15.11.2022

Part-A : Integral Calculas

Chapter : 1

18(i) $\int \frac{dx}{1+\sin x} = \int \frac{(1-\sin x)}{(1-\sin x)(1+\sin x)} dx$

$= \int \frac{1-\sin x \, dx}{1-\sin^2 x} = \int \frac{1-\sin x}{\cos^2 x} dx$

$= \int \frac{1}{\cos^2 x} dx - \int \frac{\sin x}{\cos^2 x} dx$

$= \int \sec^2 x \, dx - \int \tan x . \sec x \, dx$

$= \tan x - \sec x + c$

H.W 15(i) $\int \sqrt{1+\sin x}\, dx$

$= \int \sqrt{(\cos\frac{x}{2} + \sin\frac{x}{2})^2}\, dx$

$= \int (\cos\frac{x}{2} + \sin\frac{x}{2})\, dx$

$= \int \cos\frac{x}{2} dx + \int \sin\frac{x}{2} dx$

$= 2\sin\frac{1}{2}x - 2\cos\frac{x}{2}$

$= 2(\sin\frac{x}{2} - \cos\frac{x}{2}) + c$

Chapter: 2

Method of substitution:

15 (1) $\int \frac{e^{2x}}{e^x+1} dx$

Let, $e^x + 1 = z$; $e^x = z - 1$

$e^x . dx = dz$

$\therefore \int \frac{e^x . e^x}{e^x+1} dx$

$= \int \frac{(z-1) dz}{z}$

$= \int dz - \int \frac{dz}{z}$

$= z - \log z + c$

$= e^x + 1 - \log(e^x + 1) + c$

A. $\int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$

B. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right|$

C. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right|$

D. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log\left|\left(x+\sqrt{x^2 \pm a^2}\right)\right|$

$$\Rightarrow \int \frac{dx}{(ax+b)\sqrt{(cx+d)}}$$

Let, $cx+d = z^2 \Rightarrow x = \frac{z^2-d}{c}$

$c\,dx = 2z\,dz$

$dx = \frac{2z.dz}{c}$

$$\int \frac{\frac{2z.dz}{c}}{\left(a.\frac{z^2-d}{c}+b\right).\sqrt{z^2}}$$

$$= \frac{2}{\cancel{c}} \int \frac{z.dz}{\frac{az^2-ad+bc}{\cancel{c}}.z}$$

$$= 2\int \frac{\cancel{z}.dz}{(az^2-ad+bc).\cancel{z}}$$

$$= 2\int \frac{dz}{az^2-ad+bc}$$

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Chapter: 3 — Intregation By Parts

1. $\int (uv)\,dx = u\int v\,dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx$

2. $\int e^{ax}.\cos bx.dx = \frac{e^{ax}}{a^2+b^2}.(a\cos bx + b\sin bx)$

3. $\int e^{ax}.\sin bx.dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

4. $\int \sqrt{x^2+a^2} = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\log|(x+\sqrt{x^2+a^2})|$

$= \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\sin h^{-1}\frac{x}{a}$

5. $\int \sqrt{x^2-a^2} = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\log|(x+\sqrt{x^2-a^2})|$

$= \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\cos^{-1}\frac{x}{a}$

6. $\int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}.\sin^{-1}\frac{x}{a}$

Proof: $\int\sqrt{a^2-x^2}\,dx = I$

$x = a\sin\theta$

$dx = a\cos\theta.d\theta$

$\therefore I = \int\sqrt{a^2-x^2} = \int\sqrt{a^2-a^2\cos^2\theta}\ d\theta$

$= \int\sqrt{a^2(1-\cos^2\theta)}\ d\theta$

$= \int a.\sin\theta\, d\theta$

Proof: $\int\sqrt{a^2-x^2}\ dx = I$

Let, $x = a\sin\theta$

$dx = a\cos\theta.d\theta$

$\therefore I = \int\sqrt{a^2-x^2}dx = \int\sqrt{a^2-a^2\sin^2\theta}\ .a\cos\theta.d\theta$

$= \int\sqrt{a^2(1-\sin^2\theta)}\ .a\cos\theta.d\theta$

$= \int a.\cos\theta.a.\cos\theta.d\theta$

$= \int a^2.\cos^2\theta.d\theta$

$$= \frac{a^2}{2}\int 2\cos^2\theta \cdot d\theta$$

$$= \frac{a^2}{2}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{a^2}{2}\int d\theta + \int \cos 2\theta \cdot d\theta$$

$$= \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + c$$

$$= \frac{a^2}{2}\left[\sin^{-1}\frac{x}{a} + \frac{1}{2}\cdot \sin 2\theta\right] + c$$

$$= \frac{a^2}{2}\left[\sin^{-1}\frac{x}{a} + \frac{1}{2}\cdot 2\sin\theta \cdot \cos\theta\right] + c$$

$$= \frac{a^2}{2}\left[\sin^{-1}\frac{x}{a} + \sin\theta \cdot \sqrt{1-\sin^2\theta}\right] + c$$

$$= \frac{a^2}{2}\left[\sin^{-1}\frac{x}{a} + \frac{x}{a} \cdot \sqrt{1-\frac{x^2}{a^2}}\right] + c$$

$$= \frac{a^2}{2}\left[\sin^{-1}\frac{x}{a} + \frac{x}{a} \cdot \frac{\sqrt{a^2-x^2}}{a}\right] + c$$

$$= \frac{\cancel{a^2}}{2} \cdot \frac{x\sqrt{a^2-x^2}}{\cancel{a^2}} + \frac{a^2}{2} \cdot \sin^{-1}\frac{x}{a}$$

$$= \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

Example - 4: $\int \log(x+\sqrt{x^2+a^2})\,dx$

$$I = \int \underset{u}{\log(x+\sqrt{x^2+a^2})} \cdot \underset{v}{1}\,dx$$

$$= \log(x+\sqrt{x^2+a^2})\int dx - \int\left[\frac{d}{dx}\{\log(x+\sqrt{x^2+a^2})\}\cdot\int dx\right]dx$$

$$= \log(x+\sqrt{x^2+a^2})\cdot x - \int \frac{1}{\sqrt{x^2+a^2}}\cdot x\,dx$$

$$= \log(x+\sqrt{x^2+a^2})\cdot x - \int \frac{x}{\sqrt{x^2+a^2}}\,dx$$

put, $x^2+a^2 = z^2$

$2x\,dx = 2z\,dz$

$\therefore x\,dx = z.dz$

$$= x\log(x+\sqrt{x^2+a^2}) - \int \frac{z\,dz}{\sqrt{z^2}}$$

$$= x\log(x+\sqrt{x^2+a^2}) - \int dz$$

$$= x\log(x+\sqrt{x^2+a^2}) - z$$

$$= x\log(x+\sqrt{x^2+a^2}) - \sqrt{x^2+a^2} + c$$

Example:16 $\int \tan^{-1}\left(\frac{1+\cos x}{\sin x}\right) dx$

$= \int \tan^{-1} \frac{\cancel{2}\cos^2\frac{x}{2}}{\cancel{2}\sin\frac{x}{2}\cdot\cos\frac{x}{2}} dx$

$= \int \tan^{-1}\left(\cot\frac{x}{2}\right) dx$

$= \int \cancel{\tan^{-1}} \tan\left(\frac{\pi}{2}-\frac{x}{2}\right) dx$

$= \int\left(\frac{\pi}{2}-\frac{x}{2}\right) dx$

$= \frac{\pi}{2}\int dx - \int \frac{x}{2} dx$

$= \frac{\pi}{2}\cdot x - \frac{x^2}{4}$

$= x\frac{\pi}{2} - \frac{x^2}{4} + c$

Example-17: $\int \sin^{-1}\sqrt{\frac{x}{x+a}}\, dx$

Let $x = a\tan^2\theta \quad \therefore dx = 2a\tan\theta\cdot\sec^2\theta\cdot d\theta$

$= \int \sin^{-1}\sqrt{\frac{a\tan^2\theta}{a\tan^2\theta + a}} \times 2a\tan\theta\sec^2\theta\cdot d\theta$

$= \int \sin^{-1}(\sin\theta)\times 2a\tan\theta\cdot\sec^2\theta\cdot d\theta$

$= \int 2a\theta\cdot\tan\theta\cdot\sec^2\theta\, d\theta = a\int\theta\, d(\tan^2\theta)$

$= a(\theta\cdot\tan^2\theta - \int\tan^2\theta\, d\theta = a\theta\tan^2\theta - a\int(\sec^2\theta-1)d\theta$

$= a\theta\tan^2\theta - a\int(\sec^2\theta - 1)d\theta$

~~20.(1)~~ $= a\theta\tan^2\theta - a\tan\theta + a\theta + c$

$= \theta(a\tan^2\theta + a) - a\tan\theta + c$

$= x\tan^{-1}\sqrt{x/a} - \sqrt{ax} \cdot a \cdot \tan^{-1}\sqrt{x/a} + c$

24. (i) $\int\sqrt{5-2x+x^2}\,dx$

$=\int\sqrt{x^2-2.x.1+1^2+4}\,dx$

$=\int\sqrt{(x-1)^2+2^2}\,dx$

$=\frac{(x-1)\sqrt{(x-1)^2+2^2}}{2}+\frac{2^2}{2}\log\left(x-1+\sqrt{(x-1)^2+2^2}\right)+c$

