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Parabolic PDEs -Problem Sheet Three

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Exercise 1. *Consider the following system of PDE's.*

$$\partial_t u - \Delta u = v$$

$$-\Delta v = u + f$$

in Ω,

$$u_{|\partial\Omega} = 0, v\mid_{\partial\Omega} = 0$$

and

$$u(x,0) = u_0(x)$$

for all $x \in \Omega$.
Let $u_0 \in H(\Omega)$ *and* $f \in L_2(0,T;H^{-1}(\Omega))$. *Show that there exists a unique pair* u *and* v *having the following properties:*

(i)

$$u, v \in L_2(0,T;H^1(\Omega)),$$

$$u \in C([0,T];H(\Omega)),\ \ \partial_t u \in L_2(0,T;H^{-1}(\Omega))$$

(ii) for a.a. $t \in]0,T[$,

$$\int_\Omega (\partial_t u(x,t)W(x) + \nabla u(x,t)\cdot\nabla W(x))\,\mathrm{d}x = \int_\Omega v(x,t)W(x)\,\mathrm{d}x$$

for all $W \in H^1(\Omega)$ *and*

$$\int_\Omega \nabla v(x,t)\cdot\nabla\widetilde{W}(x)\,\mathrm{d}x = \int_\Omega \Big(u(x,t)\widetilde{W}(x) + f(x,t)\widetilde{W}(x)\Big)\,\mathrm{d}x$$

for all $\widetilde{W} \in H^1(\Omega)$.

(iii)

$$\|u(\cdot,t) - u_0(\cdot)\|_{2,\Omega} \to 0$$

as $t \to 0$.

Proof. We construct explicit solutions using the eigenvalues and eigenfunctions of the Laplace operator Δ in the domain Ω,

$$-\Delta\varphi_k = \lambda_k\varphi_k \text{ in } \Omega$$

$$\varphi_k = 0 \text{ on } \partial\Omega$$

where $k = 1, 2 \ldots$.

First, let us expand functions f and $u_0(x)$ as Fourier series, using eigenfunctions of the Laplace operator

$$f(x,t) = \sum_{k=1}^{\infty} f_k(t)\varphi_k(x) \text{ where } f_k(t) = (f(\cdot,t), \varphi_k(\cdot))$$

and

$$u_0(x) = \sum_{k=1}^{\infty} a_k\varphi_k(x) \text{ where } a_k = (a, \varphi_k).$$

By our assumptions,

$$\|u_0(x)\|^2_{L_2(\Omega)} = \sum_{k=1}^{\infty} a_k^2 < \infty.$$

$$\|f\|^2_{L_2(0,T;H^{-1})} = \int_0^T \sum_{k=1}^{\infty} \frac{1}{\lambda_k}|f_k(t)|^2 \, dt < \infty$$

where the equality for H^{-1}-norm of f follows from Problem Sheet Two, Q.5.

We are looking for a solution pair of the form

$$u(x,t) = \sum_{k=1}^{\infty} c_k(t)\varphi_k(x), \tag{0.1}$$

$$v(x,t) = \sum_{k=1}^{\infty} d_k(t)\varphi_k(x). \tag{0.2}$$

If we insert (0.1), (0.2) into the system of PDE's, then the following equations

$$\sum_{k=1}^{\infty} c_k'\varphi_k + \lambda_k c_k\varphi_k = \sum_{k=1}^{\infty} d_k\varphi_k, \tag{0.3}$$

$$\sum_{k=1}^{\infty} d_k(t)\lambda_k\varphi_k = \sum_{k=1}^{\infty} c_k(t)\varphi_k + \sum_{k=1}^{\infty} f_k\varphi_k \tag{0.4}$$

hold if we let

$$\begin{cases} c_k'(t) + \lambda_k c_k(t) = d_k(t), \\ \lambda_k d_k(t) = c_k(t) + f_k(t), \\ c_k(0) = a_k. \end{cases}$$

