PART : A

Vector

$\bar{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$|\bar{A}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Unit vector, $\hat{n} = \dfrac{\bar{A}}{|\bar{A}|} = \dfrac{a_1\hat{i} + a_2\hat{j} + a_3\hat{k}}{\sqrt{a_1^2 + a_2^2 + a_3^2}}$

#Show that, $a - 2b + 3c$, $2a + 3b - 4c$, $-7b + 10c$ are colinear.

Solve: Let, $A = a - 2b + 3c$

$B = 2a + 3b - 4c$

$C = -7b + 10c$

collinear – সব বিন্দু একই সরলরেখায় অবস্থান করে।

$AB = (2a + 3b - 4c) - (a - 2b + 3c) = a + 5b - 7c$

$BC = (-7b + 10c) - (2a + 3b - 4c) = -2a - 10b + 14c$

$= -2(a + 5b - 7c) = -2AB$

$-AC = (7b + 10c) - (a - 2b + 3c) = -a - 5b + 7c$

$= -(a + 5b - 7c)$

$= -AB$

Thus, the vector AB and BC are collinear.

Vector product:

1. Dot product / scalar product
2. Cross product / vector product

$\bar{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\bar{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

(i) Dot product: $\vec{A}\cdot\vec{B} = |\vec{A}|.|\vec{B}|\cos\theta$

(ii) Cross product: $\vec{A}\times\vec{B} = |\vec{A}||\vec{B}|\sin\theta\,\hat{n}$

$= |A||B|\sin\theta$

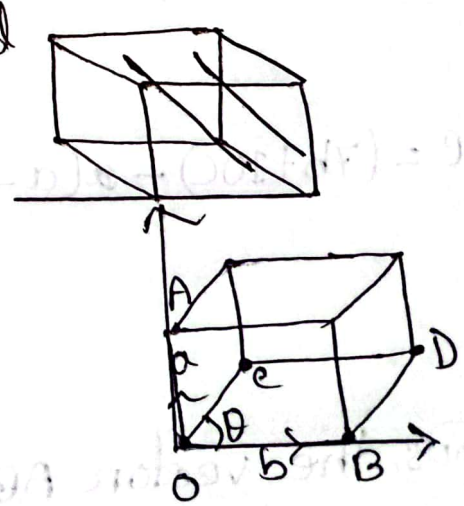
Show that $\bar{a}.(\bar{b}\times\bar{c})$ volume of the parallelopiped.

Solve: Geometrically the scalar product of three vector represent the volume of the parallelopiped whose concurrent edges are OA, OB, OC such that

$OA = a$, $OB = b$, $OC = c$

h = height of the parallelopiped

S = area of the parallelogram OBDC



$a.(b\times c) = a.(bc\sin\theta\, n)$

$= abc\sin\theta.\sin\alpha$

$= (a\sin\alpha)(bc\sin\theta)$

$= hS = V$

= Volume of the parallelopiped.

$\cancel{a \cdot c}$

$= \hat{i}\left[(a_1c_1+a_2c_2+a_3c_3)b_1-(a_1b_1+a_2b_2+a_3b_3)c_1\right]+\hat{j}\left[(a_1c_1+a_2c_2+a_3c_3)b_2-(a_1b_1+a_2b_2+a_3b_3)c_2\right]+\hat{k}\left[(a_1c_1+a_2c_2+a_3c_3)b_3-(a_1b_1+a_2b_2+a_3b_3)c_3\right]$

$= \hat{i}\{(a.c)b_1-(a.b)c_1\}+\hat{j}\{(a.c).b_2-(a.b)c_2\}+\hat{k}\{(a.c)b_3-(a.b)c_3\}$

$= (a.c)\{b_1\hat{i}+b_2\hat{j}+b_3\hat{k}\}-(a.b)(c_1\hat{i}+c_2\hat{j}+c_3\hat{k})$

$= (a.c)b-(a.b)c$

$\therefore a\times(b\times c) = (a.c)b-(a.b)c$ (proved)

⇒ $\bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}\cdot\bar{c})\cdot\bar{b} - (\bar{a}\cdot\bar{b})\cdot c$ is called vector tripple product.

⇒ $\bar{a}\cdot(\bar{b}\times\bar{c})$ এবং $\bar{a}.\bar{b}.\bar{c}$ দুইটি একই ।যার parallelopiped এর আয়তন represent করে। একে scalar tripple product বা box product বলে।

Example-14: Find the volume of the parallelopiped whose ~~volume~~ edges are represented by $a = 2i - 3j + 4k$, $b = i + 2j - k$, $c = 3i - j + 2k$.

Solve: We know,

Volume $= \bar{a}\cdot(\bar{b}\times\bar{c})$

$$\therefore (\bar{b}\times\bar{c}) = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$$

$$= i(4-1) - j(2+3) + k(-1-6)$$

$$= 3i - 5j - 7k$$

$$\bar{a}\cdot(\bar{b}\times\bar{c}) = (2i - 3j + 4k)\cdot(3i - 5j - 7k)$$

$$= 6 + 15 - 28 = -7$$

$\therefore V = 7$

The volume of the parallelopiped is 7 unit

Example-31: Prove that, $\bar{a}\times(\bar{b}\times\bar{c})+\bar{b}\times(\bar{c}\times\bar{a})+\bar{c}(\bar{a}\times\bar{b})=0$

$$LHS = \bar{a}\times(\bar{b}\times\bar{c})+\bar{b}(\bar{c}\times\bar{a})+\bar{c}(\bar{a}\times\bar{b})$$

$$=(a\cdot c)b-(ab)\cdot c+(b\cdot a)c-(b\cdot c)a+(c\cdot b)a-(c\cdot a)b$$

$$=(a\cdot c)b-(a\cdot b)\cdot c+(a\cdot b)\cdot c-(b\cdot c)\cdot a+(b\cdot c)a-(a\cdot c)b$$

