COLLEGE

LINEAR ALGEBRA

THEORY OF MATRICES

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$$\begin{bmatrix} a_{11} & a_{12} & \cdots\cdots\cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots\cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots\cdots\cdots\cdots & \cdots \\ a_{m1} & a_{m2} & \cdots\cdots\cdots\cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\mathbb{R}^3 = \{(a, b, c) | a, b, c \in \mathbb{R}\}$$

$$T : U \to V$$

$$T(u_1 + u_2) = T(u_1) + T(u_2)$$

$$T(\alpha u) = \alpha T(u)$$

$$\alpha \in \mathbb{R} \ \& \ u_1, u_2, u \in U$$

$$(AB)^+ = B^*(BB^*)^{-1} (A^*A)^{-1}A^*$$

if $(BB^*)^{-1}$ & $(A^*A)^{-1}$ exist

NAHAR BOOK DEPOT & PUBLICATIONS
BANGLABAZAR, DHAKA

List of books written by the author

Professor Md. Abdur Rahaman.

1. College Linear Algebra
2. College Modern Algebra
3. College Higher Algebra
   (Basic Algebra & Fundamentals of Mathematics)
4. College Mathematical Methods (Volume one)
   (Special Functions & Vector Analysis)
5. College Mathematical Methods (Volume two)
   (Integral Transforms & Boundary Value Problems)
6. কলেজ উচ্চতর বীজগণিত
   (Basic Algebra & Fundamentals of Mathematics)
7. উচ্চ মাধ্যমিক জ্যামিতি ও ক্যালকুলাস (এন. সি. টি. বি কর্তৃক অনুমোদিত)
8. Ideal Solution of College Linear Algebra
9. কলেজ উচ্চতর বীজগণিতের সম্পূর্ণ সমাধান।
10. উচ্চ মাধ্যমিক জ্যামিতি ও ক্যালকুলাসের আদর্শ সমাধান

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CHAPTER ONE

SYSTEMS OF LINEAR EQUATIONS

1.1 Introduction to systems of linear equations

Matrix theory is more generally known as **Linear Algebra** which is originated in the study of systems of linear equations in several unknowns (or variables) and in the attempt to find general methods for their solutions. An equation in two or more variables (or unknowns) is **linear** if it contains no terms of second degree or greater, that is, if it contains no products or powers of the variables or roots of the variables. All variables occur only to the first power. and do not appear as arguments for trigonometric, logarithmic, or exponential functions.

A straight line in the cartesion xy-plane can be represented algebraically by an equation of the form $ax + by = c$ where a, b and c are real constants (or numbers) and x and y are variables. An equation of this kind is called a **linear equation** in the variables x and y. Similarly, $ax + by + cz + d = o$ is a **linear equation** in three variables x, y and z which represents a plane in three dimensional space. In general, an equation is called **linear** if it is of the form $a_1 x_1 + a_2 x_2 + \dots \dots + a_n x_n = b$ (1) where $a_1, a_2, \dots \dots, a_n$ and b are real numbers and $x_1, x_2, \dots \dots, x_n$ are n variables which are to be determined.

If $b = o$, then (1) is called a **homogeneous linear equation** and if $b \neq o$ then (1) is called a **non-homogeneous linear equation**.

Examples of linear equations.

(i) $y - mx = o$ which is a homogeneous linear equation representing the straight line passing through the origin.

(ii) $2x + 3y = 5$ which is a non-homogeneous linear equation representing a straight line not passing through the origin.

(iii) $x + 2y + 5z = 20$ which is a non-homogeneous linear equation representing a plane.

(iv) $x_1 - x_2 - 4x_3 - 2x_4 = 3$ (Non-homogeneous)

(v) $x_1 + x_2 + \dots \dots + x_n = 1$ (Non-homogeneous).

Examples of non-linear equations

(i) $2x^2 + 3y = 1$

(ii) $x - xy = 2$

(iii) $x^2 + y^2 + 4x + 2y = 4$

(iv) $ax^2 + 2hxy + by^2 = o$

(v) $6x^2 + 13xy + 6y^2 - 5x - 5y + 1 = o$

Equation

(i) represents a parabola,

(ii) represents a hyperbola,

(iii) represents a circle,

(iv) represents a pair of straight lines passing through the origin and

(v) represents a pair of straight lines not passing through the origin.

Let IR be the set of real numbers. Then a **solution** of the linear equation $a_1x_1 + a_2x_2 + \dots \dots + a_nx_n = b$ is any n-tuple $(\alpha_1, \alpha_2, \dots \dots, \alpha_n)$ of elements of IR such that the equation is satisfied when we substitute $x_1 = \alpha_1, x_2 = \alpha_2, \dots \dots, x_n = \alpha_n$. The set of all such solutions of this linear equation is called the **solution set.**

Now we consider the following two linear equations :

$$\left.\begin{aligned} a_1 x + b_1 y &= c_1 \\ a_2 x + b_2 y &= c_2 \end{aligned}\right\} \checkmark$$

If we interpret x, y as coordinates in the xy–plane, then each of the above two linear equations represents a straight line and (α, β) is a solution of the above two equations if and only if the point P with coordinates α, β lies on both lines. Hence there are three possible cases :

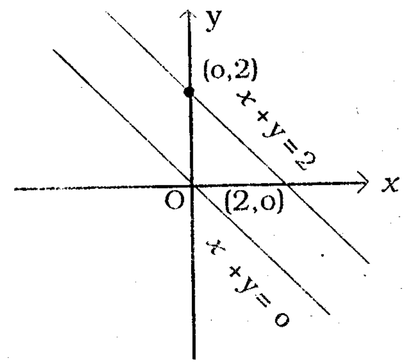
Case 1 No solution if the lines are parallel

case II Precisely one solution if they intersect

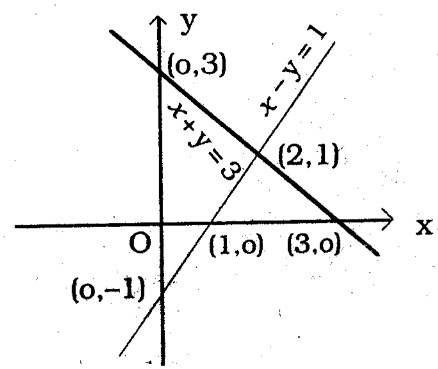
case III Infinitely many solutions if they coincide.

These cases are illustrated by the following examples :

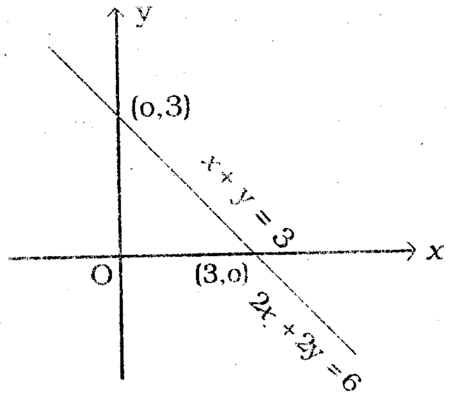
Example 1. The linear system $\left.\begin{aligned} x + y &= 2 \\ x + y &= o \end{aligned}\right\}$ has no solution, since the lines represented by these two linear equations are parallel.



Example 2. The linear system $\left.\begin{aligned} x + y &= 3 \\ x - y &= 1 \end{aligned}\right\}$ has only one solution, since the lines represented by these two equations intersect at (2, 1).



Example 3. The linear system $\left.\begin{array}{l} x+y=3 \\ 2x+2y=6 \end{array}\right\}$ has infinitely many solutions, since the lines represented by these equations coincide.



1.2 Degenerate and non-degenerate linear equations.

The general linear equation $a_1x_1 + a_2x_2 + \dots \dots + a_nx_n = b$ is also called **non-degenerate linear equation.**

A linear equation is said to be **degenerate** if it has the form $ox_1 + ox_2 + \dots\dots + ox_n = b$. That is, if every coefficient of the variable is equal to zero. The solution of such a degenerate linear equation is as follows :

(i) If the constant $b \neq o$, then the above equation has no solution.

(ii) If the constant $b = o$, then every vector $u = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is a solution of the above equation.

1.3 Solution of a non-homogeneous system of linear equations :

A system of linear equations (or a set of m simultaneous linear equations) in n variables (or unknowns) $x_1, x_2, \dots \dots, x_n$ is a set of equations of the form

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots \dots + a_{2n}x_n = b_2 \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots \dots + a_{mn}x_n = b_m \end{array}\right\} \quad (1)$$

where the coefficients a_{ij}, $i = 1, 2, \dots, m$
$j = 1, 2, \dots, n$

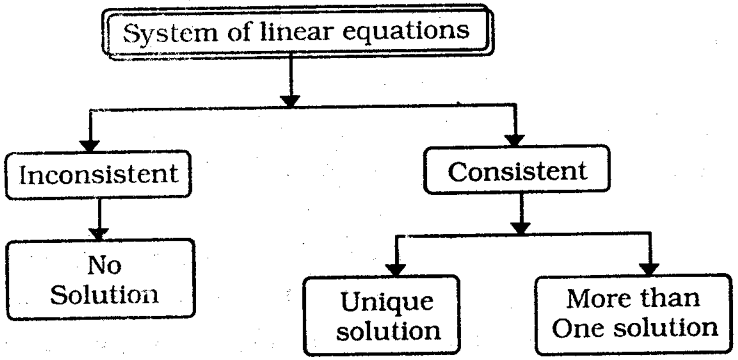
of the variables and the free terms b_i, $i = 1, 2, \dots, m$ are real numbers taken from IR, the set of real numbers. If the b_i are all zero, then the system (1) is called a **homogeneous system.** If at least one b_i is not zero, then the system (1) is called a **non-homogeneous system.** A sequence of numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ is called **solution** of the system of linear equations given by (1) if $x_1 = \alpha_1, x_2 = \alpha_2, \dots \dots, x_n = \alpha_n$ is a solution of every equation in the system (1).

A **solution vector** of (1) is a vector x whose components constitute a solution of (1). If (1) is homogeneous, it has at least the **trivial (or zero) solution** $x_1 = o, x_2 = o, \dots, x_n = o$.

If $x = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is a solution of the homogeneous system and at least one $\alpha_i \neq o$, then it is called a **non-zero** or **non-trivial solution** of the homogeneous system.

A system of linear equations is called **consistent** if it has at least one solution and **inconsistent** if it has no solution. A consistent system is called **determinate** if it has a unique solution and **indeterminate** if it has more than one solution. An indeterminate system of linear equations always has an infinite number of solutions.

Two systems of linear equations are called **equivalent** if every solution of the first system is a solution of the second and conversely (vice versa).



A straightforward method of solution, known as **Gaussian elimination** involves a successive "Whitling away" of the variables in order to isolate their values. This method is based on the following three elementary operations which alter the form of the equations, but not the solutions :

(i) Interchange a pair of equations.

(ii) Multiplying an equation through by a non–zero number.

(iii) Adding a multiple of one equation to another equation or, equivalently, if we consider a system of m linear equations in the n unknowns $x_1, x_2, \ldots, x_n$ given by (1), we can reduce it to a simpler system as follows :

Prosess 1

(i) Interchange equations so that the first unknown x_1 has a non-zero co-efficient in the first equation i, e, $a_{11} \neq 0$.

(ii) For each i>1, apply the operation

$L_i \rightarrow -a_{i1}L_1 + a_{11}L_i$

That is, replace the linear equation L_i by the equation obtained by multiplying the first equation L_1 by $-a_{i1}$, and the ith equation L_i by a_{11} and then adding. We then obtain the following system which is equivalent to the system (1).

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1 \\ a'_{2j_2}x_{j_2} + a'_{2j_2+1}x_{j_2+1} + \ldots + a'_{2n}x_n = b'_2 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ a'_{mj_2}x_{j_2} + a'_{mj_2+1}x_{j_2+1} + \ldots + a'_{mn}x_n = b'_m \end{array}\right\} \quad (2)$$

where $a_{11} \neq 0$ and x_{j_2} denotes the first unknown with $a'_{2j_2} \neq 0$ in an equation except the first. Here i> 1 and so $x_{j_2} \neq x_1$.

It is to be noted that the system (2) of equations excluding the first equation, form a subsystem which has fewer equations and fewer unknowns than the original system (1). Repeating the above process with each new smaller subsystem we obtain by induction that the system (1) is either **inconsistent** or is reducible to an equivalent system of the following form which is known as **echelon form :**

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1 \\ a'_{2j_2}x_{j_2} + a'_{2j_2+1}x_{j_2+1} + \ldots + a'_{2n}x_n = b'_2 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ a'_{rj_2}x_{j_2} + a'_{rj_2+1}x_{j_2+1} + \ldots + a'_{m}x_n = b'_r \end{array}\right\} \quad (3)$$

where $1 < _{j_2} < \ldots <_{j_r}$ and the leading co-efficients are not zero i. e, $a_{11} \neq 0, a'_{2j_2} \neq 0, \ldots, a_{rj_r} \neq 0$.

Definition : In reduced echelon form the unknowns x_i which do not appear at the beginning of any equation ($i \neq 1, j_2, \ldots j_r$) are known as **free variables.** We also note that

(i) if an equation $0x_1 + 0x_2 + \ldots + 0x_n = b$, $b \neq 0$ occurs, then the system is **inconsistent** and has no solution.

(ii) if an equation $0x_1 + 0x_n + \ldots + 0x_n = 0$ occurs, then the equation can be deleted without affecting the solution.

Process 2 : Consider the following system of m linear equations (or set of m simultaneous linear equations) in n unknowns $x_1, x_2, \ldots, x_n$

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \dots \quad \dots \quad \dots \quad \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{array}\right\} \quad (1)$$

We reduce the system (1) to a simpler system as follows :

First step Eliminations of x_1 from the second, third, ..., mth equations. We may assume that the order (rule) of the equations and the order (rule) of the unknowns in each equation such that $a_{11} \neq 0$. The variable x_1 can then be eliminated from the second, third,...,mth equations by subtracting

$\frac{a_{21}}{a_{11}}$ times the first equation from the second equation

$\frac{a_{31}}{a_{11}}$ times the first equation from the third equation

... ...

$\frac{a_{m1}}{a_{11}}$ times the first equation from the mth equation

This gives a new system of equations of the form

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \qquad c_{22}x_2 + \dots + c_{2n}x_n = b^*_2 \\ \qquad \dots \quad \dots \quad \dots \\ \qquad c_{m2}x_2 + \dots + c_{mn}x_n = b^*_m \end{array}\right\} \quad (2)$$

Any solution of the system (1) is a solution of the system (2) and conversely.

Second step Elimination of x_2 from the third, fourth, ..., mth equations in the system (2). If the co-efficients $c_{22}, c_{23}, \dots, c_{mn}$, in the system (2) are not all zero, we may assume that the order (rule) of the equations and the unknowns such that $c_{22} \neq 0$. Then we may eliminate x_2 from the third, fourth, ..., mth equations of the system (2) by subtracting

$\frac{c_{32}}{c_{22}}$ times the second equation from the third equation

$\frac{c_{42}}{c_{22}}$ times the second equation form the fourth equation

... ...

$\frac{c_{m2}}{c_{22}}$ times the second equation form the mth equation.

The further steps are now obvious. In the third step we eliminate x_3, in the fourth step we eliminate x_4 etc.

This process will terminate only when no equations are left or when the co-efficients of all the unknowns in the remaining equations are zero. We have a system of the form.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \qquad c_{22}x_2 + \dots + c_{2n}x_n = b^*_2 \\ \qquad \dots \quad \dots \quad \dots \\ \qquad c_{rs}x_r + \dots + k_m x_n = \bar{b}_r \\ \qquad\qquad o = \bar{b}_{r+1} \\ \qquad\qquad \dots\dots\dots \\ \qquad\qquad o = \bar{b}_m \end{array}\right\} \quad (3)$$

where $r \leq m$. We see that there are three possible cases

(i) No solution if $r < m$ and one of the numbers $\bar{b}_{r+1}, \dots, \bar{b}_m$ is not zero.

(ii) Precisely one solution if $r = n$ and $\bar{b}_{r+1}, \dots, \bar{b}_m$ if present are zero.

This solution is obtained by solving the nth equation of the system (3) for x_n, then the (n–1) th equation for x_{n-1} and so on up to the line.

(iii) Infinitely many solutions if $r < n$ and $\bar{b}_{r+1}, \dots, \bar{b}_m$ if present, are zero. Then any of these solutions is obtained by choosing values at pleasure for the unknowns $x_{r+1}, \dots, x_n$ solving the rth equation for x_r then (r–1) th equation for x_{r-1}, and so on up to the line.

Thus we obtain the equivalent system (with the same solutions as the system (1).

$$\left.\begin{array}{r} x_1 + 2x_2 - x_3 = 2 \\ x_2 - x_3 = 1 \\ x_2 - x_3 = 1 \end{array}\right\} \quad (2)$$

Since the second and the third equations of the new system (2) are identical, we can disregard any one of them. Hence we can simply write.

$$\left.\begin{array}{r} x_1 + 2x_2 - x_3 = 2 \\ x_2 - x_3 = 1 \end{array}\right\} \quad (3)$$

This system (3) is in echelon form and has two equations in three unknowns and so it has 3–2 = 1 free variable which is x_3 and hence it has an infinite number of solutions.

Let $x_3 = a$ (where a is arbitrary real number), then $x_2 = 1 + a$ and $x_1 = -a$, Thus the general solution is $x_1 = -a$, $x_2 = 1 + a$, $x_3 = a$, where a is any real number, Now a particular solution can be obtained by giving any value for a, Let a =1, then $x_1 = -1$, $x_2 = 2$, $x_3 = 1$ or, in other words, the 3–tuple (–1, 2, 1) is a particular solution of the given system.

Example 3. Solve the following system of linear equations:

$$\left.\begin{array}{l} 3x_1 - x_2 + x_3 = -2 \\ x_1 + 5x_2 + 2x_3 = 6 \\ 2x_1 + 3x_2 + x_3 = 0 \end{array}\right\}$$

Solution : Reduce the system to echelon form by the elementary operations. Interchange first and second equations. Then we obtain the equivalent system.

$$\left.\begin{array}{l} x_1 + 5x_2 + 2x_3 = 6 \\ 3x_1 - x_2 + x_3 = -2 \\ 2x_1 + 3x_2 + x_3 = 0 \end{array}\right\}$$

We multiply first equation by 3 and 2 and then subtract from the second and third equations respectively. Then we get the equivalent system.

$$\left.\begin{array}{r} x_1 + 5x_2 + 2x_3 = 6 \\ -16x_2 - 5x_3 = -20 \\ -7x_2 - 3x_3 = -12 \end{array}\right\}$$

We multiply second equation by $-\frac{7}{16}$ and then add with the third equation, Then we obtain equivalent system.

$$\left.\begin{array}{r} x_1 + 5x_2 + 2x_3 = 6 \\ -16x_2 - 5x_3 = -20 \\ -\frac{13}{16}x_3 = -\frac{13}{4} \end{array}\right\}$$

This system is in echelon form and has three equations in three unknowns. So the system has a unique solution. From the third equation we have $x_3 = 4$. Putting $x_3 = 4$ in the second equation, we get $x_2 = 0$, Again putting $x_2 = 0$ and $x_3 = 4$ in the first equation, we get $x_1 = -2$.

Thus $x_1 = -2$, $x_2 = 0$, $x_3 = 4$ or, in other words, the 3–tuple (–2, 0, 4) is the unique solution of the given system.

Example 4. Prove that the following system of linear equations is inconsistent :

$$\left.\begin{array}{l} x_1 + 2x_2 - 3x_3 = -1 \\ 5x_1 + 3x_2 - 4x_3 = 2 \\ 3x_1 - x_2 + 2x_3 = 7 \end{array}\right\}$$

Proof : Reduce the system to echelon form by elementary operations. We multiply first equation by 5 and 3 and then subtract from the second and third equations respectively. Then we obtain the equivalent system.

$$\left.\begin{array}{r} x_1 + 2x_2 - 3x_3 = -1 \\ -7x_2 + 11x_3 = 7 \\ -7x_2 + 11x_3 = 10 \end{array}\right\}$$

We subtract second equation from the third equation. Then we get the equivalent system.

$$\left.\begin{array}{r} x_1 + 2x_2 - 3x_3 = -1 \\ -7x_1 + 11x_3 = 7 \\ 0 + 0 = 3 \end{array}\right\}$$

or, equivalently, $\left.\begin{array}{r} x_1 + 2x_2 - 3x_3 = -1 \\ -7x_2 + 11x_3 = 7 \\ 0 = 3 \end{array}\right\}$

Thus the given system has been reduced to an echelon form and contains an equation of the form $0 = 3$ (which is not true) Hence the given system is inconsistent, i. e the system has no solution.

Example 5. Solve the following system of linear equations:

$$\left.\begin{array}{l} 2x_1 + 3x_2 + 5x_3 + x_4 = 3 \\ 3x_1 + 4x_2 + 2x_3 + 3x_4 = -2 \\ x_1 + 2x_2 + 8x_3 - x_4 = 8 \\ 7x_1 + 9x_2 + x_3 + 8x_4 = 0 \end{array}\right\} \quad (1)$$

Solution : Reduce the system to echelon form by the elementary operations. Interchange the first and third linear equations of the system (1). Then we have the equivalent system

$$\left.\begin{array}{l} x_1 + 2x_2 + 8x_3 - x_4 = 8 \\ 3x_1 + 4x_2 + 2x_3 + 3x_4 = -2 \\ 2x_1 + 3x_2 + 5x_3 + x_4 = 3 \\ 7x_1 + 9x_2 + x_3 + 8x_4 = 0 \end{array}\right\} \quad (2)$$

Let us represent the four linear equations of the system (2) by L_1, L_2, L_3 and L_4 respectively. Reduce the system to echelon form by the elementary operations. Eliminate x_1 from the second, third and fourth linear equations by the operations

$L_2 \to L_2 - 3L_1$, $L_3 \to L_3 - 2L_1$ and $L_4 \to L_4 - 7L_1$.

$$\begin{array}{rl} L_2 : 3x_1 + 4x_2 + 2x_3 + 3x_4 & = -2 \\ -3L_1 : -3x_1 - 6x_2 - 24x_3 + 3x_4 & = -24 \\ \hline L_2 - 3L_1 : -2x_2 - 22x_3 + 6x_4 & = -26 \\ L_3 : 2x_1 + 3x_2 + 5x_3 + x_4 & = 3 \\ -2L_1 : -2x_1 - 4x_2 - 16x_3 + 2x_4 & = -16 \\ \hline L_3 - 2L_1 : -x_2 - 11x_3 + 3x_4 & = -13 \\ L_4 : 7x_1 + 9x_2 + x_3 + 8x_4 & = 0 \\ -7L_1 : -7x_1 - 14x_2 - 56x_3 + 7x_4 & = -56 \\ \hline L_4 - 7L_1 : -5x_2 - 55x_3 + 15x_4 & = -56 \end{array}$$

Thus we obtain the equivalent system

$$\left.\begin{array}{l} x_1 + 2x_2 + 8x_3 - x_4 = 8 \\ -2x_2 - 22x_3 + 6x_4 = -26 \\ -x_2 - 11x_3 + 3x_4 = -13 \\ -5x_2 - 55x_3 + 15x_4 = -56 \end{array}\right\} \quad (3)$$

Divide the second linear equation of the system (3) by -2. Then we have the equivalent system

$$\left.\begin{array}{l} x_1 + 2x_2 + 8x_3 - x_4 = 8 \\ x_2 + 11x_3 - 3x_4 = 13 \\ -x_2 - 11x_3 + 3x_4 = -13 \\ -5x_2 - 55x_3 + 15x_4 = -56 \end{array}\right\} \quad (4)$$

Let us respresent the four linear equations of the system (4) by L'_1, L'_2, L'_3 and L'_4 respectively.

Apply the operations $L'_3 \to L'_3 + L'_2$ and $L'_4 \to L'_4 + 5L'_2$

$$\begin{array}{rl} L'_3 : -x_2 - 11x_3 + 3x_4 & = -13 \\ L'_2 : x_2 + 11x_3 - 3x_4 & = 13 \\ \hline L'_3 + L'_2 : 0 + 0 + 0 & = 0 \\ \text{i. e.} \quad 0 & = 0 \\ L_4 : -5x_2 - 55x_3 + 15x_4 & = -56 \\ 5L'_2 : 5x_2 + 55x_3 - 15x_4 & = 65 \\ \hline L'_4 + 5L_2 : 0 + 0 + 0 & = 9 \\ \text{i. e.} \quad 0 & = 9. \end{array}$$

Thus we obtain the equivalent system

$$\left.\begin{array}{l} x_1 + 2x_2 + 8x_3 - x_4 = 8 \\ x_2 + 11x_3 - 3x_4 = 13, \\ 0 = 0 \\ 0 = 9 \end{array}\right\} \quad (5)$$

Divide the fourth equation of the system (5) by 9 and interchange it with the third equation we get the new system.

$$\left.\begin{array}{l} x_1 + 2x_2 + 8x_3 - x_4 = 8 \\ x_2 + 11x_3 - 3x_4 = 13 \\ 0 = 1 \\ 0 = 0 \end{array}\right\} \quad (6)$$

The given system has been reduced to echelon form and contains an equation of the form $0 = 1$ which is not true; hence the given system is inconsistent i.e the system has no solution.

Theorem : Given the following system in echelon form :

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1\\
a_{2j_2}x_{j_2} + a_{2j_2+1}x_{j_2+1} + \dots + a_{2n}x_n &= b_2\\
\dots \quad \dots \quad \dots \quad \dots &\\
a_{rj_r}x_{j_r} + a_{rj_r+1}x_{j_r+1} + \dots + a_{rn}x_n &= b_r
\end{aligned}$$

where $1 < j_2 < \dots < j_r$ and $a_{11} \neq 0, a_{2j_2} \neq 0, \dots, a_{rj_r} \neq 0$

Then the solution of the given system is as follows :

There are two cases

(i) if $r = n$, i. e. if there are as many equations as unknowns then the system has a unique solution.

(ii) if $r < n$ i. e. if there are fewer equations than unknowns then we can arbitrarily assign values to $n-r$ free variables and obtain a solution of the system.

Proof : The proof is done by induction on the number of equations in the system. If $r = 1$ then we get the single linear equation $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$, where $a_1 \neq 0$.

The free variables are $x_2, \dots, x_n$. Let the free variables be assigned arbitrary values, say $x_2 = \alpha_2\ x_3 = \alpha_3, \ \dots\ , x_n = \alpha_n$.

Putting these values into the above equation and solving for x_1 we get

$$x_1 = \frac{1}{a_1}(b - a_2\alpha_2 - a_3\alpha_3 - \dots - a_n\alpha_n)$$

These values give a solution of the equation. Since putting these values we get.

$$a_1\left[\frac{1}{a_1}(b - a_2\alpha_2 - \dots - a_n\alpha_n)\right] + a_2\alpha_2 + \dots + a_n\alpha_n = b$$

or, $b = b$ which is a true statement.

Furthermore, if $r = n = 1$, then we have $ax = b$ where $a \neq 0$.

It is to be noted that $x = \frac{b}{a}$ is a solution, because $a\left(\frac{b}{a}\right) = b$ is true. Moreover if $x = \alpha$ is a solution, i. e. $a\alpha = b$, then $\alpha = \frac{b}{a}$. Thus the equation has a unique solution as desired.

New let us assume that $r > 1$ and the theorem is true for a system of $(r-1)$ equations.

$$\begin{aligned}
a_{2j_2}x_{j_2} + a_{2j_2+1}x_{j_2+1} + \dots + a_{2n}x_n &= b_2\\
\dots \quad \dots \quad \dots \quad \dots &\\
a_{rj_r}x_{j_r} + a_{rj_r+1}x_{j_r+1} + \dots + a_{rn}x_n &= b_r.
\end{aligned}$$

as a system in the unknowns $x_{j_2}, \dots, x_n$.

Now the system is in echelon form. By induction we can arbitrarily assign values to the $(n - j_2 + 1) - (r-1)$ free variables in the reduced system to obtain a solution

(say $x_{j_2} = \alpha_{j_2}, \dots, x_n = \alpha_n$)

As in case $r = 1$, these values and arbitrary values for the additional $j_2 - 2$ free variables (say $x_2 = \alpha_2, \dots, x_{j_2-1} = \alpha_{j_2-1}$). yield a solution of the equation with

$$x_1 = \frac{1}{a_{11}}(b_1 - a_{12}\alpha_2 - \dots - a_{1n}\alpha_n).$$

(Note that there are $(n-j_2 + 1)-(r-1) + (j_2-2) = n-r$ free variables). Furthermore, these values for $x_1, x_2, \dots\dots, x_n$ also satisfy the other equations, since in these equations, the co-efficients of $x_1, \dots, x_{j_2-1}$ are zero.

Now if $r = n$ then $j_2 = 2$. Thus by induction we obtain a unique solution of the subsystem and then a unique solution of the entire system. Hence the theorem is proved.

Example 6. Express the following system of linear equations in echelon form and solve it:

$$\left.\begin{aligned}
x_1 - x_2 + x_3 - x_4 + x_5 &= 1\\
2x_1 - x_2 + 3x_3 + 4x_5 &= 2\\
3x_1 - 2x_2 + 2x_3 + x_4 + x_5 &= 1\\
x_1 + x_3 + 2x_4 + x_5 &= 0
\end{aligned}\right\} \qquad (1)$$

Solution : Let us represent the four linear equations of the given system (1) by L_1, L_2, L_3 and L_4 respectively. Reduce the system to echelon form by the elementary operations. Eliminate x_1 from the second, third and fourth linear equations by the operations

$L_2 \to L_2 - 2L_1$, $L_3 \to L_3 - 3L_1$ and $L_4 \to L_4 - L_1$ respectively.

$$\begin{array}{ll}
L_2 : 2x_1 - x_2 + 3x_3 \qquad + 4x_5 & = 2 \\
-2L_1 : -2x_1 + 2x_2 - 2x_3 + 2x_4 - 2x_5 & = -2 \\
\hline
L_2 - 2L_1 : x_2 + x_3 + 2x_4 + 2x_5 & = 0 \\
L_3 : 3x_1 - 2x_2 + 2x_3 + x_4 + x_5 & = 1 \\
-3L_1 : -3x_1 + 3x_2 - 3x_3 + 3x_4 - 3x_5 & = -3 \\
\hline
L_3 - 3L_1 : x_2 - x_3 + 4x_4 - 2x_5 & = -2 \\
L_4 - L_1 : x_2 + 0 + 3x_4 + 0 & = -1.
\end{array}$$

Thus we obtain the following equivalent system ((with the same solutions as the system (1)).

$$\left.\begin{array}{l}
x_1 - x_2 + x_3 - x_4 + x_5 = 1 \\
x_2 + x_3 + 2x_4 + 2x_5 = 0 \\
x_2 - x_3 + 4x_4 - 2x_5 = -2 \\
x_2 \qquad + 3x_4 = -1
\end{array}\right\} \qquad (2)$$

Let us represent the four linear equations of the system (2) by L_1', L_2', L_3', and L_4' respectively. Eliminate x_2 from the third and the fourth linear equations by the operations

$L_3' \to L_3' - L_2'$ and $L'_4 \to L'_4 - L'_2$

$$\begin{array}{ll}
L'_3 - L'_2 : -2x_3 + 2x_4 - 4x_5 & = -2 \\
L'_4 - L'_2 : -x_3 + x_4 - 2x_5 & = -1
\end{array}$$

Thus the system (2) reduces to

$$\left.\begin{array}{l}
x_1 - x_2 + x_3 - x_4 + x_5 = 1 \\
x_2 + x_3 + 2x_4 - 2x_5 = 0 \\
-2x_3 + 2x_4 - 4x_5 = -2 \\
-x_3 + x_4 - 2x_5 = -1
\end{array}\right\} \qquad (3)$$

Dividing the third equation of the system (3) by 2 we get $-x_3 + x_4 - 2x_5 = -1$, which is identical with the fourth equation. So we can disregard one of them.

Hence the system (3) reduces to

$$\left.\begin{array}{l}
x_1 - x_2 + x_3 - x_4 + x_5 = 1 \\
x_2 + x_3 + 2x_4 + 2x_5 = 0 \\
-x_3 + x_4 - 2x_5 = -1
\end{array}\right\} \qquad (4)$$

Multiplying the third equation of the system (4) by -1 we get

$$\left.\begin{array}{l}
x_1 - x_2 + x_3 - x_4 + x_5 = 1 \\
x_2 + x_3 + 2x_4 + 2x_5 = 0 \\
x_3 - x_4 + 2x_5 = 1
\end{array}\right\} \qquad (5)$$

Now the system is in echelon form and there are only three equations in the five unknowns; hence the system has an infinite number of solutions and $5 - 3 = 2$ free variables. Since the three equations begin with the three unknowns x_1, x_2 and x_3 repectively, the other two unknowns x_4 and x_5 are free variables which may have any real values desired. To find the general solution let us say $x_4 = a$ and $x_5 = b$ where a and b are any real numbers. Putting these values in the third equation of the system (5) we get $x_3 = 1 + a - 2b$. Putting the values of x_3, x_4 and x_5 in the second equation we get

$$x_2 + 1 + a - 2b + 2a + 2b = 0$$

$$\text{or, } x_2 = -(1 + 3a)$$

Again, putting the values of x_2, x_3, x_4 and x_5 in the first equation of the system (5) we get

$$x_1 + 1 + 3a + 1 + a - 2b - a + b = 1$$

$$\text{or, } x_1 = -1 - 3a + b.$$

Hence the general solution is $x_1 = -1 - 3a + b$, $x_2 = -(1 + 3a)$, $x_3 = 1 + a - 2b$, $x_4 = a$, $x_5 = b$ where a and b are any real numbers.

1.4 Solution of a system of homogeneous linear equations.

A system of linear equations is said to be **homogeneous** if all the constant terms $b_1, b_2, \dots b_m$ of the non-homogeneous system are zero; that is, the system has the form.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0 \\ \dots \quad \dots \quad \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0 \end{array}\right\} \dots (1)$$

Every homegeneous system of linear equations is consistent Since $x_1 = 0, x_2 = 0, \dots, x_n = o$ is always a solution of the system. This solution is called the **trivial solution.** If the other solutions exist, they are called the **non-trivial solutions.** Thus the above homogeneous system can always be reduced to an equivalent homogeneous system in echelon form:

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = 0 \\ a_{2j_2}x_{j_2} + a_{2j_2+1}x_{j_2+1} + \dots + a_{2n}x_n = 0 \\ \dots \quad \dots \quad \dots \\ a_{rj_r}x_{j_r} + a_{rj_r+1}x_{j_r+1} + \dots + a_{rn}x_n = 0 \end{array}\right\} (2)$$

Hence we have the following two possibilities:

(i) if $r = n$ i. e. the number of equations is equal to the number of unknowns then the system has only the zero solution.

(ii) if $r < n$ i. e. the number of equations is less than the number of unknowns, then the system has non-zero solution.

Example 7. Find the non-trivial solution of the following system of homogeneous linear equations:

$$\left.\begin{array}{l} x_1 + x_2 + 2x_3 = 0 \\ x_2 + x_3 = 0 \\ -2x_1 + 3x_2 + x_3 = 0 \end{array}\right\} (1)$$

Solution: Reduce the system to echelon form by the elementary operations. Interchange second and third

equations of the given system (1). Then we get the equivalent system.

$$\left.\begin{array}{l} x_1 + x_2 + 2x_3 = 0 \\ -2x_1 + 3x_2 + x_3 = 0 \\ x_2 + x_3 = 0 \end{array}\right\} (2)$$

Let us represent the three linear equations of the system (2) by L_1, L_2 and L_3 respectively. Reduce the system to an echelon form by the elementary operations. Eliminate x_1 from the second equation by the operation. $L_2 \to L_2 + 2L_1$

$$\begin{array}{l} L_2 : -2x_1 + 3x_2 + x_3 = 0 \\ 2L_1 : 2x_1 + 2x_2 + 4x_3 = 0 \\ \hline L_2 + 2L_2 : 5x_2 + 5x_3 = 0 \end{array}$$

Thus we obtain the equivalent system

$$\left.\begin{array}{l} x_1 + x_2 + 2x_3 = 0 \\ 5x_2 + 5x_3 = 0 \\ x_2 + x_3 = 0 \end{array}\right\} (3)$$

Dividing the second equation of the system (3) by 5. we get the equivalent system

$$\left.\begin{array}{l} x_1 - x_2 + 2x_3 = 0 \\ x_2 + x_3 = 0 \\ x_2 + x_3 = 0 \end{array}\right\} (4)$$

Since the second and third equations are identical we can disregard one of them. Hence we have the equivalent system

$$\left.\begin{array}{l} x_1 + x_2 + 2x_3 = 0 \\ x_2 + x_3 = 0 \end{array}\right\} \text{which is in echelon form.}$$

In this echelon form there are only two equations in three unknowns, hence the system has an infinite number of non-zero solutions. The system has $3 - 2 = 1$ free variable which is x_3. Let $x_3 = a$. Thus the general solution is $x_1 = -a$, $x_2 = -a$, $x_3 = a$ or $(-a, -a, a)$, where a is any real number.

For particular solution, let $a = 1$. then $x_1 = -1$, $x_2 = -1$, $x_3 = 1$, or $(-1, -1, 1)$ is a particular solution of the system.

Example 8. Find the solution of the following system of homogeneous linear equations:

$$\left.\begin{array}{l} x_1 - x_2 - x_3 - x_4 = 0 \\ x_1 + 3x_2 - x_3 + x_4 = 0 \\ 3x_1 - 7x_2 - x_3 - 6x_4 = 0 \\ 2x_1 + 2x_2 - 2x_3 = 0 \\ 6x_1 - 2x_2 - 4x_3 - 5x_4 = 0 \end{array}\right\} \quad (1)$$

Solution: Let us represent the five linear equations of the system (1) by L_1, L_2, L_3, L_4, and L_5 respectively. Reduce the system to echelon form by the elementary operations. Eliminate x_1 from the second, third, fourth and fifth equations by the operations $L_2 \to L_2 - L_1$, $L_3 \to L_3 - 3L_1$, $L_4 \to L_4 - 2L_1$ and $L_5 \to L_5 - 6L_1$ respectively.

$L_2 - L_1$: $4x_2 + 0 + 2x_4 = 0$

L_3: $3x_1 - 7x_2 - x_3 - 6x_4 = 0$

$-3L_1$: $-3x_1 + 3x_2 + 3x_3 + 3x_4 = 0$

$L_3 - 3L_1$: $-4x_2 + 2x_3 - 3x_4 = 0$

L_4: $2x_1 + 2x_2 - 2x_3 = 0$

$-2L_1$: $-2x_1 + 2x_2 + 2x_3 + 2x_4 = 0$

$L_4 - 2L_1$: $4x_2 + 0 + 2x_4 = 0$

L_5: $6x_1 - 2x_2 - 4x_3 - 5x_4 = 0$

$-6L_1$: $-6x_1 + 6x_2 + 6x_3 + 6x_4 = 0$

$L_5 - 6L_1$: $4x_2 + 2x_3 + x_4 = 0$

Thus we obtain the equivalent system

$$\left.\begin{array}{l} x_1 - x_2 - x_3 - x_4 = 0 \\ 4x_2 + 2x_4 = 0 \\ -4x_2 + 2x_3 - 3x_4 = 0 \\ 4x_2 + 2x_4 = 0 \\ 4x_2 + 2x_3 + x_4 = 0 \end{array}\right\} \quad (2)$$

In the system (2) the second and fourth equations are identical we can disregard one of them. So, the system (2) reduces to

$$\left.\begin{array}{l} x_1 - x_2 - x_3 - x_4 = 0 \\ 4x_2 + 2x_4 = 0 \\ -4x_2 + 2x_3 - 3x_4 = 0 \\ 4x_2 + 2x_3 + x_4 = 0 \end{array}\right\} \quad (3)$$

Let us represent the four linear equations of the system (3) by L'_1, L'_2, L'_3, and L'_4 respectively. Eliminate x_2 from the third and fourth equations by the operations $L'_3 \to L'_3 + L'_2$, and $L'_4 \to L'_4 - L'_2$

$L'_3 + L'_2$: $2x_3 - x_4 = 0$

$L'_4 - L'_2$: $2x_3 - x_4 = 0$

Thus the system (3) reduces to

$$\left.\begin{array}{l} x_1 - x_2 - x_3 - x_4 = 0 \\ 4x_2 + 2x_4 = 0 \\ 2x_3 - x_4 = 0 \\ 2x_3 - x_4 = 0 \end{array}\right\} \quad (4)$$

In the system (4) the third and fourth equations are identical we can disregard one of them. Thus we obtain the equivalent system

$$\left.\begin{array}{l} x_1 - x_2 - x_3 - x_4 = 0 \\ 4x_2 + 2x_4 = 0 \\ 2x_3 - x_4 = 0 \end{array}\right\} \quad (5)$$

In this echelon form there are only three equations in four unknowns, hence the system has an infinite number of solutions and $4 - 3 = 1$ free variable which is x_4. Let $x_4 = a$, where a is any real number. Then $x_3 = \frac{a}{2}$, $x_2 = -\frac{a}{2}$ and $x_1 = a$.

Thus the general solution is $x_1 = a$, $x_2 = -\frac{a}{2}$, $x_3 = \frac{a}{2}$, $x_4 = a$.

Or, $\left(a \cdot -\frac{a}{2}, \frac{a}{2}, a\right)$

For particular solution, let $a = 2$. Then $x_1 = 2$, $x_2 = -1$, $x_3 = 1$, $x_4 = 2$ or, (2 – 1, 1, 2) is a particular solution of the given system.

Example 9. Find the solution space of the following homogeneous system of linear equations:

$$\left.\begin{array}{l} x_1 + 2x_2 + 3x_4 + x_5 = 0 \\ 2x_1 + 3x_2 + 3x_4 + x_5 = 0 \\ x_1 + x_2 + x_3 + 2x_4 + x_5 = 0 \\ 3x_1 + 5x_2 + 6x_4 + 2x_5 = 0 \\ 2x_1 + 3x_2 + 2x_3 + 5x_4 + 2x_5 = 0 \end{array}\right\} \quad \textbf{[U. P. 1987]}$$

Solution : Reduce the system to echelon form by the elementary operations. We multiply 1st equation by 2, 1, 3 and 2 and then subtract from 2nd, 3rd, 4th and 5th equations respectively. Then we have the equivalent system

$$\left.\begin{array}{l} x_1 + 2x_2 + 3x_4 + x_5 = 0 \\ -x_2 - 3x_4 - x_5 = 0 \\ -x_2 + x_3 - x_4 = 0 \\ -x_2 - 3x_4 - x_5 = 0 \\ -x_2 + 2x_3 - x_4 = 0 \end{array}\right\}$$

We subtract 2nd equation from 3rd, 4th and 5th equations. Then we have the equivalent system

$$\left.\begin{array}{l} x_1 + 2x_2 + 3x_4 + x_5 = 0 \\ -x_2 - 3x_4 - x_5 = 0 \\ x_3 + 2x_4 + x_5 = 0 \\ 0 + 0 + 0 = 0 \\ 2x_3 + 2x_4 + x_5 = 0 \end{array}\right\}$$

$$\Rightarrow \left.\begin{array}{l} x_1 + 2x_2 + 3x_4 + x_5 = 0 \\ -x_2 - 3x_4 - x_5 = 0 \\ x_3 + 2x_4 + x_5 = 0 \\ 2x_3 + 2x_4 + x_5 = 0 \end{array}\right\}$$

We multiply 3rd equation by 2 and then subtract from 4th equation. Then we have the equivatent system

$$\left.\begin{array}{l} x_1 + 2x_2 + 3x_4 + x_5 = 0 \\ -x_2 - 3x_4 - x_5 = 0 \\ x_3 + 2x_4 + x_5 = 0 \\ -2x_4 - x_5 = 0 \end{array}\right\}$$

This system is in echelon form having four equations in five unknowns. So the system has $5 - 4 = 1$ free variable which is x_5 and it has non-zero solution.

Let $x_5 = 2a$ where a is any real number. Putting $x_5 = 2a$ in the 4th equation we get $-2x_4 - 2a = 0$, that is, $x_4 = -a$. Putting $x_4 = -a$ and $x_5 = 2a$ in the 2nd & 3rd equations we get $x_2 = a$ and $x_3 = 0$. Finally putting $x_2 = a$, $x_4 = -a$ and $x_5 = 2a$ in the 1st equation. we get

$x_1 + 2a - 3a + 2a = 0$, i.e $x_1 = -a$.

Hence the solution space of the given system is

$W = \{(-a, a, o, -a, 2a) : a \in \mathbb{R}\}$.

Example 10 Determine the values of λ so that the following linear system in three variables x, y and z has (i) a unique soloution (ii) more than one solution (iii) no solution:

$$\left.\begin{array}{l} x + y - z = 1 \\ 2x + 3y + \lambda z = 3 \\ x + \lambda y + 3z = 2 \end{array}\right\} \quad \text{[D. U. H. 1987]}$$

Solution : Reduce the system to echelon form by elementary operations. We multiply first equation by 2 and 1 and then substract from the second and the third equations respectively. Then we obtain the equivalent system.

$$\left.\begin{array}{l} x + y - z = 1 \\ y + (\lambda + 2)z = 1 \\ (\lambda - 1)y + 4z = 1 \end{array}\right\}$$

We multiply second equation by $(\lambda - 1)$ and then substract from the third equation. Then we obtain the equivalent system.

$$\left.\begin{array}{l} x + y - z = 1 \\ y + (\lambda + 2)z = 1 \\ \{4 - (\lambda - 1)(\lambda + 2)\}z = 2 - \lambda \end{array}\right\}$$

Or, $$\left.\begin{array}{l} x + y - z = 1 \\ y + (\lambda + 2)z = 1 \\ (6 - \lambda - \lambda^2) = 2 - \lambda \end{array}\right\}$$

Or, $$\left.\begin{array}{l} x + y - z = 1 \\ y + (\lambda + 2)z = 1 \\ (3 + \lambda)(2 - \lambda)z = 2 - \lambda \end{array}\right\}$$

This system is in echelon form. It has a unique solution if the coefficient of z in the thrid equation is non-zero i. e. if $\lambda \neq 2$ and $\lambda \neq -3$. In case $\lambda = 2$ third equation is $0 = 0$ which is true and the system has more than one solution.

In case $\lambda = -3$, the third equation is $0 = 5$ which is not true and hence the system has no solution.

Example 11. For what values of λ and μ the following system of linear equations has (i) no solution (ii) more than one solution (iii) a unique solution :

$$\left.\begin{array}{l} x+y+z = 6 \\ x+2y+3z = 10 \\ x+2y+\lambda z = \mu \end{array}\right\}$$

[D. U. H 1980; D. U. H (Stat) 1982]

Solution: Reduce the given system to echelon form by the elementary operations. We subtract 1st equation from 2nd and 3rd equations. Then we have the equivalent system.

$$\left.\begin{array}{l} x+y+z = 6 \\ y+2z = 4 \\ y+(\lambda-1)z = \mu-6 \end{array}\right\}$$

We subtract 2nd equation from 3rd equation. Then we have the equivalent system

$$\left.\begin{array}{l} x+y+z = 6 \\ y+2z = 4 \\ (\lambda-3)z = \mu-10 \end{array}\right\} \quad (*)$$

Now from the last system (*) we have the following three cases;

(i) If $\lambda = 3$ and $\mu \neq 10$ then the 3rd equation of (*) is of the form $o = a$ where $a = \mu - 10 \neq 0$ which implies that o is equal to a non-zero real number which is not true. Thus we conclude that for $\lambda = 3$ and $\mu \neq 10$, the given system has no solution.

(ii) If $\lambda = 3$ and $\mu = 10$, then the 3rd equation of (*) vanishes & the system will be in echelon form having two equations in three unknows. so it has $3 - 2 = 1$ free variable which is z and hence for $\lambda = 3$ and $\mu = 10$, the given system will have more than one solution.

(iii) For a unique solution the coefficient of z in the 3rd equation must be non-zero i.e $\lambda \neq 3$. and μ may have any value. Thus for $\lambda \neq 3$ and μ arbitrary the given system have a unique solution.

