SCI. COMP. FOR DPHIL STUDENTS

ASSIGNMENT 2 SOLUTIONS

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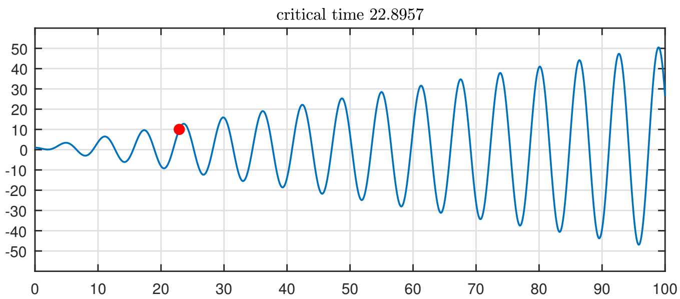
Exercise 8.1. Exploiting resonance to increase amplitude

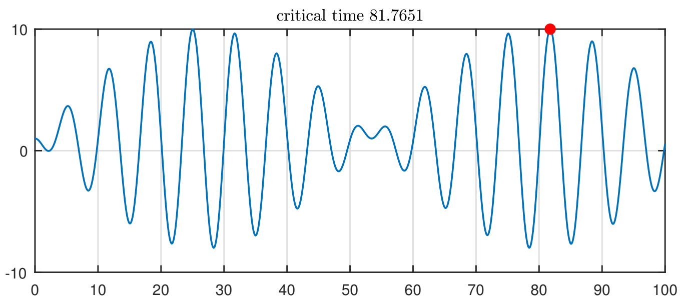
(a) For this we can just solve the problem and use roots to find the time.

```
ODEformats, format long
nu = 1; tmax = 100;
t = chebfun('t',[0 tmax]); L = chebop(0,tmax); L.lbc = [1; 0];
L.op = @(t,y) diff(y,2) + y;
f = 1 - cos(nu*t); y = L\f; plot(y)
tc = roots(y-10); tc = tc(1);
title(['critical time ' num2str(tc)])
hold on, plot(tc,y(tc),'.r','markersize',16), hold off
set(gca,'ytick',-50:10:50)
```



(b) You can do this numerically with a little hand- (or computer-) driven bisection. I think the answer is around $\nu = 0.882464$:

```
nu = .882464; tmax = 100; t = chebfun('t',[0 tmax]);
L = chebop(0,tmax); L.lbc = [1; 0];
L.op = @(t,y) diff(y,2) + y;
f = 1 - cos(nu*t); y = L\f; plot(y)
tc = roots(y-10); tc = tc(1);
title(['critical time ' num2str(tc)])
hold on, plot(tc,y(tc),'.r','markersize',16), hold off
set(gca,'ytick',-50:10:50)
```



Exercise 11.1. Cleve Moler's favorite ODE

(a) The ODE is $(y')^2 + y^2 = 1$, $y(0) = 0$, $-1 \le y \le 1$. Equivalently, the equation is $y' = \pm\sqrt{1-y^2}$. Taking the $+$ branch, we can solve by separation of variables to get $y(t) = \sin(t)$;