$=\frac{(x-1)\sqrt{(x-1)^2+4}}{2}+2\log\left(x-1+\sqrt{(x-1)^2+4}\right)+c$

(ii) $\int\sqrt{10-4x+4x^2}\,dx$

$=\int\sqrt{(2x)^2-2.2.x+1+9}\,dx$

$=\int\sqrt{(2x-1)^2+3^2}\,dx$

$=\frac{1}{2}\left[\frac{(2x-1)\sqrt{(2x-1)^2+3^2}}{2}+\frac{3^2}{2}\log\left(2x-1+\sqrt{(2x-1)^2+9}\right)\right]$

$=\frac{1}{4}\left[(2x-1)\sqrt{(2x-1)^2+9}+\frac{9}{2}\log\left[2x-1+\sqrt{(2x-1)^2+9}\right]\right]$

30(i) $\int(x-1)\sqrt{x^2-1}\,dx$

Let $x^2-1=z^2$

$2x\,dx=2z\,dz$

$\therefore x.dx=z.dz$

$$=\int x\sqrt{x^2-1}\,dx-\int\sqrt{x^2-1}\cdot 1\,dx$$

$$=\int\sqrt{z^2}\cdot x\,dx-\int\sqrt{x^2-1}\cdot dx$$

$$=\int z\cdot z\,dz-\left[\frac{x\cdot\sqrt{x^2-1}}{2}-\frac{1^2}{2}\log\left(x+\sqrt{x^2-1}\right)\right]$$

$$=\frac{z^3}{3}-\frac{x\sqrt{x^2-1}}{2}+\frac{1}{2}\log\left(x+\sqrt{x^2-1}\right)$$

$$=\frac{1}{3}(x^2-1)^{3/2}-\frac{1}{2}\left(x\sqrt{x^2-1}\right)+\frac{1}{2}\log\left(x+\sqrt{x^2-1}\right)+c$$

(11) Let, $x^2+a^2=z^2$

$$\Rightarrow 2x\,dx=2z\,dz$$

$$\therefore x\,dx=z\,dz$$

$$\int(x+b)\sqrt{x^2+a^2}\,dx$$

$$=\int x\sqrt{x^2+a^2}\,dx+\int b\sqrt{x^2+a^2}\,dx$$

$$=\int z^2\,dz+b\left[\frac{x\sqrt{x^2+a^2}}{2}+\frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right)\right]$$

$$=\frac{1}{3}\cdot z^3+b\left[\frac{x\sqrt{x^2+a^2}}{2}+\frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right)\right]+c$$

$$=(x^2+a^2)^{3/2}+b\left[\frac{x\sqrt{x^2+a^2}}{2}+\frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right)\right]+c$$

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*** 4.2

$$\int \frac{dx}{a+b\cos x}$$

$$= \int \frac{dx}{a\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)+b\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}$$ [cos2A এর সূ

$$= \int \frac{dx}{\cos^2\frac{x}{2}(a+b)+\sin^2\frac{x}{2}(a-b)}$$

$$= \int \frac{\sec^2\frac{x}{2}\,dx}{(a+b)+(a-b)\tan^2\frac{x}{2}}$$

Case-1: when $a>b$

Put, $\sqrt{a-b}\cdot\tan^2\frac{x}{2}=z$

$$\Rightarrow \frac{1}{2}\sqrt{a-b}\,\sec^2\frac{x}{2}\,dx = dz$$

$$\therefore \sec^2\frac{x}{2}\,dx = \frac{2dz}{\sqrt{a-b}}$$

$$\int \frac{\sec^2\frac{x}{2}\,dx}{(a+b)+(a-b)\tan^2\frac{x}{2}}$$

$$= \frac{2}{\sqrt{a-b}}\int \frac{dz}{(a+b)+z^2}$$

$$= \frac{2}{\sqrt{a-b}} \int \frac{dz}{(\sqrt{a+b})^2 + z^2} \qquad \left[\int \frac{dz}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$= \frac{2}{\sqrt{a-b}} \cdot \frac{1}{\sqrt{a+b}} \cdot \tan^{-1}\frac{z}{\sqrt{a+b}}$$

$$= \frac{2}{\sqrt{a^2-b^2}} \cdot \tan^{-1}\frac{\sqrt{a-b}\cdot\tan\frac{x}{2}}{\sqrt{a+b}}$$

$$= \frac{2}{\sqrt{a^2-b^2}} \cdot \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}} \cdot \tan\frac{x}{2}\right) + c$$

Case-2: $a<b$

Put $\sqrt{b-a}\tan\frac{x}{2} = z$

$$\frac{1}{2}\sqrt{b-a}\,\sec^2\frac{x}{2}\,dx = dz$$

$$\therefore \sec^2\frac{x}{2}\,dx = \frac{2dz}{\sqrt{b-a}}$$

$$\int \frac{\sec^2\frac{x}{2}dx}{(a+b)+(a-b)\tan^2\frac{x}{2}}$$

$$= \int \frac{2}{\sqrt{b-a}} \cdot \frac{dz}{(a+b)-(b-a)\tan^2\frac{x}{2}}$$

$$= \frac{2}{\sqrt{b-a}} \int \frac{dz}{(\sqrt{a+b})^2 - z^2}$$

$$= \frac{\cancel{2}}{\sqrt{b-a}} \cdot \frac{1}{\cancel{2}(\sqrt{a+b})} \cdot \log\left|\frac{\sqrt{a+b}+z}{\sqrt{a+b}-z}\right|$$

$$= \frac{1}{\sqrt{(b^2-a^2)}} \cdot \log\left|\frac{\sqrt{a+b} + \sqrt{b-a} \cdot \tan\frac{x}{2}}{\sqrt{a+b} - \sqrt{b-a} \cdot \tan\frac{x}{2}}\right| + c$$

4.8: Example-1:

$$\int \frac{dx}{\sin x + \cos x}$$

$$= \int \frac{dx}{2\sin\frac{x}{2} \cdot \cos\frac{x}{2} + \cos^2\frac{x}{2} - \sin^2\frac{x}{2}}$$

$$= \int \frac{\cancel{dz}\ \sec^2\frac{x}{2}\, dx}{2 \cdot \tan\frac{x}{2} + 1 - \tan^2\frac{x}{2}}$$

put, $\tan\frac{x}{2} = z$

$$\frac{1}{2} \cdot \sec^2\frac{x}{2} dx = dz$$

$$\therefore \sec^2\frac{x}{2}\, dx = 2dz$$

$$\int \frac{2dz}{2z + 1 - z^2}$$

$$= \int \frac{2dz}{2 - (z^2 - 2z + 1)}$$

$$= \int \frac{2dz}{(\sqrt{2})^2 - (z-1)^2}$$

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$$= \cancel{2} \cdot \frac{1}{\cancel{2}\sqrt{2}} \cdot \log\left|\frac{\sqrt{2}+z-1}{\sqrt{2}-z+1}\right|$$

$$= \frac{1}{\sqrt{2}} \cdot \log\left|\frac{\sqrt{2}+\tan\frac{x}{2}-1}{\sqrt{2}-\tan\frac{x}{2}+1}\right|$$

H.W **Example : 3**

$$\int \frac{dx}{a^2\sin^2x+b^2\cos^2x}$$

$$= \int \frac{\sec^2x\,dx}{a^2.\tan^2x+b^2}$$

put $\tan x = z$

$\sec^2x\,dx = dz$

$$\int \frac{dz}{a^2z^2+b^2}$$

$$= \frac{1}{a^2}\int \frac{dz}{z^2+\frac{b^2}{a^2}}$$

$$= \frac{1}{a^2}\int \frac{dz}{z^2+k^2} \qquad \left[k = \frac{b}{a}\right]$$

$$= \frac{1}{a^2} \cdot \frac{1}{k} \tan^{-1}\frac{z}{k}$$

$$= \frac{1}{\cancel{a}a} \cdot \frac{\cancel{a}}{b} \cdot \tan^{-1}\frac{\tan x}{\frac{b}{a}}$$

$$= \frac{1}{ab} \cdot \tan^{-1}\left(\frac{a}{b}\tan x\right)$$

H.W. Example : 4

$$\int \frac{dx}{5-13\sin x}$$

$$=\int \frac{dx}{5\left(\sin^2\frac{x}{2}+\cos^2\frac{x}{2}\right)-13.2\sin\frac{x}{2}.\cos\frac{x}{2}}$$

$$=\int \frac{\sec^2\frac{x}{2}\,dx}{5\left(\tan^2\frac{x}{2}+1\right)-26.\tan\frac{x}{2}}$$

