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α-

FEMs for PDEs - Problem Sheet 2

1. Given that α is a non-negative real number and $f \in L^2(0,1)$, consider the boundary value problem

$$-u'' + u = f(x) \quad \text{for } x \in (0,1), \qquad u(0) = 0, \quad \alpha u(1) + u'(1) = 0.$$

State the weak formulation of the problem.

Using continuous piecewise linear basis functions on a uniform mesh of size $h = 1/N$, $N \geq 2$, write down the finite element approximation to this problem and show that this has a unique solution u_h.

Show that

$$\|u - u_h\|_{H^1(0,1)} \leq Ch|u|_{H^2(0,1)},$$

where C is a positive constant.

Expand u_h in terms of the finite element basis functions ϕ_i, where $\phi_i(x) = (1 - |x - x_i|/h)_+$, $i = 1, \ldots, N$, by writing

$$u_h(x) = \sum_{i=1}^{N} U_i \phi_i(x)$$

to obtain a system of linear equations for the vector of unknowns $(U_1, \ldots, U_N)^T$.

Suppose that $\alpha = 0$, $f(x) \equiv 1$ and $h = 1/3$. Solve the resulting system of linear equations and compare the corresponding numerical solution $u_h(x)$ with the exact solution $u(x)$. $u_{exact} = 1$

2. Consider the elliptic equation

$$-\Delta u = f(x,y) \quad \text{for } (x,y) \in \Omega = (0,1)^2$$

with $f \in L^2(\Omega)$, subject to the homogeneous Dirichlet boundary condition $u = 0$ on

$$\Gamma_D = \{(x,y) \in \partial\Omega : x = 0 \text{ or } y = 0 \text{ or } y = 1\}$$

and non-homogeneous Neumann boundary condition $\frac{\partial u}{\partial x} = 1$ on

$$\Gamma_N = \{(x,y) \in \partial\Omega : x = 1\}.$$

State the weak formulation of the problem.

Consider a triangulation of Ω which has been obtained from a square mesh of spacing $h = 1/N$, $N \geq 2$, in both co-ordinate directions by subdividing each square into two triangles with the diagonal of negative slope. Using continuous piecewise linear basis functions on this triangulation, state the finite element approximation to the boundary value problem. Rewrite the finite element method as a system of linear algebraic equations and comment on the structure of the matrix.

1)

FEMs for PDEs - Problem Sheet 2

Q1 $\alpha \in \mathbb{R}$, $\alpha \geq 0$, $f \in L^2(0,1)$

$-u'' + u = f(x)$ for $x \in (0,1)$, $u(0)=0$, $\alpha u(1) + u'(1) = 0$.

- The <u>weak formulation</u>:

find $u \in H^1_{E_0}(0,1)$ s.t.

$$\int_0^1 u'v'\,dx + \int_0^1 uv\,dx + \alpha u(1)v(1) = \int_0^1 fv\,dx$$

for all $v \in H^1_{E_0}(0,1)$

where $H^1_{E_0}(0,1) := \{ v \in H^1_{E_0}(0,1) : v(0) = 0 \}$.

let $a(u,v) := \int_0^1 u'v'\,dx + \int_0^1 uv\,dx + \alpha u(1)v(1)$

and $\ell(v) := \int_0^1 fv\,dx$, then we can rewrite the problem as follows:

find $u \in H^1_{E_0}(0,1)$ s.t $a(u,v) = \ell(v)$ $\forall v \in H^1_{E_0}(0,1)$. (P)

- Now consider the finite-dim subspace $V_h \subset H^1_{E_0}(0,1) \subset H^1(0,1)$ which consists of continuous piecewise linear polynomials on a uniform mesh of size $h = \frac{1}{N}$, $N \geq 2$.

$\dim V_h = N(h)$ and $V_h = \text{span}\{\phi_1, \dots \phi_{N(h)}\}$

where ϕ_i, $i=1, \dots N(h)$ are the piecewise linear ~~function~~ basis functions with "small" support.

