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Parabolic PDEs -Problem Sheet Two

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Exercise 1. *Read the proof of Theorem 1 in E.Stein "Singular Integrals and Differentiability Properties of Functions",pp. 5–11.*

Theorem 1. *(Hardy-Littlewood Maximal Functions)*
Define $M_f(x) := \sup\{\frac{1}{|B(r)|}\int_{B(x,r)} |f(y)|\,\mathrm{d}y \colon 0 < r < \infty\}$.

(i) Let $f \in L_1(\mathbb{R}^n)$*. Then*

$$|\{x \colon M_f(x) > \alpha\}| < \frac{A(n)}{\alpha}\int_{\mathbb{R}^n} |f(x)|\,\mathrm{d}x.$$

(ii) Let $f \in L_p(\mathbb{R}^n)$ *with* $1 < p \le \infty$*. Then*

$$\|M_f\|_p \le A(n,p)\|f\|_p.$$

Proof. Here we shall prove the theorem using the following *Vitali's Covering Lemma*:

Lemma 1. *Let* E *be a measurable subset of* $\mathbb{R}^n$ *which is covered by the union of a family of balls* $\{B_j\}$*, of bounded diameter. Then from this family we can select a disjoint subsequence,* $B_1, B_2, \dots B_k, \dots,$ *(finite or infinite) so that*

$$\sum_k m(B_k) \ge Cm(E).$$

Here C *is a positive constant that depends only on the dimension* n*;* $C = 5^{-n}$ *will do.*

With the definition of M_f, and with

$$E_\alpha = \{x \colon M_f(x) > \alpha\}$$

then for each $x \in E_\alpha$ there exists a ball of centre x, which we call B_x, so that

$$\int_{B_x} |f(y)| \, \mathrm{d}y > \alpha m(B_x) \tag{0.1}$$

here by m we mean the Lebesgue measure. (0.1) implies that $m(B_x) < \frac{1}{\alpha}\|f\|_{L_1}$, for all such x. When x runs through the set E_α, the union of the corresponding B_x covers E_α. By using the *Vitali's Covering Lemma*, we can extract a sequence of mutually disjoint balls $\{B_k\}$ such that

$$\sum_{k=0}^{\infty} m(B_k) \geq C(n) m(E_\alpha). \tag{0.2}$$

Applying (0.1), (0.2) to each of the mutually disjoint balls we get

$$\int_{\bigcup B_k} |f(y)| \, \mathrm{d}y > \alpha \sum_k m(B_k) \geq \alpha C m(E_\alpha). \tag{0.3}$$

Thus (i) is proved by taking $A(n) = \frac{1}{C(n)}$.
We shall now prove the theorem for the case $1 < p \leq \infty$. The case $p = \infty$ is trivially true by taking $A(n,p) = 1$. Let us therefore suppose that $1 < p < \infty$. Define

$$f_1(x) = \begin{cases} f(x) & \text{if } |f(x)| \geq \frac{\alpha}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have successively $|f(x)| \leq |f_1(x)| + \frac{\alpha}{2}$; $M_f(x) \leq M_{f_1}(x) + \frac{\alpha}{2}$, therefore

$$\{x \colon M_f(x) > \alpha\} \subset \{x \colon M_{f_1}(x) > \frac{\alpha}{2}\},$$

and finally

$$m(E_\alpha) = m(\{x \colon M_f(x) > \alpha\}) \leq \frac{2A(n)}{\alpha} \|f_1\|_{L_1},$$

which is

$$m(E_\alpha) = m(\{x \colon M_f(x) > \alpha\}) \leq \frac{2A(n)}{\alpha} \int_{|f| > \frac{\alpha}{2}} |f| \, \mathrm{d}x. \tag{0.4}$$

We now set $g = M_f$ and λ the distribution function of g. Then we have

$$\int_{\mathbb{R}^n} (M_f)^p \, \mathrm{d}x = -\int_0^\infty \alpha^p \mathrm{d}\lambda(\alpha) = p \int_0^\infty \alpha^{p-1} \lambda(\alpha) \, \mathrm{d}\alpha.$$

In particular, because of (0.4),

$$\|M_f\|_p^p = p \int_0^\infty \alpha^{p-1} m(E_\alpha) \, \mathrm{d}\alpha \leq p \int_0^\infty \alpha^{p-1} \left(\frac{2A(n)}{\alpha} \int_{|f| > \frac{\alpha}{2}} |f(x)| \, \mathrm{d}x \right) \mathrm{d}\alpha.$$

