Math-2101

Part-B → Matrix and Linear Algebra

Matrix: A Matrix is a rectangular array / table of elements of $\mathbb{R}$ (Real numbers). It may over the complex field ($\mathbb{C}$).

Transpose of a matrix: Matrix B is called transpose of a matrix A (notation $B = A^T$) if $b_{ji} = a_{ij}$

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $\therefore A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

Different types of matrix:

① Real Matrix: A matrix A is said to be a real matrix if it satisfies the relation $A = \bar{A}$.

Example: $X = A + iB$ $\therefore \overline{X} = A - iB$

② Imaginary Matrix: A matrix A is said to be imaginary if it satisfies the relation $A = -\bar{A}$.

③ Square Matrix: A matrix with the same number of rows and columns is called a square matrix.

④ Rectangular Matrix: The number of rows and columns of a matrix need not be equal.

⑤ Diagonal Matrix: A square matrix $D=[d_{ij}]$ is diagonal if its non diagonal entities are all zero.

Example: $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

⑥ Upper-Triangular Matrix: A square matrix whose elements $a_{ij}=0$ for $i>j$, is called upper triangular Matrix. Example: $\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 7 \end{bmatrix}$

⑦ Lower-Triangular Matrix: A square matrix whose elements $a_{ij}=0$ for $i<j$, is called lower triangular Matrix. Example: $\begin{bmatrix} 5 & 0 & 0 \\ -1 & 2 & 0 \\ 3 & 7 & 1 \end{bmatrix}$

⑧ Symmetric Matrix: A matrix A is called symmetric if $A^T = A$.

Example: $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & 7 \\ -3 & 7 & 3 \end{bmatrix}$

⑨ Skew-Symmetric Matrix: A matrix equal to the negative of its transpose i.e. a square matrix such that $a_{ij}=-a_{ji}$ and in which therefore $a_{ii}=0$ is said to be skew-symmetric Matrix. Example: $\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$

⑩ Hermitian Matrix: If ~~a~~ A is a square complex, then A is said to be a Hermitian matrix of unitary matrix if $A^* = (\overline{A})^T = A$.

Example: $\begin{bmatrix} 2 & 3-2i & 3 \\ 3+2i & 5 & 2+i \\ 3 & 2-i & 0 \end{bmatrix}$

(11) <u>Skew-Hermitian Matrix</u>: If A is a square complex matrix, then A is said to be a Skew-Hermitian matrix of unitary matrix if $A^* = (\bar{A})^T = -A$

Example: $\begin{bmatrix} 5i & 4-3i & 6 \\ -4-3i & 7i & 2+i \\ -6 & -2+i & 0 \end{bmatrix}$

(12) <u>Orthogonal Matrix</u>: A real matrix A is said to be orthogonal if $AA^T = A^TA = I$, that is $A^T = A^{-1}$. There A must be square and invertable.

Example: $\begin{bmatrix} 1/9 & 8/9 & -4/9 \\ 4/9 & -4/9 & -7/9 \\ 8/9 & 1/9 & 4/9 \end{bmatrix}$

(13) <u>Idempotent Matrix</u>: If A is a matrix at $A^2 = A$ then A is said to be an idempotent matrix.

Example: $\begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$

(14) <u>Unitary Matrix</u>: Let A be a complex square matrix, then A is called unitary matrix if $AA^* = A^*A = I$ where A^* is the conjugate transpose of A.

$$A^* = (\bar{A})^T = (\overline{A^T})$$

Example: $\begin{bmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ -i/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

(15) Involutory matrix: If A is an matrix that $A^2 = I$, then A is said to be an involutory matrix.

Example: $\begin{bmatrix} 4 & 3 \\ -5 & -4 \end{bmatrix}$

(16) Nilpotent matrix: If A is an square matrix that $A^P = 0$ where P is a positive integer then A is said to be a nilpotent matrix.

Example: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\rightarrow$ Order 2.

(17) Singular Matrix: If A is a square matrix and $|A| = 0$, then A is said to be an singular matrix.

Example: $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

(18) Non-Singular Matrix: If A is a square matrix and $|A| \neq 0$ then A is said to be a non-singular matrix.

Example: $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$

(19) Normal Matrix: let A be a complex square matrix, then A is called a normal matrix if $A^*A = AA^*$ where A is the conjugate transpose of A.

Example: $\begin{bmatrix} 2+3i & 1 \\ i & 1+2i \end{bmatrix}$

(20) Equal Matrix: A matrix A is said to be an equal matrix if A is a square matrix and all the elements of A are the same.

Example: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Spur or trace of a matrix: Let $A [a_{ij}]$ be a square matrix then the trace of A, written as $tr(A)$, is the sum of the diagonal elements namely,

$$tr(A) = a_{11} + a_{22} + \cdots + a_{nn}.$$

Example: Find the diagonal and trace of the matrix.

$$A = \begin{bmatrix} 1 & 3 & 6 \\ 2 & -5 & 8 \\ 4 & 2 & 9 \end{bmatrix}$$

Sol^n: Diagonal of $A = (1, -5, 9)$

$$Tr(A) = 1 + (-5) + 9 = 5$$

Theorem-01

If A and B are two $m \times n$ matrices, then prove that

(i) $(A+B)^T = A^T + B^T$

(ii) $(A^T)^T = A$

(iii) $(\alpha A)^T = \alpha A^T \qquad \alpha \in \mathbb{R}$

Proof

(i) Let, $A = (a_{ij})$ and $B = (b_{ij})$

Here, $i = 1, 2, \cdots, m$

$j = 1, 2, \cdots, n$

Since, $(A+B)$ is defined,

Let, $C = A + B$ where, $C = (c_{ij})$ is a $m \times n$ matrix.

$$\therefore (A+B)^T = C^T$$

$$= (c_{ij})^T$$

$= (c_{ji})$

$= (a_{ji}) + (b_{ji})$

$= (a_{ij})^T + (b_{ij})^T$

$= A^T + B^T$

(ii) let, $A = (a_{ij})$ where, $i = 1, 2, ---, m$

$j = 1, 2, ---, n$

$\therefore (A^T) = (a_{ij})^T$

$= a_{ji}$

Again, $(A^T)^T = (a_{ji})^T$

$= (a_{ij})$

$= A$

(iii) let, $A = (a_{ij})$ where, $i = 1, 2, ----, m$

$j = 1, 2, ----, n$

$\therefore (A^T) = (a_{ij})^T$

$= a_{ji}$

and $(\alpha A)^T = (\alpha a_{ij})^T$

$= (\alpha a_{ji})$

$= \alpha (a_{ji})$

$= \alpha (a_{ij})^T$

$= \alpha A^T$

[Proved]

Theorem-02

If A and B are two $n \times n$ symmetric matrices then

(i) $A+B$ is symmetric i.e. $(A+B)^T = A+B$

(ii) For any scalar α, αA is symmetric

(iii) AB is symmetric, $AB=BA$

Proof

Since, the matrices A and B are symmetric.