We can rewrite the above system of ODEs as

$$\begin{cases} c_k'(t) + \mu_k c_k(t) = \frac{f_k}{\lambda_k}, \\ d_k(t) = \frac{1}{\lambda_k}\left(c_k(t) + f_k(t)\right), \\ c_k(0) = a_k \end{cases}$$

where $\mu_k = \lambda_k - \frac{1}{\lambda_k}$. The new system of ODEs has a unique solution pair

$$c_k(t) = e^{-\mu_k t}\left(a_k + \int_0^t e^{\mu_k t}\frac{1}{\lambda_k} f_k(\tau)\,\mathrm{d}\tau\right),$$

$$d_k = \frac{1}{\lambda_k}\left(c_k + f_k\right).$$

Recall that

$$c_k'(t) + \mu_k c_k(t) = \frac{f_k}{\lambda_k}. \tag{0.5}$$

We multiply (0.5) by c_k and apply Young's inequality,

$$\begin{aligned} c_k'(t)c_k(t) + \mu_k c_k^2(t) =& \frac{1}{\lambda_k}\left(f_k(t)c_k(t)\right) \\ \leq& \frac{1}{2}\frac{f_k^2(t)}{\lambda_k^2\mu_k} + \frac{1}{2}c_k^2(t)\mu_k. \end{aligned}$$

Thus,

$$(c_k^2(t))' + \mu_k c_k(t)^2 \leq \frac{f_k^2(t)}{\lambda_k^2\mu_k}.$$

Integration in t gives us

$$\begin{aligned} c_k^2(T) + \mu_k\int_0^T c_k^2(t)\,\mathrm{d}t \leq& c_k^2(0) + \int_0^T \frac{f_k(t)}{\lambda_k^2\mu_k}\,\mathrm{d}t \\ \leq& a_k^2 + \frac{1}{\mu_k\lambda_k}\int_0^T \frac{f_k^2(t)}{\lambda_k}\mathrm{d}t \\ \leq& a_k^2 + C_1\int_0^T \frac{f_k^2(t)}{\lambda_k}\mathrm{d}t \end{aligned}$$

where $C_1 = \max_k \frac{1}{\mu_k \lambda_k}$. Then

$$\begin{aligned}
\|\nabla u\|_{L_2(0,T;L_2(\Omega))} =& \int_0^T \sum_{k=1}^{\infty} \lambda_k c_k^2(t)\,\mathrm{d}t \\
\le& \max_k \frac{\lambda_k}{u_k} \int_0^T \sum_{k=1}^{\infty} \mu_k c_k^2(t)\,\mathrm{d}t \\
\le& C_2 \int_0^T \sum_{k=1}^{\infty} \mu_k c_k^2(t)\,\mathrm{d}t \\
\le& C_2 \|u_0\|_{L_2(\Omega)}^2 + C_1 C_2 \|f\|_{L_2(0,T;H^{-1})}^2 \\
<& \infty.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|\nabla v\|_{L_2(0,T;L_2(\Omega))} =& \int_0^T \sum_{k=1}^{\infty} \lambda_k d_k^2(t)\,\mathrm{d}t \\
\le& \int_0^T \sum_{k=1}^{\infty} \frac{1}{\lambda_k}(c_k + f_k)^2\,\mathrm{d}t \\
\le& \int_0^T \sum_{k=1}^{\infty} \frac{2c_k^2}{\lambda_k} + \frac{2f_k^2}{\lambda_k}\,\mathrm{d}t \\
\le& 2\|u\|_{L^2(0,T;H^{-1})}^2 + 2\|f\|_{L_2(0,T;H^{-1})}^2 \\
\le& 2C\|u\|_{L^2(0,T;L^2)}^2 + 2\|f\|_{L_2(0,T;H^{-1})}^2 \text{ (by Sobolev's embedding)} \\
<& \infty.
\end{aligned}$$