Example 12. For what values of λ, the following linear equations have a solution and solve them completely in each case :

$$\left.\begin{array}{l} x+y+z = 1 \\ x+2y+4z = \lambda \\ x+4y+10z = \lambda^2 \end{array}\right\}$$

Solution: The given system of linear equations is

$$\left.\begin{array}{l} x+y+z = 1 \\ x+2y+4z = \lambda \\ x+4y+10z = \lambda^2 \end{array}\right\}$$

Reduce the given system to echelon form by the elementary operations. We subtract 1st equation from 2nd & 3rd equations respectively. Then we have the equivalent system.

$$\left.\begin{array}{l} x+y+z = 1 \\ y+3z = \lambda-1 \\ 3y+9z = \lambda^2-1 \end{array}\right\}$$

We multiply 2nd equation by 3 and then subtract from the 3rd equation. Then we have the equivalent system

$$\left.\begin{array}{l} x+y+z = 1 \\ y+3z = \lambda-1 \\ o+o = \lambda^2-3\lambda+2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} x+y+z = 1 \\ y+3z = \lambda-1 \\ o = \lambda^2-3\lambda+2 \end{array}\right\} \quad (*)$$

If $\lambda^2 - 3\lambda + 2 \neq 0$, the given system will be inconsistent and if $\lambda^2 - 3\lambda + 2 = 0$, the above system will be in echelon form having two equations in three unknowns. So the system has $3 - 2 = 1$ free variable which is z. So the system has non – zero solution for $\lambda^2 - 3\lambda + 2 = 0$

$$\text{that is, } (\lambda - 1)(\lambda - 2) = 0$$

$$\Rightarrow \lambda = 1 \quad \text{or, } \lambda = 2$$

Thus the given system is consistent for $\lambda = 1$ and $\lambda = 2$.

Case 1 when $\lambda = 1$

The system (*) will be

$$\left.\begin{aligned} x + y + z &= 1 \\ y + 3z &= 0 \end{aligned}\right\}$$

which is in echelon form where z is a free variable.

Let z = a where a is arbitrary real number.

$$\therefore y = -3a,\ x = 1 + 2a$$

Hence the given system of linear equations has infinite number of solutions for $\lambda = 1$ In particular, let a = 1, then $x = 3$, $y = -3$, $z = 1$. is a particular solution of the given system.

Case II When $\lambda = 2$

The system (*) will be

$$\left.\begin{aligned} x + y + z &= 1 \\ y + 3z &= 1 \end{aligned}\right\}$$

which is in echelon form where z is a free variable.

Let z = b where b is any real number. $\therefore y = 1 - 3b$, $x = 2b$. Hence the system has infinite number of solutions for $\lambda = 2$. In particular, let $b = -1$, then $x = 4$, $y = -2$, $z = -1$ is a particular solution of the given system.

EXERCISES – 1

1. Which of the following systems of linear equations are inconsistent :

(i) $\left.\begin{aligned} 2x_1 + 3x_2 &= 1 \\ 4x_1 + 6x_2 &= 3 \end{aligned}\right\}$ (ii) $\left.\begin{aligned} x_1 + x_2 &= 1 \\ x_1 + 2x_2 &= 3 \\ x_1 - x_2 &= 2 \end{aligned}\right\}$

(iii) $\left.\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 2x_1 + 2x_2 + 2x_3 &= 1 \\ 3x_1 + 3x_2 + 3x_3 &= 2 \end{aligned}\right\}$ (iv) $\left.\begin{aligned} x_1 + 3x_2 + 2x_3 &= 7 \\ 2x_1 + x_2 + 3x_3 &= 8 \\ 3x_1 + 4x_2 + 6x_3 &= 16 \\ 6x_1 + 8x_2 + 11x_3 &= 20 \end{aligned}\right\}$

Answers:

(i) Inconsistent, (ii) Inconsistent, (iii) Inconsistent.

(iv) Inconsistent

2. Which of the following systems of linear equations are consistent? Find all solutions of the consistent system.

(i) $\left.\begin{aligned} 5x_1 + 3x_2 - 3x_3 &= -1 \\ 3x_1 + 2x_2 - 2x_3 &= -1 \\ 2x_1 - x_2 + 2x_3 &= 8 \end{aligned}\right\}$ (ii) $\left.\begin{aligned} 3x_1 - x_2 + 7x_3 &= 0 \\ 2x_1 - x_2 + 4x_3 &= \tfrac{1}{2} \\ x_1 - x_2 + x_3 &= 1 \\ 6x_1 - 4x_2 + 10x_3 &= 3 \end{aligned}\right\}$

Answers : (i) $x_1 = 1, x_2 = 2, x_3 = 4$

(ii) $x_1 = -3a - \frac{1}{2}, x_2 = -2a - \frac{3}{2}, x_3 = a.$

where a is any real number.

3. Express the following systems of linear equations in echelon form and solve them :

(i) $\left.\begin{aligned} x_1 - x_2 + 2x_3 + x_4 &= 0 \\ -x_1 + 3x_3 + 2x_4 &= 2 \\ 2x_1 + x_2 - x_4 &= 1 \\ 2x_1 + 2x_2 + x_3 + 3x_4 &= 14 \end{aligned}\right\}$ (ii) $\left.\begin{aligned} x_1 + 2x_2 + x_3 &= -1 \\ 6x_1 + x_2 + x_3 &= -4 \\ 2x_1 - 3x_2 - x_3 &= 0 \\ -x_1 - 7x_2 - 2x_3 &= 7 \\ x_1 - x_2 &= 1 \end{aligned}\right\}$

Answers : (i) $x_1 = 1, x_2 = 2, x_3 = -1, x_4 = 3.$

(ii) $x_1 = -1, x_2 = -2, x_3 = 4.$

4. Solve each of the following linear systems of equations :

(i) $\left.\begin{aligned} 2x_1 - 3x_2 &= -2 \\ 2x_1 + x_2 &= 1 \\ 3x_1 + 2x_2 &= 1 \end{aligned}\right\}$ (ii) $\left.\begin{aligned} 4x_1 - 8x_2 &= 12 \\ 3x_1 - 6x_2 &= 9 \\ -2x_1 + 4x_2 &= -6 \end{aligned}\right\}$

sign 6

(iii) $\left.\begin{aligned} 3x_1 + 2x_2 - x_3 &= -15 \\ 5x_1 + 3x_2 + 2x_3 &= 0 \\ 3x_1 + x_2 + 3x_3 &= 11 \\ 11x_1 + 7x_2 \quad &= -30 \end{aligned}\right\}$ (iv) $\left.\begin{aligned} x + 2y - 3z &= 6 \\ 2x - y + 4z &= 2 \\ 4x + 3y - 2z &= 14 \end{aligned}\right\}$

Assign

[D. U. S. 1980, '83]

8 **Answers :**

(i) Inconsistent

(ii) $x_1 = 3 + 2a$, $x_2 = a$, where a is any real number.

(iii) $x_1 = -4$, $x_2 = 2$, $x_3 = 7$.

(iv) $x = 2 - a$, $y = 2 + 2a$, $z = a$

where a is any real number.

5. Solve the following systems of linear equations :

(i) $\left.\begin{aligned} 2x_1 + x_2 + 3x_3 &= 0 \\ 3x_1 - 2x_2 + x_3 &= 0 \\ x_1 - 3x_2 - 2x_3 &= 0 \end{aligned}\right\}$

(ii) $\left.\begin{aligned} x_1 - x_2 + x_3 + x_4 &= 0 \\ x_1 + 2x_2 - x_3 - x_4 &= 1 \\ 2x_1 - 2x_2 + x_3 - x_4 &= 0 \\ -3x_2 + x_3 - x_4 &= -1 \end{aligned}\right\}$

Answers : (i) $x_1 = a$, $x_2 = a$, $x_3 = -a$

where a is any real number

(ii) $x_1 = \frac{2}{3}a + \frac{1}{3}$, $x_2 = -\frac{4}{3}a + \frac{1}{3}$, $x_3 = -3a$, $x_4 = a$

where a is any real number.

6. Find the solution sets of the following systems of linear equations :

(i) $\left.\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= 6 \\ x_1 - x_2 + x_3 - x_4 &= -2 \\ x_1 + 8x_2 + x_3 + 5x_4 &= 22 \\ 2x_1 + 7x_2 + 2x_3 + 4x_4 &= 20 \end{aligned}\right\}$

(ii) $\left.\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= 0 \\ 2x_1 + 3x_2 + 4x_3 \quad &= 1 \\ 3x_1 + 4x_2 \quad + x_4 &= 2 \\ 4x_1 \quad + x_3 + 2x_4 &= 3 \end{aligned}\right\}$

Answers :

(i) $x_1 = \frac{1}{3}(2 + b - 3a)$, $x_2 = \frac{1}{3}(8 - 2b)$, $x_3 = a$, $x_4 = b$

where a and b are arbitrary real numbers.

(ii) $x_1 = \frac{9}{11}$, $x_2 = -\frac{1}{11}$, $x_3 = -\frac{1}{11}$, $x_4 = -\frac{1}{11}$.

7. Which of the following systems of linear equations are consistent? Find all solutions of the consistent system :

(i) $\left.\begin{aligned} x_1 - 5x_2 + 4x_3 + x_4 &= 7 \\ 2x_1 + x_2 - 3x_3 - x_4 &= 1 \\ 3x_1 - 4x_2 + x_3 - 5x_4 &= 0 \\ x_1 + x_2 + x_3 + x_4 &= 1 \end{aligned}\right\}$ (ii) $\left.\begin{aligned} 3x_1 + 2x_2 - x_3 + 4x_4 &= 6 \\ -2x_1 + x_2 + 5x_3 + x_4 &= 0 \\ x_1 - 4x_2 + 2x_3 + 8x_4 &= 2 \\ 5x_1 + x_2 - 3x_3 - 2x_4 &= 1 \end{aligned}\right\}$

Answers :

(i) $x_1 = \frac{64}{55}$, $x_2 = \frac{-69}{55}$, $x_3 = -\frac{28}{55}$, $x_4 = \frac{8}{5}$

(ii) $x_1 = \frac{94}{679}$, $x_2 = \frac{685}{679}$, $x_3 = -\frac{30}{97}$, $x_4 = \frac{79}{97}$.

8. (i) Express the following system of linear equations in echelon form and solve it :

$\left.\begin{aligned} x + y \quad &= -3 \\ 2x - 2y - z &= -8 \\ 4x \quad - z &= -14 \\ x - 3y - z &= -5 \end{aligned}\right\}$ [D. U. P. 1980]

(ii) Reduce the following system of linear equations to an echelon form and find all solutions :

$\left.\begin{aligned} x_1 - x_2 + 2x_3 &= 5 \\ 2x_1 + x_2 - x_3 &= 2 \\ 2x_1 - x_2 - x_3 &= 4 \\ x_1 + 3x_2 + 2x_3 &= 1 \end{aligned}\right\}$ [D. U. P. 1981]
[D. U. S. 1981, 1987]

Answers : (i) $x = \frac{1}{4}(a - 14)$, $y = \frac{2 - a}{4}$, $z = a$

where a is any real number.

(ii) $x_1 = 2$, $x_2 = -1$, $x_3 = 1$.

9. (i) Reduce the system of linear equations

$$\left.\begin{array}{l} 5x_1 + 2x_2 - 7x_3 = 1 \\ 7x_1 - x_2 + 2x_3 = 0 \\ 2x_1 + 5x_2 - x_3 = 5 \end{array}\right\} \qquad \text{[D. U. S. 1982]}$$

into echelon form and hence solve it.

(ii) Use echelon method to solve the following system of linear equations :

$$\left.\begin{array}{l} x_1 + x_2 + 3 = 0 \\ 2x_1 - 2x_2 - x_3 + 8 = 0 \\ 4x_1 - x_3 + 14 = 0 \\ x_1 - 3x_2 - x_3 + 5 = 0 \end{array}\right\} \qquad \text{[D. U. S. 1984]}$$

Answers : (i) $x_1 = \frac{228}{2679}$, $x_2 = \frac{2698}{2679}$, $x_3 = \frac{29}{141}$,

(ii) $x_1 = \frac{a-14}{4}$, $x_2 = \frac{2-a}{4}$, $x_3 = a$ where a is any real number.

10. (i) Solve the following linear equations :

$$\left.\begin{array}{l} x_1 + x_2 + 2x_3 + x_4 = 5 \\ 3x_1 + 2x_2 - x_3 + 3x_4 = 6 \\ 4x_1 + 3x_2 + x_3 + 4x_4 = 11 \\ 2x_1 + x_2 - 3x_3 + 2x_4 = 1 \end{array}\right\}$$

(ii) Find all rational solutions of the following system of linear equations :

$$\left.\begin{array}{l} x_1 + x_2 - x_3 - x_4 = -1 \\ 3x_1 + 4x_2 - x_3 - 2x_4 = 3 \\ x_1 + 2x_2 + x_3 = 5 \end{array}\right\}$$

Answers : (i) $x_1 = 5a - b - 4$, $x_2 = -7a + 9$, $x_3 = a$, $x_4 = b$

where a and b are arbitrary real numbers.

(ii) $x_1 = 3a + 2b - 7$, $x_2 = -2a - b + 6$, $x_3 = a$, $x_4 = b$

where a and b are arbitrary real numbers.

11. Determine whether each system has a non-zero solution. If exists, find the non-zero solutions.

(i) $\left.\begin{array}{l} x_1 + x_2 + x_3 = 0 \\ x_1 - x_2 + x_3 = 0 \\ x_1 + x_2 - x_3 = 0 \end{array}\right\}$ (ii) $\left.\begin{array}{l} 3x_1 + 2x_2 + x_3 = 0 \\ 7x_1 - x_2 + 21x_3 = 0 \\ x_1 + 12x_2 - 37x_3 = 0 \end{array}\right\}$

Answers : (i) $x_1 = 0$, $x_2 = 0$, $x_3 = 0$.

(ii) $x_1 = -\frac{43}{17}a$, $x_2 = \frac{56}{17}a$, $x_3 = a$ where a is any real number.

12. Find the non-zero solutions of the following systems of linear homogeneous equations :

(i) $\left.\begin{array}{l} x_1 + 3x_2 + 2x_3 = 0 \\ 2x_1 - x_2 + 3x_3 = 0 \\ 3x_1 - 5x_2 + 4x_3 = 0 \\ x_1 + 17x_2 + 4x_3 = 0 \end{array}\right\}$ (ii) $\left.\begin{array}{l} x_1 - 3x_2 - 2x_3 = 0 \\ 2x_1 + x_2 + 3x_3 = 0 \\ 3x_1 - 2x_2 + x_3 = 0 \end{array}\right\}$

[D. U. P. 1986]

(iii) $\left.\begin{array}{l} 2x_1 + x_2 + x_3 - 5x_4 = 0 \\ x_1 + x_2 + x_3 - 4x_4 = 0 \\ x_1 - x_2 - x_3 + 2x_4 = 0 \end{array}\right\}$ (iv) $\left.\begin{array}{l} 2x_1 - x_2 - x_3 + x_4 = 0 \\ x_1 + 2x_2 - 3x_3 + 3x_4 = 0 \\ -x_1 - x_2 + 2x_3 - 2x_4 = 0 \end{array}\right\}$

Answers : (i) $x_1 = -\frac{11}{7}a$, $x_2 = -\frac{a}{7}$, $x_3 = a$

where a is any real number.

(ii) $x_1 = -a$, $x_2 = -a$, $x_3 = a$

where a is an arbitrary real number.

(iii) $x_1 = b$, $x_2 = -a + 3b$, $x_3 = a$, $x_4 = b$

where a and b are arbitrary real numbers.

(iv) $x_1 = a - b$, $x_2 = a - b$, $x_3 = a$, $x_4 = b$

where a and b are arbitrary real numbers.

In particular, let a = 2, b = 1 then $x_1 = 1$, $x_2 = 1$, $x_3 = 2$, $x_4 = 1$ is a particular solution of the system.

13. Find the non-trivial solutions of the following systems of linear homogeneous equations :

(i) $\left.\begin{array}{l} 2x_1 + 3x_2 - x_3 = 0 \\ x_1 - x_2 + x_3 = 0 \\ x_1 + 9x_2 - 5x_3 = 0 \end{array}\right\}$ (ii) $\left.\begin{array}{l} x_1 - x_2 + x_3 - x_4 = 0 \\ 2x_1 + x_2 - x_3 + 2x_4 = 0 \\ 2x_2 + 3x_3 + x_4 = 0 \end{array}\right\}$

(iii) $\left.\begin{array}{l} x_1 - 2x_2 + 3x_3 + x_4 = 0 \\ x_1 + x_2 - 3x_3 - x_4 = 0 \\ 2x_1 - x_2 + x_3 - x_4 = 0 \\ 2x_1 + 2x_2 - 5x_3 - 3x_4 = 0 \end{array}\right\}$

Answers : (i) $x_1 = -\frac{2a}{5}$, $x_2 = \frac{3a}{5}$, $x_3 = a$

where a is any real number.

(ii) $x_1 = -\frac{1}{3}a,\ x_2 = -a,\ x_3 = \frac{1}{3}a,\ x_4 = a$

where a is any real number.

(iii) $x_1 = \frac{4a}{3},\ x_2 = \frac{8a}{3},\ x_3 = a,\ x_4 = a$

where a is any real number.

14. Solve the following homogeneous systems of linear equations :

(i) $\left.\begin{aligned} x_1 - 2x_2 + 2x_3 &= 0 \\ 2x_1 + x_2 - 2x_3 &= 0 \\ 3x_1 + 4x_2 - 6x_3 &= 0 \end{aligned}\right\}$

(ii) $\left.\begin{aligned} 3x_1 + 4x_2 + x_3 + 2x_4 + 3x_5 &= 0 \\ 5x_1 + 7x_2 + x_3 + 3x_4 + 4x_5 &= 0 \\ 4x_1 + 5x_2 + 2x_3 + x_4 + 5x_5 &= 0 \\ 7x_1 + 10x_2 + x_3 + 6x_4 + 5x_5 &= 0 \end{aligned}\right\}$

(iii) $\left.\begin{aligned} x_1 + 2x_2 + 2x_3 - x_4 + 3x_5 &= 0 \\ x_1 + 2x_2 + 3x_3 + x_4 + x_5 &= 0 \\ 3x_1 + 6x_2 + 8x_3 + x_4 + 5x_5 &= 0 \end{aligned}\right\}$ **[D. U. H. 1986]**

(iv) $\left.\begin{aligned} x_1 + 2x_2 - 2x_3 + 2x_4 + x_5 &= 0 \\ x_1 + 2x_2 - x_3 + 3x_4 - 2x_5 &= 0 \\ 2x_1 + 4x_2 - 7x_3 + x_4 + x_5 &= 0 \end{aligned}\right\}$ **[D. U. H. 1988]**

Answers : (i) $x_1 = \frac{2a}{5},\ x_2 = \frac{6a}{5},\ x_3 = a$

where a is any real number.

(ii) $x_1 = -3a - 5b,\ x_2 = 2a + 3b,\ x_3 = a,\ x_4 = 0,\ x_5 = b$

where a and b are arbitrary real numbers.

(iii) $x_1 = -2a + 5b - 7c,\ x_2 = a,\ x_3 = -2b + 2c,\ x_4 = b,\ x_5 = c$

where a, b and c are arbitrary real numbers.

(iv) $x_1 = -2a - 4b,\ x_2 = a,\ x_3 = -b,\ x_4 = b,\ x_5 = 0$

where a and b are arbitrary real numbers.

15. (i) Find a general solution and a particular solution of the following system of homogeneous linear equations :

$\left.\begin{aligned} x_1 - 3x_2 + x_4 &= 0 \\ 2x_1 + x_2 + 3x_3 - x_4 &= 0 \\ 3x_1 - 17x_2 - 6x_3 + 7x_4 &= 0 \end{aligned}\right\}$

Answers :

General solution : $x_1 = 0,\ x_2 = \frac{a}{3},\ x_3 = \frac{2a}{9},\ x_4 = a$

where a is any real number.

Particular solution : $x_1 = 0,\ x_2 = 3,\ x_3 = 2,\ x_4 = 9.$

(ii) Solve the following homogeneous system of linear equations :

$\left.\begin{aligned} x_1 - 3x_2 + 4x_3 - x_4 &= 0 \\ 3x_1 + x_2 + 2x_3 + 4x_4 &= 0 \\ 2x_1 - 4x_2 + 6x_3 + x_4 &= 0 \\ 2x_1 + 2x_2 + 2x_4 &= 0 \\ 4x_1 - 4x_2 + 8x_3 &= 0 \end{aligned}\right\}$

Answer : $x_1 = -a,\ x_2 = x_3 = a,\ x_4 = 0$

where a is an arbitrary real number.

(iii) Solve the following system of linear equations :

$\left.\begin{aligned} x_1 + x_2 + 2x_3 + x_4 &= 5 \\ 2x_1 + 3x_2 + x_3 - x_4 &= 16 \\ x_1 + 2x_2 - 4x_3 + x_4 &= 2 \\ x_1 + x_2 + 3x_3 - 2x_4 &= 12 \end{aligned}\right\}$

Answer : $x_1 = 2,\ x_2 = 3,\ x_3 = 1,\ x_4 = -2.$

16. (i) Reduce the following system of linear equations to an echelon form and solve it :

$\left.\begin{aligned} x + 2y - 3z &= 4 \\ x + 3y + z &= 11 \\ 2x + 5y - 4z &= 13 \\ 2x + 6y + 2z &= 22 \end{aligned}\right\}$ **[D. U. S. 1980]**

(ii) Solve the following system of linear equations :

$\left.\begin{aligned} x + 2y - 3z &= 6 \\ 2x - y + 4z &= 2 \\ 4x + 3y - 2z &= 14 \end{aligned}\right\}$ **[D. U. S. 1980] (Improvement)**

using echelon form or otherwise.

(iii) Solve the following linear equations by using echelon form :

$$\left.\begin{aligned} x+2y+2z &= 2 \\ 3x-2y-z &= 5 \\ 2x-5y+3z &= -4 \\ x+4y+6z &= 0 \end{aligned}\right\}$$ **[D. U. S. 1983]**

Answers :

(i) $x=1, y=3, z=1$

(ii) $x=2-a, y=2a+2, z=a$ where a is any real number.

(iii) $x=2, y=1, z=-1$.

17. Find the general solution and also a particular solution of the following system of linear equations :

$$\left.\begin{aligned} 2x_1-x_2-x_3+3x_4 &= 1 \\ 4x_1-2x_2-x_3+x_4 &= 5 \\ 6x_1-3x_2-x_3-x_4 &= 9 \\ 2x_1-x_2+2x_3-12x_4 &= 10 \end{aligned}\right\}$$ **[D. U. S. 1989]**

Answers : General solution : $\left\{\begin{aligned} x_1 &= \tfrac{1}{2}a+b+2 \\ x_2 &= a \\ x_3 &= 5b+3 \\ x_4 &= b \end{aligned}\right.$

where a and b are any two real numbers.

Particular solution : $x_1=2, x_2=2, x_3=-2, x_4=-1$.

18. Solve the following system of linear equations :

$$\left.\begin{aligned} x_1+x_2+2x_4+3x_5 &= 5 \\ 2x_1+4x_2-x_3+5x_4+4x_5 &= -1 \\ x_1+3x_2+5x_4+2x_5 &= -3 \\ 3x_1+7x_2-3x_3+9x_4+2x_5 &= -14 \\ 2x_1+8x_2-4x_3+2x_4+7x_5 &= -10 \end{aligned}\right\}$$

Answer : $x_1=2, x_2=-3, x_3=1, x_4=0, x_5=2$.

19. Solve the following system of linear equations :

Done

$$\left.\begin{aligned} x+2y+3z &= -1 \\ 2x-y-2z &= 5 \\ 3x+3y+z &= -2 \\ 4x+3y+4z &= 3 \\ -4x+2y+4z &= -10 \end{aligned}\right\}$$

Answer : $x=2, y=-3, z=1$.

20. Solve the following systems of linear equations :

Done (i) $\left.\begin{aligned} x_1+2x_2 &= 1 \\ -x_1+x_2 &= 0 \\ 2x_1+4x_2 &= 3 \end{aligned}\right\}$ Done (ii) $\left.\begin{aligned} 2x+3y-2z &= 5 \\ x-2y+3z &= 2 \\ 4x-y+4z &= 1 \end{aligned}\right\}$ **[D.U.P. 1991]**

(iii) $\left.\begin{aligned} x_1+x_2+x_3 &= 3 \\ x_1-x_2+2x_3 &= -3 \\ 3x_1-x_2+5x_3 &= -2 \\ 2x_1-x_2-x_3 &= 4 \end{aligned}\right\}$

Answers : (i) Inconsistent (ii) Inconsistent (iii) Inconsistent.

21. Find all solutions of the following system of linear equations :

$$\left.\begin{aligned} x_1+x_2-2x_3+x_4+3x_5 &= 1 \\ 2x_1-x_2+2x_3+2x_4+6x_5 &= 2 \\ 3x_1+2x_2-4x_3-3x_4-9x_5 &= 3 \end{aligned}\right\}$$ **[D. U. H. 1989]**

Answer : $x_1=1, x_2=2a, x_3=a, x_4=-3b, x_5=b$

where a and b are arbitrary real numbers.

22. Solve the following linear equations :

$$\left.\begin{aligned} x_1+2x_2-3x_3+4x_4 &= 1 \\ 2x_1+5x_2-5x_3+6x_4 &= 3 \\ x_1+4x_2-x_3 &= 3 \\ 2x_1+3x_2-7x_3+10x_4 &= 1 \end{aligned}\right\}$$ **[D. U. H. 1987]**

Answer : $x_1=-1+5a-8b, x_2=1-a+2b, x_3=a, x_4=b$

where a and b are arbitrary real numbers.

23. Find all solutions of the following homogeneous systems of linear equations :

(i) $\left.\begin{aligned} x+2y-3z &= 0 \\ 2x+5y+2z &= 0 \\ 3x-y-4z &= 0 \end{aligned}\right\}$ **[D. U. H. 1986]**

(ii) $$\left.\begin{aligned} x_1 + 4x_2 + 5x_3 + 3x_4 &= 0\\ 2x_1 + 3x_2 + 5x_3 + x_4 &= 0\\ 3x_1 + 2x_2 + 5x_3 - x_4 &= 0\\ 4x_1 + x_2 + 5x_3 - 3x_4 &= 0 \end{aligned}\right\}$$ **[D. U. P. 1985]**

(iii) $$\left.\begin{aligned} x_1 - x_2 - x_3 - x_4 &= 0\\ x_1 + 3x_2 - x_3 + x_4 &= 0\\ 3x_1 - 7x_2 - x_3 - 6x_4 &= 0\\ 2x_1 + 2x_2 - 2x_3 &= 0\\ 6x_1 - 2x_2 - 4x_3 - 5x_4 &= 0 \end{aligned}\right\}$$ **[D. U. P. 1988]**

(iv) $$\left.\begin{aligned} 2x_1 + 2x_2 - x_3 + x_5 &= 0\\ -x_1 - x_2 + 2x_3 - 3x_4 + x_5 &= 0\\ x_1 + x_2 - 2x_3 - x_5 &= 0\\ x_3 + x_4 + x_5 &= 0 \end{aligned}\right\}$$

Answers : (i) $x = y = z = 0$, No non-zero solution.

(ii) $x_1 = b - a$, $x_2 = -(a + b)$, $x_3 = a$ and $x_4 = b$

where a and b are arbitrary real numbers.

(iii) $x_1 = 2a$, $x_2 = -a$, $x_3 = a$, $x_4 = 2a$

where a is arbitrary real number.

(iv) $x_1 = -a - b$, $x_2 = a$, $x_3 = -b$, $x_4 = 0$, $x_5 = b$

where a and b are arbitrary real numbers.

24. Reduce the following system of linear equations to echelon form and solve it :

$$\left.\begin{aligned} x_1 + x_2 + x_3 + 2x_4 + 3x_5 &= 13\\ -2x_1 - 2x_2 + x_3 - 3x_4 - 4x_5 &= 5\\ 3x_1 + 2x_2 + x_4 + 5x_5 &= 10\\ x_1 + x_2 + 2x_3 - x_4 + 9x_5 &= 18 \end{aligned}\right\}$$

Answer : $x_1 = 5 - a$, $x_2 = a$, $x_3 = 11$. $x_4 = 0$, $x_5 = -1$.

where a is any real number.

25. Solve the following system of linear equations :

$$\left.\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= 5\\ 2x_1 + x_2 + 4x_3 + x_4 &= 2\\ 3x_1 + 4x_2 + x_3 + 5x_4 &= 6\\ 2x_1 + 3x_2 + 5x_3 + 2x_4 &= 3 \end{aligned}\right\}$$

Answer : $x_1 = \frac{15}{71}$, $x_2 = \frac{-11}{71}$, $x_3 = \frac{10}{71}$, $x_4 = \frac{83}{71}$.

26. Find a general solution and also a particular solution of the following system of linear equations :

$$\left.\begin{aligned} 2x_1 + 3x_2 - x_3 + 5x_4 - 2x_5 &= 4\\ x_1 + 2x_2 + x_3 + 3x_4 - x_5 &= -1\\ 2x_1 + x_2 - 6x_3 + 7x_4 + 3x_5 &= 1\\ 5x_1 + 11x_2 + 7x_3 + 12x_4 - 10x_5 &= 4 \end{aligned}\right\}$$

Answers : $x_1 = -64 - 21a - 24b$, $x_2 = 39 + 11a + 15b$,

$x_3 = -15 - 4a - 5b$, $x_4 = a$, $x_5 = b$.

where a and b are arbitrary real numbers.

In particular $x_1 = -85$, $x_2 = 50$, $x_3 = -19$, $x_4 = 1$, $x_5 = 0$

27. Reduce the following system of linear equations into echelon form and solve it :

$$\left.\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 &= 1\\ 2x_1 + 3x_2 + 4x_3 + 5x_4 + 6x_5 &= -1\\ 3x_1 + 5x_2 + 6x_3 + 7x_4 + 4x_5 &= 2\\ 4x_1 + 7x_2 + 10x_3 + 13x_4 + 16x_5 &= 1\\ 5x_1 + 8x_2 + 9x_3 + 10x_4 + 3x_5 &= 3 \end{aligned}\right\}$$

Answer : $x_1 = -7 - 4b$, $x_2 = 7 + a + 10b$, $x_3 = -2 - 2a - 7b$, $x_4 = a$, $x_5 = b$ where a and b are arbitrary real numbers.

28. Reduce the following system of linear equations into echelon form and solve it :

$$\left.\begin{aligned} 4x_1 + 2x_2 + 5x_3 + 7x_4 + x_5 &= 2\\ x_1 + x_2 + x_3 + x_4 + 5x_5 &= 1\\ 2x_1 + 3x_2 + 4x_3 + 5x_4 + 6x_5 &= 1\\ 3x_1 + 9x_2 + 7x_3 + x_4 + 8x_5 &= 9\\ 5x_1 + x_2 + x_3 + 6x_4 + x_5 &= 0 \end{aligned}\right\}$$

Answer : $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, $x_4 = -1$, $x_5 = 0$.

29. Determine the values of λ such that the following system in unknowns x, y and z has

(i) a unique solution, (ii) no solution, (iii) more than one solution :

$$\left.\begin{aligned} \lambda x + y + z &= 1\\ x + \lambda y + z &= 1\\ x + y + \lambda z &= 1 \end{aligned}\right\}$$ **[R. U. H 1985]**

Answers : (i) $\lambda \neq 1$ and $\lambda \neq -2$, (ii) $\lambda = -2$, (iii) $\lambda = 1$.

30. Determine the values of λ such that the following system in unknowns x, y and z has (i) a unique solution, (ii) no solution (iii) more than one solution :

$$\left.\begin{aligned} x \quad\quad - 3z &= -3 \\ 2x + \lambda y - \quad z &= -2 \\ x + 2y + \lambda z &= 1 \end{aligned}\right\}$$

Answers : (i) $\lambda \neq 2$ and $\lambda \neq -5$, (ii) $\lambda = -5$, (iii) $\lambda = 2$.

31. Determine the values of λ such that the following system of linear equations in unknowns x, y and z has (i) a unique solution (ii) no solution (iii) more than one solution :

$$\left.\begin{aligned} x + y + \lambda z &= 1 \\ x + \lambda y + z &= \lambda \\ \lambda x + y + z &= \lambda^2 \end{aligned}\right\}$$

Answers : (i) $\lambda \neq -2$, $\lambda \neq 1$ (ii) $\lambda = -2$ (iii) $\lambda = 1$

32. Find out the conditions on α, β and γ so that the following systems of non-homogeneous linear equations has a solution :

(i) $$\left.\begin{aligned} x + 2y - 3z &= \alpha \\ 3x - y + 2z &= \beta \\ 2x - 10y + 16z &= 2\gamma \end{aligned}\right\}$$ (ii) $$\left.\begin{aligned} x + 2y - 3z &= \alpha \\ 2x + 6y - 11z &= \beta \\ 2x - 4y + 14z &= 2\gamma \end{aligned}\right\}$$

Answer : (i) $2\alpha = \beta - \gamma$ (ii) $5\alpha = 2\beta + \gamma$.

33. Find out the conditions on a, b and c so that the following system of non-homogeneous linear equations is consistent and also solve the system for a = 1, b = 1 and c = – 2 :

$$\left.\begin{aligned} -2x + y + z &= a \\ x - 2y + z &= b \\ x + y - 2z &= c \end{aligned}\right\}$$

Answer : Condition for consistent is a + b + c = 0 and the general solution is $x = \alpha - 1$, $y = \alpha - 1$, $z = \alpha$ where α is arbitrary real number.

A particular solution is $x = 1$, $y = 1$, $z = 2$.

MATRIX ALGEBRA

3.1 Introduction

J. J. Sylvester was the first man who introduced the word '**matrix**' in 1850 and later on in 1858 **Arthur Cayley** developed the theory of matrices in a systematic way. Matrix is a powerful tool of modern mathematics which is originated in the study of linear equations, and it has wide applications in every branch of science and especially in Physics, Chemistry, Mathematics, Statistics, Economics, and Engineerings.

3.2 Definition of matrix

A **matrix** is a rectangular array of numbers (real or complex) enclosed by a pair of brackets (or double vertical rolls) and the numbers in the array are called **the entries** or the **elements** of the matrix. that is, a rectangular array of (real or complex) numbers of the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} \end{bmatrix}$$

is called a **matrix**. The numbers $a_{11}, a_{12}, \ldots \ldots, a_{mn}$ are called the **entries** or the **elements** of the matrix. The above matrix has m rows and n columns and is called an **(m ×n) matrix** (read "m by n matrix"). The matrix of m rows and n columns is said to be of **order** "m by n" or m × n. The above matrix is also denoted by $[a_{ij}]$, $\begin{matrix} i = 1,2,\ldots,m \\ j = 1, 2, \ldots,n \end{matrix}$

The m horizontal n-tuples $(a_{11}, a_{12}, \ldots, a_{1n})$, $(a_{21}, a_{22}, \ldots, a_{2n})$, … …, $(a_{m1}, a_{m2}, \ldots, a_{mn})$ are the m rows of the matrix and the n vertical m-tuples

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \dots, \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

are its n columns. The element a_{ij}, called the **ij–entry or ij–component,** appears in the i th row and j th column.

A matrix consisting of a single row is called a **row matrix** (or **row vector**) and a matrix consisting of a single column is called a **column matrix** (or **column vertor**).

Matrices are generally denoted by capital letters A, B, X, Y etc. Square brackets [ ], or, curved brackets ( ), or, Two pairs of parallel lines || || are used for the mathematical notations of matrices. In this book we will use the notation [ ].

Examples of matrices

Example 1. $A = \begin{bmatrix} 1 & 0 & -5 \\ 2 & -3 & 7 \end{bmatrix}$ is a matrix of order 2 ×3 over the real field $\mathbb{R}$ and also over the complex field C.

The rows of A are (1, 0, – 5) and (2, –3, 7) and its columns are $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 7 \end{pmatrix}$.

Example 2. $B = \begin{bmatrix} 2 & 0 & i \\ -i & 1 & 4 \\ 1+i & -5 & 3 \end{bmatrix}$ is a matrix of order 3 × 3 over the complex field C.

The rows of B are (2, 0, i), (–i, 1, 4) and (1 + i, – 5, 3) and its columns are $\begin{pmatrix} 2 \\ -i \\ 1+i \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} i \\ 4 \\ 3 \end{pmatrix}$.

Two matrices $A = [\,a_{ij}\,]$ and $B = [\,b_{ij}\,]$ are **equal** if and only if they are identical i.e if and only if they contain the same number of rows and the same number of columns and $a_{ij} = b_{ij}$ for all values of i and j.

3.3 Addition and scalar multiplication of matrices :

Addition of matrices is defined only for the matrices having same number of rows and the same number of columns. Let A and B be the two matrices having m rows and n columns.

i. e. $A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$ and

$$B = [\,b_{ij}\,] = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix}$$

Then the sum of A and B is

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{bmatrix}$$

The multiplication of matrix by numbers (scalars) is defined as follows : The product of an (m × n) matrix A by a number k is denoted by kA or Ak and is the (m × n) matrix obtained by multiplying every element of A by k, that is,

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \dots & \dots & \dots & \dots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{bmatrix}$$

We also define $-A = (-1)A$ and $A - B = A + (-B)$.

If the matrices A, B, C are conformable for addition and if k is any scalar, then we can state that

(i) $A + B = B + A$ **(Commutative law)**

(ii) $(A + B) + C = A + (B + C)$ **(Associative law)**

(iii) $A + 0 = 0 + A = A$.

(iv) $k(A + B) = kA + kB = (A + B)k$.

where 0 is the zero matrix of the same order.

For examples, if $A = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 7 \\ -5 & 4 \end{bmatrix}$

then $A + B = \begin{bmatrix} 1+2 & (-2)+7 \\ 3+(-5) & 5+4 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -2 & 9 \end{bmatrix}$.

$$2A = \begin{bmatrix} 2.\ 1 & 2.\ (-2) \\ 2.\ 3 & 2.\ \ 5 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 6 & 10 \end{bmatrix}$$

and $A - B = \begin{bmatrix} 1-2 & -2-7 \\ 3-(-5) & 5-4 \end{bmatrix} = \begin{bmatrix} -1 & -9 \\ 8 & 1 \end{bmatrix}$.

3.4 Matrix Multiplication

Two matrices A and B are conformable for multiplication if the number of columns in A is equal to the number of rows in B.

$$\text{Let } A = [a_1\ a_2\ \ldots.\ a_n] \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\text{The } AB = [a_1 b_1 + a_2 b_2 + \ldots + a_n b_n] = \left[\sum_{i=1}^{n} a_i b_i\right]$$

Again, let the $m \times p$ matrix $A = [a_{ij}]$ and the $p \times n$ matrix $B = [b_{ij}]$, then AB is the $m \times n$ matrix $C = [c_{ij}]$ where

$$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots + a_{ip} b_{pj} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

$$\begin{cases} i = 1, 2, \ldots, m \\ j = 1, 2, \ldots, n \end{cases}$$

If the matrices A, B, C are conformable for the indicated sums and products, we have the following properties :

(i) $(AB)C = A(BC)$ **(Associative law)**

(ii) $A(B + C) = AB + AC$
(iii) $(A + B)C = AC + BC$ } **(Distributive laws)**

(iv) $k(AB) = (kA)B = A(kB)$ where k is any scalar.

Remarks : In the matrix product AB, the matrix A is called the **pre-multiplier** (or **pre-factor)** and B is called the **postmultiplier (or post-factor)**

For examples, let $A = \begin{bmatrix} 1 & -3 & 5 \\ 2 & 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -2 & 4 \\ 3 & 0 \end{bmatrix}$

$$\text{then } AB = \begin{bmatrix} 1\cdot 1 + (-3)\cdot(-2) + 5\cdot 3 & 1\cdot(-1) + (-3)\cdot 4 + 5\cdot 0 \\ 2\cdot 1 + 0\cdot(-2) + (-1)\cdot 3 & 2\cdot(-1) + 0\cdot 4 + (-1)\cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+6+15 & -1-12+0 \\ 2+0-3 & -2+0+0 \end{bmatrix} = \begin{bmatrix} 22 & -13 \\ -1 & -2 \end{bmatrix}.$$

$$BA = \begin{bmatrix} 1 & -1 \\ -2 & 4 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & -3 & 5 \\ 2 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot 1 + (-1)\cdot 2 & 1\cdot(-3) + (-1)\cdot 0 & 1\cdot 5 + (-1)\cdot(-1) \\ (-2)\cdot 1 + 4\cdot 2 & (-2)\cdot(-3) + 4\cdot 0 & (-2)\cdot 5 + 4\cdot(-1) \\ 3\cdot 1 + 0\cdot 2 & 3\cdot(-3) + 0\cdot 0 & 3\cdot 5 + 0.\ (-1) \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -3 & 6 \\ 6 & 6 & -14 \\ 3 & -9 & 15 \end{bmatrix} \quad \therefore\ AB \neq BA.$$

3.5 Transpose of a matrix

If A is an $m \times n$ matrix over the real field IR, then the $n \times m$ matrix obtained from the matrix A by writing its rows as columns and its columns as rows is called the **transpose** of A and is denoted by the symbol A^T. That is, if $A = [a_{ij}]$ is an $m \times n$ matrix then $A^T. = [a_{ji}]$ is an $n \times m$ matrix.

For examples, let $A = \begin{bmatrix} 1 & 0 & 5 & -7 \\ 2 & 3 & -1 & 6 \end{bmatrix}$, then $A^T = \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 5 & -1 \\ -7 & 6 \end{bmatrix}$.

3.6 Complex conjugate (or conjugate) of a matrix

The **conjugate** of a complex number $z = x + iy$ is the complex number $\bar{z} = x - iy$. If A is an $m \times n$ matrix over the complex field, then we say that the **conjugate** of a matrix A is the matrix $\bar{A}$ whose elements are respectively the conjugates of the elements of A. That is, if $A = [a_{ij}]$, then $\bar{A} = [\bar{a}_{ij}]$.

For examples, if $A = \begin{bmatrix} 1 & i & 1+i \\ -i & 2 & 2+3i \\ 5 & 1+2i & -5 \end{bmatrix}$

then $\bar{A} = \begin{bmatrix} 1 & -i & 1-i \\ i & 2 & 2-3i \\ 5 & 1-2i & -5 \end{bmatrix}$

Definition Real matrix

A matrix A is called **real** provided it satisfies the relation $A = \bar{A}$

Definition Imaginary matrix

A matrix A is called **imaginary** Provided it satisfies the relation $A = -\bar{A}$

3.7 Conjugate transpose of a complex matrix

The conjugate of the transpose of a given complex matrix A is said to be the **conjugate transpose** of A and is generally denoted by the symbol A^*. That is,

if $A = [a_{ij}]$ is a complex matrix, then

$$A^* = (\bar{A}^T) = [\bar{a}_{ji}].$$

For examples, if $A = \begin{bmatrix} 1 & i & 1+i \\ -i & 2 & 2+3i \\ 5 & 1+2i & -5 \end{bmatrix}$

then $A^* = \begin{bmatrix} 1 & i & 5 \\ -i & 2 & 1-2i \\ 1-i & 2-3i & -5 \end{bmatrix}$

and if $B = \begin{bmatrix} 5+7i & -3i & 2-5i \\ 4i & 1+6i & 3+2i \end{bmatrix}$

then $B^* = \begin{bmatrix} 5-7i & -4i \\ 3i & 1-6i \\ 2+5i & 3-2i \end{bmatrix}$.

3.8 Special types of matrices with examples

Square matrix : A matrix with the same number of rows and columns is called a **square matrix.**

For examples $\begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & -5 \\ 7 & -2 & 6 \end{bmatrix}$ are square matrices.

Rectangular matrix : The number of rows and columns of a matrix need not be equal.

When $m \neq n$ i.e the number of rows and columns of the array are not equal, then the matrix is known as the **rectangular matrix :**

For examples, $\begin{bmatrix} 1 & -1 & 2 & 7 \\ -2 & 3 & 5 & 0 \end{bmatrix}_{4\times 2}$ and $\begin{bmatrix} 1-i & 2i & 3 \\ i & 2 & 0 & 5 \end{bmatrix}_{4\times 2}$

are rectangular matrices.

Diagonal matrix : A square matrix whose elements $a_{ij} = 0$ when $i \neq j$ is called a **diagonal matrix.**

For examples, $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are diagonal matrices.

A diagonal matrix whose diagonal elements are all equal is called a **scalar matrix.**

Identity matrix (or **Unit matrix**) **:**

A square matrix whose elements $a_{ij} = 0$, if $i \neq j$ and $a_{ij} = 1$ if $i = j$ is called the **identity matrix** or **unit matrix**.

For examples, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ are identity matrices of order 3 and 4 respectively.

Zero matrix or **null matrix :** A matrix in which every element is zero is called a **null matrix** or a **zero matrix.**

For examples, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are zero matrices.

Upper and lower triangular matrices :

A square matrix whose elements $a_{ij} = 0$ for $i > j$ is called an **upper triangular matrix** and a square matrix whose elements $a_{ij} = 0$ for $i < j$ is called a **lower triangular matrix.**

For examples, $\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 7 \end{bmatrix}$ and $\begin{bmatrix} 1 & -i & 2 & 3 \\ 0 & -1 & i & 5 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 7 \end{bmatrix}$

are upper triangular matrices.

and $\begin{bmatrix} 5 & 0 & 0 \\ -1 & 2 & 0 \\ 3 & 7 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 & 0 \\ i & 2 & 0 & 0 \\ -2 & 5 & -1 & 0 \\ 3 & 7 & -i & 6 \end{bmatrix}$ are lower triangular matrices.

Symmetric matrix : A matrix equal to its transpose i.e a square matrix such that $a_{ij} = a_{ji}$ for $1 \le i, j \le n$ is said to be **symmetric.** In short we can say a square matrix A will be symmetric if $A^T = A$.

For examples, $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & 7 \\ -3 & 7 & 3 \end{bmatrix}$

are symmetric matrices.

λA and λB are also symmetric if λ is a scalar.

Skew-symmetric matrix : A matrix equal to the negative of its transpose i,e a square matrix such that $a_{ij} = -a_{ji}$ and in which therefore, $a_{ii} = 0$ is said to be **skew-symmetric.**

For examples, $A = \begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$

are skew-symmetric matrices.

Hermitian matrix : If $A = [a_{ij}]$ is a square matrix over the complex field and $A^* = (\bar{A}^T) = A$ i.e $a_{ij} = \bar{a}_{ji}$ for $i, j = 1, 2, \ldots, n$; then A is called a **Hermitian matrix**.

For examples, $A = \begin{bmatrix} 2 & 2-3i & 3 \\ 2+3i & 5 & 1+i \\ 3 & 1-i & 0 \end{bmatrix}$ and

$B = \begin{bmatrix} 1 & i & -1-2i \\ -i & 3 & 5+3i \\ -1+2i & 5-3i & -1 \end{bmatrix}$ are Hermitian matrices.

In this case diagonal elements of the matrices will be real numbers.

Skew-Hermitian matrix : If $A = [a_{ij}]$ is a square matrix over the complex field, and $A^* = (\bar{A}^T) = -A$ i.e $a_{ij} = -\bar{a}_{ji}$ for $i, j = 1, 2, \ldots, n$ then A is called a **skew-Hermitian matrix.**

For examples, $A = \begin{bmatrix} 2i & 2-3i & 3 \\ -2-3i & 5i & 1+i \\ -3 & -1+i & 0 \end{bmatrix}$ and

$B = \begin{bmatrix} i & 1-i & 5 \\ -1-i & 2i & i \\ -5 & i & 0 \end{bmatrix}$ are skew-Hermitian matrices.

In this case diagonal elements of the matrix will be either zero or wholly complex number.

Orthogonal matrix : A real square matrix A is said to be **orthogonal** if $AA^T = A^T A = I$.

that is, if $A^T = A^{-1}$ (the inverse of the matrix A)

For examples, $A = \begin{bmatrix} \frac{1}{9} & \frac{8}{9} & -\frac{4}{9} \\ \frac{4}{9} & -\frac{4}{9} & -\frac{7}{9} \\ \frac{8}{9} & \frac{1}{9} & \frac{4}{9} \end{bmatrix}$ and

$B = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$ are orthogonal matrices.

Idempotent matrix : A square matrix A is called an **idempotent** matrix if $A^2 = A$.

For examples, $\begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$ and $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

are idempotent matrices.

Nilpotent matrix : A square matrix A is called a **nilpotent matrix** of order n if $A^n = 0$ and $A^{n-1} \neq 0$ where n is a positive integer and 0 is the null matrix.

For examples, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$

are nilpotent matrices of order 2.