So, $A^T = A$ and $B^T = B$

(i) Now, $(A+B)^T = A^T + B^T$

$= A+B$

$\therefore (A+B)^T = A+B$

Hence, $(A+B)$ is symmetric.

(ii) Now, $(\alpha A)^T = \alpha A^T$

$= \alpha A$

Hence, αA is symmetric.

(iii) Now, $(AB)^T = B^T A^T$

$\Rightarrow (AB)^T = BA$

Hence, AB is symmetric iff → $\boxed{\text{if and only if}}$

$(AB)^T = AB$

$\therefore AB = BA$

[Proved]

Theorem - 03

For any matrix A and B. Prove that,

$$(AB)^T = B^T \cdot A^T$$

Proof

let, $A = [a_{ij}]_{m\times n}$

$B = [b_{ij}]_{n\times p}$

$\therefore A^T = [a_{ji}]_{n\times m}$

$B^T = [b_{ji}]_{p\times n}$

$\therefore B^T \cdot A^T \longrightarrow \cancel{m\times} p\times m$

$AB \longrightarrow m\times p$

$\therefore (AB)^T \longrightarrow p\times m$

$$(AB^T)_{ij} = (AB)_{ji}$$

$$= \sum_{r=1}^{n} a_{jr} \cdot b_{ri}$$

$$= \sum_{r=1}^{n} b_{ri} \cdot a_{jr}$$

$$= \sum_{r=1}^{n} (B^T)_{ir} (A^T)_{rj}$$

$$(AB)^T{}_{ij} = (B^T \cdot A^T)_{ij}$$

$$\Rightarrow AB = B^T \cdot A^T$$

[Proved]

Adjoint Matrix:

Find A^{-1} using adjoint matrix, $A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$

Sol^n

$$|A| = \begin{vmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{vmatrix} = -1 \neq 0$$

$\therefore$ A is a non-singular matrix and hence A^{-1} exist.

Cofactors of $-1 = A_{11} = \begin{vmatrix} 1 & 0 \\ -2 & 5 \end{vmatrix} = 5$

Cofactors of $2 = A_{12} = (-1)\begin{vmatrix} 2 & 0 \\ 4 & 5 \end{vmatrix} = -10$

Cofactors of $-3 = A_{13} = \begin{vmatrix} 2 & 1 \\ 4 & -2 \end{vmatrix} = -8$

Cofactors of $2 = A_{21} = (-1)\begin{vmatrix} 2 & -3 \\ -2 & 5 \end{vmatrix} = -4$

Cofactors of $1 = A_{22} = \begin{vmatrix} -1 & -3 \\ 4 & 5 \end{vmatrix} = 7$

Cofactors of $0 = A_{23} = (-1)\begin{vmatrix} -1 & 2 \\ 4 & -2 \end{vmatrix} = 6$

Cofactors of $4 = A_{31} = \begin{vmatrix} 2 & -3 \\ 1 & 0 \end{vmatrix} = 3$

Cofactors of $-2 = A_{32} = (-1)\begin{vmatrix} -1 & -3 \\ 2 & 0 \end{vmatrix} = -6$

Cofactors of $5 = A_{33} = \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} = -5$

$$\therefore \text{Adj } A = \begin{bmatrix} 5 & -10 & -8 \\ -4 & 7 & 6 \\ 3 & -6 & -5 \end{bmatrix}^T = \begin{bmatrix} 5 & -4 & 3 \\ -10 & 7 & -6 \\ -8 & 6 & -5 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{Adj } A = \frac{1}{-1}\begin{bmatrix} 5 & -4 & 3 \\ -10 & 7 & -6 \\ -8 & 6 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{bmatrix}$$

[Answer]

Row-Canonical form:

\# Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ using row-canonical form.

Sol^n

The augmented matrix of A will be

$$(A|I_3) = \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$r_2' = r_2 - r_1 \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -2 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$r_2' = r_2 + r_3 \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$r_1' = r_1 + r_2 \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 1 \\ 0 & -2 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$r_1' = r_1 - 3r_3 \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -2 \\ 0 & -2 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$r_2' = \frac{r_2}{-2} \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -2 \\ 0 & 1 & 0 & 1/2 & -1/2 & -1/2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\therefore A^{-1} = \left[\begin{array}{ccc} 0 & 1 & -2 \\ 1/2 & -1/2 & -1/2 \\ 0 & 0 & 1 \end{array}\right]$$

[Answer]

Find the inverse of the matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$$ using row canonical form.

Sol^n

The augmented matrix of A will be

$$(A|I_3) = \left(\begin{array}{ccc|ccc} 3 & 4 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right)$$

After interchanging 1st and 2nd rows we have,

$\cancel{r_2' = r_2 - 3r_1}$
$\cancel{r_3' = r_3 - 2r_1}$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 3 & 4 & -1 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{array}{l} r_2' = r_2 - 3r_1 \\ r_3' = r_3 - 2r_1 \end{array} \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right)$$

$$r_2' = r_2 - r_3 \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & -1 & -1 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right)$$

$$r_2' = (-1)r_2 \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right)$$

$$r_3' = r_3 - 5r_2 \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & -10 & \cancel{0}\ 5 & -7 & -4 \end{array}\right)$$

$$r_3' = (-1/10)r_3 \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -1/2 & 7/10 & 2/5 \end{array}\right)$$

$$r_1' = r_1 - 3r_3 \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -1/2 & 7/10 & 2/5 \end{array}\right)$$

$$\therefore A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 1 \\ -1/2 & 7/10 & 2/5 \end{pmatrix}$$

[Answer]

Solving linear equation by using matrices:

Solve the following linear equation with the help of matrices,

$$3x + 5y - 7z = 13$$
$$4x + y - 12z = 6$$
$$2x + 9y - 3z = 20$$

Sol^n

The given linear equations can be written in matrix form as

$$\begin{bmatrix} 3 & 5 & -7 \\ 4 & 1 & -12 \\ 2 & 9 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 \\ 6 \\ 20 \end{bmatrix} \longrightarrow ①$$

Let, $A = \begin{bmatrix} 3 & 5 & -7 \\ 4 & 1 & -12 \\ 2 & 9 & -3 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $L = \begin{bmatrix} 13 \\ 6 \\ 20 \end{bmatrix}$

Then equation ① reduces to $AX = L \longrightarrow ②$

Let, D be the determinent of A;

then $D = \begin{vmatrix} 3 & 5 & -7 \\ 4 & 1 & -12 \\ 2 & 9 & -3 \end{vmatrix} = 17 \neq 0$

So, A^{-1} exist.

