Homework 2

Instructions

Download this jupyer notebook (button at the top of the page or download from the Github repository). Provide your answers as Markdown text, Python code, and/or produce plots as appropriate. The notebook should run all the cells in order without errors.
Submit both the `.ipynb` and a `.pdf` to Canvas.

Make sure the `.pdf` has all the relevant outputs showing. To save as `.pdf` you can first export the notebook as `.html`, open it in a browers and then "Print to PDF".

NOTE: As we will be sharing the files for peer grading, please keep your submission anonymous.

Problem 1 (Stochastic dynamic programming)

(Adapted from Stanford AA 203)

In this problem we will explore discrete-time dynamic programming for stochastic systems; that is, systems where the result of taking a certain action is not deterministic, but instead any of a set of results may occur, according to some known probability distribution. In this case, we cannot optimize the value function directly, since even choosing a known sequence of actions will not always give in the same result. Instead, we optimize the *expected value* of the value function instead (if it's been a while since you've taken a probability class, or if you've never taken one, that Wikipedia article may be helpful).

(a) Small hand-calculation problem

Suppose we have a machine that is either running or is broken down. If it runs throughout one week, it makes a gross profit of
$100. Ifitfailsduringtheweek, grossprofitiszero. Ifitisrunningatthestartoftheweekan$
$maintenancewillcost$
20. If the machine is broken down at the start of the week, it may either be repaired at a cost of
$40, inwhichcaseitwillfailduringtheweekwithaprobabilityof0.4, oritmaybereplacedatac$
150 by a new machine; a new machine is guaranteed to run through its first week of operation. Using dynamic programming, find the optimal repair, replacement, and maintenance policy that maximizes total expected profit over four weeks, assuming a new machine at the start of the first week.

Including image of hand calculations in PDF.

(b) Larger system to solve by code

Now we consider a more complicated system and a longer time horizon.

Consider the same scenario as above, but with two additional machine states: overspeeding and destroyed. In the overspeeding state, the machine will with probability 0.5 produce $120attheendoftheweek, butwillotherwisebedestroyedandproducenorevenueforthatwee$
150, the same as if it were broken down; otherwise it will produce no revenue for that week and remain in the destroyed state. A destroyed machine may not be repaired.

Here are the state transitions possible in this new system:

- If the machine is in the "running" state at the start of the week:
  - If you do nothing (cost: $0)
    - With probability 0.3 it will produce $100 and remain in the "running" state at the end of the week.
    - With probabiity 0.63 it will produce $0 and enter the "broken down" state at the end of the week.
    - With probability 0.07 it will produce $100 and enter the "overspeeding" state at the end of the week.
  - If you maintain the machine (cost: $20):
    - With probability 0.6 it will produce $100 and remain in the "running" state at the end of the week.
    - With probabiity 0.37 it will produce $0 and enter the "broken down" state at the end of the week.
    - With probability 0.03 it will produce $100 and enter the "overspeeding" state at the end of the week.
- If the machine is in the "broken down" state at the start of the week:
  - If you do nothing (cost: $0):
    - The machine will produce $0, and will remain in the "broken down" state at the end of the week.
  - If you repair the machine (cost: $40):
    - With probability 0.6 it will produce $100 and remain in the "running" state at the end of the week.
    - With probabiity 0.37 it will produce $0 and enter the "broken down" state at the end of the week.
    - With probability 0.03 it will produce $100 and enter the "overspeeding" state at the end of the week.
  - If you replace the machine (cost: $150):
    - The new machine will produce $100 and be in the "running" state at the end of the week.

- If the machine is in the "overspeeding" state at the start of the week:
    - If you do nothing (cost: $0):
        - With probability 0.5 it will produce $120 and remain in the "overspeeding" state at the end of the week.
        - With probability 0.5 it will produce $0 and enter the "destroyed" state at the end of the week.
    - If you repair the machine (cost: $40):
        - With probability 0.6 it will produce $100 and remain in the "running" state at the end of the week.
        - With probabiity 0.37 it will produce $0 and enter the "broken down" state at the end of the week.
        - With probability 0.03 it will produce $100 and enter the "overspeeding" state at the end of the week.
- If the machine is in the "destroyed" state at the start of the week:
    - If you do nothing (cost: $0):
        - The machine will produce $0 and remain in the "destroyed" state at the end of the week.
    - If you replace the machine (cost: $150):
        - The new machine will produce $100 and be in the "running" state at the end of the week.

Suppose that by the end of the 20th week (i.e., start of the 21st week), the machine is still "running", then you can sell the machine for $200. Ifthemachineis\ ”\ overspeeding\ ”, themachinewillsellfor$120. If the machine is "broken down", the machine will sell for $30. Ifthemachineis\ ”\ destroyed\ ”, thenyoumustpayforarecyclingfeeof$50.

In the following parts, you will implement the dynamic programming algorithm to find the optimal action to take in each state in each week, as well as the optimal expected profit in each state in each week.

(b)(i) Quick hand calculation

Let's start by considering just the last week and computing the first dynamic programming step by hand. What is the value at the start of week 21? That is, what is the terminal value?

The terminal values are given for week 21. Running = 200, Overspeeding = 120, Broken Down = 30, Destroyed = 50.

(b)(ii)

Given that, what is the value at the start of week 20 and the corresponding optimal policy?

Adding a picture to the PDF file. The hand-calculated optimal policy is:

Running + Maintenance: 177.7 Breakdown + Replace: 157.7 Overspeeding + Repair: 157.7 Destroyed + Replace: 150

(b)(iii)

Now, fill in the following functions to compute the value function and optimal policy over the 20 weeks.

Print out the optimal policy and value function.

In [496…

```
import numpy as np
```

In [497…

```
# these are the states and actions of the system, and corresponding index
STATES = {"RUNNING": 0, "BROKEN_DOWN": 1, "OVERSPEEDING": 2, "DESTROYED": 3}
ACTIONS = {"NOTHING": 0, "MAINTAIN": 1, "REPAIR": 2, "REPLACE": 3}
```

In [498…

```
def construct_transition_probability_matrix(STATES, ACTIONS):
    """
    Construct the transition probability matrix for the car maintenance problem.
    The transition probability matrix is a 3D array where the first dimension
    represents the current state, the second dimension represents the next state,
    and the third dimension represents the action taken.
    """


    tpm = np.zeros((len(STATES),len(STATES),len(ACTIONS))) #4x4x4 3D array #transit

    #Runnning, Nothing
    tpm[STATES["RUNNING"]][STATES["RUNNING"]][ACTIONS["NOTHING"]] = 0.3 # running
    tpm[STATES["RUNNING"]][STATES["BROKEN_DOWN"]][ACTIONS["NOTHING"]] = 0.63 # bd
    tpm[STATES["RUNNING"]][STATES["OVERSPEEDING"]][ACTIONS["NOTHING"]] = 0.07 # os

    #Running, maintain
    tpm[STATES["RUNNING"]][STATES["RUNNING"]][ACTIONS["MAINTAIN"]] = 0.6 # running
    tpm[STATES["RUNNING"]][STATES["BROKEN_DOWN"]][ACTIONS["MAINTAIN"]] = 0.37 # bd
    tpm[STATES["RUNNING"]][STATES["OVERSPEEDING"]][ACTIONS["MAINTAIN"]] = 0.03 # os

    #Broken down, nothing
    tpm[STATES["BROKEN_DOWN"]][STATES["BROKEN_DOWN"]][ACTIONS["NOTHING"]] = 1 #brok

    #broken down, repair
    tpm[STATES["BROKEN_DOWN"]][STATES["RUNNING"]][ACTIONS["REPAIR"]] = 0.6
    tpm[STATES["BROKEN_DOWN"]][STATES["BROKEN_DOWN"]][ACTIONS["REPAIR"]] = 0.37
    tpm[STATES["BROKEN_DOWN"]][STATES["OVERSPEEDING"]][ACTIONS["REPAIR"]] = 0.03

    #Overspeeding, repair
    tpm[STATES["OVERSPEEDING"]][STATES["RUNNING"]][ACTIONS["REPAIR"]] = 0.6
    tpm[STATES["OVERSPEEDING"]][STATES["BROKEN_DOWN"]][ACTIONS["REPAIR"]] = 0.37
    tpm[STATES["OVERSPEEDING"]][STATES["OVERSPEEDING"]][ACTIONS["REPAIR"]] = 0.03

    #Replaced
    tpm[STATES["BROKEN_DOWN"]][STATES["RUNNING"]][ACTIONS["REPLACE"]] = 1 #BD

    tpm[STATES["DESTROYED"]][STATES["RUNNING"]][[ACTIONS["REPLACE"]]] = 1 #DS

    #Overspeeding, nothing
    tpm[STATES["OVERSPEEDING"]][STATES["OVERSPEEDING"]][ACTIONS["NOTHING"]] = 0.5
    tpm[STATES["OVERSPEEDING"]][STATES["DESTROYED"]][ACTIONS["NOTHING"]] = 0.5

    #Destroyed, nothing
    tpm[STATES["DESTROYED"]][STATES["DESTROYED"]][ACTIONS["NOTHING"]] = 1 #DS

    return tpm
    ########################
```

