Part-B

Chapter-3: Matrix Algebra

1. Real matrix: $A = \bar{A}$
2. Imaginary matrix: $A = -\bar{A}$
3. Square matrix: Number of rows = number of columns.
4. Rectangular matrix: number of rows and columns not equal.
5. Diagonal matrix: $a_{ij} = 0$ when $i \neq j$. (1,1), (2,2), (3,3) are not zero. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$
6. Identity matrix: $a_{ij} = 0$ when $i \neq j$, $a_{ij} = 1$ when $i = j$ (1,1), (2,2), (3,3) will be 1. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

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7. Null matrix: Every element is zero. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
8. Upper triangular matrix: For square matrix, $a_{ij} = 0$, when $i > j$. (2,1), (3,1), (3,2) will be zero.

Example: $\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 7 \end{bmatrix}$

9. Lower triangular matrix: For square matrix, $a_{ij} = 0$ when $i < j$. (1,2), (1,3), (2,3) will be zero

Example: $\begin{bmatrix} 5 & 0 & 0 \\ -1 & 2 & 0 \\ 3 & 7 & 1 \end{bmatrix}$

10. Symmetric matrix: $A^T = A$

Example: $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & 7 \\ -3 & 7 & 3 \end{bmatrix}$

11. Skew-symmetric matrix: $A^T = -A$

Example: $B = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$ $B^T = -\begin{bmatrix} 0 & +1 & -2 \\ -1 & 0 & +3 \\ +2 & -3 & 0 \end{bmatrix}$

12. Hermitian matrix: For a square matrix, $A^* = (\bar{A}^T) = A$ the A is called Hermitian matrix.

A imaginary matrix হবে A ও imaginary part এর আগের sign বদলে যাবে (plus থাকলে minus আর minus থাকলে plus).

$A = \begin{bmatrix} 2 & 2-3i & 3 \\ 2+3i & 5 & 1+i \\ 3 & 1-i & 0 \end{bmatrix}$

13. Skew-Hermitian matrix: $A^* = (\bar{A}^T) = -A$

Example: $A = \begin{bmatrix} 2i & 2-3i & 3 \\ -2-3i & 5i & 1+i \\ -3 & -1+i & 0 \end{bmatrix}$

14. Orthogonal matrix: for a square matrix $A \cdot A^T = A^T \cdot A = I$

15. Idempotent matrix: $A^2 = A$, for a square matrix.

16. Nilpotent matrix: For a square matrix, $A^n = 0$ and $A^{n-1} \neq 0$

Example: $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$

17. Unitary matrix: For a complex square matrix, if $AA^* = A^*A = I$ or $A^* = A^{-1}$, then A is a unitary matrix.

Example: $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$

3.9: Theorem-1: Prove that.

