Basis and Dimension

Linear Dependence

Definition. Let V be a vectorspace over K. The vectors $v_1, v_2, \ldots, v_m \in V$ are said to be linearly dependent over K or simply dependent if there exist scalars $a_1, a_2, \ldots, a_m \in K$, not all of them zero, such that

$$a_1 v_1 + a_2 v_2 + \cdots + a_m v_m = 0. \quad \text{— (A)}$$

Otherwise, the vectors are said to be linearly independent over K or simply independent.

Observe that the relation (A) will always hold if the a's are all 0. If this relation holds only in this case, that is

$$a_1 v_1 + a_2 v_2 + \cdots + a_m v_m = 0 \text{ only if } a_1 = 0, a_2 = 0, \ldots, a_m = 0,$$

then the vectors are linearly independent.

If the relation (A) also holds when one of the a's is not 0, then the vectors are linearly dependent.

Observe that if 0 is one of the vectors $v_1, v_2, \ldots, v_n$ say $v_1 = 0$, then the vectors must be dependent, for

$$1 \cdot v_1 + 0 v_2 + \cdots + 0 v_m = 1 \cdot 0 + 0 + \cdots + 0 = 0$$

and the coefficient of v_1 is not 0.

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On the other hand, any nonzero vector v is, by itself independent, for

$kv = 0,\ v \neq 0$ implies $k = 0$.

Ex-1. The vectors $u = (1, -1, 0)$, $v = (1, 3, -1)$ and $w = (5, 3, -2)$ are dependent since for $3u + 2v - w = 0$

ie $3(1, -1, 0) + 2(1, 3, -1) - (5, 3, -2) = (0, 0, 0)$

Ex-2. The vectors $u = (6, 2, 3, 4)$, $v = (0, 5, -3, 1)$ and $w = (0, 0, 7, -2)$ are independent. for suppose $xu + yv + zw = 0$ where x, y and z are unknown scalars.

Then $x(6, 2, 3, 4) + y(0, 5, -3, 1) + z(0, 0, 7, -2) = (0, 0, 0, 0)$

$\Rightarrow (6x, 2x + 5y, 3x - 3y + 7z, 4x + y - 2z) = (0, 0, 0, 0)$

and so by the equality of the corresponding components

$$6x = 0$$
$$2x + 5y = 0$$
$$3x - 5y + 7z = 0$$
$$4x + y - 2z = 0$$

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which implies $x=0$, $y=0$, $z=0$.
Thus $xu+yv+zw=0$ implies $x=0$, $y=0$, $z=0$. Hence u, v, and w are independent.

<u>Note</u>. The vectors in the preceding example form a matrix in echelon form

$$\begin{pmatrix} 6 & 2 & 3 & 4 \\ 0 & 5 & -3 & 1 \\ 0 & 0 & 7 & -2 \end{pmatrix}$$

Thus we have shown that the nonzero rows of the above matrix are independent. This result holds true in general.

• <u>Theorem</u>. The nonzero rows of a matrix in echelon form are linearly independent.

The vectors $v_1, v_2, \dots, v_m$ are linearly dependent if and only if one of them is a linear combination of the others.

For suppose, say, v_i is a linear combination of the others

$$v_i = a_1v_1 + \cdots + a_{i-1}v_{i-1} + a_{i+1}v_{i+1} + \cdots + a_mv_m$$

Then by adding $-v_i$ to both sides, we have

$$a_1v_1 + \cdots + a_{i-1}v_{i-1} - v_i + a_{i+1}v_{i+1} + \cdots + a_mv_m = 0,$$

where the coefficient of v_i is not zero.

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Hence the vectors are linearly dependent.

<u>Lemma.</u> The nonzero vectors $v_1, v_2, \ldots, v_m$ are linearly dependent if and only if one of them say v_i, is a linear combination of the preceding vectors

$$v_i = k_1 v_1 + k_2 v_2 + \cdots + k_{i-1} v_{i-1}$$

<u>Proof. See Book</u>.

Remarks:

(i) The set $\{v_1, v_2, \ldots, v_m\}$ ~~are~~ is called a dependent or independent set according as the vectors $v_1, v_2, \ldots, v_m$ are dependent or independent. We also define the empty set ϕ is ~~to~~ to be independent.

(ii) If two of the vectors $v_1, \ldots, v_m$ are equal say $v_1 = v_2$, then, the vectors are dependent for $v_1 - v_2 + 0v_3 + \cdots + 0v_m = 0$, and the coefficient of v_1 is not zero.

(iii) Two vectors v_1 and v_2 are dependent if and only if one of them is a multiple of the other.

(iv) A set which contains a dependent subset is itself dependent, Hence any subset of an independent set is independent.

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* Determine whether or not u and v are linearly dependent iff

(i) $u=(3,4)$, $v=(1,-3)$

(ii) $u=(4,3,-2)$, $v=(2,-6,7)$

(iii) $u=(2,-3)$, $v=(6,-9)$

(iv) $u=(-4,6,-2)$, $v=(2,-3,1)$

(v) $u=\begin{pmatrix}1 & -2 & 4\\ 3 & 0 & -1\end{pmatrix}$, $v=\begin{pmatrix}2 & -4 & 8\\ 6 & 0 & -2\end{pmatrix}$

(vi) $u=\begin{pmatrix}1 & 2 & -3\\ 6 & -5 & 4\end{pmatrix}$, $v=\begin{pmatrix}6 & -5 & 4\\ 1 & 2 & -3\end{pmatrix}$

(vii) $u=2-5t+6t^2-t^3$, $v=3+2t-4t^2+5t^3$

(viii) $u=1-3t+2t^2-3t^3$, $v=-3+9t-6t^2+9t^3$.