The double integral is evaluated by interchanging the orders of integrating and integrating first with respect to α. The inner integral is then

$$\int_0^{2|f(x)|} \alpha^{p-2}\,\mathrm{d}\alpha = \left(\frac{1}{p-1}\right)|2f(x)|^{p-1},$$

since $p > 1$. So the double integral has the value

$$\frac{2A(n)p}{p-1}\int_{\mathbb{R}^n} |f||2f|^{p-1}\,\mathrm{d}x = (A(n,p))^p \int_{\mathbb{R}^n} |f|^p\,\mathrm{d}x,$$

which proves (ii). $\square$

Exercise 2. *Consider the pressure equation*

$$\Delta p = -\mathrm{div}\mathrm{div}F,$$

where $F = (F_{ij})$ is a tensor valued field that belongs to $C_0^\infty(\mathbb{R}^3)$. Show that there exists a solution to the above pressure problem such that

$$p(x) = -\frac{1}{3}\mathrm{tr}F(x) + \lim_{\varepsilon\to 0}\frac{1}{4\pi}\int_{\mathbb{R}^3\setminus B(x,\varepsilon)} K_0(x-y) : F(y)\mathrm{d}y$$

where

$$K_0(X) = \nabla^2\left(\frac{1}{|x|}\right).$$

Proof. The solution can be written in the following form by using the Newtonian potential

$$p(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|x-y|}\mathrm{div}\mathrm{div}F(y)\,\mathrm{d}y.$$

After double integration by parts, we have

$$\begin{aligned}
p(x) =& \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|x-y|}\mathrm{div}\mathrm{div}F(y)\,\mathrm{d}y.\\
=& \lim_{\varepsilon\to 0}\frac{1}{4\pi}\int_{\mathbb{R}^3\setminus B(x,\varepsilon)} \nabla^2\left(\frac{1}{|x-y|}\right) : F(y)\,\mathrm{d}y\\
&+ \lim_{\varepsilon\to 0}\frac{1}{4\pi}\int_{\partial B(x,\varepsilon)}\frac{1}{|x-y|}\mathrm{div}F(y)\cdot n\,\mathrm{d}s\\
&+ \lim_{\varepsilon\to 0}\frac{1}{4\pi}\int_{\partial B(x,\varepsilon)} -\nabla\left(\frac{1}{|x-y|}\right)F(y)\cdot n\,\mathrm{d}s\\
=& \lim_{\varepsilon\to 0}\frac{1}{4\pi}\int_{\mathbb{R}^3\setminus B(x,\varepsilon)} \nabla^2\left(\frac{1}{|x-y|}\right) : F(y)\,\mathrm{d}y + \mathrm{I} + \mathrm{II}
\end{aligned}$$

where

$$\mathrm{I} := \lim_{\varepsilon\to 0}\frac{1}{4\pi}\int_{\partial B(x,\varepsilon)}\frac{1}{|x-y|}\mathrm{div}F(y)\cdot n\,\mathrm{d}s$$

and

$$\mathrm{II} := \lim_{\varepsilon \to 0} \frac{1}{4\pi} \int_{\partial B(x,\varepsilon)} -\nabla \left(\frac{1}{|x-y|} \right) F(y) \cdot n \, \mathrm{d}s$$

with the outward pointing normal defined as $n = (-\frac{x_1-y_1}{|x-y|}, -\frac{x_2-y_2}{|x-y|}, -\frac{x_3-y_3}{|x-y|})^T$. First note that $\mathrm{I} = \lim_{\varepsilon \to 0} \frac{1}{4\pi} 4\pi\varepsilon^2 \frac{1}{\varepsilon} \mathrm{div} F(y) \cdot (\frac{\varepsilon_1}{\varepsilon}, \frac{\varepsilon_2}{\varepsilon}, \frac{\varepsilon_3}{\varepsilon})^T = 0$ where $\varepsilon_i = y_i - x_i$ $(i = 1, 2, 3)$ for y on $\partial B(x, \varepsilon)$.