Then the <u>finite element approximation</u> to this problem is:

find $u_h \in V_h$ s.t.

$a(u_h, v_h) = \ell(v_h)$ $\forall v_h \in V_h$ (P_h).

- <u>Existence of unique solution u_h.</u>

V_h is a finite-dim subspace of a Hilbert space $\Rightarrow$ V_h is a Hilbert space.

$$a(u,u) = \int_0^1 u'u'\,dx + \int_0^1 uu\,dx + \alpha u(1)^2$$

$$\geq \|u\|^2_{H^1(0,1)}$$

$\Rightarrow a(\cdot,\cdot)$ is coercive.

$$a(u,v) = \int_0^1 u'v'\,dx + \int_0^1 uv\,dx + \alpha u(1)v(1)$$

$$\leq \|u'\|_{L^2(0,1)} \|v'\|_{L^2(0,1)} + \|u\|_{L^2(0,1)} \|v\|_{L^2(0,1)} + \alpha \sup_{x\in(0,1)}|u(x)| \sup_{x\in(0,1)}|v(x)|$$

$\leq 2\|u\|_{H^1(0,1)} \|v\|_{H^1(0,1)} + \alpha \sup_{x\in(0,1)} |u(x)| \sup_{x\in(0,1)} |v(x)|$

(as $H^1(0,1)$ is compactly embedded into $C(0,1)$) where C is a constant depending on α

$\leq C \|u\|_{H^1(0,1)} \|v\|_{H^1(0,1)}$

$\Rightarrow a(\cdot,\cdot)$ is bounded.

Also $|\ell(v)| = |\int_0^1 f v\, dx|$

$\leq \int_0^1 |fv|\, dx$

$\leq \|f\|_{L^2} \|v\|_{H^1(0,1)}$

$\Rightarrow \ell(\cdot)$ is a bounded linear functional on V_h.

By Lax-Milgram Theorem. $\exists!$ $u_h \in P_h$ s.t

$a(u_h, v_h) = \ell(v_h) \quad \forall \ v_h \in V_h$.

- Claim: $\|u-u_h\|_{H^1(0,1)} \leq Ch|u|_{H^2(0,1)}$ where C is a positive constant.

Proof:

Lemma: Let U_1 be a Banach space that is compactly embedded into a normed linear space U_0. ~~Let S_i~~ let $S_i : U_i \to \mathbb{R}_{\geq 0}$ $i=0,1$ be two bounded sublinear functionals s.t.

$\|u\|_{U_1} \leq S_0(u) + S_1(u)$.

Then $\min_{p\in P} \|u-p\|_{U_1} \leq C S_1(u) \quad \forall u \in U_1$

where $P := \ker(S_1)$.

Proof of this lemma can be found in Süli's notes.

consider $\hat{K} = (0,1)$, $U_0 = H^1(0,1)$ $U_1' = H^2(0,1)$

$S_0(\hat{u}) = \|\hat{u}\|_{H^1(0,1)}$, $S_1(\hat{u}) = |\hat{u}|_{H^2(0,1)}$.

$P = \ker(S_1) = \mathbb{P}_1 = \text{span}\{1-s, s\} =: \text{span}\{\Psi_1, \Psi_2\}$.

~~For $u_h \in V_h = P$~~

Consider the linear interpolant of $\hat{u}$

$I_{\hat{K}}\hat{u}(s) = \hat{u}(0)\Psi_1(s) + \hat{u}(1)\Psi_2(s)$

$= \hat{u}(0)(1-s) + \hat{u}(1)s$.

