Vector Spaces.

Def^n. Let K be a given field and let V be a non-empty set with rules of addition and scalar multiplication which assigns to any $u, v \in V$, a sum $u+v \in V$ and to any $u \in V$, $k \in K$, a product $ku \in V$. Then V is called a vector space over K if the following axioms hold:

[A1]: For any vectors $u, v, w \in V$, $(u+v)+w = u+(v+w)$.

[A2]: There is a vector in V, denoted by 0 and called the zero vector, for which $u+0 = u$, for any vector $u \in V$.

[A3]: For each vector $u \in V$, there is a vector in V, denoted by $-u$, for which $u+(-u) = 0$.

[A4] For any vectors $u, v \in V$, $u+v = v+u$.

[M1]: For any scalar $k \in K$ and any vectors $u, v \in V$, $k(u+v) = ku+kv$.

[M2]: For any scalars $a, b \in K$ and any vector $u \in V$, $(a+b)u = au+bu$.

[M3]: For any scalars $a, b \in K$ and any vector $u \in V$, $(ab)u = a(bu)$.

[M4]: For the unit scalar $1 \in K$, $1u = u$ for any vector $u \in V$.

N.B. The elements of V are called vectors, the zero vector 0 is unique, the negative $-u$ of u is unique and the cancellation law holds: $u+w = v+w \Rightarrow u = v$ for any scalars $u, v, w \in V$. Also, subtraction is defined by

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$\cancel{u+}$
$$u - v = u + (-v).$$

Ex:-1 Let K be an arbitrary field. The set of all n-tuples of elements of K with vector addition and scalar multiplication defined by

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n)$$
$$= (a_1+b_1, a_2+b_2, \ldots, a_n+b_n)$$

and $k(a_1, a_2, \ldots, a_n) = (ka_1, ka_2, \ldots, ka_n)$

where $a_i, b_i, k \in K$, is a vector space over K, we denote this space by K^n. The zero vector in K^n is the n-tuple of zeros,

$$0 = (0, 0, \ldots, 0),$$

If $a = (a_1, a_2, \ldots, a_n)$ & $b = (b_1, b_2, \ldots, b_n)$,

then
$$\begin{aligned} a+b &= (a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) \\ &= (a_1+b_1, a_2+b_2, \ldots, a_n+b_n) \\ &= (b_1+a_1, b_2+a_2, \ldots, b_n+a_n) \\ &= (b_1, b_2, \ldots, b_n) + (a_1, a_2, \ldots, a_n) \\ &= b+a. \end{aligned}$$

We can easily prove the others properties.

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Ex-2. Let V be the set of all $m \times n$ matrices with entries from an arbitrary field K. Then V is a vector space over K with respect to the operations of matrix addition and scalar multiplication defined by

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & & \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & & & \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{pmatrix}$$

and $$K \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} Ka_{11} & Ka_{12} & \cdots & Ka_{1n} \\ Ka_{21} & Ka_{22} & \cdots & Ka_{2n} \\ \vdots & & & \\ Ka_{m1} & Ka_{m2} & \cdots & Ka_{mn} \end{pmatrix}$$

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NOTEBOOK

* Let V be the set of all functions from a non-empty set X into a field K, for any functions $f, g \in V$ and any scalar $k \in K$, let $f+g$ and kf be functions in V defined by $(f+g)(x) = f(x) + g(x)$ and
$(kf)(x) = kf(x)$, for all $x \in X$.

Prove that V is a vector space over K.

<u>Proof.</u> Since X is non-empty, so V is non-empty. Now, we have to prove that all the axioms of a vector space hold:

[A1]: Let $f, g, h \in V$, then

$$((f+g)+h)(x) = (f+g)(x) + h(x)$$
$$= (f(x) + g(x)) + h(x) \quad \text{for all } x \in X$$

$$(f+(g+h))(x) = f(x) + (g+h)(x)$$
$$= f(x) + (g(x) + h(x)) \quad \forall x \in X.$$

But $f(x)$, $g(x)$ and $h(x)$ are scalars in the field K, where addition of scalars is associative hence

$$(f(x) + g(x)) + h(x) = f(x) + (g(x) + h(x))$$

Accordingly, $(f+g)+h = f+(g+h)$.

[A2]: Let 0 denoted the zero function, $0(x) = 0 \ \forall x \in X$. Then any function $f \in V$,

$$(f+0)(x) = f(x) + 0(x) = f(x) + 0 = f(x), \ \forall x \in X$$

Thus $f+0 = f$ and 0 is the zero vector in V.