(I) $(A^T)^T = A$

Solve: Let $A = [a_{ij}]$ $\quad i = 1, 2, \ldots m$

By definition, $\quad j = 1, 2, \ldots n$

$A^T = [a_{ij}]^T = [a_{ji}]$

Now $(A^T)^T = [a_{ji}]^T$

$= a_{ij}$

$\therefore (A^T)^T = A$

(II) $(A+B)^T = A^T + B^T$

Solve: Let, $A = [a_{ij}]$ $\quad$ where,

$B = [b_{ij}]$ $\quad i = 1, 2, \ldots m$

Then, $\quad j = 1, 2 \ldots n$

$C = A+B$ is defined as $[c_{ij}] = [a_{ij}] + [b_{ij}]$

Now, by definition of the transpose matrix of C

we have $(A+B)^T = C^T = [c_{ij}]^T = [c_{ji}] = [a_{ji}] + [b_{ji}]$

$= [a_{ij}]^T + [b_{ij}]^T$

$\therefore (A+B)^T = A^T + B^T$

(III) $(AB)^T = \cancel{AB}\, B^T.A^T$

Solve: Let,

$A = [a_{ij}]$ where, $i = 1,2,\ldots m$

$B = [b_{jk}]$ $j = 1,2\ldots n$

$k = 1,2\ldots p$

Then, $A^T = [a_{ij}]^T = [a_{ji}]$ is an $n\times m$ matrix

$B^T = [b_{jk}]^T = [b_{kj}]$ is an $p\times n$ matrix

Thus, AB is a matrix of $m\times p$ so that $(AB)^T$ is a $p\times m$ matrix. also $B^T A^T$ is a $p\times m$ matrix. Therefore, $(AB)^T$ and $B^T.A^T$ have same dimensions.

Now, $AB = [c_{ik}]$ where (i,k)th element of ~~at~~ AB is

$$c_{ik} = \sum_{j=1}^{n} a_{ij}.b_{jk}$$

where, $i = 1,2\ldots m$

$j = 1,2\ldots p$

Therefore the (i,k)th element of $(AB)^T$

$$= \sum_{j=1}^{n} a_{ij}.b_{jk}$$

$$= \sum_{j=1}^{n} a_{ji}^T.b_{kj}^T$$

$$= \sum_{j=1}^{n} b_{kj}^T.a_{ji}^T$$

$= (k,i)$th element of $B^T.A^T$

Hence, $(AB)^T = B^T.A^T$

(IV) $(\alpha A)^T = \alpha . A^T$

Solve: Let $A = [a_{ij}]$

$$\therefore A^T = [a_{ij}]^T = [a_{ji}]$$

Now, $(\alpha A)^T = [\alpha\, a_{ij}]^T$

$$= [\alpha\, a_{ji}]$$

$$= \alpha [a_{ji}]$$

$$= \alpha A^T$$

$\therefore (\alpha A)^T = \alpha A^T$

3.11

Theorem-1: Let A^* and B^* be the conjugate transpose of A and B respectively, then

(I) $(A^*)^* = A$

Solve: Let, $A^* = B$

then, $B = (\overline{A^T}) = (\overline{A})^T$

and $B^T = \{(\overline{A})^T\}^T = \overline{A}$

Again, $(\overline{B})^T = (\overline{B^T}) = \overline{\overline{A}} = A$

Therefore, $B^* = A$, since $B^* = (\overline{B})^T = (\overline{B^T})$

Hence $(A^*)^* = A$, since $B = A^*$.

(II) $(A+B)^* = A^* + B^*$

Solve: By definition,

$$(A+B)^* = (\overline{A+B})^T$$

$$= (\overline{A})^T + (\overline{B})^T \qquad \text{since } (C+D)^T = C^T + D^T$$

$$= A^* + B^* \qquad \text{since } A^* = (\overline{A})^T$$

(III) $(AB)^* = B^* . A^*$

Solve: $(AB)^* = (\overline{AB})^T$

$$= (\overline{A}\,\overline{B})^T$$

$$= (\overline{B})^T . (\overline{A})^T \qquad \text{since } (CD)^T = D^T . C^T$$

$$= B^* . A^*$$

(IV) $(KA)^* = \overline{K} . A^*$

Solve: $(KA)^* = (\overline{KA})^T$

$$= (\overline{K} . \overline{A})^T$$

$$= \overline{K} . (\overline{A})^T$$

$$= \overline{K} . A^*$$

$\therefore (KA)^* = \overline{K} . A^*$

Singular matrix: Let D be the determinant of the square matrix A, then if $D=0$ the matrix A is called the singular matrix and if $D\neq0$; the matrix A is called non-singular matrix.

Inverse matrix: If $AB=BA=I$, then A and B are inverseable and $A^{-1}=B$, $B^{-1}=A$.