$$\begin{aligned}
\|\partial_t u\|_{L_2(0,T;H^{-1})}^2 =& \int_0^T \sum_{k=1}^{\infty} \frac{|c_k(t)'|^2}{\lambda_k}\,\mathrm{d}t \\
=& \int_0^T \sum_{k=1}^{\infty} \frac{|\frac{f_k}{\lambda_k} - \mu_k c_k|^2}{\lambda_k}\,\mathrm{d}t \\
\le& \int_0^T \sum_{k=1}^{\infty} \frac{2f_k^2}{\lambda_k^3} + \frac{2|\mu_k|^2 c_k^2}{\lambda_k}\,\mathrm{d}t \\
\le& \frac{2}{\lambda_1^2} \int_0^T \sum_{k=1}^{\infty} \frac{f_k^2}{\lambda_k}\,\mathrm{d}t + \int_0^T \sum_{k=1}^{\infty} \frac{2|\mu_k|\mathrm{sgn}(\mu_k)}{\lambda_k} \mu_k c_k^2\,\mathrm{d}t \\
\le& \frac{2}{\lambda_1^2} \int_0^T \sum_{k=1}^{\infty} \frac{f_k^2}{\lambda_k}\,\mathrm{d}t + 2(1 + \frac{1}{\lambda_1^2}) \int_0^T \sum_{k=1}^{\infty} \mu_k c_k^2\,\mathrm{d}t \\
\le& \frac{2}{\lambda_1^2} \|f\|_{L_2(0,T;H^{-1})} + 2(1 + \frac{1}{\lambda_1^2})(\|u_0\|_{L^2}^2 + C_1 \|f\|_{L_2(0,T;H^{-1})}) \\
<& \infty.
\end{aligned}$$

Thus, $u, v \in L_2(0,T;H^1(\Omega))$, $\partial_t u \in L_2(0,T;H^{-1}(\Omega))$. By theorem 3.4 from the lecture notes, we know that $u \in L_2(0,T;H^1(\Omega))$ and $\partial_t u \in L_2(0,T;H^{-1}(\Omega))$ implies that $u \in C([0,T];H(\Omega))$.

To show (ii), take an arbitrary function $w \in L_2(0,T;H^{-1})$ and expand as a Fourier series

$$w(x,t) = \sum_{k=1}^{\infty} e_k(t)\varphi_k(x).$$

Note that

$$\|\nabla w\|^2_{L_2(0,T;L_2(\Omega))} = \int_0^T \sum_{k=1}^{\infty} \lambda_k e_k^2(t)\mathrm{d}t < \infty.$$

$$\int_0^T \int_\Omega \partial_t u \cdot w \mathrm{d}x\mathrm{d}t = \int_0^T \sum_{k=1}^{\infty} c_k'(t)e_k(t)\,\mathrm{d}t.$$

$$\begin{aligned}\int_0^T \int_\Omega \nabla u \cdot \nabla w \,\mathrm{d}x\, ddt &= \int_0^T \int_\Omega \sum_{k=1}^{\infty} c_k(t)e_k(t)|\nabla\varphi_k|^2 \,\mathrm{d}x\mathrm{d}t \\ &= \int_0^T \sum_{k=1}^{\infty} \lambda_k c_k(t)e_k(t)\,\mathrm{d}t.\end{aligned}$$

$$\int_0^T \int_\Omega v \cdot w \,\mathrm{d}x\mathrm{d}t = \int_0^T \sum_{k=1}^{\infty} d_k(t)e_k(t)\,\mathrm{d}t.$$

Then

$$\begin{aligned}\int_0^T \int_\Omega (\partial_t u w + \nabla u \cdot \nabla w - v \cdot w)\,\mathrm{d}x\mathrm{d}t &= \int_0^T \sum_{k=1}^{\infty} (c_k'(t) + \lambda_k c_k(t) - d_k(t))e_k \\ &=0\end{aligned}$$

since

$$c_k'(t) + \lambda_k c_k(t) = d_k(t).$$

Taking $w(x,t) = \chi(t)W(x)$ with $W \in H^1(\Omega)$ and $\chi \in C_0^1(0,T)$ for a.a. $t \in]0,T[$, the identity

$$\int_\Omega (\partial_t u(x,t)W(x) + \nabla u(x,t) \cdot \nabla W(x))\,\mathrm{d}x = \int_\Omega v(x,t)W(x)\,\mathrm{d}x$$

holds for all $W \in H^1(\Omega)$.