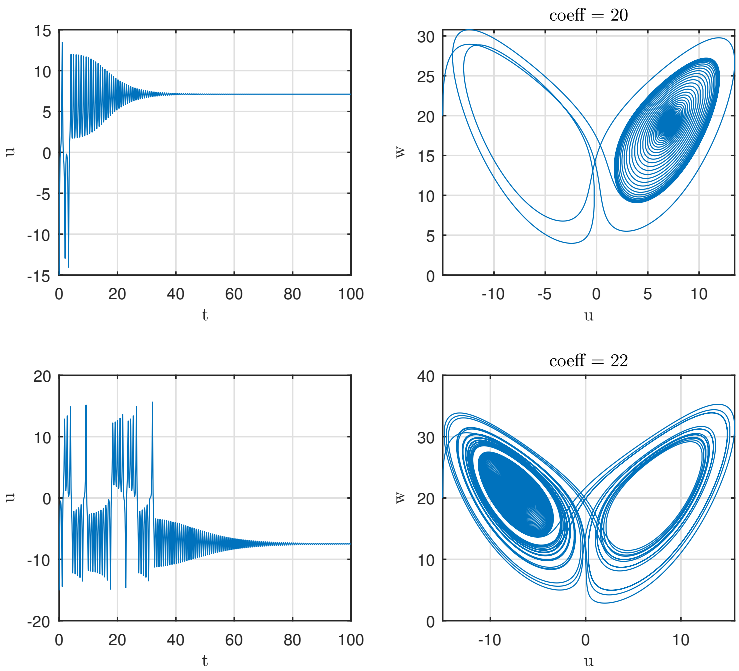
similarly the $-$ branch gives $y(t) = -\sin(t)$. These solutions continue uniquely so long as the right-hand side is smooth (the precise condition involves Lipschitz continuity), i.e., until $|y|$ reaches 1, which happens at $t = \pi/2$. So for $t \in [0, 1]$, we have exactly these two solutions.

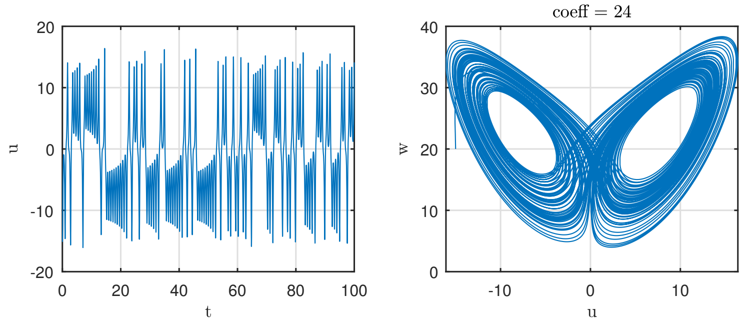
(b) The interval $t \in [0, 2]$, on the other hand, takes us past the point $t = \pi/2$ and $|y| = 1$. Let's consider the $+$ branch, with values $y(t) = \sin(t)$ for $t \le \pi/2$. For $t > \pi/2$, $y(t)$ might remain constant at value 1. Equally well, it could be 1 until some arbitrary time $t_c > \pi/2$ and then suddenly start decreasing again according to $y(t) = \sin(\pi/2 + (t - t_c))$. Thus we have infinitely many possible solutions, and another infinitely many on the $-$ branch.

Exercise 13.3. Alternative choices of the Lorenz coefficient 28

The curves suggest that $a = 20$ gives a stable fixed point, $a = 22$ gives transient chaos, and $a = 24$ gives chaos (like the standard choice $a = 28$).

```
N = chebop(0,100);
for a = [20 22 24]
  N.op = @(t,u,v,w) [diff(u)-10*(v-u); ...
    diff(v)-u*(a-w)+v; diff(w)-u*v+(8/3)*w];
  N.domain = [0 100];
  N.lbc = [-15; -15; 20]; [u,v,w] = N\0;
  subplot(1,2,1), plot(u,LW,0.5), xlabel t, ylabel u
  subplot(1,2,2), plot(u,w,LW,0.5), xlabel u, ylabel w
  title(['coeff = ' num2str(a)]), snapnow
end
```



