TOPIC NAME : DAY : TIME : DATE : 05/08/23

Math- 2101:

"Part-A"

Vector:

$$\vec{A} = \cancel{a_1 j + a_2}\ a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$|\vec{A}| = \sqrt{a_1^2+a_2^2+a_3^2}$$

$$\hat{r} = \frac{\vec{A}}{|A|} = \cancel{A\ vector}\ \frac{a_1}{\sqrt{a_1^2+a_2^2+a_3^2}}\,i + \ldots\ldots$$

Q. show that,

$a-2b+3c,\ 2a+3b-4c,\ -7b+10c$ are colinear.

(একটি সরলরেখায় অবস্থিত)

P-16,17

Chapter-2

Vector product:

1. dot product / scalar product
2. cross product / vector product.

$$\vec{A} = a_1\hat{i} + \cancel{b_1\hat{j}}\ a_2\hat{j} + a_3\hat{k}$$

$$\vec{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

$$\vec{A}.\vec{B} = |\vec{A}|\,|\vec{B}|\cos\theta = |\vec{A}|\,(|\vec{B}|\cos\theta) \rightarrow A \text{ এর উপর } B$$

$$= |\vec{B}|\,(|\vec{A}|\cos\theta) \rightarrow B \text{ এর উপর } A$$

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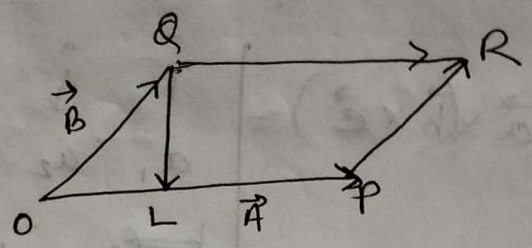
$\vec{A} \times \vec{B} = |\vec{A}|\,|\vec{B}| \sin\theta \, \hat{n}$

$|\vec{A} \times \vec{B}| = |\vec{A}|\,|\vec{B}| \sin\theta$



ΔOLQ

$\sin\theta = \dfrac{QL}{OQ}$

$\Rightarrow QL = OQ \sin\theta$

$= |\vec{B}| \sin\theta$

OPRQ

$OP \times QL$

$|\vec{A} \times \vec{B}| = |\vec{A}|\,|\vec{B}| \sin\theta$

H.W.

$\vec{a} \cdot (\vec{b} \times \vec{c})$

d ↓ scalar, ↓ vector

↓ scalar

★ prove that, $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

$\vec{a} = a_1\hat{i} + a_2\hat{j} + \cancel{a_1\hat{k}}\, a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

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$$\vec{b}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (b_2c_3 - b_3c_2)\hat{i} - (b_1c_3 - b_3c_1)\hat{j} + \hat{k}(b_1c_2 - b_2c_1)$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_2c_3 - b_3c_2 & b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1 \end{vmatrix}$$

$$\vec{a}\cdot\vec{c} = a_1c_1 + a_2c_2 + a_3c_3$$

$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

$$(\vec{a}\cdot\vec{c})\vec{b} = (a_1c_1 + a_2c_2 + a_3c_3)\cancel{(a_1b_1 + a_2b_2 + a_3b_3)}(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

$$= (a_1b_1c_1 + a_2b_1c_2 + a_3b_1c_3)\hat{i} + (a_1b_2c_1 + a_2b_2c_1 + a_3b_2c_3)\hat{j} + (a_1b_3c_1 + a_2b_3c_2 + a_3b_2c_3)\hat{k}.$$

Math-2101: 09.08.23

Vetor triple product:

(P-24): $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\cdot\vec{b} - (\vec{a}\cdot\vec{b})\cdot\vec{c}$ [প্রমাণ দেখতে হবে]

Scalar triple product $\rightarrow \vec{a}\,(\vec{b}\times\vec{c})$

$\downarrow$ Box product $\qquad \downarrow$ $[\vec{a}\ \vec{b}\ \vec{c}]$

P-33

EX: Find the volumn of the parallelopiped whose edges are represented by $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$

$\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$

$\vec{c} = 3\hat{i} - \hat{j} + 2\hat{k}$

$$\vec{b}\times\vec{c} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix} = 3\hat{i} - 5\hat{j} - 7\hat{k}$$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = 6 + 15 - 28 = -7$$

Hence the volumn of the paralledopiped is 7 unit.