Periodic matrix : A square matrix A is called **periodic** if $A^{m+1} = A$ where m is a positive integer.

If m is the least positive integer for which $A^{m+1} = A$, then A is said to be of **period** m.

For example, $A = \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$ is a periodic matrix of period 2.

Involutory matrix : A square matrix A is called an **involutory matrix** if $A^2 = I$.

For example, $A = \begin{bmatrix} 4 & 3 \\ -5 & -4 \end{bmatrix}$ is an involutory matrix.

Unitary matrix : Let A be a complex square matrix, then A is called a **unitary matrix** if $AA^* = A^*A = I$ or equivalently $A^* = A^{-1}$ where $A^* = (\bar{A})^T = (\bar{A^T})$.

For example, $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$ is a unitary matrix.

Normal matrix : Let A be a complex square matrix, then A is called a **normal matrix** if $A^*A = AA^*$ where A^* is the conjugate transpose of A.

For example, $A = \begin{bmatrix} 2+3i & 1 \\ i & 1+2i \end{bmatrix}$ is a normal matrix.

3.9 Theorems on transpose matrix

Theorem 1 If A and B are comparable matrices and A^T and B^T are the transpose matrices of A and B respectively.

then (i) $(A^T)^T = A$ (ii) $(A+B)^T = A^T + B^T$.

(iii) $(AB)^T = B^T A^T$ (iv) $(\alpha A)^T = \alpha A^T$. where α is a scalar.

Proof : (i) Let $A = [a_{ij}]_{\substack{i = 1, 2, \ldots, m \\ j = 1, 2, \ldots, n}}$

then by definition $A^T = [a_{ij}]^T = [a_{ji}]$

Now $(A^T)^T = [a_{ji}]^T = [a_{ij}] = A \quad \therefore (A^T)^T = A$.

(ii) Let $A = [a_{ij}]$ and $B = [b_{ij}]$ where $\substack{i = 1, 2, \ldots, m \\ j = 1, 2, \ldots, n}$

then $C = A + B$ is defined and $[c_{ij}] = [a_{ij}] + [b_{ij}]$.

Now by the definition of the transpose of C

we have $(A+B)^T = C^T = [c_{ij}]^T = [c_{ji}] = [a_{ji}] + [b_{ji}]$

$= [a_{ij}]^T + [b_{ij}]^T = A^T + B^T$

$\therefore (A+B)^T = A^T + B^T$.

(iii) Let $A = [a_{ij}]$ $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$

and $B = [b_{jk}]$ $j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, p$

Then $A^T = [a_{ij}]^T = [a_{ji}]$ is an $n \times m$ matrix

$B^T = [b_{jk}]^T = [b_{kj}]$ is a $p \times n$ matrix.

Thus AB is a $m \times p$ matrix so that $(AB)^T$ is a $p \times m$ matrix. Also $B^T A^T$ is a $p \times m$ matrix. Therefore, $(AB)^T$ and $B^T A^T$ have same dimensions.

Now $AB = [c_{ik}]$ where (i, k) th element of AB is

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk} \quad \text{where} \quad \begin{matrix} i = 1, 2, \ldots, m \\ k = 1, 2, \ldots, p. \end{matrix}$$

Therefore, the (k,i)th element of $(AB)^T$

$$= \sum_{j=1}^{n} a_{ij} b_{jk} = \sum_{j=1}^{n} a_{ji}^T \; b_{kj}^T$$

$$= \sum_{j=1}^{n} b_{kj}^T \; a_{ji}^T$$

$= $ (k,i)th element of $B^T A^T$

Hence $(AB)^T = B^T A^T$

(iv) Let $A = [a_{ij}]$ $\therefore A^T = [a_{ij}]^T = [a_{ji}]$

Now $(\alpha A)^T = [\alpha a_{ij}]^T = [\alpha a_{ji}] = \alpha [a_{ji}] = \alpha A^T$

$\therefore (\alpha A)^T = \alpha A^T$.

3.10 Theorems on complex conjugate of a matrix

Theorem 1. If $A = [a_{ij}]$ is any $m \times n$ complex matrix, then $\bar{\bar{A}} = A$.

Proof : By definition of complex conjugate of a complex matrix, we have $\bar{A} = [\bar{a}_{ij}]$ for all $1 \le i \le m$, $1 \le j \le n$ and $\bar{a}_{ij}$ is the complex conjugate of a_{ij} i.e $\bar{a}_{ij}$ is the entry in the i th row and j th column of $\bar{A}$. Again $\bar{\bar{A}} = [\bar{\bar{a}}_{ij}]$ for all $1 \le i \le m$, $1 \le j \le n$ and $\bar{\bar{a}}_{ij}$ is the complex conjugate of $\bar{a}_{ij}$ i.e $\bar{\bar{a}}_{ij} = a_{ij}$ (the entry in the i th row and j th column of the matrix A). Therefore, the entries of the complex conjugate of $\bar{A}$ = the corresponding entries of A. Also A and $\bar{\bar{A}}$ are matrices of the same order. Hence $\bar{\bar{A}} = A$.

Theorem 2. If A and B are two complex matrices conformable to addition, then $\overline{A+B} = \bar{A} + \bar{B}$.

Proof : Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be any two matrices of order $m \times n$. Then we have $A + B = [a_{ij} + b_{ij}]$ for all $1 \le i \le m$, $1 \le j \le n$.

Also by definition, we have $\bar{A} = [\bar{a}_{ij}]$ and $\bar{B} = [\bar{b}_{ij}]$

$\therefore \bar{A} + \bar{B} = [\bar{a}_{ij}] + [\bar{b}_{ij}] = [\overline{a_{ij} + b_{ij}}]$ (1)

for all $1 \le i \le m$, $1 \le j \le n$,

since $\bar{z}_1 + \bar{z}_2 = \overline{z_1 + z_2}$ where z_1 and z_2 are any two complex numbers.

Again $\overline{A+B}$ = complex conjugate of $[a_{ij} + b_{ij}]$

= complex conjugate of c_{ij} where $c_{ij} = a_{ij} + b_{ij}$

$= [\bar{c}_{ij}] = \left[\overline{a_{ij} + b_{ij}}\right]$ for all $1 \le i \le m$, $1 \le j \le n$.

Therefore, $\overline{A+B} = \left[\overline{a_{ij} + b_{ij}}\right]$ for all $1 \le i \le m$, $1 \le j \le n$. (2)

Since the corresponding entries of $\bar{A} + \bar{B}$ and $\overline{A+B}$ are equal and also both $\bar{A} + \bar{B}$ and $\overline{A+B}$ are matrices of order $m \times n$, from (1) and (2), we conclude that $\overline{A+B} = \bar{A} + \bar{B}$.

Theorem 3. Let $A = [a_{ij}]$ be any $m \times n$ matrix and $B = [b_{jk}]$ be any $n \times p$ matrix over the complex field i.e A and B are conformable to the product AB, then $\overline{AB} = \bar{A}\,\bar{B}$.

Proof : Since A and B are conformable to the product AB, so $AB = [a_{ij}] \times [b_{jk}] = [c_{ik}]$ where $c_{ik} = a_{ij} b_{jk}$ for all $1 \le i \le m$, $1 \le k \le p$ and $1 \le j \le n$.

Also $\bar{A} = [\bar{a}_{ij}]$ for all $1 \le i \le m$, $1 \le j \le n$

and $\bar{B} = [\bar{b}_{jk}]$ for all $1 \le j \le n$, $1 \le k < \le p$.

Now $\bar{A}\,\bar{B}$ is defined and we have $\bar{A}\,\bar{B} = [\bar{a}_{ij}] \times [\bar{b}_{jk}] = [\bar{d}_{ik}]$ (1)

where $\bar{d}_{ik} = \bar{a}_{ij}\,\bar{b}_{jk}$ for all $1 \le i \le m$, $1 \le k < \le p$ and $1 \le j \le n$.

Again $\overline{AB}$ = complex conjugate of AB

or, $\overline{AB} = [\bar{c}_{ik}]$ where $c_{ik} = a_{ij} b_{jk}$

$= \left[\overline{a_{ij} b_{jk}}\right] = [\bar{a}_{ij}\,\bar{b}_{jk}]$

$= [\bar{d}_{ik}]$ (2) since $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

for any two complex numbers z_1 and z_2.

Since $\bar{d}_{ik} = \bar{a}_{ij}\,\bar{b}_{jk}$ for all $1 \le i \le m$, $1 \le k < \le p$ and $1 \le j \le n$.

Combining (1) & (2) we conclude that $\overline{AB} = \bar{A}\,\bar{B}$.

Theorem 4. Let $A = [a_{ij}]$ be any $m \times n$ complex matrix, then $\overline{\lambda A} = \bar{\lambda}\,\bar{A}$, λ being any complex number.

Proof : By definition of the conjugate of a matrix, we have $\bar{A} = [\bar{a}_{ij}]$ for all $1 \le i \le m$, $1 \le j \le n$ and $\bar{a}_{ij}$ is the complex conjugate of a_{ij}.

Also $\overline{\lambda A} = \left[\, \overline{\lambda a_{ij}} \,\right] = [\bar{\lambda}\, \bar{a}_{ij}]$ (1) for all $1 \le i \le m$, $1 \le j \le n$.

since $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ where z_1 and z_2 are any two complex numbers.

Again $\bar{\lambda}\, \bar{A}, = [\bar{b}_{ij}]$ where $b_{ij} = \overline{\lambda a_{ij}}$ for all $1 \le i \le m$, $1 \le j \le n$.

$= [\bar{\lambda}\, \bar{a}_{ij}]$ for all $1 \le i \le m$, $1 \le j \le n$ (2)

Thus from (1) & (2), we conclude that the corresponding entries of $\overline{\lambda A}$ and $\bar{\lambda}\, \bar{A}$ are equal and they are of same order.

Hence $\overline{\lambda A} = \bar{\lambda}\, \bar{A}$.

3.11 Theorems on the conjugate transpose of a complex matrix.

Theorem 1. Let A^* and B^* be the conjugate transpose of A and B respectively, then

(i) $(A^*)^* = A$

(ii) $(A + B)^* = A^* + B^*$, A and B being conformable for addition

(iii) $(AB)^* = B^* A^*$, A and B being conformable for multiplication.

(iv) $(kA)^* = \bar{k} A^*$, k being a complex number.

Proof : (i) Let $A^* = B$, then $B = (\bar{A}^T) = (\bar{A})^T$

and $B^T = \left((\bar{A})^T\right)^T = \bar{A}$

Again $(\bar{B})^T = (\overline{B^T}) = \bar{\bar{A}} = A$

Therefore, $B^* = A$, since $B^* = (\bar{B})^T = (\overline{B^T})$

Hence $(A^*)^* = A$, since $B = A^*$.

(ii) By definition, we have

$$(A + B)^* = (\overline{A+B})^T = (\bar{A} + \bar{B})^T$$

$$= (\bar{A})^T + (\bar{B})^T \text{ since } (C + D)^T = C^T + D^T.$$

$$= A^* + B^*, \text{ since } A^* = (\bar{A})^T.$$

(iii) $(AB)^* = (\overline{AB})^T \quad = (\bar{A}\,\bar{B})^T$

$$= (\bar{B})^T (\bar{A})^T \text{ since } (CD)^T = D^T C^T.$$

$$= B^* A^*.$$

(iv) $(kA)^* = (\overline{kA})^T \quad = (\bar{k}\,\bar{A})^T = \bar{k}\,(\bar{A})^T = \bar{k} A^*$

Corollary : For any complex matrix A, $(\lambda A)^* = \lambda A^*$ where λ is a scalar.

3.12 Theorems on symmetric and skew-symmetric matrices.

Theorem 1. Every square matrix can be uniquely expressed as the sum of a symmetric matrix and a skew-symmetric matrix.

Proof : Let A be a square matrix, and A^T be its transpose. Then we have

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = B + C \text{ (say)} \qquad (1)$$

$$\text{where } B = \frac{1}{2}(A + A^T) \text{ and } C = \frac{1}{2}(A - A^T) \qquad (2)$$

$$\text{Now } B^T = \frac{1}{2}(A + A^T)^T = \frac{1}{2}(A^T + (A^T)^T)$$

$$= \frac{1}{2}(A^T + A) = B$$

$$\text{and } C^T = \frac{1}{2}(A - A^T)^T = \frac{1}{2}(A^T - (A^T)^T)$$

$$= \frac{1}{2}(A^T - A)$$

$$= -\frac{1}{2}(A - A^T) = -C.$$

Thus B is a symmetric matrix and C is a skew-symmetric matrix. Hence from (1) we conclude that a square matrix can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix.

To prove the uniqueness of the representation of (1), let if possible $A = P + Q$ (3)

where P is a symmetric matrix and Q is a skew-symmetric matrix so that $P^T = P$ and $Q^T = -Q$

Then $A^T = (P+Q)^T = P^T + Q^T = P - Q$ (4)

Adding (3) and (4) we get

$$A + A^T = 2P \quad \therefore P = \frac{1}{2}(A^T + A)$$

Again subtracting (4) from (3), we get $A - A^T = 2Q$

$$\therefore Q = \frac{1}{2}(A - A^T)$$

This establishes the uniqueness of (1).

Hence the theorem is proved.

Theorem 2. If A is a square matrix, then $A + A^T$ is symmetric and $A - A^T$ is skew-symmetric.

Proof : First portion

If A is a square matrix, then

$$(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$$

Hence by definition $A + A^T$ is symmetric.

Second portion

$$(A - A^T)^T = A^T - (A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T)$$

Hence by definition $A - A^T$ is skew-symmetric.

Theorem 3. If A is a skew-symmetric matrix then $AA^T = A^TA$ and A^2 is symmetric.

Proof : First portion

Since A is skew-symmetric, $A^T = -A$

$$\therefore AA^T = A(-A) = -A^2 \quad \text{(i)}$$

$$\text{and } A^TA = (-A)A = -A^2 \quad \text{(ii)}$$

From (i) and (ii), we get $AA^T = A^T A$.

Second portion

$$(AA^T)^T = (A^T)^T A^T = AA^T \text{ Since } (A^T)^T = A.$$

and $(A^TA)^T = A^T (A^T)^T = A^TA$

$\therefore AA^T$ and $A^T A$ are both symmetric matrices. Therefore, $-A^2$ is a symmetric matrix. Again since -1 is a scalar,

A^2 is a symmetric matrix.

Note : If A is a symmetric matrix, than kA is also a symmetric matrix where k is any scalar.

Theorem 4. If A and B are both skew–symmetric matrices of same order such that AB = BA, then AB is symmetric.

Proof : If A and B are both skew-symmetric matrices, then $A^T = -A$ and $B^T = -B$

$$\text{i.e } A = -A^T \text{ and } B = -B^T$$

Also given $AB = BA = (-B^T)(-A^T) = B^TA^T = (AB)^T$

$$\therefore (AB)^T = AB.$$

Thus AB is a symmetric matrix.

Theorem 5. If A and B are n-square symmetric matrices, then AB is symmetric if and only if A and B commute (i.e AB = BA).

Proof : Since A and B are symmetric, $A^T = A$ and $B^T = B$.

Now $(AB)^T = (BA)^T = A^TB^T = AB$

$\therefore (AB)^T = AB.$

Thus AB is a symmetric matrix.

Conversely, $AB = (AB)^T = B^TA^T = BA$

Since $B^T = B$ and $A^T = A$.

Hence A and B commute.

Remark : The matrix B^TAB is symmetric or skew-symmetric according as A is symmetric or skew-symmetric.

3.13 Theorems on Hermitian and skew-Hermitian matrices

Theorem 1. In a complex field every square matrix can be expressed uniquely as the sum of a Hermitian matrix and a skew-Hermitian matrix.

Proof : Let A be a square matrix of order n and A^* be the conjugate transpose of A. Then we can write

$$A = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*) = P + Q \text{ (say)} \qquad (1)$$

where $P = \frac{1}{2}(A + A^*)$ and $Q = \frac{1}{2}(A - A^*)$

Now $P^* = \frac{1}{2}(A + A^*)^* = \frac{1}{2}(A^* + (A^*)^*)$

$= \frac{1}{2}(A^* + A) = P$

$Q^* = \frac{1}{2}(A - A^*)^* = \frac{1}{2}(A^* - (A^*)^*)$

$= \frac{1}{2}(A^* - A)$

$= -\frac{1}{2}(A - A^*) = -Q.$

Thus P is a Hermitian and Q is a skew-Hermitian matrix. Hence from (1) we see that a square matrix A can be expressed as the sum of a Hermitian matrix P and a skew-Hermitian matrix Q.

To prove the uniqueness of representation (1) let, if possible, A be also expressible in the form $A = R + S$ (2) where R is Hermitian and S is skew-Hermitian such that

$R^* = R$ and $S^* = -S$,

Now $A^* = (R + S)^* = R^* + S^* = R - S$ (3)

Adding (2) and (3), we get $R = \frac{1}{2}(A + A^*) = P$

Subtracting (3) from (2), we get $S = \frac{1}{2}(A - A^*) = Q$.

which establishes the uniqueness of (1).

Hence the theorem is proved.

Theorem 2. If A and B are Hermitian matrices then AB + BA is Hermitian and AB − BA is skew-Hermitian.

Proof : Since A and B are Hermitian matrices, we have $A^* = A$ and $B^* = B$ where A^* and B^* are the complex conjugate transposes of A and B respectively.

Now $(AB + BA)^* = (AB)^* + (BA)^*$

$= B^*A^* + A^*B^*$

$= BA + AB = AB + BA.$

Hence AB + BA is a Hermitian matrix.

Again $(AB - BA)^* = (AB)^* - (BA)^*$

$= B^*A^* - A^*B^*$

$= BA - AB$

$= -(AB - BA)$

Hence AB − BA is a skew-Hermitian matrix.

Theorem 3. If A and B are Hermitian, then AB is Hermitian if and only if A and B commute.

Proof : Since A and B are Hermitian,

$A^* = A$ and $B^* = B$ (1)

Now $(AB)^* = B^*A^* = BA = AB$. since $AB = BA$.

Thus by definition, AB is Hermitian.

Again conversely, $AB = (AB)^* = B^*A^* = BA$

Since A and B are Hermitian

Thus A and B commute.

Theorem 4. Every square matrix can be uniquely expressed as P + iQ where p and Q are Hermitian.

Proof : Let A be the square matrix of order n and A^* be its conjugate transpose.

Then we can write

$$A = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*) = \frac{1}{2}(A + A^*) + i.\frac{1}{2i}(A - A^*)$$

$= P + iQ$ (say) (1) where $P = \frac{1}{2}(A + A^*)$ and $Q = \frac{1}{2i}(A - A^*)$

Now $P^* = \frac{1}{2}(A + A^*)^* = \frac{1}{2}\{A^* + (A^*)^*\}$

$$= \frac{1}{2}(A^* + A) = \frac{1}{2}(A + A^*) = P$$

$$Q^* = \left\{\frac{1}{2i}(A - A^*)\right\}^* = \frac{1}{2(-i)}\{A^* - (A^*)^*\}$$

$$= -\frac{1}{2i}(A^* - A)$$

$$= \frac{1}{2i}(A - A^*) = Q$$

Therefore, P and Q are Hermitian matrices. Thus from (1), we see that A can be expressed in the form P + iQ where P and Q are Hermitian matrices.

To prove the uniqueness of the representation (1) let if possible $A = R + iS$ (2) where R and S are Hermitian, that is,

$$R^* = R \text{ and } S^* = S.$$

Now $A^* = (R + iS)^* = R^* - iS^* = R - iS$ (3)

(2) + (3) gives $A + A^* = 2R$

$$\therefore R = \frac{1}{2}(A + A^*) = P$$

(2) − (3) gives $A - A^* = 2iS$

$$\therefore S = \frac{1}{2i}(A - A^*) = Q$$

which establishes the uniqueness of (1). Hence the theorem is proved.

Remarks : (1) If A is any n-square complex matrix then $A + A^*$ is Hermitian and $A - A^*$ is skew-Hermitian.

(2) If A is any n-square complex matrix then AA^* and A^*A are both Hermitian.

(3) If A is Hermitian matrix, then i A is a skew-Hermitian matrix.

(4) If A is a skew-Hermitian matrix, then i A is a Hermitian matrix.

(5) A is Hermitian if and only if $\bar{A}$ is Hermitian.

(6) A is skew-Hermitian if and only if $\bar{A}$ is skew-Hermitian.

3.14 Theorem on idempotent matrices

Theorem 1 : If A and B are idempotent matrices then AB is idempotent if AB = BA.

Proof : Since A and B are idempotent matrices

$\therefore A^2 = A$ and $B^2 = B$, Also given $AB = BA$.

Now $(AB)^2 = AB(AB) = A(BA)B = A(AB)B$

$= (A.A)(B.B) = A^2B^2 = AB.$

Hence AB is an idempotent matrix.

Theorem 2 : If A and B are idempotent matrices, then A + B will be idempotent if and only if AB = BA = 0.

Proof : Since A and B are idempotent matrices

$\therefore A^2 = A$ and $B^2 = B$.

Now if AB = BA = 0, then $(A + B)^2 = (A + B)(A + B)$

$= A^2 + AB + BA + B^2$

$= A^2 + B^2$ since $AB = BA = 0$

$= A + B$ since $A^2 = A$, $B^2 = B$.

Hence A + B is an idempotent matrix.

Again if (A + B) is an idempotent matrix, then

$(A + B)^2 = A + B$

or, $A^2 + AB + BA + B^2 = A + B$

or, $A + AB + BA + B = A + B$

or, $AB + BA = 0$

or, $AB = -BA$ (1)

Also, $AB = A^2B = AAB = A(-BA) = -(AB)A$

$= -.-(BA)A = BA^2 = BA$

$\therefore AB = BA$ (2)

Adding (1) and (2), we get $AB + AB = -BA + BA = 0$

i. e. $2AB = 0$ $\therefore AB = 0 = BA$.

Hence A + B is idempotent if and only if AB = 0 = BA.

Theorem 3. If A is an idempotent matrix, then B = I − A is idempotent and AB = 0 = BA.

Proof : We know that $AI = IA = A$

Also since A is idempotent, $A^2 = A$.

Again since I and A are square matrices, So $I - A$ is also a square matrix.

Now $(I - A)^2 = (I - A)(I - A) = I - IA - AI + A^2$

$= I - A - A + A = I - A.$

So by definition $I - A$ is an idempotent matrix.

Again $AB = A(I - A) = AI - A^2 = A - A = 0$

$\therefore AB = 0$

$BA = (I - A)A = IA - A^2 = A - A = 0$

$\therefore BA = 0.$

Theorem 4. If A and B are square matrices of order n such that $AB = A$ and $BA = B$, then A and B are idempotent.

Proof : $ABA = (AB)A = AA = A^2$ (1) since $AB = A$

Also $ABA = A(BA) = AB = A$ (2) since $BA = B$ & $AB = A$.

From (1) & (2), we get $A^2 = A$ $\therefore$ A is idempotent.

Similarly, we have

$BAB = B(AB) = BA = B$ (3) since $AB = A$ & $BA = B$

Also $BAB = (BA)B = BB = B^2$ (4) since $BA = B$

From (3) and (4) we get $B^2 = B$.

$\therefore$ B is idempotent.

3.15. Singular and non-singular matrices

Let D be the determinant of the square matrix A, then if $D = 0$, the matrix A is called the **singular matrix** and if $D \neq 0$, the matrix A is called the **non-singular matrix.**

As for examples

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ -1 & -1 & 7 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

are singular matrices,

since $D_1 = |A| = 0$, $D_2 = |B| = 0$ and $D_3 = |C| = 0$.

$$\text{Again } A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \\ 1 & -2 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & -1 & 4 \end{bmatrix}$$

are non-singular matrices,

since $D_1 = |A| = -6 \neq 0$ and $D_2 = |B| = 18 \neq 0$.

3.16 Inverse matrix

A square matrix A is said to be **invertible** if there exists a unique matrix B such that $AB = BA = I$ where I is the unit matrix. We call such a matrix B the **inverse** of A & is generally denoted by $\bar{A}^1$. Here we have to note that if B is the inverse of A, then A is the inverse of B.

Example 1. Let $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

$$\text{Then } AB = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3 & -2+2 \\ 6-6 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 4-3 & 2-2 \\ -6+6 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Therefore, A and B are invertible and are inverses of each other. That is, $\bar{A}^1 = B$ and $\bar{B}^1 = A$.

Example 2. Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

$$\text{Then } AB = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6+0-5 & -2+0+2 & 2+0-2 \\ 15-15+0 & -5+6+0 & 5-5+0 \\ 0-15+15 & 0+6-6 & 0-5+6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

$$=\begin{bmatrix} 6-5+0 & 0-1+1 & -3+0+3 \\ -30+30+0 & 0+6-5 & 15+0-15 \\ 10-10+0 & 0-2+2 & -5+0+6 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=I$$

Therefore, A and B are invertible and are inverses of each other. That is, $A^{-1} = B$ and $B^{-1} = A$.

3.17 Adjoint or adjugate matrix

$$\text{Let } A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Let D be the determinant of the matrix A,

$$\text{then } D=|A|=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Let A_{ij} ($i = 1, 2, 3, \ldots, n$ and $j = 1, 2, 3, \ldots, n$)

be the cofactors of the determinant D.

Form the matrix $[A_{ij}]$. Then the transpose of the matrix $[A_{ij}]$ is called the **adjoint** or the **adjugate matrix** of the matrix A and is generally denoted by Adj A.

$$\therefore \text{Adj}A=\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}^T=\begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

As for examples

$$\text{Let } A=\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & -1 \\ 2 & 1 & 0 \end{bmatrix} \text{ then}$$

$$\text{Adj } A=\begin{bmatrix} 1 & -2 & -5 \\ 3 & -6 & 3 \\ -11 & 4 & 1 \end{bmatrix}^T=\begin{bmatrix} 1 & 3 & -11 \\ -2 & -6 & 4 \\ -5 & 3 & 1 \end{bmatrix}.$$

3.18 Process of finding the inverse of a square matrix

$$\text{Let the matrix } A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Let D be the determinant of the matrix A. Evaluate the determinant D; if D = 0, the matrix A is **singular** and it has no inverse, if $D \neq 0$ the matrix A is **non-singular** and A^{-1} exists. Find the adjoint matrix Adj A of the matrix A; then

$$A^{-1}=\frac{1}{D}\text{ Adj } A=\frac{\text{Adj}A}{|A|}$$

$$\therefore A^{-1}=\begin{vmatrix} \frac{A_{11}}{|A|} & \frac{A_{21}}{|A|} & \cdots & \frac{A_{n1}}{|A|} \\ \frac{A_{12}}{|A|} & \frac{A_{22}}{|A|} & \cdots & \frac{A_{n2}}{|A|} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{A_{1n}}{|A|} & \frac{A_{2n}}{|A|} & \cdots & \frac{A_{nn}}{|A|} \end{vmatrix}.$$

3.19 Theorems on inverse matrix

Theorem 1 : If A^{-1} is the inverse of the n–square matrix A, then $AA^{-1} = A^{-1}A = I$, where I is the unit matrix of the same order.

Proof : Let $A = [a_{ij}]$ i, j = 1, 2, ..., n; then $A^{-1} = \frac{[A_{ji}]}{D}$ where D is the determinant of the matrix A ($D \neq 0$) and A_{ij} are the co-factors of D.

$$AA^{-1} = [a_{ij}] \times \frac{1}{D}[A_{ji}] = \frac{1}{D}[a_{ij}]\ [A_{ji}]$$

$$=\frac{1}{D}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

$$= \frac{1}{D}\begin{bmatrix} D & 0 & \dots & 0 & 0 \\ 0 & D & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & D \end{bmatrix} = \frac{D}{D}\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots \\ & 0 & 0 & \dots & 1 \end{bmatrix} = I$$

Similarly, we can show that $A^{-1} A = I$

Thus $A A^{-1} = A^{-1} A = I$.

Theorem 2. If A and B are non-singular matrices, $(AB)^{-1} = B^{-1} A^{-1}$. Also $(A^{-1})^{-1} = A$ and $(A^{-1})^T = (A^T)^{-1}$.

Proof : Since, $(AB)(B^{-1} A^{-1}) = A(BB^{-1}) A^{-1} = AIA^{-1} = AA^{-1} = I$ and $(B^{-1}A^{-1})(AB) = B^{-1} (A^{-1} A) B = B^{-1}IB = B^{-1}B = I$.

Thus $B^{-1} A^{-1}$ is the inverse of AB i.e $(AB)^{-1} = B^{-1} A^{-1}$

If A and B are two matrices such that AB = I, then $A = B^{-1}$ and $B = A^{-1}$.

Therefore, $A = B^{-1} = (A^{-1})^{-1}$ $\therefore A = (A^{-1})^{-1}$.

$A^T (A^{-1})^T = (A^{-1}A)^T = (I)^T = I$ shows that $(A^{-1})^T = (A^T)^{-1}$.

Corollary : If A_1 A_2, ..., A_n are non-singular matrices of the same order, their product is non-singular and

$$(A_1 A_2 \dots A_r)^{-1} = A_r^{-1} \dots A_2^{-1} A_1^{-1}.$$

Theorem 3. A matrix has an inverse if and only if it is non-singular. A non-singular matrix has only one inverse.

Proof : Suppose that A is of the type (m, n) and that $m \leq n$. If A has an inverse A^{-1} the products AA^{-1} and $A^{-1}A$ are both defined and hence A^{-1} must be of the form (n, m). It follows that $A^{-1} A = I_n$. If A has rank r, then we have $r \leq m \leq n \leq r$, since the rank of I_n is n and the rank of the product of two matrices is less than or equal to the rank of either factor. Hence m = n = r and A is non-singular. If $n \leq m$ the result follows similarly, using $A A^{-1} = I_m$.

Now suppose that $A = [a_{ij}]$ is non-singular

Since $|A| \neq 0$, we can let $b_{ij} = \frac{A_{ji}}{|A|}$

(i, j = 1, 2, ..., n), where A_{ij} are the co-factors of a_{ij} in A.

Let B be the matrix $[b_{ij}]$. Then

$$AB = \left[\sum_{k=1}^{n} a_{ik} \quad b_{kj}\right] = \left[\sum_{k=1}^{n} a_{ik}\frac{[A_{jk}]}{|A|}\right] = \left[\frac{|A|\,\delta_{ij}}{|A|}\right] = I$$

$$\text{Similarly, } BA = \left[\sum_{k=1}^{n} b_{ik} \, a_{kj}\right] = \left[\sum_{k=1}^{n} \frac{[A_{ki}]}{|A|} a_{kj}\right] = I.$$

Hence B is an inverse of A and every non-singular matrix A has an inverse.

Finally, we have to show that the inverse of a non-singular matrix is unique. Suppose that A is non-singular and A^{-1} is its inverse. If B is another inverses of A, then we have AB = I Multiply each side of this equation on the left by A^{-1}, then we $A^{-1} (AB) = A^{-1} I = A^{-1}$.

or, $(A^{-1}A) B = A^{-1}$

or, $IB = A^{-1}$

or, $B = A^{-1}$

Hence A has only one inverse A^{-1},

[The **rank** of a matrix A is the maximum number of linearly independent row (or column) vectors of A.]

3.20 Theorems on orthogonal matrices.

Theorem 1. If A and B are orthogonal matrices, each of order n then the matrices AB and BA are also orthogonal.

Proof : Since A and B are n-rowed orthogonal matrices

$$A^T A = AA^T = I_n \text{ and } B^T B = BB^T = I_n \quad (1)$$

The matrix product AB is also a square matrix of order n and

$$\begin{aligned}(AB)^T(AB) &= (B^T A^T)(AB) \\ &= B^T (A^T A)B \\ &= B^T I_n B \\ &= B^T B = I_n.\end{aligned}$$

Thus AB is an orthogonal matrix of order n.

Similarly, $BA\,(BA)^T = (BA)\,(A^TB^T) = B\,(AA^T)B^T$

$= BI_nB^T = BB^T = I_n$.

Hence BA is an orthogonal matrix of order n.

Theorem 2. If A is an orthogonal matrix, then A^{-1} is also orthogonal.

Proof : If A is orthogonal, we have

$AA^T = A^T A = I$ where I is the identity matrix.

or, $(AA^T)^{-1} = (A^TA)^{-1} = I^{-1} = I$

or, $(A^T)^{-1}\, A^{-1} = A^{-1}\, (A^T)^{-1} = I$

or, $(A^{-1})^T\, A^{-1} = A^{-1}(A^{-1})^T = I$, since $(A^T)^{-1} = (A^{-1})^T$

Hence A^{-1} is orthogonal by definition. That is, inverse of an orthogonal matrix is also orthogonal.

Theorem 3. Transpose of an orthogonal matrix is also orthogonal.

Proof : Here we have to show that if A is an orthogonal matrix, then A^T is also orthogonal. By definition if A is an orthogonal matrix, then we have $AA^T = A^T A = I$.

or, $(AA^T)^T = (A^T A)^T = I^T = I$

or, $(A^T)^T A^T = A^T (A^T)^T = I$

Hence A^T is orthogonal by definition.

Remark : For any square matrix A, we have if $AA^T = I$, then $A^TA = I$.

3.21 Theorems on unitary matrices

Theorem 1. If A and B are unitary matrices then AB is also a unitary matrix.

Proof : If A and B are unitary matrices, then by definition we have

$$AA^* = A^*A = I \qquad (1)$$

and $$BB^* = B^*B = I \qquad (2)$$

Now $(AB)\,(AB)^* = (AB)\,(B^*A^*)$

$= A\,(BB^*)\,A^*$

$= AIA^*$

$= AA^* = I$

Similarly, we have

$(AB)^*(AB) = (B^*A^*)\,(AB)$

$= B^*\,(A^*A)\,B$

$= B^*IB$

$= B^*B = I$

Hence AB is a unitary matrix.

Theorem 2. If A is an unitary matrix, then A^{-1} is also unitary.

Proof : By definition if A is an unitary matrix, then we have $AA^* = A^*A = I$.

$\therefore\quad (AA^*)^{-1} = (A^*A)^{-1} = I^{-1} = I$

or, $(A^*)^{-1}\, A^{-1} = A^{-1}\, (A^*)^{-1} = I$

or, $(A^{-1})^*\, A^{-1} = A^{-1}\,(A^{-1})^* = I$ Since $(A^*)^{-1} = (A^{-1})^*$

$\therefore$ A^{-1} is an unitary matrix by definition.

Theorem 3. Transpose of an unitary matrix is also unitary.

Proof : By definition if A is an unitary matrix, then $AA^* = A^*A = I$ where I is the unit matrix.

$\therefore\quad (AA^*)^T = (A^*A)^T = I^T = I$

or, $(A^*)^T A^T = A^T (A^*)^T = I$

or, $(A^T)^*A^T = A^T (A^T)^* = I$ Since $(A^*)^T = (A^T)^*$

Hence by definition A^T is an unitary matrix.

Remark : For any square matrix A, if $AA^* = I$, then $A^*A = I$.

3.22 Theorems on involutory matrix.

Theorem 1. A matrix A is **involutory** if and only if $(I - A)(I + A) = 0$.

Proof : Let $(I - A)(I + A) = 0 \Rightarrow I - A^2 = 0$

$\Rightarrow A^2 = I.$

$\Rightarrow$ A is involutory

Conversely, Let A be involutory. Then by definition, we have $A^2 = I$

$\therefore (I - A)(I + A) = I - A^2 = I - I = 0.$

Hence the theorem is proved.

Theorem 2. If A is an involutory matrix, Then $\frac{1}{2}(I + A)$ and $\frac{1}{2}(I - A)$ are idempotent.

Proof : $\left\{\frac{1}{2}(I + A)\right\}^2 = \frac{1}{4}(I^2 + 2A + A^2)$

$= \frac{1}{4}(I + 2A + I)$, since $A^2 = I$ & $I^2 = I$

$= \frac{1}{2}(I + A)$. Thus $\frac{1}{2}(I + A)$ is idempotent.

Again $\left\{\frac{1}{2}(I - A)\right\}^2 = \frac{1}{4}(I^2 - 2A + A^2)$

$= \frac{1}{4}(I - 2A + I)$ since $A^2 = I$ & $I^2 = I$

$= \frac{1}{2}(I - A)$. Thus $\frac{1}{2}(I - A)$ is idempotent.

Hence the theorem is proved.

3.23 Solution of linear equations by applying matrices

(i) When the linear system has n linear equations in n unknowns.

$$\text{Let} \quad \left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = l_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = l_2 \\ \dots \quad \dots \quad \dots \quad \dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = l_n \end{array}\right\} \qquad (1)$$

The above linear equations can be written in

matrix-form as

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{bmatrix} \qquad (2)$$

Suppose that

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } L = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{bmatrix}$$

Then equation no (2) reduces to $AX = L$ (3)

Let D be the determinant of the matrix A. Evaluate the determinant, if $D = 0$, A is **singular**, so A^{-1} does not exist and hence the system has no solution. If $D \neq 0$, A is **non-singular** So A^{-1} exists and hence the system has a solution. Now multiply both sides of (3) by A^{-1}, then we have

$A^{-1}AX = A^{-1}L$

or, $IX = A^{-1}L$ $\left\{\begin{array}{l}\text{Since } A^{-1}A = I \\ IX = X\end{array}\right.$

or, $X = A^{-1}L$

$$\text{That is,} \begin{vmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{vmatrix} = \begin{vmatrix} \frac{A_{11}}{|A|} & \frac{A_{21}}{|A|} \dots & \frac{A_{n1}}{|A|} \\ \frac{A_{12}}{|A|} & \frac{A_{22}}{|A|} \dots & \frac{A_{n2}}{|A|} \\ \dots & \dots \quad \dots & \dots \\ \frac{A_{1n}}{|A|} & \frac{A_{2n}}{|A|} \dots & \frac{A_{2n}}{|A|} \end{vmatrix} \begin{vmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{vmatrix} = \begin{vmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{vmatrix} \text{(say)}$$

Then $x_1 = m_1$, $x_2 = m_2$ ...., $x_n = m_n$ is a solution of the given system of n linear equations.

It is to be noted that the solution of the system of equations can also be found by reducing the augmented matrix of the given system to reduced row echelon form.

(ii) When the system has m linear equations in n unknowns and m < n.

Consider the following system of m linear equations in n unknowns $x_1, x_2, \dots, x_n$

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = l_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = l_2 \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = l_n \end{array}\right\} \quad (1)$$

in which m<n and a_{ij} and l_i, i = 1, 2, ..., m,
j = 1, 2, ..., n are constants (scalars).

The given system (1) can be written in matrix-form as

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_m \end{bmatrix} \quad (2)$$

Now the augmented matrix of the system is

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & : & l_1 \\ a_{21} & a_{22} & \dots & a_{2n} & : & l_2 \\ \dots & \dots & \dots & \dots & : & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & : & l_m \end{bmatrix}$$

We shall apply the elementary row transformations on the above augmented matrix to reduce it to the **reduced row echelon form.**

The reduced form of the augmented matrix will either give a solution of the given system or will indicate that the system is inconsistent.

Theorem 3.12 If A is an invertible n × n matrix, the system of linear equations $Ax = b$ has a unique solution given by $x = A^{-1}b$.

Proof : Since $A(A^{-1}b) = (AA^{-1})b = Ib = b$,

$\therefore x = A^{-1}b$ is a solution.

Conversely, suppose $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a solution,

so that $A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = b.$

Multiplying both sides of this equation by A^{-1} on the left

we get $A^{-1} A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A^{-1}b$

or, $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A^{-1}b$ since $A^{-1}A = I$.

Hence $x = A^{-1}b$ is the unique solution of the given system of linear equations $Ax = b$.

Worked out examples

Example 1.

Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 5 & 1 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & -2 \end{bmatrix}$

Find the matrices 2A, A + B and A − B.

Solution : $A = \begin{bmatrix} 1 & -2 & 3 \\ 5 & 1 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & -2 \end{bmatrix}$

$$2A = \begin{bmatrix} 2.1 & 2.(-2) & 2.3 \\ 2.5 & 2.1 & 2.(-4) \end{bmatrix} = \begin{bmatrix} 2 & -4 & 6 \\ 10 & 2 & -8 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 1 & -2 & 3 \\ 5 & 1 & -4 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2 & (-2)+3 & 3+5 \\ 5+1 & 1+4 & (-4)+(-2) \end{bmatrix} = \begin{bmatrix} 3 & 1 & 8 \\ 6 & 5 & -6 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 1 & -2 & 3 \\ 5 & 1 & -4 \end{bmatrix} + \begin{bmatrix} -2 & -3 & -5 \\ -1 & -4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+(-2) & (-2)+(-3) & 3+(-5) \\ 5+(-1) & 1+(-4) & (-4)+2 \end{bmatrix} = \begin{bmatrix} -1 & -5 & -2 \\ 4 & -3 & -2 \end{bmatrix}$$

Example 2. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$

Compute the matrix products AB and BA.

Solution : $AB = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5+0 & 0+0 \\ 0+10 & 0+5 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 10 & 5 \end{bmatrix}$

$BA = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 5+0 & 0+0 \\ 2+0 & 0+5 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 2 & 5 \end{bmatrix}$

So we see that $AB \neq BA$.

Example 3. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 1 & 3 & 2 & 5 \\ 2 & 1 & -1 & 3 \end{bmatrix}$

Calculate the product AB.

$AB = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 1 & 3 & 2 & 5 \\ 2 & 1 & -1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1+2+6 & 2+6+3 & -1+4-3 & 3+10+9 \\ 2+3+10 & 4+9+5 & -2+6-5 & 6+15+15 \end{bmatrix}$

$= \begin{bmatrix} 9 & 11 & 0 & 22 \\ 15 & 18 & -1 & 26 \end{bmatrix}$

It is to be noted that the product BA is not defined.

Example 4. If $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Prove that $A^3 + A^2 - 21A - 54I = 0$

where I is the identity matrix of order 3×3.

Proof : $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

$A^2 = AA = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

$= \begin{bmatrix} 4+4+3 & -4+2+6 & 6-12+0 \\ -4+2+6 & 4+1+12 & -6-6+0 \\ 2-4+0 & -2-2+0 & 3+12+0 \end{bmatrix} = \begin{bmatrix} 11 & 4 & -6 \\ 4 & 17 & -12 \\ -2 & -4 & 15 \end{bmatrix}$

$A^3 = A^2A = \begin{bmatrix} 11 & 4 & -6 \\ 4 & 17 & -12 \\ -2 & -4 & 15 \end{bmatrix} \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

$= \begin{bmatrix} -22+8+6 & 22+4+12 & -33-24+0 \\ -8+34+12 & 8+17+24 & -12-102+0 \\ 4-8-15 & -4-4-30 & 6+24+0 \end{bmatrix}$

$= \begin{bmatrix} -8 & 38 & -57 \\ 38 & 49 & -114 \\ -19 & -38 & 30 \end{bmatrix}$

$\therefore A^3 + A^2 - 21A - 45I$

$= \begin{bmatrix} -8 & 38 & -57 \\ 38 & 49 & -114 \\ -19 & -38 & 30 \end{bmatrix} + \begin{bmatrix} 11 & 4 & -6 \\ 4 & 17 & -12 \\ -2 & -4 & 15 \end{bmatrix}$

$-21 \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} - 45 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} -8 & 38 & -57 \\ 38 & 49 & -114 \\ -19 & -38 & 30 \end{bmatrix} + \begin{bmatrix} 11 & 4 & -6 \\ 4 & 17 & -12 \\ -2 & -4 & 15 \end{bmatrix}$

$- \begin{bmatrix} -42 & 42 & -63 \\ 42 & 21 & -126 \\ -21 & -42 & 0 \end{bmatrix} - \begin{bmatrix} 45 & 0 & 0 \\ 0 & 45 & 0 \\ 0 & 0 & 45 \end{bmatrix}$

$= \begin{bmatrix} -8+11+42-45 & 38+4-42-0 & -57-6+63-0 \\ 38+4-42-0 & 49+17-21-45 & -114-12+126-0 \\ -19-2+21-0 & -38-4+42-0 & 30+15-0-45 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$. Hence $A^3 + A^2 - 21A - 45I = O$.

Example 5. If $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 5 & -1 \\ 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 5 & 3 \\ 7 & -2 & 1 \\ 2 & 0 & -3 \end{bmatrix}$

then prove that $(AB)^T = B^TA^T$.

Proof : Given $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 5 & -1 \\ 2 & 3 & 4 \end{bmatrix}$, $\therefore A^T = \begin{bmatrix} 1 & -2 & 2 \\ 2 & 5 & 3 \\ 3 & -1 & 4 \end{bmatrix}$

Given $B = \begin{bmatrix} -1 & 5 & 3 \\ 7 & -2 & 1 \\ 2 & 0 & -3 \end{bmatrix}$, $\therefore B^T = \begin{bmatrix} -1 & 7 & 2 \\ 5 & -2 & 0 \\ 3 & 1 & -3 \end{bmatrix}$

Now $B^TA^T = \begin{bmatrix} -1 & 7 & 2 \\ 5 & -2 & 0 \\ 3 & 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ 2 & 5 & 3 \\ 3 & -1 & 4 \end{bmatrix}$

$$=\begin{bmatrix} (-1)\cdot 1+7\cdot 2+2\cdot 3 & (-1)\cdot(-2)+7\cdot 5+2\cdot(-1) & (-1)\cdot 2+7\cdot 3+2\cdot 4 \\ 5\cdot 1+(-2)\cdot 2+0\cdot 3 & 5\cdot(-2)+(-2)\cdot 5+0\cdot(-1) & 5\cdot 2+(-2)\cdot 3+0\cdot 4 \\ 3\cdot 1+1\cdot 2+(-3)\cdot 3 & 3\cdot(-2)+1\cdot 5+(-3)\cdot(-1) & 3\cdot 2+1\cdot 3+(-3)\cdot 4 \end{bmatrix}$$

$$=\begin{bmatrix} -1+14+6 & 2+35-2 & -2+21+8 \\ 5-4+0 & -10-10+0 & 10-6+0 \\ 3+2-9 & -6+5+3 & 6+3-12 \end{bmatrix}$$

$$=\begin{bmatrix} 19 & 35 & 27 \\ 1 & -20 & 4 \\ -4 & 2 & -3 \end{bmatrix} \qquad (1)$$

Again $AB=\begin{bmatrix} 1 & 2 & 3 \\ -2 & 5 & -1 \\ 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} -1 & 5 & 3 \\ 7 & -2 & 1 \\ 2 & 0 & -3 \end{bmatrix}$

$$=\begin{bmatrix} 1\cdot(-1)+2\cdot 7+3\cdot 2 & 1\cdot 5+2\cdot(-2)+3\cdot 0 & 1\cdot 3+2\cdot 1+3\cdot(-3) \\ (-2)\cdot(-1)+5\cdot 7+(-1)\cdot 2 & (-2)\cdot 5+5\cdot(-2)+(-1)\cdot 0 & (-2)\cdot 3+5\cdot 1+(-1)\cdot(-3) \\ 2\cdot(-1)+3\cdot 7+4\cdot 2 & 2\cdot 5+3\cdot(-2)+4\cdot 0 & 2\cdot 3+3\cdot 1+4\cdot(-3) \end{bmatrix}$$

$$=\begin{bmatrix} -1+14+6 & 5-4+0 & 3+2-9 \\ 2+35-2 & -10-10+0 & -6+5+3 \\ -2+21+8 & 10-6+0 & 6+3-12 \end{bmatrix}$$

$$=\begin{bmatrix} 19 & 1 & -4 \\ 35 & -20 & 2 \\ 27 & 4 & -3 \end{bmatrix}$$

$$\therefore (AB)^T=\begin{bmatrix} 19 & 35 & 27 \\ 1 & -20 & 4 \\ -4 & 2 & -3 \end{bmatrix} \qquad (2)$$

Hence from (1) and (2), we get $(AB)^T = B^T A^T$.