Now, we multiply both sides of equation ② by A^{-1} on the left, we get,

$$A^{-1} A X = A^{-1} L$$

$$\Rightarrow X = A^{-1} L \longrightarrow ③$$

[N.B. → Now find A^{-1} by adjoint matrix or row-canonical form]

Now the cofactors of D are

$A_{11} = \begin{vmatrix} 1 & -12 \\ 9 & -3 \end{vmatrix} = 105$

$A_{12} = (-1) \begin{vmatrix} 4 & -12 \\ 2 & -3 \end{vmatrix} = -12$

$A_{13} = \begin{vmatrix} 4 & 1 \\ 2 & 9 \end{vmatrix} = 34$

$A_{21} = (-1) \begin{vmatrix} 5 & -7 \\ 9 & -3 \end{vmatrix} = -48$

$A_{22} = \begin{vmatrix} 3 & -7 \\ 2 & -3 \end{vmatrix} = 5$

$A_{23} = (-1) \begin{vmatrix} 3 & 5 \\ 2 & 9 \end{vmatrix} = -17$

$A_{31} = \begin{vmatrix} 5 & -7 \\ 1 & -12 \end{vmatrix} = -53$

$A_{32} = (-1) \begin{vmatrix} 3 & -7 \\ 4 & -12 \end{vmatrix} = 8$

$A_{33} = \begin{vmatrix} 3 & 5 \\ 4 & 1 \end{vmatrix} = -17$

$\therefore \operatorname{Adj} A = \begin{bmatrix} 105 & -12 & 34 \\ -48 & 5 & -17 \\ -53 & 8 & -17 \end{bmatrix}^T$

$= \begin{bmatrix} 105 & -48 & -53 \\ -12 & 5 & 8 \\ 34 & -17 & -17 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{D} \operatorname{Adj} A$

$= \frac{1}{17} \begin{bmatrix} 105 & -48 & -53 \\ -12 & 5 & 8 \\ 34 & -17 & -17 \end{bmatrix}$

Now, from equation - 3 we have,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{17}\begin{pmatrix} 105 & -48 & -53 \\ -12 & 5 & 8 \\ 34 & -17 & -17 \end{pmatrix}\begin{pmatrix} 13 \\ 6 \\ 20 \end{pmatrix}$$

$$= \frac{1}{17}\begin{pmatrix} 17 \\ 34 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

Hence, $x = 1$, $y = 2$ and $z = 0$

[Answer]

Solve the following linear equation by using row equivalent canonical form of matrix,

$$3x + 5y - 7z = 13$$
$$4x + y - 12z = 6$$
$$2x + 9y - 3z = 20$$

Sol^n

The augmented matrix of the linear equation is

$$[AL] = \left[\begin{array}{ccc|c} 3 & 5 & -7 & 13 \\ 4 & 1 & -12 & 6 \\ 2 & 9 & -3 & 20 \end{array}\right]$$

$$\begin{array}{l} r_1' = r_1 - r_3 \Rightarrow \\ r_2' = r_2 - 2r_3 \Rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & -4 & -4 & -7 \\ 0 & -17 & -6 & -34 \\ 2 & 9 & -3 & 20 \end{array}\right]$$

$$r_3' = r_3 - 2r_1 \Rightarrow \left[\begin{array}{ccc|c} 1 & -4 & -4 & -7 \\ 0 & -17 & -6 & -34 \\ 0 & 17 & 5 & 34 \end{array}\right]$$

$$r_3' = r_2 + r_3 \Rightarrow \left[\begin{array}{ccc|c} 1 & -4 & -4 & -7 \\ 0 & -17 & -6 & -34 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$r_2' = (-1/17)\, r_2 \quad r_3' = (-1) r_3 \Rightarrow \left[\begin{array}{ccc|c} 1 & -4 & -4 & -7 \\ 0 & 1 & 6/17 & 2\ \cancel{34} \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Now, the system is in row canonical form,

Then forming the linear system,

we have, $z = 0$,

$y + 6/17\, z = 2$

$\Rightarrow y = 2$

and $x - 4y - 4z = -7$

$\Rightarrow x - 8 = -7$

$\Rightarrow x = 1$

$\therefore x = 1,\ y = 2$ and $z = 0$ is a solution of the given equations.

[Answer]

Rank of a matrix:

Let A be an arbitrary $m \times n$ matrix over a field K. The row space of A is the subspace of K^n generated by its rows and the column space of A is the subspace of K^m generated by its column. The dimensions of the row space and the column space of A are called respectively the row rank and column rank.

The rank of A, written as rank (A), is the common value of its row rank and column rank.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$ Linear Independent row

$$B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$ Linear Dependent row

Thus the rank of a matrix is the maximum number of independent rows and also the maximum number of independent columns.

Find the rank of the matrix.

$$B = \begin{pmatrix} 2 & -3 & 5 & 1 \\ 3 & 1 & -1 & 2 \\ 1 & 4 & -6 & 1 \end{pmatrix}$$

Sol^n Reduce the matrix to row echelon form by elementary row operation.

$$B = \begin{pmatrix} 2 & -3 & 5 & 1 \\ 3 & 1 & -1 & 2 \\ 1 & 4 & -6 & 1 \end{pmatrix}$$

$r_2' = r_2 - 3r_1 \Rightarrow \begin{pmatrix} 2 & -3 & 5 & 1 \\ 0 & 11 & -17 & 1 \\ 1 & 4 & -6 & 1 \end{pmatrix}$

$r_3' = 2r_3 - r_1 \Rightarrow \begin{pmatrix} 2 & -3 & 5 & 1 \\ 0 & 11 & -17 & 1 \\ 0 & 11 & -17 & 1 \end{pmatrix}$

$r_3' = r_3 - r_2 \Rightarrow \begin{pmatrix} 2 & -3 & 5 & 1 \\ 0 & 11 & -17 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

This matrix is in echelon form.
This matrix has two non zero rows.
Hence, the rank of the matrix, rank(B) = 2.

Find the rank of the matrix

$$\begin{pmatrix} 1 & 1 & 2 \\ 4 & 5 & 5 \\ 5 & 8 & 1 \\ -1 & -2 & 2 \end{pmatrix}$$

Sol^n

Reduce the matrix to row echelon form by row elementary row operation.

$$\begin{pmatrix} 1 & 1 & 2 \\ 4 & 5 & 5 \\ 5 & 8 & 1 \\ -1 & -2 & 2 \end{pmatrix}$$

$r_2' = r_2 - 4r_1$
$r_3' = r_3 - 5r_1$
$r_4 = r_1 + r_4$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 3 & -9 \\ 0 & -1 & 4 \end{pmatrix}$$

$r_3' = r_3 - 3r_2$
$r_4' = r_2 + r_4$ $\Rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Interchange r_3 and r_4 $\Rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

This matrix is in echelon form.
This matrix has ~~one~~ three non-zero rows,
Hence, the rank of the matrix, rank (M) = 3.

