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Parabolic PDEs -Problem Sheet One

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Exercise 1. *(a) Let u^2 be the volume heat potential generated by a function f. Let $p \geq 1$ and*

$$y(t) := \int_{\mathbb{R}^n} \left|u^2(x,t)\right|^p \,\mathrm{d}x.$$

Show that

$$y(t) \leq t^{\frac{p}{p'}} \int_0^t \int_{\mathbb{R}^n} |f(y,\theta)|^p \mathrm{d}y\mathrm{d}\theta = t^{\frac{p}{p'}} \int_0^t \|f(\cdot,\theta)\|_{p,\mathbb{R}^n}^p d\theta$$

for all $t \geq 0$.

(b) Let $R_0 > 0$ be so that $\operatorname{spt} f \subset B(R_0) \times]0, R_0^2[$. For any $k = 0, 1, 2, \ldots$ and for any $p > 1$, there exists a constant c depending on $n, k,$ and p only such that

$$\left|\nabla^k u^2(x,t)\right| \leq c \frac{t^{\frac{1}{p'}}}{(t+|x|^2)^{\frac{n+kp}{2p}}} \|f\|_{p,Q_+}$$

for all $t \geq 0$ and $x \in \mathbb{R}^n$ satisfying $t + |x|^2 \geq 8R_0^2$.

Proof. (a)

$$\begin{aligned}
|u^2(x,t)| &\leq \int_0^t \int_{\mathbb{R}^n} |\Gamma(x-y,t-\theta)||f(y,\theta)|dyd\theta \\
&\leq \left(\int_0^t \int_{\mathbb{R}^n} \Gamma(x-y,t-\theta)dyd\theta\right)^{\frac{1}{p'}} \left(\int_0^t \int_{\mathbb{R}^n} \Gamma(x-y,t-\theta)|f(y,\theta)|^p dyd\theta\right)^{\frac{1}{p}} \\
&\leq t^{\frac{1}{p'}} \left(\int_0^t \int_{\mathbb{R}^n} \Gamma(x-y,t-\theta)|f(y,\theta)|^p dyd\theta\right)^{\frac{1}{p}}
\end{aligned}$$

and thus integrating in x we find

$$\begin{aligned}
y(t) &\leq \int_{\mathbb{R}^n} t^{\frac{p}{p'}} \int_0^t \int_{\mathbb{R}^n} \Gamma(x-y,t-\theta)|f(y,\theta)|^p dy d\theta dx \\
&= t^{\frac{p}{p'}} \int_{\mathbb{R}^n} dx \int_0^t \int_{\mathbb{R}^n} \Gamma(x-y,t-\theta)|f(y,\theta)|^p dy d\theta \\
&= t^{\frac{p}{p'}} \int_0^t \int_{\mathbb{R}^n} |f(y,\theta)|^p dy d\theta \\
&= t^{\frac{p}{p'}} \int_0^t \|f(\cdot,\theta)\|_{p,\mathbb{R}^n}^p d\theta \text{ for all } t \geq 0
\end{aligned}$$

where the second line of the last inequality follows from Tonelli theorem and the identity

$$\int_{\mathbb{R}^n} \Gamma(x-y,t-\theta)dx = 1.$$

(b) We have

$$|\nabla^k u^2(x,t)| \leq \int_0^t \int_{B(R_0)} |\nabla^k \Gamma(x-y,t-\theta)||f(y,\theta)|dy d\theta.$$

Note that

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}.$$

We claim (will be proven later) that

$$|\nabla^k \Gamma(x,t)| \leq c \frac{1}{t^{\frac{k}{2}}}(1 + \frac{|x|^k}{t^{\frac{k}{2}}})\Gamma(x,t) \tag{0.1}$$

for some constant c, and thus by the Cauchy inequality

$$\begin{aligned}
|\nabla^k u^2(x,t)| \leq & c \left(\int_0^t \int_{B(R_0)} \Gamma(x-y,t-\theta) dy d\theta \right)^{\frac{1}{p'}} \times \\
& \left(\int_0^t \int_{B(R_0)} \Gamma(x-y,t-\theta) \frac{1}{(t-\theta)^{\frac{pk}{2}}} \left(1 + \frac{|x-y|^k}{(t-\theta)^{\frac{k}{2}}} \right)^p |f(y,\theta)|^p dy dt \right)^{\frac{1}{p}}.
\end{aligned}$$

Note that

$$\begin{aligned}
\Gamma(x-y,t-\theta)\frac{1}{(t-\theta)^{\frac{pk}{2}}} \left(1 + \frac{|x-y|^k}{(t-\theta)^{\frac{k}{2}}} \right)^p \leq & c \frac{1}{(t-\theta)^{\frac{pk}{2}+\frac{n}{2}}} \frac{1}{\left(1 + \frac{|x-y|^2}{t-\theta}\right)^{\frac{kp}{2}+\frac{n}{2}}} \\
\leq & c \frac{1}{(t-\theta+|x-y|^2)^{\frac{kp}{2}+\frac{n}{2}}}
\end{aligned}$$

Or simply

$$|\nabla^k \Gamma(x,t)| \le c\Gamma(x,t)\frac{1}{t^{\frac{k}{2}}} S_k\left(\frac{|x|}{t^{\frac{1}{2}}}\right) \tag{0.2}$$

where S_k is a polynomial of degree k. Note the second inequality follows from the fact that $e^{-z^2} S_k^p(z) \le \frac{1}{(1+z^2)^\beta}$ for all β, so we choose $\beta = \frac{n+kp}{2}$ in this case. (Exponential decays faster than the polynomial).