$\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = 0$

$\boxed{\text{formula: } \vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\cdot\vec{c}}$

$$\text{L.S.} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} + (\vec{b}\cdot\vec{a})\vec{c} - (\vec{b}\cdot\vec{c})\vec{a} + (\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}$$

$$= \cancel{(\vec{a}\cdot\vec{c})}\vec{b} - \cancel{(\vec{a}\cdot\vec{b})}\vec{c} + \cancel{(\vec{a}\cdot\vec{b})}\vec{c} - \cancel{(\vec{b}\cdot\vec{c})}\vec{a} + \cancel{(\vec{b}\cdot\vec{c})}\vec{a} - \cancel{(\vec{a}\cdot\vec{c})}(\vec{b})$$

$$= 0 \qquad [\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}]$$

P-36

P-26 # Prove that $(\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c})\cdot(\vec{c}\times\vec{a}) = (\vec{a}\cdot\vec{b}\cdot\vec{c})^2$

$\Rightarrow$

$$L.S. = (\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c})\cdot(\vec{c}\times\vec{a})$$

Let $\vec{a}\times\vec{b} = \vec{m}$

$$= \vec{m}\times(\vec{b}\times\vec{c})\cdot(\vec{c}\times\vec{a})$$

$$= \{(\vec{m}\cdot\vec{c})\vec{b} - (\vec{m}\cdot\vec{b})\vec{c}\}\cdot(\vec{c}\times\vec{a})$$

$$= [\{(\vec{a}\times\vec{b})\cdot\vec{c}\}\vec{b} - \{(\vec{a}\times\vec{b})\vec{b}\}\vec{c}]\cdot(\vec{c}\times\vec{a})$$

$$= [[\vec{a}\,\vec{b}\,\vec{c}]\cdot\vec{b} - [\vec{a}\,\vec{b}\,\vec{b}]\cdot\vec{c}]\cdot(\vec{c}\times\vec{a})$$

$$= [[\vec{a}\,\vec{b}\,\vec{c}]\cdot\vec{b} - 0\cdot\vec{c}]\,(\vec{c}\times\vec{a}) \qquad [\because \vec{a}\cdot\vec{b}\cdot\vec{b} = 0]$$

prove আসবে

$$= [\vec{a}\,\vec{b}\,\vec{c}]\,\vec{b}\cdot(\vec{c}\times\vec{a})$$

$$= [\vec{a}\,\vec{b}\,\vec{c}]\,[\vec{a}\,\vec{b}\,\vec{c}]$$

$$= [\vec{a}\,\vec{b}\,\vec{c}]^2$$

$$= R.H.S.$$

E-24 P-37 # show that if $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$ then $\vec{a}$ & $\vec{b}$ are perpendicular.

Given, $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$

$$\Rightarrow |\vec{a}+\vec{b}|^2 = |\vec{a}-\vec{b}|^2$$

$$\Rightarrow (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = (\vec{a}+\vec{b})(\vec{a}-\vec{b})$$

$$\Rightarrow |\vec{a}|^2 + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} + |\vec{b}|^2 = |\vec{a}|^2 - \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{a} + |\vec{b}|^2$$

$$\Rightarrow \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} = 0$$

$$\Rightarrow 2\vec{a}\cdot\vec{b} + 2\vec{b}\cdot\vec{a} = 0$$

$$\Rightarrow 4\vec{a}\vec{b} = 0 \quad [\because \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}]$$

$$\Rightarrow \vec{a}\cdot\vec{b} = 0$$

Hence, $\vec{a}$ & $\vec{b}$ are perpendicular.

12.08.23

Math-2101:

Vector operator:

The operator $\vec{\nabla}$ defined as follows

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

is called $\vec{\nabla}$ operator / nabla.
(Del)

Gradient:

Let $\phi(x,y,z)$ be a differentiable scalar function at all points in a region then the gradient of ϕ is written as grad ϕ or $\vec{\nabla}\phi$, is defined ~~as~~ by

$$\text{grad } \phi = \vec{\nabla}\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\phi$$

$$= \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

Divergence:

Let $\vec{V}(x,y,z) = V_1(x,y,z)\hat{i} + V_2(x,y,z)\hat{j} + V_3(x,y,z)\hat{k}$ be differentiable vector valued function at all points in a region, then divergence of $\vec{V}$ is defined by div $\vec{V}$ or $\vec{\nabla}.\vec{V}$ is defined by,

$$\text{div } \vec{V} = \vec{\nabla}.\vec{V} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right).\left(V_1\hat{i} + V_2\hat{j} + V_3\hat{k}\right)$$

$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

~~Curl:~~

Curl:

Let $\vec{v}(x,y,z) = v_1(x,y,z)\hat{i} + v_2(x,y,z)\hat{j} + v_3(x,y,z)\hat{k}$ be differentiable vector valued function at all points in a ~~reig~~ region, then the curl of $\vec{v}$ is denoted by curl $\vec{v}$ or $\vec{\nabla}\times\vec{v}$ is defined by,

$$\text{curl}\,\vec{v} = \vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \left(\frac{\delta v_3}{\delta y} - \frac{\delta v_2}{\delta z}\right)\hat{i} + \left(\frac{\delta v_3}{\delta x} - \frac{\delta v_1}{\delta z}\right)\hat{j} + \left(\frac{\delta v_2}{\delta x} - \frac{\delta v_1}{\delta y}\right)\hat{k}$$

Q. show that, curl grad $\phi = 0$

$\Rightarrow$ Let $\phi(x,y,z)$ be a scalar function. Then

$$\text{grad}\,\phi = \frac{\delta\phi}{\delta x}\hat{i} + \frac{\delta\phi}{\delta y}\hat{j} + \frac{\delta\phi}{\delta z}\hat{k}$$

L.S. = curl grad ϕ

$= \cancel{\text{curl} \times (\text{grad}\,\phi)}\ \vec{\nabla}\times(\text{grad}\,\phi)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ \frac{\delta\phi}{\delta x} & \frac{\delta\phi}{\delta y} & \frac{\delta\phi}{\delta z} \end{vmatrix} = \hat{i}\left(\frac{\delta^2\phi}{\delta y\delta z} - \frac{\delta^2\phi}{\delta z\delta y}\right) + \hat{j}\left(\frac{\delta^2\phi}{\delta x\delta z} - \frac{\delta^2\phi}{\delta\delta x\delta z}\right) + \hat{k}\left(\frac{\delta^2\phi}{\delta x\delta y} - \frac{\delta^2\phi}{\delta x\delta y}\right)$$

since, $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$

Hence, $\vec{\nabla} \times (\text{grad}\, \phi)$

$= 0$

Q. show that, $\text{div}(\text{curl}\, \vec{A}) = \vec{0}$

$\Rightarrow$ let $\vec{A} = A_1(x,y,z)\hat{i} + A_2(x,y,z)\hat{j} + A_3(x,y,z)\hat{k}$ be a vector valued function.

$$\text{curl}\, \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) + \hat{j}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) + \hat{k}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial x}\right)$$

$\therefore \text{div}(\text{curl}\, \vec{A}) = \vec{\nabla} \cdot (\text{curl}\, \vec{A})$

$$= \frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$= \frac{\partial^2 A_1}{\partial x \partial y} - \frac{\partial^2 A_2}{\partial x \partial z} + \frac{\partial^2 A_1}{\partial y \partial z} - \frac{\partial^2 A_3}{\partial y \partial x} + \frac{\partial^2 A_2}{\partial z \partial x} - \frac{\partial^2 A_1}{\partial z \partial y}$$

$$= 0$$

~~* curl = 0~~

* If curl $\vec{A} = \bar{0}$, then the fluid/flow is irrotation.

Neiver-stokes eq^n

Ex:

If $\vec{A} = \left[(axy - z^3, (a-2)x^2, (1-a)xz^2\right]$ is irrotational, the prove that, $a = 4$.

$\Rightarrow$ Since $\vec{A}$ is irrotational,

hence, curl $\vec{A} = 0$

$\Rightarrow \vec{\nabla} \times \vec{A} = \bar{0}$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy - z^3 & (a-2)x^2 & (1-a)xz^2 \end{vmatrix} = \bar{0}$$

$$\Rightarrow \left[0, -3z^2 - (1-a)z^2, 2(a-2)x - ax\right] = [0, 0, 0]$$

$\Rightarrow -3z^2 - (1-a)z^2 = 0$

$\Rightarrow -3 - 1 + a = 0$

$\therefore a = 4$

and, $2(a-2)x - ax = 0$

$\Rightarrow 2a - 4 - a = 0$

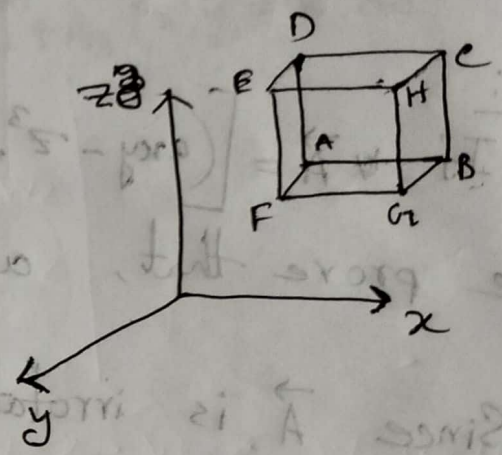
$\therefore a = 4$

14.08.23

Significance of gradient:

Let $\phi(x,y,z)$ be a scalar field, then $\vec{\nabla}\phi$ is a vector field which at a given point defines a vector whose direction is that in which the scalar field changes most rapidly and whose magnitude of the rate of change. The component of $\vec{\nabla}\phi$ in any direction is the rate of change of ϕ with respect to displacement in that direction.