Find the rank of the matrix.

$$\begin{pmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 1 & 4 & 2 & 4 & 3 \\ 2 & 7 & -3 & 6 & 13 \end{pmatrix}$$

Sol^n
Reduce the matrix to row echelon form by elementary row operation.

$$\begin{pmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 1 & 4 & 2 & 4 & 3 \\ 2 & 7 & -3 & 6 & 13 \end{pmatrix}$$

$r_2' = r_2 - r_1$
$r_3' = r_3 - r_1$ $\Rightarrow \begin{pmatrix} 1 & 3 & -2 & 5 & 4 \\ 0 & 1 & 3 & -2 & 1 \\ 0 & 1 & 4 & -1 & -1 \\ 0 & 1 & 1 & -4 & 5 \end{pmatrix}$
$r_4' = r_4 - 2r_1$

$$r_3' = r_3 - r_2, \quad r_4' = r_4 - r_2 \Rightarrow \begin{pmatrix} 1 & 3 & -2 & 5 & 4 \\ 0 & 1 & 3 & -2 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & -2 & -2 & 4 \end{pmatrix}$$

$$r_4' = r_4 + 2r_3 \Rightarrow \begin{pmatrix} 1 & 3 & -2 & 5 & 4 \\ 0 & 1 & 3 & -2 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This matrix is in echelon form

This matrix has ~~one~~ ~~n~~ three non-zero rows,

Hence, the rank of the matrix, $\text{rank}(M) = 3$.

[N:B: The row rank and the column rank of any matrix are equal.
$\rightarrow \text{rank}(A) = \text{rank}(A^T)$]

Solution of a system of linear equation:

A system of linear equation in n variables $x_1, x_2, \cdots, x_n$ is a set of equation of the form

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_n \end{aligned}\right\} \rightarrow (1)$$

where the coefficients a_{ij}, $i = 1, \cdots, m$
$j = 1, \cdots, n$

of the variables and the free terms
b_i, $i = 1, \ldots, m$ are real numbers
taken from the set of all real number ($\mathbb{R}$)

If the b_i are all zero, then the system ① is called a homogenous system.

If at least one b_i is non-zero, the system ① is called a non-homogenous system.

A Sol^n vector of ① is a vector x whose components constitute a Sol^n of ①. If ① is homogenous, it has at least the trivial (zero solution)

$$\text{Sol}^n \Rightarrow x_1 = 0,\ x_2 = 0, \ldots\ x_n = 0.$$

If $x = (\alpha_1, \alpha_2, \ldots \alpha_n)$ is a solution of the homogenous and at least one $\alpha_i \neq 0$, then it is called a non-zero or non-trivial Sol^n

A system of linear equation is called consistent if it has at least one Sol^n and inconsistent if it has no Sol^n.

A consistent system is called determinate if it has a unique Sol^n and indeterminate if it has more than one Sol^n.

$$\boxed{\text{System of linear eq}^{ns}}$$

$\boxed{\text{Inconsistent}}$ $\boxed{\text{Consistent}}$

$\boxed{\text{Determinate}}$ $\boxed{\text{Indeterminate}}$

Solve the system :

$$\left.\begin{aligned} x_1-2x_2+3x_3&=7\\ 2x_1+x_2-x_3&=1\\ x_1-x_2-x_3&=-6 \end{aligned}\right\}\longrightarrow ①$$

Solⁿ

Let us, represent the three linear equations of the system ① by L_1, L_2, L_3 respectively.

Reduce the system to echelon ~~from~~ form by elementary operations :

$$\begin{aligned} L_2 &\to -2L_1+L_2\\ L_3 &\to L_3-L_1 \end{aligned} \Rightarrow \left.\begin{aligned} x_1-2x_2+3x_3&=7\\ 5x_2-7x_3&=-13\\ x_2-4x_3&=-13 \end{aligned}\right\}\longrightarrow ②$$

$$L_3\to L_2-5L_1 \Rightarrow \left.\begin{aligned} x_1-2x_2+3x_3&=7\\ 5x_2-7x_3&=-13\\ 13x_3&=52 \end{aligned}\right\}\longrightarrow ③$$

This system is in echelon form. Now,

From L_3 (system-③) we get,

$$x_3=4$$

From L_2 (system-③) we get,

$$5x_2-28=-13$$
$$\Rightarrow x_2=3$$

From, L_1 (system-③ we get,

$$x_1-2\cdot3+3\cdot4=7$$
$$\Rightarrow x_1=1$$

$\therefore$ $x_1=1$, $x_2=3$ and $x_3=4$ [Answer]

Determine the values of λ so that the following system has (i) a unique Sol^n

(ii) more than one Sol^n

(iii) no Sol^n

$$x+y-z=1$$
$$2x+3y+\lambda z=3$$
$$x+\lambda y+3z=2$$

Sol^n

Reduce the system to echelon form by elementary operations.

Eq ② = Eq ② − Eq ① × 2
Eq ③ = Eq ③ − Eq ②

$$\Rightarrow \begin{aligned} x+y-z&=1\\ y+(\lambda+2)z&=1\\ (\lambda-1)y+4z&=1 \end{aligned}$$

Eq ③ = Eq ③ − Eq ② × $(\lambda-1)$ $\Rightarrow$

$$\begin{aligned} x+y-z&=1\\ y+(\lambda+2)z&=1\\ \{4-(\lambda-1)(\lambda+2)\}z&=2-\lambda \end{aligned}$$

or,
$$\begin{aligned} x+y-z&=1\\ y+(\lambda+2)z&=1\\ (6-\lambda-\lambda^2)z&=2-\lambda \end{aligned}$$

or,
$$\begin{aligned} x+y-z&=1\\ y+(\lambda+2)z&=1\\ (3+\lambda)(2-\lambda)z&=2-\lambda \end{aligned}$$

This system is in echelon form.

It has a unique soln if the coefficient of z in the third equation is non-zero i.e. if $\lambda \neq 2$ and $\lambda \neq -3$.

In case, $\lambda = 2$ third equation is $0=0$ which is true and the system has more than one soln.

In case, $\lambda = -3$ the third equation is $0=5$ which is not true and hence the system has no soln.

Eigenvalues and Eigenvectors:

If A is a $n \times n$ matrix, then a non-zero vector $v \in \mathbb{R}^n$ is called an eigenvector of A if Av is a scalar multiple of v. that is $AV = \lambda V$, for some scalar λ. The scalar λ is called an eigenvalue of A and v is said to be an eigenvectors of A corresponding to λ.

For 'λ' to be an eigenvalue of the matrix A, then $(\lambda I - A)$ is called the characteristic matrix. The determinant of characteristic matrix

$$|\lambda I - A| \rightarrow \text{Characteristic Polynomial of } A$$

The equation $|\lambda I - A| = 0$ is called the characteristic equation of A.

Find all eigenvalues and the corresponding eigen-vectors of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & -5 & 2 \end{bmatrix}$$

Soln

The characteristic equation of A is

$$|\lambda I - A| = 0$$

$$\Rightarrow \left| \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & -5 & 2 \end{pmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} \lambda-1 & -2 & 1 \\ 0 & \lambda+2 & 0 \\ 0 & 5 & \lambda-2 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda-1)(\lambda+2)(\lambda-2) = 0$$

$$\Rightarrow \lambda = 1, 2, -2.$$

which are the eigenvalues of A.