Example-12(b)

Let D be the determinant of the matrix

then. $D = \begin{vmatrix} 2 & -1 & 3 \\ 4 & 0 & -1 \\ 3 & 3 & 2 \end{vmatrix}$

$= 2(0+3)+1(8+3)+3(12-0)$

$= 6+11+36$

$= 53 \neq 0$

$\therefore$ So the matrix is non-singular and A^{-1} exists. Now the co-factors of D are:

$A_{11} = \begin{vmatrix} 0 & -1 \\ 3 & 2 \end{vmatrix}$ $= (0+3)$ $= 3$

$A_{12} = (-1)\begin{vmatrix} 4 & -1 \\ 3 & 2 \end{vmatrix}$ $= -1(8+3)$ $= -11$

$A_{13} = \begin{vmatrix} 4 & 0 \\ 3 & 3 \end{vmatrix}$ $= 12$

$A_{21} = (-1)\begin{vmatrix} -1 & 3 \\ 3 & 2 \end{vmatrix}$

$= -1(-2-9)$

$= 11$

$A_{22} = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix}$

$= (4-9)$

$= -5$

$A_{23} = (-1)\begin{vmatrix} 2 & -1 \\ 3 & 3 \end{vmatrix}$

$= -1(6+3)$

$= -9$

$A_{31} = \begin{vmatrix} -1 & 3 \\ 0 & -1 \end{vmatrix}$

$= (1-0)$

$= 1$

$A_{32} = \begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix} \times (-1)$

$= -1(-2-12)$

$= 14$

$A_{33} = \begin{vmatrix} 2 & -1 \\ 4 & 0 \end{vmatrix}$

$= (0+4)$

$= 4$

Therefore,

$$\text{Adj of } A = \begin{bmatrix} 3 & -11 & 12 \\ 11 & -5 & -9 \\ 1 & 14 & 4 \end{bmatrix}^T = \begin{bmatrix} 3 & 11 & 1 \\ -11 & -5 & 14 \\ 12 & -9 & 4 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{D} \text{Adj of } A$$

$$= \frac{1}{53}\begin{bmatrix} 3 & 11 & 1 \\ -11 & -5 & 14 \\ 12 & -9 & 4 \end{bmatrix}$$

Exercise-13: Row canonical form:

$A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ $\quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$AI_2 = \left[\begin{array}{cc:cc} 2 & 5 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right]$$

Interchanging R_1 and R_2 we have:

$$\left[\begin{array}{cc:cc} 1 & 3 & 0 & 1 \\ 2 & 5 & 1 & 0 \end{array}\right]$$

$R_2' = R_2 - 2R_1$

$$\left[\begin{array}{cc:cc} 1 & 3 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{array}\right]$$

$R_1' = R_1 + 3R_2$

$$\left[\begin{array}{cc:cc} 1 & 0 & 3 & -5 \\ 0 & -1 & 1 & -2 \end{array}\right]$$

$R_2' = (-1) \times R_2$

$$\left[\begin{array}{cc:cc} 1 & 0 & 3 & -5 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

$$\therefore I_2 A^{-1} = \left[\begin{array}{cc:cc} 1 & 0 & 3 & -5 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

Hence,

$$A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

Example-17:

$$\left.\begin{array}{l} 2x + y = 1 \\ x - 2y = 3 \end{array}\right\} \text{ — (1)}$$

The system of linear equations can be written in matrix as:

$$\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \text{ — (2)}$$

Let,

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \end{bmatrix} \quad L = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

From (2) we get,

$$AX = L \text{ — (3)}$$

Let D be the determinant of the matrix A, then

$$D = \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} = (-4-1)$$
$$= -5 \neq 0$$

So the matrix A is non singular and A^{-1} exists.