Significance of divergence:



Let region in space be field of with a moving compressible fluid P be the center of a small parallopiped ABCDEFGH with edges parallel to the co-ordinate axes having length dx, dy, dz.

The vector point function

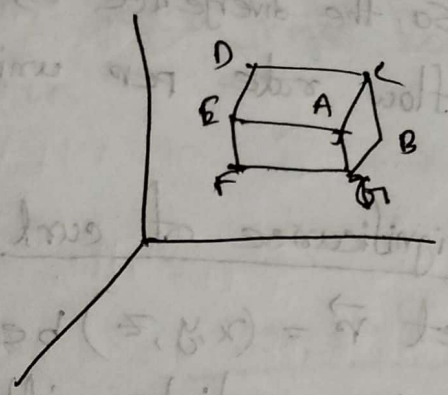
$$\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}.$$

represent the velocity at the point P.

From the Taylors theorem,

$$V_y(y+dy) = V_y + \frac{\delta V_y}{\delta y}.dy + \dots$$

$$\approx V_y + \frac{\delta V_y}{\delta y} dy$$



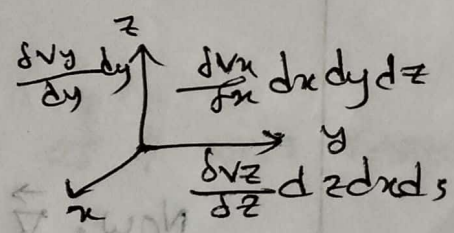
So,

The flow on the ~~area~~ faces ADEF and BCHG parallel to y axis will be $V_y \delta z \delta x$ and

$$V_y + \frac{\delta V_y}{\delta y} dy$$

The net flow on the area $dzdx$ at distance y and $y+dy$ $= V_y + \frac{\delta V_y}{\delta y} dy - V_y dz dx$

$$= \frac{\delta V_y}{\delta y} dx.dydz$$



similarly, the net flow of the rigid on the area $dxds$ and $dydz$ are $\frac{\delta V_z}{\delta z}$ and $\frac{\delta V_x}{\delta x} dxdydz$

Hence the total flux out of the volume $du = dxdydz$ is $\left(\frac{\delta V_x}{\delta x} + \frac{dy}{\delta y} + \frac{\delta V_z}{\delta z}\right) dxdydz$

$$= \vec{\nabla}.\vec{V} dx.dydz$$

$$= \text{div } \vec{V} \, \delta v$$

So, the diverjence of $\vec{v}$ is the net outword volume flow rate per unit volume.

Significance of curl:

Let $\vec{r} = (x, y, z)$ be the position vector of a moving particle with respect to O and $\vec{v} = (v_x, v_y, v_z)$ be the velocity at $\vec{r}$. If $\vec{w} = (w_x, w_y, w_z)$ be the angular velocity about O, then we have,

$$\vec{w} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ w_x & w_y & w_z \\ x & y & z \end{vmatrix}$$

$$= \left(\underbrace{w_x z - w_z y}_{x}, \underbrace{w_z x - w_x z}_{y}, \underbrace{w_x y - w_y x}_{z} \right)$$

Now, $\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ w_x z - w_z y & w_z x - w_x z & w_x y - w_y z \end{vmatrix}$

$$= i \left(\frac{\delta}{\delta y}(w_x y - w_y x) - \frac{\delta}{\delta z}(w_z x - w_z y) \right)$$

$$+ j\left(\quad \right) + \hat{k}\left(\quad \right)$$

Do/

$$= \hat{i}(w_x - 0 - 0 + w_x) + \hat{j}(\ ) + \hat{k}(\ )$$

$$= 2(w_x \hat{i} + w_y \hat{j} + w_z \hat{k}) = 2\vec{w}$$

Thus the curl of the velocity vector of a rigid body is twice of its angular velocity vector.

19.08.23

Directional derivative:

Let $\phi(x,y,z)$ be a function of x,y,z which is defined on a space curve c. Let $\phi(x+\Delta x, y+\Delta y, z+\Delta z)$ be the value of the function at the point on c and suppose Δs denotes the arc lenth of the curve between the points.

$$\frac{\Delta\phi}{\Delta s} = \frac{\phi(x+\Delta x, y+\Delta y, z+\Delta z) - \phi(x,y,z)}{\Delta s}$$

represents the averate rate of change of ϕ with respect to the arc lenth.

