\right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{i=1}^{N} \frac{h_i^4}{\pi^4} \| (f z_h)'' \|^2_{L_2(x_{i-1}, x_i)} \right)^{1/2}$$ By thm 3 from Pg 48

$$(f z_h)'' = f'' z_h + 2f' z_h' + \cancel{f z_h''} + f z_h''$$

However, z_h is a linear functional on each interval (x_{i-1}, x_i) $i = 1, 2, \ldots, N$ so $z_h'' \equiv 0$ on (x_{i-1}, x_i) $i = 1, 2, \ldots, N$.

$$\therefore \; |\ell(z_h) - \ell_h(z_h)| \leq \left(\sum_{i=1}^{N} \frac{h_i^4}{\pi^4} \| f'' z_h + 2f' z_h' \|^2_{L_2(x_{i-1}, x_i)} \right)^{\frac{1}{2}}$$

$$\leq \max_{1 \leq i \leq N} \frac{h_i^2}{\pi^2} \left(\sum_{i=1}^{N} \| f'' z_h \|^2_{L_2(x_{i-1}, x_i)} + 4 \sum_{i=1}^{N} \| f' z_h' \|^2_{L_2(x_{i-1}, x_i)} \right)^{\frac{1}{2}}$$

✓✓

$$\cancel{\leq \max_{1 \leq i \leq N} \frac{h_i^2}{\pi^2} \left(\| f'' \|^2_{L_2[0,1]} + 4 \| z_h \|^2_{L_2(x_{i-1}, x_i)} \right)}$$

$$\leq \max_{1 \leq i \leq N} \frac{h_i^2}{\pi^2} \left[\left(\sum_{i=1}^{N} \| f'' \|^2_{C[x_{i-1}, x_i]} \| z_h \|^2_{L_2(x_{i-1}, x_i)} \right) + \sum_{i=1}^{N} 4 \| f' \|^2_{C[x_{i-1}, x_i]} \| z_h' \|^2_{L_2(x_{i-1}, x_i)} \right]^{\frac{1}{2}}$$

✓ ✓

Note that $\| z_h \|_{L_2(x_{i-1}, x_i)} \leq \| z \|_{L_2(x_{i-1}, x_i)}$? Is this true? ✗

$$\sum_{i=1}^{N} \| z_h \|^2_{L_2(x_{i-1}, x_i)} \leq \| z \|^2_{L_2(0,1)} \leq \frac{1}{\pi^4} \| u - u_h \|^2_{L_2(0,1)}$$

$$\sum_{i=1}^{N} \| z_h \|^2_{L_2(x_{i-1}, x_i)} \leq \| z' \|^2_{L_2(0,1)} \leq \frac{1}{\pi^2} \| u - u_h \|^2_{L_2(0,1)}$$

$$\Rightarrow \; |\ell(z_h) - \ell_h(z_h)| \leq \max_{1 \leq i \leq N} \frac{h_i}{\pi^2} \left(\| f'' \|^2_{C(x_{i-1}, x_i)} + 4 \| f' \|_{C(x_{i-1}, x_i)} \right) \left[\frac{1}{\pi} \| u - u_h \|_{L^2(0,1)} \right)$$

$$= K_1 \max_{1 \leq i \leq N} h_i^2 \left(\| f'' \|^2_{C[x_{i-1}, x_i]} + 4 \| f' \|^2_{C[x_{i-1}, x_i]} \right)^{\frac{1}{2}} \| u - u_h \|_{L_2(0,1)}$$

$K_1 = \frac{1}{\pi^3}$. end up with $K_1 = \frac{1}{\pi^2}\left(1 + \frac{L}{\pi}\left(1 + \frac{1}{\pi^2}\right)^{1/2}\right)$.

