Matrices

Definition

Let K be an arbitrary field. A rectangular array of the form
$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ - & - & - & - \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$
where the a_{ij} are scalars in ~~of~~ K, is called a matrix over K, or simply a matrix if K is implicit.
The above matrix is also denoted by (a_{ij}), $i=1,2,\ldots,m$, $j=1,2,\ldots,n$. or simply by (a_{ij}).
The m horizontal n-tuples
$$(a_{11}, a_{12}, \ldots, a_{1n}), (a_{21}, a_{22}, \ldots, a_{2n}), \ldots, (a_{m1}, a_{m2}, \ldots, a_{mn})$$
are the rows of the matrix and the n vertical m-tuples
$$\begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \ldots \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$
are its columns.

The element a_{ij} called the ij-component, appears in the ith row and the jth column.

A matrix with m rows and n columns is called an m by n matrix or $m \times n$ matrix. The pair of numbers (m,n) is called its size or shape. $\begin{pmatrix} 1 & -2 & 3 \\ 0 & 2 & -1 \end{pmatrix}$ is a 2×3 matrix.

Matrix is an entity by itself and determinant is a single number.

Square Matrix. A matrix in which the number of rows is equal to the number of columns, is called a square matrix.

Ex: $\begin{bmatrix} 1 & 2 & 3 \\ 3 & -1 & 0 \\ 0 & 5 & 9 \end{bmatrix}_{3\times 3}$.

Thus $m\times n$ matrix A will be a square matrix if $m=n$. and it will be called a square matrix of order n.

In a square matrix, the elements $a_{11}, a_{22}, \ldots, a_{33}$ are called its diagonal elements.

The sum of the diagonal elements of a square matrix A is called the trace of A.

Ex: Let $A=\begin{pmatrix} 2 & 3 & 5 \\ 1 & 9 & 0 \\ 3 & 2 & 4 \end{pmatrix}$, then trace of $A = 2+9+4=15$.

Zero or Null Matrix. If all the elements of an $m\times n$ matrix are zero, then the matrix is called a zero matrix or null matrix and it is denoted by O. Ex. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ is the 3×2 null matrix.

Diagonal matrix. A square matrix A is said to be diagonal matrix if its all non-diagonal elements be zeros. Thus the matrix $A=(a_{ij})$ will be a diagonal matrix if $a_{ij}=0$ for $i\neq j$.

Ex: $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Scalar matrix: A square matrix, all of whose non-diagonal elements are zero and also all the diagonal elements are equal, is called a scalar matrix.

Thus the matrix $A = (a_{ij})_{n \times n}$ will be a scalar matrix if $a_{ij} = 0$ for all $i \neq j$ and $a_{ij} = k$ for all $i = j$.

Ex: $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Identity matrix or Unit matrix.

A square matrix, all of whose non-diagonal elements are zero and also all the diagonal elements are unity ~~equal~~ is called an identity matrix or a unit matrix. Thus the matrix $A = (a_{ij})_{n \times n}$ will be a unit matrix if $a_{ij} = 0$ for all $i \neq j$ and $a_{ij} = 1$ for all $i = j$.

Ex: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3 \times 3}$. It is denoted by I_n.

- **Upper triangular matrix.** If all a_{ij} for $i > j$ be zero in a square matrix, then it is called an upper triangular matrix. Thus the matrix $A = (a_{ij})_{n \times n}$ is an upper triangular matrix if $a_{ij} = 0$ for $i > j$.

ie $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$.

• Lower triangular matrix. If all a_{ij} for $i<j$ be zero in a square matrix, then it is called a lower triangular matrix. Thus the matrix $A = (a_{ij})_{n \times n}$ will be a lower triangular matrix if $a_{ij} = 0$ for all $i<j$. ie

$$A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ - & - & & - \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

• Row matrix. A matrix which has only one row is called a row matrix. $A = (1 \; 2 \; 3)$

• Column matrix. A matrix which has only one column is called a column matrix.

Ex: $A = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$.

Equal Matrix. Two matrices $A = (a_{ij})$ and $B = (b_{ij})$ are said to be equal if and only if they have the same order and each element of one is equal to the corresponding element of the other. ie Iff $a_{ij} = b_{ij}$, $i = 1, 2, \ldots, m$
$j = 1, 2, \ldots, n$.

Thus, two matrices are equal iff one is a duplicate of the other.

• Conformable matrices. Two matrices $A = (a_{ij})$ and $B = (b_{ij})$ are said to be conformable if each have the same number of rows and columns as the other. i.e If they have the same dimension or order.

Ex: $A = \begin{pmatrix} 1 & 2 \\ 5 & 9 \end{pmatrix}$ & $B = \begin{pmatrix} 3 & 5 \\ 2 & 6 \end{pmatrix}$ are conformable matrices.

Sums of Matrices. If $A = (a_{ij})$ and $B = (b_{ij})$ are two $m \times n$ matrices, their sum (difference), denoted by $A \pm B$, is defined as the $m \times n$ matrix $C = (c_{ij})$, where each element of C is the sum (difference) of the corresponding elements of A and B. Thus $A + B = (a_{ij} + b_{ij})$.

Ex: If $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ & $B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}$

then $A + B = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{pmatrix}$.

$C = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{pmatrix}$.

Law of Matrix Addition. If $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ are three matrices each of order $m \times n$, then the following law of algebra hold:

(i) Commutative law: $A + B = B + A$

(ii) Associative law: $A + (B + C) = (A + B) + C$

(iii) Scalar multiplication law: $KA = AK$. $K\cancel{(A+B) = KA + KB}$, k is a sca

(iv) Distributive law: ① $K(A+B) = KA + KB$ $\cancel{(A+B)K}$

② $(A+B)K = AK + BK$.

Proof. (i) Commutative law:

Since A and B are conformable matrices of order $m \times n$, then the matrices $A+B$ and $B+A$ also of the same order or dimensions. Thus the matrices $A+B$ and $B+A$ are defined and are conformable. Then we have

$$(i\text{-}j)\text{th element of } A+B = a_{ij} + b_{ij} = b_{ij} + a_{ij}$$
$$= (i\text{-}j)\text{th element of } B+A.$$

Hence $A+B = B+A$.

Similarly we can prove Associative law ie
$$A+(B+C) = (A+B)+C.$$

(iii) Scalar multiplication. If $A = (a_{ij})$ is $m \times n$ matrix and k be any scalar, real or complex then the product of k and A, written as kA is defined by

$$kA = \begin{pmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \hline ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{pmatrix} = (ka_{ij}) = (a_{ij}k)$$
$$= Ak.$$

ie $kA = Ak$. Hence proved.

