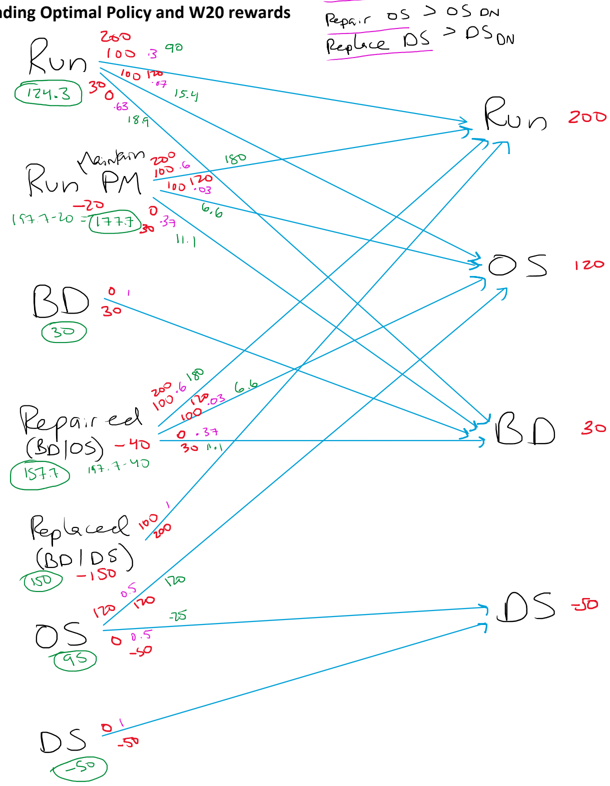
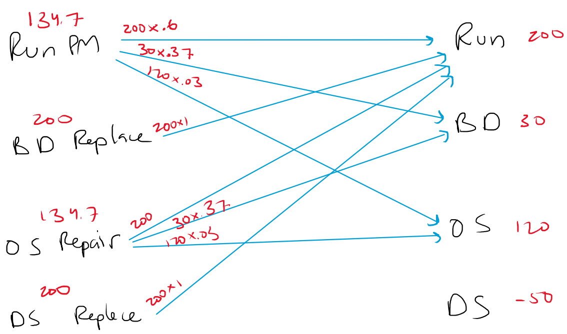
HW 2

Sunday, May 11, 2025 2:21 PM

Policy: Run Maintain Run Do Nothing
Repair BD > Replace BD > BD DN
Repair OS > OS DN
Replace DS > DS DN

P1 Finding Optimal Policy and W20 rewards





HW2 P3a

Optimal future cost

$$x_k^T P_k x_k = V_k(x_k) \to V_{k+1}(x_{k+1})$$

$$V_k(x_k) = \min_{u_k}\left[x_k^T Q x_k + u_k^T R u_k + 2x_k^T S u_k + V_{k+1}(x_{k+1})\right]$$

$$x_{k+1} = Ax_k + Bu_k$$

$$V_k(x_k) = \min_{u_k}\left[x_k^T Q x_k + u_k^T R u_k + 2x_k^T S u_k + \underline{(Ax_k+Bu_k)^T P_{k+1}(Ax_k+Bu_k)}\right]$$

$$\underbrace{(Ax_k+Bu_k)^T P_{k+1}(Ax_k+Bu_k)} = x_k^T A^T P_{k+1} A x_k + u_k^T B^T P_{k+1} B u_k + 2x_k^T A^T P_{k+1} B u_k$$

$$V_k(x_k) = \min_{u_k}\left[x_k^T (Q + A^T P_{k+1} A) x_k + u_k^T (R + B^T P_{k+1} B) u_k + 2x_k^T (S + A^T P_{k+1} B) u_k\right]$$

$$\frac{d}{du} \quad 2(R + B^T P_{k+1} B)u + 2(S + B^T P_{k+1} A)x_k$$

$$(R + B^T P_{k+1} B)u = -(S + B^T P_{k+1} A)x_k$$

$$\boxed{u_k^* = -(R + B^T P_{k+1} B)^{-1}(S + B^T P_{k+1} A)x_k}$$

$$\boxed{u_k^* = -\underbrace{\left(R + B^T P_{k+1} B\right)^{-1}\left(S + B^T P_{k+1} A\right)}_{K} x_k}$$

$$u^*_k = -K_k x_k$$

Repeated previous steps using M, H, N to help me organize

$$V_k(x_k) = \min_{u_k}\left[x_k^T\left(Q + A^T P_{k+1} A\right)x_k + u_k^T\left(R + B^T P_{k+1} B\right)u_k + 2x_k^T\left(S + A^T P_{k+1} B\right)u_k\right]$$

$$M = Q + A^T P_{k+1} A, \quad H = R + B^T P_{k+1} B, \quad N = S + A^T P_{k+1} B$$