Soln Two vectors u and v are dependent iff one is a multiple of the other.

(i) No. (ii) No. (iii) Yes, for $v=3u$.

(iv) Yes, for $u=-2v$. (v) Yes, for $v=2u$.

(vi) No. (vii) No, (viii) Yes, for $v=-3u$.

Lemma. The non-zero vectors $v_1, v_2, \ldots, v_m$ are linearly dependent if and only if one of them, say v_i, is a linear combination of the preceding vectors:

$$v_i = a_1 v_1 + a_2 v_2 + \cdots + a_{i-1} v_{i-1}.$$

Proof. Suppose $v_i = a_1 v_1 + a_2 v_2 + \cdots + a_{i-1} v_{i-1}$.

Then $a_1 v_1 + a_2 v_2 + \cdots + a_{i-1} v_{i-1} - v_i + 0 v_{i+1} + \cdots + 0 v_m = 0$ and the coefficient of v_i is not 0. Hence the v_i are linearly dependent.

Conversely, suppose the v_i are linearly dependent. Then there exists scalars $a_1, \ldots, a_m$, not all 0 such that

$$a_1 v_1 + a_2 v_2 + \cdots + a_m v_m = 0.$$

Let k be the largest integer such that $a_k \neq 0$

Then $a_1 v_1 + \cdots + a_k v_k + 0 v_{k+1} + \cdots + 0 v_m = 0$

ie, $a_1 v_1 + \cdots + a_k v_k = 0$.

Suppose $k = 1$, then $a_1 v_1 = 0$, $a_1 \neq 0$ and so $v_1 = 0$. But v_i are non-zero vectors, hence $k > 1$ and

$$v_k = -a_k^{-1} a_1 v_1 - \cdots - a_k^{-1} a_{k-1} v_{k-1}$$

Thus ie v_k is a linear combination of the preceding vectors.

Suppose $\{v_1, \ldots, v_m\}$ generates a vector space V. Prove (i) If $w \in V$, then $\{w, v_1, v_2, \ldots, v_m\}$ is linearly dependent and generates V.

(ii) If v_i is a linear combination of the preceding vectors, then $\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_m\}$ generates V.

Proof. (i) If $w \in V$, then $w = a_1 v_1 + \cdots + a_m v_m$, where ~~am~~ $a_1, a_2, \ldots, a_m$ are scalars, since $\{v_1, v_2, \ldots, v_m\}$ generates V.

~~Thus~~ $-w + a_1 v_1 + \cdots + a_m v_m = 0$,

Hence $\{w, v_1, \ldots, v_m\}$ is linearly dependent and clearly $\{w, v_1, \ldots, v_m\}$ generates V.

(ii) Suppose v_i is a linear combination of the preceding vectors, then

$$v_i = k_1 v_1 + \cdots + k_{i-1} v_{i-1}.$$

where $k_1, k_2, \ldots, k_{i-1}$ are scalars.

Again, since $\{v_i\}$ generates V, then for $u \in V$, $u = a_1 v_1 + \cdots + a_m v_m$, where $a_1, a_2, \ldots a_m$ are scalars.

$$\therefore u = a_1 v_1 + \quad + a_i (k_1 v_1 + \cdots + k_{i-1} v_{i-1}) + a_{i+1} v_{i+1} + \cdots + a_m v_m$$

$$= (a_1 + a_i k_1) v_1 + (a_2 + a_i k_2) v_2 + \cdots$$

$$+ (a_{i-1} + a_i k_{i-1}) v_{i-1} + a_{i+1} v_{i+1} + a_m v_m$$

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Thus $\{v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_m\}$ generates V.

Lemma Suppose $\{v_1, \dots, v_n\}$ generates a vector space V. If $\{w_1, \dots, w_m\}$ is linearly independent, then $m \leq n$ and V is generated by a set of the form $\{w_1, \dots, w_m, v_{i_1}, \dots, v_{i_{n-m}}\}$. Thus in particular, any $n+1$ or more vectors in V are linearly dependent.

Basis and Dimension

Definition. A vector space V is said to be of finite dimension n or to be n-dimensional, written $\dim V = n$, if there exists linearly independent vectors $e_1, e_2, \dots, e_n$ which span V. The sequence $\{e_1, e_2, \dots, e_n\}$ is called a basis of V.

Ex-1 Let K be any field. Consider the vector space K^n which consists of n-tuples of elements of K. Then the vectors

$$e_1 = (1, 0, 0, \dots, 0)$$
$$e_2 = (0, 1, 0, \dots, 0, 0)$$

$e_n = (0, 0, 0, \cdots, 0, 1)$

form a basis, called the usual basis of K^n. Thus K^n has dimension n.

Ex-2 Let V be ~~the set~~ a vector space of all 2×2 matrices. Then ~~V is a vector space~~ the vectors $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,

$e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

form a basis. Thus $\dim V = 4$.

* The vector space $\{0\}$ is defined to have dimension 0.