(iv) Distributive law: (i) $k(A+B) = kA + kB$

(ii) $(B+C)k = Bk + Ck$.

proof. We have
$$(i\text{-}j)\text{th element of } k(A+B) = k(a_{ij} + b_{ij})$$
$$= ka_{ij} + kb_{ij} =$$
$$= k \text{ times of } (i\text{-}j)\text{th element of } A \text{ and } k(i\text{-}j)\text{th}$$
$$= kA + kB.$$

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$(i\text{-}j)$th element of $(B+C)k = (b_{ij}+c_{ij})k$.

$= b_{ij}k + c_{ij}k$.

$= (i\text{-}j)$th element of B and $(c_{ij})k$ th element of C

$= Bk + Ck$.

Hence $(B+C)k = Bk + Ck$.

Associative law. $A+(B+C) = (A+B)+C$

Let A, B and C ~~are~~ be conformable matrices of the dimension $m \times n$. Then the matrices A, $B+C$, $A+B$ and C are also of the same dimensions.

Thus the matrix additions $A+(B+C)$ and $(A+B)+C$ are defined and these are conformable. ~~Hence~~

Now we have,

$(i\text{-}j)$th element of $A+(B+C) = a_{ij} + (b_{ij}+c_{ij})$

$= (a_{ij}+b_{ij}) + c_{ij}$

$= (i\text{-}j)$th element of $(A+B)+C$.

Hence $A+(B+C) = (A+B)+C$.

Product of Two Matrices.

Two matrices $A = (a_{ij})$ and $B = (b_{ij})$ are said to be conformable for the product AB if the number of columns of A is equal to the number of rows of B.

Ex. If $A = (a_{ij})_{2\times 3}$ and $B = (b_{ij})_{4\times 3}$ then AB does not exist.

* If A and B are square matrices of the same order say n, then both the products AB and BA are defined and each of these matrices is square of order n.

Ex:- If $A = \begin{pmatrix} 2 & 3 & 1 \\ & & \end{pmatrix}$

* Matrix Multiplication is, in general, not commutative. i.e $AB \neq BA$, in general.

* If A is a square matrix of order n and I_n is an identity matrix of order n, then

$AI_n = I_nA = A$.

* Associative law for matrices multiplication.

i.e $A(BC) = (AB)C$.

__Proof.__ Let $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ be matrices of order $m \times n$, $n \times p$ and $p \times q$ respectively, and hence $AB = (u_{ij})$, $BC = (v_{ij})$ are $m \times p$ and $n \times q$ matrices respectively, where

$$u_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \quad \text{——— (i)}$$

$$v_{ij} = \sum_{k=1}^{p} b_{ik} c_{kj} \quad \text{——— (ii)}$$

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Also we have

$(AB)C = (w_{ij})$, say and $A(BC) = (x_{ij})$, say

are matrices, each of which is $m \times q$ order,

where $w_{ij} = \sum_{r=1}^{p} u_{ir} c_{rj}$ ——— (3)

and $x_{ij} = \sum_{k=1}^{n} a_{ik} v_{kj}$ ——— (4)

We need to prove that $w_{ij} = x_{ij}$.

Now, $w_{ij} = \sum_{r=1}^{p} u_{ir} c_{rj} = \sum_{r=1}^{p} \left(\sum_{k=1}^{n} a_{ik} b_{kr} \right) c_{rj}$, from (1).

$= \sum_{k=1}^{n} a_{ik} \sum_{r=1}^{p} (b_{kr} c_{rj})$

$= \sum_{k=1}^{n} a_{ik} v_{kj} = x_{ij}$

Hence $A(BC) = (AB)C$.

* Multiplication of matrices is distributive with respect to matrix addition.

i.e, $A(B+C) = AB + AC$.

Proof: Let $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ be three matrices of order $m \times n$, $n \times p$ and $n \times p$ respectively. Then we shall prove that $A(B+C) = AB + AC$.

According to the assumption we have $A(B+C) = (d_{ij})$, $AB = (e_{ij})$ and $AC = (f_{ij})$ are ~~$m \times n$~~ each of order $m \times p$,

where $d_{ij} = \sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj})$, $e_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$,

$f_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$, Now $(i\text{-}j)$th element of $A(B+C) = \sum_{k=1}^{n} a_{ik}(b_{kj}+c_{kj})$

$$= \sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj} \quad \text{— (I)}$$

Also $(i\text{-}j)$th element of $AB = \sum_{k=1}^{n} a_{ik}b_{kj}$ and — (II)

$(i\text{-}j)$ th element of $AC = \sum_{k=1}^{n} a_{ik}c_{kj}$ — (III)

Hence from (I), (II) & (III), $(i\text{-}j)$th element of $A(B+C) = \sum_{k=1}^{n}$

= $(i\text{-}j)$ th element of AB + $(i\text{-}j)$ th element of AC

Hence $A(B+C) = AB + AC$.

* For any matrix A, the product AA is defined Iff A is a square matrix. Then we write $AA = A^2$.
Also we have $A^2 \cdot A = (AA)\cdot A = AAA = A^3$.
In general $A.A.A \ldots A$ (m factors) $= A^m$.

* If A and B are two matrices such that AB and A+B both defined, then prove that A and B are square matrices.

Proof. Let A be an $m \times n$ matrix. Since A+B is defined ie A and B are conformable for to addition, so B must be an $m \times n$ matrix.

Again, since AB is defined i.e. A and B are conformable to multiplication and hence the number of columns in A must be equal to the number of rows in B ie $m = n$.

Hence A and B are $m \times m$ matrices ie they are square matrices.

Definition If A is a square matrix of order n, then $A^1 = A$ and $A^{k+1} = A^k . A$, for each positive integer k.

Theorem. If A is a square matrix of order n, then for positive integral values of p and q,

(i) $A^p . A^q = A^{p+q}$

(ii) $(A^p)^q = A^{pq}$.

Proof. By the above definition, we have

$$A^p . A^1 = A^{p+1}$$

Thus (i) is true for $q = 1$ and for all values of p. The result will be established by mathematical induction. Let (i) be true for a fixed values of $q = r$ (say) and all values of p. Then we have $A^p . A^r = A^{p+r}$

Also $A^p . A^{r+1} = A^p (A^r . A^1) = (A^p . A^r) . A^1 = A^{p+r} . A^1 = A^{(p+r)+1}$

$= A^{p+(r+1)}$

Thus if (i) holds for $q = r$ and all values of p, then it also holds for $q = r+1$ & all values of p. Hence the result follows by mathematical induction.

(ii) $(A^p)^q = A^p . A^p . A^p . \ldots A^p$ (q times)

$= A^{p+p+ \cdots +p}$

$= A^{pq}$.