Example 6. Find the symmetric and skew symmetric parts of the matrix $A=\begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \\ 3 & 5 & 7 \end{bmatrix}$

Solution : Given $A=\begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \\ 3 & 5 & 7 \end{bmatrix}$ $\therefore A^T=\begin{bmatrix} 1 & 6 & 3 \\ 2 & 8 & 5 \\ 4 & 1 & 7 \end{bmatrix}$

The symmetric part of $A=\frac{1}{2}(A+A^T)$

$$=\frac{1}{2}\left(\begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \\ 3 & 5 & 7 \end{bmatrix}+\begin{bmatrix} 1 & 6 & 3 \\ 2 & 8 & 5 \\ 4 & 1 & 7 \end{bmatrix}\right)$$

$$=\frac{1}{2}\begin{bmatrix} 1+1 & 2+6 & 4+3 \\ 6+2 & 8+8 & 1+5 \\ 3+4 & 5+1 & 7+7 \end{bmatrix}=\frac{1}{2}\begin{bmatrix} 2 & 8 & 7 \\ 8 & 16 & 6 \\ 7 & 6 & 14 \end{bmatrix}=\begin{bmatrix} 1 & 4 & \frac{7}{2} \\ 4 & 8 & 3 \\ \frac{7}{2} & 3 & 7 \end{bmatrix}$$

The skew-symmetric part of $A=\frac{1}{2}(A-A^T)$

$$=\frac{1}{2}\left(\begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \\ 3 & 5 & 7 \end{bmatrix}-\begin{bmatrix} 1 & 6 & 3 \\ 2 & 8 & 5 \\ 4 & 1 & 7 \end{bmatrix}\right)$$

$$=\frac{1}{2}\begin{bmatrix} 1-1 & 2-6 & 4-3 \\ 6-2 & 8-8 & 1-5 \\ 3-4 & 5-1 & 7-7 \end{bmatrix}=\frac{1}{2}\begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -4 \\ -1 & 4 & 0 \end{bmatrix}=\begin{bmatrix} 0 & -2 & \frac{1}{2} \\ 2 & 0 & -2 \\ -\frac{1}{2} & 2 & 0 \end{bmatrix}$$

Example 7. Prove that the matrix

$$A=\frac{1}{6}\begin{bmatrix} 3 & 3 & 3 & 3 \\ 3 & -5 & 1 & 1 \\ 3 & 1 & 1 & -5 \\ 3 & 1 & -5 & 1 \end{bmatrix} \text{ is orthogonal.}$$

Proof : Given $A=\frac{1}{6}\begin{bmatrix} 3 & 3 & 3 & 3 \\ 3 & -5 & 1 & 1 \\ 3 & 1 & 1 & -5 \\ 3 & 1 & -5 & 1 \end{bmatrix}$

$$\therefore A^T=\frac{1}{6}\begin{bmatrix} 3 & 3 & 3 & 3 \\ 3 & -5 & 1 & 1 \\ 3 & 1 & 1 & -5 \\ 3 & 1 & -5 & 1 \end{bmatrix}$$

Now $AA^T=\frac{1}{36}\begin{bmatrix} 3 & 3 & 3 & 3 \\ 3 & -5 & 1 & 1 \\ 3 & 1 & 1 & -5 \\ 3 & 1 & -5 & 1 \end{bmatrix}\begin{bmatrix} 3 & 3 & 3 & 3 \\ 3 & -5 & 1 & 1 \\ 3 & 1 & 1 & -5 \\ 3 & 1 & -5 & 1 \end{bmatrix}$

$$=\frac{1}{36}\begin{bmatrix} 9+9+9+9 & 9-15+3+3 & 9+3+3-15 & 9+3-15+3 \\ 9-15+3+3 & 9+25+1+1 & 9-5+1-5 & 9-5-5+1 \\ 9+3+3-15 & 9-5+1-5 & 9+1+1+25 & 9+1-5-5 \\ 9+3-15+3 & 9-5-5+1 & 9+1-5-5 & 9+1+25+1 \end{bmatrix}$$

$$=\frac{1}{36}\begin{bmatrix} 36 & 0 & 0 & 0 \\ 0 & 36 & 0 & 0 \\ 0 & 0 & 36 & 0 \\ 0 & 0 & 0 & 36 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I. \therefore AA^T = I.$$

Similarly, $$A^TA = \frac{1}{36}\begin{bmatrix} 3 & 3 & 3 & 3 \\ 3 & -5 & 1 & 1 \\ 3 & 1 & 1 & -5 \\ 3 & 1 & -5 & 1 \end{bmatrix}\begin{bmatrix} 3 & 3 & 3 & 3 \\ 3 & -5 & 1 & 1 \\ 3 & 1 & 1 & -5 \\ 3 & 1 & -5 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I.$$

Therefore, $AA^T = A^TA = I$.

Hence A is orthogonal.

Example 8. If $A = \begin{bmatrix} 1 & 1+i \\ 2-3i & i \end{bmatrix}$ and $B = \begin{bmatrix} 2-i & i \\ 1+5i & 3 \end{bmatrix}$

then prove that $\overline{A+B} = \bar{A} + \bar{B}$.

Proof : $\bar{A} = \begin{bmatrix} 1 & 1-i \\ 2+3i & -i \end{bmatrix}$ and $\bar{B} = \begin{bmatrix} 2+i & -i \\ 1-5i & 3 \end{bmatrix}$

$$\therefore \bar{A} + \bar{B} = \begin{bmatrix} 3+i & 1-2i \\ 3-2i & 3-i \end{bmatrix} \qquad (1)$$

Again $$A + B = \begin{bmatrix} 3-i & 1+2i \\ 3+2i & 3+i \end{bmatrix}$$

$$\therefore \overline{A+B} = \begin{bmatrix} 3+i & 1-2i \\ 3-2i & 3-i \end{bmatrix} \qquad (2)$$

Thus from (1) and (2), we get

$$\overline{A+B} = \bar{A} + \bar{B}.$$

Example 9. Prove that $A = \begin{bmatrix} 2 & 2-3i & 3+5i \\ 2+3i & 3 & i \\ 3-5i & -i & 5 \end{bmatrix}$ is Hermitian.

Proof : $$\bar{A} = \begin{bmatrix} 2 & 2+3i & 3-5i \\ 2-3i & 3 & -i \\ 3+5i & i & 5 \end{bmatrix}$$

$$A^* = \bar{A}^T = \begin{bmatrix} 2 & 2-3i & 3+5i \\ 2+3i & 3 & i \\ 3-5i & -i & 5 \end{bmatrix} = A$$

$\therefore A^* = A$. Hence A is Hermitian.

Example 10. Prove that the matrix $A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$ is unitary.

Proof : Given $A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$ A^2

$$\therefore A^* = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$$

Now $$AA^* = \frac{1}{2}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1+1 & i-i \\ -i+i & 1+1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Similarly, $A^*A = I$.

Thus $AA^* = A^*A = I$

Hence A is an unitary matrix.

Example 11. Show that the matrix $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ is involutory.

Proof : $$A^2 = A \times A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} \times \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

$$=\begin{bmatrix} 25-24+0 & 40-40+0 & 0+0+0 \\ -15+15+0 & -24+25+0 & 0+0+0 \\ -5+6-1 & -8+10-2 & 0+0+1 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1 \quad \therefore A^2 = I$$

Hence the given matrix A is involutory.

Example 12 (a). Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Solution : Let D be the determinant of the matrix; then $D = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0$. So the matrix A is non-singular and hence A^{-1} exists.

Now the cofactors of D are

$A_{11} = 4$, $\quad A_{12} = -3$

$A_{21} = -2$, $\quad A_{22} = 1$

Then $\text{Adj } A = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}^T = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

$$\therefore A^{-1} = \frac{1}{D}\text{Adj } A = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}.$$

Example 12 (b). Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 0 & -1 \\ 3 & 3 & 2 \end{bmatrix}.$$

Solution : Let D be the determinant of the matrix;

$$\text{then } D = \begin{vmatrix} 2 & -1 & 3 \\ 4 & 0 & -1 \\ 3 & 3 & 2 \end{vmatrix} = 2(0+3) + 1(8+3) + 3(12-0) = 6 + 11 + 36 = 53 \neq 0.$$

So the matrix A is non-singular and A^{-1} exists. Now the cofactors of D are $A_{11} = \begin{vmatrix} 0 & -1 \\ 3 & 2 \end{vmatrix} = 3$,

$A_{12} = (-1)\begin{vmatrix} 4 & -1 \\ 3 & 2 \end{vmatrix} = -11$, $A_{13} = \begin{vmatrix} 4 & 0 \\ 3 & 3 \end{vmatrix} = 12$,

$A_{21} = (-1)\begin{vmatrix} -1 & 3 \\ 3 & 2 \end{vmatrix} = 11$. $A_{22} = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = -5$,

$A_{23} = (-1)\begin{vmatrix} 2 & -1 \\ 3 & 3 \end{vmatrix} = -9$, $A_{31} = \begin{vmatrix} -1 & 3 \\ 0 & -1 \end{vmatrix} = 1$,

$A_{32} = (-1)\begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix} = 14$, $A_{33} = \begin{vmatrix} 2 & -1 \\ 4 & 0 \end{vmatrix} = 4$.

$$\text{Therefore, Adj } A = \begin{bmatrix} 3 & -11 & 12 \\ 11 & -5 & -9 \\ 1 & 14 & 4 \end{bmatrix}^T = \begin{bmatrix} 3 & 11 & 1 \\ -11 & -5 & 14 \\ 12 & -9 & 4 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{D}\text{Adj } A = \frac{1}{53}\begin{bmatrix} 3 & 11 & 1 \\ -11 & -5 & 14 \\ 12 & -9 & 4 \end{bmatrix}.$$

Example 13. Find the inverse of the matrix $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ by using row canonical form.

Solution : $[AI_2] = \left[\begin{array}{cc:cc} 2 & 5 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right]$ Interchange first and second rows.

$\sim \left[\begin{array}{cc:cc} 1 & 3 & 0 & 1 \\ 2 & 5 & 1 & 0 \end{array}\right]$ We multiply first row by 2 and then subtract from the second row.

$\sim \left[\begin{array}{cc:cc} 1 & 3 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{array}\right]$ We multiply second row by 3 and then add with the first row.

$\sim \left[\begin{array}{cc:cc} 1 & 0 & 3 & -5 \\ 0 & -1 & 1 & -2 \end{array}\right]$ We multiply second row ,by (-1).

$\sim \left[\begin{array}{cc:cc} 1 & 0 & 3 & -5 \\ 0 & 1 & -1 & 2 \end{array}\right] = [I_2 \ A^{-1}]$

Hence A is invertible and $A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$.

Example 14. If $A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$ and

$B = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$ find $A^{-1}B$. **[D. U. P. 1985]**

Solution : Given $A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$

$$\text{Let } D = |A| = \begin{vmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{vmatrix} = -1(5+0) - 2(10-0) - 3(-4-4) = -5 - 20 + 24 = -1 \neq 0$$

So A is non-singular and hence A^{-1} exists.

$$\text{Cofactors of } -1 = A_{11} = \begin{vmatrix} 1 & 0 \\ -2 & 5 \end{vmatrix} = 5$$

$$\text{" \quad " } \; 2 = A_{12} = (-1)\begin{vmatrix} 2 & 0 \\ 4 & 5 \end{vmatrix} = -10$$

$$\text{" \quad " } \; -3 = A_{13} = \begin{vmatrix} 2 & 1 \\ 4 & -2 \end{vmatrix} = -8$$

$$\text{" \quad " } \; 2 = A_{21} = (-1)\begin{vmatrix} 2 & -3 \\ -2 & 5 \end{vmatrix} = -4$$

$$\text{" \quad " } \; 1 = A_{22} = \begin{vmatrix} -1 & -3 \\ 4 & 5 \end{vmatrix} = 7$$

$$\text{" \quad " } \; 0 = A_{23} = (-1)\begin{vmatrix} -1 & 2 \\ 4 & -2 \end{vmatrix} = 6$$

$$\text{" \quad " } \; 4 = A_{31} = \begin{vmatrix} 2 & -3 \\ 1 & 0 \end{vmatrix} = 3$$

$$\text{" \quad " } \; -2 = A_{32} = (-1)\begin{vmatrix} -1 & -3 \\ 2 & 0 \end{vmatrix} = -6$$

$$\text{" \quad " } \; 5 = A_{33} = \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} = -5.$$

$$\therefore \text{Adj } A = \begin{bmatrix} 5 & -10 & -8 \\ -4 & 7 & 6 \\ 3 & -6 & -5 \end{bmatrix}^T = \begin{bmatrix} 5 & -4 & 3 \\ -10 & 7 & -6 \\ -8 & 6 & -5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{D}\text{Adj } A = \frac{1}{-1} = \begin{bmatrix} 5 & -4 & 3 \\ -10 & 7 & -6 \\ -8 & 6 & -5 \end{bmatrix} = \begin{bmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{bmatrix}$$

$$\text{Thus } A^{-1}B = \begin{bmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -10+0-15 & -5+8-6 & 5+4+9 \\ 20+0+30 & 10-14+12 & -10-7-18 \\ 16+0+25 & 8-12+10 & -8-6-15 \end{bmatrix}$$

$$= \begin{bmatrix} -25 & -3 & 18 \\ 50 & 8 & -35 \\ 41 & 6 & -29 \end{bmatrix}.$$

Example 15. Find the inverse of the following matrix by using row canonical form : $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$

Solution :

$$[AI_3] = \left[\begin{array}{ccc:ccc} 3 & 4 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right]$$ Interchange first and second rows.

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 3 & 4 & -1 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right]$$ We multiply f rst row by 3 and 2 and then subtract from the second and third rows respectively.

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right]$$ Subtract third row from the second row.

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & -1 & -1 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right]$$ Multiply the second row by (–1).

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right]$$ We multiply second row by 5 and then subtract from the third row.

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & -10 & 5 & -7 & -4 \end{array}\right]$$ We multiply third row by $\left(-\frac{1}{10}\right)$

$$\sim\begin{bmatrix}1&0&3&:&0&1&0\\0&1&0&:&-1&1&1\\0&0&1&:&-\frac{5}{10}&\frac{7}{10}&\frac{4}{10}\end{bmatrix}\sim\begin{bmatrix}1&0&3&:&0&1&0\\0&1&0&:&-1&1&1\\0&0&1&:&-\frac{1}{2}&\frac{7}{10}&\frac{2}{5}\end{bmatrix}$$

We multiply third row by 3 and then subtract from the first row.

$$\sim\begin{bmatrix}1&0&0&:&\frac{3}{2}&-\frac{11}{10}&-\frac{6}{5}\\0&1&0&:&-1&1&1\\0&0&1&:&-\frac{1}{2}&\frac{7}{10}&\frac{2}{5}\end{bmatrix}=[I_3\ A^{-1}]$$

Hence A is invertible and $A^{-1}=\begin{bmatrix}\frac{3}{2}&-\frac{11}{10}&-\frac{6}{5}\\-1&1&1\\-\frac{1}{2}&\frac{7}{10}&\frac{2}{5}\end{bmatrix}$.

Example 16. Find the inverse of the matrix

$A=\begin{bmatrix}1&-1&0&0\\1&2&0&0\\-6&0&1&-2\\8&1&-2&1\end{bmatrix}$ by using only row transformations to reduce A to I.

Solution : $[AI_4]=\begin{bmatrix}1&-1&0&0&:&1&0&0&0\\1&2&0&0&:&0&1&0&0\\-6&0&1&-2&:&0&0&1&0\\8&1&-2&1&:&0&0&0&1\end{bmatrix}$

We subtract first row from second row. We multiply first row by 6 and then add with the third row. Also we multiply first row by 8 and then subtract from the fourth row.

$$\sim\begin{bmatrix}1&-1&0&0&:&1&0&0&0\\0&3&0&0&:&-1&1&0&0\\0&-6&1&-2&:&6&0&1&0\\0&9&-2&1&:&-8&0&0&1\end{bmatrix}$$

We multiply second row by 2 and add with the third row. We also multiply second row by 3 and then subtract from the fourth row.

$$\sim\begin{bmatrix}1&-1&0&0&:&1&0&0&0\\0&3&0&0&:&-1&1&0&0\\0&0&1&-2&:&4&2&1&0\\0&0&-2&1&:&-5&-3&0&1\end{bmatrix}$$

We multiply second row by $\frac{1}{3}$

$$\sim\begin{bmatrix}1&-1&0&0&:&1&0&0&0\\0&1&0&0&:&-\frac{1}{3}&\frac{1}{3}&0&0\\0&0&1&-2&:&4&2&1&0\\0&0&-2&1&:&-5&-3&0&1\end{bmatrix}$$

We add second row with the first row.

$$\sim\begin{bmatrix}1&0&0&0&:&\frac{2}{3}&\frac{1}{3}&0&0\\0&1&0&0&:&-\frac{1}{3}&\frac{1}{3}&0&0\\0&0&1&-2&:&4&2&1&0\\0&0&-2&1&:&-5&-3&0&1\end{bmatrix}$$

We multiply third row by 2 and then add with the fourth row.

$$\sim\begin{bmatrix}1&0&0&0&:&\frac{2}{3}&\frac{1}{3}&0&0\\0&1&0&0&:&-\frac{1}{3}&\frac{1}{3}&0&0\\0&0&1&-2&:&4&2&1&0\\0&0&0&-3&:&3&1&2&1\end{bmatrix}$$

We multiply fourth row by $\left(-\frac{1}{3}\right)$.

$$\sim\begin{bmatrix}1&0&0&0&:&\frac{2}{3}&\frac{1}{3}&0&0\\0&1&0&0&:&-\frac{1}{3}&\frac{1}{3}&0&0\\0&0&1&-2&:&4&2&1&0\\0&0&0&1&:&-1&-\frac{1}{3}&-\frac{2}{3}&-\frac{1}{3}\end{bmatrix}$$

We multiply fourth row by 2 and then add with the third row.

$$\sim\begin{bmatrix}1&0&0&0&:&\frac{2}{3}&\frac{1}{3}&0&0\\0&1&0&0&:&-\frac{1}{3}&\frac{1}{3}&0&0\\0&0&1&0&:&2&\frac{4}{3}&-\frac{1}{3}&-\frac{2}{3}\\0&0&0&1&:&-1&-\frac{1}{3}&-\frac{2}{3}&-\frac{1}{3}\end{bmatrix}=[I_4\ A^{-1}]$$

$R_3' = R_3 + 2R_4$

Hence A is invertible and

$$A^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 2 & \frac{4}{3} & -\frac{1}{3} & -\frac{2}{3} \\ -1 & -\frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

Example 17. Solve the following linear equations with the help of matrices : $\left.\begin{matrix} 2x + y = 1 \\ x - 2y = 3 \end{matrix}\right\}$ (1)

Solution : First Process : The system of linear equations can be written in matrix -form as

$$\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad (2)$$

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $L = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, then from

(2) we get $AX = L$ (3)

Let D be the determinant of the matrix A, then

$$D = \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} = -4 - 1 = -5 \neq 0.$$

So the matrix A is non-singular and hence A^{-1} exists.

Now the cofactors of D are.

$A_{11} = -2$, $A_{12} = -1$

$A_{21} = -1$, $A_{22} = 2$.

Therefore, Adj $A = \begin{bmatrix} -2 & -1 \\ -1 & 2 \end{bmatrix}^T = \begin{bmatrix} -2 & -1 \\ -1 & 2 \end{bmatrix}$ and

$$A^{-1} = \frac{1}{D} \text{adj } A = -\frac{1}{5} \begin{bmatrix} -2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & -\frac{2}{5} \end{bmatrix}$$

We multiply both sides of equation (3) by A^{-1}.

$A^{-1}AX = A^{-1}L$

or, $IX = A^{-1}L$ $\left\{\begin{matrix} \text{Since } A^{-1}A = I \\ \text{and } \quad IX = X \end{matrix}\right.$

or, $X = A^{-1}L$

$$\text{Thus } X = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & -\frac{2}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} + \frac{3}{5} \\ \frac{1}{5} - \frac{6}{5} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

or, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ Hence $\left.\begin{matrix} x = 1 \\ y = -1 \end{matrix}\right\}$.

Second process

The augmented matrix of the given system of linear equations is

$[AL] = \begin{bmatrix} 2 & 1 & : & 1 \\ 1 & -2 & : & 3 \end{bmatrix}$ Interchange first and second rows.

$\sim \begin{bmatrix} 1 & -2 & : & 3 \\ 2 & 1 & : & 1 \end{bmatrix}$ We multiply first row by 2 and then subtract from the second row.

$\sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & 5 & -5 \end{bmatrix}$ We multiply second row by $\frac{1}{5}$.

$\sim \begin{bmatrix} 1 & -2 & : & 3 \\ 0 & 1 & : & -1 \end{bmatrix}$

Now the system is in row canonical form.

Then forming linear system we have $y = -1$ and $x - 2y = 3$.

$\therefore x = 2y + 3 = -2 + 3 = 1$.

Thus the required solution of the system is $x = 1$ and $y = -1$.

Example 18. Solve the following linear equations with the help of matrices : $\left.\begin{matrix} 3x + 5y - 7z = 13 \\ 4x + y - 12z = 6 \\ 2x + 9y - 3z = 20 \end{matrix}\right\}$ **[D. U. P. 1984]**

Solution : First Process : The given linear equations can be written in matrix-form as

$$\begin{bmatrix} 3 & 5 & -7 \\ 4 & 1 & -12 \\ 2 & 9 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 \\ 6 \\ 20 \end{bmatrix} \qquad (1)$$

Suppose that $A = \begin{bmatrix} 3 & 5 & -7 \\ 4 & 1 & -12 \\ 2 & 9 & -3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $L = \begin{bmatrix} 13 \\ 6 \\ 20 \end{bmatrix}$

then the equation given by (1) reduces to $AX = L$ (2)

Let D be the determinant of the matrix A, then

$$D = \begin{vmatrix} 3 & 5 & -7 \\ 4 & 1 & -12 \\ 2 & 9 & -3 \end{vmatrix} \begin{array}{l} = 3(-3 + 108) - 5(-12 + 24) - 7(36 - 2) \\ = 315 - 60 - 238 = 17 \neq 0. \end{array}$$

So the matrix A is non-singular and hence A^{-1} exists.

We multiply both sides of equation no (2) by A^{-1} on the left,

Then we get $A^{-1}AX = A^{-1}L$

or, $IX = A^{-1}L$

or, $X = A^{-1}L$ (3) $\left\{\begin{array}{l} \text{Since } A^{-1}A = I \text{ and} \\ \qquad IX = X \end{array}\right.$

Now the cofactors of D are

$$A_{11} = \begin{vmatrix} 1 & -12 \\ 9 & -3 \end{vmatrix} = 105, \quad A_{12} = (-1)\begin{vmatrix} 4 & -12 \\ 2 & -3 \end{vmatrix} = -12,$$

$$A_{13} = \begin{vmatrix} 4 & 1 \\ 2 & 9 \end{vmatrix} = 34,$$

$$A_{21} = (-1)\begin{vmatrix} 5 & -7 \\ 9 & -3 \end{vmatrix} = -48, \quad A_{22} = \begin{vmatrix} 3 & -7 \\ 2 & -3 \end{vmatrix} = 5,$$

$$A_{23} = (-1)\begin{vmatrix} 3 & 5 \\ 2 & 9 \end{vmatrix} = -17.$$

$$A_{31} = \begin{vmatrix} 5 & -7 \\ 1 & -12 \end{vmatrix} = -53, \quad A_{32} = (-1)\begin{vmatrix} 3 & -7 \\ 4 & -12 \end{vmatrix} = 8.$$

$$A_{33} = \begin{vmatrix} 3 & 5 \\ 4 & 1 \end{vmatrix} = -17.$$

Therefore,

$$\text{Adj } A = \begin{bmatrix} 105 & -12 & 34 \\ -48 & 5 & -17 \\ -53 & 8 & -17 \end{bmatrix}^T = \begin{bmatrix} 105 & -48 & -53 \\ -12 & 5 & 8 \\ 34 & -17 & -17 \end{bmatrix}$$

$$\text{and } A^{-1} = \frac{1}{D}\text{Adj } A = \frac{1}{17}\begin{bmatrix} 105 & -48 & -53 \\ -12 & 5 & 8 \\ 34 & -17 & -17 \end{bmatrix}$$

Now from equation no (3) we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 105 & -48 & -53 \\ -12 & 5 & 8 \\ 34 & -17 & -17 \end{bmatrix}\begin{bmatrix} 13 \\ 6 \\ 20 \end{bmatrix}$$

$$= \frac{1}{17}\begin{bmatrix} 1365 & -288 & -1060 \\ -156 & +30 & +160 \\ 442 & -102 & -340 \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 1365 & -1348 \\ -156 & +190 \\ 442 & -442 \end{bmatrix}$$

$$\text{or, } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 17 \\ 34 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \text{ Hence } \left.\begin{array}{l} x = 1 \\ y = 2 \\ z = 0 \end{array}\right\}.$$

Second process : The augmented matrix of the given linear equations is

$$[AL] = \left[\begin{array}{ccc:c} 3 & 5 & -7 & 13 \\ 4 & 1 & -12 & 6 \\ 2 & 9 & -3 & 20 \end{array}\right]$$ We subtract third row from the first row. Also we multiply third row by 2 and then subtract from the second row.

$$\sim \left[\begin{array}{ccc:c} 1 & -4 & -4 & -7 \\ 0 & -17 & -6 & -34 \\ 2 & 9 & -3 & 20 \end{array}\right]$$ We multiply first row by 2 and then subtract from the third row.

$$\sim \left[\begin{array}{ccc:c} 1 & -4 & -4 & -7 \\ 0 & -17 & -6 & -34 \\ 0 & 17 & 5 & 34 \end{array}\right]$$ We add second row with third row.

$$\sim \left[\begin{array}{ccc:c} 1 & -4 & -4 & -7 \\ 0 & -17 & -6 & -34 \\ 0 & 0 & -1 & 0 \end{array}\right]$$ We multiply second row by $\left(-\frac{1}{17}\right)$ and third row by (−1).

$$\sim \left[\begin{array}{ccc:c} 1 & -4 & -4 & -7 \\ 0 & 1 & \frac{6}{17} & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Now the system is in row canonical form, Then forming the linear system, we have z = 0,

$$y + \frac{6}{17} z = 2, \; x - 4y - 4z = -7$$

or, $z = 0, y = 2, x = 8 - 7 = 1$.

Thus $x = 1$, $y = 2$, $z = 0$ is a solution of the given equations.

EXERCISES – 3

1. (a) If $A = \begin{bmatrix} 1 & 0 & 3 & 4 \\ 5 & 2 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & -1 & -2 & 5 \\ 1 & 0 & -3 & 4 \end{bmatrix}$

find the matrices 3A, A + B, A – B, 3A – 2B.

Answers : $\begin{bmatrix} 3 & 0 & 9 & 12 \\ 15 & 6 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 7 & -1 & 1 & 9 \\ 6 & 2 & -3 & 5 \end{bmatrix}$

$\begin{bmatrix} -5 & 1 & 5 & -1 \\ 4 & 2 & 3 & -3 \end{bmatrix}$ and $\begin{bmatrix} -9 & 2 & 13 & 2 \\ 13 & 6 & 6 & -5 \end{bmatrix}$.

(b) Show that the six matrices

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, B = \frac{1}{2}\begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$

$C = \frac{1}{2}\begin{bmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}, D = \frac{1}{2}\begin{bmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{bmatrix},$

$E = \frac{1}{2}\begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ satisfy the relations

$A^2 = B^2 = C^2 = I, AB = D, AC = BA = E.$

2. If $A = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 3 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Then show that (i) A (B + C) = AB + AC

(ii) (A + B) C = AC + BC. **[R. U. S. 1971]**

3. (a) If $A = \begin{bmatrix} -1 & 3 & 2 \\ 4 & -2 & 5 \\ 6 & 1 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & -1 \\ 5 & 2 & 1 \end{bmatrix}$

find the matrices AB and BA.

(b) Construct the products AB and BA.

where $A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix}$

and also show that

$AB - BA = -4C$, where $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

Answers : (a) $\begin{bmatrix} 15 & 15 & -2 \\ 25 & -4 & 11 \\ -7 & -15 & 2 \end{bmatrix}, \begin{bmatrix} -3 & 8 & -11 \\ 4 & -1 & 22 \\ 9 & 12 & 17 \end{bmatrix}$.

(b) $AB = \begin{bmatrix} -2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 2 \end{bmatrix}, BA = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & -2 \end{bmatrix}$.

4. If $A = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$

then show that $AB \neq BA$. **[R.U.P. 1974]**

5. If $A = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}, B = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$ and

$C = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ then show that

AB = BA = 0, AC = A and CA = C. **[R. U. S 1973]**

6. If $A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$ Prove that $A^2 - 3A + 2I = 0$

7. If $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$ Prove that $A^3 - 4A^2 - A + 4I = 0$.

8. If $A=\begin{bmatrix} 1 & -4 & -1 & -4 \\ 2 & 0 & 5 & -4 \\ -1 & 1 & -2 & 3 \\ -1 & 4 & -1 & 6 \end{bmatrix}$

Prove that $A^4-5A^3+9A^2-7A+2I=0$.

9. Verify that $(AB)^T=B^T A^T$ where

$$A=\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 2 & 4 & 9 \end{bmatrix} \text{ and } B=\begin{bmatrix} 0 & 1 & -1 \\ -3 & 2 & 4 \\ 1 & 1 & 0 \end{bmatrix}$$

10. If $A=\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ then show that

$A^3=A^2 . A=A. A^2=I$ and hence find A^{-1}.

[D. U. H. 1986; R. U. S. 1987]

Answer : $A^{-1}=\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$

11. Find the adjoint matrices and the inverse matrices of each of the following matrices :

(i) $A=\begin{bmatrix} 2 & -1 \\ 4 & 5 \end{bmatrix}$ (ii) $B=\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(iii) $C=\begin{bmatrix} -1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 1-i \end{bmatrix}$

Answers : (i) $\text{Adj } A=\begin{bmatrix} 5 & 1 \\ -4 & 2 \end{bmatrix}$, $A^{-1}=\frac{1}{14}\begin{bmatrix} 5 & 1 \\ -4 & 2 \end{bmatrix}$

(ii) $\text{Adj } B=\begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & -2 \\ -1 & -1 & 1 \end{bmatrix}$, $B^{-1}=\frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}$

(iii) $\text{Adj } C=\begin{bmatrix} 1+i & 0 & 0 \\ 0 & i-1 & 0 \\ 0 & 0 & -i \end{bmatrix}$, $C^{-1}=\begin{bmatrix} -1 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & \frac{1+i}{2} \end{bmatrix}$

12. (i) Find the inverse of the matrix

$$A=\begin{bmatrix} 2 & -1 & -1 \\ 1 & -2 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$ [J. U. H. 1975]

(ii) Find the inverse of the matrix

$$A=\begin{bmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{bmatrix}$$ [D. U. P. 1979]

Answers : (i) $A^{-1}=\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{6} & -\frac{5}{6} & \frac{1}{2} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{1}{2} \end{bmatrix}$

(ii) $A^{-1}=\frac{1}{419}\begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix}$

13. If $A=\begin{bmatrix} 1 & -3 & 2 \\ 4 & 1 & -1 \\ -3 & 2 & 5 \end{bmatrix}$ and $B=\begin{bmatrix} 2 & 1 & 5 \\ -1 & -2 & -2 \\ 3 & 1 & 2 \end{bmatrix}$

Find the matrices AB, BA, A^{-1}, B^{-1}, $(AB)^{-1}$. Check your results by verifying the relations.
$(AB)^{-1}=B^{-1}A^{-1}$, $AA^{-1}=I$, $B^{-1}B=I$.

Answers : $AB=\begin{bmatrix} 11 & 9 & 15 \\ 4 & 1 & 16 \\ 7 & -2 & -9 \end{bmatrix}$, $BA=\begin{bmatrix} -9 & 5 & 28 \\ -3 & -3 & -10 \\ 1 & -4 & 15 \end{bmatrix}$,

$A^{-1}=\frac{1}{80}\begin{bmatrix} 7 & 19 & 1 \\ -17 & 11 & 9 \\ 11 & 7 & 13 \end{bmatrix}$,

$B^{-1}=\frac{1}{17}\begin{bmatrix} -2 & 3 & 8 \\ -4 & -11 & -1 \\ 5 & 1 & -3 \end{bmatrix}$

$(AB)^{-1}=\frac{1}{1360}\begin{bmatrix} 23 & 51 & 129 \\ 148 & -204 & -116 \\ -15 & 85 & -25 \end{bmatrix}$

14. If $A=\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $B=\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Show that $A^3-A^2+A-2B=0$. [C. U. P. 1976]

15. If $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 1 \end{bmatrix}$

Show that $(A+B)^2 \neq A^2 + 2AB + B^2$.

[D. U. H. 1976, C. U. H. 1977]

16. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$

Find the values of $A^{-1}B$ and AB^{-1} **[D. U. P. 1976]**

Answers : $A^{-1}B = \begin{bmatrix} 1 & -1 & -\frac{7}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & \frac{2}{3} \end{bmatrix}$ $AB^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$

17. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ Find $\frac{1}{2}(A^{-1} + A)$. **[D. U. P. 1977]**

Answer : $\frac{1}{2}(A^{-1} + A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$

18. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ Prove that $A^3 = A^{-1}$.

19. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ -1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \end{bmatrix}$$

Answer : $A^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{6} & \frac{5}{6} & \frac{1}{6} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{6} & \frac{7}{6} & -\frac{1}{6} \\ 1 & 0 & -1 & 0 \end{bmatrix}$

20. Find the inverse of each of the following matrices by using only the elementary row operations (row equivalent canonical matrix) :

(i) $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 3 & -4 \\ 1 & 2 & 3 \\ 3 & -1 & -1 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 3 & 4 \\ 3 & -1 & 6 \\ -1 & 5 & 1 \end{bmatrix}$

(iv) $\begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 0 & 2 & 2 \\ 2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$ (vi) $\begin{bmatrix} 2 & 5 & 2 & 3 \\ 2 & 3 & 3 & 4 \\ 3 & 6 & 3 & 2 \\ 4 & 12 & 0 & 8 \end{bmatrix}$

Answers :

(i) $\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 7 & 17 \\ 10 & 10 & -10 \\ -7 & 11 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} \frac{31}{2} & -\frac{17}{2} & -11 \\ \frac{9}{2} & -\frac{5}{2} & -3 \\ -7 & 4 & 5 \end{bmatrix}$

(iv) $\begin{bmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{bmatrix}$ (v) $\begin{bmatrix} 0 & 1 & 0 & -2 \\ -2 & 1 & -1 & 1 \\ -1 & 1 & -1 & 0 \\ 1 & -2 & 1 & 3 \end{bmatrix}$

(vi) $\begin{bmatrix} -3 & \frac{3}{4} & \frac{5}{4} & \frac{7}{16} \\ 1 & -\frac{5}{12} & -\frac{1}{4} & \frac{5}{48} \\ 1 & -\frac{1}{12} & -\frac{1}{4} & -\frac{13}{48} \\ 0 & \frac{1}{4} & -\frac{1}{4} & \frac{1}{16} \end{bmatrix}$

21. If $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & 3 \\ 1 & -1 & 1 \end{bmatrix}$ then find $A^3 - 2A^2 - 4A - 2I$ and hence find A^{-1}.

[D. U. H. T. 1984]

Answers : O and $A^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -5 & 9 \\ 1 & -1 & 1 \\ -2 & 4 & -6 \end{bmatrix}$

22. Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

Answer : $A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$.

23. Show that the matrix

$$A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is invertible for all values of θ and find A^{-1}.

Answer : $A^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

24. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix}$

are two matrices, then find $A^{-1}B$ and AB^{-1}.

[D. U. H. T. 1981, 1987]

Answers : $A^{-1}B = \frac{1}{3}\begin{bmatrix} 3 & -3 & -11 \\ 0 & 3 & 10 \\ 0 & 0 & -2 \end{bmatrix}$, $AB^{-1} = \frac{1}{2}\begin{bmatrix} 0 & -2 & 0 \\ -2 & 4 & 0 \\ -12 & 17 & -3 \end{bmatrix}$

25. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

then prove that $(AB)^{-1} = B^{-1}.\ A^{-1}$ **[D. U. H. T. 1983, 1987]**

26. Prove that the following matrices are orthogonal :

(i) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ (ii) $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$ (iv) $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$

(v) $\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

27. If $A = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$ then show that AA^T and A^TA are both symmetric matrices.

28. Show that the matrix

$$A = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \text{ is orthogonal.}$$

29. Prove that the matrix

$$A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \text{ is unitary.}$$

30. Show that the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{bmatrix}$ is idempotent.

31. Show that the matrix $A = \begin{bmatrix} 4 & -1 & -4 \\ 3 & 0 & -4 \\ 3 & -1 & -3 \end{bmatrix}$ is involutory.

32. Show that if $A = \begin{bmatrix} 0 & 4 & 3 \\ 1 & -3 & -3 \\ -1 & 4 & 4 \end{bmatrix}$ then $A^2 = I$.

33. Show that if $A = \begin{bmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{bmatrix}$ then $A^2 = A$.

34. Prove that if $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ then $A^4 = 0$, but $A^3 \neq 0$.

35. Solve the following systems of linear equations with the help of matrices :

(i) $\left.\begin{array}{l} x + y = 5 \\ 2x + 3y = 13 \end{array}\right\}$ (ii) $\left.\begin{array}{l} x + y + z = 6 \\ x - y + z = 2 \\ 2x + y - z = 1 \end{array}\right\}$

[D. U. P. 1977]

(iii) $\left.\begin{array}{r} 3x + 2y - z = 20 \\ 2x + 3y - 3z = 7 \\ x - y + 6z = 41 \end{array}\right\}$ **[C. U. P. 1977]**

(iv) $\left.\begin{array}{r} 3x - y + z = -2 \\ x + 5y + 2z = 6 \\ 2x + 3y + z = 0 \end{array}\right\}$ (v) $\left.\begin{array}{r} x + 2y + z = 2 \\ 2x - y + 2z = -1 \\ 3x - 4y - 3z = -16 \end{array}\right\}$

Answers : (i) $x = 2; y = 3,$ (ii) $x = 1 = y = 2, z = 3.$

(iii) $x = 5, y = 6, z = 7,$ (iv) $x = -2, y = 0, z = 4,$

(v) $x = -2, y = 1, z = 2.$

36. Solve the following systems of linear equations with the help of matrices :

(i) $\left.\begin{array}{r} 5x - 6y + 4z = 15 \\ 7x + 4y - 3z = 19 \\ 2x + y + 6z = 46 \end{array}\right\}$ **[D. U. P. 1979, J. U. H. 1980]**

Answer : $x = 3, y = 4, z = 6.$

(ii) $\left.\begin{array}{r} 2x - 3y + 4z = 1 \\ 3x + 4y - 5z = 10 \\ 5x - 7y + 2z = 3 \end{array}\right\}$ **[D. U. P. 1966. J. U. H. 1978]**

Answer : $x = 2, y = 1, z = 0,$

(iii) $\left.\begin{array}{r} x + y + z = 9 \\ 2x + 5y + 7z = 52 \\ 2x + y - z = 0 \end{array}\right\}$ **[C. U. P. 1984]**

Answer : $x = 1, y = 3, z = 5.$

(iv) $\left.\begin{array}{r} x + 2y - z = -1 \\ 3x + 8y + 2z = 28 \\ 4x + 9y - z = 14 \end{array}\right\}$ **[D. U. H. 1976]**

Answer : $x = -26, y = 13, z = 1.$

(v) $\left.\begin{array}{r} x_1 + 3x_2 + 2x_3 = 5 \\ 2x_1 + x_2 + 3x_3 = 1 \\ 3x_1 + 2x_2 + x_3 = 4 \end{array}\right\}$ **[D. U. H. 1978]**

Answer : $x_1 = \frac{2}{9}, x_2 = \frac{17}{9}, x_3 = -\frac{4}{9}.$

(vi) $\left.\begin{array}{r} x + y + z = 1 \\ 2x + 5y + 5z = 2 \\ 4x + 9y + 12z = 3 \end{array}\right\}$ **[D. U. P. 1975]**

Answer : $x = 1, y = \frac{1}{3}, z = -\frac{1}{3}.$

(vii) $\left.\begin{array}{r} 2x - y - z = 6 \\ x + 3y + 2z = 1 \\ 3x - y - 5z = 1 \end{array}\right\}$ **[D. U. P. 1985]**

Answer : $x = 3, y = -2, z = 2.$

(viii) $\left.\begin{array}{r} x + y + z = 3 \\ x + 2y + 2z = 4 \\ x + 4y + 9z = 6 \end{array}\right\}$ **[D. U. H. 1983]**

Answer : $x = 2, y = 1, z = 0.$

(ix) $\left.\begin{array}{r} 2x + y + z = 3 \\ 3x - 2y + 2z = 9 \\ 4x + 3y - z = -1 \end{array}\right\}$

Answer : $x = 1, y = -1, z = 2.$

(x) $\left.\begin{array}{r} x + 2y + 3z + 4 = 0 \\ 2x + 4y + 5z + 7 = 0 \\ 3x + 5y + 6z + 10 = 0 \end{array}\right\}$ **[D. U. P. 1986]**

Answer : $x = -3, y = 1, z = -1.$

37. Solve the following systems of linear equations by using row equivalent canonical matrix (by elementary row transformations) :

(i) $\left.\begin{array}{r} 2x - y + z = 1 \\ x + 4y - 3z = -2 \\ 3x + 2y - z = 0 \end{array}\right\}$ (ii) $\left.\begin{array}{r} x + 2y + z = 2 \\ 3x + y - 2z = 1 \\ 4x - 3y - z = 3 \\ 2x + 4y + 2z = 4 \end{array}\right\}$

(iii) $\left.\begin{array}{r} x + 2y + 3z + t = 3 \\ x + y + z - t = 5 \\ x + y - z + t = -4 \\ x - y + z + t = 2 \end{array}\right\}$

Answers : (i) $x = 0, y = 1, z = 2,$ (ii) $x = 1, y = 0, z = 1,$

(iii) $x = 1, y = -1, z = 2, t = -2.$

(iv) $\left.\begin{array}{l} x+\ y+\ z+\ t=5 \\ 2x+\ y+3z-\ t=14 \\ 3x+3y-2z+2t=1 \\ 4x-2y+\ z-3t=6 \end{array}\right\}$

Answer : $x=1, y=2, z=3, t=-1$.

38. Find the inverse of $A=\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 0 & 2 & 2 \\ 2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$ [C. U. P. 1976]

Answer : $A^{-1}=\begin{bmatrix} 0 & 1 & 0 & -2 \\ -2 & 1 & -1 & 1 \\ -1 & 1 & -1 & 0 \\ 1 & -2 & 1 & 3 \end{bmatrix}$

39. Find the inverse of the matrix $A=\begin{bmatrix} 1 & 2 & 1 & 2 & 3 \\ 2 & 3 & 1 & 0 & 1 \\ 2 & 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 2 & -2 \end{bmatrix}$

by using only the elementary row operations (row equivalent canonical matrix).

Answer :

$$A^{-1}=\begin{bmatrix} 2 & -4 & 5 & -3 & -\frac{1}{2} \\ -1 & 3 & -3 & 1 & \frac{1}{2} \\ -2 & 2 & -3 & 4 & 0 \\ 0 & 1 & -1 & 0 & \frac{1}{2} \\ 1 & -2 & 2 & -1 & \frac{1}{2} \end{bmatrix}$$

40. If A and B are two symmetric matrices of the same order, then show that a necessary and sufficient condition for the matrix AB to be symmetric is that AB = BA.

41. Show that every non-singular idempotent matrix is an identity matrix.

42. If A is an idempotent matrix and A + B = I, then prove that B is an idempotent matrix and AB = BA = 0.

43. If A and B are n-rowed unitary matrices, then prove that BA is also unitary matrix.

CHAPTER SIX

VECTOR SPACES

6.1 Binary operation (or composition) on a set

It is a rule by which two elements can be combined together to give a new element. Such a rule may be addition, subtraction, multiplication and so on. The most fundamental concept for studying algebraic structures is that of binary operation on a set.

Definition : Let S be a non-empty set. Then

$S \times S = \{(a, b) : a \in S, b \in S\}$. A **binary operation** on a set S is a function (or mapping) from $S \times S$ into S; i. e if $f : S \times S \to S$, then f is said to be a **binary operation** on the set S.

Often we use the symbols $*$, o, +, $\times$, ... etc to denote the binary operations on set. Thus '$*$' will be a binary operation on S if and only if $a * b \in S$ for every a, b $\in S$ and $a*b$ is unique.

A binary operation on a set S is also sometimes called a **binary composition** in the set S. If $a*b \in S$ for every a, b $\in$ S, then we also say that S is **closed** with respect to the composition denoted by '$*$'.

A set having one or more binary operations is known as **algebraic structure.**

Now we will define some algebraic structures such as Groups, Rings, Fields and Vector Spaces using binary operations.

6.2 Definition of group with examples

A group G is a non-empty set of elements for which a binary operation $*$ is defined. This operation satisfies the following axioms :

(i) **Closure,** If a, b $\in$ G implies that $a * b \in G$.

(ii) **Associativity.** If a, b, c $\in$ G implies that

$$(a * b) * c = a * (b * c)$$

(iii) **Identity.** There exists a unique element $e \in G$ (called the **identity element**) such that $a * e = e * a = a$ for all $a \in G$.

(iv) **Inverse.** For every $a \in G$ there exists an element $a' \in G$ (called the **inverse** of a) such that $a * a' = a' * a = e$.

When the binary operation is **addition** G is called an **additive group** and when the binary operation is multiplication G is called a **multiplicative group.**

A group G is called **abelian** (or **commutative**) if for every $a, b \in G$, $a * b = b * a$.

Example 1. The set {1, – 1} is a group with respect to the binary operation of multiplication.

Example 2. The set {1, –1, i, – i} where $i = \sqrt{-1}$ is a group with respect to the binary operation of multiplication.

Example 3. The set of all integers, i. e. {..., –3, –2, –2, –1, 0, 1, 2, 3, ...} is a group with respect to the binary operation of addition.

6.3 Definition of ring with examples

A **ring** R is an **additive abelian group** with the following additional properties :

(i) The group R is **closed** with respect to the binary operation multiplication. i.e $a, b \in R \Rightarrow ab \in R$

(ii) Multiplication is **associative** i. e.

$(ab)c = a(bc)$ for all $a, b, c \in R$.

(iii) Multiplication is **distributive** with respect to addition on both the left and the right, that is,

$$\left.\begin{aligned} a(b + c) &= ab + ac \\ (b + c)a &= ba + ca \end{aligned}\right\} \text{ for all } a, b, c \in R.$$

Example 1. The set Z = {0, ± 1, ± 2, ...} is a ring under the binary operations of ordinary addition and multiplication.

Example 2. Consider the set Z = {0, 1, 2, 3, 4, 5}.

Z is ring under the binary operations of addition and multiplication modulo 6.

6.4 Definition of field with examples

A **field** is a commutative ring with unit element in which every non-zero element has a multiplicative inverse.

Examples of fields are the ring of rational numbers, the ring of real numbers and the ring of complex numbers.

6.5 Field properties of real numbers

For every pair of real numbers a and b there corresponds a real number called their **sum** and is denoted by a + b. The addition composition has the following properties :

A(1) $a + b = b + a$ for every a, b in R **(commutative).**

A(2) $(a + b) + c = a + (b + c)$ for every a, b, c in R **(Associative)**

A (3) There exists a real number, viz 0 such that

$a + 0 = 0 + a = a$ **(existense of additive identity)**

A (4) To each real number a, there corresponds a real number, viz (–a) such that $(-a) + a = a + (-a) = o$

(existence of additive inverse)

Also for every pair of real numbers a and b there corresponds a real number called their **product** and is denoted by ab. The multiplication composition has the following properties :

M (1) $ab = ba$ for every a, b in R **(Commutative)**

M (2) $(ab)c = a(bc)$ for every a, b, c in R **(Associative)**

M (3) There exists a real number, viz 1 such that for every

$a \in R$ $1a = a1 = a$. **(existence of multiplicative identity)**

M (4) To each real number $a \neq 0$, there corresponds another, viz $\frac{1}{a}$ such that $\left(\frac{1}{a}\right) a = a \left(\frac{1}{a}\right) = 1$

(existence of multiplicative inverse)

AM (1) $a(b + c) = ab + ac$ for every a, b, c in R

(Distributive law)

6.6 Definition of a vector space (or linear space)

A **vector space** over an arbitrary field F is a non-empty set V, whose elements are called **vectors** for which two operations are prescribed. The first operation, called **vector addition,** assigns to each pair of vectors u and v in V a vector denoted by u + v, called their **sum**. The second operation, called **scalar multiplication** assings to each vector v in V and each scalar α in F a vector denoted by α v which is in V. The two operations are required to satisfy the following axioms :

A (1) **Addition is commutative.**

For all vectors u, v $\in$ V, u + v = v + u.