$$\begin{aligned}
\mathrm{II} &= \lim_{\varepsilon \to 0} \frac{1}{4\pi} \int_{\partial B(x,\varepsilon)} -\nabla \left(\frac{1}{|x-y|} \right) F(y) \cdot n \, \mathrm{d}s \\
&= \lim_{\varepsilon \to 0} \frac{1}{4\pi} \int_{\partial B(x,\varepsilon)} (\frac{x_1-y_1}{|x-y|^3}, \frac{x_2-y_2}{|x_y|^3}, \frac{x_3-y_3}{|x-y|^3}) F(y) (-\frac{x_1-y_1}{|x-y|}, -\frac{x_2-y_2}{|x-y|}, -\frac{x_3-y_3}{|x-y|})^T \mathrm{d}s \\
&= \lim_{\varepsilon \to \infty} \frac{1}{4\pi} \int_{\partial B(x,\varepsilon)} - \left(\frac{(x_1-y_1)^2}{|x-y|^4} F_{11}(y) + \frac{(x_2-y_2)^2}{|x-y|^4} F_{22}(y) + \frac{(x_3-y_3)^2}{|x-y|^4} F_{33}(y) \right) \mathrm{d}s \\
&= \lim_{\varepsilon \to \infty} \frac{1}{4\pi} \int_{\partial B(x,\varepsilon)} - \left(\frac{(x_1-y_1)^2}{|x-y|^4} (F_{11}(x) + \langle y-x, \nabla F_{11}(\xi_1) \rangle) \right) \mathrm{d}s \\
&\quad - \lim_{\varepsilon \to \infty} \frac{1}{4\pi} \int_{\partial B(x,\varepsilon)} \left(\frac{(x_2-y_2)^2}{|x-y|^4} (F_{22}(x) + \langle y-x, \nabla F_{22}(\xi_2) \rangle) \right) \mathrm{d}s \\
&\quad - \lim_{\varepsilon \to \infty} \frac{1}{4\pi} \int_{\partial B(x,\varepsilon)} \left(\frac{(x_3-y_3)^2}{|x-y|^4} (F_{33}(x) + \langle y-x, \nabla F_{33}(\xi_3) \rangle) \right) \mathrm{d}s \\
&\quad \text{(by Mean Value theorem with } \xi_i \text{ in between } x \text{ and } y) \\
&= \lim_{\varepsilon \to 0} \frac{1}{4\pi} \int_{\partial B(x,\varepsilon)} - \left(\frac{(x_1-y_1)^2}{|x-y|^4} F_{11}(x) + \frac{(x_2-y_2)^2}{|x-y|^4} F_{22}(x) + \frac{(x_3-y_3)^2}{|x-y|^4} F_{33}(x) \right) \mathrm{d}s \\
&\quad - \lim_{\varepsilon \to 0} \frac{1}{4\pi} \int_{\partial B(x,\varepsilon)} \frac{(x_1-y_1)^2}{|x-y|^4} \langle y-x, \nabla F_{11}(\xi_1) \rangle \, \mathrm{d}s \\
&\quad - \lim_{\varepsilon \to 0} \frac{1}{4\pi} \int_{\partial B(x,\varepsilon)} \frac{(x_2-y_2)^2}{|x-y|^4} \langle y-x, \nabla F_{22}(\xi_2) \rangle \, \mathrm{d}s \\
&\quad - \lim_{\varepsilon \to 0} \frac{1}{4\pi} \int_{\partial B(x,\varepsilon)} \frac{(x_3-y_3)^2}{|x-y|^4} \langle y-x, \nabla F_{33}(\xi_3) \rangle \, \mathrm{d}s
\end{aligned}$$

Note that

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \frac{1}{4\pi} \int_{\partial B(x,\varepsilon)} - \left(\frac{(x_1-y_1)^2}{|x-y|^4} F_{11}(x) + \frac{(x_2-y_2)^2}{|x-y|^4} F_{22}(x) + \frac{(x_3-y_3)^2}{|x-y|^4} F_{33}(x) \right) \mathrm{d}s \\
&= - \lim_{\varepsilon \to 0} \frac{1}{4\pi} 4\pi\varepsilon^2 \frac{1}{\varepsilon^2} \left((x_1-y_1)^2 F_{11}(x) + (x_2-y_2)^2 F_{22}(x) + (x_3-y_3)^2 F_{33}(x) \right) \\
&= - \frac{1}{3} \mathrm{tr} F(x) \text{ (by symmetry of the 3-D sphere)}
\end{aligned}$$

and

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{1}{4\pi}\int_{\partial B(x,\varepsilon)}\frac{(x_1-y_1)^2}{|x-y|^4}\langle y-x,\nabla F_{11}(\xi_1)\rangle\,\mathrm{d}s\\
&\quad+\lim_{\varepsilon\to 0}\frac{1}{4\pi}\int_{\partial B(x,\varepsilon)}\frac{(x_2-y_2)^2}{|x-y|^4}\langle y-x,\nabla F_{22}(\xi_2)\rangle\,\mathrm{d}s\\
&\quad+\lim_{\varepsilon\to 0}\frac{1}{4\pi}\int_{\partial B(x,\varepsilon)}\frac{(x_3-y_3)^2}{|x-y|^4}\langle y-x,\nabla F_{33}(\xi_3)\rangle\,\mathrm{d}s\\
&\leq\lim_{\varepsilon\to 0}\frac{1}{4\pi}4\pi\varepsilon^2\frac{1}{\varepsilon^4}\varepsilon\varepsilon^2(|\nabla F_{11}(\xi_1)|+|\nabla F_{22}(\xi_2)|)+|\nabla F_{33}(\xi_3)|)\\
&=0.
\end{aligned}
$$

Thus, the required equality follows. $\square$

Exercise 3. *Let $F=(F_{ij})\in C_0^\infty(Q_T)$ and p_F is a solution to the pressure equation that is defined by the Newtonian potential show that*

$$
-\int_0^t\int_{\mathbb{R}^3}\Gamma(x-y,t-\tau)\mathrm{div}(F+p_F)(y,\tau)\,\mathrm{d}y\mathrm{d}\tau=\int_0^t\int_{\mathbb{R}^3}K_{ijl}(x-y,t-\tau)F_{jl}(y,\tau)\,\mathrm{d}y\mathrm{d}\tau
$$

for all $(x,t)\in Q_T$ with $K_{ijl}=\Phi_{,ijl}-\delta_{ij}\Phi_{,kkl}$.

