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FEMs for PDEs - Problem Sheet 3

1. Suppose that Ω is a bounded polygonal domain ~~in $\mathbb{R}^2$ with boundary~~ $\partial\Omega$. Assume that $p \in C^1(\bar{\Omega})$ and $q \in C(\bar{\Omega})$ are two functions, defined and positive on $\bar{\Omega}$, and let $f \in L_2(\Omega)$. Consider the partial differential equation

$$-\nabla \cdot (p(x,y)\nabla u) + q(x,y)u = f(x,y) \qquad \text{in } \Omega,$$

subject to the homogeneous Dirichlet boundary condition $u = 0$ on $\partial\Omega$.

Suppose further that V_h is a finite-dimensional subspace of $H_0^1(\Omega)$ (consisting, say, of continuous piecewise polynomial functions defined on a certain triangulation of Ω). Assuming that u is the unique weak solution in $H_0^1(\Omega)$ to the boundary-value problem and $u_h \in V_h$ is its finite element approximation, show that the following, so called, *Galerkin orthogonality property* holds:

$$a(u - u_h, v_h) = 0 \qquad \text{for all } v_h \in V_h, \checkmark$$

with $a(\cdot,\cdot)$ a suitable bilinear functional on $H_0^1(\Omega) \times H_0^1(\Omega)$ that you should define. ✓
Show further that there exists a positive constant c_0 such that

$$\|u - u_h\|^2_{H^1(\Omega)} \leq \frac{1}{c_0} a(u - u_h, u - u_h). \checkmark \quad \text{coercivity}$$

Apply the Galerkin orthogonality property to deduce that

$$\|u - u_h\|^2_{H^1(\Omega)} \leq \frac{1}{c_0} a(u - u_h, u - v_h) \qquad \text{for all } v_h \in V_h. \checkmark$$

Verify that there exists a positive constant c_1 such that   boundedness

$$\|u - u_h\|_{H^1(\Omega)} \leq \frac{c_1}{c_0} \min_{v_h \in V_h} \|u - v_h\|_{H^1(\Omega)}.$$

How would these results be affected (if at all) if you replaced the boundary condition $u = 0$ on $\partial\Omega$ by $u = 0$ on Γ_1 and $\frac{\partial u}{\partial \nu} = 0$ on Γ_2 where Γ_i, $i = 1, 2$, are nonempty unions of edges contained in $\partial\Omega$, $\Gamma_1 \cup \Gamma_2 = \partial\Omega$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, and ν denotes the unit outward normal vector to Γ_2? Describe ~~of a~~ physical situation that would be modelled by this mixed Dirichlet–Neumann boundary-value problem. ✓

2. Consider the two-point boundary-value problem

$$-(p(x)u')' + q(x)u = f(x), \quad x \in (0,1), \qquad u(0) = 0, \quad u(1) = 0,$$

where $p(x) \geq \tilde{c} > 0$, $q(x) \geq 0$ for $x \in [0,1]$, $p \in C^1[0,1]$, $q \in C[0,1]$ and $f \in L_2(0,1)$. Let u_h denote the continuous piecewise linear finite element approximation to u on a uniform subdivision of $[0,1]$ into subintervals of length $h = 1/N$, $N \geq 2$. Show that

$$\|u - u_h\|_{H^1(0,1)} \leq C_1 h \|u''\|_{L_2(0,1)},$$

where C_1 is a positive constant that you should specify. Show further that there is a positive constant C_2 such that

$$\|u - u_h\|_{H^1(0,1)} \leq C_2 h \|f\|_{L_2(0,1)}. \checkmark$$

Calculate the right-hand sides in these inequalities in the case when $p(x) \equiv 1$, $q(x) \equiv 0$, $f(x) \equiv 1$ for $x \in [0,1]$, and $h = 10^{-3}$.