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Theorem. Suppose S generates V and $\{v_1, \cdots, v_m\}$ is a maximal independent subset of S. Then $\{v_1, \cdots, v_m\}$ is a basis of V.

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Theorem. Let V be of finite dimension n. Then (i) Any set of $n+1$ or more vectors is linearly dependent.

(ii) Any linearly independent set is part of a basis, i.e. can be extended to a basis.

(iii) A linearly ~~idep~~ independent set with n elements is a basis.

Ex-1 The four vectors in K^4:

$(1,1,1,1), (0,1,1,1), (0,0,1,1), (0,0,0,1)$

are linearly independent, since they form a matrix echelon form.

Since $\dim K^4 = 4$, they form a basis of K^4.

Ex-2. The four vectors in $\mathbb{R}^3$,

$(7,-3,8), (4,0,-1), (1,-7,4)$ and

$(3,-5,-3)$ must be linearly dependent since they come from a vector space of dimension 3.

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Theorem. Let W be a subspace of an n-dimensional vector space V. Then $\dim W \leq n$. In particular, if $\dim W = n$, then $W = V$.

~~**Ex:** Let W be a subspace of an n-dimensional vector space V. Then $\dim W \leq$~~

Ex: Let W be a subspace of the real space $\mathbb{R}^3$. ~~Now~~ Then $\dim \mathbb{R}^3 = 3$. Hence by the preceding theorem, the dimension of W can only be $0, 1, 2$ or 3. The following cases apply:

(i) $\dim W = 0$, then $W = \{0\}$, a point.

(ii) $\dim W = 1$, then W is a line through the origin.

(iii) $\dim W = 2$, then W is a plane through the origin

(iv) $\dim V = 3$, then W is the entire space $\mathbb{R}^3$.

Theorem Let U and W be finite-dimensional subspaces of a vector space V. Then

(i) $U+W$ has finite dimension,

(ii) $\dim(U+W) = \dim U + \dim W - \dim(U\cap W)$

Proof. Since U and W are finite dimensional subspaces of V. Then clearly $U+W$ is finite dimensional.

Observe that $U\cap W$ is a subspace of both U and W. Suppose $\dim U = m$, $\dim W = n$ and $\dim(U\cap W) = r$.

Suppose $\{v_1, v_2, \ldots, v_r\}$ is a basis of $U\cap W$.

Then we can extend $\{v_i\}$ to a basis of U and to a basis of W: say

$\{v_1, v_2, \ldots, v_r, u_1, \ldots, u_{m-r}\}$

and $\{v_1, v_2, \ldots, v_r, w_1, \ldots, w_{n-r}\}$

are basis of U and W respectively.

Let $B = \{v_1, \ldots, v_r, u_1, \ldots, u_{m-r}, w_1, \ldots, w_{n-r}\}$.

Then B has exactly $m+n-r$ elements.

Thus the theorem is proved if we can show that B is a basis of $U+W$.

Since $\{v_i, u_j\}$ generates U and $\{v_i, w_k\}$ generates W, the union $B = \{v_i, u_j, w_k\}$ generates $U+W$. Thus it suffices to show that B is independent.

Suppose

$$a_1 v_1 + \cdots + a_r v_r + b_1 u_1 + \cdots + b_{m-r} u_{m-r} + c_1 w_1 + \cdots + c_{n-r} w_{n-r} = 0 \quad \text{— (1)}$$

where a_i, b_j, c_k are scalars.

Let $v = a_1 v_1 + \cdots + a_r v_r + b_1 u_1 + \cdots + b_{m-r} u_{m-r}$. — (2)

From (1), we have

$$v = -c_1 w_1 - \cdots - c_{n-r} w_{n-r} \quad \text{— (3)}$$

Since $\{v_i, u_j\}$ generates U i.e. $\{v_i, u_j\} \subset U$, then by (2) $v \in U$. and since $\{w_k\} \subset W$, by (3), $v \in W$. Thus $v \in U \cap W$.

Now, $\{v_i\}$ is a basis of $U \cap W$ and so there exists scalar $d_1, \ldots, d_r$ for which $v = d_1 v_1 + \cdots + d_r v_r$.

Hence from (3), we have

$$d_1 v_1 + \cdots + d_r v_r + c_1 w_1 + \cdots + c_{n-r} w_{n-r} = 0$$

But $\{v_i, w_k\}$ is a basis of W and so is independent. Hence $c_1 = 0, c_2 = 0, \ldots, c_{n-r} = 0$.

Substituting these values in (1), we have

$a_1v_1 + \cdots + a_rv_r + b_1u_1 + \cdots + b_{m-r}u_{m-r} = 0$

Since $\{v_i, u_j\}$ is a basis of U, so is independent. Hence $a_1 = 0, \ldots, a_r = 0,$ $b_1 = 0, \ldots, b_{m-r} = 0$.

Since the equation (1) implies that the ~~con~~ scalars a_i, b_j, c_k are all zero, therefore, $B = \{v_i, u_j, w_k\}$ is linearly independent.

Thus $\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$

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★ Let U and W be the following subspace of $\mathbb{R}^4$: $U = \{(a,b,c,d) : b+c+d = 0\}$,

$W = \{(a,b,c,d) ; a+b = 0, \ c = 2d\}$.

Find the dimension and a basis of (i) U, (ii) W, (iii) $U \cap W$.