Let, $\tan\frac{x}{2}=z$

$$\frac{1}{2}\sec^2\frac{x}{2}\,dx=dz$$

$$\int \frac{2\,dz}{5z^2+5-26z}$$

$$=\frac{2}{5}\int \frac{dz}{z^2-2.\frac{13}{5}z+1}$$

$$=\frac{2}{5}\int \frac{dz}{z^2-2.\frac{13}{5}z+\frac{13}{5}+1-\frac{12}{5}}$$

$$=\frac{2}{5}\int \frac{dz}{(z-13/5)^2-\left(\frac{12}{5}\right)^2}$$

$$=\frac{2}{5}\int \frac{du}{u^2-a^2}$$ [where, $u=z-\frac{13}{5}$, $a=\frac{12}{5}$]

$$=\frac{2}{5}\cdot\frac{1}{2a}\cdot\log\left|\frac{u-a}{u+a}\right|$$

$$= \frac{2}{5} \cdot \frac{1}{2\times\frac{12}{5}} \cdot \log\left|\frac{z-\frac{13}{5}-\frac{12}{5}}{z-\frac{13}{5}+\frac{12}{5}}\right|$$

$$= \frac{2}{\cancel{5}} \times \frac{\cancel{5}}{24} \cdot \log\left|\frac{\tan\frac{x}{2}-\frac{25}{5}}{\tan\frac{x}{2}-\frac{1}{5}}\right|$$

$$= \frac{1}{12}\log\left|\frac{5\tan\frac{x}{2}-25}{5\tan\frac{x}{2}-1}\right|$$

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Example-6:

$$\int \frac{2\sin x + 3\cos x}{3\sin x + 4\cos x}\,dx$$

Let, $2\sin x + 3\cos x = l(\text{denominator}) + m(\text{differential coefficient of denominator})$

$2\sin x + 3\cos x = l(3\sin x + 4\cos x) + m(-4\sin x + 3\cos x)$

$= 3l\sin x + 4l\cos x - 4m\sin x + 3m\cos x$

$= \sin x(3l - 4m) + \cos x(4l + 3m)$

Comparing the coefficient of $\sin x$ and $\cos x$ of both side,

$3l - 4m = 2$ —— (i)

$4l + 3m = 3$ —— (ii)

$① \times 3 + ⑪ \times 4$

$$9l - 12m = 6$$
$$16l + 12m = 12$$
$$25l = 18$$
$$\therefore l = \frac{18}{25}$$

$$4m = 3 \times \frac{18}{25} - 2$$
$$= \frac{54 - 50}{25}$$

$$m = \frac{4}{25 \times 4}$$
$$\therefore m = \frac{1}{25}$$

$$\therefore 2\sin x + 3\cos x = \frac{18}{25}(3\sin x + 4\cos x) + \frac{1}{25}(3\cos x - 4\sin x)$$

$$I = \frac{18}{25}\int dx + \frac{1}{25}\int \frac{3\cos x - 4\sin x}{3\sin x + 4\cos x}\, dx$$

$$= \frac{18}{25}x + \frac{1}{25}\log\left|(3\sin x + 4\cos x)\right| + c$$

26.

(I) $\displaystyle\int \frac{dx}{5+4\cos x}$

$a=5,\ b=4$; where $a>b$

$$\int \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}} \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}} \tan\frac{x}{2}\right)+c$$

$$\therefore \int \frac{dx}{5+4\cos x} = \frac{2}{\sqrt{25-16}} \tan^{-1}\left(\sqrt{\frac{5-4}{5+4}} \tan\frac{x}{2}\right)+c$$

$$= \frac{2}{\sqrt{9}} \tan^{-1}\left(\sqrt{\frac{1}{9}} \tan\frac{x}{2}\right)+c$$

$$= \frac{2}{3} \tan^{-1}\left(\frac{1}{3} \tan\frac{x}{2}\right)+c$$

(II) $\displaystyle\int \frac{dx}{3+5\cos x}$

$a=3,\ b=5$ where $a<b$

$$\int \frac{dx}{a+b\cos x} = \frac{1}{\sqrt{b^2-a^2}} \log \frac{\sqrt{b+a}+\sqrt{b-a}\tan\frac{x}{2}}{\sqrt{b+a}-\sqrt{b-a}\tan\frac{x}{2}}$$

$$\therefore \int \frac{dx}{3+5\cos x} = \frac{1}{\sqrt{5^2-3^2}} \log \frac{\sqrt{5+3}+\sqrt{5-3}\tan\frac{x}{2}}{\sqrt{5+3}-\sqrt{5-3}\tan\frac{x}{2}}$$

$$= \frac{1}{\sqrt{16}} \log \frac{\sqrt{8}+\sqrt{2}\cdot\tan\frac{x}{2}}{\sqrt{8}-\sqrt{2}\tan\frac{x}{2}}$$

$$= \frac{1}{4} \log \frac{\sqrt{2}\left(\sqrt{4}+\tan\frac{x}{2}\right)}{\sqrt{2}\left(\sqrt{4}-\tan\frac{x}{2}\right.}$$

$$\therefore \int \frac{dx}{3+5\cos x} = \frac{1}{4} \log \frac{2+\tan\frac{x}{2}}{2-\tan\frac{x}{2}} + c$$

31. $\int \frac{11\cos x - 16\sin x}{2\cos x + 5\sin x}\, dx$

Let, $11\cos x - 16\sin x = l(2\cos x + 5\sin x) + m(-2\sin x + 5c$

$= 2l\cos x + 5l\sin x - 2m\sin x + 5m\cos x$

$= (5l - 2m)\sin x + (2l + 5m)\cos x$

$\therefore\ 5l - 2m = -16$ —— (I)

$2l + 5m = 11$ —— (II)

(I) $\times 5$ + (II) $\times 2$

$25l - 10m = -80$

$4l + 10m = 22$

$29l = -58$

$\therefore l = -2$

$5m = 11 - 2\times(-2)$

$= 11 + 4 = 15$

$\therefore m = 3$

$$\int \frac{11\cos x - 16\sin x}{2\cos x + 5\sin x} = -2\int dx + 3\int \frac{-2\sin x + 5\cos x}{2\cos x + 5\sin x} d$$

$$= -2x + 3\log(2\cos x + 5\sin x) +$$

33. $\int \frac{6+3\sin x+14\cos x}{3+4\sin x+5\cos x}\,dx$

Let, $6+3\sin x+14\cos x = l(3+4\sin x+5\cos x)$

$+m(4\cos x-5\sin x)+c$

$= 3l+4l\sin x+5l\cos x+4m\cos x-5m\sin x+c$

$= 3l+(4l-5m)\sin x+(5l+4m)\cos x+c$

$3l+m=6$; $4l-5m=3$ — (i) ; $5l+4m=14$ — (ii)

(i) $\times 4$ + (ii) $\times 5$

$16l-20m=12$

$25l+20m=70$

$41l=82$

$\therefore l=2$

$4m=14-5\times 2$

$=14-10$

$\therefore m=1$

$\int \frac{6+3\sin x+14\cos x}{3+4\sin x+5\cos x}\,dx = 2\int dx + 1\int \frac{4\cos x-5\sin x}{3+4\sin x+5\cos x}$

$= 2x+1\log(3+4\sin x+5\cos x)+c$

Chapter - 8

29-11.22

Define Integrals

2(iii)

$$\int_1^{e^2} \frac{dx}{x(1+\log x)^2}$$

Let, $1+\log x = z$

$\frac{1}{x} dx = dz$

when, $x=1$, $z=1$

$x=e^2$, $z = 1+2=3$

$$= \int_1^3 \frac{dz}{z^2}$$

$$= \left[-\frac{1}{z}\right]_1^3$$

$$= \left[\frac{1}{z}\right]_3^1$$

$$= 1 - \frac{1}{3} = \frac{2}{3}$$

H.W 11(i)

$$\int x\log x\, dx = \log x . \int x\, dx - \int\left(\frac{d}{dx}\log x \int x\, dx\right) dx$$

$$= \log x \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} dx$$

$$= \frac{x^2}{2}\log x - \frac{x^2}{4}$$

$$\int_1^{\sqrt{e}} x\log x\, dx = \left[\frac{x^2}{2}\log x - \frac{x^2}{4}\right]_1^{\sqrt{e}}$$

$$= \left[\frac{e}{2}\log\sqrt{e} - \frac{e}{4}\right] - \left[\frac{1}{2}\log 1 - \frac{1}{4}\right]$$

$= \frac{1e}{4}\log e - \frac{e}{4} + \frac{1}{4}$

$= \frac{1}{4}e - \frac{1}{4}e + \frac{1}{4}$

$= \frac{1}{4}$

H.W 12 (iii)

$$\int_0^{\frac{1}{2}\pi} \frac{dx}{a+b\cos x}$$

$$= \frac{2}{\sqrt{a^2-b^2}} \left[\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{1}{2}x\right)\right]_0^{\frac{1}{2}\pi}$$

$$= \frac{1}{\sqrt{a^2-b^2}} \left[\cos^{-1}\frac{b+a\cos x}{a+b\cos x}\right]_2^{\frac{1}{2}\pi}$$

$$= \frac{1}{\sqrt{a^2-b^2}} \left[\cos^{-1}\frac{b+a\cos\frac{\pi}{2}}{a+b\cos\frac{\pi}{2}}\right] - \left[\cos^{-1}\frac{b+a\cos 1}{a+b\cos 1}\right]$$

$$= \frac{1}{\sqrt{a^2-b^2}} \left(\cos^{-1}\frac{b}{a} - \cos^{-1}1\right)$$

$$= \frac{1}{\sqrt{a^2-b^2}} \cos^{-1}\frac{b}{a}$$

H.W 14.

$$I=\int \log x \cdot \frac{1}{x}\, dx$$

$$= \log x \int \frac{1}{x} dx - \int \left\{ \frac{d}{dx}(\log x) \cdot \int \frac{1}{x} dx \right\} dx$$

$$= \log x \cdot \log x - \int \frac{1}{x} \cdot \log x \, dx$$

$$= (\log x)^2 - I$$

$$2I = (\log x)^2$$

$$I = \frac{1}{2}(\log x)^2$$

$$\therefore \int_a^b \frac{\log x}{x} dx = \frac{1}{2}\left[(\log x)^2\right]_a^b$$

$$= \frac{1}{2} (\log b)^2 - (\log a)^2$$

$$= \frac{1}{2}(\log b + \log a)(\log b - \log a)$$

$$= \frac{1}{2} \log\left(\frac{b}{a}\right) \cdot \log (ab)$$

H.W 18.