If $|y| < R_0$ and $\theta < R_0^2$, then

$$\begin{aligned}
t - \theta + |x-y|^2 >& t - R_0^2 + |x|^2 + |y|^2 - 2x\cdot y \\
\ge& t + |x|^2 - 2|x|R_0 - R_0^2 \\
\ge& t + |x|^2 - 2\sqrt{t+|x|^2}R_0 - R_0^2 \\
=& t + |x|^2 - 2\sqrt{t+|x|^2}R_0 + R_0^2 - 2R_0^2 \\
=& (\sqrt{t+|x|^2} - R_0)^2 - 2R_0^2 \\
\ge& \frac{t+|x|^2}{2} - 3R_0^2 \\
\ge& \frac{t+|x|^2}{8}
\end{aligned}$$

provided that $t + |x| \ge 8R_0^2$. So for those x and t, we find

$$\begin{aligned}
|\nabla^k u^2(x,t)| \le& c\frac{t^{\frac{1}{p'}}}{(t+|x|^2)^{\frac{kp+n}{2}\frac{1}{p}}}\left(\int_0^t \int_{B(R_0)} |f(y,\theta)|^p dy d\theta\right)^{\frac{1}{p}} \\
\le& c\frac{t^{\frac{1}{p'}}}{(t+|x|^2)^{\frac{n+kp}{2p}}} \|f\|_{p,Q_+}
\end{aligned}$$

Now we prove inequality (0.1): Since $|\nabla\Gamma(x,t)| \le c\left(\frac{|x|}{t}\right)\Gamma(x,t)$, the statement is true for $k=1$. Now assume that

$$|\nabla^{k-1}\Gamma(x,t)| \le c\left(\frac{1}{t}\right)^{\frac{k-1}{2}}\left(\frac{|x|^{k-1}}{t^{(k-1)/2}} + 1\right)\Gamma(x,t),$$

then

$$\begin{aligned}
|\nabla^k \Gamma(x,t)| \le& c(\frac{1}{t})^{\frac{k-1}{2}}\left(\frac{|x|^{k-1}}{t^{(k-1)/2}} + 1\right)\left(\frac{|x|}{t}\right)\Gamma(x,t) + c(\frac{1}{t})^{\frac{k-1}{2}}\left((k-1)\frac{|x|^{k-2}}{t^{(k-1)/2}}\right)\Gamma(x,t) \\
\le& c(\frac{1}{t})^{k/2}\left(\frac{|x|^k}{t^{\frac{k}{2}}} + \frac{|x|^{k-2}}{t^{(k-2)/2}} + \frac{|x|}{\sqrt{t}}\right)\Gamma(x,t) \\
\le& \tilde{c}(\frac{1}{t})^{k/2}\left(\frac{|x|^k}{t^{\frac{k}{2}}} + 1\right)\Gamma(x,t)
\end{aligned}$$

Then by mathematical induction, $|\nabla^k\Gamma(x,t)| \le c\frac{1}{t^{\frac{k}{2}}}(1+\frac{|x|^k}{t^{\frac{k}{2}}})\Gamma(x,t)$ for all $k = 0, 1, \ldots$

$\square$

Exercise 2. *Prove the identity*

$$\Gamma(x-z,t+s)=\int_{\mathbb{R}^n}\Gamma(x-y,t)\Gamma(y-z,s)dy$$

for all x and y in $\mathbb{R}^n$ and for all $t\geq 0$ and $s\geq 0$.

Proof.

$$\begin{aligned}
RHS=&\int_{\mathbb{R}^n}\Gamma(x-y,t)\Gamma(y-z,s)dy\\
=&\int_{\mathbb{R}^n}\frac{1}{(4\pi t)^{\frac{n}{2}}}\frac{1}{(4\pi s)^{\frac{n}{2}}}e^{-\frac{|x-y|^2}{4t}}e^{-\frac{|y-z|^2}{4s}}dy\\
=&\frac{1}{(4\pi(t+s))^{\frac{n}{2}}}\int_{\mathbb{R}^n}\frac{1}{(4\pi(\frac{ts}{t+s}))^{\frac{n}{2}}}e^{-\frac{1}{4ts}(sx^2-2s\langle x,y\rangle+sy^2)-\frac{1}{4ts}(ty^2-2t\langle y,z\rangle+tz^2)}dy\\
=&\frac{1}{(4\pi(t+s))^{\frac{n}{2}}}\int_{\mathbb{R}^n}\frac{1}{(4\pi(\frac{ts}{t+s}))^{\frac{n}{2}}}e^{-\frac{s+t}{4ts}y^2+\frac{2s}{4ts}\langle x,y\rangle+\frac{2t}{4ts}\langle y,z\rangle}e^{-\frac{1}{4ts}(sx^2+tz^2)}dy\\
=&\frac{1}{(4\pi(t+s))^{\frac{n}{2}}}\int_{\mathbb{R}^n}\frac{1}{(4\pi(\frac{ts}{t+s}))^{\frac{n}{2}}}e^{-\frac{|y-\frac{sx+tz}{s+t}|^2}{4(\frac{st}{t+s})}}e^{-\frac{1}{4ts}(sx^2+tz^2)-\frac{s+t}{4ts}(\frac{sx+tz}{t+s})^2}dy\\
=&\frac{1}{(4\pi(t+s))^{\frac{n}{2}}}e^{-\frac{1}{4ts}(sx^2+tz^2)-\frac{s+t}{4ts}(\frac{sx+tz}{t+s})^2}\int_{\mathbb{R}^n}\frac{1}{(4\pi(\frac{ts}{t+s}))^{\frac{n}{2}}}e^{-\frac{|y-\frac{sx+tz}{s+t}|^2}{4(\frac{st}{t+s})}}dy\\
=&\frac{1}{(4\pi(t+s))^{\frac{n}{2}}}e^{-\frac{1}{4ts}(sx^2+tz^2)-\frac{s+t}{4ts}(\frac{sx+tz}{t+s})^2}\\
=&\frac{1}{(4\pi(t+s))^{\frac{n}{2}}}e^{-\frac{|x-z|^2}{4(t+s)}}\\
=&\Gamma(x-z,t+s).
\end{aligned}$$

□

Exercise 3. *Let u^1 be a heat potential generated by smooth compactly supported initial data u_0. Prove the following facts:*

(i) $\sup_{0\leq t<\infty}\left\|\nabla u^1(\cdot,t)\right\|_{2,\mathbb{R}^n}+\left\|\nabla^2 u^1\right\|_{2,Q_+}+\left\|\partial_t u^1\right\|_{2,Q_+}\leq c\left\|\nabla u_0\right\|_{2,\mathbb{R}^n}$

(ii) $\left\|u^1(\cdot,t)\right\|^2_{2,\mathbb{R}^n}+2\left\|\nabla u^1\right\|^2_{2,Q_t}=\left\|u_0\right\|^2_{2,\mathbb{R}^n}$

for any $t>0$.

