Linear Equations

Def^n. An ~~equat~~ expression of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \quad \text{--- (1)}$$

where the $a_i, b \in \mathbb{R}$ and x_i are indeterminants or unknowns or variables, is called a linear equation over the real field $\mathbb{R}$.

The scalars a_i are called the coefficients of the x_i respectively and b is called the constant term of the equation. A set of values for the unknowns, say

$x_1 = \lambda_1, x_2 = \lambda_2, \ldots, x_n = \lambda_n$ is a solution of (1) if the set of values satisfy the equation (1). i.e. $a_1\lambda_1 + a_2\lambda_2 + \cdots + a_n\lambda_n = b$, is true. and then the set of solution $u = (\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Ex: Consider the equation ~~x~~

$x + 2y - 4z + w = 3$, then the 4-tuple $u = (3, 2, 1, 0)$ is a solution of the equation, since $3 + 2\cdot2 - 4\cdot1 + 0 = 3$ or $3 = 3$, is a true statement. However, the 4-tuple $v = (1, 2, 4, 5)$ is not a solution of the equation since $1 + 2\cdot2 - 4\cdot4 + 5 = 3$
$\Rightarrow -6 = 3$ is not a true statement.

Solutions of (1) can be easily described is obtained. There are three cases:

Case ①: One of the coefficients in ① is not zero, say $a_1 \neq 0$, Then we can rewrite the equation as

$$a_1 x_1 = b - a_2 x_2 - a_3 x_3 - \cdots - a_n x_n$$

or $$x_1 = a_1^{-1} b - a_1^{-1} a_2 x_2 - \cdots - a_1^{-1} a_n x_n$$

By arbitrarily assigning values to the unknowns $x_2, \ldots, x_n$, we obtain a value for x_1, these values form a solution of the equation ①. Furthermore, every solution of the equation can be obtained in this way. In particular, the linear equation in one variable $ax = b$, with $a \neq 0$ has the unique solution $x = a^{-1}b$.

Ex: $2x - 4y + z = 8 \Rightarrow x = 4 + 2y - \frac{1}{2}z$, If let $y = 3$, $z = 2$, then $x = 4 + 6 - 2\cdot\frac{1}{2} = 9$, i.e the 3-tuple $u = (9, 3, 2)$ is a solution of the equation.

Case-II. All the coefficients in ① are zero, but the constant is not zero, i.e the equation is of the form $0x_1 + 0x_2 + \cdots + 0x_n = b$, with $b \neq 0$

— Then the equation has no solution.

Case-III. All the coefficients in (1) are zero, and the constant ~~are~~ is also zero. i.e the equation is of the form $0x_1 + 0x_2 + \cdots + 0x_n = 0$

— Then every n-tuple of scalars in $\mathbb{R}$ is a solution of the equation.

Matrix of coefficients of a system of equations:

Let us consider, m linear equations in n unknowns as follows

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \text{---} (2)$$

These m equations can be expressed as a single matrix equation

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

or simply, $AX = B$ ——— (3)

where $A = (a_{ij})$, $X = (x_j)$ and $B = (b_i)$

$i = 1, 2, \ldots, m$, $j = 1, \ldots, n$

The system of linear equation (2) is called non-homogeneous.

Thus every solution of the system (2) is a solution of the matrix equation (3) and vice versa.

The system of linear equation (2) is said to be homogeneous if the constants $b_1, b_2, \ldots, b_m$ are all zero

i.e. The system of linear equation

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}\right\} \text{---} (4)$$

or $AX = 0$, where $A = \begin{pmatrix} \quad \end{pmatrix}$, $X = \begin{pmatrix} \quad \end{pmatrix}$, o is null matrix.

The above system always has a solution, namely the zero n-tuple $0 = (0, 0, \ldots, 0)$ called the zero or trivial solution. Any other solution, if it exists, is called a nonzero or non-trivial solution. ** If the above systems of equations (2) and (4) possess at least one solution set, then the equations are said to be consistent, otherwise they are said to be inconsistent. The fundamental relationship between the systems (2) and (4) follows

__Theorem.__ Suppose u is a particular solution of the non-homogeneous system (2) and suppose W is the general solution of the homogeneous system (4). Then

$$u + W = \{u + w : w \in W\}$$

is the general solution of the non-homogeneous system (2).

__Theorem-1.__ If the coefficient matrix A is non-singular, then the system of n-homogeneous linear equations in n unknowns has only trivial solution.