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[A3]: for any function $f \in V$, let $-f$ be the function denoted by $(-f)(x) = -f(x)$. Then

$$(f+(-f))(x) = f(x) + (-f)(x) = f(x) - f(x) = 0 = 0(x) \quad \forall x \in X.$$

Hence $f + (-f) = 0$.

[A4]: Let $f, g \in V$. Then $(f+g)(x) = f(x) + g(x)$

$$= g(x) + f(x) = (g+f)(x), \quad \forall x \in X.$$

Hence, $f + g = g + f$.

[M1]: Let $f, g \in V$ and $k \in K$. Then

$$(k(f+g))(x) = k((f+g)(x)) = k(f(x) + g(x))$$
$$= kf(x) + kg(x)$$
$$= (kf)(x) + (kg)(x)$$
$$= (kf + kg)(x) \quad \forall x \in X.$$

Hence $k(f+g) = kf + kg$.

[M2]: Let $f \in V$ and $a, b \in K$. Then

$$((a+b)f)(x) = (a+b)f(x) = af(x) + bf(x)$$
$$= (af)(x) + (bf)(x) = (af + bf)(x) \quad \forall x \in X.$$

$\therefore (a+b)f = af + bf$.

[M3]: Let $f \in V$ and $a, b \in K$. Then

$$((ab)f)(x) = (ab)f(x) = a(bf)(x) = (a(bf))(x) \quad \forall x \in X.$$

$\therefore (ab)f = a(bf)$

[M4]: Let $f \in V$. Then for the unit $1 \in K$,

$(1f)(x) = 1f(x) = f(x) \quad \forall x \in X$. Hence $1f = f$.

Since all of the axioms are satisfied, V is a vectorspace.

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NOTEBOOK

* Let V be the set of ordered pairs of real numbers: $V = \{(a,b) : a, b \in \mathbb{R}\}$. Show that V is not a vector space over $\mathbb{R}$ with respect to each of the following operations of addition in V and scalar multiplication on V:

(i) $(a,b) + (c,d) = (a+c, b+d)$
and $k(a,b) = (ka, b)$.

(ii) $(a,b) + (c,d) = (a, b)$ and $k(a,b) = (ka, kb)$.

(iii) $(a,b) + (c,d) = (a+c, b+d)$ and $k(a,b) = (k^2a, k^2b)$.

Proof.

(i) Let $r = 1$, $s = 2$, $v = (3,4)$. Then

$(r+s)v = 3(3,4) = (9,4)$

and $rv + sv = 1(3,4) + 2(3,4)$
$= (3,4) + (6,4)$
$= (9,8)$

Since $(r+s)v \neq rv + sv$, V is not vector space under the ~~ab~~ given operations of addition and scalar multiplication.

(ii) Let $v = (1,2)$, $w = (3,4)$. Then

$v + w = (1,2) + (3,4) = (1,2)$

$w + v = (3,4) + (1,2) = (3,4)$

Since $v + w \neq w + v$, V is not a vector space.

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(iii) Let $r=1$, $s=2$, $v=(3,4)$. Then

$(r+s)v = 3(3,4) = (27,36)$

$rv+sv = 1(3,4)+2(3,4)$

$= (3,4)+(12,16) = (15,20)$.

Thus $(r+s)v \neq rv+sv$, Hence V is not a vector space.

<u>Theorem</u>. Let V be a vector space over a field K. (i) for any scalar $k \in K$ and $0 \in V$, $k0=0$.

(ii) for $0 \in K$ and any vector $u \in V$, $0u=0$.

(iii) If $ku=0$, where $k \in K$ and $u \in V$, then $k=0$ or $u=0$.

(iv) for any scalar $k \in K$ and any vector $u \in V$, $(-k)u = k(-u) = -ku$.

Proof. (i) We have $0+0=0$ and hence

$k(0+0) = k0 \Rightarrow k0+k0 = k0$

Adding $-k0$ to both sides, we have

$-k0+k0+k0 = -k0+k0$

$\Rightarrow k0 = 0$

(ii) We have $0 = 0+0$, so

$\therefore u0 = u(0+0)$

$\Rightarrow u0 = u0+u0$

Adding $-u0$ to both sides we have

$-u0+u0 = -u0+u0+u0$

$\Rightarrow 0 = 0+u0$

$\Rightarrow 0 = u0$ ie $u0=0$.