Show that solution operators $S(t)$ are bounded operators on $\overset{\circ}{L}{}^1_2(\mathbb{R}^n)$ and

$$S(t)u_0\to S(t_0)u_0$$

in $\overset{\circ}{L}{}^1_2(\mathbb{R}^n)$ as $t\to t_0$.

Proof. (i) u_0 is smooth and compactly supported in $\mathbb{R}^n$, so there exists $R_0 > 0$ such that $\mathrm{spt} u_0 \in B(R_0)$. We have

$$\nabla u^1(x,t) = \int_{\mathbb{R}^n} \Gamma(x-y,t)\nabla u_0(y)dy.$$

We first show that $\nabla u^1(\cdot,t)$ is continuous at $t=0$, that is, $\|\nabla u^1(\cdot,t) - \nabla u_0(\cdot)\|_{2,\mathbb{R}^n} \to 0$ as $t \to 0$.

$$\begin{aligned}\|\nabla(u^1(\cdot,t) - u_0(\cdot))\|_{2,\mathbb{R}^n}^2 &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Gamma(x-y,t)|\nabla u_0(y) - \nabla u_0(x)|^2 dydx \\ &= I_1 + I_2\end{aligned}$$

where

$$I_1 = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n\setminus B(x,\delta)} \Gamma(x-y,t)|\nabla u_0(y) - \nabla u_0(x)|^2 dydx$$

and

$$I_2 = \int_{\mathbb{R}^n}\int_{B(x,\delta)} \Gamma(x-y,t)|\nabla u_0(y) - \nabla u_0(x)|^2 dydx$$

for some positive δ. By a change of variable, $z = y - x$, we have

$$\begin{aligned}I_1 =& \int_{\mathbb{R}^n}\int_{\mathbb{R}^n\setminus B(\delta)} \Gamma(z,t)|\nabla u_0(x+z) - \nabla u_0(x)|^2 dxdz \\ \le& c\|\nabla u_0\|_{2,\mathbb{R}^n}^2 \int_{\mathbb{R}^n\setminus B(\delta)} \Gamma(z,t)dz \\ \le& \tilde{c}\|\nabla u_0\|_{2,\mathbb{R}^n}^2 \int_{\mathbb{R}^n\setminus B(\delta/(2\sqrt{t}))} e^{-|u|^2} du \to 0 \text{ as } t \to 0\end{aligned}$$

where we made another change of variable $z = 2\sqrt{t}u$ in the last line of the above inequality. For I_2, we can do the same change of variable and get

$$\begin{aligned}I_2 =& \int_{\mathbb{R}^n}\int_{B(x,\delta)} \Gamma(x-y,t)|\nabla u_0(y) - \nabla u_0(x)|^2 dydx \\ =& \int_{B(\delta))} \Gamma(z,t)dx \int_{\mathbb{R}^n} |\nabla u_0(x+z) - \nabla u_0(x)|^2 dx \\ \le& \sup_{|z|\le\delta} \int_{\mathbb{R}^n} |\nabla u_0(x+z) - \nabla u(x)|^2 dx \to 0 \text{ as } \delta \to 0\end{aligned}$$

Thus, $\|\nabla(u^1(\cdot,t) - u_0(\cdot))\|_{2,\mathbb{R}^n} \to 0$ as $t \to 0$. Note that

$$\begin{aligned}\|\nabla u^1(\cdot,t)\|_{2,\mathbb{R}^n} &= \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} \Gamma(x-y,t)\nabla u_0(y)dy\right)^2 dx\right)^{\frac{1}{2}} \\ &\le \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Gamma(x-y,t)|\nabla u_0(y)|^2 dydx\right)^{\frac{1}{2}} \\ &\le \|\nabla u_0\|_{2,\mathbb{R}^n}\end{aligned}$$

Together with the continuity of $\nabla u^1(\cdot,t)$ at $t=0$ in $L_2(\mathbb{R}^n)$, we have

$$\sup_{0\le t<\infty}\|\nabla u^1(\cdot,t)\|_{2,\mathbb{R}^n}\le\|\nabla u_0\|_{2,\mathbb{R}^n}.$$

Now we consider the estimates for $\|\nabla^2 u^1\|_{2,Q_+}$. Here $Q_+=\mathbb{R}^n\times(0,\infty)$. Integration by parts gives

$$|\nabla^2 u^1(x,t)|\le\int_{\mathbb{R}^n}|\nabla\Gamma(x-y,t)||\nabla u_0(y)|dy.$$

Note that

$$|\nabla\Gamma(x,t)|\le c\Gamma(x,t)\left(\frac{|x|}{t}\right)\le c\Gamma(x,t)\frac{1}{\sqrt{t}}\left(\frac{|x|}{\sqrt{t}}+1\right).$$

Then by the Cauchy inequality, we have

$$|\nabla^2 u^1(x,t)|\le c\left(\int_{B(R_0)}\Gamma(x-y,t)dy\right)^{\frac12}\left(\int_{B(R_0)}\Gamma(x-y,t)\frac{1}{t}(1+\frac{|x-y|}{\sqrt{t}})^2|\nabla u_0(y)|^2dy\right)^{\frac12}.$$