Similarly,

$$\int_\Omega \nabla v \cdot \nabla w \,\mathrm{d}x = \int_\Omega \sum_{k=1}^{\infty} d_k e_k |\nabla\varphi_k|^2 \,\mathrm{d}x = \sum_{k=1}^{\infty} \lambda_k d_k(t)e_k(t)\,\mathrm{d}t.$$

$$\int_\Omega u \cdot w \,\mathrm{d}x\mathrm{d}t = \sum_{k=1}^{\infty} c_k(t) e_k(t).$$

$$\int_\Omega f \cdot w \,\mathrm{d}x\mathrm{d}t = \sum_{k=1}^{\infty} f_k(t) e_k(t).$$

It follows that

$$\int_\Omega \nabla u \cdot \nabla w - u \cdot w - f \cdot w = \sum_{k=1}^{\infty} (\lambda_k d_k(t) - c_k(t) - f_k(t)) e_k = 0.$$

Taking $w = \widetilde{W}\chi$ for $\tilde{W} \in H^1(\Omega)$ and $\chi(t) \in C_0^1(0,T)$, we obtain

$$\int_\Omega \nabla v(x,t) \cdot \nabla \widetilde{W}(x) \,\mathrm{d}x = \int_\Omega \left(u(x,t)\widetilde{W}(x) + f(x,t)\widetilde{W}(x) \right) \,\mathrm{d}x$$

for all $\widetilde{W} \in H^1(\Omega)$.

(iii) is a direct consequence of $u \in C([0,T]; H(\Omega))$ since $H(\Omega) = L^2(\Omega)$.

To prove the uniqueness of the solution, it is sufficient to prove that for $f = 0$, $u \equiv 0$ and $v \equiv 0$.

$$\partial_t u - \Delta u = v, \tag{0.6}$$

$$-\Delta v = u \tag{0.7}$$

with $u(x,0) = 0$. Testing (0.6) against u, we have

$$\frac{1}{2}\frac{d}{dt} \int_{Omega} |u|^2 \,\mathrm{d}x + \int_\Omega |\nabla u|^2 \,\mathrm{d}x = \int_\Omega vu.$$

Testing (0.7) against v, we obtain

$$\int_\Omega |\nabla v|^2 \,\mathrm{d}x = \int_\Omega uv \,\mathrm{d}x.$$

Then

$$\int_\Omega |\nabla v|^2 \,\mathrm{d}x = \frac{1}{2}\frac{d}{dt} \int_\Omega |u|^2 \,\mathrm{d}x + \int_\Omega |\nabla u|^2 \,\mathrm{d}x.$$

Note that

$$\begin{aligned}
\int_\Omega uv \,\mathrm{d}x \leq & \left(\int_\Omega |u|^2 \right)^{\frac{1}{2}} \left(\int_\Omega |v|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \\
\leq & c(\Omega) \left(\int_\Omega |u|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \left(\int_\Omega |\nabla v|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \quad \text{(by Poincaré's inequality)}
\end{aligned}$$

This shows that
$$\left(\int_\Omega |\nabla v|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \leq c(\Omega)\left(\int_\Omega |u|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}$$
which in turn also implies tat
$$\frac{d}{dt}\int_\Omega |u|^2 \,\mathrm{d}x \leq \tilde{c}(\Omega)\int_\Omega |u|^2 \,\mathrm{d}x.$$
By Gronwall's inequality, we can show that $u \equiv 0$ which in turns implies that $v \equiv 0$. □

Exercise 2. *Let smooth functions u and $g = (g_i)$ satisfy the equation*
$$\partial_t u - \Delta u = -\mathrm{div} g$$
in Q. Let $p > n + 2$. Show that there exists a constant $c = c(n, p)$ such that
$$\sup_{x \in Q(1/2)} |u(x)| \leq c(\|u\|_{p,Q} + \|g\|_{p,Q}).$$

Proof. Note that the solution satisfies
$$\int_Q u(\partial_t w + \Delta w)\,\mathrm{d}x\mathrm{d}t = \int_Q \mathrm{div} g w \,\mathrm{d}x\mathrm{d}t \tag{0.8}$$
for any $w \in C_0^\infty(Q)$. Let us pick up a cut-off function $\varphi_r(x,t) = \chi_r(t)W_r(x)$ where non-negative functions $\chi_r \in C_0^\infty(-1,1)$ and $W_r \in C_0^\infty(B)$ satisfy
$$\chi_r = \begin{cases} 1 & \text{in } (-r^2, r^2), \\ 0 & \text{outside } (-\frac{(1+r)^2}{4}, \frac{(1+r)^2}{4}), \end{cases}$$
and
$$W_r = \begin{cases} 1 & \text{in } B(r), \\ 0 & \text{outside } B(\frac{1+r}{2}) \end{cases}$$
for some $0 < r < 1$. Consider $w = \varphi_r \psi$, then
$$\partial_t w = \partial_t \varphi_r \psi + \varphi_r \partial_t \psi,$$
$$\Delta w = \Delta \varphi_r \psi + \varphi_r \Delta \psi + 2\nabla \varphi_r \nabla \psi.$$
By inserting $w = \varphi_r \psi$ into (0.8), we have
$$\int_{\mathbb{R}^n \times (-1,0)} v(\partial_t \psi + \Delta \psi)\mathrm{d}x\mathrm{d}t = -\int_{\mathbb{R}^n \times (-1,0)} (f - \varphi_r \mathrm{div} g)\psi \,\mathrm{d}x\mathrm{d}t \tag{0.9}$$