__Proof.__ Since A is non-singular, then A^{-1} exists. Now, the system of linear homo. $AX = 0$, then $A^{-1}AX = A^{-1}0$.

$\Rightarrow X = 0$ i.e. $x_1 = 0, x_2 = 0, \ldots, x_n = 0$.

Since A^{-1} is unique, the trivial solution is the only ~~soul~~ solution.

Theorem. If x_1 and x_2 are any two solutions of $AX=0$, then their linear combination $k_1x_1+k_2x_2$ is also a solution of $AX=0$, where k_1 and k_2 are any nonzero scalars.

Proof: Since x_1 and x_2 are solution of $AX=0$, then $Ax_1=0$ and $Ax_2=0$, we have

$$A(k_1x_1+k_2x_2) = k_1Ax_1+k_2Ax_2 = k_1\cdot 0+k_2\cdot 0=0$$

Hence $k_1x_1+k_2x_2$ is a solution of $AX=0$.

Nature of the solution of $AX=0$ i.e nature of the solution of the system of homogeneous eq Linear equations:

Case-I. If $r=n$, then in this case equations $AX=0$ ~~have~~ has only the trivial solution or the ~~only~~ zero solution, where r is the rank of A.

Case-II. If $r<n$, then the equations $AX=0$ will have infinitely many solutions and here ~~n-r...~~ we have $n-r$ free variables.

Solve the system of equations

$$x_1-x_2+x_3=0$$
$$x_1+2x_2-x_3=0$$
$$2x_1+x_2+3x_3=0$$

Soln We have the coefficient matrix

$$A=\begin{pmatrix}1&-1&1\\1&2&-1\\2&1&3\end{pmatrix}$$

Then $A\sim\begin{pmatrix}1&-1&1\\0&3&-2\\0&3&1\end{pmatrix}$ $R_2'=R_2-R_1$, $R_3'=R_3-2R_1$

$$\sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & -2 \\ 0 & 0 & 3 \end{pmatrix} \quad R_3' = R_3 - R_2$$

which is an Echelon form of the matrix. Here $\rho(A) = 3 = n$, the number of variables. Hence the given system of equation has only trivial solution or zero solution ie $x_1 = 0, x_2 = 0, x_3 = 0$.

* Solve the system of equations

$$x - 2y + z - w = 0$$
$$x + y - 2z + 3w = 0$$
$$4x + y - 5z + 8w = 0$$
$$5x - 7y + 2z - w = 0.$$

Soln. We can write the above system of equation in matrix form ie $AX = 0$

ie
$$\begin{pmatrix} 1 & -2 & 1 & -1 \\ 1 & 1 & -2 & 3 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Now
$$A = \begin{pmatrix} 1 & -2 & 1 & -1 \\ 1 & 1 & -2 & 3 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -2 & 1 & -1 \\ 0 & 3 & -3 & 4 \\ 0 & 9 & -9 & 12 \\ 0 & 3 & -3 & 4 \end{pmatrix} \quad \begin{matrix} R_2' = R_2 - R_1 \\ R_3' = R_3 - 4R_1 \\ R_4' = R_4 - 5R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & -2 & 1 & -1 \\ 0 & 3 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} R_3' = R_3 - 3R_2 \\ R_4' = R_4 - R_2 \end{matrix}$$

which is an Echelon form of the matrix A.

Here $\rho(A) = 2 < n = 4$, the no.s of variables.

$\therefore$ the no. of free variables $= 4-2 = 2$.

Then we have

$$x - 2y + z - w = 0$$
$$3y - 3z + 4w = 0$$

Taking $z = k_1$ and $w = k_2$, then

$$y = \frac{1}{3}(3k_1 - 4k_2)$$

and $x = \frac{2}{3}(3k_1 - 4k_2) - k_1 + k_2$

$$= k_1 - \frac{5}{3}k_2$$

where k_1 and k_2 are arbitrary constants.

Since we can assign arbitrarily values of k_1 and k_2, then given system of equations have infinitely many solutions.