Let, $\cos x = z$

$\sin x\, dx = -dz$

when, $x = 0$ then $z = 1$

" $x = \frac{3\pi}{4}$ " $z = -\frac{1}{\sqrt{2}}$

$$\int_0^{\frac{3\pi}{4}} \frac{\sin x\, dx}{1+\cos^2 x} = \int_1^{-1/\sqrt{2}} \frac{-dz}{1+z^2}$$

$$= \int_{-1/\sqrt{2}}^{1} \frac{dz}{1+z^2}$$

$$= \left[\tan^{-1} z\right]_{-\frac{1}{\sqrt{2}}}^{1}$$

$$= \left[\tan^{-1} 1 - \tan^{-1}\left(\frac{-1}{\sqrt{2}}\right)\right]$$

$$= \cancel{\tan} \frac{\pi}{4} + \tan^{-1} \frac{1}{\sqrt{2}}$$

H.W 21.(1)

$$\int_0^{\frac{\pi}{2}} \frac{dx}{4+5\sin x}$$

$$= \int_0^{\pi/2} \frac{dx}{4\left(\sin^2\frac{x}{2} + \cos^2\frac{x}{2}\right) + 5.2\sin\frac{x}{2}.\cos\frac{x}{2}}$$

$$= \int_0^{\pi/2} \frac{\sec^2 x\, dx}{4\tan^2\frac{x}{2} + 4 + 10\tan\frac{x}{2}}$$

Let $\tan\frac{x}{2} = z$

$\frac{1}{2}\sec^2\frac{x}{2}\, dx = dz$

when $x = 0,\ z = 0$

$x = \frac{\pi}{2},\ z = 1$

$$= \int_0^1 \frac{2dz}{4z^2 + 10z + 4}$$

$$= \frac{2}{4}\int_0^1 \frac{dz}{z^2 + \frac{10z}{4} + 1}$$

$$= \frac{1}{2}\int_0^1 \frac{dz}{z^2+2\cdot\frac{5}{4}\cdot z+\frac{25}{16}+1-\frac{25}{16}}$$

$$= \frac{1}{2}\int_0^1 \frac{dz}{\left(z+\frac{5}{4}\right)^2-\frac{16-25}{16}}$$

$$= \frac{1}{2}\int_0^1 \frac{dz}{\left(z+\frac{5}{4}\right)^2-\left(\frac{3}{4}\right)^2}$$

$$= \frac{1}{2}\int_0^1 \frac{1}{2\cdot 3/4}\cdot\log\frac{z+\frac{5}{4}-\frac{3}{4}}{z+5/4+3/4}$$

$$= \frac{1}{3}\left[\log\frac{\frac{4z+5-3}{4}}{\frac{4z+5+3}{4}}\right]_0^1$$

$$= \frac{1}{3}\log\left[\frac{4z+2}{4z+8}\right]_0^1$$

$$= \frac{1}{3}\left\{\log\left(\frac{4+2}{4+8}\right)-\log\left(\frac{2}{8}\right)\right\}$$

$$= \frac{1}{3}\log\frac{6}{12}-\log\frac{1}{4}$$

$$= \frac{1}{3}\log\frac{\cancel{6}^{2}}{\cancel{12}_{3}}\times 4$$

$$= \frac{1}{3}\log 2$$

22.

H.W (I) $\int_0^{\pi/2} \frac{dx}{5+3\cos x}$

$a=5;\ b=3\quad a>b$

$$I = \frac{2}{\sqrt{a^2-b^2}} \cdot \left[\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}} \cdot \tan\frac{x}{2}\right)\right] + c$$

$$= \frac{2}{\sqrt{25-9}} \tan^{-1}\left\{\sqrt{\frac{5-3}{5+3}}\left(\tan\frac{\pi}{4} - \tan 0\right)\right\} + c$$

$$= \frac{2}{\sqrt{16}} \tan^{-1}\sqrt{\frac{\cancel{2}}{\cancel{8}}\frac{1}{4}}$$

$$= \frac{1}{2}\tan^{-1}\frac{1}{2} + c$$

H.W (II) $\int_0^{\frac{\pi}{2}} \frac{dx}{3+5\cos x}$

$a=3;\ b=5\quad a<b$

$$I = \frac{1}{\sqrt{b^2-a^2}} \cdot \left[\log\frac{\sqrt{b+a} + \sqrt{b-a}\tan\frac{x}{2}}{\sqrt{b+a} - \sqrt{b-a} \cdot \tan\frac{x}{2}}\right]_0^{\pi/2}$$

$$= \frac{1}{\sqrt{25-9}}\left[\log\frac{\sqrt{5+3} + \sqrt{5-3} \cdot \tan\frac{x}{2}}{\sqrt{5+3} - \sqrt{5-3} \cdot \tan\frac{x}{2}}\right]_0^{\pi/2}$$

$$= \frac{1}{4} \cdot \left[\log\frac{\sqrt{8} + \sqrt{2} \cdot \tan\frac{x}{2}}{\sqrt{8} - \sqrt{2} \cdot \tan\frac{x}{2}}\right]_0^{\pi/2}$$

$$= \frac{1}{4}\left[\log\frac{\sqrt{2}(\sqrt{4} + 1\tan\frac{x}{2})}{\sqrt{2}(\sqrt{4} - 1\tan\frac{x}{2})}\right]_0^{\pi/2}$$

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$$= \frac{1}{4}\left[\log \frac{2+\tan\frac{x}{2}}{2-\tan\frac{x}{2}}\right]_0^{\pi/2}$$

$$= \frac{1}{4}\log\left[\cancel{2}+\tan\cancel{\frac{x}{2}} - \cancel{2}+\tan\frac{x}{2}\right]_0^{\pi/2}$$

$$= \frac{1}{4}\log 2\left(\tan\frac{\pi}{4}\right) - \tan 0$$

$$= \frac{1}{4}\log \frac{2+\tan\frac{\pi}{4}}{2-\tan\frac{\pi}{4}}$$

$$= \frac{1}{4}\log 3 - \log 1$$

$$= \frac{1}{4}\log 3 + c$$

$= RHS.$

29.

(11) $$\underset{n\to\infty}{Lt}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots\cdots+\frac{n}{n^2+n^2}\right]$$

$$= \underset{n\to\infty}{Lt}\left[\frac{\frac{1}{n}}{1+\frac{n^2}{n^2}}+\frac{\frac{1}{n}}{1+\frac{2^2}{n^2}}+\cdots\cdots+\frac{\frac{1}{n}}{1+\frac{n^2}{n^2}}\right]$$

$$= \underset{n\to\infty}{Lt}\frac{1}{n}\left[\frac{1}{1+\left(\frac{1}{n}\right)^2}+\frac{1}{1+\left(\frac{2}{n}\right)^2}+\cdots\cdots+\frac{1}{1+\left(\frac{n}{n}\right)^2}\right]$$

$$= \underset{n\to\infty}{Lt}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+\left(\frac{r}{n}\right)^2}$$

put, $\frac{1}{n}=h$

$\therefore nh = 1$

$rh = f(x).dx$

$$= \underset{h\to 0}{Lt}\, h\sum_{r=1}^{n}\frac{1}{1+(hr)^2}$$

$$= \underset{h\to 0}{Lt}\, h\int_0^1 \frac{dx}{1+x^2}$$

$$= \left[\tan^{-1} x\right]_0^1$$

$$= \tan^{-1} 1 - \tan^{-1} 0$$

$$= \frac{\pi}{4} - 0$$

$$= \frac{\pi}{4}$$

H.W (iii) $\underset{n\to\infty}{Lt} \left[\frac{1^2}{n^3+1^3} + \frac{2^2}{n^3+2^3} + \cdots + \frac{n^2}{n^3+n^3}\right]$