* If A is a square matrix of order n and O be a null matrix of the same order, then $AO = OA = O$.

* Two matrices $A = (a_{ij})$ and $B = (b_{ij})$ are said to be commutative if $AB = BA$.

If $A = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$, then A & B are commutative i.e. $AB = BA$.

* Prove that the ~~only~~ matrix which is commutative for matrix multiplication with a diagonal matrix with distinct diagonal elements ~~are~~ is diagonal matrix.

Sol^n. Let A be a diagonal matrix of order n with distinct diagonal elements,

i.e. $A = \begin{pmatrix} a_{11} & 0, & \cdots & ,0 \\ 0 & a_{22}, & \cdots & 0 \\ - & - & - & - \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$

such that $a_{ii} \neq a_{jj}$, for $i \neq j$.

Let $B = (b_{ij})$ be any matrix ~~of order n~~ which is commutes with A. ie $AB = BA$. Thus B must be a square matrix of order n.

Now $AB = BA$ If and only if

$$a_{ii} b_{ij} = b_{ij} a_{jj}, \quad i \neq j$$

$$\Rightarrow b_{ij}(a_{ii} - a_{jj}) = 0, \quad i \neq j$$

Since $a_{ii} \neq a_{jj}$, for $i \neq j$, then we have $b_{ij} = 0$, for $i \neq j$ and hence all the non-diagonal elements of B are zero. ~~That~~ Therefore B is a diagonal matrix.

* If A and B are two matrices such that AB and A+B are defined, then prove that A and B are square matrices.

Sol^n Let A be an $m \times n$ matrix. Since $A+B$ is defined i.e. A and B are conformable to addition, so B must be also an $m \times n$ matrix.

Again AB is defined i.e. A and B are conformable to multiplication and hence the number of columns in A must be equal to the number of rows in B i.e. $n = m$. Hence A and B are $m \times m$ matrices i.e square matrices.

Definition. A square matrix A is called periodic if $A^{k+1} = A$, where k is a positive integer.

If k is the least positive integer for which $A^{k+1} = A$, then A is said to ~~be~~ have period k.

Definition. A square matrix A is called idempotent if $A^2 = A$. Let I be ~~a idempotent mat~~ an identity matrix for order n, then $I_n^2 = I_n$.

If A and B are idempotent matrices, then show that AB is idempotent if A and B commute.

Soln Since A and B are idempotent, then we have $A^2 = A$ and $B^2 = B$. — (i) Also since A and B are commutative, then $AB = BA$. — (ii)

Now $(AB)^2 = (AB)(AB) = A(BA)B$ by associative law

$= A(AB)B$ by the above relation (ii),

$= (AA)(BB)$

$= A^2B^2$

$= AB$ by (i)

Hence AB is idempotent.

* If A is an idempotent and $A+B=I$, then B is idempotent and $AB=0=BA$.

Soln If A is an idempotent, then we have $A^2=A$.

Also we have $B=I-A$. Now $B^2=B\cdot B=(I-A)(I-A)$
$=I^2-IA-AI+AA=I-A-A+A^2=I-A-A+A=I-A$
$=B$. Thus B is an idempotent.

Again,
$$AB=A(I-A)=AI-AA=A-A^2=A-A=0$$
and
$$BA=(I-A)A=IA-AA=A-A^2=A-A=0$$

* Show that If A and B are matrices of order n and such that $AB=A$ and $BA=B$, then A and B are idempotent matrices.

Soln Since $AB=A$ and $BA=B$, then we have
$$ABA=(AB)A=AA=A^2 \quad \text{---(i)}$$
also
$$ABA=A(BA)=AB=A \quad \text{---(ii)}$$
Hence from (i) & (ii), A is idempotent.

Again,
$$BAB=B(AB)=BA=B \quad \&$$
$$BAB=(BA)B=BB=B^2$$
& hence B is idempotent.

* Show that the matrix $A=\begin{pmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{pmatrix}$ is idempotent

* Show that the matrix $A=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is involutary matrix

* Show that " " $A=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is a nilpotent matrix.

* $A=\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ & $B=\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, find AB. $A=\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $B=\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$

Nilpotent Matrix: A square matrix A is called nilpotent matrix if ~~order~~ $A^m = 0$, a null matrix, where m is a positive integer. If however, m is the least positive integer such that the above condition holds, then A is said to be nilpotent matrix of index m.

Ex: If $A = \begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix}$, then A is a nilpotent matrix of index 2.

Since $A^2 = AA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$

Involutory Matrix: A square matrix A is called involutory matrix if $A^2 = I$ i.e. $(A+I)(A-I) = 0$. Obviously, identity matrix is involutory.

Ex: Show that the matrix $A = \begin{pmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{pmatrix}$ is involutory.

Transpose of a matrix: The matrix obtained from a given matrix A by interchanging the rows and columns is called the transpose of A and is denoted by A' or A^t. Thus if $A = (a_{ij})$ is an $m \times n$ matrix, then the $n \times m$ matrix $B = (b_{ij})$ where $b_{ij} = a_{ji}$ is called the transpose of A.

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}_{3\times 2}$; $A' = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}_{2\times 3}$

Theorem. If the matrices A and B are conformable for the product AB, then the matrices A' and B' are also conformable for the product $B'A'$ and $(AB)' = B'A'$.

Proof. Let $A = (a_{ij})$ be an $m \times n$ matrix, ~~and~~ ~~B~~ $B = (b_{ij})$ be $n \times p$ matrix. Then we have

$A' = (a_{ij})' = (a_{ji})$ is ~~an~~ $n \times m$ matrix.

and $B' = (b_{ij})' = (b_{ji})$ is $p \times n$ matrix. Thus A' and B' is conformable for the product $B'A'$.

~~We~~ Also we have ~~Thus~~ AB is $m \times p$ matrix, and hence

$(AB)'$ is $p \times m$ matrix. But $B'A'$ is also $p \times m$ matrix. Thus the matrices $(AB)'$ and $B'A'$ are same dimensions. ~~and hence A' and B' are conformable for the product B'A'~~.

Now, let $AB = (c_{ij})$, where $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

$\therefore$ (i-j)th element of $(AB)' = \cancel{(c_{ij})'} = c_{ji}$

$$\cancel{= (c_{ji})} = \sum_{k=1}^{n} a_{jk} b_{ki}$$

$$= \sum_{k=1}^{n} a'_{kj} b'_{ik}$$

$$= \sum_{k=1}^{n} b'_{ik} a'_{kj}$$

~~Again (i-j)th element of $B'A' = \sum_{k=1}^{n} b'_{ik} a'_{kj}$~~

$$= \text{(i-j)th element of } B'A'.$$

Hence $(AB)' = B'A'$.