If $\lim_{\Delta s \to 0} \frac{\Delta\phi}{\Delta s} = \lim_{\Delta s \to 0} \frac{\phi(x+\Delta x, y+\Delta y, z+\Delta z) - \phi(x,y,z)}{\Delta s}$

exists, then it is called the directional derivative of ϕ at the point (x,y,z) along the curve c and it is given by

$$\frac{d\phi}{ds} = \frac{\delta\phi}{\delta x}\cdot\frac{dx}{ds} + \frac{\delta\phi}{\delta y}\cdot\frac{dy}{ds} + \frac{\delta\phi}{\delta z}\cdot\frac{dz}{ds}$$

$$= \left(\frac{\delta\phi}{\delta x}\hat{i} + \frac{\delta\phi}{\delta y}\hat{j} + \frac{\delta\phi}{\delta z}\hat{k}\right)\cdot\left(\frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k}\right)$$

$$= \vec{\nabla}\phi \cdot \frac{d\vec{r}}{ds}$$

where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is the position vector and

$\frac{d\vec{r}}{ds} = \vec{T}$ is the tangent vector.

* Find the directional derivative of $\phi(x,y,z) = 4xz^3 - 3x^2y^2z$ at the point $(2,-1,2)$ in the direction $2\hat{i} - 3\hat{j} + 6\hat{k}$

Here, let $\vec{a} = 2\hat{i} - 3\hat{j} + 6\hat{k}$

$$|\vec{a}| = \sqrt{2^2 + (-3)^2 + 6^2} = \sqrt{49}$$

And. $\vec{\nabla}\phi = \frac{\delta\phi}{\delta x}\hat{i} + \frac{\delta\phi}{\delta y}\hat{j} + \frac{\delta\phi}{\delta z}\hat{k}$

$$= (4z^3 - 6xy^2z)\hat{i} + (-6x^2yz)\hat{j} + (12xz^2 - 3x^2y^2)\hat{k}$$

At $(2,-1,2)$ we have,

$$\vec{\nabla}\phi = 8\hat{i} + 48\hat{j} + 84\hat{k}$$

Hence, the directional ~~de~~ directivative of ϕ in the direction of $\vec{a}$ is

$$\vec{\nabla}\phi \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$= (8\hat{i} + 48\hat{j} + 84\hat{k}) \cdot \left(\frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right)$$

$$= \frac{8 \times 2}{7} + \frac{48(-3)}{7} + \frac{84 \times 6}{7} = \frac{376}{7}.$$

If $\phi = 2xyz^2$ and $\vec{F} = [xy, -z, x^2] = xy\,\hat{i} - z\hat{j} + x^2\hat{k}$, then find

i) $\int_c \phi\, d\vec{r}$; ii) $\int_c \vec{F} \cdot d\vec{r}$; iii) $\int_c \vec{F} \times d\vec{r}$, where c is the curve $x = t^2$, $y = 2t$, $z = t^3$ from $t = 0$ to $t = 1$

Soln:

Along the curve c,

$$\phi = 2 \cdot t^2 \cdot 2t\,(t^3)^2$$
$$= 4t^9$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$= t^2\hat{i} + 2t\hat{j} + t^3\hat{k}$$

$$d\vec{r} = 2t\,dt\,\hat{i} + 2dt\,\hat{j} + 3t^2 dt\,\hat{k}$$
$$= (2t\hat{i} + 2\hat{j} + 3t^2\hat{k})\,dt$$

i) $\int_c 4t^9 (2t\hat{i} + 2\hat{j} + 3t^2\hat{k})\,dt$

$$= \int_0^1 (8t^{10}\hat{i} + 8t^9\hat{j} + 12t^{11}\hat{k})\,dt$$

$$= \left[\frac{8t^{11}}{10}\right]_0^1 \hat{i} + \left[\frac{8t^{10}}{10}\right]_0^1 \hat{j} + \left[\frac{12t^{12}}{12}\right]_0^1 \hat{k}$$

$$= \frac{8}{11}\hat{i} + \frac{8}{10}\hat{j} + \hat{k}$$

ii) $\int_c \vec{F}\, d\vec{r}$

$$\vec{F} = [xy, -z + x^2]$$
$$= [t^2 \cdot 2t, -t^3, t^4]$$
$$= [2t^3, -t^3, t^4]$$

$$= \int_0^1 [4t^4 - 2t^3 + 3t^6]\, dt$$

$$= \frac{51}{70}$$

iii) $\int_c \vec{F} \times d\vec{r} = -\frac{9}{10}\hat{i} - \frac{2}{3}\hat{j} + \frac{7}{5}\hat{k}$