The last few lines of the proof of this lemma in the lecture notes implies that

$$\|\hat{u} - I_{\hat{K}}(\hat{u})\|_{H^1(0,1)} \le C\,|\hat{u}|_{H^2(0,1)}$$

Now consider $K = (0,h)$. Define $\hat{u}(s) \in \hat{K} = (0,1)$ by

$$u(x) = u(hs) =: \hat{u}(s) \qquad s \in (0,1)$$

Then $\hat{u} \in H^2(0,1)$. Let $\hat{p} = I_{\hat{K}}(\hat{u})$ be the linear interpolant of $\hat{u}$ on the interval $\hat{K} = (0,1)$. We rescale to $(0,h)$ and define $p(x) = p(hs) = \hat{p}(s)$ the linear intepolant of u on $K = (0,h)$, denoted by $I_K(u)$. Then

$$\|u - I_K(u)\|^2_{L^2(0,h)} = \int_0^h |u(x) - p(x)|^2\,dx$$
$$= h\int_0^1 u(hs) - p(hs)|^2 ds$$
$$= h\int_0^1 |\hat{u}(s) - \hat{p}(s)|^2 ds$$
$$\le ch\int_0^1 \left|\frac{d^2}{ds^2}\hat{u}(s)\right|^2 ds$$
$$= ch\int_0^1 h^4 \left|\frac{d^2}{dx^2}u(x)\right|^2 \frac{1}{h}\,dx$$
$$= ch^4 |u|^2_{H^2(0,h)}$$

$$\Rightarrow \|u - I_K(u)\|_{L^2(0,h)} \le ch^2 |u|_{H^2(0,h)}$$

Similarly ~~$\|\nabla u - \nabla I_K(u)\|$~~

$$\|u' - I_K'(u)\|^2_{L^2(0,h)} = \int_0^h |u'(x) - p'(x)|^2 dx$$
$$= h\int_0^1 \frac{1}{h^2} |u'(hs) - p'(hs)|^2 ds$$
$$= \frac{1}{h}\int_0^1 |\hat{u}'(s) - p'(s)|^2 ds$$
$$\le \frac{C}{h}\int_0^1 \left|\frac{d^2}{ds^2}\hat{u}(s)\right|^2 ds$$
$$= \frac{C}{h}\int_0^1 h^4 \left|\frac{d^2}{dx^2}u(x)\right|^2 \frac{1}{h}\,dx$$
$$= ch^2 |u|^2_{H^2(0,h)}.$$

$$\Rightarrow \|u - I_K(u)\|_{H^1(0,h)} \le ch\,|u|_{H^2(0,h)}. \quad (*)$$

$$\|u - u_h\|_{H^1(0,1)} = \sum_{K\in T_h} \|u - u_h\|_{H^1(K)}$$

(cea's lemma)

T_h is the uniform mesh with length h

$$\le \sum_{K\in T_h} C \min_{v_h\in V_h} \|u - v_h\|_{H^1(K)}$$
$$\le \sum_{K\in T_h} C\, \|u - I_K(u)\|_{H^1(K)}$$

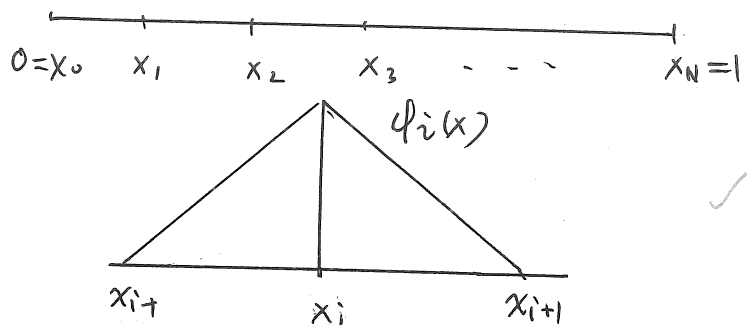
by $(*)$

$$\le C\sum_{K\in T_h} h\,|u|_{H^2(0,h)}$$
$$\le C\,h\,|u|_{H^2(0,1)}.$$

$u_h(x) = \sum_{i=1}^{N} u_i \phi_i(x)$

where ϕ_i is the ~~basis~~ finite-element basis function defined as

$$\phi_i(x) = \left(1 - \frac{|x - x_i|}{h}\right)_+ \qquad i=1, \dots N$$



Then $\cancel{a(u_h, \phi_j) = \sum_{i=1}^{N} u_i \int_0^1 \phi_i' \phi_j' dx + \int_0^1 \phi_i \phi_j dx}$