The above theorem we have

$$(ABC)' = (A(BC))' = (BC)'A' = C'B'A'.$$

In general

$$(A_1A_2 \cdots A_K)' = A_K' A_{K-1}' \cdots A_2' A_1'.$$

* If A and B are conformable matrices, then prove that (i) $(A')' = A$, (ii) $(A+B)' = A'+B'$ (iii) $(KA)' = KA'$, K is a scalar.

Proof. (i) Let $A = (a_{ij})$ be $m \times n$ matrix. Then

$A' = (a_{ij})' = (a_{ji})$ is $n \times m$ matrix.

Again, $(A')' = (a_{ji})' = (a_{ij})$ is $m \times n$ matrix.

Thus the matrices A and $(A')'$ are same dimensions. Therefore $(A')' = A$.

(ii) We shall prove that $(A+B)' = A'+B'$.

Let $A = (a_{ij})$ be $m \times n$ matrix and $B = (b_{ij})$ be $m \times n$ matrix, since A and B are conformable

Then $A+B = (a_{ij}+b_{ij})$

$\therefore (A+B)' = (a_{ij}+b_{ij})' = (a_{ji}+b_{ji})$

Again, $A' = (a_{ij})' = (a_{ji})$ be $n \times m$ matrix
$= (j-i)$th element of A',

and $B' = (b_{ij})' = (b_{ji})$ be $n \times m$ matrix
$= (j-i)$th element of B'.

Thus $(j-i)$th element of $(A+B)' = a_{ji}+b_{ji} = (j-i)$th element of A' and $(j-i)$th element of B'.

Thus $(A+B)' = A'+B'$.

(iii) Let $A = (a_{ij})$ be an $m \times n$ matrix.

$\therefore A' = (a_{ij})' = (a_{ji})$ be $n \times m$ matrix.

~~$\Rightarrow (j-i)$th element of~~

ie $(j-i)$th element of $A' = (a_{ji})$

~~Thus~~ $(kA)' = (ka_{ij})' = k(a_{ji}) = \cancel{k(j-i)} = kA'$.

ie $(kA)' = kA'$.

* If $A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 3 \\ 4 & 1 & 8 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 1 & 0 \\ 2 & -3 & 1 \\ 1 & 1 & -1 \end{pmatrix}$,

then verify $(AB)' = B'A'$.

* If $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ & $B = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}$, verify that $(AB)' = B'A'$.

Definition. A square matrix $A = (a_{ij})$ is called a symmetric matrix if $A' = A$ ie $a_{ji} = a_{ij}$ for all i & j.

Ex: Let $A = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$ $\therefore A' = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$

Hence A is a symmetric.

Definition. A square matrix $A = (a_{ij})$ is called a skew-symmetric or anti-symmetric if $A' = -A$. ie $a_{ji} = -a_{ij}$ for all i & j.

Ex: $A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$, $A' = \begin{pmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{pmatrix} = -A$.

In skew-symmetric matrix, for the diagonal elements, we have $a_{ii} = -a_{ii} \Rightarrow a_{ii} + a_{ii} = 0 \Rightarrow 2a_{ii} = 0 \Rightarrow a_{ii} = 0$. Hence all the diagonal elements of a skew symmetric matrix must be zero.

* If A and B are two symmetric matrices of the same order, then show that a necessary and sufficient condition for the matrix AB to be symmetric, is that $AB = BA$.

Proof. Since A and B are symmetric matrices, then $A' = A$ and $B' = B$. Suppose AB is symmetric. Hence $(AB)' = AB$. i.e $B'A' = AB$ i.e $BA = AB$.

Conversely, let $AB = BA$, then $(AB)' = B'A'$
$= BA$
$= AB$

hence AB is symmetric.

Theorem. Every square matrix can be uniquely expressed as the sum of a symmetric and a skew symmetric matrices.

Soln. Let A be a square matrix, then we can write $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$.

$$A = B + C \quad \text{—— (1)}$$

where $B = \frac{1}{2}(A + A')$
and $C = \frac{1}{2}(A - A')$.

Now, $B' = \left(\frac{1}{2}(A+A')\right)' = \frac{1}{2}(A+A')' = \frac{1}{2}(A'+(A')')$

$= \frac{1}{2}(A'+A) = \frac{1}{2}(A+A') = B$

Hence B is a symmetric matrix.

And $C' = \left(\frac{1}{2}(A-A')\right)' = \frac{1}{2}(A-A')' = \frac{1}{2}(A'-(A')')$

$= \frac{1}{2}(A'-A) = -\frac{1}{2}(A-A') = -C$

Ie $C' = -C$

Hence C is a skew symmetric matrix.

Hence from (1), we conclude that a square matrix can be expressed as the sum of a symmetric and a skew-symmetric matrices.

For uniqueness.

Let If possible $A = P + Q$ —— (2)

where P is symmetric and Q is skew symmetric matrices. Then we have $P' = P$ & $Q' = -Q$.

Now $A' = (P+Q)' = P' + Q' = P - Q$

Ie $A' = P - Q$ —— (3)

Then from (2) & (3), we have

$P = \frac{1}{2}(A+A') = B$

and $Q = \frac{1}{2}(A-A') = C$

Hence the representation (1) is unique.

Orthogonal Matrix. A square matrix A of order n is said to be orthogonal if $A'A = I_n = AA'$.

* Show that the matrix $\frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$ is orthogonal.

Theorem. If A and B are orthogonal matrices of order n, then the matrices AB and BA are also orthogonal matrices.

Proof. Since A and B are orthogonal matrices then we have

$$\left.\begin{aligned} A'A &= I_n = AA' \\ \text{and} \quad B'B &= I_n = BB' \end{aligned}\right\} \text{—— (1)}$$

We need to show that AB and BA are orthogonal matrices.

Now $(AB)'(AB) = (B'A')(AB) = B'(A'A)B = B'I_nB$
$= B'B = I_n$

and $(AB)(AB)' = (AB)(B'A') = (ABB')A' = AI_nA'$
$= AA' = I_n$

Hence AB is ~~a~~ orthogonal matrix.

Again, $(BA)'(BA) = A'B'BA = A'I_nA = A'A = I_n$
& $(BA)(BA)' = BAA'B' = BI_nB' = BB' = I_n$.

Hence BA is an orthogonal matrix.

Conjugate of a Matrix. If $A = (a_{ij})$ is a given matrix, then the matrix obtained on replacing all its elements by their corresponding complex conjugates, is called the conjugate of the matrix A, and it is denoted by $\bar{A} = (\bar{a}_{ij})$.