A (2) **Addition is associative**

For all vectors u, v, w $\in$V, (u + v) + w = u + (v + w)

A (3) **Existence of O (zero vector).**

There exists a vector O $\in$V such that for all v $\in$V

v + O = O + v = v.

A (4) **Exisience of negative.**

For each v $\in$ V there is a vector

– v $\in$V for which v + (– v) = (– v) + v = 0

M (1) For any scalar $\alpha \in$ F and any vectors u, v $\in$ V, α (u + v) = αu + αv

M(2) For any scalars $\alpha,\beta \in$ F and any vector v $\in$V, ($\alpha + \beta$)v = αv + βv.

M (3) For any scalars $\alpha,\beta \in$ F and any vector v $\in$ V, ($\alpha\beta$) v = α(βv),

M (4) For each v $\in$V, 1v = v

where 1 is the unit scalar and 1 $\in$ F.

For some applications it is necessary to consider vector spaces where the scalars are complex numbers rather than the real numbers. Such vector spaces are called **complex vector spaces.**

Vector spaces are also sometimes called **linear spaces.**

6.7 Examples of vector space

Example 1. $\mathbb{R}^2 = \{(a, b) \mid a, b \in \mathbb{R}\}$ is a vector space over the field F = $\mathbb{R}$ with respect to the operations of vector addition in $\mathbb{R}^2$ and scalar multiplication on $\mathbb{R}^2$.

$\mathbb{R}^2$ represents the set of all points in the plane.

Example 2. $\mathbb{R}^3 = \{(a, b, c) \mid a, b, c \in \mathbb{R}\}$ is a vector space over the field F = $\mathbb{R}$ with respect to the operations of vector addition in $\mathbb{R}^3$ and scalar multiplication on $\mathbb{R}^3$.

$\mathbb{R}^3$ represents the set of all points in space.

Example 3. Let V be any plane through the origin in $\mathbb{R}^3$. Then the points in V form a vector space under the standard addition and scalar multiplication operations for vectors in $\mathbb{R}^3$.

Example 4. For any arbitrary field F and any integer n, the set of all n-tuples $(u_1, u_2, \ldots, u_n)$ of elements of F is a vector space over F under vector addition and scalar multiplication given by $(u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n) = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$

$\alpha(u_1, u_2, \ldots, u_n) = (\alpha u_1, \alpha u_2, \ldots, \alpha u_n)$ where $u_i, v_i, \alpha \in F$.

This vector space is generally denoted by F^n. The zero vector in F^n is the n–tuple of zero i. e. 0 = (o, o, ..., o).

Example 5. Let M be the set of all m ×n matrices with entries from an arbitrary field F. Then M is a vector space over F with respect to the operations of matrix addition and scalar multiplication given by

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ b_{m1} & b_{m2} & \ldots & b_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}$$

and $\alpha \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha a_{m1} & \alpha a_{m2} & \cdots & \alpha a_{mn} \end{bmatrix}$

where a_{ij}, b_{ij} and $\alpha \in F$.

Example 6. Let V be the set of all continuous real valued functions defined on the closed interval [0, 1]. For any f, g ∈ V and $\alpha \in \mathbb{R}$, define f + g and αf by $(f + g)(x) = f(x) + g(x)$

$(\alpha f)(x) = \alpha f(x)$ for every x in [0, 1].

Under these operations V becomes a vector space over $\mathbb{R}$. (since the sum of continuous functions and scalar multiple of any continuous function are continuous)

Example 7. Let V be the set of all polynomials

$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n$ with co-efficients a_i from an arbitrary field F. Then V is a vector space over F with respect to the usual operations of addition of polynomials and multiplication by a constant.

Example 8. Let S denote the set of all pairs of positive real numbers. $u = (u_1, u_2)$, $v = (v_1, v_2)$,

Define $u + v = (u_1 v_1, u_2 v_2)$ and $\alpha u = (u_1^{\alpha}, u_2^{\alpha})$ (α is any scalar)

where u_1v_1 and u_2v_2 are the usual products of real numbers and u_1^{α} and u_2^{α} are the αth powers. S together with this prescription for addition and scalar multiplication is a vector space which we denote by M^2.

Example 9. Let $\mathbb{R}_p$ be the set of all positive real numbers and define for $x, y \in \mathbb{R}_p$ a vector sum by $x + y = xy$ where the product on the right is the usual product of numbers, If a is any number and $x \in \mathbb{R}_p$ define $a\,x = x^a$, that is, the number x raised to the a power.

We claim that with these definitions of vector addition and scalar multiplication $\mathbb{R}_p$ becomes a vector space.

Example 10. The set of all continuous functions $y = f(x)$, $-\infty < x < +\infty$ satisfying the differential equation $y'' - y' - 2y = 0$ is a vector space. (In fact any solution of this differential equation is a linear combination of $y = e^{-x}$ and $y = e^{2x}$).

Theorem 6.1 Let V be a vector space over an arbitrary field F. Then,

(i) For any scalar $\alpha \in F$ and $0 \in V$, $\alpha 0 = 0$.

(ii) For $o \in F$ and any vector $v \in V$, $ov = 0$

(iii) For $\alpha \in F$ and $v \in V$, $(-\alpha)v = \alpha(-v) = -\alpha v$.

(iv) If $\alpha v = 0$, where $\alpha \in F$ and $v \in V$, then $\alpha = o$ or $v = 0$.

Proof : (i) $\alpha 0 = \alpha(0 + 0) = \alpha 0 + \alpha 0$

Adding $-\alpha 0$ to both sides, we get

$(-\alpha 0) + \alpha 0 = (-\alpha 0) + (\alpha 0 + \alpha 0)$

$= (-\alpha 0 + \alpha 0) + \alpha 0 = 0 + \alpha 0$

or, $0 = 0 + \alpha 0 = \alpha 0 \therefore \alpha 0 = 0$.

(ii) $0 = ov + (-ov) = (o + o)v + (-ov)$

$= (ov + ov) + (-ov)$

$= (ov + (ov + -ov) = ov + 0 = ov$

$\therefore ov = 0$.

(iii) $0 = \alpha 0 = \alpha(v + (-v)) = \alpha v + \alpha(-v)$

Adding $-\alpha v$ to both sides, we get

$-\alpha v + 0 = -\alpha v + (\alpha v + \alpha(-v))$

$= (-\alpha v + \alpha v) + \alpha(-v)$

$= 0 + \alpha(-v)$

or, $-\alpha v = \alpha(-v) \therefore \alpha(-v) = -\alpha v$

Again, $0 = ov = (\alpha + (-\alpha))v = \alpha v + (-\alpha)v$

Adding $-\alpha v$ to both sides, we get

$$-\alpha v + 0 = -\alpha v + (\alpha v + (-\alpha)\, v)$$
$$= (-\alpha v + \alpha v) + (-\alpha)v$$
$$= ((-\alpha) + \alpha)v + (-\alpha)v$$
$$= ov + (-\alpha v) = 0 + (-\alpha)\, v$$

or, $-\alpha v = (-\alpha)v \quad \therefore (-\alpha)\, v = -\alpha v.$

(iv) Suppose that $\alpha v = 0$ and $\alpha \neq o$, then, there exists a scalar $\alpha^{-1} \in F$ such that $\alpha^{-1}\alpha = 1$, hence

$$v = 1v = (\alpha^{-1}\ \alpha)v = \alpha^{-1}\ (\alpha v) = \alpha^{-1}(0) = 0.$$

Definition of Cartesian or Euclidean space

Let n be a positive integer. The cartesian n–space denoted by $\mathbb{R}^n$ is the set of all sequences $(u_1, u_2, \ldots, u_n)$ of n real numbers together with two operations

$$(u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n) = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$$

and $\alpha\,(u_1, u_2, \ldots, u_n) = (\alpha u_1, \alpha u_2, \ldots, \alpha u_n)$.

In particular, $\mathbb{R}^1 = \mathbb{R}$ is the set of all real numbers with their usual addition and multiplication.

Theorem 6.2 For each positive integer n, Euclidean space $\mathbb{R}^n$ is a vector space.

Proof : We shall have to show that $\mathbb{R}^n$ satisfies all axioms of a vector space.

(i) Let $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ be in $\mathbb{R}^n$ then

$$u + v = (u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n)$$
$$= (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$$
$$= (v_1 + u_1, v_2 + u_2, \ldots, v_n + u_n) = v + u.$$ So axiom A (1) is true.

(ii) Let $u = (u_1, u_2, \ldots, u_n)$, $v = (v_1, v_2, \ldots, v_n)$ and $w = (w_1, w_2, \ldots, w_n)$ be in $\mathbb{R}^n$. Then

$$(u + v) + w = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n) + (w_1, w_2, \ldots, w_n)$$
$$= (u_1 + v_1 + w_1, u_2 + v_2 + w_2, \ldots, u_n + v_n + w_n)$$
$$= (u_1, u_2, \ldots, u_n) + (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n)$$
$$= u + (v + w).$$ So axiom A (2) holds.

(iii) Let $0 = (o, o, \ldots, o)$ be in $\mathbb{R}^n$. Then for any $u = (u_1, u_2, \ldots, u_n)$ in $\mathbb{R}^n$ we will have

$$u + 0 = (u_1, u_2, \ldots, u_n) + (o, o, \ldots, o)$$
$$= (u_1 + o, u_2 + o, \ldots, u_n + o)$$
$$= (u_1, u_2, \ldots, u_n) = u.$$

Moreover, if $v = (v_1, v_2, \ldots, v_n)$ is any vector in $\mathbb{R}^n$ such that $u + v = u$, then $(u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n) = (u_1, u_2, \ldots, u_n)$

$$\text{Therefore, } \left.\begin{array}{l} u_1 + v_1 = u_1 \text{ implies } v_1 = 0 \\ u_2 + v_2 = u_2 \text{ implies } v_2 = 0 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \\ u_n + v_n = u_n \text{ implies } v_n = 0 \end{array}\right\}$$

i. e. $v = (o, o, \ldots, o) = 0$. Thus 0 is the unit vector with property that $u + 0 = u$ and so the axiom A (3) holds.

(iv) Let $u = (u_1, u_2, \ldots, u_n)$ and set $-u = (-u_1, -u_2, \ldots, -u_n)$. Then

$$u + (-u) = (u_1, u_2, \ldots, u_n) + (-u_1, -u_2, \ldots, -u_n)$$
$$= (u_1 - u_1, u_2 - u_2, \ldots, u_n - u_n)$$
$$= (o, o, \ldots, o) = 0.$$

Moreover, if $w = (w_1, w_2, \ldots, w_n)$ is any vector in $\mathbb{R}^n$ such that $u + w = 0$, then $(u_1 + w_1, u_2 + w_2, \ldots, u_n + w_n) = (o, o, \ldots, o)$ and therefore, $u_1 + w_1 = 0$ implies $w_1 = -u_1$

$u_2 + w_2 = 0$ implies $w_2 = -u_2$

... ...

... ...

$u_n + w_n = 0$ implies $w_n = -u_n$

i. e. $w = -u$. Thus $-u$ is the unique vector with the property that $u + (-u) = 0$ and so axiom A (4) holds.

(v) Let α be a real number (scalar) and $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ be vectors in $\mathbb{R}^n$. Then

$\alpha(u + v) = \alpha (u_1 + v_1, u_2 + v_2, ..., u_n + v_n)$

$= (\alpha(u_1 + v_1), \alpha(u_2 + v_2), ..., \alpha(u_n + v_n))$

$= (\alpha u_1 + \alpha v_1, \alpha u_2 + \alpha v_2, ..., \alpha u_n + \alpha v_n)$

$= (\alpha u_1, \alpha u_2, ..., \alpha u_n) + (\alpha v_1, \alpha v_2, ..., \alpha v_n)$

$= \alpha(u_1, u_2, ..., u_n) + \alpha(v_1, v_2, ..., v_n) = \alpha u + \alpha v,$

So that axiom M(1) holds.

(vi) Let α, β be the real numbers (scalars) and $u = (u_1, u_2, ..., u_n)$ be in $\mathbb{R}^n$, Then

$(\alpha + \beta) u = ((\alpha + \beta) u_1, (\alpha + \beta) u_2, ..., (\alpha + \beta) u_n)$

$= (\alpha u_1 + \beta u_1, \alpha u_2 + \beta u_2, ..., \alpha u_n + \beta u_n)$

$= (\alpha u_1, \alpha u_2, ..., \alpha u_n) + (\beta u_1, \beta u_2, ..., \beta u_n)$

$= \alpha(u_1, u_2, ..., u_n) + \beta(u_1, u_2, ..., u_n)$

$= \alpha u + \beta u$ So axiom M (2) is satisfied.

(vii) Let α, β be real numbers (scalars) and $u = (u_1, u_2, ..., u_n)$ be in $\mathbb{R}^n$. Then $(\alpha\beta) u = \alpha\beta (u_1, u_2, ..., u_n) = (\alpha\beta u_1, ..., \alpha\beta u_n)$

$= \alpha(\beta u_1, \beta u_2, ..., \beta u_n)$

$= \alpha (\beta(u_1, u_2, ..., u_n)) = \alpha (\beta u)$

So axiom M (3) holds.

(viii) Let 1 be the unit scalar and $u = (u_1, u_2, ..., u_n)$ be in $\mathbb{R}^n$. Then

$1u = 1(u_1, u_2, ..., u_n) = (1u_1, 1u_2, ..., 1u_n)$

$= (u_1, u_2, ..., u_n) = u$

So axiom M (4) is satisfied.

Therefore, $\mathbb{R}^n$ is a vector space.

Theorem 6.3 Let V be the set of all functions from a non-empty set S into an arbitrary field F. For any functions f, g $\in$V and any scalar $\alpha \in$ F let f + g and α f be the functions in V defined by $(f + g)(x) = f(x) + g(x)$ and $(\alpha f)(x) = \alpha f(x)$ for every $x \in$ S.

Prove that V is a vector space over F.

Proof : Since S is non-empty, V is also non-empty. Now we have to show that all the axioms of a vector space hold.

(i) Let f, g$\in$V, Then

$(f + g)(x) = f(x) + g(x) = g(x) + f(x)$

$= (g + f)(x)$ for every $x \in$ S.

Thus f + g = g + f. So axiom A (1) holds.

(ii) Let f, g, h$\in$V, To show that (f + g) + h = f + (g + h), It is required to show that function (f + g) + h and the function f + (g + h)) both assign the same value to each $x \in$S. Now

$((f + g) + h)(x) = (f + g)(x) + h(x) = (f(x) + g(x)) + h(x)$.

$(f + (g + h))(x) = f(x) + (g + h)(x) = f(x) + (g(x) + h(x))$.

for every $x \in$ S.

But $f(x)$, $g(x)$ and $h(x)$ are scalars in the field F where addition of scalars is associative.

Hence $(f(x) + g(x)) + h(x) = f(x) + (g(x) + h(x))$

Accordingly, (f + g) + h = f + (g + h).

So axiom A (2) is satisfied.

(iii) Let O denote the zero function.

$0(x) = 0$ for every $x \in$S.

Then for any function f $\in$V

$(f + 0)(x) = f(x) + 0(x) = f(x) + 0 = f(x)$

for every $x \in$ S.

Thus f + 0 = f and 0 is the zero vector in V.

So axiom A (3) holds.

(iv) For any function f $\in$ V, let – f be the function defined by $(-f)\, x = -f(x)$.

Then $(f + (-f))(x) = f(x) + (-f)(x) = f(x) - f(x) = 0 = 0(x)$

for every $x \in$ S. Hence $f + (-f) = 0$

So axiom A (4) is satisfied.

(v) Let $\alpha \in F$ and $f, g \in V$.

Then $(\alpha(f + g))(x) = \alpha((f + g)(x)) = \alpha(f(x) + g(x))$

$= \alpha f(x) + \alpha g(x)$

$= (\alpha f)(x) + (\alpha g)(x) = (\alpha f + \alpha g)(x)$ for every $x \in S$.

[Since α, $f(x)$, $g(x)$ are scalars in the field F where multiplication is distributive over addition]

Hence $\alpha(f + g) = \alpha f + \alpha g$.

So axiom M (1) is satisfied.

(vi) Let $\alpha, \beta \in F$ and $f \in V$.

Then $((\alpha + \beta) f)(x) = (\alpha + \beta) f(x) = \alpha f(x) + \beta f(x)$

$= (\alpha f)(x) + (\beta f)(x)$

$= (\alpha f + \beta f)(x)$ for every $x \in S$.

Hence $(\alpha + \beta) f = \alpha f + \beta f$.

So axiom M (2) holds.

(vii) Let $\alpha, \beta \in F$ and $f \in V$.

Then $((\alpha\beta)f)(x) = (\alpha\beta) f(x) = \alpha(\beta f(x))$

$= \alpha(\beta f)(x) = (\alpha(\beta f))(x)$ for every $x \in S$.

Hence $(\alpha\beta)f = \alpha(\beta f)$ So axiom M (3) holds.

(viii) Let $f \in V$, then for the unit scalar $1 \in F$

$(1f)(x) = 1f(x) = f(x)$ for every $x \in S$.

Hence $1f = f$, So axiom M (4) is satisfied

Since all the axioms are satisfied, so V is a vector space over F.

6.8 Subspaces of a vector space

Let W be a non-empty subset of a vector space V over the field F. We call W a **subspace** of V if and only if W is a vector space over the field F under the laws of vector addition and scalar multiplication defined on V, or equivalently, W is a **subspace** of V whenever $w_1, w_2 \in W$, $\alpha, \beta \in F$ implies that $\alpha w_1 + \beta w_2 \in W$.

6.9 Examples of subspaces

Example 1. Let V be any vector space over the field F. Then the set {0} consisting of the zero vector alone and also the entire space V are subspaces of V.

Example 2. Consider the vector space $\mathbb{R}^2 = \{(a, b) \mid a, b \in \mathbb{R}\}$.

Then $W_1 = \{(a, 0) \mid a \in \mathbb{R}\}$, $W_2 = \{(0, b) \mid b \in \mathbb{R}\}$ and $W_3 = \{(a, b) \mid a = b \text{ and } a, b \in \mathbb{R}\}$ are subspaces of $\mathbb{R}^2$.

W_1 and W_2 represent the sets of all points on the x-axis and on the y-axis respectively.

Also W_3 represents the set of all points on the line $y = x$.

Example 3. Consider the vector space

$\mathbb{R}^3 = \{(a, b, c) \mid a, b, c \in \mathbb{R}\}$

Then $W_1 = \{(a, b, 0) \mid a, b \in \mathbb{R}\}$, $W_2 = \{0, b, c) \mid b, c \in \mathbb{R}\}$ and $W_3 = \{(a, o, c) \mid a, c \in \mathbb{R}\}$ are subspaces of $\mathbb{R}^3$. W_1, W_2 and W_3 represent the sets of all points in the X–Y, Y–Z and Z–X planes respectively.

Example 4. Consider the vector space

$\mathbb{R}^3 = \{(a, b, c) \mid a, b, c \in \mathbb{R}^3\}$.

Then $W_1 = \{(a, 0,0) \mid a \in \mathbb{R}\}$, $W_2 = \{(0, b, 0) \mid b \in \mathbb{R}\}$ and $W_3 = \{(0,0,c) \mid c \in \mathbb{R}\}$ are subspaces of $\mathbb{R}^3$. W_1, W_2 and W_3 represent the sets of all points on the x-axis, y-axis and z -axis respectively.

Example 5. Let M_{22} be the vector space of all square 2×2 matrices. Then the set W of all 2×2 matrices having zeroes on the main diagonal is a subspace of the vector space M_{22}.

Example 6. Let V be the vector space of all square $n \times n$ matrices. Then the set consisting of those matrices $A = [a_{ij}]$ for which $a_{ij} = a_{ji}$ called **symmetric matrices** is a subspace of V.

Example 7. Let V be the vector space of all functions of a real variable x and let W be the set of all functions $f \in V$ for which $f(5) = 0$. Suppose that $f, g \in W$ and let $h = f + g$. Then $f(5) = 0$. $g(5) = 0$ and so $h(5) = f(5) + g(5) = 0 + 0 = 0$.

It follows that $h \in W$ and so W is closed under addition. Again let $k = \alpha f$ where α is any scalar. $k(5) = \alpha f(5) = \alpha 0 = 0$ whence $k \in W$. Clearly, W is closed under scalar multiplication. Thus W is a subspace of V.

Example 8. The set of all continuous functions is a subspace of the vector space of all real valued functions.

Example 9. Let C be the set of all complex numbers. Then C is a vector space over the real field $\mathbb{R}$.

Let $W = \{ib \mid b \in R, i = \sqrt{(-1)}\}$. Then W is a subspace of C.

Theorem 6.4 W is a subspace of V if and only if

(i) W is non-empty

(ii) W is closed under vector addition i, e, $v, w \in W$ implies that $v + w \in W$.

(iii) W is colsed under scalar multiplication i, e, $v \in W$ implies that $\alpha v \in W$ for every $\alpha \in F$.

Proof : Suppose that the given three conditions (i) (ii) (iii) hold in W; then W is non-empty and closed under vector addition and scalar multiplication. Since the vectors in W belong to V axioms A(1), A(2), M(1), M(2), M(3) and M(4) hold in W. Hence we have only to show that A(3), A(4) hold in W, By condition (i) W is non-empty, say $u \in W$, then by condition (iii) $ou = 0 \in W$ and $v + 0 = v$ for every $v \in W$. Hence A(3) holds in W.

Again, if $v \in W$ then $(-1)\, v = -v \in W$ and $v + (-v) = 0$

Hence A(4) holds in W. Thus W is a subspace of V.

Conversely, if W is a subspace of V then clearly the given conditions (i), (ii), (iii) hold in W. Hence the theorem is proved.

Theorem 6.5 A non-empty subset W of a vector space V over the field F is a subspace of V if and only if

(i) $u, v \in W \Rightarrow u - v \in W$ (ii) $\alpha \in F, u \in W \Rightarrow \alpha u \in W$.

Proof : The conditions are necessary

If W is a subspace of V, then W is an abelian group with respect to vector addition.

i, e, $u, v \in W \Rightarrow u, -v \in W$

$\Rightarrow u + (-v) \in W$

$\Rightarrow u - v \in W$.

Also W must be closed under scalar multiplication, i, e $\alpha \in F$, $u \in W \Rightarrow \alpha u \in W$. Thus conditions (i) and (ii) are necessary.

The conditions are sufficient

Let w be a non-empty subset of V satisfying the two given conditions, From condition (i), we have

$u \in W \Rightarrow u - u \in W \Rightarrow O \in W$.

Thus zero vector of V belongs to W and it will also be the zero vector of W.

Now $O \in W$, $u \in W \Rightarrow O - u \in W \Rightarrow -u \in W$.

Thus additive inverse of each element of W is also in W.

Again, $u \in W$, $v \in W \Rightarrow u \in W, -v \in W$

$\Rightarrow u - (-v) \in W$

$\Rightarrow u + v \in W$.

Hence W is closed with respect to vector addition.

Since the elements of W are also the elements of V, therefore vector addition will be commutative as well as associative in W.

Hence W is an abelian group under vector addition. Also from condition (ii) W is closed under scalar multiplication. The remaining postulates of a vector space will hold in W since they hold in V of which W is a subset. Thus W is a subspace of V.

Theorem 6.6 A non-empty subset W of a vector space V over the field F is a subspace of V if and only if

$\alpha, \beta \in F$ and $u, v \in W \Rightarrow \alpha u + \beta v \in W$.

Proof : The condition is necessary

If W is a subspace of V, then W must be closed under vector addition and scalar multiplication, Therefore,

$$\alpha \in \mathbb{R}, \text{ and } u \in W \Rightarrow \alpha u \in W$$

$$\beta \in F \text{ and } v \in W \Rightarrow \beta v \in W.$$

Now $\alpha u \in W$ and $\beta v \in W \Rightarrow \alpha u + \beta v \in W$.

Thus the condition is necessary.

The condition is sufficient

Suppose that W is a non-empty subset of V satisfying the given condition, i.e $\alpha, \beta \in F$ and $u, v \in W \Rightarrow \alpha u + \beta v \in W$.

Let $\alpha = \beta = 1$, then $1 \in F$ and $u, v \in W \Rightarrow 1u + 1v \in W$

i.e, $u + v \in W$

[since $u \in W \Rightarrow u \in V$ and $1u = u \in V$.]

Thus W is closed under vector addition,

Now taking $\alpha = -1$, $\beta = o$, we see that if $u \in W$,

then $(-1)\, u + o\, u \in W \Rightarrow -(1u) + O \in W \Rightarrow -u \in W$.

Therefore, the additive inverse of each element of W is also in W.

Now $u \in W$ and $-u \in W$.

$\therefore u + (-u) \in W \Rightarrow O \in W$.

Thus zero vector of V belongs to W and it will also be the zero vector of W.

Since the elements of W are also the elements of V, therefore, vector addition will be associative as well as commutative in W, Therefore, W is an abelian group with respect to vector addition,

Now taking $v = O$, we see that if $\alpha, \beta \in F$ and $u \in W$, then $\alpha u + \beta O \in W$ i.e $\alpha u + O \in W \Rightarrow \alpha u \in W$.

Thus w is closed under scalar multiplication. The remaining postulates of a vector space will hold in W since they hold in V of which W is a subset. Hence W is a subspace of V.

Definition : Let S and T be two sets. By $S \cap T$ we mean the **interesection** of S and T, the set of all elements common to S and T, By $S \cup T$ we mean the **union** of S and T, the set of all elements which are in at least one of S and T.

Theorem 6.7 The intersection of two subspaces S and T of a vector space V is also a subspace of V.

Proof : Since S and T are subspaces of V, they are non-empty and clearly $0 \in S$ and $0 \in T$. Therefore, $0 \in S \cap T$ and hence $S \cap T \neq \emptyset$.

Now let $u, v \in S \cap T$ then $u, v \in S$ and $u, v \in T$. Since S and T are subspaces of V, $u, v \in S$ implies that $\alpha u + \beta v \in S$ where $\alpha, \beta \in F$. Similarly, $u, v \in T$ implies that $\alpha u + \beta v \in T$ where $\alpha, \beta \in F$.

Hence $u, v \in S \cap T$ implies that $\alpha u + \beta v \in S \cap T$ for $\alpha, \beta \in F$.

Therefore, $S \cap T$ is subspace of the vector space V.

It is to be noted that the union of S and T is not a subspace of V unless $S \subset T$ or $T \subset S$,

[the symbol "$\subset$" means contained in]

Theorem 5.8 The intersection of any family of subspaces of a vector space is a subspace.

Proof : Let $\{S_i \mid i \in I\}$ be any family of subspaces of a vector space V over the field F. Let $S = \bigcap_{i \in I} S_i$.

Since $0 \in S_i$ for every $i \in I$, we have $0 \in S$. Hence S is a non-empty subset of V. Now $u, v \in S$ and $\alpha, \beta \in F$ implies that $u \in S_i$, $v \in S_i$ for every $i \in I$ as $S = \bigcap_{i \in I} S_i$ and hence $\alpha u + \beta v \in S_i$ for every $i \in I$, since each S_i is a subspace of V.

$\therefore \alpha u + \beta v \in S$.

Therefore, S is a subspace of V.

WORKED OUT EXAMPLES

Example 1. Show that $S = \{(a, o, c) : a, c \in \mathbb{R}\}$ is a subspace of the vector space $\mathbb{R}^3$.

Proof : For $0 \in \mathbb{R}^3$, $0 = (o, o, o) \in S$

Since the second component of 0 is o,

Hence S is non-empty.

For any vectors $u = (a, o, c)$ and $v = (a', o, c')$ in S and any scalars (real numbers) α, β, we have

$$\alpha u + \beta v = \alpha (a, o, c) + \beta(a', o, c')$$
$$= (\alpha a, o, \alpha c) + (\beta a', o, \beta c')$$
$$= (\alpha a + \beta a', o, \alpha c + \beta c')$$

Since second component is zero.

Thus $\alpha u + \beta v \in S$ and so S is a subspace of $\mathbb{R}^3$.

Example 2. Show that

$T = \{(a, b, c, d) \in \mathbb{R}^4 : 2a - 3b + 5c - d\}$ is a subspace of $\mathbb{R}^4$.

Proof : For $0 \in \mathbb{R}^4$, $0 = (o, o, o, o) \in T$

Since $2.o - 3.o + 5.o - o = o$

Hence T is non-empty.

Suppose that $u = (a, b, c, d)$ and $v = (a', b', c', d')$ are in T then $2a - 3b + 5c - d = o$ and $2a' - 3b' + 5c' - d' = o$.

Now for any scalars (real numbers) α, β

We have $\alpha u + \beta v = \alpha(a, b, c, d) + \beta(a', b', c', d')$

$= (\alpha a, \alpha b, \alpha c, \alpha d) + (\beta a', \beta b' \beta c', \beta d')$

$= (\alpha a + \beta a', \alpha b + \beta b', \alpha c + \beta c', \alpha d + \beta d')$

Also we have $2(\alpha a + \beta a') - 3(\alpha b + \beta b') + 5(\alpha c + \beta c') - (\alpha d + \beta d')$

$= \alpha (2a - 3b + 5c - d) + \beta(2a' - 3b' + 5c' - d') = \alpha o + \beta o = o$.

Thus $\alpha u + \beta v \in T$ and so T is a subspace of $\mathbb{R}^4$.

Example 3. $W = \{(a, b, c) \mid a, b, c \in \mathbb{R}$ and $a - 2b + 3c = 5\}$ is not a subspace of $\mathbb{R}^3$.

For $0 \in \mathbb{R}^3$, $0 = (o, o, o) \notin W$, Since $o - 2.0 + 3.0 = 0 \neq 5$.

Example 4. Let V be the vector space of all square $n \times n$ matrices over the real field $F = \mathbb{R}$.

Show that W is a subspace of V where :

(i) W consists of all symmetric matrices, that is, all matrices $A = [a_{ij}]$ for which $a_{ij} = a_{ji}$. $i, j = 1, 2, 3, \ldots\ldots, n$.

(ii) W consists of all matrices which commute with a given matrix T; that is, $W = \{A \in V : AT = TA\}$. **[D. U. H. T 1986]**

Proof : (i) $O \in W$ since all entries of O are o and hence equal. Now suppose that $A = [a_{ij}]$ and $B = [b_{ij}]$ belong to W, that is, $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$.

Then for any scalars $\alpha, \beta \in F$, $\alpha A + \beta B$ is the matrix whose ij-entry is $\alpha a_{ij} + \beta b_{ij}$.

But $\alpha a_{ij} + \beta b_{ij} = \alpha a_{ji} + \beta b_{ji}$. Thus $\alpha A + \beta B$ is also a symmetric matrix. Therefore, $\alpha A + \beta B \in W$.

Hence W is a subspace of V.

(ii) $O \in W$ since $OT = O = TO$. Now suppose that $A, B \in W$; that is, $AT = TA$ and $BT = TB$.

For any scalars $\alpha, \beta \in F$

$$\begin{aligned}(\alpha A + \beta B)\,T &= (\alpha A)\,T + (\beta B)\,T \\ &= \alpha\,(AT) + \beta(BT) \\ &= \alpha\,(TA) + \beta(TB) \\ &= T\,(\alpha A) + T(\beta B) \\ &= T\,(\alpha A + \beta B).\end{aligned}$$

Therefore, $\alpha A + \beta B$ commutes with T. Thus $\alpha A + \beta B \in W$.

Hence W is a subspace of V.

Example 5. Let $V = \mathbb{R}^3$, show that W is not a subspace of V where : **[D. U. P. 1972]**

(i) $W = \{(a, b, c) \mid a \geq 0\}$ i.e W consists of those vectors of $\mathbb{R}^3$ whose first component is non-negative.

(ii) $W = \{(a, b, c) \mid a^2 + b^2 + c^2 \leq 1\}$ i.e W consists of those vectors of $\mathbb{R}^3$ whose lengths do not exceed 1.

[D. U. T. H. 1993, 94]

(iii) $W = \{(a, b, c) \mid a, b, c \in \mathbb{Q}\}$ i.e W consists of those vectors of $\mathbb{R}^3$ whose components are rational numbers.

Proof : (i) Let $v = (2, 3, 5) \in W$ and $\alpha = -3 \in \mathbb{R} = F$.

Then $\alpha v = -3\,(2, 3, 5) = (-6, -9, -15) \notin W$, since -6 is negative.

Hence W is not a subspace of V.

(ii) Let $u = (0, 1, 0)$ and $v = (0, 0, 1)$

Then $u \in W$ since $0^2 + 1^2 + 0^2 = 1 \leq 1$

and $v \in W$ since $0^2 + 0^2 + 1^2 = 1 \leq 1$.

Now for any two scalars $\alpha = \beta = 1 \in \mathbb{R} = F$,

$$\begin{aligned}\alpha u + \beta v &= 1\,(0, 1, 0) + 1\,(0, 0, 1) \\ &= (0, 1, 0) + (0, 0, 1) \\ &= (0, 1, 1) \notin W\end{aligned}$$

Since $0^2 + 1^2 + 1^2 = 2 \not\leq 1$.

Hence W is not a subspace of V.

(iii) Let $v = (3, 5, 7) \in W$ and $\alpha = \sqrt{3} \in \mathbb{R} = F$.

Then $\alpha v = \sqrt{3}\,(3, 5, 7) = \left(3\sqrt{3}, 5\sqrt{3}, 7\sqrt{3}\right) \notin W$

since its components are not rational.

Hence W is not a subspace of V.

Example 6. Let V be a vector space of all 2×2 matrices over the real field $\mathbb{R}$. Show that W is not a subspace of V where :

(i) W consists of all matrices with zero determinant;

(ii) W consists of all matrices A for which $A^2 = A$.

Proof : (i) Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

Then $A, B \in W$, since $\det(A) = 0$ and $\det(B) = 0$.

But $A + B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \notin W$

since $\det(A + B) = -1 \neq 0$.

Hence W is not a subspace of V.

(ii) The unit matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ belongs W, since

$$I^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

But $4I = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ does not belong to W

Since $(4I)^2 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix} \neq 4I$.

Hence W is not a subspace of V.

6.10 Linear combination of vectors

Let V be a vector space over the field F and let $v_1, \ldots, v_n \in V$ then any vector $v \in V$ is called a **linear combination** of $v_1, v_2, \ldots, v_n$ if and only if there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ in F such that $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = \sum_{i=1}^{n} \alpha_i v_i$.

Example 7. Consider the vectors $v_1 = (2, 1, 4)$, $v_2 = (1, -1, 3)$ and $v_3 = (3, 2, 5)$ in $\mathbb{R}^3$. Show that $v = (5, 9, 5)$ is a linear combination of v_1, v_2 and v_3.

Proof : In order to show that v is a linear combination of v_1, v_2 and v_3, there must be scalars α_1, α_2 and α_3 in F such that

$v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$

$$\begin{aligned} \text{i, e, } (5, 9, 5) &= \alpha_1 (2, 1, 4) + \alpha_2 (1, -1, 3) + \alpha_3 (3, 2, 5) \\ &= (2\alpha_1, \alpha_1, 4\alpha_1) + (\alpha_2, -\alpha_2, 3\alpha_2) + (3\alpha_3, 2\alpha_3, 5\alpha_3) \\ &= (2\alpha_1 + \alpha_2 + 3\alpha_3, \alpha_1 - \alpha_2 + 2\alpha_3, 4\alpha_1 + 3\alpha_2 + 5\alpha_3) \end{aligned}$$

Equating corresponding components and forming linear system we get

$$\left.\begin{aligned} 2\alpha_1 + \alpha_2 + 3\alpha_3 &= 5 \\ \alpha_1 - \alpha_2 + 2\alpha_3 &= 9 \\ 4\alpha_1 + 3\alpha_2 + 5\alpha_3 &= 5 \end{aligned}\right\} \qquad (1)$$

Reduce the system (1) to echelon form by elementary operations. Interchange first and second equations Then we have the equivalent system

$$\left.\begin{aligned} \alpha_1 - \alpha_2 + 2\alpha_3 &= 9 \\ 2\alpha_1 + \alpha_2 + 3\alpha_3 &= 5 \\ 4\alpha_1 + 3\alpha_2 + 5\alpha_3 &= 5 \end{aligned}\right\} \qquad (2)$$

We multiply first equation by 2 and by 4 and then subtract from the second and third equations respectively. Then we have the equivalent system

$$\left.\begin{aligned} \alpha_1 - \alpha_2 + 2\alpha_3 &= 9 \\ 3\alpha_2 - \alpha_3 &= -13 \\ 7\alpha_2 - 3\alpha_3 &= -31 \end{aligned}\right\} \qquad (3)$$

We multiply second equation by $\frac{7}{3}$ and then subtract from the third equation. Then we have the equivalent system

$$\left.\begin{aligned} \alpha_1 - \alpha_2 + 2\alpha_3 &= 9 \\ 3\alpha_2 - \alpha_3 &= -13 \\ -\tfrac{2}{3}\alpha_3 &= -\tfrac{2}{3} \end{aligned}\right\} \qquad (4)$$

From the third equation, we have $\alpha_3 = 1$. Substituting $\alpha_3 = 1$ in the second equation we get $3\alpha_2 - 1 = -13$ or, $3\alpha_2 = -12$ or, $\alpha_2 = -4$.

Again substituting $\alpha_2 = -4$, $\alpha_3 = 1$ in the first equation, we get $\alpha_1 + 4 + 2 = 9 \quad \therefore \alpha_1 = 3$

So the solution of the system is $\alpha_1 = 3$, $\alpha_2 = -4$, $\alpha_3 = 1$.

Hence $v = 3v_1 - 4v_2 + v_3$.

Therefore, v is a linear combination of v_1, v_2 and v_3.

Example 8. Is the vector $v = (2, -5, 3)$ in $\mathbb{R}^3$ is a linear combination of the vectors

$v_1 = (1, -3, 2)$, $v_2 = (2, -4, -1)$ and $v_3 = (1, -5, 7)$.

Solution : Let $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$ where α_1, α_2 and α_3 are unknown scalars.

$$\begin{aligned} \text{i.e } (2, -5, 3) &= \alpha_1 (1, -3, 2) + \alpha_2 (2, -4, -1) + \alpha_3 (1, -5, 7) \\ &= (\alpha_1, -3\alpha_1, 2\alpha_1) + (2\alpha_2, -4\alpha_2, -\alpha_2) + (\alpha_3, -5\alpha_3, 7\alpha_3) \\ &= (\alpha_1 + 2\alpha_2 + \alpha_3, -3\alpha_1 - 4\alpha_2 - 5\alpha_3, 2\alpha_1 - \alpha_2 + 7\alpha_3) \end{aligned}$$

Equating corresponding components and forming the linear system we get

$$\left.\begin{aligned} \alpha_1 + 2\alpha_2 + \alpha_3 &= 2 \\ -3\alpha_1 - 4\alpha_2 - 5\alpha_3 &= -5 \\ 2\alpha_1 - \alpha_2 + 7\alpha_3 &= 3 \end{aligned}\right\}$$

Reduce the system to echelon form by the elementary operations. We multiply 1st equation by -3 and 2 and then subtract from 2nd & 3rd equations respectively. Then we have the equivalent system

$$\left.\begin{aligned} \alpha_1 + 2\alpha_2 + \alpha_3 &= 2 \\ 2\alpha_2 - 2\alpha_3 &= 1 \\ -5\alpha_2 + 5\alpha_3 &= -1 \end{aligned}\right\}$$

We multiply 2nd equation by $\frac{5}{2}$ and then add with the 3rd equation. Then we have the equivalent system

$$\left.\begin{aligned}\alpha_1 + 2\alpha_2 + \alpha_3 &= 2\\ 2\alpha_2 - 2\alpha_3 &= 1\\ 0 + 0 &= \frac{3}{2}\end{aligned}\right\} \Rightarrow \left.\begin{aligned}\alpha_1 + 2\alpha_2 + \alpha_3 &= 2\\ 2\alpha_2 - 2\alpha_3 &= 1\\ 0 &= \frac{3}{2}\end{aligned}\right\}$$

This system has an equation of the form $0 = \frac{3}{2}$

which is not true. Hence the above system is inconsistent i.e it has no solution. Thus the vector v is not a linear combination of the given vectors v_1, v_2 and v_3.

Example 9. For which value of λ will be the vector $v = (1, \lambda, 5)$ in $\mathbb{R}^3$ is a linear combination of the vectors $v_1 = (1, -3, 2)$ and $v_2 = (2, -1, 1)$.

Solution : Let $v = \alpha_1 v_1 + \alpha_2 v_2$ where α_1 and α_2 are unknown scalars. i.e $(1, \lambda, 5) = \alpha_1 (1, -3, 2) + \alpha_2 (2, -1, 1)$

$$= (\alpha_1, -3\alpha_1, 2\alpha_1) + (2\alpha_2, -\alpha_2, \alpha_2)$$

$$= (\alpha_1 + 2\alpha_2, -3\alpha_1 - \alpha_2, 2\alpha_1 + \alpha_2)$$

Equating corresponding components and forming the linear system, we have

$$\left.\begin{aligned}\alpha_1 + 2\alpha_2 &= 1\\ -3\alpha_1 - \alpha_2 &= \lambda\\ 2\alpha_1 + \alpha_2 &= 5\end{aligned}\right\} \qquad (*)$$

Reduce the system to echelon form by the elementary operations. We multiply 1st equation by -3 and 2 and then subtract from 2nd and 3rd equations respectively. Then we have the equivalent system

$$\left.\begin{aligned}\alpha_1 + 2\alpha_2 &= 1\\ 5\alpha_2 &= \lambda + 3\\ -3\alpha_2 &= 3\end{aligned}\right\}$$

We multiply 3rd equation by $-\frac{1}{3}$. Then we have the equivalent system

$$\left.\begin{aligned}\alpha_1 + 2\alpha_2 &= 1\\ 5\alpha_2 &= \lambda + 3\\ \alpha_2 &= -1\end{aligned}\right\}$$

This system has solution for $\lambda = -8$ and the solution is $\alpha_1 = 3, \alpha_2 = -1$

i.e the system (*) has solution for $\lambda = -8$.

Hence v is a linear combination v_1 and v_2 if $\lambda = -8$.

Example 10. Write the matrix $A = \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$ as a linear combination of the matrices $A_1 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$,

$A_2 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ and $A_3 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. **[D. U. H. T. 1994]**

Solution : Set A as a linear combination of A_1, A_2 and A_3 using the unknowns α_1, α_2 and α_3.

$A = \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3$.

that is, $\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

or, $\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_1 \\ 0 & -\alpha_1 \end{bmatrix} + \begin{bmatrix} \alpha_2 & \alpha_2 \\ -\alpha_2 & 0 \end{bmatrix} + \begin{bmatrix} \alpha_3 & -\alpha_3 \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} \alpha_1 + \alpha_2 + \alpha_3 & \alpha_1 + \alpha_2 - \alpha_3 \\ 0 - \alpha_2 + 0 & -\alpha_1 + 0 + 0 \end{bmatrix}$

Equating corresponding components and forming the linear system we get

$$\left.\begin{aligned}\alpha_1 + \alpha_2 + \alpha_3 &= 3\\ \alpha_1 + \alpha_2 - \alpha_3 &= -1\\ -\alpha_2 &= 1\\ -\alpha_1 &= -2\end{aligned}\right\}$$

Hence the the solution of the system is $\alpha_1 = 2, \alpha_2 = -1, \alpha_3 = 2$

Therefore, $A = 2A_1 - A_2 + 2A_3$;

that is, A is a linear combination of A_1, A_2 and A_3.

Example 11. Write the matrix $A = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}$ as a linear combination of the matrices

$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ and $A_3 = \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}$.

Solution : Set A as a linear combination of A_1, A_2 and A_3 using the unknown scalars α_1, α_2 and α_3.

$$A = \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3$$

$$\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 & \alpha_1 \\ \alpha_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \alpha_2 & \alpha_2 \end{bmatrix} + \begin{bmatrix} 0 & 2\alpha_3 \\ 0 & -\alpha_3 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 & \alpha_1 + 2\alpha_3 \\ \alpha_1 + \alpha_2 & \alpha_2 - \alpha_3 \end{bmatrix}$$

Equating corresponding components and forming the linear system, we get

$$\left.\begin{array}{llll} \alpha_1 & & & = 3 \\ \alpha_1 & & + 2\alpha_3 & = 1 \\ \alpha_1 & + \alpha_2 & & = 1 \\ & \alpha_2 & - \alpha_3 & = -1 \end{array}\right\}$$

Hence the solution of the system is $\alpha_1 = 3$, $\alpha_2 = -2$, $\alpha_3 = -1$

Therefore, $A = 3A_1 - 2A_2 - A_3$

Thus A is a linear combination of A_1, A_2 and A_3.

6.11 Linear span of a subset of a vector space

If S is a non-empty subset of a vector space V, then L (S), the **linear span** of S, is the set of all linear combinations of finite sets of elements of S.

Example 12. The vectors $e_1 = (1, o, o)$, $e_2 = (o, 1, o)$ and $e_3 = (o, o, 1)$ generate the vector space $\mathbb{R}^3$. For any vector $(v_1, v_2, v_3) \in \mathbb{R}^3$ is a linear combination of e_1, e_2 and e_3, specifically $(v_1, v_2, v_3) = v_1 (1, o, o) + v_2 (o, 1, o) + v_3 (o, o, 1)$

$$= v_1 e_1 + v_2 e_2 + v_3 e_3.$$

Theorem 6.9 Let S be a non-empty subset of a vector space V, then L (S) is a subspace of V containing S. Furthermore, if W is any other subspace of V containing S, then L (S) $\subset$ W.

Proof : If $u \in S$, then $1u = u \in S$; hence S is a subset of L (S). Also L (S) is non-empty, since S is non-empty.

Now suppose that u, $v \in$ L (S), then

$u = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_m u_m$ and

$v = \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n v_n$, where

$u_i, v_j \in S$ and α_i, β_j are scalars in F. Thus for $\lambda, \mu \in F$,

$$\lambda u + \mu v = \lambda(\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_m u_m) + \mu(\beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n v_n)$$

$$= (\lambda\alpha_1) u_1 + (\lambda\alpha_2) u_2 + \ldots + (\lambda\alpha_m) u_m + (\mu\beta_1) v_1 + (\mu\beta_2) v_2 + \ldots + (\mu\beta_n) v_n$$

which is a linear combination of the elements of S and so is again in L (S). Hence L (S) is a subspace of V.

Now suppose that W is a subspace of V containing S and $u_1, u_2, \ldots, u_m \in S \subset W$. then all multiples $\alpha_1 u_1, \alpha_2 u_2, \ldots, \alpha_m u_m \in W$ where $\alpha_i \in F$ and hence the sum $\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_m u_m \in W$; that is; W contains all linear combinations of elements of S, Consequently, L (S) $\subset$ W. Hence the theorem is proved.

Theorem 6.10 : If S and T are subsets of a vector space V over the field F, then.

(i) $S \subseteq T \Rightarrow L(S) \subseteq L(T)$

(ii) $L(S \cup T) = L(S) + L(T)$

(iii) S is a subspace of V if and only if L (S) = S

(iv) L (L (S)) = L (S)

Proof : (i) Let $u = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n \in L(S)$ where $\{u_1, u_2, \ldots, u_n\}$ is a finite subset of S and $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$. Since $S \subseteq T$, therefore, $\{u_1, u_2, \ldots, u_n\}$ is also a finite subset of T.

So $u = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n \in L(T)$

Thus $u \in L(S) \Rightarrow u \in L(T)$. $\therefore L(S) \subseteq L(T)$

Hence $S \subseteq T \Rightarrow L(S) \subseteq L(T)$.

(ii) Let $u \in L(S \cup T)$ then

$u = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n + \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_m v_m$

where $\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_m\}$ is a finite subset of $S \cup T$ such that $\{u_1, u_2, \dots, u_n\} \subseteq S$ and $\{v_1, v_2, \dots, v_m\} \subseteq T$.

Now $\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n \in L(S)$

and $\beta_1 v_1 + \beta_2 v_2 + \dots + \beta_m v_m \in L(T)$.

Therefore, $u \in L(S) + L(T)$.

Evidently $L(S \cup T) \subseteq L(S) + L(T)$ (A).

Let w be any element of $L(S) + L(T)$

then $w = u + v$ for $u \in L(S)$ and $v \in L(T)$.

Now since u is a linear combination of a finite number of elements of S and v is a linear combination of a finite number of elements of T. Therefore, u + v will be a linear combination of a finite numbers of elements of $S \cup T$. Thus $u + v \in L(S \cup T)$.