Note that

$$\begin{aligned}\Gamma(x-y,t)\frac{1}{t}(1+\frac{|x-y|}{\sqrt{t}})^2&\le\tilde{c}\frac{1}{t^{\frac{n}{2}+1}}\frac{1}{(1+\frac{|x-y|}{\sqrt{t}})^{n+2}}\\&\le\frac{\tilde{c}}{(\sqrt{t}+|x-y|)^{n+2}}.\end{aligned}$$

Thus

$$\begin{aligned}|\nabla^2 u^1(x,t)|\le&c\left(\int_{\mathbb{R}^n}\frac{1}{(\sqrt{t}+|x-y|)^{n+2}}|\nabla u_0(y)|^2dy)\right)^{\frac12}\\\le&\frac{c}{(\sqrt{t}+|x|)^{\frac{n}{2}+1}}\|\nabla u_0\|_{2,\mathbb{R}^n}.\end{aligned}$$

This implies that

$$\begin{aligned}\|\nabla^2 u^1\|_{2,Q_+}=&\int_0^\infty|\nabla^2 u^1(x,t)|^2dt\\\le&\int_0^\infty\frac{c}{(\sqrt{t}+|x|)^{\frac{n}{2}+1}}\|\nabla u_0\|_{2,\mathbb{R}^n}dt\\\le&c\|\nabla u_0\|_{2,\mathbb{R}^n}.\end{aligned}$$

Since u^1 solves $\partial_t u^1=\Delta u^1$, we know that

$$\begin{aligned}|\partial_t u^1(x,t)|\le&|\nabla^2 u^1(x,t)|\\\le&\frac{c}{(\sqrt{t}+|x|)^{\frac{n}{2}+1}}\|\nabla u_0\|_{2,\mathbb{R}^n}.\end{aligned}$$

This gives

$$
\begin{aligned}
\|\partial_t u^1\|_{2,Q_+} &= \int_0^\infty |\partial_t u^1(x,t)|^2 dt \\
&\leq \int_0^\infty \frac{c}{(\sqrt{t}+|x|)^{\frac{n}{2}+1}} \|\nabla u_0\|_{2,\mathbb{R}^n} dt \\
&\leq c\|\nabla u_0\|_{2,\mathbb{R}^n}.
\end{aligned}
$$

Combining these estimates together, we have

$$
\sup_{0\leq t<\infty} \left\|\nabla u^1(\cdot,t)\right\|_{2,\mathbb{R}^n} + \left\|\nabla^2 u^1\right\|_{2,Q_+} + \left\|\partial_t u^1\right\|_{2,Q_+} \leq c\left\|\nabla u_0\right\|_{2,\mathbb{R}^n}.
$$

$u^1(x,t) = \int_{\mathbb{R}^n} \Gamma(x-y,t) u_0(y)\,\mathrm{d}y$ satisfies $\partial_t u^1 - \Delta u^1 = 0$. Then

$$
\begin{aligned}
0 &= \int_0^T \int_{B(R)} |\partial_t u^1 - \Delta u^1|^2 \mathrm{d}x\mathrm{d}t \\
&= \int_0^T \int_{B(R)} |\partial_t u^1|^2 + |\Delta u^1|^2 \mathrm{d}x\mathrm{d}t - 2I
\end{aligned}
$$

where $R>0$ to be taken to ∞ later and

$$
I := \int_0^T \int_{B(R)} \partial_t u^1 \Delta u^1 \mathrm{d}x\mathrm{d}t.
$$

Integrating by parts gives us:

$$
\begin{aligned}
I &= \int_0^T \int_{\partial B(R)} \partial_t u^1 \nabla u^1 \cdot \frac{x}{|x|} \mathrm{d}\sigma_R \mathrm{d}t - \int_0^T \int_{B(R)} \partial_t \nabla u^1 \cdot \nabla u^1 \\
&= \int_0^T \int_{\partial B(R)} \partial_t u^1 \nabla u^1 \cdot \frac{x}{|x|} \mathrm{d}\sigma_R \mathrm{d}t + \frac{1}{2}\int_{B(R)} |\nabla u^1|^2 - \frac{1}{2}\int_{B(R)} |\nabla u_0|^2
\end{aligned}
$$

By using the fact that $\|\nabla(u^1(\cdot,t) - u_0(\cdot))\|_{2,\mathbb{R}^n} \to 0$ as $t \to 0$.
One more integration by parts leads to

$$
\begin{aligned}
\int_0^T \int_{B(R)} |\Delta u^1|^2 \mathrm{d}x\mathrm{d}t &= \int_0^T \int_{\partial B(R)} \frac{x}{|x|} \cdot \nabla u^1 \Delta u^1 \mathrm{d}\sigma_R \mathrm{d}t - \int_0^T \int_{B(R)} \nabla u^1 \cdot \nabla \Delta u^1 \mathrm{d}x\mathrm{d}t \\
&= \int_0^T \int_{\partial B(R)} \frac{x}{|x|} \nabla u^1 \Delta u^1 - \frac{x}{|x|} \otimes \nabla u^1 : \nabla u^1) \mathrm{d}\sigma_R \mathrm{d}t + \int_0^T \int_{B(R)} |\nabla^2 u^1|^2 \mathrm{d}x\mathrm{d}t
\end{aligned}
$$

This shows that

$$
\begin{aligned}
&\int_0^T \int_{B(R)} |\partial_t u^1|^2 \mathrm{d}x\mathrm{d}t + \int_0^T \int_{B(R)} |\nabla^2 u^1|^2 \mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_{B(R)} |\nabla u^1|^2 \\
=&\frac{1}{2}\int_{B(R)} |\nabla u_0|^2 + \eta(R)
\end{aligned}
$$

where $\eta(R)$ is the sum of the surface integrals which tend to 0 as $R \to \infty$. Thus $\int_0^T \int_{\mathbb{R}^n)} |\partial_t u^1|^2 \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}^n} |\nabla^2 u^1|^2 \mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u^1|^2 \leq \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_0|^2$ for all $0 \leq T < \infty$. The desired inequality then follows.

