Aili Shao

α/β

FEMs for PDEs - Problem Sheet 6

1. Suppose that $d \in \{2,3\}$ and $\Omega \subset \mathbb{R}^d$ is a bounded open set with Lipschitz continuous boundary $\partial\Omega$. Suppose further that $\beta = (\beta_1, \dots, \beta_d)^{\mathrm{T}}$ and $f = (f_1, \dots, f_d)^{\mathrm{T}}$ are d-component vector functions, with $\beta_i \in H^1(\Omega)$, $f_i \in L^2(\Omega)$, and $\operatorname{div} \beta = 0$ a.e. on Ω.

   State the weak formulation of the following linearized Navier–Stokes boundary value-problem for the unknown velocity field $u = (u_1, \dots, u_d)^{\mathrm{T}}$ and pressure p:

$$-\Delta u + \operatorname{div}(u \otimes \beta) + \nabla p = f \qquad \text{in } \Omega, \tag{1a}$$

$$\operatorname{div} u = 0 \qquad \text{in } \Omega, \tag{1b}$$

$$u = 0 \qquad \text{on } \partial\Omega. \tag{1c}$$

   Here $u \otimes \beta$ is the $d \times d$ rank-1 matrix-function with (i,j) entry $u_i\beta_j$, and $\operatorname{div}(u \otimes \beta)$ is a d-component vector function, whose i-th entry is $\sum_{j=1}^d \frac{\partial}{\partial x_j}(u_i\beta_j)$ for $i = 1, \dots, d$.

   Show that there exists a unique weak solution $(u,p) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ to the boundary-value problem (1a), (1b), (1c).

   Show, further, that for $\beta \in H^1(\Omega)^d$ fixed, the mapping

$$f \in L^2(\Omega)^d \mapsto (u,p) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$$

   is Lipschitz continuous.

2. Suppose that X_h and M_h are finite-dimensional linear subspaces of $H_0^1(\Omega)^d$ and $L_0^2(\Omega)$, respectively, parametrized by a positive parameter $h \in (0,1)$, and consider the following approximation of problem (1a), (1b), (1c): find $u_h \in X_h$ and $p_h \in M_h$ such that

$$a(u_h, v_h) + b(v_h, p_h) = (f, v_h) \qquad \forall v_h \in X_h, \tag{2a}$$

$$b(u_h, q_h) = 0 \qquad \forall q_h \in M_h, \tag{2b}$$

   where $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are two bilinear forms on $H_0^1(\Omega)^d \times H_0^1(\Omega)^d$ and $H_0^1(\Omega) \times L_0^2(\Omega)$, that you should carefully define.

   Show that there exists a unique function

$$u_h \in V_h := \{v_h \in X_h \,:\, b(v_h, q_h) = 0 \quad \forall q_h \in M_h\}$$

   such that

$$a(u_h, v_h) = (f, v_h) \qquad \forall v_h \in V_h.$$

   Show further that

$$\|u - u_h\|_{H^1(\Omega)^d} \leq \left(1 + \frac{C_a}{c_a}\right) \inf_{v_h \in V_h} \|u - v_h\|_{H^1(\Omega)^d} + \frac{C_b}{c_a} \inf_{q_h \in M_h} \|p - q_h\|_{L^2(\Omega)},$$

   where c_a, C_a and C_b are positive constants that you should specify.

   Finally, show that if the bilinear functional b satisfies the discrete inf-sup condition on $X_h \times M_h$ with a discrete inf-sup constant $c_b > 0$, independent of h, then there exists a unique solution pair $(u_h, p_h) \in X_h \times M_h$ to the problem (2a), (2b) and in addition to the bound above on $\|u - u_h\|_{H^1(\Omega)^d}$ the following bound holds:

$$\|p - p_h\|_{L^2(\Omega)} \leq C \left(\inf_{v_h \in X_h} \|u - v_h\|_{H_0^1(\Omega)^d} + \inf_{q_h \in M_h} \|p - q_h\|_{L_0^2(\Omega)} \right),$$

   where $C = C(c_a, c_b, C_a, C_b)$ is a positive constant.

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Q1
$$\begin{cases} -\Delta \underline{u} + \operatorname{div}(\underline{u} \otimes \underline{\beta}) + \nabla P = \underline{f} & \text{in } \Omega \\ \operatorname{div} \underline{u} = 0 & \text{in } \Omega \\ \underline{u} = 0 & \text{on } \partial\Omega \end{cases}$$

We define the function spaces $X := H_0^1(\Omega)^d = \underbrace{H_0^1(\Omega) \times \cdots \times H_0^1(\Omega)}_{d \text{ times}}$

and $M := L_0^2(\Omega) = \{ q \in L^2(\Omega) : \int_\Omega q \, dx = 0 \}$.