Proof. Note that $F=(F_{ij})\in C_0^\infty$. We observe that $p_F=(-\Delta)^{-1}\mathrm{div}\mathrm{div}F$, so

$$
\|p_F\|_{L^p}\leq C(p,n)\|F\|_{L^p}\text{ for }p\in(1,\infty)
$$

and

$$
\|p_F\|_{BMO(\mathbb{R}^3)}\leq\|F\|_\infty.
$$

Also,

$$
\nabla^k p_F=O(|y|^{-1-k})\text{ as }|y|\to\infty.
$$

This implies that all the surface integrals from integration by parts are zero.

$$
-\int_0^t\int_{\mathbb{R}^3}\Gamma(x-y,t-\tau)\mathrm{div}(F+p_F)(y,\tau)\,\mathrm{d}y\mathrm{d}\tau=\mathrm{I}+\mathrm{II}
$$

where

$$
\mathrm{I}=-\int_0^t\int_{\mathbb{R}^3}\Gamma(x-y,t-\tau)\mathrm{div}F(y,\tau)\,\mathrm{d}y\mathrm{d}\tau
$$

and

$$
\mathrm{II}=-\int_0^t\int_{\mathbb{R}^3}\Gamma(x-y,t-\tau)\mathrm{div}p_F(y,\tau)\,\mathrm{d}y\mathrm{d}\tau.
$$

Integration by parts once on I gives

$$
\begin{aligned}
\mathrm{I} = &-\int_0^t \int_{\mathbb{R}^3} \Gamma(x-y,t-\tau)\mathrm{div}F(y,\tau)\,\mathrm{d}y\mathrm{d}\tau \\
= &-\int_0^t \int_{\mathbb{R}^3} \left(\Gamma(x-y,t-\tau)\delta_{ij}\frac{\partial F_{jl}}{\partial x_l}(y,\tau)\right) e_i\mathrm{d}y\mathrm{d}\tau \\
= &-\int_0^t \int_{\mathbb{R}^3} \left(\partial_{kk}\Phi(x-y,t-\tau)\delta_{ij}\frac{\partial F_{jl}}{\partial x_l}(y,\tau)\right) e_i\mathrm{d}y\mathrm{d}\tau \\
= &-\int_0^t \int_{\mathbb{R}^3} \left(\partial_{kkl}\Phi(x-y,t-\tau)\delta_{ij}F_{jl}(y,\tau)\right) e_i\mathrm{d}y\mathrm{d}\tau.
\end{aligned}
$$

Integration by parts on II, we have

$$
\begin{aligned}
\mathrm{II} = &\int_0^t \int_{\mathbb{R}^3} \Delta\Phi(x-y,t-\tau)(y,\tau)\mathrm{div}p_F\,\mathrm{d}y\mathrm{d}\tau \\
= &\int_0^t \int_{\mathbb{R}^3} \left(\Delta\Phi(x-y,t-\tau)(y,\tau)\delta_{ik}\partial_k p_F\right) e_i\,\mathrm{d}y\,\mathrm{d}\tau \\
= &-\int_0^t \int_{\mathbb{R}^3} \delta_{ik}\partial_k\Phi(x-y,t-\tau)\Delta p_F(y,\tau)\mathrm{d}y\mathrm{d}\tau \\
= &\int_0^t \int_{\mathbb{R}^3} \delta_{ik}\partial_k\Phi(x-y,t-\tau)\mathrm{div}\mathrm{div}F(y,\tau)\,\mathrm{d}y\mathrm{d}\tau \\
= &\int_0^t \int_{\mathbb{R}^3} \delta_{ik}\partial_k\Phi(x-y,t-\tau)\frac{\partial^2 F_{jl}}{\partial x_j\partial x_l}(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau \\
= &\int_0^t \int_{\mathbb{R}^3} \delta_{ik}\partial_{kjl}\Phi(x-y,t-\tau)F_{jl}(y,\tau)\,\mathrm{d}y\mathrm{d}\tau \\
= &\int_0^t \int_{\mathbb{R}^3} \partial_{ijl}\Phi(x-y,t-\tau)F_{jl}(y,\tau)\,\mathrm{d}y\mathrm{d}\tau
\end{aligned}
$$