Non-homogeneous Linear equations

Let us consider m linear equation in n unknown as follows

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right\} (A)$$

We can write this system in matrix form $AX = B$, where A the coefficient matrix $A = \begin{pmatrix} \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

$$B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

The matrix

$$(A:B) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

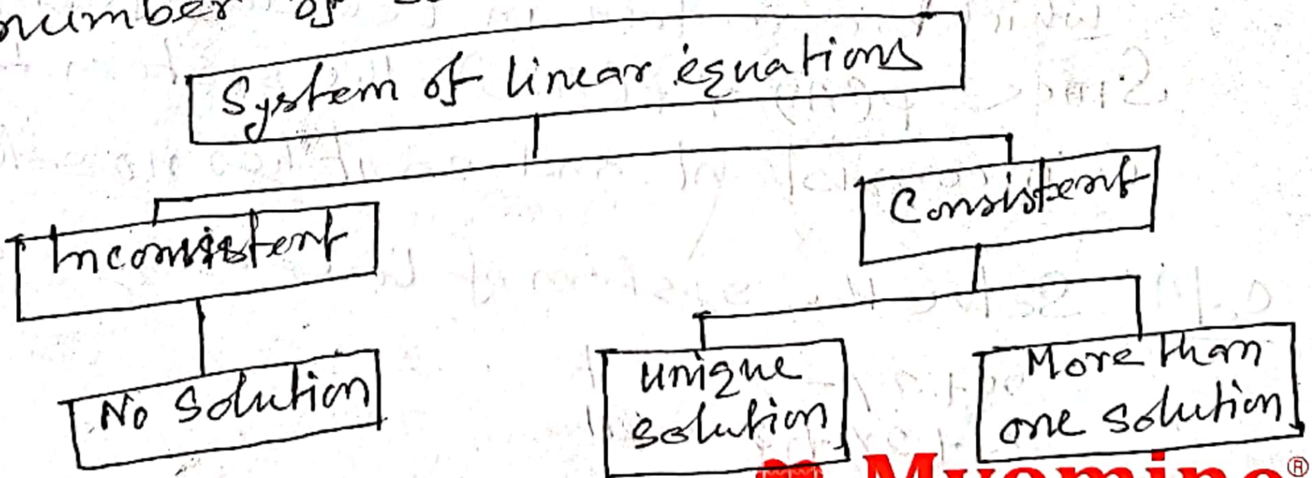
is called the augmented matrix of (A).

Nature of the solution of $AX = B$:

<u>Case-I.</u> If $\rho(A) \neq \rho(A:B)$, then the system of linear non-homogeneous will be inconsistent and it has no solution.

<u>Case-II.</u> If $\rho(A) = \rho(A:B) = n$, the number of variables, then the system has a unique solution.

<u>Case-III.</u> If $\rho(A) = \rho(A:B) < n$, the number of variables, then the system has an infinite number of solution.



* Solve the system of equations

$$2x + y - 2z + 3w = 1$$
$$3x + 2y - z + 2w = 4$$
$$3x + 3y + 3z - 3w = 5$$

Sol: We have the augmented matrix

$$(A:B) = \left(\begin{array}{cccc:c} 2 & 1 & -2 & 3 & 1 \\ 3 & 2 & -1 & 2 & 4 \\ 3 & 3 & 3 & -3 & 5 \end{array}\right)$$

$R_2' = 2R_2 - 3R_1$ & $R_3' = 2R_3 - 3R_1$

$$\sim \left(\begin{array}{cccc:c} 2 & 1 & -2 & 3 & 1 \\ 0 & 1 & 4 & -5 & 5 \\ 0 & 3 & 12 & -15 & 7 \end{array}\right)$$

$R_3' = R_3 - 3R_2$

$$\sim \left(\begin{array}{cccc:c} 2 & 1 & -2 & 3 & 1 \\ 0 & 1 & 4 & -5 & 5 \\ 0 & 0 & 0 & 0 & -8 \end{array}\right)$$

which is a matrix in Echelon form.

Since $\rho(A) \neq \rho(A:B)$, the system ~~has~~ is inconsistent. and so it has no solution.