<u>Weak formulation</u>: find a pair of functions $(\underline{u}, p) \in X \times M$

such that $a(\underline{u}, \underline{v}) + b(\underline{v}, p) = \ell_f(\underline{v}) \quad \forall \underline{v} \in X$ (1d)

$b(\underline{u}, q) = 0 \quad \forall q \in M$ (1e).

where $a(\underline{u}, \underline{v}) = \int_\Omega \nabla u : \nabla v \, dx \; - \int_\Omega (\underline{u} \otimes \underline{\beta}) : \nabla \underline{v} \, dx$

and $b(\underline{v}, p) = - \int_\Omega p \operatorname{div} v \, dx$

$\ell_f(\underline{v}) = \int_\Omega f \cdot v \, dx$

We need to check that our bilinear form $a(\cdot,\cdot)$ is well-defined.

i.e. $\int_\Omega (\underline{u} \otimes \underline{\beta}) : \nabla \underline{v} \, dx < \infty$ for $u \in H_0^1(\Omega)^d$

$v \in H_0^1(\Omega)^d$ and $\underline{\beta} \in H^1(\Omega)^d$

Recall Sobolev's embedding that. if $d = 3$,

$H^1(\Omega) \hookrightarrow L^6(\Omega)$ compactly

since $\partial\Omega$ is Lipschitz continuous.

(we can get even better result for $d = 2$).

$$\int_\Omega |(\underline{u} \otimes \underline{\beta}) : \nabla \underline{v}| \, dx \le \|u\|_{L^6(\Omega)^d} \|\beta\|_{L^6(\Omega)^d} \|\nabla V\|_{L^{\frac{3}{2}}(\Omega)^{3d}}$$

$$\frac{1}{6} + \frac{1}{6} + \frac{1}{q} = 1 \Rightarrow q = \frac{3}{2} \qquad \le C \|u\|_{L^6(\Omega)^d} \|\beta\|_{L^6(\Omega)^d} \|\nabla V\|_{L^{\frac{3}{2}}(\Omega)^d}$$

$$< \infty$$

<u>Existence of uniqueness weak solution</u>

Consider the closed linear subspace V of the Hilbert space X, defined by $V := \{ v \in X : b(\underline{v}, q) = 0 \quad \forall q \in M \}$.

By choosing a test function $\underline{v} \in V \; (\subset X)$ in (1d), we have $a(\underline{u}, \underline{v}) = \ell_f(\underline{v}) \quad \forall \underline{v} \in V$.

Now we aim to check the conditions for applying ~~Lax-Milgram~~

theorem 3.3 from lecture notes. By Poincaré's inequality, ~~① $\forall$~~ $\|V\|_{H^1(\Omega)^d}$ and $|V|_{H^1(\Omega)^d}$ are equivalent for $V \in H_0^1(\Omega)^d$. so we define $\|V\|_{H_0^1(\Omega)^d} = |V|_{H^1(\Omega)^d}$.

① $|a(\underline{u},\underline{v})| \le \int_\Omega |\nabla \underline{u} : \nabla \underline{v}|\, dx + \int_\Omega |(\underline{u} \otimes \underline{\beta}) : \nabla \underline{v}|\, dx$

$$\le \|\nabla \underline{u}\|_{L^2(\Omega)^d} \|\nabla \underline{v}\|_{L^2(\Omega)^d} + \|\underline{u}\|_{L^6(\Omega)^d} \|\underline{\beta}\|_{L^6(\Omega)^d} \|\nabla \underline{v}\|_{L^{\frac{3}{2}}(\Omega)^d}$$

$$\le |\underline{u}|_{H^1(\Omega)^d} |\underline{v}|_{H^1(\Omega)^d} + |\Omega|^{\frac{1}{6}} \|\underline{u}\|_{L^6(\Omega)^d} \|\underline{\beta}\|_{L^6(\Omega)^d} |\underline{v}|_{H^1(\Omega)^d}$$

$$\le \|\underline{u}\|_{H_0^1(\Omega)^d} \|\underline{v}\|_{H_0^1(\Omega)^d} + C(d) |\Omega|^{\frac{1}{6}} \|\underline{u}\|_{H^1(\Omega)^d} \|\underline{\beta}\|_{H^1(\Omega)^d} \|\underline{v}\|_{H_0^1(\Omega)^d}$$

$$\le \|\underline{u}\|_X \|\underline{v}\|_X + C(d)\, C_p(\Omega) |\Omega|^{\frac{1}{6}} \|\underline{u}\|_X \|\underline{\beta}\|_{H^1(\Omega)^d} \|\underline{v}\|_X$$