Now by definition $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ is an eigenvectors of A.

Corresponding to the eigenvalue λ if and only if v is a non-trivial solution of $(\lambda I - A)v = 0$

$$(\lambda I - A)v = 0$$

$$\Rightarrow \begin{bmatrix} \lambda-1 & -2 & 1 \\ 0 & \lambda+2 & 0 \\ 0 & 5 & \lambda-2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda = 1$, $\begin{bmatrix} 0 & -2 & 1 \\ 0 & 3 & 0 \\ 0 & 5 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \quad -2v_2 + v_3 = 0$$
$$3v_2 = 0$$
$$5v_2 - v_3 = 0$$

$$\Rightarrow v_2 = 0, \; v_3 = 0$$

Here v_1 is a free variable. let, $v_1 = a$, where a is any real number. So, $\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$ is the eigenvectors of A corresponding to $\lambda = 1$.

In particular, if $a = 1$, then the eigenvectors is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

For $\lambda = 2$, $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 4 & 0 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \quad v_1 - 2v_2 + v_3 = 0$$
$$4v_2 = 0$$
$$5v_2 = 0$$

$$\Rightarrow v_2 = 0$$

Here ~~let,~~ v_3 is a free variable. let, $v_3 = b$, where b is any real number. Then, $v_1 = -b$. Hence, $\begin{bmatrix} -b \\ 0 \\ b \end{bmatrix}$ is the eigenvectors of A corresponding to $\lambda = 2$.

In particular, if $b = 1$, then the eigenvector is

$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

For $\lambda = -2$, $\begin{bmatrix} -3 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 5 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow -3v_1 - 2v_2 + v_3 = 0$$
$$5v_2 - 4v_3 = 0$$

Here, v_3 is a free variable. Let, $v_3 = c$, where c is any real number. Then, $v_2 = 4c/5$, $v_1 = -c/5$. Hence, $\begin{bmatrix} -c/5 \\ 4c/5 \\ c \end{bmatrix}$ is the eigenvectors of A corresponding to $\lambda = -2$

In particular, if $c = 5$, then the eigenvector is $\begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix}$.

Diagonalization:

A square matrix A is called diagonalizable if there exist an invertible matrix P such that $P^{-1}AP$ is diagonal, the matrix P is said to diagonalize A.

Find the matrix P that diagonalizes the matrix $A = \begin{bmatrix} 1 & 4 \\ 9 & 1 \end{bmatrix}$ and also determine $P^{-1}AP$.

Sol^n

The characteristic equation of the given matrix is

$$|\lambda I - A| = 0$$

$$\Rightarrow \left| \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 4 \\ 9 & 1 \end{pmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} \lambda - 1 & -4 \\ -9 & \lambda - 1 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda - 1)^2 - 36 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 1 - 36 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda - 35 = 0$$

$$\Rightarrow \lambda = -5, 7$$

which are the eigenvalues of A.

Now, by definition $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is an eigenvectors of matrix A.

Corresponding to the eigenvalue λ if and only if v is a non-trivial solution of $(\lambda I - A)v = 0$

$$(\lambda I - A)v = 0$$

$$\Rightarrow \begin{bmatrix} \lambda - 1 & -4 \\ -9 & \lambda - 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For, $\lambda = -5$ $\begin{bmatrix} -6 & -4 \\ -9 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{matrix} -6v_1 - 4v_2 = 0 \\ -9v_1 - 6v_2 = 0 \end{matrix} \quad \Rightarrow \begin{matrix} -3v_1 - 2v_2 = 0 \\ -3v_1 - 2v_2 = 0 \end{matrix}$$

Now, $v_1 = 2$ and $v_2 = -3$ is a solution.

$\therefore \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = -5$.

For $\lambda = 7$, $\begin{bmatrix} 6 & -4 \\ -9 & 6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow 6v_1 - 4v_2 = 0$$
$$-9v_1 + 6v_2 = 0$$
$$\Rightarrow 3v_1 - 2v_2 = 0$$
$$3v_1 - 2v_2 = 0$$

Now, $v_1 = 2$ and $v_2 = 3$ is a solution.

$\therefore \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 7$.

Suppose that P is the matrix which has the above two eigenvectors as columns.

Then $P = \begin{bmatrix} 2 & 2 \\ -3 & 3 \end{bmatrix}$

$$\therefore P^{-1} = \frac{1}{12} \begin{bmatrix} 3 & -2 \\ 3 & 2 \end{bmatrix}$$

$$\therefore P^{-1}AP = \frac{1}{12} \begin{bmatrix} 3 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 9 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -3 & 3 \end{bmatrix}$$

$$= \frac{1}{12} \begin{bmatrix} 3 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -10 & 14 \\ 15 & 21 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 0 \\ 0 & 7 \end{bmatrix} = D$$

which is the diagonal matrix of the eigenvalues of the matrix A.

$\therefore$ $P = \begin{bmatrix} 2 & 2 \\ -3 & 3 \end{bmatrix}$ is the required matrix that diagonalizes the given matrix $A = \begin{bmatrix} 1 & 4 \\ 9 & 1 \end{bmatrix}$

Cayley-Hamilton Theorem:

Every square matrix satisfies its own characteristic equation $|\lambda I - A| = 0$

Using Cayley Hamilton Th^m find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Sol^n

The characteristic equation of the given matrix A is $|\lambda I - A| = 0$

$$\Rightarrow \left| \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} \lambda-1 & -2 & -2 \\ -3 & \lambda-1 & 0 \\ -1 & -1 & \lambda-1 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda-1)^3 + 2\{-3(\lambda-1)\} - 2(3+\lambda-1)$$

$$\Rightarrow (\lambda-1)^3 - 6\lambda + 6 - 4 - 2\lambda$$

$$\Rightarrow \lambda^3 - 3\lambda^2 + 3\lambda - 1 - 8\lambda + 2$$

$$\Rightarrow \lambda^3 - 3\lambda^2 - 5\lambda + 1 = 0$$

Therefore, the characteristic equation is

$$\lambda^3 - 3\lambda^2 - 5\lambda + 1 = 0$$

From Cayley Hamilton Th^m, we can write,

$$A^3 - 3A^2 - 5A + I = 0 \longrightarrow (1)$$

Multiplying (1) by A^{-1}, we get,

$$A^2 - 3A - 5I + A^{-1} = 0$$

$$\Rightarrow A^{-1} = 3A + 5I - A^2$$

Now, $A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$$3A = \begin{bmatrix} 3 & 6 & 6 \\ 9 & 3 & 0 \\ 3 & 3 & 3 \end{bmatrix}$$

$$5I = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 6 & 4 \\ 6 & 7 & 6 \\ 5 & 4 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = 3A + 5I - A^2$$

$$= \begin{bmatrix} 3 & 6 & 6 \\ 9 & 3 & 0 \\ 3 & 3 & 3 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} - \begin{bmatrix} 9 & 6 & 4 \\ 6 & 7 & 6 \\ 5 & 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & -6 \\ -2 & -1 & 5 \end{bmatrix}$$

[Answer]

Vector Space:

A vector space over an arbitrary field F is a non-empty set V, whose elements are called vectors for which two operations are prescribed. The first operation, called vector addition, assigns to each pair of vectors u and v in V a vector denoted by $u+v$ called their sum. The second operation called scalar multiplication assigns to each vector v in V and each scalar α in F a vector denoted by αv which is in V.