In [499…

```
def construct_reward_matrix(STATES, ACTIONS):
    """
    Construct the reward matrix for the car maintenance problem.
    The reward matrix is a 3D array where the first dimension
    represents the current state, the second dimension represents the next state,
    and the third dimension represents the action taken.
    """
    rm = np.zeros((len(STATES),len(STATES),len(ACTIONS))) #4x4x4 3D array #reward m

    #Runnning, Nothing
    rm[STATES["RUNNING"]][STATES["RUNNING"]][ACTIONS["NOTHING"]] = 100 # running
    rm[STATES["RUNNING"]][STATES["BROKEN_DOWN"]][ACTIONS["NOTHING"]] = 0 # bd
    rm[STATES["RUNNING"]][STATES["OVERSPEEDING"]][ACTIONS["NOTHING"]] = 100 # os

    #Running, maintain
    rm[STATES["RUNNING"]][STATES["RUNNING"]][ACTIONS["MAINTAIN"]] = 100 - 20 # runn
    rm[STATES["RUNNING"]][STATES["BROKEN_DOWN"]][ACTIONS["MAINTAIN"]] = 0 - 20 # ba
    rm[STATES["RUNNING"]][STATES["OVERSPEEDING"]][ACTIONS["MAINTAIN"]] = 100 - 20 #

    #Broken down, nothing
    rm[STATES["BROKEN_DOWN"]][STATES["BROKEN_DOWN"]][ACTIONS["NOTHING"]] = 0 #broke

    #broken down, repair
    rm[STATES["BROKEN_DOWN"]][STATES["RUNNING"]][ACTIONS["REPAIR"]] = 100 - 40
    rm[STATES["BROKEN_DOWN"]][STATES["BROKEN_DOWN"]][ACTIONS["REPAIR"]] = 0 - 40
    rm[STATES["BROKEN_DOWN"]][STATES["OVERSPEEDING"]][ACTIONS["REPAIR"]] = 100 - 40

    #Overspeeding, repair
    rm[STATES["OVERSPEEDING"]][STATES["RUNNING"]][ACTIONS["REPAIR"]] = 100 - 40
    rm[STATES["OVERSPEEDING"]][STATES["BROKEN_DOWN"]][ACTIONS["REPAIR"]] = 0 - 40
    rm[STATES["OVERSPEEDING"]][STATES["OVERSPEEDING"]][ACTIONS["REPAIR"]] = 100 - 4

    #Replaced
    rm[STATES["BROKEN_DOWN"]][STATES["RUNNING"]][ACTIONS["REPLACE"]] = 100 - 150 #B

    rm[STATES["DESTROYED"]][STATES["RUNNING"]][ACTIONS["REPLACE"]] = 100 - 150 #DS

    #Overspeeding, nothing
    rm[STATES["OVERSPEEDING"]][STATES["OVERSPEEDING"]][ACTIONS["NOTHING"]] = 120 #
    rm[STATES["OVERSPEEDING"]][STATES["DESTROYED"]][ACTIONS["NOTHING"]] = 0  #broke

    #Destroyed, nothing
    rm[STATES["DESTROYED"]][STATES["DESTROYED"]][ACTIONS["NOTHING"]] = 0 #DS

    return rm
    ########################
```

In [500…

```
def allowable_action_set(state):
    """
    Returns the set of actions that are allowed in the given state.
    """
    if state == "RUNNING":
        return ["NOTHING", "MAINTAIN"]
    elif state == "BROKEN_DOWN":
        return ["NOTHING", "REPAIR", "REPLACE"]
    elif state == "OVERSPEEDING":
        return ["NOTHING", "REPAIR"]
    elif state == "DESTROYED":
        return ["NOTHING", "REPLACE"]
```

In [501…

```
probability_matrix = construct_transition_probability_matrix(STATES, ACTIONS)
reward_matrix = construct_reward_matrix(STATES, ACTIONS)

n_weeks = 20

V = np.zeros((len(STATES), n_weeks+1))
V[:,-1] = np.array([200, 30, 120, -50])  # RUNNING, BROKEN_DOWN, OVERSPEEDING, DEST

policy = {}
for t in range(n_weeks + 1):  # Initialize policy for all time steps
    for curr_state_name, curr_state_num in STATES.items():
        policy[(curr_state_name, t)] = None  # Set a default value (None) for all s

##### FILL ME IN #####
# update value function for each state and time step
# Use the Bellman equation to update the value function

for t in reversed(range(n_weeks)): # iterating backwards
    for curr_state_name, curr_state_num in STATES.items(): #for every state in dict
        max_value = float('-inf') # initial max value starts at the minimum so it c
        best_action = None # init best actions

        #iterate through each action at state
        for action_name, action_num in ACTIONS.items(): # for every action in dict
            expected_value = 0 #init expected valye
            for next_state_name, next_state_num in STATES.items(): # for every stat

                prob = probability_matrix[curr_state_num][next_state_num][action_nu
                reward = reward_matrix[curr_state_num][next_state_num][action_num]
                expected_value += prob * (reward + V[next_state_num][t + 1]) # sum

            #total_value = reward + expected_value # Add reward /term cost/ at the

            if expected_value > max_value:
                max_value = expected_value
                best_action = action_name
        V[curr_state_num][t] = max_value #max value at timestep t
        policy[(curr_state_name, t)] = best_action #best action at tiemstep t


##################


##### UNCOMMENT THE FOLLOWING CODE #####

print(STATES)
for t in range(n_weeks):
    print("Week %i,:"%(t+1), [policy[(state, t)] for state in STATES.keys()])

for t in range(n_weeks+1):
    print("Week %i,:"%(t+1), [np.round(V[(STATES[state], t)], 2).item() for state i
{'RUNNING': 0, 'BROKEN_DOWN': 1, 'OVERSPEEDING': 2, 'DESTROYED': 3}
Week 1,: ['MAINTAIN', 'REPAIR', 'REPAIR', 'REPLACE']
Week 2,: ['MAINTAIN', 'REPAIR', 'REPAIR', 'REPLACE']
Week 3,: ['MAINTAIN', 'REPAIR', 'REPAIR', 'REPLACE']
Week 4,: ['MAINTAIN', 'REPAIR', 'REPAIR', 'REPLACE']
Week 5,: ['MAINTAIN', 'REPAIR', 'REPAIR', 'REPLACE']
Week 6,: ['MAINTAIN', 'REPAIR', 'REPAIR', 'REPLACE']
Week 7,: ['MAINTAIN', 'REPAIR', 'REPAIR', 'REPLACE']
Week 8,: ['MAINTAIN', 'REPAIR', 'REPAIR', 'REPLACE']
Week 9,: ['MAINTAIN', 'REPAIR', 'REPAIR', 'REPLACE']
Week 10,: ['MAINTAIN', 'REPAIR', 'REPAIR', 'REPLACE']
Week 11,: ['MAINTAIN', 'REPAIR', 'REPAIR', 'REPLACE']
Week 12,: ['MAINTAIN', 'REPAIR', 'REPAIR', 'REPLACE']
Week 13,: ['MAINTAIN', 'REPAIR', 'REPAIR', 'REPLACE']
Week 14,: ['MAINTAIN', 'REPAIR', 'REPAIR', 'REPLACE']
Week 15,: ['MAINTAIN', 'REPAIR', 'REPAIR', 'REPLACE']
Week 16,: ['MAINTAIN', 'REPAIR', 'REPAIR', 'REPLACE']
Week 17,: ['MAINTAIN', 'REPAIR', 'REPAIR', 'REPLACE']
Week 18,: ['MAINTAIN', 'REPAIR', 'NOTHING', 'REPLACE']
Week 19,: ['MAINTAIN', 'REPAIR', 'NOTHING', 'NOTHING']
Week 20,: ['MAINTAIN', 'REPAIR', 'REPAIR', 'REPLACE']
Week 1,: [843.74, 823.74, 823.74, 758.74]
Week 2,: [808.74, 788.74, 788.74, 723.74]
Week 3,: [773.74, 753.74, 753.74, 688.74]
Week 4,: [738.74, 718.74, 718.74, 653.74]
Week 5,: [703.74, 683.74, 683.74, 618.74]
Week 6,: [668.74, 648.74, 648.74, 583.74]
Week 7,: [633.74, 613.74, 613.74, 548.74]
Week 8,: [598.74, 578.74, 578.74, 513.74]
Week 9,: [563.74, 543.74, 543.74, 478.74]
Week 10,: [528.74, 508.74, 508.74, 443.74]
Week 11,: [493.74, 473.74, 473.74, 408.74]
Week 12,: [458.74, 438.74, 438.74, 373.74]
Week 13,: [423.74, 403.74, 403.74, 338.74]
Week 14,: [388.74, 368.74, 368.74, 303.74]
Week 15,: [353.74, 333.74, 333.74, 268.74]
Week 16,: [318.74, 298.74, 298.74, 233.74]
Week 17,: [283.74, 263.74, 263.74, 198.33]
Week 18,: [248.33, 228.33, 241.92, 162.7]
Week 19,: [212.7, 192.7, 213.85, 150.0]
Week 20,: [177.7, 157.7, 157.7, 150.0]
Week 21,: [200.0, 30.0, 120.0, -50.0]
```

Problem 2 (Value iteration)

(This problem is adapted from Stanford AA203 course)

In this problem, you will implement value iteration to compute the value function for a rescue drone that needs to deliver aid to a goal state while avoiding regions with fire and uncertain wind conditions.