$$= \underset{n\to\infty}{Lt} \frac{n^2}{n^3}\left[\frac{\left(\frac{1}{n}\right)^2}{1+\left(\frac{1}{n}\right)^3} + \frac{\left(\frac{2}{n}\right)^2}{1+\left(\frac{2}{n}\right)^3} + \cdots + \frac{\left(\frac{n}{n}\right)^2}{1+\left(\frac{n}{n}\right)^3}\right]$$

$$= \underset{n\to\infty}{Lt} \frac{1}{n}\left\{\sum_{r=1}^{n} \frac{\left(\frac{r}{n}\right)^2}{1+\left(\frac{r}{n}\right)^3}\right.$$

$$= \underset{h\to 0}{Lt} \sum_{r=1}^{n} \frac{(rh)^2}{1+(rh)^3}$$

$\frac{1}{n} = h$

$(rh) = dx. f(x)$

$$= \int_0^1 \frac{x^2}{1+x^3}\,dx$$

$1+x^3 = z$

$3x^2\,dx = dz$

$x^2\,dx = \frac{1}{3}\,dz$

when $x = 1,\ z = 2$

$x = 0,\ z = 1$

$$= \frac{1}{3}\int_1^2 \frac{dz}{z}$$

$$= \frac{1}{3}\left[\log z\right]_1^2$$

$$= \frac{1}{3}\left[\log(x^3+1)\right]_0^1$$

$$= \frac{1}{3}\log(1+1) - \frac{1}{3}\log(1+0)$$

$$= \frac{1}{3}\log 2.$$

H.W (iv) $\lim_{n\to\infty}\left[\frac{n^2}{(n+0)^3}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\cdots\frac{n^2}{(n+n)^3}\right]$

$$=\lim_{n\to\infty}\frac{n^2}{n^3}\left[\frac{1}{\left(1+\frac{0}{n}\right)^3}+\frac{1}{\left(1+\frac{1}{n}\right)^3}+\frac{1}{\left(1+\frac{2}{n}\right)^3}+\cdots+\frac{1}{\left(1+\frac{n}{n}\right)^3}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n}\frac{1}{\left(1+\frac{r}{n}\right)^3}$$

$$=\lim_{h\to 0}h\sum_{r=0}^{n}\frac{1}{(1+rh)^3}$$

$$=\int_0^1\frac{dx}{(1+x)^3}$$

$$=\left[-\frac{1}{2(1+x)^2}\right]_0^1$$

$$=\left[-\frac{1}{2\cdot(1+1)^2}\right]-\left[-\frac{1}{2\cdot(1+0)^2}\right]$$

$$=-\frac{1}{8}+\frac{1}{2}$$

$$=\frac{-1+4}{8}=\frac{3}{8}$$

(vii) $\lim_{n\to\infty}\left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right\}^{1/n}$

Let $\log y=\lim_{n\to\infty}\log\left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right\}^{1}$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log\left(1+\frac{r}{n}\right)$$

$$= \lim_{h \to 0} h \sum_{r=1}^{n} \log(1 + rh)$$

$$= \int_0^1 \underbrace{\log(1+x)}_{u} \cancel{dx} \cdot \underbrace{1}_{v} \, dx$$

$$= \left[\log(1+x) \int 1 \, dx\right]_0^1 - \int_0^1 \left\{\frac{d}{dx}\{\log(x+1)\} \cdot \int 1 \, dx\right\} dx$$

$$= \left[x \log(1+x)\right]_0^1 - \int_0^1 \frac{1}{1+x} \cdot x \, dx$$

$$= \left[x \cdot \log(1+x)\right]_0^1 - \int_0^1 \frac{x}{1+x} \, dx$$

$$= \left[x \cdot \log(1+x)\right]_0^1 - \int_0^1 1 - \frac{1}{1+x} \, dx$$

$$= \left[x \cdot \log(1+x) - x + \log(1+x)\right]_0^1$$

$$= \{1 \cdot \log(1+1) - 1 + \log(1+1)\} - \{0 \cdot \log(1+0) - 0 + \log(1+0)\}$$

$$= \log 2 - 1 + \log 2 - 0$$

$$= 2\log 2 - 1$$

$$= \log 2^2 - 1 = \log 4 - \log e$$

$$= \log\left(\frac{4}{e}\right)$$

$$\log y = \log \frac{4}{e}$$

$$\therefore y = \frac{4}{e}$$

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H.W Example : 1

$$\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

$$I=\int_0^{\pi/2} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}\,dx$$

$$=\int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$$

$$I+I=\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx+\int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$$

$$2I=\int_0^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

$$=\int_0^{\pi/2} dx$$

$$=\left[x\right]_0^{\pi/2}$$

$$=\frac{\pi}{2}-0$$

$$\therefore I=\frac{\pi}{4}$$

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Example: 2

$$\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\log 2$$

Put, $x = \tan\theta$

$dx = \sec^2\theta \cdot d\theta$

when, $x = 1$, $\theta = \frac{\pi}{4}$

$x = 0$, $\theta = 0$

$$I = \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{1+\tan^2\theta}\sec^2\theta \cdot d\theta$$

$$= \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{\sec^2\theta} \cdot \sec^2\theta \cdot d\theta$$

$$= \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta$$

$$= \int_0^{\pi/4} \log\left\{1+\tan\left(\frac{\pi}{4}-\theta\right)\right\} d\theta \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/4} \log\left\{1+\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\cdot\tan\theta}\right\} d\theta$$

$$= \int_0^{\pi/4} \log\left(1 + \frac{1 - \tan\theta}{1 + \tan\theta}\right) d\theta$$

$$= \int_0^{\pi/4} \log \frac{1 + \tan\theta + 1 - \tan\theta}{1 + \tan\theta}\, d\theta$$

$$= \int_0^{\pi/4} \log \frac{2}{1 + \tan\theta}\, d\theta$$

$$= \int_0^{\pi/4} \log 2\, d\theta - \int_0^{\pi/4} \log(1 + \tan\theta)\, d\theta$$

$$= \frac{\pi}{4} \log 2 - I$$

$$I + I = \frac{\pi}{4} \log 2$$

$$2I = \frac{\pi}{4} \log 2$$

$$\therefore I = \frac{\pi}{8} \log 2$$

6.12.22

Chapter-9: Improper Integral

Improper Integrals: If in an integral either the range is infinite or the integrand has an infinite discontinuity in the range, the integral is usually called an infinite or improper integral.

First kind: It is said to be an improper integral of first kind if the range of integration is unbounded.

$$\int_0^{\infty} e^{-x^2} dx, \quad \int_{-\infty}^{1} \frac{dx}{(2-x)^2}, \quad \int_{-\infty}^{\infty} \tan^{-1} x \, dx$$

are first kind improper integrals.

Second kind: I is said to be an improper integral of the second kind if the range of integration is finite, but the integrand $f(x)$ has one or more points of infinite discontinuity within the range of integration.

$\int_0^1 \frac{dx}{x^{\prime}}$, $\int_{-1}^2 \frac{dx}{(x-1)^2(2-x)}$ are second kind improper integral.

Mixed Kind: I is said to be an ~~or~~ improper integral of mixed kind if the range of integra is unbounded; also the integrand $f(x)$ has one o more points of infinite discontinuity within th range of integration.

$\int_0^\infty \frac{dx}{x-1}$ mixed kind improper integral.

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Convergence of $\int_a^\infty \frac{dx}{x^n}$ $(a>0)$

Let, $I = \int_a^\infty \frac{dx}{x^n}$, which is the improper integral of the first kind.

$$\int \frac{dx}{x^n} = \frac{x^{-n+1}}{-n+1} = \frac{x^{1-n}}{1-n}$$

Case: 1 - when $n>1$:

$$\lim_{X\to\infty} \int_a^X \frac{dx}{x^n} = \lim_{X\to\infty} \left[\frac{-1}{(n-1)\,x^{n-1}}\right]_a^X$$

$$= \lim_{X\to\infty} \frac{1}{n-1}\left[\frac{-1}{x^{n-1}}\right]_a^X$$

$$= \lim_{X\to\infty} \frac{1}{n-1}\left(-\frac{1}{X^{n-1}} + \frac{1}{a^{n-1}}\right)$$

$$= \frac{1}{(n-1)\,a^{n-1}}$$ which is finite.

$\therefore$ I is convergent and it's value $\frac{1}{(n-1)\,a^{n-1}}$

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Case - 2 : when $n<1$

$$\lim_{X\to\infty}\int_a^X \frac{dx}{x^n} = \left[\frac{x^{1-n}}{1-n}\right]_a^X$$

$$= \lim_{X\to\infty} \frac{X^{1-n} - a^{1-n}}{1-n}$$

$$= \infty$$

$\therefore$ I is divergent.

Case - 3 : when $n=1$

$$I = \int_a^\infty \frac{dx}{x^n} = \int_a^\infty \frac{dx}{x}$$

$$\lim_{X\to\infty}\int_a^{X} \frac{dx}{x} = \lim_{X\to\infty}\left[\log x\right]_a^X$$

$$= \lim_{X\to\infty}\left(\log X - \log a\right)$$

$$= \lim_{X\to\infty} \log\left(\frac{X}{a}\right)$$

$$= \infty$$

$\therefore$ I is divergent.