Thus

$$
\begin{aligned}
&-\int_0^t \int_{\mathbb{R}^3} \Gamma(x-y,t-\tau)\mathrm{div}(F+p_F)(y,\tau)\,\mathrm{d}y\mathrm{d}\tau \\
= &\int_0^t \int_{\mathbb{R}^3} K_{ijl}F_{jl}(y,\tau)\,\mathrm{d}y\mathrm{d}\tau
\end{aligned}
$$

with $K_{ijl} = \Phi_{,ijl} - \delta_{ij}\Phi_{,kkl}$. □

Exercise 4. *Let u^1 be the heat potential for u_0 in $\mathbb{R}^3$,i.e.,*

$$
u^1(x,t) = \int_{\mathbb{R}^3} \Gamma(x-y,t)u_0(y)\,\mathrm{d}y.
$$

Let $\sigma \geq 3$. Show that

$$
\|u^1\|_{s,r,Q_T} \leq c\|u_0\|_\sigma
$$

provided

$$\frac{2}{s}+\frac{2}{r}=\frac{3}{\sigma}$$

with a constant depending on s and σ only.

Proof. Note that $u^1 = \Gamma * u_0$. Using Young's inequality for the convolution of two functions, we get

$$\|\Gamma * u_0\|_{L^s} \leq \|\Gamma\|_{L^p}\|u_0\|_{L^\sigma} \text{ with } 1+\frac{1}{s}=\frac{1}{p}+\frac{1}{\sigma}.$$

Since $\frac{2}{s}+\frac{2}{r}=\frac{3}{\sigma}$, we have $\frac{1}{p}=1+\frac{1}{s}-\frac{1}{\sigma}=1+\frac{1}{3s}-\frac{2}{3r}$. This gives $p=\frac{3rs}{3rs+r-2s}$. Then

$$\begin{aligned}\|u^1\|_{s,r,Q_+} =&\left(\int_0^T \|\Gamma * u_0\|_{L^s}^r \,\mathrm{d}t\right)^{\frac{1}{r}}\\ \leq&\|u_0\|_\sigma \left(\int_0^T \|\Gamma\|^r_{L^{\frac{3rs}{3rs+r-2s}}}\right)^{\frac{1}{r}}.\end{aligned}$$

Now we need to show that

$$\left(\int_0^T \|\Gamma\|^r_{L^{\frac{3rs}{3rs+r-2s}}}\right)^{\frac{1}{r}} < \infty.$$

Recall that in $\mathbb{R}^3$,

$$\Gamma(x,t)=\frac{1}{(4\pi t)^{\frac{3}{2}}}\exp(-\frac{|x|^2}{4t}).$$

Then

$$\begin{aligned}\|\Gamma\|^r_{L^{\frac{3rs}{3rs+r-2s}}} =&\frac{1}{(4\pi t)^{\frac{3}{2}r}}\left(\int_{\mathbb{R}^3}\exp(-\frac{|x|^2}{4t}\frac{3rs}{3rs+r-2s})\right)^{\frac{3rs+r-2s}{3s}}\\ =&\frac{1}{(4\pi t)^{\frac{3}{2}r}}\left(4\pi t\frac{3rs+r-2s}{3rs}\right)^{\frac{3}{2}\frac{3rs+r-2s}{3s}}\\ =&(4\pi t)^{\frac{r-2s}{2s}}\left(\frac{3rs+r-2s}{3rs}\right)^{\frac{3rs+r-2s}{2s}}.\end{aligned}$$

Note that $\frac{r-2s}{2s}=\frac{r}{2s}-1>-1$ for $r>0, s>0$. Thus we can conclude that

$$\left(\int_0^T \|\Gamma\|^r_{L^{\frac{3rs}{3rs+r-2s}}}\right)^{\frac{1}{r}} = c(s,r)=c(s,\sigma)<\infty.$$

Then the required inequality follows. ☐

Proof. *Claim:* Let $\sigma \geq 3$. Show that $\|u^1\|_{s,r,Q_+} \leq C(\sigma, s)\|u_0\|_\sigma$ provided

$$\frac{3}{s} + \frac{2}{r} = \frac{3}{\sigma}$$

with a constant depending on s and σ only.
We write u for u^1 for simplicity. Testing the heat equation $\partial_t u - \Delta u = 0$ with $|u|^{\sigma-2}u$, we have

$$\int_{\mathbb{R}^3} \partial_t u u|u|^{\sigma-2} = \int_{\mathbb{R}^3} \Delta u u|u|^{\sigma-2}.$$

Note that

$$\int_{\mathbb{R}^3} \partial_t u u|u|^{\sigma-2} = \int_{\mathbb{R}^3} \partial_t(\frac{1}{\sigma}|u|^\sigma) = \frac{1}{\sigma}\frac{d}{dt}\int_{\mathbb{R}^3} |u|^\sigma$$

and

$$\int_{\mathbb{R}^3} \Delta u u|u|^{\sigma-2} = -\int_{\mathbb{R}^3} |\nabla u|^2|u|^{\sigma-2} - (\sigma-2)\int_{\mathbb{R}^3} |\nabla u|^2|u|^{\sigma-2} = -(\sigma-1)\int_{\mathbb{R}^3} |\nabla u|^2|u|^{\sigma-2}$$