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Q1 Ω is a bounded polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. $p \in C^1(\bar{\Omega})$ and $q \in C(\bar{\Omega})$ are positive on $\bar{\Omega}$. $f \in L_2(\Omega)$.

$$\begin{cases} -\nabla \cdot (p(x,y) \nabla u) + q(x,y) u = f(x,y) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

weak formulation: find $u \in H_0^1(\Omega)$ such that

(1) $$a(u,v) = \ell(v) \quad \forall v \in H_0^1(\Omega) \quad (P)$$

where $$a(u,v) := \int_\Omega p(x,y) \nabla u \cdot \nabla v \, dxdy + \int_\Omega q(x,y) uv \, dxdy$$

and $$\ell(v) := \int_\Omega f(x,y) \, v \, dxdy$$

V_h is a finite-dimensional subspace of $H_0^1(\Omega)$ consisting of continuous piecewise polynomial functions defined on a certain triangulation of Ω. Then the finite element approximation of the problem is:

find $u_h \in V_h$ such that

(2) $$a(u_h, v_h) = \ell(v_h) \quad \forall v_h \in V_h \quad (P_h).$$

For any $v_h \in V_h \subset H_0^1(\Omega)$, we have

(3) $$a(u, v_h) = \ell(v_h).$$

(3) $-$ (2) gives (by linearity of $a(\cdot,\cdot)$)

$$a(u-u_h, v_h) = 0 \quad \text{for all } v_h \in V_h$$

Note that $a(v,v) = \int_\Omega p(x,y) \nabla v \cdot \nabla v \, dxdy + \int_\Omega q(x,y) \, v \cdot v \, dxdy$

since $p \in C^1(\bar{\Omega})$ and $q \in C(\bar{\Omega})$ are positive on $\bar{\Omega}$, we may assume that $p \geq \tilde{c} > 0$ and $q \geq \tilde{d} > 0$ on $\bar{\Omega}$.

Then

$$\begin{aligned} a(v,v) &\geq \tilde{c} \int_\Omega |\nabla v|^2 \, dxdy + \tilde{d} \int_\Omega |v|^2 \, dxdy \\ &\geq \min(\tilde{c}, \tilde{d}) \, \|v\|^2_{H^1(\Omega)} \\ &= C_0 \, \|v\|^2_{H^1(\Omega)}. \end{aligned}$$

(Take $C_0 := \min(\tilde{c}, \tilde{d})$).

(4)

$$\Rightarrow \quad \|v\|^2_{H^1(\Omega)} \le \frac{1}{a_0}\, a(v,v) \quad \text{for all } v \in H_0^1.$$

For ~~all~~ $u_h \in V_h$, $\quad u - u_h \in V_h \subset H_0^1(\Omega)$.

Replacing v in the above inequality by $u - u_h$, we have

$$(5) \quad \|u - u_h\|^2_{H^1(\Omega)} \le \frac{1}{\cancel{a_0}\, c_0}\, a(u-u_h,\, u-u_h).$$

Note that $\quad a(u-u_h,\, u-u_h)$

(by linearity of $a(\cdot,\cdot)$) $\quad = a(u-u_h,\, u-v_h) + a(u-u_h,\, v_h-u_h)$

(by Galerkin orthogonality property) $\quad = a(u-u_h,\, u-v_h)$

$$\Rightarrow_{(6)} \quad \|u-u_h\|^2_{H^1(\Omega)} \le \frac{1}{c_0}\, a(u-u_h,\, u-v_h) \quad \forall\, v_h \in V_h.$$

Since p and q are continuous functions on a closed and bounded domain $\bar{\Omega}$, $\max_{\bar{\Omega}} p(x,y)$ and $\max_{\bar{\Omega}} q(x,y)$ are achieved.

Define $\quad M_p := \max_{\bar{\Omega}} p(x,y) \quad$ and $\quad M_q := \max_{\bar{\Omega}} q(x,y)$.