Show that, $\oint_c \vec{F} \cdot d\vec{r} = 2\pi$

along the circle $(x-1)^2 + y^2 = 1$, $z = 1$

where, $\vec{F} = -y\hat{i} + x\hat{j} + z\hat{k}$

$$\vec{F} \cdot d\vec{r} = (-y\hat{i} + x\hat{j} + z\hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= -y\,dx + x\,dy + z\,dz \quad \text{—— (1)}$$

circle
$x = r\cos\theta$
$y = r\sin\theta$

The parametric eq^n of the given circle is $x = 1 = \cos\theta$, $y = \sin\theta$; $0 \le \theta \le 2\pi$

$\Rightarrow x = 1 + \cos\theta$, $y = \sin\theta$; $z = 1$ $\therefore dz = 0$

$\therefore dx = -\sin\theta d\theta$, $dy = \cos\theta d\theta$

Here, $\oint_C \vec{F} \cdot \vec{dr} = \int (-\sin\theta) d\theta + (1 + \cos\theta) \cos\theta d\theta + 0$ [From ①]

$$= \int_0^{2\pi} \sin^2\theta d\theta + \cos\theta d\theta + \cos^2\theta d\theta$$

$$= \int_0^{2\pi} d\theta + \cos\theta d\theta = \left[\theta + \sin\theta\right]_0^{2\pi}$$

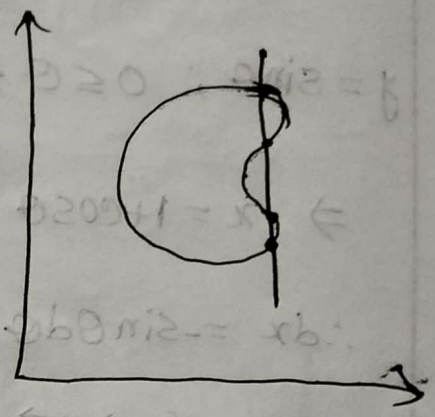
$$= 2\pi + \sin 2\pi - 0 - \sin 0$$

$$= 2\pi$$

CS CamScanner

21.08.23

Green's th^m:



Let R is a closed region in XY plane bouned by a simple closed curve c such that any straight line parallel to the co-ordinates axes cut c atmost two points.

Now, if $P(x,y)$ and $Q(x,y)$ are continuous ~~for~~ functions of x & y having continuous partial derivatives in R, then $\oint P\,dx + Q\,dy$

$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

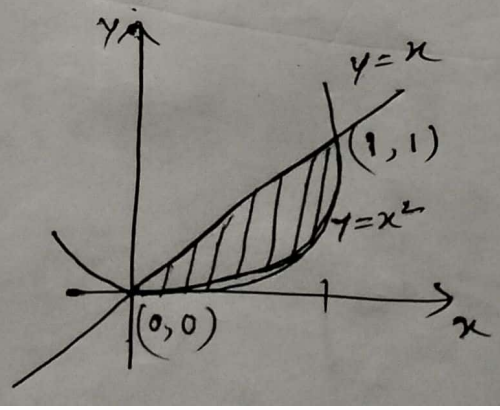
Q.

Verify the Green's th^m in the plane for $\oint_c (xy+y^2)dx + x^2 dy$; where c is the colsed curve of the region bounded by $y=x$ & $y=x^2$.

⇒

Here,

$y=x$ and $y=x^2$ meet at $(0,0)$ and $(1,1)$.



Now, along the parabola $y=x^2$ from $(0,0)$ to $(1,1)$ the line integral is ~~$\int(xy+y^2)dx$~~ $\int_0^1 (x.x^2+x^4)dx + x^2.2x\,dx$; where $y=x^2$

$$= \int_0^1 (x^3+x^4+2x^2)dx$$

$$= \frac{19}{20}$$

Now, along the ~~parabola~~ line $y=x$ from $(1,1)$ to $(0,0)$ the line integral is, $\int_1^0 (x.x+x^2)\,dx + x^2 dx$

$$= -1$$

Hence, the line integral is $= \frac{19}{20} - 1$

$$= \frac{19-20}{20} = -\frac{1}{20}$$

Here, $P = xy+y^2$, $Q = x^2$

$$\therefore \iint_R \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}\right) dx\,dy$$

$$= \iint_R (2x - x - 2y)\,dx\,dy$$

$$= \int_0^1\int_{x^2}^{x} (x-2y)\,dy\,dx = \int_0^1 \left[xy - \frac{2y^2}{2}\right]_{x^2}^{x} dx$$

