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Parabolic PDEs -Problem Sheet Four

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Exercise 1. *Assume that Ω is a bounded domain with smooth boundary in $\mathbb{R}^n$ with $n > 2$. Let $a = (a_{ij})$ be a symmetric tensor valued field satisfying the strong ellipticity condition*

$$\nu I \leq a(z) \leq \nu^{-1} I \ \ (\Leftrightarrow \nu|\xi|^2 \leq \xi \cdot a(z)\xi \leq \nu^{-1}|\xi|^2 \ \ \xi \in \mathbb{R}^n)$$

for a.a. space time points $z = (x,t) \in Q_T$ with a positive constant ν.
Suppose that $b = (b_i) \in L_{n,\infty}(Q_T)$. Here $Q = \Omega \times (0,T)$ and $T > 0$ is given.
Let $u_0 \in H := L_2(\Omega)$.
Show that there exists a number $\varepsilon > 0$ with the following property: if

$$\|b\|_{n,\infty,Q_T} < \varepsilon,$$

there exists a unique function $u\colon Q_T \to \mathbb{R}$ that is a weak solution to the following initial boundary value problem:

$$\partial_t u + Lu = 0, \tag{0.1}$$

where

$$Lu = -\mathrm{div}(a\nabla u) + b \cdot \nabla u,$$

$$u = 0 \tag{0.2}$$

on $\partial\Omega \times [0,T]$

$$u(\cdot, 0) = u_0(\cdot) \tag{0.3}$$

in Ω.

Proof. We first show the existence of the solution. Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis consisting of eigenfunctions of the Laplace operator in Ω under Dirichlet boundary conditions. We are looking for a function

$$u^N(x,t) = \sum_{k=1}^{N} c_k(t) e_k(x),$$

where unknown coefficients c_k are determined as a solution of the following linear system of ODE's

$$\int_\Omega \partial_t u^N(x,t)e_k(x) + (a\nabla u^N(x,t))\cdot\nabla e_k(x) + b(x,t)\cdot\nabla u^N(x,t)e_k(x)\,\mathrm{d}x = 0 \tag{0.4}$$

for all $k = 1, 2, \dots, N$ and for a.a. $t \in [0, T]$ with initial data $c_k(0) = a_k := (u_0, e_k)$. A solution of such ODE system always exists.
Now let us consider the energy estimates. First note that

$$\int_\Omega \partial_t u^N(x,t)u^N(x,t) + (a\nabla u^N(x,t))\cdot\nabla u^N(x,t) + b(x,t)\cdot\nabla u^N(x,t)u^N(x,t)\,\mathrm{d}x = 0. \tag{0.5}$$

Applying the ellipticity condition of a and Hölder's inequality to the term involving $b(x,t)$, we have

$$\begin{aligned}\frac{d}{dt}\|u^N(\cdot,t)\|^2_{L^2(\Omega)} + 2\nu\|\nabla u^N(\cdot,t)\|^2_{L^2(\Omega)} &\le 2\|b(\cdot,t)\|_{L^n(\Omega)}\|\nabla u^N(\cdot,t)\|_{L^2(\Omega)}\|u^N(\cdot,t)\|_{L^{\frac{2n}{n-2}}(\Omega)}\\ &\le 2C(n,\Omega)\|b(\cdot,t)\|_{L^n(\Omega)}\|\nabla u^N(\cdot,t)\|^2_{L^2(\Omega)}\end{aligned}$$

where $C(n,\Omega)$ is the Sobelev's constant depending on n and Ω. Integrating the above inequality from 0 to T, we have for a.a $t \in (0,T)$,

$$\|u^N(\cdot,t)\|^2_{L^2(\Omega))} + 2\left(\nu - C(n,\Omega)\|b\|_{L^\infty(0,T;L^n(\Omega))}\right)\|\nabla u^N\|^2_{L^2(0,T;L^2(\Omega))} \le \|u_0\|^2_{L^2(\Omega)}.$$

If $\|b\|_{L^\infty(0,T;L^n(\Omega))} < \varepsilon := \frac{\nu}{2C(n,\Omega)}$, then we have

$$\|u^N\|_{L^\infty(0,T;L^2(\Omega))} + \nu\|\nabla u^N\|_{L^2(0,T;L^2(\Omega))} \le \|u_0\|_{L^2(\Omega)}.$$