$$a(u_h, \phi_j) = \ell(\phi_j) \qquad + \cancel{\alpha \phi_i(1)} \phi_j(1)$$

$$\sum_{i=1}^{N} u_i \, a(\phi_i, \phi_j) = \ell(\phi_j)$$

In matrix form,

$$AU = b$$

where $A_{ji} = a(\phi_i, \phi_j)$, $b = (\ell(\phi_1), \dots \ell(\phi_N))^T$

and $U = (u_1, \dots u_N)^T$

$$a(\phi_i, \phi_j) = \int_0^1 \phi_i' \phi_j' dx + \int_0^1 \phi_i \phi_j dx + \alpha \phi_i(1) \phi_j(1)$$

$$= \begin{cases} \frac{2}{h} + \frac{4h}{6} & \text{if } i=j \neq N \\ -\frac{1}{h} + \frac{h}{6} & \text{if } |i-j| = 1 \\ 0 & \text{if } |i-j| \end{cases}$$

$$a(\phi_N, \phi_N) = \cancel{\frac{2}{h} + \frac{4h}{6} + \alpha} \;.\; \frac{1}{h} + \frac{2h}{6} + \alpha$$

if $i=j\neq N$, $a(\phi_i,\phi_i)=\int_0^1 \phi_i'^2\,dx+\int_0^1 \phi_i^2\,dx$

$$=\int_{x_{i-1}}^{x_i}\phi_i'^2+\int_{x_i}^{x_{i+1}}\phi_i'^2+\int_{x_{i-1}}^{x_i}\phi_i^2+\int_{x_i}^{x_{i+1}}\phi_i^2$$

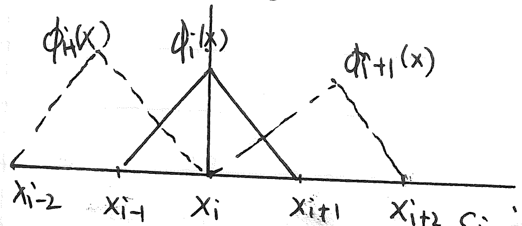
$$=\int_{x_{i-1}}^{x_i}\frac{1}{h^2}+\int_{x_i}^{x_{i+1}}\frac{1}{h^2}+\int_{x_{i-1}}^{x_i}\left(1-\frac{x_i-x}{h}\right)^2+\int_{x_i}^{x_{i+1}}\left(1-\frac{x-x_i}{h}\right)^2dx$$

$$=\frac{2}{h}+\frac{4h}{6}$$

$$a(\phi_N,\phi_N)=\int_{x_{N-1}}^{x_N}\phi_N^2\,dx+\int_{x_{N-1}}^{x_N}\phi_N^2+\alpha\,\phi_N(1)^2$$

$$=\frac{1}{h}+\frac{2h}{6}+\alpha$$

If $|i-j|=1$



$$a(\phi_i,\phi_{i+1})=\int_{x_i}^{x_{i+1}}-\frac{1}{h}\cdot\frac{1}{h}\,dx+\int_{x_i}^{x_{i+1}}\left(1-\frac{x_{i+1}-x}{h}\right)\left(1-\frac{x-x_i}{h}\right)dx$$

$$=-\frac{1}{h}+\frac{h}{6}$$

Similarly, $a(\phi_i,\phi_{i-1})=-\frac{1}{h}+\frac{h}{6}$

$$A=\begin{bmatrix} 2+\frac{4h}{6} & -\frac{1}{h}+\frac{h}{6} & & \\ -\frac{1}{h}+\frac{h}{6} & 2+\frac{4h}{6} & -\frac{1}{h}+\frac{h}{6} & 0 \\ & \ddots & \ddots & \ddots \\ 0 & & & -\frac{1}{h}+\frac{h}{6} \\ & & -\frac{1}{h}+\frac{h}{6} & \frac{1}{h}+\frac{2h}{6}+\alpha \end{bmatrix}$$