$$= \int_0^1 (x^4 - x^3)\,dx$$

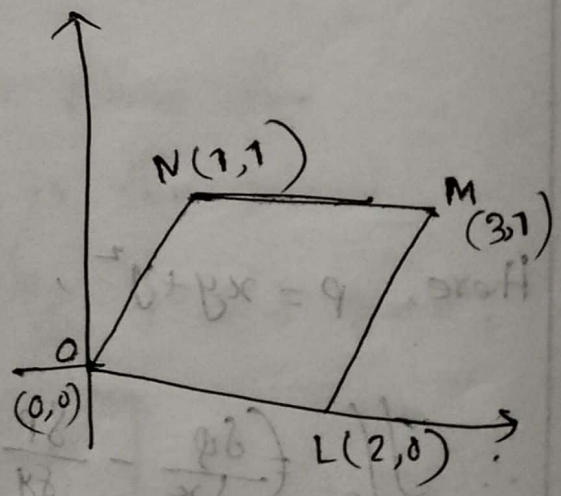
$$= \frac{1}{5} - \frac{1}{4}$$

$$= -\frac{1}{20}$$

$\therefore$ verified the Green's th^m.

Verify Greens th^m in the plane for evalute $\oint_c (3x^2 + 2y)\,dx - (x + 3\cos y)\,dy$ where c is boundary of the parallelogram having vertices at $(0,0), (2,0), (3,1), (1,1)$.

$\Rightarrow$



Along the line $OL (y = 0)$ we have,

$0 \le x \le 2$.

$$\therefore \int_{OL} p\,dx + q\,dy = \int_0^2 3x^2\,dx$$

$$= 8 \quad \text{—— (1)}$$

$\therefore$ The eq^n of the line LM is $y = x-2$

Hence,
$$\int_{LM} P\,dx + Q\,dy = \int_2^3 (3x^2 + 2x - 4)\,dx - [x - 3\cos x\,(x-2)]\,dx$$

$$= \frac{35}{2} - 3\sin 1 \quad \text{--- (ii)}$$

The eq^n of MN is $y = 1$

Hence,
$$\int_{MN} P\,dx + Q\,dy = \int_3^1 (3x^2 + 2)\,dx = -30 \quad \text{--- (iii)}$$

The eq^n of NO is $y = x$

Hence,
$$\int_{NO} P\,dx + Q\,dy = \int_1^0 (3x^2 + x - 3\cos x)\,dx$$

$$= -\frac{3}{2} + 3\sin 1 \quad \text{--- (IV)}$$

$$\therefore \oint P\,dx + Q\,dy = -6 \qquad [\text{eq}^n\ \text{(i)} + \text{(ii)} + \text{(iii)} + \text{(IV)}]$$

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dy\,dx$$

$$= \int_0^2\int_0^1 (-1-2)\,dy\,dx = -6$$

(verified)

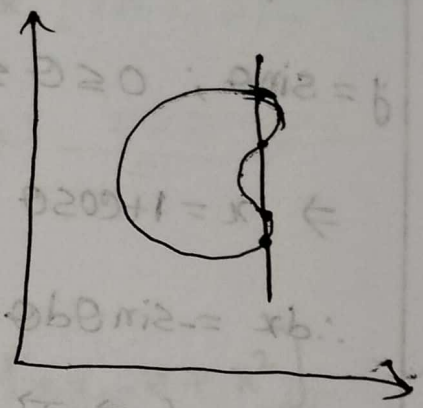
Since $\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dy\,dx$

Hence the Green's th^m is verified.

21.08.23

Green's th^m:

Let R is a closed region in XY plane bouned by a simple closed curve c such that any straight line parallel to the co-ordinates axes cut c atmost two points.



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Q.

Verify the Green's th^m in the plane for $\oint_c (xy+y^2)dx + x^2 dy$; where c is the colsed curve of the region bounded by $y=x$ & $y=x^2$.

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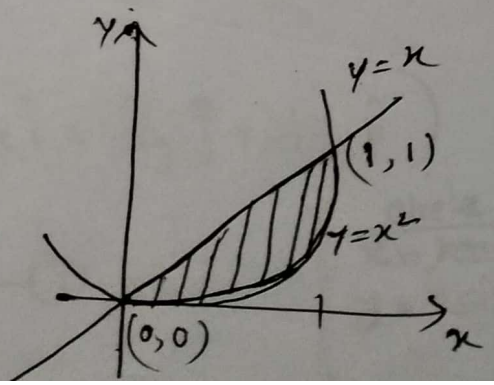

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$$= \int_0^1 (x^3+x^4+2x^2)dx$$

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Now, along the ~~parabola~~ line $y=x$ from $(1,1)$ to $(0,0)$ the line integral is, $\int_1^0 (x.x+x^2)\,dx + x^2 dx$

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$$= \int_0^1\int_{x^2}^{x} (x-2y)\,dy\,dx = \int_0^1 \left[xy - \frac{\cancel{2}y^2}{\cancel{2}}\right]_{x^2}^{x} dx$$