To establish the second estimate, we take an arbitrary function $w \in H^1_0(\Omega)$ and let $w^N(x) = \sum_{k=1}^N w_k e_k(x)$, where $w_k = (w_k, e_k)$. Then by (0.4) and by the orthogonality, we have

$$\begin{aligned}\int_\Omega \partial_t u^N(x,t)w(x)\,\mathrm{d}x &= \int_\Omega \partial_t u^N(x,t)w^N(x)\,\mathrm{d}x\\ &= \int_\Omega -a\nabla u^N(x,t)\cdot\nabla w^N(x,t) - b(x,t)\nabla u^N(x,t)w^N(x)\,\mathrm{d}x.\end{aligned}$$

Using the Poincare's inequality $\|w^N\|_{L^2(\Omega)} \le C(\Omega)\|\nabla w^N\|_{L^2(\Omega)}$ and the fact that $\|\nabla w^N\|_{L^2(\Omega)} \le \|\nabla w\|_{L^2(\Omega)}$, we derive the following estimate

$$\begin{aligned}\int_\Omega \partial_t u^N(x,t)w(x)\,\mathrm{d}x \le &\|a\|_{L^\infty(\Omega)}\|\nabla u^N(\cdot,t)\|_{L^2(\Omega)}\|\nabla w^N\|_{L^2(\Omega)}\\ &+ C(\Omega)\|b(\cdot,t)\|_{L^n(\Omega)}\|\nabla u^N(\cdot,t)\|_{L^2(\Omega)}\|\nabla w^N\|_{L^2(\Omega)}.\end{aligned}$$

This implies that

$$\|\partial_t u^N\|_{L^2(0,T;H^{-1})} \le \tilde{C}\|u_0\|_{L^2(\Omega)}.$$

Since both $\|\partial_t u^N\|_{L^2(0,T;H^{-1}(\Omega))}$ and $\|u^N\|_{L^\infty(0,T;L^2(\Omega))}$ are bounded, we may assume without loss of generality that

$$u^N \rightharpoonup u$$

in $L^2(0,T;H_0^1(\Omega))$ and

$$\partial_t u^N \rightharpoonup \partial_t u$$

in $L^2(0,T;H^{-1}(\Omega))$ (or otherwise pass to a subsequence), thus the limit function u satisfies $u \in C(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ and $\partial_t u \in L^2(0,T;H^{-1}(\Omega))$. Passing to the limit in (0.4), we have

$$\int_{Q_T} (\partial_t u(x,t) e_k(x) + (a\nabla u(x,t)) \cdot \nabla e_k(x) + b(x,t) \cdot \nabla u(x,t) e_k(x))\, \chi(t)\, \mathrm{d}x\, \mathrm{d}t = 0 \quad (0.6)$$

for all $k = 1, 2, \dots$ and for all $\chi \in L^2(0,T)$. Clearly, (0.6) can be extended from an arbitrary eigenfunction e_k to arbitrary function $w \in H_0^1(\Omega)$. This shows that u is the weak solution. Now we show that the initial data holds. We notice that

$$\left(u^N(\cdot,t) - u_0^N(\cdot), w(\cdot)\right) = \int_0^t (\partial_t u^N(\cdot,s), w(\cdot))\, \mathrm{d}s$$

for any $w \in H_0^1(\Omega)$. By Lemma 2.2 from lecture notes, we know that

$$\left(u^N(\cdot,t) - u_0^N(\cdot), w(\cdot)\right) \to (u(\cdot,t) - u_0(\cdot), w(\cdot))\,.$$

On the other hand, by the weak convergence of the derivative in time,

$$\int_0^t (\partial_t u^N(\cdot,s), w(\cdot))\, \mathrm{d}s \to \int_0^t (\partial_t u(\cdot,s), w(\cdot))\, \mathrm{d}s.$$

Hence,

$$(u(\cdot,t) - u_0(\cdot), w(\cdot)) = \int_0^t (\partial_t u(\cdot,s), w(\cdot))\, \mathrm{d}s \to 0 \text{ as } t \to 0$$

for any $w \in H_0^1(\Omega)$. But we also know that $\|u(\cdot,t) - u(\cdot,0)\|_{L^2(\Omega)} \to 0$ as $t \to 0^+$. Thus $u(\cdot,0) = u_0(\cdot)$ a.a in Ω.
To show the uniqueness of the solution, it is sufficient to show that if $u_0 = 0$, then the only weak solution is $u = 0$. By passing to the limit of the energy estimates, we get

$$\frac{d}{dt}\|u(\cdot,t)\|^2_{L^2(\Omega))} + \nu\|\nabla u(\cdot,t)\|^2_{L^2(\Omega))} \leq 0.$$