If $\alpha=0$, $f(x)=1$. $h=1/3$

Then $N=\frac{1}{h}=3$

$$\frac{2}{h}+\frac{4h}{6}=6+\frac{2}{9} \qquad \frac{1}{h}+\frac{2h}{6}=3+\frac{1}{9}$$

$$-\frac{1}{h}+\frac{h}{6}=-3+\frac{1}{18}$$

$$A=\begin{bmatrix} 6+\frac{2}{9} & -3+\frac{1}{18} & 0 \\ -3+\frac{1}{18} & 6+\frac{2}{9} & -3+\frac{1}{18} \\ 0 & -3+\frac{1}{18} & 3+\frac{1}{9} \end{bmatrix}$$

If $i\neq N$. $b_i=\ell(\phi_i)=\int_{x_i}^{x_{i+1}}1-\frac{x-x_i}{h}+\int_{x_{i-1}}^{x_i}1-\frac{x_i-x}{h}\,dx$

$$=\frac{h}{2}+\frac{h}{2}$$

$$=h.$$

$$b_N = \int_{x_{N-1}}^{x_N} 1 - \frac{x_N - x}{h}\, dx = \frac{h}{2}$$

$$\begin{bmatrix} \frac{56}{9} & -\frac{53}{18} & 0 \\ -\frac{53}{18} & \frac{56}{9} & -\frac{53}{18} \\ 0 & -\frac{53}{18} & \frac{28}{9} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{6} \end{bmatrix}$$ ✓

$u_1 = 0.2039$

$u_2 = 0.3177$

$u_3 = 0.3543$ ✓✓

$\phi_1 = (1 - 3|x - \frac{1}{3}|)_+$

$\phi_2 = (1 - 3|x - \frac{2}{3}|)_+$

$\phi_3 = (1 - 3|x - 1|)_+$

$u_h = 0.2039\phi_1 + 0.3177\phi_2 + 0.3543\phi_3$. ✓

$$\begin{cases} -u'' + u = 1 & x \in (0,1) \quad (1) \\ u(0) = 0 \quad u'(1) = 0 & \quad (2) \end{cases}$$

From (1), we know $u = C_1 e^x + C_2 e^{-x} + 1$

From (2), we know $u(0) = C_1 + C_2 + 1 = 0$

$u'(1) = C_1 e - C_2 e^{-1} = 0$

$C_1 = -\frac{1}{1+e^2}$ $\qquad C_2 = -\frac{1}{1+e^{-2}}$

$u_{exact} = -\frac{1}{1+e^2} e^x - \frac{1}{1+e^{-2}} e^{-x} + 1$

Evaluating at $x = [\frac{1}{3}, \frac{2}{3}, 1]$, we have

$u_h = \begin{pmatrix} 0.2039 \\ 0.3177 \\ 0.3543 \end{pmatrix}$ $\qquad u_{exact} = \begin{pmatrix} 0.2025 \\ 0.3156 \\ 0.3519 \end{pmatrix}$ ✓✓

$|u_h - u_{exact}| = \begin{pmatrix} 0.0014 \\ 0.021 \\ 0.0024 \end{pmatrix}$ ✓

$\|u_h - u_{exact}\|_\infty = 0.0024$.

nice work !! :)

