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Matrix Representation

Def^n Let V be a vector space over a field K. Let $\{e_1, e_2, \ldots, e_n\}$ be a basis of V and let $v \in V$, then $v = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n$.
The co-ordinate of v with respect to the basis $\{e_i\}$, denoted by

$$[v]_e = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

Linear Operator: A linear mapping $F: V \longrightarrow V$ is called a linear operator.

Matrix Representation of a linear operator:

Let $T: V \longrightarrow V$ be a linear operator. Let $\{e_1, e_2, \ldots, e_n\}$ be a basis of V. Then $T(e_1), T(e_2), \ldots, T(e_n)$ are also vectors in V. Therefore

$$T(e_1) = a_{11} e_1 + a_{12} e_2 + \cdots + a_{1n} e_n$$

$$T(e_2) = a_{21} e_1 + a_{22} e_2 + \cdots + a_{2n} e_n$$

$$\cdots \quad \cdots \quad \cdots$$

$$T(e_n) = a_{n1} e_1 + a_{n2} e_2 + \cdots + a_{nn} e_n$$

The transpose of the above matrix of coefficients is the matrix representation of T with respect to the basis $\{e_i\}$

ie, $[T]_e = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ - & - & - & - \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$.

Problem. Find the matrix representation of $T: \mathbb{R}^3 \to \mathbb{R}^3$ in the usual basis, where $T(x, y, z) = (x, y, 0)$.

Sol^n We have $\{e_1(1,0,0), e_2(0,1,0), e_3(0,0,1)\}$ is the usual basis of $\mathbb{R}^3$. Thus

$$T(e_1) = T(1,0,0) = 1.e_1 + 0.e_2 + 0.e_3 = (1,0,0)$$

$$T(e_2) = T(0,1,0) = 0.e_1 + 1.e_2 + 0.e_3 = (0,1,0)$$

$$T(e_3) = T(0,0,1) = 0.e_1 + 0.e_2 + 0.e_3 = (0,0,0)$$

$$\therefore [T]_e = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Theorem. Let $\{e_1, e_2, \dots, e_n\}$ be a basis of V and let T be any operator on V. Then for any vector $v \in V$, $[T]_e [v]_e = [T(v)]_e$.

Proof. Let $T : V \to V$ and $\{e_1, e_2, \dots, e_n\}$ be a basis of V. Then ~~for any vector~~ we have

$$T(e_1) = a_{11} e_1 + a_{12} e_2 \dots + a_{1n} e_n$$

$$T(e_2) = a_{21} e_2 + a_{22} e_2 + \dots + a_{2n} e_n$$

$$\vdots$$

$$T(e_n) = a_{n1} e_1 + a_{n2} e_2 + \dots + a_{nn} e_n$$

$$\therefore [T]_e = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \vdots & & & \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{bmatrix}$$

Let $v \in V$, then $v = \lambda_1 e_1 + \lambda_2 e_2 + \dots + \lambda_n e_n$.

and $[v]_e = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}$

$$\therefore [T]_e [v]_e = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \vdots & & & \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}\lambda_1 + a_{21}\lambda_2 + \dots + a_{n1}\lambda_n \\ a_{12}\lambda_1 + a_{22}\lambda_2 + \dots + a_{n2}\lambda_n \\ \vdots \\ \dots + a_{nn}\lambda_2 + \dots + a_{nn}\lambda_n \end{bmatrix}$$

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Again, $T(v) = T(\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n)$