$C_p(\Omega)$'s Poincaré constant

$$= \left(1 + C(d)\, C_p(\Omega) |\Omega|^{\frac{1}{6}} \|\underline{\beta}\|_{H^1(\Omega)^d}\right) \|\underline{u}\|_X \|\underline{v}\|_X$$

$$= C_a \|\underline{u}\|_X \|\underline{v}\|_X \qquad \forall\, v \in V$$

where ~~with~~ $C_a = 1 + C(d)\, C_p(\Omega) |\Omega|^{\frac{1}{6}} \|\underline{\beta}\|_{H^1(\Omega)^d}$ ✓

② $a(\underline{v},\underline{v}) = \int_\Omega \nabla \underline{v} : \nabla \underline{v}\, dx - \int_\Omega (\underline{v} \otimes \underline{\beta}) \cdot \nabla \underline{v}\, dx$

Note that $\int_\Omega (\underline{v} \otimes \beta) : \nabla \underline{v}\, dx$

$$= \int_\Omega \sum_{i=1}^d \sum_{j=1}^d v_i \beta_j \frac{\partial}{\partial x_j} v_i\, dx$$

$$= \int_\Omega - \sum_{i=1}^d \sum_{j=1}^d \frac{\partial}{\partial x_j}(v_i \beta_j)\, v_i\, dx$$

$$= \int_\Omega - \sum_{i=1}^d \sum_{j=1}^d \frac{\partial}{\partial x_j} v_i \cdot \beta_j v_i\, dx - \cancel{\int_\Omega \sum_{i=1}^d \sum_{j=1}^d v_i \frac{\partial \beta_j}{\partial x_j} v_i\, dx} \quad 0$$

since $\operatorname{div} \beta = 0$

$$= - \int_\Omega \nabla V : (\underline{V} \otimes \beta)\, dx$$

$\Rightarrow \int_\Omega (\underline{v} \otimes \beta) : \nabla \underline{v}\, dx = 0$

$\Rightarrow a(\underline{v},\underline{v}) = \int_\Omega \nabla \underline{v} : \nabla \underline{v}\, dx$

$$= \|\nabla v\|^2_{L^2(\Omega)^d}$$ ✓

$$\ge C_a \|v\|_X^2 \quad \text{with } C_a = 1 \quad \forall\, v \in V.$$

③ $|\ell_f(v)| \le \int_\Omega |f \cdot v|\, dx$

$$\le \|f\|_{L^2(\Omega)^d} \|v\|_{L^2(\Omega)^d}.$$

$$\le C_p \|f\|_{L^2(\Omega)^d} \|v\|_X \quad \text{(by Poincaré's inequality)}$$

$$\le C_\ell \|v\|_X$$

with $C_\ell = C_p \|f\|_{L^2(\Omega)^d}$.

(4) $b(\underline{v}, q) = -\int_\Omega q \operatorname{div} v \, dx$

$$\|q\|_{L^2(\Omega)} = \sup_{\operatorname{div} v \in L^2(\Omega)\setminus\{0\}} \frac{(\operatorname{div} v, q)}{\|\operatorname{div} v\|_{L^2(\Omega)}}$$

$$\leq \sup_{\underline{v} \in X\setminus\{0\}} \frac{(\operatorname{div} v, q)}{c|\underline{v}|_{H^1(\Omega)^d}}$$

~~since $\|\operatorname{div} v\|_{L^2(\Omega)}$~~
$|\underline{v}|_{H^1(\Omega)^d}$
$= \|\nabla v\|_{L^2(\Omega)^d}$
$\leq c\|\operatorname{div} v\|_{L^2(\Omega)}$
for some constant c.

$$\Rightarrow C_b \|q\|_{L^2(\Omega)} \leq \sup_{\underline{v} \in X\setminus\{0\}} \frac{(\operatorname{div} v, q)}{|v|_{H^1(\Omega)^d}}$$

$$= \sup_{\underline{v} \in X\setminus\{0\}} \frac{(\operatorname{div} v, q)}{\|v\|_X}.$$

(5) $|b(v, q)|$
$\leq \int_\Omega |q \cdot \operatorname{div} v| \, dx$
$\leq \|q\|_{L^2(\Omega)} \|\operatorname{div} v\|_{L^2(\Omega)^d}$
$\leq C_b \|q\|_{L^2(\Omega)} \|v\|_X$.

i.e. b satisfies the inf-sup conditions.