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(i) We seek a basis of the set of solutions (a,b,c,d) of the equation

$$b+c+d = 0$$

or

$$0 \cdot a + b + c + d = 0$$

The free variables are a, c and d.

set: (i) $a=1,\ c=0,\ d=0,$

(ii) $a=0,\ c=1,\ d=0$

(iii) $a=0,\ c=0,\ d=1$

to obtain the respective solutions

$v_1=(1,0,0,0)$, $v_2=(0,-1,1,0)$

$v_3=(0,-1,0,1)$.

The set $\{v_1,v_2,v_3\}$ is a basis of U and

$\dim U=3$.

(ii) We seek a basis of the of solutions

(a,b,c,d) of the system

$a+b=0$ i.e. $a+b=0$

$c=2d$ $\quad c-2d=0$

The free variables are b, d.

set: (i) $b=0,\ d=1$; (ii) $b=1,\ d=0$.

to obtain the respective solutions

$v_1=(0,0,2,1)$, $v_2=(-1,1,0,0)$.

Thus the set $\{v_1,v_2\}$ is a basis of W

and $\dim W=2$.

(iii) $U\cap W$ consists of those vectors (a,b,c,d) which satisfy the three equations

$b+c+d=0$ $a+b\quad=0$ $c-2d=0$	Here the free variable d. set $d=1$, to obtain the solution $v=(3,-3,2,1)$.

Thus $\{v\}$ is a basis of $U\cap W$ and $\dim U\cap W=1$.

1. Let V be the vector space of 2×2 symmetric matrices over K. Show that dim V = 3.

Soln. An arbitrary 2×2 symmetric matrices is of the form $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, where $a, b, c \in K$.

Setting: (i) $a = 1, b = 0, c = 0$

(ii) $a = 0, b = 1, c = 0$

(iii) $a = 0, b = 0, c = 1$

We obtain the respective matrices

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

We have to show that $\{A_1, A_2, A_3\}$ is a ~~basis~~ basis of V, ie, (i) $\{A_i\}$ generates V and (ii) is independent.

(i) For the above arbitrary matrix A in V, we have

~~$A = a$~~

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

ie $A = aA_1 + bA_2 + cA_3$.

Thus $\{A_1, A_2, A_3\}$ generates V.

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(ii) Suppose $xA_1 + yA_2 + zA_3 = 0$, where x, y, z are scalars. Then

$$x\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + z\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x & y \\ y & z \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

ie $x = 0, y = 0, z = 0$.

ie $xA_1 + yA_2 + zA_3 = 0$ implies

$x = 0, y = 0, z = 0$.

Hence $\{A_1, A_2, A_3\}$ ~~are~~ is independent.

Thus $\{A_1, A_2, A_3\}$ ~~are~~ is a basis of V

and $\dim V = 3$.

Coordinates

Let $\{e_1, \ldots, e_n\}$ be a basis of an n-dimensional vector space V over a field K, and let v be any vector in V. Since $\{e_i\}$ generates V, v is a linear combination of the e_i's. ie

$$v = a_1e_1 + \ldots + a_ne_n, \quad a_i \in K.$$

Since the e_i are independent, such a representation is unique. ie, the n scalars $a_1, a_2, \ldots, a_n$ are completely determined by the vector v and the basis $\{e_i\}$. We

These scalars the coordinates of v in $\{e_i\}$ and we call the n-tuple $(a_1, \ldots, a_n)$ the coordinates vector of v relative to $\{e_i\}$ and denote it by $[v]_e$ ~~or~~ or $[v]$.

ie $[v]_e = (a_1, a_2, \ldots, a_n)$.

Linear Mapping or Linear Transformation.

Definitions. Let X, Y and Z be sets. A function $f: X \longrightarrow Y$ involves a rule which associates to each $x \in X$, a unique element $f(x) \in Y$. X is called the domain of f, denoted by $\operatorname{dom} f$, while Y is called the codomain of f, denoted by $\operatorname{codom} f$. ~~Sometimes.~~ $f(X)$ is called the range of f, denoted by $\operatorname{Im} f$.

Given functions $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$, we can form their composition $g \circ f: X \longrightarrow Z$ which has the rule

$$(g \circ f)(x) = g(f(x)).$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left, "g\circ f"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

Ex! ① For any set X, the identity function $f: X \longrightarrow Y$ ~~d~~ has rule $f(x) = x$.

Ex-2. Let a function $f: \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = x+1, \quad f(x) = \frac{1}{x^2+1},$$

$$f(x) = \sin x, \quad f(x) = e^{5x^3}.$$

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Ex 3. Let $X = Y = \mathbb{R}^+$, the set of all positive real numbers. Then the rules define two functions

$$f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$$

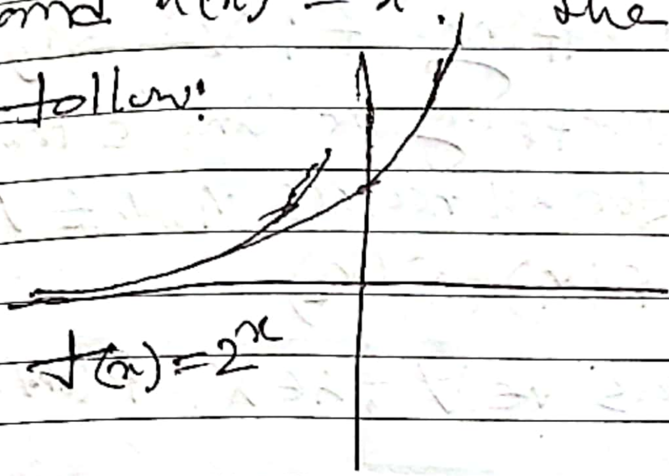
by $f(x) = \sqrt{x}$ & $f(x) = -\sqrt{x}$.