$= \alpha_1 T(e_1) + \alpha_2 T(e_2) + \cdots + \alpha_n T(e_n)$

$= \alpha_1 (a_{11} e_1 + a_{12} e_2 + \cdots + a_{1n} e_n)$

$+ \alpha_2 (a_{21} e_1 + a_{22} e_2 + \cdots + a_{2n} e_n)$

$\vdots$

$+ \alpha_n (a_{n1} e_1 + a_{n2} e_2 + \cdots + a_{nn} e_n)$

$\therefore \cancel{[T(v)]_e}$

$= \alpha_1 a_{11} e_1 + \alpha_2 a_{12} e_2 + \cdots + \alpha_n a_{1n} e_n$

$+ \alpha_2 a_{21} e_1 + \alpha_2 a_{22} e_2 + \cdots + \alpha_2 a_{2n} e_n$

$\vdots$

$+ \alpha_n a_{n1} e_1 + \alpha_n a_{n2} e_2 + \cdots + \alpha_n a_{nn} e_n$

$= (a_{11}\alpha_1 + a_{21}\alpha_2 + \cdots + a_{n1}\alpha_n) e_1$

$+ (a_{12}\alpha_1 + a_{22}\alpha_2 + \cdots + a_{n2}\alpha_n) e_2$

$+ \cdots$

$+ (a_{1n}\alpha_1 + a_{2n}\alpha_2 + \cdots + a_{nn}\alpha_n) e_n$

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$$[T(v)]_e = \begin{bmatrix} a_{11}d_1 + a_{21}d_2 + \cdots + a_{m1}d_n \\ a_{12}d_1 + a_{22}d_2 + \cdots + a_{m2}d_n \\ \cdots \\ a_{1n}d_1 + a_{2n}d_2 + \cdots + a_{nn}d_n \end{bmatrix}$$

Hence $[T]_e [v]_e = [T(v)]_e$.

Proved.

Problem. Let V be the vector space of 2×2 matrices over $\mathbb{R}$. and let $M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Let T be the linear operator on V. Defined by $T(A) = MA$. Find the trace of T.

Soln. Let $T: V \to V$ be a linear ~~operator~~ operator. and the usual basis of V is $\{e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$.

Then $T(e_1) = Me_1 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}$

$= 1.e_1 + 0.e_2 + 3.e_3 + 0.e_4$.

$T(e_2) = Me_2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix}$

$= 0.e_1 + 1.e_2 + 0.e_3 + 3.e_4$

$T(e_3) = 2.e_1 + 0.e_2 + 4e_3 + 0.e_4$

$T(e_4) = 0.e_1 + 2.e_2 + 0.e_3 + 4.e_4$.

$$\therefore [T]_e = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{pmatrix}.$$

$\therefore$ Trace of $T = 1+1+4+4$
$= 10$. (A)

Problem Let V be the vector space of 2×2 matrices over $\mathbb{R}$ and let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, T be a linear operator on V. Find matrix of T, where

(i) $T(A) = MA$, (ii) $T(A) = AM$.
(iii) $T(A) = MA - AM$.

Change of Basis

Definition. Let $\{e_1, e_2, \ldots, e_n\}$ and $\{f_1, f_2, \ldots, f_n\}$ be two basis of a vector space V. then

$$f_1 = a_{11}e_1 + a_{12}e_2 + \cdots + a_{1n}e_n$$
$$f_2 = a_{21}e_1 + a_{22}e_2 + \cdots + a_{2n}e_n$$
$$\vdots$$
$$f_n = a_{n1}e_1 + a_{n2}e_2 + \cdots + a_{nn}e_n$$

The transpose of the above matrix of coefficients is called the transition matrix from $\{e_i\}$ to $\{f_i\}$.

ie, $P = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ - & - & - & - \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}$ is the transition matrix.

<u>Theorem</u>. Let P be the transition matrix from a basis $\{e_i\}$ to a basis $\{f_i\}$ in a vector space V. Then for any vector $v \in V$, $P[v]_f = [v]_e$.

$$\Rightarrow [v]_f = P^{-1}[v]_e.$$

Proof. We have

$$P = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ - & - & - & - \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}$$

Let $v = a_1 f_1 + a_2 f_2 + \cdots + a_n f_n$

$$[v]_f = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

$$\therefore\ P[v]_f = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}d_1 + a_{21}d_2 + \cdots + a_{n1}d_n \\ a_{12}d_1 + a_{22}d_2 + \cdots + a_{n2}d_n \\ \vdots \\ a_{1n}d_1 + a_{2n}d_2 + \cdots + a_{nn}d_n \end{pmatrix} \quad \text{--- (1)}$$