Q2. $-\Delta u = f(x,y)$ for $(x,y) \in \Omega = (0,1)^2$ with $f \in L^2(\Omega)$ subject to the homogeneous Dirichlet BC. $u=0$ on

$$\Gamma_D = \{(x,y) \in \partial\Omega : x=0 \text{ or } y=0 \text{ or } y=1\}$$

and non-homogeneous Neumann BC. $\frac{\partial u}{\partial x} = 1$ on

$$\Gamma_N = \{(x,y) \in \partial\Omega : x=1\}.$$

Consider the special Sobolev space

$$H^1_{0,\Gamma_D}(\Omega) := \{v \in H^1(\Omega) : v=0 \text{ on } \Gamma_D\}.$$

question is sloppy
make sure to use trace in the future

$v|_{\Gamma_D} = 0$

weak formulation :

find $u \in H^1_{0,\Gamma_D}(\Omega)$ s.t

$$a(u,v) = \ell(v) \qquad \forall\ v \in H^1_{0,\Gamma_D}(\Omega) \quad (P)$$

where

$$a(u,v) = \int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) dx\,dy$$

$$\ell(v) = \int_\Omega f\, v\, dx\,dy + \int_{x=1} v(s)\, ds$$

We can apply the Lax-Milgram theorem with $V = H^1_{0,\Gamma_D}(\Omega)$ to show the existence and uniqueness of a weak solution to this mixed problem.

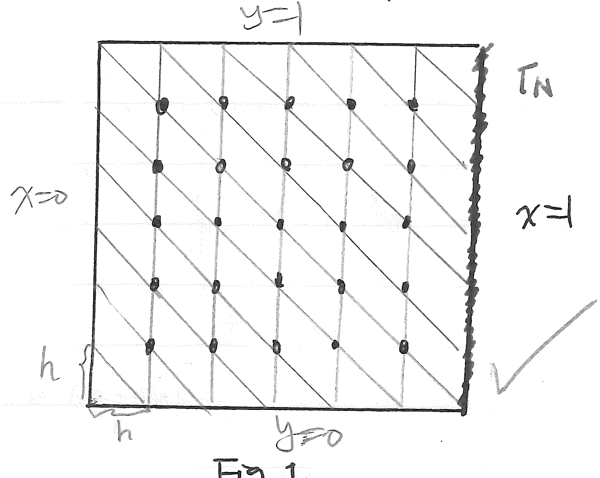


Fig 1.

Consider $V_h \subset H^1_{0,\Gamma_D}$ where V_h is a finite-dimensional subspace of H^1_{0,Γ_D} which consists of continuous piecewise linear functions associated with the triangulation stated in this problem.

Then the finite element approximation is :

find $u_h \in V_h$ s.t

$$a(u_h, v_h) = \ell(v_h) \qquad \forall\ v_h \in V_h$$

where $a(\cdot,\cdot)$ and $\ell(\cdot)$ are defined as above.

Let us suppose that the interior nodes are labelled $1, 2, \ldots N(h)$;

Let $\phi_1(x,y), \dots, \phi_{N(h)}(x,y)$ be the corresponding basis functions.

$V_h = \text{span}\{\phi_1, \dots \phi_{N(h)}\}$.

writing $$u_h(x,y) = \sum_{i=1}^{N(h)} u_i \phi_i(x,y).$$

the finite element method can be restated as follows:

find $U = (u_1, \dots u_{N(h)})^T \in \mathbb{R}^{N(h)}$ s.t.

$$\sum_{i=1}^{N(h)} u_i \left[\int_\Omega \left(\frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y} \frac{\partial \phi_j}{\partial y} \right) dx\,dy \right]$$

$$= \int_\Omega f \phi_j \, dx\,dy + \int_{x=1} \phi_j(s)\, ds.$$

Letting $A = (a_{ij})$, $F = (F_1, \dots, F_{N(h)})^T$.

$$a_{ij} = a_{ji} = \int_\Omega \left(\frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y} \frac{\partial \phi_j}{\partial y} \right) dx\,dy$$

$$F_j = \int_\Omega f \phi_j \, dx\,dy + \int_{x=1} \phi_j \, ds,$$

the finite element approximation can be written as a system of linear equations

$$AU = F.$$

Solving this, we obtain $U = (u_1, \dots, u_{N(h)})^T$, and hence the approximate solution

$$u_h(x,y) = \sum_{i=1}^{N(h)} u_i \phi_i(x,y).$$

Now we study the structure of matrix A.