Then $\quad a(w,v) = \int_\Omega p(x,y)\, \nabla w\, \nabla v\, dxdy + \int_\Omega q(x,y)\, w\, v\, dxdy$

$$\le M_p \left(\int_\Omega |\nabla w|^2\right)^{\frac12} \left(\int_\Omega |\nabla v|^2\right)^{\frac12} + M_q \left(\int_\Omega |w|^2\right)^{\frac12} \left(\int_\Omega |v|^2\right)^{\frac12}$$

$$= M_p\, \|\nabla w\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + M_q\, \|w\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}$$

I think you need a factor of $\sqrt{2}$ here

$$\le \max(M_p, M_q)\; \|w\|_{H^1(\Omega)}\, \|v\|_{H^1(\Omega)}$$

$$= C_1\, \|w\|_{H^1(\Omega)}\, \|v\|_{H^1(\Omega)} \quad \text{with } C_1 := \max(M_p, M_q)$$

$$\Rightarrow \quad a(u-u_h,\, u-v_h) \le C_1\, \|u-u_h\|_{H^1(\Omega)}\, \|u-v_h\|_{H^1(\Omega)} \quad (*)$$

By $(*)$ and (6) we have

$$\|u-u_h\|^{\cancel{2}}_{H^1(\Omega)} \le \frac{1}{c_0}\, a(u-u_h,\, u-v_h)$$

$$\le \frac{C_1}{c_0}\, \cancel{\|u-u_h\|_{H^1(\Omega)}}\, \|u-v_h\|_{H^1(\Omega)}$$

$$\Rightarrow \quad \|u-u_h\|_{H^1(\Omega)} \le \frac{C_1}{c_0}\, \|u-v_h\|_{H^1(\Omega)} \quad \forall\, v_h \in V_h.$$

By taking minimal v_h, we have

$$(7) \quad \|u-u_h\|_{H^1(\Omega)} \le \frac{C_1}{c_0} \min_{v_h \in V_h} \|u-v_h\|_{H^1(\Omega)}.$$

Now consider

$$\begin{cases} -\nabla \cdot (p(x,y)\nabla u) + q(x,y)\, u = f(x,y) & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_1 \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_2 \end{cases}$$

where Γ_i, $i=1,2$ are non-empty unions of edges contained in $\partial\Omega$, $\Gamma_1 \cup \Gamma_2 = \partial\Omega$, $\Gamma_1 \cap \Gamma_2 = \phi$ and ν denotes the unit outward normal vector to Γ_2.

Following a similar reasoning as in the case of the Dirichlet boundary value problem, we consider the special Sobolev space

$$H^1_{0,\Gamma_1}(\Omega) = \{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_1 \}.$$

and define the weak formulation of the mixed problem as follows: find $u \in H^1_{0,\Gamma_1}(\Omega)$ s.t.

$$a(u,v) = \ell(v) \quad \text{for all } v \in H^1_{0,\Gamma_1}(\Omega)$$

where $a(u,v) = \int_\Omega p(x,y)\, \nabla u \cdot \nabla v \, dx dy + \int_\Omega q(x,y)\, u \cdot v \, dx dy$

and $\ell(v) = \int_\Omega f(x,y)\, v \, dx dy$.

Now V_h is a finite-dimensional subspace of $H^1_{0,\Gamma_1}(\Omega)$ (instead of $H^1_0(\Omega)$). Then the rest of the results are all the same as before.

<u>Physical situation</u>:

Consider the heat diffusion modelled by the equation $-\nabla \cdot (p(x,y)\nabla u) + q(x,y)\, u = f(x,y)$ in the ~~dom~~ bounded polygonal domain Ω.

$u=0$ on Γ_1 means that the temperature is set to be zero on this boundary. $\frac{\partial u}{\partial \nu} = 0$ on Γ_2 means that ~~the edge~~ the boundary Γ_2 is isolated, i.e. flux of heat through this boundary is zero.