for any $\psi \in C_0^\infty(\mathbb{R}^n \times (-2,0))$ with $v := \varphi_r u$ and $f := -2\mathrm{div}(u\nabla\varphi_r) + u(\partial_t\varphi_r + \Delta\varphi_r)$. Note that since u is smooth, $\varphi_r = \chi_r(t)W_r(x)$ is smooth, we know that f is smooth. The following equality is also valid.

$$\int_Q v(\partial_t\psi + \Delta\psi)\mathrm{d}x\mathrm{d}t = -\int_Q (f - \varphi_r \mathrm{div} g)\psi \,\mathrm{d}x\mathrm{d}t \tag{0.10}$$

for any $\psi(x,t) = \chi(t)W(x)$ with an arbitrary function $\chi \in C_0^\infty(-2,0)$ and an arbitrary function $W \in C^2(\bar{B})$. To see that the later identity holds, let us notice that there is a sequence of functions $W_k \in C_0^\infty(\mathbb{R}^n)$ such that $W_k \to W$ in $C^2(\bar{B})$ and also that $v = \varphi_r u$ is zero outside B for all t, so we can easily get (0.10) from (0.9). By Theorem 3.6, there exists a unique weak solution to the problem

$$\partial_t\tilde{v} - \Delta\tilde{v} = \tilde{f} \text{ in } Q,$$

$$\tilde{v} = 0 \text{ on } \partial' Q$$

where $\tilde{f} = f - \varphi_r \mathrm{div} g$. From the same theorem, it follows that $\tilde{v}$ satisfies (0.10) provide $W = 0$ on ∂B (since we need the boundary term from integration by parts vanish). By Lemma 3.9, $\tilde{v} = v$ in Q. Now let

$$v^2(x,t) = \int_{-1}^t \int_{\mathbb{R}^n} \Gamma(x-y, t-s)\tilde{f}(y,s)\,\mathrm{d}y\mathrm{d}s.$$

As we know that v^2 satisfies

$$\int_{\mathbb{R}^n \times (-1,0)} v^2(\partial_t\psi + \Delta\psi)\,\mathrm{d}x\mathrm{d}t = -\int_{\mathbb{R}^n \times (-1,0)} \tilde{f}\psi\,\mathrm{d}x\mathrm{d}t$$

for any $\psi(x,t) \in C_0^\infty(\mathbb{R}^n \times (-2,0))$. By the uniqueness theorem proved for the volume heat potential $v^2 = v$.
Now take $\tau = \frac{1}{2}$, and let $r = \frac{1+\tau}{2} = \frac{3}{4}$. Then taking into account estimates for heat potential, we find for $(x,t) \in Q(\frac{1}{2})$,

$$\begin{aligned}
|u(x,t)| \leq & \int_{-1}^t \int_{\mathbb{R}^n} \Gamma(x-y,t-s)|\tilde{f}(y,s)|\,\mathrm{d}y\mathrm{d}s \\
\leq & c(n)\left(\int_{-1}^t \int_B \Gamma(x-y,t-s)|\tilde{f}(y,s)|^p\,\mathrm{d}y\,\mathrm{d}s\right)^{\frac{1}{p}} \\
\leq & \tilde{c}(n)\left(\int_{-1}^t \int_B \frac{1}{(t-s+|x-y|^2)^{\frac{n}{2}}}|\tilde{f}(y,s)|^p\,\mathrm{d}y\,\mathrm{d}s\right)^{\frac{1}{p}}
\end{aligned}$$