(ii) Recall that u^1 satisfies the heat equation in Q_+, that is,

$$\partial_t u^1(x,t) - \Delta u^1(x,t) = 0 \text{ with } u^1(x,0) = u_0(x).$$

Multiplying the equation by u^1 and then integrating by parts, we have

$$\int_{\mathbb{R}^n} \partial_t u^1(x,t) u^1(x,t) dx + \int_{\mathbb{R}^n} |\nabla u^1(x,t)|^2 dx = 0.$$

That is,

$$\int_{\mathbb{R}^n} \frac{1}{2}\frac{d}{dt}(u^1(x,t))^2 dx + \int_{\mathbb{R}^n} |\nabla u^1(x,t)|^2 dx = 0.$$

Multiplying by 2 on both sides gives

$$\frac{d}{dt}\|u^1(\cdot,t)\|^2_{2,\mathbb{R}^n} + 2\|\nabla u^1(\cdot,t)\|^2_{2,\mathbb{R}^n} = 0.$$

Integrating from 0 to t, we have

$$\int_0^t \frac{d}{ds}\|u^1(\cdot,s)\|^2_{2,\mathbb{R}^n} + 2\|\nabla u^1\|^2_{2,Q_t} ds = 0.$$

Thus, $\|u^1(\cdot,t)\|^2_{2,\mathbb{R}^n} - \|u^1(\cdot,0)\|^2_{2,\mathbb{R}^n} + 2\|\nabla u^1(\cdot,t)\|^2_{2,\mathbb{R}^n} = 0$. Rearranging gives

$$\|u^1(\cdot,t)\|^2_{2,\mathbb{R}^n} + 2\|\nabla u^1\|^2_{2,Q_t} = \|u_0\|^2_{2,\mathbb{R}^n}.$$

Now we show that $S(t)$ are bounded operators on $\overset{\circ}{L}{}^1_2(\mathbb{R}^n)$ and continuous with respect to t in $\overset{\circ}{L}{}^1_2(\mathbb{R}^n)$.

Let us show that for each $t > 0$, $u^1(\cdot,t) \in \overset{\circ}{L}{}^1_2(\mathbb{R}^n)$. Let φ be a standard cut-off function so that $\varphi = 1$ in $B(r)$, $\varphi = 0$ outside $B(2r)$, and $|\nabla\varphi| \leq \frac{c}{r}$ in $\mathbb{R}^n$. Then $\varphi u^1 \in C_0^\infty(\mathbb{R}^n)$ for any $r > 0$. According to (2.1.26) in the lecture notes and the Gagliardo-Nirenberg inequality, we have

$$\left\|u^1(\cdot,t)\right\|_{\frac{2n}{n-2},\mathbb{R}^n} \leq \|u_0(\cdot)\|_{\frac{2n}{n-2},\mathbb{R}^n} \leq c(n)\,\|\nabla u_0\|_{2,\mathbb{R}^n}$$

and thus

$$\begin{aligned}\int_{\mathbb{R}^n} \left|\nabla u^1(x,t) - \nabla\left(\varphi(x)u^1(x,t)\right)\right|^2 dx &\leq \int_{\mathbb{R}^n\setminus B(r)} \left|\nabla u^1(x,t)\right|^2 dx + \frac{c^2}{r^2}\int_{B(2r)\setminus B(r)} \left|u^1(x,t)\right|^2 dx \\ &\leq \int_{\mathbb{R}^n\setminus B(r)} \left|\nabla u^1(x,t)\right|^2 dx + c^2\left(\int_{B(2r)\setminus B(r)} \left|u^1(x,t)\right|^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{n}}\end{aligned}$$

converges to 0 as $r \to \infty$. Thus for every $t > 0$, $u_0 \in C_0^\infty(\mathbb{R}^n) \subset \overset{\circ}{L}{}_2^1(\mathbb{R}^n)$, and

$$u^1(\cdot, t) := S(t)u_0(\cdot) \in \overset{\circ}{L}{}_2^1(\mathbb{R}^n).$$

We also know that

$$\begin{aligned}\|\nabla(S(t)u_0)\|_{2,\mathbb{R}^n} &= \|\nabla u^1(\cdot,t)\|_{2,\mathbb{R}^n}\\ &\le \|\nabla u_0\|_{2,\mathbb{R}^n}.\end{aligned}$$

Then we can conclude that $S(t)$ are bounded operators on $\overset{\circ}{L}{}_2^1(\mathbb{R}^n)$.
Note that the family of operators $\{S(t)\}_{t\ge 0}$ satisfy the semi-group rule

$$S(t)S(s) = S(t+s)$$

for all $t, s \ge 0$. For the continuity result, it is sufficient to show that

$$S(t)u_0 \to u_0 \text{ in } \overset{\circ}{L}{}_2^1(\mathbb{R}^n) \text{ as } t \to 0$$

since

$$\begin{aligned}\|\nabla(S(t)u_0 - S(t_0)u_0)\|_{2,\mathbb{R}^n} &= \|\nabla(S(t_0)S(t-t_0)u_0 - S(t_0)S(0)u_0)\|_{2,\mathbb{R}^n}\\ &\le \|\nabla(S(t-t_0)u_0 - u_0)\|_{2,\mathbb{R}^n} \text{ (By boundedness of } S(t_0)).\end{aligned}$$