Thus $\frac{d}{dt}\|u(\cdot,t)\|^2_{L^2(\Omega)} \leq 0$ which in turn implies that $\|u(\cdot,t)\|^2_{L^2(\Omega)} \leq \|u(\cdot,0)\|^2_{L^2(\Omega)} = 0$. Thus $\|u(\cdot,t)\|_{L^2(\Omega)} = 0$ for all $t \in [0,T]$. This immediately implies that $u \equiv 0$.
We can use 'stampacchia's truncation' to avoid using Galerkin approximation. Define $b_k := \max(\min(b,k), -k)$ so that $|b_k| \leq |b|$ and $|b_k| \leq k$ a.e. in Q_T. By Theorem 2.1, Chapter 3, existence and uniqueness of $u_k \in C([0,T], L_2(\Omega)) \cap L_2(0,T;H_0^1(\Omega))$ to

$$\begin{cases} \partial_t u_k - \operatorname{div}(a\nabla u_k) + b_k \cdot \nabla u_k = 0 \text{ in } Q_T \\ b_k \mid \partial\Omega \times [0,T] = 0, u_k \mid_{t=0} = u_0. \end{cases}$$

Then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|u_k(\cdot,t)\|^2_{L_2(\Omega)} =&2\left\langle \partial_t u_k . \partial u_k \right\rangle_{H^{-1}\times H^1_0} \\
=&-2\int_\Omega (a\nabla u_k)\cdot\nabla u_k - 2\int (b_k\cdot\nabla u_k)u_k \\
\leq&-2\nu\int_\Omega |\nabla u_k(\cdot,t)|^2 + 2\|b\|_{n,\infty}\|\nabla u_k(\cdot,t)\|_{2,\Omega}\|u_k(\cdot,t)\|_{\frac{2n}{n-2},\Omega} \\
\leq&(-2\nu+2C(\Omega)\|b\|_{n,\infty})\|\nabla u_k(\cdot,t)\|^2_{2,\Omega} \\
=&-M\|\nabla u_k(\cdot,t)\|^2_{2,\Omega}
\end{aligned}$$

where $M>0$ if and only if $\|b\|_{n,\infty}<\frac{\nu}{C(\Omega)}$. This implies that

$$\sup_{0\leq t\leq T}\|u_k(\cdot,t)\|_{L_2(\Omega)} + \|\nabla u_k\|_{2,Q_T} \leq C(M)\|u_0\|_{2,\Omega} \text{ for all } k\geq 1.$$

We also have

$$\|\partial_t u_k\|_{L_2(0,T;H^{-1})} \leq C(M)\|u_0\|_{2,\Omega} \text{ for all } k\geq 1.$$

By a different version of Aubin-Lion Lemma, we know that $u_k\to u$ in $C([0,T];L_2(\Omega))$, $u_k\rightharpoonup u$ weakly in $L_2(0,T;H^1_0(\Omega))$, $\partial_t u_k\rightharpoonup\partial_t u$ weakly in $L_2(0,T;H^{-1})$, and $u(t)\to u_0$ as $t\to 0$. We can also deduce that $b_k\to b$ in $L_2(Q_T)$ by DCT since $b_k\to b$ a.e. and $|b_k|\leq g\in L_2$. □

Exercise 2. *Let Ω be a bounded domain with sufficiently smooth boundary and let $p\geq 1$. Then the space $W^{2,1}_p(Q_T)$ is compactly embedded into $W^{1,0}_p(Q_T)$.*

Proof. Recall that

$$W^{2,1}_p(Q_T)\equiv\left\{\|u\|_{W^{2,1}_p(Q_T)} = \|u\|_{L_p(0,T;W^{2,p}(\Omega))} + \|\partial_t u\|_{L_p(0,T;L_p(\Omega))} < \infty\right\}$$

and

$$W^{1,0}_p(Q_T)\equiv\left\{\|u\|_{W^{1,0}_p(Q_T)} = \|u\|_{L_p(0,T;W^{1,p}(\Omega))} < \infty\right\}$$

