7·0. EQUATION OF CONTINUITY

According of the principle of conservation of charge the net amount of charge in an isolated system remains constant. For generality let us assume that charge density is a function of time. Then the principle of conservation of charge can be stated as follows :

If the net charge crossing a surface bounding a closed volume is not zero, then the charge density within the volume must change with time in a manner that the time rate of decrease of charge within the volume equals the net rate of flow of charge out of the volume. This statement of conservation of charge can be expressed by the equation of continuity which we shall derive here.

As we are now dealing with charges in motion, let us consider that charge density, ρ, is a function of time. The transport of charge constitutes the current *i.e.*,

$$I = \frac{dq}{dt} = \frac{d}{dt}\int_V \rho \, dV, \qquad \ldots (1)$$

where we have considered that the current is extended in space of volume V closed by a surface S. The net amount of charge which crosses a unit area (normal to the direction of charge flow) of a surface in unit time is defined as the current density **J**. According to all experiments to date charge in a closed system is always *conserved*. Therefore if a net amount of current is flowing outward a closed surface, the charge contained within that volume should decrease at the rate,

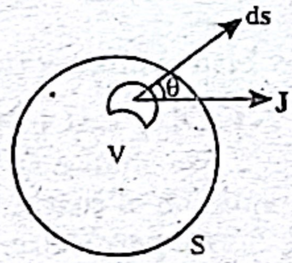

Fig. 1.

$$-\frac{dq}{dt} = I \qquad \ldots (2)$$

where I is the total current flowing through surface **S**. If **J** is the current density, then by definition, total current I will be

$$I = \oint_S \mathbf{J}.d\mathbf{S} \qquad \ldots (3)$$

From eqs. (2) and (3), we get

$$\oint_S \mathbf{J}.d\mathbf{S} = -\frac{dq}{dt}$$

$$= -\frac{d}{dt}\int_V \rho \, dV, \qquad \ldots (4)$$

using eq. (1).

Because it is ρ which is changing with time, we can write

$$\frac{d}{dt}\int_V \rho \, dV = \int_V \frac{\partial \rho}{\partial t} dV,$$

so that eq. (4), becomes

$$\oint_S \mathbf{J}.d\mathbf{S} = -\int_V \frac{\partial \rho}{\partial t} dV. \qquad \ldots (5)$$

From divergence theorem, we have

$$\oint_S \mathbf{J}.d\mathbf{S} = \int_V (\text{div } \mathbf{J}) \, dV,$$

so that eq. (5), becomes

$$\int_V (\text{div } \mathbf{J}) \, dV = -\int_V \frac{\partial \rho}{\partial t} dV$$

or
$$\int_V \left(\text{div } \mathbf{J} + \frac{\partial \rho}{\partial t} \right) dV = 0 \quad \ldots(6)$$

Since eq. (6) holds for any arbitrary volume, we can put integrand equal to zero, *i.e.*,

$$\text{div } \mathbf{J} + \frac{\partial \rho}{\partial t} = 0, \quad \ldots(7)$$

and is referred to as the *equation of continuity.* It is the mathematical expression for the conservation of charge. It states that the *'total current flowing out of some volume must be equal to the rate of decrease of charge within the volume, assuming that charge can not be created or destroyed, i.e. no sources and sinks are present in that volume'*. In case of *stationary currents*, charge density at any point within the region remains constant.

$$\frac{\partial \rho}{\partial t} = 0,$$

so that
$$\text{div } \mathbf{J} = 0$$

or
$$\vec{\nabla} \cdot \mathbf{J} = 0, \quad \ldots(8)$$

which expresses the fact that there is no net outward flux of current density $\mathbf{J}$.

7·1. DISPLACEMENT CURRENT

We shall now see how Maxwell changed the definition of total current density to adapt the equation of continuity to time dependent fields.