$$\Rightarrow \quad \|u-u_h\|^2_{L_2(0,1)} = a(u-u_h, z)$$

$$\leq \Big[K_0 \Big(\sum_{i=1}^{N} h_i^4 \|R(u_h)\|^2_{L_2(x_{i-1},x_i)} \Big)^{1/2} + K_1 \max_{1\leq i\leq N} h_i^2 \Big(\|f''\|^2_{C[x_{i-1},x_i]} + 4\|f'\|^2_{C[x_{i-1},x_i]} \Big)^{\frac{1}{2}} \Big]$$

$$\times \|u-u_h\|_{L_2(0,1)}$$

$$\Rightarrow \quad \|u-u_h\|_{L_2(0,1)} \leq K_0 \Big(\sum_{i=1}^{N} h_i^4 \|R(u_h)\|^2_{L_2(x_{i-1},x_i)} \Big)^{1/2} + K_1 \max_{1\leq i\leq N} h_i^2 \Big(\|f''\|^2_{C[x_{i-1},x_i]} + 4\|f'\|^2_{C[x_{i-1},x_i]} \Big)^{\frac{1}{2}}$$

where $K_0 := \frac{1}{\pi^2} + \frac{1}{\pi^4}$ $\quad K_1 := \frac{1}{\pi^3}$.

Given $TOL > 0$, we want to compute u_h such that $\|u-u_h\|_{L_2(0,1)} \leq TOL$.

This will hold, provided that

$$(1) \quad K_0 \Big(\sum_{i=1}^{N} h_i^4 \|R(u_h)\|^2_{L_2(x_{i-1},x_i)} \Big)^{1/2} + K_1 \max_{1\leq i\leq N} h_i^2 \Big(\|f''\|^2_{C[x_{i-1},x_i]} + 4\|f'\|_{C[x_{i-1},x_i]} \Big)^{1/2}$$

$$\leq TOL \qquad (\text{"stopping criterion"}).$$

<u>Algorithm:</u>

① Choose initial mesh. Compute u_h.
Calculate $R(u_h)$. ~~$\|f''\|_{C[0,1]}$ and $\|f'\|_{C[0,1]}$~~

② If (1) is TRUE, then END. (Check ~~the~~ "Stopping Criterion")
If (1) is FALSE, halve the mesh-size to obtain new mesh and go back to ①.

Q2 $\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} & 0<x<1 \quad 0<t\le T \\ u(0,t)=0, \ u(1,t)=0 & 0\le t\le T \\ u(x,0)=u_0(x) & 0<x<1. \end{cases}$

where $T>0$ and $u_0 \in L_2(0,1)$.

We begin by constructing a mesh on $\bar{Q} = [0,1]\times[0,T]$ $\Delta t = T/M$ $M\ge 2$. is the time step. $h=\frac{1}{N}$, $N\ge 2$ is the mesh-size in x-direction. We define the uniform mesh $\overline{Q_h}^{\Delta t}$ on $\bar{Q}$ by

$$\overline{Q_h}^{\Delta t} = \{(x_j, t^m) : x_j = jh,\ 0\le j\le N,\ t^m = m.\Delta t\ 0\le m\le M\}.$$

Let $V_h \subset H_0^1(0,1)$ denote the set of all continuous piecewise linear functions defined on the x-mesh $0=x_0<x_1<\dots<x_{N-1}<x_N=1$ which vanish at the end points $x=0$ and $x=1$.

Denote by U_h^m the finite element approximation to $u(\cdot,t^m)$ where $t^m = m\Delta t$, $0\le m\le M$.

Backward Euler Scheme:

Find $U_h^m \in V_h$ $0\le m\le M$ s.t

$$(+)\begin{cases} \left(\frac{U_h^{m+1}-U_h^m}{\Delta t}, v_h\right) + a(U_h^{m+1}, v_h) = 0 \text{ for all } v_h\in V_h \\ \text{where } a(\cdot,\cdot) \text{ is defined by} \\ \quad a(u,v) = \int_0^1 u'v'\,dx. \\ \text{and } (U_h^0 - u_0, v_h)=0 \quad \forall\ v_h\in V_h. \end{cases}$$

We can rewrite $(+)$ as

$$(U_h^{m+1}, v_h) + \Delta t\, a(U_h^{m+1}, v_h) = (U_h^m, v_h) \quad \forall v_h\in V_h$$

$0\le m\le M-1$ with $(U_h^0, v_h) = (u_0, v_h)$ $\forall v_h \in V_h$.