where $W^{2,p}$ is the standard Sobolev's space with functions having weak derivatives up to order 2 in L^p space.
If $1<p<\infty$, we can apply *Aubin-Lions Lemma* from lecture notes directly by taking $V_0=W^{2,p}(\Omega)$, $V=W^{1,p}(\Omega)$, $V_1=L^p(\Omega)$ to show that $W^{2,1}_p(Q_T)$ is compactly embedded into $L_p(0,T;W^{1,p}(\Omega))=W^{1,0}_p(\Omega)$.
Now consider the case $p=1$. $L^1(\Omega)$ is not reflexive, so we cannot apply the *Aubin-Lions Lemma* from lecture notes directly. We proceed our proof as follows. Let $\{u_m\}^\infty_{m=1}$ be a bounded sequence in $W^{2,1}_1(Q_T)$, then aim to show that there exists a subsequence $\{u_{m_k}\}^\infty_{k=1}$ converging in $W^{1,0}_1(Q_T)$. Since $\{u_m\}^\infty_{m=1}$ is a bounded sequence in $W^{2,1}_p(Q_T)$, there exists a constant $K>0$ such that

$$\|u_m\|_{L_1(0,T;W^{2,1}(\Omega))} = \int_0^T \|u_m\|_{W^{2,1}(\Omega)}\,\mathrm{d}t \leq K$$

and

$$\|\partial_t u_m\|_{L_1(0,T;L^1(\Omega))} = \int_0^T \|\partial_t u_m\|_{L^1(\Omega)}\,\mathrm{d}t \le K.$$

Note that Lemma 1.1 still holds for $V_0 = W^{2,1}(\Omega)$, $V = W^{1,1}(\Omega)$ and $V_1 = L^1(\Omega)$. That is, given $\eta > 0$, there exists $C(\eta) > 0$ such that for any $v \in W^{1,1}(\Omega)$,

$$\|v\|_{W^{1,1}(\Omega)} \le \eta\|v\|_{W^{2,1}(\Omega)} + C(\eta)\|v\|_{L^1(\Omega)}.$$

To prove this result, we assume for contradiction that for any $n \in \mathbb{N}$, there exists $v_n \in W^{2,1}(\Omega)$ such that

$$\|v_n\|_{W^{1,1}(\Omega)} > \eta\|v_n\|_{W^{2,1}(\Omega)} + n\|v_n\|_{L^1(\Omega)}.$$

After normalization, we have

$$\|v_n^{'}\|_{W^{1,1}(\Omega)} = 1 > \eta\|v_n^{'}\|_{W^{2,1}(\Omega)} + n\|v_n^{'}\|_{L^1(\Omega)}$$

where $v_n^{'} = \frac{v_n}{\|v_n\|_{W^{1,1}(\Omega)}}$. The sequence $v_n^{'}$ is bounded in $W^{2,1}(\Omega)$, thus there exists a subsequence $v_{n_j} \to v$ in $W^{1,1}(\Omega)$ and $L^1(\Omega)$ by compact embeddings. Since $n\|v_{n_j}^{'}\|_{L^1(\Omega)}$ is bounded and therefore $v_{n_j}^{'} \to 0$ in $L^1(\Omega)$. By uniqueness of limit, $v = 0$ in $W^{1,1}(\Omega)$, so $\|v\|_{W^{1,1}(\Omega)} = 0$. But $1 = \|v_{n_j}^{'}\|_{W^{1,1}(\Omega)} \to \|v\|_{W^{1,1}(\Omega)}$. This leads to contradiction.
Thanks to Lemma 1.1, for each $\eta > 0$, there exists $C(\eta) > 0$ such that for a.e $t \in [0,T]$,

$$\|u_m(\cdot,t)\|_{W^{1,1}(\Omega)} \le \eta\|u_m(\cdot,t)\|_{W^{2,1}(\Omega)} + C(\eta)\|u_m(\cdot,t)\|_{L^1(\Omega)}.$$

Integrating over $[0,T]$, we deduce that

$$\|u_m\|_{L_1(0,T;W^{1,1}(\Omega))} \le \eta K + C(\eta)\|u_m\|_{L_1(0,T;L^1(\Omega))}. \tag{0.7}$$