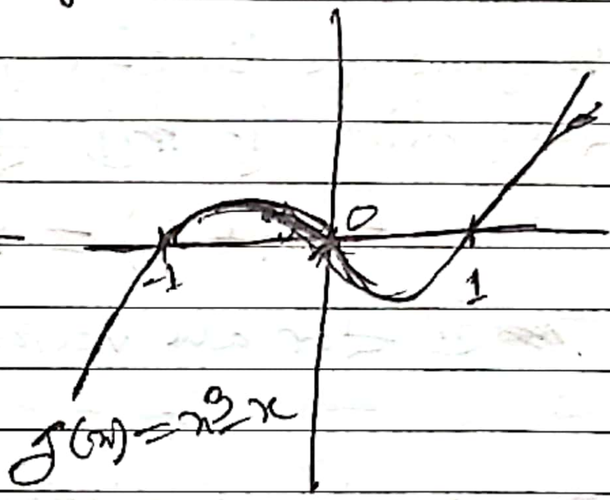
Definition. A function $f : X \longrightarrow Y$ is injective (one-to-one) if for $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

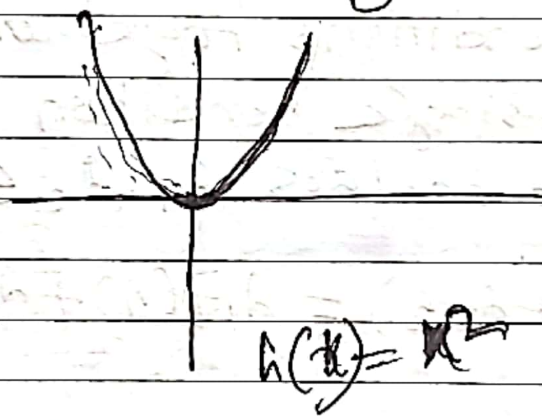
A function $f : X \longrightarrow Y$ is surjective (onto) if for every $y \in Y$, there is an $x \in X$ for which $y = f(x)$.

A function $f : X \longrightarrow Y$ is bijective (one-one and onto) if it is both injective and surjective.

Ex-1 Let $f: R \to R$, $g: R \to R$ and $h: R \to R$ be defined by $f(x) = 2^x$, $g(x) = x^3 - x$ and $h(x) = x^2$. The graphs of these mapping follow:







The mapping f is one-one, geometrically, this means that each horizontal line does not contain more than one point of f. The mapping g is onto, geometrically, this means that each horizontal line contains at least one point of g. The mapping h is neither one-one nor onto for example, 2 and -2 have the same image 4.

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Linear Mapping.

Let V and U be vector spaces over the same field K. A mapping $f: V \longrightarrow U$ is called a linear mapping or linear transformation if (i) for all vectors $v_1, v_2 \in V$,

$$f(v_1+v_2) = f(v_1)+f(v_2)$$

(ii) for all vectors $v \in V$, $\lambda \in K$, $f(\lambda v) = \lambda f(v)$

These two conditions together are equivalent to the single condition for all vectors $v_1, v_2 \in V$ and scalars $\lambda_1, \lambda_2 \in K$,

$$f(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 f(v_1) + \lambda_2 f(v_2).$$

Remark: For a linear transformation $f: V \longrightarrow U$, we always have $f(0)=0$, where the left hand 0 means the zero vector in V and the right 0 means the zero vectors in U.

Ex-1 Let A be any $m \times n$ matrix over a field K. Define $f_A: K^n \longrightarrow K^m$ by the rule $f_A(x) = Ax$.

Then for $u, w \in K^n$, $k \in K$,

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$F_A(v+w) = A(v+w) = Av + Aw = F_A(v) + F_A(w)$.

and $F_A(kv) = A(kv) = k(Av) = k\,F_A(v)$

Thus F_A is a linear transformation.

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Ex-2 Let $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the projection mapping into the xy-plane defined by

$f(x, y, z) = (x, y, 0)$. Then for any

$v = (a, b, c)$, $w = (a', b', c')$, we have

$$f(v+w) = f(v) + f(w).$$

$$= f(a,b,c) + f(a',b',c')$$

$$= (a,b,0) + (a',b',0)$$

$$= (a+a', b+b', 0+0)$$

$$= f((a,b,c) + (a',b',c'))$$

$$= f(a+a', b+b', c+c')$$

$$= (a+a', b+b', 0)$$

$$= (a,b,0) + (a',b',0)$$

$$= f(v) + f(w)$$

and for any $k \in \mathbb{R}$,

$$f(kv) = f(ka, kb, kc) = (ka, kb, 0)$$

$$= k(a, b, 0) = k f(v).$$

Hence f is linear.