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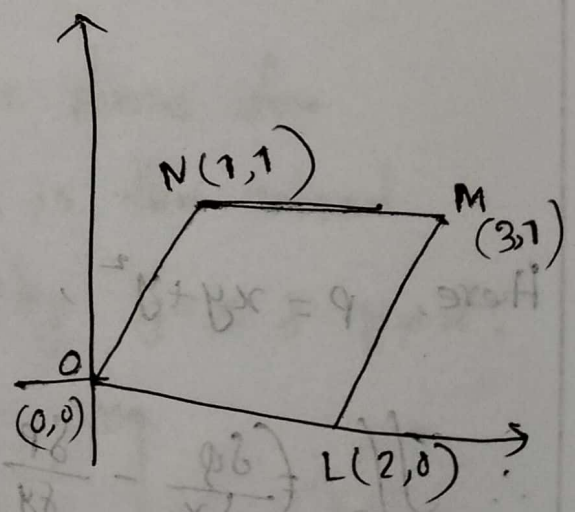

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$$= \frac{35}{2} - 3\sin 1 \quad \text{--- (I)}$$

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Hence,
$$\int_{MN} P\,dx + Q\,dy = \int_3^1 (3x^2 + 2)\,dx = -30 \quad \text{--- (II)}$$

The eq^n of NO is $y = x$

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$$\int_{NO} P\,dx + Q\,dy = \int_1^0 (3x^2 + x - 3\cos x)\,dx$$

$$= -\frac{3}{2} + 3\sin 1 \quad \text{--- (IV)}$$

$$\therefore \oint P\,dx + Q\,dy = -6 \qquad [\text{eq}^n\ \text{I} + \text{II} + \text{III} + \text{IV}]$$

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dy\,dx$$

$$= \int_0^2\int_0^1 (-1-2)\,dy\,dx = -6$$

$\therefore$ (verified)

Since $\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}\right) dy\,dx$

Hence the Green's th^m is verified.

22.08.23

Math-2101

Divergence th^m:

If V is the volumn bounded by a closed surface S and $\vec{A}$ is a vector function of position with continuous derivative, then

$$\iiint_V \vec{\nabla}.A\,dv = \iint_S \vec{A}.\vec{n}\,ds = \oiint_S \vec{A}.\vec{ds}$$

Q. Evaluate $\iint_S \vec{F}.\vec{n}\,ds$, where $\vec{F} = 6z\hat{i} + (2x+y)\hat{j} - x\hat{k}$ and S is the surface of the region bounded by $x^2+z^2=9$, $x=0, y=0$, $z=0$ and $y=8$.

$\Rightarrow$ We know,

$$\iint_S \vec{F}.\vec{n}\,ds = \iiint_V \vec{\nabla}.\vec{F}\,dv \quad \text{—(1)}$$

$\vec{\nabla}.\vec{F} = 1$

$x^2+z^2=9$

$\therefore z = \pm\sqrt{9-x^2}$

1) $$\iiint_V \vec{\nabla}.\vec{F}\,dv = \int_0^3\int_0^8\int_0^{\sqrt{9-x^2}} 1\,\cancel{dydx}\,dz\,dy\,dx.$$

$$= \int_0^3\int_0^8 [z]_0^{\sqrt{9-x^2}}\,dy\,dx$$

$$= \int_0^3\int_0^8 \sqrt{9-x^2}\,dy\,dx$$

$$= \int_0^3 (\sqrt{9-x^2})\,[y]_0^8\,dx$$

$$= 8\int_0^3 \sqrt{9-x^2}\,.dx = 8\left[\frac{x\sqrt{9-x^2}}{2} + \frac{9}{2}\sin^{-1}\frac{x}{3}\right]_0^3$$

$$= 18\pi$$

~~Q. Verify the divergenic thm for~~

Q. Show that, $\iint_S \vec{F}\cdot\vec{n}\, d\vec{s} = \pi$ where $\vec{F} = x\hat{i} - y\hat{j} + (z^2-1)\hat{k}$ and $\bar{S}$ is the closed surface, bounded by the planes $z=0, z=1$ and the cylinder $x^2+y^2=1$.

$\Rightarrow$ We know, $\iint_S \vec{F}\cdot\vec{n}\, ds = \iiint_V \vec{\nabla}\cdot\vec{F}\, dv = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}}\int_0^1 2z\, dz\, dy\, dx$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[z^2\right]_0^1 dy\, dx$$

$$= \int_{-1}^{1} \left[y\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$

$$= \int_{-1}^{1} 2\sqrt{1-x^2}\, dx$$

$$= 2\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}\frac{x}{1}\right]_{-1}^{1}$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

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