Let $\{\phi_i\}_{i=1}^{N-1}$ denote the 1-D piecewise linear finite element basis function associated with the x-mesh points x_i that is $V_h = \text{span}\{\phi_i\}$ ~~$i=$~~ for $1\le i\le N-1$. Then $U_h^{m+1} = \sum_{i=1}^{N-1} U_i^{m+1}\phi_i$

If we choose $v_h=\phi_i$ for each i. then

$$AU^{m+1} = b$$

where $U^{m+1} = [U_1^{m+1}, \dots U_{N-1}^{m+1}]^T$.

A is a symmetric positive definite matrix of size $(N-1)\times(N-1)$

with $A_{ij} = (\phi_i, \phi_j) + \Delta t\, (\phi_i', \phi_j')$

That is $A = M + \Delta t K$ where

$$M = \frac{h}{6}\begin{pmatrix} 4 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 4 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 4 & 1 & \cdots & 0 \\ \cdots & \cdots & & \cdots & & \cdots \\ 0 & 0 & \cdots & & 1 & 4 \end{pmatrix} \qquad K = \frac{1}{h}\begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \cdots & \cdots & & \cdots & & \cdots \\ 0 & & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$b_i = \sum_{j=1}^{N-1} U_j^m (\phi_j, \phi_i)$$

Note that $(u_h^{m+1}, v_h) + \Delta t\, a(u_h^{m+1}, v_h) = (u_h^m, v_h)$ $(++)$

for all $v_h \in V_h$, $0 \le m \le M-1$.

and $(u_h^0, v_h) = (u_0, v_h)$ $\forall v_h \in V_h$.

Taking the inner product of $(++)$ with u_h^{m+1}, we have

$$(u_h^{m+1}, u_h^{m+1}) + \Delta t\, a(u_h^{m+1}, u_h^{m+1}) = (u_h^m, u_h^{m+1})$$

$$\|u_h^{m+1}\|_{L_2(0,1)}^2 + \Delta t\, |u_h^{m+1}|_{H^1(0,1)}^2 \le \|u_h^{m+1}\|_{L_2(0,1)} * \|u_h^m\|_{L_2(0,1)}$$

$$\le \frac{1}{2}\|u_h^{m+1}\|_{L_2(0,1)}^2 + \frac{1}{2}\|u_h^m\|_{L_2(0,1)}^2$$

$$\Rightarrow \frac{1}{2}\|u_h^{m+1}\|_{L_2(0,1)}^2 + \Delta t\, |u_h^{m+1}|_{H^1(0,1)}^2 \le \frac{1}{2}\|u_h^m\|_{L_2(0,1)}^2$$

$$\Rightarrow \|u_h^{m+1}\|_{L_2(0,1)}^2 + 2\Delta t\, |u_h^{m+1}|_{H^1(0,1)}^2 \le \|u_h^m\|_{L_2(0,1)}^2 \quad (+)$$

Note that Poincare - Friedrichs inequality tells us that

$$\|v\|_{L_2(0,1)}^2 \le \frac{1}{2}|v|_{H^1(0,1)}^2 \quad \text{for } v \in H_0^1(0,1).$$

$$\Rightarrow \|u_h^{m+1}\|_{L_2(0,1)}^2 \le \frac{1}{2}|u_h^{m+1}|_{H^1(0,1)}^2$$

$$\Rightarrow \|u_h^{m+1}\|_{L_2(0,1)}^2 + 4\Delta t\, \|u_h^{m+1}\|_{L_2(0,1)}^2 \le \|u_h^{m+1}\|_{L_2(0,1)}^2 + 2\Delta t\, |u_h^{m+1}|_{H^1(0,1)}^2$$

$$\le \|u_h^m\|_{L_2(0,1)}^2$$

$$\Rightarrow (1+4\Delta t)\ \|u_h^{m+1}\|_{L_2(0,1)}^2 \le \|u_h^m\|_{L_2(0,1)}^2$$

$$\Rightarrow \|u_h^{m+1}\|_{L_2(0,1)}^2 \le (1+4\Delta t)^{-1}\ \|u_h^m\|_{L_2(0,1)}^2$$

$$\Rightarrow \|u_h^{m+1}\|_{L_2(0,1)} \le (1+4\Delta t)^{-1/2}\ \|u_h^m\|_{L_2(0,1)}$$

$$\le (1+4\Delta t)^{-\frac{2}{2}}\ \|u_h^{m-1}\|_{L_2(0,1)}$$

$$\le \dots$$

$$\le (1+4\Delta t)^{-\frac{m+1}{2}}\ \|u_h^0\|_{L_2(0,1)}$$

$$\Rightarrow \|u_h^m\|_{L_2(0,1)} \le (1+4\Delta t)^{-m/2}\ \|u_h^0\|_{L_2(0,1)} \qquad 1\le m\le M \qquad (*)$$