Then the continuity follows from that

$$\|\nabla(u^1(\cdot,t) - u_0(\cdot))\|_{2,\mathbb{R}^n} \to 0 \text{ as } t \to 0$$

as proved in (i) of this exercise. □

Exercise 4. *Complete the proof of Theorem 1.7 of Section 2.1.4.*
Consider the following initial boundary value problem for the heat equation

$$\partial_t u - \Delta u = f \tag{0.3}$$

in the half space $Q_T = \mathbb{R}^n_+ \times]0, T[$, *where*

$$\mathbb{R}^n_+ := \{x = (x', x_n) \colon x' = (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}, x_n > 0\},$$

$$u(x', 0, t) = 0 \tag{0.4}$$

for all $x' \in \mathbb{R}^{n-1}$ *and for all* $0 \le t \le T$ *and*

$$u(x, 0) = u_0(x) \tag{0.5}$$

for all $x \in \mathbb{R}^n$. *We call* u *a weak solution to (0.3)-(0.5) if*

$$\int_{Q_T} u\,(\partial_t w + \Delta w)\,dxdt + \int_{\mathbb{R}^n_+} u_0(x)w(x,0)dx = -\int_{Q_T} fw dxdt \tag{0.6}$$

for any $w \in C_0^\infty(\mathbb{R}^n \times] - T, T[)$ *with* $w(x', 0, t) = 0$.

Theorem 1. *(Theorem 1.7 in the lecture notes)*

I. Let $f \in L_2(Q_T)$ and $u_0 = 0$, there exists a unique function $u \in L_2(Q_T)$ satisfying the identity (0.6). Moreover, $u \in W_2^{2,1}(Q_T)$ and the following estimates are valid:

$$\|u(\cdot,t)\|^2_{2,\mathbb{R}^n_+} \leq t \int_0^t \|f(\cdot,s)\|^2_{2,\mathbb{R}^n_+} ds$$

and

$$\|\nabla u\|_{2,\infty,Q_T} + \|\partial_t u\|_{2,Q_T} + \left\|\nabla^2 u\right\|_{2,Q_T} \leq c\|f\|_{2,Q_T}.$$

II. Let $u_0 \in L_2(\mathbb{R}^n)$ and $f = 0$, there exists a unique function u that belongs to $L_2(Q_T)$ for any $T > 0$ and satisfies the identity (0.6) . Moreover, the function u satisfies the heat equation in the sense of distributions and the estimate

$$\|u(\cdot,t)\|_{2,\mathbb{R}^n_+} + \|\nabla u\|_{2,Q_t} \leq c\,\|u_0\|_{2,\mathbb{R}^n_+}$$

for all $t \geq 0$, where $Q_t = \mathbb{R}^n_+ \times]0,t[$, and is continuous as a function of $t \geq 0$ with values in $L_2\left(\mathbb{R}^n_+\right)$. In particular,

$$\|u(\cdot,t) - u_0(\cdot)\|_{2,\mathbb{R}^n_+} \to 0$$

as $t \to 0$.

III. Let $u_0 \in \overset{\circ}{L}{}^1_2\left(\mathbb{R}^n_+\right)$ and $f = 0$, there exists a unique function v that belongs to $L_2(0,T;\overset{\circ}{L}{}^1_2\left(\mathbb{R}^n_+\right))$ for any $T > 0$ and satisfies the identity (0.6). Moreover, the function u satisfies the heat equation in the sense of distributions and the estimate

$$\|\nabla v(\cdot,t)\|_{2,\mathbb{R}^n_+} + \left\|\nabla^2 v\right\|_{2,Q_t} + \|\partial_t v\|_{2,Q_t} \leq c\,\|\nabla u_0\|_{2,R^n_+}$$

for all $t \geq 0$, ∇v is continuous as a function of $t \geq 0$ with values in $L_2(\mathbb{R}^n_+)$. In particular,

$$\|\nabla v(\cdot,t) - \nabla u_0(\cdot)\|_{2,\mathbb{R}^n_+} \to 0$$

as $t \to 0$.

Proof. I. Let first prove the existence of solution. Let $f_m \in C_0^\infty(Q_T)$ be such that $f_m \to f$ in $L_2(Q_T)$, and let $\tilde{f_m}$, $\tilde{f}$ denote the odd extensions of f_m and f to the whole $\mathbb{R}^n$ respectively. Obviously, $\tilde{f_m} \in C_0^\infty(Q_T)$. Then the volume heat potential

$$u^2_m(x,t) := \int_0^t \int_{\mathbb{R}^n} \Gamma(x-y,t-\theta)\tilde{f}_m(y,\theta)dyd\theta.$$

satisfies the boundary condition $u^2_m(x',0,t) = 0$ and $u_m := u^2\mid_{\mathbb{R}^n_+}$ satisfies equality (0.6),

$$\|u_m(\cdot,t)\|^2_{2,\mathbb{R}^n_+} \leq t \int_0^t \|\tilde{f}_m(\cdot,s)\|^2_{2,\mathbb{R}^n_+} ds$$

and

$$\|\nabla u_m\|_{2,\infty,Q_T} + \|\partial_t u_m\|_{2,Q_T} + \left\|\nabla^2 u_m\right\|_{2,Q_T} \le c\|f_m\|_{2,Q_T}.$$

It follows from the estimate in (2.1.12) in section 2.1.12 that $u_m \to u$ in $L^2(Q_T)$. Passing to the limit $m \to \infty$, we complete the proof of the existence and derive all the required estimates that are true. It remains to prove the uniqueness.