$[a\cdot b=b\cdot a]$

$$=0$$

$$=RHS$$

$\therefore LHS = RHS$. (proved)

$\Rightarrow$ Prove that $(\bar{a}\times\bar{b})\times(\bar{b}\times\bar{c})\cdot(\bar{c}\times\bar{a})=(\bar{a}\times\bar{b}\cdot\bar{c})^2$

Solve:

$$LHS = (\bar{a}\times\bar{b})\times(\bar{b}\times c)\cdot(\bar{c}\times\bar{a})$$

Let, $\bar{a}\times\bar{b}=\bar{m}$

$$=\bar{m}\times(\bar{b}\times\bar{c})\cdot(\bar{c}\times\bar{a})$$

$$=\{(\bar{m}\cdot\bar{c})\cdot b-(\bar{m}\cdot\bar{b})\cdot\bar{c}\}\cdot(\bar{c}\times\bar{a})$$

$$=[\{(\bar{a}\times\bar{b})\cdot\bar{c}\}\cdot\bar{b}-\{(\bar{a}\times\bar{b})\cdot\bar{b}\}\cdot\bar{c}]\cdot(\bar{c}\times\bar{a})$$

$$=\{(\bar{a}\cdot\bar{b}\cdot\bar{c})\cdot\bar{b}-(\bar{a}\cdot\bar{b}\cdot\bar{b})\cdot\bar{c}\}(\bar{c}\times\bar{a})$$

$$=\{(\bar{a}\cdot\bar{b}\cdot\bar{c})\cdot\bar{b}-0\cdot\bar{c}\}(\bar{c}\times\bar{a})$$

$[\because \bar{a}\cdot\bar{b}\cdot\bar{b}=0]$

$$=(\bar{a}\cdot\bar{b}\cdot\bar{c})\cdot\bar{b}(\bar{c}\times\bar{a})$$

$$=(\bar{a}\cdot\bar{b}\cdot\bar{c})(\bar{a}\cdot\bar{b}\cdot\bar{c})$$

$$=[\bar{a}\cdot\bar{b}\cdot\bar{c}]^2$$

$$=RHS.$$

⇒ If $|\bar{a}+\bar{b}| = |\bar{a}-\bar{b}|$ then show that $\bar{a}$ and $\bar{b}$ are perpendicular

Solve: Given that,

$|\bar{a}+\bar{b}| = |\bar{a}-\bar{b}|$

$\Rightarrow |\bar{a}+\bar{b}|^2 = |\bar{a}-\bar{b}|^2$

$\Rightarrow |\bar{a}+\bar{b}| \cdot |\bar{a}+\bar{b}| = |\bar{a}-\bar{b}| \cdot |\bar{a}-\bar{b}|$

$\Rightarrow |\bar{a}|^2 + \bar{a}\cdot\bar{b} + \bar{b}\cdot\bar{a} + |b|^2 = |a|^2 - \bar{a}\cdot\bar{b} - \bar{b}\cdot\bar{a} + |b|^2$

$\Rightarrow 2\bar{a}\cdot\bar{b} = -2\bar{a}\cdot\bar{b}$

$\Rightarrow 4\bar{a}\cdot\bar{b} = 0$

$\therefore \bar{a}\cdot\bar{b} = 0$

Hence $\bar{a}\cdot\bar{b}$ and $\bar{b}$ are perpendicular.

⇒ Prove that $[\bar{a}\ \bar{b}\cdot\bar{b}] = 0$

Solve:

Let, $a = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$b = b_1\hat{i} + b_2\hat{j} + b_3 k$

$\bar{a}\cdot\bar{b}\cdot\bar{b} = \bar{a}\cdot(\bar{b}\times\bar{b})$

$$[\bar{b}\times\bar{b}] = \begin{vmatrix} i & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$= \hat{i}(b_2b_3 - b_2b_3) - j(b_1b_3 - b_1b_3) + \hat{k}(b_1b_2 - b_1b_2)$

$= i \cdot 0 - j \cdot 0 + k \cdot 0$

$= 0$

$\bar{a} \cdot (\bar{b} \times \bar{b}) = (a_1 \hat{i} + a_2 \hat{j} + a_3 k) \cdot 0$

$= 0$

$\therefore [\bar{a} \cdot \bar{b} \cdot \bar{b}] = 0$ (proved)

Vector operator

The operator $\vec{\nabla}$ is defined as follows:

$\vec{\nabla} = \frac{\delta}{\delta x} \hat{i} + \frac{\delta}{\delta y} \hat{j} + \frac{\delta}{\delta z} k$ is called $\vec{\nabla}$ operator or Nabla.