Q2 $-(p(x)u')' + q(x)u = f(x) \quad x \in (0,1) \quad u(0)=0, \; u(1)=0.$
where $p(x) \geq \hat{c} > 0$. $q(x) \geq 0$ for $x \in [0,1]$ $p \in C^1[0,1]$ $q \in C([0,1])$ and $f \in L_2(0,1)$.

The weak formulation of the problem is:
(P) find $u \in H_0^1(0,1)$ s.t. $a(u,v) = \ell(v)$ $\forall$ $v \in H_0^1(0,1)$.
where $a(u,v) := \int_0^1 p(x) u'v' \, dx + \int_0^1 q(x) uv \, dx$
and $\ell(v) := \int_0^1 f(x) v(x) \, dx$

We consider the finite element basis function

$$\phi_i(x) = \begin{cases} 0 & \text{if } x \leq x_{i-1} \\ \frac{x - x_{i-1}}{h} & \text{if } x_{i-1} \leq x \leq x_i \\ \frac{x_{i+1} - x}{h} & \text{if } x_i \leq x \leq x_{i+1} \\ 0 & \text{if } x_{i+1} \leq x. \end{cases}$$

~~The~~ for $i = 1, 2, \dots N-1$ we put $V_h = \text{span}\{\phi_1, \dots \phi_{N-1}\}$
V_h is an $(N-1)$-dim subspace of $H_0^1(0,1)$.

The finite element approximation of problem (P) is:
(Ph) find $u_h \in V_h$ s.t. $a(u_h, v_h) = \ell(v_h)$ $\forall$ $v_h \in V_h$.

By Céa's lemma, we have

$$\|u - u_h\|_{H^1(0,1)} \leq \frac{c_1}{c_0} \min_{v_h \in V_h} \|u - v_h\|_{H^1(0,1)}$$

where c_0 is the coercivity constant. ie. $a(u-u_h, u-u_h) \geq c_0 \|u-u_h\|^2_{H^1(0,1)}$
and c_1 the continuity constant. i.e. $a(u-u_h, u-v_h) \leq c_1 \|u-u_h\|_{H^1(0,1)} \|u-v_h\|_{H^1(0,1)}$

$$\Rightarrow \|u - u_h\|_{H^1(0,1)} \leq \frac{c_1}{c_0} \|u - v_h\|_{H^1(0,1)} \quad (\star)$$

Let $I_h u \in V_h$ denote the interpolant of u from the finite element space. Thus. $I_h u(x) = \sum_{i=1}^{N-1} u(x_i) \phi_i(x)$.

Choosing $v_h = I_h u$ in $(\star)$. we see that

$$\|u - u_h\|_{H^1(0,1)} \leq \frac{c_1}{c_0} \|u - I_h u\|_{H^1(0,1)}.$$

Assume that $u \in H^2(0,1)$, now we aim to show that

$$\cancel{\|u - I_h u\|_{H^1(0,1)}} \; \|u - I_h u\|_{H^1(0,1)} \leq \tilde{C} h \|u''\|_{L^2(0,1)}$$

for some constant $\tilde{C}$.

Consider a subinterval $[x_{i-1}, x_i]$, $1 \le i \le N$, and define
$$\zeta(x) = u(x) - I_h u(x) \quad \text{for } x \in [x_{i-1}, x_i].$$
Then $\zeta \in H^2(x_{i-1}, x_i)$ and $\zeta(x_{i-1}) = \zeta(x_i) = 0$.
Therefore, ζ can be expanded into a convergent Fourier sine-series
$$\zeta(x) = \sum_{k=1}^{\infty} a_k \sin\left(\frac{k\pi(x - x_{i-1})}{h}\right) \qquad x \in [x_{i-1}, x_i]$$
Hence $\int_{x_{i-1}}^{x_i} [\zeta(x)]^2 \, dx = \frac{h}{2} \sum_{k=1}^{\infty} |a_k|^2$.