Note that $t - s + |x-y|^2 \geq C > 0$ for any $(x,t) \in Q(\frac{1}{2})$. Thus, we have

$$|u(x,t)| \leq \tilde{c}(n)\frac{1}{C^{\frac{1}{p}}}\|\tilde{f}\|_{p,Q} \leq c(n,p)\|\tilde{f}\|_{p,Q}.$$

Recall that

$$\tilde{f} = -2\mathrm{div}(u\Delta\varphi_r) + u(\partial_t\varphi_r + \Delta\varphi_r) - \mathrm{div}g\varphi_r,$$

then

$$\|\tilde{f}\|_{p,Q} \le \tilde{c}(n,p)(\|u\|_{p,Q} + \|g\|_{p,Q}).$$

Therefore,

$$\sup_{z\in Q(\frac{1}{2})} |u(z)| \le c(\|u\|_{p,Q} + \|g\|_{p,Q})$$

where $z = (x,t)$ and the constant c depends on n and p only.
Take $\tau = \frac{1}{2}$, then $r = \frac{3}{4}$, $t - s + |x-y|^2 \ge -\frac{1}{4} + \frac{9}{16} = \frac{5}{16} > 0$ for $(x,t) \in Q(\frac{1}{2})$ and $(y,s) \in Q(\frac{7}{8}) \setminus Q(\frac{3}{4})$.

$$\begin{aligned}
u(x,t) &= \int_{-1}^{t}\int_{\mathbb{R}^n} \Gamma(x-y,t-s)\tilde{f}(y,s)\,\mathrm{d}x\mathrm{d}t \\
&= \int_{-1}^{t}\int_{\mathbb{R}^n} \Gamma(x-y,t-s)(f - \mathrm{div}g\varphi_r)\,\mathrm{d}x\mathrm{d}t \\
&= I + II
\end{aligned}$$

where

$$I = \int_{-1}^{t}\int_{\mathbb{R}^n} \Gamma(x-y,t-s)f\,\mathrm{d}x\mathrm{d}t$$

and

$$II = -\int_{-1}^{t}\int_{\mathbb{R}^n} \Gamma(x-y,t-s)\mathrm{div}g\varphi_r\,\mathrm{d}x\mathrm{d}t.$$

For I, we can follow the lecture notes to do the estimates, but φ_r in II is a problematic term since it has no derivative here and the integrand explodes around $s = t$. Integrate by parts for each $0 < s < t$ in space as φ_r is compactly supported,

$$\int_{\mathbb{R}^n} \Gamma(x-y,t-s)\mathrm{div}g\varphi_r\,\mathrm{d}x = -\int_{\mathbb{R}^n} \nabla\Gamma(x-y,t-s)\varphi_r \cdot g + \Gamma(x-y,t-s)\nabla\varphi_r \cdot g.$$

Because $\mathrm{supp}(\nabla\varphi_r) \subset Q(\frac{7}{8}) \setminus Q(\frac{3}{4})$,

$$\int_{-1}^{t}\int_{\mathbb{R}^n} \nabla\Gamma\varphi_r g\,\mathrm{d}x\,\mathrm{d}t \le \|g\|_{p,Q}\|\nabla\Gamma\varphi_r\|_{p',Q}.$$

By a change of variable, $x - y = y, t - s = s$,

$$\begin{aligned}
\|\nabla\Gamma\varphi_r\|_{p',Q}^{p'} &\lesssim \int_{t+1}^{0}\int_{B} \Gamma(y,s)^{p'}\frac{|y|^{p'}}{s^{p'}}\,\mathrm{d}y\,\mathrm{d}s \\
&\approx \int_{t+1}^{0} \frac{1}{s^{p'(1+\frac{n}{2})}}\,\mathrm{d}s \int_{B} e^{-\frac{|y|^2}{s}}|y|^{p'}\,\mathrm{d}y \\
&\approx \int_{t+1}^{0} \frac{1}{s^{p'(1+\frac{n}{2})}}\,\mathrm{d}s \int s^{\frac{n}{2}+\frac{p'}{2}} e^{-z^2}|z|^{p'}\,\mathrm{d}y \text{ (take } y = \sqrt{s}z) \\
&\approx \int_{t+1}^{0} s^{\frac{n}{2}+\frac{p'}{2}-p'(1+\frac{n}{2})}\,\mathrm{d}s.
\end{aligned}$$

The last part is integrable if and only if

$$\frac{n}{2} + \frac{p'}{2} - p'(1 + \frac{n}{2}) > -1 \Leftrightarrow p > n + 2.$$

$\square$

Exercise 3. *Assume that a function* $u \colon Q_- := \mathbb{R}^n \times]-\infty, 0[$ *has the following properties:*

(i) $u \in L_1(Q(R))$ *for any* $R > 0$.