H.W Example : 3

$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ is an first kind improper integral.

Consider the integral I_1, I_2

$$I_1 = \int_{-\infty}^{a} \frac{dx}{1+x^2}, \quad I_2 = \int_{a}^{\infty} \frac{dx}{1+x^2}$$

$$I_1 = \lim_{X \to \infty} \int_{-X}^{a} \frac{dx}{1+x^2} = \lim_{X \to \infty} \left[\tan^{-1} x\right]_{-X}^{a}$$

$$= \lim_{X \to \infty} \left\{\tan^{-1} a - \tan^{-1}(-X)\right\}$$

$$= \lim_{X \to \infty} \left(\tan^{-1} a + \tan^{-1} X\right)$$

$$= \tan^{-1} a + \frac{\pi}{2}$$

I_1 is convergent and it's value $= \tan^{-1} a + \frac{\pi}{2}$

Again, $I_2 = \lim_{X \to \infty} \int_{a}^{X} \frac{dx}{1+x^2} = \lim_{X \to \infty} \left[\tan^{-1} x\right]_{a}^{X}$

$$= \lim_{X \to \infty} \left(\tan^{-1} X - \tan^{-1} a\right)$$

$= \frac{\pi}{2} - \tan^{-1} a$

I_2 is convergent and it's value $\frac{\pi}{2} - \tan^{-1} a$

$\therefore I = I_1 + I_2$

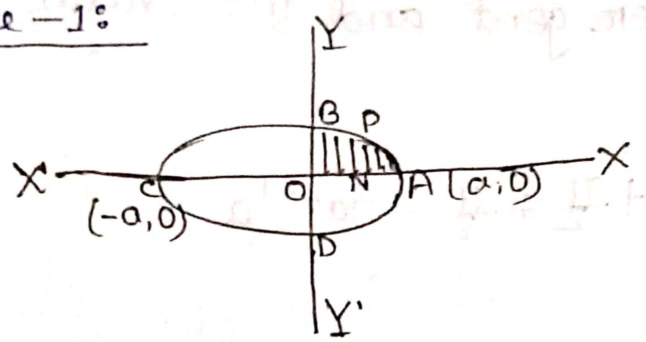
$= \tan^{-1} a + \frac{\pi}{2} + \frac{\pi}{2} - \tan^{-1} a$

$= \pi$

12.12.22

Chapter – 10

Example – 1:



Let, $x = a\sin\theta$

$dx = a\cos\theta . d\theta$

when, $x = 0$, $\theta = 0$

$x = a$, $\theta = \frac{\pi}{2}$

Given that,

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{x^2}{a^2} + 0 = 1$

$x^2 = a^2$

$\therefore x = a$

$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$

$\Rightarrow \quad = \frac{a^2 - x^2}{a^2}$

$\Rightarrow y^2 = \frac{b^2}{a^2}(a^2 - x^2)$

$\therefore y = \pm \frac{b}{a}\sqrt{(a^2 - x^2)}$

Clearly, the area bounded by the curve, the x-axis and the ordinates $x = 0$ and $x = a$, ~~is~~

the required area,

$\int_0^a y\, dx$

$= \int_0^a \frac{b}{a}\sqrt{(a^2 - x^2)}\, dx$

$= \int_0^{\pi/2} \frac{b}{a}\sqrt{a^2 - a^2\sin^2\theta} . a\cos\theta . d\theta$

$= \int_0^{\pi/2} \frac{b}{a}\sqrt{a^2(1 - \sin^2\theta)} . a\cos\theta . d\theta$

$= \frac{b}{a}\int_0^{\pi/2} a\cos\theta . a\cos\theta\, d\theta$

$= \frac{b}{a}\int_0^{\pi/2} a^2\cos^2\theta . d\theta$

$= ab\int_0^{\pi/2} \cos^2\theta . d\theta$

$$= \frac{ab}{2}\int_0^{\pi/2} 2\cos^2\theta \, d\theta$$

$$= \frac{ab}{2}\int_0^{\pi/2} (1+\cos 2\theta)\, d\theta$$

$$= \frac{ab}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{ab}{2}\left\{\left(\frac{\pi}{2} + \frac{1}{2}\sin\pi\right) - 0\right\}$$

$$= \frac{ab}{2}\cdot\frac{\pi}{2} = \frac{\pi ab}{4}$$

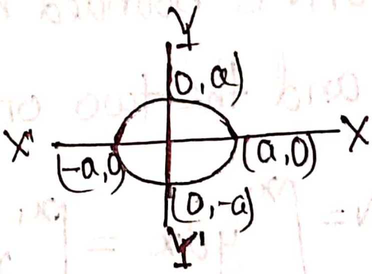
H.W.

$$x^2+y^2=a^2$$

$$\Rightarrow x^2+0=a^2$$

$$\Rightarrow x^2=a^2$$

$$\therefore x=a$$



$$x^2+y^2=a^2$$

$$y^2=a^2-x^2$$

$$\therefore y=\sqrt{a^2-x^2}$$

The required area,

$$\int_0^a y\,dx$$

$$= \int_0^a \sqrt{a^2-x^2}\,dx$$

Let,

$x = a\sin\theta$

$dx = a\cos\theta\, d\theta$

when, $x=0$, $\theta=$

$x=a$, $\theta=$

$$= \int_0^{\pi/2} \sqrt{(a^2-a^2\sin^2\theta)}\cdot a\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \sqrt{a^2(1-\sin^2\theta)}\cdot a\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} a^2\cos\theta^2\, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2} 2\cos^2\theta\, d\theta$$

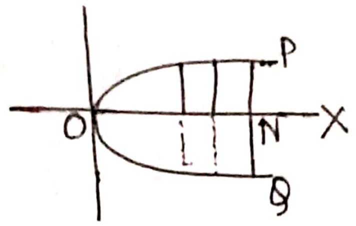
$$= \frac{a^2}{2}\int_0^{\pi/2} (1+\cos 2\theta)\, d\theta$$

$$= \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{a^2}{2}\left\{\left(\frac{\pi}{2} + \frac{1}{2}\sin\pi\right) - 0\right\}$$

$$= \frac{a^2}{2} \cdot \frac{\pi}{2} = \frac{a^2\pi}{4}$$

H.W Example-2:



The area OPN is bounded by the curve $y^2 = 4ax$, the x axis and the two ordinates $x = 0$ and $x = x_1$.

$$\therefore \text{area OPN} = \int_0^{x_1} y\,dx = \int_0^{x_1} \sqrt{4ax}\,dx$$

$$= \sqrt{4a}\int_0^{x_1} \sqrt{x}\,dx$$

$$= \sqrt{4a}\left[\frac{2}{3}x^{3/2}\right]_0^{x_1}$$

$$= \sqrt{4a}\,\frac{2}{3}\left(x_1^{3/2} - 0\right)$$

$$= \frac{2}{3}x_1 \cdot x_1^{1/2} \cdot \sqrt{4a}$$

$$= \frac{2}{3}x_1\sqrt{x_1} \cdot \sqrt{4a}$$

$$= \frac{2}{3}x_1\sqrt{4ax_1} \qquad \left[\sqrt{4ax_1} = y_1\right]$$

$$= \frac{2}{3}x_1 y_1$$

The parabola being symmetrical about x-axis, the required area POQ,

$= 2 \cdot \frac{2}{3} x_1 y_1$

$= \frac{4}{3} x_1 y_1$

H.W **Example-4:**

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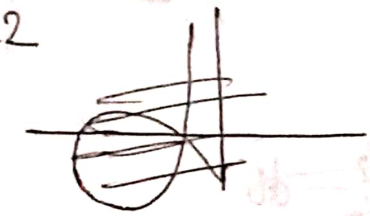
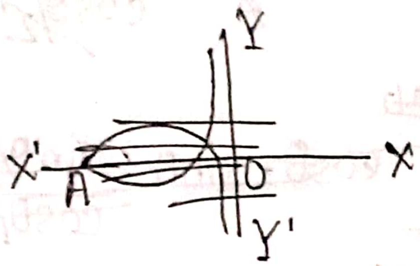
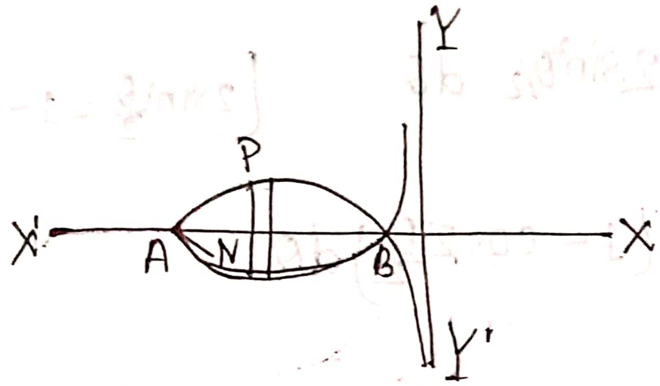