By theorem 3.3 from the lecture notes, we deduce that $\exists!$ pair $(u, p) \in X \times M$ that solves ~~(1a)~~ (1d), (1e).

$\Rightarrow \exists!$ weak solution $(u, p) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ to the BVP (1a)-(1c).

✓✓ great!

Lipschitz continuity:

(1f) $a(u_f, \underline{v}) + b(\underline{v}, P_f) = (\underline{f}, \underline{v}) \quad \forall \underline{v} \in X$

(1g) $a(u_g, \underline{v}) + b(\underline{v}, P_g) = (\underline{g}, \underline{v}) \quad \forall \underline{v} \in X$

(1f) - (1g) $\Rightarrow$

$$a(u_f - u_g, \underline{v}) + b(\underline{v}, P_f - P_g) = (\underline{f} - \underline{g}, \underline{v}) \quad \forall \underline{v} \in X.$$

Taking $v = u_f - u_g \in X$, we have

$$a(u_f - u_g, u_f - u_g) + b(u_f - u_g, P_f - P_g) = (f - g, u_f - u_g)$$

By the coercivity of $a(\cdot, \cdot)$ and inf-sup conditions of $b(\cdot, \cdot)$, we have [?] this step is dubious to me

$$C_a \|u_f - u_g\|_X^2 + C_b \|u_f - u_g\|_X \|P_f - P_g\|_{L^2(\Omega)} \leq \|f - g\|_{L^2(\Omega)^d} \cdot C(\Omega)\|u_f - u_g\|_X$$

where $C(\Omega)$ is the Poincaré's constant.

$$\Rightarrow \quad C_a \| u_f - u_g \|_X + C_b \| P_f - P_g \|_{L^2(\Omega)} \le C(\Omega) \| f - g \|_{L^2(\Omega)^d}$$

$$\Rightarrow \quad C_a \| u_f - u_g \|_X \le C(\Omega) \| f - g \|_{L^2(\Omega)}$$

and $\quad C_b \| P_f - P_g \|_{L^2(\Omega)} \le C(\Omega) \| f - g \|_{L^2(\Omega)}$

That is, for β fixed. the mapping

$$f \in L^2(\Omega)^d \to (u, p) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$$

is Lipschitz continuous.

Let (u_1, p_1) and (u_2, p_2) be solutions corresponding to $f_1, f_2 \in L^2$ (respectively).

$$a(u_1 - u_2, v) + b(v, p_1 - p_2) = (f_1 - f_2, v) \quad \forall v \in H_0^1$$

$$b(u_1 - u_2, q) = 0. \quad \forall q \in L_0^2$$

Take $v = u_1 - u_2$

$f = p_1 - p_2$

$$\| \nabla (u_1 - u_2) \|_{L^2}^2 \le C_p \| f_1 - f_2 \|_{L^2} \| \nabla (u_1 - u_2) \|_{L^2}$$

$$\Rightarrow \quad \| \nabla (u_1 - u_2) \|_{L^2} \le C_p \| f_1 - f_2 \|_{L^2}$$

$$C_b \| p_1 - p_2 \|_{L^2} \le \sup_{v \in H_0^1} \frac{b(v, p_1 - p_2)}{\| \nabla v \|_{L^2}} = \sup_{v \in H_0^1} \frac{(f_1 - f_2, v) - a(u_1 - u_2, v)}{\| \nabla v \|_{L^2}} \le \tilde{C} \| f_1 - f_2 \|_{L^2}$$

$C_0 \| q \|_{L^2} \le \| \nabla q \|_{-1} \quad \forall q \in L_0^2$.

$\nabla : L_0^2(\Omega) \to \ker(\mathrm{div})^\perp$ is an isomorphism.

$\mathrm{div} : H_0^1(\Omega) / \ker(\mathrm{div}) \to L_0^2(\Omega)$

Prove. $\forall f \in L^2(\Omega)$ with $\int_\Omega f = 0$. $\exists v \in H_0^1(\Omega)$ s.t. $\mathrm{div}\, v = f$.