Giradiant: Let $\varphi(x, y, z)$ be a differentiable scalar function at all points in a region then the gradient of φ is written as grad φ or $\vec{\nabla}\varphi$ is defined by

$$\text{grad } \vec{\nabla}\varphi = \left(\frac{\delta}{\delta x} \hat{i} + \frac{\delta}{\delta y} \hat{j} + \frac{\delta}{\delta z} \hat{k}\right) \varphi$$

$$= \frac{\delta \varphi}{\delta x} \hat{i} + \frac{\delta \varphi}{\delta y} \hat{j} + \frac{\delta \varphi}{\delta z} \hat{k}$$

Divergence: Let $\bar{v}(x,y,z) = v_1(x,y,z)\hat{i} + v_2(x,y,z)\hat{j} + v_3(x,y,z)\hat{k}$ be differentiable vector value function ~~of~~ at all the points in a reason then divergence of $\vec{v}$ is denoted by div $\bar{v}$ or $\vec{\nabla}.\vec{v}$ is defined by:

$$\text{div.}\vec{v} = \vec{\nabla}.\vec{v} = \left\{\left(\frac{\delta}{\delta x}\hat{i}\right) + \left(\frac{\delta}{\delta y}\hat{j}\right) + \left(\frac{\delta}{\delta z}\hat{k}\right)\right\}(v_1\hat{i} + v_2\hat{j} + v_3\hat{k})$$

The scalar $\nabla.v$ is the scalar product of the vectors ∇ and v.

Curl: Let $\bar{v}(x,y,z) = v_1(x,y,z)\hat{i} + v_2(x,y,z)\hat{j} + v_3(x,y,z)\hat{k}$ be differentiable vector value function at all the points in a reason then the curl of $\vec{v}$ is denoted by curl $\vec{v}$ or $\vec{\nabla}\times\vec{v}$ is defined by:

$$\vec{\nabla}\times\vec{v} = \begin{vmatrix} i & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= i\left(\frac{\delta v_3}{\delta y} - \frac{\delta v_2}{\delta z}\right) + \hat{j}\left(\frac{\delta v_1}{\delta z} - \frac{\delta v_3}{\delta x}\right) + \hat{k}\left(\frac{\delta v_2}{\delta x} - \frac{\delta v_1}{\delta y}\right)$$

⇒ Show that, curl grad $\varphi = 0$.

Solve: Let, $\varphi(x, y, z)$ be a scalar function. Then grad

$$\varphi = \hat{i}\frac{\delta\varphi}{\delta x} + \hat{j}\frac{\delta\varphi}{\delta y} + k\frac{\delta\varphi}{\delta z}$$

LHS = curl grad φ

$$= \vec{\nabla} \times (\text{grad } \varphi)$$

$$= \begin{vmatrix} i & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ \frac{\delta\varphi}{\delta x} & \frac{\delta\varphi}{\delta y} & \frac{\delta\varphi}{\delta z} \end{vmatrix}$$

$$= \hat{i}\left(\frac{\delta^2\varphi}{\delta y\,\delta z} - \frac{\delta^2\varphi}{\delta z \cdot \delta y}\right) + \hat{j}\left(\frac{\delta^2\varphi}{\delta z\,\delta x} - \frac{\delta^2\varphi}{\delta x\,\delta z}\right) + k\left(\frac{\delta^2\varphi}{\delta x\,\delta y} - \frac{\delta^2\varphi}{\delta y\,\delta x}\right)$$

$$= i \cdot 0 + \hat{j} \cdot 0 + k \cdot 0 \qquad \left[\text{since, } \frac{\delta^2\varphi}{\delta x\,\delta y} = \frac{\delta^2\varphi}{\delta y\,\delta x}\right]$$

$$= 0$$

Hence, $\vec{\nabla} \times (\text{grad } \varphi) = 0$

$\therefore$ LHS = RHS (showed)

$\Rightarrow$ Show that, $\text{div}(\text{curl}\,\bar{A}) = 0$

Solve: Let $\bar{A} = \bar{A}_1(x,y,z)\hat{i} + \bar{A}_2(x,y,z)\hat{j} + A_3(x,y,z)\hat{k}$ be a vector valued function.

$$\text{curl}\,\bar{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\delta A_3}{\delta y} - \frac{\delta A_2}{\delta z}\right) + \hat{j}\left(\frac{\delta A_1}{\delta z} - \frac{\delta A_3}{\delta x}\right) + \hat{k}\left(\frac{\delta A_2}{\delta x} - \frac{\delta A_1}{\delta y}\right)$$

$$\therefore \text{div.}(\text{curl}\,\bar{A}) = \frac{\delta}{\delta x}\left(\frac{\delta A_3}{\delta y} - \frac{\delta A_2}{\delta z}\right) + \frac{\delta}{\delta y}\left(\frac{\delta A_1}{\delta z} - \frac{\delta A_3}{\delta x}\right) + \frac{\delta}{\delta z}\left(\frac{\delta A_2}{\delta x} - \frac{\delta A_1}{\delta y}\right)$$

$$= \frac{\delta^2 A_3}{\delta x \delta y} - \frac{\delta^2 A_2}{\delta x \delta z} + \frac{\delta^2 A_1}{\delta y \delta z} - \frac{\delta^2 A_3}{\delta x \delta y} + \frac{\delta^2 A_2}{\delta x \delta z} - \frac{\delta^2 A_1}{\delta y \delta z}$$

$$= 0$$

LHS = RHS.

Hence, $\text{div}(\text{curl}\,\bar{A}) = 0$ (showed)

⇒ If vector $\bar{A} = [(axy - z^3), (a-2)x^2, (1-a)xz^2]$ is irrotational, then prove that $a=0$.