Again,

$$v = d_1 f_1 + d_2 f_2 + \cdots + d_n f_n$$

$$= d_1(a_{11}e_1 + a_{12}e_2 + \cdots + a_{1n}e_n)$$

$$+ d_2(a_{21}e_1 + a_{22}e_2 + \cdots + a_{2n}e_n)$$

$$\vdots$$

$$+ d_n(a_{n1}e_1 + a_{n2}e_2 + \cdots + a_{nn}e_n)$$

$$= (a_{11}d_1 + a_{21}d_2 + \cdots + a_{n1}d_n)e_1$$

$$+ (a_{12}d_1 + a_{22}d_2 + \cdots + a_{n2}d_n)e_2$$

$$\vdots$$

$$+ (a_{1n}d_1 + a_{2n}d_2 + \cdots + a_{nn}d_n)e_n$$

$$\therefore [V]_e = \begin{pmatrix} a_{11}\alpha_1 + a_{21}\alpha_2 + \cdots + a_{n1}\alpha_n \\ a_{12}\alpha_1 + a_{22}\alpha_2 + \cdots + a_{n2}\alpha_n \\ \cdots \\ \cdots \\ a_{1n}\alpha_1 + a_{2n}\alpha_2 + \cdots + a_{nn}\alpha_n \end{pmatrix}$$

$$\therefore [V]_e = P[V]_f.$$

Proved

Problem. Consider the following basis of $\mathbb{R}^2$: $\{e_1 = (1,0), e_2 = (0,1)\}$ and $\{f_1 = (1,2), f_2 = (2,3)\}$. Find the transition matrices P and Q from $\{e_i\}$ to $\{f_i\}$ and from $\{f_i\}$ to $\{e_i\}$ respectively. Verify that $P^{-1} = Q$.

Sol^n: Let $(a,b) \in \mathbb{R}^2$, and $(a,b) = ae_1 + be_2$.

$\therefore f_1 = (1,2) = 1 \cdot e_1 + 2 \cdot e_2$

$f_2 = (2,3) = 2 \cdot e_1 + 3 \cdot e_2$

$\therefore$ the transition matrix from $\{e_i\}$ to $\{f_i\}$ is $P = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$.

Again, let $(a,b) \in \mathbb{R}^2$ and $(a,b) = xf_1 + yf_2$

$\Rightarrow (a,b) = x(1,2) + y(2,3)$

$\Rightarrow (a,b) = (x+2y, 2x+3y)$

$\therefore x+2y=a \Rightarrow x+2y=a$
$2x+3y=b \qquad\qquad +y=2a-b$

$\therefore y=2a-b$
$x=a-2y$
$=a-4a+2b=-3a+2b$

$\therefore (a,b)=(2b-3a)f_1+(2a-b)f_2$

$\therefore e_1=(1,0)=-3f_1+2f_2$

$e_2=(0,1)=2f_1-f_2$

$\therefore$ The transition matrix from $\{f_i\}$ to $\{e_i\}$ is $Q=\begin{pmatrix}-3 & 2\\ 2 & -1\end{pmatrix}$.

Now, $PQ=\begin{pmatrix}1 & 2\\ 2 & 3\end{pmatrix}\begin{pmatrix}-3 & 2\\ 2 & -1\end{pmatrix}$
$=\begin{pmatrix}-3+4 & 2-2\\ -6+6 & 4-3\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=I$

Hence $Q=P^{-1}$.

Theorem. Let $A(V)$ be the set of all linear operator on V. Then

(i) $[T+S]_e=[T]_e+[S]_e$

(ii) $[\lambda T]_e=\lambda[T]_e$ (iii) $[ST]_e=[S]_e[T]_e$

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Eigen Values and Eigenvectors

Definition. Let $T: V \to V$ be a linear operator on a vector space V over K. A scalar $\lambda \in K$ is called an eigen value of T if there exists a non-zero vector $v \in V$ such that $T(v) = \lambda v$.

Every vectors satisfying the above relations ~~are~~ is called eigenvector of T belonging to λ.

Note: If v is an eigen vector, then ~~each~~ the scalar multiple μv is also an eigen vector, since $T(v) = \lambda v$,

$$T(\mu v) = \cancel{\lambda(\mu v)} = \mu T(v) = \mu(\lambda v)$$
$$= \lambda(\mu v).$$

Therefore μv is also an eigen vector.

*. The set of all such vectors is a subspace of V, called the eigen space of λ,

i.e $V_\lambda = \{v \in V : T(v) = \lambda v\}$, is a subspace of V.

Proof: (i) Since $T(0) = 0 = \lambda 0$, so $0 \in V_\lambda$.
Hence V_λ is non-zero.