Let $\phi_{i,j}$ denote the basis function associated with the interior node (x_i, y_j)

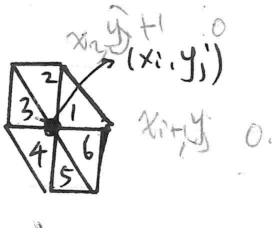


Fig 7

$$\phi_{ij}(x,y) = \begin{cases} 1 - \frac{x - x_i}{h} - \frac{y - y_j}{h} & (x,y) \in 1 \\ 1 - \frac{y - y_j}{h} & (x,y) \in 2 \\ 1 - \frac{x_i - x}{h} & (x,y) \in 3 \\ 1 - \frac{x_i - x}{h} - \frac{y_j - y}{h} & (x,y) \in 4 \\ 1 - \frac{y_j - y}{h} & (x,y) \in 5 \\ 1 - \frac{x - x_i}{h} & (x,y) \in 6 \\ 0 & \text{otherwise} \end{cases}$$

where $1, 2, \dots, 6$ denote triangles surrounding the node (x_i, y_j), as shown in the Fig. 2.

Thus
$$\frac{\partial \phi_{ij}}{\partial x} = \begin{cases} -1/h & (x,y) \in 1 \\ 0 & (x,y) \in 2 \\ 1/h & (x,y) \in 3 \\ 1/h & (x,y) \in 4 \\ 0 & (x,y) \in 5 \\ -1/h & (x,y) \in 6 \\ 0 & \text{otherwise}. \end{cases}$$

and
$$\frac{\partial \phi_{ij}}{\partial y} = \begin{cases} -1/h & (x,y) \in 1 \\ -1/h & (x,y) \in 2 \\ 0 & (x,y) \in 3 \\ 1/h & (x,y) \in 4 \\ 1/h & (x,y) \in 5 \\ 0 & (x,y) \in 6 \\ 0 & \text{otherwise}. \end{cases}$$

Since
$$\sum_{i=1}^{N-1} \sum_{j=1}^{N-1} U_{ij} \int_\Omega \left(\frac{\partial \phi_{ij}}{\partial x} \frac{\partial \phi_{kl}}{\partial x} + \frac{\partial \phi_{ij}}{\partial y} \frac{\partial \phi_{kl}}{\partial y} \right) dx\,dy$$

$$= 4U_{kl} - U_{k-1,l} - U_{k+1,l} - U_{k,l-1} - U_{k,l+1}$$

This is similar to the 5-point finite difference scheme.

$$A = \begin{bmatrix} B & -I & & 0 \\ -I & B & -I & \\ & & \ddots & -I \\ 0 & & -I & B \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -1 & & 0 \\ -1 & 4 & -1 & \\ & 0 & \ddots & -1 \\ & & -1 & 4 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

you forgot to include $\int_{x=1}$ ~~ϕ_N, tedy effects~~ BCs' effects.

A is an $(N-1)\times(N-1)$ matrix. $N(N-1)^2 \times N(N-1)^2$ It is sparse, ~~and~~ block-tridiagonal. and symmetric positive definite.

$\Rightarrow$ A is invertible.