Note that from (+), we have

$$|u_h^{m+1}|_{H^1(0,1)}^2 \le \frac{1}{2\Delta t}\|u_h^m\|_{L_2(0,1)}^2$$

$$\Rightarrow |u_h^m|_{H^1(0,1)} \le \frac{1}{\sqrt{2\Delta t}}\|u_h^{m-1}\|_{L_2(0,1)}$$

$$\le \frac{1}{\sqrt{2\Delta t}}(1+4\Delta t)^{-\frac{m-1}{2}}\|u_h^0\|_{L_2(0,1)}$$

$$\le \frac{1}{\sqrt{4\Delta t}}(1+4\Delta t)^{-\frac{m}{2}}\sqrt{2}\,(1+4\Delta t)^{\frac{1}{2}}\|u_h^0\|_{L_2(0,1)}$$

$T = M\Delta t$

$\ge 2\Delta t$ as $M \ge 2$

$$\le \frac{\sqrt{2}\,(1+2T)^{\frac{1}{2}}}{\sqrt{4\Delta t}}(1+4\Delta t)^{-\frac{m}{2}}\|u_h^0\|_{L_2(0,1)}$$

$$\alpha = \sqrt{2}\,(1+2T)^{\frac{1}{2}}$$

$\phi_{i_0}(x) = \left(1-\left|\frac{(x-i_0h)}{h}\right|\right)_+$ $\quad i_0$ is a fixed integer $\quad 1\le i_0\le N$

Let $u_h^0 = c\,\phi_{i_0}(x)$

Then $1 = \|u_h^0\|_{L_2(0,1)} = \left(\int_0^1 c^2\phi_{i_0}(x)^2dx\right)^{\frac{1}{2}} = \left(c^2\frac{2h}{3}\right)^{\frac{1}{2}}$

$$\Rightarrow c^2 = \frac{3}{2h}$$

Since u_h^0 is piecewise linear $\quad \frac{u_h^1-u_h^0}{\Delta t} \approx \cancel{\frac{\partial^2 u_h^0}{\partial x^2}}\ \cancel{=0}\ \frac{\partial^2 u_h^0}{\partial x^2} = 0$

$\Rightarrow u_h^1 = u_h^0$

$$|u_h^1|_{H^1(0,1)} = \left(\int_0^1 c^2|\phi_{i_0}'(x)|^2dx\right)^{\frac{1}{2}}$$

$$= \frac{\sqrt{3}}{\sqrt{2}\sqrt{h}}\ \frac{\sqrt{2}}{\sqrt{h}} = \frac{\sqrt{3}}{h}$$

$\Delta t = \frac{1}{6}h^2$ $\qquad \frac{\beta}{\sqrt{4\Delta t}} = \frac{\beta}{\sqrt{\frac{4}{6}h^2}}$

$$= \frac{\beta\sqrt{3}}{\sqrt{2}\,h}$$

$|u_h'|_{H^1(0,1)} = \frac{\sqrt{3}}{h}$ not quite but good attempt!

$$\Rightarrow \quad \beta = \sqrt{2} \quad < \alpha = \sqrt{2}\,(1+2T)^{\frac{1}{2}}$$

$$\Rightarrow \quad |u_h'|_{H^1(0,1)} \geq \frac{\beta}{\sqrt{4\Delta t}} \geq \frac{\beta}{\sqrt{4\Delta t}}\,(1+4\Delta t)^{-1/2}\,\|u_h^0\|_{L_2(0,1)}$$

FEMs for PDEs - Problem Sheet 5

$$\begin{cases} -u''+u=f & f\in C^2[0,1] \\ u(0)=0=u(1) \end{cases}$$

$a(u,v) = \int_0^1 u'v' \, dx + \int_0^1 uv \, dx \qquad \ell(v) = \int_0^1 fv$

$$\ell_h(v_h) = \sum_{i=1}^{N} \frac{h_i}{2}\left(f(x_{i-1}) v_h(x_{i-1}) + f(x_i) v_h(x_i) \right)$$

$$= \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} I_h(f v_h)$$

$a(u,v) = \ell(v)$.