(ii) Let $v_1, v_2 \in V_\lambda$, then $T(v_1) = \lambda v_1$, $T(v_2) = \lambda v_2$

$\therefore\ T(v_1+v_2) = \cancel{\lambda(v_1+v_2) = \lambda v_1 + \lambda v_2 = T(v_1)+T(v_2)}$

$= T(v_1) + T(v_2)$

$= \lambda v_1 + \lambda v_2$

$= \lambda(v_1+v_2)$

Hence $v_1+v_2 \in V_\lambda$.

(iii) ~~let~~ $v \in V_\lambda$, $\mu \in K$. Then $T(v) = \lambda v$.

$\therefore\ T(\mu v) = \mu T(v) = \mu(\lambda v) = \lambda(\mu v) =$

Hence $\mu v \in V_\lambda$.

Hence V_λ is a subspace of ~~of~~ V,

Theorem. Let $T: V \longrightarrow V$ be a linear operator. Then $\lambda \in K$ is an eigen~~vector~~ of T if and only if the operator $\lambda I - T$ is singular.

Proof. Let $\lambda \in K$ be an eigen value of T. Then for any non-zero vector $v \in V$,

$T(v) = \lambda v$.

$\Rightarrow T(v) = (\lambda I)(v) \Rightarrow (\lambda I)(v) - T(v) = 0$

$\cancel{\Rightarrow T(\lambda I)(v) - T(v) = 0}$

$\Rightarrow (\lambda I - T)(v) = 0$

Hence $\lambda I - T$ is singular.

Conversely, if $\lambda I - T$ is singular, then

$(\lambda I - T)(v) = 0 \Rightarrow (\lambda I)(v) - T(v) = 0$

$\Rightarrow T(v) = \lambda v$. Hence v is

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Diagonalization and Eigenvectors

Let $T: V \to V$ be a linear operator on a vector space V. $[T]$ can be represented by a diagonal matrix iff there exists a basis $\{v_1, v_2, \ldots, v_n\}$ of V, where $v_1, v_2, \ldots, v_n$ are eigenvectors. then

$$T(v_1) = \lambda_1 v_1 + 0.v_2 + \cdots + 0$$
$$T(v_2) = 0 + \lambda_2 v_2 + \cdots + 0$$
$$\vdots$$
$$T(v_n) = 0 + 0 \cdots + \lambda_n v_n$$

$$\therefore [T] = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

Theorem. A linear operator $T: V \to V$ ~~is a~~ can be represented by a diagonal matrix B iff. V has a basis consisting of eigenvectors of T.

Theorem. An n-square matrix A is similar to a diagonal matrix B if and only if ~~V has a~~ A has n linearly independent eigenvectors,

$$B = P^{-1}AP$$

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Characteristic Polynomial

Consider an n-square matrix A over a field K:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Then the matrix $tI_n - A$, where I_n is the n-square identity matrix and t is unknown, is called the characteristic matrix of A.

$$\therefore tI_n - A = \begin{pmatrix} t & 0 & \cdots & 0 \\ 0 & t & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & t \end{pmatrix} - \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} t-a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & t-a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & t-a_{nn} \end{pmatrix}$$

~~tA~~ $|tI_n - A| = \Delta_A(t)$ is a polynomial in t of degree n, is called the characteristic polynomial of A. and $\Delta_A(t) = |tI_n - A| = 0$ is called the characteristic equation of A.

* Find eigenvalues and associated nonzero eigenvectors of the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$.

Soln: Let a scalar λ and a nonzero vector $x = \begin{pmatrix} x \\ y \end{pmatrix}$ ~~We have $T(v) = \lambda v$~~

such that $AX = \lambda X$

$\Rightarrow$

[If A is an n-square matrix over K, then an eigenvalue of A means an eigenvalue of A viewed as an operator on K^n. i.e, $\lambda \in K$ is an eigenvalue of A if for some nonzero vector $v \in K^n$,

$Av = \lambda v$.]

$$AX = \lambda X$$

$$\Rightarrow \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x+2y \\ 3x+2y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix}$$

$$\Rightarrow \quad x+2y = \lambda x$$
$$3x+2y = \lambda y$$

$$\Rightarrow \left.\begin{aligned} (1-\lambda)x + 2y &= 0 \\ 3x + (2-\lambda)y &= 0 \end{aligned}\right\} \longrightarrow (1)$$

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for non-zero solution,

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(2-\lambda)-6=0$$

$\Rightarrow 2-3\lambda+\lambda^2-6=0$

$\Rightarrow \lambda^2-3\lambda-4=0$

$\Rightarrow \lambda^2-4\lambda+\lambda-4=0$

$\Rightarrow (\lambda-4)(\lambda+1)=0$

i.e. $\lambda = 4, -1$.