The required area of the loop = 2 area APB

$$= 2\int_{-2a}^{-a} \sqrt{\frac{-(x+a)^2(x+2a)}{x}}\, dx$$

Let, $x+2a = z$

$dx = dz$

$\therefore x = z-2a$

when, $x=-2a$, $z=0$

$x=-a$, $z=a$

$$= 2\int_0^a (z-2a+a)\sqrt{\frac{-z}{z-2a}}\, dz$$

$$= 2\int_0^a (z-a)\sqrt{\frac{z}{2a-z}}\, dz$$

$$= 2\int_0^{\pi/2} (2a\sin^2\tfrac{\theta}{2} - a)\cdot\sqrt{\frac{2a\sin^2\theta/2}{2a - 2a\sin^2\theta/2}}\cdot 2a\sin\tfrac{\theta}{2}\cdot\cos\tfrac{\theta}{2}\,d\theta$$

$z = 2a\sin^2\frac{\theta}{2}$

$dz = \cancel{4}^{2}a\sin\frac{\theta}{2}\cdot\frac{1}{\cancel{2}}\cos\frac{\theta}{2}\,d\theta$

when, $z=0$, $\theta = 0$

$z = a$, $\theta = \frac{\pi}{2}$

$$= 2\int_0^{\pi/2} a(2\sin^2\tfrac{\theta}{2} - 1)\cdot\sqrt{\frac{2\cancel{a}\sin^2\theta/2}{2\cancel{a}(1-\sin^2\theta/2)}}\cdot 2a\sin\tfrac{\theta}{2}\cdot\cos\tfrac{\theta}{2}\,d\theta$$

$$= 2\int_0^{\pi/2} a^2\cos\cancel{2}\cdot\frac{\theta}{\cancel{2}}\cdot\frac{\sin\theta/2}{\cos\theta/2}\,2\sin\frac{\theta}{2}\,\cancel{d\theta}\cdot\cos\frac{\theta}{2}\,d\theta$$

$$= \cancel{2a^2\int_0^{\pi/2}\cos\theta\cdot\sin\theta\cdot\frac{\sin\theta/2}{\cos\theta/2}\,d\theta}$$

$$= 2a^2\int_0^{\pi/2}\cos\theta\cdot 2\sin^2\theta/2\,d\theta \qquad \left[2\sin^2\frac{\theta}{2} = 1 - \cos 2\frac{\theta}{2}\right]$$

$$= 2a^2\int_0^{\pi/2}\cos\theta\cdot\left(1 - \cos 2\frac{\theta}{2}\right)d\theta$$

$$= 2a^2\int_0^{\pi/2}\cos\theta(1 - \cos\theta)\,d\theta$$

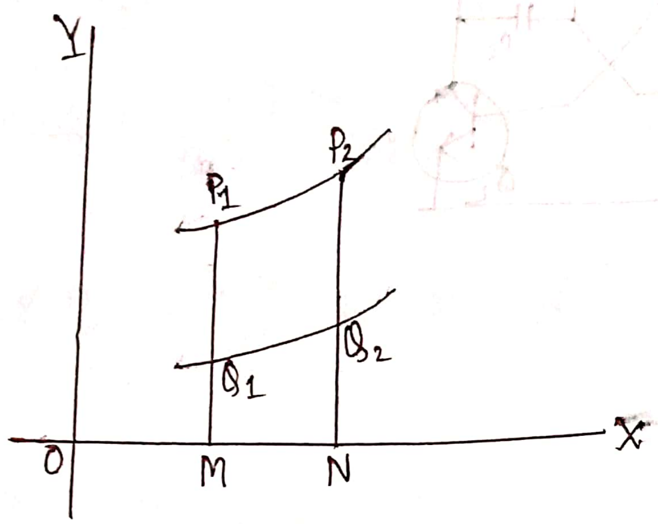
$$= \cancel{2a^2\left[\sin\theta\cdot(\theta - \sin\theta)\right]_0^{\pi/2}}$$

$$= \cancel{2a^2\left[1\cdot\left(\frac{\pi}{2} - 1\right)\right]}$$

$$= 2a^2\left(1 - \frac{\pi}{4}\right)$$

$$= \frac{1}{2}a^2(4 - \pi)$$

H.(c) Area between two given curves and two given ordinates:



Let the area required be bounded by two given curves $y=f_1(x)$ and $y=f_2(x)$ and two given ordinates $x=a$ and $x=b$, indicated by $Q_1Q_2P_2P_1Q_1$ in the figure, where $OM=a$ and $ON=b$.

Clearly area $Q_1Q_2P_2P_1Q_1$ = area P_1MNP_2 − area Q_1MNQ_2

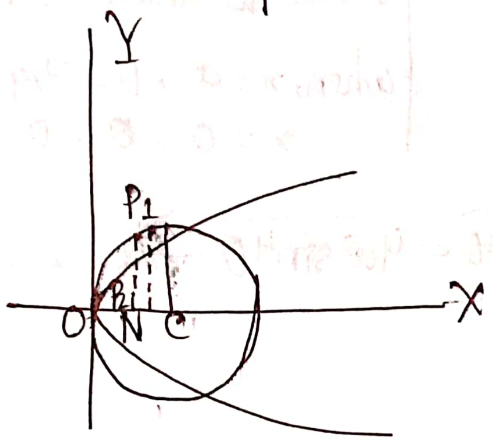
$$= \int_a^b f_1(x)\,dx - \int_a^b f_2(x)\,dx$$

$$= \int_a^b \{f_1(x) - f_2(x)\}\,dx$$

$$= \int_a^b (y_1 - y_2)\,dx$$

where y_1 and y_2 denote the ordinates of the two curves P_1P_2 and Q_1Q_2 corresponding to the same abscissa x.

Page: 324 – Example: 1



The abscissae of the common points of the curves $y^2 = ax$ and $x^2+y^2 = 2ax$ are given by $x^2 + ax = 2ax$, i.e. $x=0$ and $x=a$.

We are thus to find out the area between the curves and the ordinates $x=0$ and $x=a$ above the x-axis.

The required area is therefore

$$\int_0^a (y_1 - y_2)\,dx$$

Here, $y_1^2 = 2ax - x^2$

$y_2^2 = ax$

$$= \int_0^a \left(\sqrt{2ax - x^2} - \sqrt{ax}\right) dx$$

$$= \int_0^a \sqrt{2ax - x^2}\, dx - \sqrt{ax}\, dx$$

let,
$x = 2a\sin^2\theta$
$dx = 2a2\sin\theta.\cos\theta.d\theta$
$dx = 4a\sin\theta.\cos\theta\, d\theta$
when, $x = a$, $\theta = \pi/4$
$x = 0$, $\theta = 0$

Now

$$\int_0^a \sqrt{2ax - x^2}\,dx = \int_0^{\pi/4} \cancel{\sqrt{2a.2a\sin^2\theta - 4a^2\sin^4\theta}\; dx}$$

$$\cancel{= \sqrt{4a(1 - \sin^2\theta}}$$

$$= \int_0^{\pi/4} \sqrt{2a.2a\sin^2 a - 4a^2\sin^4\theta}\; 4a\sin\theta.\cos\theta.d\theta$$

$$= \int_0^{\pi/4} \sqrt{4a^2\sin^2\theta(1 - \sin^2\theta)}\cdot 4a\sin\theta.\cos\theta.d\theta$$

$$= \int_0^{\pi/4} \sqrt{4a^2.\sin^2\theta.\cos^2\theta}\,.4a\sin\theta.\cos\theta.d\theta$$

$$= \int_0^{\pi/4} 2a\sin\theta.\cos\theta.\,4a\sin\theta.\cos\theta\,d\theta.$$

$$= a^2 \int_0^{\pi/4} (1 - \cos 4\theta)\, d\theta$$

$$= a^2 \left[\theta - \frac{\sin 4\theta}{4} \right]_0^{\pi/4}$$

$$= \frac{\pi}{4} a^2$$

Again,

$$\int_0^a \sqrt{ax}\, dx = \sqrt{a} \left[\frac{2}{3} \cdot x^{3/2} \right]_0^a$$

$$= \frac{2}{3} \cdot a^{1/2} \cdot a^{3/2}$$

$$= \frac{2}{3} a^2$$

$\therefore$ The required area is $\frac{\pi}{4} a^2 - \frac{2}{3} a^2 = a^2 \left(\frac{\pi}{4} - \frac{2}{3} \right)$

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Example-10 :