Ex. Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping defined by $F(x,y) = (x+2, y+5)$. Then we have

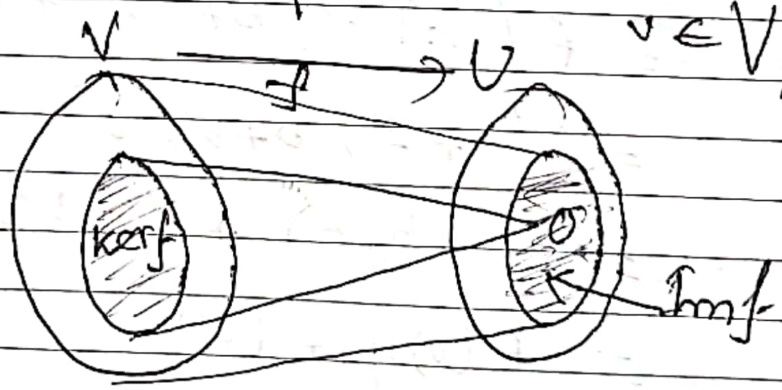
$$F(0) = F(0,0) = (0+2, 0+5) = (2,5) \neq (0,0)$$

ie, the zero vector is not mapped onto the zero vector. Hence F is not linear.

Definition. Let $f: V \to U$ be a linear transformation. Then the kernel (or null space) of f is defined by

$$\ker f = \{v \in V : f(v) = 0\}.$$

The image (or range) of f is defined by $\operatorname{Im} f = \{u \in U : u = f(v), \text{ for some } v \in V\}$.



Theorem. Let $f: V \longrightarrow U$ be a linear transformation. Then

(a) $\ker f$ is a subspace of V,

(b) $\operatorname{Im} f$ is a subspace of U.

Proof. (a) Let $v_1, v_2 \in \ker f$, then $f(v_1) = 0$ and $f(v_2) = 0$. Then

$$f(v_1+v_2) = f(v_1)+f(v_2) = 0+0$$
$$= 0$$

$\therefore v_1+v_2 \in \ker f$.

(i) $\ker f$ is nonempty, since $f(0) = \bar{0}$ ie $0 \in \ker f$.

(iii) Let $\lambda \in K$ and $v \in V$, then

$f(\lambda v) = \lambda f(v) = \lambda . 0 = 0$

ie $\lambda v \in \ker f$.

Hence $\ker f$ is a subspace of V.

(b) We have to prove that $\operatorname{Im} f$ is a subspace of U. Then (i), $\operatorname{Im} f$ is non-empty, since $f(0) = \bar{0}$ ie $\bar{0} \in \operatorname{Im} f$.

(ii) Let $u_1, u_2 \in U$, such that $u_1 = f(v_1)$ and $u_2 = f(v_2)$ for some $v_1, v_2 \in V$.

Then $u_1+u_2 = f(v_1)+f(v_2) = f(v_1+v_2)$

Thus $u_1+u_2 \in \mathrm{Im}\,F$.

(iii) $u \in$ $\lambda \in K$, $u \in \mathrm{Im}\,F$, Then

$$\lambda u = \lambda F(v), \text{ for some } v \in V$$
$$= F(\lambda v)$$

i.e. $\lambda u \in \mathrm{Im}\,F$.

Hence $\mathrm{Im}\,F$ is a subspace of U.

Singular and Non-singular Mapping:

A linear mapping $F: V \to U$ is said to be singular if there exists $v \in V$ for which $v \neq 0$ but $F(v) = 0$.

A linear mapping $F: V \to U$ is said to be non-singular if only $0 \in V$, $F(0) = \bar{0}$, $\bar{0} \in U$. i.e. $\ker F = \{0\}$.

Theorem. A linear mapping $F: V \to U$ is an isomorphism if and only if F is non-singular.

Proof. Suppose F is an isomorphism, We have to show that F is non-singular.

Since F is isomorphism, so F is one-one and hence ~~each~~ only $0 \in V$ can be mapped into $0 \in U$. Thus F is non-singular.

Conversely, Let F be non-singular.
ie $F(v) = 0$ only when $v = 0$. Then we have to show that F is isomorphism ie F is one-one.

Let $F(v_1) = F(v_2)$

$\Rightarrow F(v_1) - F(v_2) = 0$

$\Rightarrow F(v_1 - v_2) = 0$

$\because F$ is non-singular.

$\Rightarrow v_1 - v_2 = 0$ ie $v_1 = v_2$.

ie $F(v_1) = F(v_2) \Rightarrow v_1 = v_2$

hence F is one-one.

therefore F is isomorphism.

Problem. Let V be the vector space of 2×2 matrices over $\mathbb{R}$ and let $M = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}$.
Let $F : V \longrightarrow V$ be a linear mapping defined by $F(A) = MA$. Find ~~the~~ basis and the dimension of (i) kernel of F and (ii) Image of F.

Soln. We seek the solution set $\begin{pmatrix} x & y \\ s & t \end{pmatrix}$ such that $F\begin{pmatrix} x & y \\ s & t \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Then $\begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} x & y \\ s & t \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} x-s & y-t \\ -2x+2s & -2y+2t \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow$
$$x-s=0$$
$$y-t=0$$
$$-2x+2s=0$$
$$-2y+2t=0$$

or
$$x-s=0$$
$$y-t=0$$

Here s and t are two free variable hence ~~dim V = 2~~. dim (ker F) = 2.

for a basis, let $s=0$ and $t=1$ to obtain the solution $x=0, y=1, s=0, t=1$.

$\therefore$ the matrix becomes $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$.

Also, let $s=1, t=0$, to obtain the solution set $x=1, y=0, s=1, t=0$

$\therefore$ the matrix becomes $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$.