Differentiating the Fourier sine-series for ζ twice we deduce that the Fourier coefficients of ζ' are $\left(\frac{k\pi}{h}\right) a_k$ while those of ζ'' are $-\left(\frac{k\pi}{h}\right)^2 a_k$. Thus
$$\int_{x_{i-1}}^{x_i} [\zeta'(x)]^2 \, dx = \frac{h}{2} \sum_{k=1}^{\infty} \left(\frac{k\pi}{h}\right)^2 |a_k|^2.$$
$$\int_{x_{i-1}}^{x_i} [\zeta''(x)]^2 \, dx = \frac{h}{2} \sum_{k=1}^{\infty} \left(\frac{k\pi}{h}\right)^4 |a_k|^2.$$
Because $k^4 \ge k^2 \ge 1$, it follows that
$$\int_{x_{i-1}}^{x_i} [\zeta(x)]^2 \, dx \le \left(\frac{h}{\pi}\right)^4 \int_{x_{i-1}}^{x_i} [\zeta''(x)]^2 \, dx,$$
$$\int_{x_{i-1}}^{x_i} [\zeta'(x)]^2 \, dx \le \left(\frac{h}{\pi}\right)^2 \int_{x_{i-1}}^{x_i} [\zeta''(x)]^2 \, dx.$$
However, $\zeta''(x) = u''(x) - \underbrace{(I_h u)''(x)}_{} = u''(x)$ for $x \in (x_{i-1}, x_i)$.
because $I_h u$ is a linear function on this interval,
Therefore, upon summation over $i = 1, 2, \dots N$, we obtain
$$\|\zeta\|^2_{L^2(0,1)} \le \left(\frac{h}{\pi}\right)^4 \|u''\|^2_{L^2(0,1)}$$
$$\|\zeta\|^2_{L^2(0,1)} \le \left(\frac{h}{\pi}\right)^2 \|u''\|^2_{L^2(0,1)}$$

$$\Rightarrow \|u - I_h u\|^2_{H^1(0,1)} = \|u - I_h u\|^2_{L^2(0,1)} + \|(u - I_h u)'\|^2_{L^2(0,1)}$$
$$\le \frac{h^2}{\pi^2}\left(1 + \frac{h^2}{\pi^2}\right) \|u''\|^2_{L^2(0,1)}$$

$$\Rightarrow \|u - u_h\|_{H^1(0,1)} \le \frac{c_1}{c_0} \left(\cancel{\frac{h^2}{\pi^2}}\right)\frac{h}{\pi}\left(1 + \frac{h^2}{\pi^2}\right)^{1/2} \|u''\|_{L^2(0,1)}$$
$$= C_1 h \|u''\|_{L^2(0,1)}$$
where $C_1 = \frac{c_1}{c_0}\left(1 + \frac{h^2}{\pi^2}\right)^{1/2} \cdot \frac{1}{\pi}$.

Note that $\ell(u) = a(u,u) \geq C_0 \|u\|^2_{H^1(0,1)}$

$$\ell(u) = \int_0^1 f(x)\, u\, dx \leq \|u\|_{L^2(0,1)} \|f\|_{L^2(0,1)}$$

$$\Rightarrow \|u\|^{\cancel{2}}_{L^2(0,1)} \leq \|u\|^2_{H^1(0,1)} \leq \frac{1}{C_0} \|\cancel{u\|_{L^2(0,1)}} \|f\|_{L^2(0,1)}$$

$$\Rightarrow \|u\|_{L^2(0,1)} \leq \frac{1}{C_0} \|f\|_{L^2(0,1)}$$

Similarly $\|u'\| \leq \frac{1}{C_0} \|f\|_{L^2(0,1)}$

$$-p'(x)u' - p(x)u'' + q(x)\, u = f(x)$$

$$\min_{x\in[0,1]} |p(x)|\, \|u''\|_{L^2(0,1)} \leq \max_{x\in[0,1]} |p'(x)|\, \|u'\|_{L^2(0,1)} + \max_{x\in[0,1]} |q(x)|\, \|u\|_{L^2} + \|f\|_{L^2(0,1)}$$