The two operations are required to satisfy the following axioms:

A(1) Addition is commutative.

For all vectors $u, v \in V$, $u+v = v+u$.

A(2) Addition is associative.

For all vectors $u, v, \omega \in V$ $(u+v)\,\omega = u+(v+\omega)$.

A(3) Existence of $\underline{0}$ (zero vectors)

There exists a vector $\underline{0} \in V$ such that for all $v \in V$

$$v+0 = 0+v = v.$$

A(4) Existence of negative.

For each $v \in V$ there is a vector $-v \in V$ for which

$$v+(-v) = (-v)+v = 0$$

M(1) For any scalar $\alpha \in F$ and any vectors $u, v \in V$,

$$\alpha(u+v) = \alpha u + \alpha v$$

M(2) For any scalars $\alpha, \beta \in F$ and any vector $v \in V$,

$$(\alpha + \beta)v = \alpha v + \beta v.$$

M(3) For any scalars $\alpha, \beta \in F$ and any vector $v \in V$,

$$(\alpha \beta)v = \alpha v + \beta v.$$

M(4) For each $v \in V$, $1v = v$

where 1 is the unit scalar and $1 \in F$.

Theorem:

For each positive integer n. Euclidean space $\mathbb{R}^n$ is a vector space.

Proof

We shall have to show that $\mathbb{R}^n$ satisfies all axioms of a vector space.

A(1) Let $u = (u_1, u_2, ---, u_n)$ and $v = (v_1, v_2, ---, v_n)$ be in $\mathbb{R}^n$. Now

$$u + v = (u_1, u_2, --- u_n) + (v_1, v_2, --- v_n)$$
$$= (u_1 + v_1, u_2 + v_2, ---, u_n + v_n)$$
$$= (v_1 + u_1, v_2 + u_2, --- v_n + u_n)$$
$$\Rightarrow u + v = (v_1, v_2, ---, v_n) + (u_1, u_2, ---, u_n)$$
$$= v + u$$

So, Axiom A① is satisfied.

A(2) let, $u=(u_1,u_2,\cdots u_n)$ $v=(v_1,v_2,\cdots v_n)$ and $\omega=(\omega_1,\omega_2,\cdots,\omega_n)$ be in $\mathbb{R}^n$. Now,

$$(u+v)+\omega = (u_1+v_1, u_2+v_2, \cdots, u_n+v_n)+(\omega_1,\omega_2,\cdots \omega_n)$$
$$= (u_1+v_1+\omega_1, u_2+v_2+\omega_2, \cdots, u_n+v_n+\omega_n)$$
$$= (u_1,u_2,\cdots u_n)+(v_1+\omega_1, v_2+\omega_2, \cdots v_n\omega_n)$$
$$= u+(v+\omega)$$

So, Axiom A② is satisfied.

A(3) let, $\underline{0}=(0,0,\cdots 0)$ be in $\mathbb{R}^n$. Then for any $u=(u_1,u_2,\cdots u_n)$ $\in\mathbb{R}^n$. So,

$$u+0=(u_1,u_2,u_3,\cdots u_n)+(0,0,0,\cdots 0)$$
$$=(u_1+0, u_2+0, u_3+0, \cdots u_n+0)$$
$$=(u_1,u_2,u_3,\cdots u_n)$$
$$= u$$

So, Axiom A③ is satisfied.

A(4) let, $u=(u_1,u_2,\cdots u_n)$ and the set $-u=(-u_1,-u_2,\cdots,-u_n)$ be in $\mathbb{R}^n$. Now,

$$\therefore u+(-u) = (u_1,u_2,\cdots,u_n)+(-u_1,-u_2,\cancel{-u_3},\cdots,-u_n)$$
$$= (u_1-u_1, u_2-u_2, \cdots u_n-u_n)$$
$$=(0,0,\cdots 0)$$
$$=0$$

So, Axiom A④ is satisfied.

M(1) let α be a real number (scalar) and $u=(u_1,u_2,\cdots u_n)$, $v=(v_1,v_2,\cdots v_n)$ be in $\mathbb{R}^n$. Now,

$$\alpha(u+v)=\alpha\ (u_1+v_1, u_2+v_2, \cdots u_n+v_n)$$
$$= (\alpha(u_1+v_1), \alpha(u_2+v_2), \cdots \alpha(u_n+v_n))$$
$$= (\alpha u_1+\alpha v_1, \alpha u_2+\alpha v_2, \cdots \alpha u_n+\alpha v_n)$$
$$=(\alpha u_1,\alpha u_2,\cdots \alpha u_n)+(\alpha v_1,\alpha v_2,\cdots \alpha v_n)$$

$= \alpha u + \alpha v$

So, Axiom M① is satisfied.

M(2) let α, β be two scalars and $u = (u_1, u_2, \ldots, u_n)$ be in $\mathbb{R}^n$. Now,

$$(\alpha+\beta)u = (\alpha+\beta)(u_1, u_2, \ldots u_n)$$
$$= ((\alpha+\beta)u_1, (\alpha+\beta)u_2, \ldots (\alpha+\beta)u_n)$$
$$= (\alpha u_1 + \beta u_1, \alpha u_2 + \beta u_2, \ldots \alpha u_n + \beta u_n)$$
$$= (\alpha u_1, \alpha u_2, \cancel{\alpha u_n} \ldots \alpha u_n) + (\beta u_1, \beta u_2, \ldots \beta u_n)$$
$$= \alpha u + \beta u$$

So, Axiom M② is satisfied.

M(3) let α, β be two scalars and $u = (u_1, u_2, \ldots u_n)$ be in $\mathbb{R}^n$. Now,

$$(\alpha\beta)u = (\alpha\beta)(u_1, u_2, \ldots u_n)$$
$$= (\alpha\beta u_1, \alpha\beta u_2 \ldots \alpha\beta u_n)$$
$$= \alpha(\beta u_1, \beta u_2, \ldots, \beta u_n)$$
$$= \alpha(\beta u)$$

So, Axiom M③ is satisfied.