The world is represented as an $n \times n$ grid, i.e., the state space is

$$\mathcal{S} := \{(x_1, x_2) \in \mathbb{Z}_+^2 \mid |x_1, x_2 \in \{0, 1 \ldots, n-1\}\} \cup \{(\texttt{None}, \texttt{None})\}, .$$

In these coordinates, $(0,0)$ represents the bottom left corner of the map and $(n-1, n-1)$ represents the top right corner of the map. While $(\texttt{None}, \texttt{None})$ is a terminal state. For any non-terminal state, from any location $x = (x_1, x_2) \in \mathcal{S}$, the drone has four possible directions it can move in, i.e.,

$$\mathcal{A} := \{\texttt{up}, \texttt{down}, \texttt{left}, \texttt{right}\}.$$

The corresponding state changes for each action are:

- **up** : $(x_1, x_2) \mapsto (x_1, x_2 + 1)$
- **down** : $(x_1, x_2) \mapsto (x_1, x_2 - 1)$
- **left** : $(x_1, x_2) \mapsto (x_1 - 1, x_2)$
- **right** : $(x_1, x_2) \mapsto (x_1 + 1, x_2)$

There is a storm centered at $x_{\text{eye}} \in \mathcal{S}$. The storm's influence is strongest at its center and decays farther from the center according to the equation

$$\omega(x) = \exp\left(-\frac{\|x - x_{\text{eye}}\|_2^2}{2\sigma^2}\right)$$

Given its current state x and action a, the drone's next state is determined as follows:

- With probability $\omega(x)$, the storm will cause the drone to move in a uniformly random direction.
- With probability $1 - \omega(x)$, the drone will move in the direction specified by the action.
- If the resulting movement would cause the drone to leave $\mathcal{S}$, then it will not move at all. For example, if the drone is on the right boundary of the map, then moving right will do nothing.
- If the drone reaches the goal state, then the drone will always transition to a *terminal state* $(\texttt{None}, \texttt{None})$. Once in the terminal state, the drone remains in that state indefinitely regardless of the action taken.

The drone's objective is to reach $x_{\text{goal}} \in \mathcal{S}$. If the drone reaches the goal state, then it receives a reward of r_{goal} (successfully delivers aid), and a reward of r_{travel} otherwise (cost of traveling one unit). Additionally, there are some states where there is a fire. If the drone reaches a state where there is a fire, then it receives a reward of r_{fire} (drone suffers damage). Once the drone is in the terminal state, it receives zero reward (i.e., mission has terminated). The reward of a trajectory in this infinite horizon problem is a discounted sum of the rewards earned in each timestep, with discount factor $\gamma \in (0, 1)$.

To find the optimal policy to reach the goal state from any starting location, we perform value iteration. Recall that the value iteration repeats the Bellman update until convergence.

$$V(x) \leftarrow \max_{a \in \mathcal{A}} \left(\sum_{x' \in \mathcal{S}} p(x, a, x')(R(x') + \gamma V(x')) \right)$$

(a) Problem set up

Below are some helper functions. Some are filled in, others you will you need to fill in yourself.

In [502…

```
import numpy as np
import matplotlib.pyplot as plt
from ipywidgets import interact
import functools
```

In [503…

```
def is_terminal_state(state):
    """
    Check if the state is a terminal state.
    Args:
        state: Current state (row, column).
    Returns:
        True if the state is terminal, False otherwise.
    """
    return state == (None, None)


def _state_space(max_rows, max_columns):
    return [(i, j) for i in range(max_rows) for j in range(max_columns)] + [(None,


def _reward(state, fire_states, goal_states, fire_value, goal_value, travel_value):
    """
    Reward function for the grid world.
    Args:
        state: Current state (row, column).
        fire_states: List or set of fire states.
        goal_states: List or set of goal states.
        fire_value: Reward value for fire states.
        goal_value: Reward value for goal states.
    Returns:
        Reward value for the current state.
    """
    #### FILL CODE HERE ####
    if state in fire_states:
        return fire_value
    elif state in goal_states:
        return goal_value
    else:
        return travel_value
    ########################

def _transition_function(s, a, w=0,
                        max_rows=20, # number of rows
                        max_columns=20, # number of columns
                        goal_states=set([]),
                        action_set=["down", "right", "up", "left"]):
    """
    Transition function for the grid world.
    Args:
        s: Current state (row, column).
        a: Action to take.
        w: Probability of taking the action.
        max_rows: Number of rows in the grid.
        max_columns: Number of columns in the
            grid.
        action_set: List of possible actions.
    Returns:
        New state after taking the action.
    """
    i,j = s
    if is_terminal_state(s) or (s in goal_states):
        #print(f"Transitioning from state {s} to terminal (None, None)")
        return (None, None)
    if (np.random.rand(1) < w)[0]: #random movement due to storm
        a = np.random.choice(action_set)
    if a == "up":
        return (min(i+1, max_rows-1), j)
    if a == "right":
        return (i, min(j+1, max_columns-1))
    if a == "down":
        return (max(i-1, 0), j)
    if a == "left":
        return (i, max(j-1, 0))


def _compute_omega_probability(state, storm_eye, storm_sigma):
    """
    Computes the probability of a state being affected by a storm.
    Args:
        state: Current state (row, column).
        storm_eye: Center of the storm (row, column).
        storm_sigma: Standard deviation of the storm.
    Returns:
        Probability of the state being affected by the storm.
    """
    if is_terminal_state(state):
        return 0
    return np.exp(-((state[0] - storm_eye[0])**2 + (state[1] - storm_eye[1])**2) /
```

(b) Problem set up (continued)

Below are the problem parameters:

- grid size 20×20
- $x_{\text{eye}} = (10, 6)$, $\sigma = 10$
- $\mathcal{S}_{\text{goal}} = \{(19, 9)\}$
- $\mathcal{S}_{\text{fire}} = \{(10, 10), (11, 10), (10, 11), (11, 11), (13, 4), (13, 5), (14, 4), (14, 5)\}$
- $\gamma = 0.95$

- $r_{\text{fire}} = -200$
- $r_{\text{goal}} = 100$
- $r_{\text{travel}} = -1$

Also, there are some helper functions. Some are filled in, others you will you need to fill in yourself.

In [504…

```
# problem set up
max_rows, max_columns = 20, 20
fire_states = set([(10,10), (11,10), (10,11), (11,11), (13, 4), (13, 5), (14, 4), (
storm_eye = (10, 6)
storm_sigma = 10
goal_states = set([(19,9)])
gamma = 0.95
fire_value = -200
goal_value = 100
travel_value = -1
action_set=["down", "right", "up", "left"]


# fix the problem parameters in the functions to avoid passing them every time
state_space = functools.partial(_state_space, max_rows=max_rows, max_columns=max_co
reward = functools.partial(_reward, fire_states=fire_states, goal_states=goal_state
transition_function = functools.partial(_transition_function, max_rows=max_rows, ma
compute_omega_probability = functools.partial(_compute_omega_probability, storm_eye
```

In [505…

```
def probability_function(state, action, next_state, w,
                         action_set=["down", "right", "up", "left"]):
    """
    Computes the probability of transitioning to a next state given the current sta
    Args:
        state: Current state (row, column).
        action: Action to take.
        next_state: Next state (row, column).
        w: Probability of taking random action.
        action_set: List of possible actions.
    Returns:
        Probability of transitioning to the next state.

    """




    #### FILL CODE HERE ####
    # HINT: Our solution takes ~3 lines of code

    num_actions = len(action_set)

    # Intended deterministic move
    intended_next = transition_function(state, action, w=0)
    prob = 0

    # Add probability of intended move (no storm)
    if intended_next == next_state:
        prob += (1 - w)

    # Add probabilities for all possible random actions (uniformly likely if storm
    for random_action in action_set:
        random_next = transition_function(state, random_action, w=0)
        if random_next == next_state:
            prob += w / num_actions
    #print(prob)
    return prob
    #################

def get_possible_next_states(state, action_set):
    """
    Returns the set of possible next states given the current state.
    Args:
        state: Current state (row, column).
    Returns:
        Set of possible next states.
    """
    return set([transition_function(state, action, w=0) for action in action_set])

def bellman_update(value_tuple, gamma, action_set):
    """
    Performs a Bellman update on the value function.
    Args:
        value_tuple: Current value function. A tuple of (value, value_terminal).
        value: Array representing the value at each state in the grid
        value_terminal: Value of the terminal state.
        gamma: Discount factor.
        action_set: List of possible actions.
    Returns:
        Updated value_tuple and policy as a dictionary.
    """
    #### FILL CODE HERE ####

    policy = {} #init policy
    value, value_terminal = value_tuple #unpack value_tuple
    new_value = value.copy() # duplicate value_tuple, placeholder for next_state it

    for state in state_space(): #for each state
        max_value = float('-inf')  # Start at a minimum value
        best_action = None #init best action
        expected_vals_list = [] # init list of expected values

        if state == (None, None): continue # KEY for skipping the added (None, None

        #iterate through each possible action at state
        for action in action_set:
            expected_value = 0 # init expected value of possible action



            #iterate through probabilities of next state
            for next_state in get_possible_next_states(state, action_set):
                prob = probability_function(state, action, next_state, compute_omeg
                if not is_terminal_state(next_state):
                    expected_value += prob * (reward(next_state) + gamma * value[ne
                else:
                    expected_value += 0

            expected_vals_list.append(expected_value)


        new_value[state] = max(expected_vals_list)
        policy[state] = action_set[expected_vals_list.index(max(expected_vals_list)

    return  (new_value, value_terminal), policy
    ########################


def simulate(start_state, policy, num_steps):
    """
    Simulates the agent's trajectory in the grid world.
    Args:
        start_state: Starting state (row, column).
        policy: Policy to follow.
        num_steps: Number of steps to simulate.
    Returns:
        List of states visited during the simulation.
    """
    states = [start_state]
    for _ in range(num_steps):
        action = policy[start_state]
        w = compute_omega_probability(start_state)
        next_state = transition_function(start_state, action, w=w)
        if is_terminal_state(next_state):
            break
        start_state = next_state
        states.append(start_state)
    return states
```