This shows that

$$\frac{1}{\sigma}\frac{d}{dt}\int_{\mathbb{R}^3} |u|^\sigma = -(\sigma-1)\int_{\mathbb{R}^3} |\nabla u|^2|u|^{\sigma-2}.$$

Trick:

$$\nabla(|u|^{\frac{\sigma}{2}}) = \frac{\sigma}{2}\nabla u \cdot u|u|^{\frac{\sigma}{2}-2}$$

which implies that

$$|\nabla(|u|^{\frac{\sigma}{2}})|^2 = \frac{\sigma^2}{4}|\nabla u|^2|u|^{\sigma-2}.$$

Thus

$$-(\sigma-1)\int_{\mathbb{R}^3} |\nabla u|^2|u|^{\sigma-2}\,\mathrm{d}x = -\frac{4(\sigma-1)}{\sigma^2}\int_{\mathbb{R}^3} |\nabla|u|^{\frac{\sigma}{2}}|^2\,\mathrm{d}x$$

Then

$$\frac{d}{dt}\int_{\mathbb{R}^3} v^2 + 4(\frac{\sigma-1}{\sigma})\int_{\mathbb{R}^3} |\nabla v|^2 = 0 \text{ where } v = |u|^{\frac{\sigma}{2}}.$$

Integrating from 0 to t, we have

$$\int_{\mathbb{R}^3} v^2 + \frac{4(\sigma-1)}{\sigma}\int_0^t\int_{\mathbb{R}^3} |\nabla v|^2 = \int_{\mathbb{R}^3} |u_0|^\sigma = \|u_0\|^\sigma_{\sigma,\mathbb{R}^3}.$$

Now we aim to show that

$$\|v\|_{s_0,r_0,Q_T} \leq C(s_0)|v|_{2,Q_T}$$

for $s_0 \in [2,6]$ since $2^* = 6$ in $\mathbb{R}^3$, where $|v|_{2,Q_T} := \sup_{0<t<T}\|v(\cdot,t)\|_{2,\mathbb{R}^3} + \|\nabla v\|_{2,Q_T}$.
Note that

$$\begin{aligned}
\|v(\cdot,t)\|_{s_0} &\leq \|v(\cdot,t)\|^\alpha_{2,\mathbb{R}^3}\|v(\cdot,t)\|^{1-\alpha}_{6,\mathbb{R}^3} \text{ (Interpolation inequality for } L^p \text{ norms)}\\
&\leq C\|v(\cdot,t)\|^\alpha_{2,\mathbb{R}^3}\|\nabla v(\cdot,t)\|^{1-\alpha}_{2,\mathbb{R}^3} \text{ (By Gagliardo-Nirenberg inequality)}\\
&\leq C|v|^\alpha_{2,Q_T}\|\nabla v(\cdot,t)\|^{1-\alpha}_{2,\mathbb{R}^3}
\end{aligned}$$

with $\frac{1}{s_0} = \frac{\alpha}{2} + \frac{1-\alpha}{6}$. Then we have

$$\|v(\cdot,t)\|_{s_0}^{r_0} \le C(s_0)|v|_{2,Q_T}^{\alpha r_0} \|\nabla u(\cdot,t)\|_{2,\mathbb{R}^3}^{(1-\alpha)r_0}$$

which implies that

$$\begin{aligned}\|v\|_{s_0,r_0,Q_T}^{r_0} \le & C(s_0)|v|_{2,Q_T}^{\alpha r_0} \int_0^T \|\nabla v(\cdot,t)\|_{2,\mathbb{R}^3}^2 \mathrm{d}t \\ \le & C(s_0)|v|_{2,Q_T}^{r_0-2}|v|_{2,Q_T}^2 \\ = & C(s_0)|v|_{2,Q_T}^{r_0}\end{aligned}$$

by taking $(1-\alpha)r_0 = 2$ (then $\alpha r_0 = r_0 - 2$) and $\frac{3}{s_0} + \frac{2}{r_0} = \frac{3}{2}$. Then

$$\|v\|_{s_0,r_0,Q_T} \le C(s_0)C(\sigma)\|u_0\|_{\sigma,\mathbb{R}^3}^{\frac{\sigma}{2}}$$