Hence $L(S) + L(T) \subseteq L(S \cup T)$ (B)

Therefore, from (A) and (B), we get

$$L(S \cup T) = L(S) + L(T).$$

(iii) Suppose that S is a subspace of V. Then we have to prove that $L(S) = S$. Let $u \in L(S)$.

Then $u = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n$ where $\alpha_1, \alpha_2, \dots, \alpha_n \in F$ and $u_1, u_2, \dots, u_n \in S$. But S is a subspace of V.

Therefore, it is closed with respect to scalar multiplication and vector addition.

Hence $u = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n \in S$.

Thus $u \in L(S) \Rightarrow u \in S$.

Therefore, $L(S) \subseteq S$. Also $S \subseteq L(S)$.

Hence $L(S) = S$.

Conversely, suppose that $L(S) = S$. then we have to prove that S is a subspace of V;

Now we know that L(S) is a subspace of V.

Since $S = L(S)$, Therefore, S is also a subspace of V.

(iv) We know that L (S) is a subspace of V.

Therefore, by part (iii), it follows that $L(L(S)) = L(S)$.

Example 12 (a) Show that the vectors $u = (1, 2, 3)$, $v = (0, 1, 2)$ and $w = (0, 0, 1)$ generate $\mathbb{R}^3$. **[C. U.P. 1975]**

Proof : We must determine whether an arbitrary vector $v' = (a, b, c)$ in $\mathbb{R}^3$ can be expressed as a linear combination $v' = xu + yv + zw$ of the vectors u, v and w. Expressing this equation in terms of components gives.

$$\begin{aligned}(a, b, c) &= x(1, 2, 3) + y(0, 1, 2) + z(0, 0, 1)\\ &= (x, 2x, 3x) + (0, y, 2y) + (0, 0, z)\\ &= (x, 2x + y, 3x + 2y + z)\end{aligned}$$

Equating corresponding components and forming the linear system we get

$$\left.\begin{array}{l} x = a \\ 2x + y = b \\ 3x + 2y + z = c \end{array}\right\} \Rightarrow \left.\begin{array}{r} z + 2y + 3x = c \\ y + 2x = b \\ x = a \end{array}\right\}$$

The above system is in echelon form and is consistent. In fact, the system has the solution $x = a$, $y = b - 2a$, $z = c - 2b + a$

Thus u, v and w generate (span) $\mathbb{R}^3$.

Example 13. Determine whether vectors $v_1 = (2, -1, 3)$ $v_2 = (4, 1, 2)$ and $v_3 = (8, -1, 8)$ span $\mathbb{R}^3$.

Solution : We must determine whether an arbitrary vector $v = (a, b, c)$ in $\mathbb{R}^3$ can be expressed as a linear combination $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$ of the vectors v_1, v_2 and v_3. Expressing this equation in terms of components, we get

$$(a\ b, c) = \alpha_1\ (2, -1, 3) + \alpha_2\ (4, 1, 2) + \alpha_3\ (8, -1, 8)$$
$$= (2\alpha_1, -\alpha_1, 3\alpha_1) + (4\alpha_2, \alpha_2, 2\alpha_2) + (8\alpha_3, -\alpha_3, 8\alpha_3)$$
$$= (2\alpha_1 + 4\alpha_2 + 8\alpha_3, -\alpha_1 + \alpha_2 - \alpha_3, 3\alpha_1 + 2\alpha_2 + 8\alpha_3)$$

Equating corresponding components and forming the linear system, we have
$$\left.\begin{aligned} 2\alpha_1 + 4\alpha_2 + 8\alpha_3 &= a \\ -\alpha_1 + \alpha_2 - \alpha_3 &= b \\ 3\alpha_1 + 2\alpha_2 + 8\alpha_3 &= c \end{aligned}\right\}$$

This problem thus reduces to determining whether or not this system is consistent for all values of a, b and c. Now this system will be consistent for all a, b and c, if and only if the matrix of co-efficients.

$$A = \begin{bmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{bmatrix} \text{ is invertible.}$$

Now the determinant of A is

$$|A| = \begin{vmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{vmatrix} = 2(8+2) - 4(-8+3) + 8(-2-3)$$
$$= 20 + 20 - 40 = 0.$$

Hence the co-efficient matrix A is not invertible and consequently, v_1, v_2 and v_3 do not span $\mathbb{R}^3$.

Example 14. Determine whether the vectors $u_1 = (1, \frac{1}{2}, \frac{1}{4})$, $u_2 = (-2, -4, -8)$ and $u_3 = (3, 9, 27)$ generate (span) $\mathbb{R}^3$.

Solution : We must determine whether an arbitrary vector $u = (a, b, c) \in \mathbb{R}^3$ can be expressed as a linear combination $u = x_1u_1 + x_2u_2 + x_3u_3$ of the vectors u_1, u_2, and u_3.

Expressing this equation in term of components, we get

$$(a, b, c) = x_1\left(1, \frac{1}{2}, \frac{1}{4}\right) + x_2\ (-2, -4, -8) + x_3\ (3, 9, 27)$$
$$= \left(x_1, \frac{1}{2}x_1, \frac{1}{4}x_1\right) + (-2x_2, -4x_2, -8x_2) + (3x_3, 9x_3, 27x_3)$$
$$= (x_1 - 2x_2 + 3x_3, \tfrac{1}{2}x_1 - 4x_2 + 9x_3, \tfrac{1}{4}x_1 - 8x_2 + 27x_3)$$

Now form the system of linear equations by equating the corresponding components.

$$\left.\begin{aligned} x_1 - 2x_2 + 3x_3 &= a \\ \tfrac{1}{2}x_1 - 4x_2 + 9x_3 &= b \\ \tfrac{1}{4}x_1 - 8x_2 + 27x_3 &= c \end{aligned}\right\}$$

or, equivalently

$$\left.\begin{aligned} x_1 - 2x_2 + 3x_3 &= a \\ x_1 - 8x_2 + 18x_3 &= 2b \\ x_1 - 32x_2 + 108x_3 &= 4c \end{aligned}\right\} \quad (1)$$

Now we reduce the system to echelon form by the elementary transformations. We subtract first equation from the second and third equations respectively. Then we have the equivalent system

$$\left.\begin{aligned} x_1 - 2x_2 + 3x_3 &= a \\ -6x_2 + 15x_3 &= 2b - a \\ -30x_2 + 105x_3 &= 4c - a \end{aligned}\right\} \quad (2)$$

We multiply second equation by 5 and then subtract from the third equation. Thus we get the equivalent system

$$\left.\begin{aligned} x_1 - 2x_2 + 3x_3 &= a \\ -6x_2 + 15x_3 &= 2b - a \\ 30x_3 &= 4c - a - 10b + 5a \end{aligned}\right\} \quad (3)$$

which is in echelon form

From the third equation, we get $x_3 = \frac{1}{15}(2a - 5b + 2c)$

Putting the value of x_3 in the second equation,

we get $-6x_2 + 2a - 5b + 2c = 2b - a$

or, $6x_2 = 2a - 5b + 2c - 2b + a = 3a - 7b + 2c$

$\therefore x_2 = \frac{1}{6}(3a - 7b + 2c)$.

Again, putting the values of x_2 and x_3 in the first equation we get $x_1 - \frac{1}{3}(3a - 7b + 2c) + \frac{1}{5}(2a - 5b + 2c) = a$

or, $15x_1 - 15a + 35b - 10c + 6a - 15b + 6c = 15a$

or, $15x_1 = 24a - 20b + 4c$

$x_1 = \frac{1}{15}(24a - 20b + 4c)$.

Therefore, $(a, b, c) = \frac{1}{15}(24a - 20b + 4c)u_1 + \frac{1}{6}(3a - 7b + 2c)u_2 + \frac{1}{15}(2a - 5b + 2c)u_3$.

Hence u_1, u_2, u_3, span $\mathbb{R}^3$.

Verification of the result

(i) $\frac{1}{15}(24a - 20b + 4c) - \frac{1}{3}(3a - 7b + 2c) + \frac{1}{5}(2a - 5b + 2c)$

$= \frac{1}{15}(24a - 20b + 4c - 15a + 35b - 10c + 6a - 15b + 6c)$

$= \frac{1}{15}(30a - 15a + 35b - 35b + 10c - 10c = \frac{1}{15}(15a) = a.$

(ii) $\frac{1}{15}(12a - 10b + 2c) - \frac{1}{3}(6a - 14b + 4c) + \frac{1}{5}(6a - 15b + 6c)$

$= \frac{1}{15}(12a - 10b + 2c - 30a + 70b - 20c + 18a - 45b + 18c)$

$= \frac{1}{15}(30a - 30a + 70b - 55b + 20c - 20c)$

$= \frac{1}{15}(15b) = b.$

(iii) $\frac{1}{15}(6a - 5b + c) - \frac{1}{3}(12a - 28b + 8c) + \frac{1}{5}(18a - 45b + 18c)$

$= \frac{1}{15}(6a - 5b + c - 60a + 140b - 40c + 54a - 135b + 54c)$

$= \frac{1}{15}(60a - 60a + 140b - 140b + 55c - 40c)$

$= \frac{1}{15}(15c) = c.$

Hence the result is verified.

6.12 Row space and column space of a matrix

Let A be an arbitrary m ×n matrix over the real field $\mathbb{R}$:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The m rows of A are $R_1 = (a_{11}, a_{12}, \ldots, a_{1n})$

$R_2 = (a_{21}, a_{22}, \ldots, a_{2n}), \ldots, R_m = (a_{m1}, a_{m2}, \ldots, a_{mn})$.

These m rows viewed as vectors in R^n span a subspace of R^n called the **row space** of A.

Again, the columns of A are

$$C_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \cdots \\ a_{m1} \end{bmatrix} \quad C_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \cdots \\ a_{m2} \end{bmatrix} \ldots, C_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \cdots \\ a_{mn} \end{bmatrix}$$

These n columns of A viewed as vectors in R^m, span a subspace of R^m called the **column space** of A.

Definition : For any two subspaces S and T of a vector space V, their **sum** S + T is the set given by $\{u+v \mid u \in S \text{ and } v \in T\}$.

Theorem 5.11 For any two subspaces S and T of a vector space V, S + T is a subspace of V.

Proof : Clearly, $0 = 0 + 0 \in S + T$: So $S + T \neq \varnothing$

i.e. S + T is non-empty.

Let $x, y \in S + T$, $\alpha, \beta \in F$. Then $x = u_1 + v_1, y = u_2 + v_2$

where $u_1, u_2 \in S$ and $v_1, v_2 \in T$.

Thus $\alpha x + \beta y = \alpha(u_1 + v_1) + \beta(u_2 + v_2)$

$= \alpha u_1 + \alpha v_1 + \beta u_2 + \beta v_2$

$= \alpha u_1 + \beta u_2 + \alpha v_1 + \beta v_2$ (1)

Now since S and T are subspaces of V

$\alpha u_1 + \beta u_2 \in S$ and $\alpha v_1 + \beta v_2 \in T$.

Consequently, (1) gives that $\alpha x + \beta y \in S + T$.

Hence S + T is a subspace of V.

6.13 Direct sum of subspaces

The vector space V is said to be the **direct sum** of its subspaces S and T, denoted by $V = S \oplus T$ if every vector $v \in V$ can be written in one and only one way as **v = u** + w, where $u \in S$ and $w \in T$.

Theorem 5.12 The vector space V is the direct sum of its subspaces S and T if and only if (i) V = S + T and (ii) $S \cap T = \{0\}$.

Proof : Suppose that $V = S \oplus T$. Then any $v \in V$ can be uniquely written in the form v = u + w, where $u \in S$ and $w \in T$.

Thus in particular ; $V = S + T$.

Now suppose that $v \in S \cap T$, Then

(1) $v = v + 0$, where $v \in S$ and $0 \in T$.

(2) $v = 0 + v$, where $0 \in S$ and $v \in T$.

Since such a sum for v must be unique.

$\therefore$ v = o; Accordingly $S \cap T = \{0\}$.

On the other hand, suppose that $V = S + T$ and $S \cap T = \{0\}$

Let $v \in V$ since $V = S + T$, there exist $u \in S$ and $w \in T$ such that $v = u + w$. Now we have to show that such a sum is unique. If the sum is not unique, let $v = u' + w'$ where $u' \in S$ and $w' \in T$. Then $u + w = u' + w'$ and so $u - u' = w' - w$. But $u - u' \in S$ and $w' - w \in T$, hence by $S \cap T = \{0\}$, $u - u' = 0$ and $w' - w = 0$.

Therefore, $u = u'$ and $w = w'$. Thus such a sum for $v \in V$ is unique and $V = S \oplus T$.

Example 15. Let S and T be subspaces of $\mathbb{R}^4$ defined by $S = \{a, b, 0, 0) \mid a, b \in F\}$ and $T = \{(0, 0, c, d) \mid c, d \in F\}$. One can easily verify that both these subsets are subspaces of $\mathbb{R}^4$.

The zero vector of $\mathbb{R}^4$ is (0, 0, 0, 0). Then $z = (a, b, 0, 0)$
$= (0, 0, c, d)$ for some $a, b, c, d \in F$ implies that
$a = b = c = 0$. Hence $z = 0$, so that $S \cap T = \{0\}$.

Therefore, $S + T = S \oplus T$, Now every $(a_1, b_1, c_1, d_1) \in \mathbb{R}^4$ can be expressed as $(a_1, b_1, 0, 0) + (0, 0, c_1, d_1)$ with $(a_1, b_1, 0, 0) \in S$ and $(0, 0, c_1, d_1) \in T$. This gives $\mathbb{R}^4 = S \oplus T$.

EXERCISES – 6 (A)

1. Let $u = (1, 2, 1, -2)$, $v = (3, 0, 4, 1)$ and $w = (6, 3, -3, 0)$ be the vectors in $\mathbb{R}^4$. Compute the following vectors : (i) $2u + v - w$ (ii) $5u - 3v - w$.

Answers : (i) (–1, 1, 9, –3)

(ii) (– 10, 7, – 4, – 13).

2 . (i) Find $x \in \mathbb{R}^5$ such that $u + x = v$ where $u = (2, -1, 0, 3, 6)$ and $v = (0, 1, 2, -1, -2)$ **[D. U. P. 1982]**

(ii) Solve the vector equation $u + x = v$ for the following pair of vectors u and v in $\mathbb{R}^5$: $u = \left(\frac{1}{2}, 3, \frac{1}{3}, -6, 1\right)$

$v = \left(\frac{1}{2}, 4, -1, \frac{1}{3}, 1\right)$

Answers : (i) $x = (-2, 2, 2, -4, -8)$

(ii) $x = \left(0, \ 1, -\frac{4}{3}, -\frac{17}{3}, 0\right)$.

3. Let $u_1 = (-1, 3, 2, 0)$, $u_2 = (2, 0, 4, -1)$, $u_3 = (7, 1, 1, 4)$ and $u_4 = (6, 3, 1, 2)$. Find scalars $\alpha_1, \alpha_2, \alpha_3$ and α_4 such that $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \alpha_4 u_4 = (0, 5, 6, -3)$

Answer : $\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = -1. \alpha_4 = 1$.

4. Let V be a vector space over the field F. Let u and v any two vectors in V and α be any scalar in F.

Show that (i) $(-1) v = -v$

(ii) $\alpha(u - v) = \alpha u - \alpha v$.

5. Verify whether the following sets are subspaces of $V_3(\mathbb{R})$:

(i) $\{(x, 2y, 5) : x, y \in \mathbb{R}\}$

(ii) $\{(x, x + y, 3z), x, y, z \in \mathbb{R}\}$.

Answers: (i) The set is not a subspace.

(ii) The set is a subspace.

6. Show that $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 - x_2 + x_3 = 0\}$ is a subspace of $\mathbb{R}^3$. **[D. U. P. 1983]**

7. Show that $W = \{(a, b, c, d) \mid (a, b, c, d) \in \mathbb{R}^4$ and $3a - 2b - 2c - d = 0\}$, is a subspace of $\mathbb{R}^4$. **[D. U. S. 1984]**

8. Show that $W = \{(a, b, c) \in \mathbb{R}^3 \mid a+b+c = 0\}$ is a subspace of $\mathbb{R}^3$ **[D. U. H. 1986]**

9. Show that $V = \{(a, b, c, d) \in \mathbb{R}^4 \mid a + b + c + d = 0\}$ is a subspace of $\mathbb{R}^4$.

10. Which of the following are subspaces of $\mathbb{R}^3$?

(i) $S = \{(x, y, z) \in \mathbb{R}^3 \mid y - 6z = 0\}$

(ii) $T = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1\}$ **[D. U. P. 1985]**

Answers : (i) S is a subspace of $\mathbb{R}^3$

(ii) T is not a subspace of $\mathbb{R}^3$.

11. Show that $W = \{(a, b, c) \mid a, b, c \in \mathbb{R}$ and $2a - b + c = 10\}$ is not a subspace of $V_3(\mathbb{R})$. **[D. U. P. 1984]**

12. Show that each of the following subsets of the vector space $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$:

(i) $W = \{(a, b, c) \mid a + b = 0\}$; **[D. U. H. T. 1986]**

(ii) $W = \{(a, b, c) \mid b + c = 0\}$;

(iii) $W = \{(a, b, c) \mid c + a = 0\}$.

13. Let $W = \{(a, b, 1) \mid a, b \in \mathbb{R}\}$ be a subset of the vector space $\mathbb{R}^3$. Then show that W is not a subspace of $\mathbb{R}^3$.

[D. U. H. T. 1990]

14. Let $W = \{(a,b, c) \mid a \geq b\}$ be a subset of the vector space $\mathbb{R}^3$. Then show that W is not a subspace of $\mathbb{R}^3$. **[D. U. H. T. 1988]**

15. Show that $W = \{(a, a, a) \mid a \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.

[R. U. H. 1985]

16. show that each of the following subsets ~~of the vector~~ space $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$:

(i) $W_1 = \{(a, b, 0) \mid a, b \in \mathbb{R}\}$ (ii) $W_2 = \{(0, b, c) \mid b, c \in \mathbb{R}\}$.

17. Let V be a vector space of all n-square matrices over a real field $\mathbb{R}$. Show that W is a subspace of V if W consists of all skew-symmetric matrices (i.e $A^T = -A$). **[R. U. H. 1986]**

18. Show that W is a subspace of $\mathbb{R}^4$ where

(i) $W = \{(a, b, c, 0) \mid a, b, c \in \mathbb{R}\}$

(ii) $W = \{(x_1, x_2, x_3, x_4) \mid \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 = 0, \alpha_i \in \mathbb{R}\}$

19. Write the vectors (1, 0, 0) and (0, 0, 1) as linear combinations of the vectors (1, 0, −1), (0, 1, 0), (1, 0, 1)

Answers : $(1, 0, 0) = \frac{1}{2}(1, 0, -1) + 0\,(0, 1, 0) + \frac{1}{2}(1, 0, 1)$

$(0, 0, 1) = -\frac{1}{2}(1, 0, -1) + 0\,(0, 1, 0) + \frac{1}{2}(1, 0, 1)$

20. Write (5, 6, 0) as a linear combination of (−1, 2, 0), (3, 1, 2), (4, −1, 0) and (0, 1, −1) **[D. U. Prel. 1983]**

Answer : $(5, 6, 0) = 2(-1, 2, 0) + 1\,(3, 1, 2) + 1\,(4, -1, 0) + 2(0, 1, -1)$.

21. Whether or not the vector (1, 2, 6) is a linear combination of the vectors $u_1 = (2, 1, 0)$, $u_2 = (1, -1, 2)$ and $u_3 = (0, 3, -4)$ **[C. U. P. 1979]**

Answer : (1, 2, 6) can not be written as a linear combination of u_1, u_2 and u_3.

22. Write the vectors (2, 3, −7, 3) and (− 4, 6, −13, 4) as a linear combination of the vectors

$v_1 = (2, 1, 0, 3)$, $v_2 = (3, -1, 5, 2)$, $v_3 = (-1, 0, 2, 1)$

Answer : (i) $(2, 3, -7, 3) = 2v_1 - v_2 - v_3$

(ii) $(-4, 6, -13, 4) = 3v_1 - 3v_2 - v_3$.

23. Determine whether or not the vector (3, 9, −4, −2) is a linear combination of the vector set {(1, −2, 0, 3), (2, 3, 0, −1), (2, −1, 2, 1)} **[D. U. P. 1984] [D. U. H. T. 1989]**

Answer : $(3, 9, -4, -2) = 1\,(1, -2, 0, 3) + 3(2, 3, 0, -1)$
$-2(2, -1, 2, 1)$.

24. Express the polynomial $p = t^2 + 4t - 3$ over $\mathbb{R}$ as a linear combination of the polynomials

$p_1 = t^2 - 2t + 5$, $p_2 = 2t^2 - 3t$ and $p_3 = t + 3$.

Answer : $p = -3p_1 + 2p_2 + 4p_3$.

25. In the vector space $\mathbb{R}^3$ express the $v = (1, -2, 5)$ as a linear combination of the vectors vectors $v_1 = (1, 1, 1)$, $v_2 = (1, 2, 3)$ and $v_3 = (2, -1, 1)$.

Answer : $v = -6v_1 + 3v_2 + 2v_3$.

26. Determine whether (4, 2, 1, 0) is a linear combination of each of the following sets of vectors. If so find one such combination.

(i) {(1, 2, −1, 0), (1, 3, 1, 2), (6, 1, 0, 1)}

(ii) {(3, 1, 0, 1), (1, 2, 3, 1), (0, 3, 6, 6)}

(iii) {(6, −1, 2, 1), (1, 7, −3, −2), (3, 1, 0, 0), (3, 3, −2, −1)}

Answers : (i) (4, 2, 1, 0) is not a linear combination

(ii) (4, 2, 1, 0) is not a linear combination.

(iii) (4, 2, 1, 0) is a linear combination and

$(4, 2, 1, 0) = 2(6, -1, 2, 1) + 1\,(1, 7, -3, -2)$
$-3(3,1,0,0) + 0(3, 3, -2, -1)$

27. Express the matrix $A = \begin{bmatrix} 5 & 1 \\ -2 & 3 \end{bmatrix}$ as a linear combination of the matrices $A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ and $A_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. [D. U. H. T. 1988]

Answer : $A = 2A_1 + 3A_2 + 4A_3$

28. Express (if possible) the matrix $A = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}$ as a linear combination of the matrices

$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ and $A_3 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$.

[D. U. H. T. 1991; R. U. S. 1989]

Answer : A can not be expressed as a linear combination of A_1, A_2 and A_3.

29. Which of the following matrices are linear combinations of the matrices

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 2 & 4 \end{bmatrix}, C = \begin{bmatrix} 4 & -2 \\ 0 & -2 \end{bmatrix}?$$

(i) $\begin{bmatrix} 6 & 3 \\ 0 & 8 \end{bmatrix}$ (ii) $\begin{bmatrix} 6 & -1 \\ -8 & -8 \end{bmatrix}$

Answers : (i) $\begin{bmatrix} 6 & 3 \\ 0 & 8 \end{bmatrix} = 2A + B + C.$

(ii) $\begin{bmatrix} 6 & -1 \\ -8 & -8 \end{bmatrix} = 2A - 3B + C.$

30. In $\mathbb{R}^3$, let $S = \{(1, 2, 1), (3, 5, 0)\}$ and
$T = \{(1, 2, 1), (3, 5, 0), (2, 3, -1)\}$

Examine whether $\langle S \rangle = \langle T \rangle$. [D. U. P. 1982]

30(a) Show that the vectors $u_1 = (1, 1)$ and $u_2 = (1, -1)$ span $\mathbb{R}^2$.

31. Determine whether or not the following vectors span $\mathbb{R}^3$

(i) $u_1 = (1, 1, 2)$, $u_2 = (1, -1, 2)$, $u_3 = (1, 0, 1)$

(ii) $v_1 = (-1, 1, 0)$, $v_2 = (-1, 0, 1)$, $v_3 = (1, 1, 1)$

Answer : (i) u_1, u_2, u_3 span $\mathbb{R}^3$

(ii) v_1, v_2, v_3 span $\mathbb{R}^3$.

32. Show that the space generated by the vectors $u_1 = (1, 2, -1, 3)$, $u_2 = (2, 4, 1, -2)$ and $u_3 = (3, 6, 3, -7)$ and the space generated by the vectors $v_1 = (1, 2, -4, 11)$ and $v_2 = (2, 4, -5, 14)$ are equal. [C. U. P. 1978]

33.(i) Show that $\mathbb{R}^3 = \{(1, 2, 1), (2, 1, 0), (1, -1, 2)\}$

[D. U. H. T. 1992]

(ii) Find a condition on a, b, c so that $v = (a, b, c)$ is a linear combination of $v_1 = (1, -3, 2)$ and $v_2 = (2, -1, -1)$, that is, so that v belongs to span (v_1, v_2).

Answer : $a + b + c = 0$.

34. Let U and W be the subspaces of $\mathbb{R}^3$ defined by

$U = \{(a, b, c) : a = b = c\}$ and $W = \{(0, b, c)\}$

Show that $R^3 = U \oplus W$. [D. U. S. 1983]

35. Show that each of the following subsets S is a subspace of the indicated space V :

(i) $V = \mathbb{R}^3$, S is the collection of all triples (x, y, z) such that $x = y$ and $z = 0$.

(ii) $V = \mathbb{R}^4$, S is the collection of all 4–tuples (x, y, z, t) such that $x - y = z + t$.

36. Show that xz plane $W = (a, 0, c)$ in R^3 is generated by

(i) $(1, 0, 1)$ and $(2, 0, -1)$

(ii) $(1, 0, 2)$, $(2, 0, 3)$ and $(3, 0, 1)$.

37. (i) Prove that $S = \{(x, y, z, t) \in \mathbb{R}^4 \mid x + y - z + t = 0$ and $2x = y\}$ is a subspace of $\mathbb{R}^4$.

(ii) Prove that $T = \{(x_1, x_2, ..., x_n) \in R^n \mid x_1 + x_2 + ... + x_n = 0\}$ is a subspace of $\mathbb{R}^n$.

38. Let W_1 and W_2 be the subspaces of $\mathbb{R}^3$ defined by $W_1 = \{(a, b, 0) \mid a, b \in \mathbb{R}\}$ and $W_2 = \{(0, 0, c) \mid c \in \mathbb{R}\}$. Show that $\mathbb{R}^3 = W_1 \oplus W_2$.

39. Let W_1 and W_2 be the subspaces of $\mathbb{R}^3$ defined by $W_1 = \{(a, 0, 0) \mid a \in \mathbb{R}\}$ and $W_2 = \{(0, b, c) \mid b, c \in \mathbb{R}\}$.

Show that $\mathbb{R}^3 = W_1 \oplus W_2$.

40. Let S and T be subspaces of $\mathbb{R}^4$ defined by

$S = \{(a, 0, 0, 0) \mid a \in \mathbb{R}\}$ and $T = \{(0, b, c, d) \mid b, c, d \in \mathbb{R}\}$.

Show that $\mathbb{R}^4 = S \oplus T$.

41. Let $V(\mathbb{R}) = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R}\right\}$. Then show that $V(\mathbb{R}) = W_1 \oplus W_2$ where

$$W_1 = \left\{\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a, b, d \in \mathbb{R}\right\}.$$

$$W_2 = \left\{\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} : c \in \mathbb{R}\right\}.$$

42. Let V be the vector space over the field F.

Show that $V = W_1 \oplus W_2$ where W_1 and W_2 are the subspaces of symmetric and skew-symmetric matrices respectively over F.

6.14 Linear dependence and linear independence.

Definition : Let V be a vector space over the field F. The vectors $v_1, v_2, ..., v_m \in V$ are said to be **linearly dependent** over F or simply **dependent** if there exists a non-trivial linear combination of them equal to the zero vector 0.

That is, $\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_m v_m = 0$ where $\alpha_i \neq o$ for at least one i.

On the other hand, the vectors $v_1, v_2, ..., v_m$ in V are said to be **linearly independent** over F or simply **independent** if the only linear combination of them equal to 0 (zero vector) is the trivial one. that is,

$\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_m v_m = 0$ if and only if

$\alpha_1 = \alpha_2 = ... = \alpha_m = o.$

A single non-zero vector v is necessarily independent. since $\alpha v = 0$ if and only if $\alpha = o$.

Theorem 6.13 The non-zero vectors $v_1, v_2, ..., v_n$ in a vector space V are linearly dependent if and only if one of the vectors v_k is a linear combination of the preceding vectors $v_1, v_2, ..., v_{k-1}$.

Proof : If $v_k = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_{k-1} v_{k-1}$.

then $\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_{k-1} v_{k-1} + (-1) v_k + o v_{k+1} + ... + o v_n = 0$.

and hence the vectors $v_1, v_2, ..., v_n$ are linearly dependent.

Conversely, suppose that the vectors $v_1, v_2, ..., v_n$ are linearly dependent, then $\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n = 0$ where the scalars $\alpha_i \neq o$, then $\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_k v_k + 0 v_{k+1} + ... + 0 v_n = 0$

or, $\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_k v_k = 0$.

Now if k = 1, this implies that $\alpha_1 v_1 = 0$.

with $\alpha_1 \neq o$, so that $v_1 = 0$, giving a contradiction, because the v_1 is a non-zero vector. Hence $k > 1$ and we may

write $v_k = -\left(\frac{\alpha_1}{\alpha_k}\right) v_1 - \left(\frac{\alpha_2}{\alpha_k}\right) v_2 - \dots - \left(\frac{\alpha_{k-1}}{\alpha_k}\right) v_{k-1}$

giving v_k as a linear combination of $v_1, v_2, \dots, v_{k-1}$.

Thus the theorem is proved.

Lemma 1. Let V be a vector space which is spanned by a finite set of vectors, say $\{v_1, v_2, \dots, v_n\}$. Let $U = \{u_1, u_2, \dots, u_k\}$ be an independent set in V. Then $k \leq n$.

Proof : Since $\{v_1, v_2, \dots, v_n\}$ spans V, each vector in U is a linear combination of $\{v_1, v_2, \dots, v_n\}$, i.e.

$$\left.\begin{aligned} u_1 &= \alpha_{11}v_1 + \alpha_{21}v_2 + \dots + \alpha_{n1}v_n \\ u_2 &= \alpha_{12}v_1 + \alpha_{22}v_2 + \dots + \alpha_{n2}v_n \\ &\dots \quad \dots \quad \dots \quad \dots \\ u_k &= \alpha_{1k}v_1 + \alpha_{2k}v_2 + \dots + \alpha_{nk}v_n \end{aligned}\right\} \quad (1)$$

Suppose that k>n, then the system

$$\left.\begin{aligned} \alpha_{11}x_1 + \alpha_{12}x_2 + \dots + \alpha_{1k}x_k &= 0 \\ \alpha_{21}x_1 + \alpha_{22}x_2 + \dots + \alpha_{2k}x_k &= 0 \\ \dots \quad \dots \quad \dots \quad \dots \\ \alpha_{n1}x_1 + \alpha_{n2}x_2 + \dots + \alpha_{nk}x_k &= 0 \end{aligned}\right\} \quad (2)$$

has at least one non-trivial solution, say $(b_1, b_2, \dots, b_k)$

$$\left.\begin{aligned} \alpha_{11}b_1 + \alpha_{12}b_2 + \dots + \alpha_{1k}b_k &= 0 \\ \alpha_{21}b_1 + \alpha_{22}b_2 + \dots + \alpha_{2k}b_k &= 0 \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ \alpha_{n1}b_1 + \alpha_{n2}b_2 + \dots + \alpha_{nk}b_k &= 0 \end{aligned}\right\} \quad (3)$$

Then $b_1u_1 + b_2u_2 + \dots + b_ku_k$

$$= b_1(\alpha_{11}v_1 + \alpha_{21}v_2 + \dots + \alpha_{n1}v_n) +$$
$$b_2(\alpha_{12}v_1 + \alpha_{22}v_2 + \dots + \alpha_{n2}v_n)$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$+ b_k(\alpha_{1k}v_1 + \alpha_{2k}v_2 + \dots + \alpha_{nk}v_n)$$

$$= (\alpha_{11}b_1 + \alpha_{12}b_2 + \dots + \alpha_{1k}b_k)v_1 +$$
$$(\alpha_{21}b_1 + \alpha_{22}b_2 + \dots + \alpha_{2k}b_k)v_2 +$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$+ (\alpha_{n1}b_1 + \alpha_{n2}b_2 + \dots + \alpha_{nk}b_k)v_n$$
$$= 0v_1 + 0v_2 + \dots + 0v_n = 0.$$

Thus if k>n, we can choose $b_1, b_2, \dots, b_k$ not all zero such that $b_1u_1 + \dots + b_ku_k = 0$ i. e. $\{u_1, u_2, \dots, u_k\}$ is a dependent set which is a contradiction to the given condition. So we conclude that $k \leq n$. Hence the lemma is proved.

Lemma 2. Let $\{u_1, u_2, \dots, u_r\}$ be an independent set in a vector space V. Let W be the subspace spanned by $\{u_1, u_2, \dots, u_r\}$. Let v be a vector which is in V, but not in W then $\{u_1, u_2, \dots, u_r, v\}$ is an independent set.

Proof : Let $\alpha_1u_1 + \alpha_2u_2 + \dots + \alpha_r u_r + \alpha v = 0$ (1)

We will show that $\alpha_1 = \alpha_2 = \dots = \alpha_r = \alpha = o$.

If $\alpha \neq o$, solve for v. This gives.

$$v = -\frac{\alpha_1}{\alpha}u_1 - \frac{\alpha_2}{\alpha}u_2 - \dots - \frac{\alpha_r}{\alpha}u_r \quad (2)$$

The right hand side of this expressin (2) is a linear combination of $u_1, u_2, \dots, u_r$ and hence in W, but the left hand side of (2) is not in W; which is a contradiction. So we conclude that $\alpha = o$. This gives $\alpha_1u_1 + \alpha_2u_2 + \dots + \alpha_ru_r = 0$.

Since $\{u_1, u_2, \dots, u_r\}$ is an independent set, we must have $\alpha_1 = \alpha_2 = \dots = \alpha_r = o$.

Hence we have shown that $\alpha_1 = \alpha_2 = \dots = \alpha_r = \alpha = o$.

Therefore, $\{u_1, u_2, \dots, u_r, v\}$ is an independent set.

So the lemma is proved.

Theorem 6.14 The non-zero rows of a matrix in echelon form are linearly independent.

Proof : Suppose that the set of non-zero rows say $\{R_m, R_{m-1}, \ldots, R_1\}$ is linearly dependent. Then one of the rows say R_m is a linear combination of the preceding rows i. e.

$R_m = \alpha_{m+1} R_{m+1} + \alpha_{m+2} R_{m+2} + \ldots + \alpha_n R_n$ (1)

Now suppose that the kth component of R_m is its first non-zero entry. Then since the matrix is in echelon form, the kth components of $R_{m+1}, \ldots, R_n$ are all zero and so the kth component of (1) is $\alpha_{m+1} 0 + \alpha_{m+2} 0 + \alpha_{m+2} 0 + \ldots + \alpha_n 0 = 0$. But this contradicts the assumption that the kth component of R_m is not zero. Thus $R_1, R_2, \ldots, R_n$ are linearly independent.

Hence the theorem is proved.

Theorem 6.15 Let $v_1, v_1, \ldots, v_m$ be m linearly independent vectors and a vector u is a linear combination of $v_i, v_2, \ldots, v_m$ i.e. $u = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m$ where α_i are scalars. Then the above representation of u is unique.

Proof : Given $u = \alpha_1 v_1 + \alpha_1 v_2 + \ldots + \alpha_m v_m$ (1)

Suppose that the representation of u is not unique.

Then let $u = \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_m v_m$ (2)

where β_i are scalars.

Now by subtracting (2) from (1), we get

$0 = u - u = (\alpha_1 - \beta_1) + (\alpha_2 - \beta_2) v_2 + \ldots + (\alpha_m - \beta_m) v_m$

But the vectors $v_1, v_2, \ldots, v_m$ are linearly independent.

So that this implies $\alpha_1 - \beta_1 = o, \ldots, \alpha_m - \beta_m = o$

i. e, $\alpha_1 = \beta_1, \ldots, \alpha_m = \beta_m$.

This proves that $u = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m$ is the unique representation of u as a linear combination of $v_1, v_2, \ldots, v_m$.

Theorem 6.16 Supose that the set $\{v_1, v_2, \ldots, v_m\}$ is linearly independent, but the set $\{v_1, v_2, \ldots, v_m, w\}$ is linearly dependent. Then w is a linear combination of the vectors

$v_1, v_2, \ldots, v_m$

Proof : Since the set $\{v_1, v_2, \ldots, v_m, w\}$ is linearly dependent there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_m, \beta$ not all o (zero), such that

$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m + \beta w = 0.$

If $\beta = o$ then one of the α_i is not zero and

$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m = 0$. But this contradicts the hypothesis that $\{v_1, v_2, \ldots, v_m\}$ is linearly independent. Accordingly $\beta \neq o$ and so

$$w = \left(-\frac{\alpha_1}{\beta}\right) v_1 + \left(-\frac{\alpha_2}{\beta}\right) v_2 + \ldots + \left(-\frac{\alpha_m}{\beta}\right) v_m$$

That is, w is linear combination of the vectors $v_1, v_2, \ldots, v_m$.

Theorem 6.17 Let $u_1, u_2, \ldots, u_n$ be any n linearly independent vectors in a vector space V. Then any $(n + 1)$ vectors $v_1, v_2, \ldots, v_{n+1}$, each of which is a linear combination of $u_1, u_2, \ldots, u_n$ are linearly dependent.

Proof : We prove the result by induction on n. If any of the v_i' s is zero, then trivially given $(n + 1)$ vectors are linearly dependent. So we suppose that none of the v_i, s is zero. Now as v_1 and v_2 are both linear combinations of $u_1, u_2, \ldots, \ldots, u_n$ with $n = 1$, we get $v_1 = \alpha_1 u_1$, $v_2 = \alpha_2 u_1$ with $\alpha_1 \neq o$. $\alpha_2 \neq o$.

This gives $u_1 = \frac{v_1}{\alpha_1}, u_1 = \frac{v_2}{\alpha_2}$

or, $\frac{v_1}{\alpha_1} - \frac{v_2}{\alpha_2} = u_1 - u_1 = 0.$

or, $v_1 - \alpha_1 \alpha_2^{-1} v_2 = 0$

Hence v_1 and v_2 are linearly dependent.

To apply induction suppose that the result holds for any k linearly independent vectors where $k < n$. We can now write

$$v_1 = \alpha_{11} u_1 + \alpha_{12} u_2 + \ldots + \alpha_{1n} u_n$$
$$v_2 = \alpha_{21} u_1 + \alpha_{22} u_2 + \ldots + \alpha_{2n} u_n$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$v_{n+1} = \alpha_{n+11} u_1 + \alpha_{n+12} u_2 + \ldots + \alpha_{n+1n} u_n$$

for some scalars α_{ij} in F.

If $\alpha_{in} = o$ for every i = 1, 2, ..., n + 1 then each of v_i is a linear combination of (n − 1) vectors $u_1, u_2, ..., u_{n-1}$ and the induction hypothesis will yield that $v_1, v_2, ..., v_n$ are also linearly dependent and consequently $v_1, v_2 ..., v_{n+1}$ are also linearly dependent.

So we suppose that at least one of $\alpha_{in} \neq o$.

Let $\alpha_{in} \neq o$, then for each i = 2, 3, .., n + 1

$v_i - \alpha_{in}\alpha_{in}^{-1}v_1 = (\alpha_{i1} - \alpha_{in}\alpha_{11}\alpha_{1n}^{-1})u_1 +$

$(\alpha_{i2} - \alpha_{in}\alpha_{12}\alpha_{1n}^{-1})u_2 + ... + (\alpha_{i'n-1} - \alpha_{in}\alpha_{1'n-1}\alpha_{1n}^{-1})u_{n+1}$

So by induction hypothesis the vectors

$w_i = v_i - \alpha_{in}\alpha_{in}^{-1}v_1$ $(2<i<n+1)$ are linearly dependent. Consequently, there exist $\beta_2, \beta_3, ..., \beta_{n+1}$ in F, not all zero, such that $\beta_2 w_2 + \beta_3 w_3 + ... + \beta_{n+1}w_{n+1} = 0$ i. e.

$\beta_2(v_2 - \alpha_{2n}\alpha_{1n}^{-1}v_1) + \beta_3(v_3 - \alpha_{3n}\alpha_{1n}^{-1}v_1) +$

$.. + \beta_{n+1}(v_{n+1} - \alpha_{n+1'n}\alpha_{1n}^{-1}v_1) = 0$

i. e. $-(\beta_2\alpha_{2n}\alpha_{1n}^{-1} + \beta_3\alpha_{3n}\alpha_{1n}^{-1} + ... + \beta_{n+1}\alpha_{n+1'n}\alpha_{1n}^{-1})v_1$

$+ \beta_2 v_2 + \beta_3 v_3 + ... + \beta_{n+1}v_{n+1} = 0.$

This gives that $v_1, v_2, ..., v_{n+1}$ are linearly dependent. Hence the theorem is proved.

Theorem 6.18 Every set of vectors containing a dependent subset is dependent and every subset of an independent set is independent.

Proof : First portion

Suppose that the set $S = \{v_1, v_2, ..., v_m\}$

constains a dependent subset, say $\{v_1, v_2, ..., v_r\}$

Since $\{v_1, v_2, ..., v_r\}$ is linearly dependent, there exist scalars $\alpha_1, \alpha_2, ..., \alpha_r$ not all o, such that

$\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_r v_r = 0$

Hence there exist scalars $\alpha_1, \alpha_2, ..., \alpha_r$, o, .., o not all o (zero), such that $\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_r v_r + o\,v_{r+1} + ... + o v_m = 0$. Accordingly, S is dependent.

Second portion : Let $T = \{v_1, v_2, ..., v_m\}$ be an independent set then $\beta_1 v_1 + \beta_2 v_2 + ... + \beta_m v_m = 0$ implies that

$\beta_1 = \beta_2 = ... = \beta_m = o$

Now let $\{v_1, v_2, ..., v_k\}$ k < m be a subset of T, then there exist scalars $\beta_1, \beta_2, ..., \beta_k$ where each $\beta_i = o$, i = 1, 2, ..., k such that $\beta_1 v_1 + \beta_2 v_2 + ... + \beta_k v_k = 0$. Therefore, the subset $\{v_1, v_2, ..., v_k\}$ is independent.

WORKED OUT EXAMPLES

Example 16. Prove that the set of vectors

{(2, 1, 2), (0, 1, −1), (4, 3, 3)} is linearly dependent.

Proof : First Process

Set a linear combination of the given vectors equal to zero by using unknown scalars *x*, y, z :

$x(2, 1, 2) + y(0, 1, -1) + z(4, 3, 3) = (0, 0, 0)$

or, $(2x, x, 2x) + (0, y, -y) + (4z, 3z, 3z) = (0, 0, 0)$

or, $(2x + 0 + 4z, x + y + 3z, 2x - y + 3z) = (0, 0, 0)$

Equating corresponding components and forming the linear system, we get

$$\left.\begin{aligned} 2x + 0 + 4z &= 0 \\ x + y + 3z &= 0 \\ 2x - y + 3z &= 0 \end{aligned}\right\} \quad (1)$$

Reduce the system to echelon form by the elementary transformations. Interchange first and second equations. Then we get the equivalent system

$$\left.\begin{aligned} x + y + 3z &= 0 \\ 2x + 0 + 4z &= 0 \\ 2x - y + 3z &= 0 \end{aligned}\right\} \quad (2)$$

We multiply first equation by 2 and then subtract from the second and third equations respectively. Then we get the equivalent system

$$\left.\begin{aligned} x + y + 3z &= 0 \\ -2y - 2z &= 0 \\ -3y - 3z &= 0 \end{aligned}\right\} \quad (3)$$

Divide second and third equations by–2 and–3 respectively.

Then we get the equivalent system

$$\left.\begin{aligned} x + y + 3z &= 0 \\ y + z &= 0 \\ y + z &= 0 \end{aligned}\right\} \quad (4)$$

Since second and third equations are identical, we can disergard one of them. Then we have the equivalent system

$$\left.\begin{aligned} x + y + 3z &= 0 \\ y + z &= 0 \end{aligned}\right\} \quad (5)$$

This system is in echelon form and has only two non-zero equations in three unknowns, hence the system has non-zero solution. Thus the original vectors are linearly dependent.

Second process : Form the matrix whose rows are the given vectors and reduce the matrix to row echelon form by using the elementary row operations.

$$\begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & -1 \\ 4 & 3 & 3 \end{bmatrix}$$

we multiply first row by 2 and then subtract from the third row.

$$\sim \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

we subtract second row from the third row.

$$\sim \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix is in row echelon form and has a zero row; hence the given vectors are linearly dependent.

Example 17. Show that the set of vectors $\{(3, 0, 1, -1), (2, -1, 0, 1), (1, 1, 1, -2)\}$ is linearly dependent.

[D. U. S. 1984]

Proof : First process

Set a linear combination of the given vectors equal to the zero vector using unknown scalars x, y, z :

$x(3, 0, 1, -1) + y(2, -1, 0, 1) + z(1, 1, 1, -2) = (0, 0, 0, 0)$

or, $(3x, 0, x, -x) + (2y, -y, o, y) + (z, z, z, -2z) = (0, 0, 0, 0)$

or, $(3x + 2y + z, 0 - y + z, x + 0 + z, -x + y - 2z) = (0, 0, 0, 0)$

Form the homogeneous linear equations by equating the corresponding components.

$$\left.\begin{aligned} 3x + 2y + z &= 0 \\ -y + z &= 0 \\ x + 0 + z &= 0 \\ -x + y - 2z &= 0 \end{aligned}\right\} \quad (1)$$

Reduce the system to echelon form by elementary transformations. Interchange first and third equations.

Then we have the equivalent system

$$\left.\begin{aligned} x + 0 + z &= 0 \\ -y + z &= 0 \\ 3x + 2y + z &= 0 \\ -x + y - 2z &= 0 \end{aligned}\right\} \quad (2)$$

we multiply second equation by -1.

We multiply first equation by 3 and then subtract from the third equation. We also add first equation with the fourth equation. Then we get the equivalent system.

$$\left.\begin{aligned} x+0+z&=0\\ y-z&=0\\ 2y-2z&=0\\ y-z&=0 \end{aligned}\right\} \quad (3)$$

Again, divide third equation by 2. Then we get the equivalent system

$$\left.\begin{aligned} x+0+z&=0\\ y-z&=0\\ y-z&=0\\ y-z&=0 \end{aligned}\right\} \quad (4)$$

Since second, third and fourth equations are identical, we can disregard any two of them. Thus the system (4) reduces to

$$\left.\begin{aligned} x+0+z&=0\\ y-z&=0 \end{aligned}\right\} \quad (5)$$

This system is in echelon form and has only two non-zero equations in three unknowns ; hence the system has a non-zero solution. Thus the original vectors are linearly dependent.

Second Process : Form the matrix whose rows are the given vectors and reduce the matrix to row echelon form by using elementary row operations :

$$\begin{bmatrix} 3 & 0 & 1 & -1\\ 2 & -1 & 0 & 1\\ 1 & 1 & 1 & -2 \end{bmatrix}$$

Interchange first and third rows,

$$\sim\begin{bmatrix} 1 & 1 & 1 & -2\\ 2 & -1 & 0 & 1\\ 3 & 0 & 1 & -1 \end{bmatrix}$$

We multiply first row by 2 and 3 and then subtract from the second and third rows respectively.

$$\sim\begin{bmatrix} 1 & 1 & 1 & -2\\ 0 & -3 & -2 & 5\\ 0 & -3 & -2 & 5 \end{bmatrix}$$

We subtract second row from third row.

$$\sim\begin{bmatrix} 1 & 1 & 1 & -2\\ 0 & -3 & -2 & 5\\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in row echelon form and has a zero row; hence the given vectors are linearly dependent.

Example 18. Show that the vectors (2, – 1, 4) (3, 6, 2) and (2, 10, – 4) are linearly independent.