(c) Value iteration

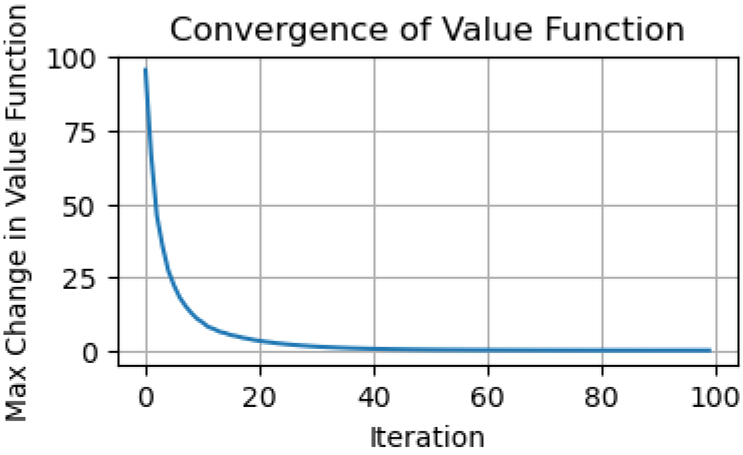
With all the building blocks all completed, you are ready to perform value iteration! Below is the value iteration loop, simulating the policy, and corresponding visualization. Run the code and visualize the results and get some intuition into how the value function changes over the iterations.

Then explore how the value and policy changes as you change different problem parameters. Share some insights/findings based on your exploration. Do these insights/findings align with your understanding?

In [506…

```
# Initialize the value function
V = (np.zeros([max_rows, max_columns]), 0)
# keep list of value functions
Vs = [V]
dV = []
num_iterations = 100 # feel free to change this value as needed
```

```
for _ in range(num_iterations):
    # perform Bellman update
    V_new, policy = bellman_update(V, gamma, action_set)
    # store the new value function
    Vs.append(V_new)
    dV.append(np.abs(V_new[0] - V[0]).max())

    # check for convergence
    if np.abs(V_new[0] - V[0]).max() < 1e-3:  #1e-3
        print("Converged!")
        break
    # update the value function
    V = V_new


start_state = (3,9) # pick a starting state
num_steps = 200 # feel free to change this value as needed
# simulate the trajectory
trajectory = simulate(start_state, policy, num_steps)

plt.figure(figsize=(4,2))
plt.plot(dV)
plt.title('Convergence of Value Function')
plt.xlabel('Iteration')
plt.ylabel('Max Change in Value Function')
plt.grid()
```



In [507…

```
def plot_policy(policy):
    for (row, col), action in policy.items():
        if row is None or col is None:
            continue
        if action == "up":
            plt.text(col + 0.5, row + 0.5, '↑', ha='center', va='center', color='bl
        elif action == "down":
            plt.text(col + 0.5, row + 0.5, '↓', ha='center', va='center', color='bl
        elif action == "left":
            plt.text(col + 0.5, row + 0.5, '←', ha='center', va='center', color='bl
        elif action == "right":
            plt.text(col + 0.5, row + 0.5, '→', ha='center', va='center', color='bl
# compute the storm strength for each state for plotting later
storm_strength = np.zeros([max_rows, max_columns])
for state in state_space():
    if not is_terminal_state(state):
        storm_strength[state] = compute_omega_probability(state)
```

In [508…

```
# visualize the value function and storm strength
@interact(iteration=(0,len(Vs)-1, 1), t=(0,len(trajectory)-1, 1))
def plot_value_function(iteration, t):
    plt.figure(figsize=(14,5))
    plt.subplot(1, 2, 1)
    plt.imshow(Vs[iteration][0], origin='lower', extent=[0, max_columns, 0, max_row
    plt.colorbar(label='Value')
    plt.title('Value Function')
    plt.xlabel('Column')
    plt.ylabel('Row')
    plt.xticks(ticks=np.arange(0.5, max_columns, 1), labels=np.arange(0, max_column
    plt.yticks(ticks=np.arange(0.5, max_rows, 1), labels=np.arange(0, max_rows))
    plt.scatter(storm_eye[1] + 0.5, storm_eye[0] + 0.5, c='cyan', s=100, label='Sto
    for fire_state in fire_states:
        plt.scatter(fire_state[1] + 0.5, fire_state[0] + 0.5, c='red', s=100)
    plt.scatter(fire_state[1] + 0.5, fire_state[0] + 0.5, c='red', s=100, label='Fi
    for goal_state in goal_states:
        plt.scatter(goal_state[1] + 0.5, goal_state[0] + 0.5, c='green', s=100, lab

    # Overlay the policy
    plot_policy(policy)
    # Plot the trajectory
    trajectory_x = [state[1] + 0.5 for state in trajectory]
    trajectory_y = [state[0] + 0.5 for state in trajectory]
    plt.plot(trajectory_x, trajectory_y, color='orange', label='Trajectory', linewi
    plt.scatter(trajectory_x[t], trajectory_y[t], color='orange', s=100, label='Cur
    plt.legend(loc="lower left", framealpha=0.6)

    plt.subplot(1, 2, 2)
    plt.imshow(storm_strength, origin='lower', extent=[0, max_columns, 0, max_rows]
    plt.colorbar(label='Storm Strength')
    plt.title('Storm Strength')
    plt.xlabel('Column')
    plt.ylabel('Row')
    plt.xticks(ticks=np.arange(0.5, max_columns, 1), labels=np.arange(0, max_column
    plt.yticks(ticks=np.arange(0.5, max_rows, 1), labels=np.arange(0, max_rows))
    plt.show()
```

```
interactive(children=(IntSlider(value=50, description='iteration'), IntSlider(value=
33, description='t', max=6…
```

Problem 3 (Linear Quadratic Regulator)

In class, we looked at the (discrete-time) non-time-varying Linear Quadratic Regular problem. Briefly, the goal is to find a sequence of control inputs $\mathbf{u} = (u_0, u_1, \dots, u_{N-1})$, for a time horizon of N time steps, that minimizes the (quadratic) cost

$$J(\mathbf{x}, \mathbf{u}) = \left[\sum_{k=0}^{N-1} \underbrace{x_k^T Q x_k + u_k^T R u_k}_{\text{Running cost}}\right] + \underbrace{x_N^T Q_N x_N}_{\text{Terminal cost}}$$

where $Q = Q^T \geq 0, k = 0, \ldots, T$, $R = R^T > 0$, and subject to linear dynamics $x_{k+1} = Ax_k + Bu_k$. Noting that there are no constraints on states and controls, aside from the fact that the system must start from the current state $x_0 = x_{\text{curr}}$ and obey the linear dynamics. If we assume the value function took the form of $V(x, k) = x^T P_k x$, then we can compute the value for any state at any time step k using the following recursion update rule

$$P_k = Q + A^T P_{k+1} A - A^T P_{k+1} B(R + B^T P_{k+1} B)^{-1} B^T P_{k+1} A, \qquad \text{for } k = N-1, \ldots, 0,$$

and the corresponding optimal gain K_k where $u_k^\star = K_k x_k$ is given by

$$K_k = -(R + B^T P_{k+1} B)^{-1} B^T P_{k+1} A, \qquad \text{for } k = 0, \ldots, N-1 \tag{1}$$

(a) LQR with cross term

Consider a slightly different case where the running cost has a cross term $2x_k^T S_k u_k$. That is,

$$\tilde{J}(\mathbf{x}, \mathbf{u}) = \left[\sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k + \underbrace{2x_k^T S u_k}_{\text{Cross term}}\right] + x_N^T Q_N x_N,$$

where $S \in \mathbb{R}^{n \times m}$, $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geq 0$ for $k = 0, \ldots, N-1$. What is the corresponding update equation for P_k and gain K_k with the cross term present? We are still considering a time-invariant case (i.e., A, B, Q, R, S are constants that do not change over time).

This was a very long hand calculation and the solution is attached to the PDF.

(b) Time-varying LQR

For standard LQR (i.e., without the cross term introduced in the previos part), what is the corresponding update equation for P_k and gain K_k if the dynamics are time-varying? That is, when $x_{k+1} = A_k x_k + B_k u_k$, where A_k and B_k are dependent on k.

Solution also attached to PDF

(c) Dynamic programming vs. convex optimization

The dynamic programming method we discussed in class is a method of analytically solving a minimization problem in closed form, minimizing the cost $J(\mathbf{x}, \mathbf{u})$ subject to the constraint $x_{k+1} = Ax_k + Bu_k$. But since the cost function and constraint set are both convex, we can also use convex optimization methods to solve the same problem numerically.

Consider an LQR problem with

$$A = \begin{bmatrix} 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0 & 0.1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.005 & 0 \\ 0 & 0.005 \\ 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$

and $R = I, Q = 3I, Q_N = 10I$, and $N = 10$. The initial state is $x_{init} = [1, 2, -0.25, 0.5]$. These data are given in the cell immediately below.