Now, cofactors of D are:

$A_{11} = -2 \qquad A_{12} = -1$

$A_{21} = -1 \qquad A_{22} = 2$

Therefore adj of $A = \begin{bmatrix} -2 & -1 \\ -1 & 2 \end{bmatrix}^T = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{D}$ Adj of $A = \frac{1}{-5}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 2/5 & 1/5 \\ 1/5 & -2/5 \end{bmatrix}$$

Multiplying both sides of eqn ③ by A^{-1}

$A^{-1}.AX = A^{-1}L$

or $IX = A^{-1}L$

$$X = A^{-1}L$$

$$X = \begin{bmatrix} 2/5 & 1/5 \\ 1/5 & -2/5 \end{bmatrix} \times \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{5} + \frac{3}{5} \\ \frac{1}{5} - \frac{6}{5} \end{bmatrix} = \begin{bmatrix} \frac{5}{5} \\ \frac{-5}{5} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Hence, $x = 1$

$y = -1$

Chapter – 4: Rank of a matrix

Echelon matrix: An echelon matrix is a matrix which have the property that if in any of its rows the first element distinct from is in the kth position, then in all the following rows there are zeros in the first k positions, or equivalently.

Rank of a matrix: The rank of a matrix is the maximum number of linearly independent rows or columns in the matrix.

Example – 4:

Given that,

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 2 & 4 & 1 & -2 & 3 \\ 3 & 6 & 2 & -6 & 5 \end{bmatrix}$$

First let us reduce the matrix A to echelon form by the elementary row operations. We multiply 1st row by 2 and 3 then subtract from 2nd and 3rd rows respectively:

$$= \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -6 & 1 \\ 0 & 0 & 5 & -12 & 2 \end{bmatrix} \qquad \begin{matrix} r_2' = r_2 - 2r_1 \\ r_3' = r_3 - 3r_1 \end{matrix}$$

We multiply 3rd row by 3:

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -6 & 1 \\ 0 & 0 & 15 & -36 & 6 \end{bmatrix}$$

We multiply 2nd row by 5 and substract from 3rd row:

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -6 & 1 \\ 0 & 0 & 0 & -6 & 1 \end{bmatrix}$$

This matrix is in row echelon from.

We substract 3rd row from the second row:

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -6 & 1 \end{bmatrix} \qquad r_2' = r_2 - r_3$$

Multiply 3rd row by $-1/6$:

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1/6 \end{bmatrix}$$

We multiply 2nd row by 1/3 :

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1/6 \end{bmatrix}$$

We add second row with the first row:

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1/6 \end{bmatrix} \quad r_1' = r_1 + r_2$$

We multiply 3rd row by 2 and substract from 1st row:

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 4/3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1/6 \end{bmatrix} \quad r_1' = r_1 - 3r_3$$

This matrix is in row ~~eche~~ reduced echelon form.

Example-5:

We apply both elementary column and row operations to the matrix A for reducing it to the normal form.

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 2 & 5 \end{bmatrix}$$

We replace c_2 and c_4 by $c_2 - 2c_1$ and $c_4 + c_1$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & -2 & 1 & 5 \\ 2 & 7 & 2 & 3 \end{bmatrix} \qquad \begin{aligned} c_2' &= c_2 - 2c_1 \\ c_4' &= c_4 + c_1 \end{aligned}$$

We replace c_2 and c_4 by $c_2 + 2c_3$ and $c_4 - 5c_3$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 2 & 11 & 2 & -7 \end{bmatrix} \qquad \begin{aligned} c_2' &= c_2 + 2c_3 \\ c_4' &= c_4 - 5c_3 \end{aligned}$$

We replace c_1 by $c_1 + c_3$ and $c_4 + \frac{7}{11} c_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ 0 & 11 & 2 & 0 \end{bmatrix}$$

We replace R_2 by $R_2 - 4R_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 11 & 2 & 0 \end{bmatrix} \qquad R_2' = R_2 - 4R_1$$

We repalce R_3 by $R_3 - 2R_2$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 11 & 0 & 0 \end{bmatrix} \qquad R_3' = R_3 - 2R_2$$

We interchange C_2 and C_3

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 11 & 0 \end{bmatrix}$$

We replace C_3 by $\frac{1}{11} C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\sim [I_3 \ 0]$ where $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Hence the rank of matrix A is 3.