(ii) $M(u) := \sup_{z_0=(x_0,t_0)\in Q_- R>0} \left(\frac{1}{|Q(R)|} \int_{Q(z_0,R)} |u - (u)_{z_0,R}|^2 \, dz \right)^{\frac{1}{2}} < \infty$, *where*

$$(f)_{z_0,R} := \frac{1}{|Q(R)|} \int_{Q(z_0,R)} f \, dz.$$

(iii) u *is a distributional solution to the heat equation in* Q_-.

Show that u *is constant in* Q_-.

Proof. Recall that $Q(z_0, R) = B(x_0, R) \times] - R^2, 0[$. Consider

$$u_R(x, t) = u(x_0 + Rx, t_0 + R^2 t) \text{ defined on } Q = B(0, 1) \times] - 1, 0[.$$

Since u is a distributional solution to the heat equation in Q_-,

$$v_R = u_R - (u_R)_{0,1}$$

is a distributional solution in Q. By using a similar argument for the proof of Q.2 (theorem 4.1 from lecture notes), we can show that

$$\sup_{(x,t)\in Q(\tau)} |\nabla v_R(x, t)| \leq C \left(\int_Q |v_R|^2 \mathrm{d}x\mathrm{d}t \right)^{\frac{1}{2}}$$

for $0 < \tau < 1$. Note that

$$\begin{aligned}
C \left(\int_Q |v_R|^2 \mathrm{d}x\mathrm{d}t \right)^{\frac{1}{2}} =& C \left(\int_Q |u_R - (u_R)_{0,1}|^2 \mathrm{d}x\mathrm{d}t \right)^{\frac{1}{2}} \\
& (\text{take } y = x_0 + Rx, s = t_0 + R^2 t) \\
=& C \left(\int_{Q(z_0,R)} \frac{1}{R^n} |u(y, s) - (u(y, s))_{z_0,t_0}|^2 \mathrm{d}y \frac{1}{R^2} \mathrm{d}s \right)^{\frac{1}{2}} \\
\leq& \frac{C}{R} \left(\frac{1}{|Q(R)|} \int_{Q(z_0,R)} |u - (u)_{z_0,t_0}|^2 \mathrm{d}y\mathrm{d}s \right)^{\frac{1}{2}} \\
\leq& \frac{CM}{R} \\
\to& 0 \text{ since } M < \infty.
\end{aligned}$$

This shows that v_R is constant on the spatial domain, i.e. v_R is constant on each fixed time $t = \tau$. Now we show that the time derivative is also zero.
Note that

$$\begin{aligned}
0 &= \int_{Q(\tau)} v_R(\partial_t w + \Delta w)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{-\infty}^{0}\int_{\Omega} v_R(\partial_t w)\,\mathrm{d}x\,\mathrm{d}t + \int_{-\infty}^{0}\int_{\Omega} v_R \Delta w\,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_{-\infty}^{0}\int_{\Omega} (\partial_t v_R) w\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

where the second term vanishes by integration by parts. Since

$$-\int_{-\infty}^{0}\int_{\Omega} (\partial_t v_R) w\,\mathrm{d}x\,\mathrm{d}t = 0$$

for arbitrary $w \in C_0^\infty(Q(\tau))$, $\partial_t v_R = 0$ on $Q(\tau)$. Together with the fact that $|\nabla v_R| = 0$ on $Q(\tau)$, v_R is constant on $Q(\tau)$. Thus $u - (u)_{z_0,t_0}$ is constant in Q_-. That is, u is constant in Q_-. $\square$

Exercise 4. *Let Ω be a bounded domain, $T > 0$, $Q_T := \Omega \times]0, T[$. Assume that $u \in C(\bar{Q}_T) \cap C^2(\Omega \times]0, T[)$ satisfies*

$$\partial_t u - Lu \leq 0,$$

where

$$Lu := a : \nabla^2 u + b \cdot \nabla u = a_{ij} u_{,ij} + b_j u_{,j}$$

with continuous symmetric tensor valued function $a = (a_{ij})$, satisfying

$$\nu \mathbb{I} \leq a(z) \leq \nu^{-1} \mathbb{I}$$

for all $z \in \overline{Q}_T$ and for some $\nu > 0$, and continuous vector valued function $b = (b_i)$, and

$$u \leq 0$$

on $\partial' Q_T$. Then

$$u \leq 0$$

in Q_T.