In [509…

```
# Problem data (given)

# These are the result of discretizing 2D double integrator dynamics with zero-orde
A = np.array([[1, 0, 0.1, 0],
              [0, 1, 0, 0.1],
              [0, 0, 1, 0],
              [0, 0, 0, 1]])

B = np.array([[0.005, 0],
              [0, 0.005],
              [0.1, 0],
              [0, 0.1]])

# LQR cost matrices
Q = 3 * np.eye(4)
R = np.eye(2)
Q_N = 10 * np.eye(4)


# Time horizon
N = 10

# Initial state
x_init = np.array([1, 2, -0.25, 0.5])
print(x_init)
```

```
[ 1.    2.   -0.25  0.5 ]
```

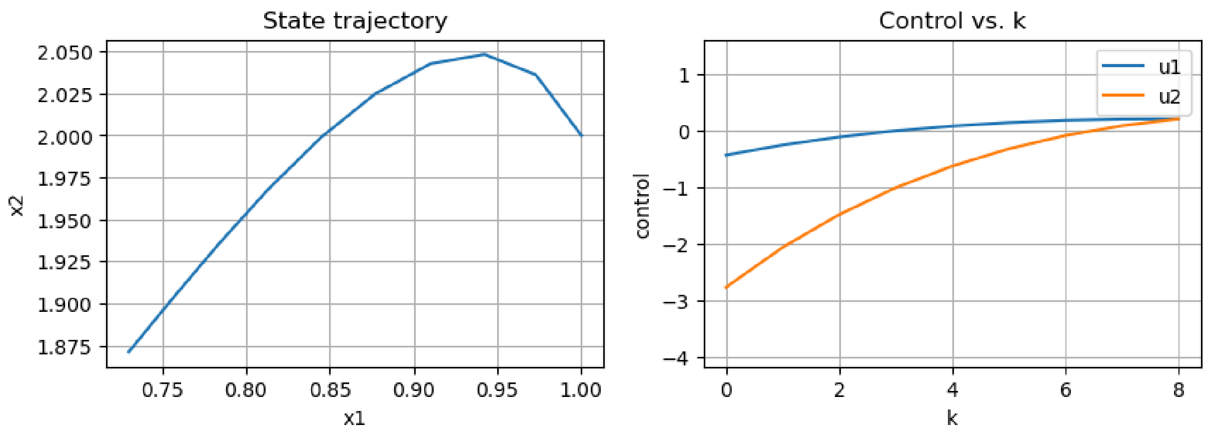
(c)(i) Solution via LQR

In the following cell, find the finite-horizon LQR controller for each time step by the iterative LQR process, and simulate the trajectory over the given time horizon with the given initial state. The provided plotting code at the end of the cell will plot the resulting state trajectory and control history.

In [510…

```
# LQR implementation goes in this cell
```

```
P_matrices = []
K_matrices = []

x_trajectory_lqr = np.zeros((N,4))
u_history_lqr = np.zeros((N-1, 2))





###### FILL CODE HERE ######

# as a result of your code, the P_matrices list defined above should contain the te
# and the K_matrices list defined above should contain the nine K matrices in incre
#
# x_trajectory_lqr should contain the complete state trajectory (with the k-th row
# the state at time k), and likewise u_history_lqr should contain the complete cont
# of u_history_lqr containing the control at time k).

P = Q_N
P_matrices.append(P)

# Backwards recursion for K and P
for k in reversed(range(N-1)):
    K = np.linalg.inv(R + B.T @ P @ B) @ (B.T @ P @ A)
    K_matrices.insert(0,K) #insert at beginning of array. First term will end up in
    P = Q + A.T @ P @ (A - B @ K)
    P_matrices.insert(0, P)

# Forward for x and u
x_trajectory_lqr[0] = x_init
for k in range(N-1):
    #print(k, x_trajectory_lqr[k])
    U = -K_matrices[k] @ x_trajectory_lqr[k] # u_k = -k_k @ x_k
    u_history_lqr[k] = U #add to back of list
    X = A @ x_trajectory_lqr[k] + B @ u_history_lqr[k]
    x_trajectory_lqr[k+1] = X
##########################################

# Provided plotting code
plt.figure(figsize=(10, 3))
plt.subplot(1,2,1)
plt.plot(x_trajectory_lqr[:,0], x_trajectory_lqr[:,1])
plt.title("State trajectory")
plt.xlabel("x1")
plt.ylabel("x2")
plt.axis("equal")
plt.grid()

plt.subplot(1,2,2)
plt.plot(range(N-1), u_history_lqr[:,0], label="u1")
plt.plot(range(N-1), u_history_lqr[:,1], label="u2")
plt.legend()
plt.title("Control vs. k")
plt.xlabel("k")
plt.ylabel("control")
plt.axis("equal")
plt.grid()
```



(c)(ii) Solution via convex optimization

In the following cell, directly find a state trajectory and control history that solves the same LQR optimization problem using cvx. The provided plotting code at the end of the cell will plot the resulting state trajectory and control history.

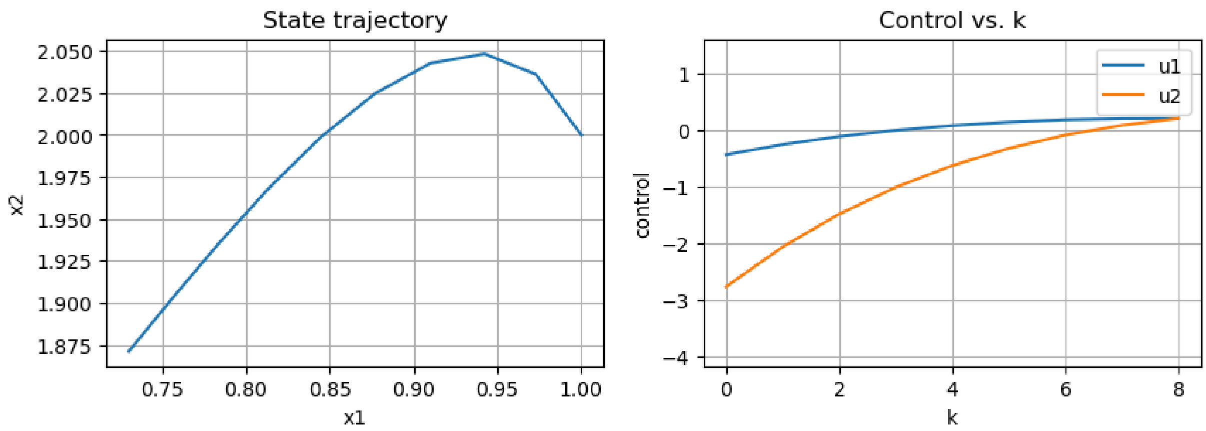
Hint: you may find the cvxpy function `quad_form` (documented here) useful. Since Q and R are both positive (semi)definite, the quadratic forms x^TQx and u^TRu are convex, but because of subtleties of the way cvxpy works, cvxpy does not immediately recognize those expressions as convex in general. `quad_form` provides additional information to cvxpy that allows it to determine the convexity of those expressions.

In [511…

```
import cvxpy as cp
```

In [512…

```
# CVX implementation goes in this cell

x_trajectory_cvx = np.zeros((N,4))
u_history_cvx = np.zeros((N-1, 2))

###### FILL CODE HERE ######

# As a result of your code, x_trajectory_cvx should contain the complete state traj
# the state at time k), and likewise u_history_cvx should contain the complete cont
# containing the control at time k).

# init x and u cp vars
x = cp.Variable((N, 4))
u = cp.Variable((N-1, 2))

cost = 0
for k in range(N-1):
    cost += cp.quad_form(x[k], Q) + cp.quad_form(u[k], R)
cost += cp.quad_form(x[N-1], Q_N)  #terminal cost

constraints = [x[0] == x_init] #first val of trajectory at x[0] is init
for k in range(N-1):
    constraints += [x[k+1] == A @ x[k] + B @ u[k]]  #dynamics Ax_k + Bu_k

prob = cp.Problem(cp.Minimize(cost), constraints)
prob.solve()

# Extract solution
x_trajectory_cvx = x.value
u_history_cvx = u.value

######################################################

# Provided plotting code
plt.figure(figsize=(10, 3))
plt.subplot(1,2,1)
plt.plot(x_trajectory_cvx[:,0], x_trajectory_cvx[:,1])
plt.title("State trajectory")
plt.xlabel("x1")
plt.ylabel("x2")
plt.axis("equal")
plt.grid()

plt.subplot(1,2,2)
plt.plot(range(N-1), u_history_cvx[:,0], label="u1")
plt.plot(range(N-1), u_history_cvx[:,1], label="u2")
plt.legend()
plt.title("Control vs. k")
plt.xlabel("k")
plt.ylabel("control")
plt.axis("equal")
plt.grid()
```



(d) Open-loop vs. closed-loop

You should have gotten the same results from LQR and cvxpy in the previous part. LQR is a method of designing a *closed-loop controller* that minimizes $J(\mathbf{x}, \mathbf{u})$. In contrast, the trajectory optimization with cvxpy is an *open-loop* method, which designs an entire trajectory in advance and provides a numerical sequence of control inputs to achieve that trajectory; notably, the sequence of control inputs is given as a fixed, independent output of the optimizer, without reference to whatever the state may be at any point along the trajectory.