From the equations $y^2=4x$, $y=x$.
we get $x^2=4x$ $\therefore x=0,4$. This shows that the line $y=x$ cuts the parabola $y^2=4x$ at the origin and at the point whose abscissa $=4$. Also if we write $y_1=4x$, $y_2=x$. Then the area of the segment in question is given by,

$$\therefore A=\int_0^4 (y_1-y_2)\,dx$$

$$=\int_0^4 (\sqrt{4x}-x)\,dx$$

$$=\int_0^4 (2x^{1/2}-x)\,dx$$

$$=\left[2.\frac{x^{1/2+1}}{1/2+1}-\frac{x^2}{2}\right]_0^4$$

$$=\left[2.\frac{x^{\frac{1+2}{2}}}{\frac{1+2}{2}}-\frac{x^2}{2}\right]_0^4$$

$$=\left[\frac{4}{3}x^{\frac{3}{2}}-\frac{1}{2}x^2\right]_0^4$$

$$= \frac{4}{3} \cdot 4^{\frac{3}{2}} - \frac{1}{2} 4^2$$

$$= \frac{32}{3} - \frac{16}{2}$$

$$= \frac{64 - 48}{6}$$

$$= \frac{16}{6}$$

$$= \frac{8}{3}$$

H.W Example: 12

Given that,

$x^2 + y^2 = 1$

$\Rightarrow y^2 = 1 - x^2$

$\Rightarrow 1 - x = 1 - x^2 \qquad [\because y^2 = 1 - x]$

$\Rightarrow x = x^2$

$\Rightarrow x^2 - x = 0$

$\Rightarrow x(x-1) = 0$

$\therefore x = 0, \quad x = 1$

At $x = 1$, the parabola has its vertex V where

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it touches the given circle. Again, when $x=0$, $y^2=1$. Hence the circle and the parabola cut again at $P(0,1)$ and $Q(0,-1)$, and in this case the 2 points are at the ends of a ~~diate~~ diameter of the given circle. The area in question is then

A = (semi circle of radius 1) + (segment VPQ of the parabola)

$$= \frac{1}{2}\pi + \int_{-1}^{+1} x\,dy$$

$$= \frac{1}{2}\pi + \int_{-1}^{1} (1-y^2)\,dy$$

$$= \frac{1}{2}\pi + \left[y - \frac{1}{3}y^3\right]_{-1}^{1}$$

$$= \frac{1}{2}\pi + \left(1-\frac{1}{3}\right) - \left(-1+\frac{1}{3}\right)$$

$$= \frac{1}{2}\pi + \left(\frac{3-1}{3}\right) - \left(\frac{-3+1}{3}\right)$$

$$= \frac{1}{2}\pi + \frac{2}{3} + \frac{2}{3}$$

$$= \frac{1}{2}\pi + \frac{4}{3}$$

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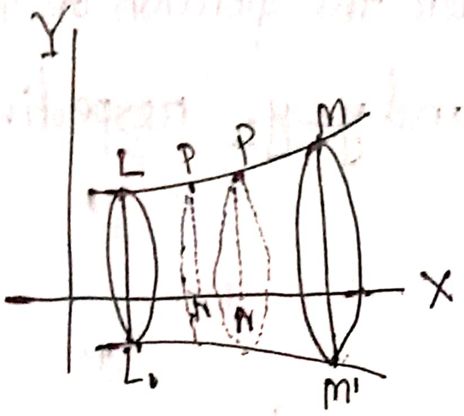
Chapter : 12



Hence, the total volume of the solid considered (bounded by $x=x_1$ and $x=x_2$) is given by

$$V = \underset{\Delta x \to 0}{Lt} \sum \pi y^2 \Delta x$$

$$= \pi \int_{x_1}^{x_2} y^2 dx$$

Hence, the required surface-area is given by.

$$S = \underset{\Delta t \to 0}{Lt} \sum (2\pi y . \Delta s)$$

$$= 2\pi \int_{S_1}^{S_2} y \, ds$$

$$= 2\pi \int_{x_1}^{x_2} y \sqrt{1+\left(\frac{dy}{dx}\right)^2} \, dx$$ [S_1, S_2 being the values of s for the points L, M]

H.W Cor: 1 When the axis of revolution is the y-axis, and we consider the portion of the solid bounded by $y=y_1$ and $y=y_2$ respectively,

$$V=\pi\int_{y_1}^{y_2} x^2\,dy$$

$$S=2\pi\int_{S_1}^{S_2} x\,ds$$

$$=2\pi\int_{y_1}^{y_2} x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\cdot dy.$$

Cor: 2 - Even if the curve revolved be given by its polar equation and the portion of the volume considered be bounded by two parallel planes perpendicular to the initial line, we may change to corresponding cartesian co-ordinates, with the initial line as the x-axis, by writing $x = r\cos\theta$, $y = r\sin\theta$

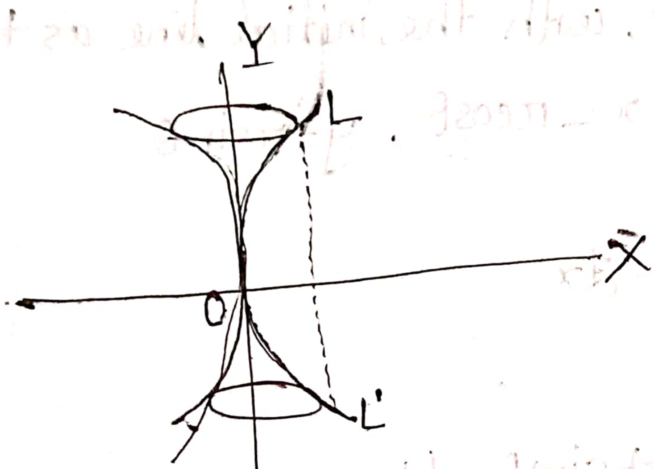
$$\therefore V = \pi \int_{x_1}^{x_2} y^2 \, dx$$

$$= \pi \int_{\theta_1}^{\theta_2} r^2 \sin^2\theta \, . \, d(r\cos\theta)$$

$$\therefore S = 2\pi \int_{s_1}^{s_2} y \, ds$$

$$= 2\pi \int_{\theta_1}^{\theta_2} r\sin\theta \sqrt{dr^2 + r^2 \, d\theta^2}$$

Example-2:



Here the axis of the ~~parabola $y^2=$~~ revolution being the y axis, and the extreme values of y being evidently $\pm 2a$,

the required volume

$$V = \pi \int_{-2a}^{2a} x^2 \, dy$$

$$= \pi \int_{-2a}^{2a} \frac{y^4}{16a^2} \, dy \quad \left[y^2 = 4ax, \; x^2 = \frac{y^4}{16a^2} \right]$$

$$= \frac{\pi}{16a^2} \left[\frac{1}{5} y^5 \right]_{-2a}^{2a}$$

$$= \frac{\pi}{16a^2}\left[\frac{1}{5}(2a)^5 - \frac{1}{5}(-2a)^5\right]$$

$$= \frac{\pi}{16a^2}\left[\frac{1}{5}(2a)^5 + \frac{1}{5}(2a)^5\right]$$

$$= \frac{\pi}{16a^2} \cdot 2 \cdot \frac{1}{5} \cdot 32a^5$$

$$= \frac{4}{5}\pi a^3$$

Also the required surface area,

$$S = 2\pi \int_{-2a}^{+2a} x\, ds = 2\pi \int_{-2a}^{2a} x \sqrt{1+\left(\frac{dx}{dy}\right)^2} \cdot dy$$

$$= 2\pi \int_{-2a}^{2a} \frac{y^2}{4a} \sqrt{1+\frac{y^2}{4a^2}}\, dy \quad \left[\because \frac{dx}{dy} = \frac{y}{2a}\right]$$

$$= 2\pi \int_{-2a}^{2a} \frac{y^2}{4a} \sqrt{\frac{4a^2+y^2}{4a^2}}\, dy$$

$$= 2\pi \int_{-2a}^{2a} \frac{y^2}{4a \cdot 2a} \sqrt{4a^2+y^2}\, dy$$

$$= \frac{2\pi}{8a^2} \int_{-2a}^{2a} y^2 \sqrt{4a^2+y^2}\, dy$$

$$= \frac{\pi}{4a^2}\int_{-\pi/4}^{\pi/4} 4a^2\tan^2\theta\sqrt{4a^2\tan^2\theta + 4a^2}$$

$$\cdot\, 2a\sec^2\theta\, d\theta$$

Let,
$y = 2a\tan\theta$
$dy = 2a\sec^2\theta\, d\theta$
when,
$y = 2a \rightarrow \theta = \frac{\pi}{4}$
$y = -2a \rightarrow \theta = -\frac{\pi}{4}$

$$= \frac{\pi}{4a^2}\int_{-\pi/4}^{\pi/4} 4a^2.\tan^2\theta\sqrt{4a^2(\tan^2\theta+1)}\, .\, 2a\sec^2\theta\, d\theta$$

$$= \frac{\pi}{4a^2}\int_{-\pi/4}^{\pi/4} 16a^2\tan^2\theta . \sec\theta . \sec^2\theta\, d\theta$$

$$= 4a^2\pi\int_{-\pi/4}^{\pi/4} \tan^2\theta . \sec^3\theta . d\theta$$

$$= 4a^2\pi\int_{-\pi/4}^{\pi/4} (\sec^3\theta - \sec^2\theta) . d\theta$$

$$= 4a^2\pi\left[\frac{1}{4}\tan\theta\sec^3\theta - \frac{1}{8}\tan\theta\sec\theta - \frac{1}{8}\log\tan\left(\frac{1}{4}\pi + \frac{1}{2}\theta\right)\right]_{-\pi/4}^{\pi/4}$$

$$= 4a^2\pi\left[\frac{3}{2}\sqrt{2} - \frac{1}{4}\log\cot\frac{1}{8}\pi\right] = \pi a^4\left[3\sqrt{2} - \log(\sqrt{2}+1)\right]$$