So it is sufficient to show that there exists a subsequence u_{m_k} which is Cauchy in $L_1(0,T;L^1(\Omega))$. We adapt a proof from Süli and Barrett [1]. First note that for $u \in W_1^{2,1}(Q_T)$, we have $\int_0^T \|u(\cdot,t)\|_{W^{2,1}(\Omega)}\,\mathrm{d}t < \infty$ and$\int_0^T \|\partial_t u(\cdot,t)\|_{L^1(\Omega)}\,\mathrm{d}t < \infty$. Note that $\int_0^T \|u(\cdot,t)\|_{W^{2,1}(\Omega)}\,\mathrm{d}t < \infty$ implies that $\int_0^T \|u(\cdot,t)\|_{L^1(\Omega)}\,\mathrm{d}t < \infty$ since $W^{2,1}(\Omega)$ is compactly embedded into $L^1(\Omega)$. This implies that $W_1^{2,1}(Q_T) \subset W^{1,1}(0,T;L^1(\Omega)) \hookrightarrow C(0,T;L^1(\Omega))$. Since $\{u_m\}_{m=1}^{\infty}$ is a bounded sequence in $C([0,T];L^1(\Omega))$ that is,

$$\max_{t\in[0,T]} \|u_m(\cdot,t)\|_{L^1(\Omega)} \le K$$

for the same K as we defined before.
Note there exists a countable dense subset G of the interval $(0,T)$ and an infinite subsequence $\{u_m\}_{m\in F}$ of the sequence $\{u_m\}_{m\in\mathbb{N}}$ where F is an infinite subset of $\mathbb{N}$ such that $\{u_m(s)\}_{m\in F}$ converges in $L^1(\Omega)$ for each $s \in G$. Details of the proof can be found in [1].
We now aim to show that $\{u_m\}_{m\in F}$ is a Cauchy sequence in $L^1(0,T;L^1(\Omega))$. By the definition of derivative in time, we have

$$\|u_m(\cdot,t) - u_m(\cdot,s)\|_{L^1(\Omega)} \le \int_s^t |\partial_\tau u_m(\cdot,\tau)|\mathrm{d}\tau \tag{0.8}$$

for any $0 \leq s \leq t \leq T$ and for all $m \in \mathbb{N}$. For $\varepsilon > 0$ and $K > 0$ as defined before, take $N(\varepsilon) = \left[\frac{3K}{\varepsilon}T\right] + 1$ where $[x]$ means the integer part of x. Let us subdivide the interval $[0, T]$ into N sub-intervals

$$[0, t_1], [t_1, t_2], \ldots, [t_{N-1}, t_N]$$

each of length $h = \frac{T}{N}$ where $t_k = kh$, $k = 0, 1 \ldots, N$. Now choose in each of the open sub-intervals (t_{k-1}, t_k), a single element $s_k \in G$, $k = 1, 2, \ldots N$ and define $G_\varepsilon := \{s_1, s_2, \ldots, s_N\}$. It follows from (0.8) that

$$\begin{aligned}
\int_{t_{k-1}}^{t_k} \|u_m(\cdot, t) - u_m(\cdot, s_k)\|_{L^1(\Omega)} \,\mathrm{d}t \leq & \int_{t_{k-1}}^{t_k} \left| \int_{s_k}^{t} \|\partial_\tau u_m(\cdot, \tau)\|_{L^1(\Omega)} \,\mathrm{d}\tau \right| \mathrm{d}t \\
\leq & h \int_{s_k}^{t} \|\partial_\tau u_m(\cdot, \tau)\|_{L^1(\Omega)} \,\mathrm{d}\tau
\end{aligned}$$

for $k = 1, 2 \ldots, N$. Summing over $k = 1, 2 \ldots N$ yields that

$$\begin{aligned}
\sum_{k=1}^{N} \|u_m - u_m(\cdot, s_k)\|_{L_1(t_{k-1}, t_k, L^1(\Omega))} = & \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \|u_m(\cdot, t) - u_m(\cdot, s_k)\|_{L^1(\Omega)} \,\mathrm{d}t \\
\leq & h \int_0^T \|\partial_t u_m\|_{L^1(\Omega)} \,\mathrm{d}t \\
\leq & Kh
\end{aligned}$$

for all $m \in \mathbb{N}$. It follows from our definition $N = N(\varepsilon)$ that $Kh < \frac{\varepsilon}{3}$. For each $u_m \in W_1^{2,1}(Q_T) \subset C([0, T]; L^1(\Omega))$, we define the piecewise constant interpolant

$$\hat{u} := \sum_{k=1}^{N} u(s_k) \chi_{(t_{k-1}, t_k)}$$

where $\chi_{(t_{k-1}, t_k)}$ is the indicator function on the interval (t_{k-1}, t_k). Thus, by triangle inequality, we have