When and why might we prefer one technique over the other? Discuss.

P.S. Run the cell below to check that the solutions found by both methods match.

In [513…

```
print("States match:", np.allclose(x_trajectory_cvx, x_trajectory_lqr))
print("Controls match:", np.allclose(u_history_cvx, u_history_lqr))
```

```
States match: True
Controls match: True
```

Problem 4 (Trajectory optimization - sequential quadratic programming)

In this problem, you will explore sequential quadratic programming, an algorithm where you successively convexify your nonlinear trajectory optmization problem about a previous solution, and reduce the problem into a quadratic program. Assuming the cost objective is already quadratic, the bulk of the convexification will be focused on linearizing the dynamics and constraints about a previous solution.

Below is an implementation of SQP with a dynamically extended simple car model avoiding *one* circular obstacle. You should read each line of the code; the following questions will be based on your understanding of the SQP algorithm and following implementation.

The dynamically extended simple car has the following dynamics:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v\cos\theta \\ v\sin\theta \\ \frac{v}{L}\tilde{\delta} \\ a \end{bmatrix}, \qquad u = \begin{bmatrix} \tilde{\delta} \\ a \end{bmatrix} \text{ where } \tilde{\delta} = \tan\delta$$

Note: We have made the substitution $\tilde{\delta} = \tan\delta$ so that the system is control affine since the mapping is bijective over $(-\pi/2, \pi/2)$

Let $g(x; x_{\text{ob}}, r)$ denote a function that measure how far away the state x is from a circular obstacle centered at x_{ob} with radius r.

You need to install a new package `dynamaxsys`. You can install it by running

```
pip install dynamaxsys==0.0.3
```

Alternatively, you can clone it locally by following the instructions here `https://github.com/UW-CTRL/dynamaxsys.git`

In [514…

```
import cvxpy as cp # import cvxpy

# in this problem, we will use the dynamaxsys library to import dynamical systems i
```

```
from dynamaxsys.simplecar import DynamicallyExtendedSimpleCar
from dynamaxsys.base import get_discrete_time_dynamics
from dynamaxsys.utils import linearize


import jax
import jax.numpy as jnp
import matplotlib.pyplot as plt
import numpy as np
import functools
import functools
from ipywidgets import interact
```

In [515…

```
# define the robot dynamics
wheelbase = 1.0
dt = 0.1
ct_robot_dynamics = DynamicallyExtendedSimpleCar(wheelbase=wheelbase) # robot dynam
dt_robot_dynamics = get_discrete_time_dynamics(ct_robot_dynamics, dt=dt) # discrete
state_dim = dt_robot_dynamics.state_dim
control_dim = dt_robot_dynamics.control_dim
```

In [516…

```
# some helper functions

# define obstacle function g(x) >= 0
# where g(x) is the distance from the obstacle
@jax.jit
def obstacle_constraint(state, obstacle, radius):
    return jnp.linalg.norm(state[:2] - obstacle[:2]) - radius

# function to simulate the discrete time dynamics given initial state and control s
@functools.partial(jax.jit, static_argnames=["dt_dynamics"])
def simulate_discrete_time_dynamics(dt_dynamics, state, controls, t0, dt):
    states = [state]
    t = t0
    for c in controls:
        state = dt_dynamics(state, c, t)
        states.append(state)
        t += dt
    return jnp.stack(states)

# jit the linearize constraint functions to make it run faster
linearize_obstacle = jax.jit(jax.vmap(jax.grad(obstacle_constraint), in_axes=[0, No
```

In [517…

```
# set up the problem parameters
planning_horizon = 10 # length of the planning horizon
num_time_steps = 50 # number of time steps to simulate
num_sqp_iterations = 15 # number of SQP iterations
t = 0. # this doesn't affect anything, but a value is needed

# control and velocity limits
v_max = 1.5
v_min = 0.
acceleration_max = 1.0
acceleration_min = -1.0
steering_max = 0.5
```

```
steering_min = -0.5



robot_radius = 0.1 # robot radius

# obstacle parameters
obstacle_location = jnp.array([1.0, 0.0]) # obstacle location
obstacle_radius = 0.5 # obstacle radius

# second obstacle parameters
obstacle_location_two = jnp.array([3.0, -0.5]) # obstacle location
obstacle_radius_two = .5 # obstacle radius

# third obstacle parameters
obstacle_location_three = jnp.array([5., 0.0]) # obstacle location
obstacle_radius_three = .20 # obstacle radius
```

(a) The inner quadratic program problem

Take a close look at the following two cells. Let x_t^{prev} and u_t^{prev} denote the state and control at time t from the *previous* SQP iteration. Write out the exact quadratic program that is being solved at each SQP iteration. Keep variables/parameters in terms of their names and don't use their numerical values. For example, use β_1 instead of 0.2.

Additionally, describe what each term in the problem represents, and the expression for them. That is, define mathematically what they are using mathematical expressions, their role within the optimization problem, and describe in words the interpretation of them.

Write the QP here....

At each iteration, a trajectory is computed as a denoted planning_horizon based on the trajectory from the previous iteration. This previous trajectory is used to compute the linearized dynamics and the constraints.

Cost term:

$$\text{Cost} = \beta_2 \left(\left(x_N^{(2)} \right)^2 + \left(x_N^{(1)} \right)^2 - x_N^{(0)} \right)$$

This term computes the cost, where beta is the decay variable. It is based on the state variables. The first term corresponds to the heading angle. A higher heading angle causes a higher cost, so it is discouraged. This encourages the vehicle to go straight. The second term corresponds to the y-position. Higher deviation from the the y-axis/ y = 0 increases cost, so the vehicle is encourages to be close to this line. The third term corresponds to the x-position. This term is negative, which means the cost decreases as the object moves to the right. This encourages forward progress toward the goals.

Trust Region:

$$\sum_{t=0}^{N} \|\mathbf{x}_t - \mathbf{x}_t^{\text{prev}}\|^2 + \sum_{t=0}^{N-1} \|\mathbf{u}_t - \mathbf{u}_t^{\text{prev}}\|^2$$

This term, summing up the differences between the current and previous state and control solutions, penalizes larger changes in the trajectory (state and control) over each iteration. Small steps.

Slack Term:

$$slackpenalty \cdot slack^2$$

Slack is how much leeway is given toward violation of constraints. In this case, the constraint is the obstacle. A high slack term means it is more costly to collid with the obstacle.

There is a fourth term that is iterated through for every timestep. this term defines the Stage Cost

Stage Cost:

$$\sum_{t=0}^{N-1} \beta_1 \cdot \text{markup}^t \cdot \left(\|\mathbf{u}_t\|^2 + (x_t^{(2)})^2 + (x_t^{(1)})^2 - x_t^{(0)} \right)$$

The stage cost calculates the cost at the current timestep. It penalizes deviations from the desired directions, like the above Cost Term, and a high control effort. The markup term at the end indicates the weight increases over time, which means there is a greater weight to this term over time. The algo looks more toward the future. This term both makes for smoother controls and for long term trajectory planning.

All four of these terms are summmed together to make the QP

In [518…

```
# set up cvxpy problem variables and parameters
xs = cp.Variable([planning_horizon+1, state_dim])  # cvx variable for states
us = cp.Variable([planning_horizon, control_dim])  # cvx variable for controls
slack = cp.Variable(1) # slack variable to make sure the problem is feasible
As = [cp.Parameter([state_dim, state_dim]) for _ in range(planning_horizon)]  # par
Bs = [cp.Parameter([state_dim, control_dim]) for _ in range(planning_horizon)] # pa
Cs = [cp.Parameter([state_dim]) for _ in range(planning_horizon)] # parameters for 

#Gs = [cp.Parameter([state_dim]) for _ in range(planning_horizon+1)] # parameters f
#hs = [cp.Parameter(1) for _ in range(planning_horizon+1)] # parameters for lineari
num_obstacles = 3# for changing num of obstacles
Gs = [cp.Parameter((num_obstacles, state_dim)) for _ in range(planning_horizon + 1)
hs = [cp.Parameter(num_obstacles) for _ in range(planning_horizon + 1)]



xs_previous = cp.Parameter([planning_horizon+1, state_dim]) # parameter for previou
us_previous = cp.Parameter([planning_horizon, control_dim]) # parameter for previou
initial_state = cp.Parameter([state_dim]) # parameter for current robot state
```

In [519…

```
# set up cvxpy problem cost and constraints
beta1 = .2 # coefficient for control effort
beta2 = 5. # coefficient for progress
beta3 = 10. # coefficient for trust region
slack_penalty = 1000. # coefficient for slack variable
markup = 1.05

objective = beta2 * (xs[-1,2]**2 + xs[-1,1]**2 - xs[-1,0]) + beta3 * (cp.sum_square
constraints = [xs[0] == initial_state, slack >= 0] # initial state and slack constr
for t in range(planning_horizon):
    objective += (beta1 * cp.sum_squares(us[t]) + beta1 * (xs[t,2]**2 + xs[t,1]**2
    constraints += [xs[t+1] == As[t] @ xs[t] + Bs[t] @ us[t] + Cs[t]] # dynamics co
    constraints += [xs[t,-1] <= v_max, xs[t,-1] >= v_min, us[t,1] <= acceleration_m
    constraints += [Gs[t] @ xs[t] + hs[t] >= -slack] # linearized collision avoidan
constraints += [xs[planning_horizon,-1] <= v_max, xs[planning_horizon,-1] >= v_min,
prob = cp.Problem(cp.Minimize(objective), constraints) # construct problem
```

In [520…

```
# initial states
robot_state = jnp.array([-1.5, -0.1, 0., 1.])  # robot starting state
robot_trajectory = [robot_state] # list to collect robot's state as it replans
sqp_list = [] # list to collect each sqp iteration
robot_control_list = []  # list to collect robot's constrols as it replans
robot_trajectory_list = [] # list to collect robot's planned trajectories

# initial robot planned state and controls
previous_controls = jnp.zeros([planning_horizon, control_dim]) # initial guess for
previous_states =  simulate_discrete_time_dynamics(dt_robot_dynamics, robot_state,
xs_previous.value = np.array(previous_states) # set xs_previous parameter value
us_previous.value = np.array(previous_controls) # set us_previous parameter value
```

(b) The planning loop

In the follow cell is the main loop where the planning at each time step occurs. At each time step, multiple SQP iterations are performed (either until convergence or for some fixed number of iterations). The cell is intentionally uncommented. Please add comments to each line of code, giving a brief description of the purpose/function of each line of code.