Proof : First Process

Set a linear combination of the given vectors equal to the zero vector using unknown scalars x, y, z :

$x(2, -1, 4) + y(3, 6, 2) + z(2, 10, -4) = (0, 0, 0)$

or, $(2x, -x, 4x) + (3y, 6y, 2y) + (2z, 10z, -4z) = (0, 0, 0)$

or, $(2x + 3y + 2z, -x + 6y + 10z, 4x + 2y - 4z) = (0, 0, 0)$

Form a homogeneous system of linear equations equating the corresponding components :

$$\left.\begin{aligned} 2x+3y+2z&=0\\ -x+6y+10z&=0\\ 4x+2y-4z&=0 \end{aligned}\right\} \quad (1)$$

Reduce the system to echelon form by the elementary transfromations. Interchange first and second equations. Then we have the equivalent system

$$\left.\begin{aligned} -x+6y+10z&=0\\ 2x+3y+2z&=0\\ 4x+2y-4z&=0 \end{aligned}\right\} \quad (2)$$

We multiply first equation by -1 and we divide third equation by 2. Then we have the equivalent system

$$\left.\begin{aligned} x - 6y - 10z &= 0 \\ 2x + 3y + 2z &= 0 \\ 2x + y - 2z &= 0 \end{aligned}\right\} \qquad (3)$$

We multiply first equation by 2 and then subtract from the second and third equations. Then the system reduces to

$$\left.\begin{aligned} x - 6y - 10z &= 0 \\ 15y + 22z &= 0 \\ 13y + 18z &= 0 \end{aligned}\right\} \qquad (4)$$

We multiply second equation by $\frac{13}{15}$ and then subtract from the third equation. Then we have the equivalent system

$$\left.\begin{aligned} x - 6y - 10z &= 0 \\ 15y + 22z &= 0 \\ -\tfrac{8}{15}z &= 0 \end{aligned}\right\} \qquad (5)$$

which is in echelon form.

In echelon form there are exactly three equations in three unknowns; hence the system has only the zero solution $x = 0$, $y = 0$, $z = 0$. Accordingly, the vectors are linearly independent.

Second Process

Form the matrix whose rows are the given vectors and reduce the matrix to row echelon form by elementary row operations.

$$\begin{bmatrix} 2 & -1 & 4 \\ 3 & 6 & 2 \\ 2 & 10 & -4 \end{bmatrix}$$

We divide third row by 2 and then interchange with the first row.

$$\sim \begin{bmatrix} 1 & 5 & -2 \\ 3 & 6 & 2 \\ 2 & -1 & 4 \end{bmatrix}$$

We multiply first row by 3 and 2 and then subtract from the second and third rows respectively.

$$\sim \begin{bmatrix} 1 & 5 & -2 \\ 0 & -9 & 8 \\ 0 & -11 & 8 \end{bmatrix}$$

We multiply second row by $\frac{11}{9}$ and then subtract from the third row.

$$\sim \begin{bmatrix} 1 & 5 & -2 \\ 0 & -9 & 8 \\ 0 & 0 & -\frac{16}{9} \end{bmatrix} \text{ which is in row echelon form.}$$

Since the echelon matrix has no zero-row, the veetors are linearly independent.

Example 19. Let u, v and w are independent vectors, Show that u + v, u − v, u − 2v + w are also independent.

[D. U. S, 1980, 1983]

Proof : Set a linear combination of the given vectors equal to the zero vector using three unknown scalars x, y, z :

$x(\mathrm{u} + \mathrm{v}) + y(\mathrm{u} - \mathrm{v}) + z(\mathrm{u} - 2\mathrm{v} + \mathrm{w}) = 0$

or, $x\mathrm{u} + x\mathrm{v} + y\mathrm{u} - y\mathrm{v} + z\mathrm{u} - 2z\mathrm{v} + z\mathrm{w} = 0$

or, $(x + y + z)\,\mathrm{u} + (x - y - 2z)\,\mathrm{v} + z\mathrm{w} = 0 \qquad (1)$

Since u, v and w are linearly independent, the co-efficients in the above relation (1) are each 0 (zero), that is,

$$\left.\begin{aligned} x + y + z &= 0 \\ x - y - 2z &= 0 \\ z &= 0 \end{aligned}\right\}$$

The only solution to the above system is $x = 0$, $y = 0$, $z = 0$. Hence the given vectors u + v, u − v and u − 2v + w are independent.

Example 20. Test the dependency of the following sets :

(i) {(1, 2, − 3), (2, 0, − 1), (7, 6, − 11)}

(ii) {(2, 0, −1), (1, 1, 0), (0, − 1, 1)} **[D. U. P. 1984]**

Solution : (i) Set a linear combination of the given vectors equal to zero vector using three unknown scalars x, y, z :

$x(1, 2, -3) + y(2, 0, -1) + z(7, 6, -11) = (0, 0, 0)$

or, $(x, 2x, -3x) + (2y, 0, -y) + (7z, 6z, -11z) = (0, 0, 0)$

or, $(x + 2y + 7z, 2x + 0 + 6z, -3x - y - 11z) = (0, 0, 0)$

Equating the corresponding components and forming the linear system, we get

$$\left.\begin{aligned} x + 2y + 7z &= 0 \\ 2x + 0 + 6z &= 0 \\ -3x - y - 11z &= 0 \end{aligned}\right\} \qquad (1)$$

Reduce the system to echelon form by the elementary operations. We multiply first equation by 2 and then subract from the second equation. We also multiply the first equation by 3 and then add with the third equation. Then we have the equivalent system

$$\left.\begin{aligned} x + 2y + 7z &= 0 \\ -4y - 8z &= 0 \\ 5y + 10z &= 0 \end{aligned}\right\} \qquad (2)$$

We divide second equation by -4 and the third equation by 5. then we have the equivalent system

$$\left.\begin{aligned} x + 2y + 7z &= 0 \\ y + 2z &= 0 \\ y + 2z &= 0 \end{aligned}\right\} \qquad (3)$$

Since second and third equations are identical, we can disregard one of them, Then the system (3) reduces to

$$\left.\begin{aligned} x + 2y + 7z &= 0 \\ y + 2z &= 0 \end{aligned}\right\} \qquad (4)$$

The system, in echelon form, has only two non-zero equations in the three unknowns; hence the system has a non-zero solution. Thus the original vectors are linearly dependent.

(ii) Set a linear combination of the given vectors equal to the zero vector using unknown scalars x, y, z :

$x(2, 0, -1) + y(1, 1, 0) + z(0 - 1, 1) = (0, 0, 0)$

or, $(2x, 0, -x) + (y, y, 0) + (0, -z, z) = (0, 0, 0)$

or, $(2x + y + 0, 0 + y - z, -x + 0 + z) = (0, 0, 0)$

Equating corresponding components, and forming the linear system we get

$$\left.\begin{aligned} 2x + y + 0 &= 0 \\ 0 + y - z &= 0 \\ -x + 0 + z &= 0 \end{aligned}\right\} \qquad (1)$$

Reduce the system to echelon form by the elementary operations. We multiply third equation by — 1 and then interchange with the first equation. Then we get the equivalent system

$$\left.\begin{aligned} x + 0 - z &= 0 \\ 0 + y - z &= 0 \\ 2x + y + 0 &= 0 \end{aligned}\right\} \qquad (2)$$

We multiply first equation by 2 and then subtract from the third equation. Then we get the equivalent system

$$\left.\begin{aligned} x + 0 - z &= 0 \\ y - z &= 0 \\ y + 2z &= 0 \end{aligned}\right\} \qquad (3)$$

We subtract second equation from the third equation. Then the system (3) reduces to

$$\left.\begin{aligned} x + 0 - z &= 0 \\ y - z &= 0 \\ 3z &= 0 \end{aligned}\right\} \qquad (4)$$

From the third equation we get $z = 0$.

Putting $z = 0$ is the second and first equations we get $y = 0$, $x = 0$ respectively. Thus $x = 0$, $y = 0$, $z = 0$, Hence the given vectors are linearly independent.

Example 21. Let V be the vector space of all 2×3 matrices over the real field $\mathbb{R}$. Show that $A, B, C \in V$:

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 & 4 \\ 4 & 5 & -2 \end{bmatrix}$$

and $C = \begin{bmatrix} 3 & -8 & 7 \\ 2 & 10 & -1 \end{bmatrix}$ are linearly dependent.

[D.U.H.T. 1986]

Proof : Let $xA + yB + zC = 0$ where $x, y, z \in \mathbb{R}$.

Then $x\begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 1 \end{bmatrix} + y\begin{bmatrix} 1 & -1 & 4 \\ 4 & 5 & -2 \end{bmatrix} + z\begin{bmatrix} 3 & -8 & 7 \\ 2 & 10 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

or, $\begin{bmatrix} x & -2x & 3x \\ 2x & 4x & x \end{bmatrix} + \begin{bmatrix} y & -y & 4y \\ 4y & 5y & -2y \end{bmatrix}$

$$+ \begin{bmatrix} 3z & -8z & 7z \\ 2z & 10z & -z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

or, $\begin{bmatrix} x+y+3z & -2x-y-8z & 3x+4y+7z \\ 2x+4y+2z & 4x+5y+10z & x-2y-z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Equating corresponding components and forming the linear system, we get

$$\left.\begin{aligned} x + y + 3z &= 0 \\ 2x + 4y + 2z &= 0 \\ -2x - y - 8z &= 0 \\ 4x + 5y + 10z &= 0 \\ 3x + 4y + 7z &= 0 \\ x - 2y - z &= 0 \end{aligned}\right\}$$

Reduce the system to echelon form by the elementary operations. We multiply 1st equation by 2, – 2, 4, 3 and 1 and then subtract from 2nd, 3rd, 4th 5th and 6th equations respectively. Then we have the equivalent system.

$$\left.\begin{aligned} x + y + 3z &= 0 \\ 2y - 4z &= 0 \\ y - 2z &= 0 \\ y - 2z &= 0 \\ y - 2z &= 0 \\ -y + 2z &= 0 \end{aligned}\right\}$$

We multiply 2nd equation by $\frac{1}{2}$ and 6th equation by -1. Then we have the equivalent system

$$\left.\begin{aligned} x + y + 3z &= 0 \\ y - 2z &= 0 \\ y - 2z &= 0 \\ y - 2z &= 0 \\ y - 2z &= 0 \\ y - 2z &= 0 \end{aligned}\right\}$$

Now 2nd, 3rd, 4th, 5th & 6th equations are identical, we can disregard any four of them.

Thus we have the equivalent system

$$\left.\begin{aligned} x + y + 3z &= 0 \\ y - 2z &= 0 \end{aligned}\right\}$$

This system is in echelon form having two equations in three unknowns. So the system has $3 - 2 = 1$ free variable which is z. Thus the system has non-zero solutions.

Let $z = 1$, then $y = 2$ and $x = -5$.

$\therefore -5A + 2B + C = 0$

Hence the given matrices A, B and C are linearly dependent.

Example 22. Let P(t) be the vector space of all polynomials of degree ≤ 3 over the real field $\mathbb{R}$. Determine whether the following polynomials in P(t) are linearly dependent or independent :

$u = t^3 + 4t^2 - 2t + 3$, $v = t^3 + 6t^2 - t + 4$ and

$$w = 3t^3 + 8t^2 - 8t + 7.$$

Solution : Set a linear combination of the given polynomials u, v and w equal to the zero polynomial using the unknown scalars x, y and z; that is, $xu + yv + zw = 0$.

Thus $x(t^3+4t^2-2t+3)+y(t^3+6t^2-t+4)+z(3t^3+8t^2-8t+7)=0$

or, $xt^3+4xt^2-2xt+3x+yt^3+6yt^2-yt+4y+3zt^3+8zt^2$
$-8zt+7z=0$

or, $(x+y+3z)t^3+(4x+6y+8z)t^2+(-2x-y-8z)t+3x+4y+7z=0$

Setting the coefficients of the powers of t each equal to 0 (zero), we get the following homogeneous linear system :

$$\left.\begin{array}{r} x+y+3z=0\\ 4x+6y+8z=0\\ -2x-y-8z=0\\ 3x+4y+7z=0 \end{array}\right\}$$

Reduce this system to echelon form by the elementary operations. We multiply 1st equation by 4, –2 and 3 and then subtract from 2nd, 3rd and 4th equations respectively. Then we have the equivalent system

$$\left.\begin{array}{r} x+y+3z=0\\ 2y-4z=0\\ y-2z=0\\ y-2z=0 \end{array}\right\}$$

Interchange 2nd and 4th equations. Then we have the equivalent system

$$\left.\begin{array}{r} x+y+3z=0\\ y-2z=0\\ y-2z=0\\ 2y-4z=0 \end{array}\right\}$$

We multiply 2nd equation by 1 and 2 and then subtract from the 3rd & 4th equations respectively. Then we have the equivalent system

$$\left.\begin{array}{r} x+y+3z=0\\ y-2z=0\\ 0+0=0\\ 0+0=0 \end{array}\right\} \Rightarrow \left.\begin{array}{r} x+y+3z=0\\ y-2z=0 \end{array}\right\}$$

This system is in echelon form having two equations in three unknowns. So the system has 3 – 2 = 1 frree variable which is z.

Hence the system has non -zero solutions. that is, $xu+yv+zw=0$ does not imply that $x=y=z=0$.

Thus the given polynomials u, v and w are linearly dependent.

EXERCISES–6(B)

1. Show that the vectors (1, 0, 0), (0, 1, 0) and (1, 1, 0) in $V_3(\mathbb{R})$ are linearly dependent. **[D.U.P. 1980]**

2. Which of the following sets of vectors in $\mathbb{R}^3$ are linearly independent?

(i) {(1, 0, 1), (–3, 2, 6), (4, 5, –2)}

(ii) {(1, –4, 2), (3, –5, 1), (2, 7, 8), (–1, 1, 1)}

Answers : (i) Independent (ii) Dependent.

3. Prove that the following sets of vectors in $\mathbb{R}^3$ are linearly independent;

(i) {(1, 0, 2), (–1, 1, 0), (0, 2, 3)}

(ii) {(1, 1, 1), (0, 1, 1,) (0, 0, 1)}

4. (i) Prove that the set of vectors (2, 1, 1), (3, – 4, 6) and (4, –9, 11) is linearly dependent in $\mathbb{R}^3$. **[D. U. S. 1983]**

(ii) Test the set {1, 0, 1), (0, 2, 2), (3, 7, 1)} for linear dependence on $\mathbb{R}^3$.

Answer : (ii) the set is linearly independent.

5. Show that the set of vectors

$S=\{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}$ in $\mathbb{R}^3$ is linearly dependent but that any set of three of them is linearly independent.

6. Test whether the following sets are linearly independent :

(i) {(0, 0, 1), (1, 1, –2), (3, 4, 1)}

(ii) {(1, –1, 3) (1, 4, 5), (2 –3, 7)}

(iii) {(1, 1, –1), (1, 2, 3), (4, 5, –3)} **[D.U.S. 1981, D.U. P. 1981]**

Answers : (i) Independent (ii) Independent (iii) dependent.

7. Examine the linear dependence of the following two subsets of $\mathbb{R}^3$:

(i) S = {(1, 0, –1), (–3, 2, 6), (4, 5, –2)}

(ii) T = (1, 1,1), (0, 0, 0), (1, 1, 0)} **[D. U. P. 1983]**

Answers : (i) S is linearly independent.

(ii) T is linearly dependent.

8. (i) Decide whether S = {(1, –4, 2), (3, –5, 1), (–1, 1, 1)} is a linearly independent subset of $\mathbb{R}^3$. **[D. U. P. 1982]**

(ii) Consider the subset {(1, 3, 2), (1, –7, –8) (2, 1, –1)} of $\mathbb{R}^3$. Test the dependence of the subset. **[D. U. P. 1979]**

Answers : (i) S is linearly independent.

(ii) The subset is linearly dependent.

9. Show that the vectors u = (6, 2, 3, 4), v = (0, 5, –3, 1) and w = (0, 0, 7, –2) are linearly independent. **[D.U. S. 1980]**

10. If the vector set $\{v_1, v_2, v_3\}$ is independent prove that (a) the set $\{v_1 + v_2 - 2v_3, v_1 - v_2 - v_3, v_1 + v_3\}$ is independent and (b) the set $\{v_1 + v_2 - 3v_3, v_1 + 3v_2 - v_3, v_2 + v_3\}$ is dependent. **[D. U. P. 1984]**

11. Determine whether or not the following vectors in $\mathbb{R}^3$ are linearly dependent or independent :

(i) v_1 = (1, – 2, 1), v_2 = (2, 1, –1), v_3 = (7, –4, 1) **[C.U.P. 1973]**

(ii) u = (1, – 2, 4, 1), v = (2, 1, 0, –3) and w = (3, – 6, 1, 4) **[C. U. P. 1978]**

Answers : (i) vectors are linearly dependent.

(ii) vectors are linearly independent.

12. Determine whether each of the following sets are linearly dependent or linearly independent :

(i) {(0, 1, 0, 1) (1, 2, 3, –1) (8, 4, 3, 2), (0, 3, 2, 0)}

(ii) {(1, 3, 2), (1, –7, –8), (2, 1, –1)} **[D. U. Prel. 1983]**

Answers : (i) vectors are linearly independent.

(ii) vectors are linearly dependent

13. (i) Show that the set {(4, 4, 0, 0), (0, 0, 6, 6) (–5, 0, 5, 5)} is linearly independent in $\mathbb{R}^4$.

(ii) Prove that the set {(1, 0, 2, 4), (0, 1, 9, 2), (–5, 2, 8, –16)} is linearly dependent in $\mathbb{R}^4$.

14. Determine whether the following sets of vectors in $\mathbb{R}^4$ are linearly independent :

(i) {(3, 0, 4, 1), (6, 2, –1, 2), (–1, 3, 5, 1), (–3, 7, 8, 3)}

(ii) {(4, – 4, 8, 0), (2, 2, 4, 0), (6, 0, 0, 2), (6, 3, – 3, 0)}

Answers : (i) The set is linearly independent.

(ii) The set is linearly independent.

15. Show that the following sets of vectors in $\mathbb{R}^3$ are linearly dependent :

(i) {(1, 1, –1), (1, 2, 3), (3, 5, –3)} **[D. U. S. 1981]**

(ii) {(2, 1, 1), (3, –4, 6) (4, – 9, 11)} **[D. U. S. 1983]**

16. Show that the following sets of vectors in $\mathbb{R}^3$ are linearly independent :

(i) {(1, –1, 3), (1, 4, 5), (2, – 3,7)} **[D. U. H. 1974]**

(ii) {(1, –2, 1), (2, 1, –1), (7, –4, 1)} **[C. U. P. 1973]**

17. Determine whether the following sets of vectors in $\mathbb{R}^3$ are linearly dependent or independent :

(i) {(1, –2, 1), (2, 1, –1), (7, –4, 1)} **[C. U. P. 1973]**

(ii) {(–1, 1, 1), (1, –4, 2) (3, –5, 1)} **[D. U. P. 1982]**

Answers : (i) Linearly dependent

(ii) Linearly independent.

18. Prove that the following set of vectors in $\mathbb{R}^3$ is linearly dependent :

$\{(3, 0, -3), (-1, 1, 2), (4, 2, -2), (2, 1, 1)\}$.

19. Which of the following sets of vectors in $\mathbb{R}^3$ is linearly dependent :

(i) $\{(2, -1, 4)\ (3, 6, 2), (2, 10, -4)\}$

(ii) $\{(1, 3, 3), (0, 1, 4), (5, 6, 3), (7, 2, -1)\}$

Answers : (i) Linearly independent (ii) Linearly dependent

20. Determine whether the following sets of vectors in $\mathbb{R}^4$ are linearly dependent or independent :

(i) $\{(1, 2, 1, -2), (0, -2, -2, 0)\ (0, 2, 3, 1), (3, 0, -3, 6)\}$

(ii) $\{(3, 0, 4, 1), (6, 2, -1, 2), (-1, 3, 5, 1), (-3, 7, 8, 3)\}$

Answers :

(i) Linearly independent (ii) Linearly independent.

21. For which real values of λ do the following vectors form a linearly dependent set in $\mathbb{R}^3$?

$v_1 = \left(\lambda, -\frac{1}{2}, -\frac{1}{2}\right)$, $v_2 = \left(-\frac{1}{2}, \lambda, -\frac{1}{2}\right)$ and $v_3 = \left(-\frac{1}{2}, -\frac{1}{2}, \lambda\right)$.

Answer : $\lambda = -\frac{1}{2}$, $\lambda = 1$.

22. Show that the set of vectors in $\mathbb{R}^4$

$\{(2, 1, -1, 1), (3, -2, 4, 5), (-1, 3, 2, 6), (4, 2, 5, 12)\}$ is linearly dependent.

23. If u, v and w are linearly independent vectors in a vector space V (F), then show that the vectors

(i) u + v, v + w, w + u

(ii) u + v, u − v, u − 2v + w

are also linearly independent.

24. Show that the vectors

$(1, 1, 2, 4), (2, -1, -5, 2), (1, -1, -4, 0)$ and $(2, 1, 1, 6)$

are linearly dependent in $\mathbb{R}^4$.

25. Let V be the vector space of all 2×2 matrices over the real field $\mathbb{R}$. Show that the matrices A, B, C $\in$ V are linearly independent where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$ **[R.U.M. Sc. P. 1988]**

26. In vector space V ($\mathbb{R}$) of all 2×2 matrices determine whether the following matrices are linearly dependent :

$$\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 3 & -1 \\ 2 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & -5 \\ -4 & 0 \end{bmatrix}.$$

Answer : Linearly dependent

27. Let V($\mathbb{R}$) be the vector space of all 2×3 matrices over the real field $\mathbb{R}$. Show that the following matrices in V($\mathbb{R}$) are linearly independent :

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & -3 \\ -2 & 0 & 5 \end{bmatrix} \text{ and } C = \begin{bmatrix} 4 & -1 & 2 \\ 1 & -2 & 3 \end{bmatrix}.$$

28. Let V_2 ($\mathbb{R}$) be the vector space of all polynomials of degree ≤ 2. Show that the set

$\{t^2 + t + 2, 2t^2 + t, 3t^2 + 2t + 2\}$ of polynomials is linearly dependent.

[D. U. P. 1987]

29. Let V($\mathbb{R}$) be the vector space of all polynomials of degree ≤ 3. Determine whether the following polynomials are linearly dependent or independent :

(i) $t^3 + 2t^2 + 4t - 1, 2t^3 - t^2 - 3t + 5, t^3 - 4t^2 + 2t + 3$

(ii) $t^3 - 3t^2 - 2t + 3, 2t^3 - 5t^2 - 5t + 7, t^3 - 2t^2 - 3t + 3$

Answers : (i) Linearly independent

(ii) Linearly independent.

6.15 Basis and dimension of a vector space

Definition : Let V be a vector space and $\{v_1, v_2, \ldots, v_n\}$ finite set of vectors in V.

We call $\{v_1, v_2, ..., v_n\}$ a **basis** for V if and only if

(i) $\{v_1, v_2, ..., v_n\}$ is linearly independent.

(ii) $\{v_1, v_2, ..., v_n\}$ spans v.

Definition : Let $e_1 = (1, 0, 0, ..., 0)$, $e_2 = (0, 1, 0, ..., 0)$, ..., $e_n = (0, 0, ..., 0, 1)$. Then $\{e_1, e_2, ..., e_n\}$ is a linearly independent set in $\mathbb{R}^n$. Since any vector $v = (v_1, v_2, ..., v_n)$ in $\mathbb{R}^n$ can be written as $v = v_1e_1 + v_2e_2 + ... + v_ne_n$, $\{e_1, e_2, ..., e_n\}$ spans $\mathbb{R}^n$. Therefore, $\{e_1, e_2, ..., e_n\}$ is a basis. It is called the **standard basis** or **usual basis** for $\mathbb{R}^n$.

Definition : A non-zero vector space V is called **finite dimensional** if it contains finite set of vectors $\{v_1, v_2, ..., v_n\}$ which forms a basis for V. If no such set exists, V is called **infinite dimensional.**

Definition : The **dimension** of a finite dimensional vector space is the number of vectors in any basis of it.

Or, equivalently, the **dimension** of a vector space is equal to the maximum number of linearly independent vectors contained in it.

Definition : If $v_1, v_2, ..., v_n$ are vectors of a vector space V such that every vector $v \in V$ can be written in the form $v = \alpha_1v_1 + \alpha_2v_2 + ... + \alpha_nv_n$ where α_i are scalars, then $v_1, v_2, ..., v_n$ is called a **generating system** of the vector space V.

Theorem 6.19 If V is a vector space of dimension n, every generating system of V contains n, but not more than n, linearly independent vectors.

Proof : Let $v_1, v_2, ..., v_m$ be any generating system of V. Let r be the greatest number of linearly independent vectors that can be chosen from $v_1, v_2, ..., v_m$. Then we may assume that $v_1, v_2, ..., v_r$ are linearly independent but that $v_1, v_2, ..., v_r, v_{r+j}$ are linearly dependent for $j = 1, 2, 3, ..., m - r$. Hence we have a non-trivial relation $\alpha_1v_1 + \alpha_2v_2 + ... + \alpha_rv_r + \alpha_{r+j}v_{r+j} = 0$ ... (1) in which $\alpha_{r+j} \neq 0$, since otherwise the above equation (1) would be a non-trival relation between $v_1, v_2, ..., v_r$ contrary to the linear independence of these vectors. Hence the above equation (1) defines v_{r+j} as a linear combination of $v_1, v_2, ..., v_r$ for $j = 1, 2, ..., m - r$. Therefore, it follows that $v_1, v_2, ..., v_r$, is also a generating system of V.

Now any set of more than r vectors of V is linearly dependent. Since the dimension of V is n, V certainly contains n linearly independent vectors. We must have $n \leq r$. But since V contains r linearly independent vectors $u_1, u_2, ..., u_r$ we also have $r \leq n$. Hence $r = n$ and the theorem is proved.

Theorem 6.20 If V is a vector space of dimension n, every basis of V contains exactly n linearly independent vectors, conversely, any n linearly independent vectors of V constitute a basis of V.

Proof : Since every basis is a generating system, it contains n but not more than n linearly independent vectors. Since the vectors of a basis are linearly independent, therefore, it contains exactly n vectors in all.

Conversely, let $v_1, v_2, ..., v_n$ be any n linearly independent vectors of V and let v be any other vector of V. Since n is the dimension of V, the $(n + 1)$ vectors $v_1, v_2, ..., v_n$, v are linearly dependent and there exist a non-trival relation of the form

$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n + \alpha v = 0$. Moreover $\alpha \neq 0$. Since otherwise we would have a non-trival relation among $v_1, v_2, \ldots, v_n$. Hence we have

$$v = -\frac{\alpha_1}{\alpha} v_1 - \frac{\alpha_2}{\alpha} v_2 - \ldots - \frac{\alpha_n}{\alpha} v_n.$$

It follows that every vector $v \in V$ is a linear combination of $v_1, v_2, \ldots, v_n$ and hence these n vectors form a generating system. Since they are linearly independent, they also form a basis of V. Hence the theorem is proved.

Theorem 6.21 Let V be a vector space of dimension n and let $v_1, v_2, \ldots, v_r$, $(r < n)$ be any r linearly independent vectors of V. Then there exist $n - r$ vectors $v_{r+1}, v_{r+2}, \ldots, v_n$ of V which together with $v_1, v_2, \ldots, v_r$ constitute a basis of V.

Proof : Since $r < n$, the vectors $v_1, v_2, \ldots, v_r$, do not generate the whole space V. Hence there exists a vector v_{r+1}, of V that is not in the space generated by $v_1, v_2, .., v_r$. The vectors $v_1, v_2, .., v_r, v_{r+1}$ are therefore linearly independent, for if

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_r v_r + \alpha_{r+1} v_{r+1} = 0.$$

We must have $\alpha_{r+1} = 0$. Since otherwise v_{r+1} would belong to the space generated by $v_1, v_2, \ldots\ldots, v_r$. Then it follows that $\alpha_1 = \alpha_2 = \ldots \alpha_r = 0$ since $v_1, v_2, \ldots, v_r$ are linearly independent.

Now if $r + 1 < n$, we can repeat this argument to obtain a vector v_{r+2} such that $v_1, v_2, \ldots, v_r, v_{r+1}, v_{r+2}$ are linearly independent and so on until we have n linearly independent vectors $v_1, v_2, \ldots, v_r, v_{r+1}, v_{r+2}, \ldots, v_n$. These n vectors constitute a basis of V. Hence the theorem is proved.

Theorem 6.22 If $\{v_1, v_2, \ldots, v_n\}$ is a basis of the vector space V, then every vector $v \in V$ can be expressed uniquely in the form $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$.

Proof : Since $\{v_1, v_2, \ldots, v_n\}$ is a basis of V, any vector $v \in V$ can be written as a linear combination of the vectors $v_1, v_2, \ldots, v_n$ i. e, $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$ where α_i are scalars. Therefore, we have only to show that the co-efficients $\alpha_1, \alpha_2, \ldots, \alpha_n$ are uniquely determined by v. Suppose that

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n v_n$$

Then $(\alpha_1 - \beta_1) v_1 + (\alpha_2 - \beta_2) v_2 + \ldots + (\alpha_n - \beta_n) v_n = 0$

Since $v_1, v_2, \ldots, v_n$ are linearly independent, it follows that $\alpha_1 - \beta_1 = 0$, $\alpha_2 - \beta_2 = 0$, $\alpha_n - \beta_n = 0$, that is, $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2, \ldots, \alpha_n = \beta_n$. Hence the theorem is proved.

Theorem 6.23 Let W be a subspace of an n-dimensional vector space V. Then dim $W < n$. In particular, if dim $W = n$ then $W = V$.

Proof : Since V is of dimension n, any $n + 1$ or more vectors are linearly dependent.

Furthermore, since a basis of W consists of linearly independent vectors, it can not contain more than n elements. Accordingly dim $W \leq n$. In particular, if $\{w_1, w_2, \ldots, w_n\}$ is a basis of W, then since it is an independent set with n elements, it is also a basis of V. Thus $W = V$, when dim $W = n$. Hence the theorem is proved.

Theorem 6.24 If S and T are subspaces of a finite dimensional vector space V over the field F then

$$\dim (S + T) = \dim S + \dim T - \dim (S \cap T)$$

Proof : Let dim $S = s$, dim $T = t$ and dim $(S \cap T) = r$.

Let $\{u_1, u_2, \ldots, u_r\}$ be a basis of $S \cap T$. Since $S \cap T$ is a subspace of S, We can extend the above basis to a basis of S, say $\{u_1, u_2, , u_r, v_1, \ldots, v_{s-r}\}$. This basis has s elements since dim $S = s$ Similarly, we can extend the basis $\{u_1, u_2 \ldots, u_r\}$ to a basis of T; say $\{u_1, u_2, \ldots, u_r, w_1, w_2, \ldots, w_{t-r}\}$

Let $A = \{u_1, u_2, \ldots, u_r, v_1, \ldots, v_{s-r}, w_1, \ldots, w_{t-r}\}$.

Clearly, A has exactly $s + t - r$ elements. Thus the theorem is proved if we can show that A is a basis of S + T. Since $\{u_i, v_j\}$ generates S and $\{u_i, w_k\}$ generates T, the union $A = \{u_i, v_j, w_k\}$ generates S + T. Now we have only to show that A is linearly independent.

Suppose that $\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_r u_r + \beta_1 u_1 + \ldots + \beta_{s-r} v_{s-r}$.

$$+ \gamma_1 w_1 + \ldots + \gamma_{t-r} w_{t-r} = 0. \qquad (1)$$

where $\alpha_i, \beta_j, \gamma_k$ are scalars in F.

Let $v = \alpha_1 u_1 + \ldots + \alpha_r u_r + \beta_1 v_1 + \ldots + \beta_{s-r} v_{s-r}$ (2)

Then form (1) we get

$v = -\gamma_1 w_1 - \gamma_2 w_2 - \ldots - \gamma_{t-r} w_{t-r}$, (3)

Since $\{u_i, v_j\} \in S$, $v \in S$ by (2) and

Since $\{w_k\} \subset T$, $v \in T$, by (3)

Accordingly $v \in S \cap T$.

Now since $\{u_i\}$ is a basis of $S \cap T$, there exist scalars $\delta_1, \ldots, \delta_r$ such that $v = \delta_1 u_1 + \delta_2 u_2 + \ldots + \delta_r u_r$. Thus by (3), we have

$\delta_1 u_1 + \alpha_2 u_2 + \ldots + \delta_r u_r + \gamma_1 w_1 + \ldots + \gamma_{t-r} w_{t-r} = 0$.

But $\{u_i, w_k\}$ is a basis of T and so is independent. Hence the above equation forces $\gamma_1 = 0, \ldots, \gamma_{t-r} = 0$. Substituting this into (1), we get $\alpha_1 u_1 + \ldots + \alpha_r u_r + \beta_1 v_1 + \ldots + \beta_{s-r} v_{s-r} = 0$.

But $\{u_i, v_j\}$ is a basis of S and so is independent. Hence the above equation forces $\alpha_1 = 0, \ldots, \alpha_r = 0, \beta_1 = 0, \ldots, \beta_{s-r} = 0$.

So the equation (1) implies that α_i, β_j and γ_k are all 0 (zero), So $A = \{u_i, v_j, w_k\}$ is linearly independent and form a basis of S + T.

Thus $\dim(S + T) = s + t - r = \dim S + \dim T - \dim(S \cap T)$.

Hence the **theorem is proved.**

Corollary : If S and T are two subspaces of a finite dimensional vector space V such that $S \cap T = \{0\}$ then $\dim(S + T) = \dim S + \dim T$.

Proof : Since $S \cap T = \{0\}$, $\dim(S \cap T) = 0$

Thus $\dim(S + T) = \dim S + \dim T - \dim(S \cap T)$ implies that $\dim(S + T) = \dim S + \dim T$. Hence the corollary is proved.

6.16 Quotient space

Definition : Let W be any subspace of a vector space V over the field F. Let v be any element of V. Then the set $W + v = \{\omega + v : \omega \in W\}$ is called a **right coset** of W in V generated by v.

Obviously, W + v and v + W are both subsets of V. Since addition in V is commutative, therefore, we have W + v = v + W. Hence we shall call W + v as simply a **coset** of W in V generated by v.

Properties : (i) $o \in V \Rightarrow W + o = W$

(ii) $\omega \in W \Rightarrow W + \omega = W$

(iii) If $W + v_1$ and $W + v_2$ are two cosets of W in V, then $W + v_1 = W + v_2 \Rightarrow v_1 - v_2 \in W$

Theorem 6.25 If W is any subspace of a vector space V over the field F, then the set V/W of all cosets $W + v_1$ where v_1 is any arbitrary element of V, is a vector space over F for the vector addition and scalar multiplication compositions defined as follows :

$(W + v_1) + (W + v_2) = W + (v_1 + v_2)$ for every $v_1, v_2 \in V$.

and $\alpha(W + v_1) = W + \alpha v_1$ where $\alpha \in F$ and $v_1 \in V$.

Proof : Let $v_1, v_2 \in V$ then $v_1 + v_2 \in V$ and also $\alpha \in F$, $v_1 \in V \Rightarrow \alpha v_1 \in V$. Therefore, $W + (v_1 + v_2) \in V/W$ and also $W + \alpha v_1 \in V/W$.

Thus V/W is closed with respect to addition of cosets and scalar multiplication as defined above.

Let $W + v_1 = W + v_1'$ where $v_1, v_1' \in V$
and $W + v_2 = W + v_2'$ where $v_2, v_2' \in V$.
Now we have $W + v_1 = W + v_1' \Rightarrow v_1 - v_1' \in W$
and $W + v_2 = W + v_2' \Rightarrow v_2 - v_2' \in W$.

Since W is a subspace of V, we have

$$\begin{aligned} v_1 - v_1' \in W, v_2 - v_2' \in W &\Rightarrow v_1 - v_1' + v_2 - v_2' \in W \\ &\Rightarrow (v_1 + v_2) - (v_1' + v_2') \in W \\ &\Rightarrow W + (v_1 + v_2) = W + (v_1' + v_2') \\ &\Rightarrow (W + v_1) + (W + v_2) = (W + v_1') + (W + v_2') \end{aligned}$$

Therefore, **addition of cosets** in V/W is well-defined.

Again, $\beta \in F$, $v_1 - v_1' \in W \Rightarrow \beta(v_1 - v_1') \in W$

$$\begin{aligned} &\Rightarrow \beta v_1 - \beta v_1' \in W \\ &\Rightarrow W + \beta v_1 = W + \beta v_1'. \end{aligned}$$

Therefore, **scalar multiplication** in V/W is also well defined.

(i) Addition is commutative

Let $W + v_1$, $W + v_2$ be any two elements of V/W.

Then $(W + v_1) + (W + v_2) = W + (v_1 + v_2) = W + (v_2 + v_1)$

$$= (W + v_2) + (W + v_1)$$

(iii) Addition is associative

Let $W + v_1$, $W + v_2$ and $W + v_3$ be any three elements of V/W.

Then $(W + v_1) + [(W + v_2) + (W + v_3)]$

$$\begin{aligned} &= (W + v_1) + [W + (v_2 + v_3)] \\ &= W + [v_1 + (v_2 + v_3)] \\ &= W + [(v_1 + v_2) + v_3] \\ &= [W + (v_1 + v_2)] + (W + v_3) \\ &= [(W + v_1) + (W + v_2)] + (W + v_3) \end{aligned}$$

(iii) Existence of additive identity

If O is the zero vector of V, then

$W + O = W \in V/W$. If $W + v_1$ is any element of V/W,

then $(W + O) + (W + v_1) = W + (O + v_1) = W + v_1$

$\therefore W + O = W$ is the additive identity;

(iv) Existence of additive inverse.

If $W + v_1$ is any element of V/W, then

$W + (-v_1) = W - v_1 \in V/w$. Also we have

$(W + v_1) + (W - v_1) = W + (v_1 - v_1) = W + O = W$.

$\therefore W - v_1$ is the additive inverse of $W + v_1$.

Thus V/W is an **abelian group** with respect to addition composition.

Further for scalar multiplication, we observe that if $\alpha, \beta \in F$ and $W + v_1, W + v_2 \in V/W$, then

(v) $\alpha[(W + v_1) + (W + v_2)] = \alpha[W + (v_1 + v_2)]$

$$\begin{aligned} &= W + \alpha(v_1 + v_2) \text{ Since } \alpha W = W \\ &= W + (\alpha v_1 + \alpha v_2) \\ &= (W + \alpha v_1) + (W + \alpha v_2) \\ &= \alpha(W + v_1) + \alpha(W + v_2). \end{aligned}$$

(vi) $(\alpha + \beta)(W + v_1) = W + (\alpha + \beta)v_1$

$$\begin{aligned} &= W + (\alpha v_1 + \beta v_1) \\ &= (W + \alpha v_1) + (W + \beta v_1) \\ &= \alpha(W + v_1) + \beta(W + v_1) \end{aligned}$$

(vii) $\alpha\beta(W + v_1) = W + (\alpha\beta)v_1 = W + \alpha(\beta v_1)$

$$= \alpha(W + \beta v_1) = \alpha[\beta(W + v_1)]$$

(viii) $1(W + v_1) = W + 1v_1 = W + v_1$ where $1 \in F$.

Thus V/W is a vector space over the field F for the addition of cosets and scalar multiplication. The vector space V/W is called the **Quotient space** of V relative to W.

The coset $W + O = W$ is the **zero vector** of this vector space.

6.17 Dimension of Quotient Space.

Theorem 6.26 If W is a subspace of a finite dimensional vector space V over the field F, then dim V/W = dim V − dim W.

Proof : Let m be the dimension of the subspace W of the vector space V. Let $S = \{w_1, w_2, \ldots, w_m\}$ be a basis of W. Since S is linearly independent subset of V, therefore, it can be extended to a basis of V.

Let $S' = \{w_1, w_2, \ldots, w_m, v_1, v_2, \ldots, v_r\}$ be a basis of V.

Then dim V = m + r.

$\therefore$ dim V − dim W = (m + r) − m = r.

So we have to prove that dim V/W = r.

Now we claim that the set of r cosets

$S_1 = \{W + v_1, W + v_2, \ldots, W + v_r\}$ is a basis of V/W.

First we have to show that S_1 is linearly independent.

The zero vector of V/W is W.

Let $\alpha_1 (W + v_1) + \alpha_2 (W + v_2) + \ldots + \alpha_r (W + v_r) = W$

$\Rightarrow (W + \alpha_1 v_1) + (W + \alpha_2 v_2) + \ldots + (W + \alpha_r v_r) = W$

$\Rightarrow W + (\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_r v_r) = W$

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_r v_r \in W$

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_r v_r = \beta_1 w_1 + \beta_2 w_2 + \ldots + \beta_m w_m$

Since $\{w_1, w_2, \ldots, w_m\}$ is a basis of W.

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_r v_r - \beta_1 w_1 - \beta_2 w_2 - \ldots - \beta_m w_m = 0$

$\Rightarrow \alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_r = 0$

Since the vectors $v_1, v_2, \ldots, v_r, w_1, w_2, \ldots, w_m$ are linearly independent. Thus the set $S_1 = \{W + v_1, W + v_2, \ldots, W + v_r\}$ is linearly independent.

Now we have to show that $L(S_1) = {}^V/_W$. i.e S_1 spans ${}^V/_W$.

Let W + v be any element of ${}^V/_W$. Then $v \in V$ can be expressed as

$v = \gamma_1 w_1 + \gamma_2 w_2 + \ldots + \gamma_m w_m + \delta_1 v_1 + \delta_2 v_2 + \ldots + \delta_r v_r$

$= w + \delta_1 v_1 + \delta_2 v_2 + \ldots + \delta_r v_r$

where $w = \gamma_1 w_1 + \gamma_2 w_2 + \ldots + \gamma_m w_m \in W$.

So $W + v = W + (w + \delta_1 v_1 + \delta_2 v_2 + \ldots + \delta_r v_r)$

$= (W + w) + \delta_1 v_1 + \delta_2 v_2 + \ldots + \delta_r v_r$

$= W + \delta_1 v_1 + \delta_2 v_2 + \ldots + \delta_r v_r$

Since $w \in W$ $\therefore W + w = W$

$= (W + \delta_1 v_1) + (W + \delta_2 v_2) + \ldots + (W + \delta_r v_r)$

$= (\delta_1 (W + v_1) + \delta_2 (W + v_2) + \ldots + \delta_r (W + v_r)$

Thus any element W + v of ${}^V/_W$ can be expressed as linear combination of $S_1 = \{W + v_1, W + v_2, \ldots, W + v_r\}$

$\therefore {}^V/_W = L(S_1)$

So S_1 is a basis of ${}^V/_W$

Thus dim ${}^V/_W$ = r = dim V − dim W.

Hence the theorem is proved.

6.18 Coordinates of a vector relative to a basis

Let $\{v_1, v_2, \ldots, v_n\}$ be a basis of an n dimensional vector space V over the field F. Then any vector $v \in V$ can be expressed uniquely in the form $v = x_1 v_1 + x_2 v_2 + \ldots + x_n v_n$

where $x_i \in F$ $(1 \le i \le n)$ and $(x_1, x_2, \ldots, x_n)$ are called the **co-ordinates** of v relative to the given basis. There is clearly one-one correspondence between the vector $v \in V$ and ordered n-tuples $(x_1, x_2, \ldots, x_n)$ with elements in F.

One can easily verify that if v, w have co-ordinates $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$ respectively and $\alpha \in F$, then v + w has co-ordinates $(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$ and αv has co-ordinates $(\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$.

6.19 Solution space of a homogeneous system of linear equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0 \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0 \end{array}\right\} \quad (1)$$

is a **homogeneous system** of m linear equations in n unknowns over the real field $\mathbb{R}$. The solution set W of (1) constitutes a subspace of $\mathbb{R}^n$ and this subspace is called the **solution space** of the system of linear equations.

As for examples

(i) The solution set $W = \{(2a, a) : a \in \mathbb{R}\}$ of $x - 2y = 0$ is a solution space and W is a subspace of $\mathbb{R}^2$.

(ii) The solution set $W = \{(-2a, -a, a)\ a \in \mathbb{R}\}$ of the linear system of equations $\left.\begin{array}{r} x - y + z = 0 \\ y + z = 0 \end{array}\right\}$ is a solution space and W is a subspace of $\mathbb{R}^3$.

Remark : The solution set W of the non-homogeneous linear system $\sum_{j=1}^{n} a_{ij} x_j = b_i \ (i = 1, 2, \dots, m)$ (2) over the real field $\mathbb{R}$ does not constitute a subspace of $\mathbb{R}^n$.

Theorem 6.27 (Without proof)

The following three statements are equivalent :

(i) The system of linear equations AX = B has a solution

(ii) B is a linear combination of the columns of A

(iii) The coefficient matrix A and the augmented matrix (A, B) have the same rank.

mid

6.20 Basis and dimension for the general solution of a homogeneous linear system.

Let W denote the general solution of a homogeneous linear system. The non zero solution vectors $u_1, u_2, \dots, u_s$ are said to form a **basis** of W if every solution vector $\omega \in W$ can be expressed uniquely as a linear combination of $u_1, u_2, \dots, u_s$. The number s of such basis vectors is called the **dimension** of W, written as dim W = s. [If W = {0}, we define dim W = 0]

6.21 Procedure of finding the basis and dimension of the solution space of a homogeneous linear system.

Let W be the general solution of a homogeneous linear system and suppose an echelon form of the system has s free variables. Let $u_1, u_2, \dots, u_s$ be the solutions obtained by setting one of the free variables equal to one (or any non-zero constant) and the remaining free variables equal to zero.

Then dim W = s and $u_1, u_2, \dots, u_s$ form a basis of W.

Theorem 6.28 The dimension of the solution space W of the homogeneous system of linear equations AX = 0 is n−r where n is the number of unknowns and r is the rank of the coefficient matrix A.

Proof : Suppose $u_1, u_2, \dots, u_r$ form a basis for the column space of A (There are r such vectors since rank of A is r). By theorem 6.27, each system $AX = u_i$ has a solution say v_i. Hence

$Av_1 = u_1, Av_2 = u_2, \dots, Av_r = u_r$ (1)

Suppose dim W = s and $w_1, w_2, \dots, w_s$ form a basis of W.

Let $B = \{v_1, v_2, \dots, v_r, w_1, w_2, \dots, w_s\}$

We claim that B is a basis of $\mathbb{R}^n$.

Thus we need to prove that B spans $\mathbb{R}^n$ and that B is linearly independent.

(i) Proof of "B spans $\mathbb{R}^n$ "

Suppose $v \in \mathbb{R}^n$ and $Av = u$. Then $u = Av$ belongs to the column space of A and hence Av is a linear combination of the u_i, say $Av = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_r u_r$ (2)

Let $v' = v - \alpha_1 v_1 - \alpha_2 v_2 - \dots - \alpha_r v_r$. Then using (1) and (2) we have $A(v') = A(v - \alpha_1 v_1 - \alpha_2 v_2 - \dots - \alpha_r u_r)$

$$= Av - \alpha_1 Av_1 - \alpha_2 Av_2 - \dots - \alpha_r Av_r$$
$$= Av - (\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_r u_r)$$
$$= Av - Av = 0$$

Thus v' belongs to the solution space W and hence v' is a linear combination of w_i, say

$$v' = \beta_1 w_1 + \beta_2 w_2 + \dots + \beta_s w_s = \sum_{j=1}^{s} \beta_j w_j.$$

$$\text{Then } v = v' + \sum_{i=1}^{r} \alpha_i v_i = \sum_{i=1}^{r} \alpha_i v_i + \sum_{j=1}^{s} \beta_j w_j$$

Thus v is a linear combination of the elements in B and hence B spans $\mathbb{R}^n$.

(ii) Proof of "B is linearly independent"

Suppose $\gamma_1 v_1 + \gamma_2 v_2 + \dots + \gamma_r v_r + \delta_1 w_1 + \delta_2 w_2 + \dots + \delta_s w_s = 0$ (3)

Since $w_j \in W$, each $Aw_j = 0$. Using this fact and (1) and (3), we get

$$0 = A(0) = A\left(\sum_{i=1}^{r} \gamma_i v_i + \sum_{j=1}^{s} \delta_j w_j\right) = \sum_{i=1}^{r} \gamma_i Av_i + \sum_{j=1}^{s} \delta_j Aw_j$$

$$= \sum_{i=1}^{r} \gamma_i u_i + \sum_{j=1}^{s} \delta_j 0 = \gamma_1 u_1 + \gamma_2 u_2 + \dots + \gamma_r u_r.$$

Since $u_1, u_2, \dots, u_r$ are linearly independent, each $\gamma_i = 0$

Substituting this in (3), we get $\delta_1 w_1 + \dots + \delta_s w_s = 0$.