M(4) let 1 be the scalar unit and $u = (u_1, u_2, \ldots u_n)$ be in $\mathbb{R}^n$,

$$\therefore 1u = 1(u_1, u_2, \ldots, u_n)$$
$$= (1u_1, 1u_2, \ldots 1u_n)$$
$$= (u_1, u_2, \ldots u_n)$$
$$= u$$

So, Axiom M④ is satisfied.

$\therefore \mathbb{R}^n$ satisfied all the axioms.

$\therefore \mathbb{R}^n$ is a vector space.

[Proved]

⟹ Set of integers is not a vector space.

In the set of integers no negative vector ~~is~~ has exist.

So, A(4) axioms is not satisfied.

So, set of integers is not a vector space.

Subspaces of a vector space:

Let, W be a non-empty subset of a vector space V over the field F. We call W a subspace of V if and only if W is a vector space over the field F under the laws of vector addition and scalar multiplication defined on V, or equivalently, W is a subspace of V whenever $w_1, w_2 \in W$, $\alpha, \beta \in F$ implies that $\alpha w_1 + \beta w_2 \in W$.

Show that $S = \{(a,0,c) : a, c \in \mathbb{R}\}$ is a subspace of the vector space $\mathbb{R}^3$.

Proof

For $0 \in \mathbb{R}^3$, $0 = (0,0,0) \in S$

Since the second component of 0 is 0.

Hence, S is non-empty.

For any vectors, $u = (a,0,c)$ and $v = (a',0,c')$ in S and any scalars (real numbers) α, β, we have,

$$\begin{aligned} &\alpha u + \beta v \\ &= \alpha(a,0,c) + \beta(a',0,c') \\ &= (\alpha a, 0, \alpha c) + (\beta a', 0, \beta c') \\ &= (\alpha a + \beta a', 0, \alpha c + \beta c') \end{aligned}$$

Since, the second component is zero.
∴ and first and third elements belong to $\mathbb{R}$.
∴ $\alpha u + \beta v \in S$.
Hence, S is subspace of the vector space $\mathbb{R}^3$.
[Showed]

$W = \{(a,b,c) \mid a,b,c \in \mathbb{R}$ and $a-2b+3c=5\}$ is not a subspace of $\mathbb{R}^3$.

Sol^n

let, $0 \in \mathbb{R}^3$, $0 = (0,0,0)$

Here, $0 = (0,0,0) \notin W$,

since, ~~0-20+3.0~~ $0-2\cdot 0+3\cdot 0 = 0 \neq 5$.

∴ W is not a subspace of $\mathbb{R}^3$.

Linear Combination of vectors:

let, V be a vector space over the field F and let, $v_1, \ldots, v_n \in V$ then any vector $v \in V$ is called a linear combination of $v_1, v_2, \ldots, v_n$ if and only if there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ in F such that $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$

$$= \sum_{i=1}^{n} \alpha_i v_i$$

Consider the vectors $v_1 = (2,1,4)$, $v_2 = (1,-1,3)$ and $v_3 = (3,2,5)$ in $\mathbb{R}^3$. Show that $v = (5,9,5)$ is a linear combination of v_1, v_2 and v_3.

Sol^n

In order to show that v is a linear combination of v_1, v_2 and v_3, there must be scalars α_1, α_2 and $\alpha_3 \in F$ such that $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$.

i.e. $(5,9,5) = \alpha_1(2,1,4) + \alpha_2(1,-1,3) + \alpha_3(3,2,5)$

$= (2\alpha_1, \alpha_1, 4\alpha_1) + (\alpha_2, -\alpha_2, 3\alpha_2) + (3\alpha_3, 2\alpha_3, 5\alpha_3)$

$= (2\alpha_1 + \alpha_2 + 3\alpha_3, \alpha_1 - \alpha_2 + 2\alpha_3, 4\alpha_1 + 3\alpha_2 + 5\alpha_3)$

That implies,

$$\left.\begin{aligned} 2\alpha_1 + \alpha_2 + 3\alpha_3 &= 5 \\ \alpha_1 - \alpha_2 + 2\alpha_3 &= 9 \\ 4\alpha_1 + 3\alpha_2 + 5\alpha_3 &= 5 \end{aligned}\right\} \rightarrow ①$$

Reduce the system to echelon form by elementary operations. Interchange 1st and 2nd equation,

$$\alpha_1 - \alpha_2 + 2\alpha_3 = 9$$
$$2\alpha_1 + \alpha_2 + 3\alpha_3 = 5$$
$$4\alpha_1 + 3\alpha_2 + 5\alpha_3 = 5$$

Eq ② = Eq ② − Eq ① × 2
Eq ③ = Eq ③ − Eq ① × 4 $\Rightarrow$

$$\alpha_1 - \alpha_2 + 2\alpha_3 = 9$$
$$3\alpha_2 - \alpha_3 = -13$$
$$7\alpha_2 - 3\alpha_3 = -31$$

Eq ③ = Eq ③ − Eq ② × $7/3$ $\Rightarrow$

$$\alpha_1 - \alpha_2 + 2\alpha_3 = 9$$
$$3\alpha_2 - \alpha_3 = -13$$
$$-2/3\,\alpha_3 = -2/3$$

From the 3rd equation, $\alpha_3 = 1$

Substituting $\alpha_3 = 1$ in equation ②,

$$3\alpha_2 - 1 = -13$$
$$\Rightarrow \alpha_2 = -4$$

Again substituting $\alpha_3 = 1$ and $\alpha_2 = -4$ in equation ①,

$$\alpha_1 + 4 + 2 = 9$$
$$\Rightarrow \alpha_1 = 3$$

$\therefore$ the solution of the system, $\alpha_1 = 3$, $\alpha_2 = -4$, $\alpha_3 = 1$

Hence, $v = 3v_1 - 4v_2 + v_3$

Therefore, v is a linear combination of v_1, v_2, v_3.

Is the vector $v = (2, -5, 3)$ in $\mathbb{R}^3$ is a linear combination of the vectors

$v_1 = (1, -3, 2)$ $v_2 = (2, -4, -1)$ and $v_3 = (1, -5, 7)$

Soln

Let, $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$ where $\alpha_1, \alpha_2, \alpha_3$ are unknown scalars.

i.e.,

$$(2, -5, 3) = \alpha_1 (1, -3, 2) + \alpha_2 (2, -4, -1) + \alpha_3 (1, -5, 7)$$
$$= (\alpha_1, -3\alpha_1, 2\alpha_1) + (2\alpha_2, -4\alpha_2, -\alpha_2) + (\alpha_3, -5\alpha_3, 7\alpha_3)$$
$$= (\alpha_1 + 2\alpha_2 + \alpha_3, -3\alpha_1 - 4\alpha_2 - 5\alpha_3, 2\alpha_1 - \alpha_2 + 7\alpha_3)$$

That implies,

$$\left.\begin{aligned} \alpha_1 + 2\alpha_2 + \alpha_3 &= 2 \\ -3\alpha_1 - 4\alpha_2 - 5\alpha_3 &= -5 \\ 2\alpha_1 - \alpha_2 + 7\alpha_3 &= 3 \end{aligned}\right\} \longrightarrow ①$$

Reduce the system-① to echelon form by the elementary operations.