$v = -\nabla \varphi$

$\Rightarrow -\Delta \varphi = f$.

$$\Rightarrow \quad \| u_1 - u_2 \|_{H_0^1} + \| p_1 - p_2 \|_{L^2} \le \tilde{C} \| f_1 - f_2 \|_{L^2}.$$

Q2 Find $u_h \in X_h$ and $p_h \in M_h$ s.t.

$$a(u_h, v_h) + b(v_h, p_h) = (f, v_h) \quad \forall\, v_h \in X_h \quad (2a)$$

$$b(u_h, q_h) = 0 \quad \forall\, q_h \in M_h \quad (2b)$$

where $a(u_h, v_h) := \int_\Omega \nabla u_h : \nabla v_h \, dx - \int_\Omega (\underline{u}_h \otimes \underline{\beta}_h) : \nabla v_h \, dx$

and $b(v_h, p_h) = \int_\Omega p_h \, \mathrm{div}(v_h) \, dx.$

$V_h := \{ v_h \in X_h : b(v_h, q_h) = 0 \quad \forall q_h \in M_h \}$

$a(u_h, v_h) = \ell_f(v_h)$ for all $v_h \in V_h$.

① $|a(u_h, v_h)| \le \int_\Omega |\nabla u_h : \nabla v_h| \, dx + \int_\Omega |(u_h \otimes \beta) : \nabla v_h|$

$\le \|\nabla u_h\|_{L^2(\Omega)^d} \|\nabla v_h\|_{L^2(\Omega)^d} + \|u_h\|_{L^6(\Omega)^6} \|\underline{\beta}\|_{L^6(\Omega)^d} \|\nabla \underline{v}_n\|_{L^{\frac{3}{2}}(\Omega)^d}$

$\le \|u_h\|_X \|v_h\|_X + C(d) |\Omega|^{\frac{1}{6}} C_p(\Omega) \|u_h\|_X \|\beta\|_{H^1(\Omega)^d} \|v_h\|_X$

$= C_a \|u_h\|_X \|v_h\|_X$

with $C_a = 1 + C(d)\, C_p(\Omega) \, |\Omega|^{\frac{1}{6}} \|\beta\|_{H^1(\Omega)^d}.$

② It also follows ~~that~~ from Q1 that $c_a \|v_h\|_X^2 \le a(v_h, v_h)$, $\forall\, v_h \in V_h$.

③ $|$ ~~$\ell_f(v)| \le$~~ $|(f, v_h)| \le \|f\|_{L^2(\Omega)^d} \|v_h\|_{L^2(\Omega)^d}$

$\le C(\Omega) \|f\|_{L^2(\Omega)^d} \|v_h\|_X.$

By Lax-Milgram Theorem, $\exists!$ $u_h \in V_h$ s.t

$$a(u_h, v_h) = (f, v_h) \quad \forall\, v_h \in V_h.$$

By taking $v = w_h \in V_h \subset X$ in (1d) in Q1 and substracting the resulting equation from (2a) with $v_h = w_h \in V_h$, we have

$$a(u - u_h, w_h) = a(u, w_h) - a(u_h, w_h)$$

$$= \ell_f(w_h) - b(w_h, p) - a(u_h, w_h)$$

$$= -b(w_h, p)$$

$$= -b(w_h, p - q_h) \quad \text{for all } q_h \in M_h.$$

Therefore, $a(u - u_h, w_h) + b(w_h, p - q_h) = 0 \quad \forall\, w_h \in V_h \quad \forall q_h \in M_h.$

For any $v_h \in V_h$. $u_h - v_h \in V_h$. Take $w_h = u_h - v_h$, then

$$a(u - u_h, u_h - v_h) + b(u_h - v_h, p - q_h) = 0$$

$$C_a \|u_n - v_n\|_X^2 \le a(u_n - v_n, u_n - v_n)$$
$$= a(u - v_n, u_n - v_n) + a(u_n - u, u_n - v_n)$$
$$= a(u - v_n, u_n - v_n) + b(u_n - v_n, \cancel{u_n - v_n})\, p - q_n)$$

Dividing by $c_a \|u_n - v_n\|_X$ on both sides, we have

$$\|u_n - v_n\|_X \le \frac{C_a}{c_a}\|u - v_n\|_X + \frac{C_b}{c_a}\|p - q_n\|_M \quad \forall q_n \in M_h.$$

C_a, c_a and C_b follow from Q1

Note that $\|u - u_n\|_X \le \|u - v_n\|_X + \|u_n - v_n\|_X$

$$\Rightarrow \|u - u_n\|_X \le \left(1 + \frac{C_a}{c_a}\right)\|u - v_n\|_X + \frac{C_b}{c_a}\|p - q_n\|_M.$$