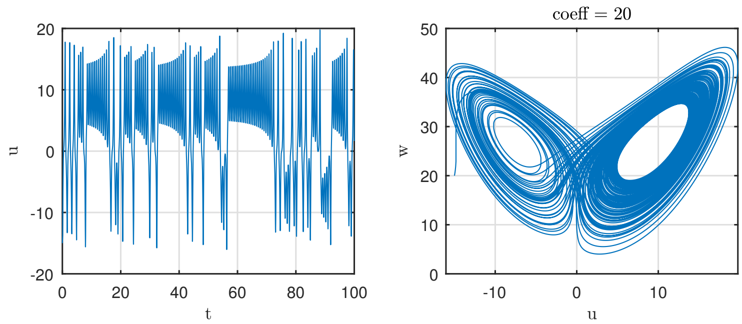


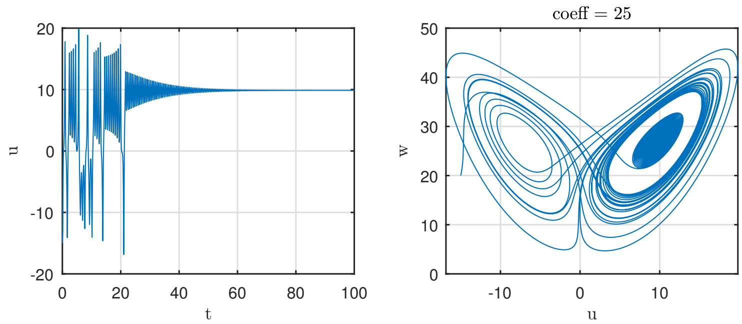
Exercise 13.4. Lorenz equations with a breeze

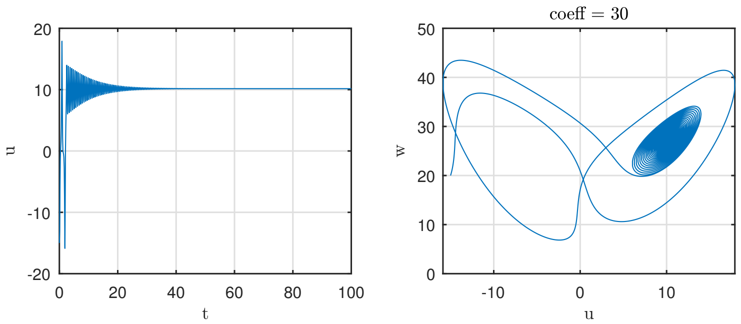
The curves suggest that $a = 30$ gives a stable fixed point, $a = 25$ gives transient chaos, and $a = 20$ gives chaos (like the standard choice $a = 0$).

Notice how visible the "breeze" is in the u-w plots, where we see most of the time being spent with values $u > 0$.

```
N = chebop(0,100);
for a = [20 25 30]
  N.op = @(t,u,v,w) [diff(u)-10*(v-u)-a; ...
    diff(v)-u*(28-w)+v; diff(w)-u*v+(8/3)*w];
  N.domain = [0 100];
  N.lbc = [-15; -15; 20]; [u,v,w] = N\0;
  subplot(1,2,1), plot(u,LW,0.5), xlabel t, ylabel u
  subplot(1,2,2), plot(u,w,LW,0.5), xlabel u, ylabel w
  title(['coeff = ' num2str(a)]), snapnow
end
```