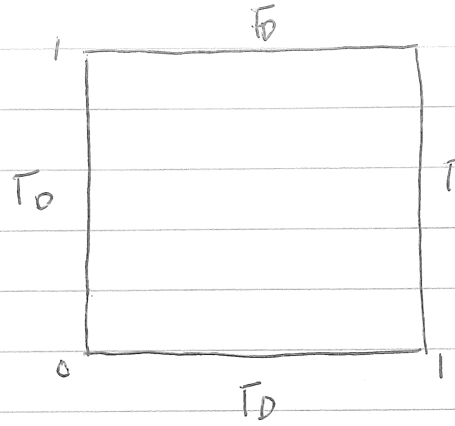
also must include F since it is non-standard due to $\int \varphi_{Nx}(1,y)\,dy$ effects

$\Rightarrow$ $AU = F$ has unique solution U.

$$\Rightarrow U_h(x,y) = \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} U_{ij}\, \phi_{ij}(x,y)$$

can be is uniquely determined

FEMs for PDEs - Problem Sheet 2.



$$\begin{cases} -\Delta u = f, & \Omega \\ u = 0, & \Gamma_D \\ \frac{\partial u}{\partial n} = 1, & \Gamma_N \end{cases}$$

Find $u \in H^1_{\Gamma_D}(\Omega)$

s.t. $a(u,v) = \ell(v) \quad \forall v \in H^1_{\Gamma_D}(\Omega)$

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v$$

$$\ell(v) = \int_\Omega f v + \int_0^1 v(1,y)\,dy$$

dim $= N(N-1)$

$$V^h = \{ \phi_{ij} \mid \begin{matrix} 1 \le i \le N \\ 1 \le j \le N-1 \end{matrix} \}$$

$$u_h = \sum_{i=1}^{N} \sum_{j=1}^{N-1} U_{ij}\, \phi_{ij}(x,y)$$

$$\phi_{ij}(x,y) = \begin{cases} 1 - \frac{x-x_i}{h} - \frac{y-y_j}{h}, & 1 \\ 1 - \frac{y-y_j}{h}, & 2 \\ 1 - \frac{x_i-x}{h}, & 3 \\ 1 - \frac{x_i-x}{h} - \frac{y_j-y}{h}, & 4 \\ 1 - \frac{y_j-y}{h}, & 5 \\ 1 - \frac{x-x_i}{h}, & 6 \\ 0 & \text{otherwise} \end{cases} \qquad \frac{\partial \phi_{ij}}{\partial x} = \begin{cases} -\frac{1}{h}, & 1 \\ 0, & 2 \\ \frac{1}{h}, & 3 \\ \frac{1}{h}, & 4 \\ 0, & 5 \\ -\frac{1}{h}, & 6 \\ 0 & \text{otherwise.} \end{cases}$$

arbitrary $\phi_{k,\ell} \rightarrow U_{k+1,\ell}\ U_{k-1,\ell}\ U_{k,\ell-1},\ U_{k,\ell+1}.$

$U_{k+1,\ell-1}\quad U_{\cancel{k+1}\atop k-1}.\ U_{\ell+1},\ U_{k,k}$

$u_{k+1,\ell} \quad \nabla \varphi_{k+1,\ell} \nabla \varphi_{k,\ell} = -\frac{1}{h^2} \xrightarrow{\int_\Omega} -1$

two triangles have area h^2

$k \in \{1, \dots N-1\}$

$$-\frac{u_{k-1,\ell} - 2u_{k,\ell} + u_{k+1,\ell}}{h^2} - \frac{u_{k,\ell-1} - 2u_{k,\ell} + u_{k,\ell+1}}{h^2}$$

$$= \frac{1}{h^2} \int_\Omega f \cdot \varphi_{k,\ell}.$$

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = u_0 & \Gamma_D \\ \frac{\partial u}{\partial n} = g & \Gamma_N \end{cases}$$

$w = u - u_0.$

$-\Delta w = f + \Delta u_0$

$w = 0$

$w = g$

$u \in H^1(\Omega) \longrightarrow v|_{\Gamma_D} \in H^{\frac{1}{2}}(\Gamma_D)$

$u_0 \in H^{1/2}(\Gamma_D)$?

$u_0 \in H_{00}^{1/2}(\Gamma_D) = \{ w|_{\Gamma_D} \mid w \in H^1_{\Gamma_N}(\Omega) \}$

constant not necessarily here