Taking inf over all $v_n \in V_n$ and all $q_n \in M_h$.

$$\Rightarrow \|u - u_n\|_X \le \left(1 + \frac{C_a}{c_a}\right)\inf_{v_n \in V_n}\|u - v_n\|_X + \frac{C_b}{c_a}\inf_{q_h \in M_h}\|p - q_h\|_{L^2(\Omega)}$$

✓✓

If the bilinear functional b satisfies the discrete inf-sup condition on $X_h \times M_h$

$$\tilde{c}_b \le \inf_{q_h \in M_h \setminus \{0\}} \sup_{v_h \in X_h \setminus \{0\}} \frac{b(v_h, q_h)}{\|v_h\|_X \|q_h\|_M}$$

(Note $\tilde{c}_b$ and c_b may not be the same).

then the existence of a unique solution pair (u_h, p_h) follows from theorem 3.3 in lecture notes with X and M replaced by X_h and M_h.

By the discrete inf-sup condition,

$$\tilde{c}_b \|q_h - p_h\|_M \le \sup_{w_h \in X_h \setminus \{0\}} \frac{b(w_h, q_h - p_h)}{\|w_h\|_X}$$

$$= \sup_{w_h \in X_h \setminus \{0\}} \frac{b(w_h, p - p_h) + b(w_h, q_h - p)}{\|w_h\|_X}$$

$$\le \sup_{w_h \in X_h \setminus \{0\}} \frac{|b(w_h, p - p_h)| + |b(w_h, q_h - p)|}{\|w_h\|_X}$$

$$= \sup_{w_h \in X_h \setminus \{0\}} \frac{|a(u - u_h, w_h)| + |b(w_h, q_h - p)|}{\|w_h\|_X}$$

$$\le C_a \|u - u_h\|_X + C_b \|p - q_h\|_M.$$

✓

$$\Rightarrow \|p-p_h\|_M \le \|p-q_h\|_M + \|p_h-q_h\|_M$$

$$\le \frac{C_a}{\tilde{C}_b}\|u-u_h\|_X + \left(1+\frac{C_b}{\tilde{C}_b}\right)\|p-q_h\|_M$$

$$\le C\left(\|u-u_h\|_X + \|p-q_h\|_M\right)$$

where $C=\max\left(\frac{C_a}{\tilde{C}_b}, 1+\frac{C_b}{\tilde{C}_b}\right)$.

Taking inf over all $q_h \in M_h$ and $v_h \in X_h$, we have

$$\|p-p_h\|_{L^2(\Omega)} \le C\left(\inf_{v_h\in X_h}\|u-v_h\|_{H_0^1(\Omega)^d} + \inf_{q_h\in M_h}\|p-q_h\|_{L_0^2(\Omega)}\right).$$

where $C=\max\left(\frac{C_a}{\tilde{C}_b}, 1+\frac{C_b}{\tilde{C}_b}\right)$. ✓

you bounded $\|u-u_h\|_{H^1}$ with $\inf_{v_h\in V_h}\|u-v_h\|_{H^1}+\ldots$

whereas question wanted $\inf_{v_h\in X_h}\|u-v_h\|_{H^1}+\ldots$

so a bit more work to do.

$$\|u-u_h\|_{H^1} + \|p-p_h\|_{L^2} \le \hat{C}_1\left[\inf_{v_h\in V_h}\|u-u_h\|_{H^1} + \inf_{q_h\in M_h}\|p-q_h\|_{L^2}\right]$$

$v_h = I_h u$.

[Orthogonal decomposition]

Let $v_h \in X_h$, and let $w_h \in V_h$ s.t. $v_h - w_h \in V_h^\perp$

$\Rightarrow \operatorname{div} w_h = 0$

$$\|u-w_h\|_{H^1} \le \|u-v_h\|_{H^1} + \|v_h-w_h\|_{H^1}$$

$$\|u_h-w_h\|_{H^1} \le \sup_{q_h\in M_h}\frac{b(v_h-w_h,q_h)}{\|q_h\|_{L^2}}$$

$$= \sup_{q_h\in M_h}\frac{b(v_h-u,q_h)}{\|q_h\|_{L^2}}$$

$$\le \tilde{C}\|u-v_h\|_{H_0^1}.$$

$$\Rightarrow \inf_{w_h\in V_h}\|u-w_h\|_{H^1} \le \tilde{C}\inf_{v_h\in X_h}\|u-v_h\|_{H_0^1}$$