Proof. Let us first suppose that we have the strict inequality

$$\partial_t u - Lu < 0 \text{ in } Q_T,$$

but there exists a point $(x_0, t_0) \in Q_T$ with $u(x_0, t_0) = \max_{\bar{Q}_T} u > 0$.
If $0 < t_0 < T$, then (x_0, t_0) belongs to the interior of Q_T and consequently, $\partial_t u = 0$ at (x_0, t_0) since u attains its maximum at this point. On the other hand, we can show that $Lu \leq 0$ at (x_0, t_0). First note that $u_{,j}(x_0) = 0$ for all $1 \leq j \leq n$. The matrix

$A = (a_{ij}(x_0))$ is symmetric and positive definite, there exists an orthogonal matrix $O = (o_{ij})$ so that $OAO^T = \text{diag}(d_1, \dots, d_n)$, $QQ^T = I$ with $d_k > 0$ for $k = 1, \dots, n$. Write $y = x_0 + O(x - x_0)$, then $x - x_0 = O^T(y - x_0)$ and so

$$u_{x_i} = \sum_{k=1}^{n} u_{y_k} o_{ki}.$$

$$u_{x_i x_j} = \sum_{k,l=1}^{n} u_{y_k} u_{y_l} o_{ki} o_{lj}$$

for $i, j = 1, 2 \dots, n$. Hence at the point x_0,

$$\begin{aligned} \sum_{i,j=1}^{n} a_{ij} u_{x_i,x_j} &= \sum_{k,l=1}^{n} \sum_{i,j=1}^{n} a_{ij} u_{y_k} u_{y_l} o_{ki} o_{lj} \\ &= \sum_{k=1}^{n} d_k u_{y_k y_k} \\ &\leq 0. \end{aligned}$$

Thus, $\partial_t u - Lu \geq 0$ at (x_0, t_0), yielding a contradiction.
Now suppose that $t_0 = T$. Then since u attains its maximum over $\overline{Q}_T$ at (x_0, t_0), $\partial_t u \geq 0$ at (x_0, t_0). We still have the inequality $\partial_t u - Lu \geq 0$ at (x_0, t_0), we deduce the contradiction again.
Now consider $u^\varepsilon(x, t) = u(x, t) - \varepsilon t$, $\varepsilon > 0$. Note that

$$\partial_t u^\varepsilon - Lu^\varepsilon = \partial_t u - Lu - \varepsilon < 0 \text{ in } Q_T.$$

Then by the above argument, we know that $\max_{\bar{Q}_T} u^\varepsilon = \max_{\partial' Q_T} u^\varepsilon$. Let $\varepsilon \to 0$ to find

$$\max_{\bar{Q}_T} u = \max_{\partial' Q_T} u \leq 0.$$

This proves that $u \leq 0$ in Q_T.
Assume for contradiction that there exist $(x_0, t_0) \in Q_T$ such that $u(x_0, t_0) > 0$, we have that $M := \sup_{Q_T} u > 0$. Set $v = e^{-t} u$, we have

$$v_{|\partial' Q_T} \leq 0 \text{ and } \partial_t v - Lv = -e^{-t} u + e^{-t} \partial_t u - e^{-t} Lu \leq -v \text{ in } Q_T$$

and $M_1 := \sup_{Q_T} v > 0$. There exists $(x_0, t_0) \in Q_T \cup (B(R) \times \{t = T\})$ such that $M_1 = v(x_0, t_0)$. By necessary conditions of maxima, we have:

$$\partial_t v(x_0, t_0) \geq 0,$$

$$Lv(x_0, t_0) \leq 0$$

by a similar argument as the orthogonal transformation one introduced in the original proof. Hence

$$0 \leq \partial_t v(x_0, t_0) - Lv(x_0, t_0) \leq -v(x_0, t_0) < 0,$$

yielding a contradiction. $\square$