Let $\lambda=4$ in (1), then $3x-2y=0$
$-3x+2y=0$

i.e. $3x-2y=0$

$\therefore X = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Let $\lambda=-1$, in (1), then $2x+2y=0$
$3x+3y=0$

i.e. $x+y=0$,

so $X = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Thus $X = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is ~~again~~ a non-zero eigenvector belonging to the eigenvalue $\lambda=4$ and every other eigenvector belonging to $\lambda=4$ is a multiple of X.

and $w = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is a nonzero eigenvector belonging to the eigenvalue $t\,\lambda=-1$, and every eigenvector belonging to $\lambda=-1$ is a multiple of w.

Cayley – Hamilton Theorem.

Every matrix is a zero of its characteristic polynomial.

Proof Let A be an arbitrary n-square matrix and let $\Delta(t)$ be its characteristic polynomial, say

$$\Delta(t) = |tI - A|$$
$$= t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0.$$

Let $B(t)$ the classical adjoint of the matrix $tI - A$. The elements of $B(t)$ are cofactors of the matrix $tI - A$ and hence are polynomials in t of degree not exceeding $n-1$. Thus

$$B(t) = B_{n-1}t^{n-1} + \cdots + B_1 t + B_0.$$

where the B_i are n-square matrices

over K which are independent of t.

But we know that

$$A \cdot \operatorname{adj} A = \operatorname{adj} A \cdot A = |A| I$$

Thus $$(tI - A) . B(t) = |tI - A| . I$$

$$\Rightarrow (tI - A)(B_{n-1} t^{n-1} + B_{n-2} t^{n-2} + \cdots + B_1 t + B_0) = (t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0) I$$

Comparing coefficient of t from both sides, we have

$$B_{n-1} = I$$

$$B_{n-2} - A B_{n-1} = a_{n-1} I$$

$$B_{n-3} - A B_{n-2} = a_{n-2} I$$

$$- \quad - \quad -$$

$$B_0 - A B_1 = a_1 I$$

$$- A B_0 = a_0 I$$

Multiplying the above matrix equations by $A^n, A^{n-1}, \ldots, A, I$ respectively,

$$A^n B_{n-1} = A^n$$

$$A^{n-1} B_{n-2} - A^n B_{n-1} = a_{n-1} A^{n-1}$$

$$A^{n-2} B_{n-3} - A^{n-1} B_{n-2} = a_{n-2} A^{n-2}$$

$$- \quad - \quad -$$

$$A B_0 - A^2 B_1 = a_1 A$$

$-AB_0 = a_0 I$

Adding the above equations,

$$0 = A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I.$$

ie $\Delta(A) = 0$.

Thus A is a zero of its characteristic polynomial.

* For the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$, verify the Cayley-Hamilton theorem.

Proof – The characteristic polynomial of the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ is

$$\Delta(t) = |tI - A| = \begin{vmatrix} t-1 & -2 \\ -3 & t-2 \end{vmatrix}$$

$$= (t-1)(t-2) - 6$$
$$= t^2 - 3t - 4$$

$\therefore \Delta(A) = A^2 - 3A - 4I$

$$= \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} - 3\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

$$-4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence the theorem is verified.

Problem For the matrix, find all eigen values and a basis for each eigen space. $A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{pmatrix}$.

If possible, find an invertible matrix P such that $P^{-1}AP$ is diagonal.

Soln The characteristic polynomial

$$\Delta(t) = |tI - A|$$

$$= \begin{vmatrix} t-1 & -2 & -2 \\ -1 & t-2 & 1 \\ 1 & -1 & t-4 \end{vmatrix}$$

$$= (t-3)^2(t-1)$$

$\therefore t = 3, 3, 1.$

Thus the eigen values are $3, 3, 1$.