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$\therefore$ Basis of the kernel $F = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}$

For the basis and dimension of $\operatorname{Im} F$.

Consider the usual basis of 2×2 spaces as

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ e_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$F(e_1) = F\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 0 \end{pmatrix} = u_1$$

$$F(e_2) = F\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix} = u_2$$

$$F(e_3) = F\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 2 & 0 \end{pmatrix} = u_3$$

$$F(e_4) = \begin{pmatrix} 0 & -1 \\ 0 & 2 \end{pmatrix} = u_4.$$

But clearly, $u_3 = \begin{pmatrix} -1 & 0 \\ 2 & 0 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ -2 & 0 \end{pmatrix}$

$= -u_1$

and $u_4 = \begin{pmatrix} 0 & -1 \\ 0 & 2 \end{pmatrix} = -\begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix} = -u_2$

Infact u_3 and u_4 are two linearly independent vectors. So $\dim(\operatorname{Im} F) = 2$

and Basis of $\operatorname{Im} f = \left\{ \begin{pmatrix} -1 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 2 \end{pmatrix} \right\}$.

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Theorem. Let V be of finite dimension and let $F: V \to U$ be a linear mapping. Then

$$\dim V = \dim(\ker F) + \dim(\operatorname{Im} F)$$

Proof Since $\dim V$ is finite, and $\ker F$ is a subspace, $\dim(\ker F)$ is also finite. Let $\dim V = n$ and $\dim(\ker F) = r \leq n$.

Thus we have to prove that ~~dim~~

$$\dim(\operatorname{Im} F) = n - r.$$

Let $\{w_1, w_2, \ldots, w_r\}$ be a basis of ~~R~~ $\ker F$. Extend the basis of $\ker F$ to a basis of V:

i.e. $\{w_1, \ldots, w_r, v_1, \ldots, v_{n-r}\}$ is a basis of V.

Let $B = \{F(v_1), F(v_2), \ldots, F(v_{n-r})\}$

We want that B is the basis of $\operatorname{Im} F$.

(i) First we have to ~~prove~~ show that B generate the space $\operatorname{Im} F$.

Let $u \in \operatorname{Im} F$. Then $u = F(v)$, for some $v \in V$. $\Rightarrow u = F(a_1 w_1 + a_2 w_2 + \cdots + a_r w_r + b_1 v_1 + \cdots + b_{n-r} v_{n-r})$

$$= a_1 F(w_1) + \cdots + a_r F(w_r) + b_1 F(v_1) + \cdots + b_{n-r} F(v_{n-r}).$$

$= a_1 . 0 + \dots + a_r . 0 + b_1 f(v_1) + \dots + b_{n-r} f(v_{n-r})$

$= b_1 f(v_1) + \dots + b_{n-r} f(v_{n-r})$.

Thus u is a linear combination of $f(v_1)$, $f_2(v_2), \dots, f(v_{n-r})$.

Therefore $f(v_1), \dots, f(v_{n-r})$ generates $\operatorname{Im} f$.

(ii) Now, we prove that $f(v_1), \dots, f(v_{n-r})$ are linearly independent.

Suppose

$$a_1 f(v_1) + \dots + a_{n-r} f(v_{n-r}) = 0$$

$$\therefore f(a_1 v_1 + \dots + a_{n-r} v_{n-r}) = 0$$

$$\Rightarrow a_1 v_1 + \dots + a_{n-r} v_{n-r} \in \ker f.$$

Since $\{w_i\}$ generates $\ker f$, there exists scalars $b_1, b_2, \dots, b_r$ such that

$$a_1 v_1 + \dots + a_{n-r} v_{n-r} = b_1 w_1 + b_2 w_2 + \dots + b_r w_r$$

$$\Rightarrow a_1 v_1 + \dots + a_{n-r} v_{n-r} - b_1 w_1 - \dots - b_r w_r = 0$$

Since $\{v_1, v_2, \dots, v_{n-r}, w_1, \dots, w_r\}$ is a basis of V, so it is ~~leaner~~ linearly independent, and hence

$$a_1 = 0, \dots, a_{n-r} = 0, b_1 = 0, \dots, b_r = 0.$$

ie, $a_1 = 0$, $a_2 = 0$, ..., $a_{n-r} = 0$

ie $F(v_1), F(v_2), \ldots, F(v_{n-r})$ are linearly independent.

Thus $\{F(v_1), \ldots, F(v_{n-r})\}$ is a basis of $\operatorname{Im} F$.

Therefore $\dim(\operatorname{Im} F) = n - r$.

$\Rightarrow \dim(\operatorname{Im} F) = \dim V - \dim(\ker F)$

ie $\dim V = \dim(\ker F) + \dim(\operatorname{Im} F)$.

* Find a basis and dimension of (i) $\operatorname{Im} F$ and (ii) $\ker F$, where

(1) $F: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by

$$F(x, y, z) = (x + 2y,\ y - z,\ x + 2z).$$

soln (i) If $\{v_1, \ldots, v_n\}$ generates V, then $\{F(v_1), \ldots, F(v_n)\}$ generates $\operatorname{Im} F$.

Consider the usual basis of $\mathbb{R}^3$,

$$\{e_1 = (1, 0, 0),\ e_2 = (0, 1, 0),\ e_3 = (0, 0, 1)\}$$

$\therefore \{f(e_1), f(e_2), f(e_3)\}$ generates $\operatorname{Im} f$.