Assume that a function $v \in L_2(Q_T)$ satisfies the identity

$$\int_{Q_T} v(\partial_t w + \Delta w)dxdt = 0 \tag{0.7}$$

for any $w \in C_0^\infty(\mathbb{R}^n \times]-T,T[)$ with $w(x',0,t) = 0$. Our aim is to show that $v = 0$ in Q_T. Let $\tilde{v}$ be the odd extension of v to $\mathbb{R}^n$. Then from (0.7) we have

$$\int_{\mathbb{R}^n \times]-T,T[} \tilde{v}(\partial_t w + \Delta w)dxdt = 0$$

for any odd function $w \in C_0^\infty(\mathbb{R}^n \times]-T,T[)$. Now, let w be an arbitrary function in $C_0^\infty(\mathbb{R}^n \times]-T,T[)$. We can present it as sum of $w_{odd}(x,t) = (w(x',x_n,t) - w(x',-x_n,t))/2$ and $w_{even} = (w(x',x_n,t) + w(x',-x_n,t))/2$. Then,

$$\int_{\mathbb{R}^n \times]-T,T[} \tilde{v}(\partial_t(w - w_{even}) + \Delta(w - w_{even})dxdt = \int_{\mathbb{R}^n \times]-T,T[} \tilde{v}(\partial_t w + \Delta w)dxdt = 0$$

for any function $w \in C_0^\infty(\mathbb{R}^n \times]-T,T[)$. By Theorem 1.3, $\tilde{v} = 0$. The first part of the theorem is proved.

II. Consider $u_{0m} \in C_0^\infty(\mathbb{R}^n_+)$ such that $u_{0m} \to u_0$ in $L_2(\mathbb{R}^n_+)$ as $m \to \infty$. Let denote the odd extensions of u_{0m} and u_0 to the whole $\mathbb{R}^n$ by $\tilde{u}_{0m}$ and $\tilde{u}_0$ respectively. Obviously $\tilde{u}_{0m} \in C_0^\infty(\mathbb{R}^n)$ and $\tilde{u}_{0m} \to \tilde{u}_0$ in $L_2(\mathbb{R}^n)$. Consider

$$u_m^1(x,t) := \int_{\mathbb{R}^n} \Gamma(x-y,t)\tilde{u}_{m0}(y)dy = S(t)\tilde{u}_{m0}.$$

Then for any $T > 0$, $u_m := u_m^1 \mid_{\mathbb{R}^n_+}$ satisfies equality (0.6) and converges to

$$u(\cdot,t) = S(t)\tilde{u}_0 \mid_{\mathbb{R}^n_+}$$

in $L_{2,\infty}(Q_t)$ for any $0 \le t \le T$. Indeed,it follows from (ii) of Exercise 3 that

$$\|u_m - u_k\|_{2,\infty,Q_t} + \|\nabla(u_m - u_k)\|_{2,Q_t} \le c\|u_{0m} - u_{0k}\|_{2,\mathbb{R}^n_+} \to 0 \text{ as } m,n \to \infty.$$

By the continuity of the semi-group $S(t)$ and (ii) of Exercise 3, all the statements in II hold for u_m for each $m \in \mathbb{N}$ apart from the uniqueness. We can thus pass to the limit $m \to \infty$ to get the required inequalities for $u(x,t)$. The uniqueness proof follows from the proof in the first part applied to Q_T with a growing T.

III. According to the definition of the space $\overset{\circ}{L}{}^1_2(\mathbb{R}^n_+)$, we can find a sequence $u_{0m} \in C_0^\infty(\mathbb{R}^n_+)$ such that $\nabla u_{0m} \to \nabla u_0$ in $L_2(\mathbb{R}^n_+)$. Let denote the odd extensions of u_{0m} and u_0 to the whole of $\mathbb{R}^n$ by $\tilde{u}_{0m}$ and $\tilde{u}_0$. Then $\tilde{u}_{0m} \in C_0^\infty(\mathbb{R}^n)$ and $\tilde{u}_0 \in \overset{\circ}{L}{}^1_2(\mathbb{R}^n)$. We have $\nabla \tilde{u}_{0m} \to \nabla \tilde{u}_0$ in $L_2(\mathbb{R}^n)$. By the Gagliardo-Nirenberg inequality, $\tilde{u}_{0m} \to \tilde{u}_0$ in $L_{\frac{2n}{n-2}}(\mathbb{R}^n)$ as well and thus in $L_2(\mathbb{R}^n)$. We know that

$$u^1_m(x,t) := \int_{\mathbb{R}^n} \Gamma(x-y,t)\tilde{u}_{m0}(y)dy = S(t)\tilde{u}_{m0} \in \overset{\circ}{L}{}^1_2(\mathbb{R}^n_+)$$

for any $t > 0$ and for any m. Now all the statements of the theorem can be deduced by passing to the limit as $m \to \infty$ and take the restriction on $\mathbb{R}^n_+$ as in the proof for II. To prove uniqueness, we let $u = v^1 - v^2$ where v^1 and v^2 both satisfy the statements of III. Then we can deduce from (0.6) the identity for the derivative $u_{,i}$:

$$\int_{Q_T} u_{,i}(\partial_t w + \Delta w)dxdt = 0$$

for any $w \in C_0^\infty(\mathbb{R}^n \times]-T,T[)$. By Theorem 1.3 in the lecture notes, $u_{,i} = 0$ for all $i = 1, 2, \dots, n$ and thus u is constant in Q_T. On the other hand, u satisfies the identity

$$\int_{Q_T} u(\partial_t w + \Delta w)dxdt = 0$$

for any $w \in C_0^\infty(\mathbb{R}^n \times]-T,T[)$. This immediately implies that $u \equiv 0$.