Let $t=1$ in the characteristic polynomial matrix $It-A$ to obtain the homogeneous system

$$\begin{pmatrix} 0 & -2 & -2 \\ -1 & -1 & 1 \\ 1 & -1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore$ $y+z=0$ or $x+y-z=0$

$x+y-z=0$ $\quad$ $y+z=0$

$x-y-3z=0$

Let z be the free variable and let $z=1$, then $x=2$, $y=-1$. Thus $u=(2,-1,1)$ form a basis of the eigen space $t=1$.

Let $t=3$, in the characteristic polynomial matrix $It-A$ to obtain the homogeneous system

$$\begin{pmatrix} 2 & -2 & -2 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ $x - y - z = 0$ i.e. $x - y - z = 0$

$x - y - z = 0$

$x - y - z = 0$

There are two independent variables say, y & z. and

(i) set $y = 1, z = 0$, then $x = 1$

(ii) set $y = 0, z = 1$, then $x = 1$

Thus $v_1 = (1, 1, 0)$ and $v_2 = (1, 0, 1)$ form a basis of the eigen space $t = 3$.

Since A has three linear independent eigen vectors, so A is diagonalizable.

Let P be the matrix whose columns are three independent eigen vectors

i.e $P = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

Then $P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

Problem For the following operator $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, find eigen values and a basis for the eigen space, where

$$T(x,y,z) = (x-y,\ 2x+3y+2z,\ x+y+2z).$$

Sol Consider the usual basis of $\mathbb{R}^3$ i.e. $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$

$$\therefore T(e_1) = T(1,0,0) = (1,2,1) = e_1 + 2e_2 + e_3$$

$$T(e_2) = T(0,1,0) = (-1,3,1) = -e_1 + 3e_2 + e_3$$

$$T(e_3) = T(0,0,1) = (0,2,2) = 0.e_1 + 2e_2 + 2e_3$$

$$\therefore A = [T] = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{pmatrix}.$$

$\therefore$ The ch. poly. $\Delta(t) = |tI - A|$

$$= \begin{vmatrix} t-1 & 1 & 0 \\ -2 & t-3 & -2 \\ -1 & -1 & t-2 \end{vmatrix}$$

$$= (t-1)(t-2)(t-3)$$

$\therefore$ The eigen values are $1, 2, 3$.

The basis of the eigen space when $t=1$, we have

$$\begin{pmatrix} 0 & 1 & 0 \\ -2 & -2 & -2 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

ie $\cancel{x}\, y = 0$ ie $y = 0$

$x+y+z=0$ $\quad$ $x+y+z=0$

$x+y+z=0$

There is one free variable say z,

let $z=-1$, then $x=+1$, $y=0$, $z=-1$.

$$\therefore v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

let $t=2$, we get

$$\begin{pmatrix} 1 & 1 & 0 \\ -2 & -1 & -2 \\ -1 & -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ $x+y=0$ or $x+y=0$

$2x+y+2z=0$. $\quad$ $2x+y+2z=0$

$x+y=0$

or $x+y=0$

$y-2z=0$

There is one free variable say, $\cancel{x}\, z$

let $\cancel{z}=1$, $\cancel{x}\, y=2$, $x=-2$

Then $v_2 = \begin{pmatrix} -2 \\ 2 \\ \end{pmatrix}$ is a basis of eigen space for $t=2$.

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For $t=3$,

$$\begin{pmatrix} 2 & 1 & 0 \\ -2 & 0 & -2 \\ -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

ie $2x+y=0$

$x+z=0$

$x+y-z=0$

ie $2x+y=0$

$y-2z=0$

let z be the free variable and we

$z=1,\ y=-2,\ x=1$

$\therefore v_3 = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ form a basis of the eigen space $t=3$.

Theorem. Similar matrices have the same ch. poly.

Proof. Two matrices A and B are said to be similar iff $B = P^{-1}AP$, where P is invertible.

$$\therefore |tI - B| = |tI - P^{-1}AP|$$
$$= |P^{-1}tIP - P^{-1}AP|$$
$$= |P^{-1}(tI - A)P|$$
$$= |P^{-1}||tI - A||P|$$
$$= |P^{-1}||P||tI - A|$$
$$= |tI - A|$$

Thus A and B have the same ch. poly.

Minimal polynomial

Let A be n-square matrix over K. From Cayley's theorem, we know, $\exists\ f(t) \ni f(A) = 0$.