Soln: Solve the system of linear equations

$$x + 2y - 3z = 4$$
$$x + 3y + z = 11$$
$$2x + 5y - 4z = 13$$
$$2x + 6y + 2z = 22$$

Sol^n We have the matrix form

$$AX = B,$$

where $A = \begin{pmatrix} 1 & 2 & -3 \\ 1 & 3 & 1 \\ 2 & 5 & -4 \\ 2 & 6 & 2 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $B = \begin{pmatrix} 4 \\ 11 \\ 13 \\ 22 \end{pmatrix}$

and the augmented matrix

$$(A:B) = \left(\begin{array}{ccc:c} 1 & 2 & -3 & 4 \\ 1 & 3 & 1 & 11 \\ 2 & 5 & -4 & 13 \\ 2 & 6 & 2 & 22 \end{array}\right)$$

$R_2' = R_2 - R_1$, $R_3' = R_3 - 2R_1$, $R_4' = R_4 - 2R_1$

$$\sim \left(\begin{array}{ccc:c} 1 & 2 & -3 & 4 \\ 0 & 1 & 4 & 7 \\ 0 & 1 & 2 & 5 \\ 0 & 2 & 8 & 14 \end{array}\right)$$

$R_3' = R_2 - R_3$, $R_4' = R_4 - 2R_2$

$$\sim \left(\begin{array}{ccc:c} 1 & 2 & -3 & 4 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

which is Echelon form

Since $\rho(A) = \rho(A:B) = 3$, the number of variables, the system is consistent and so it has a unique solution.

Then we have $2z = 2$ ie $z = 1$

$y + 4z = 7$ ie $y = 7 - 4 = 3$

$x + 2y - 3z = 4$

ie $x = 4 - 6 + 3 = 1$

Thus $x=1, y=3$ and $z=1$ or in the other words, the 3-triple $(1,3,1)$ is the unique solution of the system. —x—

A Solve the system of linear equations

$$x+2y-2z+3w=2$$
$$2x+4y-3z+4w=5$$
$$5x+10y-8z+11w=12$$

Sol^n We have the matrix form $AX=B$,

where $A=\begin{pmatrix} 1 & 2 & -2 & 3 \\ 2 & 4 & -3 & 4 \\ 5 & 10 & -8 & 11 \end{pmatrix}$ $X=\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ $B=\begin{pmatrix} 2 \\ 5 \\ 12 \end{pmatrix}$

and the augmented matrix

$$(A:B)=\left(\begin{array}{cccc:c} 1 & 2 & -2 & 3 & 2 \\ 2 & 4 & -3 & 4 & 5 \\ 5 & 10 & -8 & 11 & 12 \end{array}\right)$$

$R_2' = R_2 - 2R_1$; $R_3' = R_3 - 5R_1$

$$\sim \left(\begin{array}{cccc:c} 1 & 2 & -2 & 3 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 2 & -4 & 2 \end{array}\right)$$

$R_3' = R_3 - 2R_2$

$$\sim \left(\begin{array}{cccc:c} 1 & 2 & -2 & 3 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

which is Echelon form.

Scanned with CamScanner

Since $\rho(A) = \rho(A:B) < 4$, the number of variables, then the system is consistent and it has an infinite number of solutions.

Here $n - r = 4 - 2 = 2$ free variables, y and w and so a particular solution can be obtained by giving y and w any values.

$$\therefore \quad \cancel{z=2} \quad x + 2y - 2z + 3w = 2$$
$$z - 2w = 1.$$

Let $\cancel{z}\, w = 1$ and $y = -2$, then $z = 1 + 2 = 3$

$x - 4 - 6 + 3 = 2$ i.e. $x = 9$

Thus $x = 9, y = -2, z = 3$ and $w = 1$ is a particular or the 4-tuple $(9, -2, 3, 1)$ is a particular solution of the system.

* Investigate for what values of λ and μ, the system of equations

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + \lambda z = \mu$$

have (i) no solution (ii) a unique solution (iii) ~~an infinite~~ more than one solution.

Soln: We have the augmented matrix

$$(A:B) = \begin{pmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & \lambda & : & \mu \end{pmatrix} \xrightarrow[R_3' = R_3 - R_1]{R_2' = R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & \lambda - 1 & : & \mu - 6 \end{pmatrix}$$

$$R_3' = R_3 - R_2 \begin{pmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & \lambda-3 & : & \mu-10 \end{pmatrix}$$

which is a matrix in Echelon form.

For no solution. The system of equations will be inconsistent i.e. it has no solution if $\lambda = 3$ ~~both~~ $\mu \neq 10$.

For unique solution. The system of equations has a unique solution if $\rho(A) = \rho(A:B) = n$, the number of variable. Thus the system of equations has a unique solution if $\lambda \neq 3$.

For more than one solution. If $\lambda = 3, \mu = 10$ then we have $\rho(A) = \rho(A:B) = 2 < 3$, the no. of variables. Thus in this case the system of equation has more than one solution.