However, in this case, we are not sure whether the limit function u^1_+ is in $\overset{\circ}{L}{}^1_2(\mathbb{R}^n_+)$ or not. So the only thing we need to check here is that

$$u^1_+(x,t) = \int_{\mathbb{R}^1_+} G(x,y,t)u_0(y)dy$$

belongs to $\overset{\circ}{L}{}^1_2(\mathbb{R}^n_+)$ for each $t > 0$ provided $u_0 \in C_0^\infty(\mathbb{R}^n_+)$. To this end, let us explore how our potential behaves around the boundary $x_n = 0$. We assume that $u_0(x) = 0$ provided that $0 < x_n < 2h$. Then

$$\begin{aligned}
|u^1_+(x,t)| =& \Big|\int_{\mathbb{R}^n_+} (\Gamma(x'-y', x_n - y_n, t) - \Gamma(x'-y', x_n + y_n, t))u_0(y', y_n)dy'dy_n\Big| \\
\leq& \int_{\mathbb{R}^n_+} \Gamma(x'-y', x_n - y_n, t)(1 - \exp(-x_n y_n/t))|u_0(y)|dy \\
\leq& \frac{x_n}{t}\int_{\mathbb{R}^n_+} \Gamma(x'-y', x_n - y_n, t) y_n |u_0(y)|dy.
\end{aligned}$$

After application of the Hölder inequality,

$$|u^1_+(x,t)|^2 \leq \left(\frac{x_n}{t}\right)^2 \int_{\mathbb{R}^n_+} \Gamma(x'-y', x_n - y_n, t) y_n^2 |u_0(y)|^2 dy$$

and thus, for any positive a,

$$
\begin{aligned}
\int_0^a \int_{\mathbb{R}^{n-1}} |u_+^1(x,t)|^2 dx' dx_n \le & \frac{1}{(4\pi t)^{\frac{n}{2}} t^2} \int_0^a x_n^2 dx_n \int_{\mathbb{R}^{n-1}} dx' \int_{\mathbb{R}^n_+} e^{-|x'-y'|^2/(4t)} \times \\
& e^{-|x_n-y_n|^2/(4t)} y_n^2 |u_0(y)|^2 dy \\
\le & \frac{1}{(4\pi t)^{\frac{n}{2}} t^2} \int_0^a x_n^2 dx_n (4\pi t)^{\frac{n-1}{2}} \int_{\mathbb{R}^n_+} e^{-|x_n-y_n|^2/(4t)} y_n^2 |u_0(y)|^2 dy \\
\le & \frac{1}{(4\pi t)^{\frac{1}{2}} t^2} \int_0^a x_n^2 dx_n \int_{\mathbb{R}^n_+} y_n^2 |u_0(y,t)|^2 dy \\
= & \frac{C(u_0) a^3}{(4\pi t)^{\frac{1}{2}} t^2}.
\end{aligned}
$$

Using the odd extension and the corresponding result for the heat potential, we can show that for each $t > 0$, that

$$\|\nabla u_+^1(\cdot,t) - \nabla(\varphi(\cdot) u_+^1(\cdot,t)\|_{2,\mathbb{R}^n_+} \to 0 \text{ as } r \to \infty,$$

where $0 \le \varphi \le 1$ is a smooth function with support in $B(2r)$ such that $\varphi = 1$ in $B(r)$, and $|\nabla \varphi| \le c/r$. We need to cut-off functions φu_+^1 around $x_n = 0$. To this end, let us pick up a smooth cut-off function $0 \le \eta \le 1$ such that $\eta(t) = 1$ if $t > 2h$, $\eta(t) = 0$ if $h > t > 0$ and $|\eta'(t)| \le \frac{c}{h}$ for all $0 < t < \infty$. So, $\eta \varphi u_+^1 \in C_0^\infty(\mathbb{R}^n_+)$. It remains to show that

$$\int_{\mathbb{R}^n_+} |\nabla(\varphi u_+^1 (1-\eta))|^2 dx \to 0$$

as $h \to 0$. It follows easily from the following estimates:

$$\int_{\mathbb{R}^n_+} |\eta'|^2 \varphi^2 |u_+^1|^2 dx \le \frac{c^2}{h^2} \int_0^{2h} \int_{\mathbb{R}^{n-1}} |u_+^1|^2 dx = O(h) \text{ as } h \to 0.$$

Thus

$$
\begin{aligned}
\|\nabla u_+^1(\cdot,t) - \nabla(\varphi(\cdot) u_+^1(\cdot,t)\eta(t))\|_{2,\mathbb{R}^n_+} \le & \|\nabla u_+^1(\cdot,t) - \nabla(\varphi(\cdot) u_+^1(\cdot,t)\|_{2,\mathbb{R}^n_+} \\
& + \int_{\mathbb{R}^n_+} |\nabla(\varphi u_+^1(1-\eta))|^2 dx \\
& \to 0 \text{ as } r \to \infty, h \to 0.
\end{aligned}
$$

This implies that $u_+^1(x,t) \in \overset{\circ}{L}{}_2^1\left(\mathbb{R}^n_+\right)$ for each $t > 0$.

The key point of this proof is to show the uniqueness of solution. We claim that

$$\int_{\mathbb{R}^n \times (-T,T)} \tilde{v}_{,i} (\partial_t w + \Delta w) = 0 \text{ for all } w \in C_0^\infty(\mathbb{R}^n \times (-T,T)).$$

$$
\begin{aligned}
&\int_{\mathbb{R}^n\times(-T,T)} \tilde{v}_{,i}(\partial_t w^{even} + \Delta w^{even}) \\
=&\int_{\mathbb{R}^n\times(-T,T)} (\tilde{v}(\partial_t w^{even} + \Delta w^{even}))_{,i} \\
&-\int_{\mathbb{R}^n\times(-T,T)} \tilde{v}_{,i}(\partial_t w^{even}_{,i} + \Delta w^{even}_{,i}) \\
&= 0.
\end{aligned}
$$

Then $\tilde{v}_{,i} = 0$ for all $i = 1, \ldots, N$, we have $v = C(t)$. By the fact that

$$
\int_{\mathbb{R}^n\times(-T,T)} \tilde{v}(\partial_t w + \Delta w) = 0 \text{ for all } w \in C_0^\infty(\mathbb{R}^n \times (-T,T))
$$

and $\tilde{v} \in \overset{\circ}{L}{}^1_2$, we can conclude that $\tilde{v} \equiv 0$.

$\square$