$$V_k(x_k) = x_k^T M x_k + u_k^T H u_k + 2x_k^T N u_k$$

$$\frac{\partial V_k}{\partial u_k} = 0 + 2Hu_k + 2N^T x_k = 0$$

$$Hu_k = -N^T x_k$$

$$u_k = -N^T x_k H^{-1}$$

$$\boxed{u_k^* = -\left(S + A^T P_{k+1} B\right)\left(R + B^T P_{k+1} B\right)^{-1} x_k}$$

$$V_k(x_k) = \cancel{x_k^T} P_k \cancel{x_k} = \cancel{x_k^T} M \cancel{x_k} + u_k^T H u_k + 2x_k^T N u_k$$

$$\text{"} = \text{"} \quad + \left(-N^T x_k H^{-1}\right)^T H\left(-N^T x_k H^{-1}\right) + 2x_k^T N\left(-N^T x_k H^{-1}\right)$$

$$H^{-1} x_k^T N \cancel{H} N^T x_k \cancel{H^{-1}} - 2x_k^T N N^T x_k H^{-1}$$

$$-\cancel{x_k^T} N H^{-1} N^T \cancel{x_k}$$

$$P_k = M - NH^{-1}N^T$$

$$\boxed{P_k = Q + A^T P_{k+1} A - \left(S + A^T P_{k+1} B\right)\left(R + B^T P_{k+1} B\right)^{-1}\left(S^T + B^T P_{k+1} A\right)}$$

HW2 P3b

$$x_{k+1} = A_k x_k + B_k u_k$$

Terminal cost: $V_N(x_N) = x^T_N Q_N x_N$, $P_N = Q_N$

as

$V_k(x_k) = x_k^T P_k x_k$

$$V_k(x_k) = \min_{u_k} \left[x_k^T Q_k x_k + u_k^T R_k u_k + x_{k+1}^T Q_{k+1} x_{k+1} \right]$$

↓

$$x_{k+1}^T P_{k+1} x_{k+1}$$

"

$$(A_k x_k + B_k u_k)^T P_{k+1} (A_k x_k + B_k u_k)$$

"

$$\min_{u_k} \left[x_k^T Q_k x_k + u_k^T R_k u_k + x_k^T A_k^T P_{k+1} A_k x_k + u_k^T B_k^T P_{k+1} B_k u_k + 2 x_k^T A_k^T P_{k+1} B_k u_k \right.$$

$$x_k^T \cancel{(Q_k + A_k^T P_{k+1} A_k)} x_k + u_k^T \underbrace{(R_k + B_k^T P_{k+1} B_k)}_{H} u_k + 2 x_k^T \underbrace{A_k^T P_{k+1} B_k}_{G} u_k$$

Derivative wrt u_k

$$\frac{\partial}{\partial u_k}$$

$$0 \quad \frac{\partial}{\partial u_k}\left(u_k^T H u_k + 2 x_k^T G u_k \right)$$

↓

$$2 H u_k + 2 G^T x_k = 0$$

$$u_k^* = - G^T x_k H^{-1}$$

$$u_k^* = - A_k^T P_{k+1} B_k \left(R_k + B_k^T P_{k+1} B_k \right)^{-1} x_k$$

$$u^*_k = - K_k x_k \qquad \boxed{K_k = A_k^T P_{k+1} B_k \left(R_k + B_k^T P_{k+1} B_k \right)^{-1}}$$

$$K_k = G^T H^{-1}$$

$$V_k(x_k) = x_k^T P_k x_k$$

$$\cancel{x_k^T} P_k \cancel{x_k} = \cancel{x_k^T} \left(Q_k + A_k^T P_{k+1} A_k \right) \cancel{x_k} + u_k^T \underbrace{\left(R_k + B_k^T P_{k+1} B_k \right)}_{H} u_k + 2 x_k^T \underbrace{A_k^T P_{k+1} B_k}_{G} u_k$$

↓

$$(-K_k x_k)^T H (-K_k x_k) + 2 x_k^T G (-K_k x_k)$$

↓

$$\cancel{x_k^T} K_k^T H K_k \cancel{x_k} - 2 \cancel{x_k^T} G K_k \cancel{x_k}$$

$$P_k = (Q_k + A_k^T P_{k+1} A_k) + K_k^T H_k K_k - 2 G K_k$$

$$(G_k^T H_k^{-1})^T H_k (G_k^T H^{-1}) - 2G (G_k^T H^{-1})$$

$$(G_k^T H^{-1}) \left[(G_k^T H_k^{-1})^T H_k - 2G \right]$$

$$\cancel{H_k^{T}} G \cancel{H_k} - 2G = -G$$

$$-G_k^T H^{-1} G$$

$$\boxed{P_k = Q_k + A_k^T P_{k+1} A_k - (B_k^T P_{k+1} A_k)(R_k + B_k^T P_{k+1} B_k)^{-1}(A_k^T P_{k+1} B_k)}$$