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(iii) Suppose $ku = 0$ and $k \neq 0$, then there exists a scalar k^{-1} such that $k^{-1}k = 1$.

Hence $u = 1u = (k^{-1}k)u = k^{-1}(ku) = k^{-1}0 = 0$

ie $u = 0$.

(iv) We have

$$u + (-u) = 0$$

So, $k(u + (-u)) = k0$

$\Rightarrow ku + k(-u) = 0$

~~$\Rightarrow -ku$~~

Adding $-ku$ to both sides, we have

$\cancel{-ku + ku + k(-u)} = -ku + 0$

$\Rightarrow (-ku + ku) + k(-u) = -ku$

$\Rightarrow 0 + k(-u) = -ku$

ie $k(-u) = -ku$.

Again, $k + (-k) = 0$

$\Rightarrow \cancel{k(k + (-k)k)u = k0u}$

$\Rightarrow \cancel{k}\cdot ku + (-k)u = 0$

~~$\Rightarrow -ku$~~

$\Rightarrow -ku + ku + (-k)u = 0 + (-ku)$

$\Rightarrow 0 + (-k)u = -ku$

$\Rightarrow (-k)u = -ku$

~~Hence~~ $k(-u) = (-k)u = -ku$.

~~Hence proved~~

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NOTEBOOK

Subspace.

Let W be a subset of a vector space V over a field K. Then W is called a subspace of V if W is itself a vector space over K, with respect to the operations of vector addition and scalar multiplication on V.

Ex-1. Let V be any vector space. Then the set $\{0\}$ consisting of the zero vector alone and also the entire space V are subspaces of V.

Ex-2. Let V be the vector space $\mathbb{R}^3$. Then the set W consisting of those vectors whose third component is zero, $W = \{(a,b,0) : a,b \in \mathbb{R}\}$ is a subspace of V.

Theorem. W is a subspace of V if and only if (i) W is non-empty.

(ii) W is closed under addition; $u, v \in W$ implies $u+v \in W$,

(iii) W is closed under scalar multiplication; $v \in W$ implies $kv \in W$ for every $k \in K$.

Proof. Suppose W satisfies (i), (ii) and (iii). ~~By ie~~, Thus W is non-empty, the operations of vector addition and scalar multiplication are well defined for W. and hence the axioms $[A_1]$, $[A_4]$, $[M_1]$, $[M_2]$, $[M_3]$ and $[M_4]$ hold in W, since the vectors in W belong

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to V. Thus we have only to show that $[A_2]$ and $[A_3]$ hold in W.

By (1), W is non-empty, so let $u \in W$. Then by (iii), $0u = 0 \in W$ and $v + 0 = v$ for every $v \in W$. Hence W satisfies $[A_2]$.

Finally, if $v \in W$, then $(-1)v = -v \in W$ and $v + (-v) = 0$, hence W satisfies $[A_3]$.

Thus W is a subspace of V.

Conversely, if W is a subspace of V, then clearly W satisfies (i), (ii) and (iii).

* W is a subspace of V if and only if

(i) $W \neq \phi$

(ii) $v, w \in W$ implies $av + bw \in W$ for every $a, b \in K$.

<u>Proof</u> Suppose W satisfies (i) and (ii). Then by (1), W is non-empty. If $u, v \in W$, then by (ii), $u + v = 1.u + 1.v \in W$ and if $u \in W$ and $k \in K$, then by (ii), $ku = ku + 0.u \in W$. Thus by the above theorem W is a subspace of V.

Conversely, if W is a subspace of V, then clearly (i) and (ii) hold in W.

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NOTEBOOK

* Let $V = \mathbb{R}^3$ be a vector space, and W is the set consisting of those vectors whose third component is zero, ie

$$W = \{(a,b,0) : a,b \in \mathbb{R}\}.$$

Then show that W is a subspace of V.

<u>Proof</u> Here $0 = (0,0,0) \in W$, for $0 \in \mathbb{R}^3$, since the third component of 0 is 0.

for any vectors $v = (a,b,0)$, $w = (c,d,0)$ in W. and any scalars k, k' we have

$$\begin{aligned} kv + kw' &= k(a,b,0) + k'(c,d,0) \\ &= (ka, kb, 0) + (k'c, k'd, 0) \\ &= (ka + k'c, kb + k'd, 0) \in W \end{aligned}$$

Thus $kv + kw' \in W$ and hence W is a subspace of $V = \mathbb{R}^3$.