$\therefore f(e_1) = f(1,0,0) = (1,0,1)$

$f(e_2) = f(0,1,0) = (2,1,0)$

$f(e_3) = f(0,0,1) = (0,-1,2)$.

Thus $\{(1,0,1), (2,1,0), (0,-1,2)\}$ generates $\operatorname{Im} f$.

Now, $\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$

$\therefore \{(1,0,1), (0,1,-2)\}$ is a basis of $\operatorname{Im} f$.

Thus $\dim(\operatorname{Im} f) = 2$.

(ii) For the kernel of f,

We have $\dim(\mathbb{R}^3) = \dim(\ker f) + \dim(\operatorname{Im} f)$

$\Rightarrow \dim(\ker f) = \dim(\mathbb{R}^3) - \dim(\operatorname{Im} f)$

$= 3 - 2 = 1$.

For any vector $(x, y, z) \in \ker f$, we have

$f(x, y, z) = (0,0,0)$

$$\Rightarrow (x+2y,\ y-z,\ x+2z) = (0,0,0)$$

i.e.
$$x+2y = 0$$
$$y-z = 0$$
$$x \quad +2z = 0$$

or
$$x+2y = 0$$
$$y-z = 0$$
$$2y-2z = 0$$

or
$$x+2y = 0$$
$$y-z = 0$$

~~Here the free variable is one, say z.~~ Let z be the free variable

Let $z=1$, then $y=1$, $x=-2$.

Therefore

$\{(-2,1,1)\}$ is a basis of $\ker F$.

Problem. Let V be the vector space of $n \times n$ matrices over $\mathbb{R}$ and let M be an arbitrary matrix in V. Show that $F: V \to V$ is linear when F is defined as

(i) $T(A) = MA$ (ii) $T(A) = MA - AM$.

Sol^n (a) $T(A+B) = M(A+B) = MA + MB$

$= T(A) + T(B)$.

(b) $T(\lambda A) = M(\lambda A) = \lambda(MA) = \lambda T(A)$

$\therefore$ T is linear.

(ii) (a) $T(A+B) = M(A+B) - (A+B)M$

$= MA + MB - AM - BM$

$= MA - AM + MB - BM$

$= T(A) + T(B)$.

(b) $T(\lambda A) = M(\lambda A) - (\lambda A)M$

$= \lambda(MA - AM)$

$= \lambda T(A)$

$\therefore$ T is linear

Theorem. Let V and U are vector spaces over a field K. Let $\{v_1, \ldots, v_n\}$ be a basis of V and $u_1, \ldots, u_n$ be any arbitrary vectors in U. Then there exists a unique linear mapping $f: V \longrightarrow U$ such that $f(v_1) = u_1$, $f(v_2) = u_2, \ldots, f(v_n) = u_n$.

Proof. $\boxed{\text{Step-1}}$ Let $v \in V$, then

$$v = a_1 v_1 + \cdots + a_n v_n,$$

since $\{v_1, \ldots, v_n\}$ is a basis of V. Since the above expression is unique, so $a_1, \ldots, a_n \in K$ are unique.

We define $f: V \longrightarrow U$ by

$$f(v) = a_1 u_1 + \cdots + a_n u_n.$$

Now, $v_1 = 1.v_1 + 0.v_2 + \cdots + 0.v_n$

$$f(v_1) = 1.u_1 + 0.u_2 + \cdots + 0.u_n = u_1$$

Similarly, $v_2 = 0v_1 + 1.v_2 + 0.v_3 + \cdots + 0.v_n$

$$\therefore f(v_2) = 0.u_1 + 1.u_2 + \cdots + 0.u_n = u_2$$

$\vdots$

$$f(v_n) = u_n.$$

$\boxed{\text{Step-2}}$ Let $v, w \in V$. Suppose

$$v = a_1 v_1 + \cdots + a_n v_n$$

and $w = b_1 v_1 + \cdots + b_n v_n$

$\therefore\ v+w = (a_1+b_1)v_1 + \dots + (a_n+b_n)v_n$

Then $F(v+w) = (a_1+b_1)u_1 + \dots + (a_n+b_n)u_n$

$= (a_1u_1 + \dots + a_nu_n) + (b_1u_1 + \dots + b_nu_n)$

$= F(a_1v_1 + \dots + a_nv_n) + F(b_1v_1 + \dots + b_nv_n)$

$= F(v) + F(w)$.

And $\lambda v = \lambda a_1v_1 + \lambda a_2v_2 + \dots + \lambda a_nv_n$

$= \lambda(a_1v_1 + \dots + a_nv_n)$

$\therefore F(\lambda v) = \lambda(a_1u_1 + \dots + a_nu_n)$

$= \lambda F(v)$

$\therefore$ F is linear,

Step-3. Suppose $G: V \to U$ is another linear and $G(v_i) = u_i$, $i = 1, 2, \dots, n$.

If $v = a_1v_1 + \dots + a_nv_n$ then

$G(v) = G(a_1v_1 + \dots + a_nv_n)$

$= a_1G(v_1) + \dots + a_nG(v_n)$

$= a_1u_1 + \dots + a_nu_n$

$= F(v)$

$\therefore G(v) = F(v)\ \forall\ v \in V$

Hence $G = F$.

Therefore F is unique