Ex: $A = \begin{pmatrix} 2+3i & 1+i \\ 3-2i & 2i \end{pmatrix}$ $\quad \bar{A} = \begin{pmatrix} 2-3i & 1-i \\ 3+2i & -2i \end{pmatrix}$.

Properties of a conjugate of the matrix.

(i) $(\bar{\bar{A}}) = A$ (ii) $\overline{A+B} = \bar{A} + \bar{B}$ (iii) $\overline{KA} = \bar{K}\bar{A}$, where K is complex scalar, $\overline{KA} = K\bar{A}$, where K is scalar. (iv) $\overline{AB} = \bar{A}.\bar{B}$.

Conjugate Transpose of a matrix. The conjugate of the transpose of a given matrix A, is called the conjugate transpose of A and is denoted by A^θ. Thus $A^\theta = (\bar{A})'$.

Clearly, (i) $A^\theta = (\bar{A})' = \overline{(A')}$, (ii) $(A^\theta)^\theta = A$

~~(iii) $(A+B)^\theta = A^\theta + B^\theta = (\overline{A+B})' = (\bar{A})' + (\bar{B})'$~~

(iii) $(A+B)^\theta = (\overline{A+B})' = (\bar{A}+\bar{B})' = (\bar{A})' + (\bar{B})'$

$= A^\theta + B^\theta$.

(iv) $(KA)^\theta = \bar{K}A^\theta$ (v) $(AB)^\theta = (\overline{AB})' = (\bar{A}.\bar{B})'$

$= (\bar{B})'(\bar{A})' = B^\theta . A^\theta$.

Unitary Matrix. A square matrix A of order n is said to be unitary if $A^\theta A = I_n = AA^\theta$.

Problem. If A and B are square unitary matrices of order n, then AB and BA are also unitary matrices.

Solution. Since A and B are unitary matrices, Then $A^\theta A = AA^\theta = I_n$, ——(i) $B^\theta B = I_n = BB^\theta$ ——(ii)

Now $(AB)^\theta.(AB) = (B^\theta A^\theta)(AB) = B^\theta(A^\theta A)B = B^\theta I_n B = B^\theta B = I_n$

$(AB)(AB)^\theta = (AB)(B^\theta A^\theta) = A(BB^\theta)A^\theta = AI_nA^\theta = AA^\theta = I_n$.

Hence AB is also unitary.

Similarly BA is also unitary.

Hermitian Matrix

A square matrix $A = (a_{ij})$ is said to be hermitian matrix if $(i\text{-}j)$th element of A is equal to the complex conjugate of its $(j\text{-}i)$th element.

Thus $A = (a_{ij})$ is hermitian if $A^\theta = A$ ie $a_{ij} = \bar{a}_{ji}$ for all i & j.

Theorem. The diagonal elements of a Hermitian matrix are necessary real.

Proof. Let $A = (a_{ij})$ be a Hermitian matrix. Then by definition $a_{ij} = \bar{a}_{ji}$, for all i & j.

for diagonal elements, $a_{ii} = \bar{a}_{ii}$.

Let $a_{ii} = a+ib$, then $\bar{a}_{ii} = a-ib$, and hence

$$a+ib = a-ib$$

$$\Rightarrow 2ib = 0 \Rightarrow b = 0$$

Thus $a_{ii} = a$, which is purely real. Hence the diagonal elements of a Hermitian matrix are necessary real.

* Show that $A = \begin{pmatrix} a & b-ic \\ b+ic & d \end{pmatrix}$ is a Hermitian Matrix.

Skew-Hermitian Matrix. A square matrix $A=(a_{ij})$ is a skew Hermitian matrix if $(i\text{-}j)$th element of A is equal to the negative of the $(j\text{-}i)$th element. Thus $A=(a_{ij})$ is skew Hermitian if $A^{\theta}=-A$, i.e.

$$a_{ij}=-\bar{a}_{ji} \text{ for all } i \,\&\, j.$$

Theorem. The diagonal elements of a skew-Hermitian matrix are either purely imaginary or zero.

Proof. Let $A=(a_{ij})$ be a skew Hermitian matrix, then by the definition, we have

$$a_{ij}=-\bar{a}_{ji} \text{ for all } i \text{ and } j.$$

~~Let a~~ for diagonal elements, we have

$$a_{ii}=-\bar{a}_{ii}.$$

Now, let $a_{ii}=\alpha+i\beta$, then $\bar{a}_{ii}=\alpha-i\beta$.

Then $\alpha+i\beta=-(\alpha-i\beta)$

$\Rightarrow \alpha+i\beta+\alpha-i\beta=0$

$\Rightarrow 2\alpha=0 \Rightarrow \alpha=0$

Hence, $a_{ii}=i\beta$, which is purely imaginary and can be zero if $\beta\neq0$. Hence the diagonal elements of a skew-Hermitian matrix are either purely imaginary or zero.

\# Show that $A=\begin{pmatrix}0 & 2+i\\ -2+i & 0\end{pmatrix}$ is a skew Hermitian.

$$B=\begin{pmatrix}-i & 1-2i\\ -1-2i & 0\end{pmatrix}$$

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Theorem. Every square matrix (with complex elements) can be uniquely expressed as the sum of a Hermitian and a skew-Hermitian matrices.

Proof. Let A be a square matrix (with complex elements). Then we can write

$$A = \frac{1}{2}(A + A^\theta) + \frac{1}{2}(A - A^\theta)$$

$$= B + C \quad \text{——— (i)}$$

where $B = \frac{1}{2}(A + A^\theta)$, & $C = \frac{1}{2}(A - A^\theta)$

Now, $B^\theta = \frac{1}{2}(A + A^\theta)^\theta = \frac{1}{2}(A^\theta + (A^\theta)^\theta) = \frac{1}{2}(A^\theta + A)$

$= \frac{1}{2}(A + A^\theta) = B$

Hence B is a Hermitian Matrix.

And $C^\theta = \left(\frac{1}{2}(A - A^\theta)\right)^\theta = \frac{1}{2}(A - A^\theta)^\theta = \frac{1}{2}(A^\theta - A)$

$= -\frac{1}{2}(A - A^\theta) = -C$

Hence C is a skew-Hermitian matrix.

For Uniqueness. Let if possible, $A = P + Q$ ——— (ii)

where P is a Hermitian and Q is a skew-Hermitian matrices, so that $P^\theta = P$ and $Q^\theta = -Q$.

Now $A^\theta = (P + Q)^\theta = P^\theta + Q^\theta = P - Q$

i.e. $A^\theta = P - Q$ ——— (iii)

Adding (ii) & (iii), we have $2P = A + A^\theta \Rightarrow P = \frac{1}{2}(A + A^\theta)$
$= B$

and Subtracting (iii) from (ii), we have

$2Q = A - A^\theta \Rightarrow Q = \frac{1}{2}(A - A^\theta) = C$

Hence established the uniqueness of (i).