Solve: If $\text{curl}\bar{A} = 0$ then the fluid / flow is irrotational.
Since $\vec{A}$ is irrotational, then,

$\text{curl}\vec{A} = 0$

$\Rightarrow \vec{\nabla} \times \bar{A} = 0$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ (axy - z^3) & (a-2)x^2 & (1-a)xz^2 \end{vmatrix} = 0$$

$$\Rightarrow \left[\left\{\frac{\delta}{\delta y}(1-a)xz^2 - \frac{\delta}{\delta z}(a-2)x^2\right\}, \frac{\delta}{\delta z}(axy - z^3) \cancel{(1-a)xz^2} - \frac{\delta}{\delta x}(1-a)xz^2, \frac{\delta}{\delta x}(a-2)x^2 - \frac{\delta}{\delta y}(axy - z^3)\right] = [0, 0, 0]$$

$$\Rightarrow [0-0, -3z^2 - (1-a)z^2, 2x(a-2) - ax] = [0, 0, 0]$$

Now,

$-3z^2 - (1-a)z^2 = 0$

$\Rightarrow -3z^2 - z^2 + a = 0$

$\Rightarrow -3 - 1 + a = 0$

$\Rightarrow -4 + a = 0$

$\therefore a = 4$

Again.

$2x(a-2) - ax = 0$

$\Rightarrow 2ax - 2x - ax = 0$

$\Rightarrow ax - 4x = 0$

$\Rightarrow a - 4 = 0$

$\therefore a = 4$

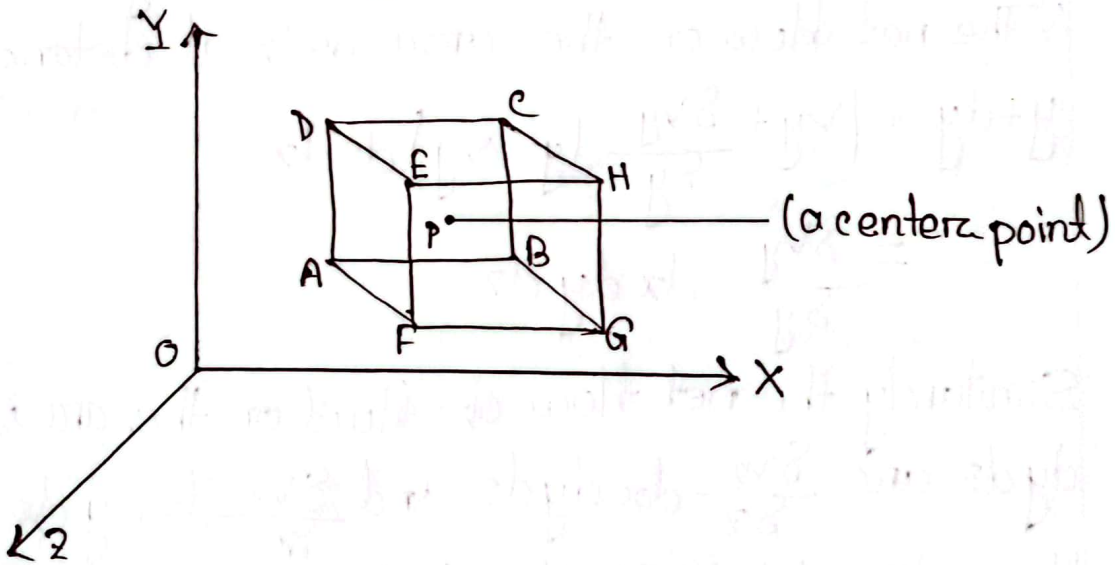
scalar

Vel

scalar

Significance of gradient: Let $\varphi(x, y, z)$ be a vector field which at a given point defines a vector whose direction is that in which the scalar field changes rapidly and whose magnitude of that rate of change. The component $\nabla\varphi$ in any direction is the rate of change of φ with respect to displacement in that direction.

Significance of divergence:



Let a region is space be filled up with a moving compressible fluid. P be the center of a small parallelopi ABCDEFGH with edges parallel to co-ordinate axes having length dx, dy, dz the vector point function $V = iv_x + jv_y + kv$ represent the velocity at the point P.

From taylor's theorem, we know:

$$v_y(y+dy) = v_y + \frac{\delta v_y}{\delta y} dy + \cdots$$

$$\approx v_y + \frac{\delta v_y}{\delta y} dy$$

So the flow on the faces ADEF and BCHG parallel to y axis will be $v_y \, dz \, dx$ and $\left(v_y + \frac{\delta v_y}{\delta y} dy\right) dz \, dx$.

$\therefore$ The net flow on the area $dz\,dx$ at distance y and y+dy $= \left(v_y + \frac{\delta v_y}{\delta y} dy - v_y\right) dz\,dx$

$$= \frac{\delta v_y}{\delta y} dx \, dy \, dz$$

Similarly the net flow of fluid on the area $dx\,dy$ and $dy\,dz$ are $\frac{\delta v_z}{\delta z} dx\,dy\,dz$ and $\frac{\delta v_x}{\delta x} dx\,dy\,dz$.

Hence, the total fluid out of the volume $dv = dx\,dy\,dz$ is $\left(\frac{\delta v_x}{\delta x} + \frac{\delta v_y}{\delta y} + \frac{\delta v_z}{\delta z}\right) dx\,dy\,dz = \cancel{div.\vec{v}.dv}\ \vec{\nabla}.\vec{v}.dx\,dy\,dz$

$\Rightarrow$ ~~div $\vec{v}$~~ $div.\vec{v}.dv$

So, the divergence of v is the net outward volume flow rate per unit volume.

