Evaluate $\oint_C (x\,dy + y\,dx)$, where C is the boundary of the rectangle $0 \leq x \leq 2, 0 \leq y \leq 1$.

Solution:

1. **Given:** $P = y, Q = x$.

$$\frac{\partial Q}{\partial x} = 1, \quad \frac{\partial P}{\partial y} = 1.$$

   So, $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - 1 = 0$.

2. **Apply Green's Theorem:**

$$\oint_C (x\,dy + y\,dx) = \iint_R 0\,dA = 0.$$

   Since the curl is zero, the line integral is also zero.

Evaluate $\oint_C (y^2\,dx + x^2\,dy)$, where C is the boundary of the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$.

Solution:

1. **Given**: $P = y^2, Q = x^2$.

$$\frac{\partial Q}{\partial x} = 2x, \quad \frac{\partial P}{\partial y} = 2y.$$

   So, $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - 2y$.

2. **Region R**: The region is the triangular area $0 \leq x \leq 1, 0 \leq y \leq 1 - x$.

3. **Apply Green's Theorem**:

$$\oint_C (y^2\,dx + x^2\,dy) = \iint_R (2x - 2y)\,dA.$$

   Using double integration:

$$\int_0^1 \int_0^{1-x} (2x - 2y)\,dy\,dx.$$

First, integrate with respect to y:

$$\int_0^{1-x} (2x - 2y)\, dy = \left[2xy - y^2\right]_0^{1-x} = 2x(1-x) - (1-x)^2.$$

Simplify:

$$2x - 2x^2 - 1 + 2x - x^2 = 4x - 3x^2 - 1.$$

Now integrate with respect to x:

$$\int_0^1 (4x - 3x^2 - 1)\, dx = \left[2x^2 - x^3 - x\right]_0^1 = 2(1) - 1 - 1 = 0.$$

Evaluate $\oint_C(-y\,dx + x\,dy)$, where C is the circle $x^2 + y^2 = 1$.

Solution:

1. **Given**: $P = -y$, $Q = x$.

$$\frac{\partial Q}{\partial x} = 1, \quad \frac{\partial P}{\partial y} = -1.$$

   So, $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - (-1) = 2$.

2. **Region R**: The region is the disk $x^2 + y^2 \leq 1$.

3. **Apply Green's Theorem**:

$$\oint_C (-y\,dx + x\,dy) = \iint_R 2\,dA.$$

   In polar coordinates, $dA = r\,dr\,d\theta$ and R is defined as $0 \leq r \leq 1, 0 \leq \theta \leq 2\pi$:

$$\int_0^{2\pi}\int_0^1 2r\,dr\,d\theta = \int_0^{2\pi}\left[r^2\right]_0^1 d\theta = \int_0^{2\pi} 2\,d\theta = 4\pi.$$

Evaluate $\oint_C (y\,dx - x\,dy)$, where C is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution:

1. **Given:** $P = y, Q = -x$.

$$\frac{\partial Q}{\partial x} = -1, \quad \frac{\partial P}{\partial y} = 1.$$

   So, $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -1 - 1 = -2$.

2. **Region R:** The region is the interior of the ellipse. Its area is $A = \pi ab$.

3. **Apply Green's Theorem:**

$$\oint_C (y\,dx - x\,dy) = \iint_R -2\,dA = -2 \cdot (\text{Area of } R) = -2 \cdot \pi ab = -2\pi ab.$$