* Let $V = \mathbb{R}^3$ be a vector space and W be the set consisting of those vectors each with the property that the sum of its components is zero.

ie $W = \{(a,b,c) : a+b+c = 0,\ a,b,c \in \mathbb{R}\}$.

Then show that W is a subspace of V.

<u>Soln</u> for any $0 \in \mathbb{R}^3$, $0 = (0,0,0) \in W$, since $0+0+0 = 0$. Hence W is non-empty.

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Suppose $v=(a,b,c)$ and $w=(a',b',c')$ belong to W, then $a+b+c=0$, $a'+b'+c'=0$.

Now for any scalars k, k',

$$kv+wk' = k(a,b,c)+k'(a',b',c')$$
$$=(ka,kb,kc)+(k'a',k'b',k'c')$$
$$=(ka+k'a',\ kb+k'b',\ kc+k'c')$$

But $ka+k'a'+kb+k'b'+kc+k'c'$
$$=k(a+b+c)+k'(a'+b'+c')$$
$$=k.0+k'.0=0$$

Thus $kv+wk' \in W$ and hence W is a subspace of V.

* Show that $T=\{(a,b,c,d): a-3b+7c-5d=0\}$ is a subspace of $\mathbb{R}^4$.

Proof. For $0\in\mathbb{R}^4$, $0=(0,0,0,0)\in T$, since $0-3.0+7.0-5.0=0$.

Hence T is non-empty.

Suppose that $u=(a,b,c,d)$ and $v=(a',b',c',d')$ in T. Then
$$a-3b+7c-5d=0$$
and $a'-3b'+7c'-5d'=0$.

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Now for any scalars α, β, we have

$$\alpha u + \beta v = \alpha(a,b,c,d) + \beta(a',b',c',d')$$
$$= (\alpha a, \alpha b, \alpha c, \alpha d) + (\beta a', \beta b', \beta c', \beta d')$$
$$= (\alpha a + \beta a', \alpha b + \beta b', \alpha c + \beta c', \alpha d + \beta d')$$

But

$$\alpha a + \beta a' - 3(\alpha b + \beta b') + 7(\alpha c + \beta c') - 5(\alpha d + \beta d')$$
$$= (\alpha a + 3\alpha b + 7\alpha c - 5\alpha d) + (\beta a' - 3\beta b' + 7\beta c' - 5\beta d')$$
$$= \alpha(a - 3b + 7c - 5d) + \beta(a' - 3b' + 7c' - 5d')$$
$$= \alpha \cdot 0 + \beta \cdot 0 = 0$$

Hence $\alpha u + \beta v \in T$, and therefore T is a subspace of $V = \mathbb{R}^4$.

* Let $V = \mathbb{R}^3$. Show that W is not a subspace of V, where

(i) $W = \{(a,b,c) : a \geq 0\}$,

(ii) $W = \{(a,b,c) : a^2 + b^2 + c^2 \leq 1\}$.

(iii) $W = \{(a,b,c) : a, b, c \in \mathbb{Q}\}$.

Proof: Consider a vector $v = (5,3,2)$. Then $v \in W$, since 5 is non-negative. For a scalar $-5 \in \mathbb{R}$, we have $-5v = -5(5,3,2)$
$= (-25, -15, -10) \notin W$,

$\therefore$ ~~$-25 \geq 0$~~

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since -25 is negative.

Hence W is not a subspace of $V = \mathbb{R}^3$.

(ii) Consider $u = (1,0,0)$ and $v = (0,1,0)$.

Then $u, v \in W$, since $1^2+0^2+0^2 = 1 \leq 1$ and $0^2+1^2+0^2 = 1 \leq 1$.

But $u+v = (1,0,0)+(0,1,0)$

$= (1,1,0) \notin W$,

since $1^2+1^2+0^2 = 2 > 1$.

Hence W is not a subspace of V.

(iii) Consider $u = (1,2,3)$, then $u \in W$.

for $k = \sqrt{2}$, $ku = (\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}) \notin W$, since its component are not rational numbers.

Hence W is not a subspace of V.

Let V be the vector space of all 2×2 matrices over the real field $\mathbb{R}$. Show that W is not a subspace of V, where

(i) W consists of all matrices with zero determinant.