Significance of curl: Let $\vec{r} = (x, y, z)$ be the position vector of a mooving particle with respect to $\vec{O}$ and $v = (v_x, v_y, v_z)$ be the velocity at $\vec{r}$. If $\vec{\omega}(\omega_x, \omega_y, \omega_z)$ be the angular velocity about $\vec{O}$, then we have:

$$\vec{V} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ x & y & z \end{vmatrix}$$

$$= (\omega_y z - \omega_z y, \omega_z x - \omega_x z, \omega_x y - \omega_y x)$$

Now, $\vec{\nabla} \times V = \begin{vmatrix} i & j & k \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ \omega_y z - \omega_z y & \omega_z x - \omega_x z & \omega_x y - \omega_y x \end{vmatrix}$

$\cancel{= i\left[\frac{\delta}{\delta x}(\omega_y z - \omega_z\right.}$

$= i\left[\frac{\delta}{\delta y}(\omega_x y - \omega_y x) - \frac{\delta}{\delta z}(\omega_z x - \omega_x z)\right] + \hat{j}\left[\frac{\delta}{\delta z}(\omega_y z - \omega_z y - \frac{\delta}{\delta x}(\omega_x y - \omega_y x)\right] + \hat{k}\left[\frac{\delta}{\delta x}(\omega_z x - \omega_x z) - \frac{\delta}{\delta y}(\omega_y z - \omega_y y)\right]$

$= \hat{i}(\omega_x + \omega_x) + \hat{j}(\omega_y + \omega_y) + \hat{k}(\omega_z + \omega_z)$

$= 2(i\omega_x + j\omega_y + k\omega_z)$

$= 2\vec{\omega}$

Thus the curl of the velocity vector of a rigid body is twice its angular velocity vector.

Directional derivative: Let $\varphi(x, y, z)$ be a function of x, y, z which is defined at a point (x, y, z) on a space curve c. Let $\varphi(x+\Delta x, y+\Delta y, z+\Delta z)$ be the value of the function at a neighboring point on c and suppose Δs denote the arc length of the curve between these two points.

Then $\frac{\Delta\varphi}{\Delta s} = \frac{\varphi(x+\Delta x, y+\Delta y, z+\Delta z) - \varphi(x, y, z)}{\Delta s}$ represents the average rate of change of φ with respect to the arc length.

If $\lim_{\Delta s \to 0} \frac{\Delta\varphi}{\Delta s} = \lim_{\Delta s \to 0} \frac{\varphi(x+\Delta x, y+\Delta y, z+\Delta z) - \varphi(x, y, z)}{}$

exists, then it is called the directional derivative of φ at the point (x, y, z) along the curve c and it is given by

$$\frac{d\varphi}{ds} = \frac{\delta\varphi}{\delta s} \cdot \frac{dx}{ds} + \frac{\delta\varphi}{\delta y} \cdot \frac{dy}{ds} + \frac{\delta\varphi}{\delta z} \cdot \frac{dz}{ds}$$

$$= \left(\frac{\delta\varphi}{\delta x} i + \frac{\delta\varphi}{\delta y} j + \frac{\delta\varphi}{\delta z} k\right) \cdot \left(\frac{dx}{ds} i + \frac{dy}{ds} j + \frac{dz}{ds} k\right)$$

$$= \nabla\varphi \cdot \frac{dr}{ds}$$

$$= \vec{\nabla}\varphi \cdot T$$

Where $r = x\hat{i} + y\hat{j} + z\hat{k}$ is the position vector and $T = \frac{dr}{ds}$ is the tangent vector at the point. So the

directional derivative is given by the componest of $\bar{\nabla}.\varphi$ in the direction of the tangent to the curve at the point (x, y, z). Again the direction derivative of $\varphi(x, y, z)$ of an arbitary point (x, y, z) in the direction a is :

$$\frac{d\varphi}{ds} = \bar{\nabla}\varphi . \frac{\bar{a}}{|a|}$$

Here $\frac{d\varphi}{ds} = \vec{\nabla}\varphi . \frac{\bar{a}}{|a|}$ is the projection of $\bar{\nabla}\varphi$ in the direction $\frac{d\vec{r}}{ds}$.

Example-35: Find the directional derivative of $\varphi(x, y, z) = 4xz^3 - 3x^2y^2z$ at the point $(2, -1, 2)$ in the direction $2\hat{i} - 3\hat{j} + 6k$.

Solve: Let, $\bar{a} = 2\hat{i} - 3\hat{j} + 6\hat{k}$

$$|\bar{a}| = \sqrt{4 + 9 + 36}$$
$$= \sqrt{49} = 7$$

Given that,

$\varphi = 4xz^3 - 3x^2y^2z$

$$\nabla.\varphi = \frac{\delta(4xz^3 - 3x^2y^2z)}{\delta x}\hat{i} + \frac{\delta(4xz^3 - 3x^2y^2z)}{dy}\hat{j} + \frac{\delta(4xz^3 - 3x^2y^2z)}{dz}\hat{k}$$

$$= (4z^3 - 6xy^2z)\hat{i} + (-6x^2yz)\hat{j} + (12xz^2 - 3x^2y^2)\hat{k}$$

At $(2,-1,2)$ we have,

$$\vec{\nabla}\cdot\varphi = \hat{i}[4.2^3-6.2.(-1)^2.2] + \hat{j}[(-6).2.(-1)^2.2] + \hat{k}[12.2.2^2 - 3.2^2.(-1)^2]$$

$$= 8\hat{i} + 48\hat{j} + 84\hat{k}$$

Hence, the directional derivative of φ in the direction of $\bar{a}$ is:

$$\bar{\nabla}\cdot\varphi \frac{\bar{a}}{|a|}$$

$$= (8\hat{i} + 48\hat{j} + 84\hat{k}) \cdot \frac{2i - 3j + 6k}{7}$$

$$= \frac{(8\times2) + 48\times(-3) + 84\times6}{7}$$

$$= \frac{16 - 144 + 504}{7}$$

$$= \frac{376}{7}$$

⇒ If $\varphi = 2xyz^2$ and $\bar{F} = [xy, -z, x^2] = xy\hat{i} - z\hat{j} + x^2\hat{k}$, then find (i) $\int_c \varphi\, d\bar{r}$ (ii) $\int_c \bar{F}\cdot d\bar{r}$, (iii) $\int_c F\times d\vec{r}$, where c is the curve $x = t^2$, $y = 2t$, $z = t^3$ from $t=0$ to $t=1$

Solve:

Along the curve c, $\varphi = 2t^2.2t.(t^3)^2$

$$= 4t^9$$ [keeping the value of x, y, z at φ]

$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$= t^2\hat{i} + 2t\hat{j} + t^3\hat{k}$

$d\bar{r} = 2t\,dt\,\hat{i} + \cancel{2dt}\ 2\,dt\,\hat{j} + 3t^2\,dt\,\hat{k}$

$= (2t\hat{i} + 2\hat{j} + 3t^2 k)\,dt$

(I) $\int_C \phi\, d\bar{r} = \int_C 4t^9 (2t\hat{i} + 2\hat{j} + 3t^2\hat{k})\,dt$

$= \int_0^1 (8t^{10}\hat{i} + 8t^9\hat{j} + 12t^{11}\hat{k})\,dt$

$= \left[\frac{8t^{11}}{11}\right]_0^1 \hat{i} + \left[\frac{8t^{10}}{10}\right]_0^1 + \left[\frac{12t^{12}}{12}\right]_0^1$

$= \frac{8}{11}\hat{i} + \frac{8}{10}\hat{j} + \hat{k}$

(II) $\int_C \bar{F}\cdot d\bar{r}$

$= \int_C (2t^3\hat{i} - t^3\hat{j} + t^4\hat{k})\cdot(2t\hat{i} + 2\hat{j} + 3t^2 k)\,dt$

$= \int_C (4t^4 - 2t^3 + 3t^6)$

$= \left[\frac{4t^5}{5} - \frac{2t^4}{4} + \frac{3t^7}{7}\right]_0^1$

$= \frac{4}{5} - \frac{1}{2} + \frac{3}{7}$

$= \frac{51}{70}$

Here,

$\bar{F} = [xy, -z, x^2]$

$= (t^2 \cdot 2t, -t^3, t^4)$

$= 2t^3, -t^3, t^4$

(11) $\int_c \bar{F} \times d\bar{r}$

Solve:

$\bar{F} = [xy, -z, x^2]$

$= [2t^3, -t^3, t^4]$

$d\bar{r} = (2t\hat{i} + 2\hat{j} + 3t^2\hat{k})dt$

$$\bar{F} \times d\bar{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2t^3 & -t^3 & t^4 \\ 2t & 2 & 3t^2 \end{vmatrix}$$

$= \hat{i}(-3t^5 - 2t^4) - \hat{j}(6t^5 - 2t^5) + \hat{k}(4t^3 + 2t^4)$

$\int_c \bar{F} \times d\bar{r} = \left[\hat{i}\left(\frac{-3t^5}{2} - \frac{2t^5}{5}\right) - \hat{j}\left(\frac{6t^6}{6} - \frac{2t^6}{6}\right) + \hat{k}\left(\frac{4t^4}{4} + \frac{2t^5}{5}\right)\right]$

$= \left[\frac{-5t^5 - 4t^5}{10}\right]_0^1 \hat{i} - \hat{j}\left[\frac{2t^6}{3}\right]_0^1 + \left[\frac{5t^4 + 2t^5}{5}\right]_0^1 \hat{k}$

$= -\frac{9}{10}\hat{i} - \frac{2}{3}\hat{j} + \frac{7}{5}\hat{k}$

⇒ Show that, $\oint_C \bar{F}\cdot d\bar{r} = 2\pi$ along the circle $(x-1)^2+y^2=1$, $z=1$, where $\bar{F} = -y\hat{i} + x\hat{j} + z\hat{k}$.

Solve:

$$\bar{F}\cdot d\bar{r} = (-y\hat{i} + x\hat{j} + z\hat{k})\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= -y\,dx + x\,dy + z\,dz$$

The parametric equation of the given circle is

$x = 1 = \cos\theta$, $y = \sin\theta$, $0 \le \theta \le 2\pi$

$\Rightarrow x = 1 + \cos\theta$, $y = \sin\theta$, $z = 1$

$\therefore dx = -\sin\theta$, $dy = \cos\theta$, $dz = 0$

Circle $(0, 0)$, r
$x = r\cos\theta$
$y = r\sin\theta$

Hence,

$$\oint_C \bar{F}\cdot d\bar{r} = \int_0^{2\pi} (-\sin\theta)(-\sin\theta)\,d\theta + (1+\cos\theta)\cos\theta\,d\theta + 0$$

$$= \int_0^{2\pi} d\theta + \cos\theta\,d\theta$$

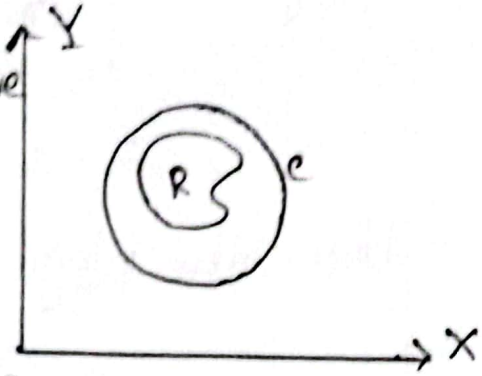
$$= [\theta + \sin\theta]_0^{2\pi}$$

$$= 2\pi + \sin 2\pi - 0 - \sin 0$$

$$= 2\pi \text{ (Ans)}$$

Green's theorem:

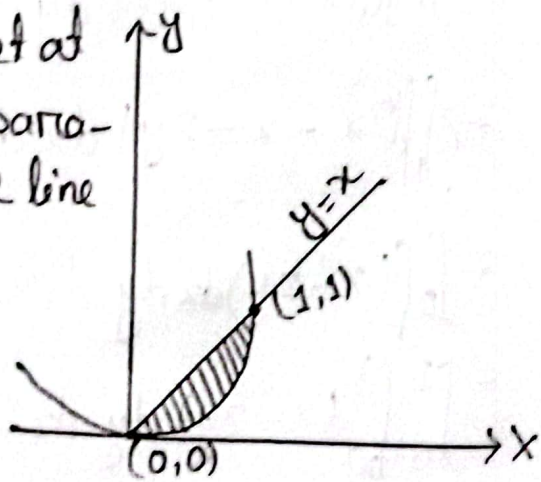
Let R, is a closed region in XY plane bounded by a simple closed curve c such that a straight line parallel to the co-ordinate axes cut c at most two points. Now, if P(x, y) and Q(x, y) are continuous function of x and y having continuous partical derivatives in R, then



$$\oint_c P\,dx + Q\,dy = \iint_R \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}\right) dx\,dy.$$

⇒ Verify the Green's th^m in the plane of $\oint_c (xy+y^2)\,dx + x^2\,dy$ where c is the closed curve of the region bounded by $y=x$ and $y=x^2$.

Solve: Here, $y=x$ and $y=x^2$ meet at (0,0) and (1,1). Now along the parabola $y=x^2$ from (0,0) to (1,1) the line integral is:



$$\int_0^1 (x.x^2 + x^4)\,dx + x^2\,2x\,dx$$

$$= \int_0^1 (x^3 + x^4 + 2x^3)\,dx$$

$$= \frac{19}{20}$$

Now, along the line $y=x$ from $(1,1)$ to $(0,0)$ the line integral is $\int_1^0 (x \cdot x + x^2)\,dx + x^2 dx$

$$= -1$$

Hence, the integral is $\frac{19}{20} - 1$

$$= \frac{19-20}{20}$$

$$= \frac{-1}{20}$$

Here,

$P = xy + y^2$, $Q = x^2$

$$\therefore \iint_R \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}\right) dx\,dy$$

$$\Rightarrow \iint_R \left(\frac{\delta (x^2)}{\delta x} - \frac{\delta (xy+y^2)}{dy}\right) dx\,dy$$

$$\Rightarrow \iint_R (2x - x - 2y)\,dx\,dy.$$

$$= \iint_R (x-2y)\,dy.\,dx$$

$$= \int_0^1 \int_{x^2}^{x} (x-2y)\,dy\,dx.$$

$$= \int_0^1 \left[xy - \frac{2y^2}{2}\right]_{x^2}^{x} dx$$

$$= \int_0^1 (x^2 - x^2 - x^3 + x^4)\,dx$$

$$= \int_0^1 (x^4 - x^3)\,dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1$$

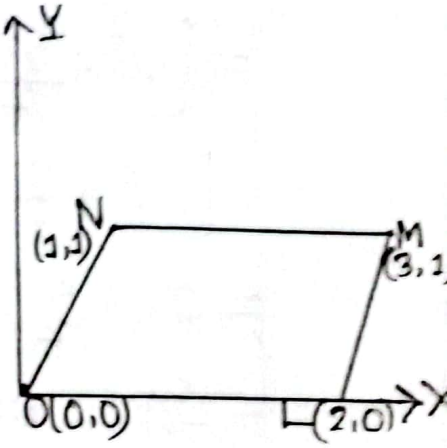
$$= \frac{1}{5} - \frac{1}{4}$$

$$= \frac{4-5}{20}$$

$$= \frac{-1}{20}$$

⇒ Verify green's th^m in the plane for evaluate $\oint (3x^2+2y)dx - (x+3\cos y)dy$ where C is boundary of the the parallelogram having vertices at (0,0),(2,0),(3,1),(1,1).