$$
\begin{aligned}
\|u_m - u_n\|_{L_1(0,T;L^1(\Omega))} \leq & \|u_m - \hat{u}_m\|_{L_1(0,T;L^1(\Omega))} + \|\hat{u}_m - \hat{u}_n\|_{L_1(0,T;L^1(\Omega))} + \|\hat{u}_n - \hat{u}_n\|_{L_1(0,T;L^1(\Omega))} \\
\leq & \sum_{k=1}^{N} \|u_m - u_m(s_k)\|_{L_1(t_{k-1},t_k;L^1(\Omega))} + \sum_{k=1}^{N} \|u_m(s_k) - u_n(s_k)\|_{L_1(t_{k-1},t_k;L^1(\Omega))} \\
& + \sum_{k=1}^{N} \|u_n - u_n(s_k)\|_{L_1(t_{k-1},t_k;L^1(\Omega))} \\
\leq & \frac{2}{3}\varepsilon + \sum_{k=1}^{N} \|u_m(s_k) - u_n(s_k)\|_{L_1(t_{k-1},t_k;L^1(\Omega))} \\
= & \frac{2}{3}\varepsilon + \sum_{k=1}^{N} h\|u_m(s_k) - u_n(s_k)\|_{L^1(\Omega)} \\
\leq & \frac{2}{3}\varepsilon + T \max_{1\leq k\leq N} \|u_m(s_k) - u_n(s_k)\|_{L^1(\Omega)} \\
= & \frac{2}{3}\varepsilon + T \max_{s\in G_\varepsilon} \|u_m(s) - u_n(s)\|_{L^1(\Omega)}
\end{aligned}
$$

for all $m, n \in \mathbb{N}$. As $\{u_m(s)\}_{m\in F}$ is a Cauchy sequence in $L^1(\Omega)$ for each $s \in G_\varepsilon$ and G_ε is a finite set (of cardinality $N = N(\varepsilon)$), it follows that $\{u_m(s)\}_{m\in F}$ is a Cauchy sequence in $L^1(\Omega)$ uniformly in $s \in G_\varepsilon$. i.e. for each $\varepsilon > 0$, there exists a positive integer $n_0 = n_0(\varepsilon) \in F$ such that

$$
\max_{s\in G_\varepsilon} \|u_m(s) - u_n(s)\|_{L^1(\Omega)} < \frac{\varepsilon}{3T}
$$

for all $m, n \in F$ such that $m, n \geq n_0$. This implies that

$$
\|u_m - u_n\|_{L^1(0,T;L^1(\Omega))} < \varepsilon
$$

for all $m, n \in F, m, n \geq n_0$. Then $\{u_m\}_{m\in F}$ is a Cauchy sequence in $L_1(0, T; L^1(\Omega))$, and thus a Cauchy sequence in $L_1(0, T; W^{1,1}(\Omega))$ by the inequality (0.7). Since $L_1(0, T; W^{1,1}(\Omega))$ is a Banach space, $\{u_m\}_{m\in F}$ converges. This completes the proof for the case $p = 1$.
An alternative approach to deal with $p = 1$. First we know that for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that $\|v\|_{W^{1,1}(\Omega)} \leq \varepsilon\|v\|_{W^{2,1}(\Omega)} + C_\varepsilon\|v\|_{L^1(\Omega)}$. Let v_k be a sequence such that $A := \sup_k \|v_k\|_{W_1^{2,1}(Q_T)} < \infty$.

$$
\begin{aligned}
\|v_k - v_l\|_{W_1^{1,0}(Q_T)} \leq & \varepsilon \int_0^T \|v_k - v_l\|_{W^{2,1}(\Omega)} \mathrm{d}t + C_\varepsilon \int_0^T \|v_k - v_l\|_{L^1(\Omega)} \mathrm{d}t \\
\leq & 2\varepsilon A + C_\varepsilon \int_0^T \|v_k - v_l\|_{L^1(\Omega)} \mathrm{d}t
\end{aligned}
$$