But

$$\begin{aligned}\|v\|_{s_0,r_0,Q_T} &= \left[\int_0^T \left(\int_{\mathbb{R}^3} |u|^{\frac{s_0\sigma}{2}} \mathrm{d}x\right)^{\frac{r_0}{s_0}} \mathrm{d}t\right]^{\frac{1}{r_0}} \\ &= \left[\int_0^T \left(\int_{\mathbb{R}^3} |u|^{s} \mathrm{d}x\right)^{\frac{\sigma}{2}\frac{r_0}{s}} \mathrm{d}t\right]^{\frac{1}{r_0}} \text{ by taking } s = \frac{s_0\sigma}{2} \\ &= \left[\int_0^T \left(\int_{\mathbb{R}^3} |u|^{s} \mathrm{d}x\right]^{\frac{r}{s}} \mathrm{d}t\right]^{\frac{\sigma}{2r}} \text{ by taking } r = \frac{\sigma r_0}{2} \\ &\le C(s)C(\sigma)\|u_0\|_{\sigma,\mathbb{R}^3}^{\frac{\sigma}{2}}.\end{aligned}$$

Therefore

$$\|u\|_{s,r,Q_T}^{\frac{\sigma}{2}} \le C(s)C(\sigma)\|u_0\|_{\sigma,\mathbb{R}^3}^{\frac{\sigma}{2}}.$$

Since the constant is independent of T, we can send $T \to \infty$ to get

$$\|u\|_{s,r,Q_+} \le C\|u_0\|_{\sigma,\mathbb{R}^3}.$$

Note that $\frac{3}{s_0} + \frac{2}{r_0} = \frac{3}{2}$, and $\frac{3\sigma}{2s} + \frac{2\sigma}{2r} = \frac{3}{2}$. Then $\frac{3}{s} + \frac{2}{r} = \frac{3}{\sigma}$. $\square$

Exercise 5. *If $f \in H^{-1}(\Omega)$, then*

$$\|f\|_{H^{-1}(\Omega)}^2 = \sum_{k=1}^{\infty} f_k^2/\lambda_k,$$

where $f_k = (f, \varphi_k)$ and λ_k and φ_k are eigenvalues and eigenfunctions of the Laplace operator in Ω under the Dirichlet boundary conditions.

Proof. Note that $H^1 := \overset{\circ}{L}{}^1_2(\Omega)$ is equivalent to the Sobolev space $H^1_0(\Omega)$ if Ω is regular enough. Let us denote the duality pairing between H^{-1} and H^1 by $\langle\cdot,\cdot\rangle$ and the standard L^2 inner product by $(\cdot,\cdot)$. Since the extension $\Delta\colon H^1 \to H^{-1}(\Omega)$ is bijective, for each $f \in H^{-1}(\Omega)$ we can find $u_f \in H^1(\Omega)$ such that the duality pairing can be defined as

$$\langle f, v\rangle = (-\Delta u_f, v)$$

for each $v \in H^1(\Omega)$ by Riesz Representation theorem. Then

$$\begin{aligned}\|f\|^2_{H^{-1}(\Omega)} =&\|u_f\|^2_{H^1(\Omega)}\\ =&(\nabla u_f, \nabla u_f).\end{aligned}$$

Since $u_f \in H^1(\Omega)$ and the set of eigenfunctions $\{\varphi_k\}_{k=1}^{\infty}$ is an orthogonal system in $H^1(\Omega)$ so that $(\varphi_i, \varphi_j) = \delta_{ij}$, we can write u_f as

$$u_f := \sum_{k=1}^{\infty}(u_f, \varphi_k)\varphi_k.$$

Then

$$\begin{aligned}\|f\|^2_{H^{-1}} =&(\nabla u_f, \nabla u_f)\\ =&\lim_{N\to\infty}\sum_{k=1}^{N}(u_f, \varphi_k)(\nabla u_f, \nabla\varphi_k)\\ =&\lim_{N\to\infty}\sum_{k=1}^{N}(u_f, \varphi_k)(u_f, -\Delta\varphi_k) \text{ (integration by parts)}\\ =&\lim_{N\to\infty}\sum_{k=1}^{N}(u_f, \lambda_k\varphi_k)(u_f, -\Delta\varphi_k)/\lambda_k\\ =&\lim_{N\to\infty}\sum_{k=1}^{N}(u_f, -\Delta\varphi_k)(u_f, -\Delta\varphi_k)/\lambda_k\\ =&\lim_{N\to\infty}\sum_{k=1}^{N}(-\Delta u_f, \varphi_k)(-\Delta u_f, \varphi_k)/\lambda_k \text{ (integration by parts)}\\ =&\lim_{N\to\infty}\sum_{k=1}^{N}\langle f, \varphi_k\rangle\langle f, \varphi_k\rangle/\lambda_k \text{ (by duality pairing definition)}\\ =&\lim_{N\to\infty}\sum_{k=1}^{N} f_k^2/\lambda_k\\ =&\sum_{k=1}^{\infty} f_k^2/\lambda_k\end{aligned}$$