(ii) W consists of all matrices A for which $A^2 = A$.

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Sol(i) Consider two matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

Since $|A| = 0$ and $|B| = 0$, so $A, B \in W$.

But $A + B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ not belong to W,

since $|A+B| = 1$. Hence W is not a subspace of V.

(ii) The unit matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ belongs to W, since $I^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$.

But $2I = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ does not belong to W, since

$$(2I)^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \neq 2I.$$

Hence W is not a subspace of V.

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NOTEBOOK

Linear Combination

<u>Definition.</u> Let V be a vector space over a field K and let $v_1, v_2, \ldots, v_n \in V$. Any vector $v \in V$ is called a linear combination of $v_1, v_2, \ldots, v_n$ if v can be expressed in the form $a_1v_1 + a_2v_2 + \ldots + a_nv_n$, where $a_1, a_2, \ldots, a_n \in K$, ie $v = a_1v_1 + a_2v_2 + \ldots + a_nv_n$.

The set of all such linear combination, denoted by $\text{span}(v_1, v_2, \ldots, v_n)$, is called the linear span of $v_1, v_2, \ldots, v_n$ and the set $\{v_1, v_2, \ldots, v_n\}$ is the generator of V.

* Express $u = (a, b, c) \in \mathbb{R}^3$ as a linear combination of the vectors $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$.

<u>Soln.</u> We have $u = (a, b, c)$

$$= a(1,0,0) + b(0,1,0) + c(0,0,1)$$

ie $u = ae_1 + be_2 + ce_3$

Hence u is a linear combination of e_1, e_2, e_3.

Thus e_1, e_2, e_3 generate the vector space $\mathbb{R}^3$.

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NOTEBOOK

* Determine whether or not the vector $v=(3,9,-4,-2)$ is a linear combination of the vectors $u_1=(1,-2,0,3)$, $u_2=(2,3,0,-1)$ and $u_3=(2,-1,2,1)$.

Sol^n: Let v be the linear combination of u_1, u_2, u_3.

Then $v = xu_1 + yu_2 + zu_3$

$$\Rightarrow (3,9,-4,-2) = x(1,-2,0,3) + y(2,3,0,-1) + z(2,-1,2,1)$$

$$\Rightarrow (3,9,-4,-2) = (x+2y+2z, -2x+3y-z, 2z, 3x-y+z)$$

$$\therefore \begin{aligned} x+2y+2z &= 3 \\ -2x+3y-z &= 9 \\ 2z &= -4 \\ 3x-y+z &= -2. \end{aligned}$$

Now, we can write the augmented matrix

$$(A:B) = \left(\begin{array}{ccc:c} 1 & 2 & 2 & 3 \\ -2 & 3 & -1 & 9 \\ 0 & 0 & 2 & -4 \\ 3 & -1 & 1 & -2 \end{array}\right)$$

$R_2' = R_2 + 2R_1$
$R_4' = R_4 - 3R_1$

$$\sim \left(\begin{array}{ccc:c} 1 & 2 & 2 & 3 \\ 0 & 7 & 3 & 15 \\ 0 & 0 & 2 & -4 \\ 0 & -7 & -5 & -11 \end{array}\right)$$

$R_4' = R_4 + R_2$

$$\sim \left(\begin{array}{ccc:c} 1 & 2 & 2 & 3 \\ 0 & 7 & 3 & 15 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & -2 & 4 \end{array}\right)$$

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$$\sim \begin{pmatrix} 1 & 2 & 2 & : & 3 \\ 0 & 7 & 3 & : & 15 \\ 0 & 0 & 2 & : & -4 \\ 0 & 0 & 0 & & 0 \end{pmatrix}$$

$\because \rho(A) = \rho(A:B) = n$, the number of variables, so the above system is consistent and has a solution. Therefore v is a linear combination of u_1, u_2, u_3.

Solving, we have

$$x + 2y + 2z = 3$$
$$7y + 3z = 15$$
$$2z = -4$$

ie $z = -2$, $y = 3$, $x = 1$.

Hence $v = u_1 + 3u_2 - 2u_3$.

* Determine whether or not the vector $v = (1, -2, 5)$ in $\mathbb{R}^3$ is a linear combination of the vectors $u_1 = (1, -3, 2)$, $u_2 = (2, -4, -1)$ and $u_3 = (1, -5, 7)$.

