# Part : B

## Chapter : 15

**Differential equation:** A differential equation is an equation involving at least one differential with or without the variables from which these differentials are derived.

Examples:

$$\frac{dy}{dx} = e^x$$

$$\left(\frac{dy}{dx}\right)^2 = ax^2 + bx + c$$

$$\frac{d^2y}{dx^2} = 0$$

**Ordinary differential equation:** An ~~differential~~ ordinary differential equation is one in which all the differentials (or derivatives) involved have reference to a single independent variable.

**Partial differential equation:** A partial differential equation is one which contains partial differentials (or derivatives) and such involves two or more independent variables.

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**Order:** The order of a differential equation is the order of the highest ~~derivatives~~ derivative in the equation.

$\frac{dy}{dx} = e^x$, $\left(\frac{dy}{dx}\right)^2 = ax^2 + bx + c$ are the first order.

$\frac{d^2y}{dx^2} = 0$, $\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^2 + 2y = 0$ are the second order.

$\left(\frac{d^3y}{dx^3}\right)^2 = x^2 \frac{dy}{dx}$ are the third order.

**Degree:** The degree of an algebraic differential equation is the degree of the derivatives of the highest order in the equation, after the equation is freed from radicals and fractions in the derivatives.

$\left(\frac{dy}{dx}\right)^2 = ax^2 + bx + c$, $\left(\frac{d^3y}{dx^3}\right)^2 = x^2 \frac{dy}{dx}$ are the second degree.

Example-1:

(i) $\frac{d^2y}{dx^2} + 4\left(\frac{dy}{dx}\right)^5 + 10y = 0$

Here, the order of the differential equation is 2 and degree of the equation is 1.

(ii) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{2/3} = \frac{d^2y}{dx^2}$

$\Rightarrow \left[1+\left(\frac{dy}{dx}\right)^2\right]^2 = \left(\frac{d^2y}{dx^2}\right)^3$

Here, the order of the differential equation is 2, while the degree of the equation is 3.

(iii) $y\left(\frac{dy}{dx}\right)^2 + 2y\frac{dy}{dx} - y = 0$

Here, the order of the differential equation is 1 and the degree of the equation is 2.

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(IV) $\sqrt{1+\left(\frac{dy}{dx}\right)^2} = 1+x$

$\Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = (1+x)^2$

Here, the order of the differential equation is 1 and degree is 2.

Example-2: Find the differential equation:

(i) $y = A\sin x + B\cos x$

$\frac{dy}{dx} = A\cos x - B\sin x$

$\Rightarrow \frac{d^2y}{dx^2} = -A\sin x - B\cos x$

$\Rightarrow \frac{d^2y}{dx^2} = -(A\sin x + B\cos x)$

$\Rightarrow \frac{d^2y}{dx^2} = -y$

$\therefore \frac{d^2y}{dx^2} + y = 0$ is the required differential equation.

(II) $y = e^{-x}(A\cos x + B\sin x)$

$\Rightarrow ye^{x} = A\cos x + B\sin x$

Differentiating with respect to $x$,

$y.e^{x} + e^{x}\frac{dy}{dx} = -A\cos x + - A\sin x$

Again differentiating with respect to $x$,

$y.e^{x} + e^{x}\frac{dy}{dx} + e^{x}.\frac{dy}{dx} + e^{x}\frac{d^2y}{dx^2} = -B\sin x - A\cos x$

$\Rightarrow e^{x}y + 2e^{x}\frac{dy}{dx} + e^{x}\frac{d^2y}{dx^2} = -(A\cos x + B\sin x)$

$\Rightarrow \cancel{e^{x}}\left(y + 2\frac{dy}{dx} + \frac{d^2y}{dx^2}\right) = -y\cancel{e^{x}}$

$\therefore 2y + 2\frac{dy}{dx} + \frac{d^2y}{dx^2} = 0$ this is the required differential eq