$a(u_h, v_h) = \ell_h(v_h)$

$$\begin{cases} -z''+z = u-u_h \\ z(0)=0=z(1) \end{cases} \qquad \|u-u_h\|_{L^2}^2 = a(u-u_h, z)$$

$$= \underbrace{a(u-u_h, z-z_h)}_{I} + \underbrace{\ell(z_h)-\ell_h(z_h)}_{II}.$$

$$I = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} R(u_h)(z-z_h)$$

$$|I| \le \sum_{i=1}^{N} \|R(u_h)\|_{L^2(x_{i-1},x_i)} \frac{h_i^2}{\pi^2} \|z''\|_{L^2(x_{i-1},x_i)}$$

$$\le \frac{1}{\pi^2}\left(\sum_{i=1}^{N} h_i^4 \|R(u_h)\|^2_{L^2(x_{i-1},x_i)} \right)^{1/2} \|z''\|_{L^2(0,1)} \quad --(1)$$

$$|II| = \left| \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (f z_h - I_h(f z_h)) \right|$$

$$\le \sum_{i=1}^{N} \sqrt{h_i} \left(\int_{x_{i-1}}^{x_i} |f z_h - I_h(f z_h)|^2 \right)^{1/2}$$

$$\le \left(\sum_{i=1}^{N} \frac{h_i^4}{\pi^4} \|\underbrace{(f z_h)''}_{\substack{\| \\ f'' z_h + 2f' z_h'}}\|^2_{L^2(x_{i-1},x_i)} \right)^{\frac{1}{2}}.$$

$$\leq \left(\sum_{i=1}^{N} \frac{h_i^4}{\pi^4} \left(\| f'' z_h \|_{L^2} + \| 2f' z_h' \|_{L^2} \right)^2 \right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{i=1}^{N} \frac{h_i^4}{\pi^4} \| f'' z \|^2_{L^2(x_{i-1}, x_i)} \right)^{\frac{1}{2}} + \left(\sum_{i=1}^{N} \frac{h_i^4}{\pi^4} \| 2f' z' \|^2_{L^2(x_{i-1}, x_i)} \right)^{\frac{1}{2}}$$

$$\leq \max_{i \in \{1,\dots,N\}} \frac{h_i^2}{\pi^2} \| f'' \|_{C[x_{i-1}, x_i]} \| z_h \|_{L^2(0,1)}$$

$$+ \max_i \frac{h_i^2}{\pi^2} \| 2f' \|_{C[x_{i-1}, x_i]} \| z_h' \|_{L^2(0,1)}$$

$$\leq \max_i \frac{h_i^2}{\pi^2} \left(\| f'' \|^2_{C[x_{i-1}, x_i]} + 4 \| f' \|^2_{C[x_{i-1}, x_i]} \right)^{\frac{1}{2}} \| z_h \|_{H^1(0,1)}$$

$$\| z_h \| \leq \| z - z_h \| + \| z \|_{H^1(0,1)}$$

$$\leq \frac{1}{\pi} \left(1 + \frac{1}{\pi}\right)^{1/2} \| z'' \|_{L^2(0,1)} + \| z \|_{H^1(0,1)}$$

Testing with z: $\| z \|_{H^1(0,1)} \leq \| u - u_h \|_{L^2(0,1)}$

Testing with $-z''$. $\Rightarrow \| z'' \|_{L^2(0,1)} \leq \| u - u_h \|_{L^2(0,1)}$