$$\Rightarrow \|u''\|_{L^2(0,1)} \leq \frac{\max_{x\in[0,1]} |p'(x)|}{\min_{x\in[0,1]} |p(x)|} \|u'\|_{L^2(0,1)} + \frac{\max_{x\in[0,1]} |q(x)|}{\min_{x\in[0,1]} |p(x)|} \|u\|_{L^2} + \frac{1}{\min_{x\in[0,1]} |p(x)|} \|f\|_{L^2(0,1)}$$

$$\leq \frac{\max_{x\in[0,1]} |p'(x)|}{\min_{x\in[0,1]} |p(x)|} \cdot \frac{1}{C_0} \|f\|_{L^2(0,1)} + \frac{\max_{x\in[0,1]} |q(x)|}{\min_{x\in[0,1]} |p(x)|} \cdot \frac{1}{C_0} \|f\|_{L^2(0,1)} + \frac{1}{\min_{x\in[0,1]} |p(x)|} \|f\|_{L^2(0,1)}$$

$$\leq \tilde{\tilde{C}} \|f\|_{L^2(0,1)}$$

were $\tilde{\tilde{C}} = \left(\frac{1}{\min_{x\in[0,1]} |p(x)|} + \frac{\max_{x\in[0,1]} |p'(x)|}{\min_{x\in[0,1]} |p(x)|} \frac{1}{C_0} + \frac{\max_{x\in[0,1]} |q(x)|}{\min_{x\in[0,1]} |p(x)|} \frac{1}{C_0} \right)$

$$\Rightarrow \|u - u_h\|_{H^1(0,1)} \leq C_1 h \|u''\|_{L^2(0,1)}$$

$$\leq C_1 \tilde{\tilde{C}} h \|u''\|_{L^2(0,1)}$$

$$= C_2 h \|f\|_{L^2(0,1)}$$

with $C_2 = \tilde{\tilde{C}} \cdot C_1$.

$p(x) \equiv 1 \quad q \equiv 0. \quad f(x) \equiv 1.$

$$\begin{cases} -u'' = 1 & x \in (0,1) \\ u(0) = u(1) = 0. \end{cases}$$

$$a(u,v) = \int_0^1 u'v' \quad \leq \|u\|_{H^1(0,1)} \|v\|_{H^1(0,1)}$$

$$c_1 = 1.$$

$$\begin{aligned} a(u,u) &= \int_0^1 |u'|^2 \\ &= \frac{1}{2}\int_0^1 |u'|^2 + \frac{1}{2}\int_0^1 |u'|^2 \\ &\geq \frac{1}{2}\int_0^1 |u'|^2 + \frac{1}{2} \|u\|^2_{L^2(0,1)} \end{aligned}$$

By Poincaré - Fredrichs from sheet 1.

$$\geq \frac{1}{2} \|u\|_{H^1(0,1)}$$

$$c_0 = \frac{1}{2}.$$

$$C_1 = \frac{c_1}{c_0}\left(1 + \frac{h^2}{\pi^2}\right)^{1/2} \cdot \frac{1}{\pi} = 2\left(1 + \frac{10^{-6}}{\pi^2}\right)^{1/2} \cdot \frac{1}{\pi}$$

$$\|u''\|_{L^2(0,1)} = \|f\|_{L^2(0,1)} = \left(\int_0^1 1\right)^{\frac{1}{2}} = 1$$

$$\Rightarrow \|u - u_h\|_{H^1(0,1)} \leq 2\left(1 + \frac{10^{-6}}{\pi^2}\right)^{1/2} \cdot \frac{10^{-3}}{\pi} \cdot \|u''\|^2_{L^2(0,1)}$$

$$\approx 6.37 \times 10^{-4}$$

$$C_2 = \tilde{\tilde{C}}\, C_1 = C_1 \quad \text{since } p(x) \equiv 1,\ p'(x) = 0 \quad q(x) \equiv 0.$$