(VI) $y = Ae^{2x} + Be^{-2x}$

$\Rightarrow \frac{dy}{dx} = 2.Ae^{x} - 2.Be^{-2x}$ [differentiating w.r.t. $x$]

$\Rightarrow \frac{d^2y}{dx^2} = 2.2Ae^{x} + 2.2Be^{-2x}$

$\Rightarrow \frac{d^2y}{dx^2} = 4(Ae^{2x} + Be^{-2x})$

$\Rightarrow \frac{d^2y}{dx^2} = 4y$

$\therefore \frac{d^2y}{dx^2} - 4y = 0$ this is the required differential equatio

Example - 3:

(I) Equation of any parabola whose axis is parallel to the x-axis is $y = Ax^2 + Bx + C$, where A, B, C are aribitary constant,

Differentiating (1) w.r.t x we get,

$$\frac{dy}{dx} \cancel{\phi} = 2Ax + B$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2A$$

$\Rightarrow \frac{d^3y}{dx^3} = 0$, this is the required differential equation of the system of parabola.

(II) Since the circles, having their centress on the x-axis, pass through the origin, if $(a, 0)$ be the centre of any member of the family of, it radius will be 'a'.

The equation of the circle is,

$$(x-a)^2 + (y-0)^2 = a^2$$

$$\Rightarrow x^2 + y^2 - 2ax = 0 \quad \text{———} \quad (1)$$

where ~~is~~ a is a parameter.

Differentiating w.r.t. $x$,

$$2x+2y\frac{dy}{dx}-2a=0$$

$$\Rightarrow 2a=2x+2y\frac{dy}{dx} \quad \text{——— (II)}$$

Eliminating 'a' between (1) and (2) we get,

$$x^2+y^2-\left(2x+2y\frac{dy}{dx}\right)x=0$$

$$\Rightarrow x^2+y^2-2x^2-2y\frac{dy}{dx}=0$$

$\therefore y^2-x^2-2y\frac{dy}{dx}=0$ is the required differential equation of the circles.

(III) Let $(\alpha,\beta)$ be the centre of any circle of the system with constant radius 'a'.

Equation of the circle is,

$$(x-\alpha)^2+(y-\beta)^2=a^2 \quad \text{——— (I)}$$

where 'a' is constant and $\alpha,\beta$ are parameters.

Differentiating both side of (1) w.r.t. $x$,

$$(x-\alpha)+(y-\beta)\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-(x-\alpha)}{(y-\beta)} \quad \text{——— (II)}$$

Differentiating (ii) w.r.t. $x$,

$$\frac{d^2y}{dx^2} = -\frac{(y-\beta)\frac{d}{dx}(x-\alpha) - (x-\alpha)\frac{dy}{dx}(y-\beta)}{(y-\beta)^2}$$

$$= -\frac{(y-\beta).1 - (x-\alpha)\frac{dy}{dx}}{(y-\beta)^2}$$

$$= -\frac{(y-\beta) - (x-\alpha)}{(y-\beta)^2} \times \frac{-(x-\alpha)}{(y-\beta)}$$ [keeping the value of $\frac{dy}{dx}$]

$$= -\frac{\{(y-\beta)^2 + (x-\alpha)^2\}}{(y-\beta)^3}$$

$$= -\frac{a^2}{(y-\beta)^3}$$ [from eqn (1)]

$$\therefore (y-\beta)^3 = -\frac{a^2}{y_2}$$

where, $y_2 = \frac{d^2y}{dx^2}$.

Again, $1+y_1^2 = 1 + \left(\frac{dy}{dx}\right)^2$

$$= \frac{a^2}{(y-\beta)^2}$$ [From eqn (1)]

or, $(y-\beta)^2 = \frac{a^2}{1+y_1^2}$

From eqn (3), (4)

$$\frac{a^4}{y_2^2} = \frac{a^6}{(1+y_1^2)^3}$$

$\Rightarrow \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2$ is the required differential equation.