KEY PART

$$(-z'' + z, -z'') = \| z'' \|^2_{L^2(0,1)} + \| z' \|^2_{L^2(0,1)}$$

$$\shortparallel \qquad \Rightarrow$$

$$(u - u_h, -z'') \leq \| u - u_h \|_{L^2(0,1)} \| z'' \|_{L^2(0,1)}$$

$$\Rightarrow \| z'' \|_{L^2(0,1)} \leq \| u - u_h \|_{L^2}$$

$$\| u - u_h \|_{L^2(0,1)} \leq K_0 \left(\sum_i h_i^4 \| R(u_h) \|^2_{L^2(x_{i-1}, x_i)} \right)^{\frac{1}{2}}$$

$$+ K_1 \max_i h_i^2 \left(\| f'' \|^2_{C[x_{i-1}, x_i]} + 4 \| f \|^2_{C[x_{i-1}, x_i]} \right)^{1/2}$$

$$K_0 = \frac{1}{\pi^2}, \quad K_1 = \frac{1}{\pi^2} \left(1 + \frac{1}{\pi} \left(1 + \frac{1}{\pi}\right)^{1/2} \right)$$

$$\varepsilon^2 \le 2\sum_i \big(k_0^2 h_i^4 \| R(u_h) \|^2_{L^2(x_{i-1}, x_i)} + k_1^2 h_i^4 (\| f'' \|^2_{C_i} + 4 \| f' \|^2_{C_i}) \big)$$

$$\underbrace{\qquad}_{R_i}$$

If $R_i > \frac{TOL^2}{N}$ refine $[x_{i-1}, x_i]$.

Q2

$\Delta t = T/M$. $\quad h = \frac{1}{N}$

$$\| u_h^{m+1} \|^2_{L^2} + 2\Delta t \, | u_h^{m+1} |^2_{H^1} \le \| u_h^m \|^2_{L^2}$$

$$\| u_h^m \|_{L^2} \le (1 + 4\Delta t)^{-m/2} \| u_h^0 \|_{L^2}$$

$$| u_h^m |_{H^1} \le \frac{\alpha}{\sqrt{1+4\Delta t}} (1 + 4\Delta t)^{-m/2} \| u_h^0 \|_{L^2}$$

$$\alpha = \sqrt{2} (1 + 4T)^{\frac{1}{2}} \quad \left(\text{or } \alpha = \sqrt{2} (1 + 2T)^{\frac{1}{2}} \right).$$

$u_h^0 = \gamma \varphi_{i_0}$ choose γ s.t. $\| u_h^0 \|_{L^2} = 1$.

$$\Rightarrow \gamma = \sqrt{\frac{3}{2h}}$$

$$u_h^1 = \sum_{j=1}^{N-1} U_j^1 \varphi_j \rightsquigarrow M U^1 + \frac{\Delta t}{h^2} S U^1 = M U^0.$$

$$M = \frac{1}{6} \begin{pmatrix} 4 & 1 & & \\ 1 & 4 & 1 & 0 \\ & 0 & & \ddots \end{pmatrix} \qquad S = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & 0 \\ & & \ddots & \\ & 0 & & \ddots \end{pmatrix}$$

choose $\frac{\Delta t}{h^2} = 1/6$

$$\Rightarrow U^1 = M U^0$$

$$\|u_h^1\|_{H^1(0,1)}^2 = \frac{1}{h} u^{1T} S u^1$$

$$= \frac{1}{h} u^{0T} MSMu^0$$

$$u_j^0 = \gamma \delta_{10,j}$$

$$\|u_h^1\|_{H^1(0,1)}^2 = \frac{1}{h} \gamma^2 \underbrace{(MSM)_{10,10}}_{\frac{30}{36}} = \frac{5}{6h^2}$$

$$\|u_h^1\|_{H^1} = \frac{\sqrt{5}}{6\sqrt{\Delta t}} = \frac{2\sqrt{5}}{6} \frac{1}{\sqrt{4\Delta t}}$$

$$= \frac{\beta}{\sqrt{4\Delta t}} \|u_h^0\|_{L^2}$$

$$\geq \frac{\beta}{\sqrt{4\Delta t}} (1+4\Delta t)^{-\frac{1}{2}} \|u_h^0\|_{L^2}.$$

$$\beta = \frac{2\sqrt{5}}{6} < \alpha$$