Eq② = Eq② − Eq① × (−3)
Eq③ = Eq③ − Eq① × 2

$$\Rightarrow \begin{aligned} \alpha_1 + 2\alpha_2 + \alpha_3 &= 2 \\ 2\alpha_2 - 2\alpha_3 &= 1 \\ -5\alpha_2 + 5\alpha_3 &= -1 \end{aligned}$$

Eq③ = Eq③ + Eq② × $\frac{5}{2}$

$$\Rightarrow \begin{aligned} \alpha_1 + 2\alpha_2 + \alpha_3 &= 2 \\ 2\alpha_2 - 2\alpha_3 &= 1 \\ 0 + 0 &= \frac{3}{2} \end{aligned}$$

$$\Rightarrow \begin{aligned} \alpha_1 + 2\alpha_2 + \alpha_3 &= 2 \\ 2\alpha_2 - 2\alpha_3 &= 1 \\ 0 &= \frac{3}{2} \end{aligned}$$

The system has equation of the form $0 = \frac{3}{2}$ which is not true. Hence, the above system is inconsistent i.e. it has no solution.
∴ V vector is not a linear combination of V_1, V_2, V_3.

Linear Transformation:

Let U and V be two vector spaces over the same field F. A linear transformation T of U into V, written as $T: U \rightarrow V$, is a transformation of U into V such that

① $T(u_1+u_2) = T(u_1) + T(u_2)$ for all $u_1, u_2 \in U$
⑪ $T(\alpha u) = \alpha T(u)$ for all $u \in U$ and all $\alpha \in F$.

In $\mathbb{R}^2$ consider the function $T:\mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x,y) = (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta)$ where θ is a fixed number and $0 \le \theta \le 2\pi$. Show that, T is a linear transformation.

$\underline{\text{Sol}^n}$

Let, $u_1 = (a_1, b_1)$ and $u_2 = (a_2, b_2)$ be two vectors in $\mathbb{R}^2$.

Then, $u_1 + u_2 = (a_1 + a_2,\ b_1 + b_2)$

$$\therefore T(u_1+u_2) = T(a_1+a_2,\ b_1+b_2)$$

$$= \left((a_1+a_2)\cos\theta - (b_1+b_2)\sin\theta,\ (a_1+a_2)\sin\theta + (b_1+b_2)\cos\theta\right)$$

$$= (a_1\cos\theta + a_2\cos\theta - b_1\sin\theta - b_2\sin\theta,\ a_1\sin\theta + a_2\sin\theta + b_1\cos\theta + b_2\cos\theta)$$

$$= (a_1\cos\theta - b_1\sin\theta + a_2\cos\theta - b_2\sin\theta,\ a_1\sin\theta + b_1\cos\theta + a_2\sin\theta + b_2\cos\theta)$$

$$= (a_1\cos\theta - b_1\sin\theta,\ a_1\sin\theta + b_1\cos\theta) + (a_2\cos\theta - b_2\sin\theta,\ a_2\sin\theta + b_2\cos\theta)$$

$$= T(a_1, b_1) + T(a_2, b_2)$$

$$= T(u_1) + T(u_2)$$

Again, for any scalar, $\alpha \in F$, and $u = (a,b) \in U$, we have,

$$T(\alpha u) = T(\alpha(a,b))$$

$$= T(\alpha a, \alpha b)$$

$$= (\alpha a\cos\theta - \alpha b\sin\theta,\ \alpha a\sin\theta + \alpha b\cos\theta)$$

$$= (\alpha(a\cos\theta - b\sin\theta),\ \alpha(a\sin\theta + b\cos\theta))$$

$= \alpha\,(a\cos\theta - b\sin\theta,\ a\sin\theta + b\cos\theta)$
$= \alpha\, T(a,b)$
$= \alpha\, T(u)$

$\therefore T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation.

[Showed]

Kernel and Range of a linear transformation:

Let T be a linear transformation of U into V. Then the kernel of T is the subset of U consisting of all $x \in U$ for which $T(x) = 0$ where $0 \in V$. The kernel of T is generally denoted by $\ker(T)$.
The range of T is the subset of V consisting of all $y \in V$ such that $T(x) = y$ for all $x \in U$. It is generally denoted by $R(T)$.

Rank and Nullity of a linear transformation:

If $T: U \to V$ is a linear transformation then dimension of the range of T is called the rank of T and the dimension of the kernel of T is called the nullity of T.

Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator defined by $T(x,y,z) = (x+2y, y-z, x+2z)$.
Find the rank and the nullity of T.

Solⁿ

$$T(x,y,z) = (x+2y,\ y-z,\ x+2z)$$

The images of the generators of $\mathbb{R}^3$ generate the Im T (Image of T).

$$T(1,0,0) = (1,0,1)$$
$$T(0,1,0) = (2,1,0)$$
$$T(0,0,1) = (0,-1,2)$$

Form the matrix whose rows are the generators of Im T and row reduced to echelon form:

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix} \quad r_2' = r_2 - 2r_1$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad r_3' = r_2 + r_3$$

Thus $\{(1,0,1), (0,1,-2)\}$ is a basis of Im T and dimension $\dim(\text{Im } T) = 2$ i.e. Rank of $T = 2$.

Now, we have to find out (x, y, z) such that

$$T(x,y,z) = (0,0,0)$$

i.e.,

$$\left.\begin{aligned} x + 2y \quad &= 0 \\ y - z &= 0 \\ x \quad + 2z &= 0 \end{aligned}\right\} \longrightarrow ①$$

Reduce the system to echelon form by elementary row transformations.

$Eq ③ = Eq ③ - Eq ①$

$$\left.\begin{aligned} x + 2y \quad &= 0 \\ y - z &= 0 \\ -2y + 2z &= 0 \end{aligned}\right\} \longrightarrow ②$$

$Eq - ③ = Eq ③ / (-2)$

$$\left.\begin{aligned} x + 2y \quad &= 0 \\ y - z &= 0 \\ y - z &= 0 \end{aligned}\right\} \longrightarrow ③$$

Since, 2^{nd} and 3^{rd} equations are identical, we can disregard one of them.

Thus, $\left.\begin{aligned} x + 2y \quad &= 0 \\ y - z &= 0 \end{aligned}\right\} \rightarrow ④$

This system is in echelon form and it has two eq^{ns} in three unknowns, hence the system has $(3-2) = 1$ free variable which is z.

let, $z = -1$, then $y = -1$ and $x = 2$.

Therefore, $\{(2,-1,-1)\}$ is a basis of $\ker(T)$ and

$$\dim \ker(T) = 1$$

$\therefore$ nullity of $T = 1$

[N.B. $\rightarrow$ $\text{Rank}(T) + \text{Nullity}(T) = \dim \mathbb{R}^3$]