However, $w_1, w_2, \dots, w_s$ are linearly independent. Thus each $\delta_j = 0$. Therefore, B is linearly independent. Accordingly, B is a basis of $\mathbb{R}^n$. Since B has r + s elements, we have r + s = n. Consequently dim W = s = n − r. Hence the theorem is proved.

Example 23. Prove that the vectors (1, 2, 0), (0, 5, 7) and (−1, 1, 3) form a basis for $\mathbb{R}^3$.

Proof : The given vectors will be a basis of $\mathbb{R}^3$ if and only if they are linearly independent and every vector in $\mathbb{R}^3$ can be written as a linear combination of (1, 2, 0) (0, 5, 7) and (−1, 1,3).

First we shall prove that the vectors are linearly independent.

For arbitrary scalars x, y, z, let

$x(1, 2, 0) + y(0, 5, 7) + z(-1, 1, 3) = (0, 0, 0)$

or, $(x, 2x, 0) + (0, 5y, 7y) + (-z, z, 3z) = (0, 0, 0)$

or, $(x - z, 2x + 5y + z, 7y + 3z) = (0, 0, 0)$.

Equating corresponding components and forming linear system, we have

$$\left.\begin{aligned} x \quad\quad - z &= 0 \\ 2x + 5y + z &= 0 \\ 7y + 3z &= 0 \end{aligned}\right\} \quad (1)$$

Reduce the system to echelon form by the elementary transformations. We multiply first equation by 2 and then subtract from the second equation.

Thus we get the equivalent system

$$\left.\begin{aligned} x - z &= 0 \\ 5y + 3z &= 0 \\ 7y + 3z &= 0 \end{aligned}\right\} \quad (2)$$

or, we can write

$$\left.\begin{aligned} x - z \quad\quad &= 0 \\ 3z + 5y &= 0 \\ 3z + 7y &= 0 \end{aligned}\right\} \quad (3)$$

We subtract second equation from the third equation. Then we get the equivalent system

$$\left.\begin{aligned} x - z &= 0 \\ 3z + 5y &= 0 \\ 2y &= 0 \end{aligned}\right\} \qquad (4)$$

This system is in echelon form and has exactly three equations in the three unknowns; hence the system has only the zero solution i. e. $x = 0$, $y = 0$, $z = 0$. So the given veotors are linearly independent.

[By the application of matrices we can also show that the given vectors are linearly independent]

To show that the given vectors span $\mathbf{R}^3$, we must show that an arbitrary vector $v = (a, b, c)$ can expressed as a linear combination $v = (a, b, c) = x(1, 2, 0) + y(0, 5, 7) + z(-1, 1, 3)$

Then forming linear system, we get

$$\left.\begin{aligned} x \quad\;\; - z &= a \\ 2x + 5y + z &= b \\ 7y + 3z &= c \end{aligned}\right\} \qquad (5)$$

Reduce this system to echelon form by this elementary operations.

We multiply first equation by 2 and then subtract from the second equation. Thus we have the equivalent system

$$\left.\begin{aligned} x \quad - z &= a \\ 5y + 3z &= b - 2a \\ 7y + 3z &= c \end{aligned}\right\} \qquad (6)$$

We subtract second equation from the third equation. Then we get the equivalent system

$$\left.\begin{aligned} x \quad - z &= a \\ 5y + 3z &= b - 2a \\ 2y \qquad &= c - b + 2a \end{aligned}\right\} \qquad (7)$$

From the third equation, we have $y = \frac{1}{2}(c - b + 2a)$

Substituting the value of y in the second equation we get $z = \frac{1}{6}(7b - 5c - 14a)$

Again substituting $z = \frac{1}{6}(7b - 5c - 14a)$ in the first equation, we get $x = \frac{1}{6}(7b - 5c - 8a)$

Therefore, $v = (a, b, c) = \frac{1}{6}(7b - 5c - 8a)(1, 2, 0)$
$+ \frac{1}{2}(c - b + 2a)(0, 5, 7) + \frac{1}{6}(7b - 5c - 14a)(-1, 1, 3)$

Thus every vector in $\mathbb{R}^3$ can be expressed as a linear combination of the vectors (1, 2, 0), (0, 5, 7) and (–1, 1, 3). Hence the vectors (1, 2, 0), (0, 5, 7) and (–1, 1, 3) form a basis of $\mathbb{R}^3$.

Example 24. (i) Extend {(2, 0, 1), (1, 1, 1)} to a basis of $\mathbb{R}^3$.

[D. U. P. 1984]

(ii) Extend the set {(3, 2, 1), (0, 1, 1)} to a basis of $\mathbb{R}^3$.

[D. U. S. 1982]

Solution : (i) First we have to show that the given set of two vectors is linearly independent. Set a linear combination of the two given vectors equal to zero by using unknown scalars x and y :

$$x(2, 0, 1) + y(1, 1, 1) = (0, 0, 0)$$

or, $(2x, 0, x) + (y, y, y) = (0, 0, 0)$

or, $(2x + y, y, x + y) = (0, 0, 0)$

Equating corresponding components and forming the linear system, we get

$$\left.\begin{aligned} 2x + y &= 0 \\ y &= 0 \\ x + y &= 0 \end{aligned}\right\} \quad \text{Thus we have } x = 0, y = 0.$$

Hence the given two vectors in $\mathbb{R}^3$ are linearly independent. So {(2, 0, 1), (1, 1, 1)} is a part of the basis of $\mathbb{R}^3$ and hence we can extend them to a basis of $\mathbb{R}^3$. Now we seek three independent vectors in $\mathbb{R}^3$ which include the given two vectors. Thus we can easily verify that (2, 0, 1), (1, 1, 1), (0, 1, 0) are linearly independent. So they form a basis of $\mathbb{R}^3$ which is an extension of the given set of vectors to a basis of $\mathbb{R}^3$.

(ii) First we have to show that the given set of the vectors is linearly independent. Set a linear combination of the two given vectors equal to zero by using unknown scalars x and y :

$$x(3, 2, 1) + y(0, 1, 1) = (0, 0, 0)$$

or, $(3x, 2x, x) + (0, y, y) = (0, 0, 0)$

or, $(3x, 2x + y, x + y) = (0, 0, 0)$

Equating corresponding components and forming the linear system, we get

$$\left.\begin{aligned} 3x \quad &= 0 \\ 2x + y &= 0 \\ x + y &= 0 \end{aligned}\right\} \quad \text{Thus we have } x = 0,\ y = 0.$$

Hence the given two vectors in $\mathbb{R}^3$ are linearly independent. So the given set of vectors is a part of the basis of $\mathbb{R}^3$ and hence we can extend them to a basis of R^3. Now we seek three independent vectors in $\mathbb{R}^3$ which include the given vectors. Thus we can easily verify that (3, 2, 1) (0, 1, 1), (1, 0, 0) are linearly independent. So they form a basis of $\mathbb{R}^3$ which is an extension of the given set of vectors to a basis of $\mathbb{R}^3$.

Example 25. Determine a basis and the dimension for the solution space of the following homogeneous system:

$$\left.\begin{aligned} x - 3y + \ \ z &= 0 \\ 2x - 6y + 2z &= 0 \\ 3x - 9y + 3z &= 0 \end{aligned}\right\} \qquad (2)$$

Solution : The given linear system is

$$\left.\begin{aligned} x - 3y + \ \ z &= 0 \\ 2x - 6y + 2z &= 0 \\ 3x - 9y + 3z &= 0 \end{aligned}\right\} \qquad (1)$$

Reduce the system to echelon form by the elementary transformations. We multiply first equation by 2 and by 3 and then subtract from the second and the third equations respectively. Then we have the equivalent system

$$\left.\begin{aligned} x - 3y + z &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}\right\} \quad \Rightarrow x - 3y + z = 0$$

This system is in echelon form and has only one non-zero equation in three unknowns. So the system has $3 - 1 = 2$ free variables which are y and z, Hence the dimension of the solution space is 2 (two).

Set (i) $y = 1,\ z = 0$ (ii) $y = 0,\ z = 1$ to obtain the respective solutions $v_1 = (3, 1, 0)$, $v_2 = (-1, 0, 1)$.

Hence the set {(3, 1, 0), (−1, 0, 1)} is a basis of the solution space.

Mid

Example 26. Find the solution space W of the following homogeneous system of linear equations :

$$\left.\begin{aligned} x + 2y - \ \ z + 4t &= 0 \\ 2x - \ \ y + 3z + 3t &= 0 \\ 4x + \ \ y + 3z + 9t &= 0 \\ y - \ \ z + \ t &= 0 \\ 2x + 3y - \ \ z + 7t &= 0 \end{aligned}\right\} \qquad \begin{aligned} &\textbf{[D. U. H. T. 1983]} \\ &\textbf{[J. U. H. 1988]} \end{aligned}$$

Solution : Reduce the given system to echelon form by the elementary operations. We multiply 1st equation by 2, 4 and 2 and then subtract from 2nd, 3nd and 5th equations respectively. Then we have the equivalent system

$$\left.\begin{aligned} x + 2y - \ \ z + 4t &= 0 \\ -5y + 5z - 5t &= 0 \\ -7y + 7z - 7t &= 0 \\ y - \ \ z + \ \ t &= 0 \\ -\ y + \ \ z - \ \ t &= 0 \end{aligned}\right\}$$

We multiply 2nd, 3rd and 5th equations by $-\frac{1}{5}, -\frac{1}{7}$ and (-1) respectively. Then we have the equivalent system

$$\left.\begin{aligned} x + 2y - z + 4t &= 0 \\ y - z + t &= 0 \\ y - z + t &= 0 \\ y - z + t &= 0 \\ y - z + t &= 0 \end{aligned}\right\}$$

Since 2nd, 3rd, 4th & 5th equations are identical, we can disregard any three of them. Then we have the equivalent system

$$\left.\begin{aligned} x + 2y - z + 4t &= 0 \\ y - z + t &= 0 \end{aligned}\right\}$$

This system is in echelon form having two equations in 4 unknowns. So the system has $4 - 2 = 2$ free variables which are z and t and hence it has non-zero solutions. Let $z = a$ and $t = b$ where a and b are arbitrary real numbers. Putting $z = a$ and $t = b$ in the 2nd equation we get $y = a - b$. Again putting the values of y, z and t in the 1st equation, we get $x = -a - 2b$. Hence the required solution space is

$W = \{(-a - 2b, a - b, a, b) : a, b \in \mathbb{R}\}$.

Example 27. Let S and T be the following subspaces of $\mathbb{R}^4$:

$S = \{(x, y, z, t) \mid y - 2z + t = 0\}$

$T = \{(x, y, z, t) \mid x - t = 0, y - 2z = 0\}$

Find a basis and the dimension of (i) S (ii) T (iii) $S \cap T$.

Solution : (i) We seek a basis of the set of respective solution (x, y, z, t) of the equation $y - 2z + t = 0$.

The free variables are x, z and t. Set (a) $x = 1, z = 0, t = 0$

(b) $x = 0, z = 1, t = 0$.

(c) $x = 0, z = 0, t = 1$, to obtain the respective solutions $u_1 = (1, 0, 0, 0), u_2 = (0, 2, 1, 0), u_3 = (0, 1, 0, 1)$

The set $\{u_1, u_2, u_3\}$ is a basis of S and dim S = 3.

(ii) We seek a basis of the set of solutions (x, y, z, t) of the equations

$$\left.\begin{aligned} x - t &= 0 \\ y - 2z &= 0 \end{aligned}\right\}$$

The free variables are z and t. Set (a) $z = 1, t = 0$, (b) $z = 0, t = 1$ to obtain the respective solutions $u_1 = (0, 2, 1, 0)$ and $u_2 = (1, 0, 0, 1)$. The set $\{u_1, u_2\}$ is a basis of T and dim T = 2.

(iii) $S \cap T$ consists of those vectors (x, y, z, t) which satisfy all conditions given in S and in T. i. e.

$$\left.\begin{aligned} y - 2z + t &= 0 \\ x - t &= 0 \\ y - 2z \quad &= 0 \end{aligned}\right\}$$

or, $$\left.\begin{aligned} x \quad - \quad t &= 0 \\ y - 2z \quad &= 0 \\ y - 2z + t &= 0 \end{aligned}\right\} \text{Subtract second equation from the third equation.}$$

Then we have $$\left.\begin{aligned} x - t &= 0 \\ y - 2z &= 0 \\ t &= 0 \end{aligned}\right\} \text{Which is in echelon form.}$$

The free variable is z. Set $z = 1$ to obtain the solution $u = (0, 2, 1, 0)$.

Thus $\{u\}$ is a basis of $S \cap T$ and dim $(S \cap T) = 1$.

Example 28. (i) Let U be the subspace of $\mathbb{R}^3$ spanned (generated) by the vectors $(1, 2, 1), (0, -1, 0)$ and $(2, 0, 2)$. Find a basis and the dimension of U.

(ii) Let W be the subspace of $\mathbb{R}^5$ spanned by the vectors (1, − 2, 0, 0, 3), (2, − 5, − 3, −2, 6), (0, 5, 15, 10, 0) and (2, 6, 18, 8,6). Find a basis and the dimension of W.

Solution : (i) Form the matrix whose rows are given vectors and reduce the matrix to row-echelon form by the elementary row operations.

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$ we multiply first row by 2 and then subtract from the third row.

$$\sim\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & -4 & 0 \end{bmatrix}$$ we multiply second row by 4 and subtract from the third row.

$$\sim\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ we multiply second row by 2 and then add with the first row.

$$\sim\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ we multiply second row by − 1

$$\sim\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix is in row–echelon form and the non-zero rows in the matrix are (1, 0, 1) and (0, 1, 0). These non-zero rows form a basis of the row space and consequently a basis of U; that is, Basis of U = {(1, 0, 1), (0, 1, 0)} and dim U = 2.

(ii) Form the matrix whose rows are the given vectors and reduce the matrix to row–echelon form by the elementary row operations :

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

Reduce this matrix to row echelon form by the elementary row operations.

we multiply first row by 2 and then subtract from the second and fourth rows respectively.

$$\sim\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & -1 & -3 & -2 & 0 \\ 0 & 5 & 15 & 10 & 0 \\ 0 & 10 & 18 & 8 & 0 \end{bmatrix}$$

We multiply second row by 5 and then add with the third row.

$$\sim\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & -1 & -3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 18 & 8 & 0 \end{bmatrix}$$

Interchange third and fourth rows

$$\sim\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & -1 & -3 & -2 & 0 \\ 0 & 10 & 18 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We multiply second row by 10 and then add with the third row.

$$\sim\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & -1 & -3 & -2 & 0 \\ 0 & 0 & -12 & -12 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We multiply second row by −1 and divide third row by −12,

$$\sim\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The above matrix is in row echelon form. The non-zero rows (vectors) in the above matrix are (1, −2, 0, 0, 3), (0, 1, 3, 2, 0) and (0, 0, 1, 1, 0). These non-zero rows form a basis for the row space and consequently a basis of W. Thus basis of W = {(1, −2, 0, 0, 3), (0, 1, 3, 2, 0), (0, 0, 1, 1, 0)} and dim W = 3.

Example 29. Let W be the subspace generated by the polynomials $p_1(t) = t^3 + 2t^2 - 2t + 1$,

$p_2(t) = t^3 + 3t^2 - t + 4$ and $p_3(t) = 2t^3 + t^2 - 7t - 7$.

Find a basis and the dimension of W.

Solution : Clearly, W is a subspace of the vector space V (F) of polynomials in t of degree ≤ 3. Thus the set $S_1 = \{1, t, t^2, t^3\}$ is a basis of V (F).

Now the coordinates of the given vectors $p_1(t)$, $p_2(t)$ and $p_3(t)$ relative to the basis S_1 are (1, 2, −2, 1), (1, 3, −1, 4) and (2, 1, −7, −7) respectively.

Forming the matrix whose rows are the above coordinate vectors, we get

$$\begin{bmatrix} 1 & 2 & -2 & 1 \\ 1 & 3 & -1 & 4 \\ 2 & 1 & -7 & -7 \end{bmatrix}$$

Reduce this matrix to row echelon form by the elementary row operations. We multiply 1st row by 1 and 2 and then subtract from 2nd & 3rd rows respectively.

$$\sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & -3 & -3 & -9 \end{bmatrix}$$

We multiply 2nd row by 3 and then add with the 3rd row.

$$\sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in row echelon form having two non-zero rows (coordinate vectors) (1, 2, −2, 1) and (0, 1, 1, 3) which will form a basis of the vector space generated by the coordinate vectors and so the set of corresponding polynomials is $\{t^3 + 2t^2 - 2t + 1, t^2 + t + 3\}$ which will form the basis of W. Thus dim W = 2.

Example 30. Let U and W be the subspaces of $\mathbb{R}^4$ generated by the set of vectors

{(1, 1, 0, −1), (1, 2, 3, 0), (2, 3, 3, −1)} and

{(1, 2, 2, −2), (2, 3, 2, −3), (1, 3, 4, −3)} respectively.

Find (i) dim (U + W) and (ii) dim (U∩W). **[D. U. H. 1998]**

Solution : (i) U + W is subspace spanned (or generated) by all given six vectors. Hence form the matrix whose rows are the given six vectors and then reduce this matrix to row echelon form by the elementary row operations :

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \\ 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix}$$

Reduce this matrix to row-echelon form by the elementary row operations.

We subtract 1st row from 2nd, 4th and 6th rows. Also we multiply 1st row by 2 and then subtract from 3rd and 5th rows.

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 4 & -2 \end{bmatrix}$$

We subtract 2nd row from 3rd, 4th & 5th rows. Also we multiply 2nd row by 2 and then subtract from 6th row.

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -2 & -4 \end{bmatrix}$$

We multiply 4th row by 1 and 2 and then subtract from 5th and 6th rows respectively.

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We interchange 3rd and 4th rows.

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in row-echelon form having three non-zero (1, 1, 0, –1), (0,1, 3, 1) and (0, 0, –1, –2) which will form a basis of U + W. Thus dim (U + W) = 3.

(ii) Let us first find the dim U and the dim W. Form the matrix whose rows are the generators of U and then reduce the matrix to row-echelon form by the elementary row operations.

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \end{bmatrix}$$

We multiply 1st row by 1 and 2 and then subtract from 2nd and 3rd rows respectively.

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix}$$

We subtract 2nd row from third row.

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in row-echelon form having two non-zero rows (1, 1, 0, –1) and (0, 1, 3, 1) which will form a basis of U.

Thus dim U = 2.

Again form the matrix whose rows are the generators of W and then reduce the matrix to row-echelon form by the elementary row operations.

$$\begin{bmatrix} 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix}$$

We multiply 1st row by 2 and 1 and then subtract from 2nd and 3rd rows respectively.

$$\sim \begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \\ 0 & 1 & 2 & -1 \end{bmatrix}$$

We add 2nd row with 3rd row.

$$\sim \begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in row-echelon form having two non-zero rows which will form a basis of W. Thus dim W = 2.

Now by theorem we have

$\dim (U + W) = \dim U + \dim W - \dim (U \cap W)$

or, $\dim (U \cap W) = \dim U + \dim W - \dim (U + W) = 2 + 2 - 3 = 1$

$\therefore \dim (U \cap W) = 1$ (one).

Example 31. Let V be the vector space of 2×2 matrices over the real field $\mathbb{R}$. Find a basis and the dimension of the subspace W of V spanned by

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & 5 \\ 1 & -1 \end{bmatrix}, C = \begin{bmatrix} 5 & 12 \\ 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}.$$

Solution : The coordinate vectors of the given matrices relative to the usual basis of V are as follows :

$[A] = (1, 2, -1, 3)$, $[B] = (2, 5, 1, -1)$, $[C] = (5, 12, 1, 1)$ and $[D] = (3, 4, -2, 5)$.

Form a matrix whose rows are the coordinate vectors and then reduce this matrix to row-echelon form by the elementary row operations and join successive matrices by the equivalence sign ~ :

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 5 & 1 & -1 \\ 5 & 12 & 1 & 1 \\ 3 & 4 & -2 & 5 \end{bmatrix}$$

We multiply 1st row by 2, 5 & 3 and then subtract from 2nd, 3rd and 4th rows respectively.

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & -7 \\ 0 & 2 & 6 & -14 \\ 0 & -2 & 1 & -4 \end{bmatrix}$$

We multiply 2nd row by 2 and -2 and then subtract from 3rd and 4th rows respectively.

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & -18 \end{bmatrix}$$

Interchange 3rd and 4th rows.

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & -7 \\ 0 & 0 & 7 & -18 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in row-echelon form having three non-zero rows $(1, 2, -1, 3)$, $(0, 1, 3, -7)$ and $(0, 0, 7, -18)$ which are linearly independent.

Hence the corresponding matrices $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 3 & -7 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 7 & -18 \end{bmatrix}$ form a basis of W and dim W = 3.

Example 32. Let V be the vector space of 2×2 matrices over the real field $\mathbb{R}$. Find a basis and the dimension of the subspace W of V spanned by the matrices

$$A = \begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}, C = \begin{bmatrix} 2 & -4 \\ -5 & 7 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & -7 \\ -5 & 1 \end{bmatrix}.$$

Solution : The coordinate vectors of the given matrices relative to the usual basis of V are as follows :

$[A] = (1, -5, -4, 2)$, $[B] = (1, 1, -1, 5)$, $[C] = (2, -4, -5, 7)$ and $[D] = (1, -7, -5, 1)$

Form a matrix whose rows are the coordinate vectors and then reduce this matrix to row-echelon form by the elementary row operations and join successive matrices by the equivalence sign ~ :

$$\begin{bmatrix} 1 & -5 & -4 & 2 \\ 1 & 1 & -1 & 5 \\ 2 & -4 & -5 & 7 \\ 1 & -7 & -5 & 1 \end{bmatrix}$$

We multiply 1st row by 1, 2, and 1 and then subtract from 2nd, 3rd and 4th rows respectively.

$$\sim \begin{bmatrix} 1 & -5 & -4 & 2 \\ 0 & 6 & 3 & 3 \\ 0 & 6 & 3 & 3 \\ 0 & -2 & -1 & -1 \end{bmatrix}$$

We multiply 2nd row by 1 and $-\frac{1}{3}$ and then subtract from 3rd & 4th rows respectively.

$$\sim \begin{bmatrix} 1 & -5 & -4 & 2 \\ 0 & 6 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in row-echelon form having two non-zero rows (1, –5, – 4, 2) and (0, 6, 3, 3) which are linearly independent. Hence the corresponding matrices

$\begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}$ and $\begin{bmatrix} 0 & 6 \\ 3 & 3 \end{bmatrix}$ form a basis of W and dim W = 2.

Example 33. Prove that the vectors $u_1 = (1, 0, 2)$, $u_2 = (-1, 1, 0)$ and $u_3 = (0, 2, 3)$ form a basis of $\mathbb{R}^3$ and find the co-ordinates of the vectors $v = (1, -1, 1)$ and $w = (-1, 8, 11)$ relative to this basis.

Proof : First Portion

The given vectors will be a basis of $\mathbb{R}^3$ if and only if they are linearly independent and every vector in $\mathbb{R}^3$ can be written as a linear combination of u_1, u_2 and u_3. First we shall prove that the vectors u_1, u_2 and u_3 are linearly independent. For arbitrary scalars x_1, x_2 and x_3

let $x_1u_1 + x_2u_2 + x_3u_3 = 0$

or, $x_1 (1, 0, 2) + x_2 (-1, 1, 0) + x_3 (0, 2, 3) = (0, 0, 0)$

or, $(x_1 - x_2, x_2 + 2x_3, 2x_1 + 3x_3) = (0, 0, 0)$

Equating corresponding components and forming the linear system, we get

$$\left.\begin{aligned} x_1 - x_2 &= 0 \\ x_2 + 2x_3 &= 0 \\ 2x_1 + 3x_3 &= 0 \end{aligned}\right\} \quad (1)$$

Reduce the system to echelon form by elementary transformations. We multiply first equation by 2 and then subtract from the third equation. Thus the above system reduces to

$$\left.\begin{aligned} x_1 - x_2 \qquad &= 0 \\ x_2 + 2x_3 &= 0 \\ 2x_2 + 3x_3 &= 0 \end{aligned}\right\} \quad (2)$$

Again we multiply second equation by 2 and then subtract from the third equation. Then we get the equivalent system.

$$\left.\begin{aligned} x_1 - x_2 \qquad &= 0 \\ x_2 + 2x_3 &= 0 \\ -\ x_3 &= 0 \end{aligned}\right\} \quad (3)$$

This system is in echelon form and has exactly three equations in three unknowns, hence the system has only the zero solution i. e. $x = 0$, $y = 0$, $z = 0$. Accordingly, the vectors are linearly independent.

To show that u_1, u_2 and u_3 span $\mathbb{R}^3$, we must show that an arbitrary vector $v = (a, b, c)$ can be expressed as a linear combination $v = x_1u_1 + x_2u_2 + x_3u_3$

or, $v = (a, b, c) = x_1 (1, 0, 2) + x_2 (-1, 1, 0) + x_3 (0, 2, 3)$

Forming the linear system, we get

$$\left.\begin{aligned} x_1 - x_2 \qquad &= a \\ x_2 + 2x_3 &= b \\ 2x_1 + \quad 3x_3 &= c \end{aligned}\right\} \quad (4)$$

Reduce this system to echelon form by the elementary transformations.

We multiply first equation by 2 and then subtract from the third equation, Thus we have the equivalent system

$$\left.\begin{aligned} x_1 - x_2 \quad &= a \\ x_2 + 2x_3 &= b \\ 2x_2 + 3x_3 &= c-2a \end{aligned}\right\} \qquad (5)$$

We multiply second equation by 2 and then subtract from the third equation. Then we get the equivalent system

$$\left.\begin{aligned} x_1 - x_2 \quad &= a \\ x_2 + 2x_3 &= b \\ -\ x_3 &= c - 2a - 2b \end{aligned}\right\} \qquad (6)$$

From the third equation, we have $x_3 = 2a + 2b - c$.

Substituting the value of x_3 in the second equation, we get $x_2 = -4a - 3b + 2c$.

Again, substituting the value of x_2 in the first equation, we get $x_1 = -3a - 3b + 2c$. Therefore,

$v = (-3a - 3b + 2c)\ u_1 + (-4a - 3b + 2c)\ u_2 + (2a + 2b - c)\ u_3$

or, $(a, b, c) = (-3a - 3b + 2c)\ (1, 0, 2) + (-4a - 3b + 2c)\ (-1, 1, 0)$
$+ (2a + 2b - c)\ (0, 2, 3)$

Thus every vector in $\mathbb{R}^3$ can be expressed as a linear combination of the vectors $(1, 0, 2)$, $(-1, 1, 0)$ and $(0, 2, 3)$.

Hence the vectors u_1, u_2 and u_3 form a basis of $\mathbb{R}^3$.

Second Portion : Let $v = (1, -1, 1) = x_1u_1 + x_2u_2 + x_3u_3$

or, $(1, -1, 1) = x_1\ (1, 0, 2) + x_2\ (-1, 1, 0) + x_3\ (0, 2, 3)$.

Forming linear system, we have

$$\left.\begin{aligned} x_1 - x_2 \quad &= 1 \\ x_2 + 2x_3 &= -1 \\ 2x_1 \quad + 3x_3 &= 1 \end{aligned}\right\} \qquad (7)$$

Solving the system, we get $x_1 = 2$, $x_2 = 1$, $x_3 = -1$.

Thus $v = (1, -1, 1) = 2u_1 + 1u_2 + (-1)\ u_3$.

So the vector v has co-ordinates $(2, 1, -1)$.

Similarly, let $w = (-1, 8, 11) = y_1u_1 + y_2u_2 + y_3u_3$

or, $(-1, 8, 11) = y_1\ (1, 0, 2) + y_2\ (-1, 1, 0) + y_3\ (0, 2, 3)$.

Forming linear system, we have

$$\left.\begin{aligned} y_1 - y_2 \quad &= -1 \\ y_2 + 2y_3 &= 8 \\ 2y_1 \quad + 3y_3 &= 11 \end{aligned}\right\} \qquad (8)$$

Solving the system, we get $y_1 = 1$, $y_2 = 2$, $y_3 = 3$

Thus $w = (-1, 8, 11) = 1u_1 + 2u_2 + 3u_3$

So the vector w has co-ordinates $(1, 2, 3)$.

Example 34. Given the vectors $(2, 1, 1)$, $(1, 3, 2)$, $(1, 3, -1)$ and $(1, -2, 3)$. Test whether they are linearly independent by **Sweep out method.**

Solution : Let $\alpha_1\ (2, 1, 1) + \alpha_2\ (1, 3, 2) + \alpha_3\ (1, 3, -1)$
$+ \alpha_4\ (1, -2, 3) = O = (0, 0, 0)$,

where α_1, α_2, α_3 and α_4 are scalars. then

$(2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4,\ \alpha_1 + 3\alpha_2 + 3\alpha_3 - 2\alpha_4,\ \alpha_1 + 2\alpha_2 - \alpha_3 + 3x_4)$ $= (0, 0, 0)$. Equating corresponding components from both sides and forming linear system, we get

$$\left.\begin{aligned} 2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &= 0 \\ \alpha_1 + 3\alpha_2 + 3\alpha_3 - 2\alpha_4 &= 0 \\ \alpha_1 + 2\alpha_2 - \alpha_3 + 3\alpha_4 &= 0 \end{aligned}\right\}$$

α_1	α_2	α_3	α_4	
2	1	1	1	$O \rightarrow R_1$
1	3	3	-2	$O \rightarrow R_2$
1	2	-1	3	$O \rightarrow R_3$
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$O \rightarrow R_4 = \frac{R_1}{2}$ 1st pivotal row
O	$\frac{5}{2}$	$\frac{5}{2}$	$-\frac{5}{2}$	$O \rightarrow R_5 = R_2 - R_4$
O	$\frac{3}{2}$	$-\frac{3}{2}$	$\frac{5}{2}$	$O \rightarrow R_6 = R_3 - R_4$
	1	1	-1	$O \rightarrow R_7 = \frac{R_5}{\frac{5}{2}}$ 2nd pivotal row
	O	-3	4	$O \rightarrow R_8 = R_6 - \frac{3}{2} R_7$
		1	$-\frac{4}{3}$	$O \rightarrow R_9 = \frac{R_8}{-3}$ 3rd pivotal row

From the pivotal rows, we have

$$\alpha_1 + \tfrac{1}{2}\alpha_2 + \tfrac{1}{2}\alpha_3 + \tfrac{1}{2}\alpha_4 = 0 \qquad \text{(i)}$$
$$\alpha_2 + \alpha_3 - \alpha_4 = 0 \qquad \text{(ii)}$$
$$\alpha_3 - \tfrac{4}{3}\alpha_4 = 0 \qquad \text{(iii)}$$

This system is in echelon form and has three equations in 4 unknowns and hence $4 - 3 = 1$ free variable which is α_4.

Thus the system has an infinite number of non-zero solutions.

Let $\alpha_4 = t$, where t is a scalar.

Then $\alpha_3 = \frac{4}{3}t$, $\alpha_2 = -\frac{1}{3}t$, and $\alpha_1 = -t$.

Since all α's are not zero, so the given vectors are linearly dependent.

Example 35. Are these vectors (2, 1, 1), (2, 4, 7), (4, −9, 11) dependent ?. Test by **Sweep out method.**

Solution : Let $\alpha_1\,(2, 1, 1) + \alpha_2\,(2, 4, 7) + \alpha_3\,(4, -9, 11) = (0, 0, 0)$

Or, $(2\alpha_1 + 2\alpha_2 + 4\alpha_3,\ \alpha_1 + 4\alpha_2 - 9\alpha_3,\ \alpha_1 + 7\alpha_2 + 11\alpha_3) = (0, 0, 0)$

Equating corresponding components from both sides and forming the linear system, we get

$$\left.\begin{aligned} 2\alpha_1 + 2\alpha_2 + 4\alpha_3 &= 0 \\ \alpha_1 + 4\alpha_2 - 9\alpha_3 &= 0 \\ \alpha_1 + 7\alpha_2 + 11\alpha_3 &= 0 \end{aligned}\right\}$$

α_1	α_2	α_3	
2	2	4	$O \rightarrow R_1$
1	4	-9	$O \rightarrow R_2$
1	7	11	$O \rightarrow R_3$
1	1	2	$O \rightarrow R_4 = \frac{R_1}{2}$ 1st pivotal row
O	3	-11	$O \rightarrow R_5 = R_2 - R_4$
O	6	9	$O \rightarrow R_6 = R_3 - R_4$
	1	$-\frac{11}{3}$	$O \rightarrow R_7 = \frac{R_5}{3}$ 2nd pivotal row
	O	31	$O \rightarrow R_8 = R_6 - 6R_7$
		1	$O \rightarrow R_9 = \frac{R_8}{31}$ 3rd pivotal row

From the pivotal rows, we have

$$\alpha_1 + \alpha_2 + 2\alpha_3 = 0 \qquad \text{(i)}$$
$$\alpha_2 - \tfrac{11}{3}\alpha_3 = 0 \qquad \text{(ii)}$$
$$\alpha_3 = 0 \qquad \text{(iii)}$$

Therefore, the system has zero solution i. e $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Hence the given vectors are linearly independent.

Example 36. Find the rank and the basis of a given set of vectors {(2, −1, 5, 4), (0, 1, 2, 3), (4, 0, 6, 1), (0, − 2, 4, 7) by using **Sweep out method.**

Solution : Rank of a given set of vectors is the number of linearly independent vectors of that set and these linear independent vectors form a basis of that set.

2	−1	5	4	$\to R_1$
0	1	2	3	$\to R_2$
4	0	6	1	$\to R_3$
0	−2	4	7	$\to R_4$
1	$-\frac{1}{2}$	$\frac{5}{2}$	2	$\to R_5 = \frac{R_1}{2}$ 1st pivotal row
0	1	2	3	$\to R_6 = R_2 - 0R_5$
0	2	−4	−7	$\to R_7 = R_3 - 4R_5$
0	−2	4	7	$\to R_8 = R_4 - 0R_5$
	1	2	3	$\to R_9\ \frac{R_6}{1}$ 2nd pivotal row
	0	−8	−13	$\to R_{10} = R_7 - 2R_9$
	0	8	13	$\to R_{11} = R_8 + 2R_9$
		1	$\frac{13}{8}$	$\to R_{12} = \frac{R_{10}}{-8}$ 3rd pivotal row
		0	0	$\to R_{13} = R_{11} - 8R_{12}$

Now the rank of the given set of vectors is equal to the number of pivotal rows = 3. A basis of the given set of vectors is $\{(1, -\frac{1}{2}, \frac{5}{2}, 2), (0, 1, 2, 3), (0, 0, 1, \frac{13}{8})\}$

Since $\alpha_1 (1, -\frac{1}{2}, \frac{5}{2}, 2) + \alpha_2(0,1,2,3) + \alpha_3 (0, 0, 1, \frac{13}{8}) = (0, 0, 0, 0)$

implies $\alpha_1 = \alpha_2 = \alpha_3 = 0$. (Zero solution).

EXERCISES – 6 (C)

1. Prove that the vectors (1, 1) and (1, 0) form a basis of $\mathbb{R}^2$.

2. (i) Prove that $\{(2, -\frac{1}{2}, 1), (3, 2, 1), (0, 1, 1)\}$ is a basis of $\mathbb{R}^3$.

(ii) Prove that $\{(1, 1, 1, 1), (0, 1, 1, 1), (0, 0, 1, 1) (0, 0, 0, 1)\}$ is a basis of $\mathbb{R}^4$.

3. (i) Extend $\{(2, 0, 0, -1), (1, 3, -1, 0)\}$ to a basis for $\mathbb{R}^4$.

[D. U. S. 1984]

(ii) Extend $\{(1, 2, 0, 3), (2, -1, 0, 0)\}$ to a basis of $\mathbb{R}^4$.

Answers : (i) $\{(1, 3, -1, 0), (2, 0, 0, -1), (0, 0, 1, 0), (0, 0, 0, 1)\}$

(ii) $\{(1, 2, 0, 3), (2, -1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$

4. Decide whether $S = \{(2, 1, 1), (1, 0, 0), (5, 1, 1)\}$ is a linearly dependent subset of $\mathbb{R}^3$. What is the dimension of the subspace spanned by S? **[D. U. S. 1982]**

Answer : S is linearly dependent and dimension of the subspace is 2.

5. The subspace U of $\mathbb{R}^4$ is spanned by the vectors (1, 0, 2, 3) and (0, 1, −1, 2) and the subspace V of $\mathbb{R}^4$ is spanned by (1, 2, 3, 4), (−1, −1, 5, 0) and (0, 0, 0, 1).

Find the dimension of U, V, $U \cap V$ and U + V.

Answers : dim U = 2, dim V = 3.

dim $(U \cap V)$ = 1, dim (U + V) = 4.

6. Find a basis for the subspace of $\mathbb{R}^4$ spanned by the given vectors :

(i) (1, 1, −4, −3), (2, 0, 2, −2), (2, −1, 3, 2)

(ii) (1, 1, 0, 0), (0, 0, 1, 1), (−2, 0, 2, 2) (0, −3, 0, 3).

Answers : (i) $\{(1, 1, -4, -3), (0, 1, -5, -2), (0, 0, 1, -\frac{1}{2})\}$

(ii) $\{(1, 1, 0, 0), (0, 1, 1, 1), (0, 0, 1, 1), (0, 0, 0, 1)\}$.

7. Find the dimension of the subspace generated by the set $\{(1, 2, 1), (3, 1, 2) (1, -3, 4)\}$ of $V_3(\mathbb{R})$. **[D. U. P. 1979]**

Answer : The dimension of the subspace is 3.

8. Let W be the subspace generated by the polynomials $v_1 = t^3 - 2t^2 + 4t + 1$, $v_2 = 2t^3 - 3t^2 + 9t - 1$,

$v_3 = t^3 + 6t - 5$, $v_4 = 2t^3 - 5t^2 + 7t + 5$.

Find the basis and dimension of W.

[C.U. P. 1973, '86 ; J. U. H. 1986]

Answer : Basis : $\{t^3 - 2t^2 + 4t + 1, t^2 + t - 3\}$ and dim W = 2.

9. Find a basis and the dimension of the subspace W of P(t) spanned by the polynomials

(i) $P_1(t) = t^3 + 2t^2 - 3t + 2$, $p_2(t) = t^3 + 2t^2 - 2t + 3$, and
$P_3(t) = 2t^3 + 3t^2 - 5t - 5$.

(ii) $P_1(t) = t^3 + t^2 - 3t + 2$, $P_2(t) = 2t^3 + t^2 + t - 4$, and
$P_3(t) = 4t^3 + 3t^2 - 5t + 2$.

Answers : (i) Basis of W = $\{t^3 + 2t^2 - 3t + 2, t^2 - t + 9, t + 1\}$.
dim W = 3

(ii) Basis of W = $\{t^3 + t^2 - 3t + 2,\ t^2 - 7t + 8, 2\}$
dim W = 3

10. Determine whether the given set of vectors is a basis for $\mathbb{R}^3$ over $\mathbb{R}$: (i) $\{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$
(ii) $\{(-1, 1, 2), (2, -3, 1), (10, -14, 0)\}$

Answers : (i) Set of vectors is a basis for $\mathbb{R}^3$.
(ii) Set of vectors is not a basis for $\mathbb{R}^3$.

11. Let $\{v_1, v_2, v_3\}$ be basis for a vector space V. Show that $\{u_1, u_2, u_2\}$ is also a basis, where $u_1 = v_1$, $u_2 = v_1 + v_2$ and $u_3 = v_1 + v_2 + v_3$.

12. Let W be the subspace of $\mathbb{R}^4$ generated by the vectors (1, –2, 5, –3), (2, 3, 1, –4) and (3, 8, –3, –5). Find a basis and the dimension of W. **[D. U. S. 1983]**

Answer : Basis $\{(1, -2, 5, -3), (0, 7, -9, 2)\}$, dim W = 2.

13. Consider the following subspaces of $\mathbb{R}^5$:
U = span $\{(1, 3, -2, 2, 3), (1, 4, -3, 4, 2), (2, 3, -1, -2, 9)\}$
W = span $\{(1, 3, 0, 2, 1), (1, 5, -6, 6, 3), (2, 5, 3, 2, 1)\}$
Find a basis and the dimension of **[D. U. H. 1987]**
(i) U + W (ii) $U \cap W$.

Answers : (i) Basis of U + W
$= \{(1, 3, -2, 2, 3), (0, 1, -1, 2, -1), (0, 0, 2, 0, -2)\}$
dim (U + W) = 3.

(ii) Basis of $U \cap W = \{(1, 4, -3, 4, 2)\}$
dim $(U \cap W) = 1$.

14. Let S and T be the following subspaces of $\mathbb{R}^4$:
$S = \{(x, y, z, t) \mid y + z + t = o\}$
$T = \{(x, y, z, t) \mid x + y = 0, z = 2t\}$
Find the basis and the dimension of
(i) S (ii) T (iii) $S \cap T$.

Answers : (i) Basis :
$\{(1, 0, 0, 0), (0, -1, 1, 0), (0, -1, 0, 1)\}$, dim S = 3.
(ii) Basis : $\{(-1, 1, 0, 0), (0, 0, 2, 1)\}$, dim T = 2.
(iii) Basis : $\{(3, -3, 2, 1)\}$, dim $(S \cap T) = 1$.

15. Let $W = \{(a, b, c) \mid a, b, c \in R \text{ and } 2a + b + 2c = 0\}$
Find a basis and dimension of W. **[D. U. Prel. 1983]**

Answer : $\{(1, 1, -1), (-1, 2, 0)\}$ is a basis of W and dim W = 2.

16. Prove that in $\mathbb{R}^3$ the vectors (1, –1, 0), (0, 1, –1) form a basis for the subspace $U = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$.

17. Find a basis of each of the following subspaces of $\mathbb{R}^3$:
(i) $U = \{(x, y, z) \mid x - z = 0\}$
(ii) $V = \{(x, y, z) \mid x = 0, \text{ and } y + z = 0\}$

Answers : (i) $\{(2, 1, 2), (1, 0, 1)\}$ is a basis of U.
(ii) $\{(0, -1, 1), (0, 1, -1)\}$ is a basis of V.

18. Let V be the vector space of all 2 ×2 matrices over the real field F. Prove that V has dimension 4 by exhibiting a basis for V which has four elements.

19. Let V be the vector space of 2 × 2 matrices over the real field $\mathbb{R}$. Find a basis and the dimension of the subspace W of V spanned by the matrices $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$

Answers : Basis of W = $\left\{\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\}$
dim W = 2.

20. Let V be the vector space of 2×2 matrices over the real field $\mathbb{R}$. Find a basis and the dimension of the subspace W of V spanned by the matrices

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & 5 \\ 1 & -1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}.$$

Answers : Basis of W = $\left\{ \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 3 & -7 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 7 & -18 \end{bmatrix} \right\}$

dim W = 3.

21. Let V be the vector space of 2×2 matrices over the real field $\mathbb{R}$. Determine whether

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

form a basis for V.

Answer : They form a basis for V.

mid

22. Find the dimension and a basis of the solution space W of the following system

$$\left.\begin{aligned} x + 2y + 2z - s + 3t &= 0 \\ x + 2y + 3z + s + t &= 0 \\ 3x + 6y + 8z + s + 5t &= 0 \end{aligned}\right\}$$

[D. U. S. 1980]

Answer : dim W = 3.

Basis : {(–2, 1, 0, 0, 0), (5, 0, –2, 1, 0), (–7, 0, 2, 0, 1)}

mid

23. Determine a basis and the dimension for the solution space of the following homogeneous system

$$\left.\begin{aligned} 2x_1 + 2x_2 - x_3 \quad\quad + x_5 &= 0 \\ -x_1 - x_2 + 2x_3 - 3x_4 + x_5 &= 0 \\ x_1 + x_2 - 2x_3 \quad\quad - x_5 &= 0 \\ x_3 + x_4 + x_5 &= 0 \end{aligned}\right\}$$

[D . U. P. 1984]

Answer : Basis : {(–1,1,0,0,0), (–1, 0, –1, 0, 1)}

dimension = 2.

24. Find a basis and the dimension of the solution space for the following homogeneous linear equations :

$$\left.\begin{aligned} x_1 + 2x_2 - x_3 + 4x_4 &= 0 \\ 2x_1 - x_2 + 3x_3 + 3x_4 &= 0 \\ 4x_1 + x_2 + 3x_3 + 9x_4 &= 0 \\ x_2 - x_3 + x_4 &= 0 \\ 2x_1 + 3x_2 - x_3 + 7x_4 &= 0 \end{aligned}\right\}$$

[D. U. P. 1984]

Answer : Basis : {(–1, 1, 1, 0), (–2, –1, 0, 1)}

dimension = 2.

mid

25. Find a basis and the dimension of the solution space for the following homogeneous linear system :

$$\left.\begin{aligned} x + 2y - z + 3s - 4t &= 0 \\ 2x + 4y - 2z - s + 5t &= 0 \\ 2x + 4y - 2z + 4s - 2t &= 0 \end{aligned}\right\}$$

[R. U. H. 1985]

Basis : {(2, –1, 0, 0, 0), (1, 0, 1, 0, 0)}

dimension = 2.

mid

26. Find a basis and the dimension of the following system of linear equations :

$$\left.\begin{aligned} x_1 + 2x_2 - 2x_3 + 2x_4 - x_5 &= 0 \\ x_1 + 2x_2 - x_3 + 3x_4 - 2x_5 &= 0 \\ 2x_1 + 4x_2 - 7x_3 + x_4 + x_5 &= 0 \end{aligned}\right\}$$

[D. U. H. 1988]
[R. U. H. 1988]

Answers : Basis : {(–2, 1, 0, 0, 0), (–4, 0, –1, 1, 0), (3, 0, 1, 0, 1)}

dimension = 3.

27. Prove that the vectors $u_1 = (1, 2, 1 -2)$, $u_2 = (0, -2, -2, 0)$, $u_3 = (0, 2, 3, 1)$ and $u_4 = (3, 0, -3, 6)$ form a basis of $\mathbb{R}^4$ and find the coordinates of the vectors v = (5, 0, – 8, –1) and w = (–9, 20, 34, – 25) relative to this basis.

Answer : v and w have coordinates [2, – 1, – 3, 1] and [3, –2, 5, –4] respectively.

28. Using **Sweep out method** prove that the following set of vectors is linearly independent :

$\{(1, 3, 2), (1, -7, -8), (-3, -1, -4)\}$.

29. Find the rank and the basis of a given set of vectors $\{(3, 1, 5), (2, 2, 1), (4, 1, 9)\}$ by using **Sweep out method.**

Answer : Rank is 3

Basis = $\{(1, \frac{1}{3}, \frac{5}{3}), (0, 1, -\frac{7}{4}), (0, 0, 1)\}$.

30. Show that the set $S = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$ is a basis of $\mathbb{R}^3$ and hence find the coordinates of the vector (a, b, c) with respect to the above basis.

Answer : $(a - b, b - c, c)$.

31. Show that if $\{u, v, w\}$ is a basis of $\mathbb{R}^3$, then $\{u + v, v + w, w + u\}$ is a basis of $\mathbb{R}^3$.

32. If W_1 is a subspace of $\mathbb{R}^4$ generated by a set of vectors $S_1 = \{(1, 1, 0, -1), (1, 2, 3, 0), (2, 3, 3, -1)\}$ and W_2 is a subspace of $\mathbb{R}^4$ generated by the set of vectors

$S_2 = \{(1, 2, 2, -2), (2, 3, 2, -3), (1, 3, 4, -3)\}$.

find (i) $\dim (W_1 + W_2)$ (ii) $\dim (W_1 \cap W_2)$.

Answers : (i) $\dim (W_1 + W_2) = 3$. (ii) $\dim (W_1 \cap W_2) = 1$.

33. Prove that the vectors $v_1 = (1, 2, 0)$, $v_2 = (0, 5, 7)$, and $v_3 = (-1, 1, 3)$ form a basis of $\mathbb{R}^3$ and find the coordinates of the vector $v = (2, 3, 1)$. **[J. U. H. 1990, 91]**

Answer : $[v] = [0, 1, -2]$.

34. Let S be the following basis of the vector space W of 2×2 real symmetric matrices :

$$\left\{\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}\right\}$$

Find the coordinate vector of the matrix $A \in W$ relative to the above basis where (a) $A = \begin{bmatrix} 1 & -5 \\ -5 & 5 \end{bmatrix}$ and (b) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

Answer : (a) $[2, -1, 1]$ (b) $[3, 1, -2]$.