Notice that

$$
\|v\|_{W^{1,1}(\Omega\times(0,T))} \leq \|v\|_{W_1^{2,1}(Q_T)}.
$$

Then we have $\sup_k \|v_k\|_{W^{1,1}(\Omega\times(0,T))} \leq A < \infty.$, by compact embedding of $W^{1,1}(\Omega \times (0,T))$in to $L_1(\Omega\times(0,T))$, we know that $v_k \to v$ in $L_1(\Omega\times(0,T))$ for some $v \in W^{1,1}(\Omega\times(0,T))$. There exists $N_{\varepsilon'}$ such that for all $k,l \geq N_{\varepsilon'}$, $\|v_k - v_l\|_{L_1(Q_T)} < \varepsilon'$ where $\varepsilon' = \frac{\varepsilon}{C_\varepsilon}$. Then for all $k,l \geq N_{\varepsilon'}$, $\|v_k - v_l\|_{W_1^{1,0}(Q_T)} \leq 2\varepsilon A + \varepsilon$ for all $l,k \geq N_\varepsilon'$. Thus v_k is a Cauchy sequence in $W_1^{1,0}(Q_T)$ which implies that $v_k \to v$ in $W_1^{1,0}(Q_T)$. □

Exercise 3. *Let a function $u \in L_{2,\infty}(Q) \cap W_2^{1,0}(Q)$ satisfies the equation*

$$\partial_t u - \mathrm{div} a \nabla u = \mathrm{div} g$$

in Q in the sense of distributions with

$$g \in C^{\alpha,\frac{\alpha}{2}}(\bar{Q})$$

for some $0 < \alpha < 1$. Here, $a = (a_{ij})$ is a symmetric positive matrix with constant entries. Show that

$$\nabla u \in C^{\alpha,\frac{\alpha}{2}}(\bar{Q}(r_1))$$

for any $0 < r_1 < 1$.

Proof. We decompose $u \in L_{2,\infty}(Q) \cap W_2^{1,0}(Q)$ as $u = w + v$ such that v solves the following initial boundary value problem

$$\partial_t v - \mathrm{div} a \nabla v = \mathrm{div} g \text{ in } Q(z_0, r)$$

with

$$v = 0 \text{ on } \partial' Q(z_0, r)$$

for $0 < r < 1$ and w is the distributional solution to the parabolic equation

$$\partial_t w - \mathrm{div}(a\nabla w) = 0.$$

Note that the estimates for linear problem is

$$\int_{Q(z_0,\rho)} |\nabla w - (\nabla w)_{z_0,\rho}|^2 \, \mathrm{d}z \leq c(n,\nu) \left(\frac{\rho}{r}\right)^{n+2+2} \int_{Q(z_0,r)} |\nabla w - (\nabla w)_{z_0,r}|^2 \, \mathrm{d}z$$

for any $0 < \rho \leq r < 1$. Therefore, we have

$$\begin{aligned}
\int_{Q(z_0,\rho)} |\nabla u - (\nabla u)_{z_0,r}|^2 \, \mathrm{d}z \leq & 2c(n,\nu) \left(\frac{\rho}{r}\right)^{n+4} \int_{Q(z_0,r)} |\nabla w - (\nabla w)_{z_0,r}|^2 \, \mathrm{d}z \\
& + 2 \int_{Q(z_0,\rho)} |\nabla v - (\nabla v)_{z_0,r}|^2 \, \mathrm{d}z \\
\leq & c(n,\nu) \left[\left(\frac{\rho}{r}\right)^{n+4} \int_{Q(z_0,r)} |\nabla u - (\nabla u)_{z_0,r}|^2 \, \mathrm{d}z + \int_{Q(z_0,r)} |\nabla v|^2 \, \mathrm{d}z \right].
\end{aligned}$$

Now we try to find the estimate of $\int_{Q(z_0,r)} |\nabla v|^2 \,\mathrm{d}z$. Note that v satisfies the energy identity,

$$\frac{1}{2}\int_{B(x_0,r)} |v(x,t)|^2 \,\mathrm{d}x + \int_{-r^2}^{0}\int_{B(x_0,r)} (a\nabla v(x,s))\cdot\nabla v(x,s)\,\mathrm{d}x\mathrm{d}s = \int_{-r^2}^{0}\int_{B(x_0,r)} \mathrm{div} g(x,s) v(x,s)\,\mathrm{d}x\mathrm{d}s,$$

so we have

$$\begin{aligned}\int_{Q(z_0,r)} |\nabla v|^2 \,\mathrm{d}z \leq & c\int_{-r^2}^{0}\int_{B(x_0,r)} \mathrm{div} g(x,s) v(x,s)\,\mathrm{d}x\mathrm{d}s \\ =& c\int_{Q(z_0,r)} \mathrm{div}(g(z)-g(z_0))v(z)\,\mathrm{d}z \\ \leq & c\int_{Q(z_0,r)} |g(z)-g(z_0)|\cdot\nabla v(z)\,\mathrm{d}z \\ \leq & cr^{\alpha}|Q(z_0,r)|^{\frac{1}{2}}\left(\int_{Q(z_0,r)} |\nabla v|^2\,\mathrm{d}z\right)^{\frac{1}{2}}\end{aligned}$$