where $f_k = \langle f, \varphi_k\rangle$. $\square$

Proof. We first show that $\left(\frac{\varphi_k}{\sqrt{\lambda_k}}\right)$ is an orthonormal basis of $H_0^1(\Omega)$. Assume that $u \in H_0^1(\Omega)$ such that $(u, \varphi_k)_{H_0^1} = 0$ for all $k \geq 1$. That is, $0 = (\nabla u, \nabla \varphi_k)_{L_2} = \lambda_k \int_\Omega u\varphi_k \mathrm{d}x$ for all $k \geq 1$. As $\{\varphi_k\}$ is complete basis in L^2, this implies that $u = 0$. That is, $\{\varphi_k\}^\perp = 0$ in $H_0^1(\Omega)$. Also note that $(\varphi_k, \varphi_k)_{H_0^1} = \lambda_k \int_\Omega \varphi_k^2 \,\mathrm{d}x = \lambda_k$. Let $f \in H^{-1}$ then by Riesz Theorem, there exists $u_f \in H_0^1(\Omega)$ such that $-\Delta u_f = f$ in the variational sense and $\|f\|_{H^{-1}}^2 = \|u_f\|_{H_0^1}^2$.

$$
\begin{aligned}
\|u_f\|_{H_0^1}^2 &= \sum_{n=1}^{\infty} |\left(u_f, \frac{\varphi_k}{\sqrt{\lambda_k}}\right)|^2 \\
&= \sum_{n=1}^{\infty} \frac{1}{\lambda_k} |\int_\Omega \nabla u_f \cdot \nabla \varphi_k|^2 \\
&= \sum_{n=1}^{\infty} \frac{1}{\lambda_k} |\langle f, \varphi_k \rangle_{H^{-1} \times H_0^1}|^2.
\end{aligned}
$$

$\square$

Exercise 6. *Let $v \in L_2(0,T;H^1(\Omega))$ and $\partial_t v \in L_2(0,T;H^{-1}(\Omega))$. Then*

$$
\|v(\cdot,t)\|^2 \leq \frac{1}{T}\|v\|_{2,Q_T}^2 + 2\|\partial_t v\|_{L_2(0,T;H^{-1}(\Omega))}\|\nabla v\|_{2,Q_T}
$$

for all $0 \leq t \leq T$.

Proof. Since $v \in L_2(0,T;H^1(\Omega))$ and $\partial_t v \in L_2(0,T;H^{-1}(\Omega))$ with $H^1 \hookrightarrow H$('$\hookrightarrow$' here means continuous embedding and $H = L_2$), we can apply Theorem 3.5 in the lecture notes to get

$$
\|v(\cdot,t)\|_H^2 - \|v(\cdot,t_1)\|_H^2 = 2\int_{t_1}^{t} (\partial_t v(\cdot,\tau), v(\cdot,\tau))_H \,\mathrm{d}\tau
$$

for any $t, t_1 \in [0,T]$. Integrating from 0 to T with respect to t_1, we have

$$
\int_0^T \|v(\cdot,t)\|_H^2 \,\mathrm{d}t_1 = \int_0^T \|v(\cdot,t_1)\|_H^2 \,\mathrm{d}t_1 + 2\int_0^T \int_{t_1}^{t} (\partial_t v(\cdot,\tau), v(\cdot,\tau))_H \,\mathrm{d}\tau \mathrm{d}t_1.
$$

This shows that

$$
\begin{aligned}
T\|v(\cdot,t)\|_H^2 =& \|v\|_{2,Q_T}^2 + 2\int_0^T \int_{t_1}^{t} (\partial_t v(\cdot,\tau), v(\cdot,\tau))_H \,\mathrm{d}\tau \mathrm{d}t_1 \\
\leq& \|v\|_{2,Q_T}^2 + 2\int_0^T \int_{t_1}^{t} \|\partial_t v(\cdot,\tau)\|_{H^{-1}(\Omega)} \|v(\cdot,\tau))\|_{H^1(\Omega)} \,\mathrm{d}\tau \mathrm{d}t_1.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|v(\cdot,t)\|_H^2 \leq & \frac{1}{T}\|v\|_{2,Q_T}^2 + \frac{2}{T}\int_0^T \int_{t_1}^t \|\partial_t v(\cdot,\tau)\|_{H^{-1}(\Omega)} \|v(\cdot,\tau))\|_{H^1(\Omega)} \,\mathrm{d}\tau \mathrm{d}t_1 \\
\leq & \frac{1}{T}\|v\|_{2,Q_T}^2 + 2\int_0^T \|\partial_t v(\cdot,\tau)\|_{H^{-1}(\Omega)} \|v(\cdot,\tau))\|_{H^1(\Omega)} \,\mathrm{d}\tau \\
\leq & \frac{1}{T}\|v\|_{2,Q_T}^2 + 2\|\partial_t v\|_{L_2(0,T;H^{-1}(\Omega))} \|v\|_{L_2(0,T;H^1(\Omega))} \\
= & \frac{1}{T}\|v\|_{2,Q_T}^2 + 2\|\partial_t v\|_{L_2(0,T;H^{-1}(\Omega))} \|\nabla v\|_{2,Q_T}.
\end{aligned}
$$

□