In [521…

```
##### ADD COMMENTS TO EXPLAIN THE SQP SOLVER #####

solver = cp.CLARABEL

for t in range(num_time_steps): # iterate through each timestep
    initial_state.value = np.array(robot_state) # init state
    sqp_solutions = [previous_states] # using previous solutions as part of traj ca

    for i in range(num_sqp_iterations): # number of iterations use to optimize traj
        As_value, Bs_value, Cs_value = jax.vmap(linearize, in_axes=[None, 0, 0, Non
        Gs_value_1 = linearize_obstacle(previous_states, obstacle_location, obstacl
        Gs_value_2 = linearize_obstacle(previous_states, obstacle_location_two, obs
        Gs_value_3 = linearize_obstacle(previous_states, obstacle_location_three, o
        hs_value_1 = jax.vmap(obstacle_constraint, [0, None, None])(previous_states
        hs_value_2 = jax.vmap(obstacle_constraint, [0, None, None])(previous_states
        hs_value_3 = jax.vmap(obstacle_constraint, [0, None, None])(previous_states

    # Combine obstacles
        Gs_value = jnp.concatenate([Gs_value_1, Gs_value_2, Gs_value_3], axis =  0)
        hs_value = jnp.concatenate([hs_value_1, hs_value_2, hs_value_3], axis = 0)


        for i in range(planning_horizon): #init dynamics values, setting up for opt
            As[i].value = np.array(As_value[i])
            Bs[i].value = np.array(Bs_value[i])
            Cs[i].value = np.array(Cs_value[i])
            #Gs[i].value = np.array(Gs_value[i])  # one obstacle
            #hs[i].value = np.array(hs_value[i:i+1]) # one obstacle
            Gs[i].value = np.array(Gs_value[i * num_obstacles : (i + 1) * num_obsta
            hs[i].value = np.array(hs_value[i * num_obstacles : (i + 1) * num_obsta

        # one obstacle
        #Gs[planning_horizon].value = np.array(Gs_value[planning_horizon]) # termin
        #hs[planning_horizon].value = np.array(hs_value[planning_horizon:planning_h
        Gs[planning_horizon].value = np.array(Gs_value[planning_horizon * num_obsta
        hs[planning_horizon].value = np.array(hs_value[planning_horizon * num_obsta


        result = prob.solve(solver=solver) # solve the QP

        previous_controls = us.value #current control is no previous control
        previous_states =  simulate_discrete_time_dynamics(dt_robot_dynamics, robot
        sqp_solutions.append(previous_states) # add traj to list of trajs within th
        xs_previous.value = np.array(previous_states) # update state parameters for
        us_previous.value = np.array(previous_controls) # update control parameters
    sqp_list.append(np.stack(sqp_solutions)) # add sqp to stack of sqps
    robot_control = previous_controls[0] # set the robot to the first control input
    robot_control_list.append(robot_control)# add the initial control input to the
    robot_state = dt_robot_dynamics(robot_state, robot_control, 0.) #simulate dynam
    robot_trajectory.append(robot_state) # add next robot state to robot trajectory
    robot_trajectory_list.append(previous_states) # add planned trajectory to list
    previous_states =  simulate_discrete_time_dynamics(dt_robot_dynamics, robot_sta


robot_trajectory = jnp.stack(robot_trajectory) # convert to jax
robot_controls = jnp.stack(robot_control_list)  # convert to jax
```

In [522…

```
# plotting the results. No need to add comments here. Just run this cell to visuali
@interact(i=(0,num_time_steps-1), j=(0,num_sqp_iterations-1))
def plot(i, j):
    fig, axs = plt.subplots(1, 2, figsize=(12, 4), gridspec_kw={'width_ratios': [2,
    # fig, axs = plt.subplots(1,2, figsize=(10, 4))
    ax = axs[0]
    robot_position = robot_trajectory[i, :2]
    circle0 = plt.Circle(robot_position, robot_radius, color='C0', alpha=0.4)
    circle1 = plt.Circle(obstacle_location, obstacle_radius, color='C1', alpha=0.4)
    circle2 = plt.Circle(obstacle_location_two, obstacle_radius_two, color='C1', al
    circle3 = plt.Circle(obstacle_location_three, obstacle_radius_three, color='C1'

    ax.add_patch(circle0)
    ax.add_patch(circle1)
    ax.add_patch(circle2)
    ax.add_patch(circle3)
    ax.plot(robot_trajectory[:,0], robot_trajectory[:,1], "o-", markersize=3, color
    ax.plot(robot_trajectory_list[i][:,0], robot_trajectory_list[i][:,1], "o-", mar
    # Plot planned trajectory for the selected SQP iteration
    planned_trajectory = sqp_list[i][j]
    ax.plot(planned_trajectory[:, 0], planned_trajectory[:, 1], "o-", markersize=3,
    ax.scatter(robot_trajectory[i:i+1,0], robot_trajectory[i:i+1,1], s=30,  color='
    ax.set_xlim([-2, 7])
    ax.grid()
    ax.legend()
    ax.axis("equal")

    ax.set_title("heading=%.2f velocity=%.2f"%(robot_trajectory[i,2], robot_traject

    ax = axs[1]
    plt.plot(robot_controls)
    plt.scatter([i], robot_controls[i:i+1, 0], label="$tan(\\delta)$", color='C0')
    plt.scatter([i], robot_controls[i:i+1, 1], label="Acceleration", color='C1')

    plt.hlines(steering_min, 0, num_time_steps-1, color='C0', linestyle='--')
    plt.hlines(steering_max, 0, num_time_steps-1, color='C0', linestyle='--')
    plt.hlines(acceleration_min, 0, num_time_steps-1, color='C1', linestyle='--')
    plt.hlines(acceleration_max, 0, num_time_steps-1, color='C1', linestyle='--')

    plt.plot(robot_trajectory[:,-1], markersize=3, color='C2')
    plt.scatter([i], robot_trajectory[i:i+1, 3], label="Velocity", color='C2')
    plt.hlines(v_min, 0, num_time_steps-1, color='C2', linestyle='--')
    plt.hlines(v_max, 0, num_time_steps-1, color='C2', linestyle='--')
    ax.set_xlim([0, num_time_steps])
    ax.set_ylim([-2, 2])
    ax.set_xlabel("Time step")
    ax.set_ylabel("Control")
    ax.set_title("Velocity, steering and acceleration")
    ax.legend()
    ax.grid()
```

```
interactive(children=(IntSlider(value=24, description='i', max=49), IntSlider(value=
7, description='j', max=14…
```

(c) Add another obstacle

Great! You have just planned a robot trajectory using the SQP algorithm! After parsing through all the code, perhaps you could have written that up yourself from scratch right? Now, edit the code above to add another obstacle of the same size centered at $(3, -0.5)$ and amend your answer to (a) to include the second obstacle.

(d) Try out different parameter values

Try out different parameter values, cost functions, obstacle size/locations, etc and see what kind of behaviors emerge. As you investigate, think about things like "Are there instances where the solution isn't very good? Why is that so?" or "How much does initial guess matter?" or "Are the obstable constraints always satisfied?" You can come up with your own questions to guide your exploration.

Share your findings/insights based on your exploration.

I think I implemented the second obstacle correctly. While my solution with the initial beta values cleared both obstacles, due to beta_1, which tightens controls and costs as time increases, the lcoation of the robot seemed to converge at around x=3, placing it right below the second obstacle. Increasing the number of timesteps and softening beta_1 allowed the robot to pass the second obstacle. The trade off is the planned trajectories appeared more variable at the start.

The solution has to be fine tuned... I think for the most part the solution won't be very good unless you tweak it just right. It is very easy to set beta_1 too high or the slack variable too low, and then you dont avoid the obstacles at all. It is an art.

At low planning horizons, it's neat to see the planned trajectory reach up toward y = 0 and then upon detecting the obstacle, trend back down.