$$\Rightarrow \|u - u_h\|_{H^1(0,1)} \leq 2\left(1 + \frac{10^{-6}}{\pi^2}\right)^{1/2} \frac{10^{-3}}{\pi} \|f\|_{L^2(0,1)}$$

$$\approx 6.37 \times 10^{-4}$$

$$\sim 0.0004.$$

Q1 $a(u,v) = \int p(x,y)\, \nabla u \cdot \nabla v + q\,u\,v$ $\qquad \ell(v) = \int_\Omega f v$

continuity constant of a

$$\|u - u_h\|_{H^1} \le \frac{C_1}{C_0} \min_{v \in V_h} \|u - v_h\|_{H^1}$$

coercivity constant.

$$\|a\| = \sup_{v,w} \frac{|a(v,w)|}{\|v\|\|w\|} \qquad a(u,v) \le C_1 \|u\|_{H^1} \|v\|_{H^1}$$

$$|a(u,v)| \le \int_\Omega |p \nabla u \cdot \nabla v + q u v| \le \underbrace{\max\{\|p\|_\infty, \|q\|_\infty\}}_{C_1} \int_\Omega |\nabla u \cdot \nabla v| + |uv|$$

$$\le C_1 \|u\|_{H^1} \|v\|_{H^1}$$

$$a(v,v) \ge \underbrace{\min\{\min_\Omega p, \min_\Omega q\}}_{C_0 > 0 \text{ since } p>0,\ q>0} \|v\|_{H^1}^2$$

Q2. $a(v,v) \ge \min_\Omega p \int_\Omega |\nabla v|^2 \ge \frac{\min p}{C_*} \int_\Omega |u|^2$

$\int_\Omega |u|^2 \le C_* \int_\Omega |\nabla u|^2$

$a(v,v) \ge \frac{\min_\Omega p}{1 + C_*} \|v\|_{H^1}^2$

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$$\lambda_{\min} = \min_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2} > 0$$

If $\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases} \Rightarrow \lambda_{\min}$ can be zero.

$$\begin{cases} -u'' = \lambda u \\ u(0) = 0 = u(1) \end{cases}$$

In (b) of Q1.

$C_* = \frac{1}{\pi^2}$.

Q2 $\|u-u_h\|_{H^1} \le \frac{1+C_*}{\min_{(0,1)} p} \max\{\|p\|_\infty, \|q\|_\infty\} \|u-I_h u\|_{H^1}$

$$\le \frac{h}{\pi}\left(1+\frac{h^2}{\pi^2}\right)\|u''\|_{L^2}$$

$$\overset{\shortparallel}{O(h)}$$

$\|u''\|_{L^2} \le C\|f\|_{L^2}$

$u'' = \frac{1}{p}(qu - p'u' - f)$

$\int_0^1 (p|u'|^2 + q|u|^2) = \int_0^1 fu$

$\min_\Omega p \int_0^1 |u'|^2 \le \int_0^1 fu \le \|f\|_{L^2}\|u\|_{L^2}$

$\le C_*^{1/2} \|f\|_{L^2}\|u'\|_{L^2}$

$\Rightarrow \|u'\|_{L^2} \le \frac{C_*^{1/2}}{\min_\Omega p}\|f\|_{L^2}$

$\frac{\min_\Omega p}{C_*}\|u\|_{L^2}^2 \le \|f\|_{L^2}\|u\|_{L^2}$

$\longrightarrow \|u\|_{L^2} \le \frac{C_*}{\min p}\|f\|_{L^2}$

$\|u-u_h\|_{H^1} \le \frac{1+C_*}{\min p} \max\{\|p\|_\infty, \|q\|_\infty\} \left(\frac{C_*\|q\|_\infty}{\min_\Omega p} + \frac{C_*^{\frac{1}{2}}\|p'\|_\infty}{\min_\Omega p} + 1\right)$

$\frac{h}{\pi}\left(1+\frac{h^2}{\pi^2}\right)^{1/2} \|f\|_{L_2}$