* Show that every square matrix can be uniquely expressed as $P+iQ$, where P, Q are Hermitian.

Proof: Let A be a square matrix. Then we can write

$$A = \frac{1}{2}(A+A^\theta) + \frac{1}{2}(A-A^\theta)$$
$$= \frac{1}{2}(A+A^\theta) + i \cdot \frac{1}{2i}(A-A^\theta)$$

ie $A = P+iQ$ ——— (I).

where $P = \frac{1}{2}(A+A^\theta)$, $Q = \frac{1}{2i}(A-A^\theta)$

Now $P^\theta = \left(\frac{1}{2}(A+A^\theta)\right)^\theta = \frac{1}{2}(A+A^\theta)^\theta = \frac{1}{2}(A^\theta+(A^\theta)^\theta)$
$= \frac{1}{2}(A^\theta+A) = \frac{1}{2}(A+A^\theta) = P$

ie $P^\theta = P$, Hence P is a Hermitian.

And $Q^\theta = \left(\frac{1}{2i}(A-A^\theta)\right)^\theta = \frac{1}{\overline{2i}}(A-A^\theta)^\theta = \frac{1}{-2i}(A^\theta-(A^\theta)^\theta)$
$= \frac{-1}{2i}(A^\theta-A) = \frac{1}{2i}(A-A^\theta) = Q$

ie $Q^\theta = Q$, Hence Q is a ~~sk~~ Hermitian.

For Uniqueness, let if possible, $A = B+iC$ ——— (II)
where B and C are Hermitian matrices, so that $B^\theta = B$ and $C^\theta = C$.

Now $A^\theta = (B+iC)^\theta = B^\theta + \bar{i}C^\theta = B^\theta - iC^\theta = B - iC$

ie $A^\theta = B-iC$ ——— (III)

(II) + (III) $\Rightarrow$ $2B = A+A^\theta \Rightarrow B = \frac{1}{2}(A+A^\theta) = P$

(II) − (III) $\Rightarrow$ $2iC = A-A^\theta \Rightarrow C = \frac{1}{2i}(A-A^\theta) = Q$

~~Hence the uniques~~ established the uniqueness of (I),

* If A and B are Hermitian, then show that AB is Hermitian iff A and B commute.
* If A is a Hermitian, then show that iA is skew-Hermitian.
* If A is a skew-Hermitian, then show that iA is a Hermitian.

• Adjoint of a Matrix

Definition. A square matrix A is said to be singular if $|A| \neq 0$ and it is called non-singular If $|A| \neq 0$.

• **Adjoint of a Matrix.** Let $A=(a_{ij})$ be a square matrix of order n and A_{ij} be the cofactor of a_{ij} in the determinant $|A|$, then the adjoint of A is defined as the transpose of the matrix (A_{ij}) and is ~~denoted~~ written ~~by~~ as adj A.

* Find the adjoint of the matrix $A=\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{pmatrix}$.

• **Theorem.** If $A=(a_{ij})$ be a square matrix of order n, then $A(\operatorname{adj} A) = (\operatorname{adj} A)A = |A| I_n$.

Proof. Since $A=(a_{ij})$ is a square matrix of order n, then we have

$$A(\operatorname{adj} A) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} |A| & 0 & \cdots & 0 \\ 0 & |A| & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & |A| \end{pmatrix} = |A| \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = |A| I_n . \quad \text{——(i)}$$

Similarly, we have $(\text{adj} A) A = |A| I_n$. ——(ii)

from (i) & (ii), we have

$$(\text{adj} A) A = A (\text{adj} A) = |A| I_n .$$

** Def^n. Let $A = (a_{ij})$ be a square matrix of order n. If M_{ij} be the ~~$(n-1) \times (n-1)$~~ submatrix of order $(n-1)$ of the matrix $A = (a_{ij})$ obtained by removing the ith row and jth column, then the determinants $|M_{ij}|$ is defined as the minor of the element a_{ij} in the determinants $|A| = |a_{ij}|$ of order n.

The cofactor C_{ij} of the element a_{ij} in the determinants $|A| = |a_{ij}|$ is defined by

$$C_{ij} = (-1)^{i+j} |M_{ij}| .$$

Theorem. (1) If $A = (a_{ij})$ be a square matrix of order n, then $|A| = |a_{ij}| = \sum_{k=1}^{n} a_{ik} C_{ik}$, where C_{ij} is the cofactor of a_{ij}.

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Theorem. If C_{ij} is the cofactor of a_{ij} in the determinant $|A| = |a_{ij}|$ of order n, then

(i) the sum of the products of the elements of the ith row with the cofactors of the corresponding elements of the kth row is zero provided $i \neq k$.

(ii) Also the sum of the products of the elements of the jth column with the cofactors of the corresponding elements of the kth column is zero — provided $j \neq k$.

i.e. (i) $\sum_{j=1}^{n} a_{ij} C_{kj} = 0$ if $i \neq k$

(ii) $\sum_{j=1}^{n} a_{ij} C_{ik} = 0$ if $j \neq k$

Hence Theorem-1 & Theorem-2, we can write

$$\sum_{j=1}^{n} a_{ij} C_{kj} = \begin{cases} |A| & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

Theorem. If A is a non-singular matrix of order n, then $|\operatorname{adj} A| = |A|^{n-1}$.

Proof. We have, from the above theorem

$$A(\operatorname{adj} A) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & & & \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} |A| & 0 & \cdots & 0 \\ 0 & |A| & \cdots & 0 \\ 0 & 0 & \cdots & |A| \end{pmatrix}$$

Taking determinant on both sides, we have

$$|A(\text{adj}A)| = \begin{vmatrix} |A| & 0 & \cdots & 0 \\ 0 & |A| & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & |A| \end{vmatrix} = |A|^n \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ & & & \\ 0 & 0 & \cdots & 1 \end{vmatrix}$$

$|A||\text{adj}A| = |A|^n$, since A is a non-singular so $|A| \neq 0$.

$\therefore \Rightarrow |\text{adj}A| = \frac{|A|^n}{|A|} = |A|^{n-1}$.

~~Theorem.~~ Problem. If A is a non-singular matrix, then

$\cancel{|\text{adj}(\text{adj}A)|}$ $|\text{adj}(\text{adj}A)| = |A|^{(n-1)^2}$

Proof. $L.H.S = |\text{adj}(\text{adj}A)|$,

Taking $\text{adj}A = B$, then $L.H.S = |\text{adj}B|$

$= |B|^{n-1}$ by the above theorem.