Soln Let v be the linear combination of u_1, u_2, u_3. Then

$$v = xu_1 + yu_2 + zu_3$$

ie $(1, -2, 5) = x(1, -3, 2) + y(2, -4, -1) + z(1, -5, 7)$

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$\Rightarrow (1,-2,5) = (x+2y+z, -3x-4y-5z, 2x-y+7z)$

$\therefore \quad x+2y+z = 1$
$3x+4y+5z = 2$
$2x-y+7z = 5$

$\Rightarrow \quad x+2y+z = 1$
$2y-2z = 1$
$5y-5z = -3$

$\Rightarrow \quad x+2y+z = 1$
$2y-2z = 1$
$0 = 11$

The above system is inconsistent and so it has no solution and hence v is not a linear combination of u_1, u_2, u_3.

* Write the vector $v = (1,-2,5)$ as a linear combination of the vectors $e_1 = (1,1,1)$, $e_2 = (1,2,3)$ and $e_3 = (2,-1,1)$.

Soln. Let v be a linear combination of $e_1, e_2, \& e_3$. Then

$$v = xe_1 + ye_2 + ze_3$$

$\Rightarrow (1,-2,5) = x(1,1,1) + y(1,2,3) + z(2,-1,1)$

$\Rightarrow (1,-2,5) = (x+y+2z, x+2y-z, x+3y+z)$

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$$\therefore \quad x + y + 2z = 1$$
$$x + 2y - z = -2$$
$$x + 3y + z = 5$$

$$\Rightarrow \quad x + y + 2z = 1$$
$$y - 3z = -3$$
$$2y - z = 4$$

$$\Rightarrow \quad x + y + 2z = 1$$
$$y - 3z = -3$$
$$5z = 10$$

Then the above system is consistent and so has a solution. Solve for the unknowns to obtain

$$x = -6 \qquad y = 3 \qquad z = 2$$

Hence $v = -6e_1 + 3e_2 + 2e_3$.

* Write the vector $v = (2, -5, 3)$ in $\mathbb{R}^3$ as a linear combination of the vectors $e_1 = (1, -3, 2)$, $e_2 = (2, -4, -1)$ and $e_3 = (1, -5, 7)$.

* Write the polynomial $v = t^2 + 4t + 3$ over $\mathbb{R}$ as a linear combination of the polynomials $e_1 = t^2 - 2t + 5$, $e_2 = 2t^2 - 3t$ and $e_3 = t + 3$.

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Soln Let v be a linear combination of e_1, e_2, e_3 using unknowns x, y, z, then

$$v = xe_1 + ye_2 + ze_3$$

$$\cancel{\Rightarrow t^2+4t+5 = x(}$$

$$t^2+4t-3 = x(t^2-2t+5)+y(2t^2-3t)+z(t+3)$$

$$\Rightarrow t^2+4t-3 = (x+2y)t^2+(-2x-3y+z)t + (5x+3z)$$

Equating the coefficients of the same power of t on both sides, we have

$$\begin{aligned} x+2y &= 1 \\ -2x-3y+z &= 4 \\ 5x \quad +3z &= -3 \end{aligned}$$

or

$$\begin{aligned} x+2y &= 1 \\ y+z &= 6 \\ -10y+3z &= -8 \end{aligned}$$

or

$$\begin{aligned} x+2y &= 1 \\ y+z &= 6 \\ 13z &= 52 \end{aligned}$$

Thus the system is consistent and so has a solution. Solve for the unknowns to obtain $z=4$, $y=2$, $x=-3$.

Thus $v = -3e_1+2e_2+4e_3$.

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* Write the matrix $E = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$ as a linear combination of the matrices $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 2 \\ 0 & -1 \end{pmatrix}$.

Sol? Set E as a linear combination of A, B, C using the unknowns x, y, z. Then

$$E = xA + yB + zC$$

ie $$\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} = x\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + z\begin{pmatrix} 0 & 2 \\ 0 & -1 \end{pmatrix}.$$

$$= \begin{pmatrix} x & x \\ x & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ y & y \end{pmatrix} + \begin{pmatrix} 0 & 2z \\ 0 & -z \end{pmatrix}.$$

$$= \begin{pmatrix} x & x+2z \\ x+y & y-z \end{pmatrix}$$

$\therefore$ $x = 3$, $x + 2z = 1$, $x + y = 1$, $y - z = -1$.

ie $x = 3$, $y = -2$, $z = -1$

Hence $E = 3A - 2B - C$.