Ampere's circuital law is

$$\oint_S \mathbf{B}.d\mathbf{l} = \mu_0 I$$

or
$$\oint_S \mathbf{H}.d\mathbf{l} = I = \int_S \mathbf{J}.d\mathbf{S}$$

Changing line integral into surface integral by Stoke's theorem,

$$\int_S \text{Curl } \mathbf{H}.d\mathbf{S} = \int_S \mathbf{J}.d\mathbf{S}$$

or
$$\text{Curl } \mathbf{H} = \mathbf{J} \quad \ldots(1)$$

Let us put it in equation of continuity which is

$$\text{div } \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

we get
$$\text{div(curl } \mathbf{H}) = -\frac{\partial \rho}{\partial t}$$

$$0 = -\frac{\partial \rho}{\partial t}.$$

That is, eq. (1) leads to steady state conditions in which charge density is not changing. Therefore, for time dependent (changing) fields eq. (1) should be modified. Maxwell suggested that the definition of total current density is incomplete and advised to add something to $\mathbf{J}$. Let it be $\mathbf{J}'$. Then eq. (1) becomes

$$\text{Curl } \mathbf{H} = (\mathbf{J} + \mathbf{J}') \quad \ldots(2)$$

In order to identify $\mathbf{J}'$, we take divergence of eq. (2). That is,

$$\text{div(curl } \mathbf{H}) = \text{div } (\mathbf{J} + \mathbf{J}')$$

$$0 = \text{div } \mathbf{J} + \text{div } \mathbf{J}'$$

or
$$\text{div } \mathbf{J}' = -\text{div } \mathbf{J} = \frac{\partial \rho}{\partial t} \quad \ldots(3)$$

We know that

$$\rho = \vec{\nabla}.\mathbf{D}$$

so that eq. (3) becomes

$$\text{div } \mathbf{J}' = \frac{\partial}{\partial t}(\vec{\nabla}.\mathbf{D})$$

$$= \vec{\nabla} . \frac{\partial \mathbf{D}}{\partial t}$$

$$= \text{div}\left(\frac{\partial \mathbf{D}}{\partial t}\right)$$

or

$$\text{div}\left(\mathbf{J}' - \frac{\partial \mathbf{D}}{\partial t}\right) = 0 \quad \text{... (4)}$$

Since eq. (4) is true for any arbitrary volume, we can put

$$\mathbf{J}' = \frac{\partial \mathbf{D}}{\partial t} \quad \text{... (5)}$$

Therefore the modified form of the ampere's law is

$$\text{curl } \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \text{... (6)}$$

We now note that :

(i) Since $\mathbf{J}'$ arises due to the variation of electric displacement $\mathbf{D}$ with time, it is termed as *displacement current density*. According to Maxwell it is just as effective as $\mathbf{J}$, the conduction current density, in producing magnetic field.

(ii) The important inference that we draw from eq. (6) is that, since displacement current, $\mathbf{J}'$, is related to the electric field vector $\mathbf{D}$ (as $\mathbf{D} = \varepsilon\mathbf{E}$), it is not possible in case of time varying fields to deal separately with electric and magnetic fields but, instead, the two fields are interlinked giving rise to electromagnetic fields. Thus $\mathbf{J}'$ results into unification of electric and magnetic phenomenon.

(iii) In a good conductor, $\mathbf{J}'$ is negligible compared to $\mathbf{J}$ at a frequency lower than light frequencies (10^{15} Hz) as is evident from the following example.

Example 1. *Show that for a conductor subject to electric field*

$$E = E_0 \cos \omega t$$

Displacement current density is negligible compared to conduction current density at frequencies less than 10^{15} c/s.

We know that

$$\mathbf{D} = \varepsilon\mathbf{E} = \varepsilon E_0 \cos \omega t$$

so that

$$\mathbf{J}' = \frac{\partial \mathbf{D}}{\partial t} = -\,\varepsilon\omega\, E_0 \sin \omega t$$

$$= \varepsilon\omega\, E_0 \cos\left(\omega t + \frac{\pi}{2}\right)$$

According to Ohm's law, conduction current density is

$$\mathbf{J} = \sigma\mathbf{E} = \sigma\, E_0 \cos \omega t$$

so that

$$\frac{\mathbf{J}'}{\mathbf{J}} = \frac{\varepsilon\omega}{\sigma} \cdot \frac{\cos(\omega t + \pi/2)}{\cos \omega t}$$

or

$$\left|\frac{\mathbf{J}'}{\mathbf{J}}\right| = \frac{\varepsilon\omega}{\sigma} = \frac{2\pi\varepsilon f}{\sigma} \approx 10^{-17} f.$$