Solve:



Along the line OL (y=0) we have $0 \le x \le 2$

$$\cancel{\oint_{OL} Pdx + Qdy = \int_0^2 3x^2 dx}$$

Here, $P = 3x^2+2y$

$Q = -(x+3\cos y)$

$$\oint_{OL} Pdx + Qdy = \int_0^2 (3x^2+2y)dx - (x+3\cos y)dy$$

$$= \int_0^2 3x^2 dx \qquad [\because y=0]$$

$$= \left[\frac{3x^3}{3}\right]^2$$

$$= 8$$

Equation of the line LM $= \frac{y-0}{0-1} = \frac{x-2}{2-3}$

$$\Rightarrow y = x-2$$

Hence,

$$\int_{LM} (Pdx + Qdy) = \int_2^3 \left[3x^2+2(x-2) - x - 3\cos(x-2)\right] dx$$

$$= \int_2^3 \left[3x^2 + 2x - 4 - x - 3\cos(x-2)\right] dx$$

$$= \int_2^3 \left[\frac{3x^3}{3} + \frac{2x^2}{2} - 4x - \frac{x^2}{2} - 3\sin(x-2)\right]_2^3$$

$$= \left[x^3 + \frac{x^2}{2} - 4x - 3\sin(x-2)\right]_2^3$$

$$= \left[(3)^3 + \frac{9}{2} - (4\times 3) - 3\sin(3-2)\right] - \left[2^3 - \frac{4}{2} - (4\times 2) - 3\sin(2-2)\right]$$

$$= 27 + \frac{9}{2} - 12 - 3\sin 1 - 8 + 2 - 8$$

$$= \frac{35}{2} - \sin 1.$$

Equation of MN : $\frac{y-3}{1-1} = \frac{x-3}{3-1}$

$$\therefore y = 1$$

$\therefore$ Along the line MN

$$\int_{MN} (P\,dx + Q\,dy) = \int_3^1 \cancel{3x^2\,dx - x - 3\cos}\,(3x^2+2)\,dx$$

$$= \left[x^3 + 2x\right]_3^1$$

$$= (1+2) - 27 - 6$$

$$= -30$$

Equation of NO : $y = x$

Along the line NO:

$$\oint P\,dx + Q\,dy = \int_1^0 (3x^2 + x - 3\cos x)\,dx$$

$$= -\frac{3}{2} + 3\sin 1$$

$$\therefore \oint_C (P\,dx + Q\,dy) = 8 + \frac{35}{2} - 3\sin 1 - 30 - \frac{3}{2} + 3\sin 1$$

$$= -6$$

Again,

$$\iint_R \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}\right) dx\,dy = \int_{x=0}^{2} \int_{y=0}^{1} (-1-2)\,dy\,dx$$

$$= -3\int_0^2 [y]_0^1\,dx$$

$$= -6$$

$$\therefore \iint_R \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}\right) dx\,dy = \oint_C (P\,dx + Q\,dy)$$

Divergence theorem of Gauses: If V is the volume bounded by a closed surface S and A is a vector function of position with continuous derivatives then

$$\iiint_V \vec{\nabla}\cdot\vec{A}\, dv = \iint_S \bar{A}\cdot\bar{n}\, ds = \oiint_S \bar{A}\cdot ds$$

Example-12: Evalute $\iint_S \bar{F}\cdot\bar{n}\, ds$ where $\bar{F} = 6z\hat{i} + (2x+y)\hat{j} - x\hat{k}$ and s is the surface of the region bounded by $x^2+z^2=9$, $x=0$, $y=0$, $z=0$ and $y=8$

Solve: We know,

$$\iint_S \vec{F}\cdot\vec{n}\cdot ds = \iiint_V \bar{\nabla}\cdot\bar{F}\, dv \quad \text{—— (1)}$$

Here, $\bar{\nabla}\cdot\bar{F} = \frac{\delta}{\delta x}(6z) + \frac{\delta}{\delta y}(2x+y) + \frac{\delta}{\delta z}(-x)$

$= 1$

From equation (1) $\Rightarrow$

$$\iint_S \bar{F}\cdot\bar{n}\cdot ds = \int_0^3\int_0^8\int_0^{\sqrt{9-z^2}} dz\, dy\, dx$$

$$= \int_0^3\int_0^8 [z]_0^{\sqrt{9-x^2}}\, dy\, dx$$

$$= \int_0^3 \int_0^8 \sqrt{9-x^2}\, dy\, dx$$

$$= \int_0^3 \sqrt{9-x^2}\, [y]_0^8\, dx$$

$$= 8 \int_0^3 \sqrt{9-x^2}\, dx$$

$$= 8 \left[\frac{x\sqrt{9-x^2}}{2} + \frac{9}{2} \sin^{-1}\left(\frac{x}{3}\right) \right]_0^3$$

$$= 8 \left[0 + \frac{9}{2} \cdot \frac{\pi}{2} - 0 \right]$$

$$= 18\pi$$

Example-20: Show that $\iint_S \bar{F} \cdot \bar{n}\, ds = \pi$ where $\bar{F} = x\hat{i} - y\hat{j} + (z^2-$ and S is the closed surface bounded by the planes $z=0$ $z=1$ and the cylinder $x^2+y^2=1$.

Solve: We know,

$$\iint_S \bar{F} \cdot \bar{n}\, ds = \iiint_V \bar{\nabla} \cdot \bar{F}\, dv$$

$$= \iiint_V \left[\frac{\delta}{\delta x}(x) + \frac{\delta}{\delta y}(-y) + \frac{\delta}{\delta z}(z^2-1) \right] dv$$

$$= \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{z=0}^{1} (1-1+2z)\, dz\, dy\, dz$$

$$= \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} [2^2]_0^1 \, dy \, dx$$

$$= \int_{x=-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \, dx$$

$$= \int_{-1}^{1} [y]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \, dx$$

$$= 2 \int_{-1}^{1} \sqrt{1-x^2} \, dx$$

$$= 2 \int \left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2} \sin^{-1} x \right]_{-1}^{1}$$

$$= \frac{2}{2} \left[\sin^{-1} 1 - \sin^{-1}(-1) \right]$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$ (proved)