$= ||\text{adj}(A)|^{n-1}$ putting the value of B.

$= (|A|^{n-1})^{n-1} = |A|^{(n-1)^2}$

Theorem. If A is a nonsingular matrix of order n, then

$\text{adj}(\text{adj}A) = |A|^{n-2}A$

Proof. We know that $(\text{adj}B)B = |B|I_n$

Taking $B = \text{adj}A$, then

$(\text{adj}(\text{adj}A))(\text{adj}A) = |\text{adj}A|I_n$

Multiplying both side by A, we have

$(\text{adj}(\text{adj}A)).(\text{adj}A)A = |\text{adj}A| I_n \cdot A$

$\Rightarrow (\text{adj}(\text{adj}A))|A| I_n = |\text{adj}A| A$

$\Rightarrow \text{adj}(\text{adj}A) I_n |A| = |A|^{n-1} \cdot A$

$\Rightarrow \text{adj}(\text{adj}A) = \dfrac{|A|^{n-1}}{|A|} \cdot A$, since A is non-singular.

$\therefore \text{adj}(\text{adj}A) = |A|^{n-2} \cdot A$

—x—

Q. Find the adjoint of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and verify the theorem $A.(\text{adj}A) = (\text{adj}A).A = |A| I_2$.

<u>Theorem</u>. If A and B are square matrices of order n, then $\text{adj}(AB) = \text{adj}B . \text{adj}A$.

<u>Proof</u>. We know that $A(\text{adj}A) = |A| I_n = (\text{adj}A).A$.

Hence $(AB)(\text{adj}AB) = |AB| I_n = (\text{adj}AB)(AB)$ —— (i)

~~Also~~ $AB(\text{adj}B . \text{adj}A) = A(B.\text{adj}B).\text{adj}A$

$= A|B| I_n \cdot \text{adj}A$

$= A|B| \text{adj}A$

$= A\,\text{adj}A . |B|$

$= |A| I_n |B|$

$= |A||B| I_n$

i.e. $AB(\text{adj}B.\text{adj}A) = |AB| I_n$ —— (ii)

and $(\text{adj}B.\text{adj}A)(AB) = \text{adj}B(\text{adj}A.A)B = \text{adj}B |A| I_n B$

$= \text{adj}B . B . |A| = |B| I_n |A| = |A||B| I_n = |AB| I_n$ —— (iii)

Hence from (i), (ii) & (iii), we have $\text{adj}(AB) = \text{adj}B . \text{adj}A$.

Theorem. (i) If A is a square matrix of order n. Then prove that $adj\, A' = (adj\, A)'$.

(ii) If A is a symmetric matrix, then prove that $adj\, A$ is also symmetric matrix.

Proof. Since A is a square matrix of order n, then each of the matrices $adj\, A'$ and $(adj\, A)'$ is also square matrix of order n.

Now, $(i\text{-}j)$th element of $(adj\, A)'$ = $(j\text{-}i)$th element of $adj\, A$

= cofactor of $(i\text{-}j)$th element in ~~of~~ $|A|$

= cofactor of $(j\text{-}i)$th element in ~~of~~ $|A'|$

= $(i\text{-}j)$th element of $adj\, A'$.

Hence $(adj\, A)' = adj\, A'$, —— (i)

(ii) Since A is a symmetric matrix, then $A' = A$, from (i), we have

$$(adj\, A)' = adj\, A' = adj\, A$$

Hence $adj\, A$ is symmetric matrix.

* If $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{pmatrix}$, find $A^2 - 2A + adj\, A$.

* ~~If A~~ Find the adjoint matrix of $A = \begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix}$ and verify the theorem $A \cdot (adj\, A) = (adj\, A) \cdot A = |A| I_n$.

Inverse of a matrix. Let $A = (a_{ij})$ be a square matrix of order n, then the matrix B, if it exists, is called the inverse of A if $AB = BA = I_n$ and it is denoted by $B = A^{-1}$. i.e. $AA^{-1} = A^{-1}A = I_n$.

A matrix $A = (a_{ij})$ having an inverse, is called an invertiable matrix.

Note. Since AB and BA are both defined and equal, then each of the matrices A and B must be a square matrix. Thus it is only a square matrix which can have an inverse. Non-square and rectangular matrix cannot ~~procees~~ possess an inverse.

However, it may be pointed out that every square matrix does not necessarily possess an inverse. It is only a non-singular matrix which always ~~proccesse~~ possess an inverse, which is seen later.

Theorem. If A is a non-singular matrix, then

(i) $|A^{-1}| = |A|^{-1}$ (ii) $A^{-1} = \dfrac{\operatorname{adj} A}{|A|}$.

Proof. Since A is a non-singular, then $|A| \neq 0$.

Then by definition, we have

$$AA^{-1} = A^{-1}A = I_n$$

Taking determinant on both sides, we have

$$|AA^{-1}| = |A^{-1}A| = |I_n|$$

$\Rightarrow |A||A^{-1}| = |A^{-1}||A| = 1$

ie $|A^{-1}| = \frac{1}{|A|}$ ie $|A^{-1}| = |A|^{-1}$.

Hence proved (i).

(iii) We know that

$$A.(adj A) = (adj A).A = |A| I_n$$

Dividing both sides by $|A|$, we have

$$\frac{A.(adj A)}{|A|} = \frac{(adj A).A}{|A|} = \frac{|A| I_n}{|A|}$$

ie $A.\frac{adj A}{|A|} = \frac{adj A}{|A|}.A = I_n$

Hence by definition $\frac{adj A}{|A|}$ is the inverse of A ie $A^{-1} = \frac{adj A}{|A|}$,

* Compute the inverse of the matrix

$$A = \begin{pmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{pmatrix}.$$

Theorem. The necessary and sufficient condition that a square matrix may possess an inverse is that it be non-singular.

Proof. Necessary condition. If B is the inverse of a square matrix A of order n, then by definition,

$$AB = BA = I_n. \text{ Then we have}$$

$$|AB| = |BA| = |I_n|$$

$$\Rightarrow |A||B| = 1, \text{ hence } |A| \neq 0 \text{ i.e. } A \text{ is nonsingular.}$$

Sufficient condition. If A is non-singular, then $|A| \neq 0$. ~~Let there~~ be another matrix B defined by $B = \dfrac{\operatorname{adj} A}{|A|}$.

Then $AB = A \cdot \dfrac{\operatorname{adj} A}{|A|} = \dfrac{A \cdot \operatorname{adj} A}{|A|} = \dfrac{|A| I_n}{|A|}$ — (v)

But we know that if A is a ~~square ma~~ ~~n×n~~ square matrix of order n, then

$$A \cdot \operatorname{adj} A = \operatorname{adj} A \cdot A = |A| I_n.$$

Hence from (v), $AB = \dfrac{|A| I_n}{|A|} = I_n$.