where in the last line we have used Hölder continuity of g and Hölder's inequality for the integral. This implies that

$$\int_{Q(z_0,r)} |\nabla v|^2 \,\mathrm{d}z \leq cr^{2\alpha+n+2}$$

since $|Q(z_0,r)| \leq cr^{n+2}$. Then we have

$$\int_{Q(z_0,\rho)} |\nabla u - (\nabla u)_{z_0,r}|^2 \,\mathrm{d}z \leq c(n,\nu)\left[\left(\frac{\rho}{r}\right)^{n+4}\int_{Q(z_0,r)} |\nabla u - (\nabla u)_{z_0,r}|^2\,\mathrm{d}z + cr^{2\alpha+n+2}\right].$$

We define

$$\Psi(z_0,\rho) = \int_{Q(z_0,\rho)} |\nabla u - (\nabla u)_{z_0,r}|^2 \,\mathrm{d}z,$$

then we can rewrite the last inequality in the following form:

$$\Psi(z_0,\rho) \leq c(n,\nu)\left(\frac{\rho}{r}\right)^{n+4}\Psi(z_0,r) + cr^{2\alpha+n+2}$$

for any $z_0 \in Q$ such that $Q(z_0,r) \subset Q$ and for any $0 < \rho \leq r < 1$. By using a generalization of Lemma 5.1 from the lecture notes, we have that

$$\Psi(z_0,r) \leq \tilde{c}r^{n+4-\gamma}\int_Q |\nabla u|^2 \,\mathrm{d}z$$

for $0 < \gamma < n+2$, $z_0 \in Q(\rho)$ and any $0 < r < 1$. Taking $\gamma = 2(1-\alpha)$, we have

$$\Psi(z_0,r) \leq \tilde{c}r^{2\alpha+n+2}\int_Q |\nabla u|^2 \,\mathrm{d}z.$$

Then we have

$$\begin{aligned}\psi(z_0, r) =& \frac{1}{|Q(r)|}\int_{Q(z_0,r)} |\nabla u - (\nabla u)_{z_0,r}| \,\mathrm{d}z \\ \leq& |Q(r)|^{-\frac{1}{2}}\Psi(z_0, r)^{\frac{1}{2}} \\ \leq& \tilde{c}r^{\alpha}\left(\int_Q |\nabla u|^2 \,\mathrm{d}z^2\right)^{\frac{1}{2}} \\ \leq& Ar^{\alpha}\end{aligned}$$

for any $0 < r < 1$ such that $Q(z_0, r) \subset Q$. Then using Proposition 4.1 from the lecture notes, we can conclude that

$$\nabla u \in C^{\alpha,\frac{\alpha}{2}}(\bar{Q}(r_1))$$

for $r_1 < 1$.

$$\begin{aligned}\int_{Q(z_0,\rho)} |\nabla u - (\nabla u)_{z_0,\rho}|^2 \,\mathrm{d}z \leq& \int_{Q(z_0,\rho)} |\nabla u - (\nabla u - (\nabla u)_{z_0,r})_{z_0,\rho}|^2 \,\mathrm{d}z \\ \leq& C(n, \nu, \varphi)\rho^2 \int_{Q(z_0,2\rho)} |\nabla^2 u|^2 \,\mathrm{d}z \\ \leq& C(n, \nu, \varphi)\rho^2 \left(\frac{2\rho}{r}\right)^{n+2} \frac{1}{r^2}\int_{Q(z_0,r)} |\nabla u - (\nabla u)_{z_0,r}|^2 \,\mathrm{d}z \\ \leq& C(n, \nu, \varphi)\left(\frac{2\rho}{r}\right)^{n+4} \int_{Q(z_0,r)} |\nabla u - (\nabla u)_{z_0,r}|^2 \,\mathrm{d}z\end{aligned}$$

for $\rho < \frac{r}{2}$. □

References

[1] J.W.Barrett, E. Süli *Reflections on Dubinskii's nonlinear compact embedding theorem*, *Publications de l'Institut Mathématique (Belgrade)* , **91(105)** (2012), 95–110