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* Show that the vectors $u=(1,2,3)$, $v=(0,1,2)$ and $w=(0,0,1)$ generate $\mathbb{R}^3$.

Sol^n: We have to show that an arbitrary vector $(a,b,c) \in \mathbb{R}^3$ is a linear combination of u, v & w.

set $(a,b,c) = xu + yv + zw$.

$\Rightarrow (a,b,c) = x(1,2,3) + y(0,1,2) + z(0,0,1)$

$\Rightarrow (a,b,c) = (x, 2x+y, 3x+2y+z)$

ie
$$x = a$$
$$2x+y = b$$
$$3x+2y+z = c$$

ie,
$$x = a$$
$$y = b-2a$$
$$z = c-3a-2(b-2a)$$
$$= c-3a-2b+4a$$
$$= a-2b+c$$

The above system on a, b, c

ie
$$3x+2y+z = c$$
$$2x+y = b$$
$$x = a$$

ie
$$z+2y+3x = c$$
$$y+2x = b$$
$$x = a$$

The above system is in echelon form and is consistent and $x=a$, $y=b-2a$ & $c = c-2b+a$ is a solution.

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Thus u, v, w generate $\mathbb{R}^3$.

* Find conditions on a, b and c so that $(a, b, c) \in \mathbb{R}^3$ belongs to the space generated by $u = (2, 1, 0)$, $v = (1, -1, 2)$ and $w = (0, 3, -4)$.

Soln Let (a, b, c) as a linear combination of u, v and w using unknows x, y & z.

$$(a, b, c) = xu + yv + zw$$

ie, $(a, b, c) = x(2, 1, 0) + y(1, -1, 2) + z(0, 3, -4)$

ie $(a, b, c) = (2x + y, x - y + 3z, 2y - 4z)$

form the equivalent system of linear equations and reduce it to echelon form

$$\begin{aligned} 2x + y &= a \\ x - y + 3z &= b \\ 2y - 4z &= c \end{aligned} \quad \text{or} \quad \begin{aligned} 2x + y &= a \\ 3x \quad + 3z &= a + b \end{aligned}$$

$$\text{or} \quad \begin{aligned} 2x + y &= a \\ 3y - 6z &= a - 2b \\ 2y - 4z &= c \end{aligned} \quad \text{or} \quad \begin{aligned} 2x + y &= a \\ 3y - 6z &= a - 2b \\ 0 &= 2a - 4b - 3c \end{aligned}$$

The vectors (a, b, c) belongs to the space generated by u, v & w iff

and only if the above system is consistent and it is consistent iff $2a-4b-3c=0$.

Note in particular, that u, v and w do not generate the whole space $\mathbb{R}^3$.

Sums and Direct sums.

Let U and W be subspaces of a vector space V. The sum of U and W, written $U+W$, is defined by ~~(consists of all sums $u+w$, where $u \in U$ and $w \in W$.)~~

$$U+W = \{u+w : u \in U, w \in W\}.$$

Note: $0 = 0+0 \in U+W$; $0 \in U$, $0 \in W$.

Suppose $u+w$ and $u'+w'$ belong to $U+W$, with $u, u' \in U$ and $w, w' \in W$. Then

$$(u+w)+(u'+w') = (u+u')+(w+w') \in U+W.$$

and for any scalar k,

$$k(u+w) = ku+kw \in U+W.$$

* Theorem: The sum $U+W$ of the subspaces U and W of V is also a subspace of V.

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Definition. The vector space V is said to be the direct sum of its subspaces U and W denoted by

$$V = U \oplus W$$

if every vector $v \in V$ can be written in one and only one way as $v = u + w$, where $u \in U$ and $w \in W$.

Ex-1 In the vector space $\mathbb{R}^3$, let U be the xy-plane and let W be the yz plane; i.e $U = \{(a, b, 0); a, b \in \mathbb{R}\}$ and

$$W = \{(0, b, c); b, c \in \mathbb{R}\}$$

Then $\mathbb{R}^3 = U + W$, since every vector in $\mathbb{R}^3$ is the sum of a vector in U and a vector in W. However, $\mathbb{R}^3$ is not the direct sum of U and W, since such sums are not unique, for example

$$(3, 5, 7) = (3, 1, 0) + (0, 4, 7)$$

and $(3, 5, 7) = (3, -4, 0) + (0, 9, 7)$.