Similarly, we have $BA = \dfrac{\operatorname{adj} A}{|A|} \cdot A = \dfrac{|A| I_n}{|A|} = I_n$

Thus we have $AB = BA = I_n$, i.e. B is the inverse of A.

Theorem. The inverse of a matrix, if it exists is unique.

Proof. Let A be a invertible matrix of order n

If possible, let the matrices B and C be its distinct inverse such that

$AB = BA = I_n$ —— ① and $AC = CA = I_n$ —— ②

Premultiplying ① by C and postmultiplying ② by B, we have

$CAB = \cancel{CBA}\ CI_n = C$ &

$\cdot \cancel{ACB} =\ CAB = I_nB' = B.$

Hence $B = C$. ie the inverse is unique.

Theorem. If A and B are non-singular matrices of the same order, then AB is also nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof. Since A and B are non-singular matrices, then $|A| \neq 0$ and $|B| \neq 0$ and hence $|AB| = |A||B| \neq 0$. Thus AB is non-singular, and we have AB has an inverse.

Now, $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n$

and $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$.

Thus $(AB)(B^{-1}A^{-1}) = I_n = (B^{-1}A^{-1})(AB)$

$\therefore (AB)^{-1} = B^{-1}A^{-1}$. Proved.

In general, if $A_1, A_2, \ldots, A_k$ are non-singular matrices of the same order, then

$(A_1A_2 \cdots A_k)^{-1} = A_k^{-1}A_{k-1}^{-1} \cdots A_2^{-1}A_1^{-1}$.

* If A is a non-singular matrix and k any positive integer, then $(A^k)^{-1} = (A^{-1})^k$.

Proof: We have $(A \cdot A \cdots A \text{ k times})^{-1} = A^{-1} \cdot A^{-1} \cdots$ k times $= (A^{-1})^k$

* If A is non-singular matrix, then (i) $(A^{-1})^{-1} = A$ (ii) $(A')^{-1} = (A^{-1})'$

Proof Since A is a non-singular matrix, then A has an inverse i.e A^{-1}. By definition, $AA^{-1} = A^{-1}A = I_n$ —— (1)

This implies that A is an inverse of A^{-1}. Thus $(A^{-1})^{-1} = A$.

(ii) Since A is non-singular, then $|A| \neq 0$.

Also we have $|A'| = |A| \neq 0$ i.e A' is also non-singular and hence A' has an inverse

By definition $AA^{-1} = A^{-1}A = I_n$.

Taking transpose on both sides, we have

$$(AA^{-1})' = (A^{-1}A)' = I_n'$$

$$\Rightarrow \cancel{A'(}(A^{-1})'A' = A'(A^{-1})' = I_n$$

i.e. A' is the inverse of $(A^{-1})'$, i.e $(A')^{-1} = (A^{-1})'$.

* Find the adjoint and inverse of each of the following

(v) $\begin{pmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{pmatrix}$ (vi) $\begin{pmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}$

* Show that the matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is its own inverse.

* If $A = \begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$, show that $A^3 = A^{-1}$.

Rank of a Matrix.

Defn. The number r is said to be the rank of the matrix A if (i) every minor of A of order $(r+1)$ is zero and (ii) A has at least one minor of order r which does not vanish.

OR. The number r is said to be the rank of the matrix A if (i) every square sub-matrix of A of order $(r+1)$ is singular and (ii) there is at least one square sub-matrix of order r which is non-singular. A zero matrix is said to have rank 0.

Ex: The rank of $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{pmatrix}$ is ~~2~~ 2, since

$\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1 \neq 0$, while $|A| = 0$.

Note: The product of two or more non-singular n-square matrices is non-singular; the product of two or more n-square matrices is singular if at least one of the matrices is singular.

~~Equivalent Matrix.~~
~~Elementary transformations.~~

(i) The rank of a matrix A is also denoted by $\rho(A)$ i.e. $r = \rho(A)$

(ii) All equivalent matrices have the same rank.

(iii) If a matrix does not possess any minor of order $(r+1)$, then $\rho(A) \leq r$.

(iv) If at least one minor of order r of the matrix A is not equal to zero, then $\rho(A) \geq r$.

(v) If every minor of order ρ of a matrix A is zero then every minor of order higher than ρ is definitely zero.

Elementary Transformation,

The following operations, called elementary transformations, on a matrix do not change either its order or its rank.

(i) The interchange of the ith and jth rows, R_{ij} (column) (C_{ij})

~~The interchange of the~~

(ii) The multiplication of every element of the ith row (ith column) by a non-zero scalar k, denoted by $R_i(k)$ ($C_i(k)$).

(iii) The addition to the elements of the ith row (ith column) of k, a scalar, times the corresponding elements of the jth row (jth column) denoted by $R_{ij}(k)$

The transformations R are called elementary row transformations; the transformations C are called elementary column transformations.

Equivalent Matrices. Two matrices A and B are called equivalent, denoted by $A \sim B$, if one can be obtained ~~by~~ from the other by a sequence of elementary transformations.

Equivalent matrices have the same order and the same rank.

$$A = \begin{pmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & -7 \end{pmatrix}$$

Normal form of a Matrix

Every non-zero matrix A of order $m \times n$ can be reduced by application of elementary row and column operations into equivalent matrix of one of the following forms :—

(i) $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ (ii) $\begin{pmatrix} I_r \\ 0 \end{pmatrix}$ (iii) $(I_r, 0)$, (iv) $\{I_r\}$,

where I_r is $r \times r$ identity matrix and 0 is null matrix of any order. These four forms are called Normal or canonical form of A.

Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{pmatrix} \sim \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}$. Hence rank of A is 2.

* Let A be a matrix of order $m \times n$ and r be its rank, then (i) if $m > r$ and $n = r$, then the normal form becomes $\begin{pmatrix} I_r \\ 0 \end{pmatrix}$.

(ii) If $m = r$ and $n > r$, then the normal form becomes $[I_r, 0]$.

(iii) If $m = n = r$, then it becomes $\{I_r\}$

(iv) If $m > r$, $n > r$, then it becomes $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$.

Echelon Form of a Matrix

A matrix is said to be in Echelon form if

(i) all the non-zero rows, if any, precede the zero rows,

(ii) the number of zero preceding the first non-zero element in a row is less than the number of such zero in the succeeding row.

(iii) the first nonzero element in a row is unity, then it is in the Echelon form.

$$A = \begin{pmatrix} 1 & 3 & 4 & 5 & 6 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$