HW 2

Sunday, May 11, 2025 2:21 PM

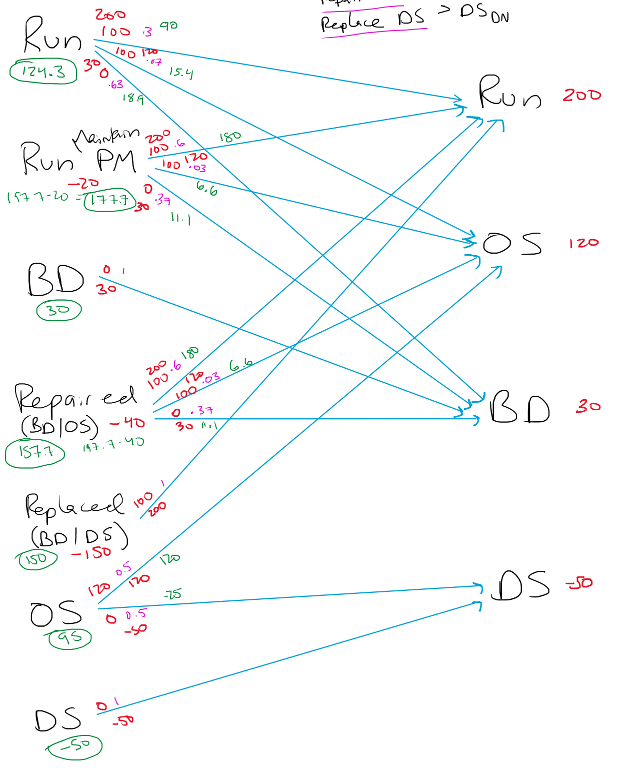
P1 Finding Optimal Policy and W20 rewards

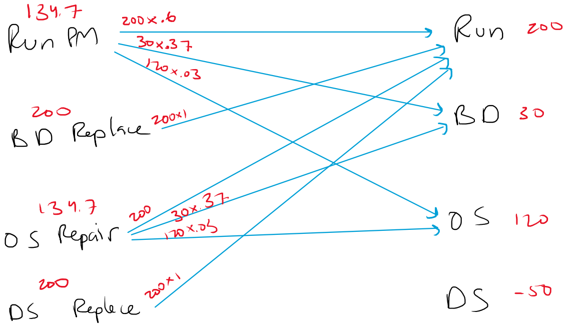
Policy: Run Maintain > Run Do Nothing

RepairBD > Replace BD > BD DN

Repair OS > OS DN

Replace DS > DS DN





HW2 P3a

Optimal future cost

$$x_k^T P_k x_k = V_k(x_k) \to V_{k+1}(x_{k+1})$$

$$V_k(x_k) = \min_{u_k}\left[x_k^T Q x_k + u_k^T R u_k + 2x_k^T S u_k + V_{k+1}(x_{k+1})\right]$$

$$x_{k+1} = Ax_k + Bu_k$$

$$V_k(x_k) = \min_{u_k}\left[x_k^T Q x_k + u_k^T R u_k + 2x_k^T S u_k + \underline{(Ax_k + Bu_k)^T P_{k+1}(Ax_k + Bu_k)}\right]$$

$$\underline{(Ax_k + Bu_k)^T P_{k+1}(Ax_k + Bu_k)} = x_k^T A^T P_{k+1} A x_k + u_k^T B^T P_{k+1} B u_k + 2x_k^T A^T P_{k+1} B u_k$$

$$V_k(x_k) = \min_{u_k}\left[x_k^T(Q + A^T P_{k+1} A)x_k + u_k^T(R + B^T P_{k+1} B)u_k + 2x_k^T(S + A^T P_{k+1} B)u_k\right]$$

$$\frac{d}{du} \qquad \cancel{2}(R + B^T P_{k+1} B)u + \cancel{2}(S + B^T P_{k+1} A)x_k$$

$$(R + B^T P_{k+1} B)u = -(S + B^T P_{k+1} A)x_k$$

$$\boxed{u_k^* = -(R + B^T P_{k+1} B)^{-1}(S + B^T P_{k+1} A)x_k}$$

$$\boxed{u^*_k = -\underbrace{(R+B^TP_{k+1}B)'(S+B'P_{k+1}A)}_{K}x_k}$$

$$u^*_k = -K_k x_k$$

Repeated previous steps using M, H, N to help me organize

$$V_k(x_k) = \min_{u_k}\left[x_k^T(Q+A^TP_{k+1}A)x_k + u_k^T(R+B^TP_{k+1}B)u_k + 2x_k^T(S+A^TP_{k+1}B)u_k\right]$$

$$M = Q+A^TP_{k+1}A, \quad H = R+B^TP_{k+1}B, \quad N = S+A^TP_{k+1}B$$

$$V_k(x_k) = x_k^TMx_k + u_k^THu_k + 2x_k^TNu_k$$

$$\frac{\partial V_k}{\partial u_k} = 0 + 2Hu_k + 2N^Tx_k = 0$$

$$Hu_k = -N^Tx_k$$

$$u_k = -N^Tx_kH^{-1}$$

$$\boxed{u^*_k = -(S+A^TP_{k+1}B)(R+B^TP_{k+1}B)^{-1}x_k}$$

$$V_k(x_k) = \cancel{x_k^T}P_k\cancel{x_k} = \cancel{x_k^T}M\cancel{x_k} + u_k^THu_k + 2x_k^TNu_k$$

$$\text{''} \quad = \quad \text{''} \quad + (-N^Tx_kH^{-1})^TH(-N^Tx_kH^{-1}) + 2x_k^TN(-N^Tx_kH^{-1})$$

$$H^{-1}x_k^TN\cancel{H}N^Tx_k\cancel{H^{-1}} - 2x_k^TNN^Tx_kH^{-1}$$

$$-\cancel{x_k^T}NH^{-1}N^T\cancel{x_k}$$

$$P_k = M - NH^{-1}N^T$$

$$\boxed{P_k = Q + A^TP_{k+1}A - (S+A^TP_{k+1}B)(R+B^TP_{k+1}B)^{-1}(S^T+B^TP_{k+1}A)}$$

HW2 P3b

$$x_{k+1} = A_kx_k + B_ku_k$$

Terminal cost: $V_N(x_N) = x^T_N Q_N x_N$, $P_N = Q_N$

as

$V_k(x_k) = x_k^T P_k x_k$

$$V_k(x_k) = \min_{u_k}\left[x_k^T Q_k x_k + u_k^T R_k u_k + x_{k+1}^T Q_{k+1} x_{k+1}\right]$$

$\downarrow$

$x_{k+1}^T P_{k+1} x_{k+1}$

"

$(A_k x_k + B_k u_k)^T P_{k+1} (A_k x_k + B_k u_k)$

"

$$\min_{u_k}\left[x_k^T Q_k x_k + u_k^T R_k u_k + x_k^T A_k^T P_{k+1} A_k x_k + u_k^T B_k^T P_{k+1} B_k u_k + 2 x_k^T A_k^T P_{k+1} B_k u_k \right.$$

$$x_k^T (Q_k + \cancel{A_k^T P_{k+1} A_k}) x_k + u_k^T \underbrace{(R_k + B_k^T P_{k+1} B_k)}_{H} u_k + 2 x_k^T \underbrace{A_k^T P_{k+1} B_k}_{G} u_k$$

Derivative wrt u_k

$\frac{\partial}{\partial u_k}$

$$0 \quad \frac{\partial}{\partial u_k}\left(u_k^T H u_k + 2 x_k^T G u_k\right)$$

$\downarrow$

$$2 H u_k + 2 G^T x_k = 0$$

$$u_k^* = -G^T x_k H^{-1}$$

$$u_k^* = -A_k^T P_{k+1} B_k (R_k + B_k^T P_{k+1} B_k)^{-1} x_k$$

$u^*_k = -K_k x_k$

$$\boxed{K_k = A_k^T P_{k+1} B_k (R_k + B_k^T P_{k+1} B_k)^{-1}}$$

$$K_k = G^T H^{-1}$$

$V_k(x_k) = x_k^T P_k x_k$

$$\cancel{x_k}^T P_k \cancel{x_k} = x_{\cancel{k}}^T (Q_k + A_k^T P_{k+1} A_k) x_{\cancel{k}} + u_k^T \underbrace{(R_k + B_k^T P_{k+1} B_k)}_{H} u_k + 2 x_k^T \underbrace{A_k^T P_{k+1} B_k}_{G} u_k$$

$\downarrow$

$$(-K_k x_k)^T H (-K_k x_k) + 2 x_k^T G (-K_k x_k)$$

$\downarrow$

$$\cancel{x^T_k} K_k^T H K_k \cancel{x_k} - 2 \cancel{x_k^T} G K_k \cancel{x_k}$$

$\downarrow$

$$P_k = \left(Q_k + A_k^T P_{k+1} A_k\right) + K_k^T H_k K_k - 2G K_k$$

$\downarrow$

$$\left(G_k^T H_k^{-1}\right)^T H_k \left(G_k^T H^{-1}\right) - 2G\left(G_k^T H^{-1}\right)$$

$$\left(G_k^T H^{-1}\right)\left[\left(G_k^T H_k^{-1}\right)^T H_k - 2G\right]$$

$\downarrow$

$$\cancel{H_k^{-1}} G \cancel{H_k} - 2G = -G$$

$\downarrow$

$$-G_k^T H^{-1} G$$

$$\boxed{P_k = Q_k + A_k^T P_{k+1} A_k - \left(B_k^T P_{k+1} A_k\right)\left(R_k + B_k^T P_{k+1} B_k\right)^{-1}\left(A_k^T P_{k+1} B_k\right)}$$