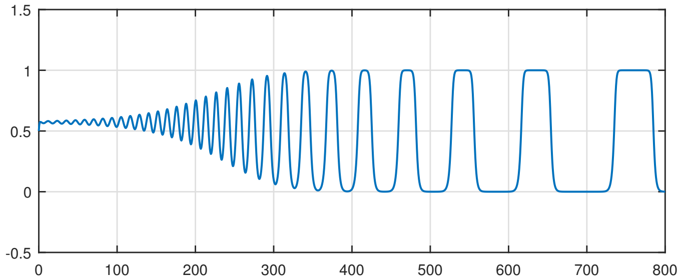




Exercise 15.5. A cyclic system of three ODEs

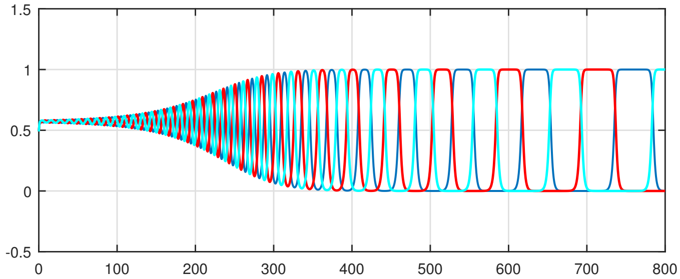
(a) Here is a plot of $u(t)$,

```
b = 0.55; c = 1.5;
N = chebop(0,800); clf
N.op = @(t,u,v,w) [ ...
    diff(u) - u.*(1 - u.^2 - b*v.^2 - c*w.^2)
    diff(v) - v.*(1 - v.^2 - b*w.^2 - c*u.^2)
    diff(w) - w.*(1 - w.^2 - b*u.^2 - c*v.^2)];
N.lbc = @(u,v,w) [u-0.5; v-0.49; w-0.49];
[u,v,w] = N\0;
plot(u,LW,1), ylim([-0.5 1.5])
```



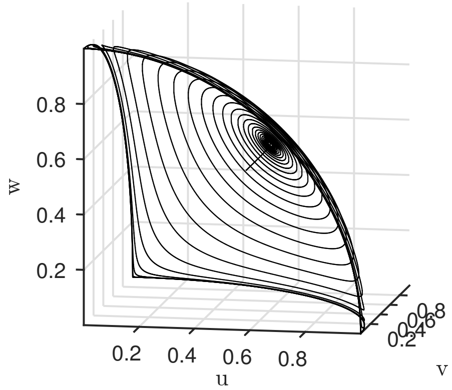
and here we add $v(t)$ and $w(t)$ to the same plot,

```
hold on, plot(v,LW,1.2,'r'), plot(w,LW,1.2,'c')
```



On a 3D plot, we can see how the orbit swings from one corner in the u, v, w octant to the next to the next. Most of the time is spent near the corners, where the velocity is low. The orbit is approaching a *heteroclinic limit cycle* between the three fixed points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.

```
hold off, plot3(u,v,w,'k',LW,.4), view(10,10), axis equal, grid on
xlabel u, ylabel v, zlabel w
```

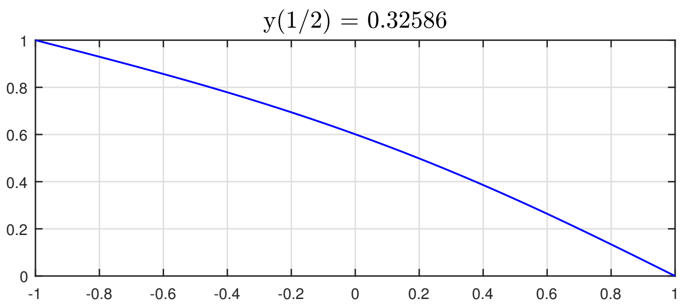


(b) One fixed point is $(u, v, w) = (0, 0, 0)$. If just two variables are zero, we find further fixed points at $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$. If just one variable is zero, we find further fixed points with $u = 0$, $w^2 = (c-1)/(bc-1)$, and $v^2 = 1 - b(c-1)/(bc-1)$ and similarly with the variables permuted. And if no variables are zero we find fixed points with u^2, v^2, and w^2 all taking the value $1/(1+b+c)$. This last is evident in the 3D plot.

Exercise 16.1. Fisher equation

(a) If we try a Chebfun BVP solution with the default initial guess, we get the solution required:

```
N = chebop(-1,1); N.lbc = 1; N.rbc = 0;
N.op = @(x,y) diff(y,2) + y - y^2;
y1 = N\0; plot(y1,'b'), grid on
title(['y(1/2) = ' num2str(y1(0.5))],FS, 12)
```



(b) Clearly for this we need a negative initial guess. My first try worked:

```
N = chebop(-1,1); N.lbc = 1; N.rbc = 0;
N.op = @(x,y) diff(y,2) + y - y^2;
x = chebfun('x');
N.init = (1-x)/2-3*cos(pi*x); y2 = N\0; plot(y2,'r'), grid on
title(['y(1/2) = ' num2str(y2(0.5))],FS, 12)
```