Ex-2. In $\mathbb{R}^3$, let U be the xy-plane and let W be the z pl...

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$U = \{(a,b,0) : a,b \in \mathbb{R}\}$ and $W = \{(0,0,c) : c \in \mathbb{R}\}$

Now, any vector $(a,b,c) \in \mathbb{R}^3$ can be written as the sum of a vector in U and a vector in W in one and only one way.

$$(a,b,c) = (a,b,0) + (0,0,c)$$

Hence $\mathbb{R}^3$ is the direct sum of U & W.

i.e, $\mathbb{R}^3 = U \oplus W$.

<u>Theorem</u>. The vector space $\mathbb{R}^3$, let U be the xy-plane

<u>Theorem</u>. The vector space V is the direct sum of its subspaces U and W if and only if

(i) $V = U + W$

(ii) $U \cap W = \{0\}$.

<u>Proof</u> Suppose $V = U \oplus W$. Then any $v \in V$ can be uniquely written in the form $v = u + w$, where $u \in U$ and $w \in W$. Thus in particular $V = U + W$.

Now suppose that $v \in U \cap W$.

Then (i), $v = v + 0$, where $v \in U$, $0 \in W$

and $v = 0 + v$, where $0 \in U$, $v \in W$

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since such a sum for v must be unique, so $v=0$. Accordingly, $U \cap W = \{0\}$

Conversely, suppose $V = U + W$ and $U \cap W = \{0\}$. Let $v \in V$. Since $V = U + W$, ~~Since $v \in V$~~ so there exist $u \in U$ and $w \in W$ such that $v = u + w$. We have to show that such a sum is unique.

Suppose that $v = u' + w'$, where $u' \in U$, $w' \in W$. ~~Then~~ $u + w = u' + w'$ and so $u - u' = w' - w$. But $u - u' \in U$ and $w' - w \in W$. Hence by $U \cap W = \{0\}$,

$u - u' = 0$ and $w' - w = \emptyset$

ie $u = u'$ and $w' = w$

Thus such a sum for $v \in V$ is unique and hence $V = U \oplus W$.

* Let U and W be the subspace of $\mathbb{R}^3$ defined by

$U = \{(a, b, c) : a = b = c\}$ and

$W = \{(0, b, c) : b, c \in \mathbb{R}\}$.

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show that $\mathbb{R}^3 = U \oplus W$.

Sol^n: First we have to show that $U \cap W = \{0\}$,

for $v = (a, b, c) \in U \cap W$ implies

$(a, b, c) \in U$ and $(a, b, c) \in W$.

Hence $a = b = c$ and $a = 0$.

ie $a = b = c = 0$

ie $v = (0, 0, 0)$

$\therefore U \cap W = \{0\}$.

We also claim that $\mathbb{R}^3 = U + W$.

for if $v = (a, b, c) \in \mathbb{R}^3$, then

$v = (a, b, c) = (a, a, a) + (0, b-a, c-a)$

where $(a, a, a) \in U$ and $(0, b-a, c-a) \in W$

Hence the conditions $U \cap W = \{0\}$

and $\mathbb{R}^3 = U + W$ implies

$\mathbb{R}^3 = U \oplus W$.

* Let V be the vector space of n-square matrices over a field $\mathbb{R}$. Let U and W be the subspaces of symmetric and antisymmetric matrices respectively, show that $V = U \oplus W$.

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Sol^n: First suppose that $V = U + W$.

Let A be any arbitrary n-square matrix. Then $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$.

We claim that $\frac{1}{2}(A + A') \in U$

and $\frac{1}{2}(A - A') \in W$.

for $\left(\frac{1}{2}(A + A')\right)' = \frac{1}{2}(A' + (A')') = \frac{1}{2}(A + A')$

ie $\frac{1}{2}(A + A')$ is symmetric

and $\left(\frac{1}{2}(A - A')\right)' = \frac{1}{2}(A - A')'$

$= \frac{1}{2}(A' - (A')') = -\frac{1}{2}(A - A')$

ie $\frac{1}{2}(A - A')$ is antisymmetric

We next show that $U \cap W = \{0\}$.

Suppose $M \in U \cap W$. Then $M \in U$ and $M \in W$. Thus $M' = M$ and $M' = -M$ which implies $M = -M$ ie $M = 0$. Hence $U \cap W = \{0\}$.

Therefore $V = U \oplus W$.